ON PIECEWISE CONTINUOUS MAPPINGS OF METRIZABLE SPACES

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ABSTRACT. Let $f : X \to Y$ be a resolvable-measurable mapping of a metrizable space $X$ to a regular space $Y$. Then $f$ is piecewise continuous. Additionally, for a metrizable completely Baire space $X$, it is proved that $f$ is resolvable-measurable if and only if it is piecewise continuous.

In an old question Lusin asked if any Borel function is necessarily countably continuous. This question was answered negatively by Keldiš [K34], and an example of a Baire class 1 function which is not decomposable into countably many continuous functions was later found by Adyan and Novikov [AN]; see also the paper of van Mill and Pol [vMP].

The first affirmative result was obtained by Jayne and Rogers [JR, Theorem 1].

**Theorem JR** (Jayne–Rogers). If $X$ is an absolute Souslin-$\mathcal{F}$ set and $Y$ is a metric space, then $f : X \to Y$ is $\Delta^0_2$-measurable if and only if it is piecewise continuous.

Later Solecki [Sol, Theorem 3.1] proved the first dichotomy theorem for Baire class 1 functions. This theorem shows how piecewise continuous functions can be found among $\Sigma^0_2$-measurable ones.

**Theorem S** (Solecki). Let $f : X \to Y$ be a $\Sigma^0_2$-measurable function from an analytic set $X$ to a separable metric space $Y$. Then precisely one of the following holds:

(i) $f$ is piecewise continuous,

(ii) one of $L$, $L_1$ is contained in $f$, where $L$ and $L_1$ are two so-called Lebesgue’s functions.

Kačena, Motto Ros, and Semmes [KMS, Theorem 1] showed that Theorem JR holds for a regular space $Y$. They also got [KMS, Theorem 8] a strengthening of Solecki’s theorem from an analytic set $X$ to an absolute Souslin-$\mathcal{F}$ set $X$.

On the other hand, Banakh and Bokalo [BB, Theorem 8.1] proved among other things that a mapping $f : X \to Y$ from a metrizable completely Baire space $X$ to a regular space $Y$ is piecewise continuous if and only if it is $\Pi^0_2$-measurable. Under some set-theoretical assumptions, examples of $\Pi^0_2$-measurable mappings which are not piecewise continuous were constructed in the work [BB].

Recently, Ostrovsky [Ost] proved that every resolvable-measurable function $f : X \to Y$ is countably continuous for any separable zero-dimensional metrizable spaces $X$ and $Y$.

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The main result of the paper (see Theorem 4) states that every resolvable-measurable mapping \( f: X \to Y \) of a metrizable space \( X \) to a regular space \( Y \) is piecewise continuous. Comparison of our result and the Banakh and Bokalo theorem shows that the condition on \( X \) is weakened but \( f \) is restricted to the class of resolvable-measurable mappings. Notice also that Theorem 4 generalizes and strengthens the Ostrovsky theorem.

In completely metrizable spaces, resolvable sets coincide with \( \Delta^0_2 \)-sets, see [Kur1, p. 418]. Lemma 6 shows that every metrizable completely Baire space has such a property. This enables us to refine the above result of Banakh and Bokalo, see Theorem 7.

Theorem 9 states that in the study of \( \Sigma^0_2 \)-measurable mappings defined on metrizable completely Baire spaces it suffices to consider separable spaces. In a sense, Theorem 9 is similar to the non-separable version of Solecki’s Theorem S.

Notation. For all undefined terms, see [Eng].

A subset \( E \) of a space \( X \) is resolvable if it can be represented as

\[
E = (F_1 \setminus F_2) \cup (F_3 \setminus F_4) \cup \ldots \cup (F_\xi \setminus F_{\xi+1}) \cup \ldots,
\]

where \( \langle F_\xi \rangle \) forms a decreasing transfinite sequence of closed sets in \( X \).

A metric space \( X \) is said to be an absolute Souslin-\( F \) set if \( X \) is a result of the \( A \)-operation applied to a system of closed subsets of \( \hat{X} \), where \( \hat{X} \) is the completion of \( X \) under its metric. Metrizable continuous images of the space of irrational numbers are called analytic sets.

A mapping \( f: X \to Y \) is said to be

- **resolvable-measurable** if \( f^{-1}(U) \) is a resolvable subset of \( X \) for every open set \( U \subset Y \),
- **\( \Delta^0_2 \)-measurable** if \( f^{-1}(U) \in \Delta^0_2(X) \) for every open set \( U \subset Y \),
- **\( \Sigma^0_2 \)-measurable** if \( f^{-1}(U) \in \Sigma^0_2(X) \) for every open set \( U \subset Y \),
- **\( \Pi^0_2 \)-measurable** if \( f^{-1}(U) \in \Pi^0_2(X) \) for every open set \( U \subset Y \),
- **countably continuous** if \( X \) can be covered by a sequence \( X_0, X_1, \ldots \) of sets such that the restriction \( f \upharpoonright X_n \) is continuous for every \( n \in \omega \),
- **piecewise continuous** if \( X \) can be covered by a sequence \( X_0, X_1, \ldots \) of closed sets such that the restriction \( f \upharpoonright X_n \) is continuous for every \( n \in \omega \).

Obviously, every piecewise continuous mapping is countably continuous. Notice that every resolvable-measurable mapping of a metrizable space \( X \) is \( \Sigma^0_2 \)-measurable because, by [Kur1, p. 362], every resolvable subset of a metrizable space \( X \) is a \( \Delta^0_2 \)-set, i.e., a set that is both \( F_\sigma \) and \( G_\delta \) in \( X \). The following example shows that there exists a \( \Delta^0_2 \)-measurable mapping which is not resolvable-measurable.

**Example.** Let \( f: \mathbb{Q} \to D \) be a one-to-one mapping of the space \( \mathbb{Q} \) of rational numbers onto the countable discrete space \( D \). Clearly, \( f \) is piecewise continuous and \( \Delta^0_2 \)-measurable. Gao and Kientenbeld [GK, Proposition 4] got a characterization of nonresolvable subsets of \( \mathbb{Q} \). In particular, they showed that there exists a nonresolvable subset \( A \) of \( \mathbb{Q} \). Since \( A = f^{-1}(f(A)) \), the mapping \( f \) is not resolvable-measurable.
The closure of a set \( A \subset X \) is denoted by \( \overline{A} \). Given a mapping \( f : X \to Y \), let us denote by \( \mathcal{I}_f \) the family of all subsets \( A \subset X \) for which there is a set \( S \in \Sigma_2^0(X) \) such that \( A \subset S \) and the restriction \( f \upharpoonright S \) is piecewise continuous. In particular, \( f \) is piecewise continuous if and only if \( X \in \mathcal{I}_f \). From [HZZ, Proposition 3.5] it follows that the family \( \mathcal{I}_f \) forms a \( \sigma \)-ideal which is \( F_\sigma \) supported and is closed with respect to discrete unions, see also [KMS].

To prove Theorem [4] we shall use the technique due to Kačena, Motto Ros, and Semmes [KMS]. Therefore, the terminology from [KMS] is applied. The sets \( A, B \subset Y \) are strongly disjoint if \( \overline{A} \cap \overline{B} = \emptyset \). Let \( f : X \to Y \) be a mapping. Put \( A^f = f^{-1}(Y \setminus \overline{A}) \). As noted in [KMS], if \( A, B \) are strongly disjoint and \( A^f, B^f \in \mathcal{I}_f \), then \( X \in \mathcal{I}_f \).

Let \( x \in X, X' \subset X \), and \( A \subset Y \). The pair \( (x, X') \) is said to be \( f \)-irreducible outside \( A \) if for every open neighborhood \( V \subset X \) of \( x \) we have \( A^f \cap X' \cap V \notin \mathcal{I}_f \). Otherwise we say that \( (x, X') \) is \( f \)-reducible outside \( A \), i.e., there exist a neighborhood \( V \) of \( x \) and a set \( S \in \Sigma_2^0(X) \) such that \( A^f \cap X' \cap V \subset S \) and \( f \upharpoonright S \) is piecewise continuous. Clearly, \( x \in \overline{A^f \cap X'} \) if \( (x, X') \) is \( f \)-irreducible outside \( A \).

**Lemma 1** ([KMS, Lemma 3]). Let \( X \) be a metrizable space and \( Y \) a regular space. Suppose \( f : X \to Y \) is a \( \Sigma_2^0 \)-measurable mapping, \( X' \) is a subset of \( X \), and \( A \subset Y \) is an open set such that \( X' \subset A^f \). Then the following assertions are equivalent:

(i) \( X' \notin \mathcal{I}_f \),

(ii) there exist a point \( x \in \overline{X'} \) and an open set \( U \subset Y \) strongly disjoint from \( A \) such that \( f(x) \in U \) and the pair \( (x, X') \) is \( f \)-irreducible outside \( U \).

**Lemma 2** ([KMS, Lemma 4]). Let \( f : X \to Y \) be a mapping of a metrizable space \( X \) to a regular space \( Y \), \( x \in X, X' \subset X, A \subset Y \), and let \( U_0, \ldots, U_k \) be a sequence of pairwise strongly disjoint open subsets of \( Y \). If \( (x; X') \) is \( f \)-irreducible outside \( A \), then there is at most one \( i \in \{0, \ldots, k\} \) such that \( (x, X') \) is \( f \)-reducible outside \( A \cup U_i \).

Recall that a set \( A \subset Y \) is relatively discrete in \( Y \) if for every point \( a \in A \) there is an open set \( U \subset Y \) such that \( U \cap A = \{a\} \).

**Lemma 3.** Let \( X \) be a metrizable space and \( Y \) be a regular space. Suppose \( f : X \to Y \) is a \( \Sigma_2^0 \)-measurable mapping which is not piecewise continuous. Then there exists a subset \( Z \subset X \) such that:

1. \( Z \) is homeomorphic to the space of rational numbers,
2. the restriction \( f \upharpoonright Z \) is a bijection,
3. the set \( f(Z) \) is relatively discrete in \( Y \),
4. \( \dim Z = 0 \).

**Proof.** Fix a metric \( \rho \) on \( X \). Denote by \( 2^{\leq \omega} \) the set of all binary sequences of finite length. The construction will be carried out by induction with respect to the order \( \preceq \) on \( 2^{\leq \omega} \) defined by

\[
s \preceq t \iff \text{length}(s) < \text{length}(t) \lor (\text{length}(s) = \text{length}(t) \land s \leq_{\text{lex}} t),
\]

where \( \leq_{\text{lex}} \) is the lexicographic order on sequences.
where $\leq_{\text{lex}}$ is the usual lexicographical order on $2^{\text{length}(s)}$. We write $s \prec t$ if $s \leq t$ and $s \neq t$.

We will construct a sequence $\langle x_s: s \in 2^{<\omega}\rangle$ of points of $X$, a sequence $\langle V_s: s \in 2^{<\omega}\rangle$ of subsets of $X$, and a sequence $\langle U_s: s \in 2^{<\omega}\rangle$ of open subsets of $Y$ such that for every $s \in 2^{<\omega}$:

1. if $t \subset s$ then $V_s \subset V_t$,
2. $V_s$ is an open ball in $X$ with the centre $x_s$ and radius $\leq 2^{-\text{length}(s)}$,
3. if $s = t^\emptyset\emptyset$ then $x_s = x_t$,
4. $f(x_s) \in U_s$,
5. $(x_t, V_t)$ is $f$-irreducible outside $A$ for every $t \leq s$, where $A = \bigcup_{u \preceq s} U_u$,
6. the family $\{V_t: t \in 2^n\}$ is pairwise strongly disjoint for every $n \in \omega$,
7. the family $\{U_t: t \leq s\}$ is pairwise strongly disjoint.

Since $f$ is not piecewise continuous, we can apply Lemma 1 with respect to $X' = X$ and $A = \emptyset$ to obtain the point $x \in X$ and the open set $U \subset Y$. Then put $x_0 = x$ and $U_0 = U$. Let $V_0 = B(x_0, 1)$ be an open ball in $X$ with the centre $x_0$ and radius 1.

Assume that $x_t, V_t$, and $U_t$ have been constructed for any $t \leq s$. Put $x_{s^0} = x_s$ and $U_{s^0} = U_s$.

Let $A = \bigcup_{t \prec s} U_t$ and $O = Y \setminus \overline{A}$. By the inductive hypothesis, the pair $(x_s, V_s)$ is $f$-irreducible outside $A$. Take a neighborhood $W$ of $x_s$ such that $\overline{W} \subset V_s$. Then $(x_s, W)$ is $f$-irreducible outside $A$ and $f^{-1}(O) \cap W = A^f \cap W \notin \mathcal{I}_f$. By Lemma 1 there exist a point $x' \in f^{-1}(O) \cap W$ and an open set $U_{x'} \subset Y$ strongly disjoint from $A$ such that $f(x') \in U_{x'}$ and the pair $(x', f^{-1}(O) \cap W)$ is $f$-irreducible outside $U_{x'}$. Notice that $x' \neq x_s$ because $f(x_s) \in A$ and $\overline{U_{x'}} \cap \overline{A} = \emptyset$. If the pair $(x', f^{-1}(O) \cap W)$ is $f$-irreducible outside $A \cup U_{x'}$, put $x^* = x'$ and $U^* = U_{x'}$.

Consider the case when the pair $(x', f^{-1}(O) \cap W)$ is $f$-reducible outside $A \cup U_{x'}$. Take a neighborhood $W'$ of $x'$ such that $\overline{W'} \subset V_s$. Let

$$O' = Y \setminus (\overline{A} \cup \overline{U_{x'}})$$

Then the pair $(x', X')$ is $f$-irreducible outside $U_{x'}$ and $X' \notin \mathcal{I}_f$. As above, by Lemma 1 there exist a point $x'' \in X'$ and an open set $U_{x''} \subset Y$ strongly disjoint from $A \cup U_{x'}$ such that $f(x'') \in U_{x''}$ and the pair $(x'', X')$ is $f$-irreducible outside $U_{x''}$. Notice that $x'' \neq x_s$ and $x'' \neq x'$. From Lemma 2 it follows that the pair $(x'', X')$ is $f$-irreducible outside $A \cup U_{x''}$. Then put $x^* = x''$ and $U^* = U_{x''}$.

Let $k = |\{t \in 2^{<\omega}: t \prec s^1\}|$, $z_0 = x^*$, and $U_0 = U^*$. Repeating the above construction, for $j = 0, \ldots, k$ recursively construct $z_j \in V_s$ and $U_j$ such that $f(z_j) \in U_j$, $U_j$ is strongly disjoint from $A_j = A \cup \bigcup_{i < j} U_i$, and the pair $(z_j, V_s \cap (A_j)^f)$ is $f$-irreducible outside $A \cup U_j$. From Lemma 2 it follows that for each $t \prec s^1$ there is at most one $j \in \{0, \ldots, k\}$ such that $(x_t, V_t)$ is $f$-reducible outside $A \cup U_j$. The pigeonhole principle implies that there exists $\ell \in \{0, \ldots, k\}$ such that the pair $(z_\ell, V_s \cap (A_\ell)^f)$ is $f$-irreducible outside $A \cup U_\ell$ and $(x_\ell, V_\ell)$ is $f$-irreducible outside $A \cup U_\ell$ for each $t \prec s^1$. Finally, set $x_{s^1} = z_\ell$ and $U_{s^1} = U_\ell$.

Since $x_{s^0}$ and $x_{s^1}$ are two distinct points from $V_s$, we can choose their neighborhoods $V_{s^0}$ and $V_{s^1}$, respectively, according to (1),(2), and (6).
One readily verifies that conditions (1)–(7) are satisfied.

The set \( Z = \bigcup \{ x_s : s \in 2^{<\omega} \} \) is countable and has no isolated points by (1) and (2). According to the Sierpiński theorem [Eng, Exercise 6.2.A], \( Z \) is homeomorphic to the space of rational numbers. By construction, the set \( \bigcup \{ f(x_s) : s \in 2^{<\omega} \} \) consists of isolated points. From conditions (4) and (5) it follows that the restriction \( f \mid Z \) is a bijection.

From conditions (1) and (2) it follows that the family \( \mathcal{V}_n = \{ V_t : t \in 2^n \} \) forms a cover of \( Z \) by open sets of diameter \(\leq 2^{1-n}\) for each \( n \in \omega \). Then
\[
\overline{Z} \subset \bigcap \{ \bigcup \{ V_t : t \in 2^n \} : n \in \omega \}.
\]

Since the family \( \mathcal{V}_n \) is finite and pairwise strongly discrete, we can find a pairwise strongly discrete open family \( \mathcal{W}_n = \{ W_t : t \in 2^n \} \) such that \( \text{diam}(W_t) < 2^{-n} \) and \( V_t \subset W_t \) for each \( t \in 2^n \). Without loss of generality, each \( \mathcal{W}_{n+1} \) is a refinement of \( \mathcal{W}_n \). Every family \( \{ W \cap Z : W \in \mathcal{W}_n \}, n \in \omega \), forms a discrete open cover of \( \overline{Z} \). From the Vopěnka theorem (see [Eng, Theorem 7.3.1]) it follows that \( \dim(\overline{Z}) = 0 \). \( \square \)

**Theorem 4.** Every resolvable-measurable mapping \( f : X \to Y \) of a metrizable space \( X \) to a regular space \( Y \) is piecewise continuous.

**Proof.** Suppose towards a contradiction that there is a resolvable-measurable mapping \( f : X \to Y \) which is not piecewise continuous. Using Lemma 3 we can find a subset \( Z \subset X \) such that \( Z \) is homeomorphic to the space of rational numbers, the restriction \( f \mid Z \) is a bijection, and \( f(Z) \) is relatively discrete. Since \( f \) is a resolvable-measurable mapping, \( f \mid Z \) is the same. On the other hand, \( f \mid Z \) fails to be resolvable-measurable as shown in Example. \( \square \)

**Corollary 5.** Let \( f : X \to Y \) be a bijection between metrizable spaces \( X \) and \( Y \) such that \( f \) and \( f^{-1} \) are both resolvable-measurable mappings. Then \( \dim X = \dim Y \).

**Proof.** Theorem 4 implies that \( X = \bigcup_{n \in \omega} A_n \), where each \( A_n \) is closed in \( X \) and each restriction \( f \mid A_n \) is continuous. Similarly, \( Y = \bigcup_{k \in \omega} B_k \), where each \( B_k \) is closed in \( Y \) and each restriction \( f^{-1} \mid B_k \) is continuous. The sequence \( (A_n \cap f^{-1}(B_k) : n \in \omega, k \in \omega) \) forms a cover of \( X \) by closed sets. Similarly, the sequence \( (f(A_n) \cap B_k : n \in \omega, k \in \omega) \) forms a cover of \( Y \) by closed sets. Since \( f \mid (A_n \cap f^{-1}(B_k)) \) is a homeomorphism, we have
\[
\dim(A_n \cap f^{-1}(B_k)) = \dim(f(A_n) \cap B_k).
\]

The corollary follows from the countable sum theorem [Eng, Theorem 7.2.1]. \( \square \)

A topological space \( X \) is called a **Baire space** if the intersection of countably many dense open sets in \( X \) is dense; or equivalently every nonempty open set in \( X \) is not of the first category. A space \( X \) is **completely Baire** if every closed subspace of \( X \) is a Baire space. Recall that \( F \subset X \) is a **boundary set** in \( X \) if its complement is dense, i.e., if \( \overline{X \setminus F} = X \).

**Lemma 6.** For a metrizable space \( X \) the following conditions are equivalent:

(i) no closed subspace of \( X \) is homeomorphic to the space \( \mathbb{Q} \) of rational numbers,
(ii) $X$ is a completely Baire space,
(iii) the $\Delta_2^0(X)$-sets coincide with the resolvable sets in $X$.

Proof. (i)$\Rightarrow$(ii): Suppose towards a contradiction that $X$ is not a completely Baire space. Then there is a closed set $F \subset X$ which is not Baire. Hence we can find a nonempty open (in $F$) set $U \subset F$ of the first category in $F$. The closure $\overline{U}$ is of the first category on itself. According to [M86, Corollary 1] (see also [DS7]) $\overline{U}$ contains a closed copy of $\mathbb{Q}$, a contradiction.

(ii)$\Rightarrow$(iii): By [Kur1, p. 362], every resolvable set in a metrizable space is a $\Delta_2^0$-set. Conversely, let $E \in \Delta_2^0(X)$ and $F$ be an arbitrary non-empty closed set. According to [Kur1, p. 99], we have to show that that either $F \cap E$ or $F \setminus E$ is not a boundary set in $F$. Otherwise, the sets $F \cap E$ and $F \setminus E$ would be of the first category in $F$ (because every boundary $F_\sigma$-set is of the first category), so their union $F = (F \cap E) \cup (F \setminus E)$ would be of the first category on $F$. This contradicts the fact that $F$ is a Baire space.

(iii)$\Rightarrow$(i): Striving for a contradiction, suppose that $X$ contains a closed set $F$ which is homeomorphic to $\mathbb{Q}$. As shown in Example, there is a nonresolvable set $A \in \Delta_2^0(F)$. The set $A$ is the same in $X$ because $F$ is closed in $X$. □

Theorem 7. Let $f : X \to Y$ be a mapping of a metrizable completely Baire space $X$ to a regular space $Y$. Then the following conditions are equivalent:

(i) $f$ is resolvable-measurable,
(ii) $f$ is piecewise continuous,
(iii) $f$ is $\Pi_2^0$-measurable.

Proof. The implication (i)$\Rightarrow$(ii) follows from Theorem [3].

(ii)$\Rightarrow$(i): By definition, there are closed sets $X_n \subset X$, $n \in \omega$, such that $\bigcup_{n \in \omega} X_n = X$ and each $f \mid X_n$ is continuous. Then

$$f^{-1}(A) = \bigcup \{X_n \cap f^{-1}(A) : n \in \omega\}$$

is an $F_\sigma$-set in $X$ for every open (or closed) set $A \subset Y$. Hence $f^{-1}(U) \in \Delta_2^0(X)$ for every open $U \subset Y$. From Lemma [4] it follows that $f^{-1}(U)$ is a resolvable set in $X$.

Banakh and Bokalo [BB, Theorem 8.1] got (ii)$\Leftrightarrow$(iii). □

Corollary 8. Let $X$ be a completely metrizable space and $Y$ a regular space. Then $f : X \to Y$ is resolvable-measurable if and only if $f$ is $\Pi_2^0$-measurable.

According to [KMS, Corollary 6], for an absolute Souslin-$\mathcal{F}$ set $X$, if $f : X \to Y$ is $\Sigma_2^0$-measurable and not piecewise continuous, then there is a copy $K \subset X$ of the Cantor space $2^\omega$ such that $f \mid K$ has the same properties. The following theorem shows that a similar statement is valid for metrizable completely Baire spaces. However, such a set $K$ from Theorem [6] need not be homeomorphic to the Cantor space. In fact, every Bernstein set is a metrizable completely Baire space but it contains no copy of the Cantor space.
Theorem 9. Let $X$ be a metrizable completely Baire space and $Y$ a regular space. If $f : X \to Y$ is $\Sigma^0_2$-measurable and not piecewise continuous, then there is a zero-dimensional separable closed set $K \subset X$ such that the restriction $f \upharpoonright K$ is the same.

Proof. Let $K = \overline{Z}$, where the set $Z \subset X$ is obtained by Lemma 3. Clearly, $f \upharpoonright K$ is $\Sigma^0_2$-measurable.

Suppose towards a contradiction that $f \upharpoonright K$ is piecewise continuous. Then there are closed sets $K_n \subset X$, $n \in \omega$, such that $\bigcup_{n \in \omega} K_n = K$ and $f \upharpoonright K_n$ is continuous. Since $K$ is a Baire space, there exists a $K_j$ with the nonempty interior $V_j$ (in $K$). Clearly, $f \upharpoonright \overline{V_j \cap Z}$ is continuous. Take a point $q \in V_j \cap Z$. Fix a neighborhood $U_q \subset Y$ of $f(q)$ such that $U_q \cap f(Z) = f(q)$. From continuity of $f \upharpoonright \overline{V_j \cap Z}$ it follows that there is a neighborhood $V \subset V_j$ (in $K$) of $q$ such that $f(V) \subset U_q$. Then $V \cap Z = \{q\}$, i.e., $q$ is an isolated point of $Z$. This contradicts the fact that the set $V_j \cap Z$ has no isolated points. \(\square\)

The last theorem yields

Theorem 10. Let $f : X \to Y$ be an $F_\sigma$-measurable mapping of a metrizable completely Baire space $X$ to a regular space $Y$. If the restriction $f \upharpoonright Z$ is piecewise continuous for any zero-dimensional separable closed subset $Z$ of $X$, then $f$ is piecewise continuous.

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