Rainbow matchings in edge-colored simple graphs

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Abstract

There has been much research on the topic of finding a large rainbow matching (with no two edges having the same color) in a properly edge-colored graph, where a proper edge coloring is a coloring of the edge set such that no same-colored edges are incident. Barát, Gyárfás, and Sárközy conjectured that in every proper edge coloring of a multigraph (with parallel edges allowed, but not loops) with $2q$ colors where each color appears at least $q$ times, there is always a rainbow matching of size $q$. Recently, Aharoni, Berger, Chudnovsky, Howard, and Seymour proved a relaxation of the conjecture with $3q - 2$ colors. Our main result proves that $2q + o(q)$ colors are enough if the graph is simple, confirming the conjecture asymptotically for simple graphs. This question restricted to simple graphs was considered before by Aharoni and Berger. We also disprove one of their conjectures regarding the lower bound on the number of colors one needs in the conjecture of Barát, Gyárfás, and Sárközy for the class of simple graphs. Our methods are inspired by the randomized algorithm proposed by Gao, Ramadurai, Wanless, and Wormald to find a rainbow matching of size $q$ in a graph that is properly edge-colored with $q$ colors, where each color class contains $q + o(q)$ edges. We consider a modified version of their algorithm, with which we are able to prove a generalization of their statement with a slightly better error term in $o(q)$. As a by-product of our techniques, we obtain a new asymptotic version of the Brualdi-Ryser-Stein Conjecture.

1 Introduction

1.1 State of the art

Transversals in Latin squares have been a central topic of study in combinatorics, dating back to the time of Euler, who studied conditions on when Latin squares can be decomposed.

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into transversals. For a survey of transversals in Latin squares, see, e.g., [28]. One of the central and longstanding conjectures in this field is the following, which is attributed to Brualdi, Ryser, and Stein.

**Conjecture 1.1** (Brualdi and Ryser [13], Stein [27]). *Every $n \times n$ Latin square has a partial transversal of size $n - 1$."

This conjecture translates into the graph theoretic statement: “In any proper edge-coloring of $K_{n,n}$ with $n$ colors, there is a rainbow matching of size $n - 1$”. To see this connection, see, e.g., [23] or [28]. Hatami and Shor [19] proved an asymptotic version of Conjecture 1.1 and the error term was further improved in a very recent result of Keevash, Pokrovskiy, Sudakov, and Yepremyan [20]. In this paper, we study generalizations (in many different aspects) of Conjecture 1.1 in the setting of rainbow matchings in edge-colored (but not only properly edge-colored) graphs. Aharoni and Berger conjectured the following generalization of Conjecture 1.1 in [2].

**Conjecture 1.2** (Aharoni and Berger [2]). *Let $G$ be a properly edge-colored bipartite multigraph with $n$ colors, having at least $n + 1$ edges of each color. Then $G$ has a rainbow matching using every color.*

In this paper, multigraphs permit parallel edges, but not loops. Many authors have attacked Conjecture 1.2 by proving it with the weaker assumption that every color class contains substantially more than $n + 1$ edges. An easy greedy argument shows that having $2n$ edges in each color suffices. Indeed, if the largest rainbow matching $M$ contains less than $n - 1$ edges, then there would be an edge with an unused color that is not adjacent to any of the edges in $M$ and the edge could be added to $M$ to get a larger rainbow matching. After subsequent effort by several authors (see, e.g., [6], [10], [14], [18], [22], and [24]), Pokrovskiy proved an asymptotic version of Conjecture 1.2 in [23], where he proved that the conclusion of the conjecture holds if there are at least $n + o(n)$ edges of each color.

It is very natural to generalize Conjecture 1.2 for non-bipartite graphs, which was considered by both Aharoni, Berger, Chudnovsky, Howard, and Seymour [3] and Barát, Gyárfás, and Sárközy [12]. An earlier version of [3] conjectured the following:

**Conjecture 1.3** (from the earlier version [4] by Aharoni, Berger, Chudnovsky, Howard, and Seymour). *Let $G$ be a properly edge-colored multigraph with $n$ colors having at least $n + 2$ edges of each color. Then $G$ has a rainbow matching using every color.*

We remark here that one cannot replace $n + 2$ by $n + 1$ in Conjecture 1.3. To see this, consider the graph $G$ that is a disjoint union of two copies of $K_4$. If we consider the unique decomposition of $G$ in 3 perfect matchings, there is no matching of size 3 using edges from different perfect matchings. The most notable works progressing towards Conjecture 1.3 are [16] by Gao, Ramadurai, Wanless, and Wormald, and [21] by Keevash and Yepremyan. The former one proved an asymptotic version of Conjecture 1.3 for simple graphs, which is analogous to the result of Pokrovskiy [23] but without the bipartiteness assumption on the underlying graph. In this paper, we provide a generalization of this result of [16].

Another way to generalize Conjecture 1.2 is to remove the assumption that each color appears as a matching. Aharoni and Berger conjectured the following in [2]:

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Conjecture 1.4 (Aharoni and Berger [2]). Let $G$ be a bipartite multigraph, with maximum degree $\Delta$, whose edges are (not necessarily properly) colored. If every color appears on at least $\Delta + 1$ edges, then $G$ has a rainbow matching using every color.

Gao, Ramadurai, Wanless, and Wormald refuted Conjecture 1.4 in [16] by constructing an example where $\Delta$ is linear in terms of the number of colors used. In this context, it will be interesting to check if such a statement holds for edge-coloring with the additional assumption of bounded (perhaps in terms of $\Delta$) maximum degree in each color class (color class refers to the subgraph formed by the edges of that color). Note that if the maximum degree of a color class is 1, that color appears as a matching. So, this will be yet another generalization of Conjecture 1.2. Our generalization (see Theorem 1.12 in the next section) of the result by Gao, Ramadurai, Wanless, and Wormald [16] implies the asymptotic version of Conjecture 1.4 for edge-coloring of not-necessarily-bipartite simple graphs, where the number of colors is bounded in terms of $\Delta$, and with a bounded degree assumption on each color class.

We now move on to a related but slightly different problem. All the problems discussed so far focused on the minimum number of edges in each color class to ensure a rainbow matching using all the colors. Alternatively, we can insist that each color class has size $n$ and discuss how many colors we need to ensure a rainbow matching of size $n$. In this context, the first result appeared by Drisko [15] in 1998, which was later revisited by Aharoni and Berger [2]. They proved the following:

Theorem 1.5 (Drisko [15]). Let $G$ be a properly edge-colored bipartite multigraph with $2n - 1$ colors having exactly $n$ edges of each color. Then $G$ has a rainbow matching using $n$ colors.

There is a unique construction achieving the bound $2n - 1$ in this theorem (see, e.g., [9]). Theorem 1.5 was strengthened by using topological methods in [8]. Barát, Gyárfás, and Sárközy [12] suggested the following stronger conjecture:

Conjecture 1.6 (Barát, Gyárfás, and Sárközy [12]). Let $G$ be a properly edge-colored multigraph with $2n - t_n$ colors and exactly $n$ edges of each color, where $t_n = 0$ for even $n$ and $t_n = 1$ for odd $n$. Then $G$ has a rainbow matching using $n$ colors.

By the same example as in the case of bipartite multigraphs, we see that $2n - 1$ is the best possible bound in Conjecture 1.6. Aharoni, Berger, Chudnovsky, Howard, and Seymour proved in [3] that it is sufficient if there are $3n - 2$ color classes in the graph $G$ (recently improved to $3n - 3$ by Aharoni, Briggs, Kim, and Kim [5]). They also conjectured that Theorem 1.5 is true even without the assumption that the edge-coloring of $G$ is proper.

Aharoni and Berger considered Conjecture 1.6 in a simple graph, and made the following general conjecture (Conjecture 3.2 in [2]) for simple hypergraphs.

Conjecture 1.7 (Aharoni and Berger [2]). Consider any $r$-uniform hypergraph, formed by a set of $2^{r-2}(s-1)+2$ matchings of size $t$, no two of which share an edge. Suppose that for each matching, all of its edges are colored with the same color, and the colors used are different for different matchings. Then, there is a matching with $t$ edges on which at least $s$ colors appear.
In this paper, we disprove this conjecture when \( r = 2 \) and \( s = t \) and make progress on the upper bound on the required number of matchings, which will be discussed in Section 1.2. To be more precise, this upper bound is obtained by establishing an asymptotic version of Conjecture 1.6 when we restrict ourselves to simple graphs \( G \) (see Theorem 1.9 in the next section).

### 1.2 Main results

We first disprove Conjecture 1.7. As mentioned in [2], the \( r = 2 \) and \( s = t \) case of Conjecture 1.7 would have given us a generalization of Conjecture 1.1. But unfortunately, we have found a counterexample to this.

**Proposition 1.8.** For all even \( t \), there exists a set of \( t + 1 \) edge-disjoint \( t \)-edge-matchings with no rainbow \( t \)-matching.

So, it is natural to find \( f(t) \) which is the minimum \( k \) such that every set of \( k \) edge-disjoint \( t \)-edge-matchings contains a rainbow \( t \)-matching. Proposition 1.8 proves that \( f(t) \geq t + 2 \) for all even \( t \). Our main result makes progress in the upper bound and establishes that \( f(t) \leq 2t + o(t) \). This provides an asymptotic version of Conjecture 1.6 in the case of simple graphs. Although we believe that the lower bound on \( f(t) \) is closer to the truth, there is a natural barrier to improve the upper bound of \( 2t + o(t) \), which we elaborate in the concluding remarks. Our main result is the following:

**Theorem 1.9.** For every \( \epsilon > 0 \), there exists \( N = N(\epsilon) \) such that whenever \( q \geq N \), for any graph \( G \) that is edge-colored with \( 2(1 + \epsilon)q \) colors such that there are at least \( q \) edges of each color and no same-colored edges are incident, there is a rainbow matching of \( G \) using \( q \) colors.

We next show that Theorem 1.9 is the best possible with respect to the number of colors in each color class. We need each color to appear at least \( q \) times, because it is possible to have no matching on \( q \) edges when each color appears \( q - 1 \) times, even if we have any number \( n \) of colors. To see this, consider the graph which is a disjoint union of \( q - 1 \) copies of \( K_{1,n} \) and color each \( K_{1,n} \) using all the colors exactly once. Although each color appears as a matching with \( q - 1 \) edges, the graph does not contain any matching with \( q \) edges. It turns out that this rigid behavior with respect to the number of colors in each color class actually poses some difficulty in approaching this problem with probabilistic methods. Nevertheless, we overcome this difficulty by proving and using the following statement, which is also of independent interest.

**Theorem 1.10.** For every \( \epsilon > 0 \), there exists \( N = N(\epsilon) \) such that whenever \( q \geq N \) the following holds. Suppose \( G \) is a bipartite graph on the vertex set with bipartition \( A \cup B \), where \( |A| = q \) and every vertex in \( A \) has degree at least \( (1 + \epsilon)q \). Suppose the edges are properly colored. Then, there always is a rainbow matching in \( G \) which uses every vertex in \( A \).

This result is best possible, because it is easy to construct examples showing that \( \epsilon = 0 \) does not work in Theorem 1.10 (to see such an example, check the first part of Section 2).
As a straightforward corollary of Theorem 1.10, we next obtain the following asymptotic version of Conjecture 1.1 (the Brualdi-Ryser-Stein conjecture), which is also best possible up to the small error term.

**Corollary 1.11.** If $K_{q,q+o(q)}$ is properly edge-colored, then there is a rainbow matching of size $q$.

Our proofs of Theorem 1.9 and Theorem 1.10 are motivated by the probabilistic approach in [16] by Gao, Ramadurai, Wanless, and Wormald. In order to better demonstrate this approach in our settings, we start with some modifications of this approach to prove the following extension of their result. We also prove a weaker bound on Theorem 1.9 with a direct application of this extension.

**Theorem 1.12.** There exists an $N$ such that whenever $q \geq N$ and $1 \leq \Delta \leq \frac{q^{\frac{1}{4}}}{(\log q)^{\frac{5}{9}}}$, the following holds. Suppose $G$ is a graph with maximum degree at most $q$ that is edge-colored with $q$ colors such that there are at least $\left(1 + \left(\frac{\Delta}{q}\right)^{\frac{1}{3}}(\log q)^{\frac{1}{3}}\right)q$ edges of each color. Suppose that at most $\Delta$ edges of the same color are incident to any vertex (so this is not necessarily a proper coloring). Then, there is a rainbow matching in $G$ which uses every color.

In contrast to the result in [16], Theorem 1.12 does not require each color class to be a matching, but the subgraph formed by any color still must have bounded degree. Our randomized algorithm to prove Theorem 1.12 is a little different compared to theirs, which makes the analysis a bit simpler and gives us a slightly better error term in the required number of edges in each color in the case when each color class is a matching. Incidentally, this is the only known result making progress on Conjecture 1.3 where the required error term is of the form $q^c$ for some constant $c < 1$, which addresses an open problem of [23]. That said, their approach probably could also be easily adapted to handle the maximum degree $\Delta$ situation.

This paper is organized as follows. We disprove Conjecture 1.7 in the next section by proving Proposition 1.8. For the next few subsequent sections, we concentrate on Theorem 1.12. Section 3 contains the randomized algorithm we use to prove Theorem 1.12 which is one of the central ideas of this paper. Later, we consider similar randomized algorithms with some slight changes to prove Theorem 1.9 and Theorem 1.10. In Section 4, we give an intuitive analysis for why the algorithm in Section 3 should work. In the subsequent couple of sections, we prove Theorem 1.12 rigorously. We mention some standard probabilistic tools in Section 5, which we use throughout this paper. In Section 7, we prove a weaker version of our main result Theorem 1.9 which we need later to prove the full version. Section 8 contains the proof of Theorem 1.9 assuming Theorem 1.10. Then, Theorem 1.10 is proved in Section 9. We finish with a few concluding remarks, which include a discussion on why the required number of colors in Theorem 1.9 might be hard to improve using probabilistic methods.

## 2 Construction for Proposition 1.8

We prove Proposition 1.8 in this section. Fix an even $t$. Consider a graph $G$ on the vertex set $A \cup B$, where $A$ and $B$ have $t$ vertices each, and recognize each of $A$ and $B$ by the group
For each \( j \in \mathbb{Z}_t \), introduce a color \( j \) with \( t \) edges, where each \( a \in A \) is adjacent to \( a + j \in B \). So we have \( t \) colors each of which is a \( t \)-matching. First, we prove that there is no rainbow \( t \)-matching using these \( t \) colors. For the sake of contradiction, assume that we have such a rainbow \( t \)-matching and fix such a matching. All colors and vertices of \( G \) have to participate in such a matching. Let \( a_jb_j \) denote the edge in color \( j \in \mathbb{Z}_t \) in the rainbow \( t \)-matching, where \( a_j \in A \) and \( b_j \in B \). Clearly, we have the following:

\[
\sum_{j \in \mathbb{Z}_t} a_j = \sum_{j \in \mathbb{Z}_t} b_j = \sum_{j \in \mathbb{Z}_t} j = \frac{t(t-1)}{2}.
\]

But by the definition of each color class \( E_j \), we should have \( \sum_{j \in \mathbb{Z}_t} (b_j - a_j) = \sum_{j \in \mathbb{Z}_t} j \), which is a contradiction for even \( t \).

Now let us introduce an extra \((t+1)\)-st color in \( G \), whose edges are the union of an arbitrary perfect matching in \( A \) and arbitrary perfect matching in \( B \). This has clearly \( t \) edges. Note that all the matchings are still disjoint, because any matching considered before contains edges between \( A \) and \( B \) only. We claim that even after introducing this extra color, we still cannot have rainbow \( t \)-matching. For the sake of contradiction, assume that we have such a rainbow \( t \)-matching. Clearly, the extra color has to participate in that, because we could not do it before. Without loss of generality, the extra color edge in the rainbow matching is between two vertices \( u, v \in A \). Now it is not possible to have a rainbow \((t-1)\)-matching on \( G \setminus \{u, v\} \) using the rest of the colors, because each edge uses a distinct vertex from each of \( A \) and \( B \). Hence, we have a contradiction.

### 3 Algorithm

In this section, we give a randomized algorithm which will be primarily used for Theorem 1.12. As we use some variants of this algorithm for proving Theorems 1.9 and 1.10, this algorithm will be referred as Algorithm throughout the paper. Assume that we are given a simple graph \( G \) with maximum degree at most \( q \) that is edge-colored with \( q \) colors such that there are \((1 + \epsilon)q \) edges of each color and at most \( \Delta \) edges of same color can be incident to any vertex. We are going to provide a randomized algorithm, which constructs a rainbow matching using almost all the colors in several iterations. Each iteration is a random procedure, where we can show that we get to the desirable state with some positive probability. Then we fix that choice of desirable state to analyze the next iteration. A single step of our algorithm will look like the following.

This algorithm is in terms of some parameter \( \delta > 0 \), which will be specified later. Define \( \tau = \frac{1}{\delta} \). We follow the algorithm below for \( \eta \tau \) steps, for some \( \eta < 1 \) to be specified later.

1. Select independently \( \delta q \) edges \( e_1, e_2, \ldots, e_{\delta q} \) uniformly at random with replacement from among the remaining edges. Denote by \( T \) the set of all selected edges.

2. Delete all the vertices corresponding to the edges in \( T \) from \( G \). Deleting vertices always also deletes all incident edges.
3. With some probability (to be specified later), independently delete each vertex in $G$. This will make sure that among Steps 2 and 3 combined, every vertex in $G$ gets deleted with the same probability.

4. For each edge $e_i \in T$, add it to the rainbow matching if $e_i$ is not incident to any edge $e_j \in T$ for $j < i$ and $e$ does not have same color as any edge $e_j \in T$ for $j < i$.

5. For each edge $e$ added to the rainbow matching in Step 4, delete all the edges of the same color as $e$ from $G$.

4 Intuitive analysis

We aim to show that if we run this randomized algorithm until we have picked almost all the colors we need, then near the end, each remaining color still has so many edges left (relative to the number of colors we still need to pick to finish) that we can conclude via the simple greedy algorithm. To this end, it is useful to track the evolution of the color class sizes. We do that by modeling the remaining color class size and vertex degrees using a system of differential equations.

Define $d_{0,v}$ to be the initial degree of $v$ in $G$. We will define two functions $s(x)$ and $g(x)$ such that after the $t$-th iteration of the algorithm we have the following:

1. Each remaining color class has size around $s_t = s(t\delta)(1 + \epsilon)q$.

2. Each surviving vertex $v$ has degree, $d_{t,v} \approx (1 - t\delta)g(t\delta)d_{0,v}$.

An explicit small value of $\epsilon$ will be picked in Section 7. For now, it can be assumed that $\epsilon < \frac{1}{10}$ (in fact, it will be much smaller than $\frac{1}{10}$). We outline a rough analysis below. Clearly $s(0) = 1$ and $g(0) = 1$. Assume that $t$-th iteration is done, and the properties are true. First of all, there will be very few edges which will be discarded in Step 4 throughout the process. In other words, the number of edges in the partial rainbow matching will be roughly $t\delta q$ after the $t$-th round.

Because of Step 3, every vertex of $G$ is deleted with the same probability among Steps 2 and 3 combined. This is done only for convenience in our analysis, so we should set that probability (denote it by $a_t$) as low as possible. In other words, we need to find the maximum probability a vertex $v$ can be deleted in Step 2, and set $a_t$ to be that maximum probability. After the $t$-th iteration, the number of remaining colors is about $(1 - t\delta)q$, and so the number of remaining edges is about $(1 - t\delta)qs_t$. Now if one picks an edge uniformly at random from $G$, then the probability (denote by $p$) that one of the edges incident to $v$ will be picked is exactly $d_{t,v}$ divided by the total number of edges of $G$, which is about $\frac{d_{t,v}}{(1 - t\delta)qs_t}$. So the probability that $v$ is not deleted in Step 2 is exactly $(1 - p) = 1 - \frac{\delta g(t\delta)d_{0,v}}{s(t\delta)(1 + \epsilon)q}$. So, we can define $a_t$ to be $\frac{\delta g(t\delta)d_{0,v}}{s(t\delta)(1 + \epsilon)q} = \gamma \frac{\delta g(t\delta)}{s(t\delta)} (note that d_{0,v} \leq q)$, where $\gamma = \frac{1}{1 + \epsilon}$.

The probability that a single vertex gets deleted among Steps 2 and 3 is $a_t$, so the probability that a pair of vertices get deleted is about $a_t^2$, which we ignore in this intuitive analysis as it is a strictly lower-order term. So, the expected number of edges deleted in
each color class among Steps 2 and 3 is approximately $2a_ts_t$. This suggests the following behavior:

$$s'(x) = -2\gamma g(x).$$

(4.1)

Next, let us estimate the change in $d_{t,v}$ to get a differential equation for $g$. Any edge incident to $v$ will be deleted in Step 5 by probability $\approx \frac{\delta q}{(1-t\delta)q}$, due to the fact that there are about $\delta q$ colors picked in Step 1, relatively few conflicts in Step 4, and the number of colors remaining is about $(1-t\delta)q$. We also know that each neighbor of $v$ is deleted among Steps 2 and 3 with probability $a_t$. So, neglecting small error terms due to lack of independence, we expect to have the following:

$$d_{t+1,v} - d_{t,v} \approx - \left[ a_t + (1-a_t)\frac{\delta q}{(1-t\delta)q} \right] d_{t,v}$$

$$- (t - 1)\delta g((t + 1)\delta) d_v - (t - t\delta) g(t\delta) d_v \approx -\delta g(t\delta) d_v - (t - t - 1)\delta g(t\delta) d_v a_t$$

$$\frac{g((t + 1)\delta) - g(t\delta)}{\delta} \approx -\gamma \cdot \frac{g(t\delta)^2}{s(t\delta)}. \quad (4.2)$$

Equation (4.2) suggests: $g'(x) = -\gamma \frac{g(x)^2}{s(x)}$. Combining with Equation (4.1), we have that $\frac{ds}{dx} = \frac{1}{2g}$, whose solution is $s = cg^2$ for some constant $c$. By the initial conditions that $g(0) = s(0) = 1$, we get that $s = g^2$. Now, Equation (4.2) implies that $g'(x) = -\gamma$. Solving this with the initial condition $g(0) = 1$, we obtain that $g(x) = 1 - \gamma x$. Hence, $s(x) = (1 - \gamma x)^2$ because $s = g^2$.

We show in the next section that the degrees of vertices and the sizes of color classes are concentrated in some sense throughout the process, which implies that we can not get stuck as long as $t \leq \eta \tau$. Moreover, after $\eta \tau$-th iteration, the number of edges in any color class $\approx (1 - \gamma \eta)^2(1 + \epsilon)q$, and the number of remaining color is about $q - \eta \tau \delta q = (1 - \eta)q$. As long as $(1 - \gamma \eta)^2(1 + \epsilon)q > 2\Delta(1 - \eta)q$, we can finish the rainbow matching greedily, which can be made true by choosing $1 - \eta$ small enough compared to $1 - \gamma$.

5 Formal analysis

The previous section motivates us to define $s(x) = (1 - \gamma x)^2$ and $g(x) = 1 - \gamma x$, where $\gamma = \frac{1}{1+\epsilon}$. From now on, we introduce error terms $\alpha_t$ and $\beta_t$ (which will be explicitly specified later and much less than $\frac{1}{100}$), which will accumulate as we run the process (i.e., as $t$ increases).

It will be easier to analyze Algorithm if all colors have same number of edges at the starting of an iteration. We will add now a 6th step in Algorithm to ensure that all the remaining color classes have same number of edges, $s_t = (1 - \alpha_t)s(t\delta)(1 + \epsilon)q$.

• 6. For each remaining color class, if the number of edges in that color class is more than $s_t$ after Step 5, then delete arbitrary edges of that color class to make sure that it has exactly $s_t$ edges.

We make sure the following two happen after each iteration:
1. The remaining color classes have size exactly:

\[ s_t = (1 - \alpha_t)s(t\delta)(1 + \epsilon)q. \]  

(5.1)

2. The degree of each survived vertex is at most:

\[ d_t = (1 + \beta_t)(1 - t\delta)g(t\delta)q. \]  

(5.2)

Note that our definitions of \( s_t \) and \( d_{t,v} \) were different in the last section. For the convenience of writing, those ideal values are denoted by a tilde symbol over the corresponding notations, which will be used in Sections 6 and 7 to estimate the error terms \( \alpha_t \) and \( \beta_t \).

If one picks an edge uniformly at random from \( G \), then the probability (denote by \( p \)) that one of the edges incident to \( v \) will be picked is at most \( d_t \) divided by the total number of edges of \( G \), which is at most \( d_t(1 - t\delta)qs \). So the probability that \( v \) is not deleted in Step 2 is exactly \( (1 - p)\delta_q \geq \frac{1}{1 + \epsilon} \). So as in Section 4, we can define \( a_t \) to be \( \gamma \cdot \delta_q \).

Let us first specify a probability for Step 3. For a fixed \( t \), at the \((t+1)\)-st iteration we delete each vertex \( v \) independently with probability \( p_v \) such that \( b_v + (1 - b_v)p_v = a_t \), where \( b_v \) is the probability that \( v \) gets deleted in Step 2.

We next give certain values to the parameters \( \epsilon, \delta, \) and \( \eta \) in terms of \( q \) and \( \Delta \) to show that the error terms \( \alpha_t \) and \( \beta_t \) are negligible. Recall that \( \Delta = O\left(\frac{q^2}{(\log q)^{8}}\right) \). Define the following:

\[ \epsilon = \left(\frac{\Delta^5}{q}\right)^\frac{1}{5}(\log q)^{3}, \quad \delta = \left(\frac{\Delta^2}{q}\right)^\frac{1}{2} \log q, \quad \eta = 1 - \left(\frac{\Delta}{q^2}\right)^\frac{1}{2}. \]

Our analysis is inspired by the “nibble” technique introduced in Rödl’s pioneering work \[26\], which was later developed more by various authors. In each iteration we make some deterministic choice by some probabilistic argument, and maintain the properties (5.1) and (5.2) throughout all iterations. Next we state few lemmas showing that such deterministic choices can be made. We need a few large-deviation inequalities for our analysis.

### 5.1 Probabilistic tools

We state some standard tools which will be used throughout the paper.

**Theorem 5.1** (Chernoff bound, see \[17\]). Let \( X = \sum_{i=1}^{n} X_i \), where \( X_i = 1 \) with probability \( p_i \) and \( X_i = 0 \) with probability \( 1 - p_i \), and all \( X_i \) are independent. Let \( \mu = \mathbb{E}(X) = \sum_{i=1}^{n} p_i \). Then for any \( 0 < \lambda < \mu \), we have that

\[ \mathbb{P}(|X - \mu| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2\mu}}. \]

We need Azuma-Hoeffding inequality (see, e.g., \[1\] and \[17\]) to prove certain concentration bound on the number of discarded edges at a single iteration, the sizes of the color classes and the degrees of vertices. More specifically, we use applications of Azuma-Hoeffding inequality in two general settings from Chapter 7 of \[11\]. We mention both of them subsequently.
For finite sets $A$ and $B$, let $\Omega = A^B$ denote the set of functions $g : B \rightarrow A$. Now define a probability measure by setting $P[g(b) = a]$ where the values of $g(b)$ are independent for different $b$. Fix a gradation $\emptyset = B_0 \subset B_1 \subset \cdots \subset B_N = B$. Let $L : A^B \rightarrow \mathbb{R}$ be a functional. Now define a martingale $X_0, X_1, \ldots, X_N$ as the following:

$$X_j(h) = \mathbb{E}(L(g) \mid g(b) = h(b) \text{ for all } b \in B_j).$$

Clearly, $X_0$ is the deterministic expected value of $L$, and $X_N$ is the random variable $L$. The values of $X_j(g)$ approach $L(g)$ as the values of $g(b)$ are exposed. We say that the functional $L$ is $l$-Lipschitz relative to the given gradation if for all $0 \leq j < m$:

$$h, h' \text{ only differ on } B_{j+1} \setminus B_j \Rightarrow |L(h) - L(h')| \leq l. \quad (5.3)$$

**Theorem 5.2.** Let $L$ satisfies the Lipschtiz condition $(5.3)$ relative to a gradation of length $m$, and let $\mu = \mathbb{E}(L(g))$. Then for all $\lambda > 0$,

$$P[|L(g) - \mu| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{2ml^2}} \quad (5.4)$$

Next we mention a special case of Theorem 7.4.3 from Chapter 7 of [11], which was originally used by Alon, Kim and Spencer in [7].

**Theorem 5.3.** Let $I$ be a finite index set. For $i \in I$, let $c_i$ and $p_i$ be positive real numbers such that $p_i \leq 1$ and $c_i \leq C$ for $i \in I$ and some $C > 0$, and let $X_i$ be mutually independent random variables defined by the following:

$$P[X_i = c_i] = p_i, \quad P[X_i = 0] = 1 - p_i$$

Let $\sigma^2$ be an upper bound on the variance of $X = \sum_{i \in I} X_i$. We then have the following:

$$P[|X - \mathbb{E}(X)| > \lambda\sigma] < 2e^{-\frac{\lambda^2}{4\sigma^2}},$$

for all positive $\lambda$ with $\lambda < \frac{2\sigma}{C}$.

### 5.2 Concentration bounds

For the convenience of writing, we define a notion “with very high probability” (in short, w.v.h.p.). We say an event $A_q$ happens w.v.h.p. to mean that the probability that $A_q$ occurs with probability at least $1 - e^{-\omega(\log q)}$, where $\omega(n)$ is any function whose ratio with $n$ tends to infinity as $n$ tends to infinity.

**Lemma 5.4.** For each fixed $t$, the number of edges discarded in Step 4 at the $(t + 1)$-st iteration is at most $\frac{35q^2}{1 - \delta^q}$ w.v.h.p.

**Proof.** For $i < j$, let $X_{i,j}$ be the indicator random variable for the event that $e_i \in T$ and $e_j \in T$ are adjacent or have the same color. Clearly, the total number of edges discarded in Step 4 is given by $X \leq \sum_{1 \leq i < j \leq \delta q} X_{i,j}$. Now for $i < j$, the probability of the event that
adjacent to e ∈ E. So, we can conclude that w.v.h.p. it holds that X = 1 is at most \( \sum_{1 \leq i < j \leq \delta q} X_{i,j} \) where \( f(e) \) is the probability that a uniformly selected edge in \( G \) is adjacent to \( e \) or have the same color as \( e \). For any fixed \( e ∈ G \), the number of edges adjacent to \( e \) is at most \( 2d_t \) and the number of edges which has the same color as \( e \) is at most \( s_t \). So, the probability that \( X_{i,j} = 1 \) is at most \( \frac{2d_t + s_t}{(1 - t\delta)q s_t} \). By using the linearity of expectation and the union bound we get the following:

\[
\mathbb{E}(X) \leq \mathbb{E} \left( \sum_{1 \leq i < j \leq \delta q} X_{i,j} \right) \leq \sum_{1 \leq i < j \leq \delta q} \mathbb{P}[X_{i,j} = 1] \\
\leq \frac{(\delta q)^2}{2} \cdot \frac{2d_t + s_t}{(1 - t\delta)q s_t} \\
\leq \frac{\delta^2 q}{2} \left( \frac{2(1 + \beta_t)(1 - t\delta)g(t\delta)q + 1}{1 - t\delta} \right) \\
\leq \frac{\delta^2 q}{2} \left( 1 - \gamma t\delta + \frac{1}{1 - t\delta} \right) \\
\leq \frac{2\delta^2 q}{1 - t\delta}.
\]

Now we use Theorem 5.2 to show that the random variable \( X \) is not much greater than its expectation w.v.h.p. In the setting mentioned right before Theorem 5.2, let \( A \) be the set of all remaining edges of \( G \) at the start of \( (t + 1) \)-st iteration, \( B = \{1, 2, \ldots, \delta q\} \), and \( \mathbb{P}[g(j) = e] \) is same for all edge \( e \). The random variable \( X \) is the functional \( L : A^B \rightarrow \mathbb{R} \) and the corresponding martingale is \( X_0 = E(L), X_1, \ldots, X_{\delta q} \), with respect to the gradation \( 0 ⊂ [1] ⊂ [2] ⊂ \ldots [\delta q] \) (where \([n]\) denotes the set \{1, 2, \ldots, n\}). If one edge is replaced by some other edge in any one of the \( \delta q \) co-ordinates, then \( L \) will be effected by at most 2, in other words, 2 is an upper-bound on the Lipschitz constant of the martingale. By the conclusion of Theorem 5.2 we can conclude that the probability that \( L \) deviates from its mean by \( \lambda = \delta^2 q \) is at most \( 2e^{-\frac{\lambda^2}{8s_t}} = 2e^{-\frac{\delta^2 q}{8}} = e^{-\omega(\log q)} \) (remember that \( \delta = \left( \frac{\delta^2}{q} \right)^\frac{1}{2} \log q \)).

So, we can conclude that w.v.h.p. it holds that \( X \leq \mathbb{E}(X) + \delta^2 q \leq \frac{2\delta^2 q}{1 - t\delta} + \delta^2 q \leq \frac{3\delta^2 q}{1 - t\delta} \).

**Corollary 5.5.** The total number of edges not added in the rainbow matching in Step 4 throughout the algorithm is at most \( 3\delta q \log \left( \frac{1}{1 - \eta} \right) \) w.v.h.p. Consequently, the number of colors left after the \( t \)-th iteration is at most \( \left( 1 - t\delta + 3\delta \log \left( \frac{1}{1 - \eta} \right) \right) q \).

**Proof.** By using Lemma 5.4, the total number is at most the following:

\[
\sum_{t=0}^{\frac{3\delta q}{1 - t\delta}} \leq 3\delta q \int_0^\eta \frac{dx}{1 - x} = 3\delta q \log \left( \frac{1}{1 - \eta} \right).
\]

**Lemma 5.6.** For each fixed \( t ≤ \eta \tau \), in the \( (t + 1) \)-st iteration, the number of edges deleted from a single color class in Steps 2 and 3 combined is at most \( 2a_t s_t + 3\delta^2 q + 2\Delta \sqrt{\delta q} \log q \) w.v.h.p.
Proof. Fix a color remaining at the start of the iteration \( t + 1 \). Let \( s_t \) denote the number of edges in that color class before starting the \((t + 1)\)-st iteration, and \( S_{t+1} \) denote the number of edges right after Steps 2 and 3 in the \((t + 1)\)-st iteration. Clearly, \( S_t - S_{t+1} \leq L + X \), where \( L \) is the random variable denoting the number of edges of the fixed color removed because of Step 2, and \( X \) is the same number because of Step 3. We bound \( L \) and \( X \) separately. Recall that the vertex \( v \) gets deleted with probability \( b_v \) in Step 2 and with probability \( p_v \) in Step 3, where \( b_v + p_v = a_t + b_v p_v \leq a_t + a_t^2 \). For convenience, let \( \mathcal{H} \) denote the set of edges in the fixed color class before starting the \((t + 1)\)-st iteration.

Firstly, we have that \( \mathbb{E}(L) \leq \sum_{u \in \mathcal{H}} (b_u + b_v) \), because the probability that an edge gets deleted due to the deletion of one of its vertices in Step 2 is at most \( b_u + b_v \) by a simple union bound. To show concentration of \( L \), we use the same sets \( A, B \), and the same distribution of \( g \) as in the proof of Lemma 5.4. For the functional \( L \), we consider the martingale considered in Theorem 5.2. If one edge is replaced by some other edge in any one of the \( \delta q \) co-ordinates, then \( L \) will be affected by at most \( 2\Delta \). So, \( 2\Delta \) is a bound on the Lipschitz constant of the martingale. Hence, using Theorem 5.2, we can conclude that the probability that \( L \) deviates from its mean by \( \lambda = \Delta \sqrt{q} \log q \) is at most \( 2e^{-\frac{\lambda^2}{2\Delta^2q}} = e^{-\omega(\log q)} \).

We analyze the behavior of \( X \) by stochastically dominating it by a simpler random variable which counts the number of edges deleted in Step 3 without deleting anything in Step 2 (this is done to avoid conditioning on \( L \)). Clearly \( X \) is upper-bounded by \( Y = \sum_{v \in I} Y_v \), where \( I \) is the collection of vertices which is adjacent to at least one edge of the fixed color, and \( Y_v \) is the random variable that equals to the number of edges of the fixed color adjacent to \( v \) if \( v \) is deleted in Step 3, else \( Y_v = 0 \). It is clear that \( \mathbb{E}(Y) = \sum_{u \in \mathcal{H}} (p_u + p_v) \). For the random variables \( Y_v \) and \( Y \) in Theorem 5.3, we have \( C = \Delta \) and \( \text{Var}(Y) = \sum_{v \in I} \text{Var}(Y_v) \leq \sum_{v \in I} a_t \Delta^2 \), and so let \( \sigma = \sqrt{2a_t \Delta} \). Next, we show that \( \lambda \) can be picked as \( \frac{\log q}{2\sqrt{2}} \) by ensuring that \( \frac{2a}{C} = 2\sqrt{2a_t \sigma} > \frac{\log q}{2\sqrt{2}} \) for all \( t \leq \eta \). For \( t \leq \eta \), we have the following by plugging in the values of \( \epsilon, \delta, \) and \( \eta \):

\[
a_t s_t = \frac{\delta q(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)} \cdot (1 - \alpha_t) s(t\delta)(1 + \epsilon) q \geq \delta (1 - \eta)(1 + \beta_t) q \\
\geq 2\delta \epsilon (1 - \eta) q \\
> \frac{(\log q)^2}{16},
\]

which shows that \( 2\sqrt{2a_t \sigma} > \frac{\log q}{2\sqrt{2}} \). So, by using Theorem 5.3, the probability that \( Y \) deviates by more than \( \lambda \sigma = \frac{3}{2} \sqrt{a_t \sigma} \log q \) from its expectation is less than \( e^{-\omega(\log q)} \).

Observe that \( \mathbb{E}(L) + \mathbb{E}(Y) \leq \sum_{u \in \mathcal{H}} (b_u + b_v + p_u + p_v) \leq 2(a_t + a_t^2) s_t \). Hence, using triangle inequality we conclude that the random variable \( L + X \) does not exceed from \( 2(a_t + a_t^2) s_t \) by more than \( 2\Delta \sqrt{q} \log q + \frac{3}{2} \sqrt{a_t \sigma} \log q \) w.v.h.p. Now, Lemma 5.6 follows by noting the following two inequalities:

\[
a_t s_t = \frac{\delta g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)} \cdot (1 - \alpha_t) s(t\delta)(1 + \epsilon) q \leq 2\delta q,
\]
\[
a_t^2 s_t = \frac{\delta^2 g(t\delta)^2(1 + \beta_t)^2}{s(t\delta)^2(1 - \alpha_t)^2} \cdot (1 - \alpha_t) s(t\delta)(1 + \epsilon) q \leq \frac{3}{2} \delta^2 q
\]
Lemma 5.7. For each fixed $t \leq \eta \tau$, at the end of $(t+1)$-st iteration, w.v.h.p. the degree of each remaining vertex $v$ is at most:

$$d_t \cdot (1 - a_t) \left( 1 - \frac{1}{\tau - t} \left( 1 - \frac{\Delta}{s_t} - \frac{4\delta \log \left( \frac{1}{1-\eta} \right)}{1 - t\delta} \right) \right) + 2\Delta \sqrt{\delta q} \log q.$$ 

Proof. First of all, Step 6 can only decrease the degrees of any vertex, so it is enough to prove Lemma 5.7 before Step 6. For each remaining vertex $v$ after $t$-th iteration, we define $D_{t+1,v}$ to be the random variable denoting the degree of $v$ after Step 5 in $(t+1)$-st iteration. Throughout the proof of this lemma, for a simple analysis we assume that Step 3 is done before Step 2 (this can be done because these two steps can be done independently). Fix a vertex $v$. Let $d$ denote the degree of $v$ after $t$-th iteration of Algorithm. Clearly, $d \leq d_t$. Let $X$ be the random variable denoting the number of neighbors of $v$ deleted in Step 3, and $L$ be the random variable denoting the number of edges $uv$ adjacent to $v$ deleted in Steps 2 (after Step 3) and 5 and $u$ is not deleted in Step 3. Clearly, $D_{t+1,v} = d - (X + L)$. So, we are interested in a good lower bound on the random variable $X + L$.

Observe that $\mathbb{E}(X) = \sum_{u \in N(v)} p_u$. Clearly, $X$ is the sum of at most $d_t$ Bernoulli random variables with probability at most $a_t$. So, by using standard Chernoff bound 5.1, the probability that $X$ deviates by more than $\sqrt{\frac{a_t d_t \log q}{2}} \leq \sqrt{\frac{\delta q \log q}{2}}$ from its expectation is at most $e^{-\omega(\log q)}$, because:

$$a_t d_t = \gamma \frac{\delta q (t\delta) (1 + \beta_t)}{s(t\delta)} (1 + \beta_t)(1 - t\delta)^g(t\delta) q \leq 2\delta q.$$

Now to facilitate the analysis of the random variable $L$, we introduce the random variable $X' = \sum_{u \in N(v)} b_u 1_{u \in N'}$, where $N'$ denote the (random) subset of $N(v)$ containing all the vertices which are not deleted in Step 3. Clearly, $\mathbb{E}(X') = \sum_{u \in N(v)} b_u (1 - p_u)$. Again by Chernoff bound 5.1, the probability that $X'$ deviates by more than $\sqrt{\frac{\delta q \log q}{2}}$ from its expectation is at most $e^{-\omega(\log q)}$. Let $F$ denote the event that both $X$ and $X'$ are within a gap of $\sqrt{\frac{\delta q \log q}{2}}$ from their corresponding expected values. Next, we show the concentration bound on $L$ conditioning on this event $F$ which holds w.v.h.p. Lemma 5.7 follows from the following claim.

Claim 5.8. Conditioned on $F$, for each fixed $t \leq \eta \tau$, at the end of Step 5 of the $(t+1)$-st iteration, w.v.h.p. the degree of each remaining vertex $v$ after $t$-th iteration is at most:

$$d_t \cdot (1 - a_t) \left( 1 - \frac{1}{\tau - t} \left( 1 - \frac{\Delta}{s_t} - \frac{4\delta \log \left( \frac{1}{1-\eta} \right)}{1 - t\delta} \right) \right) + 2\Delta \sqrt{\delta q} \log q.$$ 

Proof. Fix a vertex $v$ that survived after the $t$-th iteration. The probability that any neighbor $u$ of $v$ is deleted in Step 2 is exactly $b_u$. So, the next task remaining is to calculate the probability that an edge $uv$ will be deleted in Step 5, conditioned on the fact that the vertex $u$ did not get deleted in Step 2. For the convenience of writing, let $A$ denote the event that
an edge of the same color as $uv$ is picked in the partial rainbow matching from $T$ in Step 4. Let $B$ denote the event that $u$ does not get deleted in Step 2. We are interested in getting a good lower-bound on $\Pr[A|B]$.

The event $B$ is same as the event that none of the edges uniformly picked in Step 1 is adjacent to the vertex $u$. Hence, Conditioning on the event $B$ essentially is same as considering the uniform measure in Step 1 excluding the edges adjacent to $u$. Also note that the event $A$ contains the event $A'$ that exactly one edge $e$ with the same color as $uv$ is picked in Step 1 and no other edges adjacent to $e$ is picked in Step 1. Clearly, we can just calculate this probability for each fixed edge adjacent to $v$ and sum the probability. So, we have the following:

$$\Pr[A|B] \geq \Pr[A'|B] \geq \frac{\delta q(s_t - \Delta)}{(1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)) q^t s_t} \left(1 - \frac{2d_t + s_t}{(1 - t\delta)q^t s_t - d_t}\right)^{\delta q - 1},$$

because, the probability to pick an edge $e$ with the same color as $uv$ if we choose an edge uniformly at random among the edges which are not incident to $u$ is at least $\frac{s_t - \Delta}{(1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)) q^t s_t}$ (which follows from Corollary 5.5). Now, conditioning on the fact that such an $e$ is picked, the probability that each of the remaining $\delta q - 1$ edges is not incident to $e$ and is not of the same color, is at least $\left(1 - \frac{2d_t + s_t}{(1 - t\delta)q^t s_t - d_t}\right)^{\delta q - 1}$. Hence, we can write the following:

$$\Pr[A|B] > \frac{\delta q(s_t - \Delta)}{(1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)) q^t s_t} \left(1 - \frac{2d_t + s_t}{(1 - t\delta)q^t s_t - d_t}\right)^{\delta q}$$

$$> \frac{\delta}{1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)} \cdot \frac{s_t - \Delta}{s_t} \left(1 - \frac{\delta q(2d_t + s_t)}{(1 - t\delta)q^t s_t - d_t}\right)$$

$$> \frac{\delta}{1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)} \left(1 - \frac{\Delta}{s_t}\right) \left(1 - \frac{\delta q(2d_t + s_t)}{1 - t\delta q^t s_t}\right)$$

$$= \frac{\delta}{1 - t\delta} \cdot \frac{1 - t\delta}{1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)} \left(1 - \frac{\Delta}{s_t}\right) \left(1 - \frac{4\delta d_t}{(1 - t\delta)q^t s_t} - \frac{2\delta}{1 - t\delta}\right)$$

$$> \frac{1}{\tau - t} \left(1 - \frac{3\delta \log \left(\frac{1}{1 - \eta}\right)}{1 - t\delta}\right) \left(1 - \frac{\Delta}{s_t}\right) \left(1 - \frac{7\delta}{1 - t\delta}\right)$$

$$> \frac{1}{\tau - t} \left(1 - \frac{3\delta \log \left(\frac{1}{1 - \eta}\right)}{1 - t\delta} - \frac{\Delta}{s_t} - \frac{7\delta}{1 - t\delta}\right)$$

$$> \frac{1}{\tau - t} \left(1 - \frac{\Delta}{s_t} - \frac{4\delta \log \left(\frac{1}{1 - \eta}\right)}{1 - t\delta}\right)$$

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**Clarification of (5.5), (5.6), and (5.7):** In (5.5), we have used \((1 - t\delta)qs_t - d_t > \frac{1}{2}(1 - t\delta)qs_t\) for all \(t \leq \eta\tau\), which we show below by plugging in the the values of the parameters \(\epsilon, \delta, \) and \(\eta\) assigned in the beginning of Section 5. For \(t \leq \eta\tau\), we have the following:

\[
\frac{1}{2}(1 - t\delta)qs_t - d_t = \frac{1}{2}(1 - \alpha_t)(1 + \epsilon)(1 - t\delta)(1 - \gamma t\delta)^2 q^2 - (1 + \beta_t)(1 - t\delta)(1 - \gamma t\delta)q
\]

\[
\geq (1 - t\delta)(1 - \gamma t\delta)q \left(\frac{1}{4}(1 - \gamma t\delta)q - 2\right)
\]

\[
\geq (1 - \eta)(1 - \gamma \eta)q \left(\frac{1}{4}(1 - \gamma \eta)q - 2\right)
\]

\[
> 0,
\]

which shows that \((1 - t\delta)qs_t - d_t > \frac{1}{2}(1 - t\delta)qs_t\) for all \(t \leq \eta\tau\), as we desired. In (5.6), we have used that \(\frac{d_t}{s_t} = \frac{(1 + \beta_t)(1 - t\delta)}{(1 - \alpha_t)(1 + \epsilon)(1 - \gamma t\delta)} < 1 + \frac{1}{10}\). In (5.7), we have used the fact that \(\frac{3\delta \log \left(\frac{1}{1 - \eta}\right)}{1 - t\delta}, \Delta, \text{ and } \frac{1}{1 - t\delta}\) are all strictly less than 1 for all \(t \leq \eta\tau\), which can be verified by plugging in the values of the parameters \(\epsilon, \delta, \) and \(\eta\).

Hence, conditioned on \(F\), we have the following:

\[
\mathbb{E}(D_{t+1,v}) \leq \sum_{u \in N'} (1 - b_u) \left(1 - \frac{1}{\tau - t} \left(1 - \frac{\Delta}{s_t} - \frac{4\delta \log \left(\frac{1}{1 - \eta}\right)}{1 - t\delta}\right)\right). \tag{5.8}
\]

Note that

\[
\sum_{u \in N'} b_u = X' \geq \sum_{u \in N(v)} b_u (1 - p_u) - \frac{1}{2}\sqrt{\delta q \log q}.
\]

So, we have the following:

\[
\sum_{u \in N'} (1 - b_u) = (d - X) - \sum_{u \in N'} b_u
\]

\[
\leq \sum_{u \in N(v)} (1 - p_u) + \frac{1}{2}\sqrt{\delta q \log q} - \sum_{u \in N(v)} b_u (1 - p_u) + \frac{1}{2}\sqrt{\delta q \log q}
\]

\[
\leq \sum_{u \in N(v)} (1 - b_u)(1 - p_u) + \sqrt{\delta q \log q}
\]

\[
= d(1 - a_t) + \sqrt{\delta q \log q},
\]

which we use in (5.8) to get the following conditioned on \(F\):

\[
\mathbb{E}(D_{t+1,v}) \leq d_t(1 - a_t) \left(1 - \frac{1}{\tau - t} \left(1 - \frac{\Delta}{s_t} - \frac{4\delta \log \left(\frac{1}{1 - \eta}\right)}{1 - t\delta}\right)\right) + \sqrt{\delta q \log q}. \tag{5.9}
\]

Furthermore, conditioning on \(F\), we can prove a concentration bound on \(L\) using Theorem 5.2. We use the same sets \(A, B\) and the same probability distribution as in the proof of Lemma 5.4. For the functional \(L\), we consider the martingale considered in Theorem 5.2.
one edge is replaced by some other edge in any one of the $\delta q$ co-ordinates, then $L$ will be
effected by at most $\Delta + 2 \leq 3\Delta$, because picking a different color in the partial rainbow
matching in Step 4 can change the number of deletions of edges adjacent to $v$ in Step 5 by
at most $\Delta$ and the number of neighbors deleted in Step 2 can differ by at most 2. So, $3\Delta$
is a bound on the Lipschitz constant of the martingale. Hence, using Theorem \ref{thm:lip}, we can
conclude that the probability that $L$ deviates from its mean by at least $\lambda = \Delta \sqrt{\delta q \log q}$ is at
most $2e^{-\frac{\lambda^2}{2\delta q(3\Delta)^2}} = e^{-\omega(\log q)}$. This together with \ref{claim:prob_dev} finishes the proof of Claim \ref{claim:lip_dev}
\hfill $\square$

6 Estimating the error terms and greedy completion

In order to estimate the error terms in the parameters throughout the algorithm, let us
first estimate the error terms in the ideal expressions from the intuitive analysis. Define the
following parameters (recall that in the beginning of Section 5, we mentioned that we would
denote the ideal values of parameters by tilde symbol over the corresponding notations).

\[
\tilde{s}_t = s(t\delta)(1 + \epsilon)q. \\
\tilde{d}_t = (1 - t\delta)g(t\delta)q. \\
\tilde{a}_t = \gamma \delta g(t\delta) s(t\delta). 
\]

We have the following couple of lemmas concerning the relations between these three
ideal functions.

\textbf{Lemma 6.1.} \textit{We have that $(1 - 2\tilde{a}_t)\tilde{s}_t = \tilde{s}_{t+1} - \gamma^2 \delta^2 (1 + \epsilon)q$.}

\textit{Proof.} A routine calculation shows:

\[
(1 - 2\tilde{a}_t)\tilde{s}_t = \left(1 - \frac{2\gamma \delta g(t\delta)}{s(t\delta)}\right) s(t\delta)(1 + \epsilon)q \\
= \left(1 - \frac{2\gamma \delta}{s(t\delta)}\right) s(t\delta)(1 + \epsilon)q \\
= (1 - \gamma t\delta)^2 - 2\gamma \delta (1 - \gamma t\delta) \gamma^2 \delta^2 (1 + \epsilon)q \\
= \tilde{s}_{t+1} - \gamma^2 \delta^2 (1 + \epsilon)q.
\]

\hfill $\square$

\textbf{Lemma 6.2.} \textit{We have that $(1 - \tilde{a}_t) (1 - \frac{1}{\tau - t}) \tilde{d}_t = \tilde{d}_{t+1}$.}

\textit{Proof.} A routine calculation shows:

\[
(1 - \tilde{a}_t) (1 - \frac{1}{\tau - t}) \tilde{d}_t = \left(1 - \frac{\gamma \delta (1 - \gamma t\delta)}{1 - \gamma t\delta}\right) \left(1 - \frac{\delta}{1 - t\delta}\right) (1 - t\delta)g(t\delta)q \\
= (1 - (t + 1)\gamma \delta)(1 - (t + 1)\delta)q \\
= \tilde{d}_{t+1}.
\]

\hfill $\square$
Now we mention a couple of lemmas relating the error terms we accumulate for estimating \( d_t \) and \( s_t \) as the process goes. We will find \( y_t \) and \( z_t \) such that \( d_t \leq \tilde{d}_t + y_t \) and \( s_t \geq \tilde{s}_t - z_t \). We will then define that \( \alpha_t = \frac{\alpha}{\tilde{s}_t} \) and \( \beta_t = \frac{\beta}{\tilde{d}_t} \). So, as we promised, we will find that the accumulated error terms are negligible compared to the ideal parameter values, i.e., \( y_t \ll \tilde{d}_t \) and \( z_t \ll \tilde{s}_t \).

**Lemma 6.3.** For each \( t \leq \frac{\eta}{\hat{\beta}} \), we can choose \( y_t \) such that

\[
y_t \leq 3\Delta \log \left( \frac{1}{1 - \eta} \right) + 4\delta q \log \left( \frac{1}{1 - \eta} \right)^2 + 4t\delta^2 q \log \left( \frac{1}{1 - \eta} \right) + 2t\Delta \sqrt{\delta q} \log q.
\]

**Proof.** Starting with the conclusion from Lemma 5.7 and noting that \( a_t = \tilde{a}_t \cdot \frac{1 + \beta_t}{1 - \alpha_t} \geq \tilde{a}_t \), a routine calculation shows w.v.h.p.:

\[
D_{t+1,v} \leq d_t \cdot (1 - \tilde{a}_t) \left( 1 - \frac{1}{\tau - t} \right) \left( 1 - \frac{\Delta}{s_t} - \frac{4\delta \log \left( \frac{1}{1 - \eta} \right)}{1 - t\delta} \right) + 2\Delta \sqrt{\delta q} \log q
\]

\[
\leq (\tilde{d}_t + y_t) \left( 1 - \frac{1}{\tau - t} \right) + 1 - \tilde{a}_t \left( \frac{\Delta}{s_t} + \frac{4\delta \log \left( \frac{1}{1 - \eta} \right)}{1 - t\delta} \right) + 2\Delta \sqrt{\delta q} \log q
\]

\[
\leq \tilde{d}_t (1 - \tilde{a}_t) \left( 1 - \frac{1}{\tau - t} \right) + \tilde{d}_t \delta (1 - \tilde{a}_t) \left( \frac{\Delta}{s_t} + \frac{4\delta \log \left( \frac{1}{1 - \eta} \right)}{1 - t\delta} \right) + y_t + 2\Delta \sqrt{\delta q} \log q
\]

\[
\leq \tilde{d}_{t+1} + y_t + \frac{2\delta \Delta}{1 - \gamma t\delta} + 4\delta^2 q \frac{1 - \gamma t\delta}{1 - t\delta} \log \left( \frac{1}{1 - \eta} \right) + 2\Delta \sqrt{\delta q} \log q.
\]

So, we can choose \( y_t \) with the following recursion relation:

\[
y_{t+1} = y_t + \frac{2\delta \Delta}{1 - \gamma t\delta} + 4\delta^2 q \frac{1 - \gamma t\delta}{1 - t\delta} \log \left( \frac{1}{1 - \eta} \right) + 2\Delta \sqrt{\delta q} \log q.
\]

With \( y_0 = 0 \), this recursion gives us the following upper-bound on \( y_t \).

\[
y_t = \sum_{i=0}^{t-1} \left( \frac{2\delta \Delta}{1 - \gamma i\delta} + 4\delta^2 q \log \left( \frac{1}{1 - \eta} \right) \frac{1 - \gamma i\delta}{1 - i\delta} + 2\Delta \sqrt{\delta q} \log q \right)
\]

\[
\leq 2\Delta \int_0^{t\delta} \frac{dx}{1 - \gamma x} + 4\delta q \log \left( \frac{1}{1 - \eta} \right) \int_0^{t\delta} \frac{1 - \gamma x}{1 - x} dx + 2t\Delta \sqrt{\delta q} \log q
\]

\[
= \frac{2\Delta}{\gamma} \log \left( \frac{1}{1 - \gamma t\delta} \right) + 4\delta q \log \left( \frac{1}{1 - \eta} \right) \left( 1 - \gamma \right) \log \left( \frac{1}{1 - t\delta} \right) + \gamma t\delta + 2t\Delta \sqrt{\delta q} \log q
\]

\[
\leq 3\Delta \log \left( \frac{1}{1 - \eta} \right) + 4\delta q \log \left( \frac{1}{1 - \eta} \right)^2 + 4t\delta^2 q \log \left( \frac{1}{1 - \eta} \right) + 2t\Delta \sqrt{\delta q} \log q.
\]

\[\Box\]

**Lemma 6.4.** For each \( t \leq \frac{\eta}{\hat{\beta}} \), we have that \( y_t = o \left( \tilde{d}_t \right) \).

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Proof. Plugging in the parameter values in Lemma 6.3 we get the following:

\[ y_t \leq y_{\frac{\eta}{\delta}} \leq 100(\Delta q)^{\frac{3}{2}} (\log q)^2. \]  
\tag{6.1}

Plugging in the parameters in the expression for \( \tilde{d}_t \) we have the following:

\[ \tilde{d}_t \geq \tilde{d}_{\frac{\eta}{\delta}} = (1 - \eta)(1 - \gamma \eta)q \geq \frac{1}{2}(\Delta q)^{\frac{3}{2}} (\log q)^3. \]  
\tag{6.2}

By Equations (6.1) and (6.2), we conclude that Lemma 6.4 holds.

Lemma 6.5. For each \( t \leq \frac{\eta}{\delta} \), we can choose \( z_t \) such that

\[ z_t \leq 4t\delta^2 q + 2t\Delta \sqrt{\delta q \log q} + 2y_t \log \left( \frac{1}{1 - \eta} \right). \]

Proof. Recall that \( \alpha_t = \frac{\tilde{s}_t}{s_t} \) and \( \beta_t = \frac{y_t}{\tilde{d}_t} \). Starting with the conclusion from Lemma 5.6, a routine calculation shows the following w.v.h.p.:

\[
S_{t+1} \geq (1 - 2\alpha_t)s_t - 3\delta^2 q - 2\Delta \sqrt{\delta q \log q} \\
= (1 - 2\bar{a}_t \frac{1 + \beta_t}{1 - \alpha_t}) (1 - \alpha_t) \tilde{s}_t - 3\delta^2 q - 2\Delta \sqrt{\delta q \log q} \\
= (1 - 2\bar{a}_t)\tilde{s}_t - \alpha_t \tilde{s}_t - 2\bar{a}_t \beta_t \tilde{s}_t - 3\delta^2 q - 2\Delta \sqrt{\delta q \log q} \\
= \tilde{s}_{t+1} - \gamma^2 \delta^3 (1 + \epsilon)q - z_t - \frac{2\gamma \delta(1 + \epsilon)}{1 - t\delta} \cdot y_t - 3\delta^2 q - 2\Delta \sqrt{\delta q \log q} \\
\geq \tilde{s}_{t+1} - 4\delta^2 q - z_t - \frac{2\delta}{1 - t\delta} \cdot y_t - 2\Delta \sqrt{\delta q \log q}.
\]

So, similar to Lemma 6.3, we can choose \( z_t \) with the following recursion relation:

\[ z_{t+1} = z_t + 4\delta^2 q + 2\Delta \sqrt{\delta q \log q} + \frac{2\delta}{1 - t\delta} \cdot y_t. \]  
\tag{6.3}

With \( z_t = 0 \), the solution of this recursion relation is given by:

\[
z_t = \sum_{i=0}^{t-1} \left( 4\delta^2 q + 2\Delta \sqrt{\delta q \log q} + \frac{2\delta}{1 - i\delta} \cdot y_i \right) \\
\leq 4t\delta^2 q + 2t\Delta \sqrt{\delta q \log q} + 2y_t \int_0^{t\delta} \frac{dx}{1 - x} \\
\leq 4t\delta^2 q + 2t\Delta \sqrt{\delta q \log q} + 2y_t \log \left( \frac{1}{1 - \eta} \right).
\]

Lemma 6.6. For each \( t \leq \frac{\eta}{\delta} \), we have that \( z_t = o(\tilde{s}_t) \).
Proof. Similar to the proof of Lemma 6.4, plugging in the parameter values in Lemma 6.5, we get the following:

$$z_t \leq z_\frac{\eta}{3} \leq 500(\Delta q)^{\frac{2}{3}}(\log q)^{3}.$$  \hspace{1cm} (6.4)

Now, Lemma 6.6 can be seen to be true by noting the following:

$$\tilde{s}_t \geq \tilde{s}_\frac{\eta}{3} = (1 - \gamma \eta)^2(1 + \epsilon)q \geq \frac{1}{2}q^{\frac{1}{9}}(\Delta q)^{\frac{2}{9}}(\log q)^{6}.$$  \hspace{1cm} (6.4)

Hence, by Lemmas (6.4) and (6.6) we can conclude that we can run the algorithm of Section 3 with the specified values of parameters until $\frac{\eta}{3}$-th iteration. At the end of this iteration, the number of edges in any remaining color is at least $s_\frac{\eta}{3} \geq \frac{1}{2}q^{\frac{1}{9}}(\Delta q)^{\frac{2}{9}}\log q$. Now, it is easy to check that $s_\frac{\eta}{3} > 2\Delta r$. Hence, after the $\frac{\eta}{3}$-th iteration of the algorithm, we can greedily add edges to the partial rainbow matching constructed so far to make a rainbow matching using every color. This finishes the proof of Theorem 1.12.

7 A weaker bound on the main result, Theorem 1.9

We need some easy upper bound on the number of colors needed in Theorem 1.9 to prove the actual statement of it. Although it is possible to use the result proven in [3], which gives us a bound of $3q - 2$ on the required number of colors, we instead use Theorem 1.12 to prove a slightly weaker bound. We have two reasons to do so: (i) We want to further demonstrate the power of the randomized algorithm from Section 3, (ii) We want to keep this paper self-contained. If the readers want, they can skip this section and go to the next one assuming the statement proven in [3].

One can prove a weaker bound of $3q + o(q)$ on the number of colors in Theorem 1.9 using Theorem 1.12. However, as this section is not strictly necessary, we only prove an even weaker bound of $4q$ to keep the arguments simple. In order to do so, we start with showing an easy bound of $2q^2$. Suppose we have a properly colored graph $G$ with $2q^2$ colors where each color appears $q$ times. Consider the maximum size rainbow matching $M = \{c_1, c_2, \ldots, c_m\}$ of $G$. Assume for the sake of contradiction that $|M| = m \leq q - 1$. Let $C$ denote all the colors used to color the edges in any edge connecting two vertices participating in $M$. Clearly, $|C| \leq \left(\begin{array}{c} 2m \\ 2 \end{array}\right) < 2q^2 - 2q$. So, there are $2q$ colors $c_1, c_2, \ldots, c_{2q}$ which do not appear in any of the edges between two vertices in $M$. If there is any edge with any of these $2q$ colors which does not use any of the vertices used in $M$, then we are already done because we can obtain a larger rainbow matching by adding that edge to $M$. So, all the edges using colors from $\{c_1, c_2, \ldots, c_{2q}\}$ uses exactly one vertex in $M$. By a simple application of pigeonhole principle, for every color $c$ from $\{c_1, c_2, \ldots, c_{2q}\}$, there is an edge $e \in M$ such that there are two edges with color $c$, which are incident to $e$. By another application of pigeonhole principle, there exists an edge $e \in M$ such that there are three colors (without loss of generality, say $c_1, c_2,$
and $c_3$) with the property that there are two edges with each color in $\{c_1, c_2, c_3\}$ such that all six of these edges are incident to $e$. Now, it is easy to find two edges among these six edges such that they are not incident to each other and they have distinct colors. By observing that one obtain a larger matching by adding these two edges and subtracting $e$ from $M$, we can finish our argument to prove the desired upper bound of $2q^2$.

Next, we next show that $4q$ colors suffices for all sufficiently large $q$, as promised. Let $q$ be such that $q \geq N^2$, where $N$ comes from Theorem 1.12. Now we split the proof into two cases based on the degree sequence $d_1 \geq d_2 \geq \cdots$ of $G$. Define $k$ to be the minimum integer such that $d_k \leq 3(q - k)$, if it exists.

Case 1: If $k > q - \sqrt{q}$ or $k$ does not exist, then remove $q - \sqrt{q}$ highest degree-vertices from $G$. In the remaining graph there are at least $\sqrt{q}$ edges of each color and the number of colors is $4q \geq 2(\sqrt{q})^2$. So, by using the bound proven in the last paragraph, we can find a rainbow matching using $\sqrt{q}$ colors. Now, we can greedily add an edge adjacent to each of the removed vertices to the rainbow matching to get a rainbow matching of size $q$. This can be done in the following way. Let $v_i$ denotes the vertex with degree $d_i$. Now, the number of colors used so far is $\sqrt{q}$ and the number of vertices used in the rainbow matching is $2\sqrt{q}$. So, one can pick a neighbor of $v_{q-\sqrt{q}}$ out of more than $3\sqrt{q}$ neighbors so that neither the color of that edge nor the neighbor are used in the rainbow matching built yet. We can continue this process greedily for $i = \sqrt{q}, \sqrt{q} - 1, \ldots, 1$ respectively to find a rainbow matching of size $q$.

Case 2: If $k \leq q - \sqrt{q}$, then similarly remove $k$ highest degree-vertices from $G$. Similar to the last time, there are at least $q - k$ edges of each color in the remaining graph, and the maximum degree is at most $3(q - k)$. Now merge 4 colors at a time to make sure that there are $4(q - k)$ edges in each color. So, the total number of colors is now $q$, and each color class has maximum degree at most 4. Pick any $q - k$ colors among those $q$ colors and apply Theorem 1.12 on the graph induced by the edges using only from these $q - k$ colors. By the choice of $N$, we can find a rainbow matching using these $q - k$ colors. Finally, we can finish greedily exactly the same way as last time.

8 Proof of the main result, Theorem 1.9

If we blindly apply the algorithm of Section 3 used for Theorem 1.12 then we can only obtain a matching of size $q - o(q)$, which is not enough for Theorem 1.9. As we have discussed in the introduction, the rigid behavior in terms of the number of edges used in each color class is somewhat responsible for this type of failure of the randomized algorithm used before. It turns out that we need to be extra careful about the vertices with small degree. In order to do so, we first need to show that there cannot be too many vertices with small degree. In order to do so, we first need to show that there cannot be too many vertices with degree close to the number of colors (note that the maximum degree can be at most the number of colors, because we have a proper coloring this time).

Fix $0 < \epsilon < \frac{1}{10}$. We start with a simple graph $G$ that is properly edge-colored with $2(1 + \epsilon)q$ colors such that there are at least $q$ edges of each color, and $G$ does not have a rainbow matching of size $q$. We first apply Theorem 1.10 to show that the number of vertices with degree more than $2(1 + \theta)q$ is at most $(1 - \theta)q$, for $\theta = \frac{\epsilon}{5}$. If not, then select a set $A$ of size exactly $(1 - \theta)q$ such that the degree of any vertex in $A$ is at least $2(1 + \theta)q$. Let
\(G'\) denote the graph after deleting all the edges incident to \(A\). We have at least \(\theta q\) edges in each color in \(G'\). So, by applying the weaker bound for Theorem 1.9 established in the last section, we can find a rainbow matching \(T\) of size \(\theta q\) in \(G'\). Now in \(G\), discard all the vertices incident to \(T\) and all the edges colored with a color used in \(T\). For a fixed \(v \in A\), we discard at most \(3\theta q\) edges incident to \(v\). Hence, \(v\) still has at least \(2(1 + \theta)q - 3\theta q - (1 - \theta)q = q\) neighbors outside of \(A\). Now, with a straightforward application of Theorem 1.10 on the graph \(A \cup B\) (where \(B = V(G) \setminus A\)), we can find a rainbow matching \(T'\) using all the vertices in \(A\). As we have already discarded the vertices and colors used in \(T\), the edge set \(T \cup T'\) gives us a rainbow matching of size \(q\). Hence, from now on, we assume that the number of vertices with degree more than \(2(1 + \theta)q\) is at most \((1 - \theta)q\).

### 8.1 Algorithm

Assume that we are given a simple graph \(G\) that is properly edge-colored with \(2(1 + \epsilon)q\) colors such that there are \(q\) edges of each color. Additionally, assume that there exists \(A \subseteq V(G)\) such that \(|A| \leq (1 - \theta)q\) and the vertices outside \(A\) have degree at most \(2(1 + \theta)q\), where \(\theta = \frac{\epsilon}{2}\). Like before, we are going to provide a randomized algorithm, which constructs a rainbow matching using \(q\) colors in several iterations.

This algorithm is in terms of some parameter \(\delta > 0\), which will be specified later. Define \(\tau = \frac{1}{\delta}\). We follow the algorithm below for \(\eta\tau\) steps, for some \(\eta < \frac{1}{2}\) to be specified later.

1. Select independently \(\delta \cdot 2(1 + \epsilon)q\) edges \(e_1, e_2, \ldots, e_{2\delta(1+\epsilon)q}\) uniformly at random with replacement from among the remaining edges. Denote by \(T\) the set of all selected edges.

2. Delete all the vertices corresponding to the edges in \(T\) from \(G\). Deleting vertices always also deletes all incident edges.

3. With some probability (to be specified later), independently delete each vertices in \(G\). This will make sure that among Steps 2 and 3 combined, every vertex in \(A\) gets deleted with the same probability, and also every vertex outside of \(A\) gets deleted with the same probability (possibly different).

4. For each edge \(e_i \in T\), add it to the rainbow matching if \(e_i\) is not incident to any edge \(e_j \in T\) for \(j < i\) and \(e\) does not have same color as any edge \(e_j \in T\) for \(j < i\).

5. For each edge \(e\) added to the rainbow matching in Step 4, delete all the edges of the same color as \(e\) from \(G\).

6. For each remaining color class, if the number of edges in that color class is more than \(s_t\) after Step 5, then delete edges of that color class in the following way to make sure that it has exactly \(s_t\) edges, where \(s_t\) will be specified later. If the number of edges that need to be deleted is at most the number of edges containing at least one vertex from \(A\), then delete arbitrarily from those edges. Otherwise, delete all the edges with at least one vertex from \(A\), and then delete the rest arbitrarily from the rest of the edges of the same color.
Note that this algorithm is exactly the same as the one in Section 3 (plus Step 6 of Section 5) except Steps 1, 3, and 6. Here, we need to be more careful in deleting the vertices with smaller degree, which is the reason we are treating the vertices outside \( A \) differently in Steps 3 and 6.

### 8.2 Intuitive analysis

As before, we aim to show that if we run this randomized algorithm, then we get a rainbow matching with \( q \) edges. Like last time, define \( d_{0,v} \) to be the initial degree of \( v \) in \( G \). We will define two functions \( s(x) \) and \( g(x) \) such that after the \( t \)-th iteration of the algorithm we have the following:

1. Each remaining color class has size exactly \( s_t \), which is at least about \( s(t\delta)q \).
2. Each surviving vertex \( v \) has degree at most \( d_{t,v} \), which is approximately \((1-\delta)g(t\delta)d_{0,v}\).
3. For each remaining color class, the fraction of number of vertices remaining in \( A \) is at most \( \frac{1-\theta}{2} \).

We outline a rough analysis below. Clearly \( s(0) = 1 \) and \( g(0) = 1 \). Assume that \( t \)-th iteration is done, and the properties are true. First of all, there will be very few edges which will be discarded in Step 4 throughout the process. In other words, the number of edges in the partial rainbow matching will be roughly \( 2(1+\epsilon)t\delta q \) after the \( t \)-th round.

The Step 3 is there just to make the analysis more convenient. It ensures that every vertex of \( A \) is deleted with the same probability (call it \( a_t \), and every vertex outside of \( A \) is also deleted with the same probability (call it \( b_t \)). Like last time, we need to find the maximum probability a vertex \( v \) can be deleted in Step 2, and set \( a_t \) or \( b_t \) to be that maximum probability. After the \( t \)-th iteration, the number of remaining colors is at least \( 2(1-t\delta)(1+\epsilon)q \), and so the number of remaining edges is at least \( 2(1-t\delta)(1+\epsilon)qs_t \). Now if one picks an edge uniformly at random from \( G \), then the probability (denote by \( p \)) that one of the edges incident to \( v \) will be picked is exactly the degree of \( v \) divided by the total number of edges of \( G \), which is at most \( \frac{d_{t,v}}{2(1-t\delta)(1+\epsilon)qs_t} \) (so, \( p \leq \frac{d_{t,v}}{2(1-t\delta)(1+\epsilon)qs_t} \)). So the probability that \( v \) is not deleted in Step 2 is exactly \((1-p)^{2(1+\epsilon)q}\), which is at least about \( 1-\frac{\delta g(t\delta)d_{0,v}}{s(t\delta)q} \). So, we can define \( a_t \) to be about \( \frac{\delta g(t\delta)2(1+\epsilon)q}{s(t\delta)q} = 2(1+\epsilon)\delta \cdot \frac{g(t\delta)}{s(t\delta)} \) (note that \( d_{0,v} \leq 2(1+\epsilon)q \)). Similarly, we can define \( b_t \) to be about \( 2(1+\epsilon)\gamma \delta \cdot \frac{g(t\delta)}{s(t\delta)} \), where \( \gamma = \frac{1+\theta}{1+\epsilon} \).

The probability that a single vertex gets deleted among Steps 2 and 3 is \( a_t \) or \( b_t \). So, the expected number of edges deleted in each color class among Steps 2 and 3 is at most \( 2s_t \left( \frac{1-\theta}{2} \cdot a_t + \frac{1+\theta}{2} \cdot b_t \right) \). This suggests defining \( s(x) \) with the following behavior:

\[
\frac{ds}{dt} = -2(1+\epsilon) \left[ (1-\theta) + (1+\theta)\gamma \right] g(x).
\]  

(8.1)

Next, let us estimate the change in \( d_{t,v} \) to get a differential equation for \( g \). Any edge incident to \( v \) will be deleted in Step 5 by probability \( \approx \frac{\delta}{(1-t\delta)} \), due to the fact that there are about \( 2(1+\epsilon)\delta q \) colors picked in Step 1, relatively few conflicts in Step 4, and the number of colors remaining is about \( 2(1+\epsilon)(1-t\delta)q \). We also know that each neighbor of \( v \) is deleted
among Steps 2 and 3 with probability \( a_t \) or \( b_t \), which is at least \( b_t \). So, neglecting small error terms due to lack of independence, we can choose to pick \( d_{t,v} \) with the following behavior:

\[
\begin{align*}
    d_{t+1,v} - d_{t,v} & \approx -\left[ b_t + (1 - b_t) \frac{\delta q}{(1 - t\delta)q} \right] d_{t,v} \\
    (\tau - t - 1)\delta g((t + 1)\delta)d_v - (\tau - t)\delta g(t\delta)d_v & \approx -\delta g(t\delta)d_v - 2(1 + \epsilon)(\tau - t - 1)\delta g(t\delta)d_v b_t \\
    \frac{g((t + 1)\delta) - g(t\delta)}{\delta} & \approx -2(1 + \epsilon)\gamma \cdot \frac{g(t\delta)^2}{s(t\delta)}. \tag{8.2}
\end{align*}
\]

Equation (8.2) suggests: \( g'(x) = -2(1 + \epsilon)\gamma \frac{g(x)^2}{s(x)} \). Combining with Equation (8.1), we have that \( \frac{d\alpha}{ds} = \frac{\gamma}{(1 - \theta) + (1 + \theta)\gamma} \cdot \frac{\alpha}{q} \), whose solution is \( s = cg^M \) with \( M = \frac{(1 - \theta) + (1 + \theta)\gamma}{\gamma} \) for some constant \( c \). Similar to Section 4, by using the initial conditions that \( g(0) = s(0) = 1 \), we get \( s(x) = (1 - 2(1 + \epsilon) \cdot (1 - \theta + \theta\gamma)x)^{\frac{M}{1 - \tau}} \) and \( g(x) = (1 - 2(1 + \epsilon) \cdot (1 - \theta + \theta\gamma)x)^{\frac{1}{1 - \tau}} \).

We show in the next section that the degrees of vertices and the sizes of color classes are concentrated in some sense throughout the process, which implies that we can not get stuck as long as \( t \leq \eta\tau \) with \( \eta < \frac{1}{2(1 + \epsilon)\gamma} \) (note that we desire a reasonable gap between \( \eta \) and \( \frac{1}{2(1 + \epsilon)(1 - \theta + \theta\gamma)} \) to make room for the error terms arising from concentration inequalities). Moreover, after \( \eta\tau \)-th iteration for \( \eta > \frac{1}{2(1 + \epsilon)} \) (with a reasonable gap), the number of edges in the rainbow matching will exceed \( q \), even after accounting for the discarded edges in Step 2 throughout the process. So, we need to make sure that \( \frac{1}{2(1 + \epsilon)} < \eta < \frac{1}{2(1 + \epsilon)(1 - \theta + \theta\gamma)} \). Notice that we do not need to use any greedy algorithm at the end to finish.

### 8.3 Formal analysis

Let \( \gamma = \frac{1 + \theta}{1 + \epsilon} \) and \( M = \frac{(1 - \theta) + (1 + \theta)\gamma}{\gamma} \). Observe that \( M > 2 \), which will be used often in our analysis. Define the following two functions:

\[
\begin{align*}
    s(x) & = (1 - 2(1 + \epsilon) \cdot (1 - \theta + \theta\gamma)x)^{\frac{M}{1 - \tau}}, \\
    g(x) & = (1 - 2(1 + \epsilon) \cdot (1 - \theta + \theta\gamma)x)^{\frac{1}{1 - \tau}}.
\end{align*}
\]

Similar to last time, we introduce error terms \( \alpha_t \) and \( \beta_t \) so that we can always make sure that the following three happen after each iteration:

1. The remaining color classes have size exactly:
   \[
   s_t = (1 - \alpha_t)s(t\delta)q.
   \]
2. The degree of each survived vertex in \( A \) is at most:
   \[
   d_t = 2(1 + \beta_t)(1 - t\delta)g(t\delta)(1 + \epsilon)q.
   \]
3. The degree of each survived vertex outside \( A \) is at most:
   \[
   d'_t = 2(1 + \beta_t)(1 - t\delta)g(t\delta)(1 + \theta)q.
   \]
We also make sure that after each iteration, for each remaining color class, the fraction of number of vertices which is in $A$ is at most $\frac{1-\theta}{2}$. Next, we explicitly define $a_t$ and $b_t$. The similar analysis as done in Section 5.1 leads us to define the following:

$$a_t = 2(1+\epsilon)\delta \cdot \frac{g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)} \quad \text{and} \quad b_t = 2(1+\theta)\delta \cdot \frac{g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)}.$$  

Similar to before, we explicitly assign a probability for Step 3. For a fixed $t$, at the $(t+1)$-st iteration we delete each vertex $v \in A$ independently with probability $p_v$ such that $p'_v + (1-p'_v)p_v = a_t$, where $p'_v$ is the probability that $v$ gets deleted in Step 2. If $v \notin A$, then we pick $p_v$ such that $p'_v + (1-p'_v)p_v = b_t$.

We now pick the values of the parameters like we did last time in Section 7. Let $\epsilon$ be a fixed positive number less than $\frac{1}{10}$. Recall $\theta = \frac{\epsilon}{2}$ and $\gamma = \frac{1+\theta}{1+\epsilon}$ as introduced before. Recall from the intuitive analysis that we want $\eta$ such that $\frac{1}{2(1+\epsilon)} < \eta < \frac{1}{1+\epsilon} \cdot \frac{1+\theta}{1+\epsilon} \cdot \frac{1+\epsilon}{1+\epsilon}$. So, explicitly let us define $\eta = \frac{1}{2(1+\epsilon)} \cdot \frac{1+\theta}{1+\epsilon} \cdot \frac{1+\epsilon}{1+\epsilon}$. Let $\delta = \frac{1}{\log q}$. Note that $\epsilon$, $\theta$, $\gamma$, and $\eta$ are all fixed constants, where the latter three constants are fixed once $\epsilon$ is fixed. On the other hand, $\delta$ can be made arbitrarily small by picking sufficiently large $q$.

Next, we have a few concentration bounds similar to the ones in Section 5.2 (recall the definition of the notion of `w.v.h.p.` defined in the beginning of the same section).

**Lemma 8.1.** For each fixed $t$, the number of edges discarded in Step 4 at the $(t+1)$-st iteration is at most $\frac{8\delta^2 q}{1-2(1+\epsilon)(1-\theta+\epsilon)\delta^t}$ w.v.h.p.

**Proof.** For $i < j$, let $X_{i,j}$ be the indicator random variable for the event that $e_i \in T$ and $e_j \in T$ are incident or have the same color. Clearly, the total number of edges discarded in Step 4 is given by $X \leq \sum_{1 \leq i < j \leq 2(1+\epsilon)\delta q} X_{i,j}$. Now, by using the same argument as in Lemma 3.4, the probability that $X_{i,j} = 1$ is at most $\frac{2d_t + s_t}{2(1+\epsilon)(1-t\delta)q s_t}$. Hence, we get the following:

$$\mathbb{E}(X) \leq \mathbb{E} \left( \sum_{1 \leq i < j \leq 2(1+\epsilon)\delta q} X_{i,j} \right) = \sum_{1 \leq i < j \leq 2(1+\epsilon)\delta q} \mathbb{P}[X_{i,j} = 1] \leq \frac{(2(1+\epsilon)\delta q)^2}{2} \cdot \frac{2d_t + s_t}{2(1+\epsilon)(1-t\delta)q s_t} \leq (1+\epsilon)^2 \delta^2 q \left( \frac{4(1+\beta_t)(1-t\delta)g(t\delta)(1+\epsilon)q + 1}{(1+\epsilon)(1-t\delta)(1-\alpha_t)s(t\delta)q + (1+\epsilon)(1-t\delta)} \right) \leq (1+\epsilon)^2 \delta^2 q \left( \frac{5}{1-2(1+\epsilon)(1-\theta + \epsilon)\delta t} + \frac{1}{1-t\delta} \right) \leq \frac{7\delta^2 q}{1-2(1+\epsilon)(1-\theta + \epsilon)\delta t}.$$  

Similar to the proof of Lemma 3.4, use Theorem 5.2 to conclude that the probability that $X$ deviates from its mean by $\lambda = \delta^2 q$ is at most $2e^{-\frac{\lambda^2}{2(1+\epsilon)(1-\theta+\epsilon)}} = 2e^{-\frac{\delta^4 q}{\lambda(1+\epsilon)}} = e^{-\omega(\log q)}$.  


Consequently, the number of colors left after the \(t\)-th iteration is at most \(2\) w.v.h.p.

\[ \left(1 - t\delta + 4\delta \log \left(\frac{1}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\eta}\right)\right) \cdot 2(1 + \epsilon)q. \]

**Proof.** By using Lemma 8.1, the total number is at most the following:

\[ \sum_{t=0}^{\eta} \frac{8\delta^2 q}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)td\delta} \leq 8\delta q \int_0^{\eta} \frac{dx}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)x} = \frac{8\delta q}{2(1 + \epsilon)(1 - \theta + \theta\gamma)} \log \left(\frac{1}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\eta}\right). \]

**Corollary 8.2.** The total number of edges not added in the rainbow matching in Step 4 throughout the algorithm until \(\eta\tau\)-th iteration is at most \(8\delta q \log \left(\frac{1}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\eta}\right)\) w.v.h.p.

Consequently, the number of colors left after the \(t\)-th iteration is at most the following:

\[ \left(1 - t\delta + 4\delta \log \left(\frac{1}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\eta}\right)\right) \cdot 2(1 + \epsilon)q. \]

**Proof.** By using Lemma 8.1 the total number is at most the following:

\[ \sum_{t=0}^{\eta} \frac{8\delta^2 q}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)td\delta} \leq 8\delta q \int_0^{\eta} \frac{dx}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)x} = \frac{8\delta q}{2(1 + \epsilon)(1 - \theta + \theta\gamma)} \log \left(\frac{1}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\eta}\right). \]

**Lemma 8.3.** For each fixed \(t \leq \eta\tau\), in the \((t + 1)\)-st iteration, the number of edges deleted from a single color class in Steps 2 and 3 combined is at most \(2 \left(\frac{1 - \theta}{2} \cdot a_t + \frac{1 + \theta}{2} \cdot b_t\right) s_t + \sqrt{q} \log q\) w.v.h.p.

**Proof.** Fix a color remaining at the start of the iteration \(t + 1\). Let \(s_t\) denote the number of edges in that color class before starting the \((t + 1)\)-st iteration, and \(S_{t+1}\) denote the number of edges right after Steps 2 and 3 in the \(t\)-th iteration. We have that \(E(S_{t+1}) - s_t\) is at most \(2 \left(\frac{1 - \theta}{2} \cdot a_t + \frac{1 + \theta}{2} \cdot b_t\right) s_t + \sqrt{q} \log q\) w.v.h.p.

Let \(L\) be the random variable denoting the number of edges of the fixed color removed because of Steps 2 and 3. In the setting mentioned right before Theorem 5.2 let \(A = A' \cup \{0, 1\}\) where \(A'\) is the set of all remaining edges of \(G\) at the start of \((t + 1)\)-st iteration, and \(B = [\delta q] \cup V\) where \(V\) is the set of remaining vertices of \(G\). For \(j \in [\delta q]\), the probability \(P[g(j) = e]\) is same for all edge \(e\) and \(P[g(j) \in \{0, 1\}] = 0\), and for \(v \in V\), the probability \(P[g(v) = 1]\) is same as the probability with which the vertex \(v\) gets deleted in Step 3, and \(P[g(v) = 0] = 1 - P[g(v) = 1]\). Consider the functional \(L : A^B \rightarrow \mathbb{R}\) and the corresponding martingale is \(X_0 = E(L), X_1, \ldots, X_2(1 + \epsilon)\delta q, X_{v_1}, X_{v_2}, \ldots, X_{v_k}\), with respect to the gradation \(0 \subset [1] \subset [2] \subset \ldots \subset [2(1 + \epsilon)\delta q] \subset [2(1 + \epsilon)\delta q] \cup \{v_1\} \subset \ldots \subset [2(1 + \epsilon)\delta q] \cup V\) (where \(V = \{v_1, v_2, \ldots, v_k\}\)). If one edge is replaced by some other edge in any one of the \(\delta q\) coordinates or the decision of deleting a vertex in Step 3 is reversed, then \(L\) will be effected by at most \(2\), in other words, \(2\) is an upper-bound on the Lipschitz constant of the martingale. Use Theorem 5.2 to conclude that the probability that \(L\) deviates from its mean by \(\lambda = \sqrt{q} \log q\) is at most \(2e^{-\frac{\lambda^2}{2q\epsilon}} = e^{-\omega(\log q)}\).
**Lemma 8.4.** For each remaining vertex $v$ after $t$-th iteration, let $D_{t+1,v}$ be the random variable denoting the degree of $v$ after Step 5 in $(t+1)$-st iteration. For each fixed $t \leq \eta t$, at the end of Step 5 of the $(t+1)$-st iteration, the expected degree of each remaining vertex $v \in A$ after $t$-th iteration,

$$
\mathbb{E}(D_{t+1,v}) \leq d_t \cdot (1 - b_t) \left(1 - \frac{1}{\tau - t} \left(1 - \frac{1}{s_t} - \frac{15\delta \log \left(\frac{1}{1-2(1+\epsilon)(1-\theta+\gamma)\eta}\right)}{1 - 2(1 + \epsilon)(1 - \theta + \gamma)t\delta}\right)\right),
$$

and the expected degree of each remaining vertex $v \notin A$ after $t$-th iteration,

$$
\mathbb{E}(D_{t+1,v}) \leq d'_t \cdot (1 - b_t) \left(1 - \frac{1}{\tau - t} \left(1 - \frac{1}{s_t} - \frac{15\delta \log \left(\frac{1}{1-2(1+\epsilon)(1-\theta+\gamma)\eta}\right)}{1 - 2(1 + \epsilon)(1 - \theta + \gamma)t\delta}\right)\right).
$$

**Proof.** We only deal with the vertices in $A$, the same treatment works for other vertices as well. We proceed similarly to the proof of lemma 5.8. For a vertex $v$, the probability that any neighbor $u$ of $v$ is deleted in Steps 2 and 3 of the $(t+1)$-st iteration is either exactly $a_t$ or $b_t$, which is at least $b_t$. In order to calculate the probability that an edge $uv$ will be deleted in Step 5 conditioned on the fact that the vertex $u$ did not get deleted in Steps 2 and 3, we define the events $A$, $B$, and $A'$ as the ones defined in 5.8. We are interested in finding a good lower bound on $\mathbb{P}[A|B \cap C]$, where $C$ denotes the event that $u$ does not get deleted in Step 3.

To simplify the analysis, we make a slight change in Step 3 which will not alter the output of the original algorithm. Instead of the original Step 3, we delete each of the remaining vertices (even the ones which get deleted in Step 2) with the same probability as desired in Algorithm. Deleting a vertex in Step 3 which was already deleted in Step 2 does not do anything further. By making this simple change, if $C$ is the same event defined before ($u$ does not get deleted in Step 3), then $C$ will be mutually independent of $A$ and $B$. Hence, it is clear that $\mathbb{P}[A|B \cap C] = \mathbb{P}[A|B] / \mathbb{P}[B|C] = \mathbb{P}[A|B]$.

Now $\mathbb{P}[A|B]$ can be bounded similar to the proof of Lemma 5.8. Hence, we can finish the proof of Lemma 8.4 by the following:

$$
\mathbb{P}[A|B \cap C] = \mathbb{P}[A|B] \geq \mathbb{P}[A'|B] \geq \frac{2(1 + \epsilon)\delta q(s_t - 1)}{(1 - t\delta + 4\delta \log \left(\frac{1}{1-2(1+\epsilon)(1-\theta+\gamma)\eta}\right)) \cdot 2(1 + \epsilon)qs_t \left(1 - \frac{2d_t + s_t}{2(1 + \epsilon)(1 - t\delta)qs_t - d_t}\right)^{2(1+\epsilon)\delta q}} \cdot \frac{s_t - 1}{s_t} \left(1 - \frac{2(1 + \epsilon)\delta q(2d_t + s_t)}{2(1 + \epsilon)(1 - t\delta)qs_t - d_t}\right)
$$

$$
\geq \frac{\delta}{1 - t\delta + 4\delta \log \left(\frac{1}{1-2(1+\epsilon)(1-\theta+\gamma)\eta}\right)} \left(1 - \frac{1}{s_t}\right) \left(1 - \frac{2(1 + \epsilon)\delta q(2d_t + s_t)}{(1 + \epsilon)(1 - t\delta)qs_t}\right) (8.3)
$$

$$
= \frac{\delta}{1 - t\delta} \cdot \frac{1 - t\delta}{1 - t\delta + 4\delta \log \left(\frac{1}{1-2(1+\epsilon)(1-\theta+\gamma)\eta}\right)} \left(1 - \frac{1}{s_t}\right) \left(1 - \frac{2\delta}{(1 - t\delta)s_t}\right) = \frac{\delta}{1 - t\delta} \cdot \frac{1 - t\delta}{1 - t\delta + 4\delta \log \left(\frac{1}{1-2(1+\epsilon)(1-\theta+\gamma)\eta}\right)} \left(1 - \frac{1}{s_t}\right) \left(1 - \frac{2\delta}{(1 - t\delta)s_t}\right)
$$
\[ \frac{1}{\tau - t} \left( 1 - \frac{4\delta \log \left( \frac{1}{1-2(1+\epsilon)(1-\theta + \theta \gamma)\eta} \right)}{1 - t\delta} \right) \left( 1 - \frac{1}{s_t} \right) \left( 1 - \frac{11\delta}{1 - 2(1+\epsilon)(1-\theta + \theta \gamma)t\delta} \right) \]

(8.4)

\[ \frac{1}{\tau - t} \left( 1 - \frac{4\delta \log \left( \frac{1}{1-2(1+\epsilon)(1-\theta + \theta \gamma)\eta} \right)}{1 - t\delta} \right) \left( 1 - \frac{1}{s_t} \right) \left( 1 - \frac{11\delta}{1 - 2(1+\epsilon)(1-\theta + \theta \gamma)t\delta} \right) \]

(8.5)

\[ \frac{1}{\tau - t} \left( 1 - \frac{15\delta \log \left( \frac{1}{1-2(1+\epsilon)(1-\theta + \theta \gamma)\eta} \right)}{1 - 2(1+\epsilon)(1-\theta + \theta \gamma)t\delta} \right), \]

where in (8.3), we have used \((1-t\delta)qs_t - d_t > \frac{1}{2}(1-t\delta)qs_t \) for all \(t \leq \eta \tau\) and for sufficiently large \(q\). In (8.4), we have used that \( \frac{d_t}{s_t} = \frac{2(1+\epsilon)(1+\beta)(1-t\delta)q(t\delta)}{(1-\alpha_t)(t\delta)} < \frac{2.25(1-t\delta)}{1-2(1+\epsilon)(1-\theta + \theta \gamma)t\delta} \) and \( \frac{1}{1-t\delta} < \frac{1}{1-2(1+\epsilon)(1-\theta + \theta \gamma)t\delta} \). In (8.5), we have used the fact that \( \frac{5\delta \log \left( \frac{1}{1-t\delta} \right)}{1-t\delta}, \frac{1}{s_t}, \) and \( \frac{11\delta}{1-2(1+\epsilon)(1-\theta + \theta \gamma)t\delta} \) are all strictly less than 1 for all \(t \leq \eta \tau\).

\[ \square \]

**Lemma 8.5.** For each fixed \(t \leq \eta \tau\), at the end of the \((t+1)\)-st iteration, the degree of each remaining vertex \(v\) is at most \(E(D_{t+1,v}) + \sqrt{q} \log q \, w.v.h.p\), where \(D_{t+1,v}\) is the degree of \(v\) after Step 5 in \((t+1)\)-st iteration.

We omit the proof of this lemma entirely because the same treatment as in the proof of Lemma 8.3 using Theorem 5.2 will work here.

**Lemma 8.6.** For each fixed \(t \leq \eta \tau\), at the end of the \((t+1)\)-st iteration, for each color class, the fraction of vertices that are in \(A\) is at most \(\frac{1-\theta}{2}\) w.v.h.p.

**Proof.** Fix a color class. After the completion of \(t\)-th iteration, let \(\rho\) denotes the ratio between the number of vertices of \(A\) in this color class to the total number of vertices in this color class. Clearly, \(\rho \leq \frac{1-\theta}{2}\). First consider the case that \(\rho \leq \frac{1}{8}\). In this case, at the end of \((t+1)\)-st iteration, the fraction of vertices in \(A\) is at most the following:

\[ \frac{\rho \cdot 2s_t}{s_{t+1}} = \frac{\rho(1-\alpha_t)(1-2(1+\epsilon)(1-\theta + \theta \gamma)t\delta)^{\frac{\alpha_t}{\alpha_t+1}}}{(1-\alpha_{t+1})(1-2(1+\epsilon)(1-\theta + \theta \gamma)(t+1)\delta)^{\frac{\alpha_t}{\alpha_t+1}}} \]

\[ \leq \frac{1/8}{1-\alpha_{t+1}} \left( 1 + \frac{2(1+\epsilon)(1-\theta + \theta \gamma)\delta}{(1-2(1+\epsilon)(1-\theta + \theta \gamma)(t+1)\delta)} \right)^{\frac{\alpha_t}{\alpha_t+1}} \]

\[ \leq \frac{1}{6} \left( 1 + \frac{C(\epsilon)}{\log q} \right) \leq \frac{1}{4} \leq \frac{1-\theta}{2}, \]

where \(C(\epsilon)\) is a constant depending on \(\epsilon\) (recall that \(\theta\), \(\eta\), and \(\gamma\) were defined in terms of \(\epsilon\)).

So, we can assume that \(\rho > \frac{1}{8}\). For convenience, let \(X\) be the number of vertices from that color class that are removed in Steps 2 and 3 of the \((t+1)\)-st iteration. Similarly, let \(X_A\) be the number of vertices from that color class, which are in \(A\) and are removed in Steps 2 and 3 of the \((t+1)\)-st iteration. In order to prove Lemma 8.6, it is enough to prove that
\( \rho X \leq X_A \) w.v.h.p. (check that after Step 6, we still satisfy the statement of Lemma 8.6). The same proof of Lemma 8.3 obtains the following w.v.h.p. (note that we multiplied the expression in Lemma 8.6 with 2, because the number of vertices deleted is exactly 2 times the number of edges deleted):

\[
X \leq 4 (\rho \cdot a_t + (1 - \rho) b_t) s_t + 2\sqrt{q} \log q. \tag{8.6}
\]

Next, we find the expected value of \( X_A \). Fix a vertex \( u \in A \) such that there is an alive edge \( uv \) in the fixed color class for some vertex \( v \). We compute the probability that \( u \) will be removed from this color class after Steps 2 and 3. This probability is exactly same as the probability of the event that at least one of the vertices in \( \{ u, v \} \) gets deleted in Steps 2 and 3 combined. Denote the event that \( u \) gets deleted in Steps 2 and 3 by \( U \) (similarly for \( v \), denote the event by \( V \)). It is clear that

\[
P[U \cup V] = P[U] + P[U^c] \cdot P[V | U^c] = a_t + (1 - a_t) \cdot P[V | U^c].
\]

After the completion of \( t \)-th iteration, let \( d_{t,u} \) be the degree of the vertex \( u \) and \( |E| \) be the number of edges in \( G \). Recall that \( p_v' \) and \( p_v \) denote the probability that \( v \) gets deleted in Step 2 and Step 3 respectively. Clearly, we have that \( 1 - p_v' = \left(1 - \frac{d_{t,u}}{|E|}\right)^{2\delta(1+\epsilon)q} \). Unfortunately, we cannot use \( p_v' \) directly to estimate \( P[V | U^c] \), because the probability we care about is the probability with which \( v \) gets deleted in Steps 2 and 3 conditioned on the event that \( u \) is not deleted in Step 2. For the convenience of estimating that probability, denote by \( W \) the event that \( u \) is not deleted in Step 2. A similar argument as in the second paragraph of the proof of Lemma 8.4 shows that \( P[V | U^c] = P[V | W] \). The event \( W \) is same as the event that none of the edges uniformly picked in Step 1 is adjacent to the vertex \( u \). Hence, conditioning on the event \( W \) essentially is same as considering the uniform measure in Step 1 excluding the edges adjacent to \( u \). Let \( q_v' \) denote the probability that \( v \) gets deleted in Step 2 conditioned on \( W \). Then, we have that \( 1 - q_v' = \left(1 - \frac{d_{t,v} - 1}{|E| - d_{t,u}}\right)^{2\delta(1+\epsilon)q} \). Hence, we have the following:

\[
q_v' - p_v' = \left(1 - \frac{d_{t,v}}{|E|}\right)^{2\delta(1+\epsilon)q} - \left(1 - \frac{d_{t,v} - 1}{|E| - d_{t,u}}\right)^{2\delta(1+\epsilon)q} \\
\geq \left(1 - \frac{1}{|E|}\right)^{2\delta(1+\epsilon)q} - 1 \\
\geq - \frac{2\delta(1+\epsilon)q}{|E|} \\
\geq - \frac{2\delta(1+\epsilon)q}{2(1 - t\delta)(1 + \epsilon)q \cdot s_t} \\
\geq - \frac{\delta}{(1 - t\delta) \cdot s_t}, \tag{8.8}
\]

where in (8.7), we have used the following fact which can be proved using basic calculus. For \( m > 1 \) and \( \zeta < 1 \), the function \( f(x) = (1 - x)^m - (1 + \zeta - x)^m \) is increasing in the range \( x \in [\zeta, 1] \). Then, using Equation (8.8), we have the following:
\[ \Pr[V|W] = q'_v + (1 - q'_v)p_v \
\geq q'_v(1 - p_v) + p_v - \frac{\delta}{(1 - t\delta) \cdot s_t} \
\geq b_t - \frac{\delta}{(1 - t\delta) \cdot s_t}, \]

where in the last line, we have used the definition that \( p'_v + p_v - p'_v p_v = a_t \) or \( b_t \) depending on the fact if \( v \in A \) or not.

So, the probability that \( u \in A \) is removed from the color class is at least \( a_t + b_t - \frac{\delta}{(1 - t\delta) \cdot s_t} \). Hence, the expected number \( X_A \) of vertices in \( A \) removed from the color class in Steps 2 and 3 is at least \( (a_t + b_t - \frac{\delta}{(1 - t\delta) \cdot s_t}) \cdot 2p s_t \). Similar arguments as in Lemma 8.3 also shows the following w.v.h.p.

\[ X_A \geq 2 (a_t + b_t) \rho s_t - \frac{2\delta \rho}{1 - t\delta} - 2\sqrt{q \log q}. \quad (8.9) \]

From Equations (8.6) and (8.9), we have the following w.v.h.p.

\[
\begin{align*}
X_A - \rho \cdot X &\geq 2\rho (1 - 2\rho)(a_t - b_t)s_t - \frac{2\delta \rho}{1 - t\delta} - 4\sqrt{q \log q} \\
&\geq \frac{1}{4} \cdot \theta \cdot 2\theta \delta \cdot \frac{g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)} (1 - \alpha_t)s(t\delta)q - \frac{2\delta \rho}{1 - t\delta} - 4\sqrt{q \log q} \\
&\geq \frac{1}{2} \theta^2 (1 - \theta) g(\eta) \frac{q}{\log q} - \frac{2\log q}{1 - \eta} - 4\sqrt{q \log q} \\
&\geq \frac{1}{2} \theta^2 (1 - \theta) g(\eta) \frac{q}{\log q} - \frac{2\log q}{1 - \eta} - 4\sqrt{q \log q}. \\
\end{align*}
\]

(8.10)

From (8.10), it is clear that \( X_A - \rho \cdot X \geq 0 \) for all sufficiently large \( q \), which is what we wished to show in order to complete the proof of Lemma 8.6.

\( \square \)

### 8.4 Estimating the error terms

Similar to Section 6, we estimate the error terms in the parameters throughout the algorithm in Section 9.1. We proceed by first estimating the error terms in the ideal expressions from the intuitive analysis in Section 9.2. Define the following parameters (we denote the ideal values of parameters by tilde symbol over the corresponding notations).

\[
\begin{align*}
\tilde{s}_t &= s(t\delta)q. \\
\tilde{a}_t &= 2(1 - t\delta) g(t\delta)(1 + \epsilon)q. \\
\tilde{b}_t &= 2(1 - t\delta) g(t\delta)(1 + \theta)q. \\
\tilde{d}_t &= 2(1 + \epsilon) \delta \frac{g(t\delta)}{s(t\delta)}. \\
\tilde{d}'_t &= 2(1 + \theta) \delta \frac{g(t\delta)}{s(t\delta)}. \\
\end{align*}
\]

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We have the following couple of lemmas concerning the relations between these ideal parameters. This time their proofs are a bit technical due to the fact that the functions \( s(x) \) and \( g(x) \) are not as nice as in Section 6.

**Lemma 8.7.** There is some constant \( K \) depending only on \( \epsilon, \theta, \) and \( \eta \) such that for all \( t \leq \eta \cdot 2(1+\epsilon)q \), we have that \( (1 - (1 - \theta)\bar{a}_t - (1 + \theta)\bar{b}_t)\bar{s}_t \geq \bar{s}_{t+1} - K\delta^2 q \).

**Proof.** A routine calculation shows:

\[
\bar{s}_{t+1} - (1 - (1 - \theta)\bar{a}_t - (1 + \theta)\bar{b}_t)\bar{s}_t \\
= s((t+1)\delta)q - \left( 1 - (1 - \theta) \cdot 2(1 + \epsilon)\delta \frac{g(t\delta)}{s(t\delta)} - (1 + \theta) \cdot 2(1 + \theta)\delta \frac{g(t\delta)}{s(t\delta)} \right) s(t\delta)q \\
= (s((t+1)\delta) - s(t\delta))q + 2\delta((1 - \theta)(1 + \epsilon) + (1 + \theta)^2)g(t\delta)q. \tag{8.11}
\]

Using Mean Value Theorem on the function \( s(x) \) in the domain \([0, \eta]\), we obtain that \( s((t+1)\delta) - s(t\delta) = \delta s'(y) \) for some \( y \in [t\delta, (t+1)\delta] \). Note that \( s'(x) = -2(1 - \theta)(1 + \epsilon) + (1 + \theta)^2)g(x) \), which is an increasing function in the domain \([0, \eta]\). Hence, we can write that \( s((t+1)\delta) - s(t\delta) \leq -2\delta((1 - \theta)(1 + \epsilon) + (1 + \theta)^2)g((t+1)\delta) \). Using this in (8.11), we get the following:

\[
\bar{s}_{t+1} - (1 - (1 - \theta)\bar{a}_t - (1 + \theta)\bar{b}_t)\bar{s}_t \\
\leq -2\delta((1 - \theta)(1 + \epsilon) + (1 + \theta)^2)g((t+1)\delta) - g(t\delta))q. \tag{8.12}
\]

Similarly, using Mean Value Theorem on the function \( g(x) \) in the domain \([0, \eta]\), we obtain that \( g((t+1)\delta) - g(t\delta) = \delta g'(y) \) for some \( y \in [t\delta, (t+1)\delta] \). Using this in (8.12) like last time, it is easy to find a constant \( K \) depending on \( \epsilon, \theta, \) and \( \eta \) such that

\[
\bar{s}_{t+1} - (1 - (1 - \theta)\bar{a}_t - (1 + \theta)\bar{b}_t)\bar{s}_t \leq K\delta^2 q. \tag{8.13}
\]

\( \square \)

**Lemma 8.8.** There is some constant \( K' \) depending only on \( \epsilon, \theta, \) and \( \eta \) such that for all \( t \leq \eta \cdot 2(1+\epsilon)q \), we have that

\[
\left( 1 - \frac{1}{\tau - t} \right) \tilde{d}_t \leq \tilde{d}_{t+1} + K'\delta^2 q \quad \text{and} \quad \left( 1 - \frac{1}{\tau - t} \right) \tilde{d}'_t \leq \tilde{d}'_{t+1} + K'\delta^2 q.
\]

**Proof.** We will only show the first one, because the second one can be shown in the exactly same way. A routine calculation shows:

\[
\tilde{d}_t - \tilde{d}_{t+1} = -2 ((1 - (t+1)\delta)g((t+1)\delta) - (1 - t\delta)g(t\delta)) (1 + \epsilon)q. \tag{8.13}
\]

Now using Mean Value Theorem on the function \((1 - x)g(x)\) in the domain \([0, \eta]\), we obtain that \((1 - (t+1)\delta)g((t+1)\delta) - (1 - t\delta)g(t\delta) = \delta (-g(y) + (1 - y)g'(y)) \) for some \( y \in [t\delta, (t+1)\delta] \). Noting that for all \( y \in [t\delta, (t+1)\delta] \), we have that \( g(y) \leq g(t\delta) \) and \((1 - y)g'(y) \geq (1 - (t+1)\delta)g'((t+1)\delta) \), from (8.13) we get the following:
where \( K' \) depends only on \( \epsilon, \theta, \) and \( \eta, \) and in the last step we apply Mean Value Theorem on the function \( g'(x) \) like before. In (8.14), we use the fact that \( g'(x) = 2(1 + \theta) \frac{g(x)^2}{s(x)}. \)

\[
\begin{align*}
(1 - \tilde{b}_t) & \left( 1 - \frac{1}{\tau - t} \right) \tilde{d}_t - \tilde{d}_{t+1} \\
\leq & \tilde{d}_t - \tilde{d}_{t+1} - \frac{\delta}{1 - t\delta} \tilde{d}_t - \tilde{b}_t \left( 1 - \frac{1}{\tau - t} \right) \tilde{d}_t \\
\leq & 2\delta [g(t\delta) - (1 - (t + 1)\delta)g'((t + 1)\delta)] (1 + \epsilon)q - 2\delta g(t\delta)(1 + \epsilon)q \\
& - 2(1 + \theta)\delta \frac{g(t\delta)}{s(t\delta)}(1 - (t + 1)\delta) \cdot 2g(t\delta)(1 + \epsilon)q \\
\leq & 2\delta (1 - (t + 1)\delta) (g'((t + 1)\delta) - g'(t\delta)) (1 + \epsilon)q \\
\leq & K'\delta^2 q, \tag{8.14}
\end{align*}
\]

\[\text{Lemma 8.9. For each } t \leq \eta \tau, \text{ we can choose } y_t = C_1 t\delta^2 q, \text{ where } C_1 \text{ is a constant depending on } \epsilon. \text{ Moreover for this choice of } y_t, \text{ we have that } y_t = o(\tilde{d}_t) \text{ for all } t \leq \eta \tau.\]

\[\text{Proof.} \text{ Starting with the conclusion from Lemma [8.3}, \text{ a routine calculation shows:}

\[\begin{align*}
D_{t+1,v} & \leq d_t \cdot (1 - b_t) \left( 1 - \frac{1}{\tau - t} \right) \left( 1 - \frac{15\delta \log \left( \frac{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\eta}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\tau\delta} \right)}{s_t} \right) + \sqrt{q} \log q \\
& \leq \left( \tilde{d}_t + y_t \right) \left( 1 - b_t \left( 1 - \frac{1}{\tau - t} \right) \right) + \frac{1 - b_t}{\tau - t} \left( \frac{15\delta \log \left( \frac{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\eta}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\tau\delta} \right)}{s_t} \right) + \sqrt{q} \log q \\
& \leq \tilde{d}_t (1 - b_t) \left( 1 - \frac{1}{\tau - t} \right) + \tilde{d}_t \frac{\delta (1 - b_t)}{1 - t\delta} \left( \frac{15\delta \log \left( \frac{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\eta}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)\tau\delta} \right)}{s_t} \right) + y_t + \sqrt{q} \log q \\
& \leq \tilde{d}_{t+1} + K'\delta^2 q + y_t + C(\epsilon)\delta + \sqrt{q} \log q \\
& \leq \tilde{d}_{t+1} + C_1 \delta^2 q + y_t
\end{align*}\]

where \( C(\epsilon) \) and \( C_1 \) are constants depending on \( \epsilon, \theta, \gamma, \) and \( \eta. \) So, we can choose \( y_t \) such that \( y_{t+1} = y_t + C_1 \delta^2 q. \) Using \( y_0 = 0, \) we get \( y_t = C_1 t\delta^2 q. \)

Clearly for \( t \leq \eta \tau, \) we have that \( y_t \leq y_{\eta \tau} = O \left( \frac{q}{\log q} \right) \) and \( \tilde{d}_t \geq \tilde{d}_{\eta \tau} = \Omega(q). \) Hence, \( y_t = o(\tilde{d}_t) \) for all \( t \leq \eta \tau. \) \]

\[\text{Lemma 8.10. For each } t \leq \eta \tau, \text{ we can choose } z_t = C_2 t\delta^2 q \text{ where } C_2 \text{ is a constant depending on } \epsilon. \text{ Moreover for this choice of } z_t, \text{ we have that } z_t = o(\tilde{s}_t) \text{ for all } t \leq \eta \tau.\]

\[31\]
Proof. Recall that \( \alpha_t = \frac{s_t}{\eta} \) and \( \beta_t = \frac{y_t}{d_t} \). Starting with the conclusion from Lemma 8.3, a routine calculation shows the following:

\[
S_{t+1} \geq (1 - (1 - \theta)a_t - (1 + \theta)b_t)s_t - \sqrt{q}\log q
\]

\[
= \left(1 - (1 - \theta)\tilde{a}_t \frac{1 + \beta_t}{1 - \alpha_t} - (1 + \theta)\tilde{b}_t \frac{1 + \beta_t}{1 - \alpha_t}\right) (1 - \alpha_t)\tilde{s}_t - \sqrt{q}\log q
\]

\[
= \left(1 - (1 - \theta)\tilde{a}_t - (1 + \theta)\tilde{b}_t\right)\tilde{s}_t - \alpha_t\tilde{s}_t - \left( (1 - \theta)\tilde{a}_t + (1 + \theta)\tilde{b}_t \right) \beta_t\tilde{s}_t - \sqrt{q}\log q
\]

\[
\geq \tilde{s}_{t+1} - K\delta^2 q - z_t - \frac{2\delta}{1 - t\delta} \cdot y_t - \sqrt{q}\log q
\]

(8.15)

\[
\geq \tilde{s}_{t+1} - C_2\delta^2 q,
\]

where \( C_2 \) is a constant depending on \( \epsilon, \theta, \gamma, \) and \( \eta \). In (8.15), we use the fact that \( \left(1 - \theta\right)\tilde{a}_t + (1 + \theta)\tilde{b}_t \leq 2\tilde{a}_t\tilde{s}_t = \frac{2\delta}{1 - \eta}\tilde{d}_t \). So, we can choose \( z_t \) to such that \( z_{t+1} = z_t + C_2\delta^2 q \), which together with \( z_0 = 0 \) yields that \( z_t = C_2 t\delta^2 q \).

Clearly for \( t \leq \eta \tau \), we have that \( z_t \leq z_{\eta \tau} = O\left(\frac{q}{\log q}\right) \) and \( \tilde{s}_t \geq \tilde{s}_{\eta \tau} = \Omega(q) \). Hence, \( z_t = o(\tilde{s}_t) \) for all \( t \leq \eta \tau \).

By Lemmas 8.9 and 8.10, we can run the algorithm of Section 9.1 until \( \eta \tau \)-th iteration. At the end of this iteration, using Lemma 8.2, the number of edges picked in the rainbow matching is at least \( \eta \cdot 2(1 + \epsilon)\delta q - 8\delta q \log \left(\frac{1}{1 - 2(1 + \epsilon)(1 - \theta + \theta \gamma)}\right) > q \), finishing the proof of Theorem 1.9.

## 9 Proof of Theorem 1.10

In this section we prove Theorem 1.10. We again need to adjust the algorithm of Section 3 to apply for this theorem.

### 9.1 Algorithm

Assume that we are given a simple bipartite graph \( G \) with the the vertex set partition \( A \cup B \), \( |A| = q \), and all the vertices in \( A \) has degree at least \( (1 + \epsilon)q \). Furthermore, \( G \) is edge-colored with some colors such that no two edges of same color are incident to each other. We are going to provide a randomized algorithm, which constructs a rainbow matching using almost all the vertices in \( A \) in several iterations. For the convenience of analyzing the algorithm, we make sure that throughout the algorithm the degree of each of the remaining vertices in \( A \) is same at the start of each iteration. So, we start with deleting arbitrary edges from \( G \) to make sure that all the vertices in \( A \) has degree exactly \( (1 + \epsilon)q \).

This algorithm is in terms of some parameter \( \delta > 0 \), which will be specified later. Define \( \tau = \frac{1}{\delta} \). We follow the algorithm below for \( \eta \tau \) steps, for some \( \eta < 1 \) to be specified later.

1. Select independently \( \delta q \) edges \( e_1, e_2, \ldots, e_{\delta q} \) uniformly at random with replacement from among the remaining edges. Denote by \( T \) the set of all selected edges.
2. For each edge $e_i \in T$, add it to the rainbow matching if $e_i$ is not incident to any edge $e_j \in T$ for $j < i$ and $e_i$ does not have the same color as any edge $e_j \in T$ for $j < i$.

3. Delete all the vertices corresponding to the edges added in the rainbow matching in the last step. Deleting vertices always also deletes all incident edges.

4. With some probability (to be specified later), independently delete each vertex $v \in B$. This will make sure that among Steps 3 and 4 combined, every vertex in $B$ gets deleted with the same probability.

5. For each edge $e$ added to the rainbow matching in Step 2, delete its color class from $G$. Here, deleting a color class means deleting all the remaining edges of that color.

6. With some probability (to be specified later), independently delete each color class. This will make sure that among Steps 5 and 6 combined, every remaining color class gets deleted with the same probability.

7. At this point, every remaining vertex in $A$ will have degree at least $d_t$, where $d_t$ will be specified later. For each remaining vertex $v$ in $A$ after Step 6, delete arbitrary edges incident to $v$ to make sure that $v$ has exactly $d_t$ edges incident to it.

We remark here that we do not have the flexibility of throwing vertices from $A$ without using, because we need to use all of them at the end. As a result, we need to shift Step 4 of the algorithm in Section 3 to Step 2, and delete vertices only after adding the edges to the rainbow matching in the current iteration. This makes our analysis slightly more complicated.

### 9.2 Intuitive analysis

As before, we aim to show that if we run this randomized algorithm until we have used almost all the vertices in $A$, then near the end, each remaining vertex still has so many edges incident to it (relative to the number of vertices we still need to use to finish) that we can conclude via the simple greedy algorithm. To this end, it is useful to track the evolution of the degrees of vertices in $A$. We do that by modeling the vertex degrees using a system of differential equations.

Define $d_{0,v}$ to be the initial degree of $v$ in $G$. Note that $d_{0,v} = (1 + \epsilon)q$ for $v \in A$ and $d_{0,v} \leq q$ for $v \in B$. We will define three functions $s(x), g_1(x)$, and $g_2(x)$ such that after the $t$-th iteration of the algorithm we have the following:

1. Each surviving vertex in $A$ has degree exactly $d_t$, which is approximately $s(t\delta)(1 + \epsilon)q$.

2. Each surviving vertex $v \in B$ has degree at most $d_{t,v}$, which approximately behaves in the following way: $d_{t,v} \approx (1 - t\delta)g_1(t\delta)d_{0,v}$.

3. Each remaining color class has size is at most $s_t$, which is about $(1 - t\delta)g_2(t\delta)q$. 

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The readers would expect to see $s(t\delta)$ being used to track the trajectory of $s_t$, but instead we use it for $d_t$ because of the fact that the trajectory for $d_t$ in this section turns out to be the same as $s_t$ of Section 4. This will be clear by the end of this subsection. We outline a rough analysis below. Clearly $s(0) = 1$, $g_1(0) = 1$, and $g_2(0) = 1$. Assume that $t$-th iteration is done, and the properties are true. First of all, there will be very few edges which will be discarded in Step 2 throughout the process. In other words, the number of edges in the partial rainbow matching will be roughly $t\delta q$ after the $t$-th round.

Because of Step 4, every vertex of $B$ is deleted with the same probability among Steps 3 and 4 combined. This is done only for convenience in our analysis, so we should set that probability (denote it by $b_t$) as low as possible. In other words, we need to find the maximum probability a vertex $v \in B$ can be deleted in Step 3, and set $b_t$ to be that maximum probability. After the $t$-th iteration, the number of remaining vertices in $A$ is about $(1 - t\delta)q$, and so the number of remaining edges is about $(1 - t\delta)qd_t$. Now if one picks an edge uniformly at random from $G$, then the probability (denote by $p$) that one of the edges incident to $v \in B$ will be picked in Step 2 is exactly $d_{t,v}$ divided by the total number of edges of $G$, which is about $\frac{d_{t,v}}{(1-t\delta)qd_t}$. So the probability that $v$ is not deleted in Step 3 is about $(1-p)^q \approx 1 - \frac{\delta g_1(t\delta)d_{t,v}}{s(t\delta)(1+\epsilon)q}$. Since $d_{0,v} \leq q$, we can define $b_t$ to be about $\frac{\delta g_1(t\delta)q}{s(t\delta)(1+\epsilon)q} = \gamma \frac{\delta g_1(t\delta)}{s(t\delta)}$, where $\gamma = \frac{1}{1+\epsilon}$.

Next, we similarly provide a probability (denote it by $a_t$) with which a single color class will be deleted in Steps 5 and 6 combined. Any color class will be deleted in Step 5 by probability $\frac{\delta g_2(t\delta)}{(1-t\delta)qd_t} = \gamma \frac{\delta g_2(t\delta)}{s(t\delta)}$, due to a simple union bound using the fact that there are $\delta q$ edges picked in Step 1, relatively few conflicts in Step 2, and the number of edges remaining in the graph is at least $(1-t\delta)qd_t$. Hence, we set $a_t$ to be about $\gamma \frac{\delta g_2(t\delta)}{s(t\delta)}$.

We now analyze the change in $d_t$ to get a differential equation for $s$. Any edge will be deleted in Steps 5 and 6 combined by probability $a_t$. We also know that for every $v \in A$, each of its neighbors gets deleted among Steps 3 and 4 with probability $b_t$. So, neglecting small error terms due to lack of independence, we expect to have the following:

$$d_{t+1} - d_t \approx -d_t(a_t + b_t)$$

$$s'(x) = -\gamma (g_1(x) + g_2(x)). \quad (9.1)$$

Next, let us estimate the change in $d_{t,v}$ to get a differential equation for $g_1$. The probability that a single vertex $v \in A$ gets deleted in Step 3 is about $\frac{\delta g_1(t\delta)}{(1-t\delta)q} = \frac{\delta}{1-t\delta}$. Any edge will be deleted in Step 5 by probability $a_t$. So, neglecting small error terms due to lack of independence, we can choose $d_{t+1,v}$ so that $d_{t+1,v} = (1 - \left[\frac{\delta}{1-t\delta} + 1 - \frac{\delta}{1-t\delta}\right]a_t) d_{t,v}$. This suggests the following behavior:

$$d_{t+1,v} - d_{t,v} \approx -\left[\frac{\delta}{1-t\delta} + \left(1 - \frac{\delta}{1-t\delta}\right)a_t\right] d_{t,v} \quad (9.2)$$

$$(\tau - t - 1)\delta g_1((t+1)\delta)d_{0,v} - (\tau - t)\delta g_1(t\delta)d_{0,v} \approx -\delta g_1(t\delta)d_{0,v} - (\tau - t - 1)\delta \frac{\gamma \delta g_2(t\delta)}{s(t\delta)} g_1(t\delta)d_{0,v}$$

$$\frac{g_1((t+1)\delta) - g_1(t\delta)}{\delta} \approx -\gamma \frac{g_1(t\delta)g_2(t\delta)}{s(t\delta)}.$$

The above suggests:
\[ g'(x) = -\gamma \cdot \frac{g_1(x)g_2(x)}{s(x)}. \] (9.3)

We next estimate the change in \( s_t \) to obtain a differential equation for \( g_2 \). The probability that a single vertex \( v \in B \) gets deleted among Steps 3 and 4 is \( b_t \), and the probability that a single vertex \( v \in A \) gets deleted in Step 3 is about \( \frac{\delta q}{1-t\delta} \). So, similar to before, we can choose \( s_{t+1} \) so that

\[ s_{t+1} = (1 - \left[b_t + (1 - b_t) \frac{\delta}{1-t\delta}\right]) s_t = (1 - \left[\frac{\delta q}{1-t\delta} + (1 - \frac{\delta q}{1-t\delta}) a_t\right]) s_t, \]

where the expression is similar to (9.2). So, similarly we expect the following behavior:

\[ g_2((t + 1)\delta) - g_2(t\delta) \approx -\gamma \cdot \frac{g_1(t\delta)g_2(t\delta)}{s(t\delta)}. \] (9.4)

Equations (9.3) and (9.4) along with the initial conditions \( g_1(0) = g_2(0) \) imply that \( g_1 = g_2 = g \). Combining with Equation (9.1), we have the exact same equations and initial conditions as in Section 4. So, we get \( s(x) = (1 - \gamma x)^2 \) and \( g_1(x) = g_2(x) = 1 - \gamma x \).

We show in the next section that the degrees of vertices and the sizes of color classes are concentrated in some sense throughout the process, which implies that we can not get stuck as long as \( t \leq \eta \tau \). Moreover, after \( \eta \tau \)-th iteration, the degree of any remaining vertex in \( A \) is \( \approx (1 - \gamma \eta)^2(1 + \epsilon)q \), and the number of remaining vertices in \( A \) is about \( q - \eta \tau \delta q = (1 - \eta)q \). As long as \((1 - \gamma \eta)^2(1 + \epsilon)q > 2(1 - \eta)q\), we can finish the rainbow matching greedily, which can be made true by choosing \( 1 - \eta \) small enough compared to \( 1 - \gamma \).

### 9.3 Formal analysis

Similar to Section 5, we define that \( s(x) = (1 - \gamma x)^2 \) and \( g(x) = 1 - \gamma x \), where \( \gamma = \frac{1}{1 + \epsilon} \). We introduce error terms \( \alpha_t \) and \( \beta_t \) (which will be explicitly specified later and much less than \( \frac{1}{100} \)) such that we make sure the following three happen after each iteration:

1. The survived vertices of \( A \) have degree exactly:
   \[ d_t = (1 - \alpha_t)s(t\delta)(1 + \epsilon)q. \] (9.5)

2. The degree of each survived vertex in \( B \) and the size of each remaining color class both are at most:
   \[ s_t = (1 + \beta_t)(1 - t\delta)g(t\delta)q. \] (9.6)

The similar analysis as done in Section 5.1 leads us to define \( a_t \) and \( b_t \) both to be \( \gamma \frac{\delta s(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)} \). So from now on, we drop the notation \( b_t \) and use \( a_t \) in place of \( b_t \). We next specify probabilities for Step 4 and Step 6. For a fixed \( t \), at the \((t + 1)\)-st iteration we delete
each vertex \( v \in B \) in Step 4 independently with probability \( p_v \), such that \( p'_v + (1 - p'_v)p_v = a_t \), where \( p'_v \) is the probability that \( v \) gets deleted in Step 3. Similarly for a fixed \( t \), at the \((t + 1)\)-st iteration we delete each color class \( C \) in Step 6 independently with probability \( p_C \) such that \( p'_C + (1 - p'_C)p_C = a_t \), where \( p'_C \) is the probability with which that color class gets deleted in Step 5.

We now fix the values of the parameters of this section. Let \( \epsilon \) be a fixed positive number less than \( \frac{1}{10} \). Let \( \gamma = \frac{1}{1 + \epsilon} \) as introduced in Section 10.2. Let \( \eta = 1 - \epsilon^3 \) and \( \delta = \frac{1}{\log q} \). Similar to Section 9.3, \( \epsilon, \theta, \gamma, \) and \( \eta \) are all fixed constants, where the latter three constants are fixed once \( \epsilon \) is fixed. On the other hand, \( \delta \) can be made arbitrarily small by picking sufficiently large \( q \).

Next, we have a few concentration bounds similar to the ones in Sections 5.2 and 9.3, where we use the notion of w.v.h.p. introduced in Section 5.2.

**Lemma 9.1.** For each fixed \( t \), the number of edges discarded in Step 2 at the \((t + 1)\)-st iteration is at most \( \frac{3\delta^2 q}{1 - \delta} \) w.v.h.p.

We omit the proof of Lemma 9.1 because the exact same proof of Lemma 5.4 along with the specific calculations works for this lemma. We similarly have the same corollary as Corollary 5.5.

**Corollary 9.2.** The total number of edges not added in the rainbow matching in Step 2 throughout the algorithm is at most \( 3\delta q \log \left( \frac{1}{1 - \eta} \right) \) w.v.h.p. Consequently, the number of colors left after the \( t \)-th iteration is at \( \left( 1 - t\delta + 3\delta \log \left( \frac{1}{1 - \eta} \right) \right) q \).

**Lemma 9.3.** For each fixed \( t \leq \eta \tau \), in the \((t + 1)\)-st iteration, the number of deleted edges incident to each vertex in \( A \) in Steps 3, 4, 5, and 6 combined is at most \( 2a_t d_t + \sqrt{q} \log q \) w.v.h.p.

**Proof.** Fix a vertex \( v \in A \) at the start of the iteration \( t + 1 \). Let \( d_t \) denote the number of edges in that color class before starting the \((t + 1)\)-st iteration, and \( D_{t+1} \) denote the number of edges right after Step 6 in the \((t + 1)\)-st iteration. First, we show that \( \mathbb{E}(D_{t+1}) - d_t \geq -2a_t d_t \). For a fixed neighbor \( w \in B \) of \( v \), the probability that an edge \( vw \) gets removed due to the deletion of \( w \) in Steps 3 and 4 combined is exactly \( a_t \), and the probability that an edge \( vw \) gets deleted in Steps 5 and 6 is again exactly \( a_t \). So, the probability that \( vw \) gets deleted in Steps 3, 4, 5, and 6 combined is at most \( 2a_t \) by a simple union bound. Therefore, it follows that \( \mathbb{E}(D_{t+1}) - d_t \geq -2a_t d_t \). Now, similar to the proof of Lemma 8.3, the required concentration bound on \( D_{t+1} \) can be proved using Theorem 5.2.

Next, we show the following lemma which will help us in proving the lemmas on the concentration bounds for the degrees of survived vertices in \( B \) and the sizes of each remaining color classes.

**Lemma 9.4.** For each fixed \( t \leq \eta \tau \) and \( v \in A \), in the \((t + 1)\)-st iteration, the probability that at least an edge incident to \( v \) is picked in Step 1 but \( v \) is not deleted in Step 3 is at most \( C\delta^2 \), where \( C \) is a constant depending on \( \epsilon \).
Proof. For a vertex $w$ in the neighborhood $N(v)$ of $v$, let $X_w$ denote the event that $vw$ is picked in Step 1 and an edge incident to $v$ or $w$, or an edge with the same color as $vw$ is picked in Step 1 as well. The probability of Lemma 9.4 can be bounded by the following using a simple union bound.

$$\sum_{w \in N(v)} P[X_w] \leq d_t \cdot \frac{\delta q}{(1 - t\delta)qd_t} \cdot \left(1 - \left(1 - \frac{d_t + 2st}{(1 - t\delta)qd_t}\right)^{\delta q-1}\right) = O(\delta^2).$$ (9.7)

Lemma 9.5. For each fixed $t \leq \eta \tau$, for each remaining vertex $v \in B$ that remains after $t$-th iteration, at the end of Step 6 of the $(t + 1)$-st iteration, the expected degree of $v$ is at most

$$s_t \cdot (1 - a_t) \left(1 - \frac{\delta}{1 - t\delta}\right) + C'\delta^2,$$

where $C'$ is a constant depending on $\epsilon$.

Proof. We proceed similarly to the proof of Lemma 9.8. For any neighbor $u \in A$ of $v \in B$, we bound the probability that $u$ survives in Step 3 and the color of $uv$ is not deleted in Steps 5 and 6. Fix a neighbor $u \in A$ of $v \in B$. For the convenience of writing, let $X$ denote the event that $u$ survives in Step 3. Let $Y$ denote the event that the color of $uv$ is not deleted in Steps 5 and 6. We are interested in a good upper bound on $P[X \cap Y]$. Let $Z$ denote the event that no edge incident to $u$ is picked in Step 1, and $Z^c$ denote the complementary event of $Z$. Clearly,

$$P[X \cap Y] = P[X \cap Y \cap Z] + P[X \cap Y \cap Z^c]$$

$$\leq P[Y \cap Z] + P[X \cap Z^c]$$

$$\leq P[Z]P[Y | Z] + C\delta^2,$$ (9.8)

where in the last step, we use Lemma 9.4. Due to Corollary 9.2, the number of color classes remaining after the $t$-th iteration is at most $\left(1 - t\delta + 3\delta \log\left(\frac{1}{1 - \eta}\right)\right) q$. So, we can upper-bound the probability of $Z$ by

$$P[Z] \leq \left(1 - \frac{1}{\left(1 - t\delta + 3\delta \log\left(\frac{1}{1 - \eta}\right)\right) q}\right)^{\delta q}.$$ 

So, we have that

$$P[Z] \leq 1 - \frac{\delta q}{\left(1 - t\delta + 3\delta \log\left(\frac{1}{1 - \eta}\right)\right) q} + \frac{\delta^2 q^2}{2(1 - t\delta)^2 q^2}$$

$$\leq 1 - \frac{\delta}{1 - t\delta} + \frac{3\delta^2 \log\left(\frac{1}{1 - \eta}\right)}{(1 - t\delta)^2} + \frac{\delta^2}{2(1 - t\delta)^2}$$

$$\leq 1 - \frac{\delta}{1 - t\delta} + \frac{4\delta^2 \log\left(\frac{1}{1 - \eta}\right)}{(1 - \eta)^2},$$ (9.9)

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where in (9.9), we have used the fact that 
\[ \frac{\delta q}{(1-t\delta+3\delta \log \left(\frac{1}{1-\eta}\right))q} \geq \frac{\delta}{1-t\delta} - \frac{3\delta^2 \log \left(\frac{1}{1-\eta}\right)}{(1-t\delta)^2}, \]
which was proved in the proof of Lemma 5.8 (see, Step (5.6)). Now for the convenience of bounding \( \mathbb{P}[Y\mid Z] \), we define \( Y_1 \) and \( Y_2 \) in the following manner. \( Y_1 \) denote the event that the color of \( uv \) is not deleted in Step 5, and \( Y_2 \) denote the event that the color of \( uv \) is not deleted in Step 6. Clearly, \( Y = Y_1 \cap Y_2 \). A similar argument as in Lemma 5.8 shows that \( Y_2 \) is independent with both \( Y_2 \) and \( Z \). So, we have the following: \[
\mathbb{P}[Y\mid Z] = \mathbb{P}[(Y_1 \cap Y_2)\mid Z] = \mathbb{P}[Y_1\mid Z]\mathbb{P}[Y_2] \leq (1 - \mathbb{P}[Y_1\mid Z])\mathbb{P}[Y_2] \tag{9.11}
\]

Now, the event \( Y_1^{c} \) contains the event that exactly one edge with the same color as \( uv \) is picked in Step 1 and no other edges adjacent to that edge is picked in Step 1. Let \( t \) denote the number of edges with the same color as \( uv \) (note that \( r \leq s_t \)). Then, similar arguments and calculations as in the proof of Lemma 5.8 up to Step (5.6) shows the following: \[
\mathbb{P}[Y_1^{c}\mid Z] \geq \frac{\delta q \cdot r}{(1-t\delta + 3\delta \log \left(\frac{1}{1-\eta}\right))qd_t} \left(1 - \frac{2d_t + s_t}{(1-t\delta)qd_t - d_t}\right)^{\delta q}
\geq \frac{\delta qr}{(1-t\delta)qd_t} \left(1 - \frac{3\delta \log \left(\frac{1}{1-\eta}\right)}{1-t\delta}\right) \left(1 - \frac{7\delta}{1-t\delta}\right)
\]

By noting that \( \mathbb{P}[Y_1^{c}] \leq \frac{\delta qr}{(1-t\delta)qd_t} \) (due to a simple union bound), we have the following: \[
\mathbb{P}[Y_1^{c}\mid Z] - \mathbb{P}[Y_1^{c}] \geq - \frac{\delta qr}{(1-t\delta)qd_t} \left(\frac{3\delta \log \left(\frac{1}{1-\eta}\right)}{1-t\delta} + \frac{7\delta}{1-t\delta}\right)
\geq - \frac{\delta s_t}{(1-t\delta)d_t} \left(\frac{10\delta \log \left(\frac{1}{1-\eta}\right)}{1-t\delta}\right) \tag{9.12}
\]

A routine calculation from (9.12) shows that for \( t \leq \eta \tau \), \( \mathbb{P}[Y^{c}\mid Z] - \mathbb{P}[Y_1^{c}] \geq -O(\delta^2) \). This together with (9.8), (9.10), and (9.11) gives us the following: \[
\mathbb{P}[X \cap Y] \leq \left(1 - \frac{\delta}{1-t\delta} + \frac{4\delta^2 \log \left(\frac{1}{1-\eta}\right)}{(1-\eta)^2}\right) \left(1 - \mathbb{P}[Y_1^{c}\mid Z]\mathbb{P}[Y_2]\right) + C\delta^2
\leq \left(1 - \frac{\delta}{1-t\delta}\right) \left(1 - \mathbb{P}[Y_1^{c}]\right)\mathbb{P}[Y_2] + O(\delta^2).
\]

Finally, Lemma 9.5 follows by noting that \( (1 - \mathbb{P}[Y_1^{c}]\mathbb{P}[Y_2] = \mathbb{P}[Y_1]\mathbb{P}[Y_2] = \mathbb{P}[Y_1 \cap Y_2] = \mathbb{P}[Y] = 1 - a_t. \)

**Lemma 9.6.** For each fixed \( t \leq \eta \tau \), for each remaining color class after \( t \)-th iteration, at the end of Step 4 of the \( (t + 1) \)-st iteration the expected size of that color class is at most \( s_t \cdot (1 - a_t) \left(1 - \frac{\delta}{1-t\delta}\right) + C'\delta^2 \), where \( C' \) is a constant depending on \( \epsilon \).
The proof of Lemma 9.5 can be repeated word by word for Lemma 9.6 if $Y$ (as introduced in the proof of Lemma 9.5) is defined as the event that $v \in B$ is not deleted in Steps 3 and 4. Next using Theorem 5.2 like in the proof of Lemma 8.3, we obtain the following concentration result.

**Lemma 9.7.** For each fixed $t \leq \eta \tau$, at the end of $(t + 1)$-st iteration, the degree of each remaining vertex $v \in B$ and the size of each remaining color class both are at most $s_t \cdot (1 - \alpha_t) \left(1 - \frac{\delta}{1 - \tau \delta}\right) + C' \delta^2 + \sqrt{q} \log q$ w.v.h.p., where $C'$ is a constant depending on $\epsilon$.

### 9.4 Estimating the error terms and greedy completion

Similar to Sections 5 and 9, let us first estimate the error terms in the ideal expressions from the intuitive analysis. Define the following parameters (recall that in the beginning of Section 5, we mentioned that we would denote the ideal values of parameters by tilde symbol over the corresponding notations).

\[
\tilde{d}_t = s(t \delta)(1 + \epsilon)q,
\]
\[
\tilde{s}_t = (1 - t \delta)g(t \delta)q.
\]
\[
\tilde{a}_t = \frac{\delta g(t \delta)}{s(t \delta)}.
\]

Next, we have the following couple of lemmas concerning the relations between these three ideal functions, which are same as the ones in Section 5.

**Lemma 9.8.** We have that 
\[
(1 - 2\tilde{a}_t)\tilde{d}_t = \tilde{d}_{t+1} - \gamma^2 \delta^2 (1 + \epsilon)q.
\]

**Lemma 9.9.** We have that 
\[
(1 - \tilde{a}_t) \left(1 - \frac{1}{\tau \gamma}\right) \tilde{s}_t = \tilde{s}_{t+1}.
\]

As before, we will find $y_t$ and $z_t$ such that $d_t \geq \tilde{d}_t - y_t$ and $s_t \leq \tilde{s}_t + z_t$. Similar results as Lemma 6.3 and Lemma 6.5 hold showing that the accumulated error terms are negligible compared to the ideal parameter values, i.e., $y_t \ll \tilde{d}_t$ and $z_t \ll \tilde{s}_t$. Hence, we can run the randomized algorithm of Section 10.1 until $\eta \tau$-th iteration. Hence, at the end of $\eta \tau$-th iteration, the degree of any remaining vertex is at least \[
\sqrt{1 + \epsilon} (1 - \gamma \eta)^2 (1 + \epsilon)q
\] and the number of remaining vertices in $A$ is at most \[
q - \eta \tau \delta q + 3 \delta q \log \left(\frac{1}{1 - \eta}\right) \leq \sqrt{1 + \epsilon} (1 - \eta)q
\] (here, any constant greater than 1 would work in the places of the $\sqrt{1 + \epsilon}$ factors, this particular choice was made to make the final inequality easier). The straightforward greedy algorithm will work here as long as \[
\frac{1}{\sqrt{1 + \epsilon}} (1 - \gamma \eta)^2 (1 + \epsilon)q > 2 \sqrt{1 + \epsilon} (1 - \eta)q
\] which is equivalent to \[
\left(1 - \frac{1 - \eta^2}{1 + \epsilon}\right)^2 > 2 \epsilon^3
\] (remember that $\eta = 1 - \epsilon^3$). This can be easily checked to be true for $\epsilon < \frac{1}{10}$. Hence, the rainbow partial matching obtained from running the algorithm can be completed to a rainbow matching using all vertices in $A$, proving Theorem 1.10.
10 Concluding remarks

Our arguments can be extended to get a generalization of Theorem 1.12 for multigraphs with bounded edge multiplicities. In particular, Theorem 1.12 is true (with worse error term in the required number of edges in each color) for multigraphs with edge multiplicities at most \( q^c \) for some \( c > 0 \). Gao, Ramadurai, Wanless, and Wormald have discussed this point in more details in their paper [16]. Similar to them, our arguments seem to be hard to extend all the way to the general multigraphs with no restrictions.

We remark here that it was shown in [16] that Theorem 1.12 is true even if the number of colors is more than \( q \), for example our proof can show the statement with \( q^{1+c} \) colors for some \( c > 0 \). It will be interesting if one can prove a statement like Theorem 1.12 with any number of colors, which is the same question as in Conjecture 1.2 with some bounded degree assumption on color classes. Our probabilistic argument does not work when the number of colors is too large compared to \( q \).

It will be interesting to investigate if one can replace \( \epsilon q \) by some constant in Theorem 1.10. Most likely, to obtain such statement new idea is needed, because the probabilistic approach might not work well to find such exact results. In particular, it will be interesting to further investigate how small the ‘\( o(q) \)’ term can be made in Corollary 1.11. Our proof can only give some error term of the form \( q^c \) for some \( 0 < c < 1 \). It will be interesting to make this error term poly-logarithmic.

Another direction of research can be to improve the number of colors needed in Theorem 1.9 with a condition that the underlying graph is bipartite. In the multigraph case, the answer is completely known for the bipartite case due to Theorem 1.5 by Drisko. In this literature, some of the results seem to be easier when the underlying graph is bipartite. So, it would be interesting to resolve the problem if it is enough to have \( q + o(q) \) colors for bipartite graphs in Theorem 1.9.

Finally, we end with a discussion why improving the required number of colors in Theorem 1.9 might be hard using our approach. In order to demonstrate that we state an asymptotically tight generalization of Theorem 1.9. We start by providing a motivation for the generalization, which was first mentioned in an earlier version of [3]. Consider the setting of Theorem 1.9 where we have some number (say \( r \)) of matchings of size \( q \) each. Now what happens if we scramble the edges to obtain another system of \( r \) sets of edges (resulting color classes might not be matching anymore), each of size \( q \)? The maximum degree of the underlying graph will be still bounded by the number of colors used. Furthermore, if the scrambling was done randomly, then each class of edges will have bounded maximum degree.

**Theorem 10.1.** For all \( \epsilon > 0 \) and positive integer \( \Delta \), there exists an \( N = N(\epsilon, \Delta) \) such that whenever \( q \geq N \) for any graph \( G \) with maximum degree at most \( 2(1 + \epsilon)q \) that is edge-colored with \( 2 \left( 1 + \epsilon - \frac{\epsilon^2}{(\Delta + 3)^2} \right) q \) colors such that there are at least \( q \) edges of each color and at most \( \Delta \) edges of same color can be incident to any vertex, there is a rainbow matching of \( G \) using \( q \) colors.

Note that \( \Delta = 1 \) retrieves the original Theorem 1.9 because the condition \( \Delta = 1 \) ensures that the maximum degree of the underlying graph is at most the number of colors used. Next, we show that Theorem 10.1 is tight for \( \Delta \geq 2 \) in the sense that the constant 2 cannot
be replaced with a smaller constant in the required number of colors. More precisely, we show that there is a graph $G$ with maximum degree $2q - 2$ that is edge-colored with $2q - 3$ colors such that each color class has $q$ edges and maximum degree 2 such that $G$ does not contain any rainbow matching of size $q$. To see this, consider a decomposition of the edges of the complete graph $K_{2q - 1}$ into $2q - 1$ perfect matchings (with size $q - 1$). Select any $2q - 3$ such perfect matchings and assign distinct colors corresponding to each of these matchings. From the rest of $2q - 2$ remaining edges select any $2q - 3$ edges and color them using the $2q - 3$ colors using each color exactly once. Now, it is easy to check that we do not even have any matching of size $q$, because we have only $2q - 1$ vertices in the considered graph.

Theorem 10.1 can be proved by using the following couple of results which can be proved using arguments analogous to Sections 8 and 9. In particular, the arguments of Section 8 can be easily extended to obtain the following statement.

**Theorem 10.2.** For all $\epsilon, \theta, \alpha > 0$, positive integer $\Delta$, there exists an $N = N(\epsilon, \theta, \alpha, \Delta)$ such that whenever $q \geq N$ and $\Delta_G \geq (1 + \epsilon)q$, the following holds. Suppose that $G$ is a graph with maximum degree at most $\Delta_G$ and the number of vertices with degree at least $(1 - \theta)2q$ is at most $(1 - \theta)q$. Furthermore, $G$ is edge-colored with $\max((1 + \alpha)q, (1 - \theta^2 + \alpha)\Delta_G)$ colors such that there are at least $q$ edges of each color and at most $\Delta$ edges of same color can be incident to any vertex. Then, there is a rainbow matching of $G$ using $q$ colors.

Similarly, the arguments in Section 9 can be modified appropriately to obtain the following statement, which is a generalized version of Theorem 1.10.

**Theorem 10.3.** For all $\epsilon > 0$ and positive integer $\Delta$, there exists an $N = N(\epsilon, \Delta)$ such that whenever $q \geq N$, the following holds. Suppose $G$ is a bipartite graph on the vertex set with bipartition $A \cup B$, where $|A| = q$ and every vertex in $A$ has degree at least $(1 + \epsilon)q$. Suppose the edges are colored such that there are at most $q$ edges of each color and at most $\Delta$ edges of same color can be incident to any vertex. Then, there always is a rainbow matching in $G$ which uses every vertex in $A$.

It is clear that any proof of any significant improvement on the required number of colors in Theorem 1.9 should not work for Theorem 10.1 because Theorem 10.1 is already asymptotically tight. So, the above discussion indicates that our probabilistic approach alone cannot yield any such improvement of Theorem 1.9 and we might need to use some structural arguments taking advantage of the fact that each color appears as a matching to improve this result.

On a different note, Stein [27] made a conjecture stronger than Conjecture 1.1 which translates into the graph theoretic statement: “In any $n$-edge-coloring of $K_{n,n}$ with each color appearing exactly $n$ times, there is a rainbow matching of size $n$”. This stronger conjecture is proven to be false in [25] by Pokrovskiy and Sudakov. However, Theorem 10.3 obtains a generalized version of Corollary 1.11 in the style of Stein’s conjecture, where the coloring does not need to be proper but each color class needs to have bounded size and maximum degree.

We finish by mentioning the following problem. It will be interesting to investigate the problems of Theorem 10.1 or Theorem 1.12 without the maximum degree condition on the underlying graph.
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