Quantum derivation of the use of classical electromagnetic potentials in relativistic Coulomb excitation

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We prove that a relativistic Coulomb excitation calculation in which the classical electromagnetic field of the projectile is used to induce transitions between target states gives the same target transition amplitudes, to all orders of perturbation theory, as would a calculation in which the interaction between projectile and target is mediated by a quantized electromagnetic field.

I. INTRODUCTION

What has become the standard approach to relativisitic Coulomb excitation (RCE) was proposed by A. Winther and K. Alder (WA) in 1979 [1]. The projectile nucleus is assumed to travel along a straight-line orbit parallel to the $\hat{z}$ axis, with impact parameter $b$, at constant speed $v$. The magnitude of the impact parameter is large enough so that nuclear interactions between the target and projectile are negligible. Because of the assumed large projectile momentum, the electromagnetic impulse the projectile receives due to its interaction with the target has little effect on its trajectory, so the projectile maintains its constant speed and impact parameter throughout the collision. As the projectile passes, the target nucleus feels the time-dependent projectile electromagnetic fields, which induce transitions between the quantum states of the target. Starting from the assumption that the target is in its ground state at $t = -\infty$, WA used first-order perturbation theory to calculate the occupation probabilities of excited target states at $t = +\infty$, and used these probabilities to obtain Coulomb excitation cross-sections [2].
An important ingredient in the calculation of target transition probabilities is the interaction potential felt by the target as the projectile moves past it. WA took this to be the classical electromagnetic field of the moving projectile. A justification of this assumption can be found in the work of Alder, Bohr, Huus, Mottelson and Winther (ABHMW) [4]. These authors used the lowest-order of perturbation theory to calculate the target transition amplitude as a result of photon exchange with the projectile, in a situation in which the projectile motion was described by quantum mechanics. They found that this photon-induced transition amplitude was the same as the target transition amplitude induced by the classical electromagnetic field of the projectile, again calculated in the lowest order of perturbation theory.

It is well known [5] that the quantum-mechanical treatment of the interaction of two charged particles yields the same result whether the interaction between them is

- The classical electromagnetic field, or
- The quantized electromagnetic field, calculated up to terms of order $e^2$.

This is essentially the same result as reported by ABHMW [4].

In the 25 years since the WA paper, attempts have been made to improve their calculation of transition amplitudes by going beyond first-order perturbation theory [10, 11, 12, 13, 14, 15, 16, 17, 18]. In particular, the time-dependent Schrödinger equation that governs the occupation amplitudes of the target states has been integrated numerically from $t = -\infty$ to $t = \infty$ as a set of coupled-channel equations, within a finite set of target basis states. In these calculations, the classical electromagnetic field has been used as the interaction potential. But this clearly goes beyond the justification provided by the work of ABHMW, which established the connection between the classical and quantized field only up to first-order perturbation theory.

In this paper we consider a somewhat restricted problem. We will assume that transition charge and current densities that characterize the projectile are specified function of position and time, $\rho_P(\mathbf{r},t), J_P(\mathbf{r},t)$. In fact this is what is usually done in RCE calculations, with $\rho_P(\mathbf{r},t)$ and $J_P(\mathbf{r},t)$ taken to be the charge and current densities appropriate to a spherically-symmetric charge moving with constant speed $v$ along a straight-line trajectory with impact parameter $b$. We will not restrict the trajectory or the shape of the projectile; we only require that $\rho_P(\mathbf{r},t), J_P(\mathbf{r},t)$ be specified functions [22]. This means that we are neglecting
the effects of projectile-state changes on the coupling of the projectile to the electromagnetic field. We will see that this commonly made assumption allows us to make a much stronger statement about the connection between the classical and quantized-field treatments of the excitation of the target. In fact, we will prove that they yield the same results to all orders of perturbation theory, not just to first order.

In Section II we will develop the basic expression for the transition amplitude in terms of the time-dependent interaction potential. In Section III we will evaluate this expression in the particular case in which the interaction is provided by the quantized electromagnetic field, calculated in the Coulomb gauge, and in Section IV we will show that this result is precisely the same as if we had used a classical electromagnetic field for the interaction. In Section V we will show that this agreement also holds if we had used the Lorentz gauge for both the classical and quantized field calculations. A more accurate treatment of the coupling between the projectile and the electromagnetic field is discussed in Section VI.

II. THE TRANSITION AMPLITUDE

Consider the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = [ h_0 + v(t) ] \psi \quad \text{(II.1)}$$

where $h_0$ does not depend explicitly on $t$. This equation is expressed in the interaction representation by expanding $\psi$ in terms of the normalized eigenstates of $h_0$:

$$\psi(t) = \sum_\gamma e^{-i\frac{E_\gamma}{\hbar} t} a_\gamma(t) \Phi_\gamma, \quad \text{(II.2)}$$

with

$$h_0 \Phi_\gamma = E_\gamma \Phi_\gamma \quad \text{(II.3a)}$$

$$\langle \Phi_\gamma | \Phi_\beta \rangle = \delta_{\gamma,\beta} \quad \text{(II.3b)}$$

If $\Phi_\alpha$ is the initial state, which existed when $t \to -\infty$, then

$$a_\gamma(-\infty) = \delta_{\gamma,\alpha}. \quad \text{(II.4)}$$

The transition probability to state $\Phi_\gamma$ is given by $|a_\gamma(+\infty)|^2$. 
Because of equation (II.1), the \( a_\alpha(t) \) obey the set of coupled differential equations:

\[
i \hat{h} a_\gamma(t) = \sum_\beta e^{i\Omega_{\gamma\beta}t} [v(t)]_{\gamma\beta} a_\beta(t), \tag{II.5a}\]

with

\[
[v(t)]_{\gamma\beta} \equiv <\Phi_\gamma|v(t)|\Phi_\beta>. \tag{II.5b}
\]

\( \Omega_{\gamma\beta} \) is defined by

\[
\Omega_{\gamma\beta} \equiv \frac{E_\gamma - E_\beta}{\hbar}. \tag{II.6}
\]

The integral equation equivalent to equation (II.5a), incorporating the initial condition (II.4), is

\[
a_\gamma(t) = \delta_{\gamma,\alpha} + \int_{-\infty}^t \frac{dt'}{i\hbar} e^{i\Omega_{\gamma\alpha}t'} v_\alpha(t') a_\beta(t').
\]

This can be iterated to develop a perturbation series in powers of \( v(t) \):

\[
a_\gamma(t) = \delta_{\gamma,\alpha} + \int_{-\infty}^t \frac{dt'}{i\hbar} e^{i\Omega_{\gamma\alpha}t'} v_\alpha(t') + \sum_\beta \int_{-\infty}^t \frac{dt'}{i\hbar} e^{i\Omega_{\gamma\beta}t'} v_\beta(t') \int_{-\infty}^{t'} \frac{dt''}{i\hbar} e^{i\Omega_{\alpha\beta}t''} v_\alpha(t'') + \cdots \tag{II.7}
\]

The amplitude for a transition from \( \Phi_\alpha \) at \( t = -\infty \) to \( \Phi_\gamma \) at \( t = \infty \) is \( a_\gamma(\infty) \). Thus the perturbation series expansion of the transition amplitude can be written

\[
a_\gamma(\infty) = \delta_{\gamma,\alpha} + \sum_{n=1}^{\infty} \sum_{\beta_1, \ldots, \beta_n} \int_{-\infty}^t \frac{dt_n}{i\hbar} e^{i\Omega_{\gamma\beta_n}t_n} v_\beta(t_n) \int_{-\infty}^{t_n} \frac{dt_{n-1}}{i\hbar} e^{i\Omega_{\beta_{n-1}\beta_n}t_{n-1}} v_\beta(t_{n-1}) \cdots \tag{II.8}
\]

\[
\ldots \int_{-\infty}^{t_1} \frac{dt_1}{i\hbar} e^{i\Omega_{\alpha\beta_1}t_1} v_\beta(t_1)
\]

III. SPECIALIZATION TO THE QUANTUM THEORY OF RELATIVISTIC COULOMB EXCITATION

In the usual approach to relativistic Coulomb excitation (RCE), the target nucleus is at rest in the reference frame. The rapidly moving projectile nucleus follows a prescribed orbit, which we will leave unspecified. The projectile interacts with the electromagnetic field via its time-dependent charge and current densities \( \rho_P(r, t) \) and \( J_P(r, t) \).

The Hamiltonian \( h_0 \) of Equation (II.1) will refer to the target internal degrees of freedom, plus the free electromagnetic field. An eigenstate of \( h_0 \) will therefore be specified by a target
state $\phi_\beta$, plus a specification of the numbers of photons in each of the quantized modes of the field. If the energy of the target state $\phi_\beta$ is $\epsilon_\beta$, then a transition from target state $\phi_\alpha$ to $\phi_\beta$ with creation of a photon of momentum $\hbar\vec{q}$ will result in an energy increase of

$$(\epsilon_\beta + \hbar c \vec{q}) - \epsilon_\alpha,$$

so that the quantity $\Omega_{\beta\alpha}$ referred to in Equation (II.8) is

$$\Omega_{\beta\alpha} = \frac{(\epsilon_\beta + \hbar c \vec{q}) - \epsilon_\alpha}{\hbar} = \frac{\epsilon_\beta - \epsilon_\alpha}{\hbar} + cq \equiv \omega_{\beta\alpha} + cq.$$

Similarly, if the $\phi_\alpha \rightarrow \phi_\beta$ transition were accompanied by the absorption of a photon of momentum $\hbar\vec{q}$, we would have

$$\Omega_{\beta\alpha} = \frac{(\epsilon_\beta - \hbar c \vec{q}) - \epsilon_\alpha}{\hbar} = \frac{\epsilon_\beta - \epsilon_\alpha}{\hbar} - cq \equiv \omega_{\beta\alpha} - cq.$$

Since the calculation involves the electromagnetic potentials, we must choose a gauge. In this section and the next, we use the Coulomb, or radiation, gauge, [5, 6, 7] because it allows the simplest treatment of the quantized electromagnetic field. In Section V we will discuss the modifications required for the Lorentz gauge. In the Coulomb gauge, the vector potential is a solenoidal field

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0.$$

(III.1)

Here $\mathbf{r}$ labels points of space relative to an origin at the target center, which we assume to be fixed. We describe the photon field in terms of normal modes, each labelled by a wave vector $\mathbf{q}$. Associated with each $\mathbf{q}$ are two polarization vectors $\hat{\mathbf{e}}_{\mathbf{q},1}$ and $\hat{\mathbf{e}}_{\mathbf{q},2}$. The three vectors ($\hat{\mathbf{q}},\hat{\mathbf{e}}_{\mathbf{q},1},\hat{\mathbf{e}}_{\mathbf{q},2}$) form an orthonormal coordinate system. The field operator $\mathbf{A}(\mathbf{r})$ is expressed in terms of photon creation and annihilation operators $(a^+_{\mathbf{q},j}, a_{\mathbf{q},j})$ by the expansion

$$\mathbf{A}(\mathbf{r}) = c \sqrt{\frac{4\pi}{V}} \sum_{\mathbf{q}} \sum_{j=1}^{2} \sqrt{\frac{\hbar}{2qc}} \left[ e^{i\mathbf{q} \cdot \mathbf{r}} \hat{\mathbf{e}}_{\mathbf{q},j} a_{\mathbf{q},j} + e^{-i\mathbf{q} \cdot \mathbf{r}} \hat{\mathbf{e}}^*_{\mathbf{q},j} a^+_{\mathbf{q},j} \right].$$

(III.2)

Here $V$ is the quantization volume. The number of modes per unit $\mathbf{q}$-space volume is $V/8\pi^3$, for each polarization $\hat{\mathbf{e}}_{\mathbf{q},j}$. Since the plane waves that describe the modes are transverse ($\hat{\mathbf{e}}_{\mathbf{q},j} \cdot \hat{\mathbf{q}} = 0$), the solenoidal condition (III.1) is automatically satisfied. With the normalization given in equation (III.2), the $(a^+_{\mathbf{q},j}, a_{\mathbf{q},j})$ obey the commutation relations

$$[a_{\mathbf{q},j}, a^+_{\mathbf{q}',j'}] = \delta_{\mathbf{q},\mathbf{q}'} \delta_{j,j'}.$$  (III.3a)
\[
\begin{aligned}
\{ a_{q,j}, a_{q',j'}^+ \} &= \{ a_{q,j}^+, a_{q',j'}^+ \} = 0 \\
\text{(III.3b)}
\end{aligned}
\]

Their matrix elements with respect to states with \( n_{q,j} \) photons in the mode \( q, j \) are

\[
\begin{aligned}
<n_{q,j}'|a_{q,j}^+|n_{q,j}> &= \delta_{n_{q,j}',n_{q,j}+1} \sqrt{n_{q,j}+1} \\
<n_{q,j}'|a_{q,j}|n_{q,j}> &= \delta_{n_{q,j}',n_{q,j}-1} \sqrt{n_{q,j}} \\
\text{(III.4a)} \quad &\text{and} \\
<n_{q,j}'|a_{q,j}^+|n_{q,j}> &= \delta_{n_{q,j}',n_{q,j}+1} \sqrt{n_{q,j}+1} \\
<n_{q,j}'|a_{q,j}|n_{q,j}> &= \delta_{n_{q,j}',n_{q,j}-1} \sqrt{n_{q,j}} \\
\text{(III.4b)}
\end{aligned}
\]

In the Coulomb gauge, the full Hamiltonian for our system is

\[
\begin{aligned}
H &= h_0 - \frac{1}{c} \int d^3r \left( J_P(r,t) + J_T(r) \right) \cdot A(r) + \int d^3r \rho_T(r)r' \frac{\rho_P(r',t)}{|r-r'|} \\
H_0 &= H_T + H_\gamma \\
\text{(III.5a)} \quad &\text{with} \\
H_T &= \text{the target internal Hamiltonian, and} \\
\text{(III.5b)} \quad &H_\gamma = \sum_q \sum_{j=1}^2 \hbar c a_{q,j}^+ a_{q,j}
\end{aligned}
\]

is the free-field photon Hamiltonian. There is no term in equation (III.5a) corresponding to the kinetic energy of relative motion. We are using the usual RCE picture of the projectile moving on a prescribed classical trajectory, and therefore the projectile-target relative coordinate is not one of the degrees of freedom of the problem. The projectile charge and current densities \( (\rho_P(r',t), J_P(r',t)) \) are to be regarded as specified functions of \( r', t \), whereas the target charge and current densities are to be regarded as operator functions, to be represented in the calculation by their matrix elements \( \rho_T(\gamma,\beta)(r,t), J(\gamma,\beta)(r,t) \) with respect to the target eigenstates \( \phi_\gamma, \phi_\beta \).

The interaction to be used in Equation (II.1) is

\[
v(t) = -\frac{1}{c} \int d^3r \left[ \left( J_P(r,t) + J_T(r) \right) \cdot A(r) \right] + \int d^3r \rho_T(r) \int d^3r' \rho_P(r',t) \frac{1}{|r-r'|} \\
\text{(III.6)}
\]

It is conveniently regarded as a sum of two terms:

\[
v_0(t) \equiv \int d^3r \rho_T(r) \int d^3r' \rho_P(r',t) \frac{1}{|r-r'|} \\
\text{(III.7a)} \quad &\text{which does not change the number of photons, and} \\
v_1(t) \equiv -\frac{1}{c} \int d^3r \left[ \left( J_P(r,t) + J_T(r) \right) \cdot A(r) \right] \\
\text{(III.7b)}
\]
which changes the number of photons by \( \pm 1 \) because of the presence of \( a_{q,j} \) and \( a_{q',j'}^{+} \) in the expansion of \( A(r, t) \) (Equation (III.2)).

We are interested in a situation in which neither the initial state \( \phi_{\alpha} \) nor the final state \( \phi_{\gamma} \) has any photons\(^{23}\). Thus we exclude bremsstrahlung processes. Moreover, we assume that every photon absorbed by the target at time \( t \) was emitted by the projectile at an earlier time \( t' \), and every photon emitted by the target at time \( t \) will be absorbed by the projectile at a later time \( t' \)\(^{24}\). This implies that the creation and annihilation operators entering into the expansion (Equation (II.8)) for the transition amplitude will occur in pairs \( a_{q,j} \ldots a_{q',j'}^{+} \).

Specifically, if the \( t_p \) integrand in Equation (II.8) contained

\[
\frac{e^{i\Omega_{\mu\nu}t_p}}{i\hbar} \mathbf{\hat{e}}_{q,j} \cdot [\mathbf{J}_T]_{\mu\nu}(r) a_{q,j} e^{iq \cdot r}
\]

corresponding to absorption of a \((q, j)\) photon at the target at time \( t_p \), it must be that this photon was created at the projectile at an earlier time \( t'_p \). This means that farther to the right in the multiple integral of Equation (II.8) there occurs a factor

\[
\frac{e^{icq t'_p}}{i\hbar} \mathbf{\hat{e}}_{q,j}^{+} \cdot [\mathbf{J}_P](r', t'_p) a_{q,j}^{+} e^{-iq \cdot r'}
\]

and \( t'_p \) must be integrated from \(-\infty\) to \( t_p \). Similarly, if the \( t_p \) integrand in Equation (II.8) contained

\[
\frac{e^{i\Omega_{\mu\nu}t_p}}{i\hbar} \mathbf{\hat{e}}_{q,j}^{+} \cdot [\mathbf{J}_T]_{\mu\nu}(r) a_{q,j}^{+} e^{-iq \cdot r}
\]

corresponding to creation of a \((q, j)\) photon at the target at time \( t_p \), it must be that this photon will be annihilated at the projectile at a later time \( t'_p \). This means that farther to the left in the multiple integral of Equation (II.8) there occurs a factor

\[
\frac{e^{-icq t'_p}}{i\hbar} \mathbf{\hat{e}}_{q,j} \cdot [\mathbf{J}_P](r', t'_p) a_{q,j} e^{iq \cdot r'}
\]

and \( t'_p \) must be integrated from \( t_p \) to \( \infty \). The result of these two \( t'_p \) integrations is that the double integral over \( t_p \) and \( t'_p \) in Equation (II.8) is replaced by the single integral

\[
\int_{-\infty}^{t_p+1} \frac{dt_p}{i\hbar} e^{i\omega_{\mu\nu}t_p} w_{\mu\nu}(t_p), \quad (III.8)
\]
where $w_{\mu\nu}(t_p)$ is defined by

$$w_{\mu\nu}(t_p) \equiv \frac{4\pi}{V} \sum_q \sum_{j=1}^2 \frac{\hbar}{2qc} \quad (III.9)$$

$$\left[ \int d^3r \hat{e}^*_{q,j} \cdot [\mathbf{J}_T]_{\mu\nu}(\mathbf{r}) e^{i(cq_{tp} + q \mathbf{r})(t')}} \int_{t_p}^{t'} \frac{dt'_p}{i\hbar} \int d^3r' \hat{e}_{q,j} \cdot \mathbf{J}_F(\mathbf{r}', t'_p) e^{i(-cq_{tp} + q \mathbf{r}'')}}$$

$$+ \int d^3r \hat{e}^*_{q,j} \cdot [\mathbf{J}_T]_{\mu\nu}(\mathbf{r}) e^{i(cq_{tp} + q \mathbf{r})} \int_{-\infty}^{t_p} dt'_p \frac{dt'_p}{i\hbar} \int d^3r' \hat{e}_{q,j} \cdot \mathbf{J}_F(\mathbf{r}', t'_p) e^{i(cq_{tp} + q \mathbf{r}')}} \right]$$

Note that it is possible to group all the contributions to the $t'_p$ range in Equation (III.8) into these two intervals, $-\infty \leq t'_p \leq t_p$ and $\infty \geq t'_p \geq t_p$, only because of our assumption that the projectile current density is independent of what occurs at the projectile or target.

We can repeat this process for every $a_{q,j} \ldots a_{q,j}^+$ pair occurring in the multiple integral of Equation (II.8), thereby replacing every $t, t'$ integration ($t$ occurring at the target, $t'$ at the projectile) by a single $t$ integration. All that will remain in Equation (II.8) is integrals of the form (III.8) obtained in this way, and integrals of the form

$$\int_{-\infty}^{t_{p+1}} \frac{dt_t}{i\hbar} e^{i\omega_{\mu\nu} t_t} v_0(t_t).$$

Note that here $\Omega_{\mu\nu} = \omega_{\mu\nu}$, since $v_0(t_t)$ does not create or annihilate photons. The remaining combination of the integrals

$$\int_{-\infty}^{t_{p+1}} \frac{dt_p}{i\hbar} e^{i\omega_{\mu\nu} t_p} w_{\mu\nu}(t_p) \quad \text{and} \quad \int_{-\infty}^{t_{p+1}} \frac{dt_t}{i\hbar} e^{i\omega_{\mu\nu} t_t} v_0(t_t)$$

is equal to

$$a(\infty) = \delta_{\gamma,\alpha} + \sum_{n=1}^{\infty} \sum_{\beta \ldots \tau} \int_{-\infty}^{t_n} \frac{dt_n}{i\hbar} e^{i\omega_{\beta\tau} t_n} \tilde{v}_{\beta\tau}(t_n) \int_{-\infty}^{t_n} \frac{dt_{n-1}}{i\hbar} e^{i\omega_{\beta\tau} t_{n-1}} \tilde{v}_{\beta\tau}(t_{n-1}) \ldots (III.10)$$

$$\ldots \int_{-\infty}^{t_{p+1}} \frac{dt_p}{i\hbar} e^{i\omega_{\mu\nu} t_p} \tilde{v}_{\mu\nu}(t_p) \ldots \int_{-\infty}^{t_3} \frac{dt_2}{i\hbar} e^{i\omega_{\sigma\tau} t_2} \tilde{v}_{\sigma\tau}(t_2) \int_{-\infty}^{t_2} \frac{dt_1}{i\hbar} e^{i\omega_{\tau\alpha} t_1} \tilde{v}_{\tau\alpha}(t_1)$$

in which the time-dependent perturbation $\tilde{v}(t)$ is defined by

$$\tilde{v}(t) \equiv v_0(t) + w(t).$$

Comparison with Equation (II.8) shows that the transition amplitude for the excitation of target state $\phi_\gamma$ under the influence of the quantized electromagnetic field is precisely the same as if the target had experienced the time dependent effective field $\tilde{v}(t)$, from which all photon degrees of freedom have been removed.
In the next Section, we will discuss the physical meaning of the effective target interaction \( \tilde{v}(t) \) defined in Equation (III.11).

The structure of Equation (III.10) can be elucidated by examining the \( n = 2 \) term in detail:

\[
a_\gamma^{(2)}(\infty) = \sum_\beta \int_{-\infty}^{\infty} \frac{dt_2}{i\hbar} e^{i\omega_\gamma t_2} \left[ v_0(t_2) + w(t_2) \right]_{\gamma\beta} \int_{-\infty}^{t_2} \frac{dt_1}{i\hbar} e^{i\omega_\beta t_1} \left[ v_0(t_1) + w(t_1) \right]_{\beta\alpha}. \tag{III.12}
\]

The \( v_0(t_2)v_0(t_1) \) combination arises from the \( n = 2 \) term of Equation (II.8), when the \( v_0 \) part of \( v(t) \) (Equation (III.7a)) is used both times. The \( v_0(t_2)w(t_1) \) and \( w(t_2)v_0(t_1) \) combinations arise from the \( n = 3 \) term of Equation (II.8), when one of the three interactions is chosen to be \( v_0 \) and the other two are chosen to be photon interactions \( v_1 \) (Equation (III.7b)). The \( w(t_2)w(t_1) \) combination arises from the \( n = 4 \) term of Equation (II.8), with \( v_1 \) acting four times. This involves the exchange of two photons between projectile and target, the target interactions occurring at \( t_2, t_1 \) and the corresponding projectile interactions occurring at \( t_2', t_1' \), with both \( t_2 \) and \( t_1 \) integrated from \(-\infty\) to \( \infty \). These components of the \( n = 2 \) term of Equation (III.10) are illustrated in Figure 1,2 and 3.

IV. THE EFFECTIVE INTERACTION

If we define four fields \( \tilde{\varphi}(r, t) \) and \( \tilde{A}(r, t) \) by

\[
\tilde{\varphi}(r, t) \equiv \int d^3r' \frac{\rho_P(r', t)}{|r - r'|}, \tag{IV.1a}
\]

\[
\tilde{A}(r, t) \equiv -\frac{2\pi}{iV} \sum_q \sum_{j=1}^2 \frac{1}{q} \left[ \hat{e}_{q,j} e^{i(qct - qr)} \int_{t}^{\infty} \frac{dt'}{i\hbar} \int d^3r' \hat{e}_{q,j}^* \cdot [J_P](r', t') e^{i(-qct' + qr')} \right. \\
\left. + \hat{e}_{q,j} e^{-i(qct - qr)} \int_{-\infty}^{t} \frac{dt'}{i\hbar} \int d^3r' \hat{e}_{q,j}^* \cdot [J_P](r', t') e^{i(qct' - qr')} \right], \tag{IV.1b}
\]

then we can re-write Equations (III.11) (III.9) and (III.7a) as follows:

\[
\tilde{v}(t) = \int d^3r \left[ \rho_T(r) \tilde{\varphi}(r, t) - \frac{1}{c} J_T(r, t) \cdot \tilde{A}(r, t) \right]. \tag{IV.2}
\]

It follows immediately from these definitions of \( \tilde{\varphi}(r, t) \) and \( \tilde{A}(r, t) \) that

\[
\nabla^2 \tilde{\varphi}(r, t) = -4\pi \rho_P(r, t) \tag{IV.3a}
\]

\[
\nabla \cdot \tilde{A}(r, t) = 0. \tag{IV.3b}
\]
The latter equation is a consequence of our use of transverse photon modes \( \hat{e}_{q,j} \cdot \hat{q} = 0 = \hat{e}_{q,j}^* \cdot \hat{q} \).

To proceed, we replace the sum \( \sum_{\mathbf{q}} \) by the integral \( \frac{V}{\pi^2} \int d^3 q \). Moreover the completeness of the set \( (\hat{e}_{q,0}, \hat{e}_{q,1}, \hat{q}) \) allows us to write

\[
\mathbf{J}_P = \hat{q} \left( \hat{q} \cdot \mathbf{J}_P \right) + \sum_{j=1}^{2} \hat{e}_{q,j}^* \left( \hat{e}_{q,j} \cdot \mathbf{J}_P \right)
\]

from which we can obtain

\[
\sum_{j=1}^{2} \hat{e}_{q,j}^* \left( \hat{e}_{q,j} \cdot \mathbf{J}_P \right) = \sum_{j=1}^{2} \hat{e}_{q,j} \left( \hat{e}_{q,j}^* \cdot \mathbf{J}_P \right) = \mathbf{J}_P - \hat{q} \left( \hat{q} \cdot \mathbf{J}_P \right) = \mathbf{J}_P - \frac{q(q \cdot \mathbf{J}_P)}{q^2}.
\]

Then we can rewrite Equation (4.11) as

\[
\tilde{\mathbf{A}}(\mathbf{r}, t) = \frac{i}{4\pi^2} \int \frac{d^3 q}{q} \left[ e^{i(cqt - \mathbf{q} \cdot \mathbf{r})} \int_{-\infty}^{\infty} dt' \int d^3 r' \left( \mathbf{J}_P(\mathbf{r}', t') - \frac{q(q \cdot \mathbf{J}_P(\mathbf{r}', t'))}{q^2} \right) e^{i(qt' + \mathbf{q} \cdot \mathbf{r}')} \right]
\]

By carrying out the differentiations, one can show that

\[
\nabla^2 \tilde{\mathbf{A}}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \tilde{\mathbf{A}}(\mathbf{r}, t)}{\partial t^2} = \frac{-1}{2\pi^2 c} \int d^3 q \int d^3 r' \left( \mathbf{J}_P(\mathbf{r}', t') - \frac{q(q \cdot \mathbf{J}_P(\mathbf{r}', t'))}{q^2} \right) e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \tag{4.5a}
\]

and

\[
\frac{\partial}{\partial t} \nabla \varphi(\mathbf{r}, t) = \frac{1}{2\pi^2} \int \frac{d^3 q}{q^2} i q \int d^3 r' e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \frac{\partial \rho_P(\mathbf{r}', t)}{\partial t} \tag{4.5b}
\]

But projectile charge conservation implies that

\[
\int d^3 r' e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \frac{\partial \rho_P(\mathbf{r}', t)}{\partial t} = - \int d^3 r' e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r})} \nabla \varphi \cdot \mathbf{J}_P(\mathbf{r}', t)
\]

\[
= \int d^3 r' \mathbf{J}_P(\mathbf{r}', t) \cdot \nabla \varphi e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} = -i \int d^3 r' \mathbf{J}_P(\mathbf{r}', t) e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r})}.
\]

Thus Equation (4.5b) becomes

\[
\frac{\partial}{\partial t} \nabla \varphi(\mathbf{r}, t) = \frac{1}{2\pi^2} \int d^3 q \int d^3 r' \frac{q(q \cdot \mathbf{J}_P(\mathbf{r}', t))}{q^2} e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \tag{4.5c}
\]

which we can combine with Equation (4.5a) to get

\[
\nabla^2 \tilde{\mathbf{A}}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \tilde{\mathbf{A}}(\mathbf{r}, t)}{\partial t^2} - \frac{1}{c} \frac{\partial}{\partial t} \nabla \varphi(\mathbf{r}, t) = -\frac{1}{2\pi^2 c} \int d^3 r' \mathbf{J}_P(\mathbf{r}', t) \int d^3 q e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')}
\]
\[-\frac{1}{2\pi^2c} \int d^3r' (2\pi)^3 \delta(r - r') J_P(r', t) \]

\[
\nabla^2 \tilde{A}(r, t) - \frac{1}{c^2} \frac{\partial^2 \tilde{A}(r, t)}{\partial t^2} - \frac{1}{c} \frac{\partial}{\partial t} \nabla \tilde{\varphi}(r, t) = -\frac{4\pi}{c} J_P(r, t) \quad (IV.6)
\]

Equations (IV.3a, IV.3b) and (IV.6) show that \( \tilde{A}(r, t) \) and \( \tilde{\varphi}(r, t) \) defined in Equations (IV.1a, IV.1b) are the classical vector and scalar potentials \( A(r, t) \) and \( \varphi(r, t) \) associated (in Coulomb gauge) with projectile charge and current densities \( (\rho_P(r, t), J_P(r, t)) \). Moreover, Equation (IV.2) shows that the effective potential \( \tilde{v}(t) \) is the classical interaction of \( (\varphi(r, t), A(r, t)) \) with the target charge and current densities.

Thus we have proven that an RCE calculation in which the classical electromagnetic field of the projectile is used to induce transitions between target states gives the same target transition amplitudes, to all orders of perturbation theory, as would a calculation in which the interaction between projectile and target is mediated by a quantized electromagnetic field.

V. THE LORENTZ GAUGE

The two previous sections used the Coulomb gauge. In this section, we will describe the modifications needed if the Lorentz gauge is used.

1. In the Lorentz gauge, the full Hamiltonian is

\[
H = h_0 + \int d^3r \left( (\rho_P(r, t) + \rho_T(r)) \varphi(r) - \frac{1}{c} (J_P(r, t) + J_T(r)) \cdot A(r) \right) , \quad (V.1)
\]

which lacks the density-density interaction present in Equation (III.5a).

2. In the Coulomb gauge, the vector potential \( A \) has only two components, and they are transverse. In the Lorentz gauge, \( A \) has these two transverse components, and also a longitudinal component (along \( \hat{q} \)). Moreover, the scalar potential \( \varphi \) is also quantized. Thus, in addition to the two transverse photons of the Coulomb gauge, we have a longitudinal photon and a scalar photon. The commutation relations of the longitudinal photon creation and annihilation operators are the same as for the transverse photons (Equations (III.3a) and (III.3b)). However, the treatment of the scalar photon is more complicated. A consistent formalism for quantizing the scalar
field was developed by S.N. Gupta and K. Bleuler. For our purposes, the only manifestation of the extra complications of scalar field quantization is the presence of the two minus signs in Equation (V.2) below.

When these changes are accounted for, Equation (III.9) is replaced by

\[ \tilde{v}_{\mu\nu}(t_p) = w_{\mu\nu}(t_p) = \frac{4\pi}{V} \sum_q \frac{\hbar}{2q_c} \]  

(V.2)

\[
\begin{align*}
&\left[ - \sum_{j=1}^3 \int d^3r \hat{e}_{q,j}^* \cdot [J_T]_{\mu\nu}(r) e^{i(cqt_p - q \cdot r)} \int_{t_p}^{\infty} \frac{dt'}{i\hbar} \int d^3r' \hat{e}_{q,j} \cdot J_P(r', t_p') e^{i(-cqt_p + q \cdot r')} \\
&+ c^2 \sum_{j=1}^3 \int d^3r [\rho_T]_{\mu\nu}(r) e^{i(cqt_p - q \cdot r)} \int_{t_p}^{\infty} \frac{dt'}{i\hbar} \int d^3r' \rho_P(r', t_p') e^{i(-cqt_p + q \cdot r')} \\
&- c^2 \sum_{j=1}^3 \int d^3r [\rho_T]_{\mu\nu}(r) e^{i(-cqt_p + q \cdot r)} \int_{-\infty}^{t_p} \frac{dt'}{i\hbar} \int d^3r' \rho_P(r', t_p') e^{i(-cqt_p - q \cdot r')} \right]
\end{align*}
\]

Following the procedure of Section 4, we cast this expression in the form (IV.2), which leads to

\[
\tilde{\varphi}(r, t) \equiv -\frac{2c\pi}{iV} \sum_q \sum_{j=1}^3 \frac{1}{q} \left[ e^{i(cqt - q \cdot r)} \int_{t}^{\infty} dt' \int d^3r' \rho_P(r', t') e^{i(-cqt' + q \cdot r')} \\
+ e^{i(-cqt + q \cdot r)} \int_{-\infty}^{t} dt' \int d^3r' \rho_P(r', t') e^{i(cqt' - q \cdot r')} \right] \]  

(V.3a)

\[
\tilde{A}(r, t) \equiv -\frac{2\pi}{iV} \sum_q \sum_{j=1}^3 \frac{1}{q} \left[ \hat{e}_{q,j}^* e^{i(cqt - q \cdot r)} \int_{t}^{\infty} dt' \int d^3r' \hat{e}_{q,j} \cdot [J_P](r', t') e^{i(-cqt' + q \cdot r')} \\
+ \hat{e}_{q,j} e^{i(-cqt + q \cdot r)} \int_{-\infty}^{t} dt' \int d^3r' \hat{e}_{q,j} \cdot [J_P](r', t') e^{i(cqt' - q \cdot r')} \right] \]  

(V.3b)

The polarization sum simplifies to

\[
\sum_{j=1}^3 \hat{e}_{q,j}^* \cdot J_P = J_P,
\]

and the replacement of \( \sum_q \) by the integral \( \frac{V}{8\pi^3} \int d^3q \) still applies. When these relations are used, it is straightforward to calculate that \( \tilde{\varphi}(r, t) \) and \( \tilde{A}(r, t) \) defined by Equations (V.3a)
Equations (V.4a, V.4b and V.4c) show that $\tilde{\varphi}(r, t)$ and $\tilde{A}(r, t)$ defined in Equations (V.3a, V.3b) are the classical vector and scalar potentials $A(r, t)$ and $\varphi(r, t)$ associated (in Lorentz gauge) with projectile charge and current densities $(\rho_P(r, t), J_P(r, t))$. Thus, whether one chooses to work in the Coulomb or Lorentz gauge, in an RCE calculation of target transition amplitudes the effect of the quantized electromagnetic field can be replaced by the classical electromagnetic field of the projectile used in the classical expression (IV.2) for the interaction.

The proof of the equivalence of the quantized field and classical field treatments of Coulomb excitation given in ABHMW [4] applies only to the on-shell ($\omega = \omega_{\gamma\alpha}$) Fourier component of $\tilde{v}_{\gamma\alpha}(t)$. The reason for this apparent restriction is that ABHMW used the Coulomb gauge for the calculation of the quantized-field version of $\tilde{v}_{\gamma\alpha}(t)$, but they used the Lorentz gauge for the calculation of the classical field version. It is shown in Reference [19] that the versions of $\tilde{v}_{\gamma\alpha}(t)$ calculated in the two gauges agree in their on-shell Fourier components, but are generally different when $\omega \neq \omega_{\gamma\alpha}$. In Sections III, IV, and V, we have shown that the quantized field and classical field treatments agree for all $t$, and so for all $\omega$, if both are calculated in the same gauge.

VI. DISCUSSION

It has been emphasized that an important ingredient in our derivation is the assumption that the charge and current densities associated with the projectile are specified functions of position and time, which is equivalent to the assumption that the projectile does not change its internal state during the collision. This assumption is also made in RCE calculations involving classical electromagnetic fields. Even if both the target and projectile are allowed to undergo internal transitions, the fields that induce those transitions are always assumed to be generated by moving static spherically symmetric charge distributions, with no internal structure. To remove this assumption, we would have to calculate, at each instant, the
classical retarded electromagnetic fields that have been generated by the projectile and target since the beginning of the collision, while they have moved relative to each other and undergone changes of their internal states. These would be the electromagnetic fields that, in turn, would be used to induce further internal transitions. It is clear that this would be a very difficult calculation, even though it involved only classical fields. It would be the classical analogue of the quantized field calculation we have described in Sections II and III, if we had allowed the projectile state to change as well as the target state.

Although it would be difficult to perform the complete calculation just described, in which the electromagnetic fields are generated by the actual dynamic transition charge and current densities, it may be required for a full understanding of ultra-high-energy RCE. Baltz, Rhoades-Brown and Weneser [20] have estimated excitation probabilities for collisions between oppositely directed 100 GeV Au nuclei. They find excitation probabilities that are greater than 0.5 for grazing collisions, and greater than 0.1 out to impact parameters of about 50 fm. Thus it is important to investigate, with classical and/or quantized electromagnetic fields, the effect on excitation probabilities of internal transitions occurring within both the target and projectile.

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[21] Galetti, Kodama and Nemes [2], and Bertulani et al [3] have pointed out difficulties associated with this simple picture.
[22] Since we will neglect bremsstrahlung, our analysis only applies to situations in which the projectile does not suffer strong acceleration, which would certainly be the case in the constant-projectile-velocity scenario usually employed in RCE.
[23] To keep the notation simple, we use $\phi_{\alpha,\gamma}$ to label target states, or target states in the presence of the photon vacuum. We believe that the presence or absence of the vacuum will be clear in all situations.
[24] In particular, we do not imagine that a photon absorbed by the target was emitted by the target. Such target emission-absorption processes contribute to the renormalization of the target mass, and are not considered here.
Figure Captions

Figure 1. The $v_0(t_2)v_0(t_1)$ term of Equation (III.12). Time increases as we move upwards in the diagram. The vertical line on the left corresponds to the projectile, and the lines labelled $\alpha, \beta, \gamma$ represent states of the target.

Figure 2. The $w(t_1)v_0(t_2)$ and $v_0(t_1)w(t_2)$ terms in Equation (III.12). The curly lines represent exchanged photons. The $t'_1$ and $t'_2$ variables must be integrated from $-\infty$ to $\infty$.

Figure 3. The $w(t_1)w(t_2)$ term of Equation (III.12). The $t'_1$ and $t'_2$ variables must be integrated, independently, from $-\infty$ to $\infty$. 
Fig. 1
Fig. 2
Fig. 3