ON POSSIBILITY OF APPLYING THE QUASI-ISOTHERMAL STÄCKEL’S MODEL TO OUR GALAXY

A. O. Gromov\textsuperscript{1}, I. I. Nikiforov\textsuperscript{2}, L. P. Ossipkov\textsuperscript{1}

\textsuperscript{1} Department of Space Technologies and Applied Astrodynamics, Saint Petersburg State University, Universitetskij pr. 35, Staryj Peterhof, Saint Petersburg 198504, Russia; granat08@yandex.ru

\textsuperscript{2} Sobolev Astronomical Institute, Saint Petersburg State University, Universitetskij pr. 28, Staryj Peterhof, Saint Petersburg 198504, Russia; nii@astro.spbu.ru

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Abstract. Earlier the quasi-isothermal Stäckel’s model of mass distribution in stellar systems was considered. The expression for spatial density was found. In this work an application of this model to our Galaxy is studied. The model rotation curve was fitted to data on kinematics of neutral hydrogen. Estimations of structural and scale parameters of the model and equidensities for our Galaxy are presented.

Key words: Stellar dynamics – methods: analytical – Galaxy: mass distribution

1. INTRODUCTION

There are various methods of modelling the gravitational potential of our Galaxy and other stellar systems. One of them is based on assumption that the potential is of Stäckel’s form. After pioneering works by Kuzmin (1952, 1956) the method was further developed by de Zeeuw et al. (1986), Dejonghe & de Zeeuw (1988) and others. In this paper we study possibilities of using the quasi-isothermal potential suggested by Kuzmin et al. (1986) for constructing Stäckel’s model of our Galaxy.

It is known that the third quadratic in velocities integral of motion

\[ I_3 = (Rv_z - zv_R)^2 + z^2v_\theta^2 + z_0^2(v_z^2 - 2\Phi^*), \]  

exists for Stäckel’s potentials. Here \( R, z \) are the cylindrical coordinates; \( v_R, v_\theta, v_z \) are the projections of spatial velocity; \( z_0 \) is a scale parameter of dimension of length; a function \( \Phi^*(R, z) \) must satisfy the equations

\[ z_0^2 \frac{\partial \Phi^*}{\partial R} = z^2 \frac{\partial \Phi}{\partial R} - Rz \frac{\partial \Phi}{\partial z}, \quad z_0^2 \frac{\partial \Phi^*}{\partial z} = (R^2 + z_0^2) \frac{\partial \Phi}{\partial z} - Rz \frac{\partial \Phi}{\partial R}, \]  

where \( \Phi(R, z) \) is an axisymmetric potential. The assumption on the existence of \( I_3 \) allows to explain the observed triaxiality of velocity ellipsoid.
In this work we suppose that the potential in the equatorial plane has the following form:

\[ \Phi(R, 0) = \Phi_0 \ln \left[ 1 + \frac{\beta}{w(R)} \right], \]  
(3)

where \( \beta \in [0, +\infty) \) is a structural parameter of the model,

\[ w^2(R) = 1 + \kappa^2 R^2, \]  
(4)

\( \Phi_0 \) and \( \kappa \) are scale parameters.

This potential was proposed by Kuzmin et al. (1986) for spherical systems and was called quasi-isothermal. The potentials by Schuster–Plummer and Jaffe are limiting cases of this one (when \( \beta \to 0 \) and \( \beta \to \infty \) respectively).

Stäckel’s models of mass distribution with the quasi-isothermal potential were constructed and an analytical (though cumbersome) expression for a spatial density was found (Gromov 2012, 2013, 2014). Stäckel’s models with the Schuster–Plummer and Jaffe potentials were also studied (Gromov 2013, 2014).

2. DATA

In this study we use data on the rotation of the neutral hydrogen: five independent data sets from the whole 21-cm line profile and one set from the 21-cm tangent points (see details in Nikiforov & Petrovskaya 1994). The whole profile data contain Camm’s function values \( \Omega \equiv R_0 (\omega - \omega_{\text{LSR}}) \) in relation to \( x \equiv R/R_0 \), where \( R \) is the distance to the galactic axis, \( \omega \) and \( \omega_{\text{LSR}} \) are the angular velocities of HI at the distance \( R \) and of the Local Standard of Rest respectively, \( R_0 \) is the distance of the Sun to the Galactic center. The tangent points’ data also define in fact the dependence of \( \Omega \) on \( x \). The total number of HI data points is 239.

It should be pointed out that analyses of the whole 21-cm profiles give at any radius the velocity of rotation averaged over essentially all galactocentric circle of this radius. Therefore the resultant HI rotation curve is well smoothed over essentially all galactocentric longitudes.

A value of the linear velocity of rotation, \( v_i \), for an HI data point \((x_i, \Omega_i)\) can be calculated by the formula

\[ v_i = [\Omega_i + v_{\text{LSR}}] x_i, \]  
(5)

where \( v_{\text{LSR}} \) is the linear velocity of the LSR. Here we adopted a value for \( v_{\text{LSR}} = 220 \) km/s as a combination of constants \( v_{\text{LSR}} = R_0 \omega_{\text{LSR}} \), where \( R_0 = 8.0 \) kpc (e.g., Reid 1993; Nikiforov 2004; Nikiforov & Smirnova 2013), \( \omega_{\text{LSR}} = 27.5 \) km s\(^{-1}\) kpc\(^{-1}\) is an intermediate value between recent determinations (e.g., Feast & Whitelock 1997; Zabolotskikh et al. 2002). Since as a rule the number of points in a HI data set can be arbitrary set by authors of study, the weights of data points, \( p_i \), were adopted to be proportional to the length of interval \([x_{\text{min}}, x_{\text{max}}]\) covered by this set (see Table 1 in Nikiforov & Petrovskaya 1994).

Notice that the form of the rotation curve based on the HI data is independent of calibrations of distance scales and is specified by only the parameter \( v_{\text{LSR}} \), according to Eq. (5). Choosing a value for \( R_0 \) affects the value of \( v_{\text{LSR}} \), with our parametrization of the HI rotation curve, and scales the coefficient \( \kappa \) in Eq. (4), in any case. Other parameters of the model potential (3) do not directly depend on an adopted value of \( R_0 \).
3. ESTIMATION OF $\Phi_0$, $\beta$, $\kappa$

To construct the Stäckel’s model potential (3), (4) for our Galaxy it is necessary to estimate parameters $\Phi_0$, $\beta$, $\kappa$. We shall fit the model circular velocities, $v_c$, to observational data on the rotation of the Galaxy. An expression for the circular velocity

$$v_c^2(R) = -R \frac{\partial \Phi}{\partial R}(R, 0)$$

in the case of the quasi-isothermal potential has the following form:

$$v_c^2(R) = \Phi_0 \frac{\beta \kappa^2 R^2}{(1 + \kappa^2 R^2)^{\frac{3}{2}} \left(1 + \frac{\beta}{\sqrt{1 + \kappa^2 R^2}}\right)},$$

where $q = \frac{\beta}{\beta + 1}$. We shall use $q$ as a structural parameter instead of $\beta$ for $q \in [0, 1]$.

The model parameters were estimated by the ordinary least-squares fitting. We minimized the statistics

$$L^2 = \sum_{i=1}^{239} p_i \left[v_c(R_i) - v_i\right]^2,$$
where \( v_c(R_i) \) is the model value of circular velocity at \( R_i \) calculated by Eq. (7), \( v_i \) is the “observed” value of circular velocity calculated by Eq. (5), \( p_i \) is the weight.

We found, that minimum \( L_2^{2} \) is achieved, when \( \Phi_0 = 258.1 \pm 1.5 \) km s\(^{-2}\), \( \kappa = (0.32 \pm 0.01) (R_0/8 \) kpc\)\(^{-1} \) kpc\(^{-1} \), \( q = 1.09^{+0.008} \). With these values, the quasi-isothermal model gives the best approximation for observational data.

A comparison of the model curve of circular velocity with observational data is shown on Fig. 1, where the solid curve is the model velocity curve \( v_c(R) \) and points are the HI data. The mean error of weight unit for this solution is \( \sigma = 2.98 \) km s\(^{-1}\).

Although a polynomial model of rotation curve can reveal a finer structure of the HI rotation law with \( \sigma = 2.10 \) km s\(^{-1}\) for the same data (Nikiforov 2000), Fig. 1 demonstrates that the quasi-isothermal potential in the equatorial plane represents the main trend in the rotation low and can be used for an approximation of the rotation curve for our Galaxy.

It is worth noting that the form of rotation curve for the quasi-isothermal potential (Fig. 1) is similar to the form of the “universal” rotation curve for spiral galaxies (Persic et al. 1996). Recent results of kinematic modelling the Galactic subsystem of high-mass star forming regions (Reid et al. 2014) demonstrate that the universal rotation curve of Persic et al. (1996) is well suited to the representation of the rotation curve of the Galaxy.

4. ESTIMATION OF \( z_0 \)

The expression for density of Stäckel’s models includes the parameter \( z_0 \). But it is impossible to find the latter from the rotation curve. In principle, \( z_0 \) can be estimated from the run of total spatial density, but presently we cannot do it for any one-component Stäckel’s model. Indeed, we do not know the density run for the dark matter from observations. Hence, to estimate \( z_0 \) we are obliged to suppose that in the Solar neighborhood the potential of our model is close to potentials of other models constructed with using not only the rotation curve but other observational data as well. The model by Gardner et al. (2011) is the last of such models. Its potential is

\[
\Phi = \Phi_H + \Phi_C + \Phi_D + \Phi_g, \tag{9}
\]

where

\[
\Phi_H = \frac{1}{2} V_h^2 \ln(R^2 + z^2 + R_1^2),
\]

\[
\Phi_C = -\frac{GM_{C_1}}{\sqrt{R^2 + z^2 + R_{C_1}^2}} - \frac{GM_{C_2}}{\sqrt{R^2 + z^2 + R_{C_2}^2}},
\]

\[
\Phi_D = \sum_{i=1}^{3} \frac{-GM_{d_i}}{\sqrt{R^2 + (a_{d_i} + \sqrt{z^2 + b_i^2})^2}},
\]

\[
\Phi_g = \sum_{n=1}^{3} \frac{-GM_{g_n}}{\sqrt{R^2 + (a_{g_n} + \sqrt{z^2 + b_g^2})^2}},
\]

where \( G \) is the gravitational constant, the values of parameters \( V_h^2, R_1, M_{C_1}, M_{C_2}, R_{C_1}, R_{C_2}, M_{d_i}, a_{d_i}, b, M_{g_n}, b_g \) are given by Gardner et al. (2011).
Fig. 2. The function \( |z_0(R)| \) for the model of Gardner et al. (2011).

For any potential \( \Phi(R, z) \) it is possible to define a function

\[
\begin{align*}
|z_0^2(R)| &= \left[ 3 \frac{\partial \Phi(R, z)}{\partial R} + R \left( \frac{\partial^2 \Phi(R, z)}{\partial R^2} - 4 \frac{\partial^2 \Phi(R, z)}{\partial z^2} \right) \right]_{z=0} - R^2 \tag{10}
\end{align*}
\]

(Kuzmin 1952, Ossipkov 1975). It follows from the condition of existence of the third quadratic integral that \( |z_0^2(R)| \equiv \text{const} \) for Stäckel’s models. Fig. 2 shows \( |z_0(R)| \) for the model of Gardner et al. (2011). We see that this function is almost constant for \( R \) from 1.5 kpc to 9 kpc. The latter means that for such potential the expression can be used as an approximate integral of motion for stars moving inside this zone (with not very large \( |v_z| \)). Functions \( z_0(R) \) were constructed earlier by Einasto & Rümmel (1970) and Ossipkov (1975) for Einasto’s models of our Galaxy and M 31. These authors discussed the negativity of \( z_0^2(R) \) for large \( R \) that takes place also for the potential.

We conclude that it is possible to set \( z_0 = 5.4 \) kpc (this is a value of \( |z_0| \) at \( R_0 = 8 \) kpc for the model of Gardner et al. 2011). Earlier Ossipkov (1975) found \( z_0 = 7 \) kpc, and according to Kuzmin (1956) \( z_0 = 3.6 \) kpc. Our value of \( z_0 \) is close to \( z_0 = 4.8 \) kpc obtained by Malasidze (1973).
For constructing the quasi-isothermal model of Galaxy the obtained values of parameters were substituted in an expression for spatial density (Gromov 2012, 2013, 2014). As a result, the equidensities for central layers of Galaxy were obtained. They are shown on Fig. 3. As earlier (Gromov 2012, 2013, 2014), $\rho_0$ is a density in the center of the model, when $\kappa = 1 \text{kpc}^{-1}$, $\beta = 1$, $\Phi_0 = 1 \text{km}^2\text{s}^{-2}$.

These equidensities are similar to equidensities for Jaffe’s model, which is the limiting case ($\beta \to \infty$) of quasi-isothermal model

\[ \Phi(R, 0) = \Phi_0 \ln \left( 1 + \lambda^{-1} R^{-1} \right), \tag{11} \]

where $\lambda$ is a scale parameter.

Fig. 3 shows a drastic drop in density at equidensities which dimensions are close to ones of the Galactic triaxial bulge/bar—(3.5 : 1.4 : 1.0) kpc (see Gardner & Flynn 2010). It is possible that this resemblance is accidental. However, the direct inclusion of bulge component in the dynamic model with the use of the same H I data (Nikiforov 2001) gives a similar result: a severely limited from above estimate of the cut-off radius of bulge $a_b = 2.1^{+0.6}_{-2.1} \text{kpc}$.

It is evident that the total mass of the quasi-isothermal model is

\[ M = \Phi_0 \frac{\beta}{\kappa G}. \tag{12} \]

The minimum value of $M$ for the obtained values of parameters is $6 \times 10^{12} M_\odot$. The
maximum value of the mass for the same parameter values is unlimited, because $\beta$ can take infinity large values.

6. CONCLUSIONS

We estimated parameters of the quasi-isothermal model of mass distribution for our Galaxy using data on the rotation of the neutral hydrogen and obtained equidensities for this set of parameters.

The comparison of the model with observational data shows the relevance of the model. So, the quasi-isothermal model can be used for approximation of the gravitational potential of our Galaxy. It is possible to apply the quasi-isothermal model to external galaxies with almost flat rotation curve.

In the future a multi-component Stäckel’s model will be constructed for the Galaxy to gain a better approximation to the data.

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