Some issues related to polarized radiative transfer in a multilayer medium with a changing index of refraction

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Abstract. A couple of issues raised in a recent work on Fresnel boundary and interface conditions for polarized radiative transfer are discussed. The first issue concerns a normalization that has to be performed on the transmission matrix relating the transmitted and incident Stokes vectors at a smooth interface between two material layers with different indices of refraction. Normalization of the transmission matrix is required to ensure conservation of energy. Surprisingly, the author noted that in many published works on this topic such a normalization is not done in a correct way. The most frequent discrepancies detected in these works were either a complete absence of a normalizing factor or the lack of a multiplicative term of the form \((n_b/n_a)^2\), where \(n_b\) denotes the index of refraction of the medium to where radiation is being transmitted and \(n_a\) that of the medium of incidence. The effects of these discrepancies in the calculated reflectance and transmittance of a single water layer with a vacuum on top and a black bottom were estimated using an analytical discrete-ordinates (ADO) solution [2] that does not take polarization effects into account.

The second issue is more subtle than the first. The reflection matrix that relates the reflected and incident Stokes vectors at a smooth interface is derived from the well-known Fresnel formulas for the amplitude coefficients of reflection [3, 4]. As discussed many years ago [5], the theory is unable to determine the signs that should be used with the amplitude coefficients of reflection \(R_l\).
and $R_r$. Physically, a change of sign applied to one of these coefficients is equivalent to a change of $\pi$ in the phase of the corresponding component (parallel or perpendicular) of the reflected electric field with respect to that of the incident electric field. A recommendation on the signs to be used with the $R_l$ and $R_r$ coefficients of reflection was produced [6] in an ellipsometry symposium held in Nebraska in 1968. An undesirable feature of that sign choice—which we call “conventional”—is that the $R_l$ and $R_r$ coefficients are not consistent with each other when normal incidence is considered [6]. Even so, the conventional sign choice is the one that appears most frequently in optics and electromagnetism books [6].

If a sign change is applied to the usual form [6] of the $R_l$ coefficient, it becomes consistent with the $R_r$ coefficient when the incidence is normal to the interface. Moreover, the physical notion that a change of $\pi$ should be expected in the phase of the reflected electric field with respect to that of the incident electric field when the index of refraction of the medium of incidence is smaller than the index of refraction of the reflecting medium [7] is verified. We refer to such a choice of sign as “consistent”.

Except for one work [8], all of the works in radiative transfer that the author is aware of use the conventional sign choice. The net effect of a sign change in $R_l$ is a sign change in the (3,3), (3,4), (4,3), and (4,4) elements of the reflection matrix. While these sign changes are of almost no consequence in simple situations like the cases of no scattering and/or normal incidence (only sign changes in the calculated Stokes parameters $U$ and $V$ are expected), the same is not true when there is scattering and the beam incidence is not in the normal direction. In this case, all four Stokes parameters ($I$, $Q$, $U$, and $V$) are coupled and so completely different numerical results are expected when a sign change is applied to the $R_l$ coefficient.

In this work, we study a simple model of an atmosphere-water system with a two-fold purpose: (i) to evaluate the influence of the most frequently encountered discrepancies in the transmission matrix when polarization effects are taken into account, and (ii) to compare the calculated Stokes parameters for the two sign choices of the reflection coefficient $R_l$ mentioned above, aiming at determining the appropriate sign choice for polarized radiative transfer calculations.

2. Mathematical description of a multilayer medium subject to Fresnel conditions

Considering a medium composed of layers $k = 1, 2, \ldots, K$, we write the equation of transfer for layer $k$ as [1]

$$
\mu \frac{\partial}{\partial \tau} I_k(\tau, \mu, \phi) + I_k(\tau, \mu, \phi) = \frac{\varpi_k}{4\pi} \int_0^{2\pi} \int_{-1}^1 P_k(\mu, \mu', \phi - \phi') I_k(\tau, \mu', \phi') d\mu' d\phi',
$$

(1)

where $I_k(\tau, \mu, \phi)$ is the Stokes vector with four Stokes parameters $I_k(\tau, \mu, \phi)$, $Q_k(\tau, \mu, \phi)$, $U_k(\tau, \mu, \phi)$, and $V_k(\tau, \mu, \phi)$ as components [9], $\tau \in (a_k-1, a_k]$ is the optical variable that measures the position in layer $k$, and $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$ are, respectively, the cosine of the polar angle (measured with respect to the positive $\tau$-axis) and the azimuthal angle that define the direction of propagation of the radiation. In our notation, $a_0$ gives the location of the surface of the first layer, $a_1, a_2, \ldots, a_{K-1}$ give the locations of the interfaces between the layers, and $a_K$ the location of the surface of the last layer. To each layer, we associate an index of refraction $n_k$, an albedo for single scattering $\varpi_k$, and a phase matrix $P_k(\mu, \mu', \phi - \phi')$ that we allow to be expressed in the general form [10]

$$
P_k(\mu, \mu', \phi - \phi') = \frac{1}{2} \sum_{m=0}^{L_k} (2 - \delta_{0,m}) [C_{km}^m(\mu, \mu') \cos m(\phi - \phi') + S_{km}^m(\mu, \mu') \sin m(\phi - \phi')],
$$

(2)

where $L_k$ is a truncation parameter that defines the scattering order,

$$
C_{km}^m(\mu, \mu') = A_{km}^m(\mu, \mu') + D A_{km}^m(\mu, \mu') D,
$$

(3a)
and

\[ S_k^m(\mu, \mu') = A_k^m(\mu, \mu')D - DA_k^m(\mu, \mu'). \tag{3b} \]

In these equations, \( D = \text{diag}\{1, 1, -1, -1\} \) and

\[ A_k^m(\mu, \mu') = \sum_{l=m}^{L_k} P_l^m(\mu)B_{k,l}P_l^m(\mu'), \tag{4} \]

where

\[ B_{k,l} = \begin{pmatrix} \beta_{k,l} & \gamma_{k,l} & 0 & 0 \\ \gamma_{k,l} & \alpha_{k,l} & 0 & 0 \\ 0 & 0 & \zeta_{k,l} & -\epsilon_{k,l} \\ 0 & 0 & \epsilon_{k,l} & \delta_{k,l} \end{pmatrix} \tag{5} \]

and

\[ P_l^m(\mu) = \begin{pmatrix} F_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & F_l^m(\mu) \end{pmatrix}. \tag{6} \]

The constants \( \{\alpha_{k,l}, \beta_{k,l}, \gamma_{k,l}, \delta_{k,l}, \epsilon_{k,l}, \zeta_{k,l}\} \) are known in the literature as Greek constants. In addition, \( P_l^m(\mu) \) is a normalized version of the associated Legendre function of the first kind, and the normalized functions \( R_l^m(\mu) \) and \( T_l^m(\mu) \) are expressed in terms of the generalized spherical functions \( P_{m-2}^l(\mu) \) and \( P_{m+2}^l(\mu) \) discussed by Gel’fand and Šapiro [11]. We note that the matrix \( P_l^m(\mu) \) and its elements have been defined and denoted in slightly different ways in the literature; in this work we have followed the notation and the definitions given in a paper by Siewert [12].

The boundary and interface conditions subject to which we must solve Eq. (1) for layers \( \mu = 1, 2, \ldots, K \) have been derived from first principles [1]. Assuming that a uniform beam of radiation characterized by the Stokes vector

\[ F_0 = \begin{pmatrix} F_I \\ F_Q \\ F_U \\ F_V \end{pmatrix}, \tag{7} \]

is traveling, along a direction defined by \((\mu_0, \phi_0)\), towards the surface \( \tau = a_0 \) in an external \((\tau < a_0)\), non-participating medium with index of refraction \( n_0 \), we can write the boundary condition for the first layer as [1]

\[ I_1(a_0, \mu, \phi) = X(n_{1,0}, \mu)I_1(a_0, -\mu, \phi) + Y(n_{1,0}, \mu)F_0 \delta[f(n_{1,0}, \mu) - \mu_0]\delta(\phi - \phi_0), \tag{8} \]

for \( \mu \in (0, 1] \) and \( \phi \in [0, 2\pi] \). In this expression, we make use of some general definitions: the ratio between indices of refraction \( n_{k,k'} = n_k/n_{k'} \), the function

\[ f(n, \mu) = [1 - n^2(1 - \mu^2)]^{1/2}, \tag{9} \]

the \( 4 \times 4 \) reflection matrix

\[ X(n, \mu) = \begin{cases} G(n, \mu), & n \leq 1, \\ G(n, \mu)H[\mu - \mu_c(n)] + \Gamma(n, \mu)\{1 - H[\mu - \mu_c(n)]\}, & n \geq 1, \end{cases} \tag{10a} \]

and the \( 4 \times 4 \) transmission matrix

\[ Y(n, \mu) = \begin{cases} D(n, \mu), & n \leq 1, \\ D(n, \mu)H[\mu - \mu_c(n)], & n \geq 1. \end{cases} \tag{10b} \]
In Eqs. (10), we use the Heaviside function

\[ H(x) = \begin{cases} 
1, & x \geq 0, \\
0, & x < 0,
\end{cases} \]  

the quantity \( \mu_c(n) = (1 - 1/n^2)^{1/2} \) is the cosine of the critical angle \([3]\), and the non-zero elements of the \( 4 \times 4 \) basic matrices \( G(n, \mu) \), \( \Gamma(n, \mu) \), and \( D(n, \mu) \) are given by \([1]\)

\[
G_{11}(n, \mu) = G_{22}(n, \mu) = \frac{1}{2} \left\{ \left[ \frac{nf(n, \mu) - \mu}{nf(n, \mu) + \mu} \right]^2 + \left[ \frac{n\mu - f(n, \mu)}{n\mu + f(n, \mu)} \right]^2 \right\}, \tag{12a}
\]

\[
G_{12}(n, \mu) = G_{21}(n, \mu) = \frac{1}{2} \left\{ \left[ \frac{nf(n, \mu) - \mu}{nf(n, \mu) + \mu} \right]^2 - \left[ \frac{n\mu - f(n, \mu)}{n\mu + f(n, \mu)} \right]^2 \right\}, \tag{12b}
\]

\[
G_{33}(n, \mu) = G_{44}(n, \mu) = \frac{n\mu - f(n, \mu)}{nf(n, \mu) + \mu}, \tag{12c}
\]

\[
\Gamma_{11}(n, \mu) = \Gamma_{22}(n, \mu) = 1, \tag{13a}
\]

\[
\Gamma_{33}(n, \mu) = \Gamma_{44}(n, \mu) = 1 - \frac{2(1 - \mu^2)}{1 - (1 + 1/n^2)\mu^2}, \tag{13b}
\]

\[
\Gamma_{34}(n, \mu) = -\Gamma_{43}(n, \mu) = \frac{2\mu(1 - \mu^2)[\mu^2(n) - \mu^2]^{1/2}}{1 - (1 + 1/n^2)^2}, \tag{13c}
\]

\[
D_{11}(n, \mu) = D_{22}(n, \mu) = 2n^3 \mu f(n, \mu) \left\{ \frac{1}{[nf(n, \mu) + \mu]^2} + \frac{1}{[n\mu + f(n, \mu)]^2} \right\}, \tag{14a}
\]

\[
D_{12}(n, \mu) = D_{21}(n, \mu) = 2n^3 \mu f(n, \mu) \left\{ \frac{1}{[nf(n, \mu) + \mu]^2} - \frac{1}{[n\mu + f(n, \mu)]^2} \right\}, \tag{14b}
\]

and

\[
D_{33}(n, \mu) = D_{44}(n, \mu) = \frac{4n^3 \mu f(n, \mu)}{[nf(n, \mu) + \mu][n\mu + f(n, \mu)]}, \tag{14c}
\]

At the interfaces between the layers, we have \([1]\)

\[
I_k(a_k, -\mu, \phi) = X(n_{k,k+1}, \mu)I_k(a_k, \mu, \phi) + Y(n_{k,k+1}, \mu)I_{k+1}(a_k, -f(n_{k,k+1}, \mu), \phi) \tag{15a}
\]

and

\[
I_{k+1}(a_k, \mu, \phi) = X(n_{k+1,k}, \mu)I_{k+1}(a_k, -\mu, \phi) + Y(n_{k+1,k}, \mu)I_k[a_k, f(n_{k+1,k}, \mu), \phi], \tag{15b}
\]

for \( \mu \in (0, 1], \phi \in [0, 2\pi], \) and \( k = 1, 2, \ldots, K - 1. \) With regard to the boundary condition for the surface located at \( \tau = a_K, \) we assume that there is no radiation coming from an external \((\tau > a_K), \) non-participating medium with index of refraction \( n_{K+1}, \) and so we write \([1]\)

\[
I_K(a_K, -\mu, \phi) = X(n_{K,K+1}, \mu)I_K(a_K, \mu, \phi), \tag{16}
\]

for \( \mu \in (0, 1] \) and \( \phi \in [0, 2\pi]. \)

In order to simulate the case of an atmosphere-water system, we take \( K = 2 \) in our formulation. Since both the atmosphere \((k = 1)\) and the water \((k = 2)\) layers are taken to be described by Rayleigh scattering, we use \( c_k = 1.0, \) \( L_k = 2 \) and the Greek constants of table 1 for both \( k = 1 \) and \( k = 2. \) The beam incident on the top of the atmosphere is unpolarized,
and so we take $F_I = 1$ and $F_Q = F_U = F_V = 0$ in Eq. (7). To complete the specification of the atmosphere-water system that we wish to study, we use the data of Sommersten et al. [8]: the atmosphere has an optical depth of 0.15 and an index of refraction $n_1 = 1.0$ while the water layer has an optical depth of 1.0 and an index of refraction $n_2 = 1.338$. Above the atmosphere, a vacuum (index of refraction $n_0 = 1.0$) is assumed; at the bottom of the water layer, a black surface is simulated by assuming the presence of an external, non-participating medium with an index of refraction identical to that of the water body ($n_3 = 1.338$). The cosine of the polar angle of incidence is $\mu_0 = 1/2$.

### Table 1. The Greek constants for Rayleigh scattering.

| $l$ | $\alpha_l$ | $\beta_l$ | $\gamma_l$ | $\delta_l$ | $\epsilon_l$ | $\zeta_l$ |
|-----|-------------|------------|------------|------------|--------------|------------|
| 0   | 0           | 1          | 0          | 0          | 0            | 0          |
| 1   | 0           | 0          | 0          | 3/2        | 0            | 0          |
| 2   | 3           | 1/2        | $-\sqrt{6}/2$ | 0          | 0            | 0          |

The calculations were performed with an extension of a scalar ADO method [13]. The extension of that work [13] for the case with polarization is still in documentation phase, so the complete description of the method will appear in a future publication.

### 3. The issue concerning the transmission factor

As discussed in detail in our recent work [1], when radiation crosses an interface between two different media, conservation of energy requires that a normalizing scalar (that we call transmission factor) be applied to the transmission matrix obtained from the Fresnel formulas for the amplitude coefficients of transmission. When both indices of refraction are real, the transmission factor is given by [1]

$$f_T = \begin{cases} (n_b/n_a)^3(\cos \theta_b/\cos \theta_a), & n_a/n_b \leq 1 \text{ or } n_a/n_b > 1 \text{ with } \theta_a < \theta_c, \\ 0, & n_a/n_b > 1 \text{ with } \theta_a \geq \theta_c, \end{cases}$$

(17)

where $n_a$ and $n_b$ are the indices of refraction of the incidence and transmission media, respectively. In addition, $\theta_a$ and $\theta_b$ are, respectively, the angles of incidence and refraction measured with respect to the interface normal, and $\theta_c = \arcsin(n_b/n_a)$ is the critical angle [3].

Figures 1 through 8 compare the results of our calculations for three situations: (i) correct transmission factor $f_T$ as given by Eq. (17), (ii) top line of Eq. (17) divided by $n^2$, where $n = n_b/n_a$, and (iii) $f_T$ missing or, equivalently, top line of Eq. (17) replaced with 1. Items (ii) and (iii) replicate the most frequent discrepancies detected in the transmission matrix by our study [1]. The plots shown in these figures correspond to the scattered Stokes parameters just below the water surface, which is one of the places where the differences are most conspicuous. Figures 1 and 2 are for the azimuthal angle $\phi = \phi_0$, figures 3 through 6 for the azimuthal angle $\phi = \phi_0 + \pi/2$, and figures 7 and 8 for the azimuthal angle $\phi = \phi_0 + \pi$. The $U$ and $V$ parameters are not shown for $\phi = \phi_0$ and $\phi = \phi_0 + \pi$ because they are identically zero for these angles.

The results shown in figures 1 through 8 show that the magnitudes of all four Stokes parameters obtained when $f_T$ is ignored or replaced with $f_T/n^2$ are much smaller than the correct ones on the water side of the interface. This is in agreement with our previous conclusion [1] that the transmission of radiation to the water layer is underestimated when the transmission factor $f_T$ is not used (or used incorrectly with a missing $n^2$ term). In particular, it should be
noted that when $f_T$ is neglected the sharp peaks that appear near the cosine of the critical angle, $\mu_c(1.338) \approx 0.664$, are almost completely lost.
Finally, we note that there is apparent agreement between our results and the results presented also in graphical form by Sommersten et al. [8] for test 1 of their work. It should be pointed out that these authors define $\mu$ as the negative of our $\mu$. This causes a sign difference in the $U$ and $V$ parameters and explains why our curves look like mirror images of their curves.

4. The issue concerning the sign of one of the reflection coefficients

The amplitude coefficients of reflection by an interface are usually written as

$$R_l = \frac{\tan(\vartheta_a - \vartheta_b)}{\tan(\vartheta_a + \vartheta_b)} \quad (18a)$$

and

$$R_r = -\frac{\sin(\vartheta_a - \vartheta_b)}{\sin(\vartheta_a + \vartheta_b)}, \quad (18b)$$

where the subscripts $l$ and $r$ indicate the coefficients that correspond, respectively, to the parallel and perpendicular components of the electric field. In addition, the angle $\vartheta_a$ defines the initial direction of propagation in medium $a$ with index of refraction $n_a$ and the angle $\vartheta_b$ the direction of propagation in medium $b$ with index of refraction $n_b$ after transmission. Among 25 references compiled by Muller [6], 19 use these coefficients as given by Eqs. (18) while six use Eq. (18a) with a change in sign. The choice of signs recommended by Muller [6] follows the one used by most authors and is referred to in this paper as “conventional”. The main argument in favor of this sign choice is that it preserves the coordinate system used before and after reflection [6].

Equations (18) can be written in the alternative forms

$$R_l = \frac{n_b \cos \vartheta_a - n_a \cos \vartheta_b}{n_b \cos \vartheta_a + n_a \cos \vartheta_b} \quad (19a)$$

and

$$R_r = \frac{n_a \cos \vartheta_a - n_b \cos \vartheta_b}{n_a \cos \vartheta_a + n_b \cos \vartheta_b} \quad (19b)$$

Using Eqs. (19) and Snell’s law

$$n_a \sin \vartheta_a = n_b \sin \vartheta_b, \quad (20)$$

we can easily conclude that, for normal incidence ($\vartheta_a = 0$), $R_l$ reduces to $(n_b - n_a)/(n_b + n_a)$ while $R_r$ reduces to $(n_a - n_b)/(n_b + n_a)$. This difference in sign between the parallel and perpendicular
reflection coefficients is unjustifiable on physical grounds, since the parallel and perpendicular components of the electrical field become indistinguishable when the incidence is normal.

A sign change in either \( R_l \) or \( R_r \) means simply a change of \( \pi \) in the phase of the corresponding component of the reflected electric field with respect to that of the incident electric field. Because a change of \( \pi \) in the phase of the reflected electric field with respect to that of the incident electric field is expected, at least for normal incidence, when radiation travels from an optically rarer medium to an optically denser medium [7], the use of Eq. (18a) [and Eq. (19a)] with a sign change can be considered as the most consistent choice from a physical point of view. This is the reason why we call it the “consistent” sign choice. However, one problem with this sign choice is that it does not preserve the coordinate system upon reflection [6]. Among eleven works on radiative transfer with polarization where the choice of the signs of \( R_l \) and \( R_r \) is specified [1], only one [8] uses the consistent sign choice.

A sign change in \( R_l \) causes only a small difference in our formulation: for the conventional sign choice the \( G(n, \mu) \) and \( \Gamma(n, \mu) \) matrices are used as defined in section 2, whereas for the consistent sign choice the signs of the \((3,3)\), \((3,4)\), \((4,3)\), and \((4,4)\) elements of these matrices must be changed.

In a preliminary version [14] of our previous work [1], we have adopted the conventional sign choice. However, motivated by a comment of a referee during the review process of [1], we came to consider the possibility that the consistent sign choice could be the most adequate. This idea has also permeated the abstract submitted to CTRPM-IV. In the end, a numerical study performed with a code based on an extension of the ADO method discussed in [13] to the case with polarization, led us to conclude that the conventional sign choice is indeed the appropriate one for our formulation. We discuss that numerical study next.

The numerical study was based on the same test case defined at the end of section 2 but with a difference: \( n_2 \), the index of refraction of layer 2 (originally water), was varied towards 1.0. For consistency, \( n_3 \) was taken to be equal to \( n_2 \) for all of the six choices of \( n_2 \) so that the condition of a black surface at \( \tau = a_2 \) was preserved throughout the study. We note that when \( n_2 = 1.0 \), the studied two-layer system reduces to a single homogenous layer without Fresnel boundary/interface conditions, a well-studied case for which the issue of the appropriate sign choice for the reflection coefficient \( R_l \) does not exist.

Our numerical results for the scattered \( U \) and \( V \) parameters in layer 2, at a position just below the interface between layers 1 and 2, are given in tables 2 and 3 for the conventional choice of the \( R_l \) sign. For our purposes here, it was sufficient to consider values of \( \mu \) in the vicinity of zero and one value of the azimuthal angle for which the parameters \( U \) and \( V \) do not vanish. Tables 4 and 5 provide similar results for the choice of the \( R_l \) sign that we have called consistent.

We can see from tables 2 and 3 that the results obtained with the conventional sign choice of \( R_l \) are continuous as \( \mu \to 0 \) from above and from below, and that the results for \( n_2 = 1.0 \) are approached smoothly. On the other hand, as can be seen from tables 4 and 5, the consistent sign choice yields results for \( U \) and \( V \) that exhibit a sign flip as \( \mu = 0 \) is crossed. Moreover, it is apparent from table 4 that the \( U \) results for \( n_2 = 1.0 \) are not approached smoothly for all values of \( \mu \) shown in the table when the consistent sign choice of \( R_l \) is made. We note that the results reported for \( n_2 = 1.0 \) in tables 2 through 5 were confirmed using an independent computational implementation of the ADO method for radiative transfer with polarization [15]. The study was repeated for more challenging cases, including cases defined by larger numbers of layers and/or phase matrices with more expansion terms than the Rayleigh phase matrix. Similar results were obtained.

In conclusion, we expect the reader will agree with us that there is enough evidence in support of our claim that the conventional sign choice for the \( R_l \) coefficient [6] is the appropriate one for our formulation of polarized radiative transfer with Fresnel boundary and interface conditions.
Table 2. The behavior of $U_2(a_1, \mu, \phi_0 + \pi/2)$ as $n_2 \rightarrow 1.0$ for the conventional sign choice.

| $\mu$ | $n_2 = 1.1$ | $n_2 = 1.01$ | $n_2 = 1.001$ | $n_2 = 1.0001$ | $n_2 = 1.00001$ | $n_2 = 1.0$ |
|-------|-------------|--------------|--------------|----------------|----------------|-------------|
| -0.1  | -5.4010(-2) | -5.4582(-2)  | -5.4143(-2)  | -5.4079(-2)    | -5.4072(-2)    | -5.4072(-2) |
| -0.05 | -5.6692(-2) | -5.7862(-2)  | -5.7451(-2)  | -5.7386(-2)    | -5.7379(-2)    | -5.7378(-2) |
| -0.01 | -5.8748(-2) | -6.0505(-2)  | -6.0126(-2)  | -6.0061(-2)    | -6.0054(-2)    | -6.0053(-2) |
| -0.005| -5.9183(-2) | -6.1084(-2)  | -6.0714(-2)  | -6.0652(-2)    | -6.0647(-2)    | -6.0646(-2) |
| -0.0  | -5.9230(-2) | -6.1147(-2)  | -6.0799(-2)  | -6.0717(-2)    | -6.0712(-2)    | -6.0712(-2) |
| 0.0   | -5.9230(-2) | -6.1147(-2)  | -6.0799(-2)  | -6.0717(-2)    | -6.0712(-2)    | -6.0712(-2) |
| 0.001 | -5.9182(-2) | -6.1084(-2)  | -6.0714(-2)  | -6.0652(-2)    | -6.0647(-2)    | -6.0677(-2) |
| 0.01  | -5.8737(-2) | -6.0504(-2)  | -6.0126(-2)  | -6.0061(-2)    | -6.1287(-2)    | -6.1358(-2) |
| 0.05  | -5.6606(-2) | -5.7857(-2)  | -6.1433(-2)  | -6.0430(-2)    | -6.0134(-2)    | -6.0101(-2) |
| 0.1   | -5.3766(-2) | -5.4571(-2)  | -5.1931(-2)  | -4.9825(-2)    | -4.9630(-2)    | -4.9609(-2) |

Table 3. The behavior of $V_2(a_1, \mu, \phi_0 + \pi/2)$ as $n_2 \rightarrow 1.0$ for the conventional sign choice.

| $\mu$ | $n_2 = 1.1$ | $n_2 = 1.01$ | $n_2 = 1.001$ | $n_2 = 1.0001$ | $n_2 = 1.00001$ | $n_2 = 1.0$ |
|-------|-------------|--------------|--------------|----------------|----------------|-------------|
| -0.1  | -7.8510(-4) | -2.0199(-5)  | -3.0494(-7)  | -3.5996(-9)    | -3.8114(-11)   | 0.0         |
| -0.05 | -9.0750(-4) | -2.7002(-5)  | -4.8436(-7)  | -6.4323(-9)    | -7.1833(-11)   | 0.0         |
| -0.01 | -1.0462(-3) | -3.8121(-5)  | -9.7414(-7)  | -1.9178(-8)    | -2.815(-10)    | 0.0         |
| -0.001| -1.0875(-3) | -4.2667(-5)  | -1.3183(-6)  | -3.7398(-7)    | -9.4900(-10)   | 0.0         |
| -0.0  | -1.0923(-3) | -4.3301(-5)  | -1.3812(-6)  | -4.3336(-8)    | -1.3632(-9)    | 0.0         |
| 0.0   | -1.0923(-3) | -4.3301(-5)  | -1.3812(-6)  | -4.3336(-8)    | -1.3632(-9)    | 0.0         |
| 0.001 | -1.1366(-3) | -5.9816(-5)  | -6.7433(-6)  | -1.7489(-6)    | -5.2965(-7)    | 0.0         |
| 0.01  | -1.5355(-3) | -2.0757(-4)  | -5.3354(-5)  | -1.2031(-5)    | -6.6611(-12)   | 0.0         |
| 0.05  | -3.2563(-3) | -7.8781(-4)  | -9.9907(-8)  | -3.7190(-10)   | -4.0113(-12)   | 0.0         |
| 0.1   | -5.1878(-3) | -1.1063(-3)  | -2.4599(-8)  | -2.7124(-10)   | -2.9271(-12)   | 0.0         |

Table 4. The behavior of $U_2(a_1, \mu, \phi_0 + \pi/2)$ as $n_2 \rightarrow 1.0$ for the consistent sign choice.

| $\mu$ | $n_2 = 1.1$ | $n_2 = 1.01$ | $n_2 = 1.001$ | $n_2 = 1.0001$ | $n_2 = 1.00001$ | $n_2 = 1.0$ |
|-------|-------------|--------------|--------------|----------------|----------------|-------------|
| -0.1  | -4.3239(-2) | -5.1631(-2)  | -5.3681(-2)  | -5.4024(-2)    | -5.4066(-2)    | -5.4072(-2) |
| -0.05 | -4.4265(-2) | -5.3964(-2)  | -5.6735(-2)  | -5.7289(-2)    | -5.7368(-2)    | -5.7378(-2) |
| -0.01 | -4.3973(-2) | -5.4908(-2)  | -5.8699(-2)  | -5.9782(-2)    | -6.0012(-2)    | -6.0053(-2) |
| -0.001| -4.3423(-2) | -5.4573(-2)  | -5.8690(-2)  | -6.0055(-2)    | -6.0508(-2)    | -6.0646(-2) |
| -0.0  | -4.3279(-2) | -5.4434(-2)  | -5.8584(-2)  | -6.0025(-2)    | -6.0494(-2)    | -6.0712(-2) |
| 0.0   | -4.3279(-2) | -5.4434(-2)  | -5.8584(-2)  | -6.0025(-2)    | -6.0494(-2)    | -6.0712(-2) |
| 0.001 | -4.3423(-2) | -5.4573(-2)  | -5.8690(-2)  | -6.0055(-2)    | -6.0508(-2)    | -6.0777(-2) |
| 0.01  | -4.3978(-2) | -5.4908(-2)  | -5.8699(-2)  | -5.9782(-2)    | -6.0907(-2)    | -6.1358(-2) |
| 0.05  | -4.4253(-2) | -5.3960(-2)  | -4.4713(-2)  | -6.0358(-2)    | -6.0131(-2)    | -6.0101(-2) |
| 0.1   | -4.3230(-2) | -5.1621(-2)  | -5.1484(-2)  | -4.9809(-2)    | -4.9629(-2)    | -4.9609(-2) |
5. Concluding remarks

Two standing issues relevant to polarized radiative transfer in a multilayer medium with a layer-dependent index of refraction were analyzed. In particular, the effects of two commonly encountered discrepancies in forms of the transmission matrix used by different authors, namely the neglect of a normalization factor required for conservation of energy across an interface and the omission of a factor given by the square of the ratio between the refractive index of the interface (see, for example, [16]), we believe that the issues discussed in this work are equally important in that context.

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