High-precision computation of two-loop Feynman diagrams with Wilson fermions

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We apply the coordinate-space method by Lüscher and Weisz to the computation of two-loop diagrams in full QCD with Wilson fermions on the lattice. The essential ingredient is the high-precision determination of mixed fermionic-bosonic propagators.

At the Lattice conference of last year we presented [1] an algebraic algorithm that allows to express every one-loop lattice integral with gluon or Wilson-fermion propagator in terms of a small number of basic constants which can be computed with arbitrary high precision [2]. This was a generalization of what we did previously with purely bosonic integrals [3] and it was also an essential step in order to apply the coordinate-space method by Lüscher and Weisz [4] to higher-loop integrals with fermions.

Let us consider, in order to fix the ideas, an example of a two-loop integral at zero external momentum, like

\[ I = \int \frac{d^4 l}{(2\pi)^4} \frac{d^4 r}{(2\pi)^4} \frac{1}{D_F(l)D_F(r)D_F(l+r)} \]  

where

\[ D_F(l) = \sum_{i=1}^{4} \sin^2 l_i + \frac{r_W}{4} (l^2)^2 \]  

defines the usual Wilson propagator with momentum $\vec{l}$ at zero fermion mass with Wilson parameter $r_W$. In all our calculation we have set $r_W = 1$ but the method can be applied for any value of $r_W$. A possible strategy to evaluate such an integral amounts to replace each integration with a discrete sum over $L$ points and afterwards to extrapolate to infinite $L$. A possibility is rewriting (1) as

\[ I = \frac{1}{L^8} \sum_{l,r,l+r \neq 0} \frac{1}{D_F(l)D_F(r)D_F(l+r)} \]  

where each component $l_i$ and $r_i$ runs over the set $2\pi(n+1/2)/L$, $n = 0, \ldots, L-1$. From the sum we exclude the points such that $l + r = 0 \mod 2\pi$ where the third propagator diverges. Another possibility is to use (3) but with $l_i$ and $r_i$ running over the set $2\pi n/L$, $n = 0, \ldots, L-1$. In this case however there are more problems with the zero modes and one should exclude from the sum the terms with $l = r = l + r = 0 \mod 2\pi$. For this reason we have decided to use the first method. For our previous example we get for increasing values of $L$

\begin{align*}
L = 10 & \quad 0.000799652 \\
L = 18 & \quad 0.000848862 \\
L = 20 & \quad 0.000853822 \\
L = 26 & \quad 0.000863064
\end{align*}

Then, using an extrapolation of the form

\[ a_0 + \frac{a_1 \log L + a_2}{L^2} + \frac{a_3 \log L + a_4}{L^4} \]  

and data with $6 \leq L \leq 26$, we obtain the estimate

\[ I \approx 0.000879776 \]
which has to be compared with what we obtained by using the coordinate-space method

\[ I \approx 0.0008797779181(12) \]

Let

\[ G(p, q, \vec{x}) = \int dk \frac{e^{i\vec{k} \cdot \vec{x}}}{D^p_B(k)D^q_B(k)} \quad (5) \]

where \( D_B(k) = \hat{k}^2 \) is the usual bosonic propagator on the lattice. In the coordinate-space approach we are interested in the evaluation of the lattice sums

\[ I(p_1, q_1, \vec{a}, p_2, q_2, \vec{b}, p_3, q_3, \vec{c}) = \sum_{\vec{x}} G(p_1, q_1, \vec{x} + \vec{a})G(p_2, q_2, \vec{x} + \vec{b})G(p_3, q_3, \vec{x} + \vec{c}) \quad (6) \]

In this notation our previous example corresponds to

\[ I = I(1, 0, \vec{0}, 1, 0, \vec{0}, 1, 0, \vec{0}) \quad (7) \]

In the evaluations of these sums we make use of the following advantages:

- only four infinite lattice sums must be computed;
- the \( G(p, q, \vec{x}) \)'s can be determined with the desired precision, for a sufficiently large domain of values of \( \vec{x} \), by using our algebraic algorithm [2];
- the asymptotic expansion for large values of \( |\vec{x}| \) of the \( G(p, q, \vec{x}) \)'s is easily computed. For example

\[ G(1, 0, \vec{x}) = \frac{1}{\pi^2} \left[ \frac{1}{4x_2} - \frac{1}{x_2^2} + \frac{2x_4}{x_2^2} - \frac{10}{x_2^4} + \frac{52}{x_2^5} + \frac{160x_7}{x_2^7} - \frac{192x_9}{x_2^9} + \ldots \right] \quad (8) \]

where \( x_n = \sum_\mu x_\mu^n \).

Let us now consider how to compute sums of the type

\[ \Sigma = \sum_\Lambda f(x) \quad (9) \]

on the lattice \( \Lambda \). Of course we will not be able to sum over all the lattice. If \( |x|_1 = \sum_\mu |x_\mu| \), we will perform a sum over a domain of the type \( D_p = \{ x \in \Lambda : |x|_1 \leq p \} \). The problem is to give an estimate of the error. If \( f(x) \) decreases for large \( |x| \) as \( 1/|x|^{2k} \) we expect the sum restricted to \( D_p \) to behave as

\[ \Sigma(p) = \sum_{D_p} f(x) = \Sigma + A \frac{p}{p^{2k-4}} + \ldots \quad (10) \]

Thus we will estimate

\[ |\Sigma - \Sigma(p)| = \frac{p}{2k-4} |\Sigma(p) - \Sigma(p-1)| \quad (11) \]

Our error formula seems to work correctly. We will use this formula to estimate the error on the integration sums. We can also define an improved estimate for \( \Sigma \) by

\[ \Sigma \approx \Sigma(p) + \frac{p}{2k-4} (\Sigma(p) - \Sigma(p-1)) \quad (12) \]

Notice that now our error estimate is very conservative. The larger is \( k \) the best is the estimate. For this reason, if we know the asymptotic behaviour \( Af(x) \) of the function \( f(x) \) for large \( x \) it is convenient to write

\[ \Sigma \approx \sum_{D_p} [f(x) - Af(x)] + \sum_\Lambda Af(x) \quad (13) \]

because the difference is decreasing faster at infinity and the sum of \( Af(x) \) can be computed directly on the infinite lattice by using harmonic polynomials and \( \zeta \)-functions as explained in [2].

Coming back to our preferred example, by subtracting an increasing number of terms of the asymptotic expansion, we get the estimates

\[
\begin{align*}
I & = 0.0008798104043 \pm 0.0000008730034 \\
& = 0.0008797776858 \pm 0.000000029778 \\
& = 0.0008797779227 \pm 0.000000000410 \\
& = 0.0008797779181 \pm 0.0000000000012
\end{align*}
\]

In our work we have always used the asymptotic expansions to increase the precision of the estimates. In each case we have subtracted the asymptotic behaviour to order \( 1/|x|^{10} \); therefore the function which is summed over a finite lattice decays at least as \( 1/|x|^{12} \).
A number of checks have been performed on the table of numerical integrals that we have collected (at the moment we have a number of entries of order $10^4$). In particular

- In the case in which we restrict ourselves to purely bosonic integrals we compare perfectly with the numbers given in [4].

- Because of translation invariance, for every $\vec{v}$ on the lattice

$$I(p_1, q_1, \vec{a}, p_2, q_2, \vec{b}, p_3, q_3, \vec{c}) = I(p_1, q_1, \vec{a} + \vec{v}, p_2, q_2, \vec{b} + \vec{v}, p_3, q_3, \vec{c} + \vec{v})$$

- From the definition of the bosonic propagator one easily gets

$$\frac{1}{D_B^n(k)} = \sum_{\mu} \left( 2 - e^{ik_\mu} - e^{-ik_\mu} \right) \frac{1}{D_B^{n+1}(k)}$$

which can be used to derive relations among different $I$’s. Similarly, other relations can be obtained from the definition of the fermionic propagator.

- New identities are obtained by integration by parts, that is by using

$$\int dk_\mu \frac{\partial}{\partial k_\mu} F(k_\mu) = 0$$

All the checks we have performed are satisfied with a precision of at least $10^{-10}$

We have prepared a completely automatic procedure which evaluates Feynman diagrams at two loops. It goes through the following steps

1. Each diagram is reduced as a sum of the integrals $I$’s, previously defined in (14). A mass is added at all propagators in order to regularize the infrared divergences.

2. All the possible symmetries (cubic: translations, permutations of the axes, inversions of the axes; permutations of the three propagators) are used to reduce the number of terms.

3. Through subtractions all the terms are written as convergent sums plus product of 1-loop integrals.

4. All 1-loop integrals are expressed as in (15).

5. The convergent sums are replaced by their numerical estimate obtained from a precompiled table of lattice sums in the domain $D_{21}$.

This procedure is now being used to compute the mixing coefficients of the four fermion operators of the lattice weak Hamiltonian with the dimension 5 operators. We have chosen this computation because it starts at the two-loop level and because these coefficients have already been studied by using the momentum-space approach [5].

The computation is highly non trivial. The typical input, for each diagram, contains order $10^4$ terms and produces a final result of order $10^{-4}$. Unfortunately at the moment our results are still preliminary and we are performing all possible checks on our evaluations.

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