Beyond the $c = 1$ Barrier in Two-Dimensional Quantum Gravity

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Abstract

We introduce a simple model of touching random surfaces, by adding a chemical potential $\rho$ for “minimal necks”, and study this model numerically coupled to a Gaussian model in $d$–dimensions (for central charge $c = d = 0$, 1 and 2). For $c \leq 1$, this model has a phase transition to branched polymers, for sufficiently large $\rho$. For $c = 2$, however, the extensive simulations indicate that this transition is replaced by a cross-over behavior on finite lattices — the model is always in the branched polymer phase. This supports recent speculations that, in 2d–gravity, the behavior observed in simulations for $c \leq 1$, is dominated by finite size effects, which are exponentially enhanced as $c \to 1^+$.

1 Introduction

When conformally invariant matter $S_M(X)$ is coupled to two-dimensional quantum gravity:

$$Z(\mu) = \int Dg DX e^{-\mu \int d^2 \xi \sqrt{|g|} - S_m(X; g)},$$

(1)

this breaks down when the matter central charge $c$ becomes larger than one. We get unphysical complex critical exponents, such as the string susceptibility exponent $\gamma_s$: $Z(\mu) \sim (\mu_c - \mu)^{2-\gamma_s}$; given by the KPZ-scaling relation:

$$\gamma_s = \frac{1}{12} \left( c - 1 - \sqrt{(c-25)(c-1)} \right)$$

(2)

Hence, predictions of continuum theories become meaningless for $c > 1$. This puzzle, which is related to the occurrence of tachyons in bosonic string theories in $d > 2$, still remains a challenging problem in 2d–gravity.
Discretized models of $2d$–gravity are, on the other hand, well defined for $c > 1$, and suitable for studying this problem. Simplicial gravity, alias dynamical triangulations, is a discretization of quantum gravity with integrations over metrics replaced by all possible gluing’s of simplices into piecewise linear manifolds $T$:

$$Z = \sum_A e^{-\mu A} \sum_{T \in T(A)} Z_M .$$

(3)

$A$ is the area of the surface, $Z_M$ the (discretized) matter partition function; for example, a $d$–dimensional Gaussian model (bosonic string theory with $c = d$):

$$Z_M = \int \text{d}^d x \, \delta(x_{cm}) \, e^{-\sum_{<ij>} (\vec{x}_i - \vec{x}_j)^2} ,$$

(4)

and $T$ is an appropriate class of triangulations; different classes amount to different discretizations of the manifolds, but should yield the same continuum theory. Commonly used are combinatorial ($T_C$) and degenerate ($T_D$) triangulations.

Models of dynamical triangulations have been studied extensively, both as matrix models (for $c \leq 1$) and using numerical simulations. What have we learned so far:

- For $c \leq 1$ the models are well understood; $\gamma_s$ agrees with the KPZ–scaling and the (internal) fractal dimension of the triangulations ($A(r) \sim r^{d_H}$) is $d_H \approx 4$ (still somewhat controversial).
- For $c \gtrsim 5$ the dominant triangulations are branched polymers (bubbles glued together in a tree-like structure) with $\gamma_s = 1/2$ and $d_H = 2$.
- But, for $1 < c \lesssim 5$ the situation is still unclear. Numerical simulations indicate a smooth cross-over to the branched polymer phase as $c$ increases.

Is this due to very big finite-size effects [1], or is there a different critical behavior for $1 < c \lesssim 5$?

2 Touching random surfaces

A conjecture for the observed $c > 1$ behavior, was put forward in [2]: “For $c > 1$ the dynamical triangulation model is always in a branched polymer phase. But finite size effects are exponentially enhanced as $c \to 1^+$, due to the influence of the $c = 1$ fixed point (which becomes complex for $c > 1$).”

This is based on a large-$N$ renormalization group analysis of a matrix model including “touching” interactions:

$$Z = \int \text{d}M \, e^{-N \text{tr}(M^2 + gM^4) - x (\text{tr}(M^2))^2} .$$

(5)
For $c \leq 1$ this model has a transition to branched polymers at a critical value of the touching coupling $x$ \cite{3}. For $c > 1$, however, this fixed point moves into the complex plane; but it still influences the RG-flow’s when $c$ is not too big.

How do we verify this conjecture? We introduce a simple model of touching random surfaces, adding a chemical potential $\rho$ for minimal necks $n_m$ on the surface. As we work with degenerate triangulations, a minimal neck is a vertex connected to itself via a link (a tadpole in the dual graph). The (fixed area) partition function is:

$$Z_A(\rho) = \sum_{T \in \mathcal{T}_D} e^{\rho n_m} Z_M.$$  (6)

We have simulated this model for $c \leq 2$, using 0, 1 and 2 Gaussian models, on surfaces up to 8000 triangles. Our goal is to verify the existence of a transition to branched polymers for $c \leq 1$, and to see if this transition still exists for $c > 1$. Or, alternatively, is it replaced by cross-over behavior on finite lattices.

To study the phase structure we measure the second derivative of the free energy: $C_A = A^{-1} \partial^2 \log Z_A/\partial \rho^2$, and the string susceptibility exponent $\gamma_s$. The latter is obtained from the distribution of baby universes on the surface, using the large-$A$ behavior of the partition function: $Z_A \approx e^{\mu A} A^{\gamma_s-3}$. For $c = 1$ this behavior is modified by logarithmic corrections, $Z_A \approx e^{\mu A} A^{\gamma_s-3} \log^\alpha A$, — including them is essential to extract the correct $\gamma_s$ numerically \cite{4}.

3 Results

For $c = 0$ (pure gravity) we see a clear signal of a phase transition. There is a peak in $C_A$, which gets sharper as $A$ increases, but does not diverge (Fig. 1a). Finite size scaling of the peak ($C_A \sim c_0 + c_1 A^{\alpha/\nu d_H}$) gives: $\rho_c = 0.695(5)$ and $\alpha = -1.07(11)$, assuming hyper-scaling is valid ($\alpha = 2 - \nu d_H$). (Note that this $\nu$ is related to the
touching interaction; hence $\nu \neq 1/d_H$). At the same value of $\rho_c$ there is a sharp transition in $\gamma_s$ from its pure gravity value, $\gamma_s(PG) = -1/2$, to branched polymers, $\gamma_s(BP) = 1/2$ (Fig. 1b).

We observe a similar behavior for $c = 1$ (Figs. 2a and 2b): a non-divergent peak in $C_A$, with $\rho_c = 0.45(1)$ and $\alpha = -0.8(2)$, accompanied by a transition to branched polymers in $\gamma_s$. In this case, $\gamma_s$ is extracted using logarithmic corrections, with $\alpha$ as a free parameter. Below $\rho_c$, $\alpha \approx -1$, whereas $\alpha \approx 0$ for branched polymers.

For $c > 1$, on the other hand, the behavior is different. We still observe a peak in $C_A$ (Fig. 3a), but it saturates faster than for $c \leq 1$. In fact, $\alpha/\nu d_H < -1$, which implies, if this is a phase transition, that hyper-scaling is violated. And, more important, there is no indication of a phase transition in $\gamma_s$, only a smooth crossover to branched polymers, which seems to disappear as $A \to \infty$ (Fig. 3b). This is
independent of the corrections included in extracting $\gamma_s$. This behavior is, in our opinion, not compatible with the existence of a phase transition, and we conclude that there is only a branched polymer phase. This strongly supports the conjecture in [2] about the nature of the $c = 1$ “barrier”.

References

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