NON-ALGEBRAIC GEOMETRICALLY TRIVIAL COHOMOLOGY CLASSES OVER FINITE FIELDS

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Abstract. We give the first examples of smooth projective varieties $X$ over a finite field $\mathbb{F}$ admitting a non-algebraic torsion $\ell$-adic cohomology class of degree 4 which vanishes over $\mathbb{F}$. We use them to show that two versions of the integral Tate conjecture over $\mathbb{F}$ are not equivalent to one another and that a fundamental exact sequence of Colliot-Thélène and Kahn does not necessarily split. Some of our examples have dimension 4, and are the first known examples of fourfolds with non-vanishing $H^3_{nr}(X, \mathbb{Q}_2/\mathbb{Z}_2(2))$.

1. Introduction

Let $\mathbb{F}$ be a finite field, $\ell$ be a prime number invertible in $\mathbb{F}$, and $X$ be a smooth projective geometrically connected $\mathbb{F}$-variety. We let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$, $G := \text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ and $\overline{X} := X \times_{\mathbb{F}} \overline{\mathbb{F}}$. We have the $\ell$-adic cycle maps:

\begin{align*}
(1.1) & \quad CH^2(X) \otimes \mathbb{Z}_\ell \rightarrow H^4(X, \mathbb{Z}_\ell(2)), \\
(1.2) & \quad CH^2(X) \otimes \mathbb{Z}_\ell \rightarrow H^4(\overline{X}, \mathbb{Z}_\ell(2))^G.
\end{align*}

From the commutative diagram

\begin{equation}
\begin{array}{c}
\begin{array}{ccc}
CH^2(X) \otimes \mathbb{Z}_\ell & \rightarrow & H^4(\overline{X}, \mathbb{Z}_\ell(2))^G \\
\downarrow & & \downarrow \\
H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))) & \rightarrow & H^4(X, \mathbb{Z}_\ell(2))
\end{array}
\end{array}
\end{equation}

where the exact row is induced by the Hochschild-Serre spectral sequence, we see that the surjectivity of (1.1) implies that of (1.2). The converse holds in all known examples [2, 12, 28, 35, 36, 37, 40]. In particular, the following question had remained open.

**Question 1.1.** Is it true that, as $X$ ranges over all smooth projective geometrically connected $\mathbb{F}$-varieties, (1.1) is surjective if and only if (1.2) is surjective?

In other words: Is it possible for the integral Tate conjecture to fail, (solely) due to the existence of non-algebraic geometrically trivial classes? If so, this would lead to a new interesting obstruction to the integral Tate conjecture over $\mathbb{F}$, which vanishes over $\overline{\mathbb{F}}$.

Combined with a theorem of Saito [38, Theorem 8.6] (see also [8, Proposition 3.2]), surjectivity of (1.1) for all three-dimensional $X$ implies Colliot-Thélène’s conjecture [8, Conjecture 2.2] on zero-cycles of prime-to-$\ell$ degree on smooth projective geometrically connected varieties over global function fields of arbitrary dimensions. A positive answer to **Question 1.1** would be a significant step towards Colliot-Thélène’s conjecture, reducing surjectivity of (1.1) to that of (1.2).
The \( \ell \)-adic cycle map (1.1) is related to the third unramified cohomology group \( H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \), a stable birational invariant of \( X \) which is known to be very difficult to compute. According to a theorem of Kahn [27, Théorème 1.1], if the group \( H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \) is trivial then the cokernel of (1.1) is torsion-free, hence (1.1) is surjective if the Tate conjecture in codimension 2 is true. Colliot-Thélène and Kahn showed that the converse follows from the Beilinson conjecture and a strong form of the Tate conjecture [10, Théorème 3.18].

It is a challenging problem to construct examples with \( H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \neq 0 \) of dimension as small as possible; see e.g. [29, 30, 35] and [9, §5.4]. The known examples of smallest dimension are due to Pirutka [35] and have dimension 5. On the other hand, \( H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \) vanishes if \( \dim(X) \leq 2 \). The following question naturally arises.

**Question 1.2.** Is it true that, for every smooth projective geometrically connected \( \mathbb{F} \)-variety of dimension \( \dim(X) \in \{3, 4\} \), we have \( H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0 \)?

Codimension 2 cycles and \( H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \) also appear in the following exact sequence of Colliot-Thélène and Kahn [10, Theorem 6.8]:

\[
\begin{align*}
0 & \to \text{Ker}(CH^2(X) \to CH^2(\overline{X})) (\ell) \xrightarrow{\varphi_\ell} H^1(\mathbb{F}, H^4(\overline{X}, \mathbb{Z}_\ell(2)))_{\text{tors}} \\
& \to \text{Ker}(H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \\
& \to \text{Coker}(CH^2(X) \to CH^2(\overline{X})^G) (\ell) \to 0,
\end{align*}
\]

(1.4) where the composition of \( \varphi_\ell \) with the natural map

\[
H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2)))_{\text{tors}} \to H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2)))
\]

coincides with (1.1). The following basic question was left open.

**Question 1.3.** Is it true that, for every smooth projective geometrically connected \( \mathbb{F} \)-variety \( X \), the map \( \varphi_\ell \) of (1.4) is surjective?

If \( \varphi_\ell \) were surjective, that is, an isomorphism, the sequence (1.4) would split and give even more precise information about the Galois descent of codimension 2 cycles on \( X \). In [11], several equivalent conditions for the surjectivity of \( \varphi_\ell \) were established, and it was proved that \( \varphi_\ell \) is surjective if \( X \) is the product of a smooth projective surface and an arbitrary number of smooth projective curves.

1.1. **Non-algebraic geometrically trivial cohomology classes.** Our first theorem shows that, despite the above expectations, the answers to **Question 1.1** and **Question 1.3** are negative in general.

**Theorem 1.4.** Let \( \mathbb{F} \) be a finite field, \( \ell \) be a prime number invertible in \( \mathbb{F} \), and suppose that \( \mathbb{F} \) contains a primitive \( \ell^2 \)-th root of unity. There exists a smooth projective geometrically connected \( \mathbb{F} \)-variety \( X \) of dimension \( 2\ell + 3 \) such that:

(i) the map (1.2) is surjective whereas the map (1.1) is not, and

(ii) the homomorphism \( \varphi_\ell \) is not an isomorphism for \( X \).

More precisely, we construct examples where the image of the composition

\[
H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2)))_{\text{tors}} \to H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))) \xrightarrow{\alpha} H^4(\overline{X}, \mathbb{Z}_\ell(2))
\]

contains a class \( \alpha \) whose mod \( \ell \) reduction is not algebraic.

**Theorem 1.4** provides new examples of smooth projective varieties \( X \) satisfying \( H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \neq 0 \), though the construction only works for \( \dim(X) \geq 7 \). Using ideas from the proof of **Theorem 1.4**, we give the first examples of dimension 4.
Theorem 1.5. Let $p$ be an odd prime. There exist a finite field $\mathbb{F}$ of characteristic $p$ and a smooth projective geometrically connected fourfold $X$ over $\mathbb{F}$ for which the image of the composition
\[ H^1(\mathbb{F}, H^4(X, \mathbb{Z}_2(2))_{\text{tors}}) \to H^1(\mathbb{F}, H^3(X, \mathbb{Z}_2(2))) \to H^4(X, \mathbb{Z}_2(2)) \]
contains a non-algebraic torsion class. In particular, $H^3_{\text{nr}}(X, \mathbb{Q}_2/\mathbb{Z}_2(2)) \neq 0$.

Thus Question 1.2 has negative answer in dimension 4; the question remains open in dimension 3. It should be noted that, over the complex numbers, examples of threefolds $X$ such that $H^3_{\text{nr}}(X, \mathbb{Q}_2/\mathbb{Z}_2(2)) \neq 0$ abound. Indeed, the Trento counterexamples to the integral Hodge conjecture over $\mathbb{C}$ due to Kollár [29] have non-trivial $H^3_{\text{nr}}$. Conjecturally, no such example can exist over $\overline{\mathbb{F}}$. Indeed, a celebrated theorem of Schoen [41] states that if the rational Tate conjecture for surfaces over finite fields is true, then the integral Tate conjecture for 1-cycles over $\overline{\mathbb{F}}$ is true in arbitrary dimension.

The example of Theorem 1.5 is based on a construction of Benoist and Ottem [6, Theorem 5.3]. For such $X$, the map (1.1) is not surjective, but we do not know whether (1.2) is surjective. Using étale cobordism, we show that our methods cannot be used to find an example of dimension 3; see Proposition 4.2.

1.2. New ingredients. We now explain the new ideas that go into the proof of Theorem 1.4. Over the complex numbers, Atiyah and Hirzebruch [5] proved that all odd-degree Steenrod operations vanish on algebraic classes, and used this to construct the first counterexamples to the integral Hodge conjecture. Their examples are Godeaux-Serre varieties: quotients of smooth complete intersections by the free action of a finite group. As shown by Totaro [12, Théorème 2.1], the method of Atiyah–Hirzebruch may be adapted to give counterexamples to the integral Tate conjecture over an arbitrary algebraically closed field. This method has later been refined by Pirutka–Yagita [36]. Of course, this does not say anything about Question 1.1, as the cohomology classes of interest all vanish over $\overline{\mathbb{F}}$.

The key new idea for our proof is that one can use Steenrod operations over $\mathbb{F}$ to detect non-algebraic geometrically trivial classes. More precisely, we are able to prove the following analogue of the Atiyah–Hirzebruch criterion over a finite field.

Theorem 1.6 (Theorem 3.7). Let $i \geq 1$ be an integer, $\ell \geq i$ be a prime number, $\mathbb{F}$ be a finite field containing a primitive $\ell^2$-th root of unity $\zeta$ and $X$ be a smooth projective $\mathbb{F}$-variety. If $\alpha \in H^2(X, \mathbb{Z}/\ell)$ is an algebraic class, then all odd-degree Steenrod operations on $H^*(X, \mathbb{Z}/\ell)$ vanish on $\alpha$.

(The assumption $\ell \geq i$ already appears over $\overline{\mathbb{F}}$; see [12, Théorème 2.1]. It is vacuously true when $i = 2$, which is the case we are interested in.) The proof of Theorem 1.6, is much harder than the proof of its counterpart over algebraically closed fields and requires new ideas: the entire Section 3 is devoted to it. In particular, a key technical ingredient in our argument is the proof of the compatibility of étale Steenrod operations over $\mathbb{F}$ and $\overline{\mathbb{F}}$ with the Hochschild-Serre spectral sequence, which we prove by appealing to étale homotopy theory to reduce to the analogous (classical) problem for the Serre spectral sequence for Serre fibrations; see Theorem 3.3 and Corollary 3.5. (For reasons that will be clarified in the next paragraph, we need to work with simplicial schemes as opposed to schemes, but this does not make the proofs harder.)

In order to prove Theorem 1.4, it remains to construct examples to which Theorem 1.6 may be applied. Here we follow the general philosophy of Atiyah–Hirzebruch of using suitable smooth projective “approximations” of a classifying space $BG$, where $G$ is an algebraic group. Loosely speaking, this reduces the
problem of finding a cohomology class not killed by some odd-degree Steenrod operation to a calculation in the group cohomology of $G$. When $G$ is finite, Atiyah and Hirzebruch considered high-dimensional quotients of a general smooth projective complete intersection $Y$ by the free action of $G$. When $G$ is a reductive group, similar constructions have been carried out by Ekedahl [15] and Pirutka–Yagita [36]. (Over the complex numbers, one may take $BG$ to be a topological space. In positive characteristic, one may view $BG$ as an algebraic stack or a simplicial scheme. We follow the second approach.)

Even with Theorem 1.6 at our disposal, none of the counterexamples to the integral Tate conjecture already in the literature allows us to prove Theorem 1.4, since in those cases (1.2) is not surjective. Instead, we consider the projective linear group $G = \text{PGL}_d$. The Chow groups and singular cohomology of $BPGL_n$ are in general quite mysterious. However, when $n = \ell$ is prime they have been computed by Vistoli [51], who generalized earlier work of Vezzosi [50]. It follows that if $X$ is a sufficiently fine smooth projective approximation of $BG$, then $H^4(X, \mathbb{Z}_\ell(2)) \cong \mathbb{Z}_\ell$ and that (1.2) is surjective. Combining Vistoli’s results with Galois cohomology computations, we show that, when $F$ contains a primitive $\ell$-th root of unity, the subgroup of geometrically vanishing classes $H^4(F, H^3(X, \mathbb{Z}_\ell)) \cong \mathbb{Z}/\ell$ is generated by an $\ell$-torsion class $\alpha$. Finally, we show by topological means that the odd-degree Steenrod operation $\beta \text{P}^1$ does not vanish on the mod $\ell$ reduction of $\alpha$, and we conclude that $\alpha$ is not algebraic by Theorem 1.6.

1.3. Structure of the paper. The present paper is structured as follows. Section 2 presents some preliminaries on étale Steenrod operations. In Section 3, we prove the compatibility of Steenrod operations with the mod $\ell$ Hochschild spectral sequence (Theorem 3.3), which leads to a criterion for the existence of non-algebraic geometrically trivial $\ell$-adic cohomology classes for varieties over finite fields (Proposition 3.8). Section 4 is devoted to the proof of Theorems 1.4 and 1.5.

1.4. Two coniveau filtrations and Abel-Jacobi maps. Recently, Benoist–Ottem [6] studied the coniveau and strong coniveau filtrations over the complex numbers and gave the first examples where the two filtrations differ.

Theorems 1.4 and 1.5 provide the first examples where the two coniveau filtrations differ over $\overline{F}$. This is a consequence from the following general fact, to be proved in a subsequent paper using Jannsen’s Abel-Jacobi map over finite fields: If $X$ is a smooth projective geometrically connected variety over $\overline{F}$ and the coniveau and strong coniveau filtrations on $H^3(\overline{X}, \mathbb{Z}_\ell(2))$ agree, then $\varphi_\ell$ is an isomorphism.

However, simpler examples where the two coniveau filtrations differ may be given following the original approach of Benoist and Ottem. Indeed, we also show that the constructions and the proofs of Benoist and Ottem adapt to an arbitrary algebraically closed field. In particular, we prove a relative version of Wu’s Theorem in arbitrary characteristic.

1.5. Notation. Let $k$ be a field. We denote by $\overline{F}$ a separable closure of $k$ and $G := \text{Gal}(\overline{F}/k)$ the absolute Galois group of $k$. For a continuous $G$-module $M$, we denote by $H^i(k, M) := H^i(G, M)$ the continuous Galois cohomology of $M$. When $k$ is a finite field, we denote it by $\mathbb{F}$.

If $X$ is a $k$-scheme, we define $\overline{X} := X \times_k \overline{F}$. A $k$-variety is a separated $k$-scheme of finite type. For a smooth $k$-variety $X$ of pure dimension and an integer $i \geq 0$, we denote by $\text{CH}^i(X)$ the Chow group of codimension $i$ cycles on $X$ modulo rational equivalence. If $\ell$ is a prime invertible in $k$, the notation $H^i(X, \mathbb{Z}_\ell(m))$ will mean the $\ell$-adic continuous étale cohomology group of Jannsen [24]. This coincides with the usual étale cohomology $\varprojlim_m H^i(X, \mu_{\ell^m}^\infty)$ when the base field $k$ has finite Galois
cohomology, which is the case if \( k \) is separably closed or a finite field. We write \( H^j(X, \mathbb{Q}_l/\mathbb{Z}_l(m)) \cong \varprojlim H^j(X, \mathbb{Q}_l^m) \) for the (continuous) étale cohomology of the sheaf \( \mathbb{Q}_l/\mathbb{Z}_l(m) \). We let \( H^j_{nr}(X, \mathbb{Q}_l/\mathbb{Z}_l(m)) \) be the unramified cohomology group.

For an abelian group \( A \), an integer \( n \geq 1 \), and a prime number \( \ell \), we denote \( A[n] := \{ a \in A \mid na = 0 \} \), \( A(\ell) \) the subgroup of \( \ell \)-primary torsion elements of \( A \) and \( A_{\text{tors}} \) the subgroup of torsion elements of \( A \).

2. Preliminaries on étale Steenrod operations

2.1. Étale homotopy type. Let \( X_\bullet = (X_n)_{n \geq 0} \) be a simplicial scheme, such that \( X_n \) is a locally noetherian scheme for every \( n \geq 0 \). We will only be interested in the cases when \( X_\bullet \) is the simplicial scheme associated to a scheme \( X \), that is, \( X_n = X \) for all \( n \geq 0 \), and all degeneracy and face maps of \( X_\bullet \) are the identity, or \( X_\bullet = \mathbb{B}G \) is the simplicial classifying space of a linear algebraic group \( G \) over a field \( k \), as defined in [20, Example 1.2]. If \( x : \text{Spec}(\Omega) \to X_0 \) is a geometric point, the pair \((X_\bullet, x)\) is said to be a pointed simplicial scheme.

We let \( \text{AbSh}(X_\bullet) \) be the abelian category of abelian sheaves on the small étale site of \( X_\bullet \). If \( A \) is an object of \( \text{AbSh}(X_\bullet) \), we define the cohomology of \( A \) as

\[
H^i(X_\bullet, A) := \text{Ext}^i_{\text{AbSh}(X_\bullet)}(\mathbb{Z}, A),
\]

where \( \mathbb{Z} \) is the constant sheaf associated to \( \mathbb{Z} \); see [20, Definition 2.3].

We write \( \text{HRR}(X_\bullet) \) for the left directed category of rigid hypercoverings of \( X_\bullet \); see [18, Proposition 4.3]. We let \( \text{Et}(X_\bullet) \) be Friedlander’s étale topological type of \( X_\bullet \); see [20, Definition 4.4]. It is the pro-simplicial set (that is, the pro-object in the category of simplicial sets) defined as the functor

\[
\text{Et}(X_\bullet) : \text{HRR}(X_\bullet) \to \text{sSet}
\]

sending a rigid hypercovering \( U_\bullet \to X_\bullet \) to \( \pi(\Delta(U_\bullet)) \), that is, the simplicial set of level-wise connected components of the diagonal simplicial scheme \( \Delta(U_\bullet) \). If \((X_\bullet, x)\) is a pointed simplicial scheme, \( \text{Et}(X_\bullet) \) is naturally a pointed pro-simplicial space, that is, a pro-object in the category of pointed simplicial sets, which we denote by \( \text{Et}(X_\bullet, x) \).

Let \( A \) be an abelian group. If \( S_\bullet \) is a simplicial set, we denote by \( \mathbb{Z}(S_\bullet) \) the free simplicial abelian group on \( S_\bullet \), and by \( H^*(S_\bullet, A) \) the singular cohomology groups of \( S_\bullet \) with coefficients in \( A \), that is, the cohomology of the chain complex corresponding to \( \hom(\mathbb{Z}(S_\bullet), A) \) via the Dold-Kan correspondence; see [21, Corollary III.2.3]. We also denote by \( A \) the abelian sheaf on the small étale site of \( X \) associated to \( A \). Following [20, Definition 5.1] we define

\[
H^*(\text{Et}(X_\bullet), A) := \text{colim} H^*(\pi(\Delta(U_\bullet)), A),
\]

where on the right side we consider cohomology of simplicial sets and the colimit is indexed by \( \text{HRR}(X_\bullet) \). By [20, Proposition 5.9], there is a natural isomorphism

\[
H^*(X_\bullet, A) \cong H^*(\text{Et}(X_\bullet), A),
\]

(2.1)

If \( X \) is a \( k \)-scheme of finite type and \( X_\bullet \) is the associated simplicial scheme, we have a canonical identification \( H^*(X, A) \cong H^*(X_\bullet, A) \), and we define \( \text{Et}(X) := \text{Et}(X_\bullet) \).

2.2. Étale Steenrod operations. Steenrod operations in étale cohomology have long been known and used. Epstein [16] constructed Steenrod operations in great generality, and his definition may be applied to étale cohomology. Instead, we follow the definition of [17], based on étale homotopy theory. This approach is closer to construction of Jardine [25].
Let $\ell$ be a prime number. If $S_\bullet$ is a simplicial set, $H^*(S_\bullet, \mathbb{Z}/\ell)$ is endowed with Steenrod operations, and these operations are natural with respect to simplicial maps $S' \to S$. It follows that $H^*({\acute{\text{E}}}t(X_\bullet), \mathbb{Z}/\ell)$ is endowed with mod $\ell$ Steenrod operations: power operations $P^n: H^*(X_\bullet, \mathbb{Z}/\ell) \to H^{*+2n(\ell-1)}(X_\bullet, \mathbb{Z}/\ell), \quad n \geq 0$

and a reduced Bockstein homomorphism

$$\beta: H^*(X_\bullet, \mathbb{Z}/\ell) \to H^{*+1}(X_\bullet, \mathbb{Z}/\ell).$$

By definition, $\beta$ is the connecting homomorphism of the cohomology long exact sequence associated to the short exact sequence

$$1 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 1$$

of étale sheaves on $X_\bullet$. More generally, for every $m \geq 1$, we have a homomorphism

$$\beta_m: H^*(X_\bullet, \mathbb{Z}/\ell) \to H^{*+1}(X_\bullet, \mathbb{Z}/\ell^m)$$

obtained as the connecting homomorphism of the cohomology long exact sequence associated to the short exact sequence

$$1 \to \mathbb{Z}/\ell^m \to \mathbb{Z}/\ell^{m+1} \to \mathbb{Z}/\ell \to 1.$$

The homomorphisms $\beta_m$ form an inverse system in $m$. Passing to the inverse limit in $m$ yields the non-reduced Bockstein homomorphism

$$\tilde{\beta}: H^*(X_\bullet, \mathbb{Z}/\ell) \to H^{*+1}(X_\bullet, \mathbb{Z}/\ell).$$

Letting

$$\pi_\ell: H^*(X_\bullet, \mathbb{Z}_\ell) \to H^*(X_\bullet, \mathbb{Z}/\ell)$$

be the homomorphism of reduction modulo $\ell$, we have

$$\beta = \pi_\ell \circ \tilde{\beta}.$$ 

Since the étale Steenrod operations are defined as colimit of the simplicial Steenrod operations, the standard properties of the simplicial operations are also true for the étale operations. Therefore, the étale Steenrod operations are natural with respect to morphisms of simplicial schemes, $P^0$ is the identity, $P^n(x) = x^{\ell^n}$ if $|x| = 2n$ and $P^n(x) = 0$ if $|x| < 2n$, Cartan’s formula and the Adem relations hold.

If $\ell = 2$, we define the Steenrod squares as

$$\text{Sq}^{2n} := P^n, \quad \text{Sq}^{2n+1} := \beta P^n.$$ 

Since $P^0$ is the identity, we have $\beta = \text{Sq}^1$.

We write $P$ for the total Steenrod $\ell$-th power operation, and $\text{Sq}$ for the total Steenrod square:

$$P := \sum_{j=0}^{\infty} P^j, \quad \text{Sq} := \sum_{j=0}^{\infty} \text{Sq}^j.$$ 

Cartan’s formula translates to the fact that $P$ and $\text{Sq}$ are ring homomorphisms.

### 2.3. Specialization map.

For the proof of Theorem 1.5, it will be useful to know that the specialization map in étale cohomology is compatible with Steenrod operations.

**Lemma 2.1.** Let $R$ be a strictly henselian discrete valuation ring, $k$ be the residue field of $R$, $K$ be the fraction field of $R$, $X$ be a smooth projective $R$-scheme of finite type, and $\ell$ be a prime number invertible in $k$. Fix geometric points

$$\pi: \text{Spec}(k) \to \text{Spec}(R), \quad \pi: \text{Spec}(K) \to \text{Spec}(R).$$

Then the specialization map in étale cohomology

$$\text{sp}_{\pi, \pi}: H^*(X_\pi, \mathbb{Z}/\ell) \to H^*(X_\pi, \mathbb{Z}/\ell)$$
is compatible with Steenrod operations.

Proof. Since $R$ is strictly henselian and $X$ is smooth over $R$, there exists a section $x: \text{Spec}(R) \to X$. We set $x_\pi := x \circ \pi \in X(k)$ and $x_\tau := x \circ \tau \in X(k)$. By [20, Propositions 8.6 and 8.7], the closed embedding $X_k \hookrightarrow X$ and the open embedding $X_K \hookrightarrow X$ induce homotopy equivalences

\[
\text{holim}(\mathbb{Z}/\ell)_\infty(\text{Et}(x_\pi, x_\tau)) \cong \text{holim}(\mathbb{Z}/\ell)_\infty(\text{Et}(X, x_\tau)) \Rightarrow \text{holim}(\mathbb{Z}/\ell)_\infty(\text{Et}(X, x_\tau)),
\]

where $(\mathbb{Z}/\ell)_\infty$ is the Bousfield-Kan pro-$\ell$ completion functor, and $\text{holim}(-)$ is the Bousfield-Kan homotopy (inverse) limit functor; see [20, p. 57]. (This is a consequence of the smooth and proper base change theorems in étale cohomology.) The specialization map $\text{sp}_{\eta, \tau}$ is induced by the homotopy equivalence of pointed pro-simplicial sets $j^{-1} \circ i$, hence it commutes with Steenrod operations. \(\square\)

There is a similar result for the classifying space of a linear algebraic group. We will need it during the proof of Theorem 1.4.

Lemma 2.2. Let $G$ be a reductive group over $\mathbb{Z}$, $k$ an algebraically closed field of positive characteristic, $\ell$ be a prime invertible in $k$. Then we have a commutative square of ring homomorphisms

\[
\begin{array}{ccc}
H^*(B_k G, \mathbb{Z}_\ell) & \xrightarrow{\sim} & H^*_\text{sing}(BG(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_\ell \\
\downarrow i_\tau & & \downarrow i_\ell \\
H^*(B_k G, \mathbb{Z}/\ell) & \xrightarrow{\sim} & H^*_\text{sing}(BG(\mathbb{C}), \mathbb{Z}/\ell),
\end{array}
\]

where the horizontal maps are isomorphisms. Moreover, the bottom horizontal map is compatible with étale Steenrod operations (on the left side) and topological Steenrod operations (on the right side).

Proof. This follows from the homotopy equivalence

\[
\text{holim}(\mathbb{Z}/\ell)_\infty(\text{Et}(B_k G)) \xrightarrow{\sim} \text{holim}(\mathbb{Z}/\ell)_\infty(\text{Sing}(BG(\mathbb{C})),
\]

established in [20, Proposition 8.8], where $\text{Sing}(-)$ denotes the singular complex. \(\square\)

3. Non-algebraicity criterion

The purpose of this section is the proof of Proposition 3.8, which provides a criterion for the existence of non-algebraic geometrically trivial $\ell$-adic cohomology classes for smooth projective varieties over finite fields. The key technical ingredient for the proof of Proposition 3.8 comes from Theorem 3.3, which clarifies the behavior of Steenrod operations in the mod $\ell$ Hochschild-Serre spectral sequence. In the case of a finite field $F$, Theorem 3.3 takes a particularly simple form; see Corollary 3.5. The reader willing to take on faith the compatibility of the Steenrod operations with the Hochschild-Serre spectral sequence should skip to the statement of Corollary 3.5.

3.1. Steenrod operations and the Hochschild-Serre spectral sequence. Let $k$ be a field, $\ell$ be a prime number, let $X_\bullet = (X_n)_{n \geq 0}$ be a simplicial scheme, and suppose that $X_n$ is a $k$-scheme of finite type for all $n \geq 0$. We write $\overline{X}_\bullet := X_\bullet \times_k \overline{\mathbb{F}}$. The discussion of Section 2.2 yields Steenrod operations on $H^*(X, \mathbb{Z}/\ell)$ and on $H^*(\overline{X}, \mathbb{Z}/\ell)$. The two cohomology rings are related by the Hochschild-Serre spectral sequence

\[
E_2^{ij} := H^i(k, H^j(\overline{X}, \mathbb{Z}/\ell)) \Rightarrow H^{i+j}(X, \mathbb{Z}/\ell).
\]
This is the Grothendieck spectral sequence for the composition of the functors of Galois-equivariant global sections \( H^0(X, -) \) and of \( G \)-invariants. We may also view (3.1) as the direct limit of the spectral sequences

\[
E^{i,j}_2 := H^i(\text{Gal}(K/k), H^j((X^\bullet)_K, \mathbb{Z}/\ell)) \Rightarrow H^{i+j}(X^\bullet, \mathbb{Z}/\ell)
\]

defined in a similar way, where \( K/k \) ranges over all finite Galois subextensions of \( \overline{K}/k \).

For the proof of Theorems 1.4 and 1.5, it will be important to understand the compatibility of Steenrod operations with (3.1).

Let \( \Gamma \) be a finite group. If \( S^\bullet \) is a simplicial set with a simplicial \( \Gamma \)-action, we denote by \((S^\bullet)_{h\Gamma}\) the homotopy quotient of \( S^\bullet \) by \( \Gamma \), namely

\[(S^\bullet)_{h\Gamma} := (S^\bullet \times E\Gamma)/\Gamma,
\]

where \( E\Gamma \) is a weakly contractible simplicial set with a free \( \Gamma \)-action. The projection \((S^\bullet \times E\Gamma) \to (S^\bullet)_{h\Gamma}\) is a principal \( \Gamma \)-fibration in the sense of [20, Definition 5.3]. By construction, we have a fibration of simplicial sets

\[S^\bullet \to (S^\bullet)_{h\Gamma} \to \Gamma^\bullet,
\]

where \( \Gamma^\bullet := E\Gamma/\Gamma \) is a model for the simplicial classifying space for \( \Gamma \). We have the Serre spectral sequence

\[
E^{i,j}_2 := H^i(\Gamma, H^j(S^\bullet, \mathbb{Z}/\ell)) \Rightarrow H^{i+j}((S^\bullet)_{h\Gamma}, \mathbb{Z}/\ell),
\]

where \( H^j(S^\bullet, \mathbb{Z}/\ell) \) is viewed as a local system over \( \Gamma^\bullet \), or equivalently the Grothendieck spectral sequence

\[
E^{i,j}_2 := H^i(\Gamma, H^j(S^\bullet, \mathbb{Z}/\ell)) \Rightarrow H^{i+j}((S^\bullet)_{h\Gamma}, \mathbb{Z}/\ell)
\]

where \( H^0(\Gamma, -) \) denotes group cohomology and \( H^j(S^\bullet, \mathbb{Z}/\ell) \) is now viewed as a \( \Gamma \)-representation.

Let \( I \) be a left directed category and \( S^\bullet = ((S_i^\bullet))_{i \in I} \) is a pro-simplicial set with a \( \Gamma \)-action: \( \Gamma \) acts on every \( (S_i^\bullet) \) via simplicial automorphisms and the transition morphisms \( (S_i^\bullet) \to (S_j^\bullet) \) are \( \Gamma \)-equivariant. We define the homotopy quotient of \( S^\bullet \) by the \( \Gamma \)-action as

\[(S^\bullet)_{h\Gamma} := \left((S_i^\bullet)_{h\Gamma}\right)_{i \in I},
\]

that is, the pro-simplicial set associated to the inverse system of homotopy quotients.

By definition, \( H^*(S^\bullet, \mathbb{Z}/\ell) \) is the direct limit of \( H^j((S_i^\bullet), \mathbb{Z}/\ell) \) over \( i \in I \), and similarly for \((S^\bullet)_{h\Gamma}\). Since group cohomology commutes with direct limits, by passing to the direct limit in (3.4) we obtain a spectral sequence

\[
E^{i,j}_2 := H^i(\Gamma, H^j(S^\bullet, \mathbb{Z}/\ell)) \Rightarrow H^{i+j}((S^\bullet)_{h\Gamma}, \mathbb{Z}/\ell). \tag{3.5}
\]

**Lemma 3.1.** The rigid étale hypercoverings of \((X^\bullet)_K\) that are defined over \( k \) are cofinal in \( \text{HRR}((X^\bullet)_K) \). In particular, we have a canonical Gal\((K/k)\)-equivariant isomorphism

\[H^*(\text{Ét}((X^\bullet)_K), A) \simeq \text{colim} H^*(\pi(\Delta((U^\bullet)_K)), A),
\]

where the colimit is over \( \text{HRR}(X^\bullet) \).

**Proof.** Let \( \mathcal{C} \) be the site of étale coverings of \((X^\bullet)_K\) by simplicial schemes, and \( \mathcal{B} \) be the collection of all rigid coverings \( Y^\bullet \to (X^\bullet)_K \) that are defined over \( k \). If \( Y^\bullet \to (X^\bullet)_K \) is a rigid covering, the level-wise Galois closure \( \bar{Y}^\bullet \to Y^\bullet \) is a rigid covering of \( Y^\bullet \) defined over \( k \). Moreover, if \( \{Y^\bullet_j \to Y^\bullet\} \) is an étale covering of \( Y^\bullet \) and \( Z^\bullet \to Y^\bullet \) is a rigid covering, then \( \{Y^\bullet_j \to Y^\bullet\} \coprod \{Z^\bullet \to Y^\bullet\} \) is also an étale covering of \( Y^\bullet \). Therefore, the pair \( (\mathcal{C}, \mathcal{B}) \) satisfies the assumptions of [46, 0DAV]. The conclusion now follows from [46, 0DAV]. \( \square \)
Let \( K/k \) be a finite Galois extension, and let \( f: (X_\bullet)_K \to X_\bullet \) be the projection map. The group \( \text{Gal}(K/k) \) acts on \((X_\bullet)_K\) via its action on \( \text{Spec}(K) \). By Lemma 3.1, we may write \( \text{Ét}((X_\bullet)_K) \) as the pro-simplicial set associated to an inverse system of simplicial sets endowed with compatible \( \text{Gal}(K/k) \)-actions. Therefore, we may let \( S_\bullet = \text{Ét}((X_\bullet)_K) \) and \( \Gamma = \text{Gal}(K/k) \) in the previous discussion. The spectral sequence \((3.5)\) specializes to:

\[
E_\infty^{ij} = H^i(\text{Gal}(K/k), H^j(\text{Ét}((X_\bullet)_K), \mathbb{Z}/\ell)) \Rightarrow H^{i+j}(\text{Ét}((X_\bullet)_K)_h, \text{Gal}(K/k), \mathbb{Z}/\ell).
\]

**Lemma 3.2.** For every finite Galois extension \( K/k \), there exists an isomorphism between the spectral sequences \((3.2)\) and \((3.6)\), which is functorial with respect to \( K \).

**Proof.** If \( A_{\bullet\bullet} \) is a bisimplicial abelian group, we denote by \( M(A_{\bullet\bullet}) \) the Moore chain bicomplex corresponding to \( A_{\bullet\bullet} \) (see [21, p. 205]), and by \( \text{Tot}(M(A_{\bullet\bullet})) \) the associated total complex. (In [21], the Moore complex of \( A_{\bullet\bullet} \) is also denoted by \( A_{\bullet\bullet} \).) By definition, \( H^*(A_{\bullet\bullet}) \) is the cohomology of the chain complex \( \text{Tot}(M(A_{\bullet\bullet})) \); this is in accordance with [20, p. 20]. (Recall that the cohomology of a chain complex is computed by first applying \( \text{Hom}(-, A) \) in each degree, and then taking cohomology of the resulting cochain complex.) Since the total complex and Moore bicomplex functors are exact, this defines a \( \delta \)-functor on the category of bisimplicial abelian groups. If \( A^{\bullet\bullet} \) is a bicosimplicial abelian group, we define dually the Moore cochain bicomplex \( M(A^{\bullet\bullet}) \), and let \( H^*(A^{\bullet\bullet}) \) be the cohomology of the total complex of \( M(A^{\bullet\bullet}) \).

If \( F \) is an étale sheaf of abelian groups on a bisimplicial scheme \( Y_{\bullet\bullet} \), we denote by \( F(Y_{\bullet\bullet}) \) the bicosimplicial abelian group obtained by taking global sections levelwise. By [20, Theorem 3.8], we have an isomorphism of \( \delta \)-functors on \( \text{AbSh}(X_\bullet) \):

\[
H^*(((X_\bullet)_K), -) \cong \text{colim} \ H^*(((-)((U_{\bullet\bullet})_K)),
\]

where the colimit is over all étale hypercoverings \( U_{\bullet\bullet} \to X_\bullet \). (The right hand side is a \( \delta \)-functor because the colimit functor is exact.) Since group cohomology commutes with colimits, this induces an isomorphism of spectral sequences from \((3.2)\) to the colimit of the spectral sequences

\[
E_\infty^{ij} := H^i(\text{Gal}(K/k), H^j((\mathbb{Z}/\ell)((U_{\bullet\bullet})_K))) \Rightarrow H^{i+j}(((\mathbb{Z}/\ell)(U_{\bullet\bullet})),
\]

as \( U_{\bullet\bullet} \) ranges over all étale hypercoverings of \( X_\bullet \). The spectral sequences \((3.7)\) are defined as the Grothendieck spectral sequences for the composition of \((-)((U_{\bullet\bullet})_K)\) and of the functor of \( \text{Gal}(K/k) \)-invariants.

Let \( A_{\bullet\bullet} \) be a bisimplicial abelian group, \( \Delta(A_{\bullet\bullet}) \) be the diagonal simplicial abelian group, and \( d(A_{\bullet\bullet}) \) be the chain complex corresponding to the simplicial diagonal \( \Delta(A_{\bullet\bullet}) \) via the Dold-Kan correspondence. (In [21, p. 205], both notions are denoted by \( d(A_{\bullet\bullet}) \).) By the bissimplicial Eilenberg-Zilber Theorem [21, Theorem IV.2.3], proved by Dold and Puppe, we have a natural chain homotopy equivalence \( \text{Tot}(M(A_{\bullet\bullet})) \cong d(A_{\bullet\bullet}) \). In particular, if \( S_{\bullet\bullet} \) is a bisimplicial set, letting \( A_{\bullet\bullet} = \mathbb{Z}(S_{\bullet\bullet}) \) we get a natural chain homotopy equivalence

\[
\text{Tot}(M(\mathbb{Z}(S_{\bullet\bullet}))) \cong d(\mathbb{Z}(S_{\bullet\bullet})) = \mathbb{Z}(\Delta(S_{\bullet\bullet})).
\]

Let \( U_{\bullet\bullet} \to X_\bullet \) be an étale hypercovering, \( A \) an abelian group. We also denote by \( A \) the corresponding constant étale sheaf on \( X_\bullet \) and \( U_{\bullet\bullet} \). Then

\[
A(\Delta((U_{\bullet\bullet})_K)) = \text{Hom}(\mathbb{Z}(\pi(\Delta((U_{\bullet\bullet})_K))), A),
\]

hence by \((3.8)\) we obtain an isomorphism of \( \delta \)-functors

\[
H^*((-)((U_{\bullet\bullet})_K)) \cong H^*(\pi(\Delta((U_{\bullet\bullet})_K)), -)
\]
from the abelian category of constant abelian groups (or constant abelian étale sheaves on $U_{\bullet}$) to the abelian category of $\text{Gal}(K/k)$-modules.

Letting $A = \mathbb{Z}/\ell$, we deduce that (3.7) is naturally isomorphic to the colimit of the Grothendieck spectral sequences

$$E_2^{a,b} = H^i(\text{Gal}(K/k), H^j(\pi(\Delta((U_{\bullet})_K)), \mathbb{Z}/\ell)) \Rightarrow H^{i+j}(\pi(\Delta(U_{\bullet})), \mathbb{Z}/\ell),$$

as $U_{\bullet}$ ranges over all étale hypercoverings of $X_{\bullet}$. By [20, Corollary 4.6], in the spectral sequence (3.9) we may equivalently let $U_{\bullet}$ range over all rigid hypercoverings of $X_{\bullet}$. By construction, this is the spectral sequence (3.6).

\[ \square \]

**Theorem 3.3.** Let $k$ be a field, let $X_{\bullet} = (X_n)_{n \geq 0}$ be a simplicial scheme, such that $X_n$ is of finite type over $k$ for all $n \geq 0$, and let $\ell$ be a prime number. On the Hochschild-Serre spectral sequence (3.1) there are, for all integers $s,i \geq 0$ and all $2 \leq r \leq \infty$, homomorphisms

$$f^P E_r^{a,b} \to E_r^{a+2s(\ell-1),b}, \quad f^P \beta E_r^{a,b} \to E_r^{a+2s(\ell-1)+1,b}, \quad 0 \leq 2s \leq b$$

$$b^P E_r^{a,b} \to E_r^{a+(2s-b)(\ell-1),b}, \quad b^P \beta E_r^{a,b} \to E_r^{a+(2s-b)(\ell-1)+1,b}, \quad b \leq 2s$$

when $\ell$ is odd, and

$$f^\text{Sq}^i E_r^{a,b} \to E_r^{a,b+i}, \quad 0 \leq i \leq b$$

$$b^\text{Sq}^i E_r^{a,b} \to E_r^{a+i-b,2b}, \quad b \leq i$$

when $\ell = 2$. Here we have defined

$$t := \begin{cases} r - 1 + (2s - b)(\ell - 1), & b \leq 2s < b + r - 2, \\ (\ell(r-2) + 1), & b + r - 2 \leq 2s, \end{cases}$$

$$u := \begin{cases} r + (2s - b)(\ell - 1), & b \leq 2s < b + r - 2, \\ (\ell(r-2) + 1), & b + r - 2 \leq 2s, \end{cases}$$

$$v := \begin{cases} r + i - b, & b \leq i \leq b + r - 2, \\ 2r - 2, & b + r - 2 \leq i. \end{cases}$$

Note that $t = u = v = 2$ if $r = 2$ and $t = u = v = \infty$ if $r = \infty$.

These operations satisfy the following properties.

(i) For all $2 \leq r \leq \infty$, the operations on the $E_r$ page are determined by those on the $E_2$ page. More precisely, suppose that $x \in E_2^{a,b}$ survives to the $E_r$ page, let $[x] \in E_r^{a,b}$ be the image of $x$, and let $P: E_r^{a,b} \to E_r^{c,d}$ be one of $f^P$, $f^P \beta$, $b^P$, $b^P \beta$, $f^\text{Sq}^i$, or $b^\text{Sq}^i$. (The indices $c,d,w$ have been written out earlier in the statement.) Then $P(x)$ survives to the $E_w$ page and $[P(x)] = P([x])$.

(ii) For every integer $n \geq 0$, let

$$\{0\} = F^{n+1}H^n(X_{\bullet}, \mathbb{Z}/\ell) \subset F^nH^n(X_{\bullet}, \mathbb{Z}/\ell) \subset \cdots \subset F^0H^n(X_{\bullet}, \mathbb{Z}/\ell) = H^n(X_{\bullet}, \mathbb{Z}/\ell)$$

be the filtration of $H^n(X_{\bullet}, \mathbb{Z}/\ell)$ induced by (3.1), with projection maps

$$\rho: F^nH^{a+b}(X_{\bullet}, \mathbb{Z}/\ell) \to E_{n+1}^{a,b}.$$ 

For all integers $s,i \geq 0$, write $P^s$, $\beta P^s$ and $\text{Sq}^i$ for the étale Steenrod operations on $H^n(X_{\bullet}, \mathbb{Z}/\ell)$. Then, when $\ell$ is odd, we have commutative squares

$$\begin{array}{ccc} F^nH^{a+b+2s(\ell-1)}(X_{\bullet}, \mathbb{Z}/\ell) & \xrightarrow{\rho} & E^{a+b+2s(\ell-1)}_{n+1} \\
F^nH^{a+b}(X_{\bullet}, \mathbb{Z}/\ell) & \xrightarrow{\rho} & E_{n+1}^{a,b} \end{array}$$
when $2s \leq b$, and
\[
F^{a+(2s-b)(\ell-1)}H^{a+b+2s(\ell-1)}(X_\bullet, \mathbb{Z}/\ell) \xrightarrow{\rho} E^{\infty}_{a+(2s-b)(\ell-1),tb} \xrightarrow{\rho \rho} E^{\infty}_{\infty, tb}
\]
when $2s \geq b$. The two squares coincide for $2s = b$. We have similar commutative squares with $P^s$ replaced by $\beta P^s$. When $\ell = 2$, we have the commutative squares
\[
F^a H^{a+b+i}(X_\bullet, \mathbb{Z}/\ell) \xrightarrow{\rho} E^{\infty}_{\infty, b+i} \xrightarrow{\rho \rho} E^{\infty}_{\infty, b}
\]
when $i \leq b$ and
\[
F^{a+i-b} H^{a+i+b}(X_\bullet, \mathbb{Z}/\ell) \xrightarrow{\rho} E^{\infty}_{\infty, b+2b} \xrightarrow{\rho \rho} E^{\infty}_{\infty, b}
\]
when $i \geq b$. The two squares coincide for $i = b$.

(iii) Suppose $r = 2$. Then for all $s \geq 0$ the operations
\[
pP^s : H^a(k, H^b(X_\bullet, \mathbb{Z}/\ell)) \to H^a(k, H^{b+2s(\ell-1)}(X_\bullet, \mathbb{Z}/\ell))
\]
are induced by $P^s : H^b(X_\bullet, \mathbb{Z}/\ell) \to H^{b+2s(\ell-1)}(X_\bullet, \mathbb{Z}/\ell)$, and similarly for $P^s \beta P^s$ and $P^s \beta P^s$.

Our applications do not rely on the full strength of Theorem 3.3, but only on its simpler consequence Corollary 3.5. In particular, we will not need the operations with prescript $B$, nor the definitions of $t, u, v$, but we included them for completeness and possible future use.

In (ii), the fact that $P^s$, $\beta P^s$ and $\rho^s$ respect the spectral sequence filtration of $H^*(X_\bullet, \mathbb{Z}/\ell)$ as described in the commutative squares is part of the statement.

Proof. Let $F \to E \to B$ be a (topological) Serre fibration, and consider the Serre spectral sequence
\[
E_2^{ij} = H^j_{\text{sing}}(B, H^i_{\text{sing}}(F, \mathbb{Z}/\ell)) \Rightarrow H^{i+j}_{\text{sing}}(E, \mathbb{Z}/\ell),
\]
where $H^j_{\text{sing}}(F, \mathbb{Z}/\ell)$ is the local system of coefficients associated to the Serre fibration. We will make use of the construction of (3.10) due to Dress [14]; see also [30, pp. 225-229] or [45, Chapter 2, §5] (for $\ell = 2$). Let $\text{Sing}_*(E)$ be the singular simplicial complex with $\mathbb{Z}/\ell$ coefficients of $E$, regarded as a simplicial coalgebra via the Eilenberg-Zilber map. To the Serre fibration $F \to E \to B$, Dress associated a bisimplicial $(\mathbb{Z}/\ell)$-coalgebra $K_{**}(f)$ and an augmentation map $\lambda : K_{**}(f) \to \text{Sing}_*(E)$ such that the induced map $\lambda^* : H^*(E, \mathbb{Z}/\ell) \to H^*(\text{Tot}(K_{**}(f)))$ is an isomorphism. The first-quadrant spectral sequence for the double complex corresponding via Dold-Kan to the bisimplicial $(\mathbb{Z}/\ell)$-coalgebra $\text{Hom}(K_{**}(f), \mathbb{Z}/\ell)$ may then be identified with (3.10).

We now give references for the construction of Steenrod operations on (3.10). These references work in the setting of the spectral sequence associated to a bisimplicial $(\mathbb{Z}/\ell)$-coalgebra, so by Dress’ construction they apply to (3.10). Singer [44] constructed Steenrod operations on (3.10) when $\ell = 2$. Singer’s definition of $P^s \rho^s$ given in [44] has incorrect codomain. This has been fixed in [45, Theorem 2.15],

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where the correct \( t \) appears. When \( \ell \) is odd, Mori [33] constructed Steenrod operations on (3.10), but his definition of \( B^P \) and \( B^P \) has a similar problem. These errors were noticed by Sawka [39, End of p. 741]. In the same article, Sawka [39] defined the operations when \( \ell \) is odd. Sawka’s operations are defined as homomorphisms \( E_r \to E_r \) but with indeterminacy, see [39, pp. 739-740]. By [39, Proposition 2.5, Proposition 6.2], we may view Sawka’s operations as homomorphisms \( E_r \to E_w \), where \( w = t, u \).

We may now give references for the analogues of (i), (ii) and (iii) for (3.10). For (i), see [44, Proposition 1.2] when \( \ell = 2 \) and [39, Proposition 2.2] when \( \ell \) is odd. When \( \ell = 2 \), the proof of (ii) may be found in [44, Proposition 1.5] and [45, Theorem 2.16]. When \( \ell \) is odd, the proof of (ii) is given in [39, Proposition 2.4] and [33, Theorem 1.4 and §5].

We now consider (iii). When \( \ell = 2 \), (iii) is stated in [45, (2.77)]. The key tool for its proof is [44, I, Proposition 5.1] (or [45, Theorem 2.23 (2.60)]). An analogue of (iii) is proved for a different spectral sequence in [44, II, Proposition 5.2], but the method is general and applies to the Eilenberg-Moore spectral sequence (see [44, II, Proposition 7.2], where the author leaves the proof to the reader) and to the Serre spectral sequence [44, II, §8]. When \( \ell \) is odd, the proof is identical once we replace [44, I, Proposition 5.1] by [39, Proposition 7.1, Case 1].

Since the spectral sequence (3.4) is a special case of (3.10), it is endowed with Steenrod operations satisfying evident analogues of (i), (ii) and (iii). Moreover, the spectral sequence (3.4) is naturally isomorphic to (3.3), hence the properties just established for (3.3) hold for (3.4) as well. By the naturality of Steenrod operations, this endows the spectral sequence (3.5) with Steenrod operations that satisfy evident analogues of (i), (ii) and (iii).

By Lemma 3.2, there is a natural isomorphism of spectral sequences from (3.2) to (3.6). We use this isomorphism to define operations on (3.2) which satisfy all the properties required in the statement of Theorem 3.3. The construction of (3.2) and the operations on it is functorial with respect to finite field extensions \( K \subset K' \) such that \( K'/k \) is Galois. Passing to the direct limit, we obtain operations with the required properties on the spectral sequence (3.1).

Remark 3.4. Steenrod operations on the Serre spectral sequence (3.10) were first defined by Araki [4, Definition 2 p. 87 and end of p. 89] and independently by Vázquez [49]. We have not been able to access Vázquez’s paper. Araki’s definition is based on Serre’s construction of (3.10) using the cubical singular complex; see [42]. Araki then proved a number of properties of these operations and listed them in [4, Summary p. 89]. In particular, [4, (K) p. 90] yields (i). It is possible to prove (ii) and (iii) for (3.10) using Araki’s definition, but there seems to be no reference in the literature. Multiple authors affirm without proof that the operations defined using Dress’ and Serre’s constructions of (3.10) agree; see e.g. [44, p. 328], [39, p. 737] and [33, §5]. This is certainly true, but we could not locate a proof in the literature. This is why we do not rely on Araki’s construction for the proof of Theorem 3.3.

Suppose now that \( k = \mathbb{F} \) is a finite field. By [20, Theorem 7.7] (which follows from [13, Chapitre 7, Théorème 1.1]), the \( G \)-module \( H^j(X, \mathbb{Z}/\ell) \) is finite for all \( j \geq 0 \). Since the cohomological dimension of \( \mathbb{F} \) is equal to 1, the groups \( H^i(\mathbb{F}, H^j(X, \mathbb{Z}/\ell)) \) are trivial for all \( i \geq 2 \). Therefore, the \( E_2 \)-page of (3.1) induces natural surjections

\[
H^j(X, \mathbb{Z}/\ell) \to H^j((X, \mathbb{Z}/\ell)^G)
\]

and natural isomorphisms

\[
\text{Ker}(H^j(X, \mathbb{Z}/\ell) \to H^j((X, \mathbb{Z}/\ell)^G)) \cong H^1(\mathbb{F}, H^{j-1}((X, \mathbb{Z}/\ell)^G)).
\]
which may be rewritten as natural short exact sequences

(3.11) \[ 0 \to H^1(\mathbb{F}, H^{j-1}(\mathbb{X}, \mathbb{Z}/\ell)) \to H^j(\mathbb{X}, \mathbb{Z}/\ell) \to H^j(\mathbb{X}, \mathbb{Z}/\ell)^G \to 0. \]

**Corollary 3.5.** Let \( F \) be a finite field, \( X_\bullet = (X_n)_{n \geq 0} \) be a simplicial scheme such that \( X_n \) is of finite type over \( F \) for all \( n \geq 0 \), \( s \geq 0 \) be an integer, \( \ell \) be a prime number invertible in \( F \), and \( P \) be one of \( \mathbb{P}^s \), \( \mathbb{P}^s \times \mathbb{P}^d \) \((\ell = 2)\). For every \( j \neq 2s \), we have a commutative diagram of short exact sequences:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\mathbb{F}, H^{j-1}(\mathbb{X}, \mathbb{Z}/\ell)) & \longrightarrow & H^j(\mathbb{X}, \mathbb{Z}/\ell) & \longrightarrow & H^j(\mathbb{X}, \mathbb{Z}/\ell)^G & \longrightarrow & 0 \\
& & \downarrow{\mu^1(F, P)} & & \downarrow{\mu} & & \downarrow{\mu} & & \downarrow{\mu} \\
0 & \longrightarrow & H^1(\mathbb{F}, H^{j-d-1}(\mathbb{X}, \mathbb{Z}/\ell)) & \longrightarrow & H^{j+d}(\mathbb{X}, \mathbb{Z}/\ell) & \longrightarrow & H^{j+d}(\mathbb{X}, \mathbb{Z}/\ell)^G & \longrightarrow & 0,
\end{array}
\]

where the rows are (3.11) and \( d \) is the degree of \( P \).

**Proof.** Recall that \( \text{Sq}^{2s+1} = \text{Sq}^1 \text{Sq}^{2s} \) when \( \ell = 2 \). If \( 2s > j \), then \( P \) is zero on \( H^j(X_\bullet, \mathbb{Z}/\ell) \) and there is nothing to prove. Suppose now that \( 2s < j \). By Theorem 3.3(ii), the map \( P : H^j(X_\bullet, \mathbb{Z}/\ell) \to H^{j+d}(X_\bullet, \mathbb{Z}/\ell) \) respects the filtration coming from (3.1), hence \( P \) induces a commutative diagram as in Corollary 3.5, except that the left and right vertical maps have not been identified yet. The naturality of Steenrod operations implies that the vertical map on the right is \( P \). It remains to show that the vertical map on the left is \( H^1(\mathbb{F}, P) \). This follows from Theorem 3.3(iii) for \((a, b) = (1, j - 1)\). \( \square \)

In the rest of this paper, we only will apply (3.11) and Corollary 3.5 when \( X_\bullet \) is the constant simplicial scheme associated to a smooth projective \( F \)-variety \( X \).

### 3.2. Odd-degree operations and non-algebraic classes

The material of this subsection is well known; see [36, §2]. It differs from [12] because the ground field is finite instead of algebraically closed. The arguments of [36] use motivic cohomology and also require the existence of a primitive \( \ell^2 \)-root of unity. In order to make the current paper more self-contained and to avoid introducing motivic Steenrod operations, we provide alternative arguments which do not use motivic cohomology. This also saves us the trouble of proving the compatibility between the étale Steenrod operations defined in 2.2 and the motivic Steenrod operations, or alternatively the étale Steenrod operations defined using homological algebra as in [7] or [23].

**Lemma 3.6.** Let \( \ell \) be a prime number invertible in \( F \), and let \( \mathbb{F} \) be a finite field containing a primitive \( \ell^2 \)-th root of unity. Then for all \( n \geq 0 \) the odd-degree Steenrod operations on \( H^\bullet(\mathbb{P}^n, \mathbb{Z}/\ell) \) vanish.

**Proof.** Viewing \( \mathbb{P}^n \) as a projective bundle over \( \text{Spec}(\mathbb{F}) \), the projective bundle formula [1, Exposé VII, Théorème 2.2.1] yields a \( G \)-equivariant isomorphism of graded rings:

\[
\oplus_{i=0}^n H^i(\mathbb{P}^n, \mu_\ell^{\otimes i}) \simeq (\mathbb{Z}/\ell)[h]/(h^{n+1}), \quad |h| = 2,
\]

where on the multiplication on the left side is given by cup product.

Since \( F \) contains a primitive \( \ell \)-th root of unity, \( \mu_\ell^{\otimes i} \simeq \mathbb{Z}/\ell \) for all \( i \). We have

\[
H^\bullet(F, \mathbb{Z}/\ell) \simeq (\mathbb{Z}/\ell)[\epsilon]/(\epsilon^2), \quad |\epsilon| = 1.
\]

We also denote by \( \epsilon \in H^1(\mathbb{P}^n, \mathbb{Z}/\ell) \) the pullback of \( \epsilon \) along the structure morphism \( \mathbb{P}^n \to \text{Spec}(\mathbb{F}) \).

Consider the Hochschild-Serre spectral sequence

\[
E_{2}^{ij} := H^i(F, H^j(\mathbb{P}^n, \mathbb{Z}/\ell)) \Rightarrow H^{i+j}(\mathbb{P}^n, \mathbb{Z}/\ell).
\]
Note that $h$ maps to the generator $\overline{h} \in E_{2}^{0,2} = E_{2}^{0,2}$. Since the Hochschild-Serre spectral sequence is multiplicative and degenerates at the $E_{2}$-page, we have a ring isomorphism

$$E_{\infty} \simeq E_{2} \simeq (\mathbb{Z}/\ell)[\overline{h}, \overline{e}]/(\overline{e}^{n+1}, \overline{e}^{i}).$$

It follows that $h^ne$ is not zero in the associated graded of $H^{*}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell)$, hence $h^ne \neq 0$ in $H^{*}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell)$. We obtain a graded $\mathbb{Z}/\ell$-algebra isomorphism

$$(3.12) \quad H^{*}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell) \simeq (\mathbb{Z}/\ell)[\epsilon, h]/(\epsilon^{2}, h^{n+1}).$$

We now show that the Bockstein homomorphism

$$\beta: H^{*}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell) \to H^{*}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell)$$

is the zero homomorphism.

To show that $\beta = 0$, it suffices to show that $\beta(\epsilon) = 0$ and $\beta(h) = 0$. Since $\epsilon$ comes from $\text{Spec}(\mathbb{F})$ and $H^{2}(\mathbb{F}, \mathbb{Z}/\ell) = 0$, we have $\beta(\epsilon) = 0$. We have a commutative triangle

$$\begin{array}{ccc}
CH^{1}(\mathbb{P}^{n}_{\mathbb{F}}) & \longrightarrow & H^{2}(\mathbb{P}^{n}_{\mathbb{F}}, \mu_{\ell^{2}}) \\
& \downarrow & \\
& H^{2}(\mathbb{P}^{n}_{\mathbb{F}}, \mu_{\ell}), & \\
\end{array}$$

where the horizontal and oblique arrows are cycle maps, and where the vertical arrow is the reduction map. Since $\mathbb{F}$ contains a primitive $\ell^{2}$-th root of unity, we may rewrite this triangle as

$$\begin{array}{ccc}
CH^{1}(\mathbb{P}^{n}_{\mathbb{F}}) & \longrightarrow & H^{2}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell^{2}) \\
& \downarrow & \\
& H^{2}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell), & \\
\end{array}$$

It follows that $h$ lifts to $\mathbb{Z}/\ell^{2}$, and hence $\beta(h) = 0$. Thus $\beta$ vanishes on $H^{*}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell)$, as desired. Every odd-degree Steenrod operation is of the form $\beta P^{i}$ (for $\ell$ odd) or $Sq^{2i+1} = \beta Sq^{2i}$ (for $\ell = 2$), and hence must also vanish on $H^{*}(\mathbb{P}^{n}_{\mathbb{F}}, \mathbb{Z}/\ell)$. $\square$

**Theorem 3.7 (Theorem 1.6).** Let $i \geq 1$ be an integer, $\ell \geq i$ be a prime number, $\mathbb{F}$ be a finite field containing a primitive $\ell^{2}$-th root of unity $\zeta$ and $X$ be a smooth projective $\mathbb{F}$-variety. Let $\alpha \in H^{2i}(X, \mathbb{Z}/\ell)$ be an algebraic class, that is, a class in the image of

$$CH^{i}(X)/\ell \cong H^{2i}(X, \mu_{\ell^{i}}) \cong H^{2i}(X, \mathbb{Z}/\ell),$$

where the isomorphism on the right is induced by the choice of the root $\zeta$. Then all odd-degree Steenrod operations on $H^{*}(X, \mathbb{Z}/\ell)$ vanish on $\alpha$.

**Proof.** We adapt an argument given by Totaro over algebraically closed fields; see the proof of [12, Théorème 2.1(1)]. There are however some difficulties to overcome, as the $\mathbb{Z}/\ell$-étale cohomology of Grassmannians over $\mathbb{F}$ is not concentrated in even degrees. This is where Corollary 3.5 and Lemma 3.6 come into play.

Since $\ell$ is coprime with $(i-1)!$, by Jouanolou’s Riemann-Roch Theorem without denominators [26], every element of $CH^{i}(X)/\ell$ is a linear combination of Chern classes of vector bundles on $X$. Therefore, we may assume that $\alpha = c_{i}(E)$ is the $i$-th Chern class of a vector bundle $E \to X$.

If $\pi: \mathbb{P}(E) \to X$ is the projective bundle associated to $E \to X$, then $\pi^{*}E$ admits a rank 1 quotient bundle and $\pi^{*}: H^{*}(X, \mathbb{Z}/\ell) \to H^{*}(\mathbb{P}(E), \mathbb{Z}/\ell)$ is injective. Moreover, since pullback along $\pi$ is defined on the Chow ring and is compatible with the pullback on étale cohomology, $\pi^{*}\alpha$ is still algebraic. Iterating this procedure, we may suppose that $E$ is an iterated extension of line bundles $L_{j}$, $j = 1, \ldots, r$. Then
$c_i(E)$ is the degree $i$ symmetric polynomial in the $c_1(L_j)$. By the Cartan formula, it suffices to show that odd degree Steenrod operations vanish on the $c_1(L_j)$. We may thus assume that $E$ is a line bundle and that $i = 1$.

Let $L$ be a very ample line bundle on $X$ such that $E \otimes L$ is generated by global sections. Then

$$c_1(E) = c_1(E \otimes L) - c_1(L).$$

By the linearity of the Steenrod operations, to prove that odd operations vanish on $c_i(E)$ it suffices to show that they vanish on $c_1(E \otimes L)$ and $c_1(L)$. We may thus assume that $E$ is generated by global sections, in which case $c_1(E)$ is the pullback of a cohomology class on $\mathbb{P}^n$ for some integer $n \geq 1$. This reduces us to the case when $X = \mathbb{P}^n$. The conclusion follows from Lemma 3.6.

3.3. **Non-algebraic geometrically trivial classes.** Let $\sigma \in G$ be the Frobenius automorphism of $\mathbb{F}$; it is a topological generator of $G$. If $M$ is a finite $G$-module, there is an isomorphism $H^1(\mathbb{F}, M) \cong M/(\sigma - 1)M$, sending the class of a cocycle $\{m_g\}_{g \in G}$ to the class of $m_\sigma$ in $M/(\sigma - 1)M$. We obtain a surjection

$$M \twoheadrightarrow H^1(\mathbb{F}, M),$$

which is an isomorphism if and only if $G$ acts trivially on $M$; see [43, XIII §1, Proposition 1].

The homomorphism (3.13) is functorial in the following way. Suppose that $f : M \rightarrow N$ is a homomorphism of continuous $G$-modules, and let

$$H^1(\mathbb{F}, f) : H^1(\mathbb{F}, M) \rightarrow H^1(\mathbb{F}, N)$$

be the induced homomorphism. By definition, $H^1(\mathbb{F}, f)$ sends the class of a cocycle $\{m_g\}_{g \in G}$ to the class of $\{f(m_g)\}_{g \in G}$. It follows that we have a commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{(3.13)} & H^1(\mathbb{F}, M) \\
\downarrow f & & \downarrow H^1(\mathbb{F}, f) \\
N & \xrightarrow{(3.13)} & H^1(\mathbb{F}, N)
\end{array}$$

where the composition of the horizontal map is given by reduction modulo $\sigma - 1$.

Before stating Proposition 3.8, we recall a construction of the $\ell$-adic Hochschild-Serre spectral sequence over finite fields. Of course, this is a special case of [24]. For every integer $m \geq 1$, the Hochschild-Serre spectral sequence

$$E_2^{r,s} := H^r(\mathbb{F}, H^s(X, \mu_{\ell^m}^{\otimes i})) \Rightarrow H^{r+s}(X, \mu_{\ell^m}^{\otimes i})$$

yields compatible short exact sequences

$$(3.14) \quad 0 \rightarrow H^1(\mathbb{F}, H^{2i-1}(X, \mu_{\ell^m}^{\otimes i})) \xrightarrow{\iota} H^2(X, \mu_{\ell^m}^{\otimes i}) \rightarrow H^2(\overline{X}, \mu_{\ell^m}^{\otimes i})^G \rightarrow 0.$$

Since the abelian groups $H^{2i-1}(X, \mu_{\ell^m}^{\otimes i})$ are finite, we may pass to the inverse limit in (3.14) and obtain a short exact sequence

$$(3.15) \quad 0 \rightarrow H^1(\mathbb{F}, H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \xrightarrow{\iota} H^2(X, \mathbb{Z}_\ell(i)) \rightarrow H^2(\overline{X}, \mathbb{Z}_\ell(i))^G \rightarrow 0.$$

We will produce our counterexamples by means of the following consequence of Theorem 3.7.

**Proposition 3.8.** Let $\mathbb{F}$ be a finite field, $\ell$ be a prime number invertible in $\mathbb{F}$, $1 \leq i \leq \ell$ be an integer, and $X$ be a smooth projective $\mathbb{F}$-variety. Suppose that $\mathbb{F}$ contains a primitive $\ell^i$-root of unity. Let:

- $u \in H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))$,
- $P$ be a Steenrod operation of odd degree $d$ which does not vanish on $\pi_\ell(u)$,
- $u' \in H^1(\mathbb{F}, H^{2i-1}(\mathcal{X}, \mathbb{Z}_d(i)))$ be the image of $u$ under the map (3.13) for $M = H^{2i-1}(\mathcal{X}, \mathbb{Z}_d(i))$.
- $\alpha \in H^2(X, \mathbb{Z}_d(i))$ be the image of $u'$ under the inclusion $i$ appearing in (3.15).

Then:

(i) $\omega = 0$, and
(ii) if $G$ acts trivially on $H^{2i-1+d}(\mathcal{X}, \mathbb{Z}/\ell)$, then $\alpha$ is not algebraic.

Proof. (i) follows from the exactness of (3.15).

If $G$ acts trivially on $H^{2i-1+d}(\mathcal{X}, \mathbb{Z}/\ell)$, then (3.13) is an isomorphism for $M = H^{2i-1+d}(\mathcal{X}, \mathbb{Z}/\ell)$. Since $P(\pi_\ell(u)) \neq 0$ and $\pi_\ell(u')$ is the image of $\pi_\ell(u)$ under (3.13) for $M = H^{2i-1}(\mathcal{X}, \mathbb{Z}_d(i))$, we deduce that $H^1(\mathbb{F}, P)(u') \neq 0$. Now Corollary 3.5 implies that $P(\pi_\ell(\alpha)) \neq 0$. Since the degree $d$ of $P$ is odd, Theorem 3.7 implies that $\pi_\ell(\alpha)$ is not algebraic, hence $\alpha$ is not algebraic. This proves (ii). $\square$

4. Proof of Theorems 1.4 and 1.5

4.1. Proof of Theorem 1.5.

Proof of Theorem 1.5. Let $p$ be an odd prime number, $K$ be a $p$-adic field with ring of integers $R$ and residue field $\mathbb{F}$ is finite of characteristic $p$, and fix a field embedding $K \rightarrow \mathbb{C}$. Let $E_1$ and $E_2$ be elliptic curves over $\text{Spec}(R)$, such that $E_1$ admits a torsion point $P$ of order 4, and that $\text{Aut}_R(E_2, 0)$ admits an element $\sigma$ of order 4. One can easily construct such $E_1$ and $E_2$ when $R$ is a sufficiently large finite extension of $\mathbb{Z}_p$. (To construct $E_1$, we start from any elliptic curve $E'/\mathbb{Z}_p[i]$. Any 4-torsion point of $E'/(\mathbb{Z}_p[i])$ is defined over a finite extension $K/\mathbb{Q}_p$. We let $R$ be the integral closure of $\mathbb{Z}_p$ in $K$, we define $E_1 := (E')_R$, and we let $P$ be the closure of the 4-torsion point. For $E_2$, consider the elliptic curve $E''/\mathbb{Z}_p[i]$ given by the Weierstrass equation $y^2 = x^3 + x$. Since $R$ contains $\mathbb{Z}_p[i]$, we may define $E_2 := (E'')_R$.)

Consider the free $\mathbb{Z}/4$-action on $E_1 \times_R E_2$ over $R$, where the generator $1 \in \mathbb{Z}/4$ acts on $E_1$ by translation by $P$ and on $E_2$ as $\sigma$, and define $S := (E_1 \times_R E_2)/(\mathbb{Z}/4)$. The morphism $E_1 \times_R E_2$ given by $(u, v) \mapsto (u, v + \sigma(v))$ induces a morphism $f : S \rightarrow S$ which is finite étale of degree 2. The morphism $f$ corresponds to the composition $\tilde{\sigma} : S \rightarrow S$ induced by $(u, v) \mapsto (u, \sigma(v))$. We let $\alpha \in H^1(S, \mathbb{Z}/2)$ be the corresponding cohomology class.

By [6, Lemma 5.1] and Artin’s comparison theorem, there exists a cohomology class $\beta' \in H^1(S, \mathbb{Z}/2)$ such that $\alpha \beta' \neq 0$ and $(\beta')^2 = 0$. By the invariance of étale cohomology under extensions of algebraically closed fields [31, Corollary VI.2.6], $\beta'$ is defined over $\overline{K}$, hence over a suitable finite extension of $K'$ of $K$. Replacing $R$ by its integral closure in $K'$ and localizing at one of the maximal ideals, we may assume that $\beta'$ is defined over $K$.

Since $p \neq 2$, we have an isomorphism of $R$-group schemes $\mu_{2, R} \simeq (\mathbb{Z}/2)_R$. It follows from the Kummer sequence

$$1 \rightarrow \mu_2 \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 1$$

over $S_R$ and $\overline{S}$ that $H^1(S, \mathbb{Z}/2) = \text{Pic}(S)[2]$ and $H^1(S_R, \mathbb{Z}/2) = \text{Pic}(\overline{S_R})[2]$. Since the special fiber of $S \rightarrow \text{Spec}(R)$ is a principal divisor, the restriction map $\text{Pic}(S) \rightarrow \text{Pic}(S_R)$ is an isomorphism, hence

$$H^1(S, \mathbb{Z}/2) \rightarrow H^1(S_R, \mathbb{Z}/2)$$

is surjective.

Therefore there exists $\beta \in H^1(S, \mathbb{Z}/2)$ such that $\beta \beta' = \beta'$.

Let $E$ be another elliptic curve over $\text{Spec}(R)$. (For example, one may take $E = E_1$ or $E = E_2$.) Define $Y := (S \times_R E)/(\mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts on $S$ as $\tilde{\sigma}$ and on
E as \(-\text{Id}\). Let \(\pi: Y \rightarrow S\) be the morphism induced by the first projection, and let \(Y' \rightarrow Y\) be the double cover corresponding to \(\pi^*(\beta)\). Define \(Z \coloneqq (Y' \times_Y E)/(\mathbb{Z}/2)\), where \(\mathbb{Z}/2\) acts via the involution corresponding to the double cover \(Y' \rightarrow Y\) on the left and via \(-\text{Id}\) on the right. It is proved in [6, Proof of Proposition 5.3] that there exists \(\sigma' \in H^3_{\text{sing}}(\mathbb{Z}[\mathbb{Z}], \mathbb{Z})[2]\) such that \(\pi_2(\sigma')^2 \neq 0\). By Artin’s comparison theorem, we deduce the existence of \(\sigma \in H^3(\mathbb{Z}[\mathbb{Z}], \mathbb{Z})[2]\) such that \(\pi_2(\sigma)^2 \neq 0\). By the invariance of étale cohomology under extensions of algebraically closed fields, \(\sigma\) is defined over \(\overline{K}\). Therefore, using (4.1) and enlarging \(R\) if necessary, we may suppose that \(\sigma\) is defined over \(R\). Letting \(X\) be the special fiber of \(Z\), by Lemma 2.1 the restriction of \(\sigma\) to \(X\) satisfies the conditions of Proposition 3.8, and the conclusion follows.

Remark 4.1. We give a shorter proof of the following weakening of Theorem 1.5: For all but finitely many primes \(p\), there exist a finite field \(\mathbb{F}\) of characteristic \(p\) and a smooth projective fourfold \(X\) over \(\mathbb{F}\) such that the conclusion of Theorem 1.5 holds for \(X\).

Indeed, by [6, Proposition 5.3] and Artin’s comparison theorem, there exist a smooth projective fourfold \(Y\) over \(\mathbb{C}\) and a \(2\)-torsion class \(\sigma \in H^3(\mathbb{Z}[\mathbb{Z}], \mathbb{Z})\) such that \(\pi_2(\sigma)^2 \neq 0\). By a spreading out argument, there exist a finitely generated field extension \(K/\mathbb{Q}\), a smooth integral \(\mathbb{Z}\)-scheme of finite type \(S\) with fraction field \(K\), and a smooth projective scheme \(Z \rightarrow S\) such that \(\mathbb{Z}_C \simeq Z\). The morphism \(S \rightarrow \text{Spec}(\mathbb{Z})\) is flat, hence open. It follows that all but finitely many primes appear as residue characteristics of closed points of \(S\).

Let \(p \neq \ell\) be one such prime, \(s \in S\) be a closed point whose residue field \(k(s)\) is finite of characteristic \(p\), and set \(\mathbb{F} := k(s)\) and \(X := (Z)_{k(s)}\). By [22, Proposition 7.1.9], there exist a discrete valuation ring \(R\), with generic point \(\eta\) and closed point \(t\), and a morphism \(\text{Spec}(R) \rightarrow S\) which sends \(\eta\) to the generic point of \(S\), \(t\) to \(s\), and such that the inclusion \(K \hookrightarrow k(\eta)\) is an equality. By the invariance of étale cohomology under field extensions and the smooth and proper base change theorems, we have an isomorphism

\[
H^*(Z, \mathbb{Z}_2) \simeq H^*(Z_{\mathbb{F}}, \mathbb{Z}_2) \simeq H^*(X_{k(\eta)}, \mathbb{Z}_2) \simeq H^*(\overline{X}, \mathbb{Z}_2),
\]

where \(\overline{X} = X_{\mathbb{F}}\), and a compatible isomorphism with \(\mathbb{Z}/2\) coefficients. Let \(\alpha \in H^3(\overline{X}, \mathbb{Z}_2)\) be the image of \(\sigma\) under (4.2). By Lemma 2.1, we have \(\pi_2(\alpha)^2 \neq 0\). Replacing \(\mathbb{F}\) by a finite extension if necessary, we may suppose that \(G\) acts trivially on \(H^3(\overline{X}, \mathbb{Z}/2)\). The conclusion now follows from Proposition 3.8.

We now show that Proposition 3.8 cannot be used to find a 3-dimensional example \(X\) with \(H^3_{\text{sing}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \neq 0\).

Proposition 4.2. Let \(\mathbb{F}\) be a finite field, \(\ell\) be a prime number invertible in \(\mathbb{F}\), \(X\) be a smooth projective threefold over \(\mathbb{F}\), \(\alpha \in H^3(X, \mathbb{Z}_\ell(2))\) be a class such that \(\alpha_{\mathbb{F}} = 0\), and let \(\overline{\alpha}\) the reduction of \(\alpha\) modulo \(\ell\). Suppose that \(\mathbb{F}\) contains a primitive \(\ell^2\)-th root of unity. Then all odd-degree étale Steenrod operations on \(X\) vanish on \(\overline{\alpha}\).

Proof. In view of Corollary 3.5, it suffices to show that all odd-degree étale Steenrod operations vanish on the image of the reduction map \(H^3(\overline{X}, \mathbb{Z}_\ell) \rightarrow H^3(\overline{X}, \mathbb{Z}/\ell)\). This is clear if \(\ell\) is odd: indeed, the only non-zero Steenrod operation of odd degree \(\leq 3\) is the Bockstein, which vanishes on \(\overline{\alpha}\) because \(\overline{\alpha}\) lifts to an integral class. Therefore, we may suppose that \(\ell = 2\). In this case, the proof follows from the argument of [6, Proposition 3.6], replacing the topological Eilenberg-MacLane spectrum \(HZ\) by the \(\ell\)-adic Eilenberg-MacLane spectrum \(H\mathbb{Z}_\ell\) and the complex cobordism spectrum \(MU\) by the \(\ell\)-adic cobordism spectrum \(MU\); see [37].
4.2. Proof of Theorem 1.4. Before we begin with the proof of Theorem 1.4, we need some observations about the \( \ell \)-adic cohomology and cycle class map of the classifying space of \( \text{PGL}_{\ell} \).

Lemma 4.3. Let \( k \) be an algebraically closed field and \( \ell \) be a prime number invertible in \( k \).

(a) If \( \ell = 2 \), there exists \( u_3 \in H^3(B_k \text{PGL}_2, \mathbb{Z}/2) \) such that \( \text{Sq}^1(\pi_2(u_3)) \neq 0 \).

(b) If \( \ell \) is odd, there exists \( u_3 \in H^3(B_k \text{PGL}_\ell, \mathbb{Z}/\ell) \) such that \( \beta^1(\pi_\ell(u_3)) \neq 0 \).

Proof. By Lemma 2.2, we may suppose that \( k = \mathbb{C} \) and replace étale cohomology by singular cohomology.

(a) We have an isomorphism \( \text{PGL}_2(\mathbb{C}) \simeq \text{SO}_3(\mathbb{C}) \). Write \( H^*_\text{sing}(B\text{SO}_3(\mathbb{C}), \mathbb{Z}/2) = F_2[w_2, w_3] \), where \( w_2 \) and \( w_3 \) are the Stiefel-Whitney classes; see e.g. [47, Theorem 2.2, (2.3)]. By Wu’s formula, we have \( \text{Sq}^1(w_3) = w_3 \). Define \( u_3 = w_3 \). Since \( \text{Sq}^1 = \beta_3 \), we have \( \pi_2(u_3) = w_3 \), hence \( \text{Sq}^1(\pi_2(u_3)) = \pi_2(u_3)^2 = w_3^2 \neq 0 \).

(b) By [51, Theorem 3.6(b)] or [3, §3], we have

\[
H^3_{\text{sing}}(B\text{PGL}_\ell(\mathbb{C}), \mathbb{Z}/\ell) = \begin{cases} 
\mathbb{Z} & i = 0, \\
0 & i = 1, \\
\mathbb{Z}/\ell & i = 3, \\
\mathbb{Z} & i = 4.
\end{cases}
\]

From (4.3) and the universal coefficient theorem, we deduce that

\[
H^2_{\text{sing}}(B\text{PGL}_\ell(\mathbb{C}), \mathbb{Z}/\ell) \simeq \mathbb{Z}/\ell
\]

and that \( \pi_\ell : H^3_{\text{sing}}(B\text{PGL}_\ell(\mathbb{C}), \mathbb{Z}) \to H^3_{\text{sing}}(B\text{PGL}_\ell(\mathbb{C}), \mathbb{Z}/\ell) \) is an isomorphism.

Consider the Serre spectral sequence associated to the Serre fibration

\[
B\text{GL}_\ell(\mathbb{C}) \to B\text{PGL}_\ell(\mathbb{C}) \overset{f}{\longrightarrow} K(\mathbb{Z}, 3).
\]

We have an isomorphism \( H^*_\text{sing}(B\text{GL}_\ell(\mathbb{C}), \mathbb{Z}/\ell) \simeq (\mathbb{Z}/\ell)[c_1, \ldots, c_t] \), where \( |c_i| = 2i \).

Since \( H^2_{\text{sing}}(B\text{PGL}_\ell(\mathbb{C}), \mathbb{Z}/\ell) \neq 0 \), we have \( d_2(c_1) = 0 \) and \( d_3(c_1) = 0 \), hence

\[
f^* : H^3_{\text{sing}}(K(\mathbb{Z}, 3), \mathbb{Z}/\ell) \to H^3_{\text{sing}}(B\text{PGL}_\ell(\mathbb{C}), \mathbb{Z}/\ell)
\]

is an isomorphism. By [32, VII.4.19(2)], \( H^3_{\text{sing}}(K(\mathbb{Z}, 3), \mathbb{Z}/\ell) = (\mathbb{Z}/\ell)[v_3] \), for some \( v_3 \neq 0 \). Let \( u_3 \) be the unique element of \( H^3_{\text{sing}}(B\text{PGL}_\ell(\mathbb{C}), \mathbb{Z}) \) such that \( u_3 = f^*(v_3) \).

Recall from [32, VII.4.19(1)] that \( H^*(K(\mathbb{Z}, 2), \mathbb{Z}/\ell) = (\mathbb{Z}/\ell)[z_2] \), where \( |z_2| = 2 \).

Consider the mod \( \ell \) Serre spectral sequence associated to the Serre fibration

\[
K(\mathbb{Z}, 2) \to B\text{GL}_\ell(\mathbb{C}) \to B\text{PGL}_\ell(\mathbb{C}).
\]

We must have \( d_3(z_2) = \lambda \pi_\ell(u_3) \) for some \( \lambda \in (\mathbb{Z}/\ell)^\times \), hence by the Kudo transgression theorem [30, Theorem 6.14] the class \( z_2^{-1} \oplus \pi_\ell(u_3) \in E_2^{0,2,3} \) is transgressive and \( d_{2\ell-1}(z_2^{-1} \oplus \pi_\ell(u_3)) = -\beta_\ell \pi_\ell(u_3) \), that is, \( d_{2\ell-1}(z_2^{-1} \oplus \pi_\ell(u_3)) = -\beta_\ell \pi_\ell(u_3) \). Since \( H^*(B\text{GL}_\ell(\mathbb{C}), \mathbb{Z}/\ell) \) is concentrated in even degrees and the total degree of \( z_2^{-1} \oplus \pi_\ell(u_3) \) is odd, \( z_2^{-1} \oplus \pi_\ell(u_3) \) does not survive in the \( E_\infty \) page, hence \( \beta_\ell \pi_\ell(u_3) \neq 0 \), as desired.

Lemma 4.4. Let \( k \) be a field and \( \ell \) be a prime number invertible in \( k \).

(a) Let \( i \geq 0 \) be an integer, and suppose that \( G \) is a reductive group scheme over \( \mathbb{Z} \) and that the cycle map

\[
CH^i(B_k G) \otimes \mathbb{Z}_\ell \to H^{2i}(B_k G, \mathbb{Z}_\ell(i))
\]

is surjective. Then

\[
CH^i(B_k G) \otimes \mathbb{Z}_\ell \to H^{2i}(B_k G, \mathbb{Z}_\ell(i))
\]
is also surjective.

(b) The cycle map

\[ CH^i(B_k \text{PGL}_d) \otimes \mathbb{Z}_\ell \to H^{2i}(B_k \text{PGL}_d, \mathbb{Z}_\ell(i)) \]

is surjective for all \( i \geq 0 \).

**Proof.** (a) If \( \text{char}(k) = 0 \), this follows from the invariance of étale cohomology under extensions of algebraically closed fields. Suppose that \( \text{char}(k) = p > 0 \). By the invariance of étale cohomology under extensions of algebraically closed fields, we may suppose that \( k = \mathbb{F}_p \). There exist a \( \mathbb{G}_{\mathbb{F}_p} \)-representation \( V \) and an open subscheme \( U \subset V \) such that \( V_{\mathbb{F}_p} - U_{\mathbb{F}_p} \) and \( V_{\mathbb{Q}_p} - \mathbb{Q}_p \) have codimension \( i + 1 \) in \( V_{\mathbb{F}_p} \) and \( V_{\mathbb{Q}_p} \), respectively, and a \( \mathbb{G}_{\mathbb{F}_p} \)-torsor \( U \to B \), where \( B \) is a smooth \( \mathbb{Z}_p \)-scheme.

Let \( R := W(\mathbb{F}_p) \). Fix an algebraic closure \( \mathbb{Q}_p \) of \( \mathbb{Q}_p \) and an inclusion of \( \mathbb{Z}_p \)-algebras \( R \subset \mathbb{Q}_p \). We obtain a commutative diagram

\[
\begin{array}{ccc}
Z^i(B_{Q_p}) \otimes \mathbb{Z}_\ell & \rightarrow & H^{2i}(B_{\mathbb{Q}_p}, \mathbb{Z}_\ell(i)) \\
\downarrow & & \downarrow \\
Z^i(\text{flat}(B/\mathbb{Z}_p)) \otimes \mathbb{Z}_\ell & \rightarrow & H^{2i}(B_R, \mathbb{Z}_\ell(i)) \\
\downarrow & & \downarrow \\
Z^i(B_{\mathbb{F}_p}) \otimes \mathbb{Z}_\ell & \rightarrow & H^{2i}(B_{\mathbb{F}_p}, \mathbb{Z}_\ell(i)).
\end{array}
\]

Here \( Z^i(\text{flat}(B/\mathbb{Z}_p)) \) is the free abelian group generated by classes of integral subschemes of \( B \) which are flat (that is, dominant) over \( \mathbb{Z}_p \). The horizontal maps in the middle is defined as the inverse limit in \( n \) of the cycle maps

\[ Z^i(\text{flat}(B/\mathbb{Z}_p)) \rightarrow H^{2i}(B, \mu_{p^{2n}}^\otimes) \rightarrow H^{2i}(B_R, \mu_{p^{2n}}^\otimes) \]

of [13, Chapitre 4, §2.3] (where we take \( S = \text{Spec}(\mathbb{Z}_p) \) in the definition). The top and bottom horizontal homomorphisms are the usual cycle maps, the top vertical maps are pullbacks along the open embeddings \( B_{Q_p} \hookrightarrow B \) and \( B_{\mathbb{Q}_p} \hookrightarrow B_R \), and the bottom vertical maps are pullbacks along the closed embeddings \( B_{\mathbb{F}_p} \hookrightarrow B \) and \( B_{\mathbb{F}_p} \hookrightarrow B_R \). The top-left vertical map is surjective: indeed, if \( Z \subset B_{Q_p} \) is an integral subscheme, its closure inside \( B \) is irreducible, hence flat over \( R \).

Since \( V_{\mathbb{F}_p} - U_{\mathbb{F}_p} \) and \( V_{\mathbb{Q}_p} - \mathbb{Q}_p \) have codimension \( i + 1 \) in \( V_{\mathbb{F}_p} \) and \( V_{\mathbb{Q}_p} \), respectively, by definition we have \( CH^i(B_{Q_p}) = CH^i(B_{\mathbb{Q}_p}, G) \) and \( CH^i(B_{\mathbb{F}_p}) = CH^i(B_{\mathbb{F}_p}, G) \). Moreover, the natural morphism \( B \to B_{\mathbb{F}_p} G \) induces a commutative diagram

\[
\begin{array}{ccc}
H^{2i}(B_{\mathbb{Q}_p}, \mathbb{Z}_\ell(i)) & \sim & H^{2i}(B_{\mathbb{F}_p} G, \mathbb{Z}_\ell(i)) \\
\uparrow & & \uparrow \\
H^{2i}(B_R, \mathbb{Z}_\ell(i)) & \sim & H^{2i}(B_R G, \mathbb{Z}_\ell(i)) \\
\downarrow & & \downarrow \\
H^{2i}(B_{\mathbb{F}_p}, \mathbb{Z}_\ell(i)) & \sim & H^{2i}(B_{\mathbb{F}_p} G, \mathbb{Z}_\ell(i)).
\end{array}
\]

By [19, Corollary 2] the two vertical maps on the right are isomorphisms. The conclusion follows from (4.4).

(b) In view of (a), we may suppose that \( k = \mathbb{Q} \). The conclusion now follows from [34] when \( \ell = 2 \), and is the content of [51, Corollary 3.5] when \( \ell \neq 2 \). \( \square \)

**Lemma 4.5.** Let \( \mathbb{F} \) be a finite field, \( \ell \) be a prime invertible in \( \mathbb{F} \), and suppose that \( \mathbb{F} \) contains a primitive \( \ell \)-th root of unity. Then the \( G \)-action on the group \( H^{2i+2}(B_{\mathbb{F}}(\text{PGL}_d \times \mathbb{G}_m), \mathbb{Z}/\ell) \) is trivial.
Proof. Let \( p := \text{char}(F) \). Since \( F \) contains a primitive \( \ell \)-th root of unity, the \( G \)-action on \( H^{2i}(B_P G_m, Z/\ell) \simeq Z/\ell \) is trivial for all \( i \geq 0 \). By the Künneth formula, it thus suffices to show that \( H^{2i}(B_P PGL_\ell, Z/\ell) \) is \( G \)-invariant for all \( 0 \leq i \leq \ell + 1 \).

This is clear for \( i = 0 \).

By \cite[Theorem 3.6]{51} we have \( H^3(B_P PGL_\ell, Z_\ell) = Z/\ell \) and \( H^{2i+1}(B_P PGL_\ell, Z_\ell) = 0 \) for all \( 2 \leq i \leq \ell + 1 \). (This is sharp, as \( H^{2i+1}(B_P PGL_\ell, Z_\ell) \neq 0 \).) The Bockstein short exact sequence (see \cite[Lemma V.1.11]{31})

\[
0 \to H^2(B_P PGL_\ell, Z_\ell)/\ell \to H^2(B_P PGL_\ell, Z/\ell) \to H^{2i+1}(B_P PGL_\ell, Z_\ell)[\ell] \to 0
\]

is \( G \)-equivariant. We deduce that the reduction map

\[
H^{2i}(B_P PGL_\ell, Z_\ell)/\ell \to H^{2i}(B_P PGL_\ell, Z/\ell)
\]

is an isomorphism for all \( 2 \leq i \leq \ell + 1 \). Now \textbf{Lemma 4.4}(b) implies that \( G \) acts trivially on \( H^{2i}(B_P PGL_\ell, Z/\ell) \) for all \( 2 \leq i \leq \ell + 1 \).

Suppose now that \( i = 1 \). Then by \textbf{(4.3)} we know that \( H^2(B_P PGL_\ell, Z_\ell) = 0 \), hence by \textbf{(4.5)} we have a \( G \)-equivariant isomorphism

\[
H^2(B_P PGL_\ell, Z_\ell) \sim H^3(B_P PGL_\ell, Z_\ell).
\]

Consider the Serre spectral sequence with \( Z_\ell \) coefficients associated to the fibration \( B_P G_m \to B_P GL_\ell \to B_P PGL_\ell \):

\[
E_2^{i,j} = H^i(B_P PGL_\ell, H^j(B_P G_m, Z_\ell)) \Rightarrow H^{i+j}(B_P PGL_\ell, Z_\ell).
\]

Since the fibration is defined over \( F \), the spectral sequence is \( G \)-equivariant. Let \( h \) be a generator of \( E_2^{i,j} = H^j(B_P G_m, Z_\ell) \simeq Z_\ell(-1) \). Since \( H^3(B_P GL_\ell, Z_\ell) = 0 \), the group \( E_2^{0,3} = H^3(B_P PGL_\ell, Z_\ell) \simeq Z/\ell \) does not survive to the \( E_\infty \) page, and so it is generated by \( d_3(h) \).

Let \( \sigma \in G \) be the Frobenius endomorphism. The \( G \)-action on \( H^3(B_P G_m, Z_\ell)/\ell \) is trivial, hence \( \sigma(h) - h \) is a multiple of \( \ell h \). It follows that \( \sigma(d_3(h)) - d_3(h) \in H^3(B_P PGL_\ell, Z_\ell) \) is also a multiple of \( \ell \). Since \( H^3(B_P PGL_\ell, Z_\ell) = (Z/\ell) \cdot d_3(h) \) is \( \ell \)-torsion, this means that \( d_3(h) \) is \( G \)-invariant, that is, that \( G \) acts trivially on \( H^3(B_P PGL_\ell, Z_\ell) \).

\( \square \)

\textbf{Proof of Theorem 1.4.} By \cite[Remark (v) p. 5 and §1.1]{15}, there exist a smooth projective \( F \)-variety \( X \) of dimension \( 2\ell + 3 \) and a morphism \( \varphi : X \to B_P (PGL_\ell \times G_m) \) over \( F \) such that the pullback

\[
\varphi^* : H^*(B_P (PGL_\ell \times G_m), Z_\ell) \to H^*(X, Z_\ell)
\]

is an isomorphism in degrees \( \leq 2\ell + 2 \) and is injective with torsion-free cokernel in degree \( 2\ell + 3 \). We may assume that \( X \) and \( \varphi \) are defined over the prime field of \( X \). The cycle map

\[
\text{cl} : C H^*(B_P G_m) \to H^{2*}(B_P G_m, Z_\ell(i)) \simeq Z_\ell[\ell], \quad |\ell| = 2
\]

is an isomorphism. We have a commutative square

\[
\begin{array}{ccc}
C H^*(B_P PGL_\ell) \otimes C H^*(B_P G_m) \otimes Z_\ell & \longrightarrow & C H^*(B_P (PGL_\ell \times G_m)) \otimes Z_\ell \\
\downarrow \text{cl} \otimes \text{id} & & \downarrow \text{cl} \\
H^{2*}(B_P PGL_\ell, Z_\ell(+)) \otimes Z_\ell & \longrightarrow & H^{2*}(B_P (PGL_\ell \times G_m), Z_\ell(+)),
\end{array}
\]

where the top horizontal map is the Künneth isomorphism of \cite[Lemma 2.12(i)]{48}, the bottom horizontal map is the Künneth isomorphism in étale cohomology, and the vertical maps are the cycle class maps. We deduce from \textbf{Lemma 4.4}(b) that the
cycle map $\text{CH}^i(B_{\mathbb{F}}(\text{PGL}_\ell \times \mathbb{G}_m)) \otimes \mathbb{Z}_\ell \to H^{2i}(B_{\mathbb{F}}(\text{PGL}_\ell \times \mathbb{G}_m), \mathbb{Z}_\ell(i))$ is surjective for all integers $i \geq 0$. It now follows from the commutativity of the square

$$
\begin{array}{ccc}
\text{CH}^2(B_{\mathbb{F}}(\text{PGL}_\ell \times \mathbb{G}_m)) \otimes \mathbb{Z}_\ell & \longrightarrow & \text{CH}^2(X) \otimes \mathbb{Z}_\ell \\
\downarrow & & \downarrow \\
H^4(B_{\mathbb{F}}(\text{PGL}_\ell \times \mathbb{G}_m), \mathbb{Z}_\ell(2)) & \sim & H^4(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2))
\end{array}
$$

that the cycle map $\text{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2))$ is surjective.

From the $\ell$-adic Bockstein short exact sequence [31, Lemma V.1.11] we see that

$$
\varphi^*: H^*(B_{\mathbb{F}}(\text{PGL}_\ell \times \mathbb{G}_m), \mathbb{Z}/\ell) \to H^*(\overline{\mathcal{X}}, \mathbb{Z}/\ell)
$$

is an isomorphism in degrees $\leq 2\ell + 2$. (Here we use the fact that the pullback is injective with torsion-free cokernel in degree $2\ell + 3$.) In particular, by Lemma 4.5, $G$ acts trivially on $H^{2\ell+2}(\overline{\mathcal{X}}, \mathbb{Z}/\ell)$. Let $\varphi' := \text{pr}_1 \circ \varphi$, where

$$
\text{pr}_1: B_{\mathbb{F}}(\text{PGL}_\ell \times \mathbb{G}_m) \to B_{\mathbb{F}} \text{PGL}_\ell
$$

is the first projection. As a consequence, by the Künneth formula in étale cohomology and the injectivity of $\varphi^*$ is degrees $\leq 2\ell + 2$, the composition

$$(\varphi')^*: H^*(B_{\mathbb{F}}(\text{PGL}_\ell, \mathbb{Z}/\ell)) \to H^*(\overline{\mathcal{X}}, \mathbb{Z}_\ell)$$

is injective in degrees $\leq 2\ell + 2$.

Consider the Steenrod operation $P$ given by $P = \text{Sq}^3$ if $\ell = 2$ and $P = \beta P^1$ if $\ell \neq 2$. The degree of $P$ is equal to the odd number $2\ell - 1$. By Lemma 4.3, there exists a class $u_3 \in H^3(B_{\mathbb{F}} \text{PGL}_\ell, \mathbb{Z}_\ell(2))$ such that $P(\pi_3(u_3)) \neq 0$. Define $u := (\varphi')^*(u_3)$. Since $P(\pi_3(u_3))$ has degree $2\ell + 2$ and $(\varphi')^*$ is injective in degrees $\leq 2\ell + 2$, we have $P(\pi_3(u)) \neq 0$. Since $G$ acts trivially on $H^{2\ell+2}(\overline{\mathcal{X}}, \mathbb{Z}/\ell)$, the assumptions of Proposition 3.8 are satisfied, hence the conclusion follows from Proposition 3.8. 

\[ \square \]

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