The geometric phase and the dynamics of quantum phase transition induced by a linear quench

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Abstract
We have analysed here the role of geometric phase in the dynamical mechanism of quantum phase transition in the transverse Ising model. We have investigated the system when it is driven at a fixed rate characterized by a quench time $\tau_q$ across the critical point from a paramagnetic to ferromagnetic phase. Our argument is based on the fact that the spin fluctuation occurring during the critical slowing down causes a random fluctuation in the ground-state geometric phase at the critical regime. The correlation function of the random geometric phase determines the excitation probability of the quasiparticles, which are excited during the transition from the initial paramagnetic to the ferromagnetic phase. This helps us to evaluate the number density of the kinks formed during the transition, which is found to scale as $\tau_q^{-1/2}$. In addition, we have also estimated the spin–spin correlation at criticality.

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1. Introduction

The celebrated geometric phase (GP), commonly known as the Berry phase [1], arises when a quantum system is adiabatically transported around a closed circuit in a parameter space. In this transportation, besides the familiar dynamical phase, a GP appears when due to nonintegrability some variables fail to return to their original values. The geometrical meaning of this phase is precisely the holonomy in a Hermitian line bundle and naturally defines a $U(1)$ connection [2]. The Berry connection and the GP can be defined even if the condition of adiabaticity is relaxed [3]. Subsequently, the generalization of the Berry phase was introduced to the non-Abelian case [4] where a collection of degenerate states were cycled. An analogous phase in classical mechanics was also pointed out [5, 6]. The case with non-Hermitian evolution was also studied [7]. The interpretation [8] of the driving parameters as dynamical variables,
whose evolution was influenced by the same geometrical objects as the Berry phase, has also been noted. This made clear [9] that the reaction of the GP on the parameters takes the form of an abstract magnetic field which was explored earlier [10]. A classic connection was made [11] between the Berry phase and the GP in beams of light [12], which was formulated for two-state systems in the optical picture by Pancharatnam [13]. The concept of the Berry phase has now become very important in quantum mechanics with wide applications in various fields [14, 15]. In particular, GPs have been associated with a variety of condensed matter and solid-state phenomena [16–22]. Besides theoretical analyses, it has been a subject of experimental investigations also [14, 23–29].

A characteristic feature of all non-trivial geometrical evolutions is that they demonstrate the presence of non-analytic points in the energy spectrum where the state is degenerate. It is be mentioned here that before the discovery of the geometrical phase, the sign change associated with the molecular electronic degeneracies was understood [30, 31]. The presence of degeneracy at some point is accompanied by the curvature in its immediate neighbourhood, and a state evolving around a closed path can detect it. These aspects indicate an intuitive connection between the criticality around the degeneracy point and the GP. Recently, the GP has been studied [32–34] in the context of quantum phase transitions (QPTs) [35, 36] to investigate the regions of criticality. QPTs exhibit the crossover of ground states at a critical value of some external parameter at zero temperature and are caused by the presence of degeneracy in the energy spectrum. The GP, which measures the curvature associated with the degeneracy point, can contemplate the energy structure of the system and essentially captures certain characteristic features of QPT. Pachos and Carollo [37] have exhaustively studied the role of GP as a tool to probe QPTs in many-body systems and suggested the feasibility of the experimental realization also. The GP has also been used as a topological test to reveal QPT [34]. Recently it has been shown that the GP associated with the ground state of the XY model exhibits universality in the critical properties and obeys the scaling behaviour near the critical point [33], which established a significant connection between the GP and QPT.

The interrelation between the GP and QPT has so far been concerned with the detection of criticality through this phase and focus on equilibrium scalings of various physical quantities near the critical point. However, till now, the role of the GP in the dynamical mechanism of QPT has not been explored. It is our novel attempt here to show that the GP plays a crucial role in the dynamical mechanism of the QPT when it is induced by a linear quench, where non-equilibrium dynamics while passing through a critical point becomes relevant. The Kibble–Zurek mechanism (KZM) [38, 39], which is a universal theory of the dynamics of the second-order phase transition, has recently been explored in the zero temperature limit to study the dynamics of the QPT. The behaviour of the quantum Ising model in one dimension has been investigated during a quench caused by gradual tuning of the transverse bias field [40, 41]. The system is driven at a fixed rate characterized by the quench time \( \tau_g \) across the critical point from paramagnetic to ferromagnetic phase. In agreement with the KZM (which recognizes that evolution is approximately adiabatic far away, but becomes approximately impulse sufficiently near the critical point), the quantum state of the system after the transition exhibits a characteristic correlation length \( \xi \) proportional to the square root of the quench time \( \tau_g \): \( \xi \propto \sqrt{\tau_g} \). The inverse of this correlation length is known to determine the average density of defects formed during the transition. We focus here on the role of the GP in the dynamical mechanism of QPT induced by a linear quench and have established a nontrivial connection between the KZM and the GP. It is pointed out that the quench-induced QPT in a spin chain generates spin fluctuation, which causes the randomization of the time-dependent magnetic field so the GP acquires random values. The two-point correlation of the randomized GP corresponds to the excitation probability of the quasiparticles. From this we
have estimated the number density of defects generated in the transverse Ising model during the phase transition.

The plan of the paper is as follows. In section 2, we review the GP (Berry phase) of the ground state of the transverse Ising model as discussed in [32]. Section 3 deals with the discussion of the GP during critical slowing down induced by a linear quench. In section 4 we estimate the number density of kinks from the two-point correlation of the Berry phase factor. Section 5 is devoted to discussion of the spin–spin correlation at the critical point.

2. The geometric phase of the ground state of the transverse Ising model

The Hamiltonian of a one-dimensional transverse Ising model is given by

$$H = -J \sum_{j=-M}^{M} \sigma_x^j \sigma_x^{j+1} + \lambda \sigma_x^j$$  \hspace{1cm} (1)

with $M = \frac{N-1}{2}$, where $N$ (odd) denotes the number of sites, $\lambda$ is the external field and $\sigma$‘s are the standard Pauli matrices. We assume periodic boundary condition. In the transverse Ising model, the GP of the ground state is evaluated by applying a rotation of an angle $\alpha$ around the $z$-axis to each spin [32, 37]. A new set of Hamiltonians $H_\alpha$ is constructed from the Hamiltonian (1) as

$$H_\alpha = U(\alpha) H U^\dagger(\alpha)$$  \hspace{1cm} (2)

where

$$U(\alpha) = \prod_{j=-M}^{+M} \exp \left( \frac{i\alpha \sigma_z^j}{2} \right).$$  \hspace{1cm} (3)

We use the standard Jordan–Wigner transformation, which maps the spins to one-dimensional spinless fermions,

$$a_j = \left( \prod_{i<j} \sigma_z^i \right) \sigma_j^j$$  \hspace{1cm} (4)

and the Fourier transforms of the fermionic operator

$$d_k = \frac{1}{\sqrt{N}} \sum_j a_j \exp \left( \frac{-2\pi ijk}{N} \right) \quad \text{with} \quad k = -M, \ldots, +M$$  \hspace{1cm} (5)

such that the Hamiltonian $H$ can be diagonalized by transforming the fermionic operators in momentum space and then using the Bogoliubov transformation. This yields

$$H = \sum_k 2J \Lambda_k \left( c_k^\dagger c_k - \frac{1}{2} \right) = \sum_k \epsilon_k \left( c_k^\dagger c_k - \frac{1}{2} \right),$$  \hspace{1cm} (6)

where

$$c_k = d_k \cos \theta_k - \frac{id_k^\dagger}{2} e^{2i\alpha} \sin \theta_k$$  \hspace{1cm} (7)

is the fermionic annihilation operator in the momentum space, with

$$\cos \theta_k = \frac{\cos \frac{2\pi k}{N} - \lambda}{\Lambda_k},$$  \hspace{1cm} (8)

$$\Lambda_k = \frac{\sqrt{\lambda^2 + 4J^2}}{2},$$  \hspace{1cm} (9)

$$\epsilon_k = \sqrt{\lambda^2 + 4J^2} \cos \frac{2\pi k}{N},$$  \hspace{1cm} (10)

$$\theta_k = \frac{\Lambda_k}{\sqrt{\lambda^2 + 4J^2}},$$  \hspace{1cm} (11)

$$\Lambda_k = \frac{\sqrt{\lambda^2 + 4J^2}}{2},$$  \hspace{1cm} (12)

$$\epsilon_k = \sqrt{\lambda^2 + 4J^2} \cos \frac{2\pi k}{N},$$  \hspace{1cm} (13)

$$\theta_k = \frac{\Lambda_k}{\sqrt{\lambda^2 + 4J^2}},$$  \hspace{1cm} (14)

$$\Lambda_k = \frac{\sqrt{\lambda^2 + 4J^2}}{2},$$  \hspace{1cm} (15)

$$\epsilon_k = \sqrt{\lambda^2 + 4J^2} \cos \frac{2\pi k}{N},$$  \hspace{1cm} (16)

$$\theta_k = \frac{\Lambda_k}{\sqrt{\lambda^2 + 4J^2}},$$  \hspace{1cm} (17)
and
\[ \Lambda_k = \sqrt{\left( \lambda - \cos \left( \frac{2\pi k}{N} \right) \right)^2 + \sin^2 \left( \frac{2\pi k}{N} \right)} \]  

the energy spectrum.

The ground state \(|\psi\rangle\) of the system is expressed as
\[ |\psi\rangle = \prod_{k=1}^{M} \left( \cos \frac{\theta_k}{2} |0\rangle_k - i \exp(2i\alpha) \sin \frac{\theta_k}{2} |1\rangle_k \right), \]

where \(|0\rangle_k\) and \(|1\rangle_k\) are the vacuum and single fermionic excitation of the \(k\)th momentum mode respectively. The GP of the ground state, accumulated by varying the angle \(\alpha\) from 0 to \(\pi\), is found to be \([32, 37]\)
\[ \gamma = -i \int_0^\pi \left( g \frac{\partial}{\partial \alpha} |g\rangle \right) d\alpha = \sum_{k>0} \pi (1 - \cos \theta_k). \]

3. The ground-state geometric phase in a varying magnetic field and at criticality

We start with the review of the geometric phase of a spin-1/2 system in a varying magnetic field. The Hamiltonian of a single spin-1/2 system in the presence of an external time-dependent magnetic field is given by
\[ H(t) = \frac{1}{2} \mathbf{B}(t) \cdot \mathbf{\sigma}, \]

where \(\mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\) are the Pauli matrices and \(\mathbf{B}(t) = B_0 \mathbf{n}(\theta(t), t)\), \(\mathbf{n}(\theta(t), t)\) being the unit vector with \(\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\). If the external magnetic field is varied adiabatically, the instantaneous spin states can be expressed in the \(\sigma_z\) basis as
\[ |\uparrow_n; t\rangle = \cos \frac{\theta}{2} |\uparrow_z\rangle + e^{i\phi} \sin \frac{\theta}{2} |\downarrow_z\rangle \]
\[ |\downarrow_n; t\rangle = \sin \frac{\theta}{2} |\uparrow_z\rangle - e^{i\phi} \cos \frac{\theta}{2} |\downarrow_z\rangle, \]

where \(|\uparrow_z\rangle\), \(|\downarrow_z\rangle\) are the eigenstates of the \(\sigma_z\) operator.

For a cyclic time evolution, i.e. for \(\mathbf{B}(T) = \mathbf{B}(0)\), apart from the dynamical phase, the eigenstates acquire a GP also, and we can write
\[ |\uparrow_n(T)\rangle = e^{i\delta} e^{i\gamma} |\uparrow_n(0)\rangle, \]

where the dynamical phase \(\delta = \int_0^T B_0(t) \, dt\) and the GP \(\gamma = \oint \mathbf{A}^\top \cdot d\mathbf{\lambda}; \mathbf{\lambda}\) is the set of control parameters and \(\mathbf{A}^\top = i(\uparrow_n |\nabla_{\mathbf{n}}| \uparrow_n)\) is the so-called Berry connection. In our case, \(\mathbf{\lambda} = (\theta, \phi), \)
and a straightforward calculation shows that
\[ \gamma_{\uparrow} = -\gamma_{\downarrow} = \pi (1 - \cos \theta). \]

It may be pointed out that though the eigenenergies depend on \(B_0(t)\), the eigenstates depend on \(\mathbf{n}(\theta(t))\) only. The GP is proportional to the solid angle subtended by \(\mathbf{B}\) with respect to the degeneracy \(\mathbf{B} = 0\). It may also be noted that fluctuation in the external magnetic field will obviously induce fluctuation in the GP \((\gamma_{\uparrow} + \gamma_{\downarrow})\) of the corresponding spin eigenstate through fluctuation in \(\cos \theta\) \([42]\).

This formulation may be extended for a chain of spin-1/2 system with XY type of interactions, which exhibits QPT for a critical magnetic field. Subsequently, we can analyse the dynamics of the GP when there is a gradual slowing down of the external magnetic field.
We consider that the quantum fluctuation responsible for a QPT induces spin fluctuation [43], which eventually generates random fluctuation in the ground-state Berry phase. In the critical regime, the Berry phase of the ground state of the relevant system undergoes a random fluctuation, which is assumed to be small for a slow transition and large enough for a fast transition. In the present communication, we have used this specific feature of the GP of the ground state of the transverse Ising model during its critical slowing down and showed that the two-point correlation of the Berry phase factors may be used to determine the excitation probability of the quasiparticles and hence the density of defects formed during the transition.

Let the system governed by the Hamiltonian (1) be initially in its ground state at the large values of $\lambda \gg 1$, a paramagnet with all spins polarized along the $x$-axis. Gradually, when $\lambda$ is ramped down to very small values $\lambda \ll 1$, there may be two degenerate ground states representing ferromagnets either with all up spins or all down spins along the $z$-axis. For a large number of spins, i.e. $N \rightarrow \infty$, the energy gap at the critical point $\lambda = \lambda_c = 1$ tends to zero implying that the system is excited while passing (with a finite speed) through the critical point and the system settles down in the final state with $\lambda = 0$. As a result the system in the final state represents a quantum superposition of states as

$$|\ldots \downarrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \ldots \rangle$$

with finite domains of spins pointing up or down. The domains are separated by kinks or antikinks, where the polarization of spin changes its orientation. During the excitation, spin flips occur at different domains of the chain so that in these regions there are two nearest-neighbour spins with opposite orientations. The average density of kinks depends on the transition rate. For a slow transition, the number of kinks present is small. For a fast transition, the orientation of individual spins becomes random, uncorrelated with their nearest neighbours, and the density of kinks becomes large. We would like to explore the situation when the time-dependent field $\lambda(t)$ driving the transition is given by

$$\lambda(t)(t < 0) = -\frac{t}{\tau_q}, \quad (16)$$

where the quench time $\tau_q$ is an adjustable parameter. We may note that before the quench ($-t > \tau_q$), the system is in the symmetric (paramagnetic) phase. During the quench time $\tau_q$, it undergoes a transition towards the symmetry-broken ferromagnetic phase and finally at the end of the quench ($t = 0$) it settles down.

The total time-dependent Hamiltonian of the system may be written as

$$H(t) = H_0(t) + H_{flip}(t). \quad (17)$$

If $N$ pairs of quasiparticles are excited with momentum $k_0, -k_0$ with ($i = 1, 2, \ldots, N$) at a certain quench time $t = \tau_q$, the time-dependent spin-flip interaction is given by

$$H_{flip}(t) = -2J \sum_{k=1}^{\Delta k_0(t)} \frac{\sigma^z_i}{2} \sigma^z_i \quad (18)$$

with

$$\sigma^z_i = \downarrow \uparrow \text{ or } \uparrow \downarrow$$

and $\Delta k_0(t)$ being a time-dependent dimensionless parameter characterizing the spin-flip interaction strength. Here the summation includes the spins which get excited during the transition and finally are flipped with opposite orientations. This term corresponds to the standard Landau–Zener Hamiltonian [44]. The first term in equation (17) is given by

$$H_0(t) = -J \left( \sum_{n} \sum_{i=1}^{N_n} \sigma^z_i \sigma^z_{i+1} + \lambda(t) \sigma^z_i \right), \quad (19)$$
\(N_n\) being the number of spins between the flipped spins in the \(n\)th domain and \(n\) is the number of domains in the whole system. This corresponds to the Hamiltonian of the surrounding Ising spin chain. We have neglected the coupling between the flipped spins with the surrounding Ising chain.

In the process of critical slowing down certain spins, which are excited, undergo fluctuations around the \(z\)-axis. In fact, the spin fluctuation is instrumental in a metal–insulator transition which in zero temperature limit corresponds to QPT [43]. When a spin gets excited, it passes through the process of the change of its alignment around the \(z\)-axis until it settles with opposite orientation at the end of the quench at \(t = 0\). This process of the change of alignment of spins at the quench time can be depicted as a fluctuation of the spin when magnetic moments appear with random orientations.

It may be noted from equation (15) that during critical slowing down the fluctuation of the angle \(\theta\), associated with spin fluctuation, makes the Berry phase a fluctuating one. Indeed, when a quasiparticle associated with a spin rotates around a closed path, the Berry phase, which is given by the holonomy, corresponds to the number of magnetic flux lines enclosed by the loop. As during the quench, the time-dependent magnetic field is randomized due to the spin fluctuation, the Berry phases acquired by the spin eigenstates take random values.

To evaluate the Berry phase acquired by the spin eigenstate associated with the quasiparticle undergoing excitation near the critical point, we note that at the quench time \(\tau_q\), spins tend to align along the \(z\)-axis as the initial paramagnetic system \((\lambda \gg 1)\) transits to a ferromagnetic one \((\lambda \ll 1)\) passing through the critical point \((\lambda_c = 1)\). In fact the critical Ising model corresponds to a free fermionic field theory [45] with spins aligned along the \(z\)-axis. To determine the Berry phase at the quench time \(\gamma_{k_0}(\tau_q)\) for a spin eigenstate corresponding to a quasiparticle with momentum \(k_0\), we note that initially at \(\lambda \gg 1\), the system is in the paramagnetic state with all the spins aligned along the \(x\)-axis. We consider that the time-dependent magnetic field rotates around the \(x\)-axis with an angular velocity \(\omega_0\). At time \(t = \tau_q\) \((\tau_q = \frac{2\pi}{\omega_0})\), the spin state with momentum mode \(k_0\) picks up a GP

\[\gamma_{k_0}(\tau_q) = \pi (1 - \cos \theta_{k_0}(\tau_q))\]  

with \(\theta_{k_0}(\tau_q) = \pi/2\) indicating that at \(t = \tau_q\), the GP \(\gamma_{k_0}(\tau_q)\) acquired by the spin eigenstate is \(\pi\).

We consider that during critical slowing down \(N\) pairs of quasiparticles with momentum modes \(k_0, -k_0\) (with \(i = 1, 2, \ldots, N\)) are excited with probability \(p_{k_0}\). The corresponding spins are flipped with opposite orientations and settle down at \(t = 0\) implying \(\theta_{k_0}(t = 0) = \pi\), such that the random variable \(\gamma_{k_0}(t = 0)\) takes the value \(2\pi\) with probability \(p_{k_0}\). In essence, the Berry phase acquired by a spin eigenstate at \(t = \tau_q\) and \(t = 0\) is given by the random values

\[\gamma_{k_0}(\tau_q) = \pi, \quad \gamma_{k_0}(t = 0) = 2\pi\]  

with probability \(p_{k_0}\) and 0 with probability \(1 - p_{k_0}\). We can now estimate the number density of kinks generated in the system due to a quench-induced QPT.

### 4. Estimation of the number density of defects

For our further analysis, we define a (dimensionless) random variable

\[\phi_{k_0}(t) = \frac{\gamma_{k_0}(t)}{2\pi}\]  

such that at \(t = \tau_q\) and \(t = 0\), the Berry phase factors \(\phi_{k_0}(t)\) attain the values

\[\phi_{k_0}(t = \tau_q) = 1/2, \quad \phi_{k_0}(t = 0) = 1.\]
Let us assume that the Berry phase factor \( \phi_{k_0}(t) \) follows the simplest stochastic differential equation \[46\]
\[
\text{d}\phi_{k_0}(t) = -\omega_{k_0}\phi_{k_0}(t) \text{d}t + \text{d}\eta(t),
\]
where \( \omega_{k_0} \) is the frequency associated with the energy \( \epsilon_{k_0} \) of the quasiparticle near the critical point and \( \eta(t) \) denotes the fluctuation. Let us consider \( \eta(t) \) as the Gaussian white noise with moments
\[
\langle \text{d}\eta(t) \rangle = 0 \quad \text{(25)}
\]
\[
\langle \text{d}\eta(t) \text{d}\eta(t') \rangle = \delta(t - t') \text{d}t' \quad \text{(26)}
\]
Using equation (24) and performing the averages over the noise, the correlations of the time-dependent Berry phase factors can be evaluated \[47\]. This gives
\[
\langle \phi_{k_0}(t) \rangle = 0 \quad \text{(27)}
\]
\[
\langle \phi_{k_0}(t)\phi_{k_0}(t') \rangle = \frac{1}{2} e^{-\omega_{k_0}(t - t')} \quad \text{(28)}
\]
This leads us to derive the two-point correlation function of the Berry phase factor \( \phi_{k_0} \) during the critical slowing down from \( t = \tau_q \) to \( t = 0 \)
\[
\langle \phi_{k_0}(\tau_q)\phi_{k_0}(0) \rangle = \frac{1}{2} e^{-\omega_{k_0}\tau_q} \quad \text{(29)}
\]
The correlation is large for fast transition (small \( \tau_q \)) and small for slow transition (large \( \tau_q \)).

As \( N \) pairs of quasiparticles are excited during the critical slowing down with momentum modes \( k_0, -k_0 \) (\( i = 1, 2, \ldots, N \)) with probability \( p_{k_0} \), we note from relation (23) that the random variable \( |2\phi_{k_0}(t = \tau_q)\phi_{k_0}(t = 0)| \) picks up the value 1 with probability \( p_{k_0} \) and 0 with probability \( 1 - p_{k_0} \). This suggests the excitation probability to be given by
\[
p_{k_0} = 2 \langle \phi_{k_0}(\tau_q)\phi_{k_0}(0) \rangle \quad \text{(30)}
\]
Using equation(29), this can be explicitly written as
\[
p_{k_0} = 2 \langle \phi_{k_0}(\tau_q)\phi_{k_0}(0) \rangle = e^{-\omega_{k_0}\tau_q} = e^{-2\pi\tau_q}(\hbar = 1). \quad \text{(30)}
\]
One should note that in the final state at \( t = 0 \), the flipped spins correspond to the values of \( \theta_{k_0} = \pi \), with \( \cos \theta_{k_0} = -1 \). Now if we choose the lattice spacing \( \alpha = \frac{2\pi}{N} = 1 \), in the critical region with \( \lambda = 1 \), the constraint \( \cos \theta_{k_0} = -1 \) suggests from equation (8) that
\[
\Lambda_{k_0} = 1 - \cos k_0 \quad \text{(31)}
\]
The expression for the energy then yields
\[
\epsilon_{k_0} = 2J\Lambda_{k_0} = 2J(1 - \cos k_0) = 4J\sin^2 \frac{k_0}{2} \sim Jk_0^2 \text{ for small } k_0. \quad \text{(32)}
\]
From this, we find
\[
p_{k_0} = e^{-2\pi Jk_0^2\tau_q}(\hbar = 1). \quad \text{(33)}
\]
The number of kinks can now be evaluated as
\[
2N = \sum_{k_0} p_{k_0}. \quad \text{(34)}
\]
For large number of spins, i.e. in the thermodynamic limit as \( N \rightarrow \infty \), the expectation value of the density of kinks is given by
\[
n = \lim_{N \rightarrow \infty} \left( \frac{2N}{N} \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_{k_0} \text{d}k_0 = \frac{1}{2\pi}(2J\tau_q)^{-\frac{1}{2}}, \quad \text{(35)}
\]
which shows that the density of kinks scales as \( \tau_q^{-\frac{1}{2}} \). In this novel framework, the dynamics of the GP associated with the flipped spins helped us to estimate the density of defects produced in a system due to QPT and the result is found to be in concordance with that derived by a standard technique [41].
5. Spin–spin correlation at the critical point

We can now derive the spin–spin correlation at the critical point in terms of the space-dependent correlation of the GP. In the initial ground state, with $\lambda \to \infty$, due to symmetry $\langle \sigma^z \rangle = 0$ and the transverse magnetization $\langle \sigma^x \rangle = 1$. For a slow quench, the final magnetization, $\langle \sigma^x \rangle \to 0$, and in the final state at $\lambda = 0$, $\langle \sigma^x \rangle = 0$. It may be noted that for a certain quench time $\tau_q$, $\langle \sigma^x \rangle$ will depend on the density of kinks. In the region between a kink–antikink pair, the direction of the spin orientation remains the same corresponding to the direction of orientation of the flipped spin. If the kink–antikink pair maintains more or less the same distance from each other, the system represents a kink–antikink chain with a lattice constant $\sim \hat{\xi}$, where $\hat{\xi} = \sqrt{\tau_q}$ is the Kibble–Zurek correlation length. Indeed the expected value $\langle \sigma^x_i \rangle$ for a particular quench time $\tau_q$ is [48]

$$\langle \sigma^x_i \rangle \sim \frac{1}{2\pi \sqrt{2\tau_q}} \quad (J = 1),$$

which is valid for $\tau_q \gg 1$. We have just analysed how the variation in the direction of the spin orientation during the quench time $\tau_q$ is reflected in the GP factor $\phi_{k_0}(\tau_q)$. In fact, the direction of the spin orientation at the spatial position $r$ denoted by site $i$ near the critical point determines the GP $\gamma(r)$ acquired by the spin eigenstate after a cyclic evolution. If $r'$ denotes the spatial coordinate of the spin at site $i+R$, then we can define the two-point correlation of the Berry phase factor near the critical point as

$$C_{xx}^\gamma = \langle \phi(r)\phi(r') \rangle,$$

(37)

where $\phi(r) = \frac{1}{2\pi} \gamma(r)$. Correlation (37) effectively corresponds to the the spin–spin correlation at the critical point. As already mentioned, the Berry phase undergoes a stochastic fluctuation near the critical point. In terms of spatial variable, we consider that the Berry phase factor corresponding to the spin state at position $r$ follows the stochastic differential equation

$$d\phi(r) = -\frac{\phi(r)}{\hat{\xi}} dr + d\eta(r).$$

(38)

Here $\eta(r)$ is the fluctuation and $\hat{\xi}$ is the KZ correlation length, which is the relevant length scale in the system. We consider that $\eta(r)$ represents Gaussian distribution with white noise and satisfies the moments

$$\langle d\eta(r) \rangle = 0$$

(39)

$$\langle d\eta(r) d\eta(r') \rangle = \delta(r - r') \, dr'.$$

(40)

From this the correlation function of the Berry phase factor can be evaluated [47] and is given by

$$\langle \phi(r) \rangle = 0$$

(41)

$$\langle \phi(r)\phi(r') \rangle = \frac{1}{2} e^{-\frac{(r - r')^2}{\hat{\xi}^2}}.$$  

(42)

This implies that

$$C_{xx}^\gamma = \langle \phi(r)\phi(r') \rangle = \frac{1}{2} e^{-\frac{\bar{\hbar}}{\tau_q}} = 0.5 e^{-\frac{\bar{\hbar}}{\tau_q}} \quad (\bar{\hbar} = J = 1)$$

(43)

This can be compared with the numerical estimate of spin–spin correlation at the critical point in [48]:

$$C_{xx}^\gamma \approx 0.44 \frac{e^{-2.03 \frac{\bar{\hbar}}{\tau_q}}}{\tau_q},$$

(44)

which is valid for $\tau_q \gg 1$ and $R \gg \hat{\xi}$. 

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It is noted that in contrast with result (43), relation (42) is a generalized one and does not involve any restriction on the values of $\tau_q$ and $R$. It is observed that for very small $\tau_q$ ($\tau_q \ll 1$) the spin–spin correlation tends to zero which is consistent with the fact that for a fast transition the spins behave almost uncorrelated. Moreover, contrary to equation (43), we have found that there is a finite correlation for large $\tau_q$ and large $R$.

6. Conclusion

The KZM in the second-order phase transition suggests that the formation of defects is driven by thermal fluctuation. In the analysis of spin models, the KZ theory has been extended to study the quench-induced QPT driven by quantum fluctuation. Equivalently, we have argued that when the KZM is encompassed in the QPT of spin systems, the defect formation is driven by the fluctuation of the Berry phase factor associated with the ground state. Recently, we have studied an XY spin chain in a slowly varying time-dependent magnetic field and investigated the behaviour of the GP during a linear quench caused by a gradual turning off of the magnetic field [49]. In the present formulation, we have argued that in QPT induced by a quench the Berry phase factor undergoes a random fluctuation at criticality. The two-point correlation function of the Berry phase factor determines the dynamics of the QPT induced by a quench and controls the formation of defects. It is found that the number of kinks generated in this transition scales as $\tau_q^{-1/2}$, where $\tau_q$ is the quench time. The result is in agreement with that obtained through other formulations. Besides, the space-dependent correlation of the random Berry phase gives us the estimation of the spin–spin correlation at the critical point. It is hoped that this work may initiate investigations in relation to the role of the GP in the dynamical mechanism of QPT.

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