A linear realization for the new space-time superalgebras in ten and eleven dimensions

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Abstract

The new extensions of the Poincaré superalgebra recently found in ten and eleven dimensions are shown to admit a linear realization. The generators of the nonlinear and linear group transformations are shown to fall into equivalent representations of the superalgebra. The parametrization of the coset space $G/H$, with $G$ a given extended supergroup and $H$ the Lorentz subgroup, that corresponds to the linear transformations is presented.

1 Introduction

Recently, new extensions of the Poincaré superalgebra were found in ten and eleven dimensions [1–5]. To a large extent they can be thought of as suggesting a natural geometric framework for extended objects [6–8]. A covariant formulation of the super $p$-brane theory essentially involves a Wess–Zumino-type term. Being crucial in providing local $k$-symmetry, this term is invariant under global supersymmetry transformations up to total derivative only. As was shown in Ref. 9, this causes the effect of introducing a topological central charge into the Poincaré superalgebra (the effect also known for the massive superparticle in $d = 9$ [10]). The total derivative terms in a variation of the action can be suppressed by redefining the action functional itself (i.e., by enlarging the original configuration space and adding appropriate total derivative terms to the action). It turns out that following this course one arrives at an extension of the Poincaré superalgebra again [11, 3].

A conventional way to build a theory associated with a given Lie (super) algebra is to apply the standard group-theoretic construction [12].
case concerned, it is suffice to consider the coset space $G/H$, where $G$ is a given extended supergroup, with an element

$$\tilde{g} = e^{-ia^n P_n + i\alpha Q_n + \ldots + i\omega^{mn} M_{mn}},$$

(1)

and $H$ is the Lorentz subgroup. The symbol $i\sigma\ldots\Sigma\ldots$ denotes the sum over all generators $\Sigma\ldots$ that extend the Poincaré superalgebra $(M, P, Q)$. A point in the space

$$g(x, \theta, \psi) = e^{-ix^n P_n + i\theta^\alpha Q_n + \ldots \Sigma\ldots} \times SO(1, d - 1)$$

(2)

is parametrized by the set of coordinates $(x^m, \theta^\alpha, \psi\ldots)$ with the statistics being analogous to that of the generators.

Left multiplication with a group element

$$\tilde{g} : \ g(x, \theta, \psi) \to g(x', \theta', \psi') = \tilde{g}g(x, \theta, \psi)$$

(3)

defines, via the Baker–Campbell–Haussdorff formula, an action of the group on the coset. Invariants of the group can be used to construct a theory.

A salient feature of the new superalgebras is that the translation generators do not commute with the supertranslations [1–4]

$$[P, Q] \sim \Sigma.$$  

(4)

In view of the construction just outlined it means highly nonlinear transformation laws for the coordinates parametrizing the coset.

The purpose of this letter is to show that the extended superalgebras recently proposed in [1–3] admit a linear realization. The $\Gamma$-matrix identities that underlie the superstring theory in $d = 10$ and the supermembrane theory in $d = 11$ turn out to be important for the linearization.

Apart from the obvious technical benefits, there is an additional motivation to study the extended superalgebras in linear realization. The important observation is that, due to Eq. (4), the first Casimir operator in the algebra (see Eqs. (17) and (12) below) includes the odd generators contribution. This allows one to expect that equations extracting on-shell field irreps of the group will involve interaction. Construction of the irreps may suggest an interesting way to attack the higher spins interaction problem [13]. The results on this subject will be presented elsewhere.

In this work we use (anti)symmetrization “without strength”, i.e., $A_{[mB_n]} \equiv A_m B_n - A_n B_m$, $A_{(mB_n)} \equiv A_m B_n + A_n B_m$. The conventions adopted for $d = 11$ are presented in the Appendix.
2 Green superalgebra

Green superalgebra [1] can be understood as a global limit of Kač–Moody one [14, 2] that arises in the context of superstring theory. The commutation relations read (the usual Poincaré subalgebra is omitted)

\[
\begin{align*}
\{Q_\alpha, Q_\beta\} &= -2\Gamma_{\alpha\beta}^m P_m, & [P_m, Q_\alpha] &= -i\Gamma_{m\alpha\beta}\Sigma^\beta, \\
[M_{mn}, Q_\alpha] &= -\frac{i}{4}(\Gamma_{mn})^\beta_\alpha Q_\beta, & [M_{mn}, \Sigma^\alpha] &= \frac{i}{4}\Sigma^\beta(\Gamma_{mn})^\beta_\alpha,
\end{align*}
\]

where \(\Gamma_{mn} \equiv \Gamma_m\tilde{\Gamma}_n - \Gamma_n\tilde{\Gamma}_m\) and \(\Gamma_m\tilde{\Gamma}_n + \Gamma_n\tilde{\Gamma}_m = -2\eta_{mn}\). The Jacobi identities for Eq. (5) restrict the \(\Gamma\)-matrices to satisfy the relation

\[
\Gamma^m_{\alpha(\beta\Gamma_m\gamma\delta)} = 0
\]

which holds only in \(d = 3, 4, 6, 10\). For definiteness we examine ten-dimensional case here\(^1\). The generalization to other dimensions is straightforward.

An application of the group-theoretic construction to the case at hand results in the transformation laws [1, 3]

\[
\begin{align*}
\delta_\epsilon \theta^\alpha &= \epsilon^\alpha, & \delta_\epsilon x^m &= i\theta \Gamma^m \epsilon, \\
\delta_\epsilon \psi_\alpha &= \frac{1}{2} x^m (\epsilon \Gamma_m)_\alpha - \frac{1}{6} i(\theta \Gamma^m \epsilon)(\theta \Gamma_m)_\alpha; \quad (7a) \\
\delta_\alpha x^m &= a^m, & \delta_\alpha \psi_\alpha &= -\frac{1}{2} a^m (\theta \Gamma_m)_\alpha; \quad (7b) \\
\delta_\omega x^m &= \omega^m_n x^n, & \delta_\omega \theta^\alpha &= \frac{1}{8} \omega^{mn}(\theta \Gamma_{mn})^\alpha, \quad (7c) \\
\delta_\omega \psi_\alpha &= -\frac{1}{8} \omega^{mn}(\Gamma_{mn}\psi)_\alpha; & \delta_\sigma \psi_\alpha &= \sigma_\alpha, \quad (7d)
\end{align*}
\]

where \((x^m, \theta^\alpha, \psi_\beta)\) are the coordinates parametrizing the coset \(G/H \sim R^{10/32}\) and \((\epsilon, a, \omega, \sigma)\) are the parameters associated to the generators

\[
\begin{align*}
Q_\alpha &= i\partial_\alpha + (\theta \Gamma^m)_\alpha \partial_n + \frac{i}{2} x^n \Gamma_m \alpha\beta \partial^\beta - \frac{1}{6} (\theta \Gamma^m)_\alpha(\theta \Gamma_n)_\beta \partial^\beta, \\
P_n &= -i\partial_n + \frac{i}{2}(\theta \Gamma_n)_\alpha \partial^\alpha, \\
M_{mn} &= -i(x_m \partial_n - x_n \partial_m) + \frac{i}{4}(\theta \Gamma_{mn})^\alpha \partial_\alpha - \frac{i}{4}(\Gamma_{mn}\psi)_\alpha \partial^\alpha, \\
\Sigma^\alpha &= i\partial^\alpha.
\end{align*}
\]

\(^1\)The conventions adopted in \(d = 10\) are those of Ref. 15.
In Eq. (8) we have set \( \partial_{\alpha} \equiv \partial/\partial \theta^{\alpha} \), \( \partial^\alpha \equiv \partial/\partial \psi_{\alpha} \), \( \partial_n \equiv \partial/\partial x^n \). Note that Eq. (7a) essentially involves a nonlinear contribution.

In terms of the transformations (7) the algebra (5) acquires the form

\[
\begin{align*}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] &= \delta_a, & a^m &= 2i\epsilon_1 \Gamma^m \epsilon_2, \\
[\delta_{a}, \delta_{\epsilon}] &= \delta_{\sigma}, & \sigma_{\alpha} &= a^m (\epsilon \Gamma_m)_{\alpha}, \\
[\delta_{\omega}, \delta_{\epsilon}] &= \delta_{\epsilon_1}, & \epsilon_1^\alpha &= -\frac{1}{8} \omega^{ab} (\epsilon \Gamma_{ab})^\alpha, \\
[\delta_{\omega}, \delta_{\sigma}] &= \delta_{\sigma_1}, & \sigma_{1\alpha} &= \frac{1}{8} \omega^{ab} (\Gamma_{ab} \sigma)_{\alpha}, \\
[\delta_{\omega}, \delta_{a}] &= \delta_{\alpha_1}, & a_1^m &= -\omega^m_n a^n, \\
[\delta_{\omega_1}, \delta_{\omega_2}] &= \delta_{\omega_3}, & \omega_3^m n &= (\omega_2 \omega_1)^m n - (\omega_1 \omega_2)^m n.
\end{align*}
\]

(9)

Evaluation of this algebra turns out to be more instructive than it may seem at a glance. Actually, direct calculation with the use of Eq. (6) yields

\[
\begin{align*}
\delta_{\epsilon_1} \left( \frac{1}{2} x^n (\epsilon_2 \Gamma_n)^\alpha \right) - \delta_{\epsilon_2} \left( \frac{1}{2} x^n (\epsilon_1 \Gamma_n)^\alpha \right) &= -\frac{1}{2} i (\epsilon_1 \Gamma^n \epsilon_2) (\theta \Gamma_n)_{\alpha}, \\
\delta_{\epsilon_1} \left( -\frac{1}{6} i (\theta \Gamma^m \epsilon_2) (\theta \Gamma_n)_{\alpha} \right) - \delta_{\epsilon_2} \left( -\frac{1}{6} i (\theta \Gamma^m \epsilon_1) (\theta \Gamma_n)_{\alpha} \right) &= -\frac{1}{2} i (\epsilon_1 \Gamma^m \epsilon_2) (\theta \Gamma_n)_{\alpha},
\end{align*}
\]

(10)

which means that the linear and nonlinear terms in the variation \( \delta_{\epsilon} \psi \) make the same contribution into the first line of Eq. (9). The latter fact implies that Eq. (7) can be rewritten in the linear form without spoiling the algebra (9). The linear version for the Green transformations looks like

\[
\begin{align*}
\delta_{\epsilon} \theta^\alpha &= \epsilon^\alpha, & \delta_{\epsilon} x^m &= i \theta \Gamma^m \epsilon, & \delta_{\epsilon} \psi_{\alpha} &= \frac{2}{3} x^m (\epsilon \Gamma_m)_{\alpha}; \\
\delta_a x^m &= a^m, & \delta_a \psi_{\alpha} &= -\frac{1}{3} a^m (\theta \Gamma_m)_{\alpha}; \\
\delta_{\sigma} \psi_{\alpha} &= \sigma_{\alpha},
\end{align*}
\]

(11)

where we omitted the usual Lorentz transformations. The generators associated to Eq. (11) read

\[
\begin{align*}
Q_{\alpha} &= i \partial_{\alpha} + \left( \theta \Gamma_n \right)_\alpha \partial_n + \frac{2}{3} i x^n \Gamma_{n \alpha \beta} \partial^\beta, \\
P_n &= -i \partial_n + \frac{i}{3} \left( \theta \Gamma_n \right)_\alpha \partial^\alpha, \\
\Sigma^\alpha &= i \partial^\alpha.
\end{align*}
\]

(12)

In geometric terms, a realization of the algebra (5) by the linear transformations corresponds to the possibility to choose another parametrization
of the coset (in this context see also Ref. 16). It is straightforward to check that adopting the following parametrization:
\[
g = e^{-ix^n P_n + i \psi_\alpha \Sigma^\alpha + \frac{2}{3} i \theta^\alpha Q_\alpha e^{\frac{1}{3} i \theta^\alpha Q_\alpha}} \times SO(1, d - 1)
\] (13)
and exploiting the group-theoretic machinery one arrives just at Eq. (11).

It is interesting to note that the generators (8) and (12) fall into two equivalent representations of Green superalgebra
\[
T_{i_{\text{nonlinear}}} = ST_{i_{\text{linear}}} S^{-1},
\] (14)
where
\[
S = e^{\frac{1}{6} x^m (\theta \Gamma_m)_\alpha \partial^\alpha},
\] (15)
and we denoted \( T_i \equiv (P, Q, M, \Sigma) \). This relation can easily be checked by making use of the formula
\[
e^{-B} A e^B = \sum_{n=0}^{\infty} \frac{1}{n!} [A, B]_{(n)},
\] (16)
where \([A, B]_{(0)} \equiv A, [A, B]_{(n+1)} = [[A, B]_{(n)}, B]\).

Thus, Green superalgebra was shown to admit a linear realization.

Some remarks seem to be relevant here. First, since momenta do not commute with supertranslations the first Casimir operator in the algebra includes the odd generators contribution
\[
P^2 - i \Sigma^\alpha Q_\alpha.
\] (17)
This allows one to expect that equations extracting on-shell field irreps of the group will involve interaction. Second, the algebra (5) can naturally be extended by the new bosonic generator \( B^m \) with commutation relations
\[
[B^m, Q_\alpha] = -i \Gamma^m_{\alpha \beta} \Sigma^\beta, \quad [M_{ab}, B_m] = i \eta_{am} B_b - i \eta_{bm} B_a.
\] (18)
It is straightforward to check that the Jacobi identities hold for the full algebra. Third, apart from the algebra (5) one can realize the conjugate superalgebra (see also Ref. 11)
\[
\{Q^\alpha, Q^\beta\} = -2 \Gamma^m_{\alpha \beta} P_m, \quad [P_m, Q^\alpha] = -i \Gamma^m_{\alpha \beta} \Sigma^\beta, \quad [M_{mn}, Q^\alpha] = \frac{i}{4} Q^\beta (\Gamma_{mn})^\beta_{\alpha} \quad [M_{mn}, \Sigma_\alpha] = -\frac{i}{4} (\Gamma_{mn})_{\alpha \beta} \Sigma^\beta
\] (19)
\footnote{The explicit form for the operator \( S \) immediately follows from a comparison of Eqs. (2) and (13) with the latter being rewritten in the equivalent form \( g = e^{-ix^n P_n + i \psi_\alpha \Sigma^\alpha + \frac{2}{3} i \theta^\alpha Q_\alpha e^{\frac{1}{3} i \theta^\alpha Q_\alpha}} \times SO(1, d - 1) = e^{-ix^n P_n + i \theta^\alpha Q_\alpha + (\psi_\alpha - \frac{1}{3} x^n (\theta \Gamma_n)_\alpha) \Sigma^\alpha} \times SO(1, d - 1).}
with the chirality of the odd generators being opposed to that of the generators in Eq. (5). It would be interesting to use the doublet (5), (19) to construct a superparticle model with local $k$-symmetry along the lines of Ref. 5.

An attractive feature of the Green superalgebra is that it allows one to formulate the Green–Schwarz superstring in a manifestly supersymmetric way [11], which was proven to be important in formulating the theory on the lattice [11]. Let us show that analogous construction works in terms of the linear transformations (11). Making use of invariants of the group

$$d\theta^\alpha, \quad dx^m + i\theta\Gamma^m d\theta, \quad d\psi_\alpha = \frac{1}{2}x^m(d\theta\Gamma_m)\alpha + dx_m(\theta\Gamma^m)\alpha + \frac{1}{2}(\theta\Gamma^m d\theta)(\theta\Gamma_m)\alpha$$

one can write down a superstring action

$$S = \int d\tau d\sigma \left\{ \frac{1}{2}\sqrt{-g}g^{ij}(\partial_i x^m + i\theta\Gamma^m\partial_i \theta)(\partial_j x^m + i\theta\Gamma^m\partial_j \theta) + 
+ \lambda\epsilon^{ij}(\partial_i \psi - \frac{1}{2}x_n\partial_i\theta\Gamma^n + \partial_i x_n\theta\Gamma^n)\partial_j \theta \right\},$$

where $\epsilon^{ij} = -\epsilon^{ji}$, $\epsilon^{01} = -1$ and $\lambda$ is a relative coefficient. Passing to the Hamiltonian formalism one finds that fermionic constraints of the theory are mixture of half first and half second class constraints only when

$$\lambda = \pm 2/3,$$

which, after integrating by parts in Eq. (21), leads to the standard Green–Schwarz action.

### 3 d=11 Bergshoeff–Sezgin superalgebra

In trying to formulate a supermembrane theory with manifest supersymmetry and inspired by the $\Gamma$-matrix identities, upon which the original formulation of super $p$-brane relies [7], Bergshoeff and Sezgin suggested [2–4] new extensions of the Poincaré superalgebra in $d = 11$ (in another respect similar extensions appeared also in Refs. 18, 19, and 9). The

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3 It is straightforward to check that two possible values of $\lambda$ in Eq. (22) correspond to physically equivalent theories (21). The reason is that the sign in front of the Wess–Zumino term correlates with the type of two-dimensional projector $p^{\pm ij} = \frac{1}{2}(g^{ij} \pm \epsilon^{ij}/\sqrt{-g})$ that appears in the $k$-symmetry transformation law [17].
simplest of them reads (our conventions for \( d = 11 \) are presented in the Appendix)

\[
\begin{align*}
\{Q_\alpha, Q_\beta\} &= -2\Gamma^m_{\alpha\beta} P_m + (\Gamma_{mn}C)_{\alpha\beta} \Sigma^{mn}, \\
\{Q_\alpha, P_m\} &= -i(\Gamma_{mn}C)_{\alpha\beta} \Sigma^{n\beta}, \\
\{Q_\alpha, \Sigma^{mn}\} &= i\Gamma^{|m}_{\alpha\beta} \Sigma^{n\beta}.
\end{align*}
\]

(23)

The Jacobi identities for the algebra (23) hold due to the \( \Gamma \)-matrix identity

\[
\Gamma^m_{(\alpha\beta}(\Gamma_{mn}C)_{\gamma\delta)} = 0,
\]

(24)

the latter satisfied in \( d = 4, 5, 7 \) and 11. In this case, with the standard parametrization of the coset adopted (Eq. (2)), the group-theoretic construction gives [3]

\[
\begin{align*}
\delta_\epsilon \theta^\alpha &= \epsilon^\alpha, & \delta_\epsilon x^m &= i\theta\Gamma^m \epsilon, \\
\delta_\epsilon \Phi_{mn} &= i(\theta\Gamma_{mn}C)\epsilon, \\
\delta_\epsilon \Phi_{n\alpha} &= -\frac{1}{2}(\epsilon\Gamma_{mn}C)_{\alpha} x^m - \frac{1}{2}(\epsilon\Gamma^m)_{\alpha} \Phi_{mn} + \\
&\quad + \frac{1}{6}i(\theta\Gamma^m \epsilon)(\theta\Gamma_{mn}C)_{\alpha} + \frac{1}{6}i(\theta\Gamma_{mn}C\epsilon)(\theta\Gamma^m)_{\alpha}; \\
\delta_\epsilon x^m &= a^m, & \delta_\epsilon \Phi_{n\alpha} &= \frac{1}{2}a^m(\theta\Gamma_{mn}C)_{\alpha}; \\
\delta_\epsilon \Phi_{mn} &= \epsilon_{mn}, & \delta_\epsilon \Phi_{n\alpha} &= \frac{1}{2}(\theta\Gamma^m)_{\alpha} \epsilon_{mn}; \\
\delta_\epsilon \Phi_{n\alpha} &= \epsilon_{n\alpha},
\end{align*}
\]

(25)

(25a)

(25b)

(25c)

(25d)

where \((x^m, \theta^\alpha, \Phi_{mn}, \Phi_{n\alpha})\) are the coordinates parametrizing the coset and \((a^m, \epsilon^\alpha, \epsilon_{mn}, \epsilon_{n\alpha})\) are the parameters associated to the generators \((P_m, Q_\alpha, \Sigma^{mn}, \Sigma^{n\alpha})\) respectively. As in the previous case, the \( \delta_\epsilon \)-transformations involve non-linear contributions. However, rewriting the algebra (23) in the following form:

\[
\begin{align*}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] &= \delta_a + \delta_{\epsilon_{mn}}, & a^m &= 2i\epsilon_1 \Gamma^m \epsilon_2, & \epsilon_{mn} &= 2i\epsilon_1 \Gamma_{mn}C \epsilon_2; \\
[\delta_a, \delta_\epsilon] &= \delta_{\epsilon_{n\alpha}}, & \epsilon_{n\alpha} &= -a^m(\epsilon\Gamma_{mn}C)_{\alpha}; \\
[\delta_{\epsilon_{mn}}, \delta_\epsilon] &= \delta_{\epsilon_{n\alpha}}, & \epsilon_{n\alpha} &= -(\epsilon\Gamma^m)_{\alpha} \epsilon_{mn},
\end{align*}
\]

(26)
one finds that the linear and nonlinear terms in the variation $\delta \Phi_{n\alpha}$ make the same contribution into the first line of Eq. (26)

$$\begin{align*}
\delta_{\epsilon_1} & \left( -\frac{1}{2}x^m(\epsilon_2 \Gamma_{mn}C)_{\alpha} - \frac{1}{2}(\epsilon_2 \Gamma^m)_{\alpha} \Phi_{mn} \right) - (1 \leftrightarrow 2) = \\
& = \frac{1}{2} i(\epsilon_1 \Gamma_{mn}C \epsilon_2)(\Gamma^m \theta)_{\alpha} + \frac{1}{2} i(\epsilon_1 \Gamma^m \epsilon_2)(\theta \Gamma_{mn}C)_{\alpha},
\end{align*}$$

(27)

$$\begin{align*}
\delta_{\epsilon_1} & \left( \frac{1}{6} i(\theta \Gamma^m \epsilon_2)(\theta \Gamma_{mn}C)_{\alpha} + \frac{1}{6} i(\theta \Gamma_{mn}C \epsilon_2)(\theta \Gamma^m)_{\alpha} \right) - (1 \leftrightarrow 2) = \\
& = \frac{1}{2} i(\epsilon_1 \Gamma_{mn}C \epsilon_2)(\Gamma^m \theta)_{\alpha} + \frac{1}{2} i(\epsilon_1 \Gamma^m \epsilon_2)(\theta \Gamma_{mn}C)_{\alpha}.
\end{align*}$$

In checking Eqs. (26) and (27) the consequences of the identity (24)

$$\begin{align*}
(\epsilon_2 \Gamma_{mn}C)_{(\alpha}(\epsilon_1 \Gamma^m)_{\beta)} - (\epsilon_1 \Gamma_{mn}C)_{(\alpha}(\epsilon_2 \Gamma^m)_{\beta)} &= \\
& = (\epsilon_1 \Gamma^m \epsilon_2)(\Gamma_{nm}C)_{\alpha\beta} + (\epsilon_1 \Gamma_{nm}C \epsilon_2)\Gamma^m_{\alpha\beta},
\end{align*}$$

(28)

$$(\epsilon_1 \Gamma^m)_{\alpha}(\theta \Gamma_{mn}C \epsilon_2) + (\epsilon_1 \Gamma_{mn}C)_{\alpha}(\theta \Gamma^m \epsilon_2) - (1 \leftrightarrow 2) = \\
& = (\theta \Gamma^m)_{\alpha}(\epsilon_1 \Gamma_{mn}C \epsilon_2) + (\theta \Gamma_{mn}C)_{\alpha}(\epsilon_1 \Gamma^m \epsilon_2)$$

are to be used. This observation suggests that one can find another parametrization of the coset on which Bergshoeff–Sezgin superalgebra (23) would be linearly realized. The suitable parametrization looks like

$$g = e^{-i x^m P_n + i \Phi_{n\alpha} \Sigma^\alpha + \frac{i}{2} \Phi_{mn} \Sigma^{mn} + \frac{1}{2} i \theta \alpha Q_\alpha \Phi_{n\alpha} \Sigma^\alpha \times \text{SO}(1, d - 1) \ (29)$$

and the linear version for Eqs. (25a)-(25d) reads

$$\begin{align*}
\delta_{\epsilon} \theta^\alpha &= \epsilon^\alpha, \\
\delta_{\epsilon} x^m &= i \theta \Gamma^m \epsilon, \\
\delta_{\epsilon} \Phi_{n\alpha} &= -\frac{2}{3} x^m (\epsilon \Gamma_{mn}C)_{\alpha} - \frac{2}{3} \Phi_{mn} (\epsilon \Gamma^m)_{\alpha}; \\
\delta_{a} x^m &= a^m, \\
\delta_{a} \Phi_{n\alpha} &= \frac{1}{3} a^m (\theta \Gamma_{mn}C)_{\alpha}; \\
\delta_{\epsilon_{mn}} \Phi_{mn} &= \epsilon_{mn}, \\
\delta_{\epsilon_{mn}} \Phi_{n\alpha} &= \frac{1}{3} \epsilon_{mn} (\theta \Gamma^m)_{\alpha}; \\
\delta_{\epsilon_{n\alpha}} \Phi_{n\alpha} &= \epsilon_{n\alpha};
\end{align*}$$

(30a-d)
As already might be expected, the generators of the nonlinear transformations (25)

\[ Q_{\alpha} = i\partial_{\alpha} + (\theta \Gamma^n)_{\alpha} \partial_n + (\theta \Gamma_{mn} C)_{\alpha} \partial^{mn} - \left( \frac{i}{2} x^m (\Gamma_{mn} C)_{\alpha\beta} + \frac{1}{2} \Gamma_{\alpha\beta} \frac{i}{6} \right) \partial^{n\beta}, \]

\[ P_m = -i \partial_m - \frac{i}{2} (\theta \Gamma_{mn} C)_{\alpha} \partial^{n\alpha}, \]

\[ \Sigma^{mn} = i \partial^{[mn]} + \frac{i}{2} (\theta \Gamma^{[mn]} C)_{\alpha} \partial^{n\alpha}, \]

\[ \Sigma^{n\alpha} = i \partial^{n\alpha}, \]

and ones of the linear transformations (30)

\[ Q_{\alpha} = i\partial_{\alpha} + (\theta \Gamma^n)_{\alpha} \partial_n + (\theta \Gamma_{mn} C)_{\alpha} \partial^{mn} - \left( \frac{i}{2} x^m (\Gamma_{mn} C)_{\alpha\beta} + \frac{2}{3} i \Phi_{mn} \Gamma^{m}_{\alpha\beta} \right) \partial^{n\beta}, \]

\[ P_m = -i \partial_m - \frac{i}{3} (\theta \Gamma_{mn} C)_{\alpha} \partial^{n\alpha}, \]

\[ \Sigma^{mn} = i \partial^{[mn]} + \frac{i}{3} (\theta \Gamma^{[mn]} C)_{\alpha} \partial^{n\alpha}, \]

\[ \Sigma^{n\alpha} = i \partial^{n\alpha}, \]

belong to equivalent representations of the superalgebra (23)

\[ T_{i(\text{nonlinear})} = ST_{i(\text{linear})} S^{-1}, \] (33)

where

\[ S = e^{-\frac{1}{6}(x^m (\theta \Gamma_{mn} C)_\alpha + \Phi_{mn} (\theta \Gamma_{m} C)_\alpha)} \partial^{n\alpha}. \] (34)

In Eqs. (31)–(34) we denoted \( \partial/\partial \theta^\alpha, \partial/\partial\partial^\alpha = \partial_{\alpha}, \partial/\partial x^n = \partial_n, \partial/\partial \Phi_{n\alpha} = \partial^{n\alpha}, \partial/\partial \Phi_{mn} = \partial^{mn} \) and set \( T_i \equiv (Q_{\alpha}, P_m, \Sigma^{mn}, \Sigma^{n\alpha}) \).

4 Inclusion of \( \Sigma^{\alpha\beta} \)-charge

Close examination of invariants of the group (25) (or (30)) shows that they are not sufficient to construct a supermembrane theory with local

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\[ g = e^{-ix^n P_n + i \Phi_{\alpha\beta} \Sigma^{\alpha\beta} + \frac{1}{2} \Phi_{\alpha\beta} \Sigma^{\alpha\beta} + \frac{1}{6} \Phi_{\alpha\beta} Q_{\alpha} e^{\frac{1}{6} i \Phi_{\alpha\beta} Q_{\alpha} \times SO(1,d-1)} =}

\[ = e^{-ix^n P_n + i \Phi_{\alpha\beta} \Sigma^{\alpha\beta} + \frac{1}{2} \Phi_{\alpha\beta} \Sigma^{\alpha\beta} + i \Phi_{\alpha\beta} \Sigma^{\alpha\beta} + \frac{1}{6} \Phi_{\alpha\beta} (\theta \Gamma_{mn} C)_{\alpha} + \frac{1}{6} \Phi_{\alpha\beta} (\theta \Gamma_{mn} C)_{\alpha} \Sigma^{\alpha\beta}, \]

suggests the explicit form for the operator \( S \).
Further extension by \( \Sigma^{\alpha\beta} \)-generator was proven \([3]\) to be necessary. The structure relations of the algebra have the form\[3\]

\[
\begin{align*}
\{ Q_\alpha, Q_\beta \} &= -2 \Gamma^m_{\alpha\beta} P_m + (\Gamma_{mn} C)_{\alpha\beta} \Sigma^{mn}, \\
[ Q_\alpha, P_m ] &= -i (\Gamma_{mn} C)_{\alpha\beta} \Sigma^{mn}, \\
[ P_m, P_n ] &= i (\Gamma_{mn} C)_{\alpha\beta} \Sigma^{\alpha\beta}, \\
[ P_m, \Sigma^{np} ] &= -\frac{1}{2} i \delta_m [n \Gamma^p]_{\alpha\beta} \Sigma^{\alpha\beta}, \\
[ Q_\alpha, \Sigma^{mn} ] &= i \Gamma^m_{\alpha\beta} \Sigma^{n|\beta}, \\
\{ Q_\alpha, \Sigma^{n\beta} \} &= \left( \frac{1}{2} \Gamma^n_{\gamma\delta} \delta_{\alpha\beta} + 4 \Gamma^n_{\gamma\alpha} \delta_{\gamma\delta} \right) \Sigma^{\gamma\delta}.
\end{align*}
\]

The Jacobi identities for (35) hold due to Eq. (24). As compared to the usual Poincaré superalgebra, one finds the super two-form charge \( \Sigma^{AB} = (\Sigma^{mn}, \Sigma^{ma}, \Sigma^{\alpha\beta}) \) in Eq. (35) which was connected \([3]\) with the existence of supermembrane solution \([20]\) of \( d = 11 \) supergravity.

Although in this case the commutation relations look rather complicated, the modified algebra (35) can be linearly realized like its contraction (23). Omitting details we present here the final result.

The transformations on the coset space (with the standard parametrization (2) adopted) are given by Eq. (25) with the following transformation laws for the coordinates \( \Phi_{\alpha\beta} \) associated to the new generators \( \Sigma^{\alpha\beta} \) added (compare with Ref. 3 where another parametrization of the coset has been chosen)

\[
\begin{align*}
\delta_c \Phi_{\alpha\beta} &= \frac{i}{2} (\Phi_{m\gamma} \epsilon^\gamma) \Gamma^m_{\alpha\beta} + 2i \Phi_{m(\alpha} (\epsilon^m_{\beta)} - \\
&- \frac{1}{4} \Phi_{mn} i(\theta \Gamma^m \epsilon) \Gamma^m_{\alpha\beta} + \frac{1}{3} x^n i(\epsilon \Gamma_{nm} C)_{(\alpha} (\theta \Gamma^m)_{\beta)} + \\
&+ \frac{1}{12} x^n i(\theta \Gamma_{nm} C \epsilon) \Gamma^m_{\alpha\beta}; \\
\delta_a \Phi_{\alpha\beta} &= -a^n x^m (\Gamma_{nm} C)_{\alpha\beta} - \frac{1}{2} a^m \Phi_{mn} \Gamma^m_{\alpha\beta} - \\
&- \frac{1}{3} a^n i(\theta \Gamma_{nm} C)_{(\alpha} (\theta \Gamma^m)_{\beta)}; \tag{36a}
\end{align*}
\]

\[\text{Recently, an extension of Eq. (35) by super five-form charge has been proposed \([4]\). The M-algebra may suggest an attractive way to construct a super five-brane theory in eleven dimensions. The possibility to realize the M-algebra by linear transformations will be examined elsewhere.}\]
\[
\delta_{\epsilon_{mn}} \Phi_{\alpha\beta} = \frac{1}{2} x^n \epsilon_{nm} \Gamma^m_{\alpha\beta} - \frac{1}{3} \epsilon_{nm} i(\theta \Gamma^n)_{(\alpha}(\theta \Gamma^m)_{\beta)}; \quad (36c)
\]
\[
\delta_{\epsilon_{na}} \Phi_{\alpha\beta} = \frac{1}{2} i(\theta^n \epsilon_{n\gamma}) \Gamma^n_{\alpha\beta} - 2 i \epsilon_{n(\alpha}(\theta \Gamma^n)_{\beta)}; \quad (36d)
\]
\[
\delta_{\epsilon_{\alpha\beta}} \Phi_{\alpha\beta} = \epsilon_{\alpha\beta}. \quad (36e)
\]

Taking the linearization for Eq. (25) from Eq. (30) and supplementing it with
\[
\delta_x \Phi_{\alpha\beta} = 3 i(\Phi_{n\gamma} \epsilon^\gamma) \Gamma^n_{\alpha\beta} + 3 i \Phi_{n(\alpha} (\epsilon^n)_{\beta)}; \quad (37a)
\]
\[
\delta_a \Phi_{\alpha\beta} = -2 a^m \Phi_{mn} \Gamma^n_{\alpha\beta} + a^m x^n (\Gamma_{nm} C)_{\alpha\beta}; \quad (37b)
\]
\[
\delta_{\epsilon_{mn}} \Phi_{\alpha\beta} = -\epsilon_{mn} x^m \Gamma^n_{\alpha\beta}; \quad (37c)
\]
\[
\delta_{\epsilon_{na}} \Phi_{\alpha\beta} = 2 i(\epsilon_{n\gamma} \theta^\gamma) \Gamma^n_{\alpha\beta} - i \epsilon_{n(\alpha}(\theta \Gamma^n)_{\beta)}; \quad (37d)
\]
\[
\delta_{\epsilon_{\alpha\beta}} \Phi_{\alpha\beta} = \epsilon_{\alpha\beta}, \quad (37e)
\]
on one gets the linear version for Bergshoeff–Sezgin superalgebra (35). Making use of Eq. (28), it is straightforward to check that transformations (30), (37) satisfy the algebra
\[
[\delta_1, \delta_2] = \delta_a + \delta_{\epsilon_{mn}}, \quad a^m = 2 i \epsilon_1 \Gamma^m \epsilon_2, \quad \epsilon_{mn} = 2 i (\epsilon_1 \Gamma_{mn} C \epsilon_2);
\]
\[
[\delta_a, \delta_\epsilon] = \delta_{\epsilon_{na}}, \quad \epsilon_{na} = -a^m (\epsilon \Gamma_{mn} C)_{\alpha};
\]
\[
[\delta_{\epsilon_{mn}}, \delta_\epsilon] = \delta_{\epsilon_{na}}, \quad \epsilon_{na} = -\epsilon_{mn} (\Gamma^m \epsilon)_{\alpha};
\]
\[
[\delta_\epsilon, \delta_{\epsilon_{na}}] = \delta_{\epsilon_{\alpha\beta}}, \quad \epsilon_{\alpha\beta} = -i (\epsilon_{n\gamma} \epsilon^\gamma) \Gamma^n_{\alpha\beta} - 4 i \epsilon_{n(\alpha}(\theta \Gamma^n)_{\beta)};
\]
\[
[\delta_{a_1}, \delta_{a_2}] = \delta_{a_{\alpha\beta}}, \quad \epsilon_{\alpha\beta} = 2 a_2^m a_1^n (\Gamma_{nm} C)_{\alpha\beta};
\]
\[
[\delta_a, \delta_{\epsilon_{mn}}] = \delta_{\epsilon_{\alpha\beta}}, \quad \epsilon_{\alpha\beta} = a^m \epsilon_{mn} \Gamma^n_{\alpha\beta},
\]
which, being rewritten in terms of the generators, coincides with Eq. (35).

Thus, in this work we have demonstrated that nontrivial extensions of the Poincaré superalgebra in higher dimensions can be realized without spoiling the linear structure of the original super Poincaré transformations. We hope that analogous construction will work in the case of the $M$-algebra [4] which may suggest considerable simplification in constructing a super five-brane theory in eleven dimensions.

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Appendix

In this Appendix we present the spinor notations adopted for $d = 11$ in this work.

A conventional way to build $\Gamma$-matrices in eleven dimensions is to use those of ten-dimensional space-time. The imaginary (Majorana) representation

$$
\Gamma_{\alpha}^{\beta} = i \begin{pmatrix} 0 & \Gamma_{AB}^{M} \\ \tilde{\Gamma}_{AB}^{M} & 0 \end{pmatrix}, \quad \alpha = 1, \ldots, 32; \quad M = 0, \ldots, 9,
$$

(A.1)
is of common use in $d = 10$. Here $\Gamma_{AB}^{M}, \tilde{\Gamma}_{AB}^{M}$ are $16 \times 16$ matrices which form “chiral” representation. They are real and symmetric, obeying the algebra

$$
\Gamma_{M}^{\alpha} \Gamma_{N}^{\beta} + \Gamma_{N}^{\alpha} \Gamma_{M}^{\beta} = 2 \eta^{MN}, \quad \eta^{MN} = (+, -, -, \ldots).
$$

(A.2)
The explicit form for the matrices is

$$
\begin{align*}
\Gamma_{0} &= \begin{pmatrix} 1_{8} & 0 \\ 0 & 1_{8} \end{pmatrix}, & \tilde{\Gamma}_{0} &= \begin{pmatrix} -1_{8} & 0 \\ 0 & -1_{8} \end{pmatrix}, \\
\Gamma_{i} &= \begin{pmatrix} 0 & \gamma_{i}^{a\dot{a}} \\ \gamma_{i}^{\dot{a}a} & 0 \end{pmatrix}, & \tilde{\Gamma}_{i} &= \begin{pmatrix} 0 & \gamma_{i}^{\dot{a}a} \\ \gamma_{i}^{a\dot{a}} & 0 \end{pmatrix}, \\
\Gamma_{9} &= \begin{pmatrix} 1_{8} & 0 \\ 0 & -1_{8} \end{pmatrix}, & \tilde{\Gamma}_{9} &= \begin{pmatrix} 1_{8} & 0 \\ 0 & -1_{8} \end{pmatrix},
\end{align*}
$$

(A.3)

where $\gamma_{i}^{a\dot{a}}, \gamma_{i}^{\dot{a}a} \equiv (\gamma_{i}^{a\dot{a}})^{T}$ are $SO(8)$ $\gamma$-matrices [17]

$$
\gamma_{i}^{a\dot{a}} \gamma_{j}^{\dot{a}b} + \gamma_{j}^{a\dot{a}} \gamma_{i}^{\dot{a}b} = 2\delta^{ij}\delta_{ab}, \quad i = 1, \ldots, 8; \quad a, \dot{a} = 1, \ldots, 8.
$$

(A.4)
The properties of $\Gamma, \tilde{\Gamma}$ induce the relations for $\Gamma_{M}^{\alpha}^{\beta}$

$$
\begin{align*}
\Gamma_{M}^{N} \Gamma_{N}^{M} + \Gamma_{N}^{M} \Gamma_{M}^{N} &= 2\eta^{MN}, & \eta^{MN} &= (+, -, -, \ldots), \\
(\Gamma_{M})^{*} &= -\Gamma_{M}, & (\Gamma_{0})^{T} &= -\Gamma_{0}, & (\Gamma_{i})^{T} &= \Gamma_{i}, \quad i = 1, \ldots, 9.
\end{align*}
$$

(A.5)
The product $\Gamma_{0} \Gamma_{1} \cdots \Gamma_{9}$ is known as $\Gamma^{11}$

$$
\Gamma^{11} = \begin{pmatrix} 1_{16} & 0 \\ 0 & -1_{16} \end{pmatrix},
$$

(A.6)

and serves to extract two inequivalent irreducible spinor representations of the Lorentz group in $d = 10$ (right-handed and left-handed Weyl spinors: $\Psi_{R,L} = \frac{1}{2}(1 \pm \Gamma^{11})\Psi$ with $\Psi$ a Dirac spinor).
With the $\Gamma^{11}$ at hand, construction of $d = 11$ $\Gamma$-matrices present no special problem [21]. It is easy to check that the set

$$\Gamma^m_{\alpha \beta} \equiv [\Gamma^M_{\alpha \beta} , i\Gamma^{11}_{\alpha \beta}] , \quad m = 0, \ldots, 10; \quad \alpha = 1, \ldots, 32,$$

(A.7)
satisfies the needed algebra

$$\Gamma^m \Gamma^n + \Gamma^n \Gamma^m = 2\eta^{mn} , \quad \eta^{mn} = (+, -, -, \ldots)$$

(A.8)

and possesses the properties

$$(\Gamma^m)^* = -\Gamma^m , \quad (\Gamma^0)^T = -\Gamma^0 , \quad (\Gamma^i)^T = \Gamma^i , \quad i = 1, \ldots, 10.$$

(A.9)

Under action of the Lorentz group a $d = 11$ Dirac spinor is transformed as

$$\delta \Psi_\alpha = \frac{1}{8} \omega^{mn} (\Gamma_{mn} \Psi)_\alpha , \quad \Gamma_{mn} \equiv \Gamma_m \Gamma_n - \Gamma_n \Gamma_m .$$

(A.10)

In the imaginary representation (A.7)–(A.9) the reality condition

$$\Psi_\alpha^* = \Psi_\alpha$$

(A.11)
is compatible with Eq. (A.10) which defines a Majorana spinor.

In studying the extended superalgebras in eleven dimensions it is convenient to deal with symmetric $\Gamma$-matrices much as the chiral representation (A.2), (A.3) is of common use in $d = 10$. It is straightforward to check that the matrices

$$\Gamma^m_{\alpha \beta} \equiv \left[ \begin{array}{cc} -\tilde{\Gamma}^M_{AB} & 0 \\ 0 & \Gamma^M_{AB} \end{array} \right] , \quad \Gamma^M_{AB} \equiv \begin{pmatrix} 0 & \mathbf{1}_{16} \\ \mathbf{1}_{16} & 0 \end{pmatrix} ,$$

(A.12)

satisfy the algebra

$$\Gamma^m \tilde{\Gamma}^n + \tilde{\Gamma}^n \Gamma^m = 2\eta^{mn} .$$

(A.13)

Obviously, they are real and symmetric. In these terms one can define two spinor representations

$$\delta \Psi_\alpha = \frac{1}{8} \omega^{mn} (\Gamma_{mn})^\alpha_\beta \Psi_\beta , \quad \Gamma_{mn} \equiv \Gamma_m \tilde{\Gamma}_n - \tilde{\Gamma}_n \Gamma_m ,$$

$$\delta \chi^\alpha = \frac{1}{8} \omega^{mn} (\tilde{\Gamma}_{mn})^\alpha_\beta \chi^\beta , \quad \tilde{\Gamma}_{mn} \equiv \tilde{\Gamma}_m \Gamma_n - \Gamma_n \tilde{\Gamma}_m ,$$

(A.14)
which, however, are equivalent. The Lorentz invariant charge conjugation matrix

\[ C_{\alpha\beta} = i \begin{pmatrix} 0 & 1_{16} \\ -1_{16} & 0 \end{pmatrix}, \]  
(A.15)

\[ (\Gamma_{mn})_{\alpha\beta} C_{\beta\gamma} = C_{\alpha\beta} (\tilde{\Gamma}_{mn})_{\beta\gamma}, \]

can be used to raise or lower spinor indices. One can check also, that the following relations:

\[ (\Gamma_{mn} C)_{\alpha\beta} = (\Gamma_{mn} C^{'})_{\beta\alpha}, \]

\[ \tilde{\Gamma}^m = C\Gamma^m C^{-1} \]  
(A.16)

hold.

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