THE ESSENTIAL DIMENSION OF STACKS OF PARABOLIC VECTOR BUNDLES OVER CURVES

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Abstract. We find upper bounds on the essential dimension of the moduli stack of parabolic vector bundles over a curve. When there is no parabolic structure, we improve the known upper bound on the essential dimension of the usual moduli stack. Our calculations also give lower bounds on the essential dimension of the semistable locus inside the moduli stack of vector bundles of rank $r$ and degree $d$ without parabolic structure.

1. Introduction

Essential dimension is a numerical invariant of an algebraic or geometric object defined over a base field $k$ which roughly speaking measures its complexity in terms of the smallest number of parameters needed to define the object over $k$. It was originally defined by Reichstein and Buhler, see [BR97, Re2], in the context of finite and then algebraic groups. The definition was then rephrased into functorial language and generalized by Merkurjev. See [BF03, Mer09] for surveys on this topic.

Denote by $\text{Fields}_k$ the category of field extensions of $k$. Let $F : \text{Fields}_k \to \text{Sets}$ be a functor. We say that $a \in F(L)$ is defined over a field $K \subseteq L$ if there exists $b \in F(K)$ so that $r(b) = a$ where $r$ is the restriction $F(K) \to F(L)$.

The essential dimension of $a$ is defined to be

$$\text{ed}(a) \overset{\text{def}}{=} \min_K \text{tr-deg}_k K,$$

where the minimum is taken over all fields of definition $K$ of $a$.

The essential dimension of $F$ is defined to be

$$\text{ed}(F) = \sup_a \text{ed}(a),$$

where the supremum is taken over all $a \in F(K)$ and $K$ varies over all objects of $\text{Fields}_k$.

Reichstein and Buhler first considered the essential dimension of the functor $F_G : \text{Fields}_k \to \text{Sets}, \ K \mapsto H^1(K, G)$, where $H^1(K, G)$ is the set of isomorphism classes of $G$-torsors over $\text{Spec}(K)$. Then $\text{ed}(G) := \text{ed}(F_G)$ is a numerical invariant of the algebraic group. Upper and lower bounds on $\text{ed}(G)$ have been a topic of much study since then. See [Re1] for a survey.

2000 Mathematics Subject Classification. 14D23, 14D20.

Key words and phrases. Essential dimension, moduli stack, gerbe, twisted sheaf, parabolic bundle.

Second and third authors partially supported by NSERC. The third author acknowledges the support of a Swiss National Science Foundation International short visit research grant.
of work on essential dimension and the particular cases of the essential dimension of finite and algebraic groups.

For an algebraic stack $\mathcal{X} \to \text{Aff}_k$ we obtain a functor

$$F_{\mathcal{X}} : \text{Fields}_k \to \text{Sets}$$

which sends any $K$ to the set of isomorphism classes of objects in $\mathcal{X}(K)$. The essential dimension of $\mathcal{X}$ is defined to be the essential dimension of this functor. We denote this number by $\text{ed}_k(\mathcal{X})$. This definition was first made by Brosnan, Reichstein and Vistoli, see [BFRV], in which they studied the essential dimension of an algebraic stack. Note that this is indeed a generalization of the original definition of essential dimension. In particular, for the classifying stack of an algebraic group, $BG$, we have $\text{ed}_k(BG) = \text{ed}_k(G)$ since $F_{BG} = H^1(-, G)$. Furthermore, viewed in this light, essential dimension can be considered as a generalization of the dimension of an algebraic variety. Indeed there are two ways to define the dimension of a variety, via the transcendence degree of its function field or via Krull dimension. Essential dimension is a lift to stacks of the transcendence degree definition.

Brosnan, Reichstein and Vistoli in [BFRV] used stack-theoretic results to prove a genericity theorem computing the essential dimension of a Deligne-Mumford stack $\mathcal{X}$, of finite type with finite inertia, from the dimension of its coarse moduli space $M$ and the essential dimension of the generic gerbe $\mathcal{X}_K$, where $K$ is the function field of $M$ (see Theorem 2.1 for more details). They were then able to apply these results to obtain better bounds for the essential dimension of some algebraic and finite groups.

We fix a base field $k$ of characteristic zero that may not be algebraically closed. We further fix a smooth projective geometrically connected curve $X$ of genus $g$ at least 2 over this field. We will also assume that $X$ has at least three $k$-rational points. This technical hypothesis is needed in the proof of Theorem 8.4.

For an algebraic group $G$, we denote by $\text{Bun}_X^G$ the moduli stack of $G$-torsors over the curve $X$. We are interested in studying the essential dimension of $\text{Bun}_X^G$ for an algebraic group $G$. Note then that the essential dimension of $\text{Bun}_X^G$ is the essential dimension of the functor sending a field extension $K/k$ to the set of isomorphism classes of $G$-torsors over $X_K$.

We concentrate on the case $G = \text{SL}_r$. In this case, $\text{SL}_r$ torsors over the curve $X$ are vector bundles over $X$ with fixed isomorphism of the determinant bundle with the trivial bundle on $X$. In [DL09], we obtained upper bounds on $\text{ed}(\text{Bun}_{X_X}^\text{SL}_r)$ and the essential dimensions of some related moduli stacks such as $\text{Bun}_X^{r,d}$, the moduli stack of vector bundles over $X$ of rank $r$ and degree $d$. Our upper bounds on $\text{ed}(\text{Bun}_X^{r,d})$ depended on the genus $g$ and were quartic polynomials of the rank $r$.

In this paper, we denote by $\text{Bun}_{X_D}^{r,d}$, the moduli stack of vector bundles of rank $r$ and degree $d$ on $X$ with parabolic structure along some reduced divisor (see §3). Note that in the special case of rational weights (see §8 for definitions and more details), vector bundles on $X$ with parabolic structure along some reduced divisor $D$ were shown by Biswas in [Bis97] to correspond to certain orbifold bundles over $X$ where $D$ is the ramification divisor.

Our aim is to compute an upper bound on the essential dimension of $\text{Bun}_X^{r,d}$. When the divisor is empty, the moduli stack coincides with $\text{Bun}_{X_D}^{r,d}$, the moduli stack of vector bundles of rank $r$ and degree $d$ over $X$. Dhillon and Lemire found bounds on the essential dimension of $\text{Bun}_X^{r,d}$ in [DL09]. Our results improve the
upper bound obtained there; this is explained in Remark \[13.3\]. In particular, our
new upper bound is now a quadratic polynomial in the rank \(r\), which depends on \(g\).
Further, by carefully choosing our parabolic data we are able to find lower bounds
on the essential dimension of the semistable locus of the usual moduli stack (see §\[14\]). There was no previously known lower bound better than the trivial bound
given by the dimension of the coarse moduli space (see Theorem \[2.1\]).

In order to find an upper bound on the essential dimension of \(\mathrm{Bun}^{r,d}_X\), we first
show that \(\mathrm{Bun}^{r,d,s}_X\), namely the moduli stack of stable vector bundles of rank \(r\) and
degree \(d\) with parabolic structure along \(D\), is a Deligne Mumford stack satisfying
the conditions of the genericity theorem mentioned above. We are able to apply
this theorem to \(\mathrm{Bun}^{r,d,s}_X\) using the calculation in \[BD\] of Brauer group of the moduli
space and some facts about twisted sheaves due to Lieblich \[Lie08\].

To pass from the essential dimension of \(\mathrm{Bun}^{r,d,s}_X\) to that of \(\mathrm{Bun}^{r,d,ss}_X\), the moduli
stack of semistable vector bundles of rank \(r\) and degree \(d\) with parabolic structure
along \(D\), we use the socle filtration, a finite filtration of a semistable parabolic
bundle with polystable parabolic bundles as quotients (see §\[8\] for details). This was
the key to improving the bounds in \[DL09\] as there we used the the Jordan-H"{o}lder
filtration, a finite filtration of a semistable parabolic bundle with stable parabolic
bundles as quotients. Unlike the Jordan-H"{o}lder filtration, the socle filtration is
Galois invariant, so it exists over the base field. This sidesteps the major difficulty
in \[DL09\].

We then use the Harder-Narasimhan filtration, a finite filtration of a parabolic
bundle with semistable parabolic bundles as quotients, to pass from the essential
dimension of the semistable locus to that of the full moduli stack, \(\mathrm{Bun}^{r,d}_X\).

The other main ingredient is the correspondence set up in \[Bis97\] between parabolic
bundles and orbifold bundles. This allows us to compute extensions of parabolic
bundles in terms of orbifold bundles. Finally we need the orbifold Riemann-Roch theorem, originally proved in \[T"{o}99\], to bound the dimensions of these groups.

The key results that compute upper bounds are Theorem \[12.1\], Proposition \[11.2\],
Proposition \[13.1\] and Proposition \[13.2\]. The first two results bound the essential
dimension in terms of an auxiliary function. The last two tell us about the growth
rate of this function. The lower bound is given in Theorem \[14.1\].

In summary, we have the following (best known) bounds for the essential dimen-
sion of the moduli stack of vector bundles of rank \(r\) and degree \(d\):

**Theorem 1.1.** Let \(X\) be a smooth projective curve of genus at least 2 and with at
least 3 \(k\)-points. Suppose that \(p\) is a prime such that \(p^l\) divides \(r\) and \(d\). We have
\[
(r^2 - 1)(g - 1) + p^l - 1 + g \leq \text{ed}(\text{Bun}^{r,d}_X) \leq r^2 g.
\]

This result follows by combining Theorem \[12.1\], Theorem \[14.1\], Proposition \[13.2\]
and Proposition \[13.1\].

Upon introducing parabolic structure we obtain the following result :

**Theorem 1.2.** Suppose that \(p\) is a prime such that \(p^l\) divides \(l(D)\). We have
\[
(r^2 - 1)(g - 1) + p^l - 1 + g + \sum_{y \in D} \dim \text{Flag}_y(D) \leq \text{ed}(\text{Bun}^{r,d}_X) \leq F_{g,D}(r).
\]

The reader is referred to section 13 for the definition of the function \(F_{g,D}(r)\).
The number \(l(D)\) is defined in section 6. Once again this results from combining
Theorem \[12.1\], Theorem \[14.1\], Proposition \[13.2\] and Proposition \[13.1\].
It is worth noting at this point that the main conjecture of [CKM07] is closely related to calculating the essential dimension of the moduli stack of vector bundles via Theorem 2.2.

Acknowledgments

The authors would like to thank the referee for very useful comments.

2. Essential Dimension

In this section, we will recall some theorems from [BFRV], including the genericity theorem mentioned in the introduction, that will be needed in the future. We assume for the remainder of this section that $X/k$ is a Deligne-Mumford stack, of finite type, with finite inertia. By [KM97], such a stack has a coarse moduli space $M$. The first result that we shall need is the following theorem proved in [BFRV].

Theorem 2.1. Recall that our base field has characteristic zero. Suppose $X$ is smooth and connected. Let $K$ be the field of rational functions of $M$, and let $X_K = \text{Spec}(K) \times_M X$ be the base change. Then

$$\text{ed}_K(X) = \dim M + \text{ed}_K(X_K).$$

Proof. See [BFRV, Theorem 6.1].

The stack $X_K/K$ is called the generic gerbe. In the case where this gerbe is banded by $\mu_n$, more can be said about $\text{ed}_K(X_K)$.

Let $\mathfrak{G}$ be a gerbe over a field $K$ banded by $\mu_n$. There is an associated $\mathbb{G}_m$-gerbe over $K$, denoted by $\mathfrak{G}_m$, coming from the canonical inclusion $\mu_n \hookrightarrow \mathbb{G}_m$. It gives a torsion class in the Brauer group $\text{Br}(K)$. The index of this class is called the index of the gerbe, and is denoted by $\text{ind}_K(\mathfrak{G}) = d$. There is a Brauer-Severi variety $P/K$ of dimension $d - 1$ whose class maps to the class of $\mathfrak{G}$ via the connecting homomorphism

$$H^1(\text{Spec}(K), \text{PGL}_d) \rightarrow H^2(\text{Spec}(K), \mathbb{G}_m)$$

for the exact sequence $e \rightarrow \mathbb{G}_m \rightarrow \text{GL}_d \rightarrow \text{PGL}_d \rightarrow e$.

Let $V$ be a smooth and proper variety over $k$. The set $V(k(V))$ is the collection of rational endomorphisms of $V$ defined over $k$. Define

$$\text{cd}_k(V) = \inf \{ \dim \text{im}(\phi) \mid \phi \in V(k(V)) \}.$$

The number $\text{cd}_k(V)$ is called the canonical dimension of $V$. We recall another theorem from [BFRV].

Theorem 2.2. In the above situation, if $d > 1$, then

$$\text{ed}_K(\mathfrak{G}) = \text{cd}_K(P) + 1,$$

and

$$\text{ed}_K(\mathfrak{G}_m) = \text{cd}_K(P).$$

See [BFRV, Theorem 4.1] for a proof.

Corollary 2.3. In the above situation, if $\text{ind}(P) = p^r$ is a prime power, we have

$$\text{ed}_K(\mathfrak{G}) = \text{ind}_K(P)$$

and

$$\text{ed}_K(\mathfrak{G}_m) = \text{ind}_K(P) - 1.$$
Proof. See [Kar00] Theorem 2.1 and [Mer03]. □

3. Parabolic Bundles

Definition 3.1. A parabolic point \( x \) on \( X \) consists of a triple

\[
(x, \{k_i^{x_i} \}_{i=1}^{n(x)}, \{\alpha_i^{x_i} \}_{i=1}^{n(x)}),
\]

where \( x \) is a \( k \)-point of \( X \), the \( k_i^{x_i} \) are positive integers called the multiplicities and the \( \alpha_i^{x_i} \) are rational numbers, called the parabolic weights (or simply weights). The weights are required to satisfy the following condition:

\[
0 \leq \alpha_1^{x} < \alpha_2^{x} < \cdots < \alpha_{n(x)}^{x} < 1.
\]

Definition 3.2. A parabolic datum \( D \) on \( X \) consists of a finite collection of parabolic points \( x_j = (x_j, (k_i^{x_j})_{i=1}^{n(x_j)}, (\alpha_i^{x_j})_{i=1}^{n(x_j)}) \), so

\[
D = \{x_1, x_2, \cdots, x_s\},
\]

such that \( \sum_{i=1}^{n(x_j)} k_i^{x_j} \) is independent of \( j \). We require the points to be pairwise distinct, that is \( x_j \neq x_i \) for \( j \neq i \).

The support of the datum is defined to be the reduced divisor \( x_1 + \cdots + x_s \). We denote this divisor by \( |D| \).

Definition 3.3. Fix a parabolic datum \( D \) on \( X \). If \( S \) is a scheme then a family of parabolic bundles \( \mathcal{F}_s \) on \( X \) parameterized by \( S \) with parabolic datum \( D = \{x_1, x_2, \cdots, x_s\} \) consists of a vector bundle \( \mathcal{F} \) on \( X \times S \) together with filtrations by vector subbundles

\[
\mathcal{F}|_{x_j \times S} = F_1^{x_j}(\mathcal{F}) \supseteq F_2^{x_j}(\mathcal{F}) \supseteq \cdots \supseteq F_{n(x_j)}^{x_j}(\mathcal{F}) \supseteq F_{n(x_j)+1}^{x_j}(\mathcal{F}) = 0
\]

with \( F_i^{x_j}(\mathcal{F}) \) locally free of rank

\[
k_i^{x_j} + k_{i+1}^{x_j} + \cdots + k_{n(x_j)}^{x_j}.
\]

The weight \( \alpha_i^{x_j} \) is associated with \( F_i^{x_j}(\mathcal{F}) \). This definition forces \( \text{rk}(\mathcal{F}) = \sum_{i=1}^{n(x)} k_i^{x_j} \) for each \( x \in \text{supp}(|D|) \).

When \( S \) is reduced to a point we call \( \mathcal{F}_s \) a parabolic bundle.

Definition 3.4. Suppose that \( \mathcal{F}_s \) is a parabolic bundle with datum \( D \). A parabolic subbundle \( \mathcal{F}'_s = (\mathcal{F}', \{F_i^{x_j}(\mathcal{F}') : i = 1, \cdots, n'(x), x \in \text{supp}(|D|)\}) \) of \( \mathcal{F}_s \) is a parabolic bundle with datum \( D' \) such that

(1) \( |D| = |D'| \)
(2) \( F' \) is a subbundle of \( \mathcal{F} \)
(3) for each point \( x \) in the support, the weights \( \{\alpha_i^{x_j} \}_{i=1}^{n(x)} \) are a subset of the weights \( \{\alpha_i^{x_j} \}_{i=1}^{n(x)} \)
(4) if \( m \) is maximal so that \( F_i^{x_j}(\mathcal{F}') \subseteq F_m^{x_j}(\mathcal{F}) \), then \( \alpha_i^{x_j} = \alpha_m^{x_j} \)

Given a parabolic bundle \( \mathcal{F}_s \) on \( X \) with parabolic datum

\[
D = \{x_1, x_2, \cdots, x_s\},
\]

we define the parabolic degree of \( \mathcal{F}_s \) to be the rational number

\[
\text{par-deg}(\mathcal{F}_s) = \deg(\mathcal{F}) + \sum_{j=1}^{s} \sum_{i=1}^{n(x_j)} k_i^{x_j} \alpha_i^{x_j}.
\]
The parabolic slope is defined to be $\text{par-}\mu(\mathcal{F}_s) = \text{par-deg}(\mathcal{F}_s)/\text{rk}(\mathcal{F})$.

Denote by $k$ an algebraic closure of the ground field $k$.

We say the $\mathcal{F}_s$ is semistable (respectively, stable) if for every parabolic subbundle $\mathcal{E}_s$ of $(\mathcal{F}_k)_s$, with $0 < \text{rk}(\mathcal{E}_s) < \text{rk}(\mathcal{F}_s)$, we have

$$\text{par-}\mu(\mathcal{E}_s) \leq \text{par-}\mu((\mathcal{F}_k)_s)$$

(respectively, $\text{par-}\mu(\mathcal{E}_s) < \text{par-}\mu((\mathcal{F}_k)_s)$),

where $(\mathcal{F}_k)_s$ is the base change of $\mathcal{F}_s$.

The usual arguments show that an arbitrary parabolic bundle has a unique maximal destabilizing parabolic subbundle $\mathcal{E}_s \subseteq (\mathcal{F}_k)_s$ of maximal parabolic slope. The uniqueness implies that all the Galois conjugates $\sigma^*(\mathcal{E}_s) \subset (\mathcal{F}_k)_s$ coincide. Hence this subbundle is defined over the ground field $k$ so that a base extension is not required in the definition of semistable parabolic bundles.

**Construction 3.5.** Let $\mathcal{F}_s$ be a parabolic bundle with datum $D$. We wish to construct a bundle $\mathcal{F}_t$ for each $t \in \mathbb{R}$. Set $D = |D|$. For each $t \in \mathbb{R}$ with $0 \leq t < 1$, we construct a coherent sheaf $V_t(\mathcal{F}_s)$ supported on $D$ by letting the component of $V_t(\mathcal{F}_s)$ on $x_j$ be the subspace $F^x_t(\mathcal{F})$ of $F|_{x_j}$, where $\alpha_{i-1}^x < t \leq \alpha_i^x$; if $t > \alpha_i^x$, then the component of $V_t(\mathcal{F}_s)$ on $x_j$ is defined to be zero. Taking preimages of $V_t(\mathcal{F})$ gives a sheaf $\mathcal{F}_t$ with $\mathcal{F} \supseteq \mathcal{F}_t \supseteq \mathcal{F}(-D)$. We can extend this construction to $t \in \mathbb{R}$ by defining $\mathcal{F}_s = \mathcal{F}_s(-\lfloor t \rfloor D)$, where $s = t - \lfloor t \rfloor$.

This collection $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ is decreasing; it has jumps at rational numbers only. Also, it is periodic, more precisely, $\mathcal{F}_t(-D) = \mathcal{F}_{t+1}$, and it is left continuous. It is clear that $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ uniquely determines the parabolic bundle.

Following [Yo95] we will now define an abelian category $\text{QCoh}(X, D, N)$.

First, we consider $\mathbb{R}$ as a category using its natural structure as an ordered set. An $\mathbb{R}$-filtered sheaf is a functor

$$\mathbb{R}^{op} \to \text{QCoh}(X).$$

Morphisms of $\mathbb{R}$-filtered sheaves are just natural transformations. Given $s \in \mathbb{R}$ and an $\mathbb{R}$-filtered sheaf $\mathcal{F}_s$, we can define a shifted sheaf by setting

$$\mathcal{F}[s]_t = \mathcal{F}_{s+t}.$$

Given a quasicoherent sheaf $\mathcal{G}$ and an $\mathbb{R}$-filtered sheaf $\mathcal{F}_s$, we define the $\mathbb{R}$-filtered sheaf $\mathcal{G} \otimes \mathcal{F}_s$ by taking the tensor product pointwise, ie

$$(\mathcal{G} \otimes \mathcal{F}_s)_t = \mathcal{G} \otimes \mathcal{F}_t.$$

We identify a full abelian subcategory $\text{QCoh}(X, D, N)$ of the category of $\mathbb{R}$-filtered sheaves. The objects of this category are $\mathbb{R}$-filtered sheaves $\mathcal{F}_s$ such that

(i) we have a natural isomorphism

$$j : \mathcal{F}_s \otimes \mathcal{O}_X(-D) \to \mathcal{F}[1]_s,$$

such that the induced map

$$\mathcal{F}_{s+1} \cong \mathcal{F}_s(-D) \to \mathcal{F}_s$$

is the natural inclusion.

(ii) For $l/N < s \leq (l+1)/N$ we have that the natural maps $\mathcal{F}_{l+1/N} \to \mathcal{F}_s$ are isomorphisms.
Fix a reduced effective divisor \( D = \sum_{i=1}^{s} x_i \) on \( X \), where each \( x_i \) is a \( k \)-rational point.

Denote by \( \text{PVect}(X, D, N) \) the category of parabolic bundles with parabolic datum only inside the support of \( D \) and parabolic weights integer multiples of \( \frac{1}{N} \). The morphisms in this category are given by the following definition.

**Definition 3.6.** Suppose that \( \mathcal{F} \) and \( \mathcal{F}' \) are parabolic bundles with parabolic bundles with parabolic data \( D \) and \( D' \). Suppose that \( |D| = |D'| \). A morphism of parabolic bundles \( f_* : \mathcal{F}_* \to \mathcal{F}'_* \) is a morphism \( f : \mathcal{F} \to \mathcal{F}' \) of underlying bundles such that for every parabolic point \( x \) we have

\[
f_x(F_x^\alpha(\mathcal{F})) \subseteq F'_x(F')
\]

whenever \( \alpha_x > \alpha'_x \).

We identify \( \text{PVect}(X, D, N) \) with a full subcategory of the abelian category \( \text{QCoh}(X, D, N) \) via \ref{3.5}.

### 4. Our Stacks

Let \( D \) be a parabolic datum on \( X \). We will denote by \( \text{Bun}_{r,d}^X, D \) the moduli stack of rank \( r \) degree \( d \) parabolic bundles with datum \( D \). Note that the weights only play a role when defining stability and semistability. Hence this stack is just a fibered product of flag varieties over the moduli stack \( \text{Bun}_{r,d}^X \) of vector bundles of rank \( r \) and degree \( d \) without parabolic structure.

Fix a line bundle \( \xi \) on \( X \). We denote by \( \text{Bun}_{r,\xi}^X, D \) the moduli stack of parabolic bundles with fixed identification with \( \xi \) of the top exterior power of the underlying vector bundle. Precisely, there is a Cartesian square

\[
\begin{array}{ccc}
\text{Bun}_{r,\xi}^X, D & \longrightarrow & \text{Spec}(k) \\
\downarrow & & \downarrow \\
\text{Bun}_{r,d}^X & \longrightarrow & \text{Bun}_{1,d}^X.
\end{array}
\]

Here \( \text{Bun}_{1,d}^X \) is the moduli stack of line bundles of degree \( d \), the right vertical arrow corresponds to the line bundle \( \xi \) and the bottom horizontal arrow is the determinant map. As stability and semistability are open conditions, see \cite[page 226-228]{MSS}, there are various open substacks, \( \text{Bun}_{r,d}^{s,ss}^X, D, \text{Bun}_{r,d,ss}^X, D, \text{Bun}_{r,\xi,s}^X, D \) and \( \text{Bun}_{r,\xi,ss}^X, D \).

We explain explicitly what \( \text{Bun}_{r,\xi}^X, D \) is. The objects of the category fibered in groupoids over a scheme \( S \) consist of pairs \((\mathcal{F}, \phi)\), where \( \mathcal{F} \) is a family of parabolic bundles of rank \( r \) on \( X \times S \) and \( \phi \) is an isomorphism

\[
\phi : \wedge^r \mathcal{F} \xrightarrow{} \xi.
\]

The isomorphisms in the groupoid over \( S \) are isomorphisms of parabolic bundles compatible with the trivializations.

The stack \( \text{Bun}_{r,\xi}^X, D \) is somewhat unnatural as \( \xi \) is not a parabolic line bundle. However it is a natural stepping stone in understanding the essential dimension of the stack \( \text{Bun}_{r,d}^{s,ss}^X, D \). Below we will see that \( \text{Bun}_{r,\xi,ss}^X, D \) is a smooth Deligne-Mumford stack with finite inertia so that Theorem \ref{2.1} applies. We will be able to compute the period and index of its generic gerbe and apply Theorem \ref{2.2} to understand its essential dimension.
Proposition 4.1. The stack $\text{Bun}^{r,\xi}_{X,D}$ is smooth.

Proof. First recall that the moduli stack $\text{Bun}^r_X$ of vector bundles is smooth. If $D = \{x_1, \cdots, x_s\}$, then set

$D' = \{x_1, \cdots, x_{s-1}\}$.

The morphism forgetting one parabolic point

$$\text{Bun}^{r,\xi}_{X,D} \to \text{Bun}^{r,\xi}_{X,D'}$$

is representable with fibers flag varieties. Hence the above morphism is smooth and the result follows by induction. □

Remark 4.2. The stack $\text{Bun}^{r,d,s}_{X,D}$ is in fact a global quotient stack. For simplicity, in this remark we will assume that $D = \{x\}$ with multiplicities $k_1, \cdots, k_n$.

From [MS80, page 226], the family of stable parabolic bundles of rank $r$ and degree $d$ is a bounded family. We may find an integer $N$ so that for every $n \geq N$, we have

- $H^1(X, F(n)) = 0$, and
- $F(n)$ is generated by global sections for every vector bundle underlying a stable parabolic bundle.

Let $Q$ be the corresponding quot scheme. Let $W$ be the universal bundle on $Q \times X$. There is a flag variety $F$ over $Q$ parametrizing flags of $W$ of type $k_1, \cdots, k_n$. To give an $S$-point of $F$ is the same as giving a quotient:

$$\pi_X^* \mathcal{O}_X(-N)^m \to F$$

on $S \times X$ and a flag of $F|_{S \times \{x\}}$. There is a Zariski open subset $\Omega$ parametrizing quotients that are stable as parabolic bundles. We have

$$[\Omega/GL_m] = \text{Bun}^{r,d,s}_{X,D}$$

The stack $\text{Bun}^{r,\xi,s}_{X,D}$ is also a global quotient stack. There is a $\mathbb{G}_m$-torsor $I$ over the determinant $\xi$ locus in $\Omega$ parametrizing isomorphisms of the determinant with $\xi$ (see [LMB00, page 29]). Then

$$[I/GL_m] = \text{Bun}^{r,\xi,s}_{X,D}$$

Proposition 4.3. Let $F$ be a family of stable parabolic bundles on $\text{Spec}(R) \times X$. Then all parabolic endomorphisms of $F$ are scalar multiplication by elements of $R$.

Proof. This is well known when $R$ is a field. We will explain how to deduce this result from the case of a field.

There exists a natural inclusion

$$\epsilon : R \hookrightarrow H^0(X_R, \text{End}(F))$$

that we wish to show to be an isomorphism. By flat base change, we may assume $R = (R, m)$ is local. Via Nakayama’s Lemma we need to show that

$$\tau : R/m \hookrightarrow H^0(X_R, \text{End}(F)) \otimes_R R/m$$

is surjective. But by the field case, the composition

$$R/m \to H^0(X, \text{End}(F)) \otimes_R R/m \to H^0(X_{R/m}, \text{End}(F_{R/m}))$$

is surjective. The result follows from the base change theorem, [Har77, Ch. III, Theorem 12.11]. □
Theorem 4.4. The stack $\text{Bun}_{X,D}^{r,\xi,s}$ is a smooth Deligne-Mumford stack with finite inertia.

Proof. This follows by applying Remark 4.2 and Proposition 4.3. □

It follows that $\text{Bun}_{X,D}^{r,\xi,s}$ has a coarse moduli space that we shall denote by $M(X,D,r,\xi)^s$.

For each parabolic point $x$ in the datum $D$, we denote by $\text{Flag}_x(D)$ or

$$\text{Flag}(k_1^x, k_2^x, \cdots, k_n^x(x))$$

the flag variety determined by the multiplicities of $x$. Explicitly, if the multiplicities at $x$ are $k_1^x, k_2^x, \cdots, k_n^x(x)$, then $\text{Flag}_x(D)$ is the flag variety parametrizing flags $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n(x)$ of a fixed vector space $V_1$ with

$$\dim V_1 = \sum_{i=1}^{n(x)} k_i^x$$

Employing this notation, we compute the dimension of the moduli space to be

$$\dim M(X,D,r,\xi)^s = \dim \text{Bun}_{X,D}^{r,\xi,s} = (r^2 - 1)(g - 1) + \sum_{x \in D} \dim \text{Flag}_x(D).$$

By Proposition 4.3, this makes the stack into a gerbe banded by the $r$-th roots of unity over the moduli space.

5. Period and Index

For a parabolic datum $D = (D, \{k_i^x, \alpha_i^x : x \in \text{supp}(D), i = 1, \cdots, n(x)\})$ we define an integer

$$l(D) = \gcd(\deg(\xi), r, \{k_i^x : x \in \text{supp}(D), i = 1, \cdots, n(x)\}),$$

where the greatest common divisor is taken over the rank degree and all multiplicities of all parabolic weights at all parabolic points.

Theorem 5.1. Assume the base field is the complex numbers. The period of the gerbe

$$\text{Bun}_{X,D}^{r,\xi,s} \to M(X,D,r,\xi)^s$$

is $l(D)$.

Proof. This follows from [BD]. Note that the gerbe of splittings of the Severi-Brauer variety in [BD] is easily identified with the moduli stack. □

We will see below that the above result is true over any field of characteristic zero.

A useful tool for understanding the difference between the period and the index is the notion of a twisted sheaf. A twisted sheaf on a $\mathbb{G}_m$-gerbe $\mathcal{G} \to X$ is a coherent sheaf $\mathcal{F}$ on $\mathcal{G}$ such that inertial action of $\mathbb{G}_m$ on $\mathcal{F}$ coincides with natural module action of $\mathbb{G}_m$ on $\mathcal{F}$. We spell out the meaning of this statement in the next paragraph.

Suppose that we have a $T$-point $T \to X$ and an object $a$ of $\mathcal{G}$ above this point. Part of the data of the coherent sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}_a$ on $T$. These sheaves are required to satisfy compatibility conditions on pullbacks for morphisms in the category $\mathcal{G}$. In particular, every object $a$ of the gerbe $\mathcal{G}$ has an action of $\mathbb{G}_m$ and hence there is an action of $\mathbb{G}_m$ on $\mathcal{F}$. The above definition says that the action of $\mathbb{G}_m$ on $\mathcal{F}$ should be the same as the $\mathbb{G}_m$-action coming from the fact that $\mathcal{F}$ is an $\mathcal{O}_\mathcal{G}$-module.
Example 5.2. We have a $\mu_r$-gerbe

$$\text{Bun}_{X, D}^{r, \xi, s} \rightarrow M(X, D, r, \xi)^r.$$ 

It gives rise to a $\mathbb{G}_m$-gerbe via the natural group homomorphism $\mu_r \rightarrow \mathbb{G}_m$. We denote this gerbe by $\text{Bun}_{X, D}^{r, \xi, s, \mathbb{G}_m}$. One can describe this stack explicitly. The objects over a scheme $S$ are families $\mathcal{F}_s$ of stable parabolic bundles on $S \times X$ such that $\wedge^r \mathcal{F}$ is isomorphic to

$$\pi_S^* \mathcal{L} \otimes \pi_X^* \xi$$

for some line bundle $\mathcal{L}$ on $S$. There is a universal stable parabolic bundle $\mathcal{W}_s$ on $\text{Bun}_{X, D}^{r, \xi, s, \mathbb{G}_m} \times X$. The data that makes up $\mathcal{W}_s$ consists of a vector bundle $\mathcal{W}$ of rank $r$ on $\text{Bun}_{X, D}^{r, \xi, s, \mathbb{G}_m} \times X$. and, for each parabolic point $x$ in the datum $D$, a universal flag

$$F^x_1(\mathcal{W}) \supset \cdots \supset F^x_{n(x)}(\mathcal{W})$$

on $\text{Bun}_{X, D}^{r, \xi, s, \mathbb{G}_m} \times \{x\}.$

As the automorphism group of a stable parabolic bundle is multiplication by a scalar, we see that each of these bundles produces a twisted sheaf. Fix a $k$-rational point $y \in X$. We have on $\text{Bun}_{X, D}^{r, \xi, s, \mathbb{G}_m}$

the following twisted sheaves

(1) $W_{\text{Bun}_{X, D}^{r, \xi, s, \mathbb{G}_m}} \times \{y\}$ of rank $r$,

(2) $F^x_i(\mathcal{W})$ of rank $k^x_i + \cdots + k^x_{n(x)}$ for each $x \in D$, $1 \leq i \leq n(x)$.

There is one more twisted sheaf that will be of importance to us below. We have a projection

$$p : \text{Bun}_{X, D}^{r, \xi, s, \mathbb{G}_m} \times X \rightarrow \text{Bun}_{X, D}^{r, \xi, s, \mathbb{G}_m}.$$ 

The sheaf

$$p_* \mathcal{W} \otimes O_X(Ny)$$

is locally free for large $N$ and gives us another twisted sheaf on the stack. Using Riemann-Roch, it has rank

$$\deg(\xi) + Nr + r(1 - g).$$

We will need the following:

**Proposition 5.3.** Let $\mathfrak{G} \rightarrow \text{Spec}(K)$ be a $\mathbb{G}_m$-gerbe over a field. Then the index of $\mathfrak{G}$ divides $m$ if and only if there is a locally free twisted sheaf on $\mathfrak{G}$ of rank $m$.

**Proof.** See [Lie08, Proposition 3.1.2.1].

**Theorem 5.4.** The index equals the period for the gerbe

$$\text{Bun}_{X, D}^{r, \xi, s} \rightarrow M(X, D, r, \xi)^s.$$
Proof. Over the complex numbers the result follows from Theorem 5.1, Example 5.2 and Proposition 5.3. For a non algebraically closed field we proceed as follows. Set 
\[ e = \gcd(\deg(\xi), r, \{k^x_j : x \in \text{supp}(|D|), j = 1, \cdots, n(x)) \}. \] From Example 5.2 it follows that the index divides \( e \). After base change to an algebraically closed field we find that \( e \) is the period, using a Lefschetz principle. This means that the period of the original gerbe was larger than \( e \). However the period always divides the index. \( \square \)

6. The Essential Dimension of the Stable Locus

Let \( g \) be the genus of \( X \).

**Theorem 6.1.** Set

\[
(6.1) \quad l(D) := \gcd(\deg \xi, r, k^x),
\]

where the \( \gcd \) ranges over all possible multiplicities of all parabolic weights at all the parabolic points in the datum \( D \). If \( l(D) > 1 \) then the essential dimension of \( \text{Bun}^{r, \xi,s}_{X,D} \) is bounded above by

\[
l(D) + \dim M(X, D, r, \xi)^s = l(D) + (r^2 - 1)(g - 1) + \sum_{x \in D} \dim \text{Flag}_x(D).
\]

This upper bound is an equality when \( l(D) = p^l > 1 \) is a prime power.

For any \( l(D) \) we have

\[
ed(\text{Bun}^{r, \xi,s, G_m}_{X,D}) \leq l(D) - 1 + (r^2 - 1)(g - 1) + \sum_{x \in D} \dim \text{Flag}_x(D).
\]

This upper bound is an equality when \( l(D) \) is a prime power.

Proof. Recall that the dimension of the moduli space was computed in equation (4.1) on page 9.

Also recall that for a field extension \( L/k \), an \( L \)-point of \( \text{Bun}^{r, \xi,s, G_m}_{X,D} \) is the same as a family of stable parabolic bundles with prescribed data and determinant \( \xi \) while an \( L \)-point of \( \text{Bun}^{r, \xi,s}_{X,D} \) is such a similar family and fixed trivialization of the top exterior power.

If \( l(D) > 1 \), then the upper bounds follow from Theorem 2.2, Theorem 2.1 and Theorem 5.4. To deduce the result for \( l(D) = 1 \), observe that the gerbe

\[
\text{Bun}^{r, \xi,s, G_m}_{X,D} \rightarrow M(X, D, r, \xi)^s
\]
is neutral as the period is one. Hence there is a universal bundle on the moduli space and the theorem follows.

Suppose that \( l(D) \) is a prime power. We show that

\[
ed(\text{Bun}^{r, \xi,s, G_m}_{X,D}) = l(D) - 1 + (r^2 - 1)(g - 1) + \sum_{x \in D} \dim \text{Flag}_x(D).
\]

The assertion for \( \text{Bun}^{r, \xi,s}_{X,D} \) is similar. Denote by \( K \) the function field of the moduli space. In the case that \( l(D) \) is a prime power, applying 2.3 and Theorem 5.4 we can find a family \( \mathcal{F} \) defined over a field \( L \) containing \( K \) so that \( \text{tr-deg}_K L = l(D) - 1 \).
and this family is not compressible over $K$. If it descends to an extension $L'/k$ then $L'$ must contain $K$ by properties of the moduli space. It follows that
\[ \text{tr-deg}_K L = \text{tr-deg}_K L' \]
Hence the result in the prime power case. \qed

**Corollary 6.2.**

(i) The essential dimension of $\text{Bun}_{X, D}^{r, d, s}$ (degree $d$, and not fixed determinant) is bounded above by
\[ l(D) - 1 + \dim M(X, D, r, d) = l(D) - 1 + (r^2 - 1)(g - 1) + \sum_{x \in D} \dim \text{Flag}_x(D) + g. \]

(ii) When $l(D) = p^l$ a prime power the above inequality is an equality.

**Proof.** If $l(D) = 1$ the moduli space has a universal bundle and we are done, so we may assume $l(D) > 1$.

For (i), suppose that we have a family $\mathcal{F}$ of stable parabolic vector bundles of rank $r$ and degree $d$ over a field $K$. By adjoining at most $g$ parameters to $k$, the base field, we may assume that $\det \mathcal{F}$ is defined over our new base field $k'$ and then apply the above result. In other words, the family compresses to a field $L$ with
\[ \text{tr-deg}_k L \leq l(D) - 1 + (r^2 - 1)(g - 1) + \sum_{x \in D} \dim \text{Flag}_x(D) \]

For (ii), denote by $k(J^d(X))$ the function field of the degree $d$ Picard variety. We have on $X \otimes_k k(J^d(X))$ a line bundle $\xi$ obtained by pulling back the Poincaré bundle. We apply the above theorem to the stack $\text{Bun}_{X, D}^{r, \xi, s}$ to obtain a field extension $K/k(J^d(X))$ and a family $\mathcal{F}$ over this extension that cannot be compressed to a field extension $L/k(J^d(X))$ of smaller transcendence degree. If this family compresses to a family $\mathcal{F}'$ over $L'/k$ then by considering the map to $J^d(X)$ induced by $\det \mathcal{F}'$ we see that $L'$ contains $k(J^d(X))$. Hence the result. \qed

7. **Some Linear Algebra**

We will use the results from this section to find an upper bound on the essential dimension of the polystable locus. The results of this section allow us to pass from polystable to stable by adding appropriate parabolic structure.

Let $V$ be a finite dimensional vector space of dimension $r$. We equip $V$ with two full flags
\[ V = F^x_1 \supset F^x_2 \supset \cdots \supset F^x_r \supset F^x_{r+1} = 0 \]
and
\[ V = F^y_1 \supset F^y_2 \supset \cdots \supset F^y_r \supset F^y_{r+1} = 0 \]
with $\dim F^x_i = \dim F^y_i = r - i + 1$. We say that the flags are *generic* if
\[ \dim(F^x_i + F^y_i) = \min(r, \dim(F^x_i) + \dim(F^y_i)). \]
It is clear that generic flags exist. Fix a 1-dimensional subspace $l \subseteq V$ with
\[ \dim(l + F^x_i + F^y_j) = \min(1 + \dim(F^x_i) + \dim(F^y_j), r). \]
We say that the triple $(F^x_i, F^y_j, l)$ is *generic* if the flags are generic and the subspace $l$ satisfies the above condition. It is easy to see that generic triples exist and any generic pair of flags can be completed to a generic triple.
For a subspace $W \subseteq V$ define the degree of $W$ to be

$$\deg_V(W) = \sum_{i=1}^{r} ((i-1)(\dim(W \cap F_{x}^{i}) - \dim(W \cap F_{x}^{i+1}))$$

$$+ \sum_{i=1}^{r} ((i-1)(\dim(W \cap F_{y}^{i}) - \dim(W \cap F_{y}^{i+1}))$$

$$= \sum_{i=2}^{r} \dim(W \cap F_{x}^{i}) + \sum_{i=2}^{r} \dim(W \cap F_{y}^{i}).$$

We also need the notation

$$\deg(W) = \deg_V(W) + (r - 1) \dim(l \cap W).$$

Note that $\deg_V(W)$ only depends on the first two flags and not the line.

Let us set $d_{x}^{i}(W) = \dim(W \cap F_{x}^{i})$ and $d_{y}^{i}(W) = \dim(W \cap F_{y}^{i}).$

We define the slope of $W$ to be $\mu(W) = \deg(W)/\dim(W)$.

**Lemma 7.1.** For a generic pair of flags $\{F_{x}^{i}\}$ and $\{F_{y}^{i}\}$, We have

$$(W \cap F_{x}^{i}) \oplus (W \cap F_{y}^{r+2-i}) \subseteq W$$

so that $d_{x}^{i}(W) + d_{y}^{r+2-i}(W) \leq \dim(W)$ for all $2 \leq i \leq r$.

In the case $l \subseteq W$, we have

$$l \oplus (W \cap F_{x}^{i}) \oplus (W \cap F_{y}^{r+3-i}) \subseteq W$$

so that $d_{x}^{i}(W) + d_{y}^{r+3-i}(W) \leq \dim(W) - 1$ for all $2 \leq i \leq r + 1$. In particular, $d_{x}^{2}(W) \leq \dim(W) - 1$ and $d_{y}^{r}(W) \leq \dim(W) - 1$.

**Proof.** This follows from the definition of a generic pair of flags. \qed

**Proposition 7.2.** We have

$$\mu(V) > \mu(W)$$

for every proper subspace $W$.

**Proof.** Note that $\deg(V) = 2 \sum_{i=1}^{r-1} i + (r - 1) = r^2 - 1$ so that $\mu(V) = r - \frac{1}{r}$.

First consider the case that $W \cap l = \{0\}$. Then by Lemma 7.1 we have

$$\deg(W) = \deg_V W$$

$$= \sum_{i=2}^{r} (d_{x}^{i}(W) + d_{y}^{r+2-i}(W))$$

$$\leq (r - 1) \dim W.$$

So $\mu(W) \leq r - 1 < \mu(V)$.

Then consider the case in which $l \subseteq W$. By Lemma 7.1, we have

$$\deg(W) = \deg_V W + (r - 1)$$

$$= \left(\sum_{i=3}^{r} (d_{x}^{i}(W) + d_{y}^{r+3-i}(W))\right) + (d_{x}^{2}(W) + d_{y}^{r}(W)) + (r - 1)$$

$$\leq r(\dim(W) - 1) + (r - 1).$$

So $\mu(W) \leq r - \frac{1}{\dim(W)} < \mu(V)$. \qed
8. The Socle

Definition 8.1. Let $K$ a field containing $k$. We say that a parabolic bundle $F$ on $X_K$ is polystable if $F \otimes_K K$ is a direct sum of stable parabolic bundles of the same parabolic slope.

Proposition 8.2. Let $F_\ast$ be a semistable parabolic bundle on $X_K$ with parabolic slope $\mu$. Then there exists a unique maximal polystable subbundle with parabolic slope $\mu$. We call this bundle the socle of $F_\ast$ and write $\text{Soc}(F_\ast)$. If $F_\ast$ is defined over $K$ then so is $\text{Soc}(F_\ast)$.

Proof. Over algebraically closed fields for bundles without parabolic structure the proof can be found in [HL97]. To add parabolic structure one can use the parabolic orbifold correspondence, see below Theorem 9.2. Note that if a semistable vector bundle $F$ has a group action the uniqueness of the socle implies that the socle is preserved by that action.

To see the assertion about ground fields notice that $\text{Soc}(F_\ast)$ will always be defined over some finite Galois extension $L/K$ with Galois group $G$. The Galois action will descend to the socle, as observed above. \hfill \Box

Consider the functor

$$F = F_{X,D}^{\text{poly}, r, d} : \text{Fields}_k \rightarrow \text{Sets}$$

with

$$F(K) = \{\text{families of polystable bundles on } X_K \text{ of degree } d \text{ and rank } r\}/\sim,$$

where $\sim$ is the equivalence relation given by isomorphism of families. We will need to say something about the essential dimension of this functor. The main idea is that we can turn a polystable vector bundle into a stable one by adding parabolic structure at three points.

Let $F \in F(K)$. Choose three $k$-points $x, y, z \in X$ that are not parabolic points. (Recall that we assumed that they exist in the introduction.) Choose a trivialization of $F$ in a Zariski neighborhood of the three points. Using the trivialization we can identify the fiber over the three points with a common vector space $V$. Then $V$ is a $r$ dimensional vector space. We turn these points into parabolic points by defining full flags at $x$ and $y$ to be

$$V = F^x_1 \supset F^x_2 \supset \cdots \supset F^x_r \supset F^x_{r+1} = \{0\}$$

and

$$V = F^y_1 \supset F^y_2 \supset \cdots \supset F^y_r \supset F^y_{r+1} = \{0\}$$

so that $k^x_i = k^y_i = 1$ for all $i = 1, \cdots, r$. Choose the flag at $z$ to be

$$V = F^z_1 \supset F^z_2 = l \supset F^z_3 = 0,$$

where $l$ is a line in $V$ so that $k^z_2 = r - 1$ and $k^z_3 = 1$. The weights for $x$ and $y$ are chosen to be $\alpha^x_i = (i - 1)\epsilon$, $i = 1, \cdots, r$, and the weights for $z$ are chosen to be $\alpha^z_1 = 0$ and $\alpha^z_2 = (r - 1)\epsilon$, where $\epsilon$ is so chosen that the largest weight is smaller than 1. The corresponding parabolic points are denoted $x, y$ and $z$.

Let

$$E = \text{Flag}_x F \times_K \text{Flag}_y (F) \times_K \text{Flag}_z (F).$$

On $E \times X$ there is a universal extension of the quasiparabolic structure of $F$ to the three new points. This means that there is a parabolic bundle $E_\ast$ on $E \times X$ with
datum \(D' = D \cup \{x, y, z\}\). Note by construction, the parabolic slope of the new parabolic bundle \(E_*\) is

\[
\text{par-}\mu(E_*) = \text{par-}\mu(F_*) + \sum_{i=1}^{r} k_i x_i \alpha_i^x + \sum_{i=1}^{r} k_i y_i \alpha_i^y + \sum_{i=1}^{2} k_i z_i \alpha_i^z = \mu + \mu(V) \epsilon
\]

where \(\mu(V)\) is defined in Section 7. Any parabolic subbundle \(E'_*\) of \(E_*\) has parabolic slope

\[
\text{par-}\mu(E'_*) = \mu + \mu(W) \epsilon,
\]

where \(W\) is the common fiber of the subbundle \(E'_*\) and hence is a vector subspace of \(V\). Then by Proposition 7.2, we have that \(E_*\) is a stable parabolic bundle. There is an open subscheme \(E^* \subseteq E\) where this bundle is stable.

**Lemma 8.3.** The open subscheme \(E^*\) is not empty.

**Proof.** It suffices to show that \(E^* \otimes_K \mathcal{K}\) is not empty so we may assume that \(F = F_1 \oplus \cdots \oplus F_l\) with the \(F_i\) non isomorphic stable bundles of the same slope \(\mu\). We may find an open \(U\) of \(X_K\) which contains \(x, y\) and \(z\) such that \(F|_U\) is trivial. Then we apply the argument above to obtain the result. \(\square\)

**Theorem 8.4.** We have

\[
ed(F) \leq (r^2 - 1)(g - 1) + \sum_{x \in D} \dim \text{Flag}_x(D) + g + r^2 - 1
\]

\[
= r^2 g + \sum_{x \in D} \dim \text{Flag}_x(D).
\]

**Proof.** As above, we may add parabolic structure at \(x, y\) and \(z\) to obtain a stable parabolic bundle. Since \(l(D') = 1\) (defined in (6.1) and

\[
\dim \text{Flag}_x(D') = \dim \text{Flag}_y(D') = r(r - 1)/2, \quad \dim \text{Flag}_z(D') = r - 1
\]

we may apply Corollary 6.2 to obtain the result. \(\square\)

### 9. Orbifold Bundles and Orbifold Riemann-Roch

Let \(Y\) be a smooth projective curve with an action of the finite group \(\Gamma\) defined over \(k\). If \(E\) is a \(\Gamma\) bundle on \(Y\) the cohomology groups \(H^*(Y, E)\) are naturally representations of the group \(\Gamma\). We define \(\chi(\Gamma, Y, E)\) to be the equivariant Euler characteristic. Precisely, it is the class

\[
\chi(\Gamma, Y, E) = [H^0(Y, E)] - [H^1(Y, E)]
\]

in the \(K\)-ring of representations of \(\Gamma\). The orbifold Riemann–Roch theorem is a formula for this class.

**Theorem 9.1.** Suppose that \(k = \overline{k}\). Consider the projection

\[
\pi : Y \to X = Y/\Gamma.
\]

For each \(y \in Y\) write \(e_y\) for the ramification index of \(\pi\) at \(y\) and \(\Gamma_y\) for the isotropy group at \(y\). We have a character

\[
\chi_y : \Gamma_y \to \mathbb{G}_m
\]

coming from the action of \(\Gamma_y\) on the cotangent space \(m_y/m_y^2\).
We have
\[ |\Gamma| \chi(\Gamma, Y, \mathcal{E}) = (|\Gamma|(1 - g)\text{rk}(\mathcal{E}) + \deg(\mathcal{E})[k[\Gamma]] - \sum_{y \in Y} \sum_{d=0}^{r_y-1} d[\text{Ind}_{\Gamma_y}^\Gamma(\mathcal{E}|_y \otimes \chi^d_y)]. \]

**Proof.** See [Koc05]. This formula can also be deduced from [Tö99, Theorem 4.11] by considering the morphism of quotient stacks
\[ [Y/\Gamma] \to B\Gamma. \]

We now recall the main result of [Bis97] in the case of curves. Consider a reduced divisor \( D = \sum_{i=1}^s x_i \) on \( X \), where \( x_i \) are \( k \)-rational points.

Recall that the positive integer \( N \) was chosen in Section 3 so that the parabolic weights were integer multiples of \( 1/N \). Consider a curve \( Y \) with an action of a finite group \( \Gamma \) such that \( Y/\Gamma = X \). There is a projection \( \pi : Y \to X \). We further assume that for each \( x \in \text{supp}(D) \), we have \( \pi^*(x) = kN(\pi^*x)_{\text{red}} \) for some positive integer \( k \).

Denote by \( \text{Vect}^D_{\Gamma}(Y, N) \) the full subcategory of \( \Gamma \)-bundles on \( Y \) if and only if
\[ \bullet \text{ for all geometric points } y \text{ in } Y \text{ with } y \in \text{supp}(\pi^*(D)_{\text{red}}), \text{ and for each } \gamma \in \Gamma_y \text{ in the isotropy group, } \gamma^N \text{ acts on the fiber } W_y \text{ trivially, and} \]
\[ \bullet \text{ for all geometric points } y \text{ in } Y \text{ with } y \notin \text{supp}(\pi^*(D)_{\text{red}}), \text{ the action of the isotropy group } \Gamma_y \text{ on the fiber } W_y \text{ is trivial.} \]

**Note:** We have not asserted that such a \( Y \) exists over our base field \( k \). If such a curve \( Y \) with \( \Gamma \) action exists then we will say that it *splits* the parabolic structure on \( X \).

Recall that \( \text{PVect}(X, D, N) \) is the category of parabolic bundles with parabolic datum only inside the support of \( D \) and parabolic weights integer multiples of \( 1/N \). The category \( \text{PVect}(X, D, N) \) is a tensor category. To define the tensor product, it is convenient to think of parabolic bundles as being an appropriate family \( \{F_t\}_{t \in \mathbb{R}} \), as described in Construction 3.5. Then \( (F \otimes F')_t \) is the subsheaf of \( i^*i^*F \otimes F' \) generated by
\[ F_a \otimes F'_b, \quad a + b \geq t. \]

Here we denote the inclusion \( X \setminus D \hookrightarrow X \) by \( i \). One checks that the resulting collection \( \{F_t \otimes F'_t\}_{t \in \mathbb{R}} \) gives a bundle with parabolic datum \( D \).

With these definitions \( \text{PVect}(X, D, N) \) becomes a tensor category. The unit \( U \) for the tensor product is the parabolic bundle with \( U_0 = O_X \) and \( U_t = O_X(-D) \) for \( 0 < t < 1 \). It is readily checked that we have an associative, commutative tensor structure with unit.

**Theorem 9.2.** There is a \( k \)-linear additive equivalence of tensor categories between \( \text{PVect}(X, D, N) \) and \( \text{Vect}^D_{\Gamma}(Y, N) \).

**Proof.** This can be found in [BBN01, page 344], [Bis97] and [Bor07]. Also see below for a description of one of the functors in this equivalence.

We will denote the \( \Gamma \)-bundle associated to a parabolic bundle \( F \) by \( F_Y \).

There is a usual notion of exact sequence in the category \( \text{Vect}^D_{\Gamma}(Y, N) \). There is also a notion of exact sequence in the category \( \text{PVect}(X, D, N) \) inherited from the category \( \text{QCoh}(X, D, N) \).

**Proposition 9.3.** The equivalence in Theorem 9.2 preserves exact sequences.
Proof. We use the notation set up before Theorem 9.2. Write \( D = \sum_{i=1}^s x_i \) and \( y_i = \pi^*(x_i) \). Set \( \pi^*(x_i) = n_i y_i \).

It will be convenient to think of parabolic bundles in terms of Construction 3.5. From [Bis97], the functor \( M : \text{Vect}^D(Y, N) \to \text{PVect}(X, D, N) \) is given by the formula \( M(W) = E_t \), where

\[
E_t = (\pi^*(W) \otimes O_Y(\sum_{i} [n_i] y_i))^\Gamma.
\]

The functor is clearly additive. It suffices to remark that an equivalence of abelian categories by additive functors must preserve exact sequences.

\[ \Box \]

10. A Universal Construction

Let \( E_\ast \) and \( E'_\ast \) be parabolic bundles with parabolic datum \( D \) and \( D' \). We choose an integer \( N \) so that all the weights of both of the bundles are integer multiples of \( \frac{1}{N} \). We denote by \( \text{Ext}_{\text{par}}(E'_\ast, E_\ast) \) the Yoneda Ext group in the category \( \text{QCoh}(X, D, N, \Gamma) \), where \( \Gamma \) is chosen large enough to contain all parabolic points. Note that all such extensions will create lie inside the category \( \text{PVect}(X, D, N) \) as \( E_\ast \) and \( E'_\ast \) have underlying vector bundles. It is a \( k \)-vector space and we view it as a variety. We would like to construct a universal extension on it. (A quick check shows that exact sequences are preserved by Baer sum and scalar multiplication.)

After some finite base extension \( L/k \), there exists a group \( \Gamma \), a smooth curve \( Y \), and an action of \( \Gamma \) on \( Y \) defined over \( L \) such that \( Y/\Gamma = X \) and \( Y \) splits the parabolic structures of \( E_\ast \) and \( E'_\ast \). Therefore, there is an equivalence of categories between \( \text{PVect}(X, D, N) \) and \( \text{Vect}^D(Y, N) \). By further extension of \( L \) we may assume that all representations of \( \Gamma \) are defined over \( L \).

Proposition 10.1. Let \( \mathcal{F} \) and \( \mathcal{G} \) be \( \Gamma \) bundles on \( Y \). There exists an \( L \)-vector space \( \text{Ext}^1_\Gamma(\mathcal{F}, \mathcal{G}) \), which we view as an \( L \)-variety, and an extension of \( \Gamma \) sheaves on \( \text{Ext}^1_\Gamma(\mathcal{F}, \mathcal{G}) \times Y \),

\[
0 \to \pi^* \mathcal{G} \to \mathcal{E} \to \pi^* \mathcal{F} \to 0 \quad (E)
\]

with the following universal property: Given a scheme \( f : V \to \text{Spec } L \) and an extension

\[
0 \to f^* \mathcal{G} \to \mathcal{E}' \to f^* \mathcal{F} \to 0 \quad (E')
\]

of \( \Gamma \) sheaves on \( V \times Y \), there exists a unique \( \Gamma \)-morphism \( t : V \to \text{Ext}^1_\Gamma(\mathcal{F}, \mathcal{G}) \) with \( t^*(E) \cong (E') \).

Proof. There exists a universal extension on \( \text{Ext}^1(\mathcal{F}, \mathcal{G}) \). This follows via base change for cohomology. To obtain a universal \( \Gamma \)-extension, just restrict this extension to

\[
\text{Ext}^1_\Gamma(\mathcal{F}, \mathcal{G}) \overset{\text{defn}}{=} \text{Ext}^1(\mathcal{F}, \mathcal{G})_{\text{triv}}.
\]

This proves the proposition.

\[ \Box \]

Proposition 10.2. There exists a universal extension of parabolic bundles on

\[
\text{Ext}_{\text{par}}(E'_\ast, E_\ast).
\]

Proof. We may assume that \( L/k \) is Galois with group \( G \). Using the equivalence in Theorem 9.2 we see that there is a universal extension on the base extension \( \text{Ext}_{\text{par}}(E'_\ast, E_\ast) \otimes L \). However, the universal extension inherits a Galois action, in view of its universal property, and hence descends to \( k \).

\[ \Box \]
We need to bound the dimension of $\text{Ext}_{\text{par}}(E_s', E_s)$. The following lemma will be useful.

**Lemma 10.3.** Let $\Gamma$ be a finite group and $\Gamma_y$ a cyclic subgroup of it with generator $\gamma$. Let $V$ be a finite dimensional representation of $\Gamma_y$ on which $\gamma^N$ acts trivially and $T$ a one dimensional representation of $\Gamma$ whose restriction to $\Gamma_y$ is faithful. Then

$$\sum_{d=0}^{\vert\Gamma_y\vert-1} d \cdot \dim(\text{Ind}_{\Gamma_y}^{\Gamma} V \otimes T^d)_{\text{triv}}$$

is bounded by

$$(\dim V)\vert\Gamma_y\vert(1 - \frac{1}{N})$$

(here $W_{\text{triv}} = W^{\Gamma_y}$ is the fixed part).

**Proof.** By base change we may assume that the ground field contains all roots of unity. Let $\zeta$ be a primitive $\vert\Gamma_y\vert$-th root of unity. Write $V = \bigoplus_{i=0}^{N-1} V_{(\frac{i}{N})\Gamma_y}$, where the generator $\gamma$ acts as scalar multiplication by $\zeta^j$ on $V_j$. Note that only these weight spaces can occur as $\gamma^N$ acts trivially on $V$. Then

$$\sum_{d=0}^{\vert\Gamma_y\vert-1} d \cdot \dim(\text{Ind}_{\Gamma_y}^{\Gamma} V \otimes T^d)_{\text{triv}}$$

by Frobenius reciprocity. But $\gamma$ acts as multiplication by the scalar $\zeta^{s+d}$ on $V_s \otimes T^d$.

We see that for fixed $i$, the set of invariants $(V_{(\frac{i}{N})\Gamma_y} \otimes T^d)_{\text{triv}}$ is non-zero if and only if either $(d, i) = (0, 0)$ or $i > 0$ with $d = \vert\Gamma_y\vert(1 - \frac{i}{N})$. When $(V_{(\frac{i}{N})\Gamma_y} \otimes T^d)_{\text{triv}}$ is non-zero, $\dim((V_{(\frac{i}{N})\Gamma_y} \otimes T^d)_{\text{triv}}) = \dim(V_{(\frac{i}{N})\Gamma_y})$. So the above sum becomes

$$\sum_{i=1}^{N-1} \vert\Gamma_y\vert(1 - \frac{i}{N}) \dim(V_{(\frac{i}{N})\Gamma_y}) \leq \vert\Gamma_y\vert(1 - \frac{1}{N}) \dim(V)$$

since $V = \bigoplus_{i=0}^{N-1} V_{(\frac{i}{N})\Gamma_y}$. \(\Box\)

Recall from [Bis97] that a $\Gamma$-bundle is semistable if and only if the underlying bundle is semistable. This fact follows from the uniqueness of the Harder-Narasimhan filtration. We also need the following lemma.

**Lemma 10.4.** Let $E$ be a semistable $\Gamma$-bundle on $Y$. Then

$$\dim H^0(Y, E)_{\text{triv}} \leq \begin{cases} 0 & \text{if } \deg(E) < 0 \\ \text{rk}(E) + \frac{\deg(E)}{\vert\Gamma\vert} & \text{otherwise} \end{cases}$$

**Proof.** The assertion is obvious when the degree is negative. We induct on the degree. By extending $L$, we may find a point $y \in Y$ for which the isotropy subgroup, for the action of $\Gamma$, is trivial. We let $D$ be the divisor $\text{orb}(y)$. The result follows from the exact sequence

$$0 \rightarrow E(-D) \rightarrow E \rightarrow E_D \rightarrow 0.$$ 

Note that the twist $E(-D)$ is indeed semistable. \(\Box\)
Proposition 10.5. Let \( \mathcal{F}_* \) and \( \mathcal{G}_* \) be parabolic bundles such that \( \mathcal{F}_* \otimes \mathcal{G}_* \) is parabolic semistable and par-deg(\( \mathcal{F}_* \otimes \mathcal{G}_* \)) \( \geq 0 \). Then

\[
\dim \text{Ext}_{\text{par}}(\mathcal{F}_*, \mathcal{G}_*) \leq \text{rk}(\mathcal{F}_*)\text{rk}(\mathcal{G}_*)g + (\deg |D|)(1 - \frac{1}{N})\text{rk}(\mathcal{F}_*)\text{rk}(\mathcal{G}_*),
\]

where \( g \) is the genus of \( X \). (Recall the notation \( |D| \) from Section 5.)

Proof. We may pass to a field extension \( L/k \) so that there is a \( \Gamma \) cover \( Y \to X \) as in Theorem 9.2. Write \( \mathcal{W}_Y \) for the \( \Gamma \)-bundle associated to a parabolic bundle \( \mathcal{W}_\gamma \). We need to compute the dimension of \( \text{Ext}^1_{\mathcal{U}}(\mathcal{F}_*, \mathcal{G}_*) \). Note that the fact that

\[
\dim \text{Ext}^1_{\mathcal{U}}(\mathcal{F}_*, \mathcal{G}_*) = \dim \text{Ext}^1_{\mathcal{U}}(\mathcal{F}_* \otimes \mathcal{G}_*)
\]

allows us to pass to an algebraic closure and apply Theorem 9.1 to the \( \Gamma \) bundle \( (\mathcal{F}_*)^\vee \otimes \mathcal{G}_*^\vee \). Then

\[
\dim \text{Ext}_{\text{par}}(\mathcal{F}_*, \mathcal{G}_*) = h^1((\mathcal{F}_*)^\vee \otimes \mathcal{G}_*)_{\text{triv}}
\]

\[
= h^0((\mathcal{F}_*)^\vee \otimes \mathcal{G}_*)_{\text{triv}} - \chi(\Gamma, Y, (\mathcal{F}_*)^\vee \otimes \mathcal{G}_*)_{\text{triv}}
\]

\[
\leq \text{rk}((\mathcal{F}_*)^\vee \otimes \mathcal{G}_*)g + \frac{1}{|\Gamma|} \sum_{y \in Y} \sum_{d=0}^{e-1} d \dim \text{Ind}_{\Gamma_y}(\mathcal{F}_*)^\vee \otimes \mathcal{G}_*)_{y \otimes \chi^d}_{\text{triv}}
\]

Here we applied Lemma 10.4 under the hypothesis that the \( \Gamma \)-bundle \( (\mathcal{F}_*)^\vee \otimes \mathcal{G}_*^\vee \) is semistable of non-negative degree. This is true since, by hypothesis, the corresponding parabolic bundle \( \mathcal{F}^\vee \otimes \mathcal{G}^\vee \) is parabolic semistable with non-negative parabolic degree. We also applied Theorem 9.1 with the observation that the trivial part \( k[\Gamma]^\mathcal{U} \) of the regular representation \( k[\Gamma] \) has dimension one. Since \( \text{rk}((\mathcal{F}_*)^\vee \otimes \mathcal{G}_*)_{\text{triv}}) = \text{rk}(\mathcal{F})\text{rk}(\mathcal{G}) \), we need only bound the second term. If \( y \notin \text{supp}(\mathcal{\pi}^*(\mathcal{D}))_{\text{red}} \), then Lemma 10.3 shows that the sum corresponding to \( y \) vanishes as the isotropy group acts trivially on \( (\mathcal{F}^\vee \otimes \mathcal{G})_{y} \). If \( N \) is an integer so that all weights are integer multiples of \( \frac{1}{N} \) and \( y \in \text{supp}(\mathcal{\pi}^*(\mathcal{D}))_{\text{red}} \), then Lemma 10.3 shows that the sum corresponding to \( y \) is bounded by \( (1 - \frac{1}{N})\text{rk}(\mathcal{F})\text{rk}(\mathcal{G}) \) since in this case the kernel of the homomorphism \( \Gamma_y \to GL((\mathcal{F}^\vee \otimes \mathcal{G})_{y}) \) has order dividing \( N \). As

\[
\frac{1}{|\Gamma|} |\{y \in Y \mid y \in \text{supp}(\mathcal{\pi}^*(\mathcal{D}))_{\text{red}}\}| \leq \deg(|D|),
\]

the proof is complete. \( \square \)

Let \( \mathcal{D} \) be a parabolic datum on \( X \). We denote by \( N(\mathcal{D}) \) the smallest positive integer so that the weights in \( \mathcal{D} \) are scalar multiples of \( \frac{1}{N(\mathcal{D})} \). Set

\[
(10.1) \quad (1 - \frac{1}{N(\mathcal{D})})\deg(|D|) = M(\mathcal{D}).
\]

Corollary 10.6. Let \( \mathcal{E}_* \) be a non-stable parabolic bundle of rank \( r \) with parabolic data \( \mathcal{D} \). Let

\[
0 \subset (\mathcal{E}_1)_* \subset (\mathcal{E}_2)_* \subset \cdots \subset (\mathcal{E}_m)_* = (\mathcal{E}_*)_* \subset E_*
\]

be the Harder-Narasimhan filtration of \( \mathcal{E}_* \). Define \( (\mathcal{E}')_* := (\mathcal{E}_{m-1})_* \). Then

\[
\dim(\text{Ext}_{\text{par}}^1(\mathcal{E}/\mathcal{E}'_*), (\mathcal{E}')_*) \leq r'(r - r')(g + M(\mathcal{D})).
\]
Proof. The Harder-Narasimhan filtration of $(E' \otimes (E/E')^\vee)_s$ as a parabolic bundle is

$$0 \subset (E)_s \otimes (E/E_m-1)^\vee_s \subset (E_2)_s \otimes (E/E_m-1)^\vee_s \subset \cdots \subset (E_m-1)_s \otimes (E/E_m-1)^\vee_s = E'_s \otimes (E/E_m-1)^\vee_s.$$ 

Note that

$$\text{par-}\mu(((E/E_m-1)^\vee_s \otimes (E_i/E_{i-1})_s)) = \frac{(-r_md_i + d_mr_i)}{r_ir_m} = \text{par-}\mu((E_i/E_{i-1})_s) - \text{par-}\mu((E_m/E_{m-1})_s) > 0,$$

where $d_i$ is the parabolic degree of $(E_i/E_{i-1})_s$ and $r_i$ is its rank. So the previous proposition applies to each $((E/E_m-1)^\vee_s \otimes (E_i/E_{i-1})_s)$ since this is a semistable parabolic bundle with positive parabolic slope. We find that

$$\dim(\text{Ext}^1_{\text{par}}((E/E_m-1)_s, (E_i/E_{i-1})_s)) \leq (r_i)(r - r')(g + M(D)).$$

The result follows by a simple induction.

\[\square\]

11. The Semistable Locus

We define a function $F_{g,D} : \mathbb{N} \rightarrow \mathbb{N}$ recursively by

$$F_{g,D}(r) = \max_{s,t,r^+} F_{g,D}(s) + t^2g + st(g + M(D))$$

with $F_{g,D}(1) = g$ and $F_{g,D}(0) = 0$, where $M(D)$ is defined in [10.1].

Let us record the following:

**Lemma 11.1.** Consider positive integers $k_i$ with partitions $s_i + t_i = k_i$. Here $s_i$ and $t_i$ are nonnegative. Then

$$\dim\text{Flag}(s_1, \cdots, s_n) + \dim\text{Flag}(t_1, \cdots, t_n) \leq \dim\text{Flag}(k_1, \cdots, k_n).$$

**Proof.** Recall that

$$\dim\text{Flag}(k_1, \cdots, k_n) = \sum_{i=1}^{n-1} k_i(k_{i+1} + \cdots + k_n).$$

The result follows from this. \[\square\]

**Proposition 11.2.** We have

$$\text{ed}(\text{Bun}^{r^{ss,d}}_{X,D}) \leq F_{g,D}(r) + \sum_{x \in D} \dim(\text{Flag}_x(D)).$$

**Proof.** We proceed by induction on the rank. The case of rank one is trivial. Consider a parabolic bundle $E_s$. If $E_s$ is stable, then we are done by Corollary [6.2].

Suppose next that $\text{Soc}(E_s) = E_s$.

Then by Theorem [8.3], the bundle is defined over a field of transcendence degree at most

$$r^2g + \sum_{x \in D} \dim(\text{Flag}_x(D)).$$

In the remaining case there is an exact sequence

$$0 \rightarrow \text{Soc}(E_s) \rightarrow E_s \rightarrow F_s \rightarrow 0.$$
Suppose that \( D_1 \) and \( D_2 \) are the parabolic data for \( \text{Soc}(\mathcal{E}_s) \) and \( \mathcal{F}_s \) respectively. Lemma 11.1 shows that for every parabolic point \( x \) we have

\[
\dim \text{Flag}_x(D_1) + \dim \text{Flag}_x(D_2) \leq \dim \text{Flag}_x(D).
\]

Let the ranks of \( \text{Soc}(\mathcal{E}_s) \) and \( \mathcal{F}_s \) be \( t \) and \( s \) respectively. By Theorem 8.4 we know that \( \text{Soc}(\mathcal{E}_s) \) is defined over a field of transcendence degree at most \( t^2g + \sum_{x \in D_2} \dim(\text{Flag}_x(D_1)) \). By induction the parabolic bundle \( \mathcal{F}_s \) is defined over a field of transcendence degree \( F_g(s) + \sum_{x \in D_2} \dim(\text{Flag}_x(D_2)) \). Let the compositum of these two fields be \( K \). We have

\[
\dim \text{Ext}_{\mathfrak{par}}(\mathcal{F}_s, \text{Soc}(\mathcal{E}_s))_{\text{triv}} \leq st(g + M(D))
\]

by Proposition 10.5. Note that this result applies as \( \mathcal{F}_s' \otimes \text{Soc}(\mathcal{E}_s) \) is semistable of parabolic degree zero. To obtain the result we apply Proposition 10.2.

12. THE FULL MODULI STACK

We form a function

\[
G_{g,D}(r) = \max_{s+t=r, \ s \geq 0, \ t \geq 0} F_{g,D}(t) + G_{g,D}(s) + st(g + M(D))
\]

with \( G_{g,D}(1) = g \) and \( G_{g,D}(0) = 0 \).

**Theorem 12.1.** We have a bound

\[
\text{ed}(\text{Bun}_{X,D}^{r,d}) \leq G_{g,D}(r) + \sum_{x \in D} \dim(\text{Flag}_x(D)).
\]

**Note:** The left-hand side in the inequality does not depend upon the weights in the parabolic datum. Hence the inequality is true for all possible choices of weights.

**Proof.** We prove this theorem by using induction on the rank \( r \). The case of rank one is trivial. Consider a parabolic bundle \( \mathcal{E}_s \) and the exact sequence

\[
0 \rightarrow (\mathcal{E}_1)_s \rightarrow (\mathcal{E}_2)_s \rightarrow 0.
\]

where \((\mathcal{E}_1)_s\) is the (destabilizing) parabolic proper subbundle of maximal rank in the Harder-Narasimhan filtration of \( \mathcal{E}_s \). Suppose that \( D_1 \) and \( D_2 \) are the parabolic structures for \((\mathcal{E}_1)_s\) and \((\mathcal{E}_2)_s\) respectively. Then Lemma 11.1 shows that for every parabolic point \( x \) we have

\[
\dim(\text{Flag}_{\mathcal{E}_1}(D_1)) + \dim(\text{Flag}_{\mathcal{E}_2}(D_2)) \leq \dim(\text{Flag}_x(D)).
\]

Let the ranks of \((\mathcal{E}_1)_s\) and \((\mathcal{E}_2)_s\) be \( t \) and \( s \) respectively. By Proposition 11.2 we know that \((\mathcal{E}_2)_s\) is defined over a field of transcendence degree at most \( F_{g,D_1}(s) + \sum \dim(\text{Flag}_x(D_1)) \). By induction, the parabolic bundle \((\mathcal{E}_1)_s\) is defined over a field of transcendence degree \( F_{g,D_2}(t) + \sum_{x \in D_2} \dim(\text{Flag}_x(D_2)) \). Let the compositum of these two fields be \( K \). Note that deg\((\mathcal{D}_1)\) \( \leq \deg(\mathcal{D}) \) and \( N(D_i) \leq N(D) \) for \( i = 1, 2 \) so that \( M(D_i) \leq M(D) \) for \( i = 1, 2 \), and hence \( F_{g,D_i} \leq F_{g,D} \). Then

\[
\text{tr-deg}K \leq F_{g,D}(s) + F_{g,D}(t) + \sum_{x \in D} \dim(\text{Flag}_x(D)).
\]
We have
\[ \dim \text{Ext}_{\text{par}}((E_2)_*, (E_1)_*)_{\text{triv}} \leq \text{st}(g + M(D)) \]
by Corollary 10.6. Let \( W = \text{Ext}((E_2)_*, (E_1)_*)_{\text{triv}} \). The parabolic bundle \( E_* \) is defined over the function field \( K' \) of a subvariety of this linear variety \( W \). Then
\[ \text{trdeg} K' \leq F_{g,D}(s) + G_{g,D}(t) + \sum_{x \in \mathcal{D}} \dim(\text{Flag}_x(D)) + \text{st}(g + M(D)). \]
The result follows.

\[ \square \]

13. SOME FACTS ABOUT \( F_{g,D} \) AND \( G_{g,D} \)

Let
\[ H_{g,D,r}(t) = F_{g,D}(t) + (r - t)^2g + (g + M(D))t(r - t) \]
so that
\[ F_{g,D}(r) = \max_{0 \leq t \leq r - 1} H_{g,D,r}(t). \]

Proposition 13.1. If \( g \leq M(D) \), then for all \( r \geq 0 \), we have
\[ F_{g,D}(r) = \frac{r(r + 1)}{2}g + \frac{r(r - 1)}{2}M(D). \]
If \( g \geq M(D) \), then for all \( r \geq 0 \), we have
\[ F_{g,D}(r) = r^2g. \]

Proof. Case 1: \( g \leq M(D) \).

For \( r = 0, 1 \), this follows from definition of \( F_{g,D} \). Assume the result for \( 0 \leq t < r \) by induction. Then by the inductive hypothesis, for all \( 0 \leq t \leq r - 1 \), we have
\[ H_{g,D,r}(t) = \frac{t(t + 1)}{2}g + \frac{t(t - 1)}{2}M(D) + (r - t)^2g + (g + M(D))t(r - t). \]
Simplifying this, we find that for all \( 0 \leq t \leq r - 1 \), we have
\[ H_{g,D,r}(t) = \left( \frac{g - M(D)}{2} \right) t^2 - (r - \frac{1}{2})(g - M(D))t + r^2g. \]
Note that
\[ H_{g,D,r}(r - 1) = \frac{r(r + 1)}{2}g + \frac{r(r - 1)}{2}M(D). \]
So it suffices to prove the claim that
\[ \max_{0 \leq t \leq r - 1} H_{g,D,r}(t) = H_{g,D,r}(r - 1). \]
If \( g = M(D) \), then it is clear that \( H_{g,D,r}(t) = r^2g \) for all \( 0 \leq t \leq r - 1 \) so the claim holds in this case.
Assume then that \( g < M(D) \).
Consider the parabola that agrees with \( H_{g,D,r}(t) \):
\[ f(t) = \left( \frac{g - M(D)}{2} \right) t^2 - (r - \frac{1}{2})(g - M(D))t + r^2g. \]
Since
\[ f'(t) = (g - M(D))(t - (r - 1/2)), \]
we have $f'(t) \geq 0$ if and only if $t \leq (r-1/2)$ under the hypothesis $g - M(D) < 0$. So in particular, $f(t)$ is increasing on the interval $0 \leq t \leq r - 1$. But then since $H_{g, D, r}(t) = f(t)$, we have

$$F_{g, D}(r) = \max_{0 \leq t \leq r - 1} H_{g, D, r}(t) = H_{g, D, r}(r - 1).$$

as required.

**Case 2:** $g \geq M(D)$.

We will prove by induction on $r$ that

$$F_{g, D}(r) = H_{g, D, r}(0) = r^2g$$

if $g \geq M(D)$. The statement is true for $r = 0, 1$ by definition and since we have more generally that

$$H_{g, D, r}(0) - H_{g, D, r}(1) = (r - 1)(g - M(D)) \geq 0,$$

this shows that, in particular, we have

$$F_{g, D}(2) = H_{g, D, 2}(0) = 4g.$$

By the inductive hypothesis, we may assume that for $0 \leq t \leq r - 1$, we have

$$H_{g, D, r}(t) = t^2g + (r - t)^2g + (g + M(D))t(r - t).$$

Simplifying this, we obtain

$$H_{g, D, r}(t) = (g - M(D))(t(t - r)) + r^2g$$

by the above. Since $(g - M(D)) \geq 0$, we have $(g - M(D))(t(t - r)) \leq 0$ if $0 \leq t \leq r - 1$. So $H_{g, D, r}(t) \leq r^2g = H_{g, D, r}(0)$ if $0 \leq t \leq r - 1$. This implies that $F_{g, D}(r) = r^2g.$

Observe that

$$F_{g, D}(r) = r^2g = \frac{r(r + 1)}{2}g + \frac{r(r - 1)}{2}M(D)$$

if $g = M(D)$ so that the answers agree on the overlap. \qed

Recall that

$$G_{g, D}(r) = \max_{s + t = r, s \geq 0, t > 0, s, t \text{ integers}} F_{g, D}(t) + G_{g, D}(s) + st(g + M(D))$$

and $G_{g, D}(1) = g, G_{g, D}(0) = 0$.

**Proposition 13.2.** $F_{g, D}(r) = G_{g, D}(r)$ for all $r \geq 0$.

**Proof.** The result is true by definition for $r = 0, 1$. It suffices to prove that for all $0 < s \leq t$, we have

$$F_{g, D}(s + t) - F_{g, D}(s) - F_{g, D}(t) - (g + M(D))st \geq 0$$

using the previous proposition.

**Case 1:** $g \leq M(D)$.

Since

$$F_{g, D}(r) = \frac{r(r + 1)}{2}g + \frac{r(r - 1)}{2}M(D),$$

we find that

$$F_{g, D}(s + t) - F_{g, D}(s) - F_{g, D}(t) - (g + M(D))st = st(g + M(D)) - st(g + M(D)) = 0.$$

**Case 2:** $g > M(D)$.
Since
\[ F_{g,D}(r) = r^2 g , \]
we find that
\[ F_{g,D}(s + t) - F_{g,D}(s) - F_{g,D}(t) - (g + M(D))st = 2stg - st(g + M(D)) = st(g - M(D)) \geq 0 . \]
This proves the proposition. \( \square \)

**Remark 13.3.** The main result of [DL09] shows that
\[ \text{ed}(\text{Bun}_X^{r,ξ,ss,D}) \leq \lfloor h_g(r) \rfloor + g . \]
The function \( h_g(r) \) is defined recursively by \( h_g(1) = 1 \) and
\[ h_g(r) - h_g(r - 1) = r^3 - r^2 + \frac{r^4}{4}(g - 1) + \frac{r}{2} + \frac{r^2 g^2}{4} + \frac{1}{4} . \]
(Note: solving the recursion would produce a quartic.) Putting together Theorem 12.1, Proposition 13.1 and Proposition 13.2 for \( D = \emptyset \) and the original hypothesis \( g \geq 2 \) on the curve, we have
\[ \text{ed}(\text{Bun}_X^d) \leq r^2 g , \]
which is a substantial improvement. The main reason for the improvement is the use of the socle filtration as opposed to the Jordan-Hölder filtration.

### 14. Lower Bounds

The issue of finding lower bounds in questions on essential dimension is more subtle. We fix a rank \( r \) and denote by \( \text{Bun}_X^{r,ξ,ss,D} \) the semistable locus of the moduli stack of vector bundles of rank \( r \), determinant \( ξ \) and parabolic structure along \( D \).

We would like to find a lower bound on its essential dimension.

Suppose that \( p^l \) divides \( l(D) \) where \( p \) is a prime. (Recall the definition of \( l(D) \) from Theorem 6.1.) Let \( x \) be a \( k \)-point of \( X \); it exists by our ongoing hypothesis on \( X \). Construct a parabolic point \( x = (x, \{ p^l, r - p^l \}, \{ α_1, α_2 \}) \), where \( α_i \) are chosen sufficiently small so that if a parabolic vector bundle is semistable for the datum \( D \cup \{ x \} \) then the underlying vector bundle is semistable. It is easy to see one can do this using the definition of parabolic slope.

**Theorem 14.1.** We have
\[ \text{ed}(\text{Bun}_X^{r,ξ,ss,D}) \geq (r^2 - 1)(g - 1) + p^l + \sum_{y \in D} \dim \text{Flag}_y(D) \]
and
\[ \text{ed}(\text{Bun}_X^{r,d,ss,D}) \geq (r^2 - 1)(g - 1) + p^l - 1 + g + \sum_{y \in D} \dim \text{Flag}_y(D) . \]

**Proof.** Let \( D' = \{ x \} \cup D \) be the datum constructed above. Now the greatest common divisor of the multiplicities is \( p^l \). Hence by Theorem 6.1 as we are in the prime power case and we calculate the essential dimension of the stack \( \text{Bun}_X^{r,ξ,ss,D'} \).

(Note that Theorem 6.1 does not require that \( X \) have three \( k \)-rational points that are not parabolic.) So we have
\[ \text{ed}(\text{Bun}_X^{r,ξ,ss,D'}) = p^l + (r^2 - 1)(g - 1) + \text{Flag}_x(D') + \sum_{y \in D'} \dim \text{Flag}_y(D) . \]
The result now follows from the fibration theorem, [BFRV, Theorem 3.2], applied to the representable fibration

\[ \text{Bun}_{r,\xi,s}^{\mathbb{X},D} \rightarrow \text{Bun}_{r,\xi,s}^{\mathbb{X},D}. \]

The non-fixed determinant case is analogous via Corollary [6.2].

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