ON A BASE CHANGE CONJECTURE FOR HIGHER ZERO-CYCLES

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Abstract. We show the surjectivity of a restriction map on higher $(0,1)$-cycles for a smooth projective scheme over an excellent henselian discrete valuation ring. This gives evidence for a conjecture stated in [KEW] saying that base change holds for such schemes in general for motivic cohomology in degrees $(i,d)$ for fixed $d$ being the relative dimension over the base. Furthermore, the restriction map we study is related to a finiteness conjecture for the $n$-torsion of $CH_0(X)$, where $X$ is a variety over a $p$-adic field.

1. Introduction

Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field $K$ and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and always assume that $1/n \in k$. Let $X$ be a regular scheme, flat and projective over $\text{Spec}\mathcal{O}_K$ of fibre dimension $d$. Let $X_K$ denote the generic fibre and $X_0$ the reduced special fibre. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

In [SS, Cor. 9.5] and [EWB, App.] it is shown that for $X \to \text{Spec}\mathcal{O}_K$ smooth and projective and $k$ finite or algebraically closed, the restriction map

$$CH_1(X)_\Lambda \xrightarrow{\sim} CH_0(X_0)_\Lambda$$

is an isomorphism of Chow groups with coefficients in $\Lambda$. This result is reproven in [KEW] for more general residue fields and generalised to the case that $X_0$ is a simple normal crossings divisor. In that case one needs to replace $CH_0(X_0)$ by $H^{2d}_{cdh}(X_0, \mathbb{Z}/n\mathbb{Z}(d))$, i.e. the hypercohomology of the motivic complex $\mathbb{Z}/n\mathbb{Z}(d)$ in the cdh-topology, which is isomorphic to $CH_0(X_0)$ for $X_0/k$ smooth. The result then says that if $k$ is finite, or algebraically closed, or $(d-1)!$ prime to $m$, or $A$ is of equal characteristic, or $X/\mathcal{O}_K$ is smooth with perfect residue field $k$, then there is an isomorphism

$$CH_1(X)_\Lambda \xrightarrow{\sim} H^{2d}_{cdh}(X_0, \mathbb{Z}/n\mathbb{Z}(d))$$

which is induced by restricting a one-cycle in general position to a zero-cycle on $X_0^{sm}$. Generalising this result, the following conjecture is stated in section 10 of [KEW]:

Conjecture 1.1. The restriction homomorphism

$$\text{res} : H^{i,d}(X, \mathbb{Z}/m\mathbb{Z}) \to H^{i,d}_{cdh}(X_0, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all $i \geq 0$.

Here $H^{i,d}(X, \mathbb{Z}/m\mathbb{Z}) = H^i(X, \mathbb{Z}/m\mathbb{Z}(d))$ are the motivic cohomology groups for schemes over Dedekind rings defined in [Sp]. In this article we consider the corresponding restriction
map on higher Chow groups of zero-cycles with coefficients in $\Lambda$

$$\text{res}_{CH}^d : CH^d(X, 2d-i)_\Lambda \to CH^d(X_0, 2d-i)_\Lambda$$

for $X/O_K$ smooth which we define to be induced by the following composition:

$$\text{res}_{CH}^n : CH^n(X, m) \to CH^n(X_K, m) \xrightarrow{(-\pi)} CH^{n+1}(X_K, m+1) \xrightarrow{\partial} CH^n(X_0, m).$$

Here $\cdot(-\pi)$ is the product with $-\pi \in CH^1(K, 1) = K^\times$ defined in [Bl Sec. 5], $\pi$ is a local parameter for the discrete valuation on $K$ and $\partial$ is the boundary map coming from the localization sequence for higher Chow groups (see [Le1]). We call the composition

$$\text{sp}_{CH}^\pi : CH^n(X_K, m) \xrightarrow{(-\pi)} CH^{n+1}(X_K, m+1) \xrightarrow{\partial} CH^n(X_0, m)$$

a specialisation map for higher Chow groups. We note that $\text{res}_{CH}^d$ does not depend on the choice of $\pi$ whereas $\text{sp}_{CH}^\pi$ does. For a detailed discussion of the specialisation map see also [ADIKMP, Sec. 3].

Our main theorem is the following:

**Theorem 1.2.** Let $X/O_K$ be smooth. Then the restriction map

$$\text{res}_{CH}^d : CH^d(X, 1)_\Lambda \to CH^d(X_0, 1)_\Lambda$$

is surjective. This implies in particular the surjectivity part of conjecture [J] for the pair $(2d-1, d)$.

This implies the following corollary:

**Corollary 1.3.** Let $X/O_K$ be smooth. Then the specialisation map

$$\text{sp}_{CH}^\pi : CH^d(X_K, 1)_\Lambda \to CH^d(X_0, 1)_\Lambda$$

is surjective.

The restriction map in the degree of theorem 1.2 is of particular interest since it is related to a conjecture on the finiteness of $CH^d(X_K)[n]$ for $K$ a $p$-adic field. This is shown in section 3 as well as the injectivity for $d = 2$. Furthermore theorem 1.2 together with the main result of [KEW] may be considered as a generalization to perfect residue fields of the vanishing of the Kato homology group $KH_3(X)$ defined in [SS] where it was proven for $k$ finite or separably closed.

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2. Main result

Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field $K$ and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and always assume that $1/n \in k$. From now on let $X$ be a smooth and projective scheme over $\text{Spec} \mathcal{O}_K$ of fibre dimension $d$ in which case we also say that $X$ is of relative dimension $d$ over $\mathcal{O}_K$. Let $X_K$ denote the generic fibre and $X_0$ the reduced special fibre. By $X_{(p)}$ we denote the set of points $x \in X$ such that $\dim(\overline{\{x\}}) = p$, where $\overline{\{x\}}$ denotes the closure of $x$ in $X$.

We are going to use the following notation for Rost’s Chow groups with coefficients in Milnor K-theory (see [Ro, Sec. 5]):

$$C_p(X, m) = \bigoplus_{x \in X_{(p)}} (K^M_{m+p} k(x)) \otimes \mathbb{Z}/n\mathbb{Z}$$

$$Z_p(X, m) = \ker[\partial : C_p(X, m) \to C_{p-1}(X, m)]$$

$$A_p(X, m) = H_p(C_*(X, m))$$

We write $Z_k(X)$ for the group of $k$-cycles on $X$, i.e. the free abelian group generated by $k$-dimensional closed subschemes of $X$.

Let $\pi$ be some fixed a local parameter of $\mathcal{O}_K$. We define the restriction map

$$\text{res}_\pi : C_p(X, m) \to C_{p-1}(X_0, m+1)$$

to be the composition

$$\text{res}_\pi : C_p(X, m) \to C_{p-1}(X_K, m+1) \xrightarrow{\{-\pi\}} C_{p-1}(X_K, m+2) \xrightarrow{\partial} C_{p-1}(X_0, m+1).$$

In the above composition the map $C_p(X, m) \to C_{p-1}(X_K, m)$ is defined to be the identity on all elements supported on $X(\pi) \setminus X_0(\pi)$ and zero on $X_0(\pi)$. The map $\partial$ is defined to be the boundary map induced by the tame symbol on Milnor K-theory for discrete valuation rings. More precisely, $\partial$ is defined as follows: Let $\overline{\{x\}}$ be the subscheme corresponding to $x \in X(\pi)$. Let us assume for simplicity that $\overline{\{x\}}$ is normal. Otherwise we take the normalisation and use the norm map. Now if $y \in \overline{\{x\}}_{(p-1)}$, then $y$ defines a discrete valuation on $k(x)$. Let $\pi'$ be a local parameter of $k(x)$. Let $\overline{\partial \pi}' : K^M_{n+1} k(x) \to K^M k(y)$ be the tame symbol defined by sending $\{\pi', u_1, ..., u_n\}$ to $\{\overline{u}_1, ..., \overline{u}_n\}$, where the $\overline{u}_i$ are units in the discrete valuation ring of $k(x)$ and the $\overline{u}_i$ their images in $k(y)$. $\partial$ is defined to be the sum of all $\overline{\partial \pi}'$ taken over all $x \in X(\pi)$ and all $y \in \overline{\{x\}}_{(p-1)}$. Note that the restriction map $\text{res}_\pi$ has to be distinguished from the specialisation map

$$sp^\pi_{\gamma, \pi'} = \overline{\partial \pi}' \circ \{-\pi'\} : K^M_{n+1} k(x) \to K^M k(y).$$

$sp^\gamma_{\pi, \pi'}$ sends $\{\pi^{n_1} u_1, ..., \pi^{n_n} u_n\}$ to $\{\overline{u}_1, ..., \overline{u}_n\}$, where again the $\overline{u}_i$ are units in the discrete valuation ring of $k(x)$ and the $\overline{u}_i$ their images in $k(y)$.

The map $\text{res}_\pi$ depends on the choice of $\pi$ but the induced map on homology

$$\text{res} : A_p(X, m) \to A_{p-1}(X_0, m+1)$$
is independent of the choice. This can be seen as follows: Let \( u \in \mathcal{O}_K^x \) and \( \alpha \in C_p(X, m) \). Then \( \text{res}_u(\alpha) = \partial(-\pi u) \cdot \alpha = \partial(\{-\pi\} \cdot \alpha) + \partial(u) \cdot \alpha = \text{res}_u(\alpha) + \partial(u) \cdot \alpha \). Now if \( \alpha \in A_p(X, m) \), then \( \partial(u) \cdot \alpha = 0 \) and \( \text{res}_u(\alpha) = \text{res}_\pi(\alpha) \). In the following we will write \( \text{res} \) for \( \text{res}_\pi \), fixing a local parameter \( \pi \in \mathcal{O}_K \).

We now turn to our principle interest of study, the restriction map

\[
\text{res} : C_2(X, -1) \to C_1(X, 0).
\]

We start with the following lemma:

**Lemma 2.1.** The map \( \text{res} : C_2(X, -1) \to C_1(X, 0) \), after having fixed \( \pi \), is surjective.

**Proof.** Let \( \bar{u} \in K^1 M k(x) \) for some \( x \in X_0^{(d-1)} \). As in the proof of [SS, Lem. 7.2] we can find a relative surface \( Z \subset X \) containing \( x \) and being regular at \( x \) and such that \( Z \cap X_0 \) contains \( \{x\} \) with multiplicity 1. Let \( Z_0 = \bigcup_{i \in I} Z_0^{(i)} \cap \{x\} \) be the union of the pairwise different irreducible components of the special fiber of \( Z \) with those irreducible components different from \( \{x\} \) indexed by \( I \). Since all maximal ideals, \( m_i \) corresponding to \( Z_0^{(i)} \) and \( m_x \) corresponding to \( \{x\} \), in the semi-local ring \( \mathcal{O}_{Z, Z_0} \) are coprime, the map \( \mathcal{O}_{Z, Z_0} \to \prod_{i \in I} \mathcal{O}_{Z, Z_0}/m_i \times \mathcal{O}_{Z, Z_0}/m_x \) is surjective. Therefore we can find a lift \( u \in K^1 M k(z) \), \( z \) being the generic point of \( Z \), of \( \bar{u} \) which specialises to \( \bar{u} \) in \( K(\{x\})^x \) and to 1 in \( K(Z_0^{(i)})^x \) for all \( i \in I \).

The main result we are going to prove is the following:

**Proposition 2.2.** The restriction map \( \text{res} : A_2(X, -1) \to A_1(X, 0) \) is surjective.

It will be implied by the following key lemma:

**Key Lemma 2.3.** Let \( \xi \in \text{ker}[Z_1(X)/n \to Z_0(X)/n] \), then there is a \( \xi' \in \text{ker}[C_2(X, -1) \to C_1(X, 0)] \) with \( \partial(\xi') = \xi \).

**Proof.** (Proposition 2.2) Let \( \xi_0 \in \text{ker}[C_1(X, 0) \to C_0(X, 0)] \). By lemma 2.1 there is a \( \xi \in C_2(X, -1) \) with \( \text{res}(\xi) = \xi_0 \). As \( \partial(\text{res}(\xi)) = \partial(\text{res}(\xi)) = 0 \), key lemma 2.3 tells us that there is a \( \xi' \in \text{ker}(C_2(X, -1) \to C_1(X, 0)) \) with \( \partial(\xi') = \partial(\xi) \). As \( \text{res} \) is a homomorphism, it follows that \( \xi_0 = \text{res}(\xi - \xi') \) and \( \partial(\xi - \xi') = 0 \). Hence \( \text{res} : Z_2(X, -1) \to Z_1(X, 0) \) is surjective and the commutativity of \( \partial \) and \( \text{res} \) implies that \( \text{res} : A_2(X, -1) \to A_1(X, 0) \) is surjective.

**Proof.** (Key Lemma 2.3) We start with the case of relative dimension \( d = 1 \), i.e. \( X \) is a smooth fibered surface over \( \mathcal{O}_K \), and consider the following diagram:

\[
\begin{array}{ccc}
C_2(X, -1) = K(X)^* \otimes \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\text{res}} & C_1(X, 0) = K(X)^* \otimes \mathbb{Z}/n\mathbb{Z} \\
\partial & & \partial \\
Z_1(X)/n & \xrightarrow{\text{res}} & Z_0(X)/n
\end{array}
\]

where we write \( Z_i(X)/n \) for \( C_i(X, -i) \) which are just the cycles of dimension \( i \) modulo \( n \). The restriction map in the lowest degree \( \text{res} : Z_1(X)/n \to Z_0(X)/n \) agrees with the specialisation map on cycles defined by Fulton in [Fu, Rem. 2.3] since \( X_0 \) is a principle Cartier
divisor and $\partial_0\xi \cap \{ \xi = \pi \} = \text{ord}_{\Omega_{x_0}^{\text{rel}}} \pi$. Modifying $\xi \in \ker[Z_1(X)/n \rightarrow Z_0(X_0)/n]$ by elements equivalent to zero in $Z_1(X)/n$, we may represent it by an element $x \in \ker[Z_1(X) \rightarrow Z_0(X_0)]$.

We consider the following short exact sequence of sheaves:

$$0 \rightarrow \mathcal{O}_{X,0} \rightarrow \mathcal{M}_{X,0} \rightarrow \text{Div}(X, 0) \rightarrow 0,$$

where $\mathcal{M}_{X,0}$ (resp. $\mathcal{O}_{X,0}$) denotes the sheaf of invertible meromorphic functions (resp. invertible regular functions) relative to $\text{Spec}\mathcal{O}_K$ and congruent to 1 in the generic point of $X_0$, i.e. in $\mathcal{O}_{X,\mu}$, where $\mu$ is the generic point of $X_0$, and $\text{Div}(X, 0)$ is the sheaf associated to $\mathcal{M}_{X,0}/\mathcal{O}_{X,0}$. In other words, $\text{Div}(X, 0)(U)$ is the set of relative Cartier divisors on $U \subset X$ which specialise to zero in $X_0$. For the concept of relative meromorphic functions and divisors see [EGA], Sec. 20, 21.15.

We want to show that $(\text{Div}(X, 0)(X)/\mathcal{M}_{X,0}(X))/n = 0$.

**Claim 2.4.** $\text{Pic}(X, 0) \cong \text{Div}(X, 0)(X)/\mathcal{M}_{X,0}(X)$.

Short exact sequence (2.1) induces the following exact sequence:

$$\mathcal{O}_{X,0}(X) \rightarrow \mathcal{M}_{X,0}(X) \rightarrow \text{Div}(X, 0)(X) \rightarrow \text{Pic}(X, 0) \rightarrow H^1(X, \mathcal{M}_{X,0}(X)).$$

Now $\text{Pic}(X, 0) = H^1(X, \mathcal{O}_{X,0}(X))$ can also be described as the group of isomorphism classes of pairs $(\mathcal{L}, \psi)$ of an invertible sheaf $\mathcal{L}$ with a trivialisation $\psi : \mathcal{L}|_{X_0} \cong \mathcal{O}_{X_0}$ (see e.g. [SV], Lem. 2.1).

The following argument shows that the map $\text{Div}(X, 0)(X) \rightarrow \text{Pic}(X, 0)$ is surjective. Let $(\mathcal{L}, \psi) \in \text{Pic}(X, 0)$. The trivialisation $\psi$ gives an isomorphism $\psi : \mathcal{L} \otimes \mathcal{O}_X \mathcal{O}_{X_0} \cong \mathcal{O}_{X_0}$ and by localising an isomorphism $\psi_\mu : \mathcal{L}_\mu \otimes \mathcal{O}_{X_\mu} \mathcal{O}_{X_0,\mu} \cong \mathcal{O}_{X_0,\mu}$, where $\mu$ again denotes the generic point of $X_0$. Let $s$ denote a lift of $\psi_\mu^{-1}(1)$ under the surjective map $\mathcal{L}_\mu \rightarrow \mathcal{L}_\mu \otimes \mathcal{O}_{X_\mu} \mathcal{O}_{X_0,\mu}$. Then $s$ is a meromorphic section of $\mathcal{L}$ and the divisor $\text{div}(s) \in \text{Div}(X, 0)(X)$ maps to $(\mathcal{L}, \psi)$.

It follows that $\text{Pic}(X, 0) \cong \text{Div}(X, 0)(X)/\mathcal{M}_{X,0}(X)$. □

**Claim 2.5.** $\text{Pic}(X, 0)$ is uniquely $n$-divisible.

Since

$$\text{Pic}(X, 0) \cong \lim_{\leftarrow m} \text{Pic}(X_m, X_0) \cong \lim_{\leftarrow m} H^1(X_0, 1 + \pi \mathcal{O}_{X_m}),$$

where the first isomorphism follows from [EGA], Thm. 5.1.4, it suffices to show that $H^1(X_0, 1 + \pi \mathcal{O}_{X_m})$ is uniquely $n$-divisible. This can be seen as follows:

$$1 + \pi \mathcal{O}_{X_m} \supset 1 + \pi^2 \mathcal{O}_{X_m} \supset \ldots \supset 1$$

defines a finite filtration on the sheaf $1 + \pi \mathcal{O}_{X_m}$ with graded pieces $gr^n = (\pi)^n/(\pi)^{n+1} \cong \mathcal{O}_{X_0} \otimes (\pi)^n$. We use this filtration to define a filtration on $H^1(X_0, 1 + \pi \mathcal{O}_{X_m})$ by

$$F^n := \text{Im}(H^1(X_0, 1 + \pi^n \mathcal{O}_{X_m}) \rightarrow H^1(X_0, 1 + \pi \mathcal{O}_{X_m})).$$
The unique divisibility of $H^1(X_0, 1 + \pi\mathcal{O}_{X_0})$ follows now by descending induction from the exact sequence

$$0 \to 1 + \pi^{n+1}\mathcal{O}_{X_0} \to 1 + \pi^n\mathcal{O}_{X_0} \to gr^n \to 0,$$

the unique divisibility of $H^1(X_0, \mathcal{O}_{X_0} \otimes \pi^n)$ as a finitely generated $k$-module and the five-lemma.

It follows that Pic($X, X_0)/n \cong (\text{Div}(X, X_0)/\mathcal{M}_{X_0}^*(X))/n = 0$ and therefore that the class of $x$ in $Z_1(X)/n$, i.e. $\xi$, is in the image of $\ker[C_2(X, -1) \rightarrow \mathbb{C}_1(X_0, 0)]$ under $\partial$.

We now do the induction step for $X$ of arbitrary relative dimension $d > 1$ over $\text{Spec}\mathcal{O}_K$, assuming that the key lemma holds for relative dimension $d - 1$, using an idea of Bloch put forward in [EWB, App.]. By a standard norm argument we may from now on assume that $k$ is infinite.

As above we may represent $\xi$ by an element of $\ker[Z_1(X) \to Z_0(X_0)]$ and as in the proof of [KEW] Prop. 4.1 we may assume that $\xi$ is represented by a cycle of the form $[x] - r[y] \in \ker[Z_1(X) \to Z_0(X_0)]$ with $x$ and $y$ integral and such that $y$ is regular and has intersection number 1 with $X_0$. Let us recall the argument: First note that one can lift a reduced closed point of $X_0$ to an integral horizontal one-cycle having intersection number 1 with $X_0$. Now if $\xi = \sum_{i=1}^n n_ir_i[x_i] \in \ker[Z_1(X) \to Z_0(X_0)]$, then we lift $(x_i \cap X_0)_{\text{red}}$ to a one-cycle $y_i$ of the aforementioned type. Furthermore, we choose the same $y_i$ for all the $x_i$ intersecting $X_0$ in the same closed point. Let $r_i$ be the intersection multiplicity of $x_i$ with $X_0$. Then also $\sum_{i=1}^n n_ir_i[y_i] \in \ker[Z_1(X) \to Z_0(X_0)]$ and it suffices to show the statement for each $x_i - r_iy_i$ separately, i.e. the claim follows.

Let $\tilde{x}$ be the normalisation of $x$. Since $\mathcal{O}_K$ is excellent, $\tilde{x}$ is finite over $x$. This implies that there is an imbedding $\tilde{x} \hookrightarrow X' := X \times_{\text{Spec}\mathcal{O}_K} \mathbb{P}^N$ such that the following diagram commutes:

$$\begin{array}{ccc}
\tilde{x} & \longrightarrow & X' = X \times_{\text{Spec}\mathcal{O}_K} \mathbb{P}^N \\
\downarrow & & \downarrow \text{pr}_X \\
x & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}\mathcal{O}_K & \longrightarrow & \text{Spec}\mathcal{O}_K
\end{array}$$

Let $[\tilde{x} \cap X'_0] = r'[\tilde{z}]$ for $\tilde{z}$ an integral zero-dimensional subscheme of $X'_0$. We take a regular lift $z$ of $\tilde{z}$ in $y \times \mathbb{P}^N \subset X'$ which has intersection number 1 with $X'_0$ and get that $[\tilde{x}] - r'[z] \in \ker[Z_1(X') \to Z_0(X'_0)]$ and $\text{pr}_{X^*}([\tilde{x}] - r'[z]) = [x] - r[y] = \xi$.

We now use a Bertini theorem by Altman and Kleiman to prove key lemma 2.3 by an induction on the relative dimension of $X$ over $\mathcal{O}_K$.

**Lemma 2.6.** There exist smooth closed subschemes $Z, Z' \subset X'$ with the following properties:

1. $Z$ has fiber dimension one, $Z'$ has fiber dimension $d - 1$.
2. $Z$ contains $\tilde{x}$, $Z'$ contains $z$.
3. The intersection $Z \cap Z' \cap X'_0$ consists of reduced points.
Proof. First note that for a sheaf of ideals \( \mathcal{J} \subset \mathcal{O}_{X'} \) we have the following short exact sequence:

\[
0 \to \mathcal{J} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-[X_0]) (M) \to \mathcal{J} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(M) \to \mathcal{J} \otimes_{\mathcal{O}_{X'}} i_* \mathcal{O}_{X_0}(M) \to 0
\]

for \( i : X_0' \hookrightarrow X' \) and \( M \in \mathbb{Z} \). For \( M \gg 0 \) Serre vanishing implies that \( H^1(X', \mathcal{F}(M)) = 0 \) for \( \mathcal{F} \) coherent and therefore that the map

\[
\Gamma(\mathcal{J} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(M)) \to \Gamma(\mathcal{J} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X_0}(M))
\]

is surjective. This allows us to lift the sections on the right defining subvarieties of \( X_0 \) to sections of a twisted sheaf of ideals on \( X' \).

Let \( \mathcal{J}_\xi \) be the sheaf of ideals defining \( \tilde{x} \) and \( \mathcal{J}_z \) be the sheaf of ideals defining \( z \). Let \( p \in \tilde{x} \cap X_0' \) (\( q \in z \cap X_0' \)). Then \( \dim_{\mathcal{O}_X}(p) = d \geq 2 \) and since \( \tilde{x} \) (resp. \( z \)) is regular, we have that \( e_{\tilde{x} \cap X_0'}(p) \leq e_{\tilde{x}}(p) = \dim_{\mathcal{O}_{\tilde{x}}(p)}(\Omega_{\tilde{x}}(p)) = 1 < 2 \), where \( e_{\tilde{x}}(p) \) is the embedding dimension of \( \tilde{x} \) at \( p \) and analogously for \( q \). Therefore by [AK] Thm. 7 we can find sections in \( \sigma_1, ..., \sigma_{d+N-1} \in \mathcal{J}_\xi |_{X_0'}(M) \) (resp. \( \sigma' \in \mathcal{J}_z |_{X_0'}(M) \)) defining smooth subschemes containing \( p \) (resp. \( q \)) that intersect transversally. Let \( \sigma_1, ..., \sigma_{d+N-1} \) (resp. \( \sigma' \)) be liftings under the surjections \( \Gamma(\mathcal{J}_\xi \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(M)) \to \Gamma(\mathcal{J}_\xi \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X_0'}(M)) \) and \( \Gamma(\mathcal{J}_z \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(M)) \to \Gamma(\mathcal{J}_z \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X_0'}(M)) \). Then the complete intersections \( Z := V(\sigma_1, ..., \sigma_{d+N-1}) \) and \( Z' := V(\sigma') \) have the desired properties. \( \square \)

Using these subschemes, we can now do the induction step and finish the proof of the key lemma. Since \( Z \cap Z' \cap X_0' \) consists of reduced points, the component \( Z' \cap Z \) of \( Z \cap Z' \) that contains \( z \cap X_0' \) has intersection number 1 with \( X_0' \) and is a regular curve as it is regular over the closed point of \( \text{Spec} \mathcal{O}_K \). Now since \( Z' \) is of relative dimension \( d \) and \( z \) and \( z' \) both lie in \( Z' \) and satisfy \( \text{res}([z'] - [z]) = 0 \), we get by the induction assumption that there is a \( \xi \) with support on \( Z' \) restricting to 1 and with \( \partial(\xi) = [z'] - [z] \).

By the relative dimension one case proved in the beginning we get that for \( \tilde{x}, z' \subset Z \) and \( [\tilde{x}] - r'[z'] \), which also restricts to 0, there is a \( \xi' \) with support on \( Z \) such that \( \text{res}(\xi') = 0 \) and \( \partial(\xi') = [\tilde{x}] - r'[z'] \). It follows that \( \text{res}(\xi' + r'\xi) = 1 \) and \( \partial(\xi' + r'\xi) = [\tilde{x}] - r'[z'] \).

By the commutativity of the following diagram we get the result.

\[
\begin{array}{ccc}
C_2(X', -1) & \rightarrow & C_1(X_0', 0) \\
\downarrow & & \downarrow \\
Z_1(X')/n & \rightarrow & Z_0(X_0')/n \\
\downarrow & & \downarrow \\
C_2(X, -1) & \rightarrow & C_1(X_0, 0) \\
\downarrow & & \downarrow \\
Z_1(X)/n & \rightarrow & Z_0(X)/n
\end{array}
\]

The commutativity of the diagram follows from [Ro] Sec. 4] since all the maps in question are defined in terms of the 'four basic maps' which are compatible. \( \square \)
Corollary 2.7. The restriction map
\[ \text{res}^{CH} : CH^d(X, 1)_\Lambda \to CH^d(X_0, 1)_\Lambda \]
dondefined in the introduction is surjective.

Proof. We first show that the homology of the sequence
\[ \bigoplus_{x \in X^{(d-2)}} K^M_2(k(x)) \to \bigoplus_{x \in X^{(d-1)}} K^M_1(k(x)) \to \bigoplus_{x \in X^{(d)}} K^M_0(k(x)) \]
is isomorphic to \( CH^d(X_0, 1) \) which implies that \( A_1(X_0, 0) \cong CH^d(X_0, 1)_\Lambda \). This follows from the spectral sequence
\[ E^{p,q}_1 = \bigoplus_{x \in X_0^{(p)}} CH^{r-p}(\text{Spec}k(x), -p-q) \Rightarrow CH^r(X_0, -p-q) \]
(see [B1, Sec. 10]) for \( r = d = \dim X_0 \), the fact that \( CH^r(k(x), r) \cong K^M_p(k(x)) \) and the vanishing of \( CH^r(\text{Spec}k(x), j) \) for \( r > j \).

Using a limit argument and the localization sequence for schemes over a regular noetherian base \( B \) of dimension one constructed in [Ł1], we also get the existence of spectral sequence (2.2) for \( X/\mathcal{O}_K \). Now for the same reasons as above this spectral sequence implies that the homology of
\[ \bigoplus_{x \in X^{(d-2)}} K^M_2(k(x)) \to \bigoplus_{x \in X^{(d-1)}} K^M_1(k(x)) \to \bigoplus_{x \in X^{(d)}} K^M_0(k(x)) \]
is isomorphic to \( CH^d(X, 1) \) which implies that \( A_2(X, -1) \cong CH^d(X, 1)_\Lambda \).

The result now follows from proposition 2.2 and the compatibility of \( \text{res} \) and \( \text{res}^{CH} \). \( \square \)

Remark 2.8. The isomorphism \( A_1(X_0, 0) \cong CH^d(X_0, 1)_\Lambda \) also follows from the isomorphism \( CH(X, 1) \cong H^{p-1}(X, \mathcal{K}_p) \) for \( p \geq 0 \) and \( \mathcal{K}_p \) the K-theory sheaf (see f.e. [M Cor. 5.3]).

3. Remarks on the injectivity of \( \text{res} \)

In this section we prove the injectivity of the restriction map for \( d = 2 \) in our case and remark on implications of the conjectured injectivity.

Conjecture 3.1. The map \( \text{res} : A_2(X, -1) \to A_1(X_0, 0) \) is injective.

Proposition 3.2. Conjecture 3.1 holds for \( X/\mathcal{O}_K \) of relative dimension 2.

Proof. Let \( \Lambda := \mathbb{Z}/n \) and \( \Lambda(q) := \mu_n^{\otimes q} \). We use the coniveau spectral sequence
\[ E^{p,q}_1(X, \Lambda(c)) = \prod_{x \in X^p} H^{p+q}_x(X, \Lambda(c)) \Rightarrow H^{p+q}_{\text{ét}}(X, \Lambda(c)), \]
where \( H^*_x \) is étale cohomology with support in \( x \).

Cohomological purity (respectively absolute purity) gives isomorphisms \( H^{p+q}_x(X, \Lambda(c)) \cong H^{q-p}(k(x), \Lambda(c-p)) \) which lets us write the above spectral sequence in the following form:
\[ E^{p,q}_1(X, \Lambda(c)) = \prod_{x \in X^p} H^{q-p}(k(x), \Lambda(c-p)) \Rightarrow H^{p+q}_{\text{ét}}(X, \Lambda(c)). \]
For more details see for example [CHK]. Writing out this spectral sequence for \( X \) and \( X_0 \) respectively and using the norm residue isomorphism \( K^M_n(k)/m \cong H^n(k, \mu_m^{\otimes n}) \) for \( n \leq 2 \)
(see [MS]), we get injective edge morphisms $A_2(X, -1) \hookrightarrow H^3_{\text{ét}}(X, \Lambda(2))$ and $A_1(X_0, -1) \hookrightarrow H^3_{\text{ét}}(X_0, \Lambda(2))$ for dimensional reasons. The restriction map induces a map between these spectral sequences and therefore a commutative diagram

$$
\begin{array}{ccc}
A_2(X, -1) & \rightarrow & A_1(X_0, 0) \\
\downarrow & & \downarrow \\
H^3_{\text{ét}}(X, \Lambda(2)) & \cong & H^3_{\text{ét}}(X_0, \Lambda(2))
\end{array}
$$

whose lower horizontal morphism is an isomorphism by proper base change. It follows that $A_2(X, -1) \rightarrow A_1(X_0, 0)$ is injective.

\begin{remark}
The injectivity of $res$ would have implications for a finiteness conjecture on the $n$-torsion of $CH_0(X_K)$ for $X_K$ a smooth scheme over a $p$-adic field with finite residue field and good reduction (see for example [Co1]). More precisely, using the coniveau spectral sequence, we can see that the group $A_1(X_K, 0)$ is isomorphic to $H^{2d-1}_{\text{Zar}}(X_K, \mathbb{Z}/n(d))$ and therefore surjects onto $CH_0(X_K)[n]$. Furthermore it fits into the exact sequence (see [Ro], Sec. 5)

$$
A_2(X, -1) \rightarrow A_1(X_K, 0) \rightarrow A_1(X_0, -1) \cong CH_1(X_0)/n.
$$

Now conjecture 3.1 implies that there is a sequence of injections $A_2(X, -1) \hookrightarrow A_1(X_0, 0) \hookrightarrow H^{2d-1}_{\text{ét}}(X_0, \mathbb{Z}/n(d))$ into the finite group $H^{2d-1}_{\text{ét}}(X_0, \mathbb{Z}/n(d))$. Note that the second injection follows from the Kato conjectures. More precisely, there is an exact sequence

$$
KH_3(X_0, \mathbb{Z}/n\mathbb{Z}) \rightarrow A_1(X_0, 0) \cong CH^d(X_0, 1) \rightarrow H^{2d-1}_{\text{ét}}(X_0, \mathbb{Z}/n(d))
$$

(see [JS], Lem. 6.2) and the Kato homology group $KH_3(X_0, \mathbb{Z}/n\mathbb{Z})$ is zero due to the Kato conjectures (see [KS]). Therefore the finiteness of $CH_0(X_K)[n]$ would depend on the finiteness of $CH_1(X_0)/n$.

In the case of relative dimension 2 the finiteness of $CH_1(X_0)/n \cong \text{Pic}(X_0)/n$ can be shown using the injection $\text{Pic}(X_0)/n \hookrightarrow H_2^2(X_0, \mu_n)$ and the finiteness of $H_2^2(X_0, \mu_n)$ (see f.e. [Mi], VI.2.8]). Therefore proposition 3.2 implies in particular the finiteness of $CH_0(X_K)[n]$ for $X_K$ a smooth surface over a $p$-adic field with finite residue field and good reduction which is a well-known result by Bloch (see f.e. [Co2], Thm. 3.3.2).

\begin{remark}
In the light of remark 3.3 and the base change conjecture for higher zero-cycles stated in the introduction one might ask if

$$
CH^d(X_K, i)[n]
$$

is finite for all $i \geq 0$ for smooth schemes over $p$-adic fields.

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