Theorems and counterexamples on structured matrices

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0.1 Introduction

This thesis is devoted to several problems posed for special classes of matrices (such as GKK) and solved using structured matrices (such as Toeplitz) belonging to that class.

The topic of Chapter 1 is GKK $\tau$-matrices. This notion was introduced in the 1970’s as a response to the Taussky unification problem posed in the late 50’s, which is discussed in Section 1.1. Section 1.2 is devoted to four conjectures proposed in the 1970’s-1990’s on the stability of GKK $\tau$-matrices. They are all disproved in Section 1.3 using GKK $\tau$-matrices with additional structure (Toeplitz and Hessenberg). Further properties of the counterexample matrices, which are themselves not important for disproving the stability conjectures but seem to be worth analyzing, are taken up in Section 1.4. Section 1.5 contains a brief discussion of open problems related to the topics of earlier sections.

Chapter 2 is centered around the following problem: given a collection of matrices $(A_\alpha)$ in a special class (such as totally nonnegative) bounded in some matrix norm and such that the spectrum of $A_\alpha$ lies outside a disk of fixed radius with center at zero, determine whether the collection $(A_\alpha^{-1})$ is bounded in the same matrix norm. For any matrix norm, the answer is yes for matrices of bounded order, as is shown in Section 2.1. The next sections all deal with collections of matrices of unbounded order and the ‘simplest’ $\infty$-norm, the choice also motivated by applications. It is shown in Section 2.2 that the answer is still yes for totally nonnegative Hermitian matrices. However, the answer is no for positive definite Hermitian matrices. Section 2.3 contains a pertinent counterexample and a variation of it both based on the Hilbert matrix. A counterexample for the class of Hermitian Toeplitz matrices is obtained in Section 2.4. Finally, an interesting question of the same type arising in spline theory is discussed in Section 2.5.

0.2 Notation

The following conventions are used throughout the thesis.

To make a clear distinction between equality and equality by definition, the latter is denoted by :=. The symbol $\#$ denotes the cardinality of a set. The set $\{1, \ldots, n\}$ for $n \in \mathbb{N}$ is denoted by $\langle n \rangle$. For $p, q \in \mathbb{Z}$, let

$$p:q := \begin{cases} \{p, p+1, \ldots, q-1, q\} & \text{if } p \leq q \\ \emptyset & \text{otherwise} \end{cases}.$$  

For any $x \in \mathbb{R}$, set

$$x_+ := \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}.$$
The linear space of (column-)vectors with \( n \) entries in \( \mathbb{C} \) is denoted by \( \mathbb{C}^n \). The \( j \)th vector of the standard basis of \( \mathbb{C}^n \), i.e., the vector with 1 at the \( j \)th position and zeros elsewhere, will be denoted by \( e_j \). The linear space of all \( n \times n \)-matrices with entries in \( \mathbb{C} \) is denoted by \( \mathbb{C}^{n \times n} \). If the order of a matrix is not clear from the context, it will be indicated by the subscript \( n \times n \) or simply \( n \). The symbol \( I \) stands for the identity matrix of appropriate order. Both the zero matrix and the number zero are denoted by 0.

### 0.3 Basic matrix notions

Given a matrix \( A \in \mathbb{C}^{n \times n} \), let \( A(\alpha, \beta) \) denote the submatrix of \( A \) whose rows are indexed by \( \alpha \) and columns by \( \beta \) (\( \alpha, \beta \in \langle n \rangle \)) and let \( A[\alpha, \beta] \) denote \( \det A(\alpha, \beta) \) if \( \#\alpha = \#\beta \). For simplicity, \( A(\alpha) \) will stand for \( A(\alpha, \alpha) \) and \( A[\alpha] \) for \( A[\alpha, \alpha] \). By definition, \( A[\emptyset] := 1 \). Elements of \( A \) are denoted by \( a(i, j) \). A block diagonal matrix with (square) blocks \( A \) and \( B \) will be denoted by \( \text{diag}(A, B) := \left( \begin{array}{cc} A & 0 \\
0 & B \end{array} \right) \).

The spectrum of \( A \), i.e., the multiset of its eigenvalues (with each eigenvalue repeated according to its multiplicity), is denoted by \( \sigma(A) \). The spectral radius of \( A \) is denoted by \( \rho(A) \)(:= \( \max |\sigma(A)| \)).

The inequality \( A \geq 0 \) (\( > 0 \)) means that \( A \) is entrywise nonnegative (positive). \( A \geq B \) (\( A > B \)) means, by definition, that \( A - B \geq 0 \) (\( A - B > 0 \)).

A norm \( \| \cdot \| \) on the space \( \mathbb{C}^{n \times n} \) is a matrix norm if it satisfies the inequality

\[
\| AB \| \leq \| A \| \cdot \| B \| \quad \forall \ A, B \in \mathbb{C}^{n \times n}.
\]

A matrix norm \( \| \cdot \|_\circ \) is the operator norm subordinate to the norm \( \| \cdot \| \) on \( \mathbb{C}^n \) if

\[
\| A \| = \sup_{v \in \mathbb{C}^n \setminus \{0\}} \frac{\| Av \|}{\| v \|} \quad \forall \ A \in \mathbb{C}^{n \times n}.
\]

In particular, the \( p \)-norm \( \| \cdot \|_p \) (1 \( \leq p \leq \infty \)) on \( \mathbb{C}^{n \times n} \) is the operator norm subordinate to the \( p \)-norm

\[
\| v \|_p := \begin{cases} \left( \sum_{i=1}^n |v(i)|^p \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \max_{i=1,...,n} |v(i)| & \text{if } p = \infty \end{cases}
\]
on the space \( \mathbb{C}^n \). The condition number of an invertible matrix \( A \) (for the norm \( \| \cdot \| \)) is the product \( \| A \| \cdot \| A^{-1} \| \).

A matrix \( T \in \mathbb{C}^{n \times n} \) is Toeplitz if it has the form \( T =: (\tau(i-j))_{i,j=1}^n \) for some \( (\tau(i))_{i=1}^{n-1} \). A matrix is Hessenberg if the entries on its first subdiagonal are all equal to 1 and the entries below that subdiagonal are zero.
Chapter 1

Eigenvalues of GKK matrices

1.1 Taussky unification problem

A matrix $A$ is called totally nonnegative if $A[\alpha, \beta] \geq 0$ for all $\alpha, \beta \in \langle n \rangle$ with $\#\alpha = \#\beta$. A is called an $M$-matrix if $A = rI - P$ where $P \geq 0$ and $r > \rho(P)$. For more than a dozen other ways to define $M$-matrices, see [2]. A matrix $A$ is called a $P$-matrix if $A[\alpha] > 0$ for all $\alpha \subseteq \langle n \rangle$. $A$ is said to be sign-symmetric if

$$A[\alpha, \beta]A[\beta, \alpha] \geq 0 \quad \forall \alpha, \beta \in \langle n \rangle, \quad \#\alpha = \#\beta.$$  

$A$ is called weakly sign-symmetric if

$$A[\alpha, \beta]A[\beta, \alpha] \geq 0 \quad \forall \alpha, \beta \in \langle n \rangle, \quad \#\alpha = \#\beta = \#\alpha \cup \beta - 1. \quad (1.1)$$

The minors $A[\alpha, \beta]$ with the property (1.1) are sometimes called almost principal.

Weakly sign-symmetric $P$-matrices are also called GKK after Gantmacher, Krein, and Kotelyansky.

Let

$$l(A) := \min \sigma(A) \cap \mathbb{R},$$

with the understanding that, in this setting, $\min \emptyset = \infty$. A matrix $A$ is called an $\omega$-matrix if it has eigenvalue monotonicity in the sense that

$$l(A(\alpha, \alpha)) \leq l(A(\beta, \beta)) < \infty \quad \text{whenever} \quad \emptyset \neq \beta \subseteq \alpha \subseteq \langle n \rangle.$$  

$A$ is a $\tau$-matrix if, in addition, $l(A) \geq 0$.

Hermitian positive definite, nonsingular totally nonnegative, and $M$-matrices all enjoy positivity of principal minors, weak sign symmetry, and eigenvalue monotonicity. In fact, these properties were singled out as a response to the ‘unification problem’ for the above-mentioned three classes of matrices that stems from a research problem posed by O. Taussky [32].

\[1\] O. Taussky pointed out in [32] that similar theorems were known for some positive matrices and for positive definite Hermitian matrices, for which the then available proofs were different, and asked for a unified treatment of both cases. She gave four examples of such similar theorems, two of which illustrate common properties of totally nonnegative and positive definite Hermitian matrices. The term ‘Taussky unification problem’ was later taken to mean a much wider class of problems.
The fact that Hermitian positive definite matrices are sign-symmetric $P$-matrices (a property stronger than being GKK matrices) is standard. The eigenvalue interlacing property of Hermitian matrices ([10] or, e.g., [3, p.59]) implies their eigenvalue monotonicity. Directly from the definition, nonsingular totally nonnegative matrices are sign-symmetric with non-negative principal minors. Their eigenvalue monotonicity was proved by Friedland [19]. This fact and the spectral theory of totally nonnegative matrices show that all principal minors of a nonsingular totally nonnegative matrix are in fact positive.

The Perron-Frobenius spectral theory of nonnegative matrices shows that $M$-matrices are $P$- and $\omega$-matrices. The weak sign symmetry of $M$-matrices was proved by Carlson [8].

1.2 Stability conjectures on GKK $\tau$-matrices

Yet another property shared by Hermitian positive definite, totally nonnegative, and $M$-matrices is their positive stability. To recall, a matrix is called positive (negative) stable if its spectrum lies entirely in the open right (left) half plane. In the sequel, the term ‘positive stable’ will be usually shortened to simply ‘stable’.

Hermitian positive definite and totally nonnegative matrices are obviously stable (having only positive eigenvalues), while the stability of $M$-matrices follows from the Perron-Frobenius theory.

The natural question arising from this observation was whether some combination of the properties from Section 1.1, viz., positivity of principal minors, weak sign-symmetry, and eigenvalue monotonicity, is sufficient to guarantee stability. (None of those properties alone is sufficient, which can be checked by simple examples.)

Carlson [9] conjectured that the GKK matrices are stable and showed his conjecture to be true for $n \leq 4$.

Engel and Schneider [16] asked if nonsingular $\tau$-matrices or, equivalently, $\omega$-matrices all whose principal minors are positive (see Remark 3.7 in [16]), are positive stable. Varga [33] conjectured even more than stability, viz.

$$|\arg(\lambda - l(A))| \leq \frac{\pi}{2} - \frac{\pi}{n} \quad \forall \lambda \in \sigma(A).$$

This inequality was proven for $n \leq 3$ by Varga (unpublished) as well as by Hershkowitz and Berman [24] and for $n = 4$ by Mehrmann [28]. In his survey paper [23], Hershkowitz posed the weaker conjecture that $\tau$-matrices that are also GKK are stable.

The above conjectures were plausible not only because they were verified for matrices of small order, but also due to the following two theorems.

The first indicates that there is a certain ‘forbidden wedge’ around the negative real axis where eigenvalues of a $P$-matrix cannot lie. (The angle of the wedge depends on the order of the matrix.)

**Theorem (Kellogg [26]).** Let $A \in \mathbb{C}^{n \times n}$ be a $P$-matrix. Then

$$|\arg(\lambda)| < \pi - \frac{\pi}{n} \quad \forall \lambda \in \sigma(A).$$
The second shows that sign symmetry together with positivity of principal minors is sufficient for stability.

**Theorem (Carlson [9]).** *Sign-symmetric* $P$*-matrices are stable.*

Carlson’s elegant proof employs the Cauchy-Binet formula to show that $A^2$ is a $P$-matrix whenever $A$ is sign-symmetric, diagonal stable scaling property of $P$-matrices to conclude that $DA$ is stable for some diagonal matrix $D$ with positive diagonal, so that the homotopy $S(t):=((1-t)D+tI)A$ preserves sign symmetry as well as positivity of principal minors, hence the eigenvalues of $S(t)$ cannot cross the imaginary axis as $t$ runs from 0 to 1.

### 1.3 Counterexample

As is shown in [25], none of the proposed conjectures is true. Described below is a class of GKK $\tau$-matrices which are not even nonnegative stable, i.e., they do have eigenvalues with negative real part. This class consists of Toeplitz Hessenberg matrices $A_{n,k,t}$ of order $n$ that depend on two more parameters $k \in \mathbb{N}$ and $t \in \mathbb{R}$. In what follows, it will be shown that $A_{n,k,t}$ is a GKK $\tau$-matrix for any $t \in (0,1)$ and that $A_{2k+2,k,t}$ is unstable for sufficiently large $k$ and sufficiently small positive $t$.

Let $A$ be an infinite Toeplitz Hessenberg matrix with first row $(a_0, a_1, \ldots)$ and let $d_n$ denote its leading principal minor of order $n$. By the Laplace expansion by minors of the first row,

$$d_n = \sum_{j=1}^{n} (-1)^{n-j} a_j d_{n-j}, \quad n \in \mathbb{N}. \quad (1.2)$$

This is, in effect, an invertible lower triangular system for the $a_j$. So, for an arbitrary sequence $(d_1, d_2, \ldots)$, there exists exactly one Toeplitz Hessenberg matrix having these as its leading principal minors.

With this, let $A_{\infty,k,t}$ be the Toeplitz Hessenberg matrix with leading principal minors

$$d_n = t^{(n-k-1)_+}, \quad n \in \mathbb{N},$$

for some $t \in (0,1)$ and $k \in \mathbb{N}$. Then equation (1.2) becomes

$$\sum_{j=0}^{n-1} (-1)^j a_j t^{(n-j-k-2)_+} = t^{(n-k-1)_+}, \quad n \in \mathbb{N}. \quad (1.3)$$

Let $A_{n,k,t}$ be the leading principal submatrix of $A_{\infty,k,t}$ of order $n$.

Two observations are immediate: The first $k+1$ entries in the sequence $(d_1, d_2, \ldots)$ equal 1 and $d_{k+2} = t$, hence $a_0 = 1$, $a_j = 0$ for $1 \leq j \leq k$, and $a_{k+1} = (-1)^{k}(1-t)$. Secondly, the matrix $A_{\infty,k,t}$ is Toeplitz, so all its principal minors indexed by $j$ consecutive integers equal $t^{(j-k-1)_+}$.\n
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1.3.1 On two characteristics of $A_{\infty,k,t}$

As is well known, associated to any infinite Toeplitz matrix $T = (\tau(i-j))_{i,j=0}^{\infty}$ is its symbol $S(T)$, i.e., the Laurent series

$$S(T)(s) := \sum_{j=-\infty}^{\infty} \tau(-j)s^j.$$ 

It is also convenient to introduce the Taylor series $D(T)$

$$D(T)(s, \lambda) := \sum_{j=0}^{\infty} D_j(\lambda)s^j,$$

where

$$D_j(\lambda) := \begin{cases} 1 & \text{if } j = 0 \\ \det(T(1:j) - \lambda I_j) & \text{if } j \in \mathbb{N}. \end{cases}$$

To avoid cumbersome notation, $D_j(\lambda)$ will be used in the sequel, but one should keep in mind that $D_j(\lambda)$ also depends on $T$, i.e., for $T = A_{\infty,k,t}$, on the parameters $k$ and $t$, so should have been denoted by $D_j(\lambda, k, t)$.

**Proposition 1.**

$$S(A_{\infty,k,t})(s) = 1 + ts - (1 - t)(-s)^{k+1},$$

where

$$D(A_{\infty,k,t})(s, \lambda) = \frac{1 + (1 - t)\sum_{j=1}^{k+1} s^j}{1 + s(\lambda - t) + \lambda(1 - t)\sum_{j=2}^{k+2} s^j}. \quad (1.5)$$

**Proof.** From (1.3),

$$a_l = (-1)^l t^{l-k+1} + \sum_{j=1}^{l} (-1)^{j-l} t^{j-l-k+1} a_{j-1} \quad (1.6)$$

for all $l \in \mathbb{N}$. Let $\Phi(s) := S(A_{\infty,k,t})(s) - 1/s$. Replacing each $a_j, j \in \mathbb{N}$, by its expansion (1.6) and collecting terms with the factor $t^{j-k+1}$ for each $j$, one obtains

$$\Phi(s) = a_0 - \Phi(s) \sum_{j=1}^{\infty} t^{j-k+1}(-s)^j + \sum_{j=1}^{\infty} t^{j-k+1}(-s)^j.$$ 

Recall that $a_0 = 1$ and compute the sums in the right-hand side. This yields

$$\Phi(s) = \frac{1 + ts - (1 - t)(-s)^{k+2}}{(1 + s)(1 + ts)}$$

$$\Phi(s) = \frac{1 + ts - (1 - t)(s)^{k+1}}{(1 + s)(1 + ts)}, \quad (1.7)$$

hence

$$S(A_{n,k,t})(s) = \Phi(s) + \frac{1}{s} = \frac{(1 + s)(1 + ts)}{s(1 + ts - (1 - t)(-s)^{k+2})},$$

which, after canceling $(1 + s)$, gives (1.4).
Now observe that, similarly to (1.3), expansion of the determinant $D_n(\lambda)$ by its first row gives
\[(a_0 - \lambda)D_{n-1} + \sum_{j=1}^{n-1} (-1)^ja_jD_j(\lambda) = D_n(\lambda), \quad n \in \mathbb{N}. \tag{1.8}\]
That is, the coefficient of $s^j$ in $D(A_{\infty,k,t})(s,\lambda)$ equals the coefficient of $s^{j-1}$ in the product of $D(A_{\infty,k,t})(s,\lambda)$ and $\Phi(-s) - \lambda$ whenever $j \geq 1$. With the zeroth coefficient taken into account, this means
\[D(A_{\infty,k,t})(s,\lambda) = s(\Phi(-s) - \lambda)D(A_{\infty,k,t})(s,\lambda) + 1, \tag{1.9}\]
which, together with (1.7), implies (1.5). □

**Corollary.** The polynomials $D_j$ satisfy the recurrence relation
\[D_j(\lambda) + (\lambda - t)D_{j-1}(\lambda) + \lambda(1 - t)\sum_{l=2}^{k+2} D_{j-l}(\lambda) = 0 \quad \forall j \geq k + 2. \tag{1.9}\]

**Proof.** Compare the coefficient of $s^j$ for $j \geq k + 2$ in the numerator (zero) with that in the product of $D(A_{\infty,k,t})(s,\lambda)$ and the denominator in (1.5). □

These results will be revisited in subsection 1.3.3. Let us now show that $A_{\infty,k,t}$ is GKK.

### 1.3.2 $A_{n,k,t}$ are GKK

Since $A_{n,k,t}$ is Hessenberg, the submatrix $A_{n,k,t}(\langle n \rangle \setminus i:i+j-1)$ is block upper triangular if $1 < i \leq i + j - 1 < n$, so
\[A_{n,k,t}[\alpha \cup \beta] = A_{n,k,t}[\alpha]A_{n,k,t}[\beta] \quad \text{whenever} \quad i < j - 1 \quad \text{for all} \quad i \in \alpha, \quad j \in \beta. \tag{1.10}\]
This shows that $A_{n,k,t}$ is a $P$-matrix.

Fortunately, there is no need to verify the weak sign symmetry of a $P$-matrix by computing its almost principal minors. Instead, one makes use of the following remarkable fact.

**Theorem (Gantmacher, Krein [17, p.55], and Carlson [8]).** A $P$-matrix $A$ is GKK iff it satisfies the generalized Hadamard-Fisher inequality
\[A[\alpha]A[\beta] \geq A[\alpha \cup \beta]A[\alpha \cap \beta] \quad \forall \alpha, \beta \subseteq \langle n \rangle. \tag{1.11}\]

Since $0 < t < 1$ and
\[(x + y)_+ + (x + z)_+ \leq x_+ + (x + y + z)_+ \quad \forall x, \forall y, z \geq 0, \]
on one obtains
\[
A_{n,k,t}[i:i+j-1] \cdot A_{n,k,t}[l:l+m-1] \\
= t(j-k-1)_+ + (m-k-1)_+ \geq t(l+m-i-k-1)_+ + (i+j-l-k-1)_+ \quad \text{if} \quad l \leq i+j-1. \tag{1.12}\]

\[
A_{n,k,t}[i:i+j-1] \cdot A_{n,k,t}[l:l+m-1] \\
= A_{n,k,t}[i:i+m-1] \cdot A_{n,k,t}[l:l+j-1]
\]
Together with (1.10), (1.12) shows that \( A_{n,k,t} \) satisfies (1.11) if \( \alpha, \beta \) are sets of consecutive integers.

To prove (1.11) in general, first make a definition. Call the subsets \( \alpha, \beta \subseteq \langle n \rangle \) separated if \( |p - q| > 1 \) \( \forall p \in \alpha, q \in \beta \). Suppose \( \alpha, \beta_1, \ldots, \beta_j \subseteq \langle n \rangle \) are sets of consecutive integers, \( \beta_i \) \( (i = 1, \ldots, j) \) are pairwise separated, and

\[
\text{for any } i = 1, \ldots, j, \text{ there exist } p \in \beta_i \text{ and } q \in \alpha \text{ such that } |p - q| \leq 1. \quad (1.13)
\]

Then \( A_{n,k,t}, \alpha, \gamma \), and \( \beta := \bigcup_{i=1}^{j} \beta_i \) satisfy (1.11). Indeed, (1.11) holds for \( \alpha \) and \( \beta_1 \). If \( 1 \leq l < j \), then, assuming (1.11) for \( \alpha \) and \( \gamma_1 := \bigcup_{i=1}^{l} \beta_i \),

\[
A_{n,k,t}[\alpha \cup \gamma_1, \beta_l+1] = A_{n,k,t}[\alpha \cup \gamma_1]A_{n,k,t}[\beta_l+1] \\
\geq A_{n,k,t}[\alpha \cup \gamma_1]A_{n,k,t}[\alpha \cap \beta_1].
\]

Due to (1.13), \( \alpha \cup \gamma_1 \) is a set of consecutive integers, so an application of (1.11) yields

\[
A_{n,k,t}[\alpha \cup \gamma_1, \beta_l+1] \geq A_{n,k,t}[\alpha \cup \gamma_1]A_{n,k,t}[(\alpha \cup \gamma_1) \cap \beta_1].
\]

But \( (\alpha \cup \gamma_1) \cap \beta_1 = \alpha \cap \beta_1 \) since the sets \( \beta_i \) are pairwise disjoint. So,

\[
A_{n,k,t}[\alpha]A_{n,k,t}[\gamma_1, \beta_l+1] \geq A_{n,k,t}[\alpha \cup \gamma_1]A_{n,k,t}[\alpha \cap \gamma_1]A_{n,k,t}[\alpha \cap \beta_1] \\
= A_{n,k,t}[\alpha \cup \gamma_1]A_{n,k,t}[\alpha \cap \gamma_1].
\]

Now, given a set of consecutive integers \( \alpha \subseteq \langle n \rangle \) and any set \( \beta \subseteq \langle n \rangle \), write \( \beta = \gamma_1 \cup \gamma_2 \) where \( \gamma_1 := \bigcup_{i=1}^{l} \beta_i, \gamma_2 := \bigcup_{i=l+1}^{l+m} \beta_i, \) all \( \beta_i \) \( (i = 1, \ldots, l + m) \) are separated, and \( \beta_i \) satisfies (1.13) if and only if \( i \leq l \). Then

\[
A_{n,k,t}[\alpha]A_{n,k,t}[\beta] = A_{n,k,t}[\alpha \cup \gamma_1]A_{n,k,t}[\gamma_2] \geq A_{n,k,t}[\alpha \cup \gamma_1]A_{n,k,t}[\alpha \cap \gamma_1]A_{n,k,t}[\gamma_2] \\
= A_{n,k,t}[\alpha \cup \gamma_1 \cup \gamma_2]A_{n,k,t}[\alpha \cap \gamma_1] = A_{n,k,t}[\alpha \cup \beta]A_{n,k,t}[\alpha \cap \beta].
\]

In other words, \( A_{n,k,t} \) satisfies (1.11) if \( \alpha \subseteq \langle n \rangle \) is a set of consecutive integers and \( \beta \subseteq \langle n \rangle \) is arbitrary.

Finally, if \( \alpha_1, \alpha_2, \beta \subseteq \langle n \rangle \), the sets \( \alpha_i \) \( (i = 1,2) \) are separated, \( (1.11) \) holds for \( \alpha_1 \) and \( \beta \), and \( \alpha_2 \) is a set of consecutive integers, then (1.11) holds for \( \alpha := \alpha_1 \cup \alpha_2 \) and \( \beta \):

\[
A_{n,k,t}[\alpha]A_{n,k,t}[\beta] = A_{n,k,t}[\alpha_1]A_{n,k,t}[\alpha_2]A_{n,k,t}[\beta] \\
\geq A_{n,k,t}[\alpha \cap \beta]A_{n,k,t}[\alpha_1 \cap \beta]A_{n,k,t}[\alpha_2] \\
\geq A_{n,k,t}[(\alpha \cup \beta) \cap \alpha_2]A_{n,k,t}[(\alpha_1 \cup \beta) \cap \alpha_2]A_{n,k,t}[\alpha_1 \cap \beta] \\
= A_{n,k,t}[\alpha \cup \beta]A_{n,k,t}[(\alpha \cup \beta) \cap \alpha_2]A_{n,k,t}[\alpha_1 \cap \beta] \\
= A_{n,k,t}[\alpha \cup \beta]A_{n,k,t}[\alpha_2 \cap \beta].
\]

So, by induction on the number of ‘components’ of \( \alpha \), (1.11) holds for any \( \alpha, \beta \subseteq \langle n \rangle \). Thus, an application of the Gantmacher-Krein-Carlson Theorem concludes the proof of the following.

**Proposition 2.** The matrices \( A_{n,k,t} \) are GKK for any \( n, k \in \mathbb{N} \) and \( t \in (0,1) \). \( \square \)
1.3.3 $A_{n,k,t}$ are $\tau$-matrices

Now check that $A_{n,k,t}$ have eigenvalue monotonicity for any $n \in \mathbb{N}$ and $t \in (0,1)$. If $\langle n \rangle \supseteq \alpha = \cup_{i=1}^j \alpha_i$ is the union of separated sets of consecutive integers, then

$$\det(A_{n,k,t}(\alpha) - \lambda I) = \prod_{i=1}^j \det(A(\alpha_i) - \lambda I)$$

since $A_{n,k,t} - \lambda I$ is Hessenberg (the same observation earlier led to (1.10)). Since $A_{n,k,t} - \lambda I$ is Toeplitz, the product in the right hand side equals $\prod_{i=1}^j D_{\langle \#\alpha_i \rangle}(\lambda)$. Hence, to prove eigenvalue monotonicity of $A_{n,k,t}$ it is enough to prove it for leading principal submatrices of $A_{n,k,t}$ only, i.e., to show

$$l(A_{n+1,k,t}) \leq l(A_{n,k,t}) \quad \forall n \in \mathbb{N}.$$  

First recall that a $P$-matrix has no nonpositive real eigenvalues, since the coefficients of its characteristic polynomial strictly alternate in sign (Kellogg’s theorem gives a stronger statement, but one does not need it here). So, $l(A_{n,k,t}) > 0$ for all $n \in \mathbb{N}$. Now note that $D_n(\lambda) = (1 - \lambda)^n$ for $n = 0, \ldots, k+1$. For $n = k+2$, recall that $a_{k+1} = (-1)^k(1-t)$, so (1.8) yields $D_n(\lambda) = (1 - \lambda)^n - (1 - t)$. Hence,

$$l(A_{k+2,k,t}) = 1 - (1 - t)^{1/(k+2)} < 1 = l(A_{k+1,k,t}) = \cdots = l(A_{1,k,t})$$

and $D_j(l(A_{k+2,k,t})) < 0$ for all $j < k+2$.

On the other hand, if

$$D_j(l(A_{n,k,t})) < 0 \quad \forall j < n,$$  

then (1.9) and the positivity of $l(A_{n,k,t})$ imply that

$$D_{n+1}(l(A_{n,k,t})) = -(1-t)l(A_{n,k,t}) \sum_{j=1}^{k+1} D_{n-j}(l(A_{n,k,t})) < 0.$$  

But $D_{n+1}(0) = l^{(n-k)} > 0$, hence $D_{n+1}$ changes its sign on the interval $(0, l(A_{n,k,t}))$, hence $l(A_{n+1,k,t}) < l(A_{n,k,t})$ and (1.14) holds for $n+1$. Thus, by induction, the matrices $A_{n,k,t}$ have eigenvalue monotonicity.

**Proposition 3.** $A_{n,k,t}$ is a $\tau$-matrix for any $n, k \in \mathbb{N}$ and any $t \in (0,1)$. 

1.3.4 $A_{2k+2,k,t}$ is unstable for sufficiently large $k$ and small $t$

Now let $B_k := \lim_{t \to 0^+} A_{2k+2,k,t}$. The matrix $B_k$ is Toeplitz with first column

$$(1, 1, 0, \ldots, 0)^T$$

and first row

$$(1, 0, \ldots, 0, (-1)^k, (-1)^k, 0, \ldots, 0).$$
Show that there exists $K \in \mathbb{N}$ such that, for all $k > K$, $B_k$ has an eigenvalue $\lambda$ with $\text{Re} \lambda < 0$. As the eigenvalues depend continuously on the entries of the matrix, this will demonstrate that, for any $k > K$, there exists $t \in (0,1)$ such that the GKK $\tau$-matrix $A_{2k+2,k,t}$ has an eigenvalue with negative real part.

The polynomial $D_{2k+2}$ has a root with negative real part iff the polynomial $\psi_k$ defined by

$$\psi_k(\lambda) := \frac{D_{2k+2}(-\lambda)}{(1+\lambda)^{k-1}} = (1 + \lambda)^{k+3} - (k + 1)(1 + \lambda) + k$$

has a root with positive real part. Since

$$\psi_k(\lambda) = \lambda \sum_{j=0}^{k+1} \binom{k+3}{j} \lambda^{k+3-j-1} + 2,$$

it is, in turn, enough to show that $\eta_k$ defined by

$$\eta_k(\lambda) := \lambda^{k+3} \psi_k \left( \frac{1}{\lambda} \right) = 2\lambda^{k+2} + \sum_{j=2}^{k+3} \binom{k+3}{j} \lambda^{k+3-j}$$

has a root with positive real part.

One is now in the position to apply the classical negative stability criterion of Hurwitz to the polynomial $\eta_k$. It is more efficient, however, to use the following condition necessary for nonpositive stability due to Ando.

**Theorem (Ando [1]).** The Hurwitz matrix

$$H_k := \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & \cdots \\ a_0 & a_2 & a_4 & a_6 & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots \\ 0 & a_0 & a_2 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{d \times d}$$

of a nonpositive stable polynomial $f(x) := \sum_{j=0}^{d} a_j x^{d-j}$ of degree $d$ is totally nonnegative.

The Hurwitz matrix for the polynomial $\eta_k$ is

$$H_k := \begin{pmatrix} \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \binom{k+3}{8} & \binom{k+3}{10} & \cdots \\ 2 & \binom{k+3}{3} & \binom{k+3}{5} & \binom{k+3}{7} & \binom{k+3}{9} & \cdots \\ 0 & \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \binom{k+3}{8} & \cdots \\ 0 & 2 & \binom{k+3}{3} & \binom{k+3}{5} & \binom{k+3}{7} & \cdots \\ 0 & 0 & \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(k+2) \times (k+2)}$$

Compute the minor $H_k[2:5]$, taking out the factors $\binom{k+3}{2}, \binom{k+3}{4}, \binom{k+3}{6}$ from its second, third, and fourth columns respectively. This gives

$$H_k[2:5] = -\frac{1}{132300} (3k^3 - 49k^2 - 210k - 318)(k + 4)^2 (k + 5) \binom{k+3}{2} \binom{k+3}{4} \binom{k+3}{6}.$$
Thus, \( H_k[2:5] < 0 \) for large enough \( k \), precisely, for all \( k > 20 \). So, for \( k > 20 \), \( \eta_k \) has a zero with positive real part, therefore, \( D_{2k+2} \) has a zero with negative real part.

This completes the proof of the following.

**Theorem 1.** The GKK \( \tau \)-matrices \( A_{2k+2,k,t} \) are unstable for sufficiently large \( k \) and sufficiently small positive \( t \). \( \square \)

### 1.3.5 Numerics

To illustrate the result, consider the matrix \( A_{44,21,1/2} \), i.e., the Toeplitz matrix whose first column is

\[
(1, 1, 0, \ldots, 0)^T
\]

and first row is

\[
(1, 0, \ldots, 0, -1/2, -1/2^2, 1/2^3, -1/2^4, \ldots, -1/2^{22})
\]

and the limit matrix \( B_{21} \), with the same first column as \( A_{44,21,1/2} \) and first row equal to

\[
(1, 0, \ldots, 0, -1, -1, 0, \ldots, 0).\]

According to MATLAB, the two eigenvalues with minimal real part of the first matrix are

\[-2.809929189497896 \cdot 10^{-2} \pm 3.275076252367531 \cdot 10^{-1}i;\]

those of the second are

\[-3.420708309454068 \cdot 10^{-2} \pm 3.400425852703498 \cdot 10^{-1}i.\]

Further MATLAB calculations suggest that the matrices \( A_{28,13,t} \) are already unstable for sufficiently small \( t \).

### 1.4 More on the counterexample matrices

#### 1.4.1 The sign pattern of \( A_{n,k,t}^{-1} \)

The matrices \( A_{n,k,t} \) turn out to be quite ‘close’ to being sign-symmetric matrices for \( n \leq 2k + 2 \). Namely, all their minors of order \( n - 1 \) are nonnegative. Since \( \det A_{n,k,t} > 0 \) whenever \( t > 0 \), this property can be restated as follows.

**Proposition 4.** \( A_{n,k,t}^{-1} \) is checkerboard for \( n \leq 2k + 2 \).

The proof will make use of the Gohberg-Semencul formula for inverting a Toeplitz matrix.

**Theorem (Gohberg and Semencul [20] or, e.g., [22, p.21]).** If the Toeplitz matrix \( T \) of order \( n \) is invertible, then

\[
T^{-1} = x_0^{-1} \{ \begin{pmatrix}
    x_0 & 0 & \cdots & 0 \\
    x_1 & x_0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n-1} & x_{n-2} & \cdots & x_0
\end{pmatrix} \begin{pmatrix}
    y_0 & y_1 & \cdots & y_{n+1} \\
    0 & y_0 & \cdots & y_{n+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & y_0
\end{pmatrix} \}
\]
where \( x := (x_0, \ldots, x_{n-1})^T \) is the solution to \( Tx = e_1 \) and \( y := (y_{n+1}, \ldots, y_0)^T \) is the solution to \( Ty = e_n \) (recall that \( e_j \) denotes the \( j \)-th unit vector).

**Proof of Proposition 4.** There is nothing to prove if \( n \leq k + 1 \). The proof for \( n \geq k + 2 \) is by induction on \( n \). To avoid confusion, let us make the notation more precise. The goal is to inductively solve equations \( A_{n,k,t} x_n = e_1^{(n)} \) and \( A_{n,k,t} y_n = e_n^{(n)} \).

Let \( s := 1/t \). Prove by induction that

\[
\begin{align*}
x_l &= \begin{cases} (-1)^l s^{l+1} & \text{if } l \leq n - k - 2 \\ (-1)^l s^{n-k-1} & \text{if } l \geq n - k - 1, \end{cases} \\
y_l &= \begin{cases} s & \text{if } l = 0 \\ (-1)^l (1-s) & \text{if } -1 \leq l \leq -k - 1 \\ 0 & \text{if } l \leq -k - 2, \end{cases}
\end{align*}
\]

for \( n \geq k + 2 \). Indeed, formulas (1.16) and (1.17) hold for \( n = k + 2 \) (which can be checked by direct calculation). To justify the inductive step, one needs to show that

\[
\begin{pmatrix} 1 & B_{n,k,t} \\ e_1^{(n-1)} & A_{n-1,k,t} \end{pmatrix} \begin{pmatrix} s \\ -sX_{n-1} \end{pmatrix} = e_1^{(n)}, \quad \begin{pmatrix} 1 & B_{n,k,t} \\ e_1^{(n-1)} & A_{n-1,k,t} \end{pmatrix} \begin{pmatrix} 0 \\ Y_{n-1} \end{pmatrix} = e_1^{(n)}
\]

where \( B_{n,k,t} := A_{n,k,t}(1,2:n) \). Expanding (1.13) to order \( 2k + 2 \), one verifies that

\[
a_j = \begin{cases} (-1)^k (1-t) & \text{if } j = k + 1 \\ (-1)^j t^{j-k-2} (1-t)^2 & \text{if } k + 1 < j \leq 2k + 2. \end{cases}
\]

So,

\[
B_{n,k,t} X_{n-1} = \sum_{i=1}^{n-k-1} a_i (-1)^{k+i-1} s^{n-k-2} = s^{n-k-2} \left[ (1-\lambda) - \sum_{i=2}^{n-k-1} t^{i-2} (1-\lambda)^2 \right] = s^{n-k-2} \left[ (1-\lambda) - (1-\lambda)^2 \frac{1-t^{n-k-2}}{1-\lambda} \right] = s^{n-k-2} (1-\lambda) t^{n-k-2} = (1-\lambda),
\]

so \( s(1 - B_{n,k,t} X_{n-1}) = 1 \). Since \( e_1^{(n-1)} = A_{n-1,k,t} X_{n-1} \) by the inductive hypothesis, this justifies the transition from \( X_{n-1} \) to \( X_n \). Now,

\[
B_{n,k,t} Y_{n-1} = \sum_{i=1}^{n-k-2} a_i (-1) s^{n-k-i} (1-s) + a_{n-k-1} s
\]

\[= (-1)^{n-1} [(1-s)(1-\lambda) - \sum_{i=2}^{n-k-2} (1-\lambda)^2 t^{i-2} + t^{n-k-3} (1-\lambda)^2 s] 13\]
\[
= (-1)^{n-1} \left[ \frac{t-1}{t}((1 - \lambda) - (1 - \lambda) \frac{1}{(1 - \lambda)} + t^{n-k-3}(1 - \lambda) \frac{1}{t}} \right]
\]

and, since \(A_{n-1,k,t}Y_{n-1} = e^{(n-1)}_{n-1}\) by the inductive hypothesis, the inductive step is completed. This implies
\[
|x_l y_{-m}| \geq |x_{n-m} y_{l-n}| \quad \text{whenever} \quad m \geq 1, \ n - m > l. \quad (1.18)
\]

Indeed, if \(l - n \leq -k - 2\), the right hand side of (1.18) is zero. But if \(l \geq n - k - 1\), then \(|x_l| = |x_{n-m}|\) and \(|y_{-m}| = |y_{l-n}|\) (observe that \(l - n \neq 0\)).

Since the \((i, j)\) element of the right hand side of (1.18) has the form
\[
\sum_{l=i-\min\{i,j\}}^{i} x_l y_{i-j-l} - \sum_{l=n-j}^{n-j+\min\{i,j\}-1} x_l y_{i-j-l}
\]

and the terms in both sums have sign \((-1)^{i-j}\), the subtraction gives
\[
\begin{cases}
x_0 y_{i-j} + (-1)^{i-j} r_{i,j} & \text{if } i \leq j \\
x_{i-j} y_0 + (-1)^{i-j} r_{i,j} & \text{if } i \geq j
\end{cases}
\]

for some \(r_{i,j} \geq 0\).

So, in either case, if the \((i, j)\) element of \(A_{n,k,t}^{-1}\) is nonzero, then its sign is \((-1)^{i-j}\). \(\Box\)

### 1.4.2 The spectrum of \(A_{\infty,k,t}\)

It is well known (e.g., [18, p.21]) that the spectrum of an infinite Toeplitz matrix \(T = (\tau(i-j))\) with
\[
\sum_{j=-\infty}^{\infty} |\tau(j)| < \infty
\]

is the union of the curve
\[
C(T) = \{ S(T)(s) : s \in \mathbb{C}, \ |s| = 1 \}
\]

and those points in \(\mathbb{C} \setminus C(T)\) whose winding number with respect to \(C(T)\) is nonzero.

**Proposition 5.** The spectrum \(\sigma(A_{\infty,k,t})\) is the union of
\[
C(A_{\infty,k,t}) = \{ \frac{1 + ts}{s(1 + (1-t) \sum_{j=1}^{k+1} (-s)^j)} : s \in \mathbb{C}, \ |s| = 1 \}
\]

and the set of points in \(\mathbb{C}\) whose winding number with respect to \(C(A_{\infty,k,t})\) is nonzero. In particular, \(C(A_{\infty,k,t})\) contains a negative real point.

**Proof.** The value of the symbol \(S(A_{\infty,k,t})\) (given by (1.4)) at the point \(s = -1\) is
\[
- (1-t)/(k+2 - (k+1)t) \quad (1.21)
\]
Next, let us try to determine the limit set $\mathcal{L}(A_{\infty,k,t})$ of eigenvalues of the finite sections of $A_{\infty,k,t}$ (the matrices $A_{n,k,t}$) as $n$ tends to infinity. It is known (see, e.g., [1]) that $\mathcal{L}(T) \subseteq \sigma(T)$ for any Toeplitz matrix satisfying (1.9). Moreover, if $\mathcal{S}(T)$ is a rational function, there is a characterization of the set $\mathcal{L}(T)$ due to K. M. Day.

**Theorem (Day [12]).** Let $T := (\tau(i-j))_{i,j=0}^{\infty}$ be a Toeplitz matrix satisfying (1.9) with symbol $\mathcal{S}(T)$ that coincides with the expansion of the rational function $\frac{F}{GH}$ in the annulus \( \{ s \in \mathbb{C} : R_1 < |s| < R_2 \} \), $G$ (or $H$) being a polynomial all of whose roots lie in the set \( |s| \leq R_1 \) (\( |s| \geq R_2 \)), $F$ a polynomial having no common factors with $GH$, and $p := \deg G$.

Let $T_n := (\tau(i-j))_{i,j=0}^{n}$. Then the set

\[
\mathcal{L}(T) := \{ \lambda : \lambda = \lim \lambda_m, \ \lambda_m \in \sigma(T_{im}) \}
\]

(1.22)

coincides with

\[
\{ \lambda \in \mathbb{C} : |r_p(\lambda)| = |r_{p+1}(\lambda)| \},
\]

where $r_j(\lambda)$ are the roots of the polynomial $R := F - \lambda GH$ listed in the order of their absolute values $|r_1(\lambda)| \leq |r_2(\lambda)| \leq \cdots$.

For the problem in hand, $F(s) = 1 + ts$, $G(s) = s$, $H(s) = 1 + (1 - t) \sum_{j=1}^{k+1} (-s)^j$, $p = 1$, and $R(s) = 1 + ts - \lambda s(1 + (1 - t) \sum_{j=1}^{k+1} (-s)^j)$. It seems hopeless to seek an explicit formula for $\mathcal{L}(A_{\infty,k,t})$ for arbitrary $k$ and $t$. Even checking whether a particular point belongs to $\mathcal{L}(A_{\infty,k,t})$ turns out to be rather nontrivial.

However, one can easily make the following (negative) observation.

**Proposition 6.** The point $\{1.24\}$ does not belong to $\mathcal{L}(A_{\infty,k,t})$ for any $k \in \mathbb{N}$ and $t \in (0,1)$.

**Proof.** Let $\lambda$ equal $\{1.24\}$. If $|s| \leq 1$, then

\[
|F(s)| \geq 1 - t,
\]

(1.23)

while, by the triangle inequality,

\[
|\lambda G(s) H(s)| \leq |\lambda|(1 + (1 - t)(k + 1)) = 1 - t,
\]

(1.24)

so $R = 0$ iff inequalities (1.23) and (1.24) both become equalities iff $s = -1$. On the other hand, $\frac{d}{ds}R(-1) \neq 0$, so $s = -1$ is not a double root of $R$. Thus, $|r_1(\lambda)| < |r_2(\lambda)|$ and $\lambda \notin \mathcal{L}(A_{\infty,k,t})$.

Nevertheless, some eigenvalues of $A_{n,k,t}$ form sequences approaching points on the negative real axis as $n$ tends to infinity. This can be shown using the following result of M. Biernacki.

**Theorem (Biernacki [4] as cited in [30]).** Let $p, q \in \mathbb{N}$ be relatively prime and let $f(s) := s^{-p} + s^q - \lambda$. Then the set

\[
\{ \lambda : |r_p(\lambda)| = |r_{p+1}(\lambda)| \}
\]

is the star-shaped curve

\[
S := \{ \lambda = \varepsilon r \}
\]

where $\varepsilon$ is any $(p+q)$th root of unity and

\[
0 \leq r \leq (p + q)p^{p/(p+q)}q^{-q/(p+q)}.
\]

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Here is the more precise formulation of the above claim.

**Proposition 7.** For any \( k \in \mathbb{N} \), there exists \( t(k) \in (0, 1) \) such that the set \( \mathcal{L}(A_{\infty,k,t}) \) contains a segment on the negative real axis for any \( t \in (0, t(k)) \).

**Proof.** If \( t = 0 \), \( \tilde{R}(s) := (1 + s)R(s) = 1 + s - \lambda s(1 - (-s)^{k+2}) \). For \( \lambda \neq 0 \), the equation \( \tilde{R}(s) = 0 \) is equivalent to the equation

\[
(-s)^{-1} + (-s)^{k+2} - \mu = 0 \quad \text{where} \quad \mu := \frac{1 - \lambda}{(-\lambda)^{1/(k+3)}}.
\]

Taking \( \varepsilon := 1 \) and applying Biernacki’s theorem amounts to solving the equation

\[
1 - \lambda - (-\lambda)^{1/(k+3)}r = 0
\]

for some \( r \in (0, r_k) \) where \( r_k := (k+3)(k+2)^{-1/(k+3)} \). Since \( r_k > 2 \) for all \( k \geq 0 \), the left hand side of (1.26) changes sign between \( (\lambda =) -1 \) and \( \lambda = 0 \) for all \( r \in (2, r_k) \), therefore, there exists a segment \( \Lambda := (\lambda_{\min}, \lambda_{\max}) \subseteq (-1, 0) \) such that \(|r_1(\lambda)| = |r_2(\lambda)|\) for the polynomial \( \tilde{R} \) whenever \( \lambda \in \Lambda \).

Without loss of generality, one can assume that \( \tilde{R} \) has no double roots when \( \lambda \in \Lambda \), since that assumption can rule out only a finite number of \( \lambda \)'s. So, one can assume that such points are already excluded from \( \Lambda \).

If, for some \( \lambda \in \Lambda \), \( \tilde{R} \) had two distinct roots of the form \( s \) and \( \varepsilon |s| \), with \( \varepsilon := \pm 1 \), that would imply that one of the triangle inequalities

\[
|\left(\lambda - 1\right)s + \lambda (-s)^{k+3}| \leq |\lambda - 1| \cdot |s| + |\lambda| \cdot |s|^{k+3}
\]

\[
|\left(\lambda - 1\right)s + \lambda (-s)^{k+3}| \geq |\lambda - 1| \cdot |s| - |\lambda| \cdot |s|^{k+3}
\]

becomes equality, which is possible only if \( s^{k+2} \) is real, hence, by the condition \( \tilde{R}(s) = 0 \), \( s \) itself must be real. But the assumption that \( \tilde{R} \) has two roots \( s, -s \in \mathbb{R} \) leads to the contradictions \( 2 = 0 \) if \( k \) is even and \( (\lambda - 1)s = 0 \), hence \( \lambda = 1 \), if \( k \) is odd.

So, \( \tilde{R} \) has no roots of the form \( s, \pm |s| \) for \( \lambda \in \Lambda \).

This implies, first of all, that none of the roots of \( \tilde{R} \) with minimum absolute value equals \(-1\) when \( \lambda \in \Lambda \), so that \( r_1(\lambda), r_2(\lambda) \) are also roots of \( \tilde{R} \) with minimal absolute value. Moreover, the set of roots with the smallest absolute value consists of (possibly, several) distinct non-real conjugate pairs \( s_j, \overline{s_j} \). Since the roots of an algebraic equation are continuous functions of the coefficients, one of those pairs must stay a pair of complex conjugate roots with smallest absolute value as \( t \) runs from \( 0 \) to \( t(k) \) for some sufficiently small value \( t(k) \).

To visualize the sets \( \sigma(A_{\infty,k,t}) \) and \( \mathcal{L}(A_{\infty,k,t}) \), here are four figures, two for \( k := 3, t := 0.2 \) and two for \( k := 10, t := 0.4 \), drawn by MATLAB.
Figure 1. The curve $\mathcal{C}(A_{\infty,k,t})$ (black) and the sets $\sigma(A_{50,k,t})$ (green), $\sigma(A_{100,k,t})$ (cyan), $\sigma(A_{200,k,t})$ (blue), $\sigma(A_{400,k,t})$ (magenta), $\sigma(A_{800,k,t})$ (red). Here $k = 3$, $t = 0.2$.

Figure 2. Figure 1 zoomed in the part of $\sigma(A_{\infty,k,t})$ around the negative real axis.
Figure 3. The curve $\mathcal{C}(A_{\infty,k,t})$ (black) and the sets $\sigma(A_{50,k,t})$ (green), $\sigma(A_{100,k,t})$ (cyan), $\sigma(A_{200,k,t})$ (blue), $\sigma(A_{400,k,t})$ (magenta), $\sigma(A_{800,k,t})$ (red). Here $k = 10$, $t = 0.4$.

Figure 4. Figure 3 zoomed in the part of $\sigma(A_{\infty,k,t})$ around the negative real axis.
1.5 Open problems

The following questions appear to deserve further investigation in connection with the GKK \( \tau \)-matrix problem.

1. Can the matrices \( A_{n,k,t} \) be approximated by \( \tau \)-matrices that are strict GKK, i.e., \( P \)-matrices satisfying

\[
A[\alpha, \beta]A[\beta, \alpha] > 0 \quad \forall \alpha, \beta \in \langle n \rangle, \quad \#\alpha = \#\beta = \#\alpha \cup \beta - 1?
\]

The matrices \( A_{n,k,t} \) themselves are not strict GKK. If the answer is no,

1a. Are strict GKK matrices positive stable? Are strict GKK and \( \tau \)-matrices positive stable?

2. Given \( \alpha, \beta \subseteq \langle n \rangle \) with \( \#\alpha = \#\beta \), call the number \( \#\alpha - \#(\alpha \cap \beta) \) the dispersal of the minor \( A[\alpha, \beta] \). The counterexample from Section 1.3 shows it is not sufficient for stability of a \( P \)-matrix \( A \) that the inequalities

\[
A[\alpha, \beta]A[\beta, \alpha] \geq 0
\]

hold for all minors of dispersal \( \leq d := 1 \). Carlson’s theorem asserts that the value \( d = n \) is sufficient for stability. What minimal value of the parameter \( d \) would guarantee stability? In particular, does that value depend on \( n \)?

Also, here are two less directly related questions, which arose in the construction of the counterexample.

3. Given \( n \in \mathbb{N} \) and positive numbers \( (p_{\alpha})_{\emptyset \neq \alpha \subseteq \langle n \rangle} \) satisfying the generalized Hadamard-Fisher inequality \((1.11)\), when is there a matrix \( A \) such that \( A[\alpha] = p_{\alpha} \) for all \( \alpha \)?

4. For a matrix \( A \), let

\[
b_j := \sum_{\#\alpha = j} A[\alpha], \quad c_j := \frac{b_j}{\binom{n}{j}}, \quad j = 0, \ldots, n.
\]

When is it true that

\[
c_j^2 \geq c_{j-1}c_{j+1}, \quad j = 1, \ldots, n - 1?
\]

These inequalities are known for diagonal matrices with positive diagonal elements (e.g., \([21\text{, p.51}]\)) and go back to Newton. Since the numbers \( c_j \) are invariant under similarity, Newton’s inequalities \((1.27)\) also hold for all diagonalizable matrices with positive real eigenvalues. Do the GKK matrices satisfy \((1.27)\)?
Chapter 2

Inverses of special matrices

2.1 Bounded invertibility problem

The most general setup of the problem of this chapter is the following.

Let \( \mathcal{A} \) be a collection of matrices such that
\[
\inf_{A \in \mathcal{A}} \min \{|z| : z \in \sigma(A)\} > 0, \quad \sup_{A \in \mathcal{A}} \|A\| < \infty
\]
(2.1)
for some norm \( \| \cdot \| \). What conditions on \((A_j)\) imply
\[
\sup_{A \in \mathcal{A}} \|A^{-1}\| < \infty?
\]
(2.2)

In the easy case when the order of matrices is bounded above, the conclusion \(2.2\) holds without any additional hypothesis.

**Proposition 8.** Let \( \mathcal{A} \) be a collection of complex matrices satisfying \(2.1\) for some norm \( \| \cdot \| \) and such that
\[
\sup_{A \in \mathcal{A}} \text{order}(A) < \infty.
\]
(2.3)

Then \(2.2\) holds.

**Proof.** Without loss of generality, one can assume that all matrices \( A \in \mathcal{A} \) belong to \( \mathbb{C}^{n \times n} \) where \( n := \sup_{A \in \mathcal{A}} \text{order}(A) \). Indeed, just replace each \( A \) by
\[
\tilde{A} := \text{diag}(A, I_{(n-\text{order}(A)+1)}).
\]
Then \(2.1\) holds for the new collection \( \tilde{\mathcal{A}} \). Also, \( \tilde{\mathcal{A}} \) satisfies \(2.2\) iff \( \mathcal{A} \) satisfies \(2.2\). Next, since all norms on \( \mathbb{C}^{n \times n} \) are equivalent, one can assume that \( \| \cdot \| \) is an operator norm subordinate to a norm on \( \mathbb{C}^n \) (also denoted by \( \| \cdot \| \)).

So, suppose \( \mathcal{A} \subset \mathbb{C}^{n \times n} \) and \(2.2\) is violated. Then, by the Banach-Steinhaus theorem, there exists \( v \in \mathbb{C}^n \) and a sequence \((A_j)\) such that
\[
\|A_j^{-1}v\| \xrightarrow{j \to \infty} \infty.
\]
But the sequence \((A_j)\) is totally bounded in \( \mathbb{C}^{n \times n} \), hence contains a Cauchy subsequence. Without loss, it is \((A_j)\) itself. The limit \( A := \lim_{j \to \infty} A_j \) is invertible since
\[
\min\{|z| : z \in \sigma(A)\} = \lim_{j \to \infty} \min\{|z| : z \in \sigma(A_j)\},
\]

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hence \( \lim \| A_j^{-1}v \| = \| A^{-1}v \| < \infty \). This contradiction shows that \( 2.2 \) holds for any operator norm, hence for any norm on \( \mathbb{C}^{n \times n} \).

Next, let us consider the case when the order of matrices \( A_j \) is not bounded above and the norm in question is the \( \infty \)-norm. This question was posed by K. West \[34\] for positive definite Hermitian matrices in connection with a problem from econometrics. As is shown in Section 2.3, the answer in that case is no. However, under certain additional hypotheses the answer is yes, as is shown next.

### 2.2 Boundedly invertible collections of Hermitian matrices

One of the possible restrictions on the collection \( (A_j) \) that ensures that \( 2.2 \) holds is (uniform) bandedness of matrices \( A_j \), as follows directly from a theorem of S. Demko. To recall, a matrix \( A = (a(i,j)) \) is called banded with band width \( w \) if

\[
a(i,j) = 0 \quad \text{whenever} \quad |i - j| \geq w.
\]

**Theorem (Demko \[14\]).** Let \( A \in \mathbb{C}^{n \times n} \) be banded with band width \( w \) and satisfy conditions

\[
\| A \|_p \leq 1, \quad \| A^{-1} \|_p \leq \mu^{-1}
\]

for some \( 1 \leq p \leq \infty \) and \( \mu > 0 \). Then, with \( A^{-1} =: (\alpha(i,j))_{i,j=1}^n \), there are numbers \( K \) and \( r \in (0,1) \) depending only on \( \mu \) and \( \omega \) such that

\[
|\alpha(i,j)| \leq Kr^{i-j} \quad \forall i,j = 1, \ldots, n.
\]

In particular, for any \( 1 \leq q \leq \infty \), \( \| A^{-1} \|_q \leq C \) where the bound \( C \) depends only on \( \mu \), and \( w \).

Since the smallest eigenvalue of a Hermitian matrix is the reciprocal of the 2-norm of its inverse, the hypothesis of Demko’s theorem is satisfied for \( p := 2 \) whenever the collection \( A \) satisfies \( 2.1 \), hence the collection \( A^{-1} \) is bounded in any \( q \)-norm \( (1 \leq q \leq \infty) \), in particular, in the \( \infty \)-norm.

A different restriction that ensures boundedness of \( (A_j^{-1}) \), provided that \( A_j \)’s are Hermitian and satisfy \( 2.1 \), is the oscillatory property. Recall that a matrix is totally positive if all its minors are positive. A totally nonnegative matrix \( A \) is called oscillatory if \( A^l \) is totally positive for some \( l \in \mathbb{N} \). It is well known (see, e.g., \[17\] p.123]) that \( \sigma(A) \) consists of \( n \) distinct positive real numbers \( \lambda_1 < \cdots < \lambda_n \) and that the \( k \)th eigenvector \( v_k \) \( (Av_k = \lambda_k v_k) \) (unique up to a scalar multiple) has no zero entries and precisely \( n - k \) sign changes if \( A \) is oscillatory.

**Theorem 2.** Let \( A \in \mathbb{C}^{n \times n} \) be an oscillatory Hermitian matrix with smallest eigenvalue \( \lambda_{\text{min}} \). Then

\[
\| A^{-1} \|_{\infty} \leq \frac{\| A \|_{\infty}}{\lambda_{\text{min}}^2}.
\]

The proof will make use of the following lemma due to C. de Boor.

**Lemma (de Boor \[6\]).** Let \( A \in \mathbb{C}^{n \times n} \) be a nonsingular totally nonnegative matrix. If, for some \( x, y \in \mathbb{C}^n \),

\[
Ax = y, \quad \text{sign} x(i) = \text{sign} y(i) = (-1)^{i-1},
\]

in particular, \( \| A^{-1} \|_{\infty} \leq C \) where the bound \( C \) depends only on \( \mu \), and \( \omega \).

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then
\[ \|A^{-1}\|_\infty \leq \frac{\|x\|_\infty}{\min_i |y(i)|}. \]

Proof of Theorem 2. Since \(A\) is Hermitian, its eigenvector \(v\) corresponding to the eigenvalues \(\lambda_{\min}\) can be uniquely determined from the minimization problem
\[ v^*Av \to \min \]
once one of the entries of \(v\) is fixed. Let \(v(1) = 1\), so that \(v = [1 \ \tilde{v}]^T\). Since \(A\) is real, all the entries of \(v\) are necessarily real. Let \(A\) be partitioned conformably to \(v\):
\[ A = \begin{pmatrix} a(1, 1) & A(1, 2:n) \\ A(1, 2:n)^T & A(2:n) \end{pmatrix}. \]
Then \(v^TAv = a(1, 1) + 2A(1, 2:n)\tilde{v} + \tilde{v}^T A(2:n)\tilde{v}\) achieves its minimum at
\[ \tilde{v} := -A(2:n)^{-1}A(1, 2:n)^T. \]
By the eigenvalue interlacing property of \(A\), \(\min \sigma(A(2:n)) \geq \lambda_{\min}\). Since \(\|A(1, 2:n)^T\| \leq \|A\|_1\), this yields
\[ \|\tilde{v}\|_\infty \leq \|\tilde{v}\|_2 \leq \|A(2:n)^{-1}\| \cdot \|A(1, 2:n)^T\|_2 \leq \frac{\|A\|_\infty}{\lambda_{\min}}. \]
The same argument can be applied to the case when any other entry of \(v\) is set to be 1. Since the eigenvector \(v\) is unique up to multiplication by a scalar, this means
\[ \frac{\|v\|_\infty}{\min_i |v(i)|} \leq \frac{\|A\|_\infty}{\lambda_{\min}} \quad \text{whenever} \quad Av = \lambda_{\min}v. \]
Now apply de Boor’s lemma with \(x := v\), \(y := \lambda_{\min}v\) and obtain (2.4). \(\square\)

2.3 On shifted Hilbert matrices and their companions

However, the Hermitian property alone is not sufficient for the implication \((2.4) \implies (2.2)\), as is demonstrated by the two examples below. For those counterexamples, one needs several additional notions. The matrix \(H_n := \left(\frac{1}{i+j-1}\right)_{i,j=1}^n\) is known as the Hilbert matrix. It has been a subject of extensive studies and has served as an example of many unusual phenomena in operator theory. M.-D. Choi in [11] used what he called the companion of the Hilbert matrix, viz. the matrix \(C_n := \left(\frac{1}{\max(i,j)}\right)_{i,j=1}^n\). It turns out that the matrix \(C_n\) belongs to the special class of ultrametric matrices\(^1\) introduced by Nabben and Varga [29] as a generalization of the notion of strictly ultrametric matrices attributed by Martínez, Michon, and San Martín [27] to C. Dellacherie [13]\(^2\).

---

\(^1\)This matrix is also from the class of single-pair or (‘one-pair’) matrices due to Gantmacher and Krein [17] p.113.

\(^2\)Ultrametricity was first introduced in connection with \(p\)-adic number theory. Recall that a distance \(d\) on a space \(X\) is said to be ultrametric if it satisfies the inequality
\[ d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad \forall x, y, z \in X. \]
A matrix $A =: (a(i,j))_{i,j=1}^n$ is \textit{ultrametric} if 

$$
A = A^*, \quad A \geq 0 \\
a(i,j) \geq \min\{a(i,k), a(k,j)\} \quad \forall \ i, j, k \in \langle n \rangle \\
$$

and 

$$
a(i,i) \geq \max\{a(i,k) : k \in \langle n \rangle \setminus \{i\}\} \quad \forall i \in \langle n \rangle. \quad (2.5)
$$

If the inequality $(2.5)$ is strict for all $i \in \langle n \rangle$, $A$ is called \textit{strictly ultrametric}.

Finally, before stating the result of Martínez, Michon, and San Martín, recall that a matrix $A =: (a(i,j)) \in \mathcal{C}^{n \times n}$ is row (column) \textit{diagonally dominant} if 

$$
a(i,i) \geq \sum_{j \in \langle n \rangle \setminus \{i\}} |a(i,j)| \quad \forall i \in \langle n \rangle \quad (2.6) \\
(a(i,i) \geq \sum_{j \in \langle n \rangle \setminus \{i\}} |a(j,i)| \quad \forall i \in \langle n \rangle. \quad (2.7)
$$

If all the inequalities $(2.6)$ (the inequalities $(2.7)$) are strict, $A$ is called \textit{strictly row (column) diagonally dominant}.

\textbf{Theorem (Martínez, Michon, and San Martín [27]).} The inverse of a \textit{strictly ultrametric} matrix is a symmetric \textit{strictly diagonally} dominant $M$-matrix.

Now one can construct the following counterexample to the implication $(2.1) \implies (2.2)$.

\textbf{Proposition 9.} Let $\alpha > 0$ and let $A_n =: (\alpha I_n + C_n)^{-1}$. Then the collection $(A_n)_{n \in \mathbb{N}}$ of Hermitian positive definite matrices satisfies $(2.1)$ but does not satisfy $(2.2)$.

\textbf{Proof.} Subtracting of the $j$th column from the $j$th 1st column of the matrix $C_n$, for $j = 2, \ldots, n$, one verifies that $\det C_n > 0$ for all $n \in \mathbb{N}$. Hence $C_n$ is a Hermitian positive definite matrix. So, $\min \sigma(\alpha I_n + C_n) > \alpha$. By [11, Problem V], $\|C_n\|_2 \leq 4$, so $\max \sigma(\alpha I_n + C_n) \leq 4 + \alpha$. Thus, $\sigma(A_n) \subset [1/(\alpha + 4), 1/\alpha]$ for any $n \in \mathbb{N}$.

The matrices $C_n$ are ultrametric, hence the matrices $\alpha I_n + C_n$ are strictly ultrametric, so by the theorem of Martínez, Michon and San Martín, their inverses $A_n$ are diagonally dominant $M$-matrices. But any diagonal entry of $A_n$ is bounded above by $\|A_n\|_2 \leq 1/\alpha$, so the $\infty$-norm of $A_n$ is bounded above by $2/\alpha$. On the other hand,

$$
\|A_n^{-1}\|_{\infty} = \|\alpha I_n + C_n\|_{\infty} \approx \alpha + \ln n \underset{n \to \infty}{\longrightarrow} \infty.
$$

So, the collection $(A_n)$ satisfies $(2.1)$ but violates $(2.2)$. \hfill \Box

\textbf{Proposition 10.} For large enough $\alpha > 0$, the collection $(A_n =: \alpha I_n + H_n)^{-1}$ satisfies $(2.1)$ but does not satisfy $(2.2)$.

\textbf{Proof.} Note that $C_n - H_n \geq 0$ and estimate $\|C_n - H_n\|_{\infty}$.

$$
\|C_n - H_n\|_{\infty} = \max_{i \in \langle n \rangle} \left( \sum_{j=1}^{i} \left( \frac{1}{i} - \frac{1}{i + j - 1} \right) + \sum_{j=i+1}^{\infty} \left( \frac{1}{j} - \frac{1}{i + j - 1} \right) \right)
$$

$$
= \max_{i \in \langle n \rangle} \left( \sum_{j=1}^{i} \left( \frac{1}{i} - \frac{1}{i + j - 1} \right) + \sum_{j=i}^{2i-1} \frac{1}{j} \right) \leq \max_{i \in \langle n \rangle} \frac{2i}{i} = 2.
$$

By Dellacherie’s definition, a symmetric matrix $A \in \mathcal{C}^{n \times n}$ is \textit{ultrametric} if there exists an ultrametric distance $d$ on $\langle n \rangle$ such that 

$$
d(i,j) = d(i,k) \quad \text{iff} \quad a(i,j) = a(i,k).
$$
Recall that
\[ \|A^{-1}\| \leq \frac{\|B^{-1}\|}{1 - \|A - B\| \cdot \|B^{-1}\|} \]
whenever \( \|A - B\| \cdot \|B^{-1}\| < 1 \)
for any operators \( A, B \) and operator norm \( \| \cdot \| \). The proof of Proposition 9 demonstrated that \( \|(\alpha I_n + C_n)^{-1}\|_{\infty} \leq 2/\alpha \). Hence, if \( \alpha > 4 \), then
\[ \|C_n - H_n\|_{\infty} \|(\alpha I_n + C_n)^{-1}\|_{\infty} \leq \frac{4}{\alpha} < 1, \]
hence \( \|(\alpha I_n + H_n)^{-1}\|_{\infty} \leq \frac{2}{\alpha - 4} \). Thus, the collection \((A_n = (\alpha I_n + H_n)^{-1})\) satisfies (2.1) but violates (2.2).

\[ \boxdot \]

### 2.4 Inverses of nonnegative Hermitian Toeplitz matrices

The last example in the same spirit deals with Hermitian Toeplitz matrices.

**Proposition 11.** Let \( T_k \) be the infinite (upper triangular) Toeplitz matrix with symbol \( S(T_k)(s) = 1 + s + cs^k \), where \( c \) is any complex number with \( |c| > 2 \) (chosen to be positive if the matrices \( A_n \) must be nonnegative). Set \( A_k = T_k T_k^* \) and let \( A_{n,k} \) be the leading principal submatrix of \( A_k \) of order \( n \). Then the collection \((A_{n,k})\) of positive definite Hermitian Toeplitz matrices satisfies (2.1) but violates (2.2).

**Proof.** By the spectral theory of Toeplitz matrices, which was already discussed in Section 1.5, \( \sigma(A_k) \) equals the set of values of its symbol on the unit circle \( |s| = 1 \). Notice that \( |S(T_k)(s)| = |cs^k| - |1 + s| \geq |c| - 2 > 0 \) whenever \( |s| = 1 \), so
\[ \inf_{k \in \mathbb{N}} \min \sigma(A_k) > 0. \]

On the other hand, the matrices \( A_k \) have at most 9 nonzero diagonals with the absolute value of each nonzero term at most 2, so
\[ \sup_{k \in \mathbb{N}} \|A_k\|_{\infty} \leq 18. \]

Finally, \( A_k^{-1} = T_k^{-1} T_k^{-1} \) and the symbol \( S(T_k^{-1}) \) of \( T_k^{-1} \) is
\[ \frac{1}{S(T_k)(s)} = 1 - s + s^2 - \cdots + (-1)^{k-1} s^{k-1} + \cdots. \]

So, the \((k - 1) \times (k - 1)\) leading principal submatrix of \( A_k^{-1} \) has the form
\[ \begin{pmatrix}
1 & -1 & 1 & \cdots & (-1)^k \\
-1 & 2 & -2 & \cdots & (-1)^{k-1} 2 \\
1 & -2 & 3 & \cdots & (-1)^{k-2} 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^k & (-1)^{k-1} 2 & (-1)^{k-2} 3 & \cdots & k - 1
\end{pmatrix}, \]
hence \( \sup_{k \in \mathbb{N}} \|A_k^{-1}\|_{1} = \infty. \)

Since the matrices \( A_k \) are Hermitian, the limit set of the eigenvalues of \( A_{n,k} \) (as \( n \) tends to infinity) coincides with \( \sigma(A_k) \) (see, e.g., [7]). Since the (elementwise) limit of \( A_{n,k}^{-1} \) is \( A_k^{-1} \) by [18 p.74], the collection \((A_{n,k})\) satisfies (2.1) but not (2.2). \[ \boxdot \]
2.5 Least-squares spline projection matrices

An interesting problem of the same type arises in spline theory. Bounding the $\infty$-norm of the ($L_2$-)orthogonal projector onto splines leads to matrices of the specific form

$$A_{n,k,t}(i,j) = \frac{k}{t_{i+k} - t_i} \int B_{ik} B_{jk}. \quad (2.8)$$

Here $n, k \in \mathbb{N}$, $t$ is a nondegenerate knot sequence with $n + k$ knots, $B_{ik}$ is the $i$-th $B$-spline of order $k$ for the sequence $t$, $S_{k,t} := \text{span}\{B_{1k}, \ldots, B_{nk}\}$, and $Lf$ is the least squares approximation to $f \in L_\infty[t_1, t_{n+k}]$ by elements of $S_{k,t}$. C. de Boor [5] showed that there exists a positive constant $C_k$ such that

$$C_k \|A_{n,k,t}^{-1}\|_\infty \leq \|L\|_\infty \leq \|A_{n,k,t}^{-1}\|_\infty$$

and conjectured that, for $k$ fixed,

$$\sup_{n,t} \|A_{n,k,t}^{-1}\|_\infty < \infty. \quad (2.9)$$

This conjecture was recently proved by A. Shadrin [31] using sophisticated tools from spline theory to construct vectors $x$ and $y$ appearing in de Boor’s lemma with the min $|y(i)|$ and max $|x(i)|$ depending only on $k$ but not on the knot sequence $t$ or $n$ and conclude, using the lemma, that (2.9) holds.

The matrices $A_{n,k,t}$ are known to be oscillatory and diagonally similar to Hermitian matrices, i.e.,

$$A_{n,k,t} = D_{n,k,t} H_{n,k,t} D_{n,k,t}^{-1}$$

where $D_{n,k,t}$ are diagonal with positive diagonal entries. (For sure, the condition number of the matrices $D_{n,k,t}$ is not bounded above.) Moreover, $A_{n,k,t}$ are banded with band width $k$. Finally, the smallest eigenvalue of $H_{n,k,t}$ is known to be bounded away from zero independently of $t$ and $n$, so the collection $(A_{n,k,t})$ satisfies (2.1). (All those facts can be found in, e.g., [15, p.401–406].) However, the above conditions are not sufficient for (2.2) (that is, (2.9)) to hold. In particular, an application of Theorem 2 to the eigenvector $v_{\min}$ of $H_{n,k,t}$ corresponding to its smallest eigenvalue demonstrates that the eigenvector $D_{n,k,t} v$ of $A_{n,k,t}$ corresponding to the same eigenvalue cannot be used to prove (2.9).

Hence the following (somewhat vaguely formulated) problem:

**Problem.** Single out an additional property of the matrices $A_{n,k,t}$ to obtain a simple matrix theoretic proof of (2.9).
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