POLICY CHOICE AND BEST ARM IDENTIFICATION: ASYMPTOTIC ANALYSIS OF EXPLORATION SAMPLING

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ABSTRACT

We consider the “policy choice” problem—otherwise known as best arm identification in the bandit literature—proposed by Kasy and Sautmann (2021) for adaptive experimental design. Theorem 1 of Kasy and Sautmann (2021) provides three asymptotic results that give theoretical guarantees for exploration sampling developed for this setting. We first show that the proof of Theorem 1 (1) has technical issues, and the proof and statement of Theorem 1 (2) are incorrect. We then show, through a counterexample, that Theorem 1 (3) is false. For the former two, we correct the statements and provide rigorous proofs. For Theorem 1 (3), we propose an alternative objective function, which we call posterior weighted policy regret, and derive the asymptotic optimality of exploration sampling.

1 Introduction

Kasy and Sautmann (2021) proposes what the authors call the “policy choice” problem for adaptive treatment assignment in experiments. The goal in policy choice is to choose a policy that is the best treatment amongst a set of treatments within several waves of an experiment. To evaluate algorithms in this setting, the authors propose a metric called “policy regret.” Using this metric, they develop a dynamic programming algorithm to optimize the expected policy regret (expected social welfare), but find that the proposed algorithm is computationally intractable. In light of this, the authors propose an algorithm called “exploration sampling” and prove its asymptotic optimality (Theorem 1).

The purpose of this paper is to show, first, that Theorem 1 is incorrect. In particular, we show that the proof of Theorem 1 (1) has technical issues, proof and statement of Theorem 1 (2) is incorrect, and Theorem 1 (3) is false, which we show through a counterexample. We then provide a corrected version of Theorem 1, as well as the associated corrected lemmata, with rigorous proofs. As Theorem 1 (3) is false under the expected policy regret, we propose the posterior weighted policy regret, which we then show the asymptotic optimality of exploration sampling under this objective. We further extend the theoretical results by relaxing the assumptions on the prior specification. These results provide theoretical support and extend the applicability of exploration sampling in adaptive experiments.

This paper is organized as follows. We review the problem setting and main theoretical results in Kasy and Sautmann (2021) in Section 2. The problems with the main theorem are discussed in Section 3. In Section 4, we provide our corrected theorem (Theorem 4.1). The proof of Theorem 4.1 and the corrected lemmata used in our proof is in Appendix D and Appendix B, respectively.

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2 Problem Setting and Main Results in Kasy and Sautmann (2021)

We first review the problem setting and theoretical result of Kasy and Sautmann (2021).

Problem Setting

Suppose that there are multiple treatments, which are also called “arms” in the multi-armed bandit literature. Let \( k \geq 2 \) be the number of possible treatments. In the policy choice problem, a policymaker is interested in the expected outcome of the treatments. The outcome corresponding to each treatment is a binary random variable. For estimating the expected values, the policymaker conducts adaptive experiments with multiple waves. At the beginning of each wave, the policymaker observes the outcomes and updates treatment assignment in subsequent waves based on past observations. Upon completion of the waves, the policymaker chooses the treatment that yields the highest expected outcome, which is called policy. The goal of policy choice (Kasy and Sautmann, 2021), also known as best arm identification\(^2\) (BAI), is to propose an optimal experimental design for this setting.

The experiment consists of waves \( t = 1, \ldots, T \), where at each wave \( t \), there is a new random sample of \( N_t \) experimental units, \( i = 1, \ldots, N_t \), drawn from the population of interest. We denote the total sample size by \( M = \sum_{t=1}^{T} N_t \). For each unit \( i \) in period \( t \), the experimenter can assign one of \( k \) different treatments \( D_{i,t} \in \{ 1, \ldots, k \} \) and then observe a binary outcome \( Y_{i,t} \in \{ 0, 1 \} = \sum_{d=1}^{k} \mathbb{1}\{D_{i,t} = d\} Y_{i,t}^d \), where the potential outcome vectors \( (Y_{i,t}^1, \ldots, Y_{i,t}^k) \) for unit \( i \) in period \( t \) are i.i.d. draws. Let us assume that each treatment \( d \in \{1, \ldots, k\} \) has the average potential outcome and denote it as \( \theta^d = \mathbb{E}[Y_{i,t}^d] \). Under this setting, the goal is to choose the policy with the highest expected outcome amongst the given multiple treatments. Kasy and Sautmann (2021) denotes the true optimal treatment by \( d(1) \in \arg\max_d \theta^d \), and let \( \Delta^d = \theta^{d(1)} - \theta^d \) be the policy regret when choosing treatment \( d \in \{1, \ldots, k\} \), relative to the optimal treatment \( d(1) \). This performance metric is referred to as the simple regret in the BAI literature (Audibert, Bubeck, and Munos, 2010; Lattimore and Szepesvári, 2020).

At the beginning of wave \( t \), there are \( N_t \) units available, and the experimenter can optimize the allocation of the treatments to these \( N_t \) units. In Kasy and Sautmann (2021), the treatment assignment in wave \( t \) is summarized by the vector \( n_t = (n_t^1, \ldots, n_t^k) \) with \( \sum_{d=1}^{k} n_t^d = N_t \). For each treatment \( d \), denote the number of successes among \( n_t^d \) units in wave \( t \) by \( s_t^d = \sum_{i,t}^{N_t} \mathbb{1}\{D_{i,t} = d, Y_{i,t} = 1\} \). Let us denote the outcome of wave \( t \) by the vector \( s_t = (s_t^1, \ldots, s_t^k) \), where \( s_t^d \leq n_t^d \), which can be observed at the end of wave \( t \). We denote the cumulative versions of these terms from 1 to \( t \) by \( m_t^d = \sum_{u \leq t} s_u^d \), \( r_t^d = \sum_{u \leq t} s_u^d \), and \( m_t = (m_t^1, \ldots, m_t^k) \), \( r_t = (r_t^1, \ldots, r_t^k) \).

The policymaker holds prior belief \( \text{Beta}(\alpha_0^d, \beta_0^d) \) for treatment \( d \in \{1, \ldots, k\} \). In Kasy and Sautmann (2021), the uniform prior is used as the default for applications, i.e., \( \alpha_0^d = \beta_0^d = 1 \) for all \( d \). The posterior belief is defined by the parameters \( (\alpha_t^d, \beta_t^d) = (\alpha_0^d + r_{t-1}^d, \beta_0^d + m_{t-1}^d - r_{t-1}^d) \).

Kasy and Sautmann (2021) gives per-capita expected social welfare of policy \( d \) as

\[
\text{SW}_T(d) = \mathbb{E}[\theta^d|m_T, r_T] = \frac{\alpha_T^d + r_T^d}{\alpha_0^d + \beta_0^d + m_T^d},
\]

and proposes choosing a policy as \( d_T^* \in \arg\max_{d \in \{1, \ldots, k\}} \text{SW}_T(d) \).

Expected Policy Regret

In Kasy and Sautmann (2021), the treatment assignment algorithms are evaluated by the expected social welfare, or, equivalently, expected policy regret, which is defined for the policy \( d_T^* \) as follows:

\[
\text{R}_\theta(T) = \mathbb{E}\left[\Delta_{d_T^*}^T | \theta\right] = \sum_{d=1}^{k} \Delta^d \cdot \mathbb{P}(d_T^* = d | \theta),
\]

where \( T \) is the number of experimental waves, and the expectation is taken over all possible successes and assignment choices for treatments. This objective is identical to the expected simple regret in the BAI literature.

\(^2\)Here, “identification” simply means selecting the arm (treatment) with the highest expected reward, and not the identification problem in the econometric literature.
Exploration Sampling

In each wave $t$, we define the posterior probability that the treatment $d \in \{1, \ldots, k\}$ is the optimal treatment as

$$p^d_t = \Pr\left( d = \arg\max_{d' \in \{1, \ldots, k\}} \tilde{\theta}^{d'} \mid m_{t-1}, r_{t-1} \right),$$

where $\tilde{\theta} = (\tilde{\theta}^1, \ldots, \tilde{\theta}^k)$ is a sample drawn from the posterior. Thompson sampling can be interpreted as a method that assigns $[q^d_t N_t]$ observations to treatment $d$. Based on this idea and the result of Russo (2016), Kasy and Sautmann (2021) proposes their treatment assignment algorithm called exploration sampling. In exploration sampling, we assign $[q^d_t N_t]$ of observations to treatment $d$, where

$$q^d_t = S_t \cdot p^d_t \cdot (1 - p^d_t),$$

with the normalization term $S_t = \left( \sum_{d=1}^k p^d_t \cdot (1 - p^d_t) \right)^{-1}$. Kasy and Sautmann (2021) analyzes the theoretical properties of this algorithm and provides asymptotic guarantees.

Remark (BAI). This problem setting is known as BAI in the multi-armed bandit literature (Even-Dar, Mannor, and Mansour, 2002; Mannor and Tsitsiklis, 2004; Even-Dar, Mannor, and Mansour, 2006; Audibert, Bubeck, and Munos, 2010). Though the problem of BAI itself goes back decades, variants go as far back as the 1950s, in the context of sequential testing problems (Wald, 1945; Chernoff, 1959). Some of the earliest advances on this topic are summarized in Bechhofer, Kiefer, and Sobel (1968). Another literature on ordinal optimization has been studied in the operation research community and a modern formulation was established in the 2000s (Chen, Lin, Yücesan, and Chick, 2000; Glynn and Juneja, 2004). Most of those studies have considered the estimation of optimal allocations separately from the error rate under known optimal allocations. In the 2010s, the machine learning community reformulated the problem to synthesize both issues and explicitly discussed them. For a more detailed survey, see, for example, the Introduction section of Kaufmann, Cappé, and Garivier (2016) and Section 33 of Lattimore and Szepesvári (2020).

Main Theorem in Kasy and Sautmann (2021)

Kasy and Sautmann (2021) provides the following performance guarantees for exploration sampling.

Theorem 1 (Kasy and Sautmann, 2021). Consider exploration sampling, with fixed wave size $N_t = N \geq 1$. Assume that the optimal arm $d^{(1)} = \arg\max_{d \in \{1, \ldots, k\}} \theta^d$ is unique and that $\theta^{d^{(1)}} < 1$. As $T \to \infty$, the followings hold:

1. The share of observations $m^{d^{(1)}}_T$ assigned to the best treatment $d^{(1)}$ converges in probability to $\frac{1}{2}$, that is,

$$\frac{m^{d^{(1)}}_T}{N_T} \to 0.5$$

2. The share of observations $m^d_T$ assigned to each treatment $d \neq d^{(1)}$ converges in probability to a non-random share $p^d$, that is,

$$\frac{m^d_T}{N_T} \to p^d, \quad \forall d \neq d^{(1)},$$

where the limit assignment shares $p = (p^1, \ldots, p^k)$ (with $p^{d^{(1)}} = \frac{1}{2}$) is such that $-\frac{1}{N_T} \log p^{d^{(1)}} \Rightarrow \Gamma^*$ for some $\Gamma^* > 0$ that is constant across $d \neq d^{(1)}$.

3. Expected policy regret converges to 0 at the same rate $\Gamma^*$, that is,

$$-\frac{1}{N_T} \log R_\theta(T) \to \Gamma^*.$$ 

No algorithm with limit assignment shares $\tilde{p} \neq p$ with $\tilde{p}^{d^{(1)}} = \frac{1}{2}$ exists for which $R_\theta(T)$ goes to 0 at a faster rate than $\Gamma^*$.

While the introduction of the policy choice setting into the field of economics is laudable, as well as conducting a field experiment that applies this methodology to actual policy experiments, unfortunately, this theorem has several issues, which we will expound and correct below.
3 Incorrectness of Theorem 1 in Kasy and Saumtmann (2021)

First, we describe the incorrectness of Theorem 1 (1) and (2). Then, we provide a counterexample to Theorem 1 (3).

Incorrect Proof for Theorem 1 (1) and (2)

In their proof, Kasy and Saumtmann (2021) refers to Russo (2016) when showing posterior convergence. However, the results of Russo (2016) do not guarantee the performance of their algorithm with the Beta–Bernoulli model. This is because Assumption 1 of Russo (2016) requires the boundedness on the first derivative of the log-partition function of the reward distribution belonging to the exponential family. This assumption is violated when the parameter space of the Bernoulli models is $[0,1]$. Therefore, when using a beta prior whose support covers $[0,1]$, we cannot apply the results of Russo (2016). For a more detailed discussion, see Section 5 of Shang, de Heide, Menard, Kaufmann, and Valko (2020). This problem has already been pointed out by Russo (2016) and Shang, de Heide, Menard, Kaufmann, and Valko (2020). To show the posterior convergence rate for the Beta–Bernoulli model with the general Beta$(\alpha^d_0, \beta^d_0)$ prior, a separate proof is needed for each algorithm. For instance, Shang, de Heide, Menard, Kaufmann, and Valko (2020) proves posterior consistency when considering the top-two Thompson sampling (TTTS). This proof is non-trivial, which Kasy and Saumtmann (2021) does not show.

Additionally, the limit of the convergence is different from that of Russo (2016). In Kasy and Saumtmann (2021), $\Gamma^*$ and $\rho = (\rho^1, \ldots, \rho^k)$ are given as the optimal value and solutions of the following optimization problem:

$$\max_{w \in \mathbb{R}^k} \Gamma$$

subject to $G_d \left( w^{d(1)}, w^d \right) - \Gamma \geq 0, \ \forall d \neq d^{(1)}$, 

$$\sum_{d=1}^k w^d = 1, \ w^{d(1)} = \frac{1}{2}, \ w^d \geq 0, \ \forall d \neq d^{(1)},$$

where

$$G_d \left( w^{d(1)}, w^d \right) = \min_{x \in \left[ \theta^{d(1)}, \theta^d \right]} \left[ w^{d(1)} d_{KL} \left( x, \theta^{d(1)} \right) + w^d d_{KL} \left( x, \theta^d \right) \right],$$

and $d_{KL}(p, q) := p \log(p/q) + (1 - p) \log((1 - p)/(1 - q))$ is the KL divergence between two Bernoulli distributions. However, these terms are different from that introduced in Russo (2016), which is defined as the value $\Lambda^*$ and the solution $\Lambda = (\Lambda^1, \ldots, \Lambda^k)$ of the following optimization problem:

$$\max_{w \in \mathbb{R}^k} \Lambda$$

subject to $C_d \left( w^{d(1)}, w^d \right) - \Lambda \geq 0, \ \forall d \neq d^{(1)}$, 

$$\sum_{d=1}^k w^d = 1, \ w^{d(1)} = \frac{1}{2}, \ w^d \geq 0, \ \forall d \neq d^{(1)},$$

where

$$C_d \left( w^{d(1)}, w^d \right) = \min_{x \in \left[ \theta^{d(1)}, \theta^d \right]} \left[ w^{d(1)} d_{KL} \left( \theta^{d(1)}, x \right) + w^d d_{KL} \left( \theta^d, x \right) \right].$$

The major difference between these optimization problems is that the arguments of the Kullback–Leibler divergence are reversed. Intuitively speaking, the value $\Gamma^*$ of Glynn and Juneja (2004) characterizes the likelihood of incorrectly recommending treatment when the optimal treatment is $d^{(1)}$, whereas the value $\Lambda^*$ of Russo (2016) characterizes the likelihood of the observed data under the hypothesis that the optimal treatment is not treatment $d^{(1)}$. The latter notion is appropriate here as we are minimizing the mass of the posterior where the optimal treatment is different.

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3 In Section 5 of Shang, de Heide, Menard, Kaufmann, and Valko (2020), the authors explain that “Russo (2016) proves a similar theorem under three confining boundedness assumptions (see Russo (2016), Assumption 1) on the parameter space, the prior density and the (first derivative of the) log-normalizer of the exponential family. Hence, the theorems in Russo (2016) do not apply to the two bandit models most used in practice and considered in this paper: the Gaussian and Bernoulli model. In the first case, the parameter space is unbounded; in the latter, the derivative of the log-normalizer (which is $e^\theta/(1 + e^\theta)$) is unbounded.”

4 Sandeep Juneja and Daniel Russo have also discovered this asymmetry. Chao Qin thanks them for the discussion.
Remark (Almost sure convergence and convergence in probability). The original statement of Russo (2016) states $-\frac{1}{NT} \log \left( \sum_{d \neq d(1)} p^d_T \right) \xrightarrow{a.s.} \Lambda^*$, whereas Kasy and Sautmann (2021) incorrectly cites the result as $-\frac{1}{NT} \log p^d_T \xrightarrow{P} \Gamma^*$. Almost sure convergence describes an event that holds on each sample path, whereas convergence in probability describes an event that holds for a single $t$, which is weaker than almost sure convergence.

During the proofs, one can find many steps that essentially require pathwise discussions. For example, to apply the law of large numbers, it is required that we have infinitely many samples for almost all sample paths. The derivation of the limit rate $\Lambda^*$ in our corrected theorem is via a bound of the posterior probabilities of the form $\exp(- (1 \pm \epsilon) NT \Lambda^*)$ that holds over all $t > T_0(\epsilon)$ in Appendix E (proof of Lemma B.1).

Remark (General prior). Results in Shang, de Heide, Menard, Kaufmann, and Valko (2020) are limited to the uniform prior (i.e., Beta(1, 1)). In this paper, we extend the results in Shang, de Heide, Menard, Kaufmann, and Valko (2020) to Beta($\alpha^d_0$, $\beta^d_0$) priors for each $d \in \{1, \ldots, k\}$, where the constants $\alpha^d_0$, $\beta^d_0 > 0$ can be arbitrary.

Comments on Theorem 1 (3)

There are two ways in which Theorem 1 (3) is problematic. First, we show that the proof is incorrect. Second, we show that the statement contradicts an existing theoretical result (Carpentier and Locatelli, 2016).

Incorrect Proof of Theorem 1 (3)

Theorem 1 (3) cannot be a consequence of Theorem 1 (1) and (2). Theorem 1 (3) quantifies the asymptotic convergence rate of the expected policy regret, $R_\theta(T)$, which is equivalent to the weighted sum of the asymptotic probability of misidentification, $\sum_{d=1}^{k} \Delta^d P (d^*_T = d(\theta))$. In order to evaluate this, one needs to quantify the convergence rate of $-\frac{1}{NT} \log p^d_T$ to the optimal treatment allocation $\Gamma^*$. However, Theorem 1 (1) and (2) only state the consistency of the optimal treatment allocation, without providing the convergence rate.

A convergence in probability states that, for any $\epsilon > 0$ and $\delta > 0$, there exists $t_0(\epsilon, \delta)$, such that for all $t > t_0(\epsilon, \delta)$,

$$P \left( \left| -\frac{1}{NT} \log p^d_T - \Gamma^* \right| > \epsilon \right) \leq \delta. \quad (4)$$

From the convergence in probability, we can show that the expected policy regret converges to 0 as $T \to \infty$ ($R_\theta(T) = o(1)$), but we cannot derive Theorem 1 (3), i.e., $-\frac{1}{NT} \log R_\theta(T) \to \Gamma^*$ as $T \to \infty$, because there exist counterexamples where the speed of convergence can be insufficient.

For example, convergence in probability means that it can include the following examples of convergence: a relationship $t_0 = \max(1/\epsilon^2, 1/\delta^2)$. In this case, for any $\epsilon$ and for all $t \geq t_0(\epsilon, \delta)$, the inequality (4) holds for $\delta \geq 1/\sqrt{T}$. In this case, the probability of convergence cannot be guaranteed to be greater than $1 - 1/\sqrt{T}$. It leads to a polynomial order $1/\sqrt{T}$ of the expected policy regret. Thus, convergence in probability does not lead to the expected policy regret of the same rate.

Counterexample to Theorem 1 (3)

The result derived in Carpentier and Locatelli (2016) contradicts Theorem 1 (3). By utilizing information theoretic arguments, Carpentier and Locatelli (2016) constructs a counterexample where the expected policy regret can be strictly larger than what is expected from the results of Glynn and Juneja (2004).

The crux here is that the optimal allocation computed from $\Gamma^* = \Gamma^*(\theta)$ depends on the true parameter $\theta$. Let us hypothetically consider several policy choice problems under different parameters. We call each policy choice problem a problem instance, which is characterized by the parameter $\theta$. For an easy problem instance (i.e., $\theta$ such that $\Gamma^*(\theta)$ is large), the expected policy regret must decay faster, whereas for a hard problem instance (i.e., $\theta$ such that $\Gamma^*(\theta)$ is small) the expected policy regret decays slower. Suppose that there exist $k$ problem instances under $k$ different parameters $\theta_1, \ldots, \theta_k$, which are indexed by $1, \ldots, k$, respectively. Here, $k$ is the same as the number of treatments in Kasy and Sautmann (2021). Carpentier and Locatelli (2016) constructs a particular set of $k$ problem instances such that the expected policy regret of an algorithm converges at the rate slower than $\Gamma^*(\theta)$ for at least one of the $k$ problem instances.

In the following, we assume unit wave size $N = 1$ (and thus $M = NT = T$), which is usually adopted in the BAI literature:

The difference between $\Gamma^*$ and $\Lambda^*$ does not matter in this counterexample; Theorem 3.1 utilizes the Pinsker’s inequality, which bounds both $\Gamma^*$ and $\Lambda^*$ from below.
Theorem 3.1. (Lower bound on the expected policy regret.) There exists a problem instance $\theta$ and an infinite subsequence of integers $\{T_n\}_{n=1}^{\infty}$ such that for any $T_n$, the expected policy regret of any algorithm is lower bounded as

$$\mathbb{E}_\theta(T_n) \geq \exp \left( -\frac{C}{\log(k)} T_n^* \right)$$

for some constant $C > 0$.

We present the proof of Theorem 3.1 in Appendix A. To make the connection with the result of Kasy and Sautmann (2021) clearer, we transform the inequality in Theorem 3.1 as for any $T_n$,

$$-\frac{1}{T_n} \log \mathbb{E}_\theta(T_n) \leq \frac{C}{\log(k)} T_n^*.$$  (5)

Since $\log(k) \to \infty$ as $k \to \infty$, the expected policy regret decays arbitrarily slower than what Theorem 1 (3) claims.

4 Corrected Main Theorem

In this paper, we provide a correction to all three statements in Theorem 1. Since the asymptotic optimality on the expected policy regret, used in original Theorem 1 (3), is unattainable, we propose a new objective—posterior weighted policy regret—and derive the asymptotic optimality under this objective. We introduce the following objective

$$W_\theta(T) = \sum_{d \in \{1, \ldots, k\}} \Delta^d \cdot p_T^d = \sum_{d \in \{1, \ldots, k\}} \Delta^d \cdot \mathbb{P} \left( d = \arg\max_{d' \in \{1, \ldots, k\}} \hat{\theta}^d \mid m_{T-1}, r_{T-1} \right),$$

where the gap $\Delta^d = \theta^{d(1)} - \theta^d$ depends on the unknown true parameter $\theta = (\theta^1, \ldots, \theta^k)$, while the posterior probability above is based on the random sample $\hat{\theta} = (\hat{\theta}^1, \ldots, \hat{\theta}^k)$ drawn from the posterior. By comparing the posterior probability of making the wrong decision $\sum_{d \neq d(1)} p_T^d$, this objective takes into account the magnitude between the best policy and sub-optimal policies.

Our objective is a synthesis of the frequentist objective (policy regret) and Bayesian objective (expected posterior weight regret), as it combines the regret for not choosing the “true” policy (frequentist), which is unobserved by the policymaker, and the magnitude reflected in the posterior distribution (Bayesian), which the policy maker updates. Unlike the expected policy regret, defined in Kasy and Sautmann (2021), we can derive the optimality of the exploration sampling in view of this objective.

Theorem 4.1 (Corrected Theorem 1 of Kasy and Sautmann (2021)). Consider exploration sampling, for Bernoulli bandits with Beta($\alpha_0^d, \beta_0^d$) priors for each $d \in \{1, \ldots, k\}$, where the constants $\alpha_0^1, \ldots, \alpha_0^k, \beta_0^1, \ldots, \beta_0^k > 0$ can be arbitrary. Let the wave size be fixed $N_T = N \geq 1$. Assume that the optimal arm $d^{(1)} = \arg\max_{d \in \{1, \ldots, k\}} \theta^d$ is unique and that $\theta^{d(1)} < 1$. The following statements hold:

1. The share of observations $m_T^{d(1)} / N_T$ assigned to the best treatment $d^{(1)}$ converges almost surely to $\frac{1}{2}$, that is,

$$\frac{m_T^{d(1)}}{N_T} \xrightarrow{\text{a.s.}} \frac{1}{2}.$$ 

2. The share of observations $m_T^d / N_T$ assigned to each treatment $d \neq d^{(1)}$ converges almost surely to a non-random share $\lambda^d$, that is,

$$\frac{m_T^d}{N_T} \xrightarrow{\text{a.s.}} \lambda^d, \quad \forall d \neq d^{(1)},$$

where the limit assignment shares $\lambda = (\lambda^1, \ldots, \lambda^k)$ (with $\lambda^{d^{(1)}} = \frac{1}{2}$) is such that

$$-\frac{1}{N_T} \log \left( \sum_{d \neq d^{(1)}} p_T^d \right) \xrightarrow{\text{a.s.}} \Lambda^*,$$

which is the optimal value of the optimization problem defined in (3).
(3) The posterior weighted policy regret converges almost surely to 0 at the same rate $\Lambda^*$, that is,

$$-\frac{1}{NT} \log W_\theta(T) \xrightarrow{a.s.} \Lambda^*.$$ 

No algorithm with limit assignment shares $\hat{\Lambda} \neq \Lambda$ with $\hat{\Lambda}^{d^{(1)}} = \frac{1}{2}$ exists for which $W_\theta(T)$ goes to 0 at a faster rate than $\Lambda^*$.

To prove Theorem 4.1, we require Lemmata 2 (2), 4–6 in Kasy and Sautmann (2021) (Lemmata 1, 2 (1), and 3 are irrelevant for the corrected theorem). The original version of these lemmata has technical issues that require correction. Lemma 2 (2) in Kasy and Sautmann (2021) cites Russo (2016), which cannot be applied to Beta-Bernoulli models with Beta($\alpha_0^d$, $\beta_0^d$) priors. Lemma 4 cites Russo (2016), though, the statement is adapted from the original: while Lemma 4 supposes that $m_T^d / (NT) \xrightarrow{P} \beta$ for a constant $\beta > 0$, Russo (2016) supposes $\lim_{T \to \infty} m_T^d / (NT) = \beta$. Moreover, Lemmata 2 (2), 4–6 are incorrectly adapted from Russo (2016) regarding the convergence of random variables. To address these issues, we show the corrected versions of Lemmata 2 (2), 4–6 in Appendix B.

A Proof of Theorem 3.1

Let us define the problem complexity as

$$H(\theta) = \sum_{d \notin \text{arg max}_{d' \in \{1, \ldots, k\}} \theta^{d'}} \frac{1}{(\Delta^d)^2} = \sum_{d \notin \text{arg max}_{d' \in \{1, \ldots, k\}} \theta^{d'}} \frac{1}{(\theta^{d^{(1)}} - \theta^d)^2},$$

where $d^{(1)} \in \text{arg max}_{d' \in \{1, \ldots, k\}} \theta^{d'}$. To show Theorem 3.1, we use the following results.

**Lemma A.1.** (Lower bound on $\Gamma^*(\theta)$) Consider $\theta = (\theta^1, \ldots, \theta^k)$ such that $d^{(1)} = \text{arg max}_{d \in \{1, \ldots, k\}} \theta^d$ is unique. We have

$$\Gamma^*(\theta) \geq \frac{1}{2H(\theta)}.$$

**Proof.** Recall that $\Gamma^*(\theta)$ is the solution of the optimization problem defined in (2). We have,

$$\Gamma^*(\theta) = \max_{u} \min_{d \neq d^{(1)}} G_j \left( w^{d^{(1)}}, w^d \right)$$

$$\geq 2 \max_{u} \min_{d \neq d^{(1)}} \min_{x \in [\theta^d, \theta^{d^{(1)}}]} \left[ w^{d^{(1)}} (x - \theta^{d^{(1)}})^2 + w^d (x - \theta^d)^2 \right]$$

(by Pinsker’s inequality)

$$\geq 2 \max_{u} \min_{d \neq d^{(1)}} \min_{x \in [\theta^d, \theta^{d^{(1)}}]} \min \{ w^{d^{(1)}}, w^d \} \cdot \left[ (x - \theta^{d^{(1)}})^2 + (x - \theta^d)^2 \right]$$

$$= 2 \sum_{d \neq d^{(1)}} \min_{u = 1/2 d \neq d^{(1)}} \min_{x \in [\theta^d, \theta^{d^{(1)}}]} w^d \left[ (x - \theta^{d^{(1)}})^2 + (x - \theta^d)^2 \right]$$

(since $w^{d^{(1)}} = 1/2 \geq w^d$)

$$\geq \sum_{d \neq d^{(1)}} \min_{u = 1/2} \min_{d \neq d^{(1)}} w^{d^{(1)}} (\theta^{d^{(1)}} - \theta^d)^2$$

(since the inner minimization achieves optimality at $x = \frac{\theta^{d^{(1)}} + \theta^d}{2}$)

$$\geq \frac{1}{2} \left( \sum_{d \neq d^{(1)}} \frac{1}{(\theta^{d^{(1)}} - \theta^d)^2} \right)^{-1} = \frac{1}{2H(\theta)}.$$
where the last inequality holds by considering a specific choice \( \left( w_d^d \right)_{d \neq d(1)} \) with \( \sum_{d \neq d(1)} w_d = 1/2 \), such that
\[
w_d^d = \frac{1}{2} \left( \theta_d^{d(1)} - \theta_d^d \right)^2 \left( \sum_{d' \neq d(1)} \frac{1}{\left( \theta_{d'}^{d(1)} - \theta_{d'}^d \right)^2} \right)^{-1}, \quad \forall d \neq d(1).
\]

Next, we define \( k \) problem instances as follows. Let us denote the parameter of the problem instance \( i \in \{1, \ldots, k\} \) by \( \theta_i = (\theta_i^1, \ldots, \theta_i^k) \), i.e., the instance \( i \)'s average potential outcome of the treatment \( d \) is denoted as \( \theta_i^d \).

**Definition A.2.** (Problem instances) The first problem instance denoted by \( \theta_1 = (\theta_1^1, \ldots, \theta_1^k) \) is defined as
\[
\theta_1^1 = \frac{1}{2} \quad \text{and} \quad \theta_1^d = \frac{1}{2} - f^d \quad \forall d \neq 1,
\]
where \( f^d := \frac{d}{\pi} \).

The other \((k - 1)\) problem instances are denoted by \( \theta_i \), for \( i = 2, 3, \ldots, k \), and each \( \theta_i = (\theta_i^1, \ldots, \theta_i^k) \) is defined as the ones where the average potential outcome of the treatment \( i \) is replaced by \( \frac{1}{2} + f^i \) so that treatment \( i \) is the best treatment in the problem instance \( i \), that is,
\[
\theta_i^1 = \frac{1}{2} + f^i \quad \text{and} \quad \theta_i^d = \theta_1^d \quad \forall d \neq i.
\]

We use the following proposition.

**Proposition A.3.** (Theorem 2 in *Carpentier and Locatelli (2016)*.) Let \( k \geq 2 \). Consider the problem instances \( \{\theta_1, \ldots, \theta_k\} \) in Definition A.2. For any algorithm, for each \( T \), there exists at least one \( \theta \in \{\theta_1, \ldots, \theta_k\} \) such that
\[
R_\theta(T) \geq \frac{1}{6} \exp \left( -\frac{60T}{h^* H(\theta)} - 2\sqrt{T \log(6Tk)} \right)
\]
where \( h^* \geq \frac{3}{10} \log(k) \).

This proposition gives the following corollary.

**Corollary A.4.** Let \( k \geq 2 \). Consider the problem instances \( \{\theta_1, \ldots, \theta_k\} \) in Definition A.2. For any algorithm, there exists an instance \( \theta \in \{\theta_1, \ldots, \theta_k\} \) and an infinite subsequence of integers \( \{T_n\}_{n=1}^\infty \) such that for any \( T_n \),
\[
\frac{200T_n}{\log(k) H(\theta)} \geq 2 \sqrt{T_n \log(6T_n k)},
\]
and thus
\[
R_\theta(T_n) \geq \frac{1}{6} \exp \left( -\frac{200T_n}{\log(k) H(\theta)} - 2\sqrt{T_n \log(6T_n k)} \right) \geq \frac{1}{6} \exp \left( -\frac{400T_n}{\log(k) H(\theta)} \right).
\]

Corollary A.4 directly follows from the fact that \( \sqrt{T \log T} = o(T) \) and Proposition A.3. Corollary A.4 states that the exponent of the expected policy regret is at most \( \frac{400T_n}{\log(k) H(\theta)} \) in one of the \( k \) problem instances of Definition A.2. Combining Lemma A.1 and Corollary A.4, for any \( T_n \),
\[
-\frac{1}{T_n} \log R_\theta(T_n) \leq \frac{400}{\log(k) H(\theta)} \leq \frac{800 \Gamma^*(\theta)}{\log(k)},
\]
which implies (5) with \( C = 800 \). Therefore, the statement in Theorem 1 (3) does not hold. This concludes the proof.

**B Corrected Lemmata**

Here, we correct the lemmata associated with the corrected theorem.
Lemma B.1 (Corrected Lemma 2 (2) of Kasy and Sautmann (2021)). Consider Beta($\alpha^d_0, \beta^d_0$) priors for each $d \in \{1, \ldots, k\}$. Under any allocation rule satisfying $\frac{m^d_T}{NT} \to \frac{1}{2}$,

$$\limsup_{T \to \infty} -\frac{1}{NT} \log \left( \sum_{d \neq d(1)} p^d_T \right) \leq \Lambda^*,$$

and under any allocation rule satisfying $\frac{m^d}{NT} \to \lambda^d$ for each $d \in \{1, \ldots, k\}$,

$$\lim_{T \to \infty} -\frac{1}{NT} \log \left( \sum_{d \neq d(1)} p^d_T \right) = \Lambda^*.$$

Lemma B.2 (Corrected Lemma 4 of Kasy and Sautmann (2021). From Lemma 12 of Russo (2016) and Lemma 30 of Shang, de Heide, Menard, Kaufmann, and Valko (2020)). Consider any adaptive allocation rule. If

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} q_t^{d(1)} = \frac{1}{2}$$

then

$$\sum_{t=1}^{\infty} q_t^d \cdot \mathbb{1} \left\{ \frac{1}{T} \sum_{t=1}^{T} q_t^d > \lambda^d + \delta \right\} < \infty \quad \forall d \neq d(1), \delta > 0,$$

we have

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} q_t^{d(1)} = \lambda^d.$$

Lemma B.3 (Corrected Lemma 5 of Kasy and Sautmann (2021). From Lemma 13 of Russo (2016) and Lemma 29 of Shang, de Heide, Menard, Kaufmann, and Valko (2020)). Fix any $\xi > 0$ and $d \neq d(1)$. Under any allocation rule, if

$$\lim_{T \to \infty} \frac{m^d_T}{NT} = 1/2,$$

there exists $\xi' > 0$ and a sequence $\varepsilon_T$ with $\lim_{T \to \infty} \varepsilon_T = 0$ such that for any $T \in \mathbb{N}$,

$$\frac{m^d_T}{NT} \geq \lambda^d + \xi \Rightarrow \frac{p^d_T}{\max_{d \neq d(1)} p^d_T} \leq \exp \left( -T(\xi' + \varepsilon_T) \right),$$

almost surely.

Lemma B.4 (Corrected Lemma 6 of Kasy and Sautmann (2021)). Denote with $\overline{D}$ the arms that are sampled only a finite amount of times:

$$\overline{D} = \{ d \in \{1, \ldots, k\} : \forall t, m_t^d < \infty \}.$$

If $\overline{D}$ is empty, $p_t^{d(1)}$ converges almost surely to 1. If $\overline{D}$ is non-empty, then for every $d \in \overline{D}$, we have $\liminf_{T \to \infty} p_t^d > 0$ almost surely.

Lemma B.1 basically follows from Theorem 6 of Shang, de Heide, Menard, Kaufmann, and Valko (2020). We prove this lemma in Appendix E. We extend the result from Beta-Bernoulli bandit model with the Beta(1,1) prior to that with Beta($\alpha^d_0, \beta^d_0$) priors for each $d \in \{1, \ldots, k\}$, where the constants $\alpha^1_0, \ldots, \alpha^k_0, \beta^1_0, \ldots, \beta^k_0 > 0$ can be arbitrary.

Lemmata B.2 and B.3 are corrected citation of Russo (2016) and Shang, de Heide, Menard, Kaufmann, and Valko (2020).

We derive Lemma B.4 with the help of Lemma 28 in Shang, de Heide, Menard, Kaufmann, and Valko (2020).

C Auxiliary Results

In addition to Lemmata B.1–B.4, which correspond to Lemmata 2 (2), 4–6 in Kasy and Sautmann (2021), we additionally use the following results, Proposition C.1, C.2, and C.3, from Shang, de Heide, Menard, Kaufmann, and Valko (2020) and Kasy and Sautmann (2021). We further state and prove Lemma C.4 and C.5, which is required for the proof of Lemma B.1.
Proposition C.1 (From Lemma 4 of Shang, de Heide, Menard, Kaufmann, and Valko (2020)). There exists a random variable $W$ with $E [\exp (\lambda W)] < \infty$ for any $\lambda > 0$ such that
\[
|m_T^d - N \sum_{t=1}^{T} q_t^d| \leq W \sqrt{(NT + 1) \log(e^2 + NT)}, \text{ almost surely, } \forall T \in \mathbb{N}, d \in \{1, \ldots, k\}.
\]

Proposition C.2 (From Step 2 of the proof of Theorem 1 in Kasy and Sautmann (2021)). Under exploration sampling, for each $d \in \{1, 2, \ldots, k\}$ and all $t \in \mathbb{N}$,
\[
\frac{p_t^d}{p_t^d + 1} \leq q_t^d \leq \frac{1}{2}.
\]

Proposition C.3 (Lemma 26 of Shang, de Heide, Menard, Kaufmann, and Valko (2020)). Let $X \sim \text{Beta}(a_0, a_1)$ and $Y \sim \text{Beta}(a_2, a_3)$ such that
\[
0 < \frac{a_0 - 1}{a_0 + a_1 - 1} < \frac{a_2 - 1}{a_2 + a_3 - 1}.
\]
Then, we have
\[
\mathbb{P}(X > Y) \leq D \exp(-C),
\]
where
\[
C = \inf_{\frac{a_0 - 1}{a_0 + a_1 - 1} \leq y \leq \frac{a_2 - 1}{a_2 + a_3 - 1}} C_{a_0, a_1}(y) + C_{a_2, a_3}(y),
\]
\[
C_{a_0, a_1}(y) = (a_0 + a_1 - 1) d_{\text{KL}} \left( \frac{a_0 - 1}{a_0 + a_1 - 1}, y \right),
\]
and
\[
D = 3 + \min \left\{ C_{a_0, a_1} \left( \frac{a_2 - 1}{a_2 + a_3 - 1} \right), C_{a_2, a_3} \left( \frac{a_0 - 1}{a_0 + a_1 - 1} \right) \right\}.
\]

Proposition C.1 is used for showing almost sure convergence of the shares of observations (see Remark 3). Here, we note that from Proposition C.1, we can insist that with probability 1,
\[
\lim_{T \to \infty} \frac{m_T^d}{NT} = \lambda^d \iff \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} q_t^d = \lambda^d.
\]

Besides, to prove Lemma B.1, we show the following lemmata.

Lemma C.4 (Lower bound on the deviation probability of Beta distribution). Let $X \sim \text{Beta}(a_0, a_1)$. We have
\[
\mathbb{P}(X \geq x) \geq \exp \left( - (a_0 + a_1 - 1) d_{\text{KL}} \left( \frac{a_0 - 1}{a_0 + a_1 - 1}, x \right) \right) / a_0 + a_1,
\]
and
\[
\mathbb{P}(X \leq x) \geq \exp \left( - (a_0 + a_1 - 1) d_{\text{KL}} \left( \frac{a_0 - 1}{a_0 + a_1 - 1}, x \right) \right) / a_0 + a_1.
\]

Proof of Lemma C.4. We use following facts (see e.g., Appendix I.1 of Shang, de Heide, Menard, Kaufmann, and Valko (2020).). Let $F_{a_0, a_1}^{\text{Beta}}(x)$ be the cumulative distribution function of a Beta distribution with parameters $a_0$ and $a_1$. Similarly, let $F_{a_0, a_1}^{\text{Bin}}(x)$ be the cumulative distribution function of a Binomial distribution with parameters $a_0$ and $a_1$. We have a following relationship (Beta-Binomial trick)
\[
F_{a_0, a_1}^{\text{Beta}}(x) = 1 - F_{a_0 + a_1 - 1, a_0 - 1}^{\text{Bin}}(a_0 - 1).
\]
Therefore, we have
\[
\mathbb{P}(X \geq x) = \mathbb{P}(B(a_0 + a_1 - 1, x) \leq a_0 - 1) = \mathbb{P}(B(a_0 + a_1 - 1, 1 - x) \geq a_1),
\]
where $B(a_0, a_1)$ is a Binomial Distribution with parameters $a_0$ and $a_1$. From Sanov’s inequality (Appendix I.1 of Shang, de Heide, Menard, Kaufmann, and Valko (2020)), we have
\[
\frac{\exp \left( - n d_{\text{KL}}(x/n, p) \right)}{n + 1} \leq \mathbb{P}(B(n, p) \geq x) \leq \exp \left( - n d_{\text{KL}}(x/n, p) \right).
\]
We get
\[ P(X \geq x) = P(B(a_0 + a_1 - 1, 1 - x) \geq a_1) \]
\[ \geq \frac{1}{a_0 + a_1} \exp \left( -(a_0 + a_1 - 1) d_{KL} \left( \frac{a_1}{a_0 + a_1 - 1}, 1 - x \right) \right) \]
\[ = \frac{1}{a_0 + a_1} \exp \left( -(a_0 + a_1 - 1) d_{KL} \left( \frac{a_1}{a_0 + a_1 - 1}, x \right) \right), \]
and
\[ P(X \leq x) = P(B(a_0 + a_1 - 1, x) \geq a_0 - 1) \]
\[ \geq \frac{1}{a_0 + a_1} \exp \left( -(a_0 + a_1 - 1) d_{KL} \left( \frac{a_0 - 1}{a_0 + a_1 - 1}, x \right) \right). \]
This concludes the proof.

Then, based on Lemma C.4, we have the following lemma, which has an exponent that matches the exponent of Proposition C.3.

**Lemma C.5.** Let \( X \sim \text{Beta}(a_0, a_1) \) and \( Y \sim \text{Beta}(a_2, a_3) \) such that
\[ 0 < \frac{a_0 - 1}{a_0 + a_1 - 1} < \frac{a_2 - 1}{a_2 + a_3 - 1}. \]
Then, we have
\[ P(X > Y) \geq D \exp(-C), \]
where
\[ C = \inf_{\frac{a_0 - 1}{a_0 + a_1 - 1} \leq y \leq \frac{a_2 - 1}{a_2 + a_3 - 1}} C_{a_0, a_1}(y) + C_{a_2, a_3}(y), \]
\[ C_{a_0, a_1}(y) = (a_0 + a_1 - 1) d_{KL} \left( \frac{a_0 - 1}{a_0 + a_1 - 1}, y \right), \]
and
\[ D = \frac{1}{(a_0 + a_1)(a_2 + a_3)}. \]

**Proof of Lemma C.5.** For each \( y \in \left[ \frac{a_0 - 1}{a_0 + a_1 - 1}, \frac{a_2 - 1}{a_2 + a_3 - 1} \right] \), we have
\[ P(X > Y) \]
\[ \geq P(\{X > y\} \cap \{Y < y\}) \]
\[ \geq \exp \left( -(a_0 + a_1 - 1) d_{KL} \left( \frac{a_0 - 1}{a_0 + a_1 - 1}, y \right) + (a_2 + a_3 - 1) d_{KL} \left( \frac{a_2 - 1}{a_2 + a_3 - 1}, y \right) \right), \]
where the last inequality is from Lemma C.4. Optimizing the right hand side of the previous inequality over \( y \), we conclude the proof.

\[ \square \]

**D Proof of Theorem 4.1**

In this section, we show the proofs of Theorem 4.1 (1) and (2) in Section D.1 and Theorem 4.1 (3) in Section D.2. The proof procedure in Section D.1 follows that of Kasy and Sautmann (2021) as closely as possible.
D.1 Proof of Theorem 4.1 (1) and (2)

From Lemma B.1, under any allocation rule, given the event \( \lim_{T \to \infty} \frac{m_d}{NT} = \lambda^d \), which occurs with probability 1; we can conclude that

\[
\frac{1}{NT} \log \left( \sum_{d \not= d(1)} p_T^d \right) \xrightarrow{a.s.} \Lambda^*.
\]

To prove Theorem 4.1 (1) and (2), it suffices to show that under the exploration sampling, for each \( d \in \{1, \ldots, k\} \),

\[
\frac{m_T^d}{NT} \xrightarrow{a.s.} \lambda^d.
\]  \hspace{1cm}(8)

To show that the exploration sampling satisfies (8), we correct the following three steps in the proof of Kasy and Sautmann (2021), i.e.,

**Step 1:** Each treatment is assigned infinitely often, that is, \( m_T^d \xrightarrow{a.s.} \infty, \forall d \in \{1, \ldots, k\} \).

**Step 2:** The share of observations \( m_T^{d(1)}/(NT) \) assigned to the best treatment \( d^{(1)} \) converges to \( \lambda^{d(1)} = 1/2 \) almost surely as \( T \to \infty \).

**Step 3:** The share of observations \( m_T^d/(NT) \) assigned to each treatment \( d \neq d^{(1)} \) converges to \( \lambda^d \) almost surely as \( T \to \infty \).

We use Step 1 to show the almost sure convergence of the posterior probability. Then, because the share of observations is determined by the posterior probability, we can show Step 2 and 3, which directly implies (8).

*Proof of Theorem 4.1 (1) and (2).*

**Step 1:** Each treatment is assigned infinitely often. We show \( m_T^d \xrightarrow{a.s.} \infty \) for each \( d \in \{1, 2, \ldots, k\} \) using proof by contradiction.

Suppose that there exists \( d' \in \{1, 2, \ldots, k\} \), such that \( \lim_{T \to \infty} m_T^{d'} < \infty \). Under the exploration sampling, from Proposition 2, we have \( q_T^{d'} \geq \frac{p_T^{d'}}{p_T^{d'} + 1} \); therefore, by Lemma B.4, if \( d' \in \mathcal{D} = \{d \in \{1, \ldots, k\} : \forall T, m_T^d < \infty\} \), then \( \liminf_{T \to \infty} p_T^{d'} > 0 \), which implies that \( \sum_{T=1}^{\infty} q_T^{d'} = \infty \). By Proposition C.1, we have \( m_T^{d'} \xrightarrow{a.s.} \infty \). This causes a contradiction with probability 1. Therefore, \( m_T^d \xrightarrow{a.s.} \infty \), for all \( d \in \{1, \ldots, k\} \).

**Step 2:** The share of observations \( m_T^{d(1)}/(NT) \) assigned to the best treatment \( d^{(1)} \) converges to \( 1/2 \) almost surely as \( T \to \infty \). Because \( m_T^d \xrightarrow{a.s.} \infty \) for all \( d \in \{1, \ldots, k\} \), we have \( p_T^{d(1)} \xrightarrow{a.s.} 1 \) from Lemma B.4. Then, from Proposition C.2, we conclude that \( m_T^{d(1)}/(NT) \xrightarrow{a.s.} 1/2 \).

**Step 3:** The share of observations \( m_T^d/(NT) \) assigned to each treatment \( d \neq d^{(1)} \) converges to \( \lambda^d \) almost surely as \( T \to \infty \). Our final step is to show (8). From Proposition B.2, we can obtain this result if (6) and (7) hold almost surely.

Firstly, using Proposition C.1, \( m_T^{d(1)}/(NT) \xrightarrow{a.s.} 1/2 \) (the result of Step 2) leads to

\[
\frac{1}{T} \sum_{t=1}^{T} q_T^{d(1)} \xrightarrow{a.s.} \frac{1}{2}.
\]

Thus, (6) holds.

Next, we check that (7) holds. For \( d \neq d^{(1)} \), let us define an event \( \mathcal{F}^d \) as

\[
\mathcal{F}^d = \left\{ \lim_{T \to \infty} p_T^d = 0 \quad \text{and} \quad \forall T \in \mathbb{N}, \left| m_T^d - N \sum_{t=1}^{T} q_t^d \right| \leq W \sqrt{(NT + 1) \log(e^2 + NT)} \right\},
\]

where \( W \) is a random variable defined in Proposition C.1. This event \( \mathcal{F}^d \) occurs with probability 1 by Lemma B.4 and Proposition C.1. Because each treatment is assigned infinitely often from Step 1, \( \lim_{T \to \infty} p_T^d = 0 \) for \( d \neq d^{(1)} \) almost surely from Lemma B.4. The second element of \( \mathcal{F}^d \) occurs with probability 1 from Proposition C.1.
Under this event $\mathcal{F}^d$, for each constant $\xi > 0$, there exists $s$ such that for all $T \geq s$, we have
\[
\left| \frac{1}{T} \sum_{t=1}^{T} q^d_t - \frac{m^d_T}{NT} \right| \leq \xi.
\] (9)
This is because from the second element of $\mathcal{F}^d$, for all $T$,
\[
\left| \frac{m^d_T}{NT} - \frac{1}{T} \sum_{t=1}^{T} q^d_t \right| \leq W \frac{\sqrt{(NT + 1) \log(e^2 + NT)}}{NT}
\]
holds, which implies that for each constant $\xi > 0$, (9) holds for sufficiently large $T$.

Then, the following relationship holds almost surely:
\[
\mathbb{I} \left\{ \frac{1}{T} \sum_{t=1}^{T} q^d_t \geq \lambda^d + 2\xi \right\} \leq \mathbb{I} \left\{ \frac{m^d_T}{NT} \geq \lambda^d + \xi \right\},
\]
where for an event $\mathcal{E}$, $\mathbb{1} \{ \mathcal{E} \} = 1$ if the event $\mathcal{E}$ occurs. Under the event $\mathcal{F}^d$, the exists $t_0 > 0$ such that for all $t \geq t_0$, $p^d_t \leq 1/2$. As Kasy and Sautmann (2021) shows in Step 3 of the proof in Theorem 1, when $\max_{d \neq d(1)} p^d_t \leq 1/2$, we have
\[
q^d_t \leq 2 \frac{p^d_t}{\max_{d \neq d(1)} p^d_t}.
\]

Besides, Lemma B.3 insists that given the event $m^d_T/NT \rightarrow 1/2$, there exists $\xi' > 0$ and a sequence $\varepsilon_T$ with $\varepsilon_T \rightarrow 0$ such that for any $T \in \mathbb{N}$,
\[
\frac{m^d_T}{NT} \geq \lambda^d + \xi \implies \frac{p^d_T}{\max_{d \neq d(1)} p^d_T} \leq \exp \left( -T(\xi' + \varepsilon_T) \right).
\]

Therefore, for $d \neq d(1)$, under the event $\mathcal{F}^d$, the following inequality holds with probability 1.
\[
\sum_{t=\max\{s, t_0\}}^{T} q^d_t \mathbb{I} \left\{ \frac{1}{T} \sum_{t=1}^{T} q^d_t \geq \lambda^d + 2\xi \right\} \leq \sum_{t=\max\{s, t_0\}}^{T} q^d_t \mathbb{I} \left\{ \frac{m^d_T}{NT} \geq \lambda^d + \xi \right\}
\leq \sum_{t=\max\{s, t_0\}}^{T} \exp \left( -t(\xi' + \varepsilon_t) \right)
< \infty.
\]

Therefore, (7) holds with probability 1. By combining these results, from Lemma B.2, (8) holds. This concludes the proof.

\[\square\]

D.2 Proof of Theorem 4.1 (3)

We prove Theorem 4.1 (3) by using Lemma B.1. The proof consists of two parts: derivation of the upper bound under the exploration sampling and lower bound under any allocation rule.

**Proof of Theorem 4.1 (3).** First, we prove $-\frac{1}{NT} \log W_\theta(T) \xrightarrow{a.s.} \Lambda^*$. The logarithmic posterior policy regret can be decomposed as
\[
\log W_\theta(T) = \log \left( \sum_{d \in \{1, \ldots, k\}} \Delta^d \cdot p^d_T \right)
= \log \left( \sum_{d \neq d(1)} \Delta^d \cdot p^d_T \right) \leq \log \left( \max_{d \in \{1, \ldots, k\}} \Delta^d \sum_{d \neq d(1)} p^d_T \right).
\]
Therefore,
\[-\frac{1}{NT} \log W_\theta(T) \geq -\frac{1}{NT} \log \left( \max_{d \in \{1, \ldots, k\}} \Delta^d \right) - \frac{1}{NT} \log \left( \sum_{d \neq d^{(1)}} p_T^d \right).\]

While \(\max_{d \in \{1, \ldots, k\}} \Delta^d\) is constant, \(\sum_{d \neq d^{(1)}} p_T^d\) decays exponentially, that is, the first converges to 0 and the second term converges to constant, \(\Lambda^*\). Similarly, we can also show that
\[-\frac{1}{NT} \log W_\theta(T) \leq -\frac{1}{NT} \log \left( \min_{d \in \{1, \ldots, k\}} \Delta^d \right) - \frac{1}{NT} \log \left( \sum_{d \neq d^{(1)}} p_T^d \right).\]

By taking the limit from the lower and upper bounds, we show the statement.

Next, we prove the second statement of Theorem 4.1 (3), which is the theoretical lower bound of algorithms. From Lemma B.1, under any adaptive allocation rule satisfying \(\frac{m^{(1)}_T}{NT} \xrightarrow{a.s.} \frac{1}{2}\), \(\limsup_{T \to \infty} -\frac{1}{NT} \log \left( \sum_{d \neq d^{(1)}} p_T^d \right) \leq \Lambda^*\) almost surely. This implies that under any adaptive allocation rule satisfying \(\frac{m^{(1)}_T}{NT} \xrightarrow{a.s.} \frac{1}{2}\), \(\limsup_{T \to \infty} -\frac{1}{NT} \log W_\theta(T) \leq \Lambda^*\) almost surely. Thus, the limit of the logarithmic posterior policy regret under the exploration sampling matches the lower bound. This concludes the proof.

\[\square\]

E Proof of Lemma B.1

The proof follows similar steps to the proof of Theorem 6 in Shang, de Heide, Menard, Kaufmann, and Valko (2020). Note that the original proof has a technical issue, and we also correct it in our proof.\(^6\)

**Proof.** We restate the set \(\mathcal{D}\) for the sake of readability:
\[\mathcal{D} = \{d \in \{1, \ldots, k\} : \forall t, m_t^d < \infty\} .\]

First, we prove the first part of the Lemma B.1, a lower bound on the posterior probabilities \(\sum_{d \neq d^{(1)}} p_T^d\), by giving separate proofs for the cases where \(\mathcal{D}\) is the empty set or not.

**Case 1. A lower bound on the posterior probabilities when \(\mathcal{D}\) is not empty.** The posterior variance \(\sigma_{T, d}^2\) is computed as
\[\sigma_{T, d}^2 = \frac{\alpha_T^d \beta_T^d}{(\alpha_T^d + \beta_T^d)^2 (\alpha_T^d + \beta_T^d + 1)} = \frac{\left(\alpha_0^d + r_{T-1}^d\right) \left(\beta_0^d + m_{T-1}^d - r_{T-1}^d\right)}{\left(\alpha_0^d + \beta_0^d + m_{T-1}^d\right)^2 \left(\alpha_0^d + \beta_0^d + m_{T-1}^d + 1\right)} .\]

Thus, when \(d \in \mathcal{D}\), we have \(\liminf_{T \to \infty} \sigma_{T, d}^2 > 0\) and \(\liminf_{T \to \infty} p_T^d > 0\). This means that \(\limsup_{T \to \infty} p_T^{d^{(1)}} < 1\). Thus, we have
\[\limsup_{T \to \infty} -\frac{1}{TN} \log \left( \sum_{d \neq d^{(1)}} p_T^d \right) = \limsup_{T \to \infty} -\frac{1}{TN} \log \left( 1 - p_T^{d^{(1)}} \right) = 0 \leq \Lambda^* .\]

**Case 2. A lower bound on the posterior probabilities when \(\mathcal{D}\) is empty.** When \(\mathcal{D}\) is empty, we have
\[\max_{d \neq d^{(1)}} \mathbb{P} \left( \hat{\theta}^d \geq \hat{\theta}^{d^{(1)}} \mid m_{T-1}, r_{T-1} \right) \leq 1 - p_T^{d^{(1)}} \leq \sum_{d \neq d^{(1)}} \mathbb{P} \left( \hat{\theta}^d \geq \hat{\theta}^{d^{(1)}} \mid m_{T-1}, r_{T-1} \right) \leq (k - 1) \max_{d \neq d^{(1)}} \mathbb{P} \left( \hat{\theta}^d \geq \hat{\theta}^{d^{(1)}} \mid m_{T-1}, r_{T-1} \right) .\]  

\(^6\)In particular, transformation of Eq. (14) fixes the issue of Theorem 6 of Shang, de Heide, Menard, Kaufmann, and Valko (2020) by utilizing Lemma C.5.
Since all the arms are sampled infinitely often, there exists $t_0$ such that for all $T \geq t_0$, for all $d \neq d^{(1)}$,
\[
\frac{\alpha_0^d - 1 + r_{T-1}^d}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d} \leq \frac{\alpha_0^{d^{(1)}} - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^{d^{(1)}} + \beta_0^{d^{(1)}} - 1 + m_{T-1}^{d^{(1)}}}.
\]
For each $d \neq d^{(1)}$, define the interval,
\[
I_d = \left[ \frac{\alpha_0^d - 1 + r_{T-1}^d}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, \frac{\alpha_0^{d^{(1)}} - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^{d^{(1)}} + \beta_0^{d^{(1)}} - 1 + m_{T-1}^{d^{(1)}}} \right].
\]
Using Proposition C.3 with $a_0 = \alpha_0^d + r_{T-1}^d$, $a_1 = \beta_0^d + m_{T-1}^d - r_{T-1}^d$, $a_2 = \alpha_0^{d^{(1)}} + r_{T-1}^{d^{(1)}}$, and $a_3 = \beta_0^{d^{(1)}} + m_{T-1}^{d^{(1)}} - r_{T-1}^{d^{(1)}}$, we get
\[
\mathbb{P}\left( \hat{\theta}^d \geq \hat{\theta}^{d^{(1)}} \mid m_{T-1}, r_{T-1} \right) \leq D \exp\left( -\inf_{y \in I_d} C_{a_0, a_1}(y) + C_{a_2, a_3}(y) \right).
\]
We have
\[
D \leq 3 + (a_0 + a_1 - 1) d_{KL}\left( \frac{a_0 - 1}{a_0 + a_1 - 1}, \frac{a_2 - 1}{a_2 + a_3 - 1} \right)
= 3 + (\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d) d_{KL}\left( \frac{\alpha_0^d - 1 + r_{T-1}^d}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, \frac{\alpha_0^{d^{(1)}} - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^{d^{(1)}} + \beta_0^{d^{(1)}} - 1 + m_{T-1}^{d^{(1)}}} \right)
\leq 3 + 2TN d_{KL}\left( 0, \frac{\alpha_0^{d^{(1)}} - 1 + TN}{\alpha_0^{d^{(1)}} + \beta_0^{d^{(1)}} - 1 + TN} \right)
\leq C\left( \alpha_0^{d^{(1)}}, \beta_0^{d^{(1)}} \right) TN \log TN,
\]
with some positive constant $C\left( \alpha_0^{d^{(1)}}, \beta_0^{d^{(1)}} \right)$. We get
\[
\limsup_{T \to \infty} \frac{1}{NT} \log \left( \frac{\mathbb{P}\left( \hat{\theta}^d \geq \hat{\theta}^{d^{(1)}} \mid m_{T-1}, r_{T-1} \right)}{\exp\left( -\inf_{y \in I_d} C_{a_0, a_1}(y) + C_{a_2, a_3}(y) \right)} \right)
\leq \limsup_{T \to \infty} \frac{1}{NT} \log \left( C\left( \alpha_0^{d^{(1)}}, \beta_0^{d^{(1)}} \right) TN \log TN \right)
= 0.
\]
(11)
Using Lemma C.5 with $a_0 = \alpha_0^d + r_{T-1}^d$, $a_1 = \beta_0^d + m_{T-1}^d - r_{T-1}^d$, $a_2 = \alpha_0^{d^{(1)}} + r_{T-1}^{d^{(1)}}$, and $a_3 = \beta_0^{d^{(1)}} + m_{T-1}^{d^{(1)}} - r_{T-1}^{d^{(1)}}$, we get,
\[
\mathbb{P}\left( \hat{\theta}^d \geq \hat{\theta}^{d^{(1)}} \mid m_{T-1}, r_{T-1} \right) \geq D \exp\left( -\inf_{y \in I_d} C_{a_0, a_1}(y) + C_{a_2, a_3}(y) \right),
\]
where $D = \frac{1}{(\alpha_0^d + \beta_0^d + m_{T-1}^d)(\alpha_0^{d^{(1)}} + \beta_0^{d^{(1)}} + m_{T-1}^{d^{(1)}})}$. We get
\[
\liminf_{T \to \infty} \frac{1}{NT} \log \left( \frac{\mathbb{P}\left( \hat{\theta}^d \geq \hat{\theta}^{d^{(1)}} \mid m_{T-1}, r_{T-1} \right)}{\exp\left( -\inf_{y \in I_d} C_{a_0, a_1}(y) + C_{a_2, a_3}(y) \right)} \right)
\geq \liminf_{T \to \infty} \frac{1}{NT} \log D
\geq \liminf_{T \to \infty} \frac{1}{NT} \log \frac{1}{4(NT)^2}
\geq 0
\]
(12)
Therefore, for each $d \neq d^{(1)}$, similarly to the proof of Theorem 6 of Shang, de Heide, Menard, Kaufmann, and Valko (2020), we get

\[ 1 - p_T^{d^{(1)}} \]
\[ \geq \max_{d \neq d^{(1)}} \mathbb{P} \left( \tilde{\theta}^d \geq \tilde{\theta}^{d^{(1)}} \mid m_{T-1}, r_{T-1} \right) \]
\[ \geq \exp \left( -TN \min_{d \neq d^{(1)}} \inf_{y \in I_d} \left[ w_d d_{\text{KL}} \left( \frac{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, y \right) \right] + \frac{1}{2} d_{\text{KL}} \left( \frac{\alpha_0^d - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, y \right) \right) \]
\[ \geq \exp \left( -TN \max_{d \neq d^{(1)}} \min_{y \in I_d, \varepsilon} \left[ w_d d_{\text{KL}} \left( \frac{\alpha_0^d - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, y \right) \right] + \frac{1}{2} d_{\text{KL}} \left( \frac{\alpha_0^d - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, y \right) \right) \]

where for two real-valued sequences $(a_n)$ and $(b_n)$, $a_n \overset{\Delta}{=} b_n$ denotes logarithmic equivalence, that is,

\[ \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{a_n}{b_n} \right) = 0. \]

For each $\varepsilon > 0$, there exists $t_1(\varepsilon) > 0$ such that for all $T \geq t_1$, for all $d \neq d^{(1)}$,

\[ I_d \subset [\tilde{\theta}^d + \varepsilon, \tilde{\theta}^{d^{(1)}} - \varepsilon] = I_{d,\varepsilon}. \]

As the Kullback-Leibler divergence is uniformly continuous on $I_{d,\varepsilon}$, there exists $t_2(\varepsilon) > 0$ such that for all $T \geq t_2$,

\[ d_{\text{KL}} \left( \frac{\alpha_0^d - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, y \right) \geq (1 - \varepsilon) d_{\text{KL}}(\tilde{\theta}^d, y), \]

for all $y$ and all $d \in \{1, \ldots, k\}$. Thus, we get,

\[ 1 - p_T^{d^{(1)}} \geq \exp \left( -TN \max_{d \neq d^{(1)}} \min_{y \in I_{d,\varepsilon}} \left[ w_d d_{\text{KL}}(\tilde{\theta}^d, y) + \frac{1}{2} d_{\text{KL}}(\tilde{\theta}^{d^{(1)}}, y) \right] \right) \]

and thus,

\[ \limsup_{T \to \infty} \frac{1}{NT} \log \left( \sum_{d \neq d^{(1)}} p_T^d \right) \leq \Lambda^*. \]

This concludes the proof of the first part of Lemma B.1.

Next, we show the second part of Lemma B.1. When \( \lim_{T \to \infty} m_T^d / (NT) = \lambda_d \) for all $d \in \{1, \ldots, k\}$, we have that for each $d \in \{1, \ldots, k\}$,

\[ \lim_{T \to \infty} \inf_{y \in I_d} \left[ \frac{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}{TN} d_{\text{KL}} \left( \frac{\alpha_0^d - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, y \right) \right] + \frac{1}{2} d_{\text{KL}} \left( \frac{\alpha_0^d - 1 + r_{T-1}^{d^{(1)}}}{\alpha_0^d + \beta_0^d - 1 + m_{T-1}^d}, y \right) \]
\[ = \inf_{y \in [\tilde{\theta}^d, \tilde{\theta}^{d^{(1)}}]} \left[ \lambda_d d_{\text{KL}}(\tilde{\theta}^d, y) + \frac{1}{2} d_{\text{KL}}(\tilde{\theta}^{d^{(1)}}, y) \right] \]
\[ = \Lambda^*. \]
Therefore,

\[ 1 - p_T^{d(1)} = \exp \left( -TN \max_{w} \min_{d \neq d^{(1)}} \inf_{y \in I_{d,w}} \left[ w_d d_{KL}(\theta^d, y) + \frac{1}{2} d_{KL}(\theta^{d^{(1)}}, y) \right] \right) \]

\[ \hat{=} \exp(-TN \Lambda^*). \]

Hence, we get

\[ \lim_{T \to \infty} -\frac{1}{TN} \log \left( \sum_{d \neq d^{(1)}} p_T^d \right) = \Lambda^*. \]

This concludes the proof.

\[ \Box \]

## F Proof of Lemma B.4

The proof is analogous to the proof of Lemma 28 in Shang, de Heide, Menard, Kaufmann, and Valko (2020). We extend the result from Beta-Bernoulli bandit model with the Beta(1, 1) prior to that with the Beta(\(\alpha_0^d, \beta_0^d\)) priors for each \(d \in \{1, \ldots, k\}\), where the constants \(\alpha_0^1, \ldots, \alpha_0^k, \beta_0^1, \ldots, \beta_0^k > 0\) can be arbitrary.

**Proof.** When \(D\) is empty, then \(SW_T(d) \xrightarrow{\text{a.s.}} \theta^d\). The posterior variance \(\sigma^2_{T,d}\) is

\[ \sigma^2_{T,d} = \frac{\alpha_T^d \beta_T^d}{(\alpha_T^d + \beta_T^d)^2 (\alpha_T^d + \beta_T^d + 1)} = \frac{(\alpha_T^d + r_T^d - 1)(\beta_T^d + m_T^d - r_T^d)}{(\alpha_T^d + \beta_T^d + m_T^d - 1)^2 (\alpha_T^d + \beta_T^d + m_T^d - 1 + 1)}. \tag{15} \]

Therefore, under the event of \(D = \emptyset\), \(\sigma^2_{T,d} \xrightarrow{\text{a.s.}} 0\) (posterior concentration). When \(D\) is not empty, then from (15), we have that \(\lim \inf_{T \to \infty} \sigma^2_{T,d} > 0\). Hence \(\lim \inf_{T \to \infty} p_T^d > 0\). This concludes the proof.

\[ \Box \]

## References

Audibert, J., S. Bubeck, and R. Munos (2010): “Best Arm Identification in Multi-Armed Bandits,” in Conference on Learning Theory, pp. 41–53. 2, 3

Bechhofer, R., J. Kiefer, and M. Sobel (1968): Sequential Identification and Ranking Procedures: With Special Reference to Koopman-Darmois Populations. University of Chicago Press. 3

Carpentier, A., and A. Locatelli (2016): “Tight (Lower) Bounds for the Fixed Budget Best Arm Identification Bandit Problem,” in Conference on Learning Theory, vol. 49 of Proceedings of Machine Learning Research, pp. 590–604. 5, 8

Chen, C.-H., J. Lin, E. Yücesan, and S. E. Chick (2000): “Simulation Budget Allocation for Further Enhancing The Efficiency of Ordinal Optimization,” Discrete Event Dynamic Systems, 10(3), 251–270. 3

Chernoff, H. (1959): “Sequential Design of Experiments,” The Annals of Mathematical Statistics, 30(3), 755 – 770. 3

Even-Dar, E., S. Mannor, and Y. Mansour (2002): “PAC bounds for multi-armed bandit and Markov decision processes,” in International Conference on Computational Learning Theory, pp. 255–270. Springer. 3

Glynn, P., and S. Juneja (2004): “A large deviations perspective on ordinal optimization,” in Winter Simulation Conference, vol. 1. IEEE. 3, 4, 5

Kasy, M., and A. Sautmann (2021): “Adaptive Treatment Assignment in Experiments for Policy Choice,” Econometrica, 89(1), 113–132. 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13

Kaufmann, E., O. Cappe, and A. Garivier (2016): “On the Complexity of Best-Arm Identification in Multi-Armed Bandit Models,” Journal of Machine Learning Research, 17(1), 1–42. 3
LATIMORE, T., AND C. SZEPESVÁRI (2020): Bandit Algorithms. Cambridge University Press. 2, 3

MANNOR, S., AND J. N. TSITSIKLIS (2004): “The Sample Complexity of Exploration in the Multi-Armed Bandit Problem,” Journal of Machine Learning Research, 5, 623–648. 3

RUSSO, D. (2016): “Simple Bayesian Algorithms for Best Arm Identification,” arXiv preprint arXiv:1602.08448. 3, 4, 5, 7, 9

SHANG, X., R. DE HEIDE, P. MENARD, E. KAUFMANN, AND M. VALKO (2020): “Fixed-confidence guarantees for Bayesian best-arm identification,” in International Conference on Artificial Intelligence and Statistics, vol. 108 of Proceedings of Machine Learning Research, pp. 1823–1832. 4, 5, 9, 10, 14, 16, 17

WALD, A. (1945): “Sequential Tests of Statistical Hypotheses,” The Annals of Mathematical Statistics, 16(2), 117–186. 3