Coalescing random walks is a fundamental distributed process, where a set of particles perform independent discrete-time random walks on an undirected graph. Whenever two or more particles meet at a given node, they merge and continue as a single random walk. The coalescence time is defined as the expected time until only one particle remains, starting from one particle at every node. Despite recent progress such as that of Cooper et al., the coalescence time for graphs, such as binary trees, $d$-dimensional tori, hypercubes, and, more generally, vertex-transitive graphs, remains unresolved.

We provide a powerful toolkit that results in tight bounds for various topologies including the aforementioned ones. The meeting time is defined as the worst-case expected time required for two random walks to arrive at the same node at the same time. As a general result, we establish that for graphs whose meeting time is only marginally larger than the mixing time (a factor of $\log^2 n$), the coalescence time of $n$ random walks equals the meeting time up to constant factors. This upper bound is complemented by the construction of a graph family demonstrating that this result is the best possible up to constant factors. Finally, we prove a tight worst-case bound for the coalescence time of $O(n^3)$. By duality, our results yield identical bounds on the voter model.

Our techniques also yield a new bound on the hitting time and cover time of regular graphs, improving and tightening previous results by Broder and Karlin, as well as those by Aldous and Fill.

CCS Concepts: • Mathematics of computing → Markov processes;

Additional Key Words and Phrases: Coalescing random walks, population protocols, voter model

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1 INTRODUCTION
Coalescing random walks is a fundamental distributed process on connected and undirected graphs. The process begins with particles on some subset of the nodes in the graph. At discrete timesteps,
every particle performs one step of an independent random walk.\footnote{Throughout this article, we use the terms \textit{random walk} and \textit{particle} interchangeably, assuming that every random walk has an identifier.} Whenever two or more particles arrive at the same node at the same timestep, they merge into a single particle and continue as a single random walk. The \textit{coalescence time} is defined as the first timestep when only one particle remains. The coalescence time depends on the number and starting positions of the particles.

Studying the coalescence time is of substantial importance in distributed computing: at the heart of many distributed computing applications lie consensus protocols and leader election, such as data consistency, consolidation of replicated states, synchronization of processes and devices, and communication networks \cite{4, 6, 24, 20, 42}. Other applications of the coalescence process appear in robotics \cite{28}; here, robots perform random walks to gather samples from their environment and need to communicate these samples to all other robots. Studying the coalescence time also implies results for other interaction types of random walks including predator and prey particles, as well as annihilating particles \cite{17}.

\textit{Relationship to Consensus Protocols.} Arguably, the simplest consensus protocol achieving consensus on any undirected graph is the \textit{voter model}. Initially, every node has a distinct opinion. At every round, each node chooses synchronously one of its neighbors at random and adopts that node’s opinion. The \textit{consensus time} is defined as the time it takes until only one opinion remains. The voting process viewed backward is exactly the same as the coalescence process starting with a random walk on every node; thus, the coalescence time and consensus time have the same distribution (this duality assumes that the random walks are non-lazy). Despite recent progress by Cooper et al. \cite{14, 19} and Berenbrink et al. \cite{11}, the coalescence time and consensus time are far from being well understood—even for certain fundamental graphs as we describe in the following.

Recently, there have been several studies on variants of the voter model, most notably 2-Choices and 3-Majority, which received ample attention \cite{8–10, 15, 16, 20, 26, 29}. However, the behavior of these processes is fundamentally different, and despite their efficiency in reaching consensus on expanders and cliques, they are unsuitable on more general undirected graphs as the consensus time is exponential in some graphs.

In this article, we follow the approach of Cooper et al. \cite{14} and Hassin and Peleg \cite{30} and study the consensus time through the more tangible analysis of the coalescence time. When starting with two particles, the coalescence time is referred to as the \textit{meeting time}. Let $t_{\text{meet}}$ denote the worst-case expected meeting time over all pairs of starting nodes, and let $t_{\text{coal}}$ denote the expected coalescence time starting from one particle on every node. It is clear that $t_{\text{meet}} \leq t_{\text{coal}}$; as for an upper bound, it can be shown that $t_{\text{coal}} = O(t_{\text{meet}} \log n)$, where $n$ is the number of nodes in the graph. The main idea used to obtain the bound is that the number of surviving random walks halves roughly every $t_{\text{meet}}$ steps. A proof of the result appears implicitly in the work of Hassin and Peleg \cite{30}.

Aldous \cite{1} showed in \textit{continuous time} that the meeting time is bounded by the maximum hitting time, $t_{\text{hit}} := \max_{u,v} t_{\text{hit}}(u,v)$, where $t_{\text{hit}}(u,v)$ denotes the expected time required to hit $v$ starting from vertex $u$. We observe that the result of Aldous also holds in discrete time. Thus, this gives a bound of \( O(t_{\text{hit}} \log n) \) for the coalescing time; however, in general, \( O(t_{\text{hit}}) \) may be a loose upper bound on $t_{\text{meet}}$. In recent work, Cooper et al. \cite{14} provide results that are better than $O(t_{\text{meet}} \log n)$ for several interesting graph classes, notably expanders and power-law graphs. They show that $t_{\text{coal}} = O((\log^4 n + \| \pi \|^2_2) \cdot (1 - \lambda_2)^{-1})$, where $\lambda_2$ is the second largest eigenvalue of the transition matrix of the random walk and $\pi$ is the stationary distribution. Berenbrink et al. \cite{11} show that $t_{\text{coal}} = O(m/(d_{\text{min}} \cdot \Phi))$, where $m$ is the number of edges, $d_{\text{min}}$ is the minimum degree, and $\Phi$ is
the conductance. Their result improves on that of Cooper et al. for certain graph classes, such as cycles.

As mentioned before, despite the recent progress due to Cooper et al. [14] and Berenbrink et al. [11], for many fundamental graphs, such as the binary tree, hypercube, and the (d-dimensional) torus, the coalescing time in the discrete setting remains unsettled. We provide a rich toolkit allowing us to derive tight bounds for many graphs including all of the aforementioned ones. One of our main results establishes a relationship between the ratios $t_{coal}/t_{meet}$ and $t_{meet}/t_{mix}$, where $t_{mix} = t_{mix}(1/e)$ denotes the mixing time.\(^2\) In particular, the result shows that if $t_{meet}/t_{mix} = \Omega(\log^2 n)$, then $t_{coal} = O(t_{meet})$; however, we also provide a more fine-grained tradeoff. For almost-regular graphs,\(^3\) we bound the coalescing time by the hitting time. For vertex-transitive graphs, we show that the coalescence time, the meeting time, and the hitting time are equal up to constant factors. Finally, we prove that for any graph, the coalescence time is bounded by $O(n^3)$; it can be easily verified that this is tight by considering the barbell graph. Surprisingly, the right bound on this fundamental quantity was not known prior to this work. Unlike in the analogous case of the cover time [2] where such a bound can be easily derived, the argument in the case of coalescence time appears significantly involved.\(^4\) Prior to this work, Hassin and Peleg [30] had shown a worst-case upper bound of $O(n^3 \log n)$. We also give worst-case upper and lower bounds on the meeting time and coalescence time that are tight for general graphs and regular (or nearly-regular) graphs.

In the process of establishing bounds on the coalescence time, we develop techniques to give tight bounds on the meeting time. We apply these to various topologies such as the binary tree, torus and hypercube. We believe that these techniques might be of more general interest.

The process of coalescing random walks was first studied in continuous time; in this case, particles jump to a random neighboring node when activated according to a Poisson clock with mean 1. In the continuous time setting, Cox [22] showed that the coalescence time is bounded by $\Theta(t_{hit})$ for tori. Oliveira [37] showed that the coalescence time is $O(t_{hit})$ in general. In a different work, Oliveira [38] derived so-called mean field conditions, which are sufficient conditions for the coalescing process on a graph to behave similarly to that on the complete graph up to scaling by the expected meeting time. His main result (for non-vertex-transitive graphs) [38, Theorem 1.2] implies that $t_{coal} = O(t_{meet})$ whenever $t_{mix} \cdot \pi_{max} = O(1/\log^4 n)$. One of our main results, Theorem 1.1, implies that $t_{coal} = O(t_{meet})$ whenever $t_{mix}/t_{meet} = O(1/\log^2 n)$. Notice that since $t_{meet} = \Omega(1/\|\pi\|^2_2) \geq \Omega(1/\pi_{max})$, our condition is considerably more general; however, the results in the work of Oliveira [38] also establish mean-field behavior (i.e., when suitably scaled, the distribution of the coalescence time is similar to that on a complete graph), whereas ours are only concerned with the expected coalescence time, $t_{coal}$. Yet our result also applies to graphs where $t_{coal} \gg t_{meet}$, such as the star graph, and together with Theorem 1.2 demonstrates that the tradeoff between meeting and mixing time is the best possible.

1.1 Main Results

In this work, we provide several results relating the coalescence and meeting times to each other and to other fundamental quantities of random walks on undirected graphs. In particular, our focus is on understanding for which graphs the coalescence time is the same as the meeting time, as we know that $t_{coal}$ is always in the rather narrow interval of $[t_{meet}, O(t_{meet} \cdot \log n)]$. As a consequence of our results, we derive new and redefine existing bounds on the meeting and coalescence times for several graph families of interest. These results are summarized in Table 1.

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\(^2\)The mixing time is the first timestep at which the distribution of a random walk starting from an arbitrary node is close to the stationary distribution.

\(^3\)We call a graph almost-regular if $\deg(u) = \Theta(\deg(v))$ for all $u, v \in V$.

\(^4\) Cooper et al. [14] mistakenly stated, as a side remark, that this last result was a simple consequence of their main result.
Formal definitions of all quantities used in the following appear in Section 2. Throughout this article, we assume that random walks are lazy, meaning that w.p. 1/2 the walk stays put.

Our first main result relates $t_{\text{coal}}$ to $t_{\text{meet}}$ and $t_{\text{mix}}$. As already mentioned, the crude bound $t_{\text{coal}} = O(t_{\text{meet}} \log n)$ is well known. However, this bound can be improved substantially, as demonstrated by our following result (recall that for any graph $G$, $t_{\text{mix}} = O(t_{\text{meet}})$ [2]).

**Theorem 1.1.** For any graph $G$, we have

$$t_{\text{coal}} = O \left( t_{\text{meet}} \left( 1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \cdot \log n} \right) \right).$$

Consequently, when $t_{\text{meet}} \geq t_{\text{mix}} \log^2 n$, $t_{\text{coal}} = O(t_{\text{meet}})$.

The proof of Theorem 1.1 appears in Section 3. One interesting aspect about this bound is that it can be used to establish $t_{\text{coal}} = \Theta(t_{\text{meet}})$ even without explicit bounds on the quantities $t_{\text{meet}}$ or $t_{\text{mix}}$. Another interesting feature of our theorem is that the main result of Cooper et al. [14, Theorem 1] can be reproven by combining their work [14, Theorem 2] with Theorem 1.1 (see Proposition 5.2).

Our next main result shows that the bound in Theorem 1.1 is tight up to a constant factor, which we establish by constructing an explicit family of graphs. Interestingly, for this family of almost-regular graphs, we also have $t_{\text{hit}} \gg t_{\text{meet}}$, thus showing that $t_{\text{hit}}$ may be a rather loose upper bound for $t_{\text{coal}}$ in some cases.\(^5\)

**Theorem 1.2.** For any sequence $(\alpha_n)_{n \geq 0}$, $\alpha_n \in [1, \log^2 n]$ there exists a family of almost-regular graphs $(G_n)$, with $G_n$ having $\Theta(n)$ nodes and satisfying $\frac{t_{\text{meet}}}{t_{\text{mix}}} = \Theta(\alpha_n)$ such that

$$t_{\text{coal}} = \Omega \left( t_{\text{meet}} \cdot \left( 1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \cdot \log n} \right) \right).$$

The preceding two results show that the condition $t_{\text{meet}} \geq t_{\text{mix}} \log^2 n$ in Theorem 1.1 is best possible up to constant factors. A natural question is therefore whether in the case of structured sub-classes such as vertex-transitive graphs, or special graphs such as grids, tori, binary trees, cycles, and real-world (power-law) graphs, that better bounds can be obtained. We provide results that are tight or nearly tight in several of these cases; some of these results were previously known using other methods, and some are novel to the best of our knowledge.

In an earlier version [31], we also proved that

$$t_{\text{coal}} = O \left( t_{\text{hit}} + t_{\text{meet}} \cdot \log(\Delta/d) \right) \quad (1)$$

\(^5\)Note that the star also exhibits $t_{\text{hit}} \gg t_{\text{meet}}$. However, the star is not almost-regular.

Table 1. Summary of Bounds on the Mixing, Meeting, Coalescence, and Hitting Times for Fundamental Topologies for Discrete-Time Random Walks

| Graph     | $t_{\text{mix}}$ | $t_{\text{meet}}$ | $t_{\text{coal}}$ | $t_{\text{hit}}$ |
|-----------|------------------|-------------------|-------------------|------------------|
| Binary tree | $\Theta(n)$      | $\Theta(n \log n)$ | $\Theta(n \log n)$ | $\Theta(n \log n)$ |
| Clique    | $\Theta(1)$      | $\Theta(n)$       | $\Theta(n)$       | $\Theta(n)$      |
| Cycle     | $\Theta(n^\alpha)$ | $\Theta(n^\alpha)$ | $\Theta(n^\alpha)$ | $\Theta(n^\alpha)$ |
| Rand. $r$-reg. | $\Theta(\log n)$ | $\Theta(n)$      | $\Theta(n)$       | $\Theta(n)$      |
| Hypercube | $\Theta(\log n \log \log n)$ | $\Theta(n)$       | $\Theta(n)$       | $\Theta(n)$      |
| Path      | $\Theta(n^\alpha)$ | $\Theta(n^\alpha)$ | $\Theta(n^\alpha)$ | $\Theta(n^\alpha)$ |
| Star      | $\Theta(1)$      | $\Theta(1)$       | $\Theta(1)$       | $\Theta(1)$      |
| Torus ($d = 2$) | $\Theta(n)$ | $\Theta(n \log n)$ | $\Theta(n \log n)$ | $\Theta(n \log n)$ |
| Torus ($d > 2$) | $\Theta(n^{d/2})$ | $\Theta(n)$       | $\Theta(n)$       | $\Theta(n)$      |

\(^1\)All bounds on the mixing and hitting times appear directly or implicitly in the work of Aldous and Fill [2]. Note that for regular graphs, $t_{\text{meet}} = \Omega(n)$ (e.g., Theorem 5.1.iii). The results for the hypercube and the torus with $d > 2$ are new (see Section 6). Note that $t_{\text{coal}} = \Omega(t_{\text{meet}})$.

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for any graph $G$ with maximum degree $\Delta$ and average degree $d$. We omit the proof of this result here, since very recently Oliveira and Peres [39] proved that $t_{\text{coal}} = O(t_{\text{hit}})$ holds for all undirected graphs.

Another natural question is to express $t_{\text{meet}}$ or $t_{\text{coal}}$ solely in terms of $t_{\text{mix}}$, the spectral gap $1 - \lambda_2$, or other connectivity properties of $G$. We derive several such bounds on $t_{\text{meet}}, t_{\text{hit}},$ and $t_{\text{coal}}$.

As a by-product of our techniques, we also derive new bounds on $t_{\text{hit}}$ and $t_{\text{cov}}$, the cover time, which is the expected time until a random walk has visited all vertices in $G$ (starting from a worst-case vertex). The detailed results are given in Section 5, and here we state only one of these results.

**Theorem 1.3.** For any regular graph, it holds that

$$t_{\text{hit}} = O(n/\sqrt{1 - \lambda_2}) = O(n/\Phi),$$

where $\Phi$ is the conductance of the graph and $\lambda_2$ is the second largest eigenvalue of the transition matrix $P$ of a lazy random walk. Consequently, $t_{\text{cov}} = O(n \log n/(1 - \lambda_2)) = O(n \log n/\Phi)$. Furthermore,

$$t_{\text{meet}} \leq t_{\text{coal}} = O(n/\sqrt{1 - \lambda_2}) = O(n/\Phi).$$

We point out that so far the best possible bound on $t_{\text{coal}}$ for regular graphs has been $t_{\text{coal}} = O(n/(1 - \lambda_2))$ from the work of Cooper et al. [14].6 The best possible bound on $t_{\text{hit}}$ (and $t_{\text{cov}}$) in terms of $1 - \lambda_2$ was $t_{\text{hit}} = O(n/(1 - \lambda_2))$ and $t_{\text{cov}} = O(n \log n/(1 - \lambda_2))$ due to Broder and Karlin [12] from 1989. In all four cases, $t_{\text{meet}}, t_{\text{coal}}, t_{\text{hit}},$ and $t_{\text{cov}}$, Theorem 1.3 improves the dependency on $1/(1 - \lambda_2)$ (or, equivalently $t_{\text{mix}}$), by almost a square root (we refer the reader to Theorems 5.4 and 5.6 for further details). As a result of this improvement, we get a bound of $O(n/\Phi)$ on the hitting time, which is the best known bound on the hitting time (and cover time) in terms of the conductance and improves the bound of Aldous and Fill [2, Corollary 6.2.1] by a factor of $1/\Phi$. After an extended abstract of our work was published in SODA 2019 [32], Oliveira and Peres [39] improved versions of these bounds for non-regular graphs by showing that the maximum degree $\Delta$ in the preceding statement can be replaced by the average degree $d$.

In addition, we derive a general lower bound on $t_{\text{meet}}$ that combines the lower trivial bound, $1/\|\pi\|^2_2$, with the minimum number of collisions (see Theorem 5.1.iii).

Finally, we also provide asymptotically tight worst-case bounds on $t_{\text{meet}}$ and $t_{\text{coal}}$. We show that on any graph, the coalescence time must be at least $\Omega(\log n)$ and is no more than $O(n^2)$. For regular (and particularly vertex-transitive) graphs, these bounds become $\Omega(n)$ and $O(n^2)$. These two new upper bounds for general and regular graphs complete the picture of worst-case bounds.

**Theorem 1.4.** The following hold for graphs of the stated kind:

(i) For any graph $G$, we have $t_{\text{meet}} \in [\Omega(1), O(n^3)]$ and $t_{\text{coal}} \in [\Omega(\log n), O(n^3)]$.

(ii) For any regular graph $G$, we have $t_{\text{meet}}, t_{\text{coal}} \in [\Omega(n), O(n^2)]$.

The proof of Theorem 1.4 appears in Section 4.

1.2 Proof Ideas and Technical Contributions

When dealing with processes involving concurrent random walks, a significant challenge is to understand the behavior of “short” random walks. This challenge appears in several settings, such as in the context of cover time of multiple random walks [5, 25], where Efremenko and Reingold [25, Section 6] highlight the difficulty in analyzing the hitting time distribution before its expectation. In the context of concentration inequalities for Markov chains, Lezaud [34, p. 863] points out

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6Alternatively, the same bound as the known bound can also be derived from the bound on the conductance in the work of Berenbrink et al. [11] together with Cheeger’s inequality.
the requirement to spend at least mixing timesteps before taking any samples. Related to that, in property testing, dealing with graphs that are far from expanders has been mentioned as one of the major challenges to test their expansion (see the work of Czumaj and Sohler [23]). In our setting, we also face these generic problems and devise different methods to get a handle on the meeting time distribution before its expectation. Our approaches can be leveraged to derive new bounds on other random walk quantities such as hitting times or cover times (see Section 5).

**Bounds on \( t_{coal} \) in Terms of \( t_{mix} \) and \( t_{meet} \)**

The key ingredient in the proof of Theorem 1.1, where we express \( t_{coal} \) as a tradeoff between \( t_{meet} \) and \( t_{mix} \), is a better understanding of meeting events prior to the (expected) meeting time. More precisely, we derive a tight bound on the probability \( p_\ell \) that two random walks meet before \( \ell \) timesteps, for \( \ell \) in the range \([t_{mix}, t_{meet}]\). Arguing about meeting probabilities of walks that are much shorter than \( t_{meet} \) allows us to understand the rate at which the number of alive random walks is decreasing.

Optimistically, one may hope that starting with \( k \) random walks, as there are \( \binom{k}{2} \) possible meeting events, roughly \( \binom{k}{2} \cdot p_\ell \) meetings may have occurred after \( \ell \) timesteps. However, the non-independence of these events turns out to be a serious issue, and we require a significantly more sophisticated approach to account for the dependencies. We divide the \( k \) random walks into disjoint groups \( G_1 \) and \( G_2 \) (with \(|G_1|\) usually being much smaller than \(|G_2|\)), and walks of \( G_1 \) cannot be eliminated. The domination of the real process by the group-restricted one is established by introducing a formal concept called the immortal process at the beginning of Section 3.1. We believe that this domination might be of further use to analyze other interacting particle processes.

In the group-restricted process, we can expose the random walks of \( G_1 \) first and consider meetings with random walks in \( G_2 \) (for an illustration, see Figure 1, presented later in the article). Conditioning on a specific exposed walk in \( G_1 \), the events of the different walks in \( G_2 \) meeting
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Fig. 2. The graph described in Section 3.4 with $t_{\text{coal}} = \Omega(t_{\text{meet}} + \sqrt{t_{\text{meet}}/t_{\text{mix}}} \cdot \log n \cdot t_{\text{mix}})$.

this exposed walk are indeed independent. In fact, we will also use the symmetric case where the roles of $G_1$ and $G_2$ are switched. Thus, the problem then reduces to calculating the probability of a random walk in $G_2$ having a "good trajectory"—that is, one that many random walks in $G_1$ would meet with large enough probability.

Surprisingly, it suffices to divide trajectories into only two categories (Lemma 3.3). Although one may expect that a more fine-grained classification of trajectories would result in better bounds, this turns out not to be the case. In fact, the bound that we derive on the coalescing time in Theorem 1.1 is tight, and this is precisely due to the tightness of Lemma 3.3. The tightness is established by the following construction (cf. Figure 2). The graph is designed such that most meetings (between any two random walks) occur in a relatively small part of the graph ($G_2$ presented later in Figure 3). On average, it takes a considerable number of timesteps before random walks actually get to this part of the graph. What this implies is that for relatively short trajectories (of length significantly smaller than $t_{\text{meet}}$), it is quite likely that other random walks will not meet them (cf. Lemma 3.3). There is a bit of a dichotomy here: once a walk reaches $G_2$, it is likely that many random walks will meet it; however, a random walk not reaching $G_2$ is unlikely to be met by any other random walk.

Equipped with Theorem 1.1, we can bound $t_{\text{coal}} = \Theta(t_{\text{meet}})$ for all graphs satisfying $t_{\text{meet}}/t_{\text{mix}} \geq \log^2 n$. Therefore, the problem of bounding $t_{\text{coal}}$ reduces to bounding $t_{\text{meet}}$.

For some of the other results, including Theorem 1.2, we will need a more fine-grained approach to derive lower (or upper bounds) on the probability that two walks meet during a certain number of steps, which may or may not be smaller than the mixing time or meeting time. The starting point is the following simple observation. If we have two random walks $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$, and count the number of collisions $Z := \sum_{t=0}^{r-1} 1_{X_t = Y_t}$ before timestep $r \in \mathbb{N}$, then

$$P[Z \geq 1] = \frac{\mathbb{E}[Z]}{\mathbb{E}[Z | Z \geq 1]},$$

(2)
If we further assume that both walks start from the stationary distribution, then we have

$$
P[Z \geq 1] = \frac{\tau \cdot \|\pi\|_2^2}{\mathbb{E}[Z | Z \geq 1]}.
$$

To the best of our knowledge, this is the first application of this formula to meeting (and coalescence) times. However, we should mention that variants of this formula have been used by Cooper and Frieze in several works (e.g., [18]) to derive accurate bounds on the hitting (and cover time) on various classes of random graphs, and in the work of Barlow et al. [7] to bound the collisions of random walks on infinite graphs. Using (2), we are able to obtain several improvements to existing bounds on the meeting time, and as a consequence for coalescing time. We believe that our work further highlights the power of this basic identity.

The crux of (2) is that to lower (or upper) bound the probability that the two walks meet, we need to derive a corresponding bound on $\mathbb{E}[Z | Z \geq 1]$ (i.e., the number of collisions conditioning on the occurrence of at least one collision). Our results employ various tools to get a handle on this quantity, but here we mention one that is quite intuitive:

$$
\mathbb{E}[Z | Z \geq 1] \leq \max_{u \in V} \sum_{t=0}^{\tau-1} \sum_{v \in V} (p_{u,v}^t)^2.
$$

The inner summand $\sum_{v \in V} (p_{u,v}^t)^2$ is the probability that two walks starting from the same vertex $u$ will meet after a further $t$ steps. Thus, summing over $t$ and conditioning on the first meeting happening (i.e., the condition $Z \geq 1$) at some vertex $u$ before timestep $\tau$ yields the bound in (3).

Despite the seemingly crude nature of this bound, it can be used to derive new results for $t_{\text{hit}}$, $t_{\text{meet}}$, and $t_{\text{coal}}$ that significantly improve over the state of the art for regular graphs (see Section 5 or the last paragraph in this section for a summary).

**Worst-Case Bounds.** We proceed by establishing that $t_{\text{coal}} = O(n^3)$ on all graphs. The proof of $t_{\text{coal}} = O(n^3)$ (Theorem 1.4) follows by reducing the number of walks from $n$ to $(\Delta/d)^{O(1)} \leq \dots$
\[(n^2/|E|)^{O(1)} \text{ in } O(t_{hit})\]. We have, by Proposition 5.7, \(t_{meet} \leq 2t_{hit} + 1 = O(n \cdot |E|)\), where this last bound follows from the work of Aleliunas et al. [3].

Finally, combining the bound \(t_{meet} = O(n \cdot |E|)\) together with \(t_{coal}(S_0) = O(t_{meet} \cdot \log(|S_0|))\) (Proposition 3.4) for any set of start vertices \(S_0\) yields that after additional

\[O(t_{meet} \cdot \log(|S_0|)) = O(n \cdot |E| \cdot \log(n^2/|E|)) = O(n^3)\]

steps, the coalescing terminates. The fact that this is tight can be easily verified by considering the Barbell graph.\(^7\)

For regular graphs, the same argument as before shows that \(t_{coal} = O(n^2)\), and this is matched by the cycle, for instance.

**Bounds on \(t_{meet}\) and Other Results.** In Section 5, we derive several bounds on \(t_{meet}\). These bounds are derived more directly by (2) and/or (3), and involve other quantities such as \(\|\pi\|_2^2\) or the eigenvalue gap \(1 - \lambda_2\). One important technical contribution is to combine routine spectral methods involving the spectral representation and fundamental matrices that have been used in previous works, such as that of Cooper et al. [14] with some short-time bounds on the \(t\)-th step probabilities. This allows us to improve several bounds, not only on \(t_{meet}\) and \(t_{coal}\) but also \(t_{hit}\) and \(t_{cov}\), by significantly reducing the dependency on the spectral gap or mixing time—by almost a square root factor. As a corollary, we also derive a new bound on the cover time for regular graphs that considerably improves over the best known bound by Broder and Karlin [12].

**2 NOTATION AND PRELIMINARIES**

Throughout the article, let \(G = (V, E)\) denote an undirected, connected graph with \(|V| = n\) and \(|E| = m\). For a node \(u \in V\), \(\deg(u)\) denotes the degree of \(u\) and \(N(u) = \{v : (u, v) \in E\}\) the neighborhood of \(u\). By \(\Delta, \delta, \text{ and } d = \frac{1}{n} \sum_{u \in V} \deg(u)\), we denote the maximum, minimum, and average degree, respectively. We say \(G\) is \(\Gamma\)-approximative regular if \(\Delta/\delta = \Gamma\).

Unless stated otherwise, all random walks are assumed to be discrete time (indexed by natural numbers) and lazy—that is, if \(P\) denotes the \(n \times n\) transition matrix of the random walk, \(p_{u, u} = 1/2, p_{u, v} = 1/(2\deg(u))\) for any edge \((u, v) \in E\) and \(p_{u, v} = 0\) otherwise. We define \(p'_{u, v}\) to be the probability that a random walk starting at \(u \in V\) is at node \(v \in V\) at time \(t \in \mathbb{N}\). Furthermore, let \(p'_{u, \cdot}\) be the probability distribution of the random walk after \(t\) timesteps starting at \(u\). By \(\pi\), we denote the stationary distribution, which satisfies \(\pi(u) = \deg(u)/(2m)\) for all \(u \in V\).

Let \(d(t) = \max_u \|\pi_{u, \cdot} - \pi\|\text{TV} \text{ and } d(t) = \max_{u, v} \|p'_{u, \cdot} - p'_{v, \cdot}\|\text{TV}\), where \(\| \cdot \|\text{TV}\) denotes the total variation distance, which can be defined for any two discrete distributions \(\mu\) and \(\nu\) as \(\|\mu - \nu\|\text{TV} = \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \nu(\omega)|\). Following Aldous and Fill [2], we define the mixing time to be \(t_{mix}(\epsilon) = \min\{t \geq 0 : d(t) \leq \epsilon\}\) and for convenience we will write \(t_{mix} = t_{mix}(1/\epsilon)\). We define separation from stationarity to be \(s(t) = \min_{\epsilon} \{\epsilon : p'_{u, v}(t) \geq (1 - \epsilon)\pi(v)\}\), the first time \(u,v \in V\) and \(s(t)\) is submultiplicative (i.e., \(s(t_1 + t_2) \leq s(t_1) \cdot s(t_2)\) [33, Exercise 6.4]), so particularly non-increasing [33, Exercise 6.4]. Hence, we can define the separation threshold time \(t_{sep}\) as \(t_{sep} = \min\{t \geq 0 : s(t) \leq 1/\epsilon\}\), and, by Aldous and Fill [2, Lemma 4.11], \(t_{sep} \leq 4t_{mix}\). We write \(T_{hit}(u, v)\) to denote the first timestep \(t \geq 0\) at which a random walk starting at \(u\) hits \(v\). In particular, \(T_{hit}(u, u) = 0\). The hitting time \(h_{hit}(u, v) = \mathbb{E}[T_{hit}(u, v)]\) of any pair of nodes \(u, v \in V\) is the expected time required for a random walk starting at \(u\) to hit \(v\). Thus, \(h_{hit}(u, v)\) is the expectation of \(T_{hit}(u, v)\) and the hitting time of a graph \(h_{hit} = \max_{u, v} h_{hit}(u, v)\) is the maximum over all such pairs.

\(^7\)This \(n\)-vertex graph is constructed by taking two cliques of size \(n/4\) each and connecting them through a path of length \(n/2\).
For $A \subseteq V$, we use $t_{\text{hit}}(u, A)$ to denote the expected time required for a random walk starting to $u$ to hit some node in the set $A$. Furthermore, we define $t_{\text{hit}}(\pi, u) = \sum_{v \in V} t_{\text{hit}}(v, u) \cdot \pi(v)$. Furthermore, we define $t_{\text{avg-hit}} = \sum_{u, v \in V} \pi(u) \cdot \pi(v) \cdot t_{\text{hit}}(u, v)$.

Finally, let $T_{\text{cov}}(u)$ be the first time until a random walk starting at $u$ has visited all vertices in $G$. Then, the cover time of $G$, denoted by $t_{\text{cov}}(G)$, is defined as $t_{\text{cov}}(G) := \max_{u \in V} \mathbb{E}[T_{\text{cov}}(u)]$.

Let $t_{\text{meet}}(u, v)$ denote the expected time when two random walks starting at $u$ and $v$ first arrive at the same node at the same time, and we write $t_{\text{meet}}^n$ for the expected meeting time of two random walks starting at two independent samples from the stationary distribution. Finally, let $t_{\text{meet}} = \max_{u, v} t_{\text{meet}}(u, v)$ denoted the worst-case expected meeting time.

We define the coalescence process as a stochastic process as follows: Let $S_0 \subseteq V$ be the set of nodes for which there is initially one random walk on it, and for all $v \in S_t$ let

$$Y_v(t) = \begin{cases} u \in N(v) & \text{w.p. } \frac{1}{2|N(v)|} \\ v & \text{w.p. } \frac{1}{2} \end{cases}$$

The set of active nodes in step $t + 1$ is given by $S_{t+1} = \{Y_v(t) \mid v \in S_t\}$. The process satisfies the Markov property—that is,

$$\mathbb{P}[S_{t+1} \mid \mathcal{F}_t] = \mathbb{P}[S_{t+1} \mid S_t],$$

where $\mathcal{F}_t$ is the filtration up to time $t$, which, informally speaking, is the history of all random decisions up to time $t$. Finally, we define the time of coalescence as $T_{\text{coal}}(S_0) = \min\{t \geq 0 \mid |S_t| = 1\}$. Throughout this article, this expression w.h.p. (with high probability) means w.p. (with probability) at least $1 - n^{-\Omega(1)}$ and the expression w.c.p. (with constant probability) means w.p. $c > 0$ for some constant $c$. We use $\log n$ for the natural logarithm. Appendix A contains some known results about Markov chains that we frequently use in our proofs.

3 BOUNDING $t_{\text{coal}}$ FOR LARGE $t_{\text{meet}}/t_{\text{mix}}$

In this section, we prove Theorem 1.1, one of our main results. We refer the reader to Section 1.2 for a high-level description of the proof ideas.

3.1 Auxiliary Stochastic Process

To prove our first main result, it is helpful to consider a more general stochastic process, $P_{\text{imm}}$, called the immortal process, involving multiple independent random walks. In the immortal process, whenever several random walks arrive at the same node at the same time, a subset of them (rather than just one) may survive, whereas the remaining are merged with one of the surviving walks. To identify the random walks, we assume that each walk has a natural number (in $\mathbb{N}$) as an identifier. To define this process formally, we introduce some additional notation and definitions; then we state and prove some auxiliary lemmas. A related concept was introduced in the work of Oliveira [37, Section 3.4] under the name of “allowed killings.”

As mentioned before, we assume that every random walk $r$ has a unique identifier $\text{id}(r) \in \mathbb{N}$. We divide the ids into two groups: $G_1$, the group of immortal walks, and $G_2$, the group of the remaining (mortal) walks. Whenever two or more walks collide at a node and at least one of these walks is in $G_1$, then all walks with ids in $G_1$ survive, whereas all walks with ids in $G_2$ are killed (merged with some walk with id in $G_1$). Furthermore, if all walks have ids in $G_2$ (i.e., there are no walks with id in $G_1$), then the walk with the minimum id among these walks survives. The ids along with the assignment of ids to groups determine which of the random walks that arrive at a given node at the same time survive.
Formally, let $P_{\text{imm}}$ denote the following process:

1. At time 0, $S_0 = \{(u_r, \text{id}(r))\}$, where $u_r$ is the starting node of random walk $r$ and $\text{id}(r)$ is its identifier.

2. At time $t$, several random walks may arrive at the same node. The process $P_{\text{imm}}$ allows some subset of them to survive, whereas the rest “coalesce” with one of the surviving walks. Formally, $S_{t+1}$ is defined using $S_t$ as follows. Define the (random) next-step position of the random walk with id $i \in \mathbb{N}$ that is on node $v \in V$ to be

$$Y_{v,i}(t) := \begin{cases} u & \text{where } u \in N(v) \text{ w.p. } \frac{1}{|N(v)|} \\ v & \text{w.p. } \frac{1}{2} \end{cases}.$$ 

Let $R_v(t) := \{(Y_{v,i}(t), i) \mid (v, i) \in S_t, v \in V\}$ be the set of next-step positions (before merging happens) for random walks that were at node $v$ at time $t$. Let $\hat{R}_v(t) := \{(v, i) \mid \exists u \in V, (v, i) \in R_u(t)\}$ be the random walks that have arrived at node $v$ at timestep $t + 1$, just before merging happens. Then, merging happens w.r.t. the ids as follows:

(a) If there exists $i \in G_1$ such that $(v, i) \in \hat{R}_v(t)$ (at least one walk with id in $G_1$ arrives at $v$), then

$$S_v(t + 1) := \{(v, j) \mid (v, j) \in \hat{R}_v(t), j \in G_1\},$$

where $j = \min\{i \mid (v, i) \in \hat{R}_v(t)\}$.

(b) If there is no $i \in G_1$ such that $(v, i) \in \hat{R}_v(t)$ and $\hat{R}_v(t) \neq \emptyset$ (no walk with id in $G_1$ arrives at $v$ but at least one walk arrives at $v$), then

$$S_v(t + 1) := \{(v, j)\},$$

Finally, let

$$S_{t+1} := \bigcup_{v \in V} S_v(t + 1).$$

We now relate this more general process, $P_{\text{imm}}$, to the coalescing process defined in Section 2. Let $P$ be regarded as a special instance of $P_{\text{imm}}$ with $G_1 = \{1\}$. In process $P$, only one of several walks arriving at the same node survives and by convention the one having the smallest id is chosen. Let $(S_t)_{t=0}^{\infty}$ denote the stochastic process $P$. If we define $\hat{S}_t := \{v \mid (v, i) \in S_t\}$, then $(\hat{S}_t)_{t=0}^{\infty}$ is a coalescence process as defined in Section 2. Moreover, $P$ represented by $(S_t)_{t=0}^{\infty}$ is the coalescence process that additionally keeps track of the ids. Throughout this article, we assume that every random walk of $S_0$ is on a distinct node.

In the following, we show that the time it takes to reduce to $k$ random walks in the original process $P$ is majorized by the time it takes in $P_{\text{imm}}$ to reduce to $k$ random walks.

**Proposition 3.1.** Consider the following two processes:

1. Process $P$ is the standard process of coalescing random walks, viewed as a special case of $P_{\text{imm}}$ with $G_1 = \{1\}$ as described earlier.

2. Process $P_{\text{imm}}$ is the process defined earlier using groups $G_1$ and $G_2$, where $1 \in G_1$.

Let $T^k, T_{\text{imm}}^k$ be the stopping times given by the condition that fewer than $k$ random walks remain for the two processes, respectively. Assume both processes start with the same initial configuration—that is, the vertices occupied by walks in both processes are identical and there is only one walk per vertex in either process. Then, there exists a coupling such that

$$T^k \leq T_{\text{imm}}^k.$$
Note that this is not obvious since it could happen that a random walk of $\mathcal{G}_1$, which dies in the original process, is able to eliminate many of the other random walks hence leading to a small $T^k_{\text{imm}}$ (in comparison to $T^k$). In particular, the standard coupling, in which the random choices $P$ and $P_{\text{imm}}$ are exactly the same, fails. Instead, we use two non-trivial couplings.

**Proof of Proposition 3.1.** We will give a coupling between the moves of walks in $P_{\text{imm}}$ and $P_{\text{int}}$, a new process that is essentially intermediate between $P$ and $P_{\text{imm}}$; furthermore, we will show that the original process $P$ is essentially a restricted view of the process $P_{\text{int}}$. The process $P_{\text{int}}$ will label the walks *dead*, *alive*, and *phantom*. We emphasize that a phantom walk is not considered alive. Note that the processes $P$ and $P_{\text{imm}}$ can be viewed as processes that assign labels to each random walk of the type alive and dead.

Let $S^Q_t$ denote the set of tuples of alive walks in process $Q \in \{P, P_{\text{int}}, P_{\text{imm}}\}$ at time $t$. Let $\bar{S}^Q_t = \{v \mid (v, i) \in S^Q_t\}$ for $Q \in \{P, P_{\text{int}}, P_{\text{imm}}\}$ be the set of nodes that are occupied by at least one alive walk (there might be several in $P_{\text{imm}}$ at $t \geq 1$). To prove the proposition, we show that there exists a coupling such that for any $t \in \mathbb{N}$,

1. $S^P_t \subseteq S^{P_{\text{int}}}_t$, (5)
2. $S^{P_{\text{int}}}_t \subseteq S^{P_{\text{imm}}}_t$, (6)

implying that $|S^P_t| \leq |S^{P_{\text{imm}}}_t|$, which yields the claim since

$$T^k = \min\{t \geq 0 : |S^P_t| \leq k\} \leq \min\{t \geq 0 : |S^{P_{\text{imm}}}_t| \leq k\} = T^k_{\text{imm}}.$$  

We now define $P_{\text{int}}$. As mentioned previously, the walks in $P_{\text{int}}$ will be given three kinds of labels (alive, dead, or phantom); the dead walks do not continue ahead in time, but alive and phantom walks do.

Formally, $P_{\text{int}}$ using the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as follows. We say that walk $r$ is of type $\mathcal{G}_i$ if $\text{id}(r) \in \mathcal{G}_i$ for $i \in \{1, 2\}$. Whenever at least one walk arrives\(^8\) on a node, the following happens:

1. At least one of the walks is of type $\mathcal{G}_1$
   (a) At least one walk of type $\mathcal{G}_1$ is alive
      (i) The walk of $\mathcal{G}_1$ with the smallest id is labeled as alive (even if it was labeled phantom before)
      (ii) All other walks of type $\mathcal{G}_1$ (if there are any) are labeled as phantom
      (iii) Alive walks of type $\mathcal{G}_2$ are labeled dead (if present).
   (b) All walks of type $\mathcal{G}_1$ are phantom walks
      (i) There is no walk of type $\mathcal{G}_2$
         (A) No label is changed
         (ii) There is at least one walk of type $\mathcal{G}_2$
            (A) The walk of type $\mathcal{G}_1$ with the smallest id is labeled as alive
            (B) All other walks of type $\mathcal{G}_1$ (if there are any) are labeled as phantom
            (C) Alive walks of type $\mathcal{G}_2$ are labeled dead.
2. All walks are of type $\mathcal{G}_2$
   (a) The walk of $\mathcal{G}_2$ with the smallest id is labeled as alive
   (b) All other walks are labeled as dead.

Note that walks of $\mathcal{G}_1$ are either alive or phantom and walks of $\mathcal{G}_2$ are either alive or dead. Additionally, note that in the process $P_{\text{int}}$, there is at most one *alive* walk at any given node. Throughout

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\(^8\) Throughout, by *arrive*, we take into account that walks may arrive at a node from the same node through laziness.
Fig. 4. An illustration of couplings between the processes. The squares depict the random walks. Walks of \( G_1 \) are colored black and gray (phantom), and the nodes of \( G_2 \) are white. The blue arrows denote the moving decisions. Observe that in \( P_{\text{int}} \), a phantom becomes alive (and a walk of \( \bar{G}_2 \) is labeled dead).

In the proof, we regard the processes in two stages. In the first stage, each random walk selects a destination (possibly the same node it was on) and moves there. In the second phase, the walks are merged according to the process. Figure 4 presents an illustration.

**Couplings.** We couple all decisions by the random walks in \( P_{\text{int}} \) with the decisions of the random walks \( P_{\text{imm}} \). The coupling between the random walks’ decisions in \( P \) with those of \( P_{\text{int}} \) in the following way. For the (unique) random walk at \( v \in \bar{S}_t \) under process \( P \), we couple its transition to node \( Y_v(t+1) \) (where we possibly have \( Y_v(t+1) = v \)) with the corresponding alive walk of \( \bar{S}_t^{P_{\text{int}}} \) (there might be several walks of \( P_{\text{int}} \); however, only one is alive and we couple with this alive walk).

**Proof of (5).** We prove (5) by induction on \( t \) starting from the same initial configuration: if \( v \in \bar{S}_t^P \), then \( v \in \bar{S}_t^{P_{\text{int}}} \). Consider the inductive step from \( t \) to \( t+1 \), and assume that the claim holds at the end of round \( t \) (after merging happened). For the (unique) random walk at \( v \in \bar{S}_t^P \) under process \( P \), we coupled its transition to node \( Y_v(t+1) \) (where we possibly have \( Y_v(t+1) = v \)) with the corresponding alive walk of \( \bar{S}_t^{P_{\text{int}}} \) (there might be several walks of \( P_{\text{int}} \); however, only one is alive and we couple with this alive walk). Let \( S \) be the set of nodes to which a random walk in \( P \) moved (i.e., \( S = \{ Y_v(t+1) : v \in \bar{S}_t^P \} \)). Observe that before the merging takes place in round \( t + 1 \) (but moves have been made), there is, by the induction hypothesis and the coupling, at least one alive walk of \( P_{\text{int}} \) on each node of \( S \). Furthermore, the definition of \( P_{\text{int}} \) ensures that whenever an alive random walk moves to a node, then after merging takes place, at least one alive walk remains.

---

9We will show that such a walk always exists in \( P_{\text{int}} \); nonetheless, for completeness, we can define a suitable random decision should no such walk in \( P_{\text{int}} \) be available.

10By definition, there is actually exactly one alive walk.
Thus, our coupling ensures that if \( v \in S_{t+1}^P \), then \( v \in S_{t+1}^{P_{\text{int}}} \). In other words, if one looks at the subsets where there is an alive walk of \( P_{\text{int}} \), this is essentially the standard coalescence process. This finishes the proof of (5), and we turn to proving (6).

When starting from the same initial configuration, we will provide a coupling that satisfies the following invariants:

1. There is a bijective map from the alive and phantom walks of \( P_{\text{int}} \) to the alive walks of \( P_{\text{imm}} \) such that the following holds. All walks of \( P_{\text{int}} \) of type \( G_i \) are mapped to walks of \( P_{\text{imm}} \) of type \( G_i \), for \( i \in \{1, 2\} \).
2. Whenever a walk of type \( G_2 \) is labeled dead in \( P_{\text{imm}} \), then it is also labeled dead in \( P_{\text{int}} \) and vice versa.

At the beginning, there are no dead or phantom walks in \( P_{\text{int}} \); there are no dead walks in \( P_{\text{imm}} \); all walks are alive; and as the starting positions in \( P_{\text{imm}} \) and \( P_{\text{int}} \) are the same, an arbitrary bijective mapping may be chosen, as long as it respects node positions and walk types.

Assume the invariant holds at time \( t \). We take one random walk step for each alive or phantom random walk in \( P_{\text{int}} \). These are coupled with the corresponding walks in \( P_{\text{imm}} \), under the chosen map. Walks that are already dead are neither simulated in \( P_{\text{int}} \) nor in \( P_{\text{imm}} \). Hence, we can ensure the bijection between the walks of \( G_i \) in both processes holds at time \( t + 1 \).

We now prove the second invariant. Note that whenever a walk \( r \) of type \( G_2 \) in \( P_{\text{imm}}(P_{\text{int}}) \) is labeled dead, this implies there must have been another walk \( r' \) on the same node at the same time. Since there is a bijective map, \( r' \) must be on the same node in \( P_{\text{int}}(P_{\text{imm}}) \). We have that either \( r' \) is of type \( G_1 \) or \( r' \) is of type \( G_2 \) and that \( \text{id}(r') < \text{id}(r) \). In either case, \( r \) is also killed (labeled dead) in \( P_{\text{int}}(P_{\text{imm}}) \). Hence, we can ensure the bijection between the walks of \( G_2 \) in both processes holds at time \( t + 1 \). Thus, the invariant holds at time \( t + 1 \). By induction, and since the alive walks of \( P_{\text{int}} \) are a subset of the alive walks of \( P_{\text{imm}} \), the invariant holds throughout the process and yielding (6). This finishes the proof. \( \square \)

### 3.2 Meeting Time Distribution Prior to \( t_{\text{meet}} \)

Let \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) be independent random walks starting at arbitrary positions. For \( \tau \) a multiple of \( t_{\text{mix}} \), the following lemma gives a lower bound on the probability of intersection of the two random walks in \( \tau \) steps.

**Lemma 3.2.** Let \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) be two independent random walks starting at arbitrary positions. Let \( \text{intersect}(X_t, Y_t, \tau) \) be the event that there exists \( 0 \leq s \leq \tau \) such that \( X_s = Y_s \). Then,

\[
P\left(\text{intersect}(X_t, Y_t, 5t_{\text{mix}})\right) \geq \frac{1}{32\alpha'},
\]

where \( \alpha = t_{\text{meet}}/t_{\text{mix}} \). Furthermore, there exists a constant \( c > 0 \) such that for any \( 1 \leq b \leq \frac{c-1}{c} \cdot \alpha \), we have

\[
P\left(\text{intersect}(X_t, Y_t, c b t_{\text{mix}})\right) \geq \frac{b}{\alpha}.
\]

**Proof.** First, let \((\tilde{X}_t)_{t \geq 0}\) and \((\tilde{Y}_t)_{t \geq 0}\) be two random walks that start from two independent samples drawn from the stationary distribution and are run for \( \ell := 2[\alpha'] [t_{\text{mix}}] \) steps. Notice that \( \ell \geq 2t_{\text{meet}} \), and hence, by Markov’s inequality,

\[
P\left(\text{intersect}(\tilde{X}_t, \tilde{Y}_t, \ell)\right) \geq \frac{1}{2}.
\]

Furthermore, if we divide the interval \([1, \ell]\) into \( 2[\alpha'] \) consecutive sections of length \( [t_{\text{mix}}] \) each, the probability for a collision in each of these sections is identical and therefore the union bound...
implies
\[ P \left[ \text{intersect}(\bar{X}_t, \bar{Y}_t, t) \right] \leq 2[\alpha] \cdot P \left[ \text{intersect}(\bar{X}_t, \bar{Y}_t, t_{\text{mix}}) \right], \]  
and hence combining Equations (7) and (8) yields
\[ P \left[ \text{intersect}(\bar{X}_t, \bar{Y}_t, t_{\text{mix}}) \right] \geq \frac{1}{4[\alpha]}. \]
Consider now two independent random walks \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) starting at arbitrary positions. By applying Lemma A.5 to both walks, w.p. at least \((1 - e^{-1})^{2\alpha}\), both \(X_{t_{\text{mix}}}\) and \(Y_{t_{\text{mix}}}\) are drawn independently from the stationary distribution since \(4t_{\text{mix}} \geq t_{\text{sep}}\). Therefore,

\[ P \left[ \text{intersect}(X_t, Y_t, 5t_{\text{mix}}) \right] \geq (1 - e^{-1})^2 \cdot P \left[ \text{intersect}(\bar{X}_t, \bar{Y}_t, t_{\text{mix}}) \right] \geq (1 - e^{-1})^2 \cdot \frac{1}{4[\alpha]}. \]

Observing that for any \(\alpha \geq 1\) the preceding RHS expression is greater than \(1/(32\alpha)\) completes the proof of the first part. For the second part, we consider \(k\) blocks of length \(5t_{\text{mix}}\). Due to independence of different blocks, the probability that the two walks meet in at least one of the \(k\) blocks is at least \(1 - (1 - \frac{1}{32\alpha})^k\). We set \(k := \lceil 32b/(1 - e^{-1}) \rceil, x := 1/(32\alpha)\). We distinguish between two cases:

**Case \(k \cdot x < 1\):** We use the fact that \((1 - x)^k \leq e^{-xk} \leq 1 - (1 - e^{-1})xk\) for \(0 \leq x < 1, k \geq 0\) and \(xk \leq 1\). We derive that the probability of intersecting after \(k\) blocks is at least \(1 - (1 - \frac{1}{32\alpha})^k \geq (1 - e^{-1})k/(32\alpha) = b/\alpha\).

**Case \(k \cdot x \geq 1\):** We have \(1 - (1 - \frac{1}{32\alpha})^k \geq 1 - (1 - \frac{1}{32\alpha})^{32\alpha} \geq 1 - 1/e \geq b/\alpha\). In both cases, the second part follows.

At the heart of the proof of Theorem 1.1 lies the following lemma that analyzes the marginal distribution of the meeting time distribution. In other words, we only expose the first random walk \((X_t)_{t \geq 0}\) and look at how this affects the probability of meeting. In essence, we show that at least one of the two “orthogonal” cases hold. In Case 1 (corresponding to set \(C_1\)), there is at least a modest probability that after exposing \((X_t)\), \((Y_t)\) will intersect with significant probability. Otherwise, in Case 2 (corresponding to set \(C_2\)), there is a significant probability that after exposing \((X_t)\), \((Y_t)\) will intersect with at least a modest probability.

**Lemma 3.3.** Fix \(\tau \in \mathbb{N}\) and a graph \(G\). Let \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) be two independent random walks starting from the stationary distribution.\(^{11}\) Let \(p = P \left[ \text{intersect}(X_t, Y_t, \tau) \right]\) and let \(\mathcal{T}_\tau\) denote the set all possible trajectories of a walk of length \(\tau\) in \(G\) (including possible self-loops). We define the following two categories \(C_1\) and \(C_2\) with \(C_1 \subsetneq C_2\):

\[ C_1 := \{(z_0, \ldots, z_\tau) \in \mathcal{T}_\tau : P \left[ \exists 0 \leq s \leq \tau, Y_s = z_s \right] \geq \sqrt{p} \}, \]

\[ C_2 := \{(z_0, \ldots, z_\tau) \in \mathcal{T}_\tau : P \left[ \exists 0 \leq s \leq \tau, Y_s = z_s \right] \geq p/3 \}. \]

Then, \(P[(X_t)_{t \geq 0} \in C_1] \geq \frac{p}{\alpha}\) or \(P[(X_t)_{t \geq 0} \in C_2] \geq \sqrt{p/3}\).

Although the actual lower bounds on the probabilities appear rather crude, it turns out that the “significant probability” \(\sqrt{p}/3\) is best possible, as we demonstrate in our lower bound construction later. Remarkably, the fact that the “modest probability” is only \(p/3\) and much smaller than \(\sqrt{p}/3\) does not affect the tightness of our bound, since in Claim 3.5 we can make up for this gap in both cases through a simple amplification argument over the unexposed random walks.

\(^{11}\)The lemma also holds for arbitrary distributions; however, for the sake of simplicity, we only state this slightly less general version.
Proof. Let us suppose that $P[(X_t)_{t=0} \in C_1] < \frac{p}{3}$. We show that this implies $P[(X_t)_{t=0} \in C_2] > \frac{\sqrt{p}}{3}$. Assume for the sake of contradiction $P[(X_t)_{t=0} \in C_2] < \frac{\sqrt{p}}{3}$. We have
\[
\begin{align*}
p &= P[\text{intersect}(X_t, Y_t, \tau) ] \\
&\leq P[ (X_t)_{t=0} \in C_1] \cdot 1 + P[ (X_t)_{t=0} \in (C_2 \setminus C_1)] \cdot \sqrt{p} + P[ (X_t)_{t=0} \not\in C_2] \cdot \frac{p}{3} \\
&< \frac{p}{3} + \sqrt{p}/3 \cdot \sqrt{p} + p/3 \leq p.
\end{align*}
\]
This completes the proof.

It is well known that starting with $k$ random walks, the coalescence time is bounded by $O(t_{\text{meet}} \log k)$, and this can be deduced from the proof presented in the work of Hassin and Pellegrin [30]. For the sake of completeness, we give a self-contained proof.

**Proposition 3.4.** We have $t_{\text{coal}}(S_0) = O(t_{\text{meet}} \log |S_0|)$.

**Proof.** Let $P$ be the coalescing process (with ids) defined in Section 3.1. Recall that $G_1 = \{1\}$. Let $S_t$ be set of coalescing random walks at an arbitrary timestep $t$. In the following, we show the slightly stronger claim that the expected time to reduce the number of random walks by a constant factor is $O(t_{\text{meet}})$.

Formally, we fix an arbitrary timestep $t_0$. With $T := \min\{t \geq t_0 : |S_t| \leq 99/100 \cdot |S_{t_0}|, |S_{t_0}| \geq 100\}$ denoting the first timestep the number of coalescing random walks reduces by a factor of $99/100$, we will prove that $E[T] = O(t_{\text{meet}})$. Iterating the argument $O(\log |S_0|)$ times implies the expected time it takes to reduce to 100 random walks is $O(t_{\text{meet}} \log |S_0|)$. Note that the expected time to reduce from 100 random walks to 1 is bounded by $O(t_{\text{meet}})$. Hence, the claim $t_{\text{coal}}(S_0) = O(t_{\text{meet}} \log |S_0|)$ follows.

It remains to show that the expected number of timesteps it takes to reduce the number of random walks by a factor of $99/100$ is indeed $O(t_{\text{meet}})$.

We divide time into blocks of length $\tau := c\frac{e-1}{e} t_{\text{meet}} + 4t_{\text{mix}}$, where $c$ is the constant of Lemma 3.2—that is,
\[
P[\text{intersect}(X_t, Y_t, c\frac{e-1}{e} t_{\text{meet}})] \geq \frac{e-1}{e}.
\]
We are primarily interested in what happens at the end of the blocks (i.e., at timesteps $t_0, t_0 + \tau, t_0 + 2\tau, \ldots$). For simplicity, we will start counting time from 0 at the beginning of each block. Let $(X_t)_{t \geq 0}$ be the random walk with id 1. After $4t_{\text{mix}}$ steps, we can couple the state of the random walk $(X_t)_{t \geq 4t_{\text{mix}}}$ with a node drawn from $\pi$ w.p. at least $(1 - e^{-1})$, since $4t_{\text{mix}} \geq t_{\text{sep}}$ (see Lemma A.5). Further, note that conditioned on this coupling, the statement of Lemma 3.3 implies that $(X_t)_{t \geq 4t_{\text{mix}}} \in C_2$ w.p. at least $p/3$, where we used $C_2 \supseteq C_1$, and where $p := P[\text{intersect}(X_t, Y_t, c\frac{e-1}{e} t_{\text{meet}})] \geq \frac{e-1}{e}$ for $X_0, Y_0 \sim \pi$.

We condition on the successful coupling of $X_{4t_{\text{mix}}}$ with a node drawn from $\pi$ and that $(X_t)_{t \geq 4t_{\text{mix}}} \in C_2$, which happens w.p. at least $(1 - e^{-1})p/3 = \frac{(e-1)^2}{3e^2}$ (called event $E$). Finally, consider any random walk $(Y_t)_{t \geq 0}$ with id other than 1. Again w.p. at least $1 - e^{-1}$, we can couple $Y_{4t_{\text{mix}}}$ with a node drawn from $\pi$, and conditioned on successful coupling, $(Y_t)_{t \geq 4t_{\text{mix}}}$ meets $(X_t)_{t \geq 4t_{\text{mix}}}$ between timesteps $[4t_{\text{mix}}, \tau]$ w.p. at least $p/3$, by definition of $C_2$. Thus, conditioned on event $E$, each walk of $G_2$ vanishes w.p. $(1 - e^{-1})p/3 = \frac{(e-1)^2}{3e^2}$, and thus the expected fraction of walks killed in the $t$ timesteps is at least $\frac{(e-1)^2}{3e^2}$.

Let $Z_\ell = |S_{t_0 + \ell \tau}|$ denote the number of random walks alive at the beginning of block $\ell$:
\[
E[Z_\ell | F_{t_0 + (\ell-1) \tau}] \leq Z_{\ell-1} - (Z_{\ell-1} - 1) \cdot \frac{(e-1)^4}{9e^4} \leq Z_{\ell-1} - \frac{Z_{\ell-1}}{100}.
\]
The preceding holds as long as $Z_{\ell-1} \geq 100$. We can therefore apply Lemma A.9 with parameters $g = 99/100 \cdot S_0$ and $\beta = 99/100$ to obtain that $E[T] = O(\tau) = O(t_{\text{meet}})$, which completes the proof.

### 3.3 Upper Bound: Proof of Theorem 1.1

We commence by considering the process $P_{\text{imm}}$ defined in Section 3.1. This allows us to establish Claim 3.5 providing us with the following tradeoff. For a given period $\tau$ of length at least $t_{\text{mix}}$, we obtain a bound on the required number of periods to reduce the number of random walks by an arbitrary factor. The proof relies heavily on Lemma 3.3, which divides the walks of $G_1$ into two groups allowing us to expose the walks of $G_1$ first and then to calculate the probability of the walks of $G_2$ to intersect with them. In fact, we will also use the symmetric case where the roles of $G_1$ and $G_2$ are switched. These probabilities are derived from the time-probability tradeoff presented in Lemma 3.2. We then use Claim 3.5 to derive a bound on the number of timesteps it takes to reduce the number of walks to $2$ in $O(t_{\text{meet}})$ timesteps. Melding both phases together yields the bound of Theorem 1.1.

We now define a process $P_{\text{imm}}(S_0, k)$ with $k < |S_0|$, which is a parameterized version of the process $P_{\text{imm}}$ defined in Section 3.1:

- Let $|S_0| = k'$; there are $k'$ random walks with ids $1, \ldots, k'$ and starting nodes $v_1, \ldots, v_{k'}$. Thus, $S_0 = \{(v_i, i) \mid 1 \leq i \leq k'\}$.
- Let $G_1 = \{1, \ldots, k\}$ and $G_2 = \{k + 1, \ldots, k'\}$. Recall that by definition of $P_{\text{imm}}$, we have that if some random walks with ids in $G_1$ and some with ids in $G_2$ are present on the same node at the same time, only the ones with ids in $G_1$ survive. If all random walks have ids only in $G_1$, then all of them survive. If all random walks have ids only in $G_2$, then only the one with the smallest id survives.

We define $\text{IDs}(S_t) := \{\text{id}(r) \mid (u_r, \text{id}(r)) \in S_t\}, t \in \mathbb{N}$.

The following lemma gives the expected time it takes to reduce the number of random walks in $G_2$ from $k' - k$ to some arbitrary integer $g \geq k$: given a period of length $\tau$ and integer $g$, assuming that $k = |G_1|$ is large enough, we derive a bound on the number of periods of length $\tau$ until the walks in $G_2$ are reduced to $g$. The required size of $k$ is a function of the probability for two random walks drawn from $\pi$ intersecting after $\tau$ timesteps.

**Claim 3.5.** Let $\tau \in \mathbb{N}$, and let $(X_t)_{t=0}^{\tau}$ and $(Y_t)_{t=0}^{\tau}$ be independent random walks run for $\tau$ steps, with $X_0$ and $Y_0$ drawn independently from $\pi$. Let $p_\tau \leq P[\text{intersect}(X_t, Y_t, \tau)]$ be a lower bound on the probability of the intersection of the two walks during the $\tau$ steps. Consider an instantiation of $P_{\text{imm}}(S_0, k)$. Suppose that $k \geq \frac{3}{(1-e^{-1})p_\tau}$. For some $1 \leq g \leq |S_0| - k$, define the stopping condition $T_g = \min\{t \geq 0 \mid ||\text{IDs}(S_t) \cap G_2| \leq g\}$. Then, the expected stopping time satisfies

$$E[T_g] = O \left( (4t_{\text{mix}} + \tau) \cdot \sqrt{\frac{1}{p_\tau} \cdot (\log |G_2| - \log g)} \right).$$

We first describe the high-level proof idea before delving into the formal proof. We divide time into blocks of size $4t_{\text{mix}} + \tau$. For any random walk $(Z_t)_{t=0}^{4t_{\text{mix}} + \tau}$, we can couple its position after $4t_{\text{mix}} \geq t_{\text{sep}}$ w.c.p. with a node drawn from $\pi$. Thus, conditioning on the success of this coupling, we have, by Lemma 3.3, $P[(Z_t)_{t=4t_{\text{mix}}}^{4t_{\text{mix}}+\tau} \in C_1] \geq \frac{p_\tau}{3}$ or $P[(Z_t)_{t=4t_{\text{mix}}}^{4t_{\text{mix}}+\tau} \in C_2] \geq \frac{\sqrt{p_\tau}}{3}$. In the former case, we have that w.c.p., there is at least one random walk $r$ in $G_1$ that is, due to independence...
of the walks, in class \( C_1 \). The hypothetical extension of the trajectory of any random walk in \( r' \in G_2 \) intersects with \( r \) w.p. \( c \sqrt{\frac{1}{p_r}} / 3 \), where the constant arises due to the fact that we also need to couple the state of \( r' \) at time \( 4t_{mix} \) to a node drawn according to \( \pi \). (We need to consider the hypothetical extension because the walk \( r' \) may get eliminated sooner—this only helps us.) Thus, \( r' \) gets eliminated w.p. at least \( c \sqrt{\frac{1}{p_r}} \) for a suitable constant \( c \).

In the latter case, we have that w.p. at least \( c \sqrt{\frac{1}{p_r}} / 3 \) a random walks of \( G_2 \) is in class \( C_2 \). Every random walk in that class intersects w.c.p. with at least one of the walks of \( G_1 \). Thus, in both cases, we have that in each block, a random walk of \( G_2 \) is eliminated w.p. at least \( c \sqrt{\frac{1}{p_r}} \) for some constant \( c \). Thus, the number of random walks in \( G_2 \) decrease in expectation by a factor of \( c \sqrt{\frac{1}{p_r}} \).

**Proof.** We will consider the process in **blocks**, each consisting of \( 4t_{mix} + \tau \) timesteps. For convenience in the proof, we will restart counting timesteps from 0 at the beginning of each block; we keep track of the total number of timesteps by counting the number of blocks. Let \( C_1 \) and \( C_2 \) be as defined in Lemma 3.3. Then we perform a case analysis by considering the two possible outcomes described in Lemma 3.3 separately. We define \( Z_j = |\{S_j (t_{mix} + \tau) \cap G_2 \}| - \) that is, number of walks remaining in \( G_2 \) after \( j \) blocks of time have passed. For any \( j \geq 1 \), we will show that there exists a constant \( c > 0 \) such that

\[
\mathbb{E}[Z_j | F_{j-1}] \leq Z_{j-1} \cdot (1 - c \sqrt{\frac{1}{p_r}}).
\]

By using Lemma A.9, we get \( \mathbb{E}[T_g] = O((4t_{mix} + \tau) \cdot \frac{1}{\sqrt{p_r}} \cdot (\log |G_2| - \log g)) \) (the factor \( (4t_{mix} + \tau) \) appears as the size of the block). Recall that \( F_j \) is the filtration up to end of the \( j \)th block. In the remainder, we show that we have indeed \( \mathbb{E}[Z_j | F_{j-1}] \leq Z_{j-1} \cdot (1 - c \sqrt{\frac{1}{p_r}}) \).

**Case 1:** \( \mathbb{P}[ \{ X_i \}_{i=0}^{t} \in C_1 \} \geq \frac{p_r}{4} \)

Consider any random walk \( r \) in \( G_1 \) at the beginning of a block. Using Lemma A.5, after \( 4t_{mix} \) steps, we can couple the state of the random walk with a node drawn from \( \pi \) w.p. at least \( (1 - e^{-1}) \). Furthermore, conditioned on this coupling, the portion of the random walk between timesteps \( 4t_{mix} \) and \( 4t_{mix} + \tau \) of the walk is in class \( C_1 \) w.p. at least \( \frac{p_r}{4} \). Since \( k \geq \frac{3}{p_r (1 - e^{-1})} \), w.p. \( c_1 > 0 \), in any block, there exists a walk in \( G_1 \) that has the portion between timesteps \( 4t_{mix} \) and \( 4t_{mix} + \tau \) in \( C_1 \).

Fix a block and condition on the event that there is a walk in \( G_1 \), denoted by \( r_1 \), whose portion between timesteps \( 4t_{mix} \) and \( 4t_{mix} + \tau \) is in \( C_1 \). Consider any walk in \( G_2 \), denoted by \( r_2 \), at the beginning of the block. We want to argue that this walk \( r_2 \) has a reasonable probability of intersecting some walk in \( G_1 \) in this block of timesteps. First, consider (the possibly hypothetical continuation of \( r_2 \) ) walk \( r_2' \) for the entire length of the block. The reason for this is that if \( r_2 \) and some walk from \( G_1 \) are at the same node at the same time sometime in the block, \( r_2 \) will be eliminated in the process \( P_{\text{hmm}}(S_0, k) \); however, we can consider its hypothetical extension to the entire length of the block. Using Lemma A.5, the state of the walk \( r_2' \) at timestep \( 4t_{mix} \) can be coupled with a node drawn from \( \pi \) w.p. at least \( c_2 := 1 - e^{-1} \). Then, conditioned on successful coupling, the probability that \( r_2' \) and \( r_1 \) collide during timesteps \( 4t_{mix} \) and \( 4t_{mix} + \tau \) is at least \( \sqrt{\frac{1}{p_r}} \) (by definition of \( C_1 \) in Lemma 3.3). Thus, the probability that \( r_2 \) hits at least one walk in \( G_1 \) is at least \( c_1 \cdot c_2 \cdot \sqrt{\frac{1}{p_r}} \). Note that it is also possible for \( r_2' \) to be eliminated by another walk from \( G_2 \). In any case, we have that \( r_2 \) is eliminated w.p. at least \( c \sqrt{\frac{1}{p_r}} \) and we get

\[
\mathbb{E}[Z_j | F_{j-1}] \leq Z_{j-1} \cdot (1 - c_1 \cdot c_2 \sqrt{\frac{1}{p_r}}).
\]

**Case 2:** \( \mathbb{P}[ \{ X_i \}_{i=0}^{t} \in C_2 \} \geq \frac{\sqrt{p_r}}{3} \)

Consider a walk in \( G_2 \), denoted by \( r_2 \), at the beginning of a block; as in the previous case, we will consider a possibly hypothetical continuation \( r_2' \) of \( r_2 \). Using Lemma A.5, we can couple the state of \( r_2' \) at timestep \( 4t_{mix} \) with a node drawn from \( \pi \) w.p. at least \( 1 - e^{-1} \). Furthermore, conditioned

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on the successful coupling, w.p. at least $\frac{\pi_c}{\pi}$, the trajectory of $r'_2$ between the timesteps $4t_{\text{mix}}$ and $4t_{\text{mix}} + \tau$ is in $C_2$. Thus, w.p. at least $p := (1 - e^{-1}) \frac{\pi_c}{\pi}$, $r'_2$ has a trajectory between timesteps $4t_{\text{mix}}$ and $4t_{\text{mix}} + \tau$ that lies in $C_2$. Now consider any random walk $r_1 \in G_1$ at the beginning of the block. Again, using Lemma A.5 w.p. at least $1 - e^{-1}$, we can couple the state of the random walk at time $4t_{\text{mix}}$ with a node drawn from $\pi$. Conditioned on this, between timesteps $4t_{\text{mix}}$ and $4t_{\text{mix}} + \tau$, this random walk hits any trajectory whose portion between timesteps $4t_{\text{mix}}$ and $4t_{\text{mix}} + \tau$ lies in $C_2$. Since the random walks in $G_1$ are independent, by the definition of the immortal process, we have that any walk in $G_2$ is eliminated by the end of the block w.p. at least $c_1 \cdot p = c \sqrt{p_c}$ for some constant $c > 0$. Similarly as before, it is possible that $r_2$ is eliminated by at least one of the walks of $G_2$, which only increases the probability for $r_2$ of being eliminated. We get

$$E[Z_j \mid F_{j-1}] \leq Z_{j-1} \cdot (1 - c \sqrt{p_c}).$$

\[\square\]

In the following, we bound the time $T$ required to reduce to $2[\alpha]$ random walks. The claim follows by applying Claim 3.5 to derive a bound on $T_{\text{imm}}$ for process $P_{\text{imm}}$, and using the majorization of $T$ by $T_{\text{imm}}$ (Proposition 3.1).

**Corollary 3.6.** Consider the coalescence process starting with set $S_0$, and let $\alpha = t_{\text{meet}}/t_{\text{mix}}$. Let $T_1 = \min\{t \geq 0 \mid |S_t| \leq 2[\alpha]\}$. Then, $E[T_1] = O(t_{\text{mix}} \cdot \sqrt{\alpha} \cdot \log |S_0|)$.

**Proof.** We consider the process $P$ (defined in Section 3.1), which is identical to the coalescence process but additionally keeps track of the number of random walks and that allows only the walk with the smallest id to survive. We assume that the ids are from the set $\{1, 2, \ldots, |S_0|\}$. Let $S_0 = \{(v_1, 1), \ldots, (v_{|S_0|}, |S_0|)\}$ and $S_0 = \{i : (v, i) \in S_0\}$. We consider the process $P_{\text{imm}}(S_0, k)$ and $k = [\alpha]$. Let $T^*_1$ be the stopping time defined by $|\text{ID}_s(S_t) \cap G_2| \leq \alpha$ for the process $P_{\text{imm}}(S_0, k)$. By definition of $P_{\text{imm}}$ and Proposition 3.1, it follows that $T_{\text{imm}}$ stochastically dominates $T$. Thus, it suffices to bound $E[T_{\text{imm}}]$. Without loss of generality, we assume that $\alpha \geq \frac{6}{e - 1}$; otherwise, the claim follows directly from Proposition 3.4. We apply Lemma 3.2 with $b = 6$ and derive that for some suitable constant $c$,

$$p = \mathbb{P}[\text{intersect}(X_{t > 0}, Y_{t > 0}, 6ct_{\text{mix}})] \geq \frac{6}{\alpha}.$$

Thus, we have

$$\frac{3}{(1 - e^{-1}) \cdot p} \leq \frac{3}{\frac{1}{2} \cdot p} \leq \alpha \leq k$$

Applying Claim 3.5 with $g = \alpha$, $\tau = 6ct_{\text{mix}}$ (where $c$ is a constant as given by Lemma 3.2) and $p_c = 6/\alpha$, and observing that $k \geq \frac{3}{(1 - e^{-1})p_c}$, we get the required result.\[\square\]

In the following, we bound the time $T$ required to reduce to $2[\alpha]$ random walks to a single random walk. The proof uses the same ideas as before (Corollary 3.6); however, this time, we consider several phases, and in each, we reduce the number of random walks by a constant factor. The expected time per phase is geometrically increasing as the number of walks decreases, and the overall time is essentially dominated by the time for a constant number of random walks to meet, which is $O(t_{\text{meet}})$.

**Lemma 3.7.** Consider the coalescence process starting with set $S_0$, satisfying $|S_0| \leq 4\alpha \log \alpha$, where $\alpha = t_{\text{meet}}/t_{\text{mix}}$. Let $T_2 := \min\{t \geq 0 \mid |S_t| \leq 1\}$. Then, $E[T_2] = O(t_{\text{meet}})$.
Proof. We will consider the coalescence process in phases. Let $\ell$ be the largest integer such that $|S_0| \geq \left( \frac{4}{3} \right)^\ell$. For $j \geq 1$, the $j^{th}$ phase ends when $|S_t| < \left( \frac{4}{3} \right)^{\ell-j+1}$. The $(j+1)^{th}$ phase begins as soon as the $j^{th}$ phase ends. Note that it may be the case that some phases are empty. Let $T_2(j)$ denote the time for phase $j$ to last. We will only consider phases up to which $\ell - j + 1 \geq 32$.

Now we focus on a particular phase $j$. Let $t_j$ be the time when the $j^{th}$ phase begins, and let $S_{t_j}$ denote the corresponding set at that time. Thus, we have

$$\left( \frac{4}{3} \right)^{\ell-j+1} \leq |S_{t_j}| < \left( \frac{4}{3} \right)^{\ell-j+2}. \tag{9}$$

We consider the process $P_{\text{imm}}$ defined in Section 3.3 as follows. Define $n_j = |S_{t_j}|$. Fix a phase $j$, and define $S'_j = \{(v_1, 1), \ldots, (v_{n_j}, n_j)\}$ and $S'_0 = \{v_1, \ldots, v_{n_j}\}$. Then, consider again the set of occupied vertices (ignoring the labels) $S'_{t_j+t} = \{v \mid \exists i \in \mathbb{N}, (v, i) \in S'_j\}$ with $t \in \mathbb{N}$. Thus, phase $j$ ends when $|S'_t| = |S'_{t_j+t}| < \left( \frac{4}{3} \right)^{\ell-j+1}$. Let

$$k_j := \left\lceil \frac{|S'_0|}{2} \right\rceil$$

be the size of $G_1$, and consider the process $P_{\text{imm}}(S'_j, k_j)$ as defined in Section 3.3. Let

$$g_j := \left\lceil \frac{|S'_0| - k_j}{3} \right\rceil$$

and

$$T_2^*(j) := \min\{t \mid |\text{IDs}(S'_t) \cap G_2| \leq g_j\}.$$

We note that as long as $\ell - j + 1 \geq 32$, $g_j \geq 1$ and at time $T_2^*(j)$,

$$|S'_t| \leq g_j + k_j \leq \frac{|S'_0| - k_j}{3} + k_j = \frac{|S'_0|}{3} + \frac{2k_j}{3} \leq \frac{|S'_0|}{3} + \frac{2}{3} \leq \frac{3}{4} \cdot |S'_0|.$$

By Proposition 3.1, $T_2^*(j)$ stochastically dominates $T_2(j)$, and hence it suffices to bound $\mathbb{E}[T_2^*(j)]$. To bound $\mathbb{E}[T_2^*(j)]$, we define

$$b_j := 32\alpha \log(4/3)(\ell - j + 1)(3/4)^{\ell-j+1}.$$

Since we only consider phases with $j$ respecting $\ell - j + 1 \geq 32$, we have $b_j \leq b_{\ell-31} \leq ((e - 1)/e)\alpha$. Furthermore, we have $b_j \geq b_0 \geq 4\alpha \log(3/4)^{\ell} \geq 1$, where the last inequality follows from $(4/3)^\ell \leq |S_0| \leq 4\alpha \log(\alpha)$, which in turn follows from definition of $\ell$ and the assumed bound on $|S_0|$. Applying Lemma 3.2 with this value of $b_j$, we get that for

$$\tau_j := cb_j t_{\text{mix}},$$

for independent random walks $(X_t)_{t=0}^{\tau_j}$, $(Y_t)_{t=0}^{\tau_j}$, $\mathbb{P}[\text{intersect}(X_t, Y_t, \tau_j)] \geq p_j$, where

$$p_j := 32 \log(4/3)(\ell - j + 1)(3/4)^{\ell-j+1}.$$
and we continue by dissecting that bound. Since $b_j \geq 1$, there exists a suitably large constant $c_1$ so that $\tau_j + 4t_{\text{mix}} \leq c_1 b_j t_{\text{mix}}$. Furthermore,
\[
\frac{b_j}{\sqrt{\beta_j}} = \frac{32\alpha \log(4/3)(\ell - j + 1)(3/4)^{\ell-j+1}}{\sqrt{32 \log(4/3)(\ell - j + 1)(3/4)^{\ell-j+1}}} = O\left(\alpha \sqrt{\ell - j + 1} \cdot \left(\frac{3}{4}\right)^{(\ell-j+1)/2}\right).
\]
Observe that by definition, $|\text{IDs}(S_0^c) \cap G_2|/g_j \leq 3$, hence $\log |\text{IDs}(S_0^c) \cap G_2| - \log g_j \leq \log(3)$. Putting everything together, we get that there is a constant $c_2$ such that
\[
\mathbb{E}[T_2(j)] \leq c_2 \cdot t_{\text{mix}} \cdot \alpha \cdot \sqrt{\ell - j + 1} \cdot \left(\frac{3}{4}\right)^{(\ell-j+1)/2}.
\]
(10)

Note that since we stop when $\ell - j + 1 < 32$, there are at most $\ell - 30$ phases considered. Let $\widehat{T}$ be the random variable denoting the timestep when the last phase ends; at this point, $|S_{\widehat{T}}| = O(1)$. Therefore, using Proposition 3.4, $\mathbb{E}[T_2 - \widehat{T} | \widehat{T}] = O(t_{\text{meet}})$. However, clearly $\widehat{T}$ is stochastically dominated by $\sum_{j=30}^{\ell-30} T_2(j)$. Thus, we have
\[
\mathbb{E}[T_2] = \mathbb{E}[\widehat{T}] + \mathbb{E}[T_2 - \widehat{T} | \widehat{T}]
\leq c_2 \cdot t_{\text{mix}} \cdot \alpha \cdot \sum_{j=0}^{\ell-30} \sqrt{\ell - j + 1} \cdot \left(\frac{3}{4}\right)^{(\ell-j+1)/2} + c_3 t_{\text{meet}},
\]
(11)
\[
\leq c_2 \cdot t_{\text{mix}} \cdot \alpha + c_3 t_{\text{meet}} = O(t_{\text{meet}}).
\]
(12)

In (11), we used (10) and the fact that $\mathbb{E}[T_2 - \widehat{T} | \widehat{T}] \leq c_3 t_{\text{meet}}$ for some constant $c_3 > 0$, and in step (12), we used the fact that $\sum_{j=32}^{\ell-30} j c_j < 1$ for $c \leq 3/4$.

Thus, the first phase (Corollary 3.6) and the second phase (Lemma 3.7) take together $O(\sqrt{\alpha} \cdot \log n \cdot t_{\text{mix}} + t_{\text{meet}})$ timesteps, which yields Theorem 1.1.

### 3.4 Lower Bound: Proof of Theorem 1.2

In this section, we give a construction of a graph family to establish lower bounds on $t_{\text{coal}}(G)$ in terms of $t_{\text{meet}}(G)$ and $t_{\text{mix}}(G)$ demonstrating that Theorem 1.1 is asymptotically tight. Additionally, our construction generalizes a claim of Aldous and Fill [2, Chapter 14]: they mention that it is possible to construct regular graphs that mimic the $n$-star in the sense that the $t_{\text{meet}} = o(t_{\text{avg-hit}})$, without giving further details of the construction. Our construction shows that even the coalescence time can be significantly smaller than the average hitting time for almost-regular graphs. For our family of almost-regular graphs, there is a polynomial gap between $t_{\text{meet}}$ and $t_{\text{avg-hit}}$. More importantly, we show that these almost-regular graphs have a gap of $\sqrt{t_{\text{mix}}/t_{\text{meet}}} \cdot \log n$ between coalescing and meeting time. This shows that the bound in Theorem 1.1 is the best possible, even if we constrain $G$ to be almost-regular. We refer the reader to Section 1.2 for a high-level description of the proof ideas.

More precisely, in the proof of Theorem 1.2, we shall give an explicit construction of a graph family $G = G_n$ with $t_{\text{coal}} = \Omega(\sqrt{\alpha n} \cdot \log n \cdot t_{\text{mix}})$, where $\alpha_n = t_{\text{meet}}/t_{\text{mix}}$. For the remainder of this section, we will drop the dependence on $n$ and will simply use $G$ instead of $G_n$ and $\alpha$ instead of $\alpha_n$.

The construction of $G$ (Figure 3 presents an illustration) is based on two building blocks, $G_1$ and $G_2$. First, let $G_1 = (V_1, E_1)$ be a clique of size $\sqrt{n}$. Let $G_2 = (V_2, E_2)$ be an Erdős-Rényi random graph $G(n/\sqrt{\alpha'}, \sqrt{\alpha}/\sqrt{n})$ (the nodes having expected degree $\sqrt{n}$), where $\alpha' = \max\{\alpha, 2^{20} \cdot C^2\}$, where $C > 1$ is the universal constant of Corollary A.3. The graph $G$ is made of one copy of $G_2$, $\kappa = \sqrt{n}$ copies of $G_1$ (denoted by $G_1^1, G_1^2, \ldots, G_1^\kappa$), and a node $z$, which has an edge to $\sqrt{n}/\alpha'^2$ distinct nodes of $G_2$ and to each of the designated nodes $z^i \in V_1^i$ in $G_1^i$ for $i \in [1, \kappa]$. It is not difficult to see that
this graph is almost-regular—in fact, the maximum and minimum degree differ by a factor of at most $(1 + o(1))$, since by applying a Chernoff bound, it follows that w.h.p. that all degrees of the Erdős-Rényi random graph are within $\sqrt{n} \pm O(n^{1/4} \sqrt{\log n})$.

In Lemmas 3.12, 3.13, 3.14, and 3.15, we respectively show that $t_{\text{mix}} = O(n)$, $t_{\text{meet}} = \Theta(\alpha' n)$, $t_{\text{coal}} = \Omega(\sqrt{n} \cdot n \log n)$, and $t_{\text{avg-hit}} = \Omega(n^{3/2})$. We start with the following auxiliary lemma, which shows that the walk restricted to $V_2$ behaves similarly to the walk restricted to $V_2 \cup \{\hat{z}\}$, meaning that the walks have very similar $t$-step probabilities.

**Lemma 3.8.** Let $P$ denote the transition matrix of the random walk on $G$, $Q$ denote the transition matrix of the random walk on $G_2$, and $\hat{Q}$ denote the transition matrix of the random walk on the subgraph of $G$ induced by $V_2 \cup \{\hat{z}\}$. Let $p'_{u,v}, q'_{u,v}, \hat{q}'_{u,v}$ denote the corresponding transition probabilities for a walk starting at $u$ to end up at node $v$ after $t$ steps. Let $S^* = \{u \in V_2 \cap N(\hat{z})\}$. Then, the following statements hold:

1. For any $u, v \in V_2$, we have $\|p'_{u,v} - q'_{u,v}\|_{TV} \leq (1 + o(1)) (1 + o(1)) t/(2n)$. 
2. For any $u, v \in V_2$, we have $\|q'_{u,v} - \hat{q}'_{u,v}\|_{TV} \leq (1 + o(1)) t/(2n)$. 
3. For any $u, v \in V_2$, we have that after $t = t_{\text{mix}}(G_2)$ timesteps, $\|p'_{u,v} - p'_{u,v}\|_{TV} \leq \|o(1) + 1/e\|

**Proof.** Let $(X_t)_{t \geq 0}$ be the Markov chain with transition matrix $P$, and let $(Y_t)_{t \geq 0}$ be the Markov chain with transition matrix $Q$. We will inductively couple these two random walks starting from $X_0 = Y_0 = u$. Given that we coupled chains up to time $t - 1$, we can couple $(X_t, Y_t)$ such that $X_t = Y_t$ with an error probability

$\mathbb{P}[X_t \neq Y_t | X_{t-1} = Y_{t-1}] = \mathbb{P}[X_t \neq Y_t | X_{t-1} = Y_{t-1}, X_{t-1} \in S^*] \cdot \mathbb{P}[X_{t-1} \in S^*] + \mathbb{P}[X_t \neq Y_t | X_{t-1} = Y_{t-1}, X_{t-1} \in V_2 \setminus S^*] \cdot \mathbb{P}[X_{t-1} \in V_2 \setminus S^*]$

$\leq \mathbb{P}[X_t = \hat{z} | X_{t-1} \in S^*] p_{u,S^*}^{t-1} + 0 \leq (1 + o(1)) p_{u,S^*}^{t-1} / (2n),$

where we used the laziness of the process and the fact that the degrees are in $(1 \pm o(1)) \sqrt{n}$ for all nodes in $S_2$ by a Chernoff bound. We have, by the work of Levin et al. [33, Proposition 4.7],

$\|p'_{u,v} - p'_{u,v}\|_{TV} = \inf \{\mathbb{P}[X \neq Y] | (X, Y) \text{ is a coupling of } p'_{u,v} \text{ and } p'_{u,v}\}.$

Hence, by a union bound over $t$ steps,

$\|p'_{u,v} - p'_{u,v}\|_{TV} = \inf \{\mathbb{P}[X \neq Y] | (X, Y) \text{ is a coupling of } p'_{u,v} \text{ and } p'_{u,v}\} \leq \mathbb{P}[X_t \neq Y_t]$

$\leq (1 + o(1)) \sum_{i=1}^{t-1} p_{u,S^*}^{i} / (2n) \leq (1 + o(1)) \frac{t}{2n}.$

To prove the second part, we redefine $(X_t)_{t \geq 0}$ to be the Markov chain with transition matrix $\hat{Q}$ and the proof is identical.

We proceed with the last part. For $u, v \in V_2$, we have that after $t = t_{\text{mix}}(G_2)$ timesteps, by the triangle inequality and using that $t_{\text{mix}}(G_2) = O(1)$, by Corollary A.3, we get

$\|p'_{u,v} - \pi^{G_2}(\cdot)\|_{TV} \leq \|p'_{u,v} - q'_{u,v}\|_{TV} + \|q'_{u,v} - \pi^{G_2}(\cdot)\|_{TV}$

$\leq (1 + o(1)) \frac{t_{\text{mix}}(G_2)}{\sqrt{n}} + \|q'_{u,v} - \pi^{G_2}(\cdot)\|_{TV}$

$\leq o(1) + \|q'_{u,v} - \pi^{G_2}(\cdot)\|_{TV} \leq o(1) + 1/e,$

where the last inequality follows form the definition of mixing time. Again, by the triangle inequality, $\|p'_{u,v} - p'_{u,v}\|_{TV} \leq o(1) + 2/e.$

□
Based on Lemma 3.8, we can now bound the hitting time to reach \( \hat{z} \), which will later be used to establish the bounds on the mixing and meeting time of the whole graph \( G \). But first, we prove that the mixing time of the graph \( \hat{G} \) induced by \( V_2 \cup \{ \hat{z} \} \) is constant, and that after mixing on \( \hat{G} \), the random walk has a probability of \( \Omega(1/n) \) to hit \( \hat{z} \) in a constant number of timesteps.

**Lemma 3.9.** The following three statements hold:

(i) Let \( \hat{G} \) be the induced graph by the vertices \( V_2 \cup \{ \hat{z} \} \). Then, \( t_{\text{mix}}(\hat{G}) = O(1) \).

(ii) Let \( u \in V \setminus \{ \hat{z} \} \). Then, there exists a constant \( c \geq 1 \) such that \( \mathbb{P} \left[ T_{\text{hit}}(u, \hat{z}) \geq n/c \right] \geq 1/2 \).

(iii) Let \( u \in V \setminus \{ \hat{z} \} \). Then, \( T_{\text{hit}}(u, \hat{z}) = O(n) \).

**Proof.** We prove the statements one by one:

(i) Let \( Q \) be the transition matrix of a random walk restricted to \( G_2 \). Let \( d^Q(t) \) be the total variation distance w.r.t. the transition matrix \( Q \). Further, let \( \hat{Q} \) be the transition matrix of a random walk restricted to \( \hat{G} \). Recall that \( t_{\text{mix}}(G_2) = O(1) \), by Corollary A.3.

Fix an arbitrary \( t \in [2t_{\text{mix}}(G_2), 2t_{\text{mix}}(G_2) + 7] \). In the following, we show \( \| q^t_{u,v} - \pi^{\hat{G}}(\cdot) \|_{TV} \leq 1/e \). We first consider any start vertex \( u \in V_2 \setminus \{ \hat{z} \} \) and afterward the vertex \( u = \hat{z} \). Let \( \mathcal{D} \) be the set of distributions over \( V(\hat{G}) = V_2 \cup \{ \hat{z} \} \) assigning no probability mass to \( \hat{z} \)—that is,

\[
\mathcal{D} = \{ D' : \text{ for } u \sim D' \text{ we have } \mathbb{P}[u = \hat{z}] = 0 \}.
\]

For any such \( D' \in \mathcal{D} \), we have, by definition of the total variation distance,

\[
\| q^t_{u,D'} - \pi^{\hat{G}}(\cdot) \|_{TV} = 0 + \frac{1}{2} \sum_{v \in V_2} | q^t_{u,D'} \cdot v - \pi^{\hat{G}}(v) | + \frac{1}{2} | q^t_{u,D',\hat{z}} - \pi^{\hat{G}}(\hat{z}) |.
\]

For \( u \in V_2 \), observe that \( \pi^{\hat{G}}(u) \in \{ \pi^{G_2}(u) (1 - \zeta), \pi^{G_2}(u) (1 + \zeta) \} \) for some \( \zeta = o(1) \). By the work of Levin et al. [33, Exercise 4.1], we have the following identity for \( d^Q(t) \). Let \( \mathcal{D}^* \) be the set of all distributions over \( V(G_2) \), then

\[
d^Q(t) = \max_{D \in \mathcal{D}^*} \| q^t_{u,D} - \pi^{G_2}(\cdot) \|_{TV} \geq \max_{D' \in \mathcal{D}} \| q^t_{u,D'} - \pi^{G_2}(\cdot) \|_{TV}.
\]

Thus, for \( \delta_v := | q^t_{u,v} - q^t_{u,v} | \), we get, by using triangle inequality,

\[
\frac{1}{2} \sum_{v \in V_2} | q^t_{u,D',v} - \pi^{G_2}(v) | \leq \frac{1}{2} \sum_{v \in V_2} | q^t_{u,D',v} - \pi^{G_2}(v) | + \frac{1}{2} \sum_{v \in V_2} | \pi^{G_2}(v) - \pi^{\hat{G}}(v) | \leq \frac{1}{2} \sum_{v \in V_2} | q^t_{u,D',v} - \pi^{G_2}(v) | + \frac{1}{2} \sum_{v \in V_2} | \pi^{G_2}(v) \zeta | \leq d^Q(t) + 1/32 + \frac{\zeta}{2},
\]

where the second-last inequality is due to Lemma 3.8.ii, \( \frac{1}{4} \sum_{v \in V} | \delta_v | \leq (1+o(1))t/(2\sqrt{n}) \leq 1/32 \).

By definition of the \( t_{\text{mix}}(G_2) \) and by submultiplicativity, we have \( d^Q(t) \leq d^Q(2t_{\text{mix}}(G_2)) \leq 1/e^2 \).

Equation (14) only considers the variation distance w.r.t. \( V_2 \). For \( \hat{z} \), we have

\[
\frac{1}{2} | q^t_{u,D',\hat{z}} - \pi^{\hat{G}}(\hat{z}) | \leq (2t_{\text{mix}}(G_2) + 7)(1 + o(1))/\sqrt{n} \leq 1/32.
\]
Putting everything together, we get
\[
\|q^t_{u-D^c} - \pi(G)\|_{TV} = \frac{1}{2} \sum_{v \in V_2} |q^t_{u-D^c, v} - \pi(G)(v)| + \frac{1}{2} |q^t_{u-D^c, z} - \pi(G)(z)| \\
\leq d^G(t) + 1/32 + 1/32 + 1/32 \leq 1/e^2 + 3/32. \tag{15}
\]

Consider the random walk starting at $\tilde{z}$, and let $(X_0, X_1, \ldots)$ denote its trajectory. Observe that at time 7, we have
\[
q^7_{\tilde{z}, \tilde{z}} \leq \frac{1}{2^7} + \sum_{i \leq 7} \sum_{v \in N(\tilde{z})} \frac{q^{i-1}_{\tilde{z}, v}}{\sqrt{\alpha}} \leq \frac{1}{2^7} + \frac{1}{\sqrt{\alpha}} \leq 1/32.
\]

The set of distribution for the position of the random walk at time 7 conditioning on $X_7 \neq \tilde{z}$ gives the same distribution $D$ as defined in (15). Let $D_{\tilde{z}} \in D$ be distribution of the random walk at time 7 starting at $\tilde{z}$. Hence, by (15), we get
\[
\|q^{2t_{\text{mix}}(G_2)+7}_{\tilde{z}, \tilde{z}} - \pi(G)\|_{TV} \leq \|q^{2t_{\text{mix}}(G_2)}_{\tilde{z}, \tilde{z}} - \pi(G)\|_{TV} + q^7_{\tilde{z}, \tilde{z}} \cdot 1 \\
\leq 1 \cdot (1/e^2 + 3/32) + 1/32 \leq 1/e. \tag{16}
\]

Thus, for $t' = 2t_{\text{mix}}(G_2) + 7$, we have $\|q^{t'}_{\tilde{z}, \tilde{z}} - \pi(G)\|_{TV} \leq 1/e$. Together with (15), we conclude that for all $u \in V'$, $\|q^t_{u, \tilde{z}} - \pi(G)\|_{TV} \leq 1/e$ and by definition of $t_{\text{mix}}$ and we get $t_{\text{mix}}(\hat{G}) \leq 2t_{\text{mix}} + 7 = O(1)$.

(ii) To prove $\mathbb{P}[T_{\text{hit}}(u, \tilde{z}) \geq n/c_1] \geq 1/2$ for $u \in V_2$, we show that the random walk restricted to $\hat{G}$ does not hit $\tilde{z}$ after $n/c_1$ steps w.c.p. for some large enough constant $c_1$. By the union bound, for large constants $c_1, c_2$, we have that
\[
\mathbb{P}[T_{\text{hit}}(u, \tilde{z}) \leq n/c_1] = \mathbb{P}\left[T_{\text{hit}}(u, \tilde{z}) \leq n/c_1 \right] \leq \sum_{t=1}^{n/c_1} \frac{c_1 \log n}{2/\sqrt{n}} + \sum_{t=c_2 \log n}^{n/c_1} \frac{\hat{q}^t_{u, \tilde{z}}}{\pi(G)(u)} \lambda_2(\hat{G})^t \leq o(1) + n/c_1 \cdot (\pi(G)(\tilde{z}) + 1/n^2) \leq 1/2,
\]
where we used $\hat{q}^t_{u, \tilde{z}} \leq \pi(G)(\tilde{z}) + \sqrt{\pi(G)(u)} \lambda_2(\hat{G})^t$ (Lemma A.1).

We proceed by bounding that $\mathbb{P}[T_{\text{hit}}(u, \tilde{z}) \geq n/c_1] \geq 1/2$ for $u \in V_1$. Consider first a random walk $(\tilde{X}_t)_{t \geq 0}$ restricted to $G_1^t = G_1$ that starts at vertex $z^1$, and let $\tilde{p}$ denote the transition matrix. Furthermore, to couple the random walk $\tilde{X}_t$ restricted to $G_1$ with a random walk in $G$, we will consider the random variable $\tilde{Z} := \sum_{t=0}^{c_1} 1_{\tilde{X}_t = z^1}$. Since $G_1$ is a clique, $t_{\text{sep}}^{G_1} = O(1)$, and $\tilde{p}^{z, z_1} \leq \frac{1}{\sqrt{n}} + \lambda_2(G_1)^t$ by Lemma A.1, where $\lambda_2(G_1)$ is some constant bounded away from 1. Therefore, $\mathbb{E}[\tilde{Z}] = \sum_{t=0}^{c_1} \tilde{p}^{z_1, z_1} \leq 2\sqrt{n}/c_1$. Let $\gamma := 4 \cdot \mathbb{E}[\tilde{Z}]$. Then, by Markov’s inequality,
\[
\mathbb{P}[\tilde{Z} \geq \gamma] \leq 1/4.
\]

Consider now the straightforward coupling between a random walk $(X_t)_{t \geq 1}$ in $G$ that starts at vertex $z^1$ and the random walk $(\tilde{X}_t)_{t \geq 1}$ restricted to $G_1^t$ that starts at the same vertex. Whenever the random walk $\tilde{X}_t$ is at a vertex different from $z^1$, the random walk $X_t$ makes
the same transition. If the random walk $\tilde{X}_t$ is at vertex $z^1$, then there is a coupling so that the random walk $X_t$ makes the same transition as $\tilde{X}_t$ w.p. $\frac{2\sqrt{n} - 1}{2\sqrt{n}}$. Conditional on the event $\tilde{Z} \leq \gamma$ occurring, the random walk $\tilde{X}_t$ follows the random walk $X_t$ up until step $n/c_1$ w.p. at least

$$p := \left(\frac{2\sqrt{n} - 1}{2\sqrt{n}}\right)^Y \geq 3/4,$$

since the random walk $\tilde{X}_t$ has at most $\gamma$ visits to $z^1$. Therefore, by the union bound,

$$\mathbb{P}\left[ T^G_{\text{hit}}(u, \tilde{z}) \geq n/c_1 \right] \geq \mathbb{P}\left[ \bigcup_{t=0}^{n/c_1} X_t = \tilde{X}_t \right] \geq 1 - \mathbb{P}[\tilde{Z} > \gamma] - (1 - p) \geq 1/2$$

and the proof is complete.

(iii) We proceed by showing $t_{\text{hit}}(u, \tilde{z}) = O(n)$ for $u \in V_2$.

Let $Q$ be the transition matrix of the random walk restricted to $G_2$. Let $u \in V_2$ and $S^* = N(\tilde{z})$ be the neighbors of $\tilde{z}$ in $G_2$. For every $v \in S^*$, we have $\pi_{G_2}(v) = \frac{n}{n + (1 + o(1))\sqrt{n} \sqrt{\alpha}} \geq \frac{\sqrt{\alpha'}}{1.2n}$.

Hence, after $t = t_{\text{sep}}(G_2)$, we have that

$$q^l_{u, S^*} := \sum_{v \in S^*} q^l_{u, v} \geq \sum_{v \in S^*} \pi_{G_2}(v)(1 - e^{-1}) \geq \frac{\sqrt{n}}{\sqrt{\alpha}} \frac{\sqrt{\alpha'}}{1.2n}(1 - e^{-1}) = \frac{1 - e^{-1}}{1.2\sqrt{n}}.$$

By Lemma 3.8, we have for any $u \in V_2$ that $\|p^l_{u, \cdot} - q^l_{u, \cdot}\|_{\text{TV}} \leq (1 + o(1))t_{\text{sep}}(G_2)/(2\sqrt{n})$. To bound $T^G_{\text{hit}}(u, \tilde{z})$, we show that after $t_{\text{sep}} + 1 = O(1)$ steps, the random walk hits $\tilde{z}$ w.p. $\Omega(1/n)$.

We distinguish between two cases.

(a) For all $i \leq t$, we have $p^l_{u, S^*} \leq 1/t_{\text{sep}}(G_2)$. Thus, by Lemma 3.8.i,

$$p^l_{u, S^*} = \sum_{v \in S^*} p^l_{u, v} \geq q^l_{u, S^*} - \|p^l_{u, \cdot} - q^l_{u, \cdot}\|_{\text{TV}} \geq \frac{1 - e^{-1}}{1.2\sqrt{n}} - (1 + o(1))t_{\text{sep}}(G_2)/(2\sqrt{n})$$

$$\geq \frac{1 - e^{-1}}{1.2\sqrt{n}} - (1 + o(1))t_{\text{sep}}(G_2)2\sqrt{n} = \Omega(1/\sqrt{n}).$$

Hence, the random walk hits $\tilde{z}$ after $t_{\text{sep}}(G_2) + 1$ w.p. at least $p^l_{u, S^*} \cdot \min_{v \in S^*}\{p_{v, \tilde{z}}\} = \Omega(1/n)$.

(b) Otherwise, there exists a $t^*$ such that $p^l_{u, S^*} > 1/t_{\text{sep}}(G_2)$. Thus, the random walk hits $\tilde{z}$ after $t_{\text{sep}}(G_2) + 1$ w.p. at least $p^l_{u, S^*} \cdot \min_{v \in S^*}\{p_{v, \tilde{z}}\} = \Omega(1/n)$.

Thus, after $O(1)$ steps, the random walk hits $\tilde{z}$ w.p. $\Omega(1/n)$.

We now show a similar statement if $u \in V_1$. Let $(X_t)_{t \geq 0}$ be a random walk on $G$ starting on $u$. Observe that $X_t$ (the walk on $G$) hits $\tilde{z}$ w.p. $p^1_{u, \cdot} \cdot p^1_{\cdot, \tilde{z}} = \Omega(1/n)$ in two timesteps. Hence, for any $u \in V$, we have $\mathbb{P}\left[ T_{\text{hit}}(u, \tilde{z}) = O(1) \right] = \Omega(1/n)$. Thus, repeating this iteratively and using independence yields $t_{\text{hit}}(u, \tilde{z}) = O(n)$ for $u \in V$.

To establish a bound on the mixing time of $G$, we will make use of the following result of Peres and Sousi [41].

**Theorem 3.10** ([41]). For any $\beta < 1/2$, let $t_{\text{hit}}(\beta) = \max_{u : \pi(A) > \beta} t_{\text{hit}}(u, A)$. Then, there exist positive constants $c_\beta$ and $c'_{\beta}$ such that

$$c'_{\beta} \cdot t_{\text{hit}}(\beta) \leq t_{\text{mix}}(1/4) \leq c_{\beta} \cdot t_{\text{hit}}(\beta).$$

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In the following, we show, for any $\beta$ close enough to $1/2$, that any $A \subseteq V$ satisfying $\pi(A) \geq \beta$ must include at least a constant fraction of nodes from a constant fraction of copies of $G_1$.

**Claim 3.11.** Let $\beta = 1/2 - 10^{-3}$. For any $A \subseteq V$ with $\pi(A) \geq \beta$, define $H(A) = \{i \mid |G_i^1 \cap A| \geq |V_1|/(2e)\}$. Then, $|H(A)| \geq \kappa/(2e)$.

**Proof.** This follows from a simple pigeonhole argument. Suppose $|H(A)| < \kappa/(2e)$ was true. Then,

$$
\pi(A) \leq |H(A)| \cdot \pi(V_1) + (\kappa - |H(A)|) \cdot \left(\frac{\pi(V_1)}{2e} + \pi(z)\right) + \pi(V_2) + \pi(\hat{Z})
$$

$$
< \frac{\kappa}{2e} \cdot \pi(V_1) + \kappa \cdot \left(\frac{\pi(V_1)}{2e} + \pi(z)\right) + 1/20 < \beta \leq \pi(A),
$$

which is a contradiction, and hence the choice of $A$ must fulfill $|H(A)| \geq \kappa/(2e)$.

We are now ready to determine the mixing time of $G$. The lower bound is a simple application of Cheeger’s inequality, whereas the upper bound combines the previous lemmas with Theorem 3.10.

**Lemma 3.12.** Let $G$ be the graph described at the beginning of Section 3.4. We have $t_{\text{mix}}(G) = \Theta(n)$.

**Proof.** First we show $t_{\text{mix}} = \Omega(n)$. The conductance of $G = (V, E)$ is defined by

$$
\Phi(G) = \min_{0 < \text{vol}(U) \leq \text{vol}(V)/2} \frac{|E(U, V \setminus U)|}{\text{vol}(U)}.
$$

In particular, for $U = V_1$, we get that $\Phi(G) \leq \frac{n}{4}$. Hence, by Cheeger’s inequality and

$$
\left(\frac{1}{1 - \lambda_2(G)} - 1\right) \cdot \log \left(\frac{\varepsilon}{2}\right) \leq t_{\text{mix}}(1/e)
$$

(e.g., see [33, Chapter 12]),

$$
\frac{n}{4} \leq \Phi(G) \leq \frac{2}{1 - \lambda_2(G)} = \frac{2}{1 - \lambda_2(G)} - 2 + 2 \leq \frac{2t_{\text{mix}}}{\log \left(\frac{\varepsilon}{2}\right)} + 2.
$$

Rearranging the terms yields $t_{\text{mix}} = \Omega(n)$.

We proceed with the upper bound on the mixing time. Let $\beta = 1/2 - 10^{-3}$, and let $A \subseteq V$ be an arbitrary set satisfying $\pi(A) \geq \beta$. First, we apply Claim 3.11 to conclude that $|H(A)| \geq \kappa/(2e)$. This immediately implies that with $Z := \{z^i : i \in H(A)\}$, $|Z| \geq \kappa/(2e)$. The remainder of the proof is divided into the following three parts:

(i) Starting from any vertex $u \in V$, w.p. at least $1/2$, the random walk hits $z^*$ after $2 \max_{u \in V} t_{\text{hit}}(u, \hat{z}) = O(n)$ steps.

(ii) With constant probability $p_1 > 0$, the random walk moves from $z^*$ to a vertex in $Z$.

(iii) With constant probability $p_2 > 0$, a random walk starting from a vertex in $Z$ will hit $A$ after one step.

It is clear that combining these three results shows that w.c.p. $\frac{1}{4}p_1p_2 > 0$, a random walk starting from an arbitrary vertex $u \in V$ hits a vertex in $A$ after $O(n) + 1 + 1$ timesteps. Iterating this and using independence shows that $t_{\text{hit}}(u, A) = O(n)$, and hence by Theorem 3.10, $t_{\text{mix}} = O(n)$ as needed.

**Part (i):** Consider max$_{u \in V} t_{\text{hit}}(u, \hat{z})$. For $u \in V$, Lemma 3.9.(iii) implies $t_{\text{hit}}(u, \hat{z}) = O(n)$.

**Part (ii):** If the random walk is on $z^*$, then since $\text{deg}(z^*) = \kappa + \sqrt{n/\alpha'}$, $|Z| \geq \kappa/(2e)$, it follows that the random walk hits a vertex in $Z$ after one step w.c.p. $p_1 := \frac{|Z|}{2(\kappa + \sqrt{n/\alpha'})} > 0$.

**Part (iii):** Finally, for any $z \in Z$, we have that $p_2 = p_{z, A} = \frac{|V_1|/(2e)}{2\sqrt{n}} > 0$, and the proof is complete.

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In the following, we establish the bound on the meeting time. As it turns out, any meeting is very likely to happen on $V_2$ and it takes about $\Theta(\alpha' n)$ timesteps until both walks reach $V_2$ simultaneously. The lower bound then follows from our common analysis method (2). The upper bound combines the mixing time bound of $O(n)$ (Lemma 3.12), and once a random walk reaches a copy of $G_1$, it stays there for $\Theta(n)$ steps w.c.p. (Lemma 3.9.(ii)).

**Lemma 3.13.** Let $G$ be the graph described at the beginning of Section 3.4. We have $t_{\text{meet}}(G) = \Theta(\alpha' n)$.

**Proof.** We start by proving $t_{\text{meet}} = \Omega(\alpha' n)$. Consider two non-interacting random walks with starting positions drawn from the stationary distribution $\pi$. Let $\ell = c'\alpha' n$, for some small enough constant $c' > 0$. Let $Z_1$ be the number of collisions of the two random walks on the nodes in $V_1^1 \cup V_1^2 \cup \cdots \cup V_1^\kappa$. Let $Z_2$ be the number of collisions of the two random walks on the nodes in $V_2$. Let $Z_\ast$ be the number of collisions of the two random walks on the node $\hat{z}$.

Let $Z$ be the number of collisions of the two walks during the first $\ell$ timesteps (i.e., $Z = Z_1 + Z_2 + Z_\ast$). Using the union bound, we derive

$$
\mathbb{P}[Z \geq 1] \leq \mathbb{P}[Z_1 > 1] + \mathbb{P}[Z_2 > 1] + \mathbb{P}[Z_\ast > 1]
$$

(18)

Using an exactly analogous analysis for $Z_2$, we can upper bound $\mathbb{E}[Z_2 | Z_2 \geq 1]$ as follows:

$$
\mathbb{E}[Z_2 | Z_2 \geq 1] \geq \mathbb{E}[Z_2 | Z_2 \geq 1, \mathcal{E}_2] \cdot \mathbb{P}[\mathcal{E}_2] \geq \frac{1}{4} \min_{u \in V_2} \sum_{t=0}^{n/c-1} \sum_{v \in V_2} (p_{u,v}^t)^2 \geq \frac{1}{4} \sum_{t=0}^{n/c-1} 1/|V_2| = \frac{\sqrt{\ell}}{4c},
$$

where $\mathcal{E}_2$ is the event that for $u \in V_2$, we have $T_{\text{hit}}(u, \hat{z}) \geq n/c$ for some large enough constant $c$. Plugging everything into (18) and using $\ell = c'\alpha' n$ yields

$$
\mathbb{P}[Z \geq 1] \leq \frac{\mathbb{E}[Z_2]}{\mathbb{E}[Z_1 | Z_1 \geq 1]} + \frac{\mathbb{E}[Z_2]}{\mathbb{E}[Z_2 | Z_2 \geq 1]} + \frac{\mathbb{E}[Z_\ast]}{\mathbb{E}[Z_\ast | Z_\ast \geq 1]}
$$

(19)

\begin{align*}
&\leq \frac{\ell n (\frac{\sqrt{\ell}}{n})^2}{\sqrt{\ell}} + \frac{\ell n (\frac{\sqrt{\ell}}{n})^2}{\sqrt{\ell}} + \frac{\ell (\frac{\sqrt{\ell}}{n})^2}{1} \\
&\leq o(1) + 16c \cdot c' + o(1) \leq 1/2,
\end{align*}

for any constant $c' \in (0, \frac{1}{32c})$. This finishes the proof of $t_{\text{meet}} = \Omega(\alpha' n)$. In the remainder, we prove $t_{\text{meet}} = O(\alpha' n)$. Consider two independent walks $(X_t)_{t\geq0}$ and $(Y_t)_{t\geq0}$ on $G$, both starting from...
arbitrary nodes. Note $t_{\text{sep}} = t_{\text{sep}}(G) \leq 4t_{\text{mix}} = O(n)$ by Lemma 3.12, and
\[
p_0 := P\{ X_{t_{\text{sep}}} \in V_2 \} \cap \{ Y_{t_{\text{sep}}} \in V_2 \} \geq \left( \sum_{u \in V_2} (1 - e) \pi(u) \right)^2 = \Omega((1/\sqrt{n})^2) = \Omega(1/\alpha').
\]
We assume in the following that $\{ X_{t_{\text{sep}}} \in V_2 \} \cap \{ Y_{t_{\text{sep}}} \in V_2 \}$. We have $t_{\text{mix}}(G_2) = O(1)$, by Corollary A.3. Consider a random walk $(\tilde{X}_t)_{t \geq t_{\text{sep}}}$ restricted to $G_2$ that starts at vertex $X_{t_{\text{sep}}} \in V_2$, and let $\tilde{P}$ denote the transition matrix. Furthermore, to couple the random walk $\tilde{X}_t$ restricted to $G_2$ with a random walk in $G$, we will consider the random variable
\[
\tilde{Z} := \sum_{t=t_{\text{sep}}}^{t_{\text{sep}}+n/c-1} \sum_{z \in N(\tilde{\pi})} 1_{\tilde{X}_t = z},
\]
for $c = 32$. Thus, for any $z \in N(\tilde{\pi})$,
\[
E[\tilde{Z}] \leq t_{\text{mix}}(G_2) + \sum_{t=t_{\text{sep}}+n/c+1}^{t_{\text{sep}}+n/c-1} |N(\tilde{\pi})| (\pi^{G_2}(z) + d^{\tilde{P}}(t))
\leq t_{\text{mix}}(G_2) + |N(\tilde{\pi})| (n/c + O(1)) \leq (1 + 1/e) \sqrt{n}/c.
\]
Let $\gamma := 8(1 + 1/e) \sqrt{n}/c$. Then, by Markov’s inequality,
\[
P[\tilde{Z} \geq \gamma] \leq 1/8.
\]
Consider now the straightforward coupling between a random walk $(X_t)_{t \geq t_{\text{sep}}}$ in $G$ that starts at vertex $X_{t_{\text{sep}}} \in V_2$ and the random walk $(\tilde{X}_t)_{t \geq t_{\text{sep}}}$ restricted to $G_2$ that starts at the same vertex. Whenever the random walk $\tilde{X}_t$ is at a vertex in $V_2 \setminus N(\tilde{\pi})$, the random walk $X_t$ makes the same transition. If the random walk $\tilde{X}_t$ is at vertex $z' \in N(\tilde{\pi})$, then there is a coupling so that the random walk $X_t$ makes the same transition as $\tilde{X}_t$ w.p. $\frac{(1-v)\sqrt{n}}{2(1-v)\sqrt{n}+2}$ for some $v = o(1)$. Conditional on the event $[\tilde{Z} \leq \gamma]$ occurring, the random walk $\tilde{X}_t$ follows the random walk $X_t$ up until step $n/c$ w.p. at least
\[
p_1 := \left( \frac{2(1-v)\sqrt{n}}{2(1-v)\sqrt{n}+2} \right)^\gamma = \left( 1 - \frac{2}{2(1-v)\sqrt{n}+2} \right)^Y \geq 3/4
\]
since the random walk $\tilde{X}_t$ has at most $\gamma$ visits to $N(\tilde{\pi})$. Consider now the random walk $(\tilde{Y}_t)_{t \geq t_{\text{sep}}}$ using $\tilde{P}$ (i.e., restricted to $V_2$) starting at $Y_{t_{\text{sep}}}$ (i.e., $\tilde{Y}_{t_{\text{sep}}} = Y_{t_{\text{sep}}}$). By an analogous argument as before, we can couple $(Y_t)_{t \geq t_{\text{sep}}}$ and $(\tilde{Y}_t)_{t \geq t_{\text{sep}}}$ for $n/c$ timesteps w.p. at least $p_1$.

Furthermore, after $t_{\text{sep}}(G_2) = O(1)$ steps, we can couple $\tilde{X}_t$ and $\tilde{Y}_t$ with nodes drawn independently from $\pi^{G_2}$. Hence,
\[
p_2 := P\{ X_{t_{\text{sep}}+t_{\text{sep}}(G_2)} = \tilde{Y}_{t_{\text{sep}}+t_{\text{sep}}(G_2)} \mid \mathcal{F}_t \} \geq (1 - 1/e)^2 \| \pi^{G_2} \|_1^2 \geq \frac{\sqrt{\alpha'}}{8n}.
\]
Recall that $\alpha' \geq 2^{20}t_{\text{sep}}(G_2)^2$ by definition. Therefore, the probability that $\tilde{X}_t$ and $\tilde{Y}_t$ do not meet in the time interval $[t_{\text{sep}}(G_1), t_{\text{sep}}(G_1) + n/c - 1]$ is at most
\[
p_3 := (1 - p_2)^{\lfloor n/(t_{\text{sep}}(G_2)c) \rfloor} \leq (1 - p_2)^{2^{20}n/(\sqrt{\alpha'}c)} \leq 1/4.
\]
Therefore, by the union bound,
\[
P\{ \bigcup_{t=0}^{t_{\text{sep}}(G_1) + n/c-1} X_t = Y_t \} \geq p_0 \cdot (1 - P[\tilde{Z} \geq \gamma] - 2 \cdot (1 - p_1) - p_3) = \Omega(\alpha').
\]
Repeating this $O(1/p_3)$ times and using the independence yields that the expected meeting time is $O((t_{sep}(G_1) + n/c - 1)/p_3) = O(a'n)$ and the proof is complete. □

Finally, we analyze the coalescing time of $G$. The proof idea is to consider $\sqrt{n}$ random walks starting from $\pi$ and show that meetings only occur on $V_2$ and that at least one random walk requires $\Omega(\sqrt{\alpha'} \cdot n \log n)$ timesteps to reach $V_2$.

**Lemma 3.14.** Let $G$ be the graph described at the beginning of Section 3.4. We have $t_{\text{coal}}(G) = \Omega(\sqrt{\alpha'} \cdot n \log n)$.

**Proof.** Let $\varepsilon = 1/5$. We show that even the coalescing time of $n^\varepsilon$ random walks requires $\Omega(\sqrt{\alpha'} \cdot n \log n)$ timesteps w.c.p. Let $R$ be a collection of $n^\varepsilon$ independent (i.e., non-interacting) random walks with starting positions drawn from the stationary distribution $\pi$. We define the following three bad events:

(i) Let $E_1$ be the event that any two of the $n^\varepsilon$ random walks meet on a node in $V \setminus V_2$ in $\sqrt{\alpha'} \cdot n \log^2 n$ steps.

(ii) Let $E_2$ be the event that fewer than $n^\varepsilon/4$ random walks start on copies of $G_1$ (i.e., on nodes in $V \setminus (V_2 \cup \hat{z})$).

(iii) Let $E_3$ be the event that all random walks starting from a copy of $G_1$ require fewer than $c \cdot \sqrt{\alpha'} \cdot n \log n$ timesteps for leaving $V \setminus (V_2 \cup z^*)$ for some constant $c > 0$ to be determined later.

In the following, we show that $P[E_1] = o(1)$, $P[E_2] = o(1)$, and $P[E_3] \mid E_2 < 1/\varepsilon$, which implies, by the union bound,

$$P[E_1 \cap E_2 \cap E_3] \geq P[E_1] - (1 - P[E_2] \cap E_3)$$

$$\geq 1 - o(1) - (1 - o(1)) \cdot \left(1 - \frac{1}{\varepsilon}\right) \geq 1 - \frac{1}{2\varepsilon}.$$

Conditioning on $E_1 \cap E_2 \cap E_3$, none of the independent random walks meet on any node $V \setminus V_2$, and hence they are indistinguishable from coalescing random walks until they reach $V_2$. Therefore, it is necessary for all random walks to reach $G_2$ to coalesce. Hence, we conclude that $t_{\text{coal}}(G) = \Omega(\sqrt{\alpha'} \cdot n \log n)$, yielding the lemma.

(i) We now prove $P[E_1] = o(1)$. Consider any pair of the random walks $R$. Since both random walks start from the stationary distribution, the probability for them to meet on a node on $\hat{z}$ in a fixed step $t \geq 0$ is at most $O(1/n^2)$.

Hence, by the union bound over $\binom{n^\varepsilon}{2}$ pairs of random walks and $\sqrt{\alpha'} \cdot n \log^2 n \leq n \log^3 n$ steps, the probability of any two random walks meeting on $\hat{z}$ is at most

$$p_1 := \frac{n^\varepsilon}{2} \cdot n \log^3 n \cdot O(1/n^2) = o(1),$$

since $\varepsilon = \frac{1}{5}$. Furthermore, the probability that no two walks start on the same copy of $G_1$ is at most $p_2 := n^\varepsilon \cdot \frac{n^\varepsilon}{\sqrt{n}} = o(1)$ by the union bound.

Moreover, using a Chernoff bound together with Lemma 3.9.(ii), it follows that a random walks visits the vertex $z^*$ at most $10 \log^3 n$ times during $n \log^3 n$ steps w.p. at least $1 - n^{-2}$. By the union bound over all random walks, it follows that w.p. at least $1 - n^{-1}$, each random walk visits at most $10 \log^3 n$ different copies of $G_1$, and by construction of $G$, each such copy is chosen uniformly and independently at random among $G_1^1, G_1^2, \ldots, G_1^k$. Therefore, the probability
that there exists a copy of $G_1$ that is visited by at least two random walks in $n \log^3 n$ steps is at most

$$p_3 := n^{-1} + n^\varepsilon (10 \log^3 n + 1) \cdot \frac{n^\varepsilon (10 \log^3 n + 1)}{\sqrt{n}} = o(1).$$

Putting everything together, using the union bound, yields $\mathbb{P}[\mathcal{E}_1] \leq p_1 + p_2 + p_3 = o(1)$.

(ii) We now prove $\mathbb{P}[\mathcal{E}_2] = o(1)$. The probability $p$ for each random walk to start on a node of $V \setminus (V_2 \cup \tilde{Z})$ is $\pi(V \setminus (V_2 \cup \tilde{Z})) > 1/2$. For each of the random walks with label $1 \leq i \leq n^\varepsilon$, we define the indicator variable $X_i$ to be 1, if that random walk starts on $V \setminus (V_2 \cup \tilde{Z})$. Let $X = \sum_{i=1}^{n^\varepsilon} X_i$. We have $\mathbb{E}[X] = n^\varepsilon \cdot \mathbb{E}[X_i] \geq n^\varepsilon/2$. Since the starting positions of the $n^\varepsilon$ random walks are drawn independently, by a Chernoff bound,

$$\mathbb{P}[\mathcal{E}_2] = \mathbb{P}\left[ X \leq \frac{1}{4} n^\varepsilon \right] \leq \mathbb{P}\left[ X \leq \mathbb{E}[X] / 2 \right] \leq e^{-n^\varepsilon/16} = o(1).$$

(iii) We now prove $\mathbb{P}[\mathcal{E}_3 \mid \mathcal{E}_2] < 1/4$. From Lemma 3.9.(ii), we get that w.p. at least 1/2, a random walk starting at any node $u \in V_1$ does not leave $G_1$ (i.e., does not reach $z'$) after $c_1 n$ timesteps for some constant $c_1 > 0$. It is easy to see that the number of visits to $\tilde{Z}$ required before the random walk hits $G_2$ instead of returning to $G_1$ is w.c.p. at least $(1 - o(1)) \sqrt{\alpha^*} / 2$; this is because the fraction of edges from $\tilde{Z}$ to $G_2$ is $\sqrt{n/\alpha^*} / (\sqrt{n/\alpha^*} + (1 + o(1)) \sqrt{n})$. Using a Chernoff bound, we conclude that any random walk starting at $G_1$ does not hit $G_2$ during the first $T = c_1 \cdot \sqrt{n^3/2}$ timesteps w.c.p. $p > 0$. Thus, the probability that a random walk does not reach $G_2$ after $\lambda \cdot T$ timesteps is at least $p^\lambda$, for any integer $\lambda \geq 1$. Setting $\lambda = \varepsilon \cdot \log(1/p) \cdot \log(n/4)$, the probability that all of the at least $\frac{1}{2} n^\varepsilon$ random walks starting from $G_1$ reach $G_2$ within $\lambda \cdot T = \Omega(\sqrt{\alpha^*} \cdot n \log n)$ steps is

$$\mathbb{P}[\mathcal{E}_3 \mid \mathcal{E}_2] \leq (1 - p^\lambda)^{1/2} n^\varepsilon \leq 1/e,$

completing the proof.
4.1 General Upper Bound $t_{coal} = O(n^3)$

In this section, we establish that $t_{coal} = O(n^3)$ on all graphs, which is matched for instance by the Barbell graph.

**Theorem 4.1.** For any graph $G$, we have $t_{coal} = O(n \cdot |E| \cdot \log(|E|/n))$, so in particular, $t_{coal} = O(n^3)$.

This result was first derived in a self-contained fashion in an earlier version of this work, which gave the first tight bound on the worst-case coalescing time. Due to the subsequent work [39], Theorem 4.1 follows immediately.

**Proof of Theorem 4.1.** This follows directly from $t_{coal} = O(t_{hit})$ [39, Theorem 5.1] and the standard bound $t_{hit} \leq n \cdot 2|E|$ (cf. [3]).

4.2 General Lower Bound $t_{coal} = \Omega(\log n)$

In this section, we prove that the coalescing time of any graph is $\Omega(\log n)$. We consider a process $P'$ where there is exactly one random walk starting at each node in the graph. For every node $u \in V$ and every timestep $t \in \mathbb{N}$, we draw an independent random variable $Z_{u,t} \in \{0, 1\}$ with $\mathbb{P}[Z_{u,t} = 1] = 1/2$ and $\mathbb{P}[Z_{u,t} = 0] = 1/2$. If $Z_{u,t} = 1$, then the random walk on $u$ at time $t$ (if there is any) moves to a neighboring node chosen uniformly at random. Otherwise, $(Z_{u,t} = 0)$, the random walk on $u$ at time $t$ (if there is any), stays on the same node. It is straightforward to show that the set of nodes that have an active random walk according to this process can be coupled with the coalescence process defined in Section 2.

We show that after $c \log n$ steps, for a sufficiently small $c$, there are at least two surviving walks in this process. To do this, we simply argue that there must be at least two walks that have not left their starting position. Note that there is no way for these walks to be eliminated, because even if other walks visited one of their starting nodes, there are two nodes from which no walks can have left. The formal proof follows.

**Lemma 4.2.** For any graph $G = (V, E)$, $|V| = n$, we have $t_{coal} = \Omega(\log n)$.

**Proof.** Consider the process $P'$ defined earlier. Let $T$ be the coalescence time. Note that coalescence at time $\tau$ in $P'$ requires that for $n - 1$ nodes $u \in V$, there exists $t_u \leq \tau$ such that $Z_{u,t_u} = 1$. In symbols, let $T$ be the first point in time where all walks coalesced, then $T \geq T'$, with $T' := \min\{t' \in \mathbb{N} : |\{u : Z_{u,t_u} = 1\}| \geq n - 1\}$. Let $Y_u$ be the indicator variable which is 1 if $Z_{u,t_u} = 0$ for all $t \leq \tau := \log n/2$. The process ensures independence of the $Y_u$. Due to the laziness of the random walk, $\mathbb{P}[Y_u = 1] = 1/2^\tau = 1/\sqrt{n}$. Thus, using the independence of the $Y_u$,

$$
\mathbb{P}[T \geq \tau] \geq \mathbb{P}[T' \geq \tau] \geq \mathbb{P}\left[ \sum_{u \in V} Y_u \geq 2 \right] = \mathbb{P}[\text{Binomial}(n, 1/\sqrt{n}) \geq 2] = 1 - o(1),
$$

where Binomial$(n, p)$ denotes the binomial distribution with parameters $n$ and $p$. We conclude that $\mathbb{P}[T] = \Omega(\log n)$, which yields the claim.

4.3 Proof of Theorem 1.4

We are now ready to put all the pieces together. The upper bound on general graphs follows directly from $t_{meet} \leq t_{coal} = O(n^3)$, by Theorem 4.1. The lower bound on the meeting time holds by definition, and the lower bound on the coalescing time follows from Lemma 4.2. For the upper bound on regular graphs, we have $t_{meet} \leq t_{coal} = O(t_{hit}) = O(n^3)$ having used the standard bound $t_{hit} = O(n^3)$ for regular graphs (see [2]). The lower bound follows from $t_{coal} \geq t_{meet} \geq t_{meet}^{\pi} = \Omega(n)$, by Theorem 5.1.
5 BASIC BOUNDS

Although the focus of this work is on understanding the coalescence time, to apply our general results, we also devise some tools to obtain lower and upper bounds on $t_{\text{meet}}$. In Theorem 5.1 (Section 5.1), we establish upper and lower bounds on the meeting time in terms of $\|\pi\|_2^2 = \sum_{u \in V} \pi(u)^2$. Section 5.1 contains several additional upper bounds on $t_{\text{meet}}$ and $t_{\text{hit}}$. Through combination with other results, we also obtain new bounds on $t_{\text{coal}}$ and $t_{\text{cov}}$. A common feature of many of these bounds is a sub-linear dependence on the spectral gap $1/(1 - \lambda_2)$, which we obtain by an application of short-term bounds on the $t$-step transition probabilities.

In Proposition 5.7 (Section 5.2), we establish a discrete-time counterpart of Aldous and Fill [2, Proposition 14.5], albeit with worse constants, stating that the meeting time is at most of the order of the hitting time; on vertex-transitive graphs, these quantities are asymptotically of the same order.

5.1 Relating Meeting Time to $t_{\text{mix}}$ and $1/(1 - \lambda_2)$

We first state some basic bounds on $t_{\text{hit}}$ and $t_{\text{meet}}$, which mostly follow directly from (2) and its counterpart for the hitting times (cf. Cooper and Frieze [18]). In these bounds, we will use the following notation:

$$C_{\text{max}} := \max_{u \in V} \sum_{t=0}^{t_{\text{mix}}-1} \sum_{v \in V} (p^t_{u,v})^2,$$

$$C_{\text{min}} := \min_{u \in V} \sum_{t=0}^{t_{\text{mix}}-1} \sum_{v \in V} (p^t_{u,v})^2.$$

Note that $C_{\text{max}}$ and $C_{\text{min}}$ provide worst-case upper respective lower bounds on the expected collisions of two independent random walks of length $t_{\text{mix}}$, starting from the same vertex $u$. Similarly, we define

$$R_{\text{max}} := \max_{u \in V} \sum_{t=0}^{t_{\text{mix}}-1} p^t_{u,u}.$$

Note that $R_{\text{max}}$ is the number of expected returns of a random walk to $u$ during $t_{\text{mix}}$ steps. This quantity is more convenient to bound than $C_{\text{max}}$—for instance, it can be easily bounded by $\max_{u \in V} \pi(u) \cdot t_{\text{mix}} + \frac{1}{1 - \lambda_2}$ (cf. Lemma A.1, or also the work of Cooper et al. [14]), using $\sum_{t=0}^{\infty} \lambda_t = \frac{1}{1 - \lambda_2}$.

Before stating the next result, we recall that $G$ is $\Gamma$-approximative regular if $\Delta/\delta = \Gamma$.

**Theorem 5.1.** For any graph $G = (V, E)$, the following statements hold:

(i) For any pair of vertices $u, v \in V$,

$$t_{\text{hit}}(u, v) \leq 5e \cdot \left( \sum_{t=0}^{t_{\text{mix}}-1} p^t_{v,v} \right) / \pi(v).$$

In particular, if the graph $G$ is $\Gamma$-approximative regular, then $t_{\text{hit}}(u, v) \leq 5e \cdot \Gamma \cdot n \cdot \sum_{t=0}^{t_{\text{mix}}-1} p^t_{v,v}.$

(ii) For any pair of vertices $u, v \in V$,

$$t_{\text{meet}}(u, v) \leq \frac{5e \cdot C_{\text{max}}}{\|\pi\|_2^2}.$$
In particular, if the graph $G$ is $\Gamma$-approximative regular, then

$$t_{\text{meet}}(u, v) \leq \frac{10e \cdot (4 + \log_2(\Gamma)) \cdot R_{\text{max}}}{\|\pi\|_2^2}.$$  

(iii) For any graph $G$, it holds that

$$t_{\text{meet}} \geq t_{\text{meet}}^* \geq \frac{3}{16\|\pi\|_2^2}.$$  

In particular, if the graph $G$ is $\Gamma$-approximate regular, then

$$t_{\text{meet}} \geq t_{\text{meet}}^* = \frac{3}{16} \cdot \frac{n}{\Gamma}.$$  

Furthermore, if

$$\frac{C_{\min}}{16\|\pi\|_2^2} \cdot t_{\text{mix}} \geq 1,$$

then we have

$$t_{\text{meet}} \geq t_{\text{meet}}^* \geq \frac{C_{\min}}{16\|\pi\|_2^2}.$$  

Let us make two remarks. First, note that the second upper bound on $t_{\text{meet}}(u, v)$ depends only logarithmically on $\Gamma$. Second, the final lower bound involving $C_{\min}$ could be tightened further by having a different range of the sum in $C_{\min}$, but for simplicity we only state this version.

Proof. We begin by proving the first part. Consider one random walk $(X_t)_{t \geq 0}$, starting from an arbitrary vertex. Divide the time interval into consecutive epochs of length $t_{\text{sep}} + t_{\text{mix}}$, and let

$$Z := \sum_{t=t_{\text{sep}}}^{t_{\text{sep}}+t_{\text{mix}}-1} 1_{X_t = v}$$

denote the number of visits. Then, by the separation time,

$$\mathbb{E}[Z] \geq t_{\text{mix}} \cdot \frac{1}{e} \cdot \pi(v),$$

and (2) yields

$$\mathbb{P}[Z \geq 1] \geq \frac{t_{\text{mix}} \cdot \pi(v)}{\mathbb{E}[Z \mid Z \geq 1]}.$$  

Clearly,

$$\mathbb{E}[Z \mid Z \geq 1] \leq \max_{t=0}^{t_{\text{mix}}-1} \sum_{s=t}^{t_{\text{mix}}} P_{v, v}^{s-t} \leq \sum_{t=0}^{t_{\text{mix}}-1} P_{v, v}^t.$$  

Hence,

$$\mathbb{P}[Z \geq 1] \geq \frac{t_{\text{mix}} \cdot \pi(v)}{\sum_{t=0}^{t_{\text{mix}}-1} P_{v, v}^t} =: p.$$  

This means that in every epoch of length $t_{\text{sep}} + t_{\text{mix}} \leq 5t_{\text{mix}}$, the random walk has a probability of at least $p$ to visit vertex $v$, and this is independent of any previous epoch. Therefore, the expected number of steps until $v$ is visited is upper bounded by

$$t_{\text{hit}}(u, v) \leq 5t_{\text{mix}} \cdot \frac{1}{p} \leq 5e \cdot \frac{\sum_{t=0}^{t_{\text{mix}}-1} P_{v, v}^t}{\pi(v)}.$$  

The claim for $\Gamma$-approximate regular graphs follows from the observation that $\min_{u \in V} \pi(u) \geq 1/(\Gamma n)$. We continue with the second part. Consider two independent random walks $(X_t)_{t \geq 0}, (Y_t)_{t \geq 0}$ of length $t_{\text{sep}} + t_{\text{mix}}$ with arbitrary start vertices. Let $Z$ be the random variable counting the number of collisions between steps $t_{\text{sep}}$ and $t_{\text{sep}} + t_{\text{mix}}$—that is,

$$Z := \sum_{t=t_{\text{sep}}}^{t_{\text{sep}}+t_{\text{mix}}-1} 1_{X_t = Y_t}.$$
By linearity of expectation,

\[ E[Z] = \sum_{t = t_{sep}}^{t_{sep} + t_{mix} - 1} \sum_{u \in V} P[X_t = u] \cdot P[Y_t = u] \]  

(20)

\[ \geq \sum_{t = t_{sep}}^{t_{sep} + t_{mix} - 1} (1 - 1/e) \cdot \pi_u \cdot (1 - 1/e) \cdot \pi_u \]  

(21)

\[ \geq t_{mix} \cdot \frac{1}{e} \cdot \|\pi\|_2^2, \]  

(22)

since \((1 - 1/e) \cdot (1 - 1/e) > 1/e\). Let us now consider \(E[Z \mid Z \geq 1]\) and recall that conditioning on \(Z \geq 1\) can be regarded as jumping to the first step \(\tau := \min\{t : t_{sep} \leq t \leq t_{mix} - 1, X_t = Y_t\}\) without knowing anything about the future steps \(t > \tau\) of both walks. Therefore,

\[ E[Z \mid Z \geq 1] \leq \max_{u \in V} \sum_{t = 0}^{t_{mix} - 1} \sum_{v \in V} (p_{u,v}^t)^2 = C_{\text{max}}. \]

Plugging this into (2) and using (22), we finally arrive at

\[ P[Z \geq 1] = \frac{E[Z]}{E[Z \mid Z \geq 1]} \geq \frac{t_{mix} \cdot \frac{1}{e} \cdot \|\pi\|_2^2}{C_{\text{max}}}, \]

Hence,

\[ t_{\text{meet}}(u, v) \leq 5t_{mix} \cdot \frac{1}{p} \leq \frac{5e \cdot C_{\text{max}}}{\|\pi\|_2^2}. \]

Let us derive the result for \(\Gamma\)-approximate regular graphs. To this end, define

\[ S_i := \{u \in V : \deg(u) \in [2^{i-1}, 2^i)\}, \]

and note that \(S_0, \ldots, S_{\log_2 n}\) forms a partition of \(V\). Since the graph is \(\Gamma\)-approximate regular, at most \(4 + \log_2(\Gamma)\) of the \(S_i\)'s are non-empty. Hence, there exists a set \(S_j\) with

\[ \sum_{j \in S_j} \pi(j)^2 \geq \frac{1}{4 + \log_2(\Gamma)} \cdot \|\pi\|_2^2. \]

Let us now by \(Z_j\) denote the collisions on the set \(S_j\)—that is,

\[ Z_j := \sum_{t = t_{sep}}^{t_{sep} + t_{mix} - 1} 1\{X_t = Y_t \cap \{X_t \in S_j\}\}. \]

Then,

\[ E[Z_j] = \sum_{t = t_{sep}}^{t_{sep} + t_{mix} - 1} \sum_{u \in S_j} P[X_t = u] \cdot P[Y_t = u] \geq t_{mix} \cdot \sum_{j \in S_j} \frac{1}{e} \cdot \pi(j)^2 \geq t_{mix} \cdot \frac{1}{e} \cdot \frac{1}{4 + \log_2(\Gamma)} \cdot \|\pi\|_2^2. \]

Furthermore,

\[ E[Z_j \mid Z_j \geq 1] \leq \max_{u \in S_i} \sum_{t = 0}^{t_{mix}} \sum_{v \in S_i} (p_{u,v}^t)^2 \leq \max_{u \in S_i} \sum_{t = 0}^{t_{mix}} p_{u,v}^t \cdot 2p_{v,u}^t, \]

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having used reversibility—that is, $p^t_{u,v} \pi(u) = p^t_{v,u} \pi(v)$ and $\pi(v)/\pi(u) \leq 2$ by definition of $S_t$. Further,

$$\mathbb{E}[Z_j \mid Z_j \geq 1] \leq 2 \cdot \max_{u \in S_t} \sum_{t=0}^{t_{\text{mix}}-1} \sum_{v \in V} p^t_{u,v} \cdot p^t_{v,u}$$

$$\leq 2 \cdot \max_{u \in V} \sum_{t=0}^{t_{\text{mix}}-1} p^t_{u,u} \leq 2 \cdot \max_{u \in V} \sum_{t=0}^{t_{\text{mix}}-1} p^t_{u,u} = 2 \cdot R_{\text{max}},$$

where the last inequality holds since $p^t_{u,u}$ is non-increasing by Lemma A.1. Hence, similarly as before,

$$t_{\text{meet}}(u,v) \leq 5t_{\text{mix}} \cdot \frac{2 \cdot R_{\text{max}}}{\frac{1}{4} + \log_2(\Gamma)} \cdot \frac{\|\pi\|^2}{\|\pi\|^2} \leq \frac{10e \cdot (4 + \log_2(\Gamma)) \cdot R_{\text{max}}}{\|\pi\|^2}.$$

Finally, for the third statement, let $(X_t)_{t \geq 0}, (Y_t)_{t \geq 0}$ be two random walks starting from stationarity. Define $\tilde{Z}$ as the number of collisions between steps 0 and $\ell := \frac{1}{4\|\pi\|^2}$—that is,

$$\tilde{Z} := \sum_{t=0}^{\ell} 1_{X_t=Y_t}.$$ 

Then, $\mathbb{E}[\tilde{Z}] = \frac{1}{4\|\pi\|^2} \cdot \|\pi\|^2 = \frac{1}{4}$. By Markov’s inequality, $\mathbb{P}[\tilde{Z} \geq 1] \leq \frac{1}{4}$, and hence $t^{\pi}_{\text{meet}} \geq \frac{3}{4} \cdot \frac{1}{4\|\pi\|^2} = \frac{3}{16} \cdot \frac{1}{\|\pi\|^2}$. If the graph is $\Gamma$-approximative regular, then the maximum degree $\Delta$ satisfies $\Delta \leq \Gamma \cdot d$, where $d$ is the average degree, and so

$$\|\pi\|^2 = \sum_{u \in V} \left(\frac{\deg(u)}{2|E|}\right)^2 \leq \frac{nd}{\Gamma d} \cdot \left(\frac{\Gamma d}{nd}\right)^2 = \frac{\Gamma}{n}.$$

For the next part of the third statement, let $\overline{Z}$ be the random variable counting the number of collisions between steps 0 and $2t_{\text{mix}}$—that is,

$$\overline{Z} := \sum_{t=0}^{2t_{\text{mix}}-1} 1_{X_t=Y_t}.$$ 

Then,

$$\mathbb{E}[\overline{Z}] = 2t_{\text{mix}} \cdot \|\pi\|^2.$$

To lower bound $\mathbb{E}[\tilde{Z} \mid \overline{Z} \geq 1]$, let us write $\tilde{Z} = \tilde{Z}_1 + \tilde{Z}_2$ with $\tilde{Z}_1 := \sum_{t=0}^{t_{\text{mix}}} 1_{X_t=Y_t}$, and $\tilde{Z}_2 := \sum_{t=t_{\text{mix}}+1}^{t_{\text{mix}}+1} 1_{X_t=Y_t}$. By Observation A.8,

$$\mathbb{P}[\tilde{Z}_1 \geq 1 \mid \overline{Z} = 1] \geq \frac{1}{2}.$$ 

Therefore, by law of total expectation,

$$\mathbb{E}[\overline{Z} \mid \tilde{Z} \geq 1] \geq \frac{1}{2} \cdot \mathbb{E}[\tilde{Z} \mid \overline{Z} \geq 1] \geq \frac{1}{2} \cdot \min_{u \in V} \sum_{t=0}^{t_{\text{mix}}} \sum_{v \in V} (p^t_{u,v})^2 = \frac{1}{2} \cdot C_{\text{min}}.$$

Hence,

$$\mathbb{P}[\tilde{Z} \geq 1] \leq \frac{1}{2} \cdot C_{\text{min}} =: \rho.$$
By the precondition, we have \[ \frac{C_{\text{min}}}{16\|\pi\|_2^2 \cdot t_{\text{mix}}} \geq 1, \] so consider now \[ \left[ \frac{C_{\text{min}}}{16\|\pi\|_2^2 \cdot t_{\text{mix}}} \right] \geq 1 \] consecutive time intervals of length \( 2t_{\text{mix}} \) each. We conclude that if \( B \) denotes the total number of collisions between the walks across all the intervals,

\[
\mathbb{E}[B] \leq \frac{C_{\text{min}}}{16\|\pi\|_2^2 \cdot t_{\text{mix}}} \cdot p \leq \frac{C_{\text{min}}}{8\|\pi\|_2^2 \cdot t_{\text{mix}}} \cdot 2t_{\text{mix}} \cdot \|\pi\|_2^2 = \frac{1}{2}.
\]

Hence, by Markov’s inequality, \( \mathbb{P}[B \geq 1] \leq \frac{1}{2} \) and thus

\[
t_{\text{meet}}^\pi \geq \frac{1}{2} \left[ \frac{C_{\text{min}}}{16\|\pi\|_2^2 \cdot t_{\text{mix}}} \right] \cdot 2t_{\text{mix}} \geq \frac{C_{\text{min}}}{16\|\pi\|_2^2}.
\]

\[ \square \]

It is interesting to compare the upper bound on \( t_{\text{meet}} \) in Theorem 5.1 with the bound \( t_{\text{meet}} = O\left(\frac{1}{1-\lambda_2} \cdot \left(\frac{1}{\|\pi\|_2^2} + \log n\right)\right) \) from Cooper et al. [14, Theorem 2]. Using the trivial bound \( C_{\text{max}} \leq t_{\text{mix}} \) and \( t_{\text{mix}} = O\left(\frac{\log n}{1-\lambda_2}\right) \), we obtain \( t_{\text{meet}} = O\left(\frac{\log n}{1-\lambda_2} \cdot \|\pi\|_2^2\right) \), which is at most a \( \log n \)-factor worse. However, for certain graphs like grids or tori, one may have a better control on the \( t \)-step probabilities so that \( C_{\text{max}} \ll t_{\text{mix}} \) could be established.

**Proposition 5.2.** For any graph \( G = (V, E) \),

\[
t_{\text{coal}} = O\left(\frac{1}{1-\lambda_2} \cdot \frac{1}{\|\pi\|_2^2} + t_{\text{mix}} \cdot \log^3 n\right),
\]

which is at least as good as the bound of Cooper et al. [14, Theorem 1] and equally good if one uses the trivial bound \( t_{\text{mix}} = O\left(\frac{\log n}{1-\lambda_2}\right) \).

**Proof.** We first derive the claimed bound (by a case distinction), then compare it to the bound of Cooper et al. [14, Theorem 1]. The first case is \( t_{\text{meet}} / t_{\text{mix}} \geq \log^2 n \). In this case,

\[
t_{\text{coal}} = O(t_{\text{meet}}) = O\left(\frac{1}{1-\lambda_2} \cdot \left(\frac{1}{\|\pi\|_2^2} + \log n\right)\right),
\]

where the first inequality is by Theorem 1.1 and the second inequality by Cooper et al. [14, Theorem 2]. The second case is \( t_{\text{meet}} / t_{\text{mix}} \leq \log^2 n \), so \( t_{\text{meet}} \leq t_{\text{mix}} \cdot \log^2 n \). By Proposition 3.4, \( t_{\text{coal}} = O(t_{\text{meet}} \log n) = O(t_{\text{mix}} \log^3 n) \). Using \( t_{\text{mix}} = \Omega\left(\frac{1}{1-\lambda_2}\right) \), we can state the following (unconditional) upper bound:

\[
t_{\text{coal}} = O\left(\frac{1}{1-\lambda_2} \cdot \frac{1}{\|\pi\|_2^2} + t_{\text{mix}} \cdot \log^3 n\right).
\]

Using \( t_{\text{mix}} = O\left(\frac{\log n}{1-\lambda_2}\right) \), we derive indeed \( O\left(\frac{1}{1-\lambda_2} \cdot \left(\frac{1}{\|\pi\|_2^2} + \log^4 n\right)\right) \), which is the same bound as that of Cooper et al. [14, Theorem 1]. \[ \square \]

We will make use of the following standard result about return probabilities (related bounds were proven in the work of Aldous and Fill [2, Proposition 6.16 (iii)] and Lyons and Gharan [36, Theorem 4.9].

**Lemma 5.3 ([39, Theorem 1.2]).** Let \( G \) be any regular graph. Then, for any \( t \leq 10n^2 \),

\[ p_{u,u}^t \leq \frac{20}{\sqrt{t+1}}. \]
Theorem 5.4. For any regular graph, we have
\[ t_{\text{hit}} = O\left(\frac{n}{\sqrt{1 - \lambda_2}}\right), \]
and by Cheeger’s inequality, we obtain \( t_{\text{hit}} = O\left(\frac{n}{\Phi}\right) \), where \( \Phi \) is the conductance of \( G \).

For non-regular graphs, the best bound on the hitting time was shown in the work of Oliveira and Peres [39, Theorem 1.1], which states that for any graph \( G \) with average degree \( d \) and minimum degree \( \delta \),
\[ t_{\text{hit}} = O\left(\frac{d \cdot n}{\sqrt{1 - \lambda_2}}\right). \]

Proof. By applying first Lemma 10.2 and then Proposition 10.26 in the work of Levin et al. [33],
\[ t_{\text{hit}} \leq 2 \max_{u \in V} \sum_{v \in V} t_{\text{hit}}(v, u) \pi(v) \]
\[ = \max_{u \in V} \frac{2}{\pi(u)} \sum_{t=0}^{2\lambda_{rel}} \left( p_{t,u,u} - \pi(u) \right) =: \max_{u \in V} f(u). \]

In the following, we will upper bound \( f(u) \) for an arbitrary vertex \( u \in V \). We write \( t_{\text{rel}} = 1/(1 - \lambda_2) \).

For lazy random walks, it holds that for any \( u \in V \) and \( t \geq 1 \), \( p_{t,u,u} \geq \pi(u) \) (Lemma A.1, third statement). Hence, for any \( u \in V \),
\[ f(u) \leq \frac{2}{\pi(u)} \sum_{t=0}^{2\lambda_{rel}} \left( p_{t,u,u} - \pi(u) \right) \]
\[ \leq \frac{O(1)}{\pi(u)} \sqrt{t_{\text{rel}}} + \frac{2}{\pi(u)} 2\lambda_{rel} \sum_{k=0}^{\infty} \left( p_{k,\text{rel}+k\text{rel},u,u} - \pi(u) \right), \]  
where the bound on the first term of the last inequality follows from \( p_{k,\text{rel},u,u} \leq O\left(\frac{1}{\sqrt{t_{\text{rel}}}}\right) \) (Lemma 5.3).

We can bound the sum as follows using that \( \pi(v) = 1/n \), and that for regular graphs for any integer \( r \), it holds that \( p_{u,v}^{2r} = \sum_{v \in V} (p_{u,v}^{2r})^{2} \) as follows:
\[ \sum_{k=0}^{\infty} \left( p_{k,\text{rel}+k\text{rel},u,u} - \frac{1}{n} \right)^{2} = \sum_{k=0}^{\infty} \left( \sum_{v \in V} \left( p_{k,\text{rel}+k\text{rel},v,v} - \frac{1}{n} \right)^{2} \right) \]
\[ \leq \sum_{k=0}^{\infty} \sum_{v \in V} \left( p_{k,\text{rel}+k\text{rel},v,v} - \frac{1}{n} \right)^{2} \]
\[ \leq \sum_{k=0}^{\infty} \left( p_{k,\text{rel}+k\text{rel},u,u} - \frac{1}{n} \right)^{2} \]
\[ \leq \sum_{k=0}^{\infty} \left( p_{k,\text{rel}+k\text{rel},u,u} - \frac{1}{n} \right)^{2} \]
\[ \leq \sum_{k=0}^{\infty} \left( p_{k,\text{rel}+k\text{rel},u,u} - \frac{1}{n} \right)^{2} \]  
\[ \leq \sum_{k=0}^{\infty} \left( p_{k,\text{rel}+k\text{rel},u,u} - \frac{1}{n} \right)^{2} , \]  
\[ \quad \text{for regular graphs} \]
where we have used that for regular graphs, $P$ is symmetric (i.e., $P = P^T$); moreover, (*) follows from the the fact that $\sum_{v \in V} P_{u,v} \cdot 1 = 1$ and (**) is the Binomial theorem. Now using the spectral decomposition of $p^{k \cdot t_{rel}}$ [33, Exercise 12.4], we have

$$\left\| p^{k \cdot t_{rel}} p_{u,u}^{t_{rel}} - \frac{1}{n} \right\|^2 \leq (\lambda_2(p^{k \cdot t_{rel}}))^2 \cdot \left\| p_{u,u}^{t_{rel}} - \frac{1}{n} \right\|^2$$

since due to laziness, all eigenvalues are non-negative, and thus $\lambda_2$ is the second largest eigenvalue of $P$ (and $p^{k \cdot t_{rel}}$) in absolute value. Hence, using that $\sum_{v \in V} p_{u,v}^{t_{rel}} = 1$, we get

$$\sum_{k=0}^{\infty} \left( \frac{2(t_{rel} + k \cdot t_{rel})}{\lambda_2} - \frac{1}{n} \right) \leq \sum_{k=0}^{\infty} \lambda_2^{2k \cdot t_{rel}} \left\| p_{u,u}^{t_{rel}} - \frac{1}{n} \right\|^2$$

$$= \sum_{k=0}^{\infty} \lambda_2^{2k \cdot t_{rel}} \left( \sum_{v \in V} \left( p_{u,v}^{t_{rel}} \right)^2 - 2 \frac{1}{n} \sum_{v \in V} p_{u,v}^{t_{rel}} + \sum_{v \in V} \frac{1}{n^2} \right)$$

$$\leq \sum_{k=0}^{\infty} \lambda_2^{2k \cdot t_{rel}} \frac{2 \cdot t_{rel}}{\lambda_2}$$

$$\overset{(*)}{=} O \left( \sum_{k=0}^{\infty} \lambda_2^{2k \cdot t_{rel}} \frac{1}{\sqrt{t_{rel}}} \right)$$

$$\overset{(**)}{=} O \left( \frac{1}{\sqrt{t_{rel}}} \right),$$

(25)

where (*) follows from $p_{u,u}^{2t_{rel}} \leq O \left( \frac{1}{\sqrt{t_{rel}}} \right)$ (Lemma 5.3) and (**) follows since $g(y) = y^{1/(1-y)}$ is bounded from above by $1/e$ for any $y \in (0,1)$, and hence the sum is a geometric series. Combining (23), (24), and (25) yields the claim.

It turns out that the hitting time bound of Theorem 5.4 is tight in the sense that for any $\Phi$, there exists a graph with conductance $\Phi$ and hitting time and coalescence time of order $\Omega(n/\Phi)$.

**Proposition 5.5 ([11]).** For every $n, d \geq 3$, and constant $\Phi$, there exists a $d$-regular graph $G$ with $n$ nodes and a constant conductance such that the expected coalescing time on $G$ is $\Omega(n)$. Furthermore, for every even $n$, $\Phi > 1/n$, and constant $d$, there exists a $d$-regular graph $G$ with $\Theta(n)$ nodes and a conductance of $\Theta(\Phi)$ such that the meeting time time on $G$ is $\Omega(n/\Phi)$. Therefore, the coalescence time and hitting time are of order $\Omega(n/\Phi)$.

We will now apply hitting time bound of Theorem 5.4 to deduce results for three other quantities.

**Theorem 5.6.** For any regular graph $G$, the following three bounds hold:

$$t_{meet} = O \left( \frac{n}{\sqrt{1 - \lambda_2}} \right),$$

and

$$t_{coal} = O(t_{hit}) = O \left( \frac{n}{\sqrt{1 - \lambda_2}} \right),$$

and

$$t_{cov} = O \left( \frac{n}{\sqrt{1 - \lambda_2}} \cdot \log n \right).$$

The upper bound on $t_{meet}$ and $t_{coal}$ gives $t_{meet} = O(n^2)$ for cycles and paths, and $t_{meet} = O(n)$ on regular expanders (since $1/(1 - \lambda_2) = O(1)$). It thus improves the bound by Cooper et al. [14, Theorem 1], which states that for any regular graph, $t_{meet} = O(n/(1 - \lambda_2))$. For any regular graph,
we also improve the best-known bound on the cover time $t_{\text{cov}}$ in terms of the eigenvalue gap, which is $t_{\text{cov}} = O(n \log n / (1 - \lambda_2))$ established by Broder and Karlin [12].

**Proof.** The proof of the first part follows from Theorem 5.4 and $t_{\text{meet}} \leq 2t_{\text{hit}} + 1$ for regular graphs (Proposition 5.7). The second statement follows by combining the Theorem 5.4 with $t_{\text{cov}} = O(t_{\text{hit}})$ [39]. The third statement follows also from Theorem 5.4 and Matthew’s bound $t_{\text{cov}} = O(t_{\text{hit}} \cdot \log n)$, which holds for any graph $G$. \hfill $\square$

### 5.2 Relating Meeting Time to $t_{\text{hit}}$

In this section, we state the following proposition that relates the meeting time to the hitting time. It can be seen as an analogous version of Aldous and Fill [2, Proposition 14.5] in discrete time. The upper bound is very similar to the bound given by Coppersmith et al. [21, Theorem 2].

**Proposition 5.7.** For any graph $G = (V, E)$ and $u, v \in V$, we have

$$
(\min_{u' \in V} t_{\text{hit}}(\pi, u') + (t_{\text{hit}}(u, v) - t_{\text{hit}}(\pi, v))) / 2 \leq t_{\text{meet}}(u, v)
$$

$$
\leq \max_{u', v \in V} (t_{\text{hit}}(u, v) - t_{\text{hit}}(\pi, v)) + \max_{u' \in V} t_{\text{hit}}(\pi, u') + 1.
$$

Consequently, for any graph, we have $t_{\text{meet}} \leq 2t_{\text{hit}} + 1$, and for any vertex-transitive graph $G$, we have $t_{\text{hit}} / 2 \leq t_{\text{meet}} \leq t_{\text{hit}} + 1$.

**Proof.** We define a pair of chains $((X_t)_{t \geq 0}, (Y_t)_{t \geq 0})$ with arbitrary start vertices $X_0, Y_0 \in V$, called **sequential** random walks, by

$$
X_{t+1} = \begin{cases} v \in N(X_t) \text{ w.p. } \frac{1}{2|N(X_t)|} & \text{if } t \text{ is even} \\ X_t & \text{otherwise} \end{cases}
$$

and

$$
Y_{t+1} = \begin{cases} v \in N(Y_t) \text{ w.p. } \frac{1}{2|N(Y_t)|} & \text{if } t \text{ is odd} \\ Y_t & \text{otherwise} \end{cases}
$$

In particular, for odd $t$ (even $t$, respectively) the random walk is lazy, meaning $X_{t+1} = X_t$ (and $Y_{t+1} = Y_t$, respectively).

Consider two “non-sequential” random walks $(X'_t)_{t \geq 0}$ and $(Y'_t)_{t \geq 0}$ with $X'_0 = X_0$ and $Y'_0 = Y_0$. We will couple their decisions with the walks $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$, by setting $X'_t = X_{2t}$ and $Y'_t = Y_{2t}$ for all integers $t \geq 0$. Due to this coupling and since each random walk is lazy w.p. $1/2$,

$$
\mathbb{P} \left[ X'_t = Y'_t \mid X_{2t} = Y_{2t} \right] = 1
$$

and

$$
\mathbb{P} \left[ X'_{t+1} = Y'_{t+1} \mid X_{2t+1} = Y_{2t+1} \right] = \mathbb{P} \left[ X_{2t+2} = Y_{2t+2} \mid X_{2t+1} = Y_{2t+1} \right] = 1/2.
$$

Let $t_{\text{seq}}(u, v)$ be the meeting time of the sequential chains $X_t$ and $Y_t$—that is,

$$
t_{\text{seq}}(u, v) = \min \{t \mid X_t = Y_t, X_0 = u, Y_0 = v\}
$$

and $t_{\text{seq}} = \max_{u, v} t_{\text{seq}}(u, v)$. We seek to relate $t_{\text{meet}}$ and $t_{\text{seq}}$. Clearly, for any pair $u, v \in V$, $t_{\text{seq}}(u, v) / 2 \leq t_{\text{meet}}(u, v)$ since a meeting of $X'_t = Y'_t$ implies that $X_{2t} = Y_{2t}$.

For an upper bound on $t_{\text{meet}}$, recall that $X'_t$ and $Y'_t$ meet—that is, $X'_t = Y'_t$ w.p. at least $1/2$ whenever $X_{2t} = Y_{2t}$ or $X_{2t-1} = Y_{2t-1}$. Hence, we can consider intervals of length $t_{\text{meet}}$. In each of them, we have, independent of what happened in previous intervals, that the random walks meet w.p. at least $1/2$. Thus, $t_{\text{meet}}(u, v) \leq 2(t_{\text{seq}} + 1)/2 = t_{\text{seq}} + 1$. We conclude,

$$
t_{\text{seq}}(u, v) / 2 \leq t_{\text{meet}}(u, v) \leq t_{\text{seq}} + 1.
$$

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We proceed by deriving a lower bound on $t^\text{seq}_{\text{meet}}(u,v)$ and an upper bound on $t^\text{seq}_{\text{meet}}(u,v)$, which gives us the desired bounds on $t_{\text{meet}}$. We will make use of the following statement that is a weaker version of the original statement by Aldous and Fill [2, Proposition 3.3]. For all $u, v \in V$, we have

$$\min_{u' \in V} t_{\text{hit}}(\pi, u') \leq t^\text{seq}_{\text{meet}}(u,v) - (t_{\text{hit}}(u,v) - t_{\text{hit}}(\pi, v)) \leq \max_{u' \in V} t_{\text{hit}}(\pi, u').$$

Using (26), we derive

$$t_{\text{meet}}(u,v) \leq t^\text{seq}_{\text{meet}}(u,v) + 1 \leq \max_{u, v \in V} (t_{\text{hit}}(u,v) - t_{\text{hit}}(\pi, v)) + \max_{u' \in V} t_{\text{hit}}(\pi, u') + 1$$

and

$$t_{\text{meet}}(u,v) \geq t^\text{seq}_{\text{meet}}(u,v)/2 \geq (\min_{u' \in V} t_{\text{hit}}(\pi, u') + (t_{\text{hit}}(u,v) - t_{\text{hit}}(\pi, v))) / 2,$$

which yields the first part of the claim. This also implies that $t_{\text{meet}} \leq 2t_{\text{hit}} + 1$. In the remainder of the proof, we focus on vertex-transitive chains. By vertex transitivity, we have $t_{\text{hit}}(\pi, u) = t_{\text{hit}}(\pi, u')$ for all $u, u' \in V$, and thus using (26),

$$t^\text{seq}_{\text{meet}}(u,v) = t_{\text{hit}}(u,v).$$

Thus, putting everything together and fixing $u, v \in V$ to be the nodes maximizing $t_{\text{hit}}(u,v)$, we derive

$$t_{\text{hit}} = t_{\text{hit}}(u,v) = t^\text{seq}_{\text{meet}}(u,v) \leq t^\text{seq}_{\text{meet}} \leq 2t_{\text{meet}}.$$

Similarly, letting $u, v \in V$ be the nodes maximizing $t^\text{seq}_{\text{meet}}(u,v)$, we derive

$$t_{\text{meet}} \leq t^\text{seq}_{\text{meet}} + 1 = t^\text{seq}_{\text{meet}}(u,v) + 1 = t_{\text{hit}}(u,v) + 1 \leq t_{\text{hit}} + 1.$$

This yields Proposition 5.7. \hfill \square

5.3 Proof of Theorem 1.3

The proof follows from Theorems 5.4 and 5.6.

6 SPECIFIC GRAPHS

We present bounds on the coalescence time for specific graphs.

6.1 d-Dimensional Grids and Tori, $d > 2$

Here, the bounds on $t_{\text{meet}}$ and $t_{\text{coal}}$ follow immediately from our general results. First, recall that for any regular (or almost-regular) graph, we have $t_{\text{meet}} = \Omega(n)$ (Theorem 5.1.iii). Combining this with the fact that $t_{\text{mix}} = O(n^{2/d})$ [2], our Theorem 1.1 yields the bound $t_{\text{coal}} = \Theta(t_{\text{meet}})$. Further, it is a well-known result that $t_{\text{hit}} = O(n)$ (e.g., [33]), and using $t_{\text{meet}} \leq 2t_{\text{hit}} + 1$ (Proposition 5.7) gives $t_{\text{meet}} = O(n)$, and hence also $t_{\text{coal}} = O(t_{\text{meet}}) = O(n)$. Let us point out that one could also use the more recent result $t_{\text{coal}} = O(t_{\text{hit}})$ [39] to deduce that $t_{\text{coal}} = O(n)$ without appealing to the meeting time.

6.2 Hypercubes

First, for any regular graph, we again have $t_{\text{meet}} = \Omega(n)$ (Theorem 5.1.iii). Furthermore, since $t_{\text{meet}} \leq 2t_{\text{hit}} + 1$ and using that $t_{\text{hit}} = O(n)$ (e.g., see [35]), we get $t_{\text{meet}} = O(n)$. It is a well-known fact that $t_{\text{mix}} = O(\log n \cdot \log \log n)$ [33], so applying Theorem 1.1 yields $t_{\text{coal}} = \Theta(t_{\text{meet}}) = \Theta(n)$.

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APPENDIX

A BASIC RESULTS ABOUT MARKOV CHAINS

We will frequently use the following basic fact about lazy random walks, which in fact also holds for arbitrary reversible Markov chains. Here, we recall that a non-lazy random walk moves at each step to a neighbor chosen uniformly at random. A lazy random walk stays at the current vertex w.p. 1/2 and otherwise moves to a neighbor chosen uniformly at random.

**Lemma A.1** (cf. [33, Equation 12.13]). Let $P$ be the transition matrix of a reversible Markov chain with state space $\Omega$. Then, the following statements hold: if $P$ is irreducible, then for any two states $x, y \in \Omega$, 

$$p^t_{x,y} \leq \pi(y) + \sqrt{\frac{\pi(y)}{\pi(x)} \cdot \lambda^t},$$

where $\lambda := \max(\lambda_2, |\lambda_n|)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the $n$ real eigenvalues of the matrix $P$.

Note that the $t$-step distribution $q^t$ of a non-lazy random walk can be expressed by $q^t = q^{t-1} \cdot Q$, with transition matrix $Q := D^{-1} \cdot A$.

Next define the normalized Laplacian matrix by

$$L = I - D^{-1/2} AD^{-1/2} = I - \tilde{A},$$

where $\tilde{A} = D^{1/2} Q D^{-1/2}$. Hence,

$$D^{-1/2} \tilde{A} = Q D^{-1/2}.$$

It is easy to verify that the eigenvalues of $\tilde{A}$ and $Q$ are the identical. Indeed, if $v$ satisfies $\tilde{A} v = \lambda v$, then

$$D^{-1/2} \tilde{A} v = D^{-1/2} \lambda v = \lambda (D^{-1/2} v),$$

but also

$$(D^{-1/2} \tilde{A}) v = (Q D^{-1/2}) v = Q (D^{-1/2} v).$$

Hence, $\tilde{A}$ and $Q$ have the set of same eigenvalues $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1$. Further, let $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of $L$. Let $\mu := \max \{|1 - \mu_2|, |1 - \mu_n|\}$.

**Lemma A.2** ([13, Theorem 1.2, (2)]). Let $G$ be a graph sampled from $G(n, p)$ with $p = \omega(\log n/n)$ and $\bar{d} = (n - 1) \cdot p$. Then, w.p. $1 - o(1)$,

$$\mu = O \left( \frac{1}{\sqrt{\bar{d}}} \right).$$

By definition of the normalized Laplacian, we have the relation

$$\lambda_i = 1 - \mu_i,$$

and hence the conclusion of the preceding lemma implies

$$\max \{|\lambda_2|, |\lambda_n|\} = O \left( \frac{1}{\sqrt{\bar{d}}} \right).$$

**Corollary A.3.** Let $n_0$ be a sufficiently large constant. Let $H_{n^*}$ be a $G(n^*, p)$ graph with $p \geq (n^*)^{-c}$ for some constant $0 < c < 1$. Then, there exists a constant $C = C(c) > 0$ such that w.p. $1 - o(1)$, the lazy random walk on $G(n^*, p)$ satisfies $\max_{n^* \geq n_0} \{t_{\text{sep}}(H_{n^*}), t_{\text{mix}}(H_{n^*})\} \leq C$. 

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Proof. Using a Chernoff bound, it follows that for the random graph $H_n^*$, the following two conditions hold: $\Delta \leq (1 + o(1))p \cdot n^*$, $\delta \geq (1 - o(1))p \cdot n^*$. Further, by Lemma A.2,
\[
\max \{\lambda_2, |\lambda_n^*|\} = O(1/d).
\]
Hence, by Lemma A.1, it is easy to see that for $t \geq c_1$ being a sufficiently large constant $c_1 > 0$, for all pairs of vertices $x, y \in V$,
\[
|q_{x,y}^t - \pi(y)| \leq 1/(n^*)^2.
\]
Recall $p_{x,.}^t$ is $t$-step distribution of a lazy walk, and $d(t) = \max_{u,v} \|p_{u,.}^t - \pi\|_{TV}$, as well as $\tilde{d}(t) = \max_{u,v} \|p_{u,.}^t - p_{u,.}^t\|_{TV}$. By conditioning on the number of loops taken by a $t$-step random walk,
\[
p_{x,.}^t = \sum_{s=0}^{t} \binom{t}{s} 2^{-t} q_{x,.}^s.
\]
Further, by the triangle inequality,
\[
\|p_{x,.}^t - \pi\|_{TV} \leq \sum_{s=0}^{t} a_s \cdot \|q_{x,.}^s - \pi\|_{TV}.
\]
By choosing $t = c_2$ for another constant $c_2 > c_1$, by using a Chernoff bound for a Binomial random variable, it follows that
\[
\sum_{s=0}^{c_2-1} a_s = \mathbb{P} [ \text{Bin}(c_2, 1/2) \leq c_1 ] \leq 1/e^2.
\]
Applying this to the previous inequality yields
\[
d(t) = \|p_{x,.}^t - \pi\|_{TV} \leq \sum_{s=0}^{c_2-1} a_s \cdot 1 + \max_{s \geq c_2} \|p_{x,.}^s - \pi\|_{TV}
\leq 1/e^2 + 2n^* \cdot 1/(n^*)^2 < 1/(2e),
\]
for sufficiently large $n^*$. By Levin et al. [33, Lemma 4.10], we have $\tilde{d}(t) \leq 2d(t)$, and hence $t_{\text{mix}}(H_n^*) \leq c_2$ and $t_{\text{sep}}(H_n^*) \leq 4t_{\text{mix}}(H_n^*) \leq 4c_2$, which completes the proof. \hfill $\Box$

The following lemma will be helpful to define a coupling between distributions that are close to the stationary distribution and the exact stationary distribution. (A very similar lemma has been derived in the work of Elsässer and Sauerwald [27, Lemma 2.8].)

**Lemma A.4.** Let $\varepsilon \in (0, 1]$ be an arbitrary value. Let $Z_1$ and $Z_2$ be two probability distributions over $\{1, \ldots, n\}$ so that $\mathbb{P} [ Z_1 = i ] \geq \varepsilon \cdot \mathbb{P} [ Z_2 = i ]$ for every $1 \leq i \leq n$. Then, there is a coupling $(\tilde{Z}_1, \tilde{Z}_2)$ of $(Z_1, Z_2)$ and an event $E$ with $\mathbb{P} [ E ] \geq \varepsilon$ so that
\[
\mathbb{P} [ \tilde{Z}_1 = i \mid E ] = \mathbb{P} [ \tilde{Z}_2 = i ] \quad \text{for every} \quad 1 \leq i \leq n.
\]

**Proof.** Let $U \in [0, 1]$ be a uniform random variable. We next define our coupling $(\tilde{Z}_1, \tilde{Z}_2)$ of $Z_1$ and $Z_2$ that will depend on the outcome of $U$. First, if $U \in [0, \varepsilon)$, then we set
\[
\tilde{Z}_1 = \tilde{Z}_2 = i, \quad \text{if} \quad i \text{ satisfies } \varepsilon \sum_{k=1}^{i-1} \mathbb{P} [ Z_2 = k ] \leq U < \varepsilon \sum_{k=1}^{i} \mathbb{P} [ Z_2 = k ].
\]
For the case where $U \in (\varepsilon, 1)$, it is clear that the definition of $U$ can be extended in a way so that $\tilde{Z}_1$ has the same distribution as $Z_1$ and $\tilde{Z}_2$ has the same distribution as $Z_2$. Furthermore, notice that if $U \in [0, \varepsilon)$ happens, then $\tilde{Z}_1$ has the same distribution as $Z_2$, and $\tilde{Z}_1 = \tilde{Z}_2$. Observing that $\mathbb{P} [ U \in [0, \varepsilon) ] = \varepsilon$ completes the proof. \hfill $\Box$
The following lemma is an immediate consequence of Lemma A.4.

**Lemma A.5.** Consider a random walk \((X_t)_{t \geq 0}\), starting from an arbitrary but fixed vertex \(x_0\). Then, there is an event \(E\) with \(P[\ E\ ] \geq 1 - 1/e\) such that for every \(v \in V\),

\[
P[X_{4t_{\max}} = v | E] = \pi(v).
\]

**Proof.** Consider the random walk \((X_t)_{t \geq 0}\) after step \(s := t_{\text{sep}} \leq 4t_{\text{mix}}\). By definition of \(t_{\text{sep}}\), \(p_{u,v}^s \geq (1 - 1/e)\pi(v)\). Applying Lemma A.4, where \(Z_t\) is the distribution given by \(p_{u,v}^t\), and \(Z_s\) is the stationary distribution shows that w.p. at least \(1 - 1/e\), \(X_s\) has the same distribution as \(\pi\). If this is the case, then the same holds for \(X_{4t_{\max}}\) as well. \(\square\)

The preceding lemma shows that for \(t_{\text{meet}}\) and \(t_{\text{coal}}\), it suffices to consider the following stationary case.

**Lemma A.6.** For any graph \(G\),

\[
\max \left\{ \frac{(1/e)t_{\text{mix}}}{t_{\text{meet}}} \right\} \leq t_{\text{meet}} \leq \frac{2}{(1 - 1/e)^2} \cdot (4t_{\text{mix}} + 2t_{\text{meet}}),
\]

and similarly, \(t_{\text{coal}} \leq 4 \cdot (4t_{\text{mix}} + 2t_{\text{coal}})\), where \(t_{\text{coal}}^\pi\) is the expected coalescence time when all \(n\) walks start at positions drawn independently from the stationary distribution \(\pi\).

**Proof.** We begin by proving the lower bound on \(t_{\text{meet}}\). First, consider a coupling of two random walks, \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) that are run for \(t = e \cdot t_{\text{meet}}\) timesteps and start from arbitrary vertices \(x_0\) and \(y_0\). As soon as the two random walks are at the same vertex for some time \(t\), all future transitions are identical, i.e., \(X_t = Y_t\) for all \(t \geq s\). Then, we have

\[
d(t) = \max_{u,v} \left\| p_{u,v}^t - p_{v,u}^t \right\|_{TV} \leq P [ \cap_{s=0}^{t} \{ X_s \neq Y_s \} ] \leq \frac{1}{e},
\]

where the first inequality is due to the coupling method [33, Equation 5.5] and the second inequality follows by Markov’s inequality. The preceding inequality implies \(t_{\text{mix}} \leq e \cdot t_{\text{meet}}\). Furthermore, \(t_{\text{meet}} \leq t_{\text{mix}}\) holds by definition, and the lower bound follows.

For the upper bound, we divide the two random walks into consecutive epochs of length \(\ell := 4t_{\text{mix}} + 2t_{\text{meet}}\). For the statement, it suffices to prove that in each such epoch, regardless of the start vertices of the two random walks, a meeting occurs w.p. at least \((1 - 1/e)^2 \cdot 1/2\).

Consider the first random walk \((X_t)_{t \geq 0}\) starting from an arbitrary vertex after \(s := 4t_{\text{mix}}\) steps. By Lemma A.5, we obtain that w.p. at least \(1 - 1/e\), the distribution of \(X_s\) is equal to that of a stationary random walk. Similarly, we obtain that w.p. at least \(1 - 1/e\), the distribution of \(Y_s\) is equal to that of a stationary distribution. Hence, w.p. \((1 - 1/e)^2\), \(X_s\) and \(Y_s\) are drawn independently from the stationary distribution. In this case, it follows by Markov’s inequality that the two random walks meet before step \(s + 2t_{\text{meet}}\) w.p. at least \(1/2\). Overall, we have shown that w.p. at least \((1 - 1/e)^2 \cdot 1/2\), a meeting occurs in a single epoch. Since this lower bound holds for every epoch, independent of the outcomes in previous epochs, the upper bound on the expected time \(t_{\text{meet}}\) follows. The upper bound on \(t_{\text{coal}}\) in terms of \(t_{\text{coal}}^\pi\) is shown in exactly the same way. \(\square\)

**Lemma A.7.** For a lazy random walk on an \(n\)-vertex graph with \(n \geq 2\), we have \(t_{\text{hit}} \geq \frac{2}{\pi_{\min}} - 2\). In particular, for \(n \geq 2\), we have \(t_{\text{hit}} \geq \frac{1}{\pi_{\min}} \geq n\).

Note that \(t_{\text{hit}} \geq \frac{2}{\pi_{\min}} - 2\) is tight in the sense that the hitting time of the clique is indeed \(2(n-1) = \frac{2}{\pi_{\min}} - 2\) since the random walk moves w.p. \(1/2\) and when it moves the probability to hit the target node is \(1/(n-1)\) (assuming that the random walk is not on the target node).
Proof. Let \( u \) be a vertex attaining \( \pi_{\text{min}} = \pi(u) \). Consider the random walks \( (X_t)_{t \geq 0} \) starting at \( u \). Then it is well known (cf. [2]) that for the first return \( \tau^+(u, u) := \min \{ t > 0 : X_t = u, X_0 = u \} \), we have \( \mathbb{E}[\tau^+(u, u)] = 1/\pi(u) = 1/\pi_{\text{min}} \). By conditioning on the first step of the random walk, we obtain
\[
\frac{1}{\pi_{\text{min}}} = \mathbb{E}[\tau^+(u, u)] = 1 + \frac{1}{2} \cdot 0 + \frac{1}{2} \sum_{v \in N(u)} \frac{1}{\deg(u)} \cdot t_{\text{hit}}(v, u),
\]
and rearranging yields
\[
\frac{1}{\deg(u)} \cdot \sum_{v \in N(u)} t_{\text{hit}}(v, u) = \frac{2}{\pi_{\text{min}}} - 2.
\]
Now, by the pigeonhole principle, there exists a vertex \( v \in N(u) \) with \( t_{\text{hit}}(v, u) \geq \frac{2}{\pi_{\text{min}}} - 2 \), and the first claim follows. The second part follows from observing that if \( n \geq 2 \), we have \( \pi_{\text{min}} \leq 1/2 \), and thus \( t_{\text{hit}} \geq \frac{2}{\pi_{\text{min}}} - 2 \geq \frac{1}{\pi_{\text{min}}} \geq n \), where the last inequality follows from the simple pigeonhole principle.

Observation A.8. Consider two random walks \( (X_t)_{t \geq 0} \) and \( (Y_t)_{t \geq 0} \) starting on nodes drawn from the stationary distribution. Fix an arbitrary \( t \in \mathbb{N} \). Define the collision-counting random variables 
\[
Z_1 = \sum_{i=0}^{[t/2]} 1_{X_i = Y_i}, \quad Z_2 = \sum_{i=[t/2]+1}^{t} 1_{X_i = Y_i}, \quad \text{and} \quad Z = Z_1 + Z_2.
\]
Then, \( \mathbb{P}[Z_1 > 1 | Z \geq 1] \geq \frac{1}{2} \).

Proof. Since both nodes start from the stationary distribution, \( \mathbb{P}[Z_1 > 1] > \mathbb{P}[Z_2 > 1] \). By the union bound, \( \mathbb{P}[Z > 1] \leq \mathbb{P}[Z_1 > 1] + \mathbb{P}[Z_2 > 1] \leq 2 \cdot \mathbb{P}[Z_1 > 1] \). By law of total probability, \( \mathbb{P}[Z_1 > 1] = \mathbb{P}[Z_1 > 1 | Z \geq 1] \cdot \mathbb{P}[Z \geq 1] \). Putting everything together yields \( \mathbb{P}[Z_1 > 1 | Z \geq 1] \geq \frac{1}{2} \).

Lemma A.9. Let \( \mathcal{F}_{t-1} \) be the filtration at time \( t-1 \).\(^{12}\) Let \( (X_t)_{t \geq 0} \) be a stochastic process satisfying (i) \( \mathbb{E}[X_t | \mathcal{F}_{t-1}] \leq \beta \cdot X_{t-1} \), for some \( \beta < 1 \), and (ii) \( X_t \geq \epsilon, \text{ for all } t \geq 0 \). Let \( \tau(g) := \min \{ t \geq 0 : X_t \leq g \} \) for \( g \in (0, \alpha_0) \), then
\[
\mathbb{E} \{ \tau(g) \} \leq 2 \cdot [\log_\beta(g/(2X_0))].
\]

Proof. By the iterative law of expectation, we have
\[
\mathbb{E}[X_t] \leq \beta^t \cdot X_0.
\]
Furthermore, by Markov’s inequality, for any \( \lambda \geq 1 \),
\[
\mathbb{P} \{ \tau(g) > \lambda \cdot [\log_\beta(g/(2X_0))] \} \leq \mathbb{E} \left[ \frac{X_{\lambda \cdot [\log_\beta(g/(2X_0)]}}{g} \right] \leq \frac{\beta^{\lambda} \cdot [\log_\beta(g/(2X_0))]}{g} \leq 2^{-\lambda}.
\]
Therefore,
\[
\mathbb{E}[\tau(g)] = \sum_{i=1}^{\infty} \mathbb{P} \{ \tau(g) > i \} \leq [\log_\beta(g/(2X_0))] + \sum_{\lambda=1}^{\infty} [\log_\beta(g/(2X_0))] \cdot \mathbb{P} \{ \tau(g) > \lambda \cdot [\log_\beta(g/(2X_0))] \} \leq 2 \cdot [\log_\beta(g/(2X_0))].
\]

\(^{12}\) Informally speaking, is the history of all random decisions up to time \( t \).

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