D-brane Superpotentials and Geometric Invariants in Complete Intersection Calabi-Yau Manifolds

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Abstract

Abstract: By blowing up the ambient space along the curve wrapped by B-branes, we study the brane superpotentials and Ooguri-Vafa invariants on complete intersections Calabi-Yau threefolds. On the topological B-model side, B-brane superpotentials are expressed in terms of the period integral of the blow-up manifolds. By mirror maps, the superpotentials are generating functions of Ooguri-Vafa invariants counting holomorphic disks on the topological A-model side.

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1 Introduction

For open topological string theory, A-models admit A-branes wrapping special Lagrangian cycles on $M^*$ described by Fukaya category, and B-models admit B-branes wrapping holomorphic cycles on $M$ described by derived category of coherent sheaves [23]. Then, open mirror symmetry implies the category equivalence $\text{Fuk}(M^*) = D^b(M)$.

As the topological sector of the superpymmetric spacetime effective Lagrangian, D-brane superpotential plays an important role in describing string theory compactifications. It can be obtained by dimension reduction of the holomorphic Chern-Simons theory [31] or analysis of $A_\infty$-structure [4, 5] in derived category. The A-brane superpotential is generated by disk instantons [19, 20] and its expansion at the large volume point encodes the Ooguri-Vafa invariants counting the holomorphic disks ending on A-branes [28]. The B-brane superpotential arises from variation of mixed Hodge structure, which depends on the brane moduli and the complex structure moduli of the Calabi-Yau threefold. It satisfied the extended Picard-Fuchs differential system [3, 10, 25] and measures the obstructions to deforming D-brane in the open string direction [14].

Superpotentials and Ooguri-Vafa invariant for toric branes on compact Calabi-Yau threefolds $M$ in [3]. The B-brane geometry is captured by an auxiliary divisor $D$. The extended Picard-Fuchs equation associated to a enhanced polyhedron, arises from N=1 special geometry [9, 24], and can be obtained by Griffith-Dwork reduction method [26, 29] or GKZ-system [15, 17], governing the variation of Hodge structure of $(M, D)$ [18]. It is argued in [11, 12] that under blowing up calabi-Yau threefold $M$ along curve $\Sigma$ wrapped by B-branes, the deformation theory of the blow-up manifold $M'$ with an exceptional divisor $E$ is equivalent to the deformation theory of the original manifold $M$ with the curve $\Sigma$. Then, the B-brane superpotential can be expressed in terms of period integral of $X'$. In this work, we mainly study the B-brane superpotentials and Ooguri-Vafa invariants on compact complete intersection Calabi-Yaus (CICY) threefolds by blow-up constructions. Periods integrals are solved by GKZ-system, and then used to find mirror maps and superpotentials. Also, Ooguri-Vafa invariants are extracted from the A-brane superpotentials at large volume point.

The organization of this paper is as follows. In section 2, we review the construction of complete intersection Calabi-Yau manifolds in weighted projective space and variation of mixed Hodge structure, and outline the procedure to blow up a curve on a Calabi-Yau threefold. In section 3, for one-parameter CICYs $(M_{2,4},M_{3,4})$, we apply the blowing-up method and obtain the new mani-
folds. The Picard-Fuchs equations and their solutions are derived by GKZ hypergeometric system from toric information of the enhanced polyhedron. The superpotential are identified as double-logarithmic solutions of the Picard-Fuchs equations and Ooguri-Vafa invariants are extracted at large volume phase of A-model. The last section is a brief summary and outlook for further research. In Appendix A, we summarize the GKZ-system of blow-ups resulting from blowing up curves on $M_{2,6}, M_{3,6}, M_{4,6}$. In Appendix B, we present the Ooguri-Vafa invariants on aforementioned models.

2 Toric Geometry and Blowing up

2.1 Complete Intersection Calabi-Yau and GKZ-System

The complete intersection Calabi-Yau threefolds $M^* \subset \mathbb{P}^5_{(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5)}$ considered here are described by the reflexive polyhedron $\Delta = \Delta_1 \times \Delta_2$ [7, 8]. The mirror threefold $M_{d_1,d_2}$ is the intersection of the vanishing locus of two homogeneous polynomials of degree $d_1$ and $d_2$, captured by the combinatorial data of the dual polyhedron $\Delta^*$ with vertices $\nu^*_{\ell} = (1,0,\ldots,0), \ldots, \nu^*_5 = (0,\ldots,0,1), \nu^*_6 = (-\omega_1, -\omega_2, -\omega_3, -\omega_4, -\omega_5)$.

The vertices are partitioned into two groups $\{\nu^*_i\} = E_1 \cup E_2$, where $E_i$ contains $d_i$ vertices, known as the nef-partition. Each vertex $\nu^*_i$ of $E_i$ is extended to $\bar{\nu}^*_i = (e(i), \nu^*_i)$. With adding extra vertices $\bar{\nu}^*_{0,i} = (e(i),0)$, the new vertices $\bar{\nu}^*_i$ satisfy the relations,

$$\sum l^i \bar{\nu}^*_i, = 0, \quad l = (-d_1, -d_2; \omega_1, \omega_2, \omega_3, \omega_4, \omega_5),$$

The mirror manifold $M_{d_1,d_2}$ is defined by the following Laurent polynomials in the torus coordinates $X_i$ [6, 16],

$$P_r = a_{0,r} - \sum_{\nu^*_i \in E_r} a_i X^{\nu^*_i}, r = 1, 2 \quad (2.1)$$

The period integral of $M$,

$$\sigma(a) = \int \frac{a_1 \cdots a_6}{P_1 P_2} \prod_{i=1}^5 \frac{dX_i}{X_i},$$
is annihilated by the following GKZ-system,

\[ \mathcal{L} = \prod_{l_j > 0} \left( \frac{\partial}{\partial a_j} \right)^{l_j} - \prod_{l_j < 0} \left( \frac{\partial}{\partial a_j} \right)^{-l_j}, \]

\[ \mathcal{Z}_i = \sum_j (\bar{v}_j)^i \vartheta_j - \beta_i, \quad i = 0, \ldots, 5 \]

Here \( \beta \) is the exponent, \( \vartheta_j = a_j \frac{\partial}{\partial a_j} \) is the logarithmic derivative and \( l_j \) corresponds to the maximal triangulation of \( \Delta^* \).

For open topological string theory, the A-brane configuration on \( M^* \) is specified by \( l \) with another two relation \( l_i, i = 1, 2 \), such that \( \sum_j l_j = 0 \)\(^\text{[1, 2]} \). The corresponding B-brane geometry is described by the curve \( \Sigma \) on the mirror threefold \( M \),

\[ \Sigma : \{ P_1 = P_2 = 0 \} \cap \{ Q(D_1) = 0 \} \cap \{ Q(D_2) = 0 \}, \]

where \( Q(D) \) are divisors defined by vertices on polyhedron \( \Delta^* \). The polyhedron \( \Delta \) can be extended to a higher dimensional polyhedron \( \Delta' \), called the enhanced polyhedron, by adding vertices on the original vertices \( \Delta \). The GKZ-system associated to the enhanced polyhedron annihilates the relative period integral and is used as a way to understand brane geometry in \( \text{[3]} \).

The coordinates on open-closed moduli space is given by

\[ z_j = (-a_0)^{l_j} \prod_i a_i^{l_j}, \quad (2.3) \]

For appropriate choice of basis vector \( l_j \), solutions to the GKZ system can be written in terms of the deformed gamma series,

\[ \sigma(z; \rho) = \sum \frac{\Gamma(1 - \Sigma_j l_j(n_j + \rho_j))}{\prod_{l_j > 0} \Gamma(1 + \Sigma_j l_j(n_j + \rho_j))} \prod_k z_k^{n_j + \rho_j} \]

then we have a natural basis for the period integral,

\[ \omega_0(z) = \sigma(z; \rho)|_{\rho \to 0}, \]

\[ \omega_{1,i}(z) = \frac{\partial_{\rho_i} \sigma(z; \rho)|_{\rho \to 0}}, \]

\[ \omega_{2,i}(z) = \sum_{j,k} K_{ijk} \partial_{\rho_j} \partial_{\rho_k} \sigma(z; \rho)|_{\rho \to 0} \]

...(2.4)
and at the large complex structure point $z = 0$,

$$\omega_0(z) \sim 1 + \mathcal{O}(z),$$

$$\omega_{1,i}(z) \sim \log(z_i),$$

$$\omega_{2,i}(z) \sim \log(z_j) \log(z_k),$$

Here $\omega_{1,i}$ define the open-closed mirror map by

$$t_i(z) = \frac{\omega_{1,i}(z)}{\omega_0(z)}, \quad q_i = e^{2\pi i t_i},$$

and the special solution $\Pi = \mathcal{W}_{\text{open}}(z)$ has instanton expansion near the large volume point, containing the Ooguri-Vafa invariants of the A-brane geometry,

$$\mathcal{W}_{\text{inst}}(q) = \sum_{\beta} \sum_{k=1}^{\infty} N_{\beta} \frac{q^k \beta}{k^2}. $$

2.2 Relative Period and Variation of Mixed Hodge Structure

For the submanifold $\Sigma$, embedded by the map $i : \Sigma \hookrightarrow M$ in the Calabi-Yau threefold $M$, the space of relative forms $\Omega^* (M, \Sigma)$ is the subspace of forms $\Omega(M)$, defined as the kernel of the pullback, $i^* : \Omega(M) \to \Omega^*(\Sigma)$. Consider the exact sequence,

$$0 \to \Omega^*(M, \Sigma) \overset{i^*}{\to} \Omega^*(M) \to \Omega^*(\Sigma) \to 0$$

Then, the relative cohomology groups $H^*(X, S)$ arise from the space of closed modulo exact relative forms with respect to the de Rham differential and the long exact sequence for the three-form cohomology group is obtained,

$$H^3(M, \Sigma) \cong \ker(H^3(M) \to H^3(\Sigma)) \oplus \operatorname{coker}(H^2(M) \to H^2(\Sigma)), $$

Thus, we can represent a relative three form $\Theta$ as a pair of closed three form $\Theta$ and a closed two form $\theta$,

$$\Theta = (\Theta, \theta) \in H^3(M, \Sigma), $$

and obey the equivalence relation,

$$\Theta \sim \Theta + (d\alpha, i^*\alpha - d\beta), $$
where $\alpha$ is a two form on $M$, and $\beta$ is a one form on $\Sigma$. Similarly, the relative homology group $H_3(M, \Sigma)$ consists of relative three cycles $\Gamma$ with boundaries $\partial \Gamma$ lying in $S$. The duality pairing between $\Gamma$ and $\Theta$ is given by,

$$\int_{\Gamma} \Theta = \int_{\Gamma} \Theta - \int_{\partial \Gamma} \theta,$$

In particular, the relative period,

$$\Pi^a(z, \hat{z}) = \int_{\Gamma} \Omega,$$

where $\Omega$ is a relative three form, and $\Gamma^a$ is a relative three cycle. The moduli dependence of the period is captured by the variation of the mixed Hodge structure given by cohomology group $H^3(X, S, \mathbb{Z})$ and the filtration,

$$F_3 = H^3_3(M),$$

$$F_2 = H^3_3(M) \oplus H^2_2(M),$$

$$F_1 = H^3_3(M) \oplus H^2_2(X, \Sigma) \oplus H^{1,2}_3(M),$$

$$F_0 = H^3_3(M) \oplus H^2_2(M, \Sigma) \oplus H^{1,2}_3(M) \oplus H^{0,3}_3(M, \Sigma).$$

and the weight filtration induced from is

$$W_3 \cong H^3(M), \quad W_4 \cong H^3(M) \oplus H^2_{var}(\Sigma) \cong H^3(M, \Sigma),$$

Notice that an infinitesimal closed-string complex structure deformation $\partial z$ changes the Hodge type of a $(p, q)$-form and an infinitesimal open-string deformation $\partial u$ affects the two-form sector $H^{2}_{var}(X)$, thus the two deformations as tangent vectors in the open/closed string moduli space, act on the mixed Hodge structure as:

$$F_3 \cap W_3 \xrightarrow{\partial z} F_2 \cap W_3 \xrightarrow{\partial z} F_1 \cap W_3 \xrightarrow{\partial z} F_0 \cap W_3$$

$$F_2 \cap W_4 \xrightarrow{\partial z, \partial z} F_1 \cap W_4 \xrightarrow{\partial z, \partial z} F_0 \cap W_4$$

and the two form sector $H^2_{var}(X) \cong W_4/W_3$ constitutes a subsystem,

$$F_2 \cap (W_4/W_3) \xrightarrow{\partial z, \partial z} F_1 \cap (W_4/W_3) \xrightarrow{\partial z, \partial z} F_0 \cap (W_4/W_3)$$

which is essential to derive and solve the Picard-Fuchs equations related to the relative period.
2.3 Blowing Up

The construction and properties of blowing up a manifold along its submanifold is an integral part of this work. For more detail about blowing up, we refer to [13].

The blow-up threefold $M'$ resulting from blowing up the threefold $M$ along its submanifold $\Sigma$ is obtained by gluing local blow-ups. Consider multidisks $U_\alpha$ on $M$ with holomorphic coordinates $x_{\alpha,i}, i = 1, 2, 3$, and $V_\alpha$ specified by $x_{\alpha,1} = x_{\alpha,2} = 0$ on each $U_\alpha$. The local blow-up is defined by,

$$\tilde{U}_\alpha = \{(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}, (y_1 : y_2)) \subset U_\alpha \times \mathbb{P}^1 : x_{2,\alpha} y_1 - x_{\alpha,1} y_2 = 0\},$$

where $y_1, y_2$ are the homogeneous coordinates on $\mathbb{P}^1$. The manifold $\tilde{U}_\alpha$ is the blow-up of $U_\alpha$ along $V_\alpha$, under the projection map $\pi_\alpha : \tilde{U}_\alpha \to U_\alpha$, with the inverse image $E_\alpha = \pi_\alpha^{-1}(V_\alpha)$ an exceptional divisor of $\tilde{U}_\alpha$. The coordinates patches $U_1 = \{y_1 \neq 0\}$ and $U_2 = \{y_2 \neq 0\}$ have holomorphic coordinates

$$z_i^{(1)} = x_{\alpha,1}, \quad z_2^{(1)} = \frac{y_2}{y_1} = \frac{x_{\alpha,2}}{x_{\alpha}}, \quad y_3 = x_{\alpha,3},$$

$$z_1^{(2)} = \frac{y_1}{y_2} = \frac{x_{\alpha,1}}{x_{\alpha,2}}, \quad z_2^{(1)} = x_{\alpha,2}, \quad y_3 = x_{\alpha,3},$$

with transition function on $U_1 \cap U_2$,

$$g_{ij} = z_i^{(j)} = \frac{y_i}{y_j} = \frac{x_{\alpha,i}}{x_{\alpha,j}},$$

Let $\{U_\alpha\}$ be a collection of disks in $M$ covering $\Sigma$ such that $V_\alpha = \Sigma \cap U_\alpha$. The local blow-ups $\tilde{U}_\alpha$ can be patched up to a manifold $\tilde{U} = \cup \pi_{\alpha\beta} \tilde{U}_\alpha$ by the projection map,

$$\pi_{\alpha\beta} : \pi_{\alpha\beta}^{-1}(U_\alpha \cap U_\beta) \to \pi^{-1}_\beta(U_\alpha \cap U_\beta),$$

$M' = \tilde{U} \cup \pi M - \Sigma$, together with $\pi : X' \to X$, is called the blow-up of $M$ along $\Sigma$, with exceptional divisor $E$.

The excision theorem of cohomology [30] implies that,

$$H^3(M, \Sigma) \cong H^3(M - \Sigma) \cong H^3(M' - E) \cong H^3(M', E)$$

which leads to the equivalence between the mixed Hodge structures of $H^3(M, \Sigma)$ and the mixed Hodge structure $H^3(M', E)$ over the corresponding moduli space. Thus, the blow-up $M'$ can be used to compute superpotentials with brane geometry $(M, \Sigma)$. 

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Given $M$ and $\Sigma$ as follows,

\[ M : P_1 = P = 2 = 0, \]
\[ \Sigma : P_1 = P_2 = 0, h_1 = h_2 = 0, \]

where $h_1$ and $h_2$ are two divisors. The blow-up manifold $M'$ is given globally as the complete intersection in the total space of the projective bundle $\mathcal{W} = \mathbb{P}(\mathcal{O}(h_1) \oplus \mathcal{O}(h_2))$.

\[ P_1 = P_2 = 0, \quad Q \equiv y_1 h_2 - y_2 h_1 = 0 \tag{2.6} \]

where $(y_1, y_2) \sim \lambda (y_1, y_2)$ is the projective coordinates on the $\mathbb{P}^1$-fiber of the blow-up $X'$.

The holomorphic three form $\Omega'$ on $M'$ is pullback of the holomorphic three form on $M$ by the projection map,

\[ \Omega' = \int_{S_{P_1}} \int_{S_{Q}} \frac{h_i \omega_{P_1}}{Q} \wedge \frac{\omega}{P_1 P_2}, \quad i = 1, 2 \]

Here $\omega$ and $\omega_{P_1}$ denote the invariant holomorphic top form on the $M$ and $\mathbb{P}_1$, respectively. $S_{P_1}$ and $S_{Q}$ are small loops around $P_1 = P_2 = Q = 0$ encircling only the corresponding poles.

The period integral of $\Omega'$ is annihilated by the GKZ-system associated with the enhanced polyhedron. This GKZ-system and explicit construction of the enhanced polyhedron will be present by our examples in details.

### 3 Superpotentials and Ooguri-Vafa Invariants on one parameter CICY

In this section, we study one-parameter compact CICYs with curves wrapped by B-branes, and obtain B-brane superpotentials and Ooguri-Vafa invariants by GKZ-system.
3.1 Branes on CICY $M_{2,4}$

3.1.1 GKZ-system and Blowing Up Geometry

The threefold $M^*$ associated to A-model is described by the polyhedron $\Delta^*$ with one internal point $\nu^* = (0,0,0,0,0)$ and six vertices,

$$v_1^* = (1,0,0,0,0), \quad v_2^* = (0,1,0,0,0), \quad v_3^* = (0,0,1,0,0),$$
$$v_4^* = (0,0,0,1,0), \quad v_5^* = (0,0,0,0,1), \quad v_6^* = (-1,-1,-1,-1,-1),$$

We group the vertices into two sets $E_1 = \{v_1^*, v_2^*\}$ and $E_2 = \{v_3^*, v_4^*, v_5^*, v_6^*\}$, extend the vertices $\bar{v}^* = (\bar{v}_1^*, \bar{v}_2^*)$ and add two vertices $\nu_{0,k}^* = (\bar{v}_{0,k}^*, 0)$. This leads to the defining equation of mirror Calabi-Yau $M_{2,4}$ as the following two Laurent polynomials by equation 2.1:

\[
\begin{cases}
    P_1 = a_{0,1} - a_1X_1 - a_2X_2 \\
    P_2 = a_{0,2} - a_2X_2 - a_3X_3 - a_4X_4 - a_5X_5 - a_6(X_1X_2X_3X_4X_5)^{-1},
\end{cases}
\]

or in homogeneous coordinates [27],

\[
\begin{cases}
    P_1 = a_{0,1}x_1^4x_5x_6 - a_1x_1^4 - a_2x_2^4 \\
    P_2 = a_{0,2}x_1x_2 - a_3x_3^2 - a_4x_4^2 - a_5x_5^2 - a_6x_6^2,
\end{cases}
\]

where $a$ is complex-valued free parameter. The linear relation among vertices,

$$l = (-2, -4; 1, 1, 1, 1, 1, 1),$$

and vertex coordinates of $\Delta^*$ give rise to the GKZ-system by equation 2.2,

$$\mathcal{L}_{0,1} = \vartheta_{1,0} + \vartheta_1 + \vartheta_2, \quad \mathcal{L}_{0,2} = \vartheta_{2,0} + \sum_{i=3}^{6} \vartheta_i, \quad \mathcal{L}_i = -\vartheta_i + \vartheta_6, \quad i = 1, \ldots, 5,$$

$$\mathcal{L}_1 = \prod_{i=1}^{6} \frac{\partial}{\partial a_i} - (\frac{\partial}{\partial a_{0,1}})^2(\frac{\partial}{\partial a_{0,2}})^4,$$

with $\vartheta_i = a_i \frac{\partial}{\partial a_i}$ logarithmic derivative. Here $\mathcal{L}_{0,1}$ and $\mathcal{L}_{0,2}$ correspond to the invariance of $P_1, P_2$ under overall rescaling, $\mathcal{L}_i$ relate to the torus symmetry,

$$\mathcal{L}_i : \quad X_i \mapsto \lambda X_i, \quad (a_i, a_6) \mapsto (\lambda^{-1} a_i, \lambda a_6), \quad i = 1, \ldots, 5,$$
and \( \mathcal{L}_i \) represents the relations among monomials in \( P_1 \) and \( P_2 \),

\[
\mathcal{L}_i : \prod_{i=1}^{6} \frac{\partial}{\partial a_i} (P_1 P_2) = \left( \frac{\partial}{\partial a_{0,1}} \right)^2 \left( \frac{\partial}{\partial a_{0,2}} \right)^4 (P_1 P_2),
\]

The B-branes wrap on the curve \( \Sigma \) on \( M_{2,4} \),

\[
\{ P_1 = P_2 = 0 \} \cap \{ h_1 = a_7 X_3 + a_8 X_4 = 0 \} \cap \{ h_2 = a_9 X_3 + a_{10} X_5 = 0 \},
\]

with two extra linear relation \( l^1 = (0, 0; 0, 0, -1, 1, 0, 0), l^2 = (0, 0; 0, 0, -1, 0, 1, 0) \) among vertices. After blowing up \( M_{2,4} \) along the curve \( \Sigma \), we obtain the blow-up manifold \( M' \) as the complete intersection the projective bundle,

\[
M' : P_1 = P_2 = 0, \quad Q = y_1 (a_9 X_3 + a_{10} X_5) - y_2 (a_7 X_3 + a_8 X_4) = 0, \quad (3.2)
\]

with \( (y_1, y_2) \) are homogeneous coordinates on \( \mathbb{P}^1 \).

The GKZ-system that annihilates period integral of the blow-up \( M' \) is obtained from the GKZ-system \( 3.1 \) of \( M_{2,4} \) by observing the invariance under overall rescalling and torus symmetry of equation \( 3.2 \). Due to the presence of equation \( Q = 0 \), there are new invariance under overall scaling of \( Q = 0 \), unchanged torus symmetry \( \mathcal{Z}_i, i = 1, 2 \) and new torus symmetry,

\[
\mathcal{Z}_3' : \quad X_3 \mapsto \lambda X_3, \quad (a_3, a_6, a_7, a_9) \mapsto (\lambda^{-1} a_3, \lambda a_6, \lambda^{-1} a_7, \lambda^{-1} a_9)
\]

\[
\mathcal{Z}_4' : \quad X_4 \mapsto \lambda X_4, \quad (a_4, a_6, a_8) \mapsto (\lambda^{-1} a_4, \lambda a_6, \lambda^{-1} a_8)
\]

\[
\mathcal{Z}_5' : \quad X_5 \mapsto \lambda X_5, \quad (a_5, a_6, a_{10}) \mapsto (\lambda^{-1} a_5, \lambda a_6, \lambda^{-1} a_{10})
\]

\[
\mathcal{Z}_6' : \quad (y_1, y_2) \mapsto (\lambda y_1, \lambda^{-1} y_2).
\]

In addition, there are new algebraic relation between monomials in \( Q \),

\[
\frac{\partial (P_1 P_2)}{\partial a_4} \frac{\partial Q}{\partial a_7} = \frac{\partial (P_1 P_2)}{\partial a_3} \frac{\partial Q}{\partial a_8},
\]

\[
\frac{\partial (P_1 P_2)}{\partial a_5} \frac{\partial Q}{\partial a_9} = \frac{\partial (P_1 P_2)}{\partial a_3} \frac{\partial Q}{\partial a_{10}}.
\]
Then, the whole GKZ-system of $M'$ is:

$$\mathcal{Z}'_{0,1} = \vartheta_{0,1} + \vartheta_1 + \vartheta_2, \quad \mathcal{Z}'_{0,2} = \vartheta_{0,2} + \sum_{i=3}^{6} \vartheta_i, \quad \mathcal{Z}'_{3,0} = \sum_{i=7}^{10} \vartheta_i,$$

$$\mathcal{Z}'_1 = -\vartheta_1 + \vartheta_6, \quad \mathcal{Z}'_2 = -\vartheta_2 + \vartheta_6, \quad \mathcal{Z}'_3 = -\vartheta_3 + \vartheta_6 - \vartheta_7 + \vartheta_9,$$

$$\mathcal{Z}'_4 = -\vartheta_4 + \vartheta_6 - \vartheta_8, \quad \mathcal{Z}'_5 = -\vartheta_5 + \vartheta_6 - \vartheta_{10}, \quad \mathcal{Z}'_6 = -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10},$$

$$\mathcal{L}'_1 = \prod_{i=1}^{6} \frac{\partial}{\partial a_i} - \left( \frac{\partial}{\partial a_{0,1}} \right)^2 \left( \frac{\partial}{\partial a_{0,2}} \right)^4,$$

$$\mathcal{L}'_2 = \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_8}, \quad \mathcal{L}'_3 = \frac{\partial}{\partial a_5} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_{10}},$$

which relates to the following extended polyhedron $\Delta'$.

| $v'_i$ | $\Delta'$ | $l'_1$ | $l'_2$ | $l'_3$ |
|--------|-----------|--------|--------|--------|
| $v'_{0,1}$ | 1 0 0 0 0 0 0 0 0 | -2 | 0 | 0 |
| $v'_{0,2}$ | 0 1 0 0 0 0 0 0 0 | -4 | 0 | 0 |
| $v'_1$ | 1 0 1 0 0 0 0 0 0 | 1 | 0 | 0 |
| $v'_2$ | 1 0 0 0 1 0 0 0 0 | 1 | 0 | 0 |
| $v'_3$ | 0 1 0 0 0 1 0 0 0 | 3 | -1 | -1 |
| $v'_4$ | 0 1 0 0 0 0 1 0 0 | 0 | 1 | 0 |
| $v'_5$ | 0 1 0 0 0 0 0 1 0 | 0 | 0 | 1 |
| $v'_6$ | 0 1 0 -1 -1 -1 -1 -1 0 | 1 | 0 | 0 |
| $v'_7$ | 0 0 1 0 0 1 0 0 -1 | -1 | 1 | 0 |
| $v'_8$ | 0 0 0 0 0 0 1 0 -1 | 1 | -1 | 0 |
| $v'_9$ | 0 0 1 0 0 1 0 0 1 | -1 | 0 | 1 |
| $v'_{10}$ | 0 0 1 0 0 0 1 1 1 | 1 | 0 | -1 |

Table 1: Toric Information of the Extended Polyhedron

Here $v'_i$ are the integral vertices of the polyhedron, $l'_i$ correspond to the maximal triangulation of $\Delta'$, such that $l = l'_1 + l'_2 + l'_3$. The blue vertices denote the original vertices on the polyhedron $\Delta^*$ of $M^*$. The red vertices denote the added vertices for the brane.

The coordinates $z_i$ on the complex structure moduli space of $M'$, are given by
In terms of the logarithmic derivatives $\theta_i = z_i \frac{d}{dz_i}$, the operators $\mathcal{L}'_i$ can be rewritten as,

$$\mathcal{D}_1 = \theta_1^3 \prod_{k=0}^{2} (3\theta_1 - \theta_2 - \theta_3 - i)(\theta_1 - \theta_2)(\theta_1 - \theta_3)$$
$$- z_1 (-\theta_1 + \theta_2)(-\theta_1 + \theta_3) \prod_{i=1}^{2} \prod_{j=1}^{4} (-2\theta_1 - i)(-4\theta_1 - j),$$

$$\mathcal{D}_2 = \theta_2 (-\theta_1 + \theta_2) - z_2 (\theta_1 - \theta_2)(3\theta_1 - \theta_2 - \theta_3)$$

$$\mathcal{D}_3 = \theta_3 (-\theta_1 + \theta_3) - z_3 (\theta_1 - \theta_3)(3\theta_1 - \theta_2 - \theta_3),$$

where each operator $\mathcal{D}_i$ corresponds to a linear combination among the charge vectors $l'_1, l'_2, l'_3$.

### 3.1.2 Brane Superpotential and Disk Instantons

By equation [2.4], we solve the equation [3.5] at $z_i \to 0$. The unique series solution, as well as the fundamental period of $M_{2,4}$, is,

$$\omega_0 = 1 + 48z + 15120z^2 + 7392000z^3 + O(z^4),$$

with $z = z_1z_2z_3$. The solutions with single-logarithmic leading term,

$$\omega_{1,1} = \omega_0 \log(z_1) - 24z_1z_2 + 1260z_1^2z_2^2 - 24z_1z_3 + 256z_1z_2z_3 + 24z_1z_2^2z_3 + 1260z_1^2z_3^2$$
$$+ 24z_1z_2z_3^2 + O(z^4)$$

$$\omega_{1,2} = \omega_0 \log(z_2) + 24z_1z_3 - 24z_1z_2z_3 - 1260z_1^2z_3^2 + O(z^4)$$

$$\omega_{1,3} = \omega_0 \log(z_3) + 24z_1z_2 - 1260z_1^2z_2^2 - 24z_1z_2z_3^2 + O(z^4)$$
give rise to the flat coordinates on the open-closed moduli space by equation 2.5 and mirror map to A-model moduli space,

\[
\begin{align*}
    z_1 &= q_1 + 24q_1^2q_2 + 24q_1^2q_3 - 256q_1^2q_2q_3 + \mathcal{O}(q^4) \\
    z_2 &= q_2 - 24q_1q_2q_3 + 24q_1q_2^3q_3 + 972q_1^2q_2q_3^2 + \mathcal{O}(q^5) \\
    z_3 &= q_3 - 24q_1q_2q_3 + 972q_1^2q_2q_3^2 + \mathcal{O}(q^5).
\end{align*}
\]

Furthermore, there are also double logarithmic solutions with leading terms

\[
\frac{1}{2}\ell_1^2, \quad \frac{1}{2}\ell_2^2 + \ell_1\ell_2, \quad \frac{1}{2}\ell_3^2 + \ell_1\ell_3, \quad \ell_2\ell_3
\]

where \(\log(z_i)\)'s are abbreviated as \(\ell_i\)'s. The B-brane superpotential is given by the linear combination of these solutions,

\[
\mathcal{W} = 2(t - t_1)^2 + \sum N_{k,m,n}\text{Li}_2(q_1^k q_2^m q_3^n),
\]

where \(t = t_1 + t_2 + t_3\) is the flat coordinate of closed B-model and \(N_{l,m,n}\) are the Ooguri-Vafa invariants. First a few invariants are summarized in table 3. They are unchanged under exchange of \(m\) and \(n\), and the table is symmetric along the diagonal, which results from the symmetry of the curve \(\Sigma\) that we choose.

### 3.2 Branes on CICY \(M_{3,4}\)

The next calculation example is studied on the mirror CICY \(M_{3,4} \subset \mathbb{P}^5_{(1,1,1,1,1,2)}\). Superpotentials and invariants are obtained.

#### 3.2.1 GKZ-system and Blowing Up Geometry

The vertices on the polyhedron \(\Delta^*\) associated to \(M^*\) are as follow,

\[
\begin{align*}
    v_1^* &= (1, 0, 0, 0, 0), \quad v_2^* = (0, 1, 0, 0, 0), \quad v_3^* = (0, 0, 1, 0, 0), \\
    v_4^* &= (0, 0, 0, 1, 0), \quad v_5^* = (0, 0, 0, 0, 1), \quad v_6^* = (-1, -1, -1, -1, -2),
\end{align*}
\]

and they satisfy the linear relation,

\[
l = (-3, -4; 1, 1, 1, 1, 1, 2, 1),
\]
By the nef-partition, $E_1 = \{v_5^*, v_6^*\}$ and $\{v_1^*, v_2^*, v_3^*, v_4^*\}$, the mirror threefold $M_{3,4}$ is defined by,

\[
\begin{aligned}
P_1 &= a_{0,1} - a_5X_5 - a_6(X_1X_2X_3X_4X_5^2)^{-1} \\
P_2 &= a_{0,2} - a_1X_1 - a_2X_2 - a_3X_3 - a_4X_4,
\end{aligned}
\]

The period integral is annihilated by the GKZ-system,

\[
\begin{aligned}
\mathcal{Z}_{0,1} &= \vartheta_{0,1} + \vartheta_5 + \vartheta_6, \\
\mathcal{Z}_{0,2} &= \vartheta_{0,2} + \sum_{i=1}^4 \vartheta_i, \\
\mathcal{Z}_i &= -\vartheta_i + \vartheta_6, \quad i = 1, 2, 3, 4, \\
\mathcal{Z}_5 &= -\vartheta_5 + 2\vartheta_6,
\end{aligned}
\]

(3.7)

where $\mathcal{Z}_i$ are related to the invariance of equation $P_1 = P_2 = 0$ under overall rescaling and torus symmetry, and $\mathcal{Z}_i$ corresponds to the relations among Laurent monomials in $P_1$ and $P_2$.

Consider the following curve $\Sigma$ on $M_{3,4}$,

\[
\{P_1 = P_2 = 0\} \cap \{h_1 = a_7X_1 + a_8X_2 = 0\} \cap \{h_2 = a_9X_1 + a_{10}X_3 = 0\},
\]

described by two extra linear relation $l^1 = (0, 0; -1, 1, 0, 0, 0, 0, 0), l^2 = (0, 0; -1, 0, 1, 0, 0, 0)$ among vertices.

The blow-up manifold $M'$ from blowing up $M_{3,4}$ along the curve $\Sigma$ is defined by,

\[
M': P_1 = P_2 = 0, \quad Q = y_1h_2 - y_2h_1 = 0.
\]
The GKZ-system of $M'$ is derived from the GKZ-system of $M_{3,4}$.

\[ \mathcal{L}_{0,1} = \mathcal{L}_{0,1}, \quad \mathcal{L}_{0,2} = \mathcal{L}_{0,2}, \quad \mathcal{L}_{3,0} = \sum_{i=7}^{10} \vartheta_i, \]

\[ \mathcal{L}_1 = \mathcal{L}_1 - \vartheta_7 - \vartheta_9, \quad \mathcal{L}_2 = \mathcal{L}_2 - \vartheta_8, \quad \mathcal{L}_3 = \mathcal{L}_3 - \vartheta_{10}, \]

\[ \mathcal{L}_4 = \mathcal{L}_4, \quad \mathcal{L}_5 = \mathcal{L}_4, \quad \mathcal{L}_6 = -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \quad (3.8) \]

\[ \mathcal{L}'_1 = \mathcal{L}_1, \quad \mathcal{L}'_2 = \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_8}, \quad \mathcal{L}'_3 = \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_{10}}, \]

with the corresponding enhanced polyhedron $\Delta'$ given by,

| $\nu'_i$ | $\Delta'$ | $l'_1$ | $l'_2$ | $l'_3$ |
|--------|--------|-------|-------|-------|
| $\nu'_{0,1}$ | 1 0 0 0 0 0 0 0 0 -3 0 0 |
| $\nu'_{0,2}$ | 0 1 0 0 0 0 0 0 0 -4 0 0 |
| $\nu'_{1}$ | 0 1 0 1 0 0 0 0 0 3 -1 -1 |
| $\nu'_{2}$ | 0 1 0 0 1 0 0 0 0 0 1 0 |
| $\nu'_{3}$ | 0 1 0 0 0 0 1 0 0 0 0 1 |
| $\nu'_{4}$ | 0 1 0 0 0 0 0 1 0 2 0 0 |
| $\nu'_{5}$ | 0 1 0 -1 -1 -1 -1 -2 0 1 0 0 |
| $\nu'_{6}$ | 0 0 1 1 0 0 0 0 -1 -1 1 0 |
| $\nu'_{7}$ | 0 0 0 0 0 0 0 0 -1 1 1 0 |
| $\nu'_{8}$ | 0 0 0 1 0 0 0 0 -1 1 1 0 |
| $\nu'_{9}$ | 0 0 0 1 0 0 0 0 -1 0 1 0 |
| $\nu'_{10}$ | 0 0 1 0 0 1 0 0 1 1 0 -1 |

Table 2: Toric Information of the Extended Polyhedron
Here $\nu'_i$ are the integral vertices of the polyhedron, $l'_i$ correspond to the maximal triangulation of $\Delta^*$, s.t. $l = l'_1 + l'_2 + l'_3$. The blue and red vertices denote the original and added vertices, respectively.

The coordinates $z_i$ on the complex structure moduli space of $M'$ is obtained by

\[ z_1 = \frac{a_3^3 a_4^2 a_5^2 a_6 a_8 a_{10}}{a_1 a_2 a_3^2 a_4 a_5^2 a_6 a_7 a_9}, \quad z_2 = \frac{a_2 a_7}{a_1 a_8}, \quad z_3 = \frac{a_3 a_9}{a_1 a_{10}}, \quad (3.9) \]
The corresponding Picard-Fuchs operators $\mathcal{D}_i$ can be obtained by $\mathcal{L}_i$ operators

$$
\mathcal{D}_1 = \prod_{k=0}^{2} (3\theta_1 - \theta_2 - \theta_3 - k)(2\theta^3)(2\theta - 1)(\theta_1 - \theta_2)(\theta_1 - \theta_3) \\
\quad - z_1(-\theta_1 + \theta_2)(-\theta_1 + \theta_3) \prod_{i=1}^{3} \prod_{j=1}^{4} (-3\theta_i - i)(-4\theta_1 - j),
$$

(3.10)

$$
\mathcal{D}_2 = \theta_2(-\theta_1 + \theta_2) - z_2(3\theta_1 - \theta_2 - \theta_3)(\theta_1 - \theta_2) \\
\mathcal{D}_3 = \theta_3(-\theta_1 + \theta_3) - z_3(3\theta_1 - \theta_2 - \theta_3)(\theta_1 - \theta_3),
$$

\ldots

### 3.2.2 Brane Superpotential and Disk Instantons

The solutions of equation 3.10 underlie the mirror map and the superpotential. The series solution

$$
\omega_0 = 1 + 72z + 37800z^2 + 31046400z^3 + \mathcal{O}(z^4),
$$

with $z = z_1z_2z_3$, together with the following single logarithmic solutions,

$$
\begin{align*}
\omega_{1,1} &= \omega_0 \log(z_1) - 36z_1z_2 + 3150z_1^2z_2^2 - 36z_1z_3 + 420z_1z_2z_3 + 36z_1z_2^2z_3 \\
&\quad + 3150z_1^2z_2^2 + 36z_1z_2z_3^2 + \mathcal{O}(z^4) \\
\omega_{1,2} &= \omega_0 \log(z_2) + 36z_1z_3 - 36z_1^2z_2z_3 - 3150z_1^2z_2^2 + \mathcal{O}(z^4) \\
\omega_{1,3} &= \omega_0 \log(z_3) + 36z_1z_2 - 3150z_1^2z_2^2 - 36z_1z_2z_3^2 + \mathcal{O}(z^4)
\end{align*}
$$

give rise to the mirror map, connecting the B-model complex moduli $z$ to the A-model moduli $q$,

$$
\begin{align*}
z_1 &= q_1 + 36q_1^2q_2 - 1206q_1^3q_3^2 + 36q_1^2q_3 - 420q_1^3q_2q_3 + 1296q_1^3q_2q_3 - 36q_1^2q_2^2q_3 \\
&\quad - 1206q_1^3q_3^2 - 36q_1^2q_2q_3^2 + \mathcal{O}(q^5) \\
z_2 &= q_2 - 36q_1q_2q_3 + 36q_1q_2^3q_3 + \mathcal{O}(q^5) \\
z_3 &= q_3 - 36q_1q_2q_3 + 2502q_1^2q_2^2q_3 + 36q_1q_2q_3^3 + \mathcal{O}(q^5)
\end{align*}
$$
In addition, the superpotential is given by the linear combinations of double logarithmic solutions,

\[ \mathcal{W} = 2(t - t_1)^2 + \sum N_{k,m,n} \text{Li}_2(q_1^k q_2^m q_3^n), \]  

(3.11)

First a few invariants are summarized in table 4.

### 3.3 Branes on CICY $M_{2,6}, M_{3,6}, \text{and } M_{4,6}$

The B-brane superpotentials on CICY on $M_{2,6} \subset \mathbb{P}^5_{(1,1,1,1,3)}, M_{3,6} \subset \mathbb{P}^5_{(1,1,1,2,3)},$ and $M_{2,6} \subset \mathbb{P}^5_{(1,1,2,2,3)}$ are computed and the Ooguri-Vafa invariants on the corresponding A-model threefold $M^*$ are extracted. For brevity, certain technical details that are similar to the last two examples are omitted. The GKZ-system of blow-ups are summarized in Appendix [A] and invariants in Appendix [B].

### 4 Summary and Conclusions

In this work, we calculate the B-brane superpotentials and Ooguri-Vafa invariants with several deformations on compact complete intersection Calabi-Yau threefolds by blowing up the curve wrapped by B-branes. The complex structure moduli of the blow-up manifolds relates complex structure moduli and brane moduli on the original threefolds $M$. From observation on the defining equation of $M'$, the GKZ-system of blow-up is obtained from the GKZ-system associated to $M$, which annihilates the period matrix of $M'$. The logarithmic solutions are used to construct mirror maps and B-brane superpotentials. By multi-cover formula and mirror symmetry, the Ooguri-Vafa invariants are extracted from A-model side and interpreted as counting disk instantons. In our calculation, suitable A-brane configurations on A-model $M$ are assumed to exist. An independent computation directly in the topological A-models would further support our results by mirror symmetry.

The one-parameter CICY threefolds in this paper can be obtained by extremal transition from multi-parameter models[21, 22, 27]. It would be interesting to calculate the Ooguri-Vafa invariants on more complicated models and study the relation of the invariants on two distinct models. In addition, the arithmetic properties of the mirror maps and the superpotentials are worth to study.
Acknowledgement

This work is dedicated to our dear supervisor Prof. Fu-Zhong Yang who sadly passed away while the paper was being prepared.

A GKZ-system of Blow-ups

A.1 CICY $M_{2,6}$

The brane geometry is captured by the following linear relation,

$$l = (-2, -6; 1, 1, 1, 1, 3, 1),$$

$$l^1 = (0, 0; 0, 1, -1, 0, 0, 0), \quad l^2 = (0, 0; 0, 1, 0, -1, 0, 0),$$

The GKZ-system associated to the blow-up $M'$ is,

$$L_{1,0}' = \vartheta_{1,0} + \vartheta_1 + \vartheta_6, \quad L_{2,0}' = \vartheta_{2,0} + \sum_{i=2}^5 \vartheta_i, \quad L_{3,0} = \sum_{i=7}^{10} \vartheta_i,$$

$$L_1' = -\vartheta_1 + \vartheta_6, \quad L_2' = -\vartheta_2 + \vartheta_6 + \vartheta_7 + \vartheta_9, \quad L_3' = -\vartheta_3 + \vartheta_6 + \vartheta_8,$$

$$L_4' = -\vartheta_4 + \vartheta_6 + \vartheta_{10}, \quad L_5' = -\vartheta_5 + 3\vartheta_6, \quad L_6' = -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10},$$

$$\mathcal{L}_1' = \prod_{i=1}^4 \frac{\partial}{\partial a_i} (\frac{\partial}{\partial a_5})^3 (\frac{\partial}{\partial a_6}) - (\frac{\partial}{\partial a_{0,1}})^2 (\frac{\partial}{\partial a_{0,2}})^6,$$

$$\mathcal{L}_2' = \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_8}, \quad \mathcal{L}_3' = \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_{10}},$$

with the relation among vertices of the enhanced polyhedron $\Delta'$,

$$l_1' = (-2, -6; 1, 3, 0, 0, 3, 1, -1, 1 - 1, 1),$$

$$l_2' = (0, 0; 0, -1, 1, 0, 0, 0, 1, -1, 0, 0), \quad l_3' = (0, 0; 0, -1, 0, 1, 0, 0, 0, 0, 1, -1).$$

The B-brane superpotential is obtained by solve the GKZ-system $A.1$ and Ooguri-Vafa invariants are extracted at the large volume point of A-model as in table 5.
A.2 CICY $M_{3,6}$

The brane geometry is described by,

$$l = (-3, -6; 1, 1, 1, 2, 3, 1),$$

$$l^1 = (0, 0; 1, -1, 0, 0, 0), \quad l^2 = (0, 0; 1, 0, -1, 0, 0, 0)$$

The GKZ-system of the blow-up $M'$,

$$\mathcal{Z}^0_{0,1} = \vartheta_{1,0} + \sum_{i=1}^{3} \vartheta_i, \quad \mathcal{Z}^0_{0,2} = \vartheta_{2,0} + \sum_{i=4}^{6} \vartheta_i, \quad \mathcal{Z}^0_{3,0} = \sum_{i=7}^{10} \vartheta_i,$$

$$\mathcal{Z}^0_1 = -\vartheta_1 + \vartheta_6 - \vartheta_7 - \vartheta_9, \quad \mathcal{Z}^0_2 = -\vartheta_2 + \vartheta_6 - \vartheta_8, \quad \mathcal{Z}^0_3 = -\vartheta_3 + \vartheta_6 - \vartheta_{10},$$

$$\mathcal{Z}^0_4 = -\vartheta_4 + 2\vartheta_6, \quad \mathcal{Z}^0_5 = -\vartheta_5 + 3\vartheta_6, \quad \mathcal{Z}^0_6 = -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10}$$

$$\mathcal{L}^0_1 = \prod_{i=1}^{3} \left( \frac{\partial}{\partial a_i} \right) \left( \frac{\partial}{\partial a_4} \right)^2 \left( \frac{\partial}{\partial a_5} \right)^3 \left( \frac{\partial}{\partial a_6} \right) - \left( \frac{\partial}{\partial a_{0,1}} \right)^3 \left( \frac{\partial}{\partial a_{0,2}} \right)^6,$$

$$\mathcal{L}^0_2 = \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_8}, \quad \mathcal{L}^0_3 = \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_{10}},$$

(A.2)

with the maximal triangulation of the enhanced polyhedron corresponding to,

$$l = (-3, -6; 3, 0, 0, 2, 3, 1, -1, 1, -1, 1),$$

$$l^1 = (0, 0; -1, 1, 0, 0, 0, 1, -1, 0, 0), \quad l^2 = (0, 0; -1, 0, 1, 0, 0, 0, 0, 0, 1, -1)$$

The superpotential and invariants are given by solving the above equation [A.2].

The results are summarized in the table [A.2]

A.3 CICY $M_{4,6}$

The brane geometry is given by specifying two extra linear relation,

$$l = (-4, -6; 1, 1, 2, 3, 1),$$

$$l^1 = (0, 0; 1, -1, 0, 0, 0), \quad l^2 = (0, 0; 1, 0, 0, 0, 0, 1),$$

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The GKZ-system associated to the blow-up $M'$ is,

\[\mathcal{L}_{0,1}' = \vartheta_{0,1} + \vartheta_3 + \vartheta_4, \quad \mathcal{L}_{0,2}' = \vartheta_{0,2} + \sum_{i=4}^{6} \vartheta_i, \quad \mathcal{L}_{0,3}' = \sum_{i=7}^{10} \vartheta_i,\]

\[\mathcal{L}_1' = -\vartheta_1 + \vartheta_6 - \vartheta_7 - \vartheta_9 - \vartheta_{10}, \quad \mathcal{L}_2' = -\vartheta_2 + \vartheta_6 - \vartheta_8 - \vartheta_{10}, \quad \mathcal{L}_3' = -\vartheta_3 + 2\vartheta_6 + 2\vartheta_{10},\]

\[\mathcal{L}_4' = -\vartheta_4 + 2\vartheta_6 + 2\vartheta_{10}, \quad \mathcal{L}_5' = -\vartheta_5 + 3\vartheta_6 + 3\vartheta_{10}, \quad \mathcal{L}_6' = -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10}\]

\[\mathcal{L}_1' = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} \left( \frac{\partial}{\partial a_3} \right)^2 \left( \frac{\partial}{\partial a_4} \right)^2 \left( \frac{\partial}{\partial a_5} \right)^3 \frac{\partial}{\partial a_6} - \left( \frac{\partial}{\partial a_{0,1}} \right)^4 \left( \frac{\partial}{\partial a_{0,2}} \right)^6,\]

\[\mathcal{L}_2' = \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_8}, \quad \mathcal{L}_3' = \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_{10}},\]

(A.3)

with the maximal triangulation of the extended polyhedron,

\[l = (-4, -6; 3, 0, 2, 2, 3, 0, -1, 1, -1, 1),\]

\[l^1 = (0, 0; -1, 1, 0, 0, 0, 0, 1, -1, 0, 0), \quad l^2 = (0, 0; 1, 0, 0, 0, 0, -1, 0, 0, 1, -1),\]

The corresponding Ooguri-Vafa invariants on $M_{4,6}'$ are listed in table 7.
## B Ooguri-Vafa Invariants

| $k = 1$ | $n = 0$ | 1   | 2    | 3     | 4     |
|---------|---------|------|------|-------|-------|
| $m = 0$ | 32      | 288  | −96  | 16    | 0     |
|         | 1       | 288  | 192  | −144  | 0     |
|         | 2       | −96  | −144 | 0     | 0     |
|         | 3       | 16   | 0    | 0     | 0     |
|         | 4       | 0    | 0    | 0     | 0     |

| $k = 2$ | $n = 0$ | 1   | 2    | 3     | 4     |
|---------|---------|------|------|-------|-------|
| $m = 0$ | 160     | −1248| −10272| 4416  | −2112 |
|         | 1       | −1248| 8128 | 9120  | −9600 |
|         | 2       | −10272| 9120 | 10272 | −15456|
|         | 3       | 4416 | −9600| −15456| 11136 |
|         | 4       | −2112| 2912 | 2808  | 0     |

| $k = 3$ | $n = 0$ | 1   | 2    | 3     | 4     |
|---------|---------|------|------|-------|-------|
| $m = 0$ | 1952    | −18336| 101760| 861024| −378624|
|         | 1       | −18336| 144384| −691584| −1089280|
|         | 2       | 101760| −691584| 2445312| 3711808|
|         | 3       | 861024| −1089280| 3711808| 3466560|
|         | 4       | −378624| 1283136| −2563968| −4092928|

Table 3: Ooguri-Vafa Invariants on $M_{2,4}$
\begin{table}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\multicolumn{1}{|c|}{k = 1} & \multicolumn{1}{|c|}{n = 0} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{|c|}{3} & \multicolumn{1}{|c|}{4} \\
\hline
\multicolumn{1}{|c|}{m = 0} & 48 & 432 & -144 & 24 & 0 \\
\multicolumn{1}{|c|}{1} & 432 & 288 & -216 & 0 & 0 \\
\multicolumn{1}{|c|}{2} & -144 & -216 & 0 & 0 & 0 \\
\multicolumn{1}{|c|}{3} & 24 & 0 & 0 & 0 & 0 \\
\multicolumn{1}{|c|}{4} & 0 & 0 & 0 & 0 & 0 \\
\hline
\multicolumn{1}{|c|}{m = 0} & 408 & -3312 & -27048 & 11616 & -5400 \\
\multicolumn{1}{|c|}{1} & -3312 & 21600 & 29328 & -26496 & 8208 \\
\multicolumn{1}{|c|}{2} & -27048 & 29328 & 31680 & -42960 & 7656 \\
\multicolumn{1}{|c|}{3} & 11616 & -26496 & -42960 & 28416 & 0 \\
\multicolumn{1}{|c|}{4} & -5400 & 8208 & 7656 & 0 & 0 \\
\hline
\multicolumn{1}{|c|}{m = 0} & 8208 & -80640 & 466272 & 3910320 & -1721376 \\
\multicolumn{1}{|c|}{1} & -80640 & 646848 & -3157056 & -5718912 & 5931072 \\
\multicolumn{1}{|c|}{2} & 466272 & -3157056 & 11239776 & 17265264 & -11883456 \\
\multicolumn{1}{|c|}{3} & 3910320 & -5718912 & 17265264 & 16301088 & -19212576 \\
\multicolumn{1}{|c|}{4} & -1721376 & 5931072 & -11883456 & -19212576 & 13049280 \\
\hline
\end{tabular}
\caption{Ooguri-Vafa Invariants on $M_{3,4}$}
\end{table}
$k = 1$

| $m = 0$ | $n = 0$   | 1        | 2        | 3        | 4        |
|---------|-----------|----------|----------|----------|----------|
| 0       | 160       | 1440     | -480     | 80       | 0        |
| 1       | 1440      | 960      | -720     | 0        | 0        |
| 2       | -480      | -720     | 0        | 0        | 0        |
| 3       | 80        | 0        | 0        | 0        | 0        |
| 4       | 0         | 0        | 0        | 0        | 0        |

$\begin{align*}
 k = 2 \\
 m = 0 & \\
 & 5504 \quad -47328 \quad -383040 \quad 164160 \quad -73440 \\
 & -47328 \quad 292800 \quad 362400 \quad -347520 \quad 117600 \\
 & -383040 \quad 362400 \quad 432480 \quad -574560 \quad 113160 \\
 & 164160 \quad -347520 \quad -574560 \quad 385920 \quad 0 \\
 & -73440 \quad 117600 \quad 113160 \quad 0 \quad 0
\end{align*}$

$\begin{align*}
 k = 3 \\
 m = 0 & \\
 & 432160 \quad -4503840 \quad 27511680 \quad 228161760 \quad -100550400 \\
 & -4503840 \quad 35859456 \quad -175001472 \quad -305181440 \quad 317586240 \\
 & 27511680 \quad -175001472 \quad 594892800 \quad 907979840 \quad -616498560 \\
 & 228161760 \quad -305181440 \quad 907979840 \quad 866126400 \quad -1014325760 \\
 & -100550400 \quad 317586240 \quad -616498560 \quad -1014325760 \quad 685572480
\end{align*}$

Table 5: Ooguri-Vafa Invariants on $M_{2,6}$
\begin{table}[h]
\centering
\begin{tabular}{c|ccccc}
\multicolumn{1}{l|}{$k = 1$} & \multicolumn{5}{c}{$n = 0$} \\
\multicolumn{1}{l|}{$m$} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\
0 & 240 & 2160 & -720 & 120 & 0 \\
1 & 2160 & 1440 & -1080 & 0 & 0 \\
2 & -720 & -1080 & 0 & 0 & 0 \\
3 & 120 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 \\
\hline
\multicolumn{1}{l|}{$k = 2$} & \multicolumn{5}{c}{$n = 0$} \\
\multicolumn{1}{l|}{$m$} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\
0 & 13800 & -123120 & -993240 & 424800 & -186120 \\
1 & -123120 & 781920 & 1230480 & -984960 & 326160 \\
2 & -993240 & 1230480 & 1337760 & -1625040 & 299520 \\
3 & 424800 & -984960 & -1625040 & 979200 & 0 \\
4 & -186120 & 326160 & 299520 & 0 & 0 \\
\hline
\multicolumn{1}{l|}{$k = 3$} & \multicolumn{5}{c}{$n = 0$} \\
\multicolumn{1}{l|}{$m$} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\
0 & 1815120 & -19514880 & 122876640 & 1013165424 & -446689440 \\
1 & -19514880 & 160125120 & -801541440 & -1655884416 & 1502625600 \\
2 & 122876640 & -801541440 & 2808833760 & 4414021680 & -2977110720 \\
3 & 1013165424 & -1655884416 & 4414021680 & 4250616480 & -4938103200 \\
4 & -446689440 & 1502625600 & -2977110720 & -4938103200 & 3233476800 \\
\end{tabular}
\caption{Ooguri-Vafa Invariants on $M_{3,6}$}
\end{table}
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\multicolumn{1}{|c|}{\(k = 1\)} & \multicolumn{1}{|c|}{\(n = 0\)} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{|c|}{3} & \multicolumn{1}{|c|}{4} \\
\hline
\(m = 0\) & 480 & 4320 & -1440 & 240 & 0 \\
\hline
1 & 4320 & 2880 & -2160 & 0 & 0 \\
\hline
2 & -1440 & -2160 & 0 & 0 & 0 \\
\hline
3 & 240 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 \\
\hline
\hline
k = 2 & \(m = 1\) & 64560 & -603360 & -4849680 & 2068800 & -883440 \\
\hline
1 & -603360 & 3948480 & 7689120 & -5293440 & 1715040 \\
\hline
2 & -4849680 & 7689120 & 7781760 & -8715360 & 1497840 \\
\hline
3 & 2068800 & -5293440 & -8715360 & 4656000 & 0 \\
\hline
4 & -883440 & 1715040 & 1497840 & 0 & 0 \\
\hline
\hline
\multicolumn{1}{|c|}{\(k = 3\)} & \multicolumn{1}{|c|}{\(m = 0\)} & 19966560 & -222979680 & 1458023040 & 11939377824 & -5265987840 \\
\hline
1 & -222979680 & 1896606720 & -9795738240 & -23758008576 & 19015358400 \\
\hline
2 & 1458023040 & -9795738240 & 35610071040 & 59297784000 & -39000389760 \\
\hline
3 & 11939377824 & -23758008576 & 59297784000 & 57246494400 & -65195381760 \\
\hline
4 & -5265987840 & 19015358400 & -39000389760 & -65195381760 & 40631529600 \\
\hline
\end{tabular}
\caption{Ooguri-Vafa Invariants on \(M_{4,6}\)}
\end{table}

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