The zeros and poles of the partition function

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ABSTRACT: In this paper, we consider the physical meaning of the zeros and poles of partition function. We consider three different systems, including the harmonic oscillator in one dimension, Riemann zeta function and the quasinormal modes of black hole.

KEYWORDS: Partition function, Riemann zeta function, complex analysis, quasinormal mode.
1. Introduction

The partition function plays a fundamental role in physics. In statistics mechanics, if we know the partition function of the system, we can calculate all thermal quantities of this system. In quantum field theory, we can get the correlate function from the generating function. Even in AdS/CFT correspondence, the dictionary relate the partition functions of two sides. Usually the partition function is real function of real variables, and is positive.

A good understanding of the thermodynamics properties of a system can be obtained by studying the complex zeros of its partition function. For example, the famous Lee-Yang theorem\cite{1} said that the zeros of the partition function of ferromagnetic spin-1/2 Ising model with two-spin interaction lie on the imaginary H axis. This imply that the ferromagnetic Ising model cannot have a phase transition in a finite (real) magnetic field.

The zeros of the partition function occur in other place, such as in Riemann zeta function\cite{2}, and in quasinormal modes of the black hole\cite{3}. In this paper, we want to investigate what we can learn about those systems from the zeros(poles) of their partition functions.
2. Three systems

First let’s consider the simplest system, the 1D harmonic oscillator. The energy level of the system are

\[ E_n = (n + 1/2)\hbar\omega = (n + 1/2)E_0. \quad n = 0, 1, 2, \ldots \]  

(2.1)

The partition function is

\[ Z(\beta) = \sum_{n=0}^{\infty} \exp(-\beta E_n) = \frac{\exp(-1/2\beta E_0)}{1 - \exp(-\beta E_0)}. \]  

(2.2)

Obviously the partition function has poles at \( \beta_n = 2\pi in/E_0 \), and from complex analysis we know that this partition can be expressed as follows:

\[ Z(\beta) = \frac{\exp(-1/2\beta E_0)}{\beta E_0} \prod_{n \neq 0} \left(1 - \frac{\beta E_0}{2\pi ni}\right) \exp(-\beta E_0) = \frac{\exp(-1/2\beta E_0)}{\beta E_0} \prod_{n > 0} \left(1 + \frac{\beta^2 E_0^2}{4\pi^2 n^2}\right). \]  

(2.3)

That is, the energy levels and the poles are dual to each other. In this simple system, we observe the relation \( \Delta E \Delta \beta = 2\pi i \). But on the other hand, if the energy changed into \( E_n = nE_0 \), the poles are unchanged, so the poles alone can’t determine the energy level.

Next let’s consider the Riemann zeta function,

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\text{ (prime)}} \frac{1}{1 - p^{-s}}. \]  

(2.4)

This function can be considered as the partition function of the Connes-Bost system or the primon gas system. The energy level of those system is \( E_n = \ln(n) \), \( n = 1, 2, \ldots \). On the other hand, from the zeros of the zeta function, we can get,

\[ \zeta(\beta) = \sum_{n=1}^{\infty} \exp(-\beta \ln n) = \exp(\frac{\gamma + \ln \pi}{2} - \beta \ln 2) \prod_{\rho} (1 - \beta \rho) \prod_{n > 0} (1 + \frac{\beta}{2n}) \exp(-\frac{\beta}{2n}). \]  

(2.5)

where \( \gamma \) is the Euler constant, \( \rho \) are the nontrivial zeros of the zeta function, and the famous Riemann hypothesis state that the real part of all \( \rho \) is 1/2.

Then we get another dual system, the energy level \( E_n = \ln(n) \), \( n = 1, 2, \ldots \) and the zeros \( \rho_n \). This duality can also be seen from the Riemann-Weil explicit formula.
relating the prime numbers $p$ and the imaginary part of the Riemann zeros $\rho_n$. This duality is similar to Selberg’s duality \cite{6} between the lengths of the primitive orbits and the eigenvalues of the Laplace-Beltrami operator on a compact Riemann surface with negative curvature. We also have the celebrated Selberg’s trace formula.

Consider a system with the energy level $E_n$ of the form $\rho_n = 1/2 + iE_n$. In Berry and Keating’s opinion \cite{7}, this system correspondence to a quantum chaos system, and $lnp$ are closed periods orbits for this system, $lnn$ the pseudoperiods.

We investigate the other system, the quasinormal modes (QNMs) of black hole physics. The QNMs can be used to get the one-loop correction for quantum gravity in AdS/CFT correspondence \cite{8}. But we will show that we can get more then just the one-loop correction from the QNMs. In a semiclassical quantization of gravity the partition function can be written schematically as

$$Z = \sum_{g^*} \det(-\nabla_{g^*}^2)^{\pm 1} \exp(-S_E(g^*)). \tag{2.6}$$

Here $g^*$ are saddle points of the Euclidean gravitational action $S_E$. In this paper, we just consider the case which there is only one such saddle point. In paper ?? it was derived that the determinant in \ref{2.6} can be formula in terms of the quasinormal mode of the spacetime, or

$$Z_B = \exp(\text{Pol}(\Delta)) \prod_{z^*} \frac{\sqrt{z^* z^*}}{2\pi T} \prod_{n \geq 0} (n + \frac{i z^*}{2\pi T})^{-1} (n - \frac{i z^*}{2\pi T})^{-1}. \tag{2.7}$$

where $z^*$ are QNMs.

In our opinion, we conjecture that the QNMs are zeros or poles of the full quantum gravity partition function. Then the partition function can be written as

$$Z = \exp(-S_E(g^*)) \prod_{z^*} (1 - \frac{z}{z^*}). \tag{2.8}$$

Though the zeros only contribute to the one-loop correction, from above examples we know that those zeros are related to the energy level (or periods orbits) of this system. They are dual to each other. In many case, the QNMs are equally spaced asymptotic. And this fact may uncover that the energy level of the quantum gravity
are equally spaced asymptotic, just as the area eigenvalues in loop quantum gravity,
\[ A = 8\pi\gamma\hbar G \sum_n \sqrt{j_n(j_n + 1)}. \]

3. Conclusion

The partition function determine the properties of the system, and the poles and zeros of the partition function partly determine the partition function, though not completely. In this paper, we investigate three system: the 1D harmonic oscillator, the Riemann zeta function and the quasinormal modes of black hole. For those system, we know some properties from the mathematical or physical point of view independently. But if we combine those two view, maybe we can get more information.

Acknowledgments

This work was partly done at Beijing Normal University. This research was supported in part by the Project of Knowledge Innovation Program (PKIP) of Chinese Academy of Sciences, Grant No. KJCX2.YW.W10

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