The mapping class group and the Meyer function for plane curves

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Abstract

For each \( d \geq 2 \), the mapping class group for plane curves of degree \( d \) will be defined and it is proved that there exists uniquely the Meyer function on this group. In the case of \( d = 4 \), using our Meyer function, we can define the local signature for 4-dimensional fiber spaces whose general fibers are non-hyperelliptic compact Riemann surfaces of genus 3. Some computations of our local signature will be given.

Introduction.

Let \( \Sigma_g \) be a closed oriented \( C^\infty \)-surface of genus \( g \geq 0 \) and let \( \Gamma_g \) be the mapping class group of \( \Sigma_g \), namely the group of all isotopy classes of orientation preserving diffeomorphisms of \( \Sigma_g \).

In [12] W. Meyer discovered and studied a cocycle \( \tau_g: \Gamma_g \times \Gamma_g \to \mathbb{Z} \). For the sake of the reader a brief definition of \( \tau_g \) will be given in Appendix. This cocycle is called Meyer’s signature cocycle. In his paper W. Meyer showed that the cohomology class \([\tau_g] \in H^2(\Gamma_g; \mathbb{Z})\) is torsion for \( g = 1, 2 \) and has infinite order for \( g \geq 3 \), and gave an explicit formula for the unique \( \mathbb{Q} \)-valued 1-cochain of \( \Gamma_1 \) cobounding \( \tau_1 \) using the Rademacher function ([12] p.259 Satz 4). Since the hyperelliptic mapping class group \( \Gamma^H_g \), a subgroup of \( \Gamma_g \), was shown to be \( \mathbb{Q} \)-acyclic by F. Cohen[5] and N. Kawazumi[9] independently, it was known to specialists that there exists the unique 1-cochain of \( \Gamma^H_g \) cobounding \( \tau_g \) restricted to \( \Gamma^H_g \). In [7] H. Endo directly showed the existence and the uniqueness of such a 1-cochain \( \phi^H_g: \Gamma^H_g \to \mathbb{Z}_{2g+1} \) using a finite presentation of \( \Gamma^H_g \) by J. Birman-H. Hilden[3]. He also defined the local signature for hyperelliptic fibrations using \( \phi^H_g \), and studied the geometry of hyperelliptic fibrations; for example, he derived a signature formula for such fibrations over a closed surface. His formula originates from Y. Matsumoto[11] Theorem 3.3 where genus 2 fibrations are discussed. For the study of the function \( \phi^H_g \), see also T. Morifuji’s paper[13].

The purpose of the present paper is to give another interesting example of these phenomena; the Meyer function on the mapping class group for plane curves.

For \( d \geq 2 \) a group \( \Pi(d) \) and a homomorphism \( \rho: \Pi(d) \to \Gamma_g \),where \( g = \frac{1}{2}(d - 1)(d - 2) \), will be constructed. The group \( \Pi(d) \) can be considered as the fundamental group of the classifying space for isotopy classes of continuous families of non-singular plane curves of degree \( d \); the precise meaning of this statement will be given in Theorem [6.1] later.

The main results of this paper are Theorem 4.1 and Theorem 4.2. As a consequence of them it follows that the pull back \( \rho^*[\tau_g] \) vanishes in the rational cohomology \( H^2(\Pi(d); \mathbb{Q}) \) and there exists the unique 1-cochain \( \phi^d: \Pi(d) \to \mathbb{Q} \) such that \( \delta \phi^d = \rho^* \tau_g \). \( \phi^d \) will be called the Meyer function for plane curves of degree \( d \).
This is similar to the case of $\Gamma_1, \Gamma_2$, and $\Gamma^H_g$, but we remark that the homomorphism $\rho$ seems no more injective nor surjective. In fact, for $d = 4$ we will see in Proposition 6.3 that $\rho$ is surjective but has non-trivial kernels. In this sense our result is different from the works of W. Meyer and H. Endo where subgroups of $\Gamma_g$ are considered.

While they did explicit computations of $\tau_g$ for certain relators of the mapping class groups to prove the vanishing of $[\tau_g]$, our method depends on the vanishing of $[\tau_g]$ pulled back to the cohomology of a fundamental group of the complement of a hypersurface in a complex vector space, which will be stated in Proposition 3.1 and proved using the definition of Meyer’s signature cocycle and the standard argument in differential topology; the way from Proposition 3.1 to the vanishing of $[\tau_g]$ pulled back to $H^2(\Pi(d); \mathbb{Q})$ are elementary. Since this needs no explicit computations of $\tau_g$, we believe that our method has its own meaning to grasp the conceptual reason of the vanishing of $[\tau_g]$ and can be applied to other cases in the future.

Our study of the vanishing of $\rho^*[\tau_g] \in H^2(\Pi(d); \mathbb{Q})$ has a connection with localization of the signature of 4-dimensional fiber spaces, that is a recent hot topic studied in various fields such as topology, algebraic geometry, and complex analysis (see T. Ashikaga-H. Endo[1] and T. Ashikaga-K. Konno[2]).

As an application of our study, especially $d = 4$, we define the local signature for the set of all fiber germs of 4-dimensional fiber spaces whose general fibers are non-hyperelliptic compact Riemann surfaces of genus 3 by using our 1-cochain $\phi^4$ of $\Pi(4)$. The fact that any non-hyperelliptic compact Riemann surface of genus 3 can be realized as a smooth quartic curve in $\mathbb{P}^2$ by the canonical embedding, is crucial.

In this case of non-hyperelliptic family of genus 3, T. Ashikaga-K. Konno [2] and K. Yoshikawa [15] have already defined local signature independently. The definition of [2] is algebro geometric and that of [15] is complex analytic. We compute some examples of values of our local signature, defined by topological way, and observe that they coincide with those computed in [2] and [15].

1 Definitions

Throughout this paper, $d$ denotes a fixed integer $\geq 2$. Let $V^d$ be the complex vector space of homogeneous polynomials of degree $d$ in the determinates $x, y, z$, and $z$, and let $\mathbb{P}(d) = \mathbb{P}(V^d)$ be the projectivization of $V^d$. By taking the set of monomials $\{x^{(k)}y^{m(k)}z^{n(k)}\}_{k=0}^N$ of degree $d$, where $N = \frac{1}{2}(d+2)(d+1) - 1$, each element of $V^d$ can be uniquely written as the form

$$\Phi = \sum_{k=0}^N a_k x^{(k)}y^{m(k)}z^{n(k)},$$

where $a_k \in \mathbb{C}$. We denote the corresponding homogeneous coordinates of $\mathbb{P}(d)$ by $[a_0 : a_1 : \cdots : a_N]$. Each element $a \in \mathbb{P}(d)$ determines an algebraic curve $C_a \subset \mathbb{P}^2$ of degree $d$. Later we also denote by $C_F$ the algebraic curve defined by $F \in V^d \setminus \{0\}$. We believe this use of notation does not confuse the reader. Let $D$ be the set of points $a \in \mathbb{P}(d)$ such that the corresponding curve $C_a$ is singular. $D$ is called the discriminant locus and is well-known to be irreducible and of codimension 1. For a proof, see also the remark after the proof of Proposition 2.1 in this paper.

There is an action of $GL(3; \mathbb{C})$ on $V^d$ given by

$$(A \cdot F)(x, y, z) = F((x, y, z) \cdot ^tA^{-1}),$$

where $A \in GL(3; \mathbb{C})$. This action intertwines with the algebraic action of $GL(3; \mathbb{C})$ on $\mathbb{P}(d)$. For $A \in GL(3; \mathbb{C})$, we define $A \cdot a = [a_0 : a_1 : \cdots : a_N]$. When $A$ is a member of $SL(3; \mathbb{C})$, $A \cdot a$ is a homogenous coordinate of a point in $\mathbb{P}(d)$. When $A$ is not in $SL(3; \mathbb{C})$, $A \cdot a$ is not a homogenous coordinate of a point in $\mathbb{P}(d)$.
where \( A \in GL(3; \mathbb{C}) \) and \( F \in V^d \). Here \(^t A\) is the transpose of the matrix \( A \). This action induces the action of \( PGL(3) \) on \( \mathbb{P}(d) \), \( D \), and \( \mathbb{P}(d) \setminus D \).

Let \( EPGL(3) \to BPGL(3) \) be the universal principal \( PGL(3) \) bundle. We denote by \( \Pi(d) \) the fundamental group of the Borel construction \( (\mathbb{P}(d) \setminus D)_{PGL(3)} = EPGL(3) \times_{PGL(3)} (\mathbb{P}(d) \setminus D) \) and call this group the mapping class group for plane curves of degree \( d \).

For \((e, a) \in EPGL(3) \times (\mathbb{P}(d) \setminus D)\), we denote by \([e, a]\) the element of \((\mathbb{P}(d) \setminus D)_{PGL(3)}\) represented by \((e, a)\). This notation concerning Borel construction will be used several times.

Let \( \mathcal{F} \) (resp. \( \mathcal{F} \)) be the hypersurface in \( \mathbb{P}(d) \times \mathbb{P}^2 \) (resp. \( (\mathbb{P}(d) \setminus D) \times \mathbb{P}^2) \) defined as the zero set of \( \Phi \) considered as a bi-homogeneous polynomial in \( a_0, \ldots, a_N \) and \( x, y, z \). Then the restriction of the first projection \( p: \mathcal{F} \to \mathbb{P}(d) \setminus D \) is a family of non-singular plane curves of degree \( d \) whose fiber over \( a \in \mathbb{P}(d) \setminus D \) is \( C_a \). Since the diagonal action of \( PGL(3) \) on \( \mathbb{P}(d) \times \mathbb{P}^2 \) preserves \( \mathcal{F} \) and \( p \) is \( PGL(3) \)-equivariant, we have a family of Riemann surfaces \( p_a: \mathcal{F}_{PGL(3)} \to (\mathbb{P}(d) \setminus D)_{PGL(3)} \). We denote the topological monodromy (see Appendix) of this family by \( \rho: \Pi(d) \to \Gamma_g \), where \( g = \frac{1}{2}(d - 1)(d - 2) \). Note that the genus of a non-singular plane curve of degree \( d \) is given by \( \frac{1}{2}(d - 1)(d - 2) \).

In section 4 we will prove that the rational cohomology class \( \rho^*[\tau_g] \in H^2(\Pi(d); \mathbb{Q}) \) vanishes and compute the abelianization of \( \Pi(d) \). In section 6 we will prove that the space \((\mathbb{P}(d) \setminus D)_{PGL(3)}\) is the classifying space of the set of all isotopy classes of continuous families of non-singular plane curves of degree \( d \).

### 2 The discriminant locus

In this section we investigate the discriminant locus \( D \), which also can be described in terms of dual variety as follows. For generality of dual variety, see [8] or [10]. Let \( \mathbb{P}(d)^\vee \) be the dual projective space of \( \mathbb{P}(d) \), i.e., the space of all hyperplanes of \( \mathbb{P}(d) \). We denote by \([\alpha^0: \alpha^1: \cdots: \alpha^N]\) the homogeneous coordinates of \( \mathbb{P}(d)^\vee \) corresponding to the homogeneous coordinates \([a_0: a_1: \cdots: a_N]\) of \( \mathbb{P}(d) \); \( \alpha = [\alpha^0: \alpha^1: \cdots: \alpha^N] \) is the hypersurface of \( \mathbb{P}(d) \) defined by

\[
\alpha^0a_0 + \alpha^1a_1 + \cdots + \alpha^N a_N = 0.
\]

The Veronese embedding \( v: \mathbb{P}^2 \to \mathbb{P}(d)^\vee \) is defined by

\[
v([x: y: z]) = [x^{\ell(0)}y^{m(0)}z^{n(0)}; \cdots; x^{\ell(N)}y^{m(N)}z^{n(N)}].
\]

Since the dual of \( \mathbb{P}(d)^\vee \) is canonically isomorphic to \( \mathbb{P}(d) \), each element \( a \in \mathbb{P}(d) \) determines the hypersurface of \( \mathbb{P}(d)^\vee \) which we denote by \( H_a \). We set

\[
\mathcal{X}' := \{(a, \alpha) \in \mathbb{P}(d) \times \mathbb{P}(d)^\vee ; \alpha \in v(\mathbb{P}^2) \text{ and } H_a \text{ is tangent to } v(\mathbb{P}^2) \text{ at } \alpha \}.
\]

Then the image of \( \mathcal{X}' \) by the first projection is just \( D \), i.e., \( D \) is the dual variety of \( v(\mathbb{P}^2) \).

Let \( \mathcal{X} \) be the analytic subset of \( \mathbb{P}(d) \times \mathbb{P}^2 \) defined by the equations

\[
\Phi = \Phi_x = \Phi_y = \Phi_z = 0,
\]

where \( \Phi_x \) is the partial derivative of \( \Phi \) with respect to \( x \), etc. Thus if \((a, p)\) is a point of \( \mathcal{X} \), then \( a \) is a point of \( D \) and \( p \) is a singular point of \( C_a \). Then we see that \( \mathcal{X} \to \mathcal{X}' \), \((a, p) \mapsto (a, v(p))\) is an isomorphism. \( \mathcal{X}' \) has the structure of fiber bundle over \( v(\mathbb{P}^2) \) whose fiber
over $\alpha \in v(\mathbb{P}^2)$ is the set of all hyperplanes in $\mathcal{X}'$ tangent to $v(\mathbb{P}^2)$ at $\alpha$, which is isomorphic to a $(N - 3)$-dimensional projective space. From this point of view it is clear that $\mathcal{X}$ is non-singular (see also [8, p.30], but for later consideration we give here an alternative proof using coordinate description.

**Proposition 2.1.** $\mathcal{X}$ is non-singular.

**Proof.** Let $(a^0, [x_0: y_0: z_0])$ be a point of $\mathcal{X}$. We will show $\mathcal{X}$ is non-singular at this point. Since the action of $\text{PGL}(3)$ on $\mathbb{P}(d) \times \mathbb{P}^2$ preserves $\mathcal{X}$, we may assume that $[x_0: y_0: z_0] = [0: 0: 1]$. Take a polynomial representative $F \in V^d$ of $a^0$, then the coefficient of $z^d$ of $F$ is zero because $[0: 0: 1] \in C_{a^0}$. Moreover, $F$ cannot be written as the form

$$F = (ax + by)^{d-1},$$

where $(a, b) \neq (0, 0)$ because $[0: 0: 1]$ is a singular point of $C_{a^0}$. Therefore there is a monomial $x^k y^m z^n$ which is different from $z^d$, $x z^{d-1}$, and $y z^{d-1}$ such that the coefficient of $x^k y^m z^n$ of $F$ is not zero. By a rearrangement of indices we may assume that $k = 0$ and $a_1$, $a_2$, and $a_3$ correspond to monomials $z^d$, $x z^{d-1}$, and $y z^{d-1}$, respectively. Then setting $a_0 = 1$ and $z = 1$, we have an inhomogeneous coordinates $(a_1, \ldots, a_N, x, y)$ of $\mathbb{P}(d) \times \mathbb{P}^2$ near $(a^0, [0: 0: 1])$. In this local coordinate system $\mathcal{X}$ is defined by the equations

$$\Psi = \Psi_x = \Psi_y = 0,$$

where $\Psi = \Phi(1, a_1, \ldots, a_N, x, y, 1)$. Now the Jacobian matrix of $(\Psi, \Psi_x, \Psi_y)$ at $(a^0, [0: 0: 1])$ is

$$J = \begin{pmatrix}
\Psi_{a_1} & \Psi_{a_2} & \Psi_{a_3} & \cdots & \Psi_x & \Psi_y \\
\Psi_{x,a_1} & \Psi_{x,a_2} & \Psi_{x,a_3} & \cdots & \Psi_{xx} & \Psi_{xy} \\
\Psi_{y,a_1} & \Psi_{y,a_2} & \Psi_{y,a_3} & \cdots & \Psi_{yx} & \Psi_{yy}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & \Psi_{xx} & \Psi_{xy} \\
0 & 0 & 1 & \cdots & 0 & \Psi_{yx} & \Psi_{yy}
\end{pmatrix},$$

we see that the rank of $J$ is 3. This shows that $\mathcal{X}$ is non-singular at $(a^0, [0: 0: 1])$. □

Let $\pi: \mathcal{X} \rightarrow D \subset \mathbb{P}(d)$ be the first projection. The above proof shows that $(a^0, [0: 0: 1])$ is a regular point of $\pi$ if and only if

$$\det \begin{pmatrix}
\Psi_{xx} & \Psi_{xy} \\
\Psi_{yx} & \Psi_{yy}
\end{pmatrix} \neq 0$$

at $(a^0, [0: 0: 1])$. By an argument like the Morse lemma, we can take a coordinate system $(X, Y)$ of $\mathbb{P}^2$ centered at $[0: 0: 1]$ such that $C_{a^0}$ is locally given by the equation $X^2 + Y^2 = 0$. Thus $[0: 0: 1]$ is a nodal singularity. This holds for other points of $\mathcal{X}$; $(a, p) \in \mathcal{X}$ is a regular point of $\pi$ if and only if $p$ is a nodal singularity of $C_a$.

Let $E$ be the union of singular points of $D$ and the $\pi$-image of critical points of $\pi$. $E$ is a proper analytic subset of $D$ by Sard’s theorem.

Here we give a short proof that $D$ is irreducible and of codimension 1. At first, $\mathcal{X} \cong \mathcal{X}'$ is non-singular and connected hence irreducible. Therefore, $D = \pi(\mathcal{X})$ is also irreducible. On the other hand, $D$ is at most $N - 1$ dimensional because $D$ is a proper analytic subset
of \( \mathbb{P}(d) \). Let \( a \) be a point of \( D \setminus E \) and take a point \((a, p)\) in the fiber \( \pi^{-1}(a) \). Then \( D \) is smooth around \( a \) and the differential of \( \pi \) at \((a, p)\) is of maximal rank \( N - 1 \). This shows \( D \) is indeed \( N - 1 \) dimensional. Note that \( E \) is at most \( N - 2 \) dimensional.

In the next lemma we shall describe the hyperplane of \( \mathbb{P}(d) \) tangent to \( D \) at a point in \( D \setminus E \).

**Lemma 2.2.** Let \((a^0, [x_0 : y_0 : z_0])\) be a point of \( \mathcal{X} \) and suppose that \( a^0 \in D \setminus E \). Then the hyperplane \( T_{a^0} \) tangent to \( D \) at \( a^0 \) is given by

\[
T_{a^0} = \left\{ \left[ \xi_0 : \xi_1 : \cdots : \xi_N \right] \in \mathbb{P}(d) : \sum_{k=0}^{N} \xi_k x_0^{\ell(k)} y_0^{m(k)} z_0^{n(k)} = 0 \right\}.
\]

Moreover, \([x_0 : y_0 : z_0]\) is the unique singular point of \( C_{a^0} \).

**Proof.** To prove the first part, we may assume \( a^0 = z_0 = 1 \) and take an inhomogeneous coordinate system \((a_1, \ldots, a_N, x, y)\) of \( \mathbb{P}(d) \times \mathbb{P}^2 \) near \((a^0, [x_0 : y_0 : 1])\). Since \( a^0 \) is a non-singular point of \( D \) and \((a^0, [x_0 : y_0 : 1])\) is a regular point of \( \pi \), we have \( T_{a^0} D = \tilde{\pi}_* \left( T_{(a^0, [x_0 : y_0 : 1])} \mathcal{X} \right) \), where \( \tilde{\pi} : T_{(a^0, [x_0 : y_0 : 1])} \mathbb{P}(d) \times \mathbb{P}^2 \to T_{a^0} \mathbb{P}(d) \) is the differential of the first projection \( \tilde{\pi} : \mathbb{P}(d) \times \mathbb{P}^2 \to \mathbb{P}(d) \) and we regard \( T_{(a^0, [x_0 : y_0 : 1])} \mathcal{X} \) (resp. \( T_{a^0} D \)) as the subspace of \( T_{(a^0, [x_0 : y_0 : 1])} \mathbb{P}(d) \times \mathbb{P}^2 \) (resp. \( T_{a^0} \mathbb{P}(d) \)).

Now the Jacobian matrix \( J \) appeared in the proof of Proposition 2.1 has the form

\[
J = \begin{pmatrix}
\ell(1) & m(1) & \cdots & \ell(N) & m(N) & 0 & 0 \\
* & \cdots & * & \Psi_{xx} & \Psi_{xy} \\
* & \cdots & * & \Psi_{yx} & \Psi_{yy}
\end{pmatrix}
\]

at \((a^0, [x_0 : y_0 : 1])\). The rank of this matrix is \( 3 \), because \( \det \begin{pmatrix} \Psi_{xx} & \Psi_{xy} \\ \Psi_{yx} & \Psi_{yy} \end{pmatrix} \neq 0 \) at \((a^0, [x_0 : y_0 : 1])\) by \( a^0 \notin E \) and there is an index \( i \) such that \( x_0^{\ell(i)} y_0^{m(i)} \neq 0 \). Therefore

\[
T_{(a^0, [x_0 : y_0 : 1])} \mathcal{X} = \left\{ \sum_{k=1}^{N} \xi_k \frac{\partial}{\partial a_k} + \xi_{N+1} \frac{\partial}{\partial x} + \xi_{N+2} \frac{\partial}{\partial y} : J \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{N+2} \end{pmatrix} = 0 \right\}
\]

and

\[
T_{a^0} D = \tilde{\pi}_* \left( T_{(a^0, [x_0 : y_0 : 1])} \mathcal{X} \right) = \left\{ \sum_{k=1}^{N} \xi_k \frac{\partial}{\partial a_k} : \sum_{k=1}^{N} \xi_k x_0^{\ell(k)} y_0^{m(k)} = 0 \right\}.
\]

Interpreting this equation in terms of homogeneous coordinates of \( \mathbb{P}(d) \), we obtain the desired description of \( T_{a^0} \). The latter statement of the lemma follows from the form of \( T_{a^0} \) just proved and the injectivity of the Veronese embedding. \( \square \)

Combining the remark after the proof of Proposition 2.1 we can say more about the curve \( C_{a^0} \):

**Lemma 2.3.** Let \( a^0 \in D \setminus E \) and \([x_0 : y_0 : z_0]\) be as in Lemma 2.2. Then \([x_0 : y_0 : z_0]\) is a nodal singularity of \( C_{a^0} \), and \( C_{a^0} \) is irreducible except for \( d = 2 \). Thus if \( d \geq 3 \) the topological type of \( C_{a^0} \) is Lefschetz singular fiber of type I, that is obtained by pinching a non-separating simple closed curve on \( \Sigma_g \) into a point.
Proof. We only have to show the irreducibility of \(C_\alpha\) for \(d \geq 3\). If \(C_\alpha\) is reducible it has two irreducible components \(C_1\) and \(C_2\) with degrees \(d_1\) and \(d_2\), and they intersect transversely at one point. We have \(d_1d_2 = 1\) by Bézout’s theorem, but this contradicts to \(d_1 + d_2 = d \geq 3\).

The projective space \(\mathbb{P}(d)\) can be regarded as the set of all complex lines through the origin in \(V^d\). Let \(\widetilde{D}\) (resp. \(\widetilde{E}\)) be the union of all lines in \(D\) (resp. \(E\)). In the coordinate system \((a_0, \ldots, a_N)\) of \(V^d\), the tangent space of \(\widetilde{D}\) at \(F \in \widetilde{D} \setminus \widetilde{E}\) is given by

\[
T_F\widetilde{D} = \left\{ \sum_{k=0}^{N} \xi_k \frac{\partial}{\partial a_k} ; \sum_{k=0}^{N} \xi_k x_0^{\ell(k)} y_0^{m(k)} z_0^{n(k)} = 0 \right\},
\]

where \([x_0 : y_0 : z_0]\) is the singular point of \(C_F\). This follows from Lemma 2.2.

We shall prove a useful lemma which will be used in the next two sections. Let \(\widetilde{F}\) be the family of algebraic curves over \(V^d \setminus \{0\}\) defined as in the case of \(\tilde{F}\) over \(\mathbb{P}(d)\).

**Lemma 2.4.** Let \(B\) be a \(C^\infty\)-manifold of dimension \(s \geq 2\) and \(j : B \to V^d\) a \(C^\infty\)-map such that \(j(B) \subset V^d \setminus \widetilde{E}\) and \(j\) is transverse to \(\widetilde{D}\). Then the total space \(j^*\widetilde{F}\) of the pull back of the family \(\widetilde{F}\) by \(j\) is a \(C^\infty\)-manifold.

**Proof.** \(j^*\widetilde{F}\) is given by

\[
j^*\widetilde{F} = \{(b, p) \in B \times \mathbb{P}^2 ; \Phi(j(b), p) = 0\}
\]

and it is easy to see that if \((b^0, p_0) \in j^*\widetilde{F}\) and \(p_0\) is a smooth point of \(C_{j(b^0)}\) then \(j^*\widetilde{F}\) is smooth at \((b^0, p_0)\).

Suppose \((b^0, p_0) \in j^*\widetilde{F}\) and \(p_0 = [x_0 : y_0 : z_0]\) is the singular point of \(C_{j(b^0)}\). Note that we have \(j(b^0) \in \widetilde{D}\). Let \((j_0, j_1, \ldots, j_N)\) denote the \(N+1\)-tuples of smooth functions on \(B\) determined by \(j\) and the coordinate system \((a_0, a_1, \ldots, a_N)\) of \(V^d\). By the assumption of transversality and the description of \(T_{j_0(b)}\widetilde{D}\) given above, we can choose a suitable local coordinate system \((b_1, \ldots, b_s)\) of \(B\) around \(b_0\) such that complex numbers

\[
\sum_{k=0}^{N} \frac{\partial j_k(b^0)}{\partial b_1} x_0^{\ell(k)} y_0^{m(k)} z_0^{n(k)} \quad \text{and} \quad \sum_{k=0}^{N} \frac{\partial j_k(b^0)}{\partial b_2} x_0^{\ell(k)} y_0^{m(k)} z_0^{n(k)}
\]

are linearly independent over the real numbers. From this we can conclude that \(j^*\widetilde{F}\) is smooth at \((b^0, p_0)\). This completes the proof. \(\square\)

We remark that in holomorphic category one can say more; if \(B\) is a complex manifold of complex dimension \(\geq 1\) and \(j\) is holomorphic, \(j^*\widetilde{F}\) has a complex structure as a hypersurface in \(B \times \mathbb{P}^2\).

### 3 The 1-cochain of \(\pi_1(V^d \setminus \widetilde{D})\)

Let \(\chi_1 : \pi_1(V^d \setminus \widetilde{D}) \to \pi_1(\mathbb{P}(d) \setminus D)\) be the homomorphism induced by the projection map \(V^d \setminus \widetilde{D} \to \mathbb{P}(d) \setminus D\) and let \(\chi_2 : \pi_1(\mathbb{P}(d) \setminus D) \to \Pi(d)\) be the homomorphism induced by the inclusion map \(\mathbb{P}(d) \setminus D \to (\mathbb{P}(d) \setminus D)_{PGL(3)}, a \mapsto [e_0, a]\) where \(e_0\) is the base point of
Proof. We first show that $c$ is well defined and $\delta c = \tilde{\rho}^{*}\tau_{g}$, i.e., $c$ is a cobounding cochain for $\tilde{\rho}^{*}\tau_{g}$.

Proposition 3.1. The above definition of $c$ is well defined and $\delta c = \tilde{\rho}^{*}\tau_{g}$, i.e., $c$ is a cobounding cochain for $\tilde{\rho}^{*}\tau_{g}$.

Proof. We first show that $c$ is well defined. Let $\ell_{0}$ and $\ell_{1}$ are $C^{\infty}$-maps from $S^{1}$ to $V^{d} \setminus \tilde{D}$, and suppose that they represent the same element of $\pi_{1}(V^{d} \setminus \tilde{D})$. Then there exists a $C^{\infty}$-homotopy $H : S^{1} \times [0, 1] \to V^{d} \setminus \tilde{D}$ such that $H(\cdot, 0) = \ell_{0}$ and $H(\cdot, 1) = \ell_{1}$.

Regard the 2-sphere $S^{2}$ as the boundary of the unit disk $D^{2}$ in $\mathbb{R}^{2}$. $D^{2}$ has the natural orientation induced by that of $\mathbb{R}^{2}$ and this induces the orientation of $S^{1}$ by counter clockwise manner. Let $\ell : S^{1} \to V^{d} \setminus \tilde{D}$ be a $C^{\infty}$-map. Since $V^{d} \setminus \tilde{E}$ is simply connected we can extend $\ell$ to a $C^{\infty}$-map $\tilde{\ell} : D^{2} \to V^{d} \setminus \tilde{E}$. We may assume that $\tilde{\ell}$ is transverse to $\tilde{D}$. By Lemma 2.4 $\tilde{\ell}^{*}\tilde{F}$ is a compact 4-dimensional $C^{\infty}$-manifold with boundary and has the natural orientation induced by the orientation of $D^{2}$ and that of the fibers, which have the natural orientations as compact Riemann surfaces. Set

$$c([\ell]) := \text{Sign}(\tilde{\ell}^{*}\tilde{F}),$$

where $[\ell]$ denotes the element of $\pi_{1}(V^{d} \setminus \tilde{D})$ represented by $\ell$ and the right hand side is the signature of $\tilde{\ell}^{*}\tilde{F}$.

Since $\pi_{2}(V^{d} \setminus \tilde{E}) = 0$, we can extend $\tilde{H}$ to a $C^{\infty}$-map $\tilde{H} : D^{2} \to V^{d} \setminus \tilde{E}$ transverse to $\tilde{D} \setminus \tilde{E}$. Then $H^{*}\tilde{F}$ is a $C^{\infty}$-manifold with boundary $\tilde{H}^{*}\tilde{F}$. Since the signature of the boundary of a manifold is zero, we have by the Novikov additivity of the signature

$$\text{Sign}(\tilde{\ell}_{0}^{*}\tilde{F}) - \text{Sign}(\tilde{\ell}_{1}^{*}\tilde{F}) = 0,$$

so $c$ is well defined.

We next show the latter part. Let $\ell_{0}$ and $\ell_{1}$ be $C^{\infty}$-maps from $S^{1}$ to $V^{d} \setminus \tilde{D}$. We will show

$$c([\ell_{0}]) + c([\ell_{1}]) - c([\ell_{0}][\ell_{1}]) = \tilde{\rho}^{*}\tau_{g}([\ell_{0}],[\ell_{1}]).$$

(1)

Let $P$ denote the pair of pants; this is the 2-sphere $S^{2}$ with the interior of the three disjoint closed disks removed. We also choose two of three boundary components of $P$ and denote them by $S^{1}_{0}$ and $S^{1}_{1}$, respectively. $S^{1}_{0}$ and $S^{1}_{1}$ have the natural orientations induced by that of $P$ and can be naturally identified with $S^{1}$. Since $P$ has the homotopy type of the bouquet $S^{1} \vee S^{1}$, we can construct a $C^{\infty}$-map $L : P \to V^{d} \setminus \tilde{D}$ such that the restriction of $L$ to $S^{1}_{0}$ (resp. $S^{1}_{1}$) are exactly same as $\ell_{0}$ (resp. $\ell_{1}$).
We notice that the restriction of $L$ to the remaining boundary component of $P$ with the natural orientation is homotopic to the inverse of the composition loop $\ell_0 \cdot \ell_1$. We also have $\text{Sign}(L^*\mathcal{F}) = -\tilde{\rho}^* \tau_g([\ell_0], [\ell_1])$ by the definition of Meyer’s signature cocycle $\tau_g$. Using some extensions $\tilde{\ell}_i$ of $\ell_i$ for $i = 0$ and 1, and an extension $\tilde{\ell}_0 \cdot \tilde{\ell}_1$ of $\ell_0 \cdot \ell_1$, $L$ extends to a $C^\infty$-map $\tilde{L}: S^2 \to V^d \setminus \tilde{E}$. Moreover $\tilde{L}$ extends to a map $\tilde{L}: D^3 \to V^d \setminus \tilde{E}$ transverse to $\tilde{D}$. We have $\text{Sign}(L^*\mathcal{F}) = 0$ since $L^*\mathcal{F}$ is the boundary of $\tilde{L}^*\mathcal{F}$ hence we obtain by the Novikov additivity

$$0 = \text{Sign}(\tilde{L}^*\mathcal{F}) = \text{Sign}(\tilde{\ell}_0^* \mathcal{F}) + \text{Sign}(\tilde{\ell}_1^* \mathcal{F}) - \text{Sign}(\tilde{\ell}_0 \cdot \tilde{\ell}_1^* \mathcal{F}) + \text{Sign}(L^*\mathcal{F}),$$

that is just the equation $\mathbf{(I)}$.

\section{Main theorems}

In this section we shall state and prove the main results of this paper. In section 1 we defined the group $\Pi(d)$ and the homomorphism $\rho: \Pi(d) \rightarrow \Gamma_g$.

\begin{thm}
\label{thm:rho}
$\rho^*[\tau_g] = 0 \in H^2(\Pi(d); \mathbb{Q})$.
\end{thm}

\begin{thm}
\label{thm:h1}
The first homology group of $\Pi(d)$ is given as follows:

$$H_1(\Pi(d); \mathbb{Z}) = \begin{cases} 
\mathbb{Z}/3(d - 1)^2\mathbb{Z} & \text{if } d \equiv 0 \mod 3, \\
\mathbb{Z}/(d - 1)^2\mathbb{Z} & \text{if } d \equiv 1 \text{ or } 2 \mod 3.
\end{cases}$$

In particular, we have $H^1(\Pi(d); \mathbb{Q}) = 0$.
\end{thm}

As an immediate consequence of these theorems, it follows that there exists the unique 1-cochain $\phi^d: \Pi(d) \rightarrow \mathbb{Q}$ such that $\delta\phi^d = \rho^* \tau_g$. We will call $\phi^d$ the Meyer function for plane curves of degree $d$.

The rest of this section will be devoted to the proof of these theorems. In Proposition 3.1 we have showed that $\tilde{\rho}^*[\tau_g] = 0 \in H^2(\pi_1(V^d \setminus \tilde{D}); \mathbb{Z})$. Thus Theorem 4.1 follows from the following:

\begin{lem}
The homomorphism

$$\chi^*: H^2(\Pi(d); \mathbb{Q}) \rightarrow H^2(\pi_1(V^d \setminus \tilde{D}); \mathbb{Q})$$

induced by $\chi$, introduced in section 3, is injective.
\end{lem}

\begin{proof}
Recall that $\chi$ is the composition of $\chi_1$ and $\chi_2$. We first consider $\chi_1$. Let $\xi \in H^2(\mathbb{P}(d); \mathbb{Q})$ denote the first Chern class of the principal $\mathbb{C}^*$ bundle $V^d \setminus \{0\} \rightarrow \mathbb{P}(d)$. Then the restriction of $\xi$ to $\mathbb{P}(d) \setminus D$ is zero, for the first Chern class $c_1([D]) \in H^2(\mathbb{P}(d); \mathbb{Q})$ of the line bundle $[D]$ determined by the divisor $D$ of $\mathbb{P}(d)$ is a multiple of $\xi$ and of course the restriction of $c_1([D])$ to $\mathbb{P}(d) \setminus D$ is zero.

By the Gysin sequence

$$H^0(\mathbb{P}(d) \setminus D; \mathbb{Q}) \xrightarrow{\cup [\xi]} H^2(\mathbb{P}(d) \setminus D; \mathbb{Q}) \rightarrow H^2(V^d \setminus \tilde{D}; \mathbb{Q})$$

we have

$$H^2(\mathbb{P}(d) \setminus D; \mathbb{Q}) = 0$$

if $d \equiv 1 \text{ or } 2 \mod 3$.

Thus, $\chi^*$ is injective.
\end{proof}
of the principal \( \mathbb{C}^* \) bundle \( V^d \setminus \widetilde{D} \to \mathbb{P}(d) \setminus D \) we see that \( H^2(\mathbb{P}(d) \setminus D; \mathbb{Q}) \to H^2(V^d \setminus \widetilde{D}; \mathbb{Q}) \) is injective. Therefore
\[
\chi_1^*: H^2(\pi_1(\mathbb{P}(d) \setminus D); \mathbb{Q}) \to H^2(\pi_1(V^d \setminus \widetilde{D}); \mathbb{Q})
\]
is also injective.

We next consider \( \chi_2 \). By the homotopy exact sequence of the \( \mathbb{P}(d) \setminus D \) bundle \( (\mathbb{P}(d) \setminus D)_{\text{PGL}(3)} \to \text{BPGL}(3), [e, a] \mapsto \pi(e) \) where \( \pi \) denotes the projection map \( \text{EPGL}(3) \to \text{BPGL}(3) \), we have an exact sequence
\[
\mathbb{Z}/3\mathbb{Z} \cong \pi_2(\text{BPGL}(3)) \to \pi_1(\mathbb{P}(d) \setminus D) \xrightarrow{\chi_2} \Pi(d) \to 1.
\]
This implies that
\[
\chi_2^*: H^2(\Pi(d); \mathbb{Q}) \to H^2(\pi_1(\mathbb{P}(d) \setminus D); \mathbb{Q})
\]
is isomorphic. Since \( \chi^* = \chi_1^* \circ \chi_2^* \), the lemma follows. \( \square \)

We next proceed to Theorem 4.2. In the following we consider (co)homology with coefficients in \( \mathbb{Z} \). We need the following two lemmas:

**Lemma 4.4.** Let \( a^0 \in \mathbb{P}(d) \setminus D \) and denote by \( \mathbb{P} \) the set of all projective lines in \( \mathbb{P}(d) \) through \( a^0 \). Then there exist a non-empty Zariski open subset \( U \subset \mathbb{P} \subset \mathbb{P} \) such that each element of \( U \) does not meet \( E \) and is transverse to \( D \).

**Proof.** Consider the projection with center \( a^0 \)
\[
f: D \to \mathbb{P}, \quad f(a) = \text{the line through } a^0 \text{ and } a.
\]
Note that for \( a \in D \setminus E \), \( f \) is critical at \( a \) if and only if \( f(a) \) is contained in the hyperplane \( T_a \) appeared in Lemma 2.2, namely \( f(a) \) is not transverse to \( D \) at \( a \).

\( \mathbb{P} \) is a \((N - 1)\)-dimensional projective space and \( f(E) \) is a proper algebraic set in \( \mathbb{P} \) since \( \dim E \leq N - 2 \). Let \( K \) denote the set of all critical values of \( f \circ \pi: \mathcal{X} \to \mathbb{P} \). \( K \) contains all critical values of \( f|_{D \setminus E} \) since \( \pi|_{\pi^{-1}(D \setminus E)}: \pi^{-1}(D \setminus E) \to D \setminus E \) is biholomorphic by Lemma 2.2, and is algebraic and proper because \( K \) is nowhere dense in \( \mathbb{P} \) by Sard’s theorem. Therefore if we set
\[
U := \mathbb{P} \setminus (f(E) \cup K),
\]
\( U \) has the desired property. \( \square \)

**Lemma 4.5.** Let \( a^0 \) and \( U \) be as in Lemma 4.4. For each \( Q \in U \) the invariants of the complex surface \( M = \{(a, p) \in Q \times \mathbb{P}^2; p \in C_a\} \) is given as follows:
\[
c_1^2(M) = -d^2 + 9, \quad c_2(M) = d^2 + 3, \quad \text{Sign}(M) = 1 - d^2.
\]

**Proof.** Since \( Q \cong \mathbb{P}^1 \) we can regard \( M \) as a smooth hypersurface in \( \mathbb{P}^1 \times \mathbb{P}^2 \) determined by a \((1, d)\) homogeneous polynomial. For \( i = 1 \) and \( i = 2 \) respectively, we denote by \( \xi_i \in H^2(\mathbb{P}^1 \times \mathbb{P}^2; \mathbb{Z}) \) the pull back of the first Chern class of \( O(1) \) by \( H^2(\mathbb{P}^1; \mathbb{Z}) \to H^2(\mathbb{P}^1 \times \mathbb{P}^2; \mathbb{Z}) \) induced by the projection \( \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1 \). Here \( O(1) \) denotes the dual of the tautological line bundle over \( \mathbb{P}^1 \). The first Chern class of the line bundle \([M]\) determined by the divisor
$M$ of $\mathbb{P}^1 \times \mathbb{P}^2$ is $c_1([M]) = \xi_1 + d\xi_2$. Therefore by the adjunction formula, the first Chern class of $M$ is

$$c_1(M) = (c_1(\mathbb{P}^1 \times \mathbb{P}^2) - c_1([M]))|_M$$

$$= (2\xi_1 + 3\xi_2 - (\xi_1 + d\xi_2))|_M$$

$$= (\xi_1 + (3 - d)\xi_2)|_M.$$ 

Then the Chern number $c_1^2(M)$ is computed as follows:

$$c_1^2(M) = \langle c_1(M)^2, \mu_M \rangle$$

$$= \langle (\xi_1 + (3 - d)\xi_2)c_1([M]), \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \rangle$$

$$= \langle (\xi_1 + (3 - d)\xi_2)^2(\xi_1 + d\xi_2), \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \rangle$$

$$= \langle (-d^2 + 9)\xi_1\xi_2, \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \rangle$$

$$= -d^2 + 9.$$ 

Here $\mu_M$ (resp. $\mu_{\mathbb{P}^1 \times \mathbb{P}^2}$) denotes the fundamental homology class of $M$ (resp. $\mathbb{P}^1 \times \mathbb{P}^2$) and $\langle -, - \rangle$ denotes the Kronecker pairing between cohomology and homology. We next compute $c_2(M)$. Again by the adjunction formula, the second Chern class of $M$ is

$$c_2(M) = c_2(\mathbb{P}^1 \times \mathbb{P}^2)|_M - c_1(M) \cdot c_1([M])|_M$$

$$= (3\xi_2^2 + 6\xi_1\xi_2 - (\xi_1 + (3 - d)\xi_2)(\xi_1 + d\xi_2))|_M$$

$$= (3\xi_1\xi_2 + (d^2 - 3d + 3)\xi_2^2)|_M,$$

and the Chern number which will also be denoted by $c_2(M)$ is

$$c_2(M) = \langle c_2(M), \mu_M \rangle$$

$$= \langle (3\xi_1\xi_2 + (d^2 - 3d + 3)\xi_2^2)c_1([M]), \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \rangle$$

$$= \langle (3\xi_1\xi_2 + (d^2 - 3d + 3)\xi_2^2)(\xi_1 + d\xi_2), \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \rangle$$

$$= \langle (d^2 + 3)\xi_1\xi_2^2, \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \rangle$$

$$= d^2 + 3.$$ 

Finally by the Hirzebruch signature theorem we have $\text{Sign}(M) = \frac{1}{3}(c_1^2(M) - 2c_2(M)) = 1 - d^2.$

Let $a^0$ and $Q \in U$ be as in Lemma 1.5. Using the above two lemmas we can compute the first homology group of $\pi_1(\mathbb{P}(d) \setminus D)$:

**Proposition 4.6.** $H_1(\pi_1(\mathbb{P}(d) \setminus D); \mathbb{Z}) = \mathbb{Z}/3(d - 1)^2\mathbb{Z}.$

**Proof.** The first projection $g: M \to Q$ is a family of algebraic curves, whose all singular fibers are of type I by Lemma 2.3. Since the Euler contribution (see [4]p.118, (11.4)Proposition) of a singular fiber of type I is $+1$, the number of singular fibers of $g: M \to Q \cong \mathbb{P}^1$ is

$$c_2(M) - 2(2 - 2g) = d^2 + 3 - 2\left(2 - 2 \cdot \frac{1}{2}(d - 1)(d - 2)\right) = 3(d - 1)^2.$$ 

Now consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{Z} \cong H_0(\mathbb{P}(d)) & \longrightarrow & H_2(\mathbb{P}(d), \mathbb{P}(d) \setminus D) \\
\| & & \| \\
H^{2N-2}(\mathbb{P}(d)) & \overset{i^*}{\longrightarrow} & H^{2N-2}(D) \cong \mathbb{Z}.
\end{array}$$

$$H_1(\mathbb{P}(d) \setminus D) \longrightarrow 0$$
Here the vertical isomorphisms are Poincaré duality and the first horizontal sequence is a part of the homology exact sequence of the pair \((P(d), P(d) \setminus D)\), and \(i^*\) is induced by the inclusion \(D \hookrightarrow P(d)\). For a generator of \(H_2(P(d))\) we can choose \([Q]\). We can conclude this generator is mapped to \(3(d-1)^2\) times a generator of \(H^{2N-2}(D)\) in the above diagram, because \((3)\) shows that \(Q\) and \(D\) intersect transversely in \(3(d-1)^2\) points. This completes the proof, since \(H_1(P(d) \setminus D) \cong H_1(\pi_1(P(d) \setminus D); \mathbb{Z})\) is isomorphic to the cokernel of \(H_2(P(d)) \to H_2(P(d), P(d) \setminus D)\).

Now we start the proof of Theorem \[4.2\] Let \(F_0 \in V^d \setminus \bar{D}\) be a base point and \(d^0\) the image of \(F_0\) under the map \(V^d \setminus \bar{D} \to P(d) \setminus D\). We consider the maps \(\lambda: GL(3; \mathbb{C}) \to V^d \setminus \bar{D}, A \mapsto A \cdot F_0\) and \(\tilde{\lambda}: PGL(3) \to P(d) \setminus D, \tilde{\lambda} \mapsto \tilde{\lambda} \cdot d^0\). Since the isomorphism \(\pi_2(BPGL(3)) \cong \pi_1(PGL(3))\) induced by the homotopy exact sequence of the universal \(PGL(3)\) bundle \(E_{PGL(3)} \to BPGL(3)\) is compatible with \((2)\) and \(\tilde{\lambda}_*: \pi_1(PGL(3)) \to \pi_1(P(d) \setminus D)\), we have an exact sequence of group homology

\[\mathbb{Z}/3\mathbb{Z} \cong H_1(\pi_1(PGL(3))) \mapsto H_1(\pi_1(P(d) \setminus D)) \cong \mathbb{Z}/3(d-1)^2\mathbb{Z} \xrightarrow{\bar{\lambda}_*} H_1(\Pi(d)) \to 0.\]

Therefore we must compute the map \(\bar{\lambda}_*\) to determine \(H_1(\Pi(d))\).

For this purpose, we consider the following exact sequence

\[\mathbb{Z} \cong H_1(\mathbb{C}^*) \to H_1(V^d \setminus \bar{D}) \to H_1(P(d) \setminus D) \to 0\]

induced by a part of the homotopy exact sequence of the principal \(\mathbb{C}^*\) bundle \(V^d \setminus \bar{D} \to P(d) \setminus D\). We have \(H_1(V^d \setminus \bar{D}) \cong \mathbb{Z}\) (see \([3]\) Chapter 4, Corollary(1.4)). Let \(\gamma\) be the generator of \(H_1(\mathbb{C}^*)\) represented by the loop \(\gamma(t) = e^{2\pi\sqrt{-1}t}, 0 \leq t \leq 1\). By Proposition \[4.6\] we see that the image of \(\gamma\), which is represented by the loop \(t \mapsto e^{2\pi\sqrt{-1}t} \cdot F_0\), is \(3(d-1)^2\) times a generator of \(H_1(V^d \setminus \bar{D})\). On the other hand the loop

\[t \mapsto \begin{pmatrix} e^{2\pi\sqrt{-1}t} & 0 & 0 \\ 0 & e^{2\pi\sqrt{-1}t} & 0 \\ 0 & 0 & e^{2\pi\sqrt{-1}t} \end{pmatrix}, \quad 0 \leq t \leq 1\]

in \(GL(3; \mathbb{C})\), representing \(3\) times a generator of \(H_1(GL(3; \mathbb{C})) \cong \mathbb{Z}\), is mapped to the loop \(t \mapsto (e^{2\pi\sqrt{-1}t})^{-d} \cdot F_0\) by \(\lambda\). Hence in the commutative diagram

\[\mathbb{Z} \cong H_1(GL(3; \mathbb{C})) \xrightarrow{\lambda_*} H_1(V^d \setminus \bar{D}) \twoheadrightarrow \mathbb{Z}/3\mathbb{Z} \cong H_1(PGL(3)) \xrightarrow{\bar{\lambda}_*} H_1(P(d) \setminus D)\]

we have \(\lambda_*(1) = \pm d(d-1)^2 \in \mathbb{Z} \cong H_1(V^d \setminus \bar{D})\) so we can conclude \(\bar{\lambda}_*(1 \text{ mod } 3) = \pm d(d-1)^2 \text{ mod } 3(d-1)^2\). This completes the proof of Theorem \[4.2\]

5 The value of the Meyer function

By Proposition \[4.6\] we have \(H^1(\pi_1(P(d) \setminus D); \mathbb{Q}) = 0\). Therefore \(\bar{\phi}^d := \phi^d \circ \chi_2\) is the unique 1-cochain of \(\pi_1(P(d) \setminus D)\) satisfying \((\rho \circ \chi_2)^* \tau_9 = \delta \bar{\phi}^d\). In this section we will compute the value of \(\bar{\phi}^d\) on a special element in \(\pi_1(P(d) \setminus D)\) so called lasso.
We first explain what a lasso is. Let $M$ be a connected complex manifold of dimension $m$ and $N$ an irreducible hypersurface of $M$. Then the inclusion $M \setminus N \hookrightarrow M$ induces the following exact sequence:

$$1 \hookrightarrow \langle \sigma \rangle \twoheadrightarrow \pi_1(M \setminus N) \twoheadrightarrow \pi_1(M) \twoheadrightarrow 1.$$ 

Here $\langle \sigma \rangle$ denotes the normal closure of an element $\sigma$ of $\pi_1(M \setminus N)$, which is described in the following. Let $p$ be a non-singular point of $N$ and $(z_1, \ldots, z_m)$ a local coordinate system of $M$ around $p$ such that $N$ is defined by $z_1 = 0$. For a sufficiently small $\varepsilon > 0$, consider a loop defined in this coordinate system by

$$[0, 1] \to M \setminus N, \ t \mapsto (\varepsilon e^{2\pi\sqrt{-1}t}, 0, \ldots, 0)$$

based at $q = (\varepsilon, 0, \ldots, 0)$. Joining this loop with a path from the base point of $M \setminus N$ to $q$, we get an element $\sigma$ of $\pi_1(M \setminus N)$. Since $N$ is irreducible, the conjugacy class of $\sigma$ in $\pi_1(M \setminus N)$ is independent of choices of $p$ and a local coordinate system. Each element of this conjugacy class is called a lasso around $N$.

Returning to $\pi_1(\mathbb{P}(d) \setminus D)$, $D$ is an irreducible hypersurface of $\mathbb{P}(d)$. Let $\sigma^d \in \pi_1(\mathbb{P}(d) \setminus D)$ be a lasso around $D$. Since $\tilde{\phi}^d$ is a class function (see Lemma 8.2 in Appendix), the values of $\tilde{\phi}^d$ on the conjugacy class of $\sigma^d$ is constant.

**Proposition 5.1.** For $d \geq 3$,

$$\tilde{\phi}^d(\sigma^d) = -\frac{d + 1}{3(d - 1)}.$$

**Proof.** Choose $a^0$ and $Q \in U$ as in Lemma 4.5. In the proof of Proposition 4.6 we see that $Q$ meets $D$ transversely in $3(d - 1)^2$ points. Let $Q \cap D = \{q_1, \ldots, q_{3(d-1)^2}\}$ and let $D_i$ ($i = 1, \ldots, 3(d - 1)^2$) be a small closed 2-disk in $Q$ such that $q_i \in \operatorname{Int} D_i$ and $D_i \cap D_j = \emptyset$ for $i \neq j$. We fix a base point of $Q_0 := Q \setminus \bigcup_{i=1}^{3(d-1)^2} \operatorname{Int} D_i$ and for each $i = 1, \ldots, 3(d - 1)^2$, choose a based loop $\sigma_i$ in $Q_0$ such that $\sigma_i$ is free homotopic to the loop traveling once the boundary $\partial D_i$ by counter clockwise manner. Note that regarded as an element in $\pi_1(\mathbb{P}(d) \setminus D)$, $\sigma_i$ is a lasso around $D$ hence we have $\tilde{\phi}^d(\sigma_i) = \tilde{\phi}^d(\sigma^d)$.

Let $g: M \to Q$ be as in the proof of Proposition 4.6 and set $M_0 := g^{-1}(Q_0)$ and $M_i := g^{-1}(D_i), i = 1, \ldots, 3(d - 1)^2$. By Meyer’s signature formula ([12] Satz 1) and the equation $(\rho \circ \chi_2)^* \tau_g = \delta \tilde{\phi}^d$, we obtain

$$\operatorname{Sign}(M_0) = \sum_{i=1}^{3(d-1)^2} \tilde{\phi}^d(\sigma_i) = 3(d - 1)^2 \tilde{\phi}^d(\sigma^d).$$

Since the topological type of $g^{-1}(q_i)$ is Lefschetz singular fiber of type I, we have $\operatorname{Sign}(M_i) = 0$. We compute by the Novikov additivity and Lemma 4.6

$$1 - d^2 = \operatorname{Sign}(M) = \operatorname{Sign}(M_0) + \sum_{i=1}^{3(d-1)^2} \operatorname{Sign}(M_i) = 3(d - 1)^2 \tilde{\phi}^d(\sigma^d).$$

This completes the proof. \qed
In the rest of this section we consider the remaining case \( d = 2 \). Since \( V^2 \) is the set of quadratic forms each element of \( V^2 \) can be expressed by a \( 3 \times 3 \) symmetric matrix \( S \). In this viewpoint \( V^2 \setminus \tilde{D} \) is the space of non-singular symmetric matrices and the action of \( GL(3; \mathbb{C}) \) on \( V^2 \setminus \tilde{D} \) is given by

\[
A \cdot S = A^{-1} \cdot S \cdot A^{-1}, \quad A \in GL(3; \mathbb{C}).
\]

Since this action is transitive and the isotropy group of the unit matrix is the complex orthogonal group \( O_3(\mathbb{C}) = \{ A \in GL(3; \mathbb{C}) : t^A \cdot A = I \} \), we have

\[
V^2 \setminus \tilde{D} \cong GL(3; \mathbb{C}) / O_3(\mathbb{C}).
\]

Also we have

\[
\mathbb{P}(2) \setminus D \cong PGL(3) / SO_3(\mathbb{C}),
\]

where \( SO_3(\mathbb{C}) = \{ A \in O_3(\mathbb{C}) : \det A = 1 \} \) is regarded as a subgroup of \( PGL(3) \) by the injection \( SO_3(\mathbb{C}) \hookrightarrow PGL(3) \) induced by the projection \( GL(3; \mathbb{C}) \to PGL(3) \). Therefore, we obtain

\[
(\mathbb{P}(2) \setminus D)_{PGL(3)} = EPGL(3) \times_{PGL(3)} (\mathbb{P}(2) \setminus D) \\
\cong EPGL(3) / SO_3(\mathbb{C}) = BSO_3(\mathbb{C}) \simeq BSO_3.
\]

The last homotopy equivalence holds because the natural inclusion \( SO_3 \hookrightarrow SO_3(\mathbb{C}) \) is homotopy equivalence. In particular, we have

\[
\Pi(2) \cong \pi_1(BSO_3) = 1.
\]

6 The universal property of \( (\mathbb{P}(d) \setminus D)_{PGL(3)} \)

In this section we will show the universal property of the space \( (\mathbb{P}(d) \setminus D)_{PGL(3)} \). In the latter part of the section, we consider the case \( d = 4 \) more detail; in particular, we prove that \( p : \Pi(4) \to \Gamma_3 \) is surjective.

We first make some definitions. Let \( \iota : X \to P \) be a continuous map and \( h : P \to B \) a \( \mathbb{P}^2 \) bundle whose structure group is \( PGL(3) \). We call \( \xi = (X, \iota, P, h, B) \) a family of non-singular plane curves of degree \( d \) if

1. \( p := h \circ \iota : X \to B \) is a continuous family of compact Riemann surfaces of genus \( g = \frac{1}{2}(d - 1)(d - 2) \), and

2. for each \( b \in B \), the restriction \( \iota|_{X_b} : X_b \to P_b \) is a holomorphic embedding where \( X_b = p^{-1}(b) \) and \( P_b = h^{-1}(b) \).

For each \( b \in B \), the image \( \iota(X_b) \subset P_b \cong \mathbb{P}^2 \) is a non-singular plane curve of degree \( d \). Two such families \( \xi_i = (X^i, \iota^i, P^i, h^i, B), i = 0, 1 \), are called isotopic if there exists a family of non-singular curves of degree \( d \) over \( B \times [0, 1] \), denoted by \( \tilde{\xi} = (X, \tilde{\iota}, \tilde{P}, \tilde{h}, B \times [0, 1]) \), such that for \( i = 0, 1 \), the restriction of \( \tilde{\xi} \) to \( B \times \{ i \} \) is isomorphic to \( \xi_i \), i.e., for \( i = 0, 1 \), there
exists a homeomorphism $\Psi_i: P^i \to \tilde{P}|_{B \times \{i\}}$ and $\psi_i: X^i \to \tilde{X}|_{B \times \{i\}}$ such that the diagram

\[
\begin{array}{ccc}
X^i & \xrightarrow{\psi_i} & \tilde{X}|_{B \times \{i\}} \\
\downarrow{\iota_i} & & \downarrow{\tilde{i}} \\
P^i & \xrightarrow{\Psi_i} & \tilde{P}|_{B \times \{i\}} \\
\downarrow{h_i} & & \downarrow{\tilde{h}} \\
B & \longrightarrow & B \times \{i\},
\end{array}
\]

where the last horizontal arrow is the homeomorphism $B \to B \times \{i\}$ given by $b \mapsto (b, i)$, commutes and $\Psi_i$ (resp. $\psi_i$) maps each fiber $P^i_b$ (resp. $X^i_b$) onto $\tilde{h}^{-1}(b, i)$ (resp. $(\tilde{h} \circ \tilde{i})^{-1}(b, i)$) biholomorphically.

For a given space $B$, we denote by $\mathcal{PC}_d(B)$ the set of all isotopy classes of families of non-singular plane curves of degree $d$ over $B$. $\mathcal{PC}_d(\bullet)$ is contravariant; for a given continuous map $f: B' \to B$ we have a natural map $\mathcal{PC}_d(B) \to \mathcal{PC}_d(B')$ which assigns the isotopy class of $\xi$ the isotopy class of the pull back of $\xi$ by $f$, which will be denoted by $f^*\xi$. In fact, the isotopy class of $f^*\xi$ is uniquely determined by the homotopy class $[f] \in [B', B]$.

Among such families of non-singular plane curves of degree $d$, there is a universal one. Consider the inclusion map $\mathcal{F} \hookrightarrow (\mathbb{P}(d) \setminus D) \times \mathbb{P}^2$ and the first projection $(\mathbb{P}(d) \setminus D) \times \mathbb{P}^2 \to \mathbb{P}(d) \setminus D$. For simplicity, we write $Y$ instead of $(\mathbb{P}(d) \setminus D) \times \mathbb{P}^2$. Since these maps are $PGL(3)$-equivariant, we obtain

$$\iota_u: \mathcal{F}_{PGL(3)} \to Y_{PGL(3)}$$

and

$$h_u: Y_{PGL(3)} \to (\mathbb{P}(d) \setminus D)_{PGL(3)}.$$ 

The map $p_u := h_u \circ \iota_u$ is the same as the map defined in section 1 and

$$\xi_u := (\mathcal{F}_{PGL(3)}, \iota_u, Y_{PGL(3)}, h_u, (\mathbb{P}(d) \setminus D)_{PGL(3)})$$

is a family of non-singular plane curves of degree $d$. The next theorem says that $(\mathbb{P}(d) \setminus D)_{PGL(3)}$ is the classifying space for the functor $\mathcal{PC}_d(\bullet)$ and $\xi_u$ is the universal family.

**Theorem 6.1.** For any space $B$, the map

$$\eta: [B, (\mathbb{P}(d) \setminus D)_{PGL(3)}] \to \mathcal{PC}_d(B)$$

which assigns the homotopy class of $f: B \to (\mathbb{P}(d) \setminus D)_{PGL(3)}$ the isotopy class of the pull back $f^*\xi_u$, is bijective.

In the following we shall construct the inverse of $\eta$.

Let $\xi = (X, \iota, P, h, B)$ be given. We divide the argument in three steps.

**Step 1.** We first consider the case when $P$ is trivial: suppose that $P = B \times \mathbb{P}^2$. Then for each $b \in B$, $\iota(X_b) \subset \{b\} \times \mathbb{P}^2 \cong \mathbb{P}^2$ is a non-singular plane curve of degree $d$, so the defining equation of $\iota(X_b)$ in $\mathbb{P}^2$ is uniquely determined as an element of $\mathbb{P}(d) \setminus D$. Denoting it by $Eq(b)$, we obtain a map

$$Eq: B \to \mathbb{P}(d) \setminus D.$$
**Lemma 6.2.** The map $Eq$ is continuous.

**Proof.** Regard $\mathbb{P}^2$ as the set of all complex lines through the origin in $\mathbb{C}^3$. Then the holomorphic line bundle $\mathcal{O}(d)$ over $\mathbb{P}^2$ is given by

$$\mathcal{O}(d) = \mathcal{O}(1)^\otimes d = \bigcup_{\ell \in \mathbb{P}^2} \text{Hom}(\ell, \mathbb{C})^\otimes d.$$ 

Let $p_2: B \times \mathbb{P}^2 \to \mathbb{P}^2$ be the second projection and consider the pull back $L := (p_2 \circ i)^* \mathcal{O}(d)$. $L \to X$ is a continuous family over $B$ of holomorphic vector bundles. Now $H^0(\mathbb{P}^2; \mathcal{O}(d))$ is canonically isomorphic to $V^d$ and for each $b \in B$ there is the natural homomorphism

$$\sigma_b: V^d \cong H^0(\mathbb{P}^2; \mathcal{O}(d)) \to H^0(\iota(X_b); \mathcal{O}(d)|_{\iota(X_b)}) \cong H^0(X_b; L_b),$$

where $L_b$ is the restriction of $L$ to $X_b$. Combining all $\sigma_b, b \in B$ together, we obtain a homomorphism of vector bundles

$$\sigma: B \times V^d \to \bigcup_{b \in B} H^0(X_b; L_b).$$

We see that for each $b \in B$, $\sigma_b$ is surjective and its kernel is 1-dimensional generated by the defining equation of $\iota(X_b)$, i.e., $Eq(b) = \ker \sigma_b$. This shows that $Eq$ is continuous.

We also define a $PGL(3)$-equivariant continuous map $\Psi: B \times PGL(3) \to \mathbb{P}(d) \setminus D$ by

$$\Psi(b, g) = g \cdot Eq(b).$$

Here we regard $B \times PGL(3)$ as the trivial principal $PGL(3)$ bundle with left $PGL(3)$ action.

**Step 2.** We next consider the general case $\xi = (X, \iota, P, h, B)$. Let $\{U_i\}_{i \in I}$ be an open covering of $B$ trivializing $h: P \to B$: There is an isomorphism $\varphi_i: h^{-1}(U_i) \to U_i \times \mathbb{P}^2$ for each $i$ and a system of transition functions $g_{ij}: U_i \cap U_j \to PGL(3)$ for each $(i, j)$ satisfying $U_i \cap U_j \neq \emptyset$, such that

$$(\varphi_i \circ \varphi_j^{-1})(b, p) = (b, g_{ij}(b) \cdot p), \ b \in U_i \cap U_j, \ p \in \mathbb{P}^2.$$ 

As in step 1, we have a continuous map $Eq^i: U_i \to \mathbb{P}(d) \setminus D$ and a $PGL(3)$-equivariant map $\Psi^i: U_i \times PGL(3) \to \mathbb{P}(d) \setminus D$ for each $i$. Let $Q(\xi)$ be a principal $PGL(3)$ bundle over $B$ associated to $h: P \to B$: namely $Q(\xi)$ is constructed from the disjoint union $\bigsqcup_{i \in I} U_i \times PGL(3)$ by identifying $(b, g) \in U_i \times PGL(3)$ with $(b, g \cdot g_{ij}(b)) \in U_j \times PGL(3)$ where $b \in U_i \cap U_j$. We have $g_{ij}(b) \cdot Eq^i(b) = Eq^j(b)$ for $b \in U_i \cap U_j$ because $g \cdot C_a = C_{ga}$ for $g \in PGL(3)$, $a \in \mathbb{P}(d) \setminus D$. Therefore piecing all $\Psi^i, \ i \in I$ together, we obtain a $PGL(3)$ equivariant map $\Psi: Q(\xi) \to \mathbb{P}(d) \setminus D$ and a continuous map

$$\Psi_{PGL(3)}: Q(\xi)_{PGL(3)} \to (\mathbb{P}(d) \setminus D)_{PGL(3)}.$$ 

Note that $Q(\xi)$ and $\Psi$ are determined up to isomorphism over $B$.

**Step 3.** The natural map

$$T: Q(\xi)_{PGL(3)} = EPGL(3) \times_{PGL(3)} Q(\xi) \to PGL(3) \setminus Q(\xi) \cong B$$
is a homotopy equivalence because this is an $EPGL(3)$ bundle. Taking a homotopy inverse map $\zeta : B \to Q(\xi)_{PGL(3)}$ of $T$, we set
\[ \theta(\xi) := [\Psi_{PGL(3)} \circ \zeta]. \]
Here $[\xi]$ denotes the element of $\mathcal{PC}_d(B)$ represented by $\xi$ and $[\Psi_{PGL(3)} \circ \zeta]$ denotes the element of $[B, (\mathbb{P}(d) \setminus D)_{PGL(3)}]$ represented by $\Psi_{PGL(3)} \circ \zeta$. It is easy to see that
\[ \theta : \mathcal{PC}_d(B) \to [B, (\mathbb{P}(d) \setminus D)_{PGL(3)}] \]
is well defined.

Before starting the proof of Theorem 6.1, we describe the above construction applied to the family $\xi_u$. In the below, $e \in EPGL(3)$, $g, h \in PGL(3)$ and $a \in \mathbb{P}(d) \setminus D$. We can write
\[ Q(\xi_u) \cong (\mathbb{P}(d) \setminus D) \times PGL(3) \] (4)
where the action of $PGL(3)$ on $(\mathbb{P}(d) \setminus D) \times PGL(3)$ is diagonal, i.e.,
\[ g \cdot (a, h) = (g \cdot a, g \cdot h), \]
and the left action of $PGL(3)$ on the right hand side of (4) is given by
\[ g \cdot [e, (a, h)] = [e, (a, h \cdot g^{-1})]. \]
The $PGL(3)$-equivariant map $\Psi_u : Q(\xi_u) \to \mathbb{P}(d) \setminus D$ defined as in step 2 is given by
\[ \Psi_u([e, (a, g)]) = g^{-1} \cdot a, \]
and moreover, the induced map $Q(\xi_u)_{PGL(3)} \to (\mathbb{P}(d) \setminus D)_{PGL(3)}$ has a section $s_u$ given by
\[ s_u([e, a]) = [e, (a, 1)] \].

Proof of Theorem 6.1. We first prove $\eta \circ \theta = id_{PC_d(B)}$. Let $\xi = (X, \iota, P, h, B)$ be given. By construction, there is the canonical isomorphism $T^*\xi \to \Psi^*_{PGL(3)}\xi_u$ as families of non-singular plane curves of degree $d$ over $Q(\xi)_{PGL(3)}$. Thus we have
\[ (\Psi_{PGL(3)} \circ \zeta)^*\xi_u = \zeta^*\Psi^*_{PGL(3)}\xi_u = \zeta^*T^*\xi = (T \circ \zeta)^*\xi. \]

Since $T \circ \zeta$ is homotopic to the identity map of $E$, this shows $\eta \circ \theta = id_{PC_d(B)}$.

We next show $\theta \circ \eta = id_{[B, (\mathbb{P}(d) \setminus D)_{PGL(3)}]}$. Let $f : B \to (\mathbb{P}(d) \setminus D)_{PGL(3)}$ be a continuous map.

Starting from the family $f^*\xi$ and tracing the construction of $\theta$, we construct the map
\[ \Psi_{PGL(3)} : Q(f^*\xi_u)_{PGL(3)} \to (\mathbb{P}(d) \setminus D)_{PGL(3)}. \]
$Q(f^*\xi_u)_{PGL(3)}$ is naturally isomorphic to the pull back of $Q(\xi_u)_{PGL(3)} \to (\mathbb{P}(d) \setminus D)_{PGL(3)}$ by $f$. Thus pulling back the section $s_u$, we obtain a map $\zeta' := f^*s_u : B \to Q(f^*\xi_u)_{PGL(3)}$ such that $T \circ \zeta' = id_B$ and $\Psi_{PGL(3)} \circ \zeta' = f$. Then $\zeta'$ is a homotopy inverse of $T$ and $\theta \circ \eta([f]) = [\Psi_{PGL(3)} \circ \zeta'] = [f]$, so we obtain $\theta \circ \eta = id_{[B, (\mathbb{P}(d) \setminus D)_{PGL(3)}]}$. \hfill \square

We call any representative of $\theta(\xi)$ the classifying map for the family $\xi$.  

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For $d = 4$, we don’t have to consider $\mathbb{P}^2$ bundles. Recall that a non-hyperelliptic Riemann surface $C$ of genus 3 can be realized as a non-singular plane curve of degree 4 (=plane quartic) by the canonical embedding. This means that the canonical map

$$\iota_C : C \rightarrow \mathbb{P}(H^0(C; K_C)^\vee)$$

where $H^0(C; K_C)$ is the space of holomorphic 1-forms on $C$, is an embedding and if we identify $\mathbb{P}(H^0(C; K_C)^\vee)$ with $\mathbb{P}^2$ by a choice of a basis of $H^0(C; K_C)$, the image of $C$ is a non-singular plane curve of degree 4. The defining equation of the image is uniquely determined as an element of $\mathbb{P}(4) \setminus D$.

Let $p : X \rightarrow B$ be a continuous family of compact Riemann surfaces of genus 3. We call it a non-hyperelliptic family of genus 3 if the complex structure of each fiber $p^{-1}(b), b \in B$ is non-hyperelliptic. Two such families $p_i : X_i \rightarrow B$ are called isotopic if there exists a non-hyperelliptic family of genus 3 over $B \times [0, 1]$ such that for $i = 0, 1$, its restriction to $B \times \{i\}$ is isomorphic to $p_i : X_i \rightarrow B$ as continuous family of Riemann surfaces over $B \cong B \times \{i\}$.

For a given space $B$, we denote by $\mathcal{N}H_3(B)$ the set of all isotopy classes of non-hyperelliptic families of genus 3 over $B$. Then the forgetful functor

$$\mathcal{P}C_4(\bullet) \rightarrow \mathcal{N}H_3(\bullet)$$

defined by an obvious manner, is bijective. For, let $p : X \rightarrow B$ be a given non-hyperelliptic family of genus 3. Set $\Lambda_X := \bigcup_{b \in B} H^0(p^{-1}(b); K_b)$, where $H^0(p^{-1}(b); K_b)$ denotes the space of holomorphic 1-forms on $p^{-1}(b)$. This has the structure of complex vector bundle over $B$. Projectivising the dual of $\Lambda_X$, we obtain a $\mathbb{P}^2$ bundle

$$h' : P' = \bigcup_{b \in B} \mathbb{P}(H^0(p^{-1}(b); K_b)^\vee) \rightarrow B,$$

and piecing the fiberwise canonical maps $\iota_{X_b}, b \in B$ together, we get a map $\iota : X \rightarrow P'$. Then we obtain an element $\xi = (X, \iota, P', h', B) \in PC_4(\bullet)$. This correspondence gives the inverse of (5).

We continue the consideration of the case $d = 4$. We next prove that:

**Proposition 6.3.** The homomorphism $\rho : \Pi(4) \rightarrow \Gamma_3$ is surjective.

Combining this with Theorem 4.1 which implies that $\rho^* : H^2(\Gamma_3; \mathbb{Q}) \rightarrow H^2(\Pi(4); \mathbb{Q})$ is not injective, we see that the order of the kernel of $\rho$ is infinite.

**Proof of Proposition 6.3.** Let $T_3$ be the Teichmüller space of compact Riemann surfaces of genus 3 and $H_3$ the hyperelliptic locus of $T_3$; namely the set of marked Riemann surfaces whose complex structure is hyperelliptic. $H_3$ is a complex analytic closed submanifold of codimension 1 with infinitely many components (see [14],p.259-260). In particular, $T_3 \setminus H_3$ is path connected.

We recall; there is a holomorphic family $\pi : V_3 \rightarrow T_3$ called the universal Teichmüller curve, whose fiber over the marked Riemann surface $[f, C]$ is isomorphic to $C$; the mapping class group $\Gamma_3$ acts on $T_3$ and $V_3$, and $\pi$ is equivariant with respect to these actions; it is well known that the quotient space $\Gamma_3 \setminus T_3$ is the Riemann moduli space. Since the action of $\Gamma_3$ on $T_3$ preserves $H_3$, $\Gamma_3$ also acts on $T_3 \setminus H_3$ and its inverse image by $\pi$. Restricting $\pi$ to $T_3 \setminus H_3$ and taking the Borel construction, we obtain a non-hyperelliptic family of genus 3 over $(T_3 \setminus H_3)_{\Gamma_3}$.
It is not difficult to see that this family also have the universal property which the family \( p_u \) over \((\mathbb{P}(4) \setminus D)_{\text{PGL}(3)}\) has. Therefore, \((T_3 \setminus H_3)_{\Gamma_3}\) is homotopy equivalent to \((\mathbb{P}(4) \setminus D)_{\text{PGL}(3)}\), hence its fundamental group is isomorphic to \(\Pi(4)\).

By the homotopy exact sequence of the \(T_3 \setminus H_3\) bundle \((T_3 \setminus H_3)_{\Gamma_3} \to B\Gamma_3 = K(\Gamma_3, 1)\) we obtain an exact sequence

\[
\Pi(4) \simeq \pi_1((T_3 \setminus H_3)_{\Gamma_3}) \xrightarrow{\rho'} \pi_1(B\Gamma_3) = \Gamma_3 \to \pi_0(T_3 \setminus H_3).
\]

We notice that the homomorphism \(\rho'\) just coincides with the topological monodromy over \((T_3 \setminus H_3)_{\Gamma_3}\), and \(\pi_0(T_3 \setminus H_3)\) is one point. This shows \(\rho'\) is surjective, so \(\rho\) is.

**7 Local signature for 4-dimensional non-hyperelliptic fibration of genus 3**

As an application, we will define the local signature for the set of all fiber germs of 4-dimensional fiber spaces whose general fibers are non-hyperelliptic Riemann surfaces of genus 3, using the Meyer function \(\phi^4\). This local signature is used to derive a signature formula for a class of 4-dimensional fiber spaces, whose general fibers are non-hyperelliptic Riemann surfaces of genus 3.

Let \(\Delta\) be a closed oriented 2-disk and \(p\) its center. A 4-tuple \(F = (E, \pi, \Delta, p)\) is called a fiber germ of non-hyperelliptic family of genus 3 if

1. \(E\) is a \(C^\infty\) manifold of dimension 4 and \(\pi: E \to \Delta\) is a \(C^\infty\) map,
2. the restriction of \(\pi\) to \(\Delta \setminus \{p\}\) is a non-hyperelliptic family of genus 3.

Note that \(E\) has the natural orientation and compact, hence its signature \(\text{Sign}(E)\) is defined. Two such germs \((E, \pi, \Delta, p)\) and \((E', \pi', \Delta', p')\) are called equivalent if there exist a smaller disk \(\Delta_0 \subset \Delta\) (resp. \(\Delta_0' \subset \Delta'\)) whose center is \(p\) (resp. \(p'\)), and there exist orientation preserving diffeomorphisms \(\varphi: (\Delta_0, p) \to (\Delta_0', p')\) and \(\varphi': \pi^{-1}(\Delta_0) \to \pi'^{-1}(\Delta_0')\) such that \(\varphi \circ \pi = \pi' \circ \varphi'\) and

\[
\varphi|_{\pi^{-1}(\Delta_0 \setminus \{p\})}: \pi^{-1}(\Delta_0 \setminus \{p\}) \to \pi'^{-1}(\Delta_0' \setminus \{p'\})
\]

maps each fiber biholomorphically.

Let \(\mathcal{NH}_3\) denote the set of all equivalence classes of such 4-tuples. We denote the element of \(\mathcal{NH}_3\) also by \(F = (E, \pi, \Delta, p)\). For \(F = (E, \pi, \Delta, p) \in \mathcal{NH}_3\), \(\gamma\) denotes the element of \(\pi_1(\Delta \setminus \{p\})\) traveling once the boundary \(\partial \Delta\) by counter clockwise manner. We denote by \(F^0\) the restriction of \(\pi: E \to \Delta\) to \(\Delta \setminus \{p\}\). \(F^0\) is a non-hyperelliptic family of genus 3 and can be considered as an element of \(\mathcal{PC}_4(\Delta \setminus \{p\})\) in view of (5).

**Definition 7.1.** Define \(\text{loc.sig}^Q: \mathcal{NH}_3 \to \mathbb{Q}\) by

\[
\text{loc.sig}^Q(F) := \phi^4(\theta(F^0)_*(\gamma)) + \text{Sign}(E).
\]

Here, \(\theta(F^0)_*\) is the homomorphism from \(\pi_1(\Delta \setminus \{p\})\) to \(\Pi(4)\) induced by the classifying map \(\theta(F^0)\) for \(F^0\). It is assumed that suitable base points of \(\Delta \setminus \{p\}\) and \((\mathbb{P}(4) \setminus D)_{\text{PGL}(3)}\) are chosen. Since \(\phi^4\) is a class function, we don’t have to care about base point so we omit it.

We call a triple \((E, \pi, B)\) a 4-dimensional non-hyperelliptic fibration of genus 3 if
1. $E$ (resp. $B$) is a closed oriented $C^\infty$-manifold of dimension 4 (resp. 2) and $\pi: E \to B$ is a $C^\infty$-map,

2. there exist finitely many points $b_1, \ldots, b_n \in B$ such that the restriction of $\pi$ to $B \setminus \{b_1, \ldots, b_n\}$ is a non-hyperelliptic family of genus 3.

For $i = 1, \ldots, n$, we obtain an element of $\mathcal{NH}_3$ by restricting $\pi$ to a small closed disk neighborhood of $b_i$. We denote it by $\mathcal{F}_i$. Then, we obtain

**Theorem 7.2** (The signature formula). Let $(E, \pi, B)$ be a 4-dimensional non-hyperelliptic fibration of genus 3. Then

$$\text{Sign}(E) = \sum_{i=1}^n \text{loc}.\text{sig}^Q(\mathcal{F}_i).$$

**Proof.** For $i = 1, \ldots, n$, take a small closed 2-disk $D_i$ around $b_i$ so that they don’t intersect each other. Then $\mathcal{F}_i = (\pi^{-1}(D_i), \pi, D_i, b_i)$. We denote by $\mathcal{F}_i^0$ the restriction of $\pi$ to $D_i \setminus \{b_i\}$ and set $B_0 := B \setminus \bigcup_{i=1}^n \text{Int}D_i$. By Meyer’s signature formula, we get

$$\text{Sign}(\pi^{-1}(B_0)) = \sum_{i=1}^n \phi^4(\theta(\mathcal{F}_i^0)_*(\gamma)).$$

Using the Novikov additivity, we compute

$$\text{Sign}(E) = \text{Sign}(\pi^{-1}(B_0)) + \sum_{i=1}^n \text{Sign}(\pi^{-1}(D_i))$$

$$= \sum_{i=1}^n \phi^4(\theta(\mathcal{F}_i^0)_*(\gamma)) + \sum_{i=1}^n \text{Sign}(\pi^{-1}(D_i))$$

$$= \sum_{i=1}^n \text{loc}.\text{sig}^Q(\mathcal{F}_i).$$

\[ \square \]

**Corollary 7.3.** Let $g: E \to B$ be a non-hyperelliptic family of genus 3 over a closed oriented surface $B$. Then $\text{Sign}(E) = 0$.

We compute some examples. Comparing the following computations with those in [2] and [13], we see that their values coincide.

**Singular fiber of type I.** Let $\Delta \subset \mathbb{P}(4)$ be a closed 2-disk intersecting with $D$ only in its center $p \in \Delta$ transversely. Let $\pi_I: E_I \to \Delta$ be the restriction of $\mathcal{F} \to \mathbb{P}(4)$ to $\Delta$. Then $E_I$ is smooth by Lemma 2.4 and $\mathcal{F}_I = (E_I, \pi_I, \Delta, p)$ is a fiber germ of non-hyperelliptic family of genus 3. By Lemma 2.3 the topological type of $\pi_I^{-1}(p)$ is Lefschetz singular fiber of type I, therefore we also call $\mathcal{F}_I \in \mathcal{NH}_3$ a singular fiber germ of type I. The signature of $E_I$ is 0 and by definition, the inclusion $\Delta \setminus \{p\} \hookrightarrow \mathbb{P}(4) \setminus D \hookrightarrow (\mathbb{P}(4) \setminus D)_{\text{PGL}(3)}$ is the classifying map for $\mathcal{F}_I^0$ and the boundary of $\Delta$ is a lasso about $D$. Therefore, by Proposition 5.1, we have
Proposition 7.4.

\[
\text{loc.}\sig^Q(\mathcal{F}_I) = -\frac{5}{9}.
\]

Hyperelliptic fiber. Let \( F \in V^4 \setminus \{0\} \) be a polynomial such that \( C_F \) intersects with the non-singular conic \( C : yz - x^2 = 0 \) in 8 points, and let \( \Delta \) be a small closed 2-disk around 0 \( \in \mathbb{C} \) with the complex coordinate \( s \). Let \( S_F \) be the hypersurface in \( \Delta \times \mathbb{P}^2 \) defined by the equation

\[(yz - x^2)^2 + s^2 F(x, y, z) = 0.
\]

\( S_F \) is singular along \( C' = \{0\} \times C \). Blowing up \( \Delta \times \mathbb{P}^2 \) along \( C \), let \( \widetilde{S}_F \) be the proper transform of \( S_F \) and \( \pi: \widetilde{S}_F \to \Delta \) the composition of \( \widetilde{S}_F \to S_F \) and the first projection \( S_F \to \Delta \). Then \( \widetilde{S}_F \) is non-singular and the exceptional divisor \( \pi^{-1}(0) \) is a non-singular hyperelliptic curve of genus 3 with a natural projection onto \( C' \cong \mathbb{P}^1 \), which is a double cover.

Choose \( \Delta \) small enough so that the singular fiber of \( \pi \) is \( \pi^{-1}(0) \) only. Set \( \mathcal{F}_h = (\widetilde{S}_F, \pi, \Delta, 0) \) and call this fiber germ a hyperelliptic germ. Let \( \ell_h \) be the corresponding loop in \( \mathbb{P}(4) \setminus D \) defined by

\[\ell_h(t) = (yz - x^2)^2 + (\varepsilon e^{2\pi\sqrt{-1}t})^2 F(x, y, z), \quad 0 \leq t \leq 1,\]

where \( \varepsilon \) is the radius of \( \Delta \).

Proposition 7.5.

\[
\text{loc.}\sig^Q(\mathcal{F}_h) = \bar{\phi}^4([\ell_h]) = \frac{4}{9}.
\]

Proof. We first note that \( \text{loc.}\sig^Q(\mathcal{F}_h) = \bar{\phi}^4([\ell_h]) \) since a hyperelliptic germ is topologically trivial.

The set \( W \) of all polynomials in \( V^4 \) such that the corresponding curve intersects with \( C \) in 8 points is a non-empty Zariski open subset of \( V^4 \). Since \( [\ell_h] \) and \( \text{Sign}(S_F) \) does not change under any small perturbation of \( F \) in \( V^4 \), it suffices to show the proposition for a particular element of \( W \). But by the same reason as in Lemma 4.4 there is actually an element \( F \in W \) such that the map

\[\mathbb{P}^1 \to \mathbb{P}(4), \quad [w_0: w_1] \mapsto w_0^2(yz - x^2)^2 + w_1^2 F(x, y, z),\]

does not meet \( E \) and is transverse to \( D \), except at \( [w_0: w_1] = [1:0] \). Then for this choice of \( F \), the complex surface \( S \) in \( \mathbb{P}^1 \times \mathbb{P}^2 \) defined by the equation

\[w_0^2(yz - x^2)^2 + w_1^2 F(x, y, z) = 0,\]

has singularities only along the conic \( \{[1:0]\} \times C \). After blowing up \( \mathbb{P}^1 \times \mathbb{P}^2 \) along this conic, we obtain the proper transform \( \widetilde{S} \) of \( S \). By the choice of \( F \), \( \widetilde{S} \) is non-singular. The composition of \( \widetilde{S} \to S \) and the first projection \( S \to \mathbb{P}^1 \) is a family of algebraic curves whose all singular fiber germs are singular fiber germ of type I except the fiber germ around \([1:0]\), and the fiber germ around \([1:0]\) is a hyperelliptic germ. The invariants of \( \widetilde{S} \) are computed as: \( c_1^2(\widetilde{S}) = -6 \), \( c_2(\widetilde{S}) = 18 \), and \( \text{Sign}(\widetilde{S}) = -14 \).
Now the number of singular fiber germs of type I is equal to the total Euler contribution
\[ 18 - 2(2 - 2 \cdot 3) = 26. \]

Note that a hyperelliptic germ, which is topologically trivial, does not contribute to the Euler number. By Theorem 7.2 and Proposition 7.4 we have
\[ -14 = -\frac{5}{9} \times 26 + \text{loc.sig}(\mathcal{F}_h), \]

hence \( \text{loc.sig}(\mathcal{F}_h) = \frac{4}{9}. \)

**Singular fiber of type II.** Let \( \Delta \) be as in the previous example, and let \( S \) be the surface in \( \Delta \times \mathbb{P}^2 \) defined by
\[ z^3x + y^2x^2 + y^4 + s^6x^4 = 0. \]

\( S \) has an isolated singularity at \( p_0 = (0, [1 : 0 : 0]) \) so called a singularity of type \( \tilde{E}_8 \). The inverse image \( C_2 \) of \( 0 \in \Delta \) by the first projection \( p_1: S \to \Delta \) is a curve of geometric genus 2 with one cusp singularity.

Let \( \varpi: \tilde{S} \to S \) be the minimal resolution of the singularity of \( S \) at \( p_0 \). Then the exceptional curve is a non-singular elliptic curve \( C_1 \) with self intersection number \(-1\). If \( \Delta \) is small enough, \( \mathcal{F}_{II} = (\tilde{S}, p_1 \circ \varpi, \Delta, 0) \) is a fiber germ of non-hyperelliptic family of genus 3. The topological type of the singular fiber \((p_1 \circ \varpi)^{-1}(0)\) is obtained by the disjoint union of \( C_1 \) and \( C_2 \) by identifying a point of \( C_1 \) with the cusp singularity of \( C_2 \), that is, Lefschetz singular fiber of type II. We call \( \mathcal{F}_{II} \) a singular fiber germ of type II.

Let \( \ell_{II} \) be the corresponding loop in \( \mathbb{P}(4) \setminus D \) defined by
\[ \ell_{II}(t) = z^3x + y^2x^2 + y^4 + (\varepsilon e^{2\pi \sqrt{-1} t})^6x^4, \quad 0 \leq t \leq 1. \]

**Proposition 7.6.**
\[ \text{loc.sig}(\mathcal{F}_{II}) = \frac{1}{3}, \quad \tilde{\phi}^4(\ell_{II}) = \frac{4}{3}. \]

**Proof.** In this case \( \tilde{\phi}^4(\ell_{II}) = \text{loc.sig}(\mathcal{F}_{II}) + 1 \) because the intersection form of \( \tilde{S} \) is given by \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) hence \( \text{Sign}(\tilde{S}) = -1 \).

We perturb \( S \) slightly by adding a higher term about \( s \); consider the surface in \( \Delta \times \mathbb{P}^2 \) defined by
\[ z^3x + y^2x^2 + y^4 + s^6x^4 + s^mF(x, y, z) = 0, \]

where \( m \) is an integer \( \geq 7 \) and \( F \) is a polynomial in \( V^4 \). The singularity of this surface remains at the origin and is still of type \( \tilde{E}_8 \). Taking the minimal resolution of this singularity and taking \( \Delta \) to be smaller if needed, we obtain a new fiber germ \( \mathcal{F}'_{II} \) and a new loop \( \ell'_{II} \) in \( \mathbb{P}(4) \setminus D \). This perturbation does not influence the value of \( \tilde{\phi}^4 \) and the topology of the fiber neighborhood of the singular fiber. So it suffices to compute \( \text{loc.sig}(\mathcal{F}'_{II}) \).

Let \( S' \) be the complex surface in \( \mathbb{P}^1 \times \mathbb{P}^2 \) defined by the equation
\[ w_0^m(z^3x + y^2x^2 + y^4) + w_0^{m-6}w_1^6x^4 + w_1^mF(x, y, z) = 0, \]

and let \( \tilde{S}' \to S' \) be the minimal resolution of the singularity of \( S' \) at \( p_0 = ([1 : 0], [1 : 0 : 0]) \). If a generic \( F \) is chosen, then \( \tilde{S}' \) is non-singular and the singular fiber germs of the family
of algebraic curves $\tilde{S}' \to S' \to \mathbb{P}^1$ are all of type I except the fiber germ around $[1 : 0]$, and the fiber germ around $[1 : 0]$ is $\mathcal{F}'_{II}$. The invariants of $\tilde{S}'$ are computed as: $c_2^k(\tilde{S}') = 9m - 17$, $c_3(\tilde{S}') = 27m - 19$, and $\text{Sign}(\tilde{S}') = -15m + 7$.

Now the number of singular fiber germs of type I is equal to

$$27m - 19 - 2(2 - 2 \cdot 3) - 1 = 27m - 12.$$ 

This time, the fiber over $[1 : 0]$ do contribute to the Euler number.

By the signature formula,

$$-15m + 7 = -\frac{5}{9} \cdot (27m - 12) + \text{loc.sign}^\mathbb{Q}(\mathcal{F}'_{II}).$$

Thus we obtain $\text{loc.sign}^\mathbb{Q}(\mathcal{F}_{II}) = \text{loc.sign}^\mathbb{Q}(\mathcal{F}'_{II}) = \frac{1}{3}$. $\square$

### 8 Appendix

In this appendix we give a definition of Meyer’s signature cocycle in the form used in the present paper and review its properties. For details, see W. Meyer’s original paper [12].

We first explain the topological monodromy of surface bundles. Let $\pi: E \to B$ be an oriented $\Sigma_g$ bundle whose structure group is the group of all orientation preserving diffeomorphisms of $\Sigma_g$. Choose a base point $b_0 \in B$ and fix an identification $\phi: \Sigma_g \xrightarrow{\simeq} \pi^{-1}(b_0)$. For each based loop $\ell: [0, 1] \to B$ the pull back $\ell^*(E) \to [0, 1]$ of $\pi: E \to B$ by $\ell$ is trivial. Hence there exist a trivialization $\Phi: \Sigma_g \times [0, 1] \to \ell^*(E)$ such that $\Phi(x, 0) = \phi(x)$. By assigning the isotopy class of $\Phi(x, 1)^{-1} \circ \phi$ to the homotopy class of $\ell$, we obtain a map $\chi: \pi_1(B, b_0) = \pi_1(B) \to \Gamma_g$. This map becomes a homomorphism under the conventions; 1) for any two mapping classes $f_1$ and $f_2$, the multiplication $f_1 \circ f_2$ means that $f_2$ is applied first, 2) for any two homotopy classes of based loops $\ell_1$ and $\ell_2$, their product $\ell_1 \cdot \ell_2$ means that $\ell_1$ is traversed first. $\chi$ is called the topological monodromy of $\pi: E \to B$ and determined up to inner automorphisms of $\Gamma_g$.

Let $P$ denote the pair of pants, i.e., $P = S^2 \setminus \bigcup_{i=1}^{3} \text{Int}D_i$ where $D_i, i = 1, 2, 3$ are the three disjoint closed disks in the 2-sphere $S^2$. Choose a base point $p_0 \in \text{Int}P$ and fix a based loop $\ell_1$ and $\ell_2$ such that $\ell_1$ is free homotopic to the loop traveling the boundary $\partial D_i$ by counter clockwise manner ($i = 1, 2$). For $(f_1, f_2) \in \Gamma_g \times \Gamma_g$, we can construct an oriented $\Sigma_g$ bundle $E(f_1, f_2)$ over $P$ such that the topological monodromy $\chi: \pi_1(P) \to \Gamma_g$ sends $[\ell_i]$ to $f_i$ for $i = 1, 2$. (If $g \geq 2$, the isomorphism class of this bundle is unique.) $E(f_1, f_2)$ is a compact $C^\infty$-manifold of dimension 4 and has the natural orientation induced by the orientation of $P$ and that of the fibers. Then the signature of $E(f_1, f_2)$ is defined and we set

$$\tau_g(f_1, f_2) := -\text{Sign}(E(f_1, f_2)).$$

This turns out to be well defined even when $g = 1$, and $\tau_g: \Gamma_g \times \Gamma_g \to \mathbb{Z}$ is called Meyer’s signature cocycle. The basic properties of $\tau_g$ are

1) $\tau_g(f_1 f_2, f_3) + \tau_g(f_1, f_2) = \tau_g(f_1, f_2 f_3) + \tau_g(f_2, f_3);$
2) $\tau_g(f_1, 1) = \tau_g(1, f_1) = \tau_g(f_1, f_1^{-1}) = 0;$
3) $\tau_g(f_1^{-1}, f_2^{-1}) = -\tau_g(f_1, f_2);$
4) $\tau_g(f_1, f_2) = \tau_g(f_2, f_1);$
\( \tau_g(f_3 f_1 f_3^{-1}, f_3 f_2 f_3^{-1}) = \tau_g(f_1, f_2), \)

where \( f_1, f_2, \) and \( f_3 \) are elements of \( \Gamma_g. \)

For an oriented \( \Sigma_g \) bundle \( \pi: E \to B \) and a choice of base point \( b_0 \) of \( B, \) we obtain a 2-cocycle \( \chi^* \tau_g \) of \( \pi_1(B) = \pi_1(B, b_0) \) by pulling back \( \tau_g \) by the topological monodromy \( \chi: \pi_1(B) \to \Gamma_g. \) Although \( \chi \) is determined only up to conjugacy, \( \chi^* \tau_g \) is uniquely determined by the property (5) of \( \tau_g \) above. Moreover, \( \chi^* \tau_g \) does not depend on the choice of base point of \( B \) in the following sense: suppose \( b_0' \in B \) and \( b_0 \) are in the same path component of \( B \) then under any isomorphism \( \pi_1(B, b_0) \cong \pi_1(B, b_0') \) using a path from \( b_0 \) to \( b_0' \), two cocycles of \( \pi_1(B, b_0) \) and \( \pi_1(B, b_0') \) defined as the pull back of \( \tau_g \) by topological monodromies, correspond to each other.

Let \( G \) be a group and \( \varphi: G \to \Gamma_g \) a homomorphism.

**Definition 8.1.** A \( \mathbb{Q} \)-valued 1-cochain \( \phi: G \to \mathbb{Q} \) is called a Meyer function with respect to the pull back \( \varphi^* \tau_g \) of \( \tau_g \) by \( \varphi \) if it satisfies \( \delta \phi = \varphi^* \tau_g, \) i.e., \( \phi \) cobounds the 2-cocycle \( \varphi^* \tau_g. \)

If a Meyer function exists on \( G, \) the cohomology class \( \varphi^*[\tau_g] \in H^2(G; \mathbb{Z}) \) is torsion. The following properties of \( \phi \) are easily derived by the above properties of \( \tau_g \) (see also [7, Proposition 3.1]).

**Lemma 8.2.** If \( \phi \) is a Meyer function with respect to \( \varphi^* \tau_g, \) we have

1. \( \phi(xy) = \phi(x) + \phi(y) - \varphi^* \tau_g(x, y); \)
2. \( \phi(1) = 0; \)
3. \( \phi(x^{-1}) = -\phi(x); \)
4. \( \phi(yxy^{-1}) = \phi(x), \)

where \( x, y \in G. \)

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