1. Introduction

A variety of results in complex geometry and mathematical physics depend upon the analysis of holomorphic sections of high powers $L^\otimes N$ of holomorphic line bundles $L \to M$ over compact Kähler manifolds ([A] [Bis] [Bis.V] [Bou.1] [Bou.2] [B.G] [D] [Don] [G] [G.S] [K] [Ji] [T] [W]). The principal tools have been Hörmander’s $L^2$-estimate on the $\bar{\partial}$-operator over $M$ [T], the asymptotics of heat kernels $k_N(t,x,y)$ for associated Laplacians [Bis] [Bis.V] [D] [Bou.1] [Bou.2] [G], the method of stationary phase for formal functional integrals [A] [W] and the microlocal analysis of Szegö and Bergman kernels [B.F.G] [B.S] [B.G].

In this note we wish to apply the latter methods, specifically the Boutet de Monvel-Sjöstrand parametrix for the Szegö kernel, to a problem in complex geometry. Our purpose is to prove the following theorem:

**Theorem 1.** Let $M$ be a compact complex manifold of dimension $n$ (over $\mathbb{C}$) and let $(L,h) \to M$ be a positive hermitian holomorphic line bundle. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g := \text{Ric}(h)$. For each $N \in \mathbb{N}$, $h$ induces a hermitian metric $h_N$ on $L^\otimes N$. Let $\{S^0_N, \ldots, S^d_N\}$ be any orthonormal basis of $H^0(M, L^\otimes N)$, with respect to the inner product $\langle s_1, s_2 \rangle_{h_N} = \int_M h_N(s_1(z), s_2(z)) dV_g$. Here, $dV_g = \frac{1}{n!} \omega^n_g$ is the volume form of $g$. Then there exists a complete asymptotic expansion:

$$\sum_{i=0}^{d_N} ||S^i_N(z)||^2_{h_N} = a_0 N^n + a_1(z) N^{n-1} + a_2(z) N^{n-2} + \ldots$$

for certain smooth coefficients $a_j(z)$ with $a_0 = 1$. More precisely, for any $k$

$$\left| \sum_{i=0}^{d_N} ||S^i_N(z)||^2_{h_N} - \sum_{j<k} a_j(z) N^{n-j} \right| C^k \leq C_{R,k} N^{n-R}.$$

Above, $\text{Ric}(h)$ is the Ricci curvature of $h$, given locally by $\frac{-\pi}{2^2} \bar{\partial} \partial a$ where $a = ||e_L||_h$ is the positive function locally representing $h$ in a local holomorphic frame $e_L$.

This theorem has a number of corollaries. First it implies that for sufficiently large $N$, there are no common zeroes of the sections $\{S^0_N, \ldots, S^d_N\}$. Hence one can define the holomorphic map

$$\phi_N : M \to \mathbb{C}P^{d_N}, \quad z \mapsto [S^0_N(z), \ldots, S^d_N(z)]$$ (1)

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where \([S^N_0(z), \ldots, S^N_d(z)]\) denotes the line thru \((S^N_0(z), \ldots, S^N_d(z))\) as defined in a local holomorphic frame. Since all the components transform by the same scalar under a change of frame, the line is well-defined. As is well-known \([3,4]\), \(\phi_N\) is equivalent to the invariantly defined map

\[
\tilde{\phi}_N : M \to PH^0(M, L^\otimes N)^*, \quad z \mapsto H_z := \{ s \in H^0(M, L^\otimes N) : s(z) = 0\}. \tag{2}
\]

Secondly it gives an asymptotic formula for the distortion function between the metrics \(h_N\) and \(h_{FS,N}\), where \(h_{FS,N}\) is the Fubini-Study metric on \(L^\otimes N\) induced by \(\tilde{\phi}_N\). The following result was simultaneously proved in the case of abelian varieties by G.Kempf \([K]\) and S.Ji \([J]\) (with \(C^0\) convergence) and for general projective varieties (with \(C^4\) convergence) by G.Tian \([T]\), Lemma 3.2(i)). Heat kernel proofs were later found by T. Bouche \([B1, B2]\) and J.P. Demailly \([D]\).

**Corollary 2.** Let \(G\) be any Riemannian metric on \(M\), endow \(H^0(M, L^\otimes N)\) with the Hermitian inner product induced by \((G,h_N)\) and define the map \(\tilde{\phi}_N\) as above. Identify \(L^\otimes N\) with the pull-back \(\tilde{\phi}_N^*O(1)\) of the hyperplane bundle \(O(1) \to PH^0(M, L^\otimes N)\), and let \(h_{FS,N}\) be the pullback of the standard Hermitian metric on \(O(1)\). Then

\[
\frac{h_{N,z}}{h_{FS,N,z}} = (\frac{N}{2\pi})^n |\alpha_1(z) \ldots \alpha_n(z)| + O(N^{-n-1}).
\]

where \(\alpha_1(z), \ldots, \alpha_n(z)\) are the eigenvalues \(\text{Ric}(h)\) with respect to \(G\).

If we take the background metric \(G\) to be the Kähler metric associated to the Kähler form \(\omega_g = \text{Ric}(h)\), then the curvature eigenvalues are all equal to one.

Third, it implies:

**Corollary 3.** Let \(\omega_{FS}\) denote the Fubini-Study form on \(G\mathbb{P}^d\). Then:

\[
|| \frac{1}{N} \phi_N^*(\omega_{FS}) - \omega_g ||_{C^k} = O\left(\frac{1}{N}\right)
\]

for any \(k\).

This statement for \(k \leq 2\) was the principal result of Tian \([T]\), Theorem A). The map \(\tilde{\phi}_N\) depends on the choice of \(\{S^N_0, \ldots, S^N_d\}\) but it is easily seen that \(\phi_N^*(\omega_{FS})\) does not. This result follows formally from the preceding one by taking the curvature of both sides of the asymptotic formula.

Our result strengthens the previous ones in two ways: First, it shows that the convergence of \(\frac{1}{N} \phi_N^*(\omega_{FS}) \to \omega_g\) takes place in the \(C^\infty\)-topology and not just in \(C^2\). This was conjectured in \([T]\). Second, it shows that this convergence is just the first term of a complete asymptotic expansion. If only the principal terms are desired, then the proof could be simplified further: as in \([2]\), which treats an analogous problem in the context of Zoll manifolds, one could obtain the principal terms by the symbol calculus of Toeplitz operators. But we believe that the lower order terms should also be of interest.

The proof begins by expressing the maps \(\phi_N\) and \(\tilde{\phi}_N\) in terms of an associated equivariant map \(\Phi_N\) on the unit circle bundle \(X\) of the dual line bundle \(L^*\) with respect to the induced metric \(h\). Roughly, this converts the holomorphic geometry of \(L\) to the CR geometry of \(X\). Since \(X\) is the boundary of the strictly pseudo-convex domain \(D = \{ v \in L^* : |v|_h < 1 \} \subset L^*\) it has a Szegő kernel \(\Pi(x,y)\) which projects \(L^2(X)\) to
the Hardy space $H^2(X)$ of boundary values of holomorphic functions in $D$. Under the natural $S^1$ action of $X \to M$, $H^2(X)$ splits up into weight spaces $H^2_N(X)$ and one has a canonical isomorphism $s \mapsto \tilde{s} : H^0(M, L^\otimes N) \to H^2_N(X)$, where we write a point of $X$ as $(z, u), u \in L^*_z, |u|_h = 1$ and where $\langle \cdot , \cdot \rangle$ is the pairing between $L$ and $L^*$. So the basis $\{S^N_0, \ldots , S^N_{d_N}\}$ of $H^0(M, L^\otimes N)$ corresponds to an orthonormal basis $\{\tilde{S}^N_0, \ldots , \tilde{S}^N_{d_N}\}$ of $H^2_N(X)$. One then gets an associated CR map of $X$ into $H^2_N(X)^*$. Expressed invariantly in terms of the orthogonal projection $\Pi_N$ onto $H^2_N(X)$ it is defined by:

$$\Phi_N(x) = \Pi_N(x, \cdot) : X \to H^2_N(X)^*.$$  
(3)

Using the canonical isomorphism above we get an essentially equivalent map $\Phi_N : X \to H^0(M, L^\otimes N)^*$. (We note that these maps are well-defined even when the set $Z_N$ of common zeroes of the sections is non-empty.) Thus for $N \gg 0$ we get the diagram:

$$\begin{array}{ccc}
X & \Phi_N & H^0(M, L^\otimes N)^* \\
\pi \downarrow & \downarrow & \downarrow \\
M & \Phi_N & PH^0(M, L^\otimes N)^*
\end{array}$$
(4)

where $\pi$ and $\rho$ are the canonical projections. The top arrow is well-defined for all $N$.

After unravelling the identifications, we find (§1) that $\sum_{i=0}^d S^N_i(z) = \Pi_N(x, x)$ and that $\nabla_{S^N_i} \omega_{FS} = \omega_g + \frac{i}{2} \partial \bar{\partial} \log \Pi_N(x, x)$ for any $x$ with $\pi(x) = z$. The second term on the right side is an $S^1$ invariant form so we have identified it with a form on $M$. The theorem is therefore equivalent to the statement that $\Pi_N(x, x)$ has a complete asymptotic expansion as $N \to \infty$ which can be differentiated any number of times. This will follow by applying the method of stationary phase to Boutet de Monvel-Sjöstrand’s parametrix for the Szegö projector $\Pi(x, y)$ (see §3).

The analysis of Szegö kernels should have other applications in complex geometry. By studying the off-diagonal of $\Pi_N(x, y)$ one can show that the maps $\Phi_N$ are embeddings, thus obtaining an analytic proof of the Kodaira embedding theorem. In a forthcoming paper [S.Z], B.Shiffman and the author also use the Szegö kernels to show (among other things) that the zeroes of a ‘random section’ of $H^0(M, L^\otimes N)$ become uniformly distributed as $N \to \infty$. There are also some potential analogues in the almost complex setting. In a recent paper [B.U], Borthwick-Uribe conjecture some results on Szegö kernels in the almost complex setting which seem very close to what is proved here in the complex setting. In part their motivation (as well as ours) was to reinterpret some constructions of Donaldson [D] from the viewpoint of semiclassical analysis.

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## 2. From line bundle to circle bundle

The purpose of this section is to convert the statements of Theorem 1 and Corollaries 2 - 3 into statements about $\Pi_N$. Before doing so let us recall why one exists in this context and establish some notation.
2.1. The CR setting. Let $O(1) \to \mathbb{CP}^n$ denote the hyperplane section line bundle and let $\langle \cdot, \cdot \rangle$ denote its natural Hermitian metric. Let $M \subset \mathbb{CP}^n$ be a non-singular projective variety, let $L$ denote the restriction of $O(1)$ to $M$ and let $h$ denote the restriction of $\langle \cdot, \cdot \rangle$ to $L$. The following proposition is well-known (it was originally observed by Grauert in the 50’s):

**Proposition 4.** Let $D = \{(m, v) \in L^* : h(v, v) \leq 1\}$. Then $D$ is a strictly pseudoconvex domain in $L$.

Here $L^*$ is the dual line bundle to $L$. The boundary of $D$ is a principal $S^1$ bundle $X \to M$ whose defining function is given by

$$\rho : L^* \to \mathbb{R}, \quad \rho(z, \nu) = 1 - |\nu|^2$$

where $\nu \in L^*_z$ and where $|
u|_z$ is its norm in the metric induced by $h$. That is, $D = \{\rho > 0\}$.

In a local coframe $e^i_z$ over $U \subset M$ we may write $\nu = \lambda e^i_z$ and then $|\nu|^2_z = a(z)|\lambda|^2$ where $a(z) = |e^i_z|^2_z$ is a positive smooth function on $U$. Thus in local holomorphic coordinates $(z, \lambda)$ on $L^*$ the defining function is given by $\rho = 1 - a(z)|\lambda|^2$. We will denote the $S^1$ action by $r_\theta x$ and its infinitesimal generator by $\frac{\partial}{\partial \theta}$. We note that $\rho$ is $S^1$-invariant.

Let us denote by $T'D, T''D \subset TD \otimes \mathbb{C}$ the holomorphic, resp. anti-holomorphic subspaces and define $d'f = df|_{T'}$, $d''f = df|_{T''}$ for $f \in C^\infty(D)$. Then $X$ inherits a CR structure $TX \otimes \mathbb{C} = T' \oplus T'' \oplus \mathbb{C} \mathbb{R}$. Here $T'X$ (resp. $T''X$) denotes the holomorphic (resp. anti-holomorphic vectors) of $D$ which are tangent to $X$. They are given in local coordinates by vector fields $\sum a_j \frac{\partial}{\partial z_j}$ such that $\sum a_j \frac{\partial}{\partial z_j} \rho = 0$. A local basis is given by the vector fields $Z_j^k = \frac{\partial}{\partial z_j} - (\frac{\partial}{\partial x_k})^{-1} (\frac{\partial}{\partial x_j}) \frac{\partial}{\partial x_k}$ ($j \neq k$).

The Cauchy-Riemann operator on $X$ is defined by

$$\bar{\partial}_h : C^\infty(X) \to C^\infty(X, (T'')^*), \quad \bar{\partial}_h f = df|_{T''}.$$\hspace{1cm} (6)

In terms of the local basis above, it is given by

$$\bar{\partial}_h f = \sum_{j \neq k} Z_j^k f d\bar{z}_j|_{T''}.$$\hspace{1cm} (7)

Also associated to $X$ are

- the contact form $\alpha = \frac{i}{n} d'|_X = -\frac{i}{n} d''|_X$
- the volume form $d\mu = \alpha \wedge (d\alpha)^n$
- the Levi form $L_\rho(z) = \sum \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} z_j \bar{z}_k.$ \hspace{1cm} (8)
- the Levi form on $X$ $L_X = L_\rho|_{T' \oplus T'' \cap TX}$

which are independent of the choice of $\rho$. The Levi form on $X$ is related to $d\alpha = \pi^* \omega_g$ by: $L_X(V, W) = d\alpha(V, \bar{W})$. Since $\omega_g$ is Kähler, $D$ is a strictly pseudoconvex domain.

The Hardy space $H^2(X)$ is the space of boundary values of holomorphic functions on $D$ which are in $L^2(X)$, or equivalently $H^2 = (\ker \bar{\partial}_h) \cap L^2(X)$. The $S^1$ action commutes with $\bar{\partial}_h$, hence $H^2(X) = \oplus_{N=1}^\infty H^2_N(X)$ where $H^2_N(X) = \{ f \in H^2(X) : f(r_\theta x) = e^{iN\theta} f(x)\}$.
A section $s$ of $L$ determines an equivariant function $\hat{s}$ on $L^* - 0$ by the rule: $\hat{s}(z, \lambda) = (\lambda, s(z))$ ( $z \in M, \lambda \in L^*_z$). It is clear that if $r \in \mathfrak{g}^*$ then $\hat{s}(z, r\lambda) = r\hat{s}$. We will usually restrict $\hat{s}$ to $X$ and then the equivariance property takes the form: $\hat{s}(r_\theta x) = e^{i\theta} \hat{s}(x)$. Similarly, a section $s$ of $L^\otimes N$ determines an equivariant function $\hat{s}_N$ on $L^* - 0$: put $\hat{s}_N(z, \lambda) = (\lambda^{\otimes N}, s_N(z))$ where $\lambda^{\otimes N} = \lambda \otimes \lambda \otimes \cdots \otimes \lambda$. The following proposition is well-known:

**Proposition 5.** The map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^\otimes N)$ and $H^0_N(X)$.

As above, we let $\Pi_N : L^2(X) \to H^2_N(X)$ denote the orthogonal projection. Its kernel is defined by

$$\Pi_N f(x) = \int_X \Pi_N(x, y) f(y) d\mu(y). \quad (9)$$

This definition differs from that of [B.S] in using $d\mu$ as the reference density.

### 2.2. Line bundles and maps to projective space

Since the definitions of the various maps $\phi_N, \hat{\phi}_N, \Phi_N, \tilde{\Phi}_N$ involve some identifications, we pause to recall some basic facts about maps to projective space [G.H., I.4].

Let $E \to M$ denote a holomorphic line bundle. Since we are interested in $E = L^\otimes N$ for large $N$ we may assume that not all sections $s \in H^0(M, E)$ vanish at any point $z \in M$. Then the space of sections vanishing at $z$ forms a hyperplane $H_z$ in $H^0(M, E)$ and one can define a map $\iota_E : M \to \mathbf{P}(H^0(M, E))^*$ by $z \mapsto H_z$. Here $\mathbf{P}(H^0(M, E))^*$ denotes the dual projective space of linear functionals on $H^0(M, E)$ modulo scalar multiplication.

Now equip $E$ with a Hermitian metric $h$ and $M$ with a volume form, and let $\langle \cdot, \cdot \rangle$ denote the induced inner product on $H^0(M, E)$. Then choose an orthonormal basis $\{s_0, \ldots, s_m\}$ of $H^0(M, E)$ with respect to $\langle \cdot, \cdot \rangle$. Also, choose a local holomorphic frame $e_E$ and write $s_j = f_j e_E$. Then the point of $\mathbf{P}^m$ with homogeneous coordinates $[f_0(z), \ldots, f_m(z)]$ is independent of $e_E$ and defines a map $\phi_E : M \to \mathbf{CP}^m$. The same basis also gives an identification $\mathbf{P}^m = \mathbf{P}(H^0(M, E))^*$ by writing a linear functional in the dual basis $\{s_0^*, \ldots, s_m^*\}$ of $H^0(M, E)^*$. We observe that under this identification, $\phi_E \equiv \iota_E$ for $H_z = \{\sum_j a_j s_j : \sum_j a_j f_j(z) = 0\} = \ker (\sum_j f_j(z) s_j^*) \leftrightarrow [f_0(z), \ldots, f_m(z)]$.

Next, recall that the Kähler form $\omega_{FS}$ of the Fubini-Study metric $g_{FS}$ on $\mathbf{CP}^m$ is given in homogeneous coordinates $[w_0, \ldots, w_m]$ by $\omega_{FS} = \frac{1}{2\pi} \partial \bar{\partial} \log(\sum_{j=0}^m |w_j|^2)$. Hence

$$\phi_E^* \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\sum_{j=0}^m |f_j|^2). \quad (10)$$

It is easy to see that this form is independent of the choice of orthonormal basis.

### 2.3. The maps $\phi_N, \hat{\phi}_N, \Phi_N, \tilde{\Phi}_N$

Now let us return to our setting. We fix a local holomorphic section $e_L$ of $L$ over $U \subset M$. It induces sections $e_L^N$ of $L^\otimes N|_U$ and we write $S_N^1(z) = f^N_1(z) e_L^N(z)$ for a holomorphic function $f^N_1$ on $U$. By the above we have:

$$\phi_N^* (\omega_{FS}) = \phi_N^* (\omega_{FS}) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\sum_{j=0}^d |f^N_j|^2). \quad (11)$$
The definition is independent of the choice of \( e_L^N \).

Since \( S_i^N \) is a holomorphic section of \( L^\otimes N \), \( \hat{S}_i^N \) is an equivariant CR function of level \( N \), i.e. \( \hat{S}_i^N \in H_2^N(X) \). We will need a series of formulae relating expressions in \( S_i^N \) to expressions in \( \hat{S}_i^N \). Let us introduce local coordinates \((z, \theta)\) on \( X \) on the domain \( U \) of the unitary frame \( \left\{ \frac{\partial}{\partial z} \right\} \) by \((z, \theta) \mapsto (z, \rho \frac{\partial}{\partial \rho}) \).

**Proposition 6.** \( \|S_i^N(z)\|_{h_N}^2 = |\hat{S}_i^N(x)|^2 \) for any \( x \) with \( \pi(x) = z \).

**Proof:**

By definition,
\[
\hat{S}_i^N(z, u) = \langle u^N, S_i^N(z) \rangle = f_i^N(z) \langle u^N, e_L^N(z) \rangle = f_i^N(z) a(\hat{S}_i^N(z) \langle u, e_L^N(z) \rangle)^N.
\]

(12)

In the above local coordinates, we get \( \hat{S}_i^N(z, \theta) = f_i^N(z)a(\theta)\hat{\omega} e^{iN\theta} \). Hence \( |\hat{S}_i^N(z, \theta)|^2 = (z)^N \langle f_i^N(z) \rangle^2 \). This obviously equals \( |\hat{S}_i^N(z)|^2 \|\hat{S}_i^N\|_{h_N}^2 \).

**Proposition 7.** \( \{\hat{S}_i^N\} \) is an orthonormal basis of \( H_2^N(X) \).

**Proof:**

Let \( dV_g = \omega_g^n \) be the volume form of \((M, g)\). Then we have:
\[
\langle S_\alpha^N, S_\beta^N \rangle := \int_M h_N(S_\alpha^N, S_\beta^N) dV_g = \int_M a^N(z) f_j^N(z) \tilde{f}_j^N(z) dV_g
\]

where \( d\mu = \alpha \wedge d\alpha \). In the latter step we use that \( \alpha \wedge d\alpha = d\theta \wedge \pi^* \omega_g \). This follows from the fact \( d\alpha = \pi^* \omega \) and that \( \alpha = d\theta + \eta \) where \( \eta \) only involves \( dz, \bar{\bar{z}} \).

We further have:

**Proposition 8.** \( \frac{1}{N} \varphi_N \omega_{FS} = \omega_g + \frac{iN}{2\pi N} \partial \bar{\partial} \log(\sum_{j=0}^{d_N} |S_j^N|^2) = \frac{1}{2\pi N} \partial \bar{\partial} \log(\sum_{j=0}^{d_N} |\hat{S}_j^N|^2) + \omega_g \).

**Proof:**

The first statement follows by writing \( \|S_j^N(z)\|_{h_N}^2 = a_j^N(z) |f_j^N(z)| \) and using that \( \frac{1}{2\pi N} \partial \bar{\partial} \log a_j^N(z) = \omega_g \). To prove the second statement, we note that \( \sum_{j=0}^{d_N} |\hat{S}_j^N|^2 \) and \( \partial \bar{\partial} \log(\sum_{j=0}^{d_N} |\hat{S}_j^N|^2) \) are \( S^1 \)-invariant and hence may be identified with functions on \( M \). In the latter case, this uses the fact that the \( S^1 \) action commutes with \( \partial \bar{\partial} \), i.e. acts by CR automorphisms. The statement then follows from the general fact that \( \pi_\ast \partial \bar{\partial} \pi^* f = \partial \bar{\partial} f \) for any \( f \in C^\infty(M) \), where \( \pi_\ast F \) denotes the function on \( M \) corresponding to an \( S^1 \)-invariant function \( F \) on \( X \).

Now let us rewrite these relations in terms of the Szegö projectors \( \Pi_N \).

**Proposition 9.** \( \frac{1}{N} \varphi_N \omega_{FS} = \omega_g + \frac{iN}{2\pi N} \partial \bar{\partial} \log \Pi_N(x, x) \).

**Proof:**
We first observe that
\[ \Pi_N(x, y) = \sum_{i=0}^{d_N} \hat{S}_N^i(x) \hat{S}_N^*(y) \] (13)
or, in local coordinates,
\[ \Pi_N(z, \theta, w, \theta') = a(z) \overline{a(w)} N e^{iN(\theta - \theta')} \sum_{i=0}^{d_N} f_i^N(z) \overline{f_i^N(w)}. \] (14)
Hence we have
\[ \sum_{d_N} ||S_j^N(z)||^2_{h_N} = \Pi_N(z, 0, z, 0) \] (15)
Together with Proposition 8 this completes the proof. \[ \square \]

Corollary 10. The statement of Corollary 2 is equivalent to:
\[ ||\partial_b \overline{\partial_b} \log \Pi_N(x, x)||_{C^k} = O(1). \]

3. Parametrix for the Cauchy-Szegö kernel

Now we recall the necessary background on the Szegö kernel \( \Pi(x, y) \) for a strictly pseudoconvex domain. The following theorem states that it is a Fourier integral operator of complex type, or more precisely a Toeplitz operator in the sense of Boutet de Monvel-Guillemin [B.G]. The notation below differs from [B.S] in that \( n + 1 = \dim \mathbb{C}D \).

Theorem 11. [B.S] Theorem 1.5 and \( \S 2.c \). Let \( \Pi(x, y) \) be the Szegö kernel of the boundary \( X \) of a bounded strictly pseudo-convex domain \( \Omega \) in a complex manifold \( L \). Then there exists a symbol \( s \in S^n(X \times X \times \mathbb{R}^+) \) of the type
\[ s(x, y, t) \sim \sum_{k=0}^{\infty} t^{n-k}s_k(x, y) \]
so that
\[ \Pi(x, y) = \int_{0}^{\infty} e^{it\psi(x, y)} s(x, y, t) dt \]
where the phase \( \psi \in C^\infty(D \times D) \) is determined by the following properties:

- \( \psi(x, x) = \frac{1}{i} \rho(x) \) where \( \rho \) is the defining function of \( X \).
- \( d_x^* \psi \) and \( d_y^* \psi \) vanish to infinite order along the diagonal.
- \( \psi(x, y) = -\overline{\psi(y, x)} \).

More precisely, the phase is determined up to a function which vanishes to infinite order at \( x = y \). The integrals are regularized by taking the principal value (see [B.S]). The second condition states that \( \psi(x, y) \) is almost analytic. Roughly speaking, \( \psi \) is obtained by Taylor expanding \( \rho(z, \bar{z}) \) and replacing all the \( \bar{z} \)'s by \( \bar{w} \)'s. More precisely, the Taylor expansion of \( \psi \) near the diagonal is given by
\[ \psi(x + h, x + k) = \frac{1}{i} \sum \frac{\partial^{\alpha + \beta} \rho}{\partial z^\alpha \partial \bar{z}^\beta}(x) \frac{h^\alpha k^\beta}{\alpha! \beta!}. \]
We note that Theorem 1.5 of [B.S] is stated only in the case where \( L = \mathcal{Q}^n \). But in [B.S] (§2.c, especially (2.17)) it is extended to general complex manifolds.

The simplest example is that of the unit ball in \( \mathcal{Q}^{n+1} \) in which case the above formula has the form

\[
K_{\partial B}(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1}} = \int_0^\infty e^{it\psi_B(z, w)}t^n dt
\]

with \( \psi_B(z, w) = 1 - \langle z, w \rangle \).

The above result states that \( \Pi \) is a Fourier integral operator with complex phase, in the class \( I_0^c(X \times X, \mathcal{C}^+) \), where \( \mathcal{C}^+ \) is the canonical relation \( \mathcal{C}^+ \subset T^*X \times T^*X \) generated by the phase \( t\psi(x, y) \) on \( X \times X \times \mathbb{R}^+ \). Its critical points are the solutions of \( \frac{d}{dt}(t\psi) = 0 \), i.e. \( \psi = 0 \) and on the diagonal of \( X \times X \) one has

\[
d_x \psi = -d_y \psi = \frac{1}{i} d'\rho |x|.
\] (16)

In particular, the real points of \( \mathcal{C}^+ \) consist in the diagonal of \( \Sigma^+ \times \Sigma^+ \) where \( \Sigma^+ = \{(x, r\alpha) : r > 0\} \) is the cone generated by the contact form \( \alpha = \frac{1}{i} d'\rho \). In the terminology of [B.G], \( \Pi \) is a Toeplitz structure on the symplectic cone \( \Sigma^+ \).

The principal term \( s_0(x, y) \) was also determined in [B.S (4.10)], using that \( \Pi \) is a projection. On the diagonal one has

\[
s_0(x, x) d\mu(x) = \frac{1}{4\pi^n} (\det L_x) |d\rho| dx
\] (17)

where \( L_x = L_{\rho|T^\prime \oplus T'' \cap TX} \) is the restriction of the Levi form to the maximal complex subspace of \( TX \).

4. Proof of the Theorem

The weight space projections \( \Pi_N \) are Fourier coefficients of \( \Pi \) and hence may be expressed as:

\[
\Pi_N(x, y) = \int_0^\infty \int_{S^1} e^{-iN\theta} e^{it\psi(r\theta x, y)} s(r\theta x, y, t) dt d\theta
\] (18)

where \( r\theta \) denotes the \( S^1 \) action on \( X \). Changing variables \( t \mapsto Nt \) gives

\[
\Pi_N(x, y) = N \int_0^\infty \int_{S^1} e^{iN(-\theta + t\psi(r\theta x, y))} s(r\theta x, y, tN) dt d\theta.
\] (19)

From the fact that \( \text{Im} \psi(x, y) \geq C(d(x, X) + d(y, X) + |x - y|^2 + O(|x - y|^3) \) (see [B.S], Corollary (1.3)) it follows that the phase

\[
\Psi(t, \theta; x, y) = t\psi(r\theta x, y) - \theta.
\] (20)

has positive imaginary part. Here, \( d(x, X) \) is the distance from \( x \) to \( X \) and \( |x - y| \) is a local Euclidean metric. It follows that the integral is a complex oscillatory integral. Before analysing its asymptotics we simplify the phase. As above, we choose a local holomorphic co-frame \( e_L^* \), put \( a(z) = |e_L^*|^2 \), and write \( \nu \in L^*_x \) as \( \nu = \lambda e_L^* \). In the associated coordinates \( (x, y) = (z, \lambda, w, \mu) \) on \( X \times X \) we have:

\[
\rho(z, \lambda) = a(z)|\lambda|^2, \quad \psi(z, \lambda, w, \mu) = \frac{1}{i} a(z, w)\lambda\mu
\] (21)
where \( a(z, w) \) is an almost analytic function on \( M \times M \) satisfying \( a(z, z) = a(z) \). On \( X \) we have \( a(z)|\lambda|^2 = 1 \) so we may write \( \lambda = a(z)^{-\frac{1}{2}} e^{i\phi} \). Similarly for \( \mu \). So for \( (x, y) = (z, \phi, w, \phi') \in X \times X \) we have
\[
\psi(z, \phi, w, \phi') = \frac{1}{i} (1 - \frac{a(z, w)}{i a(z) a(w)}) e^{i(\phi - \phi')}. \tag{22}
\]

4.1. **Proof of Theorem 1.** On the diagonal \( x = y \) we have \( \psi(r \theta x, x) = \frac{1}{i} (1 - \frac{a(z, z)}{a(z)}) e^{i\theta} = \frac{1}{i} (1 - e^{i\theta}) \). So
\[
\Psi(t, \theta; x, x) = \frac{1}{i} (1 - e^{i\theta}) - \theta. \tag{23}
\]
We have
\[
d_\theta \Psi = \frac{1}{2} \theta - e^{i\theta}, \quad d_t \Psi = t e^{i\theta} - 1 \tag{24}
\]
so the critical set is \( C = \{(x, t, \theta) : \theta = 0, t = 1\} \). The Hessian \( \Psi'' \) on the critical set equals
\[
\begin{pmatrix}
0 & 1 \\
1 & i
\end{pmatrix}
\]
so the phase is non-degenerate and the Hessian operator is given by \( L_{\Psi} = \langle (\Psi''(1, 0)^{-1} D, D) = 2 \frac{\partial^2}{\partial \theta^2} - i \frac{\partial}{\partial \theta} \rangle \). It follows by the stationary phase method for complex oscillatory integrals that
\[
\Pi_N(x, x) \sim N \frac{1}{\sqrt{\det(N \Psi''(1, 0)/2\pi i)}} \sum_{j, k=0}^{\infty} N^{n-k-j} L_j s_k(x, x) \tag{25}
\]
where \( L_j \) is a differential operator of order \( 2j \) defined by
\[
L_j s_k(x, x) = \sum_{\nu - \mu = j, 2\nu \geq 3\mu} \frac{1}{2^{\nu} \nu! \mu!} L_\nu [t \phi^\nu(t, \theta) g^\mu(t, \theta)]_{t=1, \theta=0} \tag{26}
\]
with \( g(t, \theta) \) the third order remainder in the Taylor expansion of \( \Psi \) at \( (t, \theta) = (1, 0) \). More precisely, for any \( m \geq 0 \), one has by \([\text{H I, Theorem 7.7.5}]\) that
\[
|\Pi_N(x, x) - N \frac{1}{\sqrt{\det(N \Psi''(1, 0)/2\pi i)}} \sum_{j+k<R} N^{n-k-j} L_j s_k(x, x)| \tag{27}
\]
\[
\leq C N^{n-R} \sum_{k<R, |\alpha| \leq 2R - 2k} \|D^\alpha s_k\|_\infty.
\]
Note that the hypotheses of \([\text{H I, loc.cit.}]\) are satisfied since the phase has a non-negative imaginary part and since its critical points are real and independent of \( x \). Note also that the expansion can be differentiated any number of times. After some rearrangement, the series has the form
\[
\Pi_N(x, x) = N^n C_n s_0(x, x) + N^{n-1} a_1(x, x) + \ldots \tag{28}
\]
where \( C_n \) is a universal constant depending only on \( n \) and where the coefficients \( s_0(x, x) \), \( a_1(x, x) \) depend only on the jets of the terms \( s_k \) along the diagonal. From the description above of the leading coefficient \( s_0(x, x) \) we have (for some other universal constant \( C_n' \))
\[
\Pi_N(x, x) d\mu(x) = N^n C_n \alpha \wedge \omega^n + O(N^{n-1}). \tag{29}
\]
Relative to the Riemannian volume measure $dV_g$, the coefficient is a (non-zero) constant $a_0$ times $N^n$. Comparing to the leading term of the Riemann-Roch polynomial gives that $a_0 = 1$, concluding the proof of (a). \[\square\]

4.2. **Proof Corollary 2.** The new element here is that an arbitrary metric $G$ on $M$, or more precisely its volume form $dV_G$, is used to define orthogonality of sections. We may express $dV_G = J_G \omega^n$ for some positive $J_G \in C^\infty(M)$ and then express $d\mu_G := d\theta \wedge \pi^* dV_G = J_G \alpha \wedge d\alpha^n$. Let $\Pi^G : L^2(X, d\mu_G) \to H^2(X, d\mu_G)$ denote the corresponding orthogonal projection and let $\Pi^G_N$ denote the Fourier components. The Boutet de Monvel-Sjöstrand parametrix construction applies to $\Pi^G$ just as well as to $\Pi$, the only difference lying in their symbols. The principal symbol for $\Pi^G_N$ equals $|\langle d\rho \rangle| \det_{L^G_X}$ where $\det_{L^G_X}$ is the determinant relative to $d\mu_G$, that is, $\det_{L^G_X} = \omega^n_{d\mu_G} = J_G^n$. Clearly this is equal to the determinant of $\omega = \text{Ric}(h)$ relative to $dV_G$. Hence we get

$$\Pi^G_N(x, x) = C_n N^n \text{Ric}(h)(x)[1 + O\left(\frac{1}{N}\right)].$$ \hspace{1cm} (30)

Corollary 2 follows immediately from this and from

$$|\xi|^2_{F.S.N} = \frac{|\xi|^2}{|S_0^N(x)|^2 + \ldots + |S_N^N(x)|^2} \quad \text{for} \quad \xi \in \mathbb{E}^N_x.$$ \hspace{1cm} (31)

It is equivalent to the statement (cf.,[Bou.1, Theoreme Principal] [D, §4])

$$\sum |S_j^N(x)|^2 \sim N^n (2\pi)^{-n} |\alpha_1(x) \ldots \alpha_n(x)|$$

where $\alpha_j(z)$ are the eigenvalues of $ic(L) = \text{Ric}(h)$ relative to $G$

4.3. **Proof of Corollary 3.** Because $\Phi_N$ is a CR map, the asymptotics of the derivatives follow immediately from the asymptotics of $\Pi_N(x, x)$. Indeed, $\partial_b \bar{\partial}_b \log \Pi_N(x, x) = \partial_b \bar{\partial}_b \log \Pi_N(x, y)|_{y=x}$.

By (a) we have

$$\log \Pi_N(x, x) = \log(N^n s_0(x, x)[1 + N^{-1} \frac{s_0}{s_0} + \ldots])$$

$$= n \log N + \log s_0(x, x) + \log[1 + N^{-1} \frac{s_0}{s_0} + \ldots]) = n \log N + \log s_0(x, x) + O\left(\frac{1}{N}\right).$$ \hspace{1cm} (32)

By differentiating the expansion we get

$$\partial_b \bar{\partial}_b \log \Pi_N(x, x) = \partial_b \bar{\partial}_b \log s_0(x, x) + O\left(\frac{1}{N}\right) = O(1).$$ \hspace{1cm} (33)

\[\square\]

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