TECHNIQUES FOR CLASSIFYING NONNEGATIVELY CURVED LEFT-INARIANT METRICS ON COMPACT LIE GROUPS

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Abstract. We provide techniques for studying the nonnegatively curved left-invariant metrics on a compact Lie group. For “straight” paths of left-invariant metrics starting at bi-invariant metrics and ending at nonnegatively curved metrics, we deduce a nonnegativity property of the initial derivative of curvature. We apply this result to obtain a partial classification of the nonnegatively curved left-invariant metrics on SO(4).

1. Introduction

What are the nonnegatively curved left-invariant metrics on a compact Lie group? All such metrics on SO(3) and U(2) were classified in [1]. These classifications make use of various techniques that only work in low dimensions. For higher-dimensional groups, the situation becomes complicated, and it is evident that more tools are necessary to approach the problem effectively.

In [6], Tapp proposed to classify the nonnegatively curved left-invariant metrics on SO(4) by using infinitesimal techniques. He considered a path of left-invariant metrics starting at a bi-invariant metric, and aimed to classify the possibilities for the initial derivative such that the path looks nonnegatively curved near the bi-invariant metric. In this article, we provide a method of passing from this local derivative information to global information about the entire family of left-invariant metrics. As a consequence, we are able to transform Tapp’s restrictions on the derivative of the path into necessary conditions for a left-invariant metric to be nonnegatively curved.

Let \( G \) be a compact Lie group, with bi-invariant metric \( h_0 \) and Lie algebra \( \mathfrak{g} \). Any left-invariant metric \( h \) on \( G \) is determined entirely by its restriction to \( \mathfrak{g} \), and there is always some \( h_0 \)-self-adjoint positive definite \( \Phi: \mathfrak{g} \to \mathfrak{g} \) such that
\[
h(X, Y) = h_0(\Phi X, Y)
\]
holds for \( X, Y \in \mathfrak{g} \). In this manner a smoothly varying family \( h_t \) of left-invariant metrics on \( G \) can be identified with a smoothly varying family \( \Phi_t \) of endomorphisms of \( \mathfrak{g} \). In this article we are particularly interested in inverse-linear variations of the bi-invariant metric \( h_0 \). We say that \( \Phi_t \) is inverse-linear if it is of the form \( \Phi_t = (I - t\Psi)^{-1} \) for some \( \Psi: \mathfrak{g} \to \mathfrak{g} \); of course \( \Psi \) is then self-adjoint. Notice that \( \Psi \) is the derivative of \( \Phi_t \) at \( t = 0 \).

When \( \Phi_t \) is an inverse-linear variation and \( X, Y \in \mathfrak{g} \) have been fixed, we define \( \kappa_t^\Psi \) to be the unnormalized sectional curvature of \( \Phi_t^{-1} X \) and \( \Phi_t^{-1} Y \) with respect
to the metric $h_t$. We omit the superscript $\Psi$ when no confusion will arise. For fixed $t_0$, $h_{t_0}$ is nonnegatively curved if and only if $\kappa(t_0) \geq 0$ for all $X, Y \in \mathfrak{g}$.

**Definition 1.1.** The inverse-linear path $\Phi_t$ (or the endomorphism $\Psi$) is *infinitesimally nonnegative* if for any $X, Y \in \mathfrak{g}$ there is some $\varepsilon > 0$ such that the function $\kappa$ associated to $X$ and $Y$ satisfies $\kappa(t) \geq 0$ when $t \in [0, \varepsilon)$; if one choice of $\varepsilon$ holds for all pairs $X, Y$, then $\Phi_t$ is *locally nonnegative*.

The notion of an infinitesimally nonnegative endomorphism first arose in [6], where Tapp considers the infinitesimally nonnegative endomorphisms of $\mathfrak{so}(4)$ in an effort to classify the nonnegatively curved metrics on SO(4).

Given a left-invariant metric $h$ on $G$ with matrix $\Phi$, there is a unique inverse-linear path $\Phi_t$ with $\Phi_1 = \Phi$. Most of the inverse-linear paths we are interested in are constructed in this manner, by specifying the left-invariant metric at time $t = 1$. Such a path is called *nonnegative* if the metric $h_t$ is nonnegatively curved for $0 \leq t \leq 1$. The following conjecture of Tapp provides substantial motivation for considering inverse-linear variations.

**Conjecture 1.2 ([6]).** Let $h$ be a nonnegatively curved left-invariant metric on $G$. The unique inverse-linear path from $h_0$ to $h$ is nonnegative.

This conjecture is particularly interesting since it is not even known if the space of nonnegatively curved left-invariant metrics on $G$ is path-connected. The conjecture is known to be true when $G$ is one of SO(3) or U(2) (this follows from the complete classification of nonnegatively curved left-invariant metrics on these spaces in [1]), and it is true for all known left-invariant metrics on SO(4) ([6]). While a proof of the conjecture eludes us, we do have the following theorem.

**Theorem 1.3.** Let $h$ be a nonnegatively curved left-invariant metric on $G$. The unique inverse-linear path from $h_0$ to $h$ is infinitesimally nonnegative.

The techniques we develop to prove this theorem could potentially be useful in a proof of Tapp’s conjecture. This theorem is also interesting in its own right, since it can be used in a classification of the left-invariant metrics of nonnegative curvature on $G$. In light of this theorem, one can first classify the infinitesimally nonnegative endomorphisms $\Psi$, and then determine which endomorphisms $(I - \Psi)^{-1}$ correspond to nonnegatively curved metrics. Classifying the infinitesimally nonnegative endomorphisms of $\mathfrak{g}$ seems to be an easier question than classifying the nonnegatively curved left-invariant metrics on $G$.

We will calculate the Taylor series of $\kappa(t)$ at 0 to deduce a formula for $\kappa(t)$ in Section 2 and this is the essential ingredient to the proof of Theorem 1.3. We prove our main theorem in Section 3. In Section 4 we use the Taylor series for $\kappa(t)$ to classify which subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ can be expanded while maintaining nonnegative curvature. In Section 5 we apply our main theorem to Tapp’s partial classification of the infinitesimally nonnegative endomorphisms of $\mathfrak{so}(4)$ to obtain a partial classification of the nonnegatively curved left-invariant metrics on SO(4). We conclude the paper in Section 6 by examining what happens when the bi-invariant metric on $G$ is changed.

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2. A Formula for $\kappa(t)$

Throughout this section, we let $\Phi_t = (I - t\Psi)^{-1}$ be a fixed inverse-linear variation from the bi-invariant metric $h_0$ on $G$. We also fix two vectors $X, Y \in \mathfrak{g}$. As described in the introduction, we obtain a function $\kappa(t)$ determined by $\Phi_t$, $X$, and $Y$. The domain of $\kappa$ is the set of all $t$ such that $\Phi_t$ corresponds to a metric on $G$; this set is always an open interval of $\mathbb{R}$, determined by the set of eigenvalues of $\Psi$. Certain elements of $\mathfrak{g}$ will appear frequently in what follows, so to simplify the exposition we introduce the Lie algebra elements

$$A^\Psi = [\Psi X, Y] + [X, \Psi Y],$$
$$B^\Psi = [\Psi X, \Psi Y],$$
$$C^\Psi = [\Psi X, Y] + [\Psi Y, X],$$
$$D^\Psi = \Psi^2[X, Y] - \Psi A^\Psi + B^\Psi.$$

As with $\kappa^\Psi(t)$, we omit the superscripts when no confusion will arise. Our main tool throughout the article is an explicit formula for $\kappa(t)$. If $Z_1, Z_2 \in \mathfrak{g}$, we put

$$\langle Z_1, Z_2 \rangle = h_0(Z_1, Z_2), |Z_1|^2 = h_0(Z_1, Z_1), \text{ and } |Z_1|_{h_t}^2 = h_t(Z_1, Z_1) = \langle \Phi_t Z_1, Z_1 \rangle.$$

**Theorem 2.1.** For any $t$ in the domain of $\kappa$,

$$\kappa(t) = \alpha + \beta t + \gamma t^2 + \delta t^3 - \frac{3}{4} t^4 : |D|_{h_t}^2,$$

where

$$\alpha = \frac{1}{4} |[X, Y]|^2,$$
$$\beta = -\frac{3}{4} \langle \Psi [X, Y], [X, Y] \rangle,$$
$$\gamma = -\frac{3}{4} |[X, Y]|^2 + \frac{3}{2} \langle \Psi [X, Y], A \rangle - \frac{1}{2} \langle [X, Y], B \rangle - \frac{1}{4} |A|^2 + \frac{1}{4} |C|^2 - \langle [X, X], [\Psi Y, Y] \rangle,$$
$$\delta = -\frac{3}{4} \langle \Psi^3 [X, Y], [X, Y] \rangle + \frac{3}{2} \langle \Psi^2 [X, Y], A \rangle - \frac{3}{2} \langle \Psi [X, Y], B \rangle - \frac{3}{4} \langle \Psi A, A \rangle - \frac{1}{4} \langle \Psi C, C \rangle + \langle \Psi [\Psi X, X], [\Psi Y, Y] \rangle + \langle A, B \rangle.$$
This formula is valid whenever $|t| < \|\Psi\|^{-1}$, where $\|\Psi\| = \sup_{|X|=1} |\Psi X|$ is the operator norm of $\Psi$.

Proof. In [5], Püttmann shows that the unnormalized sectional curvature of vectors $Z_1, Z_2 \in g$ with respect to a left-invariant metric $h$ whose matrix with respect to $h_0$ is $\Phi$ is given by

$$k_h(Z_1, Z_2) = \frac{1}{2} \langle [\Phi Z_1, Z_2] + [Z_1, \Phi Z_2], [Z_1, Z_2] \rangle - \frac{3}{4} ||Z_1, Z_2||_h^2$$

$$+ \frac{1}{4} \langle [Z_1, \Phi Z_2] + [Z_2, \Phi Z_1], \Phi^{-1}([Z_1, \Phi Z_2] + [Z_2, \Phi Z_1]) \rangle$$

$$- \langle [Z_1, \Phi Z_1], \Phi^{-1}[Z_2, \Phi Z_2] \rangle.$$

(2.2)

It follows that

$$\kappa(t) = k_{h_t}(\Phi^{-1}_tX, \Phi^{-1}_tY)$$

$$= \frac{1}{2} \langle [X, \Phi^{-1}_tY] + [\Phi^{-1}_tX, Y], [\Phi^{-1}_tX, \Phi^{-1}_tY] \rangle$$

$$- \frac{3}{4} \langle \Phi_t[\Phi^{-1}_tX, \Phi^{-1}_tY], [\Phi^{-1}_tX, \Phi^{-1}_tY] \rangle$$

$$+ \frac{1}{4} \langle [\Phi^{-1}_tX, Y] + [\Phi^{-1}_tY, X], \Phi_t^{-1}([\Phi^{-1}_tX, Y] + [\Phi^{-1}_tY, X]) \rangle$$

$$- \langle [\Phi^{-1}_tX, X], \Phi_t^{-1}[\Phi^{-1}_tY, Y] \rangle$$

$$= I_1 - I_2 + I_3 - I_4.$$

Using the expression $\Phi_t^{-1} = I - t\Psi$, we can easily simplify $I_1, I_3,$ and $I_4$. We find

$$I_1 = ||[X, Y]|^2 - \frac{3t}{2} \langle [X, Y], A \rangle + t^2\langle [X, Y], B \rangle + \frac{1}{2} |A|^2 \rangle - \frac{t^3}{2} \langle A, B \rangle$$

$$I_3 = \frac{t^2}{4} |C|^2 - \frac{t^3}{4} \langle C, \Psi C \rangle$$

$$I_4 = t^3 \langle [\Psi X, X], [\Psi Y, Y] \rangle - t^4 \langle [\Psi X, X], [\Psi Y, Y] \rangle.$$

To calculate $I_2$, notice that if $|t| < \|\Psi\|^{-1}$, then

$$\Phi_t = \sum_{n=0}^{\infty} t^n \Psi^n,$$

where the convergence is in the space of endomorphisms of $g$ with the operator norm. Using this expression we calculate

$$\frac{4}{3} I_2 = \langle \Phi_t([X, Y] - tA + t^2 B), [X, Y] - tA + t^2 B \rangle$$

$$= \sum_{n=0}^{\infty} t^n \langle \Psi^n [X, Y] - t\Psi^n A + t^2 \Psi^n B, [X, Y] - tA + t^2 B \rangle$$

$$= \sum_{n=0}^{\infty} t^n \langle [\Psi^n [X, Y], [X, Y]] - 2t[\Psi^n [X, Y], A]$$

$$+ t^2([\Psi^n A, A] + 2[\Psi^n [X, Y], B]) - 2t^3 [\Psi^n A, B] + t^4 [\Psi^n B, B] \rangle$$
\[
||[X, Y]|^2 + t(\langle \Psi [X, Y], [X, Y] \rangle - 2\langle [X, Y], A \rangle) \\
+ t^2(\langle \Psi^2 [X, Y], [X, Y] \rangle - 2\langle \Psi [X, Y], A \rangle + |A|^2 + 2\langle [X, Y], B \rangle) \\
+ t^3(\langle \Psi^3 [X, Y], [X, Y] \rangle - 2\langle \Psi^2 [X, Y], A \rangle + \langle \Psi A, A \rangle \\
+ 2\langle [X, Y], B \rangle - 2\langle A, B \rangle) \\
+ \sum_{n=4}^{\infty} t^n \langle \Psi^{n-4} D, D \rangle.
\]

Combining the different terms proves the result. \[\square\]

Notice that the power series of \(\kappa(t)\) would have been much messier if we were considering the unnormalized sectional curvature of \(X\) and \(Y\) with respect to \(h_t\) instead of the unnormalized sectional curvature of \(\Phi^{-1}X\) and \(\Phi^{-1}Y\).

When \(|t| < ||\Psi||^{-1}\), observe that
\[
\Gamma(t) = -\frac{3}{4} t^3 \langle \Phi D, D \rangle = -\frac{3}{4} t^4 \langle \Phi_0 D, D \rangle = -\frac{3}{4} t^4 |D|_{h_t}^2.
\]

This proves that Equation 2.1 holds for small \(t\). Therefore to complete the proof of Theorem 2.1, all we must do is prove that \(\kappa(t)\) and \(|D|_{h_t}^2\) are analytic.

**Lemma 2.3.** The function \(\kappa(t)\) is analytic on its domain of definition.

**Proof.** Assume that \(t_0\) is such that \(\Phi_{t_0}\) corresponds to a metric on \(G\). We show that \(\gamma\) is locally a power series at \(t_0\). Recalling Pütmann’s Formula 2.2, it is clear that we must only prove that
\[
||[\Phi^{-1}_t X, \Phi^{-1}_t Y]|^2_{h_t}
\]
can be expressed as a power series near \(t_0\). Since \(\Psi\) is \(h_0\)-self-adjoint, it can be diagonalized; say \(\Psi = \text{diag}(a_1, \ldots, a_d)\). We then have
\[
\Phi_t = \text{diag}\left(\frac{1}{1-a_1 t}, \ldots, \frac{1}{1-a_d t}\right) \\
\Phi_t = \text{diag}\left(\frac{1}{1-a_1 t_0} \sum_{n=0}^{\infty} \left(\frac{a_i}{1-a_i t_0}\right)^n (t - t_0)^n\right)^d \\
\Phi_t = \Phi_{t_0} \sum_{n=0}^{\infty} \Phi_{t_0}^n \Psi^n (t - t_0)^n,
\]
with convergence whenever \(|t - t_0|\) is sufficiently small. We can use this expression for \(\Phi_t\) together with the identity \(\Phi^{-1}_t = I - t_0 \Psi - (t - t_0) \Psi\) to expand \(||[\Phi^{-1}_t X, \Phi^{-1}_t Y]|^2_{h_t}\) as a power series as in the proof of Proposition 2.2. \[\square\]

The analyticity of \(|D|_{h_t}^2\) also follows from Equation 2.3, completing the proof of Theorem 2.1.

3. The main theorem

We now use the formula for \(\kappa(t)\) from the previous section to prove our main theorem. First, we use the power series of \(\kappa(t)\) to rephrase what it means for \(\Psi\) to be infinitesimally nonnegative.
Proposition 3.1. An $h_0$-self-adjoint endomorphism $\Psi: g \to g$ is infinitesimally nonnegative if and only if whenever $X, Y \in g$ commute we have either

1. $\kappa'''(0) \geq 0$
2. $\kappa'''(0) = 0$ and $D = 0$.

In the second case, $\kappa$ is identically zero.

Notice that $\kappa''''(0) = 6\delta$, so that $\kappa''''(0)$ is a relatively simple sum of inner products of elements of $g$. This equivalent definition of infinitesimally nonnegative is often more useful than our original definition, especially in applications.

Proof. It is clear that $\Psi$ is infinitesimally nonnegative if and only if either all coefficients of the Taylor series of $\kappa(t)$ at 0 are zero or the first nonzero coefficient is positive. When $X$ and $Y$ do not commute, we have $\alpha > 0$.

Suppose instead that $X, Y \in g$ commute. Notice that $\alpha = \beta = \gamma = 0$; in particular $\gamma = 0$ follows from the observation that $-\frac{3}{4} |D|^2 h_t$. As $h$ is nonnegatively curved, $0 \leq \kappa(1) = \frac{1}{6} \kappa''''(0) - \frac{3}{4} t^4 \cdot |D|^2 h_t$.

Therefore $\kappa''''(0) \geq 0$, and if $\kappa''''(0) = 0$ then $D = 0$. By Proposition 3.1, $\Psi$ is infinitesimally nonnegative.

With this equivalent definition of infinitesimally nonnegative, we are prepared to prove our main theorem.

Theorem 1.3. Let $h$ be a nonnegatively curved left-invariant metric on $G$. The unique inverse-linear path from $h_0$ to $h$ is infinitesimally nonnegative.

Proof. Let $\Phi_t$ be the unique inverse-linear path from $h_0$ to $h$, and fix commuting $X, Y \in g$. By Theorem 2.1

$$\kappa(t) = \frac{1}{6} \kappa''''(0) t^3 - \frac{3}{4} t^4 \cdot |D|^2 h_t.$$

As $h$ is nonnegatively curved,

$$0 \leq \kappa(1) = \frac{1}{6} \kappa''''(0) - \frac{3}{4} t^4 \cdot |D|^2 h_t.$$

Therefore $\kappa''''(0) \geq 0$, and if $\kappa''''(0) = 0$ then $D = 0$. By Proposition 3.1, $\Psi$ is infinitesimally nonnegative.

As an example of how Theorem 1.3 can be applied to transform infinitesimal results into global ones, consider the following result from [6].

Lemma 3.2 ([6, Lemma 2.6]). Assume that $\Psi$ is infinitesimally nonnegative. Let $p_0$ be the eigenspace of $\Psi$ corresponding to the smallest eigenvalue. If $X \in p_0$, $Y \in g$ and $[X, Y] = 0$, then $[X, \Psi Y] \in p_0$.

This lemma was the main tool used by Tapp to prove rigidity statements about the family of infinitesimally nonnegative endomorphisms of $so(4)$. Applying Theorem 1.3, we derive the following global result.
Lemma 3.3. Assume that $\Phi$ is the matrix of a nonnegatively curved metric. Let $p_0$ be the eigenspace of $\Phi$ corresponding to the smallest eigenvalue of $\Psi$. If $X \in p_0$, $Y \in g$ and $[X, Y] = 0$, then $[X, \Phi^{-1}Y] \in p_0$.

Proof. This follows immediately from Lemma 3.2 since $\Psi = I - \Phi^{-1}$ is infinitesimally nonnegative and $p_0$ is the eigenspace of $\Psi$ corresponding to the smallest eigenvalue. \[\Box\]

We note that this result can also be derived directly from Puttmann’s Formula 2.2.

4. Enlarging subalgebras

Perhaps the simplest type of inverse-linear variation is one which gradually scales vectors in a subalgebra $h$ of $g$. In this section, we let $H \subset G$ be a Lie subgroup of the Lie group $G$ with Lie algebra $h \subset g$. It is known that shrinking vectors in $h$ yields nonnegatively curved metrics (see, for example, [3]). On the other hand, expanding vectors in $h$ does not always produce nonnegatively curved metrics, but does when $h$ is abelian [4]. In this section, we use the power series formula from the previous section to determine when the subalgebra $h$ can be expanded while maintaining nonnegative curvature.

For $Z \in g$, denote by $Z_h$ and $Z_p$ the projections of $Z$ onto $h$ and its $h_0$-orthogonal complement $p$. When we discuss scaling $h$ by a factor $\lambda > 0$, we are referring to the metric $h(X, Y) = h_0(\lambda X^h + X^p, Y)$, so that the square of the norm of a vector in $h$ is scaled by a factor of $\lambda$.

Let $\Psi(Z) = Z_h$, so $\Phi_t = (I - t\Psi)^{-1}$ is the inverse-linear variation which gradually expands vectors in $h$. If $h$ is abelian, it is easy to use the formulas for the coefficients of the power series of $\kappa(t)$ in tandem with the analyticity of $\kappa$ to prove

$$\kappa(t) = \frac{1}{4} |[X, Y]|^2 - \frac{3}{4} |[X, Y]^h|^2 \cdot \frac{t}{1 - t} \quad (\infty < t < 1).$$

From this formula we can show that enlarging $h$ by a factor of up to $4/3$ always preserves nonnegative curvature, a result which first appeared in [4]. In fact, the particularly nice form of $\kappa(t)$ allows us to prove a stronger statement.

Theorem 4.1. Scaling the abelian subalgebra $h \subset g$ preserves nonnegative curvature if and only if no vector in $[g, g]$ is expanded by more than $4/3$.

Proof. By Equation 4.1, the metric $h_t$ is nonnegatively curved if and only if

$$|Z_h|^2 \cdot \frac{t}{1 - t} \leq \frac{1}{3} |Z|^2$$

holds for all $Z \in [g, g]$. As

$$|Z|^2_{h_t} = \langle \Phi_t Z, Z \rangle = \langle Z + \frac{t}{1 - t} Z^h, Z \rangle = |Z|^2 + |Z^h|^2 \cdot \frac{t}{1 - t},$$

we find that Inequality 4.2 is equivalent to requiring that $|Z|^2_{h_t} \leq (4/3) \cdot |Z|^2$ holds for all $Z \in [g, g]$.

If $[g, g] \cap h \neq 0$, this theorem says that $h$ can be by a factor up to $4/3$. At the other extreme, if $[g, g] \perp h$ then we find that $h$ can be expanded up by an arbitrary amount. This was already known, since if $h$ is orthogonal to $[g, g]$ then
\( \mathfrak{h} \) is contained in the center of \( \mathfrak{g} \). This rescaling then stays within the family of bi-invariant metrics on \( \mathfrak{g} \).

When \( \mathfrak{h} \) is not abelian, things are not quite so simple. In this case the power series simplifies to

\[
\kappa(t) = \frac{1}{4}||[X, Y]|^2 - \frac{3}{4}||[X, Y]^h|^2t + \frac{3}{4}|B|^2t^2 - \frac{1}{4}|B|^2t^3 - \frac{3}{4}||[X^p, Y^p]^h||^2 \cdot \frac{t^2}{1-t}.
\]

We can use this formula to classify exactly which subalgebras of \( \mathfrak{g} \) can be enlarged a small amount while maintaining nonnegative curvature.

**Theorem 4.2.** Expanding the subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) by a small amount preserves nonnegative curvature if and only if there exists a constant \( c \) such that \( ||[X, Y]| \leq c \cdot ||[X, Y]|| \) holds for all \( X, Y \in \mathfrak{g} \).

We omit the lengthy but easy proof for the reason that we do not know if there are any interesting examples of subalgebras for which the latter condition holds. It clearly holds when \( \mathfrak{h} \) is either abelian or an ideal of \( \mathfrak{g} \) (or the sum of an ideal and a disjoint abelian subalgebra), but it is already known that such subalgebras can be enlarged while maintaining nonnegative curvature.

5. **An Application to \( \text{SO}(4) \)**

In this section we apply our main theorem to Tapp’s partial classification of the infinitesimally nonnegative endomorphisms of \( \mathfrak{so}(4) \) to give a partial classification of the nonnegatively curved left-invariant metrics on \( \text{SO}(4) \). Let \( G = \text{SO}(4) \), so \( \mathfrak{g} = \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \). When we give \( \mathfrak{so}(3) \) the bi-invariant metric

\[
\langle Z_1, Z_2 \rangle = \frac{1}{2} \cdot \text{trace}(Z_1 Z_2^T),
\]

\( \mathfrak{g} \) inherits the bi-invariant product metric \( h_0 \) from its factors. We call a vector in \( \mathfrak{g} \) singular if it is in either \( \mathfrak{g}_1 \) or \( \mathfrak{g}_2 \).

The known examples of left-invariant metrics of nonnegative curvature are discussed thoroughly in [6, Section 3]. These metrics can be grouped into three categories: product metrics, metrics which come from a torus action, and metrics which come from an \( S^3 \)-action. Our main result in this section is the following theorem.

**Theorem 5.1.** Let \( \Phi \) be the matrix of a nonnegatively curved left-invariant metric \( h \). If \( \Phi \) has a singular eigenvector, then either \( h \) is a product metric or \( h \) comes from a torus action. In either case, \( h \) is a known example of a metric of nonnegative curvature.

Since the nonnegatively curved left-invariant metrics on \( \text{SO}(3) \) have been classified, the product metrics are well-understood.

The metrics arising from torus actions are only a little more complicated than the product metrics. Let \( \{A_1, A_2, A_3\} \) and \( \{B_1, B_2, B_3\} \) be \( h_0 \)-orthonormal bases of \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \), respectively. After scaling \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) by factors \( c \) and \( d \), respectively, then enlarging the abelian subalgebra \( \tau = \text{span}\{A_3, B_1\} \) by \( 4/3 \), then further altering the metric on \( \tau \) via a canonical \( T^2 \)-action on \( G \) (using the method of Cheeger [2]),
one obtains a nonnegatively curved metric $h$ with matrix

$$
\Phi = \begin{pmatrix}
c & 0 & 0 & 0 & 0 & 0 \\
0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & a_1 & a_3 & 0 & 0 \\
0 & 0 & a_3 & a_2 & 0 & 0 \\
0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & 0 & 0 & 0 & d \\
\end{pmatrix}
$$

with respect to the basis $\{A_1, A_2, A_3, B_1, B_2, B_3\}$. The only restriction on $\Phi$, coming from the fact that the final alteration only shrinks vectors (see [6, Section 3.2]), is that the norm on $\tau$ determined by the matrix

$$
\begin{pmatrix}
a_1 & a_3 \\
a_3 & a_2 \\
\end{pmatrix}
$$

is bounded above by the norm determined by

$$
\begin{pmatrix}
\frac{4}{3} \cdot c & 0 \\
0 & \frac{4}{3} \cdot d \\
\end{pmatrix}.
$$

The most difficult part of the proof of Theorem 5.1 was already completed in [6, Theorem 4.1], which is an infinitesimal version of the theorem. We call an endomorphism $\Psi$ of $\mathfrak{g}$ a product if $\Psi(\mathfrak{g}_1) \subset \mathfrak{g}_1$ and $\Psi(\mathfrak{g}_2) \subset \mathfrak{g}_2$.

**Theorem 5.2 ([6, Theorem 4.1]).** Let $\Psi$ be an infinitesimally nonnegative endomorphism with a singular eigenvector. Then either $\Psi$ is a product or $\Psi$ is of Form 5.1.

**Proof of Theorem 5.2.** Since $h$ is nonnegatively curved, $\Psi = I - \Phi^{-1}$ is infinitesimally nonnegative. As $\Phi$ has a singular eigenvector, so does $\Psi$. According to Theorem 5.2, either $\Psi$ is a product or $\Psi$ can be written in Form 5.1. Clearly if $\Psi$ is a product then $\Phi$ is a product, which means $h$ is a product metric. If instead $\Psi$ has Form 5.1 then so does $\Phi$.

Assume $\Phi$ has Form 5.1, we must prove that $\Phi$ comes from a torus action. Permuting some basis vectors if necessary, we may assume that $A_1, A_2, A_3$ and $B_1, B_2, B_3$ behave like the quaternions $i, j, k$ with respect to their Lie bracket structure. Denote by $\tilde{h}$ the metric on $\tau$ corresponding to the matrix

$$
\begin{pmatrix}
\frac{4}{3} \cdot c & 0 \\
0 & \frac{4}{3} \cdot d \\
\end{pmatrix}.
$$

We must prove that

$$
|\alpha A_3 + \beta B_1|^2 \leq |\alpha A_3 + \beta B_1|^2_{\tilde{h}}
$$

holds for all $\alpha, \beta \in \mathbb{R}$.

Consider the unnormalized sectional curvature of the vectors $\alpha A_1 + \beta B_2$ and $A_2 + B_3$ with respect to $h$. We have

$$
[\Phi(\alpha A_1 + \beta B_2), A_2 + B_3] = \alpha c A_3 + \beta d B_1 \\
[\alpha A_1 + \beta B_2, \Phi(A_2 + B_3)] = \alpha c A_3 + \beta d B_1 \\
[\alpha A_1 + \beta B_2, A_2 + B_3] = \alpha A_3 + \beta B_1,
$$
and therefore by Puttman's Formula 2.2
\[ k_h(\alpha A_1 + \beta B_2, A_2 + B_3) = \langle \alpha c A_3 + \beta d B_1, \alpha A_3 + \beta B_1 \rangle - \frac{3}{4}|\alpha A_3 + \beta B_1|^2_h \]
\[ = \frac{3}{4}(|\alpha A_3 + \beta B_1|^2_h - |\alpha A_3 + \beta B_1|^2_h). \]

Since \( h \) is nonnegatively curved, this proves the required inequality. \( \square \)

We have therefore completed the classification of nonnegatively curved metrics with singular eigenvectors. The rest of the classification is still a difficult problem. Among all the remaining left-invariant metrics, we must still locate the metrics which come from \( S^3 \)-actions. Noticing that there are three \( \Phi \)-invariant 2-dimensional abelian subalgebras of \( g \) whenever \( \Phi \) comes from an \( S^3 \)-action ([6, Section 3.3]), Tapp looked for such subalgebras in the infinitesimal version of the problem.

**Theorem 5.3** ([6, Theorem 4.4]). If \( \Psi \) is an infinitesimally nonnegative endomorphism of \( g \), then \( g \) has a \( \Psi \)-invariant 2-dimensional abelian subalgebra.

Applying our main theorem, we immediately have the following result.

**Theorem 5.4.** If \( h \) is nonnegatively curved, then \( g \) has a \( \Phi \)-invariant 2-dimensional abelian subalgebra.

We can actually say slightly more, by mimicking the argument of [6, Theorem 4.2].

**Theorem 5.5.** There are orthonormal bases \( \{A_1, A_2, A_3\} \) and \( \{B_1, B_2, B_3\} \) of the two factors of \( g = g_1 \oplus g_2 \) such that with respect to the basis \( \{A_1, B_1, A_2, B_2, A_3, B_3\} \), \( \Phi \) has the form
\[
\Phi = \begin{pmatrix}
a_1 & a_3 & 0 & 0 & 0 & 0 \\
a_3 & a_2 & 0 & 0 & 0 & 0 \\
0 & 0 & b_1 & b_3 & \lambda & 0 \\
0 & 0 & b_3 & b_2 & 0 & \mu \\
0 & 0 & \lambda & 0 & c_1 & c_3 \\
0 & 0 & 0 & \mu & c_3 & c_2 
\end{pmatrix}.
\]

**Proof.** In [6, Theorem 4.2], Tapp proves that \( \Psi \) can be written in this form. The only properties of \( \Psi \) which are used are that there is a 2-dimensional \( \Psi \)-invariant abelian subalgebra and that \( \Psi \) is self-adjoint. Since Theorem 5.4 implies that \( \Phi \) has these properties as well, the proof carries over to \( \Phi \) immediately. \( \square \)

If it is the case that \( \lambda = \mu = 0 \), then there are three \( \Phi \)-invariant 2-dimensional abelian subalgebras.

6. Changing the bi-invariant metric

Suppose that the unique inverse-linear path from the bi-invariant metric \( h_0 \) to the left-invariant metric \( h \) is nonnegative. If \( h_1 \) is another bi-invariant metric, must the unique inverse-linear path from \( h_1 \) to \( h \) also be nonnegative? If Tapp's Conjecture 1.2 is true, then the answer must be yes. A proof of this statement could also be a logical step towards proving the conjecture, as it would imply that Conjecture 1.2 is equivalent to the seemingly weaker statement that if \( h \) is nonnegatively curved then there is some bi-invariant metric \( h_0 \) such that the unique inverse-linear path
from $h_0$ to $h$ is nonnegative. In this section, we address the case where $h_1$ is a scalar multiple of $h_0$.

**Theorem 6.1.** If $\lambda > 0$ and $h$ is any left-invariant metric, then

1. the unique inverse-linear path from $h_0$ to $h$ is nonnegative if and only if the unique inverse-linear path from $\lambda h_0$ to $h$ is nonnegative.
2. The unique inverse-linear path from $h_0$ to $h$ is locally nonnegative if and only if the unique inverse-linear path from $\lambda h_0$ to $h$ is locally nonnegative.
3. The unique inverse-linear path from $h_0$ to $h$ is infinitesimally nonnegative if and only if the unique inverse-linear path from $\lambda h_0$ to $h$ is infinitesimally nonnegative.

Let $\Phi$ be the matrix of the left-invariant metric $h$ with respect to $h_0$. Let $\Theta$ be the matrix of $h$ with respect to the bi-invariant metric $\lambda h_0$. When we let $\Psi = I - \Phi^{-1}$ and $\Upsilon = I - \Theta^{-1}$, we see that $\Phi_t = (I - t\Psi)^{-1}$ is the unique inverse-linear path from $h_0$ to $h$ and $\Theta_t = (I - t\Upsilon)^{-1}$ is the unique inverse-linear path from $\lambda h_0$ to $h$. As

$$h_0(\Phi X, Y) = h(X, Y) = \lambda h_0(\Theta X, Y),$$

we find that $\Theta = \lambda^{-1}\Phi$. Theorem 6.1 will follow immediately from our next proposition.

**Proposition 6.2.** With the notation of the previous paragraph,

$$\kappa^\Upsilon(t) = \lambda(1 - (1 - \lambda)t)^3 \cdot \kappa^\Psi \left( \frac{\lambda t}{1 - (1 - \lambda)t} \right) \quad (0 \leq t \leq 1).$$

**Proof.** By analyticity, it is enough to show that the above equality holds for all sufficiently small $t$. We therefore assume that $t$ is small enough that the various power series which appear in the following proof are convergent. Since $\Theta = \lambda^{-1}\Phi$, we find

$$\Upsilon = I - \lambda \Phi^{-1} = I - \lambda(I - \Psi) = (1 - \lambda)I + \lambda\Psi.$$

A very long, but entirely straightforward calculation using this equality now shows

$$\begin{align*}
\alpha^\Upsilon &= \lambda \alpha^\Psi \\
\beta^\Upsilon &= -3(1 - \lambda)\lambda \alpha^\Psi + \lambda^2 \beta^\Psi \\
\gamma^\Upsilon &= 3(1 - \lambda)^2 \lambda \alpha^\Psi - 2(1 - \lambda)\lambda^2 \beta^\Psi + \lambda^3 \gamma^\Psi \\
\delta^\Upsilon &= -(1 - \lambda)^3 \lambda \alpha^\Psi + (1 - \lambda)^2 \lambda^2 \beta^\Psi - (1 - \lambda)\lambda^3 \gamma^\Psi + \lambda^4 \delta^\Psi,
\end{align*}$$

(6.2)

from which we conclude that

$$\lambda^{-1} \kappa^\Upsilon(t) = \alpha^\Psi(1 - (1 - \lambda)t)^3 + \beta^\Psi t\lambda(1 - (1 - \lambda)t)^2 + \gamma^\Psi t^2 \lambda^2(1 - (1 - \lambda)t) + \delta^\Psi t^3 \lambda^3 + \lambda^{-1} \Gamma^\Upsilon(t).$$

Dividing both sides of this equation by $(1 - (1 - \lambda)t)^3$, we see

$$\frac{\kappa^\Upsilon(t)}{\lambda(1 - (1 - \lambda)t)^3} = \kappa^\Psi \left( \frac{\lambda t}{1 - (1 - \lambda)t} \right) - \Gamma^\Psi \left( \frac{\lambda t}{1 - (1 - \lambda)t} \right) + \frac{\Gamma^\Upsilon(t)}{\lambda(1 - (1 - \lambda)t)^3},$$
and we must only prove that the last two terms above cancel. Now $D^\Upsilon = \lambda^2 D^\Psi$, so applying the binomial theorem to $\Upsilon^{n-4} = ((1 - \lambda)I + \lambda \Psi)^{n-4}$ we have

$$\Gamma^\Upsilon(t) = \lambda^5 \sum_{n=4}^\infty t^n \langle \Upsilon^{n-4} D^\Psi, D^\Psi \rangle$$

$$= \lambda^5 \sum_{n=4}^\infty \sum_{k=0}^{n-4} t^n \binom{n-4}{k} (1 - \lambda)^{n-4-k} \lambda^k \langle \Psi^k D^\Psi, D^\Psi \rangle.$$  

This double sum is absolutely convergent for all sufficiently small $t$. Thus we may interchange the order of summation, and we find

$$\Gamma^\Upsilon(t) = \lambda \sum_{k=0}^\infty t^{k+4} \lambda^{k+4} \langle \Psi^k D^\Psi, D^\Psi \rangle \sum_{n=0}^\infty t^n \binom{n+k}{k} (1 - \lambda)^n$$

$$= \lambda \sum_{k=0}^\infty t^{k+4} \lambda^{k+4} \langle \Psi^k D^\Psi, D^\Psi \rangle \left( \sum_{n=0}^\infty t^n (1 - \lambda)^n \right)^{k+1}$$

$$= \lambda \sum_{k=0}^\infty t^{k+4} \lambda^{k+4} \langle \Psi^k D^\Psi, D^\Psi \rangle \cdot \frac{1}{(1 - (1 - \lambda)t)^{k+1}}.$$  

From this we conclude

$$\frac{\Gamma^\Upsilon(t)}{\lambda(1 - (1 - \lambda)t)^3} = \Gamma^\Psi \left( \frac{\lambda t}{1 - (1 - \lambda)t} \right)$$

holds for all sufficiently small $t$, as was to be shown. \(\square\)

Proof of Theorem 6.1. This follows from Proposition 6.2 since

$$t \mapsto \frac{\lambda t}{1 - (1 - \lambda)t}$$

is an endpoint-fixing self-homeomorphism of the unit interval. \(\square\)

While our proof of Theorem 6.1 is nice in that it proves all three parts of the theorem at once, it also hides the original intuition behind the argument. Notice that when $X$ and $Y$ commute, Equation 6.2 implies that $\delta^\Upsilon = \lambda^4 \delta^\Psi$. Since also $D^\Upsilon = \lambda^2 D^\Psi$, we may conclude that $\Psi$ is infinitesimally nonnegative if and only if $\Upsilon$ is infinitesimally nonnegative by appealing to Proposition 3.1. This proves the third part of Theorem 6.1. We originally found this relation between $\delta^\Upsilon$ and $\delta^\Psi$, which suggested that we look for a relation between $\kappa^\Upsilon$ and $\kappa^\Psi$. This culminated in deriving Equation 6.1 for all sufficiently small $t$, which is enough to prove the second part of the theorem. Finally we noticed that $\kappa(t)$ is analytic, which proves the first part of the theorem.

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