Geometry of foliations and flows I: Almost transverse pseudo-Anosov flows and asymptotic behavior of foliations

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March 29, 2022

Abstract

Let \( F \) be a foliation in a closed 3-manifold with negatively curved fundamental group and suppose that \( F \) is almost transverse to a quasigeodesic pseudo-Anosov flow. We show that the leaves of the foliation in the universal cover extend continuously to the sphere at infinity, therefore the limit sets of the leaves are continuous images of the circle. One important corollary is that if \( F \) is a Reebless, finite depth foliation in a hyperbolic manifold, then it has the continuous extension property. Such finite depth foliations exist whenever the second Betti number is non zero. The result also applies to other classes of foliations, including a large class of foliations where all leaves are dense and infinitely many examples with one sided branching. One key tool is a detailed understanding of asymptotic properties of almost pseudo-Anosov singular 1-dimensional foliations in the leaves of \( F \) lifted to the universal cover.

1 Introduction

A 2-dimensional foliation in a 3-manifold is called Reebless if it does not have a Reeb component: a foliation of the solid torus so that the boundary is a leaf and the interior is foliated by plane leaves spiralling towards the boundary. As such the boundary leaf does not inject in the fundamental group level and is compressible. Novikov \([\text{No}]\) showed that Reebless foliations and the underlying manifolds have excellent topological properties. This result was extended by Rosenberg \([\text{Ros}]\), Palmeira \([\text{Pa}]\) and many others.

The goal of this article is to analyse geometric properties of foliations. Let \( F \) be a Reebless foliation in \( M^3 \) with negatively curved fundamental group. Reebless implies that \( M \) is irreducible \([\text{Ros}]\). In this article we will not make use of Perelman’s fantastic results \([\text{Pe1, Pe2, Pe3}]\), which if confirmed imply that the manifold is hyperbolic. Reebless foliations exist for instance whenever \( M \) is irreducible, orientable and the second homology of \( M \) is not finite \([\text{Gal, Ga3}]\). They also exist in much more generality by work of Roberts \([\text{Ro1, Ro2, Ro3}]\), Thurston \([\text{Th5}]\) and many others.

Let \( M^3 \) be closed, irreducible with negatively curved fundamental group. The universal cover is canonically compactified with a sphere at infinity (denoted by \( S^2_\infty \)), with compactification a closed ball \([\text{Be-Mc}]\). The covering transformations act by homeomorphisms in the compactified space. Let \( \tilde{F} \) be the lifted foliation to the universal cover \( \tilde{M} \). The leaves of \( \tilde{F} \) are topological planes \([\text{No}]\) and they are properly embedded. Hence they only limit in the sphere at infinity. For hyperbolic manifolds, the relationship between objects in hyperbolic 3-space (isometric to \( \tilde{M} \)) and their limit sets in the sphere at infinity is central to the theory of such manifolds \([\text{Th1, Th2, Mar}]\). The same

*Reseach partially supported by NSF grants DMS-0296139 and DMS-0305313.
†Mathematics Subject Classification. Primary: 53C23, 57R30, 37D20; Secondary: 57M99, 53C12, 32Q05, 57M50.
is true if $\pi_1(M)$ is negatively curved \([\text{Gr} \ \text{Ch-Ha}]\). There is a metric in $M$ so that all leaves of $\mathcal{F}$ are hyperbolic (that is constant curvature $-1$) \([\text{Ca}]\) and so the universal cover of each leaf of $\mathcal{F}$ is isometric to the hyperbolic plane ($\mathbb{H}^2$). The *continuous extension question* asks whether these leaves extend continuously to the sphere at infinity, that is: given the inclusion map from a leaf $F$ of $\tilde{\mathcal{F}}$ to $\tilde{M}$ is there a continuous extension to a map $F \cup \partial_\infty F$ to $\tilde{M} \cup S^2_\infty$? Here $\partial_\infty F$ is the ideal boundary of $F$ which is homeomorphic to a circle. If this is true we say that $\mathcal{F}$ has the *continuous extension property*. In that case the restriction of the map to $\partial_\infty F$ expresses the limit set of $F$ as the continuous image of a circle, showing it is locally connected.

In this article we prove the continuous extension property for a very large class of foliations. A pseudo-Anosov flow is *almost transverse* to a foliation if an appropriate blow up of the flow is transverse to the foliation. A blow up is obtained by replacing a (possibly empty) collection of singular orbits by a union of annuli. Another property that is important is a metric property: A flow is *quasigeodesic* if it is uniformly efficient up to a bounded multiplicative distortion in measuring distances in the universal cover. This is extremely important for manifolds with negatively curved fundamental group \([\text{Th1 Th3 Gr}]\). Our main result is the following:

**Main theorem** – Let $\mathcal{F}$ be a foliation in $M^3$ closed, atoroidal. Suppose that $\mathcal{F}$ is almost transverse to a quasigeodesic pseudo-Anosov flow $\Phi$, which has some prong singularity (that is, not a topological Anosov flow). This implies that $M$ has negatively curved fundamental group. Then $\mathcal{F}$ has the continuous extension property. Therefore the limits sets of leaves of $\tilde{\mathcal{F}}$ are locally connected. The set of foliations almost transverse to a flow is open in the Hirsch topology of foliations.

Notice that the hypothesis imply that $\mathcal{F}$ is transversely orientable. Since $M$ has a singular pseudo-Anosov flow then $M$ is irreducible and the stable/unstable foliations of $\Phi$ split to genuine laminations in $M$. A fundamental result of Gabai and Kazez \([\text{Ga-Ka}]\) then implies that $M$ has negatively curved fundamental group. For simplicity of statements we usually use the group negative curvature hypothesis, but in most places that could be substituted by the atoroidal hypothesis.

Notice that it is not necessary to assume that $\mathcal{F}$ is Reebless – we prove that the condition of being almost transverse to a pseudo-Anosov flow implies that $\mathcal{F}$ is Reebless.

As a first consequence we prove the continuous extension property for all Reebless finite depth foliations in hyperbolic 3-manifolds. Roughly a foliation is *finite depth* if all leaves are proper and the leaves are perfectly fitted along the cutting surfaces of a hierarchy of the manifold. In particular there are compact leaves. These foliations exist whenever the second homology is not finite.

**Corollary A** – Let $\mathcal{F}$ be a Reebless finite depth foliation in $M^3$ closed hyperbolic. Then $\mathcal{F}$ has the continuous extension property. In particular the limit sets of the leaves are all locally connected.

Hence any orientable, hyperbolic 3-manifold with non finite second homology has such a foliation with the continuous extension property. Conjecturally any closed, hyperbolic 3-manifold has a finite cover with positive first Betti number. This would imply there is always a foliation with the continuous extension property in a finite cover. The proof of corollary A is simple given previous results: If necessary take a double cover and assume that $\mathcal{F}$ is transversely oriented. Then Mosher and Gabai proved that such $\mathcal{F}$ is almost transverse to a pseudo-Anosov flow $\Phi$ \([\text{Mo5}]\). We proved, jointly with Mosher that these flows are quasigeodesic \([\text{Fe-Mo}]\). The main theorem then implies corollary A. The result in \([\text{Fe-Mo}]\) is only for finite depth foliations: the proof depends heavily on the existence of a compact leaf, $M$ being hyperbolic and the direct association with a hierarchy. By Thurston’s geometrization theorem \([\text{Th1 Th2 Mor}]\) it suffices to assume that $M$ is atoroidal.

**Corollary B** – There are infinitely many foliations with all leaves dense which have the continuous extension property. Many of these have one sided branching.
Foliations with all leaves dense can be obtained for example starting with finite depth foliations and doing small perturbations — keeping it still almost transverse to the same quasigeodesic pseudo-Anosov flow. A construction is carefully explained by Gabai [Ga3], providing infinitely many examples with dense leaves to which corollary B applies. The examples occur whenever the second Betti number of $M$ is non zero. In fact whenever a foliation $\mathcal{F}$ satisfies the hypothesis of the main theorem, then any $\mathcal{F}'$ sufficiently close to $\mathcal{F}$ will also be transverse to the same flow. By the main theorem again, $\mathcal{F}'$ will have the continuous extension property. This perturbation feature of the main theorem is not shared by any previous result concerning the continuous extension property.

A foliation $\mathcal{F}$ is $\mathbb{R}$-covered if the leaf space of $\tilde{\mathcal{F}}$ is homeomorphic to the real numbers. Equivalently this leaf space is Hausdorff. A foliation which is not $\mathbb{R}$-covered has branching, that is there are non separated points in the leaf space. This leaf space is oriented (being a simply connected, perhaps non Hausdorff 1-manifold) and there is a notion of branching in the positive or negative directions. If it branches only in one direction the foliation is said to have one sided branching. Foliations with one sided branching, where all leaves are dense and the foliation is transverse to a suspension pseudo-Anosov flow (which is quasigeodesic) were constructed by Meigniez [Me]. This provides infinitely many examples with one sided branching to which corollary B applies. The examples occur whenever the second pseudo-Anosov flow. A construction is carefully explained by Gabai [Ga3], providing infinitely many examples with dense leaves to which corollary B applies. The examples occur whenever the second Betti number of $M$ is non zero. In fact whenever a foliation $\mathcal{F}$ satisfies the hypothesis of the main theorem, then any $\mathcal{F}'$ sufficiently close to $\mathcal{F}$ will also be transverse to the same flow. By the main theorem again, $\mathcal{F}'$ will have the continuous extension property. This perturbation feature of the main theorem is not shared by any previous result concerning the continuous extension property.

A very important tool in the proof of the main theorem is an analysis of the topological structure of the pseudo-Anosov flow. Let $\Phi_1$ be the original pseudo-Anosov flow almost transverse to $\mathcal{F}$. To make the flow transverse to $\mathcal{F}$ one needs in general to blow up a collection of singular orbits into a collection of flow saturated annuli so that each boundary is a closed orbit of the new flow $\Phi$. The blown up flow is called an almost pseudo-Anosov flow (see section 3). If $\Phi$ is the lifted flow to the universal cover $\tilde{M}$ and $\mathcal{O}$ is its orbit space, then $\mathcal{O}$ is homeomorphic to the plane $\mathbb{R}^2$ — this is true for pseudo-Anosov and almost pseudo-Anosov flows. When one blows up some singular orbits into a collection of joined annuli, the stable/unstable singular foliations also blow up. The two new singular foliations $\Lambda^s, \Lambda^u$ are everywhere transverse to each other except at the singularities and the blown up annuli. The blown up annuli are part of both singular foliations. Since $\mathcal{F}$ is transverse to the blown up foliations, then the stable/unstable foliations $\Lambda^s, \Lambda^u$ induce singular 1-dimensional foliations in the leaves of $\mathcal{F}$ and hence also in the leaves $\tilde{\mathcal{F}}$. The behavior of this is described in the following result, which is of independent interest also:

**Theorem C** — Let $\mathcal{F}$ be a foliation with hyperbolic leaves in $M^3$ closed. Let $\Phi_1$ be a pseudo-Anosov flow almost transverse to $\mathcal{F}$ and let $\Phi$ be a corresponding almost pseudo-Anosov flow transverse to $\mathcal{F}$. Let $\Lambda^s, \Lambda^u$ be the stable/unstable 2-dimensional foliations of $\Phi$ and $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ the lifts to $\tilde{M}$. Given $F$ leaf of $\tilde{\mathcal{F}}$, let $\tilde{\Lambda}^s_F, \tilde{\Lambda}^u_F$ be the induced singular 1-dimensional foliations in $F$. Then

- For every ray $l$ in a leaf of $\tilde{\Lambda}^s_F$ or $\tilde{\Lambda}^u_F$, then $l$ limits in a single point of $\partial_\infty F$.
- If the stable/unstable foliations $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ of $\Phi$ have Hausdorff leaf space, then the leaves of $\tilde{\Lambda}^s_F, \tilde{\Lambda}^u_F$ are uniform quasigeodesics in $F$, the bound is independent of the leaf. In general the leaves of $\Lambda^s, \Lambda^u$ are not quasigeodesic.
- Any non Hausdorffness (of say $\tilde{\Lambda}^s_F$) is associated to a Reeb annulus in a leaf of $\mathcal{F}$ and when projected to $M$ it either projects to or spirals to a Reeb annulus.
- The set of ideal points of leaves of $\tilde{\Lambda}^s_F$ is dense in $\partial_\infty F$ and similarly for $\tilde{\Lambda}^u_F$.
- If two rays of the same leaf of $\tilde{\Lambda}^s_F$ limit to the same ideal point in $\partial_\infty F$ then this leaf of $\tilde{\Lambda}^s_F$ does not contain a singularity and the region in $F$ bounded by the leaf projects in $M$ to a set in a leaf of $\mathcal{F}$ which is either contained in or asymptotic to a Reeb annulus.
This is a completely general result: one does not need negatively curved fundamental group of $M$ or any metric properties of $\Phi$. Theorem C is a key tool used in the proof of the main theorem.

The article is organized as follows: In the next section we discuss previous results on the continuous extension property and the strategy of the proof of the main theorem and theorem C. In section 3 we present basic definitions and results concerning pseudo-Anosov flows and almost pseudo-Anosov flows. In section 4 we analyse projections to the orbit space. In sections 5 and 6 we analyse the singular foliations $\tilde{\Lambda}^F, \tilde{\Lambda}^u$ and asymptotic properties of their leaves, proving theorem C. In section 7 we prove the continuous extension property — the main theorem. In the final section we comment on general relationships between foliations and Kleinian groups.

In a subsequent article we analyse other important consequences of quasigeodesic behavior for flows and foliations [Fe10].

### 2 Historical remarks and strategy of proofs

Here we review what is known about the continuous extension property. In a seminal work, Cannon and Thurston [Ca-Th] proved this property when $F$ is a fibration over the circle. Previously Thurston showed that a fibering manifold is hyperbolic when the monodromy of the fibration is pseudo-Anosov [Th1, Th3, Th4, Bl-Ca]. Since the fundamental group of a leaf of $F$ is a normal subgroup of the fundamental group of $M$, then every limit set of a leaf of $\tilde{F}$ is the whole sphere. In this way Cannon and Thurston produced many examples of group invariant Peano curves.

Another extremely important class of foliations is the following: A foliation is proper if the leaves never limit on themselves — this is in the foliation sense and it means that a sufficiently small transversal to a given leaf only meets the leaf in a single point. In particular leaves are not dense. In a proper foliation there are compact leaves which are said to have depth 0. The depth of a leaf is inductively defined to be $i$ (for finite $i$) if $i - 1$ is the maximum of the depths of leaves in the (foliation) limit set of the leaf. A foliation has finite depth if it is proper and there is a finite upper bound to the depths of all leaves.

Gabai proved that whenever a compact 3-manifold $M$ is irreducible, orientable and the second homology group $H_2(M, \partial M, \mathbb{Z})$ is not finite, then there is a Reebless finite depth, foliation associated to each non torsion homology class [Ga1, Ga3]. The foliation is directly associated to a hierarchy of the manifold and as such is strongly connected with the topological structure of the manifold. These results had several fundamental consequences for the topology of 3-manifolds [Ga1, Ga2, Ga3]. If $M$ is hyperbolic (or atoroidal), then one important question is whether these finite depth foliations have the continuous extension property.

Subsequently Gabai and Mosher showed [Mo5] that any Reebless finite depth foliation in a closed, atoroidal 3-manifold admits a pseudo-Anosov flow $\Phi$ which is almost transverse to it. Roughly a flow is pseudo-Anosov if it has transverse hyperbolic dynamics — even though it may have finitely many singularities. It has stable and unstable two dimensional foliations which in general are singular. The term almost transverse means that one may need to blow up one singular orbit (or more) into a finite collection of joined annuli to make the flow transverse to the foliation. See detailed definitions and comments in section 3. Since $F$ has a compact leaf and $M$ is atoroidal, then Thurston [Th1, Th3] proved that $M$ is in fact hyperbolic.

We proved, jointly with Mosher, that these pseudo-Anosov flows almost transverse to finite depth foliations in hyperbolic 3-manifolds are quasigeodesic [Fe-Mo]. This means that flow lines are uniformly efficient in measuring distance in relative homotopy classes, or equivalently, uniformly efficient in measuring distance in the universal cover. This was first proved by Mosher [Mo4] for a class of flows transverse to some examples of depth one foliations obtained by handle constructions. Another concept is that of quasi-isometric behavior: a foliation (perhaps singular) is quasi-isometric if its
leaves are uniformly efficient in measuring distance in the universal cover. There are no non singular 2 dimensional quasi-isometric foliations in closed 3-manifolds with negatively curved fundamental group \([\text{Fe1}]\). As for singular foliations the situation is quite different and there are examples. The stable/unstable singular foliations of the quasigeodesic flows above may be quasi-isometric \([\text{Fe6}]\) and may not \([\text{Mo5}, \text{Fe6}]\). In general quasi-isometric behavior of \(\Lambda^s\) (or \(\Lambda^u\)) implies that \(\Phi\) is quasigeodesic.

If both the stable and unstable foliations are quasi-isometric and the flow is actually transverse (as opposed to being almost transverse) to the finite depth foliation then we proved in \([\text{Fe6}]\) that \(\mathcal{F}\) has the continuous extension property. We stress that this result only applies to finite depth foliations – the proof depends, amongst other things, on induction in the depth. To use this result we needed to check the quasi-isometric behavior of \(\Lambda^s, \Lambda^u\) and transversality between \(\mathcal{F}\) and \(\Phi\). This was very tricky and we could only do that for some depth one foliations. More to the point, it is known that these conditions do not always hold for general finite depth foliations. Corollary A proves the continuous extension property for all finite depth foliations: there are no restrictions on the depth of the foliation, or about transversality of the flow with the foliation or quasi-isometric behavior of \(\Lambda^s, \Lambda^u\).

The continuous extension property was also proved for another class of foliations: A foliation \(\mathcal{F}\) is uniform if any two leaves in the universal cover are a bounded distance apart, the bound depends on the individual leaves. Hence \(\mathcal{F}\) is \(\mathbb{R}\)-covered. Thurston \([\text{Th5}]\) proved that there is a large class of uniform foliations. If in addition \(\pi_1(M)\) is negatively curved, then Thurston \([\text{Th5}]\) proved there is a pseudo-Anosov flow transverse to \(\mathcal{F}\). From this it is easy to prove that the flow has quasi-isometric stable/unstable foliations. In this case it also easily implies that the foliation \(\mathcal{F}\) has the continuous extension property. The arguments are a very clever generalization of the fibering situation.

Notice that in all the previous results, there is a pseudo-Anosov flow \(\Phi\) transverse to \(\mathcal{F}\) and so that the stable/unstable foliations of \(\Phi\) are quasi-isometric singular foliations. Both of these properties were crucial in all proofs. The main theorem implies all the previous results about the continuous extension property and it has the unique feature that it applies to an open set of foliations.

The main theorem can potentially be widely applicable because of the abundance of pseudo-Anosov flows almost transverse to foliations: Thurston proved this for fibrations \([\text{Th4}]\). It is also true for all \(\mathbb{R}\)-covered foliations \([\text{Fe7}, \text{Cal1}]\) and Calegari proved it for all foliations with one sided branching \([\text{Cal2}]\), all minimal foliations \([\text{Cal3}]\) and many other foliations \([\text{Cal3}]\). The missing ingredient is the quasigeodesic property of these pseudo-Anosov flows which is needed to apply the main theorem. In general the quasigeodesic property for a pseudo-Anosov flow (or an arbitrary flow) is very hard to obtain. This property is only known when the pseudo-Anosov flow is almost transverse to a foliation of one of the following types: finite depth or uniform. One main goal in the study of a pseudo-Anosov flow \(\Phi\) in \(M\) with Gromov hyperbolic fundamental group is to decide whether \(\Phi\) is quasigeodesic. There are many examples where it is not quasigeodesic \([\text{Fe2}]\).

Why almost transversality and not just transversality? In many cases (\(\mathbb{R}\)-covered, one sided branching) the pseudo-Anosov flow is actually transverse to \(\mathcal{F}\). But for finite depth foliations (which have two sided branching), there are many examples where it is impossible to make the pseudo-Anosov flow transverse to the foliation \([\text{Mo5}]\) and one can only get almost transversality. We will have more comments about this in section 3. Finite depth foliations are extremely important as they are strongly connected to the topology of the underlying manifold. Also, in some sense, foliations with two sided branching are probably more common than foliations which are either \(\mathbb{R}\)-covered or with one sided branching.

We also remark that in all previous results concerning the continuous extension property, theorem C was a crucial property on which the whole analysis hinged. In the previous situations, the analysis
of leaves of $\tilde{\Lambda}_s^u, \tilde{\Lambda}_u^s$ was either trivial or substantially easier. In particular in these situations the leaves of $\tilde{\Lambda}_s^u, \tilde{\Lambda}_u^s$ were always uniform quasigeodesics, which simplified subsequent proofs considerably. Such is not the case in general. In particular if $\Lambda^s, \Lambda^u$ do not have Hausdorff leaf space, then a priori the leaves of $\tilde{\Lambda}_s^u, \tilde{\Lambda}_u^s$ do not have to be quasigeodesic. The proof of theorem C works for any pseudo-Anosov flow almost transverse to a foliation with hyperbolic leaves. The proof uses the denseness of contracting directions for foliations as proved by Thurston \cite{Th6, Th7} when he introduced the universal circle for foliations — even though we do not directly use the universal circle here. The basic idea is: if any ray of (say) $\tilde{\Lambda}_s^u$ does not limit to a single point in $\partial_\infty F$ then it limits in a non trivial interval of $\partial_\infty F$. Zoom into this interval and analyse the situation in the limit. This is actually the easiest statement to prove in theorem C. The facts about rays with same ideal point and non Hausdorffness are much trickier, but they will be essential in the analysis of the main theorem. The results of theorem C are also used in other contexts, for example to analyse rigidity of pseudo-Anosov flows almost transverse to a given foliation. This will be explored in a future article \cite{Fe10}.

The proof of the main theorem has 2 parts: given a leaf $F$ of $\tilde{F}$, one first constructs an extension to the ideal boundary and then show it is continuous. To define the extension, one uses the foliations $\tilde{\Lambda}_s^u, \tilde{\Lambda}_u^s$ as they hopefully define a basis neighborhood of an ideal point $p$ of $F$. The best situation is that the leaves of $\Lambda^s, \Lambda^u$ which contain these leaves of $\tilde{\Lambda}_s^u, \tilde{\Lambda}_u^s$ define basis neighborhoods of unique points in $S^2_\infty$, hence defining the image of $p$ in $S^2_\infty$. There are several difficulties here: first the leaves of $\Lambda^s, \Lambda^u$ are not quasigeodesics in $F$, so much more care is needed. Another problem is that the foliations $\Lambda^s, \Lambda^u$ in general do not have Hausdorff leaf space. This keeps recurring throughout the proof. A further difficulty is that if intersections of leaves $L_i$ of say $\Lambda^s$ with a leaf $F$ of $\tilde{F}$ escape in $F$ as $i$ converges to infinity, it does not follow that the $L_i$ themselves escape compact sets in $\tilde{M}$. Consequently there are several cases to be analysed.

Another fact that is important for the analysis of the main theorem and theorem C is the following: Let $\Theta$ be the projection map from $\tilde{M}$ to $\mathcal{O}$ (projects flow lines to points). A leaf of $\tilde{F}$ intersects an orbit of $\Phi$ at most once defining an injective projection of $F$ to $\Theta(F)$. The projection $\Theta(F)$ is equal to $\mathcal{O}$ if and only if the foliation is $\mathbb{R}$-covered. An important problem here is to determine the boundary $\Theta(F)$ as a subset of $\mathcal{O}$. This turns out to be a collection of subsets of stable/unstable leaves in $\mathcal{O}$. This result is different than what happens for pseudo-Anosov flows transverse to foliations and its proof is much more delicate. This is analysed in section 4.

### 3 Preliminaries: Pseudo-Anosov flows and almost pseudo-Anosov flows

Let $\Phi$ be a flow on a closed, oriented 3-manifold $M$. We say that $\Phi$ is a pseudo-Anosov flow if the following are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x,t)$ is $C^1$, it is not a single point, and the tangent vector bundle $D_t\Phi$ is $C^0$.

- There is a finite number of periodic orbits $\{\gamma_i\}$, called singular orbits, such that the flow is “topologically” smooth off of the singular orbits (see below).

- The flowlines are tangent to two singular transverse foliations $\Lambda^s, \Lambda^u$ which have smooth leaves off of $\gamma_i$ and intersect exactly in the flow lines of $\Phi$. These are like Anosov foliations off of the singular orbits. This is the topologically smooth behavior described above. A leaf containing a singularity is homeomorphic to $P \times I/f$ where $P$ is a $p$-prong in the plane and $f$ is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$. In a stable leaf, $f$ contracts towards towards the prongs and in an unstable leaf it expands away from the prongs. We restrict to $p$ at least 2, that is, we do not allow 1-prongs.

- In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backwards asymptotic.
Basic references for pseudo-Anosov flows are \cite{Mo3, Mo5} and for 3-manifolds \cite{He}.

**Notation/definition:** The singular foliations lifted to \( \tilde{M} \) are denoted by \( \tilde{\Lambda}^s, \tilde{\Lambda}^u \). If \( x \in M \) let \( \tilde{W}^s(x) \) denote the leaf of \( \tilde{\Lambda}^s \) containing \( x \). Similarly one defines \( \tilde{W}^u(x) \) and in the universal cover \( \hat{W}^s(x), \hat{W}^u(x) \). Similarly if \( \alpha \) is an orbit of \( \Phi \) define \( W^s(\alpha) \), etc... Let also \( \tilde{\Phi} \) be the lifted flow to \( \tilde{M} \).

We review the results about the topology of \( \tilde{\Lambda}^s, \tilde{\Lambda}^u \) that we will need. We refer to \cite{Fe4, Fe6} for detailed definitions, explanations and proofs. The orbit space of \( \tilde{\Phi} \) in \( \tilde{M} \) is homeomorphic to the plane \( \mathbb{R}^2 \) \cite{Mo0, Mo5} and is denoted by \( \mathcal{O} \cong \tilde{M} / \tilde{\Phi} \). Let \( \Theta : \tilde{M} \to \mathcal{O} \cong \mathbb{R}^2 \) be the projection map. If \( L \) is a leaf of \( \tilde{\Lambda}^s \) or \( \tilde{\Lambda}^u \), then \( \Theta(L) \subset \mathcal{O} \) is a tree which is either homeomorphic to \( \mathbb{R} \) if \( L \) is regular, or is a union of \( p \)-rays all with the same starting point if \( L \) has a singular \( p \)-prong orbit. The foliations \( \tilde{\Lambda}^s, \tilde{\Lambda}^u \) induce 1-dimensional foliations \( \mathcal{O}^s, \mathcal{O}^u \) in \( \mathcal{O} \). Its leaves are \( \Theta(L) \) as above. If \( L \) is a leaf of \( \tilde{\Lambda}^s \) or \( \tilde{\Lambda}^u \), then a sector is a component of \( \tilde{M} - L \). Similarly for \( \mathcal{O}^s, \mathcal{O}^u \). If \( B \) is any subset of \( \mathcal{O} \), we denote by \( B \times \mathbb{R} \) the set \( \Theta^{-1}(B) \). The same notation \( B \times \mathbb{R} \) will be used for any subset \( B \) of \( \tilde{M} \): it will just be the union of all flow lines through points of \( B \).

**Definition 3.1.** Let \( L \) be a leaf of \( \tilde{\Lambda}^s \) or \( \tilde{\Lambda}^u \). A slice of \( L \) is \( l \times \mathbb{R} \) where \( l \) is a properly embedded copy of the reals in \( \Theta(L) \). For instance if \( L \) is regular then \( L \) is its only slice. If a slice is the boundary of a sector of \( L \) then it is called a line leaf of \( L \). If \( a \) is a ray in \( \Theta(L) \) then \( A = a \times \mathbb{R} \) is called a half leaf of \( L \). If \( \zeta \) is an open segment in \( \Theta(L) \) it defines a flow band \( L_1 \) of \( L \) by \( L_1 = \zeta \times \mathbb{R} \). Same notation for the foliations \( \mathcal{O}^s, \mathcal{O}^u \) of \( \mathcal{O} \).

If \( F \in \tilde{\Lambda}^s \) and \( G \in \tilde{\Lambda}^u \) then \( F \) and \( G \) intersect in at most one orbit. Also suppose that a leaf \( F \in \tilde{\Lambda}^s \) intersects two leaves \( G, H \in \tilde{\Lambda}^u \) and so does \( L \in \tilde{\Lambda}^s \). Then \( F, L, G, H \) form a rectangle in \( \tilde{M} \) and there is no singularity in the interior of the rectangle \cite{Fe6}. There will be two generalizations of rectangles: 1) perfect fits = rectangle with one corner removed and 2) lozenges = rectangle with two opposite corners removed. We will also denote by rectangles, perfect fits, lozenges and product regions the projection of these regions to \( \mathcal{O} \cong \mathbb{R}^2 \).

**Definition 3.2.** (\cite{Fe2, Fe4, Fe6}) Perfect fits - Two leaves \( F \in \tilde{\Lambda}^s \) and \( G \in \tilde{\Lambda}^u \), form a perfect fit if \( F \cap G = \emptyset \) and there are half leaves \( F_1 \) of \( F \) and \( G_1 \) of \( G \) and also flow bands \( L_1 \subset L \in \tilde{\Lambda}^s \) and \( H_1 \subset H \in \tilde{\Lambda}^u \), so that the set

\[
\overline{F_1} \cup \overline{F_1} \cup \overline{F_1} \cup \overline{G_1}
\]

separates \( M \) and forms an a rectangle \( R \) with a corner removed: The joint structure of \( \tilde{\Lambda}^s, \tilde{\Lambda}^u \) in \( R \) is that of a rectangle with a corner orbit removed. The removed corner corresponds to the perfect of \( F \) and \( G \) which do not intersect.

We refer to fig. \( 4 \) for perfect fits. There is a product structure in the interior of \( R \): there are two stable boundary sides and two unstable one. An unstable leaf intersects one stable boundary side (not in the corner) if and only if it intersects the other stable boundary side (not in the corner). We also say that the leaves \( F, G \) are asymptotic.

**Definition 3.3.** (\cite{Fe2, Fe4, Fe6}) Lozenges - A lozenge is a region of \( \tilde{M} \) whose closure is homeomorphic to a rectangle with two corners removed. More specifically two points \( p, q \) form the corners of a lozenge if there are half leaves \( A, B \) of \( \tilde{W}^s(p), \tilde{W}^u(p) \) defined by \( p \) and \( C, D \) half leaves of \( \tilde{W}^s(q), \tilde{W}^u(q) \) so that \( A \) and \( D \) form a perfect fit and so do \( B \) and \( C \). The region bounded by the lozenge is \( R \) and it does not have any singularities. The sides are not contained in the lozenge, but are in the boundary of the lozenge. See fig. \( 5 \).
There are no singularities in the lozenges, which implies that $R$ is an open region in $\tilde{M}$. There may be singular orbits on the sides of the lozenge and the corner orbits.

Two lozenges are adjacent if they share a corner and there is a stable or unstable leaf intersecting both of them, see fig. (c). Therefore they share a side. A chain of lozenges is a collection $\{C_i\}, i \in I$, where $I$ is an interval (finite or not) in $\mathbb{Z}$; so that if $i, i+1 \in I$, then $C_i$ and $C_{i+1}$ share a corner, see fig. (c). Consecutive lozenges may be adjacent or not. The chain is finite if $I$ is finite.

Definition 3.4. Suppose $A$ is a flow band in a leaf of $\tilde{\Lambda}^s$. Suppose that for each orbit $\gamma$ of $\tilde{\Phi}$ in $A$ there is a half leaf $B_\gamma$ of $\tilde{W}^u(\gamma)$ defined by $\gamma$ so that: for any two orbits $\gamma, \beta$ in $A$ then a stable leaf intersects $B_\beta$ if and only if it intersects $B_\gamma$. This defines a stable product region which is the union of the $B_\gamma$. Similarly define unstable product regions.

There are no singular orbits of $\tilde{\Phi}$ in $A$.

We abuse convention and call a leaf $L$ of $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$ is called periodic if there is a non trivial covering translation $g$ of $\tilde{\Phi}$ in $A$ such that $g(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of $\Phi$. In the same way an orbit $\gamma$ of $\tilde{\Phi}$ is periodic if $\pi(\gamma)$ is a periodic orbit of $\Phi$.

We say that two orbits $\gamma, \alpha$ of $\tilde{\Phi}$ (or the leaves $\tilde{W}^s(\gamma), \tilde{W}^s(\alpha)$) are connected by a chain of lozenges $\{C_i\}, 1 \leq i \leq n$, if $\gamma$ is a corner of $C_1$ and $\alpha$ is a corner of $C_n$.

If $C$ is a lozenge with corners $\beta, \gamma$ and $g$ is a non trivial covering translation leaving $\beta, \gamma$ invariant (and so also the lozenge), then $\pi(\beta), \pi(\gamma)$ are closed orbits of $\tilde{\Phi}$ which are freely homotopic to the inverse of each other.

Theorem 3.5. Let $\Phi$ be a pseudo-Anosov flow in $M^3$ closed and let $F_0 \neq F_1 \in \tilde{\Lambda}^s$. Suppose that there is a non trivial covering translation $g$ with $g(F_i) = F_i, i = 0, 1$. Let $\alpha_i, i = 0, 1$ be the periodic orbits of $\tilde{\Phi}$ in $F_i$ so that $g(\alpha_i) = \alpha_i$. Then $\alpha_0$ and $\alpha_1$ are connected by a finite chain of lozenges $\{C_i\}, 1 \leq i \leq n$ and $g$ leaves invariant each lozenge $C_i$ as well as their corners.

A chain from $\alpha_0$ to $\alpha_1$ is called minimal if all lozenges in the chain are distinct. Exactly as proved in for Anosov flows, it follows that there is a unique minimal chain from $\alpha_0$ to $\alpha_1$ and also all other chains have to contain all the lozenges in the minimal chain.

The main result concerning non Hausdorff behavior in the leaf spaces of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ is the following:  

Theorem 3.6. Let $\Phi$ be a pseudo-Anosov flow in $M^3$. Suppose that $F \neq L$ are not separated in the leaf space of $\tilde{\Lambda}^s$. Then the following facts happen:

- $F$ is periodic and so is $L$. 

Figure 1: a. Perfect fits in $\tilde{M}$, b. A lozenge, c. A chain of lozenges.
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Figure 2: The correct picture between non separated leaves of $\tilde{\Lambda}^s$.

- Let $F_0, L_0$ be the line leaves of $F, L$ which are not separated from each other. Let $V_0$ be the sector of $F$ bounded by $F_0$ and containing $L$. Let $\alpha$ be the periodic orbit in $F_0$ and $H_0$ be the component of $(W^u(\alpha) - \alpha)$ contained in $V_0$. Let $g$ be a non trivial covering translation with $g(F_0) = F_0$, $g(H_0) = H_0$ and $g$ leaves invariant the components of $(F_0 - \alpha)$. Then $g(L_0) = L_0$.

- Let $\beta$ be the periodic orbit in $L_0$. As $g(\beta) = \beta$, $g(\alpha) = \alpha$, then $\pi(\alpha), \pi(\beta)$ are closed orbits of $\Phi$ which are freely homotopic in $M$. By theorem 3.5 $F_0$ and $L_0$ are connected by a finite chain of lozenges $\{A_i\}, 1 \leq i \leq n$. Consecutive lozenges are adjacent. They all intersect a common stable leaf $C$. There is an even number of lozenges in the chain, see fig. 2.

- In addition let $B_{F,L}$ be the set of leaves non separated from $F$ and $L$. Put an order in $B_{F,L}$ as follows: Put an orientation in the set of orbits of $C$ contained in the union of the lozenges and their sides. If $R_1, R_2 \in B_{F,L}$ let $\alpha_1, \alpha_2$ be the respective periodic orbits in $R_1, R_2$. Then $W^u(\alpha_i) \cap C \neq \emptyset$ and let $a_i = W^u(\alpha_i) \cap C$. We define $R_1 < R_2$ in $B_{F,L}$ if $a_1$ precedes $a_2$ in the orientation of the set of orbits of $C$. Then $B_{F,L}$ is either order isomorphic to $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$; or $B_{F,L}$ is order isomorphic to the integers $\mathbb{Z}$.

- Also if there are $Z, S \in \tilde{\Lambda}^s$ so that $B_{Z,S}$ is infinite, then there is an incompressible torus in $M$ transverse to $\Phi$. In particular $M$ cannot be atoroidal. Also if there are $F, L$ as above, then there are closed orbits $\alpha, \beta$ of $\Phi$ which are freely homotopic to the inverse of each other.

- Up to covering translations, there are only finitely many non Hausdorff points in the leaf space of $\tilde{\Lambda}^s$.

Notice that $B_{F,L}$ is a discrete set in this order. For detailed explanations and proofs, see [Fe4, Fe6].

Theorem 3.7. ([Fe6]) Let $\Phi$ be a pseudo-Anosov flow. If there is a stable or unstable product region, then $\Phi$ is topologically conjugate to a suspension Anosov flow. In particular $\Phi$ is non singular.

Proposition 3.8. Let $\varphi$ be a (topological) Anosov flow so that every leaf of its stable foliation $\tilde{\Lambda}^s$ intersects every leaf of its stable foliations $\tilde{\Lambda}^u$. Then $\varphi$ is topologically conjugate to a suspension Anosov flow. In particular $M$ fibers over the circle with fiber a torus and Anosov monodromy.

Proof. This result is proved by Barbot [Ba1] when $\varphi$ is a smooth Anosov flow. That means it is $C^1$ and it has also strong stable/unstable foliations and contraction on the level of tangent vectors along the flow. Here we only have the weak foliations and orbits being asymptotic in their leaves. With proper understanding all the steps carry through to the general situation.

Lift to a finite cover where $\Lambda^s, \Lambda^u$ are transversely orientable. A cross section in the cover projects to a cross section in the manifold (after cut and paste following Fried [Fr]) and so we can prove the result in the cover.
First, the flow \( \varphi \) is expanding: there is \( \epsilon > 0 \) so that no distinct orbits are always less than \( \epsilon \) away from each other. Inaba and Matsumoto then proved that this flow is a topological pseudo-Anosov flow [In-Ma]. The main thing is the existence of a Markov partition for the flow. This implies that if \( F \) is a leaf of \( \tilde{\Lambda}^s \) which is left invariant by \( g \), then there is a closed orbit of \( \varphi \) in \( \pi(F) \) and all orbits are asymptotic to this closed orbit. Similarly for \( \tilde{\Lambda}^u \).

What this means is the following: consider the action of \( \pi_1(M) \) in the leaf space of \( \tilde{\Lambda}^s \) which is the reals. Hence we have a group action in \( \mathbb{R} \). Let \( g \) in \( \pi_1(M) \) which fixes a point. There is \( L \) in \( \tilde{\Lambda}^s \) with \( g(L) = L \). So there is orbit \( \gamma \) of \( \tilde{\varphi} \) with \( g(\gamma) = \gamma \). Let \( U \) be the unstable leaf of \( \tilde{\varphi} \) with \( \gamma \) contained in \( U \). Then \( g(U) = U \). If \( g \) is associated to the positive direction of \( \gamma \) then \( g \) acts as a contraction in the set of orbits of \( U \) with \( \gamma \) as the only fixed point. Since every leaf of \( \tilde{\Lambda}^u \) intersects every leaf of \( \tilde{\Lambda}^s \) then the set of orbits in \( U \) is equivalent to the set of leaves of \( \tilde{\Lambda}^s \). This implies the important fact:

**Conclusion** - If \( g \) is in \( \pi_1(M) \) has a fixed point in the leaf space of \( \tilde{\Lambda}^s \) then it is of hyperbolic type and has a single fixed point.

Using this topological characterization Barbot [Ba1] showed that \( G = \pi_1(M) \) is metabelian, in fact he showed that the commutator subgroup \( [G,G] \) is abelian. In particular \( \pi_1(M) \) is solvable. This used only an action by homeomorphisms in \( \mathbb{R} \) satisfying the conclusion above. Barbot [Ba1] also proved that the leaves of \( \tilde{\Lambda}^s, \tilde{\Lambda}^u \) are dense in \( M \).

Plante [Pl1], showed that if \( \mathcal{F} \) a minimal foliation in \( \pi_1(M) \) solvable then \( \mathcal{F} \) is transversely affine: there is a collection of charts \( f_i : U_i \to \mathbb{R}^2 \times \mathbb{R} \), so that the transition functions are affine in the second coordinate. Using this Plante [Pl1] constructs a homomorphism

\[
C : \pi_1(M) \to \mathbb{R}
\]

which measures the logarithm of how much distortion there is along an element of \( \pi_1(M) \). This is a cohomology class. Every closed orbit \( \gamma \) of \( \varphi \) has a transversal fence which is expanding - this implies that \( C(\gamma) \) is positive. Plante then refers to a criterion of Fried [Ft] to conclude that \( \varphi \) has a cross section and therefore it is easily seen that \( \varphi \) is topologically conjugate to a suspension Anosov flow. This finishes the proof of the proposition. \( \square \)

**Blown up orbits and almost pseudo-Anosov flows**

We now describe almost pseudo-Anosov flows. The description is taken from [Mo5]. First we describe the blow up of a singular orbit. Let \( \gamma \) be a singular orbit of a pseudo-Anosov flow \( \varphi \) and let \( C \) be a small disk transverse to \( \varphi \) through a point \( p \) of \( \gamma \). The set \( \tilde{W}^s(\gamma) \cup \tilde{W}^u(\gamma) \) intersects \( C \) in a tree \( T \), which is made of a single common vertex and 2\( n \) prongs, if the stable/unstable leaves \( \tilde{W}^s(\gamma), \tilde{W}^u(\gamma) \) have \( n \) prongs. The stable and unstable prongs alternate in \( C \). Orient the prongs so that stable prongs point towards \( p \) and unstable prongs point away from \( p \). Hence \( T \) is an oriented tree. Up to topological conjugacy we may assume in polar coordinates centered at \( p \) that

\[
T = \{ (r, \theta) \mid \theta = k\pi/n, k = 0, 1, ..., 2n - 1 \}.
\]

Choose a Poincaré section for \( \gamma \), that is, a smaller disk \( C' \) so that every point of \( C' \) flows forward to a point in \( C \). This defines a continuous map \( f : C' \to C \) which fixes \( p \) and is a homeomorphism into its image. This \( f \) sends the corresponding pieces of \( T \) into \( T \), contracting stable direction and expanding unstable directions. There is a rotation or reflection \( R \) in \( C \) so that \( R \) prescribes where the sectors of \( C \) defined by \( T \) go to under \( f \). For instance \( R \) may be a rotation by \( 2\pi/n \).

We first blow up \( f \). Let \( D \) be a small subdisk of \( C' \) containing \( p \) in its interior. Let \( T^s \) be an oriented tree in \( C \) which agrees with \( T \) outside \( D \) and which is invariant under \( R \) as above and such that each vertex \( v \) of \( T^s \) is “pseudo-hyperbolic”, meaning as you go around the edges incident to \( v \),
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the orientations of the edges of $T^*$ alternate pointing toward and away from $v$. Each vertex of $T^*$ must have $2i$ edges for some $i \geq 2$. The point here is that $T^*$ is created from $T$ by replacing $p$ with a finite subtree. There are finitely many ways to choose $T^*$, up to compactly supported isotopy. The new edges of $T^*$ created by the blow up are called the finite edges of $T^*$.

Given this data there is a $C^0$ perturbation $f^*$ of $f$, and a continuous map $h : C \to C$ so that

- $f^*$ leaves $T^*$ invariant.
- For each edge $E$ of $T^*$, the first return map of $f^*$ acts as a translation on $\text{int}(E)$ moving points in the direction of the orientation on $E$.
- $h$ collapses the finite edges of $T^*$ to the point $p$ and $h$ is otherwise 1 to 1.
- $h$ is a semiconjugacy from $f^*$ to $f$, that is, $f \circ h = h \circ f^*$.
- $h$ is close to the identity map in the sup norm and $h$ equals the identity on $C - D$.

We say that $f^*$ is obtained from $f$ by dynamically blowing up the pseudo-hyperbolic fixed point $p$. Each choice of $T^*$ determines a unique $f^*$ up to conjugacy by isotopy. There are therefore finitely many ways to dynamically blow up a pseudo-hyperbolic fixed point, up to conjugation by isotopy.

Now define a dynamic blow up of $\gamma$ by altering the flow $\varphi$ near $\gamma$ as follows. Let $D'$ so that $f(D') \subset D$. Alter $\varphi$ so that the first return map $f : D' \to D$ is replaced by $f^* : D' \to D$. This has the effect of altering the generating vector field of $\varphi$ inside the “mapping torus” $T_\varphi = \{\varphi(x,s) \mid x \in D', 0 \leq s \leq t(x)\}$ and leaving the vector field unaltered outside of $T_\varphi$. Let $\varphi^*$ be the new flow.

The orbit $\gamma$ gets blown up into a collection of flow invariant annuli. In each annulus $A$, the boundary components are closed orbits of $\varphi^*$ which are isotopic to $\gamma$ as oriented orbits. In the interior of $A$ orbits move from one boundary to the other, as given by the orientation of the corresponding edge of $T^*$. There is a blow down map $\xi : M \to M$ which is homotopic to the identity and isotopic to the identity in the complement of the annuli. It sends the collection of annuli into $\gamma$. The map $\xi$ sends orbits of $\varphi^*$ to orbits of $\varphi$ and preserves orientation.

**Definition 3.9.** Let $\varphi$ be a pseudo-Anosov flow in $M^3$ closed. Then $\varphi^*$ is an almost pseudo-Anosov flow associated to $\varphi$ if $\varphi^*$ is obtained from $\varphi$ by dynamically blowing some singular orbits of $\varphi$.

The reason for considering almost pseudo-Anosov flows is as follows. All of the constructions of pseudo-Anosov flows transverse to foliations are in fact constructions of a pair of laminations — stable and unstable — which are transverse to each other and to the foliation [Th4, Mo5, Fe7, Cal1, Cal2]. The intersection of the laminations is oriented producing a flow in this intersection. One then collapses the complementary regions to the laminations to produce transverse singular foliations and a pseudo-Anosov flow. The transversality problem occurs in this last step, the blow down of complementary regions. In certain situations, for example for finite depth foliations, one cannot guarantee transversality of the flow and foliation after the blow down. See extended explanations by Mosher in [Mo5].

The necessity of almost transversality as opposed to transversality was first discovered by Mosher in [Mo1] for positioning surfaces with respect to certain Z invariant flows (in non compact manifolds). This was further explored in [Mo2] where it was analysed when it is necessary to blow up a pseudo-Anosov flow before it became transverse to a given surface.

Given a dynamic blow up $\varphi^*$ of $\varphi$ with corresponding blow down map $\xi$, then $\xi^{-1}$ of $\Lambda^s, \Lambda^u$ are singular foliations, which now have some flow invariant annuli. These are the annuli that come from blowing up some singular orbits — they are called the blown up annuli. We still denote by $\Lambda^s, \Lambda^u$
the blown up stable/unstable foliations of an almost pseudo-Anosov flows. They are transverse to each other except at the blown up annuli. The same notation is used for $\tilde{\Lambda}^s, \tilde{\Lambda}^u$, etc..

Because $\varphi^*$ is transverse to the foliation $\mathcal{F}$, then $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ are transverse to $\tilde{\mathcal{F}}$ and induce singular 1-dimensional foliations $\tilde{\Lambda}^s_\ast, \tilde{\Lambda}^u_\ast$ in any leaf $F$ of $\mathcal{F}$.

The objects perfect fits, lozenges, product regions, etc.. all make sense in the setting of almost pseudo-Anosov flows: they are just the blow ups of the same objects for the corresponding pseudo-Anosov flows. Since the interior of these objects does not have singularities, the blow up operation does not affect these interiors. There may be singular orbits in the boundary which get blown into a collection of annuli. All the results in this section still hold for almost pseudo-Anosov flows, with the blow up operation. For example if $F, L$ in $\tilde{\Lambda}^s$ are not separated from each other, then they are connected by an even number of lozenges all intersecting a common stable leaf. Since parts of the boundary of these may have been blown into annuli, there is not a product structure in the closure of the union of the lozenges, but there is still a product structure of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ in the interior.

4 Projections of leaves of $\tilde{\mathcal{F}}$ to the orbit space

Notation – In some of the proofs and arguments that follow we will be working with an almost pseudo-Anosov flow transverse to a foliation $\mathcal{F}$. In those cases, for notational simplicity we denote the almost pseudo-Anosov flow by $\Phi$ and a corresponding blow down pseudo-Anosov flow by $\Phi_1$. Hence $\Phi$ is some blow up of $\Phi_1$. This is different than the notation $\varphi, \varphi^*$ from the last section.

Let then $\Phi$ be an almost pseudo-Anosov flow which is transverse to a foliation $\mathcal{F}$. This implies that $\mathcal{F}$ is Reebless – we provide a proof of this at the end of this section. An orbit of $\Phi$ intersects a leaf of $\tilde{\mathcal{F}}$ at most once – because the leaves of $\tilde{\mathcal{F}}$ are properly embedded and $\tilde{\Phi}$ is transverse to $\tilde{\mathcal{F}}$. Hence the projection $\Theta : F \rightarrow \Theta(F)$ is injective. We want to determine the set of orbits a leaf of $\tilde{\mathcal{F}}$ intersects – in particular we want to determine the boundary $\partial \Theta(F)$. As it turns out, $\partial \Theta(F)$ is composed of a disjoint union of slice leaves in $O^s, O^u$.

We assume throughout this section that $\Phi$ is blow down minimal with respect to being transverse to $\mathcal{F}$. This means that no blow down of some flow annuli of $\Phi$ produces a flow transverse to $\mathcal{F}$.

Since $\Phi$ is transverse to $\mathcal{F}$, there is $\epsilon > 0$ so that if a leaf $F$ of $\tilde{\mathcal{F}}$ intersects an orbit of $\tilde{\Phi}$ at $p$ then it intersects every orbit of $\tilde{\Phi}$ which passes $\epsilon$ near $p$ and the intersection is also very near $p$. To understand $\partial \Theta(F)$ one main ingredient is that when considering pseudo-Anosov flows, then flow lines in the same stable leaf are forward asymptotic. So if $F$ intersects a given orbit in a very future time then it also intersects a lot of other orbits in the same stable since in future time they converge. In the limit this produces a stable boundary leaf of $\Theta(F)$. The blow up operation disturbs this: it is not true that orbits in the same stable leaf of an almost pseudo-Anosov flow are forward asymptotic; when they pass arbitrarily near a blow up annulus the orbits are distorted and their distance can increase enormously. This is the key difficulty in this section. Hence we first analyse the blow up operation more carefully.
Notation – Given $\Phi$ an almost pseudo-Anosov flow, let $\Phi'$ be a corresponding pseudo-Anosov flow associated to $\Phi$. The term $\tilde{W}^s(x)$ will denote the stable leaf of $\tilde{\Phi}$ or $\tilde{\Phi}'$, where the context will make clear which one it is.

Recall that $\pi : \tilde{M} \to M$ denotes the universal covering map.

We will start with $\Phi'$ and understand the blow up procedure. The blown up annuli come from singular orbits. The lift annuli are the lifts of blown up annuli to $\tilde{M}$. Their projections to $O$ are called blown segments. If $L$ is a blown up leaf of $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$ the components of $L$ minus the lift annuli are called the prongs. A quarter associated to an orbit $\gamma$ of $\Phi'$ is the closure of a connected component of $\tilde{M} - (\tilde{W}^u(\gamma) \cup \tilde{W}^s(\gamma))$. Its boundary is a union of $\gamma$ and half leaves in the stable and unstable leaves of $\gamma$. We will be interested in a neighborhood $V$ of $\gamma$ in this quarter which projects to $M$ near the closed orbit $\pi(\gamma)$. We will understand the blow up in the projection of a quarter. Glueing up different quarters gives the overall picture of the blow up operation. In the projected quarter $\pi(V)$ in $M$ there is a cross section to the flow $\Phi'$. The orbits across the cross section are determined by which stable and unstable leaf they are in. The return map on the stable direction is a contraction and an expansion in the unstable direction. Any contraction is topologically conjugate to say $x \to x/2$ and an expansion is conjugate to $x \to 2x$. Hence the local return map is topologically conjugate to

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

a linear map. The whole discussion here is one of topological conjugacy. The flow is conjugate to $(x, y, 0) \to (2^{-t}x, 2^ty, t)$. Think of the blow up annulus as the set of unit tangent vectors to $\gamma$ associated to the quarter region. The flow in the annulus is given by the action of $DV_t$ on the tangent vectors. It has 2 closed orbits (the boundary ones corresponding to the stable and unstable leaves). The other orbits are asymptotic to the stable closed orbit in negative time and to the unstable closed orbit in positive time. This makes it into a continuous flow in this blown up part. For future reference recall this fact that in a blow up annulus the boundary components are orbits of the flow and in the interior the flow lines go from one boundary to the other without a Reeb annulus picture (there is a cross section to the flow in the annulus). Do this for each quarter region that is blown up. One can then glue up the 2 sides of the appropriate annuli because they are all of the same topological picture (using the standard model above). This describes the blown up operation in a quarter. There is clearly a blow down map which sends orbits of the blown up flow $\Phi$ to orbits of $\Phi'$ and collapses connected unions of annuli into a single $p$-prong singular orbit.

We quantify these: let $\epsilon$ very small so that any two orbits of $\Phi'$ which are always less than $\epsilon$ apart in forward time, then they are in the same stable leaf. Let $Z$ the union of the singular orbits of $\Phi'$ which are blown up. Let $\epsilon' < \epsilon$ and let $U$ be the $\epsilon'$ tubular neighborhood of $Z$. Let $U'$ (resp. $U$) be the $\epsilon'/2$ (resp. $\epsilon'$) tubular neighborhood of $Z$. Choose the blow up map to be the identity in the complement of $U'$, that is the blown up annuli are also contained in $U'$. The blow down map is then an isometry of the Riemannian metric outside $U'$. Choose the blow down to move points very little in $U'$. Isotope $\tilde{F}$ so that it is transverse to the flow $\Phi$. We are now ready to analyse $\partial \Theta(F)$.

Proposition 4.1. Let $F$ in $\tilde{F}$. Then $\Theta(F)$ is an open subset of $O$. Any boundary component of $\Theta(F)$ is a slice of a leaf of $O^s$ or $O^u$. If it is a slice of $O^s$, then as $\Theta(F)$ approaches $I$, the corresponding points of $F$ escape in the positive direction. Similarly for unstable boundary slices.

Proof. First notice that since $F$ is transverse to $\tilde{\Phi}$ then $\Theta(F)$ is an open set. Hence $\partial \Theta(F)$ is disjoint from $\Theta(F)$. The important thing is to notice that the metric is the same outside the small neighborhood $U'$ of the blown up annuli. If two points are in the same stable leaf, then their orbits under the blow down flow $\Phi'$ are asymptotic in forward time. The same is true for $\Phi$, for big enough
time if the point is outside $U$. This is because the points of the corresponding orbits of $\Phi_1$ will be both outside $U'$ — this is the reason for the construction of two neighborhoods $U', U$. The following setup will be used in all cases.

**Setup** — Let $v$ in $\partial \Theta(F)$ and $v_i$ in $\Theta(F)$ with $v_i$ converging to $v$. Let $p_i$ in $F$ with $\Theta(p_i) = v_i$ and let $w$ in $\bar{M}$ with $\Theta(w) = v$. Let $D$ be an any small disk in $\bar{M}$ transverse to $\tilde{\Phi}$ with $w$ in the interior of $D$. For $i$ big enough $v_i$ is in $\Theta(D)$ so there are $t_i$ real numbers with $p_i = \tilde{\Phi}_{t_i}(w_i)$ and $w_i$ are in $D$. As $v$ is not in $\Theta(F)$, then $|t_i|$ grows without bound. Without loss of generality assume up to subsequence that $t_i \to +\infty$. Same proof if $t_i \to -\infty$. We will prove that there is a slice leaf $L$ of $\tilde{W}^s(w)$ so that $\Theta(L) \subset \partial \Theta(F)$ and $F$ goes up as it “approaches” $L$. The stable/unstable leaves here are those of the almost pseudo-Anosov flow and they may have blown up annuli.

One transversality fact used here is the following: for each $\epsilon$ sufficiently small, there is $\epsilon' > 0$ so that if $x, y$ are $\epsilon$ close in $M$ then the $\Phi$ leaf through $x$ intersects the $\Phi$-orbit through $y$ in a point $\epsilon'$ close to $x$. The $\epsilon'$ goes to 0 as $\epsilon$ does.

**Case 1** — Suppose that $w$ is not in a blown up leaf.

First we show that we can assume no $w_i$ is in $\tilde{W}^s(w)$. Otherwise up to subsequence assume all $w_i$ are in $\tilde{W}^s(w)$. The orbits through $w_i$ and $w$ start out very close and aside from the time they stay in $\pi^{-1}(U)$ they are always very close. Let $B$ be the component of the intersection of $F$ with the flow band from $\tilde{\Phi}_R(w_i)$ to $\tilde{\Phi}_R(w)$ in the stable leaf $\tilde{W}^s(w)$, which contains $p_i$. Then $B$ does not intersect $\tilde{\Phi}_R(w)$ so it has to either escape up or down. If it escapes down it will have to intersect a small segment from $w_i$ to $w$ and hence so does $F$. For $i$ big enough $w_i$ is arbitrarily near $w$, so transversality of $F$ and $\tilde{\Phi}$ then implies that $F$ will intersect $\tilde{\Phi}_R(w)$ near $w$, contradiction see fig. 4.

We now consider the case that $B$ escapes up. If the forward orbit through $w$ is not always in $\pi^{-1}(U)$ then at those times outside of $\pi^{-1}(U)$ it will be arbitrarily close to $\tilde{\Phi}_R(w_i)$ and transversality implies again that $F$ intersects $\tilde{\Phi}_R(w)$. If the forward orbit of $w$ always stays in $\pi^{-1}(U)$ the same happens after the blow down so the blow down orbit is in the stable leaf of the singular orbit which is being blown up. This does not happen in case 1.

We can now assume that all $w_i$ are in a sector of $O^s(v)$ with $l$ the boundary of this sector and $L = l \times R$, the line leaf of $\tilde{W}^s(w)$ which is the boundary of this sector.

Let now $q$ in $l$. We will show that $q$ is in $\partial \Theta(F)$ so $l \subset \partial \Theta(F)$. There is a segment $[q, v]$ contained in $l$. Choose $x$ in $L$ with $\Theta(x) = q$. Let $\alpha$ be a segment in $\tilde{W}^s(w)$ transverse to the flow lines and going from $x$ to $w$. Notice that $x, w$ are in $\tilde{M}$ and $q, v$ are in the orbit space $O$. Let $x_i$ converging to $x$ and $x_i$ in $\tilde{W}^s(w_i)$. We can do that since all $w_i$ are in the same sector of $\tilde{W}^s(w)$. Choose segments $\alpha_i$ from $x_i$ to $w_i$ in $\tilde{W}^s(w_i)$ and transverse to the flowlines of $\tilde{\Phi}$ in $\tilde{W}^s(w_i)$.

**Claim** — For every orbit $\gamma$ of $\tilde{\Phi}$ intersecting $\alpha_i$ in $y$ then $\gamma$ intersects $F$ in $\tilde{\Phi}_s(y)$ where $s$ converges to $\infty$ as $i \to \infty$.

Suppose there is $a_0 > 0$ so that for some $i_0$ then

$$\tilde{\Phi}_{[a_0, t_i]}(w_i) \subset \pi^{-1}(U) \quad \text{for all} \quad i \geq i_0$$

Then $\tilde{\Phi}_{[a_0, \infty]}(w)$ is contained in the closure of $\pi^{-1}(U)$. As seen before this implies that $w$ is in a blown up stable leaf, which is not the hypothesis of case 1. Therefore up to subsequence, there are arbitrary big times $s_i$ between 0 and $t_i$ so that $\tilde{\Phi}_{s_i}(w_i)$ is not in $\pi^{-1}(U)$. Hence $\tilde{\Phi}_R(x_i)$ is very close to $\tilde{\Phi}_{s_i}(w_i)$ and since $F$ cannot escape up or down then $F$ intersects $\tilde{\Phi}_R(x_i)$. Hence the segment $[\Theta(x_i), v_i]$ of $O^s(v)$ is contained in $\Theta(F)$ and so $[x, v]$ is contained in the closure of $\Theta(F)$. Also the
time $s$ so that $\tilde{\Phi}_s(y)$ hits $F$ goes to $\infty$, hence $[x,v]$ cannot intersect $\Theta(F)$ — else there would be bounded times where it intersects $F$, by transversality. We conclude that $[x,v] \subset \partial \Theta(F)$, hence $l \subset \partial \Theta(F)$ as desired. If there is a sequence $z_i$ in $F$ escaping down with $\Theta(z_i)$ converging to a point in $l$, then by connectedness there is one intersecting a compact middle region — this would force an intersection of $F$ with $l \times \mathbb{R}$ which is impossible.

This finishes the proof of case 1. In this case we proved there is a line leaf $l$ of $\Theta(L)$ with $l \subset \partial \Theta(F)$ and $F$ escapes up as $\Theta(F)$ approaches $l$.

Case 2 — $w$ is in a blown up leaf, but $F$ does not intersect a lift annulus in $\tilde{W}^s(w)$.

Refer to the setup above. As before we first show we can assume $w_i$ are not in $\tilde{W}^s(w)$. Otherwise, up to subsequence assume all $w_i$ are in $\tilde{W}^s(w)$. Since $F$ does not intersect lift annuli in $\tilde{W}^s(w)$, then $w_i$ are all in prongs of $\tilde{W}^s(w)$. Up to subsequence we can assume they are all in the same prong $C$ of $\tilde{W}^s(w)$ which has boundary an orbit $\gamma$ of $\Phi$. It follows that $w$ is in $\gamma$. All the orbits in $C$ are forward asymptotic to $\gamma$, even in the blown up situation. The strangling necks analysis of case 1 shows that $F$ will be forced to intersect $\tilde{\Phi}_R(w)$. This cannot occur.

Hence assume all $v_i$ are in a sector of $\mathcal{O}^s(v)$ bounded by a line leaf $l$. Let $L$ be $l \times \mathbb{R}$. Let $q$ be a point in $l$ and choose $x, \alpha, x_i$ and $\alpha_i$ as in the proof of case 1. Choose a small disc $D$ which is transverse to $\Phi$ and has $\alpha$ in its interior. For $i$ big enough then $D$ intersects lift annuli only in $\tilde{W}^s(x)$. This is because the union of the blown annuli forms a compact set in $M$, so either $\alpha$ intersects a lift annulus, in which case there is no other lift annulus nearby or $D$ is entirely disjoint from lift annuli. From now on the arguments of case 1 apply perfectly. This shows that $\Theta(L)$ is contained in $\tilde{\Theta}(F)$, it is disjoint from $\Theta(F)$ and so it is $\partial \Theta(F)$ and $F$ escapes up as it approaches $L$. This finishes the proof of case 2.

Now we need to understand what happens when $F$ intersects a lift annulus in general. We separate that in a special case. We need the following facts before addressing this case. A lift annulus $W$ through $b$ is contained in $\tilde{W}^s(b)$ and $\tilde{W}^u(b)$ so there is not stable/unstable flow directions in $W$. However there are still such directions in $\partial W$, because one attracts nearby orbits of $\tilde{\Phi}$ in $W$ and the other one repels nearby orbits in $W$. In this generalized sense the first one is stable and the second one is unstable. In this sense if $a$ is in an endpoint of a blown segment, then all local components of $\mathcal{O}^s(a) - a, \mathcal{O}^u(a) - a$ near $a$ are either generalized stable or unstable. With this understanding there is an even number of such components and they alternate between generalized stable and unstable. Some local components of $\mathcal{O}^s(a) - a$ are also local components of $\mathcal{O}^u(a) - a$ if they are blown segments. One key thing to remember is that generalized stable and unstable alternate.

Case 3 — Suppose that $F$ intersects some lift annulus $A$ contained in $\tilde{W}^s(w)$.

Then $F$ does not intersect both boundary orbits of $A$. Otherwise collapse $\pi(A)$ to a single orbit, still keeping $\Phi$ transverse to $\mathcal{F}$. This contradicts that $\Phi$ is blow down minimal with respect to $\mathcal{F}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{a. A strangling neck is being forced, b. A slice in a leaf of $\mathcal{O}^s$ or $\mathcal{O}^u$. $x_i = \Theta(z_i)$.
}
\end{figure}
Hence either $F \cap A$ is contained in the interior of $A$ or it intersects only one boundary leaf.

Assume without loss of generality that $F$ escapes up in one direction. This defines an orbit $\gamma$ of $\Phi$ with $a = \Theta(\gamma)$ in $\partial \Theta(F)$. The orbit $\gamma$ has to be in the boundary of the lift annulus $A$. This is because an interior orbit is asymptotic to both boundary orbits and hence would intersect $F$. We now look at the picture in $O$. Consider the stable leaf $O^s(a)$. Notice that $\Theta(F)$ intersects $\Theta(A)$. From the point of view of $\gamma$, orbits in $A$ move away from $\gamma$ in future time, that is $A$ is an unstable direction from $\gamma$. This means that $\Theta(A)$ is generalized unstable as seen from $a$. It follows that there are two generalized stable sides of $O^s(a)$ one on each side of $\Theta(A)$ which are the closest to $\Theta(A)$. Choose one side, start at $a$ and follow along $O^s(a)$ if needed through blown segments and eventually into a prong in $O^s(a)$ so as to produce a piece of a line leaf of $O^s(a)$ in that direction. This path is regular on the side associated to $\Theta(A)$ and defines a half leaf $l_1$ of $O^s(a)$. Similarly define $l_2$ in the other direction, see fig. b. Let $l$ be the union of $l_1$ and $l_2$. Then $l$ is a slice leaf of $O^s(a)$ but is not a line leaf since $\Theta(A)$ is in $O^s(a)$ and is not in $l$.

Claim – $l$ is contained in $\partial \Theta(F)$ and $F$ escapes positively as $\Theta(F)$ approaches $l$.

Let $b$ in $l_1$ with $b$ not in blown segment, that is, $b$ in a prong. Choose $b_1$ in $O^s(b)$, with $b_1 \to b$ and in that component of $O - l$. Let $D$ be an embedded disc in $\tilde{M}$ which is transverse to $\tilde{\Phi}$ and projects to $O$ to a neighborhood of the arc $\xi$ in $l_1$ from $a$ to $b$. Let $y_i$ in $D$ with $\Theta(y_i) = b_i$, $y_i \to y$ with $\Theta(y) = b$. Assume that $y$ is not in $\pi^{-1}(U)$. Choose $b$ so that it is not in the unstable leaf of one singular orbit, hence $\tilde{W}^s(y)$ does not contain lift annuli. In addition choose $y_i$ so that $\tilde{W}^s(y_i)$ does not contain lift annuli either.

Choose points $u_j$ in $F \cap A$ so that $\Theta(u_j) = a_j$ converges to $a$. For each $j$ the set $\Theta(F)$ contains a small neighborhood $V_j$ of $\Theta(u_j)$ with $V_j$ converging to $a$ when $j$ converges to infinity. Notice that $a$ is not in $V_j$ as $a$ is not in $\Theta(F)$. The leaves $O^s(b_i)$ are getting closer and closer to $l_1$ and $\Theta(A)$. For $j$ fixed there is $i$ big enough so that $O^s(b_i)$ intersects $V_j$. Let

$$z_i \in F \cap \tilde{W}^s(y_i) \text{ with } \Theta(z_i) \in V_j$$

here $i$ depends on $j$. Let $z_i = \tilde{\Phi}_{t_i}(r_i)$ with $r_i$ in $D$. By choosing $j$ and $i$ converging to infinity we get that $\Theta(z_i)$ converges to $a$ and we can ensure that the arc of $D \cap \tilde{W}^s(y_i)$ between $r_i$ and $y_i$ is converging to the arc $\eta$ of $\tilde{W}^s(\gamma) \cap D$ with $\Theta(\eta) = \xi$. We can also choose $V_j$ small enough so that $t_i$ converges to $+\infty$.

The orbits $\tilde{\Phi}_R(y_i), \tilde{\Phi}_R(r_i)$ are very close in the forward direction as long as they are outside $\pi^{-1}(U)$. Since $\tilde{W}^s(y_i)$ does not contain lift annuli then for times $s$ converging to infinity $\tilde{\Phi}_s(y_i)$ is not in $\pi^{-1}(U)$. Consider the flow band $C$ in $\tilde{W}^s(y_i)$ between $\tilde{\Phi}_R(r_i)$ and $\tilde{\Phi}_R(y_i)$. The leaf $F$ intersects $\tilde{\Phi}_R(r_i)$ in $\tilde{\Phi}_{t_i}(r_i)$ with $t_i$ converging to infinity. Then an analysis exactly as in case 1 considering strangling necks and the arcs $B$ in that proof, shows that $F \cap \tilde{W}^s(y_i)$ cannot escape up before intersecting $\tilde{\Phi}_R(y_i)$.

Suppose that $F$ escapes down before intersecting $\tilde{\Phi}_R(y_i)$. We show that this is impossible. Since $F \cap \tilde{W}^s(y_i)$ has points $z_i$ in the forward direction from $D$ and points in the backwards direction from $D$ it follows that $F \cap \tilde{W}^s(y_i)$ must intersect $D$ in at least a point $q_i$. Up to subsequence we may assume that $q_i$ converges to $q$ in $\tilde{W}^s(y)$. This will be an iterative process. Let $u_1 = q$. It is crucial to notice that in the flow band of $\tilde{W}^s(y)$ between $\tilde{\Phi}_R(y)$ and $\gamma$ the flow lines tend to go closer to $\gamma$, that is, either they project to closed orbits freely homotopic to $\pi(\gamma)$ or they are asymptotic to one of these orbits moving closer to $\gamma$. We now consider the component of $F \cap \tilde{W}^s(y)$ containing $u_1$ and follow it towards $\gamma$. This component does not intersect $\gamma$ and by the above it can only escape down in $\tilde{W}^s(y)$. As it escapes down it produces points $c_i$ in $\tilde{W}^s(r_i)$ and as before produces points $c_i'$ in $D$. 
which up to subsequence converge to c in D ∩ F. By construction c is not u_1 and its orbit is closer to γ. Let u_2 = c. We can iterate this process. Notice the u_i cannot accumulate in D, or else all the corresponding points of F are in a compact set of M. On the other hand the process does not terminate. This produces a contradiction.

The contradiction shows that in fact the arc Θ(C) is in Θ(F) which implies that ξ = Θ(η) is contained in Θ(F). As the time to hit F from D grows with i, this shows that Θ(F) does not intersect ξ and hence ξ is contained in Θ(F). As b is arbitrary this shows that l ⊂ ∂Θ(F) and F escapes up as Θ(F) approaches l. This finishes the analysis of case 3.

**Case 4** — w is in a blown up stable leaf and F intersects some lift annulus A in \( \widetilde{W}^s(w) \).

The difference from case 3 is that in case 3 we obtained a slice boundary l of Θ(F) — but in our situation we do not yet know if it contains Θ(w) and whether it is a stable or unstable. Here we prove it is a stable slice and it contains Θ(w).

Recall the setup: \( v = Θ(w) \) is in ∂Θ(F) and there are \( v_i \) in Θ(F) with \( v_i \) converging to v and with \( p_i \in (v_i × R) ∩ F \). Also \( p_i = \tilde{Φ}_{t_i}(w_i) \) with \( w_i \) converging to w in \( \widetilde{M} \) and \( t_i \) diverging to infinity.

Let ξ be the blown segment Θ(A).

The analysis of case 3 shows that Θ(F) contains the interior of Θ(A). Suppose first that v is in ξ. Then v is in the boundary of ξ and by case 3 again F escapes up or down when Θ(F) approaches a slice which contains v. If it escapes up, then the slice is a stable slice and we obtain the desired result in this case. We now show that F does not escape down. Let l be the unstable slice in ∂Θ(F) associated to this. Then l cuts in half a small disk neighborhood of v in O. The set Θ(F) intersects only one component of the complement, the one which intersects ξ. As F escapes down when Θ(F) approaches l, then for all points in Θ(F) near θ v the corresponding point in F is flow backwards from D. This contradicts the fact that \( t_i \) is converging to infinity. Therefore F cannot escape down as it approaches l.

We can now assume that v is not in ξ. By changing ξ if necessary assume that ξ is the blown segment in \( O^s(v) \) intersected by Θ(F) which is closest to v. Let z be the endpoint of ξ separating the rest of ξ from v in \( O^s(v) \).

We first show that z is not in Θ(F). Suppose that is not the case and let b the intersection point of z × R and F. Since ξ is the last blown segment of \( O^s(v) \) between ξ and v intersected by Θ(F) and Θ(F) contains an open neighborhood of z, it follows that v is in a prong B of \( O^s(v) \) with endpoint z. Let \( τ \) be the component of \( F ∩ \widetilde{W}^s(b) \) containing b. Since F does not intersect v × R then it escapes. As the region between \( \tilde{Φ}_R(b) \) and z × R (should it be v × R instead of z × R? previous report???) is a prong, then F cannot escape up. As seen in the arguments of case 3, F cannot escape down either. This shows that z cannot be in Θ(F).

It follows that F escapes either up or down as Θ(F) approaches z. Suppose first that it escapes up. Then we are in the situation of case 3 and we produce a stable slice l in ∂Θ(F) with F going up as Θ(F) approaches l. If v is not in l then l separates v from Θ(F). This contradicts \( v_i \) in Θ(F) with \( v_i \) converging to v. Hence v is in l with F escaping up as Θ(F) approaches l. This is exactly what we want finishing the analysis in this case.

The last situation is F escaping down in A as Θ(F) approaches z. By case 3 there is a slice leaf l in \( O^u(z) \) with l contained in ∂Θ(F) and F escaping down as Θ(F) approaches l. We want to show that this case cannot happen. Notice that the blown segments of \( O^u(z) \) are exactly the same as the blown segments of \( O^s(z) \). The sets \( O^u(z), O^u(z) \) differ only in the prongs and as they go around the collection of blown segments. The collection of all prongs in \( O^u(z), O^u(z) \) also alternates between stable and unstable as it goes around the union of the blown segments.

Suppose first that v is in l. This contradicts F escaping down and \( t_i \) → ∞. Finally suppose that v is not in l. We claim that in this case l separates v from Θ(F). Let α be the path in \( O^u(v) \) from
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If $\alpha$ only intersects $l$ in $z$, then the separation property follows because $l_1$ and $l_2$ contain the local components of $O^s(z) \cup O^u(z) - z$ which are closest to $\Theta(A)$. This was part of the construction of $l$ in case 3. Here the $\xi$ is generalized stable at $z$ and $l_1, l_2$ are generalized unstable at $p$. The path from $z$ to $v$ in $O^s(v)$ cannot start in $\xi$ or $l_1$ or $l_2$, hence $l$ separates $\Theta(F)$ from $v$.

If on the other hand $\alpha \cap l = \delta$ is not a single point, then it is a union of blown segments. Let $u$ be the other endpoint of $\delta$. By regularity of $l_1$ and $l_2$ on the $\Theta(F)$ side it follows that each blown up segment in $\delta$ has flow direction away from $z$. Hence $\delta$ is generalized stable at $u$. Therefore the closest component of $O^s(u) \cup O^u(u) - u$ on the $\Theta(F)$ side is generalized unstable and that is contained in $l$. In this case it also follows that $l$ separates $v$ from $\Theta(F)$. As seen before this is a contradiction.

This finishes the proof of proposition 4.1.

This has an important consequence that will be used extensively in this article.

**Proposition 4.2.** Let $F$ in $\tilde{F}$ and $L$ in $\tilde{L}^s$ or $\tilde{L}^u$. Then the intersection $F \cap L$ is connected.

**Proof.** By transversality of $F$ and $\Phi$, the intersection $C = \Theta(F) \cap \Theta(L)$ is open in $\Theta(L)$. Suppose there are 2 disjoint components $A, B$ of $C$. Then there is $v$ in $\partial A$ with $v$ separating $A$ from $B$. There are $v_i$ in $A$ with $v_i$ converging to $v$. By the previous proposition $F$ escapes up or down in $A \times \mathbb{R}$ as $\Theta(F)$ approaches $v$. Assume wlog that $F$ escapes up. Then there is a slice leaf $l$ of $O^s(v)$ with $l \subset \partial \Theta(F)$ and $F$ escapes up as $\Theta(F)$ approaches $l$. Since $l$ and $\Theta(F)$ are disjoint then $B$ is disjoint from $l$. In addition $v$ separates $B$ from $A$ in $\Theta(L)$. It follows from the construction of the slice $l$ as being the closest to $A$, that $l$ separates $A$ from $B$. Hence $\Theta(F)$ cannot intersect $B$, contrary to assumption. This finishes the proof.

As promised, we now prove that $\mathcal{F}$ being almost transverse to a pseudo-Anosov flow implies that $\mathcal{F}$ is Reebless.

**Proposition 4.3.** Let $\mathcal{F}$ be a foliation almost transverse to a pseudo-Anosov flow $\Phi_1$ and transverse to a corresponding almost pseudo-Anosov flow $\Phi$. Then $\mathcal{F}$ is Reebless.

**Proof.** Suppose that $\mathcal{F}$ is not Reebless and consider a Reeb component which is a solid torus $V$ bounded by a torus $T$. Assume that the flow $\Phi$ is incoming along $T$.

Recall that there are some singular orbits of $\Phi_1$ which blow up into a collection of flow annuli of $\Phi$. Suppose that $V$ intersects one of these annuli $A$. Then since $\Phi$ is incoming along $T$, the torus $T$ cannot intersect the closed orbits in $\partial A$. Hence it intersects the interior of $A$, say in a point $p$ and the forward orbit of $p$ will limit in a closed orbit which is contained in the interior of $V$.

If on the other hand $V$ does not intersect these blown annuli then the blow down operation does not affect the flow in $V$. That means we can assume that $\Phi_1$ is equal to $\Phi$ in $V$. Since orbits of $\Phi_1$ are trapped inside $V$ once they enter $V$, the shadow lemma for pseudo-Anosov flows [Han, Man, Mol], shows that there is also a periodic orbit of $\Phi_1$ (and hence also of $\Phi$) in $V - T$. Notice that the shadow lemma is for pseudo-Anosov flows and not for almost pseudo-Anosov flows and that is why we split the analysis into 2 cases.

In any case there is a closed orbit $\gamma$ of $\Phi$ contained in the interior of $V$. Consider the generalized stable/unstable local leaves at $\gamma$. Since $\Phi$ is incoming along $T$, the generalized unstable leaves have to be contained in $V$. We eventually obtain that a whole half leaf of $W^u(\gamma)$ is contained in $V$. A lift $\tilde{V}$ to $\tilde{M}$ is homeomorphic to $D^2 \times \mathbb{R}$, because closed orbits of $\Phi$ are not null homotopic. The procedure above produces a half leaf of $\tilde{W}^u(\gamma)$ contained in $\tilde{V}$. This contradicts the fact that $\tilde{W}^u(\gamma)$ is properly embedded [Ga-Oc]. This shows that $\mathcal{F}$ is Reebless.
5 Asymptotic properties in leaves of the foliation

Let \( \Phi \) be an almost pseudo-Anosov flow transverse to a foliation \( \mathcal{F} \) with hyperbolic leaves. Let \( \Lambda^s, \Lambda^u \) be the singular foliations of \( \Phi \). Given leaf \( F \) of \( \tilde{\mathcal{F}} \) let \( \tilde{\Lambda}^s_F, \tilde{\Lambda}^u_F \) be the induced one dimensional singular foliations in \( F \). In this section we study asymptotic properties of rays in \( \tilde{\mathcal{F}} \). First we mention a result of Thurston [Th5] concerning contracting directions, which for convenience we state for 3-manifolds:

**Theorem 5.1.** (Thurston) Let \( \mathcal{F} \) be a codimension one foliation with hyperbolic leaves in \( M^3 \) closed. Then for every \( x \) in any leaf \( F \) of \( \mathcal{F} \) and every \( \epsilon > 0 \) there is a dense set of geodesic rays of \( \mathcal{F} \) starting at \( x \) such that: for any such ray \( r \) there is a transversal \( \beta \) to \( \mathcal{F} \) starting at \( x \) so that any leaf \( L \) intersecting \( \beta \) and any \( y \) in \( r \), then the distance between \( y \) and \( L \) is less than \( \epsilon \). If there is not a holonomy invariant transverse measure whose support contains \( \pi(F) \) then one can show that the directions are actually contracting, that is: if \( y \) escapes in \( r \) then the distance between \( y \) and \( L \) converges to 0. Finally if \( \pi(F) \) is not closed one can choose the \( \beta \) above to have \( x \) in the interior.

There is a carefully written published version of this result in [Ca-Du]. The directions above where distance to nearby \( L \) goes to 0 are called contracting directions. The other ones where distance is bounded by \( \epsilon \) are called \( \epsilon \) non expanding directions.

The goal of this section is to show that given a leaf \( L \) of \( \tilde{\mathcal{F}} \) and a ray \( l \) of \( \tilde{\Lambda}^s_L \), then \( l \) converges to a single point in \( \partial_\infty L \). We first analyse the non \( R \)-covered case. The proof is very involved and is done by way of contradiction. Later we deal with the \( R \)-covered situation. This result is a natural extension of a result by Levitt [Le] who proved that if \( \mathcal{G} \) is a foliation with prong singularities in a closed hyperbolic surface, then in the universal cover, an arbitrary ray converges to a single point in the circle at infinity. The situation for non compact leaves of foliations is much more delicate.

**Proposition 5.2.** Let \( \Phi \) be an almost pseudo-Anosov flow transverse to a foliation \( \mathcal{F} \) in \( M^3 \) closed and \( \mathcal{F} \) with hyperbolic leaves. Suppose that \( \mathcal{F} \) is not \( R \)-covered. Given a leaf \( L \) of \( \tilde{\mathcal{F}} \) and an arbitrary ray \( l \) in a leaf of \( \tilde{\Lambda}^s_L \) or \( \tilde{\Lambda}^u_L \) then \( l \) limits to a single point in \( \partial_\infty L \). The limit depends on the ray \( l \).

*Proof.* We do the proof for \( \tilde{\Lambda}^s_L \). Let \( \epsilon \) positive so that if \( p \) in \( \tilde{M} \) is less than \( \epsilon \) from a leaf \( F \) of \( \tilde{\mathcal{F}} \), then the flow line through \( p \) intersects \( F \) less than \( 2\epsilon \) away from \( p \). Let \( l \) be a ray in \( \tilde{\Lambda}^s_L \). Because \( \mathcal{F} \) and \( \Phi \) are transverse, \( L \) is properly embedded in \( \tilde{M} \) and leaves of \( \tilde{\mathcal{F}} \) are properly embedded, it follows that \( l \) is a properly embedded ray in \( L \). Therefore it can only limit in \( \partial_\infty L \).

Suppose by way of contradiction that \( l \) limits on 2 distinct points \( a_0, b_0 \) in \( \partial_\infty L \). Fix \( p \) a basepoint in \( L \). Since \( l \) limits in \( a_0, b_0 \), there are compact arcs \( l_i \) of \( l \) with endpoints which converge to \( a_0, b_0 \) respectively in \( L \cup \partial_\infty L \) and so that the distance from \( l_i \) to \( p \) in \( L \) converges to infinity. Also we can assume that the \( l_i \) converges to a segment \( v \) in \( \partial_\infty L \), where \( v \) connects \( a_0, b_0 \). This is in the Hausdorff topology of closed sets in \( L \cup \partial_\infty L \), which is a closed disk.

The key idea is to bring this situation to a compact part of \( \tilde{M} \). Choose a sequence \( p_i \) at bounded distance from points in \( l_k \), so that that \( p_i \) converges to a point \( a \) in the interior of \( v \). The bound depends on the sequence. Up to subsequence assume that there are converging translations \( g_i \) in \( \pi_1(\tilde{M}) \) and a point \( p_0 \) in \( \tilde{M} \) so that \( g_i(p_i) \) converges to \( p_0 \) in \( \tilde{M} \).

We claim that the set of possible limits \( p_0 \) obtained as above projects to a sublamination of \( \mathcal{F} \). Clearly if \( g_i(p_i) \) converges to \( p_0 \) and \( q \) is in the same leaf \( L_0 \) of \( \tilde{\mathcal{F}} \) as \( p_0 \), then the distance from \( p_0 \) to \( q \) is finite and there are \( q_i \) in \( L \) with \( d_L(q_i, p_i) \) bounded and \( g_i(q_i) \) converging to \( q \). Also \( q_i \) converges to \( a \) in \( \partial_\infty L \). In addition if a sequence of such limits \( c_j \) converges to \( c_0 \) then a diagonal process shows that \( c_0 \) is also obtained as a single limit. This proves the claim. Choose a minimal sublamination \( L \).

A leaf \( F \) of \( \tilde{\mathcal{F}} \) is isometric to the hyperbolic plane. A *wedge* \( W \) in \( F \) with corner \( b \) and ideal set an interval \( B \subset \partial_\infty F \) is the union of the rays in \( F \) from \( b \) with ideal point in \( B \). The angle of the
wedge is the angle that the boundary rays of $W$ make at $b$. For any such sequence $p_i$ as above, then the visual angle at $p_i$ subintended by the arc $v$ in $\partial_\infty L$ grows to $2\pi$. Therefore the angle of wedge with corner $p_i$ and ideal set $\partial_\infty L - v$ converges to 0. This is called the bad wedge.

Assume up to subsequence that $g_i(p_i)$ is converging to $p_0$ in a leaf $L_0$ of $\overline{F}$ and that the directions of the bad wedges with corners $g_i(p_i)$ in $g_i(L)$ are converging to the direction $r_0$ of $L_0$. Let $c$ be the ideal point of $r_0$ in $\partial_\infty L_0$.

Suppose first that $\pi(L_0)$ is not compact — we shall see briefly that this is in fact always the case. Thurston’s theorem shows that the set of two sided contracting directions (or $\epsilon$ non expanding directions) in $L_0$ is dense in $\partial_\infty L_0$. We will use these to transport a lot of the structure of $\tilde{\Lambda}^*_{L_0}$ to nearby leaves. Choose $s_0, s_1$ to be rays in $L_0$ defining contracting directions (or $\epsilon$ non expanding directions) very near $r_0$ so that together they form a small wedge $W$ in $L_0$ with corner $p_0$. There is an interval of leaves near $L_0$ so that any such leaf $V$ is less than $\epsilon$ away from $s_0, s_1$. Then a flow line of $\overline{\Phi}$ through any point in $s_0$ or $s_1$ intersects $V$ less than $2\epsilon$ away. So $s_0$ flows to a curve in $V$, where we can assume it has geodesic curvature very close to 0, if $\epsilon$ is sufficiently small. It is therefore a quasigeodesic with a well defined ideal point. The same happens for $s_1$ and the flow images $u_0, u_1$ of $s_0, s_1$ in $V$ define a generalized wedge $W'$ in $V$. The ideal points $e_0, e_1$ of $u_0, u_1$ are close and bound an interval $I$ which is almost all of $\partial_\infty g_i(L)$.

By construction $g_i(l)$ is a ray which limits in an interval of $\partial_\infty g_i(L)$ which contains $I$ in its interior if $i$ is big enough. There are then subarcs $\tau_j$ of $g_i(l)$ with endpoints $a_j, b_j$ in $u_0, u_1$ respectively so that $a_j$ converges to $e_0$ and $b_j$ converges to $e_1$ and $\tau_j$ converges to $I$, see fig. Here $i$ is fixed and $j$ varies. Since $a_j, b_j$ are in $u_0, u_1$ then they flow (by $\overline{\Phi}$) to points in $L_0$. The images in $L_0$ are in the same leaf of $\tilde{\Lambda}^*$. By proposition these images are in the same leaf of $\tilde{\Lambda}^*_{L_0}$. Hence the whole segment $\tau_j$ flows into $L_0$.

The point $p_0$ flows into $p'$ in $g_i(L)$ under the flow. The arc $\tau_j$ together with subarcs or $u_0, u_1$ from $a_j, b_j$ to $p'$ bound a disc $D_j$ in $g_i(L)$. The arguments above show that the boundary of $D_j$ flows into $L_0$ bounding a disc $B_j$. The segments of $\overline{\Phi}$ connecting points in $\partial D_j$ to points in $\partial B_j$ produce an annulus $C_j$. Then $D_j \cup C_j \cup B_j$ is an embedded sphere in $\tilde{M}$ and hence bounds an embedded ball. Since orbits of $\overline{\Phi}$ are properly embedded in $\tilde{M}$, it follows that all orbits of $\overline{\Phi}$ intersecting $D_j$ will also intersect $B_j$. Hence there is product flow in this ball. Since this is true for all $j$ then the union of the $D_j$ flows into $L_0$. The union of the $D_j$ is the closure of $g_i(L) - W'$. The image is contained in the closure of the closure of $L_0 - W$ in $L_0$ — call the closure $J$.

We claim that the image is in fact $J$. All the $\tau_j$ are in the same leaf of $\tilde{\Lambda}^*$ and hence all their flow images in $L_0$ also are. Since rays of $\tilde{\Lambda}^*_{L_0}$ are properly embedded in $L_0$ then when $j$ converges to infinity the images of $\tau_j$ in $L_0$ escape compact sets. This shows the claim. Therefore the flow produces a homeomorphism between the closure of $L_0 - W$ and the closure of $g_i(L) - W'$. Clearly
the same is true for any leaf in the interval associated to the contracting (non expanding) directions \( s_0, s_1 \). In particular we have the following conclusions:

**Conclusion** — In any limit leaf \( L_0 \) with a limit direction \( r_0 \) of bad wedges the following happens: Let \( c \) be the ideal point of \( r_0 \) and \( A \) a closed interval of \( \partial_\infty L_0 - \{c\} \). Then there is a leaf \( l \) of \( \Lambda^g_{L_0} \) with compact subsegments \( l_i \) so that the endpoints of \( l_i \) converge to the endpoints \( a, b \) of \( A \) and \( l_i \) converges to \( A \). In particular \( l_i \) escapes compact sets. There are also subsegments \( v_i \) with both endpoints converging to \( a \) and so that \( v_i \) converges to sets in \( \partial_\infty L_0 \) which contain \( A \). Finally for sufficiently near leaves there is a wedge in \( L_0 \) which forms a product flow region with these nearby leaves.

To get the second assertion above just follow \( l \) beyond the endpoint of \( l_i \) near \( b \) until it returns near \( a \) again. As a preliminary step to obtain proposition 5.2 we prove the following:

**Lemma 5.3.** For any limit \( g_i(p_i) \) converging to \( p_0 \), the distinguished direction of the bad wedge associated to \( g_i(p_i) \) converges to a single direction at \( p_0 \). In the second case this direction varies continuously with the leaves in \( \mathcal{L} \).

**Proof.** Suppose there are subsequences \( q_i, p_i \) converging to points in \( \mathcal{F} \) but the directions of the wedges converge to \( r_0, r_1 \) distinct geodesic rays in \( L_0 \). We will first show that there is an interval of leaves of \( \mathcal{F} \) so that the flow \( \Phi \) is a product flow in this region.

Using the limit direction \( r_0 \) we produce a wedge \( W \) in \( L_0 \) so that the closure of \( L_0 - W \) is part of a product flow region with nearby leaves of \( \mathcal{F} \). Using the other limit direction \( r_1 \) we produce a flow product region associated to another wedge region \( W_s \) disjoint from \( W - p_0 \). Together they produce a global product structure of the flow in a neighborhood of \( L_0 \).

This shows that there is a neighborhood \( N \) of \( L_0 \) in the leaf space of \( \mathcal{F} \) so that the flow is a product flow in \( N \). In particular there is no non Hausdorffness of \( \mathcal{F} \) in this neighborhood. This is a very strong property as we shall see below. It implies a global product structure of the flow.

Notice that the structure of \( \Lambda^{\text{g}_i}\mathcal{L} \) in \( g_i(L) - W' \) flows over to \( L_0 \). In particular there are many rays of \( \Lambda^{\text{g}_i}_{L_0} \) which do not have a single limit in \( \partial_\infty L_0 \). This implies that \( \pi(L_0) \) is not compact. This is because Levitt [Lc] proved that given any singular foliation with prong singularities in a closed hyperbolic surface \( R \), then the rays of the lift to \( \bar{R} \) all have unique limit points in the ideal boundary. This shows that the minimal laminations \( \mathcal{L} \) is not a compact leaf and hence it has no compact leaves.

Consider the neighborhood \( N \) as above. Consider the translates \( g(N) \) where \( g \) runs through all elements of the fundamental group. Let \( P \) be the component of the union containing \( N \). It is easy to see that the set \( P \) is precisely invariant: if \( g \) is in \( \pi_1(M) \) and \( g(P) \) intersects \( P \) then \( g(P) \) is equal to \( P \). In addition \( \mathcal{F} \) restricted to \( P \) has leaf space homeomorphic to \( \mathcal{R} \) because of the product flow property. We are assuming that \( N \) is open.

Suppose first that \( P \) is not all of \( M \), hence \( \partial P \) is a non empty collection of leaves of \( \mathcal{F} \). Let \( C \) be the projection of \( P \) to \( M \). Then \( C \) is open, saturated by leaves of \( \mathcal{F} \). Notice that \( g(P) \) does not intersect \( \partial P \) for any \( g \) in \( \pi_1(M) \) for otherwise \( g(P) \) intersects \( P \) and so \( g(P) = P \). It follows that \( \pi(\partial P) \) is disjoint from \( C \) hence \( C \) is a proper open, foliated subset of \( M \).

Dippolito [D] developed a theory of such open, saturated subsets. Let \( \overline{C} \) be the metric completion of \( C \). There is an induced foliation in \( \overline{C} \), which we will also denote by \( \mathcal{F} \). Then

\[
\overline{C} = V \cup \bigcup_{1}^{n} V_i
\]
where $V$ is compact and may be all of $\mathcal{C}$. Each nonempty $V_i$ is an $I$-bundle over a non compact surface with boundary, so that $\mathcal{F}$ is a foliation transverse to the $I$-fibers. Each component of the intersection $\partial V_i \cap V$ is an annulus (or Moebius band) with induced foliation transverse to the $I$ fibers.

In our situation with $\Phi$ transverse to the flow, if $V$ is not $\mathcal{C}$, we can choose $V$ big enough so that the flow is transverse to $\mathcal{F}$ in each $V_i$ and induces an $I$-fibration there.

Let $\tilde{R}$ be a component of $\partial P$ and $R$ the projection to $M$, so $R$ is component of $\partial C$. Parametrize the leaves of $\tilde{F}$ in $P$ as $F_t$, $0 < t < 1$ with $t$ increasing with flow direction. A leaf in the boundary of $P$ which is the limit of leaves in $P$ which are limiting from the positive side above has to be the limit of $F_t$ as $t$ goes to $0$:

Suppose that $S$ is in the boundary of $P$ and there are $x_i$ in $F_{t_i}$ with $t_i$ converging to $t_0 > 0$ and $x_i$ converging to $x$ in $S$. Then $S$ and $F_{t_0}$ are not separated from each other. For $i$ big enough the flow line through $x_i$ will intersect $S$ and therefore this flow line will not intersect $F_{t_0}$.

This contradicts the fact that $F_{t_0}$ and $F_{t_i}$ have a flow product structure.

Suppose then that $\tilde{R}$ is a limit of $F_t$ where $t$ converges to $0$. Suppose first that $R$ is compact. Suppose there are $t_i$ converging to $0$ so that $F_{t_i}$ are in $\mathcal{L}$. Then since $\mathcal{L}$ is a closed subset of $M$ it follows that $\tilde{R}$ is in $\mathcal{L}$ and so $R$ is in $\mathcal{L}$. But $R$ is closed, contradicting the fact that $\mathcal{L}$ has no closed leaves.

There is then $a > 0$ which is the smallest $a$ so that $F_a$ is in $\mathcal{L} - \text{ notice that } \mathcal{L}$ has leaves in $P$. For any $g$ in $\pi_1(R)$ then $g(N) \cap N$ is not empty hence $g(N) = N$. It follows that $g(F_a) = F_b$ for some $b$. If $b$ is not $a$ then by taking $g^{-1}$ if necessary we may assume that $b < a$. But as $F_b$ is in $\mathcal{L}$, this contradicts the definition of $a$. Hence $g(F_a) = F_a$ for any $g$ in $\pi_1(R)$. This implies that $\pi(F_a)$ is a closed surface, again contradiction.

We conclude that $R$ is not compact, hence it eventually enters some $V_i$ (the point here is that $V$ is not $\mathcal{C}$). The flow restricted to any component of $\partial V_i \cap \mathcal{C}$ goes from one component to the other in the annulus. This implies that all $\pi(F_i)$ intersect this annulus. There is then a leaf $B$ of $\mathcal{L}$ which enters $V_i$. Going deeper and deeper in this non compact $I$-bundle will produce a limit point which is not in $C$. This shows the very important fact that $\mathcal{L}$ is not contained in $C$ and therefore

$$\mathcal{E} = \mathcal{L} \cap (M - C) \neq \emptyset$$

In addition $\mathcal{E}$ is not equal to $\mathcal{L}$ since $\mathcal{L}$ has leaves in $C$ and $(M - C)$ is closed. Hence $\mathcal{E}$ is a non trivial, proper sublamination of $\mathcal{L}$. This contradicts the fact that $\mathcal{L}$ is a minimal lamination.

This shows that the assumption $P \neq \tilde{M}$ is impossible. Hence $P = \tilde{M}$, which implies the flow $\Phi$ produces a global product picture of $\tilde{F}$ and in particular $\mathcal{F}$ is $\mathcal{R}$-covered, contrary to assumption.

This shows the limits of the bad wedges are unique directions in the limit leaves. It also shows that they vary continuously from leaf to leaf, for otherwise one obtains bad wedges in very near leaves which have definitely separated directions. The same proof above then applies. This finishes the proof of lemma 5.3.

Continuation of the proof of proposition 5.2

By the previous lemma we know that limit directions of bad wedges are unique and they vary continuously in leaves of $\mathcal{L}$. These unique directions are distinguished in their respective leaves.

We first show that any complementary region of $\mathcal{L}$ (if any) is an $I$-bundle with a product flow.

Lift to a double cover if necessary to assume that $M$ is orientable. Assume this is the original foliation $\mathcal{F}$, flow $\Phi$, etc.. Let $Z$ be a leaf of $\mathcal{L}$. Since $Z$ has a distinguished ideal point, then the fundamental group of $\pi(Z)$ can be at most $\mathbb{Z}$. Since there is a transverse flow and $M$ is orientable this implies that $\pi(Z)$ is either a plane or an annulus.

Let $U$ be a complementary region of $\mathcal{L}$ with boundary leaves $R_1, R_2, R_3, \ldots$. As explained before the completion of $U$ has a compact thick part and the non compact arms which are in thin, $I$-bundle regions. Suppose first that $R_1$ is a plane. There is a big disk $D$ so that $R_1 - D$ is contained in the
thin arms and flows across $U$ to another boundary components of $U$. By connectedness it flows into a single boundary component $R_2$ of $U$. Then $\partial D$ flows into a curve $\gamma$ in $R_2$ which is null homotopic in $M$. The flow segments in $M$ produce an annulus $C$ in the completion of $U$. Since $\mathcal{F}$ is Reeble then $\gamma$ bounds a disk $D'$ in $R_2$ and so $R_2$ is a plane. The union $D \cup C \cup D'$ is an embedded sphere in $M$ which bounds a ball $B$. Since orbits of $\Phi$ are properly embedded in $M$, it follows that the flow has to a product flow in $B$ as well. This shows that flow is a product in the completion of $U$.

Suppose now that each $R_i$ is an annulus. Let $F$ be a lift of $R_1$ to $\tilde{M}$ with $F$ in the boundary of a component $\tilde{U}$ of $\pi^{-1}(U)$. In $R_1$ there are two disjoint open annuli $A_1, A_2$ contained in the thin arms so that $B = R_1 - (A_1 \cup A_2)$ is a closed annulus in the core. Then $A_1, A_2$ flow into two annuli leaves $R_2, R_3$ in the boundary of $U$. Lifting to $F = \tilde{R}_1$ we see leaves of $\tilde{\Lambda}_F^s$ limiting in an interval of $\partial_\infty F$ with very small complement (near the distinguished ideal point of $F$). This implies they will have points in the lifts $\tilde{A}_1, \tilde{A}_2$ of $A_1, A_2$ to $F$. This shows that $\tilde{A}_1, \tilde{A}_2$ are in the same leaf of $\tilde{F}$. This implies that $R_2 = R_3$. In the same way a half of the infinite strip $\tilde{B}$ flows into $\tilde{R}_2$. Since $B$ is compact, then all of $B$ flows into $R_2$. This implies that the region $U$ is an I-bundle. It is also easy to show that the flow is a product in this I-bundle.

This implies that we can collapse this complementary region along flow lines to completely eliminate it. This is because in the universal cover we are eliminating product regions of the flow and the asymptotic behavior is still preserved in the remaining regions. This can be done to all complementary regions and therefore we can assume there are no complementary regions, that is $\mathcal{L} = \mathcal{F}$ or that $\mathcal{F}$ is minimal.

Let $F_1, F_2$ be leaves of $\tilde{F}$ which are not separated from each other. Consider leaves $F$ of $\tilde{F}$ which are very close to points in both $F_1$ and $F_2$. As stated in the conclusion in the beginning of the proof of this theorem, there is a wedge of $F$ which flows into $F_1$ and similarly for $F_2$. Hence there are half planes $E_1, E_2$ of $F$ which flow into $F_1, F_2$. As $F_1, F_2$ are not separated this implies that $E_1, E_2$ are disjoint. Fix a point $w$ in $F$ and a big enough radius $r$ so that the disk $D$ of radius $r$ around $w$ intersects both $E_1, E_2$. Again as seen in the conclusion above there is an arc $l$ in a leaf of $\tilde{\Lambda}_F^s$ so that both endpoints of $l$ are outside $D$ and in $E_1$ and so that $l$ is entirely outside $D$ and as seen from $p$ the visual measure of $l$ is almost $2\pi$. This implies that $l$ intersects $E_2$. Since the endpoints of $l$ are in $E_1$, which flows to $F_1$, then proposition 4.2 implies that the whole arc $l$ flows into $F_1$. The points of $l$ in $E_2$ will also flow to $F_2$. This is a contradiction.

This contradiction finishes the proof of proposition 5.2.

Next we analyse the $\mathbb{R}$-covered situation which has interest on its own:

**Theorem 5.4.** Let $\mathcal{F}$ be an $\mathbb{R}$-covered foliation and $\Phi$ be a pseudo-Anosov flow almost transverse to $\mathcal{F}$. Then $\Phi$ is actually transverse to $\mathcal{F}$. In addition for any leaf $F$ of $\tilde{\mathcal{F}}$ and for any ray $l$ in $\tilde{\Lambda}_F^s$, it converges to a unique ideal point in $\partial_\infty F$. The limit usually depends on $l$.

*Proof.* If $\Phi$ is not transverse to $\mathcal{F}$, let $\Phi^*$ be an almost pseudo-Anosov flow which is transverse to $\mathcal{F}$ and is a blow up of $\Phi$. Notice this is not the same notation as in proposition 5.2 — here we prove $\Phi$ is equal to $\Phi^*$. There is flow annulus $A$ of $\Phi^*$ with closed orbits $\gamma_1, \gamma_2$ in the boundary, so that $A$ blows down to a single orbit of $\Phi$.

The foliation induced by $\mathcal{F}$ in $A$ has leaves which spiral to at least one boundary component — which they do not intersect. Lifting this picture to the universal cover one obtains an orbit of $\tilde{\Phi}^*$ which does not intersect every leaf of $\tilde{\mathcal{F}}$. This means that the flow $\tilde{\Phi}^*$ is not regulating for $\tilde{\mathcal{F}}$ [Th6, Th7]. We also say that $\Phi^*$ does not regulate $\mathcal{F}$. In [Fe9] we analysed a similar situation and proved the following: if $\mathcal{Y}$ is a pseudo-Anosov flow transverse to an $\mathbb{R}$-covered foliation and $\mathcal{Y}$ is not regulating, then $\mathcal{Y}$ is an $\mathbb{R}$-covered Anosov flow. The same arguments work with an almost pseudo-Anosov flow transverse to an $\mathbb{R}$-covered foliation. This shows that $\Phi^*$ is an $\mathbb{R}$-covered Anosov flow.
and has no (topological) singularities. In particular $\Phi^*$ is equal to $\Phi$, that is the original flow is already transverse to $\mathcal{F}$. This proves the first assertion of the theorem.

Assume by way of contradiction that there is $L'$ in $\hat{\Lambda}^s$ and $l$ in $\hat{\Lambda}_s^{u, \partial}$ which does not converge to a single point in $\partial_\infty L'$. As in the proof of theorem 5.2, we construct a minimal sublamination $\mathcal{L}$ of $\mathcal{F}$ such that: for every $L$ in $\hat{\mathcal{L}}$ there is an ideal point $u$ in $\partial_\infty L$ so that for every closed segment $J$ in $\partial_\infty L - \{u\}$ there is a ray $l$ of $\hat{\Lambda}_s^u$ which has subsegments limiting to $J$. As shown in the proof of theorem 5.2 $\mathcal{L}$ cannot be a compact leaf.

Suppose first that every leaf of $\mathcal{F}$ is a plane. Then Rosenberg [Res] proved that $M$ is the 3-dimensional torus $T^3$. This manifold is a Seifert fibered space. In this case Brittainham [Br1] proved that an essential lamination is isotopic to one which is either vertical (a union of Seifert fibers) or horizontal (transverse to the fibers). So after isotopy assume $\mathcal{L}$ has one of these types. If $\mathcal{L}$ has a vertical leaf $B$, then geometrically it is a product of the reals with the circle. Hence it is an Euclidean leaf and in the universal cover it has polynomial growth of area. If $\mathcal{L}$ has a horizontal leaf $B$, then because the fibration is a product, there is a projection to a $T^2$ fiber, which distorts distances by a bounded amount. Again the same growth properties hold. But the leaves of $\mathcal{F}$ are hyperbolic, which is a contradiction. We conclude that $M$ cannot be $T^3$.

Let then $F$ in $\hat{\mathcal{L}}$ with $\pi(F)$ not simply connected. Let $g$ in $\pi_1(M)$ non trivial with $g(F) = F$ and $\xi$ be the axis of $g$ in $F$. At least one ideal point of $\xi$, call it $u$, is not the direction of a fixed limit of bad wedges. Then as explained before there is a ray $l$ of $\hat{\Lambda}_s^u$ and segments $l_i$ of $l$, bounded by $a_i, b_i$ both points in $\xi$, so that $l_i$ escapes compact sets and converges to a non trivial segment in $\partial_\infty F$. We may assume that $l_i \cap \xi = \{a_i, b_i\}$ and also that all $l_i$ are in the same side of $\xi$. Let $e_0$ be the translation length of $g$ in $F$.

If the distance from $a_i$ to $b_i$ along $\xi$ is bigger than $e_0$ then this produces a contradiction as follows: There is an integer $n$ so that $g^n(a_i)$ is in the open segment $(a_i, b_i)$ of $\xi$ and $g^n(b_i)$ is outside of the closed segment $[a_i, b_i]$. Since the arc $l_i$ only intersects $\xi$ in $a_i, b_i$, then $l_i$, together with $[a_i, b_i]$ bounds a closed disk in $F$ and $g^n(a_i)$ is in $(a_i, b_i)$. But $g^n(b_i)$ is outside and $g^n(l_i)$ is also on this side of $\xi$, so this produces a transverse self intersection of $\hat{\Lambda}_s^u$. If $g^n(l_i)$ is contained in the leaf $\nu$ which contains $l_i$, then $g^n(\nu) = \nu$ and this produces infinitely many singularities in $\nu$, which is impossible. Hence $g^n(l_i)$ is not in $\nu$ and the transverse intersection is impossible. The same arguments deal with the case that $l_i$ intersects $\xi$ in other points besides $a_i, b_i$.

We conclude that the distance in $\xi$ from $a_i$ to $b_i$ is bounded. Up to subsequence we may assume there are integers $n_i$ so that $g^{n_i}(a_i)$ converges to $a_0$ and $g^{n_i}(b_i)$ converges to $b_0$, both limits in $\xi$ of course. Since the lengths of $g^{n_i}(l_i)$ are converging to infinity, it follows that $a_0, b_0$ are not in the same leaf of $\hat{\Lambda}_s^u$. By proposition 4.2 it follows that $a_0, b_0$ are not in the same leaf of $\hat{\Lambda}_s^u$. But for each $i$, the pair of points $g^{n_i}(a_i), g^{n_i}(b_i)$ is in the same leaf of $\hat{\Lambda}_s^u$. This implies that the leaf space of $\hat{\Lambda}_s^u$ is not Hausdorff.

First of all this implies that $\Phi$ is regulating for $\mathcal{F}$, for otherwise the aforementioned result from [Fed] shows that $\Phi$ is an $\mathbb{R}$-covered Anosov flow – in particular $\hat{\Lambda}_s$ has Hausdorff leaf space. Also by theorem 4.3 the fact that $\hat{\Lambda}_s$ has non Hausdorff leaf space implies that there are closed orbits $\alpha, \beta$ of $\Phi$ so that $\alpha$ is freely homotopic to the inverse of $\beta$. Let $h$ be a covering translation associated to $\alpha$ and $\tilde{\alpha}, \tilde{\beta}$ lifts of $\alpha, \beta$ to $\tilde{\mathcal{M}}$ which are left invariant by $h$. Without loss of generality assume that $h$ acts in $\tilde{\alpha}$ sending points forwards. As $\alpha \cong \beta^{-1}$ this implies that $h$ acts on $\beta$ taking points backwards. But since both of them intersects all leaves of $\tilde{\mathcal{F}}$ (by the regulating property) then as seen from $\tilde{\alpha}$ the translation $h$ acts increasingly in the leaf space of $\tilde{\mathcal{F}}$, with opposite behavior when considering $\tilde{\beta}$. This is a contradiction, which shows that this cannot happen. This finishes the proof of theorem 5.2.
Corollary 5.5. Let $Φ$ be an almost pseudo-Anosov flow transverse to a foliation $F$ with hyperbolic leaves in $M^3$ closed. For any leaf $L$ of $\tilde{F}$ and any ray $l$ of $\tilde{Λ}^s_F$ or $\tilde{Λ}^u_F$, then $l$ converges to a single point in $\partial_\infty L$.

Remark - Group invariance and compactness of $M$ are both essential here. For example start with a nicely behaved singular foliation of $H^2$, so that all rays converge. It could be a foliation by geodesics or for instance the lift of the stable singular foliation associated to a suspension. Fix a base point $p$. Now rotate the leaves at a distance $d$ of $p$ by an angle $d$. In this situation all rays limit in all points of $\partial_\infty L$, in fact they spiral indefinitely into it. Another operation is to fix a ray through $p$ and then distort the rest more and more one way and the other way. Here we have the leaves getting closer and closer to segments in $\partial_\infty F$ which are complementary to the ideal point associated to the ray.

6 Properties of leaves of $\tilde{Λ}^s_F, \tilde{Λ}^u_F$ and their ideal points

In this section $Φ$ is an almost pseudo-Anosov flow transverse to a foliation $F$. As in the previous section there is no restriction on $M$ here. In the previous section we proved that for any ray $r$ of a leaf of $\tilde{Λ}^s_F$ or $\tilde{Λ}^u_F$, then it has a unique ideal point in $\partial_\infty F$. The notation for this ideal point will be $r_\infty$. We now analyse further properties of leaves of $\tilde{Λ}^s_F$ and their ideal points. Analogous results hold for $\tilde{Λ}^u_F$.

First we want to show that if $E$ is a fixed leaf of $\tilde{Λ}^s$ (or $\tilde{Λ}^u$) then the ideal points in $\partial_\infty F$ of rays of $E \cap F$ vary continuously with $F$. In order to do that we first put a topology on the union of ideal boundaries of an interval of leaves. Let $p$ in $F$ leaf of $\tilde{F}$ and $τ$ a transversal to $\tilde{F}$ with $p$ in the interior. For any $L$ in $\tilde{F}$ intersecting $τ$, the ideal boundary is in $1$-$1$ correspondence with the unit tangent bundle to $L$ at $τ \cap L$: ideal points correspond to rays in $L$ starting at $L \cap τ$. This is a homeomorphism. This puts a topology in

$$ \mathcal{A} = \bigcup \{ \partial_\infty L \mid L \cap τ \neq \emptyset \} $$

making it into an annulus homeomorphic to $\bigcup \{ T_q^\tau \tilde{F}, q \in τ \}$ as a subspace of the unit tangent bundle of $M$. This topology in $\mathcal{A}$ is independent of the choice of transversal $τ$. The following definition/result is proved in [Fe7] or [Cal].

Definition 6.1. (markers) Given a foliation $F$ by hyperbolic leaves of $M^3$ closed, then there is $ε > 0$ so that: Let $v$ be a geodesic ray in a leaf $F$ so that it is associated to a contracting (or $ε$ non expanding) direction of $F$. For any leaf $L$ sufficiently near $F$, then all the points of $v$ flow into $L$ and define a curve denoted by $v_L$. Then $v_L$ has a unique ideal point denoted by $a_L$. The union $m$ of the $a_L$ is called a marker and is a subset of $A = \bigcup \{ \partial_\infty L \}$. Then $m$ is an embedded curve in $A$ in the topology defined above.

In addition the markers are dense in $A$ in the following sense: Let $z$ in $\partial_\infty F$ and $a_i, b_i$ in $\partial_\infty F$ which are in markers associated to contracting (non expanding) directions on a fixed side of $F$. Suppose that the sequence of open intervals $(a_i, b_i)$ in $\partial_\infty F$ contains $z$ and converges to $z$ as $i$ converges to infinity. Let $α_i, β_i$ be the markers in that side of $\partial_\infty F$ containing $a_i, b_i$ respectively. Let $L_i$ in $\tilde{F}$ be a sequence of leaves converging to $F$ and on that side of $F$ so that $\partial_\infty L_i$ intersects both $α_i$ and $β_i$. In the annulus $A$ of circles at infinity, consider the rectangle $R_i$ bounded by $(a_i, b_i)$ in $\partial_\infty F$, the parts of $α_i, β_i$ between $\partial_\infty F$ and $\partial_\infty L_i$ and the small segment in $\partial_\infty L_i$ bounded by $\partial_\infty L_i \cap α_i$ and $\partial_\infty L_i \cap β_i$. Then the sets $R_i$ converge to $z$ as $i$ converges to infinity. This is proved in [Fe7].

From now on the $ε$ is chosen small enough to also satisfy the conclusions of the definition above and also that any set in $\tilde{M}$ of diameter less than $10ε$ is in a product box of $\tilde{F}$ and $\tilde{Φ}$. Given a curve $ζ$
in a leaf $F$ with starting point $p$ and limiting on a unique point $q$ in $\partial_\infty F$, let $\zeta^*$ denote the geodesic ray of $F$ with same starting and ideal points.

**Lemma 6.2.** Let $E$ be a leaf of $\tilde{\Lambda}^s$ and $p$ the starting point of the ray $r$ of $E \cap F$. Assume that $r$ does not have any singularity. For any $L$ near $F$, then $E \cap L$ has a ray $r_L$ which is near $r$. The ideal points of $r_L$ in $\partial_\infty L$ vary continuously with $L$ in the topology of $A$ defined above.

**Proof.** We do the proof for say the positive side of $F$. We consider $r$ without singularity or else we would have to check the 2 exterior rays in $\tilde{\Lambda}^s_F$ emanating from $p$. We can always get a subray of $r$ which has no singularities.

Let $u = r_\infty$. Choose contracting (or $\epsilon$ non expanding) directions in both sides of $u$, with ideal points very close to $u$. Let them be defined by geodesic rays $r_0, r_1$ starting at $p$. There is $\tau$ a small flow segment starting at $p$ and in that side of $F$ so that for any $L$ intersecting $\tau$, then $L$ is asymptotic to $F$ along the $r_0, r_1$ rays, or at least always $\leq \epsilon$ from $F$. Hence $r_0, r_1$ flow along $\tilde{\Phi}$ to $L$. Let $s_0, s_1$ be the flow images in $L$. The $\epsilon$ is also chosen small enough so that $s_0, s_1$ have geodesic curvature very small (this $\epsilon$ depends only on $M$ and $F$). In particular the curves $s_0, s_1$ are a small bounded distance (depending only on $\epsilon$) from the corresponding geodesic arcs $\tilde{s}_0^*, s_1^*$. Let the ideal points of $s_0, s_1$ in $\partial_\infty L$ be denoted by $v_0, v_1$ and let $J_L$ be the small closed interval in $\partial_\infty L$ bounded by $v_0, v_1$. Then $v_0, v_1$ are in the markers associated to $r_0, r_1$ respectively and so they vary continuously with $L$.

Consider $\xi = E \cap L$ and the rays $l$ of $\xi$ starting at $\tau \cap L$ and containing some points which flow back to points in $r$. It may be that $\xi$ has singularities — even if $r$ does not — but there are only finitely many such rays. We want to prove that the ideal point of any such is in $J_L$. As the rectangles $R_i$ defined above converge to $u$ in $\mathcal{A}$ this will prove the continuity property of the lemma.

Choose $d > 0$ so that outside of a disk $D$ of radius $d$ in $F$, then $r$ is in the small wedge $W$ of $F$ defined by $r_0, r_1$, see fig. 4. Choose $\tau$ small enough so that if $L$ intersects $\tau$, then the entire disk $D$ is $\epsilon$ near $L$. Let $V$ be the closure in $F$ of $W - D$. The boundary $\partial V$ consists of subrays of $r_0, r_1$ and an arc in $\partial D$. Therefore all points in $\partial V$ are less than $\epsilon$ from $L$ and flow to $L$ under $\tilde{\Phi}$ with image a curve $\gamma$. This curve contains subrays of $s_0, s_1$ and it is properly embedded in $L$. Points of $F$ near $\partial V$ also flow to $L$ so there is a unique component $U$ of $L - \gamma$ which has some points flowing back to points in $V$. We want to show that the ray $l$ is eventually contained in $U$.

Let $r_{init}$ be the subarc of $r$ between $p$ and the last point $c_0$ of $r$ in $D$. As $p$ and $c_0$ flow into $L$, then proposition 4.2 shows that the entire arc $r_{init}$ flows into $L$ and let $\delta$ be its image in $L$. As $r$ is singularity free, then so is $\delta$ and hence $\delta$ is contained in any ray $l$ of $E \cap L$ in that direction. After $c_0$ the curve $r$ enters $V$ and so $l$ must enter $U$ after $\delta$. If after that the ray $l$ exits $U$ then it must cross $\partial U = \gamma$ in some point, call it $c_1$. But $c_1$ flows back to $F$ and one can apply proposition 4.2 again in the backwards direction to show that $c_1$ has to flow to a point in $r$. This contradicts the choice of $c_0$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Leaf in wedge defined by markers.}
\end{figure}
This shows that \( l \) is eventually entirely contained in \( U \) and therefore \( t_\infty \) is a point in \( J_L \). This shows the continuity property as desired and finishes the proof of the lemma.

Now we have a property which will be crucial to a lot of our analysis.

**Proposition 6.3.** Suppose that \( F \) is not topologically conjugate to the stable foliation of a suspension Anosov flow. Then the set of ideal points of rays of \( \tilde{\Lambda}_L^s \) is dense in \( \partial_\infty F \).

**Proof.** Suppose that there is \( F \) in \( \tilde{F} \) so that the set of ideal points in \( \tilde{\Lambda}_L^s \) is not dense in \( \partial_\infty F \). Let \( J \) be an open interval in \( \partial_\infty F \) free of such ideal points. Choose \( p_i \) in \( F \), \( p_i \) converging to a point in \( J \). The visual angle of \( J \) as seen from \( p_i \) converges to \( 2\pi \), so the complementary wedge \( W_i \) with corner \( p_i \) has angle which converges to zero. Up to subsequence assume that \( g_i(p_i) \) converges to \( p_0 \) in a leaf \( L \) of \( \tilde{F} \) and the small wedges \( g_i(W_i) \) converge to a geodesic ray \( s \) in \( L \) with ideal point \( z \).

**Claim** — In \( L \) all the rays of \( \tilde{\Lambda}_L^s \) converge to \( z \).

Suppose there is \( x \) different from \( z \) which is an ideal point of a ray \( r \) in \( \tilde{\Lambda}_L^s \). Then \( r \) is contained in \( \tilde{W}^s(c_0) \) for some \( c_0 \) in \( \tilde{M} \) and for \( g_i(F) \) sufficiently near \( L \) then \( \tilde{W}^s(c_0) \) intersects \( g_i(F) \). Any ray of \( \tilde{W}^s(c_0) \cap g_i(F) \) which is near \( r \) will have ideal point near \( x \) in the topology of corresponding annulus \( A \) of ideal circles near \( \partial_\infty L \). This is a consequence of the previous lemma. But \( g_i(W_i) \) converges to \( r \) in this topology of \( A \), so the sets \( g_i(\partial_\infty F - J) \) converge to \( z \) in \( A \). There are no ideal points of leaves of \( \tilde{\Lambda}_L^s \) in \( g_i(J) \). This contradicts the fact that the ideal points above are very near \( x \) and proves the claim.

The proof of the proposition is similar to that of theorem 5.24. As in that theorem consider the set of possible limits \( g_i(p_i) \) as above. This projects to a lamination in \( M \) and let \( L \) be a minimal sublamination. The claim shows that each leaf of \( \tilde{L} \) has a distinguished ideal point towards which all rays of \( \tilde{\Lambda}_L^s \) converge. The arguments in the claim also prove that if \( \tau \) is a transversal to \( \tilde{F} \), then the ideal points of leaves of \( \tilde{L} \) intersecting \( \tau \) vary continuously in the corresponding ideal annulus. Because of the distinguished ideal point property, then each leaf of \( \tilde{L} \) has fundamental group at most \( Z \). If needed lift to a double cover so that all leaves of \( F \) are orientable. Hence a leaf of \( L \) is either a plane or an annulus.

Consider a complementary component \( U \) of \( L \) and a boundary leaf \( A \) of \( U \). If \( A \) is a plane then as in the proof of theorem 5.22 the region \( U \) is an I-bundle over \( A \) and the flow \( \Phi \) is a product in \( U \). This region can be collapsed away.

Suppose now that \( A \) is an annulus. Assume that flow lines through \( A \) flow into \( U \). Again we want to show that \( U \) is a product region. As in the proof of theorem 5.22 let \( A_1, A_2 \) be two noncompact, disjoint annuli in \( A \) with \( A - (A_1 \cup A_2) \) a compact annulus and \( A_1, A_2 \) contained in the thin I-bundle region. Then \( A_1, A_2 \) flow entirely into leaves \( B \) and \( C \) in \( \partial U \). Suppose first that \( B, C \) are different. Lift to the universal cover to produce lifts \( \tilde{U}, A, A_1, A_2, B, C \). Then \( \tilde{A}_1, \tilde{A}_2 \) are disjoint half planes of \( \tilde{A} \) which flow positively respectively into \( B \) and \( C \). Let \( g \) be the generator of the isotropy group of \( \tilde{A} \), which has fixed points in \( z, x \) where \( z \) is the distinguished ideal point in \( \tilde{A} \). The argument will show there is a leaf in \( \tilde{\Lambda}_A^s \) which also has ideal point in \( x \), contradiction.

From a point in \( \tilde{A}_1 \) draw a geodesic segment of \( \tilde{A} \) to a point in \( \tilde{A}_2 \). Let \( p \) be the first point of this segment which does not flow positively into \( \tilde{B} \). Then \( \Theta(p) \) is in the boundary of \( \Theta(\tilde{B}) \). Also points in the segment near \( p \) flow to \( \tilde{B} \) in positive time, hence there is a slice leaf \( l \) of \( \Theta^*(\Theta(p)) \) which is in the boundary of \( \Theta(\tilde{B}) \). Notice that every point in \( l \) is a limit of points in \( \Theta(B) \) on that side. The set \( (l \times \mathbb{R}) \) intersects \( \tilde{A} \) in at least \( p \); if \( l \) is contained in \( \Theta(A) \) then it generates a properly embedded copy of the reals in a leaf \( s \) of \( \tilde{\Lambda}_A^s \) otherwise the part that is contained in \( \Theta(A) \) also does. Every
point of $s$ is a limit of points that flow positively into $\tilde{B}$. Therefore no point in $s$ can flow positively in $\tilde{C}$ or else we would have points flowing both in $\tilde{B}$ and $\tilde{C}$.

This shows that the leaf $s$ of $\tilde{\Lambda}^s$ is a bounded distance from the axis $r$ of $g$. Iterate $s$ by powers of $g$ acting with $z$ as an expanding fixed point. The iterates $g^n(s)$ with $n > 0$ are all distinct. Either they are all nested or they are disjoint. If they are not nested since they all have to be in a bounded distance neighborhood of the axis of $g$ and have both endpoints in $z$, then eventually they will have two points which are far along the leaf, but close in $\Lambda$. By Euler characteristic reasons, this would force a center or one prong singularity, which is impossible. Hence they are nested, increasing and they limit to a leaf of $\tilde{\Lambda}^s$ which has ideal limit points in $z$ and $x$. This is a contradiction. This shows that $B = C$. In fact the same arguments show that all of the points in $A$ flow into $B$, since that happens for the complement of a compact annulus in $A$ and then the arguments above apply here. Hence $U$ is a product region. Therefore we can collapse $F$ to a minimal foliation.

As in theorem 5.2, we can then show that $F$ is $R$-covered. Suppose this is not the case and let $F_1, F_2$ be non separated leaves. Let $L_i$ in $\tilde{F}$ leaves converging to both $F_1, F_2$. Let $u_1, u_2$ be the distinguished ideal points in $\partial_\infty F_1, \partial_\infty F_2$ respectively. Let $a_1, b_1$ be points in $\partial_\infty F_1$ very near $u_1$ and on opposite sides of $u_1$ and which are in markers associated to contracting or $\epsilon$ non expanding directions in $F_1$ associated to the $L_i$ side. Let $r_1$ be the geodesic in $F_1$ with ideal points $a_1, b_1$. Similarly for $F_2$ producing $a_2, b_2, r_2$. For $i$ big enough $L_i$ is at least $\epsilon$ far from all points in $r_1, r_2$. Therefore $r_1$ flows (by $\tilde{\Phi}$) into a curve $s_1$ in $L_i$ and $r_2$ flows into $s_2$. This implies that $s_1, s_2$ are disjoint in $L_i$. Also $s_1$ has ideal points $a_1', b_1'$ which are in markers containing $a_1, b_1$ respectively (this is using a transversal to $\tilde{F}$ through a point in $F_1$). Similarly $s_2$ has ideal points $a_2', b_2'$ in markers containing $a_2, b_2$ (using transversal to $\tilde{F}$ through a point in $F_2$). As $s_1, s_2$ are disjoint then $a_1', b_1'$ do not link $a_2', b_2'$ in $\partial_\infty L_i$, see fig. 4.

The ideal point $a_1'$ cannot be in a marker to $\partial_\infty F_1$ and to $\partial_\infty F_2$ at the same time since they are non separated leaves. Hence the points $a_1', b_1', a_2', b_2'$ are all distinct. Let $J_1$ be the interval of $\partial_\infty L_i$ bounded by $a_1', b_1'$ and not containing the other points and similarly define $J_2$. For simplicity we are omitting the dependence of $J_1, J_2$ on $L_i$ (or on $i$). Now consider $E$ a leaf of $\tilde{\Lambda}^s$ intersecting $F_1$. Then $E \cap F_1$ has a ray with ideal point $u_1$, which is in the interval $(a_1, b_1)$ of $\partial_\infty F_1$. The proof of lemma 5.2 shows that if $L_i$ is close enough to $F_1$ then the ideal points of the corresponding rays of $(E \cap L_i)$ have to be in $J_1$. In the same way using $F_2$ one shows that the distinguished ideal point has to be in $J_2$. Since $J_1, J_2$ are disjoint, this is a contradiction. This shows that $F$ is $R$-covered.

Since $F$ is $R$-covered then theorem 5.4 implies that $\Phi$ can be chosen to be a pseudo-Anosov flow. Also as $F$ is $R$-covered we can choose a transversal $\tau$ intersecting all the leaves of $\tilde{F}$. This shows that the union of all the circles at infinity has a natural topology making it into a cylinder $\cal{A}$. This situation of $R$-covered foliations is carefully analysed in [Fe]. The fundamental group of $M$ acts in $\cal{A}$ by homeomorphisms. The union of the distinguished ideal points of leaves of the distinct leaves of $\tilde{F}$ is a continuous curve $\zeta$ in $\cal{A}$ which is group invariant.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{pushingIdealPointsNear.png}
\caption{Pushing ideal points near.}
\end{figure}


\section*{Properties of leaves of \( \bar{\Lambda}_s^s, \bar{\Lambda}_s^u \) and their ideal points}

Suppose first that \( \mathcal{F} \) admits a holonomy invariant transverse measure. Since \( \mathcal{F} \) is minimal then the transverse measure has full support. Under these conditions Imanishi \cite{Im} proved that \( M \) fibers over the circle with fiber a closed surface. In addition \( \mathcal{F} \) is approximated arbitrarily near by a fibered structure. The pseudo-Anosov flow is also transverse to these nearby fibrations and so the same situation occurs for the fibrations: there is a global invariant curve in the cylinder at infinity. Since now there are compact leaves, this is impossible.

We conclude that there is no holonomy invariant transverse measure. Therefore Thurston’s theorem shows the existence of contracting directions and not just \( \epsilon \) non expanding directions. So the markers are associated to contracting directions. If \( \zeta \) intersects a marker \( m \), that corresponds to a direction in a leaf of \( \bar{\mathcal{F}} \) which is contracting. Under the flow \( \Phi \) this gets reflected in the contracted leaves nearby, that is the marker is contained in \( \zeta \). Since \( \mathcal{F} \) is minimal and \( \zeta \) is \( \pi_1(M) \) invariant, this shows that the entire curve \( \zeta \) is a marker associated to contracting directions. The results from \cite{Fe7} apply here, in particular lemma 3.17 through proposition 3.21 of \cite{Fe7}: they show that no other direction in \( \bar{\mathcal{F}} \) (outside of \( \zeta \)) is a contracting direction. By Thurston’s theorem again, there would be a holonomy invariant transverse measure, contradiction.

Therefore \( \zeta \) has no contracting directions. The same analysis of \cite{Fe7} now shows that for any leaf \( F \) in \( \bar{\mathcal{F}} \) and every direction other than the distinguished direction, then it is a contracting direction. In fact it is a contracting direction with any other leaf of the foliation.

This is a very interesting situation. Let \( a_F \) be the distinguished ideal point of \( F \) leaf of \( \bar{\mathcal{F}} \). Consider a one dimensional foliation in \( \bar{M} \) whose leaves are geodesics in leaves \( \tilde{F} \) of \( \bar{\mathcal{F}} \) and which have one ideal point \( a_F \). Let \( \xi \) be the flow which is unit speed tangent to this foliation and moves towards the ideal point \( a_F \).

This is a flow in \( \bar{M} \). Clearly in each leaf of \( \bar{\mathcal{F}} \), it is a smooth flow. If \( q_i \) in \( L_i \) of \( \bar{\mathcal{F}} \) converge to \( q \) in \( L \), then the geodesics of \( L_i \) with ideal point \( a_{L_i} \) converge to the geodesic through \( q \) in \( L \) with ideal point \( a_L \). This is because the ideal points \( a_F \) vary continuously with \( F \) and \( q_i \) converges to \( q \) – this is the local trivialization of the union of the circles at infinity using the tangent bundles to a transversal. Hence \( \xi \) varies continuously.

Since \( \zeta \) is group invariant, this induces a flow in \( M \), which is tangent to the foliation \( \mathcal{F} \). Clearly it is smooth along the leaves of \( \mathcal{F} \) and usually just continuous in the transverse direction.

This flow is a topological Anosov flow: the stable foliation is just the original foliation \( \mathcal{F} \). The unstable foliation: Let \( p \) in leaf \( L \) of \( \bar{\mathcal{F}} \), let \( \gamma \) be the flow line of \( \xi \) through \( p \). Then \( \gamma \) has positive ideal point \( a_L \) and negative ideal point \( v \). As explained above \( v \) is in a marker \( m \) which is associated to a contracting direction and so that \( m \) intersects all ideal circles. For each \( F \) in \( \bar{\mathcal{F}} \), let \( m_F \) be the intersection of \( m \) and \( \partial_{\infty}F \). Let \( \gamma_F \) be the geodesic in \( F \) with ideal points \( a_F \) and \( m_F \). Let \( E_p \) be the union of these \( \gamma_F \). Then all orbits of \( \xi \) in \( E_p \) are backwards asymptotic by construction. By construction the \( E_p \) are either disjoint or equal as \( p \) varies in \( \bar{M} \) and they form a group invariant foliation in \( \bar{M} \). This is the unstable foliation. Hence \( \xi \) is a topologically Anosov flow. Notice that in the universal cover every stable leaf intersects every unstable leaf and vice versa.

By proposition \ref{prop:topconj} it follows that \( \xi \) is topologically conjugate to a suspension Anosov flow. The foliation \( \mathcal{F} \) is then topologically conjugate to the stable foliation of this flow. This finishes the proof of this proposition.

\begin{remark}
\end{remark}

The hypothesis is necessary. Suppose that \( \mathcal{F} \) is the stable foliation of a suspension Anosov flow, \( \xi \) so that it is transversely orientable. Perturb the flow slightly so that flow lines are still tangent to the original unstable foliation of \( \xi \). The new flow, call it \( \Phi \) is transverse to \( \mathcal{F} \), it has the same unstable foliation as \( \xi \) but different stable foliation. The flow \( \Phi \) is not regulating for \( \mathcal{F} \). The intersections of leaves of \( \Lambda^s \) with leaves \( F \) of \( \bar{\mathcal{F}} \) are all horocycles with the same ideal point.
Proof. This means that \( q \subset N_{2 \delta_0}(r) \) in \( F \). We do not know if the converse holds. Suppose the lemma is not true. Then there are \( F_i \) leaves of \( \tilde{\Lambda}^s_F \) and \( p_i \) in \( r^*_i \) so that \( B_{2i}(p_i) \) (in \( F_i \)) does not intersect \( r_i \). There is one side of \( r^*_i \) in \( F_i \) so that \( r_i \) goes around that side, see fig. a. Let \( q_i \) inside a half disk of \( B_{2i}(p_i) \) with \( B_{i}(q_i) \) tangent to \( r^*_i \) and \( \partial B_{2i}(p_i) \), see fig. a.

As usual up to subsequence there are \( g_i \) in \( \pi_1(M) \) with \( g_i(q_i) \) converging to \( q_0 \) in \( L \) leaf of \( \tilde{F} \) and so that the geodesic segments \( \zeta_i \) from \( g_i(q_i) \) to \( g_i(p_i) \) in \( F_i \) converge to a geodesic ray \( s \) in \( L \). Choose two markers with points \( u_0, u_1 \) in \( \partial_\infty L \) very close to \( s_\infty \) and on opposite sides of it. The markers are associated to the side of \( L \) where the \( g_i(F_i) \) are limiting to. Let \( s_0, s_1 \) be the geodesic rays of \( L \) starting at \( q_0 \) and with ideal points \( u_0, u_1 \). For \( i \) big enough \( g_i(F_i) \) is \( \epsilon \) close to both \( s_0 \) and \( s_1 \) and so these two rays flow (under \( \Phi \)) to curves \( s'_0, s'_1 \) in \( g_i(F_i) \). The ideal points \( u'_0, u'_1 \) of \( s'_0, s'_1 \) are in the markers above.

For \( i \) big enough the ray \( g_i(r_i) \) has a subray which goes around \( g_i(B_i(q_i)) \) in \( g_i(F_i) \) and has ideal

\[ \delta \]
point in the small segment of $\partial_{\infty}g_i(F_i)$ defined by $u'_0, u'_1$, see fig. b. Since $s'_0, s'_1$ flows back to $L$ this figure flows back to $L$ producing a ray $l_i$ of $\Lambda^s_F$ which goes around a big disk in $L$ centered at $q_0$ and has ideal point in the small segment bounded by $u_0, u_1$, see fig. c. As $i$ goes to infinity, these $l_i$ escape to infinity in $L$ because bigger and bigger disks in $g_i(F_i)$ flow to $L$. This implies that there is no ideal point of a ray of $\Lambda^s_F$ outside the small segment of $\partial_{\infty}L$ bounded by $u_0, u_1$. This contradicts the previous proposition that such ideal points are dense in $\partial_{\infty}L$.

This finishes the proof of the lemma. 

Lemma 6.5. The limit points of rays of $\Lambda^s_F$ vary continuously in $\partial_{\infty}F$ except for the non Hausdorffness in the leaf space of $\Lambda^s_F$.

Proof. Suppose that $p_i$ converges to $p$ in $F$, with respective rays $r_i$ converging to the ray $r$ of $\Lambda^s_F$. Let $l$ be the leaf of $\Lambda^s_F$ through $p$. Up to subsequence assume the $r_i$ are all in the same sector of $l$ defined by $p$ and that they form a nested sequence of rays. Then the ideal points $(r_i)_\infty$ form a monotone sequence in $\partial_{\infty}F$. Perhaps some ideal points are the same. If $(r_i)_\infty$ does not converge to $r_\infty$ there is an interval $v$ in $\partial_{\infty}F$, between the limit and $r_\infty$. Since the ideal points are dense in $\partial_{\infty}F$, there is $w$ leaf of $\Lambda^s_F$ with $w_\infty$ in $v$. Therefore there is $l'$ not separated from $l$ with $r_i$ converging to $l'$ as well. In this fashion we can go from $l$ to $l'$. This shows that if there is no leaf of $\Lambda^s_F$ non separated from $l$ in that side and in the direction the rays $r_i$ go, then the limit points vary continuously.

We analyse a bit further the non Hausdorffness. In the setup above there are subrays of $r_i$ with points converging to a point in $l'$ and we can restart the analysis with $l'$ instead of $l$. If there are finitely many leaves non separated from $l$ and $l'$ we can assume that $l, l'$ are consecutive. Then they have subrays which share an ideal point. If $m$ is the last leaf non separated from $l, l'$ in the direction the rays $r_i$ go to, then there is a ray $\zeta$ of $m$ so that there are subrays of $r_i$ with points converging to a point in $\zeta$ and $(r_i)_\infty$ converges to $\zeta_\infty$. If there are infinitely many such leaves non separated from $l$, then we can order them as $\{l_j\}, j \in \mathbb{N}$ all in the direction the rays $r_i$ go to. The ideal points of $l_j$ form a monotone sequence in $\partial_{\infty}F$ which converge to a point $u$ in $\partial_{\infty}F$. The arguments above show that $(r_i)_\infty$ converges to $u$.

Our next goal is to analyse the non Hausdorffness in the leaf space of $\Lambda^s_F$. We also want to understand when can the ideal points of two different rays of the same leaf of $\Lambda^s_F$ be the same. A Reeb annulus is an annulus $A$ with a foliation so that the boundary components are leaves and every leaf in the interior is a topological line which spirals towards the two boundary components in the same direction. In the universal cover the lifted foliation does not have Hausdorff leaf space. The lifted foliation to the universal cover is called a Reeb band.

Definition 6.6. (spike region) A stable spike region in a leaf $F$ of $\widetilde{\mathcal{F}}$ is a closed $\Lambda^s_F$ saturated set $\mathcal{E}$ satisfying:

- There are finitely many boundary leaves of $\mathcal{E}$ which are line leaves of $\Lambda^s_F$. The ideal points of consecutive rays in the boundary of $\mathcal{E}$ are the same, otherwise they are distinct (like an ideal polygon).

- The region $\mathcal{E}$ is a bounded distance from the ideal polygon with these vertices. The bound is not universal in $\widetilde{\mathcal{F}}$.

- There is an ideal point $z$ of $\mathcal{E}$ so that every leaf in the interior of $\mathcal{E}$ has both ideal points equal to $z$. In addition the leaves in the interior are nested. The finitely many leaves in the boundary are all non separated from each other and they are limits of the interior leaves.
There is no singularity of $\tilde{Λ}^s_F$ in the interior of $E$. If there is a singularity of $\tilde{Λ}^s_F$ in a boundary leaf $τ$ of $E$ then the interior of $E$ is contained in the sector defined by the line leaf $τ$.

Similarly define an unstable spike region. A spike region is either a stable or unstable one.

**Proposition 6.7.** Let $E$ be a leaf in $\tilde{F}$ and $v$ a slice of a leaf $v_0$ of $\tilde{Λ}^s_E$. Suppose that both ideal points of $v$ are the same. Then $v$ is contained in the interior of a stable spike region $B$ of $E$. In addition either $B$ projects to a Reeb annulus in a leaf of $F$ or for any two consecutive rays in $\partial B$, the region between them projects to a set asymptotic to a Reeb annulus in a leaf of $F$. Similarly for $\tilde{Λ}^u_F$.

**Proof.** We do the proof for $\tilde{Λ}^s_F$. Let $v$ be a slice as above with ideal point $x$ in $\partial_\infty E$. Let $C$ be the region bounded by $v$ in $E$ which only limits in $x$. First we assume assume that $v$ is a line leaf of some leaf of $\tilde{Λ}^s_F$. We will show that the region $C$ as it approaches $x$, projects to a set in $M$ which limits to a Reeb annulus in a leaf of $F$. The process will be done in a series of steps. The proof of this proposition is very long with several intermediate results and lemmas.

Choose $z_0$ in $v$ and let $e_1,e_2$ be the rays of $v$ defined by $z_0$. Let $ζ^*$ be the geodesic ray of $E$ starting at $z_0$ and with ideal point $x$. Then $ζ^*$ is contained in the $2δ_0$ neighborhood of $e_1$ or $e_2$, where $δ_0$ is the constant of lemma 5.4. It follows that we can choose $p_i,q_i$ in $e_1,e_2$ respectively with $p_i,q_i$ converging to $x$ in $E∪\partial_\infty E$ and also $d_{E}(p_i,q_i) < 4δ_0$. Let $e_1^i$ be the subray of $e_1$ starting at $p_i$ and $e_2^i$ the subray of $e_2$ starting at $q_i$. Up to subsequence there are $p_0,q_0$ in $\tilde{M}$ and are $g_i$ in $\pi_1(\tilde{M})$ with $g_i(p_i), g_i(q_i)$ converging to $p_0$, $q_0$ respectively. The distance condition implies $p_0,q_0$ are in the same leaf of $\tilde{F}$, let $F$ be this leaf. Then $g_i(E)$ converges to $F$ and perhaps other leaves as well.

For $i$ big enough the flowlines of $Φ$ through $g_i(p_i), g_i(q_i)$ go through to $u_i$ and $v_i$ in $F$. Also $u_i → p_0, v_i → q_0$. If the leaf of $\tilde{Λ}^s_F$ through $p_0$ contains $q_0$ then for $i$ big enough the arcs in leaves of $\tilde{Λ}^s_F$ from $u_i$ to $v_i$ will have bounded length and bounded diameter. The same will happen for the arcs of of $g_i(v)$ between $g_i(p_i)$ and $g_i(q_i)$, contradiction. Hence $p_0,q_0$ are not in the same leaf of $\tilde{Λ}^s_F$.

Let $l$ be the leaf of $\tilde{Λ}^s_F$ through $p_0$ and $r$ be the one through $q_0$. Let $L,R$ be leaves of $\tilde{Λ}^s$ containing $l$ and $r$ respectively. Since the intersection of a leaf of $\tilde{Λ}^s$ with $F$ is connected, then $L$ and $R$ are distinct and also are not separated from each other in the leaf space of $\tilde{Λ}^s$.

For simplicity assume that the leaves of $\tilde{Λ}^s$ through $u_i$ form a nested collection with $i$.

The first goal is to show that we can choose $l, r$ line leaves of $\tilde{Λ}^s_F$ as above so that they also share an ideal point. Let $β_i$ be a ray in the leaf of $\tilde{Λ}^s_F$ through $u_i$ starting at $v_i$ and containing points in the flowlines which go to the ray $g_i(e^i_1)$. Similarly let $γ_i$ be a subray in the same leaf starting at $v_i$ and associated with the ray $g_i(e^i_2)$. Let $C_1$ (resp. $C_2$) be the collection of line leaves of $\tilde{Λ}^s_F$ that $β_i$ (resp. $γ_i$) converges to, including the ray of $l$ (resp. $r$). Let $C$ be the collection of all line leaves of $\tilde{Λ}^s_F$ which are non separated from $l,r$. Then $C$ contains $C_1$ and $C_2$. For any element $τ$ in $C$, let $B(τ)$ be the leaf of $\tilde{Λ}^s$ containing it. All of the $B(τ)$ are not separated from each other, and they are in the set of leaves $B$ of $\tilde{Λ}^s$ non separated from both $L,R$. By theorem 3.6 the set $B$ has a linear order, making it order isomorphic to either $Z$ or a finite set. This induces an order in $C$ where we can choose this so that an arbitrary element of $C_1$ is bigger than any element in $C_2$.

If there are finitely many elements in $C_1$ let $l'$ be the last one and let $ξ_1$ be the ideal point of the ray of $l'$ corresponding to the direction of the rays $β_i$. Otherwise the ideal points of the leaves in $C_1$ form a weakly monotone sequence in $\partial_\infty F$ and let $ξ_1$ be the limit of this sequence. Similarly define $ξ_2$ associated to $r$, see fig. 9 a.

Fix a basepoint $x_0$ in $F$. The first thing to prove is the following:

**Lemma 6.8.** $ξ_1 = ξ_2$. 

Properties of leaves of $\tilde{\Lambda}^s_F, \tilde{\Lambda}^u_F$ and their ideal points

Proof. Suppose by way of contradiction that this is not true. Choose 2 markers very near $\xi_1$ bounding an interval $J_1$ in $\partial_\infty F$ with $\xi_1$ in the interior and similarly choose markers near $\xi_2$ and interval $J_2$ so that $J_1, J_2$ are disjoint. Let $W_1$ be the wedge of $F$ centered at the point $x_0$ with ideal set $J_1$ and $W_2$ the wedge of $F$ centered also at $x_0$ with ideal set $J_2$. For $i$ big enough both boundaries of $W_1$ and $W_2$ flow into $g_i(E)$.

Suppose first that there is a last leaf $l'$ in $C_1$. Then $l'$ has a ray which is eventually contained in a strictly smaller wedge $W'_1$, since the ideal point of $l'$ is $\xi_1$. Now choose a big disk $D$ of $F$ centered in $x_0$. Let $N_1$ be the closure of $W_1 - D$. Choose $D$ big enough so that $l'$ enters $N_1$ through $\partial D$ and is then entirely in $W'_1$. For $i$ big enough $\beta_i$ will be close to $l'$ for a long distance. By Lemma 6.23 the ideal points of $\beta_i$ converge to $\xi_1$ as $i$ converges to infinity, since $l'$ is the last leaf non separated from $l$ in that side. The ideal point is in the limit set of the subwedge $W'_1$. If the rays $\beta_i$ keep exiting $W_1$ then since they are trapped by $l'$ and $\beta_{i_0}$ (for some $i_0$), it follows that the sequence $\beta_i$ has additional limits besides the leaves in $C_1$, contradiction. Therefore for big enough $i$, the $\beta_i$ enters $N_1$ through $\partial D$ and stays in $N_1$ from then on.

We want to get the same result when $C_1$ is infinite. In that case let $\{\nu_j, ~j \in \mathbb{N}\}$ be the leaves in $C_1$ ordered with same ordering as in $C_1$ and $\nu_1 = l$. Since the leaves $\nu_i$ are non separated from each other they cannot accumulate anywhere in $F$ as $i \to \infty$ and the leaves $\nu_j$ escape compact sets as $j$ grows. The ideal points of $\nu_j$ are also converging to $\xi_1$. By density of ideal points of $\tilde{\Lambda}^s_F$ in $\partial_\infty F$ the leaves $\nu_j$ cannot be getting closer to non trivial intervals in $\partial_\infty F$. This implies that there is $j_0$ so that for

$$j \geq j_0, \quad \nu_j \text{ is very close to } \xi_1 \text{ in } F \cup \partial_\infty F$$

and so contained in $W_1$. Now an argument entirely similar as in the case $C_1$ finite implies that for $i$ big enough then $\beta_i$ has subrays entirely contained in $N_1$. The same holds for $\gamma_i$ producing subrays entirely contained in the corresponding set $N_2$ — the disk $D$ may need to be bigger to satisfy all these conditions.

There is $a_1 > 0$ and $i_0$ so that for $i \geq i_0$ then except for the initial segment of length $a_1$ then $\beta_i$ is entirely contained in $N_1$ and similarly for $\gamma_i$ and $N_2$. Choose $k_0$ big enough so that $D$ is $\epsilon$ close to $g_k(E)$ for any $k \geq k_0$. Then $D$ flows in $g_k(E)$ under $\Phi$ and so do $\partial W_1, \partial W_2$. For $i$ bigger than both $i_0, k_0$ the ray $\beta_i$ flows into the ray $g_i(e_1)$ (notice these do not have singularities). The ray $g'(e_1)$ has to be in the generalized wedge which is bounded by the image of $\partial W_1$ in $g_i(E)$. Similarly for $\gamma_i$. This argument is done in Lemma 6.22. These two generalized wedges have disjoint ideal sets in $\partial_\infty g_i(E)$. Therefore $g_i(e_1^1)$ and $g_i(e_2^1)$ do not have the same ideal points. This is a contradiction because $e_1, e_2$ have the same ideal point in $\partial_\infty E$.

This proves that $\xi_1 = \xi_2$. \qed
Continuation of the proof of proposition 6.4

The first part of the proof was this: in $E$ zoom in towards an ideal point $x$ of $\partial_\infty E$ and use covering translations $g_i$ of $\hat{M}$ to map back these points near a point in $\hat{M}$ which is in a leaf $F$. We will redo this process starting with $F$. By taking translates of $F$ we will limit to a leaf $F^*$. The difference is that now we have leaves of $\hat{Λ}^*_F$ which are non separated from each other. These non separated leaves are much better suited to perturbation arguments as seen below.

The lemma shows that $\xi_1 = \xi_2$ and this implies that the ideal points of $\beta_i, \gamma_i$ are all the same and equal to $\xi_1$. Let $\xi = \xi_1$. The $\beta_i, \gamma_i$ are rays in leaves of $\hat{Λ}^*_F$ and contained in $F$. Let $\mu$ be the geodesic ray in $F$ starting at $x_0$ (the basepoint in $F$) with ideal point $\xi$. Since $(\beta_i)_\infty = (\gamma_i)_\infty = \xi$, then lemma 6.4 implies that for $z$ in $\mu$ far enough from $p_0$, we can choose a point in $\beta_i$ which is less than $2\delta_0$ away from $z$ in $F$. Call this point $b_i(z)$. Similarly define $c_i(z)$ in $\gamma_i$. This is for any $i \in \mathbb{N}$.

For each $z$ we may take a subsequence of the $b_i(z)$ which converges in $F$ and the limit is denoted by $b(z)$. Similarly define $c(z)$. By definition of $C_1$ the point $b(z)$ has to be in one of the leaves of $C_1$ and similarly for $c(z)$. The $b(z), c(z)$ are not uniquely defined and most likely do not vary continuously with $z$.

**Lemma 6.9.** There is at least one element $\zeta$ of $C_1$ which has ideal point $\xi$. Similarly for $C_2$.

**Proof.** If there are finitely elements in $C_1$ then the last one satisfies this property. Suppose then there are infinitely many elements in $C_1$. As $z$ varies in $\mu$, then so does $b(z)$. If there are $z$ escaping in $\mu$ so that $b(z)$ is in the same element $\zeta$ of $C_1$ then $\zeta$ has an appropriate ray with ideal point $\xi$. In this case we are done.

Otherwise we can find $z_k$ in $\mu$ converging to $\xi$ so that $b(z_k)$ are in leaves $\nu_{m(k)}$ of $C_1$ which are all distinct. We can choose $z_k$ so that the $m(k)$ increases with $k$. In the same way we have $c(z_k)$ in distinct elements of $C_2$. Let

$$B_k = \tilde{W}^s(b(z_k)), \quad C_k = \tilde{W}^s(c(z_k)),$$

both in $\mathcal{B}$

Recall that $\mathcal{B}$ is the set of leaves of $\hat{Λ}^*$ non separated from both $L, R$. As the length from $b(z_k)$ to $c(z_k)$ in $F$ is bounded by $4\delta_0$, then up to subsequence assume $\pi(b(z_k)), \pi(c(z_k))$ converge in $M$. For $n, k$ big enough there is $h_{nk}$ covering translation of $\hat{M}$ so that $h_{nk}(b(z_n))$ is very close to $b(z_k)$ and $h_{nk}(c(z_n))$ is very close to $c(z_k)$. Suppose $n >> k$, let $h = h_{nk}$ for simplicity. Then $B_k$ has a point $b(z_k)$ very close to $b(b(z_n)) \in h(B_n)$ and similarly $c(z_k)$ in $C_k$ very close to $h(c(z_n)) \in h(C_n)$. But $B_k$ is non separated from $C_k$ and similarly for $h(B_n), h(C_n)$, so the only way this can happen is that

$$h(B_n) = B_k, \quad h(C_n) = C_k$$

This implies that $h$ sends the set of leaves non separated from $B_k, C_k$ to itself, that is $h$ acts on the set $\mathcal{C}$ and therefore acts on $\mathcal{B}$ as well. Notice that $B_k < B_n$ in the order of $\mathcal{B}$ because $n > k$ and $C_k \geq C_n$ (the $C_k$ could be all the same, but if they are not then they decrease in the order). Since $h(B_n) = B_k$ then $h$ acts as a decreasing translation in the ordered set $\mathcal{B}$. But since $h(C_n) = C_k$ then $h$ acts as a non decreasing translation. These two facts are incompatible.

This implies that we have to have at least one element in $\mathcal{C}_1$ with ideal point $\xi$. The same happens for $\mathcal{C}_2$. This finishes the proof of the lemma.

Since the sequence $\{\beta_i\}$ also converges to the leaf $\zeta$ of $\hat{Λ}^*_F$ we can rename the objects and assume that $l = \zeta$ and $p_0$ is a point in $l$. This can be accomplished by choosing different points $p_i$ in the ray $e_1$. Similarly do the same thing in the other direction. We state this conclusion:

**Conclusion** – There are $p_i, q_i$ in $e_1, e_2$ respectively, escaping these rays, so that $d_E(p_i, q_i) < 4\delta_0$ and there are covering translations $g_i$ so that: $g_i(p_i)$ converges to $p_0$, $g_i(q_i)$ converges to $q_0$, both in $F$.
and in rays \( l, r \) of \( \tilde{\Lambda}_F^s \). Also \( l, r \) converge to the same ideal point \( \xi \) in \( \partial_{\infty} F \) and \( l, r \) are not separated in the leaf space of \( \tilde{\Lambda}_F^s \).

We will continue this perturbation approach. We want to show that the region in \( F \) “between” \( l \) and \( r \) projects to a Reeb annulus of \( \mathcal{F} \) in \( M \). Let then \( z_i \) in \( l \) converging to \( \xi \) and \( w_i \) in \( r \) converging to \( \xi \), so that \( d_F(z_i, w_i) \) is always less than \( 4\delta_0 \). Up to subsequence assume there are \( h_i \) covering translations with

\[
\begin{align*}
    h_i(z_i) & \to z_0, \\
    h_i(w_i) & \to w_0
\end{align*}
\]

Notice that \( h_i(L), h_i(R) \) are non separated from each other and \( h_i(L) \to \tilde{W}^*(z_0), h_i(R) \to \tilde{W}^*(w_0) \).

The argument in the previous lemma then implies that \( h_i(L) = h_j(L), h_i(R) = h_j(R) \) for all \( i, j \) at least equal to some \( i_0 \). Discard the first \( i_0 \) terms and postcompose \( h_i \) with \( (h_{i_0})^{-1} \) (that is \( (h_{i_0}^{-1} \circ h_i) \)) and assume that this is the original sequence \( h_i \). This implies that \( h_i(L) = L, h_i(R) = R \) for all \( i \).

So the \( h_i \) are all in the intersection of the isotropy groups of \( L \) and \( R \). This group is generated by a covering translation \( h \). Therefore there are \( n_i \) with \( h_i = h^{n_i} \). Since \( h_i(z_i) \to z_0 \) and the \( \{z_i, \ | \ i \in \mathbb{N}\} \) do not accumulate in \( \tilde{M} \) then \( |n_i| \to \infty \). In addition since \( L, R \) are not separated from each other, then \( h \) preserves each individual line leaf, slice and possible lift annulus of \( L \).

Let \( F^* \) be the leaf of \( \tilde{\mathcal{F}} \) containing \( z_0, w_0 \). Then \( h_i(F) \) converges to \( F^* \). Since \( z_i \) is in \( L \) and \( h_i(L) = L \) then \( h_i(z_i) \) is in \( L \) and so \( z_0 \) is in \( L \). It follows that \( L \) intersects \( F^* \).

Up to subsequence and perhaps taking the inverse of \( h \), assume that \( n_i \) converges to \( +\infty \). If \( h(F) = F \), then since \( h(L) = L \) this produces a closed leaf of \( \Lambda^s \cap \mathcal{F} \) in \( \pi(F) \). Similarly \( h(R \cap L) = R \cap L \) so produces another closed leaf in \( \pi(F) \) and together bound an annulus in \( \pi(F) \) with a sequence of leaves of \( \Lambda^s \cap \pi(F) \) converging to the boundary leaves. By Euler characteristic reasons, there can be no singularities inside the annulus, so we conclude that the annulus in \( \pi(F) \) has a Reeb foliation.

Now assume that \( h(F) \) is not equal to \( F \). Let \( \mathcal{H} \) be the leaf space of \( \tilde{\mathcal{F}} \). This is a one dimensional manifold, which is simply connected, but usually not Hausdorff [Ba2]. The element \( h \) acts on \( \mathcal{H} \). An analysis of group actions on simply connected non Hausdorff spaces was inatiily done by Barbot in [Ba2] and subsequently in [Ro-St] [Fe8]. One possibility is that \( h \) acts freely in \( \mathcal{H} \). Then \( h \) has an axis \( \tau \) in \( \mathcal{H} \) which is invariant under \( h \). In general this axis is not properly embedded, see [Fe8]. Since all the \( h^{n_i}(F) \) intersect a common transversal, then the analysis in [Ba2] shows that \( F \) has to be in the axis of \( h \) and \( h^n(F) \) converges to a collection of non separated leaves. In this case we get that \( F^* \) and \( h(F^*) \) are non separated from each other.

The other situation is that \( h \) has fixed points in \( \mathcal{H} \). In general the set of fixed points of \( h \) is not a closed set, but the set of points \( z \) in \( \mathcal{H} \) so that \( z \) and \( h(z) \) are not separated in \( \mathcal{H} \) is a closed subset \( Z \) of \( \mathcal{H} \) [Ba2] [Ro-St]. None of the images of \( F \) under \( h \) can be in \( Z \), so \( F \) is in a component of \( \mathcal{H} - Z \). Then \( h \) permutes these components. In addition \( h \) preserves an orientation in \( \mathcal{H} \) since \( \mathcal{F} \) is transversely orientable. Since \( h^{n_i}(F) \) all intersect a common transversal then they have all to be in the same component \( U \) of \( \mathcal{H} - Z \). Let \( i_0 \) be the smallest positive integer so that \( h^{i_0}(U) = U \).

It follows that all \( n_i \) are multiples of \( i_0 \). Since \( h^{n_i}(F) \) converges to \( F^* \) then the leaf \( F^* \) is in the boundary of the component \( U \) and \( h^{i_0}(F^*) = F^* \).

The only remaining case to be analysed is that \( h \) acts freely and \( h^n(F) \) converges to \( F^* \) with \( h(F^*) \) non separated from \( F^* \). In this particular case we prove this is not possible, that is:

Claim \(- h(F^*) = F^* \).

Suppose this is not true. The leaves \( h(F^*), F^* \) are not separated in \( \mathcal{H} \). This implies that \( \Theta(F^*) \) and \( \Theta(h(F^*)) \) are disjoint subsets of \( \mathcal{O} \), see fig. [10] Therefore there are boundary leaves separating them. But \( L \) intersects both \( F^* \) and \( h(F^*) \) as \( L \) intersects \( F^* \) and is invariant under \( h \). Therefore both \( \Theta(F^*) \) and \( \Theta(h(F^*)) \) intersect the same stable leaf \( \Theta(L) \).
Suppose that there is a stable boundary component of $\Theta(F^*)$ separating it from $\Theta(h(F^*))$. Then it has to be a slice of $\Theta(L)$ as this set intersects both of them. It would not be a line leaf of $\Theta(L)$. But as remarked before, $h$ leaves invariant all the slices, line leaves and lift annuli of $L$ and this contradicts $\Theta(h(F^*))$ being disjoint from $\Theta(F^*)$. This implies there is an unstable boundary component of $\Theta(F^*)$ separating it from $\Theta(h(F^*))$, see fig. 10.

In the same way $\Theta(R)$ intersects both $\Theta(F^*)$ and $\Theta(h(F^*))$. Let $L_i = \tilde{W}^s(u_i)$. Recall from the beginning of the proof of proposition 6.7 that $u_i, v_i$ are points in $F$ with $u_i$ converging to $p_0$ in $L$ and $v_i$ converging to $q_0$ in $R$. Then $\Theta(L_i)$ converges to $\Theta(L) \cup \Theta(R)$ (maybe other leaves as well). So $\Theta(L_i)$ intersects $\Theta(F^*)$ and $\Theta(h(F^*))$ for $i$ big enough. The intersection of $\Theta(L_i)$ with at least one of $\Theta(F^*)$ or $\Theta(h(F^*))$ cannot be connected, see fig. 10. This contradicts proposition 4.2. This contradiction implies that $h(F^*) = F^*$ and proves the claim.

So far we have proved the following: in any case there is $i_0$ a positive integer so that if $f = h^{i_0}$ then $f(F^*) = F^*$. As $f(L) = L$ then $f(F^* \cap L) = F^* \cap L$ and similarly $f(F^* \cap R) = F^* \cap R$. This produces an annulus $B$ in $\pi(F^*)$ with a Reeb foliation. The region of $F^*$ bounded by $F^* \cap R$ and $F^* \cap L$ bounds a band $B$ which is a bounded distance from a geodesic in $F^*$ and projects to a Reeb annulus in a leaf of $F$.

But to prove proposition 6.7 we really want these facts for $F$ and not just $F^*$. That is, we want a region in $\pi(E)$ which spirals towards a Reeb annulus. This turns out to be true: $\pi(E)$ has points converging to $\pi(F)$ and $\pi(F)$ has points converging to a Reeb annulus in $\pi(F^*)$. Since the annulus is compact, it turns out the second step is unnecessary. This depends on an analysis of holonomy of the foliation $F$ near the annulus in $\pi(F^*)$ as explained below.

Claim — The point $\pi(p_0)$ of $\pi(F)$ is in the boundary of a Reeb annulus of $F$ contained in $\pi(F)$. This implies that $F = F^*$.

The point $z_0$ is in $F^* \cap L$. Then $\pi(z_0)$ is in $\pi(F^* \cap L) = \alpha$ which is a closed curve since $h^{i_0}$ leaves invariant both $F^*$ and $L$ and their intersection is connected. Previous arguments in the proof imply that for $i$ big enough $h_i(z_i)$ is in the same local sheet of $\tilde{\Lambda}$ as $z_0$. Hence the points $\pi(z_i)$ are in $W^s(\pi(z_0)) = \pi(L)$ and converge to $\pi(z_0)$. This shows that $\pi(F \cap L)$ is asymptotic to $\alpha$ in the direction corresponding to the projection of the direction of escaping $z_i$ in the ray of $F \cap L$. Namely $\alpha$ has contracting holonomy (of $F$) in the side the $\pi(z_i)$ are converging to and eventually $\pi(z_i)$ is in the domain of contraction of $\alpha$.

This means that the direction of $F$ associated to the ideal point $\xi$ is a contracting direction towards $F^*$. The rays in the leaves $F^* \cap L$, $F^* \cap R$ in $F^*$ are a bounded distance from a geodesic ray in $F^*$ with same ideal point. The contraction above implies that the corresponding rays $F \cap L$, $F \cap R$ of $F$ are also a bounded distance from a ray in $F$ with ideal point $\xi$. 
Now recall the points $p_i$ in $E$. We have $g_i(p_i)$ very close to $p_0$ in the leaf $l$ of $\Lambda^s_F$. Also $\pi(l)$ is eventually in a region contracting towards a Reeb annulus of $\mathcal{F}$. Hence if $i$ is big enough the $g_i(p_i)$ will also be in this region. The leaf of $\Lambda^s \cap \mathcal{F}$ through $\pi(p_i)$ will be contracted towards the Reeb annulus in that direction. This implies that the limit of the $\pi(p_i)$ is already in a Reeb annulus, consequently the limit of the $g_i(p_i)$ is already in a Reeb band.

It now follows that $\pi(F) = \pi(F^*)$. That means that the second perturbation procedure (from points in $F$ to points in $F^*$) in fact does not produce any new leaf. This implies that up to covering translations then the leaf $E$ is asymptotic to $F$ in the direction of the ideal point $x$ in $\partial_\infty E$. Let $V$ be the region of $E$ bounded by $\upsilon$ with ideal point $x$. Then outside of a compact part it projects very near a Reeb annulus in $\pi(F)$ and so this tail of $V$ has no singularity of the foliation $\Lambda^s_E$. By Euler characteristic reasons it follows that the interior of $V$ has no singularities in the compact part of $V$ also. In fact the arguments show that the tail of $V$ flows into the interior of a Reeb band in a nearby leaf $U$ of $\mathcal{F}$. Then leaves of $\Lambda^s_E$ near $\upsilon$ on the outside of $V$ will also flow to interior of the Reeb band in $U$. Therefore there are appropriate 2 rays on the outside so that they will be in the same leaf of $\Lambda^s_U$ and hence in the same leaf of $\Lambda^s$. It follows that in $E$ the leaf $\upsilon$ is also approximated on the outside by a leaf which has a line leaf with both ideal points the same. This implies that $\upsilon$ has no singularities.

So far we proved the following:

**Conclusion** – Let $\upsilon$ be a slice of $\Lambda^s_E$ with two rays converging to the same ideal point $x$ of $\partial_\infty E$ and $V$ is the region of $E$ bounded by $\upsilon$. Then $\upsilon$ has no singularity of $\Lambda^s_E$ and neither does $V$. Also $\pi(V)$ is either contained in or asymptotic to a Reeb annulus in a leaf of $\mathcal{F}$ and so $E$ is asymptotic to a Reeb band in a leaf $F$ in the direction $x$.

Continuation of the proof of proposition 6.7

What we want to prove is that in $E$ itself the region $V$ is contained in the interior of a spike region. Notice it is not true in general that $\pi(V)$ is contained in a Reeb annulus, only that it is asymptotic to a Reeb annulus. For instance start with a leaf of $\mathcal{F}$ having a Reeb annulus and blow that into an I-bundle. Then produce holonomy associated to the core of the Reeb annulus. Then one produces Reeb bands asymptotic to but not contained in Reeb annuli.

Since $V$ is asymptotic to the Reeb band in $F$, it turns out that (after rearranging by covering translations) that $E$ intersects both $L$ and $R$ leaves of $\Lambda^s$. Their intersection produces two leaves $e_L, e_R$ of $\Lambda^s_E$ which are not separated from each other and which have the same ideal point $x$. There are then leaves $l_i$ of $\Lambda^s_E$ all with ideal point $x$ and which converge to $e_L \cup e_R$. This follows from the fact that in $F$ the same is true and $E$ is asymptotic to $F$ in that direction, plus the connectivity of the intersection of $E$ with leaves of $\Lambda^s$. 

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Figure 11: a. $l_i$ converging to non separated leaves $e_L, e_Z, e_Y, e_R$ of $\Lambda^s_E$, b. Nested families and identifications of ideal points.
Now the sequence $l_i$ can converge to other leaves as well, all of which will be non separated from $e_L,e_R$. The set of limits is an ordered set and the any other leaf is between $e_L$ and $e_R$. By theorem 6.6 there are only finitely many of them. We refer to fig. 11 a, where for simplicity we consider there are 4 leaves in the limit: $e_L,e_Y,e_Z,e_R$ contained in leaves $L,Z,Y,R$ of $\tilde{\Lambda}^s$. These leaves of $\tilde{\Lambda}^s$ are non separated from each other and form an ordered set. Let $\xi$ be the region of $E$ which is the union of the region bounded by all the $l_i$ plus the boundary leaves, which are non separated from $e_L,e_R$. Clearly every leaf in the interior has ideal point $x$ and has no singularity. We want to show that $\xi$ is a spike region.

Any two consecutive leaves of $\partial \xi$ in this ordering will have rays with same ideal point and leaves $l_i$ converging to them. This situation is important on its own and is analysed in the following proposition:

**Proposition 6.10.** Suppose $v_1,v_2$ are non separated leaves in $\tilde{\Lambda}^s_G$ for some $G$ leaf of $\tilde{F}$. Suppose there are no leaves non separated from $v_1,v_2$ in between them. Then the corresponding rays of $v_1,v_2$ have the same ideal point in $\partial_s G$. In addition they are a bounded distance from a geodesic ray of $G$ with same ideal point. In $M$ this region either projects to or is asymptotic to a Reeb annulus.

**Proof.** We do the essentially the same proof as in the case of leaves of $\tilde{\Lambda}^s_G$ with same ideal points, except that we go in the direction of the non Hausdorffness. Because there are no non separated leaves in between $v_1,v_2$, then the corresponding rays have the same ideal point. Choose $w_i,y_i$ in these rays of $v_1,v_2$ and escaping towards the ideal point and so that $d_G(w_i,y_i)$ is less than $4\delta_0$. We do the limit analysis using $f_i(w_i),f_i(y_i)$ converging in $\tilde{M}$. Because $v_1,v_2$ are non separated it follows that $f_i(w_i),f_j(w_j)$ are in the same stable leaf (of $\tilde{\Lambda}^s$) for $i,j$ big enough. Hence we can readjust so that they are all in the same stable leaf and similarly for $f_i(y_i)$. The same arguments as before show that that region of $G$ between $v_1,v_2$ projects in $M$ to set in a leaf of $\mathcal{F}$ which is either contained in or asymptotic to a Reeb annulus. The results follow. In general nothing can be said about the other direction in the leaves $v_1,v_2$: in particular it does not follow at all that the other rays of $v_1,v_2$ have to have the same ideal point.

Given this last proposition then for any two consecutive rays in $\partial \mathcal{E}$ it follows that they are a bounded distance from a geodesic ray in $E$. All that is needed to show that $\mathcal{E}$ is a spike region is to prove that the ideal points of the rays in the boundary are distinct except for consecutive rays.

Suppose there are other identifications of ideal points of leaves in the boundary of $\mathcal{E}$. Then there is at least one line leaf $\tau$ in the boundary of $\mathcal{E}$ so that $\tau$ has identified ideal points. Our analysis so far shows that $\tau$ is in the interior of another region similar to the one constructed above so that all leaves have just one common ideal point. Since the $l_i$ limit on $\tau$, then the ideal point of $\tau$ has to be $x$. In addition the leaves in this new region have to be nested. But if the $l_i$ together with $\tau$ are a nested family of leaves of $\tilde{\Lambda}^s_F$, then the $\tau$ is outside the $l_i$ hence the region in $E$ bounded by $\tau$ enclosed the whole region $\mathcal{E}$, see fig. 11 b. There is at least one other leaf $\tau'$ in $\partial \mathcal{E}$. The same arguments we used for $\tau$ can be applied to $\tau'$. But it is impossible that the $l_i$ are also nested with the $\tau'$, see fig. 11 b.

This shows that the ideal points of $\mathcal{E}$ are distinct except as mandated by consecutive rays. In addition any line leaf in the boundary of $\mathcal{E}$ has distinct ideal points and rays which are a bounded distance from geodesic rays. It follows that the whole leaf is a bounded distance from a geodesic in $E$. This shows that $\mathcal{E}$ is a spike region. This finishes the proof of proposition 6.7.

Finally in the case $\tilde{\Lambda}^s$ has Hausdorff leaf space one can say much, much more about metric properties of leaves of $\tilde{\Lambda}^s_F$. 


Proposition 6.11. Suppose that $\Phi$ is an almost pseudo-Anosov flow transverse to a foliation $\mathcal{F}$ with hyperbolic leaves. Suppose that $\tilde{\Lambda}^s$ has Hausdorff leaf space. Then there is $k_0 > 0$ so that for any $F$ leaf of $\tilde{\Lambda}^s$, then the slice leaves of $\tilde{\Lambda}_F^s$ are uniform $k_0$-quasigeodesics.

Proof. If there is a leaf $F$ of $\tilde{\mathcal{F}}$ and a slice leaf of $\tilde{\Lambda}_F^s$ with only one ideal point, then the proof of proposition 6.7 shows that there are leaves of $\tilde{\Lambda}^s$ non separated from each other. This is impossible.

Suppose now that for any integer $i$, there are $x_i$ in $\tilde{M}$, and in leaves $F_i$ of $\tilde{\mathcal{F}}$ with $x_i$ in line leaves $l_i$ of $\tilde{\Lambda}_F^s$, with distance from $x_i$ to $l_i^*$ in $F_i$ going to infinity. Here $l_i^*$ is the geodesic in $F_i$ with same ideal points as $l_i$. Up to covering translations assume $x_i$ converges to $x$. Also assume all $x_i$ are in the same sector of $\tilde{\Lambda}^s$ defined by $x$. Since $l_i$ converges to $l$, the arguments in lemmas 6.5 and 6.2 would show that the ideal points of $l$ are the same. This was just disproved above.

Given that, the line leaves are within some global distance $a_0$ of the respective geodesics in their leaves. It is well known that these facts imply that the slice leaves of $\tilde{\Lambda}_F^s$ are uniform quasigeodesics. For a proof of this well known fact see for example [Fe-Mo].

7 Continuous extension of leaves

The purpose of this section is to prove the main theorem: the continuous extension property for leaves of foliations which are almost transverse to quasigeodesic singular pseudo-Anosov flows in atoroidal 3-manifolds. As seen before this implies that $M$ has negatively curved fundamental group.

Suppose first that $\Phi$ is an almost pseudo-Anosov flow which is transverse to a foliation $\mathcal{F}$ with hyperbolic leaves in a general closed 3-manifold $M$. Given a leaf $F$ of $\tilde{\mathcal{F}}$ we introduce geodesic “laminations” in $F$ coming from $\tilde{\Lambda}_F^s, \tilde{\Lambda}^s$. We only work with the stable foliation, similar results hold for the unstable foliation. Assume that a leaf $l$ of $\tilde{\Lambda}_F^s$ is not singular. If both ideal points are the same let $l^*$ be empty. Otherwise let $l^*$ be the geodesic with same ideal points as $l$. If $l$ is singular, then no line leaves of $l$ have the same ideal point by proposition 6.7. For each line leaf $e$ of $l$ let $e^*$ be the corresponding geodesic and $l^*$ their union. Let now $\tau_F^s$ be the union of these geodesics of $F$. Leaves of $\tilde{\Lambda}_F^s$ do not have transverse intersections and therefore the same happens for leaves of $\tau_F^s$.

Suppose that $\tilde{\Lambda}_F^s$ has non separated leaves $l, v$ which are not in the boundary of a spike region. Then there are $l_i$ converging to $l \cup v$ (and maybe other leaves as well), but $l_i^*$ does not converge to $l^* \cup v^*$. Notice none of the limit leaves can have identified ideal points, because then they would be in the interior of a spike region (proposition 6.7) and have a neighborhood which is product foliated. Let $\overline{\tau}_F^s$ be the closure of $\tau_F^s$. Then $\overline{\tau}_F^s$ is a geodesic lamination in $F$. Similarly define $\tau_F^{\pm}, \overline{\tau}_F^{\pm}$. In a complementary region $U$ of $\overline{\tau}_F^s$ associated to non Hausdorffness, there is one boundary component which is added (a leaf of $\tau_F^s - \overline{\tau}_F^s$) and which is the limit of the $l_i^*$ as above. All of the other boundary leaves of the region are associated to the non separated leaves of $\tilde{\Lambda}_F^s$ and are in $\tau_F^s$.

Lemma 7.1. The new leaves in $\overline{\tau}_F^s$ (that is those in $\overline{\tau}_F^s - \tau_F^s$) come from non Hausdorff pairs $(l, v)$ of $\tilde{\Lambda}_F^s$ as in the description above.

Proof. Let $e_i$ in $\tau_F^s$ converging to $e$ not in $\tau_F^s$. Then choose $l_i$ line leaves in $\tilde{\Lambda}_F^s$ with $e_i = l_i^*$. Given $u$ a point in $e$, there is $u_i$ in $l_i^*$ very close to $u$. Then there are $p_i$ in $l_i$ which are $2\delta_0$ close to $u_i$. Up to subsequence assume that $p_i$ converges to $p_0$ and let $l$ be the line leaf of $\tilde{\Lambda}_F^s$ that the sequence $l_i$ converges to. Since the $l_i^*$ converges to $e$ which is not in $\overline{\tau}_F^s$ and $l^*$ is in $\overline{\tau}_F^s$, it follows that $l_i^*$ does not converge to $l^*$. By lemma 6.5 this is associated to a non Hausdorff situation: $l_i$ converging to $l$ and other leaves as well and $l^*$ is the added leaf associated to this non Hausdorffness. This finishes the proof of the lemma.
Lemma 7.2. The complementary regions of $\tau_F^*$ are ideal polygons associated to singular leaves and non Hausdorff behavior of $\tilde{\Lambda}^s$. If $M$ is atoroidal then these regions are finite sided ideal polygons.

Proof. Let $x$ be in a complementary region $U$ of $\tau_F^*$. Let $e$ be a leaf in the boundary $\partial U$. Let $I$ be the interval of $\partial_\infty F - \partial e$ containing other ideal points of $U$. Suppose first that $e$ is an actual leaf of $\tau_F^*$, which comes from a line leaf $l$ of $\tilde{\Lambda}_F^s$. It may be that $l$ is contained in a singular leaf $z$ of $\tilde{\Lambda}_F^s$, which is singular on the $x$ side. This means that $z$ has ideal points in $I$. In that case $x$ is in the complementary region obtained by splitting $z$. This region must be $U$. Otherwise $l$ is not singular on the side containing $x$ and we may assume there are $l_i$ leaves of $\tilde{\Lambda}_F^s$ with ideal points in the closure of $I$ in $\partial_\infty F$, with $l_i$ converging to $l$. If the ideal points of $l_i$ converge to that of $l$ then eventually $l_i^*$ separates $x$ from $e$ and $x$ is not in the complementary region $U - \text{impossible}$. Hence the ideal points of $l_i$ do not converge to $\partial e$ and there is non Hausdorffness and a complementary region in that side of $l$. Then $x$ needs to be in this complementary region (which is $U$) and $e$ is a boundary leaf of $U$ which comes from a line leaf of $\tau_F^*$.

Suppose now that $e$ is an added leaf. There are $l_i$ leaves of $\tilde{\Lambda}_F^s$ with $e_i = l_i^*$ converging to $e$ on the side opposite to $x$, otherwise $x$ is not in $U$. Then $l_i$ converges to more than one leaf of $\tilde{\Lambda}_F^s$ producing non Hausdorff behavior and a complementary region with $e$ in its boundary. The $x$ is in the region associated to this non Hausdorff behavior, so the complementary region must be $U$.

If there is a complementary region of $\tau_F^*$ with infinitely many sides then it is associated to non Hausdorff behavior and so there are leaves $l_i$ of $\tilde{\Lambda}_F^s$ converging to infinitely many distinct leaves of $\tilde{\Lambda}_F^s$. Then there is $L$ leaf of $\tilde{\Lambda}^s$ which is non separated from infinitely many other leaves. Theorem 3.6 implies that there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$, contradiction. This finishes the proof.

We now turn to the continuous extension property.

Theorem 7.3. (Main theorem) Let $\mathcal{F}$ be a foliation in $M^3$ closed, atoroidal. Suppose that $\mathcal{F}$ is almost transverse to a quasigeodesic, singular pseudo-Anosov flow $\Phi_1$ and transverse to an associated almost pseudo-Anosov flow $\Phi$. Singular means $\Phi_1$ is not a topological Anosov flow. Then for any leaf $F$ of $\mathcal{F}$, the inclusion map $\Psi : F \to \tilde{M}$ extends to a continuous map

$$\Psi : F \cup \partial_\infty F \to \tilde{M} \cup S^2_\infty$$

The map $\Psi$ restricted to $\partial_\infty F$, gives a continuous parametrization of the limit set of $F$, which is then locally connected.

Proof. The hypothesis imply that $\pi_1(M)$ is negatively curved. Difficulties in the proof of this result are that $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ may have non Hausdorff leaf space [Mo65, Fe66] and so $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ can have non Hausdorff leaf space. This implies that the leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ cannot be uniform quasigeodesics in $F$. In addition the leaves of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ are not quasi-isometrically embedded in $\tilde{M}$. The proof is done in two steps: first we define an extension and then we show that it is continuous.

The proof will fundamentally use the fact that $\Phi_1$ is a quasigeodesic pseudo-Anosov flow. It was proved in [Fe-Mo] that this implies that $\Phi$ is a quasigeodesic flow as well. From now on we use the stable/unstable foliations $\Lambda^s, \Lambda^u$ of $\Phi$. First we need to review some facts about quasigeodesic almost pseudo-Anosov flows. If $\gamma$ is an orbit of $\Phi$ then it is a quasigeodesic and hence has unique distinct ideal points $\gamma_-$ and $\gamma_+$ in $S^2_\infty$ corresponding to the positive and negative flow directions [Th1, Gr, Gh-Ha, CDP]. Hence given $x$ in $\tilde{M}$ define

$$\eta_+(x) = \gamma_+, \quad \eta_-(x) = \gamma_-, \quad \eta_+(x) \neq \eta_-(x),$$
where $\gamma$ is the $\Phi$ flowline through $x$. If $L$ is a leaf of $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$ and $a$ is a limit point of $L$ in $S^2_{\infty}$, then there is an orbit $\gamma$ of $\Phi$ contained in $L$ with either $\gamma_-=a$ or $\gamma_+=a$, that is, any limit point of $L$ is a limit point of one of its flow lines. Also any such $L$ in $\tilde{\Lambda}^s$ is Gromov negatively curved and has an intrinsic ideal boundary $\partial L$ consisting of a single forward ideal point and distinct negative ideal points for each flow line. The set $L \cup \partial_\infty L$ is a natural compactification of $L$ in the Gromov sense. For instance if $L$ is a non singular leaf, then $L \cup \partial_\infty L$ is a closed disk. In this case the foliation by flow lines in $L$ is equivalent to the foliation in $H^2$ by geodesics sharing a fixed point in $S^1_{\infty}$.

A very important fact for us is that the inclusion

$$\kappa : L \to \tilde{M}$$

extends to a continuous map $\kappa : L \cup \partial L \to \tilde{M} \cup S^2_{\infty}$.

This follows from the fact that $\Phi$ is quasigeodesic. This works for any $L$ in $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$. If $L$ is in $\tilde{\Lambda}^s$ there is a unique distinguished ideal point in $S^2_{\infty}$ denoted by $L_+$ which is the forward limit point in $S^2_{\infty}$ of any flow line in $L \subset \tilde{M}$. In addition $\Lambda^s$ is a quasi-isometric singular foliation, then the extension $\kappa$ is always a homeomorphism into its image, but this is not true if $\Lambda^s$ is not quasi-isometric. Similarly for $L$ in $\tilde{\Lambda}^u$.

Throughout the proof we fix a unique identification of $\tilde{M} \cup S^2_{\infty}$ with the closed unit ball in $\mathbb{R}^3$. The Euclidean metric in this ball induces the visual distance in $\tilde{M} \cup S^2_{\infty}$. Then $diam(B)$ denotes the diameter in this distance for any subset $B$ of $\tilde{M} \cup S^2_{\infty}$. A notation used throughout here is the following: if $A$ is a subset of a leaf $F$ of $\tilde{F}$, then $\tilde{A}$ is its closure in $F \cup \partial_\infty F$.

We now extend an extension $\Psi : \partial_\infty F \to S^2_{\infty}$.

Case 1 – Suppose that $v$ in $\partial_\infty F$ is not an ideal point of a ray in $\tilde{\Lambda}^s_F$ or in $\tilde{\Lambda}^u_F$.

Since $\pi_1(M)$ is negatively curved, then complementary regions of $\tau^u_F$ are finite sided ideal polygons. Hence there are $e_i$ in $\tau^u_F$ so that $\{e_i \cup \partial e_i\}$, $i \in \mathbb{N}$ define a neighborhood basis of $v$ (in $F \cup \partial_\infty F$) and $\{e_i\}$ forms a nested sequence. Here $\partial e_i$ are the ideal points of $e_i$ in $\partial_\infty F$. We say that the $\{e_i\}$ define a neighborhood basis at $v$. Assume that no two $e_i$ share an ideal point – possible because of hypothesis. If $e_i$ is in $\tau^s_F - \tau^u_F$ then it is the limit of leaves in $\tau^s_F$ and by adjusting the sequence above we can assume that $e_i$ is always in $\tau^s_F$. Let $l_i \in \tilde{\Lambda}^s_F$ with $l_i^+ = e_i$ and $l_i$ leaves of $\tilde{\Lambda}^s$ with $l_i \subset L_i$.

Similarly there are $c_i$ in $\tau^u_F$ defining a neighborhood basis of $v$. Up to subsequence we may assume that $e_1, c_1, e_2, c_2, \ldots$ are nested and none of them have any common ideal points (in $F \cup \partial_\infty F$) and $c_i$ is in $\tau^u_F$. Let $b_i \in \tilde{\Lambda}^u_F$ with $b_i^+ = c_i$ and $B_i$ leaves of $\tilde{\Lambda}^u$ with $b_i \subset B_i$.

At this point we need the following result:

**Lemma 7.4.** Let $L$ leaf of $\tilde{\Lambda}^s$, $B$ leaf of $\tilde{\Lambda}^u$ and $F$ leaf of $\tilde{F}$ so that $F$ intersects both $L$ and $B$: $l = L \cap F$, $b = B \cap F$. Suppose that $b$ and $l$ are disjoint in $F$. Then $L$ does not intersects $B$ in $\tilde{M}$.

**Proof.** Suppose not. Recall that $\Theta(L), \Theta(B)$ are finite pronged, non compact trees and they intersect in a compact subtree. The union is also a finite pronged tree. In addition $\Theta(L \cap B)$ is connected. The sets $\Theta(l), \Theta(b)$ are disjoint in this union. Let $x$ be a boundary point of $\Theta(l)$ which is either in $\Theta(L \cap B)$ or separates $\Theta(L \cap B)$ from $\Theta(l)$ in this union, see fig. 12 a.

Let $\gamma = x \times \mathbb{R}$, an orbit of $\Phi$. The first possibility is that $F$ escapes up as $\Theta(F)$ approaches $x$. Then $\gamma$ is a repelling orbit with respect to the $\Theta(l)$ side, see fig. 12 b and $\gamma$ is in the boundary of a lift annulus $A$. This means that $\Theta(l)$ is a generalized unstable prong from the point of view of $x$. By proposition there is a stable slice $r$ of $\Theta^s(x)$ with $r$ contained in $\partial \Theta(F)$ and $F$ escapes up as $\Theta(F)$ approaches $r$, see fig. 12 a. The two sides of $r$ are the closest generalized prongs to $\Theta(l)$ on either side of $\Theta(l)$. This implies that $r$ separates $\Theta(b)$ from $\Theta(F)$ see fig. 12 a. Then $\Theta(b)$ cannot be contained in $\Theta(F)$, contradiction.
The second option is that $F$ escapes down as $\Theta(F)$ approaches $x$ along $\Theta(l)$. Here there is a slice $r$ of $O^u(x)$ with $r$ contained in $\partial \Theta(F)$ and the closest to $\Theta(l)$ on both sides of $\Theta(l)$. Either $\Theta(b) \subset r$ or $r$ separates $\Theta(b)$ from $\Theta(F)$. In any case $\Theta(b)$ does not intersect $\Theta(F)$, again a contradiction. This finishes the proof of the lemma.

Claim – Both $L_i$ and $B_i$ escape in $\tilde{M}$.

Notice $e_i \cap c_j = \emptyset$ for any $i, j$. If $l_i \cap b_j$ is non empty with $j > i$, then the nesting property above implies that $b_{i+1}, b_{i+2}, \ldots, b_j$ all have to intersect. Since there is a global upper bound on the number of prongs of leaves of $\tilde{\Lambda}_s, \tilde{\Lambda}_u$, this can happen for only finitely many times. Up to taking a further subsequence we may assume that all the $l_i, b_j$ are disjoint.

The lemma shows that $L_i \cap B_j$ is empty for any $i, j$, and they form nested sequences of leaves in $\tilde{M}$. Suppose that the sequence $\{L_i\}$ does not escape compact sets. Then there is $L$ in $\tilde{\Lambda}_s$ which is a limit of $L_i$ (and possibly other leaves as well). Let $\alpha$ be an orbit in $L$ which is not in a lift annulus. Then $\tilde{W}^u(\alpha)$ is transverse to $L$ in $\alpha$ and hence intersects $L_i$ for $i$ big enough. Since the $L_i, B_j$ are nested this would force $\tilde{W}^u(\alpha)$ to intersect $B_j$ for $j$ big enough, contradiction. It follows that both $L_i$ and $B_j$ escape compact sets as $i, j \to \infty$.

Let $r$ be a geodesic ray in $F$ with ideal point $v$. For each $i$, there is a subray of $r$ contained in the component of $F - l_i$ which is in a small neighborhood of $v$. Hence $\Psi(r)$ has a subray which is contained in the corresponding component $V_i$ of $\tilde{M} - L_i$. These components $V_i$ form a nested sequence. The ray $\Psi(r)$ can only limit in the limit set of $V_i$. We need the following lemma which will be a key tool throughout the proof.

Lemma 7.5. (basic lemma) Let $\{Z_i\}$ be a sequence of leaves or line leaves or slices or any flow saturated sets in leaves of either in $\Lambda^s$ or $\Lambda^u$ (not all leaves $Z_i$ need to be in the same singular foliation). If the sets $Z_i$ escape compact sets in $\tilde{M}$, then up to taking a subsequence $\overline{Z_i}$ converges to a point in $S^2_{\infty}$.

Proof. Let $Y_i$ be the leaf of $\tilde{\Lambda}_s$ or $\tilde{\Lambda}_u$ which contains $Z_i$. Up to subsequence assume $Y_i \in \tilde{\Lambda}_s$. The statement is equivalent to $\text{diam}(Z_i)$ converges to 0. Otherwise up to subsequence we can assume $\text{diam}(Z_i) > a_0$ for some $a_0$ and all $i$ and hence no subsequence can converge to a single point in $S^2_{\infty}$. Then there is $p_i$ in $Z_i$ with visual distance from $p_i$ to $(Y_i)_+$ is bigger than $a_0/2$. Notice that $(Y_i)_+$ is a point in $\overline{Z_i}$. Let $\gamma_i$ the orbit of $\Phi$ through $p_i$. If $\gamma_i$ is very close to $(Y_i)_+$ then the geodesic with these ideal points has very small visual diameter. Since $\gamma_i$ is a global bounded distance from this geodesic $\text{Gr}$, $\text{Gh-Ha}$, $\text{CDP}$, the same is true for $\gamma_i$ contradiction to the choice of $p_i$. Hence the geodesic above intersects a fixed compact set in $\tilde{M}$ and so does $\gamma_i$. This contradicts the fact that $Z_i$ escape compact sets in $\tilde{M}$ and finishes the proof.

We claim that the limit sets of $V_i$ above shrink to a single point in $S^2_{\infty}$. The limit sets form a weakly monotone decreasing sequence, because the $L_i$ are nested and so are the $V_i$. If the limit set
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does not have diameter going to zero, then there are points in the limit set of \( L_i \) which are at least \( 2\delta_1 \) apart for some fixed \( \delta_1 > 0 \). By the previous lemma the \( L_i \) cannot escape compact sets in \( \widetilde{M} \), contradiction. Since the limit sets of \( V_i \) shrank to a point in \( S^2_{\infty} \), let \( \Psi(v) \) be this point. Clearly \( \Psi(r') \) limits to this point and so does \( \Psi(r'') \) for any other geodesic ray \( r'' \) in \( F \) with ideal point \( v \).

**Case 2** — Suppose that \( v \) is an ideal point of a leaf of \( \widetilde{\Lambda}_F^s \) or \( \widetilde{\Lambda}_F^u \).

Let \( l \) be a ray in say \( \widetilde{\Lambda}_F^s \) which limits on \( v \) and \( r \) a geodesic ray on \( F \) with ideal point \( v \). Then \( l \) is contained in \( L \) leaf of \( \widetilde{\Lambda}_F^s \). Either \( \Theta(l) \) escapes in \( \Theta(L) \) or limits to a point \( x \) in \( \Theta(L) \).

Consider the first case. Then in the intrinsic geometry of \( L \), the ray \( l \) converges to the positive ideal point of \( L \), hence in \( \widetilde{M} \cup S^2_{\infty} \), the image \( \Psi(l) \) converges to \( L_+ \). In the other option let \( \beta = x \times R \), an orbit of \( \Phi \). As \( l \) escapes in \( F \) then in \( L \) it either escapes up or down. If it escapes down then it converges to the negative ideal point of \( \beta \) in \( L \cup \partial_\infty L \) and hence \( \Psi(l) \) converges to \( \beta_- \). Otherwise \( l \) escapes up in \( L \) as \( \Theta(l) \) approaches \( x \). In this case \( \beta \) is in the boundary of a lift annulus and \( l \) converges to the positive ideal point in \( L \cup \partial_\infty L \) and so \( \Psi(l) \) converges to \( L_+ \) again.

The remaining case is that \( \Theta(l) \) escapes in \( \Theta(L) \). Then as seen in \( L \cup \partial_\infty L \) the ray \( l \) converges to the positive ideal point \( p \) of all flow lines in \( L \). Hence \( \Psi(l) \) converges to \( \kappa(p) = L_+ \). Let \( \Psi(v) \) be the limit point in any case. Similarly if \( l \) is a ray of \( \widetilde{\Lambda}_F^u \).

Every point in \( r \) is \( 2\delta_0 \) close to a point in \( l \) in \( F \), hence the limit of \( \Psi(r) \) in \( \widetilde{M} \cup S^2_{\infty} \) is the same as that of \( l \). If \( l' \) is another ray of \( \widetilde{\Lambda}_F^s \) or \( \widetilde{\Lambda}_F^u \) converging to \( v \), then it will have points boundedly close to \( r \) which escape in \( l' \) and therefore \( \Psi(l') \) has the same ideal point in \( S^2_{\infty} \). Therefore \( \Psi(v) \) is well defined.

This finishes the construction of the extension of \( \Psi \) to \( \partial_\infty F \).

**Proof of continuity of the extension** —

**Case 1** — \( v \) is not an ideal point of a ray in \( \widetilde{\Lambda}_F^s \) or \( \widetilde{\Lambda}_F^u \).

Let \( r \) be a geodesic ray in \( F \) with ideal point \( v \). Recall the extension construction. There are \( l_i \) in \( \widetilde{\Lambda}_F^s \) shrinking to \( v \) in \( F \cup \partial_\infty F \) and similarly \( b_i \) in \( \widetilde{\Lambda}_F^u \), assumed to be nested with the \( l_i \). Let \( \{l_i^*\} \) define a neighborhood basis of \( v \) in \( F \cup \partial_\infty F \). Let \( L_i \) in \( \Lambda^s \) with \( l_i \subset L_i \), and \( b_i \subset B_i \in \Lambda^u \) as in the construction case 1. Then as seen in the construction, the \( L_i, B_i \) escape in \( \widetilde{M} \). Let \( U_i \) be the component of \( F - l_i \) containing a subsay of \( r \) and \( V_i \) the component of \( \widetilde{M} - L_i \) containing \( U_i \).

Notice that \( \Psi(U_i) \subset V_i \). Let now \( z \) in \( U_i \) with the closure taken in \( F \cup \partial_\infty F \) and \( V_i \) the closure of \( V_i \) in \( \widetilde{M} \cup S^2_{\infty} \). Then \( U_i \) is a neighborhood of \( v \) in \( F \cup \partial_\infty F \). If \( z \) is \( \in \Psi(U_i) \) then using either of the constructions in the extension part shows that \( z \) is a limit of points in \( \Psi(U_i) \subset V_i \). As seen in the construction arguments the diameter of \( V_i \) in the visual distance is converging to 0. Hence we obtain continuity of \( \Psi \) at \( v \). This finishes the proof in this case.

**Case 2** — \( v \) is an ideal point of a ray of \( \widetilde{\Lambda}_F^s \) or \( \widetilde{\Lambda}_F^u \).

This case is considerably more complicated, with several possibilities.

**Case 2.1** — \( v \) is an ideal point of \( \widetilde{\Lambda}_F^s \), but not of \( \widetilde{\Lambda}_F^u \) (or vice versa).

Suppose the first option occurs. There is \( l \) ray in \( \Lambda^s \) with ideal point \( v \). We may assume that \( l \) is not in a leaf of \( \Lambda^s \) with same ideal points. Otherwise we can choose \( l \) to be one of the boundary leaves of the corresponding spike region. Since \( v \) is not an ideal point of \( \Lambda^u \), there are \( g_i \) line leaves in \( \Lambda^s_F \) defining a basis neighborhood system at \( v \). Let \( g_i \) be contained in \( G_i \) leaves of \( \Lambda^u \). Let \( L \) in \( \Lambda^s \) containing \( l \). If \( G_i \) escapes in \( \widetilde{M} \) as \( i \to \infty \), then as seen in case 1, we are done. Let then \( G_i \) converge to the finite set of leaves.
We can assume that $G_i \cap l \neq \emptyset$ for all $i$, $G_i$ is non singular and the sequence $\{G_i\}, i \in \mathbb{N}$ is nested.

**Case 2.1.1** — Suppose that $L$ intersects $\mathcal{V}$, say $L \cap H_1 \neq \emptyset$.

Then $l$ escapes down as $\Theta(l)$ approaches $\Theta(L \cap H_1)$. Otherwise $L \cap H_1$ is in the boundary of a lift annulus $A$ and $l$ has a subray contained in this lift annulus. But then $A$ is also contained in the unstable leaf $\tilde{W}^u(L \cap H_1)$ and so $G_i$ cannot intersect $l$, contradiction. As $l$ escapes down in $L$, then the ideal point of $\Psi(l)$ is $(L \cap H_1)_-$ which is equal to $(H_1)_-$, the negative ideal point of $H_1$.

Since the values of $\Psi(p)$ for $p$ in $\partial \infty F$ are obtained as limits of values in $\Psi(F)$, then we only need to show that if $z_k$ is in $F$ and $z_k$ converges to $p$ as $k \to \infty$, then $\Psi(z_k)$ converges to $\Psi(p)$. Suppose this is not the case.

By taking a subray if necessary, we may assume that $l$ does not intersect a lift annulus and hence it is transverse to the unstable foliation $\tilde{\Lambda}^u_F$ in $F$. Parametrize the leaves of $\tilde{\Lambda}^u_F$ intersected by $l$ as $\{g_t, t \in \mathbb{R}_+\}$, contained in $G_t \in \tilde{\Lambda}^u$ (by an abuse of notation think of the $G_t$ as a discrete subcollection of the $G_t, t \in \mathbb{R}_+$). Let

$$\mathcal{U} = \bigcup_{t > 0} G_t$$

No $g_t$ (or leaf of $\tilde{\Lambda}^u_F$) has ideal point $v$ in $\partial \infty F$. As $\{g_t\}, i \in \mathbb{N}$ converges to $v$ in $F \cup \partial \infty F$, then $g_t$ escapes compact sets in $F$ as $t \to \infty$ and the ideal points of $g_t$ converge to $v$ on either side of $v$. Up to subsequence assume that all of the elements of the sequence $\{z_k\}$ are either entirely contained in $\mathcal{U}$ or disjoint from $\mathcal{U}$.

**Situation 1** — Suppose that $z_k$ is not in $\mathcal{U}$ for any $k$.

Since $z_k$ is very close to $v$ in the compactification $F \cup \partial \infty F$ and $g_t$ converges to $v$ in $F \cup \partial \infty F$ when $t \to \infty$, then there are $t, s$ with $z_k$ between $g_t$ and $g_s$ (in $F$). Notice $z_k$ is not in any of them. Now there is a unique time $t_k$ so that exactly at that time $\Psi(z_k)$ switches from being in one side of $G_t$ in $\tilde{M}$ to the other (equivalently compare the $z$ and $g_t$ in $F$). In particular, either there is a line leaf $L_{t_k}$ of $G_{t_k}$ which separates $\Psi(z_k)$ from all the other $G_t$, see fig. 13a, or there is a leaf $L_{t_k}$ non separated from $G_{t_k}$ with $\Psi(z_k)$ either in $L_{t_k}$ or $L_{t_k}$ separates $\Psi(z_k)$ from all $G_t$, see fig. 13b. This can be seen in the leaf space of $\tilde{\Lambda}^u$, which is a non Hausdorff tree [Fe8, Ga-Ka, Ro-St].

**Claim** — In the Gromov-Hausdorff topology of closed sets of $\tilde{M} \cup S^2_\infty$, the sets $\tilde{L}_{t_k}$ converge to $(H_1)_-$ as $k \to \infty$.

If $L_{t_k}$ is a line leaf of $G_{t_k}$, then $(L_{t_k})_- = (G_{t_k})_-$. If $L_{t_k}$ is not separated from $G_{t_k}$ then also $(L_{t_k})_- = (G_{t_k})_-$. This is because there are $E_i$ leaves of $\tilde{\Lambda}^u$ with $E_i$ converging to $L_{t_k} \cup G_{t_k}$. So there are $x_i, y_i$ in $E_i$ with $x_i \to x$, $y_i \to y$ and $x \in L_{t_k}$, $y \in G_{t_k}$. Then

$$\eta_-(x_i) \to \eta_-(x) = \eta_-(L_{t_k}), \quad \eta_-(y_i) \to \eta_-(y) = \eta_-(G_{t_k}) \quad \text{and} \quad \eta_-(x_i) = \eta_-(y_i).$$
The last equality occurs because \( x_i, y_i \) are in the same unstable leaf \( E_i \). Therefore \((L_{tk})_+\) converges to \((H_1)_-\) when \( k \to \infty \). Suppose that \( \overline{L}_{tk} \) does not converge to \((H_1)_-\) in \( \bar{M} \cup S^2_\infty \). Since

\[
(L_{tk})_+ \text{ converges to } (H_1)_-,
\]

then lemma 7.3 shows that \( L_{tk} \) does not escape compact sets in \( \bar{M} \). So it limits to some \( u \) in \( \bar{M} \) and up to subsequence we may assume there \( u_k \) in \( L_{tk} \) with \( u_k \) converging to \( u \). The first possibility is that the \( L_{tk} \) are subsets of the leaves \( G_{tk} \). This implies that \( \bar{\Phi}_R(u) \) is in the limit of the sequence of leaves \( G_{tk} \) (in \( \bar{M} \)), so it is contained in \( \mathcal{V} \). The second possibility is \( L_{tk} \) non separated from \( G_{tk} \) so \( L_{tk} \) is between \( G_{tk-1} \) and \( G_{tk+1} \) hence \( u \) is again in the limit of the \( G_t \) so \( u \) is in \( \mathcal{V} \). The leaves \( H_j \) in \( \mathcal{V} \) are non singular in the side the \( G_t \) are limiting on, so there is a neighborhood of \( u \) on that side of \( H_j \) which has no singularities hence the \( u_k \) will be in \( \mathcal{U} \) for \( k \) big enough. This contradicts the hypothesis in this case.

This shows that \( \overline{L}_{tk} \) converges to \((H_1)_-\) in \( \bar{M} \cup S^2_\infty \). Also \( L_{tk} \) either contains \( \Psi(z_k) \) or separates it from a base point in \( \bar{M} \). It follows that \( \Psi(z_k) \) converges to \((H_1)_-\), which is what we wanted to prove. This finishes the analysis in situation 1.

**Situation 2** — For all \( k \) assume that \( \Psi(z_k) \) is in \( \mathcal{U} \).

Let \( t_k \) with \( \Psi(z_k) \) in \( G_{tk} \), hence \( z_k \) is in \( G_{tk} \cap F = g_{tk} \). Then \((\Psi(z_k))_- = (G_{tk})_-\) converges to \((H_1)_-\). We want to show that \( \Psi(z)_- \) converges to \((H_1)_-\). Otherwise there is \( q \) in \( S^2_\infty \) different from \((H_1)_-\) and a subsequence, still denoted by \( \Psi(z_k) \) so that \( \Psi(z_k) \) converges to \( q \). As in the claim of situation 1 above, \( \bar{\Phi}_R(z_k) \) does not escape compact sets in \( \bar{M} \) and there is \( z \) in \( \bar{M} \), so that up to another subsequence, we may assume that \( \bar{\Phi}_R(\Psi(z_k)) \) converges to \( \bar{\Phi}_R(z) \). Since \( \Psi(z_k) \) is in \( G_{tk} \), then \( z \) is in \( \mathcal{V} \), say \( z \) is in \( H_j \). Let \( p = \Theta(z) \). At this point notice that \( F \) does not intersect any leaf \( H_i \) in \( \mathcal{V} \). If it did, say in \( w \) then \( F \) intersects the nearby leaves \( G_t \) (for any \( t \) big enough) near \( w \). This would imply \( F \cap G_t = g_t \) does not escape compact sets in \( F \), contradiction. Therefore \( p = \Theta(z) \) is in \( \partial \Theta(F) \). Let \( x_k \) in \( g_{tk} \cap l \). Then \( \Theta(x_k) \) converges to a point in \( \Theta(H_1 \cap L) \). There are segments \( b_k \) in \( F \cap G_{tk} = g_{tk} \) from \( x_k \) to \( z_k \). Then \( \Theta(b_k) \) converges to a ray in \( \Theta(H_j) \) and a ray in \( \partial \Theta(H_j) \subset \mathcal{O}^u(p) \) and possibly other unstable leaves. Then there is a ray in \( \mathcal{O}^u(p) \) contained in \( \partial \Theta(F) \). This implies that \( F \) escapes down as \( \Theta(F) \) approaches this ray of \( \Theta(H_j) \). Hence \( \Psi(z_k) \) is getting closer to \( z_- \) which is \((H_j)_-\), which is also equal to \((H_1)_-\). This is what we wanted to prove anyway.

This finishes the proof of case 2.1.1, that is, when \( L \) intersects \( \mathcal{V} \).

**Lemma 7.6.** Let \( A \) in \( \tilde{\Lambda}^u \), \( B \) in \( \tilde{\Lambda}^s \) satisfying: there are \( R_i \) leaves of \( \tilde{\Lambda}^u \) intersecting \( B \) with \( R_i \) converging to \( A \) and \( R_i \cap B \) escaping compact sets in \( B \). Then \( A_- \) is equal to \( B_+ \).

**Proof.** Since \( R_i \) converges to \( A \) then \((R_i)_-\) converges to \( A_- \). Also \( R_i \) intersects \( B \) so \((R_i)_- = (R_i \cap B)_-\). As \( R_i \cap B \) escapes compact sets in \( B \) then in the intrinsic geometry of \( B \), the \( R_i \cap B \) converges to the positive ideal point of \( B \). This implies that \((R_i \cap B)_-\) converges to \( B_+ \). This implies the result.

**Case 2.1.2** — \( L \) does not intersect \( \mathcal{V} \).

Then \( \Theta(l) \) escapes in \( \Theta(L) \) and so \( \Psi(l) \) converges to \( L_+ \). By the previous lemma, this is also equal to \((H_1)_-\). From this point on, the proof is the same as in case 2.1.1. This finishes the proof of case 2.1.

**Case 2.2** — \( \nu \) is an ideal point of both \( \tilde{\Lambda}^s_F \) and \( \tilde{\Lambda}^u_F \).

**Case 2.2.1** — For any ray \( l \) of \( \tilde{\Lambda}^s_F \) and \( e \) of \( \tilde{\Lambda}^u_F \) with \( l_\infty = e_\infty = \nu \), then \( l \) does not intersect \( e \).
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Let $l', e'$ be rays as above. We may assume that $l', e'$ do not have any singularities. Parametrize the leaves of $\Lambda^u_F$ intersecting $e'$ as $\{l_t, t \geq 0\}$ where $l_t \cap e'$ converges to $v$ in $F \cup \partial_F$ as $t$ converges to infinity.

Since $l'$ limits on $v$ and is disjoint from $e'$, then $l'$ is on a side defined by $e'$. We will prove continuity of $\Psi$ at $v$ from the other side of $e'$. The point $p_t = l_t \cap e'$ disconnects $l_t$. For simplicity we only consider those $l_t$ with $l_t \subset L \in \Lambda^u$ and $L$ non singular. Let $l_t^1$ be the component of $(l_t - p_t)$ in the $e'$ side containing $l'$ union with $p_t$. Let $l_t^2$ be the other component of $(l_t - p_t)$ union with $p_t$, see fig. 14 a.

The $l_t^1$ are rays (here we use $L$ non singular - but this is just a technicality) and $(l_t^1)_\infty$ are not equal $v$ by hypothesis. They cannot escape compact sets of $F$ since $l'$ with ideal point $v$ is on that side of $e'$. Hence as $t$ converges to infinity $l_t^1$ converges to a leaf $l$ of $\Lambda^u_F$ with a ray (also denoted by $l$) with ideal point $v$ and maybe some other leaves as well. The leaf $l$ either shares a subray with $l'$ or separates $l'$ from $e'$. Let $e' \subset E$ leaf of $\Lambda^u$ and $l \subset L$, leaf of $\Lambda^u$.

**Case 2.2.1.1** - $l_t^2$ escapes in $F$ as $t \to \infty$.

Let $b_t$ be the ideal point of $l_t^2$. Then $b_t \not= v$. Let $L_t^2$ be the union of $\Phi_R(p_t)$ and the component of $L_t - \Phi_R(p_t)$ containing $l_t^2$. If $L_t^2$ escapes in $\tilde{M}$, then the arguments in case 1 show continuity of $\Psi$ at $v$ in the side of $e'$ not containing $l'$.

Now assume that $L_t^2$ converges to $R_1 \cup \ldots \cup R_m$ leaves of $\Lambda^u$ with union $\mathcal{R}$. Notice $F$ may intersect some of these leaves or not. If $\Theta(\Psi(p_t))$ does not escape in $\Theta(E')$, then one of the $R_i$, call it $R_1$, is a leaf intersecting $E'$. As seen in the arguments for case 2.1.1, $F$ escapes up in this direction so $\Psi(p_t)$ converges to $(R_1)_+$. If $\Theta(\Psi(p_t))$ escapes in $\Theta(E')$, then lemma 7.6 shows that $\Psi(p_t)$ also converges to $(R_1)_+$. This is equal to $(R_j)_+$ for any $j$.

Suppose there are $t_k \to \infty$ and $z_k$ in $l_t^2$ with $\Psi(z_k)$ not converging to $(R_1)_+$. Here there is no need to assume that $L_{t_k}$ is non singular. Then there $q$ in $S_\infty^2$, $q \not= (R_1)_+$ and a subsequence still denoted by $\Psi(z_k)$ converges to $q$. As before there is $z$ in $\tilde{M}$ so that up to another subsequence, still denoted by $\Phi_R(z_k)$, then $\Phi_R(z_k)$ converges to $\Phi_R(z)$ and hence $z$ is in $\mathcal{R}$, say in $R_i$. Then $\Phi_R(z_k)$ are near $\Phi_R(z)$ and since a ray of $\Theta(R_i)$ is in $\partial \Theta(F)$, then this is stable boundary. So $F$ escapes up as $\Theta(F)$ approaches $\Theta(z)$ and hence $\Psi(z_k)$ converges to $(R_i)_+$. This is equal to $(R_1)_+$. The arguments of Case 2.1.1, situation 1 then show continuity of $\Psi$ at $v$ on this side of $e'$. This finishes the analysis of case 2.2.1.1.

**Case 2.2.1.2** - The $l_t^2$ limit to $r$ in $F$ as $t \to \infty$.

Choose the leaf $r$ with a ray which has ideal point $v$. Then the leaves $r, l$ are not separated from each other in the leaf space of $\Lambda^u_F$. Since $r, l$ have ideal point $v$ and there is no leaf of $\Lambda^u_F$ non separated from $r, l$ and between them, proposition 6.10 shows that the region bounded by these rays of $r, l$ with ideal point $v$ projects in $M$ to a set asymptotic to a Reeb annulus. It follows that in $F$...
this region is a bounded distance from a geodesic ray with ideal point $v$. Now we restart the process with the ray $r$ of $\tilde{\Lambda}^s$ instead of $e'$ of $\tilde{\Lambda}^u$. Let $\{b_t, t \geq 0\}$ be a parametrization of the leaves of $\tilde{\Lambda}^u$ through the corresponding points $x_t$ of $r$. If the components of $(b_t - x_t)$ on the side opposite of $e'$ escapes compact sets in $F$, then the analysis of case 2.2.1.1 shows continuity of $\Psi$ at $v$ in that side of $r$. Since $r$ and $e'$ are a bounded distance from each other in $F$, this shows continuity of $\Psi$ at $v$ on that side of $e'$.

Otherwise this process keeps being repeated. Let $A_0 = L$, $A_1$ be the leaf of $\tilde{\Lambda}^s$ containing $r$. If the process above does not stop, we keep producing $A_i$ in $\tilde{\Lambda}^s$, so that they all disjoint and $A_i$ is not separated from $A_{i+1}$. By theorem 3.6 up to covering translations there are only finitely many leaves of $\tilde{\Lambda}^s$ which are not separated from some other leaf of $\tilde{\Lambda}^s$. There is then $m > n$ and $h$ covering translation with $h(A_n) = A_m$. Let $f$ be the generator of the joint stabilizer of $A_0, A_1$. This is non trivial by theorem 3.6 Then $f$ preserves all the prongs of $A_1$ and therefore leaves invariant all the $A_i$. Hence $h^{-1}fh(A_n) = A_n$ and so $h^{-1}fh = f^a$ for some integer $a$. This implies there is a $Z \oplus Z$ in $\pi_1(M)$, see detailed arguments in [Fe8]. This is a contradiction.

There is then a last leaf $l_y$ (of $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$) obtained from this process. The arguments of case 2.2.1.1 show continuity of $\Psi$ at $v$ on the other side of $l_y$. The region between $e'$ and $l_y$ is composed of a finite union of regions between non separated rays of $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$. They are all a bounded distance from a geodesic ray with ideal point $v$, so the whole region also satisfies this property. It follows that this region can only limit in $\Psi(v)$ as well and this proves continuity of $\Psi$ at $v$ in that side of $e'$.

An entirely similar analysis shows continuity of $\Psi$ at $v$ from the side of $l'$ not containing $e'$.

What remains to be analysed is the region of $F$ between the rays $l'$ and $e'$. Consider first the case that there is pair of non separated leaves in the chain from $l'$ to $e'$. Then as seen before the region between $l'$ and $e'$ is a bounded distance (the bound is not uniform) from a geodesic ray with ideal point $v$. This is not the case a priori if there is no non Hausdorffness involved. In this case the region between $l'$ and $e'$ may not have bounded thickness in $F$ and hence it is unclear whether its image under $\Psi$ can only limit in $\Psi(v)$. We analyse this case now.

In this last case parametrize the leaves of $\tilde{\Lambda}^u$ intersecting the ray $l$ of $\tilde{\Lambda}^u$ as $\{e_t | t \geq 0\}$. Since $l_t$ converges to $l$, then for big enough $t$, the leaves $l_t, e_t$ intersect — let $u_t$ be their intersection point, see fig. 14 b. Now define $l_t^*$ to be the component of $l_t - u_t$ intersecting $e$ and $e_t^*$ the component of $e_t - u_t$ intersecting $l$. Since $e'$ is on that side of $l$, the $e_t$ cannot escape and converge to a leaf $e$ of $\tilde{\Lambda}^u$ with an ideal point $v$. Let $e \subset E$ leaf of $\tilde{\Lambda}^s$.

Recall that $L_1$ is the leaf of $\tilde{\Lambda}^s$ containing $u_0^*$ and similarly let $L_1^*$ be the leaf of $\tilde{\Lambda}^u$ containing $u_t^*$. Let $L_1^*$ be the component of $L_t - \tilde{\Phi}(u_t)$ containing $l_t^*$ and similarly define $E_t^*$. In this remaining case the $l_t^*$ escape in $F$ and so do the $e_t^*$. Hence $\mu_t = l_t^* \cup \{u_t\} \cup e_t^*$ defines a shrinking neighborhood system of $v$ in $F \cup \partial_{\infty} F$. Consider the set

$$B_t = L_t^* \cup \tilde{\Phi}(u_t) \cup E_t^*$$

We want to show that $B_t$ converges to $L_1$ in the topology of closed sets of $\tilde{M} \cup S_{\infty}^2$.

First consider $L_t^* \cap E$ which intersects $F$ in $(l_t^* \cap e)$. If $L_t^* \cap E$ does not escape compact sets in $E$ then it limits to an orbit $\gamma$ contained in a leaf $H$ of $\Lambda^s$. Then $L, H$ are not separated from each other. But for $t$ big enough then $E_t$ is near enough $E$ and will intersect $H$ as well. This contradicts $E_t \cap L$ is not empty and $L, H$ non separated. Hence $L_t^* \cap E$ escapes in $E$ and similarly $E_t^* \cap L$ escapes in $L$. Hence $L, E$ form a perfect fit. This implies that $L_1 = E_\infty$. Also $\Psi(e)$ limits to $E_\infty$ and $\Psi(l)$ limits to $L_+ = E_\infty$.

The set $L_t^*$ contains $(L_t^* \cap E)_{++}$ and this converges to $E_\infty$ when $t \to \infty$. This is because $(L_t^* \cap E)$ escapes in $E$. If $L_t$ does not converge to $E_\infty$ in $\tilde{M} \cup S_{\infty}^2$, then we find $t_k \to \infty$ and $x_k \in L_{t_k}$ with
$x_k$ converging to $x$ not equal to $E_-$. Since $(x_k)_+ = (L_{t_k})_+$ converges to $E_-$, then up to subsequence assume $\Phi_R(x_k)$ converges to $\Phi_R(z)$ for some $z$ in $M$. Then $z$ is in a leaf $H$ of $\tilde{\Lambda}^s$ which is non separated from $L$.

The leaf $H$ does not intersect $F$, because $l_t^s$ escapes in $F$ by hypothesis in this final situation. It follows that $\Theta(H)$ has a ray contained in $\partial \Theta(F)$ and so this is stable boundary of $\Theta(F)$. Hence $F$ escapes up as $\Theta(F)$ approaches $\Theta(H)$ and consequently $\Psi(x_k)$ limits to $H_+ = L_+ = E_-$ which is what we wanted anyway. This shows that $\tilde{T}_i$ converges to $E_-$ in $\tilde{M} \cup S_\infty^2$.

Analytising the sets $E_i^s$ in the same manner we obtain that $\tilde{E}_i$ converges to $L_i$ as $t \to \infty$ as well. This implies that $\tilde{B}_t$ converges to $L_+ = \Psi(v)$. Since $B_t \cap F = \mu_t$ and the $\mu_t$ define a neighborhood basis of $v$ in $F \cup \partial \infty F$, this shows continuity of $\Psi$ at $v$. This finishes the proof of case 2.2.1.2 and hence of case 2.2.1.

Case 2.2.2 — There are rays $l$ of $\tilde{\Lambda}_F^u$ and $e$ of $\tilde{\Lambda}_F^u$ starting at $u_0$ and having the ideal point $v$.

We will first prove continuity on the side of $e$ not containing a subray of $l$. There will be an iteration of steps. Before we start the analysis we want to get rid of some problems as described now. Suppose that there are $\alpha_0, \beta_0$ leaves of $\tilde{\Lambda}_F^u$ (or leaves of $\tilde{\Lambda}_F^u$) which have non separated rays converging to $v$ in $\partial \infty F$ and on that side of $e$. Suppose there are infinitely many of these on that side of $e$. Let them be $\alpha_i, \beta_i$ and $G_i$ in $\tilde{\Lambda}^s$ containing $\alpha_i$. Each region $B$ between $\alpha_0$ and any $\alpha_i$ is a bounded distance from a geodesic ray in $F$ with ideal point $v$. The image $\Psi(B)$ then can only limit in $\Psi(v)$. If the $G_i$ do not escape in $M$ then they converge to a leaf $G$ of $\tilde{\Lambda}^s$. Let $A$ be an unstable leaf intersecting $G$ tranversely. For $i$ big enough then $A$ intersects $G_i$ transversely, which is impossible, as it would intersect $\alpha_i$ and $\beta_i$ and these are not separated. Hence the the $G_i$ escapes in $\tilde{M}$. Then as seen in case 1, there is continuity of $\Psi$ at $v$ in that side of $\alpha_1$.

Another situation is when there are leaves $\alpha_i$ in that side of $e$ with two rays with ideal point $v$. Then they are in the interior of a spike region $B$ with one boundary $g$ with ideal point $v$. If there are infinitely many of these, where none of the $\alpha_i$ are nested with each other, then let $G_i$ in $\tilde{\Lambda}^s$ containing $\alpha_i$. As in the previous paragraph, the $G_i$ have to escape in $\tilde{M}$ and we have continuity in that side of $\alpha_1$.

Therefore we can assume there are only finitely many occurrences of spike regions or non separated leaves with ideal point on this side of $e$. If there is any of these let $e_0$ be the last ray in that side coming from such occurrences. Otherwise let $e_0$ be the ray given $e$ by the hypothesis in this case. For simplicity assume that $e_0$ is a ray in $\tilde{\Lambda}_F^u$, the other case being similar. Let $e_0 \subset E_0 \in \tilde{\Lambda}^u$.

Parametrize the ray of $e_0$ as $\{p_t \mid t \geq 0\}$ with $p_t$ converging to $e$ as $t \to \infty$. Let $l_t$ be the leaf of $\tilde{\Lambda}_F^s$ through $p_t$ and $L_t$ in $\tilde{\Lambda}^s$ with $l_t \subset L_t$. If $L_t$ escapes $\tilde{M}$ as $t \to \infty$ then as seen before we have continuity of $\Psi$ at $v$ in that side of $e_0$. So now suppose that $L_t$ converges to $A_1 \cup ..., A_m$, leaves of $\tilde{\Lambda}^s$. This case is considerably more involved, with several possibilities.

Claim — $\Psi(e_0)$ converges to $(A_i)_+ \ (\text{notice the } (A_i)_+, 1 \leq i \leq m \text{ are all equal})$.

If $E_0$ intersects some $A_i$, say $A_1$, then as seen in case 2.1.1, $F$ escapes positively along $\Psi(e_0)$ as $\Theta(F)$ approaches $A_1$. This implies that $\Psi(e_0)$ converges to $(E_0 \cap A_1)_+ = (A_1)_+$. If $E_0$ does not intersect any $A_i$ then $\Psi(e_0)$ converges to $(E_0)_- = (A_1)_+$. This proves the claim.

Let $l_{t_1}^1$ be the component of $(l_t - p_t)$ in the side of $e_0$ we are considering. We are really interested in the behavior for $t \to \infty$, so we may assume $p_t$ is not singular and there is only one such component.

Suppose first that no $l_{t_1}^1$ has a ray with ideal point $v$ and that $l_{t_1}^1$ escapes in $F$ as $t \to \infty$. In this case it is easy to show continuity of $\Psi$ at $v$ and in this side of $e_0$: Suppose there are $x_i$ in $l_{t_i}^1$ with $t_i \to \infty$ and $\Psi(x_i) \not\to (A_i)_-$. Since $(x_i)_+$ converges to $(A_i)_+$ then up to subsequence assume that $(x_i)_- \to b \not= (A_i)_+$. Up to subsequence $\Phi_R(x_i) \to \Phi_R(x)$. Then $x$ is in some $A_i$ say $x \in A_2$. But $F$ escapes positively as $\Theta(F)$ approaches $\Theta(A_2)$, so $\Psi(x_i) \to (A_i)_+$, as we wanted. Then as in case
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2.1.1 This implies continuity.

There are 2 other options: 1) There is no $t$ with $l^1_t$ with an ideal point $v$ and $l^1_t$ does not escape in $F$; and 2) There is $t$ with $l^1_t$ having ideal point $v$. These two options interact and intercalate in appearance as explained below:

**Situation 1** — There is no $t$ with $l^1_t$ with ideal point $v$ and $l^1_t$ does not escape in $F$.

There could be several leaves of $\tilde{\Lambda}^s_F$ in the limit of $l^1_t$ as $t \to \infty$ but there is a single leaf, call it $g$ with ideal point $v$. If there is more than one such leaf with ideal point $v$, then there would have to be one with two rays with ideal point $v$. This leaf would be in a spike region and it is separated from any other leaf in $\tilde{\Lambda}^s_F$, contradiction. Let $g$ be contained in a leaf $G$ of $\tilde{\Lambda}^u$.

Parametrize the ray $g$ as $\{q_t \mid t \geq 0\}$, with $q_t \to v$ as $t \to \infty$. Let $s_t$ be the unstable leaf of $\tilde{\Lambda}^u_F$ through $q_t$. Let $s^1_t$ be the component of $(s_t - q_t)$ on the side of $g$ opposite to $e_0$ and $s^2_t$ the other component. Then $s^2_t$ cannot have ideal point $v$: for $t$ big enough it intersects $l^1_t$, see fig. 15 a. Then $s^2_t$ converges to $e_0$. By hypothesis there are no more occurrences of non separated leaves of $\tilde{\Lambda}^s_F$ with ideal point $v$ on that side of $e_0$, which implies that $s^1_t$ cannot limit to a leaf of $\tilde{\Lambda}^u_F$ at $t \to \infty$ (it would distinct but non separated from $e_0$). Hence the $s^1_t$ have to escape compact sets in $F$. If $s^1_t$ does not have an ideal point at $v$ for any $t$, then the previous analysis shows continuity of $\Psi$ at $v$ in that side of $g$.

Hence assume there is some $t_0$ so that $s^1_{t_0}$ has ideal point $v$, see fig. 15 a. Then for $t$ bigger than $t_0$ all ideal points of $s^1_t$ are $v$. Let $s^1_{t_0}$ be contained in a leaf $S$ of $\tilde{\Lambda}^u$ and $s_t$ contained in $S_t$ leaf of $\tilde{\Lambda}^u$. Since

$$l^1_t \to g, \quad s^2_t \to e_0 \quad \text{when} \quad t \to \infty,$$

then $L_t \to G, \quad S_t \to E_0, \quad \text{when} \quad t \to \infty$.

It follows that $E_0, G$ form a perfect fit, see fig. 15 b. Hence $(E_0)_- = G_+$. If $\Theta(s^1_{t_0})$ is a ray in $\Theta(S)$ then $\Psi(s^1_{t_0})$ converges to $S_-$. But $\Psi(s^1_{t_0})$ also converges to

$$\Psi(v) = (E_0)_- = G_+ = (G \cap S)_+. $$

Let $\gamma_0 = G \cap S$, an orbit of $\tilde{\Phi}$ in $G$. The above equations imply that

$$(\gamma_0)_+ = (G \cap S)_+ = (E_0)_- = \Psi(v) = S_- = (\gamma_0)_-,$$
which is a contradiction. Hence \( \Theta(s^1_t) \) is not a ray and has an endpoint \( x_1 \) in \( \Theta(S) \). Let \( \gamma_1 = x_1 \times \mathbb{R} \). Let \( H = \tilde{\Lambda}^s(\gamma_1) \). But \( F \) does not intersect \( H \). If \( F \) escapes down as \( \Theta(F) \) approaches \( x_1 \), then \( \Psi(v) = (\gamma_1)_- \). But then
\[
(\gamma_0)_- = (\gamma_1)_- = \Psi(v) = (\gamma_0)_+ \n\]
contradiction. This implies that \( F \) escapes up as \( \Theta(F) \) approaches \( x_1 \). Hence \( \Theta(H) \) has a ray in \( \partial \Theta(F) \). Therefore \( \Psi(s^1_t) \) limits to \( (\gamma_1)_+ \). This implies that \( (\gamma_0)_+ = (\gamma_1)_+ \), where \( \gamma_0, \gamma_1 \) are distinct orbits of \( \tilde{\Phi} \) in the same unstable leaf \( S \). This is dealt with by the following theorem proved in [Pe5]:

**Theorem 7.7.** (Pe5) Let \( \Phi \) be a quasigeodesic almost pseudo-Anosov flow in \( M^3 \) with \( \pi_1(M) \) negatively curved. Suppose there is an unstable leaf \( V \) of \( \Lambda^u \) and different orbits \( \beta_0, \beta_1 \) in \( V \) with \( (\beta_0)_+ = (\beta_1)_+ \). Then \( C_0 = \tilde{\Lambda}^s(\beta_0), C_1 = \tilde{\Lambda}^s(\beta_1) \) are both periodic, invariant under a nontrivial covering translation \( f \), and the periodic orbits in \( C_0, C_1 \) are connected by an even chain of lozenges all intersecting \( V \).

**Remark** – This result is case 2 of theorem 5.7 of [Pe5]. In that article the proof is done for quasigeodesic Anosov flows in \( M^3 \) with \( \pi_1(M) \) negatively curved. The proof goes verbatim to the case of pseudo-Anosov flows. The singularities make no difference. By the blow up operation, the same holds for almost pseudo-Anosov flows.

The theorem implies that \( G, H \) are in the boundary of a chain of adjacent lozenges all intersecting \( S \). The first lozenge, call it \( C \) has one stable side contained in \( G \) and an unstable side \( D_1 \) which makes a perfect fit with \( G \). Suppose first \( D_1 \) is in the side of \( S \) opposite to \( E_0 \), see fig. 16 a. The other unstable side of \( C \) is a leaf \( D_2 \) which intersects \( G \) on the other side of \( S \). Hence \( G \) is some \( S_c \) with \( c > t_0 \). Then \( S_c \cap F = s_c \) is a leaf of \( \tilde{\Lambda}^u_F \) and \( \Psi(s_c) \) has ideal point \( \Psi(v) \). Notice that \( \Theta(s_c) \) (which is contained in \( \Theta(F) \)) escapes in \( \Theta(F) \) – otherwise it would produce stable/unstable boundary in \( \Theta(F) \) before it hits \( \Theta(H) \) and \( \Theta(F) \) could not limit on \( \Theta(H) \), impossible. Hence \( \Psi(s_c) \) limits to \( (S_c)_- \) which is equal to \( \Psi(v) \). Then
\[
(S_c \cap G)_- = (S_c)_- = \Psi(v) = G_+ \n\]
which contradicts the orbit \( S_c \cap G \) being a quasigeodesic.

It follows that the perfect fits with \( G \) occurs in the \( E \) side of \( S \), see fig. 16 b. Here \( \Theta(H), \Theta(D_1) \) are contained in the boundary of \( \Theta(F) \). We now look at the region \( B \) in \( F \) bounded by \( s_{t_0} = S \cap F \) and \( e_0 = E_0 \cap F \).

**Claim 1** – The image \( \Psi(B) \) can only limit in \( \Psi(v) \).

The region \( \Psi(B) \) is contained in the region \( \mathcal{E} \) of \( \tilde{\mathcal{M}} \) which is bounded by \( E, D_1 \) (maybe other unstable leaves non separated from \( D_1 \) as well), \( H \) and \( S \), see fig. 16 b. Notice that \( F \) does not intersect \( D_1 \) or any leaf non separated from \( D_1 \) which is beyond \( D_1 \). Otherwise \( b_0 = (D_1 \cap F) \) is contained in \( B \) and non separated from \( e_0 \), so it would have both ideal points \( v \). Then it would be contained in the interior of a spike region and could not be non separated from another leaf – impossible. On the other hand since \( \Theta(H) \) has a line leaf in the stable boundary of \( \Theta(F) \), then \( \Theta(D_1) \) has a line leaf in the unstable boundary of \( \Theta(F) \). Hence \( F \) escapes down as \( \Theta(F) \) approaches \( \Theta(D_1) \).

Let \( z_k \) in \( B \) escaping in \( F \) and hence converging to \( v \) in \( \partial_\infty F \). Suppose that \( \Psi(z_k) \) does not converge to \( \Psi(v) \). Given that \( z_k \) escapes \( F \) and the structure of the region \( \mathcal{E} \), it follows that up to subsequence either \( \tilde{W}^u(z_k) \) converges to \( D_1 \) or \( \tilde{W}^s(z_k) \) converges to \( H \). Suppose that \( \tilde{W}^s(z_k) \) converges to \( H \). In that case \( (z_k)_+ \) converges to \( H_+ = \Psi(v) \). Then as seen before if \( (z_k)_- \) does not converge to \( \Psi(v) \) we can assume up to subsequence \( \tilde{\Phi}_R(z_k) \) converges to \( \tilde{\Phi}_R(z) \). Then \( z \) is in a leaf.
non separated from $H$ and since $\Psi(z_k)$ has to be in $E$ then $z$ can only be in $H$. As $F$ escapes up as $\Theta(F)$ approaches $\Theta(H)$ then $\Psi(z_k)$ converges to $H_+ = \Psi(v)$. The case $\overline{W}^u(z_k)$ converges to $D_1$ leads to $\Phi_R(z_k)$ converging to $\Phi_R(z)$ with $z$ in unstable leaf non separated from $D_1$. As $F$ escapes down as $\Theta(F)$ approaches these unstable leaves, then $\Psi(z_k)$ converges to $(D_1)_- = \Psi(v)$. Since this works for any subsequence of $z_k$, then $\Psi(z_k)$ has to converge to $\Psi(v)$ always. This proves claim 1.

Let $G_0 = G$. Notice that $G$ is periodic and connected to $H$ by an even chain of lozenges. We consider the ray $s_{t_0} = S \cap F$ which has ideal point $v$. Parametrize it as $\{z_t \mid t \geq 0\}$. Let $y_t$ be the leaf of $\Lambda^u_F$ through $z_t$ and $y_{t^1}$ the component of $(y_t - z_t)$ in the side opposite to $e_0$. The ray $s_{t_0}$ has the same behavior as the original ray $e_0$. Hence we obtain continuity in that side of $s_{t_0}$ unless $y_{t^1}$ converges to a leaf $\mu$ of $\Lambda^u_F$ with ideal point $v$. Let $G_1$ in $\Lambda^s$ with $\mu \subset G_1$. Then $G_1$ is non separated from $H$, see fig. 16 b and therefore connected to it by a chain of lozenges. It follows that $G_1$ is connected to $G_0$ by a chain of lozenges. As in the proof of claim 1, the region $B_1$ of $F$ between $e_0$ and $(F \cap G_1)$ has image $\Psi(B_1)$ which can limit only in $\Psi(v)$.

We restart the process with $g_1 = G_1 \cap F$ instead of $g$. The leaves of $\tilde{\Lambda}^u_F$ through points of $g_1$ already converge to the unstable leaf $(D_3 \cap F)$ of $\Lambda^u_F$ (Fig. 16 b). The leaf $(D_3 \cap F)$ cannot be non separated from any other leaf of $\Lambda^u_F$ in that side of $(D_3 \cap F)$. It follows that the unstable leaves intersected by $g_1$ escape in $F$. The only case to be analysed is that some of these unstable leaves have ideal point $v$. This brings the process exactly to the situation of some $s^1_t$ of $\Lambda^u_F$ having ideal point $v$ as described before (it was $s^1_{t_0}$). So this would produce $H_1$ of $\tilde{\Lambda}^s$ with similar properties as $H$. This process can now be iterated. As in claim 1 the region of $F$ between $g_i$ and $g_{i+1}$ maps to $M$ to a region which can only limit in $\Psi(v)$.

We show that this process has to stop. Otherwise consider $G_i$ leaves of $\tilde{\Lambda}^s$ which are all connected to $G_0$ by a chain of lozenges. The $G_i$ are all non separated from some other leaf of $\tilde{\Lambda}^s$. Hence there are $G_i, G_j$ project to the same stable leaf in $M$. There is a covering translation $h$ taking $G_i$ to $G_j$. If $f$ is a generator of the isotropy group of $G_0$ leaving all sectors invariant, then it leaves invariant all lozenges in any chain starting in $G_0$ so leaves invariant all the $G_i$. As before this leads to $h^{-1} f h = f^n$ for some $n$ in $Z$ and to a $Z \oplus Z$ in $\pi_1(M)$. This is disallowed. Therefore the process finishes after say $j$ steps and we obtain continuity of $\Psi$ at $v$ in that side of $g_j = G_j \cap F$. As seen above the region between $s_{t_0}$ and $g_j$ maps by $\Psi$ into a region that can only limit in $\Psi(v)$. This proves continuity of $\Psi$ at $v$ in that side of $e_0$. This finishes the analysis of situation 1.

**Situation 2** — There is $l^1_{t_0}$ with ideal point $v$.

Recall the setup before the analysis of situation 1. Let $\{u_t \mid t \geq 0\}$ be the collection of unstable leaves intersected by the ray $l^1_{t_0}$. The analysis is extremely similar to the analysis of situation 1, which shows all cases produce continuity in the first step except when $u_t$ converges to a leaf $u$ of $\tilde{\Lambda}^u_F$ with ideal point $v$. Then consider the stable leaves intersecting $u$. The analysis of situation 1
shows continuity unless there is a stable leaf with ideal point \( v \). From now on the analysis is exactly the same as in situation 1, with unstable replaced by stable and vice versa.

So far we proved continuity of \( \Psi \) at \( v \) from the side of \( e_0 \) opposite to \( l \). The same works for the other side of \( l \), producing \( l_0 \) with similar properties as \( e_0 \). We now must consider the regions between \( e_0 \) and \( e \), between \( e \) and \( l \) and between \( l \) and \( l_0 \).

First consider the region between \( e \) and \( e_0 \), which occurs only when they are distinct. This implies that the ray \( e_0 \) is a bounded distance from a geodesic ray in \( \tilde{F} \) with ideal point \( v \). Let \( \{ \mu_t \mid t \geq 0 \} \) be a parametrization of the stable leaves of \( \Lambda^s_F \) through \( e \). Let \( \mu_1^1 \) be the component of \( (\mu_1^1 - e) \) in the side of \( e \) we are considering. If some \( \mu_1^1 \) has ideal point \( v \) then both ideal points of \( \mu_t \) are \( v \) and \( \mu_t \) is inside a spike region. The same is true for \( e \) and so \( e \) is a bounded distance from a geodesic ray in \( F \) with ideal point \( a \). Hence the region between \( e \) and \( e_0 \) is a bounded distance from a geodesic ray and we are finished in this case.

The remaining case to be analysed here is that \( \mu_1^1 \) has no ideal point \( v \). Then \( \mu_1^1 \) does not escape \( F \) as \( t \to \infty \), because \( e_0 \) is in that side of \( e \). So \( \mu_1^1 \) converges to a leaf \( \mu \) which has ideal point \( v \). Now consider a parametrization \( \{ \nu_t \mid t \geq 0 \} \) of the unstable leaves intersected by \( \mu \). Then \( \nu_t \) converges to the leaf \( e \). If it converges to some other leaf, then \( e \) is a bounded distance from a geodesic ray in \( F \) and we are done. Otherwise it must be that some \( \nu_t \) has ideal point \( v \). Therefore we exactly in the setup analysed in situation 1 above.

This shows continuity of \( \Psi \) for the region between \( e \) and \( e_0 \) and similarly for the region between \( l \) and \( l_0 \).

Finally we analyse the region \( B \) between \( e \) and \( l \). First notice there is no singularity in the interior of \( B \). Otherwise there would be a line leaf in \( B \) and hence a leaf with both endpoints \( v \). It would have to be part of a spike region and the spike region does not have any singularities in its interior.

Parametrize the leaves of \( \Lambda^s_F \) through \( l \) as \( \{ e_s \mid s \geq 0 \} \) and similarly those of \( \Lambda^u_F \) through \( e \) as \( \{ l_t \mid t \geq 0 \} \). Let \( L \subseteq L \) leaves of \( \tilde{\Lambda}^s \) with \( l \subset L \), \( l_t \subset L_t \) and similarly define \( E, E_t \). There are 2 possibilities:

1) Product case

Any \( l_t \) intersects every \( e_s \) and vice versa. Equivalently \( \Lambda^u_F, \Lambda^s_F \) define a product structure in the region \( B \) bounded by \( l_0 \) and \( e_0 \). If the \( L_t \) escapes in \( \tilde{M} \) as \( t \to \infty \), then there is a stable product region defined by a segment in \( L_0 \). But then theorem \( \ref{product} \) implies that \( \Phi \) is topologically conjugate to a suspension, contradiction. It follows that the \( L_t \) converge to \( H_1 \cup \ldots H_m \) as \( t \to \infty \). Since the \( l_t \) are stable leaves, it follows that \( F \) escapes up as \( \Theta(F) \) approaches \( \Theta(H_i) \). This implies that \( \Psi(e) \) limits to \( (H_i)_+ \) which is then equal to \( \Psi(v) \). Similarly \( E_s \) converges to \( V_1 \cup \ldots V_n \) and \( F \) escapes down as \( \Theta(F) \) approaches \( \Theta(V_j) \). Hence \( \Psi(l) \) limits to \( (V_j)_- = \Psi(v) \). If some \( H_i \) intersects some \( V_j \), then

\[
(V_j \cap H_i)_+ = (H_i)_+ = \Psi(v) = (V_j)_- = (V_j \cap H_i)_-,
\]

contradiction. Let now \( \{ z_k \} \) be a sequence in \( B \) converging to \( v \). The product structure implies that up to subsequence we may assume that either \( \tilde{W}^s(z_k) \) converges to \( H_i \) or \( \tilde{W}^u(z_k) \) converges to \( V_j \). This is analysed carefully in Claim 1 above, which shows that \( \Psi(z_k) \) must converge to \( \Psi(v) \). This shows continuity of \( \Psi \) when restricted to the region \( B \).

2) Non product case.

There are \( t, u > 0 \) with \( l_t \cap e_u = \emptyset \). Consider one such \( u \). Let \( a \) be the infimum of \( t \) with \( l_t \cap e_u = \emptyset \). Now let \( b \) be the infimum of \( u \) with \( l_u \cap e_a = \emptyset \). Then \( l_a \cap e_b = \emptyset \), but for any \( 0 \leq t < a \) and \( 0 \leq u < b \) one has \( l_t \cap e_u \not\in \emptyset \). Since \( l_a \cap e_b = \emptyset \), then \( l_a \cap E_b = \emptyset \). It follows that \( l_a, E_b \) form a perfect fit, see fig. \( \ref{perfectfit} \). a. If \( \Theta(l_a) \) does not escape in \( \Theta(l_a) \), then there would be unstable boundary of \( \Theta(F) \) in the
limit and that would keep $F$ from intersecting $E_b$, contradiction. Hence $\Theta(l_a)$ escapes in $\Theta(L_a)$ and $\Theta(e_b)$ escapes in $\Theta(E_b)$. Hence $\Psi(l_a)$ limits to $(L_a)_+$ and $\Psi(e_b)$ limits to $(E_b)_-$. Also $l_a, e_b$ limit to $v$ in $\partial_{\infty} F$.

Let $p_t = l_t \cap E$. If $\Theta(p_t)$ escapes in $\Theta(E)$, then $\Psi(e)$ converges to $E_-$. Notice that $\Psi(e)$ converges to $\Psi(v)$ so:

$$E_- = \Psi(v) = (L_a)_+ = (L_a \cap E)_+$$

contradiction. It follows that $\tilde{\Phi}_R(p_t)$ converges to $\tilde{\Phi}_R(x)$ with $x$ in $E$. Also $F$ has to escape up as $\Theta(F)$ approaches $\Theta(x)$ -- same as in Situation 1 above. Hence $\Psi(e)$ limits to $x_+$. So

$$x_+ = \Psi(v) = (L_a)_+ = (p_a)_+$$

Let $X = \tilde{W}^s(x)$. Then $x, p_a$ are in 2 distinct orbits of $E$ with the same positive ideal point. Therefore theorem [C] implies that $L_a, X$ are connected by an even chain of lozenges all intersecting $E$. Let $C$ be the first lozenge. It has a stable side in $L_a$ and one unstable side, call it $D_1$ which makes a perfect fit with $L_a$. Suppose first that $D_1$ is in the component of $\tilde{M} - E$ opposite to $E_b$. Then the other unstable side of $C$, call it $D_2$ has to intersect $L_a$ in the other side of $E$. Then $D_2$ must be some $E_t$, let it be $E_{y'}$, see fig. 17 a. Then $\Theta(e_{y'})$ has to escape in $\Theta(E_{y'})$ or else one produces stable boundary to $\Theta(F)$ and $\Theta(F)$ cannot limit to $\Theta(x)$ contradiction. Hence $\Psi(e_{y'})$ converges to $\Psi(v)$ and also to $(E_{y'})_-$. But then

$$(E_{y'} \cap L_a)_- = (E_{y'})_- = \Psi(v) = (E_b)_- = (L_a)_+$$

again a contradiction.

This implies that $D_1$ is on the side of $E$ containing $E_b$, see fig. 17 b.

If there are only 2 lozenges in the chain from $L_1$ to $X$, then $D_1$ also makes a perfect fit with $X$. Otherwise there are $D_2, ..., D_l$ all non separated from $D_1$ and so that $D_i$ makes a perfect fit with $X$ and the $D_j$ are all in the boundary of the chain of lozenges. As seen in claim 1 above, $F$ cannot intersect any $D_j$ ($1 \leq j \leq l$), but all $\Theta(D_j)$ are contained in the unstable boundary of $\Theta(F)$. Also $F$ escapes down as $\Theta(F)$ limits to $\Theta(D_j)$. The set $\Theta(X)$ also has a line leaf which is a stable boundary of $\Theta(F)$ and $F$ escapes up when $\Theta(F)$ approaches $\Theta(X)$.

The same discussion applies to $L$, so there is $y$ in $L$, $Y = \tilde{W}^u(y)$ with $\Theta(Y)$ having a line leaf in the unstable boundary of $\Theta(F)$ and $F$ escapes down accordingly. There are $C_1, ..., C_n$ leaves in $\Lambda^u$, all non separated from each other and in the boundary of the lozenges in the chain from $E_b$ to $Y$ so that $C_1$ makes a perfect fit with $E_b$ and $C_n$ makes a perfect fit with $Y$, see fig. 17 b. Finally $\Theta(C_j)$ has a line leaf in the stable boundary of $\Theta(F)$ and $F$ escapes up accordingly.

Let $E$ be the region in $\tilde{M}$ bounded by

\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure17.png}
\caption{a. Reaching before, b. Reaching at the exact time.}
\end{figure}\]
§8. Foliations and Kleinian groups

Then $E \cap F$ is exactly the region $B$ bounded by the rays $e$ and $l$. Let $z_k$ in $B$ escaping to $v$. Then the region $E$ shows that up to subsequence one of the following must occur:

1) $\tilde{W}^s(z_k)$ converges to either $X$ or $C_1$. The analysis of claim 1 above shows that $\Psi(z_k)$ converges to either $X_+$ or $(C_1)_+$ both of which are equal to $\Psi(v)$.

2) $\tilde{W}^u(z_k)$ converges to either $Y$ or $D_1$. Here $\Psi(z_k)$ converges to either $Y_-$ or $(D_1)_-$ both of which are equal to $\Psi(v)$.

In any case this shows continuity of $\Psi$ in the region $B$. This finishes the non product case.

This finishes the proof of theorem 7.3, the continuous extension theorem.

8 Foliations and Kleinian groups

There are many similarities between foliations in hyperbolic 3-manifolds and Kleinian groups. We refer to [Mi, Can, Mar] for basic definitions concerning degenerate and non degenerate Kleinian groups, in particular singly and doubly degenerate groups.

If the foliation is $R$-covered then the limit set of any leaf in $\tilde{M}$ is the whole sphere $S^2_{\infty}$. This corresponds to doubly degenerate surface Kleinian groups Th1, Mi, Can, Mar, Bon. There is always a pseudo-Anosov flow which is transverse to the foliation Fe7, Cal1. If the flow is quasigeodesic then the results of this article imply that the foliation has the continuous extension property.

If the foliation has one sided branching, say branching down, then limit sets of leaves can only have domain of discontinuity “above” Fe3. Let $F$ in $\tilde{F}$ and $\Lambda_F$ its limit set. If $p$ is not in $\Lambda_F$, the $p$ is said to be above $F$ if there is a neighborhood $V$ of $p$ in $\tilde{M} \cup S^2_{\infty}$, so that $V \cap \tilde{M}$ is on the positive side of $F$. This corresponds to simply degenerate surface Kleinian groups Th1, Mi, Can. There are examples of foliations with one sided branching transverse to suspension pseudo-Anosov flows provided by Meigniez Me. Suspension flows are always quasigeodesic flows Ze. The results of this article show the continuous extension property for such foliations. Under these conditions, the limit sets are locally connected, the continuous extension provides parametrizations of these limit sets.

Finally if there is branching in both directions, then there can be domain of discontinuity above and below leaves. This corresponds to non degenerate Kleinian groups Th1, Mi, Can. These occur for example in the case of finite depth foliations, where the depth 0 leaves are not virtual fibers Fe6.

There are many interesting questions:

**Question 1** – Given a foliation $F$, is it $R$-covered if and only if for every $F \in \tilde{F}$ then the limit set $\Gamma_F$ is $S^2_{\infty}$?

The forward direction is true. The backwards direction is true if there is a compact leaf Go-Sh. In addition if there is one leaf with limit set the whole sphere then all leaves have limit set the whole sphere Fe3 – whether $F$ is $R$-covered or not.

**Question 2** – Given $F$ an $R$-covered foliation, is there a quasigeodesic transverse pseudo-Anosov flow?

This is true in the case of slitherings or uniform foliations as defined by Thurston Th5. Examples are fibrations, $R$-covered Anosov flows and many others. There is always a transverse pseudo-Anosov flow, the question is whether it is quasigeodesic.

**Question 3** – Is there domain of discontinuity of $\Lambda_F$ only above $F$ if and only if $F$ has one sided branching in the negative direction?

This occurs for the examples constructed by Meigniez Me.
**Question 4** – Are the pseudo-Anosov flows constructed by Calegari [Cal2] and transverse to one sided branching foliations quasigeodesic?

**Question 5** – If $\mathcal{F}$ has 2 sided branching is there always domain of discontinuity above and below? Is there a quasigeodesic pseudo-Anosov flow almost transverse to $\mathcal{F}$?

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