Spectrum of the hydrogen atom in Snyder space
in a semiclassical approximation

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Abstract

We study the spectrum of the hydrogen atom in Snyder space in a semiclassical approximation based on a generalization of the Born-Sommerfeld quantization rule. While the corrections to the standard quantum mechanical spectrum arise at first order in the Snyder parameter for the $l = 0$ states, they are of second order for $l \neq 0$. This can be understood as due to the different topology of the regions of integration in phase space.

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1. Introduction

Quantum mechanics (QM) with modified commutation relations has attracted much attention [1,2], since it implies the existence of a minimal observable length, in accordance with most models of quantum gravity, that predict a minimal length of the order of the Planck scale [3].

A particularly interesting example of deformation of the canonical commutation relations is given by the nonrelativistic Snyder model [4,5], which is based on the Euclidean three-dimensional version of the commutation relations originally proposed by Snyder [6], in his search for a divergenceless field theory. These commutation relations read

\[
[x_i, p_j] = i(\delta_{ij} + \beta^2 p_i p_j), \quad [x_i, x_j] = i\beta^2 J_{ij}, \quad [p_i, p_j] = 0, \quad (1)
\]

where \( J_{ij} = x_i p_j - x_j p_i \) are the generators of rotations and \( \beta \) is a parameter with the dimension of inverse momentum. The model admits an \( SO(4) \) symmetry in phase space, generated by the generators \( J_{ij} \) and \( x_i \). Its implications have been investigated by several authors both in its classical [7-8] and quantum version [4-5] and some simple systems like the harmonic oscillator have been exactly solved.

The one-dimensional version of the Snyder quantum mechanics coincides with one of the favourite models of deformed QM, its only nontrivial commutator being

\[
[x, p] = i(1 + \beta^2 p^2). \quad (2)
\]

This has been the subject of even wider investigations [2,4,9-12], showing in particular that it implies the existence of a minimal resolution attainable from measures of length, \( \Delta x \geq \beta \).

In this letter, we deal with the problem of the hydrogen atom in Snyder space. This problem has been studied in several papers [10-14]. In one dimension, it is possible to obtain exact results for the energy spectrum [10-12], while in the more interesting three-dimensional case, only perturbative solutions have been found [13-14]. Moreover, while in one dimension the corrections to the spectrum are of order \( \beta \) [10-12], in three dimensions they start from order \( \beta^2 \) [13-14], and hence the zero angular momentum sector of the theory, which coincides with the 1D problem, cannot be obtained as a smooth limit of the higher angular momentum sector for \( l \to 0 \). This fact seems to have been disregarded in previous investigations\(^2\).

The reason of this discrepancy lays presumably in the fact that the three-dimensional Hamiltonian usually postulated for the Snyder model does not preserve its full \( SO(4) \) symmetry, but only its rotational subgroup, and hence the degeneracy in the angular momentum of the QM hydrogen spectrum is lost in this case. The full symmetry could be restored by modifying the potential in a suitable way [15,8].

\(^1\) The result of [10], where the first corrections are found at order \( \beta^2 \), is not correct, see [11].

\(^2\) Actually, perturbative calculations give divergent results for \( l = 0 \) [14]. This appears as a confirmation that in this case the perturbative expansion does not start from order \( \beta^2 \) terms.
In this paper, we obtain an exact solution for the spectrum of the hydrogen atom in Snyder space in a semiclassical approximation, using the Bohr-Sommerfeld (BS) quantization rule [16]. In this way the problem is essentially reduced to a classical one. We remark that in the 1D case and also in 3D ordinary QM this calculation gives the exact spectrum. We do not know whether this is true also for 3D Snyder QM, but certainly one obtains at least a good approximation.

To perform the calculation, polar momentum coordinates must be used. These are not common in classical mechanics, but are the equivalent of those used in the momentum representation of QM, and are simply the dual of the usual polar position coordinates. The use of a momentum representation is standard in the investigation of the Snyder dynamics, because it gives the most natural representation of the Snyder commutation relations [3,4].

In standard QM, the investigation of the hydrogen atom in momentum representation has a long history. It was first investigated by Fock [17]. The one-dimensional case had later been considered in [18].

2. The semiclassical approximation

The semiclassical approximation starts from the study of the classical dynamics, where the commutators are replaced by Poisson brackets. Since the symplectic structure of the Snyder model is noncanonical, its classical dynamics is best written in Hamiltonian form. The standard choice for the Hamiltonian is identical to the classical one,

\[
H = \frac{p^2}{2m} - \frac{e^2}{r},
\]

where \( p^2 \) is the square of the momentum and \( r = \sqrt{x^2} \). This Hamiltonian is invariant under the rotation group \( SO(3) \), but not under the full \( SO(4) \) Snyder group [15]. However, it is the one that is usually adopted in this context. Due to the rotational invariance, the angular momentum is conserved and hence the classical orbits are confined to a plane, as in Newtonian mechanics, and the quantum spectrum will be independent on the magnetic quantum number. In future, we plan to study more general Hamiltonians that preserve the full Snyder group.

The difference between Snyder and Newtonian mechanics resides in the symplectic structure, that is of course modified. For example, in cartesian coordinates, the Snyder symplectic form can be written as

\[
d\omega_S = - \left( \delta_{ij} - \frac{\beta^2 p_i p_j}{1 + \beta^2 p^2} \right) x_i dp_j.
\]

This leads to a different classical motion in the two cases.

To calculate the spectrum of the 3D hydrogen atom in a semiclassical approximation, we have to generalize the standard BS quantization condition to noncanonical systems. We require that the phase integral over each degree of freedom is an integer,

\[
\int d\omega_i = 2\pi n_i,
\]
where \( d\omega = \sum_i d\omega_i \) is the symplectic form relative to the model under study. In the Newtonian case (5) reduces to the standard rule since \( d\omega_{N,i} = p_i dx_i \).

Before discussing the 3D problem, we briefly review the results obtained in 1D in [11], using a slightly different approach from ours. Of course, in 1D the rotational invariance is absent and the problem simplifies.

The 1D Hamiltonian is

\[
H = \frac{p^2}{2m} - \frac{e^2}{x} = -E,
\]

with \( x > 0 \), where the constant \(-E\) is the total energy (the minus sign is inserted because we are considering bound states, so that \( E \) is positive). The Newtonian symplectic form is simply \( d\omega_N = pdx \) and the BS condition reads

\[
\oint pdx = -\oint xdp = 2\pi n. \tag{7}
\]

In the following, we shall use the second form, because it is more apt to a generalization to the Snyder case.

On a closed orbit, starting from the origin, \( p \) goes from \( \infty \) to 0 and in the way back from 0 to \(-\infty\), while (6) yields \( x = 2me^2/(p^2 + 2mE) \). One has then

\[
\oint d\omega_N = -\oint xdp = \int_{-\infty}^{\infty} \frac{2me^2}{p^2 + 2mE} \, dp = \pi \sqrt{\frac{2me^4}{E}}. \tag{8}
\]

Imposing the BS condition gives the spectrum

\[
E_n = \frac{me^4}{2n^2}, \tag{9}
\]

which coincides with the exact result [18].

In the Snyder case, the calculation is exactly the same, but the symplectic form is

\[
d\omega_S = -\frac{xdp}{1 + \beta^2 p^2}. \tag{10}
\]

The generalized BS quantization condition reads then

\[
\int_{-\infty}^{\infty} \frac{2me^2 dp}{(p^2 + 2mE)(1 + \beta^2 p^2)} = \frac{\sqrt{2me^4} \pi}{\sqrt{E(1 + \beta \sqrt{2mE})}} = 2\pi n. \tag{11}
\]

This again coincides with the exact condition found in [11]. Note that a ”miraculous” cancellation turns the dependence of the corrections on \( \beta^2 \), that one would expect from the integral in (11), into a dependence on \( \beta \). More explicitly, an expansion in powers of \( \beta \) gives

\[
E_n = \frac{me^4}{2n^2} \left( 1 - \frac{2\beta me^2}{n} + \ldots \right). \tag{12}
\]
3. The hydrogen atom in three dimensions

In three dimensions, the BS rule requires that the integral along an orbit of each term of the symplectic form must be an integer. The calculation is best performed using spherical coordinates in the space of momenta. The BS condition applied to the angular variables gives as usual the quantization of the $z$-component $m$ and of the norm $l$ of the angular momentum.

For what concerns the radial part, one can simplify the calculation, recalling that because of the conservation of the angular momentum, valid both in the Newtonian and Snyder cases, the classical motion is confined to a plane. We shall therefore use polar momentum coordinates on a plane, defined as [19]

\[
\begin{align*}
p_\rho &= \sqrt{p_1^2 + p_2^2}, \quad p_\theta = \arctan \frac{p_2}{p_1}, \\
\rho &= \frac{p_1 x_1 + p_2 x_2}{\sqrt{p_1^2 + p_2^2}}, \quad J = J_{12} = x_1 p_2 - x_2 p_1.
\end{align*}
\]

These are duals of the polar position coordinates. In particular, $p_\rho$ is the norm of the momentum $p$ and $\rho$ is not the radial coordinate, but rather its projection along $p$. $J$ is the angular momentum, that is conserved for spherically symmetric potentials and takes the value $l$.

The previous coordinates obey the Poisson brackets

\[
\begin{align*}
\{p_\rho, p_\theta\} &= 0, & \{\rho, J\} &= 0, & \{p_\rho, J\} &= 0, \\
\{p_\rho, \rho\} &= 1 + \beta^2 p_\rho^2, & \{p_\theta, J\} &= 1, & \{p_\theta, \rho\} &= 0,
\end{align*}
\]

from which one can easily obtain the symplectic structure $d\omega$, which maintains a simple form when passing from Newtonian to Snyder mechanics [19]. In fact,

\[
d\omega_N = -(\rho dp_\rho + J dp_\theta) \rightarrow d\omega_S = -\left(\frac{\rho dp_\rho}{1 + \beta^2 p_\rho^2} + J dp_\theta\right). \tag{14}
\]

However, in these coordinates the Hamiltonian takes an unusual form,

\[
H = \frac{p_\rho^2}{2m} - \frac{e^2}{\sqrt{\rho^2 + J^2 / p_\rho^2}} = -E \tag{15}
\]

where $E$ is the conserved energy.

For $l = 0$, all the formulae reduce to the ones valid in 1D, and the spectrum is given by (12). If $l \neq 0$, instead,

\[
\rho = \pm l \sqrt{-p_\rho^4 + \left(4m^2e^4 - 4mE\right)p_\rho^2 - 4m^2E^2} \over p_\rho(p_\rho + 2mE), \tag{16}
\]
and $\rho$ is real in the interval $z_- \leq z \leq z_+$, where we have defined $z = p_\rho^2$, and

$$z_\pm = 2m \left( \frac{me^4}{l^2} - E \pm \frac{e^2}{l} \sqrt{\frac{m^2e^4}{l^2} - 2mE} \right). \quad (17)$$

In the Newtonian case, the radial integral reads

$$\oint -\rho dp_\rho = l \int_{z_-}^{z_+} \frac{\sqrt{(z-z_-)(z_+-z)}}{z(z+2mE)} \, dz = \pi \left( \sqrt{\frac{2me^4}{E}} - 2l \right). \quad (18)$$

Notice that, contrary to the 1D case, the integral extends over a finite region of phase space. Equating the result to $2\pi n$ and defining a new quantum number $n' = n + l$, one obtains the well-known spectrum $E_{n'} = \frac{me^4}{2n'^2}$.

In the Snyder case, the only change is in the symplectic form. We have now

$$\oint -\rho dp_\rho = l \int_{z_1}^{z_2} \frac{\sqrt{(z-z_1)(z_2-z)}}{z(z+2mE)(1+\beta^2 z)} \, dz$$

$$= \pi \left[ \frac{\sqrt{2me^4}}{\sqrt{E(1-2\beta^2 mE)}} - l \left( 1 + \frac{4\beta^2 m^2 e^4}{l^2(1-2\beta^2 mE)} \right) \right]. \quad (19)$$

Equating (19) to $2\pi n$, one finds that the corrections to the Newtonian result are now of order $\beta^2$, namely

$$E_{n'l} = \frac{me^4}{2n'^2} \left[ 1 + \frac{2\beta^2 m^2 e^4}{n'} \left( l - \frac{1}{n'} - \frac{1}{l} \right) + \ldots \right]. \quad (20)$$

Note that, as expected, due to the breaking of the Snyder symmetry, the spectrum is no longer degenerate in the angular quantum number $l$. The result is in good agreement with the perturbative calculation performed in [14], that predicts

$$E_{n'l} = \frac{me^4}{2n'^2} \left[ 1 + \frac{2\beta^2 m^2 e^4}{n'} \left( l - \frac{1}{n'} - \frac{1}{l} + \frac{1}{l} \right) + \ldots \right].$$

We remark that the perturbative result breaks down for $l = 0$. This is a signal that in that case the first order corrections are not of order $\beta^2$.

The different behaviour of the spectrum for nonvanishing angular momentum with respect to $l = 0$ seems to arise because the region of integration in the phase integrals is topologically different in the two cases and cannot be smoothly deformed. The corrections have opposite sign and different order of magnitude in the two cases; we are not aware of similar behaviour in other physical systems.

It would be interesting to study the symmetry group of the Snyder hydrogen atom, and to investigate if suitably modifying the potential in order to recover the symmetry under the full Snyder group, also the standard hydrogen atom symmetry is recovered, so that the difference between $l = 0$ and $l \neq 0$ disappears. This topic is presently under study.
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