Kac meets Johnson and Lindenstrauss: a memory-optimal, fast Johnson-Lindenstrauss transform

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Abstract

Based on the Kac random walk on the orthogonal group, we present a fast Johnson-Lindenstrauss transform: given a set $X$ of $n$ points in $\mathbb{R}^d$ and an error parameter $\epsilon$, this is a linear transformation $\Psi : \mathbb{R}^d \rightarrow \mathbb{R}^{O(\epsilon^{-2} \log n)}$ such that $\|\Psi x\|_2 \in (1 - \epsilon, 1 + \epsilon) : \|x\|_2$ for all $x \in X$, and such that for each $x \in X$, $\Psi x$ can be computed in time $O(d \log d + \min\{d \log n, \epsilon^{-2} \log n \log^3(n \log^3(\epsilon^{-1} \log n))\}$ with only a constant amount of memory overhead. In some parameter regimes, our algorithm is best known, and essentially confirms a conjecture of Ailon and Chazelle.

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1 Introduction

The celebrated lemma of Johnson and Lindenstrauss (henceforth abbreviated as the JL lemma) [6] guarantees that any set of $n$ points in Euclidean space $\mathbb{R}^d$ can be (linearly) embedded in $k = O(\epsilon^{-2}\log n)$ dimensions in a manner such that the norms of all points are preserved up to a multiplicative factor of $(1 \pm \epsilon)$. The value of $k$ in the JL lemma is known to be optimal (up to a constant) [9, 3].

The original proof of Johnson and Lindenstrauss achieved such an embedding by projecting the points onto a random $k$ dimensional subspace of $\mathbb{R}^d$. Such a projection amounts to multiplying each vector by a dense $k \times d$ matrix, and hence, takes time $\Omega(d\epsilon^{-2}\log n)$ for each vector. Since the JL lemma is a widely used primitive in numerous ‘high-dimensional problems’, obtaining optimal (asymptotically, in terms of the dimension of the range) embeddings which can be computed faster is an intensely studied problem of considerable interest (see, e.g., the references in [2, 8]).

One approach to obtaining such faster embeddings has been to select an embedding matrix randomly from a distribution over sparse matrices. This was first investigated by Achlioptas [1], who constructed a probability distribution over matrices which have at least two third of their entries equal to zero. Dasgupta, Kumar, and Sarlós [5] constructed a probability distribution over matrices where each column has at most $s = \tilde{O}(\epsilon^{-1}\log^3(n))$ non-zero entries (the $\tilde{O}$ hides factors which are polylogarithmic in $\epsilon$ and $\log n$) – in particular, this shows that a given vector $v$ can be embedded in time $O(s \cdot \text{NNZ}(v))$, where $\text{NNZ}(v)$ denotes the number of non-zero entries of $v$. The value of the sparsity parameter $s$ was improved by Kane and Nelson [8] to $s = O(\epsilon^{-1}\log n)$, which is optimal up to an overall multiplicative factor of $O(\log(1/\epsilon))$ [10].

For embedding general dense vectors i.e. those with $\Omega(d)$ non-zero entries, the scheme of Kane and Nelson runs in time $\Theta(d\epsilon^{-1}\log n)$, which improves on the ‘naive’ runtime of $O(d\epsilon^{-2}\log n)$, but can still be quite large. In an influential work (predating the developments summarized in the previous paragraph, except for [1]), Ailon and Chazelle [2] introduced a Fast Johnson-Lindenstrauss Transform (FJLT), which runs in time $O(d\log d + \min\{d\epsilon^{-2}\log n, \epsilon^{-2}\log^3 n\})$ (of course, by combining this with the algorithm of Kane and Nelson, one may improve the first term in the minimum to $d\epsilon^{-1}\log n$). They accomplish this via a linear embedding of the form $\Phi = \text{PHD}$, where $D$ is a (random) diagonal $\{-1, +1\}$-valued matrix, $H$ is the normalized Walsh-Hadamard matrix, and $P$ is a sparse $k \times d$ matrix whose non-zero entries are i.i.d. Gaussians. The key points in the analysis of the runtime of this algorithm are that multiplication with $H$ can be performed in time $O(d\log d)$ (via the FFT), whereas multiplication with $P$ can be performed in time $\text{NNZ}(P)$.

Ailon and Chazelle also proposed an alternative FJLT, which they conjecture to be “at least as good as the one described in this paper, yet much more elegant”, that is based on the so-called Kac random walk (introduced by Mark Kac in 1956 [7]) on the orthogonal group $\mathcal{O}(d)$. Kac’s random walk on $\mathcal{O}(d)$ evolves as follows: set $U_0 = \text{Id}$, and at time $t > 0$, choose two random coordinates $1 \leq i_t < j_t \leq d$ and a random angle $\theta_t \in [0, 2\pi)$, and set $U_{t+1} = R_{i_t,j_t,\theta_t}U_t$, where $R_{i,j,\theta}$ is a rotation of the $(i, j)$-plane by angle $\theta$. Note that even though $R_{i,j,\theta} \in \mathcal{O}(d)$ is a $d \times d$ matrix, it has only 4 non-zero entries. Consequently, for a fixed vector $x \in \mathbb{R}^d$, the computation of $R_{i,j,\theta}x$ is extremely efficient and can be done in constant time with a constant amount of memory overhead since it simply amounts to the transformation:

$$
\begin{pmatrix}
x_i \\
x_j
\end{pmatrix} \mapsto \begin{pmatrix}
x_i \cos \theta - x_j \sin \theta \\
x_i \sin \theta + x_j \cos \theta
\end{pmatrix}
$$
and all other coordinates of \( x \) are unchanged. In [2], the following version of FJLT based on Kac random walk was proposed: for \( x \in X \), where \( X \subset \mathbb{R}^d \) is a given set with \( |X| = n \), compute \( U_T x \) (iteratively), and return the projection onto the first \( O(\epsilon^{-2} \log n) \) coordinates. Ailon and Chazelle conjectured that \( T \) can be taken to be \( O \left( d \log d + \text{poly}(\log n, \epsilon^{-1}) \right) \) in order for \( U_T \) to preserve the norm of all \( x \in X \), up to a multiplicative factor of \((1 \pm \epsilon)\), with probability at least 2/3.

Although the Kac random walk has a long history and has been investigated by many researchers (see, e.g., the references in [11, 12]), there are long-standing fundamental open problems on the behavior of the walk. The conjectured optimal bound for total variation mixing time for Kac walk is \( \Theta(d^2 \log d) \), whereas the current best bound upper bound for mixing time in total variation norm is only \( O(d^4 \log d) \) by Pilipczuk and Smith [12]. In the context of dimensionality reduction, it was recently shown by Choromanski et al. [4] that for fixed vectors \( x, y \in \mathbb{R}^d \), the images \( \hat{x}, \hat{y} \) under \( O(d \log d) \) steps of the Kac walk, followed by a projection onto the first \( k \) coordinates, satisfy

\[
\mathbb{E}[(\langle \hat{x}, \hat{y} \rangle - \langle x, y \rangle)^2] = O(k^{-1}).
\]

This lends further evidence to the conjecture in [2], although without a sharp concentration estimate for the random variable \( \langle \hat{x}, \hat{y} \rangle \), such a bound cannot be used for the purpose of analyzing an FJLT.

Theorem 1.1, the main result of this paper, essentially confirms the conjecture of Ailon and Chazelle.

### 1.1 The Augmented Kac walk

Before describing our algorithm, we define an ‘augmented’ version of Kac’s walk. For \( s \in [0, T-1] \), sample \( 1 \leq i_s < j_s \leq d \) and \( \theta_s \in [0, 2\pi) \) uniformly a random, and define

\[
R_T := R_{i_{T-1}, j_{T-1}, \theta_{T-1}} \cdots R_{i_0, j_0, \theta_0}.
\]

The matrix \( R_T \) above is the usual Kac walk on \( \mathcal{O}(d) \) run for \( T \)-steps. For \( d \in \mathbb{N} \), let \( D_d \) denote a random diagonal matrix, whose diagonal entries are drawn independently from \( \{-1, 1\} \) with probability \( 1/2 \) each. For \( q \in (0, 1) \), let \( B_d(q) \) denote a random diagonal matrix, whose diagonal entries are independently \( 1 \) with probability \( q \) and \( 0 \) otherwise. We define the augmented Kac walk to be

\[
P(d, q, T) := \sqrt[T]{\frac{1}{q}} \cdot B_d(q) \cdot R_T \cdot D_d.
\]

Note that the Augmented Kac walk still retains the attractive feature of Kac’s walk of being implementable with a constant amount of memory.

Consider the sequences \( \{d_i\}_{i=0}, \{T_i\}_{i=0}, \{q_i\}_{i=0} \) given by

- \( d_0 := d, d_{i+1} := 2c_{4.1} \sqrt{d_i \log d_i \cdot \frac{\log n}{\epsilon}} \log \left( \frac{\log n}{\epsilon} \right) \) for \( i \geq 0 \);
- \( T_i := C d_i \log d_i \) for all \( i \geq 0 \);
- \( q_i := c_{4.1} \log n \log \left( \frac{\log n}{\epsilon} \right) \sqrt{\frac{\log d_i}{d_i}}, 1 \) for all \( i \geq 0 \),

where \( C \) is a constant chosen we can choose depending on the desired success probability, and \( c_{4.1} \) is a positive absolute constant. Let \( M \) be the smallest index \( i \) for which \( d_{i+1} \geq \frac{d}{2} \). Then, with sufficiently high probability, it follows (see Section 4.3 later) that an independent random collection

\[
\dim(\text{Range}(P(d_i, q_i, T_i))) \leq \dim(\text{Range}(B_{d_i}(q_i))) \leq d_{i+1}
\]
for all $0 \leq i \leq M - 1$. Therefore, by using an arbitrary injection from the coordinate vectors in $\text{Range}(B_{d_i}(q_i))$ to the first $d_{i+1}$ coordinate vectors, we can make sense of the composition

$$P := P(d_M, q_M, T_M) \circ \cdots \circ P(d_0, q_0, T_0).$$

### 1.2 Algorithm and Main Result

The following is our algorithm for implementing a fast JL transform. Our algorithm runs the usual Kac’s walk in the ‘big-data’ regime, and runs the augmented walk in the alternate regime where $\epsilon \sqrt{d}$ dominates $\log n$.

**Algorithm 1:** Fast JL via the Augmented Kac walk

```plaintext
if $\log n = \Omega(\epsilon \sqrt{d})$ then
    # The “big-data” regime where $\log n$ dominates $\epsilon \sqrt{d}$; run the usual Kac random walk for $O(d \log n)$ steps.
    Take $T \geq 20d \log n$, $K \geq 4C_{3.3} \log n/\epsilon^2$ and set
    $$\Psi := \sqrt{\frac{d}{K}} \cdot \text{Proj}_{[K]} R_T.$$ 
else
    # The regime where $\epsilon \sqrt{d}$ dominates $\log n$; run the augmented Kac walk
    Take $T \geq 20d_M \log n$, $K \geq 4C_{3.3} \log n/\epsilon^2$. Sample $R_T, P$ independently and set
    $$\Psi := \sqrt{\frac{d_M}{K}} \cdot \text{Proj}_{[K]} \cdot R_T \cdot P$$
end if
```

**Theorem 1.1.** There exist positive absolute constants $A_{1.1}, C_{1.1}, c_{1.1}$ for which the following holds. Let $d, n, T, \epsilon, C > 0$ satisfy $C \geq C_{1.1}$, $n \geq d \geq C_{1.1}$, $\epsilon \in (0, C_{1.1}^{-1})$, and $\epsilon^{-2} \log n \leq d$. Then, Algorithm 1 runs in time

$$T \leq A_{1.1} \left(Cd \log d + \min \left\{Cd \log n, C^2 \epsilon^{-2} \log^3 n \log^3 (\epsilon^{-1} \log n) \right\}\right),$$

and outputs a linear map $\Psi : \mathbb{R}^d \to \mathbb{R}^k$, where $k = k(n, \epsilon) \leq C_{1.1} \epsilon^{-2} \log n$, such that for any fixed set $X \subset \mathbb{R}^d$ of size $|X| = n$, the inequalities

$$(1 - \epsilon)\|x\|_2 \leq \|\Psi x\|_2 \leq (1 + \epsilon)\|x\|_2$$

hold simultaneously for all $x \in X$ with probability at least

$$1 - O \left(\log^{-1.1} C \left(\frac{\log n}{\epsilon}\right)\right).$$

Moreover, for each $x \in \mathbb{R}^d$, $\Psi x$ can be computed in time $O(T)$, and with only a constant amount of additional memory.

**Remark 1.2.**

- The condition $\epsilon^{-2} \log n \leq d$ does not lead to any loss of generality; indeed, if $\epsilon^{-2} \log n > d$, we may simply take $k = d$ and $\Psi$ to be the identity map. The condition $n \geq d$ also does not lead to any loss of generality, because we can always work in the subspace spanned by the points in $X$. 

3
• The first term in our min improves upon the \( \epsilon^{-1}d \log n \) runtime in [8]. To our knowledge, this is the fastest asymptotic runtime in the ‘big data’/‘high-accuracy’ regime of \( \log n > \epsilon \sqrt{d} \). Moreover, as the proof of Theorem 1.1 shows, this runtime is achieved by the standard Kac’s walk.

• The second term in our min exceeds that in [2] by a factor of \( \log^3(\epsilon^{-1} \log n) \). We believe that with more careful analysis, this can be improved.

• Our proof also shows that if one wishes to embed the points (approximately isometrically) into a slightly larger number of dimensions \( \epsilon^{-2} \cdot \log^2 n \cdot \log^3(\epsilon^{-1} \log n) \), then this may be done via a linear mapping \( \Psi' \) such that \( \Psi'x \) can be computed in time \( O(d \log d) \) with a constant amount of additional memory. In fact, the proof of Theorem 1.1 proceeds by first constructing such a \( \Psi' \).

1.3 Notation and organization

Since we will typically reserve subscripts to denote the time parameter of a stochastic process, we will denote the \( i \)th coordinate of a vector \( v \in \mathbb{R}^d \) by \( v[i] \). For instance, the \( d-1 \) dimensional unit sphere in \( \mathbb{R}^d \), which we denote by \( S^{d-1} \), is defined by

\[
S^{d-1} := \left\{ X \in \mathbb{R}^d : \sum_{i=1}^{d} X[i]^2 = 1 \right\}.
\]

We denote by \( \mu(S^{d-1}) \) the usual uniform probability measure on the sphere.

As is standard, we will use \( [d] \) to denote the discrete interval \( \{1, \ldots, d\} \). For a subset \( S \subseteq [d] \), we will use \( \text{Proj}_S \) to denote the orthogonal projection of \( \mathbb{R}^d \) onto the coordinate subspace corresponding to \( S \). By default, logarithms have base 2. We will also omit floors and ceilings when they make no essential difference. Finally, we will always assume that \( n \geq d \) and assume without loss of generality \( X = \{v_1, \ldots, v_n\} \subset S^{d-1} \).

The remainder of this paper is organized as follows. In Section 2.1, we collect some basic facts about sub-gaussian random variable; in Section 2.2, we collect some basic facts about random points on the high-dimensional sphere (whose standard proofs are deferred to Appendix A); in Section 3, we present a key estimate from the work of Pillai and Smith (who proved a sharp bound for the total variation mixing time of the Kac walk on the sphere) [11] regarding the one-step contraction behavior of the Kac walk, and in Section 3.1, we prove our main technical tool. In Section 4.1, we give a faster Johnson-Lindenstrauss transform for the ‘big data/high accuracy’ regime, in Section 4.2, we present a dimensionality reduction primitive using the Kac walk (which is very much inspired by the work of Ailon and Chazelle), and finally, in Section 4.3, we present and analyze the Augmented Kac walk, thereby completing the proof of Theorem 1.1. Appendix B reproduces the short and elementary proof of Lemma 3.2 from [11] for the reader’s convenience, thereby making the present paper essentially self-contained.

1.4 Acknowledgements

V.J. would like to thank Mehtaab Sawhney for helpful comments on the first draft of this paper. N.S.P. gratefully acknowledges the support from ONR.
2 Preliminaries

In this section, we will collect some tools and auxiliary results which will be used in the proof of our main result.

2.1 Concentration inequalities

A sub-gaussian random variable is one whose sub-gaussian norm, defined below, is finite.

**Definition 2.1** (See Definition 2.5.6 in [13]). The sub-gaussian norm of a random variable $X$, denoted by $\|X\|_{\psi^2}$, is defined by

$$\|X\|_{\psi^2} := \inf\{t > 0 : \mathbb{E}\exp(X^2/t^2) \leq 2\}.$$ 

The next lemma shows that tails of a sub-gaussian random variable are dominated by suitable gaussian tails (which also explains the terminology).

**Lemma 2.2** (See Proposition 2.5.2 in [13]). If $X$ is a sub-gaussian random variable, then for all $t \geq 0$,

$$\Pr(|X| \geq t) \leq 2 \exp\left(-\frac{t^2}{C_{2.2}\|X\|_{\psi^2}^2}\right).$$

We will also make extensive use of the following generalization of the classical concentration inequality of Hoeffding.

**Lemma 2.3** (General Hoeffding’s inequality, see Theorem 2.6.3 in [13]). Let $X[1], \ldots, X[N]$ be independent, mean zero, sub-gaussian random variables, and $a = (a[1], \ldots, a[N]) \in \mathbb{R}^N$. Then, for every $t \geq 0$, we have

$$\Pr\left(\left|\sum_{i=1}^N a[i]X[i]\right| \geq t\right) \leq 2 \exp\left(-\frac{c_{2.3}t^2}{K^2\|a\|_2^2}\right),$$

where $K = \max_i \|X[i]\|_{\psi^2}$, and $c_{2.3} > 0$ is an absolute constant.

2.2 Random vectors on the sphere

The following sharp concentration of the norm of a random sub-gaussian vector will be crucial for us.

**Lemma 2.4** (See Theorem 3.1.1 in [13]). Let $X[1], \ldots, X[d] \in \mathbb{R}^d$ be a random vector with independent, sub-gaussian coordinates $X[i]$, that satisfy $\mathbb{E}[X[i]^2] = 1$. Then,

$$\|\|X\|_2 - \sqrt{d}\|\|_{\psi^2} \leq C_{2.4}K^2,$$ 

where $K = \max_i \|X[i]\|_{\psi^2}$ and $C_{2.4}$ is an absolute constant.

Combining this with Lemma 2.2, we have:

**Corollary 2.5**. Let $X = (X[1], \ldots, X[d]) \in \mathbb{R}^d$ be a random vector with independent, sub-gaussian coordinates $X[i]$, that satisfy $\mathbb{E}[X[i]^2] = 1$. Then, for all $\epsilon \geq 0$,

$$\Pr\left(\|X\|_2 \notin [\sqrt{d} \cdot (1 - \epsilon, 1 + \epsilon)]\right) \leq 2 \exp\left(-\frac{\epsilon^2d}{C_{2.5}K^4}\right),$$ 

where $K = \max_i \|X[i]\|_{\psi^2}$, and $C_{2.5} > 0$ is an absolute constant.
We will also need the following two immediate corollaries, the first controlling the norm of the projection of a random vector, and the second controlling the largest coordinate of a random vector. These statements are well-known, although we were unable to locate a precise reference, and have in any case decided to include complete proofs in Appendix A for the reader’s convenience.

**Corollary 2.6.** Let \( X = (X[1], \ldots, X[d]) \) be uniformly distributed on \( S^{d-1} \). Fix \( S \subseteq [d] \), and let \( X[S] := \text{Proj}_S X \). Then, for all \( \epsilon \in (0, 1/2) \),

\[
\Pr \left( \|X[S]\|_2 \notin \sqrt{\frac{|S|}{d}} \cdot [1 - \epsilon, 1 + \epsilon] \right) \leq 4 \exp \left( -\frac{\epsilon^2 |S|}{C_{2.6}} \right),
\]

where \( C_{2.6} > 0 \) is an absolute constant.

**Corollary 2.7.** Let \( X = (X[1], \ldots, X[d]) \) be uniformly distributed on \( S^{d-1} \). Fix \( i \in [d] \). Then, for all \( \ell \in [0, \sqrt{d}] \)

\[
\Pr \left( \|X[i]\| \geq \ell/\sqrt{d} \right) \leq 4 \exp \left( -\ell^2/C_{2.7} \right).
\]

### 3 Coupling and Contraction estimates for the Kac walk on \( S^{d-1} \)

The Kac random walk may be viewed as a discrete-time Markov chain \( \{X_t\}_{t \geq 0} \) on \( S^{d-1} \) defined as follows: at every step \( t \), choose two coordinates \( 1 \leq i_t < j_t \leq d \) and an angle \( \theta_t \in [0, 2\pi) \) uniformly at random, and set

\[
\begin{align*}
X_{t+1}[i_t] &= \cos(\theta_t)X_t[i_t] - \sin(\theta_t)X_t[j_t] \\
X_{t+1}[j_t] &= \sin(\theta_t)X_t[i_t] + \cos(\theta_t)X_t[j_t] \\
X_{t+1}[k] &= X_t[k] \quad k \notin \{i, j\}.
\end{align*}
\]

Let \( F : [d] \times [d] \times [0, 2\pi) \times S^{d-1} \to S^{d-1} \) be the map associated with this representation, so that \( X_{t+1} = F(i_t, j_t, \theta_t, X_t) \). In this section, we describe a coupling of two copies of Kac walks \( X_t, Y_t \) so that the distance between them goes to zero exponentially quickly.

**Definition 3.1** (Proportional coupling, see Definition 3.1 in [11]). Define a coupling of two copies \( \{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0} \) of Kac’s walk as follows. Fix \( X_0, Y_0 \in S^{d-1} \). Let \((i_0, j_0, \theta_0)\) be the update variables used by \( X_1 \) in Equation (4). Choose \( \varphi \in [0, 2\pi) \) uniformly at random among all angles that satisfy

\[
\begin{align*}
X_1[i_0] &= \sqrt{X_0[i_0]^2 + X_0[j_0]^2} \cos \varphi, \\
X_1[j_0] &= \sqrt{X_0[i_0]^2 + X_0[j_0]^2} \sin \varphi.
\end{align*}
\]

As noted in [11], if \( X_0[i_0] = X_0[j_0] = 0 \), then all angles \( \varphi \) satisfy this equation; otherwise, there is a unique such \( \varphi \), and the value of \( \varphi - \theta_0 \mod 2\pi \) does not depend on \( \theta_0 \).

Then, choose \( \theta'_0 \in (0, 2\pi) \) uniformly among the angles that satisfy

\[
\begin{align*}
F(i_0, j_0, \theta'_0, Y_0)[i_0] &= \sqrt{Y_0[i_0]^2 + Y_0[j_0]^2} \cos \varphi, \\
F(i_0, j_0, \theta'_0, Y_0)[j_0] &= \sqrt{Y_0[i_0]^2 + Y_0[j_0]^2} \sin \varphi,
\end{align*}
\]

and set \( Y_1 = F(i_0, j_0, \theta'_0, Y_0) \). Note that this coupling forces \( Y_1 \) to be as close as possible to \( X_1 \) in the Euclidean distance (for instance, in two dimensions, we always have \( X_1 = Y_1 \) under this coupling, and in more than two dimensions, it still forces the points \((0, 0), (X_1[i_0], X_1[j_0]), (Y_1[i_0], Y_1[j_0])\) to be collinear).

Now, continue this process starting from \( (X_1, Y_1) \) instead of \( (X_0, Y_0) \).
The following key lemma shows that, under the coupling described above, the distance (interpreted suitably) between two copies of Kac’s walk decreases exponentially fast.

**Lemma 3.2** (See Lemma 3.3 in [11]). Fix $X_0, Y_0 \in \mathbb{S}^{d-1}$. For $t \geq 0$, couple $(X_{t+1}, Y_{t+1})$ conditional on $(X_t, Y_t)$ according to the coupling in Definition 3.1. Then, for any $t \geq 0$, Kac’s walk on $\mathbb{S}^{d-1}$ satisfies

$$
\mathbb{E}_{\{(i_s, j_s, \theta_s, \theta'_s)\}_{s=0}^{t-1}} \left[ \sum_{i=1}^{d} (X_t[i]^2 - Y_t[i]^2)^2 \right] \leq 2 \left( 1 - \frac{1}{2d} \right)^t \leq 2e^{-t/2d}.
$$

For the reader’s convenience, we include the complete (short) proof of this lemma in Appendix B.

### 3.1 Norm of coordinate projections of Kac’s walk

In this section, we present our key technical result, Proposition 3.3, which controls the norm of the projection, as well as the size of the largest coordinate, of the image of a (fixed) vector under Kac’s walk. The underlying idea behind the proof is the following: as Corollary 2.6 and Corollary 2.7 show, these conclusions hold for a uniformly random vector on the sphere with very high probability. If we run the Kac walk starting from a uniformly distributed point $Y_0 \in \mathbb{S}^{d-1}$, then $Y_t$ will also be uniformly distributed. On the other hand, Lemma 3.2 shows that $Y_t$ and $X_t$ get sufficiently close in the pseudometric used in Lemma 3.2 for statements about squares of the coordinates to be ‘transferred’ from $Y_t$ to $X_t$.

**Proposition 3.3.** Fix $X_0 \in \mathbb{S}^{d-1}$. Then, for any $\epsilon \in (1/d, 1/2)$, any $C \geq 2$, and $k \in [d]$, and any $t \geq 0$, Kac’s walk on $\mathbb{S}^{d-1}$ satisfies the following:

1. $\Pr \left( \sum_{i=1}^{k} X_t[i]^2 \notin \frac{k}{d} \cdot [1 - \epsilon, 1 + \epsilon] \right) \leq 2d^4 \exp(-t/2d) + 4(t + 1) \exp \left( -\frac{\epsilon k}{C3.3} \right),$

2. $\Pr \left( \max_i |X_t[i]| \geq C \sqrt{\log \frac{d}{t}} \right) \leq 2d^4 \exp(-t/2d) + 4(t + 1) \exp \left( -\frac{C^{2} \log d}{C3.3} \right),$

where $C3.3$ is an absolute constant.

**Proof.** From Lemma 3.2, we have for any $t \geq 0$ that

$$
\mathbb{E}_{Y_0 \sim \mu(\mathbb{S}^{d-1})} \mathbb{E}_{\{(i_s, j_s, \theta_s, \theta'_s)\}_{s=0}^{t-1}} \left[ \sum_{i=1}^{d} (X_t[i]^2 - Y_t[i]^2)^2 \right] \leq 2e^{-t/2d}.
$$

Therefore, by Markov’s inequality,

$$
\Pr_{Y_0 \sim \mu(\mathbb{S}^{d-1}), \{(i_s, j_s, \theta_s, \theta'_s)\}_{s=0}^{t-1}} \left( \sum_{i=1}^{d} (X_t[i]^2 - Y_t[i]^2)^2 \geq \frac{1}{d^2} \right) \leq 2d^4 e^{-t/2d}.
$$

In particular, letting $\mathcal{B}_1$ denote the event that

$$
\max_i |X_t[i]^2 - Y_t[i]^2| \geq \frac{1}{d^2},
$$

we see that

$$
\Pr_{Y_0 \sim \mu(\mathbb{S}^{d-1}), \{(i_s, j_s, \theta_s, \theta'_s)\}_{s=0}^{t-1}} (\mathcal{B}_1) \leq 2d^4 e^{-t/2d}.
$$
Next, let $B_2$ denote the event that
\[
\sqrt{\sum_{i=1}^{k} Y_s[i]^2} \notin \sqrt{\frac{k}{d}} \cdot [1 - \epsilon, 1 + \epsilon] \quad \text{for some } s \in [t],
\]
and let $B_3(C)$ denote the event (parameterized by $C \geq 1$) that
\[
\max_i |Y_s[i]| \geq C \sqrt{\frac{\log d}{d}} \quad \text{for some } s \in [t].
\]
Since $Y_0 \sim \mu(S^{d-1})$ implies that $Y_s \sim \mu(S^{d-1})$ for all $s \in [t]$, it follows from Corollary 2.6 (respectively Corollary 2.7) and the union bound that
\[
\Pr_{Y_0 \sim \mu(S^{d-1}), \{(i, j_s, \theta_s, \theta'_s)\}_{s=0}^{t-1}}(B_2) \leq 4(t + 1) \exp \left( -\frac{C^2k}{2.6} \right),
\]
\[
\Pr_{Y_0 \sim \mu(S^{d-1}), \{(i, j_s, \theta_s, \theta'_s)\}_{s=0}^{t-1}}(B_3(C)) \leq 4(t + 1) \exp \left( -\frac{C^2 \log d}{2.7} \right).
\]
By the definition of coupling, we have
\[
\Pr_{\{(i, j_s, \theta_s)\}_{s=0}^{t-1}} \left( \sum_{i=1}^{k} X_t[i]^2 \notin \frac{k}{d} \cdot [1 - 4\epsilon, 1 + 4\epsilon] \right) = \Pr_{Y_0 \sim \mu(S^{d-1}), \{(i, j_s, \theta_s, \theta'_s)\}_{s=0}^{t-1}} \left( \sum_{i=1}^{k} X_t[i]^2 \notin \frac{k}{d} \cdot [1 - 4\epsilon, 1 + 4\epsilon] \right)
\]
\[
\leq \Pr \left( \left\{ \sum_{i=1}^{k} X_t[i]^2 \notin \frac{k}{d} \cdot [1 - 4\epsilon, 1 + 4\epsilon] \right\} \cap B_1^c \cap B_2^c \right) + \Pr(B_1) + \Pr(B_2) = \Pr(B_1) + \Pr(B_2)
\]
\[
\leq 2d^4 \exp(-t/2d) + 4(t + 1) \exp \left( -\frac{C^2k}{2.6} \right).
\]
Here, the third line uses the fact that
\[
\Pr \left( \left\{ \sum_{i=1}^{k} X_t[i]^2 \notin \frac{k}{d} \cdot [1 - 4\epsilon, 1 + 4\epsilon] \right\} \cap B_1^c \cap B_2^c \right) = 0,
\]
since $B_1^c \cap B_2^c$ implies that
\[
\sum_{i=1}^{k} X_t[i]^2 \in \left[ \sum_{i=1}^{k} Y_t[i]^2 - \frac{k}{d^2}, \sum_{i=1}^{k} Y_t[i]^2 + \frac{k}{d^2} \right]
\]
\[
\in \left[ \frac{k}{d} \cdot (1 - \epsilon)^2 - \frac{k}{d^2}, \frac{k}{d} \cdot (1 + \epsilon)^2 + \frac{k}{d^2} \right]
\]
\[
\in \left[ \frac{k}{d} \cdot [1 - 3\epsilon, 1 + 4\epsilon] \right].
\]
Finally, rescaling $\epsilon$ proves assertion 1.

The proof of assertion 2 is similar. By the definition of coupling, we have
\[
\Pr_{\{(i, j_s, \theta_s)\}_{s=0}^{t-1}} \left( \max_i |X_t[i]| \geq \sqrt{2C} \sqrt{\frac{\log d}{d}} \right) = \Pr_{Y_0 \sim \mu(S^{d-1}), \{(i, j_s, \theta_s, \theta'_s)\}_{s=0}^{t-1}} \left( \max_i |X_t[i]| \geq \sqrt{2C} \sqrt{\frac{\log d}{d}} \right).
\]
\[ \Pr \left( \max_i |X_t[i]| \geq \sqrt{2C} \sqrt{\frac{\log d}{d}} \cap B_i^c \cap B_3(C)^c \right) + \Pr(B_1) + \Pr(B_3(C)) = \Pr(B_1) + \Pr(B_3(C)) \leq 2d^4 \exp(-t/2d) + 4(t + 1) \exp \left( -C^2 \log d / C_{2.7} \right). \]

Here, the third line uses the fact that

\[ \Pr \left( \max_i |X_t[i]| \geq \sqrt{2C} \sqrt{\frac{\log d}{d}} \cap B_i^c \cap B_3(C)^c \right) = 0, \]

since \( B_i^c \cap B_3(C)^c \) implies that

\[ \max_i |X_t[i]|^2 \leq \max_i |Y_t[i]|^2 + \frac{1}{d^2} \]
\[ \leq C^2 \cdot \frac{\log d}{d} + \frac{1}{d^2} \]
\[ \leq 2C^2 \cdot \frac{\log d}{d}. \]

Rescaling \( C \) completes the proof. \( \square \)

## 4 Proof of Theorem 1.1

With the preliminary results of the previous section in hand, we are ready to prove Theorem 1.1.

### 4.1 The ‘big data’ regime: \( \log n = \Omega(\epsilon \sqrt{d}) \)

In this regime, it follows from Proposition 3.3 and the union bound that simultaneously, for all \( \ell \in [n] \),

\[ \|\Psi v\ell\|_2^2 \in [1 - \epsilon, 1 + \epsilon], \]

except with probability at most

\[ O\left( \exp(-T/4d) + \exp(-\epsilon^2 K/2C_{3.3}) \right), \]

as desired.

### 4.2 The Augmented Kac walk regime: \( \log n = O(\epsilon \sqrt{d}) \)

We begin by analyzing our dimension reduction primitive \( P(d, q, T) \).

**Proposition 4.1.** Fix \( v_1, \ldots, v_n \in \mathbb{S}^{d-1} \), where \( n \geq d \). There exist positive absolute constants \( C_{4.1} \) and \( c_{4.1} \) such that for any \( C \geq C_{4.1} \) and \( T := Cd \log d, \quad q := \frac{C \log n}{c_{4.1}} \sqrt{\frac{\log d}{d}}, \) except with probability at most

\[ O \left( d^{-c_{4.1} C} + n^{-2} \right) \]

the following hold:

1. \( \|P(d, q, T)v\ell\|_2^2 \in [1 - \epsilon, 1 + \epsilon] \) for all \( \ell \in [n] \),
2. Trace$(B_d(q)) \leq 2dq$.

**Remark 4.2.** The main content of this proposition is the first conclusion; the second conclusion is an immediate consequence of the concentration of sums of i.i.d. Bernoulli random variables.

**Proof.** We begin by showing that, with very high probability,

$$
\|R_T D_d v_\ell\|_\infty \leq C\sqrt{\frac{\log n \log d}{d}}
$$

for all $\ell \in [n]$. To see this, note first that, by Proposition 3.3 and the union bound, except with probability at most

$$
O \left( d^5 \exp \left( -\frac{C \log d}{2} \right) + C d^2 \log d \exp \left( -\frac{C^2 \log d}{C^3} \right) \right), \quad (5)
$$

we have that

$$
\max_{i,j} |R_T(i,j)| \leq C\sqrt{\log d}. \quad (5)
$$

Restrict to any such realisation of $R_T$ (i.e. we consider the intersection with the event that the above inequality is satisfied for $R_T$). Then, for each $j \in [d]$,

$$(R_T D_d v_\ell)_j \sim R_T(j,1)r_1(v_\ell)_1 + \cdots + R_T(j,d)r_d(v_\ell)_d,$$

where $r_1, \ldots, r_d$ are independently $\pm 1$ with probability $1/2$ each. In particular, $\mathbb{E}[(R_T D_d v_\ell)_j] = 0$. Since each $r_i$ is sub-gaussian with sub-gaussian norm at most 2, it follows from Lemma 2.3 that

$$
\Pr \left( \left| (R_T D_d v_\ell)_j \right| \geq \frac{4C \sqrt{\log n \log d}}{C^{2.3}} \right) \leq 2 \exp \left( -4 \log n \right). \quad (6)
$$

Next, let us restrict to the event that for all $\ell \in [n]$

$$
\|R_T D_d v_\ell\|_\infty \leq \frac{4C \sqrt{\log n \log d}}{C^{2.3}}. \quad (6)
$$

By Equation (6) and the union bound, this event holds except with probability at most

$$
2nd \exp \left( -4 \log n \right) \leq 2n^{-2}. \quad (7)
$$

Let $w_\ell := R_T D_d v_\ell$. Since $R_T$ and $D_d$ are both orthogonal matrices, it follows that $w_\ell \in \mathbb{S}^{d-1}$. Now,

$$
\|B_d(q)w_\ell\|_2^2 \sim b_1(w_\ell)_1^2 + \cdots + b_d(w_\ell)_d^2,
$$

where $b_1, \ldots, b_d$ are independent random variables, which are 1 with probability $q$ and 0 otherwise. In particular,

$$
\mathbb{E} \left[ \|B_d(q)w_\ell\|_2^2 \right] = q,
$$

and since each $b_i - q$ has sub-gaussian norm at most 2, it follows from Lemma 2.3 that

$$
\Pr \left( \|B_d(q)w_\ell\|_2^2 \geq q \epsilon \right) \leq 2 \exp \left( -\frac{\epsilon^2 d}{64C^2 \log n \log d} \right), \quad (8)
$$
where we have used that
\[
\sum_{i=1}^{d}(w_i)^4 \leq \|w\|_\infty^2 \|w\|_2^2 \leq \frac{16C^2}{c_{2.3}} \cdot \frac{\log n \log d}{d}.
\]
Hence, by the union bound, the event in Equation (8) holds for all \(w_i\) simultaneously, except with probability at most
\[
2n \exp \left( -\frac{c_{2.3}^2 \epsilon^2 d}{64C^2 \log n \log d} \right) = 2n \exp \left( -\frac{c_{2.3} \log n}{c_{2.3}^2 16} \right).
\] (9)
Moreover, by Lemma 2.3
\[
\Pr \left( \text{Trace}(B_d(q)) \geq 2dq \right) \leq \Pr \left( \left| \sum_{i=1}^{d} (b_i - q) \right| \geq dq \right)
\leq 2 \exp \left( -\frac{c_{2.3} dq^2}{4} \right)
\leq 2 \exp \left( -\frac{c_{2.3} C^2 \log d}{4} \right).
\] (10)
Therefore, taking \(C_{4.1}\) to be sufficiently large and \(c_{4.1}\) to be sufficiently small (depending on \(c_{2.3}, C_{3.3}\)) and using Equations (5), (7), (9) and (10) finishes the proof.

4.3 Putting everything together: the Augmented Kac walk

Let \(\{d_i\}_{i=0}^\infty, \{T_i\}_{i=0}^\infty, \{q_i\}_{i=0}^\infty, \text{ and } M\) be as in Section 1.1. Then, it is readily seen that:
(a) \(\sum_{i=0}^{M} T_i \leq 2Cd \log d;\)
(b) \(d_M \leq 16 \frac{C^2}{c_{4.1}} \left( \frac{\log n}{\epsilon} \right)^2 \log^2 \left( \frac{\log n}{\epsilon} \right) \log d_M.\) In particular, for \(d\) sufficiently large,
\[
d_M \leq 40 \frac{C^2}{c_{4.1}} \left( \frac{\log n}{\epsilon} \right)^2 \log^3 \left( \frac{\log n}{\epsilon} \right);
\]
(c) \(d_{M-1} \geq 16C^2 \left( \frac{\log n}{\epsilon} \right)^2.\) In particular, for \(d\) sufficiently large, \(M \leq 4 \log \left( \frac{\log n}{\epsilon} \right).\)

Recall that Algorithm 1 samples \(P(d_0, q_0, T_0), \ldots, P(d_M, q_M, T_M)\) independently. By Proposition 4.1 (applied with \(\epsilon' = \epsilon \cdot \log^{-1}(\log n/\epsilon)\)) and the union bound (with \((c)\) controlling the size of the union), except with probability at most
\[
O \left( \log \left( \frac{\log n}{\epsilon} \right) d_M^{-c_{4.1}C} \right) = O \left( \left( \frac{\log n}{\epsilon} \right)^{-c_{4.1}C/2} \right),
\] (11)
\(P(d_i, q_i, T_i)\) satisfies the conclusions of Proposition 4.1 for all \(0 \leq i \leq M.\) For any such realization of \(\{P(d_i, q_i, T_i)\}_{i=0}^{M}\), it follows from the second conclusion of Proposition 4.1 that \(\dim(\text{Range}(P(d_i, q_i, T_i))) \leq \dim(\text{Range}(B_d(q_i))) \leq d_{i+1}\) for all \(0 \leq i \leq M - 1.\) Therefore, by using an arbitrary injection from the coordinate vectors in \(\text{Range}(B_d(q_i))\) to the first \(d_{i+1}\) coordinate vectors, we can make sense of the composition
\[
P := P(d_M, q_M, T_M) \circ \cdots \circ P(d_0, q_0, T_0).
\]
Note that
(i) \( \dim(\text{Range}(P)) \leq d_M \leq 40 \cdot \frac{C^2}{4.1} \left( \frac{\log n}{\epsilon} \right)^2 \log^3 \left( \frac{\log n}{\epsilon} \right) \) by (b).

(ii) For any vector \( v \in S^{d-1} \), it takes time \( O(\sum_{i=0}^{M} T_i) = O(Cd \log d) \) to compute \( Pv \) by (a).

(iii) By using Proposition 4.1 for each \( P(d_i, q_i, T_i) \) with error parameter \( \epsilon' := \epsilon \cdot \log^{-1} \left( \frac{\log n}{\epsilon} \right) \), it follows that for any \( \ell \in [n] \) and all sufficiently small \( \epsilon \),

\[
\|Pv_\ell\|_2^2 \in [(1 - \epsilon')^M, (1 + \epsilon')^M] \in [1 - 5\epsilon, 1 + 5\epsilon].
\]

Recall that

\[
\Psi := \sqrt{\frac{d_M}{K}} \cdot \text{Proj}_{[K]} \cdot R_T \cdot P,
\]

with \( K \geq 4C_{3.3} \log n / \epsilon^2 \) and \( T \geq 20d_M \log n \). Then, as before, except with probability at most

\[
O \left( \exp \left( -T/4d_M \right) + \exp \left( -\epsilon^2 K / 2C_{3.3} \right) \right),
\]

we have that for all \( \ell \in [n] \),

\[
\|\Psi v_\ell\|_2^2 \in [1 - \epsilon, 1 + \epsilon] \cdot \|Pv_\ell\|_2^2 \in [1 - \epsilon, 1 + \epsilon] \cdot [1 - 5\epsilon, 1 + 5\epsilon].
\]

Finally, rescaling \( \epsilon \), and using the probability estimates Equation (11) and Equation (12) gives the desired conclusion of Theorem 1.1.

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A Proofs of Corollary 2.6 and Corollary 2.7

Proof of Corollary 2.6. Note that $X \sim Y/\|Y\|_2$, where $Y = (Y[1], \ldots, Y[d]) \in \mathbb{R}^d$ is a random vector with independent, standard Gaussian coordinates $Y[i]$. In particular, $\|X[S]\|_2 \sim \|Y[S]\|_2/\|Y\|_2$.

From Corollary 2.5, we have

$$\Pr \left( \|Y\|_2 \notin \sqrt{d} \cdot [1 - \epsilon, 1 + \epsilon] \right) \leq 2 \exp \left( -\frac{\epsilon^2 d}{C_{2.5} \|Y[1]\|_p^4} \right),$$

and

$$\Pr \left( \|Y[S]\|_2 \notin \sqrt{|S|} \cdot [1 - \epsilon, 1 + \epsilon] \right) \leq 2 \exp \left( -\frac{\epsilon^2 |S|}{C_{2.5} \|Y[1]\|_p^4} \right).$$

Hence, by the union bound, it follows that except with probability at most $4 \exp \left( -\frac{\epsilon^2 |S|}{C_{2.5} \|Y[1]\|_p^4} \right)$,

$$\|X[S]\|_2 \in \sqrt{\frac{|S|}{d} \cdot \left[ 1 - \epsilon, 1 + \epsilon \right]} \in \sqrt{\frac{|S|}{d} \cdot [1 - 2\epsilon, 1 + 5\epsilon]}.$$

Finally, rescaling $\epsilon$ completes the proof.

Proof of Corollary 2.7. As in the proof of Corollary 2.6, $X \sim Y/\|Y\|_2$ and

$$\Pr \left( \|Y\|_2 \in \sqrt{d} \cdot [1/2, 3/2] \right) \leq 2 \exp \left( -\frac{d}{4C_{2.5} \|Y[1]\|_p^4} \right).$$

Moreover,

$$\Pr (|Y[i]| \geq \ell) \leq 2 \exp(-\ell^2/2).$$

Hence, by the union bound, it follows that except with probability at most

$$4 \exp \left( -\frac{\ell^2}{\max \{4C_{2.5} \|Y[1]\|_p^4, 2\}} \right),$$

we have $|X[i]| \leq 2\ell/\sqrt{d}$. Finally, rescaling $t$ completes the proof.
Lemma 3.2

Let $A_t[i] = X_i[i]^2$ and $B_t[i] = Y_i[i]^2$ for all $t \geq 0$ and $i \in [d]$. We calculate,

$$\mathbb{E}\left[\sum_{k=1}^{d} (A_1[k] - B_1[k])^2\right] = \frac{2}{d(d-1)} \sum_{1 \leq i < j \leq d} \mathbb{E}\left[\sum_{k=1}^{d} (A_1[k] - B_1[k])(i_0, j_0) = (i, j)\right]$$

$$= \frac{2}{d(d-1)} \frac{(d-1)(d-2)}{2} \sum_{k=1}^{d} (A_0[k] - B_0[k])^2$$

$$+ \frac{2}{d(d-1)} \sum_{i<j} \mathbb{E}\left[((A_0[i] + A_0[j]) \cos^2 \varphi - (B_0[i] + B_0[j]) \cos^2 \varphi)^2\right]$$

$$+ \frac{2}{d(d-1)} \sum_{i<j} \mathbb{E}\left[((A_0[i] + A_0[j]) \sin^2 \varphi - (B_0[i] + B_0[j]) \sin^2 \varphi)^2\right]$$

$$= \frac{d-2}{d} \sum_{k=1}^{d} (A_0[k] - B_0[k])^2$$

$$+ \frac{4}{d(d-1)} \mathbb{E}[\cos^4 \varphi] \sum_{i<j} ((A_0[i] + A_0[j]) - (B_0[i] + B_0[j])^2$$

$$= \left(1 - \frac{2}{d}\right) \sum_{k=1}^{d} (A_0[k] - B_0[k])^2 + \frac{3}{2d(d-1)} \sum_{i<j} ((A_0[i] + A_0[j]) - (B_0[i] + B_0[j])^2$$

$$= \left(1 - \frac{2}{d}\right) \sum_{k=1}^{d} (A_0[k] - B_0[k])^2 + \frac{3}{2d(d-1)} \sum_{i<j} ((A_0[i] - B_0[i])^2 + (A_0[j] - B_0[j])^2)$$

$$+ \frac{3}{d(d-1)} \sum_{i<j} (A_0[i] - B_0[i])(A_0[j] - B_0[j])$$

$$= \left(1 - \frac{2}{d}\right) \sum_{k=1}^{d} (A_0[k] - B_0[k])^2 + \frac{3}{2d} \sum_{k=1}^{d} (A_0[k] - B_0[k])^2$$

$$+ \frac{3}{d(d-1)} \sum_{i<j} (A_0[i] - B_0[i])(A_0[j] - B_0[j])$$

$$= \left(1 - \frac{1}{2d}\right) \sum_{k=1}^{d} (A_0[k] - B_0[k])^2$$

$$+ \frac{3}{2d(d-1)} \left(\sum_{k=1}^{d} (A_0[k] - B_0[k])\right)^2 - \sum_{k=1}^{d} (A_0[k] - B_0[k])^2$$

$$= \left(1 - \frac{1}{2d} - \frac{3}{2d(d-1)}\right) \sum_{k=1}^{d} (A_0[k] - B_0[k])^2,$$

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where the last equality uses \( \sum_{k=1}^{d} A_0[k] = 1 = \sum_{k=1}^{d} B_0[k] \). Thus, we have

\[
\mathbb{E} \left[ \sum_{k=1}^{d} (A_1[k] - B_1[k])^2 \right] \leq \left( 1 - \frac{1}{2d} \right).
\]

For \( t \geq 0 \), let \( \mathcal{F}_t \) denote the \( \sigma \)-algebra generated by the random variables \( X_0, \ldots, X_t \) and \( Y_0, \ldots, Y_t \). Repeatedly applying the previous inequality, we have for all \( t \geq 0 \) that

\[
\mathbb{E} \left[ \sum_{k=1}^{d} (A_t[k] - B_t[k])^2 \right] = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{k=1}^{d} (A_t[k] - B_t[k])^2 \mid \mathcal{F}_{t-1} \right] \right] \\
\leq \left( 1 - \frac{1}{2d} \right) \mathbb{E} \left[ \sum_{k=1}^{d} (A_{t-1}[k] - B_{t-1}[k])^2 \right] \\
\leq \left( 1 - \frac{1}{2d} \right)^t \sum_{k=1}^{d} \mathbb{E} \left[ (A_0[k] - B_0[k])^2 \right] \\
\leq 2 \left( 1 - \frac{1}{2d} \right)^t
\]

and the proof is finished.