ON FINITE GENERATION OF HOLOMORPHIC FUNCTIONS ON RIEmann SURFACES

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Abstract. Let $M^2$ be a complete noncompact Riemann surface with nonnegative curvature, we show that the ring of holomorphic functions with polynomial growth is finitely generated. The first proof of such result was given by Li and Tam [12]. We provide a short proof here.

1. Introduction

In [16], Yau proposed the uniformization conjecture for complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature. In his words, “the question is to demonstrate that every noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex euclidean space. If we only assume the nonnegativity of bisectional curvature, the manifold should be biholomorphic to a complex vector bundle over a compact Hermitian symmetric space.” See also [5][15]. For this sake, it was further asked in [16] whether or not the ring of the holomorphic functions with polynomial growth is finitely generated, and whether or not the dimension of the spaces of holomorphic functions of polynomial growth is bounded from above by the dimension of the corresponding spaces of polynomials on $\mathbb{C}^n$. The latter question was affirmed by Ni [14] with the assumption that the manifold has maximum volume growth. By using same technique in [14], Chen, Fu, Le and Zhu [2] removed the maximal volume growth condition. Very recently, the author [6] gave a proof under the much weaker condition that the holomorphic sectional curvature is nonnegative.

In the Riemannian case, there have been many articles on estimating the dimension of the harmonic functions of polynomial growth. For instance, Colding and Minicozzi [3] proved that for complete manifolds $M^m$ with nonnegative Ricci curvature, the dimension of harmonic functions with polynomial growth is finite. In [10], Li produced an elegant short proof. However, the example of Donnelly [4] shows that the sharp inequality comparing with the Euclidean space is not true. The sharp upper bound is only obtained either when $d = 1$ or $m = 1$ in [11][12] by Li and Tam. The rigidity part for $d = 1$ is due to Li[8] and Cheeger-Colding-Minicozzi[1]. Li and Wang [13] showed that when the sectional curvature is nonnegative and manifold has maximal volume growth, an asymptotic sharp estimate is valid. Thus there is a subtle difference between the Riemannian case and the Kähler case. One can refer to [7][9] for nice survey of these results.

In this note we study the finite generation conjecture of Yau.
Definition. Let $M$ be a complete noncompact Kähler manifold. Let $O(M)$ denote the ring of holomorphic functions on $M$. For any $d \geq 0$, define

$$O_d(M) = \{ f \in O(M) | \lim_{r \to \infty} \frac{|f(x)|}{r^d} < \infty \}.$$ 

Here $r$ is the distance from a fixed point on $M$. If $f \in O_d(M)$, we say $f$ is of polynomial growth with order $d$.

Theorem. Let $M^2$ be a complete noncompact Riemann surface with nonnegative curvature. Then the ring $|O_d(M)|$ is finitely generated.

The first proof of such result was given by Li and Tam [12]. We will provide an alternative proof below.

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2. Proof of the theorem

Proof. We may assume $M$ is conformal to $\mathbb{C}$, otherwise $M$ is flat and the conclusion is obvious.

Lemma 1. Let $f \in O_d(M)$ which is not identically 0, then the number of vanishing points on $M$ is at most $d$ counting multiplicities.

Proof. We assume by contradiction that $f$ vanishes at $x_1, \ldots, x_{d+1}$ on $M$. Let $p \in M$ by a fixed point. Define $r_i(x) = \text{dist}(x, x_i)$. We may assume $\text{dist}(x_i, p) \leq 1$ for each $1 \leq i \leq d + 1$ by rescaling. For $R > 2$ and small $\epsilon > 0$, consider

$$F_\epsilon(x) = \sum_{i=1}^{d+1} (1 - \epsilon) \log r_i - (d + 1)(1 - \epsilon) \log(R - 1) + \log M(R) - \log |f(x)|$$

where $M(R)$ is the maximum of $|f(x)|$ for $x \in B(p, R)$.

Claim 1. $F_\epsilon(x) \geq 0$ in $B(p, R)$.

Proof. First observe that $F_\epsilon(x) \geq 0$ for $x \in \partial B(p, R)$. Here we have used the triangle inequality $d(x, x_i) \geq d(x, p) - d(p, x_i) \geq d(x, p) - 1$. Observe that if $x$ is sufficiently close to some $x_i$, $F_\epsilon(x) \gg 0$, this is due to the vanishing order of $f$ at $x_i$. We have $\partial \partial \log |f(x)| = 0$ when $f(x) \neq 0$ and by Laplacian comparison, $\Delta \log r_i \leq 0$ in the distribution sense. Therefore $\Delta F_\epsilon(x) \leq 0$ in the distribution sense. Thus the minimum cannot be assumed in the interior of $B(p, R)$ (the minimum cannot be at the vanishing points of $f$, since at those points, $F_\epsilon = +\infty$). Since $F_\epsilon(x) \geq 0$ on the boundary, the claim is proved. \hfill \Box

Now let $\epsilon \to 0$, we find that on $B(p, R)$,

$$F(x) = \sum_{i=1}^{d+1} \log r_i - (d + 1) \log(R - 1) + \log M(R) - \log |f(x)| \geq 0.$$ 

Pick a fixed point $x \in B(p, R)$ such that $f(x) \neq 0$, thus $M(R) \geq (R - 1)^{d+1} \frac{|f(x)|}{r_1 r_2 \cdots r_{d+1}} \geq C(R - 1)^{d+1}$. If we take $R \to \infty$, this contradicts that $f \in O_d(M)$. 


Let $f \in O_d(M)$ and $z_1, \ldots, z_k$ be the vanishing points of $f(x)$. Recall $M$ is biholomorphic to $\mathbb{C}$, consider $g(x) = \frac{f(x)}{(x-z_1)(x-z_2)\cdots(x-z_k)}$. It is easy to see $g(x) \in O_d(M)$. Now $g$ has no vanishing points on $M$. For any positive integer $i$, $g_i = g^i$ is well defined. Note $|g_i(x)| \leq \max(1, |g(x)|)$, thus $g_i \in O_d(M)$. If $g$ is not a constant, then for all $i \geq 1$, $g_i$ are all linearly independent. Therefore we have a linear combination of $g_i$ such that the vanishing order at $p$ is greater than $d$. This contradicts Lemma 1. Therefore $g$ is a constant and $f(x) = c(x-z_1)(x-z_2)\cdots(x-z_k)$. This means that for any $h \in O_d(M)$, $h$ is a polynomial on $\mathbb{C}$. Note that $\lim_{z \to \infty} |\frac{1}{(z-z_1)(z-z_2)\ldots(z-z_k)}| < \infty$. Thus $z \in O_d(M)$. Therefore the ring $\{O_d(M)\}$ is generated by $z$ if $\{O_d(M)\}$ is not identically 0.

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\section*{References}

[1] J. Cheeger, T. Colding and W. Minicozzi. II, \textit{Linear growth harmonic functions on complete manifolds with nonnegative Ricci curvature}, Geom. Funct. Anal. 5 (1995), no. 6, 948-954.

[2] B.-L. Chen, X.-Y. Fu, Y. Le, X.-P. Zhu, \textit{Sharp dimension estimates for holomorphic function and rigidity}, Trans. Amer. Math. Soc. 358(2006), no. 4, 1435-1454.

[3] T. Colding and W. Minicozzi., II, \textit{Harmonic functions on manifolds}, Ann. of Math.(2) 146(1997), no. 3, 725-747.

[4] H. Donnelly, \textit{Harmonic functions on manifolds of nonnegative Ricci curvature}, Internat. Math. Res. Notices, 2001, no. 8, 429-434.

[5] R. E. Green and H. Wu, \textit{Analysis on noncompact Kähler manifolds}, Proc. Symp. Pure Math. 30(1977), 69-100.

[6] G. Liu, \textit{Sharp estimates for holomorphic functions on Kähler manifolds and dimension estimate}, available on arxiv.

[7] P. Li, \textit{Harmonic functions on complete Riemannian manifolds}, Adv. Lect. Math.(ALM), 7, Int. Press, Somerville, MA, 2008.

[8] P. Li, \textit{Linear growth harmonic functions on Kähler manifolds with nonnegative Ricci curvature}, Math. Res. Lett., 2 (1995), 79-94.

[9] P. Li, \textit{Curvature and function theory on Riemannian manifolds}, Survey in Differential Geometry vol. VII, International Press, Cambridge, 2000, 71-111.

[10] P. Li, \textit{Harmonic sections of polynomial growth}, Math. Res. Lett 4(1997), 35-44.

[11] P. Li and L.-F. Tam, \textit{Linear growth harmonic functions on a complete manifold}, J. Diff. Geom. 29 (1989), 421-425.

[12] P. Li and L.-F. Tam, \textit{Complete surfaces with finite total curvature}, J. Diff. Geom. 33(1991), 139-168.

[13] P. Li and J. Wang, \textit{Counting massive sets and dimension of harmonic functions}, J. Diff. Geom. 53(1999), no. 2, 237-278.

[14] L. Ni, \textit{A monotonicity formula on complete Kähler manifolds with nonnegative bisectional curvature}, Jour of AMS, Vol 17, No. 4, 909-946.

[15] Y. T. Siu, \textit{Pseudoconvexity and the problem of Levi}, Bull. Amer. Math. Soc. 84(1978), 481-512.

[16] S. T. Yau, \textit{Open problems in geometry}, Lectures on Differential Geometry, by Schoen and Yau 1 (1994), 365-404.

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