On the Hilbert scheme of curves in higher-dimensional projective space

Barbara Fantechi* — Rita Pardini*

Abstract

In this paper we prove that, for any \( n \geq 3 \), there exist infinitely many \( r \in \mathbb{N} \) and for each of them a smooth, connected curve \( C_r \) in \( \mathbb{P}^r \) such that \( C_r \) lies on exactly \( n \) irreducible components of the Hilbert scheme \( \text{Hilb}(\mathbb{P}^r) \). This is proven by reducing the problem to an analogous statement for the moduli of surfaces of general type.

1 Introduction

It is well-known that the Hilbert scheme parametrizing subschemes of \( \mathbb{P}^r \) can be singular at points corresponding to smooth curves as soon as \( r \geq 3 \); actually Mumford [M] gave an example of an everywhere singular irreducible component. If \( r = 3 \), it has been proven in [EHM] that the open subset of the Hilbert scheme parametrizing smooth curves in \( \mathbb{P}^3 \) with given genus and degree can have arbitrarily many components when the genus and the degree grow (in fact, they prove that no polynomial estimate on the number of such components holds).

Our main result is the following:

**Theorem 4.4.** Let \( n \geq 3 \) be an integer. Then there exist infinitely many integers \( r \), and for each of them a smooth, irreducible curve \( C_r \subset \mathbb{P}^r \) such that \( C_r \) lies exactly on \( n \) components of the Hilbert scheme of \( \mathbb{P}^r \).

The idea of the proof is very simple. Firstly, we modify a construction of [FP] to obtain a regular surface \( S \) of general type which lies on \( n \) components

*Both authors are members of GNSAGA of CNR.
of the moduli space; secondly, we consider a suitable pluricanonical embedding of this surface and intersect its image with a high-degree hypersurface $F$ to construct the curve $C$ we are interested in. Finally, we prove that all embedded deformations of $C$ are induced by embedded deformations of $F$ and $S$.

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2 Notation and preliminaries

All varieties will be assumed smooth and projective over the complex numbers unless the contrary is explicitly stated. A variety $Y$ will be called regular if $H^1(Y, \mathcal{O}_Y) = 0$. If $F$ is a sheaf on $Y$, let $h^i(Y, F) = \dim H^i(Y, F)$. If $t$ is a real number, we denote its integral part by $\lfloor t \rfloor$. Let $\zeta_3 = \exp(2\pi i/3)$.

In this paper we will be concerned with abelian covers of a very special type; we collect here the necessary notational set-up.

Let $n$ be an integer $\geq 2$, and let $G = \mathbb{Z}_3^n$, $G^*$ its dual; let $e_1, \ldots, e_n$ be the canonical basis of $G$, and $\chi_1, \ldots, \chi_n$ the dual basis of $G^*$ (i.e., $\chi_j(e_i) = 1$ if $i \neq j$ and $\chi_j(e_j) = \zeta_3$). Let $e_0 = -(e_1 + \ldots + e_n)$. Let $I = \{0, \ldots, n\}$, and to each $i \in I$ associate the pair $(H_i, \psi_i)$ where $H_i$ is the cyclic subgroup of $G$ generated by $e_i$, and $\psi_i \in H_i^*$ is the character such that $\psi_i(e_i) = \zeta_3$.

Let $Y$ be a smooth projective variety, and $(G, I)$ as above: a $(G, I)$-cover of $Y$ is a normal variety $X$ and a Galois cover $f : X \to Y$ with Galois group $G$ and (nonempty) branch divisors $D_i$ (for $i \in I$) having $(H_i, \psi_i)$ as inertia group and induced character (see [P] for details).

Lemma 2.1 To give a smooth $(G, I)$-cover of $Y$ is equivalent to giving line bundles $L$ and $F_j$, for $j = 1, \ldots, n$, together with smooth nonempty divisors $D_i \in |M_i|$ (where $M_0 = L$ and, for $i \geq 1$, $M_i = L - 3F_i$) such that the union of the $D_i$'s has normal crossings.

Proof. From [P] we know that the cover is determined by its reduced building data, divisors $D_i$ for $i \in I$ and line bundles $L_j$ for $j = 1, \ldots, n$ satisfying the relation $3L_j \equiv D_j + 2D_0$. Letting $M_i = \mathcal{O}(D_i)$, and putting $M_0 = L$, $F_j = L - L_j$, the equations become precisely $M_j = L - 3F_j$. □

As the natural map $\bigoplus_{i \in I} H_i \to G$ is surjective, the covers we consider will be totally ramified. For $\chi \in G^*$, let as usual $L_{\chi}^{-1}$ be the corresponding
eigensheaf in the direct sum decomposition of $f_*O_X$; in the above notation, we will have (for $\chi = \chi_1^{\alpha_1} \cdots \chi_n^{\alpha_n}$):

$$L_\chi = n_\chi L - \sum_{j=1}^{n} \alpha_j F_j,$$

(2.1.1)

where $n_\chi = -[(\alpha_1 - \cdots - \alpha_n)/3]$. In particular note that $n_\chi \geq 1$ when $\chi \neq 1$, and $n_\chi = 1$ if and only if $1 \leq \sum \alpha_j \leq 3$. We will write $L_j$ instead of $L_{\chi_j}$.

Recall from [P], proof of proposition 4.2 on page 208, that

$$3K_X = \pi^*(3K_Y + 2(n + 1)L - 6 \sum F_j).$$

(2.1.2)

We now recall some results from [FP] in a simplified form (fit for our situation). For details and proofs see [FP], §5.

**Remark 2.2** (1) Let $\mathcal{Y} \to B$ be a smooth projective morphism (with $B$ a smooth, connected quasiprojective variety) together with an isomorphism between $\mathcal{Y}_o$ and $Y$ for some $o \in B$, and assume that $Y$ is regular and that the morphism $\mathcal{Y} \to B$ has a section $\sigma$. Let $L$ be a line bundle on $Y$; assume that $c_1(L)$ is kept fixed by the monodromy action of $\pi_1(B, o)$ on $H^2(Y, \mathbb{Z})$. Then for each $b \in B$ there is a canonical induced class $c_1(L_b)$ on $\mathcal{Y}_b$. If, for all $b \in B$, the class $c_1(L_b)$ is of type $(1, 1)$, then $L$ can be extended to a line bundle $\mathcal{L}$ over $\mathcal{Y}$, flat over $B$; this extension is unique if we require that its restriction to $\sigma(B)$ be trivial. This follows by applying the results on p. 20 of [MF], and by noting that the relative Picard scheme of $\mathcal{Y}$ over $B$ is étale over $B$ since all fibres are smooth and regular (it is surjective as $c_1(L_b)$ is always of type $(1, 1)$); the condition on the monodromy action implies then that the component of the relative Picard scheme containing $[L]$ is in fact isomorphic to $B$. Let $L_b$ be the restriction of $\mathcal{L}$ to $\mathcal{Y}_b$.

(2) If $h^0(\mathcal{Y}_b, L_b)$ is either constant in $b$, or if it only assumes the values 1 (for $b \in Z$) and 0, then there is a (nonunique) quasiprojective variety $W^L \to B$ such that $W_b^L$ is canonically isomorphic to $H^0(\mathcal{Y}_b, L_b)$; $W^L$ is smooth and irreducible in the former case, while in the latter it is the union of one component isomorphic to $B$ and another being the total space of a line bundle over $Z$ (compare with [FP], theorem 5.8 and remark 5.11).
Assumption 2.3 Let $S = \{(i, \chi) \in I \times G^*|\chi|_{H_i} \neq \psi_i^{-1}\}$. Let $X \to Y$ be a smooth $(G, I)$-cover as in lemma 2.1, and $Y \to B$ be a smooth projective morphism (with $(B, o)$ a pointed space, and $Y_o$ isomorphic to $Y$), such that remark 2.2 (1) applies to $Y \to B$, for the line bundle $L$ and for each of the $F_j$’s. Assume moreover that remark 2.2 (2) applies for the line bundles $M_i - L_\chi$ for $(i, \chi) \in S$, yielding varieties $W^{i,\chi}$: let $W$ be the fibred product of the $W^{i,\chi}$ over $B$. Finally, assume that the germ of $B$ at $o$ maps smoothly to the base of the Kuranishi family of $Y$, and that the cohomology groups $H^1(Y, L^{-1}_\chi)$ and $H^1(Y, T_Y \otimes L^{-1}_\chi)$ vanish for each $\chi \in G^* \setminus 1$.

Theorem 2.4 Assume that assumption 2.3 holds, and let $w \in W$ be a point over $o \in B$ corresponding to sections $s_{i,\chi}$ such that $s_{i,\chi} = 0$ if $\chi \neq 1$, and $s_{i,1}$ defines $D_i$ for $i = 0, \ldots, n$. Assume also that $X$ has ample canonical class. One can construct a family of natural deformations of $(G, I)$-covers $X \to W$; the induced map from the germ of $w$ in $W$ to the Kuranishi family of $X$ is smooth (and, in particular, surjective). Moreover, the flat, projective morphism $X \to W$ defines a rational map from $W$ to the moduli of surfaces with ample canonical class, regular at $w$; this map is dominant on each irreducible component of the moduli containing $X$.

Proof. Let $L$ (resp. $F_j$) be the line bundle induced by $L$ (resp. $F_j$) on $W$; as we define $M_0$ to be $L$, $L_j$ to be $L - F_j$ and $M_j$ to be $L - 3F_j$ for $j = 1, \ldots, n$, there are global, canonical isomorphisms $\phi_j : 3L_j \to M_j + 2M_0$. By [FP], theorem 5.12, the germ of $W$ at $w$ maps smoothly to the base of the Kuranishi family of $X$.

If $M$ is an irreducible component of the moduli containing $[X]$, by the previous result the image of $W$ contains an open set in $M$ (in the strong topology), hence it cannot be contained in a closed subset (in the Zariski topology) and is therefore dominant. 

3 Moduli of surfaces of general type

The aim of this section is the proof of theorem 3.9, i.e., the explicit construction of regular surfaces with ample canonical class lying on arbitrarily many components of the moduli. This construction can be carried out in a much
more general setting (see remark 3.12); we consider only the case needed for our applications, since it is easier to describe.

For $S$ a smooth projective surface and $x_1, \ldots, x_n$ pairwise distinct points of $S$, we let $B\ell(S; x_1, \ldots, x_n)$ denote the surface obtained by blowing up $S$ at $x_1, \ldots, x_n$.

**Construction 3.1** Let $S$ be a regular surface, $x_0 \in S$, $n$ a positive integer; let $B = B(S, n)$ be the variety parametrizing data $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ where the $x_i$’s are pairwise distinct points in $S$ (for $i = 0, \ldots, n$), the $y_i$’s are pairwise distinct points in $B\ell(S; x_1, \ldots, x_n)$, such that $y_i$ is not infinitely near to $x_j$ for $i \neq j \geq 1$ and none of the $y_i$’s lies over $x_0$. $B$ is a smooth quasiprojective variety, which is naturally isomorphic to an open subset of the product of $n$ copies of $S \times S$ blown up along the diagonal. Let $Y \to B$ be the smooth projective family such that $Y_b$, the fibre of $Y$ over the point $b$, is isomorphic to $B\ell(B\ell(S; x_1, \ldots, x_n); y_1, \ldots, y_n)$ for $b = (x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Note that the morphism $Y \to B$ has a section, given by mapping $b \in B$ to the inverse image of $x_0$ in $Y_b$.

**Lemma 3.2** Assume that $S$ is rigid. Let $B^0$ be the open set in $B$ where $\text{Aut}(S)$ acts freely (the action being the natural one). Then if $b \in B^0$, the natural map from the germ of $B$ in $b$ to the Kuranishi family of $Y_b$ is smooth of relative dimension $h^0(S, T_S)$.

**Proof.** The proof is easy and left to the reader. $\square$

**Remark 3.3** For any $b \in B$, $b = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, there is a canonical isomorphism

$$NS(Y_b) = NS(S) \oplus \mathbb{Z}e'_1 \oplus \ldots \oplus \mathbb{Z}e'_n \oplus \mathbb{Z}e''_1 \oplus \ldots \oplus \mathbb{Z}e''_n,$$

where $e'_i$ is the pullback from $B\ell(S; x_1, \ldots, x_n)$ of the class of the exceptional divisor over $x_i$, and $e''_i$ is the class of the exceptional divisor over $y_i$. We will consider this isomorphism fixed, and denote this group by $NS$. We also let $f_i$ denote $e'_i - e''_i$. Since $S$ is regular, so are all the $Y_b$’s and we will not need to distinguish between line bundles and their Chern classes.
Definition 3.4 Let \( L \in NS \), \( G = \mathbb{Z}_3^n \) as in \( \S 2 \); for \( \chi \in G^* \), let \( L_\chi \in NS \) be defined by equation (2.1.1), with \( F_i = f_i \). Let \( B_L \) be the open subset of \( B \) consisting of the \( b \)'s such that

1. the cohomology groups \( H^1(\mathcal{Y}_b, L_\chi^{-1}) \), \( H^1(\mathcal{Y}_b, T_{\mathcal{Y}_b} \otimes L_\chi^{-1}) \) are zero for each \( \chi \in G^* \setminus 1 \);
2. the line bundles \( L \) and \( L - 3F_j \) are very ample on \( \mathcal{Y}_b \), for \( j = 1, \ldots, n \);
3. the line bundles \( L - K_{\mathcal{Y}_b} \) and \( L - 3F_j - K_{\mathcal{Y}_b} \) are ample on \( \mathcal{Y}_b \), for \( j = 1, \ldots, n \);
4. the line bundle \( 3K_Y + 2(n + 1)L - 6 \sum F_j \) is ample on \( \mathcal{Y}_b \).

Note that the first condition is needed to ensure that assumption 2.3 is satisfied; the second allows one to choose smooth divisors in the linear systems \( |L| \) and \( |L - 3F_j| \) meeting transversally; the third implies that these linear systems have constant dimension when \( b \) varies; and the fourth ensures, in view of equation (2.1.2), that the cover so obtained has ample canonical class (recall that the pullback of an ample line bundle via a finite map is again ample).

Lemma 3.5 Let \( Y \) be a smooth surface containing \( m \) disjoint irreducible curves \( C_1, \ldots, C_m \), such that \( C_i^2 < 0 \). Then:

1. for any choice of nonnegative integers \( a_1, \ldots, a_m \), the linear system \( |a_1C_1 + \ldots + a_mC_m| \) contains only the divisor \( a_1C_1 + \ldots + a_mC_m \);

2. for any choice of nonnegative integers \( a_1, \ldots, a_{m-1} \), and for any \( b > 0 \), the linear system \( |a_1C_1 + \ldots + a_{m-1}C_{m-1} - bC_m| \) is empty.

Proof. (1) We prove the theorem by induction on \( a_1 + \ldots + a_m \), the case where this sum is zero being trivial. Assume without loss of generality that \( a_1 \geq 1 \), and let \( C \in |a_1C_1 + \ldots + a_mC_m| \); then \( C \cdot C_1 = a_1C_1^2 < 0 \), hence \( C \) must have a common component with \( C_1 \); therefore \( C = C_1 + C' \), \( C' \in |(a_1 - 1)C_1 + \ldots + a_mC_m| \), and by induction the proof is complete.

(2) Assume that there exists \( C' \in |a_1C_1 + \ldots + a_{m-1}C_{m-1} - bC_m| \). Then \( C + bC_m \in |a_1C_1 + \ldots + a_{m-1}C_{m-1}| \), contradicting (1). \( \square \)
Corollary 3.6 Let $b \in B$, $b = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and let $a_1, \ldots, a_m$ be nonnegative integers. Then the line bundle $a_1 f_1 + \ldots + a_m f_m$ on $Y_b$ is effective if and only if $y_i$ is infinitely near to $x_i$ for every $i$ such that $a_i > 0$, and in this case it has only one section.

Proof. If $y_i$ is infinitely near to $x_i$, then $f_i$ is a $(-2)$ curve; otherwise it is the difference of two disjoint $(-1)$ curves. In the former case lemma 3.3 (1) applies and in the latter case 3.5 (2) applies. □

Notation 3.7 We will denote by $E$ the closed subset of $B$ consisting of the points $b$ such that $f_i$ is effective on $Y_b$ for $i = 1, \ldots, n$.

Lemma 3.8 Let $L \in NS$ be a line bundle and assume that $E \cap B_L \neq \emptyset$. Then assumption 2.3 holds for the restriction of $Y \to B$ to $B_L$; applying theorem 2.4 yields a quasiprojective variety $W$. In this case $W$ is the union of $2^n$ smooth irreducible components $W_A$, indexed by subsets $A \subset \{1, \ldots, n\}$. The dimension of $W_A$ and $W_{A'}$ are equal if $\#A = \#A'$, and in particular one has:

$$\dim W_A - \dim W_\emptyset = \frac{1}{6}(\#A^3 + 6\#A^2 - \#A).$$

The $W_A$’s have a nonempty intersection.

Proof. The verification that assumption 2.3 holds is easy and we leave it to the reader. For $A \subset \{1, \ldots, n\}$, let $E_A = \{b \in B_L| f_i \text{ is effective for } i \in A\}$ and let $W_A \subset W$ be defined by

$$W_A = \{(b, s_{i,\chi})| b \in E_A \text{ and } s_{i,\chi} = 0 \text{ for } \chi \neq 1 \text{ and } i \notin A\}.$$ 

It is easy to check that $W_A$ is smooth over $E_A$ of dimension $1/6(\#A^3 + 6\#A^2 + 5\#A)$; on the other hand $E_A$ is smooth of codimension $\#A$ in $B_L$. Finally, $W$ is the union of the $W_A$’s, which are easily seen to be irreducible components. Moreover, the intersection of the $W_A$’s is clearly equal to $W_\emptyset$ intersected with the inverse image of $E \cap B_L$. □

Theorem 3.9 Let $L \in NS$ and $b \in E \cap B_L \cap B^0$. Let $f : X \to Y = Y_b$ be a smooth $(G, I)$-cover with building data $(D_i, L_\chi)$; let $w \in W_b$ be a point corresponding to a choice of equations $s_i \in \mathcal{O}_Y(D_i)$ defining $D_i$, with $s_{i,\chi} = 0$
for all $\chi \neq 1$. Then the natural map from the germ of $W$ in $w$ to the Kuranishi family of $X$ is smooth. In particular, the Kuranishi family of $X$ is the union of $2^n$ irreducible components, $n + 1$ of which have pairwise different dimension. Moreover, the surface $X$ lies on exactly $n + 1$ components of the moduli space, having pairwise different dimensions.

**Proof.** The first statement is a straightforward application of theorem 2.4, in view of the previous lemma. To prove the second, note that the map from $W$ to the moduli space factors through the action of the symmetric group on $n$ letters, $\Sigma_n$. The quotient $W/\Sigma_n$ has exactly $n + 1$ irreducible components of pairwise different dimensions. By the previous result, each of these components dominates a component of the moduli; the $n + 1$ components so obtained must all be distinct, as they have different dimensions. \qed

**Remark 3.10** If $L \in NS$ is sufficiently ample, then the intersection $E \cap B_L \cap B^0$ is nonempty, hence the theorem applies yielding infinitely many surfaces with different invariants.

**Corollary 3.11** Given integers $n \geq 2$ and $m \geq 5$, for infinitely many values of $r$ there exists a smooth, regular surface $X$ in $\mathbb{P}^r$ such that $O_X(1) = mK_X$ and $X$ lies on exactly $n + 1$ irreducible components of the Hilbert scheme.

**Proof.** Let $X$ be a regular surface, with ample $K_X$, lying on exactly $n + 1$ irreducible components of the moduli; let $M$ be the union of the irreducible components of the moduli space of surfaces with ample canonical class which contain $[X]$. Let $r = h^0(X, mK_X) - 1$; infinitely many such $X$’s (with distinct values of $r$) can be constructed by applying theorem 3.9, in view of remark 3.10.

Fix an $m$-canonical embedding of $X$ in $\mathbb{P}^r$. Every small embedded deformation of $X$ in $\mathbb{P}^r$ is again a smooth surface, $m$-canonically embedded as $X$ is regular. Let $H$ be the union of the irreducible components of the Hilbert scheme of $\mathbb{P}^r$ containing $[X]$, and $H^0$ be the open dense subset of $H$ parametrizing smooth, $m$-canonically embedded surfaces.

The natural map $H^0 \to M$ is dominant, and each fibre is irreducible of dimension $(r + 1)^2 - 1$; in fact, the fibre over $[X']$ is the set of bases of $H^0(X', mK_{X'})$ modulo the action of the finite group $Aut(X')$ (and modulo the obvious $\mathbb{C}^*$-action).
In particular there is an induced bijection between irreducible components of $M$ and of $H$, which increases the dimension by $(r + 1)^2 - 1$. \ terminate}

Remark 3.12 The constructions in this section generalize easily to the case where $Y$ is neither regular nor rigid and $G$ is any abelian group. In fact, they also work if the dimension of $Y$ is bigger than 2 (using a suitable, modified form of lemma 3.3).

4 The Hilbert scheme of curves in $\mathbb{P}^r$

In this section we apply the results on the Hilbert scheme of surfaces to the Hilbert scheme of curves. We first introduce some notation. If $Z \subset \mathbb{P}^r$ is a subscheme, we will denote by $\text{Hilb}(Z)$ the union of the irreducible components of the Hilbert scheme of $\mathbb{P}^r$ containing $[Z]$; we let $H(Z)$ be the germ of $\text{Hilb}(Z)$ at $[Z]$. Note that if $F$ is a hypersurface of degree $l$, $\text{Hilb}(F)$ is naturally isomorphic to $\mathbb{P}(H^0(\mathbb{P}^r, \mathcal{O}(l)))$.

Lemma 4.1 Let $X \subset \mathbb{P}^r = \mathbb{P}$ be a smooth surface, $F \subset \mathbb{P}$ be a smooth hypersurface of degree $l$ transversal to $X$, and let $C = X \cap F$. Then there is a natural isomorphism $N_{C/F}|_C \cong N_{X/F}|_C \oplus \mathcal{O}_C(l)$.

Proof. We have a natural diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & N_{C/X} & \rightarrow & N_{C/F} & \rightarrow & N_{X/F}|_C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & N_{F/F}|_C & \rightarrow & N_{C/F}|_C & \rightarrow & 0 \\
\end{array}
\]
However \( N_{F/IP}|_C = N_{C/X} = \mathcal{O}_C(l) \), and it is easy to check that the morphism \( N_{C/X} \to N_{F/IP}|_C \) in the diagram is the identity. This implies that the natural map \( N_{C/F} \to N_{X/IP}|_C \) is also an isomorphism, hence the claimed splitting. \( \Box \)

**Proposition 4.2** Let \( X \subset IP^r = IP \) be a smooth, regular, projectively normal surface. Let \( F \) be a smooth hypersurface of degree \( l \) in \( IP \) cutting \( X \) transversally along a curve \( C \), and let \( U \subset Hilb(X) \times Hilb(F) \) be the open set of pairs \((X',F')\) such that \( X' \) and \( F' \) are smooth and transversal and \( X' \) is projectively normal. If \( l \gg 0 \), then for every \((X',F') \in U\) the map \( H(X') \times H(F') \to H(C') \) induced by intersection is smooth, where \( C' = X' \cap F' \).

**Proof.** The germ of the Hilbert scheme \( H(Z) \) represents the functor of embedded deformations of \( Z \) in \( IP \); when \( Z \) is smooth, this functor has tangent (resp. obstruction) space \( H^0(Z,N_Z/IP) \) (resp. \( H^1(Z,N_Z/IP) \)). Let \((X',F') \in U\), and \( C' = X' \cap F' \). The map \( H(X') \times H(F') \to H(C') \) induces natural maps on tangent and obstruction spaces; to prove the required smoothness it is enough to prove that the induced maps are surjective on tangent spaces and injective on obstruction spaces. Note that \( H^i(C',N_{C'/IP}) = H^i(C',N_{X'/IP}|_{C'}) \oplus H^i(C',N_{C'/IP}|_{C'}) \) by lemma 4.1. Via this isomorphism, the maps we are interested in are induced by the long exact sequences associated to:

\[
0 \to N_{X'/IP} \otimes \mathcal{I}_{C' \subset X'} \to N_{X'/IP} \to N_{X'/IP}|_{C'} \to 0
\]

\[
0 \to N_{F'/IP} \otimes \mathcal{I}_{C' \subset X'} \to N_{F'/IP} \to N_{F'/IP}|_{C'} \to 0.
\]

Therefore it is enough to prove that

\[
H^1(X',N_{X'/IP} \otimes \mathcal{I}_{C' \subset X'}) = 0
\]

and that \( H^0(F',N_{F'/IP}) \to H^0(C, N_{F'/IP}|_{C'}) \) is surjective (remark that \( N_{F'/IP} = \mathcal{O}_{F'}(l) \), hence \( H^1(F', N_{F'/IP}) = 0 \) by Kodaira vanishing).

For the claimed surjectivity, note that there is a commutative diagram

\[
\begin{array}{ccc}
H^0(IP, \mathcal{O}_IP(l)) & \to & H^0(X', \mathcal{O}_{X'}(l)) \\
\downarrow & & \downarrow \\
H^0(F', \mathcal{O}_{F'}(l)) & \to & H^0(C', \mathcal{O}_{C'}(l))
\end{array}
\]
As \( X' \) is projectively normal, the upper horizontal arrow is onto, and as \( X' \) is regular, the right vertical arrow is onto. Hence the lower horizontal arrow is also onto.

To prove the vanishing, as \( \mathcal{I}_{C'} \subset X' = \mathcal{O}_{X'}(-l) \), it is enough to prove that \( H^1(X', N_{X'/\mathbb{P}}(-l)) = 0 \) if \( l \) is sufficiently large. For any given \( X' \), this follows immediately from the definition of ampleness; on the other hand it is easy to prove (by a standard semicontinuity argument) that in fact an \( l_0 \) can be found such that the claimed vanishing holds for all \( l \geq l_0 \) and for all \( X' \in \text{Hilb}(X) \).
\( \square \)

**Proposition 4.3** Let \( X \subset \mathbb{P}^r = \mathbb{P} \) be a smooth surface and let \( F \) be a smooth hypersurface of degree \( l \) meeting \( X \) transversally in a smooth curve \( C \). Let \( U \subset \text{Hilb}(X) \times \text{Hilb}(F) \) be the open set of pairs \( (X', F') \) such that \( X' \cap F' \) is a smooth curve. If \( l \gg 0 \), then each fibre of the map \( U \rightarrow \text{Hilb}(C) \) given by \( (X', F') \mapsto X' \cap F' \) is contained in a fibre of the projection \( U \rightarrow \text{Hilb}(X) \). In other words, each curve contained in the image of \( U \) in \( \text{Hilb}(C) \) lies on exactly one surface in \( \text{Hilb}(X) \).

**Proof.** Let \( X \rightarrow \text{Hilb}(X) \) be the universal family. Inside the product \( X \times X \), consider the diagonal subvariety \( \mathcal{I} \) consisting of the pairs \( (x, x) \). Let \( W \) be the locus of \( \text{Hilb}(X) \times \text{Hilb}(X) \) over which the map \( \mathcal{I} \rightarrow \text{Hilb}(X) \times \text{Hilb}(X) \) has one-dimensional fibres. One may choose a stratification \( \{ W_j \} \) of \( W \) such that each of the restricted families \( \mathcal{I}_j \rightarrow W_j \) is flat. Thus, one has induced maps from \( W_j \) to the Hilbert scheme of one-dimensional subschemes of \( \mathbb{P} \). Since the union of the images of the \( W_j \)’s is contained in a finite number of components of the Hilbert scheme, the degree of the curves contained in the intersection of two distinct surfaces of \( \text{Hilb}(X) \) is bounded by an integer \( l_0 \). Therefore it is enough to choose \( l > l_0 \). \( \square \)

**Theorem 4.4** Let \( n \geq 3 \) be an integer. Then there exist infinitely many integers \( r \), and for each of them a smooth, irreducible curve \( C_r \subset \mathbb{P}^r \) such that \( C_r \) lies exactly on \( n \) components of the Hilbert scheme of \( \mathbb{P}^r \).

**Proof.** By corollary 3.1, for infinitely many values of \( r \) one can construct a regular surface \( X \) of general type, embedded in \( \mathbb{P}^r \) by a complete \( m \)-canonical system, such that \( X \) lies on exactly \( n \) components of the Hilbert scheme of \( \mathbb{P}^r \), having pairwise different dimensions. By [AS], \( X \) is projectively normal.
in $\mathbb{P}^r$ if $m \gg 0$: in fact, by the theorem on page 362 together with the fact that if $K_X$ is ample then $5K_X$ is very ample, it is enough to assume $m \geq 11$. Choose an integer $l$ with $l \gg 0$, such that propositions [4.2] and [4.3] hold for $l$.

Let $F$ be a smooth hypersurface of degree $l$ meeting $X$ transversally. Let $U \subset \text{Hilb}(X) \times \text{Hilb}(F)$ be the locus of pairs $(X', F')$ where both are smooth and meeting transversally, and $X'$ is projectively normal. $U$ is the union of $n$ irreducible components of pairwise different dimensions, each of them being the inverse image of an irreducible component of $\text{Hilb}(X)$.

Let now $C = C_t$ be the intersection of $X$ and $F$, and consider the natural map $U \to \text{Hilb}(C)$ given by $(X', F') \mapsto X' \cap F'$. By Proposition [4.2] this morphism is dominant and smooth on its image $V$. By [4.3] there is an induced morphism $V \to \text{Hilb}(X)$, which is also dominant and smooth on its image. Therefore there is a natural bijection between the irreducible components of $\text{Hilb}(X)$ and those of $\text{Hilb}(C)$. $\square$

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