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The nonlinear Fokker-Planck equation: comparison of the classical and quantum (boson and fermion) characteristics

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Abstract. A “free energy” functional $F$ is studied for the one-dimensional Fokker-Planck equation (FPE). Using $F$, a notion of distance between a global Maxwellian function $M$, which is a stationary explicit solution of the FPE, and an arbitrary solution $f$ (with the same total density $\rho$ at the fixed moment $t$) is introduced. In this way we generalize the important Kullback-Leibler distance. A comparison between classical and quantum characteristics of the FPE solutions is made.

1. Introduction.
We consider the one-dimensional nonlinear Fokker-Planck equation (FPE)

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial^2 v} - v \frac{\partial f}{\partial x} + \frac{\partial}{\partial \rho}[vf(1+\rho k)],$$

(1.1)

where $t \in \mathbb{R}$ stands for time, $x \in \Omega$ stands for the space coordinate, $v \in \mathbb{R}$ is velocity, $\Omega \subset \mathbb{R}$ is a bounded domain of the $x$-space, and $\mathbb{R}$ denotes the real axis. This equation (sometimes called Kramers equation) is a modification of the classical linear FPE to include quantum effects, and it turns into the classical FPE for the case that $k = 0$. Equation (1.1) is also a kinetic model for bosons ($k > 0$) and fermions ($k < 0$) [$1,3$]. We introduce a notion of distance between a global Maxwellian function $M$, which is a stationary explicit solution of the FPE, and an arbitrary solution $f$ (with the same total density $\rho$ at the fixed moment $t$) and so generalize the Kullback-Leibler distance [$4$], which was fruitfully used before (see some references in [7]). Our approach enables us to treat also the non-homogeneous case, while the Kullback-Leibler distance is used only for the homogeneous case. Next, we compare classical and quantum characteristics of the FPE solutions. We follow the ideas and scheme of the previous papers [$5,6$].

2. Extremal problem
It is required that the distribution function $f(t, x, v)$ satisfies the FPE (1.1) and inequalities

$$f(t, x, v) > 0, \quad 1 + kf > 0.$$  

(2.1)
We recall some well-known notions connected with the FPE. The entropy of $f$ is introduced by the formula
\[ S(t, f, k) = - \int_\Omega \int_\mathbb{R} \Phi(f) dv dx, \tag{2.2} \]
where
\[ \Phi(f) = f \log f \quad \text{for} \quad k = 0, \tag{2.3} \]
\[ \Phi(f) = f \log f - \frac{1}{k} (1 + kf) \log(1 + kf) + f \quad \text{for} \quad k \neq 0. \tag{2.4} \]

Here the entropy $S(t, f, 0)$, which we denote also by $S_c$, corresponds to the classical case $k = 0$.

The density $\rho(t, x)$, total density $\rho(t)$, energy $E(t, x)$, and total energy $E(t)$ are defined via formulas:
\[ \rho(t, x) = \int_\mathbb{R} f(t, x, v) dv, \quad \rho(t) = \int_\Omega \rho(t, x) dx, \tag{2.5} \]
\[ E(t, x) = \int_\mathbb{R} \frac{v^2}{2} f(t, x, v) dv, \quad E(t) = \int_\Omega \int_\mathbb{R} \frac{v^2}{2} f(t, x, v) dv dx. \tag{2.6} \]

We assume that the domain $\Omega$ is bounded and so its volume (i.e., length in our case) is bounded too:
\[ \text{Vol}(\Omega) = V_\Omega < \infty. \tag{2.7} \]

The "free energy" functional is given by the formula
\[ F(f) = (F(f))(t) = \lambda E(t) + S(t), \quad \lambda = -1/T, \tag{2.8} \]
where $S(t)$ and $E(t)$ are defined by formulas (2.2) and (2.6), respectively. The parameters $\lambda = -1/T$, $t$, and $\rho(t)$ are fixed.

Now we use the calculus of variations (see [2]) and find the function $f_{\text{max}}$ which maximizes the functional (2.8) under additional condition
\[ \rho = \int_\Omega \rho(t, x) dx. \tag{2.9} \]

The corresponding Euler’s equation takes the form
\[ \lambda \frac{v^2}{2} - \log f + \log(1 + kf) + \mu = 0. \tag{2.10} \]

From the last relation we obtain
\[ f/(1 + kf) = Ce^{-v^2/(2T)}. \tag{2.11} \]

Formula (2.11) implies that
\[ f = M_k = \frac{Ce^{-v^2/(2T)}}{1 - kCe^{-v^2/(2T)}}. \tag{2.12} \]

It is required that the distribution $M_k$ is positive, that is,
\[ C > 0, \quad -\infty < kC < 1, \tag{2.13} \]
and further we assume that (2.13) holds. Moreover, (2.13) yields also the positivity of $1 + kM_k$, and so the both inequalities in (2.1) hold for $M_k$. The constant $C$ is derived from the relation

$$V \Omega \int_\mathbb{R} \frac{Ce^{-v^2/2T}}{1 - kCe^{-v^2/2T}} dv = \rho.$$ (2.14)

Because of (2.2), (2.6) and (2.8), $F$ admits an integral representation:

$$F = \int_\Omega \int_\mathbb{R} \Psi(f) dv dx, \quad \Psi := \lambda v^2 f - \Phi.$$ (2.15)

Thus, in view of (2.1), (2.3), (2.4) and (2.15) we have the inequality

$$\delta^2 \Psi = -\frac{1}{f(1 + kf)} < 0.$$ (2.16)

Therefore, we see that the statement below is proved.

**Proposition 2.1** Under condition (2.16), the functional (2.8) attains its maximum on the function $M_k$, which is given by the formulas (2.12) and (2.14), that is,

$$G(f) = F(M_k) - F(f) > 0, \quad \text{for} \quad f \neq M_k.$$ (2.17)

**Remark 2.1** From (2.17) we see that $G$ can be considered as a generalization of the Kullback-Leibler distance. Hence, $G$ and $F$ are interesting characteristics of the solutions of the Fokker-Planck equation.

### 3. Comparison of the classical and quantum characteristics

1. The value of $C$ in the Maxwellian $M_k$ (see (2.12)) is essential in many considerations. Taking into account (2.14) we derive

$$\sqrt{2\pi T} V \Omega C L_{1/2}(kC) = \rho,$$ (3.1)

where

$$L_{n/2}(z) = \frac{2}{(2T)^{n/2} \Gamma(n/2)} \int_0^\infty \frac{r^{n-1} e^{-r^2/2T}}{1 - kCe^{-r^2/2T}} dr = \sum_{m=1}^{\infty} \frac{z^{m-1}}{m^{n/2}}, \quad \Gamma(1/2) = \sqrt{\pi}.$$ (3.2)

For the classical case $k = 0$, formulas (3.1) and (3.2) imply

$$C = C_0 = \rho / (\sqrt{2\pi TV \Omega}).$$ (3.3)

Using (3.2) we get the statement below.

**Proposition 3.1** The function $L_{1/2}(z)$ monotonically increases for $0 < z < 1$, and

$$L_{1/2}(1) = \infty.$$ (3.4)

In view of Proposition 3.1 we have:

**Corollary 3.1** If $k > 0$ (boson case), then equation (3.1) has one and only one solution $C$ such that $C > 0$ and $kC < 1$.

Next, we consider the fermion case, that is, the case that $k < 0$. 

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Proposition 3.2 If \( k < 0 \), then the function \( CL_{1/2}(kC) \) monotonically increases with respect to \( C > 0 \). Furthermore, we have \( CL_{1/2}(kC) \rightarrow \infty \) for \( C \rightarrow \infty \).

Corollary 3.2 If \( k < 0 \) (fermion case) then equation (3.1) has one and only one solution \( C \) such that \( C > 0 \).

Consider now the energy for the global Maxwellian \( M_k \):

\[
E_k = E(M_k) = \int_{\Omega} \int_{\mathbb{R}} \frac{v^2 C e^{-v^2/(2T)}}{1 - k C e^{-v^2/(2T)}} dv dx /2 = V_\Omega C \int_0^\infty \frac{r^2 C e^{-r^2/(2T)}}{1 - k C e^{-r^2/(2T)}} dr.
\]

Because of (2.13), the inequality \( kC < 1 \) is valid and the energy is well-defined. Formulas (3.1), (3.2) and (3.5) imply that

\[
E_k = \rho TL_{3/2}(kC) / 2L_{1/2}(kC), \quad E_0 = E_c = \rho T / 2.
\]

Proposition 3.3 If \( k > 0 \) (boson case), we have

\[
E_k < E_c.
\]

Proof. Taking into account (3.2), we derive \( L_{3/2}(kC)/L_{1/2}(kC) < 1 \) for \( k > 0 \). Hence, in view of (3.6) the inequality (3.7) holds.

2. To calculate the entropy \( S(M_k, k) \) we recall (2.12) and use equalities

\[
M_k = g/(1 - kg), \quad 1 + kM_k = (1 - kg)^{-1}, \quad g := Ce^{-v^2/(2T)}
\]

to simplify the expression \( \Phi(f) \) from (2.2):

\[
\Phi(M_k) = M_k(1 + \log g) + (1/k) \log(1 - kg) \quad \text{for} \quad k \neq 0,
\]

\[
\Phi(M_k) = M_k \log g \quad \text{for} \quad k = 0, \quad M_0 = g.
\]

Substitute \( \log g = \log C - (1/2T)v^2 \) into (3.9) or (3.10), substitute (3.9) or (3.10), respectively, into (2.2) and use definitions (2.5) and (2.6) of \( \rho \) and \( E \) to get

\[
S(M_k, k) = \frac{1}{T} E_k - (1 + \log C) \rho - \frac{1}{k} \int \log(1 - kg) dv
\]

\[
\text{for} \quad k \neq 0,
\]

\[
S(M_k, k) = S_c = \frac{1}{T} E_c - \rho \log C_0 \quad \text{for} \quad k = 0 \quad (E_c = E_0).
\]

Using again the definition (2.6) of energy and integrating by parts, we rewrite (3.11):

\[
S(M_k, k) = \frac{1}{T} E_k - (1 + \log C) \rho + \frac{2E_k}{T} \quad \text{for} \quad k \neq 0.
\]

From (3.12), (3.13) and the second relation in (3.6), we see that

\[
S(M_k, k) - S_c = \frac{3}{T} (E_k - E_c) - \rho \log(C/C_0).
\]

Recall also (2.8) to derive from (3.14) that

\[
F_k - F_c = \frac{2}{T} (E_k - E_c) - \rho \log(C/C_0) \quad (F_c = F_0 = F(M_0)).
\]
Lemma 3.1 The constant $C$ is bounded for the small values of $|k|$.

Proof. According to (3.2) we have $1 = L_{1/2}(0) < L_{1/2}(kr)$ for $k > 0$, $r > 0$, and so the inequality $C_0 < C_0 L_{1/2}(kr)$ is immediate. Hence, in view of (3.1), (3.3) and Proposition 3.1 the function $h(r) = \sqrt{2\pi TV_{\Omega}} L_{1/2}(kr)$ is increasing and satisfies relations

$$h(C) = \rho, \quad h(C_0) > \rho, \quad \text{i.e.,} \quad C < C_0 \quad \text{for} \quad k > 0.$$ 

Next, let $-2C_0^{-1} < k < 0$. Then, formula (3.2) yields

$$2L_{1/2}(2kC_0) > 2L_{1/2}(-1) > L_{1/2}(0) = 1.$$ 

Therefore, we have $2C_0 L_{1/2}(2kC_0) > C_0$, which, in view of (3.1), (3.3) and Proposition 3.2, implies $C < 2C_0$. □

From (3.1) and (3.3) we see that

$$z = kC_0/L_{1/2}(z), \quad z := kC.$$ 

(3.16)

We note that

$$\left| \frac{d}{dz} \left( \frac{kC_0}{L_{1/2}(z)} \right) \right| < 1 \quad \text{for} \quad |z| < 1$$

and small values of $|k|$. Since $C$ is bounded, it follows that $|z| < 1/2$ for the sufficiently small values of $k$. Thus, we apply the iteration method to the equation (3.16) and derive

$$C = C_0 + O(k), \quad k \to 0.$$ 

(3.17)

Taking into account (3.16), (3.17) and the series expansion (3.2) we get

$$C/C_0 = 1/L_{1/2}(kC) = 1 - (kC_0)/\sqrt{2} + O(k^2).$$ 

(3.18)

Moreover, from (3.18) we see that

$$\log(C/C_0) = -kC_0/\sqrt{2} + O(k^2).$$ 

(3.19)

Using relations (3.2), (3.6) and (3.17), we derive

$$E_k - E_c = -\frac{k\rho TC_0}{2^{5/2}} + O(k^2), \quad k \to 0.$$ 

(3.20)

Because of (3.14), (3.15), (3.19) and (3.20), we get the next proposition.

Proposition 3.4 For $k \to 0$, we have equality (3.20) as well as the equalities below:

$$S(M_k, k) - S_c = \frac{k\rho C_0}{2^{5/2}} + O(k^2),$$ 

(3.21)

$$F_k - F_c = \frac{k\rho C_0}{2^{3/2}} + O(k^2).$$ 

(3.22)

Corollary 3.3 Let $k_1 < 0 < k_2$ be small. Then

$$S(M_{k_1}, k_1) < S_c < S(M_{k_2}, k_2), \quad F_{k_1} < F_c < F_{k_2}.$$ 

(3.23)
4. Conclusion
The "free energy" functional \( F \) on the solutions of FPE was studied. We see that the quantum effect for the both entropy and "free energy" differs in sign for bosons and fermions for the case of one space variable. It would be of interest to check the case of several space variables as well as to apply the distance \( G(f) \) generated by the "free energy" functional (see Proposition 2.1) to the study of asymptotics of the solutions of FPE.

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References
[1] Carrillo J A, Rosado J and Salvarani F 2008 1D nonlinear Fokker-Planck equations for fermions and bosons Appl. Math. Lett. 21 148
[2] Dolbeault J 1994 Kinetic models and quantum effects: A modified Boltzmann equation for Fermi-Dirac particles Arch. Ration. Mech. Anal. 127 101
[3] Frank T D 2005 Nonlinear Fokker-Planck equations (Berlin: Springer)
[4] Kullback S and Leibler R A 1951 On information and sufficiency Ann. Math. Stat. 22 79
[5] Sakhnovich L A 2011 Comparison of Thermodynamic Characteristics in the Ordinary Quantum and Classical Approaches Physica A 390 3679
[6] Sakhnovich L A 2011 The Boltzmann equation and corresponding extremal problems Preprint arXiv:1106.3254.
[7] Villani C 2005 XIVth international congress on mathematical physics Selected papers ed J-C Zambrini (Hackensack, N.J: World Scientific) pp 130-144