Two-photon exchange amplitudes for the elastic ep scattering at $Q^2 = 2.5$ GeV$^2$
from the experimental data

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We extract two-photon exchange amplitudes for the elastic electron-proton scattering at $Q^2 = 2.5$ GeV$^2$ from the unpolarized cross-section and recent polarization transfer measurements. There are three independent amplitudes, but only one of them, $\mathcal{G}_M$, can be determined with a reasonable accuracy (about 10%). The result is in good agreement with theoretical predictions. Rough estimates for two other amplitudes are obtained.

**Introduction.** In last years, a lot of experimental and theoretical effort was made to study two-photon exchange (TPE) in the elastic electron-proton scattering. This activity was motivated by the discovery of the problem in the proton form factor measurements: values of form factor ratio $G_E/G_M$ obtained by Rosenbluth separation and polarization transfer methods were in strong disagreement. It is now widely accepted that the discrepancy is caused by TPE, but still no direct experimental observations of TPE exist.

Previously, several attempts were made to extract values of TPE amplitudes from available experimental data in more or less model-independent way \cite{1,2}. However, lack of precise data and/or theoretical understanding of TPE prevented from obtaining sufficiently accurate estimates for the amplitudes. In particular, it was difficult to determine the dependence of TPE amplitudes on the kinematical parameter $\varepsilon$, since the $\varepsilon$ dependence of polarization observables was not known experimentally.

Recently, a search for TPE effects in polarization observables was reported \cite{3}. In this experiment, ratio of transverse and longitudinal proton polarization components (polarization ratio) was measured with significantly improved precision for $Q^2 = 2.5$ GeV$^2$ and wide range of the parameter $\varepsilon$.

In the present paper we use latest experimental data to determine TPE amplitudes at $Q^2 = 2.5$ GeV$^2$ following the ideas of Ref. \cite{2} with some improvements (described below). We will try to obtain as much information on TPE amplitudes as possible, while avoiding unnecessary assumptions. In our analysis we only assume that TPE is the sole reason for the discrepancy between cross-section and polarization data, and rely on the following experimental and theoretical facts:

1. The reduced cross-section exhibits no or small non-linearity in $\varepsilon$ \cite{4,5} also verified in the present work).

2. The polarization ratio does not vary significantly with $\varepsilon$ \cite{6}.

3. TPE amplitudes must vanish at $\varepsilon \to 1$ (because they can be represented by convergent dispersion integral; $\varepsilon \to 1$ implies $s \to \infty$, where $s$ is c.m. energy squared).

The latter point was missing in Ref. \cite{2}.

**Analysis of cross-section data.** There were no recent cross-section measurements in the $Q^2$ region of our interest, so we have to use older data. These data were analyzed before, but for our paper to be self-contained we repeat such an analysis here. We have selected data in the range $2.2$ GeV$^2 < Q^2 < 2.8$ GeV$^2$ \cite{5}. The corresponding reduced cross-sections were first multiplied by $(1 + Q^2/0.71$GeV$^2)^3$, to eliminate most of the $Q^2$ dependence. Then, since in Born approximation the reduced cross-section is

$$\sigma_R = \tau G_M^2 + \varepsilon G_E^2$$

where $G_E$ and $G_M$ are electric and magnetic form factors, $\tau = Q^2/4M^2$ and $M$ is proton mass, the resulting values were fitted with the function

$$A + B\varepsilon + C(Q^2 - 2.5\text{ GeV}^2)$$

The last term takes into account $Q^2$ dependence of $\tau G_M^2$ term in $\sigma_R$. We obtain rather acceptable fit with $\chi^2 = 39$ for $28$ d.o.f., indicating that the linearity of $\sigma_R$ in $\varepsilon$ is indeed supported by the data. We will mainly need the quantity

$$R_{LT}^2 = \tau B/A$$

which would be equal to $(G_E/G_M)^2$ at $Q^2 = 2.5$ GeV$^2$ in Born approximation. We obtain $R_{LT}^2 = 0.1020 \pm 0.0057$, in agreement with the results Ref. \cite{3} (0.1015). In further calculations we use the first value.

**Extraction of TPE amplitudes.** We will use mostly the same notation as in Ref. \cite{2}. We denote particle momenta according to

$$e(k) + p(p) \to e(k') + p(p')$$

and define $q = p' - p$, $P = (p + p')/2$, $K = (k + k')/2$, $Q^2 = -q^2$. In presence of TPE, elastic electron-proton scattering amplitude has the form

$$\mathcal{M} = \frac{4\pi\alpha}{Q^2} \bar{u} \gamma_{\mu} u \bar{U'} \left( \hat{F}_1 \gamma_{\mu} - \hat{F}_2 [\gamma_{\mu}, \gamma_{\nu}] \frac{q_{\nu}}{4M} + \hat{F}_3 K_{\nu} \gamma_{\nu} \frac{p_{\mu}}{M^2} \right) U$$

where $\alpha$ is fine structure constant, $u, u'$ ($U, U'$) are initial and final electron (proton) spinors, and $\hat{F}_i$ are scalar amplitudes.
invariant amplitudes. It is convenient to introduce linear combinations \cite{2}

\[
\begin{align*}
G_E &= \tilde{F}_1 - \tau \tilde{F}_2 + \nu \tilde{F}_3/4M^2 = G_E + \delta G_E \\
G_M &= \tilde{F}_1 + \tilde{F}_2 + \nu \tilde{F}_3/4M^2 = G_M + \delta G_M \\
\delta G_3 &= \nu \tilde{F}_3/4M^2 = \delta G_3
\end{align*}
\]

where \( \nu = 4PK \) and prefix \( \delta \) indicates TPE contribution. The TPE amplitudes \( \delta G_i \) are complex, but only their real parts contribute to the observables discussed here. Everywhere below, speaking of the amplitudes, we will mean their real parts. Neglecting terms of order \( \alpha^2 \), the reduced cross-section and polarization ratio can be written as

\[
\begin{align*}
\sigma_R &= G_M^2 \left\{ \tau + \varepsilon R_0 G_E + 2\varepsilon R_0 \frac{\delta G_E}{G_E} \right\} \\
R &= R_0 \left\{ 1 + \varepsilon \frac{\delta G_E}{G_E} - \frac{\delta G_M}{G_M} - \varepsilon (1 - \varepsilon) \frac{\delta G_3}{1 + \varepsilon G_M} \right\}
\end{align*}
\]

where \( R_0 = G_E/G_M \). Note that our definition of \( R \) does not include a factor of \( \mu \approx 2.793 \), thus \( R \) (and \( R_0 \)) is rather small quantity (\( \approx 0.25 \) for \( Q^2 = 2.5 \text{ GeV}^2 \)). Utilizing this fact we will neglect last term in Eq.\( (7) \) (the validity of this approximation will be checked afterwards). Then, as it was argued in Ref. \( [2] \), the observed cross-section linearity in \( \varepsilon \) forces us to parameterize TPE amplitude \( \delta G_M \) as a linear function of \( \varepsilon \). To vanish at \( \varepsilon \to 1 \), it must have the form

\[
\delta G_M/G_M = a(1 - \varepsilon)
\]

Then we have

\[
\sigma_R = G_M^2 \left\{ \tau + \varepsilon R_0 + 2\varepsilon (1 - \varepsilon) \right\}
\]

and the cross-section slope is

\[
R_L^2 = \frac{R_0^2 - 2\varepsilon a}{\tau(1 + 2\varepsilon)}
\]

from which we obtain

\[
a = \frac{R_0^2 - R_L^2}{2(\tau + R_L)}
\]

Together with Eq.\( (7) \), this fully determines the amplitude \( \delta G_M \). As a first approximation, we replace \( R_0^2 \) by experimental value of polarization ratio, \( R = 0.6923 \pm 0.0058 \) \cite{3}, and obtain numerically

\[
a = -0.0250 \pm 0.0035
\]

Thus extracted amplitude \( \delta G_M/G_M \) is shown in Fig.\( 1 \) with 1\sigma error band. The theoretical prediction \cite{7,8} is also shown and agrees rather well with our result.

Now we will take a closer look on the polarization ratio \( R \), which allows us to get some information about the amplitude \( \delta G_E \). First, we note that the last term in Eq.\( (7) \) should be very small, because the factor \( \varepsilon (1 - \varepsilon)/(1 + \varepsilon) \) is not greater than 0.18 for \( 0 < \varepsilon < 1 \). Thus we are left with

\[
R = R_0 \left\{ 1 + \varepsilon \frac{\delta G_E}{G_E} - \frac{\delta G_M}{G_M} \right\}
\]

Since both \( \delta G_M \) and \( \delta G_E \) vanish at \( \varepsilon \to 1 \), we obviously have

\[
R_0 = R_{\varepsilon = 1}, \quad \delta G_E - R_0 \delta G_M = R - R_0
\]

The experiment says that, at \( Q^2 = 2.5 \text{ GeV}^2 \), there is no significant variation of \( R \) with \( \varepsilon \) \cite{3}. This implies

\[
R \approx R_0 \quad \text{and} \quad \delta G_E \approx R_0 \delta G_M
\]

(which also justifies using \( R \) instead of \( R_0 \) in Eq.\( (7) \)). Now we can cross-check that the amplitude \( \delta G_E \) has small impact on the cross-section. Using Eqs.\( (9,12,16) \), we calculate the corresponding correction (the last term of Eq.\( (7) \)) and subtract it from the data. Then we repeat cross-section fitting and extraction of \( \delta G_M \), as described above. We obtain \( a = -0.0248 \), i.e. practically no change with respect to \( (13) \).

Eqs.\( (10) \) are, of course, a rough estimate. In particular, they do not allow to estimate the uncertainty of \( \delta G_E \). An accurate \( G_E \) extraction with estimation of uncertainties requires determination of the small quantity \( \delta R = R - R_0 \), for which the present data hardly suffice. Nevertheless, note that theoretical calculations also show relative smallness of \( \delta R \), which arises from significant cancellation between proton and \( \Delta \) resonance contributions (Fig.\( 2 \)).

In the experiment \cite{2}, one more quantity was measured: longitudinal polarization of the final proton, \( P_l \). In principle, this data could help to determine the remaining amplitude \( \delta G_3 \).

The TPE correction to \( P_l \) is given by

\[
\delta P_l = -2\varepsilon P_l \left\{ \frac{R_0^2 \delta R}{\varepsilon R_0^2 + \tau} + \frac{\varepsilon}{1 + \varepsilon} \frac{\delta G_3}{G_M} \right\}
\]
where $\delta R = R - R_0$ is TPE correction to polarization ratio according to Eq. (14). As $R_0^2 \ll 1$, the first term is negligible, and the deviation of $P_l$ from its Born value is governed by the amplitude $\delta G_3$. However, two available data points are too few to make any statements about $\varepsilon$ dependence of $\delta G_3$. We can only compute $\delta G_3$ at the $\varepsilon$ values of experimental data. The results are shown in Fig. 3. Obviously, their precision is insufficient to make a meaningful comparison with theory. The obtained values are compatible with zero, and do not contradict theoretical estimates as well.

**Conclusions.** In summary, we have tried to extract TPE amplitudes for the elastic electron-proton scattering at $Q^2 = 2.5$ GeV$^2$ solely from the experimental data on cross-sections and polarization observables. Having defined three independent amplitudes $\delta G_M$, $\delta G_E$ and $\delta G_3$, we found that the effect of these amplitudes on the observables is “decoupled”: the cross-section is mainly influenced by $\delta G_M$, the polarization ratio — by $\delta G_E$ and $\delta G_M$, longitudinal polarization component — by $\delta G_3$. The amplitude $\delta G_M$ can be extracted with approximately 10% accuracy and is in good agreement with theoretical calculations. The weakness of polarization ratio variation with $\varepsilon$ implies approximate equality $\delta G_E/G_E \approx \delta G_M/G_M$. As to the amplitude $\delta G_3$, present experimental data are consistent with $\delta G_3 = 0$. To allow for more accurate extraction of $\delta G_E$ and $\delta G_3$, further polarization measurements at different $\varepsilon$ are clearly needed. It is also worth noting that momentum transfer value $Q^2 = 2.5$ GeV$^2$ was not good choice for the experiment [3]; just in this region TPE correction to polarization ratio is especially small because elastic and $\Delta$ resonance contributions almost cancel each other.

Recently, a preprint [9] appeared, in which the same problem was considered. However, authors have used certain parameterization of polarization component $P_l$, which, to our opinion, is not well motivated by experimental data. Their results strongly differ from ours as well as from theoretical calculations.