The curvature of semidirect product groups associated with two-component Hunter–Saxton systems

Martin Kohlmann

Institute for Applied Mathematics, University of Hannover, D-30167 Hannover, Germany

E-mail: kohlmann@ifam.uni-hannover.de

Received 11 February 2011, in final form 12 April 2011
Published 4 May 2011
Online at stacks.iop.org/JPhysA/44/225203

Abstract

In this paper, we study two-component versions of the periodic Hunter–Saxton equation and its μ-variant. Considering both equations as a geodesic flow on the semidirect product of the circle diffeomorphism group Diff(S) with a space of scalar functions on S we show that both equations are locally well posed. The main result of this paper is that the sectional curvature associated with the 2HS is constant and positive and that 2μHS allows for a large subspace of positive sectional curvature. The issues of this paper are related to some of the results for 2CH and 2DP presented in Escher et al (2011 J. Geom. Phys. 61 436–52).

PACS numbers: 61.30.Gd, 47.10.A−, 05.45.−a
Mathematics Subject Classification: 37K65, 58B25, 58D05

1. Introduction

The Hunter–Saxton (HS) equation
\[ u_{xx} + 2u_x u_{xx} + uu_{xxx} = 0, \quad x \in \mathbb{S}, \quad t > 0, \] (1)
was derived in [18] as a model for propagation of orientation waves in a massive nematic liquid crystal director field. The function \( u(t,x) \) depends on a space variable \( x \) and a slow time variable \( t \). In recent years, the question of the existence and regularity of solutions to (1) as well as integrability properties have been examined in great detail, cf, e.g., [3, 6, 19, 37]. The HS equation can be regarded as the \( \alpha \to \infty \) limit of the Camassa–Holm (CH) equation
\[ m_t = -(m_t u + 2m u_x), \quad m = (1 - \alpha \partial_x^2)u = u - \alpha u_{xx}, \] (2)
which was introduced to model the shallow-water medium-amplitude regime for wave motion over a flat bed [4]; more precisely, the HS equation is equivalent to
\[ m_t = -(m_t u + 2m u_x), \quad m = (-\partial_x^2)u = -u_{xx}. \] (2)
Alternatively, the HS equation can be regarded as the high-frequency or short-wave limit \((x, t) \mapsto (\epsilon x, \epsilon t), \) for \( \epsilon \to 0, \) of the CH equation, cf [9, 19]. Similar to the CH, the HS
equation comes up from Langrange’s variational principle with the Lagrangian
\[ L_{HS}(u) = \frac{1}{2} \int_S u_x^2 \, dx, \]
cf [19], which differs from the Lagrangian for CH only in a term proportional to \( u^2 \) under the integral sign. The Euler–Lagrange equation for the modified Lagrangian
\[ L_{2HS}(u, \rho) = \frac{1}{2} \int_S u_x^2 \, dx + \frac{1}{2} \int_S \rho^2 \, dx \]
is the 2HS equation
\[
\begin{align*}
m_t &= -m_x u - 2mu_x - \rho \rho_x, \\
\rho_t &= - (\rho u)_x,
\end{align*}
\]
an integrable two-component extension of (1) which reduces to (1) for \( \rho = 0 \).

We note that the two-component Hunter–Saxton (2HS) equation is a particular case of the Gurevich–Zybin system pertaining to nonlinear one-dimensional dynamics of dark matter as well as nonlinear ion-acoustic waves, cf [33]. Also, it is related to the two-component Camassa–Holm (2CH) system (which reads (3) with \( m = u - u_{xx}, \quad [7, 13] \)) via the short-wave limit. Note that for the 2CH equation, the second variable \( \rho \) can be interpreted as the fluid’s density and its evolution equation in fact is the continuity equation for velocity and fluid density. The 2HS equation is a special case of the two-parameter family of evolution equations (3)
\[
\begin{align*}
m_t &= -m_x u - au_x, \\
m &= -u_{xx}, \\
\rho &= -(\rho u)_x, \\
\end{align*}
\]
from which it emerges for the choice \((a, \kappa) = (2, 1)\). The system (4) is of great importance for a variety of problems occurring in mathematical physics: a special case of the two-parameter family (4) is the Proudman–Johnson equation [32, 34] for \( \rho = 0 \) and \( a = -1 \); this equation is obtained from the incompressible 2D Euler equations by a special ansatz for the stream function. Further relations to, e.g., the inviscid Kármán–Batchelor flow [5, 17], which is derived from the 3D axisymmetric Euler equations, or the famous Constantin–Lax–Majda equation [8] are explained in [35]. We also refer the reader to [35] for an extensive discussion of well-posedness and blow-up for the general system (4).

What makes the 2HS system particularly interesting is its potential exhibition of nonlinear phenomena as breaking waves and peakons [7]. In [21], the authors derive the 2HS equation as the \( N = 2 \) supersymmetric extension of the CH equation. They also work out the bi–Hamiltonian formulation and a Lax pair representation for the 2HS equation. Concerning geometry, the 2HS equation can be regarded as an Euler equation on the superconformal algebra of contact vector fields on the 1|2-dimensional supercircle. Finally, some explicit solutions of (3), such as bounded traveling waves, are shown in [21].

Interesting variants of equations (1) and (3) are obtained by using \( m = \mu(u) - u_{xx} \) instead of \( m = -u_{xx} \) in (2) and (3), where \( \mu(u) \) is the mean of \( u : \mathbb{S} \rightarrow \mathbb{R} \). This way we obtain the \( \muHS \) equation
\[
0 = u_{xxx} + 2u_x u_{xx} + uu_{xxx} - 2\mu(u)u_x
\]
and its two-component extension
\[
\begin{align*}
0 &= u_{xxx} + 2u_x u_{xx} + uu_{xxx} - 2\mu(u)u_x - \rho \rho_x, \\
\rho_t &= - (\rho u)_x.
\end{align*}
\]
In [20], it is explained that the \( \mu_{HS} \) equation models nematic liquid crystals with a preferred direction of the director field, e.g., coming from an external magnetic field acting on dipoles and turning them to this direction. Our motivation for studying the two-component extension \( 2\mu_{HS} \) comes from the observation that the generalized system possesses an integrable structure [38] and the expectancy that similar relationships between \( \mu_{HS} \) and \( 2\mu_{HS} \) compared to HS and \( 2HS \) might hold true. Both (5) and (6) are integrable equations related to the isospectral problem

\[
\begin{aligned}
\psi_{xx} + (m\lambda + \rho^2 \lambda^2)\psi &= 0, \\
\psi_t &= -\left(\frac{1}{2\lambda} + \mu\right)\psi_x + \frac{1}{2}\psi u_x,
\end{aligned}
\]

i.e. \( \psi_{1,tt} = \psi_{xx} \) and \( \lambda_t = 0 \) imply the \( 2\mu_{HS} \) equation (6). Equations (5) and (6) follow from the variational principle with the Lagrangians

\[
L_{\mu_{HS}}(u) = \frac{1}{2} \int_{S} u\left(\mu - \partial^2 x\right)\mu \, dx
\]

and

\[
L_{2\mu_{HS}}(u, \rho) = \frac{1}{2} \int_{S} u\left(\mu - \partial^2 x\right)\mu \, dx + \frac{1}{2} \int_{S} \rho^2 \, dx.
\]

While there are some well-posedness results for the \( \mu_{HS} \) in [20], there is no fully developed existence and regularity theory for its two-component version up to now. A first attempt to construct global weak solutions is presented in [28].

We now turn to the geometric theory behind (1), (3), (5) and (6) which will be relevant for this paper. Equations (1) and (5) have been studied as geodesic flow on the circle diffeomorphism group \( H^s \text{Diff} (\mathbb{S}) \) of Sobolev class \( H^s \) [20, 23]. We provide some elementary results about recasting Euler equations as geodesic equations on infinite-dimensional Lie groups in the appendix of this paper. The strategy is to define a right-invariant metric on \( H^s \text{Diff} (\mathbb{S}) \), induced at the identity by the inertia operators \(-\partial^2 x\) and \(\mu - \partial^2 x\) respectively, and to prove its compatibility with an affine connection which is given canonically in terms of the Christoffel operator for the respective equation. This implies the existence of a geodesic flow and first local well-posedness results\(^1\). Moreover, computations of the curvature tensor and the sectional curvature for both equations have been performed. The meaningful results are that the sectional curvature is strictly positive for both equations and that HS has constant positive sectional curvature which motivates that, in this case, \( H^s \text{Diff} (\mathbb{S}) \) is locally isometric to an \( L_2 \)-sphere. This sphere interpretation has various geometric consequences as explained in [24, 25]. Furthermore, the positivity of sectional curvatures is related to stability issues for the geodesics, cf the appendix.

In this paper, we extend some of the above results to the integrable two-component extensions (3) and (6). In a first step, we comment on how to choose a suitable configuration manifold for \( 2HS \) and \( 2\mu_{HS} \); here, we will make use of semidirect products. Then, we give a proof of the fact that the curvature tensor for \( 2HS \) is of the same form as for HS with the only difference that the geometric objects contained therein have to be replaced by their two-component analogs. For \( 2\mu_{HS} \) we first obtain a curvature formula which is similar to the formula for the one-component \( \mu_{HS} \). Second, we compute the sectional curvature in several two-dimensional directions and establish a positivity result. The paper is organized as follows:

Section 2 explains the geometric structure of \( 2HS \) and \( 2\mu_{HS} \); here we recall some results about the corresponding one-component equations and semidirect products which we

\(^1\) For HS one has to factorize \( H^s \text{Diff} (\mathbb{S}) \) by a subgroup as we will explain in the following.
will need in the following. The main goal of this section is to establish the existence of a unique local geodesic flow and local well-posedness for both equations. Section 3 is about the curvature of the product group associated with the 2HS. In section 4, we express the sectional curvature for $2\mu$HS in terms of its Christoffel map and present a large subspace of positive sectional curvature.

2. Geometric aspects of 2HS and $2\mu$HS

Let us denote by $H^s_{\text{Diff}}(\mathbb{S})$ the group of orientation-preserving $H^s$ diffeomorphisms of $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ where $H^s = H^s(\mathbb{S})$ is the Sobolev space of order $s > 0$ on the circle. Note that, for any $s > 3/2$, $H^s_{\text{Diff}}(\mathbb{S})$ is a topological group and a smooth Hilbert manifold modeled on the space $H^s$, but not a Lie group [23, 31]. We have $TH^s_{\text{Diff}}(\mathbb{S}) \cong H^s_{\text{Diff}}(\mathbb{S}) \times H^s(\mathbb{S})$.

2.1. The HS equation

Let us rewrite the HS equation as an autonomous system in terms of the local flow $X(t) = (\varphi(t), \varphi_t(t)) \in TH^s_{\text{Diff}}(\mathbb{S})$, $s > 5/2$, for the time-dependent $H^s$ vector field $u(t, \cdot)$ on $\mathbb{S}$. The usual starting point is to compute $\varphi_{tt}$. By the chain rule and from the relation $\varphi_t = u \circ \varphi$, we obtain

$$\varphi_{tt} = (u_t + uux) \circ \varphi.$$  

We now have to replace $u_t$ by using equation (1). But since equation (1) only includes $u_{txx}$, we differentiate twice with respect to $x$ to obtain

$$\partial_x^2(u_t + uu_x) = u_{txx} + 3u_uu_{xx} + uu_{xxx} = u_xu_{xx} = \frac{1}{2} \partial_xu_x^2.$$  

(7)

Now we are in need of the inverse of the operator $A = -\partial_x^2$.

**Lemma 1.** Let $A$ be the operator $-\partial_x^2$ with domain

$$D(A) = \{ f \in H^s(\mathbb{S}); f(0) = 0 \}, \quad s > 5/2.$$  

For any $s > 5/2$, $A$ is a topological isomorphism

$$D(A) \to \left\{ f \in H^{s-2}(\mathbb{S}); \int_{\mathbb{S}} f(x) \, dx = 0 \right\}$$  

with the inverse

$$(A^{-1}f)(x) = -\int_0^x \int_0^y f(z) \, dz \, dy + x \int_0^y f(z) \, dz \, dy.$$  

By Sobolev’s imbedding theorem $u_x \in C(\mathbb{S})$ and hence the right-hand side of (7) is a function with zero mean. Thus, it is in the domain of $A^{-1}$ and we conclude

$$\varphi_{tt} = -\frac{1}{2} \left[ A^{-1} \partial_x(\varphi \circ \varphi^{-1})^2 \right] \circ \varphi.$$  

Note also that $u_x + uu_x$ must belong to the domain of $A$ which suggests that we will need the assumption $u(0) = 0$. Setting

$$\Gamma(u, v) = -\frac{1}{2} A^{-1}(u_xv_x)_x,$$  

(8)

we obtain a symmetric bilinear operator $H^s(\mathbb{S}) \times H^s(\mathbb{S}) \to H^s(\mathbb{S})$. In fact, this Christoffel map is smooth which enables the following geometric approach, established in [25].
Let Rot(\mathbb{S}) \subset H^1 Diff(\mathbb{S}) be the subgroup of rotations \( x \mapsto x + d \) for some \( d \in \mathbb{R} \). We denote by \( H^1 \Diff(\mathbb{S})/\Rot(\mathbb{S}) \) the space of right cosets \( \Rot(\mathbb{S}) \circ \varphi = \{ \varphi(\cdot) + d; d \in \mathbb{R} \} \) for \( \varphi \in H^1 \Diff(\mathbb{S}) \) and set \( M^\prime = \{ \varphi \in H^1 \Diff(\mathbb{S}); \varphi(0) = 0 \} \). We have
\[
M^\prime = \{ \text{id} + u; u \in H^1, u_x > -1, u(0) = 0 \}
\]
and thus \( M^\prime \) is an open subset of the closed hyperplane
\[
\text{id} + E^\prime := \text{id} + \{ u \in H^1; u(0) = 0 \} \subset H^1.
\]
Writing the elements of \( H^1 \Diff(\mathbb{S})/\Rot(\mathbb{S}) \) as \( \varphi \), the map \( [\varphi] \mapsto \varphi - \varphi(0) \) establishes a diffeomorphism \( H^1 \Diff(\mathbb{S})/\Rot(\mathbb{S}) \rightarrow M^\prime \), showing in this way that \( M^\prime \) is a global chart for \( H^1 \Diff(\mathbb{S})/\Rot(\mathbb{S}) \). Furthermore, all tangent spaces \( T_\varphi M^\prime \) can be identified with \( E^\prime \). Next, we define a right-invariant metric on \( H^1 \Diff(\mathbb{S})/\Rot(\mathbb{S}) \) by setting
\[
\langle U, V \rangle_\varphi = \frac{1}{4} \int_\mathbb{S} (U \circ \varphi^{-1}) A(V \circ \varphi^{-1}) \, dx = \frac{1}{4} \int_\mathbb{S} U_x V_x \varphi_x^{-1} \, dx \tag{9}
\]
for tangent vectors \( U, V \), \( \varphi \in \Rot(\mathbb{S}) \), \( (U, V) \in \mathbb{S} \), \( (\cdot, \cdot)_\varphi \) at the identity, induced by the operator \( A \) defined in lemma \ref{lemma:HS}, is the \( H^1 \)-metric and that our definition of \( A \) ensures that \( (\cdot, \cdot)_\varphi \) is indeed a positive definite inner product. Furthermore, the metric (9) is compatible with the affine connection \( \nabla \) defined locally by
\[
\nabla_X Y(\varphi) = DY(\varphi) \cdot X(\varphi) - \Gamma(\varphi; Y(\varphi), X(\varphi)), \tag{10}
\]
where \( \Gamma(\varphi; \cdot, \cdot) = R_\varphi \circ \Gamma(\text{id}; \cdot, \cdot) \circ R_\varphi^{-1} \) is the smooth Christoffel map for the HS equation with \( \Gamma(\text{id}; u, v) = -\frac{1}{2} A^{-1}(u_x v_x)_x \). As proved in [25], the geodesics of the \( H^1 \) right-invariant metric are described by the HS equation. Let \( J \subset \mathbb{R} \) be an open interval and let \( \varphi : J \rightarrow H^1 \Diff(\mathbb{S}) \) be a smooth curve. Then, the curve \( u : J \rightarrow T_\text{id} H^1 \Diff(\mathbb{S}) \) defined by \( u : t \mapsto \varphi_t \circ \varphi^{-1} \) satisfies the HS equation (1) if and only if the curve \( [\varphi] : J \rightarrow H^1 \Diff(\mathbb{S})/\Rot(\mathbb{S}) \) given by \( [\varphi] : t \mapsto [\varphi(t)] \) is a geodesic with respect to \( \nabla \). The geodesics in \( H^1 \Diff(\mathbb{S})/\Rot(\mathbb{S}) \) can be found explicitly by the method of characteristics. For \( u_0 \in T_\text{id} M^\prime \) with \( (u_0, u_0) = 1 \) the unique geodesic \( \varphi : [0, T^*(u_0)) \rightarrow M^\prime \) with \( \varphi(0) = \text{id} \) and \( \varphi(0) = u_0 \) is given by
\[
\varphi(t) = \text{id} - \frac{1}{2} (A^{-1} \partial_t (u_0^2))(1 - \cos 2t) + \frac{1}{2} u_0 \sin 2t,
\]
where the maximal time of existence is
\[
T^*(u_0) = \frac{\pi}{2} + \arctan \left( \frac{1}{2} \min_{x \in \mathbb{S}} u_0(x) \right) \leq \pi/2.
\]
Observe that the associated solution \( u = \varphi_t \circ \varphi^{-1} \in C([0, T^*); E^\prime) \cap C^1([0, T); E^\prime) \) of the HS is not unique; the set of solutions is
\[
\{ t \mapsto u(t, \cdot - c(t) + c'(0)) \mid t \in C([0, T); H^1(\mathbb{S})) \cap C^1([0, T); H^1(\mathbb{S})) \}
\]
where \( T \leq T^* \) is the maximal time of existence, \( c : [0, T) \rightarrow \mathbb{R} \) is an arbitrary \( C^1 \)-function with \( c(0) = c'(0) = 0 \) and if \( T < T^* \), then \( |c(t)| \rightarrow \infty \) as \( t \rightarrow T \) from below. Further geometric aspects of the HS equation are discussed in [23, 24].

2.2. The \( \mu \)HS equation
The inertia operator for \( \mu \)HS is \( A = \mu - \partial_x^2 \), where \( \mu(u) = \int_\mathbb{S} u \, dx \). The following lemma can be found in [27].

\footnote{The factor 1/4 is introduced to obtain that the sectional curvature for HS is identically equal to 1.}
Lemma 2. For $s \geq 2$, the linear operator $A = \mu - \partial_x^2 : H^s \to H^{s-2}$ is a topological isomorphism with the inverse

$$(A^{-1} f)(x) = \left(\frac{1}{2} x^2 - \frac{1}{2} x + \frac{13}{12}\right) \int_0^1 f(a) \, da + \left(x - \frac{1}{2}\right) \int_0^1 \int_0^a f(b) \, db \, da - \int_0^b \int_0^a f(b) \, db \, da + \int_0^1 \int_0^a f(c) \, dc \, db \, da$$

and Green's function

$$(A^{-1} f)(x) = \int_S g(x - x') f(x') \, dx', \quad g(x - x') = \frac{1}{2} (x - x')^2 - \frac{1}{2} |x - x'| + \frac{13}{12}.$$ 

Let $s > 5/2$. The Christoffel operator $\Gamma = \Gamma_{id}$ for equation (5) is

$$\Gamma(u, v) = -A^{-1} \left(\mu(u)v + \mu(v)u + \frac{1}{2} u_x v_x\right),$$

since $\mu$HS can be written as $u_t + uu_x = \Gamma(u, u)$. The bilinear map

$$\langle U, V \rangle_\phi = \mu(U \circ \phi^{-1}) \mu(V \circ \phi^{-1}) + \int_S \frac{U}{\phi_x} \frac{V}{\phi_x} \, dx, \quad U, V \in T_\phi H^s \text{Diff}(\mathbb{S})$$

defines a right-invariant inner product on $H^s \text{Diff}(\mathbb{S})$ and the pair $(H^s \text{Diff}(\mathbb{S}), \langle \cdot, \cdot \rangle)$ is a Riemannian manifold. The $\mu$HS Christoffel map depends smoothly on $\phi$ and defines a Riemannian covariant derivative on $H^s \text{Diff}(\mathbb{S})$ via (10), compatible with the right-invariant metric $\langle \cdot, \cdot \rangle$. In consequence, the $\mu$HS possesses a unique geodesic flow $\varphi \in H^s \text{Diff}(\mathbb{S})$. As a geodesic equation on $H^s \text{Diff}(\mathbb{S})$, the $\mu$HS reads $\varphi_{tt} = \Gamma_{\varphi}(\varphi_t, \varphi_t)$ in Lagrangian coordinates. We also conclude that $\mu$HS is locally well posed in $H^s$ for $s > 5/2$, cf [20].

2.3. Generalities on semidirect products

Let $G$ be a Lie group and $V$ be a vector space. If $G$ acts on the right on $V$, one defines

$$(g_1, v_1)(g_2, v_2) = (g_1g_2, v_2 + v_1g_2)$$

and with this product, $G \times V$ becomes a Lie group (the semidirect product of $G$ and $V$) which is denoted as $G \ltimes V$. It is easy to see that $(e, 0)$ is the neutral element, where $e$ denotes the neutral element of $G$, and that $(g, v)$ has the inverse $(g^{-1}, -vg^{-1})$. To obtain the Lie bracket on the Lie algebra $g \ltimes V$, we consider the inner automorphism

$$I_{(g,v)}(h, w) = (g, v)(h, w)(g, v)^{-1} = (g h g^{-1}, -v g^{-1} + (w + v h) g^{-1}).$$

Writing $v \xi$ for the induced infinitesimal action of $g$ on $V$, i.e. the map

$$V \times g \ni (v, \xi) \mapsto v \xi := \frac{d}{dt} v g(t) \bigg|_{t=0},$$

g($t$) being a curve in $G$ starting from $e$ in the direction of $\xi$, we obtain

$$\text{Ad}_g(v \xi)(w) = (\text{Ad}_g \xi)(w + v \xi) g^{-1},$$

and hence

$$[\xi_1, v_1, \xi_2, v_2] = \text{ad}_{(g_1, v_1)}(\xi_1, v_1) = ([\xi_1, \xi_2], v_2 \xi_1 - v_1 \xi_2).$$

For our purposes, we consider the semidirect product of the orientation-preserving diffeomorphisms $H^s \text{Diff}(\mathbb{S})$ with $H^{s-1}$; the structure of our equations motivates to enforce the second component to have one order less regularity than the first, cf also [11].
We will use the notations $H^sG$ and $H^sG_0$ for the Lie groups $H^s\text{Diff}(S)\ltimes H^{s-1}(S)$ and $[H^s\text{Diff}(S)/\text{Rot}(S)]\ltimes E^{s-1}(S)$, respectively. The group product in these groups is defined by
\[(\varphi_1, f_1)(\varphi_2, f_2) := (\varphi_1 \circ \varphi_2, f_2 + f_1 \varphi_2),\]
where $\circ$ denotes the group product in $H^s\text{Diff}(S)$ (i.e. composition) and $f \varphi := f \circ \varphi$ is a right action of $H^s\text{Diff}(S)$ on the scalar functions on $S$. The neutral element is $(\text{id}, 0)$ and $(\varphi, f)$ has the inverse $(\varphi^{-1}, - f \circ \varphi^{-1})$. The above calculations show that
\[\text{Ad}_{(\varphi, f)}(u, \rho) = (\text{Ad}_u u, f_u u + \rho) \circ \varphi^{-1},\]
\[\text{ad}_{(\varphi, f)}(u, \rho) = (\text{ad}_u u, f_u u - \rho)\]
and
\[[u_1, u_2], (v_1, v_2)] = ([u_1, v_1], v_2 u_1 - u_2 v_1),\]
where $[u_1, v_1] = v_1 u_1 - u_1 v_1$ is the Lie bracket induced by right-invariant vector fields on $H^s\text{Diff}(S)$. Observe that $T H^sG \simeq H^sG \ltimes (H^s \ltimes H^{s-1})$ and $T H^sG_0 \simeq H^sG_0 \ltimes (E^s \ltimes E^{s-1})$. For further details about semidirect product groups we refer to [15, 16].

2.4. The 2HS equation

Inspired by the results for 2CH and 2DP in [11], we work with the configuration space
\[H^sG_0 := \{H^s\text{Diff}(S)/\text{Rot}(S)\} \ltimes E^{s-1}(S), \quad s > 5/2.\]
The Lagrangian for 2HS motivates to define the right-invariant metric on $H^sG_0$ which equals the $H^1$-metric for the first component plus the $L_2$-metric for the second component at the identity $(\text{id}, 0)$:
\[\langle U, V \rangle_{(\varphi, f)} = \{u_1 \circ \varphi^{-1}, v_1 \circ \varphi^{-1}\}_M + \{u_2 \circ \varphi^{-1}, v_2 \circ \varphi^{-1}\}_L,\]
for $U = (u_1, u_2), V = (v_1, v_2) \in T_{(\varphi, f)}H^sG_0$. Let $A$ and $\Gamma^0$ as in lemma 1 and (8). With the 2HS Christoffel map $\Gamma$ on $E^s \ltimes E^{s-1}$ given by
\[\Gamma(X, Y) := \left(\begin{array}{c}
\Gamma^0_{M}(X_1, Y_1) - \frac{1}{2} A^{-1}(X_2 Y_2), \\
- \frac{1}{2}(X_1 Y_2 + Y_1 X_2)
\end{array}\right),\]
we define the map $\Gamma : (M^s \times E^{s-1}) \times (E^s \times E^{s-1})^2 \rightarrow E^s \times E^{s-1},$
\[\Gamma_{(\varphi, f)}(X, Y) = \Gamma((\varphi, f); X, Y) = \Gamma(X \circ \varphi^{-1}, Y \circ \varphi^{-1}) \circ \varphi.\]
Finally, we introduce the affine connection
\[\nabla_X Y(\varphi, f) = DY(\varphi, f) \cdot X(\varphi, f) - \Gamma((\varphi, f); Y(\varphi, f), X(\varphi, f)).\]
The proof of the following proposition uses standard arguments and it is omitted for the reader’s convenience.

**Proposition 3.** Let $s > 5/2$. Let $H^sG_0 = [H^s\text{Diff}(S)/\text{Rot}(S)]\ltimes E^{s-1}(S)$ and let $\Gamma$ be the Christoffel map defined in (13) and (14). Then $\Gamma$ defines a smooth spray on $H^sG_0$, i.e. the map
\[$(\varphi, f) \mapsto \Gamma_{(\varphi, f)} : H^sG_0 \rightarrow \mathcal{L}_{\text{sym}}^2(E^s \ltimes E^{s-1}; E^s \ltimes E^{s-1})$\]
is smooth. Moreover, the metric $(\cdot, \cdot)$ defined in (12) is a smooth (weak) Riemannian metric on $H^sG_0$, i.e. the map
\[$(\varphi, f) \mapsto (\cdot, \cdot)_{(\varphi, f)} : H^sG_0 \rightarrow \mathcal{L}_{\text{sym}}^2(T_{(\varphi, f)}H^sG_0; \mathbb{R})$\]
is a smooth section of the bundle $L_2^{\text{sym}}(TH^sG_0; \mathbb{R})$. Finally, the connection $\nabla$ and the metric $\langle \cdot, \cdot \rangle$ are compatible in the sense that 

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

for all vector fields $X, Y, Z$ on $H^sG_0$.

As a consequence, we obtain a unique local geodesic flow $(\varphi(t), f(t)) \in H^sG_0$ for 2HS satisfying

$$(\varphi(t), f(t)) = \Gamma_{(\varphi, f)}((\varphi_0, f_0), (\varphi_2, f_2)).$$

**Theorem 4.** Let $s > 5/2$. Then there exists an open interval $J$ centered at 0 and an open neighborhood $U$ of $(0, 0) \in E^s \times E^{s-1}$ such that for each $(u_0, \rho_0) \in U$ there exists a unique solution $(\varphi, f) \in C^\infty(J, H^sG_0)$ of (16) satisfying $(\varphi(0), f(0)) = (\text{id}, 0)$ and $(\varphi(t), f(t))(u_0, \rho_0) = (u(t), \rho(t))$. Furthermore, the solution depends smoothly on the initial data in the sense that the local flow $\Phi : J \times U \to H^sG_0$ defined by $\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))$ is a smooth map.

Writing the Cauchy problem for 2HS in the form

$$\begin{cases}
    u_t + uu_x = -\frac{1}{2}A^{-1}(u_x^2 + \rho^2)_x, \\
    \rho_t + u\rho_x = -\rho u_x, \\
    (u(0), \rho(0)) = (u_0, \rho_0),
\end{cases}$$

(17)

Theorem 4 implies that equation (17) in terms of the Euclidean variables $u = \varphi_t \circ \varphi^{-1}$ and $\rho = f_t \circ \varphi^{-1}$ is locally well posed.

**Corollary 5.** Suppose $s > 5/2$. Then, for any $(u_0, \rho_0) \in E^s \times E^{s-1}$ there exists an open interval $J$ centered at 0 and a unique solution $(u, \rho) \in C(J, E^s \times E^{s-1}) \cap C^1(J, E^{s-1} \times E^{s-2})$ of the Cauchy problem (17) which depends continuously on the initial data $(u_0, \rho_0)$.

A similar result is true in the $C^n$-category with $n \geq 2$.

2.5. The 2μHS equation

We define a right-invariant metric on $H^sG$, $s > 5/2$, which equals the inner product induced by $\mu - \partial_x^2$ for the first component plus the $L_2$ inner product for the second one at the identity, i.e.

$$\langle U, V \rangle_{(\varphi, f)} = \mu(u_1 \circ \varphi^{-1})\mu(v_1 \circ \varphi^{-1}) + \int_S \frac{u_{1x}v_{1x}}{\varphi_x} \, dx + \int_S u_{2x}v_{2x} \, dx$$

(18)

for any $U = (u_1, u_2)$, $V = (v_1, v_2) \in T_{(\varphi, f)}H^sG$. With $\Gamma^0$ as in (11) we define a right-invariant Christoffel map for 2μHS by

$$\Gamma^0_{(\text{id}, 0)}(X, Y) = \begin{pmatrix}
    \Gamma^0_{(\text{id}, 0)}(X_1, Y_1) - \frac{1}{2}(\mu - \partial_x^2)^{-1}(X_2Y_2)_x \\
    -\frac{1}{2}(X_1Y_2 + X_2Y_1)_x
\end{pmatrix},$$

(19)

and an affine connection via (15). Then the following is easy to derive.

**Proposition 6.** Let $s > 5/2$. Let $H^sG = H^s\text{Diff}(\mathbb{S}) \otimes H^{s-1}(\mathbb{S})$ and let $\Gamma$ be the Christoffel map defined in (19). Then $\Gamma$ defines a smooth spray on $H^sG$, i.e. the map

$$(\varphi, f) \mapsto \Gamma_{(\varphi, f)} : H^sG \to L_2^{\text{sym}}(H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}))$$
is smooth. Moreover, the metric \(\langle \cdot, \cdot \rangle\) defined by (18) is a smooth (weak) Riemannian metric on \(H^G\), i.e. the map
\[
(\varphi, f) \mapsto \langle \cdot, \cdot \rangle_{(\varphi, f)} : H^G \to L^2_{\text{sym}}(T H^G; \mathbb{R})
\]
is a smooth section of the bundle \(L^2_{\text{sym}}(TH^G; \mathbb{R})\). Finally, the connection \(\nabla\) is a Riemannian covariant derivative, compatible with \(\langle \cdot, \cdot \rangle\).

We thus know that the \(2\mu\)HS is a re-expression of the geodesic flow of the connection \(\nabla\) defined in (15) on the product \(H^G\). The geodesic equation reads (16). We have the following local well-posedness result.

**Theorem 7.** Let \(s > 5/2\) and let \(\Gamma\) be the \(2\mu\)HS Christoffel map. Then, there exists an open interval \(J\) centered at 0 and an open neighborhood \(U\) of \((0, 0) \in H^s(S) \times H^{s-1}(S)\) such that for each \((u_0, \rho_0) \in U\) there exists a unique solution \((\varphi, f) \in C^\infty(J, H^G)\) of (16) satisfying \((\varphi(0), f(0)) = (id, 0)\) and \((\varphi(0), f(0)) = (u_0, \rho_0)\). Furthermore, the solution depends smoothly on the initial data in the sense that the local flow \(\Phi : J \times U \to H^G\) defined by \(\Phi(t, u_0, \rho_0) = (\varphi(t; u_0, \rho_0), f(t; u_0, \rho_0))\) is a smooth map.

We write the Cauchy problem for \(2\mu\)HS in the form
\[
\begin{cases}
  u_t + uux = -\left(\mu - \partial_x^2\right)^{-1} \left(\frac{1}{2}u_x^2 + 2\mu(u)u + \frac{1}{2}\rho^2\right), \\
  \rho_t + u\rho_x = -\rho u_x, \\
  (u(0), \rho(0)) = (u_0, \rho_0).
\end{cases}
\]
(20)

It follows from theorem 7 that \(2\mu\)HS is locally well posed in \(H^s \times H^{s-1}\) for \(s > 5/2\).

**Corollary 8.** Suppose \(s > 5/2\). Then, for any \((u_0, \rho_0) \in C^1(S) \times C^0(S)\) there exists an open interval \(J\) centered at 0 and a unique solution
\[
(u, \rho) \in C(J, H^s(S) \times H^{s-1}(S)) \cap C^1(J, H^{s-1}(S) \times H^{s-2}(S))
\]
of the Cauchy problem (20) which depends continuously on the initial data \((u_0, \rho_0)\).

The previous results hold with the obvious changes also in the \(C^n\)-category, \(n \geq 2\).

### 3. The curvature of \(H^G_0\) associated with the 2HS

Let us denote by
\[
R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z
\]
the curvature tensor of \(H^G_0\) equipped with the right-invariant metric (12). In the following theorem, we compute an explicit formula for \(R\) and show that the sectional curvature
\[
S(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}
\]
has the constant positive value \(1/4\).

**Theorem 9.** The curvature tensor for the 2HS equation on \(H^G_0\), \(s > 5/2\), equipped with the right-invariant metric (12), for vector fields \(X, Y, Z\), is given by
\[
4R(X, Y)Z = X(Y, Z) - Y(X, Z).
\]
In particular, the sectional curvature for 2HS is constant and equal to 1/4.
Proof. We have the following local formula for $R$ in terms of the Christoffel map (14):

$$R(X,Y)Z = D_1 \Gamma^p(Z,X)Y - D_1 \Gamma^p(Z,Y)X + \Gamma^p(\Gamma^p(Z,Y),X) - \Gamma^p(\Gamma^p(Z,X),Y),$$

for any vector fields $X,Y,Z$ on $H\mathcal{G}$, where $D_1$ denotes differentiation with respect to $p=(\varphi,f)$. By right invariance of $\Gamma$, i.e.

$$\Gamma^p(X,Y) \circ \psi = \Gamma^p(\varphi \circ X, \varphi \circ Y),$$

it holds that

$$R(X,Y)Z \circ \varphi^{-1} = R(u,v)w$$

if $X = u \circ \varphi$, $Y = v \circ \varphi$ and $Z = w \circ \varphi$. Therefore, it suffices to consider the curvature at $(\text{id},0)$. We write $\Gamma = \Gamma_{\text{id},0}$ and denote the components of $u$ by $u_1$ and $u_2$ similarly for $v, w$. A lengthy but straightforward computation shows that

$$R(u,v)w = D_1 \Gamma(u,v)w - D_1 \Gamma(w,v)u + \Gamma(\Gamma(w,v),u) - \Gamma(\Gamma(w,u),v),$$

In the first component, we have the terms

$$-\Gamma^0(u_1,v_1,u_1) + \frac{1}{2}A^{-1}(w_2u_1v_1)_x - \Gamma^0(u_2,v_1,u_1) + \frac{1}{2}A^{-1}(w_2u_1v_2)_x + \Gamma^0(u_1,v_1,u_1) + \frac{1}{2}A^{-1}(w_2u_1v_2)_x + \Gamma^0(u_1,v_1,u_1) - \frac{1}{2}A^{-1}(w_2u_1v_2)_x - \Gamma^0(\Gamma^0(w_1,v_1),u_1) + \frac{1}{2}A^{-1}(w_2u_1v_2)_x + \Gamma^0(\Gamma^0(w_1,v_1),u_1) + \frac{1}{2}A^{-1}(w_2u_1v_2)_x - \Gamma^0(\Gamma^0(w_1,v_1),u_1) - \frac{1}{2}A^{-1}(w_2u_1v_2)_x,$$

Using that

$$\partial_x A^{-1} \partial_x = \mu - 1$$

and the relation

$$\Gamma^0(\Gamma^0(w_1,v_1),u_1) - \Gamma^0(\Gamma^0(w_1,u_1),v_1) = -\frac{1}{2}u_1 \mu(\partial_x v_1 v_1) + \frac{1}{2}v_1 \mu(\partial_x u_1 u_1),$$

cf [23], we see that these terms equal

$$\frac{1}{2}A^{-1} \partial_x [(w_1,v_1)_x, u_1] + (u_1,v_1)_x, w_1 - (w_1,u_1)_x, v_1 = (v_1,u_1)_x, w_1$$

To see that the terms with $A^{-1} \partial_x$ cancel out, we use that $u_1(0) = v_1(0) = 0$ so that

$$\frac{1}{2}w_1 u_1 u_1 v_1 = \frac{1}{2}u_1 w_1 v_1 = -A^{-1} \partial_x \left( \frac{1}{2}w_1 u_1 v_1 - \frac{1}{2}u_1 w_1 v_1 \right) = \frac{1}{2}A^{-1} \partial_x (w_1 u_1 v_1 - u_1 w_1 v_1) + u_1 w_1 v_1 - u_1 w_1 v_1,$$

which coincides up to sign with the first row terms in (21), and

$$\frac{1}{2}w_2 u_1 v_1 = \frac{1}{2}w_2 v_1 u_1 = -A^{-1} \partial_x \left( \frac{1}{2}w_2 u_1 v_1 - \frac{1}{2}w_2 v_1 u_1 \right) = \frac{1}{2}A^{-1} \partial_x (w_2 u_1 v_1 - v_1 w_2 u_1) + w_2 u_2 v_1 - u_2 w_2 v_1.$$
Using $A^{-1} \partial_t v_1 = -v_1$ and $A^{-1} \partial_t^2 u_1 = -u_1$, the first component terms (21) thus reduce to

$$\frac{1}{2} u_1 (\mu(w_1, v_1) + \mu(w_2, v_2)) - \frac{1}{2} v_1 (\mu(w_1, u_1) + \mu(w_2, u_2)),$$

which is the desired expression. The second component terms are

$$\frac{1}{2} [(w_1, v_1), u_2 + u_1, w_2, v_1] + \frac{1}{2} [(w_1, u_1), w_2 + w_1, u_2, v_1] - \frac{1}{2} v_1 [w_1, u_2 + u_1, w_2]$$

$$- \frac{1}{2} [((w_1, u_1) + v_2 + v_1, w_2, u_1] - \frac{1}{2} [(v_1, u_1) + w_2 + w_1, u_2, u_1]$$

$$+ \frac{1}{2} u_1 [w_1, v_2 + v_1, w_2] - \frac{1}{2} (\Gamma_1(u, v), u_2 + u_1, \Gamma_2(w, v)) + \frac{1}{2} (\Gamma_1(w, u), v_2 + v_1, \Gamma_2(w, u))$$

(22)

and with $A^{-1} \partial_t = \mu - 1$ we can simplify the last row terms

$$\Gamma_1(u, v) = \frac{1}{2} w_1, u_1 - \frac{1}{2} \mu(w_1, v_1) + \frac{1}{2} w_2 v_2 - \frac{1}{2} \mu(w_2, v_2)$$

and

$$\Gamma_1(u, v) = \frac{1}{2} w_1, u_1 + \frac{1}{2} \mu(w_1, v_1) + \frac{1}{2} w_2 v_2 - \frac{1}{2} \mu(w_2, v_2).$$

It is now easy to see that the terms in (22) reduce to

$$\frac{1}{4} u_2 (\mu(w_1, v_1) + \mu(w_2, v_2)) - \frac{1}{4} v_2 (\mu(w_1, u_1) + \mu(w_2, u_2))$$

so that we obtain

$$R(u, v) w = \frac{1}{4} u \langle v, w \rangle - \frac{1}{4} v \langle u, w \rangle.$$

By the definition of the sectional curvature, we have

$$S(u, v) = \frac{(R(u, v) w, u)}{\langle u, v \rangle \langle v, u \rangle - \langle u, w \rangle^2} = \frac{1}{4}.$$

\[\square\]

**Remark 10.** Since Lenells [23] used a different scaling for the $H^1$-metric, he came to the result that the sectional curvature for the HS is identically equal to 1. Note carefully that we have only used that $u_1$ and $v_1$ vanish at zero; a corresponding assumption on the second components is not necessary in the above proof.

The above result is of particular interest concerning the stability of the geodesic flow associated with the 2HS system, cf the appendix. The very recent paper [36] explains that the geodesic flow for the 2HS system allows a continuation (beyond the breaking time of the associated solution $(u, \rho)$ of the original equation) on the space $\mathcal{M}^0_{AC} = \mathcal{M}^0 \otimes \mathcal{H}^0(\mathbb{S})$, where $\mathcal{M}^0_{AC}$ is the set of nondecreasing absolutely continuous functions $\varphi : [0, 1] \to [0, 1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. Precisely, the flow variables $(\varphi, f) \in \mathcal{M}^0_{AC}$ for 2HS satisfy the geodesic equation $(\varphi_t, f_t) = \Gamma_{\varphi, f_t}(\varphi_t, f_t, (\varphi_t, f_t))$ for any positive $t$. Interestingly, the space $\mathcal{M}^0_{AC}$ is bijective to the open subset

$$\mathcal{U}^0 = \left\{ f \in H^0(\mathbb{S}) : \|f\|_{H^0} = 1, f > 0 \text{ a.e. on } \mathbb{S} \right\}$$

of the $L_2$ unit sphere via the mapping

$$f \mapsto \varphi(x) = \int_0^1 f^2(y) \, dy.$$

Lenells [23, 24] used the constance and positivity of the sectional curvature associated with HS as a motivation to continue the geodesic flow on the infinite-dimensional sphere $\mathcal{M}^0_{AC}$, keeping in mind the well-known fact that any finite-dimensional Riemannian manifold with constant positive curvature is locally isometric to a sphere. In the interim, Wunsch [36] already adopted this approach for the 2HS system so that our work may serve retroactively as a motivation for considering the problem of extending the flow variables on a suitable configuration space (which in fact works by multiplying the sphere $\mathcal{M}^0_{AC}$ with $H^0(\mathbb{S})$).
4. The curvature of $H^G$ associated with the $2\mu$HS

In this section, $R$ and $S$ stand for the curvature tensor and the unnormalized sectional curvature for the $2\mu$HS equation.

**Theorem 11.** The unnormalized sectional curvature $S(u,v)$ for the $2\mu$HS equation is given by

$$S(u,v) = \langle \Gamma(u,v), \Gamma(u,v) \rangle - \langle \Gamma(u,u), \Gamma(v,v) \rangle - 3\mu(u_1,v_1)^2.$$ 

**Proof.** The proof is similar to the proof of proposition 5.1 in [11]; the single difference is that the additional terms involving $\Gamma^0$ in (5.4) of [11] do not cancel out but give $-3\mu(u_1,v_1)^2$ as explained in [20]. \[ \Box \]

In the following, we write $S_1$, $S_2$ to distinguish between the sectional curvature for the one-component $\mu$HS and its two-component extension. In [20], the authors prove that $S_1(u_1,v_1)$ is always positive for any two orthonormal vectors $u_1$ and $u_2$ (with respect to the scalar product induced by $\mu - \partial_x^2$). Since we have

$$S_2 \left( \left( \begin{array}{c} u_1 \\ 0 \end{array} \right), \left( \begin{array}{c} v_1 \\ 0 \end{array} \right) \right) = S_1(u_1,v_1),$$

we see that $S_2$ is positive in the $H^4$ $\text{Diff}(S)$ direction. To find a large subspace of positive sectional curvature for $2\mu$HS with non-trivial second component we compute $S_2(u,v)$ for

$$u = \left( \cos k_1 x \cos k_2 x, \cos l_1 x \cos l_2 x \right), \quad v = \left( \cos l_1 x \cos l_2 x, \cos k_1 x \cos k_2 x \right),$$

where $k_i \neq l_i \in 2\pi \mathbb{N}, i = 1, 2$; this is inspired by calculations in [11] and [26] where the authors show that the sectional curvature for CH is positive for any pair of trigonometric functions. Note that

$$S_2(u,v) = S_1(u_1,v_1) + \frac{1}{4} \int_S (u_2v_2)_s (\mu - \partial_x^2)^{-1}(u_2v_2)_s \, dx - \int_S \Gamma^0(u_1,v_1)(u_2v_2)_s \, dx$$

$$+ \frac{1}{4} \int_S (u_1v_2 + v_1u_2)^2 \, dx - \frac{1}{4} \int_S (u_2^2)_s (\mu - \partial_x^2)^{-1}(v_2^2)_s \, dx$$

$$+ \frac{1}{2} \int_S \Gamma^0(u_1,u_1)(v_2^2)_s \, dx + \frac{1}{2} \int_S \Gamma^0(v_1,v_1)(u_2^2)_s \, dx - \int_S u_1u_2v_1v_2 \, dx$$

$$= S_1(u_1,v_1) + \sum_{j=1}^4 I_j,$$ \hspace{1cm} (23)

where

$$I_1 = \frac{1}{4} \int_S (u_2v_2)_s (\mu - \partial_x^2)^{-1}(u_2v_2)_s \, dx,$$

$$I_2 = -\frac{1}{4} \int_S (u_2^2)_s (\mu - \partial_x^2)^{-1}(v_2^2)_s \, dx,$$

$$I_3 = -\int_S \Gamma^0(u_1,v_1)(u_2v_2)_s \, dx + \frac{1}{2} \int_S \Gamma^0(u_1,u_1)(v_2^2)_s \, dx + \frac{1}{2} \int_S \Gamma^0(v_1,v_1)(u_2^2)_s \, dx,$$

$$I_4 = \frac{1}{4} \int_S (u_1v_2 + v_1u_2)^2 \, dx - \int_S u_1u_2v_1v_2 \, dx.$$

We write $A = \mu - \partial_x^2$ and apply the identity

$$A^{-1}\partial_x A^{-1}\partial_x = \partial_x^2 A^{-1} = \mu - 1.$$
Using integration by parts and the orthogonality relations for trigonometric functions we find
\[
S_1(u_1, v_1) = (\Gamma^0(u_1, v_1), \Gamma^0(u_1, v_1)) - (\Gamma^0(u_1, u_1), \Gamma^0(v_1, v_1)) - 3\mu(u_1, v_1)^2
\]
\[
= -\frac{1}{2} \int_S A^{-1} [\partial_x (u_{1x} v_{1x})] A \Gamma^0_u(u_1, v_1) \, dx + \frac{1}{2} \int_S A^{-1} [\partial_x (u_{1x}^2)] A \Gamma^0(u_1, v_1) \, dx
\]
\[
= \frac{1}{2} \int_S u_{1x} v_{1x} A \Gamma^0(u_1, v_1) \, dx - \frac{1}{2} \int_S u_{1x}^2 A \Gamma^0(v_1, v_1) \, dx
\]
\[
= -\frac{1}{4} \int_S u_{1x} v_{1x} (A^{-1} \partial_x^2)(u_{1x} v_{1x}) \, dx + \frac{1}{4} \int_S u_{1x}^2 (A^{-1} \partial_x^2)(v_{1x}^2) \, dx
\]
\[
= \frac{1}{4} \mu(u_{1x}^2) \mu(v_{1x}^2)
\]
\[
= -\frac{1}{16} \kappa_1^2 \ell_1^2.
\]
(24)

Our choice of \(k_1\) and \(l_1\) implies that the one-component sectional curvature is strictly positive. All we have to show is that the second-component terms do not contribute negative terms which make the total sectional curvature negative. Similar computations show that the terms \(I_1\) and \(I_2\) in (23) are
\[
I_1 = -\frac{1}{4} \int_S u_2 v_2 \partial_x^2 A^{-1} u_2 v_2 \, dx = \frac{1}{4} \int_S u_2 v_2 (1 - \mu)(u_2 v_2) \, dx = \frac{1}{4} \int_S u_2^2 v_2^2 \, dx
\]
and
\[
I_2 = \frac{1}{4} \int_S u_2^2 \partial_x^2 A^{-1} v_2^2 \, dx = \frac{1}{4} \int_S u_2^2 (\mu - 1) v_2^2 \, dx = -\frac{1}{4} \int_S u_2^2 v_2^2 \, dx + \frac{1}{16}.
\]

Since
\[
-\int_S \Gamma^0(u_1, v_1)(u_2 v_2) \, dx = \frac{1}{2} \int_S A^{-1}(u_{1x} v_{1x})_x(u_2 v_2) \, dx
\]
\[
= \frac{1}{2} \int_S [(1 - \mu)(u_{1x} v_{1x})] u_2 v_2 \, dx
\]
\[
= \frac{1}{2} \int_S u_{1x} u_2 v_{1x} v_2 \, dx
\]
we find that
\[
I_3 + I_4 = \frac{1}{2} \int_S \Gamma^0(u_1, u_1)(v_2^2) \, dx + \frac{1}{2} \int_S \Gamma^0(v_1, v_1)(u_2^2) \, dx + \frac{1}{4} \int_S (u_{1x}^2 v_2^2 + v_{1x}^2 u_2^2) \, dx
\]
\[
= \frac{1}{4} \mu(u_{1x}^2) \mu(v_2^2) + \frac{1}{4} \mu(v_{1x}^2) \mu(u_2^2)
\]
\[
= \frac{1}{16} (k_1^2 + \ell_1^2).
\]

It follows from (23) and (24) that
\[
S_2(u, v) = \frac{1}{16} (1 + k_1^2 + \ell_1^2 + k_1^2 \ell_1^2) > \frac{1}{16}.
\]

Our calculation also shows that the sectional curvature is equal to 1/16 in the direction of the second component since
\[
S_2 \left( \begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right) = I_1 + I_2 = \frac{1}{16}.
\]

We have thus shown the following proposition.
Proposition 12. Let \( s > \frac{5}{2} \). Let \( S(u, v) := \langle R(u, v)v, u \rangle \) be the unnormalized sectional curvature on \( H^sG \) associated with the \( 2\mu \)HS equation. Then

\[
S(u, v) > \frac{1}{16}
\]

for all vectors \( u, v \in T_{id,0}H^sG \), of the form

\[
u = \left( \begin{array}{c}
\cos k_1 x \\
\cos k_2 x
\end{array} \right), \quad u = \left( \begin{array}{c}
\cos l_1 x \\
\cos l_2 x
\end{array} \right), \quad k_i \neq l_i \in \{2\pi, 4\pi, \ldots\}.
\]

Moreover, the normalized sectional curvature satisfies

\[
\frac{S(u, v)}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2} = \frac{1}{4}
\]

for all vectors \( u, v \in T_{id,0}H^sG \) of the form

\[
u = \left( \begin{array}{c}
0 \\
\cos k_2 x
\end{array} \right), \quad u = \left( \begin{array}{c}
0 \\
\cos l_2 x
\end{array} \right), \quad k_2 \neq l_2 \in \{2\pi, 4\pi, \ldots\}.
\]

It is explained in [20] that the first-component configuration space \( H^s \text{Diff}(S) \) can be thought of as a solid torus with a cross-section isomorphic to \( H^s \text{Diff}(S)/\mathbb{S} \). The fact that the one-component sectional curvature is positive is proved by a decomposition of the tangent space \( T_{id}H^s \text{Diff}(S) = U \oplus V \) corresponding to the decomposition \( u = \tilde{u} + \mu(u) \), where \( \mu(\tilde{u}) = 0 \), i.e. \( U \) are the zero mean functions and \( V \simeq \mathbb{R} \) are the constants. It is an open problem and a task for further research which geometric interpretations for the group \( H^sG \) associated with the \( 2\mu \)HS system can be given and which conclusions for the (non)existence of solutions of \( 2\mu \)HS can be drawn. While the present section shows the existence of subspaces of positive sectional curvature for \( 2\mu \)HS, one could ask whether \( S_2 \) is always positive, for arbitrary second-component functions, or whether there are directions of strictly negative sectional curvature.

Acknowledgments

The author thanks Jonatan Lenells (Baylor University, Waco) and Joachim Escher (Leibniz University, Hannover) for bringing the above problems to his attention. A cordial thank for useful remarks that helped to improve the manuscript goes to the anonymous referees.

Appendix. Generalities on geodesic flows on infinite-dimensional Lie groups and their stability properties

In this appendix, we will survey the most important results of the seminal papers [1, 10] which are relevant for the purposes of the paper at hand. It goes back to Arnold’s work [1] to model both Euler’s equation for a rotating rigid body and Euler’s equation for an ideal fluid on a Lie group with an invariant metric. The Lie group for the rotating rigid body is the matrix group \( SO(3) \), whereas the motion of an ideal fluid is modeled on the diffeomorphism group \( \text{Diff}(M) \) of volume-preserving diffeomorphisms of a certain manifold \( M \). While the matrix group \( SO(3) \) has finite dimension and is equipped with a left-invariant metric, the group \( \text{Diff}(M) \) is an infinite-dimensional Lie group which is equipped with a right-invariant metric. The geometric viewpoint is not only aesthetically appealing, but is also very useful for the study of well-posedness and stability issues. In view of the results of this paper, we provide a general overview about the geometric picture for ideal fluids (which corresponds to a right-invariant formulation), with a focus on the stability of the geodesic flow.
Let $G$ be a (not necessarily finite dimensional) Lie group with Lie algebra $\mathfrak{g} \simeq T_e G$, where $e$ denotes the unit element. We assume that there is an invertible linear operator $A : \mathfrak{g} \to \mathfrak{g}^*$ which is, for historical reasons, going back to Euler’s work on the rigid body motion, called an *inertia operator*. We also assume that $\mathfrak{g}^* \simeq \mathfrak{g}$ (which can often be achieved by considering a suitable subspace of $\mathfrak{g}^*$) so that $A$ can in fact be regarded as an automorphism of $\mathfrak{g}$. Let $R_g : G \to G$ denote the right translation map on $G$. We obtain a right-invariant metric $\rho_A$ on $G$ by setting

$$\rho_A(u, v) = (A[DR_{g^{-1}}u], DR_{g^{-1}}v),$$

for all $u, v \in T_e G$, where $(\cdot, \cdot)$ denotes the dual pairing on $\mathfrak{g}^* \times \mathfrak{g}$. If $G$ is infinite dimensional, the map $\rho_A$ defines in general only a weak Riemannian metric on $G$, i.e. the natural topology on any tangent space $T_e G$ is stronger than the topology induced by the metric $\rho_A$, cf [10]. Let $\text{ad}^* u$ denote the dual operator (with respect to $\rho_A$) of the natural action of the Lie algebra on itself given by $\text{ad}_u : \mathfrak{g} \to \mathfrak{g}, v \mapsto [u, v]$, and define the bilinear and symmetric map

$$B_e : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, B_e(u, v) = \frac{1}{2}(\text{ad}_u^* v + \text{ad}_v^* u).$$

Next, we introduce an affine connection on $G$ given by

$$\nabla_{\xi_u} \xi_v = \frac{1}{2}[\xi_u, \xi_v] + B(\xi_u, \xi_v), \quad \text{(A.1)}$$

where $\xi_u$ is the right-invariant vector field on $G$ with value $u$ at $e$ and $B$ denotes the right-invariant tensor field with value $B_e$ at the identity. Let $g(t)$ be a smooth part in $G$ and define its Eulerian velocity, which lies in the Lie algebra $\mathfrak{g}$, by

$$u(t) = DR_{g^{-1}} \dot{g}(t).$$

The crucial point is that $g(t)$ is a geodesic for the connection $\nabla$ if and only if its Eulerian velocity satisfies the Euler equation

$$u_t = -B(u, u);$$

see [12] for instance.

Interestingly, the above formalism also works the other way round. Starting from an Euler equation $u_t = -B(u, u)$ with quadratic right-hand side, defined on the Lie algebra $\mathfrak{g}$ of some Lie group $G$, and defining the connection $\nabla$ in terms of the operator $B$ as in (A.1), one sees that the Euler equation re-expresses a geodesic flow on the Lie group $G$. Nevertheless, it is not clear that there is a right-invariant metric $\rho_A$ on $G$, induced by some inertia operator $A$, such that the connection $\nabla$ is compatible with the metric in the sense that

$$X(\rho_A(Y, Z)) = \rho_A(\nabla_X Y, Z) + \rho_A(\nabla_X Z, Y),$$

for vector fields $X, Y, Z$ on $G$. For vector fields $X, Y, Z$ on $G$. Recall that the equations under discussion in the main body of this paper are metric in the sense that they allow for a Riemannian structure.

Let $x(t)$ denote a geodesic in $G$ and consider the geodesic variation $x(t, s)$ with the associated variation vector field

$$\frac{dx(t, s)}{ds} \bigg|_{s=0} = \xi(t) \in T_{x(t)} G.$$

It is well known that $\xi$ is a solution of the Jacobi equation

$$\frac{D^2 \xi}{Dt^2} = -R(\xi, v)v,$$

where $D/Dt$ is the covariant derivative, $R$ is the curvature tensor associated with the connection $\nabla$ and $v = \dot{x}(t)$ is the velocity field. By a decomposition of the variation vector $\xi$ into components parallel and perpendicular to the velocity $v$, Arnold showed that the Jacobi...
equation for the perpendicular component (which is henceforth also denoted as $\xi$ for simplicity) can be written in the form

$$\frac{D^2 \xi}{Dt^2} = -\nabla U,$$

$$U = \frac{S}{2} \partial_A(\xi, \xi) \partial_A(v, v);$$

here $S$ is the sectional curvature of the two-dimensional subspace of $T_{x(t)}G$ spanned by $v$ and $\xi$. If the norm of $v$ is equal to 1 (which can be achieved by a parametrization of the geodesic by arc length), the Jacobi equation for $\xi$ reduces to the harmonic oscillator equation with the potential energy $U$ equal to the product of the curvature in the direction spanned by the velocity vector and the normal component of the variation with the square of length of this normal component. From this, it is obvious that $S < 0$ implies an exponential divergence of the geodesics starting near $x(0)$, whereas for $S > 0$, convergence of the nearby geodesics is expected; cf [2] for further details. This motivates our research for subspaces of positive sectional curvature in the main body of the paper. Note that curvature computations for geometric evolution equations have a long tradition. They have already been carried out in the 1980s [14, 29] and in Misiołek’s paper [30] about the CH equation; see also [22] for a more recent paper.

References

[1] Arnold V I 1966 Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits Ann. Inst. Fourier (Grenoble) 16 319–61
[2] Arnold V I 1989 Mathematical Methods of Classical Mechanics (Graduate Texts in Mathematics vol 60) (New York: Springer)
[3] Beals R, Sattinger D H and Szmigielski J 2001 Inverse scattering solutions of the Hunter–Saxton equation Appl. Anal. 78 255–69
[4] Camassa R and Holm D D 1993 An integrable shallow water equation with peaked solitons Phys. Rev. Lett. 71 1661–4
[5] Chae D 2008 On the blow-up problem for the axisymmetric 3D Euler equations Nonlinearity 21 2053–60
[6] Constantin A 2005 Global solutions of the Hunter–Saxton equation SIAM J. Math. Anal. 37 996–1026
[7] Constantin A and Ivanov R 2008 On an integrable two-component Camassa–Holm shallow water system Phys. Lett. A 372 7129–32
[8] Constantin A, Lax P and Majda A 1985 A simple one-dimensional model for the three-dimensional vorticity equation Commun. Pure Appl. Math. 38 715–24
[9] Dai H H and Pavlov M 1998 Transformations for the Camassa–Holm equation, its high-frequency limit and the Sinh–Gordon equation J. Phys. Soc. Japan 67 3655–7
[10] Ebin D G and Marsden J 1970 Groups of diffeomorphisms and the motion of an incompressible fluid Ann. Math. 92 102–63
[11] Escher J, Kohlmann M and Lenells J 2011 The geometry of the two-component Camassa–Holm and Degasperis–Procesi equations J. Geom. Phys. 61 436–52
[12] Escher J and Kolev B The Degasperis–Procesi equation as a non-metric Euler equation arXiv:0908.0508v1
[13] Escher J, Lechtenfeld O and Yin Z 2007 Well-posedness and blow-up phenomena for the 2-component Camassa–Holm equation Discrete Contin. Dyn. Syst. 19 493–513
[14] Freed D 1988 The geometry of loop groups J. Diff. Geom. 28 223–76
[15] Holm D D, Marsden J E and Ratiu T S 1998 The Euler–Poincaré equations and semidirect products with applications to continuum theories Adv. Math. 137 1–81
[16] Holm D D and Tronci C 2009 Geodesic flows on semidirect-product Lie groups: geometry of singular measure-valued solutions Proc. R. Soc. A 465 457–76
[17] Hou T Y and Li C 2008 Dynamic stability of the three-dimensional axisymmetric Navier–Stokes equations with swirl Commun. Pure Appl. Math. 61 661–97
[18] Hunter J K and Saxton R 1991 Dynamics of director fields SIAM J. Appl. Math. 51 1498–521
[19] Hunter J K and Zheng Y 1994 On a completely integrable nonlinear hyperbolic variational equation Physica D 79 361–86
[20] Khesin B, Lenells J and Misiołek G 2008 Generalized Hunter–Saxton equation and the geometry of the group of circle diffeomorphisms Math. Ann. 342 617–756
[21] Lechtenfeld O and Lenells J 2009 On the $N = 2$ supersymmetric Camassa–Holm and Hunter–Saxton equations J. Math. Phys. 50 012704
[22] Lenells J 2007 Riemannian geometry on the diffeomorphism group of the circle Ark. Mat. 45 297–325
[23] Lenells J 2007 The Hunter–Saxton equation describes the geodesic flow on a sphere J. Geom. Phys. 57 2049–64
[24] Lenells J 2007 Weak geodesic flow and global solutions of the Hunter–Saxton equation Discrete Contin. Dyn. Syst. 18 643–56
[25] Lenells J 2008 The Hunter–Saxton equation: a geometric approach SIAM J. Math. Anal. 40 266–77
[26] Lenells J, Misiołek G and Preston S C 2009 Curvatures of right-invariant Sobolev metrics on diffeomorphism groups and their Euler–Arnold equations (unpublished)
[27] Lenells J, Misiołek G and Tiğlay F 2010 Integrable evolution equations on spaces of tensor densities and their peakon solutions Commun. Math. Phys. 299 129–61
[28] Liu J and Yin Z 2010 Global weak solutions for a periodic two-component μ-Hunter–Saxton system arXiv:1012.5452v3 [math.AP]
[29] McKeaen H P 1982 Curvature of an ∞-dimensional manifold related to Hill’s equation J. Diff. Geom. 17 523–9
[30] Misiołek G 1998 A shallow water equation as a geodesic flow on the Bott-Virasoro group J. Geom. Phys. 24 203–8
[31] Misiołek G 2002 Classical solutions of the periodic Camassa–Holm equation Geom. Funct. Anal. 12 1080–104
[32] Okamoto H 2009 Well-posedness of the generalized Proudman–Johnson equation without viscosity J. Math. Fluid Mech. 11 46–59
[33] Pavlov M V 2005 The Gurevich–Zybin system J. Phys. A: Math. Gen. 38 3823–40
[34] Proudman I and Johnson K 1962 Boundary-layer growth near a rear stagnation point J. Fluid Mech. 12 161–8
[35] Wunsch M 2010 The generalized Hunter–Saxton system SIAM J. Math. Anal. 42 1286–304
[36] Wunsch M 2011 Weak geodesic flow on a semi-direct product and global solutions to the periodic Hunter–Saxton system arXiv:1101.5483v1 [math.AP]
[37] Yin Z 2004 On the structure of solutions to the periodic Hunter–Saxton equation SIAM J. Math. Anal. 36 272–83
[38] Zou D 2010 A two-component μ-Hunter–Saxton equation Inverse Problems 26 085003