Principal chiral model scattering and the alternating quantum spin chain

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Abstract

We consider the critical alternating quantum spin chain with $q^+$, $q^-$ spins. Using the Bethe ansatz technique we find explicit expressions for the $S$-matrix of the model. We show that in the limit that $q_{\pm} \to \infty$ our results coincide with the ones obtained for the principal chiral model level one, for the LL (RR) LR scattering. We also study the scattering of the bound states of the model and we recover the results of the XXZ (sine-Gordon) model.

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1 Introduction

In the famous paper [1] the principal chiral model (PCM) with $SU(2)$ symmetry with Lagrangian density

$$L_0 = \frac{1}{2\alpha_0} tr(\partial_\mu g^{-1}(x)\partial^\mu g(x))$$

(1)

were firstly shown to be equivalent to a Gross-Neveu (GN) fermionic model

$$\tilde{L}_0 = \sum_{a=1}^N \bar{\psi}_a \gamma_\mu \partial^\mu \psi_a + \lambda_0 \left( \sum_{a=1}^N \bar{\psi}_a \gamma_\mu \tau^i \psi_a \right)^2$$

(2)

in the limit of infinite number $N$ of fermionic colors ($\tau^i$ are Pauli matrices here). Secondly, in order to avoid ambiguities related to the axial anomaly, the fermionic model was conjectured to be equivalent to the single fermionic field with isotopic spin $S$ ($N = 2S$). The lattice analogue of the last model is the spin $S$ quantum spin chain exactly solvable by the Bethe ansatz method. This scheme was successfully applied to PCM and GN models with different symmetries [2], [3], [4], [5]. Another modification of this scheme can be used for analysis of the PCM with topological term [4]

$$L_1 = \frac{1}{2\alpha_0} tr(\partial_\mu g^{-1}(x)\partial^\mu g(x)) + n\epsilon_{\mu\nu} \int_0^1 dtd^2x \, tr \left( g^{-1}\dot{g}g^{-1}\partial^\mu gg^{-1}\partial^\nu g \right)$$

(3)

where in the second term $g(x, t)$ interpolates between $g(x, 0) = 1$ and $g(x, 1) = g(x)$ and $n$ is an integer. This model was shown to be equivalent to fermionic GN model in the limit of infinite number of fermions, with broken chiral symmetry, i.e. different number of colors for left and right fermions:

$$\tilde{L}_1 = \sum_{a=1}^{q+} u_a^+ \partial_+ u_a + \sum_{a=1}^{q-} v_a^+ \partial_- v_a + \lambda_0 \sum_{a=1}^{q+} \sum_{b=1}^{q-} \left( u_a^+ \lambda^i u_a \right) \left( v_b^l \lambda^j v_b \right)$$

(4)

where $u_a(v_a)$ are the left(right) Dirac fermions, $\lambda_0 = -\frac{1}{2\alpha_0}$, $\lambda^i$ - the generators of symmetry group and the difference of fermionic color $a$ numbers is $q_+ - q_- = n$. Here we are focused on the case where $q_+ - q_- = 1$, which corresponds to the PCM level one. The corresponding statistical model for this case is the critical alternating spin chain with $\frac{q_+ + q_-}{2}$, $\frac{q_+ - q_-}{2}$ spins (see e.g. [4]). We are going to find the explicit $S$-matrix for the spin chain, using the Bethe ansatz method, and we will reproduce the proposed massless $S$-matrix for the model [3], see [3].
2 Bethe ansatz equations

2.1 Ground state and excitations

We focus here on the alternating spin chain with \( \frac{q_+}{2}, \frac{q_-}{2} \) spins. In order to introduce a mass scale to our system we consider the chain with inhomogeneities \([8], [9]\), namely

\[ \omega_j = (-)^j \frac{1}{\alpha_0}, \]  

where \( j \) is \( q_{\pm} \) and let us also consider that \( q_+ \) is even (obviously \( q_- \) is odd). In other words in the sites of the chain with \( q_+ \) spin \( \omega_{q_+} = \frac{1}{\alpha_0} \) whereas in the sites with \( q_- \) spin \( \omega_{q_-} = -\frac{1}{\alpha_0} \). Following the standard Bethe ansatz technique \([5], [8]-[14]\) for the model (3), one gets Bethe equations in the form

\[ e_{q_+}(\lambda_\alpha - \frac{1}{\alpha_0})^{N_+} e_{q_-}(\lambda_\alpha + \frac{1}{\alpha_0})^{N_-} = - \prod_{\beta=1}^{M} e_2(\lambda_\alpha - \lambda_\beta) \]  

where

\[ e_n(\lambda; \nu) = \frac{\sinh \mu(\lambda + \frac{in}{2})}{\sinh \mu(\lambda - \frac{in}{2})} \]  

where \( \nu = \frac{\pi}{\mu} \) is the anisotropy parameter. The spin, energy and momentum of a state are characterized by the set of quasi particles with rapidities (BA roots) \( \lambda_\alpha \), \([7], [12], [8], [15]\)

\[ E = -\frac{\mu}{2\pi} \sum_{j=1}^{M} \sum_{n=q_+,q_-} \frac{\sin \mu n}{\sinh \mu(\lambda_j - \frac{(-)^n}{\alpha_0} + \frac{in}{2}) \sinh \mu(\lambda_j - \frac{(-)^n}{\alpha_0} - \frac{in}{2})} \]  

\[ P = \frac{1}{2i} \sum_{j=1}^{M} \sum_{n=q_+,q_-} \log \frac{\sinh \mu(\lambda_j - \frac{(-)^n}{\alpha_0} + \frac{in}{2})}{\sinh \mu(\lambda_j - \frac{(-)^n}{\alpha_0} - \frac{in}{2})} \]  

\[ S^z = N_- q_- + N_+ q_+ - M. \]  

We consider here the special case where \( N_+ = N_- = N \). We are interested in the limit \( N_+, q_+, \nu \to \infty \). The thermodynamic limit \( N_+ \to \infty \) of equation (3) can be studied by standard methods. The string hypothesis says that solutions of (3) in the thermodynamic limit are grouped into strings of length \( n \) with the same real part and equidistant imaginary parts

\[ \lambda^{(n,j)}_\alpha = \lambda^n_\alpha + \frac{i}{2}(n + 1 - 2j) + i \frac{\pi}{4\mu} (1 - \nu), \quad j = 1, 2, ..., n \]  

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where $\lambda_\alpha^n$ is real and $v = \pm 1$ determines the parity of the string (positive negative parity string). We assume that $\nu > q_\pm$. The ground state consists of two Dirac “filled seas” by strings of length $q_\pm$. Therefore the Bethe ansatz equations for the ground state become,

$$\prod_{j=q_+q_-} X_{nj}(\lambda_\alpha^n - \frac{(-)^j}{\alpha_0})^N = \prod_{j=q_+q_-} \prod_{\beta=1}^{M_j} E_{nj}(\lambda_\alpha^n - \lambda_\beta^j)$$

(12)

where $n$ can be $q_\pm$, and

$$X_{nm}(\lambda) = e_{|n-m+1|}(\lambda)e_{|n-m+3|}(\lambda) \cdots e_{(n+m-3)}(\lambda)e_{(n+m-1)}(\lambda)$$

$$E_{nm}(\lambda) = e_{|n-m|}(\lambda)e_{n-|m+2|}(\lambda) \cdots e_{2(n+m-2)}(\lambda)e_{n+m}(\lambda).$$

(13)

For what follows it is necessary to introduce the following notations

$$g_n(\lambda; \nu) = e_n(\lambda \pm \frac{i\pi}{2\mu}) = \frac{\cosh(\lambda + \frac{i\pi}{2\mu})}{\cosh(\lambda - \frac{i\pi}{2\mu})},$$

(14)

$$G_{nm}(\lambda) = g_{|n-m|}(\lambda)g_{|n-m+2|}(\lambda) \cdots g_{2(n+m-2)}(\lambda)g_{n+m}(\lambda),$$

(15)

$$a_n(\lambda; \nu) = \frac{1}{2\pi} \frac{d}{d\lambda} i \log e_n(\lambda; \nu), \quad b_n(\lambda; \nu) = \frac{1}{2\pi} \frac{d}{d\lambda} i \log g_n(\lambda; \nu)$$

(16)

and the Fourier transforms of $a_n, b_n$ are given by

$$\hat{a}_n(\omega; \nu) = \frac{\sinh((\nu - n)\frac{\omega}{2})}{\sinh(\frac{\omega}{2})} \quad 0 < n < 2\nu,$$

(17)

$$\hat{b}_n(\omega; \nu) = -\frac{\sinh(\frac{\nu\omega}{2})}{\sinh(\frac{\omega}{2})} \quad 0 < n \leq \nu, = -\frac{\sinh((n - 2\nu)\frac{\omega}{2})}{\sinh(\frac{\nu}{2})} \quad \nu < n < 3\nu.$$
\[ \hat{B}_{nm}(\omega) = -\frac{2\coth(\frac{\omega}{2}) \sinh\left(\frac{m\omega}{2}\right) \sinh\left(\frac{n\omega}{2}\right)}{\sinh\left(\frac{\omega}{2}\right)}. \]  

(22)

Finally, the energy \( E \) takes the form

\[ E = -\sum_{i,j=q_+,q_-}^{N} \sum_{\alpha=1}^{M_i} Z_{ij} (\lambda^\alpha_i - \frac{(-)^j}{\alpha_0}). \]  

(23)

Now we are going to solve the Bethe ansatz equations. We take the logarithm and the derivative of the Bethe ansatz equations and we also use the Maclaurin expansion

\[ \sum_{j=1}^{N} f(\lambda_j) \sim N \int_{-\infty}^{\infty} f(\lambda) \sigma(\lambda) d\lambda - \sum_{j=1}^{u} f(\lambda_j), \]  

(24)

where \( \sigma \) is the density of the state and \( u \) is the number of holes. Then for the densities of the ground state (no holes) we obtain the following integral equations,

\[ \sigma^n_0(\lambda) = \sum_{j=q_+,q_-} Z_{nj}(\lambda - \frac{(-)^j}{\alpha_0}) - \sum_{j=q_+,q_-} (A_{nj} \ast \sigma^n_0)(\lambda) \]  

(25)

where \( \ast \) stands for the convolution and \( n \) can be \( q_+ \) or \( q_- \). We can easily solve the last equations (25) and we find that

\[ \sigma^n_0(\lambda) = s(\lambda - \frac{(-)^n}{\alpha_0}) = \frac{1}{2\cosh\left(\pi (\lambda - \frac{(-)^n}{\alpha_0})\right)}, \]  

(26)

where its Fourier transform is

\[ \hat{s}(\omega) = \sum_{i,j=q_+,q_-} (\hat{Z}_{ij} \hat{R}_{nj})(\omega) = \frac{1}{2\cosh(\frac{\omega}{2})}. \]  

(27)

Here \( R \) is the inverse of the kernel \( K \) of the system of the linear equations (25), in particular,

\[ \hat{K}_{nm}(\omega) = (1 + \hat{A}_{nm}(\omega)) \delta_{nm} + \hat{A}_{nm}(\omega)(1 - \delta_{nm}) \]  

(28)

\[ \hat{R}_{nm}(\omega) = \frac{1}{\det K} \sum_{j=q_+,q_-} ((1 + \hat{A}_{jj}(\omega)) \delta_{nm}(1 - \delta_{nj}) - \hat{A}_{nm}(\omega)(1 - \delta_{nm})), \]  

(29)

where the determinant of \( K \) is, in terms of trigonometric functions,

\[ \det K = \frac{4\coth^2(\frac{\omega}{2}) \sinh\left(\frac{\nu - q_+}{2}\right) \sinh\left(\frac{\nu - q_-}{2}\right) \sinh\left(\frac{\omega}{2}\right)}{\sinh\left(\frac{\omega}{2}\right)}. \]  

(30)

Now we consider the low lying excitations of the model. There can be holes in both seas of \( q_\pm \) strings and also strings of other length and parity (see equation \( \text{[1]} \)). Let us consider
the state with $u_\pm$ holes in the $q_\pm$ sea. The contribution of the holes modifies the densities of the states in the following way (see (24)),

$$\sigma^n(\lambda) = \sigma^n(\lambda) + \frac{1}{N} \sum_{i=q_+q_-} u_i \sum_{\alpha=1} K_1^{ni}(\lambda - \lambda^\alpha_i),$$

where $\sigma^n(\lambda) = s(\lambda - \frac{(-)^n}{\alpha_0})$

$$K_1^{ni}(\omega) = \sum_{j=q_+q_-} (\hat{A}_{ij} \hat{R}_{nj})(\omega),$$

In particular, $K$ in (31) have the following explicit form in terms of trigonometric functions

$$\hat{K}_1^{q_+q_+}(\omega) = \frac{\sinh((\tilde{\nu} - 2)\omega)}{2\cosh(\frac{1}{2})\sinh((\tilde{\nu} - 1)\omega)}, \quad \hat{K}_1^{q_+q_-}(\omega) = \frac{1}{2\cosh(\frac{1}{2})}$$

with $\tilde{\nu} = \nu - q_-$, and also

$$\hat{K}_1^{q_-q_+}(\omega) = \frac{1}{2\cosh(\frac{1}{2})}, \quad \hat{K}_1^{q_-q_-}(\omega) = \frac{\sinh((q_+ - 2)\omega)}{2\cosh(\frac{1}{2})\sinh((q_+ - 1)\omega)}$$

The energy of the state with $u_\pm$ holes in the $q_\pm$ seas is given (see (23)) by

$$E = E_0 + \sum_{j=q_+q_-} u_j \sum_{\alpha=1} \epsilon^j(\lambda^\alpha_j),$$

where $E_0$ is the energy of the ground state and $\epsilon^\pm(\lambda)$ is the energy of the hole in $q_\pm$ sea (we write $\epsilon^\pm$ instead of $\epsilon^q_\pm$ for simplicity). Finally, we compute the spin of the holes, and we can see that the spin of a hole in $q_+$ sea is from (10) $s^+ = \frac{1}{2}$, where $f$ is just a factor $f = 1 + \frac{u_\pm}{\nu - 1}$ which is obviously 1 for $\nu \to \infty$. Therefore, we consider the spin of the hole to be $\frac{1}{2}$ for what follows. The spin of a hole in $q_-$ sea is see (10) $s^- = 0$. Finally, we conclude that the hole in the $q_\pm$ sea is a particle like excitation with energy $\epsilon^\pm$, momentum $p^\pm$ ($e^n(\lambda) = \frac{1}{\pi} \frac{d}{d^\lambda} \rho^n(\lambda)$)

$$e^n(\lambda) = \frac{1}{2\cosh(\pi(\lambda - \frac{(-)^n}{\alpha_0})), \quad p^n(\lambda) = -\frac{\pi}{4} + \frac{1}{2} \tan^{-1}\left(\sinh\pi(\lambda - \frac{(-)^n}{\alpha_0})\right),$$

and spin $s^+ = \frac{1}{2}, s^- = 0$. We can easily check that in the scaling limit where $\lambda << \frac{1}{\alpha_0}$ the energy and momentum become from (26) and (36), see also [8, 9],

$$\epsilon^+(\lambda) = p^+(\lambda) \sim e^{-\frac{\lambda}{\alpha_0}} e^{\pi\lambda}, \quad \epsilon^-(\lambda) = -p^-(\lambda) \sim e^{-\frac{\lambda}{\alpha_0}} e^{-\pi\lambda}.$$
These are the energy and momentum of the “right” and “left” movers respectively (see e.g. [6]). The factor $e^{-\frac{\pi}{\alpha_0}}$ provides a mass scale for the system.

As we mentioned before there are also states that consist of strings. There exist two types of strings as we see from equation (11) (positive and negative parity). The presence of the string gives rise to some extra terms in the Bethe equations which give the $1/N$ contribution to the density, in particular the Bethe ansatz equations in the presence of a positive parity $l$-string become,

$$\prod_{j=q_+, q_-} X_{nj}(\lambda^n_{\alpha} - \frac{(-)^j}{\alpha_0})^N = \prod_{j=q_+, q_-} \prod_{\beta=1} M_j E_{nj}(\lambda^n_{\alpha} - \lambda^j_{\beta})E_{nl}(\lambda^n_{\alpha} - \lambda^0_0)$$

(38)

where $\lambda_0$ is the real center of the string. The densities of the state are

$$\sigma^n(\lambda) = \sum_{j=q_+, q_-} Z_{nj}(\lambda - \frac{(-)^j}{\alpha_0}) - \sum_{j=-,+} (A_{nj} * \sigma^j(\lambda)) - \frac{1}{N} A_{nl}(\lambda - \lambda_0),$$

(39)

we solve the system of the linear equations and we find that:

$$\sigma^n(\lambda) = e^n(\lambda) - \frac{1}{N} \hat{K}^{nl}_+(\lambda - \lambda_0),$$

(40)

where

$$\hat{K}^{nl}_+(\omega) = \sum_{j=q_+, q_-} (\hat{A}_{lj}\hat{R}_{nj})(\omega).$$

(41)

We have to consider two different cases for the positive parity strings. In particular, we choose the following strings, namely, the $l = q_+ + 1$ and $l = q_- - 1$. For each string we obtain different densities, i.e. for the $l = q_+ + 1$ string,

$$\hat{K}^{q_++(q_+ + 1)}(\omega) = -\frac{\sinh((\tilde{\nu} - 2)\frac{\omega}{2})}{\sinh((\tilde{\nu} - 1)\frac{\omega}{2})}, \quad \hat{K}^{q_-+(q_+ + 1)}(\omega) = 0$$

(42)

and for $l = q_- - 1$

$$\hat{K}^{q_++(q_- - 1)}(\omega) = 0, \quad \hat{K}^{q_-+(q_- - 1)}(\omega) = -\frac{\sinh((q_+ - 2)\frac{\omega}{2})}{\sinh((q_+ - 1)\frac{\omega}{2})}.$$
we find that $E = E_0$, therefore we conclude that this is a state with zero energy contribution. The spin (10) of the $q_+ + 1$ string is $s = -1$ (again we omit the factor $f$ for the spin) and the spin of the $q_- - 1$ string is zero. Finally, we study the state with an one negative parity string. The Bethe ansatz equations then become

$$\prod_{j=q_+,q_-} X_{nj} (\lambda^n_{\alpha} - \frac{(-)^j}{\alpha_0}) N = \prod_{j=q_+,q_-} \prod_{\beta=1}^{M_j} E_{nj} (\lambda^n_{\alpha} - \lambda_{\beta}^n) G_{n1} (\lambda^n_{\alpha} - \lambda_0)$$

(45)

Similarly to the previous case the densities of the state are

$$\sigma^n(\lambda) = \epsilon^n(\lambda) - \frac{1}{N} K^n_{-\lambda_0}(\lambda - \lambda_0),$$

(46)

where

$$\hat{K}^{n1}_{-\lambda_0}(\omega) = \sum_{j=q_+,q_-} (\hat{B}_{1j} \hat{R}_{nj})(\omega).$$

(47)

Then the $\frac{1}{N}$ contribution of the densities becomes

$$\hat{K}^{q_+1}_{\omega}(\omega) = \frac{\sinh(\frac{\omega}{2})}{\sinh((\tilde{\nu} - 1)\frac{\omega}{2})}, \quad \hat{K}^{q_-1}_{\omega}(\omega) = 0.$$ 

(48)

The energy for the string is given by

$$E = - \sum_{i,j=q_+,q_-} \sum_{\alpha=1}^{M_j} Z_{ij} (\lambda^n_{\alpha} - \frac{(-)^j}{\alpha_0}) - b_{q_+} (\lambda_0 - \frac{1}{\alpha_0}) - b_{q_-} (\lambda_0 + \frac{1}{\alpha_0}),$$

(49)

and one can show from (49), (46) and (24) that the negative string has also zero energy contribution. Again the spin of this string is $s = -1$. These are all the necessary states to describe the scattering processes for the specific model. Note that the inhomogeneities modify only the energy and the momentum of the model. However, the $\frac{1}{N}$ contributions to the densities, which essentially determine the scattering amplitudes, remain unaffected by the presence of the inhomogeneities. In other words the computation of the $S$-matrix is not affected by the presence or not of inhomogeneities in the spin chain. Therefore, the $S$-matrix holds true for any $\lambda$, and we do not have to consider the scaling limit.

### 2.2 Physical $S$ matrix

Having studied the excitations of the model in the previous section we are ready to compute the complete $S$-matrix. To do so we follow the formulation developed by Korepin, and later
by Andrei and Destri [10]. First we have to consider the so called quantization condition.

\[ (e^{2iNp}S - 1)|\tilde{\lambda}_1, \tilde{\lambda}_2\rangle = 0 \]  

(50)

where \( p \) is the momentum of the particle, the hole in our case. For the case of two holes in \( q_+ \) sea (this is the triplet state \( s = 1 \)), we compare the integrated density (31) with the quantization condition. Having also in mind that,

\[ \epsilon^n(\lambda) = \frac{1}{\pi} \frac{d}{d\lambda} p^n(\lambda) \]  

(51)

we end up with the following expression for the scattering amplitude

\[ S_0(\tilde{\lambda}) = \exp\{2\pi i \sum_{i=1}^{Np} K_{1}^{q_+\bar{q}_+}(\lambda - \tilde{\lambda}_i)\} \]  

(52)

where \( \tilde{\lambda} = \tilde{\lambda}_1 - \tilde{\lambda}_2 \) and \( \tilde{\lambda}_1, \tilde{\lambda}_2 \) are the rapidities of the holes. In terms of trigonometric functions (33)

\[ S_0(\tilde{\lambda}) = \exp\{-2\pi i \int_{-\infty}^{\infty} e^{-i\omega\tilde{\lambda}} \frac{\sinh\left((\tilde{\nu} - 2)\frac{\lambda}{2}\right)}{2\cosh\left(\frac{\lambda}{2}\right) \sinh\left((\tilde{\nu} - 1)\frac{\lambda}{2}\right)} d\omega\} = \]

\[ \prod_{j=1}^{\infty} \frac{\Gamma\left(\frac{\lambda_j + 2j - 2}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 1}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 1}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 2}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 1}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 2}{\nu - 1} + 1\right)}{\Gamma\left(\frac{\lambda_j + 2j - 1}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 1}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 1}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 1}{\nu - 1} + 1\right) \Gamma\left(\frac{\lambda_j + 2j - 1}{\nu - 1} + 1\right)}. \]  

(53)

To get the singlet we consider the state with two holes in the \( q_+ \) sea and the \( q_+ + 1 \) string. Looking at the spins of the hole in \( q_+ \) sea and the spin of the \( q_+ + 1 \) string we conclude that this is indeed a state with spin zero. Note that the real center of the string is in the middle of the two holes \( (\lambda_0 = \frac{\tilde{\lambda}_1 + \tilde{\lambda}_2}{2}) \). We conclude from (31), (33) and (40), (42) that

\[ S_+ (\tilde{\lambda}) = \exp\{-2\pi i \int_{-\infty}^{\infty} K_{1}^{q_+\bar{q}_+}(\lambda - \lambda_0)\} S_0(\tilde{\lambda}) \]  

(54)

and after some algebra we get,

\[ S_+ (\tilde{\lambda}) = \frac{\sinh\left(\frac{\pi}{2(\tilde{\nu} - 1)} (\tilde{\lambda} + i)\right)}{\sinh\left(\frac{\pi}{2(\tilde{\nu} - 1)} (\tilde{\lambda} - i)\right)} S_0(\tilde{\lambda}). \]  

(55)

However, we need another state of spin zero see [17], and this is the state with two \( q_+ \) holes and one negative parity string with its real center in the middle of the holes. For which we obtain (40)-(48)

\[ S_- (\tilde{\lambda}) = \frac{\cosh\left(\frac{\pi}{2(\tilde{\nu} - 1)} (\tilde{\lambda} + i)\right)}{\cosh\left(\frac{\pi}{2(\tilde{\nu} - 1)} (\tilde{\lambda} - i)\right)} S_0(\tilde{\lambda}). \]  

(56)
The equations (53), (55) and (56) give the XXZ S-matrix (sine-Gordon S-matrix, provided that \( \beta^2 = 8\pi(1 - \frac{1}{\bar{\nu}}) \) and \( \bar{\lambda} = \frac{\theta}{\pi} \) [9], [12], [17], [18]. Moreover, we have holes in the \( q_- \) sea, which are particles with zero spin. The scattering between two such holes (see (31), (34)) is described by the following S-matrix

\[
S'_0(\bar{\lambda}) = \exp\left\{ -\int_{-\infty}^{\infty} e^{-i\omega\bar{\lambda}} \frac{\sinh\left(\frac{(q_+ - 2)\omega}{2}\right)}{2\cosh\left(\frac{\omega}{2}\right)\sinh\left(\frac{(q_+ - 1)\omega}{2}\right)} \frac{d\omega}{\omega} \right\}, \tag{57}
\]

which, in the limit \( q_\pm \to \infty \), corresponds to the XXX scattering amplitude [4], namely,

\[
S'_0(\bar{\lambda}) = \exp\left\{ -\int_{-\infty}^{\infty} e^{-i\omega\bar{\lambda}} \frac{e^{-\frac{\omega}{2}}}{2\cosh\left(\frac{\omega}{2}\right)} \frac{d\omega}{\omega} \right\} = \frac{\Gamma\left(i\frac{\bar{\lambda}}{2}\right)\Gamma\left(\frac{1}{2} - i\frac{\bar{\lambda}}{2}\right)}{\Gamma\left(-i\frac{\bar{\lambda}}{2}\right)\Gamma\left(\frac{1}{2} + i\frac{\bar{\lambda}}{2}\right)}. \tag{58}
\]

If we add a \( q_+ + 1 \) string in the above state the only thing that changes is the spin of the state which becomes -1, the scattering amplitude remains unaffected. There is one more state which completes the correspondence to the XXX S-matrix: the state with two holes in the \( q_- \) sea and a positive parity \( q_- - 1 \) string (spin zero state). For such state the change in \( \sigma'^- \) density gives rise to an extra phase, namely,

\[
S'_+(\bar{\lambda}) = \frac{\bar{\lambda} + i}{\bar{\lambda} - i} S'_0(\bar{\lambda}). \tag{59}
\]

The last two equations give the XXX S-matrix (see e.g [7], [8]). The equations (53) - (59) give actually the correct \( S_{XXZ} \times S_{XXX} \) matrix [9]. In the limit \( \bar{\nu} \to \infty \) the XXZ matrix reduces to the XXX S-matrix, i.e. the matrix given by (58), (59). Therefore we end up with the \( SU(2)_L \times SU(2)_R \) S-matrix [1], [18], [19]. Finally, let us consider the state that consists of two holes in \( q_+ \) sea and two holes in \( q_- \) sea (we consider even number of holes in each sea). The scattering between the two different holes is described (31), (33),

\[
S_{LR}(\bar{\lambda}) = \exp\left\{ -\int_{-\infty}^{\infty} e^{-i\omega\bar{\lambda}} \frac{1}{2\cosh\left(\frac{\omega}{2}\right)} \frac{d\omega}{\omega} \right\}, \tag{60}
\]

which can be written in the following form

\[
S_{LR}(\bar{\lambda}) = \tanh\left(\frac{\pi}{2}(\bar{\lambda} - i)\right), \tag{61}
\]

the same result (up to an overall factor) is obtained for the spin 1/2, 1 alternating isotropic spin chain see [13]. The last result (61) agrees with the proposed S-matrix for the left right scattering [4]. We identified the \( SU(2)_L \times SU(2)_R \) S-matrix and the \( S_{LR} \) matrix as well. This is the proposed S-matrix that describes the massless scattering for the PCM level one.
3 Bound states

It is well known for the XXZ model that for specific values of the anisotropy parameter there exist bound states of “kink-anti kink”. The same happens for the alternating spin chain as well. In particular, for the $S_{XXZ}$ matrix (53), (55), (56) and for $q_+ + 1 > \nu$, ($\tilde{\nu} < 2$ which corresponds to $\beta^2 < 4\pi$) there exist poles in the physical strip, $\lambda_p = i - i k (\tilde{\nu} - 1)$, $k = 1, 2, \ldots < \frac{1}{\tilde{\nu} - 1}$ see also e.g. [18] for a more detailed analysis. We are going to study here the first breather $k = 1$ which corresponds to a negative parity string of length one. For $\tilde{\nu} < 2$ (obviously here we keep $\tilde{\nu}$ finite) the Fourier transform of $B_{1q+}$ changes, namely

$$\hat{B}_{1q+}(\omega) = \frac{2 \sinh \left( (\nu - q_+) \frac{\omega}{2} \right) \cosh \left( (\nu - 1) \frac{\omega}{2} \right)}{\sinh \left( \frac{\omega}{2} \right)}. \quad (62)$$

Having that in mind and also the equations (46), (47) we conclude that the $\frac{1}{N}$ contribution of the densities for the breather state are,

$$\hat{K}^{q+1}(\omega) = - \frac{\cosh \left( (\tilde{\nu} - 1) \frac{\omega}{2} \right)}{\cosh \left( \frac{\omega}{2} \right)}, \quad \hat{K}^{q-1}(\omega) = \frac{\cosh \left( (\tilde{\nu} - 2) \frac{\omega}{2} \right)}{\cosh \left( \frac{\omega}{2} \right)}. \quad (63)$$

The negative parity one string for $\tilde{\nu} < 2$ is a state with positive energy and zero spin and it corresponds to a breather. In particular the energy of a breather is see (19), (44)

$$\epsilon_1(\lambda) = \frac{1}{2} \left( \frac{1}{\cosh \left( \pi \left( \lambda - \frac{1}{\alpha_0} - \frac{i(\tilde{\nu} - 2)}{2} \right) \right)} + \frac{1}{\cosh \left( \pi \left( \lambda - \frac{1}{\alpha_0} + \frac{i(\tilde{\nu} - 2)}{2} \right) \right)} \right), \quad (64)$$

where

$$\hat{\epsilon}_1(\omega) = \frac{\cosh \left( (\tilde{\nu} - 2) \frac{\omega}{2} \right)}{\cosh \left( \frac{\omega}{2} \right)}. \quad (65)$$

it is indeed positive and it is the same as in the case of the critical XXZ model in the “attractive regime” (see e.g. [12], [20]). In the scaling limit $\lambda << \frac{1}{\alpha_0}$ the energy and the momentum of the first breather become,

$$\epsilon_1(\lambda) = p_1(\lambda) \sim 2 \sin \left( \frac{\pi}{2} (\tilde{\nu} - 1) \right) e^{-\frac{\pi \omega}{\alpha_0}} e^{\pi \lambda}, \quad (66)$$

this is a dispersion relation for massless particle where the factor $2 \sin \left( \frac{\pi \omega}{2} (\tilde{\nu} - 1) \right) e^{-\frac{\pi \omega}{\alpha_0}}$ is the corresponding mass scale.  

\footnote{In the XXZ critical model with inhomogeneities $\omega_j = (-)^j \Theta$ (see e.g. [3], [8]) one can show that the mass of the first breather is exactly $2 \sin \left( \frac{\pi \omega}{2} (\tilde{\nu} - 1) \right) 2 e^{-\pi \Theta}$, which is also the mass of the sine-Gordon first breather, provided that $2 e^{-\pi \Theta}$ is the “kink” mass.}
We can write the Bethe ansatz equations for a state with two breathers,
\[ \prod_{j=q_+,q_-} g_j(\lambda - (-)^j) = \prod_{j=q_+,q_-} \prod_{\alpha=1}^{M_i} G_{ij}(\lambda - \lambda_{\alpha}^j) \prod_{i=1}^{2} e_2(\lambda - \lambda_i). \] (67)

The density will be
\[ -\tilde{\sigma}(\lambda) = b_{q_+}(\lambda - \frac{1}{\alpha_0}) + b_{q_-}(\lambda + \frac{1}{\alpha_0}) - \sum_{j=q_+,q_-} (B_{1j} \ast \sigma^j)(\lambda) - \frac{1}{N} \sum_{i=1}^{2} a_2(\lambda - \lambda_i) \] (68)
which we can write in the form,
\[ \tilde{\sigma}(\lambda) = \epsilon_1(\lambda) + \frac{1}{N} \sum_{i=1}^{2} K_b(\lambda - \lambda_i). \] (69)

The Fourier transform of $B_{1q_-}$ is given by equation (22), whereas the Fourier transform of $B_{1q_+}$ is given by (62). Therefore the Fourier transform of the $K_b$, after some algebra, is
\[ \hat{K}_b(\omega) = -\frac{\cosh\left( (2\tilde{\nu} - 3)\frac{\omega}{2} \right)}{\cosh(\frac{\omega}{2})}. \] (70)

To study the scattering between two breathers we consider the following quantization condition for a breather
\[ (e^{2Np_1 S - 1}|\lambda_1, \lambda_2) = 0 \] (71)
where $p_1$ is the momentum of the breather, then if we compare the last equation with (69)(after we integrate it) we conclude that
\[ S_b(\lambda) = \exp\left\{ \int_{-\infty}^{\infty} e^{-i\omega\lambda} \frac{\cosh\left( (2\tilde{\nu} - 3)\frac{\omega}{2} \right)}{\cosh(\frac{\omega}{2})} d\omega \right\} = \frac{\sinh(\pi \lambda) + i \sin(\pi (\tilde{\nu} - 1))}{\sinh(\pi \lambda) - i \sin(\pi (\tilde{\nu} - 1))}. \] (72)

The result agrees with the one obtained in XXZ (sine-Gordon) model, see e.g. [12], [18], [20]. Here, we restricted ourselves to the case of the “elementary” breather only. The above analysis can be generalized, in a similar way, for any general n-breather.

4 Discussion

We considered the alternating critical quantum spin chain with $\frac{q_+}{2}, \frac{q_-}{2}$ spins. We studied the excitations of the model and their scattering. We showed that in the limit that $q_\pm \to \infty$
the S-matrix we found coincides with the proposed one [6] for the PCM level one. The natural next step is the generalization of the above method for PCM model at any level. In this case, one expects a relevance of alternating fused RSOS models by analogy to [19]. A Bethe ansatz analysis similar to the one presented here, was done by Wiegmann [21] for the reproduction of massive S-matrix for O(3) σ model with θ = 0. It was done on the basis of special parameters limit of the spin S XXZ chain. In this context the interesting problem is a relation of the alternating spin chain to the massless S-matrix of O(3) σ model with θ = π [6] and to its relatives, recently proposed in [22] for other symmetric space σ-models with θ term. It would be also interesting to study the thermodynamics of generic alternating spin chain (the case of spin 1/2 spin 1 chain was studied in [23], [24], [25]) in order to see its relation to the thermodynamics of the corresponding field theoretical model, which was studied by methods of TBA. We hope to report on these issues in a future work.

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