ON LAGRANGIAN EMBEDDINGS OF CLOSED NON-ORIENTABLE 3-MANIFOLDS

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Abstract. We prove that for any compact orientable connected 3-manifold with torus boundary, a concatenation of it and the direct product of the circle and the Klein bottle with an open 2-disk removed admits a Lagrangian embedding into the standard symplectic 6-space. Moreover, minimal Maslov number of the Lagrangian embedding is equal to 1.

1. Introduction

In this paper, we study the existence problem of a Lagrangian embedding into the standard symplectic space $\mathbb{R}^6_{st} = (\mathbb{R}^6, \sum_{j=1}^3 dx_j \wedge dy_j)$. Starting with M. Gromov’s discovery of the technique of pseudo-holomorphic curves [14], a number of necessary conditions for the existence of a Lagrangian embedding have been proven. A typical example for the standard symplectic space $\mathbb{R}^6_{st}$ is a partial classification of Lagrangian submanifolds proved by K. Fukaya: a closed orientable connected prime 3-manifold $L$ admits a Lagrangian embedding into $\mathbb{R}^6_{st}$ if and only if there exists a non-negative integer $g$ such that $L$ is diffeomorphic to the product $S^1 \times \Sigma_g$, where $\Sigma_g$ is the closed orientable connected surface of genus $g$ [12]. On the other hand, few sufficient conditions for the existence of a Lagrangian embedding were known. Recently, Y. Eliashberg and E. Murphy established the resolving theory of Lagrangian intersections and proved the $h$-principle for Lagrangian embeddings with a concave loose Legendrian boundary [11]. This $h$-principle has applications to the existence of a Lagrangian embedding of closed manifolds. For the standard symplectic space $\mathbb{R}^6_{st}$, T. Ekholm, Y. Eliashberg, E. Murphy, and I. Smith gave the following application: for a closed orientable connected 3-manifold $L$, there exists a Lagrangian embedding of the connected sum $L \# (S^1 \times S^2)$ into $\mathbb{R}^6_{st}$ [7]. The theory of loose Legendrian embeddings developed by E. Murphy [17] played a central role in the resolving theory and in the application. Here we give another application of the results of [11].

We introduce some notations and conventions before the statement. For a non-negative integer $g$, we denote by $N_{2g}$ the closed non-orientable connected surface of Euler characteristic $-2g$. We fix an embedded closed 2-disk $D^2_N$ in the Klein bottle $N_0$. We also fix an identification of the compact surface $N_0 \setminus \text{Int} D^2_N$ with the compact surface obtained by the orientation-reversing 0-surgery on the unit closed 2-disk $D^2$. This identification induces a diffeomorphism $\partial(S^1 \times (N_0 \setminus \text{Int} D^2_N)) \rightarrow \partial(S^1 \times D^2)$ between 2-tori. For a closed orientable connected 3-manifold $M$ and an
embedded 2-torus \( T \subset M \) bounding a solid torus with a parameterization \( S^1 \times D^2 \), we denote by \( M_T \) the closed non-orientable connected 3-manifold

\[
(M \setminus \text{Int}(S^1 \times D^2)) \cup (S^1 \times (N_0 \setminus \text{Int} D^2_N))
\]

concatenated along their boundaries by the above diffeomorphism. Our main result is the following.

**Theorem 1.1.** Let \( M \) be a closed orientable connected 3-manifold and \( T \subset M \) an embedded 2-torus bounding a solid torus with a parameterization \( S^1 \times D^2 \). Then, there exists a Lagrangian embedding \( M_T \to \mathbb{R}^6 \) of minimal Maslov number 1. In particular, for a closed orientable connected 3-manifold \( L \) and a non-negative integer \( g \), there exists a Lagrangian embedding \( L \# (S^1 \times N_{2g}) \to \mathbb{R}^6 \) of minimal Maslov number 1.

Our proof is similar to that of the above application [7], concatenating a Lagrangian filling and a Lagrangian cap. The existence of a Lagrangian cap is a consequence of the results of [11]. In [7], a Lagrangian filling of a loose Legendrian 2-sphere is constructed. The new part of this paper is a construction of a Lagrangian filling of a loose Legendrian 2-torus.

**Theorem 1.2.** A loose Legendrian 2-torus of vanishing Maslov class in the standard contact space \( \mathbb{R}^5_{\text{st}} \) admits a Lagrangian filling \( S^1 \times (N_0 \setminus \text{Int} D^2_N) \) of minimal Maslov number 1.

**Remark 1.3.** By Murphy’s \( h \)-principle for loose Legendrian embeddings [17], a loose Legendrian 2-torus of vanishing Maslov class in the standard contact space \( \mathbb{R}^5_{\text{st}} \) is unique up to Legendrian isotopy, see [17, Appendix A].

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2. **Proof of Theorems**

2.1. **Construction of a Lagrangian filling.** In this section, we prove Theorem 1.2. First, we recall some background on Lagrangian cobordisms.

**Definition 2.1.** Let \((Y, \alpha)\) be a coorientable contact manifold and \(\mathbb{R} \times Y\) its symplectization equipped with the symplectic structure \(d(e^t \alpha)\), where \(t\) is the coordinate of \(\mathbb{R}\). For Legendrian submanifolds \(\Lambda_-\) and \(\Lambda_+\) of \((Y, \alpha)\), a **Lagrangian cobordism** from \(\Lambda_-\) to \(\Lambda_+\) is a properly embedded Lagrangian submanifold \(L\) in the symplectization \(\mathbb{R} \times Y\) such that

\[
L \cap (-\infty, t_-] \times Y = (-\infty, t_-] \times \Lambda_- \quad \text{and} \quad L \cap [t_+, \infty) \times Y = [t_+, \infty) \times \Lambda_+
\]

for real constants \(t_-\) and \(t_+\) with \(t_- < t_+\). A Lagrangian cobordism is **exact** if the Lagrangian submanifold \(L\) is exact, i.e. the 1-form \(e^t \alpha\big|_L\) is exact. A Lagrangian cobordism is called a **Lagrangian cap** (resp. **Lagrangian filling**) if \(\Lambda_+ = \emptyset\) (resp. \(\Lambda_- = \emptyset\)). Immersed Lagrangian cobordism, cap, and filling and their exactness are defined in a similar way.
Proposition 2.3. The proof of Theorem 1.2.

Definition 2.2. Let $L_0$ be a Lagrangian cobordism from $\{t_0\} \times \Lambda_0$ to $\{t_1\} \times \Lambda_1$ and $L_1$ a Lagrangian cobordism from $\{t_1\} \times \Lambda_1$ to $\{t_2\} \times \Lambda_2$. A concatenation of $L_0$ and $L_1$ along $\{t_1\} \times \Lambda_1$ is a Lagrangian cobordism from $\{t_0\} \times \Lambda_0$ to $\{t_2\} \times \Lambda_2$ defined by the union $L_0 \cup L_1$. A concatenation of immersed Lagrangian cobordisms is defined in a similar way.

For integers $m = 1$ and 2, we denote by $\mathbb{R}_{st}^{2m+1}$ the standard contact space $(\mathbb{R}^{2m+1}, \alpha_{st} = dz - \sum_{j=1}^{m} y_j dx_j)$ and by $\mathbb{R} \times \mathbb{R}_{st}^{2m+1}$ its symplectization equipped with the symplectic structure $d(e^t \alpha_{st})$. We fix the parameterizations of the standard Legendrian unknot

$$K_1: S^1 \to \mathbb{R}_{st}^3 : \theta \mapsto \left(\sin \theta, -\sin 2\theta, \frac{2}{3} \cos^3 \theta\right)$$

and its stabilization

$$K_2: S^1 \to \mathbb{R}_{st}^3 : \theta \mapsto \left(\sin \theta, \sin 4\theta, \frac{4}{3} \cos^3 \theta - \frac{8}{5} \cos^5 \theta\right),$$

see Figure 1.

![Figure 1](image-url)  

**Figure 1.** The fronts of the standard Legendrian unknot and its stabilization.

The following construction of a Lagrangian filling of $K_2$ is the main part of the proof of Theorem 2.2.

**Proposition 2.3.** There exists a Lagrangian filling $N_0 \setminus \text{Int} \, D^3_{K_1} \to \mathbb{R} \times \mathbb{R}_{st}^3$ of $K_2$ of minimal Maslov number 1.

**Proof.** First, we construct an immersed Lagrangian cobordism from $K_1$ to $K_2$ with exactly one double point in $\mathbb{R} \times \mathbb{R}_{st}^3$ as follows. We pick a smooth cutoff function $\rho_1: [0, 1] \to [0, 1]$ such that

- $\rho_1(s) = 0$ and $\rho_1(1 - s) = 1$ if $0 \leq s \leq \frac{1}{3}$, and
- $\rho_1'(s) > 0$ if $\frac{1}{3} < s < \frac{2}{3}$.

Then, for a positive integer $n$, we define another cutoff function $\rho_2: [0, n] \to [0, 1]$ by $\rho_2(t) = \rho_1(\frac{t}{n})$. Using the cutoff function $\rho_2$, we consider the homotopy

$$k_{tr}: [0, n] \times S^1 \to \mathbb{R}^2 : (t, \theta) \mapsto \left(\sin \theta, \frac{2}{3} \cos^3 \theta + \rho_2(t) \left(\frac{2}{3} \cos^3 \theta - \frac{8}{5} \cos^5 \theta\right)\right)$$

from the front of $K_1$ to that of $K_2$. The monotonicity of $\rho_1$ implies that there is the unique tangent point $(0, 0) = k_{tr}(T, 0) = k_{tr}(T, \pi)$, where $T$ is defined by the equation $\rho_2(T) = \frac{2}{3}$. Solving the differential equation $y = \frac{\partial}{\partial t}$ on $\mathbb{R}_{st}^3$, the front homotopy $k_{tr}$ lifts to the Legendrian regular homotopy $f: [0, n] \times S^1 \to \mathbb{R}_{st}^3$,

$$f(t, \theta) = \left(\sin \theta, -\sin 2\theta + \rho_2(t)(\sin 4\theta + \sin 2\theta), \frac{2}{3} \cos^3 \theta + \rho_2(t) \left(\frac{2}{3} \cos^3 \theta - \frac{8}{5} \cos^5 \theta\right)\right).$$
Its trace \( \text{Tr}(f) : [0, n] \times S^1 \to \mathbb{R} \times \mathbb{R}^3_{\text{st}} : (t, \theta) \mapsto (t, f(t, \theta)) \) has exactly one double point

\[
(T, 0, 0, 0) = \text{Tr}(f)(T, 0) = \text{Tr}(f)(T, \pi)
\]
corresponding to the tangent point \((0, 0)\). We can check that its self-intersection number is equal to \(-1\). Perturbing the trace \( \text{Tr}(f) \), we construct the Lagrangian immersion \( \tilde{f}_{-1} : [0, n] \times S^1 \to \mathbb{R} \times \mathbb{R}^3_{\text{st}} \):

\[
\tilde{f}_{-1}(t, \theta) = \left( t, \sin \theta - \sin 2\theta + \rho_2(t)(\sin 4\theta + \sin 2\theta), \right.
\]

\[
\left. \frac{2}{3} \cos^3 \theta + (\rho_2(t) + \rho_2'(t)) \left( \frac{2}{3} \cos^3 \theta - \frac{8}{5} \cos^5 \theta \right) \right).
\]

Choosing the integer \( n \) sufficiently large, the derivative \( \rho_2' \) can be arbitrarily small, and hence there is one-to-one correspondence between the double point of \( \tilde{f}_{-1} \) and that of \( \text{Tr}(f) \) preserving self-intersection number. We denote by \( q \) the double point of self-intersection number \(-1\) of the Lagrangian immersion \( \tilde{f}_{-1} \). The image \( \tilde{f}_{-1}([0, n] \times S^1) \) is an immersed Lagrangian cobordism from \( K_1 \) to \( K_2 \), since \( \tilde{f}_{-1}([0, \varepsilon] \times S^1) = [0, \varepsilon] \times K_1 \) and \( \tilde{f}_{-1}((n - \varepsilon, n] \times S^1) = (n - \varepsilon, n] \times K_2 \) for a sufficiently small positive constant \( \varepsilon \).

We recall that the standard Legendrian unknot \( \{0\} \times K_1 \) admits a Lagrangian filling by 2-disk in \((\infty, 0] \times \mathbb{R}^3_{\text{st}}\). Concatenating this Lagrangian filling and the immersed Lagrangian cobordism \( \tilde{f}_{-1} \) along \( \{0\} \times K_1 \), we construct an immersed Lagrangian filling \( \tilde{h}_{-1} : D^2 \to \mathbb{R} \times \mathbb{R}^3_{\text{st}} \) of \( K_2 \) with exactly one double point \( q \). Resolving the double point \( q \) by Polterovich’s Lagrangian surgery \([15]\), we obtain a Lagrangian filling \( h_0 : N_0 \setminus \text{Int} D^2_N \to \mathbb{R} \times \mathbb{R}^3_{\text{st}} \) of \( K_2 \). Although there are two choices of the surgery depending on an order of the two sheets at the double point \( q \), each choice yields the same result in this dimension, see \([15]\).

Next, we compute the difference of Maslov potentials on the two sheets at the double point \( q \) of the Lagrangian cobordism \( \tilde{f}_{-1} \) and then show minimal Maslov number of the Lagrangian filling \( h_0 \) is equal to \( 1 \). A similar computation was made in \([9\text{, Section 2.2}]\). It suffices to consider the case \( \rho_1 : [0, 1] \to [0, 1] \) is the identity and \( n = 7 \). In fact, although the identity is not a cutoff function, the linear homotopy from the cutoff function \( \rho_1 \) to the identity can be realized by a Lagrangian regular homotopy of the Lagrangian immersion \( \tilde{f}_{-1} \). Moreover, a change of the integer \( n \) can also be realized by a Lagrangian regular homotopy of \( \tilde{f}_{-1} \). A straightforward computation shows that the Lagrangian immersion \( \tilde{f}_{-1} \) has exactly one double point \( q = \tilde{f}_{-1}(4, 0) = \tilde{f}_{-1}(4, \pi) \) if \( \rho_1 \) is the identity and \( n = 7 \). In this case, the Lagrangian immersion \( \tilde{f}_{-1} : [0, 7] \times S^1 \to \mathbb{R} \times \mathbb{R}^3_{\text{st}} \) is of the form

\[
\tilde{f}_{-1}(t, \theta) = \left( t, \sin \theta - \sin 2\theta + \frac{t}{7}(\sin 4\theta + \sin 2\theta), \right.
\]

\[
\left. \frac{2}{3} \cos^3 \theta + \frac{t+1}{7} \left( \frac{2}{3} \cos^3 \theta - \frac{8}{5} \cos^5 \theta \right) \right).
\]

For the computation, we choose the path \( l : [0, 1] \to [0, 7] \times S^1 : s \mapsto (4, \pi s) \), the reference Lagrangian subspace \( P_0 = (\tilde{f}_{-1})_* T_{(4,0)}([0, 7] \times S^1) \), the symplectic structure \( d(e^c \alpha_{\text{st}}) = e^c (dt \wedge (dz - ydx) + dx \wedge dy) \) on the symplectization \( \mathbb{R} \times \mathbb{R}^3_{\text{st}} \), and its symplectic 4-frame \( \{ \frac{\partial}{\partial t}, \frac{\partial}{\partial z}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \} \). Then the Lagrangian 2-frame
\((\tilde{f}_{-1}), \frac{\partial}{\partial t}, (\tilde{f}_{-1}), \frac{\partial}{\partial t}\) along the path \((\tilde{f}_{-1} \circ l)([0, 1])\) is of the form

\[
(\tilde{f}_{-1}), \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{1}{7}\left(\frac{2}{3}\cos^3(\pi s) - \frac{8}{5}\cos^5(\pi s)\right)\frac{\partial}{\partial z} + \frac{1}{7}(\sin(4\pi s) + \sin(2\pi s))\frac{\partial}{\partial y},
\]

\[
(\tilde{f}_{-1}), \frac{\partial}{\partial \theta} = \frac{1}{7}\cos(\pi s)(\sin(4\pi s) + \sin(2\pi s))\frac{\partial}{\partial z} + \cos(\pi s)\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)
+ \frac{2}{7}(8\cos(4\pi s) - 3\cos(2\pi s))\frac{\partial}{\partial y}.
\]

Taking their components, we define the paths of matrices

\[
X(s) = \begin{pmatrix} 1 & 0 \\ 0 & \cos(\pi s) \end{pmatrix}
\]
and

\[
Y(s) = \begin{pmatrix} \frac{1}{7}\frac{2}{3}\cos^3(\pi s) - \frac{8}{5}\cos^5(\pi s) \\ \frac{1}{7}(\sin(4\pi s) + \sin(2\pi s)) \\ \frac{1}{7}(8\cos(4\pi s) - 3\cos(2\pi s)) \end{pmatrix}.
\]

The difference of Maslov potentials on the two sheets at the double point \(q\) for the reference Lagrangian subspace \(P_0\) is computed by counting points through the angle

\[
\arg(\det(X(0) + iY(0))^2) = 2\arg(\det(X(0) + iY(0)))
\]
on the path \(\det(X(s) + iY(s))^2 : [0, 1] \to \mathbb{C} \setminus \{0\}\). This counting is invariant under a homotopy of paths \([0, 1] \to \mathbb{C} \setminus \{0\}\) relative to the boundary. In order to count, we look at the path \(\det(X(s) + iY(s)) : [0, 1] \to \mathbb{C} \setminus \{0\}\) being of the form

\[
\frac{1}{49}\left(\frac{704}{3}\cos^9(\pi s) - \frac{640}{3}\cos^7(\pi s) + \frac{1388}{15}\cos^5(\pi s) - \frac{32}{3}\cos^3(\pi s) + 49\cos(\pi s)\right)
+ \frac{i}{7}\left(-\frac{8}{5}\cos^5(\pi s) + \frac{386}{3}\cos^4(\pi s) - 140\cos^2(\pi s) + 22\right).
\]

Using the expression, we can show that

\[
\begin{align*}
(1) & \quad \det(X(0) + iY(0)) = \frac{1}{49}\left(115 + 136i\right); \\
(2) & \quad \det(X(\frac{1}{2}) + iY(\frac{1}{2})) = \frac{1}{22}; \\
(3) & \quad \det(X(1) + iY(1)) = \frac{1}{150}\left(-115 + 136i\right); \\
(4) & \quad \Re(\det(X(s) + iY(s))) = -\Re(\det(X(1 - s) + iY(1 - s))) \text{ if } 0 \leq s \leq \frac{1}{2}; \\
(5) & \quad \Im(\det(X(s) + iY(s))) = \Im(\det(X(1 - s) + iY(1 - s))) \text{ if } 0 \leq s \leq \frac{1}{2}; \text{ and} \\
(6) & \quad \Re(\det(X(s) + iY(s))) > 0 \text{ if } 0 \leq s < \frac{1}{2}.
\end{align*}
\]

Actually, the equalities \((1) - (5)\) are straightforward. The inequality \((6)\) can be shown by using the inequality of arithmetic and geometric means to estimate the first three terms. These properties \((1) - (6)\) imply that there exists a homotopy of paths \([0, 1] \to \mathbb{C} \setminus \{0\}\) from \(\det(X(s) + iY(s))\) to the counterclockwise circular arc, from \(\det(X(0) + iY(0))\) to \(\det(X(1) + iY(1))\), relative to the boundary. Moreover, its rotation angle \(\varphi_1\) satisfies \(\frac{\pi}{7} < \varphi_1 < \frac{\pi}{2}\), see Figure 2. This homotopy induces a homotopy of paths \([0, 1] \to \mathbb{C} \setminus \{0\}\) from \(\det(X(s) + iY(s))^2\) to the counterclockwise circular arc, from \(\det(X(0) + iY(0))^2\) to \(\det(X(1) + iY(1))^2\), relative to the boundary. Furthermore, its rotation angle \(2\varphi_1\) satisfies \(\frac{\pi}{7} < 2\varphi_1 < \pi\). Therefore, the difference of Maslov potentials on the two sheets at the double point \(q\) for the reference Lagrangian subspace \(P_0\) is equal to \(\frac{1}{2}\).

Combining the above discussion with the construction of Polterovich’s Lagrangian 1-handle \([0, 1] \times S^1\), we can see that minimal Maslov number of the Lagrangian filling \(\tilde{h}_0\) is equal to 1. In fact, Polterovich’s Lagrangian surgery \([18]\) resolves the double point \(q\) of \(\tilde{f}_{-1}\) and creates two loops. The one is the meridian loop \(\{0\} \times S^1\)
of the Lagrangian 1-handle whose Maslov index is equal to zero. The other one is the orientation-reversing loop obtained by smoothing the path \( f_{-1} \circ l \) along the path \([0,1] \times \{ \text{pt} \}\) of the Lagrangian 1-handle. By this smoothing, the path \( \det(X(s) + iY(s))^2 \) extends to a loop \( S^1 \to C \setminus \{0\} \). The rotation angle \( \varphi_2 \) of the extended part of this loop \( S^1 \to C \setminus \{0\} \) is coming from the path \([0,1] \times \{ \text{pt} \}\) of the Lagrangian 1-handle, and hence satisfies \( -2\pi < \varphi_2 < 2\pi \), see [18, Section 2]. Therefore, we obtain the estimate

\[
-\frac{3}{2}\pi < 2\varphi_1 + \varphi_2 < 3\pi.
\]

We recall that the Maslov index of a loop is odd if and only if the loop is orientation-reversing, so we also have

\[
2\varphi_1 + \varphi_2 \in \{2\pi(2k + 1) \mid k \in \mathbb{Z}\}.
\]

We conclude that the rotation angle \( 2\varphi_1 + \varphi_2 \) of the orientation-reversing loop is equal to \( 2\pi \), and thus its Maslov index is equal to 1. □

**Proof of Theorem 1.2.** We claim that the front \( S^1 \)-spinning [13] of the Lagrangian filling \( \tilde{h}_0 \) constructed in Proposition 2.3 is the desired one. First, composing a parallel transformation on the \( x \)-coordinate direction by a sufficiently large positive constant, we modify the Lagrangian filling \( \tilde{h}_0 : N_0 \setminus \text{Int} D^2_0 \to \mathbb{R} \times \mathbb{R}^3_{\text{st}} : p \mapsto (\tilde{t}(p), \tilde{x}(p), \tilde{y}(p), \tilde{z}(p)) \) to satisfy \( \tilde{x} > 0 \). Applying the front \( S^1 \)-spinning construction [13] to \( \tilde{h}_0 \), we obtain the Lagrangian filling \( F : S^1 \times (N_0 \setminus \text{Int} D^2_0) \to \mathbb{R} \times \mathbb{R}^5_{\text{st}}, \)

\[
F(\theta, p) = (\tilde{t}(p), \tilde{x}(p) \cos \theta, \tilde{y}(p) \cos \theta, \tilde{x}(p) \sin \theta, \tilde{y}(p) \sin \theta, \tilde{z}(p)).
\]

Then the Legendrian torus boundary \( \Sigma^{S_1} K_2 = F(\partial S^1 \times D^2) \subset \{ n \} \times \mathbb{R}^5_{\text{st}} \) is the front \( S^1 \)-spinning [8] of \( K_2 \), and hence is loose [5] in the sense of [17]. Moreover, its Maslov class vanishes. In fact, the Maslov index of the loop \( F(\{ \text{pt} \} \times \partial D^2) = K_2 \), the vanishing of the Maslov index is straightforward. □

**Remark 2.4.** The Lagrangian fillings \( \tilde{h}_0 \) and \( F \) are non-exact since the Legendrian boundary of \( \tilde{h}_0 \) is stabilized [3, 6] and that of \( F \) is loose [17].
2.2. Construction of a Lagrangian embedding. In this section, we prove Theorem 1.1. Concatenating the Lagrangian filling in Theorem 1.2 and a Lagrangian cap, we construct the desired Lagrangian embedding. We start with a construction of an immersed Lagrangian cap for the particular case. For a non-negative integer $g$, we fix an embedded closed 2-disk $D_g^2$ in the closed surface $\Sigma_g$.

Lemma 2.5. Let $L$ be a closed orientable connected 3-manifold, $g$ a non-negative integer, and $M'$ the connected sum

$$L\#(S^1 \times (\Sigma_g \setminus \text{Int } D_g^2)).$$

Then, the 3-manifold $M'$ can be realized as an immersed Lagrangian cap of the loose Legendrian torus $\Sigma M'$ and of the self-intersection number zero modulo two.

Proof. We note that the existence of a Lagrangian immersion $M' \to \mathbb{R} \times \mathbb{R}_st^5$ is equivalent to the triviality of the complexified tangent bundle $TM' \otimes \mathbb{C}$ by Gromov–Lees $h$-principle for Lagrangian immersions [15, 16]. In this case, the parallelizability of $M'$ implies the latter condition. Moreover, we can choose a Lagrangian immersion $M' \to \mathbb{R} \times \mathbb{R}_st^5$ to be an immersed Lagrangian cap of $\Sigma M'$ as follows.

The parallelizability of $M'$ allows us to take a Lagrangian homomorphism $TM' \to T(\mathbb{R} \times \mathbb{R}_st^5)$ such that its Gauss map $M' \to U(3)/O(3)$ is constant, where $U(3)/O(3)$ is the Lagrangian Grassmannian. Its restriction on the boundary $\partial M'$ is homotopic to the Lagrangian homomorphism $dF|_{S^1 \times \partial D^2}$ defined by the Lagrangian filling $F$ constructed in Theorem 1.2 as Lagrangian homomorphisms, since the Maslov class of $\Sigma M' / K_2$ vanishes. Therefore, there exists a Lagrangian homomorphism $\Phi: TM' \to T([n, \infty) \times \mathbb{R}_st^5)$ that is an extension of $dF|_{S^1 \times \partial D^2}$. We denote by $\phi: M' \to [n, \infty) \times \mathbb{R}_st^5$ the underlying map of $\Phi$. Using the contractibility of $\mathbb{R} \times \mathbb{R}_st^5$, we may assume that the map $\phi$ is a smooth extension of $F|_{S^1 \times \partial D^2}$. Then the relative cohomology class $[\phi^* d(e^t \alpha_{st})] \in H^2(M', \partial M'; \mathbb{R})$ vanishes by the property $H^2(\mathbb{R} \times \mathbb{R}_st^5; \mathbb{R}) = 0$. By the construction, we can choose the formal Lagrangian immersion $(\phi, \Phi)$ so that

- $d\phi = \Phi$ on a small neighborhood $U$ of the boundary $\partial M' = S^1 \times \partial D_g^2$;
- $\phi(U) \cap [n, n + \varepsilon) \times \mathbb{R}_st^5 = [n, n + \varepsilon) \times \Sigma M'$ for a small positive constant $\varepsilon$.

Applying the relative version of Gromov–Lees $h$-principle for Lagrangian immersions [15, 16], see also [10, Theorem 16.3.2], to the formal Lagrangian immersion $(\phi, \Phi)$, we obtain an immersed Lagrangian cap $\hat{\phi}: M' \to [n, \infty) \times \mathbb{R}_st^5$ of $\Sigma M' / K_2$.

We show that the Lagrangian immersion $\hat{\phi}: M' \to [n, \infty) \times \mathbb{R}_st^5$ can be chosen to have the self-intersection number zero modulo two. If the self-intersection number of $\hat{\phi}$ is equal to one modulo two, we modify it as follows. There exists a Lagrangian immersion $G$ of the 3-sphere to a Darboux ball in $\mathbb{R} \times \mathbb{R}_st^5$ of the self-intersection number zero modulo two [11]. We may assume that

- the image $G(S^3)$ is contained in $[n, \infty) \times \mathbb{R}_st^5$;
- $\hat{\phi}$ and $G$ intersect transversely at exactly two points.

In fact, we can deform the Lagrangian immersion $G$ to satisfy these conditions by a parallel transformation and a small perturbation as a Lagrangian immersion. Applying Polterovich’s Lagrangian surgery [18] to one intersection, we construct the connected sum $\hat{\phi}\# G: M' \to [n, \infty) \times \mathbb{R}_st^5$ of the Lagrangian immersions such that

$$(\hat{\phi}\# G)(M') \cap [n, n + \varepsilon'] \times \mathbb{R}_st^5 = \hat{\phi}(M') \cap [n, n + \varepsilon'] \times \mathbb{R}_st^5$$
for a small positive constant \( \varepsilon' \). Thus the image \((\tilde{\phi} \# G)(M')\) is an immersed Lagrangian cap of \( \Sigma_{S^1}K_2 \) and its self-intersection number is equal to zero modulo two. \( \square \)

**Proof of Theorem 1.1.** We first prove the particular case. We denote by \( \tilde{\phi} \) the Lagrangian immersion constructed in the proof of Lemma 2.5. We deform the Lagrangian immersion \( \phi \) to a formal Lagrangian embedding of \( M' \) into \( [n, \infty) \times \mathbb{R}^5_{st} \) relative to a small neighborhood of the loose Legendrian boundary \( \Sigma_{S^1}K_2 \). Applying [4, Theorem 2.2] to this formal Lagrangian embedding, we get a Lagrangian cap \( \tilde{\phi}_0 : M' \to [n, \infty) \times \mathbb{R}^5_{st} \) of \( \Sigma_{S^1}K_2 \). Concatenating the Lagrangian filling \( F \) and the Lagrangian cap \( \tilde{\phi}_0 \) along \( \Sigma_{S^1}K_2 \), we obtain a Lagrangian embedding \( L\#(S^1 \times N_{2g}) \to \mathbb{R} \times \mathbb{R}^5_{st} \) of minimal Maslov number 1. We recall that the symplectization \( \mathbb{R} \times S^5_{st} \) of the standard contact sphere \( S^5_{st} \) is symplectomorphic to the symplectic manifold \( \mathbb{R}^5_{st} \setminus \{0\} \). The construction is done by composing a symplectic embedding \( \mathbb{R} \times \mathbb{R}^5_{st} \to \mathbb{R} \times S^5_{st} \subset \mathbb{R}^5_{st} \) induced by a contact embedding \( \mathbb{R}^5_{st} \to S^5_{st} \).

We can similarly prove the general case. In fact, for a closed orientable connected 3-manifold \( M \) and an embedded 2-torus \( T \subset M \) bounding a solid torus with a parameterization \( S^1 \times D^2 \), the 3-manifold \( M \setminus \text{Int}(S^1 \times D^2) \) is parallelizable and its boundary is diffeomorphic to a 2-torus. The proof of Lemma 2.5 depends only on these properties, so the 3-manifold \( M \setminus \text{Int}(S^1 \times D^2) \) can also be realized as an immersed Lagrangian cap of \( \Sigma_{S^1}K_2 \) and of the self-intersection number zero modulo two. The existence of such an immersed Lagrangian cap allows us to apply the same construction. \( \square \)

**Remark 2.6.** Let \( N \) be a closed non-orientable connected 3-manifold with trivial complexified tangent bundle \( TN \otimes \mathbb{C} \to N \) and \( M \) a 3-manifold as in Theorem 1.1. Then, the connected sum \( M_T \# N \) also admits a Lagrangian embedding into \( \mathbb{R}^5_{st} \). Actually, we can construct an immersed Lagrangian cap \((M \setminus \text{Int}(S^1 \times D^2)) \# N \) of \( \Sigma_{S^1}K_2 \) by taking the connected sum of the Lagrangian cap \( M \setminus \text{Int}(S^1 \times D^2) \) and a Lagrangian immersion \( N \to \mathbb{R} \times \mathbb{R}^5_{st} \) in a way similar to the construction of \( \tilde{\phi} \# G \) in the proof of Lemma 2.5. The rest of the construction is the same to the proof of Theorem 1.1. On the other hand, there exists a compact 3-manifold with trivial complexified tangent bundle and torus boundary that cannot be realized as an immersed Lagrangian cap of \( \Sigma_{S^1}K_2 \). The direct product of the circle and the Möbius band is an example of such a 3-manifold. In fact, the boundary of the Möbius band is homotopic to twice the centered orientation-reversing loop, so its Maslov index can not be zero for any Lagrangian immersion.

**Remark 2.7.** By [4, Theorem 2.2], we can choose the Lagrangian cap \( M \setminus \text{Int}(S^1 \times D^2) \) to be exact. In particular, \( \text{Theorem 1.1} \) for the case \( L = S^1 \) gives a Lagrangian embedding \( S^1 \times N_{2g} \to \mathbb{R} \times \mathbb{R}^5_{st} \) being a concatenation of the Lagrangian filling \( F \) and an exact Lagrangian cap along the loose Legendrian torus \( \Sigma_{S^1}K_2 \). The particular case of \( \text{Theorem 1.1} \) is a consequence of the existence of this Lagrangian embedding. On the other hand, if \( g \geq 1 \) then no Lagrangian embedding \( S^1 \times \Sigma_g \to \mathbb{R} \times \mathbb{R}^5_{st} \) can be a concatenation of a Lagrangian filling and an exact Lagrangian cap by [4, Proposition 1.4] and [2, Theorem 1.1].
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References

[1] M. Audin. Fibrés normaux d’immersions en dimension double, points doubles d’immersions lagrangiennes et plongements totalement réels. Comment. Math. Helv. 63 (1988), no. 4, 593–623.

[2] M. S. Borman and M. McLean. Bounding Lagrangian widths via geodesic paths. Compos. Math. 150 (2014), no. 12, 2143–2183.

[3] Y. Chekanov. Differential algebra of Legendrian links. Invent. Math. 150 (2002), 441–483.

[4] G. Dimitroglou Rizell. Exact Lagrangian caps and non-uniruled Lagrangian submanifolds. Ark. Mat. 53 (2015), no. 1, 37–64.

[5] G. Dimitroglou Rizell and R. Golovko. On homological rigidity and flexibility of exact Lagrangian endocobordisms. Internat. J. Math. 25 (2014), no. 10, 1450098, 1–24.

[6] T. Ekholm. Rational symplectic field theory over $\mathbb{Z}_2$ for exact Lagrangian cobordisms. J. Eur. Math. Soc. 10 (2008), no. 3, 641–704.

[7] T. Ekholm, Y. Eliashberg, E. Murphy, and I. Smith. Constructing exact Lagrangian immersions with few double points. Geom. Funct. Anal. 23 (2013), no. 6, 1772–1803.

[8] T. Ekholm, J. Etnyre, and M. Sullivan. Non-isotopic Legendrian submanifolds in $\mathbb{R}^{2n+1}$. J. Differential Geom. 71 (2005), no. 1, 85–128.

[9] T. Ekholm and I. Smith. Exact Lagrangian immersions with a single double point. J. Amer. Math. Soc. 29 (2016), no. 1, 1–59.

[10] Y. Eliashberg and N. Mishachev. Introduction to the h-Principle. Graduate Studies in Mathematics 48, American Mathematical Society (2002).

[11] Y. Eliashberg and E. Murphy. Lagrangian caps. Geom. Funct. Anal. 23 (2013), no. 5, 1483–1514.

[12] K. Fukaya. Application of Floer homology of Lagrangian submanifolds to symplectic topology. NATO Sci. Ser. II Math. Phys. Chem. 217 (2006), 231–276.

[13] R. Golovko. A note on the front spinning construction. Bull. Lond. Math. Soc. 46 (2014), no. 2, 258–268.

[14] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. Invent. Math. 82 (1985), no. 2, 307–347.

[15] M. Gromov. Partial Differential Relations. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 9. Springer Verlag, Berlin (1986).

[16] J. A. Lees. On the classification of Lagrange immersions. Duke Math. J. 43 (1976), no. 2, 217–224.

[17] E. Murphy. Loose Legendrian Embeddings in High Dimensional Contact Manifolds. Preprint, arXiv:1201.2215.

[18] L. Polterovich. The surgery of Lagrange submanifolds. Geom. Funct. Anal. 1 (1991), no. 2, 198–210.

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