Nanowire Superinductance Fluxonium Qubit

T. M. Hazard,1,∗ A. Gyenis,1,∗ A. Di Paolo,2,∗ A. T. Asfaw,1 S. A. Lyon,1 A. Blais,2,3 and A. A. Houck1

1Department of Electrical Engineering, Princeton University, Princeton, New Jersey 08544
2Institut quantique and Département de Physique, Université de Sherbrooke, Sherbrooke J1K 2R1 QC, Canada
3Canadian Institute for Advanced Research, Toronto, ON, Canada

(Dated: May 4, 2018)

Disordered superconducting materials provide a new capability to implement novel circuit designs due to their high kinetic inductance. Here, we realize a fluxonium qubit in which a long NbTiN nanowire shunts a single Josephson junction. We explain the measured fluxonium energy spectrum with a nonperturbative theory accounting for the multimode structure of the device in a large frequency range. Making use of multiphoton Raman spectroscopy, we address forbidden fluxonium transitions and observe multilevel Autler-Townes splitting. Finally, we measure lifetimes of several excited states ranging from $T_1 = 620$ ns to $T_1 = 20$ µs, by applying consecutive π-pulses between multiple fluxonium levels. Our measurements demonstrate that NbTiN is a suitable material for novel superconducting qubit designs.

The development of superinductors [1–5] has received significant interest due to their potential to provide noise protection in superconducting qubit architectures [6–8]. Moreover, Josephson junction (JJ) based circuits embedded in inductive environments are known to be immune to charge noise [1], and to flux noise in the limit of large inductances [9–12]. Previous works have demonstrated JJ array implementations of superinductances achieving values exceeding 300 nH [1, 2]. Despite this remarkable progress, the superinductances that have been so far reported in the literature are still small compared to those needed for intrinsic protection [17, 8, 11, 12].

A thin-film nanowire built from a disordered superconductor constitutes an alternative approach to reach the superinductance regime. High-kinetic inductance superconducting materials, such as NbTiN and TiN, have been studied in the context of microwave detectors [13–15], parametric amplifiers [16–18] and rfSQUID qubits [19, 20]. In the nanowire, the inertia of the Cooper pair condensate is manifested as the kinetic inductance of the superconducting wire, and can be expressed as

$$L_k = \left( \frac{m}{2e^2n_s} \right) \left( \frac{l}{wd} \right), \quad (1)$$

where $m$ is the free electron mass, $e$ is the electron charge and $n_s$ is the density of Cooper pairs [14, 21]. The second bracketed term in Eq. (1) contains a geometric factor dependent on the length $l$, width $w$, and thickness $d$ of the nanowire. By choosing a disordered superconductor with a low $n_s$ and fabricating a sufficiently long and thin wire, the kinetic inductance can be made large enough to reach the superinductance regime.

A nanowire superinductor has the advantage of reproducible nanofabrication, given that the wire geometry is defined only via precision lithography. In addition, recent measurements have demonstrated that such films can have a vanishingly small self-Kerr nonlinearity [22, 23]. However, when a nanowire is sufficiently long to achieve the superinductance limit, the presence of stray ground capacitance and the large kinetic inductance lower the frequencies of the self-resonant modes of the device. Here, as in the case of long JJ arrays [2], the multimode structure of the device needs to be taken into account in the circuit Hamiltonian.

In this letter, we demonstrate a fluxonium circuit integrating a NbTiN thin-film nanowire superinductance and explore the effects of the nanowire modes on the qubit spectrum. Our work is the first to explore the multimode structure of a superinductance-based qubit in a nonperturbative regime, bridging the gap between the lumped element circuit approximation and the full multimode description [24, 25].

A simplified circuit schematic of the nanowire superinductance fluxonium is shown in Fig. 1 (a). In contrast to standard fluxonium devices where a lumped element inductor shunts the JJ [1, 3–5, 20–22], our circuit model takes into account the fact that the superinductor is a high-impedance transmission line. Here, we present data from measurements of two devices with different nanowire widths and equal lengths. The nanowire is fabricated by etching a wire of length of 730 µm and average width of 110 (40) nm, into a 15 nm thick film of NbTiN for device 1 (2), and a single Al/AlOx/Al JJ connects the two ends of the superinductor together. The fluxonium qubit is capacitively coupled to a lumped element Nb resonator, with a bare resonance frequency $\omega_0/2\pi = 6.08$ GHz and a loaded quality factor of $Q = 8,400$. An optical image of the completed device is shown in Fig. 1 (c). The sample is attached to the base plate of a dilution refrigerator with a mixing chamber temperature of 12 mK.

We explore the energy levels of the nanowire fluxonium as a function of the external magnetic flux ($\Phi_{\text{ext}}$), by performing two-tone spectroscopy measurements. Shown in Fig. 2 the spectroscopy data is obtained by monitoring the amplitude of the transmitted power at the dressed

---

* These authors contributed equally to this work.
FIG. 1. (a) A simplified circuit diagram for the nanowire superinductance fluxonium qubit displaying the first antisymmetric standing wave nanowire mode which is shown in blue. ψ(x,t) denotes the flux (field) operator as function of the dimensionless coordinate x = x/l and time t. An off-chip magnetic coil generates the magnetic flux ((Φ)ext) that is threaded through the nanowire and the JJ. C_n and C_0 are the coupling capacitances to the readout resonator and to ground, respectively. (b) The first few fluxonium eigenstate wave functions plotted for Φ/ϕ_0 = 0, π, 2π, respectively, the total ground capacitance, inductance and the inter-well fluxons, such as |g0⟩→|g_1⟩. While the plasmon transitions are almost flux-independent flat line at 16.3 GHz for Device 1. To capture the interplay between these modes on the qubit spectrum, we develop a multimode theory which considers the distributed nature of the nanowire and goes beyond the lumped element circuit approximation.

FIG. 2. Two-tone spectroscopy of device 1 (a) and device 2 (b) as a function of external magnetic field. The experimentally measured transition frequencies are indicated by the blue markers. The result of a least squares fit to the two-mode Hamiltonian in Eq. 3 and detailed in Ref. 28, is shown with red dashed lines corresponding to the fluxonium spectrum and with purple dashed lines indicating sideband transitions 29.

The nanowire is described as a homogeneous transmission line with distributed capacitance ε = C_nw/2l and inductance ℓ = L_nw/2l, where C_nw, L_nw and 2l are, respectively, the total ground capacitance, inductance and length of the nanowire. Defining the flux (field) operator ψ(x,t) in terms of the dimensionless coordinate x = x/l, the Lagrangian of the nanowire can be written as

$$ L_{nw} = \int_{-1}^{1} dx \frac{C_nw/2}{2} \psi(x,t)^2 - \frac{1}{2L_nw/2} \psi(x,t)^2. \quad (2) $$

Additionally, we consider gate (C_g) and possibly unintentional ground capacitances (C_0) placed at the two ports of the device (x_p = ±1) with respective driving voltages {V_{x_p}}, as shown in Fig. 1(a). The Lagrangian for the inductively shunted JJ therefore reads

$$ L = \sum_{x_p} C_g \left( \psi(x_p,t) - V_{x_p} \right)^2 + \frac{C_0}{2} \psi(x_p,t)^2 \] + \frac{C_J}{2} \delta \psi(t)^2 + E_J \cos(\delta \psi(t)/\varphi_0), \quad (3) $$

Here, we directly observe the second antisymmetric self-resonant nanowire mode in the spectroscopic data as an almost flux-independent flat line at 16.3 GHz for Device 1. Cavity frequency while sweeping a second spectroscopic tone, ωspec/2π. Labeling the energy eigenstates within a single potential well as |g0⟩, |e0⟩, |f0⟩, ..., where the index i indicates the potential well to which these belong (see Fig. 1(b)), the fluxonium transitions can be classified in two types: the intra-well plasmons, such as |g0⟩→|e0⟩, and the inter-well fluxons, such as |g0⟩→|g_1⟩. While the plasmon transitions are almost flux-insensitive excitations, the fluxon transitions are highly dependent on flux. The parity selection rules of the fluxonium circuit allow for direct transitions between adjacent plasmon states by absorption of a single photon, e.g. |g0⟩→|e0⟩. However, the direct transition |g0⟩→|f0⟩ can only be completed via a two-photon process, in which |e0⟩ serves as an intermediate virtual state. We note that fluxon transitions are strongly suppressed due to the large effective mass and large barrier between wells in the regime of parameters of our devices also known as heavy fluxonium 9,10. As a consequence, such excitations are most clearly visible where they hybridize with the plasmon energy levels.

Although the measured spectra show similar characteristics to the previously studied fluxonium devices,
where \[ \delta \tau(t)/\varphi_0 = (\Delta \psi(t) + \Phi_{\text{ext}})/\varphi_0, \] (4)
is the junction gauge-invariant phase difference, with \( \Delta \psi(t) = \psi(1, t) - \psi(-1, t) \) and junction energy (inductance) \( E_J (L_J = E_J/\varphi_0^2) \) [28, 29].

So far, our model considers an infinite number of degrees of freedom. To obtain a tractable description, we first map Eq. (3) into the Lagrangian of an infinite number of nonlinearly interacting normal modes, which is then approximated by truncating to a finite number. This simplification is based on the following two assumptions. First, the normal modes have a definite symmetry with respect to the center of the nanowire \( x = 0 \) (symmetric or antisymmetric). To obtain the fluxonium Hamiltonian, we exclusively account for the antisymmetric modes, given that the complementary set remains uncoupled to the JJ. Second, we select those normal modes which predominantly contribute to the qubit spectrum in an upper-bounded frequency range, based on the mode frequencies and effective impedances. Details on this approximation are provided in Ref. [28].

We find that the first two antisymmetric modes are sufficient to explain the measured fluxonium spectrum with a two-mode Hamiltonian of the form

\[
H_{\text{two-mode}} = \frac{(q_0 - q_{\theta_0})^2}{2C_0} + \frac{\phi_0^2}{2L_0} + \frac{(q_1 - q_{\theta_1})^2}{2C_1} + \frac{\phi_1^2}{2L_1} - \frac{\phi_0 q_0}{L_J} - E_J \cos \left( \frac{\phi_0 + \phi_1}{\varphi_0} + \Phi_{\text{ext}}/\varphi_0 \right),
\]
(5)

where \( \tilde{C}_i, \tilde{L}_i \) and \( q_{\theta_i} \) are, respectively, the effective capacitance, inductance and offset charge corresponding to the \( i = \{0, 1\} \) antisymmetric modes. The results of the experiment are reproduced within a dispersive theory of the complete Hamiltonian of the device, including Eq. (5), the lumped element resonator Hamiltonian and their capacitive coupling. Details regarding the spectrum fit can be found in Ref. [28]. From a least squares fit to Eq. (5), we extract inductances of \( 120 \) nH and \( 314 \) nH for device 1 and 2, respectively. These values are comparable to previously reported superinductances on JJ array-based fluxonium devices [11, 19]. We note that the effect of coherent quantum phase slips (CQPS) has not been considered in Eq. (5), as it is found to be negligible in the measured spectrum [27].

We employ coherent Raman spectroscopy to further map out the level structure of the nanowire fluxonium and measure the lifetime of the \( |g_{-1}\rangle \) fluxon state. Due to the diminishing dipole element between the \( |g_0\rangle \) and \( |g_{-1}\rangle \) states, we transfer the ground state population between the neighboring wells using the intermediate \( |h_0\rangle \) state, which is located at the top of the wells and has spectral weight in both wells. We apply three coherent and simultaneous drives of frequencies \( \omega_{\theta_{1,2}}/2\pi, \omega_{3}/2\pi \) and \( \omega_{\phi}/2\pi \), respectively targeting the \( |g_0\rangle \to |f_0\rangle \) (two-photon), the \( |f_0\rangle \to |h_0\rangle \) (one-photon) and the \( |h_0\rangle \to |e_{-1}\rangle \) (one-photon) transitions (see Fig. 3 (a)). As discussed in more detail below, the \( |e_{-1}\rangle \) state is short-lived; it decays quickly into the \( |g_{-1}\rangle \) state, thus allowing the transfer population from the \( |g_0\rangle \) state to \( |g_{-1}\rangle \) by the proposed three-drive scheme.

We model the system with a four-level Hamiltonian including three classical drives under the rotating wave approximation [28], which in a convenient rotating frame takes the form

\[
H = \begin{bmatrix}
0 & h\Omega_0/2 & 0 & 0 \\
0 & h\Omega_0/2 & h\Omega_3/2 & 0 \\
0 & 0 & h(\Delta_0 + \Delta_3) & h\Omega_3/2 \\
0 & 0 & 0 & h(\Delta_0 + \Delta_3 - \Delta_1)
\end{bmatrix},
\]
(6)

where the Rabi amplitudes, \( \Omega_i \), characterize the strength of the drives, and the detunings are defined as \( h\Delta_0 = E_{f_0} - E_{g_0} - 2\hbar\omega_{\phi}, \ h\Delta_3 = E_{h_0} - E_{f_0} - \hbar\omega_{3} \) and \( h\Delta_1 = E_{h_0} - E_{e_{-1}} - \hbar\omega_{\phi} \). In the limit of large detuning (\( \Delta_i \gg \Omega_i \)), the requirement for the three drives to stimulate Raman transitions between \( |g_0\rangle \) and \( |e_{-1}\rangle \) is: \( 2\hbar\omega_{\phi} + \hbar\omega_{3} - \hbar\omega_{0} = E_{e_{-1}} - E_{g_0} \).

The set of coherent drives is found with the following procedure. First, we turn off the third drive \( (\Omega_3 = 0) \), and simultaneously vary \( \omega_{\phi}/2\pi \) and \( \omega_{3}/2\pi \) around the \( |g_0\rangle \to |f_0\rangle \) and \( |f_0\rangle \to |h_0\rangle \) transitions. As shown in Fig. 3 (b), we observe a vertical band corresponding to the \( |g_0\rangle \to |f_0\rangle \) at \( 7.8 \) GHz, and a diagonal band with a slope of \( \omega_{\phi}/\omega_{3} = -1/2 \), which corresponds to the Raman transition between the \( |g_0\rangle \) and \( |h_0\rangle \) states, satisfying \( 2\hbar\omega_{3} + \hbar\omega_{\phi} = E_{h_0} - E_{g_0} \). Around the resonance condition \( (2\hbar\omega_{\phi} \approx E_{f_0} - E_{g_0} \) and \( \hbar\omega_{3} \approx E_{h_0} - E_{f_0} \)), the two bands exhibit an avoided crossing, which is the hallmark of the Autler-Townes doublet previously observed in other superconducting qubits [32, 33]. Next, we fix the frequency of the first tone at \( \Delta_{0}/2\pi = 20 \) MHz, turn on the third drive and simultaneously scan the frequencies \( \omega_{3}/2\pi \) and \( \omega_{\phi}/2\pi \). Figure 3 (c) displays the resulting Autler-Townes mechanism between levels, where the Raman transition is manifested here with a slope of \( \omega_{\phi}/\omega_{3} = +1 \), corresponding to the three-drive Raman condition. This method allows us to experimentally determine the energy levels of the fluxonium qubit using population transfer. We find excellent agreement between the theoretical prediction and our data by solving the corresponding master equation of the four-level system [28].

With complete information regarding the energy of the fluxonium excited states, we determine the relaxation rates of the first fluxon and plasmon excitations by performing time-resolved measurements [36]. To this end, we start applying a \( \pi \)-pulse at the \( |g_0\rangle \to |e_0\rangle \) plasmon transition frequency. Due to the large wave function overlap, the dipole element between these states is large, which contributes to relatively short relaxation times. By preparing the system in \( |e_0\rangle \) and measuring its state after \( t_{\text{wait}} \), we obtain the energy relaxation time \( T_1 = 620 \) ns. Based on the calculated matrix element between \( |g_0\rangle \) and \( |e_0\rangle \), the quality factor of the resonator and the value of
$C_g$, we find that this $T_1$ is limited by spontaneous emission into the measurement apparatus [29].

In order to measure the lifetime of the $|g_{-1}\rangle$ fluxon state, we use the frequency values obtained from the Autler-Townes spectroscopy and perform a pulse sequence which consists of three sequential $\pi$-pulses at the transition frequencies $(E_{f_{0}} - E_{g_{0}})/\hbar$, $(E_{h_{0}} - E_{f_{0}})/\hbar$ and $(E_{h_{0}} - E_{e_{-1}})/\hbar$ to prepare the system in the $|e_{-1}\rangle$ state. At the end of this procedure, the system is allowed to relax into the $|g_{-1}\rangle$ state, on the time scale of the plasmon $T_1 \sim 600 \text{ ns}$. On a longer timescale, the system relaxes back to $|g_{0}\rangle$. For $t_{\text{wait}} \gg T_{1|e_{0}}$, the reduction in $|g_{-1}\rangle$ population follows an exponential decay with $T_{1|g_{-1}} = 20 \mu s$. The increase in $T_{1|g_{-1}}$, compared to $T_{1|e_{0}}$, indicates that the disjunct fluxon wave functions do offer an increased protection against relaxation.

In conclusion, we have fabricated and measured a nanowire superinductance fluxonium qubit, where the superinductor element of the device is formed by a thin NbTiN wire. By mapping out the energy levels of the device, we found that the transition energy levels are modified due to the presence of the self-resonant modes of the nanowire, which is well explained in the framework of a nonperturbative multimode theory. As the modes of the nanowire strongly depend on the parasitic and stray capacitances of the wire, using a shorter wire with higher sheet inductance (for example high quality granular aluminum films with $L_k = 2 \mu \text{H} / \square$ [22, 23]), or integrating the fluxonium into a 3D cavity or waveguide [38], could reduce unwanted capacitances and help to push the nanowire self-resonant modes to higher frequencies.

I. ACKNOWLEDGMENTS

We thank Andrei Vrajitoarea, Zhaoci Leng and Jérôme Bourassa for useful discussions. Research supported by the Army Research Office Grant No. W911NF-15-1-0421 and the Princeton Center for Complex Materials DMR-142052. This work was undertaken thanks in part to funding from NSERC and the Canada First Research Excellence Fund.

[1] V. E. Manucharyan, J. Koch, L. I. Glazman, and M. H. Devoret, Science 326, 113 (2009).

[2] N. A. Mashuk, I. M. Pop, A. Kamal, Z. K. Minev, and M. H. Devoret, Phys. Rev. Lett. 109, 137002 (2012).
Device fabrication begins with sputtering 15 nm of NbTiN onto a 500 μm thick C-plane sapphire substrate. A small patch of NbTiN, where the nanowire will be fabricated later in the process, is protected with Microposit™ S1811 photoresist and the remain-
der layer of NbTiN is removed with an SF₆/Ar dry etch. For the lumped element readout resonator and transmission line, 200 nm of Nb is sputtered over the areas of the chip which had no NbTiN and subsequently patterned and etched with another SF₆/Ar dry etch. Next, a layer of

Appendix A: Supplementary Materials

1. Fabrication details
ZEP520A (1:1 dilution in anisole) e-beam resist is spun on the chip and the nanowire pattern is exposed and developed with standard e-beam lithography techniques. Finally, an MMA/PMMA bilayer e-beam resist is placed on the chip and the JJ layer is patterned with e-beam lithography. To ensure metallic contact between the JJ and the NbTiN nanowire, a high-voltage Ar ion beam milling process is used to remove the native oxide layer formed on the surface of the NbTiN film. The JJ layer is fabricated with a double angle evaporation of 30 nm and 60 nm of Al, with a 15 minute oxidation step in between the first and second evaporation angles to form the oxide layer of the junction.

2. Properties of NbTiN film

Measurements on similarly processed films show a superconducting critical temperature of $T_c \sim 13$ K. Room temperature resistance measurements are performed on 7 different nanowires with $l = 100 \, \mu m$ and varying widths, ranging from 50 to 900 nm. From these measurements, we extract a room temperature sheet resistivity of $R = 97 \pm 5 \, \Omega/\square$. From scanning electron microscope images and the resistance measurements of several test structures, we infer nanowire widths of $110 \pm 5$ nm and $40 \pm 5$ nm for the two devices, giving inductances of 0.15 nH $\mu m^{-1}$ and 0.4 nH $\mu m^{-1}$ (both have a fixed length of 730 $\mu m$). These values of kinetic inductance are similar to those observed by other groups [39].

3. $T_1$ estimation for plasmon

We estimate the relaxation rate for the $|g_0\rangle \rightarrow |e_0\rangle$ plasmon by calculating the admittance $Y(\omega)$ across the fluxonium [37]. The relaxation rate can be expressed as

$$\Gamma_{ge} = 2\pi \left( \omega_{ge} R_Q C_c \right)^2 \frac{R_Q}{R_{Purcell}} |\phi_{ge}|^2 \times \omega_{ge} \left( 1 + \coth \left( \frac{\hbar \omega_{ge}}{2k_b T} \right) \right),$$

(\text{A1})

where $\omega_{ge}$ is the frequency of the plasmon transition, $R_Q$ is the quantum resistance, $C_c$ is the coupling capacitance, $R_{Purcell}$ is the impedance shunting the two ends of the fluxonium ($Y(\omega)^{-1}$), $\phi_{ge}$ is the dipole matrix element overlap between states $|g_0\rangle$ and $|e_0\rangle$, $h$ is the Planck constant, $k_b$ is the Boltzmann constant and $T$ is the temperature of the device. When using the calculated and measured parameters, we expect the Purcell limit to be $\Gamma_{ge} \sim 1.5$ MHz. This value is in agreement with the measured $T_1$ of 620 ns.

4. Modeling of the Autler-Townes splitting

We model the system with a four-level Hamiltonian which, in the $(|g_0\rangle, |f_0\rangle, |h_0\rangle, |e_{-1}\rangle)$ energy eigenbasis and in the absence of external drives, reads

$$H_0 = e_{g_0} |g_0\rangle \langle g_0| + E_{f_0} |f_0\rangle \langle f_0| + E_{h_0} |h_0\rangle \langle h_0| + E_{h_{-1}} |e_{-1}\rangle \langle e_{-1}|,$$

(A2)

where the groundstate energy is chosen to be $E_{g_0} = 0$, and the energies of excited levels satisfy the relations $E_{g_0} < E_{f_0} < E_{h_0} > E_{h_{-1}}$ (see Fig. 3(a) in the main text). We work in a semiclassical picture where the external drives $\omega_a/2\pi, \omega_\beta/2\pi, \omega_\gamma/2\pi$ with respective Rabi frequencies $\Omega_\alpha, \Omega_\beta, \Omega_\gamma$ introduce a nonzero coupling exclusively between neighboring energy levels. In the rotating wave approximation, this situation is described by the interaction Hamiltonian

$$H_{int} = \frac{1}{2} h \Omega_\alpha \left( \sigma_{10} e^{-i2\omega_a t} + \sigma_{01} e^{i2\omega_a t} \right) + \frac{1}{2} h \Omega_\beta \left( \sigma_{21} e^{-i\omega_\beta t} + \sigma_{12} e^{i\omega_\beta t} \right) + \frac{1}{2} h \Omega_\gamma \left( \sigma_{32} e^{i\omega_\gamma t} + \sigma_{23} e^{-i\omega_\gamma t} \right),$$

(A3)

where $\sigma_{ij} = |i\rangle \langle j|$, with $|i\rangle$ and $|j\rangle$ as basis states, and noting that $\omega_a$ is a two-photon drive. Since $E_{h_0} > E_{h_{-1}}$, the time-dependent phase corresponding to the third term in Eq. (A3) ($\omega_\gamma$) has opposite sign. Moreover, the total Hamiltonian of the system is defined as $H = H_0 + H_{int}$

$$H = \begin{bmatrix}
0 & \frac{\hbar \Omega_\alpha}{2} e^{-i2\omega_a t} & 0 & 0 \\
\frac{\hbar \Omega_\alpha}{2} e^{i2\omega_a t} & E_{f_0} & \frac{\hbar \Omega_\beta}{2} e^{i\omega_\beta t} & 0 \\
0 & \frac{\hbar \Omega_\beta}{2} e^{-i\omega_\beta t} & E_{h_0} & \frac{\hbar \Omega_\gamma}{2} e^{i\omega_\gamma t} \\
0 & 0 & \frac{\hbar \Omega_\gamma}{2} e^{-i\omega_\gamma t} & E_{e_{-1}}
\end{bmatrix}.$$  

(A4)

We now move to the rotating frame of the classical drives by applying the unitary $U = |g_0\rangle \langle g_0| + e^{i2\omega_a t} |f_0\rangle \langle f_0| + e^{i(2\omega_\alpha + \omega_\beta) t} |h_0\rangle \langle h_0| + e^{i(2\omega_\alpha - \omega_\gamma) t} |e_{-1}\rangle \langle e_{-1}|$, and thus transforming the Hamiltonian Eq. (A3) to

$$\tilde{H} = \begin{bmatrix}
0 & \frac{1}{2} \hbar \Omega_\alpha & 0 & 0 \\
\frac{1}{2} \hbar \Omega_\alpha & h\Delta_\alpha & 0 & 0 \\
0 & \frac{1}{2} \hbar \Omega_\beta & h(\Delta_\alpha + \Delta_\beta) & 0 \\
0 & 0 & \frac{1}{2} \hbar \Omega_\gamma & h(\Delta_\alpha + \Delta_\beta - \Delta_\gamma)
\end{bmatrix}.$$  

(A5)

Here, the detunings are $h\Delta_\alpha = E_{f_0} - 2h\omega_\alpha, h\Delta_\beta = E_{h_0} - E_{f_0} - h\omega_\beta$ and $h\Delta_\gamma = E_{h_0} - E_{e_{-1}} - h\omega_\gamma$. Furthermore, we account for dissipation in the system with a Lindblad master equation of the form

$$\dot{\rho} = -\frac{i}{\hbar}[\tilde{H}, \rho] + \sum_j \left[ c_j \rho c_j^\dagger - \frac{1}{2} \{c_j \rho, c_j \} \right],$$

(A6)
where the collapse operators $c_j$ are defined as $c_j = \sqrt{\Gamma_{ij}} \sigma_{ij}$, for given energy levels $i$ and $j$. The steady state solution of Eq. (A6) is numerically obtained and the maximal excited state population, $\max (\rho_{j0} + \rho_{0j} + \rho_{-1,-1})$, is shown with dashed lines in Fig. 3 (b) and (c).

5. Spectrum Characterization

a. Multimode Hamiltonian

In this section, we outline the theory developed to treat the full multimode structure of the device in Fig. 1 (a). Our derivation is inspired on ideas introduced in Refs. [30, 31]. For simplicity, we assume the absence of disorder in gate and ground capacitances, although the effect of a small amount of disorder is discussed below.

Considering the fluxonium Lagrangian in Eq. (3), we now introduce justified approximations to reduce the problem to that of two nonlinearly interacting bosonic fields, which is then numerically diagonalized to fit the qubit spectrum in Fig. 2.

The circuit normal modes are a convenient basis where the fluxonium Hamiltonian is diagonal to second order in the Josephson nonlinearity. In addition, the symmetry of such modes unequivocally identifies the degrees of freedom effectively coupled to the JJ, for which the Josephson nonlinearity needs to be taken into account. In consequence, writing the wave equation which holds in the bulk of the nanowire $\partial^2_t \psi(x,t) = \omega_{nw}^2 \psi(x,t)$, with $\omega_{nw} = 1 / \sqrt{(C_{nw}/2)(L_{nw}/2)}$, we search for normal mode solutions of the form

$$\psi_m(x,t) = u_m(x) \xi_m(t),$$  \hspace{1cm} (A7)

satisfying $u''_m(x) = -k_m^2 u_m(x)$ and $\ddot{\xi}_m(t) = -\omega_m^2 \xi_m(t)$.

Here, $k_m$ is a dimensionless wave vector and $\omega_m = \omega_{nw} k_m$ (linear dispersion). The mode frequencies are determined by the field boundary conditions that we derive taking the continuous limit of the discretized field $\psi(x,t) \rightarrow \{\phi(x_n)\}$ equations of motion, where $\phi(x_n)$ is defined on a lattice with $2N + 1$ points in $[-1,1]$. To this end, we consider the lattice Lagrangian linearized for $V_{\pm 1} \varphi_{ext} \rightarrow 0$, for which the equations of motion simply read

$$C \ddot{\phi} + L^{-1} \phi = 0.$$  \hspace{1cm} (A8)

Here, $C$ and $L^{-1}$ are, respectively, the capacitance (and inverse) inductance matrices for the lattice model, while $\phi$ is the corresponding $2N + 1$-dimensional node-flux vector. As illustrated in Fig. 1 (a), the transmission line is modeled as a chain of $2N$ LC resonators, with single nodes connected to ground by a capacitance $C_n = \Delta x C_{nw}/2$ and neighboring nodes coupled by an inductance $L_b = \Delta x L_{nw}/2$. Projecting Eq. (A8) in the $\{\phi(\pm 1)\}$ basis and taking the continuous limit $\Delta x \rightarrow 0$ and $N \rightarrow \infty$ with $N \Delta x = 1$, we find the field boundary conditions

$$C_{bc} \ddot{\psi}_{bc} + L_{bc}^{-1} \psi_{bc} = 0,$$  \hspace{1cm} (A9)

where $\psi_{bc} = (\psi(1,t), \psi(-1,t))^T$, $C_{bc} = C_S \mathbb{I} - C_J \sigma_x$, and $L_{bc} = \left(\frac{1}{L_{nw}/2}\partial_x + \frac{1}{L_J} \mathbb{I} - \frac{1}{L_J} \sigma_x\right)$. Here, we have defined identity ($\mathbb{I}$) and Pauli-X ($\sigma_x$) matrices, the capacitances $C_S = C_p + C_J$ and $C_p = C_q + C_0$, and the operators $(x, \partial_x)$, which are evaluated at the boundaries $x = \pm 1$ according to $\psi(\pm 1, t)$. We note that the systematic procedure used here to derive the field boundary conditions can be applied to any circuit integrating distributed elements.

We now consider $\psi(x,t)$ to be the normal mode solution in Eq. (A7), and parametrize the mode function by

$$u_m(x) = A_m \cos(k_m x) + B_m \sin(k_m x),$$  \hspace{1cm} (A10)

where $A_m, B_m$ are constants to be determined. With this choice, Eq. (A9) can be rewritten in the form $M(A_m, B_m)^T = 0$, where $M$ is a coefficient matrix (omitted for brevity). Nontrivial solutions to this homogeneous system of equations follow from the condition $\det(M) = 0$, implying

$$\frac{C_p}{2} L_{nw} \omega_m^2 + k_m \tan k_m = 0,$$  \hspace{1cm} (A11)

or

$$\frac{C_p}{2} L_{nw} \omega_m^2 + k_m \tan k_m = 0.$$  \hspace{1cm} (A12)

Eq. (A11) and Eq. (A12) allow us to find the mode frequencies $\{\omega_m\}$, which are then plugged back into Eq. (A9) to determine the mode function Eq. (A10). We stress that Eq. (A11) does not includes any of the Josephson junction parameters. In fact, this equation determines the frequency of symmetric nanowire modes, which have zero voltage difference across the JJ: $\Delta u_m = u_m(1) - u_m(-1) = 0$. In contrast, Eq. (A12) depends on $C_J$ and $L_J$, and determines the frequency of antisymmetric modes which do couple to the junction ($\Delta u_m \neq 0$). This fundamental difference is discussed in more detail below.

We are now in position to expand the field in the normal mode basis, as

$$\psi(x,t) = \sum_m u_m(x) \xi_m(t),$$  \hspace{1cm} (A13)

where, in principle, the sum over $m$ is extended to all circuit modes. Making use of the orthogonality relations [30, 31].
\[ \int_{-1}^{1} dx \left( \frac{C_{vw}}{2} u_m(x) u_n(x) + C_J \Delta u_m \Delta u_n + \sum_{x_p} C_p u_m(x_p) u_n(x_p) = C_m \delta_{mn} \right), \quad (A14) \]

and

\[ \int_{-1}^{1} dx \frac{1}{L_{vw}/2} u_m'(x) u_n'(x) + \frac{\Delta u_m \Delta u_n}{L_J} = \delta_{mn}, \quad (A15) \]

where \( L_{-1} = C_m \omega_m^2 \), Eq. \( \{A13\} \) in Eq. \( \{3\} \) gives the circuit Lagrangian in the normal mode basis

\[ L = \sum_m \frac{C_m}{2} \dot{\phi}_m^2 + \sum_m C_g u_m(x_p) V_{x_p} \dot{\phi}_m - \frac{\xi_m^2}{2L_m} + E_J \left[ \cos(\delta \phi/\phi_0) + (\delta \phi/\phi_0)^2/2 \right], \quad (A16) \]

where explicit time dependence has been omitted and \( \delta \phi/\phi_0 \) is defined in Eq. \( \{4\} \). We note that Eq. \( \{A16\} \) is diagonal to second order in the Josephson potential and for \( \Phi_{ext} = 0 \), as a consequence of our normal mode basis choice. Writing

\[ \Delta \psi = \sum_m \Delta u_m \xi_m, \quad (A17) \]

we verify that symmetric modes do not couple to the Josephson nonlinearity, thus behaving as a collection of noninteracting harmonic oscillators. Accordingly, we disregard symmetric modes in our treatment and consider the set \( \{\tilde{m}\} \), consisting of antisymmetric modes for which \( \Delta u_{\tilde{m}} \neq 0 \). With the change of variables \( \phi_{\tilde{m}} = \Delta u_{\tilde{m}} \xi_{\tilde{m}} \), we rewrite Eq. \( \{A16\} \) as

\[ L = \sum_{\tilde{m}} \frac{\tilde{C}_{\tilde{m}}}{2} \tilde{\phi}_{\tilde{m}}^2 + \sum_{x_p} C_g u_{\tilde{m}}(x_p) V_{x_p} \tilde{\phi}_{\tilde{m}} - \frac{\tilde{\phi}_{\tilde{m}}^2}{2L_{\tilde{m}}} \]

\[ + \sum_{\tilde{m} < \tilde{n}} \frac{1}{L_{\tilde{J}}} \phi_{\tilde{m}} \phi_{\tilde{n}} + E_J \cos(\delta \phi/\phi_0), \quad (A18) \]

where \( \delta \phi/\phi_0 \) conserves the definition in Eq. \( \{4\} \) with the replacement \( \Delta \phi/\phi_0 = \sum_{\tilde{m}} \phi_{\tilde{m}}/\phi_0 \). Here,

\[ \tilde{C}_{\tilde{m}} = C_{m}/\Delta u_{\tilde{m}}^2, \quad \tilde{L}_{-1} = \frac{1}{\Delta u_{\tilde{m}}^2} \int_{-1}^{1} dx \frac{u_{\tilde{m}}'(x)^2}{L_{vw}/2}, \quad (A19) \]

denote the mode \( \tilde{m} \) effective capacitance and inductance, respectively. The multimode Hamiltonian

\[ H = \sum_{\tilde{m}} H_{\tilde{m}}^{(0)} + H_{\text{int}}, \quad (A20) \]

follows immediately from Eq. \( \{A18\} \), and includes the set of noninteracting terms \( \{H_{\tilde{m}}^{(0)}\} \), with

\[ H_{\tilde{m}}^{(0)} = \left( \frac{q_{\tilde{m}} - g_{\tilde{m}}}{2C_{\tilde{m}}} \right)^2 + \frac{\phi_{\tilde{m}}^2}{2L_{\tilde{m}}}, \quad (A21) \]

where \( q_{\tilde{m}} = \sum x_p C_g u_{\tilde{m}}(x_p) V_{x_p}/\Delta u_{\tilde{m}} \), and the interaction potential

\[ H_{\text{int}} = - \sum_{\tilde{m} < \tilde{n}} \frac{\phi_{\tilde{m}} \phi_{\tilde{n}}}{L_{\tilde{J}}} - E_J \cos \left( \sum_{\tilde{m}} \phi_{\tilde{m}}/\phi_0 + \Phi_{ext} \phi_0 \right). \quad (A22) \]

Eq. \( \{A20\} \) is approximated into a tractable Hamiltonian making use of a frequency and effective impedance hierarchy of the normal modes. Indeed, if these are sorted in frequency as \( \omega_0 < \omega_1 < \ldots < \omega_n \), it is possible to see that \( Z_0 \gg Z_1 \gg \ldots \gg Z_n \). Therefore, as the frequency range of interest is bounded, a good approximation for Eq. \( \{A20\} \) can be obtained taking into account a finite number of modes covering such a spectral range, and considering the rest of the modes in a vacuum state. We note that vacuum fluctuations are strongly suppressed for high-frequency modes, thanks to their vanishing effective impedance. In our particular case, the experiment probes frequencies in \( \{0, \omega_{max}\} \), with \( \omega_0 < \omega_{max} < \omega_1, \omega_{max} \ll \omega_2 \). Therefore, we approximate the device’s Hamiltonian by the two-mode Hamiltonian in Eq. \( \{5\} \), which includes the two first antisymmetric modes. As shown in Fig. \( \{2\} \), we find excellent agreement between the Eq. \( \{5\} \) diagonalization and the measured fluxonium spectrum. Moreover, we verify that the inclusion of the third antisymmetric normal mode in the fluxonium Hamiltonian does not modify appreciably the qubit spectrum. Regarding device 1, an estimation of a dispersive-like coupling strength for the first order mode-mode interaction gives \( g^2/\Delta < 100 \text{ kHz} \) between the first and third JJ modes, and \( g^2/\Delta < 1 \text{ kHz} \) between the second and third JJ modes, while the same quantities are negligible for device 2.

We note that the symmetry of the self-resonant nanowire modes is lost in presence of circuit disorder. However, if the amount of disorder is small \( (< 10\%) \), one can still work in the symmetric-antisymmetric normal mode basis, deriving a capacitive coupling between the two sets of modes proportional to the amount of disorder. Therefore, the effect of symmetric modes could be taken into account within a dispersive (thus perturbative) theory, as it was previously done in the literature \( \{1, 42\} \). However, due to the generality of our fit routine (see appendix \( \{A5\} \)), we do not find necessary to consider such a dispersive shift (which adds a fit parameter) to obtain a high-accuracy agreement between theory and experiment.
b. Fluxonium Spectrum

As the two-tone spectroscopy experiment probes the qubit spectrum in presence of the resonator, our data includes the Lamb shift contribution arising as a consequence of the dispersive coupling between the fluxonium and the resonator \[43\].

Lamb and dispersive shifts can be computed by means of the bare qubit level structure using the framework developed in \[42\], for any qubit-resonator system in the dispersive regime. Equivalently, such quantities can be obtained from full diagonalization of the transversally coupled qubit-resonator Hamiltonian. In this work, we use the second approach to compute the qubit spectrum.

We assume a readout resonator of nominal frequency \(\omega_r\) and impedance \(Z_r\), according to the measured resonator mode frequency and specifications. Considering first, a single mode \(\hat{m}\) in Eq. (A21), and making use of the antisymmetry of the mode function, the corresponding voltage coupling operator –as derived from the offset charge term– takes the form

\[ -q\hat{m}(C_g/\tilde{C}_{\hat{m}})(V_1 - V_{-1})/2. \]

In the present case, the weak fluxonium-resonator coupling Hamiltonian is obtained by replacing the voltage difference \((V_1 - V_{-1})\) by the resonator voltage operator \(V_r = \sqrt{\hbar Z_r/2}(a + a^\dagger)\), where \(a (a^\dagger)\) is the photon annihilation (creation) operator. Therefore, in a two-mode approximation for the fluxonium qubit, we consider have the complete Hamiltonian

\[ H = H_r + H_{\text{two-mode}} + H_{r0} + H_{r1}, \]  

(A23)

where \(H_r = \omega_r a^\dagger a\) denotes the resonator Hamiltonian, \(H_{\text{two-mode}}\) is given in Eq. (5), and \(H_{r\hat{m}}\)

\[ H_{r\hat{m}} = -q\hat{m}(C_g/\tilde{C}_{\hat{m}})\sqrt{\hbar Z_r/2}(a + a^\dagger)/2 \]  

(A24)

is the coupling Hamiltonian between the resonator and the \(\hat{m}\)th fluxonium mode.

From Eq. (A23), the Lamb-shifted \(i\)th qubit energy-level is identified by the energy of the Eq. (A23) eigenstate exhibiting maximum overlap with \(|0, i\rangle\) (0 resonator excitations, \(i\) fluxonium excitations). The fluxonium parameters \(C_p, C_J, L_J, C_{nw}, L_{nw}\), and the fluxonium-resonator coupling capacitance \(C_c\), are considered input variables for the qubit spectrum fit in Fig. 2. For both devices, the resulting fit values give junction plasma frequencies differing in \(\sim 4\%\), comparable total ground capacitances \(C_{nw} + C_p\) and similar fluxonium-resonator coupling capacitances.