Abstract

Standart Coherent State Systems have an analysis based on lattices (von Neumanns’s lattices) in terms of wich they are classified, looking at the size of the minimum cell, by: complete, overcomplete and not complete. In this work we analyze overcomplete systems with a geometrical viewpoint (holomorphic line-bundles). We apply the method to evaluate the degeneracy of the lowest Landau level.

I. INTRODUCTION

In this work, we are going to investigate the question of overcompletenes of a Coherent State System using the framework of line bundles. By the von Neumann construction (von Neumann lattices) we know that complete systems are related to a certain type of lattice (what we have called complete lattice) and overcomplete systems are related to another one (overcomplete lattice). Analyzing the representation of the Weyl-Heisenberg group induced by characters on these lattices, we associate to a complete system a theta function and with the overcomplete system a set of theta functions with characteristic.

Since theta functions are the sections of line bundles over a complex torus, and since these line bundles can be seen as the holomorphic quantization of a classical system, we have a direct interpretation of the overcompleteness of the system: an overcomplete system (associated with an overcomplete lattice) is a set of quantizations of this classical system. This is expected since to have overcomplete systems we’ve had to consider more than one state by Planck cell. We apply this result to analyze the degeneracy of the lowest Landau level by using the Riemann-Roch theorem in the appropriate line bundle.

In the section 2 we present briefly the question of the overcompletenes. In section 3 we remind some aspects of representation of the Weyl-Heisenberg group induced by characters on the lattice and how it’s related to theta functions. In the section 4 we associate this
result to holomorphic quantizations of the classical system. And in the last section we make
the application of the method to Landau levels.

II. THE OVERCOMPLETENESS OF C.S.S.

Let \( w \) be the Weyl-Heisenberg algebra generated by \( \{ \hat{p}, \hat{q}, I \} \), with the usual commutation
relations:

\[
\begin{align*}
[\hat{p}, \hat{q}] &= i\hbar I, \\
[\hat{q}, I] &= [\hat{p}, I] = 0
\end{align*}
\]

We have the representation in terms of the creation and annihilation operators:

\[
\begin{align*}
\alpha a &= \frac{(\hat{q} - i\hat{p})}{\sqrt{2\hbar}}, \\
\alpha a^+ &= \frac{(\hat{q} + i\hat{p})}{\sqrt{2\hbar}}, \\
\alpha &= \frac{(2\hbar)^{-1/2}}{(Q + iP)}.
\end{align*}
\]

We represent \( x \) also by \( x = (t, \vec{v}) \) or \( x = (t, \vec{\alpha}) \) where \( \vec{\alpha} = \alpha a^+ - \bar{\alpha}a, \vec{v} = (P, Q) \) is a point
in the \( V \times V \) plane and \( t \) is the central term of the algebra.

The elements of the Weyl-Heisenberg Group \( W \) is obtained by the exponentiation map:

\[
\exp(x) = \exp(itI) \exp(\alpha a^+ - \bar{\alpha}a),
\]

and because of the commutation relations, we have:

\[
\exp A \exp B = \exp\left(\frac{1}{2}[A, B]\right) \exp(A + B)
\]

for \( A, B \in w \).

We use also to represent the above equation as:

\[
(t, \vec{v}).(t', \vec{v'}) = \left(t + t' + \frac{1}{2}B(\vec{v}, \vec{v'}), \vec{v} + \vec{v'}\right)
\]

where \( B : V \times V \to \mathbb{R} \) is the alternating bilinear form defined by \( B(\vec{v}, \vec{v'}) = [\vec{v}, \vec{v'}] \).

Observe that we have used (abusing on notation) the same symbols \( (t, \vec{v}) \) to indicate
both, elements of the algebra \( w \) and the group \( W \).

Let \((T_\lambda, \mathcal{H})\) be a irreducible unitary representation of \( W \) on the Hilbert space \( \mathcal{H} \) (in the
next section we are going to review a bit of representation theory of \( W \)). Given an element
\((t, \vec{\alpha})\) of \( W \), we denote the action on a given vector \(|v\rangle\) by \( T_\lambda(t, \vec{\alpha}) |v\rangle \).

Coherent states are vectors \(|\vec{\alpha}\rangle\) of \( \mathcal{H} \), generated by the action of elements of \( W \) in the
form \((0, \vec{\alpha})\) on a fixed vector \(|\Psi_0\rangle\) of \( \mathcal{H} \), and the set of such a vectors form a coherent state
system (C.S.S.). In the case that the vector \(|\Psi_0\rangle\) is a vacuum state \(|0\rangle\), we call the system
by a standard coherent state system. There are other alternative but equivalent definitions
of C.S.S. (see [1] and [7]).
Since, we’ve assumed the representation $T_\lambda$ is irreducible, the set $\{|\vec{\alpha}\rangle\}$ generates the whole space $\mathcal{H}$. Actually, the system $\{|\vec{\alpha}\rangle\}$ is called an overcomplete system, what means we don’t have mutual orthogonality between all the vectors of the set:

$$\langle \vec{\alpha} | \vec{\beta} \rangle = \langle 0 | T_\lambda(0, \alpha^+) T_\lambda(0, \beta^-) | 0 \rangle = \exp (i \Im(\beta \overline{\alpha}) \langle 0 | T_\lambda(0, \beta^--\alpha^-) | 0 \rangle)$$

$$| \langle \vec{\alpha} | \vec{\beta} \rangle |^2 = | \langle 0 | T_\lambda(0, \beta^- - \alpha^-) | 0 \rangle |^2 = \rho(\beta^- - \alpha^-)$$

In fact, $\rho(\vec{\alpha} - \vec{\beta})$ can be not identically zero.

In terms of projectors, we have that the projectors $|\alpha\rangle\langle\alpha|$ are not mutually orthogonal projectors.

To find a orthogonal basis for the C.S.S. von-Neumann announced the existence of a countable orthogonal basis $\{|\alpha_k\rangle\}$ within $\{|\alpha\rangle\}$ when we consider a lattice in the $\alpha$-plane $V$.

Let us take two non-colinear vectors $\{w_1, w_2\}$ such that $B(w_1, w_2) = 2i \Im(w_1 \overline{w}_2) \neq 0$ and let us consider vectors in the form $\alpha_m = m_1 w_1 + m_2 w_2$ / $m_1, m_2 \in \mathbb{Z}$.

von Neumann stated that:

i)The system $\{|\alpha_m\rangle\}$ is going to be over complete if $\Im(w_1 \overline{w}_2) < \pi$, and it remains over complete even if we remove a finite number of vectors from $\{|\alpha_m\rangle\}$

ii)for $\Im(w_1 \overline{w}_2) > \pi$, the system is not complete,

iii)for $\Im(w_1 \overline{w}_2) = \pi$ the system is complete and remains complete even if we remove a single vector from $\{|\alpha_m\rangle\}$.

Now we have two remarks: First, since $\Im(w_1 \overline{w}_2)$ is two times the area of the triangle of vertices $(0, w_1, w_1 + w_2)$, when we consider the vectors $\{|\alpha_m\rangle\}$ with $\Im(w_1 \overline{w}_2) = \pi$, we are considering one state by cell of area $\Im(w_1 \overline{w}_2)$, that means (in our normalization), one state by Planck cell (see [1] and [7]).

Second, we can express $\exp(i \Im(w_1 \overline{w}_2))$ as $T_\lambda(w_1) T_\lambda(w_2) T_\lambda(w_1 + w_2)^{-1} = \exp(i \Im(w_1 \overline{w}_2))$. In geometric quantization (section 4) approach, this term is a holonomy term (see fig.1) of a holomorphic line bundle that make the geometric quantization of the system.

So, asserting that $\exp(i \Im(w_1 \overline{w}_2)) = \pi$, is equivalent to impose the Bohr first quantization for the system (see for instance [7]).

From this two remarks we wish to investigate the meaning of the overcompleteness of the C.S.S.

III. REPRESENTATIONS OF WYEL-HEISENBERG GROUP ON THE LATTICE

Let us consider a complete C.S.S., given by a ”complete” lattice
\[ L = \{ \alpha_m = m_1 w_1 + m_2 w_2 / \text{Im}(w_1 \bar{w}_2) = \pi, m_1, m_2 \in \mathbb{Z} \} \quad (7) \]

If we consider in \( V \) the alternating bilinear form \( B : V \times V \rightarrow \mathbb{R}, B(v, w) = \text{Im}(v \bar{w}) \), we observe that, if we restrict \( B \) to \( L \times L \), the image of \( B \) lies in \( \pi \mathbb{Z} \) and for no other vector \( v \in V, v \not\in L \), we can have \( B(v, \alpha_m) \in \pi \mathbb{Z} \) for any vector \( \alpha_m \in L \).

For a non-complete lattice:

\[ L = \{ \alpha_m = m_1 w_1 + m_2 w_2 / \text{Im}(w_1 \bar{w}_2) > \pi, m_1, m_2 \in \mathbb{Z} \}, \quad (8) \]

if for instance we consider \( \text{Im}(w_1 \bar{w}_2) = k \pi \) for \( k \) a positive integer, \( k \neq 1 \), we have a set of vectors \( \{ v_m = m_1/k w_1 + m_2/k w_2, m_1, m_2 \in \mathbb{Z}(\mod k \mathbb{Z}) \} \), such that \( B(v_m, \alpha_m) \rightarrow \pi \mathbb{Z} \) for any \( \alpha_m \in L \).

In the first case we say that the lattice is self-dual, and in the second case we consider a lattice \( L' = L \cup \{ v_m \} \) dual to \( L \), such that \( B : L' \times L \rightarrow \pi \mathbb{Z} \).

We can describe also \( L' \) as lattice generated by \( \{ w'_1 = w_1/k, w'_2 = w_2/k \} \) such that:

\[ L' = \left\{ m_1 w'_1 + m_2 w'_2 / \text{Im}(w'_1 \bar{w}'_2) = \frac{\pi}{k^2} (< \pi), m_1, m_2 \in \mathbb{Z} \right\}. \quad (9) \]

So, the dual lattice of a ”non-complete” lattice is an ”overcomplete lattice related to a overcomplete system.

We are going to focus now representations of the Weyl-Heisenberg group induced by characters on the lattice.

A character of a Lie Group is a continuous complex valued function \( \chi \) on \( G \) such that \( | \chi(g) | = 1 \), and \( \chi(gg') = \chi(g) \chi(g') \) for \( g, g' \in G \). The associated infinitesimal character is the linear form \( \chi \) in the Lie Algebra \( \text{Lie} G \) of \( G \) characterized by \( \chi(\exp(A)) = \exp(\chi(A)) \).

Let \( \chi \) be a character of some closed subgroup \( H \) of a group \( G \). Let \( \mathcal{H}_\chi \) denote the Hilbert space consisting of all functions \( f \) on \( G \) satisfying the following conditions:

(a) \( f \) is Borel measurable on \( G \)

(b) \( f(hg) = \chi(h)f(g) \) for \( g \) in \( G \) and \( h \) in \( H \)

(c) The integral \( \int_M | f(g) |^2 \, dg \) is finite, for \( M = G/H \)

The norm in \( \mathcal{H}_\chi \) is given by:

\[ \| f \|^2 = \int_M | f(g) |^2 \, dg. \]

Observe that since \( | \chi(h) | = 1 \), \( | f |^2 \) is constant on every coset \( Hg \), and the expression above make sense.

To every \( g \) in \( G \), there is associated an unitary operator \( \pi_\chi(g) \) on \( \mathcal{H}_\chi \) by:

\[ (\pi_\chi(g)f)(g') = f(gg') \quad (10) \]

The pair \( (\pi_\chi, \mathcal{H}_\chi) \) is a representation of \( G \), called representation induced by the character \( \chi \) of \( H \).

Observe that the C.S.S. is a representation of \( W \) induced by characters on the center \( Z \) of \( W \). The characters of \( Z \) are given by the formula:
\[ \chi_\lambda(t, 0) = \exp 2\pi i(\lambda t) \quad \text{(11)} \]

where \( \lambda \) runs \( \mathbb{R} \), and the infinitesimal character associated to \( \chi_\lambda \) is the linear form on \( \tilde{z} \) (= LieZ) is given by:

\[ \chi'_\lambda = 2\pi i\lambda \quad \text{(12)} \]

We have the classification of irreducible unitary representations of \( W \) given by the Stone-von Neumann theorem which asserts that:

(a) For every \( \lambda \neq 0 \), there is, up to unitary equivalence, exactly one irreducible representation \( (\pi, \mathcal{H}) \) satisfying \[ \text{(10)} \]

(b) The case \( \lambda = 0 \) corresponds to the representations which are trivial on the center \( Z \) of \( W \). They are the one-dimensional representations given by the characters \( \chi_u \) of \( W \), given by:

\[ \chi_u(t, v) = \exp(2\pi iB(v, u)) \quad \text{(13)} \]

What we are going to analyze are the representations of \( W \) induced by characters defined on a discrete lattice of \( \Gamma_L \) on \( W \), \( \Gamma_L = \{(t, \alpha_m) = m_1w_1 + m_2w_2, t, w_1, w_2 \in \mathbb{Z}\} \), the group which elements are given by the lattice \( L \) generated by \( (w_1, w_2) \).

The motivation to analyze these representation is pointed on [11] in the analysis of completeness of the system, (in the case that \( \Im(m_1\bar{w}_2) = 2\pi \)) where looking at the expression:

\[ T_\lambda(\vec{\alpha}_m)T_\lambda(\vec{\alpha}_{m'}) = T_\lambda(\vec{\alpha}_m + \vec{\alpha}_{m'}) = T_\lambda(\vec{\alpha}_{m'})T_\lambda(\vec{\alpha}_m) \quad \text{(14)} \]

the author ask for a common eigen-distribution for the operators \( \{T_\lambda(\vec{\alpha}_m)\} \).

If we start considering the base vectors \( w_1, w_2 \), we should have for the expression of this eigen-distribution:

\[ T_\lambda(w_i) \mid \Theta \rangle = \exp(\pi i\varepsilon) \mid \Theta \rangle \quad \text{(15)} \]

because of the unitarity, where \( 0 \leq \varepsilon < 2 \), for \( i = 1, 2 \).

For a generic element \( \vec{\alpha}_m = m_1w_1 + m_2w_2 \) of \( L \), we should have:

\[ T_\lambda(\vec{\alpha}_m) \mid \Theta \rangle = \exp \pi i\lambda(m_1\varepsilon_1 + m_2\varepsilon_2 + m_1m_2) \mid \Theta \rangle \quad \text{(16)} \]

The general form for a character in \( \Gamma_L \) [3] is:

\[ \chi_{p,F}(t, \vec{\alpha}_m) = \exp(\pi ipt) \exp\left(\frac{1}{2}F(\vec{\alpha}_m)\right) \quad \text{(17)} \]

where \( p \) runs the integers and the function \( F(\vec{\alpha}_m) \) should satisfy the following congruence:
\[ F(v_1 + v_2) = F(v_1) + F(v_2) + pB(v_1, v_2) \pmod{2} \quad (18) \]

such that we have:

\[ T_\lambda(\alpha_m) | \theta \rangle = \exp \pi i F(\alpha_m) | \theta \rangle \quad (19) \]

The general result of Cartier \cite{3} is:

Given a representation \( D_{L,p,F} = (\pi(W), \mathcal{H}) \) induced by a character \( \chi_{p,F} \) of \( \Gamma_L \), this representation is irreducible if and only if \( L \) is a self-dual lattice (a "complete" lattice), associated with a complete C.S.S.. In this case \( D_{L,p,F} \) is isomorphic to the representation induced by \( \chi_F \) (item (a) of the Stone-von Neumann theorem).

If \( L \) is not complete, to ever \( \lambda' \) in \( L' \pmod{L} \), that is, given elements of the dual lattice \( L' \) modulo the lattice \( L \), we have an operator that commutes with the induced representation \( D_{L,p,F} \). So, if we have \([L' : L] = e^2\), we have \( e \)-operators that commute with the representation, a direct sum of \( e \)-copies of the irreducible one (when \( L \) is self-dual).

\[ \mathcal{H} = \bigoplus_{i=1}^e \mathcal{H}_i \quad (20) \]

Another result asserts that the invariance equation \cite{14} has, for a given character \( F \) and up to constant multipliers, one unique solution in \( \mathcal{H}_{-\infty} \) (the dual space of \( C^\infty \)-functions \( \mathcal{H}_\infty \)) in the case that \( L \) is self-dual; and, if \([L' : L] = e^2\), the equation has \( e \) linear independent solutions, generating a \( e \)-dimensional subspace of \( \mathcal{H}_{-\infty} \).

We can contemplate both results looking at the solutions of this equation that lie in the ring of Jacobi theta functions, when we consider the holomorphic representation (the Fock-Bargmann representation) of the distributions \{ \( | \Theta \rangle \_\varepsilon \} \), solutions of equation \cite{19}.

We define a complex structure \( J \) in \( V \), such that \( J^2 = -1 \) and \( B(Jv, Jv') = B(v, v') \), \( B(v, Jv) \geq 0 \).

We consider the complexification of \( V \) to \( V_\mathbb{C} \) and the natural extension of \( B \) and \( J \) to their complexified version.

We have an unique hermitian form \( H \) such that:

\[ H(v, v') = B(v, Jv') + iB(v, v') \quad (21) \]

We consider now the representation \( D_{L,\lambda,F} = (\pi, \mathcal{L}^2) \) induced by a character \( \chi_{p,F} \) of \( \Gamma_L \) over the \( \mathcal{L}^2 \)-holomorphic functions on \( V_\mathbb{C} \), with respect to the Kahler potential \( -\pi \lambda H \), such that:

\[ (\phi, \phi') = \int_V e^{-\pi\lambda H(v, v)} \phi(v)\phi^*(v)dv \quad (22) \]

In the case of self-dual lattices (for \( \lambda = 1 \)); the action of \( W \) in this representation is:

\[ (U_v \phi)(v') = e^{-\pi [\frac{H(v,v')}{2} + H(v,v')]} \phi(v + v') \quad (23) \]

and the invariance equation \cite{18} takes the form:

\[ \phi(v + \lambda) = \phi(v) \exp \left\{ \pi \left[ \frac{1}{2} H(\lambda, \lambda) + H(\lambda, v) + iF(\lambda) \right] \right\} \quad (24) \]
for $\lambda \in L$.

This equation has one solution on the ring of the theta functions. For $[L' : L] = e^2 \neq 1$, we will have a set of the $e$-solution of the equation, $\{\Theta_m, m \in L' \mod L\}$. These are the theta functions with characteristic $m$.

These solutions can be generated acting with the $A_\lambda$ operators ($\lambda \in L' \mod L$) on $\Theta$ (the solution of 19 when $[L' : L] = 1$). Since $\{A_\lambda\}$ commute with the group $\Gamma_L$, the resulting functions are also solutions of 19.

The linear independence of the $\Theta_m$ can be verified evaluating $\langle \Theta_{m,L}, \Theta_{m',L} \rangle \propto \delta_{L+m,L+m'}$.

More about theta functions can be seen for instance in [16] and [6].

As we've said, the theta functions are the holomorphic realizations on $L^2$ of the distributions, solutions of 19. Each $\Theta_{m,L}$ function is going to be related to a Hilbert space (eq. 20) and to a single lattice of complete type but with the origin dislocated, since the periodicity of the theta functions is given by 19, and to generated all of them we have translated the original $\Theta$ by steps on the $L'(modL)$ lattice.

All we considered is easily generalized to 2n-dimension (2 p-variables and 2 q-variables) by using theta functions of many variables. In the next section we are going to consider this generalization.

IV. GEOMETRIC QUANTIZATION AND C.S.S.

To have a physical picture of this result, we're going to associate to each $\Theta_{m,L}$ function a line bundle over the torus $T = V/L$, the geometric (holomorphic) quantization over $T$ (see for instance [17], [15] or [8]).

We start with a 2n-dimensional manifold $(M, \omega)$, the phase space of the classical system, and we define a complex line-bundle with connection $(L, \nabla) \xrightarrow{\pi} (M, \omega)$.

The wave functions $\{\Psi\}$ are going to be sections $\sigma : M \to \mathbb{C}$ on the line bundle, and the operators $\{\hat{f}\}$ over $\mathcal{H}$, corresponding to the classical quantities $\{f : M \to \mathbb{R} / f \in C^\infty(M)\}$ are going to be operators that act in the sections of $L$.

The connection $\nabla$ can be defined by a connection one form $\alpha$, that vanishes in the horizontal vector fields ($\alpha(Y) = 0$) as follow:

$$\nabla_X \sigma = 2\pi i \sigma^* \alpha(X) \sigma$$

where $\sigma^*$ is the pull-back applied to the one form $\sigma$, and $\alpha$ is a section.

The line-bundle with connection $(L, M) \xrightarrow{\pi} (M, \omega)$ is a pre-quantization of $(M, \omega)$ if $d\alpha = -(2\pi\hbar)^{-1} \pi^* \omega$, that is, the curvature $d\alpha$ is projected in the sympletic form $\omega$.

Such a line bundle do exist if and only if $(2\pi\hbar)^{-1} \omega$ define a deRham cohomology class over $\mathbb{Z}$. This condition is equivalent to Bohr-Sommerfeld quantization [2].

For this, let us take a closed path in $M$, and let lift it to $L$ by $\Psi$. The holonomy term is given by $\exp \left( i/\hbar \int_{\gamma} \theta \right) = \exp \left( i/\hbar \int_{S} \omega \right)$ where $\partial S = \gamma$.

So the wave function is well defined over $\gamma$ if $i/\hbar \int_{S} \omega$ is $2\pi \mathbb{Z}$-valued.

Observe that the phase term pointed in the second remark of section 1, that the wave function obtain when we circuit around the triangle of vertex $(0, \alpha, \alpha + \beta)$ is a holonomy term, and because of this the von-Neumann condition to a lattice be complete is equivalent to the Bohr-Sommerfeld quantization.
What we want to investigate is the overcompleteness of a C.S.S. in terms of line bundles. This relation is easily obtained if we focus the last result we have obtained, the solutions of $19$ expressed in terms of the theta functions with characteristic. This is what we wish to consider from now.

In what follow where we write line bundles we are talking about holomorphic line bundles. For details see, for instance $[4]$. Given a open cover $\{U_\alpha\}$ of $M$, a line bundle $(\mathcal{L}, \mathcal{M})$ can described by a collection of transition functions $\{g_{\alpha \beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ that satisfy:

$$g_{\alpha \beta}g_{\beta \alpha} = 1 \quad (26)$$

$$g_{\alpha \beta}g_{\beta \gamma}g_{\gamma \alpha} = 1 \quad (27)$$

Transition functions can be defined in terms of local trivializations $\{\varphi_\alpha\}$ by:

$$g_{\alpha \beta} = \varphi_\alpha \varphi_\beta^{-1} \quad (28)$$

Two sets of trivializations $\{\varphi_\alpha\}, \{\varphi'_\alpha\}$ define the same line bundle if $\varphi'_\alpha = \varphi_\alpha \cdot f_\alpha$, $f_\alpha \in \mathcal{O}^*(U_\alpha)$. The equations $26$ and $27$ state that $\{g_{\alpha \beta}\}$ is a Čech cocycle. And, by the last paragraph, two cocycles $\{g_{\alpha \beta}\}, \{g'_{\alpha \beta}\}$ give the same line bundle if they differ by a Čech coboundary, that is, the set of line bundles on $M$ is just by $H^1(M, \mathcal{O}^*)$.

We are going to need now certain conceptions from sheaves cohomology sequences. Given an exact sequence of sheaves:

$$... \to \mathcal{Z} \to \mathcal{O} \to \mathcal{O}^* \to ... \quad (29)$$

we have the long sheaves cohomology exact sequence:

$$...H^1(M, \mathcal{Z}) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \to H^2(M, \mathcal{Z}) \to H^2(M, \mathcal{O})... \quad (30)$$

We have the boundary map in the cohomology:

$$H^1(M, \mathcal{O}^*) \xrightarrow{\delta} H^2(M, \mathcal{Z}) \quad (31)$$

The image $\delta(\mathcal{L}) = c^1(\mathcal{L})$ of $\mathcal{L}$ in $H^2(M, \mathcal{Z})$ is the (first) Chern class of $\mathcal{L}$. We have that:

$$c^1(\mathcal{L}) = \left[\frac{i}{2\pi} \Theta\right] \in H^2_{DR}(M) \quad (32)$$

where $\Theta$ is the curvature of $\mathcal{L}$ and $H^2_{DR}(M)$ is the second deRham cohomology group (for details see $[4]$).

Consequently with this result and the first part of this section we see that each Chern class $c^1(\mathcal{L})$ gives a quantization of the system.

In the case that $M = T = V/L$, we are going to see how theta function are related to all these facts.

Let $\mathcal{L} \to M = V/L$ be a line bundle over the complex torus $M$, and let $\pi^* \mathcal{L}$ be the pullback of $\mathcal{L}$ to $V$. Since any line bundle over $V$ is trivial we can find global trivializations:

$$\varphi : \pi^* \mathcal{L} \to V \times \mathbb{C} \quad (33)$$
For \( z \in V, \lambda \in L \), the fibers of \( \pi^* \) at \( z \) and \( z + \lambda \) are both identified with the fiber of \( L \) at \( \pi(Z) \), and comparing the trivialization \( \varphi \) at \( z \) and \( z + \lambda \) we have automorphism of \( \mathbb{C} \), given as multiplication by a nonzero complex number \( e_\lambda(z) \), and we obtain a collection of functions:

\[
\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda \in L}
\]

called set of multipliers for \( L \).

This functions \( e_\lambda \) satisfy a compatibility relation:

\[
e_\lambda(z + \lambda) e_\lambda(z) = e_\lambda(z + \lambda') e_\lambda(z) = e_{\lambda + \lambda'}(z)
\]

for all \( \lambda, \lambda' \in L \).

It’s possible to show that any line bundle \((L, M)\) can be given by a set of multipliers \( \{e_\lambda(z)\} \) and that up to a translation in \( M \) all line bundles is determined by its Chern class.

In the prove of these results we fix the multipliers to be:

\[
e_\lambda(z) = 1, \quad e_{\lambda + \alpha}(z) = e^{-2\pi i z_\alpha} \quad \alpha = 1, \ldots, n
\]

For a basis \( \lambda_1, \ldots, \lambda_n \) for \( L \) and a dual system of coordinates such that the curvature \( \omega \) is given by:

\[
\omega = \sum_{\alpha=1}^n \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}
\]

where \( e_\alpha = \delta_\alpha^{-1} \lambda_\alpha, \alpha = 1, \ldots, n \).

The Chern class in this trivialization will be \( c_1(L) = [\omega] \).

Now we want to consider the set of line bundles having a given positive the same Chern class. For any \( \mu \in M \), the translation \( \tau_\mu : M \to M \) is homotopic to the identity and hence for any line bundle \((L, M)\):

\[
c_1(\tau_\mu^* L) = c_1(L)
\]

Actually it’s possible to prove that any line bundle having the same Chern class as \( L \) must be a translate of \( L \).

If the multipliers of \( L \) is like \( 36 \), the set of multipliers of \( L' = \tau_\mu^* L \) is going to be:

\[
e_\lambda'\alpha(z) = e_{\lambda\alpha}(z + \mu) \equiv 1
\]

\[
e_{\lambda + \alpha}'(z) = e_{\lambda + \alpha}(z + \mu)
\]

\[
= e^{-2\pi i (z_\alpha + \mu_\alpha)}
\]

Now, for a given section \( \tilde{\theta} \) of \( L \) over \( U \subset M \), \( \theta = \varphi^*(\pi^* \tilde{\theta}) \) is an analytic function of \( \pi^{-1}(U) \) satisfying:
\[ \theta(z + \lambda_\alpha) = \theta(z) \]  
\[ \theta(z + \lambda_{n+\alpha}) = e^{-2\pi iz\alpha} \theta(z) \]  
(41)

and conversely any such function defines a section of \( \mathcal{L} \).

Now for \( \mu = \frac{1}{2} \sum Z_{\alpha\alpha} e_\alpha \), let \( \mathcal{L}' = \tau_\mu^* \mathcal{L} \), with the multipliers given by equations \([39]\) and \([40]\).

If \( \tilde{\theta}' \) are global sections of \( \mathcal{L}' \), we will have holomorphic functions on \( V \), just like the functions given in equations \([41]\) and \([42]\):

\[ \theta' = (z + \lambda_\alpha) = \theta'(z) \]  
\[ \theta(z + \lambda_{n+\alpha}) = e^{-2\pi iz\alpha - \pi i Z_{\alpha\alpha}} \]  
(43)

These are the equations for the theta functions. The matrix \( Z_{\alpha\beta} \) constitute part of the so called period matrix (see \([4]\)).

So we see how the theta functions are related with line bundles for a given Chern class. To obtain the theta functions with characteristic considered in the previous section we just have to consider translations \( \tau_\mu^* \mathcal{L} \) (where \( \mathcal{L} \) is a line bundle with Chern class equal to one, related to the original theta function) of a fixed size, the size of the minimum cell related to \( \mathcal{L}' \), the dual lattice of a non-complete lattice (eq. \([19]\)). These translations is going to operate just like the \( A_\lambda \) operators we mentioned in the previous section.

Another way to see these theta functions is to consider, in the first case (complete case), a principally polarized complex torus (associated to a self-dual or a complete lattice eq. \([7]\)) and the theta function is the only global section of it. In the second case, we consider a polarized torus (associated to a overcomplete lattice, eq. \([9]\)) with polarization given by the Chern class (eq. \([37]\) \( c_1(\mathcal{L}) = \Pi \delta_\alpha \)). The set of theta functions is going to be the global sections of this torus. This is equivalent to consider the overcomplete lattice itself and to associate with each theta function with characteristic a complete lattice belonging to the overcomplete one.

More about this can be seen in \([4]\) and \([5]\).

V. APPLICATION AND REMARKS

We have seen how we can associate certain types of overcomplete C.S.S. to a set of quantizations of the classical system. To have a more concrete picture of this scenario, let us consider the hamiltonian \( H_0 = \frac{1}{2}(\hat{p}^2 + \hat{q}^2 - \hbar) \) and let \( | \Psi_0 \rangle \) be the ground state of \( H_0 \).

If we map \( | \Psi_0 \rangle \) in a coherent state \( | \alpha \rangle = D(\alpha) | \Psi_0 \rangle \), this coherent state is going to be the ground state of the conjugated hamiltonian \( H_\alpha = D(\alpha) H_0 D(\alpha)^{-1} \) of \( H_0 \), that is:

\[ H_\alpha | \alpha \rangle = 0 \]  
(45)

If we have translation invariance in the problem we could have these coherent states representing degenerated states.

This is the case in Landau levels. In \([7]\) we have a phase space approach to the problem where the coherent states are used as a basis for the propagator kernel. In this work the authors, using the Riemann-Roch theorem, obtain the degeneracy of the lowest Landau level \( n + 1 - g \) where \( n \) is an integer number expressing the normalized magnetic charge plus the
Euler characteristic of the surface, and \( g \) the genus of the surface. This result has been already obtained without mention of coherent states by pure geometrical arguments in [10].

In the present work the Riemann Roch theorem can be used directly. First we have to observe that if we consider a polarized torus originated by the “overcomplete” lattice, that is, with a minimum cell with area less then \( \pi \) (eq. 9), the Riemann Roch theorem gives (see [3]):

\[
\dim H^0(M, \mathcal{O}(L)) = \Pi_{\alpha} \delta_{\alpha},
\]

that is exactly the number of theta functions with characteristic or equivalently the number of Hilbert spaces in the direct summation [20].

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