Abstract

In the first part of this article we show for some examples of surfaces of general type in toric 3-folds how to construct minimal and canonical models by toric methods explicitly. The examples we study turn out to be surfaces of general type, namely so called Kanev surfaces and Todorov surfaces. We show how properties of our examples of surfaces could be derived directly from properties of some polytopes and we compute the singularities of their canonical models.

1 Introduction

In this article we study some surfaces of general type that arise as hypersurfaces in toric 3-folds. This article illustrates results from the articles ([Bat20] and [Gie21]) via concrete examples.

We start with some lattice polytope $\Delta$ and a Laurent polynomial $f$ which has $\Delta$ as its Newton polytope and is nondegenerate with respect to $\Delta$. For this we take $\Delta$ to be a 3-dimensional canonical Fano polytope, that is $\Delta$ contains just $(0,0,0)$ as an interior lattice point and the vertices of $\Delta$ are primitive lattice vectors. Additionally since there are still quite a lot of such polytopes we restrict to those with

$$\dim F(\Delta) = 3,$$

where $F(\Delta)$ denotes the Fine interior of $\Delta$ (see Definition 2.5). $F(\Delta)$ is a rational polytope that can be constructed from the Newton polytope $\Delta$. The restriction to $\dim F(\Delta) = 3$ has the effect that if we let $Z_f := \{ f = 0 \}$ \subset
$(\mathbb{C}^*)^3$, then $Z_f$ is birational to a surface of general type.
There are 49 canonical Fano polytopes $\Delta$ with $\dim F(\Delta) = 3$. Then we ask how to find a minimal model of $Z_f$, that is a smooth projective surface $Y$, which is birational to $Z_f$ and with $K_Y$ nef.
For this we recall in section 2 the Definition of the Fine interior $F(\Delta)$, the canonical closure $C(\Delta)$, the Minkowski sum

$$\tilde{\Delta} := C(\Delta) + F(\Delta)$$

and of a simplicial fan $\Sigma$ refining the normal fan of $\Sigma_{\Delta}$. These constructions are necessary in order to specify the toric variety in which a minimal model of the surface $Z_f$ is contained. To be more precise we take the projective toric varieties to the normal fans of $F(\Delta)$, $C(\Delta)$ and $\tilde{\Delta}$ as well as the projective toric variety to the fan $\Sigma$ and call them $\mathbb{P}_{F(\Delta)}$, $\mathbb{P}_{C(\Delta)}$, $\mathbb{P}_{\tilde{\Delta}}$ and $\mathbb{P}_{\Sigma}$. Then we obtain a diagram

$$\begin{array}{c}
\mathbb{P}_{\Sigma} \\
\pi \\
\mathbb{P}_{\tilde{\Delta}} \\
\rho \end{array} \quad \begin{array}{c}
\mathbb{P}_{C(\Delta)} \\
\theta \\
\mathbb{P}_{F(\Delta)}
\end{array}$$

of birational toric morphisms, where $\theta$ is birational since we assume $\dim F(\Delta) = 3$. Let $Z_{\Sigma}$ denote the closure of $Z_f$ in $\mathbb{P}_{\Sigma}$, then $Z_{\Sigma}$ gets a minimal surface of general type and of the closure $Z_{F(\Delta)}$ of $Z_f$ in $\mathbb{P}_{F(\Delta)}$ gets a canonical model of $Z_{\Sigma}$. Since we always assume $f$ to be nondegenerate with respect to its Newton polytope $\Delta$ we are able to compute the singularities of $Z_{F(\Delta)}$ and of the closure $Z_{\tilde{\Delta}} \subset \mathbb{P}_{\tilde{\Delta}}$ of $Z_f$ at least if $\Delta$ is canonically closed, that is $\Delta = C(\Delta)$.

Apart from just computing the singularities we also illustrate these results as well as the 49 polytopes in some pictures. Besides we provide some combinatorial classification of these 49 polytopes, that is we bring them onto a normal form and divide them into 5 classes $a), b), c), d)$ and $e)$ which could be studied separately.
In section 4 we study invariants of the resulting minimal surfaces and in this way identify our surfaces with so called Kanev surfaces in the cases $a)$ and $b)$ and Todorov surfaces in the cases $c), d)$ and $e)$.
Kanev surfaces (also known as Kunev or Kynev surfaces) have invariants
\[ p_g(Y) = 1, \quad K_Y^2 = 1, \]
and were studied in ([Cat78], [Us87], [Tod80]). Todorov surfaces have invariants
\[ p_g(Y) = 1, \quad q(Y) = 0, \quad K_Y^2 = 2 \]
and were studied in ([CD89]). Both of these surfaces became interesting from the point of view of Torelli type Theorems: Some of them for example fail the infinitesimal Torelli Theorem.

We remark that the plurigenera of \( Z_\Sigma \) could be calculated via 2 different methods: Either via general formulas for plurigenera of minimal algebraic surfaces of general type or by counting lattice points and interior lattice points on \( F(\Delta) \). It is known that for \( Y \) a Kanev or Todorov surface \( 2K_Y \) is basepointfree ([Cat78], [CD89]). If \( Y = Z_\Sigma \) lies in the toric 3-fold \( \mathbb{P}_\Sigma \) this is gotten for free by the adjunction formula and an argument using a resolution of singularities of \( \mathbb{P}_\Sigma \) (see [Gie21, Prop.6.2]). We further deal with Kanev’s original example of a Kanev surface.

2 Combinatorial classification of 49 canonical Fano polytopes \( \Delta \) with \( \dim F(\Delta) = 3 \)

2.1 General background and the Fine interior

Notation (Toric setting):

- \( M \cong \mathbb{Z}^3 \): A 3-dimensional lattice with dual lattice \( N, T := \text{Hom}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^3 \) the torus. We identify \( M \) with the lattice of characters of \( T \).
- \( \Delta \subset M_\mathbb{R} := M \otimes \mathbb{R} \): A 3-dimensional lattice polytope.
- \( \Sigma_\Delta \): The normal fan to the polytope \( \Delta \) and more generally \( \Sigma_F \) denotes the normal fan to a rational polytope \( F \subset M_\mathbb{R} \).
- For \( \Sigma \) a fan in \( N_\mathbb{R} \) let \( \Sigma[i] \) denote the set of \( i \)-dimensional cones of \( \Sigma \) or for \( i = 1 \) also the set of generators of the 1-dimensional cones.
- \( \mathbb{P}_\Delta \): The toric variety to the polytope \( \Delta \), defined via its normal fan, more generally \( \mathbb{P}_\Sigma \): The toric variety to a fan \( \Sigma \).
- \( \langle v_0, ..., v_n \rangle \): The convex span of the lattice points \( v_0, ..., v_n \).
Recall that to a ray $\nu_i \in \Sigma_{\Delta}[1]$ is associated a torus invariant divisor $D_i$ and that the canonical divisor of $\mathbb{P}_\Delta$ is given by

$$K_{\mathbb{P}_\Delta} = - \sum_{\nu_i \in \Sigma_{\Delta}[1]} D_i.$$  

(2)

In this article we just deal with integral divisors. We are mainly interested in hypersurfaces in toric varieties and for this we take $\Delta$ to be the Newton polytope of a Laurent polynomial $f$:

**Definition 2.1.** Let $f$ be a Laurent polynomial with presentation

$$f = \sum_{m \in A} a_m z^m, \quad a_i \in \mathbb{C}$$

(3)

for some finite set $A \subset M$. The convex span of the $m \in A$ with $a_m \neq 0$ is called the Newton polytope of $f$.

$L(\Delta)$ denotes the set of Laurent polynomials $f$ as in (3) with Newton polytope $\Delta$ and

$$l(\Delta) := \dim\mathbb{C} L(\Delta) = |M \cap \Delta|$$

the number of lattice points in $\Delta$. Let $l^*(\Delta)$ be the number of interior lattice points of $\Delta$ and $Z_f$ the zero set $\{f = 0\} \subset T$.

**Definition 2.2.** In the above situation $f$ is called $\Delta$-regular or nondegenerate with respect to $\Delta$ if $Z_f$ is smooth and for every face $\Gamma \subset \Delta$ the variety $Z_f|_\Gamma$ is smooth as well, where

$$f|_\Gamma := \sum_{m \in \Gamma \cap M} a_m z^m.$$ 

For $f \in L(\Delta)$ we denote the closure of $Z_f$ in the toric variety $\mathbb{P}_\Delta$ by $Z_\Delta$. The nondegeneracy means that $Z_\Delta$ intersects the toric strata of $\mathbb{P}_\Delta$ transversally in a subset of codimension 1. For $f$ nondegenerate we sometimes also call $Z_\Delta$ nondegenerate. Given a complete 3-dimensional fan $\Sigma$ we write $Z_{\Sigma}$ for the closure of $Z_f$ in $\mathbb{P}_\Sigma$ and if $\Sigma = \Sigma_F$ is the normal fan of some rational polytope $F$ we write $Z_F$ for $Z_{\Sigma_F}$. We call $Z_{\Sigma}$ nondegenerate (with respect to $\Sigma$) if it intersects the toric strata of $\mathbb{P}_\Sigma$ transversally.
Construction 2.3. For $\nu \in N$ let

$$\text{ord}_\Delta(\nu) := \min_{m \in \Delta \cap M} \langle m, \nu \rangle,$$

then $\Delta$ has a facet presentation

$$\Delta = \{ x \in \mathbb{R}^n \mid \langle x, \nu_i \rangle \geq \text{ord}_\Delta(\nu_i) \}.$$  \hfill (4)

with $\nu_1, \ldots, \nu_s \in \Sigma_\Delta[1]$ the inner facet normals. By ([Bat20, Prop.5.1]) $Z_\Delta$ is linear equivalent to the following torus invariant divisor

$$Z_\Delta \sim_{\text{lin}} \sum_{\nu_i \in \Sigma_\Delta[1]} \text{ord}_\Delta(\nu_i) D_i.$$  \hfill (5)

Definition 2.4. A lattice polytope $\Delta \subset \mathbb{R}^n$ is said to be

- canonical, if $l^*(\Delta) = 1$,
- Fano, if the vertices of $\Delta$ are primitive lattice vectors.

By the classification in ([Kas10]) there are still 674,688 canonical Fano 3-topes, but these polytopes could be further divided into classes in terms of their Fine interior:

Definition 2.5. ([Rei87, Appendix to §4])

The Fine interior of $\Delta$ is defined as

$$F(\Delta) := \{ x \in \mathbb{R}^n \mid \langle x, \nu \rangle \geq \text{ord}_F(\nu) + 1, \nu \in N \setminus \{0\} \}$$

Remark 2.6. In order to construct the Fine interior $F(\Delta)$ of $\Delta$ we have to move every hyperplane which touches some face of $\Delta$ one into the interior of $\Delta$. In general it is not enough to move just the hyperplanes defining facets one into the interior (see Figure 1). However finitely many hyperplanes will always be enough. In fact define the support $S_F(\Delta)$ of $F(\Delta)$ to $\Delta$ as follows:

Definition 2.7. The set of lattice points $\nu \in N \setminus \{0\}$ with

$$\text{ord}_{F(\Delta)}(\nu) = \text{ord}_F(\nu) + 1$$

is called the support of $F(\Delta)$ to $\Delta$ and is denoted by $S_F(\Delta)$. Then we have
Figure 1: Illustration of the construction of the Fine interior $F(\Delta)$ from $\Delta$.

**Proposition 2.8.** ([Bat20, Prop.1.11])

$$S_F(\Delta) \subset \langle \nu_i | \nu_i \in \Sigma[1] \rangle$$

In particular there are only finitely many hyperplanes touching $\Delta$ and after moving into the interior also touching $F(\Delta)$.

**Remark 2.9.** For $n = 2$ the Fine interior of $\Delta$ always equals the convex span of the interior lattice points of $\Delta$, but for $n \geq 3$ the Fine interior $F(\Delta)$ is in general only a rational polytope, i.e. the vertices have rational coordinates ([Bat17, Prop.2.9, Rem. 2.10]).

**Definition 2.10.** ([Bat20, Def.1.13])

The polytope

$$C(\Delta) := \{ x \in M_{\mathbb{R}} | \langle x, \nu \rangle \geq \text{ord}_\Delta(\nu) \quad \forall \nu \in S_F(\Delta) \}$$

is called the canonical closure of $\Delta$. We call $\Delta$ canonically closed if $C(\Delta) = \Delta$.

**Remark 2.11.** Certainly the canonical closure $C(\Delta)$ contains $\Delta$ and for $\dim(\Delta) = 2$ with $F(\Delta) \neq \emptyset$ we even have equality $C(\Delta) = \Delta$ (see [Bat20, Prop.2.4]). In general we have ([Bat20, Prop.1.17(b), Cor.1.19]):

$$F(C(\Delta)) = F(\Delta), \quad C(C(\Delta)) = C(\Delta), \quad (6)$$

that is $C(\Delta)$ is canonically closed and still has the same Fine interior as $\Delta$.

**Construction 2.12.** We restrict in this article to canonical Fano 3-topes $\Delta$ with

$$\dim F(\Delta) = 3.$$ 

There are just 49 such polytopes left, listed in ([Sch18, Appendix A.3]) and they share the following properties:
• If $\Delta$ is canonically closed, then every facet of $\Delta$ has distance 1 to the origin (=unique interior lattice point) except from one facet, which we call $\Delta_{can}$ with inner facet normal $\nu_{can}$, and which has distance 2 to the origin, that is

$$\Delta = \{ x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle \geq -1 \quad \nu_i \in \Sigma[1] \setminus \nu_{can}, \langle x, \nu_{can} \rangle \geq -2 \}$$

• There are 46 polytopes $\Delta$ with $l^*(\Delta_{can}) = 2$ and 3 polytopes with $l^*(\Delta_{can}) = 3$.

Thus $F(\Delta)$ has the origin as a vertex and exactly one facet, which we call $F(\Delta)_{can}$ opposite to $(0, 0, 0)$. For $\Delta \neq C(\Delta)$ there might exist more than one facet having distance 2 to the origin (see for example the facet $\langle a_2, b, d \rangle$ in Table 1 ID:547524).

It follows from the first point that if $\Delta = C(\Delta)$ then

$$K_{Z_{\Delta}} = (Z_{\Delta} + K_{P_{\Delta}})|_{Z_{\Delta}} = D_{can}|_{Z_{\Delta}}$$

could be identified with a curve on $D_{can}$ with Newton polytope $\Delta_{can}$. This curve is smooth for $f$ sufficiently nondegenerate.

### 2.2 Further classification of the 49 polytopes

We brought the 49 polytopes onto a normal form and showed by observation the following:

**Proposition 2.13.** Among the 49 polytopes there are just 5 isomorphy types for the Fine interior. For all 49 examples

$$2 \cdot F(\Delta)_{can} := \{ 2 \cdot t | t \in F(\Delta)_{can} \}$$

is inscribed into $\Delta_{can}$ with the same shape as $\Delta_{can}$ (see the pictures below). Thus the 5 different types of $F(\Delta)$ correspond to 5 different types of $\Delta_{can}$.

For the first 46 polytopes $\Delta$ with $l^*(\Delta_{can}) = 2$ there are two types of $\Delta_{can}$: For 20 polytopes the facet $\Delta_{can}$ looks like in picture a) and for 26 polytopes the facet looks like in picture b). We list these polytopes in the Tables 1 and 2.
For the remaining 3 polytopes with $l^*(\Delta) = 3$ the facet $\Delta_{\text{can}}$ looks like in the pictures c), d) and e).

This result suggests to consider all polytopes $\Delta$ among the 49 with given $F(\Delta)$ at once. We found by observation:

**Proposition 2.14.** In each of the 5 classes there is exactly one maximal polytope with respect to the inclusion of sets.

**Remark 2.15.** For $\Delta$ one of the maximal polytopes up to translation $\Delta$ coincides with $6 \cdot F(\Delta)$ in the cases a) an b) and with $4 \cdot F(\Delta)$ in the cases c), d) and e).

We picture the polytopes in the classes a) and b) in Figures 2 and 3.

The following construction appeared first in ([Bat20, Cor.4.4]):

**Construction 2.16.** Let $\Delta \subset M_\mathbb{R}$ be a 3-dimensional lattice polytope with Fine interior $F(\Delta) \neq \emptyset$. Let

$$\tilde{\Delta} := C(\Delta) + F(\Delta)$$
be the Minkowski sum. In this way the normal fan $\Sigma_\Delta$ gets the coarsest refinement of $\Sigma_{C(\Delta)}$ and $\Sigma_{F(\Delta)}$, even if $F(\Delta)$ is not full dimensional ([CLST1, Prop.6.2.13]) and thus we get morphisms

$$
\begin{array}{ccc}
\mathbb{P}_\Delta & \xrightarrow{\rho} & \mathbb{P}_{C(\Delta)} \\
\downarrow \theta & \quad & \downarrow \\
\mathbb{P}_{F(\Delta)} & \xleftarrow{\rho} & \mathbb{P}_\Delta
\end{array}
$$

where $\rho$ is birational and $\theta$ is birational if $\dim F(\Delta) = 3$. We prove by observation

**Proposition 2.17.** For $\Delta$ out of the 49 polytopes $\Sigma_{C(\Delta)}[1] = \Sigma_\Delta[1]$, i.e. $\rho : \mathbb{P}_\Delta \to \mathbb{P}_{C(\Delta)}$ is an isomorphism in codimension 1 and thus $Z_{C(\Delta)} \cong Z_\Delta$.

**Proposition 2.18.** For $\Delta$ one of the 5 maximal lattice polytopes among the 49 polytopes we have $\Delta = C(\Delta)$ and

$$\mathbb{P}_{F(\Delta)} \cong \mathbb{P}_\Delta \cong \mathbb{P}_{C(\Delta)}.$$

**Proof.** The reason for the latter is that by Remark 2.15 the polytopes $\Delta$ and $F(\Delta)$ have the same normal fan. \hfill $\square$

**Theorem 2.19.** [Bat20, Thm.4.3]
We have $\Sigma_\Delta[1] \subset S_F(\Delta)$. In particular we may choose a simplicial fan $\Sigma$ with $\Sigma[1] = S_F(\Delta)$ refining $\Sigma_\Delta$.

3 Construction of some minimal/canonical surfaces of general type in toric 3-folds

3.1 A toric framework for constructing minimal and canonical models of hypersurfaces

In this section we summarize some results from ([Bat20]). The results of this section stay true in any dimension. In order to speak about minimal (canonical) models, we should give here a Definition of terminal (canonical) singularities of algebraic varieties. But we will just need this notions for toric varieties (in fact toric 3-folds) and algebraic surfaces. In these two cases the situation could be simplified:
Figure 2: The 11 canonically closed polytopes out of 20 polytopes in the first class a). The polytopes are ordered in rows descendingly by their number of lattice points (The maximal polytope is additionally put in the first row on the left).
Figure 3: The 15 canonically closed polytopes out of 26 polytopes in the second class b) with the same convention on the rows as in case a) and also with the maximal polytope put in the first row on the left.
Definition 3.1. \cite{Rei83} (1.11)
Consider a fan $\Sigma$ in $\mathbb{N}_R$ and a cone $\sigma$ of $\Sigma$. Then $\sigma$ is called canonical (of index $j \in \mathbb{N}$), if there exists a primitive vector $m \in M$ such that
\[ \langle m, \nu_i \rangle = j \quad \text{for } \nu_i \in \sigma[1] \]
and
\[ \langle m, n \rangle \geq j \quad \text{for } n \in \sigma \cap N, n \notin \{0\} \cup \sigma[1] \quad (8) \]
$\sigma$ is called terminal if we have strict inequality in (8). The fan $\Sigma$ is called canonical (terminal) if all its cones are canonical (terminal).

According to \cite{Rei83} (1.12) we have the following result:

Theorem 3.2. The toric variety $\mathbb{P}_\Sigma$ has at most canonical (terminal) singularities if and only if the fan $\Sigma$ is canonical (terminal).

For normal algebraic surfaces we have

Proposition 3.3. \cite[Thm.4.5]{KM98}
A normal algebraic surface $Y$ has canonical (terminal) singularities if and only if it has at most rational double points (is smooth). We choose the abbreviation R.D.P. for rational double points.

Definition 3.4. A (partial) resolution of singularities $\pi : X \rightarrow Y$ of normal varieties is called crepant if
\[ K_X = \pi^*(K_Y) . \]
Given a normal projective surface $Y$, we write $\kappa(Y)$ for the Kodaira dimension of $Y$, which by Definition is the Kodaira dimension of a resolution of singularities of $Y$.

Given a normal algebraic surface $Y$ with $\kappa(Y) \geq 0$ a smooth projective surface $Y_{\text{min}}$ birational to $Y$ and with $K_{Y_{\text{min}}}$. nef is called a minimal model of $Y$.

If $\kappa(Y) = 2$ a projective surface $Y_{\text{can}}$ birational to $Y$ with at most R.D.P. and with $K_{Y_{\text{can}}}$. ample is called a canonical model of $Y$.

Remark 3.5. If $Y$ is a surface with $\kappa(Y) \geq 0$ there exists up to isomorphism a unique minimal model of $Y$ and if $\kappa(Y) = 2$ there exists up to isomorphism a unique canonical model of $Y$. In the latter case there is a birational morphism $\phi : Y_{\text{min}} \rightarrow Y_{\text{can}}$ from the minimal model $Y_{\text{min}}$ to the canonical model $Y_{\text{can}}$, given by contracting every curve $C$ with $K_Y.C = 0$ \cite[Ch.7, Cor.2.3]{BHPV04}. By a minimal or canonical model of $Z_f$ we mean a minimal or canonical model of $Z_\Delta$. 

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Theorem 3.6. ([Bat20, Thm.6.2])
Given a 3-dimensional lattice polytope $\Delta \subset M_\mathbb{R}$ with $k := \dim(F(\Delta)) \geq 0$, the Kodaira dimension of $Z_\Delta$ equals
\[ \kappa(Z_\Delta) = \min(k, 2). \]

For nondegenerate toric hypersurfaces these birational models could be constructed explicitly:

Theorem 3.7.
The toric variety $P_\Delta$ has at most canonical singularities and $P_\Sigma$ has at most terminal singularities.

Theorem 3.8. ([Bat20, Thm.5.4], [Gie21, Thm.6.3])
The closure $Z_\Sigma$ of $Z_f$ gets a minimal model of $Z_f$ and if $\dim \Delta = \dim F(\Delta) = 3$ the closure $Z_{F(\Delta)}$ of $Z_f$ gets a canonical model of $Z_f$ and the morphism $Z_\Sigma \to Z_{F(\Delta)}$ coincides with the birational morphism $\phi$ from the minimal to the canonical model.

Remark 3.9. Thus for our 49 polytopes $\Delta$ with $\dim F(\Delta) = 3$ we have $\kappa(Z_\Delta) = 2$. By properties of canonical models of surfaces of general type the morphism $Z_\Sigma \to Z_{F(\Delta)}$ is crepant.

3.2 The singularities of $Z_\Delta$ and $Z_{F(\Delta)}$ for $\Delta$ canonically closed

Construction 3.10. (The singularities of $Z_\Delta$)
The hypersurface $Z_\Delta$ has, just as the other birational models, only isolated singularities and, assuming $\Delta = C(\Delta)$, these come from the one-dimensional singular locus of $P_\Delta$ intersected with $Z_\Delta$. For this we use that if $\Delta = C(\Delta)$ by nondegeneracy $Z_\Delta$ does not pass through the torus fixed points of $P_\Delta$ ([Tre10 Prop.5.1.3]).

Let $\tau \in \Sigma_\Delta[2]$ with associated orbit $O_\tau \cong \mathbb{C}^*$ be such that $O_\tau$ is contained in the singular locus of $P_\Delta$. Locally around $x \in O_\tau \cap Z_\Delta$ the toric variety $P_\Delta$ looks like $O_\tau \times P$ where $P$ is a normal 2-dimensional toric variety (compare ([Bat94])).

Then since $Z_\Delta$ intersects $O_\tau$ transversally, it has the same singularity at $x$ as $P$. But a normal toric variety has at most $A_k$-singularities ([CLSI11 Ch.10]).
Note that by ([Bat94]) $P$ is gotten by using the minimal 2-dimensional sub-lattice $N(\tau) \subset N$ containing $\tau \cap N$ and taking the affine toric variety to the cone $\tau \subset N(\tau)_\mathbb{R}$. Concretely if $\tau$ has generators $\nu_i$ and $\nu_j$, then writing

$$\nu_i - \nu_j = k \cdot \text{(primitive vector)}$$

$P$ has a singularity of type $A_{k-1}$ at $x \in O_\tau$. If $\tau$ corresponds to an edge $\Gamma = \langle s, t \rangle$ of $C(\Delta)$ then $Z_\Delta$ which by Proposition 2.17 is isomorphic to $Z_{C(\Delta)}$ intersects $O_\tau$ in $l$ points, where

$$s - t = l \cdot \text{(primitive vector)}.$$ 

Else $O_\tau$ contracts to a torus fixed point on $\mathbb{P}_{C(\Delta)}$ and $Z_\Delta$ does not intersect $O_\tau$.

**Construction 3.11. (The singularities of $Z_{F(\Delta)}$)**

We make the observation that in all of our examples the rays of $\Sigma$ which do not belong to $\Sigma_{F(\Delta)}[1]$ lie on the boundary on in the interior of only one 3-dimensional cone, which we call $\sigma$.

This allows us to illustrate the situation with some pictures, by taking a cross-section through this cone $\sigma$ (see Figure 4). Here between $\Sigma_\Delta$ and $\Sigma$ we picture only those rays which lie on an edge in the cross-section and skip those refining a 3-dimensional cone of $\Sigma_\Delta$. For if $\Delta = C(\Delta)$ then $Z_\Delta$ is nondegenerate with respect to $\Delta$, thus it does not pass through the torus fixed points of $\mathbb{P}_\Delta$.

Any additional ray $\rho$, that is any point in the cross-section, introduces a toric divisor $E_\rho$ on $\mathbb{P}_\Sigma$. This divisor $E_\rho$ intersects $Z_\Sigma$ in finitely many $(-2)$-curves and if $\rho$ refines the interior of $\sigma$, then $E_\rho \cap Z_\Sigma$ gets contracted to the torus fixed point $p$ corresponding to $\sigma$. By choosing a connected neighborhood $U$ of $p$ such that $p$ is a deformation retract of $U$ we get that $E_\rho \cap Z_\Sigma$ is a deformation retract of the preimage of $U$, thus it is connected and consists of just one $(-2)$-curve.

In this way we see that the diagram with vertices the points in the interior of $\sigma$ in the Figures 5 and 6 constitutes the Dynkin diagram of the singularity of $Z_{F(\Delta)}$ at the torus fixed point to $\sigma$. In these figures we use the following notations.

In a) :

$n_1 := n_{pab} = (-1, 3, 1)$, $n_2 := n_{pad} = (2, -3, 1)$, $n_3 := n_{pbd} = (0, 0, -1)$.

In b) :
\[ n_1 := n_{pab} = (2, 1, -2), n_2 := n_{pbc} = (0, -1, -1), n_3 := n_{pcd} = (-1, -1, 1), \]
\[ n_4 := n_{pad} = (0, 1, 2), \]
where for example \( n_{pab} \) is the inner facet normal to the facet \( \langle p, a, b \rangle \).

**Figure 4:** On the left \( \sigma \) in class \( a \) is pictured and on the right a refinement of \( \sigma \) as a cone is pictured, that is the added points are just rational multiples of the cone generators. This refinement shows \( \Sigma_{\Delta} \).

**Example 3.12.** Consider the polytope \( \langle a, b, d, b_1, c_2, d_1 \rangle \) in the class \( a \) (in the third row the third from the left): The orbit to the cone spanned by \( n_1 \) and \( n_2 \) corresponds to the edge \( \langle a, d \rangle \) and does not come from an edge of \( C(\Delta) \), therefore we omit the two lattice points between \( n_1 \) and \( n_2 \) in the Figure 5. The cones corresponding to the edges \( \langle a, c_2 \rangle, \langle b_1, c_2 \rangle, \langle c_2, d_1 \rangle \) yield \( A_2 \) singularities. Since \( a - c_2, b_1 - c_2, c_2 - d_1 \) are primitive lattice vectors in total we get three \( A_2 \) singularities. On \( Z_{F(\Delta)} \) we get one singularity of type \( A_8 \).

**Remark 3.13.** Note there are examples in the list where \( Z_{F(\Delta)} \) has a singularity of type \( D_5, D_7 \) or \( E_6 \). If \( \Delta \) is not canonically closed \( \Delta \) has the same support vectors as its canonical closure ([Bat94, Cor.1.18]). Thus all support vectors lie again on the cone \( \sigma \) and this might help in determining the singularities.

### 4 Kanev and Todorov surfaces

**Notation:** Let \( Y \) be a complex algebraic surface with at most R.D.P., let
Figure 5: A cross-section of the cone $\sigma$ for all canonically closed polytopes in a) with the refinement $\Sigma_{\Delta}$ and the position of the additional vectors from $S_F(\Delta)$ lying on an edge in the cross-section and yielding singularities of $Z_{\Delta}$. The order is the same as in Figure 2.
Figure 6: A cross-section of the cone $\sigma$ for all canonically closed polytopes in the class $b$) with the same convention as in Figure 5 and the same order as in Figure 3.
• \( p_g(Y) := h^0(Y, \mathcal{O}_Y(K_Y)) \) the geometric genus of \( Y \)
• \( q(Y) := h^0(Y, \Omega^1_Y) \) the irregularity of \( Y \)
• \( \chi(Y, \mathcal{O}_Y) \): The euler characteristic of the structure sheaf
• \( e(Y) \): The topological euler number.

**Construction 4.1.** By ([Gie21, Thm.4.2, Prop.5.13]) we have for our 49 examples
\[
p_g(Z_\Sigma) = l(F(\Delta)) = l^*(\Delta) = 1,
\]
and \( q(Z_\Sigma) = 0 \). In ([Sch18, Appendix 3]) the euler number of \( Y := Z_\Sigma \) has already been computed for all of the 49 polytopes. Namely \( e(Y) = 23 \) in \( a) \) and \( b) \) and \( e(Y) = 22 \) in \( c) \), \( d) \) and \( e) \). Then we may deduce from Noether’s formula ([BHPV04, Ch.1, Thm.(5.5)])
\[
2 = 1 - q(Y) + p_g(Y) = \chi(Y, \mathcal{O}_Y) = \frac{1}{12}(K^2_Y + e(Y))
\]
that \( K^2_Y = 1 \) or \( K^2_Y = 2 \). Minimal surfaces \( Y \) with
\[
p_g(Y) = 1, \quad q(Y) = 0, \quad K^2_Y = 1,
\]
are known as **Kanev surfaces** (compare ([Cat78])), whereas minimal surfaces \( Y \) with
\[
p_g(Y) = 1, \quad q(Y) = 0, \quad K^2_Y = 2,
\]
are known as **Todorov surfaces**.

We obtain

**Theorem 4.2.** Let \( \Delta \) be a polytope out of the 49 examples. Then \( Y := Z_\Sigma \) gets a Kanev surface in \( a) \) and \( b) \) and a Todorov surface in \( c) \), \( d) \) and \( e) \).

\[ \square \]

**Remark 4.3.** Note that \( K^2_Y = 2 \) if and only if \( l^*(\Delta_{can}) = 3 \) and else \( K^2_Y = 1 \) and \( l^*(\Delta_{can}) = 2 \). This is no accident since by the adjunction formula
\[
2p_g(K_{Z_\Delta}) - 2 = 2 \cdot K^2_{Z_\Delta} \Rightarrow K^2_{Z_\Delta} = p_g(K_{Z_\Delta}) - 1.
\]
Note that by Construction 2.12 at least if \( \Delta = C(\Delta) \), and by ([Gie21, Thm.4.2]) we have
\[
p_g(K_{Z_\Delta}) = l^*(\Delta_{can}).
\]
But it seems to be tedious to deduce the analogous formula for \( K_{Z_\Sigma} \) from this.
4.1 Relating properties of Kanev/Todorov surfaces in general and as toric hypersurfaces

By ([Bomb73, section 8]) a Kanev surface \( Y \) is simply connected and the condition \( q(Y) = 0 \) in the definition is actually superfluous. We may also compute the Hodge numbers of \( Y \): The only nonzero Hodge numbers are

\[
h^{0,0} = h^{2,2} = 1, \quad h^{0,2} = h^{2,0} = 1, \quad h^{1,1} = 19
\]

Similarly for Todorov surfaces the only nonzero Hodge numbers are

\[
h^{0,0} = h^{2,2} = 1, \quad h^{2,0} = h^{0,2} = 1, \quad h^{1,1} = 18.
\]

The following result is well known ([BHPV04, Ch.7, Cor.(5.4)]).

**Proposition 4.4.** Let \( Y \) be a minimal or canonical surface of general type, then we have the following formula for the plurigenera

\[
P_n(Y) := h^0(Y, \mathcal{O}_Y(nK_Y)) = \chi(Y, \mathcal{O}_Y) + \frac{n(n-1)}{2}K_Y^2 \quad \text{for } n \geq 2
\]

**Remark 4.5.** In the case of toric hypersurfaces Theorem 4.2 in [Gie21] does not only compute \( p_g(Y) \) but all plurigenera \( P_n(Y) \) for \( n \geq 1 \) by the formula

\[
P_n(Y) = l(n \cdot F(\Delta)) - l^*(((n-1)F(\Delta)).
\]

**Example 4.6.** (Kanev’s original example)

The first Kanev surface was found in ([Kan76]) via the following Fermat polynomial

\[
f(x_0, ..., x_3) := x_0^6 + x_1^6 + x_2^6 + x_3^6,
\]

Let \( X := \{ f = 0 \} \subset \mathbb{P}^3, G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \). Then \( G \) acts on \( \mathbb{P}^3 \) via

\[
g = (-1, \epsilon) : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \epsilon x_1 : \epsilon^2 x_2 : -x_3)
\]

where \( \epsilon := e^{2\pi i} \), and \( G \) leaves \( X \) invariant. The quotient \( X/G \) is a singular hypersurface in the toric variety \( \mathbb{P}^3/G \). Letting \( Y_{\text{min}} \) denote the minimal resolution of \( X/G \), \( Y_{\text{min}} \) gets a Kanev surface.

By [BKS19] Example 6.1] the hypersurface \( (X/G) \cap T \) has Newton polytope:

\[
\Delta' := \langle (1, 2, 4), (1, 0, 0), (1, 4, 2), (-2, -4, -5) \rangle
\]
Figure 7: The maximal polytope in the class \(a\) on the left and in the class \(b\) on the right. In both cases \(l(\Delta) = 18\).

This polytope is isomorphic to the maximal polytope in \(a\). Note that in this example as well as in the example \(c\) the toric variety \(\mathbb{P}_{\tilde{\Delta}}\) is almost a weighted projective space, more precisely a fake weighted projective space, that is \(\text{rk Cl}(\mathbb{P}_{\tilde{\Delta}}) = 1\).

Let \(Y := Z_\Sigma\) be a Todorov surface embedded in the toric 3-fold \(\mathbb{P}_\Sigma\) and consider the (rational) map \(\psi_{nK_Y}\) associated to the complete linear system \(\mathbb{P}H^0(Y, nK_Y)\). By ([Gie21, Prop.6.2]), since \(Y\) sits in a toric 3-fold, all these maps are in fact morphisms and in particular taking \(n = 2\) we get a morphism \(\psi_{2K_Y}: Y \to \mathbb{P}^3\).

**Proposition 4.7.** In all 3 cases c), d) and e) the image \(\psi_{2K_Y}(Y)\) is a quadric cone. The canonical divisor \(K_Y\) defines a hyperelliptic curve of genus 3 respectively.

In c) \(Y\) has 2 singularities of type \(A_3\), in d) \(Y\) has 4 singularities of type \(A_1\) and in e) \(Y\) has 2 singularities of type \(A_2\). The fundamental group of \(Y\) is \(\mathbb{Z}/2\mathbb{Z}\).

**Proof.** According to the pictures c), d) and e) in Construction 2.12 if we let \(2 \cdot F(\Delta)_{\text{can}} \cap M = \{y_1, y_2, y_3\}\) chosen as in the pictures, we get in all 3 cases
Figure 8: The polytopes $c)$, $d)$ and $e)$ and their cone $\sigma$ from left to right

c) \quad \Delta = \langle a = (2, 1, 5), p = (-2, -1, -3), b = (2, 0, 1), d = (2, 2, 1) \rangle \\
\quad F(\Delta) = \langle (0, 0, 0), (1, 1/2, 2), (1, 1/4, 1), (1, 3/4, 1) \rangle \\

d) \quad \Delta = \langle a = (2, -1, 3), b = (2, 0, 1), c = (2, -1, -1), d = (2, -2, 1), p = (-2, 1, -1) \rangle \\
\quad F(\Delta) = \langle (0, 0, 0), (1, -1/2, 2), (1, -1/2, 0), (1, -3/4, 1/2), (1, -1/4, 1/2) \rangle \\

e) \quad \Delta = \langle a = (2, 0, 1), b = (2, 1, -1), c = (2, 4, -3), d = (2, 1, 1), p = (-2, -2, 1) \rangle \\
\quad F(\Delta) = \langle (0, 0, 0), (1, 3/2, -1), (1, 3/4, 0), (1, 1/2, 0), (1, 3/4, -1/2) \rangle
the relation $y_2^2 = y_1 y_3$. Since by ([Gie21 Thm.4.2]) for $y_0 := (0,0,0)$

$$H^0(Y, 2K_Y) \cong \langle y_0, y_1, y_2, y_3 \rangle$$

in all 3 cases $\psi_{2K_Y}(Y) \subset \mathbb{P}^3$ is a quadric cone. The canonical curve $K_Y$ is hyperelliptic since in all cases we have a fibering $D_{can} \to \mathbb{P}^1$ given by projecting onto the axis $\langle a, c \rangle$ and the hypersurface $Y$ intersects a general fibre in 2 points, as could be checked with a Newton polytope type argument. The computation of the singularities follows from Figure 8 and the result on the fundamental group follows from ([CD89]). \qed
Table 1: polytopes such that $\Delta_{\text{can}}$ has 3 vertices sorted as in Figure 2 from the top to the bottom and from left to right. The arrows indicate that the polytopes are not canonically closed and the ID of the canonical closure is the polytope above the arrows (e.g. ID5389063 has canonical closure ID546219)

$$F(\Delta) = \langle (0, 0, 0), (1, 1/3, 0), (1, 2/3, 0), (1, 1/2, -1/2) \rangle$$

$p := (-4, -2, 1), a_2 := (-2, -1, 0), c_2 := (-2, -1, 1), b_1 := (-1, -1, 1),$  
$d_1 := (-1, 0, 1), a_1 := (0, 0, -1), 0 := (0, 0, 0), c_1 := (0, 0, 1),$  
$ab := (1, 0, 0), bc := (1, 0, 1), ad := (1, 1, 0), cd := (1, 1, 1),$  
b := (2, 0, 1), a := (2, 1, -2), ac_1 := (2, 1, -1), ac_2 := (2, 1, 0),
c := (2, 1, 1), d := (2, 2, 1)$

| ID     | spanning set for polytope $\Delta$ | number of points in $\Delta$ | $\text{sing. of } Z_\Delta$ | $\text{sing. of } Z_{F(\Delta)}$ |
|--------|-----------------------------------|-------------------------------|-------------------------------|-----------------------------------|
| 547444 | $\Delta_{\text{can}}, p$          | 18                            | $3A_2$                        | $3A_2$                            |
| 474457 | $\Delta_{\text{can}}, a_2, c_2, d_1, b_1$ | 17                            | $2A_2$                        | $3A_2$                            |
| $\Rightarrow$ 545932 | $\Delta_{\text{can}}, a_2, c_2$          | 15                            |                               |                                   |
| $\Rightarrow$ 532384 | $\Delta_{\text{can}}, a_2, c_2, d_1$  | 16                            |                               |                                   |
| $\Rightarrow$ 532606 | $\Delta_{\text{can}}, a_2, d_1, b_1$  | 16                            |                               |                                   |
| 483109 | $\Delta_{\text{can}}, d_1, b_1, c_2, a_1$ | 16                            | $A_2, 3A_1$                   | $A_5, A_2$                        |
| 534669 | $\Delta_{\text{can}}, c_2, d_1, a_1$ | 15                            | $2A_2, A_1$                   | $A_5, A_2$                        |
| 534866 | $\Delta_{\text{can}}, b_1, a_1, d_1$ | 15                            | $3A_2, A_1$                   | $E_6, A_2$                        |
| 534667 | $\Delta_{\text{can}}, c_2, d_1, b_1$ | 15                            | $3A_2$                        | $A_8$                            |
| 546062 | $\Delta_{\text{can}}, b_1, a_2$    | 15                            | $2A_2, A_1$                   | $3A_2$                           |
| 546205 | $\Delta_{\text{can}}, a_1, c_2$    | 14                            | $A_3, A_2$                    | $A_5, A_2$                        |
| 546219 | $\Delta_{\text{can}}, c_1, a_2$    | 14                            | $2A_2$                        | $3A_2$                           |
| $\Rightarrow$ 547324 | $\Delta_{\text{can}}, a_2$          | 11                            |                               |                                   |
| $\Rightarrow$ 546863 | $\Delta_{\text{can}}, a_2, bc$     | 12                            |                               |                                   |
| $\Rightarrow$ 539063 | $\Delta_{\text{can}}, a_2, bc, cd$ | 13                            |                               |                                   |
| 536498 | $\Delta_{\text{can}}, b_1, ad, c_2$ | 14                            | $A_3, A_2$                    | $A_8$                            |
| 537834 | $\Delta_{\text{can}}, ab, ad, c_2$ | 13                            | $A_4$                         | $A_8$                            |
| $\Rightarrow$ 547325 | $\Delta_{\text{can}}, c_2$         | 11                            |                               |                                   |
| $\Rightarrow$ 546862 | $\Delta_{\text{can}}, ab, c_2$     | 12                            |                               |                                   |
| $\Rightarrow$ 546663 | $\Delta_{\text{can}}, ad, c_2$     | 12                            |                               |                                   |
Table 2: polytopes and Fine interior in the class $b$) sorted as in Figure 3 from top to bottom, left to right. (with the same convention as in Table 1).

$F(\Delta) = \langle (0, 0, 0), (1, -1, 1/2), (1, -2/3, 1/3), (1, -1/2, 1/2), (1, -2/3, 2/3) \rangle$

$p := (-4, 3, -2), c_2 := (-2, 2, -1), a_2 := (-2, 1, -1), b_1 := (-1, 1, 0)$

d_1 := (-1, 1, -1), 0 := (0, 0, 0), a_1 := (0, -1, 0), c_1 := (0, 1, 0), cd := (1, 0, 0),
ad := (1, -1, 0), ab := (1, -1, 1), bc := (1, 0, 1), ac_2 := (2, -1, 1),
ac_1 := (2, -2, 1), d := (2, -1, 0), c := (2, 0, 1), a := (2, -3, 1), b := (2, -1, 2)

| ID  | spanning set for polytope $\Delta$ | number of points in $\Delta$ | sing. of $Z_\Delta$ | sing. of $Z_{F(\Delta)}$ |
|-----|------------------------------------|-------------------------------|---------------------|--------------------------|
| 545317 | $\Delta_{can}, p$ | 18 | $3A_1$ | $3A_1$ |
| 354912 | $\Delta_{can}, c_2, a_2, d_1, b_1$ | 17 | $2A_1$ | $A_2, 2A_1$ |
| ⇒ 535513 | $\Delta_{can}, c_2, a_2$ | 15 | $3A_1$ | $A_4, A_1$ |
| ⇒ 481575 | $\Delta_{can}, c_2, a_2, d_1$ | 16 | $4A_1$ | $A_3, 2A_1$ |
| 372528 | $\Delta_{can}, d_1, b_1, c_2, a_1$ | 16 | $4A_1$ | $D_5, A_1$ |
| 372973 | $\Delta_{can}, b_1, d_1, a_2, c_1$ | 15 | $3A_1$ | $A_3, 2A_1$ |
| ⇒ 490511 | $\Delta_{can}, b_1, d_1, a_2$ | 15 | $3A_1$ | $A_4, A_1$ |
| 388701 | $\Delta_{can}, a_1, d_1, b_1$ | 15 | $A_2, A_1$ | $A_4, A_1$ |
| ⇒ 499287 | $\Delta_{can}, a_1, d_1, b_1$ | 15 | $A_2, A_1$ | $A_6$ |
| 490485 | $\Delta_{can}, a_1, d_1, b_1$ | 15 | $A_3, 2A_1$ | $D_7$ |
| 535952 | $\Delta_{can}, a_2, c_1$ | 14 | $A_2, 2A_1$ | $A_4, 2A_1$ |
| 536013 | $\Delta_{can}, a_1, c_2$ | 14 | $A_2, A_1$ | $A_4, A_1$ |
| 495637 | $\Delta_{can}, d_1, c_2, ab$ | 14 | $A_2, A_1$ | $A_6$ |
| ⇒ 539313 | $\Delta_{can}, a_1, d_2$ | 13 | $A_3, 2A_1$ | $D_7$ |
| 499291 | $\Delta_{can}, a_1, d_1$ | 14 | $A_2, 2A_1$ | $A_4, 2A_1$ |
| ⇒ 538356 | $\Delta_{can}, a_2, c_1$ | 13 | $A_2, A_1$ | $A_6$ |
| 501298 | $\Delta_{can}, c_2, ab, ad$ | 13 | $A_2, A_6$ | |
| ⇒ 547246 | $\Delta_{can}, c_2$ | 11 | $A_2, A_6$ | |
| ⇒ 540602 | $\Delta_{can}, c_2, ab$ | 12 | $A_2, A_6$ | |
| 501330 | $\Delta_{can}, a_2, bc, cd$ | 13 | $A_2, A_6$ | |
| ⇒ 547240 | $\Delta_{can}, a_2$ | 11 | $A_2, A_6$ | |
| ⇒ 540663 | $\Delta_{can}, a_2, bc$ | 12 | $A_2, A_6$ | |
References

[Bat94] V. V. Batyrev, Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties, Journal of Algebraic Geometry 3 (1994), no. 3, 493–535.

[Bat17] V. V. Batyrev, The stringy Euler number of Calabi-Yau hypersurface in toric varieties and the Mavlyutov duality, Pure Appl. Math. Q. 13 (2017), no. 1, 1–47.

[Bat20] V. V. Batyrev, Canonical models of toric hypersurfaces, (2020), arXiv:2008.05814v1 [mathAG]

[Bomb73] E. Bombieri, Canonical models of surfaces of general type, Publications Mathématiques de l'IHÉS, tome 42 (1973), 171–219.

[BHPV04] W. Barth, K. Hulek, C. Peters, A. Van de Ven, Compact Complex Surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, (2004).

[BKS19] V. V. Batyrev, A. M. Kasprzyk, and K. Schaller, On the Fine interior of three-dimensional canonical Fano polytopes, arXiv:1911.12048 [math.AG]

[Cat78] F. Catanese, Surfaces with $K^2 = p_g = 1$ and their period mapping, Algebraic Geometry Summer Meeting, Copenhagen, (1978), 1 – 29.

[CD89] F. Catanese, O. Debarre, Surfaces with $K^2 = 2, p_g = 1, q = 0$, (1989).

[CLS11] D. A. Cox, J. B. Little and H. K. Schenck, Toric varieties, Graduate Studies in Mathematics, 124, Amer. Math. Soc., Providence, RI, (2011).

[Gie21] J. Giesler, The plurigenera and birational models of nondegenerate toric hypersurfaces.

[Kan76] V. Kanev, An example of a simply connected surface of general type for which the local torelli theorem does not hold (1976).

[Kas10] A. M. Kasprzyk, Canonical toric Fano threefolds, Can. J. Math. 62 (6) (2010) 1293–1309.
[KM98] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, Vol. 134 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge (1998).

[Rei83] M. Reid, *Decomposition of toric morphisms*, in Arithmetic and geometry, Progr. Math. **36**, Birkhäuser Boston, Boston, MA (1983), 395–418.

[Rei87] M. Reid, *Young person’s guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, (1987), pp. 345–414.

[Sch18] K. Schaller, *Stringy Invariants of Algebraic Varieties and Lattice Polytopes*, Ph.D. thesis, Eberhart-Karls-Universität Tübingen, (2018).

[Tod80] A. Todorov, *Surfaces of general type with $p_g = 1$ and $(K.K) = 1.I$*, Annales scientifiques de l’École Normale Supérieure, Serie 4, Volume 13 no. 1, (1980), 1-21.

[Tre10] J. Treutlein, *Birationale Eigenschaften generischer Hyperflächen in algebraischen Tori*, Dissertation (2010).

[Us87] S. Usui, *Type I Degeneration of Knef surfaces*, Théorie de Hodge - Luminy, Juin 1987, Astérisque no. **179-180**, 183–243.