Mapping Among Line Elements

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Abstract

In this paper, we revisit our paper "Matrix Riccati Equations, Kaluza-Klein, Finsler Spaces, and Mapping Among Manifolds’ [1]. We will build mapping among generalized quadratic Hamiltonians and we construct Calabi’s Riemannian Line Elements for non-quadratic and generalized Hamiltonians. As an application, we use conformally flat forms of two general pseudo-Riemannian line elements embedded in two flat manifolds and obtain an analytical and exact solution of the mapping between these two manifolds as well as an infinite set of exact solutions of the associated matrix Riccati equation.
1 Introduction

Mapping among manifolds can be useful to solving systems of differential equations.

This paper is organized as follows, in Sec. 2 and 3, we describe some results of mapping presented in a previous paper [1]. In Sec. 4, we generalize mapping among generalized quadratic Hamiltonians [2]. In Sec. 5, we show conformally flat forms of a general pseudo-Riemannian line element [1]. In Sec. 6, we embed an n-dimensional conformally flat form of a pseudo-Riemannian manifold in a flat manifold of dimension $n + 2$ [1]. In Sec. 7, we construct Calabi’s Riemannian line elements for non-quadratic and generalized Hamiltonians. In Sec. 8, we obtain an analytical and exact solution of mapping between two generalized pseudo-Riemannian manifolds. In Sec. 9, we construct an infinite set of exact solutions of the associated matrix Riccati equation. In Sec. 10, we summarize the main results of this paper.

2 Modified Hamiltonian Formalism

In this section we consider some results presented in [1].

Consider a time-dependent Hamiltonian $H(\tau)$ in which $\tau$ is an affine parameter, proper-time, for example.

Let us define $2n$ variables, which will be called $\xi^j$, with index $j$ running from 1 to $2n$ so that we have $\xi^j \in (\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$ in which $q^j$ and $p^j$ are coordinates and momenta, respectively.

We now define the Hamiltonian by

$$H(\tau) = \frac{1}{2}H_{ij}\xi^i\xi^j,$$  \hspace{1cm} (2.1)

in which $H_{ij}$ is a symmetric matrix.

We impose that the Hamiltonian obeys the Hamilton equation

$$\frac{d\xi^i}{d\tau} = J^{ik}\frac{\partial H}{\partial \xi^k}.$$  \hspace{1cm} (2.2)

Equation (2.2) introduces the symplectic J given by

$$ \begin{pmatrix} O & I \\ -I & O \end{pmatrix} $$  \hspace{1cm} (2.3)
in which $O$ and $I$ are the $n \times n$ zero and identity matrices, respectively. We now make a linear transformation from $\xi^j$ to $\eta^j$ given by

$$\eta^j = T^j_k \xi^k,$$  \hspace{1cm} (2.4)

in which $T^j_k$ could be a non-symplectic matrix and the new Hamiltonian is given by

$$Q = \frac{1}{2} Q_{ij} \eta^i \eta^j,$$  \hspace{1cm} (2.5)

in which $Q_{ij}$ is a symmetric matrix. The matrices $H_{ij}$, $Q_{ij}$, and $T^j_k$ obey the following system

$$\frac{dT^i_j}{d\tau} + T^i_k J^{kl} X_{lj} = J^{im} \frac{dt}{d\tau} Y_{ml} T^j_k,$$  \hspace{1cm} (2.6)

in which $2X_{lj} = \frac{\partial H_{ij}}{\partial \xi^l} \xi^i + 2H_{lj}$, $2Y_{ml} = \frac{\partial Q_{il}}{\partial \eta^m} \eta^i + 2Q_{ml}$, $t$ and $\tau$ can be the proper-times of the particle in two different manifolds. Consider $X_{lj} = Z_{lj}$ and $\frac{dt}{d\tau} Y_{ml} = \overline{Y}_{ml}$. Then, (2.6) can be rewritten in the following matrix form

$$\frac{dT}{d\tau} + TJZ = JY \overline{T},$$  \hspace{1cm} (2.7)

in which $T$, $Z$ and $\overline{Y}$ are $2n \times 2n$ matrices as

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$  \hspace{1cm} (2.8)

with similar expressions for $Z$ and $\overline{Y}$. Let us write (2.7) as follows

$$\dot{T}_1 = \overline{Y}_3 T_1 + \overline{Y}_4 T_3 + T_2 Z_1 - T_1 Z_3,$$  \hspace{1cm} (2.9)

$$\dot{T}_2 = \overline{Y}_3 T_2 + \overline{Y}_4 T_4 + T_2 Z_2 - T_1 Z_4,$$  \hspace{1cm} (2.10)

$$\dot{T}_3 = -\overline{Y}_1 T_1 - \overline{Y}_2 T_3 + T_3 Z_1 - T_3 Z_3,$$  \hspace{1cm} (2.11)

$$\dot{T}_4 = -\overline{Y}_1 T_2 - \overline{Y}_2 T_4 + T_3 Z_2 - T_3 Z_4.$$  \hspace{1cm} (2.12)

Now consider

$$\dot{S}_1 = \overline{Y}_3 S_1 + \overline{Y}_4 S_3,$$  \hspace{1cm} (2.13)

$$\dot{S}_2 = \overline{Y}_3 S_2 + \overline{Y}_4 S_4,$$  \hspace{1cm} (2.14)
\[ S_3 = -\bar{Y}_1 S_1 - \bar{Y}_2 S_3, \quad (2.15) \]
\[ S_4 = -\bar{Y}_1 S_2 - \bar{Y}_2 S_4, \quad (2.16) \]

and
\[ \dot{R}_1 = R_2 Z_1 - R_1 Z_3, \quad (2.17) \]
\[ \dot{R}_2 = R_2 Z_2 - R_1 Z_4, \quad (2.18) \]
\[ \dot{R}_3 = R_4 Z_1 - R_3 Z_3, \quad (2.19) \]
\[ \dot{R}_4 = R_4 Z_2 - R_3 Z_4. \quad (2.20) \]

The systems (2.9)-(2.12), (2.13)-(2.16), and (2.17)-(2.20) can be placed on a compact form as follows
\[ \dot{S} = J\bar{Y} S \quad (2.21) \]
\[ \dot{R} = -RJZ, \quad (2.22) \]
\[ T = SAR, \quad (2.23) \]

In which matrix \( A \) is constant and \( 2n \times 2n \).

A more explicit form for (2.23) is given by
\[ T_1 = (S_1 a + S_2 b) R_1 + (S_1 d + S_2 c) R_3, \quad (2.24) \]
\[ T_2 = (S_1 a + S_2 b) R_2 + (S_1 d + S_2 c) R_4, \quad (2.25) \]
\[ T_3 = (S_3 a + S_4 b) R_1 + (S_3 d + S_4 c) R_3, \quad (2.26) \]
\[ T_4 = (S_3 a + S_4 b) R_2 + (S_3 d + S_4 c) R_4, \quad (2.27) \]
in which \( a, b, c, \) and \( d \) are constant \( n \times n \) matrices.

From the theory of first-order differential equation systems [3], it is well known that each systems (2.15)-(2.20) have solutions in the region, in which \( Z_{lj} \) and \( \bar{Y}_{ml} \) are continuous functions.


## 3 Generalized Mapping Among Manifolds

In this section, we present more results from [1], in which the matrix Riccati equation was introduced.

We consider a time-dependent function $H$ in which $\tau$ is an affine parameter. Let us define $2n$ variables, which will be called $\xi_j$, with index $j$ running from 1 to $2n$ so that we have $\xi_j \in (\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$, in which $q^j$ and $p^j$ may or may not be the usual coordinates and momenta, respectively.

We now define the function by

$$H(\tau) = \frac{1}{2} H_{ij} \xi^i \xi^j,$$

in which $H_{ij}$ is a symmetric matrix.

Consider the following system

$$\frac{d\xi^i}{d\tau} = C_{ik} \frac{\partial H}{\partial \xi^k}.$$  \tag{3.2}

The equation (3.2) introduces the $2n \times 2n$ matrix $C$ given by

$$
\begin{pmatrix}
C_1 & C_2 \\
C_3 & C_4
\end{pmatrix}
$$

in which the $C_i$ are $n \times n$ matrices, which can be functions of usual coordinates and momenta.

We now make a linear transformation from $\xi_j$ to $\eta^i$ given by

$$\eta^i = T^i_k \xi^k,$$  \tag{3.4}

in which $T^i_k$ could be a non-sympletic matrix and the new function is given by

$$Q = \frac{1}{2} Q_{ij} \eta^i \eta^j,$$  \tag{3.5}

in which $Q_{ij}$ is a symmetric matrix.

Let us consider that (3.5) obeys the following equation

$$\frac{d\eta^i}{dt} = B^{ik} \frac{\partial Q}{\partial \eta^k},$$  \tag{3.6}
in which $B$ is given by

\[
\begin{pmatrix}
B_1 & B_2 \\
B_3 & B_4
\end{pmatrix}
\] (3.7)

in which the $B_i$ are $n \times n$ matrices, which can be functions of usual coordinates and momenta.

The matrices $H_{ij}$, $Q_{ij}$, and $T^j_k$ obey the following system

\[
\frac{dT^i_j}{d\tau} + T^i_k C^{kl} X_{lj} = B^{im} \left( \frac{dt}{d\tau} Y_{ml} \right) T^j_k,
\] (3.8)
in which $2X_{lj} = \frac{\partial H_{ij}}{\partial \xi^l} \xi^i + 2H_{ij}$ and $2Y_{ml} = \frac{\partial Q_{ij}}{\partial \eta^m} \eta^i + 2Q_{ij}$, and $t$ and $\tau$ can be the proper-times of the particle in two different manifolds.

Consider $X_{ij} = Z_{ij}$ and $\frac{dt}{d\tau} Y_{lm} = \overline{Y}_{lm}$. Then (3.8) can be rewritten in the following matrix form

\[
\frac{dT}{d\tau} + TCZ = BYT,
\] (3.9)
in which $T$ are $2n \times 2n$ matrices as

\[
\begin{pmatrix}
T_1 & T_2 \\
T_3 & T_4
\end{pmatrix}
\] (3.10)

with similar expressions for $Z$ and $\overline{Y}$.

Let us consider the matrices, $T$, $S$, $A$, $R$, in which each matrix can be functions of coordinates and momenta,

\[
\begin{pmatrix}
T_1 & T_2 \\
T_3 & T_4
\end{pmatrix}
\] (3.11)

\[
\begin{pmatrix}
S_1 & S_2 \\
S_3 & S_4
\end{pmatrix}
\] (3.12)

\[
\begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{pmatrix}
\] (3.13)

\[
\begin{pmatrix}
R_1 & R_2 \\
R_3 & R_4
\end{pmatrix}
\] (3.14)

Now consider the following matricial equation,

\[
T = SAR.
\] (3.15)
The derivative of (3.15) is given by

$$\dot{T} = \dot{S}AR + T\dot{A}R + TS\dot{R}$$  \hfill (3.16)

Let us define two systems of matrix Riccati Equations [2],

$$\dot{S} = B\bar{Y}S + D + SAF,$$  \hfill (3.17)
$$\dot{R} = -RCZ + E + GAR.$$  \hfill (3.18)

For the particular case in which

$$D + SAF = 0,$$  \hfill (3.19)
$$E + GAR = 0,$$  \hfill (3.20)
$$B = C = J,$$  \hfill (3.21)

and A is a constant matrix, the systems (3.17)-(3.18) are reduced to the systems (2.21)-(2.22).

Replacing (3.17) and (3.18) in (3.16) and assuming that

$$S\dot{A}R + DAR + SAE + S[A(F + G)A]R = 0,$$  \hfill (3.22)

we have the following simplification of (3.16), given by

$$\frac{dT}{d\tau} + TCZ - B\bar{Y}T = 0.$$  \hfill (3.23)

Equation (3.9) is the motion equation, which was obtained from the equations of motion and is identical to (3.23).

From the theory of first-order differential equation systems [3], it is well known that system (3.23) has a solution in the region in which $Z_{ij}$ and $Y_{ml}$ are continuous functions. If S and R are non-singular matrices, we can multiply (3.22) by $S^{-1}$ on the left side and by $R^{-1}$ on the right side, obtaining the matrix Riccati

$$\dot{A} + S^{-1}DA + AER^{-1} + A(F + G)A = 0.$$  \hfill (3.24)

Notice that the transformation (3.4) is directly associated with matrix Riccati differential equations (3.23), (3.17), (3.18), and (3.24).
4 Generalized Quadratic Hamiltonians

In this section, we generalize important results obtained by Leach in [2], in which mapping is constructed among generalized quadratic Hamiltonians.

As in Section 3, we consider a time-dependent function $H(\tau)$ in which $\tau$ is an affine parameter.

Let us define $2n$ variables, which will be called $\xi^j$, with index $j$ running from 1 to $2n$ so that we have $\xi^j \in (\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$, in which $q^j$ and $p^j$ may or may not be the usual coordinates and momenta, respectively.

We now define the function by

$$H(\tau) = \frac{1}{2} H_{ij}(\tau) \xi^i \xi^j + G_i(\tau) \xi^i + D(\tau), \tag{4.1}$$

in which $H_{ij}(\tau)$ is a symmetric matrix, $G_i(\tau)$ are components of a vector, and $D(\tau)$ is a scalar function.

Consider the following system

$$\frac{d\xi^i}{d\tau} = C_{ik} \frac{\partial H}{\partial \xi^k}. \tag{4.2}$$

Equation (4.2) introduces the $2n \times 2n$ matrix $C$ given by

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \tag{4.3}$$

in which the $C_i$ are $n \times n$ matrices, which can be functions of usual coordinates and momenta.

We now make a linear transformation from $\xi^j$ to $\eta^j$ given by

$$\eta^j = T^j_k \xi^k + r^j, \tag{4.4}$$

in which $r^j$ is a vector, $T^j_k$ can be a non-sympletic matrix and the new function is given by

$$Q(t) = \frac{1}{2} Q_{ij}(t) \eta^i \eta^j + E_i(t) \eta^i + F(t), \tag{4.5}$$

in which $Q_{ij}(t)$ is a symmetric matrix, $E_i(t)$ are components of a vector and $F(t)$ is a scalar function.

Let us consider that (4.5) obeys the following equation

$$\frac{d\eta^i}{dt} = K^{ik} \frac{\partial Q}{\partial \eta^k}. \tag{4.6}$$
in which \( K \) is given by
\[
\begin{pmatrix}
K_1 & K_2 \\
K_3 & K_4
\end{pmatrix}
\] (4.7)
and \( K_i \) are \( nxn \) matrices, which can be functions of usual coordinates and momenta.

The matrices \( H_{ij}, Q_{ij}, \) and \( T^j_k \) obey the following systems
\[
\frac{dT^i_j}{d\tau} + T^i_k C^{klt} \tilde{Z}_{lj} = K^{im}(\frac{dt}{d\tau})\tilde{Y}_{ml} T^j_k, \quad (4.8)
\]
\[
\frac{dr^i}{d\tau} = K^{im}\frac{dt}{d\tau}(\tilde{Y}_{ml} + E_m) - T^i_k C^{klt} \tilde{G}_l. \quad (4.9)
\]

The matrices \( \tilde{Z}_{lj}, \tilde{Y}_{ml}, \) and the vectors \( \tilde{E}_m, \tilde{G}_l \) obey the following equations
\[
2\tilde{Z}_{lj} = \frac{\partial H_{ij}}{\partial \xi^i} \xi^j + 2H_{lj} + 2\frac{\partial G_j}{\partial \xi^i}, \quad (4.10)
\]
\[
2\tilde{Y}_{ml} = \frac{\partial Q_{il}}{\partial \eta^m} \eta^i + 2Q_{ml} + 2\frac{\partial E_j}{\partial \xi^l}, \quad (4.11)
\]
\[
\tilde{E}_m = E_m + \frac{\partial F}{\partial \eta^m}, \quad (4.12)
\]
\[
\tilde{G}_j = G_j + \frac{\partial D}{\partial \xi^j}. \quad (4.13)
\]

Let us consider the matrices, \( T, S, A, R, \) in which each matrix can be functions of coordinates and momenta,
\[
\begin{pmatrix}
T_1 & T_2 \\
T_3 & T_4
\end{pmatrix} \quad (4.14)
\]
\[
\begin{pmatrix}
S_1 & S_2 \\
S_3 & S_4
\end{pmatrix} \quad (4.15)
\]
\[
\begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{pmatrix} \quad (4.16)
\]
\[
\begin{pmatrix}
R_1 & R_2 \\
R_3 & R_4
\end{pmatrix} \quad (4.17)
\]
Now consider the following matrix equation,
\[ T = SAR. \]  \hfill (4.18)

The derivative of (4.18) is given by
\[ \dot{T} = \dot{S}AR + T\dot{AR} + TS\dot{R}. \]  \hfill (4.19)

As in Section 3, let us define two systems of matrix Riccati Equations [3],
\[ \dot{S} = BYS + D + SAF, \]  \hfill (4.20)
\[ \dot{R} = -RCZ + E + GAR. \]  \hfill (4.21)

Replacing (4.20) and (4.21) in (4.19), and assuming that
\[ S\dot{AR} + D\dot{AR} + SAE + S[A(F + G)A]R = 0, \]  \hfill (4.22)

we have the following simplification of (4.19) given by
\[ \frac{dT}{d\tau} + T\dot{C}Z - KYT = 0. \]  \hfill (4.23)

Equation (3.9) is a matrix Riccati equation obtained from the equations of motion and is identical to (4.23).

The main difference between the treatment used in Section 3 and this one is the transformation law (3.4) to (4.4), and the presence of equation (4.9).

With definitions (4.10) and (4.11) we can use the same arguments as in section (3) for the case of generalized quadratic Hamiltonians.
5 Calabi’s Line Elements

In this section we consider an important result obtained by Calabi, \[4\]. We use a development given in \[5\].

For one convex surface \(u(x^i)\) Calabi defined a metric by

\[
G_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j},
\]

with line element given by

\[
ds^2 = G_{ij} dx^i dx^j.
\]

We call (5.2) Calabi’s Line Element.

There is a class of Lagrangians and Hamiltonians that are conservative and behave like surfaces. However, we can generally associate the positive power of a Hamiltonian \(H\) with the Calabi’s line element (5.2) as follows

\[
G_{ij} = \frac{\partial^2 H^n}{\partial x^i \partial x^j},
\]

with \(n = 1, 2, 3, \ldots\).

When \(H\) is on the Hamilton-Jacobi form, the line element (5.2) could be flat. From (5.2), we have Riemann’s and Ricci’s tensors,

\[
R^\alpha_{\mu\sigma\nu} = \partial_\nu \Gamma^\alpha_{\mu\sigma} - \partial_\sigma \Gamma^\alpha_{\mu\nu} + \Gamma^\eta_{\mu\sigma} \Gamma^\alpha_{\eta\nu} - \Gamma^\eta_{\mu\nu} \Gamma^\alpha_{\sigma\eta},
\]

\[
R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}.
\]

It is ease to show that

\[
R_{hijk} = G^{lm}(\Gamma_{ijm} \Gamma_{hkl} - \Gamma_{ikm} \Gamma_{hjl}),
\]

in which \(\Gamma_{ijm}\) is the Christoffel symbol of the first kind.

Calabi developed a Riemannian Geometry.

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We are only interested in (5.2) and in (5.3) because, from (5.2), we have the following Lagrangian and Hamiltonian,

\[ L = G_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}, \]  
\[ (5.7) \]

and

\[ H = G^{ij} \frac{dp_i}{ds} \frac{dp_j}{ds}, \]  
\[ (5.8) \]
in which \( L = H \).

We can use Calabi’s line elements to transform non-quadratic Hamiltonians on quadratic Hamiltonians as (5.8). Information based on second derivatives can be important for many non-quadratic Hamiltonians, in which the use of the Calabi’s line elements may be justified.

### 6 Conformally Flat Forms of a Line Elements

In this section, we describe the conformally flat form of line elements in local coordinates presented in [1].

Let us write a metric and its associated line element as follows

\[ G_{\Lambda\Pi} = E^{(A)}_{\Lambda} E_{\Pi}^{(B)} \eta_{(A)(B)}, \]  
\[ (6.1) \]

and

\[ ds^2 = G_{\Lambda\Pi} du^\Lambda du^\Pi, \]  
\[ (6.2) \]
in which \( \eta_{(A)(B)} \) and \( E_{\Lambda}^{(A)} \) are respectively flat metric and vielbein components.

We choose each \( \eta_{(A)(B)} \) as plus or minus Kronecker’s delta function.

Let us define

\[ \overline{Z}^{(A)} = E^{(A)}_{\Lambda} u^\Lambda. \]  
\[ (6.3) \]

From (6.3), we have

\[ u^\Lambda = E^{(A)}_{(A)} \overline{Z}^{(A)}, \]  
\[ (6.4) \]

with

\[ du^\Lambda = E^{(A)}_{(A)} d\overline{Z}^{(A)} + dE^{(A)}_{(A)} \overline{Z}^{(A)}. \]  
\[ (6.5) \]

We can write (6.5) in a compact form

\[ du^\Lambda = d(E_{(A)} \overline{Z}^{(A)}). \]  
\[ (6.6) \]
Substituting (6.6) in (6.2), we have
\[ ds^2 = G_{\Lambda \Pi} du^\Lambda du^\Pi = \eta_{(A)(B)} d\bar{Z}^{(A)} d\bar{Z}^{(B)} \]
\[ + G_{\Lambda \Pi} \bar{Z}^{(A)} \bar{Z}^{(B)} d(E^A_{(A)}) d(E^\Pi_{(B)}) \]
\[ + 2 \bar{Z}^{(A)} E^\Pi_{(B)} d(E^A_{(A)}) d(E^\Pi_{(A)}) \].

Let us put (6.7) in the following form
\[ ds^2 = \eta_{(A)(B)} d\bar{Z}^{(A)} d\bar{Z}^{(B)} + G_{\Lambda \Pi} \bar{Z}^{(A)} \bar{Z}^{(B)} \frac{d(E^A_{(A)})}{ds} \frac{d(E^\Pi_{(B)})}{ds} \]
\[ + 2 \bar{Z}^{(A)} E^\Pi_{(B)} \frac{ds}{ds} d(E^A_{(A)}) d(E^\Pi_{(A)}) \] ds^2. \hspace{1cm} (6.8)

From a simple calculation, we have
\[ \eta_{(A)(B)} d\bar{Z}^{(A)} d\bar{Z}^{(B)} = [1 - G_{\Lambda \Pi} \bar{Z}^{(A)} \bar{Z}^{(B)} \frac{d(E^A_{(A)})}{ds} \frac{d(E^\Pi_{(B)})}{ds} \]
\[ + 2 \bar{Z}^{(A)} E^\Pi_{(B)} \frac{ds}{ds} d(E^A_{(A)}) d(E^\Pi_{(A)}) \]] ds^2. \hspace{1cm} (6.9)

We now define the function
\[ \exp(-2\sigma) = [1 - G_{\Lambda \Pi} \bar{Z}^{(A)} \bar{Z}^{(B)} \frac{d(E^A_{(A)})}{ds} \frac{d(E^\Pi_{(B)})}{ds} \]
\[ + 2 \bar{Z}^{(A)} E^\Pi_{(B)} \frac{ds}{ds} d(E^A_{(A)}) d(E^\Pi_{(A)}) \]]. \hspace{1cm} (6.10)

Multiplying (6.11) by \exp(2\sigma), we obtain
\[ ds^2 = G_{\Lambda \Pi} du^\Lambda du^\Pi = \exp(2\sigma) \eta_{(A)(B)} d\bar{Z}^{(A)} d\bar{Z}^{(B)}. \hspace{1cm} (6.10)\]

Let us define
\[ \exp(2\Phi) = U^{-2}, \hspace{1cm} (6.11)\]
in which
\[ U = [1 + \frac{1}{4} K \eta_{(A)(B)} \bar{Z}^{(A)} \bar{Z}^{(B)}]. \hspace{1cm} (6.12)\]

Multiplying (6.11) by (6.14) and \exp(-2\sigma), we have
\[ \exp(2\Phi) \eta_{(A)(B)} d\bar{Z}^{(A)} d\bar{Z}^{(B)} = \exp(-2\sigma) G_{\Lambda \Pi} du^\Lambda du^\Pi \hspace{1cm} (6.13)\]

We conclude that (6.12) is the line element of a pseudo-Riemannian metric in a conformally flat form and (6.15) is a line element of a pseudo-Riemannian metric of constant curvature as a function of the metric \(G_{\Lambda \Pi}\).
7 Embedding a Conformally Flat Manifold in Flat Manifolds

In this section, we consider the embedding of (6.10) using a procedure also presented in [1].

Let us rewrite (6.10) as follows

\[ ds^2 = G_{\Lambda\Pi} du^\Lambda du^\Pi = \exp(2\sigma) \eta_{(A)(B)} d\overline{Z}^{(A)} d\overline{Z}^{(B)}. \]  

(7.1)

Defining the transformation of coordinates by,

\[ y^{(A)} = \exp(\sigma) \overline{Z}^{(A)}, \]  

(7.2)

with \((A) = (1, 2, 3, ..., n),\)

\[ y^{n+1} = \exp(\sigma) \eta_{(A)(B)} \overline{Z}^{(A)} \overline{Z}^{(B)} - \frac{1}{4}, \]  

(7.3)

and

\[ y^{n+2} = \exp(\sigma) \eta_{(A)(B)} \overline{Z}^{(A)} \overline{Z}^{(B)} + \frac{1}{4}. \]  

(7.4)

But

\[ \eta_{(A)(B)} \overline{Z}^{(A)} \overline{Z}^{(B)} = G_{\Lambda\Pi} u^\Lambda u^\Pi. \]  

(7.5)

Using (6.3) and (7.5) in (7.2), (7.3) and (7.4),

\[ y^{(A)} = \exp(\sigma) E^{(A)}_{\Lambda} u^\Lambda, \]  

(7.6)

with \((A) = (1, 2, 3, ..., n),\)

\[ y^{n+1} = \exp(\sigma) (G_{\Lambda\Pi} u^\Lambda u^\Pi - \frac{1}{4}), \]  

(7.7)

and

\[ y^{n+2} = \exp(\sigma) (G_{\Lambda\Pi} u^\Lambda u^\Pi + \frac{1}{4}). \]  

(7.8)

It is easy to see that

\[ \eta_{AB} y^{A} y^{B} = 0, \]  

(7.9)

in which

\[ \eta_{AB} = (\eta_{(A)(B)}, \eta_{(n+1)(n+1)}, \eta_{(n+2)(n+2)}). \]  

(7.10)
with
\[ \eta_{(n+1),(n+1)} = 1, \]  
and
\[ \eta_{(n+2),(n+2)} = -1. \]

From a simple calculation, we can verify that the line elements are given by
\[ ds^2 = G_{\Lambda\Pi} du^\Lambda du^\Pi = \exp(2\sigma) \eta^{(A)(B)} dz^{(A)} dz^{(B)} = \eta_{AB} dy^A dy^B. \]  

From (7.13), we see that an n-dimensional manifold in local coordinates can be put in a conformally flat form and embedded in an (n + 2)-dimensional flat manifold.

We can associate the following three Hamiltonians to the line element (7.13)
\[ Q(t) = \frac{1}{2} G^{\Lambda\Pi} P_\Lambda P_\Pi, \]  
\[ \tilde{H} = \frac{1}{2} \exp(2\sigma) \eta^{(A)(B)} P_\Lambda P_\Pi, \]  
and
\[ H = \frac{1}{2} \eta_{AB} P_\Lambda P_\Pi. \]

Mapping among Hamiltonians of the form (7.16) is simpler than mapping among the forms (7.14) or (7.15).

8 Solutions of Mapping Among Manifolds

In this Section, we construct an exact solution of mapping among two pseudo-Riemannian manifolds.

Let us consider the pseudo-Riemannian line element (7.13),
\[ ds^2 = G_{\Lambda\Pi} du^\Lambda du^\Pi = \exp(2\sigma) \eta^{(A)(B)} dz^{(A)} dz^{(B)} = \eta_{AB} dy^A dy^B. \]  

We can associate the following three Hamiltonians to the line element (8.1)
\[ Q = \frac{1}{2} G^{\Lambda\Pi} p_\Lambda p_\Pi, \]  
\[ \tilde{H} = \frac{1}{2} \exp(2\sigma) \eta^{(A)(B)} p_\Lambda p_\Pi, \]  
and
\[ H = \frac{1}{2} \eta_{AB} p_\Lambda p_\Pi. \]
and
\[ H = \frac{1}{2} \eta^{AB} P_A P_B. \]  
(8.4)

It is easy to show that
\[ Q = \bar{H} = H. \]  
(8.5)

Let us consider pseudo-Riemannian manifolds with dimension \( n = 4 \), so that, in the Hamiltonians (8.3), (8.4) and (8.5) we have indexes \( (A) \) and \( \Lambda \) running from 1 to 4 and index \( A \) running from 1 to 6.

The three Hamiltonians are quadratic and we can use the results developed in section 3. However the \( Z \) and \( \bar{Y} \) matrices in (3.9) are not constants for \( Q \) and \( \bar{H} \). This makes it difficult to obtain analytical and exact solutions. In practice, only numerical computation is feasible. But for \( H \), given by (8.4), \( Z \) and \( \bar{Y} \) are constant matrices and it is possible to calculate exact solutions for (8.4). We could then use these solutions together with the results developed in section 7 and obtain exact solutions of (8.2) and (8.3). In other words, exact solutions of (8.4) can be used to get exact solutions of (8.2) and (8.3).

Let us consider two Hamiltonians on the form (8.4) as follows
\[ H = \frac{1}{2} \eta^{AB} P_A P_B. \]  
(8.6)
\[ \bar{H} = \frac{1}{2} \bar{\eta}^{AB} \bar{P}_A \bar{P}_B. \]  
(8.7)

In Section 2 we put \( \frac{dt}{d\tau} Y_{ml} = \bar{Y}_{ml}. \)

Let us split the matrices into blocks.

For \( Y \) and \( Z \) we have,
\[ \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \]  
(8.8)
\[ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \]  
(8.9)

For (8.7) we have \( Y_1 = Y_2 = Y_3 = 0 \), and \( Y_4 = diag(\bar{\eta}^{AB}) \).

For (8.6) we have \( Z_1 = Z_2 = Z_3 = 0 \), and \( Z_4 = diag(\eta^{AB}) \).

Let us rewrite (3.8) as follows,
\[ \frac{dT_{ij}}{d\tau} + T_{ik} J^{kl} Z_{lj} = J^{im} \frac{dt}{d\tau} Y_{ml} T_{kj}, \]  
(8.10)
in which we choose (3.21), $B = C = J$. Then the equations of motion will be given by,

$$\frac{dT_1}{d\tau} = \frac{dt}{d\tau} Y_4 T_3,$$

(8.11)

$$\frac{dT_2}{d\tau} = \frac{dt}{d\tau} Y_4 T_4 - T_1 Z_4,$$

(8.12)

$$\frac{dT_3}{d\tau} = 0,$$

(8.13)

and

$$\frac{dT_4}{d\tau} = -T_3 Z_4.$$

(8.14)

The exact solutions of the systems (8.11)-(8.14) are given by

$$T_3 = \text{const.},$$

(8.15)

$$T_1 = Y_4 T_3 t,$$

(8.16)

$$T_2 = -Y_4 T_3 Z_4 (t \tau),$$

(8.17)

$$T_4 = T_3 Z_4 \tau.$$  

(8.18)

Notice that the matrix $T$ are $(12 \times 12)$, and the matrices $T(i), Z(4)$ and $Y(4)$ are $(6 \times 6)$.

Let us rewrite (3.4) as follows

$$\eta^A = T^A_B \xi^B,$$

(8.19)

in which $\eta^A = (Y^A, P_A)$, and

$$\bar{Y}^A = T_1^A_B Y^B + T_2^A_{(B+6)} P_B$$

(8.20)

$$\bar{P}^A = T_3^{(A+6)}_B Y^B + T_4^{(A+6)}_{(B+6)} P_B$$

(8.21)

in which we have used a convenient notation for matrices elements.

Inverting (7.6) we will have for local coordinates of two pseudo-Riemannian manifolds,

$$u^A = \exp(-\sigma) E^A_{(A)} Y^{(A)},$$

(8.22)

and

$$\bar{u}^A = \exp(-\bar{\sigma}) E^A_{(A)} \bar{Y}^{(A)},$$

(8.23)
in which we recall that indexes \((A)\) and \(\Lambda\) running from 1 to 4 and index \(A\) running from 1 to 6.

Let us consider a subset of (8.20) given by

\[
\bar{Y}^{(A)} = T_1^{(A)} B Y^B + T_2^{(A)} (B+6) P_B \tag{8.24}
\]

Substituting (8.24) into (8.23) we get,

\[
\bar{u}^\Lambda = \exp(-\bar{\sigma}) \bar{E}^{\Lambda}_{(A)} [T_1^{(A)} B Y^B + T_2^{(A)} (B+6) P_B]. \tag{8.25}
\]

Let us define the following \((4\times6)\) matrices

\[
W_1^{\Lambda} B = \exp(-\bar{\sigma}) \bar{E}^{\Lambda}_{(A)} T_1^{(A)} B \tag{8.26}
\]
and

\[
W_2^{\Lambda} (B+6) = \exp(-\bar{\sigma}) \bar{E}^{\Lambda}_{(A)} T_2^{(A)} (B+6), \tag{8.27}
\]

then we can rewrite (8.25) as follows,

\[
\bar{u}^\Lambda = W_1^{\Lambda} B Y^B + W_2^{\Lambda} (B+6) P_B. \tag{8.28}
\]

From Hamilton’s equation,

\[
\bar{p}_\Pi = \bar{G}_{\Pi\Lambda} \frac{d\bar{u}^\Lambda}{dt}. \tag{8.29}
\]

Using (8.28) in (8.29) and defining the following matrices elements

\[
M_1^{\Lambda} B = \frac{d[\exp(-\bar{\sigma}) E^{\Lambda}_{(A)}]}{dt} T_2^{\Lambda} B, \tag{8.30}
\]
\[
M_3^{\Lambda} B = \exp(-\bar{\sigma}) E^{\Lambda}_{(A)} Y_3 T_3^{(A+6)} B, \tag{8.31}
\]
\[
M_2^{\Lambda} (B+6) = \frac{d[\exp(-\bar{\sigma}) E^{\Lambda}_{(A)}]}{dt} T_2^{(A)} (B+6), \tag{8.32}
\]
\[
M_4^{\Lambda} (B+6) = \exp(-\bar{\sigma}) E^{\Lambda}_{(A)} Y_4 T_4^{(A+6)} (B+6). \tag{8.33}
\]

Let us set,

\[
W_3^{\Lambda} B = M_1^{\Lambda} B + M_3^{\Lambda} B \tag{8.34}
\]
\[
W_4^{\Lambda} (B+6) = M_2^{\Lambda} (B+6) + M_4^{\Lambda} (B+6). \tag{8.35}
\]
Then equation (8.29) can be rewritten as
\[ \bar{p}_\Pi = \bar{G}_{\Pi \Lambda} [W^\Lambda_3 B Y^B + W^\Lambda_{(B+6)} P_B]. \] (8.36)

\( \Lambda \) and \( \Pi \) are tensor indices, so we can use the metric tensor as follows
\[ W^\Lambda_3 \Lambda B = \bar{G}_{\Pi \Lambda} W^\Lambda_3 \Lambda B, \] (8.37)
\[ W^\Lambda_{4(B+6)} = \bar{G}_{\Pi \Lambda} W^\Lambda_{4(B+6)}. \] (8.38)

Then we can rewrite (8.36) as
\[ \bar{p}_\Pi = W^\Lambda_3 \Lambda B Y^B + W^\Lambda_{4(B+6)} P_B. \] (8.39)

In order to analyze the meaning of (8.28) and (8.39), it is necessary to rewrite the line elements of two different manifolds, their corresponding Hamiltonians as well (8.28) and (8.39), as follows
\[ ds^2 = G_{\Lambda \Pi} d\bar{u}^\Lambda d\bar{u}^\Pi, \] (8.40)
\[ ds^2 = \bar{G}_{\Lambda \Pi} d\bar{u}^\Lambda d\bar{u}^\Pi, \] (8.41)
\[ Q = \frac{1}{2} G^{\Lambda \Pi} \bar{p}_\Lambda \bar{p}_\Pi, \] (8.42)
\[ \bar{Q} = \frac{1}{2} \bar{G}^{\Lambda \Pi} \bar{p}_\Lambda \bar{p}_\Pi, \] (8.43)
\[ \bar{u}^\Lambda = W^\Lambda_1 \Lambda B Y^B + W^\Lambda_{(B+6)} P_B. \] (8.44)

and
\[ \bar{p}_\Pi = W^\Lambda_3 \Lambda B Y^B + W^\Lambda_{4(B+6)} P_B. \] (8.45)

Equations (8.44) and (8.45) are transformations of coordinates and momenta of two phase spaces associated with two different manifolds given by the line elements (8.40) and (8.41).
9 Solutions of Matrices Riccati Equations

In this Section, we construct a set with an infinite number of exact solutions of matrix Riccati equations.

Let us rewrite the system (8.11)-(8.14),

\[ T_3 = \text{const.}, \]  
\[ T_1 = Y_4 T_3 t, \]  
\[ T_2 = -Y_4 T_3 Z_4 (t\tau), \]  
\[ T_4 = -T_3 Z_4 \tau. \]

We remember that the matrices \( T(i), \) \( Z(4) \) and \( Y(4) \) are (6X6).

Let us introduce the matrices \( T(i), \) \( Z(4) \) and \( Y(4) \) and decompose each of them as the matrix \( T \) given by (3.10). For such, we identify \( L_i = (S_i, R_i, D_i, E_i) \), in which we have index \( i \) running from 1 to 4,

\[
\begin{pmatrix}
  L_1 & L_2 \\
  L_3 & L_4
\end{pmatrix}
\]

Let us introduce the solutions of the matrices \( S, R, D \) and \( E \),

\[ S_1 = Y_4 S_3 t, \]  
\[ S_2 = Y_4 S_4 t, \]  
\[ R_2 = -R_1 Z_4 \tau. \]  
\[ R_4 = -R_3 Z_4 \tau. \]  
\[ D_1 = Y_4 D_3 t, \]  
\[ D_2 = Y_4 D_4 t, \]  
\[ E_2 = -E_1 Z_4 \tau, \]  
\[ E_4 = -E_3 Z_4 \tau. \]

and

\[ E_4 = -E_3 Z_4 \tau. \]
The system (9.6)-(9.13) is quite general, but the choice of some solutions for matrices S and R, for example, affects the solutions of matrices D and E because the systems (3.17) and (3.18).

For a better understanding, let us suppose systems (3.17) and (3.18) reduce into systems (2.21) and (2.22). In this case, it is easily seen that the matrices $S_3$, $S_4$, $R_1$ and $R_3$ are necessarily constant as are matrices $A$ in $T = SAR$.

As we can always choose $S$ and $R$ as being non-singular, it will always be possible to use the system (3.22) in the form (3.24).

Substituting the system (9.6)-(9.13) in (3.22), assuming the matrices $S_3$, $S_4$, $R_1$, $R_3$ as constant and $A_1$, $A_2$, $A_3$, $A_4$ as not constant, we have,

\begin{align*}
A_1 &= -(S_3)^{-1}S_4A_3 + (S_3)^{-1}\int_a^b f(\tau)\,d\tau + V_1, \quad (9.14) \\
A_2 &= -(S_3)^{-1}S_4A_4 + (S_3)^{-1}\int_a^b g(\tau)\,d\tau + V_2, \quad (9.15)
\end{align*}

in which matrices $V_1$ and $V_2$ are constant and functions $f(\tau)$ and $g(\tau)$ are arbitrary. From these conditions, we conclude matrices $A_3$, $A_4$ are arbitrary.

Let us consider a second set of solutions for the matrices $A_i$,

\begin{align*}
A_1 &= -A_2R_3(R_1)^{-1} + (R_1)^{-1}\int_a^b l(\tau)\,d\tau + U_1, \quad (9.16) \\
A_3 &= -A_4R_3(R_1)^{-1} + (R_1)^{-1}\int_a^b m(\tau)\,d\tau + U_2, \quad (9.17)
\end{align*}

in which matrices $U_1$ and $U_2$ are constant and functions $l$, $m$ are arbitrary.

The same occurs with matrices $A_2$, $A_4$, they are also arbitrary.

If we replace (9.1)-(9.4), (9.6)-(9.13) and (9.14)-(9.15) (or (9.16)-(9.17)) on $T = SAR$, we get an equality.

The choice $S_3$, $S_4$, $R_1$ and $R_3$ constants was arbitrary and allowed the calculation of exact solutions of some matrix equations. From the point of view of numerical computation, it will be possible a set of solutions much larger than we have obtained.
The choice $S_3$, $S_4$, $R_1$ and $R_3$ constants will simplify the D and E matrices as follows,

$$D_3 = -[S_3(A_1F_1 + A_2F_3) + S_4(A_3F_1 + A_4F_3)], \quad (9.18)$$

$$D_4 = -[S_3(A_1F_2 + A_2F_4) + S_4(A_3F_2 + A_4F_4)], \quad (9.19)$$

$$E_1 = -[G_1(A_1R_1 + A_2R_3) + G_2(A_3R_1 + A_4R_3)], \quad (9.20)$$

$$E_3 = -[G_3(A_1R_1 + A_2R_3) + G_4(A_3R_1 + A_4R_3)]. \quad (9.21)$$

in which $F_i$ and $G_i$ are arbitrary matrices and $A_i$ are given by (9.14)-(9.15) or (9.16)-(9.17).
10 Concluding Remarks

One objective of this paper was to obtain information about a system of differential equations in which the solutions are unknown. For such, we use another system whose solutions are known. However, W matrices are functions of coordinates and momenta of the two systems, one unknown and the other known. This restricts our initial objective, but there is a large number of systems of differential equations in which only the coefficients of derivatives are known and this might be a good opportunity to use this formalism. On the other hand, for well-known situations, such as Schwarzschild and Reissner-Nordstron metrics, we will be able to construct the mapping between these important geometries.

Another important objective of this paper was to offer a set with an infinite number of exact solutions of the Riccati quadratic matrix equation. It was also shown that this could be associated with a set with an infinite number of Hamiltonians. It is always possible to associate matrices Riccati equations with a set of infinite number of Hamiltonians.
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