Parametric Modeling and Contact Analysis of Skew Spiral Bevel Gears

WANG Baichao1, ZHANG Litong1(corresponding author), Yu Xinping2, Zhao Xiongfei3, Zhang Hongming3, Wang Jun3, Zhang Xuzhong3

1Changchun University of Science and Technology, Changchun 130022, China
2Changchun Institute of Equipment and Engineering, Changchun 130021, China
3Inner Monggolia Fist Machinery Group Co., LTD

Abstract: According to the principle of spherical involute generation, the mathematical model of skew spiral bevel gear tooth flank is derived, according to the mathematical model of the tooth flank, a solid model of the bevel gear is constructed, and then the meshing equation is derived by regulation that the relative motion velocity vector is perpendicular to the common normal line at the meshing point for a pair of gears. Finally, the contact line of skew spiral bevel gears meshing is analyzed, and the relative relationship between the meshing contact line and the tooth flank generation line are derived to provide a theoretical model for design, machining and assembly of gear.

1. Introduction

Bevel gears as important driving element of transfer intersect axis motion and power drive are widely used for power machinery in the field of vehicles, aerospace, petroleum, metallurgy, etc. Skew bevel gears have many advantages such as high strength, good gear tooth coincident coefficient, smooth operation, high bearing capacity [1-2]. As a kind of bevel gears the skew bevel gears have all the advantages of bevel gears. With the development of computer technology, the modeling and simulation of bevel gears is researched more and more. The establishment of an accurate bevel gear tooth flank model is the basis and prerequisite for product design, meshing analysis and dynamic analysis [3].

Many universities and research institutions at home and abroad have conducted extensive research on the establishment of bevel gear model, but most of modeling research for tooth flank are based on Gleason gear machining methods [4]. The tooth flank model is constructed according to the four boundary information points of the tooth flank. Ji Zhenhai and others through researched on the involute bevel gear starting with the meshing law of tooth profile, a tooth flank model of a spiral bevel gear that is independent of the machining method is proposed [5-8].

This paper analyzes the forming principle of spherical involute skew bevel gears, and through the principle of coordinate transformation to establish a tooth flank parameter model of involute and generation lines. Then, the meshing equation is deduced according to the law that the relative motion velocity vector is perpendicular to the common normal at the meshing point for a pair of gears. Finally, the skew spiral bevel gears meshing contact line is analyzed to derive the relative position between the meshing contact line and the tooth flank. The relationship provides a theoretical model for gear design, machining and assembly, then the meshing equation is derived by regulation that the relative motion velocity vector is perpendicular to the common normal line at the meshing point for a pair of gears.
2. The mathematical model of tooth flank

In this paper, the generative principle of spherical involute and the base circle plane relative to the base cone in the tooth flank generation process are used for relatively pure rolling, the spherical involute helicoidal surface is formed by the curve relative rolling of basic circle plane that is basic kinematic relations, through the principle of coordinate transformation, the tooth flank parameter model of involutes and generation lines are established.

As shown in Fig.1, the base cone angle of the gear is $\delta_b$, the element length of base cone is $R_b$, the generating angle of small circle (base cone) is $\phi$, the generating angle of the circle plane $Q$ is $\gamma$, the coordinate system of tooth blank $S$ (O-X, Y, Z) is fixedly connected with the tooth blank, the coordinate system $S_q$(O-Xq,Yq, Zq) is fixedly connected with the surface $Q$, the Zq is along the direction of the base cone element, and the circle Q on the plane Xq-Zq.

The circle center coincides with the base cone vertex on the base circle plane $Q$, and remains tangent to the base cone. When the plane $Q$ is generated relative to the base cone, the tooth flank of skew spiral bevel gears are formed by the sweep path of the oblique straight line $ML$ on the plane $Q$.

![Fig.1 Coordinate transformation of tooth flank](image)

The straight line $ML$ on the plane $Q$ that generates the skew bevel gears tooth flank is the generation line of the left-handed gear tooth flank. As shown in Fig.1, the parametric equation of the generation line $ML$ in the coordinate system $S_q$ is:

$$
\begin{align*}
  x_{q} &= x_{q}(\cos\gamma\cos\phi + \sin\gamma\sin\phi\sin\delta_b) + y_{q}\sin\phi\cos\delta_b \\
  y_{q} &= 0 \\
  z_{q} &= \rho\sin\theta
\end{align*}
$$

When the base cone counterclockwise rotates around the axis Z, the left-hand left tooth flank is generated by the circle plane $Q$ that clockwise rotates around the $y_q$ axis at a certain speed ratio, the coordinate equation is used to transform the above linear equation into the coordinate system $S$. The tooth flank equation is:

$$
\begin{align*}
  x_{u} &= x_{u}(\cos\gamma\cos\phi + \sin\gamma\sin\phi\sin\delta_b) + y_{u}\sin\phi\cos\delta_b \\
  y_{u} &= x_{u}(\cos\gamma\sin\phi + \sin\gamma\cos\phi\sin\delta_b) + y_{u}\cos\phi\cos\delta_b \\
  z_{u} &= x_{u}\sin\gamma\sin\delta_b - y_{u}\sin\delta_b + z_{q}\cos\gamma\cos\delta_b
\end{align*}
$$

The matrix $M_{ll}$ transferred from the coordinate system $S_q$(Oq-xq,yq,zq) to the tooth blank coordinate system $S$(O-x,y,z) that is a transformation matrix of two coordinate systems, namely:

$$
M_{ll} = \begin{bmatrix}
\cos\gamma\cos\phi + \sin\gamma\sin\phi\sin\delta_b & \sin\phi\cos\delta_b & -\sin\gamma\cos\phi + \cos\gamma\sin\phi\sin\delta_b \\
-\cos\gamma\sin\phi + \sin\gamma\cos\phi\sin\delta_b & \cos\phi\cos\delta_b & \sin\gamma\sin\phi + \cos\gamma\cos\phi\sin\delta_b \\
\sin\gamma\cos\delta_b & -\sin\delta_b & \cos\gamma\cos\delta_b
\end{bmatrix}
$$
Then, during generation line polar radius and spherical involute of the tooth flank are generating, the tooth flank vector equation of base cone angle \( \Phi \) on the left-hand left tooth is:

\[ r_n(\rho, \phi) = x_n(\rho, \phi)i + y_n(\rho, \phi)j + z_n(\rho, \phi)k \]  

(4)

3. Meshing contact analysis

The transmission of the spherical involute spiral bevel gears belongs to the category of space meshing. When the gear pair is meshing transmission, the two gears rotate around their respective axis with certain speed ratio, and so it belong to a single degree of freedom meshing. When the two gears are meshing with each other, the axial surface is tangent to the pitch cone with two purely rolling of two gears, and the pitch cone vertex of the two bevel gears intersect at one point. The meshing coordinate system is shown in Fig.2.

![Fig.2 Meshing analysis of space](image)

It is assumed that the contact point of the two tooth flanks is \( p \) during the meshing process of gear, and the common normal is \( n \) that pass through this point and tangent to tangent plane. In order to ensure the two tooth flanks can be meshed correctly and continuous without interference, it is necessary to meet the relative velocity vector of the contact point and the common normal line \( n \) of this point are perpendicular to each other, the meshing equation of the two gears is as follows:

\[ v^{(12)} \cdot n = 0 \]

(5)

In the formula, \( v^{(12)} \) is the relative velocity vector of two tooth flanks contact point, and \( n \) is the common normal vector of the two tooth flank contact points.

3.1 Relative speed solution

In Fig.4, \( S(o-x,y,z) \) and \( S_p(o_p-x_p,y_p,z_p) \) are space-fixed coordinate systems, which are the initial position coordinate system of gear 1 and gear 2, \( S_1(o_1-x_1,y_1,z_1) \) is the fixed coordinate system of gear 1, the coordinate system \( S_1(o_1-x_1,y_1,z_1) \) is obtained by gear 1 that rotating \( \Phi_1 \) angle around its axis with angular speed \( \omega_1 \), and in the same way that \( S_2(o_2-x_2,y_2,z_2) \) is the fixed coordinate system of gear 2, the coordinate system \( S_2(o_2-x_2,y_2,z_2) \) is obtained by gear 2 rotating \( \Phi_2 \) angle around its axis with angular speed \( \omega_2 \). The coordinate conversion formula is as follows:

The coordinate transformation matrix from \( S_1 \) to \( S \) is:

\[ M_{1i} = \begin{bmatrix} \cos \Phi_1 & -\sin \Phi_1 & 0 & 0 \\ \sin \Phi_1 & \cos \Phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

(6)

The coordinate transformation matrix from \( S_2 \) to \( S_p \) is:
The coordinate transformation matrix from Sp to S is:

\[
M_{sp} = \begin{bmatrix}
\cos \phi_1 & -\sin \phi_1 & 0 & 0 \\
\sin \phi_1 & \cos \phi_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\tag{7}
\]

The coordinate transformation matrix from S2 to S1 is:

\[
M_{12} = \begin{bmatrix}
\cos \phi_2 & -\sin \phi_2 & 0 & 0 \\
\cos \Sigma \sin \phi_2 & \cos \Sigma \cos \phi_2 & \sin \Sigma & 0 \\
-\sin \Sigma \sin \phi_2 & -\sin \Sigma \cos \phi_2 & \cos \Sigma & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\tag{8}
\]

The rotational angular velocity vectors of the two gears are:

\[
\omega^{(1)} = -\omega_1 \mathbf{k}
\]
\[
\omega^{(2)} = \omega_p \mathbf{k}_p
\]
\[
= \omega_2 \sin \Sigma \mathbf{j} + \omega_2 \cos \Sigma \mathbf{k}
\tag{10}
\]

In the coordinate system S, the vector equation of the contact point p is:

\[
\overrightarrow{op} = r^{(1)} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}
\tag{11}
\]

The vector equation of two gears contact point p is respectively represented as:

\[
v^{(1)} = \omega^{(1)} \times r^{(1)}
\]
\[
= -\omega_1 \mathbf{k} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})
\]
\[
= -\omega_1 y\mathbf{i} + \omega_1 x\mathbf{j}
\tag{12}
\]

\[
v^{(2)} = \omega^{(2)} \times r^{(2)}
\]
\[
= (\omega_2 \sin \Sigma \mathbf{j} + \omega_2 \cos \Sigma \mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})
\]
\[
= (-y\omega_2 \cos \Sigma + z\omega_2 \sin \Sigma) \mathbf{i} + x\omega_2 \cos \Sigma \mathbf{j} - x\omega_2 \sin \Sigma \mathbf{k}
\tag{13}
\]

In this paper, the included angle between two gears axis are calculated by 90° unless otherwise specified, so the relative speed of the two gears at point p is:

\[
v^{(12)} = v^{(1)} - v^{(2)}
\]
\[
= (y\omega_2 \cos \Sigma - z\omega_2 \sin \Sigma) \cdot \mathbf{i} + (x\omega_1 - x\omega_2 \cos \Sigma) \cdot \mathbf{j} + x\omega_1 \sin \Sigma \cdot \mathbf{k}
\]
\[
= (-y\omega_1 - z\omega_2) \cdot \mathbf{i} + (x\omega_1) \cdot \mathbf{j} + x\omega_2 \cdot \mathbf{k}
\tag{14}
\]

3.2 Solving the equation of the common normal line

According to the previously derived tooth flank equation, the gear surface equation of gear 2 in the coordinate system S2 can be derived. Take the left-hand gear left tooth flank as an example, the tooth flank equation \( \Sigma_2 \) is:

\[
r_2^{(2)}(\rho, \phi) = x_2(\rho, \phi)\mathbf{i}_2 + y_2(\rho, \phi)\mathbf{j}_2 + z_2(\rho, \phi)\mathbf{k}_2
\tag{15}
\]

The common normal vector equation \( n^{(2)} \) at the contact point of the gear 2 tooth flank is the cross-product of the tooth flank equation after the partial differential for its two parameters:
3.3 Determination of the meshing equation

Using the coordinate transformation matrix M01, in the coordinate system S1 the relative velocity equation of the tooth flank contact point is respectively represented as follows:

\[
\begin{align*}
v_2^{(12)} &= v_{x_2}^{(12)}i_2 + v_{y_2}^{(12)}j_2 + v_{z_2}^{(12)}k_2
\end{align*}
\]  

(18)

The spherical involute skew bevel gears meshing equation can be obtained by the simultaneous formulas (5), (15), and (18):

\[
\begin{align*}
v_2^{(12)}n &= v_{x_2}^{(12)}n_{x_2} + v_{y_2}^{(12)}n_{y_2} + v_{z_2}^{(12)}n_{z_2} = 0
\end{align*}
\]  

(19)

According to requirement the coordinate equations in the coordinate systems Sp, S1 and S2 can be obtained by using coordinate transformation matrices.

In coordinate system S1 the tooth flank equation of gear 2 is shown in equation (17), and the tooth flank is transformed to the tooth flank equation in the S1 coordinate system by the coordinate transformation matrix M10, M02 as described above:

\[
\begin{align*}
r_1^{(2)}(\rho, \phi) &= x_1^{(2)}i_1 + y_1^{(2)}j_1 + z_1^{(2)}k_1
\end{align*}
\]  

(20)

4. Shape analysis of contact line

Considering the complexity of the conjugate curve surface analysis, the generation of skew bevel gears tooth flank and the meshing transmission angle are described in this paper, and then the analysis of the gear surface conjugate is expanded.

As described above, during the generation process of tooth flank, the circle plane Q relatively purely roll around the base cone, the conical spiral involute tooth flank is formed by curve cluster that generated from curve sweeping on the circle plane Q. The relative rolling motion that the circle plane Q rotates around the base cone can be divided into the base cone’s rotation and circle plane Q’s revolution, which rotates around itself axis and its center respectively, the rotation angle speed is $\omega_t$, the revolution angle speed is $\omega_q$, the basic cone angle of the gear is $\delta_b$, apparently has the formula:

\[
\sin \delta_b = \frac{m_s}{m_l}
\]  

(21)

A pair of bevel gears meshing transmission process as Fig 5, two bevel gear axis X1 and X2 intersect at point O, the big circle plane T on spherical is the bevel gear conical tangent plane of two bevel gears, and tangent line is OM. The two base cones of the circular plane Q and the bevel gear are respectively tangent to the straight line segment OU1 and OU2, the pitch cone angle and the base cone angle of the two gears are $\delta_1, \delta_2$ and $\delta_1, \delta_2$ respectively. When this pair of gears rotate around their respectively angular with speeds $\omega_{t1}, \omega_{t2}$ around their respective axis, it is obvious that meet their speed ratios requirement:

\[
\frac{\omega_{t1}}{\omega_{t2}} = \frac{\sin \delta_2}{\sin \delta_1}
\]  

(22)

According to described generation principle of the tooth flank, the circle plane Q can be set as the base plane of the gear 2 tooth flank $\Sigma_2$, and the generation line of tooth flank must be on the base plane, and the rotation speed $\omega_q$ of the base plane meet the formula (21). Now in order to prove the
circle plane Q is the base plane of the gear 1 tooth flank $\Sigma_1$, which conjugate with gear $\Sigma_2$, it is only need to prove that the rotational speed $\omega_1$ between $\omega_q$ and gear1 meet the formula as follows:

$$\sin \delta_{bl} = \frac{\sigma_{q1}}{\sigma_{b1}}$$

(23)

Fig.3 The illustration of bevel gears transmission

Since the base cone is located on the inner side of the pitch cone, and the tooth flank of spherical involute spiral bevel gear is generated based on the base cone, the generated tooth flank that take the spherical involute curve as a tooth profile curve outside the base cone, which must be intersect with pitch cone surface. Those two gears have a meshing point P on two pitch cones tangent line, according to the definition of spiral bevel gears spiral angle, we can see the angle of intersect is spiral angle $\beta$, which is tangent surface angle or the normal surface angle of two intersecting curve, the spherical trigonometric sine formula is:

$$\sin \delta = \frac{\sin \delta_b}{\sin \delta}$$

(24)

According to the basic condition of the space curve conjugate: at the meshing point the spiral angle is equal, the direction is opposite. It can be obtained that a pair of conjugate gears meet formula at the meshing point P:

$$\frac{\sin \delta_{bl}}{\sin \delta} = \frac{\sin \delta_{b2}}{\sin \delta}$$

(25)

Simultaneous (22), (24), (25):

$$\frac{\sigma_{q1}}{\sigma_{b1}} = \frac{\sin \delta_{b2}}{\sin \delta_{bl}}$$

(26)

Therefore, the generated equation (25) can be obtained according to equation (23), the circle plane Q is also the generated plane of the gear 1 tooth flank $\Sigma_1$, and the tooth flank generation line is on the Q circle plane as well.

The generation line of the gear 2 on the Q circle plane is the straight line M2M’2, and the generation line of the gear 1 is the unknown curve M1M’1, so the tooth flank $\Sigma_2$ and the tooth flank $\Sigma_1$ are respectively generated by the M2M’2, M1M’1. It only have two cases to ensure that the two tooth flanks are always tangent each other.Firstly, when two gears coincide with each other, the curve M1M’1 is also a straight line, the tangent line coincide with the generation line for two tooth flanks, the same generation line be adopt to generate a pair of tooth flank meshing surface, which pass through the tangent plane t of the two meshing tooth flanks tangent line, and perpendicular to the plane Q. Due to the two tooth flanks on both sides of the plane t, the two tooth flanks in line contact and do not interfere in the meshing process. The generated conjugate gear 1 by M1M’1 is a bevel gear as well.

In the other case, the curve M1M’1 is located on the same side of M2M’2, and tangent to M2M’2 at a point, and then no curvature interference occurs on the tooth flank, theoretically, the generated pair of gears is a point contact conjugate engagement.
Therefore, the curve M1M’1 can be a circular arc segment, and the generated bevel gear is a skew bevel gear. Theoretically, under this premise the conjugate generation line can become arbitrary shape when meet above the conditions, but the shape selections have direct effects on machining process for cutting tool. Line contact conjugate gears are suitable for light load and precision transmission applications. This article focuses on the in-depth research of skew bevel gears machining, so the conjugate generation line is selected as the analysis object. Through the above analysis, it is obvious that the instantaneous meshing curve of the two gears are always on the plane Q, and two tooth gears's the tangent circle radius of the tooth flank are equal, and the gear line is also the tooth flank meshing contact line, so the plane Q is also called the meshing plane.

Through the previously established tooth flank mathematical model and meshing model, a pair of conjugate meshing skew bevel gears are obtained by CNC machine, it was used to verify the correctness of the model, as shown in Fig.7.

5.Conclusion
By using the basic generated principle of spherical involute spiral bevel gears tooth flank, a mathematical model of the double-parameter tooth flank is established on the coordinate transformation theory, which about the polar radius $\rho$ of tooth flank generation line and base cone rotation angle $\Phi$ generation process of the spherical involute. According to the established tooth flank equation. Spiral bevel gear meshing equation was established. Starting with the tooth flank generation of skew bevel gears and the angle of meshing transmission, the flank conjugate was analyzed and generated. It was found that tooth flank line contact meshing conjugate of skew spiral bevel gears were skew bevel gears as well, and the tangent circle radius of the tooth flank generation line was equal.

Acknowledgments
This project is supported by Research Natural Science Fund of Jilin Province(20180101325JC) and Fund of Luquan.

References
[1] Gonzalez-Perez, Ignacio, Fuentes, Alfonso, Hayasaka, Kenichi, Analytical determination of basic machine-tool settings for generation of spiral bevel gears from blank data[J], ASME J. Mech. Des., 132, 2010, pp. 1-11.
[2] Faydor L. Litvin, Alfonso Fuentes, Kenichi Hayasaka, Design, manufacture, stress analysis, and experimental tests of low-noise high endurance spiral bevel gears[J], *Mechanism and Machine Theory*, No. 41, 2006, pp. 83–118.

[3] Fan Qi, Ronald S. DaFoe, John W. Swanger, Higher-Order Tooth Flank Form Error Correction for Face-Milled Spiral Bevel and Hypoid Gears[J], *ASME J. Mech. Des.*, no. 130, 2008, p. 072601.

[4] Vilmos Simon, Generation of Hypoid Gears on CNC Hypoid Generator[J], journal of mechanical design, Vol.133, No.12, 2011, pp.1-9.

[5] Yi-Pei Shih, A novel ease-off flank modification methodology for spiral bevel and hypoid gears[J], *Mechanism and Machine Theory*, 45, 2010, pp.1108–1124.219-226.

[6] Yang Z J, Hong Z B, Wang B C, et al. New tooth profile design of spiral bevel gears with spherical involute [J]. *Intl. J. Adv. Comput. Technolog.*, 2012, 4(19): 462-469.

[7] Hong Z B, Yang Z J, Zhang X C, et al. New modeling method of spiral bevel gears with spherical involute based on CATIA [C]. *Proc. SPIE Int. Soc. Opt. Eng.*, 2011, 7997: 79970Y.

[8] Gleason Seminar. Spiral bevel and hypoid gear technology update [R]. Beijing: Gleason Corporation, 2007.

[9] Litvin FL, Gutman Y. Methods of Synthesis and Analysis of Hypoid Gear Drives of ‘Formate’ and ‘Helixform’[J], *ASME J. of Mech. Design*, 1981, 103:83-113.