Coherent States for Quantum Compact Groups

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Dedicated to Professor L.D. Faddeev on his 60th birthday

Abstract

Coherent states are introduced and their properties are discussed for all simple quantum compact groups. The multiplicative form of the canonical element for the quantum double is used to introduce the holomorphic coordinates on a general quantum dressing orbit and interpret the coherent state as a holomorphic function on this orbit with values in the carrier Hilbert space of an irreducible representation of the corresponding quantized enveloping algebra. Using Gauss decomposition, the commutation relations for the holomorphic coordinates on the dressing orbit are derived explicitly and given in a compact $R$–matrix formulation (generalizing this way the $q$–deformed Grassmann and flag manifolds). The antiholomorphic realization of the irreducible representations of a compact quantum group (the analogue of the Borel–Weil construction) are described using the concept of coherent state. The relation between representation theory and non–commutative differential geometry is suggested.

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1 Introduction

It is difficult to overestimate the importance of the concept of coherent states in theoretical and mathematical physics. They found many various applications in quantum optics, quantum field theory, quantum statistical mechanics and another branches of physics as well as in some purely mathematical problems [19, 29]. The last–named include the Lie group representations, special functions, automorphic functions, reproducing kernels etc. In the Lie group representation theory there is a remarkable relation between the geometry on the coadjoint orbits and the irreducible representations, which is reflected by the method of orbits (geometric quantization) due to Kirillov, Kostant and Souriau [43]. On the other hand the concept of coherent states leads naturally to the Berezin’s quantization scheme [5]. The important sources of both methods are induced representations and the Borel–Weil theory. The intrinsic relationship between the geometric and Berezin quantization has been established. There are many papers devoted to this subject (e.g. [27, 32] and many others). Recently the coherent states were used to construct examples of non–commutative manifolds [12].

The first papers [1, 38], which can be viewed as those generalizing the coherent states to the quantum groups appeared even before the formal birth of quantum groups [9]. Number of papers followed subsequently ([13] and many others). Nevertheless no definition seems to be completely satisfactory. The coherent states are introduced mainly for the simplest quantum groups (q–deformations of the Heisenberg–Weyl, $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1)$ algebras) in a rather straightforward way which does not suggest a proper generalization to the more general case. Moreover, these states are assumed to be elements of the representation space for an irreducible representation of the corresponding quantized enveloping algebra and they do not reflect the whole underlying Hopf algebra structure.

According to the general philosophy of non–commutative geometry it would be more natural assume the coherent state as a function on an appropriate $q$–homogeneous space of the corresponding quantum group (dual to the quantized enveloping algebra) with values in the representation space. We hope that such more sophisticated generalization of the coherent states method to the case of quantum groups could be of interest not only for the representation theory but also for potential applications of quantum groups in physics. Many important ingredients needed for this generalization are already prepared. First of all the representation theory of quantum groups [13, 26, 34] and the method of induced representations are well developed [28]. The deformations of manifolds playing an important role in the Lie group representation theory (such as Schubert cells, flag and Grassmanan manifolds) are also known [21, 22, 42, 2, 10] through different approaches. Further there is the notion of the quantum dressing transformation [31] the substituent for the coadjoint action from the classical case, which is important already for classical groups [35], has interesting applications in physics [4] and is closely related to the notions of generalized Pontryagin dual and the Iwasawa decomposition [25]. Finally there is also proper definition of the quantum momentum map [24]. One of the
expected results of the coherent state approach for quantum groups would be to put all this ingredients together in a natural way. The second expected result would be a variant of the $q$–generalization of the Borel–Weil theory which follows more closely to the classical Borel–Weil construction as this one described in [13] for the case of $U_q(n)$. Finally as in the classical case it is natural to achieve a link between the representation theory and the non-commutative differential geometry on quantum groups [15]. We hope to meet this goals in the present paper.

The present paper prolong some ideas from the papers [17, 40], but now the leading idea is a proper definition of the coherent state for quantum groups, using all rich structure contained in the quantum double [9, 30].

The paper is organized as follows. Section 2 contains a very brief account of the classical theory. Section 3 has a preliminary character: some basic notions of the quantum group theory are recalled. Section 4 adapts to our purposes some well known results from the representation theory of quantum groups. Section 5 which contains the definition of coherent state for the compact quantum group and discusses its basic properties is one of the important parts of the paper. Here we would like to mention that likewise in the classical case we can start in the definition (5.1) of the coherent state $\Gamma$ from any Hopf algebra and any state in the carrier Hilbert space $\mathcal{H}$ of some its irreducible representation $\tau$ whenever (5.1) does make sense. Nevertheless the restriction to the quantized universal enveloping $U_h(\mathfrak{k})$ for $\mathfrak{k}$ compact and the choice of the lowest (or equivalently the highest) weight state $e_\lambda$ are the most relevant for the rest of the paper. This section also contains a definition of the (quantum) isotropy subgroup $K_0 \subset K$ ($K$ is the spectrum of the Hopf algebra $\mathcal{A}_q(K)$ dual to the $U_h(\mathfrak{k})$) of $e_\lambda$. Our coherent state can be then naturally viewed as a function on $q$–homogeneous space $K_0\backslash K$ with values in the representation space $\mathcal{H}_\lambda$ for the lowest weight representation $\tau_\lambda$ of $U_h(\mathfrak{k})$ corresponding to the lowest weight $\lambda$. Section 6 contains a detailed description of the canonical element $\rho$ (universal $R$–matrix) of the quantum double (particularly inspired by [11]) which makes possible a more explicit expression for the coherent state $\Gamma$ and a definition of holomorphic coordinates on a general quantum dressing orbit. Explicit commutation relations for the holomorphic coordinates in the $R$–matrix formulation are derived in Section 7. They present a compact generalization of the definition relations for the quantized flag manifold. Section 8 describes the antiholomorphic realization of the irreducible representation $\tau_\lambda$ which is most close to the classical Borel–Weil theory. The presentation of Section 8 can also be, if wished, reinterpreted as a non–commutative version of the Berezin quantization. Finally in Section 9 we make an attempt to relate the representation theory to the non–commutative differential geometry, which as we hope could be helpful for understanding the non–commutative version of the method of orbits.

Let us make on this place few comments on some points not included in the paper.

The discussion of Sections 8 and 9 is done using the local coordinates on an appropriate cell of the dressing orbit. There is no doubt that a globalization using the quantum Weyl elements is possible. As in the classical case it has to lead to
a "quantization condition" for the quantum dressing orbit and to an interpretation
of the elements of \( \mathcal{H}_\lambda \) as antiholomorphic sections of an appropriate quantum line
bundle [7, 41].

There is also no doubt that Section 9 could be formulated purely in terms of
the holomorphic coordinates \( z \) and their conjugates \( z^* \). However, this requires an
explicit description of the restriction of the bicovariant differential calculus on the
quantum group \( K \) to the quantum homogeneous space \( K_0 \backslash K \). An introduction of
the partial derivatives \( \partial_{z^*} \) with respect to the antiholomorphic coordinates would it
make possible to interpret the formula (9.13) expressed only through coordinates \( z^* \)
and partial derivatives \( \partial_{z^*} \) as a natural Fock space representation of \( \mathcal{U}_h(\mathfrak{g}) \).

It is also natural to think about limiting cases of our construction. The limit
\( q \to 1 \) gives of course the classical scheme recalled in Section 2. Nevertheless as
in the classical case [37, 5] there is a second type of limit leading to the classical
dressing orbits with their natural Poisson structure. This kind of limit is achieved
by using the sequence of irreducible representations corresponding to the sequence
of lowest weights \( n\lambda \). A rescaling of \( q \to q^{1/n} \) and a subsequent limit \( n \to \infty \) gives
the desired result.

## 2 The classical scheme

Let us start from recalling the classical situation [29]. Denote by \( G \) a simple and
simply connected complex Lie group and by \( K \subset G \) its compact form. Let \( \mathcal{T}^\lambda \)
be an irreducible unitary representation of \( K \) in \( \mathcal{H}_\lambda \) corresponding to a minimal weight
\( \lambda \). \( \mathcal{T}^\lambda \) extends unambiguously as a holomorphic representation of \( G \) in \( \mathcal{H}_\lambda \) (Weyl
unitary trick). Let \( e_\lambda \in \mathcal{H}_\lambda \) be a normalized weight vector and set
\[
\Gamma : G \to \mathcal{H}_\lambda : g \mapsto \mathcal{T}^\lambda(g^{-1}) e_\lambda .
\]
The vector–valued function \( \Gamma \) is a coherent state in the sense of Perelomov. Denote
further by \( K_0 \subset K \) respectively \( P_0 \subset G \) the isotropy subgroups of the point \( \mathbb{C} e_\lambda \in \mathbb{P}(\mathcal{H}_\lambda) \). This means that there exists a character \( \chi \) of \( P_0 \), unitary on \( K_0 \subset P_0 \), such that
\[
\mathcal{T}^\lambda(k) e_\lambda = \chi(k) e_\lambda , \text{ for } k \in P_0 .
\]
The mapping
\[
\mathcal{H}_\lambda \ni u \mapsto \langle \Gamma(\cdot), u \rangle \in C^\infty(K)
\]
is injective and so one embeds this way \( \mathcal{H}_\lambda \) into the vector space of \( \chi \)-equivariant
functions on \( K \). Sending \((g, k) \in K \times K_0 \) to \( k^{-1}g \in K \) we get a principal bundle
\( K \to K_0 \backslash K \) and using the 1–dimensional representation \( \chi \) one associates to it a line
bundle over the base space \( K_0 \backslash K = P_0 \backslash G \). Hence \( \chi \)-equivariant functions on \( K \)
are identified with sections in this line bundle. Set
\[
w_\lambda := \langle e_\lambda , \mathcal{T}^\lambda e_\lambda \rangle \in C^{\text{hol}}(G) .
\]
The function \( w_\lambda \) is \( \chi \)-equivariant on \( K \) and thus determines a trivialization of the
line bundle over the cell given by \( w_\lambda(g) \neq 0 \). The Gauss decomposition provides a
standard way to choose holomorphic coordinates \( \{z_j\} \) on this cell. Vectors \( u \) from \( \mathcal{H}_\lambda \) are then represented by polynomials \( \psi_u := w^{-1}_\lambda \langle \Gamma, u \rangle \) in the variables \( \{z_j^*\} \) and so the representation \( T^\lambda \) acts in the space of antiholomorphic functions living on the cell. Finally we also recall that every operator \( B \in \text{Lin}(\mathcal{H}_\lambda) \) is represented by its symbol \( \sigma(B) \in C^0(K_0 \backslash K) \) or, this is same, by a real analytic \( K_0 \)-invariant function on \( K \),

\[
\sigma(B) := \{ g \mapsto \langle \Gamma(g), B \Gamma(g) \rangle \}.
\]

The mapping \( B \mapsto \sigma(B) \) is injective \cite{19, 37}.

The aim of the present paper is to demonstrate that this scheme applies also for quantum groups.

## 3 Preliminaries, notation

Let us recall some basic notions related to the duality and the dressing transformation for quantum groups \cite{17}. Concerning the deformation parameter we set \( q = e^{-h}, \ h > 0 \). An important role plays the duality between the quantum groups \( K_q \) and \( A_Nq \) following from the Iwasawa decomposition \( G = K \cdot A_N \). The deformed enveloping algebra \( \mathcal{U}_h(k) \) is the \( \ast \)-Hopf algebra dual to \( \mathcal{A}_q(K) \). \( \mathcal{A}_q(AN) \) is identical to \( \mathcal{U}_h(k) \) as an algebra and opposite as a coalgebra. We note also that \( \mathcal{A}_q(G) \) is the same Hopf algebra as \( \mathcal{A}_q(K) \) but the compact form is equipped in addition with the \( \ast \)-involution. We shall also denote by \( \mathcal{U}_h(g) \) the Hopf algebra \( \mathcal{U}_h(k) \) when having forgotten about the \( \ast \)-operation. We denote by \( T, U \) and \( \Lambda \) the vector corepresentations for \( \mathcal{A}_q(G) \), \( \mathcal{A}_q(K) \) and \( \mathcal{A}_q(AN) \), respectively. The \( \ast \)-algebras \( \mathcal{A}_q(K) \) and \( \mathcal{A}_q(AN) \) are defined by the well known relations \cite{33}

\[
RU_1U_2 = U_2U_1R, \ U^\ast = U^{-1},
\]

\[
R\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1R, \ \Lambda_1^\ast R^{-1}\Lambda_2 = \Lambda_2 R^{-1}\Lambda_1^\ast,
\]

(3.1)

and for the \( B_l \), \( C_l \) and \( D_l \) series also by

\[
CU^lC^{-1} = U^{-1}, \  C\Lambda^lC^{-1} = \Lambda^{-1}.
\]

Here \( R \) is the standard R-matrix \cite{14}, and \( C \) is given in \cite{33}. The pairing between \( \mathcal{A}_q(AN) \) and \( \mathcal{A}_q(K) \) is given by \cite{17}

\[
\langle \Lambda_1; U_2 \rangle = R_{21}^{-1}, \ \langle \Lambda_1^\ast; U_2 \rangle = R_{12}^{-1}.
\]

Let us introduce the canonical element

\[
\rho = \sum x_s \otimes a_s \in \mathcal{A}_q(AN) \otimes \mathcal{A}_q(K),
\]

with \( \{x_s\} \) and \( \{a_s\} \) being mutually dual bases. Its basic properties are (\( S \) is the antipode, \( \Delta \) is the comultiplication)

\[
\rho^\ast = \rho^{-1} = (\text{id} \otimes S)\rho,
\]
\[(\Delta \otimes \text{id})\rho = \rho_{23}\rho_{13}, \quad (\text{id} \otimes \Delta)\rho = \rho_{12}\rho_{13}. \tag{3.2}\]

Using \(\rho\) one defines the dressing transformation as a coaction
\[
\mathcal{R} : \mathcal{A}_q(AN) \to \mathcal{A}_q(AN) \otimes \mathcal{A}_q(K) : u \mapsto \rho(u \otimes 1)\rho^{-1}. \tag{3.3}\]

The identification of the algebras \(\mathcal{U}_h(\mathfrak{t})\) and \(\mathcal{A}_q(AN)\) plays in this situation the role of the classical momentum mapping. It is explicitly given by \(\Lambda = S(L^+)\) and \(\Lambda^* = L^-\), where \(L^\pm\) are the matrices of \([33]\). We note that in the literature one often identifies the dressing transformation with the quantum adjoint action,
\[
\text{Ad}_x u = \sum x^{(1)} S x^{(2)}, \quad \text{with} \quad \Delta x = \sum x^{(1)} \otimes x^{(2)}. \tag{3.4}\]

However the both notions are closely related since
\[
(id \otimes \langle x, \cdot \rangle) \mathcal{R} u = \text{Ad}_x u, \tag{3.5}\]
where \(x \in \mathcal{U}_h(\mathfrak{t})\) and \(u \in \mathcal{A}_q(AN) \equiv \mathcal{U}_h(\mathfrak{t})\). The dressing transformation can be calculated explicitly on the elements of the matrix \(\Lambda^* \Lambda\),
\[
\mathcal{R}(\Lambda^* \Lambda) = U^* \Lambda^* \Lambda U, \tag{3.6}\]
provided on the RHS one identifies \(\mathcal{A}_q(AN)\) with \(\mathcal{A}_q(AN) \otimes 1\) and similarly for \(\mathcal{A}_q(K)\).

4 \hspace{1em} The ”vacuum” functional

According to the results of Rosso and Lusztig \([34, 28]\), to every lowest weight \(\lambda\) from the weight lattice there is related a unique irreducible \(*\)-representation \(\tau_\lambda\) of \(\mathcal{U}_h(\mathfrak{t})\) acting in \(\mathcal{H}_\lambda\), \(\dim \mathcal{H}_\lambda < \infty\), and correspondingly a unitary corepresentation of \(\mathcal{A}_q(K)\), \(T^\lambda = (\tau_\lambda \otimes \text{id})\rho \in \text{Lin}(\mathcal{H}_\lambda) \otimes \mathcal{A}_q(K)\). In what follows, \(e_\lambda\) stands again for a normalized weight vector.

Let us define the ”vacuum” functional \(\langle \cdot \rangle\) on \(\mathcal{U}_h(\mathfrak{t})\),
\[
\langle x \rangle := \langle e_\lambda, \tau_\lambda(x) e_\lambda \rangle. \tag{4.1}\]

**Proposition 4.1.** It holds
\[
\langle x \rangle = \langle x, w_\lambda \rangle, \quad \text{where} \quad w_\lambda := \langle e_\lambda, T^\lambda e_\lambda \rangle \in \mathcal{A}_q(K). \tag{4.2}\]

This means that \(\langle \cdot \rangle\) if viewed as an element from \(\mathcal{A}_q(K)\), the dual space to \(\mathcal{U}_h(\mathfrak{t})\), is equal to \(w_\lambda\).

**Proof.** One can verify (4.2) easily using the identity
\[(\text{id} \otimes \langle x, \cdot \rangle)\rho = x, \quad x \in \mathcal{U}_h(\mathfrak{t}) \equiv \mathcal{A}_q(AN).\]

Let us note at this place that, likewise in the classical case,
Proposition 4.2. It holds
\begin{equation}
w_{\lambda_1+\lambda_2} = w_{\lambda_1}w_{\lambda_2} = w_{\lambda_2}w_{\lambda_1},
\end{equation}
and so it is enough to determine \( w_{\lambda} \) only for the fundamental weights \( \lambda = \omega_j \). Furthermore (\( \varepsilon \) is the counit),
\begin{equation}
Sw_{\lambda} = w_{\lambda}^*, \quad \varepsilon(w_{\lambda}) = 1.
\end{equation}

Proof. To see (4.3) it suffices to observe that \( \mathcal{H}_{\lambda_1+\lambda_2} \) can be identified with the cyclic submodule \( \mathcal{M} \) in \( \mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2} \) corresponding to the cyclic vector \( e_{\lambda_1} \otimes e_{\lambda_2} \) with respect to the representation \( (\tau_{\lambda_1} \otimes \tau_{\lambda_2}) \circ \Delta \). Since
\begin{equation}
\mathcal{T}^{\lambda_1+\lambda_2} = \mathcal{T}^{\lambda_2}_{23} \mathcal{T}^{\lambda_1}_{13} |\mathcal{M}|,
\end{equation}
we have
\begin{equation}
w_{\lambda_1+\lambda_2} = \langle e_{\lambda_1} \otimes e_{\lambda_2}, \mathcal{T}^{\lambda_1+\lambda_2} e_{\lambda_1} \otimes e_{\lambda_2} \rangle
= \langle e_{\lambda_2}, \mathcal{T}^{\lambda_2} e_{\lambda_2} \rangle \langle e_{\lambda_1}, \mathcal{T}^{\lambda_1} e_{\lambda_1} \rangle
= w_{\lambda_2}w_{\lambda_1}. \quad \blacksquare
\end{equation}

Using the identification \( \mathcal{U}_h(t) \equiv \mathcal{A}_q(AN) \) one can also describe the "vacuum" functional in the following way. It holds
\begin{equation}
\tau_{\lambda}(\Lambda) e_{\lambda} = A_{\lambda} e_{\lambda},
\end{equation}
where \( A_{\lambda} \) is a positive diagonal matrix fulfilling the \( RA_{\lambda}A_{\lambda} \) equation and possibly also \( CA_{\lambda}^t C^{-1} = A_{\lambda}^{-1} \). Besides, the relation (3.1) enables one to define a normal ordering on \( \mathcal{A}_q(AN) \) by requiring the elements of the matrix \( \Lambda^* \) to stand to the left and those of the matrix \( \Lambda \) to stand to the right. It doesn’t matter that this ordering prescription is not quite unambiguous since the subalgebras generated by the entries of \( \Lambda^* \) and \( \Lambda \), respectively, intersect in the Cartan elements. We have
\begin{equation}
\langle 1 \rangle = 1, \quad \langle \Lambda^* \rangle = \langle \Lambda \rangle = A_{\lambda},
\end{equation}
and
\begin{equation}
\langle x_{i_1} \ldots x_{i_k} \rangle = \langle x_{i_1} \rangle \ldots \langle x_{i_k} \rangle,
\end{equation}
provided the product \( x_{i_1} \ldots x_{i_k} \) is normally ordered.

5 The quantum coherent state

The following definition is very analogous to the classical case and is crucial for the rest of the paper.
Definition 5.1. We define the quantum coherent state as the element
\[ \Gamma := (\mathcal{T}^{\lambda})^{-1}(e_\lambda \otimes 1) \] (5.1)

\[ = (\tau_\lambda \otimes S)\rho \cdot (e_\lambda \otimes 1) \in \mathcal{H}_\lambda \otimes \mathcal{A}_q(K). \]

\(\Gamma\) should be interpreted as a quantum function on \(K\) with values in \(\mathcal{H}_\lambda\). Thus one can relate to every vector \(u \in \mathcal{H}_\lambda\) a quantum function on \(K\),
\[ u \mapsto \langle \Gamma, u \rangle := \langle e_\lambda, (\cdot) u \rangle \otimes \text{id} \] (5.2)

Furthermore, the operators in \(\mathcal{H}_\lambda\) can be again represented by their symbols, \(\sigma : \text{Lin}(\mathcal{H}_\lambda) \to \mathcal{A}_q(K) : B \mapsto \langle \Gamma, B \Gamma \rangle\). (5.3)

Proposition 5.2. The mapping \(\sigma\) is injective.

Proof. The proof goes through as in the classical case \([19, 37]\). Let us sketch it. \(\sigma(B) = 0\) means that
\[ \langle e_\lambda, \mathcal{T}^{\lambda} B S(\mathcal{T}^{\lambda}) e_\lambda \rangle = 0. \]

Applying \(k\)-times the comultiplication to the LHS, pairing with the elements \(X_{i_1}^+ \cdots X_{i_k}^+\) and using the fact that \(e_\lambda\) is the lowest weight vector and that \((X_i^\pm)^* = X_i^\mp\) we obtain
\[ \langle \tau_\lambda(X_{i_1}^+) \cdots \tau_\lambda(X_{i_k}^+) e_\lambda, B e_\lambda \rangle = 0. \]

Since the vectors \(\tau_\lambda(X_{i_1}^+) \cdots \tau_\lambda(X_{i_k}^+) e_\lambda\) span \(\mathcal{H}_\lambda\) it follows that \(B e_\lambda = 0\). Applying instead the comultiplication \((k+1)\)-times one finds that the same argument is valid also provided \(B\) is replaced \(\mathcal{T}^{\lambda} B S(\mathcal{T}^{\lambda}) \in \text{Lin}(\mathcal{H}_\lambda) \otimes \mathcal{A}_q(K)\) and so \(B S(\mathcal{T}^{\lambda}) e_\lambda = 0\). The same reasoning as above gives \(B = 0\).

The symbol can be extended naturally as a mapping from \(U_h(\mathfrak{g}) \equiv \mathcal{A}_q(AN)\) to \(\mathcal{A}_q(K)\) by putting
\[ \sigma = (\langle \cdot \rangle \otimes \text{id}) \circ \mathcal{R}. \] (5.3)

Now we are ready to define the isotropy subgroup as a *-Hopf algebra \(\mathcal{A}_q(K_0)\) with the vector representation \(U_0\) and the projection ("restriction morphism") \(p_0 : \mathcal{A}_q(K) \to \mathcal{A}_q(K_0), \ p_0(U) = U_0\). We require
\[ (\langle \cdot \rangle \otimes p_0) \circ \mathcal{R} = \langle \cdot \rangle 1 \] (5.4)

as a morphism from \(\mathcal{A}_q(AN)\) to \(\mathbb{C} \otimes \mathcal{A}_q(K_0) \equiv \mathcal{A}_q(K_0)\), i.e.,
\[ p_0(\sigma(Y)) = \langle Y \rangle 1, \quad \text{for} \quad Y \in \mathcal{A}_q(AN) . \] (5.5)

According to (3.6) and (4.6),
\[ \sigma(\Lambda^\ast \Lambda) = U^{-1} A_\lambda^2 U. \] (5.6)
Consequently, in addition to the equations
\[RU_{01}U_{02} = U_{02}U_{01}R, \quad U_0^* = U_0^{-1},\]  
and for \(B_l, C_l, D_l\) also \(CU_0^tC^{-1} = U_0^{-1}\), \(U_0\) should fulfill
\[U_0^{-1}A_\lambda^2 U_0 = A_\lambda^2.\]  
(5.8)

The condition (5.8) is formally the same as in the classical case. In fact it amounts in annihilation of some entries of the matrix \(U\) when taking the projection \(p_0(U) = U_0\).

For the enveloping algebra \(U_h(t_0)\) this means that there exists a subset \(\Pi_0\) of the set of simple roots \(\Pi\) so that \(U_h(t_0)\) is generated by all Cartan elements \(H_i\) and only by those elements \(X_i^\pm\) for which \(\alpha_i \in \Pi_0\).

Thus on the dual level we have an injection \(U_h(t_0) \hookrightarrow U_h(t)\). An element \(X\) from \(U_h(t)\) belongs to \(U_h(t_0)\) if and only if
\[\langle X, f \sigma(Y) g \rangle = \langle \Delta X, f \otimes g \rangle \langle Y \rangle\]  
holds for every \(Y \in U_h(t)\) and \(f, g \in A_q(K)\). Letting \(f = g = 1\) we have (c.f. (3.5))
\[\langle \text{Ad}_X Y \rangle = \langle X, \sigma(Y) \rangle = \varepsilon(X) \langle Y \rangle.\]  
(5.10)

Let us substitute the elements \(H_i\) and \(X_i^+\) for \(Y\) in (5.10). Using \(\tau_\lambda(H_i) e_\lambda = \lambda(H_i) e_\lambda\) and \(\tau_\lambda(X_i^-) e_\lambda = 0\) we find that (5.10) is true for all Cartan elements \(H_i\) and only for those elements \(X_i^+\) which fulfill
\[\langle Y X_i^+ \rangle = \langle \tau_\lambda(Y^*) e_\lambda, \tau_\lambda(X_i^+) e_\lambda \rangle = 0.\]

(5.11)

Putting \(Y^* = X_{i_1}^+ \ldots X_{i_k}^+\) we conclude that the condition on the subset \(\Pi_0 \subset \Pi\) is:
\[\alpha_i \in \Pi_0 \quad \text{iff} \quad \tau_\lambda(X_i^+) e_\lambda = 0.\]

It follows immediately that there exists a character \(\chi\) on \(U_h(t_0)\) such that
\[\tau_\lambda(X) e_\lambda = \chi(X) e_\lambda, \quad \text{for} \quad X \in U_h(t_0).\]  
(5.12)

Pairing the both sides with \(e_\lambda\) one finds that \(\chi(\cdot)\) is the restriction of the ”vacuum” functional \(\langle \cdot \rangle\). Considering \(\chi\) as an element from \(A_q(K_0)\) we deduce that
\[\chi = p_0(w_\lambda) \quad \text{and} \quad \Delta \chi = \chi \otimes \chi.\]  
(5.13)

Moreover, using (5.13), (4.4) and the relation \(m \circ (S \otimes \text{id}) \circ \Delta = \varepsilon\) we have
\[S \chi = \chi^* = \chi^{-1}.\]  
(5.14)

Let us now check the equivariance property. First note that from (5.3) and the relation \((\text{id} \otimes \Delta) R = (R \otimes \text{id}) R\) it follows
Proposition 5.3.

\[ \Delta \circ \sigma = (\sigma \otimes \text{id}) \circ \mathcal{R}, \quad (5.15) \]

and hence, by (5.5),

\[ (p_0 \otimes \text{id}) \Delta \sigma(Y) = 1 \otimes \sigma(Y). \quad (5.16) \]

This means that every symbol \( \sigma(Y) \in \mathcal{A}_q(K) \) is left \( K_0 \)-invariant, i.e., \( \sigma(Y) \in \mathcal{A}_q(K_0 \backslash K) \).

Concerning the equivariance of the coherent state itself we have

Proposition 5.4. It holds

\[ (p_0 \otimes \text{id}) \Delta \langle \Gamma, u \rangle = \chi \otimes \langle \Gamma, u \rangle. \quad (5.17) \]

 Particularly, putting \( u = e_\lambda \), we have

\[ (p_0 \otimes \text{id}) \Delta w_\lambda = \chi \otimes w_\lambda. \quad (5.18) \]

So the quantum function \( w_\lambda^{-1}(\Gamma, u) \) is left \( K_0 \)-invariant and belongs to some completion of the algebra \( \mathcal{A}_q(K_0 \backslash K) \) since we admit \( w_\lambda \) to be invertible.

Proof. First note that (5.12) can be rewritten dually as

\[ (\text{id} \otimes p_0) T^\lambda \cdot (e_\lambda \otimes 1) = e_\lambda \otimes \chi. \]

Hence, using the unitarity of \( T^\lambda \) and \( \chi \), we have for any \( u \in \mathcal{H}_\lambda \),

\[ \langle e_\lambda, (\cdot) u \otimes p_0 \rangle T^\lambda = \langle e_\lambda, u \rangle \chi. \]

It follows that

\[ (p_0 \otimes \text{id}) \Delta \langle \Gamma, u \rangle = (\langle e_\lambda, (\cdot) \rangle \otimes p_0 \otimes \text{id}) T^\lambda_{12} T^\lambda_{13} \]

\[ = (\langle e_\lambda, (\cdot) u \rangle \otimes \text{id}) T^\lambda \]

\[ = \chi \otimes \langle \Gamma, u \rangle. \blacksquare \]

6 Canonical element for the double

The complex structure on the quantized homogeneous space \( K_0 \backslash K \) is introduced the same way as in the classical case. Namely, the subalgebra of \( \mathcal{A}_q(K_0 \backslash K) \) consisting of holomorphic functions coincides with \( \mathcal{A}_q(P_0 \backslash G) \). Here \( \mathcal{A}_q(P_0) \) is the Hopf algebra dual to \( U_h(p_0) \), the Hopf subalgebra in \( U_h(g) \) generated by all \( H_i, X_i^- \) and those \( X_i^+ \) for which \( \alpha_i \in \Pi_0 \). Since the condition (5.12) clearly extends to all \( X \in U_h(p_0) \)
it is easy to verify that for every $u \in H$, $\langle u, \Gamma \rangle (w_\lambda^*)^{-1}$ is a holomorphic quantum function. Thus one can represent vectors from $H$ by antiholomorphic functions,

$$u \mapsto \psi_u := w_\lambda^{-1} \langle \Gamma, u \rangle.$$  \hfill (6.1)

This mapping is injective as one can show using the same reasoning as in the case of the symbol (Sec. 5). It is desirable to introduce quantum (non-commutative) local holomorphic coordinates $z_j$ on $K_0 \setminus K$ and consequently to express $\psi_u = \psi_u(z_j^*)$ as an polynom in $z_j^*$. To this end we shall employ the Gauss decomposition.

Denote by $b_+ \subset g$ the Borel subalgebras and by $\mathfrak{h} = b_+ \cap b_-$ the Cartan subalgebra. It is known that the Hopf algebras $U_h(b_+)$ and $U_h(b_-)^{op\Delta}$ are mutually dual and that the dual quantum double for $U_h(b_+)$ can be identified as an algebra with $U_h(g) \otimes U_h(h)$. To have this identification also for the coalgebras one has to twist the comultiplication in $U_h(g) \otimes U_h(h)$ using the element $\exp(\sum H_i^0 \otimes H_i^0)$ with $\{H_i^0\}$ being any orthonormal basis in $\mathfrak{h}$ [31]. According to the terminology we have adopted here the dual quantum double means twisted multiplication while the quantum double means twisted comultiplication. Thus on the dual level one obtains for the corresponding algebras of quantum functions,

$$A_q(B_-) \otimes A_q(B_+) \simeq A_q(G) \otimes_{\text{twist}} A_q(H).$$ \hfill (6.2)

The vector corepresentations $L^{(\pm)}$ and $J$ of the quantum groups $(B_\pm)_q$ and $H_q$, respectively, fulfill the corresponding $RXX$-equations and possibly also the deformed orthogonality condition. For a proper choice of the set $\Pi$ of simple roots, $L^{(\pm)}$ is upper (lower) triangular and $J$ is diagonal. The isomorphism in (6.2) is given by

$$T \equiv T \otimes 1 = L^{-} \otimes L^{+}, \quad J \equiv 1 \otimes J = (\text{diag } L^{-})^{-1} \text{diag } L^{+},$$ \hfill (6.3)

and the twisted comultiplication on the RHS is determined by

$$\text{diag}(R) T_1 J_2 = J_2 T_1 \text{diag}(R).$$ \hfill (6.4)

This structure has turned out to be very helpful in construction of the universal R-matrix $R^u \in U_h(g) \otimes U_h(g)$. [18, 23] First by fixing a maximal Weyl element one orders the set $\Delta^+$ of positive roots as $(\beta_1, \ldots, \beta_d)$, $d = |\Delta^+|$. To each root $\beta_j$ there are related elements $E(j) \in U_h(b_+)$ and $F(j) \in U_h(b_-)$ so that the elements

$$E(d)^{n_d} \ldots E(1)^{n_1} H_l^{m_l} \ldots H_1^{m_1},$$ \hfill (6.5)

$n_i, m_i \in \mathbb{Z}_+$, form a basis in $U_h(b_+)$. The vectors $H_i$ can be replaced by any elements forming a basis in $\mathfrak{h}$ and a similar assertion is valid also for $U_h(b_-)$. In the limit $h \downarrow 0$ the elements $E(j)$ and $F(j)$ become the root vectors $X_{\beta_j} \in n_+$ and $X_{-\beta_j} \in n_-$, respectively. We recall that the universal R-matrix can be written in the form [20]

$$R^u = \exp_{q_d} (\mu_d F(d) \otimes E(d)) \ldots \exp_{q_1} (\mu_1 F(1) \otimes E(1)) \exp(\kappa),$$ \hfill (6.6)

where $\exp_q$ are the $q$-deformed exponential functions, $\mu_j$ are some coefficients depending on the parameter $\hbar$ and $\kappa$ is some element from $U_h(h) \otimes U_h(h)$. 

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Equipped with this knowledge we are able to reveal the structure of the canonical element for the double $\mathcal{A}_q(AN) \otimes \mathcal{A}_q(K)$. We make use of the fact that $\mathcal{A}_q(AN) \simeq \mathcal{U}_h(\mathfrak{g})^{op\Delta}$ is a factoralgebra of $\mathcal{U}_h(\mathfrak{b}_-)^{op\Delta} \otimes_{\text{twist}} \mathcal{U}_h(\mathfrak{b}_+)^{op\Delta}$ and $\mathcal{A}_q(K) \simeq \mathcal{U}_h(\mathfrak{g})^\tau$ is a subalgebra in $\mathcal{U}_h(\mathfrak{b}_+)^{op} \otimes \mathcal{U}_h(\mathfrak{b}_-)^{op\Delta}$. The canonical element $\tilde{\rho}$ in

$$\left(\mathcal{U}_h(\mathfrak{b}_-)^{op\Delta} \otimes_{\text{twist}} \mathcal{U}_h(\mathfrak{b}_+)^{op\Delta}\right) \otimes \left(\mathcal{U}_h(\mathfrak{b}_+)^{op} \otimes \mathcal{U}_h(\mathfrak{b}_-)^{op\Delta}\right)$$

(6.7)
can be decomposed as follows

$$\tilde{\rho} = \sum (e_j \otimes e_k) \otimes (f^i \otimes f_k)$$
$$= \sum (e_j \otimes 1 \otimes f^i \otimes 1) \cdot (1 \otimes e^k \otimes 1 \otimes f_k)$$
$$= \tilde{R}_{13} \tilde{R}_{24}.$$ 

Here $\{e_j\}$, $\{e^k\}$, $\{f^i\}$ and $\{f_k\}$ stand for bases in the corresponding factors, $\{e_j\}$ and $\{f^i\}$ are dual and the same is assumed about $\{e^k\}$ and $\{f_k\}$, the dot in the third member of equalities (6.8) indicates multiplication in the double and $\tilde{R}'$ is obtained from $\tilde{R}$ by reversing the order of multiplication. To express $\rho$ we shall use again bases of the type (6.5). In our notation the elements $F(j)$, $E(j)$, $\tilde{F}(j)$ and $\tilde{E}(j)$ belong in this order to the individual factors in (6.7). Factorizing off the redundant Cartan elements we obtain finally

**Proposition 6.1.** The canonical element for the quantum double $\mathcal{A}_q(AN) \otimes \mathcal{A}_q(K)$ has the form

$$\rho = \exp_{q_d}(\mu_d F(d) \otimes \tilde{E}(d)) \ldots \exp_{q_1}(\mu_1 F(1) \otimes \tilde{E}(1)) \exp(\kappa)$$
$$\times \exp_{q_1}(\mu_1 E(1) \otimes \tilde{F}(1)) \ldots \exp_{q_d}(\mu_d E(d) \otimes \tilde{F}(d)).$$

To proceed further in this analysis we note that the maximal Weyl element can be chosen so that there exists $p \in \mathbb{Z}_+, \ p \leq d$, such that the vectors $X_{-\beta_1}, \ldots, X_{-\beta_p}$, $H_1, \ldots, H_d$, $X_{\beta_1}, \ldots, X_{\beta_p}$ form a basis of $\mathfrak{p}_0$. Then $X_{\beta_{p+1}}, \ldots, X_{\beta_d}$ form a basis of a nilpotent subalgebra $\mathfrak{n}_0$ and $\mathfrak{g} = \mathfrak{p}_0 \oplus \mathfrak{n}_0$. Notice that in the generic case $\Pi_0 = \emptyset$ and hence $p = 0$, $\mathfrak{p}_0 = \mathfrak{b}_-$ and $\mathfrak{n}_0 = \mathfrak{n}_+$. This means that all elements $F(j)$ belong to $\mathcal{U}_h(\mathfrak{p}_0)$ while $\tilde{E}(j)$ belongs to $\mathcal{U}_h(\mathfrak{p}_0)$ only for $j = 1, \ldots, p$. Consequently,

$$\tau_\lambda(F(j)) e_\lambda = 0, \quad \text{for} \ j = 1, \ldots, d,$$
$$\tau_\lambda(E(j)) e_\lambda = 0, \quad \text{for} \ j = 1, \ldots, p.$$ 

Let $\tau$ designate the irreducible representation of $\mathcal{U}_h(\mathfrak{g})$ corresponding to the vector corepresentation $T$ of $\mathcal{A}_q(G)$, $T = (\tau \otimes \text{id}) \rho$. We have
Corollary 6.2. $T$ can be written as a product,

$$T = \Lambda_{(-)} Z,$$

where

$$
\Lambda_{(-)} = (\tau \otimes \text{id}) \exp_{q_d}(\mu_d F(d) \otimes \tilde{E}(d)) \ldots \exp_{q_1}(\mu_1 F(1) \otimes \tilde{E}(1)) \exp(\kappa)
\times \exp_{q_1}(\mu_1 E(1) \otimes \tilde{F}(1)) \ldots \exp_{q_p}(\mu_p E(p) \otimes \tilde{F}(p)),
$$

$$Z = (\tau \otimes \text{id}) \exp_{q_{p+1}}(\mu_{p+1} E(p + 1) \otimes \tilde{F}(p + 1)) \ldots \exp_{q_d}(\mu_d E(d) \otimes \tilde{F}(d)).$$

The matrix $\Lambda_{(-)}$ is block lower triangular, $Z$ is block upper triangular and the blocks on the diagonal of $Z$ are unit matrices.

Remark. The splitting into the blocks is determined by decomposition of $\mathfrak{g}_0 = \text{complexification of } \mathfrak{k}_0$ into the direct sum of simple subalgebras and an Abelian subalgebra and it will be described more explicitly in the next section. In the generic case of $\Pi_0 = \emptyset$, $\mathfrak{g}_0 = \mathfrak{h}$ and the matrices $\Lambda_{(-)}$ and $Z$ are simply lower and upper triangular.

Notice that the entries of $Z$ are expressed as polynomials in $d - p = \dim_{\mathbb{C}}(P_0 \backslash G)$ noncommutative variables $\tilde{F}(p + 1), \ldots, \tilde{F}(d)$ and can be considered as local holomorphic coordinates on the orbit. Next we are going to derive explicit commutation relations for them.

Recalling the definition (5.1) of the coherent state $\Gamma$ and using the relations (6.9), (6.10), we obtain

$$
\Gamma = \exp_{q_d}^{-1}(\mu_d \tau_{\lambda}(E(d) \otimes \tilde{F}(d)) \ldots \exp_{q_{p+1}}^{-1}(\mu_{p+1} \tau_{\lambda}(E(p + 1)) \otimes \tilde{F}(p + 1)) \cdot (e_{\lambda} \otimes w_{\lambda}^*) ,
$$

since

$$
w_{\lambda}^* = ((e_{\lambda}, \tau_{\lambda}(\cdot)e_{\lambda}) \otimes \text{id}) \rho^{-1} = \exp(( (e_{\lambda}, \tau_{\lambda}(\cdot)e_{\lambda}) \otimes \text{id}) \kappa)
$$

Thus we find again that, for every $u \in \mathcal{H}_\lambda$, $\psi_u$ given by (6.1) is an antiholomorphic quantum function and should be expressible in the variables $z^*$.

## 7 Quantum holomorphic coordinates on a general dressing orbit

We start from the decomposition $T = \Lambda_{(-)} Z$. Let now $p_0$ stand for the ”restriction” morphism $\mathcal{A}_q(G) \to \mathcal{A}_q(P_0)$. First we shall verify that the entries of $Z$ are left $P_0$–invariant quantum functions. We have

$$(p_0 \otimes \text{id}) \Delta T = (p_0 \otimes \text{id}) \Delta \Lambda_{(-)} \cdot (p_0 \otimes \text{id}) \Delta Z.$$

At the same time,

$$(p_0 \otimes \text{id}) \Delta T = p_0(T) \otimes T = (p_0(T) \otimes \Lambda_{(-)})(1 \otimes Z).$$
Since the decomposition into a product of block lower triangular and block upper triangular matrices, the latter having unit diagonal blocks, is unambiguous we find by comparing that
\[(p_0 \otimes \text{id}) \Delta Z = 1 \otimes Z.\] (7.1)

To derive commutation relations for the matrix elements of \(Z\) one can again employ the Gauss decomposition. This time we have in mind the isomorphism (6.2), (6.3). We are going to enumerate the matrix elements in the vector representation by weights. This is possible since for all four principal series \(A\), \(B\), \(C\), \(D\), the weights of the vector representations are simple. Every weight belongs either to the Weyl group orbit of the corresponding fundamental weight or is zero (only for the series \(B\)). We shall use the standard ordering on the set of weights: \(\sigma > \nu\) iff \(\sigma \neq \nu\) and \(\sigma - \nu = \sum m_i \alpha_i\), with \(m_i \in \mathbb{Z}_+\) (\(0 \in \mathbb{Z}_+\)). Set
\[W_0 = \bigoplus_{\alpha_i \in \Pi_0} \mathbb{Z}_+ \alpha_i.\] (7.2)

We shall write simply \(L = (L_{\sigma\nu})\) instead of \(L^{(+)}\). Thus \(L_{\sigma\nu} = 0\) whenever \(\sigma < \nu\) (pay attention, the ordering on weights is reversed in comparison with the standard enumeration of weights). Further we introduce a matrix \(A\) by
\[A_{\sigma\nu} = \begin{cases} L_{\sigma\nu}, & \text{if } \sigma - \nu \in W_0, \\ 0, & \text{otherwise}. \end{cases}\] (7.3)

Comparing (6.3) and (6.11) we obtain
\[Z = A^{-1}L.\] (7.4)

Next we recall a useful property of the \(R\)-matrix. Namely, \(R_{\sigma\tau,\mu\nu} \neq 0\) implies \(\sigma - \mu = \nu - \tau\), \(\sigma \leq \mu\), \(\tau \geq \nu\), and one of the following three possibilities happens:
(i) \(\sigma = \mu, \tau = \nu\),
(ii) \(\sigma = \nu < \tau = \mu\),
(iii) \(\sigma = -\tau < \mu = -\nu\).

We continue by deriving some auxiliary relations. The first one is

**Lemma 7.1.** It holds
\[\Delta A = A \hat{\otimes} A, \quad \text{in} \quad \mathcal{A}_q(B_+),\] (7.5)
and consequently
\[R A_1 A_2 = A_2 A_1 R.\] (7.6)

**Proof.** In the equality
\[\Delta L_{\sigma\nu} = \sum L_{\sigma\xi} \otimes L_{\xi\sigma},\]
the nonzero summands should fulfill \( \sigma \geq \xi \geq \nu \). To obtain (7.5) it is enough to notice that then \( \sigma - \nu \in \mathcal{W}_0 \) implies \( \sigma - \xi, \xi - \nu \in \mathcal{W}_0 \).

The relation (7.6) is the same as

\[
\langle Y, R A_1 A_2 - A_2 A_1 R \rangle = 0, \quad \text{for all } Y \in \mathcal{U}_h(b^+).
\]

The last equality can be deduced from the following facts. This relation is valid provided \( A \) is replaced by \( L \). Clearly \( \langle X^+_i, A \rangle = 0 \) whenever \( \alpha_i \notin \Pi_0 \) and so

\[
\langle Y_1 X^+_i Y_2, A \rangle = 0, \quad \text{for } \alpha_i \notin \Pi_0 \text{ and any } Y_1, Y_2 \in \mathcal{U}_h(b^+).
\]

Finally,

\[
\langle H_i, A \rangle = \langle H_i, L \rangle, \quad \text{for all } i,
\]

\[
\langle X^+_i, A \rangle = \langle X^+_i, L \rangle, \quad \text{provided } \alpha_i \in \Pi_0. \quad \square
\]

By annulating some entries of the R-matrix we define another matrix \( Q = Q_{12} \),

\[
Q_{\sigma\tau,\mu\nu} = R_{\sigma\tau,\mu\nu}, \quad \text{provided } \tau - \nu = \mu - \sigma \in \mathcal{W}_0, \quad (7.7)
\]

\[
= 0, \quad \text{otherwise}.
\]

**Lemma 7.2.** It holds

\[
QL_1 A_2 = A_2 L_1 Q \quad (7.8)
\]

and

\[
QA_1 A_2 = A_2 A_1 Q. \quad (7.9)
\]

**Proof.** To show (7.8) assume in the equality

\[
\sum_{\xi\eta} R_{\sigma\tau,\xi\eta} L_{\xi\mu} L_{\eta\nu} = \sum_{\xi\eta} L_{\tau\eta} L_{\sigma\xi} R_{\xi\eta,\mu\nu},
\]

that \( \tau - \nu \in \mathcal{W}_0 \). The nonzero summands on the both sides should fulfill \( \tau \geq \eta \geq \nu \) whence \( \tau - \eta, \eta - \nu \in \mathcal{W}_0 \). Thus we obtain

\[
\sum_{\xi\eta} Q_{\sigma\tau,\xi\eta} L_{\xi\mu} A_{\eta\nu} = \sum_{\xi\eta} A_{\tau\eta} L_{\sigma\xi} Q_{\xi\eta,\mu\nu}, \quad (7.10)
\]

It remains to verify validity of (7.10) also for \( \tau - \nu \notin \mathcal{W}_0 \). Again, the nonzero summands on the both sides of (7.10) should satisfy \( \tau - \eta, \eta - \nu \notin \mathcal{W}_0 \). But \( \mathcal{W}_0 \) is additive and so this can never happen.

Let us show (7.9). Assume in (7.10) that \( \mu \geq \sigma \). The nonzero summands on the LHS should fulfill \( \xi - \sigma \in \mathcal{W}_0 \) and \( \xi \geq \mu \geq \sigma \) whence \( \xi - \mu \in \mathcal{W}_0 \). Analogously for
the RHS we have $\mu - \xi \in W_0$ and $\mu \geq \sigma \geq \xi$ whence $\sigma - \xi \in W_0$. Thus we obtain in this case
\[
\sum_{\xi \eta} Q_{\sigma \tau, \xi \eta} A_{\xi \mu} A_{\eta \nu} = \sum_{\xi \eta} A_{\tau \eta} A_{\sigma \xi} Q_{\xi \eta, \mu \nu}.
\]
(7.11)

Next assume in (7.10) that $\sigma - \mu \in W_0$. The nonzero summands on the LHS should fulfill $\xi - \sigma \in W_0$ whence, owing to the additivity, $\xi - \mu \in W_0$. Analogously for the RHS we have $\mu - \xi \in W_0$ and hence $\sigma - \xi \in W_0$. Also in this case we arrive at (7.11). It remains to verify (7.11) for $\sigma > \mu$ but $\sigma - \mu \not\in W_0$. Now the nonzero summands on the LHS of (7.11) should fulfill $\xi - \sigma$, $\xi - \mu \in W_0$. But this can never happen since then $\mu < \sigma \leq \xi$ and $\sigma - \mu$ would belong to $W_0$. Analogously on the RHS, it never happens that, at the same time, $\mu - \xi$ and $\sigma - \xi$ belong to $W_0$. ■

The final relation we shall need is

**Lemma 7.3.** It holds
\[
A_2^{-1}Z_1A_2 = Q^{-1}Z_1Q.
\]
(7.12)

*Proof.* One can verify (7.12) by using in (7.8) the substitution $L = AZ$ and the equality (7.9),
\[
A_2A_1Z_1Q = QA_1A_2A_2^{-1}Z_1A_2 = A_2A_1QA_2^{-1}Z_1A_2.
\]

Now we are able to state the desired commutation relation.

**Proposition 7.4.** The matrix $Z$ obeys the equality
\[
RQ_{12}^{-1}Z_1Q_{12}Z_2 = Q_{21}^{-1}Z_2Q_{21}Z_1R.
\]
(7.13)

*Proof.* To prove (7.13) use the substitution $L = AZ$ in the $RLL$–equation,
\[
RA_1A_2(A_2^{-1}Z_1A_2)Z_2 = A_2A_1(A_2^{-1}Z_2A_1)Z_1R,
\]
and apply (7.6) and (7.12),
\[
A_2A_1RQ_{12}^{-1}Z_1Q_{12}Z_2 = A_2A_1Q_{21}^{-1}Z_2Q_{21}Z_1R.
\]

This result should be completed by the relations following from the q-deformed orthogonality condition.

**Proposition 7.5.** For the series $B$, $C$ and $D$, the matrix $Z$ fulfills also
\[
\delta_{jk} = \sum_s (Z_2C_2QZ_2^tQ^{-1}C_2^{-1})_{kj,ss}.
\]
(7.14)

*Proof.* Since $CL^tC^{-1} = L^{-1}$, $CA^tC^{-1} = A^{-1}$, we have
\[
C(AZ)^tC^{-1} = Z^{-1}A^{-1} = Z^{-1}CA^tC^{-1}.
\]
(7.15)

Furthermore, multiplying (7.12) by $C_2^{-1}$ from the left and by $C_2$ from the right one obtains
\[
A_2^tZ_1(A_2^t)^{-1} = \tilde{Q}^{-1}Z_1\tilde{Q}, \quad \text{where} \quad \tilde{Q} = C_2^{-1}QC_2.
\]
Using this relation one can derive for the matrix elements
\[
[(AZ)^t(A^{-1})]_{jk} = \sum_s [A^t Z_1(A^{-1})]_{ss,jk}
= \sum_s (\tilde{Q}^{-1} \tilde{Z}_1)_{ss,jk}
\]

Consequently,
\[
[ZC(AZ)^t(A^{-1}C^{-1})]_{jk} = \sum_{stu} (ZC)_{js}[\tilde{Q}^t Z_1(\tilde{Q}^{-1})]_{st,uu}(C^{-1})_{tk}
= \sum_s (Z_2 C^2 Q Z_1^t Q^{-1} C^{-1})_{kj,ss}.
\]

In view of (7.15) we have arrived at the sought relations.

In the generic case ($\Pi_0 = \emptyset$) the dressing orbit is nothing but the flag manifold. In this case $Q_{12} = Q_{21} = \text{diag} R$ and $Z$ is an upper triangular matrix with units on the diagonal. The relation (7.13) can be simplified since $\text{diag} R$ commutes with $R$,
\[
R Z_1 \text{diag}(R) Z_2 = Z_2 \text{diag}(R) Z_1 R. \quad (7.16)
\]

For the series $A$, i.e., $K = SU_q(N)$ we have
\[
R_{jk,st} = \delta_{js} \delta_{kt} + (q - q^{\text{sgn}(k-j)}) \delta_{jt} \delta_{ks},
Q_{jk,st} = q^{\delta_{jk}} \delta_{js} \delta_{kt},
\]

and the relation (7.16) can be rewritten for the individual matrix entries as
\[
q^{\delta_{ks}} z_{js} z_{kt} - q^{\delta_{jt}} z_{kt} z_{js} = (q^{\text{sgn}(k-j)} - q^{\text{sgn}(s-t)}) q^{\delta_{js}} z_{ks} z_{jt}. \quad (7.17)
\]

The relations (7.17) are already known \cite{12, 3}. Originally they were obtained by expressing the entries $z_{jk}$ by means of the q-minors ($j < k$),
\[
z_{jk} = \left| T_{1\ldots j}^{-1} \right| q \left| T_{1\ldots j-1,k} \right| q.
\]

But this derivation seems to be rather tedious and doesn’t suggest the compact form (7.16).

8 Representation acting in a space of antiholomorphic functions

Let us denote by $C_\lambda$ the algebra of quantum holomorphic functions living on the cell. This means that $C_\lambda$ is generated by the entries of $Z$ fulfilling (7.13) and possibly
also relations following from the deformed orthogonality condition. \( C^*_\lambda \) stands for the algebra of antiholomorphic functions determined by the adjoint relations. We know that every vector \( u \in H_\lambda \) is represented by an element \( \psi_u \equiv \psi_u(z^*) \) from \( C^*_\lambda \) (c.f. (6.1)), the mapping \( u \mapsto \psi_u \) is linear and injective and the lowest weight vector is sent to the unit. Denote by \( \mathcal{M}_\lambda \subset C^*_\lambda \) the image of \( H_\lambda \). We wish to transcribe the representation \( \tau_\lambda \) as acting in \( \mathcal{M}_\lambda \), but without introducing a special symbol for this new realization. We recall that both \( A_q(K) \) and \( A_q(K_0 \setminus K) \) become left \( U_h(k) \)-modules provided one relates to every element \( Y \in U_h(k) \) the left–invariant map \( \xi_Y \) on \( K_q \),

\[
\xi_Y \cdot f = (\text{id} \otimes \langle Y, \cdot \rangle) \Delta f, \quad f \in A_q(K).
\]

(8.1)

Then \( C^*_\lambda \) becomes a left \( U_h(k) \)-module with respect to the action

\[
(Y, f) \mapsto w_\lambda^{-1} \xi_Y \cdot (w_\lambda f).
\]

(8.2)

Proposition 8.1. \( \mathcal{M}_\lambda \) is the cyclic \( U_h(k) \)-submodule in \( C^*_\lambda \) with the cyclic vector 1, i.e.,

\[
\tau_\lambda(Y)\psi = w_\lambda^{-1} \xi_Y \cdot (w_\lambda \psi), \quad \text{for } Y \in U_h(k), \psi \in \mathcal{M}_\lambda.
\]

(8.3)

Proof. The proof is done by the following chain of equalities,

\[
w_\lambda \tau_\lambda(Y)\psi_u = \langle \Gamma, \tau_\lambda(Y)u \rangle = \langle e_\lambda, \tau_\lambda(\cdot) \tau_\lambda(Y)u \rangle \otimes \text{id}\rho
\]

\[
= \langle (e_\lambda, \tau_\lambda(\cdot)u) \otimes \text{id} \otimes \langle Y, \cdot \rangle \rangle \rho_{12} \rho_{13}
\]

\[
= (\text{id} \otimes \langle Y, \cdot \rangle) \Delta (\langle e_\lambda, \tau_\lambda(\cdot)u \rangle \otimes \text{id}) \rho.
\]

In the third equality we have used the identity

\[
\tau_\lambda(Y) = (\tau_\lambda(\cdot) \otimes \langle Y, \cdot \rangle)\rho.
\]

Finally we are going to show that the reproducing kernel can be introduced also in the quantum case and the scalar product in \( \mathcal{M}_\lambda \) can be expressed with its help. Let \( \eta \) designates the Haar measure on \( A_q(K) \). We have the orthogonality relations

\[
\eta(\langle u_1, T^\lambda v_1 \rangle^* \langle u_2, T^\lambda v_2 \rangle) = M_\lambda^{-1} \langle v_1, v_2 \rangle \langle u_1, \tau_\lambda(\gamma^{-1}) u_2 \rangle,
\]

(8.4)

where

\[
\gamma = \exp\left(-\frac{h}{2} \sum_{\alpha > 0} H_\alpha \right), \quad M_\lambda = \text{tr} \tau_\lambda(\gamma^2).
\]

Letting \( u_1 = u_2 = e_\lambda \) in (8.4) we get

\[
\langle u, v \rangle = c_\lambda \eta(\langle e_\lambda, T^\lambda u \rangle^* \langle e_\lambda, T^\lambda v \rangle),
\]

(8.5)

where \( c_\lambda = M_\lambda \langle e_\lambda, \tau_\lambda(\gamma^{-1}) e_\lambda \rangle \). Consequently,

\[
\langle u, v \rangle = c_\lambda \eta(\psi_u^* w_\lambda^* w_\lambda \psi_v).
\]

(8.6)
Set now
\[ \Psi(z) = \Gamma (w^*_\lambda)^{-1} \in \mathcal{H}_\lambda \otimes \mathcal{C}_\lambda, \] (8.7)
and define the reproducing kernel as
\[ K(\zeta^*,z) := \langle \Psi(\zeta), \Psi(z) \rangle \in \mathcal{C}_\lambda^* \otimes \mathcal{C}_\lambda. \] (8.8)

Here \( \zeta^* \) stands for the generators in \( \mathcal{C}_\lambda^* \) and \( z \) for those in \( \mathcal{C}_\lambda \).

It holds
\[ \langle u,v \rangle = c_\lambda \eta(\psi_u(z^*)K(z^*,z)^{-1}\psi_v(z^*)). \] (8.9)

It is enough to notice that \( K(z^*,z) \in \mathcal{C}_\lambda^* \cdot \mathcal{C}_\lambda \) is equal to \((w^*_\lambda w_\lambda)^{-1}, \)
\[ K(z^*,z) = w_\lambda^{-1} \langle e_\lambda, T^\lambda(T^\lambda)^{-1} e_\lambda \rangle (w^*_\lambda)^{-1} = w_\lambda^{-1}(w^*_\lambda)^{-1}. \]

Furthermore, substituting \( \Psi(\zeta) \) for \( u \) in (8.9) we obtain
\[ \psi(\zeta^*) = c_\lambda \eta_z(K(\zeta^*,z) K(z^*,z)^{-1}\psi(z^*)), \] for every \( \psi \in \mathcal{C}_\lambda^*. \) (8.10)

9 Representations and non-commutative differential geometry

We shall use the summation rule through this Section. All indices are running from 1 to \( N \), \( N \) being the dimension of the vector representation. With some abuse of notation we shall no more distinguish between the element \( X \in \mathcal{U}_h(\mathfrak{t}) \) and the corresponding left–invariant mapping \( \xi_X \) (8.1). We keep only the \( \cdot \) to indicate the action of \( \mathcal{U}_h(\mathfrak{t}) \) on \( \mathcal{A}_q(K) \). The following notions and facts concerning the differential calculus on \( \mathcal{A}_q(K) \) will be useful [45, 16, 8]. Let us denote as \( M_{ijkl} \) the following family of quantum functions on \( K \)
\[ M_{ijkl} = S^{-1}(U_{ij})U_{ik}. \]

Let also
\[ f_{ijkl} = S^2(L^l_{ji})S(L^j_{ki}) \]
be a family of elements of \( \mathcal{U}_h(\mathfrak{t}) \). We shall denote by \( \mathcal{E} \) the free left module over \( \mathcal{A}_q(K) \) with generators denoted by \( \Omega_{ij} \). Let us introduce the right multiplication, the right coaction \( \delta_R \) and the left coaction \( \delta_L \) of \( \mathcal{A}_q(K) \) on \( \mathcal{E} \) by
\[ a_{ij}\Omega_{ij}b = a_{ij}f_{ijkl} \cdot b_{ijkl}, \]
\[ \delta_R(a_{ij}\Omega_{ij}) = \Delta a_{ij}(\Omega_{kl} \otimes M_{klij}), \]
\[ \delta_L(a_{ij}\Omega_{ij}) = \Delta a_{ij}(1 \otimes \Omega_{ij}), \]
for \( a_{ij}, b \in \mathcal{A}_q(K) \). Then the triple \( (\mathcal{E}, \delta_R, \delta_L) \) is an \( \mathcal{A}_q(K) \)–bicovariant bimodule in the sense of [13]. If we introduce quantum functionals \( \chi_{ij} \in \mathcal{U}_h(\mathfrak{t}) \) by
\[ \chi_{ij} = \delta_{ij} - L^{-1}_{im} S(L^+_{mj}), \] (9.1)
then the mapping $d \colon \mathcal{A}_q(K) \to \mathcal{E}$

$$da = \Omega_{ij} \chi_{ij} \cdot a, \quad a \in \mathcal{A}_q(K)$$

(9.2)
defines a bicovariant first–order differential calculus on $\mathcal{A}_q(K)$, which extends uniquely to the exterior differential calculus on $\mathcal{A}_q(K)$. The linear space $\text{inv}_{\mathcal{E}}$ spanned by $\Omega_{ij}$'s is the space of left invariant one–forms. Let us denote as $\text{inv}_X$ the dual linear space of left–invariant vector fields spanned by $\chi_{ij}$'s. The linear space $\text{inv}_X$ is closed under the q-commutator

$$[X, Y]_q = \text{Ad}_X Y$$

and the comultiplication on $\chi_{ij}$ reads

$$\Delta \chi_{ij} = \chi_{ij} \otimes 1 + O_{ijkl} \otimes \chi_{kl}$$

$$O_{ijkl} = L^{-1}_{ik} S^{-1}(L^{+}_{ij}).$$

(9.3)

In the following we shall use freely the Cartan calculus on quantum groups developed in [36, 2], where the inner derivation $i_{\xi}$ and the Lie derivative $L_{\xi}$ of a general $n$–form along a general vector field $\xi$ have been introduced. Let us mention, without going into details, that the linear space of left–invariant vector fields $\text{inv}_X$ can be used to freely generate an $\mathcal{A}_q(K)$–bicovariant bimodule $\mathcal{X}$ of general vector fields on $\mathcal{A}_q(K)$. The right action of $\mathcal{A}_q(K)$ then coincides on $\text{inv}_X$ with the right dressing action $R$ (3.3). We refer the reader for more information to the above mentioned papers. Here we follow the conventions of [36].

For any quantum function $a \in \mathcal{A}_q(K)$ let us introduce the left–invariant one–form

$$\Theta_L^a = da^{(2)} S^{-1}(a^{(1)}) = \Omega_{ij} \langle \chi_{ij}, a \rangle$$

(9.4)
as well as the right invariant form

$$\Theta_R^a = da^{(1)} S(a^{(2)}) = \Theta_L^{a^{(2)}} a^{(1)} S(a^{(3)}) = \Omega_{kl} S(U_{li}) U_{jk} \langle \chi_{ij}, a \rangle.$$ (9.5)

Let $X \in \text{inv}_X$, then we have for its symbol $\sigma(X)$

$$\sigma(X) = i_X \Theta_R^{w_\lambda}. $$

(9.6)

This equality is a consequence of a chain of identities which employs the rules of [36]

$$\sigma(X) = \langle X, w^{(2)}_\lambda \rangle w^{(1)}_\lambda S(w^{(3)}_\lambda) = (i_X \Theta_L^{w^{(2)}_\lambda}) w^{(1)}_\lambda S(w^{(3)}_\lambda) = i_X \Theta_R^{w_\lambda}. $$

Applying the differential $d$ to the equality (9.6), making use of the identity

$$\mathcal{L}_X = i_X d + di_X,$$

which remains to be valid also in the quantum case and using the fact that

$$\mathcal{L}_X \omega = 0$$
for $X$ left–invariant and $\omega$ right–invariant we obtain immediately
\[ d\sigma(X) = -i_X d\Theta_R^{w\lambda}. \] (9.7)

Let $Y \in \mathcal{X}$ be now another left–invariant vector field. We have
\[-i_Y i_X d\Theta_R^{w\lambda} = \sigma([Y, X]_q), \] (9.10)
which follows from an application of $i_Y$ to the equality (9.7):
\[-i_Y i_X d\Theta_R^{w\lambda} = i_Y d\sigma(X) = Y \cdot \sigma(X) = \sigma(\text{Ad}_Y X). \]

The third equality in the above chain is a direct consequence of definitions adopted in [36].

Here we would like note the following. Let us assume the image $\sigma(U_h(\mathfrak{t}) \subset \mathcal{A}_q(K_0 \backslash K)$ under symbol mapping $\sigma$ equipped with a new product $\ast$, which respects the algebra structure of $U_h(k)$
\[ \sigma(X) \ast \sigma(Y) = \sigma(XY) \quad \text{for } X, Y \in U_h(k), \]
which is just the Berezin quantization prescription for the symbols in the classical case. Then from (5.15) it follows immediately that the mapping $\sigma$ is a quantum momentum map in the sense of [26] and we can rewrite (9.10) in the form
\[-i_Y i_X d\Theta_R^{w\lambda} = \sigma(Y(1)) \ast \sigma(X) \ast \sigma(S(Y(2))). \] (9.11)

Using the expression of the right invariant form $\Theta_R^{w\lambda}$ with the help of the left invariant forms $\Omega_{ij}$ following from (9.5) we obtain an alternative definition of the isotropy subgroup $K_0$. Instead of (5.4) we may equivalently require the invariance of $\Theta_R^{w\lambda}$ with respect to the left coaction of $K_0$
\[ (p_0 \otimes id)\delta_L \Theta_R^{w\lambda} = 1 \otimes \Theta_R^{w\lambda}. \] (9.12)

Let us denote for convenience by $\mathcal{Z} \in \mathcal{H}_\lambda \otimes \mathcal{A}_q(K_0 \backslash K)$ the unnormalized coherent state $\mathcal{Z} = \Gamma(w^*_\lambda)^{-1}$ and let the expressions $d\mathcal{Z}$ and $d\Gamma$ have the obvious meaning of differentiating with respect to the second factor in $\mathcal{H}_\lambda \otimes \mathcal{A}_q(K)$. Let us also introduce a new one–form $\Theta^{w\lambda} \equiv \Theta_R^{w\lambda} - dw_\lambda(w_\lambda)^{-1}$. Like in the classical case the one–forms $\Theta_R^{w\lambda}$ and $\Theta^{w\lambda}$ can be expressed through the coherent states $\Gamma$ and $\mathcal{Z}$ as
\[ \Theta_R^{w\lambda} = \langle d\Gamma | \Gamma \rangle \]
and
\[ \Theta^{w\lambda} = w_\lambda \langle d\mathcal{Z} | \mathcal{Z} \rangle w^*_\lambda, \]
respectively.

Now we are prepared to give a formula for the action of the elements $\chi_{ij}$ in the irreducible $\ast$–representation $\tau_\lambda$ of $U_h(\mathfrak{t})$, which directly generalizes the geometric quantization prescription for the action of generators of $U(\mathfrak{t})$ in the irreducible
representation of $K$ corresponding to a minimal weight $\lambda$. Starting from formula (8.3) and using (9.3) we have

$$\tau_\lambda(\chi_{ij})\psi = w_\lambda^{-1}(\chi_{ij} \cdot w_\lambda)\psi + w_\lambda^{-1}(O_{ijkl} \cdot w_\lambda)\chi_{kl} \cdot \psi,$$

which can be finally rewritten making use of the following identities

$$(\chi_{ij} \cdot w_\lambda)w_\lambda^{-1} = i_{\chi_{ij}} dw_\lambda w_\lambda^{-1} = \sigma(\chi_{ij}) - i_{\chi_{ij}} \Theta w_\lambda$$

in the form

$$\tau_\lambda(\chi_{ij})\psi = w_\lambda^{-1}(O_{ijkl} \cdot w_\lambda)\chi_{kl} \cdot \psi + w_\lambda^{-1}(\sigma(\chi_{ij}) - i_{\chi_{ij}} \Theta w_\lambda)w_\lambda \psi. \quad (9.13)$$

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