Invited Comment

Gravity as the square of Yang–Mills?*

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Abstract
In these lectures we review how symmetries of gravitational theories may be regarded as originating from those of ‘Yang–Mills squared’. We begin by motivating the idea that certain aspects of gravitational theories can be captured by the product, in some sense, of two distinct Yang–Mills theories, particularly in the context of scattering amplitudes. We then introduce a concrete dictionary for the covariant fields of (super)gravity in terms of the product of two (super) Yang–Mills theories. The dictionary implies that the symmetries of each (super) Yang–Mills factor generate the symmetries of the corresponding (super)gravity theory: general covariance, p-form gauge invariance, local Lorentz invariance, local supersymmetry, R-symmetry and U-duality.

Keywords: gravity, gauge theory, supersymmetry, symmetry

(Some figures may appear in colour only in the online journal)

1. MJD: tribute to Dick Arnowitt
It was thanks to Dick Arnowitt that I spent eleven wonderful years 1988–1999 here at Texas A&M. My wife Lesley and I have not forgotten the kindness shown by Dick and Young-In when we first arrived here from England. It was a privilege to work in the first rate Theoretical Physics Group that Dick had built.

2. Introduction
The idea that gravitational physics can be understood in terms of gauge theory has reoccurred a number of times, in a variety of guises. The most conceptually straight-forward approach is to regard gravity as the gauge theory of Lorentz, Poincaré or de Sitter symmetries [1–5]. The holographic principle [6, 7], concretely realized through the AdS/CFT correspondence [8–10], represents a more subtle realization of this notion, with profound consequences for our understanding of both gauge and gravity theories. Here we appeal to a third and, at least superficially, independent incarnation:

\[
gravity = \text{Yang–Mills} \times \text{Yang–Mills.} \tag{2.1}
\]

At first sight this is a radical proposal; Einstein’s general relativity describes gravity as the dynamics of spacetime, while Yang–Mills theories, as used to describe the strong, weak and electromagnetic forces, play out on spacetime. General relativity and Yang–Mills theory are seemingly worlds apart in almost every regard, from their fundamental degrees freedom to their basic symmetries. In particular, the Yang–Mills theories underlying the standard model are renormalizable, predictive quantum field theories, in stark contrast to perturbative quantum gravity.

Despite their differences, however, there already exist some fascinating hints that gravity, at least in some regimes, may be related to the square of Yang–Mills theory. String theory provided the first example in the form of the Kawai–Lewellen–Tye (KLT) relations, which connect tree-level amplitudes of closed strings to sums of products of open string amplitudes [11]. More recently, invoking Bern–Carrasco–Johansson (BCJ) colour-kinematic duality [12] it has been conjectured [13] that the on-mass-shell momentum-
space scattering amplitudes for gravity are the ‘double-copy’
of gluon scattering amplitudes in Yang–Mills theory to all orders in perturbation theory.

The recent renaissance in amplitude calculations has been
principally driven by the ‘on-shell paradigm’. Starting with
Lagrangian field theory we learnt how to compute simple
amplitudes to low orders in perturbation theory. The factorial
growth in complexity with loop order quickly renders traditio
nal approaches impractical. Searching for computational
efficiency, over time various generic amplitude structures (on-
shell recursion relations, generalized unitarity cuts, Grass-
mannians, scattering equations...) were uncovered, even-
tually allowing the Lagrangian ladder to be kicked away. For
an overview of these developments see [14]. This freedom led
to the discovery of new features of amplitudes, not visible
from the original Lagrangian perspective. BCJ colour-kine-
matic duality falls into this class of surprises. Conversely,
having climbed so high we can no longer see where we can from;
the full significance and implications of BCJ duality
remain unclear. Can we climb back down by some other route
and understand the origin of these remarkable dualities? The
basic idea reviewed here is to build a dictionary expressing
the covariant fields of (super) gravity as the product, in a well-
deﬁned sense, of two arbitrary (super) Yang–Mills theories.

2.1. Motivation

We begin by sketching the BCJ colour-Kinematic duality and
the double copy procedure [12, 13, 15]. For a more detailed
account of this topic the reader is referred to the reviews
[14, 16]. This will not only better motivate (2.1), but also
inform our ﬁeld theory constructions in the subsequent sections.

Let us consider the n-point L-loop amplitude of Yang–
Mills theory with an arbitrary gauge group. Converting all
four-point contact terms into s, t or u channel trivalent pole
diagrams by inserting propagators 1 = s/t = t/u = u/w, we have

\[ A_n^L = \frac{i^4 g_n^{n-2+2L}}{\mathcal{S}_L} \sum_{i \in \text{trivalent graphs}} \int_{r_{n-1}}^{r_L} \frac{d\rho_n}{(2\pi)^{2S_i}} \rho_n^{n_i-\delta_{n_i}}. \]  

(2.2)

The sum is over all n-point L-loop graphs i with only trivalent
vertices. \( c_i \) denotes the kinematic factor of graph i, composed
of gauge group structure constants. \( n_i \) denotes the kinematic
factor of graph i. It is a polynomial of Lorentz-invariant
contractions of polarization vectors and momenta. The \( p_{\mu_i}^2 \)
are the propagators for each graph i. \( S_i \) is the dimension of the
automorphism group of graph i.

The set of n-point trivalent graphs can be organized into
triples i, j, k such that they differ in only one propagator. For
such a triple the three distinct propagators are embedded in
the same graph, connected to the same four incoming edges,
but in the s, t, u channel for (say) i, j, k, respectively. For
such a triple the colour factors will obey a Jacobi identity

\[ c_i + c_j + c_k = 0. \]  

(2.3)

and consequently the generalized gauge transformations

\[ n_i \rightarrow n_i + s\Delta, \quad n_j \rightarrow n_j + t\Delta, \quad n_k \rightarrow n_k + u\Delta, \]  

(2.4)

leave the amplitude (2.2) invariant [12]. It was proposed in
[12] that one can arrange the diagrams, using the generalized
gauge transformations if necessary, to display a colour-
kinematic duality:

\[ c_i - c_j - n_i + n_j = 0. \]  

(2.5)

A reorganization admitting this surprising relationship
between colour and kinematic data was shown to exist for all
n-point tree-level amplitudes in [15]. Although there is as
yet no proof, the colour-kinematic duality is conjectured to
hold, with highly non-trivial evidence, at any loop level, thus
going beyond the KLT relations [12, 13]. While it is clear that
the colour factors should obey Jacobi identities (by deﬁni
tion), it is not at all obvious that the kinematic factors should
play by the same rules!

This suggests that there is in fact some underlying
kinematic algebra mirroring the properties of conventional
Lie algebras, as described in [17, 18]. In general, this hidden
algebra cannot be made manifest at the Lagrangian level,
however for the self-dual sector it can be identiﬁed as a dif-
feomorphism Lie algebra, which determines the kinematic
numerators of generic tree-level maximally helicity violating
amplitudes [17].

More remarkable still is the double-copy prescription
[12, 13, 15]. Assuming one has found a colour-kinematic
duality respecting representation of the n-point L-loop gluon
amplitude, the equivalent n-point L-loop graviton amplitude
is obtained by simply replacing each colour factor, \( c_i \), with a
second kinematic factor, \( \tilde{n}_i \), as depicted in figure 1. Exam-
ining the unitary cuts of the gravity amplitude obtained via
the double-copy is sufﬁcient to prove it reproduces the correct
result, assuming colour-kinematic duality is satisﬁed in one of
the Yang–Mills factors. These ideas are seamlessly extended
to supersymmetric theories. In particular, the square of the
amplitudes of the maximally supersymmetric \( N = 4 \) super
Yang–Mills theory yield amplitudes of the maximally
supersymmetric \( N = 8 \) supergravity theory.

The double-copy picture is not only conceptually com-
PELLING but also computationally powerful, bringing pre-
viously intractable calculations with in reach. This has pushed
forward dramatically our understanding of divergences in
perturbative quantum gravity, revealing a number of unex-
pected features and calling into question previously accepted
arguments regarding ﬁniteness. For instance, the four-point
graviton amplitude in \( N = 8 \) supergravity has been shown to
be finite to four loops [19], contradicting some early expec-
tations [20]. It has since been shown that this cancellation can
be accounted for by supersymmetry and \( E_7(7) \) U-duality [21–
24]. The consensus, however, is that at seven loops any
would-be cancellations cannot be ‘consequences of super-
symmetry in any conventional sense’ [21]. Unfortunately,
seven loops in \( N = 8 \) supergravity remains beyond reach but
by decreasing the amount of supersymmetry these arguments apply at lower loop order. Indeed, the four-point amplitude of supergravity has been shown to be finite to four loops, contrary to all expectations based on standard symmetry arguments [25]. There are ‘enhanced cancellations’ at work and the conclusion that supergravity will diverge at seven loops is thrown into doubt. Although the majority opinion is that supergravity will diverge at some loop order, there is something deeper at work we have yet to understand fully and question remains very much open.

### 3. Covariant field dictionary

These developments raise the question: to what extent, or in what sense, can one regard gravity as the square of Yang–Mills? Is there a deeper connection underlying the amplitude relations? One approach to addressing such questions is to build a dictionary at the level of fields, as opposed to on-shell states or amplitudes. In a sense this runs contrary to the ‘on-shell paradigm’ that took us here. Going back off-shell may nonetheless be instructive. This approach has been examined at the level of Lagrangians in [13, 17] and classical solutions in [26]. Here instead we focus on expressing the covariant fields of (super)gravity in terms of the product of (super) Yang–Mills fields. The first consistency check such a dictionary must pass is at the level of symmetries. As we shall review, the gravitational symmetries of general covariance, $R$-symmetry and U-duality follow from those of Yang–Mills at linearized approximation.

Much of the squaring literature invokes a mysterious product:

$$A_{\mu}(x) \otimes \tilde{A}_{\nu}(x).$$

(3.1)

Here, $A_{\mu}$ and $\tilde{A}_{\nu}$ are the gauge potentials of two distinct Yang–Mills theories, which we will refer to as left (no tilde) and right (tilde), respectively. They can have arbitrary and independent non-Abelian gauge groups $G$ and $\tilde{G}$. Reading off the meaning of “$\otimes$” from the tensor product branching rules of the appropriate spacetime little group representations or corresponding string states one can consistently match the symmetries. See in particular [27, 28]. Here, we are instead seeking a concrete definition of “$\otimes$” at the level of field theory which is valid whether or not there is an underlying string interpretation. This raises two immediate questions: (i) Where do the gauge indices go? (ii) Does it obey the Leibnitz rule?

Guided by the structure of the amplitude relations and requirements of symmetry we introduced a covariant product rule in [29]:

$$f \otimes g := f * g$$

(3.2)

Let us review the ingredients in (3.2). The $*$ product denotes a convolutive inner tensor product with respect to the Poincaré group combined with a Killing form $\langle \cdot, \cdot \rangle : g \otimes g \rightarrow \mathbb{R}$,

$$[f * g](x) = \int d^D y \langle f(y) g(x - y) \rangle.$$  

(3.3)
We have further introduced the ‘spectator’ field $\Phi$, a $G \times \tilde{G}$ bi-adjoint valued scalar. The convolution reflects the fact that the amplitude relations are multiplicative in momentum space. It turns out to be essential for reproducing the local symmetries of (super)gravity from those of the two (super) Yang–Mills factors. The Killing form accounts for the gauge groups, while the spectator field allows for arbitrary and independent $G$ and $\tilde{G}$. It fact, the appearance of $\Phi$ is quite natural from the perspective of amplitude relations. Its necessity was identified by Hodges in the context of twistor-theory [30]. From the perspective presented in section 2.1, rather than sending $c_i \rightarrow \tilde{n}_i$, doubling the kinematics and removing the colour, one could also send $n_i \rightarrow \bar{c}_i$, doubling the colour and removing the kinematics. In [31, 32] this was shown at tree-level to yield the amplitudes of a global $G \times \tilde{G}$ bi-adjoint scalar field theory with cubic interaction term

$$\mathcal{L}_{int} = -\sum_{ijk} \tilde{F}^{ij} \Phi^{\mu} \Phi_{\mu} \Phi^{\mu'}. \quad (3.4)$$

The transformation rules of $\Phi$ are fixed by this theory. A scalar field also appeared independently, but in close analogy to our spectator field, in the double-copy construction of Kerr–Schild gravity solutions from Yang–Mills solutions in [26].

4. $\mathcal{N} = 1$ supergravity

Having introduced the covariant product, let us now work through the simplest example exhibiting all the local symmetries of interest. We consider the product of a left $\mathcal{N} = 1$ and a right $\mathcal{N} = 0$ theories at linearized level:

- Off-shell $\mathcal{N} = 1$ Yang–Mills multiplet with (4 + 4) bosonic + fermionic degrees of freedom and gauge group $G$:

$$A_\mu, \quad \psi, \quad D. \quad (4.1)$$

- Off-shell off-shell $\tilde{\mathcal{N}} = 0$ Yang–Mills multiplet with (3 + 0) bosonic + fermionic degrees of freedom and gauge group $\tilde{G}$:

$$\tilde{A}_\nu. \quad (4.2)$$

Without making any assumptions regarding the dynamics this yields the (12 + 12) new-minimal $\mathcal{N} = 1$ supergravity multiplet [33]:

$$g_{\mu \nu}, \quad B_{\mu \nu}, \quad \psi_\mu, \quad V_\mu, \quad (4.3)$$

where general covariance, two-form gauge invariance, local supersymmetry and local chiral symmetry follows from the left/right gauge symmetries.

The ‘gravity = Yang–Mills × Yang–Mills’ dictionary and symmetry transformations are most concisely expressed in the superfield formalism. Hence, we consider:

1. A left $\mathcal{N} = 1$ real vector superfield

$$V(x, \theta, \bar{\theta}) = (\chi + i \phi, \bar{\chi}, \bar{\phi})$$

transforming under local supergauge, non-Abelian global $G$ and global super-Poincaré:

$$\delta V = \sum_{\text{local Abelian supergauge}} \Lambda + \tilde{\Lambda} + \sum_{\text{global non-Abelian } G} \delta A_{\mu} \frac{1}{2} \bigg( \nabla \bar{\chi} + i \bar{\phi} + i \sigma^\rho \partial_{\rho} \chi + \frac{i}{2} \theta^2 \sigma^\rho \partial_{\rho} \phi \bigg) \bigg( \nabla \bar{\chi} + i \bar{\phi} + i \sigma^\rho \partial_{\rho} \chi + \frac{i}{2} \theta^2 \sigma^\rho \partial_{\rho} \phi \bigg) \quad (4.5)$$

where $\Lambda(x, \theta, \bar{\theta})$ is a chiral superfield of supergauge parameters

$$\Lambda(x, \theta, \bar{\theta}) = B + \sqrt{2} \theta \zeta + \theta^2 \kappa + i \theta \phi \partial_{\phi} \partial_{\bar{\phi}} - \frac{i}{\sqrt{2}} \theta^2 \sigma^\rho \partial_{\rho} \phi + \frac{1}{4} \theta^2 \sigma^\rho \partial_{\rho} \phi \quad (4.6)$$

2. A right $\tilde{\mathcal{N}} = 0$ Yang–Mills potential $\tilde{A}_\nu$ transforming under local gauge, non-Abelian global $\tilde{G}$ and global Poincaré:

$$\delta \tilde{A}_\nu = \sum_{\text{local Abelian gauge}} \frac{\partial_{\mu} \sigma}{2} A_{\mu} \tilde{X} + \sum_{\text{global Poincaré}} \delta (\tilde{A}_{\mu} \tilde{X}) + \delta (A_{\mu} \bar{X}) \quad (4.7)$$

3. The spectator bi-adjoint scalar $\Phi$ field transforming under non-Abelian global $G \times \tilde{G}$ and global Poincaré:

$$\delta \Phi = -[\Phi, \tilde{X}] + \sum_{\text{global Poincaré}} \delta (\Phi \bar{X}) \quad (4.8)$$

The gravitational symmetries are reproduced here from those of Yang–Mills by invoking the gravity/Yang–Mills
dictionary for fields and supergauge parameters:

\[
\begin{align*}
\text{Fields} & \quad \varphi^\alpha = V \Phi^\alpha \hat{A}_\alpha, \text{real superfield} \\
\text{Paras} & \quad \phi = V \Phi \hat{\lambda}, \text{chiral superfield} \\
\mathcal{S}_\alpha & = \Lambda \Phi \hat{A}_\alpha, \text{chiral superfield.}
\end{align*}
\]

(4.8)

Varying the gravitational superfield

\[
\varphi^\alpha (x, \theta, \bar{\theta}) = C_\alpha + i \theta \chi^\alpha + i \bar{\theta} \bar{\chi}^\alpha + i \theta^2 \bar{F}^\alpha - i \theta \bar{\sigma}^\alpha \bar{\sigma}_\alpha \chi^\alpha \\
+ i \bar{\theta} \bar{\sigma}^\alpha \chi^\alpha + \frac{1}{2} \bar{\sigma}^\alpha \bar{\sigma}_\alpha \chi^\alpha \\
+ \frac{1}{2} \bar{\theta} \bar{\sigma}^\alpha \chi^\alpha
\]

via the dictionary

\[
\delta \varphi^\alpha = \delta V \Phi^\alpha \hat{A}_\alpha + V \Phi \delta \Phi^\alpha \hat{A}_\alpha + V \Phi \delta \hat{A}_\alpha
\]

(4.10)

we obtain

\[
\delta \varphi^\alpha = \mathcal{S}_\alpha + \bar{\mathcal{S}}_{\bar{\alpha}} + \partial_\alpha \phi + \delta_{(\alpha, \lambda, \gamma)} \varphi^\gamma.
\]

(4.11)

This is the complete set of transformation rules for the new-minimal superfield at linearized approximation. Note, this derivation makes use of

(4.12)

\[
\langle [X, Y], Z \rangle = \langle X, \{Y, Z\} \rangle
\]

and, crucially, the convolution property

\[
\partial_\alpha (f * g) = \partial_\alpha f * g = f * \partial_\alpha g.
\]

(4.13)

To summarize, we have obtained the field content (4.9) and transformation rules (4.11) at linearized approximation of new-minimal $\mathcal{N} = 1$ supergravity [34, 35]. Hence, the local gravitational symmetries of general covariance, two-form gauge invariance, local supersymmetry and local chiral symmetry follow from those of Yang–Mills at linear level.

Introducing field equations we should match the on-shell content of the tensor product of spacetime little group representations. This is done covariantly by including the ghost sector in the dictionary [27, 28]. The $12 + 12$ multiplet splits with respect to superconformal transformations into an $8 + 8$ conformal supergravity multiplet plus a $4 + 4$ conformal tensor multiplet

\[
\begin{array}{c}
\begin{array}{c}
5 + 3 + 1 + 3 \\
4 + 2 + 4 + 2
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
5 + 3 \\
4 + 4
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
3 + 1 \\
2 + 2
\end{array}
\end{array}
\]

(4.14)

in terms of $\text{SO}(3)$ representations. Since the left (anti) ghost is a chiral superfield the ghost-antighost sector gives a compensating $4 + 4$ chiral (dilaton) multiplet [27, 28], yielding old-minimal $12 + 12$ supergravity [36, 37] coupled to a tensor multiplet, which, with the conventional two-derivative Lagrangian, correctly corresponds to the on-shell content obtained by tensoring left/right helicity states.

---

**Table 1.** U-dualities (global symmetries) of M-theory ($D = 11$, $\mathcal{N} = 1$ supergravity) compactified on an $n$-torus.

| $n$-torus | U-duality | $\mathcal{G}$ | $\mathcal{H}$ |
|----------|-----------|--------------|-------------|
| 1        | SO(1, 1, $\mathbb{Z}$) | SO(1, 1, $\mathbb{R}$) | $-$ |
| 2        | SL(2, $\mathbb{Z}$) $\times$ SO(1, 1, $\mathbb{Z}$) | SL(2, $\mathbb{R}$) $\times$ SO(1, 1, $\mathbb{R}$) | SO(2, $\mathbb{R}$) |
| 3        | SL(2, $\mathbb{Z}$) $\times$ SL(3, $\mathbb{Z}$) | SL(2, $\mathbb{R}$) $\times$ SL(3, $\mathbb{R}$) | SO(2, $\mathbb{R}$) $\times$ SO(3, $\mathbb{R}$) |
| 4        | SL(5, $\mathbb{Z}$) | SL(5, $\mathbb{R}$) | SO(5, $\mathbb{R}$) |
| 5        | SO(5, 5, $\mathbb{Z}$) | SO(5, 5, $\mathbb{R}$) | SO(5, 5, $\mathbb{R}$) |
| 6        | $E_{6(6)}$($\mathbb{Z}$) | $E_{6(6)}$($\mathbb{R}$) | USp(8) |
| 7        | $E_{7(7)}$($\mathbb{Z}$) | $E_{7(7)}$($\mathbb{R}$) | SU(8) |
| 8        | $E_{8(8)}$($\mathbb{Z}$) | $E_{8(8)}$($\mathbb{R}$) | SO(16, $\mathbb{R}$) |

---

5. Extended supersymmetry and U-duality

This minimally supersymmetric example does not fully address the issue of U-duality [38], which, in context of string/M-theory, is of fundamental importance. U-duality manifests itself in supergravity, the low energy effective limit of string/M-theory, in the form of non-compact global symmetries, $\mathcal{G}$, acting nonlinearly on the scalar fields [39]. In all cases obtained from ‘Yang–Mills $\times$ Yang–Mills’ the scalars parametrize a symmetric space $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is the maximal compact subgroup of $\mathcal{G}$ [40]. The U-dualities and corresponding global symmetries for M-theory compactified on an $n$-torus are summarized in table 1. Note, we will also use the term U-duality to refer to $\mathcal{G}$. The question of global symmetries from squaring Yang–Mills has also been addressed in [41–44], particularly in the context of scattering amplitudes.

As made clear by table 1, U-duality becomes increasingly manifest as one descends in dimension. Thus, to fully expose the structure of U-duality with respect to squaring we should consider the product in $D = 3$ of left Yang–Mills theories with $\mathcal{N} = 1, 2, 4, 8$ and right Yang–Mills theories with $\mathcal{N} = 1, 2, 4, 8$. This was done in [49]. The result revealed a rather intriguing mathematical structure. The U-duality algebras obtained make up the Freudenthal–Rosenfeld–Tits magic square [50–52] as given in table 2. As we shall explain this surprise has an elegant explanation, but first we must spend some time on the magic square itself.

Note, the real forms appearing in table 2 are not unique; there are numerous possibilities as described in [53]. They also play a role in supergravity. In particular, the $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{O}$ rows of one such magic square (distinct from table 2) describe the U-dualities of the aply named magic supergravities in $D = 5, 4, 3$, respectively [54–56]. It should be emphasized, however, that the appearance of the magic square here is unrelated to these constructions.

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*We stop at $D = 3$, which has $E_{6(6)}$ U-duality, the largest finite dimensional exceptional Lie algebra. One can continue to $D = 2, 1, 0$, invoking the infinite dimensional extended algebras $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ [45–48]. Although we will not discuss these cases here, it would be interesting to investigate whether they can be understood from the perspective of Yang–Mills squared.*
5.1. Division algebras and the magic square

In this section we follow closely [57, 58]; we refer the reader to these works for more detailed explanations and proofs. An algebra \( \mathbb{A} \) defined over \( \mathbb{R} \) with identity element \( e_0 \) is said to be \textit{composition} if it has a non-degenerate quadratic form\(^5\) \( n : \mathbb{A} \to \mathbb{R} \) such that

\[
n(ab) = n(b)n(a), \quad \forall a, b \in \mathbb{A},
\]

where we denote the multiplicative product of the algebra by juxtaposition. Regarding \( \mathbb{R} \subset \mathbb{A} \) as the scalar multiples of the identity \( \mathbb{R}e_0 \) we may decompose \( \mathbb{A} \) into its ‘real’ and ‘imaginary’ parts \( \mathbb{A} = \mathbb{R} \oplus \mathbb{A}' \), where \( \mathbb{A}' \subset \mathbb{A} \) is the subspace orthogonal to \( \mathbb{R} \). An arbitrary element \( a \in \mathbb{A} \) may be written \( a = \text{Re}(a) + \text{Im}(a) \). Here \( \text{Re}(a) \in \mathbb{R}e_0, \text{Im}(a) \in \mathbb{A}' \) and

\[
\text{Re}(a) = \frac{1}{2}(a + \bar{a}), \quad \text{Im}(a) = \frac{1}{2}(a - \bar{a}),
\]

where we have defined conjugation using the bilinear form,

\[
\bar{a} := \langle a, e_0 \rangle e_0 - a, \quad \langle a, b \rangle := n(a + b) - n(a) - n(b).
\]

A composition algebra \( \mathbb{A} \) is said to be \textit{division} if it contains no zero divisors,

\[
ab = 0 \quad \Rightarrow \quad a = 0 \quad \text{or} \quad b = 0,
\]

in which case \( n \) is positive semi-definite and \( \mathbb{A} \) is referred to as a normed division algebra. Hurwitz’s celebrated theorem states that there are exactly four normed division algebras [59]: the reals, complexes, quaternions and octonions, denoted respectively by \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \). They may be constructed via the Cayley–Dickson doubling procedure, \( \mathbb{A}' = \mathbb{A} \oplus \mathbb{A} \) with multiplication in \( \mathbb{A}' \) defined by

\[
(a, b)(c, d) = (ac - db, \bar{a}d + cb).
\]

With each doubling a property is lost as summarized here:

\begin{table}[h]
\centering
\caption{The magic square of U-duality algebras obtained from the product of two Yang–Mills theories in \( D = 3 \) spacetime dimensions.}
\begin{tabular}{|l|l|l|l|l|}
\hline
\( A' \otimes A \) & 1 & 2 & 4 & 8 \\
\hline
\( \mathfrak{s}(2, \mathbb{R}) \) & \( \mathfrak{su}(2, 1) \) & \( \mathfrak{sp}(4, 2) \) & \( \mathfrak{l}_{4(2-20)} \) & \\
\hline
\( \mathfrak{s}(2, 1) \) & \( \mathfrak{su}(2, 1) \times \mathfrak{su}(2, 1) \) & \( \mathfrak{su}(4, 2) \) & \( \mathfrak{e}_{6(-14)} \) & \( \mathfrak{e}_{7(-5)} \) & \( \mathfrak{e}_{8(8)} \) \\
\hline
\end{tabular}
\end{table}

On doubling the octonions, \( \mathbb{S} \cong \mathbb{O} \oplus \mathbb{O} \), the division property fails and we will not consider such cases here. Note that, while the octonions are not associative they are alternative:

\[
[a, b, c] := (ab)c - a(bc) \quad (5.5)
\]

is an alternating function under the interchange of its arguments. This property is crucial for supersymmetry.

An element \( a \in \mathbb{O} \) may be written \( a = a^e e_0 \), where \( a = 0, \ldots, 7, a^e \in \mathbb{R} \) and \( \{e_i\} \) is a basis with one real \( e_0 \) and seven \( e_i, i = 1, \ldots, 7 \), imaginary elements. The octonionic multiplication rule is

\[
e_{a}e_{b} = (\delta_{ab}e_{c} + \delta_{0b}\delta_{ac} - \delta_{0a}\delta_{bc} + C_{abc})e_c, \quad (5.6)
\]

where \( C_{abc} \) is totally antisymmetric and \( C_{0bc} = 0 \). The non-zero \( C_{ijk} \) are given by the Fano plane. See figure 2.

There are three symmetry algebras on \( \mathbb{A} \) that we will make use of:

(1) The \textit{norm preserving} algebra is defined as

\[
\mathfrak{s}(\mathbb{A}) := \{A \in \text{Hom}_{\mathbb{R}}(\mathbb{A}) \mid \langle AA, b \rangle = 0, \quad \forall a, b \in \mathbb{A}\},
\]

yielding

\[
\mathfrak{s}(\mathbb{R}) \cong \mathbb{R}, \quad \mathfrak{s}(\mathbb{C}) \cong \mathfrak{s}(2), \quad \mathfrak{s}(\mathbb{H}) \cong \mathfrak{s}(3) \oplus \mathfrak{s}(3), \quad \mathfrak{s}(\mathbb{O}) \cong \mathfrak{s}(8). \quad (5.8)
\]

(2) The \textit{triality} algebra of \( \mathbb{A} \) is defined as

\[
\text{tri}(\mathbb{A}) := \{(A, B, C) \in \mathfrak{s}(\mathbb{A}) \oplus \mathfrak{s}(\mathbb{A}) \oplus \mathfrak{s}(\mathbb{A}) \mid A(ab) = B(a)b + aC(b), \quad \forall a, b \in \mathbb{A}\},
\]

yielding

\[
\text{tri}(\mathbb{R}) \cong \mathbb{O}, \quad \text{tri}(\mathbb{C}) \cong \mathfrak{s}(2) \oplus \mathfrak{s}(2), \quad \text{tri}(\mathbb{H}) \cong \mathfrak{s}(3) \oplus \mathfrak{s}(3) \oplus \mathfrak{s}(3), \quad \text{tri}(\mathbb{O}) \cong \mathfrak{s}(8). \quad (5.10)
\]

\[^5\text{A quadratic norm on a vector space } V \text{ over a field } \mathbb{R} \text{ is a map } n : V \to \mathbb{R} \text{ such that: } (1) \quad n(\lambda a) = \lambda^2 n(a), \lambda \in \mathbb{R}, a \in V \quad \text{and} \quad (2) \quad (a, b) = n(a + b) - n(a) - n(b) \text{ is bilinear.}\]
One can regard the triality algebra as a generalized form of the derivation algebra defined as

\[ \text{det}(A) = \{ A \in \text{Hom}_\Bbbk(A) \mid A(ab) = A(a)b + aA(b) \}, \]

which for \( A = \emptyset \) gives the smallest exceptional Lie algebra

\[ \text{det}(\emptyset) \cong \emptyset, \]

\[ \text{det}(\Bbbk) \cong \emptyset, \]

\[ \text{det}(\Bbbh) \cong \mathfrak{so}(3), \]

\[ \text{det}(\emptyset^2) \cong \emptyset_{2(-14)}. \]

(5.11)

This provides the first example of a division algebraic description of an exceptional Lie algebra. In fact, the entire Freudenthal magic square can be realized in terms of the division algebras. The magic square was the result of an effort to give a unified and geometrically motivated description of Lie algebras, including the remaining exceptional cases of \( \Bbbk, \emptyset_6, \emptyset_7, \emptyset_8 \). The classical Lie algebras \( \mathfrak{so}(n), \mathfrak{su}(2), \mathfrak{sp}(n) \) are very naturally captured by \( \Bbbk, \emptyset, \Bbbh \) geometrical structures, respectively. There are a number of ways of articulating this idea, but perhaps the most concise is in terms of the isometries of projective geometries:

\[ \text{Isom}(\Bbbk \Bbbp^n) \cong \mathfrak{so}(n + 1), \]

\[ \text{Isom}(\emptyset \Bbbp^n) \cong \mathfrak{su}(n + 1), \]

\[ \text{Isom}(\Bbbh \Bbbp^n) \cong \mathfrak{sp}(n + 1). \]

(5.12)

This sequence is rather suggestive; we might expect octonionic projective geometries to yield exceptional Lie algebras. Despite non-associativity it was shown by Moufang [60] that one can consistently construct the octonionic projective line and plane, denoted \( \emptyset \Bbbp^1 \) and \( \emptyset \Bbbp^2 \), respectively.

The latter is often referred to as the Cayley plane. We cannot go beyond \( n = 2 \) for the octonions, which in this context reflects the fact that there is indeed just a finite set of exceptional Lie algebras not belonging to any countably infinite family. The \( \emptyset \Bbbp^1 \) example is constructed in direct analogy with the real, complex and quaternionic cases. It is diffeomorphic to \( S^8 \). The octonionic plane has a more intricate structure. An element \( (a, b, c) \in \Bbbk \) with \( \mathbf{n}(a) + \mathbf{n}(a) + \mathbf{n}(c) = 1 \) and \( (ab)c = a(bc) \) gives a point in \( \emptyset \Bbbp^2 \) (the line through the origin containing \( (a, b, c) \) in \( \Bbbk \)).

It is not difficult to show the space of such elements is a 16-dimensional real manifold embedded in \( \emptyset \Bbbp^3 \) through eight real constraints: \( \mathbf{n}(a) + \mathbf{n}(a) + \mathbf{n}(c) = 1 \) and \( (ab)c = a(bc) \). The lines in \( \emptyset \Bbbp^2 \) are copies of \( \emptyset \Bbbp^1 \) and there is a duality relation sending points/lines into points/lines preserving the incidence structure. Borel showed that \( F_{4(-24)} \) is the isometry group of a 16-dimensional projective plane, which is none other than \( \emptyset \Bbbp^2 \). One can show that the points and lines in \( \emptyset \Bbbp^2 \) are in one-to-one incidence preserving correspondence with trace 1 and 2 projectors in the Jordan algebra of \( 3 \times 3 \) octonionic Hermitian matrices \( \mathfrak{H}(3)(\emptyset) \) (treating projectors as propositions the incidence relation in \( \mathfrak{H}(3)(\emptyset) \) is given by implication) [61]. Then \( F_{4(-24)} = \text{Isom}(\emptyset \Bbbp^2) \) follows automatically from the result of Chevalley and Schafer that \( F_{4(-24)} = \text{Aut}(\mathfrak{H}(3)(\emptyset)) \), the group preserving the Jordan product with Lie algebra \( \text{det}(\mathfrak{H}(3)(\emptyset)) \) [62]. In summary, the sequence in (5.13) is continued to include

\[ \text{Isom}(\emptyset \Bbbp^2) \cong \text{det}(\mathfrak{H}(3)(\emptyset)) \cong F_{4(-24)}. \]

Since \( F_{4(-24)} \) acts transitively on the space of trace one projectors and the stabiliser of a given trace one projector is isomorphic to Spin(9) we have

\[ \emptyset \Bbbp^2 \cong F_{4(-24)}/\text{Spin}(9). \]

(5.14)

The Cayley plane is a homogenous symmetric space with \( T_\emptyset(\emptyset \Bbbp^2) \cong \emptyset \Bbbp^2 \), which carries the spinor representation of Spin(9); under \( F_{4(-24)} \cong \text{Spin}(9) \) we have

\[ 52 \to 36 + 16, \]

(5.15)

or in a more division algebraic form

\[ F_{4(-24)} \cong \mathfrak{so}(9) + 16 \]

\[ \cong \mathfrak{so}(\emptyset \oplus \emptyset) + \emptyset^2 \]

\[ \cong \mathfrak{so}(\emptyset) + \emptyset + \emptyset + \emptyset. \]

(5.16)

The three \( \emptyset \) terms in the final line transform in the three triality related eight-dimensional representations of \( \mathfrak{so}(8) \), the vector, spinor and conjugate spinor. It is this triality relation which implies that \( \text{tr}(\emptyset) \cong \mathfrak{so}(\emptyset) \).

Seemingly inspired by the trivial identity \( \emptyset \cong \emptyset \Bbbk \oplus \emptyset \)

Boris Rosenfeld [52] proposed a natural extension of this

\[ \text{One way to understand this is in terms of Jordan algebras. Points in } \emptyset \Bbbp^2 \]

\[ \text{are bijectively identified with trace 1 projectors in } \emptyset \Bbbp^2, \text{ the Jordan algebra of } 3 \times 3 \text{ octonionic Hermitian matrices. However, for } m \times m \text{ octonionic Hermitian matrices do not form a Jordan algebra.} \]

\[ \text{Non-associativity, however, implies that the line through the origin containing the point } (a, b) \text{ is not given by } \{(sx, ob|x \in \emptyset)\}, \text{ unless } x = 1 \text{ or } y = 1. \text{ This obstacle is easily avoided as all non-zero octonions have an inverse; } (a, b) \text{ is equivalent to } (b^{-1}a, 1) \text{ or } (1, a^{-1}b) \text{ for } b \neq 0 \text{ or } a \neq 0, \text{ giving two charts with a smooth transition function on their overlap. See [58].} \]
Table 3. The magic square given by the Tits’ construction.

| ⊙   | R     | C     | H    | O    |
|-----|-------|-------|------|------|
| R   | su(2) | su(3) | sp(6) | f_{4(-52)} |
| C   | su(3) | su(3) × su(3) | su(6) | ε_{6(-78)} |
| H   | sp(6) | su(6) | so(12) | ε_{7(-133)} |
| O   | f_{4(-52)} | ε_{6(-78)} | ε_{7(-133)} | ε_{8(-248)} |

construction

\[ \mathcal{I} = \mathcal{S}(\mathbb{C} \otimes \mathbb{O}) \cong \epsilon_{6(-78)}, \]
\[ \mathcal{I} = \mathcal{S}(\mathbb{H} \otimes \mathbb{O}) \cong \epsilon_{7(-133)}, \]
\[ \mathcal{I} = \mathcal{S}(\mathbb{O} \otimes \mathbb{O}) \cong \epsilon_{8(-248)}, \]

(5.18)

thus giving a uniform geometric description for all Lie algebras. The would-be tangents spaces \( \mathcal{A} \otimes \mathbb{O} \) have the correct dimensions and representation theoretic properties. However, it is not actually possible to construct projective spaces over \( \mathbb{H} \otimes \mathbb{O} \) and \( \mathbb{O} \otimes \mathbb{O} \) using the logic applied to \( \mathbb{O} \), essentially because they do not yield Jordan algebras, unlike \( \mathbb{C} \otimes \mathbb{O} \). They nonetheless can be identified with Riemannian geometries with isometries \( E_{6(-133)} \) and \( E_{8(-248)} \), respectively. Indeed, the Lie algebra decompositions

\[ f_{4(-52)} \cong \mathfrak{so}(\mathbb{R} \otimes \mathbb{O}) + (\mathbb{R} \otimes \mathbb{O})^2, \]
\[ \epsilon_{6(-78)} \cong \mathfrak{so}(\mathbb{C} \otimes \mathbb{O}) + \mathfrak{u}(1) + (\mathbb{C} \otimes \mathbb{O})^2, \]
\[ \epsilon_{7(-133)} \cong \mathfrak{so}(\mathbb{H} \otimes \mathbb{O}) + \mathfrak{sp}(1) + (\mathbb{H} \otimes \mathbb{O})^2, \]
\[ \epsilon_{8(-248)} \cong \mathfrak{so}(\mathbb{O} \otimes \mathbb{O}) + (\mathbb{O} \otimes \mathbb{O})^2, \]

(5.19)

naturally suggest the identifications

\[ \mathcal{I} = \mathcal{S}(\mathbb{R} \otimes \mathbb{O}) \cong F_{4(-52)}/\mathrm{Spin}(9), \]
\[ \mathcal{I} = \mathcal{S}(\mathbb{C} \otimes \mathbb{O}) \cong E_{6(-78)}/[(\mathrm{Spin}(10) \times U(1))/\mathbb{Z}_4], \]
\[ \mathcal{I} = \mathcal{S}(\mathbb{H} \otimes \mathbb{O}) \cong E_{7(-133)}/[(\mathrm{Spin}(10) \times \mathbb{Sp}(1))/\mathbb{Z}_2], \]
\[ \mathcal{I} = \mathcal{S}(\mathbb{O} \otimes \mathbb{O}) \cong E_{8(-248)}/[(\mathrm{Spin}(16))/\mathbb{Z}_2], \]

(5.20)

with tangent spaces

\( (\mathbb{R} \otimes \mathbb{O})^2, (\mathbb{C} \otimes \mathbb{O})^2, (\mathbb{H} \otimes \mathbb{O})^2, (\mathbb{O} \otimes \mathbb{O})^2 \) carrying the appropriate spinor representations. Using the Tits’ construction \([51]\) the isometry algebras are given by the natural generalization of (5.14),

\[ f_{4(-52)} = \det(\mathbb{R}) \oplus \det(\mathbb{J}_3(\mathbb{O}))+\mathrm{Im}\mathbb{R} \otimes \mathcal{J}'_3(\mathbb{O}), \]
\[ \epsilon_{6(-78)} = \det(\mathbb{C}) \oplus \det(\mathbb{J}_3(\mathbb{O}))+\mathrm{Im}\mathbb{C} \otimes \mathcal{J}'_3(\mathbb{O}), \]
\[ \epsilon_{7(-133)} = \det(\mathbb{H}) \oplus \det(\mathbb{J}_3(\mathbb{O}))+\mathrm{Im}\mathbb{H} \otimes \mathcal{J}'_3(\mathbb{O}), \]
\[ \epsilon_{8(-248)} = \det(\mathbb{O}) \oplus \det(\mathbb{J}_3(\mathbb{O}))+\mathrm{Im}\mathbb{O} \otimes \mathcal{J}'_3(\mathbb{O}), \]

(5.21)

where \( \mathcal{J}' \) denotes the subset of traceless elements in \( \mathcal{J} \). Generalising further, the Tits’ construction defines a Lie algebra

\[ \mathcal{M}(\mathcal{A}_1, \mathcal{A}_2) = \det(\mathcal{A}_1) \oplus \det(\mathbb{J}_3(\mathcal{A}_2))+\mathrm{Im}\mathbb{A}_1 \otimes \mathcal{J}'_3(\mathbb{A}_2), \]

(5.22)

for an arbitrary pair \( \mathcal{A}_1, \mathcal{A}_2 \), \( \mathcal{A}_2 = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \), which yields the (compact) magic square given in table 3. The ‘magic’ is that table 3 symmetric about the diagonal despite the apparent asymmetry of (5.22).

To obtain a magic square with the non-compact real forms that follow from squaring Yang–Mills, as given in table 2, one can use a Lorentzian Jordan algebra \([53]\).

\[ \mathcal{M}(\mathcal{A}_1, \mathcal{A}_2) = \det(\mathcal{A}_1) \oplus \det(\mathbb{J}_3(\mathcal{A}_2))+\mathrm{Im}\mathbb{A}_1 \otimes \mathcal{J}'_3(\mathbb{A}_2). \]

(5.23)

Later we shall see that Yang–Mills squared gives an alternative form of (5.23), based on the Barton–Sudbery triality construction \([57]\), that is manifestly symmetric in \( \mathcal{A}_1, \mathcal{A}_2 \) \([49, 63]\). This symmetric form reflects the fact that the squaring procedure is itself symmetric on interchanging the left and right theories.

5.2. Division algebras and Yang–Mills theories

In the two previous sections we saw that the ‘square’ of \( D = 3 \) super Yang–Mills theories and the ‘square’ of division algebras both led to the magic square of Freudenthal. Surely this is no coincidence. Indeed, there is a long history of work connecting supersymmetry, spacetime and the division algebras \([53–56, 58, 64–93]\), which as we shall review underlies this magical meeting.

Perhaps the most direct link from division algebras to spacetime symmetries comes via the Lie algebra isomorphism of Sudbery \([69]\),

\[ \mathfrak{sl}(2, \mathbb{A}) \cong \mathfrak{so}(1, 1 + \dim \mathbb{A}), \]

(5.24)

which identifies \( D = 3, 4, 6, 10 \) as algebraically special. This is itself tied to the earlier observation of Kugo and Townsend \([68]\) that the existence of minimal super Yang–Mills multiplets in only \( D = 3, 4, 6, 10 \) is related to the uniqueness of \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \). This was followed-up by a number of authors \([94–98]\), sharpening the correspondence between supersymmetry and division algebras. The final case of \( D = 10 \) was developed most carefully in \([87]\), where the link between supersymmetry and the alternativity of \( \mathbb{O} \) was emphasized.

Puling together these ideas, it was shown in \([99]\) that \( N \)-extended super Yang–Mills theories in \( D = n + 2 \) dimensions are completely specified (the field content, Lagrangian and transformation rules) by selecting an ordered pair of division algebras \( \mathbb{A}_n \) for the spacetime dimension and \( A_{nN} \) for the degree of supersymmetry, where the subscripts denote the dimension of the algebras.

Consequently, the dual appearances of the magic square in \( D = 3 \), or equivalently for \( A_n = \mathbb{R} \), can be explained by the observation that \( D = 3, N = 1, 2, 4, 8 \) Yang–Mills theories can be formulated with a single Lagrangian and a single set of transformation rules, using fields valued in \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \),
respectively \cite{49}. Tensoring an \(\mathbb{A}\)-valued \(D = 3\) super Yang–Mills multiplet with an \(\mathbb{A}\)-valued \(D = 3\) super Yang–Mills multiplet yields a \(D = 3\) supergravity multiplet with fields valued in \(\mathbb{A} \otimes \mathbb{A}\), making a magic square of \(U\)-dualities appear rather natural.

Let us now review in some more detail these constructions. The Lagrangian for \((n + 2)\)-dimensional \(N = 1\) super Yang–Mills with gauge group \(G\) over the division algebra \(\mathbb{A}_n\) is given \cite{99} by
\[
\mathcal{L}(\mathbb{A}_n) = -\frac{1}{4} F_{\mu \nu}^{A} F^{A \mu \nu} - \text{Re}(i \lambda^A \bar{\partial}_\mu D_\mu A^A), \quad \lambda \in \mathbb{A}_n^2,
\] (5.25)
where the covariant derivative and field strength are given by the usual expressions
\[
D_\mu A^A = \partial_\mu A^A + g g^B_{BC} A^B A^C, \\
F_{\mu \nu}^A = \partial_\mu A_{\nu}^A - \partial_\nu A_{\mu}^A + g g^B_{BC} A_\mu^B A_{\nu}^C,
\] (5.26)
with \(A = 0, \ldots, \text{dim}[G]\). The \(\{\sigma^\mu\}\) are a basis for \(\mathbb{A}_n\)-valued Hermitian matrices—the straightforward generalization of the usual complex Pauli matrices \cite{79, 80, 99} to all four normed division algebras, satisfying the usual Clifford algebra relations. We can use these to write the supersymmetry transformations:
\[
\delta A^A_\mu = \text{Re}(i \lambda^A \bar{\partial}_\mu \epsilon), \quad \delta \lambda^A = \frac{1}{4} F_{\mu \nu}^{A} \sigma^{\mu \nu} (\bar{\delta} \epsilon).
\] (5.27)
Note, since the octonions are non-associative the ordering of the parentheses is important. Moreover, the components \(\lambda^A\) are anti-commuting; we are dealing with the algebra of octonions defined over the Grassmanns and we cannot rely on the usual spinor identities to hold automatically. However, everything goes through, thanks principally to alternativity.

By dimensionally reducing these theories using the Dixon-halving techniques of \cite{99}, we arrive at the Lagrangian for super Yang–Mills in \(D = n + 2\) with \(N\) supersymmetries associated with spacetime \(\mathbb{A}_n\) viewed as a subalgebra of \(\mathbb{A}_n\). The resulting Lagrangian is:
\[
\mathcal{L}(\mathbb{A}_n, \mathbb{A}_n N) = -\frac{1}{4} F_{\mu \nu}^{A} F^{A \mu \nu} - \frac{1}{2} (D_\mu \phi^A A^{\nu} \phi^A) - \text{Re}(i \lambda^A \bar{\partial}_\mu D_\mu A^A) - g g^B_{BC} \text{Re}(i \lambda^A \bar{\partial}_\mu D_\mu A^A) - \frac{1}{16} g g^B_{BC} f^{BDE} (\phi^B \phi^D) (\phi^C \phi^F),
\] (5.28)
where \(\lambda \in \mathbb{A}_n^2\) (so we have \(N\) spacetime spinors, each valued in \(\mathbb{A}_n^2\)) and \(\phi\) is a scalar field taking values in \(\phi \in \mathbb{A}_n^2\), the subspace of \(\mathbb{A}_n N\) orthogonal to the \(\mathbb{A}_n\) subalgebra. The \(\{\bar{\sigma}^\mu\}\) are still a basis for \(\mathbb{A}_n\)-valued Hermitian matrices, again, with \(\mathbb{A}_n\) viewed as a division subalgebra of \(\mathbb{A}_n N\). As noted in \cite{99}, the overall (spacetime little group plus internal) symmetry of the \(N = 1\) theory in \(D = n + 2\) dimensions is given by the triality algebra, \(\text{tri}(\mathbb{A}_n)\). If we dimensionally reduce these theories we obtain super Yang–Mills with \(N\) supersymmetries whose overall symmetries are given by
\[
\text{sym}(\mathbb{A}_n, \mathbb{A}_n N) := \{ (A, B, C) \in \text{tr}(\mathbb{A}_n N) \ |
\left[ [A, \sigma_0(\mathbb{A}_n N)]_\mathbb{A} = 0, \forall A \not\in \sigma_0(\mathbb{A}_n N) \right] \},
\] (5.29)
where \(\sigma_0(\mathbb{A}_n N)\) is the subalgebra of \(\sigma(\mathbb{A}_n N)\) that acts as orthogonal transformations on \(\mathbb{A}_n \subseteq \mathbb{A}_n N\). The division algebras used in each dimension and the corresponding \(\text{sym}\) algebras are summarized in table 4.

Let us take \(D = 3\) as a concrete example. The \(N = 8\) Lagrangian is given by
\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu}^{A} F^{A \mu \nu} - \frac{1}{2} D_\mu \phi^A A^{\nu} \phi^A + i \lambda^A \gamma^\mu D_\mu \lambda^A - \frac{1}{2} g g^B_{BC} f^{BCD} (\phi^B \phi^D) (\phi^C \phi^F) - g g^B_{BC} \phi^B \phi^C \lambda^A \Gamma^i_{ab} \phi^F, \]
(5.30)
where \(\Gamma^i_{ab}, i = 1, \ldots, 7, a, b = 0, \ldots, 7,\) belongs to the SO (7) Clifford algebra. The key observation is that this gamma matrix can be represented by the octonionic structure constants
\[
\Gamma^i_{ab} = i (\delta_{bi} \delta_{0ai} - \delta_{bi} \delta_{ai} + C_{iab}),
\] (5.31)
which allows us to rewrite the action over octonionic fields. If we replace \(\mathbb{O}\) with a general division algebra \(\mathbb{A}\), the result is \(N = 1, 2, 4, 8\) over \(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\):
\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu}^{A} F^{A \mu \nu} - \frac{1}{2} D_\mu \phi^A A^{\nu} \phi^A + i \lambda^A \gamma^\mu D_\mu \lambda^A - \frac{1}{2} g g^B_{BC} f^{BCD} (\phi^B \phi^D) (\phi^C \phi^F) + i g g^B_{BC} \phi^B \phi^C \lambda^A \lambda^F - \frac{1}{16} g g^B_{BC} f^{BDE} (\phi^B \phi^D) (\phi^C \phi^F),
\] (5.32)
where \(\phi = \phi^a e^a\) is an \(\mathbb{A}\)-valued scalar field, \(\lambda = \lambda^a e^a\) is an \(\mathbb{A}\)-valued two-component spinor and \(\bar{\lambda} = \bar{\lambda}^a e^a\).

The supersymmetry transformations in this language are given by
\[
\delta \lambda^A = \frac{1}{2} (F^{A \mu} + \varepsilon^{\mu} \phi^A) \sigma_\mu \epsilon - \frac{1}{4} g g^B_{BC} \phi^B (\phi^C \epsilon), \quad \delta A^A_\mu = i \left( \bar{\epsilon} \gamma_\mu \lambda^A - \lambda^A \gamma_\mu \epsilon \right), \quad \delta \phi^A = i \left[ \bar{\epsilon} \gamma_\mu \lambda^A - \lambda^A \gamma_\mu \epsilon \right],
\] (5.33)
where \(\epsilon\) is an \(\mathbb{A}\)-valued two-component spinor and \(\sigma_\mu\) are the generators of SL(2, \(\mathbb{R}\)) \(\cong 0\). The form of the first term in the \(\lambda^A\) transformation also highlights the vector’s status as the missing real part of the Im \(\mathbb{A}\)-valued scalar field. Indeed, in the free \(g = 0\) theory one may dualize the vector to a scalar to obtain a full \(\mathbb{A}\)-valued field.

Now consider the product of two division algebraic multiplets:
\[(1) \text{ A left } N = \text{ dim } \mathbb{A} \text{ multiplet} \]
\[\{ A_\mu \in \text{ Re } \mathbb{A}, \quad \phi \in \text{ Im } \mathbb{A}, \quad \lambda \in \mathbb{A}\}.
\] (5.34)
Since \( S \approx \sim S \), we can consider the familiar \( R \)-symmetries in \( D = 4: U(1), U(2) \) and \( SU(4) \) for \( N = 1, 2, 4 \), respectively. Note that the symmetries in \( D = 3 \) are entirely internal and that they include the \( R \)-symmetry as a subgroup (these are actually the symmetries of the theories after dualising the vector to a scalar).

| \( A_\mathbb{R} \setminus A_{nN} \) | \( O \) | \( H \) | \( C \) | \( R \) |
|---|---|---|---|---|
| \( O \) | \( so(8)_{ST} \) | \( so(4)_{ST} \oplus sp(1) \oplus sp(1) \) | \( so(4)_{ST} \oplus sp(1) \) | \( \emptyset \) |
| \( H \) | \( so(2)_{ST} \oplus su(4) \) | \( so(2)_{ST} \oplus sp(1) \oplus so(2) \) | \( so(2) \oplus so(2) \) | \( \emptyset \) |
| \( C \) | \( so(8) \) | \( so(2) \oplus sp(1) \) | \( so(2) \oplus so(2) \) | \( \emptyset \) |

The \( \mathcal{H} \) algebra in terms of the left/right super Yang–Mills theories in a division algebraic language.

(2) A right \( \tilde{N} = \dim \tilde{A} \) multiplet

\[
\begin{align*}
\{ \tilde{A}_\nu & \in \text{Re } \tilde{A}, \quad \tilde{\varphi} \in \text{Im } \tilde{A}, \quad \tilde{\chi} \in \tilde{A} \}.
\end{align*}
\]  

We obtain the field content of an \( (\mathcal{N} + \tilde{\mathcal{N}}) \)-extended supergravity theory valued in both \( \tilde{A} \) and \( \tilde{\mathcal{A}} \):

\[
\begin{align*}
g_{\mu\nu} & \in \mathbb{R}, \quad \Psi_{\mu} \in \left( \tilde{A} \oplus \tilde{\mathcal{A}} \right), \quad \varphi, \chi \in \left( \tilde{A} \otimes \tilde{\mathcal{A}} \right) \end{align*}
\]  

The \( \mathbb{R} \)-valued graviton and \( \tilde{A} \oplus \tilde{\mathcal{A}} \)-valued gravitino carry no degrees of freedom. The \( \tilde{A} \otimes \tilde{\mathcal{A}} \)-valued scalar and Majorana spinor each have \( 2(\dim A \times \dim A) \) degrees of freedom.

The \( \mathcal{H} \) algebra then follows immediately in this division algebraic language. The left and right factors each come with a commuting copy of the triality algebra, \( \text{tri}(\tilde{A}) \oplus \text{tri}(\tilde{\mathcal{A}}) \). However, the \( \tilde{A} \otimes \tilde{\mathcal{A}} \) doublet in (5.36) form irreducible representations of \( R \)-symmetry. The corresponding generators must themselves transform under \( \text{tri}(\tilde{A}) \oplus \text{tri}(\tilde{\mathcal{A}}) \) consistently, implying they are elements of \( \tilde{A} \otimes \tilde{\mathcal{A}} \). This follows, formally, from the left/right supersymmetries. The conventional infinitesimal supersymmetry variation of the left \( \otimes \) right states correctly gives the infinitesimal supersymmetry variation on the corresponding supergravity states \([28, 29, 41]\). Seeking, instead, internal bosonic transformations on the supergravity multiplet suggests starting from the rather unconventional tensor product of the left and right supercharges, \( \tilde{Q} \otimes \bar{Q} \). See figure 3. This follows, at least formally, from the observation \( \tilde{Q} \otimes \bar{Q} \in \tilde{A} \otimes \tilde{\mathcal{A}} \), where we are explicitly suppressing the spacetime representation space. Note, these are ‘pseudo-supersymmetry’ transformations since they do not change the mass dimension of the component fields. For an explicit construction see \([40]\). In summary, we have in total:

\[
\mathfrak{h} \left( \tilde{A}, \tilde{\mathcal{A}} \right) := \text{tri}(\tilde{A}) \oplus \text{tri}(\tilde{\mathcal{A}}) + \tilde{A} \otimes \tilde{\mathcal{A}},
\]

This Lie algebra, see \([63]\) for the commutators, yields the maximal compact subalgebras of the corresponding \( U \)-dualities, given in table 5.

The \( U \)-dualities \( G \) are realized non-linearly on the scalars, which parametrize the symmetric spaces \( G/\mathcal{H} \). This can be understood using the identity relating \( \tilde{A} \otimes \tilde{\mathcal{A}} \) to \( G/\mathcal{H} \).

\[
\left( \tilde{A} \otimes \tilde{\mathcal{A}} \right)^p \cong G/\mathcal{H}.
\]

The scalar fields may be regarded as points in division-algebraic projective planes. The tangent space \( T_p(G/\mathcal{H}) \cong \mathfrak{g} \oplus \mathfrak{h} \) implies the scalars carry the \( p \)-representation of \( \mathcal{H} \). The tangent space at any point of \( \mathfrak{g} \otimes \mathfrak{h} \) is just \( \mathfrak{g} \otimes \mathfrak{h} \), the required representation space of \( \mathcal{H} \). Since \( G/\mathcal{H} \) is a symmetric space, the \( U \)-duality Lie algebra is given by adjoining the scalar representation space \( \mathfrak{g} \otimes \mathfrak{h} \) to figure 3.

\[
\mathfrak{m} \left( \tilde{A}, \tilde{\mathcal{A}} \right) := \text{tri}(\tilde{A}) \oplus \text{tri}(\tilde{\mathcal{A}}) + \left( \tilde{A} \otimes \tilde{\mathcal{A}} \right)^2.
\]

This has a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) graded Lie algebra structure uniquely determined by the left/right super Yang–Mills factors and yields precisely the magic square. See \([63]\) for a full account of the commutation relations. The triality construction described in \([57]\) is isomorphic to (5.39) as a vector space, but has a different Lie algebra structure, as reflected in the distinct real forms appearing in each case. In conclusion, the product of division algebras and super Yang–Mills theories both lead to the magic square, as depicted in figure 4.

For \( D = n + 2 \), we begin with a pair of Yang–Mills theories with \( \mathcal{N} \) and \( \tilde{\mathcal{N}} \) supersymmetries written over the division algebras \( A_{nN} \) and \( A_{n\tilde{N}} \), respectively, described by (5.28). In terms of spacetime little group representations we may then write all the bosons of the left (right) theory as a single element \( b \in A_{nN} \) (\( \tilde{b} \in A_{n\tilde{N}} \)), and similarly for the fermions \( f \in A_{nN} \) (\( \tilde{f} \in A_{n\tilde{N}} \)). After tensoring we arrange the resulting supergravity fields into a bosonic doublet and a
fermionic doublet

\[ \begin{pmatrix} b \otimes \bar{b} \\ f \otimes \bar{f} \end{pmatrix}, \quad F = \begin{pmatrix} b \otimes \bar{f} \\ f \otimes \bar{b} \end{pmatrix}. \tag{5.40} \]

just as we did in \( D = 3 \). The algebra (5.37) acts naturally on these doublets. However, a diagonal \( \mathfrak{so}(\mathbb{A}_n)_{\text{ST}} \) subalgebra of this corresponds to spacetime transformations, so we must restrict \( \mathfrak{h}(\mathbb{A}_n, \mathbb{A}_n) \) to the subalgebra that commutes with \( \mathfrak{so}(\mathbb{A}_n)_{\text{ST}} \). Heuristically, we identify a diagonal spacetime subalgebra \( \mathbb{A}_n \) in \( \mathbb{A}_n \otimes \mathbb{A}_n \) and require that it is preserved by the global isometries, which picks out a subset in \( \text{Isom}(\mathbb{A}_n \otimes \mathbb{A}_n) \). Imposing this condition selects the U-duality algebra of the \( D = n + 2 \), \( (\mathbb{N} + \bar{\mathbb{N}}) \)-extended
supergravity theory obtained by tensoring $\mathcal{N}$ and $\tilde{\mathcal{N}}$ super Yang–Mills theories. The Lie algebras are given by the magic pyramid formula:

$$\mathfrak{mmpyramid}(A_n, A_{n,N}, A_{n,\tilde{N}})$$

$$= \{ a \in \mathfrak{m}(A_{n,N}, A_{n,\tilde{N}}), a \in \mathfrak{so}(A_n)_{ST}, [(a, a)_{ST}] = 0 \} \quad (5.41)$$

The terminology is made clear by the pyramid of corresponding U-dualities groups presented in figure 5. The base of the pyramid in $D = 3$ is the $4 \times 4$ Freudenthal magic square, while the higher levels are comprised of a $3 \times 3$ square in $D = 4$, a $2 \times 2$ square in $D = 6$ and Type II supergravity at the apex in $D = 10$.

Let us conclude with some comments on the product of theories other than super Yang–Mills. Particularly interesting examples are provided by the superconformal multiplets in $D = 3, 4, 6$. In a manner directly analogous to the magic pyramid the tensor product of left and right superconformal theories yields the ‘conformal pyramid’, described in [63]. It has the remarkable property that its faces are also given by the Freudenthal magic square, as depicted in figure 6. In particular, consider the maximal spine one encounters the famous exceptional sequence $E_{8(8)}$, $E_{7(7)}$, $E_{6(6)}$, but where $E_{6(6)}$ belongs to the $D = 6$, $(4, 0)$ theory proposed by Hull as the superconformal limit of M-theory compactified on a 6-torus [100–102]. This pattern suggests the existence of some highly exotic $D = 10$ theory with $F_{(4)}$ U-duality group. The existence of such a theory would be more than a little surprising and there is a (slightly) more conventional interpretation of the conformal pyramid, including its $F_{(4)}$ tip, but for theories in $D = 3, 4, 5, 6$, as described in [63].

The product of conformal theories in the context of amplitudes has been considered previously in, for example, [42, 103–106]. In particular, the maximally supersymmetric $D = 3, \mathcal{N} = 8$ Bagger–Lambert–Gustavsson (BLG) Chern–Simons–matter theory [107–109] has been shown to enjoy a colour-kinematic duality reflecting its three-algebra structure [105]. The ‘square’ of BLG amplitudes yields those of $\mathcal{N} = 16$ supergravity. Since $\mathcal{N} = 16$ supergravity is the unique theory with 32 supercharges in three dimensions it is also the ‘square’ of the $\mathcal{N} = 8$ Yang–Mills theory. The square of the amplitudes in both cases agree, despite their distinct structures [106].

In $D = 6$ one might expect relations between the ‘square’ of the $\mathcal{N} = (2, 0)$ tensor multiplet and the $\mathcal{N} = (4, 0)$ theory proposed by Hull [100–102], as discussed in [42]. Of course, amplitudes are generically not well-defined in these cases, but one can make some precise statements in terms of the tree-level S-matrix in particular regimes, as discussed in [103, 104]. For example, in the absence of additional degrees of freedom all tree-level amplitudes of the $(2, 0)$ tensor multiplet vanish [103]. The $D = 5, \mathcal{N} = 4$ super Yang–Mills theory squares to give the amplitudes of $D = 5, \mathcal{N} = 8$ supergravity. However, being non-renormalizable it ought to be regarded as a superconformal $D = 6, \mathcal{N} = (2, 0)$ theory compactified on a circle of radius $R = \sqrt{5}/2\pi^2$. At linearized level Hull’s $(4, 0)$ theory follows from the square the $(2, 0)$ theory [110] and gives $\mathcal{N} = 8$, $D = 5$ supergravity when compactified on a circle [101]. From this perspective the $(2, 0) \times (2, 0) = (4, 0)$ identity constitutes an, as yet ill-defined, M-theory up-lift of the maximally supersymmetric $D = 5$ squaring relation.

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