Wasserstein space as state space of quantum mechanics and optimal transport

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Abstract. In this work, we are in the position to view a measurement of a physical observable as an experiment in the sense of probability theory. To every physical observable, a sample space called the spectrum of the observable is therefore available. We have investigated the Wasserstein spaces over the spectrums of a quantum observables, i.e. the space of all probability measures defined on the suitable $\sigma$-algebra of subsets of the spectrum equipped with certain metric. It has been shown here the existence of one-to-one correspondence between the set of all quantum states and the set of all distribution functions defined on the spectrum of an observable. Thanks to the correspondence, every quantum state can be represented by a probability measure associated with the corresponding distribution function. It has been shown that the Wasserstein space over the spectrum of an observable is homeomorphic to the Wasserstein space over the spectrum of another observable. Such homeomorphism is referred to as generalized Fourier transform. By using the work of von Renesse, it has been shown that the Newtonian dynamics of the system can be formulated in the Wasserstein space over the spectrum of each observable which is equivalent to the Schroedinger dynamics, starting from the solution of Newton equation of motion in the Wasserstein space over the spectrum of a certain observable.

1. Introduction
A typical mathematical model serving as a representation of quantum mechanics consists four ingredients. The four ingredients are as follows:

- A quantum state space. It is a set of mathematical objects representing quantum states of a physical system under consideration. A state of the system contains all physical informations about the system. The set has to have a suitable structure to accommodate the features of quantum mechanics.

- An observable algebra. It is a set of mathematical objects representing observables or physical quantities which is relevant to the system.

- A rule of valuation. The rule guides us how to compute the probability $P_\sigma^O(U)$ that the measurement of an observable $O$ yields a value in the set $U$ whenever the system occupies a state $\sigma$.

- A local action of the group $(\mathbb{R}, +)$ on the state space. It describes the dynamics of the system. The local action is in correspondence with a differential equation called the equation of motion of the system.
In early days of quantum mechanics, Dirac has proposed a pragmatic formulation of quantum mechanics compromising the mathematical rigor [2]. He made use of vectors to represent quantum states. The set of all vectors representing quantum states was assumed to have a complex vector space structure. The vector space structure accommodates the superposition principle of states. He also introduced a scalar product in the vector space. The rigorous and complete mathematical formulation of quantum mechanics was given by von Neumann [5] in which he involved infinite dimensional complex separable Hilbert space as state space and the set of all self-adjoint operators acting on the Hilbert space as observable algebra. Afterwards both equivalent formulations are referred to as Dirac-von Neumann Axiom. The axiom for a single particle system was firstly formulated in the terms of self-adjoint operators acting on Hilbert space whose elements of unit length (unit vector) up to a scalar multiplication represent quantum states. The state space of the formulation is therefore a projective Hilbert space. The statistical aspect or the rule of valuation of the formulation is reflected in the formulae of expectation value of an observable represented by a self-adjoint operator \( \hat{A} \) in form of scalar product \( \langle \psi | \hat{A} | \psi \rangle \), when the system occupies a state represented by vector \( |\psi \rangle \). The evolution of the system is governed by the Schroedinger equation the solution of which is a curve in the state space. The next mathematical formulation of the axiom is for describing an ensemble, and is given in the terms self-adjoint operators and density operators acting on a separable Hilbert space. The states of the system are represented by the density operators and the observables of the system are represented by the self-adjoint operators. In the formulation, the statistical aspect is reflected in the rule of calculation of the expectation value of an observable \( \hat{A} \) as \( Tr(\hat{\rho} \hat{A}) \) when the system is in a state \( \hat{\rho} \). The dynamics of the system is described by Liouville-von Neumann equation. Furthermore, the most general formulation of Dirac-von Neumann axiom is given in the terms of \( C^* \)-algebra [7]. The observables of the quantum mechanical system are represented by self-adjoint elements of the \( C^* \)-algebra. The set of all linear positive normalized functionals defined on the algebra serves as state space of the formulation. The rule of the valuation of the formation is described by the expectation value given by the formulae \( \omega(\hat{A}) \) for an observable \( \hat{A} \) when the system is in a state represented by the functional \( \omega \). The dynamics of the system is formulated in the form of the so-called algebraic dynamics, i.e., one-parameter group of automorphisms of the observable algebra. The one-parameter group of automorphisms leads then to the equation of motion governing the evolution of state of the system.

In all of the formulations mentioned above the rule of the calculation of expectation values (and in turn the calculation of its standard deviation) plays an important role relating the theory to the experimental realm. We regard therefore a measurement of a physical observable as an experiment in the sense of probability theory. Thus, every physical observable is equipped with a sample space whose elements are the possible outputs of a measurement of the observable. The sample space can be an arbitrary set having various structures. Like in the case of an experiment in the sense of probability theory in which the outputs of the experiment are not necessary real numbers, so is the outputs of a measurement of a physical observable. Real numbers emerging in a measurement of an observable act as representatives of the possible outputs of the measurement, i.e. the elements of the sample space associated to the observable. The set of all real numbers representing the sample space is the range of a random variable. The sample space associated to an observable is called the spectrum of the observable. If \( A \) is an observable, then the spectrum of the observable will be denoted by \( \Sigma(A) \). In the traditional formulation of quantum mechanics, the spectrum of an observable is supposed to be the set of all eigenvalues of the self-adjoint operator representing the observable. Therefore, the nature of the spectrum of an observable depends on the observable. For instance, there are two kinds of spectrum, the continuous spectrum and the discreet one. Every eigenvalues contained in the spectrum of an observable corresponds to a subspace which is called eigensubspace. We can pick a basis in every eigensubspace and then form the union of all the bases to construct a basis for the whole
state space, i.e. a basis containing eigenvectors of the observable. The operator representing 
a physical observable should be complete in the sense that a maximal linear independent set 
constructed from its eigenvectors must be a basis for the whole state space. When the spectrum 
of the observable is countable (discrete), then a basis for the state space constructed from its 
eigenvectors is countable. On the other hand, if the spectrum of the observable is uncountable, 
then a corresponding basis is also uncountable. In the first case every state vector is represented 
by a column matrix of infinite rows, while in the second case every state vector is represented 
by a wave function defined on the spectrum of the observable. Every column vector or wave 
function representing a quantum state determines a probability measure on a suitable \(\sigma\)-algebra 
on subsets of the spectrum. More then one column vectors or wave functions can correspond to 
a single probability measure. However, not every probability measure on the spectrum of the 
observable can be obtained from a column vector or a wave function. The set \(P(\Sigma(A))\) of all 
probability measures on the spectrum \(\Sigma(A)\) of an observable \(A\) together with a suitable metric 
form a complete metric space \(P^2(\Sigma(A))\) which is called Wasserstein space. Let \(B\) be any other 
observable. It can be shown (and will be shown here) that the space \(P^2(\Sigma(B))\) is isomorphic to 
\(P^2(\Sigma(A))\) as metric spaces as well as convex spaces.

In this work, we are in the position to view a measurement of a physical observable as an 
experiment in the sense of probability theory. To every physical observable, a sample space 
called the spectrum of the observable is therefore available. We will investigate the Wasserstein 
spaces over the spectrums of a quantum observables, i.e. the space of all probability measures 
defined on the suitable \(\sigma\)-algebra of subsets of the spectrum equipped with certain metric. We 
will prove the existence of one-to-one correspondence between the set of all quantum states and 
the set of all distribution functions defined on the spectrum of an observable. Thanks to the 
correspondence, every quantum state can be represented by a probability measure associated 
with the corresponding distribution function. We will also prove that the Wasserstein space 
over the spectrum of an observable is homeomorphic to the Wasserstein space over the spectrum 
of another observable. Such homeomorphism is referred to as generalized Fourier transform. 
By using the work of von Renesse, we can formulate the Newtonian dynamics of the system in 
the Wasserstein space over the spectrum of each observable which is equivalent to Schroedinger 
dynamics, starting from the solution of Newton equation of motion in the Wasserstein space 
over the spectrum of a certain observable.

2. Wasserstein spaces over the spectrums of observables

Firstly, suppose that the spectrum \(\Sigma(A)\) of observable \(A\) is continuous and admits a metric 
\(d\) so that \((\Sigma(A), d)\) is a Polish space. The Wasserstein metric \(W_2^A\) in the set \(P(\Sigma(A))\) of all 
probability measures on \(\Sigma(A)\) is defined by

\[
W_2^A(\mu, \nu) := \left[ \sup_{\pi \in \Gamma(\mu, \nu)} \int_{\Sigma(A) \times \Sigma(A)} d(x, y)^2 d\pi(x, y) \right]^{1/2},
\]

for every \(\mu, \nu \in P(\Sigma(A))\), where \(\Gamma(\mu, \nu)\) is the set of all product measures of \(\mu\) and \(\nu\). Suppose 
that the spectrum \(\Sigma(A)\) also admits a \(\sigma\)-finite measure \(\lambda\) (perhaps Lebesgue measure). Let 
\(D(\Sigma(A))\) be the set of all density or distribution functions, i.e., non-negative functions \(\rho\) on 
\(\Sigma(A)\) so that

\[
\int_{\Sigma(A)} \rho d\lambda = 1.
\]

Every function \(\rho \in D(\Sigma(A))\) defines a probability measure \(\mu_\rho\) on \(\Sigma(A)\) according to

\[
\mu_\rho(E) := \int_E \rho d\lambda,
\]
for every Borel set \( E \subset \Sigma(A) \). The correspondence is not one-to-one as two density functions \( \rho, \rho' \in D(\Sigma(A)) \) which are different only in null sets lead to the same probability measure. Two density functions \( \rho \) and \( \rho' \) are said to be equivalent if both functions define the same probability measures. The set \( \mathcal{P}^2(\Sigma(A)) \), namely the set of all probability measures corresponding to equivalent classes of density functions, is a dense set in the Wasserstein space \( \mathcal{P}^2(\Sigma(A)) \) [3].

Now let \( L^2(\Sigma(A), d\lambda) \) be the Hilbert space of all equivalent classes of square \( \lambda \)-integrable complex functions defined on the spectrum \( \Sigma(A) \). If \( a \in \Sigma(A) \) is an element of the spectrum of the observable \( A \), then the value of a wave function \( \psi \) in \( a \), i.e. \( \psi(a) \) is the "component" of a vector \( |\psi\rangle \) contained in an abstract separable Hilbert space in the direction of an eigenvector \( |a\rangle \) belonging to \( a \). If \( \psi \in L^2(\Sigma(A), d\lambda) \) is a normalized wave function, then \( |\psi|^2 \in D(\Sigma(A)) \). There is therefore a correspondence \( |\psi\rangle \mapsto |\psi|^2 \) from the set of all quantum states into the set of all density functions defined on the spectrum \( \Sigma(A) \). If \( |\psi|^2 \) and \( |\varphi|^2 \) are equal almost everywhere, then the associated wave functions \( \psi \) and \( \varphi \) are different only in their phase factors. It means that both wave functions represent the same quantum state. Furthermore, whenever \( \rho \in [\rho] \) a density function defined on \( \Sigma(A) \), then \( \sqrt{\rho} \exp(i\theta) \), is the quantum state containing wave functions \( \sqrt{\rho} \exp(i\theta) \), with \( \theta \in \mathbb{R} \). The mapping \( |\psi\rangle \mapsto ||\psi||^2 \) from the set of all quantum states onto the set of all equivalent classes of density functions is therefore bijective. In turn, the bijection \( |\psi\rangle \mapsto ||\psi||^2 \) determines an injection

\[
|\psi\rangle \mapsto \mu_\psi^A,
\]

from the set of all quantum states into the Wasserstein space \( \mathcal{P}^2(\Sigma(A)) \), where

\[
\mu_\psi^A(E) = \int_E |\psi|^2 d\lambda,
\]

for every Borel set \( E \subset \Sigma(A) \). It is easy to show that the Wasserstein space \( \mathcal{P}^2(\Sigma(A)) \) is a convex set in which all extremal points are Dirac measures. Note that the Dirac measure \( \delta_a \) centred at \( a \), for every \( a \in \Sigma(A) \), is not contained in \( \mathcal{P}^D(\Sigma(A)) \).

Secondly, let \( B \) be another observable with discreet spectrum \( \Sigma(B) = \{b_i|i \in I\} \), where \( I \) is an appropriate index set for the elements of the spectrum. Assuming the completeness of the self-adjoint operator \( \hat{B} \) associated to the observable \( B \), there exists an orthonormal basis \( \{\varphi_{\alpha}|i \in I, \alpha \in J_i\} \) constructed from eigenvectors of \( \hat{B} \), where \( J_i \) is the index set associated to the degeneracy of eigenvalue \( a_i \). According to standard quantum mechanics the probability that the measurement of \( B \) yields \( b_i \in \Sigma(B) \) when the system occupies state \( |\psi\rangle \) is given by

\[
P(b_i, |\psi\rangle) = \sum_{\alpha \in J_i} |\langle \varphi_{\alpha}|\psi\rangle|^2.
\]

It defines then a probability measure \( \mu_B^\psi \) on the suitable \( \sigma \)-algebra \( \mathcal{F}(\Sigma(B)) \) on \( \Sigma(B) \) by

\[
\mu_B^\psi(E) = \sum_{i \in I, b_i \in E} P(b_i, |\psi\rangle) = \sum_{i \in I, b_i \in E} \sum_{\alpha \in J_i} |\langle \varphi_{\alpha}|\psi\rangle|^2,
\]

for every \( E \in \mathcal{F}(\Sigma(B)) \). Let \( \mathcal{P}(\Sigma(B)) \) denote the set of all probability measures on \( \Sigma(B) \). It is also a convex set in which every extremal point is a Dirac measure. For every \( b_i \in \Sigma(B) \), the associated Dirac measure \( \delta_{b_i} \) is given by

\[
\delta_{b_i} = \mu_B^{\psi_i},
\]

where \( |\varphi_i\rangle \) is an arbitrary element of eigensubspace \( \mathcal{E}_i \) associated to \( b_i \).
The unitary isomorphism $\psi \mapsto (\langle \varphi_{i,\alpha} | \psi \rangle)_{i \in I, \alpha \in \Lambda}$ form $L^2(\Sigma(A), d\lambda)$ onto the standard Hilbert space $l^2$ of all square summable sequences of complex numbers induces then a bijection $F : \mu_\psi^A \mapsto \mu_\psi^B$ from the set $\mathcal{P}^D(\Sigma(A))$ of all probability measures on the spectrum $\Sigma(A)$ defined by quantum states onto the set $\mathcal{P}^D(\Sigma(B))$ of all probability measures on the spectrum $\Sigma(B)$ defined by quantum states. The bijection $F$ then induces a metric $w_2^B$ on $\mathcal{P}^D(\Sigma(B))$ defined by

$$w_2^B(\mu_\psi^B, \mu_\phi^B) := W_2^A(\mu_\psi^A, \mu_\phi^A), \tag{7}$$

for every $\mu_\psi^B, \mu_\phi^B \in \mathcal{P}^D(\Sigma(B))$. Then, the set $\mathcal{P}^D(\Sigma(B))$ equipped with the metric $w_2^B$ is a metric space so that $F$ and its inverse $F^{-1}$ are continuous isometries. It can be shown that the space $(\mathcal{P}^D(\Sigma(B)), w_2^B)$ is not complete as metric space. Let $(\mathcal{P}^2(\Sigma(B)), W_2^B)$ be the completion of $(\mathcal{P}^D(\Sigma(B)), w_2^B)$. Next we consider the standard extension lemma the proof of which can be found in standard book of topology as follows.

**Lemma 1** Let $(X, d_1)$ be a metric space and $V \subset X$ be a dense set. Furthermore, let $(Y, d_2)$ be a complete metric space and $f : V \to Y$ be a uniformly continuous mapping. Then there exists a unique mapping $\bar{f} : X \to Y$ which is an extension of $f$ to the whole of $X$.

The above lemma leads then to the existence of the generalized Fourier transformation as dictated in the following theorem.

**Theorem 1** There exists a homeomorphism map $\mathfrak{F} : \mathcal{P}^2(\Sigma(A)) \to \mathcal{P}^2(\Sigma(B))$ from the Wasserstein space $\mathcal{P}^2(\Sigma(A))$ over the spectrum of observable $A$ onto the space $\mathcal{P}^2(\Sigma(B))$ over the spectrum of observable $B$.

3. The Spectrums of Observables

A consequence of our view on the nature of measurement of an observable is the assumption of the existence of a set called the spectrum of the observable. It serves as sample space of the observable. It contains all possible outcomes of a measurement concerning the observable. In standard quantum mechanics, the spectrum of an observable is supposed to be the set of all eigenvalues of the self-adjoint operator representing the observable. Therefore, the spectrum of an observable in standard quantum mechanics are sets of numbers. The self-adjointness of the operator representing an observable leads to the fact that all eigenvalues of the operator are real numbers. The spectrums of some observables are obtained by solving the corresponding eigenvalue equations. On the other hand, the spectrums of the other observables are not obtained by solving eigenvalue problems. Instead of solving the corresponding eigenvalue equations, we merely pick some suitable sets of numbers to be their spectrum whose elements are assumed to obey the eigenvalue equations. A well known example is the case of position in which we merely assume that the real line to be the spectrum of each coordinate $x_i \ (i = 1, 2, 3)$ regarded as an observable. However, note that the possible outputs of the measurement of the position of a particle are points in space. Thus, the spectrum of the position of the particle is the set of all points in space. Indeed, to every point in space we assign a three numbers (i.e., the coordinates) to label the point. However, a point in space is not the three numbers assigned to it. Moreover, the assignment of three numbers to every point in space is not unique. The space where the particle is "living" is a differentiable manifold. Each coordinate function of a coordinate system containing in the differentiable structure is merely a random variable defined on an open set of the spectrum.

Let $\Sigma(A)$ be the spectrum of an observable $A$ which is a $n$-dimensional differentiable manifold and let $(U, \varphi)$ be a coordinate system on $\Sigma(A)$. The domain of a coordinate system is measurable with respect to Borel $\sigma$-algebra generated by open sets on $\Sigma(A)$ as it is an open set. The coordinate map $\varphi$ can be written as $\varphi = (\varphi^1, \cdots \varphi^n)$, where $\varphi^i : U \to \mathbb{R}$ are continuous
functions. A coordinate system \( (U, \varphi) \) is said to support the probability measure \( \mu \) if \( \mu(U^C) = 0 \). Since \( \varphi \) is a homeomorphism, each coordinate function \( \varphi^i \) is continuous and therefore it is a measurable function on \( U \). Every coordinate function \( \varphi^i \) is a random variable.

When the system occupies the state represented by \( \mu \), then the expectation value of the measurement of \( A \) in the \( \mu \)-supporting coordinate system \( (U, \varphi) \) is given by

\[
\varphi^{-1} \left( \langle A \rangle_{\mu}^{(U, \varphi)} \right) = \varphi^{-1} \left( \int_U \varphi^1 d\mu, \int_U \varphi^2 d\mu, \cdots, \int_U \varphi^n d\mu \right) \in \Sigma(A). \tag{8}
\]

The definition of the expectation value is well-defined because

\[
\int_U \varphi^i d\mu \in \varphi^i(U),
\]

for every \( i \in \{1, 2, \ldots, n\} \). If \( (\bar{U}, \bar{\varphi}) \) is another \( \mu \)-supporting coordinate system, then it is easy to show that

\[
\bar{\varphi}^{-1} \left( \langle A \rangle_{\mu}^{(\bar{U}, \bar{\varphi})} \right) = \varphi^{-1} \left( \langle A \rangle_{\mu}^{(U, \varphi)} \right). \tag{9}
\]

Thus, we can obtain the expectation value by making use of equation (8) in whichever \( \mu \)-supporting coordinate system we use. The expectation value of the measurement of \( A \) is written then as \( \langle A \rangle_{\mu} \). The coordinate deviation \( \Delta_{\mu}^{(V, \psi)} \) of an arbitrary coordinate system \( (V, \psi) \) relative to state \( \mu \) and a Riemannian metric \( g \) on \( \Sigma(A) \) is defined by

\[
\Delta_{\mu}^{(V, \psi)} = d_g \left( \psi^{-1} \left( \langle A \rangle_{\mu}^{(V, \psi)} \right), \langle A \rangle_{\mu} \right), \tag{10}
\]

where \( d_g \) is the geodesic metric on \( \Sigma(A) \) induced by \( g \) according to

\[
d_g(a, a') = \int_0^1 \sqrt{g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))} d\tau, \tag{11}
\]

where \( \gamma : (0, 1) \to \Sigma(A) \), with \( \gamma(0) = a \) and \( \gamma(1) = a' \), is the geodesic connecting \( a \) and \( a' \).

4. Dynamics: Schroedinger Equation and Newton Second Law of Motion

In the standard Schroedinger position representation of non-relativistic quantum mechanics, a state of of a single particle living in three dimensional space \( \mathbb{R}^3 \) is represented by a wave function \( \psi(r, t) \) which is a solution of Schroedinger equation. The wave function \( \psi \) can be written as

\[
\psi = \sqrt{\rho} \exp \left( \frac{imS_{\psi}}{\hbar} \right), \tag{12}
\]

where \( \rho = |\psi|^2 \) is the probability density associated to \( \psi \) and \( S_{\psi} \) a suitable function of position and time. It was shown by Madelung [4] that the Schroedinger equation is equivalent mathematically to a couple of equations:

\[
\frac{\partial \rho_{\psi}}{\partial t} + \nabla \cdot (\rho_{\psi} \nabla S_{\psi}) = 0, \tag{13}
\]

and

\[
\frac{\partial S_{\psi}}{\partial t} + \frac{1}{2} (\nabla S)^2 + \frac{F}{m} - \frac{\hbar^2}{2m^2} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} = 0, \tag{14}
\]

where \( F \) is the potential energy. Equation (13) is the continuity equation describing the flow of probability, whereas equation (14) can be regarded as Hamilton-Jacobi equation for the action \( S \).
of the particle. It is to notice that whether the above Madelung equations is equivalent physically to the Schrödinger equation is still under discussion [8].

The continuity equation can be generalised to the case of all probability measures on $\mathbb{R}^3$ as

$$\frac{\partial \mu_t}{\partial t} + \nabla \cdot (\mu_t \nabla S) = 0,$$  \hspace{1cm} (15)

where $S$ is a scalar field defined on $\mathbb{R}^3$. All the derivative appear in the above equation are weak derivative, i.e., for every test function $\varphi \in S_c(\mathbb{R}^3, \mathbb{R})$ we have

$$\int_{\mathbb{R}^3} \frac{\partial \varphi}{\partial t} \, d\mu_t + \int_{\mathbb{R}^3} (\nabla S \cdot \nabla \varphi) \, d\mu_t + \int_{\mathbb{R}^3} (\varphi \nabla^2 S) \, d\mu_t = 0.$$  \hspace{1cm} (16)

Now let $\Sigma(A)$ be the spectrum of an observable $A$ which is a $n$-dimensional differentiable manifold and $g$ a Riemannian tensor metric on $\Sigma(A)$. The continuity equation for probability measure on $\Sigma(A)$ is a generalisation of equation (15) and is given by

$$\frac{\partial \mu_t}{\partial t} + \nabla^g \cdot (\mu_t \mathbf{V}) = 0,$$  \hspace{1cm} (17)

where $\mathbf{V}$ is a vector field on $\Sigma(A)$. The last equation can be read in two different manners. Firstly, given a vector field $\mathbf{V}$ on $\Sigma(A)$, we find a local flow $(t, \mu) \mapsto \mu_t$ satisfying the equation. The local flow describes the dynamics of the system on $\mathcal{P}_2(\Sigma(A))$ generated by the vector field $\mathbf{V}$. Secondly, given a local flow $(t, \mu) \mapsto \mu_t$, we find a vector field $\mathbf{V}$ on $\Sigma(A)$ generating the flow.

Now consider a particle ‘living’ on a differentiable Riemannian manifold $(M, g)$. Let $\hat{\mathbf{R}}$ be the position observable of particle and $\Sigma(\hat{\mathbf{R}}) = M$. The Wasserstein metric $W^A_2(\mu, \nu)$ in $\mathcal{P}^2(M)$ given in equation (1) can be written as

$$W^A_2(\mu, \nu) := \left[ \inf_{\phi \in \Phi(\mu, \nu)} \int_0^1 \int_M |\nabla \phi_t(q)|^2 \mu(q) dq \, dt \right]^{1/2},$$  \hspace{1cm} (18)

where $\Phi(\mu, \nu)$ is the set of all pairs $(\phi, \bar{\mu})$ of curves $\phi \in C^\infty((0,1), M)$ and $\bar{\mu} \in C^\infty((0,1), \mathcal{P}^2(M))$ so that

$$\bar{\mu} = -div(\nabla \phi_t \bar{\mu}_t),$$  \hspace{1cm} (19)

with $t \in (0,1)$, $\bar{\mu}_0 = \mu$, and $\bar{\mu}_1 = \nu$ [1]. Then the metric $W^A_2(\mu, \nu)$ is associated to a formal Riemannian structure on $\mathcal{P}^2(M)$ given by

$$T_\mu \mathcal{P}^2(M) = \{ \psi : M \rightarrow \mathbb{R} : \int_M \psi(q) dq = 0 \}$$  \hspace{1cm} (20)

and

$$\|\psi\|^2_{T_\mu \mathcal{P}^2(M)} = \int_M |\nabla \phi(q)|^2 d\mu_t,$$  \hspace{1cm} (21)

for $\psi = -(\mu \nabla \phi)$, where $T_\mu \mathcal{P}^2(M)$ is the tangent space of $\mathcal{P}^2(M)$ at $\mu$ for every $\mu \in \mathcal{P}^2(M)$.

Furthermore, consider a dynamical system associated with the Riemannian metric on $\mathcal{P}^2(M)$ which is given by

$$\nabla^W_\mu \bar{\mu} = -\nabla^W F(\mu),$$  \hspace{1cm} (22)

where $\nabla^W$ is Levi-Civita connection on $\mathcal{P}^2(M)$ associated with the above defined Riemannian structure. Equation (22) can be regarded as the Newton second law of motion describing the probabilistic dynamics of a particle on the Riemannian manifold $M$. The right hand side of the
equation gives the 'force' generated by the potential energy $F$ and the left hand side gives the 'acceleration' of the particle. If the potential energy $F$ is of the form

$$F(\mu) = \int_M V(q)\mu(dq) + \frac{\hbar^2}{8} \int_m |\nabla \ln \mu|^2 d\mu(q)$$

(23)

where $V(q)$ is a Schroedinger potential, von Renesse [6] has shown that a solution $\mu(t)$ of the Newton equation of motion, i.e. equation (22), leads to a solution $\Psi(q,t)$ of the standard Schroedinger equation according

$$\Psi(q,t) = \sqrt{\mu(q,t)} \exp\left(\frac{i}{\hbar} S(q,t)\right),$$

(24)

where

$$\dot{S}(q,t) = S(q,t) + \int_0^t L_F(S_\sigma, \mu_\sigma) d\sigma,$$

(25)

$$L_F(\psi) = \frac{1}{2} \|\psi\|_{\mathcal{P}_2(M)}^2 - F(\mu),$$

(26)

$\mu(q,t)$ is the probability density associated to $\mu_t$, and $S(q,t)$ is the so-called velocity potential of the flow $\mu$ satisfying $\int_M S d\mu = 0$.

Once the solution of (22) is obtained, the dynamics of the system described in the Wasserstein space over the spectrum of another observable can be determined by using the generalized Fourier transform depicted in Theorem 1. Let $B$ be another observable and $\mathfrak{F} : \mathcal{P}_2(M) \to \mathcal{P}_2(\Sigma(B))$ is the generalized Fourier transform. If $\mu : (a,b) \rightarrow \mathcal{P}_2(M)$ is the solution of (22), then $\mathfrak{F} \circ \mu$ is the curve in $\mathcal{P}_2(\Sigma(B))$ describing the same dynamics of the system. From the Fourier transform $\mathfrak{F}$ and the Newtonian equation of motion in the $\mathcal{P}_2(M)$ with the potential $F(\mu)$ given by (23), we also obtain the Newtonian equation of motion in the Wasserstein space $\mathcal{P}_2(\Sigma(B))$ over the spectrum $\Sigma(B)$ of the observable $B$ by using suitable maps derived from our generalized Fourier transform $\mathfrak{F}$ as follows

$$\nabla_{\mu_A}^{WA} \mu_A = -\nabla^{WA} F_A(\mu_A),$$

(27)

where

$$\nabla_{\mu_A}^{WA} \mu_A = \left[\left(\mathfrak{F} \circ \nabla^W \mathfrak{F}^{-1}\right) \circ \mathfrak{F}^{-1}\right](\mu_A) \circ \mathfrak{F}^{-1}$$

(28)

and

$$\nabla^{WA} F_A(\mu_A) = \left[\left(\mathfrak{F} \circ \nabla^W \mathfrak{F}^*\right) F_A(\mu_A)\right] \circ \mathfrak{F}^{-1}.$$  

(29)

By using the regularity of the generalized Fourier transform $\mathfrak{F}$ and the Equation (27), the properties of the dynamics $\mathfrak{F} \circ \mu$ in $\mathcal{P}_2(\Sigma(B))$ can be studied further.

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