CLEFT EXTENSIONS OF WEAK HOPF ALGEBRAS

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Abstract. In this paper we study the theory of cleft extensions for a weak bialgebra $H$. Among other results, we determine when two unitary crossed products of an algebra $A$ by $H$ are equivalent and we prove that if $H$ is a weak Hopf algebra, then the categories of $H$-cleft extensions of an algebra $A$, and of unitary crossed products of $A$ by $H$, are equivalent.

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Introduction

The initial motivation for this work comes from our study of the (co)homology of crossed products in weak contexts started in [14], and the present paper provides a theoretical basis for that study. A usual technique for calculating the Hochschild and cyclic homologies of crossed products is to build mixed complexes simpler than the canonical one, which give these homologies. It is convenient that these complexes are provided with filtrations whose associated spectral sequences generalize the classic ones of Hochschild-Serre and Feigin and Tygan. Such complexes exist under very mild conditions. For instance in [9] appropriated complexes were obtained for the Brzeziński’s crossed products introduced in [8], and it is possible to show that these results remain valid in the more general context considered in [3], [4] and [12]. However, if one wants to compute the second face of these spectral sequences it is necessary to restrict themselves to the crossed products with invertible cocycle (or, which is equivalent, to the cleft extensions). For crossed products of algebras with Hopf algebras there is a classical notion of invertible cocycle which works very well. For the crossed products of algebras by weak Hopf algebras introduced
the notions of algebra, coalgebra, module and comodule in $C$ and constraints $\lambda$ are associative and the coalgebras are coassociative. Given a unitary algebra $A$ by the coherence Mac Lane Theorem it suffices to prove the results when $\phi$ is invertible. In this paper we consider a symmetric category $\mathcal{C}$ with split idempotents and we study the weak crossed products with invertible cocycle of an algebra $A$ by a weak Hopf algebra $H$ in $\mathcal{C}$.

Our terminology differs slightly from that used in [1], [2], [3], [4], [5], [11] and [16]. The main differences are two. First that, opposite to the made out in that papers, in the definition of crossed product system we assume that $\nabla_\chi \circ \mathcal{F} = \mathcal{F}$ (this allow us simplify some arguments and it is free of cost because can be achieved simply by replacing $\mathcal{F}$ with $\nabla_\chi \circ \mathcal{F}$); and second, that for us, the crossed product associated with a crossed product system with preunit $(A, V, \chi, \mathcal{F}, \nu)$, in which $\mathcal{F}$ is a cocycle satisfying the twisted module condition, is $A \times_\chi V$ instead of $A \otimes_\chi V$ (but both points of view are equivalent because of Theorem 1.12(9)).

This paper is organized as follows: In Section 1 first we recall the basic properties of the preunits of an algebra; then we review the basic properties of the very general notion of crossed products of algebras by objects of $\mathcal{C}$ introduced in [5] and we do a quick review of the notion of Weak Hopf Algebra. In Section 2 we begin the study of the crossed products of algebras by weak Hopf algebras introduced in [5]. The main results are Propositions 2.7, 2.8, 2.22 and 2.26, and Corollaries 2.24 and 2.25. In Section 3 we extend to the setting of arbitrary weak bialgebras the concept of equivalence of crossed products introduced in [5, Section 3] for the case of cocommutative weak Hopf algebras. Opposite to the made out in that paper we do not require that the cocycle be invertible. In Section 4 we continue the study started in Section 2 investigating the consequences that $A$ is a weak $H$-module algebra. The main results are Propositions 4.3, 4.6, 4.10 and 4.12. In Section 5 under the assumption that $H$ is a weak Hopf algebra, we prove that each crossed product of a weak $H$-module algebra $A$ by $H$ with invertible cocycle, is an $H$-cleft extension of $A$. Finally, in Section 6 we prove that each $H$-cleft extension is isomorphic to a crossed product with invertible cocycle, of $H$ by a weak $H$-module algebra. Combining the results of this and the previous section, we obtain that for each weak Hopf algebra $H$ and each algebra $A$, the categories of unitary crossed products of $A$ by $H$ with invertible cocycle, such that $A$ is a weak $H$-module algebra, is equivalent to the category of $H$-cleft extensions of $A$.

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1 Preliminaries

Throughout this paper $\mathcal{C} = (\mathcal{C}, \otimes, K, \alpha, \lambda_\lambda, \lambda_\nu)$ is a symmetric category with split idempotents. We use the words arrow, map and morphisms as synonyms. Given objects $U$, $V$ and $W$ in $\mathcal{C}$ and a map $g: V \to W$, we write $U \otimes g$ for $\text{id}_U \otimes g$ and $g \otimes U$ for $g \otimes \text{id}_U$. By our assumptions, for each idempotent morphism $\phi: V \to V$ in $\mathcal{C}$ there exist and object $\phi(V)$ and maps $\iota_\phi: \phi(V) \to V$ and $\rho_\phi: V \to \phi(V)$, that we fix once and for all, such that $\rho_\phi \circ \iota_\phi = \phi$ and $\alpha \circ \rho_\phi = \text{id}_V$. By the coherence Mac Lane Theorem it suffices to prove the results when $\phi$ is strict. In other words, we can assume without loose of generality that the associativity constraint $\alpha$ and the unit constraints $\lambda_\lambda$ and $\lambda_\nu$ are identities, and we do it. We assume that the reader is familiar with the notions of algebra, coalgebra, module and comodule in $\mathcal{C}$. We also assume that the algebras are associative and the coalgebras are coassociative. Given a unitary algebra $A$ and a counitary
coalgebra $C$, we let

$$\mu: A \otimes A \to A, \quad \eta: K \to A, \quad \Delta: C \to C \otimes C \quad \text{and} \quad \epsilon: C \to K$$

denote the multiplication, the unit, the comultiplication and the counit, respectively, specified with a subscript if necessary.

Given an algebra $A$ and a coalgebra $C$ we consider $\text{Hom}(C, A)$ endowed with the convolution product $\alpha * \beta := \mu \circ (\alpha \otimes \beta) \circ \Delta$. It is well known that $\text{Hom}(C, A)$ is a monoid with unit $\eta \circ \epsilon$.

We use the nowadays well known graphic calculus for symmetric categories. As usual, morphisms will be composed from top to bottom and tensor products will be represented by horizontal concatenation from left to right. The identity map of an object will be represented by a vertical line. Given an algebra $A$, the diagrams

$$\begin{array}{c}
\begin{array}{c}
\gamma, \\
\delta
\end{array}
\end{array}$$

stand for the multiplication map, the unit, the action of $A$ on a left $A$-module and the action of $A$ on a right $A$-module, respectively. Similarly, given a coalgebra $C$, the diagrams

$$\begin{array}{c}
\begin{array}{c}
\omega, \\
\rho
\end{array}
\end{array}$$

stand for the comultiplication map, the counit, the coaction of $C$ on a left $C$-comodule and the coaction of $C$ on a right $C$-comodule, respectively. The natural isomorphism $c$ will be represented by the diagram

$$\begin{array}{c}
\begin{array}{c}
\circledS
\end{array}
\end{array}$$

and a map from a tensor product $V_1 \otimes \cdots \otimes V_m$ of $m$ objects to a tensor product $W_1 \otimes \cdots \otimes W_n$ of $n$ objects, will be represented by a rectangular box with rounded edges, having $m$ entries up and $n$ outputs down and the name of the arrow inside. We will make the following exceptions: given a twisted space $(A, V, \chi)$, we will use the diagram

$$\begin{array}{c}
\begin{array}{c}
\chi
\end{array}
\end{array}$$

to represent the map $\chi$, and given $f: H \otimes H \to A$, we will use the diagrams

$$\begin{array}{c}
\begin{array}{c}
\chi
\end{array}
\end{array}$$

and

$$\begin{array}{c}
\begin{array}{c}
\chi
\end{array}
\end{array}$$

to represent the map $f$ and its convolution inverse if $f$ is convolution invertible.

1.1 Preunits

In this subsection we give a quick review (without proofs) of the notion of preunit introduced in [10] (see also [11]) and its basic properties.

Definition 1.1. Let $\nabla: B \to B$ be an idempotent morphism. We say that an associative product $\mu_B: B \otimes B \to B$ is normalized with respect to $\nabla$ if $\nabla \circ \mu_B = \mu_B = \mu_B \circ (\nabla \otimes \nabla)$.

Definition 1.2. Let $B$ be an associative algebra. A preunit of $\mu_B$ is a morphism $\nu: K \to B$ such that $\mu_B \circ (B \otimes \nu) = \mu_B \circ (\nu \otimes B)$ and $\nu = \mu \circ (\nu \otimes \nu)$.

Note that if $k$ is a field and $C$ is the category of $k$-vector spaces, then $\nu: K \to B$ is a preunit of $\mu_B$ if and only if $\nu(1)$ is a central idempotent of $B$.

Remark 1.3. Let $B$ be an associative algebra. If $\nu$ is a preunit of $\mu_B$, then the morphism $\nabla_\nu: B \to B$, defined by $\nabla_\nu := \mu_B \circ (B \otimes \nu)$, is an idempotent that satisfies

$$\mu_B \circ (\nabla_\nu \otimes B) = \nabla_\nu \circ \mu_B = \mu_B \circ (B \otimes \nabla_\nu) = \mu_B \circ (\nabla_\nu \otimes \nabla_\nu) \quad \text{and} \quad \nabla_\nu \circ \nu = \nu.$$
From this and the associativity of $\mu_B$ it follows that the morphism $\mu^B_B: B \otimes B \to B$, defined by $\mu^B_B := \nabla_\nu \circ \mu_B$, is an associative product that is normalized with respect to $\nabla_\nu$. Moreover $\nu$ is a preunit of $\mu^B_B$ and $\mu^B_B \circ (B \otimes \nu) = \nabla_\nu$.

Remark 1.4. Let $\ast$ be an associative algebra and let $\nu$ be a preunit of $\mu_B$. Write $B := \nabla_\nu(B)$, $\iota_\nu := \iota_\nu$ and $p_\nu := \rho_\nu$. The map $\mu_B: B \otimes B \to B$, given by $\mu_B := p_\nu \circ \mu_B \circ (\iota_\nu \otimes \iota_\nu)$, is an associative product with unit $\eta_B := p_\nu \circ \nu$. Moreover, $\mu_B = p_\nu \circ \mu^B_B \circ (\iota_\nu \otimes \iota_\nu)$, where $\mu^B_B$ is as in Remark 1.3; the map $p_\nu: B \to B$ is multiplicative and the map $\iota_\nu: B \to B$ satisfies the equality $\iota_\nu \circ \mu_B = \mu^B_B \circ (\iota_\nu \otimes \iota_\nu)$.

Remark 1.5. Let $\nabla: B \to B$ be an idempotent arrow and let $B := \nabla(B)$. If $\mu_B: B \otimes B \to B$ is an associative product with unit $\eta_B$, then the map $\mu_B: B \otimes B \to B$, given by $\mu_B := \nu \circ \mu_B \circ (p_\nu \circ \rho_\nu)$, is an associative product which is normalized with respect to $\nabla$ and $\nu := \iota_\nu \circ \eta_B$ is a preunit of $\mu_B$ such that $\nabla = \nabla_\nu$.

1.2 General weak crossed products

In this subsection we recall a very general notion of crossed product developed in [3] and [11], and we review its basic properties.

Definition 1.6. A triple $(A, V, \chi)$, consisting of an associative unitary algebra $A$ in $\mathcal{C}$, an object $V$ of $\mathcal{C}$ and a morphism $\chi: V \otimes A \to A \otimes V$, is a twisted space if

$$\chi \circ (V \otimes \mu_A) = (\mu_A \otimes V) \circ (\chi \otimes A).$$

(1.1)

In such a case we say that $\chi$ is a twisting map.

From here to Definition 1.8 inclusive, we assume that $(A, V, \chi)$ is a twisted space. Note that $A \otimes V$ is a non unitary $A$-bimodule in $\mathcal{C}$ via the left and right actions $\rho_l$ and $\rho_r$ given by $\rho_l := \mu_A \otimes V$ and $\rho_r := (\mu_A \otimes V) \circ (A \otimes \chi)$, respectively. Clearly $\rho_l$ is an unitary action. We let $\nabla_\chi$ denote the left and right $A$-linear idempotent endomorphism of $A \otimes V$, defined by $\nabla_\chi := \rho_r \circ (A \otimes \eta_A)$. Set $A \times V := \nabla_\chi(A \otimes V)$, $\iota_\chi := \iota_\nu \otimes \chi$ and $p_\chi := p_{\nu \otimes \chi}$. Note that $A \times V$ is an unitary $A$-bimodule via the left and right actions $\tilde{\rho}_l$ and $\tilde{\rho}_r$ given by $\tilde{\rho}_l := p_{\chi} \circ \rho_l \circ (\iota_\chi \otimes A)$ and $\tilde{\rho}_r := p_{\chi} \circ \rho_r \circ (A \otimes \iota_\chi)$, respectively. Moreover, $\iota_\chi$ and $p_\chi$ are $A$-bimodule morphisms.

Definition 1.7. A tuple $(A, V, \chi, \mathcal{F})$ is a crossed product system if $\mathcal{F}: V \otimes V \to A \otimes V$ is a morphism such that $\nabla_\chi \circ \mathcal{F} = \mathcal{F}$.

Definition 1.8 ([3] Definitions 2.4 and 2.5). We say that $\mathcal{F}$ is a cocycle that satisfies the twisted module condition if

$$\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (1) at (-1,0){$x$};
\node (2) at (1,0){$y$};
\node (3) at (0,1){$\chi$};
\draw (1) to (3);
\draw (2) to (3);
\end{tikzpicture}
\end{array}
\end{align*}
\quad \text{and} \quad
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (1) at (-1,0){$x$};
\node (2) at (1,0){$y$};
\node (3) at (0,1){$\chi$};
\node (4) at (1,1){$\mathcal{F}$};
\draw (1) to (3);
\draw (2) to (3);
\draw (3) to (4);
\end{tikzpicture}
\end{array}
\end{align*}
\ .$$

More precisely, the first equality says that $\mathcal{F}$ satisfies the twisted module condition and the second one says that $\mathcal{F}$ is a cocycle.

From here to Definition 1.11 inclusive $(A, V, \chi, \mathcal{F})$ is a crossed product system.

Notation 1.9. We let $A \otimes \mathcal{F} V$ and $A \times \mathcal{F} V$ denote the objects $A \otimes V$ and $A \times V$, endowed with the multiplication maps $\mu_{A \otimes \mathcal{F} V}$ and $\mu_{A \times \mathcal{F} V}$, defined by $\mu_{A \otimes \mathcal{F} V} := (\mu_A \otimes V) \circ (\mu_A \otimes \mathcal{F}) \circ (A \otimes \chi \otimes V)$ and $\mu_{A \times \mathcal{F} V} := p_{\chi} \circ \mu_{A \otimes \mathcal{F} V} \circ (\iota_\chi \otimes \iota_\chi)$.
respectively. For the sake of brevity we will write $E$ instead of $A \times \xi V$, $\mathcal{E}$ instead of $A \otimes \xi V$, $\mu_E$ instead of $\mu_{A \times \xi V}$ and $\mu_\mathcal{E}$ instead of $\mu_{A \otimes \xi V}$. We consider $\mathcal{E} \otimes \mathcal{E}$ and $E \otimes E$ as $A$-bimodules in the natural way.

**Definition 1.10.** We say that $E$ is the crossed product of $A$ by $V$ associated with $\chi$ and $\mathcal{F}$ if $\mathcal{F}$ a cocycle that satisfies the twisted module condition.

**Definition 1.11.** Let $\nu: K \to A \otimes V$ be an arrow. The tuple $(A, \chi, \mathcal{F}, \nu)$ is a crossed product system with preunit if

\[
(\mu_A \otimes V) \circ (A \otimes \mathcal{F}) \circ (\chi \otimes V) \circ (V \otimes \nu) = \nabla_\chi \circ (\eta \otimes V), \tag{1.2}
\]

\[
(\mu_A \otimes V) \circ (A \otimes \mathcal{F}) \circ (\nu \otimes V) = \nabla_\chi \circ (\eta \otimes V), \tag{1.3}
\]

\[
(\mu_A \otimes V) \circ (A \otimes \chi) \circ (\nu \otimes A) = (\mu_A \otimes V) \circ (A \otimes \nu). \tag{1.4}
\]

Let $(A, \nu, \chi, \mathcal{F}, \nu)$ be a crossed product system with preunit and let

\[
\nabla_\nu: A \otimes V \to A \otimes V, \quad j_\nu: A \to \mathcal{E}, \quad \nu: A \to E \quad \text{and} \quad \gamma: V \to E
\]

be the arrows defined by

\[
\nabla_\nu := \mu_E \circ (\mathcal{E} \otimes \nu), \quad j_\nu := (\mu_A \otimes V) \circ (A \otimes \nu), \quad \nu := p_\chi \circ j_\nu \quad \text{and} \quad \gamma := p_\chi \circ (\eta_A \otimes V).
\]

Except for items 4), 7) and the assertions about the right $A$-linearity in items 3), 5) and 6), whose proofs we leave to the reader, the following result is [11] Remark 3.10, one implication of Theorem 3.11 and Corollary 3.12.

**Theorem 1.12.** Let $(A, \nu, \chi, \mathcal{F}, \nu)$ be a crossed product system with preunit. If $\mathcal{F}$ is a cocycle that satisfies the twisted module condition, then the following facts hold:

1. $\mu_E$ is a left and right $A$-linear associative product, that is normalized with respect to $\nabla_\chi$.
2. $\nu$ is a preunit of $\mu_E$, $\nabla_\nu \circ \nu = \nu$ and $\nabla_\nu = \nabla_\chi$.
3. $\mu_E$ is left and right $A$-linear, associative and has unit $\eta_E := p_\chi \circ \nu$.
4. The maps $\iota_\chi$ and $p_\chi$ are multiplicative.
5. $j_\nu$ is left and right $A$-linear, multiplicative, and satisfies $\nabla_\nu \circ j_\nu = j_\nu$.
6. $\nu$ is left and right $A$-linear, multiplicative and unitary.
7. $\mu_E \circ (\nu \otimes E) = \bar{\rho}_\nu$ and $\mu_E \circ (E \otimes \nu) = \bar{\rho}_\nu$.
8. $\chi = \mu_E \circ (\eta \otimes V \otimes j_\nu)$ and $\mathcal{F} = \mu_E \circ (\eta \otimes V \otimes \eta \otimes V)$.
9. $\chi = \iota_\chi \circ \mu_E \circ (\gamma \otimes \mathcal{F})$ and $\mathcal{F} = \iota_\chi \circ \mu_E \circ (\gamma \otimes \gamma)$.

**Definition 1.13.** Let $(A, \nu, \chi, \mathcal{F}, \nu)$ be a crossed product system with preunit. If $\mathcal{F}$ a cocycle that satisfies the twisted module condition, then we say that the algebra $E$ is the unitary crossed product of $A$ by $V$ associated with $\chi$, $\mathcal{F}$ and $\nu$. We also say that $\chi$ and $\mathcal{F}$ are the twisting map and the cocycle of $E$, respectively.

**Remark 1.14.** By Theorem 1.12(6) and the fact that $p_\chi$ is left $A$-linear and $\gamma = p_\chi \circ (\eta_A \otimes V)$,

\[
\mu_E \circ (\nu \otimes \gamma) = \bar{p}_\nu \circ (A \otimes \gamma) = p_\chi. \tag{1.5}
\]

**Remark 1.15.** For each crossed product system with preunit $(A, V, \chi, \mathcal{F}, \nu)$, we have

\[
\mu_E \circ (\gamma \otimes \nu) = \mu_E \circ (\nu \otimes \gamma) \circ \chi \quad \text{and} \quad \mu_E \circ (\gamma \otimes \gamma) = \mu_E \circ (\nu \otimes \gamma) \circ \mathcal{F}.
\]
1.3 Weak Hopf Algebras

Weak bialgebras and weak Hopf algebras are generalizations of bialgebras and Hopf algebras, introduced in [5,7], in which the axioms about the unit, the counit and the antipode are replaced by weaker properties. For the convenience of the reader, in this subsection we collect without proofs the properties of weak Hopf algebras in an symmetric tensor category \( C \), with split idempotents, that we need in this paper. All the results considered by us were established in [5] and [10], or they are immediate consequence of results obtained in those papers. In spite of that in [6] and [10] the authors work in the setting of finite dimensional vector spaces, all the results in this subsection (and in this paper) are valid in the context of symmetric tensor categories.

**Definition 1.16.** A weak bialgebra in \( C \) is an object \( H \) endowed with an unitary algebra and a counitary coalgebra structure, such that:

1. \( \Delta \circ \mu = (\mu \otimes \mu) \circ c \circ (\Delta \otimes \Delta) \),
2. \( \epsilon \circ \mu \circ (\mu \otimes H) = (\mu \otimes \epsilon \circ \mu) \circ (H \otimes \Delta \otimes H) = (\epsilon \circ \mu \otimes \epsilon \circ \mu) \circ (H \otimes \Delta_{\text{op}} \otimes H) \),
3. \( (\Delta \otimes \eta) \circ \Delta \circ \eta = (\Delta \otimes \eta \otimes \Delta \otimes \eta) \circ (H \otimes \mu \otimes H) \circ (\Delta \otimes \eta \circ \Delta \otimes \eta) = (H \otimes \mu_{\text{op}} \otimes H) \circ (\Delta \otimes \eta \circ \Delta \otimes \eta) \),

where \( \Delta_{\text{op}} := c \circ \Delta \) and \( \mu_{\text{op}} := \mu \circ c \).

Let \( H \) be a weak bialgebra. We denote by \( \Pi_L, \Pi_R, \Pi^L \) and \( \Pi^R \) the endomorphisms of \( H \) defined by

\[
\Pi_L := (\epsilon \circ \mu \otimes H) \circ (H \otimes c) \circ (\Delta \otimes \eta \otimes H), \\
\Pi_R := (H \otimes \epsilon \circ \mu) \circ (c \otimes H) \circ (H \otimes \Delta \otimes \eta), \\
\Pi^L := (H \otimes \epsilon \circ \mu) \circ (\Delta \circ \eta \otimes H), \\
\Pi^R := (\epsilon \circ \mu \otimes H) \circ (\Delta \circ \eta \circ \Delta \otimes \eta).
\]

A direct computation shows that \( \Pi^L \) and \( \Pi^R \) are idempotent morphisms (see, for instance, [6,10]). For each \( X \in \{ L, R \} \), set \( H^X := \Pi^X(H), \ i_X := i_{H^X} \) and \( p_X := p_{H^X} \).

**Proposition 1.17.** The following equalities hold:

\[
\Pi_L \circ \Pi_L = \Pi_L, \quad \Pi_L \circ \Pi^R = \Pi^R, \quad \Pi^R \circ \Pi^L = \Pi^L, \\
\Pi_L \circ \Pi^L = \Pi^L, \quad \Pi^L \circ \Pi^R = \Pi^R, \quad \Pi^R \circ \Pi^L = \Pi^L, \\
\Pi^L \circ \Pi^R = \Pi^R, \quad \Pi^R \circ \Pi^L = \Pi^L.
\]

**Remark 1.18.** By Proposition 1.17 the maps \( \bar{p}_R : H \rightarrow H^L \) and \( \bar{p}_L : H \rightarrow H^R \), defined by

\[
\bar{p}_R := p_L \circ \Pi^R \quad \text{and} \quad \bar{p}_L := p_R \circ \Pi^L,
\]

satisfy \( u_L \circ \bar{p}_R = \Pi^R, \quad \bar{p}_R \circ u_L = \text{id}_{H^L}, \quad u_R \circ \bar{p}_L = \Pi^L \) and \( \bar{p}_L \circ u_R = \text{id}_{H^R} \). So, \( \Pi^L \) and \( \Pi^R \) also are idempotents.

**Remark 1.19.** An immediate computation shows that \( \text{id} \ast \Pi^R = \text{id} \ast \Pi^L = \text{id} \).

**Remark 1.20.** For each \( X \in \{ L, R \} \), the maps \( \Pi^X \) and \( \Pi^X \) are unitary and counitary.

**Proposition 1.21.** The following equalities hold:

1. \( \epsilon \circ \mu = \epsilon \circ \mu \circ (H \otimes \Pi^L) = \epsilon \circ \mu \circ (\Pi^R \otimes H) = \epsilon \circ \mu \circ (\Pi^R \otimes \Pi^L) \),
2. \( \Delta \circ \eta = (H \otimes \Pi^L) \circ \Delta \circ \eta = (\Pi^R \otimes \Pi^L) \circ \Delta \circ \eta \),
3. \( \epsilon \circ \mu = \epsilon \circ \mu \circ (H \otimes \Pi^L) = \epsilon \circ \mu \circ (\Pi^R \otimes \Pi^L) = \epsilon \circ \mu \circ (\Pi^R \otimes \Pi^L) \),
4. \( \Delta \circ \eta = (H \otimes \Pi^R) \circ \Delta \circ \eta = (\Pi^L \otimes \Pi^R) \circ \Delta \circ \eta \).

**Proposition 1.22.** The following equalities hold:

1. \( \mu \circ (\Pi^L \otimes \Pi^L) = \Pi^L \circ \mu \circ (\Pi^L \otimes \Pi^L) = \Pi^L \circ \mu \circ (\Pi^L \otimes H) \),
Proposition 1.26. Let the following equalities hold:
$$p = \Delta = (\Pi^L \otimes \Pi^L) \circ \Delta = (\Pi^L \otimes H) \circ \Delta \circ \Pi^L,$$
$$\mu = (\Pi^R \otimes \Pi^R) \circ \Delta = (H \otimes \Pi^R) \circ \Delta \circ \Pi^R.$$ 

Remark 1.23. From Remark 1.20 and Proposition 1.22 it follows that $H^R$ and $H^L$ are unitary associative algebras via the multiplication maps $p_R \circ \mu \circ (\iota_R \otimes \iota_R)$ and $p_R \circ \mu \circ (\iota_L \otimes \iota_L)$, respectively. Moreover the arrows $\iota_R$ and $\iota_L$ are algebra morphisms.

Proposition 1.24. It is true that $\mu \circ \circ (\Pi^L \otimes \Pi^R) = \mu \circ (\Pi^L \otimes \Pi^R).$

Proposition 1.25. The following equalities hold:
$$\mu \circ (H \otimes \Pi^L) = (\epsilon \circ \mu \otimes H) \circ (H \otimes c) \circ (\Delta \otimes H),$$
$$\mu \circ (\Pi^R \otimes H) = (H \otimes \epsilon \circ \mu) \circ (c \otimes H) \circ (\Delta \otimes H),$$
$$\mu \circ (\Pi^R \otimes H) = (H \otimes \epsilon \circ \mu) \circ (c \otimes H) \circ (\Delta \otimes H).$$

Proposition 1.26. The following equalities hold:
$$H \otimes \Pi^L \circ \Delta = (\mu \otimes H) \circ (H \otimes c) \circ (\Delta \otimes \eta \otimes H),$$
$$(\Pi^L \otimes H) \circ \Delta = (H \otimes \mu) \circ (\Delta \otimes \eta \otimes H),$$
$$(\Pi^R \otimes H) \circ \Delta = (H \otimes \mu) \circ (\Delta \otimes \eta \otimes H),$$
$$(H \otimes \Pi^R) \circ \Delta = (\mu \otimes H) \circ (H \otimes \Delta \otimes \eta).$$

Proposition 1.27. The following equalities hold:
$$\Pi^L \circ \mu = \Pi^L \otimes \mu \circ (H \otimes \Pi^L),$$
$$\Pi^R \circ \mu = \Pi^R \otimes \mu \circ (\Pi^R \otimes H),$$
$$\Delta \circ \Pi^L = (H \otimes \Pi^L) \circ \Delta \circ \Pi^L,$$
$$\Delta \circ \Pi^R = (\Pi^R \otimes H) \circ \Delta \circ \Pi^R.$$
2 Crossed products by weak bialgebras

In this section we begin the study of the crossed product of an algebra $A$ by a weak bialgebra $H$, introduced in [3].

2.1 Basic properties

**Definition 2.1.** An arrow $\rho \otimes A \rightarrow A$ is a weak measure of $H$ on $A$ if 
\[ \rho \circ (H \otimes \mu_A) = \mu \circ (\rho \otimes \rho) \circ (H \otimes c \otimes A) \circ (\Delta \otimes A \otimes A). \]

From here to the end of this section $\rho$ is a weak measure. Let $\chi_{\rho}: H \otimes A \rightarrow A \otimes H$ be the morphism defined by $\chi_{\rho} := (\rho \otimes H) \circ (H \otimes c) \circ (\Delta \otimes A \otimes A)$.

It is well known that $(A, H, \chi_{\rho})$ is a twisted space. Moreover $(A \otimes \epsilon) \circ \chi_{\rho} = \rho$. Let $A \rtimes H$, $\nabla_{\chi_{\rho}}$, $p_{\chi_{\rho}}$ and $i_{\chi_{\rho}}$ be as in Subsection 2.2. Set $\nabla_{\rho} := \nabla_{\chi_{\rho}}$, $p_{\rho} := p_{\chi_{\rho}}$ and $i_{\rho} := i_{\chi_{\rho}}$. By definition
\[ \nabla_{\rho} = (\mu \otimes H) \circ (A \otimes \rho \otimes H) \circ (A \otimes H \otimes c) \circ (A \otimes \Delta \otimes \eta_A). \]

Furthermore, we know that $\chi_{\rho} = \nabla_{\rho} \circ \chi_{\rho}$, that $\nabla_{\rho}$ is left and right $A$-linear and idempotent and that $p_{\rho}$ and $i_{\rho}$ are left and right $A$-linear.

Given a morphism $f: H \otimes H \rightarrow A$, we define $F_f: H \otimes H \rightarrow A \otimes H$ by $F_f := (f \otimes \mu) \circ \Delta_{H=2}$. 

**Remark 2.2.** A direct computation using, that $c$ is natural, $\Delta$ and $\Delta_{H=2}$ are coassociative and item 1) of Definition 1.10 shows that

\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
\chi_{\rho} & = & \chi_{\rho} \\
\nabla_{\rho} & = & \nabla_{\rho} \\
\end{array}
\end{array}
\end{align*}

Remark 2.3. It is evident that $A \otimes H$ is a comonoidal $H$-module via the map $\delta_{A \otimes H} := A \otimes \Delta$. Moreover, by the first equality in Remark 2.2, the map $\nabla_{\rho}$ is $H$-colinear. A direct computation using this shows that $A \times H$ is a comonoidal $H$-comodule via $\delta_{A \times H} := (p_{\rho \otimes H} \circ \delta_{A \otimes H} \circ i_{\rho}$ and that $p_{\rho}$ and $i_{\rho}$ are $H$-colinear maps.

**Proposition 2.4.** The following conditions are equivalent:

1) $f = (A \otimes \epsilon) \circ F_f$ and $F_f = \nabla_{\rho} \circ F_f$.

2) $f = \mu_A \circ (A \otimes \rho) \circ (F_f \otimes \eta_A)$.

**Proof.** 1) $\Rightarrow$ 2) We have
\[ f = (A \otimes \epsilon) \circ F_f = (A \otimes \epsilon) \circ \nabla_{\rho} \circ F_f = \mu_A \circ (A \otimes \rho) \circ (F_f \otimes \eta_A). \]

2) $\Rightarrow$ 1) Left to the reader (use the second equality in Remark 2.2). 

Consequently, if $f$ satisfies Proposition 2.4 2), then $(A, H, \chi_{\rho}, F_f)$ is a crossed product system. If this is the case, then we set
\[ E = A \times_{\rho} V := A \times_{\chi_{\rho}} F_f V \quad \text{and} \quad \mathcal{E} = A \otimes_{\rho} V := A \otimes_{\chi_{\rho}} F_f V. \]

**Definition 2.5.** A map $f: H \otimes H \rightarrow A$ is a cocycle that satisfies the twisted module condition if

\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
\chi_{\rho} & = & \chi_{\rho} \\
\nabla_{\rho} & = & \nabla_{\rho} \\
\end{array}
\end{array}
\end{align*}

More precisely, the first equality says that $f$ satisfies the twisted module condition and the second one says that $f$ is a cocycle.

The following result was established in the proof of [11, Theorem 4.2]. For the convenience of the reader we provide a diagrammatic proof.

**Proposition 2.6.** If $f = (A \otimes \epsilon) \circ F_f$, then the following assertions are true:

1. $f$ satisfies the twisted module condition if and only if $F_f$ does it.
2. $f$ is a cocycle if and only if $F_f$ is.

**Proof.** We prove item (1) and leave the proof of item (2) to the reader. Composing the first equality of Definition 1.8 with $A \otimes \epsilon$ and using that $\rho = (A \otimes \epsilon) \circ \chi \rho$ and $f = (A \otimes \epsilon) \circ F_f$, we obtain that if $F_f$ satisfies the twisted module condition, then $f$ also satisfies the twisted module condition. Conversely, if this happens, then we have

\[
\begin{align*}
\chi \rho \chi \rho F_f &= \chi \rho \chi \rho f \\
&= \chi \rho \chi \rho F_f
\end{align*}
\]

where the first equality holds by the very definition of $F_f$; the second one, by the first equality in Remark 2.2; the third one, since $c$ is natural; the fourth one, since $f$ satisfies the twisted module condition and $c$ is natural; the fifth one, by the second equality in Remark 2.2 and the last one, by the very definition of $\chi \rho$. □

**Proposition 2.7.** If $f : H \otimes H \to A$ satisfies $f = (A \otimes \epsilon) \circ F_f$, then

\[f \circ (\mu \otimes H) \circ (H \otimes \Pi^R \otimes H) = f \circ (H \otimes \mu) \circ (H \otimes \Pi^R \otimes H).\]

**Proof.** By the hypothesis, Proposition 1.29 and the fact that $c$ is natural and $\mu_H$ is associative

\[
\begin{align*}
\Pi^R \chi &= \Pi^R \chi \\
&= \Pi^R \chi \\
&= \Pi^R \chi
\end{align*}
\]

as desired. □

**Proposition 2.8.** If $f : H \otimes H \to A$ satisfies $(\epsilon \circ \mu \otimes f) \circ \Delta_{H^2} = f$, then

\[f \circ (\mu \otimes H) \circ (H \otimes \Pi^L \otimes H) = f \circ (H \otimes \mu) \circ (H \otimes \Pi^L \otimes H).\]

**Proof.** By the hypothesis, Proposition 1.29 and the fact that $c$ is natural and $\mu$ is associative

\[
\begin{align*}
\Pi^L \chi &= \Pi^L \chi \\
&= \Pi^L \chi \\
&= \Pi^L \chi
\end{align*}
\]
as desired.

**Proposition 2.9** ([16, Remark 3.5]). Let \( \nu: K \to A \otimes H \) be a map. The following assertions are equivalent:

1. \( \nu = (A \otimes \Pi^L) \circ \nu \).
2. \((A \otimes \Delta) \circ \nu = (A \otimes \mu \otimes H) \circ (A \otimes H \otimes \Delta) \circ (\nu \otimes \eta)\).
3. \((A \otimes \Delta) \circ \nu = (A \otimes \mu \otimes H) \circ (A \otimes c \otimes H) \circ (A \otimes H \otimes \Delta) \circ (\nu \otimes \eta)\).

**Proof.** (1) \( \Rightarrow \) (2) and (3) By Proposition 1.28(3) and the fact that \( \Pi^L \) is idempotent, we have

\[
\nu = \nu \Pi^L = \nu \Pi^L = \nu \Pi^L \quad \text{and} \quad \nu = \nu \Pi^L = \nu \Pi^L = \nu \Pi^L,
\]
as desired.

(2) \( \Rightarrow \) (1) Since \( \Delta \) is counitary this follows composing \( A \otimes \varepsilon \otimes H \) to the equality in item (2) and using Proposition 1.21(2).

(3) \( \Rightarrow \) (1) Mimic the proof of (2) \( \Rightarrow \) (1).

The following result was established in the proof of [11, Theorem 4.2]. For the convenience of the reader we provide a diagrammatic proof.

**Proposition 2.10.** Let \( \nu: K \to A \otimes H \) and \( f: H \otimes H \to A \) be maps. Assume that \( f \) satisfies the first equality in Proposition 2.4(1) and that \( \nu = (A \otimes \Pi^L) \circ \nu \). The following assertions hold:

1. The map \( \nu \) satisfies (1.2) if and only if \( \rho (H \otimes \eta_A) = \mu_A \circ (\rho \otimes f) \circ (H \otimes c \otimes H) \circ (\Delta \otimes \nu) \).
2. The map \( \nu \) satisfies (1.3) if and only if \( \rho (H \otimes \eta_A) = \mu_A \circ (A \otimes f) \circ (\nu \otimes H) \).

**Proof.** (1) Assume that \( \rho (H \otimes \eta_A) = \mu_A \circ (\rho \otimes f) \circ (H \otimes c \otimes H) \circ (\Delta \otimes \nu) \). Then, by the coassociativity of \( \Delta \), Definition 1.16(1), the fact that \( \mu \) is unitary and \( c \) is natural, Propositions 1.21(2) and 2.7 and condition (3) in Proposition 2.9

which is condition (1.2). In order to prove the converse, it suffices to apply \( A \otimes \varepsilon \) to the equality in (1.2) and use the first equality Proposition 2.4(1).

(2) Mimic the proof of item (1).

Let \( \nu: K \to A \otimes H \) be the map \( \nu := \nabla_\rho \circ (\eta_A \otimes \eta) \). Clearly

\[
\nu = \chi_\rho \circ (\eta \otimes \eta_A), \quad \nu = (A \otimes \Pi^L) \circ \nu \quad \text{and} \quad \eta_\rho \circ \gamma = \nabla_\rho \circ (\eta_A \otimes H) = \chi_\rho \circ (H \otimes \eta_A), \quad (2.7)
\]

as desired.
where $\gamma$ is as above of Theorem 1.12. Note that
\[(A \otimes \epsilon) \circ \nu = (A \otimes \epsilon) \circ \chi_\nu \circ (\eta \otimes \eta_A) = \rho \circ (\eta \otimes \eta_A)\] (2.8)
and that by Remark 2.3 the map $\gamma$ is $H$-colinear.

**Theorem 2.11** ([10] Proposition 3.6]). Let $f: H \otimes H \to A$ be a map. If
\begin{enumerate}
  \item $f = \mu_A \circ (A \otimes \rho) \circ (f \otimes \mu \otimes A) \circ (\Delta_{H\otimes^2} \otimes \eta_A)$,
  \item $f$ is a cocycle that satisfies the twisted module condition,
  \item $\rho \circ (H \otimes \eta_A) = \mu_A \circ (\rho \otimes f) \circ (H \otimes c \otimes H) \circ (\Delta \otimes \nu)$,
  \item $\rho \circ (H \otimes \eta_A) = \mu_A \circ (\Delta \otimes f) \circ (\nu \otimes H)$,
  \item $\nu = \mu_A \circ (A \otimes \chi_\nu) \circ (\nu \otimes A) = (\mu_A \otimes H) \circ (A \otimes \nu)$,
\end{enumerate}
then
\begin{enumerate}
  \item $\nu$ satisfies condition 1) in Proposition 2.6,
  \item $\mu_E$ is left and right $A$-linear, associative and normalized with respect to $\nabla_\rho$,
  \item $\nu$ is a preunit of $\mu_E$, \(\nabla_\nu \circ \nu = \nu\) and $\nabla_\nu = \nabla_\rho$ (consequently, $\nu = \nu_0$ and $p_\nu = p_\rho$),
  \item $\mu_E$ is left and right $A$-linear, associative and has unit $\eta_E := p_\nu \circ \nu$,
  \item the morphism $j_\nu: A \to E$, defined by $j_\nu := (\mu_A \otimes H) \circ (A \otimes \nu)$ is left and right $A$-linear, multiplicative and satisfies $\nabla_\nu \circ j_\nu = j_\nu$,
  \item the morphism $j_\nu: A \to E$, defined by $j_\nu := p_\nu \circ j_\nu$, is left and right $A$-linear, multiplicative and unitary,
  \item $\chi_\rho = \nu \circ \mu_E \circ (\gamma \otimes \nu)$ and $F_I = \nu \circ \mu_E \circ (\gamma \otimes \gamma)$.
\end{enumerate}

**Proof.** Item (6) follows from the second equality in (2.7). Thus, by condition s (1), (3), (4) and (5), and Propositions 2.4 and 2.10 the tuple $(A, H, \chi_\rho, F_I, \nu)$ is a crossed product system with preunit. Moreover, by condition (2) and Proposition 2.8 the map $F_I$ is a cocycle that satisfies the twisted module condition. So, we can apply Theorem 1.12 in order to finish the proof. \(\square\)

From items (8) and (9) of Theorem 2.11 it follows that
\[\gamma \circ \eta = p_\rho \circ (\eta_A \otimes \eta) = p_\nu \circ (\eta_A \otimes \eta) = p_\nu \circ \nu = \eta_E.\]
Moreover, by items (10) and (11) of Theorem 2.11 and equality (2.8),
\[\nu \circ j_\nu = \nabla_\nu \circ j_\nu = j_\nu \quad \text{and} \quad (A \otimes \epsilon) \circ j_\nu = \mu_A \circ (A \otimes \rho) \circ (A \otimes \eta \otimes \eta_A).\] (2.9)
Consequently, if $\rho \circ (\eta \otimes \eta_A) = \eta_A$, then $j_\nu$ and $j_\nu'$ are monomorphisms. When the hypotheses of this theorem are fulfilled, we say that $E$ is the unitary crossed product of $A$ by $H$ associated with $\rho$ and $f$.

Given a (not necessarily counitary) right $H$-comodule $B$ with coaction $\delta_B$, we consider $B \otimes B$ as a right $H$-comodule via $\delta_{B \otimes B} = (B \otimes B) \otimes (B \otimes c \otimes H) \circ (\delta_B \otimes \delta_B)$.

**Proposition 2.12.** Under the hypothesis of Theorem 2.11 the maps $\mu_E$ and $\nu_E$ are $H$-colinear.

**Proof.** For $\mu_E$ this is [11] Equality (19)], and for $\mu_E$ it follows easily from the colinearity of $\mu_E$. \(\square\)

For each $n \in \mathbb{N}_0$, let $\mu_n: H^\otimes^n \to H$ be the map recursively defined by
\[\mu_0 := \id_H \quad \text{and} \quad \mu_{n+1} := \mu_n \circ (\mu \otimes H^\otimes^n) \quad \text{for } n \in \mathbb{N}_0.\]
We define the maps $u_n: H^\otimes^n \to A$ and $v_n: H^\otimes^n \to A$ by
\[u_1 := \rho \circ (H \otimes \eta_A), \quad u_{n+1} := u_n \circ \mu_n \quad \text{and} \quad v_{n+1} := u_1 \circ (H \otimes v_n) \quad \text{for } n \in \mathbb{N}_0,\]
where $\rho: H \otimes A \to A$ denotes the weak measure of $H$ on $A$.

**Remark 2.13.** Since $\rho$ is a weak measure the maps $u_n$ and $v_n$ are idempotent.

**Proposition 2.14.** For all $n \in \mathbb{N}$, the equality $(v_n \otimes \epsilon) * v_{n+1} = v_{n+1}$ holds.

**Proof.** For $n = 1$ we have

$$ (v_1 \otimes \epsilon) * v_2 = v_2. $$

Clearly the same argument works for an arbitrary $n$. $\square$

**Remark 2.15.** The equality in Proposition 2.14(1) says that $f = f * u_2$.

**Remark 2.16.** Let $f: H \otimes H \to A$ be a map. Arguing as in the proof of Proposition 2.14 we see that if $u_2 * f = f$, then $(\epsilon \circ \mu \otimes f) \circ \Delta_{H \otimes 2} = f$.

**Proposition 2.17.** If $f: H \otimes H \to A$ satisfies the twisted module condition, then $f * u_2 = v_2 * f$.

**Proof.** In fact, we have

$$ f = f = f, $$

where the first equality holds by the cocycle condition and the second one, since $c$ is natural. $\square$

**Definition 2.18.** A map $g: H \otimes H \to A$ is normal if $g \circ (\eta \otimes H) = g \circ (H \otimes \eta) = u_1$.

**Remark 2.19.** Assume that $v_2 * f = f$. Then,

$$ (v_1 \otimes \epsilon) * f = (v_1 \otimes \epsilon) * v_2 * f = v_2 * f = f; $$

and the last one, since $v_2 * f = f$. So, the equalities in items (3) and (4) of Theorem 2.11 hold if and only if $f$ is normal.

From here to the end of this subsection we assume that the hypotheses of Theorem 2.11 are satisfied.
Lemma 2.20. The following equality holds:
\[ \mu_E \circ (\gamma \otimes \gamma) \circ (\mu \otimes \mu) \circ (H \otimes H \otimes S \otimes H) \circ (H \otimes \Delta \otimes \eta \otimes H) = \mu_E \circ (\gamma \otimes \gamma). \]

Proof. We have

\[ S \gamma \gamma = S f \nu \gamma = S f \nu \gamma = S f \nu \gamma = f \nu \gamma = f \nu \gamma = \gamma \gamma, \]

where the first and last equalities hold by Remark 1.15; the second one, by item 1) of Definition 1.16, Propositions 1.32(1) and 1.29; the third one, by the associativity of \( \mu \) and the coassociativity of \( \Delta \); the fourth one, by Definition 1.30(1) and Proposition 1.21(2); and the fifth one, by Definition 1.16(1) and the fact that \( \mu \) is unitary. \( \square \)

Lemma 2.21. The following assertions hold:

1. \( (\Delta \circ \mu \otimes H) \circ (H \otimes \Delta \circ \eta) = (H \otimes \mu \otimes H) \circ (\Delta \otimes \Delta \circ \eta), \)
2. \( (H \otimes \Delta \circ \mu) \circ (\Delta \circ \eta \otimes H) = (H \otimes \mu \otimes H) \circ (\Delta \circ \eta \otimes \Delta). \)

Proof. (1) By items (1) and (3) of Definition 1.16 and the fact that \( \mu \) is associative and unitary,

(2) This follows from item (1) by symmetry. \( \square \)

Proposition 2.22. The map \( \mu_E \circ (E \otimes \mu_E) \circ (\gamma \otimes \eta \otimes \gamma) \) equalize

\[ (H \otimes A \otimes \mu) \circ (\mu \otimes c \otimes H) \circ (H \otimes H \otimes S \otimes A \otimes H) \circ (H \otimes \Delta \circ \eta \otimes A \otimes H) \] and \( \text{id}_{H \otimes A \otimes H}. \)

Proof. We have

\[ S \gamma \gamma = S f \nu \gamma = S f \nu \gamma = S f \nu \gamma = \gamma \gamma. \]
where the first and last equality hold by Remark 1.15, the second one, by Lemma 2.21(1); the third one, since \( c \) is natural and \( \mu_E \) is associative; and fourth one, by Lemma 2.20 and the associativity of \( \mu_E \).

\[ \tag{□} \]

Proposition 2.23. The following assertions hold:

1. \( \mu_E \circ (\gamma \otimes \gamma \circ \Pi^L) = \gamma \circ \mu \circ (H \otimes \Pi^L) \) and \( \mu_E \circ (\gamma \circ \Pi^L \otimes \gamma) = \gamma \circ \mu \circ (\Pi^L \otimes H) \).
2. \( \mu_E \circ (\gamma \otimes \gamma \circ \Pi^R) = \gamma \circ \mu \circ (H \otimes \Pi^R) \) and \( \mu_E \circ (\gamma \circ \Pi^R \otimes \gamma) = \gamma \circ \mu \circ (\Pi^R \otimes H) \).

Proof. (1) We have

\[ \tag{□} \]

Corollary 2.24. The equality \( \mu_E \circ (\gamma \otimes \gamma \circ \Pi^L \otimes \Pi^R) = \mu_E \circ c \circ (\gamma \otimes \gamma) \circ (\Pi^R \otimes \Pi^L) \) holds.

Proof. This follows from Propositions 1.24 and 2.23.

Corollary 2.25. The equalities \( \gamma * (\gamma \circ \Pi^R) = (\gamma \circ \Pi^L) * \gamma = \gamma \) hold.

Proof. By Proposition 2.23 and Remark 1.19 we have

\[ \tag{□} \]

Proposition 2.26. The following equalities hold:

\( \rho_0 (H \otimes \rho) \circ \eta_0 (H \otimes \mu \otimes A) \circ (H \otimes S) \circ \Delta \eta_0 \otimes H \otimes A) = \rho_0 (H \otimes \rho) \circ (S \otimes H) \circ \Delta \eta_0 \otimes H \otimes A) = \rho \).

Proof. We have

\[ \tag{□} \]
where the first and second equalities hold because $\rho$ is a measure; the third one, by the fact that $f$ is normal by Proposition 2.17 and Remark 2.19; the fourth one, by the associativity of $\mu_H$ and the fact that $\rho$ is a measure; and the last one, by Definition 1.30(1) and the fact that $\Pi_L \circ \eta = \eta$. The proof of the remaining equality is similar. □

2.2 Right $H$-comodule algebras

Proposition 2.27. Let $B$ be a right $H$-comodule which is also an unitary algebra. If $\mu_B$ is an $H$-colinear map, then the following assertions are equivalent:

1. $(B \otimes \Delta) \circ \delta_B \circ \eta_B = (B \otimes \mu \otimes H) \circ (\delta_B \otimes \Delta) \circ (\eta_B \otimes \eta)$.
2. $(B \otimes \Delta) \circ \delta_B \circ \eta_B = (B \otimes \mu \otimes H) \circ (B \otimes c \otimes H) \circ (\delta_B \otimes \Delta) \circ (\eta_B \otimes \eta)$.
3. $(B \otimes \Pi^R) \circ \delta_B = (\mu_B \otimes H) \circ (B \otimes \delta_B) \circ (B \otimes \eta_B)$.
4. $(B \otimes \Pi^L) \circ \delta_B = (\mu_B \otimes H) \circ (B \otimes c) \circ (\delta_B \otimes B) \circ (\eta_B \otimes B)$.
5. $(B \otimes \Pi^R) \circ \delta_B \circ \eta_B = \delta_B \circ \eta_B$.
6. $(B \otimes \Pi^L) \circ \delta_B \circ \eta_B = \delta_B \circ \eta_B$.

Proof. See [10]. □

Definition 2.28. An unitary algebra $B$, which is also a right $H$-comodule, is a right $H$-comodule algebra if $\mu_B$ is $H$-colinear and the equivalent statements of the previous proposition are satisfied.

The following result and its proof were kindly communicated to us by José Nicanor Alonso Álvarez y Ramón González Rodríguez. It is interesting in itself and allows to simplify the proof of Proposition 2.30, but only under the assumption that $H$ is a weak Hopf algebra.

Proposition 2.29. Let $B$ be a right $H$-comodule which is also an unitary algebra. Assume that $\mu_B$ is an $H$-colinear map. If $H$ is a weak Hopf algebra, then the equivalent items of the previous proposition are fulfilled.

Proof. By the hypotheses, Proposition 1.25 and Definition 1.30(1), we have

\[ \Pi^L = \Pi^R = \eta_B, \]

as desired. □

Proposition 2.30 (Comodule algebra structure on an unitary crossed product $E$). Each unitary crossed product $E$ is a weak $H$-comodule algebra via the coaction introduced in Remark 2.3.

Proof. By Proposition 2.12 we know that $\mu_E$ is $H$-colinear. We next prove that the equality in item (1) of Proposition 2.27 is satisfied. That is

\[ (E \otimes \Delta) \circ \delta_E \circ \eta_E = (E \otimes \mu \otimes H) \circ (\delta_E \otimes \Delta) \circ (\eta_E \otimes \eta). \] (2.10)
By the fact that $\nabla_\rho \circ \nu = \nu$, condition (2) in Proposition 2.3 and the first equality in Definition 1.14(2), we have

$$\nu = \nu' = \nu' = \nu' = \nu' = \nu' = \nu' = \nu' = \nu'.$$

This finishes the proof, since the left side in equality (2.10) is the map represented by the first diagram, and the right side is the map represented by the last diagram. □

**Definition 2.31.** Let $B$ be a right $H$-comodule algebra. An integral of $B$ is a morphism of right $H$-comodules $\gamma : H \to B$. If moreover $\gamma(1) = 1_B$, then $\gamma$ is a total integral.

**Definition 2.32.** An integral $\gamma$ is convolution invertible if there exists a morphism $\gamma^{-1} : B \to B$ such that

$$\gamma^{-1} \ast \gamma = \gamma \ast \Pi^R, \quad \gamma \ast \gamma^{-1} = \gamma \ast \Pi^L$$

and $(\gamma \ast \Pi^R) \ast \gamma^{-1} = \gamma^{-1} = \gamma^{-1}$.

Clearly $\gamma^{-1}$ is unique.

**Definition 2.33.** Let $B$ be a right $H$-comodule algebra and $j : A \to B$ an algebra monomorphism. We say that $(B,j)$ is an extension of $A$ by $H$ if $j$ is the equalizer of $\delta_B$ and $(B \otimes \Pi^L) \circ \delta_B$, and we say that a extension $(B,j)$ is $H$-cleft if there exists a convolution invertible total integral $\gamma : H \to B$ such that $\gamma \circ \Pi^L$ factorizes through $j$.

In the sequel $\gamma$ will be called a cleaving map associated with the $H$-cleft extension $(B,j)$.

**Definition 2.34.** Let $(B,j)$ and $(B',j')$ be extensions of $A$ by $H$. An arrow $\Phi : B \to B'$ is a morphism of extensions if $\Phi$ is a $H$-colinear algebra morphism and $\Phi \circ j = j'$.

Let $\Phi$ be as in the previous definition. If $(B,j)$ is an $H$-cleft extension with cleaving map $\gamma$ and inverse $\gamma^{-1}$, then $(B',j')$ is an $H$-cleft extension with cleaving map $\Phi \circ \gamma$ and inverse $\Phi \circ \gamma^{-1}$.

Moreover, in this case, $\Phi$ is an isomorphism.

**Definition 2.35.** Two extensions of $A$ by $H$ are equivalent if they are isomorphic.

3 Equivalence of weak crossed products

Let $H$ be a weak bialgebra and let $A$ be a unitary algebra. Let $\rho$ and $\rho'$ be weak measures of $H$ on $A$, and let $f : H \otimes H \to A$ and $f' : H \otimes H \to A$ be maps. Assume that both pairs $(\rho, f)$ and $(\rho', f')$ satisfy the hypotheses of Theorem 2.11. In this section we set $E := A \otimes H$, $E' := A \times H$, $E'' := A \otimes H$, and $E''' := A \times H$. Recall that the preunits of $E$ and $E'$ are the maps defined by $\nu = \chi_\rho \ast (\eta \otimes \eta_A)$ and $\nu' = \chi_{\rho'} \ast (\eta \otimes \eta_A)$, respectively, while the units of $E$ and $E'$ are the maps $\eta_E = p_\rho \circ \nu$ and $\eta_{E'} = p_{\rho'} \circ \nu'$. In the sequel we let $u_1 : H \to A$ and $u_1' : H \to A$ denote the maps defined by $u_1 := \rho \circ (H \otimes \eta_A)$ and $u_1' := \rho' \circ (H \otimes \eta_A)$. Moreover, for each map $\phi : H \to A$, we let $L(\phi) : A \otimes H \to A \otimes H$ denote the left $A$-linear and right $C$-colinear map defined by $L(\phi) := (\mu_A \otimes H) \circ (A \otimes \phi \circ H) \circ (A \otimes \Delta)$.

**Theorem 3.1.** If $\phi : H \to A$ is a map that satisfies

1. $\phi = u_1 \ast \phi = \phi \ast u_1'$,
2. There exists $\phi' : H \to A$ such that $\phi \ast \phi' = u_1$ and $\phi' \ast \phi = u_1'$,
3. $(\phi \otimes H) \circ \Delta \circ \eta = \nu'$,
(4) $\mu A \circ (A \otimes \phi) \circ \chi_\rho = \mu A \circ (\phi \otimes \rho') \circ (\Delta \otimes A)$,

(5) $\mu A \circ (A \otimes \phi) \circ F_f = \mu A \circ (\mu A \otimes f') \circ (A \otimes \chi_\rho' \otimes H) \circ (\phi \otimes H \otimes \phi \otimes H) \circ (\Delta \otimes \Delta)$,

then the map $\Phi_\phi : E \to E'$, defined by $\Phi_\phi := p_\rho' \circ L(\phi) \circ \iota_\rho$, is a left $A$-linear and right $H$-colinear isomorphism of unitary algebras. Conversely, if $\Phi : E \to E'$ be a left $A$-linear and right $H$-colinear isomorphism of unitary algebras, then the map $\phi := (A \otimes \epsilon) \circ \iota_\rho' \circ \Phi \circ p_\rho \circ (\eta_A \otimes H)$ satisfy statements (1)–(5). Moreover, the correspondences $\Phi \mapsto \phi_\phi$ and $\phi \mapsto \Phi_\phi$ are inverse one of each other.

In order to prove this result, we first establish a sequence of Lemmas.

Lemma 3.2. For each map $\phi : H \to A$, the following facts hold:

1. $\phi = u_1 * \phi$ if and only if $L(\phi) = L(\phi) \circ \nabla_\rho$.
2. $\phi = \phi * u_1'$ if and only if $L(\phi) = \nabla_\rho' \circ L(\phi)$.

Proof. Left to the reader.

Lemma 3.3. Let $\phi : H \to A$ and $\phi' : H \to A$ be maps and let $\Phi : E \to E'$ and $\Phi' : E' \to E$ be the maps $\Phi := \Phi_\phi$ and $\Phi' := \Phi_\phi'$. Assume that $\phi = u_1 * \phi = \phi * u_1'$ and $\phi' = u_1' * \phi' = \phi' * u_1$.

Then the following facts hold:

1. $\phi * \phi' = u_1$ if and only if $\Phi' \circ \Phi = \text{id}_E$.
2. $(\phi \otimes H) \circ \Delta \circ \eta = \nu'$ if and only if $\Phi \circ \eta_E = \eta_{E'}$.
3. The maps $\chi := \chi_\rho$ and $\chi' := \chi_\rho'$ have the following property:

\[ \chi_\phi = \chi_{\phi'} \quad \text{if and only if} \quad \chi_{\phi} = \chi_{\phi'} \]

(3.11)

Proof. We prove that last item and left the other ones, which are easier, to the reader. By item (3), the definition of $L(\phi)$ and the second equality in Remark 2.2, we have
Using this, the coassociativity of $\Delta$, the fact that $c$ is natural and the first equality in Remark 2.2, we obtain that, if the first equality in the statement is true, then

\[
\begin{align*}
\phi & = \phi \\
F_f \circ L(\phi) & = F_f \circ L(\phi) \\
\chi & = \chi \\
\end{align*}
\]

which proves the second equality in the statement. Conversely, if this equality is true, then, again by (3.11), we have

\[
\begin{align*}
\phi & = \phi \\
F_f \circ L(\phi) & = F_f \circ L(\phi) \\
\chi & = \chi \\
\end{align*}
\]

which proves the first equality in the statement. Assume again that the second equality in the statement is true. Then, we have

\[
\begin{align*}
\phi & = \phi \\
F_f \circ L(\phi) & = F_f \circ L(\phi) \\
\chi & = \chi \\
\end{align*}
\]

which proves the last equality in the statement. Finally, if this equality is true, then

\[
\begin{align*}
\phi & = \phi \\
F_f \circ L(\phi) & = F_f \circ L(\phi) \\
\chi & = \chi \\
\end{align*}
\]

which proves the second equality in the statement. \qed

**Proof of Theorem 3.1.** Assume that $\phi$ satisfies statements 1)–5) and set $\Phi := \Phi$. The map $\Phi$ is left $A$-linear and right $H$-colinear because $p_\rho$, $L(\phi)$ and $\iota_\rho$ are left $A$-linear and right $H$-colinear maps and $\Phi = p_\rho \circ L(\phi) \circ \iota_\rho$. We next prove that it is also an isomorphism of unitary algebras.
By Lemma 3.2 and the fact that \( \mu_E \) is normalized with respect to \( \nabla_p \) and \( \mu_{E'} \) is normalized with respect to \( \nabla_{p'} \), we have

\[
\begin{array}{ccc}
\mu_E & = & \mu_{E'} \\
\Phi & = & \Phi' \\
\nabla_p & = & \nabla_{p'} \\
L(\phi) & = & L(\phi')
\end{array}
\]

and

\[
\begin{array}{ccc}
\mu_E & = & \mu_{E'} \\
\Phi & = & \Phi' \\
\nabla_p & = & \nabla_{p'} \\
L(\phi) & = & L(\phi')
\end{array}
\]

Consequently \( \Phi \) is multiplicative if and only if \( L(\phi) \) is. But this is true, because, by items (3) and (4) of Lemma 3.3 and the fact that \( L(\phi) \) is left \( A \)-linear,

\[
\begin{array}{ccc}
\chi := \chi_p & \Rightarrow & \chi' := \chi_{p'} \\
\end{array}
\]

where \( \chi := \chi_p \) and \( \chi' := \chi_{p'} \). It remains to prove that the morphism \( \Phi \) is unitary and invertible. But \( \Phi \) is unitary by item (2) of Lemma 3.3 and using item (1) of Lemma 3.3 we obtain that \( \Phi \) is invertible (with inverse given by the map \( \Phi_\sigma := p_p \circ L(\phi) \circ i_{p'} \)).

Conversely, assume that \( \Phi: E \to E' \) is an unitary algebra isomorphism that is left \( A \)-linear and right \( H \)-colinear and set \( \phi = \phi_1 := \left( A \otimes \epsilon \right) \circ i_{p'} \circ \Phi \circ p_p \circ \left( \eta_A \otimes H \right) \). Since \( p_p \circ \nabla_p = p_p \) and \( p_{p'} \), \( \Phi \) and \( i_{p'} \) are left \( A \)-linear and left \( H \)-colinear,

\[
\begin{array}{ccc}
\nabla_p & = & \nabla_{p'} \\
L(\phi) & = & L(\phi') \\
\Phi & = & \Phi' \\
\end{array}
\]

which by Lemma 3.2 proves that \( \phi = u_1 \ast \phi \). A similar computation proves that \( \phi = \phi \ast u_1' \). So, item (1) is fulfilled. Items (2) and (3) follow from items (1) and (2) of Lemma 3.3 since

\[
\Phi = \Phi' = \Phi_\sigma \quad \text{and} \quad \Phi' = \Phi_\sigma = \Phi_\sigma',
\]

where \( \Phi_\sigma' \) is the inverse of \( \Phi \) and \( \phi' := \phi_\sigma := \left( A \otimes \epsilon \right) \circ i_p \circ \Phi' \circ p_{p'} \circ \left( \eta_A \otimes H \right) \). Note now that

\[
L(\phi) \circ \nu = L(\phi) \circ \nabla_p \circ \left( \eta_A \otimes \eta \right) = L(\phi) \circ \left( \eta_A \otimes \eta \right) = \left( \phi \otimes H \right) \circ \Delta \circ \eta = \chi_{p'} \circ \left( \eta \otimes \eta_A \right) = \nu',
\]

where the second equality holds by Lemma 3.2; the third one, by the definition of \( L(\phi) \); the fourth one, by Lemma 3.3(2); and the last one, by the first equality in (3.7). Hence,

\[
L(\phi) \circ \nu' = L(\phi) \circ \left( \mu_A \otimes H \right) \circ \left( A \otimes \nu \right) = \left( \mu_A \otimes H \right) \circ \left( A \otimes L(\phi) \right) \circ \left( A \otimes \nu \right) = \left( \mu_A \otimes H \right) \circ \left( A \otimes \nu' \right) = \nu',
\]
where the second equality holds because \( L(\phi) \) is left \( A \)-linear. Consequently, by the first equality in Theorem 1.12(8) and the fact that \( L(\phi) \) is multiplicative and both \( L(\phi) \) and \( \mu_E \) are left \( A \)-linear, we have

\[
\begin{align*}
L(\phi) &= f \quad (\phi) \\
L(\phi) &= f \quad (\phi) \\
L(\phi) &= f \quad (\phi) \\
L(\phi) &= f \quad (\phi) \\
L(\phi) &= f \quad (\phi) \\
L(\phi) &= f \quad (\phi) \\
L(\phi) &= f \quad (\phi) \\
L(\phi) &= f \quad (\phi) \\
\end{align*}
\]  

which by Lemma 3.3(3) implies that item (4) is fulfilled. Using now equality (3.13) and the fact that \( L(\phi) \) is multiplicative, we obtain that

\[
\begin{align*}
\mathcal{L}(\phi) &= \mathcal{L}(\phi) \\
\mathcal{L}(\phi) &= \mathcal{L}(\phi) \\
\mathcal{L}(\phi) &= \mathcal{L}(\phi) \\
\mathcal{L}(\phi) &= \mathcal{L}(\phi) \\
\mathcal{L}(\phi) &= \mathcal{L}(\phi) \\
\mathcal{L}(\phi) &= \mathcal{L}(\phi) \\
\mathcal{L}(\phi) &= \mathcal{L}(\phi) \\
\mathcal{L}(\phi) &= \mathcal{L}(\phi) \\
\end{align*}
\]  

which by Lemma 3.3(4) proves that item (5) is true.

Finally, the first calculations made out in (3.12) and the fact that

\[
\phi = (A \otimes \epsilon) \circ L(\phi) \circ (\eta_A \otimes H) = (A \otimes \epsilon) \circ \iota_{\phi} \circ \Phi \circ \rho \circ (\eta_A \otimes H) = \phi_{\Phi},
\]

show that the correspondences \( \Phi \mapsto \Phi_{\phi} \) and \( \phi \mapsto \Phi_{\phi} \) are inverses one of each other. \( \square \)

**Definition 3.4.** Two crossed products \( E := A \times_f^\rho H \) and \( E' := A \times_{f'}^{\rho'} H \) are said to be **equivalent** if there exists an algebra isomorphism \( \Phi : E \to E' \) that is left \( A \)-linear and right \( H \)-colinear.

**Remark 3.5.** Theorem 3.1 show that the notion of equivalence of crossed product of algebras by weak Hopf algebras given in this section reduce to the one introduced in [5] above Proposition 3.2.

### 4 Weak crossed products of weak module algebras

Let \( H \) be a weak bialgebra. In this section \( E \) is the unitary crossed product of \( A \) by \( H \) associated with a weak measure \( \rho \) and a map \( f : H \otimes H \to A \). Thus, we assume that the hypotheses of Theorem 2.11 are fulfilled. In particular, \( H, \mu, \gamma \) and \( \gamma \) are as in that theorem. We study the consequences that \( A \) is a left weak \( H \)-module algebra.

**Proposition 4.1.** Let \( A \) be an unitary algebra. If \( \rho : H \otimes A \to A \) satisfies

1. \( \rho \circ (\eta \otimes A) = \text{id}_A \),
2. \( \rho \circ (H \otimes \mu_A) = \mu_A \circ (\rho \otimes \rho) \circ (H \otimes c \otimes A) \circ (\Delta \otimes A \otimes A) \),
3. \( \rho \circ (\mu \otimes \eta_A) = \rho \circ (H \otimes \rho) \circ (H \otimes H \otimes \eta_A) \),

then the following assertions are equivalent:

4. \( \rho \circ (\Pi^L \otimes A) = \mu \circ (\rho \otimes A) \circ (H \otimes \eta_A \otimes A) \).
5. \( \rho \circ (\Pi^R \otimes A) = \mu \circ (\rho \otimes A) \circ (H \otimes \eta_A \otimes A) \).
6. \( \rho \circ (\Pi^L \otimes \eta_A) = \rho \circ (H \otimes \eta_A) \).
7. \( \rho \circ (\Pi^R \otimes \eta_A) = \rho \circ (H \otimes \eta_A) \).
8. \( \rho \circ (H \otimes \rho) \circ (H \otimes \mu \otimes A) = (\rho \otimes \epsilon) \circ (H \otimes c) \circ (H \otimes \mu \otimes A) \circ (\Delta \otimes H \otimes \eta_A) \).
An unitary algebra $A$ endowed with a map $\rho: H \otimes A \to A$ that satisfies items (1), (2), (3) and (4) of the previous proposition is called a left weak $H$-module algebra. In this case we say that $\rho$ is a weak left action of $H$ on $A$. If also $\rho \circ (\mu \otimes A) = \rho \circ (H \otimes H \otimes \eta_A)$, then we say that $\rho$ is an action and that $A$ is a left $H$-module algebra.

Proposition 4.2. Let $A$ be an unitary algebra, and let $\rho: H \otimes A \to A$ be a map. If $H$ is a weak Hopf algebra and $\rho$ satisfies conditions 1)–3) of the above proposition, then the conditions 4)–9) of that propositions are satisfied.

Proof. By the hypotheses, Proposition 1.26 and Definition 1.30(1), we have

$\rho \circ (H \otimes \rho) \circ (H \otimes \eta_A) = (\epsilon \otimes A) \circ (\mu \otimes \rho) \circ (H \otimes c \otimes A) \circ (\Delta \otimes H \otimes \eta_A)$.

From here to the end of this subsection, $A$ is a left weak $H$-module algebra. Hence, by the equality in Proposition 4.1(3), we have $v_n = u_n$ for all $n \in \mathbb{N}$.

Proposition 4.3. The pair $(A, j_\nu)$ is the equalizer of $\delta_E$ and $(E \otimes \Pi_L) \circ \delta_E$.

Proof. Let $g: X \to E$ be an arrow such that $\delta_E \circ g = (E \otimes \Pi_L) \circ \delta_E \circ g$. Then,

where the first equality holds since $\nabla_\nu$ is $H$-colinear, $\Delta$ is counitary and $\nabla_\nu \circ i_\nu = i_\nu$; the second one, since $\Delta = \epsilon \otimes \rho$ and $\delta_E \circ g = (E \otimes \Pi_L) \circ \delta_E \circ g$; the third one, by the very definition of $\Pi_L$; the fourth one, by the fact that $\Delta$ is counitary and Propositions 1.21(2) and 1.29; the fifth one, by Proposition 1.21(4) and the equalities in items (3) and (5) of Proposition 4.1 and the sixth one by the associativity of $\mu_A$ and the definition of $j_\nu$. Since $j_\nu = i_\nu \circ j_\nu$, and $i_\nu$ is a monomorphism, this implies that

$g = j_\nu \circ \mu_A \circ (A \otimes \rho) \circ (i_\nu \otimes A) \circ (g \otimes \eta_A)$.

So, $g$ factorizes through $j_\nu$. Since $j_\nu$ is a monomorphism because $(A \otimes \epsilon) \circ i_\nu \circ j_\nu = \text{id}_A$, this proves the assertion.
Remark 4.4. By the definition of $\Pi^L$, the fact that the fact that $c$ is natural and $\Delta$ is counitary, and Propositions 1.21(2) and 1.29

Proposition 4.5. We have $\nabla_\nu = (\rho \otimes \mu) \circ (H \otimes c \otimes H) \circ (\Delta \otimes A \otimes H) \circ (\eta \otimes A \otimes H)$.

Proof. By Remark 4.4 Theorem 2.11(8), equality (2.6), the fact that $c$ is natural, the equalities in items (5) and (7) of Proposition 4.1, and Proposition 1.29, we have

as desired.

Proposition 4.6. The equality $\gamma \circ \Pi^L = j_\nu \circ \rho \circ (H \otimes \eta A)$ holds.

Proof. To begin note that by items (10) and (11) of Theorem 2.11, we have $i_\nu \circ j_\nu = \nabla_\nu \circ j_\prime_\nu = j_\prime_\nu$. Using this, Theorem 2.11(8), the first and last equality in (2.7), Proposition 1.29 the equalities in items (3) and (4) of Proposition 4.1, and the definition of $j_\prime_\nu$, we obtain that

Since $i_\nu$ is a monomorphism, the statement follows from these equalities.

Proposition 4.7. The equality $\mu_E \circ (\gamma \otimes j_\nu) \circ (\Pi^R \otimes A) = \mu_E \circ (j_\nu \otimes \gamma) \circ (A \otimes \Pi^R) \circ c$ holds.

Proof. We have

where the first and fifth equalities hold by Remark 1.15, the second one, by Propositions 1.17 and 1.27, the third one, by the equality in Proposition 1.16(5); the fourth one, by Theorem 2.11(11) and the associativity of $\mu_E$; and the last one, again by Theorem 2.11(11).
5 Weak crossed products with invertible cocycle are cleft

In this section we assume that $H$ is a weak Hopf algebra, $A$ is a left weak $H$-module algebra and $E$ is the unitary crossed product of $A$ by $H$ associated with a weak left action $\rho$ and a map $f: H \otimes H \to A$. Thus, we assume that the hypotheses of Theorem 2.11 and the equalities in the items of Proposition 4.1 are fulfilled. The aim of this section is to prove that the crossed products with invertible cocycle are $H$-cleft extensions.

5.1 Regular maps

Proposition 5.1. Let $g, g' \in \text{Hom}(H^\otimes n, A)$. If $g * g' = g * g = u_n$, then

1. $g * u_n = u_n * g$ and $g' * u_n = u_n * g'$,
2. $(u_n * g) * (u_n * g') = u_n$ and $(u_n * g') * (u_n * g) = u_n$.

Proof. By symmetry we only must prove the first equalities in items (1) and (2). But

$g * u_n = g * (g' * g) = (g * g') * g = u_n * g$ and $(u_n * g) * (u_n * g') = (u_n * u_n) * (g * g') = u_n$,

as desired. □

Proposition 5.2. Let $F_1$, $F_2$ and $F_\rho$ be the morphisms from $H^\otimes 3$ to $A$ defined by $F_1 := f \circ (\mu \otimes H)$, $F_2 := f \circ (H \otimes \mu)$ and $F_\rho := \rho \circ (H \otimes f)$. The following facts hold:

1. $F_1 * u_3 = F_1$ and $u_3 * F_1 = F_1$,
2. $F_2 * u_3 = F_2$ and $u_3 * F_2 = F_2$,
3. $F_\rho * u_3 = F_\rho$ and $u_3 * F_\rho = F_\rho$.

Proof. (1) By the fact that $f * u_2 = f$ and Definition 1.16(1),

Similarly, $u_3 * F_1 = F_1$.

(2) Mimic the proof of item (1).

(3) By the fact that $f * u_2 = f$ and the equalities in item (2) and (3) of Proposition 4.1.

Similarly, $u_3 * F_\rho = F_\rho$. □
Proposition 5.3. The equality $f \circ (H \otimes \mu \circ (H \otimes \Pi^L)) = f \circ (H \otimes \mu \circ (H \otimes \Pi^L))$ holds.

Proof. It suffices to prove that 

$$f \circ (H \otimes \mu) \circ (H \otimes H \otimes \Pi^L) = \mu_A \circ (A \otimes u_1) \circ (F_f \otimes H) = f \circ (H \otimes \mu) \circ (H \otimes H \otimes \Pi^L). \quad (5.14)$$

The first equality is true since

where the first equality holds because $f = v_2 * f$ by Proposition 2.7; the second one, by the fact that $f$ is normal and the equality in Definition 1.16(1); the third one, by Propositions 1.21(2), 1.28(3), 2.7, and 2.8; the fourth one, by Proposition 2.8 and the fact that $f$ is normal; and the last one, by the fact that $f$ is normal and the equalities in items (3) and (6) of Proposition 4.1. The proof of the second equality in (5.14) is similar, but we must use that $\Pi^L = \Pi^R \circ \Pi^L$ and Propositions 2.7 and 2.8 instead of Propositions 1.21(2), 1.28(3), 2.7, and 2.8.

\[\square\]

Remark 5.4. We have

where the first equality holds by items (3), (6) and (9) of Proposition 4.1; the second one, by condition (1) of Definition 1.16; the third one, by the coassociativity of $\Delta_{H \otimes H}$; and the last one, since $c$ is natural.

Proposition 5.5. Let $F_\epsilon: H^{\otimes 3} \to A$ be the morphism defined by $F_\epsilon \equiv f \otimes \epsilon$. The following equality holds:

$$F_\epsilon \ast u_3 = u_3 \ast F_\epsilon = f \circ (H \otimes \mu) \circ (H^{\otimes 2} \otimes \Pi^L).$$
Similarly Proposition 5.9. Let $f$ be as in Proposition 5.2 and let $F = f \circ (H \otimes \mu) \circ (H^\otimes \otimes 1^L)$. A similar argument proves that

$$F \ast u_3 = f \circ (H \otimes \mu) \circ (H^\otimes \otimes 1^L)$$

By Proposition 5.8 this finishes the proof. \hfill \Box

**Definition 5.6.** A map $g : H^\otimes n \to A$ is regular if $g \ast u_n = g$ and there exists $g' \in \text{Hom}(H^\otimes n, A)$ such that $g \ast g' = g' \ast g = u_n$.

We let $\text{Reg}(H^\otimes n, A)$ denote the subset of $\text{Hom}(H^\otimes n, A)$ consisting of the regular maps. It is clear that $\text{Reg}(H^\otimes n, A)$ is closed under convolution product. Moreover, by Remark 2.13 and Proposition 5.1 we can assume that $g' = g' \ast u_n$ (and we do it). So, $\text{Reg}(H^\otimes n, A)$ is a group and $g'$ is the inverse of $g$. In the sequel we will write $g^{-1}$ instead of $g'$.

**Definition 5.7.** We say that the cocycle $f$ is **invertible** if it is regular.

**Proposition 5.8.** Let $F_1$, $F_2$ and $F_\rho$ be as in Proposition 5.2. If $f$ is invertible, then $F_1$, $F_2$ and $F_\rho$ are regular maps. Moreover

$$F_1^{-1} = f^{-1} \circ (\mu \otimes 1), \quad F_2^{-1} = f^{-1} \circ (1 \otimes \mu) \quad \text{and} \quad F_\rho^{-1} = \rho \circ (H \otimes f^{-1}).$$

**Proof.** By Proposition 5.2(1) we know that $F_1 \ast u_3 = u_3$. Moreover, since $\Delta$ is multiplicative and $f^{-1} \ast f = u_2$, we have

$$(f^{-1} \circ (\mu \otimes 1)) \ast F_1 = (f^{-1} \ast f) \circ (\mu \otimes H) = u_2 \circ (\mu \otimes H) = u_3.$$ 

Similarly, $F_1 \ast (f^{-1} \circ (\mu \otimes H)) = u_3$. So, $F_1 \in \text{Reg}(H^\otimes n, A)$ and $F_1^{-1} = f^{-1} \circ (\mu \otimes H)$. A similar argument proves that $F_2$ is regular and $F_2^{-1} = f^{-1} \circ (1 \otimes \mu)$. Next we prove the assertion about $F_\rho$. The fact that $F_\rho \ast u_3 = F_\rho$ it follows from Proposition 5.2(3). By the equalities in items (2) and (3) of Proposition 5.1 and the fact that $f^{-1} \ast f = u_2$, we have

$$(\rho \circ (H \otimes f^{-1})) \ast F_\rho = \rho \circ (H \otimes f^{-1} \ast f) = \rho \circ (H \otimes u_2) = u_3.$$ 

Similarly $F_\rho \ast (\rho \circ (H \otimes f^{-1})) = u_3$. So, $F_\rho \in \text{Reg}(H^\otimes n, A)$ and $F_\rho^{-1} = \rho \circ (H \otimes f^{-1})$. \hfill \Box

**Proposition 5.9.** Let $F_\epsilon$ be as in Proposition 5.2 and let $F_\epsilon' := f^{-1} \otimes \epsilon$. If $f$ is invertible, then the equalities $F_\epsilon' \ast F_\epsilon' = F_\epsilon' \ast F_\epsilon = u_2 \otimes \epsilon$ hold.

**Proof.** Since $f \ast f^{-1} = u_2$, we have $F_\epsilon \ast F_\epsilon' = (f \ast f^{-1}) \otimes \epsilon = u_2 \otimes \epsilon$. A similar argument, using that $f^{-1} \ast f = u_2$, shows that $F_\epsilon' \ast F_\epsilon = u_2 \otimes \epsilon$. \hfill \Box
Remark 5.10. Let $F_i$ be as in Proposition \[5.5\] and let $F'_i$ be as in Proposition \[5.9\]. Assume that $f$ is invertible. By Remark \[2.13\] and Propositions \[2.14\], \[5.5\] and \[5.9\]

$$F'_i * u_3 * \hat{F}_i = F'_i * u_3 * F_i = F'_i * u_3 * F_i = F'_i * F_i * u_3 = (u_2 \otimes \epsilon) * u_3 = u_3$$

and

$$\hat{F}_i * F'_i * u_3 = u_3 * F_i * F'_i * u_3 = u_3 * (u_2 \otimes \epsilon) * u_3 = u_3 * u_3 = u_3,$$

where $\hat{F}_i := u_3 * F_i$. Consequently, $\hat{F}_i \in \text{Reg}(H^3, A)$ and $\hat{F}_i^{-1} = F'_i * u_3$.

Proposition 5.11. Let $F_1$, $F_2$ and $F'_2$ be as in Proposition \[5.4\] and let $\hat{F}_i$ be as in Remark \[5.10\]. If $f$ is invertible, then $F_1, F_2, F'_2, \hat{F}_i \in \text{Reg}(H^3, A)$ and $F_2 * F_1^{-1} = F'_2 * \hat{F}_i$.

Proof. The cocycle condition reads $F_p * F_2 = F_1 * F_i$, where $F_i$ is as in Proposition \[5.5\]. Since $F_i * F_1 = F_i * u_3 * F_1 = F_i * F_1$, this implies that $F_p * F_2 = F_i * F_1$. This finishes the proof because, by Proposition \[5.8\] and the previous remark, $F_1, F_2, F'_2, \hat{F}_i \in \text{Reg}(H^3, A)$.

Proposition 5.12. Assume that $f$ is invertible. If $f$ is normal, then $f^{-1}$ is also.

Proof. in fact,

$$f^{-1} \circ (H \otimes \eta) = ((f \circ (\mu \otimes \eta)) \circ f^{-1}) \circ (H \otimes \eta) = (f * f^{-1}) \circ (H \otimes \eta) = u_2 \circ (H \otimes \eta) = u_1,$$

where the first equality holds since $f^{-1} = u_2 * f^{-1}$ and $u_2 = u_1 \circ \mu = f \circ (\mu \otimes \eta)$; the second one, by Proposition \[2.7\] and the third one, since $f * f^{-1} = u_2$. Similarly, $f^{-1} \circ (\eta \otimes H) = u_1$.

Proposition 5.13. Assume that $f$ is invertible. Then

$$f^{-1} \circ (\mu \otimes H) \circ (H \otimes \Pi^R \otimes H) = f^{-1} \circ (H \otimes \mu) \circ (H \otimes \Pi^R \otimes H)$$

and

$$f^{-1} \circ (\mu \otimes H) \circ (H \otimes \Pi^L \otimes H) = f^{-1} \circ (H \otimes \mu) \circ (H \otimes \Pi^L \otimes H).$$

Proof. By Remark \[2.14\] and Propositions \[2.14\], \[2.17\] and \[2.8\].

Definition 5.14. When $f$ is invertible we define $\gamma^{-1} : H \to E$ by

$$\gamma^{-1} := \mu_E \circ (\mu \otimes \gamma) \circ Q,$$

where $Q := (f^{-1} \otimes H) \circ (H \otimes \epsilon) \circ (\Delta \otimes H) \circ (S \otimes H) \circ \Delta$.

Remark 5.15. Set $f := f^{-1}$. Note that

$$Q = \begin{array}{c}
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\end{array}.$$
Proof. For legibility in the diagrams we set $\bar{\gamma} := \gamma^{-1}$. We have

\[
\begin{align*}
\Pi R \gamma \bar{\gamma} & = \gamma \bar{\gamma} = \gamma \bar{\gamma} = \gamma \bar{\gamma} = \gamma \bar{\gamma} = \gamma \bar{\gamma} = \gamma, \\
where the first equality holds by the definition of $\Pi R$; the second one, by Propositions 1.21(2) and 1.29; the third one, by Definition 5.14; the fourth one, by Propositions 1.21(2) and 1.29; the fifth one, since $S$ is antimultiplicative; the sixth one, by Proposition 2.22; the seventh one, since $\gamma \circ \eta = \eta_E$; and the last one, by Definition 5.14 and Remark 5.15. \qed
\end{align*}
\]

Remark 5.17. Assume that $f$ is invertible and set $\bar{f} := f^{-1}$ and $\bar{F}_\rho := F_\rho^{-1}$. We have

\[
\begin{align*}
S \bar{F}_\rho \hat{F} & = S \Pi \bar{f} f \chi \rho f \\
where the first equality holds by Proposition 5.5; the second one, by the definition of $\chi \rho$; the third one, by the first equality in Proposition 1.26; the fourth one, by Proposition 5.13; the fifth one, by Definition 1.16(1) and the fact that $\mu$ is unitary; and the last one, by the definition of $Q$. \qed
\end{align*}
\]

Lemma 5.18. We have $(F_2 * \bar{F}_1) \circ (H \otimes S \otimes H) \circ (\mu \otimes \Delta) \circ (\Pi R \otimes \Delta) = \rho \circ (\Pi R \otimes u_1)$. 

Proof. We compute

\[
\begin{align*}
\end{align*}
\]
where the first equality holds by the very definition of $F_2 \cdot \bar{F}_1$; the second one, by Definition 1.16; the discussion below Definition 1.30 and the fact that $\Delta$ is coassociative and $\mu$ is associative; the third one, by Definition 1.30, and the last one, by the fact that $f$ is normal, and Propositions 1.21(2), 1.28(4), 2.7, 2.8, 5.12 and 2.16. So, we are reduce to prove that the last diagram represent the function at the right hand of the equality in the statement. But this is true, since

\[
\begin{align*}
\Pi_L \Pi_R &= \Pi_L \Pi_R \\
\Pi_R &= \Pi_R \\
\Pi_R &= \Pi_L \\
\Pi_R &= \Pi_R \\
\Pi_R &= \Pi_R \\
\Pi_R &= \Pi_R,
\end{align*}
\]

where the first equality holds by the equality in Proposition 1.11(3) and the coassociativity of $\Delta$; the second one, by the equalities in items (2) and (4) of Proposition 1.11 and the associativity of $\mu$; the third one, by the equality in Proposition 1.11(2) and the fact that $\mu$ is unitary; the fourth one, by the equality in Proposition 1.11(3); the fifth one, by Remark 1.19; and the sixth one, since $u_1$ is idempotent. □

**Proposition 5.19.** Let $\gamma$ be as above of Theorem 1.12. If $f$ is invertible, then $\gamma^{-1} \ast \gamma = \gamma \circ \Pi^R$ and $\gamma \ast \gamma^{-1} = \gamma \circ \Pi^L$.

**Proof.** For the legibility in the diagrams we set $\bar{f} := f^{-1}$, $\bar{\gamma} := \gamma$, $\bar{F}_1 := F_1^{-1}$ and $\bar{F}_\rho := F_\rho^{-1}$. First we prove that $\bar{\gamma} \ast \gamma = \gamma \circ \Pi^R$. Let $Q$ be as in Definition 5.14. By the third equality in Remark 2.2 and the fact that $\bar{f} \ast f = u_2$, $\Delta$ is multiplicative and $S \ast \text{id} = \Pi^R$, we have

\[
\begin{align*}
Q &\ast f = S \\
Q &\ast f = S \\
Q &\ast f = S \\
Q &\ast f = S \\
Q &\ast f = S \\
Q &\ast f = \bar{F}_\rho.
\end{align*}
\]

Using this, the associativity of $\mu_E$, Remark 1.15 and Theorem 2.11(11), we obtain

\[
\begin{align*}
\gamma &\ast \bar{\gamma} = Q \\
\gamma &\ast \bar{\gamma} = Q \\
\gamma &\ast \bar{\gamma} = Q \\
\gamma &\ast \bar{\gamma} = Q \\
\gamma &\ast \bar{\gamma} = Q \\
\gamma &\ast \bar{\gamma} = \bar{F}_\rho.
\end{align*}
\]
as desired. We next prove that $\gamma \ast \bar{\gamma} = \gamma \circ \Pi^L$. By Remark 1.15, Theorem 2.11(11) and the associativity of $\mu_F$, 

Moreover, we have

where the first equality holds by the definition of $F_f$; the second one, by the first equality in Remark 2.2 and the fact that, by Remark 5.15

the third one, since $\Delta$ is coassociative and $c$ is natural; the fourth one, by Remark 5.17 the equality in Definition 1.30(1), the definition of $\Pi^L$ and the fact that $\Delta$ is coassociative; the fifth one, by Proposition 1.29 and the fact that $\Delta$ is counitary; and the sixth one, by Propositions 5.11. Consequently,

where the third equality holds by Lemma 5.18, the fourth one, by the fact that $c$ is natural and the equalities in items (3) and (6) of Proposition 4.1; the fifth one, by Proposition 1.29; the sixth one, since $\mu$ is unitary; the seventh one, by the definitions of $u_1$ and $\chi_{\rho}$; the eighth one, by Remark 1.15 and the last one, by Theorem 2.11(11). □
Proposition 5.20. The equality $\delta_E \circ \gamma \circ \Pi^L = (E \otimes \Pi^L) \circ \delta_E \circ \gamma \circ \Pi^L$ holds.

Proof. We have

$$\delta_E \circ \gamma \circ \Pi^L = (\gamma \otimes H) \circ \Delta \circ \Pi^L = (\gamma \otimes \Pi^L) \circ \Delta \circ \Pi^L = (H \otimes \Pi^L) \circ \delta_E \circ \gamma \circ \Pi^L,$$

where the first and last equalities hold since $\gamma$ is right $H$-colinear; and the second one, by Proposition 1.27.

Theorem 5.21. Let $A$ be a weak $H$-module algebra with weak action $\rho$ and let $f: H \otimes H \to A$ be a map. Assume that the hypotheses of Theorem 2.11 are fulfilled and let $E$ be the unitary crossed product of $A$ by $H$ associated with $\rho$ and $f$. If $f$ is convolution invertible, then $(E, \gamma)$ is $H$-cleft.

Proof. By Propositions 5.16, 5.19 and 5.20.

Remark 5.22. Let $E' = A \times_{\rho'} H$ be another unitary crossed product satisfying the hypotheses of Theorem 5.21. Let $\nu'$ be the preunit of $A \otimes_{\rho'} H$. By Proposition 4.3 and the fact that an unitary algebra morphism $\Phi: E \to E'$ is left $A$-linear if and only if $\Phi \circ j_\nu = j_{\nu'}$, the crossed products $E$ and $E'$ are equivalent if and only if the extensions $(E, j_\nu)$ and $(E', j_{\nu'})$ are equivalent.

Proposition 5.23. The equality

$$\mu_E \circ (j_\nu \otimes \gamma^{-1}) \circ c = \mu_E \circ (\gamma^{-1} \otimes j_\nu) \circ (H \otimes \rho) \circ (\Delta \otimes \gamma^{-1})$$

holds.

Proof. By the legibility in the diagrams we set $\bar{\gamma} := \gamma^{-1}$. We have

where the first equality follows by the equality in Proposition 4.1(2); the second one, from Propositions 4.9 and 5.19; the third one, by the first equality in Remark 2.2 and the associativity of $\mu_E$; and the last one, by the first equality in Remark 1.15. Using this and Propositions 4.7, 5.16 and 5.19 we obtain

as desired.
CLEFT EXTENSIONS OF WEAK HOPF ALGEBRAS

6 CLEFT EXTENSIONS ARE WEAK CROSSED PRODUCTS WITH INVERTIBLE COCYCLE

Let $H$ be a weak bialgebra. In this Section we prove that each $H$-cleft extension is isomorphic to a crossed product with invertible cocycle, of $H$ by a weak $H$-module algebra. In a final remark we prove that when $H$ is a weak Hopf algebra, the category of unitary crossed products of $A$ by $H$ with invertible cocycle, such that $A$ a weak $H$-module algebra, and the category of $H$-cleft extensions of $A$ are equivalent.

Let $(E, j)$ be a cleft extension of $A$ by $H$ and let $\gamma: H \to E$ be a convolution invertible total integral. Let $\Upsilon: H \otimes E \to E \otimes H$ be the map defined by

$$\Upsilon := (E \otimes \mu) \circ (c \otimes H) \circ (H \otimes \delta_E).$$

It is well known that $(E, H, \Upsilon)$ is a weak entwining data and $(E, \mu_E, \delta_E)$ is a weak entwined module (for the definitions of weak entwining data and weak entwined module see [3, Definitions 3.1 and 3.2]). Let $q_{\gamma^{-1}}: E \to E$ and $w_E^\gamma: A \otimes H \to E$ be the maps defined by

$$q_{\gamma^{-1}} = \mu_E \circ (E \otimes \gamma^{-1}) \circ \delta_E \quad \text{and} \quad w_E^\gamma = \mu_E \circ (j \otimes \gamma),$$

respectively. By [3, 3.8] there exists a morphism $p_{\gamma^{-1}}: E \to A$ such that $q_{\gamma^{-1}} = j \circ p_{\gamma^{-1}}$. Let $\tilde{w}_{\gamma^{-1}}: E \to A \otimes H$ be the morphism defined by $\tilde{w}_{\gamma^{-1}} = (p_{\gamma^{-1}} \otimes H) \circ \delta_E$. By [3, 3.10] we know that $w_E^\gamma \circ \tilde{w}_{\gamma^{-1}} = \id_E$, so that the map $\Omega_E := \tilde{w}_{\gamma^{-1}} \circ w_E^\gamma$ is an idempotent.

Remark 6.1. Set $\bar{\gamma} := \gamma^{-1}$. Since $\gamma$ is right $H$-colinear, we have

$$q_E^\gamma \circ \gamma = \gamma \ast \bar{\gamma}.$$

Remark 6.2. We have

$$w = \bar{w} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma},$$

where the first and fourth equality holds since $(E, H, \Upsilon)$ is an entwined module; the second one, by the equality in Proposition 2.27(3) and the fact that, since by Proposition 1.17 and the fact $j$ is an extension,

$$\delta_E \circ j = (E \otimes \Pi^L) \circ \delta_E \circ j = (E \otimes \Pi^R) \circ \delta_E \circ j,$$

the third one, since $\mu_E$ is associative; and the last one, since $\mu_E$ is unitary.

Remark 6.3. Set $w = w_E^\gamma$. We have

$$w = \bar{w} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma} = \gamma \ast \bar{\gamma},$$

where the first, fourth, fifth and last equalities hold by the definition of $w$; the second one, since $j$ is multiplicative, the third one, since $\mu_E$ is associative; the sixth one, by Remark 6.2; and the seventh one, since $\gamma$ is right $H$-colinear. Consequently $w$ is left $A$-linear and right $H$-colinear.

Proposition 6.4. The equality $p_{\gamma^{-1}}^E \circ \mu_E \circ (j \otimes E) = \mu_A \circ (A \otimes p_{\gamma^{-1}}^E)$ holds.
Proof. Set $\bar{\gamma} := \gamma^{-1}$, $q := q^E_{\bar{\gamma}^{-1}}$ and $p := p^E_{\gamma^{-1}}$. We have

$$
\begin{align*}
\gamma & = \bar{\gamma} = \gamma \quad \text{where the first and fourth equality are true by the very definition of $q$; the second one, by Remark 6.2; the third one, by the associativity of $\mu_E$; and the last one, because $q = j \circ p$ and $j$ is an algebra morphism. Since $q = j \circ p$ and $j$ is a monomorphism, this finishes the proof.}
\end{align*}
$$

Remark 6.5. Set $p := p^E_{\gamma^{-1}}$ and $\tilde{w} := \tilde{w}^M_{\gamma^{-1}}$. We have

$$
\begin{align*}
\tilde{w} & = p = \tilde{w} = 0 = \tilde{w} = \tilde{w} = \tilde{w} \quad \text{where the first, fourth, fifth and the last equalities hold by the very definition of $\tilde{w}$; the second one, by Remark 6.2; the third one, by Proposition 6.4; and the sixth one, since $\delta^E$ is coassociative. Consequently, $\tilde{w}$ is left $A$-linear and right $H$-colinear.}
\end{align*}
$$

Proposition 6.6. The map $\tilde{\mu} : A \otimes A \otimes H \otimes H \to A \otimes H$, defined by

$$
\begin{align*}
\tilde{\mu} & := \tilde{w}^E_{\gamma^{-1}} \circ \mu_E \circ (w^E_{\gamma} \otimes w^E_{\gamma}),
\end{align*}
$$

is an associative product which is normalized with respect to $\Omega_E$. Moreover $\tilde{\mu}$ is left $A$-linear, $\tilde{\nu} := \tilde{w}^E_{\gamma^{-1}} \circ \eta_E$ is a preunit of $\tilde{\mu}$ and $\Omega_E = \nabla_{\tilde{\nu}}$.

Proof. By Remark 1.5 we know that $\tilde{\mu}$ is an associative product that is normalized respect to $\Omega_E$, that $\tilde{\nu}$ is a preunit of $\tilde{\mu}$ and that $\Omega_E = \nabla_{\tilde{\nu}}$. Set $w := w_{\gamma}$ and $\tilde{w} := \tilde{w}^M_{\gamma^{-1}}$. Since, by the associativity of $\mu_E$ and Remarks 6.3 and 6.5

$$
\begin{align*}
\begin{align*}
\tilde{w} \tilde{w} & = \tilde{w} \tilde{w} \tilde{w} = \tilde{w} \tilde{w} \tilde{w} = \tilde{w} \tilde{w} \tilde{w},
\end{align*}
\end{align*}
$$

the morphism $\tilde{\mu}$ is left $A$-linear. We next prove that $\tilde{\mu}$ is right $H$-colinear. By Remarks 6.3 and 6.5 we know that $w$ and $\tilde{w}$ are right $H$-colinear. Since $\mu_E$ is also right $H$-colinear, we have

$$
\begin{align*}
\begin{align*}
\tilde{w} \tilde{w} & = \tilde{w} \tilde{w} \tilde{w} = \tilde{w} \tilde{w} \tilde{w} = \tilde{w} \tilde{w} \tilde{w},
\end{align*}
\end{align*}
$$

as desired.

Theorem 6.7. Let $\tilde{\mu}$ and $\tilde{\nu}$ be as in Proposition 6.6. The morphisms

$$
\begin{align*}
\rho : H \otimes A \to A \quad \text{and} \quad f : H \otimes H \to A,
\end{align*}
$$

defined by

$$
\begin{align*}
\rho & := (A \otimes \epsilon) \circ \tilde{\mu} \circ (\eta_A \otimes H \otimes j^E),
\end{align*}
$$

and

$$
\begin{align*}
f & := (A \otimes \epsilon) \circ \tilde{\nu} \circ (\eta_A \otimes H \otimes \eta_A \otimes H),
\end{align*}
$$

respectively.
where $f_\rho := (\mu_A \otimes H) \circ (A \otimes \nu)$, satisfy the following properties:

1. $\rho$ is a weak measure of $H$ on $A$,
2. $f$ is a cocycle that satisfies the twisted module condition,
3. $f = \mu_A \circ (A \otimes \rho) \circ (f \otimes \mu \otimes A) \circ (\Delta_H \otimes H \otimes \eta_A)$,
4. $\rho \circ (H \otimes \eta_A) = \mu_A \circ (\rho \otimes f) \circ (H \otimes \circ \otimes H) \circ (\Delta \otimes \nu)$,
5. $\rho \circ (H \otimes \eta_A) = \mu_A \circ (A \otimes f) \circ (\nu \otimes H)$,
6. $(\mu_A \otimes H) \circ (A \otimes \rho \otimes H) \circ (A \otimes H \otimes c) \circ (A \otimes \Delta \otimes A) \circ (\nu \otimes A) = (\mu_A \otimes H) \circ (A \otimes \nu)$.

Moreover

$$\tilde{\mu} = \mu_{A \otimes H}, \quad E \simeq A \times^\rho H \quad \text{and} \quad \gamma = w_\gamma \circ (\eta_A \otimes H).$$

Proof. Since $\tilde{\nu} = \tilde{w}^{E}_{\gamma^{-1}} \circ \eta_E = (p^{E}_{\gamma^{-1}} \otimes H) \circ \delta_E \circ \eta_E$, from the equality in Proposition 2.27(6), it follows that $\tilde{\nu} = (A \otimes \Pi^E) \circ \nu$. Thus, by Theorems 3.11 and 4.2, in order to prove the result it suffices to note that, by Proposition 6.10, the map $\tilde{\mu}$ is left $A$-linear, right $H$-linear and associative, the map $\tilde{\nu}$ is a preunit of $\tilde{\mu}$; and $\tilde{\mu}$ is normalized with respect to $V_p$. \hfill $\Box$

Remark 6.8. Set $w := w_E$, $\tilde{w} := \tilde{w}^{E}_{\gamma^{-1}}$, and $p := p^{E}_{\gamma^{-1}}$. By Remark 6.5

$$f_\rho = \begin{array}{ll}
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Proposition 4.11 also are satisfied. For the legibility in the diagrams we set \( P := P^{E-1} \), and \( q := q^{E-1} \).

For the equality in Proposition 4.11(1) we have
\[
\rho \circ (\eta \otimes A) = p \circ \mu_E \circ (\gamma \otimes j) \circ (\eta \otimes A) = p \circ j = \text{id}_A,
\]
where the first equality holds by Remark 6.9, the second one, since \( \gamma \circ \eta = \eta_E \); and the last one, by Remark 6.8. We next prove that the equality in Proposition 4.1(3) is true. We have
\[
\begin{align*}
\text{where the first and last equality hold since } j \text{ and } \mu_E \text{ are unitary and } j \circ p = q; \text{ the second, fourth and sixth one, by Remark 6.1 and the fact that } \gamma \circ \gamma^{-1} = \gamma = \Pi^2; \text{ the third one, by Proposition 2.23(1); and the fifth one, by Proposition 1.27. This, combined with Remark 6.8 and the fact that } j \circ p = q \text{ and } j \text{ is a monomorphism, proves Proposition 4.1(3). Finally, the equality in Proposition 4.1(6) is satisfied, since}
\end{align*}
\]

\[
\begin{align*}
\text{where the first and last equalities hold since } j \text{ and } \mu_E \text{ are unitary; the second one, by Remark 6.1 and the fact that } \gamma \circ \gamma^{-1} = \gamma \circ \Pi^2; \text{ and the third one, since } p \circ q = q.
\end{align*}
\]

Next we are going to prove that \( f \) is regular. By Theorem 6.13 we know that \( f \circ u_2 = f \).
Moreover, by Proposition 2.17 and the equality in Proposition 4.1(3), we also have \( u_2 \circ f = f \).
Let \( \sigma_E : H \otimes H \to E \) be the morphism defined by
\[
\sigma_E := \mu_E \circ (\mu_E \otimes \gamma^{-1}) \circ (\gamma \otimes \gamma) \circ (\Delta \otimes \gamma).
\]
Clearly \( \sigma_E = (\mu_E \circ (\gamma \otimes \gamma)) \circ (\gamma^{-1} \circ \mu) \).
By Remark 6.8 and II Proposition 1.17 we know that \( u_2 = p_{\gamma^{-1}} \circ \gamma \circ \mu \) and \( \sigma_E = j \circ f \).

Note that
\[
(q_{\gamma^{-1}} \circ \gamma \circ \mu) \circ \sigma_E = (j \circ u_2) \circ (j \circ f) = j \circ (u_2 \circ f) = j \circ f = \sigma_E
\]
and
\[
\sigma_E \circ (q_{\gamma^{-1}} \circ \gamma \circ \mu) = (j \circ f) \circ (j \circ u_2) = j \circ (f \circ u_2) = j \circ f = \sigma_E.
\]

Remark 6.12. Since \( \sigma_E \) and \( q_{\gamma^{-1}} \circ \gamma \) factorize through \( j \),
\[
\delta_E \circ \sigma_E = (E \otimes \Pi^2) \circ \delta_E \circ \sigma_E \quad \text{and} \quad \delta_E \circ q_{\gamma^{-1}} \circ \gamma = (E \otimes \Pi^2) \circ \delta_E \circ q_{\gamma^{-1}} \circ \gamma.
\]

Lemma 6.13. Let \( \sigma_{E}^{-1} : H \otimes H \to E \) be the map defined by \( \sigma_{E}^{-1} := (\gamma \circ \mu) \circ (\mu_E \circ \gamma) \circ (\gamma^{-1} \circ \gamma^{-1}) \).
The following equalities hold:
\[
\sigma_E \circ \sigma_{E}^{-1} = q_{\gamma^{-1}} \circ \gamma \circ \mu \quad \text{and} \quad \sigma_{E}^{-1} \circ \sigma_E = q_{\gamma^{-1}} \circ \gamma \circ \mu \quad (6.15)
\]
Proof. First we show that the first equality in (6.15) is satisfied. Since \( \mu \) is comultiplicative,

\[ (\gamma^{-1} \circ \mu) \ast (\gamma \circ \mu) = (\gamma^{-1} \ast \gamma) \circ \mu = \gamma \circ \Pi \circ \mu. \]

We claim that

\[ (\mu_E \circ (\gamma \otimes \gamma)) \ast (\gamma \circ \Pi \circ \mu) = \mu_E \circ (\gamma \otimes \gamma). \]

In fact, this is true since

\[ \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma, \]

where the first equality holds since \( \mu_E \) is associative; the second one, by Proposition 2.23(2); the third one, by Proposition 1.28(2); the fourth one, by Remark 1.19 and the fact that \( \Delta \) is coassociative and \( c \) is natural; the fifth one, by Proposition 1.25; the sixth one, by Proposition 1.32; and the last one, by Lemma 2.20. For the sake of legibility in the following diagrams we set \( \bar{\gamma} := \gamma^{-1} \) and \( q := q_{E}^{E-1}. \) In order to end the proof of the first equality in (6.15) it suffices to note that

\[ \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma, \]

where the first equality holds since \( c \) is natural and \( \mu_E \) is associative; the second and sixth one, since \( \gamma \ast \bar{\gamma} = \gamma \ast \Pi \); the third one, by Proposition 2.23(1); the fourth and seventh one, by Proposition 1.25; the fifth one, since \( \Delta \) is coassociative and \( c \) is natural; and the last one, by Proposition 1.27 and the fact that \( \gamma \circ \Pi_{L} = \gamma \circ \bar{\gamma}. \) We next prove the second equality in (6.15). To begin with we have

\[ \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma = \gamma \ast \Pi \gamma, \]

where the first equality holds since \( c \) is natural and \( \mu_E \) is associative; the second and sixth one, since \( \bar{\gamma} \ast \bar{\gamma} = \gamma \circ \Pi \); the third one, by Proposition 2.23(2); the fourth and seventh one, by Proposition 1.25; the fifth one, since \( \Delta \) is coassociative and \( c \) is natural; and the last one, by...
Proposition 1.27. Thus the proof of the second equality in (6.15) follows, because

\[
\begin{align*}
\begin{array}{ccc}
\sigma_E^{-1} \circ (q_{\gamma^{-1}} \circ \gamma \circ \mu) &= \sigma_E^{-1} \circ \sigma_E \circ \sigma_E^{-1} = (q_{\gamma^{-1}} \circ \gamma \circ \mu) \circ \sigma_E^{-1} = \sigma_E^{-1}, \\
\end{array}
\end{align*}
\]

where the first and third equalities hold by Definition 1.16(1); the second one, by Corollary 2.25 and the last one, by Remark 6.14.

Remark 6.14. We have

\[
\sigma_E^{-1} \circ (q_{\gamma^{-1}} \circ \gamma \circ \mu) = (q_{\gamma^{-1}} \circ \gamma \circ \mu) \circ \sigma_E^{-1} = (q_{\gamma^{-1}} \circ \gamma \circ \mu) \circ \sigma_E^{-1} = \sigma_E^{-1},
\]

where the last equality follows from the definition of \( \sigma_E^{-1} \) and the fact that, by Remark 6.1, equality \( \gamma \circ \gamma^{-1} = \gamma \circ \Pi^L \) and Corollary 2.25

\[
(q_{\gamma^{-1}} \circ \gamma \circ \mu) \circ (\gamma \circ \mu) = ((q_{\gamma^{-1}} \circ \gamma) \circ \mu) = ((\gamma \circ \Pi^L) \circ \gamma) \circ \mu = \gamma \circ \mu.
\]

Remark 6.15. Set \( \tilde{\sigma}_E := \sigma_E^{-1} \). We have

\[
\begin{align*}
\begin{array}{ccc}
\tilde{\sigma}_E \circ \sigma_E^{-1} &= \sigma_E^{-1} \circ \tilde{\sigma}_E = \sigma_E^{-1} \circ \sigma_E^{-1} = \sigma_E^{-1}, \\
\end{array}
\end{align*}
\]

where the first and fifth equality hold by Proposition 1.22(1) and Remark 6.12; the second one, since \( \delta_E \) is multiplicative; the third one, by Lemma 6.13; the fourth one, by Remark 6.12 and the last one, by Remark 6.13 and the fact that \( \delta_E \) is multiplicative.

Theorem 6.16. The cocycle \( f \) is invertible.

Proof. In order to abbreviate expressions we set \( U = \delta_E \circ \sigma_E \), \( \bar{U} = \delta_E \circ \sigma_E^{-1} \) and \( N = \delta_E \circ q_{\gamma^{-1}} \circ \gamma \circ \mu \). We have

\[
\bar{U} = N \ast \bar{U} = ((E \otimes \Pi^L) \circ \bar{U}) \ast U \ast \bar{U} = ((E \otimes \Pi^L) \circ \bar{U}) \ast N = (E \otimes \Pi^L) \circ \bar{U},
\]

where the first equality holds by Remark 6.14 and the fact that \( \delta_E \) is multiplicative; the second one, by the first part of Remark 6.15, the third one, by Lemma 6.14 and the fact that \( \delta_E \) is multiplicative; and the last one, by the second part of Remark 6.15. Consequently \( \sigma_E^{-1} \) factorize through \( j \). Let \( f^{-1} : H \otimes H \to A \) be such that \( \sigma_E^{-1} = j \circ f^{-1} \). Since \( j \) is a monomorphism and

\[
j \circ (f \ast f^{-1}) = (j \circ f) \ast (j \circ f^{-1}) = \sigma_E \ast \sigma_E^{-1} = q_{\gamma^{-1}} \circ \gamma \circ \mu = j \circ p_{\gamma^{-1}} \circ \gamma \circ \mu
\]

and

\[
j \circ (f^{-1} \ast f) = (j \circ f^{-1}) \ast (j \circ f) = \sigma_E^{-1} \ast \sigma_E = q_{\gamma^{-1}} \circ \gamma \circ \mu = j \circ p_{\gamma^{-1}} \circ \gamma \circ \mu,
\]

we obtain that \( f \ast f^{-1} = p_{\gamma^{-1}} \circ \gamma \circ \mu \) and \( f^{-1} \ast f = p_{\gamma^{-1}} \circ \gamma \circ \mu \), as desired. □
Theorem 6.17. Let \((E, \phi)\) be a \(H\)-cleft extension of \(A\) by \(H\) and let \(\rho, f\) and \(\tilde{\nu}\) be as in Theorem 6.7. Then \(A\) is a weak \(H\)-module algebra via \(\rho\), the hypotheses of Theorem 2.11 are fulfilled, the cocycle \(f\) is invertible and \(E \cong A \times_H f^\rho H\).

Proof. By Theorem 6.7, Proposition 6.11 and Theorem 6.16. \(\Box\)

Remark 6.18. From Theorem 5.21, Remark 5.22 and Theorem 6.17 it follows that if \(H\) is a weak Hopf algebra and \(A\) is an algebra, then the category of unitary crossed products of \(A\) by \(H\) with invertible cocycle and \(A\) a weak \(H\)-module algebra, and the category of \(H\)-cleft extensions of \(A\), are equivalent.

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