SOME REMARKS CONCERNING
VOEVODSKY’S NILPOTENCE CONJECTURE

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Abstract. In this note we extend Voevodsky’s nilpotence conjecture from smooth projective schemes to the broader setting of smooth proper dg categories. Making use of this noncommutative generalization, we then address Voevodsky’s original conjecture in the case of quadric fibrations, intersection of quadrics, linear sections of Grassmannians, and Moishezon manifolds.

1. Introduction and statement of results

Let $k$ be a base field and $F$ a field of coefficients of characteristic zero.

Voevodsky’s nilpotence conjecture. In a foundational work [32], Voevodsky introduced the smash-nilpotence equivalence relation $\sim_{\otimes \text{nil}}$ on algebraic cycles and conjectured its agreement with the classical numerical equivalence relation $\sim_{\text{num}}$. Concretely, given a smooth projective $k$-scheme $X$, he stated the following:

Conjecture $V(X)$: $\mathbb{Z}^*(X)_F / \sim_{\otimes \text{nil}} = \mathbb{Z}^*(X)_F / \sim_{\text{num}}$.

Thanks to the work of Kahn-Sebastian, Matsusaka, Voevodsky, and Voisin (see [14, 27, 32, 33] and also [1, §11.5.2.3]), the above conjecture holds in the case of curves, surfaces, and abelian 3-folds (when $k$ is of characteristic zero).

Noncommutative nilpotence conjecture. A dg category $A$ is a category enriched over dg $k$-vector spaces; see §2.1. Following Kontsevich [16, 17, 18], $A$ is called smooth if it is perfect as a bimodule over itself and proper if for any two objects $x, y \in A$ we have $\sum_i \dim H^i A(x, y) < \infty$. The classical example is the unique dg enhancement $\text{perf}_{dg}(X)$ of the category of perfect complexes $\text{perf}(X)$ of a smooth projective $k$-scheme $X$; see Lunts-Orlov [22]. As explained in §2.3-2.4, the Grothendieck group $K_0(A)$ of every smooth proper dg category $A$ comes endowed with a $\otimes$-nilpotence equivalence relation $\sim_{\otimes \text{nil}}$ and with a numerical equivalence relation $\sim_{\text{num}}$. Motivated by the above conjecture, we state the following:

Conjecture $V_{NC}(A)$: $K_0(A)_F / \sim_{\otimes \text{nil}} = K_0(A)_F / \sim_{\text{num}}$.

Our first main result is the following reformulation of Voevodsky’s conjecture:

Theorem 1.1. Conjecture $V(X)$ is equivalent to conjecture $V_{NC}(\text{perf}_{dg}(X))$.

Theorem 1.1 shows us that when restricted to the commutative world, the noncommutative nilpotence conjecture reduces to Voevodsky’s original conjecture. Making use of this noncommutative viewpoint, we now address Voevodsky’s nilpotence conjecture in several cases.
Quadric fibrations. Let $S$ be a smooth projective $k$-scheme and $q : Q \to S$ a flat quadric fibration of relative dimension $n$ with $Q$ smooth. Recall from Kuznetsov [20] (see also [5]) the construction of the sheaf $C_0$ of even parts of the Clifford algebra associated to $q$. Recall also from loc. cit. that when the discriminant divisor of $q$ is smooth and $n$ is even (resp. odd) we have a discriminant double cover $\tilde{S} \to S$ (resp. a square root stack $\tilde{S}$) equipped with an Azumaya algebra $B_0$. Our second main result allows us to decompose conjecture $V(Q)$ into simpler pieces:

**Theorem 1.2.** The following holds:

(i) We have $V(Q) \Leftrightarrow V_{NC}(\text{perf}_{dg}(S, C_0)) + V(S)$.

(ii) When the discriminant divisor of $q$ is smooth and $n$ is even, we have

$$V(Q) \Leftrightarrow V(\tilde{S}) + V(S).$$

As a consequence, $V(Q)$ holds when $\dim(S) \leq 2$, and becomes equivalent to $V(\tilde{S})$ when $S$ is an abelian 3-fold and $k$ is of characteristic zero.

(iii) When the discriminant divisor of $q$ is smooth and $n$ is odd, we have $V(Q) \Leftrightarrow V_{NC}(\text{perf}_{dg}(\tilde{S}, B_0)) + V(S)$. As a consequence, $V(Q)$ becomes equivalent to $V_{NC}(\text{perf}_{dg}(\tilde{S}, B_0))$ when $\dim(S) \leq 2$. This latter conjecture holds when $S$ is a curve or a rational surface and $k$ is algebraically closed.

Intersection of quadrics. Let $X$ be a smooth complete intersection of $r$ quadric hypersurfaces in $\mathbb{P}^m$. The linear span of these $r$ quadrics gives rise to a hypersurface $Q \subset \mathbb{P}^{r-1} \times \mathbb{P}^m$, and the projection into the first factor to a flat quadric fibration $q : Q \to \mathbb{P}^{r-1}$ of relative dimension $m - 1$.

**Theorem 1.3.** The following holds:

(i) We have $V(X) \Leftrightarrow V_{NC}(\text{perf}_{dg}(\mathbb{P}^{r-1}, C_0))$.

(ii) When the discriminant divisor of $q$ is smooth and $m$ is odd, we have

$$V(X) \Leftrightarrow V(\widehat{\mathbb{P}^{r-1}}).$$

As a consequence, $V(X)$ holds when $r \leq 3$.

(iii) When the discriminant divisor of $q$ is smooth and $m$ is even, we have

$$V(X) \Leftrightarrow V_{NC}(\text{perf}_{dg}(\widehat{\mathbb{P}^{r-1}}, B_0)).$$

This latter conjecture holds when $r \leq 3$ and $k$ is algebraically closed.

**Remark 1.4.** (Relative version) Theorem 1.3 has a relative analogue with $X$ replaced by a generic relative complete intersection $X \to S$ of $r$ quadric fibrations $Q_i \to S$ of relative dimension $m - 1$; consult [5, Def. 1.2.4] for details. Items (i), (ii), and (iii), hold similarly with $\mathbb{P}^{r-1}$ replaced by a $\widehat{\mathbb{P}^{r-1}}$-bundle $T \to S$, with $V(\widehat{\mathbb{P}^{r-1}})$ replaced by $V(\widehat{T}) + V(S)$, and with $V_{NC}(\text{perf}_{dg}(\widehat{\mathbb{P}^{r-1}}, B_0))$ replaced by $V_{NC}(\text{perf}_{dg}(\widehat{T}, B_0)) + V(S)$, respectively. Note that thanks to the relative item (ii), conjecture $V(X)$ holds when $r = 2$ and $S$ is a curve.

Linear sections of Grassmanians. Following Kuznetsov [21], consider the following two classes of schemes:

(i) Let $X$ be a generic linear section of codimension $r$ of the Grassmanian $\text{Gr}(2, W)$ (with $W = \mathbb{P}^6$) under the Plücker embedding, and $Y$ the corresponding dual linear section of the cubic Pfaffian $\text{Pf}(4, W^*)$ in $\mathbb{P}(\Lambda^2 W^*)$.

For example when $r = 3$, $X$ is a Fano 5-fold; when $r = 4$, $X$ is a Fano 4-fold; and when $r = 6$, $X$ is a $K3$ surface of degree 14 and $Y$ a Pfaffian cubic 4-fold. Moreover, $X$ and $Y$ are smooth whenever $r \leq 6$. 

(ii) Let $X$ be a generic linear section of codimension $r$ of the Grassmannian $\text{Gr}(2, W)$ (with $W = k^{8\times7}$) under the Plücker embedding, and $Y$ the corresponding dual linear section of the cubic Pfaffian $\text{Pf}(4, W^*)$ in $\mathbb{P}(\mathbb{A}^2 W^*)$.

For example when $r = 5$, $X$ is a Fano 5-fold; when $r = 4$, $X$ is a Fano 4-fold; when $r = 8$, $Y$ is a Fano 4-fold; and when $r = 9$, $Y$ is a Fano 5-fold. Moreover, $X$ and $Y$ are smooth whenever $r \leq 10$.

**Theorem 1.5.** Let $X$ and $Y$ be as in the above classes (i)-(ii). Under the assumption that $X$ and $Y$ are smooth, we have $V(X) \leftrightarrow V(Y)$. This conjecture holds when $r \leq 6$ (class (i)), and when $r \leq 6$ and $8 \leq r \leq 10$ (class (ii)).

**Moishezon manifolds.** A Moishezon manifold $X$ is a compact complex manifold such that the field of meromorphic functions on each component of $X$ has transcendence degree equal to the dimension of the component. As proved by Moishezon [28], $X$ is a smooth projective $\mathbb{C}$-scheme if and only if it admits a Kähler metric. In the remaining cases, Artin [3] showed that $X$ is a proper algebraic space over $\mathbb{C}$.

Let $Y = \mathbb{P}^2$ be one of the non-rational conic bundles described by Artin and Mumford in [4], and $X \rightarrow Y$ a small resolution. In this case, $X$ is a smooth (non necessarily projective) Moishezon manifold.

**Theorem 1.6.** Conjecture $V_{NC}(\text{perf}_{dg}(X))$ holds for the above resolutions.

## 2. Preliminaries

### 2.1. Dg categories.** A differential graded (=dg) category $\mathcal{A}$ is a category enriched over dg $k$-vector spaces; consult Keller [15] for details. For example, every (dg) $k$-algebra $A$ gives naturally rise to a dg category $\mathcal{A}$ with a single object. Let $\text{dgcat}$ be the category of small dg categories. Recall from [15, §3] the construction of the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$. This triangulated category admits arbitrary direct sums and we will write $\mathcal{D}_c(\mathcal{A})$ for the full subcategory of compact objects. A dg functor $\mathcal{A} \rightarrow \mathcal{B}$ is called a Morita equivalence if it induces an equivalence $\mathcal{D}(\mathcal{A}) \simto \mathcal{D}(\mathcal{B})$. Finally, let us write $\mathcal{A} \otimes \mathcal{B}$ for the tensor product of dg categories.

### 2.2. Perfect complexes.** Given a stack $\mathcal{X}$ and a sheaf of $\mathcal{O}_\mathcal{X}$-algebras $\mathcal{G}$, let $\text{Mod}(\mathcal{X}, \mathcal{G})$ be the Grothendieck category of sheaves of (right) $\mathcal{G}$-modules, $\mathcal{D}(\mathcal{X}, \mathcal{G}) := \mathcal{D}(\text{Mod}(\mathcal{X}, \mathcal{G}))$ the derived category of $\mathcal{G}$, and $\text{perf}(\mathcal{X}, \mathcal{G})$ the subcategory of perfect complexes. As explained in [15, §4.4], the derived category $\mathcal{D}_\text{dg}(\mathcal{E}x)$ of an abelian (or exact) category $\mathcal{E}x$ is defined as the (Drinfeld’s) dg quotient $\mathcal{C}_\text{dg}(\mathcal{E}x)/\mathcal{A}_\text{dg}(\mathcal{E}x)$ of the dg category of complexes over $\mathcal{E}x$ by its full dg subcategory of acyclic complexes. Hence, let us write $\mathcal{D}_\text{dg}(\mathcal{X}, \mathcal{G})$ for the dg category $\mathcal{D}_\text{dg}(\mathcal{E}x)$ with $\mathcal{E}x := \text{Mod}(\mathcal{X}, \mathcal{G})$ and $\text{perf}_{\text{dg}}(\mathcal{X}, \mathcal{G})$ for the full dg subcategory of perfect complexes.

**Lemma 2.1.** Let $X$ be a smooth projective $k$-scheme and $\text{perf}(X) = \langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle$ a semi-orthogonal decomposition. In this case, the dg categories $\mathcal{T}_i^{\text{dg}}$ (where $\mathcal{T}_i^{\text{dg}}$ stands for the dg enhancement of $\mathcal{T}_i$ induced from $\text{perf}_{\text{dg}}(X)$) are smooth and proper.

**Proof.** Let $\text{Ho(dgcat)}$ be the localization of $\text{dgcat}$ with respect to the class of Morita equivalences. The tensor product of dg categories gives rise to a symmetric monoidal structure on $\text{dgcat}$ which descends to $\text{Ho(dgcat)}$. Moreover, as proved in [8, Thm. 5.8], the smooth and proper dg categories can be characterized as those objects of $\text{Ho(dgcat)}$ which are dualizable. Note that the canonical inclusion $\mathcal{T}_i^{\text{dg}} \hookrightarrow \text{perf}_{\text{dg}}(X)$ and projection $\text{perf}_{\text{dg}}(X) \rightarrow \mathcal{T}_i^{\text{dg}}$ dg functors express $\mathcal{T}_i^{\text{dg}}$...
as a direct factor of $\text{perf}_{\text{dg}}(X)$ in $\text{Ho}(\text{dgcat})$. Hence, since $\text{perf}_{\text{dg}}(X)$ is smooth and proper, we conclude that $T'_{\text{dg}}$ is also smooth and proper.

2.3. $\otimes$-nilpotence equivalence relation. Let $\mathcal{A}$ be a dg category. An element $[M]$ of the Grothendieck group $K_0(\mathcal{A}) := K_0(D_c(\mathcal{A}))$ is called $\otimes$-nilpotent if there exists an integer $n > 0$ such that $[M]^\otimes n = 0$ in $K_0(\mathcal{A}^\otimes n)$. This gives rise to a well-defined equivalence relation $\sim_{\otimes \text{nil}}$ on $K_0(\mathcal{A})$ and on its $F$-linearization $K_0(\mathcal{A})_F$.

2.4. Numerical equivalence relation. Let $\mathcal{A}$ be a smooth proper dg category. As explained in [24, §4], the pairing $(M, N) \mapsto \sum_i (-1)^i \dim \text{Hom}_{D_c(\mathcal{A})}(M, N[i])$ gives rise to a well-defined bilinear form $\chi(-, -)$ on $K_0(\mathcal{A})$. Moreover, the left and right kernels of $\chi(-, -)$ are the same. An element $[M]$ of the Grothendieck group $K_0(\mathcal{A})$ is called numerically trivial if $\chi([M], [N]) = 0$ for all $[N] \in K_0(\mathcal{A})$. This gives rise to an equivalence relation $\sim_{\text{num}}$ on $K_0(\mathcal{A})$ and on $K_0(\mathcal{A})_F$.

2.5. Motives. We assume the reader is familiar with the categories of Chow motives $\text{Chow}(k)_F$ and numerical motives $\text{Num}(k)_F$; see [1, §4]. The Tate motive will be denoted $F(1)$. In the same vein, we assume some familiarity with the categories of noncommutative Chow motives $\text{NChow}(k)_F$ and noncommutative numerical motives $\text{NNNum}(k)_F$; consult the surveys [25, §2-3] [30, §4] and the references therein. In particular, we have $\text{Hom}_{\text{NChow}(k)_F}(\underline{\mathbb{L}}, \mathcal{A}) \simeq K_0(\mathcal{A})_F$.

3. ORBIT CATEGORIES AND $\otimes$-NILPOTENCE

Let $\mathcal{C}$ be an $F$-linear additive rigid symmetric monoidal category.

**Orbit categories.** Given a $\otimes$-invertible object $O \in \mathcal{C}$, recall from [29, §7] the construction of the orbit category $\mathcal{C}_{/\otimes O}$. It has the same objects as $\mathcal{C}$ and morphisms

$$\text{Hom}_{\mathcal{C}_{/\otimes O}}(a, b) := \oplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a, b \otimes O^\otimes j).$$

The composition law is induced from $\mathcal{C}$. By construction, $\mathcal{C}_{/\otimes O}$ is $F$-linear, additive, and comes equipped with a canonical projection functor $\pi : \mathcal{C} \to \mathcal{C}_{/\otimes O}$. Moreover, $\pi$ is endowed with a natural 2-isomorphism $\pi \circ (- \otimes O) \cong \pi$ and is 2-universal among all such functors. As proved in [29, Lem. 7.3], $\mathcal{C}_{/\otimes O}$ inherits from $\mathcal{C}$ a symmetric monoidal structure making $\pi$ symmetric monoidal. On objects it is the same. On morphisms it is defined as the unique bilinear pairing

$$\oplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a, b \otimes O^\otimes j) \times \oplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(c, d \otimes O^\otimes j) \longrightarrow \oplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a \otimes c, (b \otimes d) \otimes O^\otimes j)$$

which sends the pair $(a \xrightarrow{f} b \otimes O^\otimes r, c \xrightarrow{g} d \otimes O^\otimes s)$ to

$$(f \otimes g)(r+s) : a \otimes c \xrightarrow{f \otimes g} b \otimes O^\otimes r \otimes d \otimes O^\otimes s \simeq (b \otimes d) \otimes O^\otimes (r+s).$$

**$\otimes$-nilpotence.** The $\otimes_{\text{nil}}$-ideal of $\mathcal{C}$ is defined as

$$\otimes_{\text{nil}}(a, b) := \{ f \in \text{Hom}_{\mathcal{C}}(a, b) \mid f^\otimes n = 0 \text{ for } n \gg 0 \}.$$

By construction, $\otimes_{\text{nil}}$ is a $\otimes$-ideal. Moreover, all its ideals $\otimes_{\text{nil}}(a, a) \subset \text{Hom}_{\mathcal{C}}(a, a)$ are nilpotent; see [2, Lem. 7.4.2 (ii)]. As a consequence, the $\otimes$-functor $\mathcal{C} \to \mathcal{C}_{/\otimes_{\text{nil}}}$ is not only $F$-linear and additive but moreover conservative. Furthermore, since idempotents can be lifted along nilpotent ideals (see [6, §III Prop. 2.10]), $\mathcal{C}_{/\otimes_{\text{nil}}}$ is idempotent complete whenever $\mathcal{C}$ is idempotent complete.
Compatibility. Let $\mathcal{C}$ be a category and $O \in \mathcal{C}$ a $\otimes$-invertible objects as above.

**Proposition 3.1.** There exists a canonical $F$-linear additive $\otimes$-equivalence $\theta$ making the following diagram commute:

$$
\begin{array}{ccc}
C/\otimes_{\text{nil}} & \otimes \longrightarrow & C/\otimes O \\
\downarrow & & \downarrow \\
(C/\otimes_{\text{nil}})/\otimes O & \sim & (C/\otimes O)/\otimes_{\text{nil}}.
\end{array}
$$

**Proof.** The existence of the $F$-linear additive $\otimes$-functor $\theta$ follows from the fact that

$$
\mathcal{C} \longrightarrow \mathcal{C}/\otimes_O \longrightarrow (\mathcal{C}/\otimes O)/\otimes_{\text{nil}}
$$

vanishes on the $\otimes_{\text{nil}}$-ideal and also from the natural 2-isomorphism between $(3.3)$ $\circ$ $\sim$ $(3.3)$. Note that the functor $\theta$ is the identity on objects and sends $\{[f_j]_{j \in \mathbb{Z}}\}$ to $\{[f_j]_{j \in \mathbb{Z}}\}$. Clearly, it is full. The faithfulness is left as an exercise. \(\square\)

4. $\otimes$-nilpotence of motives

By construction, the categories $\text{Chow}(k)_F$ and $\text{NChow}(k)_F$ are $F$-linear, additive, rigid symmetric monoidal, and idempotent complete. Let us denote by

$\text{Voev}(k)_F := \text{Chow}(k)_F/\otimes_{\text{nil}}$ and $\text{NVoev}(k)_F := \text{NChow}(k)_F/\otimes_{\text{nil}}$

the associated quotients. They fit in the following sequences

$\text{Chow}(k)_F \rightarrow \text{Voev}(k)_F \rightarrow \text{Num}(k)_F \quad \text{NChow}(k)_F \rightarrow \text{NVoev}(k)_F \rightarrow \text{NNum}(k)_F$.

The relation between all these motivic categories is the following:

**Proposition 4.1.** There exist $F$-linear additive fully-faithful $\otimes$-functors $R, R_{\otimes_{\text{nil}}}, R_N$ making the following diagram commute:

$$
\begin{array}{ccc}
\text{Chow}(k)_F & \xrightarrow{\pi} & \text{Chow}(k)_F/\otimes_{(1)} \xrightarrow{R} \text{NChow}(k)_F \\
\downarrow & & \downarrow \\
\text{Voev}(k)_F & \xrightarrow{\pi} & \text{Voev}(k)_F/\otimes_{(1)} \xrightarrow{R_{\otimes_{\text{nil}}}} \text{NVoev}(k)_F \\
\downarrow & & \downarrow \\
\text{Num}(k)_F & \xrightarrow{\pi} & \text{Num}(k)_F/\otimes_{(1)} \xrightarrow{R_N} \text{NNum}(k)_F.
\end{array}
$$

**Proof.** The outer commutative square, with $R, R_N$ $F$-linear additive fully-faithful $\otimes$-functors, was built in [23, Thm. 1.13]. Consider now the “zoomed” diagram:

$$
\begin{array}{ccc}
\text{Chow}(k)_F/\otimes_{(1)} & \xrightarrow{\pi} & \text{Chow}(k)_F/\otimes_{(1)} \xrightarrow{R} \text{NChow}(k)_F \\
\downarrow & & \downarrow \\
\text{Voev}(k)_F/\otimes_{(1)} & \xrightarrow{\pi} & (\text{Chow}(k)_F/\otimes_{(1)})/\otimes_{\text{nil}} \xrightarrow{R_{\otimes_{\text{nil}}}} \text{NVoev}(k)_F \\
\downarrow & & \downarrow \\
\text{Num}(k)_F/\otimes_{(1)} & \xrightarrow{\pi} & \text{Num}(k)_F/\otimes_{(1)} \xrightarrow{R_N} \text{NNum}(k)_F.
\end{array}
$$

By definition, $R_{\otimes_{\text{nil}}} := R/\otimes_{\text{nil}} \circ \theta$. Hence, since $R$ is an $F$-linear additive fully-faithful $\otimes$-functor, we conclude that $R_{\otimes_{\text{nil}}}$ is also an $F$-linear additive fully-faithful
\(\otimes\)-functor. The commutativity of the bottom squares of diagram (4.3) follows from the fact that \(\text{Num}(k)_{F/\otimes F(1)}\) identifies with the quotient of \(\text{Chow}(k)_{F/\otimes F(1)}\) by its largest \(\otimes\)-ideal \(\mathcal{N}\); consult [23, Prop. 3.2] for details. \(\square\)

5. Proof of Theorem 1.1

Note first that we have the following natural isomorphisms

\[
\text{Hom}_{\text{Voev}(k)_{F/\otimes F(1)}}(\text{Spec}(k), X) \simeq \mathcal{Z}^* \otimes_{\otimes \text{nil}} F/\otimes \text{nil}
\]

\[
\text{Hom}_{\text{Num}(k)_{F/\otimes F(1)}}(\text{Spec}(k), X) \simeq \mathcal{Z}^* \otimes_{\otimes \text{num}} F/\otimes \text{num}.
\]

As a consequence, conjecture \(V(X)\) becomes equivalent to the injectivity of

\[
(5.1) \quad \text{Hom}_{\text{Voev}(k)_{F/\otimes F(1)}}(\text{Spec}(k), X) \rightarrow \text{Hom}_{\text{Num}(k)_{F/\otimes F(1)}}(\text{Spec}(k), X).
\]

Given a smooth and proper dg category \(\mathcal{A}\), we have also natural isomorphisms

\[
\text{Hom}_{\text{Voev}(k)_{F/\otimes F(1)}}(\underline{k}, \mathcal{A}) \simeq K_0(\mathcal{A})_{F/\otimes \text{nil}} \quad \text{Hom}_{\text{Num}(k)_{F/\otimes F(1)}}(\underline{k}, \mathcal{A}) \simeq K_0(\mathcal{A})_{F/\otimes \text{num}}.
\]

Hence, conjecture \(V_{NC}(\mathcal{A})\) becomes equivalent to the injectivity of

\[
(5.2) \quad \text{Hom}_{\text{Voev}(k)_{F/\otimes F(1)}}(\underline{k}, \mathcal{A}) \rightarrow \text{Hom}_{\text{Num}(k)_{F/\otimes F(1)}}(\underline{k}, \mathcal{A}).
\]

Now, recall from [29, Thm. 1.1] that the image of \(X\) under the composed functor \(R \circ \pi\) identifies naturally with the noncommutative Chow motive \(\text{perf}_{dg}(X)\). Similarly, the image of \(\text{Spec}(k)\) under \(R \circ \pi\) identifies with \(\text{perf}_{dg}(\text{Spec}(k))\) which is Morita equivalent to \(\underline{k}\). As a consequence, since the functors \(R_{\otimes \text{nil}}\) and \(R_N\) are fully-faithful, the bottom right-hand side square of diagram (4.2) gives rise to the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Voev}(k)_{F/\otimes F(1)}}(\text{Spec}(k), X) & \rightarrow & \text{Hom}_{\text{Num}(k)_{F/\otimes F(1)}}(\underline{k}, \text{perf}_{dg}(X)) \\
(5.1) & & (5.2) \\
\text{Hom}_{\text{Num}(k)_{F/\otimes F(1)}}(\text{Spec}(k), X) & \rightarrow & \text{Hom}_{\text{Num}(k)_{F/\otimes F(1)}}(\underline{k}, \text{perf}_{dg}(X)).
\end{array}
\]

Using the above reformulations of conjectures \(V\) and \(V_{NC}\), we conclude finally that conjecture \(V(X)\) is equivalent to conjecture \(V_{NC}(\text{perf}_{dg}(X))\).

6. Proof of Theorem 1.2

**Item (i).** As proved by Kuznetsov in [20, Thm. 4.2], one has the following semi-orthogonal decomposition

\[
\text{perf}(Q) = \langle \text{perf}(S,C_0), \text{perf}(S)_1, \ldots, \text{perf}(S)_n \rangle
\]

with \(\text{perf}(S)_i := q^* \text{perf}(S) \otimes \mathcal{O}_{Q/S}(i)\). Note that \(\text{perf}(S)_1 \simeq \text{perf}(S)\) for every \(i\).

Using [7, Prop. 3.1], one then obtains a direct sum decomposition in \(\mathcal{N}_{\text{Chow}}(k)_F\):

\[
(6.1) \quad \text{perf}_{dg}(Q) \simeq \text{perf}_{dg}(S,C_0) \oplus \text{perf}_{dg}(S) \oplus \cdots \oplus \text{perf}_{dg}(S),
\]

where \(\text{perf}_{dg}(S,C_0)\) stands for the dg enhancement of \(\text{perf}(S,C_0)\) induced from \(\text{perf}_{dg}(Q)\). Note that thanks to Lemma 2.1, the dg category \(\text{perf}_{dg}(S,C_0)\) is smooth and proper. Since the inclusion of categories \(\text{perf}(S,C_0) \hookrightarrow \text{perf}(Q)\) is of Fourier-Mukai type (see [20, Prop. 4.9]), its kernel \(\mathcal{K} \in \text{perf}(S \times Q, C_0^{op} \otimes \mathcal{O}_X)\) gives rise to a Fourier-Mukai Morita equivalence \(\Phi_{dg}: \text{perf}_{dg}(S,C_0) \rightarrow \text{perf}_{dg}(S,C_0)\). Hence, we can replace in the above decomposition (6.1) the dg category \(\text{perf}_{dg}(S,C_0)\) by the...
The proof follows now automatically from Theorem 1.1.

**Item (ii).** As proved by Kuznetsov in [20, Prop. 3.13], \( \text{perf}(S, \mathcal{C}_0) \) is Fourier-Mukai equivalent to \( \text{perf}(\widehat{S}, B_0) \). Hence, the above equivalence (6.2) reduces to

\[
V_{NC}(\text{perf}_{dg}(Q)) \iff V_{NC}(\text{perf}_{dg}(\widehat{S}, B_0)) + V_{NC}(\text{perf}_{dg}(S)).
\]

Since \( B_0 \) is a sheaf of Azumaya algebras and \( F \) is of characteristic zero, the canonical dg functor \( \text{perf}_{dg}(\widehat{S}) \to \text{perf}_{dg}(\widehat{S}, B_0) \) becomes an isomorphism in \( \text{NChow}(k) \); see [31, Thm. 2.1]. Consequently, conjecture \( V_{NC}(\text{perf}_{dg}(\widehat{S}, B_0)) \) reduces to conjecture \( V_{NC}(\text{perf}_{dg}(\widehat{S})) \). The proof follows now from Theorem 1.1.

**Item (iii).** As proved by Kuznetsov in [20, Prop. 3.15], \( \text{perf}(S, \mathcal{C}_0) \) is Fourier-Mukai equivalent to \( \text{perf}(\widehat{S}, B_0) \). Hence, the above equivalence (6.2) reduces to

\[
V_{NC}(\text{perf}_{dg}(Q)) \iff V_{NC}(\text{perf}_{dg}(\widehat{S}, B_0)) + V_{NC}(\text{perf}_{dg}(S)).
\]

The proof of the first claim follows now from Theorem 1.1.

Let us now prove the second claim, which via (6.4) is equivalent to the proof of \( V_{NC}(\text{perf}_{dg}(Q)) \). Assume that \( k \) is algebraically closed and that the relative dimension \( n \) is greater than 2. Thanks to the work of Graber-Harris-Starr [11], the fibration \( q : Q \to S \) has a regular section when \( S \) is a curve. When \( S \) is a rational surface, Tsen’s Theorem (see [12, Part I]), combined with the fact that \( K(S) \) is a \( C_2 \) field, implies that \( q \) has also a rational section. Using such section, one can perform reduction by hyperbolic splitting in order to obtain a conic bundle \( q' : Q' \to S \); consult [5, §1.3] for details\(^1\). The sheaf \( \mathcal{C}_0' \) of even parts of the associated Clifford algebra is such that the categories \( \text{perf}(S, \mathcal{C}_0) \) and \( \text{perf}(S, \mathcal{C}_0') \) are Fourier-Mukai equivalent; see [5, Rk. 1.8.9]. As a consequence, thanks to (6.2) and the fact that \( \dim(S) \leq 2 \), we obtain the following equivalence:

\[
V_{NC}(\text{perf}_{dg}(Q)) \iff V_{NC}(\text{perf}_{dg}(Q')).
\]

When \( S \) is a curve, \( Q' \) is a surface. Hence, conjecture (6.5) holds. On the other hand, when \( S \) is a rational surface, the Chow motive of the conic bundle \( Q' \) admits a decomposition into Lefschetz motives and submotives of curves; consult Gorchinskiy-Guletskii [10, §5-6] for details. This implies also conjecture (6.5).

### 7. Proof of Theorem 1.3

**Item (i).** As proved by Kuznetsov in [20, Thm. 5.5], we have a Fourier-Mukai equivalence \( \text{perf}(X) \simeq \text{perf}(\mathbb{P}^{r-1}, \mathcal{C}_0) \) when \( m - 2r + 1 = 0 \), the following semi-orthogonal decomposition

\[
\text{perf}(X) = \langle \text{perf}(\mathbb{P}^{r-1}, \mathcal{C}_0), \mathcal{O}_X(1), \ldots, \mathcal{O}_X(m - 2r + 1) \rangle,
\]

when \( m - 2r + 1 > 0 \), and a dual semi-orthogonal decomposition of \( \text{perf}(\mathbb{P}^{r-1}, \mathcal{C}_0) \) (containing a copy of \( \text{perf}(X) \)) and exceptional objects) when \( m - 2r + 1 < 0 \). The proof of the case \( m - 2r + 1 = 0 \) is clear. Let us now prove the case \( m - 2r + 1 > 0 \);

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\(^1\)The smoothness of the discriminant divisors ensures the genericity of \( Q \to S \) and therefore a rational section is enough to perform hyperbolic splitting; see [5, Lem. 1.8].
the proof of the case \( m - 2r + 1 < 0 \) is similar. Using [7, Prop. 3.11], one obtains the following direct sum decomposition in \( \text{NChow}(k)_F \)

\[
\text{perf}_{dg}(X) \simeq \text{perf}_{dg}(\mathbb{P}^{r-1}, \mathcal{C}_0) \oplus \left( \text{perf}_{dg}(k) \oplus \cdots \oplus \text{perf}_{dg}(k) \right) .
\]

Thanks to Lemma 2.1, the dg category \( \text{perf}_{dg}(\mathbb{P}^{r-1}, \mathcal{C}_0) \) is smooth and proper\(^2\).

Since \( \text{perf}(\mathbb{P}^{r-1}, \mathcal{C}_0) \to \text{perf}(X) \) is of Fourier-Mukai type (see [20, Prop. 4.9]), an argument similar to the one of the proof of Theorem 1.2(i) shows us that

\[
V_{NC}(\text{perf}_{dg}(X)) \Leftrightarrow V_{NC}(\text{perf}_{dg}(\mathbb{P}^{r-1}, \mathcal{C}_0)) + V_{NC}(\text{perf}_{dg}(k)).
\]

The proof follows now automatically from Theorem 1.1.

**Item (ii)-(iii).** The proofs are similar to those of items (ii)-(iii) of Theorem 1.2.

### 8. Proof of Theorem 1.5

As proved by Kuznetsov in [21, §10-11], whenever \( X \) and \( Y \) are smooth projective \( k \)-schemes as in the classes (i)-(ii), one of the following three situations occurs:

(a) there is a semi-orthogonal decomposition \( \text{perf}(X) = \langle \text{perf}(Y), \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle \), with the \( \mathcal{E}_i \)'s exceptional bundles on \( X \);

(b) there is a semi-orthogonal decomposition \( \text{perf}(Y) = \langle \text{perf}(X), \mathcal{E}_1', \ldots, \mathcal{E}_n' \rangle \), with the \( \mathcal{E}_i' \)'s exceptional bundles on \( Y \);

(c) there is a Fourier-Mukai equivalence between \( \text{perf}(X) \) and \( \text{perf}(Y) \).

Therefore, equivalence \( V(X) \Leftrightarrow V(Y) \) is clear in situation (c). Since the inclusions of categories \( \text{perf}(Y) \hookrightarrow \text{perf}(X) \) (situation (a)) and \( \text{perf}(X) \hookrightarrow \text{perf}(Y) \) (situation (b)) are of Fourier-Mukai type, a proof similar to the one of Theorem 1.2(i) shows us that equivalence \( V(X) \Leftrightarrow V(Y) \) also holds in situations (a)-(b).

Let us now focus on class (i). The smooth projective \( k \)-schemes \( X \) and \( Y \) are of dimensions \( 8 - r \) and \( r - 2 \), respectively. Hence, \( V(Y) \) holds when \( r \leq 4 \) and \( V(X) \) when \( r = 6 \). When \( r = 5 \), \( X \) (and \( Y \)) is a Fano 3-fold. As explained by Gorchinskiy and Guletskii in [10, §5], the Chow motive of \( X \) admits a decomposition into Lefschetz motives and submotives of curves. This implies that \( V(X) \) also holds.

Let us now focus on the class (ii). The smooth projective \( k \)-schemes \( X \) and \( Y \) are of dimensions \( 10 - r \) and \( r - 4 \), respectively. Hence, \( V(Y) \) holds when \( r \leq 6 \) and \( V(X) \) when \( r \geq 8 \). This achieves the proof.

### 9. Proof of Theorem 1.6

Thanks to the work of Cossec [9], the conic bundle \( Y \to \mathbb{P}^2 \) has a natural structure of quartic double solid \( Y \to \mathbb{P}^3 \) ramified along a quartic symmetroid \( D \).

Via the natural involution on the resolution of singularities of \( D \), one hence obtains an Enriques surface \( S \); consult [9, §3] for details. As proved by Zube in [34, §5], one has moreover a semi-orthogonal decomposition

\[
\text{perf}(S) = \langle \mathcal{T}_S, \mathcal{E}_1, \ldots, \mathcal{E}_{10} \rangle,
\]

with the \( \mathcal{E}_i \)'s exceptional objects. Let us denote by \( \mathcal{T}^{dg}_S \) the dg enhancement of \( \mathcal{T}_S \) induced from \( \text{perf}_{dg}(S) \). Thanks to Lemma 2.1, \( \mathcal{T}^{dg}_S \) is smooth and proper. Hence,

\(^2\)In the case \( m - 2r + 1 < 0 \), these properties follow from the existence of a fully faithful Fourier-Mukai functor \( \text{perf}(\mathbb{P}^{r-1}, \mathcal{C}_0) \to \text{perf}(Q) \), with \( Q \subset \mathbb{P}^{r-1} \times \mathbb{P}^m \) a smooth hypersurface.
since $S$ is a surface, an argument similar to the one of the proof of Theorem 1.2(i) shows us that conjecture $V_{NC}(T_{S}^{dg})$ holds.

Now, recall from Ingalls and Kuznetsov [13, §5.5] the construction of the Fourier-Mukai functor $\Phi : \text{perf}(S) \to \text{perf}(X)$ whose restriction to $T_{S}$ is fully-faithful. As proved in [13, Prop. 3.8 and Thm. 4.3], one has a semi-orthogonal decomposition

$$\text{perf}(X) = \langle \Phi(T_{S}), E'_{1}, E'_{2} \rangle,$$

with the $E'_{i}$’s exceptional objects. As a consequence, we obtain the equivalence

$$V_{NC}(\text{perf}_{dg}(X)) \iff V_{NC}(\Phi(T_{S})^{dg}),$$

where $\Phi(T_{S})^{dg}$ stands for the dg enhancement of $\Phi(T_{S})$ induced from $\text{perf}_{dg}(X)$.

Since the kernel $K$ of the above Fourier-Mukai functor $\Phi$ gives rise to a Morita equivalence $\Phi_{dg}^{K} : T_{S}^{dg} \to \Phi(T_{S})^{dg}$, we conclude that the conjecture (9.1) also holds. This achieves the proof.

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