Nonperturbative Localization\footnote{This research was partially supported by NSF grant DMS-0070755.}

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Abstract

Study of fine spectral properties of quasiperiodic and similar discrete Schrödinger operators involves dealing with problems caused by small denominators, and until recently was only possible using perturbative methods, requiring certain small parameters and complicated KAM-type schemes. We review the recently developed nonperturbative methods for such study which lead to stronger results and are significantly simpler. Numerous applications mainly due to J. Bourgain, M. Goldstein, W. Schlag, and the author are also discussed.

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1. Introduction

Consider an operator acting on on $\ell^2(\mathbb{Z}^d)$ defined by

$$H_\lambda = \Delta + \lambda V,$$

where $\Delta$ is the lattice tight-binding Laplacian

$$\Delta(n, m) = \begin{cases} 1, & \text{dist}(n, m) = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and $V(n, m) = V_n \delta(n, m)$ is a potential given by $V_n = f(T_1^{n_1} \cdots T_d^{n_d} \theta)$, $\theta \in \mathbb{T}^d$, where $T_i \theta = \theta + \omega_i$, and $\omega$ is an incommensurate vector. In certain cases $\Delta$ may also be replaced by a long-range Laplacian. Replacing $T_i$'s with other commuting ergodic transformations would give a general framework of ergodic Schrödinger operators ([21]; see Sec. 8. for an example of this kind) but we will mostly focus on the quasiperiodic (QP) operators that have been intensively studied in Physics and Mathematics literature. For another review of some recent developments in this
area see [7]. The questions of interest are the nature and structure of the spectrum, behaviour of the eigenfunctions, and particularly the quantum dynamics: properties of the time evolution $\Psi_t = e^{itH} \Psi_0$ of an initially localized wave packet $\Psi_0$.

Of particular importance is the phenomenon of Anderson localization (AL) which is usually referred to the property of having pure point spectrum with exponentially decaying eigenfunctions. A somewhat stronger property of dynamical localization (see Sec. 7.) indicates the insulator behavior, while ballistic transport, which for $d = 1$ follows from the absolutely continuous (ac) spectrum, indicates the metallic behavior.

Operators with ergodic potentials always have spectra (and pure point (p.p.) spectra, understood as closures of the set of eigenvalues) constant for a.e. realization of the potential. Moreover, the p.p. spectrum of operators with ergodic potentials never contains isolated eigenvalues, so p.p. spectrum in such models is dense in a certain closed set. An easy example of an operator with dense pure point spectrum is $H_\infty$ which is operator (1.1) with $\lambda^{-1} = 0$, or pure diagonal. It has a complete set of eigenfunctions, characteristic functions of lattice points, with eigenvalues $V_j$. $H_\lambda$ may be viewed as a perturbation of $H_\infty$ for small $\lambda^{-1}$. However, since $V_j$ are dense, small denominators $(V_i - V_j)^{-1}$ make any perturbation theory difficult, e.g. requiring intricate KAM-type schemes.

The probabilistic KAM-type scheme was developed by Fröhlich and Spencer [26] for random potentials ($V_n$ are i.i.d.r.v.’s) in the multi-dimensional case, and is called multi-scale analysis. It was significantly modified, improved, and widely applied in the later years by a number of authors, most notably [24]. An alternative method for random localization was found by Aizenman-Molchanov [2] and later further developed by Aizenman and coauthors. While still requiring certain large parameters this method relies on direct estimates of the Green’s function rather than a step-by-step perturbation scheme.

For QP potentials none of the above methods work, as, among other reasons, they do not allow rank-one perturbations, nor Wegner-type estimates. The situation here is more difficult and the theory is far less developed than for the random case. With a few exceptions the results are confined to the 1D case, and also 1-frequency case ($b = 1$) has been much better developed than that of higher frequencies.

One might expect that $H_\lambda$ with $\lambda$ small can be treated as a perturbation of $H_0 = \Delta$, and therefore have ac spectrum. It is not the case though for random potentials in $d = 1$, where AL holds for all $\lambda$. Same is expected for random potentials in $d = 2$ (but not higher). Moreover, in 1D case there is strong evidence (numerical, analytical, as well as rigorous [8]) that even models with very mild stochasticity in the underlying dynamics have point spectrum for all values of $\lambda$ like in the random case (e.g. $V_n = \lambda f(n^\sigma \alpha + \theta)$ for any $\sigma > 1$). At the same time, for QP potentials one can in many cases show ac spectrum for $\lambda$ small as well as pure point spectrum for $\lambda$ large (see below), and therefore there is a metal insulator transition in the coupling constant. It is an interesting question whether quasiperiodic potentials are the only ones with metal-insulator transition in 1D.
2. Perturbative vs nonperturbative

It is probably fair to say that much of the theory of quasiperiodic operators has been first developed around the almost Mathieu operator, which is

$$ H_{\lambda,\omega,\theta} = \Delta + \lambda f(\theta + n\omega) $$

(2.1)

acting on $\ell^2(\mathbb{Z})$, with $f: \mathbb{T} \to \mathbb{T}; \ f(\theta) = \cos(2\pi\theta)$. The first KAM-type approaches, in both large and small coupling regimes, were developed for this or similar models [23, 5]. The perturbative proofs of complete Anderson localization for $\lambda > \lambda(\omega)$ large are due to Sinai [41] and Fröhlich-Spencer-Wittwer [27], and both applied to cos-type $f$. For $\lambda$ small Chulaevsky-Delyon [19] proved pure ac spectrum using duality and the construction of Sinai [41]. Eliasson (see [25] for a review) developed alternative KAM-type arguments for both large and small $\lambda$ for the case of real-analytic (actually, somewhat more general) class of $f$ in (2.1).

The common feature of the perturbative approaches above is that, besides all of them being rather intricate multi-step procedures, they rely extensively on eigenvalue and eigenfunction parametrization and perturbation arguments.

The common feature of the perturbative results in the quasiperiodic setting is that they provide no explicit estimates on how large (or small) the parameter $\lambda$ should be, and, more importantly, $\lambda$ clearly depends on $\omega$ at least through the constants in the Diophantine characterization of $\omega$.

In contrast, the nonperturbative results allow effective (in many cases even optimal) and, most importantly, independent of $\omega$, estimates on $\lambda$. We will take the latter property (uniform in $\omega$ estimates on $\lambda$) as a definition of a NP result.

Recently developed nonperturbative methods are also quite different from the perturbative ones, in that they do not employ multi-scale schemes: usually only a few (from one to three) sufficiently large scales are involved, do not use the eigenvalue parametrization, and rely instead on direct estimates of the Green’s function. They are also significantly less involved, technically. One may think that in these latter respects they resemble the Aizenman-Molchanov method for random localization. It is, however, a superficial similarity, as, on the technical side, they are still closer to and do borrow certain ideas from [26, 24].

Several results that satisfy our definition of nonperturbative appeared prior to the recent developments, and were all related to the almost Mathieu operator (see [32] for a review). In [33, 34] AL was proved for $\lambda > 15$, and existence of p.p. component for $\lambda > 2$. The latter papers, while introducing some of the ingredients of the recent nonperturbative methods, did not take advantage of the positivity of the Lyapunov exponents which proved very important later.

3. Lyapunov exponents

Here for simplicity we consider the quasiperiodic case, although the definition of the Lyapunov exponents and some of the mentioned facts apply more generally to the 1D ergodic case.
For an energy $E \in \mathbb{R}$ the Lyapunov exponent $\gamma(E)$ is defined as

$$
\gamma(E) = \lim_{n \to \infty} \frac{1}{n} \int_{0}^{1} \ln \| M_k(\theta, E) \| d\theta,
$$

where

$$
M_k(\theta, E) = \prod_{n=k-1}^{0} \begin{pmatrix} E - \lambda f(\omega n + \theta) & -1 \\ 1 & 0 \end{pmatrix}
$$

is the $k$-step transfer-matrix for the eigenvalue equation $H \Psi = E \Psi$.

We will be interested in the regime when Lyapunov exponents are positive for all energies in a certain interval intersecting the spectrum. It is well known and fairly easy to see that if this condition holds for all $E \in \mathbb{R}$, there is no ac component in the spectrum for a.e. $\theta$ (it is actually true for all $\theta$ [39]). Positivity of Lyapunov exponents, however, does not imply exponential decay of eigenfunctions (in particular, not for the Liouville $\omega$ [3] nor for the resonant $\theta \in \mathbb{T}^b$ [36]).

NP methods, at least in their original form, stem to a large extent from estimates involving the Lyapunov exponents and exploiting their positivity.

The general theme of the results on positivity of $\gamma(E)$, as suggested by perturbation arguments, is that the Lyapunov exponents are positive for large $\lambda$. This was first established by Aubry-Andre [1] for the almost Mathieu operator with $\lambda > 2$. Their proof was made rigorous in [3]. Another proof, exploiting the subharmonicity, was given by Herman [31], and applied to trigonometric polynomials $f$. The lower bound in [31] was in terms of the highest coefficient of the trigonometric polynomial and therefore this did not easily extend to the real analytic case. All the subsequent proofs, however, were also based on subharmonicity. Sorets-Spencer [42] proved that for nonconstant real analytic potentials $v$ on $\mathbb{T}$ ($b = 1$) one has $\gamma(E) > \frac{1}{2} \ln \lambda$ for $\lambda > \lambda(v)$ and all irrational $\omega$. Another proof was given in [11], where this was also extended to the multi-frequency case ($b > 1$) with, however, the estimate on $\lambda$ dependent on the Diophantine condition on $\omega$. Finally, Bourgain [10] proved that Lyapunov exponents are continuous in $\omega$ at every incommensurate $\omega$ (for $b > 1$; for $b = 1$ this was previously established in [16]), and that led to the following Theorem which is the strongest result in this general context up to date:

**Theorem 1** [10] Let $f$ be a nonconstant real analytic function on $\mathbb{T}^b$, and $H$ given by (1.1). Then, for $\lambda > \lambda(f)$, we have $\gamma(E) > \frac{1}{2} \ln \lambda$ for all $E$ and all incommensurate vectors $\omega$.

### 3.1. Corollaries of positive Lyapunov exponents

**The almost Mathieu operator** On one hand the almost Mathieu operator, while simple-looking, seems to represent most of the nontrivial properties expected to be encountered in the more general case. On the other hand it has a very special feature: the duality (essentially a Fourier) transform maps $H_\lambda$ to $H_{1/\lambda}$, hence $\lambda = 2$ is the self-dual point. Aubry-Andre [1] conjectured that for this model, for irrational $\omega$ a sharp metal-insulator transition in the coupling constant $\lambda$ occurs at the critical
value of coupling $\lambda = 2$: the spectrum is pure point for $\lambda > 2$ and pure ac for $\lambda < 2$. A second, related, conjecture was that the dual of ac spectrum is pure point and vice versa. Both conjectures were modified based on the results of [3, 37, 36]. The first modified conjecture stated pure point spectrum for Diophantine $\omega$ and a.e. $\theta$ for $\lambda > 2$ and pure ac spectrum for $\lambda < 2$ for all $\omega, \theta$. As for the duality, the question, after some prior developments, was resolved in [28] where it was shown that the dual of point spectrum is ac spectrum (the proof applied in a more general context), and it was used (together with [38, 30]) to prove that the spectrum is purely singular continuous at $\lambda = 2$ for a.e. $\omega, \theta$.

As with the KAM methods, the almost Mathieu operator was the first model where the positivity of Lyapunov exponents was effectively exploited:

**Theorem 2** [32] Suppose $\omega$ is Diophantine and $\gamma(E, \omega) > 0$ for all $E \in [E_1, E_2]$. Then the almost Mathieu operator has Anderson localization in $[E_1, E_2]$ for a.e. $\theta$.

The condition on $\theta$ in [32] was actually explicit (arithmetic) and close to optimal. This, combined with the mentioned results on the Lyapunov exponents for the almost Mathieu operator [31] and duality [28] led to the following corollary:

**Corollary 3** The almost Mathieu operator $H_{\omega, \lambda, \theta}$ has

1° [32] for $\lambda > 2$, Diophantine $\omega \in \mathbb{R}$ and almost every $\theta \in \mathbb{R}$, only pure point spectrum with exponentially decaying eigenfunctions.

2° [28] for $\lambda = 2$, and a.e. $\omega, \theta \in \mathbb{R}$ purely singular-continuous spectrum.

3° [32, 28] for $\lambda < 2$, Diophantine $\omega \in \mathbb{R}$ and a.e. $\theta \in \mathbb{R}$, purely ac spectrum.

Precise arithmetic descriptions of $\omega, \theta$ are available. Thus the Aubry-Andre conjecture is settled at least for almost all $\omega, \theta$. One should mention, however, that while 1° is almost optimal, both 2° and 3° are expected to hold for all $\theta$ and all $\omega \notin \mathbb{Q}$, and such extension remains a challenging problem (see [40]).

The method in [32], while so far the only nonperturbative available allowing precise arithmetic conditions, uses some specific properties of the cosine. It extends to certain other but rather limited situations. A much more robust method was developed by Bourgain-Goldstein [11], which allowed them to extend (a measure-theoretic version of) the above result to the general real analytic as well as the multi-frequency case. Note, that essentially no results were previously available for the multifrequency case, even perturbative.

**Theorem 4** [11] Let $f$ be non-constant real analytic on $\mathbb{T}^b$ and $H$ given by (2.1). Suppose $\gamma(E, \omega) > 0$ for all $E \in [E_1, E_2]$ and a.e. $\omega \in \mathbb{T}^b$. Then for any $\theta$, $H$ has Anderson localization in $[E_1, E_2]$ for a.e. $\omega$.

Combining this with Theorem 1 one obtains [10] that for $\lambda > \lambda(f)$, $H$ as above satisfies Anderson localization for a.e. $\omega$.

One very important ingredient of the method of [11] is the theory of semi-algebraic sets that allows one to obtain polynomial algebraic complexity bounds for certain “exceptional” sets. Combined with measure estimates coming from the large
deviation analysis of \( \frac{1}{n} \ln ||M_n(\theta)|| \) (using subharmonic function theory and involving approximate Lyapunov exponents), this theory provides necessary information on the geometric structure of those exceptional sets. Such algebraic complexity bounds also exist for the almost Mathieu operator [32] and are actually sharp albeit trivial in this case due to the specific nature of the cosine.

4. Without Lyapunov exponents

While having led to significant advances, Lyapunov exponents have obvious limitations, as any method, based on them, is restricted to 1D nearest neighbor Laplacians. It turns out that the above methods can be extended to obtain NP results in certain quasi-1D situations where Lyapunov exponents do not exist.

For the next Theorem let \( H \) be an operator (1.1) defined on \( \ell^2(S) \) where \( S = \mathbb{Z} \times S_0 \), is a strip. \( S_0 \) here is a finite set with a metric, and \( \text{dist}((n, s), (n', s')) = |n - n'| + \text{dist}(s, s') \). Let \( V_{(n, s)} = f_s(\theta + n \omega), \theta \in \mathbb{T} \).

**Theorem 5** [14] Assume \( f_s, s \in S_0 \), are non-constant real analytic functions on \( \mathbb{T} \). Then for any \( \theta \in \mathbb{T} \) and \( \lambda > \lambda(f_s) \), operator \( H \) has AL for a.e. \( \omega \).

The following nonperturbative Theorem deals with the case of small coupling:

**Theorem 6** [15] Let \( H \) be an operator (2.1), where \( f \) is real analytic on \( \mathbb{T} \) and \( \omega \) is Diophantine. Then, for \( \lambda < \lambda(f) \), \( H \) has purely ac spectrum for a.e. \( \theta \).

We note that an analogue of this Theorem does not hold in the multi-frequency case (see next section). Theorem 6 is a result on non-perturbative localization in disguise as it was obtained using duality [28] from a localization Theorem for a dual model which has in general a long-range Laplacian and was in turn obtained by an extension of the method of [32]. A certain measure-theoretic version of it by the method of [11] allowing non-local Laplacians but leading only to continuous spectrum is also available [6]. Theorem 5 was obtained by an extension of the method of [11]. Both Theorems above rely on large deviations for the quantities of the form \( \frac{1}{n} \ln |\det(H - E)|_\Lambda \) and path-determinant expansion for the matrix elements of the resolvent [15]. The methods developed in [15] apply also to certain other situations with long-range Laplacians, for example the kicked rotor model (see Sec. 8.).

5. Multidimensional case: \( d > 1 \)

As mentioned above, there are very few results in the multidimensional lattice case \( (d > 1) \). Essentially, the only result that existed before the new developments was a perturbative Theorem - an extension by Chulaevsky-Dinaburg [20] of Sinai’s [41] method to the case of operator (1.1) on \( \ell^2(\mathbb{Z}^d) \) with \( V_n = \lambda f(n \cdot \omega), \omega \in \mathbb{R}^d \), where \( f \) is a cos-type function on \( \mathbb{T} \). Recently, Bourgain [6] obtained this result for real analytic \( f \) by a nonperturbative method. Note that since \( b = 1 \), this avoids most serious difficulties and is therefore significantly simpler than the general multidimensional case.
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**Theorem 7** [20, 6] For any \( \epsilon > 0 \) there is \( \lambda(f, \epsilon) \), and, for \( \lambda > \lambda(f, \epsilon) \), \( \Omega(\lambda, f) \subset \mathbb{T}^d \) with \( \text{mes}(\Omega) < \epsilon \), so that for \( \omega \notin \Omega \), operator (1.1) with \( V_n \) as above has Anderson localization.

This should be confronted with the following Theorem of Bourgain [8]

**Theorem 8** [8] Let \( d = 2 \) and \( f(\theta) = \cos 2\pi \theta \) in \( H = H_\omega \) defined as above. Then for any \( \lambda \) measure of \( \omega \) s.t. \( H_\omega \) has some continuous spectrum is positive.

Therefore for large \( \lambda \) there will be both \( \omega \) with complete localization as well as those with at least some continuous spectrum. This shows that nonperturbative results do not hold in general in the multi-dimensional case!

A similar (in fact, dual) situation is observed for 1D multi-frequency (\( d = 1; \; b > 1 \)) case at small disorder. One has, by duality:

**Theorem 9** [20, 19] Let \( H \) be given by (2.1) with \( \theta, \omega \in \mathbb{T}^b \) and \( f \) real analytic on \( \mathbb{T}^b \). Then for any \( \epsilon > 0 \) there is \( \lambda(f, \epsilon) \) s.t. for \( \lambda < \lambda(f, \epsilon) \) there is \( \Omega(\lambda, f) \subset \mathbb{T}^b \) with \( \text{mes}(\Omega) < \epsilon \) so that for \( \omega \notin \Omega \), \( H \) has purely ac spectrum.

**Theorem 10** [8] Let \( d = 1, b = 2 \) and \( f \) be a trigonometric polynomial on \( \mathbb{T}^2 \) with a non-degenerate maximum. Then for any \( \lambda \) measure of \( \omega \) s.t. \( H_\omega \) has some point spectrum, dense in a set of positive measure, is positive.

Therefore, unlike the \( b = 1 \) case (see Theorem 6), nonperturbative results do not hold for absolutely continuous spectrum at small disorder.

### 6. Perturbative results by NP methods

While the above demonstrates the limitations of the NP results, the nonperturbative methods have been applied to significantly simplify the proofs and obtain new perturbative results that previously have been completely beyond reach.

We refer the reader for a description of many such applications to [7, 6]. In particular, new results on the construction of QP solutions in Melnikov problems and nonlinear PDE’s, obtained by using certain ideas developed for NP quasi-periodic localization (e.g. the theory of semi-algebraic sets) are presented there.

We will only mention here a theorem by Bourgain-Goldstein-Schlag that is the only one so far treating a “true” \( d > 1 \) situation. Note that here \( d = 2 \), and the reasons why it has not yet been extended to higher dimensions are not just technical, but conceptual (there are certain purely arithmetic difficulties).

**Theorem 11** [12] Let \( d = b = 2 \) and \( f \) be real analytic on \( \mathbb{T}^2 \) such that all functions \( f(\theta_1, \cdot), f(\cdot, \theta_2), (\theta_1, \theta_2) \in \mathbb{T}^2 \) are nonconstant. Then for any \( \epsilon > 0 \) there is \( \lambda(f, \epsilon) \) s.t. for \( \lambda > \lambda(f, \epsilon) \) there is \( \Omega(\lambda, f) \subset \mathbb{T}^d \) with \( \text{mes}(\Omega) < \epsilon \) so that for \( \omega \notin \Omega \), operator (1.1) with \( V_n = \lambda f(n_1 \omega_1, n_2 \omega_2) \) has Anderson localization.
7. Dynamical localization

Anderson localization does not in itself guarantee absence of quantum transport, or nonspread of an initially localized wave packet, as characterised, e.g., by boundedness in time of moments of the position operator [22] ([35] for an example of physical model with coexistence of exponential localization and quantum transport). Considering for simplicity the second moment

\[ \langle x^2 \rangle_T = \frac{1}{T} \int_0^T \sum_n |\Psi_t(n)|^2 n^2 dt, \]

we will say that \( H \) exhibits dynamical localization (DL) if \( \langle x^2 \rangle_T < \text{Const} \). We will say that the family \( \{H_\theta\}_{\theta \in \mathbb{T}} \) exhibits strong DL if \( \int_{\mathbb{T}} d\theta \sup_t \langle x^2 \rangle_t < \text{Const} \). We note that the results mentioned below will hold with more restrictive definitions of DL (involving the higher moments of the position operator) as well. DL implies p.p. spectrum by RAGE theorem (see, e.g. [21]), so it is a strictly stronger notion.

It turns out that nonperturbative methods allow for such dynamical upgrades as well. For the almost Mathieu operator we have

**Theorem 12** [29] For \( \lambda > 2 \) and Diophantine \( \omega \) (as in [32]) strong DL holds.

While proved in [29] with a slightly more restrictive condition on \( \omega \), Theorem 12 holds as stated by a result of [16]. For the results obtained by methods stemming from the approach of [11] one has

**Theorem 13** [14] In Theorems 4,5,7,11 dynamical localization also holds.

This also applies to other results on nonperturbative localization, e.g. [13].

8. Quantum kicked rotor

The quantum kicked rotor was introduced in [18] as a model in quantum chaos. It is given by the time-dependent Schrödinger equation on \( L^2(\mathbb{T}) \)

\[ i \frac{\partial \phi}{\partial t} = a \frac{\partial^2 \phi}{\partial^2 \theta} + ib \frac{\partial \phi}{\partial \theta} + V(t, \theta) \phi \]

where \( V(t, \theta) = \kappa \cos 2\pi \theta \sum_{n \in \mathbb{Z}} \delta(t - n) \). It represents quantization of the Chirikov standard map, and a conjecture (e.g.[4]) was that for a.e. \( a, b \) the solution \( \phi \) is almost-periodic in time, thus demonstrating “quantum suppression of chaos” (as some chaos is expected for the standard map). Such almost periodicity follows from dynamical localization for the Floquet operator \( W : L^2(\mathbb{T}) \to L^2(\mathbb{T}) \) defined by \( W \phi(t, \theta) = \phi(t + 1, \theta) \). W is a unitary operator that in Fourier representation can be written as a product \( U \cdot S \) where \( S(n, m) = S(n - m) \) is a Toeplitz operator with very fast decay of \( S(n) \) and \( D(n, m) = D(n) \delta(n, m) \) is a diagonal operator with \( D(n) = \exp(2\pi i (T^n x)_2) \), with \( T \) being the skew shift of the torus \( T(x_1, x_2) = (x_1 + \omega, x_2 + x_1) \), and \( x_1, x_2, \omega \) determined by \( a, b \). The nonperturbative methods (particularly, the method for skew-shift dynamics localization in [13] and a long-range method [15]) were further developed for this model to obtain
Theorem 14 [9] For any $\epsilon > 0$ and any fixed $b$ there is $\kappa(\epsilon)$, and for $\kappa < \kappa(\epsilon)$, $\Omega(\kappa) \subset \mathbb{T}$ with $\text{mes}(\Omega) < \epsilon$, s.t. for $a \notin \Omega(\kappa)$, operator $W$ has DL.

Exploiting the multiplicative nature of $W$ one also obtains a nonperturbative counterpart:

Theorem 15 [17] There is $\kappa_0 > 0$ such that for any $b$, operator $W$ has dynamical localization for $\kappa < \kappa_0$ and a.e. $a$.

This confirms the “quantum suppression of chaos” conjecture for small $\kappa$.

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