Bit-Optimal Lempel-Ziv compression

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Abstract. One of the most famous and investigated lossless data-compression scheme is the one introduced by Lempel and Ziv about 40 years ago [23]. This compression scheme is known as “dictionary-based compression” and consists of squeezing an input string by replacing some of its substrings with (shorter) codewords which are actually pointers to a dictionary of phrases built as the string is processed. Surprisingly enough, although many fundamental results are nowadays known about upper bounds on the speed and effectiveness of this compression process (see e.g. [12, 16] and references therein), “we are not aware of any parsing scheme that achieves optimality when the LZ77-dictionary is in use under any constraint on the codewords other than being of equal length” [16, pag. 159]. Here optimality means to achieve the minimum number of bits in compressing each individual input string, without any assumption on its generating source. In this paper we provide the first LZ-based compressor which computes the bit-optimal parsing of any input string in efficient time and optimal space, for a general class of variable-length codeword encodings which encompasses most of the ones typically used in data compression and in the design of search engines and compressed indexes [14, 17, 22].

1 Introduction

The problem of lossless data compression consists of compactly representing data in a format that can be faithfully recovered from the compressed file. Lossless compression is achieved by taking advantage of the redundancy which is often present in the data generated by either humans or machines. One of the most famous lossless data-compression scheme is the one introduced by Lempel and Ziv in the late 70s [23, 24]. It has been “the solution” to lossless compression for nearly 15 years, and indeed many (non-)commercial programs are currently based on it—like gzip, zip, pkzip, arj, rar, just to cite a few. This compression scheme is known as dictionary-based compression, and consists of squeezing an input string S[1, n] by replacing some of its substrings (phrases) with (shorter) codewords which are actually pointers to a dictionary being either static (in that it has been constructed before the compression starts) or dynamic (in that it is built as the input string is compressed). The well-known LZ77 and LZ78 compressors, proposed by Lempel and Ziv in [23, 24], and all their numerous variants [17], are interesting examples of dynamic dictionary-based compression algorithms. In LZ77, and its variants, the dictionary consists of all substrings occurring in the previously scanned portion of the input string, and each codeword consists of a triple ⟨d, ℓ, c⟩ where d is the relative offset of the copied phrase, ℓ is its length and c is the single (new) character following it. In LZ78, the dictionary is built upon phrases extracted from the previously scanned prefix

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of the input string, and each codeword consists of a pair \((i, c)\) where \(i\) is the identifier of the copied phrase in the dictionary and \(c\) is the character following that phrase in the subsequent suffix of the string.

Many theoretical and experimental results have been dedicated to LZ-compressors in these thirty years (see e.g. [17] and references therein); and, although today there are alternative solutions to the problem of lossless data compression potentially offering better compression bounds (e.g., Burrows-Wheeler compression and Prediction by Partial Matching [22]), dictionary-based compression is still widely used in everyday applications because of its unique combination of compression power and speed. Over the years dictionary-based compression has also gained importance as a general algorithmic tool, being employed in the design of compressed text indexes [14], or in universal clustering tools [3], or in designing optimal prefetching mechanisms [21].

Surprisingly enough some important problems on the combinatorial properties of the LZ-parsing are still open, and they will be the main topic of investigation of this paper. Take the classical LZ77 and LZ78 algorithms. They adopt a greedy parsing of the input string: namely, at each step, they take the longest dictionary phrase which is a prefix of the currently unparsed string suffix. It is well-known [16] that greedy parsing is optimal with respect to the number of phrases in which \(S\) can be parsed by any suffix-complete dictionary (like LZ77); and a small variation of it (called flexible-parsing [13]) is optimal for prefix-complete dictionaries (like LZ78). Of course, the number of parsed phrases influences the compression ratio and, indeed, various authors [23, 24, 12] proved that greedy parsing achieves asymptotically the (empirical) entropy of the source generating the input string \(S\). However, these fundamental results have not yet closed the problem of optimally compressing \(S\) because the optimality in the number of parsed phrases is not necessarily equal to the optimality in the number of bits output by the final compressor. Clearly, if the phrases are compressed via an equal-length encoder, like in [17, 23, 12], then the produced output is bit optimal. But if one aims for higher compression by using variable-length encoders for the parsed phrases (see e.g. [22, 7]), then the bit-length of the compressed output produced by the greedy-parsing scheme is not necessarily optimal.

As an illustrative example, consider the LZ77-compressor and assume that the copy of the \(i\)th phrase occurs very far from the current unparsed position, and thus its \(d\)-value is large. In this case it could be probably more convenient to renounce to the maximality of that phrase and split it into (several) smaller phrases which possibly occur closer to that position and thus can be encoded in fewer bits overall. Several solutions are indeed known for determining the bit-optimal parsing of \(S\), but they are either inefficient [18, 8] taking \(\Theta(n^2)\) time and space in the worst case, or approximate [9], or they rely on heuristics [11, 19, 2, 4, 8] which do not provide any guarantee on the time/space performance of the compression process. This is the reason why Rajpoot and Sahinalp stated in [16, pag. 159] that “We are not aware of any on-line or off-line parsing scheme that achieves optimality when the LZ77-dictionary is in use under any constraint on the codewords other than being of equal length”.

Motivated by these poor results, we address in this paper the question posed by Rajpoot and Sahinalp by investigating a general class of variable-length codeword encodings which are typically used in data compression and in the design of search engines and compressed indexes [14, 17, 22]. We prove that the classic greedy-parsing scheme deploying these encoders may be far from the bit-optimal parsing by a multiplicative factor \(\Omega(\log n / \log \log n)\), which is indeed unbounded asymptotically (Section 2, Lemma 1). This result is obtained by considering an infinite family of strings \(S\) of increasing length \(n\) and low empirical entropy, and by showing that for these strings copying the longest phrase, as LZ77 does, may be dramatically inefficient.
We notice that this gap between LZ77 and the bit-optimal compressor strengthen the results proved by Kosaraju and Manzini in [12], who showed that the compression rate of LZ77 converges asymptotically to the kth order empirical entropy $H_k(S)$ of string $S$, and this rate is dominated for low entropy strings by an additive term $O(\frac{\log \log n}{\log n})$ which depends on the string length $n$ and not on its compressibility $H_k(S)$. These are properly the strings for which the bit-optimal parser is much better than LZ77, and thus closer to the entropy of $S$.

Given these premises, we investigate and design an LZ-based compressor that computes the bit-optimal parsing for those variable-length integer encoders in efficient time and optimal space, in the worst case (Section 5, Theorem 2). Due to space limitations, we will detail our results only for the LZ77-dictionary, and defer the discussion on other dictionary-based schemes (like LZ78) to the last Section 6. Technically speaking, we follow [18] and model the search for a bit-optimal parsing of an input string $S[1,n]$, as a single-source shortest path problem (shortly, SSSP) on a weighted DAG $G(S)$ consisting of $n$ nodes, one per character of $T$, and $e$ edges, one per possible LZ77-parsing step. Every edge is weighted according to the length in bits of the codeword adopted to compress the corresponding LZ77-phrase. Since LZ-codewords are tuples of integers (see above), we consider in this paper a class of codeword encoders which satisfy the so called increasing cost property: the larger is the integer to be encoded, the longer is the codeword. This class encompasses most of the encoders frequently used in the literature to design data compressors [7], compressed full-text indexes [14] and search engines [22]. We prove new combinatorial properties for the SSSP problem formulated on the graph $G(S)$ weighted according to these encoding functions and show that, unlike [18] (for which $e = \Theta(n^2)$ in the worst case), the computation of the SSSP in $G(S)$ can be restricted onto a subgraph $\tilde{G}(S)$ whose size is provably smaller than the complete graph (see Theorem 1). Actually, we show that the size of $\tilde{G}(S)$ is related to the structural features of the integer-encoding functions adopted to compress the LZ-phrases (Lemma 3). Finally, we design an algorithm that computes the SSSP of $\tilde{G}(S)$ without materializing that subgraph all at once, but by creating and exploring its edges on-the-fly in optimal $O(1)$ amortized time per edge and using $O(n)$ optimal space overall. As a result, our LZ77-compressor achieves optimal compression ratio, by using optimal $O(n)$ working space and taking time proportional to $|\tilde{G}(S)|$ (hence, it is optimal in its size).

If the LZ-phrases are encoded with equal-length codewords, our approach is optimal in compression ratio and time/space performance, as it is classically known [17]. But if we consider the more general (and open) case of the variable-length Elias or Fibonacci codes, as it is typical of data compressors and compressed indexes [7, 14], then our approach is optimal in compression ratio and working-space occupancy, and takes $O(n \log n)$ time in the worst case. Most other variable-length integer encoders fall in this case too (see e.g. [22]). To the best of our knowledge, this is the first result providing a positive answer to Rajpoot-Sahinalp’s question above!

The final Section 6 will discuss variations and extensions of our approach, considering the cases of a bounded compression-window and of other suffix- or prefix-complete dictionary construction schemes (like LZ78).

2 On the Bit-Optimality of LZ-parsing

Let $S[1,n]$ be a string drawn from an alphabet $\Sigma$ of size $\sigma$. We will use $S[i]$ to denote the $i$th symbol of $S$; $S[i:j]$ to denote the substring (also called the phrase) extending from the $i$th to the $j$th symbol in $S$ (extremes included); and $S_i = S[i:n]$ to denote the $i$-th suffix of $S$. 

Dictionary-based compression works in two intermingled phases: parsing and encoding. Let \( w_1, w_2, \ldots, w_{i-1} \) be the phrases in which a prefix of \( S \) has been already parsed. The parser selects the next phrase \( w_i \) as one of the phrases in the current dictionary that prefix the remaining suffix of \( S \), and possibly attaches to this phrase few other following symbols (typically one) in \( S \). This addition is sometimes needed to enrich the dictionary with new symbols occurring in \( S \) and never encountered before in the parsing process. Phrase \( w_i \) is then represented via a proper reference to the dynamic dictionary, which is built during the parsing process. The well-known \( \text{LZ77} \) and \( \text{LZ78} \) compressors [23, 24], and all their variants [17], are incarnations of the above compression scheme and differentiate themselves mainly by the way the dynamic dictionary is built, by the rule adopted to select the next phrase, and by the encoding of the dictionary references.

In the rest of the paper we will concentrate on the \( \text{LZ77} \)-scheme and defer the discussion of \( \text{LZ78} \)'s, and other approaches, to the last Section 6. In \( \text{LZ77} \) and its variants, the dictionary consists of all substrings starting in the scanned prefix of \( S \), and thus consists (implicitly) of a quadratic number of phrases which are (explicitly) represented via a (dynamic) indexing data structure that takes linear space in \(|S|\) and efficiently searches for repeated substrings. The \( \text{LZ77} \)-dictionary satisfies two main properties: Prefix Completeness—i.e. for any given dictionary phrase, all of its prefixes are also dictionary phrases—and Suffix Completeness—i.e. for any given dictionary phrase, all of its suffixes are also dictionary phrases. At any step, the \( \text{LZ77} \)-parser proceeds by adopting the so-called longest match heuristic: that is, \( w_i \) is taken as the longest phrase of the current dictionary which prefixes the remaining suffix of \( S \). This will be hereafter called greedy parsing. The classic \( \text{LZ77} \)-parser finally adds one further symbol to \( w_i \) (namely, the one following the phrase in \( S \)) to form the next phrase of \( S \)'s parsing.

In the rest of our paper, and without loss of generality, we consider the \( \text{LZ77} \)-variant which avoids the additional symbol per phrase, and thus represents the next phrase \( w_i \) by the integer pair \((d_i, \ell_i)\) where \( d_i \) is the relative offset of the copied phrase \( w_i \) within the prefix \( w_1 \cdots w_{i-1} \) and \( \ell_i \) is its length (i.e. \(|w_i|\)). We notice that every first occurrence of a new symbol \( c \) is encoded as \((0, c)\). Once phrases are identified and represented via pairs of integers, their components are compressed via variable-length integer encoders which will eventually produce the compressed output of \( S \) as a sequence of bits.

In order to study and design bit-optimal parsing schemes, we therefore need to deal with integer encoders. Let \( f \) be an integer-encoding function that maps any integer \( x \in [n] \) into a (bit-)codeword \( f(x) \) whose length (in bits) is denoted by \(|f(x)|\). In this paper we consider variable-length encodings which use longer codewords for larger integers:

**Property 1 (Increasing Cost Property).** For any \( x, y \in [n] \) it is \( x \leq y \) iff \(|f(x)| \leq |f(y)|\).

This property is satisfied by most practical integer encoders [22], as well by equal-length codewords, Elias codes (i.e. gamma, delta, and their derivatives [6]), and Fibonacci’s codes [7]. Therefore, this class encompasses all encoders typically used in the literature to design data compressors [17], compressed full-text indexes [14] and search engines [22].

Given two integer-encoders \( f \) and \( g \) (possibly \( f = g \)) which satisfy the Increasing Cost Property 1, we denote by \( \text{LZ}_{f,g}(S) \) the compressed output produced by the greedy-parsing strategy in which we have used \( f \) to compress the distance \( d_i \), and \( g \) to compress the length \( \ell_i \) of any parsed phrase \( w_i \). \( \text{LZ}_{f,g}(S) \) thus encodes any phrase \( w_i \) in \(|f(d_i)| + |g(\ell_i)|\) bits. We

\[1\] Notice that it admits the overlapping between the current dictionary and the next phrase to be constructed.
have already noticed that $LZ_{f,g}(S)$ is not necessarily bit optimal, so we will hereafter use $OPT_{f,g}(S)$ to denote the $(f,g)$-optimal parser, namely the one that parses $S$ into a sequence of phrases which are drawn from the LZ77-dictionary and which minimize the total number of bits produced by their encoders $f$ and $g$. Given the above observations it is immediate to infer that $|LZ_{f,g}(S)| \geq |OPT_{f,g}(S)|$. However this bound does not provide us with any estimate of how much worse the greedy parsing can be with respect to $OPT_{f,g}(S)$. In what follows we identify an infinite family of strings for which the compressed output of the greedy parser is a multiplicative factor $\Omega(\log n/\log \log n)$ worse than the bit-optimal parser. This result shows that the ratio $\frac{|LZ_{f,g}(S)|}{|OPT_{f,g}(S)|}$ is indeed asymptotically unbounded, and thus poses the serious need for a $(f,g)$-optimal parser, as clearly requested by [16].

Our argument holds for any choice of $f$ and $g$ from the family of encoding functions that represent an integer $x$ with a bit string of size $\Theta(\log x)$ bits. (Thus the well-known Elias’ and Fibonacci’s coders belong to this family.) Taking inspiration from the proof of Lemma 4.2 in [12], we consider the infinite family of strings $S_l = ba^1 c^2 ba^2 ba^3 \ldots ba^l$, parameterized in the positive value $l$. The LZ77-parser partitions $S_l$ as

$$(b) (a) (a^{l-1}) (c) (c^{2^l-1}) (ba) (ba^2) (ba^3) \ldots (ba^l)$$

where the symbols forming a parsed phrase have been delimited within a pair of brackets. LZ77 thus copies the latest $l$ phrases from the beginning of $S_l$ and takes at least $l|f(2^l)| = \Theta(l^2)$ bits. Let us now consider a more parsimonious parser, called $rOPT$, which selects the copy of $ba^{i-1}$ (with $i > 1$) from its immediately previous occurrence:

$$(b) (a) (a^{l-1}) (c) (c^{2^l-1}) (b) (a) (ba) (a) (ba^2) (a) \ldots (ba^{l-1}) (a)$$

$rOPT(S_l)$ takes $|g(2^l)| + |g(l)| + \sum_{i=2}^{l} [ |f(i)| + |g(i)| + f(0)] + O(l) = O(l \log l)$ bits. Since $OPT(S_l) \leq rOPT(S_l)$, we can conclude that

$$\frac{|LZ_{f,g}(S_l)|}{|OPT_{f,g}(S_l)|} \geq \frac{|LZ_{f,g}(S_l)|}{|rOPT(S_l)|} \geq \Theta \left( \frac{l}{\log l} \right)$$

(1)

Since $|S_l| = 2^l + l^2 - O(l)$, we have that $l = \Theta(\log \log n)$ for sufficiently long strings. Using this estimate into Inequality 1 we finally obtain:

**Lemma 1.** There exists an infinite family of strings such that, for any of its elements $S$, it is $|LZ_{f,g}(S)| \geq \Theta(|S|/\log \log |S|) / |OPT_{f,g}(S)|$.

On the other hand, we can prove that this lower bound is tight up to a $\log \log |S|$ multiplicative factor, by easily extending to LZ77-dictionary and Property 1, a result proved in [10] for static dictionaries. Precisely, we can show that $\frac{|LZ_{f,g}(S)|}{|OPT_{f,g}(S)|} \leq \frac{|f(n)| + |g(n)|}{f(0) + |g(0)|}$, which is upper bounded by $O(\log n)$ because $|S| = n$, $|f(n)| = |g(n)| = \Theta(\log n)$ and $|f(0)| = |g(0)| = O(1)$.

## 3 Bit-Optimal Parsing and SSPP-problem

Following [18], we model the design of a bit-optimal LZ77-parsing strategy for a string $S$ as a Single-Source Shortest Path problem (shortly, SSPP-problem) on a weighted DAG $G(S)$ defined

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Recall the variant of LZ77 we are considering in this paper, which uses just a pair of integers per phrase, and thus drops the char following that phrase in $S$. 

as follows. Graph $G(S) = (V, E)$ has one vertex per symbol of $S$ plus a dummy vertex $v_{n+1}$, and its edge set $E$ is defined so that $(v_i, v_j) \in E$ iff (1) $j = i+1$ or (2) the substring $S[i : j-1]$ occurs in $S$ starting from a (previous) position $p < i$ (clearly $i < j$ and thus $G(S)$ is a DAG). Every edge $(v_i, v_j)$ is labeled with the pair $(d_{i,j}, \ell_{i,j})$ where

- $d_{i,j} = i - p$ is the distance between $S[i : j-1]$ and the position $p$ of its right-most copy in $S$. We set $d_{i,j} = 0$ whenever $j = i+1$.
- We set $\ell_{i,j} = j - i$, if $j > i+1$, or $\ell_{i,j} = S[i]$ otherwise.

It is easy to see that the edges outgoing from $v_i$ denote all possible parsing steps that can be taken by any parsing strategy which uses a LZ77-dictionary. Hence, there exists a one-to-one correspondence between paths from $v_1$ to $v_{n+1}$ in $G(S)$ and parsings of the whole string $S$: any path $\pi = v_1 \cdots v_{n+1}$ corresponds to the parsing $P_{\pi}(S)$ represented by the phrases labeling the edges traversed by $\pi$. If we weight every edge $(v_i, v_j) \in E$ with an integer $c(v_i, v_j) = |f(d_{i,j})| + |g(\ell_{i,j})|$ which accounts for the cost of encoding its label (phrase) via the encoding functions $f$ and $g$, then the length in bits of the encoded parsing $P_{\pi}(S)$ is equal to the cost of the weighted path $\pi$ in $G(S)$. We have therefore reduced the problem of determining $OPT_{f,g}(S)$ to the problem of computing the SSSP of $G(S)$ from $v_1$ to $v_{n+1}$.

Given that $G(S)$ is a DAG, its shortest path from $v_1$ to $v_{n+1}$ can be computed in $O(|E|)$ time and space. In the worst case (take e.g. $S = a^n$), this is $\Theta(n^2)$, and thus it is inefficient and unusable in practice [18, 9]. In what follows we show that the computation of the SSSP can be actually restricted to a subgraph of $G(S)$ whose size is provably $o(n^2)$ in the worst case, and typically $O(n \log n)$ for most known integer-encoding functions. Then we will design efficient and sophisticated algorithms and data structures that will allow us to generate this subgraph on-the-fly by taking $O(1)$ amortized time per edge and $O(n)$ space overall. These algorithms will be therefore time-and-space optimal for the subgraph in hand!

4 An useful, small, subgraph of $G(S)$

We use $FS(v)$ to denote the forward star of a vertex $v$, namely the set of vertices pointed to by $v$ in $G(S)$; and we use $BS(v)$ to denote the backward star of $v$, namely the set of vertices pointing to $v$ in $G(S)$. By construction of $G(S)$, for any $v_j \in FS(v_i)$ it is $i < j$; so that, all of the edges are oriented rightward, and in fact $G(S)$ is a DAG. We can actually show a stronger property on the distribution of the indices of the vertices in $FS(v)$ and $BS(v)$, namely that they form a contiguous range.

**Fact 1** Given a vertex $v_i$, it is $FS(v_i) = \{v_{i+1}, \ldots, v_{i+x-1}, v_{i+x}\}$ and $BS(v_i) = \{v_{i-y}, \ldots, v_{i-2}, v_{i-1}\}$. Note that $x, y$ are smaller than the length of the longest repeated substring in $S$.

**Proof:** By definition of $(v_i, v_{i+x})$, string $S[i : i+x-1]$ occurs at some position $p < i$ in $S$. Any prefix $S[i : k-1]$ of $S[i : i+x-1]$ also occurs at that position $p$, thus $v_k \in FS(v_i)$. The bound on $x$ immediately derives from the definition of $(v_i, v_{i+x})$. A similar argument derives the property on $BS(v_i)$.

This actually means that if an edge does exist in $G(S)$, then they do exist also all edges which are nested within it and are incident into one of its extremes. The following property relates the indices of the vertices $v_j \in FS(v_i)$ with the cost of their connecting edge $(v_i, v_j)$, and not surprisingly shows that the smaller is $j$ (i.e. shorter edge), the smaller is the cost of encoding the parsed phrase $S[i : j-1]$.

\[ c(v_i, v_j) = |f(d_{i,j})| + |g(\ell_{i,j})|, \text{ if the edge does exist, otherwise we set } c(v_i, v_j) = +\infty. \]
Definition 1. An edge \((v_i, v_j) \in E\) is called
- \(d\)-maximal iff the next edge from \(v_i\) takes more bits to encode its distance. Namely, we have that \(|f(d_{i,j})| < |f(d_{i,j+1})|\).
- \(\ell\)-maximal iff the next edge from \(v_i\) takes more bits to encode its length. Namely, we have that \(|g(l_{i,j})| < |g(l_{i,j+1})|\).

Overall, we say that edge \((v_i, v_j)\) is maximal if it is either \(d\)-maximal or \(\ell\)-maximal: thus \(c(v_i, v_j) < c(v_i, v_{j+1})\).

Now, we wish to count the number of maximal edges outgoing from \(v_i\). This number clearly depends on the integer-encoding functions \(f\) and \(g\) (which satisfy Property 1). Let \(Q(f, n)\) (resp. \(Q(g, n)\)) be the number of different codeword lengths generated by \(f\) (resp. \(g\)) when applied to integers in the range \([n]\). We can partition \([n]\) into contiguous sub-ranges \(I_1, I_2, \ldots, I_{Q(f,n)}\) such that the integers in \(I_i\) are mapped by \(f\) to codewords (strictly) shorter than the codewords for the integers in \(I_{i+1}\). Similarly, \(g\) partitions the range \([n]\) in \(Q(g,n)\) contiguous sub-ranges.

Lemma 2. There are at most \(Q(f, n) + Q(g, n)\) maximal edges outgoing from any vertex \(v_i\).

Proof: By Fact 1, vertices in \(FS(v_i)\) have indices in a range \(R\), and by Fact 2, \(c(v_i, v_j)\) is monotonically non-decreasing as \(j\) increases in \(R\). Moreover we know that \(f\) (resp. \(g\)) cannot change more than \(Q(f, n)\) (resp. \(Q(g, n)\)) times, so that the statement follows.

In order to speed up the computation of a SSSP connecting \(v_1\) to \(v_{n+1}\) in \(G(S)\), we construct a subgraph \(\tilde{G}(S)\) which is provably smaller than \(G(S)\) and contains one of those SSSP. Next theorem shows that \(\tilde{G}(S)\) can be formed by taking just the maximal edges of \(G(S)\).

Theorem 1. There exists a shortest path in \(G(S)\) connecting \(v_1\) to \(v_{n+1}\) and traversing only maximal edges.

Proof: By contradiction assume that every such shortest paths contains at least one non-maximal edge. Let \(\pi = v_{i_1}v_{i_2}\ldots v_{i_k}\), with \(i_1 = 1\) and \(i_k = n + 1\), be one of these shortest paths, and let \(\gamma = v_{i_1}\ldots v_{i_r}\) be the longest initial subpath of \(\pi\) which traverses only maximal edges. Assume w.l.o.g. that \(\pi\) is the shortest path maximizing the value of \(|\gamma|\). We know that \((v_{i_r}, v_{i_{r+1}})\) is a non-maximal edge, and thus we can take the maximal edge \((v_{i_r}, v_{j})\) that has the same cost. By definition of maximal edge, it is \(j > i_{r+1}\); furthermore, we must have \(j < n + 1\) because we assumed that no path is formed only by maximal edges. Thus it must exist an index \(i_h \geq i_r\) such that \(j \in [i_h, i_{h+1}]\), because indices in \(\pi\) are increasing given that \(G(S)\) is a DAG. Since \((v_{i_h}, v_{i_{h+1}})\) is an edge of \(\pi\), by Fact 1 follows that it does exist the edge \((v_j, v_{i_{h+1}})\), and by Fact 2 on \(BS(v_{i_{h+1}})\) we can conclude that \(c(v_j, v_{i_{h+1}}) \leq c(v_{i_h}, v_{i_{h+1}})\). Consequently, the path \(v_{i_1}\ldots v_{i_r}v_jv_{i_{h+1}}\ldots v_{i_h}\) is also a shortest path but its longest initial subpath of maximal edges consists of \(|\gamma| + 1\) vertices, which is a contradiction!
Optimal-Parser($S[1,n]$)
1. $C[1] = 0$; $P[1] = 1$;
2. for each $i \in [2,n+1]$ do $C[i] = +\infty$; $P[i] = NIL$
3. for $i = 1$ to $n$ do
4. generate on-the-fly all maximal edges in $FS(v_i)$;
5. for any $(v_i,v_j)$ maximal do
6. if $C[j] > C[i] + c(v_i,v_j)$ then $C[j] = C[i] + c(v_i,v_j)$; $P[j] = i$

Fig. 1. Algorithm to compute the SSSP of the subgraph $\tilde{G}(S)$.

Theorem 1 implies that the distance between $v_1$ and $v_{n+1}$ is the same in $G(S)$ and $\tilde{G}(S)$, with the advantage that computing distances in $\tilde{G}(S)$ can be done faster and in reduced space, because of its smaller size. In fact, Lemma 2 implies that $|FS(v)| \leq Q(f,n) + Q(g,n)$, so that

Lemma 3. Subgraph $\tilde{G}(S)$ consists of $n+1$ vertices and at most $n(Q(f,n) + Q(g,n))$ edges.

For Elias’ codes [6], Fibonacci’s codes [7], and most practical integer encoders used for search engines and data compressors [17, 22], it is $Q(f,n) = Q(g,n) = O(\log n)$. Therefore $|\tilde{G}(S)| = O(n \log n)$ and hence it is provably smaller than the complete graph built and used by the previous papers [18, 9, 8].

The next technical step consists of achieving time efficiency and optimality in working space, because we cannot construct $\tilde{G}(S)$ all at once. In the next sections we design an algorithm that generates $\tilde{G}(S)$ on-the-fly as the computation of its SSSP goes on, and pays $O(1)$ amortized time per edge and no more than $O(n)$ optimal space overall. In some sense, this algorithm is optimal for the identified sub-graph $\tilde{G}(S)$.

5 A bit-optimal parser

From a high level, our solution proceeds as in Figure 1, where a variant of a classic linear-time algorithm for SSSP over a DAG is reported [5, Section 24.2]. In that pseudo-code entries $C[i]$ and $P[i]$ hold, respectively, the shortest path distance from $v_1$ to $v_i$ and the predecessor of $v_i$ in that shortest path. The main idea of Optimal-Parser consists of scanning the vertices of $\tilde{G}(S)$ in topological order, and of generating on-the-fly and relaxing (Step 6) the edges outgoing from a vertex $v_i$ only when $v_i$ becomes the current vertex. The correctness of Optimal-Parser follows directly from Theorem 24.5 of [5] and our Theorem 1.

The key difficulty in this process consists of how to generate on-the-fly and efficiently (in time and space) the maximal edges outgoing from vertex $v_i$. We will refer to this problem as the forward-star generation problem, and use FSG for brevity. In the next section we show that, when $\sigma \leq n$, FSG takes $O(1)$ amortized time per edge and $O(n)$ space in total. As a result (Theorem 2), Optimal-Parser requires $O(n(Q(f,n) + Q(g,n)))$ time in the worst case, since the main loop is repeated $n$ times and we have no more than $Q(f,n) + Q(g,n)$ maximal edges per vertex (Lemma 2). The space used is that for the FSG-computation plus the two arrays $C$ and $P$; hence, it will be shown to be $O(n)$ in total. In case of a large alphabet $\sigma > n$, we need to add $T_{sort}(n,\sigma)$ time because of the sorting/remapping of $S$’s symbols into $[n]$. 


Thus must have longer distance from \( S \) values assumed by the encoding function. This takes \( \ell \)-optimal time and space in the worst case.

### 5.1 On-the-fly generation of \( d \)-maximal edges

We concentrate only on the computation of the \( d \)-maximal edges, because this is the hardest task. In fact, we know that the edges outgoing from \( v_i \) can be partitioned in no more than \( Q(f, n) \) groups according to the distance from \( S[i] \) of the copied string they represent (proof of Lemma 2). Let \( I_1, I_2, \ldots, I_{Q(f, n)} \) be the intervals of distances such that all distances in \( I_k \) are encoded with the same number of bits by \( f \). Take now the \( d \)-maximal edge \((v_i, v_{h_k})\) for the interval \( I_k \). We can infer that substring \( S[i : h_k - 1] \) is the longest substring having a copy at distance within \( I_k \), because, by Definition 1 and Fact 2, any edge following \((v_i, v_{h_k})\) denotes a longer substring which must lie in a subsequent interval (by \( d \)-maximality of \((v_i, v_{h_k})\)), and thus must have longer distance from \( S[i] \).

Once \( d \)-maximal edges are known, the computation of the \( \ell \)-maximal edges is then easy because it suffices to further decompose the edges between successive \( d \)-maximal edges, say between \((v_i, v_{h_k - 1})\) and \((v_i, v_{h_k})\), according to the distinct values assumed by the encoding function \( g \) on the lengths in the range \([h_{k-1}, \ldots, h_k - 1]\). This takes \( O(1) \) time per \( \ell \)-maximal edge, because it needs some algebraic calculations and can then infer the corresponding copied substring as a prefix of \( S[i : h_k - 1] \).

So, let us concentrate on the computation of \( d \)-maximal edges outgoing from vertex \( v_i \). This is based on two key ideas. The first idea aims at optimizing the space usage by achieving the optimal \( O(n) \) working-space bound. It consists of proceeding in \( Q(f, n) \) passes, one per interval \( I_k \) of possible \( d \)-costs for the edges in \( \tilde{G}(S) \). During the \( k \)th pass, we logically partition the vertices of \( \tilde{G}(S) \) in blocks of \(|I_k|\) contiguous vertices, say \( v_{i_k}, v_{i_k+1}, \ldots, v_{i_k+|I_k|-1} \), and compute all \( d \)-maximal edges which spread from that block and have distance within \( I_k \) (thus the same \( d \)-cost \( c(I_k) \)). These edges are kept in memory until they are used by Optimal-Parser, and discarded as soon as the first vertex of the next block, i.e. \( v_{i_k+|I_k|} \), needs to be processed. The next block of \(|I_k|\) vertices is then fetched and the process repeats. Actually, all passes are executed in parallel to guarantee that all \( d \)-maximal edges of \( v_i \) are available when processing it. There are \( n/|I_k| \) distinct blocks, each vertex belongs to exactly one block at each pass, and all of its \( d \)-maximal edges are considered in some pass (because they have \( d \)-cost in some \( I_k \)). The space is \( \sum_{k=1}^{Q(f, n)} |I_k| = O(n) \) because we keep one \( d \)-maximal edge per vertex at any pass.

The second key idea aims at computing the \( d \)-maximal edges for that block of \(|I_k|\) contiguous vertices in \( O(|I_k|) \) time and space. This is what we address in the rest of this paper, because its solution will allow us to state that the time complexity of \textsc{FSG} is \( \sum_{k=1}^{Q(f, n)} \sum_{|I_k|=1}^{n/|I_k|} O(|I_k|) = O(n Q(f, n)) \), namely \( O(1) \) amortized time per \( d \)-maximal edge. Combining this fact with the above observation on the computation of the \( \ell \)-maximal edges, we get Theorem 2.

So, let us assume that the alphabet size \( \sigma \leq n \), and consider the \( k \)th pass of \textsc{FSG} in which we assume that \( I_k = [i, r] \). Recall that all distances in \( I_k \) can be \( f \)-encoded in the same number of, say, \( c(I_k) \) bits. Let \( B = [i, i+|I_k|-1] \) be the block of (indices of) vertices for which we wish to compute on-the-fly the \( d \)-maximal edges of cost \( c(I_k) \). This means that the \( d \)-maximal edge from vertex \( v_h, h \in B \), represents a phrase that starts at \( S[h] \) and has a copy whose starting position is in the window \( W_h = [h-r, h-l] \). Thus the distance of that copy can be \( f \)-encoded in \( c(I_k) \) bits, and so we will say that the edge has \( d \)-cost \( c(I_k) \). Since this computation must
be done for all vertices in \( B \), it is useful to consider \( W_B = W_i \cup W_{i+|I_k|-1} \) which merges the first and last window and thus spans all positions that can be the (copy-)reference of any \( d \)-maximal edge outgoing from \( B \). Note that \(|W_B| = 2|I_k| \) (see Figure 2).

The following fact is crucial to efficiently compute all these \( d \)-maximal edges via proper indexing data structures:

**Fact 3** If there exists a \( d \)-maximal edge outgoing from \( v_h \) having \( d \)-cost \( c(I_k) \), then this edge can be found by determining a position \( s \in W_h \) whose suffix \( S_s \) shares the maximum longest common prefix (shortly, lcp) with \( S_h \).

**Proof:** Among all positions \( s \) in \( W_h \) take one whose suffix \( S_s \) shares the maximum lcp with \( S_h \), and let \( q \) be the length of this lcp. Of course, there may exist many such positions, we take just one of them. Then the edge \((v_h, v_{h+q+1})\) has \( d \)-cost \( c(I_k) \) and is \( d \)-maximal. In fact any other position \( s' \in W_h \) induces the edge \((v_h, v_{h+q'+1})\), where \( q' < q \) is the length of the lcp shared between \( S_{s'} \) and \( S_h \). This edge cannot be \( d \)-maximal because its \( d \)-cost is still \( c(I_k) \) but its length is shorter. \(\square\)

In the rest of the paper we will call the position \( s \) of Fact 3 maximal position for vertex \( v_h \). The maximal position for \( v_h \) does exist only if \( v_h \) has a \( d \)-maximal edge of cost \( c(I_k) \). Therefore we design an algorithm which computes the maximal positions of every vertex \( v_h \) in \( B \) whenever they do exist, otherwise it will assign an arbitrary position to \( v_h \). The net result will be that we will be generating a supergraph of \( G(S) \) which is still guaranteed to have the size stated in Lemma 2 and can be created efficiently in \( O(|I_k|) \) time and space, as we required above.

Fact 3 has related the computation of maximal positions for the vertices in \( B \) to lcp-computations between suffixes in \( B \) and suffixes in \( W_B \). Therefore it is natural to resort some indexing data structure, like the compact trie \( T_B \), built over the suffixes of \( S \) which start in the range of positions \( B \cup W_B \). The \( T_B \) takes \( O(|B| + |W_B|) = O(|I_k|) \) space, and that is within our required space bounds. We then notice that the maximal position \( s \) for a vertex \( v_h \) in \( B \) having \( d \)-cost \( c(I_k) \) can be computed by finding the leaf of \( T_B \) which is labeled with an index \( s \in W_h \) and has the deepest lowest common ancestor (shortly, lca) with the leaf labeled \( h \). We need to answer this query in \( O(1) \) amortized time per vertex \( v_h \), since we aim at achieving an \( O(|I_k|) \) time complexity over all vertices in \( B \). However, this is tricky. In fact this is not the classic lca-query because we do not know \( s \), which is actually the position we are searching for. Furthermore, since the leaf \( s \) is the closest one to \( h \) in \( T_B \) among the leaves with index in \( W_h \), one could think to use proper predecessor/successor queries on a suitable dynamic set of suffixes in \( W_h \). Unfortunately, this would take \( \omega(1) \) time because of well-known lower bounds [1]. Therefore, answering this query in constant (amortized) time.

**Fig. 2.** The figure shows the interval \( B = [i, j] \) with \( j = i + |I_k| - 1 \), and the window \( W_B \) and its two halves \( W_B' \) and \( W_B'' \).
per vertex requires to devise and deploy proper structural properties of the trie $T_B$ and the problem at hand. This is what we do in our algorithm, whose underlying intuition follows.

Let $u$ be the lca of the leaves $h$ and $s$ in $T_B$. For simplicity, we assume that interval $W_h$ strictly precedes $B$ and that $s$ is the unique maximal position for $v_h$ (our algorithm deals with these cases too, see the proof of Lemma 4). We observe that $h$ must be the smallest index that lies in $B$ and labels a leaf descending from $u$ in $T_B$. In fact assume, by contradiction, that a smaller index $h' < h$ does exist. By definition $h' \in B$ and thus $v_h$ would not have a $d$-maximal edge of $d$-cost $c(I_k)$ because it could copy from the closer $h'$ a possibly longer phrase, instead of copying from the farther set of positions in $W_h$. This observation implies that we have to search only for one maximal position per node $u$ of $T_B$, and this position refers to the vertex $v_{a(u)}$ whose index $a(u)$ is the smallest one that lies in $B$ and labels a leaf descending from $u$. Computing $a$-values clearly takes $O(|T_B|) = O(|I_k|)$ time and space.

Now we need to compute the maximal position for $v_{a(u)}$, for each node $u \in T_B$. We cannot traverse the subtree of $u$ searching for the maximal position of $v_{a(u)}$, because this would take quadratic time complexity. Conversely, we define $W'_B$ and $W''_B$ to be the first and the second half of $W_B$, respectively, and observe that any window $W_h$ has its left extreme in $W'_B$ and its right extreme in $W''_B$. (See Figure 2 for an illustrative example.) Therefore the window $W_{a(u)}$ containing the maximal position $s$ for $v_{a(u)}$ overlaps both $W'_B$ and $W''_B$. So if $s$ does exist, then $s$ belongs to either $W'_B$ or to $W''_B$, and leaf $s$ descends from $u$. Therefore, the maximum (resp. minimum) among the elements in $W'_B$ (resp. $W''_B$) that label leaves descending from $u$ must belong to $W_{a(u)}$. This suggests to compute $\min(u)$ and $\max(u)$ as the rightmost position in $W'_B$ and the leftmost position in $W''_B$ that label leaves descending from $u$, respectively. These values can be computed in $O(|I_k|)$ time by a post-order visit of $T_B$.

We are now ready to compute $mp[h]$ as the maximal position for $v_h$, if it exists, or otherwise set $mp[h]$ arbitrarily. We initially set all $mp$’s entries to nil; then we visit $T_B$ in post-order and perform, at each node $u$, the following checks whenever $mp[a(u)] = \text{nil}$:

- If $\min(u) \in W_{a(u)}$, set $mp[a(u)] = \min(u)$.
- If $\max(u) \in W_{a(u)}$, set $mp[a(u)] = \max(u)$.

At the end of the visit, if $mp[a(u)]$ is still $\text{nil}$ we set $mp[a(u)] = a(\text{parent}(u))$ whenever $a(u) \neq a(\text{parent}(u))$. This last check is needed (proof below) to manage the case in which $S[a(u)]$ can copy the phrase starting at its position from position $a(\text{parent}(u))$ and, additionally, we have that $B$ overlaps $W_B$ (which may occur depending on $f$). Since $T_B$ has size $O(|I_k|)$, the overall algorithm requires $O(|I_k|)$ time and space in the worst case, as required. The following lemma proves its correctness:

**Lemma 4.** For each position $h \in B$, if there exists a $d$-maximal edge outgoing from $v_h$ and having $d$-cost $c(I_k)$, then $mp[h]$ is equal to its maximal position.

**Proof:** Recall that $B = [i, i + |I_k| - 1]$ and consider the longest path $\pi = u_1 u_2 \ldots u_z$ in $T_B$ that starts from the leaf $u_1$ labeled with $h \in B$ and goes upward until the traversed nodes satisfy $a(u_j) = h$, here $j = 1, \ldots, z$. By definition of $a$-value, we know that all leaves descending from $u_z$ and occurring in $B$ are labeled with an index which is larger than $h$. Clearly, $a(\text{parent}(u_z)) < h$ (if any). There are two cases for the final value stored in $mp[h]$.

Suppose that $mp[h] \in W_h$. We want to prove that $mp[h]$ is the index of the leaf which has the deepest lca with $h$ among all the other leaves in $W_h$. Let $u_x \in \pi$ be the node in which the value of $mp[h]$ is assigned (it is $a(u_x) = h$). Assume that there exists at least another index in $W_h$ whose leaf has a deeper lca with leaf $h$. This lca must lie on $u_1 \ldots u_{x-1}$, say
u_i$. Since $W_h$ is an interval having its left extreme in $W'_B$ and its right extreme in $W''_B$, the value $\max(u_i)$ or $\min(u_i)$ must lie in $W_h$ and thus the algorithm has set $mp[h]$ to one of these positions, because of the post-order visit of $T_B$. Therefore $mp[h]$ must be the index of the leaf having the deepest lca with $h$, and thus by Fact 3 is its maximal position (if any).

Now suppose that $mp[h] \notin W_h$ and, thus, it cannot be a maximal position for $v_h$. We have to prove that it does not exist a $d$-maximal edge outgoing from the vertex $v_h$ with cost $c(I_k)$. Let $S_h$ be the suffix in $W_h$ having the maximum lcp with $S_h$, and let $l$ be the lcp-length. Values $\min(u_i)$ and $\max(u_i)$ do not belong to $W_h$, for any node $u_i \in \pi$ (with $a(u_i) = h$), otherwise $mp[h]$ would have been assigned with an index in $W_h$ (contradicting the hypothesis). The value of $mp[h]$ remains nil up to node $u_z$. This implies that no suffix descending from $u_z$ starts in $W_h$ and, in particular, $S_z$ does not descend from $u_z$. Therefore, the lca between leaves $h$ and $s$ is a node in the path from parent($u_z$) to root, and the lcp$(S_{\pi(parent(u_z))}, S_h) \geq lcp(S_z, S_h) = l$. Since $a(parent(u_z)) < a(u_z)$ and belongs to $B$, it is nearer to $h$ than any other position in $W_h$, and shares a longer prefix with $S_h$. So we found longer edge from $v_h$ with smaller $d$-cost. This implies that $v_h$ has no $d$-maximal edge of cost $c(I_k)$ in $G(S)$.

We are left with the problem of building $T_B$ in $O(|I_k|)$ time and space, thus a time complexity which is independent of the length of the indexed suffixes and the alphabet size. In Appendix A we show how to achieve this result by deploying the fact that the above algorithm does not make any assumption on the ordering of $T_B$, because it just computes (sort of) lca-queries on its structure. This is the last step to prove Theorem 2.

6 Conclusions

Our parsing scheme can be extended to variants of LZ77 which deploy parsers that refer to a bounded compression-window (the typical scenario of gzip and its derivatives [17]). In this case, LZ77 selects the next phrase by looking only at the most recent $w$ input symbols. Since $w$ is usually a constant chosen as a power of 2 of few Kbs [17], the running time of our algorithm becomes $O(nQ(g,n))$, since $Q(f,w)$ is a constant. We notice that the remaining term could further be refined by considering the length $\ell$ of the longest repeated substring in $S$, and state the time complexity as $O(nQ(g,\ell))$. If $S$ is generated by an ergodic source [20] and $g$ is taken to be the classic Elias’ code, then $Q(g,\ell) = O(\log \log n)$ so that the complexity of our algorithm results $O(n \log \log n)$ time and $O(n)$ space for this class of strings.

We finally notice that, although we have mainly dealt with the LZ77-dictionary, the techniques presented in this paper could be extended to design efficient bit-optimal compressors for other on-line dictionary construction schemes, like LZ78. Intuitively, we can show that Theorem 1 still holds for any suffix- or prefix-complete dictionary under the hypothesis that the codewords assigned to each suffix or prefix of a dictionary phrase $w$ are shorter than the codewords assigned to $w$ itself. In this case the notion of edge maximality (Definition 1) can be generalized by calling an edge $(v_i, v_j)$ maximal if all longer edges, say $(v_i, v_{j'})$ with $j < j'$, have not larger cost, namely $w_{i,j'} \leq w_{i,j}$. In this case, we can provide an efficient bit-optimal parser for the LZ78-dictionary (details in the full paper).

The main open question is to extend our results to statistical encoding functions like Huffman or Arithmetic coders applied on the integral range $[n]$ [22]. They do not necessarily satisfy Property 1 because it might be the case that $|f(x)| > |f(y)|$, whenever the integer $y$ occurs more frequently than the integer $x$ in the parsing of $S$. We argue that it is not trivial to
design a bit-optimal compressor for these encoding-functions because their codeword lengths change as it changes the set of distances and lengths used in the parsing process.

Practically, we would like to implement our optimal parser motivated by the encouraging experimental results of [8], which have improved the standard LZ77 by a heuristic that tries to optimize the encoding cost of just phrases' lengths.

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APPENDIX A. Building $T_B$ optimally

We are left with the problem of building $T_B$ in $O(|I_k|)$ time and space, thus a time complexity which is independent of the length of the indexed suffixes and the alphabet size. We show how to achieve this result by deploying the crucial fact that the algorithm of Section 5.1 to compute the $d$-maximal edges does not make any assumption on the ordering of edges in $T_B$, because it just computes (sort of) lca-queries on its structure. This is the last step needed to complete the proof of Theorem 2, and we give its algorithmic details here.

At preprocessing time we build the suffix array of the whole string $S$ and a data structure that answers constant-time 1cp-queries between pair of suffixes (see e.g. [15]). These data structures can be built in $O(n)$ time and space, when $\sigma = O(n)$. For larger alphabets, we need to add $T_{sort}(n, \sigma)$ time, which takes into account the cost of sorting the symbols of $S$ and re-mapping them to $[n]$ (see Theorem 2).

Let us first assume that $B$ and $W_B$ are contiguous and form the range $[i, i + 3|I_k| - 1]$. If we had the sorted sequence of suffixes starting in $S[i, i + 3|I_k| - 1]$, we could easily build $T_B$ in $O(|I_k|)$ time and space by deploying the above 1cp-data structure. Unfortunately, it is unclear how to obtain from the suffix array of the whole $S$, the sorted sub-sequence of suffixes starting in the range $[i, i + 3|I_k| - 1]$ by taking $O(|B| + |W_B|) = O(|I_k|)$ time (notice that these suffixes have length $\Theta(n - i)$). We cannot perform a sequence of predecessor/successor queries because they would take $\omega(1)$ time each [1]. Conversely, we resort the key observation above that $T_B$ does not need to be ordered, and thus devise a solution which builds an unordered $T_B$ in $O(|I_k|)$ time and space, passing through the construction of the suffix array of a transformed string. The transformation is simple. We first map the distinct symbols of $S[i, i + 3|I_k| - 1]$ to the first $O(|I_k|)$ integers. This mapping does not need to reflect their lexicographic order, and thus can be computed in $O(|I_k|)$ time by a simple scan of those symbols and the use of a table $M$ of size $\sigma < n$. Then, we define $S^M$ as the string $S$ which has been transformed by re-mapping some of the symbols according to table $M$ (namely, those occurring in $S[i, i + 3|I_k| - 1]$). We can prove that

Lemma 5. Let $S_i, \ldots, S_j$ be a contiguous sequence of suffixes in $S$. The remapped suffixes $S^M_i, \ldots, S^M_j$ can be lexicographically sorted in $O(j - i + 1)$ time.

Proof: Consider the string of pairs $w = \langle S^M[i], b_i \rangle \ldots \langle S^M[j], b_j \rangle \$, where $b_h$ is 1 if $S^M_h > S^M_{h+1}$, -1 if $S^M_h < S^M_{h+1}$, or 0 if $h = j$. The ordering of the pairs is defined component-wise, and we assume that $\$ is a special “pair” larger than any other pair in $w$. For any pair of indices $p, q \in [1 \ldots j - i]$, it is $S^M_{p+i} > S^M_{q+i}$ iff $w_p > w_q$. In fact, suppose that $w_p > w_q$ and set $r = 1cp(w_p, w_q)$. We have that $w[p + r] = \langle S^M[p + i + r], b_{p+i+r} \rangle > \langle S^M[q + i + r], b_{q+i+r} \rangle = w[q + i + r]$. Hence $S^M_{p+i+r} > S^M_{q+i+r}$, by definition of the $b$'s. Therefore $S^M_{p+i} > S^M_{q+i}$, since their first $r$ symbols are equal. This implies that sorting $S^M_i, \ldots, S^M_j$ reduces to computing the suffix array of $w$, and this takes $O(|w|)$ time given that the alphabet size is $O(|w|)$ [15]. Clearly, $w$ can be constructed in that time bound because comparing $S^M_z$ with $S^M_{j+1}$ takes $O(1)$ time via an 1cp-query on $S$ (using the proper data structure above) and a check at their first mismatch. □

Lemma 5 allows us to generate the compact trie of $S^M_i, \ldots, S^M_{i+3|I_k|-1}$, which is equal to the (unordered) compacted trie of $S_i, \ldots, S_{i+3|I_k|-1}$ after replacing every $\#B$ assigned by $M$ with its original symbol in $S$. We finally notice that if $B$ and $W_B$ are not contiguous (as instead we assumed above), we can use a similar strategy to sort separately the suffixes in
$B$ and the suffixes in $W_B$, and then merge these two sequences together by deploying the $1cp$-data structure mentioned at the beginning of this section.