Contradictory uncertainty relations

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We show within a very simple framework that different measures of fluctuations lead to uncertainty relations resulting in contradictory conclusions. More specifically we focus on Tsallis and Rényi entropic uncertainty relations and we get that the minimum uncertainty states of some uncertainty relations are the maximum uncertainty states of closely related uncertainty relations, and vice versa.

This family includes the Shannon entropy in the limit $q \to 1$

$$S_{q \to 1}(A) = - \sum_j p_j \ln p_j. \quad (2)$$

This is also closely related to Rényi entropy, that we will express as $\sum_{j} p_j^{q} \,

$$R_q(A) = \left( \sum_j p_j^{q} \right)^{1/(1-q)} \quad , \quad (3)$$

so that for Gaussian-like variables $R_q(A) \propto \Delta A$. This takes its minimum $R_q(A) = 1$ when all the probability is concentrated in a single outcome $p_j = \delta_{j,k}$, while the maximum $R_q(A) = N$ occurs when all the outcomes are equally probable $p_j = 1/N$, where $N$ is the number of outcomes.

The Tsallis entropies also include the variance $(\Delta A)^2$ of two-outcome observables within two-dimensional spaces, with $A$ represented by the Hermitian operator

$$A = |a\rangle\langle a| - |\bar{a}\rangle\langle \bar{a}|, \quad (4)$$

with $\langle a|\bar{a} \rangle = 0$, since for $q = 2$ we have

$$S_2(A) = 2p_a(1-p_a) = \frac{1}{2}(\Delta A)^2, \quad (5)$$

with $p_a = \langle a|\rho|a \rangle$ for any state $\rho$.

These measures may enter in uncertainty relations for two observables $A, B$ via nontrivial lower bounds on different combinations of these entropies. The most frequent combinations considered in the literature $[4-6]$ are of the sum of Tsallis entropies

$$\Sigma_q = S_q(A) + S_q(B), \quad (6)$$

the product of Rényi entropies

$$\Pi_q = R_q(A)R_q(B), \quad (7)$$

and the combination of Tsallis entropies proposed in Ref. $U_q [6]

$$U_q = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B). \quad (8)$$

Introduction.– Uncertainty relations is a rather basic issue in quantum physics. This point has been mostly addressed in terms of the product of variances of the corresponding Hermitian operators representing observables. Nevertheless, there are situations where such formulation is not satisfactory enough and alternative approaches are required. For example: (i) variance is not always a well behaved estimator of fluctuations beyond Gaussian statistics $[1]$, (ii) in finite-dimensional systems there are no nontrivial lower bounds for the product of variances, since the variance of an observable can vanish while the variance of any other is bounded from above $[2]$, (iii) for periodic variables such as angle and phase variance is ambiguous and rather useless by strongly depending on the angle/phase window $[3]$, and (iv) there are observables not easily represented by Hermitian operators $[3]$. This has prompted the introduction of alternative measures of fluctuations and uncertainty relations $[2, 3]$.

The question addressed in this work is that different assessments of fluctuations may lead to uncertainty relations resulting in contradictory conclusions. This holds even within the same family of uncertainty measures, such as Tsallis and Rényi entropies $[3, 4]$. These contradictions are quite relevant given the importance of quantum uncertainty relations, from fundamental issues to metrological applications.

Tsallis and Rényi entropies.– For definiteness we will consider the Tsallis entropies $[2]$ $S_q(A) = \frac{1 - \sum_j p_j^q}{q - 1}$, \quad (1)

where $p_j$ is the probability of the outcome $j$ of the observable $A$, and $q$ is a real parameter. Note that $S_q(A)$ is always nonnegative. This is a suitable measure of uncertainty. Minimum uncertainty $S_q(A) = 0$ holds when all the probability is concentrated in a single outcome $p_j = \delta_{j,k}$ for any $k$, so that $\sum_j p_j^q = 1$. Maximum uncertainty occurs when all the outcomes are equally probable $p_j = 1/N$ where $N$ is the number of outcomes.
For the sake of symmetry we are going to consider the same parameter \( q \) for both \( A \) and \( B \).

In this work we are not interested in the precise lower bounds of \( \Sigma_q \), \( \Pi_q \), or \( U_q \). Instead we are worried by contradictions between the conclusions derived from different choices of \( q \) for the same family of entropy combinations.

Two-dimensional observables. – To reveal contradictions as simply as possible we consider a two-dimensional system and two observables \( A, B \) with outcomes \( A = (a, \neg a), B = (b, \neg b) \), and probabilities \( p_k, k = a, \neg a, b, \neg b \), given by projection of the system state \( |\psi\rangle \) (assumed pure for simplicity) on the corresponding vectors \( |k\rangle \)

\[
p_k = |\langle k | \psi \rangle|^2, \tag{9}
\]

with \( p_{\neg k} = 1 - p_k \) and \( \langle -k | k \rangle = 0 \).

In the general case the states \( |a\rangle \) and \( |b\rangle \) will not be orthogonal so that

\[
|b\rangle = \cos \delta |a\rangle + \sin \delta |\neg a\rangle \tag{10}
\]

For definiteness we will consider \( \pi/4 \geq \delta \geq 0 \) since otherwise we may exchange \( a \leftrightarrow \neg a \) for example.

The case \( \delta = \pi/4 \) corresponds to typical complementary observables, so that for \( |\psi\rangle = |b\rangle \) there is \( p_{\neg a} = p_a = 1/2 \) and vice versa. For example this is the case of two orthogonal components of an 1/2 spin, say \( A = \sigma_z \) and \( B = \sigma_x \), where \( \sigma_x, \sigma_z \) are the corresponding Pauli matrices.

Comparison of uncertainty relations for complementary observables. – For definiteness let us consider system states in the form

\[
|\psi\rangle = \cos \theta |a\rangle + \sin \theta |\neg a\rangle, \tag{11}
\]

where \( \theta \) is a variable, so that

\[
p_a = \cos^2 \theta, \quad p_b = \cos^2(\theta - \delta). \tag{12}
\]

In Figs. 1, 2, and 3 we plot \( \Pi_q, U_q, \) and \( \Sigma_q \) as functions of \( \theta \) for \( \delta = \pi/4 \) and several values of \( q \). It can be appreciated that in all the cases there is a maximum/minimum exchange depending on the value of \( q \). Moreover, in Fig. 4 we plot the second derivative of \( \Pi_q, U_q, \) and \( \Sigma_q \) at \( \theta = \delta/2 = \pi/8 \)

\[
F'' = \frac{d^2 F}{d \theta^2} \bigg|_{\theta = \delta/2}, \quad F = \Pi_q, \Sigma_q, U_q, \tag{13}
\]

as functions of \( q \) showing the change from maximum (negative \( F'' \)) to minimum (positive \( F'' \)). For example for \( \Sigma_q \) the exchange holds for \( q \) between \( q = 2 \) and \( q = 3 \).

The states disputing the maxima and minima are

\[
|\psi_{\theta = \delta/2}\rangle \propto |a\rangle + |b\rangle, \tag{14}
\]

that maximizes the product of probabilities \( p_a p_b \) with \( p_a = p_b \) and \( S_A(A) = S_B(B) \) \[10\],

\[
|\psi_{\theta = \delta/2 + \pi/4}\rangle \propto |\neg a\rangle + |b\rangle, \tag{15}
\]

that maximizes the product of probabilities \( p_{\neg a} p_{\neg b} \) with \( p_{\neg a} = p_{\neg b} \) and \( S_A(\neg a) = S_B(\neg b) \).

![FIG. 1: Plot of \( \Pi_q = R_q(A)R_q(B) \) as a function of \( \theta \) for \( \delta = \pi/4 \) and \( q = 0.5 \) (dashed line), \( q = 1 \) (solid line), \( q = 2 \) (dotted line) and \( q = 3 \) (dash-dotted line).](image1.png)

![FIG. 2: Plot of \( U_q \) as a function of \( \theta \) for \( \delta = \pi/4 \) and \( q = 1, 1.5, 2 \).](image2.png)

![FIG. 3: Plots of \( \Sigma_q = S_A(A) + S_B(B) \) as a function of \( \theta \) for \( \delta = \pi/4 \) and \( q = 1.8, 2, 2.5 \).](image3.png)
new features. In Figs. 5, 6, and 7 we plot $\Pi_q$, $U_q$, and $\Sigma_q$ as functions of $\theta$ for $\delta = 0.7$ and several values of $q$. It can be appreciated that a local maximum at $\theta = \delta/2$ for lower $q$ becomes absolute minimum for larger $q$. Accordingly, the absolute minima at $\theta = 0, \delta$ for low $q$ are no longer minima for larger $q$. In Fig. 8 we plot the second derivative $\Pi_q$, $U_q$, and $\Sigma_q$ at $\theta = \delta/2$ as functions of $q$, showing the change from maximum (negative $F''$) to minimum (positive $F''$).

Moreover, in Figs. 9 and 10 it can be appreciated that for $\Pi_q$ and $U_q$ the absolute maximum for low $q$ at $\theta = \delta/2 + \pi/4$ becomes a local minimum for larger $q$.

Discussion.– The above plots show that maximum uncertainty states can become minimum uncertainty states.
and vice versa, depending on the measure of uncertainty employed, even with choices within the same family of measures. To some extent is natural that different measures lead to different extremes. However, it seems paradoxical and counterintuitive that the conclusions may be contradictory to the extent of exchanging maxima and minima.

Despite its long history, uncertainty relations may still provide surprising results worth investigating. For example, recently it has been put forward that there are fluctuations measures that seemingly lead to no uncertainty relation for complementary observables [8, 9].

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