Universality in quantum chaos and the one parameter scaling theory

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We adapt the one parameter scaling theory (OPT) to the context of quantum chaos. As a result we propose a more precise characterization of the universality classes associated to Wigner-Dyson and Poisson statistics which takes into account Anderson localization effects. Based also on the OPT we predict a new universality class in quantum chaos related to the metal-insulator transition and provide several examples. In low dimensions it is characterized by classical superdiffusion or a fractal spectrum, in higher dimensions it can also have a purely quantum origin as in the case of disordered systems. Our findings open the possibility of studying the metal insulator transition experimentally in a much broader type of systems.

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Technological advances in recent years in the control of quantum coherence have supposed an important stimulus to the field of ‘quantum chaos’, namely, the study of the effect of classical chaos in quantum mechanics. A cornerstone result in quantum chaos is the so called Bohigas- Giannoni-Schmit (BGS) conjecture [1] that states quantum spectra of classically chaotic systems are universally correlated according to the Wigner-Dyson random matrix ensembles (WD) [2] for scales comparable to the typical scale \( \delta E \) [5, 6].

Feature is usually referred to as dynamical localization effects that localize eigenstates in momentum space. However, quantum diffusion is eventually arrested due to destructive interference effects that localize eigenstates in momentum space. In this limit spectral correlations are described by Poisson statistics, namely, the spectrum is uncorrelated.

Despite its unquestionable success, these conjectures are not always verified. Exceptions include the Harper model [4], and the ubiquitous kicked rotor [5]. In the latter, for short time scales, both quantum and classical motion are diffusive in momentum space. However, quantum diffusion is eventually arrested due to destructive interference effects that localize eigenstates in momentum space. In this limit spectral correlations are described by Poisson not by WD statistics. This counter-intuitive feature is usually referred to as dynamical localization [3, 6].

Deviations from the BGS conjecture are also expected for eigenvalue separations \( \delta E \sim h/E \) due to weak localization effects [2]. The typical scale \( t_E = \lambda^{-1} \log h \) is the Ehrenfest time with \( \lambda \) the classical Lyapunov exponent. However, these corrections, though universal, do not really invalidate the BGS conjecture since \( \delta E \sim h/t_E \gg \Delta \).

From the previous discussion it is clear that the main deviation from the semiclassical picture contained in the BGS conjecture is due to localization effects. This fact rises the following questions: Why in some quantum chaotic systems localization is only weak but in others is strong enough to induce a metal-insulator transition? Is it possible to give a more precise relation between classical motion and quantum features? Are there any other universality classes in the context of non-random Hamiltonians? The aim of this paper is to address these questions within the framework of the one parameter scaling theory [8] introduced originally in the context of disordered systems [9].

A key concept in this theory is the dimensionless conductance \( g \) introduced by Thouless [10]. It is defined either as i) the sensitivity of a given quantum spectrum to a change of boundary conditions in units of the mean level spacing \( \Delta \approx 1/L^d \), or ii) \( g = E_c/\Delta \) where, \( E_c \), the Thouless energy, is an energy scale related to the diffusion time to cross the sample. In the metallic limit \( E_c = hD_{\text{clas}}/L^2 \) \( (D_{\text{clas}} \text{ is the classical diffusion constant}) \) and therefore \( g \approx L^{-d-2} \). On the other hand if the particle is exponentially localized due to destructive interference, \( g \approx e^{-L/\xi} \) where \( \xi \) is the localization length and \( L \) is the system linear size.

The change of \( g(L) \) with the system size, \( \beta(g) = \frac{\partial \log g(L)}{\partial \log L} \), provides information about localization. For \( L \to \infty \), \( \beta(g) = d-2 > 0 \) in a metal (without quantum corrections) and \( \beta(g) = \log(g) < 0 \) in an insulator. The OPT is based on the following two simple assumptions: i) The \( \beta(g) \) function is continuous and monotonic; ii) The change in the conductance with the system size only depends on the conductance itself.

With this simple input the OPT is able to predict three universality classes in the \( L \to \infty \) limit: For \( d > 2 \) and disorder sufficiently weak, \( g \to \infty \). The system has metallic like features: Eigenfunctions are delocalized in real space, and the spectrum is correlated according to WD statistics. For \( d \leq 2 \), or \( d > 2 \) and strong disorder,
$g \to 0$. The system is an insulator: Eigenfunctions are exponentially localized and the spectrum is correlated according to Poisson statistics. For $d > 2$, a metal-insulator transition takes place at a certain density of impurities and energy. It is characterized by a size independent dimensionless conductance $g = g_c$, namely, $\beta(g_c) = 0$. Eigenstates at the transition are multifractals and the spectral correlations are universal \[11\], but different from WD and Poisson statistics. We now study how the OPT can be adapted to the context of quantum chaos.

One parameter scaling theory in quantum chaos.-The first problem is the very definition of the Thouless energy. In disordered systems it is estimated after an average over many disorder realizations. In quantum chaos no such ensemble average can be carried out. However, a Thouless energy can still be defined provided: i) The ensemble average is replaced by an average over initial conditions; ii) The classical phase space is homogeneous with no islands of stability. This condition guarantees that the Thouless energy does not depend on the initial conditions chosen.

The localization problem in quantum chaos is defined in momentum space, not in real space. This can be understood as follows: In a classically integrable systems the number of conserved quantities (canonical momenta) is equal to the dimensionality of the system. In quantum mechanics each of these canonical momenta becomes a good quantum number which labels the state. An integrable system is thus localized in momentum space in the sense that there exists a basis of momentum eigenstates in which the Hamiltonian is diagonal and consequently the spectrum is uncorrelated (Poisson statistics). As classical symmetries are reduced, the Hamiltonian is no longer diagonal in any momentum basis and eventually a transition from Poisson to WD statistics is expected.

After these clarifications we are ready to define $g$ in non-random Hamiltonians. As the classical dynamics is by no means restricted to standard diffusion, we study the more general case $\langle p^2 \rangle \sim t^\mu$ ($\ldots$ stands for average over initial conditions) with $\mu > 0$. The Thouless energy is given by $E_c \propto N^{-2/\mu}$ where $N$ is the system size, namely, the size of the basis of momenta eigenstates in which we express the Hamiltonian. The mean level spacing is in many cases given by $\Delta \sim 1/N^d$, but there are important exceptions: i) for periodic potentials, the Bloch theorem applies, the spectrum is continuous, and $\Delta = 0$; ii) for systems whose eigenstates are exponentially localized, $\Delta \neq 0$ even in the $N \to \infty$ limit; iii) in systems with a singular continuous spectrum the scaling with the system size may be anomalous $\Delta \propto N^{-d/d_c} (d_c \neq 1)$. A precise definition of $d_c$ may depend on the system in question. In the Harper model \[4\], $d_c \approx 1/2$ stands for the Hausdorff dimension of the spectrum. We are now ready to define the dimensionless conductance in quantum chaos. In cases i) and ii) above, $g \to \infty$ (metal) and $g = 0$ (insulator) respectively. In case iii) (including $d_c = 1$),

$$g(N) = \frac{E_c}{\Delta} = N^{\gamma_{\text{class}}}, \quad \gamma_{\text{class}} = \frac{d}{d_c} - \frac{2}{\mu} \quad (1)$$

The running of $g$ is thus described by $\beta(g) = \frac{\partial \log g(L)}{\partial \log L} = \gamma_{\text{class}} = \frac{d}{d_c} - \frac{2}{\mu}$. Under the assumptions of the OPT and using Eq. \[11\] we propose the following alternative definition of universality class in quantum chaos: i) If $\gamma_{\text{class}} > 0$, eigenfunctions are delocalized as in a metal and the spectral correlations are described by WD statistics. ii) If $\gamma_{\text{class}} < 0$, eigenfunctions are localized as in a insulator and the spectral correlations are described by Poisson statistics. iii) If $\gamma_{\text{class}} = 0$, eigenfunctions are multifractal, a metal insulator transition takes places and the spectral correlations are universally described by critical statistics \[11\] \[12\].

Several remarks are in order: i) Universality is restricted to scales of the order of the mean level spacing; ii) If $d_c \neq 1$, or for a continuous spectrum $\Delta = 0$, it may not be possible to carry out a statistical analysis of the spectral correlations. In these cases our classificatory scheme still holds but the characterization of the system as a metal or an insulator must rely directly on the eigenfunctions statistics or the transport properties; iii) Quantum destructive interference may modify $g$. For instance, all eigenstates are localized in a 2$d$ disordered system despite the fact that semiclassically $g \propto L^{d-2}$ is constant as at the Anderson transition; iv) In order to define the Thouless energy uniquely the moments of the classical distribution of probability must be described by a single scale.

Stability of the semiclassical predictions.- The previous discussion highlights the need of a clear understanding of the effect of Anderson localization on $\beta(g)$. We focus on the $d = 1$ and $d_c = 1$ case, the generalization $d > 1$ is straightforward. We investigate the quantum dynamics of a classical diffusion process described by the following fractional Fokker-Planck equation \[13\],

$$\left(\frac{\partial}{\partial t} - D_{\text{class}} \frac{\partial^{2\mu}}{\partial q^{2\mu}}\right) P(p,t) = \delta(p)\delta(t)$$

where $D_{\text{class}}$ is the classical diffusion coefficient and $\mu$ is a real number. The moments of the distribution $P(p,t)$ (if well defined) are $\langle |p|^k \rangle \propto t^{k/2\mu}$, with $k$ a real number. The classical propagator in Fourier space ($t \leftrightarrow \omega$, $p \leftrightarrow q$) is given by $K_0(q,\omega) = \frac{1}{D_{\text{class}}q^{2\mu/\nu}}$ where $\nu$ stands for the spectral density. One-loop corrections due to interference effects \[13\] take the form $K^{-1}(q) = K_0^{-1}(q) - \frac{(\pi\nu)^2}{2} \int (dk) \frac{\omega^2}{\langle k^2 \rangle^{\mu/2\nu}}$. The quantum diffusion coefficient $D_{\text{quant}}$ to this order is easily obtained by performing the integral above:

$$D_{\text{quant}} = D_{\text{class}} - C\ln(qL) \quad 2/\mu = 1$$

$$D_{\text{quant}} = D_{\text{class}} - Cq^{1-2/\mu} \quad 0 < 2/\mu < 1 \quad (2)$$
the quantum properties of an 1d Hamiltonian with classical dynamics such that \( \langle p^k \rangle \propto t^k \) resemble those of a disordered conductor at the metal-insulator transition. In summary, the semiclassical predictions of a 1d system whose classical dynamic is anomalous are stable under quantum corrections. In higher dimensions this might change. For instance, in \( d > 2 \), we expect deviations due to localization for \( \gamma_{\text{class}} > 0 \).

Examples. - We now test the predictions of the OPT in different deterministic systems.

1d Kicked rotor with classical singularities. - We study the quantum dynamics of, \( \mathcal{H} = \frac{p^2}{2} + V(q) \sum_q \delta(t - nT) \) for different non-analytic potentials \[14, 15\]: \( V(q) = \epsilon |q|^\alpha \) and \( V(q) = \epsilon \log(|q|) \) with \( q \in [-\pi, \pi], \alpha \in [-1, 1] \) and \( \epsilon \) a real number. The classical evolution is dictated by the map \( p_{n+1} = p_n - \frac{\partial V(q_n)}{\partial q_n}, q_{n+1} = q_n + Tp_{n+1} \) (mod \( 2\pi \)). In Ref. \[13\] it was found that for \( \alpha > 0.5 \) classical diffusivity is normal. Therefore, \( \langle p^2 \rangle \propto t \) and \( \gamma_{\text{class}} = -1 < 0 \). Thus the OPT predicts Poisson statistics even though the classical dynamics is chaotic. It is remarkable that by using scaling arguments dynamical localization can be predicted without having to map the problem onto a 1d Anderson model.

For \(-0.5 < \alpha < 0.5\), classical diffusion is anomalous, \( \langle p^k \rangle \propto t^{k(1-\alpha)} \). Using Eq. (3)

\[ \beta(g) = -\frac{\alpha}{1-\alpha}. \]

For \( \alpha > 0 \) the OPT predicts Poisson statistics no matter what the effect of quantum corrections is. For \( \alpha < 0 \) we expect WD statistics since quantum corrections do not modify qualitatively \( \beta(g) \). For \( \alpha = 0 \) (log singularity), we expect an Anderson transition. Therefore dynamical localization can be overcome, even in one dimension, if classical diffusion is fast enough; i.e. \( \mu > 2 \).

These predictions have been tested by studying the quantum evolution operator \( \mathcal{U} \) over a period \( T \) in a basis of plane waves \( |n\rangle \), \( \langle m|\mathcal{U}|n\rangle = \frac{1}{N^{(1-2\mu)N^2/N}} \sum_l e^{i\phi(l,m,n)} \)

where \( \phi(l,m,n) = 2\pi(l + \theta_0)(m - n)/N - ilV(2\pi(l + \theta_0)/N), l = -(N - 1)/2, \ldots, (N - 1)/2 \) and \( 0 \leq \theta_0 \leq 1 \; \theta_0 \) is a parameter depending on the boundary conditions \( \theta_0 = 0 \) for periodic boundary conditions). The eigenvalues and eigenvectors of \( \mathcal{U} \) are computed by using standard diagonalization techniques. In Fig. 1a it is observed that, in agreement with the perturbative analysis, the classical time dependence of \( \langle p^k \rangle \) is not modified by quantum corrections for \( \alpha \leq 0 \). Thus a genuine Anderson transition is expected for \( \alpha = 0 \). The scale invariance of the spectrum and the analysis of the level statistics (see Fig. 1b) fully confirms the theoretical prediction.

3d Kicked rotor. - We study the quantum dynamics of a 3d kicked rotor \[16\] with a smooth potential (the 2d version was studied in Ref. \[17\]): \( \mathcal{H} = \frac{1}{2}(\tau_1 p_1^2 + \tau_2 p_2^2 + \tau_3 p_3^2) + V(q_1, q_2, q_3) \sum_n \delta(t - nT) \) with \( V(q_1, q_2, q_3) = k \cos(q_1) \cos(q_2) \cos(q_3) \) and \( \tau_1, \tau_2, \tau_3 \) incommensurate. The spectrum of the evolution matrix

where \( C \) is a different constant for each case. The importance of the quantum effects depends strongly on the value of \( 2/\mu \).

In the region \( 1 < 2/\mu < 2 \), quantum corrections diminish the value of the classical diffusion constant. A renormalization group analysis shows that the semiclassical prediction \( \beta(g) < 0 \) for \( L \to \infty \) still holds but the transition to localization is faster due to quantum corrections. In the region \( 0 < 2/\mu < 1 \) quantum corrections are subleading with respect to the classical term. As a consequence the semiclassical prediction \( \beta(g) > 0 \) is not altered by quantum localization effects. For \( 2/\mu = 1 \), the logarithmic behavior resembles superficially that of 2d disordered system, \( D_{\text{quant}} = D_{\text{class}} - C \ln(\Lambda/l) \) where \( \ln(\Lambda/l) \) is the inverse momentum \( q^{-1} \). Therefore, the corrections to the bare coupling constant are small for small momentum \( q \ll 1/L \). Qualitatively this implies the absence of eigenstate localization. For a rigorous proof based on the evaluation of \( \beta(g) \) including higher order terms we refer to \[13\] and references therein. Therefore

FIG. 1: (Color online) (a) Comparison of the classical and quantum \( \langle |p|^4 \rangle \propto t^{4(1-\alpha)} \) for the case of a 1d kicked rotor with \( V(q) \propto \log |q|, |q|^{\alpha} \). In agreement with the analytical results the classical time dependence is not modified by quantum corrections for \( \alpha \leq 0 \). (b) Spectral rigidity \( \Delta_3(L) \) for \( V(q) = 10 \log |q| \). The level statistics in this case have all the signatures of a metal insulator transition such as scale invariance and linear \( \Delta_3(L) \sim \chi L/15 \) with \( \chi \approx 0.33 < 1 \).
was obtained by evolving a quantum state $|\psi(0)\rangle$ and performing the Fourier transform of $\langle \psi(t)|\psi(0)\rangle$. Classically the diffusion $(p^2) \propto t$ is normal provided that the classical phase space is fully chaotic. Quantum dynamics depends strongly on $k$. In analogy with a 3d disordered system, we expect destructive interference stop the classical diffusion for sufficiently small $k$. In the opposite limit, quantum effects are small and diffusion persists. A careful finite size scaling analysis [12] has confirmed this picture [16]. We have found a metal-insulator transition at $k = k_c \approx 3.3$. According to the OPT, since $\beta(g) = 0$ at the transition, quantum diffusion must be anomalous $(p^2) \propto t^{2/3}$. Likewise, level statistics are described by WD (Poisson) statistics in the limits $k \gg (\ll) k_c$. For $k = k_c$ it is expected spectral correlations be similar to those of a 3d disordered system at the transition. As is shown in Fig. 2, the numerical results fully agree with these theoretical predictions.

Harper model: The 1d Harper model [3], $\mathcal{H} = \cos(p) + \lambda \cos(2\pi sx)$, with $s$ irrational, undergoes a metal insulator transition at $\lambda = 2$. Classically the system is integrable. However, the quantum motion is diffusive $\langle x^2 \rangle \sim t^{2d_H}$ with $d_H$ the Hausdorff spectral dimension. With this information and Eq. [14] we can compute the dimensionless conductance and $\beta(g)$. As was expected $\beta(g) = 0$ and $g = g_c$ is size independent as it is expected at the metal insulator transition. Our simple method predicts correctly the metal insulator transition in this model as well. In conclusion we have investigated under what conditions the one parameter scaling theory can be utilized in quantum chaos. We have utilized it to determine the number of universality classes in quantum chaos and propose a more accurate definition of them. The universality class related to the metal insulator transition has been investigated in detail. We have tested our theoretical predictions in different kicked rotors and the Harper model. Our findings open the possibility of studying the metal insulator transition experimentally in a much broader class of systems.

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