The Proof of a Conjecture Relating Catalan Numbers to an Averaged Mandelbrot-Möbius Iterated Function

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Abstract: In 2021, Mork and Ulness studied the Mandelbrot and Julia sets for a generalization of the well-explored function \( \eta_A(z) = z^2 + \lambda \). Their generalization was based on the composition of \( \eta_A \) with the Möbius transformation \( \mu(z) = \frac{1}{z} \) at each iteration step. Furthermore, they posed a conjecture providing a relation between the coefficients of (each order) iterated series of \( \mu(\eta_A(z)) \) (at \( z = 0 \)) and the Catalan numbers. In this paper, in particular, we prove this conjecture in a more precise (quantitative) formulation.

Keywords: fractal; Mandelbrot set; Julia set; Möbius transformation; iterated function; Catalan numbers

1. Introduction

Let \( \eta : \mathbb{C} \to \mathbb{C} \) be a monic complex polynomial of degree \( d \geq 2 \). We denote by \( \eta^j \) the \( j \)-th iterate of \( \eta \), that is,

\[
\eta^j(z) = \eta(\eta^{j-1}(z)), \quad j = 1, 2, 3, \ldots
\]

The filled-in Julia set of \( \eta \) is defined as

\[
K(\eta) = \{ z \in \mathbb{C} : \eta^j(z) \text{ does not diverge} \}
\]

and the Julia set \( J(\eta) \) of the function \( \eta \) is defined to be the boundary of the set \( K(\eta) \), i.e.,

\[
J(\eta) = \partial K(\eta) \text{ (see, e.g., [1])}.
\]

In this work, we are interested in a modified version of the “classical” filled-in Julia set \( K(\eta) \) and the Julia set \( J(\eta) \) of functions in the quadratic family \( \eta(\lambda) = (z^2 + \lambda)_{\lambda \in \mathbb{C}} \). We observe that the Mandelbrot set \( M(\eta) \) is the fractal defined as

\[
M(\eta) = \{ \lambda \in \mathbb{C} : J(\eta) \text{ is connected} \}.
\]

We point out that there is a more “workable” way of considering the Mandelbrot set (we refer to [2], Theorem 14.14 for a proof of the usually referred fundamental theorem of the Mandelbrot set):

\[
\lambda \in M(\eta) \iff \eta^j_\lambda(0) \text{ does not diverge}.
\]

Some other recent results related to the Mandelbrot set can be found for example in [3–10].

In 2019, Mork et al. [11] constructed filled-in Julia sets for a lacunary function \( \eta_{N,k}(z) = \sum_{n=1}^{N} z^{P_n(k)} \), where \( (P_n(k))_n = \left( \frac{1}{2} (k n^2 - k n - 2) \right)_n \) is the sequence of centered \( k \)-gonal numbers and \( k \) is any positive integer (for more facts and history of lacunary functions see, e.g., [12–14]).

In 2021, Mork et al. [15] followed up on the aforementioned article and considered a generalization of the filled-in Julia sets and their corresponding Mandelbrot sets by...
composing the lacunary function $\eta(z) = \sum_{n=1}^{N} z^{P_k(n)}$ with a fixed Möbius transformation $\mathcal{M}(z) = e^{i \theta} \frac{z-a}{\overline{z-a}}$ (with $(\theta, a) \in \mathbb{R} \times \mathbb{D}$, where $\mathbb{D}$ denotes the the unit disc) at each iteration step. More precisely

$$h_j^i(z; a, k, N, \theta) = \mathcal{M}(\eta_{\lambda A}(\mathcal{M}(\eta_{\lambda}(\cdots \mathcal{M}(\eta_{\lambda}(z) \cdots))))).$$

Very recently, Mork and Ulness [16] continued the previous line of research by dealing with the so-called $j$-averaged Mandelbrot set which is a set generated by iterating a function obtained by composing the function $\eta_{\lambda}$ and the Möbius transformation $\mu_A(z) = \frac{az+b}{cz+d}$, where $A = (a, b, c, d) \in \mathbb{C}^4$. Thus,

$$h_j^i(z; A) = \mu_A(\eta_{\lambda A}(\cdots \mu_A(\eta_{\lambda}(\cdots \mu_A(\eta_{\lambda}(z) \cdots))))).$$

The name “$j$-averaged” is used here since the points of the resulting fractal are colored according to the total number of members of the following sequence of iterations $(\mathcal{H}_n)_{0 \leq n \leq j}$ that escaped from the circle with radius 2 (the concrete algorithm for coloring of points of this fractal you can find in Appendix 1 of [16]), see Figure 1,

$$(\mathcal{H}_n)_{0 \leq n \leq j} = \{h^0(0; A), h^1(0; A), \ldots, h^j(0; A)\}$$

$$= \{0, \mu_A(\eta_{\lambda}(0)), \ldots, (\mathcal{M}(\mu_A(\eta_{\lambda}(\cdots \mu_A(\eta_{\lambda}(0) \cdots)))))\}. $$

![Image of fractal sets](image)

**Figure 1.** The $j$-averaged Mandelbrot sets for $A = (0, 0.5, 1, 0), \lambda = x + iy, \text{ with } x \in [-1.2, 0.8], y \in [-1, 1], j = 1, 2, 3, 4$ (the first row from the left to the right) and for and $j = 5, 7, 10, 100$ (the second row from the left to the right). We used functions in the software Mathematica® (see [17]) that are defined in Appendix 1 of [16].

Mork and Ulness ([16] Theorem 1) proved that the $j$-averaged Mandelbrot set for the Möbius transformation $\mu_A$ with $A = (0, 1, 1, 0)$ has threefold rotational symmetry and dihedral mirror symmetry. Additionally, they raised a conjecture (see [16], Conjecture 2) concerning the coefficients of these iterations. Before stating their conjecture, we introduce some basic notations.

Let $\lambda \in \mathbb{D}$ be a non-zero complex number. Define the function $H(z, \lambda)$ by $H(z, \lambda) := \mu_A(\eta_{\lambda}(z))$, with $A = (0, 1, 1, 0)$. Therefore,

$$H(z, \lambda) = \frac{1}{z^2 + \lambda}.$$
Observe that the \( n \)-th iteration of \( H \) at \( z = 0 \) is a function of \( \lambda \), say \( h_n(\lambda) \), which satisfies the relations:

\[
h_0(\lambda) = 0, \ h_1(\lambda) = H(0, \lambda) = \frac{1}{\lambda} \quad \text{and} \quad h_{n+1}(\lambda) = H(h_n(\lambda), \lambda), \quad \text{for} \ n \geq 1. \tag{1}
\]

The sequence \( (C_n)_{n \geq 0} \) of the Catalan numbers, which is called the sequence A000108 in the OEIS [18], is often defined with the help of the central binomial coefficient \( \binom{2n}{n} \) by

\[
C_n = \frac{1}{n+1} \binom{2n}{n},
\]

thus, its first terms are in Table 1.

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| \( C_n \) | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | 16,796 | 58,786 | 208,012 | 742,900 | 2,674,440 |

which can lead us to the following recurrence relation (it was first discovered by Euler in 1761; for more facts, see [19])

\[
C_n = \frac{4n - 2}{n + 1} C_{n-1}, \quad \text{for} \ n \geq 1,
\]

with the initial condition \( C_0 = 1 \). Sometimes the sequence \( (C_n)_{n \geq 0} \) is defined on the basis of the generating function \((1 - \sqrt{1 - 4x})/(2x)\), as the following holds (for \( |x| < 1/4 \))

\[
\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} x^n = \frac{2}{1 + \sqrt{1 - 4x}}.
\]

The aim of this paper is to obtain a (quantitative) result for the coefficients of the power series of \( h_n(\lambda) \) which implies the Mork–Ulness’ conjecture (qualitative version). More precisely,

**Theorem 1.** For all \( n \geq 1 \), we have

\[
h_n(\lambda) = \frac{1 - (-1)^n}{2\lambda} + (-1)^n \sum_{i=1}^{\lfloor n/2 \rfloor} C_{i-1} \lambda^{3i-1} + O(\lambda^{3\lfloor n/2 \rfloor + 2}), \tag{2}
\]

where \( C_n \) is the \( n \)-th Catalan number.

**Remark 1.** We remark that Mork and Ulness [16] posed a slightly different conjecture. In fact, we can express their question by defining \( h_n^{(1)}(\lambda) \) and \( h_n^{(2)}(\lambda) \) as

\[
h_n^{(1)}(\lambda) := \lim_{n \to \infty} h_{2n+1}(\lambda) = \frac{1}{\lambda} - \sum_{i \geq 0} C_i \lambda^{3i+2}
\]

and

\[
h_n^{(2)}(\lambda) := \lim_{n \to \infty} h_{2n}(\lambda) = \sum_{i \geq 0} C_i \lambda^{3i+2}.
\]

They also asserted that these functions should converge in the whole unit disk (or the punctured one for \( h_n^{(1)}(\lambda) \)). However, this is not true (this is expected because of the exponential nature of Catalan numbers). For example, the simple bound \( \binom{2n}{n} \geq 4^n /(2n + 1) \), which comes from the fact that \( 4^n = (1 + 1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \), implies that \( C_n > 4^n/ (3n^2) \) (some other bounds can be found in ([19] Chapter 2) and [20]) and so if \( |\lambda| > 1/\sqrt{2} \approx 0.793 \) then \( |C_n \lambda^{3n+2}| \geq \)
3^{-1} |\lambda|^2 (2^n/n^2) (\sqrt{2|\lambda|})^{3n} > |\lambda|^2/3 \text{ (for } n \geq 4) \text{ yielding the divergence of } h_\infty^{(2)}(\lambda). \text{ In order to compute the radius of convergence, say } r, \text{ of } h_\infty^{(2)}(\lambda), \text{ one can write this function as } h_\infty^{(2)}(\lambda) = \sum_{n \geq 0} a_n \lambda^n, \text{ where }
\begin{align*}
a_n &= \begin{cases} 
C_{(n-2)/3}, & \text{if } n \equiv 2 \pmod{3}, \\
0, & \text{if } n \not\equiv 2 \pmod{3}.
\end{cases}
\end{align*}

Thus, \(1/r = \limsup_{n \to \infty} \sqrt[n]{a_n}\) and, by using \(C_n \approx 4^n/(n^{3/2}\sqrt{\pi})\) (which comes from the Stirling formula \(n! \approx \sqrt{2\pi n}(n/e)^n\)), we obtain
\[
\frac{1}{r} = \limsup_{n \to \infty} \sqrt[n]{a_n} = \limsup_{n \to \infty} \frac{3n/2\sqrt{\pi}}{4^n} = \frac{3\sqrt{4}}{4}.
\]

Therefore, \(B(0, 1/\sqrt[4]{4})\) is the disk of convergence of \(h_\infty^{(2)}(\lambda)\) (observe that \(r = 1/\sqrt[4]{4} \approx 0.6299\)).

2. Auxiliary Results

Before proceeding further, we shall present some useful tools related to the previous sequences.

Our first ingredient provides a useful form to the Laurent series of \(h_n(\lambda)\).

**Lemma 1.** For any \(n \geq 1\), there exists a power series \(P_n(\lambda)\) such that
\[
h_n(\lambda) = \begin{cases} 
\lambda^2 + \lambda^5 P_n(\lambda^3), & \text{if } n \text{ is even;} \\
\frac{1}{\lambda} + \lambda^2 P_n(\lambda^3), & \text{if } n \text{ is odd.}
\end{cases}
\]

**Proof.** By definition in (1), \(h_n(\lambda)\) satisfies the following recurrence relation
\[
h_{n+1}(\lambda) = \frac{1}{(h_n(\lambda))^2 + \lambda},
\]
with \(h_0(\lambda) = 0\) (since \(h_1(\lambda) = H(0, \lambda) = 1/\lambda\)). Now, by defining \(f_n(\lambda) := \lambda h_n(\lambda)\) and using the previous recurrence, we obtain
\[
f_{n+1}(\lambda)/\lambda = \frac{1}{(f_n(\lambda)/\lambda)^2 + \lambda}
\]
and so
\[
f_{n+1}(\lambda) = \frac{\lambda^3}{(f_n(\lambda))^2 + \lambda^3},
\]
with \(f_0(\lambda) = 0\). We claim that \(f_n(\lambda) = g_n(\lambda^3)\) for some rational function \(g_n(\lambda)\), where \(n\) is any positive integer. Indeed, we can proceed by induction on \(n\). For \(n = 1\), we can take \(g_1(\lambda) = 1\). Suppose (by induction hypothesis) that \(f_n(\lambda) = g_n(\lambda^3)\), for some formal power series \(g_n(\lambda)\), then, by (3), we have
\[
f_{n+1}(\lambda) = \frac{\lambda^3}{(g_n(\lambda^3))^2 + \lambda^3} = g_{n+1}(\lambda^3),
\]
where \( g_{n+1}(\lambda) \) can be chosen by satisfying the recurrence

\[
  g_{n+1}(\lambda) = \frac{\lambda}{(g_n(\lambda))^2 + \lambda},
\]

with \( g_0(\lambda) = 0 \). The inductive process is finished. Observe that, since \( h_n(\lambda) = \lambda^{-1} g_n(\lambda^3) \), then it suffices to prove that

\[
  g_n(\lambda) = \begin{cases} 
    \lambda + O(\lambda^2), & \text{if } n \text{ is even;} \\
    1 + O(\lambda), & \text{if } n \text{ is odd.} 
  \end{cases} \tag{4}
\]

The proof is also by induction on \( n \) (more precisely, a double induction). For the basis cases, we have \( g_1(\lambda) = 1 = 1 + O(\lambda) \) and

\[
  g_2(\lambda) = \frac{\lambda}{1+\lambda} = \lambda(1 - \lambda + \lambda^2 - \cdots) = \lambda + O(\lambda^2),
\]

where we used that for \(|\lambda| < 1\), one has \((1 + \lambda)^{-1} = \sum_{k=0}^{\infty} (-\lambda)^k\) (in general, it holds that \((1 + O(1))^{-1} = 1 + O(1))\). Suppose that (4) is valid for all \( n \in [1, 2k]\). Then,

\[
  g_{2k+1}(\lambda) = \frac{\lambda}{(g_{2k}(\lambda))^2 + \lambda} = \frac{\lambda}{(\lambda + O(\lambda^2))^2 + \lambda} = \frac{\lambda}{\lambda + O(\lambda^2)} = \frac{1}{1 + O(\lambda)} = 1 + O(\lambda),
\]

where we used \( O(\lambda^r) + O(\lambda^s) = O(\lambda^\min\{r,s\}) \), since \(|\lambda| < 1\).

Now, we use the previous fact

\[
  g_{2k+2}(\lambda) = \frac{\lambda}{(g_{2k+1}(\lambda))^2 + \lambda} = \frac{\lambda}{(1 + O(\lambda))^2 + \lambda} = \frac{\lambda}{1 + O(\lambda)} = \lambda(1 + O(\lambda)) = \lambda + O(\lambda^2).
\]

This completes the induction proof of (4).

Therefore, since \(|\lambda| < 1\), we can write

\[
  g_n(\lambda) = \begin{cases} 
    \lambda + \sum_{i \geq 2} c_i \lambda^i, & \text{if } n \text{ is even;} \\
    1 + \sum_{i \geq 1} c_i \lambda^i, & \text{if } n \text{ is odd}
  \end{cases}
\]

and so

\[
  h_n(\lambda) = \frac{1}{\lambda} g_n(\lambda^3) = \begin{cases} 
    \lambda^2 + \sum_{i \geq 2} c_i \lambda^{3i-1}, & \text{if } n \text{ is even;} \\
    \frac{1}{\lambda} + \sum_{i \geq 1} c_i \lambda^{3i-1}, & \text{if } n \text{ is odd.}
  \end{cases}
\]

This completes the proof. \( \square \)
Remark 2. Note that, by using Lemma 1, we can write

\[ h_n(\lambda) = a_{-1,n} \lambda^{-1} + a_{0,n} \lambda^2 + a_{1,n} \lambda^3 + \cdots = \sum_{k=1}^{\infty} a_{k,n} \lambda^{3k+2} \in \mathbb{R}[\lambda], \]  

(5)

where \( a_{-1,n} = (1 - (-1)^n) / 2 \), i.e., \( a_{-1,n} \) is 1 if \( n \) is odd and 0 if \( n \) is even. In particular, \( h_n(\lambda) \) is an analytic function in some neighborhood of \( \lambda = 0 \), when \( n \) is even, and for \( n \) odd, \( h_n(\lambda) \) has a simple pole at \( \text{origin} \) (with residue equal to 1).

Remark 3. Another viewpoint of Lemma 1 (and consequently, of Remark 2) is that the \( k \)-th derivative of \( h_n(\lambda) = 0 \) as \( \lambda \to 0 \), for any \( k \equiv 0 \) or 1 (mod 3). This fact can also be proved by a harder (but maybe theoretically useful) combination of induction, the generalized Chain Rule (Faà di Bruno’s formula) and the fact that all odd order derivatives of \( H(\lambda) := H(z, \lambda) \) vanish (for fixed \( \lambda \)) at \( z = 0 \). This last assertion follows from Cauchy’s integral formula. Indeed, we have

\[ H^{(2n+1)}(0) = \frac{(2n+1)!}{2\pi i} \int_{\Gamma} \frac{H(\omega)}{\omega^{2n+2}} d\omega = \frac{(2n+1)!}{2\pi i} \int_{\gamma_R} \frac{1}{(\omega^2 + \lambda)\omega^{2n+2}} d\omega, \]

where \( \gamma_R \) is the circle \( \gamma(t) := Re^{it}, \) for \( t \in [0, 2\pi] \) and \( 0 < R < |\lambda| \). Now, we can use the partial fraction decomposition to deduce that

\[ \frac{1}{(\omega^2 + \lambda)\omega^{2n+2}} = \frac{A}{\omega + \sqrt{\lambda}} - \frac{A}{\omega - \sqrt{\lambda}} + \frac{B}{\omega^{3n+2}}, \]

for computable constants \( A \) and \( B \). Hence, again by the Cauchy integral formula, we have

\[ \int_{\gamma_R} \frac{1}{(\omega^2 + \lambda)\omega^{2n+2}} d\omega = 2A\pi i f(0) - 2A\pi i f(0) + 2B\pi i f^{(2n+1)}(0), \]

where \( f(z) = 1 \), for all \( z \). Thus, \( H^{(2n+1)}(0) \) is equal to zero as claimed.

Now we show the important connection of the sequence \( (a_{k,n})_{k \geq 0} \) to the Catalan numbers. For the simplicity of notation, we use the following notation in the rest of the text:

\[ a_{k,n} = \begin{cases} d_k, & \text{for odd } n; \\ e_k, & \text{for even } n. \end{cases} \]

Lemma 2. Let \( (C_k)_{k \geq 0} \) be the Catalan sequence. We have

(i) If \( (d_k)_{k \geq 0} \) is defined by the recurrence,

\[ d_{k+1} = -(C_1d_k + \cdots + C_{k+1}d_0) - C_{k+2}, \]

with \( d_0 = -C_0 \), then \( d_k = -C_k \), for all \( k \geq 0 \).

(ii) If \( (e_k)_{k \geq 1} \) is defined by the recurrence,

\[ e_{k+1} = C_0 e_k + \cdots + C_{k-1} e_1 + C_k, \]

with \( e_1 = C_1 \), then \( e_k = C_k \), for all \( k \geq 1 \).

Proof. Let us recall that Catalan numbers satisfy the Segner recurrence relation (see, e.g., [19], p. 117)

\[ C_{i+1} = \sum_{j=0}^{i} C_j C_{i-j}, \]

(6)

with \( C_0 = 1 \).
(i). We shall proceed by induction on \( k \). For \( k = 0 \), one has \( d_0 = -C_0 \) (by definition). Suppose \( d_t = C_t \) for all \( t \in [0, k] \). Then,

\[
\begin{align*}
d_{k+1} &= -(C_1d_k + \cdots + C_{k+1}d_0) - C_{k+2} \\
&= C_1C_k + \cdots + C_{k+1}C_0 - C_{k+2} \\
&= \left( C_{k+2}C_{k+1} \right) - C_{k+2} \\
&= -C_{k+1}
\end{align*}
\]

which completes the proof (where we used (6)).

(ii). Again by induction on \( k \), the basis case \( e_1 = C_1 \) follows by definition. Assume now that \( e_t = C_t \), for all \( t \in [1, k] \). Then, by the recurrence for \( (e_t) \) together with the induction hypothesis, we obtain

\[
\begin{align*}
e_{k+1} &= C_0 e_k + \cdots + C_{k-1}e_1 + C_k \\
&= C_0 C_k + \cdots + C_{k-1}C_1 + C_k \\
&= \left( C_{k+1}C_k \right) + C_k \\
&= C_{k+1}
\end{align*}
\]

which finishes the proof (where we used again (6)). \( \Box \)

The next lemma gives a helpful recurrence for \( C_n \), depending on the parity of \( n \). The proof follows by induction together with (6) (we leave the details to the readers).

**Lemma 3.** Let \( (C_n)_{n \geq 0} \) be the Catalan sequence. Then,

\[
C_{2n} = 2 \sum_{j=1}^{n} C_{j-1}C_{2n-j} \quad \text{and} \quad C_{2n+1} = C_n^2 + 2 \sum_{j=1}^{n} C_{j-1}C_{2n-j+1},
\]

for all \( n \geq 0 \) (with \( C_0 = 1 \)).

Now, we are ready to deal with the proof.

### 3. The Proof of the Theorem 1

First, observe that (2) can be rewritten for any as

\[
h_{2j}(\lambda) = C_0\lambda^2 + \cdots + C_{j-1}\lambda^{3j-1} + O(\lambda^{3j+2}) \tag{7}
\]

and

\[
h_{2j+1}(\lambda) = \frac{1}{\lambda} - (C_0\lambda^2 + \cdots + C_{j-1}\lambda^{3j-1}) + O(\lambda^{3j+2}), \tag{8}
\]

where we adopt the convention that \( C_0\lambda^2 + \cdots + C_{j-1}\lambda^{3j-1} = 0 \) for \( j = 0 \).

Now, we want to prove the following fact:

**Claim.** It holds that

\[
(C_0\lambda^2 + \cdots + C_{n-1}\lambda^{3n-1})^2 = C_1\lambda^4 + \cdots + C_n\lambda^{3n+1} + O(\lambda^{3n+2}) \tag{9}
\]

for a non-negative integer \( n \).

**Proof.** The proof is by induction on \( n \). The identity is true for \( n = 0 \), since \( C_1 = C_0^2 \). Suppose that (9) holds, then one has
\( (C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1} + C_n \lambda^{3n+2})^2 = (C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})^2 + 2(C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})C_n \lambda^{3n+2} + (C_n \lambda^{3n+2})^2. \)

Since we desire to evaluate the identity up to \( O(\lambda^{3n+5}) \), then
\[
2(C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})C_n \lambda^{3n+2} + (C_n \lambda^{3n+2})^2 = 2C_0 C_n \lambda^{3n+4} + O(\lambda^{3n+7}). \quad (10)
\]

On the other hand, in the induction hypothesis
\[
(C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})^2 = C_1 \lambda^4 + \cdots + C_n \lambda^{3n+1} + O(\lambda^{3n+2}),
\]
the terms of order \( \lambda^{3n+4} \) were neglected (since we were interested in \( O(\lambda^{3n+2}) \)). Thus, we can improve the previous identity by considering these terms (note that this procedure does not affect the induction hypothesis). Additionally, since the sum of two numbers, which are congruent to 2 modulo 3, is congruent to 1 modulo 3, there is no term of magnitude \( \lambda^{3n+3} \) in \( (C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})^2 \). Let us also suppose that \( n \) is odd (the even case is carried out along the same lines). We then have
\[
(C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})^2 = C_1 \lambda^4 + \cdots + C_n \lambda^{3n+1}
+ 2(C_1 C_{n-1} + \cdots + C_{(n-1)/2} C_{(n+1)/2}) \lambda^{3n+4} + O(\lambda^{3n+5}). \quad (11)
\]

Now, we combine (10) and (11) together with Lemma 3 to arrive at
\[
(C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1} + C_n \lambda^{3n+2})^2 = (C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})^2
+ 2(C_0 \lambda^2 + \cdots + C_{n-1} \lambda^{3n-1})C_n \lambda^{3n+2} + (C_n \lambda^{3n+2})^2
= C_1 \lambda^4 + \cdots + C_n \lambda^{3n+1}
+ 2(C_1 C_{n-1} + \cdots + C_{(n-1)/2} C_{(n+1)/2}) \lambda^{3n+4} + O(\lambda^{3n+5})
+ 2C_0 C_n \lambda^{3n+4} + O(\lambda^{3n+7})
= C_1 \lambda^4 + \cdots + C_n \lambda^{3n+4} + O(\lambda^{3n+5})
\]
which finishes the proof of the claim.

Now, we return to the proof of (2). Again, the proof is by induction on \( n \). For the basis case, we have
\[
h_1(\lambda) = \frac{1}{\lambda}
\]
and, by Lemma 1,
\[
h_2(\lambda) = \lambda^2 + O(\lambda^5).
\]
Suppose that (2) is true for \( h_n(\lambda) \) with \( n \in [1, 2j] \). Then, by the recurrence relation for \( (h_n(\lambda))_n \) together with the induction hypothesis, we infer that
\[
h_{2j+1}(\lambda) = \frac{1}{(h_2(\lambda))^2 + \lambda} = \frac{1}{(C_0 \lambda^2 + \cdots + C_{2j-1} \lambda^{2j-1} + O(\lambda^{2j+2}))^2 + \lambda}.
\]
However, we can use (9) to write
\[
(C_0 \lambda^2 + \cdots + C_{j-1} \lambda^{3j-1} + O(\lambda^{3j+2}))^2 + \lambda = C_0 \lambda + C_1 \lambda^4 + \cdots + C_{2j} \lambda^{3j+1} + O(\lambda^{3j+2}). \quad (12)
\]
From Lemma 1 and Remark 2, one has
\[
h_{2j+1}(\lambda) = \frac{1}{\lambda} + d_0 \lambda^2 + d_1 \lambda^5 + \cdots + d_{j-1} \lambda^{3j-1} + O(\lambda^{3j+2}).
\]
Thus, the coefficients \(d_0, d_1, \ldots, d_{j-1}\) satisfy the following equality
\[
1 \equiv (C_0\lambda + C_1\lambda^4 + \cdots + C_k\lambda^{3j+1} + O(\lambda^{3j+2})) \left( \frac{1}{\lambda} + d_0\lambda^2 + d_1\lambda^5 + \cdots + d_{j-1}\lambda^{3j-1} + O(\lambda^{3j+2}) \right)
\]
and so
\[
1 \equiv 1 + \lambda \left( \sum_{i=0}^{j-1} d_i\lambda^{3i+2} \right) + C_1\lambda^4 \left( \frac{1}{\lambda} + \sum_{i=0}^{j-2} d_i\lambda^{3i+2} \right) + \cdots + C_k\lambda^{3j+1} \left( \frac{1}{\lambda} \right) + O(\lambda^{3j+1}).
\]
By reordering this sum, we obtain
\[
0 \equiv (d_0 + C_1)\lambda^3 + (d_1 + C_1d_0 + C_2)\lambda^6 + \cdots + (d_{j-1} + C_1d_{j-2} + \cdots + C_j)\lambda^3 + O(\lambda^{3j+1}).
\]
Therefore, \(d_0 = -C_1 = -C_0\) and
\[
d_t = -(C_1d_{t-1} + \cdots + C_td_0) - C_t,
\]
for all \(t \in [1, j-1]\). By Lemma 3 (i), we conclude that \(d_t = -C_t\), for all \(t \in [1, j-1]\) which yields that
\[
h_{2j+1}(\lambda) = \frac{1}{\lambda} - (C_0\lambda^2 + C_1\lambda^5 + \cdots + C_{k-1}\lambda^{3j-1}) + O(\lambda^{3j+2})
\]
as desired.

Thus, we determine that (2) holds for \(h_n(\lambda)\) for all \(n \in [1, 2j+1]\). To finish the proof, we must prove that (2) is also true for \(n = 2j+2\). First, one has that
\[
h_{2j+2}(\lambda) = \frac{1}{(h_{2j+1}(\lambda))^2 + \lambda} = \frac{1}{((1/\lambda) - (C_0\lambda^2 \cdots + C_{j-1}\lambda^{3j-1}) + O(\lambda^{3j+2}))^2 + \lambda}.
\]
However, by (9) and after a straightforward calculation, we arrive at
\[
(h_{2j+1}(\lambda))^2 + \lambda = \frac{1}{\lambda^2} - (C_0\lambda + C_1\lambda^4 + \cdots + C_{j-1}\lambda^{3j-2}) + C_k\lambda^{3j+1} + O(\lambda^{3j+2}). \quad (13)
\]
Now, we use Lemma 1 (and Remark 2) to write
\[
h_{2j+2}(\lambda) = e_0\lambda^2 + e_1\lambda^5 + \cdots + e_j\lambda^{3j+2} + O(\lambda^{3j+5}),
\]
where \(e_0 = 1\). Hence,
\[
1 \equiv \left( \frac{1}{\lambda^2} - (C_0\lambda + \cdots + C_{j-1}\lambda^{3j-2}) + C_k\lambda^{3j+1} + O(\lambda^{3j+2}) \right) (\lambda^2 + e_1\lambda^5 + \cdots + e_j\lambda^{3j+2} + O(\lambda^{3j+5})).
\]
Thus,
\[
1 \equiv 1 + \lambda^{-2} \left( \sum_{i=1}^{j} e_i\lambda^{3i+2} \right) - C_0\lambda \left( \sum_{i=0}^{j} e_i\lambda^{3i+2} \right) + \cdots + C_{j-1}\lambda^{3j-2} \cdot \lambda^2 + O(\lambda^{3j+3})
\]
which can be re-written as
\[
0 \equiv (e_1 - 1)\lambda^3 + (e_2 - C_0e_1 - C_1)\lambda^6 + \cdots + (e_j - e_{j-1} - C_1e_{j-2} - \cdots - C_{j-2}e_{j-1} + C_{j-1})\lambda^3 + O(\lambda^{3j+3}).
\]
We then deduce that \(e_1 = 1 = C_1\) and
\[
e_t = C_0e_{t-1} + C_1e_{t-2} + \cdots + C_{j-2}e_1 - C_{j-1}.
\]
for all $t \in [1, j]$. By Lemma 3 (ii), we have $e_t = C_t$, for all $t \in [1, j]$, yielding that

$$h_{2j+2}(\lambda) = C_0\lambda^2 + C_1\lambda^3 + \cdots + C_{j}\lambda^{3j+2} + O(\lambda^{3j+5}).$$

The proof is then complete. $\square$

4. Conclusions

This paper is devoted to the proof of a conjecture formulated by Mork and Ulness ([16], Conjecture 4.2). Roughly speaking, they computationally observed the relation between the coefficients of $h_n(\lambda)$ (the $n$-th iteration of $1/(z^2 + \lambda)$ at $z = 0$) and the Catalan sequence $(C_k)_k$. Indeed, we prove a quantitative version of their conjecture by showing that the sequence $\left( h_n(\lambda) - \left( \frac{1-(-1)^n}{2\lambda} + (-1)^n \sum_{i=1}^{[n/2]} C_{i-1}\lambda^{3i-1} \right) \right)_n$ tends to zero (with order $|\lambda|^{3[n/2]+2}$) as $n \to \infty$.

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