On asymptotic stability of moving kink for relativistic Ginzburg-Landau equation

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Abstract

We prove the asymptotic stability of the moving kinks for the nonlinear relativistic wave equations of the Ginzburg-Landau type in one space dimension: starting in a small neighborhood of the kink, the solution, asymptotically in time, is the sum of a uniformly moving kink and dispersive part described by the free Klein-Gordon equation. The remainder decays in a global energy norm. Crucial role in the proofs play our recent results on the weighted energy decay for the Klein-Gordon equations.

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1 Introduction

We prove the asymptotic stability of moving kink for relativistic nonlinear wave equations with two-well potentials of Ginzburg-Landau type. We consider the equations

\[ \ddot{\psi}(x, t) = \psi''(x, t) + F(\psi(x, t)), \quad x \in \mathbb{R} \]  

(1.1)

where \( \psi(x, t) \) is a real solution, and \( F(\psi) = -U'(\psi) \).

**Condition U1.** The potential \( U \in C^2(\mathbb{R}) \) satisfies the following conditions with some \( a > 0 \):

\[
\begin{align*}
U \in C^4 & \text{ in vicinity of points } \pm a; \quad U(\psi) > 0 \text{ for } \psi \neq \pm a; \\
U''(\pm a) = m^2 > 0; & \quad U^{(k)}(\pm a) = 0 \text{ for } 0 \leq k \leq 13, \quad k \neq 2
\end{align*}
\]  

(1.2)

In the vector form, equation (1.1) reads

\[
\begin{aligned}
\dot{\psi}(x, t) &= \pi(x, t) \\
\dot{\pi}(x, t) &= \psi''(x, t) + F(\psi(x, t)), \quad x \in \mathbb{R}
\end{aligned}
\]  

(1.3)

Formally it is a Hamiltonian system with the Hamilton functional

\[
\mathcal{H}(\psi, \pi) = \int \left[ \frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + U(\psi(x)) \right] dx
\]  

(1.4)

The corresponding stationary equation reads

\[ s''(x) - U'(s(x)) = 0 \]  

(1.5)

There are a constant solutions of the stationary equation: \( s(x) \equiv \pm a \) - stable stationary solutions. There is also a "kink", i.e. a finite energy solution \( s(x) \) to (1.5) such that

\[ s(x) \to \pm a \quad \text{as} \quad x \to \pm \infty \]  

(1.6)

The condition U1 implies that \( (s(x) \mp a)'' \sim m^2 (s(x) \mp a) \) for \( x \to \pm \infty \), hence

\[ s(x) \mp a \sim Ce^{-m|x|}, \quad x \to \pm \infty \]  

(1.7)

The system (1.3) is translation-invariant and admits the kink or soliton solutions

\[ Y_{q,v}(t) = (\psi_v(x - vt - q), \pi_v(x - vt - q)), \]  

(1.8)

for all \( q, v \in \mathbb{R} \) with \( |v| < 1 \), where

\[ \psi_v(x) = s(\gamma x) \]  

(1.9)

and \( \gamma \) is the Lorentz contraction \( \gamma = 1/\sqrt{1-v^2} \). The states \( S_{q,v} := Y_{q,v}(0) \) form the solitary manifold

\[ \mathcal{S} := \{ S_{q,v} : q, v \in \mathbb{R}, |v| < 1 \}. \]  

(1.10)
The linearized operator near soliton solution $Y_{q,v}(t)$ is (see Section 4, formula (4.20))

$$A_v = \left( \Delta - m^2 - V_v(y) \right) \nabla v \left( \frac{1}{v} \nabla \right), \quad \nabla = \frac{d}{dx}, \quad \Delta = \frac{d^2}{dx^2}$$

where

$$V_v(x) = -F'(\psi_v(x)) - m^2 = U''(\psi_v(x)) - m^2$$

By (1.7), we have

$$V_v(x) \sim C(s(\gamma x) \pm a)^{12} \sim Ce^{-12m\gamma|x|}, \quad x \to \pm \infty \quad (1.12)$$

In Section 4 we show that the spectral properties of the operator $A_v$ are determined by the corresponding properties of its determinant which is the Schrödinger operator

$$H_v = -(1 - v^2)\Delta + m^2 + V_v$$

The spectral properties of $H_v$ are identical for all $v \in (-1, 1)$ since the relation $V_v(x) = V_0(\gamma x)$ implies

$$H_v = T_v^{-1}H_0T_v, \quad \text{where} \quad T_v : \psi(x) \mapsto \psi(x/\gamma) \quad (1.14)$$

This equivalence manifests the relativistic invariance of the equation (1.3). The continuous spectrum of the operator $H_v$ coincides with $[m^2, \infty)$. The point 0 belongs to the discrete spectrum with corresponding eigenfunction $\psi'_v(x)$. By (1.6) and (1.9) we have $\psi'_v(x) = \gamma s'_v(x) > 0$ for $x \in \mathbb{R}$. Hence, $\psi'_v(x)$ is the groundstate, and all remaining discrete spectrum is contained in $(0, m^2]$.

For $\alpha \in \mathbb{R}$, $p \geq 1$, and $l = 0, 1, 2, \ldots$ let us denote by $W_{\alpha, p}^{l, L^2}$, the weighted Sobolev space of the functions with the finite norm

$$\|\psi\|_{W_{\alpha, p}^{l, L^2}} = \sum_{k=0}^{l} \|(1 + |x|)^{\alpha}\psi^{(k)}\|_{L^p} < \infty$$

and $H_{\alpha}^l := W_{\alpha, p}^{l, L^2}$, so $H_{\alpha}^0 = L^2_{\alpha}$ are the Agmon’s weighted spaces.

**Definition 1.1.** (cf. [7, 13]) Resonance is a nonzero solution $\psi \in L^2_{-1/2-0}(\mathbb{R}) \setminus L^2(\mathbb{R})$ to $H_v \psi = m^2 \psi$.

We assume the following spectral condition

**Condition U2.** For any $v \in (-1, 1)$

i) The operator $H_v$ has only eigenvalue 0.

ii) The edge point $m^2$ of the continuous spectrum is neither eigenvalue nor resonance for the Schrödinger operator $H_v$.

We use the Condition U2 ii) to prove the boundedness of the resolvent of the operator $A_v$ at the edge points $\pm im/\gamma$ of its continuous spectrum. The examples of the potentials $U(\psi)$ satisfies the Conditions U1-U2 will be published elsewhere not to overburden the exposition.

Our main results are the following asymptotics

$$(\psi(x,t), \pi(x,t)) \sim (\psi_{v,\pm}(x - v_{\pm}t - q_{\pm}), \pi_{v,\pm}(x - v_{\pm}t - q_{\pm})) + W_0(t)\Phi_{\pm}, \quad t \to \pm \infty \quad (1.15)$$
for solutions to (1.3) with initial states close to the kink. Here $W_0(t)$ is the dynamical group of the free Klein-Gordon equation, $\Phi_{\pm}$ are the corresponding asymptotic states, and the remainder converges to zero $\sim t^{-1/2}$ in the global energy norm of the Sobolev space $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Let us comment on previous results in this field.

- **The Schrödinger equation** The asymptotics of type (1.15) were established for the first time by Soffer and Weinstein [20, 21] (see also [16]) for nonlinear $U(1)$-invariant Schrödinger equation with a potential for small initial states and sufficiently small nonlinear coupling constant.

  The results have been extended by Buslaev and Perelman [2] to the translation invariant 1D nonlinear $U(1)$-invariant Schrödinger equation. The novel techniques [2] are based on the "separation of variables" along the solitary manifold and in the transversal directions. The symplectic projection allows to exclude the unstable directions corresponding to the zero discrete spectrum of the linearized dynamics. Similar techniques were developed by Miller, Pego and Weinstein for the 1D modified KdV and RLW equations, [14, 15]. The extensions to higher dimensions were obtained in [5, 10, 19, 24].

- **The Klein-Gordon equation** The asymptotics of type (1.15) were extended to the nonlinear 3D Klein-Gordon equations with a potential [22], and for translation invariant system of the 3D Klein-Gordon equation coupled to a particle [9].

- **The 3D Ginzburg-Landau equation** Recently, the asymptotic stability of the kink and asymptotics of type (1.15) were proved for 3D relativistic Ginzburg-Landau equation with initial data which differ from the kink on a compact set [4]. The equation differs from the 1D equation (1.1) by the additional 2D Laplacian. The additional Laplacian improves the decay for the corresponding free Klein-Gordon equation that provides the needed dispersive decay for the transversal dynamics.

- **The 1D Ginzburg-Landau equation** For 1D relativistic nonlinear Ginzburg-Landau equations (1.1) the orbital stability of the kinks has been proved in [8].

The case of the moving kink of 1D Ginzburg-Landau equations (1.1) was not considered previously. We develop the general strategy of [2], [3] and [9]: symplectic projection onto the solitary manifold, the modulation equations, the linearization of the transversal equations, etc. We also apply the Taylor expansion of the nonlinearity from [3]. However, the remaining part of our approach is new. Let us comment on our basic novelties.

I. The decay $\sim t^{-3/2}$ for the linearized transversal dynamics relies on our novel approach [11] to 1D Klein-Gordon equation. The slow decay $\sim t^{-1/2}$ for the free 1D Klein-Gordon equation corresponds to the presence of the resonances at the edge points of the continuous spectrum. We overcome the difficulty developing our result [11] where we have identified the slow decaying component with the contribution of the resonances. More precisely, we prove that the contribution of the high energy spectrum decays like $\sim t^{-3/2}$, in the weighted energy norms.

II. The decay for the low energy component relies on new asymptotics (A. 2) of the resolvent at the edge points of the continuous spectrum. The corresponding asymptotics in [11] follows similarly to known results of Murata [13]. In present paper, the proof of the asymptotics is more complicate and relies on a novel trick (Lemma A.8).

III. The "virial type" estimate (B. 2) for the nonlinear Ginzburg-Landau equation (1.1) is new
relativistic version of the bound [2, (1.2.5)] for the nonlinear Schrödinger equations.

IV. We establish an appropriate version (4.32) of $L^1 \to L^\infty$ estimates.

Both estimates (B. 2) and (4.32) play crucial role in obtaining the bounds for the majorants.

Our paper is organized as follows. In Section 2 we formulate the main theorem. In Section 3 we introduce the symplectic projection onto the solitary manifold. The linearized equation is defined in Section 4. In Section 5 we split the dynamics in two components: along the solitary manifold and in the transversal directions. In Section 6 the modulation equations for the parameters of the soliton are displayed. The time decay of the transversal component is established in Sections 7-11. Finally, in Section 12 we obtain the soliton asymptotics (1.15).

In Appendix A we prove the weighted energy decay for the linearized equation.

2 Main results

2.1 Existence of dynamics

We consider the Cauchy problem for the Hamilton system (1.3) which we write as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R} : \quad Y(0) = Y_0. \quad (2.1)$$

Here $Y(t) = (\psi(t), \pi(t))$, $Y_0 = (\psi_0, \pi_0)$, and all derivatives are understood in the sense of distributions. To formulate our results precisely, let us first introduce a suitable phase space for the Cauchy problem (2.1).

Definition 2.1. i) $E_\alpha := H_\alpha^1 \oplus L_\alpha^2$ is the space of the states $Y = (\psi, \pi)$ with finite norm

$$\|Y\|_{E_\alpha} = \|\psi\|_{H_\alpha^1} + \|\pi\|_{L_\alpha^2} < \infty \quad (2.2)$$

ii) The phase space $\mathcal{E} := \mathcal{S} + E$, where $E = E_0$ and $\mathcal{S}$ is defined in (1.10). The metric in $\mathcal{E}$ is defined as

$$\rho_\mathcal{E}(Y_1, Y_2) = \|Y_1 - Y_2\|_E, \quad Y_1, Y_2 \in \mathcal{E} \quad (2.3)$$

iii) $W := W_0^{2,1} \oplus W_0^{1,1}$ is the space of the states $Y = (\psi, \pi)$ with the finite norm

$$\|Y\|_W = \|\psi\|_{W_0^{2,1}} + \|\pi\|_{W_0^{1,1}} < \infty \quad (2.4)$$

Obviously, the Hamilton functional (1.4) is continuous on the phase space $\mathcal{E}$. The existence and uniqueness of the solutions to the Cauchy problem (2.1) follows by methods [12, 17, 23]:

Proposition 2.2. i) For any initial data $Y_0 \in \mathcal{E}$ there exists the unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to the problem (2.1).

(ii) For every $t \in \mathbb{R}$, the map $U(t) : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{E}$.

(iii) The energy is conserved, i.e.

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \quad t \in \mathbb{R} \quad (2.5)$$
2.2 Solitary manifold and main result

Let us consider the solitons (1.9). The substitution to (1.3) gives the following stationary equations,
\[ -v\psi'(y) = \pi_v(y), \]
\[ -v\pi'_v(y) = \psi''(y) + F(\psi(y)). \]

**Definition 2.3.** A soliton state is \( S(\sigma) := (\psi_v(x - b), \pi_v(x - b)), \) where \( \sigma := (b, v) \) with \( b \in \mathbb{R} \) and \( v \in (-1, 1). \)

Obviously, the soliton solution (1.8) admits the representation \( S(\sigma(t)), \) where
\[ \sigma(t) = (b(t), v(t)) = (vt + q, v). \]

**Definition 2.4.** A solitary manifold is the set \( S := \{S(\sigma) : \sigma \in \Sigma := \mathbb{R} \times (-1, 1)\}. \)

The main result of our paper is the following theorem

**Theorem 2.5.** Let the potential \( U \) satisfy the condition \( U_1, \) and \( Y(t) \) be the solution to the Cauchy problem (2.1) with an initial state \( Y_0 \in E \) which is close to a kink \( S(\sigma_0) = S_{\phi_0, \psi_0}: \)
\[ Y_0 = S(\sigma_0) + X_0, \quad d_0 := \|X_0\|_{E_{\beta \cap W}} \ll 1 \]
where \( \beta > 5/2. \) Assume that the spectral condition \( U_2 \) holds. Then for \( d_0 \) sufficiently small the solution admits the asymptotics:
\[ Y(x, t) = (\psi_v(x - v_\pm t - q_\pm), \pi_v(x - v_\pm t - q_\pm)) + W_0(t)\Phi_\pm + r_\pm(x, t), \quad t \to \pm \infty \]
where \( \Phi_\pm \in E, \) and \( W_0(t) \) is the dynamical group of the free Klein-Gordon equation, while
\[ \|r_\pm(t)\|_E = \mathcal{O}(|t|^{-1/2}) \]
It suffices to prove the asymptotics (2.9) for \( t \to +\infty \) since the system (1.3) is time-reversible.

3 Symplectic projection

3.1 Symplectic structure and hamiltonian form

The system (2.1) reads as the Hamilton system
\[ \dot{Y} = JD\mathcal{H}(Y), \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = (\psi, \pi) \in E, \]
where \( D\mathcal{H} \) is the Fréchet derivative of the Hamilton functional (1.4). Let us identify the tangent space of \( E, \) at every point, with the space \( E. \) Consider the symplectic form \( \Omega \) on \( E \) defined by
\[ \Omega(Y_1, Y_2) = \langle Y_1, JY_2 \rangle, \quad Y_1, Y_2 \in E, \]
For any Definition 3.3.

\[ \langle Y_1, Y_2 \rangle := \langle \psi_1, \psi_2 \rangle + \langle \pi_1, \pi_2 \rangle \]

and \( \langle \psi_1, \psi_2 \rangle = \int \psi_1(x)\psi_2(x)dx \) etc. It is clear that the form \( \Omega \) is non-degenerate, i.e.

\[ \Omega(Y_1, Y_2) = 0 \quad \text{for every} \quad Y_2 \in E \quad \Rightarrow \quad Y_1 = 0. \]

**Definition 3.1.**

i) The symbol \( Y_1 \parallel Y_2 \) means that \( Y_1 \in E, \ Y_2 \in E, \) and \( Y_1 \) is symplectic orthogonal to \( Y_2, \) i.e. \( \Omega(Y_1, Y_2) = 0. \)

ii) A projection operator \( P : E \to E \) is said to be symplectic orthogonal if \( Y_1 \parallel Y_2 \) for \( Y_1 \in \text{Ker} \ P \) and \( Y_2 \in \text{Range} \ P. \)

### 3.2 Symplectic projection onto solitary manifold

Let us consider the tangent space \( T_{S(\sigma)}S \) of the manifold \( S \) at a point \( S(\sigma). \) The vectors

\[
\tau_1 = \tau_1(v) := \partial_b S(\sigma) = (-\psi_v'(y), -\pi_v'(y)) \\
\tau_2 = \tau_2(v) := \partial_c S(\sigma) = (\partial_c \psi_v(y), \partial_c \pi_v(y))
\]

(3.3)

form a basis in \( T_{S(\sigma)}S. \) Here \( y := x - b \) is the “moving frame coordinate”. Let us stress that the functions \( \tau_j \) are always regarded as functions of \( y \) rather than those of \( x. \) Formula (1.9) implies that

\[ \tau_j(v) \in E_\alpha, \quad v \in (-1, 1), \quad j = 1, 2, \quad \forall \alpha \in \mathbb{R}. \] (3.4)

**Lemma 3.2.** The symplectic form \( \Omega \) is nondegenerate on the tangent space \( T_{S(\sigma)}S \) i.e. \( T_{S(\sigma)}S \)

is a symplectic subspace.

**Proof.** Let us compute the vectors \( \tau_1 \) and \( \tau_2. \) Recall that \( \psi_v(y) = s(\gamma y) \) and \( \pi_v = -v\psi_v'(y) = -v\gamma s'(\gamma y) \) with \( \gamma = 1/\sqrt{1-v^2}. \) Then

\[
\tau_1 = (\tau_1^1, \tau_1^2) = (-\gamma s'(\gamma y), v\gamma^2 s''(\gamma y)) \\
\tau_2 = (\tau_2^1, \tau_2^2) = (vy\gamma^3 s'(\gamma y), -\gamma^3 s'(\gamma y) - v^2 y\gamma^4 s''(\gamma y))
\]

Therefore

\[ \Omega(\tau_1, \tau_2) = \langle \tau_1^1, \tau_2^1 \rangle - \langle \tau_1^2, \tau_2^2 \rangle = \gamma^4 \{s'(\gamma y), s'(\gamma y)\} > 0 \] (3.5)

\[ \square \]

Now we show that in a small neighborhood of the soliton manifold \( S \) a “symplectic orthogonal projection” onto \( S \) is well-defined. Let us introduce the translations \( T_q : (\psi(x), \pi(x)) \mapsto (\psi(x-q), \pi(x-q)), \ q \in \mathbb{R}. \) Note that the manifold \( S \) is invariant with respect to the translations.

**Definition 3.3.** For any \( \overline{\varphi} < 1 \) denote by \( \Sigma(\overline{\varphi}) = \{ \sigma = (b, v) : b \in \mathbb{R}, |v| \leq \overline{\varphi} \}. \)

Let us note that \( S \in E_\alpha \) with \( \alpha < -1/2. \)
Lemma 3.4. Let $\alpha < -1/2$ and $\overline{\nu} < 1$. Then

i) there exists a neighborhood $O_\alpha(S)$ of $S$ in $E_\alpha$ and a mapping $\Pi : O_\alpha(S) \rightarrow S$ such that $\Pi$ is uniformly continuous on $O_\alpha(S)$ in the metric of $E_\alpha$,

$$\Pi Y = Y \quad \text{for} \quad Y \in S, \quad \text{and} \quad Y - S \upharpoonright T_S \subseteq S,$$

where $S = \Pi Y$. \hfill (3.6)

ii) $O_\alpha(S)$ is invariant with respect to the translations $T_q$, and

$$\Pi T_q Y = T_q \Pi Y, \quad \text{for} \quad Y \in O_\alpha(S) \quad \text{and} \quad q \in \mathbb{R}. \hfill (3.7)$$

iii) For any $\overline{\nu} < 1$ there exists an $r_\alpha(\overline{\nu}) > 0$ s.t. $S(\sigma) + X \in O_\alpha(S)$ if $\sigma \in \Sigma(\overline{\nu})$ and $\|X\|_{E_\alpha} < r_\alpha(\overline{\nu})$.

Proof. We have to find $\sigma = \sigma(Y)$ such that $S(\sigma) = \Pi Y$ and

$$\Omega(Y - S(\sigma), \partial_\sigma S(\sigma)) = 0, \quad j = 1, 2. \hfill (3.8)$$

Let us fix an arbitrary $\sigma^0 \in \Sigma$ and note that the system (3.8) involves two smooth scalar functions of $Y$. Then for $Y$ close to $S(\sigma^0)$, the existence of $\sigma$ follows by the standard finite dimensional implicit function theorem if we show that the $2 \times 2$ Jacobian matrix with elements $M_{ij}(Y) = \partial_{\sigma^j} \Omega(Y - S(\sigma^0), \partial_\sigma S(\sigma^0))$ is non-degenerate at $Y = S(\sigma^0)$. First note that all the derivatives exist by (3.4). The non-degeneracy holds by Lemma 3.2 and the definition (3.3) since $M_{ij}(S(\sigma^0)) = -\Omega(\partial_{\sigma^j} S(\sigma^0), \partial_\sigma S(\sigma^0)).$ Thus, there exists some neighborhood $O_\alpha(S(\sigma^0))$ of $S(\sigma^0)$ where $\Pi$ is well defined and satisfies (3.6), and the same is true in the union $O'_\alpha(S) = \cup_{\alpha \in \Sigma} O_\alpha(S(\sigma^0))$. The identity (3.7) holds for $Y, T_q Y \in O'_\alpha(S)$, since the form $\Omega$ and the manifold $S$ are invariant with respect to the translations. It remains to modify $O'_\alpha(S)$ by the translations: we set $O_\alpha(S) = \cup_{b \in \mathbb{B}} T_b O'_\alpha(S).$ Then the second statement obviously holds.

The last two statements and the uniform continuity in the first statement follow by translation invariance and the compactness arguments. \hfill \square

We refer to $\Pi$ as symplectic orthogonal projection onto $S$.

## 4 Linearization on the solitary manifold

Let us consider a solution to the system (1.3), and split it as the sum

$$Y(t) = S(\sigma(t)) + X(t) \hfill (4.1)$$

where $\sigma(t) = (b(t), v(t)) \in \Sigma$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (\psi, \pi)$ and $X = (\Psi, \Pi)$. Then (4.1) means that

$$\begin{align*}
\psi(x, t) &= \psi_{\Psi}(x - b(t)) + \Psi(x - b(t), t) \\
\pi(x, t) &= \pi_{\Psi}(x - b(t)) + \Pi(x - b(t), t)
\end{align*} \hfill (4.2)$$

Let us substitute (4.2) to (1.3), and linearize the equations in $X$. Setting $y = x - b(t)$ which is the “moving frame coordinate”, we obtain that

$$\begin{align*}
\dot{\psi} &= \dot{b}\partial_y \psi_y(y) - \dot{y} \partial_y \psi_y(y) + \dot{\Psi}(y, t) - \dot{b} \Psi(y, t) = \pi_v(y) + \Pi(y, t) \\
\dot{\pi} &= \dot{b}\partial_y \pi_v(y) - \dot{y} \partial_y \pi_v(y) + \dot{\Pi}(y, t) - \dot{b} \Pi(y, t) \\
&= \pi''_v(y) + \Psi''(y, t) + F(\psi_v(y) + \Psi(y, t)) \hfill (4.3)
\end{align*}$$
Using the equations (2.6), we obtain from (4.3) the following equations for the components of the vector $\mathbf{X}(t)$:

$$
\dot{\Psi}(y, t) = \Pi(y, t) + b\dot{\Psi}(y, t) + (b - v)\dot{\psi}_v(y) - \dot{v}\partial_v\psi_v(y)
$$

$$
\dot{\Pi}(y, t) = \Psi''(y, t) + b\Pi(y, t) + (b - v)\pi'_v(y) - \dot{v}\partial_v\pi_v(y) + F(\psi_v(y) + \Psi(y, t)) - F(\psi_v(y))
$$

We can write the equations (4.4) as

$$
\dot{\mathbf{X}}(t) = A(t)\mathbf{X}(t) + T(t) + \mathcal{N}(t), \quad t \in \mathbb{R}
$$

(4.5)

where $T(t)$ is the sum of terms which do not depend on $X$, and $\mathcal{N}(t)$ is at least quadratic in $X$. The linear operator $A(t) = A_{v, w}$ depends on two parameters, $v = v(t)$, and $w = b(t)$ and can be written in the form

$$
A_{v, w} \left( \begin{array}{c} \Psi \\ \Pi \end{array} \right) := \left( \begin{array}{cc} w\nabla & 1 \\ \Delta + F'(\psi_v) & w\nabla \end{array} \right) \left( \begin{array}{c} \Psi \\ \Pi \end{array} \right) = \left( \begin{array}{cc} w\nabla & 1 \\ \Delta - m^2 - V_v(y) & w\nabla \end{array} \right) \left( \begin{array}{c} \Psi \\ \Pi \end{array} \right)
$$

(4.6)

where

$$
V_v(y) = -F'(\psi_v) - m^2
$$

(4.7)

Furthermore, $T(t) = T_{v, w}$ and $\mathcal{N}(t) = \mathcal{N}(\sigma, X)$ are given by

$$
T_{v, w} = \left( \begin{array}{c} (w - v)\psi_v' - \dot{v}\partial_v\psi_v \\ (w - v)\pi_v' - \dot{v}\partial_v\pi_v \end{array} \right), \quad \mathcal{N}(\sigma, X) = \left( \begin{array}{c} 0 \\ N(v, \Psi) \end{array} \right)
$$

(4.8)

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t), v(t))$, $X = X(t)$, and

$$
N(v, \Psi) = F(\psi_v + \Psi) - F(\psi_v) - F'(\psi_v)\Psi
$$

(4.9)

**Remark 4.1.**

i) The term $A(t)\mathbf{X}(t)$ in the right hand side of equation (4.5) is linear in $X(t)$, and $\mathcal{N}(t)$ is a high order term in $X(t)$. On the other hand, $T(t)$ is a zero order term which does not vanish at $X(t) = 0$ since $S(\sigma(t))$ generally is not a kink solution if (2.7) fails to hold (though $S(\sigma(t))$ belongs to the solitary manifold).

ii) Formulas (3.3) and (4.8) imply:

$$
T(t) = -(w - v)\tau_1 - \dot{v}\tau_2
$$

(4.10)

and hence $T(t) \in T_{S(\sigma(t))} \mathcal{S}$, $t \in \mathbb{R}$. This fact suggests an unstable character of the nonlinear dynamics along the solitary manifold.

**4.1 Linearized equation**

Here we collect some Hamiltonian and spectral properties of the operator $A_{v, w}$. First, let us consider the linear equation

$$
\dot{\mathbf{X}}(t) = A_{v, w}\mathbf{X}(t), \quad t \in \mathbb{R}
$$

(4.11)

with arbitrary fixed $v \in (-1, 1)$ and $w \in \mathbb{R}$. Let us define the space $E^+ := H^2(\mathbb{R}) \oplus H^1(\mathbb{R})$. 

Lemma 4.2. i) For any \( v \in (-1,1) \) and \( w \in \mathbb{R} \) equation (4.11) can be represented as the Hamiltonian system (cf. (1.4)),

\[
\dot{X}(t) = JD_H(X(t)), \quad t \in \mathbb{R}
\]  

(4.12)

where \( D_H \) is the Fréchet derivative of the Hamiltonian functional

\[
H_{v,w}(X) = \frac{1}{2} \int \left[ |\Pi|^2 + |\Psi'|^2 + (m^2 + V_v)|\Psi|^2 \right] dy + \int \Pi w \Psi' dy
\]  

(4.13)

ii) The energy conservation law holds for the solutions \( X(t) \in C^1(\mathbb{R}, E^+) \),

\[
H_{v,w}(X(t)) = \text{const}, \quad t \in \mathbb{R}
\]  

(4.14)

iii) The skew-symmetry relation holds,

\[
\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \quad X_1, X_2 \in E
\]  

(4.15)

Proof. i) The equation (4.11) reads as follows,

\[
\frac{d}{dt} \left( \begin{array}{c} \Psi \\ \Pi \end{array} \right) = \left( \begin{array}{c} \Pi + w\Psi' \\ \Psi'' - (m^2 + V_v)\Psi + w\Pi' \end{array} \right)
\]  

(4.16)

The equations correspond to the Hamilton form since

\[
\Pi + w\Psi' = D_H H_{v,w}, \quad \Psi'' - (m^2 + V_v)\Psi + w\Pi' = -D_\Psi H_{v,w}
\]

ii) The energy conservation law follows by (4.12) and the chain rule for the Fréchet derivatives:

\[
\frac{d}{dt} H_{v,w}(X(t)) = \langle D_H(X(t)), \dot{X}(t) \rangle = \langle D_H(X(t)), JD_H(X(t)) \rangle = 0, \quad t \in \mathbb{R}
\]  

(4.17)

since the operator \( J \) is skew-symmetric by (3.1), and \( D_H(X(t)) \in E \) for \( X(t) \in E^+ \).

iii) The skew-symmetry holds since \( A_{v,w}X = JD_H(X) \), and the linear operator \( X \mapsto D_H(X) \) is symmetric as the Fréchet derivative of a real quadratic form.

Lemma 4.3. The operator \( A_{v,w} \) acts on the tangent vectors \( \tau = \tau_j(v) \) to the solitary manifold as follows,

\[
A_{v,w}[\tau_1] = (v - w)\tau'_1, \quad A_{v,w}[\tau_2] = (w - v)\tau'_2 + \tau_1
\]  

(4.18)

Proof. In detail, we have to show that

\[
A_{v,w} \left( \begin{array}{c} -\psi'_v \\ -\pi'_v \end{array} \right) = \left( \begin{array}{c} (v - w)\psi''_v \\ (v - w)\pi''_v \end{array} \right), \quad A_{v,w} \left( \begin{array}{c} \partial_v \psi_v \\ \partial_v \pi_v \end{array} \right) = \left( \begin{array}{c} (w - v)\partial_v \psi'_v \\ (w - v)\partial_v \pi'_v \end{array} \right) + \left( \begin{array}{c} -\psi'_v \\ -\pi'_v \end{array} \right)
\]

Indeed, differentiate the equations (2.6) in \( b_j \) and \( v_j \), and obtain that the derivatives of soliton state in parameters satisfy the following equations,

\[
-v\psi''_v = \pi'_v, \quad -v\pi''_v = \psi'''_v + F'(\psi_v)\psi'_v
\]  

(4.19)

\[
-\psi'_v - v\partial_v \psi'_v = \partial_v \pi'_v, \quad -\pi'_v - v\partial_v \pi'_v = \partial_v \psi''_v + F'(\psi_v)\partial_v \psi_v
\]

Then (4.18) follows from (4.19) by definition of \( A_{v,w} \) in (4.6)
Now we consider the operator $A_v = A_{v,v}$ corresponding to $w = v$:

$$A_v := \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v(y) & v\nabla \end{pmatrix} \quad (4.20)$$

In that case the linearized equation has the following additional specific features. The continuous spectrum of the operator $A_v$ coincides with

$$\Gamma := (-i\infty, -im/\gamma] \cup [im/\gamma, i\infty) \quad (4.21)$$

From (4.18) it follows that the tangent vector $\tau_1(v)$ is the zero eigenvector, and $\tau_2(v)$ is the corresponding root vector of the operator $A_v$, i.e.

$$A_v[\tau_1(v)] = 0, \quad A_v[\tau_2(v)] = \tau_1(v) \quad (4.22)$$

**Lemma 4.4.** Zero root space of operator $A_v$ is two-dimensional for any $v \in (-1, 1)$.

**Proof.** It suffices to check that the equation $A_v u(v) = \tau_2(v)$ has no solution in $L^2 \oplus L^2$. Indeed, the equation reads

$$\begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v(y) & v\nabla \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v\gamma^3 y s'(\gamma y) \\ -\gamma^3 s'(\gamma y) - v^2\gamma^4 y s''(\gamma y) \end{pmatrix} \quad (4.23)$$

From the first equation we get $u_2 = v\gamma^3 y s'(\gamma y) - v\nabla u_1$. Then the second equation implies that

$$H_v u_1 = -\gamma^3(1 + v^2)s'(\gamma y) - 2v^2\gamma^4 y s''(\gamma y) \quad (4.24)$$

where $H_v$ is the Schrödinger operator defined in (1.13). Setting $u_1 = -\frac{1}{2}v^2\gamma^5 y^2 s'(\gamma y) + \tilde{u}_1$, we reduce the equation to

$$H_v \tilde{u}_1 = -\gamma^2 \psi_v' \quad (4.25)$$

i.e. $\tilde{u}_1$ is the root function of the operator $H_v$ since $\psi_v'$ is eigenfunction. However, this is impossible since $H_v$ is selfadjoint operator.

**Lemma 4.5.** The operator $A_v$ has only eigenvalue $\lambda = 0$.

**Proof.** Let us consider the eigenvalues problem for operator $A_v$:

$$\begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v(y) & v\nabla \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

From the first equation we have $u_2 = -(v\nabla - \lambda)u_1$. Then the second equation implies that

$$(H_v - \lambda^2 + 2v\lambda \nabla)u_1 = 0 \quad (4.26)$$

Hence, for $v = 0$ the operator $A_0$ has only eigenvalue $\lambda = 0$ by Condition U2 i).

Further, let us consider the case $v \neq 0$. Taking the scalar product with $u_1$, we obtain

$$\langle H_v u_1, u_1 \rangle - \lambda^2 \langle u_1, u_1 \rangle = 0$$
Hence, $\lambda^2$ is real since the operator $H_v$ is selfadjoint. The nonzero eigenvalues can bifurcate either from the point $\lambda = 0$ or from the edge points $\pm im/\gamma$ of the continuous spectrum of the operator $A_v$. Let us consider each case separately.

1) The point $\lambda = 0$ cannot bifurcate since it is isolated, and the zero root space is two dimensional by Lemma 4.4.

2) The bifurcation from the edge points also is impossible. Indeed, the bifurcated eigenvalue $\lambda \in (-im/\gamma, im/\gamma)$ is pure imaginary because $\lambda^2$ is real. Hence, (4.26) is equivalent to

$$\left(H_v + \gamma^2 \lambda^2\right)p = 0$$

where $p(x) = e^{\gamma^2 v \lambda x} u_1(x) \in L^2$ that is forbidden by Condition U2 i) since $-\gamma^2 \lambda^2 \in (0, m^2)$.

By the same arguments we obtain

**Lemma 4.6.** The equation

$$\left(H_v + \frac{m^2}{\gamma^2} \pm \frac{i2vm}{\gamma} \nabla\right)\psi = 0$$

has no nonzero solution $\psi \in L^2_{-1/2-0}$.

**Proof.** The equation (4.28) is equivalent to

$$(H_v - m^2)p = 0, \quad \text{where} \quad p(x) = e^{\pm iv\gamma x} \psi(x)$$

The last equation has no nonzero solution $p \in L^2_{-1/2-0}$ by Condition U2 ii).

### 4.2 Decay for the linearized dynamics

Let us consider the linearized equation

$$\dot{X}(t) = A_v X(t), \quad t \in \mathbb{R}$$

where $A_v = A_{v,v}$ is given in (4.20) with $V_v$ is defined in (4.7).

**Definition 4.7.** For $|v| < 1$, denote by $P^d_v$ the symplectic orthogonal projection of $E$ onto the tangent space $T_{S(\sigma)} S$, and $P^c_v = I - P^d_v$.

Note that by the linearity,

$$P^d_v X = \sum p_{jl}(v) \tau_j(v) \Omega(\tau_l(v), X), \quad X \in E$$

with some smooth coefficients $p_{jl}(v)$. Hence, the projector $P^d_v$, in the variable $y = x - b$, does not depend on $b$.

Next decay estimates will play the key role in our proofs. The first estimate follows from our assumption U2 by Theorem 3.9 of [11] since the condition of type [11, (3.1)] holds in our case.

**Theorem 4.8.** Let the condition U2 hold, and $\beta > 5/2$. Then for any $X \in E^\beta$, the weighted energy decay holds

$$\|e^{A_v t} P^c_v X\|_{E_{-\beta}} \leq C(1 + t)^{-3/2} \|X\|_{E^\beta}, \quad t \in \mathbb{R}$$
In [11] we consider the Klein-Gordon equation corresponding to \( v = 0 \). In Appendix A we extend the result of [11] to the modified Klein-Gordon equation with any \( v \in (-1, 1) \).

**Corollary 4.9.** For \( \sigma > 5/2 \) and for \( X \in E_\beta \cap W \)

\[
\|(e^{A_vt}P_v^c X)_1\|_{L^\infty} \leq C(1 + t)^{-1/2}(\|X\|_W + \|X\|_{E_\beta}), \quad t \in \mathbb{R}
\]

(4.32)

Here \((\cdot)_1\) stands for the first component of the vector function.

**Proof.** Let us apply the projector \( P_v^c \) to both sides of (4.29):

\[
P_v^c \dot{X} = A_v P_v^c X = A_v^0 P_v^c X + V_v P_v^c X
\]

(4.33)

where

\[
A_v^0 = \begin{pmatrix}
v\nabla & 1 \\
\Delta - m^2 & v\nabla
\end{pmatrix}, \quad V = \begin{pmatrix}
0 & 0 \\
-V_v & 0
\end{pmatrix}
\]

Hence, the Duhamel representation gives,

\[
e^{A_v t} Y = e^{A_v^0 t} Y + \int_0^t e^{A_v^0 (t-\tau)} V e^{A_v \tau} Y d\tau, \quad Y = P_v^c X, \quad t > 0.
\]

(4.34)

Let us note, that \( e^{A_v^0 t} Z = e^{A_v^0 t} T_v Z \), where \( T_v Z(x, t) = Z(x + vt, t) \). Then (4.34) reads

\[
e^{A_v t} Y = e^{A_v^0 t} T_v Y + \int_0^t e^{A_v^0 (t-\tau)} T_v \left[ V e^{A_v \tau} Y \right] d\tau, \quad t > 0
\]

(4.35)

Applying estimate (265) from [18], the Hölder inequality and Proposition 4.8 we obtain

\[
\|(e^{A_v t} Y)_1\|_{L^\infty} \leq C(1 + t)^{-1/2} \|T_v Y\|_W + C \int_0^t (1 + t - \tau)^{-1/2} \|T_v [V (e^{A_v \tau} Y)_1]\|_{W_{0,1}^{1,1}} d\tau
\]

\[
= C(1 + t)^{-1/2} \|Y\|_W + C \int_0^t (1 + t - \tau)^{-1/2} \|V (e^{A_v \tau} Y)_1\|_{W_{0,1}^{1,1}} d\tau
\]

\[
\leq C(1 + t)^{-1/2} \|X\|_W + C \int_0^t (1 + t - \tau)^{-1/2} \|e^{A_v \tau} P_v^c X\|_{E_{-\sigma}} d\tau
\]

\[
\leq C(1 + t)^{-1/2} \|X\|_W + C \int_0^t (1 + t - \tau)^{-1/2} (1 + \tau)^{-3/2} \|X\|_{E_{\sigma}} d\tau
\]

\[
\leq C(1 + t)^{-1/2} (\|X\|_W + \|X\|_{E_{\sigma}})
\]
4.3 Taylor expansion for nonlinear term

Now let us expand $N(v, \Psi)$ from (4.9) in the Taylor series

$$N(v, \Psi) = N_2(v, \Psi) + N_3(v, \Psi) + \ldots + N_{12}(v, \Psi) + N_R(v, \Psi) = N_I(v, \Psi) + N_R(v, \Psi)$$  (4.36)

where

$$N_j(v, \Psi) = \frac{F^{(j)}(\psi_v)}{j!} \Psi^j, \quad j = 2, \ldots, 12$$  (4.37)

and $N_R$ is the remainder. Recall that $k = 12$ is the order of the zero of $F(\psi)$ at $\psi = \pm a$ by U1. Hence, the functions $F^j(\psi_v(y))$, $2 \leq j \leq 12$ decrease exponentially as $|y| \to \infty$ by (1.7) and we obtain

$$\|N_I\|_{W^{1,1}_0} = \mathcal{R}(\|\Psi\|_{L^\infty}) \|\Psi\|_{L^\infty}^{10}$$  (4.38)

For the remainder $N_R$ we have

$$|N_R| = \mathcal{R}(\|\Psi\|_{L^\infty})|\Psi|^{13}$$  (4.39)

where $\mathcal{R}(A)$ is a general notation for a positive function which remains bounded as $A$ is sufficiently small.

**Lemma 4.10.** The bounds hold

$$\|N_R\|_{W^{1,1}_0} = \mathcal{R}(\|\Psi\|_{L^\infty}) \|\Psi\|_{L^\infty}^{10}$$  (4.40)

$$\|N_R\|_{L^2_{y,2+\nu}} = \mathcal{R}(\|\Psi\|_{L^\infty})(1 + t)^{4+\nu} \|\Psi\|_{L^\infty}^{12}, \quad 0 < \nu < 1/2$$  (4.41)

**Proof.** Step i) By the Cauchy formula,

$$\tilde{N}_R(x, t) = N_{12}(x, t) + N_R(x, t) = \frac{\Psi^{12}(x, t)}{(12)!} \int_0^1 (1 - \rho)^{11} F^{(12)}(\psi_v + \rho \Psi(x, t)) d\rho$$  (4.42)

Therefore,

$$\|\tilde{N}_R\|_{L^1} = \mathcal{R}(\|\Psi\|_{L^\infty}) \int |\Psi|^{12} dx = \mathcal{R}(\|\Psi\|_{L^\infty}) \|\Psi\|_{L^\infty}^{10} \|\Psi\|_2^2 = \mathcal{R}(\|\Psi\|_{L^\infty}) \|\Psi\|_{L^\infty}^{10}$$

since $\|\Psi\|_{L^2} \leq C(d_0)$ by the results of [8].

Differentiating (4.42), we obtain

$$\tilde{N}'_R = \frac{\Psi^{12}}{(12)!} \int_0^1 (1 - \rho)^{11} (\psi_v' + \rho \Psi') F^{(13)}(\psi_v + \rho \Psi) d\rho + \frac{\Psi^{11} \Psi'}{(11)!} \int_0^1 (1 - \rho)^{11} F^{(12)}(\psi_v + \rho \Psi) d\rho$$

Hence,

$$\|\tilde{N}'_R\|_{L^1} = \mathcal{R}(\|\Psi\|_{L^\infty}) [\|\Psi\|_{L^\infty}^{12} + \|\Psi\|_{L^\infty}^{10} \int |\Psi(x)\Psi'(x)| dx] \leq \mathcal{R}(\|\Psi\|_{L^\infty}) \|\Psi\|_{L^\infty}^{10}$$
On asymptotic stability of moving kinks for relativistic Ginzburg-Landau equation

since \( \int |\Psi(x)\Psi'(x)|dx \leq \|\Psi\|_{L^2}\|\Psi'\|_{L^2} \leq C(d_0) \). Finally, note that

\[
\|\Psi_{12}\|_{W^{1,1}_0} = \mathcal{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}^{\delta} \]

Then (4.40) follows.

**Step ii)** The bound (4.39) implies

\[
\|N_R\|_{L^2}^{5/2+\nu} = \mathcal{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}^{12}\|\Psi\|_{L^2}^{5/2+\nu} \]

We will prove in Appendix B that

\[
\|\Psi(t)\|_{L^2}^{5/2+\nu} \leq C(d_0)(1+t)^{4+\nu} \tag{4.43} \]

Then (4.41) follows.

\[\square\]

5 Symplectic decomposition of the dynamics

Here we decompose the dynamics in two components: along the manifold \( S \) and in transversal directions. The equation (4.5) is obtained without any assumption on \( \sigma(t) \) in (4.1). We are going to choose \( S(\sigma(t)) := \Pi Y(t) \), but then we need to know that

\[
Y(t) \in \mathcal{O}_\alpha(S), \quad t \in \mathbb{R} \tag{5.1} \]

with some \( \mathcal{O}_\alpha(S) \) defined in Lemma 3.4. It is true for \( t = 0 \) by our main assumption (2.8) with sufficiently small \( d_0 > 0 \). Then \( S(\sigma(0)) = \Pi Y(0) \) and \( X(0) = Y(0) - S(\sigma(0)) \) are well defined. We will prove below that (5.1) holds with \( \alpha = -\beta \) if \( d_0 \) is sufficiently small. First, we choose \( \alpha = -\beta \) such that

\[
|v(0)| \leq \overline{\nu} \tag{5.2} \]

Denote by \( r_-(\overline{\nu}) \) the positive number from Lemma 3.4 iii) which corresponds to \( \alpha = -\beta \). Then \( S(\sigma) + X \in \mathcal{O}_{-\beta}(S) \) if \( \sigma = (b,v) \) with \( |v| < \overline{\nu} \) and \( \|X\|_{E_{-\beta}} < r_{-\beta}(\overline{\nu}) \). Therefore, \( S(\sigma(t)) = \Pi Y(t) \) and \( X(t) = Y(t) - S(\sigma(t)) \) are well defined for \( t \geq 0 \) so small that \( \|X(t)\|_{E_{-\beta}} < r_{-\beta}(\overline{\nu}) \). This is formalized by the standard definition of the “exit time”. First, we introduce the “majorants”

\[
m_1(t) := \sup_{s \in [0,t]} (1+s)^{3/2}\|X(s)\|_{E_{-\beta}} \tag{5.3} \]

\[
m_2(t) := \sup_{s \in [0,t]} (1+s)^{1/2}\|\Psi(s)\|_{L^\infty} \tag{5.3} \]

Here \( X = (X_1, X_2) = (\Psi, \Pi) \). Let us denote by \( \varepsilon \in (0, r_{-\beta}(\overline{\nu})) \) a fixed number which we will specify below.

**Definition 5.1.** \( t_* \) is the exit time

\[
t_* = \sup \{t \in [0, t_*] : m_j(s) < \varepsilon, \quad j = 1, 2, \quad 0 \leq s \leq t \} \tag{5.4} \]

Let us note that \( m_j(0) < \varepsilon \) if \( d_0 \ll 1 \). One of our main goals is to prove that \( t_* = \infty \) if \( d_0 \) is sufficiently small. This would follow if we show that

\[
m_j(t) < \varepsilon/2, \quad 0 \leq t < t_* \tag{5.5} \]
6 Modulation equations

In this section we present the modulation equations which allow to construct the solutions \( Y(t) \) of the equation (2.1) close at each time \( t \) to a kink i.e. to one of the functions described in Definition 2.3 with time varying ("modulating") parameters \((b, v) = (b(t), v(t))\). We look for a solution to (2.1) in the form

\[
Y(t) = S(\sigma(t)) + X(t)
\]

by setting

\[
S(\sigma(t)) = \Pi Y(t)
\]

which is equivalent to the symplectic orthogonality condition of type (3.7),

\[
\mathcal{X}(t) \mid \mathcal{T}_{S(\sigma(t))} S, \quad t < t_*
\] (6.1)

The projection \( \Pi Y(t) \) is well defined for \( t < t_* \) by Lemma 3.4 iii). Now we derive the "modulation equations" for the parameters \( \sigma(t) = (b(t), v(t)) \). For this purpose, let us write (6.1) in the form

\[
\Omega(X(t), \tau_j(t)) = 0, \quad j = 1, 2
\] (6.2)

where the vectors \( \tau_j(t) = \tau_j(\sigma(t)) \) span the tangent space \( \mathcal{T}_{S(\sigma(t))} S \). It would be convenient for us to use some other parameters \((c, v)\) instead of \( \sigma = (b, v) \), where \( c(t) = b(t) - \int_0^t v(\tau) d\tau \) and

\[
\dot{c}(t) = \dot{b}(t) - v(t) = w(t) - v(t)
\] (6.3)

Lemma 6.1. Let \( Y(t) \) be a solution to the Cauchy problem (2.1), and (4.1), (6.2) hold. Then the parameters \( c(t) \) and \( v(t) \) satisfy the equations

\[
\dot{c} = \frac{\Omega(\tau_1, \tau_2) \Omega(N, \tau_2) + \Omega(X, \partial_v \tau_1) \Omega(N, \tau_2) - \Omega(X, \partial_v \tau_2) \Omega(N, \tau_1)}{D}
\] (6.4)

\[
\dot{v} = \frac{-\Omega(\tau_1, \tau_2) \Omega(N, \tau_1) - \Omega(X, \tau'_2) \Omega(N, \tau_1) - \Omega(X, \tau'_1) \Omega(N, \tau_2)}{D}
\] (6.5)

where

\[
D = \Omega^2(\tau_1, \tau_2) + O(\|X\|_{E-\beta})
\]

Proof. Differentiating the orthogonality conditions (6.2) in \( t \) we obtain

\[
0 = \Omega(X, \tau_j) + \Omega(X, \dot{\tau}_j) = \Omega(A_{v,w} X + T + N, \tau_j) + \Omega(X, \tau_j), \quad j = 1, 2
\] (6.6)

First, let us compute the principal (i.e. non-vanishing at \( X = 0 \)) term \( \Omega(T, \tau_j) \). By (4.10), one has

\[
\Omega(T, \tau_1) = -\dot{\epsilon} \Omega(\tau_2, \tau_1) = \dot{\epsilon} \Omega(\tau_1, \tau_2); \quad \Omega(T, \tau_2) = -\dot{\epsilon} \Omega(\tau_1, \tau_2)
\] (6.7)

Second, let us compute \( \Omega(A_{v,w} X, \tau_j) \). The skew-symmetry (4.15) implies that \( \Omega(A_{v,w} X, \tau_j) = -\Omega(X, A_{v,w} \tau_j) \). Then by (4.18) we have

\[
\Omega(A_{v,w} X, \tau_1) = \Omega(X, \dot{\epsilon} \tau'_1)
\] (6.8)

and similarly,

\[
\Omega(A_{v,w} X, \tau_2) = -\Omega(X, \dot{\epsilon} \tau'_2 + \tau_1) = -\Omega(X, \dot{\epsilon} \tau'_2) - \Omega(X, \tau_1) = -\Omega(X, \dot{\epsilon} \tau'_2)
\] (6.9)

since \( \Omega(X, \tau_1) = 0 \).
Finally, let us compute the last term $\Omega(X, \dot{\tau}_j)$. For $j = 1, 2$ one has $\dot{\tau}_j = \dot{b} \partial_b \tau_j + \dot{\nu} \partial_\nu \tau_j = \dot{\nu} \partial_\nu \tau_j$ since the vectors $\tau_j$ do not depend on $b$ according to (3.3). Hence,

$$\Omega(X, \dot{\tau}_j) = \Omega(X, \dot{\nu} \partial_\nu \tau_j)$$

(6.10)

As the result, by (6.7)-(6.10), the equation (6.6) becomes

$$0 = \dot{c} \Omega(X, \tau'_1) + \dot{\nu} \left( \Omega(\tau_1, \tau_2) + \Omega(X, \partial_\nu \tau_1) \right) + \Omega(N, \tau_1),$$

$$0 = -\dot{c} \left( \Omega(X, \tau'_2) + \Omega(\tau_1, \tau_2) \right) + \dot{\nu} \Omega(X, \partial_\nu \tau_2) + \Omega(N, \tau_2)$$

Since $\Omega(\tau_1, \tau_2) \neq 0$ by (3.5) then the determinant $D$ of the system does not vanish for small $\|X\|_{E-\beta}$ and we obtain (6.4)-(6.5).

**Corollary 6.2.** Formulas (6.4)-(6.5) imply

$$|\dot{c}(t)|, |\dot{\nu}(t)| \leq C(\nu, d_0) (1 + |t|)^{1/2}, \quad |\Omega(\tau_1, \tau_2)| \leq C(\nu) \|X(t)\|_{E-\beta}^2,$$  

$$0 \leq t < t_* \quad (6.11)$$

## 7 Decay for the transversal dynamics

In Section 12 we will show that our main Theorem 2.5 can be derived from the following time decay of the transversal component $X(t)$:

**Proposition 7.1.** Let all conditions of Theorem 2.5 hold. Then $t_* = \infty$, and

$$\|X(t)\|_{E-\beta} \leq \frac{C(\nu, d_0)}{(1 + |t|)^{3/2}}, \quad \|\Psi(t)\|_{L^\infty} \leq \frac{C(\nu, d_0)}{(1 + |t|)^{1/2}}, \quad t \geq 0 \quad (7.1)$$

We will derive (7.1) in Sections 11 from our equation (4.5) for the transversal component $X(t)$. This equation can be specified using Corollary 6.2. Indeed, (4.10) implies that

$$\|T(t)\|_{E_{\beta} \cap W} \leq C(\nu) \|X(t)\|_{E-\beta}, \quad 0 \leq t < t_* \quad (7.2)$$

by (6.11) since $w - v = \dot{c}$. Thus (4.5) becomes the equation

$$\dot{X}(t) = A(t)X(t) + T(t) + \mathcal{N}_I(t) + \mathcal{N}_R(t), \quad 0 \leq t < t_* \quad (7.3)$$

where $A(t) = A_{\nu(t), \nu(t)}, T(t)$ satisfies (7.2), and

$$\|\mathcal{N}_I(t)\|_{E_{\beta} \cap W} \leq C(\nu) \|\Psi\|_{L^\infty} \|X\|_{E-\beta},$$

$$\|\mathcal{N}_R\|_{E_{\nu/2+\nu}} \leq C(\nu) (1 + t)^{4+\nu} \|\Psi\|_{L^\infty}^2, \quad 0 < \nu < 1/2 \quad 0 \leq t < t_* \quad (7.4)$$

by (4.38), (4.41)-(4.40). In all remaining part of our paper we will analyze mainly the equation (7.3) to establish the decay (7.1). We are going to derive the decay using the bounds (7.2) and (7.4), and the orthogonality condition (6.1).

Let us comment on two main difficulties in proving (7.1). The difficulties are common for the problems studied in [2]. First, the linear part of the equation is non-autonomous, hence...
we cannot apply directly the methods of scattering theory. Similarly to the approach of [2], we reduce the problem to the analysis of the \textit{frozen} linear equation,

\[ \dot{X}(t) = A_1 X(t), \quad t \in \mathbb{R} \]  

(7.5)

where \( A_1 \) is the operator \( A_{v_1} \) defined by (4.6) with \( v_1 = v(t_1) \) for a fixed \( t_1 \in [0, t_\star) \). Then we estimate the error by the method of majorants.

Second, even for the frozen equation (7.5), the decay of type (7.1) for all solutions does not hold without the orthogonality condition of type (6.1). Namely, by (4.22) the equation (7.5) admits the \textit{ secular solutions} \[ X(t) = C_1 \tau_1(v) + C_2 [\tau_1(v)t + \tau_2(v)] \] 

(7.6)

which arise also by differentiation of the soliton (1.8) in the parameters \( q \) and \( v \) in the moving coordinate \( y = x - v_1 t \). Hence, we have to take into account the orthogonality condition (6.1) in order to avoid the secular solutions. For this purpose we will apply the corresponding symplectic orthogonal projection which kills the “runaway solutions” (7.6).

\textbf{Remark 7.2.} The solution (7.6) lies in the tangent space \( T_{S(\sigma)} S \) with \( \sigma = (b_1, v_1) \) (for an arbitrary \( b_1 \in \mathbb{R} \)) that suggests an unstable character of the nonlinear dynamics \textit{along the solitary manifold} (cf. Remark 4.1 ii))

\textbf{Definition 7.3.} Denote by \( \mathcal{X}_v = \mathcal{P}_v^c E \) the space symplectic orthogonal to \( T_{S(\sigma)} S \) with \( \sigma = (b, v) \) (for an arbitrary \( b \in \mathbb{R} \)).

Now we have the symplectic orthogonal decomposition \( E = T_{S(\sigma)} S + \mathcal{X}_v, \quad \sigma = (b, v) \) 

(7.7)

and the symplectic orthogonality (6.1) can be written in the following equivalent forms,

\[ \mathcal{P}_v^d X(t) = 0, \quad \mathcal{P}_v^c X(t) = X(t), \quad 0 \leq t < t_\star \]  

(7.8)

\textbf{Remark 7.4.} The tangent space \( T_{S(\sigma)} S \) is invariant under the operator \( A_v \) by (4.22), hence the space \( \mathcal{X}_v \) is also invariant by (4.15): \( A_v X \in \mathcal{X}_v \) on a dense domain of \( X \in \mathcal{X}_v \).

\section{Frozen Form of Transversal Dynamics}

Now let us fix an arbitrary \( t_1 \in [0, t_\star) \), and rewrite the equation (7.3) in a “frozen form”

\[ \dot{X}(t) = A_1 X(t) + (A(t) - A_1) X(t) + T(t) + \mathcal{N}_I(t) + \mathcal{N}_R(t), \quad 0 \leq t < t_\star \]  

(8.1)

where \( A_1 = A_{v(t_1), v(t_1)} \) and

\[ A(t) - A_1 = \begin{pmatrix} (w(t) - v(t_1)) \nabla & 0 \\ 0 & (w(t) - v(t_1)) \nabla \end{pmatrix} \]

The next trick is important since it allows us to kill the “bad terms” \( (w(t) - v(t_1)) \nabla \) in the operator \( A(t) - A_1 \).
**Definition 8.1.** Let us change the variables \((y, t) \mapsto (y_1, t) = (y + d_1(t), t)\) where

\[
d_1(t) := \int_{t_1}^{t} (w(s) - v(t_1)) ds, \quad 0 \leq t \leq t_1
\]  

(8.2)

Next define

\[
\tilde{X}(t) = (\Psi(y_1 - d_1(t), t), \Pi(y_1 - d_1(t), t))
\]

(8.3)

Then we obtain the final form of the “frozen equation” for the transversal dynamics

\[
\tilde{X}(t) = A_1 \tilde{X}(t) + \tilde{T}(t) + \tilde{N}_I(t) + \tilde{N}_R(t), \quad 0 \leq t \leq t_1
\]

(8.4)

where \(\tilde{T}(t), \tilde{N}_I(t)\) and \(\tilde{N}_R(t)\) are \(T(t), N_I(t)\) and \(N_R(t)\) expressed in terms of \(y_1 = y + d_1(t)\). At the end of this section, we will derive appropriate bounds for the “remainder terms” in (8.4). Let us recall the following well-known inequality: for any \(\alpha \in \mathbb{R}\)

\[(1 + |y + x|)^\alpha \leq (1 + |y|)^\alpha (1 + |x|)^{|\alpha|}, \quad x, y \in \mathbb{R}\]  

(8.5)

**Lemma 8.2.** For \(f \in L^2_\alpha\) with any \(\alpha \in \mathbb{R}\) the following estimate holds:

\[
\|f(y_1 - d_1)\|_{L^2_\alpha} \leq \|f\|_{L^2_\alpha} (1 + |d_1|)^{|\alpha|} \quad d_1 \in \mathbb{R}
\]

(8.6)

**Proof.** One has by (8.5)

\[
\|f(y_1 - d_1)\|_{L^2_\alpha}^2 = \int |f(y_1 - d_1)|^2 (1 + |y_1|)^{2\alpha} dy_1 = \int |f(y)|^2 (1 + |y + d_1|)^{2\alpha} dy \leq \\
\int |f(y)|^2 (1 + |y|)^{2\alpha} (1 + |d_1|)^{2|\alpha|} dy \leq (1 + |d_1|)^{2|\alpha|}\|f\|_{L^2_\alpha}^2
\]

and the lemma is proved. \(\Box\)

**Corollary 8.3.** The following bounds hold for \(0 \leq t \leq t_1\) by (7.2) and (7.4):

\[
\begin{align*}
\|\tilde{T}(t)\|_{E_\beta} & \leq C(\overline{v})(1 + |d_1(t)|)^{\beta}\|X\|_{L^2_{E_\beta}}, \quad \|\tilde{T}(t)\|_{W} \leq C(\overline{v})\|X\|_{E_\beta}^2 \\
\|\tilde{N}_I(t)\|_{E_\beta} & \leq C(\overline{v})(1 + |d_1(t)|)^{\beta}\|\Psi\|_{L^\infty}\|X\|_{E_\beta}, \quad \|\tilde{N}_I(t)\|_{W} \leq C(\overline{v})\|\Psi\|_{L^\infty}\|X\|_{E_\beta} \\
\|\tilde{N}_R\|_{E_{5/2+\nu}} & \leq C(\overline{v})(1 + |d_1(t)|)^{5/2+\nu}(1 + t)^{4+\nu}\|\Psi\|_{L^\infty}^2, \quad 0 < \nu < 1/2 \\
\|\tilde{N}_R\|_{W} & \leq C(\overline{v})\|\Psi\|_{L^\infty}^{10}
\end{align*}
\]

(8.7)

9 **Integral inequality**

The equation (8.4) can be written in the integral form:

\[
\tilde{X}(t) = e^{A_1 t} \tilde{X}(0) + \int_0^t e^{A_1(t-s)} [\tilde{T}(s) + \tilde{N}_I(s) + \tilde{N}_R(s)] ds, \quad 0 \leq t \leq t_1
\]

(9.1)

We apply the symplectic orthogonal projection \(P_1^c := P_{\epsilon(t_1)}\) to both sides, and get

\[
P_1^c \tilde{X}(t) = e^{A_1 t} P_1^c \tilde{X}(0) + \int_0^t e^{A_1(t-s)} P_1^c [\tilde{T}(s) + \tilde{N}_I(s) + \tilde{N}_R(s)] ds
\]
We have used here that $P_1^c$ commutes with the group $e^{A_1 t}$ since the space $X_1 := P_1^c E$ is invariant with respect to $e^{A_1 t}$ by Remark 7.4. Applying (4.31) we obtain that

$$\|P_1^c \tilde{X}(t)\|_{E_{-\beta}} \leq \frac{C\|\tilde{X}(0)\|_{E_{\beta}}}{(1 + t)^{3/2}} + C \int_0^t \frac{\|\tilde{T}(s) + \tilde{N}_I(s) + \tilde{N}_R(s)\|_{E_{\beta}}}{(1 + |t - s|)^{3/2}} ds$$

Hence, for $5/2 < \beta < 3$ and $0 \leq t \leq t_1$ the bounds (8.7) imply

$$\|P_1^c \tilde{X}(t)\|_{E_{-\beta}} \leq \frac{C(\bar{d}_1(0))}{(1 + t)^{1/2}} \|X(0)\|_{E_{\beta}} + C \int_0^t \frac{\|\tilde{T}(s) + \tilde{N}_I(s) + \tilde{N}_R(s)\|_{E_{\beta} \cap W}}{(1 + |t - s|)^{1/2}} ds$$

+ $C(\bar{d}_1(t)) \int_0^t \frac{\|X(s)\|_{E_{-\beta}}^2 + \|\Psi(s)\|_{L^\infty} \|X(s)\|_{E_{-\beta}} + (1 + s)^{3/2 + \beta} \|\Psi(s)\|_{L^\infty}^{12}}{(1 + |t - s|)^{1/2}} ds$

where $\bar{d}_1(t) := \sup_{0 \leq s \leq t} |d_1(s)|$. Similarly, (4.32) and (8.7) imply

$$\|(P_1^c X(t))_1\|_{L^\infty} \leq \frac{C \|\tilde{X}(0)\|_{E_{\beta} \cap W}}{(1 + t)^{1/2}} + C \int_0^t \frac{\|\tilde{T}(s) + \tilde{N}_I(s) + \tilde{N}_R(s)\|_{E_{\beta} \cap W}}{(1 + |t - s|)^{1/2}} ds$$

$$\leq \frac{C(\bar{d}_1(0))}{(1 + t)^{1/2}} \|X(0)\|_{E_{\beta} \cap W}$$

$$+ C(\bar{d}_1(t)) \int_0^t \frac{\|X(s)\|_{E_{-\beta}}^2 + \|\Psi(s)\|_{L^\infty} \|X(s)\|_{E_{-\beta}} + (1 + s)^{3/2 + \beta} \|\Psi(s)\|_{L^\infty}^{12} + \|\Psi(s)\|_{L^\infty}^{10}}{(1 + |t - s|)^{1/2}} ds$$

Lemma 9.1. For $t_1 < t_*$ we have

$$|d_1(t)| \leq C \varepsilon^2, \quad 0 \leq t \leq t_1$$

Proof. To estimate $d_1(t)$, we note that

$$w(s) - v(t_1) = w(s) - v(s) + v(s) - v(t_1) = \dot{c}(s) + \int_s^{t_1} \dot{v}(\tau) d\tau$$

by (6.3). Hence, the definitions (8.2), (5.3), and corollary 6.2 imply that

$$|d_1(t)| = \left| \int_t^{t_1} (w(s) - v(t_1)) ds \right| \leq \int_t^{t_1} \left( |\dot{c}(s)| + \int_s^{t_1} |\dot{v}(\tau)| d\tau \right) ds$$

$$\leq C m_1^2(t_1) \int_t^{t_1} \frac{1}{(1 + s)^3} + \int_s^{t_1} \frac{d\tau}{(1 + \tau)^2} ds \leq C m_1^2(t_1) \leq C \varepsilon^2, \quad 0 \leq t \leq t_1$$

Now (9.2) and (9.3) imply that for $t_1 < t_*$ and $0 \leq t \leq t_1$

$$\|P_1^c \tilde{X}(t)\|_{E_{-\beta}} \leq \frac{C \|X(0)\|_{E_{\beta}}}{(1 + t)^{3/2}}$$

$$+ C \int_0^t \frac{\|X(s)\|_{E_{-\beta}}^2 + \|\Psi(s)\|_{L^\infty} \|X(s)\|_{E_{-\beta}} + (1 + s)^{3/2 + \beta} \|\Psi(s)\|_{L^\infty}^{12}}{(1 + |t - s|)^{3/2}} ds$$
\[ \| (P_1^d \tilde{X}(t))_1 \|_{L^\infty} \leq \frac{C \| X(0) \|_{E_0 \cap W}}{(1+t)^{1/2}} \] (9.8)

\[ + C \int_0^t \| X(s) \|_{E_{-,\beta}} + \| \Psi(s) \|_{L^\infty} \| X(s) \|_{E_{-,\beta}} + (1+s)^{3/2+\beta} \| \Psi(s) \|_{L^\infty}^{12} + \| \Psi(s) \|_{L^\infty}^{10} \frac{ds}{(1+|t-s|)^{1/2}} \]

10 Symplectic orthogonality

Finally, we are going to change \( P_1^c \tilde{X}(t) \) by \( X(t) \) in the left hand side of (9.7) and (9.8). We will prove that it is possible since \( d_0 \ll 1 \) in (2.8).

**Lemma 10.1.** For sufficiently small \( \varepsilon > 0 \), we have for \( t_1 < t_* \)

\[ \| X(t) \|_{E_{-,\beta}} \leq C \| P_1^c \tilde{X}(t) \|_{E_{-,\beta}}, \quad 0 \leq t \leq t_1 \]

\[ \| \Psi(t) \|_{L^\infty} \leq 2 \| (P_1^c \tilde{X}(t))_1 \|_{L^\infty}, \quad 0 \leq t \leq t_1 \]

where the constant \( C \) does not depend on \( t_1 \).

**Proof.** The proof is based on the symplectic orthogonality (7.8), i.e.

\[ P_{v(t)}^d X(t) = 0, \quad t \in [0,t_1] \] (10.1)

and on the fact that all the spaces \( \mathcal{X}(t) := P_{v(t)}^c E \) are almost parallel for all \( t \).

Namely, we first note that \( \| \Psi(t) \|_{L^\infty} = \| \tilde{\Psi}(t) \|_{L^\infty} \), and \( \| X(t) \|_{E_{-,\beta}} \leq C \| \tilde{X}(t) \|_{E_{-,\beta}} \) by Lemma 8.2, since \( |d_1(t)| \leq \text{const} \) for \( t \leq t_1 \leq t_* \) by (9.4). Therefore, it suffices to prove that

\[ \| \tilde{\Psi}(t) \|_{L^\infty} \leq 2 \| (P_1^c \tilde{X}(t))_1 \|_{L^\infty}, \quad \| \tilde{X}(t) \|_{E_{-,\beta}} \leq 2 \| P_1^c \tilde{X}(t) \|_{E_{-,\beta}}, \quad 0 \leq t \leq t_1 \] (10.2)

This estimate will follow from

\[ \| (P_1^d \tilde{X}(t))_1 \|_{L^\infty} \leq \frac{1}{2} \| \tilde{\Psi}(t) \|_{L^\infty}, \quad \| P_1^d \tilde{X}(t) \|_{E_{-,\beta}} \leq \frac{1}{2} \| \tilde{X}(t) \|_{E_{-,\beta}}, \quad 0 \leq t \leq t_1 \] (10.3)

since \( P_1^c \tilde{X}(t) = \tilde{X}(t) - P_1^d \tilde{X}(t) \). To prove (10.3), we write (10.1) as

\[ \tilde{P}_{v(t)}^d \tilde{X}(t) = 0, \quad t \in [0,t_1] \] (10.4)

where \( \tilde{P}_{v(t)}^d \tilde{X}(t) \) is \( P_{v(t)}^d X(t) \) expressed in terms of the variable \( y_1 = y + d_1(t) \). Hence, (10.3) follows from (10.4) if the difference \( P_1^d - \tilde{P}_{v(t)}^d \) is small uniformly in \( t \), i.e.

\[ \| P_1^d - \tilde{P}_{v(t)}^d \| < 1/2, \quad 0 \leq t \leq t_1 \] (10.5)

It remains to justify (10.5) for small enough \( \varepsilon > 0 \). In order to prove the bound (10.5), we will need the formula (4.30) and the following relation which follows from (4.30):

\[ \tilde{P}_{v(t)}^d \tilde{X}(t) = \sum p_{j\ell}(v(t)) \tilde{r}_j(v(t)) \Omega(\tilde{\eta}(v(t)), \tilde{X}(t)) \] (10.6)
where \( \tilde{\tau}_j(v(t)) \) are the vectors \( \tau_j(v(t)) \) expressed in the variables \( y_1 \). In detail (cf. (3.3)),

\[
\begin{align*}
\tilde{\tau}_1(v) &:= (v'\psi_1(y_1 - d_1(t)), -\pi_1'(y_1 - d_1(t))) \\
\tilde{\tau}_2(v) &:= (\partial_v \psi_2(y_1 - d_1(t)), \partial_v \pi_2(y_1 - d_1(t)))
\end{align*}
\]

\hspace{1cm} (10.7)

where \( v = v(t) \). Since \( \tau'_j \) are smooth and rapidly decaying at infinity functions, then Lemma 9.1 implies

\[
\|\tilde{\tau}_j(v(t)) - \tau_j(v(t))\|_{E_\beta} \leq C\varepsilon^2, \quad 0 \leq t \leq t_1, \quad j = 1, 2
\]

\hspace{1cm} (10.8)

Furthermore,

\[
\tau_j(v(t)) - \tau_j(v(t_1)) = \int_{t}^{t_1} \dot{v}(s)\partial_v \tau_j(v(s))ds,
\]

and therefore

\[
\|\tau_j(v(t)) - \tau_j(v(t_1))\|_{E_\beta} \leq C \int_{t}^{t_1} |\dot{v}(s)| ds, \quad 0 \leq t \leq t_1.
\]

\hspace{1cm} (10.9)

Similarly,

\[
|p_{ji}(v(t)) - p_{ji}(v(t_1))| = \left| \int_{t}^{t_1} \dot{v}(s)\partial_v p_{ji}(v(s))ds \right| \leq C \int_{t}^{t_1} |\dot{v}(s)| ds, \quad 0 \leq t \leq t_1,
\]

\hspace{1cm} (10.10)

since \( |\partial_v p_{ji}(v(s))| \) is uniformly bounded by (5.2). Further,

\[
\int_{t}^{t_1} |\dot{v}(s)| ds \leq Cm_1^2(t_1) \int_{t}^{t_1} \frac{ds}{(1 + s)^3} \leq C\varepsilon^2, \quad 0 \leq t \leq t_1.
\]

\hspace{1cm} (11.1)

Hence, the bounds (10.5) will follow from (4.30), (10.6) and (10.8)-(10.10) if we choose \( \varepsilon > 0 \) small enough. The proof is completed. \boxed{}

11 Decay of transversal component

Here we prove Proposition 7.1.

\hspace{1cm} Step i) We fix \( \varepsilon > 0 \) and \( t_* = t_*(\varepsilon) \) for which Lemma 10.1 holds. Then the bounds of type (9.7) and (9.8) holds with \( \|P_i^d \tilde{X}(t)\|_{E_{-\beta}} \) and \( \|(P_i^d \tilde{X}(t))_1\|_{L^\infty} \) in the left hand sides replaced by \( \|\tilde{X}(t)\|_{E_{-\beta}} \) and \( \|\tilde{\Psi}(t)\|_{L^\infty} \):

\[
\begin{align*}
\|\tilde{X}(t)\|_{-\beta} &\leq \frac{C\|\tilde{X}(0)\|_{E_\beta}}{(1 + t)^{3/2}} + C \int_{0}^{t} \frac{\|\tilde{X}(s)\|_{E_{-\beta}}^2 + \|\tilde{\Psi}(s)\|_{L^\infty} \|\tilde{X}(s)\|_{E_{-\beta}} + (1 + s)^{3/2+\beta} \|\tilde{\Psi}(s)\|_{L^\infty}^2}{(1 + |t - s|)^{3/2}} ds \\
\|\tilde{\Psi}(t)\|_{L^\infty} &\leq \frac{C\|\tilde{X}(0)\|_{E_\beta \cap W}}{(1 + t)^{1/2}} \quad (11.1)
\end{align*}
\]

\[
\begin{align*}
&\quad + C \int_{0}^{t} \frac{\|\tilde{X}(s)\|_{E_{-\beta}}^2 + \|\tilde{\Psi}(s)\|_{L^\infty} \|\tilde{X}(s)\|_{E_{-\beta}} + (1 + s)^{3/2+\beta} \|\tilde{\Psi}(s)\|_{L^\infty}^2 + \|\tilde{\Psi}(s)\|_{L^\infty}^0}{(1 + |t - s|)^{1/2}} ds \\
&\quad (11.2)
\end{align*}
\]
for \(0 \leq t \leq t_1\) and \(t_1 < t_*\). This implies an integral inequality for the majorants \(m_1\) and \(m_2\). Namely, multiplying both sides of (11.1) by \((1 + t)^{3/2}\), and taking the supremum in \(t \in [0, t_1]\), we obtain

\[
m_1(t_1) \leq C\|X(0)\|_{E, \phi} + C \sup_{t \in [0, t_1]} \int_0^t \frac{(1 + t)^{3/2} ds}{(1 + |t - s|)^{3/2}} \left[ \frac{m_1^2(s)}{(1 + s)^3} + \frac{m_1(s)m_2(s)}{(1 + s)^2} + \frac{m_2^2(s)(1 + s)^{3/2 + \beta}}{(1 + s)^6} \right]
\]

for \(t_1 < t_*\). Taking into account that \(m(t)\) is a monotone increasing function, we get

\[
m_1(t_1) \leq C\|X(0)\|_{E, \phi} + C \left[m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^2(t_1)\right] I_1(t_1), \quad t_1 < t_* \tag{11.3}
\]

where

\[
I_1(t_1) = \sup_{t \in [0, t_1]} \int_0^t \frac{(1 + t)^{3/2}}{(1 + |t - s|)^{3/2} (1 + s)^{9/2 - \beta}} \leq T_1 < \infty, \quad t_1 \geq 0, \quad 5/2 < \beta < 3
\]

Therefore, (11.3) becomes

\[
m_1(t_1) \leq C\|X(0)\|_{E, \phi} + C T_1 \left[m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^2(t_1)\right], \quad t_1 < t_* \tag{11.4}
\]

Similarly, multiplying both sides of (11.2) by \((1 + t)^{1/2}\), and taking the supremum in \(t \in [0, t_1]\), we get

\[
m_2(t_1) \leq C\|X(0)\|_{E, \phi; W} + C \left[m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^2(t_1) + m_2^1(t_1)\right] I_2(t_1), \quad t_1 < t_* \tag{11.5}
\]

where

\[
I_2(t_1) = \sup_{t \in [0, t_1]} \int_0^t \frac{(1 + t)^{1/2}}{(1 + |t - s|)^{1/2} (1 + s)^{9/2 - \beta}} \leq T_2 < \infty, \quad t_1 \geq 0, \quad 5/2 < \beta < 3
\]

Therefore, (11.5) becomes

\[
m_2(t_1) \leq C\|X(0)\|_{E, \phi; W} + C T_2 \left[m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^2(t_1) + m_2^1(t_1)\right] \quad t_1 < t_* \tag{11.6}
\]

Inequalities (11.4) and (11.6) imply that \(m_1(t_1)\) and \(m_2(t_1)\) are bounded for \(t_1 < t'_*\), and moreover,

\[
m_1(t_1), \ m_2(t_1) \leq C\|X(0)\|_{E, \phi; W}, \quad t_1 < t_* \tag{11.7}
\]

since \(m_1(0) = \|X(0)\|_{E, \phi}\) and \(m_2(0) = \|\Psi(0)\|_{L^\infty}\) are sufficiently small by (2.8).

**Step ii)** The constant \(C\) in the estimate (11.7) does not depend on \(t_*\) by Lemma 10.1. We choose \(d_0\) in (2.8) so small that \(\|X(0)\|_{E, \phi; W} < \varepsilon/(2C)\). It is possible due to (2.8). Finally, this implies that \(t_* = \infty\), and (11.7) holds for all \(t_1 > 0\) if \(d_0\) is small enough.

## 12 Soliton asymptotics

Here we prove our main Theorem 2.5 under the assumption that the decay (7.1) holds. The estimates (6.11) and (7.1) imply that

\[
|\dot{c}(t)| + |\ddot{c}(t)| \leq \frac{C_1(\pi, d_0)}{(1 + t)^3}, \quad t \geq 0 \tag{12.1}
\]
Therefore, \( c(t) = c_+ + \mathcal{O}(t^{-2}) \) and \( v(t) = v_+ + \mathcal{O}(t^{-2}), \ t \to \infty \). Similarly,

\[
b(t) = c(t) + \int_{0}^{t} v(s) ds = v_+ t + q_+ + \alpha(t), \quad \alpha(t) = \mathcal{O}(t^{-1}) \tag{12.2}\]

We have obtained the solution \( Y(x, t) = (\psi(x, t), \pi(x, t)) \) to (1.3) in the form

\[
Y(x, t) = Y_{v(t)}(x - b(t), t) + X(x - b(t), t) \tag{12.3}
\]

where we define now \( v(t) = \dot{b}(t) = v_+ + \dot{\alpha}(t) \). Since

\[
\|Y_{v(t)}(x - b(t), t) - Y_{v_+}(x - v_+ t - q_+, t)\|_E = \mathcal{O}(t^{-1}),
\]

it remains to extract the dispersive wave \( W_0(t) \Phi_+ \) from the term \( X(x - b(t), t) \). Substituting (12.3) into (1.3) we obtain by (2.6) the inhomogeneous Klein-Gordon equation for the \( X(x - b(t), t) \):

\[
\dot{X}(y, t) = A_v^0 X(y, t) + R(y, t), \quad 0 \leq t \leq \infty \tag{12.4}
\]

where \( y = x - b(t) \), and

\[
A_v^0 = \begin{pmatrix} v
\n\Delta - m^2
\n1
\n\end{pmatrix}, \quad R(t) = \begin{pmatrix} \dot{v} \partial_v \psi_v \\ \dot{v} \partial_{v_+} \psi_v + F(\Psi + \psi_v) - F(\psi_v) + m^2 \Psi
\end{pmatrix}
\]

Now we change the variable \( y \mapsto y_1 = y + \alpha(t) + q_+ \). Then we obtain the “frozen” equation

\[
\dot{X}(t) = A_+ \dot{X}(t) + \tilde{R}(t), \quad 0 \leq t \leq \infty, \tag{12.5}
\]

where \( \dot{X}(t) \) and \( \tilde{R}(t) \) are \( X(t) \) and \( R(t) \) of \( y = y_1 - \alpha(t) - q_+ \), and

\[
A_+ = \begin{pmatrix} v_+ \n\Delta - m^2
\n1
\n\end{pmatrix} \tag{12.6}
\]

Equation (12.5) implies

\[
\dot{X}(t) = W_+(t) \dot{X}(0) + \int_{0}^{t} W_+(t - s) \tilde{R}(s) ds \tag{12.7}
\]

where \( W_+(t) = e^{A_+ t} \) is the integral operator with integral kernel

\[
W_+(y_1 - z, t) = W_0(y_1 - z + v_+ t, t) = W_0(x - z, t)
\]

since by (12.2)

\[
y_1 + v_+ t = y + \alpha(t) + q_+ + v_+ t = x - b(t) + \alpha(t) + q_+ + v_+ t = x
\]

Hence, equation (12.7) implies

\[
X(x - b(t), t) = W_0(t) \dot{X}(0) + \int_{0}^{t} W_0(t - s) \tilde{R}(s) ds \tag{12.8}
\]
Let us rewrite (12.8) as

\[ X(x - b(t), t) = W_0(t) \left( \tilde{X}(0) + \int_0^\infty W_0(-s) \tilde{R}(s) ds \right) - \int_t^\infty W_0(t - s) \tilde{R}(s) ds = W_0(t) \Phi_+ + r_+(t) \]

To establish the asymptotics (2.9), it suffices to prove that

\[ \Phi_+ = \tilde{X}(0) + \int_0^\infty W_0(-s) \tilde{R}(s) ds \in E \quad \text{and} \quad \|r_+(t)\|_E = O(t^{-1/2}) \]  

(12.9)

Assumption (2.8) implies that \( \tilde{X}(0) \in E \). Let us split \( \tilde{R}(s) \) as the sum

\[ \tilde{R}(s) = \left( \frac{\dot{v}\partial_v \tilde{\psi}_v}{\dot{v}\partial_v \tilde{\pi}_v} \right) + \left( 0 \ F(\tilde{\psi}_v + m^2) \ F(\tilde{\psi}_v) \right) = \tilde{R}'(s) + \tilde{R}''(s) \]

By (12.1), we obtain

\[ \|\tilde{R}'(s)\|_E = O(s^{-3}) \]  

(12.10)

Let us consider \( \tilde{R}'' = (0, \tilde{R}''_{2}) \). We have

\[ \tilde{R}''_2 = F(\tilde{\psi}_v + m^2) \tilde{\psi}_v - F(\tilde{\psi}_v) + m^2 \tilde{\psi} + \tilde{N}(v, \tilde{\Psi}) = -\tilde{V}_v \tilde{\psi}_v + \tilde{N}(v, \tilde{\Psi}) \]

By (1.12) and (7.1), we obtain

\[ \|\tilde{V}_v \tilde{\psi}(t)\|_{L^2} \leq C \|\tilde{\Psi}(t)\|_{L^2_{\beta}} \leq C(\varpi, d_0)(1 + |t|)^{-3/2} \]  

(12.11)

since \( |q_+ + \alpha(t)| \leq C \). Finally, (7.1), (7.4), and (8.6) imply

\[ \|\tilde{N}(v, \tilde{\Psi}(t))\|_{L^2} \leq C(\varpi, d_0)(1 + |t|)^{-3/2} \]  

(12.12)

Hence, (12.11)-(12.12) imply

\[ \|\tilde{R}''(s)\|_E = O(s^{-3/2}) \]  

(12.13)

and (12.9) follows by (12.10) and (12.13).

### A Proof of Theorem 4.8

We develop the spectral approach of Agmon, Jensen and Kato [1, 6] which relies on the spectral Fourier-Laplace representation

\[ \mathbf{P}^\alpha \Psi(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \left[ R(\lambda + 0) - R(\lambda - 0) \right] \Psi_0 d\lambda, \quad t \in \mathbb{R} \]  

(A. 1)

where \( \Gamma \) is the contour (4.21), and on the investigation of the behavior of the resolvent \( R(\lambda) = (A_v - \lambda)^{-1} \) at the edge points \( \lambda = \pm \mu = \pm im/\gamma \) of the continuous spectrum and high energy behavior at \( |\lambda| \to \infty \), \( \lambda \in \Gamma \). For the case \( v = 0 \), the development is done in [11]. In general case \( v \in (-1,1) \), the investigation of high energy behavior is similar. It remains to prove the asymptotics for the resolvent and its derivatives at the edge points \( \lambda = \pm \mu \). Denote by \( \mathcal{L}(E_\beta, E_{-\beta}) \) the Banach space of bounded linear operators from \( E_\beta \) to \( E_{-\beta} \). In this section we prove
Proposition A.1. The asymptotics hold

\[ R(\lambda) = B_0^\pm + \mathcal{O}(\nu^{1/2}) \]
\[ R'(\lambda) = \mathcal{O}(\nu^{-1/2}) \]
\[ R''(\lambda) = \mathcal{O}(\nu^{-3/2}) \]

\[ \nu := \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma \] (A. 2)

in the norm \( \mathcal{L}(E_\beta, E_{-\beta}) \) with \( \beta > 5/2 \), where \( B_0^\pm \) does not depend on \( \lambda \).

The Proposition allows estimate the low energy component by Lemma 10.2 from [6] that finishes the proof of Theorem 4.8.

Remark A.2. All spectral properties of the operator \( A_v \) should be similar to \( A_0 \) due to the relativistic invariance of equation (1.1). However technically is easier to obtain the properties by direct calculation.

A. 1 Free resolvent

Here we study the resolvent of the free operator \( A_v^0 \) corresponding to \( A_v = A_{v,v} \) from (4.20) with \( V_v(y) = 0 \),

\[ R_0(\lambda) = (A_v^0 - \lambda)^{-1} = \left( \begin{array}{cc} v\nabla - \lambda & 1 \\ \Delta - m^2 & v\nabla - \lambda \end{array} \right)^{-1}, \quad \text{Re} \lambda > 0 \]

In the Fourier space we obtain

\( \left( \begin{array}{cc} -(ivk + \lambda) & 1 \\ -(k^2 + m^2) & -(ivk + \lambda) \end{array} \right) ^{-1} = [(ivk + \lambda)^2 + k^2 + m^2]^{-1} \left( \begin{array}{cc} -(ivk + \lambda) & -1 \\ k^2 + m^2 & -(ivk + \lambda) \end{array} \right) \)

Taking the inverse Fourier transform, we obtain the resolvent

\[ R_0(\lambda) = \left( \begin{array}{cc} v\nabla - \lambda & -1 \\ -\Delta + m^2 & v\nabla - \lambda \end{array} \right) G_0(\lambda) = \left( \begin{array}{cc} (v\nabla - \lambda)G_0(\lambda) & -G_0(\lambda) \\ 1 - (v\nabla - \lambda)^2G_0(\lambda) & (v\nabla - \lambda)G_0(\lambda) \end{array} \right) \] (A. 3)

where \( G_0(\lambda) \) is the integral operator with the kernel

\[ G_0(\lambda, y - y') = F_{k-y-y'}^{-1} \frac{1}{k^2 + m^2 + (ivk + \lambda)^2}, \quad y, y' \in \mathbb{R} \] (A. 4)

which is well defined since the denominator in (A. 4) does not vanish for \( \text{Re} \lambda > 0 \). Let us compute \( G_0(\lambda, y - y') \) explicitly. The fundamental solution of the operator \(-\Delta + \zeta\) is well-known

\[ F_{k-y}^{-1} \frac{1}{k^2 + \zeta} = \frac{e^{-\sqrt{\zeta}|z|}}{2\sqrt{\zeta}} \] (A. 5)

where \( \text{Re} \sqrt{\zeta} > 0 \) for \( \text{Re} \zeta > 0 \). For general \( v, |v| < 1 \) the denominator in (A. 4) reads

\[ k^2 + m^2 + (ivk + \lambda)^2 = (1 - v^2)k^2 + 2ivk\lambda + \lambda^2 + m^2 = (1 - v^2)(k + \frac{iv\lambda}{1 - v^2})^2 + \lambda^2\gamma^2 + m^2 \] (A. 6)
Therefore (A. 5) implies that
\[
G_0(\lambda, z) = \frac{1}{2\pi} \int \frac{e^{-ikz}dk}{(1-v^2)(k+i\gamma^2v\lambda)^2 + \lambda^2\gamma^2 + m^2} = \frac{e^{-\gamma^2v\lambda z}}{2\pi} \int \frac{e^{-ikz}dk}{(1-v^2)k^2 + \lambda^2\gamma^2 + m^2}.
\]
(A. 7)

where we choose \( \text{Re}(\lambda) > 0 \). Let us note that
\[
0 < \text{Re}(v\lambda) < \text{Re}(\sqrt{\lambda^2 - \mu^2}), \quad \text{Re} \lambda > 0.
\]
This inequality follows from the fact that the fundamental solution decays exponentially by the Paley-Wiener arguments since the quadratic form (A. 6) does not vanish in a complex neighborhood of the real axes \( \mathbb{R} \) for \( \text{Re} \lambda > 0 \).

Let us consider the behavior of \( G_0(\lambda) \) near \( \lambda = \pm \mu \). Expanding in the Taylor series at the point \( \nu := \lambda \mp \mu \), we obtain that
\[
G_0(\lambda, z) = G_0^\pm(0, z) + G_1^\pm(z) + r(\lambda, z), \quad \nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma
\]
where
\[
G_0^\pm(0, z) = \frac{e^{\mp \gamma^2v\mu z}}{2\sqrt{\pm \mu}}, \quad G_1^\pm(z) = -\frac{\gamma^2 e^{\mp \gamma^2v\mu z} |z|}{2}
\]
(A. 8)
and
\[
|r(\lambda, z)| \leq C(1 + |z|^2)\sqrt{|\lambda \mp \mu|}, \quad \lambda \to \pm \mu, \quad \lambda \in \mathbb{C} \setminus \Gamma
\]
Hence the asymptotics hold
\[
G_0(\lambda) = G_0^\pm \frac{1}{\sqrt{\nu}} + G_1^\pm + \mathcal{O}(\sqrt{\nu}), \quad \nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma
\]
(A. 9)
in the norm \( \mathcal{L}(E_\beta, E_{-\beta}) \) with \( \beta > 5/2 \), where
\[
G_j^\pm = \text{Op}[G_j^\pm(y - y')] \in \mathcal{L}(H^0_{\beta}, H^2_{-\beta}), \quad \beta > 1/2 + j, \quad j = 0, 1
\]

A. 2 Perturbed resolvent

Now we construct the resolvent of the perturbed equation. We use the formula
\[
R(\lambda) = (1 + R_0(\lambda)\mathbf{V}_v)^{-1}R_0(\lambda)
\]
(A. 10)
where \( \mathbf{V}_v \) is the matrix
\[
\mathbf{V}_v = \begin{pmatrix}
0 & 0 \\
-V_v & 0
\end{pmatrix}
\]
(A. 11)
On asymptotic stability of moving kinks for relativistic Ginzburg-Landau equation

We have
\[(1 + R_0(\lambda)V_v)^{-1} = \begin{pmatrix} 1 + G_0(\lambda)V_v & 0 \\ -(v\nabla - \lambda)G_0(\lambda)V_v & 1 \end{pmatrix}^{-1} = \begin{pmatrix} (1 + G_0(\lambda)V_v)^{-1} & 0 \\ (v\nabla - \lambda) \left(1 - (1 + G_0(\lambda)V_v)^{-1}\right) & 1 \end{pmatrix}\]
(A.12)

Denote \(G(\lambda) = (H_v - \lambda^2 + 2\nu\lambda\nabla)^{-1}\), where \(H_v = -(1 - \nu^2)\Delta + m^2 + V_v = H_{v,0} + V_v\). Since \(G_0(\lambda) = (H_{v,0} - \lambda^2 + 2\nu\lambda\nabla)^{-1}\), then \(G(\lambda) = (1 + G_0(\lambda)V_v)^{-1}G_0(\lambda)\). Hence, (A.10) implies
\[R(\lambda) = \begin{pmatrix} (1 + G_0(\lambda)V_v)^{-1} & 0 \\ (v\nabla - \lambda) \left(1 - (1 + G_0(\lambda)V_v)^{-1}\right) & 1 \end{pmatrix} = \begin{pmatrix} (v\nabla - \lambda)G(\lambda) & -G(\lambda) \\ 1 - (v\nabla - \lambda)^2G(\lambda) & (v\nabla - \lambda)G(\lambda) \end{pmatrix}\]
(A.13)
since
\[(v\nabla - \lambda) \left(1 - (1 + G_0V_v)^{-1}\right)(v\nabla - \lambda)G_0(\lambda) + (\Delta + m^2)G_0(\lambda) = 1 - (v\nabla - \lambda)^2G(\lambda)\]

### A.3 Boundedness of perturbed resolvent

**Lemma A.3.** The families \(\{G(\mu + \varepsilon), \mu + \varepsilon \in \mathbb{C}\setminus \overline{\Gamma}, |\varepsilon| < \delta\}\) and \(\{G(-\mu + \varepsilon), -\mu + \varepsilon \in \mathbb{C}\setminus \overline{\Gamma}, |\varepsilon| < \delta\}\) are bounded in the operator norm of \(L(H^0_\beta, H^2_{-\beta})\) for any \(\beta > 1/2\) and sufficiently small \(\delta\).

**Proof.** We adopt the arguments from the proof of [13, Theorem 7.2]. For concreteness we consider the first family and suppose that the family is unbounded in \(L(H^0_\beta, H^2_{-\beta})\) for some \(\beta > 1/2\). Then there are \(f \in H^0_\beta\) and \(\varepsilon_j\) such that
\[\varepsilon_j \to 0, \quad \mu + \varepsilon_j \not\in \mathbb{C}\setminus \overline{\Gamma} \quad \text{and} \quad \|G(\mu + \varepsilon_j)f\|_{H^2_{-\beta}} \to \infty \quad \text{as} \quad j \to \infty\]

Setting \(u_j = G(\mu + \varepsilon_j)f/\|G(\mu + \varepsilon_j)f\|_{H^2_{-\beta}}\) and \(f_j = f/\|G(\mu + \varepsilon_j)f\|_{H^2_{-\beta}}\) we obtain that
\[(H_v - (\mu + \varepsilon_j)^2 + 2\nu(\mu + \varepsilon_j)\nabla)u_j = f_j, \quad \|f_j\|_{H^0_\beta} \to 0 \quad \text{as} \quad j \to \infty\]
(A.14)

Let us rewrite the last equation as
\[u_j = \frac{1}{\sqrt{\varepsilon_j}}G_0^+(f_j - V_vu_j) + \left(G_0(\mu + \varepsilon_j) - \frac{1}{\sqrt{\varepsilon_j}}G_0^+\right)(f_j - V_vu_j)\]
(A.15)
where \(G_0^+\) is defined in (A.8). Since \(\|u_j\|_{H^2_{-\beta}} = 1\), we may assume that
\[u_j \to u\]
(A.16)
weakly in \(H^2_{-\beta}\).
**Step i)** Let us consider the last term of (A. 15). By (A. 9), the operator $G_0(\mu + \varepsilon_j) - \frac{1}{\sqrt{\varepsilon_j}} G_0^+ \varepsilon_j$ converges as $\varepsilon_j \to 0$, $\mu + \varepsilon_j \in \mathbb{C} \setminus \Gamma$ in the space of the bounded operators from $H^0_\beta$ to $H^2_\beta$, hence
\[
\|(G_0(\mu + \varepsilon_j) - \frac{1}{\sqrt{\varepsilon_j}} G_0^+) f_j\|_{H^2_\beta} \to 0, \quad j \to \infty
\]  
(A. 17)
Further, $(G_0(\mu + \varepsilon_j) - \frac{1}{\sqrt{\varepsilon_j}} G_0^+) V_v$ is a compact operator from $H^2_\beta$ to $H^2_\beta$. Therefore the last term of (A. 15) converge strongly in $H^2_\beta$ as $j \to \infty$.

**Step ii)** Further, we consider the second term $\frac{1}{\sqrt{\varepsilon_j}} G_0^+(f_j - V_v u_j)$ of (A. 15). By (A. 16) and (A. 17) this term converges weakly in $H^2_\beta$. On the other hand,
\[
G_0^+(f_j - V_v u_j) = \frac{1}{2\sqrt{2\mu}} (f_j - V_v u_j, e^{i\gamma \nu x}) e^{i\gamma \nu x}
\]
bys (A. 8). Hence,
\[
\frac{1}{\sqrt{\varepsilon_j}} (f_j - V_v u_j, e^{i\gamma \nu x}) \to a, \quad j \to \infty
\]  
(A. 18)
for some constant $a$. This shows that the second term also converges strongly in $H^2_\beta$.

**Step iii)** Now (A. 16) holds in the norm of $H^2_\beta$. Hence $\|u\|_{H^2_\beta} = 1$. On the other hand (A. 14) implies
\[
(H_v - \mu^2 + 2v_\mu \nabla) u = 0
\]
that contradicts Lemma 4.6 since $u \neq 0$.

Formulas (A. 13) implies

**Corollary A.4.** The families $\{R(\pm \mu + \varepsilon), \pm \mu + \varepsilon \in \mathbb{C} \setminus \Gamma, |\varepsilon| < \delta\}$ are bounded in the operator norm of $\mathcal{L}(E_\beta, E_{-\beta})$ for any $\beta > 1/2$ and sufficiently small $\delta$.

**Corollary A.5.** For any $\beta > 1/2$ and $k = 0, 1, 2, \ldots$ the operators $(1 + G_0(\lambda)V_v)^{-1}$ and $(1 + V_v G_0(\lambda))^{-1}$ are bounded in $\mathcal{L}(H^k_\beta, H^k_\beta)$ and in $\mathcal{L}(H^k_{\beta}, H^k_{\beta})$ respectively for $|\lambda \mp \mu| < \delta$, $\lambda \in \mathbb{C} \setminus \Gamma$ with $\delta$ sufficiently small.

**Corollary A.6.** The bounds hold
\[
(1 + G_0(\lambda)V_v)^{-1} G_0^+ = \mathcal{O}(\sqrt{\nu}), \quad G_0^+(1 + V_v G_0(\lambda) \mu^{1/2})^{-1} = \mathcal{O}(\sqrt{\nu}) \quad \nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma \quad (A. 19)
\]
in $\mathcal{L}(H^0_{\beta}, H^2_{-\beta})$ with $\beta > 3/2$.

**Proof.** The asymptotics (A. 9) implies
\[
G(\lambda) = (1 + G_0(\lambda)V_v)^{-1} G_0(\lambda) = (1 + G_0(\lambda)V_v)^{-1} (G_0^+ \frac{1}{\sqrt{\nu}} + \mathcal{O}(1)) \quad \nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma
\]
\[
G(\lambda) = G_0(\lambda)(1 + V_v G_0(\lambda))^{-1} = (G_0^+ \frac{1}{\sqrt{\nu}} + \mathcal{O}(1))(1 + V_v G_0(\lambda))^{-1}
\]
Hence, the boundedness $G(\lambda)$, $(1 + G(\lambda)V_v)^{-1}$ and $(1 + V_v G(\lambda))^{-1}$ at the points $\lambda = \pm \mu$ in corresponding norms imply the bounds (A. 19).

□
Corollary A.7. i) The bound holds
\[
\|(1 + G_0(\lambda)V_v)^{-1}[e^{\mp\gamma^2\nu y}]\|_{H^2_{-\beta}} = O(\sqrt{\nu}), \quad \nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma, \quad \beta > 3/2 \quad (A. 20)
\]

ii) For any \( f \in H^0_{\beta} \) with \( \beta > 3/2 \)
\[
\int e^{\pm\gamma^2\nu y'}[(1 + V_vG_0(\lambda))^{-1}f](y')dy' = O(\sqrt{\nu}), \quad \nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma \quad (A. 21)
\]

A. 4 Bounds for derivatives

Here we derive the bounds for the first and second derivatives of \( R(\lambda) \) at the points \( \lambda = \pm \mu \).

First we estimate \( G'_0(\lambda) \) and \( G''_0(\lambda) \).

Lemma A.8. The bounds hold
\[
G'_0(\lambda) = O(\nu^{-1/2}) \quad \left| \begin{array}{c}
\nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma
\end{array} \right. \quad (A. 22)
\]
\[
G''_0(\lambda) = O(\nu^{-3/2}) \quad \left| \begin{array}{c}
\nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma
\end{array} \right. \quad (A. 23)
\]
in the norm \( L(H^0_{\beta}, H^2_{-\beta}) \) with \( \beta > 5/2 \).

Proof. Differentiating (A. 9) we obtain
\[
g'_0(\lambda) = \frac{1}{2}G'^+_0(\nu^{-3/2} + O(\nu^{-1/2})), \quad g''_0(\lambda) = \frac{3}{4}G'^+_0(\nu^{-5/2} + O(\nu^{-3/2})), \quad \nu = \lambda \mp \mu \to 0, \quad \lambda \in \mathbb{C} \setminus \Gamma \quad (A. 24)
\]
in the norm \( L(H^0_{\beta}, H^2_{-\beta}) \) with \( \beta > 5/2 \). Further we use the identities
\[
g' = (1 + G_0V_v)^{-1}g'_0(1 + V_vG_0)^{-1}, \quad g'' = \left[ (1 + G_0V_v)^{-1}g''_0 - 2g'V_vg'_0 \right](1 + V_vG_0)^{-1}
\]
Hence (A. 20)-(A. 21) implies (A. 22) since by (A. 8)
\[
G'^+_0(y - y') = \frac{1}{2\sqrt{\pm2\mu}}e^{\mp\gamma^2\nu y}e^{\pm\gamma^2\nu y'}
\]

Finally, Lemma A.8 and formula (A. 13) imply second and third lines of (A. 2). The first line follows by integration of the second.

B Virial type estimates

Here we prove the weighted estimate (4.43). Let us recall that we split the solution \( Y(t) = (\psi(\cdot,t), \pi(\cdot,t)) = S(\sigma(t)) + X(t) \), and denote \( X(t) = (\Psi(t), \Pi(t)), \quad (\Psi_0, \Pi_0) := (\Psi(0), \Pi(0)). \)

Our basic condition (2.8) implies that for some \( \nu > 0 \)
\[
\|X_0\|_{E_{5/2+\nu}} \leq d_0 < \infty \quad (B. 1)
\]
Proposition B.1. Let the potential $U$ satisfy conditions $U_{1}$, and $\Psi_{0}$ satisfy (B.1). Then the bounds hold

$$\|\Psi(t)\|_{L_{2}/2+\nu}^{2} \leq C(\tau, d_{0})(1 + t)^{4+\nu}, \quad t > 0$$

We will deduce the proposition from the following two lemmas. The first lemma is well known. Denote

$$e(x, t) = \frac{|\pi(x, t)|^{2}}{2} + \frac{|\psi'(x, t)|^{2}}{2} + U(\psi(x, t)).$$

Lemma B.2. For the solution $\psi(x, t)$ of Klein-Gordon equation (1.1) the local energy estimate holds

$$\left\{\int_{a_{1}-t}^{a_{2}+t} e(x, t) \, dx\right\} \leq \int_{a_{1}-t}^{a_{2}+t} e(x, 0) \, dx \quad \text{for} \quad a_{1} < a_{2}, \quad t > 0.$$  \hfill (B.3)

Proof. The estimate follows by standard arguments: multiplication of the equation (1.1) by $\dot{\psi}$ and integration over the trapezium $ABCD$, where $A = (a_{1} - t, 0)$, $B = (a_{1}, t)$, $C = (a_{2}, t)$, $D = (a_{2} + t, 0)$. Then (B.3) is obtained after partial integration using that $U(\psi) \geq 0$. \hfill \square

Lemma B.3. For any $\sigma \geq 0$

$$\int_{x-b}^{x+b}(1 + |x-b|^{\sigma})e(x,t)dx \leq C(\sigma)(1 + t + |b|)^{\sigma+1}\int_{x-b}^{x+b}(1 + |x|^{\sigma})e(x,0)dx.$$  \hfill (B.4)

Proof. By (B.3)

$$\int_{x-b}^{x+b}(1 + |y|^{\sigma})\left(\int_{y+b-1}^{y+b+t} e(x, t)dx\right)dy \leq \int_{x-b}^{x+b}(1 + |y|^{\sigma})\left(\int_{y+b-1-t}^{y+b+t} e(x, 0)dx\right)dy$$

Hence,

$$\int_{x-b}^{x+b+1}(1 + |y|^{\sigma})dy \leq C(\sigma)(1 + t + |b|)^{\sigma+1}\int_{x-b}^{x+b+1}(1 + |x|^{\sigma})dy.$$  \hfill (B.5)

Obviously,

$$\int_{x-b}^{x+b+1}(1 + |y|^{\sigma})dy \geq C(\sigma)(1 + |x - b|^{\sigma})$$  \hfill (B.6)

with some $c(\sigma) > 0$. On the other hand,

$$\int_{x-b}^{x+b+1+t}(1 + |y|^{\sigma})dy \leq (2t + 1)(1 + t + |b| + |x|)^{\sigma} \leq C(1 + t + |b|)^{\sigma+1}(1 + |x|^{\sigma})$$  \hfill (B.7)

since $\sigma \geq 0$. Finally, (B.5)-(B.7) imply (B.4). \hfill \square
Proof of Proposition B.1 First, we verify that

\[ U_0 = \int (1 + |x|^{5+2\nu})U(\psi_0(x))dx < C(d_0), \quad \psi_0(x) = \psi(x, 0) \quad (B. 8) \]

Indeed, \( \psi_0(x) = \psi_{v_0}(x - q_0) + \Psi_0(x) \) is bounded since \( \Psi_0 \in H^1(\mathbb{R}) \). Hence \( U1 \) implies that

\[ |U(\psi_0(x))| \leq C(d_0)(\psi_0(x) \pm a)^2 \leq C(d_0)\left( (\psi_{v_0}(x - q_0) \pm a)^2 + \Psi_0^2(x) \right) \]

and then (B. 8) follows by (1.6) and (B. 1). Further, we have

\[ \|\Psi(t)\|^2_{L_{5/2+\nu}^2} = \int (1 + |y|^{5+2\nu})\left( \int_{0}^{t} (\Psi(y, s)ds - \Psi_0(y)) \right)^2 dy \leq 2d_0^2 + 2t \int (1 + |y|^{5+2\nu})dy \int_{0}^{t} \Psi^2(y, s)ds \]

Using the bounds (6.11) we obtain

\[ |\dot{v}(s)| \leq C(\overline{\tau})\|\Psi(s)\|_{L_2}^2 \leq C(\overline{\tau}, d_0), \quad |\dot{b}(s)| = |\dot{c}(s) + v(s)| \leq |\dot{c}(s)| + 1 \leq C(\overline{\tau}, d_0) \]

\[ |\dot{b}(s)| = \left| \int_{0}^{s} \dot{b}^{(s)}ds - b(0) \right| \leq C(\overline{\tau}, d_0)s + |q_0| \quad (B. 10) \]

Hence (4.2) implies that

\[ \dot{\Psi}^2(y, s) = \left[ \dot{b}(s)\dot{\psi}'(y + b(s), s) + \pi(y + b(s), s) - \dot{\psi}' + \dot{\psi}_v(y) \right]^2 \leq C(\overline{\tau}, d_0)\left( (\dot{\psi}'(y + b(s), s))^2 + \pi^2(y + b(s), s) + (\dot{\psi}_v(y))^2 \right) \leq C(\overline{\tau}, d_0)\left( e(y + b(s), s) + (\dot{\psi}_v(y))^2 \right) \quad (B. 11) \]

Substituting (B. 11) into (B. 9) and changing variables we obtain by (B. 4), (B. 10) and (B. 8)

\[ \|\Psi(t)\|^2_{L_{5/2+\nu}^2} \leq 2d_0^2 + C(\overline{\tau}, d_0)t \int_{0}^{t} \left( \int (1 + |x - b(s)|^{5+2\nu})e(x, s)dx + C(\overline{\tau}) \right)ds \]

\[ \leq 2d_0^2 + C(\overline{\tau}, d_0)t^2 + C(\overline{\tau}, d_0)t \int_{0}^{t} (1 + |x|^{5+2\nu})e(x, 0)dx \int_{0}^{t} (1 + s + |b(s)|)^{6+2\nu}ds \]

\[ \leq 2d_0^2 + C(\overline{\tau}, d_0)t^2 + C(\overline{\tau}, d_0)(1 + t)^{8+2\nu}\left[ \|X_0\|^2_{L_{5/2+\nu}^2} + U_0 \right] \leq C(\overline{\tau}, d_0)(1 + t)^{8+2\nu} \]

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