CONVERGENCE OF THE TWO-DIMENSIONAL RANDOM
WALK LOOP SOUP CLUSTERS TO CLE

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ABSTRACT. We consider the random walk loop soup on the discrete half-plane
$H := \mathbb{Z} \times \mathbb{N}$ corresponding to a central charge $c \in (0, 1]$. We look at the clusters
discrete loops and show that the scaling limit of the outer boundaries of
outermost clusters is the $CLE_\kappa$ loop ensemble, with the same relation between
$\kappa$ and $c$ as in the continuum Brownian setting.

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1. Introduction

One can naturally associate to a wide class of Markov processes an infinite measure on time-parametrized loops. Roughly speaking, given a locally compact second-countable space $S$, a Markov process $(X_t)_{0 \leq t \leq \zeta}$ on $S$, defined up to a killing time $\zeta \in (0, +\infty)$, with transition densities $p_t(x, y)$ with respect some $\sigma$-finite measure $m(dy)$ and with bridge probability measure $\mathbb{P}_{x, t}^\gamma(\cdot)$, where the bridges are conditioned on $\zeta > t$, the loop measure associated to $X$ is

$$
\mu(\cdot) = \int_{x \in S} \int_{t > 0} \mathbb{P}^\gamma_{x, t}(\cdot) p_t(x, x) \frac{dt}{t} m(dx)
$$

See [3] for the precise setting and definition. A Poisson ensemble of Markov loops conditioned on $S$ is second-countable space measure on time-parametrized loops. Roughly speaking, given a locally compact chain of loops $S$.

We will consider loop soups in the following settings:

- On the continuum half-plane $\mathbb{H} = \{ \Im(z) > 0 \} \subset \mathbb{C}$ we will consider the loop soups associated to the Brownian motion on $\mathbb{H}$ killed at hitting the boundary $\mathbb{R}$ and denote them $\mathcal{L}^\alpha_n$. These two-dimensional Brownian loop soups were introduced by Lawler and Werner in [5] and used by Sheffield and Werner in [9] to give a construction of Conformal Loop Ensembles (CLE). In (1.1) we use the same normalisation of the loop measure as in [3], [9], [2] or [4]. However, contrary to what is claimed in [9], the intensity parameter $\alpha$ does not equal the so called central charge $c$. Actually

$$
\alpha = \frac{c}{2}
$$

The $\frac{c}{2}$ factor was pointed out by Werner in a private communication. It also appears in the Lawler’s work [6].

- On the discrete rescaled half-plane $H_n := \left( \frac{1}{n} \mathbb{Z} \right) \times \left( \frac{1}{n} \mathbb{N} \right)$ we will consider the loop soups associated to the nearest neighbours Markov jump process with uniform transition rates and killed at hitting the boundary $\frac{1}{n} \mathbb{Z} \times \{0\}$. We will denote these loop soups $\mathcal{L}^\alpha_n$. The loop soups associated to Markov jump processes on more general electrical networks were studied by Le Jan in [2]. If one forgets the parametrisation by continuous time and the ”loops” that visit only one vertex, these are exactly the random walk loop soups studied by Lawler and Trujillo Ferreras in [4].

- We will use the metric graphs $\tilde{H}_n$ associated to $\mathbb{H}$: each ”discrete” edge $\{(\frac{i}{n}, \frac{j}{n}), (\frac{i+1}{n}, \frac{j}{n})\}$ or $\{(\frac{i}{n}, \frac{j}{n}), (\frac{i}{n}, \frac{j+1}{n})\}$ is replaced by a continuous line of length $\frac{1}{n}$. Let $(B^0_t)_{0 \leq t < \zeta_n}$ be the Brownian motion on $\tilde{H}_n$ killed at reaching the boundary, that is to say the vertices $\frac{1}{n} \mathbb{Z} \times \{0\}$ and all the lines joining $(\frac{i}{n}, 0)$ to $(\frac{i}{n}, \frac{1}{2} n)$. $(B^0_t)_{0 \leq t < \zeta_n}$ converges in law to the Brownian motion on the half-plane $\mathbb{H}$ killed at reaching $\mathbb{R}$. We will denote by $\mathcal{L}^\alpha_n$ the loop soups associated to $(B^0_t)_{0 \leq t < \zeta_n}$. The loop soups on metric graphs were first considered in [3]. We will use the metric graphs because at intensity parameter $\alpha = \frac{c}{2}$ the probability that two points belong to the same cluster of loops can be explicitly expressed using a metric graph Gaussian free field.
Indeed the clusters of loops are then exactly the sign clusters of the Gaussian free field (§).

The discrete loops \( L^H_n \) can be deterministically recovered from the metric graph loops \( \tilde{L}^H_n \). The first are the trace on the vertices of the latter. In particular each cluster of \( L^H_n \) is contained in a cluster of \( \tilde{L}^H_n \), but the clusters of \( \tilde{L}^H_n \) may be strictly larger (§).

c = 1 is known to be the critical central charge for the Brownian loop percolation on \( \mathbb{H} \) (or any other simply connected proper subset of \( \mathbb{C} \)). This means that the critical intensity parameter is \( \alpha = \frac{1}{2} \). For \( \alpha > \frac{1}{2} \) \( L^H_n \) has only one cluster everywhere dense in \( \mathbb{H} \). If \( \alpha \in (0, \frac{1}{2}] \) there are infinitely many clusters and each is bounded (§). It was shown in § that for discrete or metric graph Brownian loop soups on \( \mathbb{H} \), respectively \( \tilde{H}_n \), there are no unbounded clusters of loops if \( \alpha \in (0, \frac{1}{2}] \). In all these settings, for \( \alpha \in (0, \frac{1}{2}] \), we will consider the collection of outer boundaries of outermost clusters (not surrounded by any other cluster) and denote it \( F_{\text{ext}}(L^S_n) \), where \( S = \mathbb{H}, H_n \) or \( \tilde{H}_n \). Next we give the formal definition of \( F_{\text{ext}}(L^S_\alpha) \). We consider the set of all points in \( \mathbb{H} \) visited by a loop in \( L^S_\alpha \) and take its complementary in \( \mathbb{H} \). This complementary has only one unbounded connected component. We take the boundary in \( \mathbb{H} \) of this connected component (by definition it does not intersect \( \mathbb{R} \)). The element of \( F_{\text{ext}}(L^S_\alpha) \) are the connected components of this boundary. We will call the elements of \( F_{\text{ext}}(L^S_\alpha) \) contours. The contours are two by two disjoint and non nested. See next picture for a representation of \( F_{\text{ext}}(L^\alpha_h) \).

The contours in \( F_{\text{ext}}(L^S_\alpha), \alpha \in (0, \frac{1}{2}] \), are non self-intersecting loops, and are equal in law to a Conformal Loop Ensemble \( CLE_{\kappa}, \kappa \in (\frac{8}{3}, 4] \) (§). The relation between \( \alpha \) and \( \kappa \) is given by

\[
2\alpha = c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}
\]

We will denote by \( \kappa(\alpha) \) the value of \( \kappa \) corresponding to a particular intensity parameter \( \alpha \).

We will show that both \( F_{\text{ext}}(L^H_\alpha) \) and \( F_{\text{ext}}(L^\tilde{H}_n) \) converge in law to \( F_{\text{ext}}(L^\alpha_h) \) for \( \alpha \in (0, \frac{1}{2}] \). Observe that \( \kappa(\frac{1}{2}) = 4 \) and \( F_{\text{ext}}(L^\alpha_{\tilde{H}_n}) \) and \( CLE_4 \) are both related to the Gaussian free field. \( F_{\text{ext}}(L^\alpha_{\tilde{H}_n}) \) is the collection of outer boundaries of outermost sign clusters of a GFF on the metric graph \( \tilde{H}_n \) (§) and the \( CLE_4 \) loops.
are in some sense zero level lines of the continuum GFF on \( \mathbb{H} \) with zero boundary conditions on \( \mathbb{R} \) (\[10\]).

Next we define the notion of convergence we will use. \( d_H \) will be Hausdorff distance on the compact subsets of \( \mathbb{H} \). We introduce the distance \( d_H \) on finite sets of compact subsets of \( \mathbb{H} \):

\[
d_H(K, K') = \begin{cases} 
+\infty & \text{if } |K| \neq |K'| \\
\min_{\sigma \in B_{ij}(K, K')} \max_{K \in K} d_H(K, \sigma(K)) & \text{otherwise}
\end{cases}
\]

Given \( z \in \mathbb{H} \) we will denote by

\[ F_{ext}(\mathcal{L}_\alpha^n)(z) \]

the contour of \( F_{ext}(\mathcal{L}_\alpha^n) \) that contains or surrounds \( z \), whenever it exists. It exists a.s. in the case \( S = \mathbb{H} \). Given \( z_1, \ldots, z_j \in \mathbb{H} \) we will denote

\[ F_{ext}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] := \{ F_{ext}(\mathcal{L}_\alpha^n)(z_i) | 1 \leq i \leq j \} \]

By the convergence in law of \( F_{ext}(\mathcal{L}_\alpha^n) \) and \( F_{ext}(\mathcal{L}_\alpha^n) \) to \( F_{ext}(\mathcal{L}_\alpha^\mathbb{H}) \) we mean that for any \( z_1, \ldots, z_j \in \mathbb{H} \) \( F_{ext}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \) and \( F_{ext}(\mathcal{L}_\alpha^\mathbb{H})[z_1, \ldots, z_j] \) converge in law to \( F_{ext}(\mathcal{L}_\alpha^\mathbb{H})[z_1, \ldots, z_j] \) for the distance \( d_H \).

In the article \[1\] Van de Brug, Camia and Lis consider clusters of rescaled two-dimensional random walk loops that are not too small. Given \( T > 0 \) let \( \mathcal{L}_\alpha^{n,T} \) be the subset of \( \mathcal{L}_\alpha^n \) consisting of random walk loops that do at least \( T \) jumps. In \[1\] it is almost shown that for \( \theta \in (\frac{16}{15}, 2) \) and \( \alpha \in (0, \frac{1}{2}) \), \( F_{ext}(\mathcal{L}_\alpha^{n,T}) \) converge in law to a \( CLE_{\kappa(\alpha)} \) process in the sense described previously. The result uses the approximation of "not too small" Brownian loops by "not too small" random walk loops obtained by Lawler and Trujillo Ferreras in \[4\]. However the authors in \[1\] consider the loop soups only on bounded domains and their lattice approximations. We will fill the small gap to extend their result to the half-plane. Observe that in \[1\] the authors use the same normalisation of the measure on loops as we but do the widespread mistake to consider that the intensity parameter of the loop soups equals the central charge.

From above considerations one deduce that the limiting (in law) loops of \( F_{ext}(\mathcal{L}_\alpha^n) \) and \( \text{a fortiori} \) of \( F_{ext}(\mathcal{L}_\alpha^{n,T}) \) are at least as big as \( CLE_{\kappa(\alpha)} \) loops. We have a "lower bound”. To conclude the convergence we need an "upper bound”. We will construct such an "upper bound” for \( F_{ext}(\mathcal{L}_\alpha^n) \), deduce the convergence to \( CLE_\alpha \) of \( F_{ext}(\mathcal{L}_\alpha^n) \) and \( F_{ext}(\mathcal{L}_\alpha^\mathbb{H}) \). Then from this we will deduce the desired convergences for \( \alpha \in (0, \frac{1}{2}) \). Next we explain how the "upper bound" will be constructed.

We will concentrate on the case \( \alpha = \frac{1}{2} \). We will additionally introduce two Poisson point processes of excursions on \( \mathbb{H} \) and on \( \mathbb{H} \). First we consider \( \mathbb{H} \). Let \( x \in \frac{1}{n} \mathbb{Z} \times \{0\} \). Let \( \nu_{exc}^{\mathbb{H}}(x \rightarrow (-\infty, 0]) \) be the measure on excursions of the metric graph Brownian motion \( B^{\mathbb{H}} \) from \( x \) to a point in \( \frac{1}{n} \mathbb{Z} \times \{0\} \). It is defined as follows:

Let \( \mathcal{P}_{x+ic}^{\mathbb{H}}(\cdot, \mathbb{P}_{\zeta_n}^{\mathbb{H}} \in \frac{1}{n} \mathbb{Z} \times \{0\}) \) be the law of a sample path of \( B^{\mathbb{H}} \), started at \( x+ic \), restricted to the event \( \mathcal{B}_{\zeta_n}^{\mathbb{H}} \in \frac{1}{n} \mathbb{Z} \times \{0\} \) (we do not condition and the total mass is \( <1 \)). Then

\[
\nu_{exc}^{\mathbb{H}}(x \rightarrow (-\infty, 0]) = \lim_{\varepsilon \rightarrow 0} \mathbb{P}_{x+ic}^{\mathbb{H}} \left( \cdot, \mathbb{P}_{\zeta_n}^{\mathbb{H}} \in \frac{1}{n} \mathbb{Z} \times \{0\} \right)
\]

Let \( q \in (1, +\infty) \) and \( x \in ((\frac{1}{n} \mathbb{Z}) \cap [1, q]) \times \{0\} \). We will similarly denote by \( \nu_{exc}^{\mathbb{H}}(x \rightarrow [1, q]) \) the measure on excursions from \( x \) to \( ((\frac{1}{n} \mathbb{Z}) \cap [1, q]) \times \{0\} \). Let

\[
(1.3) \quad \nu_{exc}^{\mathbb{H}}((-\infty, 0]) := \frac{8\pi}{n} \sum_{x \in \frac{1}{n} \mathbb{Z} \times \{0\}} \nu_{exc}^{\mathbb{H}}(x \rightarrow (-\infty, 0])
\]
\begin{equation}
\tilde{\nu}_{\text{exc}}^\mathbb{H}([1, q]) := \frac{8\pi}{n} \sum_{x \in ((\frac{1}{n} \mathbb{Z}) \cap [1, q]) \times \{0\}} \nu_{\text{exc}}^\mathbb{H}(x \to [1, q])
\end{equation}

$\nu_{\text{exc}}^\mathbb{H}((-\infty, 0])$ is a measure on excursions from and to $\frac{1}{n} \mathbb{Z} \times \{0\}$. $\nu_{\text{exc}}^\mathbb{H}([1, q])$ is a measure on excursion from and to $((\frac{1}{n} \mathbb{Z}) \cap [1, q]) \times \{0\}$. Both measures are invariant under time reversal.

As $n$ tends to infinity, $\nu_{\text{exc}}^\mathbb{H}((-\infty, 0])$ and $\nu_{\text{exc}}^\mathbb{H}([1, q])$ have limits which are measures on Brownian excursions in $\mathbb{H}$, from and to $(-\infty, 0] \times \{0\}$ respectively. We will denote them by $\nu_{\text{exc}}^\mathbb{H}((-\infty, 0])$ respectively $\nu_{\text{exc}}^\mathbb{H}([1, q])$. For $x, y \in \mathbb{R}$, let $\mathbb{P}^\mathbb{H}_{x,y}()$ be the probability measure on Brownian excursions from $x$ to $y$ in $\mathbb{H}$. Then

\begin{align*}
\nu_{\text{exc}}^\mathbb{H}((-\infty, 0]) &= 2 \int_{-\infty}^{0} \int_{-\infty}^{0} \mathbb{P}^\mathbb{H}_{x,y} \frac{dxdy}{(y-x)^2} \\
\nu_{\text{exc}}^\mathbb{H}([1, q]) &= 2 \int_{1}^{q} \int_{1}^{q} \mathbb{P}^\mathbb{H}_{x,y} \frac{dxdy}{(y-x)^2}
\end{align*}

In general, given $a < b \in \mathbb{R}$, we will use the notation

$$
\nu_{\text{exc}}^\mathbb{H}([a, b]) := 2 \int_{a}^{b} \int_{a}^{b} \mathbb{P}^\mathbb{H}_{x,y} \frac{dxdy}{(y-x)^2}
$$

We will consider on $\tilde{\mathbb{H}}_n$ three independent Poisson point processes:

- a loop soup $\mathcal{L}_{\tilde{\mathbb{H}}_n}$
- a Poisson point process of excursions of intensity $\nu_{\text{exc}}^\mathbb{H}((-\infty, 0])$, $u > 0$, denoted by $\mathcal{E}_u^\mathbb{H}((-\infty, 0])$
- a Poisson point process of excursions of intensity $\nu_{\text{exc}}^\mathbb{H}([1, q])$, $v > 0$, denoted by $\mathcal{E}_v^\mathbb{H}([1, q])$

We will consider the following event: either an excursion from $\mathcal{E}_u^\mathbb{H}((-\infty, 0])$ intersects an excursion from $\mathcal{E}_v^\mathbb{H}([1, q])$ or an excursion from $\mathcal{E}_u^\mathbb{H}((-\infty, 0])$ and from $\mathcal{E}_v^\mathbb{H}([1, q])$ intersect a common contour of $\mathcal{L}_{\tilde{\mathbb{H}}_n}$. We will denote by $p_{u,v}^\mathbb{H}(\cdot, q)$ the probability of this event. The second condition of intersecting a common cluster is equivalent to intersecting a common contour in $\mathcal{F}_{\text{exc}}(\mathcal{L}_{\tilde{\mathbb{H}}_n})$.

Similarly we will consider on $\mathbb{H}$ three independent Poisson point processes:

- a loop soup $\mathcal{L}_\alpha^\mathbb{H}$, $\alpha \in (0, \frac{1}{2}]$
- a Poisson point process of excursions of intensity $\nu_{\text{exc}}^\mathbb{H}((-\infty, 0])$, $u > 0$, denoted by $\mathcal{E}_u^\mathbb{H}((-\infty, 0])$
- a Poisson point process of excursions of intensity $\nu_{\text{exc}}^\mathbb{H}([1, q])$, $v > 0$, denoted by $\mathcal{E}_v^\mathbb{H}([1, q])$

Then we will consider the event when either an excursion from $\mathcal{E}_u^\mathbb{H}((-\infty, 0])$ intersects an excursions from $\mathcal{E}_v^\mathbb{H}([1, q])$ or an excursion from $\mathcal{E}_u^\mathbb{H}((-\infty, 0])$ and one from $\mathcal{E}_v^\mathbb{H}([1, q])$ intersect a common cluster of $\mathcal{L}_\alpha^\mathbb{H}$. This event is schematically represented on the figure 2. We denote by $p_{\alpha,u,v}^\mathbb{H}(q)$ its probability.
In the section 2 we will compute $\tilde{p}_{\frac{1}{2},u,v}^n(q)$ using the duality with the Gaussian free field, and compute its limit as $n$ tends to 0. In section 3, for an arbitrary value of $v$ and a particular value $u_0(\alpha)$ of $u$ (depending on $\alpha$) we will establish a differential equation in $q$ for $1 - \tilde{p}_{\frac{1}{2},u_0,v}^\infty(q)$. Using this we will show that

$$\lim_{n\to+\infty} \tilde{p}_{\frac{1}{2},u_0,v}^n(q) = \tilde{p}_{\frac{1}{2},u_0,v}^\infty(q)$$

This will be our "upper bound". In the section 4 we will prove the convergences to CLE.

2. Computation of $p_{\frac{1}{2},u,v}^n(q)$

Let $G = (V,E)$ be a connected undirected graph. $V$ is countable and each vertex is of finite degree. Each edge $\{x, y\}$ is endowed with a positive conductance $C(x, y) > 0$. We also consider a metric graph $\tilde{G}$ associated to $G$ where each edge $\{x, y\}$ is replaced by a continuous line of length

$$r(x, y) = \frac{1}{2}C(x, y)^{-1}$$

Let $B_{\tilde{G}}$ be the Brownian motion on the metric graph $\tilde{G}$. Let $F$ be a subset of $V$. Let $\zeta_F$ be the first time $B_{\tilde{G}}$ hits $F$. Let measure $\mu_{\tilde{G},\times F}$ be the measure on loops associated to $(B_{\tilde{G}}^t)_{0 \leq t < \zeta_F}$ the Brownian motion killed at reaching $F$. It is defined according to (1.1). See [8] for details. Let $\mathcal{L}_{\alpha,\times F}$ be the Poisson point process of intensity $\alpha \mu_{\tilde{G},\times F}$.

$B_{\tilde{G}}$ has a time-space continuous family of local times $L_z^\xi(B_{\tilde{G}})$. The Green’s function of the killed Brownian motion $(B_{\tilde{G}}^t)_{0 \leq t < \zeta_F}$ satisfies

$$G_{\tilde{G},\times F}(z, z') = G_{\tilde{G},\times F}(z', z) = \mathbb{E}_z \left[ \int_0^{\zeta_F} L_{z^t}^{\xi}(B_{\tilde{G}}) \right]$$

Just as $B_{\tilde{G}}$, a loop $\gamma \in \mathcal{L}_{\alpha,\times F}$ has a family of continuous local times $L_z^\xi(\gamma)$. We will denote by $t_{\gamma}$ the total life-time of the loop $\gamma$. The occupation field $\hat{\mathcal{L}}_{\alpha}$ is defined as

$$\hat{\mathcal{L}}_{\alpha} = \sum_{\gamma \in \mathcal{L}_{\alpha,\times F}} L_{t_{\gamma}}^\xi(\gamma)$$
It is a continuous field. The clusters of \( \mathcal{L}^{\tilde{G}, \times F}_\alpha \) are delimited by the zero set of the occupation field.

At intensity parameter \( \alpha = \frac{1}{2} \) the occupation field \((\tilde{C}_z)_{z \in \tilde{G} \setminus F}\) is related to the Gaussian free field \((\phi_z)_{z \in \tilde{G} \setminus F}\) with zero mean and covariance function \(G^{\tilde{G}, \times F}\). Given \( z \in \tilde{G} \setminus F \) such that \( \tilde{C}_z > 0 \) we denote by \( C_\frac{1}{2}(z) \) the cluster of \( \mathcal{L}^{\tilde{G}, \times F}_\frac{1}{2} \) that contains \( z \). We introduce a countable family \((\sigma(C\frac{1}{2}_z(z)))_{z \in \tilde{G} \setminus F}\) of i.i.d. random variables, independent of \( \mathcal{L}^{\tilde{G}, \times F}_\frac{1}{2} \) conditionally on the clusters, which equal \(-1\) or \(1\) with equal probability. There is an equality in law (see [3]):

\[
(\phi_z)_{z \in \tilde{G} \setminus F} \overset{(d)}{=} \left( \sigma(C\frac{1}{2}_z(z)) \right) \sqrt{\frac{2\tilde{C}_z}{\pi}} \quad \forall z \in \tilde{G} \setminus F.
\]

Let \( x, y \in V \setminus F \). Let \( C^{eq}(x, y), \chi^{eq}(x,y)(x), \chi^{eq}(x,y)(y) \) be the quantities defined by

\[
\begin{pmatrix}
G^{\tilde{G}, \times F}(x, x) & G^{\tilde{G}, \times F}(x, y) \\
G^{\tilde{G}, \times F}(x, y) & G^{\tilde{G}, \times F}(y, y)
\end{pmatrix}^{-1} = \begin{pmatrix}
\chi^{eq}(x,y)(x) + C^{eq}(x, y) & -C^{eq}(x, y) \\
-C^{eq}(x, y) & \chi^{eq}(x,y)(y) - C^{eq}(x, y)
\end{pmatrix}
\]

Then \( C^{eq}(x, y) > 0, \chi^{eq}(x,y)(x), \chi^{eq}(x,y)(y) \geq 0, (\chi^{eq}(x,y)(x), \chi^{eq}(x,y)(y)) \neq (0, 0) \). \( C^{eq}(x, y) \) is the equivalent conductance between \( x \) and \( y \) given that all points in \( F \) have the same (electrical) potential.

Let \( N_{\frac{1}{2}}(x, y) \) the number of loops in \( \mathcal{L}^{\tilde{G}, \times F}_\frac{1}{2} \) that visit both \( x \) and \( y \).

**Lemma 2.1.** Let \( u, v > 0 \) and \( x, y \in V \setminus F \).

\[
P \left( \mathcal{C}_\frac{1}{2}(x) \neq \mathcal{C}_\frac{1}{2}(y) \middle| \tilde{C}_\frac{1}{2} = u, \tilde{C}_\frac{1}{2} = v, N_{\frac{1}{2}}(x, y) = 0 \right) = e^{-2C^{eq}(x,y)\sqrt{uv}}
\]

**Proof.** If \( N_{\frac{1}{2}}(x, y) > 0 \) then \( \mathcal{C}_\frac{1}{2}(x) = \mathcal{C}_\frac{1}{2}(y) \). Thus

\[
P \left( \mathcal{C}_\frac{1}{2}(x) \neq \mathcal{C}_\frac{1}{2}(y) \middle| \tilde{C}_\frac{1}{2} = u, \tilde{C}_\frac{1}{2} = v, N_{\frac{1}{2}}(x, y) = 0 \right)
\]

The value of the denominator

\[
P \left( N_{\frac{1}{2}}(x, y) = 0 \middle| \tilde{C}_\frac{1}{2} = u, \tilde{C}_\frac{1}{2} = v \right)
\]

depends only on \( u, v \) and on \( G^{\tilde{G}, \times F}(x, x), G^{\tilde{G}, \times F}(y, y), G^{\tilde{G}, \times F}(x, y) \) (or equivalently \( C^{eq}(x, y), \chi^{eq}(x,y)(x), \chi^{eq}(x,y)(y) \)). This a general property of the loop soups (see [2], especially chapter 7).

As for the numerator, it can be computed using the duality with the Gaussian free field (2.2):

\[
P \left( \mathcal{C}_\frac{1}{2}(x) \neq \mathcal{C}_\frac{1}{2}(y) \middle| \tilde{C}_\frac{1}{2} = u, \tilde{C}_\frac{1}{2} = v \right)
\]

\[
= 1 - \mathbb{E} \left[ \text{sgn}(\phi_x)\text{sgn}(\phi_y)||\phi_x|| = \sqrt{2u}, ||\phi_y|| = \sqrt{2v} \right]
\]

\[
= \frac{e^{-2C^{eq}(x,y)\sqrt{uv}}}{\cosh(2C^{eq}(x,y)\sqrt{uv})}
\]
It follows that the probability (2.3) that we want to compute only depends on \( u, v \) and on \( C_{eq}(x, y), \chi_{eq}(x, y) \). Thus it is the same if we replace \( \tilde{G} \) by the interval
\[
I = \left( -\frac{1}{2} C_{eq}(x, y)^{-1}, \frac{1}{2} C_{eq}(x, y)^{-1} \right)
\]
the Brownian motion on \( \tilde{G} \) by the Brownian motion on \( I \) killed at endpoints, and the points \( x \) and \( y \) by 0 and 1 respectively. According to the computation made in [8], we get (2.3).

By the way we also get that
\[
P\left( N_\frac{1}{2}(x, y) = 0 \left| \tilde{\mathcal{L}}_x = u, \tilde{\mathcal{L}}_y = v \right. \right) = \cosh(2 C_{eq}(x, y) \sqrt{uv})^{-1}
\]

\[
\square
\]

In [2], chapter 7, there is a combinatorial representation of \( C_{eq}(x, y) \). Given \( z \in V \), we will denote
\[
\lambda(z) := \sum_{z' \in V, z' \sim z} C(z, z')
\]
where the sum is over the neighbours of \( z \) in the (discrete) graph \( G \). Then
\[
C_{eq}(x, y) = \lambda(x) \sum_{j \geq 1} \sum_{(z_0, \ldots, z_j) \in (V \setminus F)^{j+1}} \prod_{i=1}^{j} \frac{C(z_{i-1}, z_i)}{\lambda(z_{i-1})}
\]
The sum is over all the discrete nearest neighbour paths joining \( x \) to \( y \), that avoid \( F \) and only visit \( x \) and \( y \) at endpoints. The above equality can be rewritten as
\[
C_{eq}(x, y) = \sum_{z \in V, z \sim x} C(x, z) P_z(\tilde{B} \tilde{G} \text{ hits y before F or x})
\]

(2.5)

Next we return to the metric graph half-plane \( \tilde{H}_n \). Let \( a > 0 \). Let \( \tilde{G}_{n,a}(q) \) be the metric graph obtained from \( \tilde{H}_n \) by identifying the following vertices:
- All the vertices in \( \left( \left( \frac{1}{n} \mathbb{Z} \right) \cap [-a, 0] \right) \times \{0\} \) are identified into a single vertex \( \triangleleft_{n}(a) \).
- All the vertices in \( \left( \left( \frac{1}{n} \mathbb{Z} \right) \cap [1, q] \right) \times \{0\} \) are identified into a single vertex \( \triangleright_{n}(q) \).

See the following picture.

![Fig.3: Illustration of points identified into \(<_{n}(a)\) and \(\triangleright_{n}(q)\).](image)

As the length of the line joining \( \left( \frac{1}{n}, \frac{1}{n} \right) \) to \( \left( \frac{1+n}{n}, \frac{1}{n} \right) \) or \( \left( \frac{1}{n}, \frac{1}{n} \right) \) to \( \left( \frac{1}{n}, \frac{1+n}{n} \right) \) is \( \frac{1}{n} \), the corresponding conductance is according (2.1) equal to \( \frac{1}{2} \). Let \( C_{eq,n,a}^\| (q) \) be the equivalent conductance between \( \triangleleft_{n}(a) \) and \( \triangleright_{n}(q) \) when all the points in \( \left( \frac{1}{n} \mathbb{Z} \times \{0\} \right) \)
other than those identified to \(<_n(a)\) or \(>_n(q)\) have the same electrical potential. According to (2.5):

\[
C_{n,a}^{eq}(q) = \frac{n}{2} \sum_{i=1}^{[nq]} \mathbb{P}_{\left(\frac{i}{n}, \frac{1}{n}\right)} \left(B_{n, i}^{\mathbb{H}} \text{ hits } \left(\frac{1}{n}\right) \times \{0\} \text{ in } [-a, 0] \times \{0\}\right)
\]

As \(a\) tends to infinity, \(C_{n,a}^{eq}(q)\) increases and converges to

\[
(2.6) \quad C_n^{eq}(q) = \frac{n}{2} \sum_{i=1}^{[nq]} \mathbb{P}_{\left(\frac{i}{n}, \frac{1}{n}\right)} \left(B_{n, i}^{\mathbb{H}} \text{ hits } \left(\frac{1}{n}\right) \times \{0\} \text{ in } (-\infty, 0] \times \{0\}\right)
\]

**Lemma 2.2.** For all \(n \in \mathbb{N}^*\) and \(x_0 > 0\), \(C_n^{eq}(q) < +\infty\). Moreover

\[
\lim_{n \to +\infty} \frac{1}{n} C_n^{eq}(q) = \frac{1}{8\pi} \log(q)
\]

**Proof.** Let \(G^\mathbb{H}(\cdot, \cdot)\) be the Green’s function of the simple random walk on \(\mathbb{H} = \mathbb{Z} \times \mathbb{N}\) killed at hitting \(\mathbb{Z} \times \{0\}\). Let \(i, j \in \mathbb{Z}\). Then

\[
\mathbb{P}_{\left(\frac{i}{n}, \frac{1}{n}\right)} \left(B_{n, i}^{\mathbb{H}} \text{ hits } \left(\frac{1}{n}\right) \times \{0\} \text{ in } \left(\frac{j}{n}, 0\right)\right) = \frac{1}{4} G^\mathbb{H}(i, 1), (j, 1) = \frac{1}{4} G^\mathbb{H}(0, 1), (j - i, 1)
\]

Indeed to go from \((\frac{i}{n}, \frac{1}{n})\) to \((\frac{j}{n}, 0)\) the moving particle needs to reach \((\frac{j}{n}, \frac{1}{n})\), possibly make excursions from and to this point without hitting \((\frac{j-1}{n} \mathbb{Z}) \times \{0\}\), and then with probability \(\frac{1}{4}\) transition to \((\frac{j}{n}, 0)\). Replacing in (2.6) we get that

\[
C_n^{eq}(q) = \frac{n}{8} \sum_{i=1}^{[nq]} \sum_{j=0}^{+\infty} G^\mathbb{H}(0, 1), (i + j, 1)
\]

According to the asymptotic expansion given in [7], section 8.1.1,

\[
G^\mathbb{H}(0, 1), (j, 1) = \frac{1}{\pi j^2} + O \left(\frac{1}{j^3}\right)
\]

This means that \(C_n^{eq}(q) < +\infty\) and that

\[
\frac{1}{n} C_n^{eq}(q) = \frac{1}{8\pi} \sum_{i=1}^{[nq]} \sum_{j=0}^{+\infty} \frac{1}{(i + j)^2} + O \left(\sum_{i=1}^{[nq]} \sum_{j=0}^{+\infty} \frac{1}{(i + j)^3}\right)
\]

\[
= \frac{1}{8\pi} \sum_{i=1}^{[nq]} \frac{1}{i} + O \left(\sum_{i=1}^{[nq]} \frac{1}{i^2}\right)
\]

\[
= \frac{1}{8\pi} \log(q) + O \left(\frac{1}{n}\right)
\]

\(\square\)

Let \(\mathbb{H} \mathbb{Z}_{\nu_{\mathbb{H} \mathbb{Z}}([-a, 0])}\) be the measure on excursions \(\mathbb{H} \mathbb{Z}_{\nu_{\mathbb{H} \mathbb{Z}}([-\infty, 0])}\) restricted to the excursions from and to \([-a, 0] \times \{0\}\). Let \(\mathbb{L}_{n, q, a}^{\mathbb{H}}(q)\) be the loop soup associated to the Brownian motion on the metric graph \(\mathbb{G}_{n, a}(q)\), killed at hitting \((\frac{1}{n} \mathbb{Z}) \times \{0\}\) outside the points identified to \(<_n(a)\) or \(>_n(q)\). Let \((\mathbb{L}_{n, a, q, a}^z)^\mathbb{G}_{\nu_{\mathbb{G}_{n, a}(q)}}(q)\) be the occupation field of \(\mathbb{L}_{n, a, q, a}^{\mathbb{H}}(q)\). Let \(\mathcal{N}_n(<_n(a),>_n(q))\) be the number of loops in \(\mathbb{L}_{n, a, q, a}^{\mathbb{H}}(q)\) joining \(<_n(a)\) to \(>_n(q)\).
Lemma 2.3. Let \( a, \alpha, u, v > 0 \). We consider \( \mathcal{L}_{\alpha}^{\tilde{G}_{n,a}(q)} \) conditioned on \( \tilde{\mathcal{L}}_{n,a,q,\alpha}^{\tilde{G}_{n,a}(q)} = u, \tilde{\mathcal{L}}_{n,a,q,\alpha}^{\tilde{G}_{n,a}(q)} = v \) and \( \mathcal{N}_\alpha(\langle u(a), \rangle_n, q) = 0 \). Then \( \mathcal{L}_{\alpha}^{\tilde{G}_{n,a}(q)} \) consists of three independent families of loops:

- The loops that visit neither \( \langle u(a) \rangle \) nor \( \langle v(a) \rangle \). These are the same as the loops in \( \mathcal{L}_{\alpha}^{\tilde{G}_{n,a}(q)} \).
- The loops that visit \( \langle u(a) \rangle \). The excursions these loops make outside \( \langle u(a) \rangle \) form a Poisson point process of intensity \( \frac{1}{q} u \nu^{\mathbb{R}}_{\text{exc}}([-a, 0]) \).
- The loops that visit \( \langle v(a) \rangle \). The excursions these loops make outside \( \langle v(a) \rangle \) form a Poisson point process of intensity \( \frac{1}{q} v \nu^{\mathbb{R}}_{\text{exc}}([1, q]) \).

Proof. This follows from universal properties of loop soups. See for instance [2].

The factor \( \frac{1}{q} \) in \( \frac{1}{q} u \nu^{\mathbb{R}}_{\text{exc}}([-a, 0]) \) and \( \frac{1}{q} v \nu^{\mathbb{R}}_{\text{exc}}([1, q]) \) comes from the normalisation factor \( \frac{\pi}{q} \) in the definition of \( \nu^{\mathbb{R}}_{\text{exc}}([-a, 0]) \) (1.3) and \( \nu^{\mathbb{R}}_{\text{exc}}([1, q]) \) (1.3).

Proposition 2.4. Let \( u, v > 0, q > 1 \) and \( n \geq 1 \).

\[
(2.7) \quad P_{\tilde{\mathcal{L}}_{\alpha}^{\tilde{G}_{n,a}(q)}(q)} = 1 - e^{-2C_n^u(q)\frac{\pi}{q}}
\]

\[
(2.8) \quad \lim_{n \to +\infty} P_{\tilde{\mathcal{L}}_{\alpha}^{\tilde{G}_{n,a}(q)}(q)} = 1 - q^{-2\sqrt{uv}}
\]

Proof. Let \( a > 0 \). Consider three independent Poisson point processes:

- a loop soup \( \mathcal{L}_{\tilde{G}_{n,a}(q)} \)
- a P.p.p of excursions of intensity \( u \nu^{\mathbb{R}}_{\text{exc}}([-a, 0]) \)
- a P.p.p of excursions of intensity \( v \nu^{\mathbb{R}}_{\text{exc}}([1, q]) \)

The probability for the two P.p.p. of excursions to be connected either directly or through a cluster of \( \mathcal{L}_{\tilde{G}_{n,a}(q)} \) equals, according lemma 2.3, the probability for \( \langle u(a) \rangle \) and \( \langle v(a) \rangle \) to be in the same cluster of \( \mathcal{L}_{\tilde{G}_{n,a}(q)} \) conditionally on \( \tilde{\mathcal{L}}_{n,a,q,\alpha}^{\tilde{G}_{n,a}(q)} = \frac{8\pi}{n} u \) and \( \tilde{\mathcal{L}}_{n,a,q,\alpha}^{\tilde{G}_{n,a}(q)} = \frac{8\pi}{n} v \) and \( \mathcal{N}_\alpha(\langle u(a), \rangle_n, q) = 0 \). According to lemma 2.1 this probability equals

\[
1 - e^{-2C_n^u(q)\frac{\pi}{q}}
\]

Taking the limit as \( a \) tends to infinity we get (2.7). Using lemma 2.2 we get the limit (2.8).

3. Computation of \( P_{\tilde{\mathcal{L}}_{\alpha}^{\tilde{G}_{n,a}(q)}(q)} \)

On the continuum upper half plane \( \mathbb{H} \) we consider two independent Poisson point processes:

- a Brownian loop soup \( \mathcal{L}_{\alpha}^{\tilde{G}_{\mathbb{H}}(q)} \), \( 0 < \alpha \leq \frac{1}{2} \)
- a P.p.p. of Brownian excursions from and to \((-\infty, 0] \times \{0\}, \mathcal{E}_{\alpha}^{\mathbb{H}}((-\infty, 0]), u > 0 \).

We will consider the clusters made out of loops in \( \mathcal{L}_{\alpha}^{\tilde{G}_{\mathbb{H}}(q)} \) and excursions in \( \mathcal{E}_{\alpha}^{\mathbb{H}}((-\infty, 0]) \). Among these clusters we only take the clusters that contain at least one excursion and consider the right boundary of the rightmost cluster. This boundary is a non self-intersecting curve joining \( \mathbb{R} \) to infinity. It can be formally defined as follows. Take the clusters that contain at least one excursion. The curve minus its starting point on \( \mathbb{R} \) is the boundary in \( \mathbb{H} \) of the closure in \( \mathbb{H} \) of the set of points visited by the above clusters.
CONVERGENCE OF THE RANDOM WALK LOOP SOUP CLUSTERS TO CLE

All the excursions $E_u((−∞, 0])$ are located left to the curve and there are only clusters made of loops right to it. According to [11] and [14] this boundary curve is an $SLE(κ, ρ)$ starting from 0, where $κ$ is given by (1.2) and $ρ$ by

$$u = \frac{(ρ + 2)(ρ + 6 − κ)}{4κ}$$

We will define

$$u_0(α) := \frac{6 − κ(α)}{2κ(α)}$$

See next picture.

Fig.4: Illustration of the curve separating clusters with loops and excursions on the left from the clusters with only loops on the right.

For $u = u_0(α)$, $ρ = 0$ and $SLE(κ, ρ)$ is a chordal $SLE_{κ}$ curve starting from 0. For a description of $SLE$ processes see [12]. We will denote by $(ξ_t)_{t ≥ 0}$ this curve.

$ξ_0 = 0$. It does not return to $R$ at positive times. There is only one conformal map $g_t$ that sends $H \setminus ξ([0, t])$ (half-plane minus the curve up to time $t$) onto $H$ and that is normalized at infinity $z → ∞$ as

$$g_t(z) = z + \frac{a}{z} + o(z^{-1})$$

The Loewner flow $(g_t)_{t ≥ 0}$ satisfies the differential equation

$$\frac{∂g_t(z)}{∂t} = 2 \frac{g_t(z) − √κW_t}{g_t(z) − √κW_t}$$

where $(W_t)_{t ≥ 0}$ is a standard Brownian motion on $R$.

$p_{κ,u_0(α),v}^H([1, q])$ is the probability that an excursion from $E_u^H([1, q])$ intersects an independent $SLE_{κ(α)}$ curve. The excursions $E_v^H([1, q])$ satisfy a one-sided conformal restriction property ([13]): if $K$ is a compact subset of $C$ that does not intersect $[1, q] × {0}$ and such that $H \setminus K$ is simply connected, if $f$ is a conformal map from $H \setminus K$ onto $H$ such that $f(1) < f(q) ∈ R$, then the probability that $E_v^H([1, q])$ does not intersect $K$ equals

$$\left( \frac{f'(1)f'(q)}{(f(q) − f(1))^2} \right)^v$$

Moreover conditionally on this event the law of $f(E_v^H([1, q]))$ is $E_v^H([f(1), f(q)])$ up to a change of parametrization of the excursions. From this conformal restriction property immediately follows:

**Lemma 3.1.** Let $κ ∈ (0, 4]$. Let $(ξ_t)_{t ≥ 0}$ be an $SLE_{κ}$ with the driving Brownian motion $(√κW_t)_{t ≥ 0}$ and the Loewner flow $(g_t)_{t ≥ 0}$. Denote by $g_t'$ the derivative of $g_t$ with respect the complex variable:

$$g_t'(z) = \frac{∂g_t(z)}{∂z}$$
Denote by $\tilde{p}_{\kappa,v}(q)$ the probability that and independent family of excursions $\mathcal{E}_v^\mathbb{H}([1,q])$ does not intersect $\xi$. Then the conditional probability of the event that $\mathcal{E}_v^\mathbb{H}([1,q])$ does not intersect $\xi$ conditionally on $(\xi_s)_{0 \leq s \leq t}$ (or equivalently conditionally on $(W_s)_{0 \leq s \leq t}$) and on not intersecting $(\xi_s)_{0 \leq s \leq t}$ equals

$$\tilde{p}_{\kappa,v} = \frac{g_t(q) - \sqrt{KW_t}}{g_t(1) - \sqrt{KW_t}}$$

(3.2)

The conditional probability of the event that $\mathcal{E}_v^\mathbb{H}([1,q])$ does not intersect $\xi$ conditionally on $(\xi_s)_{0 \leq s \leq t}$ is

$$(3.3) \quad \left( \frac{g'_t(1)g'_t(q)}{(g_t(q) - g_t(1))^2} \right)^v \tilde{p}_{\kappa,v} \left( \frac{g_t(q) - \sqrt{KW_t}}{g_t(1) - \sqrt{KW_t}} \right)$$

In particular for all $t \geq 0$

$$(3.4) \quad \tilde{p}_{\kappa,v}(q) = E \left[ \left( \frac{g'_t(1)g'_t(q)}{(g_t(q) - g_t(1))^2} \right)^v \tilde{p}_{\kappa,v} \left( \frac{g_t(q) - \sqrt{KW_t}}{g_t(1) - \sqrt{KW_t}} \right) \right]$$

**Proof.** (3.2) is the conditional probability that $g_t(\mathcal{E}_v^\mathbb{H}([1,q]))$ does not intersect $(g_t(\xi_{t+s}))_{s \geq 0}$. To express it we used the fact that $g_t(\mathcal{E}_v^\mathbb{H}([1,q]))$ has the same law as $\mathcal{E}_v^\mathbb{H}([1,\bar{t}])$ starting from $\sqrt{KW_t}$. In (3.3) we multiplied the conditional probability that $\mathcal{E}_v^\mathbb{H}([1,q])$ does not intersect $(\xi_s)_{0 \leq s \leq t}$ and the conditional probability that $g_t(\mathcal{E}_v^\mathbb{H}([1,q]))$ does not intersect $(g_t(\xi_{t+s}))_{s \geq 0}$. □

Next we derive a differential equation in $q$ satisfied by $\tilde{p}_{\kappa,v}(q)$ on $(1, +\infty)$:

**Proposition 3.2.** Let $\kappa \in (0, 4]$ and $v > 0$. $\tilde{p}_{\kappa,v}(q)$ is the unique solution on $(1, +\infty)$ to the differential equation

$$(3.5) \quad f'' + \frac{1}{(q-1)^2} \left( \frac{2 - \frac{4}{\kappa}}{q - \frac{4}{\kappa}} \right) f' - \frac{4v}{\kappa q f} = 0$$

with the boundary conditions

$$(3.6) \quad f(1) = 1 \quad f(+\infty) = 0$$

**Proof.** We can get the a priori $C^\infty$ regularity on $(1, +\infty)$ of $\tilde{p}_{\kappa,v}(q)$ using the equality (3.3). Fix $t > 0$. The joint law of $(W_t, g_t(1), g_t(q), g'_t(1), g'_t(q))$ has a density depending on $q$ (and of course on $\kappa$ and $t$ but we won’t use this):

$$F(q, w, z_1, z_2, \bar{z}_1, \bar{z}_2) dw dz_1 dz_2 d\bar{z}_1 d\bar{z}_2$$

Thus

$$\tilde{p}_{\kappa,v}(q) = \int_{\sqrt{KW} < z_1 < z_2, \sqrt{KW} < \bar{z}_1 < \bar{z}_2} \left( \frac{\bar{z}_1 \bar{z}_2}{(z_2 - z_1)^2} \right)^v \tilde{p}_{\kappa,v} \left( \frac{z_2 - \sqrt{KW}}{z_1 - \sqrt{KW}} \right) F(q, w, z_1, z_2, \bar{z}_1, \bar{z}_2) dw dz_1 dz_2 d\bar{z}_1 d\bar{z}_2$$

$F(q, w, z_1, z_2, \bar{z}_1, \bar{z}_2)$ has $q$-derivatives of all order and each can be bounded by an integrable function uniformly for $q$ varying in a compact subset of $(1, +\infty)$. We will detail this in an upcoming version of this preprint. It follows that $\tilde{p}_{\kappa,v}(q)$ is $C^\infty$.

Let $q > 1$. Let

$$R_t := \frac{\kappa v(q'_t(q) - q'_t)}{(g_t(q) - g_t(1))^2}, \quad q_t := \frac{g_t(q) - \sqrt{KW_t}}{g_t(1) - \sqrt{KW_t}}$$

$R_t$ has bounded variation (in $t$). Let

$$M_t := R_t^v \tilde{p}_{\kappa,v}(q_t)$$
According the lemma 3.1, $(M_t)_{t \geq 0}$ is a martingale. We will get the equation (3.5) by applying Itô’s formula to the martingale $(M_t)_{t \geq 0}$.

$$dM_t = R_t^v \left( \frac{\nu p_{\kappa,v}(q_t)}{R_t^v} \frac{dR_t^v}{R_t^v} + p_{\kappa,v}'(q_t) dq_t + \frac{1}{2} p_{\kappa,v}''(q_t)(dq_t)^2 \right)$$

Denote $\mathbb{P} := \{\exists (z) \geq 0\}$. For $z \in \mathbb{P} \setminus \{0, t\}$:

$$\frac{\partial g_t(z)}{\partial t} - \frac{\partial}{\partial z} \left( \frac{\partial g_t(z)}{\partial t} \right) = \frac{\partial}{\partial z} \left( \frac{2}{g_t(z) - \sqrt{r} W_t} \right) = -\frac{2g_t(z)}{(g_t(z) - \sqrt{r} W_t)^2}$$

Thus

$$dR_t = \left( \frac{-2g_t'(1)g_t(q)}{(g_t(1) - \sqrt{r} W_t)^2(g_t(q) - g_t(1))^2} + \frac{2}{(g_t(q) - \sqrt{r} W_t)^2} \right) dt$$

Further

$$dq_t = \sqrt{\kappa} \left( \frac{-1}{g_t(1) - \sqrt{r} W_t} + \frac{g_t(q) - \sqrt{r} W_t}{(g_t(1) - \sqrt{r} W_t)^2} \right) dW_t$$

Finally

$$dM_t = R_t^v p_{\kappa,v}^t(q_t) \frac{\sqrt{\kappa} (q_t - 1) q_t^2}{g_t(q_t) - \sqrt{r} W_t} dW_t + \frac{R_t^v (q_t - 1)}{(g_t(q_t) - \sqrt{r} W_t)^2} \times$$

$$\times \left( \frac{\kappa}{2} (q_t - 1) q_t^2 p_{\kappa,v}'(q_t) + q_t ((\kappa - 2)q_t - 2)p_{\kappa,v}'(q_t) - 2 \nu (q_t - 1) \bar{p}_{\kappa,v}(q_t) \right) dt$$

Since $(M_t)_{t \geq 0}$ is a martingale

$$\frac{\kappa}{2} (q_t - 1) q_t^2 p_{\kappa,v}''(q_t) + q_t ((\kappa - 2)q_t - 2)p_{\kappa,v}'(q_t) - 2 \nu (q_t - 1) \bar{p}_{\kappa,v}(q_t) = 0$$

which gives the equation (3.5).

Let’s show that there is only one solution to the ODE (3.5) satisfying the boundary conditions (3.6). If there are two different, they are not proportional and all other solutions are linear combinations of these two. These would mean that all the solutions of (3.5) converge to 0 at $+\infty$. We need to show that there is a solution that does not. Indeed let $f$ be a solution with initial conditions $f(2) > 0$, $f'(2) > 0$. Then $f$ is strictly increasing on $[2, +\infty)$ (which implies that $f(+\infty) > 0$). If this
was not the case there would have been a point \( q_0 \in (2, +\infty) \) such that \( f(q_0) > 0, f'(q_0) = 0 \) and for all \( q \) in an interval \([q, q + \varepsilon] \), \( f(q) \leq f(q_0) \). But then there is a contradiction because
\[
f(q) = f(q_0) + \frac{1}{2} f''(q_0) (q - q_0)^2 + o((q - q_0)^2)
\]
\[
= f(q_0) + \frac{2\varepsilon}{\kappa q^2} f(q_0) (q - q_0)^2 + o((q - q_0)^2) > f(q_0)
\]
\[ \square \]

**Proposition 3.3.** Let \( q > 1, \; v > 0 \).
\[
\lim_{n \to +\infty} p^n_{1/2, u_0(1/2), v}(q) = p^n_{1/2, u_0(1/2), v}(q) = 1 - q^{-\sqrt{\varepsilon}}
\]

**Proof.** Let \( \alpha \in (0, \frac{1}{2}] \). By definition
\[
p^n_{\alpha, u_0(\alpha), v}(q) = 1 - p_{\kappa(\alpha), v}(q)
\]
According to proposition 2.4
\[
\lim_{n \to +\infty} p^n_{\alpha, u_0(\alpha), v}(q) = 1 - q^{-2\sqrt{\varepsilon}}
\]
We need to show that \( q^{-2\sqrt{\varepsilon}} \) satisfies the ODE (3.3), with \( \kappa = \kappa(\alpha) \), for \( \alpha = \frac{1}{2} \).
\[
(3.7) \quad \left( \frac{d^2}{dq^2} + \frac{1}{(q-1)q} \left( \left( 2 - \frac{4}{\kappa(\alpha)} \right) q - \frac{4v}{\kappa(\alpha)} q^2 \right) \right) (q^{-2\sqrt{\varepsilon}}) =
\]
\[
\left( 2\sqrt{u_0(\alpha)v} (2\sqrt{u_0(\alpha)v} + 1) - \left( 2 - \frac{4}{\kappa(\alpha)} \right) 2\sqrt{u_0(\alpha)v} - \frac{4v}{\kappa(\alpha)} \right) q^{-2\sqrt{\varepsilon}} - 4\sqrt{u_0(\alpha)v} \left( 1 - \frac{4}{\kappa(\alpha)} \right) q^{-2\sqrt{\varepsilon}}
\]
\[ \square \]

4. Convergence to CLE

In this section we prove the convergence results. In this preprint we will give the main ideas. More detailed proofs will appear in upcoming versions.

Let \( Q_l := (-l, l) \times (0, l) \). Let \( \mathcal{L}^{\alpha, n_0} \) be the loops in \( \mathcal{L}^{\alpha, n_0} \) that are contained in \( Q_l \) and do at least \( T \) jumps. Let \( \mathcal{L}^{\alpha, n_0}_Q \) be the Brownian loops in \( \mathcal{L}^{\alpha, n_0} \) that are contained in \( Q_l \). From [1] follows that for \( \alpha \in (0, \frac{1}{2}], \; l > 0 \) and \( \theta \in (\frac{16}{9}, 2) \), \( \mathcal{F}_{ext}(\mathcal{L}^{\alpha, n_0}_Q) \) converges in law to \( \mathcal{F}_{ext}(\mathcal{L}^{\alpha, n_0}_Q) \).

**Lemma 4.1.** Let \( \alpha \in (0, \frac{1}{2}] \) and \( \theta \in (\frac{16}{9}, 2) \). \( \mathcal{F}_{ext}(\mathcal{L}^{\alpha, n_0}_Q) \) converges in law to \( \mathcal{F}_{ext}(\mathcal{L}^{\alpha, n_0}_Q) \).

**Proof.** Let \( z_1, \ldots, z_j \in H \). To deduce that \( \mathcal{F}_{ext}(\mathcal{L}^{\alpha, n_0}_Q)[z_1, \ldots, z_j] \) converges in law to \( \mathcal{F}_{ext}(\mathcal{L}^{\alpha, n_0}_Q)[z_1, \ldots, z_j] \) from the result of [1] we need only to show that
\[
\lim_{l \to +\infty} \lim_{n \to +\infty} \inf \mathbb{P}(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}^{\alpha, n_0}_Q)[z_1, \ldots, z_j] \text{ contained in } Q_l) = 1
\]
Let \( \varepsilon > 0 \). There is \( l_0 > 0 \) such that
\[
\mathbb{P}(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}^{\alpha, n_0}_Q)[z_1, \ldots, z_j] \text{ contained in } Q_{l_0}) \geq 1 - \varepsilon
\]
Denote
\[
\partial Q_l := (\{-l\} \times (0, l)) \cup ((\{l\} \times (0, l)) \cup ([l, l] \times \{l\})
\]
There is \( l_1 > l_0 \) such that
\[
\Pr(\exists \gamma \in \mathcal{L}_\alpha^n, \gamma \cap Q_{l_0} \neq \emptyset, \gamma \cap \partial_H Q_{l_1} \neq \emptyset) \leq \varepsilon
\]

Then
\[
\lim_{n \to +\infty} \Pr(\text{Contours of } \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n \cap Q_{l_0})[z_1, \ldots, z_j] \text{ contained in } Q_{l_0})
\]
\[
= \Pr(\text{Contours of } \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \text{ contained in } Q_{l_0})
\]
\[
\geq \Pr(\text{Contours of } \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \text{ contained in } Q_{l_0}) \geq 1 - \varepsilon
\]

According the approximation of [4]
\[
\lim_{n \to +\infty} \Pr(\exists \gamma \in \mathcal{L}_\alpha^n, \gamma \cap Q_{l_0} \neq \emptyset, \gamma \cap \partial_H Q_{l_1} \neq \emptyset)
\]
\[
= \Pr(\exists \gamma \in \mathcal{L}_\alpha^n, \gamma \cap Q_{l_0} \neq \emptyset, \gamma \cap \partial_H Q_{l_1} \neq \emptyset) \leq \varepsilon
\]

But
\[
\Pr(\text{Contours of } \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \text{ contained in } Q_{l_0}) \geq
\]
\[
= \Pr(\exists \gamma \in \mathcal{L}_\alpha^n, \gamma \cap Q_{l_0} \neq \emptyset, \gamma \cap \partial_H Q_{l_1} \neq \emptyset)
\]
\[
\text{Thus}
\]
\[
\liminf_{n \to +\infty} \Pr(\text{Contours of } \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \text{ contained in } Q_{l_0}) \geq 1 - 2\varepsilon
\]

From now on \( \theta \in \left( \frac{16}{3}, 2 \right) \) will be fixed. \( \alpha \) will belong to \((0, \frac{1}{2})\). For \( z_0 \in \mathbb{H} \), we define
\[
\delta_{\alpha, n}(z_0) := \max\{d(z, \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)(z_0))|z \in \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)(z_0)\}
\]
By \( z \in \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)(z_0) \) we mean that \( z \) is a point on the contour \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)(z_0) \). The random variable \( \delta_{\alpha, n}(z_0) \) is defined only when \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)(z_0) \) is defined, which happens with probability converging to 1.

**Lemma 4.2.** Assume that \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n) \) does not converge in law to \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n) \). Then there is \( z_{\alpha, 0} \in \mathbb{H} \) such that \( \delta_{\alpha, n}(z_{\alpha, 0}) \) does not converge in law to 0.

**Proof.** If \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n) \) does not converge in law to \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n) \) then by definition there are \( z_1, \ldots, z_j \in \mathbb{H} \) such that \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \) does not converge in law to \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \). To the contrary \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \) does converge in law to \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \). Since each contour of \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \) is surrounded by a contour of \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)[z_1, \ldots, z_j] \), one of \( \delta_{\alpha, n}(z_i) \) must not converge in law to 0.

Let \( z_{\alpha, 0} \) be defined by the previous lemma under the non convergence assumption. The set
\[
\{z \in \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)(z_{\alpha, 0})|d(z, \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)(z_{\alpha, 0})) = \delta_{\alpha, n}(z_{\alpha, 0}) \land 1\}
\]
is non-empty (when \( \delta_{\alpha, n}(z_{\alpha, 0}) \) defined) because \( \mathcal{F}_{\text{ext}}(\mathcal{L}_\alpha^n)(z_{\alpha, 0}) \) is connected and compact. Let \( Z_{\alpha, n} \) be a random point in the above set, for instance the maximum for the lexicographical order.
Lemma 4.3. Assume that $F_{ext}(L^a_{ext})$ does not converge in law to $F_{ext}(L^a_{n})$. Then there is a sub-sequence of indices $n_{a,0}$ such that the joint law of

$$(F_{ext}(L^a_{n_{a,0}}(z_{a,0}), Z_{a,n_{a,0}}))$$

has a limit when $n_{a,0} \to +\infty$, the law of

$$(F_{ext}(L^a_{\infty})(z_{a,0}), Z_{a})$$

satisfying the property that with positive probability $Z_{a}$ is not contained or surrounded by $F_{ext}(L^a_{\infty})(z_{a,0})$.

Proof. $\delta_{a,n}(z_{a,0})$ does not converge in law to 0. This means that there is $\varepsilon > 0$ and a sub-sequence of indices $n'$ such that

$$(4.1) \forall n', P(d(Z_{a,n'}, F_{ext}(L^a_{n'})(z_{a,0})) \geq \varepsilon) \geq \varepsilon$$

The sub-sequence of random variables

$$(F_{ext}(L^a_{n'})(z_{a,0}), Z_{a,n'})$$

is tight. Indeed the first component of the couple converges in law and the second is by definition at distance at most 1 from the first. Thus there is a sub-sequence of indices $n_{a,0}$ out of $n'$ such that there is a convergence in law. $F_{ext}(L^a_{n_{a,0}}(z_{a,0}))$ converges in law $F_{ext}(L^a_{\infty})(z_{a,0})$. Let $Z_{a}$ the limit in law $Z_{a,n_{a,0}}$.\(\blacksquare\)

Moreover a.s. $Z_{a}$ cannot be in the interior surrounded by $F_{ext}(L^a_{n})(z_{a,0})$ because $Z_{a,n}$ is non surrounded by $F_{ext}(L^a_{n})(z_{a,0})$.

From now on $(z_{j})_{j \geq 1}$ will be a fixed everywhere dense sequence in $\mathbb{H}$.

Lemma 4.4. Assume that $F_{ext}(L^a_{ext})$ does not converge in law to $F_{ext}(L^a_{n})$. Then there is a family of sub-sequences of indices $n_{a,j}$ such that

- $n_{a,0}$ is given by lemma 4.3
- $n_{a,j+1}$ is a sub-sequence of $n_{a,j}$.
- The random variable

$$(F_{ext}(L^a_{n_{a,j}}(z_{a,0}, z_1, \ldots, z_j), Z_{a,n_{a,j}}))$$

converges in law as $n_{a,j} \to +\infty$ and the limit defines the joint law of

$$(F_{ext}(L^a_{\infty})(z_{a,0}, z_1, \ldots, z_j), Z_{a})$$

- The family of joint laws $(F_{ext}(L^a_{n_{a,j}})(z_{a,0}, z_1, \ldots, z_j), Z_{a})_{j \geq 1}$ is consistent in the sense that the law on $(F_{ext}(L^a_{n_{a,j}})(z_{a,0}, z_1, \ldots, z_j), Z_{a})$ induced by the law of $(F_{ext}(L^a_{n_{a,j}})(z_{a,0}, z_1, \ldots, z_{j+1}), Z_{a})$ is the same as the one given by the convergence. In particular the law on $(F_{ext}(L^a_{n_{a,j}})(z_{a,0}), Z_{a})$ is the one given by lemma 4.3.
- The family of laws of $(F_{ext}(L^a_{n_{a,j}})(z_{a,0}, z_1, \ldots, z_j), Z_{a})_{j \geq 1}$ uniquely defines a law on $(F_{ext}(L^a_{n_{a,j}}), Z_{a})$.

Proof. The consistency of law follows from the fact that $n_{a,j+1}$ is a sub-sequence of $n_{a,j}$. A contour loop in $F_{ext}(L^a_{n_{a,j}})$ almost surely surrounds one of the $z_j$ points. Thus the fact that a consistent family of laws on $(F_{ext}(L^a_{n_{a,j}})(z_{a,0}, z_1, \ldots, z_j), Z_{a})_{j \geq 1}$ uniquely defines a law on $(F_{ext}(L^a_{n_{a,j}}), Z_{a})$ follows from the Kolmogorov extension theorem.

Next we explain how we extract $n_{a,j+1}$ out of $n_{a,j}$. By construction the sub-sequence $(F_{ext}(L^a_{n_{a,j}}(z_{a,0}, z_1, \ldots, z_j), Z_{a,n_{a,j}}))$ converges in law as $n_{a,j} \to +\infty$ and defines a joint law on $(F_{ext}(L^a_{n_{a,j}})(z_{a,0}, z_1, \ldots, z_j), Z_{a})$. Moreover we have the
convergence in law of $F_{\alpha}(L_\alpha^{a_0,a_j,n_{a_0,j}})(z_{j+1})$ to $F_{\alpha}(L_\alpha^{a_0,a_j})(z_{j+1})$. Thus the subsequence $(F_{\alpha}(L_\alpha^{a_0,a_j,n_{a_0,j}}))[z_{a_0},z_1,\ldots,z_{j+1},Z_{a_0,a_{a_0,j}}]$ is tight and one can extract a subset of indices $n_{a_0,j+1}$ such that it converges in law. The limit law is a law on $(F_{\alpha}(L_\alpha^{a_0,a_j}(z_{a_0},z_1,\ldots,z_{j+1}),Z_{a_0})$.

\[\square\]

**Theorem 1.** $F_{\alpha}(L_\alpha^{a_0})$ and $F_{\alpha}(L_\alpha^{a_0})$ converge in law as $n \to +\infty$ to $F_{\alpha}(L_\alpha^{a_0})$, that is to say to a CLE on $\mathbb{H}$.

**Proof.** It is enough to prove the convergence of $F_{\alpha}(L_\alpha^{a_0})$. Indeed we already have the convergence for $F_{\alpha}(L_\alpha^{a_0})$ and each contour $F_{\alpha}(L_\alpha^{a_0})$ is comprised between the contour $F_{\alpha}(L_\alpha^{a_0})$ and the contour $F_{\alpha}(L_\alpha^{a_0})$.

Assume that $F_{\alpha}(L_\alpha^{a_0})$ does not converge in law to $F_{\alpha}(L_\alpha^{a_0})$. Let $z_{\frac{1}{2},0}$ be the point defined by lemma 12 and $n_{\frac{1}{2},0}$ the sub-sequences defined by lemma 13. We also consider the joint law of $(F_{\alpha}(L_\alpha^{a_0}),Z_{\alpha,n})$ defined by 13

For $u,v > 0$ and $q > 1$ we consider additional independent Poisson point processes of excursions $\mathcal{E}_u^{H}((-\infty,0])$ and $\mathcal{E}_v^{H}([1,q])$. Let $A_{\frac{1}{2},u,v}(q)$ be the event that is satisfied if either an excursion from $\mathcal{E}_u^{H}((-\infty,0])$ and one from $\mathcal{E}_v^{H}([1,q])$ intersect each other or both intersect a common contour from $F_{\alpha}(L_\alpha^{a_0})$. By definition

$$P(A_{\frac{1}{2},u,v}(q)) = P_{\frac{1}{2},u,v}(q)$$

Let $A_{\frac{1}{2},u,v}(q)$ be the event that is satisfied if one of the following conditions holds:

- An excursion from $\mathcal{E}_u^{H}((-\infty,0])$ and one from $\mathcal{E}_v^{H}([1,q])$ intersect each other.
- An excursion from $\mathcal{E}_u^{H}((-\infty,0])$ and one from $\mathcal{E}_v^{H}([1,q])$ intersect a common contour from $F_{\alpha}(L_\alpha^{a_0})$.
- An excursion from $\mathcal{E}_u^{H}((-\infty,0])$ intersects $F_{\alpha}(L_\alpha^{a_0})(z_{\frac{1}{2},0})$ and an excursion from $\mathcal{E}_v^{H}([1,q])$ hits or surrounds $Z_{\frac{1}{2}}$.
- An excursion from $\mathcal{E}_u^{H}([1,q])$ intersects $F_{\alpha}(L_\alpha^{a_0})(z_{\frac{1}{2},0})$ and an excursion from $\mathcal{E}_v^{H}((-\infty,0])$ hits or surrounds $Z_{\frac{1}{2}}$.

Since with positive probability $Z_{\frac{1}{2}}$ is not contained or surrounded by $F_{\alpha}(L_\alpha^{a_0})(z_{\frac{1}{2},0})$

$$P(A_{\frac{1}{2},u,v}(q)) > P(A_{\frac{1}{2},u,v}(q)) = P_{\frac{1}{2},u,v}(q)$$

See next picture for the illustration of $A_{\frac{1}{2},u,v}(q) \setminus A_{\frac{1}{2},u,v}(q)$. 
Let $j \geq 1$. The events $A_{\frac{1}{2},u,v}(q,j)$ respectively $A_{\frac{1}{2},u,v}(q,j)$ are defined similarly to $A_{\frac{1}{2},u,v}(q)$ respectively $A_{\frac{1}{2},u,v}(q)$ where the condition of $\mathcal{E}_{v}^n([-\infty,0])$ and $\mathcal{E}_{v}^n([1,q])$ intersecting a common contour of $\mathcal{F}_{ext}(L_{2}^n)$ is replaced by the condition of intersecting a common contour of $\mathcal{F}_{ext}(L_{2}^n)[z_{2,0},z_{1,\ldots,z}]$. Then

$$\lim_{j \rightarrow +\infty} \mathbb{P}(A_{\frac{1}{2},u,v}(q,j)) = \mathbb{P}(A_{\frac{1}{2},u,v}(q)) \quad \lim_{j \rightarrow +\infty} \mathbb{P}(A_{\frac{1}{2},u,v}(q,j)) = \mathbb{P}(A_{\frac{1}{2},u,v}(q))$$

We will denote by $A_{\frac{1}{2},u,v}(q,j)$ and $A_{\frac{1}{2},u,v}(q,j)$ the events defined similarly to $A_{\frac{1}{2},u,v}(q,j)$ and $A_{\frac{1}{2},u,v}(q,j)$ by doing the following replacements:

- $\mathcal{E}_{v}^n((-\infty,0])$ replaced by $\mathcal{E}_{v}^n((-\infty,0])$ and $\mathcal{E}_{v}^n([1,q])$ replaced by $\mathcal{E}_{v}^n([1,q])$
- $Z_n$ replaced by $Z_n$
- $\mathcal{F}_{ext}(L_{2}^n)$ replaced by $\mathcal{F}_{ext}(L_{2}^n)$ and $\mathcal{F}_{ext}(L_{2}^n)[z_{2,0},z_{1,\ldots,z}]$ replaced by $\mathcal{F}_{ext}(L_{2}^n)[z_{2,0},z_{1,\ldots,z}]$

$\mathcal{F}_{ext}(L_{2}^n)[z_{2,0},z_{1,\ldots,z}]$ converges in law to $\mathcal{F}_{ext}(L_{2}^n)[z_{2,0},z_{1,\ldots,z}]$, the P.p.p. $\mathcal{E}_{v}^n((-\infty,0])$ to $\mathcal{E}_{v}^n((-\infty,0])$ and $\mathcal{E}_{v}^n([1,q])$ to $\mathcal{E}_{v}^n([1,q])$. Moreover in the limit, if an excursion intersects a contour loop in $\mathcal{F}_{ext}(L_{2}^n)[z_{2,0},z_{1,\ldots,z}]$ then a.s. it goes in the interior surrounded by the loop. Thus the intersection still holds for small deformations of the excursion and of the contour. Thus for all $j \geq 1$ we have the convergence

$$\lim_{n \rightarrow +\infty} \mathbb{P}(A_{\frac{1}{2},u,v}(q,j)) = \mathbb{P}(A_{\frac{1}{2},u,v}(q,j))$$

From lemma [4] follows that

$$\lim_{n \rightarrow +\infty} \mathbb{P}(A_{\frac{1}{2},u,v}(q,j)) = \mathbb{P}(A_{\frac{1}{2},u,v}(q,j))$$

Each contour of $\mathcal{F}_{ext}(L_{2}^n)$ is surrounded by a contour of $\mathcal{F}_{ext}(L_{2}^n)$ and $Z_{2,n}$ belongs to $\mathcal{F}_{ext}(L_{2}^n)(z_{2},0)$. Thus on the event $A_{\frac{1}{2},u,v}(q,j)$, an excursion from $\mathcal{E}_{v}^n((-\infty,0])$ and one from $\mathcal{E}_{v}^n([1,q])$ either intersect each other or intersect a common contour from $\mathcal{F}_{ext}(L_{2}^n)[z_{2,0},z_{1,\ldots,z}]$. Thus

$$\mathbb{P}(A_{\frac{1}{2},u,v}(q,j)) \geq \mathbb{P}(A_{\frac{1}{2},u,v}(q,j))$$
Let $u$ be equal to $u_0(\frac{1}{2})$. Then
\[ p^{n_j}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q) = \lim_{n_j \to +\infty} p_{\frac{1}{2},u_0(\frac{1}{2}),v}(q) \geq \lim_{n_j \to +\infty} P(A^{0,n_j}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q,j)) = P(A^{0}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q,j)) \]

Taking the limit as $j \to +\infty$ we get
\[ p^{n_j}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q) \geq \lim_{j \to +\infty} P(A^{0}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q,j)) = P(A^{0}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q)) > P(A^{0}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q)) = p^{n_j}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q) \]

which is a contradiction. It follows that $F_{ext}(\mathcal{L}_{\frac{1}{2}}^{H})$ converges in law to $F_{ext}(\mathcal{L}_{\frac{1}{2}}^{E})$. □

**Theorem 2.** Let $\alpha \in (0, \frac{1}{2})$. $F_{ext}(\mathcal{L}_{\alpha}^{H})$ and $F_{ext}(\mathcal{L}_{\alpha}^{E})$ converge in law as $n \to +\infty$ to $F_{ext}(\mathcal{L}_{\alpha}^{H})$, that is to say to a $CLE_{\alpha}(\kappa)$ on $\mathbb{H}$.

**Proof.** As for theorem 1 it is enough to prove that $F_{ext}(\mathcal{L}_{\alpha}^{H})$ converges in law to $F_{ext}(\mathcal{L}_{\alpha}^{H})$. Let’s assume that this is not the case. Let $z_{\alpha,0}$ be the point and $n_{\alpha,0}$ the sub-sequence defined by lemma 4.2. We also consider the joint law of $(\mathcal{L}_{\alpha}^{H}, Z_{\alpha})$ defined by 4.3. Let $\hat{z} \in \mathbb{H}$, $\hat{z} \neq z_{\alpha,0}$.

Let $\alpha := \frac{1}{2} - \alpha$. We take $\mathcal{L}_{\alpha}^{H}$ independent from $(\mathcal{L}_{\alpha}^{H}, Z_{\alpha})$ and $\mathcal{L}_{\alpha}^{H}$ independent from $(\mathcal{L}_{\alpha}^{H}, Z_{\alpha,n})$. We define $\mathcal{L}_{\frac{1}{2}}^{H}$ and $\mathcal{L}_{\alpha}^{H}$ as unions of two independent Poisson point processes:

\[ \mathcal{L}_{\frac{1}{2}}^{H} = \mathcal{L}_{\alpha}^{H} \cup \mathcal{L}_{\alpha}^{H} \quad \mathcal{L}_{\alpha}^{H} = \mathcal{L}_{\alpha}^{H} \cup \mathcal{L}_{\alpha}^{H} \]

Let $A_{\alpha}$ be the event defined by $F_{ext}(\mathcal{L}_{\frac{1}{2}}^{H})(z_{\alpha,0}) = F_{ext}(\mathcal{L}_{\frac{1}{2}}^{H})(\hat{z})$. Let $A_{\alpha}^{+}$ be the event which holds if one of the below conditions is satisfied:

- $F_{ext}(\mathcal{L}_{\alpha}^{H})(z_{\alpha,0}) = F_{ext}(\mathcal{L}_{\alpha}^{H})(\hat{z})$.
- $F_{ext}(\mathcal{L}_{\alpha}^{H})(\hat{z})$ surrounds $Z_{\alpha}$.

Since $\mathcal{L}_{\alpha}^{H}$ is independent from $(\mathcal{L}_{\alpha}^{H}, Z_{\alpha})$ and with positive probability $Z_{\alpha}$ is in the exterior of $F_{ext}(\mathcal{L}_{\alpha}^{H})(z_{\alpha,0})$

\[ P(A_{\alpha}^{+}) > P(A_{\alpha}) \]

Next is an illustration of $A_{\alpha}^{+} \setminus A_{\alpha}$.

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Fig. 6: Illustration of $A_{\alpha}^{+} \setminus A_{\alpha}$. 
Let $A^n_\alpha$ and $A^{n,+}_\alpha$ be the events defined similarly to $A_\alpha$ and $A^+_\alpha$ where the contours $F_{\text{ext}}(L^{\mathbb{H}}_\frac{1}{2})(z_\alpha,0)$, $F_{\text{ext}}(L^{\mathbb{H}}_\frac{1}{2})(\tilde{z})$ and $F_{\text{ext}}(L^{\mathbb{H}}_{\alpha})(\tilde{z})$ are replaced by $F_{\text{ext}}(L^{\mathbb{H}}_{\bar{\alpha}})(z_\alpha,0)$, $F_{\text{ext}}(L^{\mathbb{H}}_{\bar{\alpha}})(\tilde{z})$ and $F_{\text{ext}}(L^{\mathbb{H}}_{\bar{\alpha}})(\tilde{z})$ respectively and $Z_\alpha$ is replaced by $Z^{n}_{\alpha}$. Since $Z^{n}_{\alpha}$ is on the contour $F_{\text{ext}}(L^{\mathbb{H}}_{\bar{\alpha}})(z_\alpha,0)$ we have the equality $A^{n,+}_{\alpha} = A^{n}_{\alpha}$. From theorem 1 follows that
\[
\lim_{n \to +\infty} \mathbb{P}(A^{n}_{\alpha}) = \mathbb{P}(A_\alpha)
\]
On the other hand
\[
\lim_{n \to +\infty} \mathbb{P}(A^{n,o,+}_{\alpha}) = \mathbb{P}(A^{+}_{\alpha}) > \mathbb{P}(A_\alpha)
\]
which is a contradiction. It follows that $F_{\text{ext}}(L^{\mathbb{H}}_{\bar{\alpha}})$ converges in law to $F_{\text{ext}}(L^{\mathbb{H}}_{\bar{\alpha}})$. □

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