ZETA FUNCTIONS OF INTEGRAL NILPOTENT QUIVER REPRESENTATIONS

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Abstract. We introduce and study multivariate zeta functions enumerating subrepresentations of integral quiver representations. For nilpotent such representations defined over number fields, we exhibit a homogeneity condition that we prove to be sufficient for local functional equations of the generic Euler factors of these zeta functions. This generalizes and unifies previous work on submodule zeta functions including, specifically, ideal zeta functions of nilpotent (Lie) rings and their graded analogues.

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1. Introduction

1.1. Quivers, (sub-)representations, and zeta functions. Representations of quivers over fields have been extensively studied; see, e.g., [11,14]. Of particular importance in this extensive field is the geometry of quiver Grassmannians, viz. algebraic varieties parameterizing the subrepresentations of a given quiver representation with a fixed dimension vector (e.g. [7]). Despite their arithmetic significance (e.g. [10]), representations of quivers over rings seem to have received much less attention. In this paper we study multivariate zeta functions enumerating the subrepresentations of integral representations of finite quivers. In the current section we define these terms.

A quiver is a directed graph, formally a quadruple $Q = (Q_0, Q_1, h, t)$, where $Q_0$ and $Q_1$ are finite sets, called the set of vertices and arrows of $Q$, respectively, and $h, t : Q_1 \to Q_0$ are maps assigning to each arrow its head and tail, respectively. Arrows $\varphi \in Q_1$ with $t(\varphi) = h(\varphi)$ are called loops.

For a ring $R$, an $R$-representation

$$V = V_Q = (\mathcal{L}_\iota, f_\varphi)_{\iota \in Q_0, \varphi \in Q_1}$$

of $Q$ consists of a family of $R$-modules $\mathcal{L}_\iota$ indexed by the vertices $\iota \in Q_0$ and a family of linear maps $f_\varphi : \mathcal{L}_{t(\varphi)} \to \mathcal{L}_{h(\varphi)}$ indexed by the arrows $\varphi \in Q_1$. We call $V$ finitely generated resp. free of finite rank if all the $R$-modules $\mathcal{L}_\iota$ are finitely generated resp. free.
of finite rank (which may vary with \( \ell \in Q_0 \)). A representation over a ring of integers of a global or local field is called integral. Given an \( R \)-algebra \( S \), we write
\[
V(S) = V \otimes_R S = (\mathcal{L}_\ell \otimes_R S, f_\varphi \otimes_R \text{id}_S)
\]
for the resulting \( S \)-representation of \( Q \). If \( S \) is also an \( R' \)-algebra, we write \( V(S)_{R'} \) for the corresponding restriction of scalars. If \( R = \mathcal{O} \), the ring of integers of a global number field, and \( p \in \text{Spec}(\mathcal{O}) \setminus \{0\} \) is a non-zero prime ideal of \( \mathcal{O} \), then we write \( \mathcal{O}_p \) for the completion of \( \mathcal{O} \) at \( p \).

If \( V \) is an \( R \)-representation of \( Q \) which is free of finite rank, say \( n_\ell = \text{rk}_R \mathcal{L}_\ell \) for \( \ell \in Q_0 \), we call \((n_\ell)_{\ell \in Q_0} \) the rank vector and \( \text{rk} V = \sum_{\ell \in Q_0} n_\ell \) the (total) rank of \( V \). If \( n_\ell = 1 \) for all \( \ell \in Q_0 \), then \( V \) is said to be thin.

Let \( R \) be a ring and \( V = (\mathcal{L}_\ell, f_\varphi) \) be an \( R \)-representation of \( Q \). Given submodules \( \Lambda_\ell \leq \mathcal{L}_\ell \) for all \( \ell \in Q_0 \) such that \( f^\prime_\varphi(\Lambda_{\ell(\varphi)}) \leq \Lambda_{h(\varphi)} \) for all \( \varphi \in Q_1 \), where \( f^\prime_\varphi := f_\varphi|_{\Lambda_{\ell(\varphi)}} \), we call
\[
V' = (\Lambda_\ell, f'_\varphi)_{\ell \in Q_0, \varphi \in Q_1}
\]
a subrepresentation of \( V \), written \( V' \leq V \). We call \( V' \) of finite index in \( V \) if \( |\mathcal{L}_\ell : \Lambda_\ell| < \infty \) for all \( \ell \in Q_0 \).

For \( m = (m_\ell)_{\ell \in Q_0} \in \mathbb{N}^{Q_0} \), write
\[
am(V) = \# \left\{ V' \leq f V \mid \forall \ell \in Q_0, |\mathcal{L}_\ell : \Lambda_\ell| = m_\ell \right\}
\]
for the number of subrepresentations of \( V \) of index \( m \).

Assume that \( \nam(V) < \infty \) for all \( m \). The \textit{(multivariate representation) zeta function} associated with \( V \) is the Dirichlet generating series
\[
\zeta_V(s) = \sum_{V' \leq f V} \prod_{\ell \in Q_0} |\mathcal{L}_\ell : \Lambda_\ell|^{-s_\ell} = \sum_{m \in \mathbb{N}^{Q_0}} \nam(V) \prod_{\ell \in Q_0} m_\ell^{-s_\ell},
\]
enumerating the finite-index subrepresentations of \( V \). Here each \( s_\ell \) is a complex variable and \( s = (s_\ell)_{\ell \in Q_0} \).

We will sometimes turn to a specific univariate specialization of \( \zeta_V(s) \). We set
\[
|V : V'| = \prod_{\ell \in Q_0} |\mathcal{L}_\ell : \Lambda_\ell| < \infty.
\]
For \( m \in \mathbb{N} \), write
\[
am(V) = \# \left\{ V' \leq f V \mid |V : V'| = m \right\} = \sum_{\{m\sum_{m_\ell = m}\}} \nam(V)
\]
for the number of subrepresentations of \( V \) of index \( m \). The \textit{(univariate representation) zeta function} associated with \( V \) is the Dirichlet generating series
\[
\zeta_V(s) = \sum_{m \geq 1} \nam(V)m^{-s} = \zeta_V(s, \ldots, s)
\]
obtained by substitution all variables \( s_\ell \) with a single complex variable \( s \). We trust that the slight abuse of notation caused by denoting both the uni- and multivariate functions by \( \zeta_V \) will not cause confusion. Note that, trivially, multi- and univariate representation zeta functions coincide in the case of loop quivers, i.e. when \( |Q_0| = 1 \); cf. Section \ref{univ}.

As we shall explain in Section \ref{univ}, the class of univariate representation zeta functions associated with quiver representations coincides with the class of submodule zeta functions, enumerating sublattices invariant under a set of linear operators. This class of zeta function was pioneered by Solomon (\ref{sol}) and, much more recently, further developed by Rossmann (\ref{ross}); see Section \ref{univ}. We argue, however, that the interpretation
in terms of quiver representations affords new perspectives on (graded) submodule zeta functions, even in the univariate case.

We will soon focus on representations \( V \) where the numbers \( a_m(V) \) are finite for all \( m \) and, moreover, grow at most polynomially. This holds in particular if \( R \) is the ring of integers of a global or local field and \( V \) is free of finite rank. Under these assumptions the formal Dirichlet generating series \( \zeta_V(s) \) converges absolutely for all \( s \in \mathbb{C} \) with \( \Re(s) > \text{rk} V \). Indeed, it is clear that then \( a_m(V) \) is bounded above by the number of sublattices of \( R^{\text{rk} V} \) of index \( m \); the claim follows from the well-known facts recalled in Example 1.1. For further properties of univariate representation zeta functions of quiver representations, see Section 1.3.4.

If \( R = \mathbb{O} \) is a ring of integers of a global field and \( V \) is finitely generated, then its zeta function satisfies the formal Euler product

\[
\zeta_V(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathbb{O}):(0)} \zeta_V(\mathbb{O}_\mathfrak{p})(s).
\]

It follows from deep general results that each of the local zeta functions \( \zeta_V(\mathbb{O}_\mathfrak{p})(s) \) is a rational function in \( q^{-s} \), \( t \in \mathbb{Q}_0 \), where \( q \) is the residue field cardinality of the compact discrete valuation ring \( \mathbb{O}_\mathfrak{p} \). In the univariate case this follows essentially from \([17]\); the arguments extend to the multivariate situation and the case of general number fields.

**Example 1.1.** Assume that \( Q_0 \) is a singleton, represented by an \( R \)-module \( \mathcal{L} \), and \( f_\varphi = 0 \) for all \( \varphi \in Q_1 \). The representation zeta function \( \zeta_V(s) \) simply enumerates the \( R \)-submodules of finite index in \( \mathcal{L} \). If further \( R = \mathbb{O} \), the ring of integers of a number field \( K \), and \( \mathcal{L} \cong \mathbb{O}^n \), it is well-known (see, e.g., \([17\), Proposition 1.1]) that

\[
\zeta_V(s) = \prod_{i=0}^{n-1} \zeta_K(s-i),
\]

where \( \zeta_K(s) \) is the Dedekind zeta function of \( K \). Note that \( \zeta_V(s) \) converges precisely if \( \Re(s) > n = \text{rk} V \), as the abscissa of convergence of \( \zeta_K(s) \) is equal to 1.

Locally, i.e. when \( R = \mathbb{O} \) is a compact discrete valuation ring (cDVR) of residue field cardinality \( q \), we obtain (with \( t = q^{-s} \))

\[
\zeta_V(\mathbb{O})(s) = \zeta_{\mathbb{O}^n}(s) = \prod_{i=0}^{n-1} \frac{1}{1-q^it}.
\]

**Example 1.2.** Let \( L \) be a Lie ring (i.e. \( \mathbb{Z} \)-Lie algebra), with (Lie) generators \( x_1, \ldots, x_d \). We may interpret \( L \) as a representation of the loop quiver \( Q = \mathbb{L}_d \) with one vertex and \( d \) loops. Indeed, if \( Q_0 \) is a singleton and \( |Q_1| = d \), then \( V = (\mathbb{L}, \text{ad} x_1, \ldots, \text{ad} x_d) \) is a \( \mathbb{Z} \)-representation of \( \mathbb{L}_d \). The zeta function of \( V \) coincides with the *ideal zeta function* of \( L \) introduced in \([17]\); cf. Section 1.3.1 Example 1.1 is the special case of the abelian Lie ring.

1.2. Nilpotent quiver representations and the homogeneity condition. Our main result concerns integral nilpotent quiver representations satisfying a certain homogeneity condition, which we now explain. Recall that \( Q = (Q_0, Q_1, h, t) \) is a quiver and \( V = (\mathcal{L}_n, f_\varphi) \) is an \( R \)-representation of \( Q \).

A path \( w = \varphi_1 \cdots \varphi_n \) in \( Q \) of length \( n \geq 1 \) is a sequence of arrows \( \varphi_1, \ldots, \varphi_n \) such that \( t(\varphi_i) = h(\varphi_{i+1}) \) for \( 1 \leq i \leq n-1 \). One calls \( h(w) = h(\varphi_1) \) the head of \( w \) and \( t(w) = t(\varphi_n) \) the tail of \( w \). We also say that \( w \) is a path from \( t(w) \) to \( h(w) \). Any vertex \( x \in Q_0 \) of \( Q \) is considered as a path of length 0 with head \( x \) and tail \( x \), and denoted by \( w_x \).
For a path \( w = \varphi_1 \cdots \varphi_n \), write \( f_w = f_{\varphi_1} \cdots f_{\varphi_n} \) and set
\[
w(V) = f_w(V) = f_w(\mathcal{L}_i(w)) \leq \mathcal{L}_h(w).
\]
If \( w_c \) is a path of length 0, then \( w_c(V) = \mathcal{L}_x \).

We say that \( V \) is nilpotent if there exists \( t \in \mathbb{N} \) such that \( w(V) = 0 \) for any path \( w \) of length \( t \). If \( V \) is nilpotent, then the unique \( c \in \mathbb{N} \) such that \( w(V) = 0 \) for any path \( w \) of length \( c \) but not for all paths of length \( c - 1 \) is called the nilpotency class of \( V \). Note that our definition of nilpotency ensures that all representations of acyclic quivers, viz. quivers without oriented cycles, are nilpotent.

Set
\[
\mathcal{L} := \bigoplus_{i \in \mathbb{N}_0} \mathcal{L}_i.
\]
Let \( \delta_{i,j} \) denote the usual Kronecker delta. For any \( \varphi \in Q_1 \) we extend \( f_\varphi \in \text{Hom}(\mathcal{L}_{t(\varphi)}, \mathcal{L}_{b(\varphi)}) \) to an endomorphism \( e_\varphi \in \text{End}(\mathcal{L}) \) by setting, for \( i \in \mathbb{N}_0 \),
\[
e_\varphi \mathcal{L}_i = \delta_{i,t(\varphi)} f_\varphi.
\]
Informally speaking, we obtain the endomorphism \( e_\varphi \) by trivial extension of the homomorphism \( f_\varphi \).

Set
\[
\mathcal{E} = \mathcal{E}(V) := \langle e_\varphi \mid \varphi \in Q_1 \rangle \leq \text{End}(\mathcal{L}).
\]
We recursively define the upper centralizer series of \( V \), viz. the flag
\[
(Z_i)_{i \in \mathbb{N}_0} = \left( \left( Z_{i+1}/Z_i \right)_{i \in \mathbb{N}_0, \varphi \in Q_1} \right)_{i \in \mathbb{N}_0}
\]
of subrepresentations of \( V \), by setting \( Z_0 = 0 \) and, for \( i \in \mathbb{N}_0 \),
\[
Z_{i+1}/Z_i = \text{Cent}_\mathcal{E}(V/Z_i) := \{ x + Z_i \in V/Z_i \mid x \mathcal{E} \subseteq Z_i \}
\]
and \( Z_{i+1} := Z_i \cap \mathcal{L}_e \). One can check that \( V \) is nilpotent if \( Z_k = V \) for some \( k \in \mathbb{N}_0 \). In this case, the number \( c = c(V) = \min \{ k \in \mathbb{N}_0 \mid Z_k = V \} \) is the nilpotency class of \( V \) (or, equivalently, of \( \mathcal{E} \)).

In this article we consider finitely generated integral nilpotent quiver representations \( V \) over a global ring of integers \( R = \mathcal{O} \), say, satisfying the following assumption.

Assumption 1.3. For each \( i \in \mathbb{N}_0 \), there exist free \( \mathcal{O} \)-submodules \( \mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,c} \subseteq \mathcal{L}_i \) such that
\[
\mathcal{L}_i = \bigoplus_{j=1}^{c} \mathcal{L}_{i,j} \quad \text{and} \quad Z_{i,i} = \bigoplus_{j > c-i} \mathcal{L}_{i,j}.
\]

Remark 1.4. Assumption 1.3 is closely analogous to \cite{40} Assumption 1.1. As the latter, it is satisfied automatically if \( \mathcal{O} \) is a principal ideal domain. It is only made for notational convenience; see \cite{40} Remark 1.1.

For \( i \in [c]_0 = \{ 0, 1, 2, \ldots, c \} \) we set
\[
\mathcal{L}_i := \bigoplus_{i \in \mathbb{N}_0} \mathcal{L}_{i,i} \quad \text{and} \quad Z_i := \bigoplus_{i \in \mathbb{N}_0} Z_{i,i}.
\]
We further set \( \mathcal{L}_{i,0} = \mathcal{L}_{i,c+1} = \{ 0 \} \).

For \( i \in \mathbb{N}_0 \) and \( i \in [c]_0 \), let \( n = \text{rk } V \), \( n_i = \text{rk}_\mathcal{O} \mathcal{L}_i \), \( n_{i,i} = \text{rk}_\mathcal{O} \mathcal{L}_{i,i} \),
\[
N_{i,i} = \text{rk}_\mathcal{O} \bigoplus_{j \in [c-i]} \mathcal{L}_{i,j} = \sum_{j \in [c-i]} n_{i,j} = \text{rk}_\mathcal{O} (\mathcal{L}_i/Z_{i,i}),
\]
and
\[
N_i = \sum_{i \in \mathbb{N}_0} N_{i,i} = \text{rk}_\mathcal{O} (\mathcal{L}/Z_i).
\]
Note that
\[ n = N_0 = \sum_{i \in Q_0} n_i = \sum_{i,j} n_{i,j}. \]

Our main results concern generic Euler factors of zeta functions of quiver representations which satisfy a certain condition. The following generalizes \cite[Condition 1.1]{40}.

**Condition 1.5** (homogeneity). The nilpotent associative algebra \( E = E(V) \subseteq \text{End}_0(\mathcal{L}) \) is generated by elements \( c_1, \ldots, c_d \) such that, for all \( k \in [d] = \{1, \ldots, d\} \) and \( j \in [c] \),
\[ \mathcal{L}_j c_k \subseteq \mathcal{L}_{j+1}. \]

**Remark 1.6.** As in \cite{40}, Condition \( 1.5 \) is satisfied if \( E \) is cyclic (i.e. one may choose \( d = 1 \)) or if \( c \leq 2 \). It is also stable under taking direct sums of representations. The impact of this fundamental operation on the associated zeta function is, in general, poorly understood.

The following is our main theorem.

**Theorem 1.7.** Assume that \( E = E(V) \subseteq \text{End}_0(\mathcal{L}) \) satisfies Condition \( 1.5 \). Then, for almost all prime ideals \( p \) of \( \mathfrak{O} \) and all finite extensions \( \mathfrak{O} \) of \( \mathfrak{O}_p \), with residue field cardinality \( q^f \), the following functional equation holds:
\[ (1.5) \quad \zeta_{\mathcal{O}}(s) \mid_{q \rightarrow q^{-1}} = (-1)^n q^f \left( \sum_{i \in Q_0} \left( \frac{n_i}{2} \right) - \left( \sum_{i \in Q_0} s_i (\sum_{i=0}^{n-1} \mathcal{N}_{i,j}) \right) \right) \zeta_{\mathcal{O}}(s). \]

In particular,
\[ (1.6) \quad \zeta_{\mathcal{O}}(s) \mid_{q \rightarrow q^{-1}} = (-1)^n q^f \left( \sum_{i \in Q_0} \left( \frac{n_i}{2} \right) - \left( \sum_{i=0}^{n-1} \mathcal{N}_i \right) \right) \zeta_{\mathcal{O}}(s). \]

As in \cite{40}, a version of the model-theoretic transfer principle \cite{9} implies the following immediate consequence in positive characteristic.

**Corollary 1.8.** For almost all prime ideals \( p \) of \( \mathfrak{O} \) and all finite extensions \( \mathfrak{O} \) of \( \mathfrak{O}_p \), with maximal ideal \( \mathfrak{P} \) and residue field cardinality \( |\mathfrak{O}/\mathfrak{P}| = q^f \), say, the following functional equation holds:
\[ (1.7) \quad \zeta_{\mathcal{O}/\mathfrak{P}[T]}(s) \mid_{q \rightarrow q^{-1}} = (-1)^n q^f \left( \sum_{i \in Q_0} \left( \frac{n_i}{2} \right) - \left( \sum_{i=0}^{n-1} \mathcal{N}_{i,j} \right) \right) \zeta_{\mathcal{O}/\mathfrak{P}[T]}(s). \]

In particular,
\[ \zeta_{\mathcal{O}/\mathfrak{P}[T]}(s) \mid_{q \rightarrow q^{-1}} = (-1)^n q^f \left( \sum_{i \in Q_0} \left( \frac{n_i}{2} \right) - \left( \sum_{i=0}^{n-1} \mathcal{N}_i \right) \right) \zeta_{\mathcal{O}/\mathfrak{P}[T]}(s). \]

**Example 1.9.** For \( c = 1 \) we have \( E = 0 \), whence the homogeneity condition \( 1.5 \) is trivially satisfied. The functional equations \( 1.5 \) resp. \( 1.7 \) follow trivially from inspection of the formula
\[ \zeta_{\mathcal{O}}(s) = \prod_{i \in Q_0} \zeta_{\mathcal{O}^n_i}(s_i) \]
(cf. \( 1.3 \)), valid for any cDVR \( \mathfrak{o} \), regardless of its characteristic.

**Remark 1.10.** The operation \( q \rightarrow q^{-1} \) in \( 1.5 \) calls for some explanation. If there exists a single rational function \( W(X, Y) \in \mathbb{Q}(X, Y) \) such that \( \zeta_{\mathcal{O}}(s) = W(q, q^{-s}) \) for cDVRs \( \mathfrak{D} \) whose residue characteristics avoid a finite number of primes (depending on \( V \)), where \( q^{-s} = (q^{-s})_{i \in Q_0} \), then the functional equation \( 1.5 \) means that
\[ W(X^{-1}, Y^{-1}) = (-1)^a X^b Y^c W(X, Y) \]
for suitable \( a, b \in \mathbb{N}_0 \) and \( c \in \mathbb{N}_0^{Q_0} \). It is easy to exhibit small examples of quiver representations which violate this hypothesis; see, for instance, Proposition \( 3.13 \).
general interpretation of the symmetry expressed in (1.5) therefore refers to Denef-type formulae for the zeta functions \( \zeta_{V(\mathcal{O})}(s) \), viz. finite sums involving rational functions \( W_i(X, Y) \in \mathbb{Q}(X, Y) \) as above, but also the numbers of \( \mathcal{O}/\mathfrak{P} \)-rational points of (the reductions modulo \( \mathfrak{p} \) of) finitely many smooth projective varieties associated with the representation \( V \). The precise definition of the operation \( q \rightarrow q^{-1} \) involves the inversion of the Frobenius eigenvalues whose alternating sums yield the relevant numbers of rational points, by the Weil conjectures. That this operation is well-defined, i.e. independent of the choice of Denef-type formula, follows from (the straightforward multivariate refinements of the arguments given in) \cite{29} Section 4. We refer to \cite{40} Remark 1.7 for further details.

**Remark 1.11.** The finitely many prime ideals \( \mathfrak{p} \) of \( \mathcal{O} \) we are forced to disregard in Theorem 1.7 are essentially those for which a chosen principalization of ideals of \( a \)—in general very complicated—algebraic variety has bad reduction modulo \( \mathfrak{p} \); cf. Section 1.2. We know of no bounds on the size or shape of this finite set of “bad” prime ideals. Examples show, however, that it is not just an artefact of our method of proof, but non-empty in general.

Condition 1.6 is stable under taking direct products; see also \cite{40} Remark 1.8. Our methodology seems to give us no handle, however, on the sets of prime ideals which are bad for a direct product of representations in terms of the sets of bad prime ideals of the factors involved. Interesting specific questions arise in the context of base extension and restriction of scalars of quiver representations over global rings of integers. Assume, to be specific, that \( \mathcal{O} \hookrightarrow \mathcal{O}' \) is an extension of global rings of integers. Recall that by \( V(\mathcal{O}')_\mathcal{O} \) we denote the restriction of scalars to \( \mathcal{O} \) of the \( \mathcal{O}' \)-representation \( V(\mathcal{O}') = V \otimes_\mathcal{O} \mathcal{O}' \) obtained by extension of scalars. For a non-zero prime ideal \( \mathfrak{p} \in \text{Spec}(\mathcal{O}) \), write \( \mathfrak{p}\mathcal{O}' = \prod_{i=1}^s \mathfrak{P}_i^{e_i} \). Then \( V(\mathcal{O}') \otimes_\mathcal{O} \mathcal{O}_\mathfrak{p} = \times_{i=1}^s V(\mathcal{O}'_{\mathfrak{P}_i})_{\mathcal{O}_p} \), where \( V(\mathcal{O}'_{\mathfrak{P}_i})_{\mathcal{O}_p} \) denotes—in analogy to the above—the restriction of scalars to \( \mathfrak{p} \) of the \( \mathcal{O}'_{\mathfrak{P}_i} \)-representation \( V(\mathcal{O}'_{\mathfrak{P}_i}) \). Thus

\[
(1.8) \quad \zeta_{V(\mathcal{O}')}_{\mathcal{O}}(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O}) \setminus \{(0)\}} \zeta_{V(\mathcal{O}') \otimes_\mathcal{O} \mathcal{O}_p}(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O}) \setminus \{(0)\}} \zeta_{\times_{i=1}^s V(\mathcal{O}'_{\mathfrak{P}_i})_{\mathcal{O}_p}}(s).
\]

The complexity of the Euler factors in (1.8) may grow dramatically with the degree of the extension \( \mathcal{O}'/\mathcal{O} \), even for “small” representations \( V \), such as the one in Example 1.12. Large further classes of explicit examples are covered in \cite{4} Theorem 1.2 (=\cite{5} Theorem 2.2), albeit only in the univariate case. In general, it seems plausible that the bad prime ideals of \( \zeta_{V(\mathcal{O}')}_{\mathcal{P}_i}(s) \) are either ramified in \( \mathcal{O}'/\mathcal{O} \) or lie above bad primes of \( \zeta_{V(\mathcal{O})}(s) \).

1.3. **Submodule zeta functions.** In this section we explain how univariate quiver representation zeta functions may, equivalently, be seen as submodule zeta functions, and how Theorem 1.7 generalizes previous results about them. We put particular focus on (graded) ideal zeta functions of (anti-)commutative rings. As arguably these classes of zeta functions arise from the most natural counting problems to which the framework of quiver representation zeta functions applies, these provide natural motivation for the vantage point developed in this paper.

1.3.1. **Submodule zeta functions and loop quiver representations.** Let \( V \) be a finitely generated module over a ring \( R \) and \( \Omega \subseteq \text{End}_R(V) \) be a set of \( R \)-endomorphisms of \( V \); see \cite{28} Section 2.2. The (submodule) zeta function of \( \Omega \) acting on \( V \) is the Dirichlet
generating series enumerating the finite-index $\Omega$-invariant submodules $U$ of $V$:
$$
\zeta_{\Omega \to V}(s) = \sum_{U \leq V} |V : U|^{-s}.
$$

If the associative algebra generated by $\Omega$ in $\text{End}_R(V)$ can be generated by $d$ elements $c_1, \ldots, c_d$, then clearly $\zeta_{\Omega \to V}(s) = \zeta_{(\Omega(c_i))_{i=1}^d}(s)$. Submodule zeta functions are therefore representation zeta functions of loop quivers, viz. quivers $L_d$ with one vertex and $d$ loops. Recall that, in this setup, the distinction between multi- and univariate representation zeta functions is mute.

The case $d = 1$ has been treated exhaustively by Rossmann. In [20, Theorem A] he gives a fully explicit formula for the submodule zeta function of any integral $L_1$-representation in terms of Dedekind zeta functions of number fields and combinatorial data. These two types of ingredients reflect the Jordan decomposition of the endomorphism representing the unique loop into a semi-simple and a nilpotent part.

For $d \geq 2$, an important class of examples arises from ideal zeta functions of rings, defined as follows. Let $L$ be a commutative or anti-commutative ring (for instance, a Lie ring), of finite additive rank. It is easy to see that the ideal zeta function $\zeta_L^\Omega(s)$ of $L$, enumerating the (two-sided) ideals of finite index in $L$, is the zeta function of the adjoint representation $\text{ad}(L) \subseteq \text{End}_\mathbb{Z}(L)$ of $L$:
$$
\zeta_L^\Omega(s) = \zeta_{\text{ad} L \to L}(s).
$$

If $L$ is $d$-generated as a ring, then $\zeta_L^\Omega(s)$ is the zeta function associated with a representation of $L_d$ and thus fits into the general framework developed in this paper. The representation is nilpotent if and only if the ring $L$ is nilpotent.

**Example 1.12 (Heisenberg).** Consider the $\mathbb{Z}$-representation $V = (\mathbb{Z}^3, f_1, f_2)$ of $L_2$ by $\mathbb{Z}^3$ and the two endomorphisms
$$
f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

One checks easily that, for any cDVR $\mathfrak{o}$, subrepresentations of $V(\mathfrak{o})$ are in fact exactly the ideals of the Heisenberg Lie ring $\mathfrak{h}(\mathfrak{o}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. In fact,
$$
\zeta_{\mathfrak{V}(\mathfrak{o})}(s) = \zeta_{\mathfrak{h}(\mathfrak{o})}(s) = \frac{1}{(1 - q^{-s})(1 - q^{1-s})(1 - q^{2-3s})};
$$
cf. [17, Section 8]. The endomorphisms $f_i$ are the linear maps $\text{ad} x_i$, written with respect to the $\mathbb{Z}$-basis $(x_1, x_2, y)$ of $\mathfrak{h}(\mathbb{Z}) = \langle x_1, x_2, y \mid [x_1, x_2] = y, y \text{ central}\rangle_\mathbb{Z}$.

The behaviour of the representation $V$ under base extension and restriction of scalars (cf. Remark [11]) is almost completely understood. Indeed, Schein and the second author computed the Euler factors $\zeta_{V(\mathfrak{O}) \otimes \mathbb{Z}_p}(s)$ in [18] for all rational primes $p$ which are unramified in the ring of integers $\mathfrak{O}$ in in [31]; in [32] they computed formulae for the non-split case. [31, Conjecture 1.4] would imply that the bad primes are exactly the ramified ones.

It has long been known that nilpotent submodule zeta functions may or may not satisfy the kind of local functional equation established in Theorem [17]. (One of the smallest examples where it fails is the filiform nilpotent Lie ring $\text{Fil}_4$; see Section 2.1.) The homogeneity condition [17] was first proposed by the second author as a sufficient
criterion for such functional equations: [40, Theorem 1.2] is the special case of Theorem 1.7 for loop quivers.

1.3.2. Graded submodule and graded ideal zeta functions. Let \( \mathcal{V} \) and \( \Omega = \langle c_1, \ldots, c_d \rangle \) be as in Section 1.3.1 and fix an \( R \)-module decomposition \( \mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_a \), not necessarily compatible with \( \Omega \). The associated (graded submodule) zeta function \( \zeta_{\Omega \rightarrow \mathcal{V}}^{gr}(s) \) of \( \Omega \) acting on \( \mathcal{V} \) is the Dirichlet generating series enumerating graded (or homogeneous) \( \Omega \)-invariant submodules of \( \mathcal{V} \); cf. [28, Remark 3.2]:

\[
\zeta_{\Omega \rightarrow \mathcal{V}}^{gr}(s) = \sum_{\mathcal{U} \subseteq s} |\mathcal{V} : \mathcal{U}|^{-s}.
\]

Graded submodule zeta functions, too, may be seen as zeta functions of quiver representations. Indeed, let \( \mathcal{Q} \) be the quiver with vertices \( Q_0 = \{1, \ldots, a\} \) and arrows \( Q_1 = \{ \varphi_{thk} \mid (t, h) \in [a]^2, k \in [d]\} \), viz. \( d \) arrows between any two vertices, represented by modules \( \mathcal{L}_h = \mathcal{V}_h \) for \( h \in [a] \) and, for \( t \in [a] \) and \( k \in [d] \), morphisms \( f_{\varphi_{thk}} = \pi_h(\mathcal{e}_k|\mathcal{V}_t) \), where \( \pi_h \) denotes the projection \( \mathcal{V} \rightarrow \mathcal{V}_h \). Then \( \zeta_{\Omega \rightarrow \mathcal{V}}^{gr}(s) \) is the univariate zeta function associated with this quiver representation over \( R \); the multivariate one yields the obvious multivariate refinement of \( \zeta_{\Omega \rightarrow \mathcal{V}}^{gr}(s) \).

Just as in the ungraded case discussed in Section 1.3.1 important examples arise from Dirichlet generating functions enumerating certain ideals. Let \( L \) be a ring as in Section 1.3.1 with a fixed \( \mathbb{Z} \)-module decomposition \( L = L_1 \oplus \cdots \oplus L_a \), not necessarily compatible with the multiplication of \( L \). Then the graded ideal zeta function \( \zeta_L^{gr}(s) \) of \( L \) with respect to \( L = \bigoplus_{h=1}^{a} L_h \), enumerating the graded ideals of finite index in \( L \) with respect to this decomposition, is the graded submodule representation of \( \text{ad} L \) acting on \( L = \bigoplus_{h=1}^{a} L_h \), i.e.

\[
\zeta_L^{gr}(s) = \zeta_{\text{ad} L \rightarrow L}(s).
\]

Assume now that \( L \) is a nilpotent Lie ring of finite additive rank \( n \) and nilpotency class \( c \) with lower central series \( (\gamma_i(L))_{i=1}^{c} \). For \( i \in [c] \), set \( L_i = \gamma_i(L)/\gamma_{i+1}(L) \). The associated graded Lie ring is \( \text{gr} L = \bigoplus_{i=1}^{c} L_i \). The graded ideal zeta function \( \zeta_L^{gr}(s) \) of \( L \) is the graded ideal zeta function of \( \text{gr} L \) with respect to the decomposition \( \text{gr} L = \bigoplus_{i=1}^{c} L_i \).

Theorem 1.7 implies that all the questions raised in [28, Question 10.2] have positive answers provided that Condition 1.5 is satisfied. We collect further consequences of Theorem 1.7 pertaining to (graded) ideal zeta functions in Section 2. The examples explained in Section 2.1 show, in particular, that the problem of counting graded ideals may be homogeneous in the sense of Condition 1.5 even when the problem of counting all ideals of finite index is not. In Corollary 2.1 we record, specifically, the generic functional equations of the local graded ideal zeta functions associated with the free nilpotent Lie ring \( f_{c,d} \) on \( d \) generators and of nilpotency class \( c \), for all \( d \) and \( c \).

Example 1.13 (graded Heisenberg). Consider the \( \mathbb{Z} \)-representation \( \mathcal{V} = (\mathbb{Z}^2, \mathbb{Z}, f_1, f_2) \) with \( \mathcal{L}_1 = \langle x_1, x_2 \rangle, \mathcal{L}_2 = \langle y \rangle \), and the two endomorphisms

\[
f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
One checks easily that, for any cDVR \( \mathfrak{o} \), subrepresentations of \( V(\mathfrak{o}) \) are in fact exactly the grade ideals of the Heisenberg Lie ring \( \mathfrak{h}(\mathfrak{o}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathfrak{o} \\ 0 & \mathfrak{o} & 0 \end{pmatrix} \). In fact,

\[
\zeta_{V(\mathfrak{o})}(s) = \zeta_{\mathfrak{h}(\mathfrak{o})}(s) = \frac{1}{(1 - q^{-s_1})(1 - q^{1-s_1})(1 - q^{-(2s_1+s_2)})}.
\]

### 1.3.3. Quiver representation zeta functions as submodule zeta functions

While univariate quiver representation zeta functions afford new perspectives on (graded) submodule zeta functions, the latter actually comprise the former. Indeed, given a quiver \( \mathcal{Q} \) and a ring \( R \), let \( \mathbf{P} \) be the path algebra of \( \mathcal{Q} \) over \( R \), i.e. the \( R \)-algebra generated by the paths in \( \mathcal{Q} \) with multiplication induced by concatenation of paths. Then \( \mathbf{P} \cong N \oplus R[Q_0] \), where \( N \) is the ideal generated by paths of positive length. Given a representation \( V \) of \( \mathbf{P} \) with underlying module \( \mathcal{L} \), the endomorphism algebra \( \mathcal{E} = \mathcal{E}(V) \) is the image of the natural map \( N \to \text{End}(\mathcal{L}) \) induced by \( V \). Clearly, \( \mathcal{E} \) is nilpotent if and only if only if \( V \) is. If we map paths of length zero, viz. vertices \( i \in Q_0 \), to the projections \( \pi_i : \mathcal{L} \to \mathcal{L}_i \) of \( \mathcal{L} \) onto the direct summands \( \mathcal{L}_i \) (followed by inclusion into \( \mathcal{L} \) ), then the representation zeta function \( \zeta_V(s) \) is the submodule zeta function associated with the image of (all of) \( \mathbf{P} \) within \( \text{End}(\mathcal{L}) \). We thank Tobias Rossmann for pointing this out to us.

### 1.3.4. Analytic properties

Univariate local submodule zeta functions are known to be expressible in terms of the \( p \)-adic cone integrals introduced in [15]; see [25, Theorem 2.6(ii)]. This implies, in particular, that univariate analogues of Euler products such as [1,2] have rational abscissa of convergence and allow for some meromorphic continuation; see [15, Theorem 1.5(1)] for the case \( R = \mathbb{Z} \) (the proof extends easily to general rings of integers). A Tauberian theorem thus gives asymptotic estimates for the partial sums \( \sum_{i \leq m} a_i(V) \) in terms of the position and order of the right-most pole of \( Z_V(s) \); cf. [15, Theorem 1.5(2)]. It would be of interest to explore extensions of these results to the multivariate setting. While there are results on the domain of meromorphy of multivariate Euler products (see, for instance, [12]), applications to counting functions akin to quiver representation zeta functions seem to be thin on the ground; see [12] for results on bivariate representation and conjugacy class zeta functions.

### 1.3.5. Further refinements

The multivariate representation zeta function associated with a quiver representation [1,1] could be further refined to take into account (aspects of) the elementary divisor types of the lattices \( \Lambda_i \). For the “abelian” case \(|Q_0| = 1, |Q_1| = 0 \); cf. Example [1,1] this is done in [24]; see also [8]. It would be of interest to determine to what extent results such as Theorem 1.7 hold in this even finer setup. The methodology of the current paper does not seem appropriate for this task in general.

### 1.4. Methodology and organization

Our proof of Theorem 1.7 is based on generalizations of techniques and results from [39] and [40]. Section 4 develops \( p \)-adic machinery which we use to prove the result in Section 5.

In [10] the problem of enumerating submodules invariant under “homogeneous” nilpotent algebras of endomorphisms was approached using results of [39]. To this end, the problem was reformulated in terms of integer-valued “weight functions” on the vertex sets of the affine Bruhat-Tits buildings associated with groups of the form \( GL_n(K) \) for a local field \( K \), viz. homothety classes of lattices inside \( K^n \). These weight functions were then shown to be amenable to versions of a very general “blueprint result” from [39], establishing functional equations for certain \( p \)-adic integrals. This result has been used
and developed extensively to prove such functional equations for various kinds of zeta functions associated with groups, rings, and modules; see, for instance, [11, 35, 27, 19] for related work.

In Section 5 we rephrase the problem of enumerating subrepresentations of homogeneous nilpotent quiver representations in terms of weight functions on sets of tuples of full p-adic lattices, generalizing those introduced in [30]. In Section 4 we generalize the p-adic blueprint result from [39] with a view towards these generalized weight functions.

As many of their precursors, the p-adic (i.e. local) integrals in Section 4 are obtained by localizing globally defined data. Our results about these integrals typically only apply to all but finitely many bad places, which explains the need to exclude finitely many prime ideals in Theorem 1.7. Corollary 1.8 follows immediately from the formulae for the generic (“good”) places by invoking the transfer principle from model theory, which justifies the “transfer” between formulae for p-adic integrals in characteristic zero and their analogues in positive characteristic, provided the residue field characteristic is sufficiently large.

In Sections 4.1 and 5.1 we provide informal overviews of the two sections which, taken together, form the technical core of this paper.

In Sections 4 and 5 we discuss our main result Theorem 1.7 in some special contexts, viz. (graded) ideal zeta functions of nilpotent Lie rings and certain specific quiver representations which do not come from this classical setup. While some of the examples we develop in Section 4 are of a combinatorial nature, others give an inkling of the subtlety of the variation of the Euler factors of a global quiver representation zeta function with the place: general formulae involve, in an essential way, the numbers of rational points of algebraic varieties over finite fields. This notwithstanding, both sections may be read independently of Sections 4 and 5.

1.5. Notation. By $R$ we denote a ring. Usually, it will either be the ring of integers $\mathcal{O}$ of a global field or a compact discrete valuation ring (cDVR), either of characteristic zero (such as the completion $\mathcal{O}_p$ of $\mathcal{O}$ at a non-zero prime ideal $p$ of $\mathcal{O}$) or of positive characteristic (such as the ring of formal power series $\mathbb{F}_q[[T]]$ over a finite field $\mathbb{F}_q$). We write $p$ for the maximal ideal of a cDVR $\mathcal{O}$ and $q$ resp. $p$ for its residue field’s cardinality resp. characteristic. By $v$ or $v_p$ we denote the (normalized) p-adic valuation on $\mathcal{O}$, but also, by extension, on vectors and matrices over $\mathcal{O}$: if $x = (x_1, \ldots, x_a) \in \mathcal{O}^a$, then $v(x) = \min \{v(x_i) \mid i \in \{1, \ldots, a\}\}$. Occasionally we refer to a uniformizer $\pi$ of $\mathcal{O}$, viz. an element $\pi \in \mathcal{O}\setminus\mathcal{O}^\times$. In general, however, the notation $p^n$ refers to the Cartesian product $p \times \cdots \times p$ with $m$ factors. We trust that the respective contexts will prevent misunderstandings. By $K$ we denote a field, usually a global or local field such as the field of fractions of $R$.

Given matrices $A$ and $B$ over $\mathcal{O}$ with the same number of columns, we write $A \leq B$ if each row of $A$ is contained in the $\mathcal{O}$-row span of $B$.

Throughout, $\mathbb{Q}$ will be a quiver, with vertices $Q_0$ and arrows $Q_1$. Often we will write $a = |Q_0|$ and $b = |Q_1|$. Special classes of quivers discussed include, among others, loop quivers $L_d$, Kronecker quivers $K_n$, star quivers $S_n$ and their duals $S_n^*$. By $V$ we will denote a representation of a quiver over a ring $R$.

We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers and set $X_0 = X \cup \{0\}$ for a subset $X \subset \mathbb{N}$. Given $n \in \mathbb{N}_0$, we write $[n] = \{1, \ldots, n\}$; for $m, n \in \mathbb{N}_0$ we write $[m, n] = \{m + 1, m + 2, \ldots, n\}$. The power set of a set $X$ is denoted $\mathcal{P}(X)$. We write $I = \{i_1, \ldots, i_\ell\} \subset \mathbb{N}_0$ to stress that $i_1 < \cdots < i_\ell$. We write $t = q^{-s}$, where $s$ is a
complex variable. Given numbers $x_1, x_2, \ldots$ and multiplicities $e_1, e_2 \in \mathbb{N}_0$ we set

$$(x_1^{(e_1)}, x_2^{(e_2)}, \ldots) = \left(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots\right)_{e_1 \text{ times } e_2 \text{ times}}.$$ 

Given a property $\varphi$, the “Kronecker delta” $\delta_\varphi$ is equal to 1 if $\varphi$ holds and equal to 0 otherwise.

2. Graded ideal zeta functions of nilpotent Lie rings

In this section we discuss our main result in the light of some special classes of graded ideal zeta functions of nilpotent Lie rings. For simplicity and to ease comparison with the existing literature on these zeta functions we restrict ourselves to univariate zeta functions.

2.1. Fil$_4$ vs. M$_4$

Consider the class-4-nilpotent Lie ring

$$\text{Fil}_4 := \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle.$$ 

(By convention, all commutator relations other than those following from the given ones are assumed to be trivial.) It is known that its local ideal zeta functions $\zeta_{\text{Fil}_4}^p(s)$ do not satisfy a functional equation of the form described in Theorem 1.7; see [16, Theorem 2.39] and [40, Example 4.1]. Informally speaking, the homogeneity condition is violated by the “inhomogeneity” of the map $\text{ad} x_2$. Recall from Section 1.3.1 that this is an example of a zeta function of a representation of $L_2$:

$$\begin{array}{c}
\text{ad} x_1 \\
\downarrow \\
\text{Fil}_4 \cong \mathbb{Z}^5 \\
\uparrow \\
\text{ad} x_2
\end{array}$$

Consider now the associated graded Lie ring

$$\text{gr Fil}_4 = \langle x_1, x_2 \rangle \oplus \langle x_3 \rangle \oplus \langle x_4 \rangle \oplus \langle x_5 \rangle =: \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4.$$

(By slight abuse of notation we continue to write here $x_i \in \gamma_j(\text{Fil}_4)$ for its image in $\gamma_j(\text{Fil}_4)/\gamma_{j+1}(\text{Fil}_4)$.) The graded ideal zeta function $\zeta_{\text{gr Fil}_4}^p(s)$ is the zeta function of the following quiver representation:

$$
\begin{array}{c}
\text{ad} x_2|_{\mathcal{L}_1} \\
\downarrow \\
\mathcal{L}_1 \\
\uparrow \\
\text{ad} x_1|_{\mathcal{L}_1} \\
\downarrow \\
\mathcal{L}_2 \\
\uparrow \\
\text{ad} x_2|_{\mathcal{L}_2} \\
\downarrow \\
\mathcal{L}_3 \\
\uparrow \\
\text{ad} x_1|_{\mathcal{L}_3} \\
\downarrow \\
\mathcal{L}_4 \\
\uparrow \\
\text{ad} x_1|_{\mathcal{L}_4}
\end{array}$$

However, the “inhomogeneous arrow” $\text{ad} x_2|_{\mathcal{L}_2}$ is redundant, as obviously $\text{ad} x_2|_{\mathcal{L}_2} = (\text{ad} x_1)^2|_{\mathcal{L}_2}$. Omitting it yields the quiver representation modelling the graded ideal zeta function of the maximal class Lie ring

$$(2.1) \quad M_4 := \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 \rangle.$$ 

The latter is easily seen to satisfy the conditions of Theorem 1.7. For the simple explicit formula of the graded ideal zeta function, see [28, Proposition 3.5], where $M_4$ goes by the name $\mathfrak{m}(4)$.}
2.2. Graded ideal zeta functions of free nilpotent Lie rings. In [22] we investigated graded ideal zeta functions \( \zeta_{f,c,d}(s) \) associated with the free nilpotent Lie rings \( f_{c,d} \) on (Lie) generators \( x_1, \ldots, x_d \), of nilpotency class \( c \), for various rings \( R \). By the above, these zeta functions may be interpreted as zeta functions of integral (adjoint) representations of quivers on \( x \) along rational functions, valid for all residue characteristics, spelling out the cases associated with \( F \) representations of quivers above, these zeta functions may be interpreted as zeta functions of integral (adjoint) local functional equations for \( c \). In our application, as shown in [22], the vertices \( v_i \) \((i \in [c])\) are represented by free \( R \)-modules of ranks \( W_d(i) = \frac{1}{r} \sum_{j|i} \mu(j)d^{i/j} \), viz. \( \mathcal{L}_i := \gamma_i(f_{c,d})/\gamma_{i+1}(f_{c,d}) \otimes \mathbb{Z} R \), where \( \mu \) denotes the Möbius function; the arrows are represented by the maps \( ad_xk|_{\mathcal{L}_i} : \mathcal{L}_i \to \mathcal{L}_{i+1} \), where \( k \in [d] \).

In [22] Theorem 1.1] we recorded a formula for the graded ideal zeta functions \( \zeta_{f,c,d}(s) \), valid for all cDVRs \( \mathfrak{a} \). The relevant representations are all of the quiver \( F_{3,3} \) (see Figure 1), with rank vectors \((W_3(1), W_3(2), W_3(3)) = (3,3,8) \). The paper also contains explicit formulae for all \( c \leq 2 \) and \((c,d) \in \{(3,3), (3,2), (4,2)\} \). For larger values of \( c \) and \( d \), explicit computations seem currently out of reach.

We made several general conjectures about graded ideal zeta functions associated with free nilpotent Lie rings in [22, Section 6]. The one pertaining to local functional equations is implied by Theorem 1.7.

**Corollary 2.1.** [22, Conjecture 6.2] For almost all primes \( p \) and all cDVRs \( \mathfrak{a} \) of residue field cardinality \( q \) and residue field characteristic \( p \),

\[
\zeta_{f,c,d}(s)|_{q^{-1}} = (q-1)^r q^{\sum_{i=1}^c \left( \left( \binom{W_d(i)}{2} - (c+1-i)W_d(i) \right) \right)} \zeta_{f,c,d}(s).
\]

Indeed, the adjoin representation is clearly homogeneous in the sense of Condition [1,4, Corollary 2.1] is a graded analogue of [40, Theorem 4.4].

2.3. Some Lie rings of maximal class and their amalgams. Given an (integer) partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r \), with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \), consider the class-\( \lambda_1 \)-nilpotent Lie ring

\[
\mathcal{L}_\lambda = \langle x_0, \{x_{i,j} \}_{i \in [r], j \in [\lambda_i]} | \forall i \in [r], j \in [\lambda_i-1] : [x_0, x_{i,j}] = x_{i,j+1} \rangle \mathbb{Z},
\]
on \( 1 + r \) Lie generators and of \( \mathbb{Z} \)-rank \( 1 + \sum_{i=1}^r \lambda_i \); see also [40, Section 4.3]. For \( r = 1 \) and \( \lambda_1 \geq 2 \) this yields the Lie ring \( M_{\lambda_1} \) of maximal class \( \lambda_1 \) described in [16, p. 99]; for \( M_4 \), see [21]. In general, \( \mathcal{L}_\lambda \) is obtained by amalgamating several such Lie rings along \( x_0 \). Consider the special case \( \lambda = e_{(r)} = (e, \ldots, e) \) of rectangular partitions. In [21, Chapter 4], the first author investigated graded ideal zeta functions \( \zeta_{f,c,d}(s) \) associated with \( \mathcal{L}_{e_{(r)}} \). He derived explicit (if involved) combinatorial formulæ for these rational functions, valid for all residue characteristics, spelling out the cases \((e, r) \in \{(3,2), (3,3), (3,4), (4,2), (5,2)\} \) in detail. By the above, these zeta functions may be interpreted as zeta functions of integral (adjoint) representations of quivers on \( c \) vertices \( v_1, \ldots, v_c \), represented by free \( R \)-modules of ranks \( r + \delta_{1,1} \), with exactly \( 1 + r \delta_{1,1} \) arrows between \( v_i \) and \( v_{i+1} \) for all \( i \in [c-1] \). In [21, Theorem 4.2.12], he established general local functional equations for \( c = 3 \). In fact, his combinatorial approach easily extends to cover near rectangular partitions \( \lambda = (e_{(r_1)}, 1^{(r_2)}) \), where \( r_1, r_2 \in \mathbb{N}_0 \). Theorem 1.7 yields the following.
Corollary 2.2. Let $c \in \mathbb{N}$ and $r_1, r_2 \in \mathbb{N}_0$. For all cDVRs $\mathfrak{o}$ of residue field cardinality $q$

$$
\zeta_{\mathcal{L}_{\mathfrak{o}}(c^{(r_1)}, 1^{(r_2)})}(s)|_{q \to q^{-1}} = 
\left(-1\right)^{1+cr_1+r_2}q^{\left(\binom{r_1+1}{2} + (c-1)(\binom{r_2}{2})\right)}(c^{r_1+1}r_1+r_2)\zeta_{\mathcal{L}_{\mathfrak{o}}(c^{(r_1)}, 1^{(r_2)})}(s).
$$

Indeed, as in [34, Lemma 4.7] one checks that the adjoint representation is homogeneous in the sense of Condition 1.5 if and only if $\lambda$ is a near rectangle. Graded ideal zeta functions of Lie rings arising from non-near rectangular partitions thus do not fall in the remit of the current paper’s methodology. Corollary 2.2 is a graded analogue of [40, Theorem 4.8]. As in the “ungraded” case treated in [40, Question 4.9] it is natural to ask whether the “near rectangle”-condition is necessary for local functional equations for the graded ideal zeta functions $\zeta_{\mathcal{L}_{\mathfrak{o}}(s)}(s)$.

2.4. Further examples. Univariate quiver representation zeta functions being submodule zeta functions (cf. Section 1.3.3) makes them amenable to the functionality of Rossmann’s computer algebra package Zeta ([30]). In [28, Section 10], Rossmann reports numerous computations of graded ideal zeta functions obtained with Zeta. It seems noteworthy that the formulae listed in [28, Table 2] satisfy the relevant functional equation if and only if (!) they satisfy Condition 1.5 with respect to the bases suggested in [20, Section 5.1]. These observations support the speculation that homogeneity may also be a necessary condition for local functional equations. In Section 3.1 we discuss a class of nilpotent quiver representations where this is provably the case, at least in the univariate setup; see Theorem 3.2.

3. INTEGRAL NILPOTENT QUIVER REPRESENTATIONS BEYOND NILPOTENT LIE RINGS

In this section we discuss examples of univariate quiver representation zeta functions not arising from the “classical” context of (graded) ideal zeta functions of nilpotent Lie rings. Whenever these examples exhibit functional equations of the form (1.6), they are explained by Theorem 1.7. Section 3.1 explores connections between integral thin representations of Hasse quivers of posets and $P$-partitions, a combinatorial concept. Representations of star quivers star in Section 3.2, their duals in Section 3.3; representations of Kronecker quivers are the subject of Section 3.4. Throughout, $\mathfrak{o}$ denotes a cDVR of arbitrary characteristic, with maximal ideal $\mathfrak{p}$ and residue field cardinality $q$.

3.1. Integral thin representations of Hasse quivers and $P$-partitions. The exposition in this section leans closely on [34, Sec. 13.5], to which we refer for further details. Let $P$ be a partially ordered set (or poset) of cardinality $n$. Without loss of generality $P$ is a natural partial order on $[n]$, i.e. if $i <_P j$ then $i < j$ for all $i, j \in P$. Recall that a $P$-partition of $m \in \mathbb{N}_0$ is an order-reversing (!) map $\sigma : P \to \mathbb{N}_0$ satisfying $|\sigma| := \sum_{i \in P} \sigma(i) = m$. (The labelling $\omega$ referred to in [34] is subsumed by our identification of $P$ with $[n]$.) We write $a_{m, P} := \#\left\{ \sigma \mid |\sigma| = m \right\}$ for the number of $P$-partitions of $m$. As in [34, (3.62)] we set

$$
G_P(X) := \sum_{m=0}^{\infty} a_{m, P}X^m,
$$

for a variable $X$. The following result is [34, Theorem 3.15.7]. It establishes that this generating function is, in fact, rational and expresses it in terms of the major index $\text{maj}$ on $S_n$ (see [34, Sec. 1.4]), restricted to the subset $L(P) \subseteq S_n$ of linear extensions of $P$. 
Figure 2. The star quiver $S_4$

\[ \text{Theorem 3.1 (R. Stanley).} \]

\[ G_P(X) = \frac{\sum_{\pi \in \Pi(P)} X^\text{maj}(\pi)}{\prod_{i=1}^{n}(1 - X^i)}. \]

Recall further that $P$ satisfies the $\delta$-chain condition if, for all $x \in P$, all maximal chains in the principal dual order ideals $\{x' \in P \mid x' \geq_P x\}$ have the same length. For $x \in P$, let $\delta(x)$ denote the length of a longest chain in $P$ starting at $x$. Set further $\delta(P) = \sum_{x \in P} \delta(x)$. The following result is [34, Theorem 3.15.16].

\[ \text{Theorem 3.2 (R. Stanley). The poset } P \text{ satisfies the } \delta \text{-chain condition if and only if the following functional equation holds:} \]

\[ G_P(X^{-1}) = (-1)^n X^{\delta(P)} G_P(X). \]

In fact, $P$-partitions may be viewed as integral thin quiver representations of Hasse quivers of posets. Indeed, let $Q_P$ be the Hasse quiver of $P$. It has vertices $Q_P = \{0\}$ and arrows $Q_{P,1} = \{(i,j) \mid i <_P j\}$, where $i <_P j$ means that $j$ covers $i$ in $P$, i.e. $i <_P j$ and $\exists k \in P : i <_P k <_P j$. Consider the thin $\mathbb{Z}$-representation $V_P$ of $Q_P$ where all arrows are represented by identity maps. Then, for every cDVR $\mathfrak{o}$ with maximal ideal $\mathfrak{p}$, we find that

\[ (3.1) \quad \zeta_{V_P(\mathfrak{o})}(s) = \sum_{(e_1,\ldots,e_a) \in \mathbb{N}_0^a} \prod_{\mathfrak{e}_j \leq e_j \text{ if } i <_P j} k=1^{n} |\mathfrak{o} : \mathfrak{o}^{e_k}|^{-s} = \sum_{m=0}^{\infty} a_{m,P} t^m = G_P(t). \]

It is easy to check that the endomorphism algebra $\mathcal{E}(V_P)$ is homogeneous in the sense of Condition [1,3] if and only if $P$ satisfies the $\delta$-chain condition. Our Theorem [1,7] thus implies (even a multivariate version of) the “only if”-part of Theorem [3,2]. We find it remarkable that, at least in this combinatorially tightly costrained setup, the homogeneity condition is also necessary for a generic local functional equation. It is of great interest to determine the precise extension of this phenomenon, both in the multivariate and univariate setup.

3.2. Star quivers. Let $a \in \mathbb{N}$ and consider the quiver $S_a$ consisting of $a$ vertices $v_1,\ldots,v_a$ and $a - 1$ arrows, all pointing away from the central vertex $v_1$, describing a “star with $a - 1$ rays”. In the terminology of Section 3.1 $S_a$ is the Hasse quiver of the poset obtained from an antichain on $a - 1$ vertices, augmented by a minimal element $0$.

We consider $\mathfrak{o}$-representations $V_{m,a}(\mathfrak{o})$ of $S_a$, where every vertex is represented by $\mathfrak{o}^m$ and every arrow by the identity map.

3.2.1. $m = 1$. In this case, we obtain a family of thin $\mathfrak{o}$-representations of $S_a$. Recall, say from [2,3], Carlitz’ $q$-Eulerian polynomial

\[ C_{a-1}(x,q) = \sum_{w \in S_{a-1}} x^{\text{des}(w)} q^{\text{maj}(w)} \in \mathbb{Z}[x,q]. \]
where des is the descent statistic and maj the major index on the symmetric group $S_{a-1}$; see, for instance, [31, Sec. 1.4].

**Proposition 3.3.**

\[
\zeta_{V_{1,a}}(s) = \frac{C_{a-1}(t, t)}{\prod_{i=1}^{a}(1 - t^i)}.
\]

**Proof.** We have

\[
\zeta_{V_{1,a}}(s) = \sum_{r=0}^{\infty} t^r \left( \frac{1 - t^{r+1}}{1 - t} \right)^{a-1} = \frac{C_{a-1}(t, t)}{\prod_{i=1}^{a}(1 - t^i)},
\]

where the first equality follows from the definition of $\zeta_{V_{1,a}}(s)$ and MacMahon’s ([23, §462, Vol. 2, Ch. IV, Sect. IX] see also [31, (2.3)]) implies the second one.

Alternatively, the statement follows easily by combining Theorem 3.1 and 3.1. □

**Remark 3.4.**

(1) Note that $\zeta_{V_{1,a}}(s)$ has a pole of order $a$ at $t = 1$ and $(1 - t)^a \zeta_{V_{1,a}}(s)|_{t=1} = \frac{1}{a}$.

(2) The coefficients $n_{a,i}$ of the generating function $\zeta_{V_{1,a}}(s) = \sum_{i=0}^{\infty} n_{a,i} t^i$ are the numbers of compositions of $i$ into $a$ parts whose first part is maximal.

**Example 3.5.** The following formulae were obtained using Maple\(^1\):

\[
\begin{align*}
(1) \quad & \zeta_{V_{1,1}}(s) = \frac{1}{1-t} \\
(2) \quad & \zeta_{V_{1,2}}(s) = \frac{1}{(1-t)(1-t^2)} \\
(3) \quad & \zeta_{V_{1,3}}(s) = \frac{1}{(1-t)(1-t^2)(1-t^3)} \\
(4) \quad & \zeta_{V_{1,4}}(s) = \frac{1}{(1-t)(1-t^2)(1-t^3)(1-t^4)} \\
(5) \quad & \zeta_{V_{1,5}}(s) = \frac{1}{(1-t)(1-t^2)(1-t^3)(1-t^4)(1-t^5)} \\
(6) \quad & \zeta_{V_{1,6}}(s) = \frac{1}{(1-t)(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)} \\
(7) \quad & \zeta_{V_{1,7}}(s) = \frac{N_{1,7}(q, t)}{\prod_{i=1}^{7}(1-t^i)},
\end{align*}
\]

where

\[
N_{1,7}(q, t) = 1 + 5t^2 + 14t^3 + 19t^4 + 24t^5 + 40t^6 + 66t^7 + 80t^8 + 76t^9 + 70t^{10} + 76t^{11} + 80t^{12} + 66t^{13} + 40t^{14} + 24t^{15} + 19t^{16} + 14t^{17} + 5t^{18} + t^{20}.
\]

3.2.2. $m = 2.$ Given an integer partition $\lambda = (\lambda_1, \lambda_2)$, let $\zeta_{\lambda, o}(s)$ be the Dirichlet polynomial enumerating the $o$-submodules of the finite $o$-module $o/p^{\lambda_1} \times o/p^{\lambda_2}$; if $o = \mathbb{Z}_p$, this is just the “subgroup zeta function” enumerating the subgroups of the finite abelian $p$-group $C_{p^{\lambda_1}} \times C_{p^{\lambda_2}}$. It is easy to write down an explicit formula for $\zeta_{\lambda, o}(s)$—in principle also for a general partition—using, e.g., a formula due to Birkhoff; see, for example, [31, Section 2.5]. It is obviously a polynomial in $q$ and $t = q^{-s}$. Given a lattice $\Lambda \leq o^2$ of finite index, we write $\lambda(o^2/\Lambda) = \lambda$ if $o^2/\Lambda$ has type $\lambda$.

**Proposition 3.6.**

\[
\zeta_{V_{2,a}}(s) = \sum_{r_0=0}^{\infty} t^{2r_0} \left( (\zeta_{(r_0, r_0), o}(s))^{a-1} + (1 + q^{-1}) \sum_{r_1=1}^{\infty} (qt)^{r_1} (\zeta_{(r_0+r_1, r_0), o}(s))^{a-1} \right).
\]

\(^1\)Maple is a trademark of Waterloo Maple Inc.
for local zeta functions of star quiver representations of the form

\[ \zeta(s^2 : \Lambda|^{-s} = \sum_{\Lambda \leq s^2} \prod_{i \in [a]} |\sigma^2 : \Lambda_i|^{-s} = \sum_{\Lambda \leq s^2} |\sigma^2 : \Lambda_1|^{-s} \left( \zeta(s^2/\Lambda_1, \sigma(s)) \right)^{a - 1}. \]

The statement now follows from the well-known formula \( (r_0, r_1) \in \mathbb{N}_0 \)

\[ \# \{ \Lambda \leq s^2 \mid \lambda(s^2/\Lambda) = (r_0 + r_1, r_0) \} = \begin{cases} 1, & \text{if } r_1 = 0, \\ (1 + q^{-1})q^{-1}, & \text{if } r_1 > 0. \end{cases} \]

Example 3.7. The following formulae were obtained using Maple.

1. \( \zeta_{V_2^1}(s)(s) = \frac{1}{(1-t)(1-qt)} \)
2. \( \zeta_{V_2^2}(s)(s) = \frac{1}{(1-t)(1-t^2)(1-qt)(1-qt^2)} \)
3. \( \zeta_{V_2^2}(s)(s) = \frac{1}{(1-t)(1-t^2)(1-t^3)(1-qt)(1-qt^2)(1-qt^3)} \)
4. \( \zeta_{V_2^4}(s)(s) = \frac{N_{2,4}(q,t)}{(1-t)^2(1-t^5)(1-t^4)(1-qt^2)(1-qt^3)(1-qt^4)(1-qt^5)} \)
   where

\[ N_{2,4}(q,t) = 1 - t + 3t^2 + qt^2 - t^3 + q^2t^4 - qt^4 + t^4 + qt^6 + qt^7 - 3q^2t^7 - 2q^3t^7 - 5qt^8 - 2q^3t^9 + q^2t^9 + 2qt^9 - 2q^4t^{10} - q^3t^{10} + 2q^2t^{10} + 5q^4t^{11} - q^4t^{12} + 3q^3t^{12} + 2q^2t^{12} + 5q^4t^{13} - q^4t^{14} - q^5t^{15} + q^4t^{15} - q^3t^{15} + q^5t^{16} - q^4t^{17} - 3q^5t^{17} + q^5t^{18} - q^5t^{19}. \]

Remark 3.8. Viewing \( (3) \) as a \((q\text{-}\text{analogue of a) generalization of MacMahon’s formula})\), it remains a challenge to compute and interpret combinatorially formulae for local zeta functions of star quiver representations of the form \( V_{m,a}(\sigma) \).

3.3. Dual star quivers. For \( a \in \mathbb{N} \) let \( S_a^* \) be the dual of the quiver \( S_a \), consisting of \( a \) vertices \( v_1, \ldots, v_a \) and \( a - 1 \) arrows, all pointing towards the central vertex \( v_1 \). In the terminology of Section 3.1, \( S_a^* \) is the Hasse quiver of the poset obtained from an antichain on \( a - 1 \) vertices, augmented by a maximal element \( \bar{1} \).

3.3.1. We first consider dual representations \( V_{m,a}^*(\sigma) \) of the representations \( V_{m,a}(\sigma) \) introduced in Section 3.2. Recall that every vertex is represented by \( \sigma^m \) and every arrow by the identity map. It turns out that—in contrast to the zeta functions \( \zeta_{V_{m,a}(\sigma)}(s) \) discussed in Section 3.2—the associated zeta functions have a rather simple form.

Proposition 3.9.

\[ \zeta_{V_{m,a}^*(\sigma)}(s) = \zeta_{\sigma^m}(a s) \zeta_{\sigma^m}(s)^{a - 1}. \]
This follows immediately from the observation that
\[
\zeta_{V^{\ast}_{m,a}(\mathfrak{o})}(s) = \sum_{\Lambda_i \leq \mathfrak{o}^m, i \in [a], \Lambda_j \leq \mathfrak{a}^n, j \in [2, a]} |\mathfrak{o}^m : \Lambda_i|^{-s} = \sum_{\Lambda_i \leq \mathfrak{o}^m} |\mathfrak{o}^m : \Lambda_i|^{-s} \prod_{j=2}^{a} \sum_{\Lambda_j \leq \mathfrak{a}^n} |\Lambda_j : \mathfrak{a}^n|^{-s}. \]

\[\square\]

3.3.2. The following example hints at the general fact that, even in small examples, geometric as well as arithmetic considerations need to complement the combinatorial arguments we have encountered so far. This phenomenon will also feature in Section 3.3 and foreshadows the general situation set out in Sections 4 and 5.

Consider the $\mathbb{Z}$-representation $V$ of $\mathcal{S}_4^*$ described in Figure 4.

**Figure 4.** An integral representation of the dual star quiver $\mathcal{S}_4^*$

\[
\begin{array}{c}
\mathbb{Z} \\
\downarrow \text{id, id} \\
(0, \text{id}) \\
\downarrow \mathbb{Z}^2 \\
\downarrow \text{id, 0} \\
\mathbb{Z} \\
\end{array}
\]

**Proposition 3.10.**
\[
\zeta_{V(\mathfrak{o})}(s) = \frac{1 + 2t^3 - 2t^4 + t^7}{(1 - t)^3(1 - t^3)(1 - t^4)(1 - qt^3)}.
\]

**Sketch of proof.** For $a_1, a_2, a_3 \in \mathbb{N}_0$, let $m_1 = p^{a_1}, m_2 = p^{a_2}, m_3 = p^{a_3}$. Clearly
\[
\zeta_{V(\mathfrak{o})}(s) = \sum_{(a_1, a_2, a_3) \in \mathbb{N}_0^3, \Lambda \leq \mathfrak{o}^2} |\mathfrak{o}^2 : \Lambda|^{-s} \cdot \prod_{i=1}^{3} m_i^{a_i} = \sum_{(a_1, a_2, a_3) \in \mathbb{N}_0^3, \Lambda \leq \mathfrak{o}^2 \text{ maximal under } (\ast)} |\mathfrak{o}^2 : \Lambda|^{-s} \cdot \prod_{i=1}^{3} m_i^{a_i}.
\]

Assume that the maximal lattice $\Lambda \leq \mathfrak{o}^2$ is the row span of the matrix $M \in \text{Mat}_2(\mathfrak{o})$, encoding coordinates of vectors with respect to the standard basis. As in Section 5.3, we write $M = D\alpha^{-1}$ for $D = \text{diag}(\pi^r, 1)$ for $r \in \mathbb{N}_0$ and $\alpha = (\alpha_{ij}) \in \text{GL}_2(\mathfrak{o})$. Without loss of generality we may assume that $r > 0$. Condition $(\ast)$ is equivalent to
\[
\text{diag}(m_1, m_2, m_3) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \leq D\alpha^{-1},
\]

viz.
\[
m_1\alpha_{11} \equiv m_2\alpha_{21} \equiv m_3(\alpha_{11} + \alpha_{21}) \equiv 0 \mod p^r.
\]

Consider $\alpha = (\alpha_{11} : \alpha_{21})$ as an element of $\mathbb{P}^1(\mathfrak{o}/p^r)$. Essentially we need to count solutions to the congruence $XY(X + Y) \equiv 0 \mod \mathbb{P}^1(\mathfrak{o}/p^r)$. We proceed by a case distinction according to the reduction of $\alpha$ modulo $p$. 
For the \( q + 1 - 3 \) points of \( \mathbb{P}^1(F_q) \) which are not reductions modulo \( p \) of solutions to this congruence, the congruence (3.1) is only satisfied if \( m_1 \equiv m_2 \equiv m_3 \equiv 0 \mod p \). Noting that there are \( q^{-1} \) lattices with elementary divisor type \((r,0)\) below every one of type \((1,0)\), this leads to a geometric progression

\[
Z_{\text{gen}}(q,t) := \frac{1}{(1-t)^3} \cdot \frac{t^4}{1 - qt^4}.
\]

For the three solutions of \( XY'(X+Y) \equiv 0 \mod p \) we obtain the following. Counting lattices “below” a fixed solution \( \alpha_0 \in \mathbb{P}^1(F_q) \) yields the geometric progression

\[
Z_{\text{exc}}(q,t) := \frac{1}{(1-t)^3} \sum_{r=1}^{\infty} \sum_{A=1}^{r} \# \{ x \in \mathcal{O}/p^r \mid x \equiv \alpha_0 \mod p^r, v(x) = A \} t^{4r-A}.
\]

But, for \( 1 \leq A \leq r \),

\[
\sum_{A=1}^{r} \# \{ x \in \mathcal{O}/p^r \mid x \equiv \alpha_0 \mod p^r, v(x) = A \} = \begin{cases} 1, & \text{if } A = r, \\ (1 - q^{-1})q^{r-A}, & \text{if } A < r. \end{cases}
\]

A quick calculation yields \( Z_{\text{exc}}(q,t) = \frac{1}{(1-t)^3} \cdot \frac{t^3(1-t^4)}{(1-t^3)(1-t^4)} \).

We conclude by computing

\[
\zeta_{V(\mathcal{O})}(s) = \frac{1}{1-t^5} (1 + (q + 1 - 3)Z_{\text{gen}}(q,t) + 3Z_{\text{exc}}(q,t))
\]

\[
= \frac{1 + 2t^3 - 2t^4 - t^5}{(1-t^3)(1-t^4)(1-t^5)}. \tag*{□}
\]

3.4. Kronecker quivers. For \( b \in \mathbb{N}_0 \), consider the so-called Kronecker quiver \( K_b \) consisting of two vertices and \( b \) arrows between them, all in the same direction. Let \( R \) be a global ring of integers or a \( \mathbb{C} \)DVR.

3.4.1. \( b = 1 \). An \( R \)-representation \( V \) of the Kronecker quiver \( K_1 \)

\[
\bullet \quad \cdots \quad \bullet
\]

is given by a map \( \varphi \in \text{Hom}_R(R^{n_1}, R^{n_2}) \) for \( n_1, n_2 \in \mathbb{N}_0 \). The following lemma, which is similar to [17] Lemma 6.1, is a simple consequence of the rank-nullity theorem; we omit its proof. We denote by \( \text{im}(\varphi)^{\text{iso}} \) the \textit{isolator} of \( \text{im}(\varphi) \) in \( R^{n_2} \), viz. the largest submodule \( \Lambda \leq R^{n_2} \) containing \( \text{im}(\varphi) \) such that \( \Lambda/\text{im}(\varphi) \) is torsion.

**Lemma 3.11.** Assume that the image of \( \varphi \) has rank \( i \). Then

\[
\zeta_V(s) = |\text{im}(\varphi)^{\text{iso}} : \text{im}(\varphi)|^{-s}.
\]

\[
\left( \prod_{j=1}^{i} \zeta_{R}(2s-j+1) \right) \left( \prod_{k=i+1}^{n_2} \zeta_{R}(s-k+1) \right) \left( \prod_{l=1}^{n_1} \zeta_{R}(s-l+1) \right).
\]

**Remark 3.12.** We note that, if \( R = \mathbb{C} \) is a global ring of integers, then the Euler product \( \zeta_V(s) = \prod_{\mathfrak{p} \in \text{Spec}(O) \setminus \{ 0 \}} \zeta_{V(\mathcal{O})}(s) \) is \textit{almost uniform}: there exists a single rational function \( W_{n_1,n_2,i}(X,Y) \in \mathbb{Q}(X,Y) \), depending only on the rank vector \((n_1,n_2)\) and \( i = \text{rk}(\text{im}(\varphi)) \), such that, for almost all prime ideals \( \mathfrak{p} \) of \( \mathcal{O} \), we have

\[
\zeta_{V(\mathcal{O})}(s) = W_{n_1,n_2,i}(q\mathfrak{p},q\mathfrak{p}^{-s}).
\]

(This is an immediate consequence of the fact that \( \zeta_{R}(s) = \prod_{\mathfrak{p} \subseteq \mathfrak{p}} \frac{1}{1-q_{\mathfrak{p}}^{-s}} \). We shall see in the next section that this phenomenon is the exception, rather than the rule, for zeta functions of representations of Kronecker quivers \( K_n \) for \( n > 1 \).
3.4.2. \( b = 2 \). Consider the following \( \mathbb{Z} \)-representation \( V \) of the Kronecker quiver \( K_2 \)

\[
\mathbb{Z}^2 \xrightarrow{f_1 \ f_2} \mathbb{Z}^2
\]

with maps \( f_1 = \text{id} \), \( f_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

**Proposition 3.13.** Let \( \mathfrak{o} \) be a \( \mathbb{Z} \)-DVR of odd residue field cardinality \( q \). Then

\[
\zeta_{V(\mathfrak{o})}(s) = \begin{cases} 
\frac{(1+t^2)(1-t^3)}{(1-t)(1-t^2)(1-q^2t)(1-q^9t)}, & \text{if } q \equiv 1 \mod(4), \\
\frac{(1+t^2)(1-t^3)}{(1-t)(1-t^2)(1-q^8t)(1-q^9t)}, & \text{if } q \equiv 3 \mod(4).
\end{cases}
\]

**Sketch of proof.** As in the proof of Proposition 3.10 write

\[
\zeta_{V(\mathfrak{o})}(s) = \sum_{\Lambda_1, \Lambda_2 \leq \mathfrak{o}^2, \, f_i(\Lambda_1) \leq \Lambda_2, \, i \in \{1,2\}} |\alpha_1^2 : \Lambda_1|^{-s} |\alpha_2^2 : \Lambda_2|^{-s} \\
= \frac{1}{1-t^2} \sum_{\Lambda_1, \Lambda_2 \leq \mathfrak{o}^2, \, \Lambda_2 \text{ maximal} \, (s), f_i(\Lambda_1) \leq \Lambda_2, \, i \in \{1,2\}} |\alpha_1^2 : \Lambda_1|^{-s} |\alpha_2^2 : \Lambda_2|^{-s}
\]

Assume that the maximal lattice \( \Lambda_2 \leq \mathfrak{o}^2 \) is the row span of the matrix \( M = D\alpha^{-1} \in \text{Mat}_2(\mathfrak{o}) \), with \( D = \text{diag}(\pi^r,1) \) for \( r \in \mathbb{N}_0 \) and \( \alpha = (\alpha_{ij}) \in \text{GL}_2(\mathfrak{o}) \) as before. Condition \((\ast)\) is equivalent to

\[
\Lambda_1 \left( \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix} \right) \equiv 0 \mod(\mathfrak{p}^r).
\]

If \( q \equiv 3 \mod(4) \), then the matrix is invertible and the index is \( q^{2r} \), leading to a factor

\[
\zeta_{\mathfrak{o}^2}(s) \left( 1 + (1 + q^{-1})q^{t^2} \right) = \frac{1 + q^t}{(1-t)(1-q^t)(1-q^{2t})(1-q^{3t})}. \]

If \( q \equiv 1 \mod(4) \), then since \(-1\) is a square in \( \mathfrak{o}/\mathfrak{p} \), the matrix’ determinant splits into two distinct linear forms; a mild modification of the proof of Proposition 3.10 yields the result. \( \square \)

3.4.3. \( b = 3 \). Let \( M_1 = M_2 = \mathbb{Z}^3 \), with \( \mathbb{Z} \)-bases \((x_1, x_2, x_3)\) resp. \((y_1, y_2, y_3)\). Let \( D \) be a non-zero integer and consider the triple \( f = (f_1, f_2, f_3) : M_1 \to M_2 \) defined by

\[
(f_j(x_i))_{1 \leq i,j \leq 3} = \begin{pmatrix} Dy_3 & y_1 & y_2 \\ y_1 & y_3 & 0 \\ y_2 & 0 & y_1 \end{pmatrix} =: M(y).
\]

We thus obtain a \( \mathbb{Z} \)-representation \( V = (M_1, M_2, f) \) of the Kronecker quiver \( K_3 \). Note that \( \det(M(y)) \) defines the elliptic curve \( E \) given by

\[
Y^2 = X^3 - DX.
\]

**Proposition 3.14.** Let \( \mathfrak{o} \) be a \( \mathbb{Z} \)-DVR of residue field cardinality \( q \) with \( (q,2D) = 1 \). Then

\[
\zeta_{V(\mathfrak{o})}(s) = W_1(q, q^{-s}) + |E(\mathbb{F}_q)| W_2(q, q^{-s}),
\]

where

\[
W_1(q,t) = \frac{1 + (q+1)(t^4 + t^5) + qt^9}{(1-t)(1-t^2)(1-q^2t)(1-q^2t^4)(1-q^2t^5)(1-t^6)},
\]

\[
W_2(q,t) = \frac{(1-t^2)^2(1+qt^5)}{(1-t)(1-qt^2)(1-q^2t^2)(1-q^2t^4)(1-q^2t^5)(1-t^6)}.
\]

In particular, for these rings \( \mathfrak{o} \), the following functional equation holds:

\[
\zeta_{V(\mathfrak{o})}(s) \bigg|_{q \to q^{-1}} = q^3 \left( \frac{3}{2} \right)^{3} (6+3) \zeta_{V(\mathfrak{o})}(s).
\]
**Sketch of proof.** Analogous to [36, p. 1031]. For the functional equation, observe that

\[
|E(\mathbb{F}_q)|_{q \rightarrow q^{-1}} = q^{-1}|E(\mathbb{F}_q)|,
\]

\[
W_1(q^{-1}, t^{-1}) = q^6t^9W_1(q, t),
\]

\[
W_2(q^{-1}, t^{-1}) = q^7t^9W_2(q, t).
\]

Note that this example is not (finitely, let alone almost) uniform, as the function \( p \mapsto |E(\mathbb{F}_p)| \) is not. In a very similar way, one obtains close analogues of [37, Theorem 3].

4. **Functional equations for a class of \( p \)-adic integrals**

In [39], the second author studied a family of multivariate \( p \)-adic integrals generalizing Igusa’s local zeta functions, proving a general “blueprint result” on functional equations for such integrals. In this section we prove a generalization of this result. In Section 5 we will use it to prove Theorem 1.7.

4.1. **Informal overview.** In the following we stick closely—often, to ease comparison, verbatim or with only minimal modifications—to the notation of [39, Section 2]. Before we give details we discuss, in an informal and cursory manner, the main differences.

One motivation for studying the class of integrals defined in [39] was to capture algebraically defined integer-valued “weight functions” on the vertex set \( \mathcal{V}_n \) of the affine Bruhat-Tits building associated with a group of the form \( \text{GL}_n(K) \), where \( K \) is a local field. For this we considered the natural action of the group \( \text{GL}_n(\mathfrak{o}) \), where \( \mathfrak{o} \) is the valuation ring of \( K \), on the set \( \mathcal{V}_n \). It is well-known that \( \mathcal{V}_n \) may be interpreted as the set of full lattices inside \( K^n \) up to homothety. Note that \( \text{GL}_n(\mathfrak{o}) \) is the stabilizer of the homothety class of the “standard lattice” \( \mathfrak{o}^n \subset K^n \) under the natural action. By the elementary divisor theorem, orbits are parameterized by matrices in Smith normal form. The valuations of their diagonal entries were encoded by one set of variables (“diagonal variables” \( x \)), the entries of the diagonalizing matrices by another set of variables (“matrix variables” \( y \)). Crucially, the counting problems considered could all be described by evaluating polynomials \( f(x, y) \) which were monomial in \( x \) and became “locally monomial” in \( y \) after a Hironaka resolution of singularities. If the dependency on the matrix variables defined subvarieties of the quotient (flag) variety \( \text{GL}_n/B \), where \( B \) is a Borel subgroup of \( \text{GL}_n \), then the Weil conjectures for their reductions modulo the maximal ideal of \( \mathfrak{o} \) translated, after a fair bit of work, into the desired functional equations. For details, see [39].

In the current paper we treat, more generally, \( |Q_0| \)-tuples of lattices \( \Lambda_\iota \subseteq K^m_\iota \), indexed by the vertices of a quiver \( Q \), up to simultaneous (!) homothety; see Section 5 for details. The \( p \)-adic integrals covered by our new blueprint result are specifically designed to solve counting problems which may be expressed in terms of polynomial functions \( f(\langle x_\iota \rangle_{\iota \in Q_0}, \langle y_\iota \rangle_{\iota \in Q_0}) \) which are, again, monomial in the “diagonal variables” \( \langle x_\iota \rangle \) (one set of variables for each vertex of \( Q \)) and whose dependency on the “matrix variables” \( \langle y_\iota \rangle \) defines projective subvarieties of the flag variety \( \times_{\iota \in Q_0} \text{GL}_m/B_\iota \). In this paper’s main application of the new blueprint, viz. to the proof of Theorem 1.7 in Section 5 \( (n_\iota)_{\iota \in Q_0} \) will be the rank vector of the representation \( V \) of the quiver \( Q \).

In the case that \( a := |Q_0| = 1 \) we all but recover the setup and results of [39]: cf. Remark 4.1.

4.2. **A new blueprint result.** Let \( K \) be a local field with residue field characteristic \( p > 0 \). Let \( \mathfrak{o} = \mathfrak{o}_K \) denote the valuation ring of \( K \), \( \mathfrak{p} = \mathfrak{p}_K \) the maximal ideal of \( \mathfrak{o} \), and \( \bar{K} \) the residue field \( \mathfrak{o}/\mathfrak{p} \). The cardinality of \( \bar{K} \) will be denoted by \( q \).
For $x \in K$, let $v(x) = v_p(x) \in \mathbb{Z} \cup \{\infty\}$ denote the $p$-adic valuation of $x$, and $|x| := q^{-v(x)}$. For a finite set $S$ of elements of $K$, we set $\|S\| := \max\{|s| : s \in S\}$. Fix $k, m, a \in \mathbb{N}$ and $n_1, \ldots, n_a \in \mathbb{N}_0$. For each $\kappa \in [k]$, let $(f_{\kappa,j})_{j \in J_\kappa}$ be a finite family of finite sets of polynomials in $K[y_1, \ldots, y_m]$, and let $x_1, \ldots, x_{1,n_1}, \ldots, x_{a,1}, \ldots, x_{a,n_a}$ be independent variables. Set $n = n_1 + \cdots + n_a$. Also, for each $h \in [a]$, we fix nonnegative integers $e_{h,\kappa,j}$ for $\kappa \in [k]$, and $\kappa \in [k]$, we set

$$g_{\kappa,I}(x,y) = \bigcup_{j \in J_\kappa} \left( \prod_{h \in [a]} \prod_{i \in I_h} x_{h,i}^{e_{h,\kappa,j}} \right) f_{\kappa,j}(y),$$

where

$$(4.1) \quad I_h^* = I_h \cup \{n_h\}.$$

Let $W \subseteq \mathfrak{o}^m$ be a subset which is a union of cosets modulo $\mathfrak{p}^m = \mathfrak{p} \times \cdots \times \mathfrak{p}$ and $s = (s_1, \ldots, s_k)$ be independent complex variables. With $l = l(I) = \sum_{h \in [a]} |I_h|$ and $W_a = \mathfrak{o}^m \setminus \mathfrak{p}^a$ we define

$$Z_{W,K,I}^{gr}(s) := \int_{p^l \times W_a \times W} \prod_{\kappa \in [k]} \|g_{\kappa,I}(x,y)\|^s_{\kappa} \, |dx_I||dy|,$$

where

$$|dx_I| = \bigwedge_{h \in [a]} \bigwedge_{i \in I_h^*} dx_{h,i},$$

is the Haar measure on $K^{l+a}$ normalized so that $\mathfrak{o}^{l+a}$ has measure 1 (and thus $\mathfrak{p}^{l+a}$ has measure $q^{-l-a}$), and $|dy| = |dy_1 \wedge \cdots \wedge dy_m|$ is the (normalized) Haar measure on $K^m$.

**Remark 4.1.** Setting $a = 1$ all but recovers the integral $Z_{W,K,I}(s)$ defined on [39, p. 1191]. Indeed, in this case, $Z_{W,K,I=1}^{gr}(s) = (1-q^{-1}) Z_{W,K,I}(s)$. The factor $1-q^{-1}$ reflects the occurrence of the factor $W_1 = \mathfrak{o} \setminus \mathfrak{p}$ in the domain of integration, which does not feature in the integrand. Similar reasoning explains the apparent mismatch between the special cases of Theorems [12, 13] and [12] for $a = 1$ and their respective counterparts in [39].

We now assume that the polynomials constituting the sets $f_{\kappa,j}(y)$ are in fact defined over a number field $F$. As in [39], we may consider the local zeta functions $Z_{W,K,I}(s)$ for all non-archimedean completions $K$ of $F$. Also, recall the definition of a principalization $(Y,h)$ with good reduction modulo $\mathfrak{p}$. Specifically, let $(Y,h) : Y \to \mathbb{A}^m$ be a principalization of the ideal

$$\mathfrak{J} = \prod_{\kappa \in [k]} \prod_{j \in J_\kappa} (f_{\kappa,j}),$$

where $(f)$ denotes the ideal generated by the finite set $f$ of polynomials, with numerical data $(N_{\kappa,j}, \nu_t)_{t \in T, \kappa \in [k], j \in J_\kappa}$. Recall that, informally speaking, the numerical data—both the $N_{\kappa,j}$ and the $\nu_t$ are non-negative integers—keep track of the multiplicities of the irreducible components $E_t, t \in T$, of the (reduced) $h$-preimage of the scheme defined by $\mathfrak{J}$ and the transform of the Haar measure under $h$, respectively; for further details, see [39, Sec. 2.1].

**Theorem 4.2.** Suppose that all the sets $f_{\kappa,j}$ are integral (i.e., contained in $\mathfrak{o}[y]$) and do not define the zero ideal modulo $\mathfrak{p}_K$, and that $(Y,h)$ has good reduction modulo $\mathfrak{p}_K$. Then

$$Z_{W,K,I}^{gr}(s) = \frac{(1-q^{-1})^{l+a}}{q^m} \sum_{U \subseteq T} c_{U,W}(q-1)^{|U|} \Xi_{U,I}(q,s),$$
Theorem 4.3. Suppose that, in addition to the above assumptions, none of the ideals $t$-measure zero. Let $\mu$ be a $\Gamma$-invariant measure on $B_0 = B_0(F)$ for almost all completions $q$ of $F$. Note that the Haar measure $\mu'$ on $\Gamma_1 \times \cdots \times \Gamma_a$ coincides with the additive Haar measure $\mu$ induced from $\sigma^2 \times \cdots \times \sigma^2$ (and normalized such that $\mu(\sigma^2 \times \cdots \times \sigma^2) = 1$). This implies that all the cosets of a finite-index subgroup $\Gamma_1' \times \cdots \times \Gamma_a' \leq \Gamma_1 \times \cdots \times \Gamma_a$ have measure $\mu(\Gamma_1)/[\Gamma_1 : \Gamma_1'] \cdots \mu(\Gamma_a)/[\Gamma_a : \Gamma_a']$, with $\mu(\Gamma_h) = (1 - q^{-1}) \cdots (1 - q^{-n_h})$ for each $h \in [a]$.

Theorem 4.3. Suppose that, in addition to the above assumptions, none of the ideals $(f_{\kappa j})$ is equal to the zero ideal modulo $p_K$, and that $(Y, h)$ has good reduction modulo $p_K$. Then

$$Z^\Gamma_\Gamma(s) = \frac{(1 - q^{-1})^{l+a+n}}{\sum_{a \in [a]} \binom{n_a}{l}} \sum_{U \subseteq T} \mu(U) \Xi_{U, I}(q, s).$$

Proof. The proof is analogous to that of Theorem 2.2. Recall that $\Gamma = \Gamma_1 \times \cdots \times \Gamma_a$. For each $h \in [a]$, we write $\Gamma_h$ as a disjoint union of sets $\Gamma_h = \{x_h \in \Gamma_h \mid \bar{x}_h \in B_h(\mathcal{F}_q)\}$. Thus

$$Z^\Gamma_\Gamma(s) = \sum_{\sigma \in \Pi_{a \in [a]} S_{n_a}} Z^\Gamma_{\Gamma, \sigma} \Xi_{\Gamma, I}(s).$$

There is an obvious map $\gamma : \Gamma \to G/B(K)$, and, by our invariance assumption on the ideals $(f_{\kappa j})$, the value of the integrand of $Z^\Gamma_\Gamma(s)$ at a point $(x, y) \in p' \times W_a \times \Gamma$ only depends on $x$ and $\gamma(y)$. By taking the measure $\omega$ on $G/B(K)$ which induces the Haar measure on the product of unit balls $\alpha^{(n_h)}$ of each affine chart satisfying
Corollary 4.4. For \( V \in V \omega \), obtain rational points of \( \mathcal{D} \) follows from Theorem 4.2, just as \([39, \text{Theorem 2.2}]\) follows from \([39, \text{Theorem 2.1}]\).

Proof. This is analogous to \([39, \text{Corollary 2.1}]\). It follows immediately from the formula

Thus

Proposition 4.5. Let \( L_{\sigma}(r), \sigma \in [s], \tau \in [t] \), be \( \mathbb{Z} \)-linear forms in independent variables \( r_1, \ldots, r_t, r_{i+1}, \ldots, r_{i+a} \) and \( X_1, \ldots, X_{i+a}, Y_1, \ldots, Y_a \) independent variables. For \( r \in \mathbb{N}_0^{i+a} \) set

Define further

Then

\[
Z^\omega(X, Y) = \sum_{r \in \mathbb{N}_0^t \times (\mathbb{N}_0^{i+a} \setminus N^a)} X^r m_r(Y) \quad \text{and} \quad Z(X, Y) = \sum_{r \in \mathbb{N}_0^t \times (\mathbb{N}_0^{i+a} \setminus N^a)} X^r m_r(Y).
\]

\[
Z^\omega(X^{-1}, Y^{-1}) = (-1)^{i+a-1} Z(X, Y).
\]
Proof. Let
\[ Z_1'(X, Y) = \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a} X^r m_r(Y) \quad \text{and} \quad Z_2'(X, Y) = \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a} X^r m_r(Y), \]
so that
\[ Z'(X, Y) = Z_1'(X, Y) - Z_2'(X, Y). \]
By [39, Proposition 2.1]
\[ Z_2'(X^{-1}, Y^{-1}) = (-1)^{l+\alpha} Z_2(X, Y), \]
where
\[ Z_2(X, Y) := \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a} X^r m_r(Y). \]
Similarly, [39, Proposition 2.1] and the inclusion-exclusion principle give
\[ Z_1'(X^{-1}, Y^{-1}) = \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a} X^{-r} m_r(Y^{-1}) \]
\[ = \sum_{J \subseteq [\alpha]} \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a \atop r_{i+j} = 0 \text{ iff. } j \notin J} X^{-r} m_r(Y^{-1}) \]
\[ = \sum_{J \subseteq [\alpha]} (-1)^{|l+|J|} \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a \atop r_{i+j} = 0 \text{ if } j \notin J} X^r m_r(Y) \]
\[ = (-1)^{l+\alpha} \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a} X^r m_r(Y). \]
Thus, indeed,
\[ Z'(X^{-1}, Y^{-1}) = Z_1'(X^{-1}, Y^{-1}) - Z_2'(X^{-1}, Y^{-1}) \]
\[ = (-1)^{l+\alpha} \left( \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a} X^r m_r(Y) \right) - (-1)^{l+\alpha} \left( \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a} X^r m_r(Y) \right) \]
\[ = (-1)^{l+\alpha} \sum_{r \in \mathbb{N}^d \times \mathbb{N}^a} X^r m_r(Y) = (-1)^{l+\alpha-1} Z(X, Y). \] □

Corollary 4.6. For all $I \in \prod_{h \in [\alpha]} \mathcal{P}([n_h - 1])$, $V \subseteq T$,
\[ \Xi_{V, I}(q, s)|_{q \rightarrow q^{-1}} = (-1)^{|V|+\alpha} \sum_{W \subseteq V, J \subseteq I} \Xi_{W, J}(q, s). \]

We record the following simple fact, whose proof is a simple computation.

Lemma 4.7. For all $U \subseteq T$, $J \in \prod_{h \in [\alpha]} \mathcal{P}([n_h - 1])$,
\[ \sum_{V \subseteq U} (-1)^{|V| \cdot (1 - q^{-1})^{|V|}} \sum_{W \subseteq V} \Xi_{W, J}(q, s) = q^{-|U|} \sum_{V \subseteq U} (-1)^{|V| \cdot (q - 1)^{|V|}} \Xi_{V, J}(q, s). \]

The following definition is analogous to [39, (13)]:
\[ b_U(q^{-1}) := q^{-\sum_{h \in [\alpha]} (n_h - 1)^{|U_h|}} b_U(q). \]

The following set of “inversion properties” generalizes [39, Theorem 2.3].
Theorem 4.8 (inversion properties). Under the assumption of Theorem 4.3 for all \( I \in \prod_{h \in [a]} \mathcal{P}([n_{h} - 1]) \),

\[
\widetilde{Z}_{I}^{\varphi}(s) \bigg|_{q^{-1}} = (-1)^{|I|+a-1} \sum_{J \subseteq I} \widetilde{Z}_{J}^{\varphi}(s).
\]

Proof. Starting with the expression (4.2) for \( \widetilde{Z}_{I}^{\varphi}(s) \) in Corollary 4.1 we obtain, by combining Corollaries 4.6, 4.3, and 4.7, that indeed

\[
\widetilde{Z}_{I}^{\varphi}(s) \bigg|_{q^{-1}} = \left( \frac{q^{\sum_{h \in [a]} |J_h|}}{|G/B(F_q)|} \right) \sum_{U \subseteq T} b_U(q^{-1}) \sum_{V \subseteq U} (-1)^{|U \setminus V|} (1-q^{-1})^{|V|} \cdot (-1)^{|I|+a-1} \sum_{W \subseteq V, J \subseteq I} \Xi_{W,J}(q,s)
\]

\[
= (-1)^{|I|+a-1} \sum_{J \subseteq I} \left|G/B(F_q)\right|^{-1} \sum_{U \subseteq T} b_U(q) \sum_{V \subseteq U} (-1)^{|U \setminus V|} (1-q^{-1})^{|V|} \cdot \sum_{W \subseteq V} \Xi_{W,J}(q,s)
\]

\[
= (-1)^{|I|+a-1} \sum_{J \subseteq I} \widetilde{Z}_{J}^{\varphi}(s).
\]

\[\square\]

4.3. Local functional equations. For \( I = (I_1, \ldots, I_a) \in \prod_{h \in [a]} \mathcal{P}([n_{h} - 1]) \) we define \( I^c = (I_1^c, \ldots, I_a^c) \in \prod_{h \in [a]} \mathcal{P}([n_{h} - 1]) \). By the same slight abuse of notation, we extend other set-theoretic operations componentwise when we write, for instance, \( I \cup J \) for \( (I_1 \cup J_1, \ldots, I_a \cup J_a) \), where \( I, J \in \prod_{h \in [a]} \mathcal{P}([n_{h} - 1]) \). Assume throughout this section that the assumptions of Theorem 4.3 are satisfied. The Inversion Properties of Theorem 4.8 imply the following result, which is analogous to [38, Lemma 7]:

Lemma 4.9. For all \( I \in \prod_{h \in [a]} \mathcal{P}([n_{h} - 1]) \)

\[
\sum_{J \supseteq I} \widetilde{Z}_{J}^{\varphi}(s) \bigg|_{q^{-1}} = (-1)^{|I|-1} \sum_{J \supseteq I^c} \widetilde{Z}_{J}^{\varphi}(s).
\]

Proof. Recall the identity \( n = \sum_{h \in [a]} n_{h} \). Set \( 0^0 = 1 \). By Theorem 4.8

\[
\sum_{J \supseteq I} \widetilde{Z}_{J}^{\varphi}(s) \bigg|_{q^{-1}} = \sum_{J \supseteq I} (-1)^{\left(\sum_{h \in [a]} |J_h|\right)+a-1} \sum_{S \subseteq J} \widetilde{Z}_{S}^{\varphi}(s) = \sum_{R \in \prod_{h \in [a]} \mathcal{P}([n_{h} - 1])} C_R \widetilde{Z}_{R}^{\varphi}(s),
\]

where, for each \( R \) (and setting \( 0^0 = 1 \)),

\[
C_R = \sum_{J \supseteq (R \cup I)} (-1)^{\left(\sum_{h \in [a]} |J_h|\right)+a-1}
\]

\[
= (-1)^{\left(\sum_{h \in [a]} |R_h \cup J_h|\right)+a-1} \sum_{S \subseteq (R \cup I)^c} (-1)^{\sum_{h \in [a]} |S_h|}
\]

\[
= (-1)^{\left(\sum_{h \in [a]} |R_h \cup J_h|\right)+a-1} \delta_{R \supseteq I^c}
\]

\[
= (-1)^{n-1} \delta_{R \supseteq I^c}.
\]
Indeed,

\[ R \supseteq I^c \iff \forall h \in [a] : R_h \supseteq I_h^c \]
\[ \iff \forall h \in [a] : R_h \cup I_h = [n_h - 1] \]
\[ \iff \forall h \in [a] : (R_h \cup I_h)^c = \emptyset \]
\[ \iff \sum_{h \in [a]} |(R_h \cup I_h)^c| = 0 \]
\[ \iff \sum_{h \in [a]} (R_h \cup I_h) = n - a. \] 

Now we prove the functional equation. Given an element \( w \in S_b \) of the symmetric group on letters \( 1, \ldots, b \) we write

\[ \text{Des}(w) := \{ i \in [b - 1] : w(i) > w(i + 1) \} \]

for the descent type of \( w \). Similarly, given \( w = (w_1, \ldots, w_a) \in \prod_{h \in [a]} S_{n_h} \) we write

\[ \text{Des}(w) := (\text{Des}(w_1), \ldots, \text{Des}(w_a)) \in \prod_{h \in [a]} \mathcal{P}([n_h - 1]) \]

for the descent type of \( w \). By \( \ell(w_h) \) we denote the Coxeter length of \( w_h \in S_{n_h} \), i.e. the length of a shortest word for \( w_h \) in the standard generators for \( S_{n_h} \), by \( w_{0,h} \) the longest element in \( S_{n_h} \), both for \( h \in [a] \). We also set \( w_0 = (w_{0,1}, \ldots, w_{0,a}) \in \prod_{h \in [a]} S_{n_h} \). We recall the standard identities (cf. [18, Section 1.8])

\[ \text{Des}(w_h, w_{0,h}) = \text{Des}(w_h)^c, \quad \ell(w_h) + \ell(w_h, w_{0,h}) = \binom{n_h}{2} = \ell(w_{0,h}). \]

By slight abuse of notation we write \( \ell(w) = \sum_{h \in [a]} \ell(w_h) \), specifically

\[ \ell(w_0) = \sum_{h \in [a]} \ell(w_{0,h}) = \sum_{h \in [a]} \binom{n_h}{2} \]

and \( \ell(ww_0) = \sum_{h \in [a]} \ell(w_h w_{0,h}) \). Let

\[ (4.4) \quad \overline{Z}^{\mathfrak{g}_+}(s) = \sum_{I \in \prod_{h \in [a]} \mathcal{P}([n_h - 1])} \binom{n}{I} q_I^{-1} Z^{\mathfrak{g}_+}(s), \]

where, for \( I = (I_1, \ldots, I_h) \), we define

\[ \binom{n}{I}_X := \prod_{h \in [a]} \binom{n_h}{I_h}_X \]

in terms of \( X \)- multinomial coefficients: if \( I_h = \{i_{h,1}, \ldots, i_{h,h}\} \subseteq [n_h - 1] \), then

\[ (4.5) \quad \binom{n_h}{I_h}_X = \binom{n_h}{i_{h,1}}_X \binom{i_{h,1}}{i_{h,2}}_X \cdots \binom{i_{h,h-1}}{i_{h,1}}_X \binom{n_h}{I_h}_X = \sum_{w \in S_{n_h}} X^{\ell(w)} \in \mathbb{Z}[X]. \]

\textbf{Theorem 4.10} (local functional equations).

\[ \overline{Z}^{\mathfrak{g}_+}(s) \bigg|_{q \rightarrow q^{-1}} = (-1)^{n-1} q^{\sum_{h \in [a]} \binom{n_h}{2}} \overline{Z}^{\mathfrak{g}_+}(s). \]
Proof. Using Lemma 4.9 and the Coxeter-group theoretic facts collected above we obtain
\[
\widetilde{Z}^{\mathfrak{gr}}(s)\bigg|_{q \to q^{-1}} = \sum_{\mathcal{I} \in \mathcal{P}([n_h-1])} \left( \frac{n}{\mathcal{I}} \right) q^{-1} \left. \widetilde{Z}^{\mathfrak{gr}}(s) \right|_{q \to q^{-1}}
\]
for every \( n_h \geq 1 \) and \( s \in \mathbb{C} \). By (4.4)
\[
\sum_{w \in S_{n_h}} q^{-\ell(w)} \sum_{J \supseteq \text{Des}(w)} \left. \widetilde{Z}^{\mathfrak{gr}}(s) \right|_{q \to q^{-1}}
\]
and
\[
q^{\sum_{h \in [n]} \binom{h}{2}} \sum_{w \in S_{n_h}} q^{-\ell(ww_0)} (-1)^{n-1} \sum_{J \supseteq \text{Des}(ww_0)} \left. \widetilde{Z}^{\mathfrak{gr}}(s) \right|_{q \to q^{-1}}
\]
for every \( n_h \geq 1 \) and \( s \in \mathbb{C} \). The task of enumerating submodules of \( \mathfrak{g} \)-modules is parameterized by means of the Bruhat-Tits building associated with the \( \mathfrak{p} \)-adic group \( \text{GL}_n(K_\mathfrak{p}) \). The vertices of this complex, viz. homothety classes of full \( \mathfrak{p} \)-adic lattices inside \( K_\mathfrak{p}^n \), are parameterized by means of the action of the group \( \Gamma = \text{GL}_n(O) \) on the building. The technical challenge overcome in Section 4 was to describe the submodule condition in terms of polynomial functions on \( \Gamma \) which, if “homogeneity” holds, were amenable to the \( \mathfrak{p} \)-adic methodology developed in this section.

Remark 4.11. One may compare Theorem 4.10 with [4, Theorem 3.8] (or, equivalently, [5, Theorem 1.6]). This result establishes that “generalized Igusa functions”—combinatorially defined rational functions introduced in [4, Definition 3.5] (= [5, Definition 1.5])—satisfy functional equations akin to those established in Theorem 4.10. Its proof rests on the technical [4, Proposition 3.10], an apparent analogue of Lemma 4.9. It remains a challenge to determine whether generalized Igusa functions may be expressed via (monomial) \( \mathfrak{p} \)-adic integrals that fit into the remit of the \( \mathfrak{p} \)-adic methodology developed in this section.

5. Functional equations for local quiver representation zeta functions

In this section we apply Theorem 4.10 to prove Theorem 1.7, a functional equation for the generic multivariate local zeta functions associated with nilpotent integral quiver representations satisfying the homogeneity condition 1.5.

5.1. Informal overview. The arguments developed in this section are largely analogous to those of [40]. To facilitate comparison, we follow the notation and terminology from [40] closely. We briefly discuss the main differences.

Throughout, let \( \mathcal{O} \) be a global ring of integers, \( \mathfrak{p} \) be a non-zero prime ideal of \( \mathcal{O} \). We write \( \mathfrak{O} = \mathfrak{O}_\mathfrak{p} \) and \( K = K_\mathfrak{p} \) for the field of fractions of \( \mathfrak{O} \).

As discussed in Section 1.3.1 the setup of [40] are nilpotent representations of loop quivers. In particular, all arrows have the same head and tail, namely the unique vertex, represented by a single module \( \mathcal{L}(\mathfrak{O}) \cong \mathfrak{O}^n \). The task of enumerating submodules is approached by controlling the simplicial subcomplex they define of the Bruhat-Tits building associated with the \( \mathfrak{p} \)-adic group \( \text{GL}_n(K_\mathfrak{p}) \). The vertices of this complex, viz. homothety classes of full \( \mathfrak{p} \)-adic lattices inside \( K_\mathfrak{p}^n \), are parameterized by means of the action of the group \( \Gamma = \text{GL}_n(\mathfrak{O}) \) on the building. The technical challenge overcome in Section 4 was to describe the submodule condition in terms of polynomial functions on \( \Gamma \) which, if “homogeneity” holds, were amenable to the \( \mathfrak{p} \)-adic integration machinery of [39].

In the current paper we consider subrepresentations of nilpotent representations of general quivers. Here, the need to keep track of heads and targets of various arrows ramps up complexity. Indeed, instead of a single lattice, we consider compatible tuples \( (\Lambda_i)_{i \in Q_0} \) of lattices \( \Lambda_i \leq \mathcal{L}_i(\mathfrak{O}) \cong \mathfrak{O}^{n_i} \). In analogy to the approach in [40] for \( |Q_0| = 1 \), we express the subrepresentation condition in terms of polynomial functions on \( \prod_{i \in Q_0} \Gamma_\mathfrak{L}_i \), where \( \Gamma_i = \text{GL}_{n_i}(\mathfrak{O}) \). The generalization of [39] developed in Section 4 is tailor-made to deal with these functions provided the homogeneity condition 1.5 holds.
To prove Theorem 1.7 we are looking to establish the functional equation (1.5) for almost all zeta functions \( \zeta_{V(o)}(s) \), enumerating the finite-index \( o \)-subrepresentations \( V' \) of \( V(o) \), written \( V' \leq V(o) \). Recall that each such subrepresentation is of the form \( V' = (\Lambda_i, f'_\varphi)_{i \in Q_0, \varphi \in Q_1} \), for \( o \)-modules \( \Lambda_i \) of ranks \( n_k \). We write \( \Lambda = (\Lambda_i)_{i \in Q_0} \). Note that such tuples may be identified with graded \( o \)-submodules of \( \mathcal{L}(o) = \bigoplus_{i \in Q_0} \mathcal{L}_i(o) \) (see (1.4)) via the map \( (\Lambda_i)_{i \in Q_0} \mapsto \bigoplus_{i \in Q_0} \Lambda_i \). Clearly the property of being the support of a subrepresentation is really a property of the integral members of the (simultaneous!) homothety class \( [\Lambda] = \{ x\Lambda \mid x \in K_p^a \} \) of \( \Lambda \) in \( (K_p^a)_{i \in Q_0} \): either all elements of \( [\Lambda] \) support \( V(o) \)-representations or none does. By slight abuse of notation we write \( [\Lambda] \leq V(o) \) in the former case and set
\[
\text{SubRep}_V(o) = \{ [\Lambda] \mid \Lambda \leq V(o) \}.
\]
Evidently, every homothety class \( [\Lambda] \) of \( o \)-tuples of \( o \)-sublattices \( \Lambda_i \subset K_p^a \) contains a unique maximal integral element \( \Lambda_{\text{max}} \), i.e., \( \Lambda_{\text{max}} \leq \mathcal{L}(o) \), but \( \mathfrak{p}^{-1}\Lambda_{\text{max}} \not\leq \mathcal{L}(o) \). As the intersection of \( [\Lambda] \) with the set of all \( o \)-tuples of \( o \)-sublattices equals \( \{ \mathfrak{p}^m \Lambda_{\text{max}} \mid m \in \mathbb{N}_0 \} \) it thus suffices—in principle—to describe the elements of \( \text{SubRep}_V(o) \) and to control their maximal integral members’ indices in \( \mathcal{L}(o) \). Recall that \( a = |Q_0| \).

Indeed,\n\[
(5.1) \quad \zeta_{V(o)}(s) = \frac{1}{1-q} \sum_{[\Lambda] \in \text{SubRep}_{V(o)}} \prod_{[\Lambda] \leq [\Lambda]} |\mathcal{L}_h(o) : \Lambda_h|^{-s_h}.
\]
For each \( h \in [a] \), keeping track of the indices \( |\mathcal{L}_i(o) : \Lambda_i| \) for each unique maximal element \( \Lambda = \Lambda_{\text{max}} \) is easy (see (5.3)), so the problem of computing the right-hand side of (5.1) is to identify \( \text{SubRep}_{V(o)} \) as a subset of the set \( \mathcal{V} \) of all homothety classes \( [\Lambda] \).

In the case \( a = 1 \)—the case treated in [40]—this set of homothety classes may be identified with the vertices of the affine Bruhat-Tits building associated with the group \( \text{GL}_n(K_p) \). In general, the disjoint union of the buildings associated with the groups \( \text{GL}_n(K_p) \) may serve as a geometric model. We will not pursue this vantage point.

The proof now follows the lines of that of [40] Theorem 1.2], with \( \text{SubRep} \), \( \Lambda \), and \( \mathcal{V} \) taking the places of \( \text{SubMod} \), \( \Lambda \), and \( \mathcal{V}_n \), respectively.

5.2. Cocentral bases. We identify \( Q_0 \) with \([a]\). For \( h \in [a] \) and \( i \in [c] \), we write, as introduced in Section 1.2,
\[
n_{h,i} = \text{rk}_0 \mathcal{L}_{h,i}, \quad N_{h,i} = \sum_{j \leq c-i} n_{h,j}, \quad \text{and} \quad N_i = \sum_{h \in [a]} N_{h,i}.
\]
An \( \mathcal{O} \)-basis \( e_h = (e_{h,1}, \ldots, e_{h,n_h}) \) of \( \mathcal{L}_h \) is called cocentral if
\[
Z_{h,i} = Z_i \cap \mathcal{L}_h = \langle e_h, N_{h,i+1}, \ldots, e_{h,n_h} \rangle_0
\]
for all \( i \in [c] \). An \( \mathcal{O} \)-basis \( e = ((e_1), \ldots, (e_a)) \) of \( \mathcal{L} \) is called cocentral if \( e_h \) is cocentral for all \( h \in [a] \). By Assumption 1.3, cocentral bases clearly exist. Condition 1.5 is equivalent to the following condition.

**Condition 5.1.** There exist generators \( c_1, \ldots, c_d \) of \( \mathcal{E} \) and a cocentral \( \mathcal{O} \)-basis \( e \) of \( \mathcal{L} \) such that, for all \( k \in [d] \), the matrix \( C_k \) representing \( c_k \) with respect to \( e \) (acting from the right on row vectors) has the form
\[
C_k = \left( c_k^{(th)} \right)_{t,h \in [a]} \in \text{Mat}_n(\mathcal{O})
\]
for blocks $C_k^{(th)}$ which have the form

$$C_k^{(th)} = \left( \left( c_k^{(th)} \right)^{(ij)} \right)_{i,j \in [c]} \in \text{Mat}_{n_i \times n_k}(0)$$

for blocks $\left( c_k^{(th)} \right)^{(ij)} \in \text{Mat}_{n_i \times n_k,3}(0)$ which are zero unless $j = i + 1$.

**Remark 5.2.** Condition [4.5] is equal to [40] Condition 1.1, but Condition [5.1] is a proper generalization of [40] Condition 2.1, to which it specializes in the case $a = |Q_0| = 1$.

### 5.3. Lattices, matrices, and the subrepresentation condition

Let $\mathbf{e}$ be a cocentral $\mathcal{O}$-basis of $\mathcal{L}$ as in Condition [5.1] It yields an $\mathcal{O}$-basis of $\mathcal{L}(\mathfrak{o})$, which we also denote by $\mathbf{e}$ and which allows us to identify $\mathcal{L}(\mathfrak{o})$ with $\mathfrak{o}^n$ and $\mathcal{E}(\mathfrak{o})$ with matrices $C_1, \ldots, C_d$ representing the $\mathcal{O}$-linear operators $c_1, \ldots, c_d$.

As in Section [4.2] we write $\Gamma_h = \text{GL}_{n_h}(\mathfrak{o})$ for $h \in [a]$ and set $\Gamma = \Gamma_1 \times \cdots \times \Gamma_a$. A full $\mathfrak{o}$-sublattice $\Lambda$ of $\mathcal{L}(\mathfrak{o})$ may be identified with a coset $\Gamma M$ for a matrix $M \in \text{GL}_n(K_p) \cap \text{Mat}_n(\mathfrak{o})$, whose rows encode the coordinates with respect to $\mathbf{e}$ of a set of generators of $\Lambda$. If $\Lambda$ is graded, then $M$ is a block diagonal matrix

$$(5.3) \quad M = \text{diag}(M_1, \ldots, M_a),$$

where $M_h \in \text{GL}_{n_h}(K_p) \cap \text{Mat}_{n_h}(\mathfrak{o})$ for all $h \in [a]$. If $\Lambda = (\Lambda_h)_{h \in [a]}$, then the coset $\Gamma_h M_h$ is identified with the full $\mathfrak{o}$-sublattice $\Lambda_h$ of $\mathcal{L}_h(\mathfrak{o})$. Let $\pi$ be a uniformizer of $\mathfrak{o}$. By the elementary divisor theorem, for each $h \in [a]$ there exist

$$I_h = \{i_{h,1}, \ldots, i_{h,n_h}\} \subseteq [n_h - 1], \quad r_{h,n_h} \in \mathbb{N}_0, \quad \mathbf{r}_h, I_h = (r_{h,i_1}, \ldots, r_{h,i_{n_h}}) \in \mathbb{N}_{n_h}^n,$$

all uniquely determined by $\Lambda_h$, and $\alpha_h \in \Gamma_h$ such that $M_h = D_h \alpha_h^{-1}$, where

$$(5.4) \quad D_h = \pi^{r_{h,n_h}} \text{diag}\left( \left( \pi^{\sum_{i=1}^{r_{h,i}}(i_{h,1})}\right)^{(i_{h,1})}, \left( \pi^{\sum_{i=1}^{r_{h,i}}(i_{h,1})}\right)^{(i_{h,2}-i_{h,1})}, \ldots, \left( \pi^{r_{h,1}}\right)^{(i_{h,n_h}-i_{h,n_h-1})}, 1^{(n_h-i_{h,n_h})} \right) \in \text{Mat}_{n_h}(\mathfrak{o}).$$

We write $\nu([\Lambda_h]) = (I_h, \mathbf{r}_h, I_h)$. Note that $r_{h,n_h} = \nu(M_h)$, the $\mathbf{p}$-adic valuation of the matrix $M_h$. We also write

$$\nu(\Lambda) = (I, \mathbf{r}),$$

where

$$I = (I_1, \ldots, I_a) \in \prod_{h \in [a]} \mathcal{P}([n_h - 1])$$

and, setting $l = \sum_{h \in [a]} l_h = \sum_{h \in [a]} |I_h|$ as in Section [4.2],

$$\mathbf{r} = (r_{1,1}, \ldots, r_{a,n_a}, r_{1,n_1}, \ldots, r_{a,n_a}) \in \mathbb{N}_{l}^1 \times \cdots \times \mathbb{N}_{l}^a \times \mathbb{N}_0^a = \mathbb{N}_l \times \mathbb{N}_0^a.$$

Recall that $I_{h}^* := I_h \cup \{n_h\}$ for $h \in [a]$. Obviously, for each $h \in [a]$

$$|\mathcal{L}_h(\mathfrak{o}) : \Lambda_h| = q^{\nu(\text{det} \ D_h)} = q^{\sum_{h \in [a]} r_{h,n_h}}.$$

We call $\Lambda$ maximal if $\mathbf{r} \in \mathbb{N}_l \times (\mathbb{N}_0^a \setminus \mathbb{N}^a)$ and denote by $\Lambda_{\text{max}}$ the unique maximal element of $[\Lambda]$. We set

$$\nu([\Lambda]) := \nu([\Lambda_{\text{max}}]) \in \prod_{h=1}^{a} \mathcal{P}([n_h - 1]) \times \left( \mathbb{N}_l \times (\mathbb{N}_0^a \setminus \mathbb{N}^a) \right).$$

In the sequel we will often toggle between lattices $\Lambda$ (resp. $\Lambda_h$) and representing matrices $M$ (resp. $M_h$), extending notation for lattices to matrices representing them. We write, for instance, $[M]$ for the homothety class $[\Lambda]$ of the lattice $\Lambda$ determined by $M$. 

---

**Remark 5.2.** Condition [4.5] is equal to [40] Condition 1.1, but Condition [5.1] is a proper generalization of [40] Condition 2.1, to which it specializes in the case $a = |Q_0| = 1$.
that, for all $m \leq V(\omega)$. The following follows trivially from the block diagonal structure of $M$.

**Lemma 5.3.** For $k \in [d]$ and $t, h \in [a]$ let $C_k^{(th)} \in \text{Mat}_{n_k \times n_k}(\mathbb{O})$ be as in (5.2). Then

$$M \leq V(\omega) \iff \forall k \in [d] : MC_k \leq M \iff \forall h, t \in [a], k \in [d] : M_k C_k^{(th)} \leq M_h.$$  

Recall that $c$ is the nilpotency class of $L$. For $h \in [a]$, define the diagonal matrix

$$\delta_h := \text{diag}\left( (\pi^{c-1})^{(\alpha_h, 1)}, \ldots, (\pi)^{(\alpha_h, c-1)}, 1^{(\alpha_h, c)} \right) \in \text{Mat}_{n_h}(\omega)$$

and set

$$(5.6) \quad \delta := \text{diag}(\delta_1, \ldots, \delta_a) \in \text{Mat}_n(\omega)$$

The following is a trivial consequence of Condition 5.1.

**Lemma 5.4.** If $c > 1$, then $\forall h, t \in [a], k \in [d] : \delta_h C_k^{(th)} \delta_h^{-1} = \pi C_k^{(th)}$.

For $h, t \in [a], r \in [n_t]$ and $k \in [d]$, write $(e_{t,r})c_k = \sum_{i=1}^{n_t} e_{t,r,i}$ for $\lambda_{t,r,k}^{(i)} \in \mathbb{O}$. Then $C_k^{(th)}$ satisfies $(C_k^{(th)})_{r,i} = \lambda_{t,r,k}^{(i)}$ for $r \in [n_t]$ and $i \in [n_h]$. Let $Y_h = (Y_1, \ldots, Y_n)$ be independent variables and set

$$\mathcal{R}^{(th)}(Y_h) = \left( \sum_{i=1}^{n_h} \lambda_{t,r,k}^{(i)} Y_i \right)_{r,k} \in \text{Mat}_{n_t \times d}(\mathbb{O}[Y_h]).$$

Note that $c = 1$ if and only if $\forall h, t \in [a] : \mathcal{R}^{(th)}(Y_h) = 0$. In this case, Theorem 1.7 holds (cf. Example 1.9), so we may assume $c > 1$. For $i \in [n_h]$, we write $\alpha_h[i]$ for the $i$-th column of a matrix $\alpha_h \in \Gamma_h$, so that $\mathcal{R}^{(th)}(\alpha_h[i]) \in \text{Mat}_{n_t \times d}(\omega)$. The following lemma is verified by a trivial computation.

**Lemma 5.5.** For all $h, t \in [a]$, $\alpha_h \in \Gamma_h$, $\Delta \in \text{Mat}_{n_t}(\omega)$, and $D_h$ as in (5.1),

$$(\forall k \in [d] : \Delta C_k^{(th)} \alpha_h \leq D_h) \iff (\forall i \in [n_h] : \Delta \mathcal{R}^{(th)}(\alpha_h[i]) \equiv 0 \text{ mod } (D_h)_{ii}).$$

We set, for $h \in [a]$

$$\tau(h) := \sum_{i \in \Gamma^1_h} r_{h,i}, \quad \tau(M) := \sum_{h \in [a]} \tau(h), \quad \tau'(h) := \tau(M) - \tau(h).$$

**Proposition 5.6.** Given $M$ as in (5.3), there exists a unique $\hat{m}_1 = \hat{m}_1(M) \in \mathbb{N}_0$ such that, for all $m \in \mathbb{N}_0$,

$$M \delta^m \leq V(\omega) \text{ if and only if } m \geq \hat{m}_1.$$  

In particular, $M \leq V(\omega)$ if and only if $\hat{m}_1 = 0$. Moreover, $\hat{m}_1 \leq \tau(M)$.
Proof. For \( h \in [a] \), write \( M_h = D_h \alpha_h^{-1} \) as above. Using Lemmas 6.3, 5.4, and 5.5 we obtain

\[
M \delta^m \leq V(\mathfrak{o})
\]

\( \Leftrightarrow \forall h, t \in [a], k \in [d] : M_h \delta_k^m C_k^{(th)} \leq M_h \delta_k^m \)

\( \Leftrightarrow \forall h, t \in [a], k \in [d] : \pi^m M_h C_k^{(th)} \leq M_h \)

\( \Leftrightarrow \forall h, t \in [a], k \in [d] : \pi^m M_h C_k^{(th)} \alpha_h \leq D_h \)

\( \Leftrightarrow \forall h, t \in [a], i \in [n_h] : \pi^m D_t \alpha_t^{-1} \tau(\alpha_h[i]) \equiv 0 \pmod{(D_h)_{ii}} \)

\( \Leftrightarrow \forall h, t \in [a], i \in [n_h] : \pi^m D_t \alpha_t^{-1} \tau(\alpha_h[i]) \pi \sum_{i \in [n_h]} (d(n_h-i_h, h, i)) \equiv 0 \pmod{\pi^r(h)} \)

\[
\text{diag} \left( \left( \tau(h, i) \right)_{i \in [n_h]}, \ldots \right) \equiv 0 \pmod{\pi^r(M)}.
\]

In the last congruence, we may replace \( \alpha_t^{-1} \) by the adjoint matrix \( \alpha_t^{\text{adj}} \). Setting, for \( i \in [n_h] \) and \( r \in [n_t] \),

\[
\tau(\alpha_t, \alpha_h) = \alpha_t^{\text{adj}} \tau(\alpha_h[i]),
\]

\[
v_{tr}(\alpha_t, \alpha_h) = \min \left\{ v \left( \tau(\alpha_t, \alpha_h)[r, \sigma] \right) : i \leq i, \rho \geq r, \sigma \in [d] \right\},
\]

and

\[
m^{(th)}(M) = \min \left\{ \tau(h, i) \sum_{r \in \mathcal{R}^*_{th}} r_{t, r} + \sum_{i \in [n_h]} \left( v_{tr}(\alpha_t, \alpha_h) \right) \mid i \in [n_h], r \in [n_t] \right\},
\]

\[
m_1(M) = \min_{h, t \in [a]} \{ \tau'(h) + m^{(th)}(M) \},
\]

we may rephrase the above equivalence as follows:

\[
M \delta^m \leq V(\mathfrak{o}) \Leftrightarrow \forall h, t \in [a] : m \geq \tau(M) - \left( \tau'(h) + m^{(th)}(M) \right)
\]

\[
\Leftrightarrow m \geq \tau(M) - m_1(M) =: \tilde{m}_1(M).
\]

Definition 5.7. For a lattice \( \Lambda \) corresponding to a coset \( \Gamma M \), we set \( \tilde{m}_1([\Lambda]) = \tilde{m}_1(M) \).

5.4. \( \delta \)-equivalence. Recall the diagonal matrix \( \delta \) defined in (5.6).

Definition 5.8. Lattice classes \([\Lambda], [\Lambda'] \in \mathcal{V}\) are called \( \delta \)-equivalent, written \([\Lambda] \sim [\Lambda']\), if there exists \( m \in \mathbb{Z} \) such that \([\Lambda] = [\Lambda']^m\).

Just as in [40], we will use the terms lattice class for a homothety class of lattices and \( \delta \)-class for a \( \sim \)-equivalence class of lattice classes. The proof of Proposition 5.9 shows that in each \( \delta \)-class \( \mathcal{C} \) there is a unique lattice class \([\Lambda_0]\) such that \([\Lambda_0] \delta^m \) \( \leq V(\mathfrak{o}) \) if and only if \( m \in \mathbb{N}_0 \). We shall say that \([\Lambda_0]\) generates \( \mathcal{C}_{\geq 0} \) and write \( \Lambda_{0, \text{max}} \) for the unique maximal element of \([\Lambda_0]\). Setting

\[
\mathcal{C}_{\geq 0} = \{ [\Lambda_0] \delta^m \mid m \geq 0 \} = \mathcal{C} \cap \text{SubRep}_V(\mathfrak{o}),
\]

\[
\mathcal{C}_{< 0} = \{ [\Lambda_0] \delta^m \mid m < 0 \} = \mathcal{C} \setminus \mathcal{C}_{\geq 0},
\]

we obtain a partition \( \mathcal{C} = \mathcal{C}_{\geq 0} \cup \mathcal{C}_{< 0} \). For \( h \in [a] \), let \( M_{h,c} \in \text{Mat}_{n_h \times n_h,c}(\mathfrak{o}) \) denote the matrix comprising the last \( n_{h,c} \) columns of \( M_h \); one may also see this as a matrix representation of the lattice \( \mathcal{L}_{h,c}(\mathfrak{o}) \).
Lemma 5.9. For almost all prime ideals \( p \), the following holds for all \( M \in \text{GL}_n(K_p) \cap \text{Mat}_n(\mathcal{O}_p) \): if \( M \leq V(\mathcal{O}_p) \), then \( v(M) = v(M\delta) \).

Proof. Recall that \( M = \text{diag}(M_1, \ldots, M_n) \). We proceed by induction on \( c \), including the case \( c = 1 \). Indeed, for this base case the statement holds trivially (and for all \( p \)) as \( \delta = \text{Id}_n \). Assume thus that \( c \geq 2 \) and that the induction hypothesis holds.

We claim that for almost all \( p \) and all \( M \), the minimal \( p \)-valuation of the entries of \( M \) is equal that of the last block columns \( M_{h,c} \) of \( M_h \); if \( \pi \) divides \( M_{h,c} \) for all \( h \in [a] \), then it divides the whole matrix \( M \). Given \( p \), set \( \sigma = \mathcal{O}_p \).

For \( h \in [a] \), let

\[
M'_h := (M'_h(i,j))_{i,j \in [2,c]} \in \text{Mat}_{n_h - N_{h,c-1}}(\mathcal{O}),
\]
defining the lattice \( \Lambda_h \cap Z_{h,c-1}(\mathcal{O}) \), where \( Z_{h,c-1}(\mathcal{O}) = Z_{h,c-1} \otimes \mathcal{O} \mathcal{O} \). Also let \( M' := \text{diag}(M'_1, \ldots, M'_n) \in \text{Mat}_{n - N_c}(\mathcal{O}) \), defining the lattice \( \Lambda \cap Z_{c-1}(\mathcal{O}) \). By induction hypothesis, \( M' \) has the desired property that if \( \pi \) divides the last block columns \( M'_{h,c} \) for all \( h \in [a] \), \( \pi \) divides the whole matrix \( M' \). The statement now follows as in \([40\text{, Lemma 2.6}]\). \( \square \)

Assume from now that \( p \) satisfies the conclusions of Lemma 5.9. For \( \mathcal{C} \in \mathcal{V}/\sim \), define

\[
\Xi_{\mathcal{C} \supseteq \sigma}(s) = \sum_{[\Lambda] \in \mathcal{C} \supseteq \sigma} \prod_{h \in [a]} |L_h(\mathcal{O}) : \Lambda_h|^{-s_h}.
\]

Let \( \Lambda_{0,\max,h} = \Lambda_{0,\max} \cap L_h(\mathcal{O}) \). The following is proven just as its analogue \([40\text{, Corollary 2.7}]\). \( \Box \)

Corollary 5.10. For every \( \mathcal{C} \in \mathcal{V}/\sim \),

\[
\Xi_{\mathcal{C} \supseteq \sigma}(s) = \frac{1}{1 - q^{-\sum_{h \in [a]} s_h \sum_{i=1}^{N_h} N_{h,i}}} \prod_{h \in [a]} |L_h(\mathcal{O}) : \Lambda_{0,\max,h}|^{-s_h}.
\]

Proof. For all \( m \in \mathbb{N}_0 \) we have \( \Lambda_{0,\max} \delta^m = (\Lambda_{0,\max} \delta^m)_{\max} \) by Lemma 5.9. Hence

\[
|L(\mathcal{O}) : \Lambda_{0,\max} \delta^m| = \prod_{h \in [a]} |L_h(\mathcal{O}) : \Lambda_{0,\max,h} \delta_{h}^m|
\]

and therefore

\[
\Xi_{\mathcal{C} \supseteq \sigma}(s) = \sum_{[\Lambda] \in \mathcal{C} \supseteq \sigma} \prod_{h \in [a]} |L_h(\mathcal{O}) : \Lambda_h|^{-s_h}
\]

\[
= \sum_{m=0}^{\infty} \prod_{h \in [a]} |L_h(\mathcal{O}) : \Lambda_{0,\max,h} \delta_h^m|^{-s_h}
\]

\[
= \sum_{m=0}^{\infty} \prod_{h \in [a]} |L_h(\mathcal{O}) : \Lambda_{0,\max,h}|^{-s_h} q^{-s_h \sum_{i=1}^{N_h} N_{h,i}}
\]

\[
= \left( \sum_{m=0}^{\infty} q^{-m \sum_{h \in [a]} s_h \sum_{i=1}^{N_h} N_{h,i}} \prod_{h \in [a]} |L_h(\mathcal{O}) : \Lambda_{0,\max,h}|^{-s_h} \right)
\]

\[
= \frac{1}{1 - q^{-\sum_{h \in [a]} s_h \sum_{i=1}^{N_h} N_{h,i}}} \prod_{h \in [a]} |L_h(\mathcal{O}) : \Lambda_{0,\max,h}|^{-s_h}. \quad \Box
\]
Definition 5.11. Given $M = \text{diag}(M_1, \ldots, M_n) \in \text{GL}_n(K_p) \cap \text{Mat}_n(\mathfrak{o})$ as in (5.3) corresponding to a maximal lattice $\Lambda$, define

$$m_2([\Lambda]) = \min \{ v(M_{h,c}) \mid h \in [a] \}.$$  

We set $\tilde{\delta} = \pi^{c-1} \delta^{-1} = \text{diag}(\tilde{\delta}_1, \ldots, \tilde{\delta}_a)$, which gives $\tilde{\delta}_h = \pi^{c-1} \delta_h^{-1}$ for $h \in [a]$. Note that $\det \tilde{\delta} = \pi^{\sum_{i=1}^{c-1}(n_h-N_{h,i})}$, det $\tilde{\delta}_h = \pi^{\sum_{i=1}^{c-1}(n_h-N_{h,i})}$, and $c < 0 = \{ [\Lambda] \delta_m \mid m > 0 \}$.

Lemma 5.12. With $w([\Lambda]) := (c-1)\tilde{m}_1([\Lambda]) - m_2([\Lambda])$ we have

$$\Xi_{c<0}(s) := \sum_{\Lambda \in \mathcal{C}_{\leq 0}} \prod_{h \in [a]} |\mathcal{L}_h(\mathfrak{o}) : \Lambda_h(\mathfrak{o})|^{-s_h} q^{-s_h n_h w([\Lambda])}.$$  

Proof. Analogous to [40, Lemma 2.10], we observe that $v(M \delta \tilde{m}_1([\Lambda])) = m_2([\Lambda])$. Hence the matrix $M \pi^{c-1} \tilde{m}_1([\Lambda]) - m_2([\Lambda])$ corresponds to $\Lambda_{0,\max} \tilde{m}_1([\Lambda])$, whence for each $h$, $M_h \pi^{c-1} \tilde{m}_1([\Lambda]) - m_2([\Lambda])$ corresponds to $\Lambda_{0,\max, h} \tilde{m}_1([\Lambda])$.

Thus we have

$$\Xi_{c<0}(s) = \sum_{m=1}^{\infty} \prod_{h \in [a]} |\mathcal{L}_h(\mathfrak{o}) : \Lambda_{0,\max, h} \delta_h|^m |^{-s_h} q^{-s_h m \sum_{i=1}^{c-1}(n_h-N_{h,i})}$$  

$$= \sum_{m=1}^{\infty} \prod_{h \in [a]} |\mathcal{L}_h(\mathfrak{o}) : \Lambda_{0,\max, h}|^{-s_h} q^{-s_h m \sum_{i=1}^{c-1}(n_h-N_{h,i})} \prod_{h \in [a]} |\mathcal{L}_h(\mathfrak{o}) : \Lambda_{0,\max, h}|^{-s_h}$$  

$$= \left( \sum_{m=1}^{\infty} q^{-m \sum_{h \in [a]} s_h \sum_{i=1}^{c-1}(n_h-N_{h,i})} \prod_{h \in [a]} |\mathcal{L}_h(\mathfrak{o}) : \Lambda_{0,\max, h}|^{-s_h} \right) \prod_{h \in [a]} |\mathcal{L}_h(\mathfrak{o}) : \Lambda_{0,\max, h}|^{-s_h}$$

For later reference we record another formula for the invariant $m_2$. Setting, for $h \in [a]$ and $i \in [n_h]$

$$v_{h,i}^{(2)}(\alpha_h) := \min \left\{ v \left( (\alpha_h^{\text{adj}})_{i,\sigma} \right) \mid i \geq i, \sigma \in [N_{h,1}, n_{h,c}] \right\}$$

and

$$m_{h,2}([\Lambda]) := \min \left\{ \sum_{i \in \mathcal{I}_h} r_{h,i}, \sum_{i \in \mathcal{I}_h} r_{h,i} + v_{h,i}^{(2)}(\alpha_h) \mid i \in [n_h] \right\}$$

we obtain

$$m_2([\Lambda]) = \min \{ m_{h,2}([\Lambda]) \mid h \in [a] \}.$$  

Finally, let

$$A_{\text{SubRep}}(s) := \sum_{\Lambda \in \mathcal{V}} \prod_{h \in [a]} |\mathcal{L}_h(\mathfrak{o}) : \Lambda_h|^{-s_h} q^{-s_h n_h w([\Lambda])}$$

$$= \sum_{\Lambda \in \mathcal{V}} \prod_{h \in [a]} |\mathcal{L}_h(\mathfrak{o}) : \Lambda_h|^{-s_h} q^{-s_h n_h ((c-1)\tilde{m}_1([\Lambda]) - m_2([\Lambda]))}.$$
Then one argues as in [10, p. 19] that
\[ \zeta_{V(\alpha)}(s) = \frac{1}{1 - q} \frac{1 - q^{-\sum_{h \in [a]} s_h \sum_{i=0}^{n_h-1} (n_h - N_{h,i})}}{1 - q^{-\sum_{h \in [a]} s_h n_h}} A_{\text{SubRep}}(s). \]
Since
\[ \frac{1}{1 - q} \frac{1 - q^{-\sum_{h \in [a]} s_h \sum_{i=0}^{n_h-1} (n_h - N_{h,i})}}{1 - q^{-\sum_{h \in [a]} s_h n_h}} \bigg|_{q \to q^{-1}} = \frac{-q^{-\sum_{h \in [a]} s_h \sum_{i=0}^{n_h-1} N_{h,i}}}{1 - q^{-\sum_{h \in [a]} s_h n_h}}, \]
it suffices to show that \( A_{\text{SubRep}}(s) \) satisfies the functional equation
\[ (5.7) \quad A_{\text{SubRep}}(s) \bigg|_{q \to q^{-1}} = (-1)^{n-1} q^{\sum_{h \in [a]} (n_h^2)} A_{\text{SubRep}}(s). \]
To compute \( A_{\text{SubRep}}(s) \) we need, given a lattice class \([\Lambda] \in V\) with \( \nu([\Lambda]) = (I, r) \in \prod_{h=1}^{a} \mathcal{P}([n_h - 1]) \times (N^d \setminus (N^d \setminus \mathbb{N}))\), to keep track of the quantity
\[ q^{-\sum_{h \in [a]} s_h \left( \left( \sum_{i=0}^{n_h-1} r_{h,i} \right) + n_h \nu([\Lambda]) \right)} \]
\[ = q^{-\sum_{h \in [a]} s_h \left( \left( \sum_{i=0}^{n_h-1} r_{h,i} \right) + n_h((c-1)N_1([\Lambda]) - m_2([\Lambda])) \right)} \]
\[ = q^{-\sum_{h \in [a]} s_h \left( \left( \sum_{i=0}^{n_h-1} r_{h,i} + (c-1) \right) - n_h((c-1)m_1([\Lambda]) + m_2([\Lambda])) \right)} \]
Here we used (5.5), Lemma 5.12 and the fact that
\[ \bar{m}_1([\Lambda]) = \tau(M) - m_1([\Lambda]) = \left( \sum_{h \in [a]} \sum_{i \in I_h^s} r_{h,i} \right) - m_1([\Lambda]). \]
To this end we define, given \((I, r)\) as above, for \( m = (m_1, m_2) \in \mathbb{N}_0^2 \),
\[ \mathcal{N}_{I,r,m} = |\{ [\Lambda] \in V \mid \nu([\Lambda]) = (I, r), m_i([\Lambda]) = m_i, i \in \{1, 2\} \}| \]
and set
\[ (5.8) \quad A_I^{\text{SubRep}}(s) = \sum_{m=(m_1, m_2) \in \mathbb{N}_0^2} \mathcal{N}_{I,r,m} q^{\sum_{h \in [a]} s_h n_h ((c-1)m_1 + m_2)}, \]
so that
\[ A_{\text{SubRep}}(s) = \sum_{I \subseteq [a]} \prod_{h \in [a]} \mathcal{P}([n_h - 1]) A_I^{\text{SubRep}}(s). \]

5.5. \textit{p-Adic integration.} To establish the functional equation (5.7) (and thus (1.5)) we express the function \( A_{\text{SubRep}}(s) \) in terms of suitable substitutions of multivariate functions of the form \( Z^{\mathfrak{g}}(s) \) (see (4.4)). Theorem 1.7 will then follow from Theorem 4.10.

To this end, let \( \mathbf{x} = (x_1, \ldots, x_a) \in \mathbb{P}^t \times \mathcal{W}_a \) with \( x_h = (x_{h,1}, \ldots, x_{h,i}, \ldots, x_{h,n_h}) \) for \( h \in [a] \). (Recall that \( \mathcal{W}_a = \mathbb{P}^c \setminus \mathbb{P}_a \).) Let further \( \mathbf{y} = (y_1, \ldots, y_a) \in \Gamma_1 \times \cdots \times \Gamma_a \) with \( y_h = (y_{h,i})_{i \in [n_h]} \) for \( h \in [a] \). We define sets of polynomials, for \( h, t \in [a] \), and \( i \in [n_h], r \in [n_t], \)
\[ f_{i,r}^{(th)}(y) = \left( \left( \mathbf{y}_{t,r}^{(th)}(y_t, y_h) \right)_{\rho}, \ 0 \leq \sigma \leq r, \ \rho \in [d] \right), \]
\[ f_{h,i}^{(2)}(y) = \left( y_h^{(ad)}, \ 0 \leq i, \ \sigma \in [n_h, n_{h,c}] \right), \]
and set, for $I \in \prod_{h \in [a]} \mathcal{P}([n_h - 1])$,

$$g_I^{(th)}(x, y) = \left\{ \prod_{i \in I_h^*} x_{h,i} \right\} \cup \left\{ \prod_{i \in [n_h], r \in [n_i]} \left( \prod_{p \in I_h^{(s)}} x_{r \in p} \prod_{i \in I_h^*} x_{h,i}^{k_{i,r}} \right) f_I^{(th)}(y),$$

$$g_I^{(1)}(x, y) = \bigcup_{h \in [a]} \left( \prod_{k \in [a] \setminus \{h\}} \prod_{\lambda \in I_h^*} x_{k,\lambda} \right) g_I^{(th)}(x, y),$$

and

$$g_I^{(h,2)}(x, y) = \left\{ \prod_{i \in I_h^*} x_{h,i} \right\} \cup \left\{ \prod_{i \in [n_h]} \left( \prod_{i \in I_h^*} x_{h,i}^{k_{i,r}} \right) f_I^{(2)}(y),$$

$$g_I^{(2)}(x, y) = \bigcup_{h \in [a]} g_I^{(h,2)}(x, y),$$

and, for $\kappa_h \in [n_h]$,

$$g_{\kappa_h, I_h^*}(x, y) = \left\{ \prod_{i \in I_h^*} x_{h,i}^{\kappa_h} \right\}.$$

The ideals generated by the sets $f_I^{(th)}(y)$ and $f_I^{(2)}(y)$ are all $\mathcal{B}(F)$-invariant. Informally speaking, to see this one needs to check that the ideals generated by the relevant matrix entries do not change when $y$ is replaced by an element in the coset $y\mathcal{B}(F)$; the entries themselves, however, may change, of course.

With these data we define the $p$-adic integral

$$Z^\text{SubRep}_I(s) = Z^\text{SubRep}_I \left( (s_{1,\iota_1})_{i_1 \in I_h^*}, \ldots, (s_{a,\iota_a})_{i_a \in I_h^*}, s^{(1)}_n, s^{(2)}_n \right) :=
\int_{p^l \times W_a \times \Gamma} \left\| g_I^{(1)}(x, y) \right\|^{s^{(1)}_n} \left\| g_I^{(2)}(x, y) \right\|^{s^{(2)}_n} \prod_{\kappa \in \prod_{h \in [a]} [n_h]} \prod_{\kappa \in \prod_{h \in [a]} [n_h]} \left\| g_{\kappa_h, I_h^*}(x, y) \right\|^{s_{h,\kappa_h}} |dx| |dy|.$$

Here, $s$ is a vector of complex variables; note, however, that $s_{h,\kappa_h}$ occurs on the right-hand side if and only if $\kappa_h \in I_h^*.$

It now remains to prove that, for each $I \in \prod_{h \in [a]} \mathcal{P}([n_h - 1])$, the generating function $A^\text{SubRep}_I(s)$ is indeed obtainable from the $p$-adic integral $Z^\text{SubRep}_I(s)$ by a suitable specialization of the variables $s$. We start by measuring the sets on which the integrand of $Z^\text{SubRep}_I(s)$ is constant. More precisely we set, for $m = (m_1, m_2) \in \mathbb{N}_0^2$ and $r \in \mathbb{N}^l \times (\mathbb{N}_0^a \setminus \mathbb{N}^a),$

$$\mu^{\text{SubRep}}_{I, r, m} :=
\mu \left\{ (x, y) \in p^l \times W_a \times \Gamma \mid \forall h \in [a], i \in I_h^* : v(x_{h,i}) = r_{h,i}, m(x, y) = (m_1, m_2) \right\},$$
where \( m(x, y) = (m(x, y)_1, m(x, y)_2) \) and, for \( h, t \in [a] \),
\[
m(x, y)^{(h, t)} = \min \left\{ \tau(h), \sum_{r \leq \rho \in I_s^a} v(x_{t, \rho}) + \sum_{i > t \in I_s^a} v(x_{h, i}) + v_{1, i}^{(t, h)}(y) \mid i \in [n_h], r \in [n_t] \right\},
\]
\[
m(x, y)_1 = \min_{h, t \in [a]} \left\{ \tau'(h) + m(x, y)^{(h, t)} \right\},
\]
\[
m(x, y)_{h, 2} = \min \left\{ \tau(h), \sum_{i \leq \iota \in I_s^a} v(x_{h, i}) + v^{(2)}_{1, i}(y) \mid i \in [n_h] \right\}
\]
\[
m(x, y)_2 = \min_{h \in [a]} \left\{ m(x, y)_{h, 2} \right\}.
\]

Then, by design, (cf. (4.2))
\[
(5.9) \quad Z_{I}^{\text{SubRep}}(s) = \frac{1}{(1 - q^{-1})^{1+\mu(\Gamma)}} \sum_{r \in \mathbb{N}_0^a \times (\mathbb{N}_0^a \setminus \mathbb{N}_0^b)} q^{-\sum_{h \in [a]} s_h \sum_{i \leq \iota \in I_s^a} r_{h, i}(i(n_h - 1) + 1)} \sum_{m = (m_1, m_2) \in \mathbb{N}_0^2} \mu_{I, r, m}^{\text{SubRep}} q^{-s_h(m_1 - 1) - s^{(2)}_h m_2}.
\]

The numbers \( \mu_{I, r, m}^{\text{SubRep}} \) are closely related to the natural numbers \( N_{I, r, m}^{\text{SubRep}} \) we are looking to control.

Lemma 5.13.
\[
(5.10) \quad N_{I, r, m}^{\text{SubRep}} = \frac{(\frac{d}{d})^{-1} q^{-1}}{(1 - q^{-1})^{1+\mu(\Gamma)}} \mu_{I, r, m}^{\text{SubRep}} q^{(s_h - 1) - (c - 1)m_1 + m_2}.
\]

Proof. Analogous to [29] Lemma 3.1. \( \square \)

Thus, combining (5.8), (5.10), and (5.9), we obtain
\[
A_{I}^{\text{SubRep}}(s) \quad = \quad \sum_{r \in \mathbb{N}_0^a \times (\mathbb{N}_0^a \setminus \mathbb{N}_0^b)} q^{-\sum_{h \in [a]} s_h \sum_{i \leq \iota \in I_s^a} r_{h, i}(i(n_h - 1) + 1)} \sum_{m = (m_1, m_2) \in \mathbb{N}_0^2} N_{I, r, m}^{\text{SubRep}} q^{-s_h n_h((c - 1)m_1 + m_2)}
\]
\[
= \frac{(\frac{d}{d})^{-1} q^{-1}}{(1 - q^{-1})^{1+\mu(\Gamma)}} \sum_{r \in \mathbb{N}_0^a \times (\mathbb{N}_0^a \setminus \mathbb{N}_0^b)} q^{-\sum_{h \in [a]} s_h \sum_{i \leq \iota \in I_s^a} r_{h, i}(i(n_h - 1) - i(n_h - 1) - 1)} \cdot \sum_{m = (m_1, m_2) \in \mathbb{N}_0^2} \mu_{I, r, m}^{\text{SubRep}} q^{-s_h n_h((c - 1)m_1 + m_2)}
\]
\[
= \left( \frac{d}{d} \right) \frac{Z_{I}^{\text{SubRep}}}{q^{-1}} \left( (s_h(t_h + n(c - 1)) - t_h(n_h - t_h) - 1)_{h \in [a]} - \sum_{h \in [a]} s_h n_h(c - 1), - \sum_{h \in [a]} s_h n_h \right).
\]

The functional equation (5.7) now follows from Theorem 4.10. This completes the proof of Theorem 4.17.

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