Acoustic Zitterbewgung of the shear stress field in two-dimensional phononic crystals with electroreological material

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Abstract: Theoretically the paper confirmed the existence of the acoustic Zitterbewgung of the shear stress field in two-dimensional phononic crystals, in which epoxy insects arranged in square are embedded in an electroreological material matrix. The shear storage modulus, the loss modulus of electroreological material, and the band gaps of the phononic crystals are changed with electric field strength. Therefore, the oscillatory period and amplitude of the acoustic Zitterbewgung of the shear stress field can be controlled by external voltages. This could be of importance in designing an experiment to investigate the acoustic Zitterbewgung of the shear stress field in two-dimensional phononic crystals with electroreological material matrix.

1. Introduction
Electrorheological (ER) fluids consist of a type of colloidal suspensions constituting dielectric particles dispersed in insulating oil. The prominent features of ER fluids are that it can become into a state of solid at once (1-10 ms) when it is subject to an externally applied electric field with moderate strength (a few kV/mm), and a stiffness of ER fluids changes with the external field strength. This transformation from fluid changing into solid is a reversible process. Instantaneously an external field is removed, the original state of fluid come back again. Such marvelous features enable ER fluids to be ER-based active dampers and clutches [1-4]. The novel feature of the ER has attracted growing researchers’ interest [5-7]. Hong et al. investigated the propagations of low-frequency acoustic waves in a flexible thin ER layer [8]. Yeh and Chen studied the wave propagations of the ER layers arranged in periodic sandwich beam [9]. Yeh also placed the epoxy cylinders arranged in square and triangular lattice in an ER matrix, and controlled the band gaps of this smart system by applied different electric fields [10].

Zitterbewgung (ZB) refers to the rapid trembling motion of a free Dirac electron, originating from the interference between the states with positive and negative energies [11, 12]. The amplitude of ZB is approximately up to the Compton wavelength, \( \frac{h}{mc} \approx 10^{-12} m \) and is extremely short [13], so the ZB oscillations cannot be directly observed for a Dirac electron. Lately, some studies have shown that the ZB behavior do not uniquely exist in Dirac electrons, it may exist in non-relativistic electrons in some solid systems [14-24], e.g., carbon nanotubes [14,15], graphenes [16–19], spintronic systems [20,21], and ultracold atomic systems [22–24].
Xiangdong Zhang et al (2008) demonstrated both experimentally and theoretically an extremal transmission phenomenon in the classical 2D PCs systems [25], regarded as an acoustic analogue to the (ZB) of the relativistic electrons. The ZB extends to a dispersion relation with two neighboring branches touched as a pair of cones (where the cone cluster point is regard as Dirac point), similar to the ZB oscillations in narrow-gap solids [26-29]. Yun Wang et al (2010) reported theoretical results of the longitudinal acoustic ZB in ordinary PCs without involving Dirac Points [30]. In their work, they demonstrated that the acoustic ZB oscillations are originated from the interference of Bloch modes in the neighboring dispersion branches, and not involved with Dirac point. This method is different from the previous studies on classical systems resorting to a framework of quantum mechanics.

In this paper, the acoustic ZB of shear stress field in two-dimensional (2D) phononic crystals with ER materials in which epoxy cylinders arranged in square lattice are embedded, are investigated. The plane-wave-expansion (PWE) [31] method is employed to analyze the shear stress field wave equations in PCs. The effects of various electric fields for the 2D PCs with an ER matrix on ZB of the shear stress field are calculated. The investigation in this paper is to provide the based guidelines to control ZB effect of, is to use the external voltages regulate the maximal amplitude and the oscillatory period and time of ZB.

2. Theory
The 2D PCs with a square array of epoxy insects embedded in ER material are shown in figure 1. The corresponding Brillouin zone is also shown in figure 1. Due to the particularity of ER materials in which the shear stress uniquely exists (the positive strain extremely small), the equation for inhomogeneous solids can be simplified as follows [10].

\[ \frac{\partial^2 p}{\partial t^2} = \rho c_t^2 \nabla \cdot (\nabla p) \]

where \( \rho \) is the mass density, \( c_t \) is transverse speed of acoustic wave, \( \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} \) is the two-dimensional nabla. The shear stress field \( p(\vec{r}, t) \) in Eq. (1) can be written by the Bloch theorem as follows:

\[ p(\vec{r}, t) = e^{-i\omega t} e^{i\vec{k} \cdot \vec{r}} p_{\vec{k}}(\vec{r}) \]

Where \( \vec{k} \) is restricted within the first Brillouin zone(BZ). In view of the periodical systems, using the PWE method, the parameters \( 1/\rho(\vec{r}) \) and \( 1/\rho c_t^2 (\equiv \mu(\vec{r})) \) can be expanded in the two-dimensional Fourier series.

\[ h(\vec{r}) = \sum_{\vec{G}} h_{\vec{G}} e^{i\vec{G} \cdot \vec{r}} \]

where \( h(\vec{r}) \) stands for \( 1/\rho(\vec{r}) \), \( 1/\tau(\vec{r}) \), or \( P_{\vec{k}}(\vec{G}) \), then, substitution Eq. (3) into Eq. (1) yields, We can get the intrinsic equation

\[ \sum_{\vec{G}} \left[ \omega^2 \mu_{\vec{G} + \vec{G} - \vec{k}}^{-1} - \rho c_t^2 - \rho_{\vec{G} - \vec{k}}^{-1} \right] P_{\vec{k}}(\vec{G}) = 0 \]

where \( \omega(\vec{k}) \) are the eigenfrequencies, \( P_{\vec{k}}(\vec{G}) \) are the eigenvectors, and \( \vec{G} = 2\pi a (n_x \hat{i} + n_y \hat{j}) \) are the reciprocal vectors of square lattice. We take the value of integer \( n_x, n_y \) from -10 to 10 (i.e. 441 plane waves), the results have shown a very good convergence. So, we can get the dispersion relations and Bloch eigenmodes by solving Eq. (4). The Fourier coefficients \( h_{\vec{G}} \) in Eq. (3) can be written as:
where \( A, B \) label the scattering object and the matrix ER material, respectively.\( I(\vec{G}) = \frac{2\mathcal{F}(GR)}{GR} \) is the structure factor associated with the scattering shape of cylinder, the filling ratio is \( f \).

In order to simplify the research, we just consider the sonic waves transporting along \( \Gamma X \) direction (corresponding to \( X \) direction in real space) and pay more attention on the shear stress field 

\[
p(\vec{r}, t) = e^{-i\omega t} P_0^{\rho'(\vec{G})} e^{i(k+\vec{G})\cdot\vec{r}} \quad \text{at} \quad y = 0.
\]

In this work, we construct a classical acoustic wave \( F(x, t) \) propagating inside the 2D PCs with ER material. We can decompose it in reciprocal spaces[30],

\[
F(x, t) = \sum_{m} F_m(x, t) = \int \sum_{m} \sum_{m} Q_m(k) P_m(k, x) e^{-i\omega_0(k)t} \, dk
\]

(6)

where the right side of Eq. (3),

\[
\sum_{m} F_m(x, t) = \int \sum_{m} \sum_{m} Q_m(k) P_m(k, x) e^{-i\omega_0(k)t} \, dk
\]

(6)

can be treated as superimposed result of contribution from the first band and the second band with dispersion relation \( \omega_k = \omega_0(k) \), \( m \) represent the first and second band, \( P_m(k, x) \) and \( Q_m(k) \) are the Bloch eigenmodes and weight function for a given wave vector \( \vec{k} \), respectively. The shear stress intensity field changing with time can be shown as

\[
|F(x, t)|^2 = \sum_{m} |F_m(x, t)|^2 + \sum_{m<n} I_{m,n}(x, t)
\]

(7)

where \( I_{m,n}(x, t) = F_m(x, t) F_n^*(x, t) + \text{c.c.} \), “c.c.” represents the complex conjugate. The first part contributes from the first and the second band independently, the second part from the interaction of the first and the second band. Eq. (6) and Eq. (7) imply that a wave packet with finite spatial extension can be constructed by giving a suitable weight function, for example, Gaussian function 

\[
Q(k) = \exp\left[-4\pi (k - k_0)^2 / \Delta k^2 \right], \quad \text{where} \quad k_0 \text{ and } \Delta k \text{ determining the center and the width of wave packet in reciprocal space}.
\]

In order to demonstrate the oscillations more clearly, the centroid of a general wave packet is defined by

\[
\vec{X}_c(t) = C_N^{-1} \int x |F(x, t)|^2 \, dx
\]

(8)
where $C_N = \int |F(x,t)|^2 \, dx$ is a time dependent normalization factor. It includes three components

\begin{align}
\bar{X}_1(t) &= C_N^{-1} \int x |F_1(x,t)|^2 \, dx \\
\bar{X}_2(t) &= C_N^{-1} \int x |F_2(x,t)|^2 \, dx \\
\bar{X}_3(t) &= C_N^{-1} \int xI_{1,2}(x,t) \, dx
\end{align}

(9a, 9b, 9c)

where $\bar{X}_1(t)$ and $\bar{X}_2(t)$ are the spatial space of the wave packet contributed by the first and the second band, respectively, and the last part $\bar{X}_3(t)$ are derived from the interaction of the modes on the both bands. When the wave vector is narrow enough, the interference term can be approximated by

$$\bar{X}_2(t) \approx C_N^{-1} e^{-i(\omega_1(k_x)+\omega_2(k_x))}\int dx\int dk_x dk_y [xG(k_y)G(k_x)P_1(k_y,k_x)P_2^*(k_y,k_x)] + c.c$$

(10)

which indicates an oscillatory time dependence with frequency $\omega_D = \omega_1(k_0) - \omega_2(k_0)$.

3. Numerical results and discussions

A structure of PCs consisting of epoxy circular cylinders embedded in an ER matrix to form the 2D PC, where lattice spacing is $a$, with scaling the spatial extension by $a$, wave vector by $\pi/a$, frequency by $2\pi v_T/a$, and time by $a/c_T$. We use the dimensionless quantities in numerical calculations. In this study, the band structure of the 2D for the shear stress field is calculated. The elastic parameters of the materials are as follows the density $\rho = 31018.1 \times 10^3$ kg/m$^3$, the longitudinal velocities $c_l = 2.54$ km/s and the shear velocities $c_s = 1.16$ km/s respectively for epoxy.

ER fluids are suspension of extremely fine non-conducting particles (up to 50 micrometers diameter) in an electrically insulating fluid. The apparent viscosity of these fluids changes reversibly by an order of up to 100,000 in response to an electric field, and this effect is called the Winslow effect, after its discoverer the American inventor Willis Winslow, who obtained a US patent on the effect in 1947 [32, 33].

The relations of the modulus of ER materials to applied electric field were presented by Yalcintas and Coulter et al [34]. The changeable shear modulus of ER materials is given as follows

$$\mu_v = \mu_1 + \mu_2$$

(11)

where the shear storage modulus $\mu_1 = 50000 \cdot E^2$, the loss modulus $\mu_2 = 2600 \cdot E + 1700$, $v_v = 0.5$, $\rho_v = 1700$ kg/m$^3$, where $E$ is the electric field in kV/mm, $v_v$, $\rho_v$ is the Poisson’s ratio and mass density of ER material, respectively, $C_T = \sqrt{\mu_v/\rho_v}$, where $CT$ are shear sound velocities of ER materials.

The first two bands for the phononic crystals with ER material at various applied electric fields along the $\Gamma X$ direction (with respect to $X$ direction in real space) are shown in figure 2. Electric fields affect the band gaps between the first and the second bands, and control the group velocities of the first and second bands. These group velocities are plotted in figure 2. With the increase of applied electric field strength, the normalized band gaps decrease, and the group velocities also decrease.
It is visualized that the ZB of the shear stress intensity distributions occurs in the first several time units in PCs with an ER matrix under the applied electric field $E = 1.0, 2.0, 3.0, 4.0\ \text{kV/mm}$ in figure 3. It is clearly exhibited that the external voltages enable the time of ZB to last long time. It is helpful to observe the ZB in experiment that the bigger voltages applied and the longer time (the amplitude) of ZB obtained. The bigger voltages applied, the smaller angles between $|P_1(x, t)|^2$ and $|P_2(x, t)|^2$ got. Therefore, the ZB of shear stress field also come from the interference between the Bloch modes in the first and second bands, and the external voltages can control the time of ZB and angles. Because of two opposite group velocities (can be seen form figure 2) of the first and second dispersion branches, the external voltages applied cause time of the wave-packet (located $k_0 = 0.9$) splitting into two sub-wave-packets long as time go on, and induce the angels between $|P_1(x, t)|^2$ and $|P_2(x, t)|^2$ becoming small.

To exhibit the three components of the trembling motion in details for time evolutions of the shear stress intensity $|P(x, t)|^2$ under the applied electric field $E = 1.0\ \text{kV/mm}$, its three components $I_1, I_2(x, t), |P_1(x, t)|^2, |P_2(x, t)|^2$ for a incident wave packet ($k_0 = 0.9, \Delta k = 0.1, f = 0.5$) transporting inside the 2D PCs systems with an ER matrix are exhibited in figure 4. It is obviously shown that the amplitudes of $|P_1(x, t)|^2$ and $|P_2(x, t)|^2$ remain stable throughout but the notable ZB of $I_1, I_2(x, t)$ decreases with time passing away.

In order to find the reason of trembling motion, the trace of the whole wave packet and its three parts are displayed under the different applied electric field $E = 1.0, 2.0, 3.0, 4.0\ \text{kV/mm}$ in figure 5. The component $X_1(t)$ and $X_2(t)$ depict the traces of the two wavelet packets propagating along $+x$ and $-x$ directions, respectively, showing no observable oscillations and the component $X_z(t)$ shows oscillatory behavior. The positive (negative) group velocity of the first (second) band causes $X_1(t)$ ($X_2(t)$) moving up (down). $X_z(t)$ and $X_c(t)$ occur superposing at the beginning of short time, but the time of overlay becomes long with external voltages increasing. $X_x(t)$ and $X_c(t)$ propagate in the same direction. The
Fig. 3 Time evolutions of the shear stress intensity $|P(x, t)|^2$ for a incident wave packet ($k_0 = 0.9, \Delta k = 0.1, f = 0.5)$ transporting inside the 2D PCs systems with an ER matrix under the applied electric field $E=1.0, 2.0, 3.0, 4.0$ kV/mm. The reason for this syntropy is that the value of the first band’s group velocity almost equal to one of the second, which causes time of ZB to last for long time at the beginning and leads to $\hat{\chi}_c(t)$, and $\hat{\chi}_z(t)$ having the same phase.
Fig. 4 The three components of the trembling motion (a) Time evolutions of the shear stress intensity $|P(x, t)|^2$ and its three components (b) $I_1(x, t)$, (c) $|P_1(x, t)|^2$, and (d) $|P_2(x, t)|^2$ for a incident wave packet ($k_0 = 0.9, \Delta k = 0.1, f = 0.5$) transporting inside the 2D PCs systems with an ER matrix under the applied electric field $E = 0.5$kV/mm.

The time dependence of the interaction term $\bar{X}(t)$ under the different applied electric field ($E=1.0, 2.0, 3.0, 4.0$ kV/mm) is presented in figure 6. It is observed that the amplitudes of the oscillatory decay slowly with the increasing of electric field intensity. The periods of the ZB are increasing with the increasing of electric field intensity, which is resulted from the decreasing of gap frequency between the first and second bands caused by the increasing of electric field intensity.
Fig. 5 The trace of the wave packet ($k_0 = 0.9$, $\Delta k = 0.1$, $f=0.5$) $\tilde{X}_c(t)$, accompanied with three components, i.e., $\tilde{X}_1(t)$, $\tilde{X}_2(t)$, and $\tilde{X}_3(t)$ under different applied electric field ($E = 1.0, 2.0, 3.0, 4.0$ kV/mm).

Fig. 6 The time evolution of the interaction term $\tilde{X}_c(t)$ for the incident wave packet ($k_0 = 0.9$, $\Delta k = 0.1$, $f = 0.5$) under the different applied electric field ($E = 1.0, 2.0, 3.0, 4.0$ kV/mm).

4. Conclusions
Based on a classical theoretical description, we display that an acoustic wave packet may experience a ZB oscillation of the shear stress field in 2D PCs with ER materials. The trembling motion in 2D PCs with an ER material matrix still stems from the interference of the Bloch modes in the neighboring dispersion branches, and does’nt need to involve Dirac Points. The modulus of ER materials affected by the applied electric field intensity change the interference of the first and second’s band, so it is easily that the electric field intensity control the time and the maximal amplitudes and the oscillatory periods of the shear stress intensity’s ZB. It is helpful in experiment to obverse the acoustic Zitterbewegung of the shear stress field in the 2D PCs systems with ER materials. With the increasing of electric field...
intensity the amplitudes of the oscillatory decay slowly and time of trembling motion lasts for long, which provides an advantage to be easily observed in experiment for PCs with ER materials.

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