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On the thermo-elastostatics of heterogeneous materials

II. Analyze and generalization of some basic hypotheses and propositions

Abstract One considers linearly thermoelastic composite media, which consist of a homogeneous matrix containing a statistically homogeneous random set of ellipsoidal uncoated or coated inclusions. Effective properties (such as compliance and thermal expansion) as well as the first statistical moments of stresses in the phases are estimated for the general case of nonhomogeneity of the thermoelastic inclusion properties. At first, one shortly reproduces both the basic assumptions and propositions of micromechanics used in most popular methods, namely: effective field hypothesis, quasi-crystallite approximation, and the hypothesis of “ellipsoidal symmetry”. The explicit new representations of the effective thermoelastic properties and stress concentration factor are expressed through some building blocks described by numerical solutions for both the one and two inclusions inside the infinite medium subjected to both the homogeneous and inhomogeneous remote loading. The method uses as a background the new general integral equation proposed in the accompanied paper and makes it possible to abandon the basic concepts of micromechanics mentioned above. The results of this abandonment are quantitatively estimated for some modeled composite reinforced by aligned continuously inhomogeneous fibers. Some new effects are detected that are impossible in the framework of a classical background of micromechanics.

Keywords: A. microstructures, B. inhomogeneous material, B. elastic material.

1. Introduction

The prediction of the behavior of composite materials in terms of the mechanical properties of constituents and their microstructure is a central problem of micromechanics, which is evidently reduced to the estimation of stress fields in the constituents. Appropriate, but by no means exhaustive, references for the estimation of effective elastic moduli of statistically homogeneous media are provided by the reviews [1-7]. It appears today that variants of the effective medium method by Kröner [8] and by Hill [9], and the mean field method [10], [11] are the most popular and widely used methods. Recently a new method has become known, namely the multiparticle effective field method (MEFM) was put forward and developed by the author (see for references Buryachenko [6]). The MEFM is based on the theory of functions of random variables and Green’s functions. Within this method one constructs a hierarchy of statistical moment equations for conditional averages of the stresses in the inclusions. The hierarchy is then cut by introducing the notion of an effective field. This way the interaction of different inclusions is taken into account. Thus, the MEFM does not make use of a number of hypotheses which form the basis of the traditional one-particle methods.

However, a diversity of micromechanical methods and their specific formulations astonish our imagination only at first glance. We will see that most popular methods are based just on a few basic concepts of micromechanics. Effective field hypothesis is apparently the most fundamental, most prospective, and most exploited concept of micromechanics. This concept has directed a development of micromechanics over the last sixty years and made a contribution to their progress incompatible with any another concept. The notion of an effective field in which each particle is located is a basic concept of such powerful...
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methods in micromechanics as the methods of self-consistent fields and effective fields (for references see [6], [7], [12]. The idea of this concept dates back to Mossotti [13] and Clausius (in the dielectric context), Lorenz (in the refractivity context), and Maxwell (in the conductivity context). Markov [14] and Scaife [15] presented comprehensive reviews of the 150-year history of this concept accompanied by some famous formulae with extensive references. Mossotti [13] (especially Clausius) pioneered the introduction of the effective field concept as a local homogeneous field acting on the inclusions and differing from the applied macroscopic one. Among a few hypotheses used by Mossotti [13], the most important one was in fact the quasi-crystalline approximation proposed 100 years later by Lax [16] in a modern concise form. The concept of the effective field in combination with subsequent assumptions was introduced in a modern formalized form in the physics of multiple scattering of waves (see, e.g., [16-18]). Walpole [19] pioneered the application of the concept to the static of composites under the name uniform image field. Effective field technique was intensively applied in micromechanics of random and periodic structure composites (for references see, e.g., [6], [7]) as well in micromechanics of multiple interacting cracks under the name traction or pseudo-load [20]. Buryachenko and Rammerstorfer [21] has drawn the conclusion that the effective field concept is used (either explicitly or implicitly) in most popular methods of micromechanics such as, e.g., the effective medium method and their modifications, differential scheme, Mori-Tanaka method, and, needless to say, the MEFM.

The idea of effective field and quasi-crystalline approximation were added by the hypothesis of "ellipsoidal symmetry" for the distribution of inclusions attributed to Willis [22]. As a tool for concrete applications of the concepts mentioned, the Eshelby [23] solution was used although the Eshelby’s theorem has a fundamental conceptual sense (it will be shown in the current paper) rather than only an analytical solution of some particular problem for the ellipsoidal homogeneous inclusion. All these concepts creating the framework and background of modern statistical analytical micromechanics were transformed by the use of both the additional assumptions and sophisticated analytical and numerical tools to a few particular methods. However, we will show in this paper that the effective field hypothesis (also called the hypothesis H1a) is a central one and other concepts play a satellite role providing the conditions for application of the effective field hypothesis. Moreover, we will show that all mentioned hypotheses are not really necessary and can be relaxed.

The outline of the study is as follow. In Section 2 we recall the basic concepts defining the background of micromechanics. The interconnection between the different concepts and their essence are established. In Section 3 the auxiliary problem for one inclusion in the infinite matrix is presented for a general remote loading. The new general integral equation obtained in an accompanying paper by Buryachenko [24], henceforth referred to as (I), is presented in Section 4 through the operator forms of the particular solutions for both one and two interacting inclusions. This equation is solved by the iteration method in the framework of the quasi-crystallite approximation but without basic hypotheses of classical micromechanics such as both the effective field hypothesis and "ellipsoidal symmetry" assumption. In Section 5 we qualitatively explain the advantages of the new approach with respect to the classic ones and demonstrate the corrections of popular propositions obtained in the framework of the old background of micromechanics. Quantitative estimations of results of the abandonment of the central hypothesis H1a are presented in Section 6.

2. Preliminaries. Basic assumptions and propositions of micromechanics

2.1 General integral representations and notations

For the sake of brevity of the current presentation, the basic equations of thermoelasticity, the homogeneous boundary conditions (2.51), statistical description of the composite microstructure, assumption,
and notations exploited in the current paper are presented in the accompanied paper by Buryachenko [24] and the interested reader is referred to this publication, henceforth referred to as (I).

In this section we will shortly reproduce both the basic assumptions and propositions of micromechanics in the form adopted for subsequent presentation. In most detail we will consider the mentioned concepts as applied to the MEFM based on some mathematical approximations for solving the infinite systems of integral equations involved, although other methods exploiting these concepts will also be discussed.

For simplicity, we will consider only statistically homogeneous media (described, as a particular case, in Section 2.2 in I) subjected to the homogeneous boundary conditions (2.5I). If elastic properties of the comparison medium and matrix coincide (3.20I) then the known general integral equation in terms of stresses (see, e.g., [6],I)

\[
\sigma(x) = \langle \sigma \rangle + \int \Gamma(x - y)\eta(y) - \langle \Gamma(x - y)\eta(y) \rangle dy, \tag{2.1}
\]

\[
\sigma(x) = \langle \sigma \rangle + \int \Gamma(x - y)\eta(y) - \langle \eta(y) \rangle dy, \tag{2.2}
\]

can be much easier to solve because the stress-strain fields can be studied inside the inhomogeneities but not in the matrix; here \( \eta = M_1\sigma + \beta_1 \) and \( \Gamma \) are the strain polarization tensor (3.14I) and the Green stress tensor (3.5I), respectively. Buryachenko [6, I] proved that for no long-range order assumed, and for \( x \in w \) considered in Eqs. (2.1) and (2.2) and removed far enough from the boundary \( \Gamma (a \ll |x - y|, \forall y \in \Gamma) \), the right-hand side absolutely convergent integrals in (2.1) and (2.2) do not depend on the shape and size of the domain \( w \), and they can be replaced by the integrals over the whole space \( R^d \). With this assumption we hereafter omit explicitly denoting \( R^d \) as the integration domain in the equations. The new exact equation (2.1) forming a new background of micromechanics yields the known approximate one (2.2) only at some additional assumptions [(3.23I) or (3.24I), see for details I].

The solution of Eqs. (2.1) and (2.2) by the use of different assumptions provides the estimations of both the effective compliance \( M^* \) and the effective eigenstrains \( \beta^* \) governed by the overall constitutive relation

\[
\langle \varepsilon \rangle = M^*(\sigma) + \beta^* \tag{2.3}
\]

and defined by general relations

\[
M^* = M^{(0)} + \langle M_1 B^* \rangle, \tag{2.4}
\]

\[
\beta^* = \beta^{(0)} + \langle B^* \beta_1 \rangle, \tag{2.5}
\]

where \( B^* = B^*(x) (x \in v) \) is a local stress concentration tensor in the inhomogeneities obtained under pure mechanical loading \( \beta = 0 \)

\[
\sigma(x) = B^*(x)(\sigma) \quad \text{for} \quad x \in v. \tag{2.6}
\]

Analysis of Eqs. (2.1)-(2.6) leads to the universally accepted

**Proposition 1.** For statistically homogeneous media subjected to the homogeneous boundary conditions, linear elastic effective properties \( M^* \) and \( \beta^* \) depend only on stress distributions inside the inhomogeneities \( v \) but not inside the matrix \( v^{(0)} \).

Let the inclusions \( v_1, \ldots, v_n \) be fixed and we define two sorts of effective fields \( \sigma_i(x) \) and \( \tilde{\sigma}_{1,...,n}(x) \) \( (i = 1, \ldots, n; \ x \in v_1, \ldots, v_n) \) by the use of the rearrangement of Eq. (2.1) [or (2.2)] in the following form (see
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for the earliest references of related manipulations [6]):

\[ \sigma(x) = \overline{\sigma}(x) + \int \Gamma(x - y)V_i(y)\eta(y)dy, \]

\[ \overline{\sigma}_i(x) = \overline{\sigma}_{1,\ldots,n}(x) + \sum_{j \neq i} \int \Gamma(x - y)V_j(y)\eta(y)dy, \]

\[ \overline{\sigma}_{1,\ldots,n}(x) = \langle \sigma \rangle(x) + \int \{ \Gamma(x - y)\eta(y)V(y; v_1, x_1; \ldots; v_n, x_n) - \langle \Gamma(x - y)\eta(y) \rangle \} dy, \quad (2.7) \]

for \( x \in v_i, \ i = 1, 2, \ldots, n \); here \( V(y; v_1, x_1; \ldots; v_n, x_n) \) is a random indicator function of inclusions \( x \in v \) under the condition that \( x_i \neq x_j \) if \( i \neq j \) \((i, j = 1, \ldots, n)\). Then, considering some conditional statistical averages of the general integral equation (2.1) leads to an infinite system of new integral equations \((n = 1, 2, \ldots)\)

\[ \langle \sigma|v_1, x_1; \ldots; v_n, x_n \rangle(x) = \langle \sigma \rangle(x) + \int \{ \Gamma(x - y)\eta(y)V(y; v_1, x_1; \ldots; v_n, x_n) - \langle \Gamma(x - y)\eta(y) \rangle \} dy. \quad (2.8) \]

Since \( x \in v_1, \ldots, v_n \) in the \( n \)-th line of the system can take the values of the inclusions \( v_1, \ldots, v_n \), the \( n \)-th line actually contains \( n \) equations. The definitions of the effective fields \( \overline{\sigma}(x), \overline{\sigma}_{1,\ldots,n}(x) \) as well as their statistical averages \( \langle \overline{\sigma}(x), \overline{\sigma}_{1,\ldots,n}(x) \rangle \) are nothing more than notation convenience for different terms of the infinite systems (2.7) and (2.8), respectively. The physical meaning of these fields and their graphic illustrations are presented in Ref. [6].

2.2 Approximate effective field hypothesis

In order to simplify the exact system (2.8) we now apply the so-called effective field hypothesis which is the main approximate hypothesis of many micromechanical methods:

**Hypothesis 1a, H1a.** Each inclusion \( v_i \) has an ellipsoidal form and is located in the field (2.7a)

\[ \overline{\sigma}_i(y) = \overline{\sigma}(x_i) \quad (y \in v_i) \quad (2.9) \]

which is homogeneous over the inclusion \( v_i \).

In some methods (such as, e.g., the MEFM) this basic hypothesis H1a is complimented by a satellite hypothesis:

**Hypothesis 1b, H1b** The perturbation introduced by the inclusion \( v_i \) at the point \( y \notin v_i \) is defined by the relation

\[ \int \Gamma(y - x)V_i(x)\eta(x)dx = \tilde{v}_i\Gamma_i(y - x_i)\eta. \quad (2.10) \]

Hereafter \( \eta_i \equiv \langle \eta(x)V_i(x) \rangle_{(i)} \) is an average over the volume of the inclusion \( v_i \) (but not over the ensemble), \( \langle(\cdot) \rangle_{(i)} \equiv \langle(\cdot)(i) \rangle_{(i)} \), and \( (x \in v_i, \ y \in v_j) \)

\[ T_{ij}(x - x_i)\equiv \begin{cases} \frac{-1}{(\tau_{ij})^{-1}}Q_{ij} & \text{for } x \in v_i, \\ (\tau_{ij})^{-1} \int \Gamma(x - y)V_i(y)dy & \text{for } x \notin v_i; \end{cases} \quad T_{ij}(x_j - x_i) = \langle T_i(y - x_i) \rangle_{(j)}, \quad (2.11) \]

where the tensor \( Q_{ij} \) is associated with the well-known Eshelby tensor by \( S_i = I - M^{(0)}Q_i \). For a homogeneous ellipsoidal inclusion \( v_i \) the standard assumption (2.9) (see, e.g., [6], [7]) yields the assumption
(2.10), otherwise the formula (2.10) defines an additional assumption. The tensors $T_{ij}(x_i - x_j)$, proposed by Willis and Acton [25] for identical spherical inclusions have an analytical representation for spherical inclusions of different size in an isotropic matrix (see for references [6]) regardless of whether the inclusions are coated or uncoated.

According to hypothesis H1a and in view of the linearity of the problem there exist constant fourth and second-rank tensors $B_i(x)$, $R_i(x)$ and $C_i(x)$, $F_i(x)$, such that

$$
\sigma(x) = B_i(x)\sigma(x_i) + C_i(x), \quad \eta_i(x) = R_i(x)\sigma(x_i) + F_i(x), \quad x \in v_i,
$$

where $v_i \subset v^{(i)}$ and $\eta_i(x) = \bar{v}_i M_1^{(i)}(x) B_i(x)$. $F_i(x) = \bar{v}_i [M_1^{(i)}(x) C_i(x) + \beta_1(x)]$. According to Eshelby's theorem there are the following relations between the averaged tensors (2.12) $\sigma_i = \eta_i(x) = \tau_i Q_i^{-1}(I - B_i)$. $F_i = -\tau_i Q_i^{-1} C_i$, where $g_i \equiv \langle g(x) \rangle_{(i)}$ $(g$ stands for $B, C, R, F)$. It should be mentioned that the field $\sigma_i(x_i)$ can vary with the location of the center $x_i$ of the inclusion considered, but the field $\sigma(y) (y \in v_i)$ is homogeneous over the inclusion $v_i$. Because of this the application of Eshelby's theorem is correct.

For example, for the homogeneous ellipsoidal domain $v_i$ with

$$
M_1^{(i)}(x) = M_1^{(i)} = \text{const}, \quad \beta_1^{(i)}(x) = \beta_1^{(i)} = \text{const} \quad \text{at} \ x \in v_i,
$$

we obtain

$$
B_i = \left((I + Q_i M_1^{(i)})^{-1}, \quad C_i = -B_i Q_i \beta_1^{(i)}.
$$

In the general case of coated inclusions $v_i$, the tensors $B_i(x)$ and $C_i(x)$ can be found by the transformation method by Dvorak and Benveniste [26] (see for references and details [6], [27]).

Using hypothesis H1 (combining the hypotheses H1a and H1b), the system (2.7$_2$) for $k$ fixed inclusions with fixed values $\sigma_{1,...,k}(x) (x \in v_i, i = 1, \ldots, k)$ on the right-hand side of the equations becomes algebraic when the solution (2.12) for one inclusion in the field $\sigma_i(x_i)$ $(i = 1, \ldots, k)$ is applied

$$
R_i \sigma_i(x_i) + F_i = \sum_{j=1}^{k} Z_{ij} \left\{ R_j \sigma_{1,...,k}(x_j) + F_j \right\},
$$

where the matrix $Z^{-1}$ has the elements $(Z^{-1})_{ij}$

$$
(Z^{-1})_{ij} = B_{ij} - (1 - \delta_{ij}) R_j T_{ij} (x_i - x_j), \quad (i, j = 1, \ldots, n).
$$

2.3 Closing effective field hypothesis and effective properties

Different methods can be employed (see for details [6]) to truncate the hierarchy (2.8) considered as a system of coupled equations. One begins with the last hierarchy item which has the most heterogeneities held fixed, because this equation does not depend on the other. The solution obtained presents the forcing term in the next equation up the hierarchy. The unconditionally average field is finally obtained by going step by step up the hierarchy. For termination of the hierarchy of statistical moment equations (2.8) we will use the closing effective field hypothesis:

Hypothesis 2a, H2a) For a sufficiently large $n$, the system (2.8) is closed by the assumption

$$
\langle \sigma_{1,...,n+1}(x) \rangle_i = \langle \sigma_{1,...,n}(x) \rangle_i, \quad \text{where the right-hand-side of the equality does not contain the index} \ j \neq i \ (i = 1, \ldots, n; \ j = 1, \ldots, n + 1; \ x \in v_i).
$$
The hypothesis H2a rewritten in terms of stresses $\sigma(x)$, $(x \in v_i)$ is a standard closing assumption (see e.g. [1], [28]) degenerating to the “quasi-crystalline” approximation [16] at $n = 1$ (see for analysis also Subsection 2.4).

In the framework of the hypothesis H1, substitution of the solution (2.12), and (2.15) (at $k = 2$) for binary interacting inclusions into the first equation of the system (2.8) at $n = 1$ and at the effective field hypothesis H2a) with the first order approximation:

$$
\langle \sigma_{1,q}(x) \rangle_j = \langle \sigma(x) \rangle_j = \text{const.} \quad x \in v_j (j = i, q).
$$

leads to the solution $(x \in v_i)$

$$
\langle \sigma_j \rangle_i (x) = \int T_q(x - x_q)Z_{qi}\varphi(v_q, x_q; v_i, x_i)dx_q(R_i\langle \sigma_j \rangle_i + F_i)
+ \int \left[ T_q(x - x_q)Z_{qq}\varphi(v_q, x_q; v_i, x_i) - \Gamma(x - x_q)n^{(q)} \right] (R_q\langle \sigma_j \rangle_q + F_q)dx_q,
$$

(2.18)

where the matrix elements $Z_{qi}$, $Z_{qq}$ are nondiagonal elements and diagonal ones of the binary interaction matrix $Z$ (2.16) for the two inclusions $v_q$ and $v_i$; hereafter the conditional probability density $\varphi(v_q, x_q; v_i, x_i)$ and probability density $\varphi(v_q, x_q) \equiv n^{(q)}$ are described in (I). Averaging the result obtained (2.18) over the inclusion $v_i$ yields the final representation for both the statistical average stress field and effective properties

$$
\langle \sigma \rangle_i (x) = B_i(x)R_i^{-1}\sum_{j=1}^{N} Y_{ij}(R_j\langle \sigma \rangle + F_j) - F_i + C_i(x),
$$

(2.19)

$$
M^* = M^{(0)} + \sum_{i,j=1}^{N} Y_{ij}R_jn^{(i)} , \quad \beta^* = \beta^{(0)} + \sum_{i,j=1}^{N} Y_{ij}F_jn^{(i)}
$$

(2.20)

where the matrix $Y$ determines the action of the surrounding inclusions on the considered one and has an inverse matrix $Y^{-1}$ given by

$$
(Y^{-1})_{ij} = \delta_{ij} \left[ I - R_i \sum_{q=1}^{N} \left] T_{iq}(x_i - x_q)Z_{qi}\varphi(v_q, x_q; v_i, x_i)dx_q \right. 
- R_i \left. \int \left[ T_{iq}(x_i - x_q)Z_{qq}\varphi(v_q, x_q; v_i, x_i) - T_i(x_i - x_q)n^{(q)} \right] dx_q. \right]
$$

(2.21)

Buryachenko [6] demonstrated that the MEFM includes in particular cases the well-known methods of mechanics of strongly heterogeneous media (such as the effective medium and the mean field methods).

2.4 Quasi-crystalline approximation

Hypothesis H2a rewritten in terms of stresses $\sigma(x)$ degenerates to the “quasi-crystalline” approximation by Lax [16] which in our notations has two equivalent forms

Hypothesis 2b, H2b, “quasi-crystalline” approximation. It is supposed that the mean value of the effective field at a point $x \in v_i$ does not depend on the stress field inside surrounding heterogeneities $v_j \neq v_i$:

$$
\langle \sigma_i (x) | v_i, x_i; v_j, x_j \rangle = \langle \sigma_i \rangle , \quad x \in v_i, \quad \text{or} \quad Z_{ij} = I \delta_{ij}.
$$

(2.22)
Therefore, the matrix \( Y^{-1} \) can be reduced to (see Ref. [29])

\[
(Y^{-1})_{ij} = I_{ij} - R_i \int [T_{ij}(x_i - x_j)\varphi(v_j, x_j; v_i, x_i) - T_i(x_i - x_j)n^{(ij)}]dx_j.
\] (2.23)

The principal difference between the hypotheses (2.17) and (2.22) are discussed in Chapter 9 in [6]).

**Note.** It should be mentioned that the hypotheses H2a and H2b are not conceptually dependent on the hypothesis H1 and can be applied in general case even if the hypothesis H1 is violated (see for details Subsection 5.3).

2.5 **Hypothesis of “ellipsoidal symmetry” of composite structure**

To make further progress, the hypothesis of “ellipsoidal symmetry” for the distribution of inclusions attributed to Willis [22] (see also Khoroshun [28], [31], Buryachenko and Parton [29], Ponte Castaneda and Willis [32]) is widely used:

**Hypothesis 3, H3, “ellipsoidal symmetry”.** The conditional probability density function \( \varphi(v_j, x_j | x_i) \) depends on \( x_j - x_i \) only through the combination \( \rho = \| (a_{ij}^0)^{-1}(x_j - x_i) \| \):

\[
\varphi(v_j, x_j | x_i) = h(\rho), \quad \rho = \| (a_{ij}^0)^{-1}(x_j - x_i) \|,
\] (2.24)

where the matrix \( (a_{ij}^0)^{-1} \) (which is symmetric in the indexes i and j, \( a_{ij}^0 = a_{ji}^0 \)) defines the ellipsoid excluded volume \( v_{ij}^0 = \{ x : \| (a_{ij}^0)^{-1}x \|^2 < 1 \} \).

A pair distribution function has “ellipsoidal symmetry” but with an ellipsoid shape differing from the one that defines the inclusion shape. Although the assumed statistics may not be exactly realized in any particular composite, the results of effective moduli estimations are explicit and simple to use. It is crucial for the analyst to be aware of their reasonable choice of the shape of “ellipsoidal” spatial correlation of inclusion location (see Chapter 18 in Ref. [6]). For spherical inclusions the relation (2.24) is realized for a statistical isotropy of the composite structure. It is reasonable to assume that \( (a_{ij}^0)^{-1} \) identifies a matrix of affine transformation that transfers the ellipsoid \( v_{ij}^0 \) being the “excluded volume” (“correlation hole”) into a unit sphere and, therefore, the representation of the matrix \( Y_{ij} \) can be simplified:

\[
(Y^{-1})_{ij} = I_{ij} - R_i Q_{ij}^0
\] (2.25)

where \( Q_{ij}^0 = Q(v_{ij}^0) \) is a constant for the ellipsoidal domain \( v_{ij}^0 \) with the indicator function \( V_{ij}^0 \). For the sake of simplicity of the subsequent calculation we will usually assume that the shape of “correlation hole” \( v_{ij}^0 \) does not depend on the inclusion \( v_j \); \( v_{ij}^0 = v_i^0 \) and \( Q_{ij}^0 = Q_i^0 = Q(v_i^0) \).

Substitution of the representation (2.25) into Eqs. (2.19) and (2.20) completes the problem of effective properties estimations. The hypothesis H3 is widely used for micromechanical structures described also by the indicator function \( V(x; v_1, x_1; \ldots; v_n, x_n) \) (see, e.g. Willis [1], [22]; Ponte Castañeda and Willis [32]) rather than only by the conditional probability density \( \varphi(v_q, x_q | v_i, x_i) \) (2.24).

**Note.** A popular point of view is that the hypothesis of “ellipsoidal symmetry” (2.24) is exploited just for some simplification of the representation (2.23) reduced to (2.25). However, we will demonstrate in Section 5 that the destination of the hypothesis H3 is more fundamental and directed towards providing of conditions for applying of the hypothesis H1. The use of the satellite hypothesis H3 has no sense without the hypothesis H1.

**Proposition 2.** If the hypotheses H1, H2b, H3 hold for the statistically homogeneous medium and homogeneous boundary conditions then the effective properties \( M^* \) and \( \beta^* \) do not depend on the size of the correlation hole \( v_i^0 \) and the conditional probability density (2.24).
We can reach this conclusion by simple analyses of the final representations (2.19), (2.20), (2.23), and (2.25) as well as of analogous representations obtained by other methods in the framework of the hypotheses H1, H2b, and H3 (see, e.g., [1], [22], [28]). Here we keep in mind, first of all, the MEF which is equivalent for aligned identical ellipsoidal inclusions to the Mori-Tanaka [10] method. Moreover, Markov [33] demonstrated that for homogeneous ellipsoidal inclusions, the estimations by the MEF coincide with the variational estimates obtained by Ponte Castañeda and Willis [32] who also exploited the hypotheses H1, H2b, and H3. It will be demonstrated in Section 6 that satisfiability of the hypotheses H2b and H3 without H1 can lead to dependence of $M^*$ and $\beta^*$ on both the size of the correlation hole $\nu^0$ and the binary correlation function.

3. A single inclusion subjected to inhomogeneous prescribed effective field

3.1 Operator representation for solution obtained by the VIE method

In the current section we will present a slightly modified solution of a satellite problem (see [6] where additional references can be found) which is adapted for estimation of effective properties of composites in Section 6. Namely, let the inclusions $v_i$ be fixed and loaded by the inhomogeneous effective field $\bar{\sigma}_i(x)$:

$$\sigma(x) = \bar{\sigma}_i(x) + \int \Gamma(x - y)V_i(y)\eta(y)dy,$$

(3.1)

A known quadrature method for obtaining an approximate solution of Eq. (3.1) is to evaluate the volume integrals with a Gauss quadrature formula. Then the corresponding equations at the Gauss points will contain a singular term that results when the field point $x$ and source point $y$ coincide: $x = y$ (3.1). The difficulties with the troublesome singularities can be avoided if a rearrangement of Eq. (3.1) is performed in the spirit of a subtraction technique used in the modified quadrature method (see, e.g., [34])

$$\eta(x) = M_1(x)\bar{\sigma}(x) + \beta_1(x) + M_1(x) \int V_{i0}(y)\Gamma(x - y)dy\eta(x)$$

$$+ M_1(x) \int \Gamma(x - y)\eta(y) - V_{i0}(y)\eta(x)]dy, \quad x \in v_i,$$

(3.2)

where $v_i$, perhaps, is not an ellipsoid. The equation is valid for any domain $x \in v_{i0}$ with the indicator function $V_{i0}(x)$. We assume that the first integral in (3.2) is easily computable

$$- Q_{i0}(x) = \int V_{i0}(y)\Gamma(x - y)dy.$$

(3.3)

In a general case of the inclusion shape $v_i$ the function $Q_{i0}(x)$ can be found numerically, e.g. by finite element analysis (see, e.g., [35]). For ellipsoidal domain $v_{i0} \supset v_i$, the first integral on the right-hand-side of (3.2) is known and is associated with the well-known Eshelby tensor by ($x \in v_i \subset v_{i0}, \quad y \in v_{i0}$)

$$S_{i0} = I - M^{(0)}Q_{i0}, \quad Q_{i0} \equiv - \bar{v}_{i0}(\Gamma(x - y))_{i0} = \text{const.}$$

(3.4)

Hereafter

$$g_i \equiv \langle g(y) \rangle_i = \bar{v}_i^{-1} \int g(y)V_i(y)dy, \quad g_{i0} \equiv \langle g(y) \rangle_i^{0} = (\bar{v}_{i0})^{-1} \int g(y)V_{i0}(y)dy$$

(3.5)

denotes averaging of some tensor $g(y)$ over the volume of the regions $y \in v_i \subset v_{i0}$ and $y \in v_{i0}$, respectively; in so doing, $x \in v_i$ is fixed. The assumption of an ellipsoidal shape of the domain $v_{i0}$ was used
only to obtain analytical representation of the integral (3.4). This is because the tensor \((\mathbf{\Gamma}(x-y))_{i0}\) is homogeneous for \(x \in v_i\), \(y \in v_{i0}\) for an ellipsoid. For nonellipsoidal inclusions \(v^{n-e}\) one could assume that in some parts of the region \(v_i^c \subset v_i\) the properties \(M_i(x) = 0, \beta_i(x) = 0\), i.e. it is sufficient to replace a real nonellipsoidal inclusion \(v^{n-e} = v_i \setminus v_i^c\) by a fictitious ellipsoid (with smallest possible volume) and call it the inclusion \(v_i\) with a “coating” \(v_i^c\). In so doing, at the estimation of the second integral in (3.2) we keep in mind that \(\int \mathbf{\Gamma}(x-y)(V_i(y)-V_{i0}(y))dy = Q_{i0} - Q_i(x)\) (\(x \in v_i\)). With the nonessential restriction on the shape of the inclusion \(v_i\) mentioned above, we can consider without loss of generality an ellipsoidal inclusion \(v_{i0} = v_i\); in so doing \(M_i(y) \equiv 0, \beta_i(y) \equiv 0\) at \(y \in v_i^c \subset v_i\). Then Eq. (3.2) can be rewritten in the equivalent compact form

\[
\eta(x) = \mathbf{\eta}_i(x) + \int K_i(x,y)[\eta(y) - \eta(x)]dy, \quad x \in v_i, \tag{3.6}
\]

where \(\mathbf{\eta}_i(x) = \mathbf{E}_i(x)\mathbf{\tau}(x) + \mathbf{H}_i(x), (x \in v_i)\) is called the effective strain polarization tensor in the inclusion \(v_i\), and (no sum on \(i\))

\[
K_i(x,y) = \mathbf{E}_i(x)\mathbf{\Gamma}(x-y)V_i(y), \quad \mathbf{E}_i(x) = M_i(x)[I + Q_{i0}(x)M_i(x)]^{-1}, \quad \mathbf{H}_i(x) = [I + M_i(x)Q_{i0}(x)]^{-1}\beta_i(x). \tag{3.7-3.9}
\]

We rewrite Eq. (3.6) in symbolic form:

\[
\eta = \mathbf{\eta}_i + \mathbf{\mathcal{K}}\eta, \tag{3.10}
\]

where

\[
(\mathbf{\mathcal{K}}\eta)(x) = \int \mathbf{\mathcal{K}}_i(x,y)\eta(y)dy \tag{3.11}
\]

defines the integral operator \(\mathbf{\mathcal{K}}_i\) with the kernel formally represented as

\[
\mathbf{\mathcal{K}}_i(x,y) = K_i(x,y) - \delta(x-y)\int V_i(z)K_i(x,z)dz. \tag{3.12}
\]

We formally write the solution of Eq. (3.10) as

\[
\eta = \mathbf{\mathcal{L}}_i\mathbf{\eta}_i, \tag{3.13}
\]

where the inverse operator \(\mathbf{\mathcal{L}}_i = (I - \mathbf{\mathcal{K}}_i)^{-1}\) will be constructed by the iteration method based on the recursion formula

\[
\eta^{[k+1]} = \mathbf{\eta}_i + \mathbf{\mathcal{K}}_i\eta^{[k]} \tag{3.14}
\]

and (3.14) with arbitrary continuous \(\eta^{[0]}(x)\) converges to a unique solution \(\eta\) if the norm of the integral operator \(\mathbf{\mathcal{K}}_i\) is sufficiently “small” (less than 1), and the problem is reduced to the computation of the integrals involved, the density of which is given. In

\[
\eta^{[0]}(x) = \mathbf{\eta}_i(x), \tag{3.15}
\]

which is exact for a homogeneous ellipsoidal inclusion subjected to remote homogeneous stress field \(\mathbf{\tau}(x) = \mathbf{\tau} = \text{const}\). The sequence \(\{\eta^{[k]}\}\) (3.14) converges to a unique solution \(\eta\) if the norm of the integral operator \(\mathbf{\mathcal{K}}_i\) turns out to be small. In
effect the iteration method (3.14) transforms the integral equation problem (3.14) into the linear algebra problem in any case.

We will introduce the linear operators $\mathcal{L}_i^v$ and $\mathcal{L}_{ii}^v$ describing a perturbation of the stress field inside and outside the inclusion $v_i$ ($x \in R^d$)

$$\int \Gamma(x - y)V_i(y)\eta(y)dy = \sigma(x) - \sigma_i(x) \equiv \mathcal{L}_i^v(\sigma_i)(x) \equiv \mathcal{L}_{ii}^v(\eta)(x), \quad (3.16)$$

$$\mathcal{L}_i^v(\sigma_i)(x) = \int \Gamma(x - y)L_i*(E_i\sigma + H_i)(y)V_i(y)dy, \quad (3.17)$$

$$\mathcal{L}_{ii}^v(\eta)(x) = \int \Gamma(x - y)\eta(y)V_i(y)dy. \quad (3.18)$$

The right-hand side of Eqs. (3.17) and (3.18) can be also estimated in the spirit of subtraction technique (3.2) according to the next scheme

$$\mathcal{L}(g)(x) = \int \Gamma(x - y)g(y)V_i(y)dy$$

$$= Q_i(x)g(x) + \int \Gamma(x - y)[g(y) - g(x)]V_i(y)dy, \quad (3.19)$$

$$\mathcal{L}(g)(x) = Q_i(x_m)g(x_m) + \int \Gamma(x - y)[g(y) - g(x_m)]V_i(y)dy, \quad (3.20)$$

for $x \in v_i$ and $x \notin v_i$, respectively; here $x_m = \arg \min_y |x - y| (y \in v_i, x \notin v_i)$, $\mathcal{L} = \mathcal{L}_i^v, \mathcal{L}_{ii}^v$, $g = \sigma_i, \eta$, and the tensor $Q_i(x)$ is defined analogously to Eq. (3.3).

We constructed the solution (3.16) for a perturbation of the stress field inside and outside the inclusion $v_i$ in the operator form obtained by the method of volume integral equation (VIE) for an arbitrary effective field $\sigma_i(x)$, ($x \in v_i$). However, this operator could be created by any another numerical method such as, e.g. the finite element analysis (FEA). The main difficulty in such a case is a generation of prescribed effective field $\sigma_i(x)$, ($x \in v_i$) which will be considered in the next subsection.

### 3.2 Creation of prescribed stresses by the FEA

Construction of the operator $\mathcal{L}_i^v$ (3.16) anticipates a creation of prescribed effective field $\sigma_i(x)$, ($x \in v_i$) in the absence of the inclusion $v_i$. In the case of the FEA employment, it can be done by prescribing of either some boundary condition at the boundary of a large sample or some eigenstress inside this sample. We will only consider the second way.

The problem is to find a fictitious $\beta_1(x)$ generating a prescribed stress $\sigma(x)$ in an arbitrary fictitious ellipsoidal inclusion $x \in v_0$ (which has no connection with the correlation hole $\nu_{ij}^0$) with the elastic modulus $M(x) \equiv M^{(0)}$ and an indicator function $V_0$.

$$\sigma(x) = \int \Gamma(x - y)\beta_1(y)V_0(y)dy. \quad (3.21)$$

The equation (3.21) can be recast in the form

$$\beta_1(x) = -Q_0^{-1}\sigma(x) + Q_0^{-1}\int \Gamma(x - y)[\beta_1(y) - \beta_1(x)]V_0(y)dy \quad (3.22)$$
Except for notations, the Fredholm integral equation of the second kind (3.22) coincides with the direct equation for estimation of stresses \( \sigma(x) \) produced by the field \( \overline{\sigma}(x) \) inside the heterogeneity \( x \in v_0 \)

\[
\sigma(x) = B_0 \overline{\sigma}(x) + B_0 \int \Gamma(x - y) M_1(y) \sigma(y) - M_1(x) \sigma(x) \right) V_0(y) dy
\]

(3.23)

Indeed, Eq. (3.22) is reduced to Eq. (3.23) (with \( \beta_1 \equiv 0 \)) by the replacement of the notations: \( \beta_1 \rightarrow M_1 \sigma, -Q_0^{-1} \overline{\sigma} \rightarrow M_1 B_0 \overline{\sigma}, Q_0^{-1} \Gamma \rightarrow M_1 B_0 \Gamma, \) where \( B_0 = B_0(v_0) \) is defined analogously to (2.14).

In the case of the successive approximations method, we need to evaluate the right-hand side of the equation

\[
\beta_1^{[n+1]}(x) = -Q_0^{-1} \overline{\sigma}(x) + Q_0^{-1} \int \Gamma(x - y) [\beta_1^{[n]}(y) - \beta_1^{[n]}(x)] V_0(y) dy
\]

(3.24)

with the usual use of the driving term as an initial approximation

\[
\beta_1^{[0]}(x) = -Q_0^{-1} \overline{\sigma}(x).
\]

(3.25)

The volume integral equation (3.21) is reduced to the regular representation, which has no singularities and can be also presented in the form adopted for using of the FEA

\[
\beta_1^{[n+1]}(x) = Q_0^{-1} \overline{\sigma}(x) + \beta_1^{[n]}(x) - Q_0^{-1} R_0 \beta_1^{[n]}(x)
\]

(3.26)

where the operator

\[
R_0 \beta_1^{[n]}(x) = \int \Gamma(x - y) \beta_1^{[n]}(y) V_0(y) dy
\]

(3.27)

presents the stresses produced by the intermediate eigenstresses \( \beta_1^{[n]}(x) \) in the inclusion \( x \in v_0 \) (see Eq. (3.24)). Obviously, the mentioned stresses can be easy estimated by the FEA. Thus, an operator representation of the solution (3.21)

\[
\beta(x) = \Gamma^{-1} * \overline{\sigma}(x) = \int \Gamma^{-1}(x - y) \overline{\sigma}(y) V_0(y) dy
\]

(3.28)

can be considered as found by the FEA (or by any other numerical method providing a solution of the regular integral Eq. (3.22). The next step for FEA utilization is obvious. We introduce the real inclusion \( v_i \) into the fictitious ellipsoid \( v_0 \) (such that all desirable area for the stress estimation is placed inside \( v_0 \)) and estimate the real stresses \( \sigma(x) \) which can be considered as found in Eq. (3.16).

The VIE and FEA methods have a series of advantages and disadvantages (considered, e.g., in Ref. [6]), and it is crucial for the analyst to be aware of their range of applications.

4. Estimation of both the effective field and effective elastic moduli

The new general integral equation (2.1) can be rewritten in terms of the operator representation \( \mathcal{L}_v \)

(3.16)

\[
\sigma(x) = \langle \sigma \rangle(x) + \int [\mathcal{L}_v[\langle \eta \rangle(x) - \langle \eta \rangle)](x) dy
\]

(4.1)

while conditional averaging of Eqs. (2.7_2) and (2.7_3) leads to the following representation for the mean of the effective field in the fixed inhomogeneity \( x \in v_i \)

\[
\langle \overline{\sigma} \rangle_i(x) = \langle \sigma \rangle(x) + \int [\mathcal{L}_q[\langle \sigma \rangle(x), v_1, x_1]) - Q_1^{[q]}(x)](x) dx_q,
\]

(4.2)
where \( \langle \sigma \rangle_{q} (x) = \langle \sigma \rangle_{q} (y) \) is a conditional statistical average of \( \sigma (y) \) varying along the fixed heterogeneity \( y \in v_{q} \) at the fixed \( v \) while \( \langle \sigma \rangle_{q} = \langle \sigma \rangle_{q} (y) \) is a statistical average \( \sigma (y) \) inside the heterogeneity \( y \in v_{q} \). No confusion will arise hereafter in definition of the operator \( D (L_{q}^{\sigma}, L_{q}^{0}) \) with the kernel \( D(x, y) \) on the inhomogeneous functions \( g(x) \) (e.g., \( g(y) = \langle \sigma \rangle_{q} (y) \), \( y \in V_{k} \)).

The operator \( D \) is reduced to the tensor \( D(x) \) on the constant functions \( g(x) = g \equiv \text{const} \ (x \in V_{k}) \)

\[
D(g)(x) = D(x)g, \quad D(x) = \int \mathcal{D}(x, y) V_{k}(y) dy.
\]

The integral in the right-hand side of Eq. (4.2) can be decomposed as

\[
\langle \sigma \rangle_{q}(x) = \langle \sigma \rangle(x) + J, \quad J = J_{1} + J_{2} + J_{3},
\]

where

\[
J_{1} = \int L_{q}^{\sigma}((\sigma); v_{i}, x_{i}) \varphi(v_{i}, x_{i}; v_{i}, x_{i}) dx_{q},
\]

\[
J_{2} = \int L_{q}^{\sigma}((\sigma); x_{q}) \varphi(v_{q}, x_{q}; v_{i}, x_{i}) - n_{q}^{(q)}(x_{q})[1 - V_{q}^{0}(x_{q})] dx_{q},
\]

\[
J_{3} = - \int L_{q}^{\sigma}((\sigma); x_{q}) n_{q}^{(q)}(x_{q}) V_{q}^{0}(x_{q}) dx_{q}.
\]

The absolutely convergent integral in Eq. (4.2) is decomposed in Eq. (4.5) just for subsequent presentation obviousness; because of this, the absolute convergences of integrals \( J_{1}, J_{2}, \) and \( J_{3} \) are not considered.

In the framework of the quasi-crystalline approximation (2.22) \( (y \in v_{q}) \)

\[
\langle \sigma \rangle_{q}(x) = \langle \sigma \rangle(x) + \int L_{q}^{\sigma}((\sigma); x_{q}) \varphi(v_{q}, x_{q}; v_{i}, x_{i}) - n_{q}^{(q)}(x_{q}) dx_{q},
\]

(4.11)

and Eq. (4.5) is simplified \( (J = J_{2} + J_{3}) \)

(4.12)

For statistically homogeneous media when \( n(x_{q}) = n^{(q)} \equiv \text{const} \), it is logical to assume the acceptance of the additional hypothesis of “ellipsoidal symmetry” (2.24), which leads to the same simplification as in Subsection 2.4 \( (J_{2} = 0) \)

\[
\int L_{q}^{\sigma}((\sigma); x_{q}) \varphi(v_{q}, x_{q}; v_{i}, x_{i}) - n_{q}^{(q)}(x_{q})[1 - V_{q}^{0}(x_{q})] dx_{q} = 0,
\]

which can be presented in an equivalent form exploiting Green’s function

\[
\int \Gamma(x - y)[(M_{1} \sigma)(y) + \beta_{1}(y)] \varphi(v_{q}, x_{q}; v_{i}, x_{i}) - n_{q}^{(q)} V_{q}^{0}(y)[1 - V_{q}^{0}(y)] dy = 0.
\]

Then Eq. (4.5) leads to \( (J = J_{3}) \)

(4.13)

(4.14)
The accuracies of the assumptions (4.9), (4.10) and (4.12) will be estimated in Section 6. However, we will perform subsequent solution of Eq. (4.11) rather than Eq. (4.14) by keeping in mind that the hypothesis H3 (2.24) is accepted.

Obviously, the regular integral equation (4.11) has no singularities and can be solved by a direct quadrature method with formal representation of the solution

\[ \langle \sigma \rangle_i(x) = T_i * \langle \sigma \rangle(x) \]  

(4.15)

Although the direct quadrature method usually causes no problems of accuracy, for a large number of unknown variables \( N \) its \( O(N^3) \) cost dependence can lead to surprisingly long computing time. The obvious way of reducing this cost is to construct an iterative scheme which will be considered now. Namely, Eq. (4.11) will be solved by the iteration method when the initial constant effective stress \( \langle n \rangle \) of unknown variables

\[ \langle \sigma \rangle_q(x) = B_q(x)(\overline{\sigma})_q + C_q(x), \quad \langle \eta \rangle_q(x) = R_q^v(x)q_0 + F_q^v(x), \]  

(4.16)

in a general case of inhomogeneity of \( v_q \); here \( R_q^v = \bar{v}_q^{-1}R_q, F_q^v = \bar{v}_q^{-1}F_q \). In such a case, the right-hand side of Eq. (4.11) generates inhomogeneous field \( \langle \sigma_i \rangle(x) \). For elimination of this difficulty, we will use the additional condition of the effective field hypothesis H1b (2.10), when a perturbation introduced by the inhomogeneity \( v_q \) is defined by the strain polarization tensor \( \langle \eta \rangle \) averaged over the volume \( v_q(x \in R^d) \)

\[ \mathcal{L}_q^v(\langle \overline{\sigma} \rangle_q)(x) = \bar{v}_qT_q(x - x_q)\langle \eta \rangle_q. \]  

(4.17)

Then Eq. (4.11) in the framework of the hypothesis H3 (2.24) is reduced to the classical representation for the effective field \( \langle \sigma_i \rangle(x) \)

\[ \langle \sigma_i \rangle(x) = \langle \sigma \rangle + Q_i^q(x) \sum_{q=1}^{n} c(q) \langle \eta \rangle_q, \]  

(4.18)

which is homogeneous just for an additional assumption of an ellipsoidal shape of the excluded volume \( v_q \). Combining the averaged Eqs. (4.16) and (4.18) leads to the final representations for the averaged tensors of both the effective field and strain polarization

\[ \langle \overline{\sigma}_i \rangle^{[0]} = \langle \sigma \rangle + Q_i^q[I - (R^vQ^0V)]^{-1}(\langle R^vV \rangle \langle \sigma \rangle + \langle F^vV \rangle), \]  

(4.19)

\[ \bar{v}_i\langle \sigma \rangle_i^{[0]}(x) = B_i(x)\langle \sigma \rangle + C_i(x) + B_i(x)Q_i^q[I - (R^vQ^0V)]^{-1}(\langle R^vV \rangle \langle \sigma \rangle + \langle F^vV \rangle), \]  

(4.20)

\[ \langle \eta \rangle_i^{[0]}(x) = R_i(x)\langle \sigma \rangle + F_i(x) + R_i(x)Q_i^q[I - (R^vQ^0V)]^{-1}(\langle R^vV \rangle \langle \sigma \rangle + \langle F^vV \rangle), \]  

(4.21)

which will be considered as the initial approximation of the next equations \( (x \in v_i, \ y \in v_q) \)

\[ \langle \overline{\sigma}_i \rangle^{[n+1]}(x) = \langle \sigma \rangle + \int \mathcal{L}_q^v(\langle \eta \rangle_q^{[n]}(x)[\varphi(v_q, x_q; v_i, x_i) - n(q)\langle x_q \rangle]d\mathbf{x}_q, \]  

(4.22)

\[ \langle \eta \rangle_i^{[n+1]}(y) = R_q \langle \overline{\sigma}_i \rangle^{[n+1]}(y) + F_q^v(y), \]  

(4.23)

where \( R_q = M_1(I + \mathcal{L}_q^v) \) and Eq. (4.22) in the case of the assumption (4.12) is reduced to the following one

\[ \langle \overline{\sigma}_i \rangle^{[n+1]}(x) = \langle \sigma \rangle - \int \mathcal{L}_q^v(\langle \eta \rangle_q^{[n]}(x)n(q)\langle x_q \rangle V_q^0(x_q)d\mathbf{x}_q. \]  

(4.24)
The system (4.22) and (4.23) can be formally presented in an operator form \( \langle \eta \rangle (x) = (\sigma) + KC(\langle \eta \rangle)(x) \) (the indexes are dropped for simplicity). It suggests the Neumann series form for the solution \( \eta \) of (4.22) and (4.23) [compare with the solution (4.15)]

\[
\langle \eta \rangle_i(x) \equiv \lim_{n \to \infty} \langle \eta^{[n]} \rangle_i(x) = R_i^*(x) (\sigma) + F_i^*(x),
\]

(4.25)

which yields the final representations for the effective properties

\[
M^* = M^{(0)} + (R^*V), \quad \beta^* = \beta^{(0)} + (F^*V).
\]

(4.26)

A convergence of the sequence \( \langle \eta^{[n]} \rangle_i(x) \) (4.25) is analyzed analogously to the sequence (3.14).

In Eq. (4.26) we used an obvious connection between the phase average \( \langle gV \rangle \) (\( g = \sigma, \varepsilon, \eta \)) and the averages inside the representative inclusions \( v_k \in \varepsilon^{(k)} (k = 1, \ldots, N) \)

\[
\langle gV \rangle = \sum_{k=1}^{N} \varepsilon^{(k)} \langle g^k \rangle_k,
\]

(4.27)

which is only fulfilled for statistically homogeneous media subjected to the homogeneous boundary conditions. If any of these conditions is broken then it is necessary to consider a generalization of Eq. (4.27) in the form of Eq. (3.29)\( \text{I} \). However, the mentioned class of nonlocal problems is beyond the scope of the present paper.

5. Qualitative analysis of some basic hypotheses and propositions

5.1 Analysis of the proposition 1 and hypothesis H1

The hypothesis H1 is widely used (explicitly or implicitly) for the majority of the methods of micromechanics even if the term “effective field hypothesis” is not indicated. For example, Buryachenko [6] demonstrated that hypothesis H1 is exploited in the effective medium method, generalized self-consistent method, differential methods, Mori-Tanaka method, the MEFM, conditional moments method, variational methods, and others. These are a lot of other methods using the hypothesis H1 differ one from another by some additional specific assumptions.

It should be mentioned, that the domain of the operator \( L^\beta_q(\langle \eta^{[n]}_q \rangle)(x) \) (3.18) is a whole space \( x \in \mathbb{R}^d \), and, because of this, some points of the area \( x \in v_i \) in Eq. (4.22) can be uncovered by the heterogeneities \( v_q \) and, therefore, the effective stress \( \langle \sigma \rangle_i^{[n+1]}(x) \) (4.22) will depend on the stress perturbations \( L^\beta_q(\langle \eta^{[n]}_q \rangle)(x) \) in the vicinity \( x \in v_i \setminus v_q \) of the area \( v_q \) rather than only on stress distributions in the inhomogeneity \( v_q \) and \( \max_r |x - x_q| = 3a (x \in v_i \setminus v_q) \) for the identical spherical inhomogeneities of the radius \( a \) with an isotropic statistically distribution of their centers. Thus, we obtain a fundamental conclusion that effective moduli in general depend not only on the stress distribution inside the inhomogeneities but also on the stresses in the vicinities of inhomogeneities (compare with the proposition 1). However, if our estimations utilize Eq. (2.2) containing only average strain polarization \( \eta_q \) rather than \( \langle \Gamma(x - y)\eta_q(y) \rangle_q \) in Eq. (4.1) as a renormalizing item then an influence of stresses in the vicinities of inhomogeneities is degenerated. At the same time, using Eq. (4.1) leads to the necessity of evaluation of stresses in the inhomogeneity vicinities even for a statistically homogeneous field of ellipsoidal homogeneous inclusions (it will be quantitatively demonstrated in Section 6). Moreover, a fundamental deficiency of Eq. (2.2) is the dependence of the renormalizing item \( \Gamma(x - x_q) \) on the average stress polarization tensor \( \langle \eta \rangle \) while a corresponding item \( \langle \Gamma(x - y)\eta_q(y) \rangle_q(x_q) (y \in x_q) \) in the new Eq. (4.1) explicitly depends
on details distribution \( \langle \eta_i q, x_i \rangle (y) \ (y \in x_i) \). Because of this, the averaging methods used in Eq. (2.2) as a starting element conserve the mentioned deficiency of Eq. (2.2) (at least in some elements of these methods). For example, Chen and Acrivos [36] have estimated the effective elastic moduli through the accurate evaluation of binary interactions of inclusions without the hypothesis H1, e.i. in our notations the operator \( \mathcal{L}_q^i (\langle \sigma_i, v_i, x_i \rangle_q) (x) \) (4.2) was estimated at the condition \( \langle \sigma_{i,q} \rangle \equiv \langle \sigma \rangle \) [compare with Eq. (2.17)]. However, the operator \( \mathcal{L}_q^i (\langle \sigma \rangle) (x) \) (4.2) was estimated at the approximation (4.17) that implicitly implies the use of both the hypothesis H1 and Eq. (2.2) (see for details Subsection 10.2.2 in [6]).

On the other hand, although the method (4.18)-(4.21) allows the inhomogeneous statistically averaged tensor \( \langle \eta_i q \rangle (y) \ (y \in v_q) \) (4.21), but Eq. (4.21) containing the item \( R_i^q Q^i V \) generated by the second summand in the left-hand side of Eq. (4.18) depending only on the average stress polarization tensor \( \langle \eta \rangle \). As a consequence of this the final classical representations of the effective properties (2.19), (2.20) and (2.25) depend only on average stress concentrator factors \( R_i \) and \( F_i \) while the effective properties (4.26) explicitly depend on the inhomogeneous tensors \( R_i (x) \) and \( F_i (x) \) as well as on detailed distribution \( \langle \eta_i q \rangle (y) \ (y \in v_q) \) (4.23).

Moreover, the detected explicit dependence of the effective properties (4.26) on the detailed stress concentrator factors \( R_i (x) \) and \( F_i (x) \) rather than on the average values \( R_i \) and \( F_i \), allows for an abandonment of the hypothesis H1b [or (4.17)] whose accuracy is questionable for the inhomogeneous (e.g., coated) inclusions. In such a case the statistical average effective field estimated by Eq. (4.22) is found to be inhomogeneous that discards the hypothesis H1a. Quantitative estimations of the result of this abandonment of the hypothesis H1 will be performed in Section 6 in the framework of the hypothesis H3 for some particular cases of fiber composites.

5.2 Analysis of the proposition 2 and hypotheses H1b and H3

As was mentioned, forfeiting of the effective field hypothesis H1a by the additional perturbation hypothesis (4.17) leads to Eq. (4.18) providing homogeneity of the effective field estimation for the ellipsoidal excluded volume \( v_q^0 \). Moreover this estimation of the effective field (and, therefore, of effective moduli) is invariant with respect to the size of the ellipsoidal excluded volume \( v_q^0 \). However, the additional hypothesis (4.17) is exactly fulfilled only for the homogeneous ellipsoidal inhomogeneity \( v_i \). For both the inhomogeneous and nonellipsoidal inclusions the equality (4.17) is just an approximation and the new general equation (4.1) has an advantage with respect to the popular one (2.2) only based on average strain polarization \( \eta_i \) rather than \( (\mathcal{F} (x-y) \eta(y) \rangle_q \). Then the size of the excluded volume \( v_q^0 \) will impact on the effective field (4.22). Indeed, if the radius of the excluded volume \( v_q^0 \) in Fig. 1 increases from \( 2a \) to \( 3a \) then the long distance of influence zone of the inhomogeneity \( v_q \) on the effective field \( \langle \sigma_q \rangle (x) \) will increase from the value \( |x-x_q| = 3a \) (as in Fig. 1) till \( |x-x_q| = 4a \). This influence will be quantitatively estimated in the next section.

A popular explanation of acceptance of the “ellipsoidal symmetry” hypothesis (2.24) is that this hypothesis just simplifies Eq. (2.23) reducing this equating to Eq. (2.25) which does not contain the integrals. In a similar manner, a destination of the assumption of the ellipsoidal shape of the excluded volume \( v_q^0 \) in the hypothesis H3 is that this hypothesis just simplifies Eq. (4.18) by the use of analytical known tensor \( Q_q^0 \) (expressed through the Eshelby tensor \( S_q^0 \) (3.4)) which is exploited instead of a general tensor \( Q_q^0 (x) \) found numerically (see e.g. Subsection 4.7.4 in the book [6]). However, the both mentioned assumptions of the hypothesis H3 have a fundamental conceptual sense rather than only an analytical solution of some particular problem. Exploiting the Eshelby tensor concept in Eq. (4.18) (and in the MEFM) is based on the ellipsoidal shape of the correlation hole \( v_q^0 \) rather than on the inclusion shape \( v_q \). An abandonment of either the assumption of the \( v_q^0 \)’s ellipsoidal shape or “ellipsoidal symmetry” hypothesis (2.24) with necessarily leads to the inhomogeneity of the effective field \( \sigma_q \), acting on the
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inclusion \( v_i \) that is prohibited for the classical version of the MEFM. However, Buryachenko [6] (see Section 9.4) proposed a method for solution of Eq. (4.18) based on the general integral equation (2.2). Namely, acting on Eq. (4.18) by the operator \( B = (I - \Gamma M_1)^{-1}V \) yields

\[
\langle \eta \rangle_i(x) = R_i(x)\langle \sigma \rangle + F_i(x) + RQ^0_i(x) \sum_{q=1}^{N_i} \langle \eta_q \rangle \epsilon^{(q)}. \tag{5.1}
\]

where the numerical estimation scheme of the tensor \( RQ^0_i(x) \) for any shape of \( v_0^0 \) was considered in Subsection 9.4.4 in Ref. [6]. Volume averaging of Eq. (5.1) over the heterogeneity \( v_i \) and summation over the inclusion number \( i \) lead to the average strain polarization tensor \( \langle \eta \rangle \) and, therefore, gives the representations for the effective properties

\[
M^* = M^{(0)} + [I - \langle RQ^{0v}V \rangle]^{-1}\langle R^vV \rangle, \tag{5.2}
\]

\[
\beta^* = \beta^{(0)} + [I - \langle RQ^{0v}V \rangle]^{-1}\langle F^vV \rangle. \tag{5.3}
\]

If (and only if) the correlation hole \( v_0^0 \) is chosen as an ellipsoid (homothetical, for example, to \( v_i \)) then the tensor \( Q^0_i(x) = Q_0^0 \equiv \text{const.} \) constants, \( RQ^0_i(x) = \bar{v}_i(v_0^0)^{-1}R_i(x)Q_0^0, BQ^0_i(x) = B_0^0(x)Q_0^0, \) and, therefore, Eqs. (5.2) and (5.3) are reduced to the known representations (2.20) and (2.25) with the constant tensor \( Q_0^0 \) depending on the orientation of the correlation hole \( v_0^0 \). An advantage of Eqs. (4.22)–(4.26) with respect to Eqs. (5.2) and (5.3) are that Eqs. (5.2) and (5.3) are fundamentally limited by analysis of statistically homogeneous media subjected to the homogeneous boundary conditions while the system (4.22) and (4.23) can be easily generalized to the statistically inhomogeneous media. Moreover, the method (5.1)–(5.3) conserves a fundamental deficiency of the general integral Eq. (2.2) containing the renormalizing item defining only by the averaged strain polarization tensor \( \langle \eta \rangle \). Estimation of the tensor \( RQ^0_i(x) \) in an auxiliary model problem with homogeneous fictitious eigenstrain in \( v_0^0 \) implies that influence of surrounding inclusions \( v_q \ (x_q \in v_0^0) \) is defined by the average strain polarization tensors \( \langle \eta \rangle_q \langle x \rangle \) rather than its detailed distribution \( \langle \eta \rangle_q(x) \ (x \in v_q) \). Impact of the last improvement on the estimated effective properties will be considered in Section 6 for the circle \( v_0^0 \) although, of course, analysis of any shape of \( v_0^0 \) present no additional difficulties for the new method (4.22)-(4.23) as opposed to the method (5.1)-(5.3) requiring evaluation of the supplementary tensor \( RQ^0_i(x) \).

5.3 Analyses of the hypotheses H2a and H2b

As it was noted in Subsection 2.3, the hypotheses H2a and H2b are not conceptually dependent on the hypothesis H1 and can be applied in general case even if the hypothesis H1 is violated. Indeed,
at obtaining of Eq. (2.18) we already used the hypothesis \( H_1 \) because Eq. (2.18) contains the objects \( R_q(\sigma_q) + F_q \) instead of their operator generalization \( \mathcal{Z}_q(\eta)(x) \) (3.18) which does not use the hypothesis \( H_1 \). However, even in this case the effective field \( \langle \sigma \rangle_i(x) \) (2.18) is a heterogeneous function of the coordinate \( x \in v_i \). In actual truss, a subsequent averaging of Eq. (2.18) over the inclusion \( v_i \) is tantamount to a secondary using of the hypothesis \( H_1 \) that is not necessary and can be avoided. However, such an inhomogeneity of \( \langle \sigma \rangle_i(x) \) (2.18) is beyond the scope of the current study and will be analyzed in other publications. It is correctly noted that application to Eq. (2.18) of the simplified hypothesis \( H_2a \) is tantamount to a secondary using of the hypothesis \( H_1 \) and reduces Eq. (2.18) to Eq. (4.18) with subsequent obtaining of the known representations for the effective properties (2.20) and (2.25). However, the eventual abandonment of the hypothesis \( H_1 \) can be done before the use of the hypotheses either \( H_2a \) or \( H_2b \) as it was performed in Eq. (4.2). Then the following solution of Eq. (4.2) by the use of the hypotheses either \( H_2a \), \( H_2b \) or \( H_3 \) does not lead to the necessity of using the hypothesis \( H_1 \) that will be quantitatively demonstrated in Section 6 at some numerical examples.

6. Numerical results

With the non-essential restriction on space dimensionality \( d \) and the shape of inhomogeneities we will consider 2-\( D \) problems for composites reinforced by cylindrical infinite fibers. The domains of inclusions \( v_i \) are discretized along the polar angle and the radius in the local polar coordinate system with the centers \( x_i \). Then the points

\[
\left\{ (r, \varphi) \mid (p-1) \frac{2\pi}{l} < \varphi < (q-1) \frac{a_i}{m} < r < (q+1) \frac{a_i}{m} \right\}
\]  

(6.1)

\( (p = 1, 2, \ldots, l; q = 1, 2, \ldots, m) \) represent the elements of \( \Gamma_i^{pp} \) of the meshes \( \Omega_i \) \( (i = 2, \ldots, n) \) that is not optimized, but is efficient. Moreover, the square meshes

\[
\left\{ (x_1, x_2) \mid (p-1) \frac{a_i}{I} < x_1 < (q+1) \frac{a_i}{I} \right\},
\]  

(6.2)

where \( x_1, x_2 \) are local coordinates with origins at the fiber centers, will be used for stress estimation inside and outside the fiber. We will use piecewise-constant elements of the meshes which are not very cost-efficient but are very easy for computer programming, and the discretization (6.2) permits the analysis of nonregular inclusion shapes. For simplicity estimation of integrals involved we will utilize the basic numerical integrations formulas of Simpson’s rule and trapezoidal rule for the uniform (6.1)-(6.2) and nonuniform grids considered below, respectively.

We detected that in the concrete examples of high matrix-inclusion elastic contrast considered and some others, the standard popular iterative schemes (3.14) may diverge or converge very slowly (i.e. the iteration scheme (3.12) does not work in general) so that an implementation of the improved algorithm proposed in this paper becomes more complicated. In such a case, following Refs. [37], [38], we introduced the subsidiary grid of the support points \( \zeta_j \) at the centers of each elements additionally to the nodal points \( s_j \) at the apexes of elements. After determining in this way at all the points \( \zeta_j \) the values of the function \( \eta^{[1]}(\zeta_j) \), we find its values at the nodal points \( \eta^{[1]}(s_j) \) by linear interpolation, and so on (see details in Ref. [38]). Moreover, instead of the point Jacobi iteration method displayed in Eq. (3.12) we use the accelerated Liebmann method (called also extrapolated Gauss-Seidel method) which is usually “faster” than the point Jacobi method, and has the computational advantage that it does not require the simultaneous storage of the two iterations \( \eta^{[k+1]} \) and \( \eta^{[k]} \) (see, e.g., [39]). The convergence of the scheme (3.14) is provided by their modification \( \eta^{[k+1]} = \frac{1}{2}[\eta^{[i]} + \eta^{[i]} + \mathcal{K}_i \eta^{[k]}] \) (see for details Ref. [38]).
It should be mentioned that in forthcoming numerical examples we will use only the iteration scheme described above. Comparative analysis of this scheme with other known iteration schemes is beyond the scope of the current paper. Moreover, although the convergence of this method was rigorously proved in Ref. [38] for the elastic problems of an arbitrary dimensions, we will demonstrate its effectiveness only for 2-D problems; the analysis of 3-D problems is beyond the scope of this paper.

We consider a pure mechanical problem ($\beta \equiv 0$) and assume the matrix is epoxy resin ($L^{(0)} = (3k^{(0)}, 2\mu^{(0)})$, $k^{(0)} = 3.83$ GPa and $\mu^{(0)} = 1.27$ GPa) which contains identical circular glass fibers ($L^{(1)} = (3k^{(1)}, 2\mu^{(1)})$, $k^{(1)} = 34.3$ GPa and $\mu^{(1)} = 31.3$ GPa). If the pair distribution function $g(x_i - x_m) \equiv \varphi(v_i, x_i, v_m, x_m)$ depends on $|x_m - x_i|$ it is called the radial distribution function (RDF). Two alternative RDFs of inclusion will be examined (see Refs. [40], [41])

$$g(x_i - x_q) \equiv \varphi(v_i, x_i, v_q, x_q)/n^q = H(r - 2a), \quad \text{(6.3)}$$

where $H$ denotes the Heaviside step function, $r \equiv |x_i - x_q|$ is the distance between the nonintersecting inclusions $v_i$ and $v_q$, and $c$ is the volume fraction of fibers of the radius $a$. The formula (6.4) takes into account a neighboring order in the distribution of the inclusions.

At first we will perform our evaluations for composites with homogeneous fibers described by the RDF (6.3) in the framework of the hypotheses H2b and H3. Influence of the effective field hypothesis H1 is considered by comparison of statistical averages of stresses in the fibers estimated by the classical approach $\langle \sigma \rangle_i^{\text{old}}(x) \equiv \text{const.}$ (4.15), (4.18) as well as by the proposed one $\langle \sigma \rangle_i^{\text{new}}(x)$ (4.22), (4.23). We considered a volume fraction of fibers $c = 0.65$ and evaluated the stress perturbations $L_\eta^q(\langle \eta \rangle_q)(x)$ (4.22) in the vicinity $\{x : \max_i |x - x_q| = 3a\}$ of the area $v_q$ rather than only a stress distributions in the inhomogeneity $x \in v_q$. Then $\langle \sigma \rangle_i^{\text{old}}$ and $\langle \sigma \rangle_i^{\text{new}}(x)$ differ from one another no more than 0.09% that coincides with a computational error realized in the method (4.22), (4.23) for two different meshes (6.2) with $l = 15$ and $l = 30$. Thus, we qualitatively proved that in the considered example both methods the old (4.15), (4.18) and new (4.22), (4.23) ones which are based on the classical (2.2) and new (4.1) general integral equations, respectively, lead to the same numerical results. This conclusion quantitatively confirms the Proposition 1) establishing an equivalentness of Eqs. (2.2) and (4.1) for statistically homogeneous fields of homogeneous ellipsoidal heterogeneities subjected to the homogeneous boundary conditions. Now we will consider an influence of incorrect using of the operator $L_\eta^q(\langle \eta \rangle_q)(x)$ (4.22) when only $x \in v_q$ are considered, which means that stress perturbations introduced by the moving inhomogeneity $v_q$ in their vicinity $\{x : a < |x - x_q| < 3a\}$ are neglected. For this purpose, the means of stress concentrator factors ($\beta \equiv 0$)

$$\langle \sigma \rangle_i^q(x) = B^*(x)\langle \sigma \rangle,$$

(6.5)

defined analogously to Eq. (4.25), will be estimated. In Fig. 2 the components $B_{2211}^a(x)$ demonstrating maximum dependence on $x = (x_1, 0)^T$ are presented for the initial $[B_{2211}^{a0}(x) = B_{2211}^{\text{old}} = \text{const.}]$, second $[B_{2211}^{a2}(x)]$, forth $[B_{2211}^{a4}(x)]$, and tenth $[B_{2211}^{a10}(x)]$ iterations of stress concentrator factor. A fast convergence of the proposed iteration method can be seen: the tenth iteration differs from the ninth, fourth, and initial approximations by 0.013%, 0.084%, and 2.9%, respectively. In so doing, the difference 2.9% essentially exceeds the possible errors of both the calculations and iteration scheme. Thus, for statistically homogeneous fields of homogeneous circle inclusions subjected to the homogeneous boundary conditions, the old and new approaches based on the backgrounds in the form of Eqs. (2.2) and (4.1), respectively, lead to equivalent results. Thus, in the case of the background (4.1) we must estimate the stress pertur-
bation in the vicinity \( \{|x - x_q| < 3a\} \) of the moving inhomogeneity \( v_q \). This statement contradicts to the proposition 1 obtained at the use of the old background (2.2).

We are expected to get a larger difference of the backgrounds (2.2) and (4.1) for composites reinforced by either nonellipsoidal or inhomogeneous inclusions demonstrating essentially inhomogeneous stress distribution inside inclusions even in the framework of the hypothesis \( \text{H1} \). The interphase is usually the product of processing conditions involved in composite manufacture. In relation to this problem, an application of the concept of functionally graded materials by Hirai et al. [42] for description of the interphase whose moduli may vary continuously is worthy of notice. Along this line one may, for instance, refer to the works [43-47] concerned with the spatially nonuniform properties of interphase. Just for concreteness, we assume that fibers contain the cores of the radius \( a < a_c \) with the constant moduli \( L^{(1)} \equiv \text{const} \) while the moduli \( L^{\text{int}}(x) \) in the interphase with the coating thickness \( h = a - a_c \) are taken to vary linearly with the radial distance \( r = |x| \):

\[
L^{\text{int}}(r) = L^{(0)} + (L^{(1)} - L^{(0)})(a - r)/h.
\]

For demonstration of maximum difference between the old and new approaches, we will consider in detail a thick coating with the relative coating thickness \( h/a_c = 0.5 \) although other ratios \( h/a_c \) will be also analyzed in a few comparative examples. At first, we will analyze results obtained in the framework the hypotheses \( \text{H2b} \) and \( \text{H3} \) for the RDF (6.3). In Fig. 3 the iterations \( B_{1111}^{[k]}(x) \) \( (k = 0, 2, 4, 12) \) at the axis \( x = (x_1, 0)^T \) are presented for \( c = \pi a^2 n = 0.65 \). The initial approximation \( B_{1111}^{[0]}(x) \) corresponding to the classical estimation (4.20) and using the old background (2.2) reveals their essential inhomogeneity (14%) even in the framework of the effective field hypothesis \( \text{H1} \). The new background (4.1) allow the use of this inhomogeneity for refinement of the renormalizing item in Eq. (4.14) without exploiting of the hypothesis \( \text{H1} \). The twelfth iteration \( B_{1111}^{[12]}(x) \) differs from the initial approximation \( B_{1111}^{[0]}(x) \) by 10.7% while the 12th and 11th iterations are distinguished from one another by 0.9%. Of even greater difference of results obtained for the backgrounds (2.2) and (4.1) is observed for the component \( B_{1122}^{[k]}(x) \) \( (k = 0, 2, 4, 12) \) at \( x = (x_1, 0)^T \) in Fig. 4. Indeed, \( B_{1122}^{[0]}(x) > 0 \) at any \( x = (x_1, 0)^T \) while \( B_{1122}^{[k]}(x) < 0 \) at \(-0.75a < x_1 < 0.75a \). Again, the proposed iteration method converges rapidly and we contend that 12th iteration provides a difference from 11th iteration of 0.2%.
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Stress concentrator factors

Normalized coordinates

Fig. 3: $B^{[k]}_{1111}(x_1)$ vs $x_1/a$ in inhomogeneous fiber: curves 1, 2, 3, 4 for $k = 0, 2, 4, 12$, respectively.

So much prominent and systematic differences of the old and new approaches are based on the abandonment from effective field hypothesis $H_1$ in the new approach. We estimated a tensor of effective stress concentrator factor

$$\langle \sigma \rangle(x) = \mathbf{B}^{[k]}(x) \langle \sigma \rangle, \quad (x = (x_1, 0)^\top)$$

and presented the components of their $k$-th approximations $\mathbf{B}^{[k]}_{1111}(x)$ and $\mathbf{B}^{[k]}_{1122}(x)$ ($k = 0, 2, 4, 12$) in Figs. 5 and 6, respectively. $\mathbf{B}^{[12]}_{1111}(x)$ differs from both the classical $\mathbf{B}^{[0]}_{1111}(x)$ (4.15), (6.7) and $\mathbf{B}^{[11]}_{1111}(x)$ on $8.1\%$ and $0.03\%$, respectively, while $\mathbf{B}^{[12]}_{1111}(x)$ varies along $x_1$ over $2.2\%$. However, we can observe in Fig. 6 a significantly more dramatic situation with the component $\mathbf{B}^{[k]}_{1122}(x)$ where all iterations differ by a sign from the classical one $\mathbf{B}^{[0]}_{1122}(x)$ (4.15), (6.7) almost at all values $|x_1| < a$.

We now turn our attention to the analysis of the size of the circle excluded volume $v^0_i$ with the radius $a^0$ on the stress concentrator factor $\mathbf{B}^{[k]}(x)$ also for the radial distribution function (6.3) reducing Eq. (4.22) to Eq. (4.24). We will compare the estimation of $\mathbf{B}^{[k]}(x)$ carried out for $a^0 = 3a$ with previously obtained results for $a^0 = 2a$ (see Figs. 3 and 4). The components $\mathbf{B}^{[12]}_{1111}(x)$ and $\mathbf{B}^{[k]}_{1122}(x)$ ($x = (x_1, 0)^\top$) are presented in Figs. 7 and 8, respectively, for $k = 0$ (curves 1) and $k = 12$ for both $a^0 = 2a$ (curves 2) and $a^0 = 3a$ (curves 3). The RDF (6.3) provides the “ellipsoidal symmetry” hypothesis $H_3$ (2.24) and, because of this, the classical representations for $\mathbf{B}^{[0]}(x)$ is invariant to the size of $v^0_i$ while $\mathbf{B}^{[12]}_{1111}(x)$ and $\mathbf{B}^{[12]}_{1122}(x)$ estimated by the new approach (4.24) for $a^0 = 2a$ and $a^0 = 3a$ differ at $x_1 = 0$ one from another by $3.1\%$ and $50\%$, respectively. Finally, we compare the influence of the RDF (6.3) and (6.4) at $a^0 = 2a$ on estimation of $\mathbf{B}^{[k]}(x)$. Needless to mention that $\mathbf{B}^{[0]}(x)$ (4.20) is invariant with respect to the RDF while $\mathbf{B}^{[12]}_{1111}(x)$ and $\mathbf{B}^{[12]}_{1122}(x)$ estimated for the RDF (6.3) (curve 2) and (6.4) (curves 4) are distinguished by $3.7\%$ and $33\%$, respectively. The indicated differences demonstrating fundamentally new effects inherent in the new approach (4.22)-(4.25) far exceed the iteration error between 11th and 12th iterations which are less than $0.03\%$.
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Effective stress concentrator factors

Fig. 5: Effective field concentrator factors $\overline{B}_{1111}^{[k]}(x_1)$ vs $x_1/a$ in inhomogeneous fiber: curves 1, 2, 3, 4 for $k = 0, 2, 4, 12$, respectively.

Effective stress concentrator factors

Fig. 6: Effective field concentrator factors $\overline{B}_{1122}^{[k]}(x_1)$ vs $x_1/a$ in inhomogeneous fiber: curves 1, 2, 3, 4 for $k = 0, 2, 4, 12$, respectively.

Effective stress concentrator factors

Fig. 7: $B_{1111}^{[k]}(x_1)$ vs $x_1/a$ for the different $a^0$ and RDF: curves 1 ($k = 0$), 2 [RDF (6.3), $a^0 = 2a$, $k = 12$], 3 [RDF (6.3), $a^0 = 3a$, $k = 12$], 4 [RDF (6.4), $a^0 = 2a$, $k = 12$].

Effective stress concentrator factors

Fig. 8: $B_{1122}^{[k]}(x_1)$ vs $x_1/a$ for the different $a^0$ and RDF: curves 1 ($k = 0$), 2 [RDF (6.3), $a^0 = 2a$, $k = 12$], 3 [RDF (6.3), $a^0 = 3a$, $k = 12$], 4 [RDF (6.4), $a^0 = 2a$, $k = 12$].

Just for completeness, we will estimate an influence of the interphase thickness $h$ (6.6) on the stress concentrator factor $B^{[0]}(x)$ and $B^{[12]}(x)$ ($x = (x_1, 0)^T$) for the RDF (6.3) with $a^0 = 2a$. In addition to Figs. 3 and 4 displaying the results for $h/a^c = 0.5$, we are demonstrating the similar estimations $B_{1122}^{[k]}(x)$ ($k = 0, 12$) for $h/a^c = 0.1, 0.25,$ and $h/a^c = 1$ in Fig. 9. As can be seen, $B_{1122}^{[0]}(a)$ and $B_{1122}^{[12]}(a)$ differ one from another by 20.5%, 30.3%, and 33.1% for $h/a^c = 0.1, 0.25,$ and $h/a^c = 1$, respectively. The similar
differences for the components $B_{1111}^*[0](a)$ and $B_{1111}^{[12]}(a)$ are 6.0%, 7.2%, and 8.5%, respectively.

We complete our numerical analysis by estimation of isotropic effective moduli $L^* = 2k^*_2 N_1 + 2\mu^*_2 N_2$ ($N_1 = \delta \otimes \delta/2$, $N_2 = I - N_1$). For the fiber composites it is the plane-strain bulk modulus $k^{(0)}_2$ (and $k^*_2$) – instead of the 3-D bulk modulus $k^{(0)}_3$ – that plays the significant role: $k^{(0)}_2 = k^{(0)}_3 + \mu^{(0)}_3 / 3$, $\mu^{(0)}_2 = \mu^{(0)}_3 \cdot \mu^*/\mu^{(0)}_2$ are presented in Fig. 10 for both the classical approach [corresponding to the stress concentrator factors $B^*[0](x)$] and new one [corresponding to the 12th iteration $B^*[12](x)$]. As can be seen, the distinctions between two approaches equal 3.8% and 12.0% for $c = 0.65$ for $\mu^*/\mu^{(0)}_2$ and $k^*/k^{(0)}_2$, respectively. In so doing, the stress concentrator factors in these approaches at the point $x_1 = a$ of fibers can differ on 30% and, moreover, these estimations for the different approaches can have the different signs at other same domains of $v_1$ (see Fig. 4). Thus, stress concentrator factors are significantly more sensitive values to the choice of the approach than effective elastic moduli.

7. Conclusion

We have proposed the new background of micromechanics based on the new general integral equation (4.1) which does not use the central concept of classical micromechanics such as effective field hypothesis H1. The eventual abandonment from hypothesis H1 has made a rejection of the satellite hypothesis H3 possible. If statistical averages of stresses in the heterogeneities can be considered as homogeneous ones then the new approach is degenerated into the classical approach (4.19)-(4.21). However such an assumption is approximately appropriate only for statistically homogeneous fields of homogeneous ellipsoidal inhomogeneities subjected to homogeneous boundary conditions and fulfilled at the conditions of quasi-crystallite approximation 3. If any of the indicated conditions is broken then an appearing inhomogeneity of stress fields in the inclusions lead to one of two possible sources of inhomogeneities of the effective field which, in turn, generates an additional inhomogeneity of stress fields inside inclusions and so on. For example, if all above-listed conditions are satisfied but the closing hypothesis H2b is replaced by H1, then the effective homogeneity of stress fields can be degenerated into the classical approach (4.19)-(4.21).
by the hypothesis H2a taking the binary interaction of inclusions into account then the first sort of inhomogeneity of the effective field $\mathbf{\sigma}(\mathbf{x})$ is generated by the binary interaction of inclusions [see the item $L^\gamma_q(\mathbf{\sigma}; \mathbf{v}_i, \mathbf{x}_i, \mathbf{y}_i)(\mathbf{x})$ in Eq. (4.6)] even if this interaction is approximately estimated through the matrix $Z$ as in Eq. (2.18). However, even in the framework of hypothesis H2b, the replacement of ellipsoidal homogeneous inclusions by either the nonellipsoidal homogeneous ones or inhomogeneous (e.g. coated) ellipsoidal inclusions with necessity leads to the second sort of effective field inhomogeneity produced by the fundamentally new renormalizing item $L^\gamma(\eta)(\mathbf{x})$ (4.1). This new renormalizing item is directly dependent [in opposite to the classical Eq. (2.2)] on inhomogeneity of stress fields inside the inclusions that has lead to detection of fundamentally new effects in micromechanics such as dependence of stress concentrator factors estimated (see Figs. 7 and 8) on both the RDF and size of the excluded volume even in the framework of hypotheses H2b and H3.

The modeling and simulation of random nano- and microstructures are becoming more and more ambitious due to the advances in modern computer software and hardware that is stimulated by a real challenge of modern material science and technology. The researches can forget about restrictions of analytical solutions (such as, e.g., Eshelby tensor and hypothesis H1) and use the numerical solutions which they need. It is expected to get all the more differences between the old and new approaches than inhomogeneity of the stress concentrator factors $B_i(\mathbf{x})$ ($\mathbf{x} \in \mathbf{v}_i$) would be larger. So, for the square inclusion with the smoothed vertexes and the finite cylindrical fiber the components $B_i(\mathbf{x})$ can vary by factors of four and ten (compare with Fig. 2), respectively (see Subsections 4.2.4 and 18.3.2, respectively, in Ref. [6]). Another source of stress inhomogeneity inside the inclusions is a continuous variation of their mechanical properties such as in either cylindrically or spherically anisotropic particles (see, e.g., [48]). However, probably the most often investigated reason of such a stress inhomogeneity is an imperfect interphase (including sliding, debonding, cohesive phenomena, see for references, e.g., [6]). These interphases may represent weak interfacial layer due to imperfect bonding between the two phases and inter-diffusion and/or chemical interaction zones (with properties varying through the thickness and/or along the surface) at the interphase between the two phases. The thickness of interphase investigated usually ranges from $h/a^c = 0.01$ for the conventional composites to $h/a^c = 2$ for nanocomposites. The significance of interphase effects becomes important in nanocomposites due to their high surface-to-volume ratios. An alternative approach taking into account interfacial effect is based on the concept of surface stress and surface tension (see, e.g., [49], [50]). To the author’s knowledge, in tens of publications dedicated to the influence of interphase on effective properties, the methods usually based on the hypothesis H1 (such as, e.g., the Mori-Tanaka scheme and MEF) are exploited. Now all these estimations can be improved in the framework of the new approach as we did it in Figs. 2-10 [compare the results obtained for the initial $B^{10}(\mathbf{x})$ and 12th $B^{12}(\mathbf{x})$ iterations].

Other possible directions of successful applications of the proposed approach are three classes of problems where inhomogeneities of stress distributions in the inclusions are generated by the nonlocal effects even for homogeneous ellipsoidal inclusions. The first two classes of these problems are described by both the special features of applied loading (statistically homogeneous media subjected to inhomogeneous boundary conditions) and the special features of microstructure (FGMs, clustered materials, bounded media, contact of microinhomogeneous media, macro-heterogeneity inside microinhomogeneous medium, see for details and references [6]). In both cases, the known methods are based on the general integral equation (2.2) for the statistically inhomogeneous media when $\langle \eta \rangle(\mathbf{y}) \neq \text{const.}$ (see for details and references [6]). However, Eq. (2.2) is just an approximation obtained from the exact Eq. (4.1) at the assumption (3.23l). Using of more general Eq. (4.1) instead of the approximative Eq. (2.2) opens up great opportunities for detection of new effects in nonlocal micromechanics. The mentioned problems imply an estimation of nonlocal effective properties for composites through their constituents exhibit local constitutive properties. A new inverse problem is initiated by investigation of nanocomposites and formulated
as an estimation of local effective properties through the nonlocal mechanical properties of constituents. This problem was solved by Buryachenko [6] (see Section 18.2) in the framework of hypothesis H1. However, it is well known in the context of micropolar elasticity that the strains are non-uniform even for the homogeneous elastic properties of the ellipsoidal inclusion subjected to the homogeneous remote loading. This sort of inhomogeneity is an encouragement for generalization of Eq. (4.1) to the composites which constituents are described by the nonlocal constituent laws. A subsequent step is the adoption of the new approach proposed in this paper for analysis of the generalized Eq. (4.1). However, more detailed consideration of nonlocal effects mentioned is beyond the scope of the current study and will be analyzed in other publications.

Acknowledgments:

This work was partially supported by the Visiting Professor Program of the University of Cagliari and the Eppley Foundation for Research.

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