Consistency Conditions for $p$-Form Fields Localization on Braneworlds: Hodge Duality

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Abstract

In a recent work (Eur.Phys.J.C 80 (2020) 5, 432) the present authors obtained general stringent conditions on the localization of fields in branewords by imposing that its zero mode must satisfy Einstein equation. In this manuscript we continue this study by considering free $p$-form fields. These fields have an equivalency between a $p$-form and a $(D-p-2)$-form, provided by the Hodge duality transformation. This extra symmetry will impose a new consistency condition that both, the field and its dual, must be localized simultaneously on the brane. We apply the consistency conditions for some particular 6D braneworld models. We find the important conclusion that the localization of a free vector (1-form) field, attained for many models in the literature, must be ruled out. We also consider the intersecting brane model in an arbitrary codimension and find that the above conditions exclude many of the $p$-form fields. This result points to the fact that symmetries of fields can be used to rule out its localization.

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1 Introduction

In braneworld context, our 4-dimensional spacetime is regarded as a hypersurface (3-brane) embedded in a higher dimensional bulk. Among the most popular models are those proposed by Randall-Sundrum (RS) [1][2]. These models became very attractive because gravity can be confined on a delta-like 3-brane, and thus, Newton’s law of gravitation can be recovered. In addition to gravitational aspects, other important points related to the Standard Model (SM) fields can also be studied. Although RS consider all the SM fields previously confined on the 3-brane, further studies have shown that most these fields, propagating freely on the bulk, are not localized on the brane [3–5]. This fact gave rise to another line of study related to the localization of the Standard Model fields in braneworld scenarios.

After the success of the RS models, other proposals of braneworlds with localized gravity have been presented. In 5D for example, it was proposed smooth versions (thick branes) of RS-II, where the 3-brane is generated by a scalar fields propagating on the bulk [6][7]; thick brane models with inner structure [8][9]; branes generated by purely geometric quantities [10]; or still, braneworlds in a cosmological context, where the 3-brane has a Robertson-Walker metric [11][12]. In addition to these, other solutions in higher-dimensional scenarios were proposed. The 6D models, for example, deserve special attention. This because, that dimensional configuration is the only other one that has analytical solutions for the metric generated by topological defects. The most common in the literature are those where the 3-brane is generated by string-like or vortex defects. Generally, the metric for such cases is given by something like,

$$ds^2 = g_MdX^MdX^N = A^2(r)|g_{\mu\nu}(x)|dx^\mu dx^\nu + B^2(r)d\theta^2 + dr^2,$$

(1)

where, \( r \in [0, \infty) \) and \( \theta \in [0, 2\pi) \) are the extra dimensions. The above metric is considered the vacuum solution for the Einstein’s equations obtained from an action like,

$$S_{\text{grav.}} = \int d^4x dr d\theta \sqrt{-g} \left[ \frac{1}{2\kappa^2} (R - 2\Lambda) + \mathcal{L}_b \right],$$

(2)

which, \( g \) is the determinant of \( g_{MN} \) and \( \mathcal{L}_b \) is related to the brane generation process. Reference [13] was the first to present a metric solution with the above features by assuming \( \Lambda = 0 \). They showed that, although their solution had a naked singularity at \( r = 0 \), it can yet provide a gravitational theory on
the brane consistent with the observed. Soon after, in Refs. [14, 15], the authors found a non-singular solution valid for the exterior of a string-like topological defect. But now, in the presence of a negative cosmological constant (\(AdS_6\)). The metric for this model is given by (1) with,

\[
A^2(r) = \exp (-kr), \quad B^2(r) = R_0^2 A^2(r).
\]

Another solution with the feature mentioned above was obtained in Refs. [16, 17]. In these models, the metric is valid outside and inside the string-like defect. For this, the warp factors \(A(r)\) and \(B(r)\) in (1) are given by,

\[
A^2(r) = \exp [-kr + \tanh (kr)], \quad k^2 B^2(r) = \tanh^2 (kr) A^2(r).
\]

Unlike the previous cases, this metric provides a natural thickness for the string-like defect (3-brane). Beyond this, the bulk geometry is an asymptotically \(AdS_6\) space, which is a desirable characteristic in the study of gravity and matter fields localization. In addition to these string-like models, other, also in 6D, were proposed. In Refs. [18, 19], the 3-brane is generated by the intersection of two delta-like 4-brane. The metric for this model is given by,

\[
ds^2 = \frac{1}{(1 + k_1 |y| + k_2 |z|)^2} \left[ \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 + dz^2 \right],
\]

Here, we have also an asymptotically \(AdS_6\) bulk space, with the two extra dimensions, \(y\) and \(z\), infinitely large. There are yet other models in a higher-dimensional configuration. For example, braneworld generated by the intersection of an arbitrary number of delta-like branes [20], beyond other proposals [21, 24].

For all the models mentioned above, confinement of other fields, beside the gravitational one, is always an important point to be verified [25, 29]. A particular class of fields that we can highlight are the \(p\)-form fields. Among them, \(U(1)\) vector field (1-form) has an important status in particle physics. In addition to the 1-form, the Kalb-Ramond field (2-form) has received some attention, for example, in string field theory [30, 31]. The confinement study of these two fields and other \(p\)-form already was widely performed in the literature [32, 48]. As well as other related issues, such as: \(p\)-form fields been used to provide the stabilization of the radius in RS-I model [49]; or, to introduce torsion in RS-I scenarios [50, 52]; or still, to generate inflation or gravitational wave [53, 54]; among other issues [55, 58]. All confinement studies are based on the finite integral argument and this approach is used for any field. It is based on the possibility of factoring an action like the following,

\[
S = \int d^4x d^{D-4}z \sqrt{-g^{(D)}} \mathcal{L}_{(\text{matter})}^{(D)},
\]

into an effective action on the 3-brane and an integral in the coordinates of the extra dimensions, \textit{i.e.},

\[
S = \int d^Dz f(z) \int d^4x \sqrt{-g^{(4)}} \mathcal{L}_{(\text{matter})}^{(4)} = K \int d^4x \sqrt{-g^{(4)}} \mathcal{L}_{(\text{matter})}^{(4)}.
\]

Thus, the theory will be well-defined, \textit{i.e.}, the field will be confined on the brane, when the integral in the extra coordinates \(K\) is finite. This argument is commonly used as a sufficient condition to affirm that a field is localized. For an arbitrary free \(p\)-form fields, for example, Refs. [59, 60] presented some general results for codimension 1 models. By using the finite integral argument, it is possible to confine in a \(q\)-brane only the \(p\)-form in which \(p\) satisfies the conditions \(p < \frac{q+1}{2}\) or \(p > \frac{q+2}{2}\). Thus, only a scalar field (0-form) and a 3-form could be localized on a 3-brane in this codimension 1 analysis. Here, we can ask
whether this localization procedure is really sufficient to provide a consistent localization of these \( p \)-form fields. Refs. [61–63] shown that, by exploring the Hodge symmetry, for example, the values of \( p \) must satisfy not only the finite integral requirement, but also other conditions obtained from that symmetry. Ref. [63] goes further and, by using Einstein’s equations, the authors obtain an additional requirement for the localization in codimension 1 RS models be consistent. In a broader study performed by us in Ref. [64], we explored the Einstein’s equations to get some general conditions that any Standard Model field must satisfy to provide a consistent localization on the brane. These consistency conditions obtained by explore other aspects of the theory, became important because they demystify the finite integral argument as sufficient requirement to states that the field is confined on the brane.

In this direction, we propose to study the localization of an arbitrary \( p \)-form field for some codimension 2 braneworld models. Thus, we wish to obtain constraints on the values allowed of \( p \), namely, by using the Hodge transformation and also the Einstein’s equations. Therefore, this manuscript intends to reinforce, for a different gravitational scenario, that the finite integral argument is a necessary, but not sufficient, condition to provide a consistent field localization in braneworlds. This work is organized as follows: in section (2), we will develop the general setup of the localization procedure for a \( p \)-form field in codimension 2. After, in section (3), we will explore the Hodge duality and the Einstein’s equations as consistency conditions for the localization study of these fields, also for codimension 2 models. In section (4), we will apply the consistency conditions for some specific braneworld models. To finish, let us discuss, in section (5), briefly the consistency of the \( p \)-form field localization for an arbitrary codimension case. Conclusions are left to the section (6).

## 2 \( p \)-Form Fields Localization - Codimension 2 Case

In this section, let us discuss the \( p \)-form field localization by using the conventional finite integral argument. It is worth mentioning that this procedure, commonly used in the literature, is performed by considering the vacuum metric only as a background. In this way, the bulk geometry is assumed does not change by the presence of the matter field, so that backreaction effects are ignored. Below, in next sections, let us discuss under what conditions this assumption is consistent. In these first sections, we will focus on codimension 2 scenarios to perform a simpler and clearer discussion.

Let us start by considering an arbitrary codimension two braneworld metric given by,

\[
ds^2 = g_{MN}dx^Ndx^M = e^{2\sigma(y)}\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + \tilde{g}_{jk}(y)dy^j dy^k,
\]

where, the warp factor \( \sigma(y) \) and the metric components \( \tilde{g}_{jk}(y) \) depend on the extra dimensions coordinates \( y^j \). Throughout the manuscript, capital indexes \( M, N \) run on all dimensions \( D = d+2 \); coordinates \( x^\mu \) span the brane and \( \mu, \nu = (1, 2, \ldots, d) \); and coordinates \( y^j \) are related to the extra dimensions with \( j, k = (1, 2) \). In this scenario, the metric (8) is completely generic and, at first, we do not need to know how it was generated. However, let us assumed that the Einstein-Hilbert action which gives the vacuum solution (8) is in the shape,

\[
S_{\text{grav.}} = \int d^dx d^2y \sqrt{-g} \left[ \frac{1}{2\kappa} (R - 2\Lambda) + L_b(y) \right],
\]

where, \( \Lambda \) is the cosmological constant and \( L_b(y) \) is some function related to the process of braneworld generation.
With the vacuum metric solution previously defined, let us start the localization study of a $p$-form field in such background from an action like that below,

$$\int d^4xd^2y \sqrt{-g} H(y) F_{N_1...N_{p+1}} F^{N_1...N_{p+1}},$$  \hspace{1cm} (10)

where $F_{N_1...N_{p+1}} = \partial_{[N_1} A_{N_2...N_{p+1}]}$, and $A_{N_1...N_p}$ is the $p$-form field. In equation (10), we are writing the scalar function $H(y)$ to include in the discussion, in addition to the free case, some interacting cases presented in the literature. From this action, the equation of motion (EOM) can be written as,

$$\partial_{\mu_1}[\sqrt{-g} H(y) F^{\mu_1 N_2...N_{p+1}}] + \partial_{k}[\sqrt{-g} H(y) F_{k N_2...N_{p+1}}] = 0.$$  \hspace{1cm} (11)

Now, by using the metric (8) and the assumption of codimension two, we get,

$$\frac{1}{\sqrt{-g}} \partial_{\rho}[\sqrt{-g} F^{\rho \mu_1...\mu_p}] + \frac{e^{2\sigma}}{H(y)} \partial_{k}[\tilde{H}(y) \tilde{g}^{kj} (\partial_{j} A^{\mu_1...\mu_p} + (-1)^p \partial_{[\mu_1} A^{\mu_2...\mu_p]}_{j})] = 0,$$  \hspace{1cm} (12)

$$\frac{1}{\sqrt{-g}} \partial_{\rho}[\sqrt{-g} \left( \partial_{m} A^{\mu_1...\mu_{p-1} - \rho} + (-1)^p \partial^{[\rho} A^{\mu_1...\mu_{p-1}]}_{m} \right)] = 0,$$

$$- \frac{\tilde{g}_{lm}}{H(y)} \partial_{k} \left[ e^{2\sigma} \tilde{H}(y) \tilde{g}^{kl} \tilde{g}^{ji} (\partial_{j} A^{\mu_1...\mu_{p-1}}_{l})^2 \right] = 0,$$

$$\partial_{\rho} \left[ \sqrt{-\tilde{g}} \left( \partial^{[\rho} A^{\mu_1...\mu_{p-2}]}_{jkl} + \partial^{[\rho} A^{\mu_1...\mu_{p-2}]}_{jkl} \right) \right] = 0,$$  \hspace{1cm} (13-14)

where $\tilde{H}(y) \equiv e^{[d-2p] \sigma} \sqrt{g} H(y), \tilde{g}$ and $\tilde{g}$ are the determinants of $\tilde{g}_{\mu\nu}(x)$ and $\tilde{g}_{jk}(y)$, respectively, and the index contractions in Eqs. (12-14) are performed by using $\tilde{g}_{\mu\nu}(x)$ or $\tilde{g}_{jk}(y)$.

By proposing the particular solution $A_{k\mu_1...\mu_{p-1}}(x, y) = A_{\mu_1...\mu_{p-1}}(x) \partial_{k} \chi(y)$ and using the gauge symmetry to redefine the components $A_{\mu_1...\mu_p}(x, y)$, we can obtain the EOM’s below,

$$\frac{1}{\sqrt{-g}} \partial_{\rho}[\sqrt{-g} F^{\rho \mu_1...\mu_p}] + \frac{e^{2\sigma}}{H(y)} \partial_{k}[\tilde{H}(y) \tilde{g}^{kj} \partial_{j} A^{\mu_1...\mu_p}] = 0,$$  \hspace{1cm} (15)

$$\partial_{k} \left[ e^{2\sigma} \tilde{H}(y) \tilde{g}^{kl} \tilde{g}^{ji} \partial_{j} A^{\mu_1...\mu_{p-1}]}_{l} \right] = 0,$$

$$\partial_{\rho} \left[ \sqrt{-\tilde{g}} \left( \partial^{[\rho} A^{\mu_1...\mu_{p-2]}}_{jkl} + \partial^{[\rho} A^{\mu_1...\mu_{p-2]}_{jkl} \right) \right] = 0,$$  \hspace{1cm} (16-17)

Therefore, we reduce the system of equations (12-14) to the system (15-17) only with two effective fields, namely, a $p$-form $A_{\nu_1...\nu_p}$, and a $(p-2)$-form $A_{\mu_1...\mu_{p-2} j}$. We would like to stress that the particular ansatz for the components $A_{k\mu_1...\mu_{p-1}}$ does not imply the loss of generality of the next results, as we will see later. By proposing $A_{\nu_1...\nu_p}(x, y) = A_{\nu_1...\nu_p}(y) \xi(y)$ and $A_{\mu_1...\mu_{p-2} j}(x, y) = A_{\mu_1...\mu_{p-2} j}(x) \psi(y)$, we can write the separated equations,

$$\frac{1}{\sqrt{-g}} \partial_{\rho}[\sqrt{-g} F^{\rho \mu_1...\mu_p}(x)] = m^2 A^{\mu_1...\mu_p}(x),$$  \hspace{1cm} (18)

$$\partial_{\rho} \left[ \sqrt{-g} \partial^{[\rho} A^{\mu_1...\mu_{p-2]}]}_{jkl} \right] = 0,$$  \hspace{1cm} (19)

$$- \frac{e^{2\sigma}}{\tilde{H}(y)} \partial_{k}[\tilde{H}(y) \tilde{g}^{kj} \partial_{j} \xi(y)] = m^2 \xi(y),$$  \hspace{1cm} (20)

$$\partial_{k} \left[ e^{2\sigma} \tilde{H}(y) \tilde{g}^{kj} \tilde{g}^{ji} \epsilon_{ij} \psi(y) \right] = 0,$$  \hspace{1cm} (21)
with $\epsilon_{li} = -\epsilon_{il}$ and $\epsilon_{12} = 1$. In this framework, the effective Lagrangian can be written as,

$$\mathcal{L}_{(eff)}(x) = -\int d^2 y \tilde{H}(y) \left[ \xi^2 e^{-2\sigma} \left( \frac{1}{2(p+1)!} F_{\mu_1,\ldots,\mu_{p+1}}^2 + \frac{1}{2p!} m^2 A_{\mu_1,\ldots,\mu_p}^2 \right) + \frac{1}{(p-1)!} \psi^2 e^{2\sigma} g^{11} g^{22} F_{\mu_1,\ldots,\mu_{p-1}}^2 \right].$$  \hspace{1cm} (22)

Where $F_{\mu_1,\ldots,\mu_{p-1}} = F_{\mu_1,\ldots,\mu_{p-1}}(x) = \partial_{[\mu_1} A_{\mu_2,\ldots,\mu_{p-1}]}(x)$. Therefore, from a bulk $p$-form field, we obtained, for that codimension 2 scenarios, an effective $p$-form and also a $(p-2)$-form field on the brane. We will restrict ourselves to the study of the zero-mode case, thus, let us consider $m^2 = 0$ in equation (22), and the effective Lagrangian on the brane gets,

$$\mathcal{L}_{(eff)}^0(x) = -\frac{1}{2(p+1)!} F_{\mu_1,\ldots,\mu_{p+1}}^2(x) K_1 - \frac{1}{(p-1)!} F_{\mu_1,\ldots,\mu_{p-1}}^2(x) K_2,$$  \hspace{1cm} (23)

which,

$$K_1 = \int d^2 y \tilde{H}(y) \xi^2 e^{-2\sigma} = \int d^2 y e^{[d-2p-2] \sigma} \sqrt{g} H(y) \xi^2,$$  \hspace{1cm} (24)

$$K_2 = \int d^2 y \tilde{H}(y) \psi^2 e^{2\sigma} g^{11} g^{22} = \int d^2 y e^{[d-2p+2] \sigma} \sqrt{g} H(y) \psi^2 g^{11} g^{22}. \hspace{1cm} (25)$$

From this, the fields will be localized on the brane when the integrals $K_j$ are finite. Later, we will use some specifics braneworld models to obtain the values of $p$ that can be confined. In doing this, let us discuss the free case [$H(y) = 1$] and also some cases where $H(y)$ is a known function of the warp factor.

This is the procedure commonly used in the literature to study field localization, and the analysis of integrals $K_j$ is considered as a sufficient condition to states that a field is confined on the brane. Below, let us explore two other aspects of the $p$-form theory to show that this procedure is necessary, but not sufficient, to provide a consistent localization study.

### 3 Consistency Conditions

In Ref. [63] the authors study the consistency of the $p$-form localization with the Hodge duality symmetry and also with the Einstein’s equations. Such study was carried out for codimension 1 models like Randall-Sundrum. Recently, we performed a similar study, where we discussed the consistency of the localization procedure with Einstein’s equations for field of different spin [64]. The results were applied for some models, by considering the free field and also some interacting cases (localization mechanisms). For many of these cases, the localization procedure was inconsistent, reinforcing, once again, that the finite integral argument is not sufficient to state that a field is confined. Here, we will use these tools to study the consistency of the procedure performed in last section for codimension two.

#### 3.1 Einstein’s equations

Up to this point, we focus our attention in the action (10) without worrying about the effects of that matter field on the background metric. In fact, the procedure described in equations (6) and (7) is performed by considering the bulk metric obtained in vacuum. Let us consider the full Einstein-Hilbert action,

$$S = S_{\text{grav.}} - \frac{1}{2(p+1)!} \int d^d x d^2 y \sqrt{-g} H(y) F_{M_1,\ldots,M_{p+1}} F^{M_1,\ldots,M_{p+1}},$$  \hspace{1cm} (26)
and, from this, we will analyze under what conditions the vacuum metric solution obtained only from $S_{\text{grav.}}$ is consistent with the solutions obtained for (20) and (21). Here, $S_{\text{grav.}}$ is the action related to the generation of the braneworld model in the vacuum like that in Eq. (2). From Eq. (26), we get the following equation of motion,

$$G_{MN} + g_{MN} \Lambda = \kappa \left( T_{MN}^{(b)} + T_{MN}^{(\text{mat})} \right),$$

(27)

where $T_{MN}^{(b)}$ is the energy-momentum tensor that generate the braneworld. The stress tensor $T_{MN}^{(\text{mat})}$ is related to the differential $p$-form field, and it is given by,

$$T_{MN}^{(\text{mat})} = \frac{1}{p!} H(y) F_{M_{M_{2}..M_{p+1}}}^{-1} F_{N_{M_{2}..M_{p+1}}} - \frac{1}{2(p+1)!} g_{MN} H(y) F_{M_{1}..M_{p+1}} F_{M_{1}..M_{p+1}}.$$

(28)

Here, we are considering that the presence of the $p$-form does not change the shape of the bulk metric. However, as this field can be localized, the metric at the brane must be modified from $\eta_{\mu\nu}$ to $\hat{g}_{\mu\nu}(x)$. This modification of the metric at the brane lead to the following changes for the quantities $G_{\mu\nu}$ and $G_{jk}$,

$$G_{\mu\nu}^{(\text{vacuum})}(y) = -\eta_{\mu\nu} f(y) \rightarrow G_{\mu\nu}(x,y) = \hat{G}_{\mu\nu}(x) - \hat{g}_{\mu\nu}(x) f(y),$$

(29)

and

$$G_{jk}^{(\text{vacuum})}(y) = -\eta_{jk} f(y) \rightarrow G_{jk}(x,y) = -\frac{1}{2} e^{-2\sigma} \hat{g}_{jk} \tilde{R}(x) - \tilde{g}_{jk} f(y).$$

(30)

And also, the quantity $T_{MN}^{(b)}$ changes as $T_{\mu\nu}^{(b)}(y) = \eta_{\mu\nu} h(y) \rightarrow T_{\mu\nu}^{(b)}(x,y) = \hat{g}_{\mu\nu}(x) h(y)$ and $T_{jk}^{(b)}(y)$ does not change. Thus, for the components $(\mu, \nu)$, equation (27) stays in the shape,

$$\hat{G}_{\mu\nu}(x) = \kappa H(y) T_{\mu\nu}^{(\text{mat})}(x,y).$$

(31)

Where, the above result (31) was obtained by considering that the vacuum metric is still valid. In this way, by consistency reasons, energy-momentum tensor must satisfy,

$$H(y) T_{\mu\nu}^{(\text{mat})}(x,y) = T_{\mu\nu}^{(\text{mat})}(x),$$

(32)

for the localization procedure to be consistent with vacuum metric solution. This result shows that backreaction effects can be ignored for some field configurations, as discussed in more details in Ref. [63]. In other words, the vacuum metric (8), obtained for the action $S_{\text{grav.}}$, can be used to study fields localization, consistently with Einstein’s equations, if the condition (32) is satisfied. Otherwise, backreaction effects must necessarily be taken into account.

### 3.2 Hodge transformation

The action (10), for the free case, presents a symmetry provided by a specific transformation of the $p$-form fields, namely, the Hodge transformation. It is a well-known topic that, Hodge transformation relates a $q$-form to one $(D - q)$-form, which $D$ is the space dimension where the $q$-form is defined. By using the properties of the binomial coefficient, it is possible to show that both differential forms have the same number of independent components and, therefore, the same degrees of freedom. In Ref. [65], the authors

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4 This calculus can be found in more details in Ref. [64].
show that, by considering this Hodge transformation in a space without topological obstructions, there must be an indirect equivalence among a \( p \)-form field \( A_{[p]} \) and a \((D - p - 2)\)-form field \( B_{[D-p-2]} \). To see this, we can start from the Lagrangian,

\[
\mathcal{L} = -\frac{1}{2(p+1)!} F_{M_1\ldots M_{p+1}} F^{M_1\ldots M_{p+1}},
\]

with \( F_{M_1\ldots M_{p+1}} \), the components of \( F_{[p+1]} = dA_{[p]} \), where ‘\( d \)’ is the exterior derivative. Let us consider the Hodge transformation,

\[
(\ast F)^{M_1\ldots M_{D-p-1}} = \frac{(-1)^{(p+1)(D-p-1)}}{(p+1)! \sqrt{-g}} \varepsilon^{M_1\ldots M_{D-p-1} N_1\ldots N_{p+1}} F_{N_1\ldots N_{p+1}},
\]

where, \((\ast F)^{M_1\ldots M_{D-p-1}} \) are the components of \((\ast F)_{[D-p-1]} = dB_{[D-p-2]} \), whith this, it is easy to obtain the symmetry,

\[
\mathcal{L} = -\frac{1}{(p+1)!} F_{M_1\ldots M_{p+1}} F^{M_1\ldots M_{p+1}} = -\frac{1}{(D-p-1)!} (\ast F)^{N_1\ldots N_{D-p-1}} (\ast F)^{N_1\ldots N_{D-p-1}}.
\]

Thus, we get the indirect equivalence, provided by the Hodge transformation, between the bulk \( p \)-form \( A_{[p]} \) and the bulk \((D-p-2)\)-form \( B_{[D-p-2]} \). This is the main result presented in that subsection, and it gives us an important consistency condition for the localization study of a free \( p \)-form field. This because, Eq. (35) states that the localization must always comprise both values of \( p \), namely, that related to the \( p \)-form and its bulk \((D-p-2)\)-form. Apparently, this symmetry is also valid for the action (10) whether the function \( H(y) \) is invariant under Hodge transformation. However, as we will see below, for the interacting case, there are some contradictions between the results obtained from EOM and the Hodge symmetry. Thus, the below results will be considered valid only for the free theory. Now, let us discuss two consequences of the symmetry (35), and how they can be used as consistency conditions for the free \( p \)-form field localization in codimension 2.

(i) - **Statement**: Localization procedure must comprise both fields, the bulk \( p \)-form and its bulk dual \((D-p-2)\)-form.

This is an immediate consequence of equation (35). As the Hodge transformation is a symmetry of theory, the action obtained for (35) should be the same for both dual fields. This is possible only if the integral in extra dimensions is finite for both fields, for the \( p \)-form and also for its bulk dual \((D-p-2)\)-form.

(ii) - **Statement**: Hodge duality transformation must be valid even after the dimensional reduction.

To see this, let us use the equation (34). From such relation, we can write for codimension 2,

\[
(\ast F)^{\mu_1\ldots \mu_{d-p-1} l m} = -\frac{(-1)^{(p+1)(D-p-1)}}{(p+1)! \sqrt{-g}} \varepsilon^{\mu_1\ldots \mu_{d-p-1} \nu_1\ldots \nu_{p+1} l m} F_{\nu_1\ldots \nu_{p+1}},
\]

\[
(\ast F)^{\mu_1\ldots \mu_{d-1} l} = -\frac{(-1)^{(p+1)(D-p-1)}}{(p+1)! \sqrt{-g}} \varepsilon^{\mu_1\ldots \mu_{d-1} \nu_1\ldots \nu_{p} l m} F_{\nu_1\ldots \nu_{p} l m},
\]

\[
(\ast F)^{\mu_1\ldots \mu_{d-p+1} l} = -\frac{(-1)^{(p+1)(D-p-1)}}{(p+1)! \sqrt{-g}} \varepsilon^{\mu_1\ldots \mu_{d-p+1} \mu_1\ldots \mu_{p+1} l m} F_{\nu_1\ldots \nu_{p+1} l m}.
\]
Thus, it is easy to see that the relation between the brane Greek index satisfy the Hodge transformation prescription if we define the Levi-Civita on the brane as $\epsilon^{\mu_1...\mu_d} \equiv \epsilon^{\mu_1...\mu_d (12)}$.

Let us analyze these two statements for the configuration discussed in section (2). In doing this, we will use the following expressions obtained from $\mathcal{F}_{N_1...N_{p+1}} = \partial_{[N_1} A_{N_2...N_{p+1}]}$,

\begin{align*}
F_{\mu_1...\mu_{p+1}}(x, y) &= \xi(y) F_{\mu_1...\mu_{p+1}}(x), \\
F_{\kappa_{p+1}}(x, y) &= \partial_\kappa A_{\mu_1...\mu_{p+1}} = \partial_\kappa \xi(y) A_{\mu_1...\mu_{p+1}}(x), \\
F_{\mu_1...\mu_{p-1}2}(x, y) &= \psi(y) \partial_{\mu_1} A_{\mu_2...\mu_{p-1}} = \psi(y) \tilde{F}_{\mu_1...\mu_{p-1}}(x).
\end{align*}

First from (ii), by using these expressions in equations (36) and (37), we obtain

\begin{align*}
\frac{\psi^*(y) g^{11 \nu_{p2}} \sqrt{g}}{e^{2(d-p-1)\sigma}} (\ast F)^{\mu_1...\mu_{d-p-1}}_{12}(x) & \propto -\frac{\xi(y)}{e^{d\sigma}} \frac{1}{\sqrt{-g}} \epsilon^{\mu_1...\mu_{d-p-1} \nu_1...\nu_{p+1}^{12}} F_{\nu_1...\nu_{p+1}}(x), \\
e^{-2(d-p)\sigma} \partial^{\beta} \xi^*(y) B^{\mu_1...\mu_{d-p}}(x) & \propto -\frac{1}{\sqrt{-g}} \epsilon^{\mu_1...\mu_{d-p} \nu_1...\nu_{p+1}^{12}} \partial_{\nu_1...\nu_p} \xi(y) A_{\nu_1...\nu_p}(x).
\end{align*}

Where $\psi^*$ and $\xi^*$ are related to the Hodge dual $(D-p-2)$-form fields. From expression (42), by considering $\psi^*$ and $\xi$ non-zero, we can factor out the function of extra dimensions, i.e.,

\begin{align*}
\psi^*(y) g^{11 \nu_{p2}} \sqrt{g} e^{-2(d-p-1)\sigma} & \propto \xi(y) e^{-d\sigma}.
\end{align*}

Thus, the function $\psi^*(y)$, obtained from to the components $B_{\nu_1...\nu_{d-p-2}12}$, is closely related to the solution $\xi(y)$ obtained for $A_{\nu_1...\nu_p}$. Figure (1) shows a schematic picture of this equivalency relation between the bulk fields, and also between their components (effective fields on the brane). Such relation between the effective fields is a consequence of the configuration used in section (2), especially, of that solution proposed to the components $A_{\nu_1...\nu_{p+k}}$. However, the main results presented in statements (i) and (ii) are independent of this particular configuration of field. Therefore, this two consistency conditions still have a general validity. About item (i), we should ask us if the function $\xi(y)$ that localize the $p$-form $A_{\nu_1...\nu_p}$ also allows to localize the field $B_{\nu_1...\nu_{d-p-2}12}$, which is a component of a bulk $(D-p-2)$-form. Otherwise, the $p$-form field localization for such values of $p$ is not consistent with this symmetry. We still do not know what values of $p$ are allowed to confine. In fact, zero-mode solutions for Eqs. (20) and (21), and the analysis of the integrals in equations (24) and (25) depend on each specific braneworld model. In view this, we will discuss in more details these two points later in the section (4). The relation (44) indicates that the Hodge symmetry should not be valid for the interacting case, i.e., with $H(y)$. This because, for the such case, the solution for $\psi^*/\psi$, obtained from (21), depends on the function $H(y)$ and (44) does not. In view this, the consistency conditions obtained from Hodge transformation will be applied only for the free case.
4 Applications

Now, we will specify some codimension 2 braneworld models founded in the literature to apply the above results. In fact, let us use the metrics (3), (4) and (5) presented in the introduction. Let us perform the discussion by considering $H(y) = e^{\lambda y}$, thus, later, we can discuss the free case ($\lambda = 0$) and also this particular interacting case ($\lambda \neq 0$).

(A) - Let us start by considering that model presented in Refs. [14,15]. For this case, the extra dimensions live in the range $y^1 = r \in [0, \infty)$ and $y^2 = \theta \in [0, 2\pi)$, and the metric is given by,

$$ds^2 = e^{-kr} dx^\mu dx_\mu + R_0^2 e^{-kr} d\theta^2 + dr^2,$$

where, $k$ is a positive constant. By using this metric in Eqs. (20), for the zero-mode ($m^2 = 0$), and (21), we get,

$$\partial_r \left[ e^{-\frac{1}{2}[d-2p+1+\lambda]kr} \partial_r \xi_0(r) \right] = 0,$$

$$\partial_r \left[ e^{-\frac{1}{2}[d-2p+1+\lambda]kr} \psi(r) \right] = 0.$$

From these equations of motion, we obtain the following zero-mode solutions,

$$\xi_{0,(1)}(r) = c_1,$$

$$\xi_{0,(2)}(r) = c_2 e^{\frac{1}{2}[d-2p+1+\lambda]kr},$$

$$\psi(r) = c_3 e^{\frac{1}{2}[d-2p+1+\lambda]kr}.$$

In addition to this, $K_j$ integrals in equations (24) and (25) will be written as,

$$K_1 = R_0 \int dr d\theta e^{-\frac{1}{2}[d-2p-1+\lambda]kr} \xi_0^2(r),$$

$$K_2 = R_0 \int dr d\theta e^{-\frac{1}{2}[d-2p+1+\lambda]kr} \psi^2(r).$$

Thus, with all these, we can discuss what $p$-form fields can be confined. By using the zero-mode solutions for $\xi_0(r)$, the integral (51) will be finite if: (A-a) $d - 2p - 1 + \lambda > 0$, for $\xi_{0,(1)}$; or, (A-b) $d - 2p + 3 + \lambda < 0$, for $\xi_{0,(2)}$. About equation (52), it will be finite when $d - 2p + 1 + \lambda < 0$. Therefore, if the case (A-b) is satisfied, the integral $K_2$ is automatically finite, in this way, localization of $\xi_{0,(2)}$ implies in the localization of $\psi$. Thus, for the subsequent discussion, we will consider only those values of $p$ obtained from $K_1$.

These results show that, in a 3-brane ($d = 4$), the $p$-form fields confined are those which $p \leq 1$ and $p \geq 4$ for the free case ($\lambda = 0$). Already for the interacting case, any $p$-form can be localized on a 3-brane by choose $\lambda > 2p - 3$ or $\lambda < 2p - 7$. Note that, in the interacting case, for a fixed value of $p$, we must use the condition $\lambda > 2p - 3$ with the solution $\xi_{0,(1)}$, or $\lambda < 2p - 7$ with $\xi_{0,(2)}$. And also, it is not possible to confine both solutions simultaneously.

Consistency Test

First, let us consider the consistency with the Hodge transformation applied only for the free case
(\lambda = 0). For \( d = 4 \), we should have confined a scalar (0-from), a vector (1-form) and the \( p \)-form fields with \( p \geq 4 \).

And, by consistency reasons with Hodge transformation, this localization procedure must include the \( p \)-form and its bulk dual \((4-p)\)-form simultaneously. Therefore, we must have the 0-form confined together with its bulk dual 4-form; and also, the 1-form with its bulk dual 3-form. Here, we already obtain an important result, the free vector field (1-form) should be ruled out since its bulk dual 3-from is not localized. These are the main results about the consistency with Hodge transformation. Let us check the items (i) and (ii) for the scalar field. By using the metric (45), \( d = 4 \) and \( p = 0 \), in Eqs. (42), (43) and (44), we get

\[
(\ast \hat{F})^{\mu_1 \mu_2 \mu_3}_{\mu_4}(x) = \frac{1}{\sqrt{-g}} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \hat{F}_{\mu_5}(x),
\]

\[
\partial_r \xi^*_0(r) \hat{B}^{\mu_1 \mu_2 \mu_3 \mu_4}(x) = 0,
\]

\[
\psi^*(r) \propto \xi_0(r) e^{3 \sigma} = c_1 e^{3 \sigma}.
\]

Here, we use the constant solution to confine the scalar field and the non-constant solution to localize the 4-form. By looking at the Eq. (54), we get that the 4-form components \( \hat{B}^{\mu_1 \mu_2 \mu_3 \mu_4}(x) = 0 \). From Eq. (53), we get the Hodge transformation on the brane, as expected, and it relates an effective 0-form and an effective 2-form field, a well-known equivalency relation in 4D. Equation (55) gives us the function \( \psi^*(r) \) related to the effective 2-form \( \hat{B}^{\mu_1 \mu_2}(x) \) on the 3-brane. As expected, this solution for \( \psi^*(r) \) is exactly that obtained in (50) by considering \( d = 4 \), \( \lambda = 0 \) and \( p = 4 \). As we saw previously, the value \( p = 4 \) is confined. Thus, the localization of the free scalar field is completely consistent with Hodge symmetry.

About the consistency condition (32) obtained from Einstein’s equations, let us first apply it to the free case. By using the relation (28), we can write for the zero-mode,

\[
T_{\mu \nu}^{(\text{mat})}(x,r) = e^{-2p\sigma(r)} \xi_0^2(r) \left[ \frac{1}{p!} \hat{F}_{\mu \nu \rho \sigma} \hat{F}^{\rho \sigma} \hat{F}^{\mu_2 \mu_3 \mu_4} \hat{F}_{\mu_5-} \hat{F}^{\mu_5-} - \frac{1}{2(p+1)!} \hat{g}_{\mu \nu} \hat{F}_{\mu_1- \nu \mu_5} \hat{F}^{\mu_1- \mu_5} \right](x) + e^{-2(p-1)\sigma(r)} \psi^2(r) R_0^2 \left[ \frac{1}{(p-2)!} \hat{F}_{\mu \nu \rho \sigma} \hat{F}^{\rho \sigma} \hat{F}^{\mu_2 \mu_3 \mu_4} \hat{F}_{\mu_5-} \hat{F}^{\mu_5-} - \frac{1}{2(p-1)!} \hat{g}_{\mu \nu} \hat{F}_{\mu_1- \nu \mu_5} \hat{F}^{\mu_1- \mu_5} \right](x).
\]

As we discussed above, at first in 6D, is possible to confine the fields with \( p \leq 1 \) for the constant solution of \( \xi_0 \) in Eq. (48). For this case, the components \( A_{\mu_1 \ldots \mu_p-2r \theta} \) does not exist and the stress tensor in (56) gives us,

\[
T_{\mu \nu}^{(\text{mat})}(x,r) = e^{-2p\sigma(r)} c_1^2 \left[ \frac{1}{p!} \hat{F}_{\mu \nu \rho \sigma} \hat{F}^{\rho \sigma} \hat{F}^{\mu_2 \mu_3 \mu_4} \hat{F}_{\mu_5-} \hat{F}^{\mu_5-} - \frac{1}{2(p+1)!} \hat{g}_{\mu \nu} \hat{F}_{\mu_1- \nu \mu_5} \hat{F}^{\mu_1- \mu_5} \right](x).
\]

Thus, we conclude that relation (32) will be satisfied only for \( p = 0 \). Again, the vector field is ruled out by consistency reasons. For the non-constant solution of \( \xi_0 \) in Eq. (50), we saw that the allowed values of \( p \) are that which \( p \geq 4 \). For a 3-brane (\( d = 4 \)), we get that the components \( A_{\mu_1 \ldots \mu_p} \) (with\( p \geq 4 \)) are not a dynamical field and the energy-momentum tensor (56) gives us,

\[
T_{\mu \nu}^{(\text{mat})}(x,r) = e^{-2(p-1)\sigma(r)} \psi^2(r) R_0^2 \left[ \frac{1}{(p-2)!} \hat{F}_{\mu \nu \rho \sigma} \hat{F}^{\rho \sigma} \hat{F}^{\mu_2 \mu_3 \mu_4} \hat{F}_{\mu_5-} \hat{F}^{\mu_5-} - \frac{1}{2(p-1)!} \hat{g}_{\mu \nu} \hat{F}_{\mu_1- \nu \mu_5} \hat{F}^{\mu_1- \mu_5} \right](x).
\]
From the above expression and by using the solution (55), we get that the relation (32) will be satisfied only for \( p = 4 \). Thus, the Einstein’s equations provide that only the 0-form and its bulk dual 4-form can be consistently confined on the brane in total agreement with Hodge transformation.

By considering the interacting case, we get a relation similar to that in Eq. (56), namely,

\[
\begin{align*}
T^{\text{mat}}_{\mu \nu}(x, r) &= e^{[\lambda-2p]\sigma(r)} \xi_0^2(r) \left[ \frac{1}{p!} \hat{F}_{\mu_2 \ldots \mu_p+1} \hat{F}_\nu \mu_2 \ldots \mu_p+1 - \frac{1}{2(p+1)!} \hat{g}_{\mu \nu} \hat{F}_{\mu_1 \ldots \mu_p+1} \hat{F}_{\mu_1 \ldots \mu_p+1} \right](x) \\
+ &\frac{1}{\lambda-2(p-1)]\sigma(r)} \hat{F}_{\mu_2 \ldots \mu_p+1} \hat{F}_\nu \mu_2 \ldots \mu_p+1 - \frac{1}{2(p-1)!} \hat{g}_{\mu \nu} F_{\mu_1 \ldots \mu_p-1} \hat{F}_{\mu_1 \ldots \mu_p-1} \right](x). \quad (59)
\end{align*}
\]

Here, the condition (32) will be satisfied only if \( e^{[\lambda-2p]\sigma(r)} \xi_0^2(r) \) and \( e^{[\lambda-2(p-1)]\sigma(r)} \hat{F}_\nu \mu_2 \ldots \mu_p+1 \) are constants. However, the solutions obtained for \( \xi_0 \) and \( \psi \) in Eqs. (48–50) do not allow that this two relations be satisfied simultaneously for any value of the parameter \( \lambda \). In this way, let us consider \( \psi(r) = 0 \) and, with this, we can write,

\[
\begin{align*}
T^{\text{mat}}_{\mu \nu}(x, r) &= e^{[\lambda-2p]\sigma(r)} \xi_0^2(r) \left[ \frac{1}{p!} \hat{F}_{\mu_2 \ldots \mu_p+1} \hat{F}_\nu \mu_2 \ldots \mu_p+1 - \frac{1}{2(p+1)!} \hat{g}_{\mu \nu} \hat{F}_{\mu_1 \ldots \mu_p+1} \hat{F}_{\mu_1 \ldots \mu_p+1} \right](x). \quad (60)
\end{align*}
\]

Therefore, equation (32) will be satisfied if \( \lambda = 2p \), for the constant solution \( \xi_{0,(1)} \), or \( \lambda = 2(p-d-1) \), for the non-constant solution \( \xi_{0,(2)} \). Thus, Einstein’s equations fix the value of \( \lambda \) and it is completely consistent with the localization conditions \( \lambda > 2p - 3 \) for \( \xi_{0,(1)} \), or \( \lambda < 2p - d - 3 \) for \( \xi_{0,(2)} \). For the interacting case, the Hodge symmetry is not valid and, therefore, we do not need worrying in to confine simultaneously the \( p \)-form and its bulk dual.

(B) - Now, let us consider the model presented in Refs. [16,17]. Just like the previous case, extra coordinates belong to the range \( y^1 = r \in [0, \infty) \) and \( y^2 = \theta \in [0, 2\pi) \), but the metric is given by,

\[
\begin{align*}
\text{ds}^2 &= e^{-kr+\tanh(kr)} \left( dx^\mu dx_\mu + \frac{1}{k^2} \tan^2(kr) d\theta^2 \right) + dr^2.
\end{align*}
\]

By using this metric, equations (20) and (21) for the zero-mode give us,

\[
\begin{align*}
\partial_r \left[ e^{1/2(d-2p+1+\lambda)} [-kr+\tanh(kr)] \tan(kr) \partial_r \xi_{0,(1)} \right] &= 0, \quad (62) \\
\partial_r \left[ e^{1/2(d-2p+1+\lambda)} [-kr+\tanh(kr)] \tan^{-1}(kr) \psi(r) \right] &= 0. \quad (63)
\end{align*}
\]

Beyond this, the integrals of extra dimensions in Eqs. (24) and (25) can be written as,

\[
\begin{align*}
K_1 &\propto \int dr e^{(d-2p-1+\lambda)\sigma(r)} \tan(kr) \xi_{0,(1)}^2(r), \quad (64) \\
K_2 &\propto \int dr e^{(d-2p+1+\lambda)\sigma(r)} \tan^{-1}(kr) \psi^2(r). \quad (65)
\end{align*}
\]

From Eqs. (62) and (63), we can obtain the following solutions,

\[
\begin{align*}
\xi_{0,(1)}(r) &= c_1, \quad (66) \\
\xi_{0,(2)}(r) &= c_2 \int r e^{-1/2(d-2p+1+\lambda)} [-kr+\tanh(kr)] \tan(kr) dr', \quad (67) \\
\psi(r) &= c_3 e^{-1/2(d-2p+1+\lambda)} [-kr+\tanh(kr)] \tan(kr). \quad (68)
\end{align*}
\]
Figure 2: Plot of the integrand in $K_1$ with the zero-mode solutions (66) and (67)*.

Figure (2) shows a plot of the resultant function in the integral (64) for the constant solution (66) [Blue line] and for the non-constant solution (67) [Orange line]. In fact, for the non-constant solution, we performed an asymptotic analysis of the integral (67). First, the non-constant solution is a regular function in all the range of $r$, except for $r \to 0$. In this limit, it goes to $\xi_{0,(2)}(r \to 0) \propto \ln(r)$ and it is the ‘solution’ used to plot the Figure (2). Therefore, despite the fact of the solution $\xi_{0,(2)}(r)$ to be divergent when $r \to 0$, the function in the integral $K_1$ is regular in this limit. Thus, the convergence of the integral $K_1$, for both solutions, is determined by its behavior in the limit $r \to \infty$. In doing this, the localization conditions are given by, (B-a) $2p - \lambda < d - 1$, for the constant solution $\xi_{0,(1)}(r)$ and (B-b) $2p - \lambda > d + 3$ by using the asymptotic non-constant solution $\xi_{0,(2)}(r \to \infty) \propto e^{(d-2p+1+\lambda)kr}$. About the integral (65), by using the solution (68), we get the condition $2p - \lambda > d + 1$. Again, just like the case (A), the localization of $\xi_{0,(2)}(r)$ ensure the localization of $\psi(r)$. Beyond this, the conditions (B-a) and (B-b) are the same of last case, thus, we get the same conclusions obtained in case (A) about the values of $p$ confined.

**Consistency Test**

Let us start by the consistency conditions (i) and (ii) obtained from Hodge transformation for the free case ($\lambda = 0$). For this, we should have confined a $p$-form and its bulk dual ($4 - p$)-form on a 3-brane. From conditions (B-a) and (B-b) obtained above, we get the values of $p$ confined for this setup, namely, $p \leq 1$ and $p \geq 4$. Again, the vector field (1-form) can be confined by the *finite integral* argument. However, the localization of this field will be consistent with Hodge duality only if the bulk 3-form field is also localized. As we can see from above results, the value $p = 3$ is not confined, therefore, the free vector field should be ruled out for this case too. For the other value, namely $p = 0$, we should have a 4-form field confined. Again, this is satisfied, and the 0-form field localization is consistent with Hodge transformation. By using the metric (61), $d = 4$ and $p = 0$, in Eqs. (42), (43) and (44), we get the equations obtained in (53) and (54), but now, for $\psi^*(r)$ it is obtained,

$$\psi^*(r) \propto \xi_{0,(1)}(r)e^{2\sigma\sqrt{g_{22}}} = c_1e^{3\sigma(r)}\tanh(kr).$$

(69)

Here, we use the constant solution to confine the scalar field and the non-constant solution to localize the 4-form. Once again, equation (69) agree with the function $\psi^*(r)$ related to the effective 2-form
\( \hat{B}_{\mu_1\mu_212}(x) \) obtained from Eq. (68) and by considering the above setup. As already mentioned, the interacting case does not satisfy the Hodge transformation symmetry.

About the consistency condition (32), we can use the relation (28) and write for the zero-mode,

\[
T_{\mu\nu}^{(\text{mat})}(x, r) = e^{[\lambda - 2p(1) + \lambda]_1} \xi_0^2 \left[ \frac{1}{p!} \hat{F}_{\mu\nu} \hat{F}_{\mu\nu} + \frac{1}{2(p+1)!} \hat{g}_{\mu\nu} \hat{F}_{\mu_1..\mu_{p+1}} \hat{F}_{\mu_1..\mu_{p+1}} \right](x)
\]

As we already discussed, for the free case, it is possible to confine the fields with \( p \leq 1 \) for the constant solution of \( \xi_0 \). For this case, the components \( A_{\mu_1..\mu_{p-2}r} \) does not exist and the stress tensor in (70) gives us

\[
T_{\mu\nu}^{(\text{mat})}(x, r) = e^{-2\sigma(r)} \xi_0^2 \left[ \frac{1}{p!} \hat{F}_{\mu_2..\mu_{p+1}} \hat{F}_{\mu_2..\mu_{p+1}} - \frac{1}{2(p+1)!} \hat{g}_{\mu\nu} \hat{F}_{\mu_1..\mu_{p+1}} \hat{F}_{\mu_1..\mu_{p+1}} \right](x). \tag{71}
\]

Thus, we conclude that relation (32) will be satisfied only for \( p = 0 \). For the non-constant solution of \( \xi_0 \), we saw that the allowed values of \( p \) are those which \( p \geq 4 \). For a 3-brane \( (d = 4) \), we get that the components \( A_{\mu_1..\mu_p} \) are not dynamical fields and the energy-momentum tensor (70) gives us,

\[
T_{\mu\nu}^{(\text{mat})}(x, r) = e^{-2\sigma(r)} \xi_0^2 \left[ \frac{1}{p!} \hat{F}_{\mu\nu} \hat{F}_{\mu\nu} + \frac{1}{2(p+1)!} \hat{g}_{\mu\nu} \hat{F}_{\mu_1..\mu_{p+1}} \hat{F}_{\mu_1..\mu_{p+1}} \right](x). \tag{72}
\]

By using solution (69), we get that the relation (32) will be satisfied only for \( p = 4 \). Thus, we obtained, again, that only the 0-form and its bulk dual 4-form fields can be confined consistently in total agreement with Hodge transformation.

For the interacting case, the localization of the constant solution \( \xi_{0,(1)} \) on a 3-brane can be attained for \( 2p - \lambda < 3 \). This relation does not allow the confinement of the \( \psi(r) \), thus, this sector must be ruled out from equation (70). By doing this, we get,

\[
T_{\mu\nu}^{(\text{mat})}(x, r) = e^{[\lambda - 2p(1)]_1} \xi_0^2 \left[ \frac{1}{p!} \hat{F}_{\mu\nu} \hat{F}_{\mu\nu} + \frac{1}{2(p+1)!} \hat{g}_{\mu\nu} \hat{F}_{\mu_1..\mu_{p+1}} \hat{F}_{\mu_1..\mu_{p+1}} \right](x). \tag{73}
\]

From this, the relation (32) is satisfied if \( \lambda = 2p \) which is consistent with the localization condition \( 2p - \lambda < 3 \). On the other hand, the localization on a 3-brane also can be obtained for \( 2p - \lambda > 7 \), when we use the solutions \( \xi_{0,(2)} \) and \( \psi(r) \). For this, we can perform an asymptotic analysis. In this limit, we get \( \xi_{0,(2)} = \psi(r) \) and, by using these asymptotic solutions in (70), is easy to show that equation (32) cannot be satisfied for both sectors simultaneously. In fact, when we consider \( \xi_{0,(2)} \) for any limit of \( r \), the condition (32) cannot be satisfied for any choice of \( \lambda \). This result lead us to conclude that only the constant solution \( \xi_{0,(1)} \) can be really consistent for the interacting case in this model.
(C) - The last model considered here will be that generated by intersecting delta-like branes \[18,19\]. For this, the extra dimensions belong to the range \(y^1, y^2 \in (-\infty, \infty)\), and the metric can be written as

\[
ds^2 = \frac{1}{(1 + k_1 |y^1|^2 + k_2 |y^2|^2)^2} \left[ ds^\mu ds_\mu + (dy^1)^2 + (dy^2)^2 \right]. \tag{74}\]

Now, the equation of motion (16) for the zero-mode gives,

\[
\partial_j \left[ e^{[d-2p+\lambda]\sigma(y^j)} \partial^j \xi_0(y^j) \right] = 0, \tag{75}
\]

\[
\partial_k \left[ e^{[d-2p+\lambda]\sigma(y^j)} \delta^{kl} \xi_0(y^j) \right] = 0. \tag{76}
\]

and the solutions are given by

\[
\xi_{0,(1)}(y) = c_0 \quad \text{and} \quad \xi_{0,(2)}(y) = c_1 e^{-[d-2p+\lambda+1]\sigma}, \tag{77}
\]

\[
\psi(y) = c_2 e^{-[d-2p+\lambda]\sigma}. \tag{78}
\]

Beyond this, the effective Lagrangian for this sector in Eq. (21) gives the localization integral,

\[
K_1 = \int dy^1 dy^2 e^{[d-2p+\lambda]\sigma} \xi_0^2, \tag{79}
\]

\[
K_2 = \int dy^1 dy^2 e^{[d-2p+\lambda]\sigma} \psi^2. \tag{80}
\]

For this case, the integral \(K_1\) will be finite when: (C-a) \(2p - \lambda < d - 2\), by considering the constant solution in (77); and (C-b) \(2p - \lambda > d + 4\), by considering the non-constant solution \(\xi_{0,(2)}\) in (77). About the integral \(K_2\), it will be finite for \(2p - \lambda > d + 2\), by using the solution (78). This condition is always satisfied by using the condition (C-b) above. By considering the free case and a 3-brane \((d = 4)\), we can confine only a 0-form, for the constant solution \(\xi_{0,(1)}\), and \(p \geq 5\), for the non-constant solution \(\xi_{0,(2)}\).

**Consistency Test**

Here, the Hodge symmetry applied for the free case eliminate all the values of \(p\). This because, the 0-form field does not have its bulk dual, 4-form, confined, and the same for any other field. We mentioned in subsection (3.2), the Hodge duality is valid only if the space is topologically trivial. Thus, as the Ricci scalar for this intersecting model have Dirac delta ‘functions’, the space does not satisfy the above requirement. Thus, the indirect equivalence between the \(p\)-form fields, provided by Hodge transformation, is not valid for that model. In this way, statements (i) and (ii) cannot be used as consistency conditions here.

Nevertheless, we still can discuss the consistency of Einstein’s equations without consider the Hodge duality consistency. In doing this, the energy-momentum tensor gives us

\[
T_{\mu \nu}^{(\text{mat})}(x,r) = e^{[\lambda-2p]\sigma(y)} \xi_0^2(y) T_{\mu \nu}^{[p]}(x) + e^{[\lambda-2p]\sigma(y)} \psi^2(y) T_{\mu \nu}^{[p-2]}(x). \tag{81}
\]

Thus, the localization can be consistent if \(\xi_{0,(1)}(y) = c_1, \psi(y) = 0\) and \(\lambda = 2p\). On the other hand, by using the non-constant solutions, we get, \(e^{[\lambda-2p]\sigma(y)} \xi_0^2(y) = e^{-[10+\lambda-2p]\sigma}\) and \(e^{[\lambda-2p]\sigma(y)} \psi^2(y) = \ldots\)
$e^{-[8+\lambda-2p]\sigma}$. Therefore, from this, it is not possible to do both sectors consistent simultaneously. However, by doing $\psi = 0$, the consistency with Einstein’s equations can be obtained for $\lambda = 2p - 10$. Just like the other models, this consistency analysis fix the parameter $\lambda$. For the free case ($\lambda = 0$), the fields allowed are only a 0-form and a 5-form fields.

5 Arbitrary Codimension Case

In this section, let us generalize the discussion for higher codimension scenarios. Unfortunately, today there are few models with this feature in the literature, even so, this analysis will make the discussion more general. To perform the calculus, we will use a metric given by,

$$ds^2 = g_{MN}dx^M dx^N = e^{2\sigma(y)}\tilde{g}_{\mu \nu}(x)dx^\mu dx^\nu + \bar{g}_{jk}(y)dy^j dy^k,$$

(82)

where, the warp factor $\sigma(y)$ and $\bar{g}_{jk}(y)$ depend on the extra dimensions coordinates $y^j$. Now, capital indexes $M, N$ run on all dimensions $D = d + n$; Greek indexes are related to the brane and run on $\mu, \nu = (1, \ldots, d)$; and, Latin indexes are related to the extra dimensions, i.e., $j, k = (1, \ldots, n)$.

The action for a free $p$-form field in such background will be written as,

$$S = -\frac{1}{2(p+1)!} \int d^d x d^n y \sqrt{-\tilde{g}} F_{N_1 \ldots N_{p+1}} F^{N_1 \ldots N_{p+1}},$$

(83)

where $F_{N_1 \ldots N_{p+1}} = \partial (N_1 \bar{A}_{N_2 \ldots N_{p+1}})$, with $\bar{A}_{N_1 \ldots N_p}$ the $p$-form field. From the action (83), we can obtain the below equation of motion,

$$\partial_{v_1} \left( \sqrt{-\tilde{g}} F^{v_1}_{\nu_2 \nu_3 \ldots \nu_{p+1}} \right) + \partial_{k_1} \left[ \sqrt{-\tilde{g}} F^{k_1}_{\nu_2 \nu_3 \ldots \nu_{p+1}} \right] = 0.$$  

(84)

We are interested in the values of $p$ confined on the brane. Thus, to get this, we can choose only the components $A_{\nu_1 \ldots \nu_p}$ non-zero without loss of generality. In this setup, the above equations of motion can be reduced as,

$$\begin{align*}
\partial_{v_1} \left( \sqrt{-\tilde{g}} F^{v_1 \nu_2 \ldots \nu_{p+1}} \right) + \partial_{k} \left[ \sqrt{-\tilde{g}} g^{v_1 \mu_2 \ldots \nu_{p+1}} g^{\nu_{p+2} \mu_{p+1}} \partial_{v_{p+1}} A_{\mu_2 \ldots \mu_{p+1}} \right] &= 0, \\
\partial_{v_1} \left( \sqrt{-\tilde{g}} g^{v_1 \nu_2 \ldots \nu_{p}} \partial_{\nu} A_{\mu_2 \ldots \mu_{p}} \right) &= 0.
\end{align*}$$

(85)  

(86)

Now, the components will be split as $A_{\mu_1 \ldots \mu_p} = A_{\mu_1 \ldots \mu_p}(x)\xi(y)$, and the above equation gives us,

$$\partial_{v_1} \left( \sqrt{-\tilde{g}} F^{v_1 \nu_2 \ldots \nu_{p+1}}(x) \right) = m^2 \sqrt{-\tilde{g}} A^{\nu_2 \ldots \nu_{p+1}}(x),$$

(87)

$$\sqrt{-\tilde{g}} \frac{e^{-(d-2p-2)\sigma}}{\sqrt{-\tilde{g}}} \partial_k \left[ \sqrt{-\tilde{g}} e^{(d-2p)\sigma} \tilde{g}^{kj} \partial_{v} \xi \right] = m^2 \xi(y),$$

(88)

$$\partial_{v_1} \left[ \sqrt{-\tilde{g}} A^{v_1 \ldots \nu_p}(x) \right] \partial_k \xi(y) = 0.$$  

(89)

In equations (88) and (89), the index contraction are performed with brane metric $\hat{g}_{\mu \nu}(x)$. From Eq. (89), we can obtain a gauge condition $\partial_{v_1} \left[ \sqrt{-\tilde{g}} A^{v_1 \ldots \nu_p}(x) \right] = 0$. The equation (89) has a constant solution for the zero-mode $\xi_0(y)$, however, the non-constant solution is model dependent. Beyond this, localization conditions can be obtained only if we know the warp factor $\sigma(y)$. In view this, to study the zero-mode
localization and the values of $p$ allowed, will be used the braneworld model presented in Ref. [20]. For this case, the metric is given by,

$$ds^2 = e^{2\sigma(y)} \left[ dx_\mu dx^\mu + \delta_{jk} dy^j dy^k \right] = \left( 1 + k \sum_{j=1}^{n} |y_j| \right)^{-2} \left[ dx_\mu dx^\mu + \delta_{jk} dy^j dy^k \right], \quad (90)$$

By using this, Eq. (89) can be solved for the massless mode ($m^2 = 0$) and the solutions are given by,

$$\xi_{0,(1)}(y) = c_1, \quad (91)$$

$$\xi_{0,(2)}(y) = c_2 e^{-(d+n-2p-1)\sigma(y)}. \quad (92)$$

With these solutions, integral in extra dimensions for the zero-mode obtained from Eq. (83) can be written as,

$$K = \int d^n y e^{(d+n-2p-2)\sigma(y)} \xi_0^2. \quad (93)$$

Thus, the localization conditions for the free case are given by (a) $p < \frac{d-2}{2}$, for the constant zero-mode solution $\xi_{0,(1)}(y)$ and (b) $p > \frac{d+2n}{2}$, for the non-constant zero-mode solution $\xi_{0,(2)}(y)$. With these values of $p$, we can discuss the consistency of localization with Hodge transformation and the Einstein’s equations.

**Consistency test**

Here, just like the case (C), the topology of the space is non-trivial and the Hodge duality symmetry cannot be applied as consistency conditions. About the consistency with the Einstein’s equations, the discussion performed in section (3.2) is also valid here. In fact, as only the 0-form is confined for a 3-brane, we will get for the zero-mode an energy-momentum tensor given by,

$$T^{(\text{mat})}_{\mu\nu}(x,r) = \xi_0^2(y)T_{\mu\nu}(x). \quad (94)$$

And, as we see in the item (C) of last section, the localization of the scalar field is carried out with the constant solution, in order that the condition (32) is satisfied. Therefore, the scalar field confinement is consistent with Einstein’s equations.

**6 Conclusions**

Up today, the study of fields localization in braneworld was performed by considering the finite integral argument as a sufficient condition to states that a field is confined for such model. Recently, Ref. [64] performed an analysis for fields of different spin and the authors shown that the finite integral argument is a necessary, but not sufficient, condition to affirm that a field is consistently localized. For free $p$-form fields, Ref. [63] carried out a study for codimension 1 RS models. And, by considering other aspects of the theory, for example, the Hodge symmetry and Einstein’s equations, it was obtained a similar conclusion, the finite integral argument is not sufficient to provide a consistent localization.

In this manuscript, we performed a discussion about the consistency of the localization procedure commonly used to study fields confinement applied to $p$-form fields. By exploring an indirect equivalency between a $p$-form and a $(D - p - 2)$-form, provided by Hodge transformation, it was shown that the localization procedure will be consistent only if both fields are confined simultaneously on the brane. Next,
by using Einstein’s equations, we show that the stress tensor of these \( p \)-form fields must satisfy a specific relation presented in Eq. (32), both results obtained for a generic codimension 2 scenario. In section 4, these general results were applied for some 6D braneworld models. We consider first the model presented in Refs. [14][15], and for this case, the integral of localization, (51), gives us (A-a) \( d - 2p - 1 + \lambda > 0 \) and (A-b) \( d - 2p + 3 + \lambda < 0 \). As discussed in that section, by considering a 3-brane and the free case \( (\lambda = 0) \), the above conditions allow us to confine a bulk \( p \)-form, with \( p = 0, 1 \) and \( p = 4 \). In this way, by analyze only the localization integrals, a vector field (1-form) seems confined on a 3-brane. However, when the consistency conditions, provided by the Hodge transformation and the Einstein’s equations, are considered, we conclude that the free vector field cannot be consistently localized on the 3-brane. First, by Hodge transformation, the condition (i), presented in subsection (3.2), states that the confinement must comprise the 1-form and its bulk dual 3-form. Thus, as the conditions (A-a) or (A-b) with \( \lambda = 0 \) do not include the 3-form field, the confinement of this field is not consistent. This result is confirmed by using the Einstein’s equation, in fact, from this point, is consistent only the localization of the bulk 0-form and its bulk Hodge dual 4-form fields. We discuss also, for that model, the interacting case \( (\lambda \neq 0) \).

Now, the Hodge symmetry is not valid, but we can still explore the consistency with Einstein’s equation. As a main result, the condition (32) determine the coupling parameter, namely, it gets \( \lambda = 2p \). Similar results are also obtained for the models presented in Refs. [16][17], where the metric is given by Eq. (61).

We discussed still the other model generated by delta-like brane intersecting, namely, that models presented in Refs. [19][20]. In these models, the extra coordinates are both infinitely large, and the integral of localization for the free case gives us, for a 3-brane, the conditions \( p < 1 \) and \( p > 2 + n \), where \( n \) is the number of extra dimensions. In subsection (3.2), the case (C), performed for \( n = 2 \), allows the localization only of a 0-form and a \( p \)-form with \( p > 4 \). With this, When we consider the consistency conditions, obtained from Hodge symmetry and Einstein’s equations, the first conclusion is that all free \( p \)-form field must be ruled out by consistency reasons with the Hodge transformation, even that the 0-form be consistent with Einstein’s equation. However, we would like to stress that the Hodge duality is valid only when the space is trivial [65]. In this way, the models of delta-like brane intersecting can present a non-trivial space, and thus, the Hodge symmetry would not be valid. Therefore, the localization of a scalar field can be considered consistent because Hodge symmetry does not apply and there is no inconsistency with Einstein’s equations. Here, we would like stress again the results obtained for the vector field. For this field, and by using only the Einstein’s equation, Ref. [64] already has shown that the free vector field localization cannot be consistent for any dimensional setup. Here, by exploring other aspects, the Hodge symmetry, we get the same conclusion. Therefore, there is really a necessity of a mechanism that provide the confinement of the \( U(1) \) gauge field. This is an important result, considering the variety of works in the literature which states, by using only the finite integral argument, that a free \( U(1) \) gauge field can be localized in 6D naturally [19][67][72]. And, as showed in this manuscript, it is not really true. It is worth noting that the consistency conditions obtained from Hodge transformation and Einstein’s equation are completely independent of each other. Nevertheless, the conclusions obtained of them about the values of \( p \) confined are in total agreement of each other. This reinforces the results obtained in Ref. [64] on the consistency conditions provided by Einstein’s equation and applied for other fields. Now, we can ask if there may be other symmetries that can be used as consistency conditions for the localization study of these other fields, but this is a topic for future works.
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