INDUCED SEMI RIEMANNIAN STRUCTURES ON NULL SUBMANIFOLDS

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ABSTRACT. In this paper, we induce a semi-Riemannian metric on the $r$–null submanifold. We establish the links between the null geometry and basics invariants of the associated semi-Riemannian geometry on $r$-null submanifold and semi-Riemannian constructed from a semi-Riemannian ambient.

1. INTRODUCTION

One of the most fundamental problems in submanifold theory is the problem of isometric immersibility. The embedding problem had been around since Riemann in 1854. Soon after Riemann introduced the notion of a manifold, Schl"afli conjectured that every Riemannian manifold could be locally considered as a submanifold of an Euclidean space with dimension $\frac{n(n+1)}{2}$ in (1873). This was later proved in different steps by Janet (1926), E. Cartan revised Janet’s paper with the same title in (1927). This result of Cartan-Janet implies that every Einstein $n$–manifold $(n \geq 3)$ can be locally isometrically embedded in an Euclidean $\frac{n(n+1)}{2}$–space $E^{\frac{n(n+1)}{2}}$. In 1956 J.Nash proved that every closed Riemannian $n$–manifold can be isometrically embedded in an Euclidean $m$–space $E^{m}$ with $m = \frac{1}{2}n(3n + 11)$ and he proved also that every non-closed Riemannian $n$-manifold can be isometrically embedded in $E^{m}$ with $m = \frac{1}{2}n(n+1)(3n + 11)$. R.E.Green improved Nash’s result in (1970) and proved that every non-compact Riemannian $n$-manifold can be isometrically embedded in the Euclidean $m$–space $E^{m}$ with $m = 2(2n+1)(3n+7)$. In 1965, A.Friedman proved that any $n$–dimensional semi-Riemannian manifold of index $q$ can be isometrically embedded in a semi-Euclidean space of dimension $\frac{1}{2}n(n+1)$ and index $\geq q$.

In 1970 Clarke and Green proved that any semi-Riemannian $n$–manifold $M_q^n$ with index $q$ can be isometrically embedded in a semi- Euclidean $m$–space $E_{q}^{m}$, for $m$ and $q$ large enough. Moreover, this embedding may be taken inside any given open set in $E_{q}^{m}$. The problem of discovering simple sharp relationships between intrinsic and extrinsic invariants of a Riemannian submanifold becomes one of the most fundamental problems in submanifold theory. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the scalar curvature and the Ricci curvature.

In [9], the author immersed a null hypersurface equipped with the Riemannian metric (induced on it by the rigging) into a Riemannian manifold suitably constructed on the

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Lorentzian manifold and she established the basic relationships between the main extrinsic invariants and the main intrinsic invariants named Chen-Ricci inequality of the null hypersurface in the Lorentzian manifold. This rigging technique has been introduced in [6].

In the present paper, we induce a semi-Riemannian metric on the null submanifold and we immersed isometrically the null submanifold endowed with semi-Riemannian metric in semi-Riemannian manifold. We then establish links between the null geometry and basics invariants of the associated nondegenerate geometry such as linear connection, the curvature tensor, Ricci curvature such that is symmetric and sectional curvature. The remaining of this paper is organized as follows. Section (2) contains most of the prerequisites material of null submanifolds in semi-Riemannian Manifold. The normalization and the associated semi-Riemannian structure on a normalized null submanifold using Rigging techniques are introduced and discussed in Section (3). The relashionship between the null and the associated semi-Riemannian geometry is considered in Section (4).

2. PRELIMINARIES ON NULL SUBMANIFOLDS

Let \((\overline{M}, \overline{g})\) be a real \((n + k)\)-dimensional semi-Riemannian manifold of constant index \(q \in \{1, \cdots, n + k - 1\}\) where \(n > 1, k \geq 1\). Suppose \(M\) is a \(n\)-dimensional submanifold of \(\overline{M}\). In case \(\overline{g}\) is non-degenerate on \(T_x M\), then \(T_x M\) and \(T_x M^\perp\) are complementary orthogonal vector subspaces of \(T_x \overline{M}\), a part of the normal vector bundle \(T \overline{M}\) with certain nice properties [4]. Used a screen distribution \(S(\overline{T M})\) on \(M\) and a screen vector bundle \(S(T M^\perp)\) over \(M\) to construct a transversal bundle \(tr(T M)\). They obtained the structure equations of \(M\) that relate the curvature tensor of \(\overline{M}\) with the curvature tensor of the linear connections induced on the vector bundles involved in the study. If the mapping \(\text{Rad}(TM)\) is a smooth distribution with constant rank \(r > 0\), then, it is said the radical (null, lightlike) distribution on \(M\). Also, \(g\) is called \(r\)-null \((r\)-lightlike, \(r\)-degenerate\) metric on \(M\) the submanifold \(M\) is said to be \(r\)-lightlike \((r\)-null, \(r\)-degenerate\) submanifold of \(\overline{M}\), with nullity degree \(r\) and is simply called null (lightlike) submanifold. Any complementary (and hence orthogonal) distribution \(S(TM)\) of \(\text{Rad}(TM)\) in \(TM\) is called a screen distribution. For a fixed screen distribution \(S(TM)\) on \(M\), the tangent bundle splits as

\[
TM = \text{Rad}TM \oplus_{\text{orth}} S(TM).
\]

Certainly, \(S(TM)\) is not unique, however it is canonically isomorphic to the factor vector bundle \(TM^* = TM/\text{Rad}TM\) considered by Kupeli [7]. \(S(TM)\) can be constructed by using the local equations of the submanifold and therefore it enables us to obtain the main induced geometrical objects: induced connection, second fundamental form, shap operator, · · · A screen transversal vector bundle \(S(TM^\perp)\) on \(M\) is any (semi-Riemannian)
complementary vector bundle of $\text{Rad}(TM)$ in $TM^\perp$. It is obvious that $S(TM^\perp)$ is non-degenerate with respect to $\overline{g}$ and $TM^\perp$ has the following orthogonal direct decomposition

$$TM^\perp = \text{Rad}TM \oplus_{\text{orth}} S(TM^\perp). \quad (2.2)$$

$S(TM)$ and $S(TM^\perp)$ are called a screen distribution and a screen transversal vector bundle of $M$ respectively. As $S(TM)$ is not degenerate, let $S(TM)^\perp$ be its complementary orthogonal in $\overline{T M}|_M$. Then we have the following decomposition

$$\overline{T M}|_M = S(TM) \oplus_{\text{orth}} S(TM)^\perp. \quad (2.3)$$

Note that $S(TM^\perp)$ is a vector subbundle of $S(TM)^\perp$ and since both are non-degenerate we have the following orthogonal direct decomposition

$$S(TM)^\perp = S(TM^\perp) \oplus_{\text{orth}} S(TM^\perp)^\perp.$$

Since the theory of null submanifold $M$ is mainly based on both $S(TM)$ and $S(TM^\perp)$, a null submanifold is denoted by $(M, g, S(TM), S(TM^\perp))$ and we have four sub-cases with respect to the dimension and codimension of $M$ and rank of $\text{Rad}TM$:

(a) $r$–null(lightlike) if $r < \min(n, k);
(b) coisotropic if $r = k < n$ (hence $S(TM^\perp) = \{0\}$);
(c) isotropic if $r = n < k$, (hence $S(TM) = \{0\}$);
(d) totally null if $r = n = k$, (hence $S(TM) = \{0\} = S(TM^\perp)$).

We have seen from the above that the normal bundle $TM^\perp$ is orthogonal but not a complement to $TM$, since it intersects the null tangent bundle $\text{Rad}TM$. This creates a problem as a vector of $T_x\overline{M}$ cannot be decomposed uniquely into a component tangent of $T_xM$ and a component of $T_xM^\perp$. Therefore, the standard definition of second fundamental forms and the Gauss-Weingarten formulae do not work. To deal with this anomaly, one of the techniques used consists to split the tangent bundle $\overline{T M}$ into four non-intersecting complementary (but not orthogonal) vectors bundle.

**Theorem 2.1.** [4] Let $(M, g, S(TM), S(TM^\perp))$ be an $r$–lightlike submanifold of $(\overline{M}, \overline{g})$ with $r > 1$. Suppose $\mathcal{U}$ is a coordinate neighbourhood of $M$ and $\{\xi_i\}; i \in \{1, \cdots, r\}$ a basis of $\Gamma(\text{Rad}TM)|_\mathcal{U}$. Then there exist smooth sections $\{N_i\}$ of $S(TM^\perp)^\perp|_\mathcal{U}$ such that

$$\overline{g}(N_i, \xi_i) = \delta_{ij}, \quad (2.4)$$

and

$$\overline{g}(N_i, N_j) = 0 \quad (2.5)$$

for any $i, j \in \{1, ..., r\}$.

It follows that there exists a null transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. By using (2.4) and (2.2) it is easy to check that $B = \{\xi_1, \cdots, \xi_r, N_1, \cdots, N_r\}$ is a basis of $\Gamma S(TM^\perp)^\perp|_\mathcal{U}$. The set of local sections $\{N_i\}$ is not unique even if one use the same vector bundle in general. Let $tr(TM)$ be complementary called (but not orthogonal) vector bundle to $TM$ in $\overline{T M}|_M$. Then the following hold

$$tr(TM) = ltr(TM) \oplus_{\text{orth}} S(TM^\perp), \quad (2.6)$$
\[ T\overline{\mathcal{M}}|_M = S(TM) \oplus \text{Orth}(\text{Rad}TM \oplus \text{ltr}(TM)) \oplus \text{Orth}S(TM^\perp) = TM \oplus \text{ltr}(TM). \] (2.7)

Let \( \nabla \) be the Levi-Civita connection on \( \overline{\mathcal{M}} \). As \( TM \) and \( \text{ltr}(TM) \) are complementary sub-bundles of \( T\overline{\mathcal{M}}|_M \), the Gauss and Weingarten formulae are

\[
\nabla_X Y = \nabla_X Y + h(X, Y), \quad (2.8)
\]

\[
\nabla_X V = -A_V X + \nabla^t_X V, \quad (2.9)
\]

\( \forall X, Y \in \Gamma(TM), V \in \Gamma(\text{ltr}(TM)). \) \( \nabla \) and \( \nabla^t \) are linear connections on \( TM \) and the vector bundle \( \text{ltr}(TM) \) called the induced linear connection and the transversal linear connection on \( M \) respectively.

\( \nabla \) is torsion-free linear connection. The components \( \nabla_X Y \) and \( -A_V X \) belong to \( \Gamma(TM) \), \( h(X, Y) \) and \( \nabla^t_X V \) to \( \Gamma(\text{ltr}(TM)) \). Also \( h \) is a \( \Gamma(\text{ltr}(TM)) \)-valued symmetric bilinear form on \( \Gamma(TM) \) called the second fundamental form of \( M \) with respect to \( \text{ltr}(TM) \). \( A \) is a \( \Gamma(TM) \)-valued bilinear form defined on \( \Gamma(\text{ltr}(TM)) \times \Gamma(TM) \) called shape operator of \( M \). From the geometry of Riemannian submanifolds \([3]\) and non-degenerate submanifolds \([12]\), it is known that the induced connection on a non-degenerate submanifold is a Levi-Civita connection. Unfortunately, in general, this is not true for a null submanifold.

\[
(\nabla_X g)(X, Y) = \overline{g}(h^l(X, Y) + \overline{g}(h^s(X, Z), Y)) \quad (2.10)
\]

\[
(\nabla^t_X g)(V, V') = -\{\overline{g}(A_V X, V') + \overline{g}(A_{V'} X, V)\} \quad (2.11)
\]

for all \( X, Y, Z \in \Gamma(TM), V, V' \in \Gamma(\text{ltr}(TM)). \) Thus, it follows that the induced connection \( \nabla \) is not a Levi-Civita connection.

According to the decomposition \([2, 6]\), let \( L \) and \( S \) denote the projection morphisms of \( \text{ltr}(TM) \) onto \( \text{ltr}(TM) \) and \( S(TM^\perp) \) respectively, \( h^l = L \circ h \), \( h^s = S \circ h \), \( D^l_X V = L(\nabla^l_X V), D^s_X V = S(\nabla^s_X V) \). The transformations \( D^l \) and \( D^s \) do not define linear connections but Ōtsuki connections on \( \text{ltr}(TM) \) with respect to the vector bundle morphisms \( L \) and \( S \). Then we have

\[
\nabla_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y) \quad (2.12)
\]

\[
\nabla_X N = -A_N X + D^l_X N + D^s(X, N) \quad (2.13)
\]

\[
\nabla_X W = -A_W X + D^l(X, W) + \nabla^s_X W \quad (2.14)
\]

\( \forall X, Y \in \Gamma(TM), N \in \Gamma(\text{ltr}(TM)) \) and \( W \in \Gamma(S(TM^\perp)) \). Since \( \overline{\nabla} \) is a metric connection, using \([2, 12]\)-\([2, 14]\) we have

\[
\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y) \quad (2.15)
\]

\[
\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X). \quad (2.16)
\]

As \( h^l \) and \( h^s \) are \( \Gamma(\text{ltr}(TM)) \)-valued and \( \Gamma(S(TM^\perp)) \)-valued respectively, we call them the null second fundamental form and the screen second fundamental form of \( M \).

Suppose \( M \) is either r-null with \( r < \min\{m, k\} \) or coisotropic. Then, using the decomposition \([2, 1]\) we get
\[ \nabla_X Y = \nabla^*_X (PY) + h^*(X, PY) \] (2.17)

\[ \nabla_X \xi = -A^*_\xi X + \nabla^{\ast t}_X \xi \] (2.18)

for any \( X, Y \in \Gamma(TM) \) and \( \xi \in \Gamma(Rad(TM)) \), where \( \{\nabla^*_X PY, A^*_\xi X\} \) and \( \{h^*(X, PY), \nabla^{\ast t}_X \xi\} \) belong to \( \Gamma(S(TM)) \) and \( \Gamma(Rad(TM)) \) respectively. It follows that \( \nabla^* \) and \( \nabla^{\ast t} \) are linear connections on \( S(TM) \) and \( Rad(TM) \) respectively. On the other hand, \( h^* \) and \( A^* \) are \( \Gamma(Rad(TM)) \)-valued and \( \Gamma(S(TM)) \)-valued bilinear forms on \( \Gamma(TM) \times \Gamma(S(TM)) \) and \( \Gamma(Rad(TM)) \times \Gamma(TM) \) Called the second fundamental forms of distributions \( S(TM) \) and \( Rad(TM) \) respectively.

For any \( \xi \in \Gamma(Rad(TM)) \) consider the linear operator

\[ A_\xi : \Gamma(TM) \to \Gamma(S(TM)); A_\xi X = A^*(\xi, X), \forall X \in \Gamma(TM) \]

and call it the shape operator of \( S(TM) \) with respect to \( \xi \). Also, call \( \nabla^* \) and \( \nabla^{\ast t} \) the induced connections on \( S(TM) \) and \( Rad(TM) \) respectively.

The second fundamental form and shape operator of non-degenerate submanifold of a semi-Riemannian manifold are related by means of the metric tensor field. Contrary in case of null submanifolds there are interrelations between geometric objects induced by \( tr(TM) \) on one side and geometric objects induced by \( S(TM) \) on the other side. More precisely,

\[ \overline{g}(h^l(X, PY)) = g(A^*_\xi X, PY), \overline{g}(h^*(X, PY), N) = g(A_N X, PY), \overline{g}(h^l(X, \xi), \xi) = 0, A^*_\xi \xi = 0. \] (2.19)

From (2.19) as \( h^l \) is symmetric, it follows that the shape operator of \( S(TM) \) is a self-adjoint operator on \( S(TM) \).

Next, consider a coordinate neighbourhood \( \mathcal{U} \) of \( M \) and let \( \{N_i, W_\alpha\} \) be a basis of \( \Gamma(tr(TM)|_M) \) where \( N_i \in \Gamma(ltr(TM)|_M) \), \( i \in \{1, ..., r\} \) and \( W_\alpha \in \Gamma(S(TM^\perp)|_\mathcal{U}) \), \( \alpha \in \{r + 1, ..., k\} \). Then (2.12) becomes

\[ \nabla_X Y = \nabla_X Y + \sum_{i=1}^{r} h^l_i(X, Y) N_i + \sum_{\alpha=r}^{k} h^*_\alpha(X, Y) W_\alpha. \] (2.20)

We call \( \{h^l_i\} \) and \( \{h^*_\alpha\} \) the local null second fundamental forms and the local screen second fundamental forms of \( M \) on \( \mathcal{U} \).

We recall the equations of Gauss, Codazzi and Ricci which play an important role in studying differential geometry of non-degenerate submanifolds(See [4] for details). Let \( \overline{R} \) and \( \overline{R} \) denote the Riemannian curvature tensors of \( \nabla \) and \( \nabla \) on \( M \) and \( M \) respectively. The Gauss equation are given by

\[ \overline{R}(X, Y) Z = R(X, Y) Z + A_{h^l(X, Z)} Y - A_{h^l(Y, Z)} X + A_{h^*(X, Z)} Y - A_{h^*(Y, Z)} X + (\nabla_X h^l(Y, Z)) - (\nabla_Y h^l)(X, Z) + D^l(X, h^*(Y, Z)) \]

\[ -D^l(Y, h^*(X, Z)) + (\nabla_X h^*(Y, Z)) - (\nabla_Y h^*)(X, Z) + D^*(X, h^l(Y, Z)) - D^*(Y, h^*(X, Z)) \]
\(\forall X, Y, Z, U \in \Gamma(TM)\). Therefore

\[
\overline{\mathcal{R}}(X, Y, PZ, PU) = R(X, Y, Z, PU) + \overline{g}(h^*(Y, PU), h^!(X, Z)) - \overline{g}(h^!(X, PU), h^*(Y, Z)) - \overline{g}(h^*(Y, PU), h^*(X, Z))
\]

(2.21)

Throughout, we consider that the submanifold is coisotropic that is \(h^* = 0\). Let the differential 1–forms \(\omega_i, i = 1, \cdots, r(0 < r < \min\{m, n\})\) defined by:

\[
\omega_i(X) = \overline{g}(X, N_i), \forall X \in \Gamma(TM|_U), i \in \{1, \ldots, r\}.
\]

(2.22)

Then any vector \(X\) on \(M\) is expressed on \(U\) as follows

\[
X = PX + \sum_{i=1}^{r} \omega_i(X)\xi_i
\]

(2.23)

where \(P\) is the projection morphism of \(\Gamma((TM))\) onto \(\Gamma(S(TM))\).

3. Normalization and Induced Semi-Riemannian Metric

Let \(\varpi\) denote the 1–form which satisfying \(\varpi(.) = \overline{g}(N, .)\). Then, take

\[
\omega = f^*\varpi
\]

(3.1)

to be its restriction to \(M\) and the map \(f : M \rightarrow \overline{M}\) the inclusion map. Throughout, a screen distribution on \(M\) is denote by \(S(N)\). We define the associated semi-Riemannian metric on \(\overline{M}\) as

\[
\overline{g} = \overline{g} - \varpi \otimes \varpi \quad \text{and} \quad \overline{g} = f^*\omega = g - \omega_{i=1} \otimes \omega_{j=1}.
\]

(3.2)

**Lemma 3.1.** Let \((M^n, g)\) be a \(r\)-null submanifold in semi-Riemannian manifold \((\overline{M}^{n+1}, \overline{g})\). Then, \(\overline{g}\) is nondegenerate.

**Proof.** Take \(V_x = \sum_{i=1}^{r} \xi_i\), then \(\sum_{i=1}^{r} \overline{g}(U_x, \xi_i) = 0 = \sum_{i=1}^{r} g(u, \xi_i) - \sum_{i=1}^{r} \omega_i(U)\omega_i(\xi_i), \sum_{i=1}^{r} \omega_i(U) = 0\). Then

\[
U \in S(N).
\]

(3.3)

\[
0 = \overline{g}(U, V) = g(U, V) \Rightarrow U \in T_xM^\perp.
\]

(3.4)

From (1) and (2) \(\Rightarrow U \in S(N_x) \cap T_xM^\perp = \{0\}\) and then \(U = 0\) which show \(\overline{g}\) is nondegenerate. \(\blacksquare\)

With \(\sum_{i=1}^{r} \omega_i = \overline{g}(N_i, .)\). It is nothing to see that

\[
\overline{g}(\xi_i, \xi_j) = -\delta_{ij}, \omega_i(.) = \overline{g}(\xi_i, .).
\]

(3.5)

**Lemma 3.2.** Let \((M, g, S(N))\) be a \(r\)-null submanifold in semi-Riemannian manifold, then \((M, \overline{g})\) is a semi-Riemannian manifold of index \(q - 1\). In particular, \((M, \overline{g})\) is Riemannian if \((\overline{M}, \overline{g})\) is Lorentzian and \((M, \overline{g})\) is Lorentzian if \((\overline{M}, \overline{g})\) is of index 2.

**Proof.** Take \((e_2, \ldots, e_{n+1})\) an orthonormal basis of \(S(N)\) for \(\overline{g}\) and note that \((\xi, e_2, \ldots, e_{n+1})\) is an orthonormal basis of \(TM\) for the rigged metic. Elsewhere, it is a fact that \(S(N)\) is a non degenerate distribution of index \(q - 1\) (see [4]). Since \(\overline{g}\) agrees with \(g\) on \(S(N)\) and \(\overline{g}(\xi, \xi) = -\delta_{ij}\), it follows that \((M, \overline{g})\) is a semi-Riemannian manifold of index \(q - 1\). \(\blacksquare\)
Lemma 3.3. For all $X, Y, Z \in \Gamma(TM)$, we have

\[
(\nabla_X \bar{g})(Y, Z) = \sum_{i=1}^{r} \omega_i(Y)[g(A^*_i X, PZ) - g(A_N i X, PZ)] \\
+ \sum_{i=1}^{r} \omega_i(Z)[g(A^*_i X, PY) - g(A_N i X, PY)] \\
+ 2 \sum_{i=1}^{r} \tau_i (X) \omega_i(Y) \omega_i(Z).
\]

(3.6)

Let $(M, g, S(N))$ be a normalized null submanifold of a semi-Riemannian manifold, then the integral curves of the rigged vector field $\xi$ are pregeodesic but not geodesic in general. The following lemma shows that in case the normalization is a conformal vector field, then $\xi$ is $\bar{g}$-geodesic.

Lemma 3.4. Let $(M, g, S(N))$ be a conformal normalized null submanifold of a semi-Riemannian manifold such that $\xi_i$ is $\bar{g}$-geodesic and $\sum_{i=1}^{r} \tau_i (\xi_i) = 0$.

**Proof.** Firstly, note that $\sum_{i=1}^{r} \tau_i (\xi_i) = \sum_{i=1}^{r} \bar{g}(\nabla_{\xi_i} N_i, \xi_i) N_i$ being conformal, there exists a function $\lambda$ such that $\bar{g}(\nabla_X N_i, Y) + \bar{g}(\nabla_Y N_i, X) = \lambda \bar{g}(X, Y)$. Take $X = Y = \sum_{i=1}^{r} \xi_i$ to get $2 \sum_{i=1}^{r} \bar{g}(\nabla_{\xi_i} N_i, \xi_i) = \sum_{i=1}^{r} \lambda \bar{g}(\xi_i, \xi_i) = 0$; hence $\sum_{i=1}^{r} \tau_i (\xi_i) = \sum_{i=1}^{r} \bar{g}(\nabla_{\xi_i} N_i, \xi_i) = 0$. It follows that $\sum_{i=1}^{r} \nabla_{\xi_i} \xi_i = - \sum_{i=1}^{r} \tau_i (\xi_i) \xi_i = 0$. \qed

Definition 3.5. A normalized null submanifold $(M, g, N)$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to have a conformal screen \([1]\) if there exists a non vanishing smooth function $\varphi$ on $M$ such that $A_N = \varphi A^*_\xi$ holds.

This is equivalent to saying that $g(A_N X, PY) = \varphi g(A^*_\xi X, Y)$ for all tangent vector fields $X$ and $Y$. The function $\varphi$ is called the conformal factor.

4. RELATION BETWEEN THE NULL AND THE ASSOCIATED SEMI-RIEMANNIAN GEOMETRY

The main focus of this section lies on deriving jump formulas for the various curvature quantities, that is, how the Riemann and Ricci tensor and scalar curvature of course the reason why this is of a particular interest lies in physics, mainly general relativity, where such formulas might find applications due to the Einstein field equations. In this respect, using the lemma (3.3) and as connections $\nabla$ and $\bar{\nabla}$ are torsion free, we can first prove the relation between the Levi-Civita connection of the non-degenerate associate metric $\bar{g}$ and the rigged connection $\nabla$ in the following.
Proposition 4.1. Let \((M^n, g)\) be a \(r\)-normalized coisotropic null submanifold in semi-Riemannian Manifold with rigged vector field \(\xi\). Then, for all \(X, Y \in \Gamma(TM)\), we have

\[
\nabla_X Y = \nabla_X Y + \frac{1}{2} \sum_{i=1}^{r} \left[ 2 \langle A_{\xi_i}^* X, Y \rangle - \langle A_{N_i} X, Y \rangle - \langle A_{N_i} Y, X \rangle \right] + \sum_{i=1}^{r} \omega_i(X) r^{N_i}(Y) + \sum_{i=1}^{r} \omega_i(Y) r^{N_i}(X) \xi_i + \sum_{i=1}^{r} \omega_i(X)(i_X d\omega_i)^{2\xi_i} + \sum_{i=1}^{r} \omega_i(Y)(i_Y d\omega_i)^{2\xi_i}.
\]

In particular for a closed normalization,

\[
\nabla_X Y = \nabla_X Y + \frac{1}{2} \sum_{i=1}^{r} \left[ 2 \langle A_{\xi_i}^* X, Y \rangle - \langle A_{N_i} X, Y \rangle - \langle A_{N_i} Y, X \rangle \right] + \sum_{i=1}^{r} \omega_i(X) r^{N_i}(Y) + \sum_{i=1}^{r} \omega_i(Y) r^{N_i}(X) \xi_i.
\]

Let \(\tilde{R}\) and \(R\) denote the Riemann curvature tensors of \(\nabla_\alpha\) and \(\nabla\) respectively. Using proposition 4.1, we prove the following.

Proposition 4.2. Let \((M, g^n, N)\) be a \(r\)-closed normalized coisotropic null submanifold of semi-Riemannian Manifold with rigged vector field \(\xi\). Then, for all \(X, Y, W \in \Gamma(TM)\) and \(U \in TM^\perp\) we have,

\[
\tilde{g}(\tilde{R}(X, Y)Z, PW) = \langle R(X, Y)Z, PW \rangle + \frac{1}{2} \sum_{i=1}^{r} \left[ 2g(A_{\xi_i}^* X, Z) - \sum_{i=1}^{r} \langle A_{N_i} X, Z \rangle - \sum_{i=1}^{r} \langle A_{N_i} Z, X \rangle \right] + r^{N_i}(X) \sum_{i=1}^{r} \omega_i(Z) + r^{N_i}(Z) \sum_{i=1}^{r} \omega_i(X) g(A_{\xi_i}^* Y, W)
\]

\[
-2g(A_{\xi_i}^* Y, Z) - \sum_{i=1}^{r} \langle A_{N_i} Y, Z \rangle - \sum_{i=1}^{r} \langle A_{N_i} Z, Y \rangle
\]

\[
+r^{N_i}(Y) \sum_{i=1}^{r} \omega_i(Z) + r^{N_i}(Z) \sum_{i=1}^{r} \omega_i(Y) g(A_{\xi_i}^* X, W) \rangle, \tag{4.2}
\]

\[
\tilde{g}(\tilde{R}(X, Y)Z, U) = -\langle R(X, Y)U, PZ \rangle
\]

\[
-\frac{1}{2} \sum_{i=1}^{r} \left[ \langle A_{N_i} U, Y \rangle g(A_{\xi_i}^* X, Z) - \langle A_{N_i} U, X \rangle g(A_{\xi_i}^* Y, Z) \right]
\]

\[
-\frac{1}{2} \sum_{i=1}^{r} \left[ r^{N_i}(X) g(A_{\xi_i}^* Y, Z) - r^{N_i}(Y) g(A_{\xi_i}^* X, Z) \right] \omega_i(U), \tag{4.3}
\]
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