AN OPTIMAL MAINTENANCE STRATEGY FOR MULTI-STATE SYSTEMS BASED ON A SYSTEM LINEAR INTEGRAL EQUATION AND DYNAMIC PROGRAMMING

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Abstract. An optimal preventive maintenance strategy for multi-state systems based on an integral equation and dynamic programming is described herein. Unlike traditional preventive maintenance strategies, this maintenance strategy is formulated using an integral equation, which can capture the system dynamics and avoid the curse of dimensionality arising from complex semi-Markov processes. The linear integral equation of the system is constructed based on the system kernel. A numerical technique is applied to solve this integral equation and obtain all of the mean elapsed times from each reliable state to each unreliable state. An analytical approach to the optimal preventive maintenance strategy is proposed that maximizes the expected operational time of the system subject to the total maintenance budget based on dynamic programming in which both backward and forward search techniques are used to search for the local optimal solution. Finally, numerical examples concerning two different scales of systems are presented to demonstrate the performance of the strategy in terms of accuracy and efficiency. Moreover a sensitivity analysis is provided to evaluate the robustness of the proposed strategy.

1. Introduction. To increase operational availability and reliability or to reduce losses in industrial systems, it is imperative for systems to perform maintenance or repair activities periodically or when the system condition level falls below a predetermined threshold. In general, performance of maintenance activities is always

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guided by a certain type of maintenance strategy. An efficient maintenance strategy can enhance the system reliability, prolong the mean time to failure (MTTF) and reduce the failure risk and maintenance and operation costs. Thus, design of a satisfactory maintenance strategy is vital to manufacturing enterprises. The key to designing a suitable maintenance strategy lies in understanding the system dynamics and constructing an accurate model.

The system dynamics are primarily reflected in the natural degradation process of a system. The Markov or semi-Markov process is a powerful tool for modeling the degradation process of systems, particularly multi-state systems (MSSs). Multiple researchers have evaluated the availability or reliability of a system using Markov or semi-Markov frameworks. Soro et al. [33] considered a continuous-time Markov process for evaluating the availability of multi-state degrading systems that receive minimal repairs and imperfect preventive maintenance. Csenki [5] addressed the interval reliability of a repairable system in which the system was modeled using an irreducible finite semi-Markov process. Kayedpour et al. [15] developed an integrated algorithm to solve the reliability design problem by considering instantaneous availability in which repairable components and the selection of configuration strategies are based on Markov processes. Additional information on reliability analyses and maintenance strategies for MSSs based on Markov or semi-Markov processes can be found in other research works (cf. [6, 10, 17, 19, 23, 24, 25, 41]).

Among the studies mentioned above, the integral equation approach was used in reliability analysis of a system in a previous study Csenki [5] in which a set of integral equations was constructed to measure the interval reliability of a two-unit parallel system under the assumption that the deterioration process followed a finite semi-Markov process. Typically, the integral equation technique is a natural tool for application in reliability modeling if the system under consideration is regenerative because the technique can satisfactorily characterize the probability of a system state transition between two states during a certain time interval \( t \). For a certain selection of the simplest cases, the Laplace transform method can be applied to derive an analytical solution of the system. However, in general cases, the system of integral equations can be solved only by numerical methods. Rubino and Sericola [29] considered fault-tolerant systems with operational periods in which arbitrary probability distributions for the holding times in each state of the system were analysed using an integral equation technique. Christer and Jack [2] proposed an integral equation approach for calculating the exact and asymptotic estimates of the expected costs in stochastic replacement modeling over finite time horizons. Subsequently, Jack [14] extended the previous work to evaluate the expected cost of repair replacement policies over finite and infinite time horizons with the use of an integral equation technique. Gurov and Utkin [12] addressed the time-dependent availability of repairable m-out-of-n redundant and cold standby systems in which the system model based on integral equations was constructed using a transition state diagram. From the published papers mentioned above, it is clear that integral equation approaches are primarily used in maintainable systems with a particular structure, such as two-unit parallel systems, cold standby systems, or m-out-of-n systems. To the best of our knowledge, in the last decade, the studies on maintenance policies or reliability analyses based on integral equation techniques have only rarely been published.

For multi-state deteriorating systems, many researchers working in the field of reliability and maintenance focus primarily on the optimization issue of the system
performance and have achieved a significant number of valuable results by various optimization approaches. For example, El-Neweihi and Proschan [9] give a more comprehensive survey on models of multi-state deteriorating systems based on their previous survey in 1980. Levitin and Lisnianski [18] presented a hybrid approach combining a universal generating function (UGF) and a genetic algorithm (GA) to solve reliability optimization problems for MSSs. Lisnianski [20] presented an extended block diagram method for a multi-state system reliability assessment where the presented method was based on the combination of random processes and the universal generating function technique, which can drastically reduce the number of states of the multi-state model. Cheng et al. [1] presented an opportunistic maintenance model for a continuously deteriorating series system consisting of two kinds of units with economical dependence, in which the Gamma process and Poisson process were used to describe the deterioration failure and random failure respectively. Sheu et al. [31] investigated optimal preventive maintenance (PM) policies for multi-state systems where a recursive approach was proposed to compute the time-dependent distribution of the multi-state system. Lisnianski et al. [21] conducted a sensitivity evaluation for an aging MSS under minimal repair in which in order to avoid “curse of dimensionality” in computational procedure, authors proposed a new method based on an LZ-transform of the discrete-state continuous-time Markov process and on Ushakov’s Universal Generating Operator. Koutras et al. [16] presented a general model for multi-state deteriorating system with condition-based maintenance, in which multiple kinds of imperfect maintenance were considered and multi-objective optimization problems were formulated and solved for optimal maintenance policies. For more information about optimization problems and models of multi-state systems, one can refer to the works of [3, 4, 8, 13, 26, 32] and references therein. Generally, due to the current high complexity of industrial systems, the constructed maintenance models are particularly complex. Therefore, intractable optimization issues associated with maintenance strategies are often encountered. Dynamic programming approaches, including the value iteration algorithm, policy iteration, and modified policy iteration, are typically used to solve these challenging optimization problems. For instance, Zhou et al. [42] proposed a scheduling algorithm for preventive imperfect maintenance of a multi-unit system based on dynamic programming. By extending the work in [41], Neves et al. [25] used the value iteration algorithm to identify an optimal repair rule that minimizes the system operation cost. Gu et al. [11] considered two single-machine scheduling problems with a new type of aging effect in which two optimal algorithms based on dynamic programming were proposed to solve NP-hard problems. Srinivasan and Parlikad [34] developed a finite-horizon discrete-time value iteration algorithm for solving the semi-Markov decision process with partial information. Xia et al. [37] proposed an optimal condition-based preventive maintenance policy that take advantage of multiple attribute value theory and dynamic programming with consideration of maintenance effects and environmental conditions. For some highly complex maintenance models, the direct use of a dynamic programming approach is likely to suffer from the curse of dimensionality. To avoid this problem, a dynamic programming approach combined with other techniques such as heuristic information or a local search is expected to perform best. For this reason, the dynamic programming approach combined with forward and backward
local search techniques is applied to identify the optimal maintenance decision during the system lifetime. To the best of our knowledge, this combination in the field of optimal maintenance has never been introduced.

In addition to the system dynamics, the maintenance cost must be considered in the maintenance strategy as another key factor. Many models and strategies treat minimization of the expected long-run cost per unit time as the objective function when designing an optimal maintenance strategy (cf. [23, 30, 35, 36, 40]). However, many realistic repairable systems including productive systems, control systems and chemical systems often result in devastation when they fail. In such a scenario, the system must be discarded as useless when the “age” exceeds the specified years or the condition level falls below the critical level. Thus, the objective of minimizing the expected long-run cost per unit time is not appropriate for these systems when designing the maintenance strategy. In addition, certain productive enterprises prefer to invest a fixed amount of budget to maintain the system. For these two practical reasons, it is more rational for particular complex systems to treat maintenance costs as constraints rather than as objectives. Therefore, in this study, we treat the maintenance cost as a constraint. It is worth mentioning that the issue studied in this paper comes from a practical flattener control system of a steel plant. Furthermore, the developed optimal strategy has been applied to this flattener control system. In addition to flattener control systems, this optimal strategy can also apply to other maintainable systems that the total maintenance budget is determined in advance.

In this paper, a maintenance model of the MSS based on a continuous-time semi-Markov process is constructed. The system lifetime is described by a general probability distribution, and the dynamics are characterized by the kernel function. The system linear integral equations are constructed to deduce the transition probability of a random number of steps from a “better” state downgrading to a “worse” state in a natural deterioration process. Due to the complexity of the constructed integral equations, a modified numerical method is applied to solve these equations. The optimal maintenance strategy is developed to maximize the mean system operating time to failure subject to the maintenance cost.

The novelties of the paper lie in that the system integral equation with coherent summation structure, which is different from the classical integral equation, is constructed to describe the system natural deterioration process. To solve this integral equation, a modified numerical quadrature method is presented. In addition, the dynamic programming method combined with forward and backward local search strategies is first developed to determine the optimal maintenance decision.

The remainder of the paper is organized as follows. Section 2 presents system description including system states, maintenance actions and maintenance costs. Section 3 analyses the system deterioration process using a linear integral equation and gives an optimization problem. Section 4 develops an optimal maintenance strategy based on dynamic programming. Section 5 provides a numerical study and performs sensitivity analysis to illustrate the performance of the developed strategy. Section 6 concludes this paper with a brief summary.

2. Problem formulation.

2.1. System states. The MSS considered in this study deteriorates with age and usage and is subject to external shocks. The system state set is defined as $\Gamma = \{1, \cdots, n\}$, where two extreme states 1 and $n$ denote “as good as new” and complete
failure, respectively. The other intermediate states $2, \cdots, n - 1$ are ordered to reflect their relative level of deterioration in ascending worse order. Due to internal natural degradation and external shocks, the transition is always from a state to another worse state, but not necessarily consecutive. To fulfill the requirement of system reliability, a certain type of PM action must be performed as soon as the system state falls below the critical state $k$. After the PM action is completed, the system can be restored to any better state. To make a distinction among the operational states in the sense of reliability, the set $\Gamma$ is partitioned into three subsets: $R$, $U$ and $F$ which fulfill that any two of them are pairwise disjoint, where $R$ is the subset of the reliable states, $U$ is the subset of the unreliable states, and $F$ is the subset of the total failure state, i.e., $R = \{1, \cdots, k\}$, $U = \{k + 1, k + 2, \cdots, n - 1\}$ and $F = \{n\}$. The system alternates between $R$ and $U$ indefinitely under the effect of deterioration and maintenance up to the time that a total failure occurs.

2.2. Maintenance actions. According to the classification of maintenance by [27], maintenance can be classified into five widely accepted categories in accordance with its impact on the system condition: perfect, minimal, imperfect, worse and worst. Perfect maintenance refers to maintenance actions (or replacement actions) that can restore the system operating state to a “as good as new” state. Minimal maintenance means maintenance actions that improve the system condition by one stage. For example, when the system is in deterioration state $s$ just after the minimal action the system is returned to state $s - 1$. The assumption of perfect maintenance is only applicable to highly complex systems [38]. In most cases, maintenance does not restore the system back to the state of “as good as new”, or the state just improved by one stage. It is more realistic to consider that maintenance restores a system to a certain intermediate state between these two states, and that action is called imperfect maintenance. In this paper, perfect, minimal, and imperfect maintenances are considered. To measure the quality of these three types of maintenance, the following equation applies:

$$Z(X_t, a) = \begin{cases} 1 & \text{if } a \text{ is a perfect maintenance action} \\ x \in \{2, \cdots, k - 1, k\} & \text{if } a \text{ is an imperfect maintenance action} \\ k & \text{if } X_t = k + 1 \text{ and } a \text{ is a minimal maintenance action} \end{cases}$$

(1)

where $X_t \in U$ for any $t \in [0, \infty)$, $a$ denotes a maintenance action, $Z(X_t, a) \in R$ represents the state to which the system is restored from state $X_t$ just after the completion of $a$.

Let $S_p$, $S_i$ and $S_m$ denote the sets of perfect, imperfect and minimal maintenance actions, respectively. From Eq.(1), it follows that $S_p = \{X_t - 1 | \forall X_t \in U\}$, $S_i = \{X_t - X_{t+1} | \forall X_t \in U, X_{t+1} \in R \& X_{t+1} \neq 1\}$ and $S_m = \{1 | X_t = k + 1, X_{t+1} = k\}$. Therefore, the maintenance actions space, denoted by $A$, is: $A = S_p \cup S_i \cup S_m$. Note that $S_p$, $S_i$ and $S_m$ are not disjoint.

2.3. Maintenance costs. Another important factor to consider when designing a PM strategy is maintenance cost. In general, the amount of maintenance cost incurred during a PM interval is related to the level that the system is restored. In other words, the higher the degree of system recovery, the higher the maintenance cost. Let $c_{ij}$ denote the maintenance cost corresponding to the maintenance action that makes the system transit from state $i \in U$ to state $j \in R$. For a given state $i \in U$, the inequality $c_{ij_1} > c_{ij_2}$ makes sense for any two states $j_1, j_2 \in R$ satisfying $j_1 < j_2$. 

In fact, for most enterprises or factories, especially large-scale enterprises, the operational and maintenance costs incurred, respectively, during the system operation and maintenance activities are separated. In this situation, it is not appropriate to consider these two types of cost together when designing a maintenance strategy. Therefore, only maintenance cost is considered and it is treated as a constraint in the optimal maintenance strategy, i.e.,

$$\sum c_{ij} \leq C$$  \hfill (2)

where C is a positive constant representing the total maintenance budget.

3. Model analysis.

3.1. Construction of the system linear integral equation. Let $\phi_{ij}(t)$ \ $(i \in \mathbb{R}, j \in \mathbb{U} \text{ and } t > 0)$ denote the probability that the system that is currently in state $i$ will be in state $j$ after time span $t$. Note that $\phi_{ij}(t)$ is the transition probability with arbitrary steps rather than exactly one step during time span $t$. According to semi-Markov theory, in case of $j > i$, the following equation holds,

$$\phi_{ij}(t) = \sum_{k=i}^{j} \int_{0}^{t} \phi_{kj}(t-\tau)dQ_{ik}(\tau) \quad \forall j \in \{i+1, \cdots, n-1\}$$  \hfill (3)

where $Q_{ik}(t)$ is the entry of $Q(t)$ of the $i$th row and the $k$th column in which $Q(t)$ is the system transition probability matrix, i.e., the system kernel function of the semi-Markov process.

Conversely, in case of $j = i$, i.e., the system is still in state $i$ after a period of time $t$, then $\phi_{ij}(t)$ is:

$$\phi_{ij}(t) = 1 - F_i(t)$$  \hfill (4)

where $F_i(t)$ is the cumulative distribution function (CDF) of the sojourn time $t$ in state $i$.

Note that the case $j > i$ is incompatible with the case $j = i$. Therefore, $\phi_{ij}(t)$ can be expressed in a unified form. That is

$$\phi_{ij}(t) = \delta_{ij}(1 - F_i(t)) + (1 - \delta_{ij}) \sum_{k=i}^{j} \int_{0}^{t} \phi_{kj}(t-\tau)dQ_{ik}(\tau)$$  \hfill (5)

where $\delta_{ij}$ is an indicator variable whose value is

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

For more detailed discussion on system integral equations, one can refer to the work in [19].

Any transition always occurs from a given state to another worse state rather than to itself in the natural deterioration of the system. In this scenario, the indicator variable $\delta_{ij}$ is equal to zero such that Eq.(5) can be further simplified as follows:

$$\phi_{ij}(t) = \sum_{k=i}^{j} \int_{0}^{t} \phi_{kj}(t-\tau)q_{ik}(\tau)d\tau$$  \hfill (6)

where $q_{ik}(\tau)d\tau = dQ_{ik}(\tau)$.  

Letting $\eta = t - \tau$ and substituting this into $\int_0^t \phi_{kj}(t - \tau)q_{ik}(\tau)d\tau$ yields:

$$\int_0^t \phi_{kj}(t - \eta)q_{ik}(\eta)d\eta = \int_0^t q_{ik}(t - \eta)\phi_{kj}(\eta)d\eta$$

(7)

Thus, Eq.(7) can be rewritten as follows:

$$\phi_{ij}(t) = \sum_{k=i}^j \int_0^t q_{ik}(t - \eta)\phi_{kj}(\eta)d\eta$$

(8)

3.2. Solution of the integral equation. Eq.(8) is generally incompatible with deriving the analytical expression of $\phi_{ij}(t)$ because Eq.(8) essentially belongs to the second type of linear Volterra integral equation in which the function $q_{ik}(t - \tau)$ is referred to as the kernel. To solve Eq.(8), a numerical quadrature method (cf. [7, 28]) appears to be a feasible alternative. However, conventional numerical quadrature methods are not capable of solving the kind of integral equations with coherent summation structure. Therefore, we modified a conventional numerical quadrature method to make it applicable to Eq.(8). The modified method consists of four steps described as follows.

Step 1: The integral interval $[0, t]$ is partitioned into $L$ subintervals with the same length, i.e., the length of each subinterval is equal to $h = \frac{t}{L}$, $(L \geq 1)$, and let $\eta_0 = 0, \eta_l = \eta_0 + lh, (l = 1, 2, \cdots, L)$.

Step 2: $\int_0^t q_{ik}(t - \eta)\phi_{kj}(\eta)d\eta$ is first converted to the summation of $L + 1$ terms:

$$\int_0^t q_{ik}(t - \eta)\phi_{kj}(\eta)d\eta = \left[ \sum_{l=1}^{L-1} q_{ik}(t - \eta_l)\phi_{kj}(\eta_l) + \frac{1}{2}(q_{ik}(t - \eta_0)\phi_{kj}(\eta_0) + q_{ik}(t - \eta_L)\phi_{kj}(\eta_L)) \right] h$$

(9)

Substituting Eq.(9) into Eq.(8) yields the following:

$$\phi_{ij}(t) = h \sum_{k=i}^j \sum_{l=1}^{L-1} q_{ik}(t - \eta_l)\phi_{kj}(\eta_l) + \frac{1}{2} h \sum_{k=i}^j q_{ik}(t - \eta_0)\phi_{kj}(\eta_0) +$$

$$\frac{1}{2} h \sum_{k=i}^j q_{ik}(t - \eta_L)\phi_{kj}(\eta_L)$$

(10)

Because the time $t$ is independent of $\eta$, it is feasible to discretize $t$ in the form of $t_m = \eta_m (m = 0, 1, \cdots, L)$. Thus, Eq.(10) can be rewritten as follows:

$$\phi_{ij}(t_m) = h \sum_{k=i}^j \sum_{l=1}^{L-1} q_{ik}(t_m - \eta_l)\phi_{kj}(\eta_l) + \frac{1}{2} h \sum_{k=i}^j q_{ik}(t_m - \eta_0)\phi_{kj}(\eta_0) +$$

$$\frac{1}{2} h \sum_{k=i}^j q_{ik}(t_m - \eta_L)\phi_{kj}(\eta_L)$$

(11)
Let $\phi_{ijm}$ and $q_{ikmL}$ denote $\phi_{ij}(t_m)$ and $q_{ik}(t_m - \eta_l)$, respectively, and Eq.(11) can be simplified as follows:

$$\phi_{ijm} = h \sum_{k=1}^{j} \sum_{l=1}^{m} q_{ikmL} \phi_{kjl} + \frac{1}{2} h \sum_{k=1}^{j} q_{ikm0} \phi_{kjl} + \frac{1}{2} h \sum_{k=1}^{j} q_{ikmL} \phi_{kjl}$$  \hspace{1cm} (12)

Now, we investigate the values of $\phi_{kjl}$ and $q_{ikmL}$ respectively in the second and the third terms of right-hand side of Eq.(12). Firstly, due to $t_m = \eta_m$ and $\eta_0 = 0$, it follows that $\phi_{kjl}(0) = 0$ for each $k \in [i, \cdots, j]$. Therefore, the second term in the right-hand side of Eq.(12) is equal to 0. Secondly, due to $q_{ikmL} = q_{ik}(t_m - \eta_l)$ and $m \leq L$ for any $m \in [1, \cdots, L]$, $q_{ik}(t_m - \eta_l) = 0$ holds for each $k \in [i, \cdots, j]$. Therefore, the third term in the right-hand side of Eq.(12) is also equal to 0. Thus, Eq.(12) can be further simplified as:

$$\phi_{ijm} = h \sum_{k=1}^{j} \sum_{l=1}^{m} q_{ikmL} \phi_{kjl} \quad m = 0, 1, \cdots, L$$  \hspace{1cm} (13)

**Step 3:** Direct solution of Eq.(13) is impossible, because the variables $\phi_{kjl}$ ($k = i+1, \cdots, j$) on the right-hand side of Eq.(13) are dependent on the solution $\phi_{ijj}$. To address this issue, the numerical calculation on the terms indexed by $k$ in reverse order is applied. Consequently, the terms indexed by $k = j$ are first considered. According to Eq.(4), the following equation applies:

$$\phi_{jjm} = 1 - F_{jm} \quad m = 0, 1, \cdots, L$$  \hspace{1cm} (14)

where $F_{jm} = F_j(t_m)$. Because the lifetime distribution function $F_j(t)$ in state $j$ is a known function, $\phi_{jjm}$ in each discrete point $t_m$ ($m = 0, 1, \cdots, L$) can be obtained. Another case of $i = j - 1$ is investigated in a similar manner. According to Eq.(13), a set of $L$ linear algebraic equations in $L$ unknowns can be constructed as follows:

$$\begin{align*}
\phi_{(j-1)j0} &= 0 \\
\phi_{(j-1)j1} - h \sum_{k=j-1}^{j} \sum_{l=1}^{1} q_{(j-1)k1} \phi_{kjl} &= 0 \\
\phi_{(j-1)j2} - h \sum_{k=j-1}^{j} \sum_{l=1}^{2} q_{(j-1)k2} \phi_{kjl} &= 0 \\
& \vdots \\
\phi_{(j-1)j(L-1)} - h \sum_{k=j-1}^{j} \sum_{l=1}^{L-1} q_{(j-1)k(L-1} \phi_{kjl} &= 0
\end{align*}$$  \hspace{1cm} (15)

Substituting $\phi_{(j-1)j0} = 0$ and $\phi_{jjm}$ ($m = 1$) into the second equation of Eq.(15), $\phi_{(j-1)j1}$ is derived. Substituting $\phi_{(j-1)j0}$, $\phi_{(j-1)j1}$ and $\phi_{jjm}$ ($m = 1, 2$) in the third equation of Eq.(15) yields $\phi_{(j-1)j2}$. Similarly, all of the remaining discrete values $\phi_{(j-1)j3}$, $\cdots$, $\phi_{(j-1)jL}$ can be obtained.

**Step 4:** Similar to the calculation of $\phi_{(j-1)jj}$, the other solutions, i.e., $\phi_{(j-2)jj}$, $\phi_{(j-3)jj}$, $\cdots$, $\phi_{jjm}$ ($m = 0, 1, \cdots, L$) can be obtained.

3.3. **PM objective function.** In maintenance strategies, one of major factors to consider is the system performance indicator, e.g., reliability, availability or safety. In this study, the system reliability is treated as the performance indicator which is defined as the current state. That is, within operation duration, the reliability of the system must stay above the critical state $k$, i.e., $X_t \in \mathbf{R}$. PM action is triggered whenever the system reliability falls below $k$:

$$\frac{(X_t \in \mathbf{U})}{a}$$  \hspace{1cm} (16)
where, the symbol “/” denotes the trigger, i.e., if the condition of the left-hand side of “/” applies, then the action of the right-hand side of “/” is triggered.

The whole operation process of the system from initial state 1 to ultimate failure state \( n \) under a series of maintenance actions is described in the following. Initially, the system is in a “as good as new” state, i.e., state 1. Due to factors such as aging effect, usage, wearout and external shocks, the system deteriorates over time and will fall into \( U \) at any time \( t \in [0, \infty) \). Once falling into \( U \), according to Eq.(16) a certain type of maintenance action is performed instantaneously. After that, the system can be restored to any state belonging to \( R \). The system will undergo such a process repeatedly, and ultimately be in failure state \( n \), i.e., breakdown, which is shown in Fig.1.

\[
1 \xrightarrow{a_{1}X_{1}(t)} X_{1} / a_{1} \xrightarrow{a_{2}X_{2}(t)\phi_{2}X_{2}X_{1}X_{2}(t)} X_{2} / a_{2} \xrightarrow{a_{3}X_{3}(t)\phi_{3}X_{3}X_{2}X_{3}(t)} X_{3} / a_{3} \cdots
\]

**Figure 1.** The whole operation process of the system

In Fig.1, both sides of arrows are two system states belonging to either \( R \) and \( U \), or \( U \) and \( R \) respectively. The upper part of the long arrows is the transition probability between two sides of states. As reflected, the operation process of the system consists of a series of operation cycles in which the operation cycle is defined as the time span from the beginning of any state belonging to \( U \) to the next beginning of the state belonging to \( U \). In other words, the operation cycle is the time span between two adjacent maintenance actions. For example, the \( q \)th operation cycle, denoted by \( T_{q} \), is \( T_{q} = t_{q+1} - t_{q} \). Note that \( T_{q} \) for all \( q \in \mathbb{N}_{+} \) is a random variable, meaning that the value of each \( T_{q} \) probably differs from the others. In fact, an operation cycle consists of two stage: the first is the maintenance stage, the second is the system natural deterioration stage, which is shown in Fig.2.

**Figure 2.** An operation cycle consisting of the maintenance and deterioration stages

As shown in Fig.2, in some operation cycle if the current state is \( k + 2 \), then a certain maintenance action needs to be performed instantaneously. After that the system is restored to any state belonging to \( R \). Later, the system again falls into \( U \).
with time goes by. In view of the above description, the system mean operational time from state 1 to failure state $n$ can be expressed:

$$T(t,a_{t_q}) = E[\psi_0] + \sum_{q \in \mathbb{N}_+} \left[ \theta_q + E[\psi_q] \right] + E[\psi_{last}]$$  \hspace{1cm} (17)

where $T(t,a_{t_q})$ is the system mean operational time under a series of maintenance actions, $E[\cdot]$ stands for the mathematical expectation. $\psi_q$ is a random variable representing the deterioration time from a state belonging to $\mathbb{R}$ to another state belonging to $\mathbb{U}$ in the $q$th cycle. $\theta_q$ refers to the PM time in the $q$th cycle. Actually, $\theta_q$ and $\psi_q$ correspond to the durations of the first and second stages, respectively, in the $q$th cycle.

In addition, it is worth mentioning that the CDF of $\psi_q$ is $1 - \phi_{Z(X_t,a_{t_q})X_{t_{q+1}}}(t)$. Thus, $E[\psi_q]$ can be expressed as:

$$E[\psi_q] = \int_0^\infty \psi_q d(1 - \phi_{Z(X_t,a_{t_q})X_{t_{q+1}}}(t))$$  \hspace{1cm} (18)

For the feasibility of computation, it is necessary to convert Eq.(18) into its discrete form by replacing $\phi_{Z(X_t,a_{t_q})X_{t_{q+1}}}(t)$ with $\phi_{Z(X_t,a_{t_q})X_{t_{q+1}}}(t_m)(m = 0, 1, \cdots, L)$. Consequently, Eq.(18) can be rewritten as:

$$E[\psi_q] = \lim_{L \to \infty} \sum_{m=0}^L (1 - \phi_{Z(X_t,a_{t_q})X_{t_{q+1}}}(t_m))t_m$$  \hspace{1cm} (19)

where, $L$ needs to be set to a large number in the numerical calculations.

The objective function of maximizing the system operating time from initial state 1 to failure state $n$ under a series of maintenance actions subject to the total budget can be formulated as follows:

$$\max T(t,a_{t_q})$$

$$\text{s.t.} \sum_{q \in \mathbb{N}} c_{t_q} \leq C$$  \hspace{1cm} (20)

4. **Optimal maintenance strategy.** The difficulty of the optimization issue of Eq.(20) lies in the interaction that exists between two adjacent maintenance actions, $a_{t_q}$ and $a_{t_{q+1}} (\forall q \in \mathbb{N})$, because the maintenance action chosen at the decision time $t_{q+1}$ relies on the state $Z(X_t,a_{t_q})$ which is associated with $a_{t_q}$, meaning that $a_{t_{q+1}}$ is affected by $a_{t_q}$ indirectly. To solve the optimization issue of Eq.(20), a finite horizon dynamic programming method is employed. For convenience of presentation, the notations involved in the optimal maintenance strategy are listed in the following:

**Nomenclature:**

$q$ the counter denoting the $q$th operation cycle

tq the decision time of the $q$th operation cycle

$A_q^*$ the vector of order $n - k - 1$ denoting the maximal time span per unit cost in the $q$th operation cycle for all $X_{t_{q+1}} \in \mathbb{U}$.

$A^*$ the vector of order $n - k - 1$ denoting the maximal time span of each optimal path of the current operation cycle

$B_q^*$ the vector of order $n - k - 1$ denoting the maximal time span in the $q$th operation cycle for all $X_{t_{q+1}} \in \mathbb{U}$.

$B^*$ the vector of order $n - k - 1$ denoting the maximal time span of each optimal path from the first operation cycle to the current one
$C_q^*$ the vector of order $n - k - 1$ denoting the cost corresponding to $A_q^*$ in the $q$th operation cycle for all $X_{t_{q+1}} \in U$.

$C^*$ the vector of order $n - k - 1$ denoting the maintenance costs of each optimal path from the first operation cycle to the current one.

$D^*$ the list consisting of $n - k - 1$ sub-lists used to store each optimal path from the first operation cycle to the current one.

$\Theta_{t_q}$ the maintenance time matrix of dimension $(n - k - 1) \times k$ at decision time $t_q$ in which each entry $\theta_{X_{t_q}, Z(X_{t_q}, a_{t_q})}$ denotes the maintenance time of action $a_{t_q}$ that makes the system transition from state $X_{t_q} \in U$ to state $Z(X_{t_q}, a_{t_q}) \in \mathbb{R}$.

$\Psi_{t_q}$ the deterioration time matrix of dimension $k \times (n - k - 1)$ at decision time $t_q$ in which each entry $\psi_{Z(X_{t_q}, a_{t_q}), X_{t_{q+1}}}$ denotes the natural deterioration time of the system from state $Z(X_{t_q}, a_{t_q}) \in \mathbb{R}$ to state $X_{t_{q+1}} \in U$.

$C_{t_q}$ the maintenance cost matrix of dimension $(n - k - 1) \times k$ at decision time $t_q$ in which each entry $c_{X_{t_q}, Z(X_{t_q}, a_{t_q})}$ denotes the maintenance cost of action $a_{t_q}$ that makes the system transition from state $X_{t_q} \in U$ to state $Z(X_{t_q}, a_{t_q}) \in \mathbb{R}$.

$\zeta^*_{X_{t_{q+1}}}$ the maximal time span per unit cost in the $q$th operation cycle for state $X_{t_{q+1}} \in U$.

The steps of the proposed optimal strategy are described as follows.

**Step 1.** Set initial conditions, i.e., $q = 1$, $A^* = (0)_{(n - k - 1)}$, $B^* = (0)_{(n - k - 1)}$, $C^* = (0)_{(n - k - 1)}$, and $D^* = \{\emptyset\}$

**Step 2.** Identify the matrices $\Theta_{t_q}$, $E[\Psi_{t_q}]$ and $C_{t_q}$, i.e.,

\[
\Theta_{t_q} = \left[\theta_{X_{t_q}, Z(X_{t_q}, a_{t_q})}\right]_{(n - k - 1) \times k}
\]

\[
E[\Psi_{t_q}] = \left[\mathbb{E}\left[\psi_{Z(X_{t_q}, a_{t_q}), X_{t_{q+1}}}\right]\right]_{k \times (n - k - 1)}
\]

and

\[
C_{t_q} = \left[c_{X_{t_q}, Z(X_{t_q}, a_{t_q})}\right]_{(n - k - 1) \times k}
\]

where $X_{t_q}, X_{t_{q+1}} \in U$ and $Z(X_{t_q}, a_{t_q}) \in \mathbb{R}$.

**Step 3.** Determine $\zeta^*_{X_{t_{q+1}}}$ using the backward search technique at the decision time $t_q$, i.e.,

\[
\zeta^*_{X_{t_{q+1}}} = \max \|\theta_{X_{t_q}, \bullet} + E[\psi_{\bullet, X_{t_{q+1}}}]\|_{\infty} \cdot c_{X_{t_q}, \bullet}^{-1} \quad \text{for} \quad X_{t_q} = k + 1, \ldots, n - 1
\]

where $\|\cdot\|_{\infty}$ represents the $\infty$ norm of a vector, $\theta_{X_{t_q}, \bullet}$ denotes the $X_{t_q}$th row vector, $\psi_{\bullet, X_{t_{q+1}}}$ refers to the $X_{t_{q+1}}$th column vector and $c_{X_{t_q}, \bullet}^{-1}$ is defined as

\[
\left(c_{X_{t_q}, 1}^{-1}, c_{X_{t_q}, 2}^{-1}, \ldots, c_{X_{t_q}, k}^{-1}\right)
\]

This procedure is shown in Fig.3.

Evaluate $(\theta_q + E[\psi_q])_{X_{t_{q+1}}}$ and the optimal maintenance cost $c^*$ corresponding to $\zeta^*_{X_{t_{q+1}}}$, i.e.,

\[
(\theta_q + E[\psi_q])_{X_{t_{q+1}}} = \arg \max_{\Theta} \|\theta_{X_{t_q}, \bullet} + E[\psi_{\bullet, X_{t_{q+1}}}]\|_{\infty} \cdot c_{X_{t_q}, \bullet}^{-1} \quad \text{for} \quad X_{t_q} = k + 1, \ldots, n - 1
\]

\[
c^* = \arg \max_{\Theta} \|\theta_{X_{t_q}, \bullet} + E[\psi_{\bullet, X_{t_{q+1}}}]\|_{\infty} \cdot c_{X_{t_q}, \bullet}^{-1} \quad \text{for} \quad X_{t_q} = k + 1, \ldots, n - 1
\]
Figure 3. Backward search technique for every state $X_{t+1} \in U$

where $\odot$ denotes $\theta_{X_{t_k}, Z(X_t, a_{t_k})} \in \Theta_{t_q}$ and $\psi_{Z(X_t, a_{t_k}), X_{t_k+1}} \in \Psi_{t_q}$, $\odot$ denotes $c \in C_{t_q}$.

Assign $\zeta_{X_{t_q+1}}^*$, $(\theta_q + E[\psi_q])_{X_{t_q+1}}^*$ and $c^*$ for all $X_{t_q+1} \in U$ to $A_q^*$, $B_q^*$ and $C_q^*$, respectively, i.e.,

$$A_q^* = (\zeta_{k+1}^*, \zeta_{k+2}^*, \ldots, \zeta_{n-1}^*)$$

$$B_q^* = \left( (\theta_q + E[\psi_q])_{k+1}^*, (\theta_q + E[\psi_q])_{k+2}^*, \ldots, (\theta_q + E[\psi_q])_{n-1}^* \right)$$

$$C_q^* = (c_{k+1}^*, c_{k+2}^*, \ldots, c_{n-1}^*)$$

Identify the optimal paths, i.e., the triple states $(X_{t_q}, Z(X_t, a_{t_k}), X_{t_q+1})^*$, corresponding to $\zeta_{X_{t_q+1}}^*$ for all $X_{t_q+1}$:

$$(X_{t_q}, Z(X_t, a_{t_k}), X_{t_q+1})^* \arg \max_{\oplus} \|(\theta_{X_{t_k}, \bullet} + E[\psi_{\bullet, X_{t_k+1}}]) \cdot e_{X_{t_k, \bullet}}^{-1} \| \infty$$

for $X_{t_k+1} = k + 1, \ldots, n - 1$

where $\oplus$ denotes $X_{t_q} \in U$, $Z(X_t, a_{t_k}) \in R$ and $X_{t_{k+1}} \in U$.

Store them in List $D$. Note that each element of list $D$ is a sub-list used to store all of the end-to-end optimal paths of each operation cycle.

Actually, after every $\zeta_{X_{t_q+1}}^*$ for $X_{t_q+1} \in U$ is identified, two situations may exist. One situation is one-to-one correspondence between $X_{t_{k+1}}$ and $X_{t_k}$, which is shown in Fig.4(a). The other situation does not feature one-to-one correspondence, meaning that there exist at least two states $X_{t_{q+1}}$ corresponding to one state $X_{t_k}$, as shown in Fig.4(b).
In the second situation, the forward search technique (shown in Fig. 5) is used in all of the remaining states \(X_{tq}\) (e.g., state \(k+1\) in Fig. 4(b)) to identify \(\zeta_{X_{tq}}\) and the corresponding optimal path:

\[
\zeta^*_X = \max \left\{ \| (\theta_{X_{tq}} + E[\psi_{X_{tq+1}}]) \cdot c_{X_{tq}}^{-1} \| \right\}
\]

for \(X_{tq} \in \mathbb{R}\)

\[
(X_{tq}, Z(X_t, a_{tq}), X_{tq+1})^* = \arg \max \left\{ \| (\theta_{X_{tq}} + E[\psi_{X_{tq+1}}]) \cdot c_{X_{tq}}^{-1} \| \right\}
\]

for \(X_{tq+1} = k+1, \cdots, n-1; \quad X_{tq} \in \mathbb{R}\)

where \(\oplus\) denotes \(X_{tq} \in \mathbb{U}, Z(X_t, a_{tq}) \in \mathbb{R}\) and \(X_{tq+1} \in \mathbb{U}\). \(\mathbb{R}\) denotes the set of the remaining states. Each optimal path \((X_{tq}, Z(X_t, a_{tq}), X_{tq+1})^*\) is appended into the tail of corresponding sub-list of \(D\).

The aim of considering the second situation is to ensure that each state \(X_{tq} \in \mathbb{U}\) can be traversed in the procedure of dynamic programming.

**Step 4.** Update the vectors \(A^*, B^*\) and \(C^*\) i.e., \(A^* \leftarrow A^* + A^*_q, B^* \leftarrow B^* + B^*_q\) and \(C^* \leftarrow C^* + C^*_q\). Judge whether each element of \(C^*\) is greater than or equal to the total maintenance budget. If yes, the optimal search on this optimal path is over. Otherwise, the optimal search is continued. Update List \(D\) by

\[
D \leftarrow \text{Append} \left( D_{X_{tq}}, (X_{tq}, Z(X_{tq}, a_{tq}), X_{tq+1})^* \right)
\]
Step 5. Set \( q = q + 1 \), and go to Step 2 until all elements of \( C^* \) are greater than or equal to the budget \( C \).

Step 6. Calculate the mean time spans from the initial state 1 to each state \( X_{t_1} \in U \) and from each state of the last cycle to the failure state \( n \), i.e.,

\[
E[\Psi_{t_1}] = (E[\psi_{1,k+1}] , E[\psi_{1,k+2}] , \cdots , E[\psi_{1,n-1}])
\]

\[
E[\Psi_{last}] = (E[\psi_{k+1,n}] , E[\psi_{k+2,n}] , \cdots , E[\psi_{n-1,n}])
\]

![Figure 6: Time-spans from state 1 to state \( X_{t_1} \in U \) and from \( X_{last} \in U \) to failure state \( n \)](image)

Step 7. Update \( B^* \) using \( E[\Psi_{t_0}] \) and \( E[\Psi_{last}] \), i.e., \( B^* \leftarrow B^* + E[\Psi_{t_0}] + E[\Psi_{last}] \), and store the paths \((1, X_{t_1})\) and \((X_{last}, n)\) in the header and tail, respectively, of the corresponding sub-list of \( D \). The maximal expected operating time of the system from state 1 to failure state \( n \) is determined as follows:

\[
T^*(t, a_t) = \|B^*\|_{\infty}
\]

and the optimal strategy (i.e., the optimal path) can be written as follows:

\[
D^* = \arg \|B^*\|_{\infty}
\]

The schematic overview of this optimal procedure is shown in the following Algorithm 1.

**Algorithm 1** Optimal maintenance strategy based on dynamic programming

1. Initialization: \( q = 1, A^* = (0)_{1 \times (n-k-1)} \), \( B^* = (0)_{1 \times (n-k-1)} \), \( C^* = (0)_{1 \times (n-k-1)} \), and \( D = \{\{\}, \cdots, \{\}\} \), \( n \rightarrow k-1 \)
2. While \( \min\{c_1, \cdots, c_{n-k-1}\} < C \)
3. Construct matrices \( \Theta_{t_0} \), \( E[\Psi_{t_0}] \) and \( C_{t_0} \):
   \[
   \Theta_{t_0} = \left[ \theta_{X_{t_0}, Z(X_{t_0,a_{t_0}})} \right]_{(n-k-1) \times k}
   E[\Psi_{t_0}] = \left[ E[\psi Z(X_{t_0,a_{t_0}}, X_{t_0+1})] \right]_{k \times (n-k-1)}
   C_{t_0} = \left[ c_{X_{t_0}, Z(X_{t_0,a_{t_0}})} \right]_{(n-k-1) \times k}
   \]
4. Calculate the optimal vectors \( \zeta_{X_{t+1}}^* \), \( (\theta_{t+1} + E[\psi_{t+1}])^*_{X_{t+1}} \), and \( c^* \):
   \[
   \zeta_{X_{t+1}}^* = \max \| (\theta_{t+1} + E[\psi_{t+1}]) \cdot c_{X_{t+1}}^{-1} \|_{\infty} \text{ for } X_{t+1} = k + 1, \cdots, n - 1
   \]
   \[
   (\theta_{t+1} + E[\psi_{t+1}])^*_{X_{t+1}} = \arg \max \| (\theta_{t+1} + E[\psi_{t+1}]) \cdot c_{X_{t+1}}^{-1} \|_{\infty} \text{ for } X_{t+1} = k + 1, \cdots, n - 1
   \]
(\theta_{X_{t_q}} \bullet + E[\psi_{X_{t_q+1}}]) \cdot c_{X_{t_q}}^{-1} \|_\infty \quad \text{for } X_{t_q} = k + 1, \ldots, n - 1
\end{align*}

5: Assign $\zeta^{c*}_{X_{t_q+1}}$, $(\theta_{q} + E[\psi_{q}])^{c*}_{X_{t_q+1}}$ and $c^*$ for all $X_{t_q+1} \in U$ to $A^*_q$, $B^*_q$ and $C^*_q$, respectively, i.e.,

\begin{align*}
A^*_q &= (\zeta^{c*}_{k+1}, \zeta^{c*}_{k+2}, \ldots, \zeta^{c*}_{n-1}) \\
B^*_q &= (\theta_{q} + E[\psi_{q}])^{c*}_{k+1}, (\theta_{q} + E[\psi_{q}])^{c*}_{k+2}, \ldots, (\theta_{q} + E[\psi_{q}])^{c*}_{n-1}) \\
C^*_q &= (c^{c*}_{k+1}, c^{c*}_{k+2}, \ldots, c^{c*}_{n-1})
\end{align*}

6: Identify the optimal paths corresponding to $\zeta^{c*}_{X_{t_q+1}}$ for all $X_{t_q+1}$, i.e.,

\begin{align*}
(X_{t_q}, Z(X_{t}, a_{t_q}), X_{t_q+1})^* &= \arg \max \| (\theta_{X_{t_q}} \bullet + E[\psi_{X_{t_q+1}}]) \cdot c_{X_{t_q}}^{-1} \|_\infty \\
&\quad \text{for } X_{t_q+1} = k + 1, \ldots, n - 1
\end{align*}

7: Store each $(X_{t_q}, Z(X_{t}, a_{t_q}), X_{t_q+1})^*$ in List D.
8: Judge whether or not there exists the case of not one-to-one correspondence. If yes, the forward search technique is used for all of the remaining states $X_{t_q} \in R$ to identify $\zeta^{c*}_{X_{t_q}}$ and the corresponding optimal path, i.e.,

$$
\zeta^{c*}_{X_{t_q}} = \max \| (\theta_{X_{t_q}} \bullet + E[\psi_{X_{t_q+1}}]) \cdot c_{X_{t_q}}^{-1} \|_\infty \\
\quad \text{for } X_{t_q+1} = k + 1, \ldots, n - 1; \quad X_{t_q} \in R
$$

$$
(X_{t_q}, Z(X_{t}, a_{t_q}), X_{t_q+1})^* = \arg \max \| (\theta_{X_{t_q}} \bullet + E[\psi_{X_{t_q+1}}]) \cdot c_{X_{t_q}}^{-1} \|_\infty \\
\quad \text{for } X_{t_q+1} = k + 1, \ldots, n - 1; \quad X_{t_q} \in R
$$

and append them into the tail of corresponding sub-lists of $D$. Else, continue.

9: Update the vectors $A^*$, $B^*$ and $C^*$, i.e., $A^* \leftarrow A^* + A^*_q$, $B^* \leftarrow B^* + B^*_q$ and $C^* \leftarrow C^* + C^*_q$.
10: Judge whether each element of $C^*$ is greater than or equal to the total maintenance budget $C$. If yes, the optimal search on this optimal path is over. Otherwise, the optimal search is continued.
11: Update List $D$ by

$$
D \leftarrow \text{Append } (D_{X_{t_q}}, (X_{t_q}, Z(X_{t}, a_{t_q}), X_{t_q+1})^*)
$$

12: Set $q = q + 1$
13: end while
14: Determine all time-spans in cycle $t_0$ and in cycle $t_{last}$, i.e.,

$$
E[\psi_{t_1}] = (E[\psi_{1,k+1}], E[\psi_{1,k+2}], \ldots, E[\psi_{1,n-1}])
$$

$$
E[\psi_{t_{last}}] = (E[\psi_{k+1,n}], E[\psi_{k+2,n}], \ldots, E[\psi_{n-1,n}])
$$

15: Update $B^*$ by $B^* \leftarrow B^* + E[\psi_{t_0}] + E[\psi_{t_{last}}]$ and store all paths $(1, X_{t_i})$ and $(X_{t_{last}}, n)$ in the header and tail, respectively, of the corresponding sub-list.
16: Determine $T^*(t, a_t)$ and the corresponding $D^*$, i.e., $T^*(t, a_t) = \| B^* \|_\infty$

$$
D^* = \arg \| B^* \|_\infty
$$
5. **Numerical experiments.** In this section, two different scales of maintainable systems, i.e., a small-scale system and a large-scale complex system, are investigated to validate the proposed strategy. Moreover, the comparing experiments concerning the running time of Algorithm 1 on both the small-scale and large-scale systems are also performed to evaluate the performance of the optimal strategy. Furthermore, a sensitivity analysis is conducted to investigate the robustness of this strategy. Maintenance cost is randomly generated using uniform distributions. More specifically, maintenance costs corresponding to each type of maintenance actions are generated as follows:

- \( c_{k+1,k} \sim U[1, 3] \) if the maintenance action is a minimal maintenance
- \( c_{ij} \sim U[3, 10], (i \in U, j \in R) \) if the maintenance action is an imperfect maintenance
- \( c_{i1} \sim U[10, 13], (i \in U) \) if the maintenance action is a perfect maintenance

where \( U[a, b] \) represents the uniform distribution between \( a \) and \( b \). Note that the randomly generated variables \( c_{ij} \) and \( c_{i1} \) must meet the inequality (2). This manner of generating maintenance cost follows the practice of López-Santana et al. [22].

The total budget is set to 30 and 100 for the small-scale system and the large-scale system, respectively.

The PM time \( \theta_q \) of the \( q \)th cycle is also randomly generated using uniform distributions, i.e.,

- \( \theta_q \sim U[8, 24] \) if the maintenance action is a minimal maintenance
- \( \theta_q \sim U[24, 48], (i \in U, j \in R) \) if the maintenance action is an imperfect maintenance
- \( \theta_q \sim U[4, 12], (i \in U) \) if the maintenance action is a perfect maintenance

5.1. **Numerical examples for both small-scale and large-scale systems.**

Consider a maintainable small-scale system whose state space is \( \Gamma = \{1, 2, \ldots , 9\} \). According to the description of Section 2, state 1 denotes the perfect state, and state 9 the failure state. The other states are intermediate states. Assume that state 4 is the critical state, i.e., \( R = \{1, 2, 3, 4\}, U = \{5, 6, 7, 8\} \). The transition among states is governed by the semi-Markov process with the kernel \( Q(t) \), i.e.,

\[
Q(t) = F(t) \ast W
\]

where \( F(t) \ast W \) means that \( F(t) \) multiplies each entry of matrix \( W \), \( F(t) \) is a certain type of distribution function, and \( W \) is the coefficient matrix with the same dimension of \( Q(t) \), whose value is

\[
W = \begin{pmatrix}
0 & 0.35 & 0.21 & 0.16 & 0.12 & 0.07 & 0.05 & 0.03 & 0.01 \\
0 & 0 & 0.47 & 0.19 & 0.12 & 0.1 & 0.06 & 0.04 & 0.02 \\
0 & 0 & 0.53 & 0.17 & 0.12 & 0.1 & 0.05 & 0.03 & 0.01 \\
0 & 0 & 0 & 0 & 0.61 & 0.15 & 0.11 & 0.09 & 0.07 \\
0 & 0 & 0 & 0 & 0 & 0.69 & 0.14 & 0.1 & 0.07 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.77 & 0.13 & 0.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.88 & 0.12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
The determination of $W$ is described in [23]. To validate the wide applicability of the proposed optimal strategy, two different types of distribution, i.e., a Weibull distribution and a more general distribution, are considered in cases 1 and 2, respectively.

**Case 1.** In this case, the Weibull distribution, which is widely used in diverse branches of reliability theory and engineering, is used to express the sojourn times among states in which the scale and shape parameters $\lambda$ and $\alpha$ are set to 0.4 and 2, respectively, i.e., the CDF is,

$$F(t) = 1 - e^{-(0.4t)^2}$$

In numerical calculation, the integral upper limit $t$ and the number of subintervals $L$ are set to 10 and 50, respectively. Therefore, the step size $h = 0.2$, and the number of quadrature points is 51. All $\phi_{ij}(t_m)$ for $i \in R$, $j \in U$ and $m = 0, \cdots, L$ can be determined. To show some obtained results with respect to $\phi_{ij}(t_m)$, several $\phi_{ij}(t_m)$ randomly chosen, e.g., $\phi_{13}(t_m), \phi_{26}(t_m), \phi_{38}(t_m)$ and $\phi_{47}(t_m)$, are depicted in Fig.7.

![Images of graphs](image1.png)

**Figure 7.** Discrete values of $\phi_{13}(t_m), \phi_{26}(t_m), \phi_{38}(t_m), \phi_{47}(t_m)$ under the Weibull distribution with parameters $\lambda = 0.4$ and $\alpha = 2$

Using Eq.(19), $E[\Psi_q]$ is derived. In order to ensure the fidelity of the strategy, the accuracy of numerical calculation should be as high as possible. For this reason, the asymptotic errors with respect to $E[\Psi_q]$ by increasing $L$ from 30 to 50 were investigated. The absolute and relative asymptotic errors of numerical solution are
defined as:

- absolute asymptotic error (AAE): \[ |E[\psi_{ij}]_L - E[\psi_{ij}]_{L-1}| \]
- relative asymptotic error (RAE): \[ \frac{|E[\psi_{ij}]_L - E[\psi_{ij}]_{L-1}|}{E[\psi_{ij}]_L} \times 100\% \]

respectively. Taking \( E[\psi_{1,3}] \) as an example, the experiment results are shown in Table 1.

**Table 1. Comparison of the absolute and relative asymptotic errors**

| \( L \) | 30   | 31   | 32   | 33   | 34   | 35   | 36   |
|--------|------|------|------|------|------|------|------|
| \( E[\psi_{1,3}]_L \) | 9.6186 | 9.6059 | 9.5939 | 9.5827 | 9.5721 | 9.5621 | 9.5527 |
| absolute error | 0.0127 | 0.0119 | 0.0112 | 0.0105 | 0.0099 | 0.0094 |
| relative error  | 0.13\% | 0.12\% | 0.12\% | 0.11\% | 0.10\% | 0.098\% |

Table 1 reflects that both of AAE and RAE of numerical calculation tends to zero for large values of \( L \). For example, when choosing \( L = 50 \), the RAE is remarkably small, which is well within an acceptable range.

Substituting \( E[\Psi_q] \), \( \Theta_{t_q} \), \( C_{t_q} \) and budget \( C \) into Algorithm 1 yields the ultimate optimum results, i.e.,

\[ T^*(t, a_t) = 401.4816 \]
\[ D^* = (1 \, 7 \, 3 \, 8 \, 2 \, 8 \, 3 \, 5 \, 4 \, 8 \, 2 \, 6 \, 3 \, 6 \, 3 \, 9) \]

It is necessary to briefly account for these two results to make them easier to understand. \( T^*(t, a_t) \) represents the maximum of the system lifetime under a series of maintenance actions which are guided by the optimal maintenance strategy. \( D^* \) consists of a series of the system states, which corresponds to \( T^*(t, a_t) \).

**Case 2.** Without loss of generality, we construct a more general distribution function used to express the distribution of the sojourn time of each state in this case, i.e., \( F(t) = 1 - (1 + \lambda t)e^{-\lambda t} \), where \( \lambda \) is set to 1.2. The other parameters involved in the numerical calculation were set to the same values as those in Case 1. Consequently, all of \( \phi_{ij}(t_m) \), \( E[\Psi_q] \), \( E[\Psi_q] \), \( \Theta_{t_q} \), \( C_{t_q} \) were derived. Running Algorithm 1 with these parameters yields:

\[ T^*(t, a_t) = 353.0478 \]
\[ D^* = (1 \, 8 \, 4 \, 7 \, 3 \, 6 \, 3 \, 7 \, 2 \, 5 \, 4 \, 6 \, 4 \, 9) \]

In order to evaluate the performance of the optimal maintenance strategy for large-scale complex systems, a system with state space \( \Gamma = \{1, 2, \cdots, 50\} \) is considered. Assume that state 30 is the critical state and state 50 is the failure state, meaning that \( R = \{1, \cdots, 30\} \), \( U = \{31, \cdots, 49\} \) and \( F = \{50\} \). The kernel
function $Q(t)$ is a matrix of dimension $50 \times 50$ which was randomly generated by programming in MATLAB 7.0 for which it satisfies the following constraints:

$$\sum_{j=1}^{50} Q_{ij}(t) = 1 \quad \forall i \in \Gamma$$
$$Q_{ij}(t) \in (0, 1] \quad \forall i, j \in \Gamma \quad i < j$$
$$Q_{ij}(t) = 0 \quad \forall i, j \in \Gamma \quad i \geq j$$
$$Q_{ij}(t) > Q_{ik}(t) \quad \forall i, j, k \in \Gamma \quad j < k$$

The other parameters, i.e., $C_{tq}$, $C$ and $\Theta_{tq}$, were set to the same values as those in Case 1. Likewise, the Weibull and constructed general distributions were used to represent the distribution of sojourn time for each state of this large-scale system in the following cases 3 and 4, respectively.

**Case 3.** In this case, the parameters of the Weibull distribution, $\alpha$ and $\lambda$, as well as all of the other parameters involved in the numerical calculation were set to the same values as those in Case 1. Then using Eq.(19), $E[\Psi_q]$ is derived. Substituting $E[\Psi_q]$, $\Theta_{tq}$, $C_{tq}$ and budget $C$ into Algorithm 1 yields ultimate $T^*$ and $D^*$, i.e.,

$$T^*(t, a_t) = 710.6497$$
$$D^* = (1 42 10 34 26 32 14 40 27 45 2 35 14 46 11 40 26 48 19 46 2 44 29 45 5 50)$$

**Case 4.** In this case, the distribution function is the same as that in Case 2 and the parameter $\lambda$ is also set to 1.2. The other parameters involved in numerical calculation were set to the same values as those in Case 2. Running Algorithm 1 with these parameters yields the ultimate outcomes as follows:

$$T^*(t, a_t) = 652.609$$
$$D^* = (1 41 5 34 20 35 2 31 7 33 15 32 26 34 16 46 26 34 29 44 14 49 8 50)$$

Figs.7 depicts the curves of four $\phi_{ij}(t_m)$ randomly chosen in Case 1. From the shape of these four curves, it is clear that the trend of the system deterioration between any two states is nearly the same for the Weibull distribution, although the deterioration rate is distinct in different stage. Fig.7 also shows that all of $\phi_{ij}(t)$ tend toward convergence, which implies that the deviation between numerical solution and analytical solution could become arbitrarily small in the calculation of $E(\phi_{ij}(t))$ given that the number of discrete points is sufficiently large. This is also validated by the analysis of asymptotic errors in Case 1.

Additionally, from the obtained optimal strategy $D^*$ of each case, i.e., a series of states alternated between the reliable state set and the unreliable one, it is concluded that the maintenance actions selected at decision times are mostly preventive maintenance. This can be explained by the fact that the probability that the mean time per unit cost of a PM is bigger than that of minimal and perfect maintenance is relatively high. On the other hand, it is also concluded that the developed optimal maintenance strategy is applicable for both small-scale and large-scale systems with either of the Weibull or the constructed general distribution.

### 5.2 Running time of Algorithm 1.

To evaluate the performance of the developed strategy in terms of computational time, we run Algorithm 1 in MATLAB (version 7.0) 50 times for cases 1 to 4. The experiment results are shown in the following:
Figure 8. The running time of Algorithm 1 for the small-scale system with the Weibull distribution corresponding to Case 1.

Figure 9. The running time of Algorithm 1 for the small-scale system with the general distribution corresponding to Case 2.

Figure 10. The running time of Algorithm 1 for the large-scale system with the Weibull distribution corresponding to Case 3.

Figure 11. The running time of Algorithm 1 for the large-scale system with the general distribution corresponding to Case 4.

Table 2. Comparison of running times of Algorithm 1 for cases 1 to 4.

| Running time | The small-scale system | The large-scale system |
|--------------|------------------------|------------------------|
|              | Weibull | general | Weibull | general |
| maximum time | 0.0057 | 0.0051 | 0.089 | 0.093 |
| minimum time | 0.0043 | 0.0038 | 0.058 | 0.062 |
| mean time    | 0.0052 | 0.0048 | 0.0751 | 0.0747 |
Comparing Figs. 8∼11, it is clear that running times of Algorithm 1 for the small-scale and large-scale systems with either of the Weibull or general distributions are no more than 0.1 seconds. As shown in Table 2, for the small-scale system the mean running times of Algorithm 1 are 0.0052 seconds and 0.0048 seconds for the Weibull distribution and the general distribution, respectively. For the large-scale system the mean running time are 0.0751 seconds and 0.0747 seconds for these two distributions, respectively. These results reveal that the running time of Algorithm 1 is satisfactory even for the large-scale system with 50 states. Therefore, it is fair to say that the developed optimal maintenance strategy is high efficient in terms of computational time.

5.3. Sensitivity analysis. To test the robustness of the developed strategy, we consider a possible source of error in the input parameters, i.e., the parameters of the system lifetime distribution. The influence of the system lifetime distribution parameters on both the maximum of the system mean lifetime and the optimal solution is investigated. Errors of $e = \pm 10\%$ and $e = \pm 5\%$ for the parameters $\lambda$ and $\alpha$ of different distribution functions are considered, as shown in Table 3.

| Errors | Weibull (Case 1 & Case 3) | General (Case 2 & Case 4) |
|--------|--------------------------|---------------------------|
|        | $\lambda$ | $\alpha$ | $\lambda$ |
| $-10\%$ | 0.36 | 1.8 | 1.08 |
| $-5\%$ | 0.38 | 1.9 | 1.14 |
| 0%     | 0.4  | 2  | 1.2  |
| 5%     | 0.42 | 2.1 | 1.26 |
| 10%    | 0.44 | 2.2 | 1.32 |

The results shown in Tables 4-7 are obtained by varying one parameter while fixing the other parameters.

| errors of $\lambda$ and $\alpha$ | $\lambda$ | $\alpha$ | $T^*(t, a_{t_q})$ | errors of $T^*(t, a_{t_q})$ | $D^*$ |
|----------------------------------|-----------|-----------|-----------------|-----------------------------|-------|
| $-10\%$                         | 0.36      | 1.8       | 400.80          | -0.17%                      | $\{1 7 3 8 2 8 3 5 4 8 2 6 3 6 3 9\}$ |
| $-5\%$                          | 0.38      | 1.9       | 401.16          | -0.08%                      | $\{1 7 3 8 2 8 3 5 4 8 2 6 3 6 3 9\}$ |
| 0%                               | 0.4       | 2         | 401.48          | 0%                          | $\{1 7 3 8 2 8 3 5 4 8 2 6 3 6 3 9\}$ |
| 5%                               | 0.42      | 2.1       | 401.80          | 0.08%                       | $\{1 7 3 8 2 8 3 5 4 8 2 6 3 6 3 9\}$ |
| 10%                              | 0.44      | 2.2       | 402.04          | 0.14%                       | $\{1 7 3 8 2 8 3 5 4 8 2 6 3 6 3 9\}$ |
Table 5. Sensitivity analysis of $\lambda$ on $T^*(t, a_{tq})$ and $D^*$ for the small-scale system with the general distribution (Case 2)

| errors of $\lambda$ | $\lambda$ | $T^*(t, a_{tq})$ | errors of $T^*(t, a_{tq})$ | $D^*$ |
|---------------------|-----------|------------------|-----------------------------|-------|
| -10%                | 1.08      | 352.60           | -0.13%                      | (1 8 4 7 3 6 3 7 2 5 4 6 4 9) |
| -5%                 | 1.14      | 352.84           | -0.06%                      | (1 8 4 7 3 6 3 7 2 5 4 6 4 9) |
| 0%                  | 1.2       | 353.05           | 0%                          | (1 8 4 7 3 6 3 7 2 5 4 6 4 9) |
| 5%                  | 1.26      | 353.26           | 0.06%                       | (1 8 4 7 3 6 3 7 2 5 4 6 4 9) |
| 10%                 | 1.32      | 353.44           | 0.11%                       | (1 8 4 7 3 6 3 7 2 5 4 6 4 9) |

Table 6. Sensitivity analysis of $\lambda$ and $\alpha$ on $T^*(t, a_{tq})$ and $D^*$ for the large-scale system with the Weibull distribution (Case 3)

| errors of $\lambda$ and $\alpha$ | $\lambda$ | $\alpha$ | $T^*(t, a_{tq})$ | errors of $T^*(t, a_{tq})$ | $D^*$ |
|-----------------------------------|-----------|----------|------------------|-----------------------------|-------|
| -10%                              | 0.36      | 1.8      | 710.29           | -0.05%                      | (1 4 2 10 34 26 32 14 40 27 45 2 35 14 46 11 40 26 48 19 46 2 44 29 45 5 50) |
| -5%                               | 0.38      | 1.9      | 710.43           | -0.03%                      | (1 4 2 10 34 26 32 14 40 27 45 2 35 14 46 11 40 26 48 19 46 2 44 29 45 5 50) |
| 0%                                | 0.4       | 2        | 710.65           | 0%                          | (1 4 2 10 34 26 32 14 40 27 45 2 35 14 46 11 40 26 48 19 46 2 44 29 45 5 50) |
| 5%                                | 0.42      | 2.1      | 710.80           | 0.02%                       | (1 4 2 10 34 26 32 14 40 27 45 2 35 14 46 11 40 26 48 19 46 2 44 29 45 5 50) |
| 10%                               | 0.44      | 2.2      | 710.93           | 0.04%                       | (1 4 2 10 34 26 32 14 40 27 45 2 35 14 46 11 40 26 48 19 46 2 44 29 45 5 50) |

Table 7. Sensitivity analysis of $\lambda$ on $T^*(t, a_{tq})$ and $D^*$ for the large-scale system with the general distribution (Case 4)

| errors of $\lambda$ | $\lambda$ | $T^*(t, a_{tq})$ | errors of $T^*(t, a_{tq})$ | $D^*$ |
|---------------------|-----------|------------------|-----------------------------|-------|
| -10%                | 1.08      | 652.28           | -0.05%                      | (1 4 1 5 3 4 20 35 2 31 7 33 15 32 26 34 16 46 26 34 29 44 14 49 8 50) |
| -5%                 | 1.14      | 652.48           | -0.02%                      | (1 4 1 5 3 4 20 35 2 31 7 33 15 32 26 34 16 46 26 34 29 44 14 49 8 50) |
| 0%                  | 1.2       | 652.61           | 0%                          | (1 4 1 5 3 4 20 35 2 31 7 33 15 32 26 34 16 46 26 34 29 44 14 49 8 50) |
| 5%                  | 1.26      | 652.81           | 0.03%                       | (1 4 1 5 3 4 20 35 2 31 7 33 15 32 26 34 16 46 26 34 29 44 14 49 8 50) |
| 10%                 | 1.32      | 652.94           | 0.05%                       | (1 4 1 5 3 4 20 35 2 31 7 33 15 32 26 34 16 46 26 34 29 44 14 49 8 50) |

Tables 4-7 reflect the influence of parameters $\lambda$ and $\alpha$ on the objective function $T^*(t, a_{tq})$ and the optimal strategy $D^*$ for two different types of lifetime distribution. The parameters $\lambda$ and $\alpha$ influence maximum of the system mean lifetime $T^*(t, a_{tq})$, but do not influence the optimal strategy $D^*$. As shown, the maximum percentage
errors in terms of \( T^*(t, a) \) for cases 1, 2, 3 and 4 are \(-0.17\%\), \(-0.13\%\), \(-0.05\%\) and \(\pm 0.05\%\), respectively. These results reveal that the parameters of the lifetime distributions have a weak influence on the maximum of the system mean lifetime. In reality, the parameters of these two lifetime distributions embody failure rate. Therefore, the maximum of the system mean lifetime obtained by the proposed optimal strategy is slightly sensitive to the failure rate. The percentage errors of \( T^*(t, a) \) versus the percentage errors of both \( \lambda \), \( \alpha \) of the Weibull distribution and the \( \lambda \) of the general distribution are plotted in Fig.12, which reflects that the percentage error of \( T^*(t, a) \) increases slightly and approximately linearly as the percentage errors of both \( \lambda \), \( \alpha \) of the Weibull distribution and \( \lambda \) of the general distributions increase in the range of \([-0.1, 0.1]\), regardless of the system scale.

6. Conclusions. In this work, the maintenance model of MSSs based on a continuous-time semi-Markov process has been developed. The dynamics of this model are investigated using the linear integral equation as well as the Markov kernel function. The modified numerical method is provided for solving the system linear integral equation. An dynamic programming algorithm combined with forward and backward local search techniques is provided to find the optimal maintenance strategy. The modified numerical method is validated satisfactory in terms of calculation accuracy by a numerical example. The major contribution of the determined optimal strategy lies in the fact that it can cope with the coupling issue resulting from the adjacent decision variables and can be applicable to different types of the system lifetime (the Weibull distribution and a more general distribution), which are demonstrated by the outcomes of 4 cases. Furthermore, the optimal strategy is high efficient in terms of computational time even for large-scale complex systems with 50 states. In addition, the optimal strategy is not sensitive to the parameters of system lifetime distributions, which validates the high robustness of the strategy. Therefore, the developed maintenance model and the determined optimal strategy are instrumental in making maintenance decisions for managers or designers.

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