AN IMPROVED METHOD FOR RECURSIVELY COMPUTING UPPER BOUNDS FOR TWO-COLOUR RAMSEY NUMBERS

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ABSTRACT. The two-colour Ramsey number $R(m, n)$ is the least natural number $p$ such that any graph of order $p$ must contain either a clique of size $m$ or an independent set of size $n$. We exhibit a method for computing upper bounds for $R(m, n)$ recursively, using known upper bounds of $R(\cdot, \cdot)$ with lower values for at least one of the arguments. We also give an example of how this method could be used to improve several of the best known bounds that are available in the literature (which however soon will be obsolete due to a forthcoming work).

1. Introduction

We say that a graph $G$ is a $(m, n)$-graph if it contains no clique of size $m$ and no independent set of size $n$. If the order of $G$ is $p$, then we say that $G$ is a $(m, n; p)$-graph. The two-colour Ramsey number $R(m, n)$ is defined to be the least natural number $p$ such that there are no $(m, n; p)$-graphs. A list of bounds on these numbers are maintained in a dynamic survey authored by Radziszowski in [5].

We present a new method for computing upper bounds on Ramsey numbers $R(m, n)$ recursively from known upper bounds of $R(m_0, n_0)$, where $m_0 \leq m$ and $n_0 \leq n$ with at least one of the inequalities being strict. We will also show how this method can be effectively used to improve the upper bounds on several of the upper bounds of $R(m, n)$ listed in [5]. These improvements will soon however be obsolete due to a forthcoming work by Angeltveit and McKay (see Remark 1).

The new method presented in this paper is an enhancement of the method of [4] to derive bounds on the minimum edge numbers of $(m, n; p)$-graphs. These may then be used to obtain stronger results on $(m, n + 1)$- and $(m + 1, n)$-graphs.

2. The new method

Let $e(m, n; p)$ and $E(m, n; p)$ denote the minimum and maximum number of edges in a $(m, n; p)$-graph, respectively. We will denote the complement of a graph $G$ by $G \bar{\text{G}}$. The subgraph of $G$ induced by the neighbours of a vertex $v$ is $G^+_v$, while the subgraph that is induced by the vertices that are not adjacent to $v$ is $G^-_v$. $N(K_3; G)$ is the number of triangles in $G$ and $N(K_3; G, v)$ is the number of triangles in $G$ that contain the vertex $v$. $n_d$ denotes the number of vertices in $G$ that has degree $d$, and $d_v$ the degree of the vertex $v$.

The methods used to prove the following two theorems are similar to those used to prove Lemma 2. In particular we use Goodman’s lemma for counting the total

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number of triangles in \( G \) and \( \overline{G} \) (see \[2\] Lemma 1)), which states that if \( p \) is the order of \( G \) then
\[
N(K_3; G) + N(K_3; \overline{G}) = \binom{p}{3} - \frac{1}{2} \sum_v d_v (p - d_v - 1).
\]

**Theorem 1.** Let \( p, \alpha, \beta, \gamma, \delta \) be as in Lemma 2. Then
\[
(p - 1)(p - 2) \leq \max_{d \in [p - 1 - \delta, \gamma]} \left( \binom{p - d - 1}{2} + 2\Delta(m, n, p, d) + 3d(p - d - 1) \right),
\]
where
\[
\Delta(m, n, p, d) = E(m - 1, n; d) - e(m, n - 1; p - d - 1).
\]

**Proof.** Let \( G \) be a \((m, n; p)\)-graph. Clearly \( d_v \in I := [p - 1 - \delta, \gamma] \) for all vertices \( v \in V(G) \), since \( G^+_v \) is a \((m - 1, n; d_v)\)-graph and \( G^-_v \) is a \((m, n - 1; p - 1 - d_v)\)-graph. Note that \( N(K_3; \overline{G}, v) \leq \binom{p - d_v - 1}{2} - e(m, n - 1; p - d_v - 1) \) and \( N(K_3; G, v) \leq E(m - 1, n; d_v) \). Thus we have
\[
3N(K_3; \overline{G}) + 3N(K_3; G) \leq \sum_{d \in I} \left( \binom{p - d - 1}{2} + \Delta(m, n, p, d) \right) n_d.
\]
By a straightforward application of Goodman’s lemma we get
\[
p(p - 1)(p - 2) \leq \sum_{d \in I} \left( 2\binom{p - d - 1}{2} + 2\Delta(m, n, p, d) + 3d(p - d - 1) \right) n_d,
\]
and the lemma follows. \( \square \)

The minimal edge numbers, \( e(m, n; p) \), have been studied previously. For low values of \( m, n \) and \( p \) the exact value of these are known. The following lemma will however give us sufficiently good bounds for \( e(m, n; p) \) (and maximal edge numbers \( E(m, n; p) \)) for use in Theorem 1 to derive the new bounds listed in Table 1 and 2.

**Theorem 2.** Let \( p, \alpha, \beta, \gamma, \delta \) be as in Lemma 2. Then
\[
e(m, n; p) \geq \max \left\{ \frac{p(p - \delta - 1)}{2}, \frac{A - \sqrt{A^2 - B}}{12} \right\}
\]
\[
E(m, n; p) \leq \min \left\{ \frac{p\gamma}{2}, \frac{A + \sqrt{A^2 - B}}{12} \right\},
\]
where
\[
A = (\alpha - \beta + 3(p - 1))p, \quad B = 12p^2(p - 1)(p - \beta - 2).
\]

**Proof.** Let \( G \) be a \((m, n; p)\)-graph. The bounds \( e(m, n; p) \geq \frac{p(p - \delta - 1)}{2} \) and \( E(m, n; p) \leq \frac{p\gamma}{2} \) are clear since \( d_v \in [p - 1 - \delta, \gamma] \) for all \( v \in V(G) \). Using the bounds \( 3N(K_3; G) \leq \sum_{v \in V(G)} \frac{d_v}{2} \) and \( 3N(K_3; \overline{G}) \leq \sum_{v \in V(G)} \frac{\beta(p - 1 - d_v)}{2} \) together with Goodman’s theorem we get
\[
6\binom{p}{3} - \beta p(p - 1) \leq \sum_v ((\alpha - \beta + 3(p - 1))d_v - 3d_v^2). \tag{1}
\]
Therefore, by the handshaking lemma and the Cauchy-Schwartz inequality, we have
\[
6\binom{p}{3} - \beta p(p - 1) \leq 2(\alpha - \beta + 3(p - 1))e - 12e^2/p, \quad \text{which we can rewrite as} \quad 0 \leq -12e^2 + 2Ae - B/12, \quad \text{which gives us that} \quad e \in \left[ (A - \sqrt{A^2 - B})/12, (A + \sqrt{A^2 - B})/12 \right]. \tag{2} \] \( \square \)
3. Example: Application to small Ramsey numbers as given in the literature

We will now show how the method described in the previous section could be applied to obtain improved upper bounds for some small Ramsey numbers. Note, however, that although this is an improvement on the best upper bounds in the literature these are not the best known bounds (see Remark 1).

We will also use some previously known methods to compute upper bounds and indicate where in the example calculation these are sufficient. The following recursive classical bound, which appears in [3], will be used as the most trivial way to obtain new bounds.

Lemma 1 (Greenwood and Gleason [3]). \( R(m,n) \leq R(m-1,n) + R(m, n-1) \), with strict inequality if both \( R(m-1,n) \) and \( R(m, n-1) \) are even.

The essence of [4, Theorem 1], in the context that we will use it, is the following lemma.

Lemma 2 (Huang, Wang, Sheng, Yang, Zhang and Huang [4]). If \( p \leq R(m, n) - 1 \), \( \alpha \geq R(m-2, n) - 1 \), \( \beta \geq R(m, n-2) - 1 \), \( \gamma \geq R(m-1, n) - 1 \) and \( \delta \geq R(m, n-1) - 1 \), then

\[
(p-1)(p-2-\alpha) \leq \max_{d \in [p-1-\delta, \gamma]} (-3d^2 + (\alpha - \beta + 3(p-1))d + (\beta - \alpha)(p-1)).
\]

A selection of the upper bounds we can obtain with the methods presented in this paper have been summarised in the following tables. Boldface indicates that the bound is improved compared to the one in [5] and the superscript letters indicate by what method the bound has been obtained.

Remark 1. Angeltveit and McKay [1] have improved on many of the upper bounds that are listed in [5], completely independently of this paper. Their improvements are stronger and therefore the bounds listed in the tables below are generally not the best known. Their work is however still in progress, and has not yet been published.

The numbers actually listed in the below table may therefore be considered obsolete. The method used to compute them is however interesting, and this example calculation is made to illustrate its potential.

| m | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|----|----|----|----|----|----|
| n | 142 | 215 | 316 | 442 | 629 | 846 | 1102 | 1442 | 1832 |
| 6 | 48 | 165 | 298 | 405 | 780 | 1171 | 1782 | 2549 | 3526 | 4927 | 6614 |
| 7 | 539 | 1029 | 1711 | 2775 | 4518 | 6821 | 10017 | 14841 | 20928 |
| 8 | 1865 | 3576 | 6582 | 12643 | 23327 | 45488 | 80231 | 139767 | 236772 | 385139 |
| 9 | | | | | | | | | | | |
| 10 | | | | | | | | | | | |

Table 1. Upper bounds of \( R(m, n) \). Boldface indicates improved bounds (compared to [5]). Superscript is a if Lemma 1 is sufficient, b if Lemma 2 is sufficient but c if this bound requires the new methods from Theorems 1 and 2. Diagonal entries improved by using [3, Theorem 5.1]. Also see Remark 1.
Table 2. Upper bounds of $R(m,n)$. Same conventions as for Table 1.

| $m$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|-----|----|----|----|----|----|----|----|----|
| 4   | 514° | 615° | 720° | 851° | 988° | 1120° | 1300° | 1476° |
| 5   | 2321° | 2916° | 3576° | 4397° | 5350° | 6381° | 7651° | 9074° |
| 6   | 8745° | 11096° | 14903° | 19637° | 24272° | 30177° | 37497° | 46374° |

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