Schubert Classes in the Equivariant K-Theory
and Equivariant Cohomology
of the Grassmannian

Victor Kreiman

March 2, 2022

Abstract

We give positive formulas for the restriction of a Schubert Class to a
$T$-fixed point in the equivariant K-theory and equivariant cohomology of
the Grassmannian. Our formulas rely on a result of Kodiyalam-Raghavan
and Kreiman-Lakshmibai, which gives an equivariant Gröbner degenera-
tion of a Schubert variety in the neighborhood of a $T$-fixed point of the
Grassmannian.

Contents

1 Introduction 1
2 Statement of Results 4
3 Equivariant K-Theory in Affine Spaces 6
4 The Class of an Opposite Schubert Variety 8
5 Families of Nonintersecting Paths on Young Diagrams 11
6 From Path Families to Semistandard Young Tableaux 17
7 Computing $N_S$ 22

1 Introduction

The group $T$ of diagonal matrices in $GL_n(\mathbb{C})$ acts on the Grassmannian $Gr_{d,n}$,
with fixed point set indexed by $I_{d,n}$, the $d$ element subsets of $\{1, \ldots, n\}$. For
$\beta \in I_{d,n}$, denote the corresponding $T$-fixed point by $e_\beta$. The $T$-equivariant
embedding $e_{\beta} \overset{i}{\to} Gr_{d,n}$ induces restriction homomorphisms $i_{*}^{K}$ in $T$-equivariant K-theory and $i_{*}^{T}$ in $T$-equivariant cohomology:

\[ K_{T}^{*}(Gr_{d,n}) \overset{i_{*}^{K}}{\to} K_{T}^{*}(e_{\beta}) \cong R(T) = \mathbb{C}[t_{1}^{\pm 1}, \ldots, t_{n}^{\pm 1}] \]

\[ H_{T}^{*}(Gr_{d,n}) \overset{i_{*}^{T}}{\to} H_{T}^{*}(e_{\beta}) \cong \mathbb{C}[t^{*}] = \mathbb{C}[t_{1}, \ldots, t_{n}] \]

where $R(T)$ is the representation ring of $T$ and $t$ is the Lie algebra of $T$. The image of an element $z$ of $K_{T}^{*}(Gr_{d,n})$ or $H_{T}^{*}(Gr_{d,n})$ under restriction to $e_{\beta}$ is denoted by $z|_{e_{\beta}}$. The product maps

\[ K_{T}^{*}(Gr_{d,n}) \to \prod_{\beta \in I_{d,n}} K_{T}^{*}(e_{\beta}), \quad z \mapsto \prod_{\beta \in I_{d,n}} z|_{e_{\beta}} \]

\[ H_{T}^{*}(Gr_{d,n}) \to \prod_{\beta \in I_{d,n}} H_{T}^{*}(e_{\beta}), \quad z \mapsto \prod_{\beta \in I_{d,n}} z|_{e_{\beta}} \]

are both injective. Thus, an element of $K_{T}^{*}(Gr_{d,n})$ or $H_{T}^{*}(Gr_{d,n})$ is determined by its restrictions to all $e_{\beta}$, $\beta \in I_{d,n}$.

The Schubert varieties of the Grassmannian are in bijection with the $T$-fixed points, and thus are also indexed by $I_{d,n}$. For $\alpha \in I_{d,n}$, denote the Schubert variety by $X_{\alpha}$, and the corresponding Schubert classes in $K_{T}^{*}(Gr_{d,n})$ and $H_{T}^{*}(Gr_{d,n})$ by $[X_{\alpha}]_{K}$ and $[X_{\alpha}]_{H}$ respectively. In this paper, we obtain formulas for $[X_{\alpha}]_{K}|_{e_{\beta}}$ and $[X_{\alpha}]_{H}|_{e_{\beta}}$.

Various formulas already exist for $[X_{\alpha}]_{K}|_{e_{\beta}}$ and $[X_{\alpha}]_{H}|_{e_{\beta}}$. Letting $G_{\alpha}(r, t)$ and $S_{\alpha}(r, t)$ denote the double Grothendieck and double Schubert polynomials \([25, 26]\) respectively for $\alpha$,

\[ [X_{\alpha}]_{K}|_{e_{\beta}} = G_{\alpha}(\beta(t), t) \quad \text{and} \quad [X_{\alpha}]_{H}|_{e_{\beta}} = S_{\alpha}(\beta(t), t), \tag{1} \]

where on the right sides of both equations we view $\alpha$ and $\beta$ as Grassmannian permutations rather than $d$-tuples. Letting $d_{\alpha, \beta}^{\gamma}$ and $c_{\alpha, \beta}^{\gamma}$ denote the linear structure constants for the Schubert classes in the equivariant K-theory and equivariant cohomology respectively of the Grassmannian,

\[ [X_{\alpha}]_{K}|_{e_{\beta}} = d_{\alpha, \beta}^{\gamma} \quad \text{and} \quad [X_{\alpha}]_{H}|_{e_{\beta}} = c_{\alpha, \beta}^{\gamma}. \tag{2} \]

Hence the various formulas for $G_{\alpha}(r, t)$ and $S_{\alpha}(r, t)$ (see \([1, 2, 3, 5, 6, 7, 12, 13, 14, 23, 24, 25, 26, 27, 29]\), for example) and for $c_{\alpha, \beta}^{\gamma}$ (see \([15, 30]\)) can be used to compute $[X_{\alpha}]_{K}|_{e_{\beta}}$ and $[X_{\alpha}]_{H}|_{e_{\beta}}$. The main features of our formulas are that they satisfy positivity conditions, they are obtained via a Gröbner degeneration, and they are expressed in terms of semistandard set-valued tableaux.

**Posititivity**

Griffeth and Ram \([10]\) conjecture that the structure constants $d_{\alpha, \beta}^{\gamma}$ for $G/B$, where $G$ is any symmetrizable Kac-Moody group, can be expressed as $(-1)^{l(\alpha)+l(\beta)-l(\gamma)}$ times a sum of products of terms of the form $e^{\theta} - 1$ or $e^{\theta}$,
where $\theta$ is a positive root. We prove and realize this positivity conjecture for $d^\alpha\beta$, Grassmannian $G/P$. Our formula involves only terms of the form $e^\theta - 1$ (in our case, $t_b/t_a - 1, b > a$).

Graham [9] proves that the structure constants $c^\gamma_{\alpha,\beta}$ for $G/B$, where $G$ is any symmetrizable Kac-Moody group, can be expressed as sums of products of positive roots. We realize this condition for $c^\beta_{\alpha,\beta}$, Grassmannian $G/P$. Knutson and Tao [15] realize this positivity condition for all $c^\gamma_{\alpha,\beta}$, Grassmannian $G/P$. Their formula, when restricted to structure constants of the form $c^\beta_{\alpha,\beta}$, is expressed in terms of different combinatorial objects than ours, and also expresses the quantity $c^\beta_{\alpha,\beta}$ in terms of different sums of monomials in the positive roots.

Gröbner Degeneration

Our proof relies on a result of Kodiyalam and Raghavan [19], Kreiman and Lakshmibai [12], and Kreiman [20], which gives an equivariant Gröbner degeneration of a local neighborhood of $X_\alpha$ centered at $e_\beta$ to a reduced union $W_{\alpha,\beta}$ of coordinate subspaces $W_1, \ldots, W_q$ of an affine space whose coordinates are characters of $T$. Our strategy is to use this result and the inclusion-exclusion principle to deduce that

$$[X_\alpha]_{K|e_\beta} = [W_{\alpha,\beta}]_K = \sum_{j=1}^q (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq q} [W_{i_1} \cap \cdots \cap W_{i_j}]_K = \sum_S N_S [W_S]_K,$$

where each $W_S$ is an intersection $W_{i_1} \cap \cdots \cap W_{i_j}$, and the integer coefficient $N_S$ accounts for the fact that $W_S$ can in general be expressed as an intersection of $W_i$'s in more than one way. Since $W_S$ is itself a coordinate subspace, $[W_S]_K$ is easily computed.

Gröbner degenerations are used to obtain the double Grothendieck and double Schubert polynomials for all permutations by Knutson and Miller [14], and for vexillary permutations by Knutson, Miller, and Yong [12], [13]. In [16], [19], and [20], only Grassmannians (i.e., Grassmannian permutations) are studied. However, this allows the authors to degenerate at the local level, which in this paper results in the positivity of the restriction formulas. The methods and results of this paper have been extended to the case of Symplectic Grassmannians by Kreiman [21] (see also Ikeda [11]), by using a Gröbner degeneration of Ghorpade-Raghavan [8].

Semistandard Set-Valued Tableaux

Semistandard set-valued tableaux are generalizations of semistandard Young tableaux. These objects were introduced by Buch [4], who used them to give a formula for the linear structure constants for products of Schubert classes in the K-theory of the Grassmannian. Buch also expressed Grothendieck polynomials
for Grassmannian permutations in terms of semistandard set-valued tableaux. Knutson, Miller, and Yong [12, 13] give several formulas for double Grothendieck and double Schubert polynomials for vexillary permutations in terms of flagged set-valued tableaux.

In Sections 5 and 6, we discuss three equivalent combinatorial models: certain semistandard Young tableaux, ‘families of nonintersecting paths on Young diagrams’, and ‘subsets of Young diagrams’. Although the three models are equally suitable for expressing our equivariant cohomology formula, we find the tableau model to be the simplest one for deriving and expressing the equivariant K-theory formula. Each $W_i$ is naturally indexed by a semistandard Young tableau $P_i$, and each $W_{i_1} \cap \cdots \cap W_{i_j}$ is naturally indexed by the ‘union’ $S = P_{i_1} \cup \cdots \cup P_{i_j}$, which is a set-valued tableau. The $S$ for which $N_S \neq 0$ are precisely those which are semistandard.

The families of nonintersecting paths which we use appeared first in Krattenthaler [17, 18] and subsequently in [16], [19], and [20]. The subsets of Young diagrams, which were discovered independently by Ikeda-Naruse, are similar to RC graphs or reduced pipe dreams [2, 6, 14] for Grassmannian permutations.

The paper is organized as follows. In Section 2, we state our formulas for $[X_\alpha]_K|e_\beta$ and $[X_\alpha]_K|e_\beta$. In Section 3, we present basic definitions and properties of equivariant K-theory and equivariant cohomology for affine spaces and affine varieties. In Section 4, we give the main arguments for the proof of our formulas (Proposition 2.2), omitting the proofs of two lemmas. In Sections 5 and 6, we prove the first of these two lemmas, by translating a result of [16], [19], and [20] into the language of semistandard Young tableaux. In Section 7, we prove the second lemma, which computes $N_S$.

Acknowledgements. I would like to thank W. Graham, P. Magyar, and M. Shimozono for helpful discussions and suggestions.

2 Statement of Results

Let $d$ and $n$ be fixed positive integers, $0 < d < n$. Let $I_{d,n}$ be the set of $d$-element subsets of $\{1, \ldots, n\}$, where we always assume the entries of such a subset are listed in increasing order. We define the complement of $\alpha = \{\alpha(1), \ldots, \alpha(d)\} \in I_{d,n}$ by $\alpha' = \{1, \ldots, n\} \setminus \alpha$ and the length of $\alpha$ by $l(\alpha) = (\alpha(1)-1)+\cdots+(\alpha(d)-d)$. Let $J_{d,n}$ be the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$ such that $n-d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0$. There is a standard bijection $\pi : I_{d,n} \to J_{d,n}$ given by $\pi(\{\alpha(1), \ldots, \alpha(d)\}) = (\alpha(d)-d, \ldots, \alpha(1)-1)$. Let $\alpha, \beta \in I_{d,n}$ be fixed.

The Grassmannian $Gr_{d,n}$ is the set of all $d$-dimensional complex subspaces of $\mathbb{C}^n$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$. Define $e_\alpha = \text{Span}\{e_{\alpha(1)}, \ldots, e_{\alpha(d)}\} \in Gr_{d,n}$. Consider the opposite standard flag, whose $i$-th space is $F_i = \text{Span}\{e_n, \ldots, e_{n-i+1}\}$, $i = 1, \ldots, n$. The Schubert variety $X_\alpha$ (which is sometimes called an opposite Schubert variety) is defined by incidence.
relations:
\[ X_\alpha = \{ V \in Gr_{d,n} \mid \dim(V \cap F_i) \geq \dim(e_\alpha \cap F_i), \ i = 1, \ldots, n \}. \]

A Young diagram is a collection of boxes arranged into a left and top justified array. If the \( i \)-th row of a diagram has \( \lambda_i \) boxes, \( i = 1, \ldots, r \), then we say that the shape of the diagram is the partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \). A set-valued tableau is an assignment of a nonempty set of positive integers to each box of a diagram. The entries of a set-valued tableau \( S \) are the positive integers in the boxes. If a positive integer occurs in more than one box of \( S \), then we consider the separate occurrences of the positive integer to be distinct entries of \( S \). A Young tableau is a special type of set-valued tableau in which each box contains a single entry.

A set-valued tableau is said to be semistandard if all entries of any box \( B \) are less than or equal to all entries of the box to the right of \( B \) and strictly less than all entries of the box below \( B \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
1 & 2,3 & 3 & 4,6,7 & 7,9 \\
\hline
2 & 4,5,7 & 8,9 & & \\
\hline
4,6 & 8 & & & \\
\hline
\end{array}
\]

Figure 1: A semistandard set-valued tableau

If \( \mu = (\mu_1, \ldots, \mu_h) \) is any partition, then a set-valued tableau \( S \) is said to be on \( \mu \) if, for every entry \( x \) of \( S \), \( x \leq h \) and
\[
x + c(x) - r(x) \leq \mu_x, \tag{3}
\]
where \( r(x) \) and \( c(x) \) are the row and column numbers of the box containing \( x \). Note that the condition \( x \leq h \) is required for \( \mu_x \), and thus (3), to be well-defined.

**Example 2.1.** Let \( \lambda = (2,1) \), \( \mu = (4,4,2,1) \). The following list gives all semistandard set-valued tableaux on \( \mu \) of shape \( \lambda \):
Denote the set of semistandard set-valued tableaux on $\mu$ of shape $\lambda$ by $\text{SSVT}_{\lambda,\mu}$ and the set of semistandard Young tableaux on $\mu$ of shape $\lambda$ by $\text{SSYT}_{\lambda,\mu}$.

**Proposition 2.2.** Let $\lambda = \pi(\alpha)$, $\mu = \pi(\beta)$. Then

(i) $[X_\alpha]_{\kappa|e_\beta} = (-1)^{l(\alpha)} \prod_{S \in \text{SSVT}_{\lambda,\mu}} \left( \frac{t_{\beta(d+1-x)}}{t_{\beta(x+c(x)-r(x))}} - 1 \right)$.

(ii) $[X_\alpha]_{\kappa|e_\beta} = \sum_{S \in \text{SSYT}_{\lambda,\mu}} \prod_{x \in S} (t_{\beta(d+1-x)} - t_{\beta(x+c(x)-r(x))})$.

**Example 2.3.** Consider $\text{Gr}_{3,6}$, $\alpha = \{1, 3, 5\}$, $\beta = \{2, 5, 6\}$. Then $l(\alpha) = 3$, $\pi(\alpha) = (2, 1, 0)$, $\pi(\beta) = (3, 3, 1)$. The semistandard set-valued tableaux on $\pi(\beta)$ of shape $\pi(\alpha)$ are:

\begin{align*}
1 & \quad 1 \\
2 & \\
\end{align*}

\begin{align*}
1 & \quad 2 \\
2 & \\
\end{align*}

\begin{align*}
1 & \quad 1, 2 \\
2 & \\
\end{align*}

Therefore,

$[X_\alpha]_{\kappa|e_\beta} = -(t_6 - t_1) \left( t_5 - 1 \right) \left( t_5 - 1 \right) \left( t_4 - 1 \right) \left( t_3 - 1 \right) \left( t_3 - 1 \right) \left( t_3 - 1 \right) \left( t_3 - 1 \right) \left( t_1 - 1 \right) \sum_{S \in \text{SSYT}_{\lambda,\mu}} \prod_{x \in S} (t_{\beta(d+1-x)} - t_{\beta(x+c(x)-r(x))})$.

**Remark 2.4.** In Proposition 2.2(i), the condition that $S$ is on $\mu$ implies that each term in the product is of the form $t_b/t_a - 1$, $b > a$, and in Proposition 2.2(ii), the condition that $S$ is on $\mu$ implies that each term in the product is of the form $t_b - t_a$, $b > a$. Indeed, for $\mu = \pi(\beta)$, one can show that $\mu_j = \# \{ i \in \{1, \ldots, n-d\} | \beta'(i) < \beta(d+1-j) \}$, $j = 1, \ldots, d$. Therefore $i \leq \mu_j \iff \beta'(i) < \beta(d+1-j)$. Substituting $i = x+c(x)-r(x)$, $j = x$, we obtain: $[X_\alpha]_{\kappa|e_\beta} = -(t_6 - t_1) \left( t_5 - 1 \right) \left( t_4 - 1 \right) \left( t_3 - 1 \right) \left( t_3 - 1 \right) \left( t_3 - 1 \right) \left( t_3 - 1 \right) \left( t_3 - 1 \right) \left( t_1 - 1 \right)$.

3 **Equivariant K-Theory in Affine Spaces**

The equivariant $K$-theory $K^*_T(V)$ of an algebraic variety $V$ with a $T$ action is defined to be the Grothendieck group of equivariant coherent sheaves of $\mathcal{O}_V$ modules. If $Y \subset V$ is a $T$-stable closed subvariety, then we define $[Y]_{\kappa|}$ to be the class of the structure sheaf $\mathcal{O}_Y$ of $Y$.

In this section we assume that $V$ is the affine space $\mathbb{C}^n$. In this case, the notion of coherent sheaves of $\mathcal{O}_V$ modules can be replaced by that of finitely generated $\mathbb{C}[V]$ modules. If $Y$ is a $T$-stable closed subvariety of $V$, then $[Y]_{\kappa}$ is just $[\mathbb{C}[Y]]_{\kappa}$. We also have that $K^*_T(V) \cong K^*_T(0)$, which can be identified with the representation ring of $T$, $R(T) = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.
For any (possibly infinite dimensional) $T$-module $L$, define
\[ \text{Char}(L) \in \mathbb{C}[t_1^{-1}, \ldots, t_n^{-1}] \]
to be the character of $L$ under the $T$ action (one also views \text{Char}(L) as the $\mathbb{Z}^n$-graded Hilbert function of $L$, where each character of $L$ is graded by its $T$-weight). For $d \in \mathbb{Z}^n$, define $\mathbb{C}[V](-d)$ to be $\mathbb{C}[V]$ with modified $T$-action: the characters of $\mathbb{C}[V](-d)$ are the same as those of $\mathbb{C}[V]$, but with weight $t^d$ times greater. We use the standard identification
\[ [\mathbb{C}[V](-d)]_k = t^d = \frac{\text{Char}(\mathbb{C}[V](-d))}{\text{Char}(\mathbb{C}[V])}. \]

Let $Y$ be a $T$-stable closed subvariety of $V$. There is a free equivariant resolution
\[ 0 \to \mathcal{E}_r \to \cdots \to \mathcal{E}_1 \to \mathbb{C}[Y] \to 0, \quad \text{where} \quad \mathcal{E}_i = \bigoplus_{j=1}^{u_i} \mathbb{C}[V](-d_{ij}). \]

Since
\[ [\mathcal{E}_i]_k = \sum_{j=1}^{u_i} [\mathbb{C}[V](-d_{ij})]_k = \sum_{j=1}^{u_i} \frac{\text{Char}(\mathbb{C}[V](-d_{ij}))}{\text{Char}(\mathbb{C}[V])} = \frac{\text{Char}(\mathcal{E}_i)}{\text{Char}(\mathbb{C}[V])}, \]
it follows that
\[ [Y]_k = \sum_{i=1}^{r} (-1)^{i+1}[\mathcal{E}_i]_k = \sum_{i=1}^{r} (-1)^{i+1} \frac{\text{Char}(\mathcal{E}_i)}{\text{Char}(\mathbb{C}[V])} = \frac{\text{Char}(\mathbb{C}[Y])}{\text{Char}(\mathbb{C}[V])}. \quad (4) \]

**Example 3.1.** Let $V = \mathbb{C}^3$, and let $y_1, y_2, y_3 \in \mathbb{C}[V]$ be the standard coordinate functions on $V$. Suppose that $T = (\mathbb{C}^*)^4$ acts on $V$, and hence on $\mathbb{C}[V]$, and suppose that for $t = \text{diag}(t_1, t_2, t_3, t_4) \in T,$
\[ t(y_1) = \frac{t_4}{t_1} y_1, \quad t(y_2) = \frac{t_2}{t_3} y_2, \quad t(y_3) = t_1^{-2} y_3. \]

Then
\[ \text{Char}(\mathbb{C}[V]) = \sum_{i,j,k=0}^{\infty} \left( \frac{t_4}{t_1} \right)^i \left( \frac{t_2}{t_3} \right)^j (t_1^{-2})^k = \frac{1}{(1 - \frac{t_4}{t_1})(1 - \frac{t_2}{t_3})(1 - t_1^{-2})}. \]

Let $Y \subset V$ be the $y_1$-axis. Then
\[ \text{Char}(\mathbb{C}[Y]) = \sum_{i=0}^{\infty} \left( \frac{t_4}{t_1} \right)^i = \frac{1}{(1 - \frac{t_4}{t_1})} = \frac{1}{(1 - \frac{t_4}{t_1})(1 - \frac{t_2}{t_3})(1 - t_1^{-2})}. \]

Therefore, by (4), $[Y]_k = (1 - \frac{t_4}{t_1})(1 - t_1^{-2}).$

Let $y_1, \ldots, y_m \in \mathbb{C}[V]$ be the standard coordinate functions on $V = \mathbb{C}^m$. We denote by $V(\{y_{j_1}, \ldots, y_{j_k}\})$ the coordinate subspace of $V$ defined by the vanishing of $y_{j_1}, \ldots, y_{j_k}$.
Lemma 3.2. Let $\chi_i, i = 1, \ldots, m$ be characters of $T$. Suppose that $T$ acts on $V$, and hence on $\mathbb{C}[V]$, and suppose that $t(y_i) = \chi_i(t)y_i$, $t \in T$, $i = 1, \ldots, m$.

(i) If $W \subset V$ is the coordinate subspace $W = V(\{y_j, \ldots, y_k\})$, then

$$[W]_K = (1 - \chi_{j_1}(t)) \cdots (1 - \chi_{j_k}(t)).$$

(ii) If $W \subset V$ is the union of coordinate subspaces $W_1, \ldots, W_q$, then

$$[W]_K = \sum_{j=1}^{q} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq q} [W_{i_1} \cap \cdots \cap W_{i_j}]_K.$$

Proof. (i) is an easy generalization of Example 3.1.

(ii) By the inclusion-exclusion principle,

$$\text{Char}(W) = \text{Char}(W_1 \cup \cdots \cup W_q)$$

$$= \sum_{j=1}^{q} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq q} \text{Char}(W_{i_1} \cap \cdots \cap W_{i_j}).$$

The result now follows from (4).

Note that each $W_{i_1} \cap \cdots \cap W_{i_j}$ in (6) is itself a coordinate subspace, so (5) can be used to compute its class.

4 The Class of an Opposite Schubert Variety

The Plücker map $\text{pl} : \text{Gr}_{d,n} \to \mathbb{P}(\wedge^d \mathbb{C}^n)$ is defined by $\text{pl}(W) = [w_1 \wedge \cdots \wedge w_d]$, where $\{w_1, \ldots, w_d\}$ is any basis for $W$. It is well known that $\text{pl}$ is a closed immersion. Thus $\text{Gr}_{d,n}$ inherits the structure of projective variety, as does $X_\alpha \subset \text{Gr}_{d,n}$.

Reduction to an Affine Variety

Under the Plücker map, $e_\beta$ maps to $[e_{\beta_1} \wedge \cdots \wedge e_{\beta_d}] \in \mathbb{P}(\wedge^d \mathbb{C}^n)$. Define $p_\beta$ to be homogeneous (Plücker) coordinate $[e_{\beta_1} \wedge \cdots \wedge e_{\beta_d}]^* \in \mathbb{C}[\mathbb{P}(\wedge^d \mathbb{C}^n)]$. Let $O_\beta$ be the distinguished open set of $\text{Gr}_{d,n}$ defined by $p_\beta \neq 0$. Then $O_\beta$ is isomorphic to the affine space $\mathbb{C}^{(n-d)}$, with $e_\beta$ the origin. Indeed, $O_\beta$ can be identified with the space of matrices in $M_{n \times d}$ in which rows $\beta_1, \ldots, \beta_d$ are the rows of the $d \times d$ identity matrix, and rows $\beta'_1, \ldots, \beta'_{n-d}$ contain arbitrary elements of $\mathbb{C}$. Under this identification, the rows of $O_\beta$ are indexed by $\{1, \ldots, n\}$, and the columns by $\beta$.
Example 4.1. Let $d = 3$, $n = 7$, $\beta = \{2, 5, 7\}$. Then $\beta' = \{1, 3, 4, 6\}$, and

$$O_\beta = \left\{ \begin{pmatrix} y_{12} & y_{15} & y_{17} \\ 1 & 0 & 0 \\ y_{32} & y_{35} & y_{37} \\ y_{42} & y_{45} & y_{47} \\ 0 & 1 & 0 \\ y_{62} & y_{65} & y_{67} \\ 0 & 0 & 1 \end{pmatrix}, y_{ab} \in \mathbb{C} \right\}. $$

The space $O_\beta$ is $T$-stable, and for $t = \text{diag}(t_1, \ldots, t_n) \in T$ and coordinate functions $y_{ab} \in \mathbb{C}[O_\beta]$, 

$$t(y_{ab}) = \frac{t_b}{t_a} y_{ab}. \quad (7)$$

The equivariant embeddings $e_\beta \rightarrow^k O_\beta \rightarrow^{k^*} \text{Gr}_{d,n}$ induce homomorphisms 

$$K^*_T(\text{Gr}_{d,n}) \rightarrow^{k^*} K^*_T(O_\beta) \rightarrow^{j^*} K^*_T(e_\beta).$$

The map $j^*$ is an isomorphism, identifying $K^*_T(O_\beta)$ with $K^*_T(e_\beta)$. Define $Y_{\alpha,\beta} = X_\alpha \cap O_\beta$. We have 

$$[X_\alpha]_{k^*} = j^* \circ k^*([X_\alpha]_k) = j^*([k^{-1}X_\alpha]_k) = j^*([Y_{\alpha,\beta}]_k) = [Y_{\alpha,\beta}]_k. \quad (8)$$

Applying analogous arguments for equivariant cohomology, we obtain 

$$[X_\alpha]_u = [Y_{\alpha,\beta}]_u. \quad (9)$$

Reduction to a Union of Coordinate Subspaces

Let $\lambda = \pi(\alpha)$, $\mu = \pi(\beta)$. Let $\text{SVT}_{\lambda,\mu}$ denote the set of all set-valued tableaux (not necessarily semistandard) of shape $\lambda$ on $\mu$. For $S \in \text{SVT}_{\lambda,\mu}$, define 

$$W_S = V(\{y_{\beta'(x+c(x)-r(x)),\beta(d+1-x)}, x \in S\}),$$

a coordinate subspace of $O_\beta$. Define 

$$W_{\alpha,\beta} = \bigcup_{P \in \text{SSYT}_{\lambda,\mu}} W_P. \quad (11)$$

The following lemma, whose proof appears in Section 6, reduces our problem to computing the class of a union of coordinate subspaces.

**Lemma 4.2.** $[Y_{\alpha,\beta}]_k = [W_{\alpha,\beta}]_k$.

Let $R$ and $S$ be two set-valued tableaux of shape $\lambda$. Define the union $R \cup S$ to be the set-valued tableau of shape $\lambda$ whose entries in each box are the unions of the entries of $R$ and the entries of $S$ in that box. If $R$ and $S$ are both on $\mu$, then $R \cup S$ is on $\mu$, and $W_{R \cup S} = W_R \cap W_S$. We say that $R$ is **contained in**
S, and write $R \subseteq S$, if each entry in each box of $R$ is also an entry in the same box of $S$. In this case, if $S$ is on $\mu$, then $R$ is on $\mu$.

Let $S$ be a semistandard tableau of shape $\lambda$. Define $\text{SSYT}(S)$ to be the set of semistandard Young tableaux of shape $\lambda$ which are contained in $S$, and define $q_S = |\text{SSYT}(S)|$. Define $N_{S,j}$ to be the number of $j$ element subsets of $\text{SSYT}(S)$ whose unions equal $S$, and define $q_S = \sum_{j=1}^{qs} (-1)^{j+1} N_{S,j}$.

**Lemma 4.3.** $[W_{\alpha,\beta}]_{\kappa} = \sum_{S \in \text{SVT}_{\lambda,\mu}} N_S \prod_{x \in S} \left( 1 - \frac{t_\beta(d+1-x)}{t_\beta(x+c(x)-r(x))} \right)$.

**Proof.** Let $P_1, \ldots, P_q$ be an enumeration of $\text{SSYT}_{\lambda,\mu}$. Note that for any $S \in \text{SVT}_{\lambda,\mu}$, $\text{SSYT}(S) \subseteq \text{SSYT}_{\lambda,\mu}$; thus $q_S \leq q$. By (11) and Lemma 3.2(ii),

$$[W_{\alpha,\beta}]_{\kappa} = \left[ W_{P_1} \cup \cdots \cup W_{P_q} \right]_{\kappa}$$

$$= \sum_{j=1}^{q} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq q} \left[ W_{P_{i_1} \cap \cdots \cap W_{i_j}} \right]_{\kappa}$$

$$= \sum_{j=1}^{q} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq q} \left[ W_{P_{i_1} \cup \cdots \cup P_{i_j}} \right]_{\kappa}$$

$$= \sum_{j=1}^{q} (-1)^{j+1} \sum_{S \in \text{SVT}_{\lambda,\mu}} \sum_{P_{i_1} \cup \cdots \cup P_{i_j} = S} \left[ W_{S} \right]_{\kappa}$$

$$= \sum_{j=1}^{q} (-1)^{j+1} \sum_{S \in \text{SVT}_{\lambda,\mu}} N_{S,j} \left[ W_{S} \right]_{\kappa}$$

$$= \sum_{S \in \text{SVT}_{\lambda,\mu}} \sum_{j=1}^{qs} (-1)^{j+1} N_{S,j} \left[ W_{S} \right]_{\kappa}$$

$$= \sum_{S \in \text{SVT}_{\lambda,\mu}} N_S \left[ W_{S} \right]_{\kappa}.$$

By (11), (7), and Lemma 3.2(i), for each $S \in \text{SVT}_{\lambda,\mu}$,

$$[W_{S}]_{\kappa} = \prod_{x \in S} \left( 1 - \frac{t_\beta(d+1-x)}{t_\beta(x+c(x)-r(x))} \right).$$

For set-valued tableau $S$, define $|S|$ (resp. $\|S\|$) to be the total number of entries (resp. boxes) of $S$. The proof of the following Lemma appears in Section 7.
Lemma 4.4. (i) If $S$ is semistandard, then $N_S = (-1)^{|S|+\|S\|}$.
(ii) If $S$ is not semistandard, then $N_S = 0$.

Proof of Proposition 2.2. (i) Combining the preceding reductions and lemmas:

\[
[X_\alpha|_{e_\beta}] = [Y_{\alpha,\beta}] = 
\sum_{S \in SVT_{\lambda,\mu}} N_S \prod_{x \in S} \left(1 - \frac{t_{\beta}(d+1-x)}{t_{\beta'}(x+c(x)-r(x))}\right)
\]

\[
= \sum_{S \in SSVT_{\lambda,\mu}} (-1)^{|S|+\|S\|} \prod_{x \in S} \left(1 - \frac{t_{\beta}(d+1-x)}{t_{\beta'}(x+c(x)-r(x))}\right)
\]

\[
= (-1)^{l(\alpha)} \sum_{S \in SSVT_{\lambda,\mu}} (-1)^{|S|} \prod_{x \in S} \left(1 - \frac{t_{\beta}(d+1-x)}{t_{\beta'}(x+c(x)-r(x))}\right)
\]

\[
= (-1)^{l(\alpha)} \sum_{S \in SSVT_{\lambda,\mu}} \prod_{x \in S} \left(t_{\beta'}(x+c(x)-r(x)) - 1\right).
\]

(ii) By (1) and [14], Lemma 1.1.4, $[X_{\alpha,\beta}]|_{e_\beta}(t)$ equals the sum of the lowest degree terms of $[X_{\alpha,\beta}]|_{e_\beta}(1 - t)$. One checks that the sum of the lowest degree terms of

\[
[X_{\alpha,\beta}]|_{e_\beta}(1 - t) = (-1)^{l(\alpha)} \sum_{S \in SSVT_{\lambda,\mu}} \prod_{x \in S} \left(1 - \frac{t_{\beta}(d+1-x)}{1 - t_{\beta'}(x+c(x)-r(x))}\right)
\]

equals

\[
\sum_{S \in SSVT_{\lambda,\mu}} \prod_{x \in S} \left(t_{\beta'}(d+1-x) - t_{\beta'}(x+c(x)-r(x))\right).
\]

5 Families of Nonintersecting Paths on Young Diagrams

In this section, we introduce a set $F_{\lambda,\mu}$ of families of nonintersecting paths on Young diagrams, which we then show to be identical to the set $F'_{\lambda,\mu}$ of families of nonintersecting paths which appear in [16], [17], [18], [19], and [20], despite the fact that $F_{\lambda,\mu}$ and $F'_{\lambda,\mu}$ are defined quite differently. Thus one can view the result of this section as giving an alternate way of defining or expressing the path families of [16], [17], [18], [19], and [20]. In terms of this alternate definition, one can more easily see the equivalence between the path families and the other combinatorial models in Section 6.

Let $\mu$ be a partition and $D_\mu$ the corresponding Young diagram. We denote by $(i, j)$ the box of $D_\mu$ at row $i$ (from the top), column $j$ (from the left). We impose
the following order on boxes of $D_{\mu}$: if $(i, j), (k, l) \in D_{\mu}$, then $(i, j) \leq (k, l)$ if $i \leq k$ and $j \leq l$.

A path on $D_{\mu}$ is a path of contiguous boxes of the Young diagram $D_{\mu}$ which

(i) moves only up or to the right, and

(ii) begins on the lowest box of a column and ends on the rightmost box of a row.

Note that a path on $D_{\mu}$ may consist of only one box. In this case, the box must be a lower right corner of $D_{\mu}$. We define the greatest lower bound of a path $P$, or $\text{glb}(P)$, to be the greatest lower bound of all the boxes of $P$. Explicitly, if the left endpoint of $P$ is the box $(i, j)$ and the right endpoint is the box $(k, l)$, then $\text{glb}(P)$ is the box $(k, j) \in D_{\mu}$. In particular, the endpoints of $P$ determine $\text{glb}(P)$. We impose the following order on paths on $D_{\mu}$: if $P, P'$ are paths on $D_{\mu}$, then $P \leq P'$ if $\text{glb}(P) \leq \text{glb}(P')$.

Denote by $\mathcal{F}_{\mu}$ the set of families of nonintersecting paths on $D_{\mu}$. Let $F \in \mathcal{F}_{\mu}$. Define the support of $F$, $\text{Supp}(F)$, to be the set of all boxes in all paths of $F$.

**Example 5.1.** A family $F = \{P_1, P_2, P_3, P_4\}$ of nonintersecting paths on $D_{\mu}$, $\mu = (9, 9, 9, 8, 6, 6, 3, 1)$. We have $\text{glb}(P_1) = (1, 1), \text{glb}(P_2) = (3, 5), \text{glb}(P_3) = (5, 7), \text{glb}(P_4) = (9, 2)$, and $P_1 \leq P_2 \leq P_3, P_1 \leq P_4$.

**Lemma 5.2.** $\text{Supp}(F)$ uniquely determines the paths of $F$.

Proof. In other words, if $\text{Supp}(F') = \text{Supp}(F)$, then $F' = F$. Our proof is by decreasing induction on the number $m$ of paths of $F$. If $P$ is a minimal path of $F$, then one sees that $P$ must also be a path of $F'$. In particular, the result for $m = 1$ holds. If $m > 1$, then $\text{Supp}(F' \setminus \{P\}) = \text{Supp}(F') \setminus \text{Supp}(\{P\}) = \text{Supp}(F \setminus \{P\})$ and $F \setminus \{P\}$ has $m - 1$ paths. Thus by induction $F' \setminus \{P\} = F \setminus \{P\}$, which completes the proof.

Let $\lambda \leq \mu$, i.e., $\lambda_i \leq \mu_i$ for all $i$. Then $D_{\lambda}$ naturally embeds in $D_{\mu}$ in such a way that both Young diagrams share the same top left corner. In this way, $D_{\lambda}$ can be viewed as a subset of $D_{\mu}$.
We denote this ladder move by $(\text{twist}(P))$. We call the inverse of a ladder move a reverse ladder move.

**Proof.** By decreasing induction on $P$ paths of $D_\mu \setminus D_\lambda$. One can form a path $P$ on $D_\mu$ consisting of boxes $(i,j) \in D_\mu \setminus D_\lambda$ such that $(i-1,j-1) \not\in D_\mu \setminus D_\lambda$. This path moves along an upper right boundary of $D_\mu$. Now $D' = D_\lambda \cup P$ is a Young diagram contained in $D_\mu$, and $D_\mu \setminus D'$ has fewer boxes than $D_\mu \setminus D_\lambda$. Thus by induction there exists $F' \in F_\mu$ such that $\text{Supp}(F') = D_\mu \setminus D' = (D_\mu \setminus D_\lambda) \setminus P$. Set $F = F' \cup \{P\}$. 

By Lemma 5.3, the family $F$ of Lemma 5.3 is uniquely determined. We call this family the top family on $D_\mu$ for $\lambda$ and denote it by $F_{\lambda,\mu}^\text{top}$.

**Example 5.4.** The family $F_{\lambda,\mu}^\text{top}$, where $\mu = (7,6,6,3,3)$, $\lambda = (3,2)$.

If $F \in F_\mu$, then we define twist$(F) = \{\text{glb}(P) \mid P$ is a path of $F\}$, and we call this the twist of $F$. (In Example 5.1 twist$(F) = \{(1,1),(3,5),(5,7),(9,2)\}$.)

**Lemma 5.5.** If twist$(F_{\nu,\mu}^\text{top}) = \text{twist}(F_{\lambda,\mu}^\text{top})$, then $\nu = \lambda$.

**Proof.** By decreasing induction on $\mu$. Let $b \in D_\mu$ be a maximal element of twist$(F_{\lambda,\mu}^\text{top})$, and thus also of twist$(F_{\nu,\mu}^\text{top})$. Then $b = \text{glb}(P)$ for some maximal path $P$ of $F_{\lambda,\mu}^\text{top}$, which runs along the lower-right boundary of $D_\mu$. Likewise, $b = \text{glb}(P')$ for some maximal path $P'$ of $F_{\nu,\mu}^\text{top}$, which runs along the lower-right boundary of $D_\mu$. Since $P'$ and $P$ are paths with the same greatest lower bound, and they both run along the lower right boundary of $D_\mu$, $P' = P$.

Now $D_\mu \setminus P = D_\mu'$, for some $\mu' < \mu$. We have that $F_{\lambda,\mu}^\text{top} = F_{\lambda,\mu'}^\text{top} \setminus \{P\}$, $F_{\nu,\mu'}^\text{top} = F_{\nu,\mu}^\text{top} \setminus \{P\}$, and thus twist$(F_{\nu,\mu'}) = \text{twist}(F_{\nu,\mu}) \setminus \{b\} = \text{twist}(F_{\lambda,\mu}) \setminus \{b\} = \text{twist}(F_{\lambda,\mu'})$. Therefore $\nu = \lambda$ follows by induction.

Suppose that $D_\mu$ contains the four boxes $(i,j)$, $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$ (which make up a square), and that $F$ contains $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$, but not $(i,j)$. One checks that these three boxes must lie on the same path $P$ of $F$. We apply a ladder move to $F$ by altering $P$ as follows: the box $(i+1,j+1)$ of $P$ is removed and replaced with the box $(i,j)$, thus obtaining a new path $P'$ on $D_\mu$. The resulting path $P'$, combined with the paths of $F$ other than $P$, form a new family of nonintersecting paths $F'$ on $D_\mu$. We denote this ladder move by $F \rightarrow F'$. Note that a ladder move is invertible. We call the inverse of a ladder move a reverse ladder move.
Example 5.6. Let $F$ be the following family of nonintersecting paths on $D_{\mu}$, $\mu = (6, 5, 5, 4, 4, 1)$:

The following two ladder moves can be applied to $F$:

1. 

2. 

A family of nonintersecting paths on $D_{\mu}$ for $\lambda$ is an element of $F_{\mu}$ which can be obtained by applying a succession of ladder moves to $F_{\lambda, \mu}^{\text{top}}$. The set of all families of nonintersecting paths on $D_{\mu}$ for $\lambda$ is denoted by $F_{\lambda, \mu}$.

Example 5.7. The following diagram shows $F_{\lambda, \mu}$, as well as all possible ladder moves, where $\mu = (4, 4, 3, 3, 1)$, $\lambda = (2, 1)$. 

![Diagram showing families of nonintersecting paths and ladder moves]
The Set $\mathcal{F}_{\lambda,\mu}$ of Path Families

We say that a set of boxes $S \subset D_\mu$ is a twisted chain if for any two boxes $p, p'$ in $S$:

1. $p$ and $p'$ lie on different rows and different columns of $D_\mu$.
2. Either $p < p'$, $p' < p$, or $\{p, p'\}$ has no upper bound in $D_\mu$.

**Example 5.8.** The shaded boxes form a twisted chain in $D_\mu$.

![Diagram](image)

Note that, for example, $\{(10,2), (5,4)\}$ has least upper bound $(10,4)$, which is not in $D_\mu$.

It follows from the definitions that if $F \in \mathcal{F}_\mu$, then twist$(F)$ is a twisted chain of $D_\mu$. Define

$$\mathcal{F}_{\lambda,\mu} = \{ F \in \mathcal{F}_\mu \mid \text{twist}(F) = \text{twist}(F_{\lambda,\mu}^{\top}) \}.$$  

Ultimately we will be interested in the set $\mathcal{F}_{\lambda,\mu}''$ of families of nonintersecting paths on $D_\mu$, which is defined below. We introduce $\mathcal{F}_{\lambda,\mu}'$ in order to break up the proof that $\mathcal{F}_{\lambda,\mu} = \mathcal{F}_{\lambda,\mu}''$ into two parts: $\mathcal{F}_{\lambda,\mu} = \mathcal{F}_{\lambda,\mu}'$ and $\mathcal{F}_{\lambda,\mu}' = \mathcal{F}_{\lambda,\mu}''$.

**Lemma 5.9.** $\mathcal{F}_{\lambda,\mu}' = \mathcal{F}_{\lambda,\mu}''$.

**Proof.** We remark that applying a ladder move to any family in $\mathcal{F}_{\lambda,\mu}$ does not alter its twist. Thus ladder moves preserve $\mathcal{F}_{\lambda,\mu}'$. Also $\mathcal{F}_{\lambda,\mu}^{\top} \in \mathcal{F}_{\lambda,\mu}'$. Since $\mathcal{F}_{\lambda,\mu}$ is by definition the smallest subset of $\mathcal{F}_\mu$ preserved by ladder moves which contains $\mathcal{F}_{\lambda,\mu}^{\top}$, $\mathcal{F}_{\lambda,\mu} \subset \mathcal{F}_{\lambda,\mu}'$.

Next suppose that $F \in \mathcal{F}_{\lambda,\mu}'$. By applying a succession of reverse ladder moves to $F$, $\mathcal{F}_{\nu,\mu}^{\top}$ can be obtained, for some $\nu \leq \mu$. Since reverse ladder moves do not alter twist, twist$(\mathcal{F}_{\nu,\mu}^{\top}) = \text{twist}(F) = \text{twist}(F_{\lambda,\mu}^{\top})$. By Lemma 5.5 $\nu = \lambda$. Thus $F$ can be obtained from $\mathcal{F}_{\lambda,\mu}^{\top}$ by applying ladder moves, which implies $F \in \mathcal{F}_{\lambda,\mu}$. Hence $\mathcal{F}_{\lambda,\mu}' \subset \mathcal{F}_{\lambda,\mu}$.

Let $\lambda \leq \mu$, and define $\alpha = \pi^{-1}(\lambda)$, $\beta = \pi^{-1}(\mu)$. In [20], it is shown that there exists a unique twisted chain $S_{\lambda,\mu} = \{ (x_1, y_1), \ldots, (x_t, y_t) \} \subset D_\mu$ such that $\alpha = \beta \setminus \{ \beta(d+1-x_1), \ldots, \beta(d+1-x_t) \} \cup \{ \beta'(y_1), \ldots, \beta'(y_t) \}$. Define

$$\mathcal{F}_{\lambda,\mu}'' = \{ F \in \mathcal{F}_\mu \mid \text{twist}(F) = S_{\lambda,\mu} \}.$$
It follows from the definitions that $F''_{\lambda,\mu} = F'_{\lambda,\mu}$ if and only if twist($F'_{\lambda,\mu}$) = $S_{\lambda,\mu}$.

We first prove this for the case where $D_\mu \setminus \mathring{D}_\lambda$ consists of a single path.

**Lemma 5.10.** Let $P$ be a path which moves along the lower right border of $D_\mu$, so that $D_\mu \setminus \{P\} = D_\nu$, for some $\nu < \mu$. Then twist($F'_{\nu,\mu}$) = $S_{\nu,\mu}$.

**Example 5.11.** Let $d = 5$, $n = 9$. Let $\mu = (4, 4, 3, 3, 1)$, and let $P$ be the path on $D_\mu$, shown below. Then $D_\mu \setminus P = D_\nu$, where $\nu = (4, 2, 2, 1, 1)$. We have $F'_{\nu,\mu} = \{P\}$, and twist($F'_{\nu,\mu}$) = $\text{glb}(P) = (2, 2)$. Let $\beta = \pi^{-1}(\mu) = \{2, 5, 6, 8, 9\}$, $\alpha = \pi^{-1}(\nu) = \{2, 3, 5, 6, 9\}$. Then $\alpha = \beta \setminus 8 \cup 3 = \beta(5 + 1 - 2) \cup \beta'(2)$, so $S_{\nu,\mu} = (2, 2)$, which agrees with twist($F'_{\nu,\mu}$).

![Diagram](image)

**Proof of Lemma 5.10.** For $i = 1, \ldots, d$, define $i = d + 1 - i$; thus if a box is on row $i$ of $D_\mu$ counting from the top, then it is on row $i$ counting from the bottom.

Let $(x, y) = \text{glb}(P) = \text{twist}(F'_{\nu,\mu})$. Let $\beta = \pi^{-1}(\mu)$ and $\alpha = \beta \setminus \beta(\overline{x}) \cup \beta'(y)$. By definition, $S_{\pi(\alpha),\mu} = (x, y)$. We shall obtain the result by showing that $\pi(\alpha) = \nu$.

Note that

$$D_\mu = \{(u, v) \in \{1, \ldots, d\} \times \{1, \ldots, n - d\} \mid \beta(\overline{x}) > \beta'(v)\}.$$ 

Let $w = \min\{u \mid \beta(u) > \beta'(y)\}$, and note that $w \leq \overline{x}$. We have

$$\alpha = \{\beta(1), \ldots, \beta(w - 1), \beta'(y), \beta(w), \ldots, \beta(\overline{x} - 1), \beta(\overline{x}), \beta(\overline{x} + 1), \ldots, \beta(d)\}.$$ 

Therefore,

$$\alpha(i) = \begin{cases} 
\beta(i), & i < w \text{ or } i > \overline{x} \\
\beta(i - 1), & w < i \leq \overline{x} \\
\beta'(y), & i = w 
\end{cases}.$$

We have

$$(\pi(\alpha))_i = \alpha(i) - (i) = \begin{cases} 
\beta(\overline{i}) - \overline{i}, & \overline{i} < w \text{ or } \overline{i} > \overline{x} \\
\beta(\overline{i} - 1) - \overline{i}, & w < \overline{i} \leq \overline{x} \\
\beta'(y) - \overline{i}, & \overline{i} = w 
\end{cases} = \begin{cases} 
\mu_i, & \overline{i} < w \text{ or } \overline{i} > \overline{x} \\
\mu_{i+1} - 1, & w < \overline{i} \leq \overline{x} - 1 \\
y - 1, & \overline{i} = w 
\end{cases} = \nu_i,$$ 

where $\mu_i$, $\nu_i$, and $y$ are defined as in Lemma 5.8 and 5.9.
where we use the facts that $\beta(i)-(i) = \mu_i$, $\beta(i-1)-(i) = \beta(i+1)-(i+1)-1 = \mu_{i+1}-1$, and $\beta'(y)-y = w-1$.

Lemma 5.12. For any $\lambda \leq \mu$, $\text{twist}(F_{\lambda,\mu}^{\text{top}}) = S_{\lambda,\mu}$.

Proof. Define $\beta = \pi^{-1}(\mu)$, $\alpha = \pi^{-1}(\lambda)$, and let $\{(x_1,y_1),\ldots,(x_t,y_t)\} = \text{twist}(F_{\lambda,\mu}^{\text{top}})$. We wish to show that $\alpha = \beta \setminus \{\beta(d+1-x_1),\ldots,\beta(d+1-x_t)\} \cup \{\beta'(y_1),\ldots,\beta'(y_t)\}$.

We use decreasing induction on $t$, the number of paths in $F_{\lambda,\mu}^{\text{top}}$. For $t=1$, the result is identically Lemma 5.10. Suppose that $t > 1$. Let $P$ be a maximal path of $F_{\lambda,\mu}^{\text{top}}$. Then $P$ runs along the lower right border of $D_\mu$, so $D_\mu \setminus P = D_\nu$, for some $\nu < \mu$. Define $\gamma = \pi^{-1}(\nu)$, and assume that $(x_t,y_t) = \text{glb}(P)$ (re-indexing if necessary). By Lemma 5.10,

$$\gamma = \beta \setminus \beta(d+1-x_t) \cup \beta'(y_t).$$

(12)

Now $F_{\lambda,\nu}^{\text{top}} = F_{\lambda,\mu}^{\text{top}} \setminus \{P\}$, whose twist is equal to $\{(x_1,y_1),\ldots,(x_{t-1},y_{t-1})\}$. Thus by induction,

$$\alpha = \gamma \setminus \{\beta(d+1-x_1),\ldots,\beta(d+1-x_{t-1})\} \cup \{\beta'(y_1),\ldots,\beta'(y_{t-1})\}.$$  

(13)

Combining (12) and (13), we arrive at the result. $\square$

From Lemma 5.12, we obtain

Lemma 5.13. $F_{\lambda,\mu}'' = F_{\lambda,\nu}''$.

Now Lemmas 5.9 and 5.13 imply

Lemma 5.14. $F_{\lambda,\mu}'' = F_{\lambda,\mu}$. 

6 From Path Families to Semistandard Young Tableaux

Lemma 4.2 follows from a result of [16], [19], and [20] which gives an equivariant Gröbner degeneration of a Schubert variety in the neighborhood of a $T$-fixed point to a union of coordinate subspaces. However, whereas the result of [16], [19], and [20] is expressed in terms of the set $F_{\lambda,\mu}'$ of families of nonintersecting paths on Young diagrams, our results, and in particular Lemma 4.2, require the semistandard Young tableaux SSYT$_{\lambda,\mu}$. Most the previous section and this one are taken up in showing the equivalence between these two combinatorial models. In the previous section we showed that $F_{\lambda,\mu}' = F_{\lambda,\mu}$, in this section, we introduce a new model, ‘subsets of Young diagrams’. We will be interested in a certain set of subsets of the Young diagram $D_\mu$, which we denote by $D_{\lambda,\mu}$. We show that $F_{\lambda,\mu}$ and SSYT$_{\lambda,\mu}$ are both equivalent to $D_{\lambda,\mu}$, and thus are equivalent to each other. We summarize the steps we take in showing the equivalence between $F_{\lambda,\mu}'$ and SSYT$_{\lambda,\mu}$ as follows:

$$F_{\lambda,\mu}' = F_{\lambda,\mu} \leftrightarrow D_{\lambda,\mu} \leftrightarrow \text{SSYT}_{\lambda,\mu}.$$
The subsets $D_{\lambda,\mu}$, which were discovered independently by Ikeda-Naruse, are similar to RC graphs or reduced pipe dreams \cite{2,6,14} for Grassmannian permutations.

**Subsets of Young Diagrams**

Let $\mu$ be a partition and $D_{\mu}$ the corresponding Young diagram. Lemma \cite{2} tells us that a family of nonintersecting paths on $D_{\mu}$ is completely characterized by its support. This suggests that in order to study families of nonintersecting paths on $D_{\mu}$, it suffices to study the supports of these path families (or the complements in $D_{\mu}$ of their supports). This motivates the following definitions.

A **subset** of $D_{\mu}$ is a set of boxes in $D_{\mu}$. Let $D$ be a subset of $D_{\mu}$. Suppose that $D_{\mu}$ contains the four boxes $(i, j)$, $(i+1, j)$, $(i, j+1)$, and $(i+1, j+1)$ (which make up a square), but of these four boxes, $D$ only contains $(i, j)$. Then a **ladder move** removes $(i, j)$ from $D$ and replaces it with $(i+1, j+1)$, thereby obtaining a new subset $D'$ of $D_{\mu}$. We denote this ladder move by $D \rightarrow D'$. Note that a ladder move is invertible. We call the inverse of a ladder move a **reverse ladder move**.

**Example 6.1.** Let $D$ be the following subset of $D_{\mu}$, $\mu = (6, 5, 5, 4, 4, 1)$ (boxes of $D$ are shaded):

![Young Diagram Example](image)

The following two ladder moves can be applied to $F$:

1. 

2. 

Let $\lambda$ be a partition with $\lambda \leq \mu$, i.e., $\lambda_i \leq \mu_i$ for all $i$. Embed the Young diagram $D_{\lambda}$ in $D_{\mu}$ in such a way that both $D_{\lambda}$ and $D_{\mu}$ share the same top left corners. The subset of $D_{\mu}$ consisting of all boxes in this embedded Young diagram is called the **top subset** of $D_{\mu}$ for $\lambda$ and denoted by $D_{\lambda,\mu}^{\text{top}}$.
Example 6.2. The subset \( D_{\lambda,\mu}^{\text{top}} \), where \( \mu = (7, 6, 6, 6, 3, 3) \), \( \lambda = (3, 2) \).

A subset of \( D_{\mu} \) for \( \lambda \) is a subset of \( D_{\mu} \) which can be obtained by applying a succession of ladder moves to the top subset of \( D_{\mu} \) for \( \lambda \). The set of all subsets of \( D_{\mu} \) for \( \lambda \) is denoted by \( D_{\lambda,\mu} \).

Example 6.3. The following diagram shows \( D_{\lambda,\mu} \), as well as all possible ladder moves, where \( \mu = (4, 4, 3, 3, 1) \), \( \lambda = (2, 1) \).

---

Semistandard Young Tableaux

A ladder move on a semistandard Young tableau is an operation which increments one of the entries of the tableau by 1 and results in a semistandard Young tableau. Note that a ladder move is invertible. We call the inverse of a ladder move a reverse ladder move. Recall that a semistandard Young tableau is on \( \mu \) if each of its entries satisfies (3). Let \( \lambda \leq \mu \). Define the top semistandard Young tableau on \( \mu \) of shape \( \lambda \) to be the (semistandard) Young tableau of shape \( \lambda \) whose \( i \)-th row is filled with \( i \)'s. This definition does not depend on \( \mu \).

The set of all semistandard Young tableaux on \( \mu \) of shape \( \lambda \), \( \text{SSYT}_{\lambda,\mu} \), is precisely the set of semistandard Young tableaux on \( \mu \) which can be obtained by applying sequences of ladder moves to the top semistandard Young tableaux on \( \mu \) of shape \( \lambda \). This follows from the facts that (i) by applying a sequence of reverse ladder moves to any semistandard Young tableau of shape \( \lambda \), the top semistandard Young tableaux on \( \mu \) of shape \( \lambda \) can be obtained, and (ii) reverse ladder moves preserve (3).
Example 6.4. The following diagram shows SSYT\(_{\lambda,\mu}\), as well as all possible ladder moves, where \(\mu = (4, 4, 3, 3, 1)\), \(\lambda = (2, 1)\).

![Diagram showing SSYT\(_{\lambda,\mu}\) and ladder moves](image)

The Equivalences \(\mathcal{F}_{\lambda,\mu} \leftrightarrow \mathcal{D}_{\lambda,\mu}\) and \(\mathcal{D}_{\lambda,\mu} \leftrightarrow \text{SSYT}_{\lambda,\mu}\)

Recall that

\[
\begin{align*}
\mathcal{F}_{\mu} & \quad \text{the set of families of nonintersecting paths on } D_{\mu} \\
\mathcal{F}_{\lambda,\mu} & \quad \text{the set of families of nonintersecting paths on } D_{\mu} \text{ for } \lambda \\
\mathcal{F}_{\lambda,\mu}^{\text{top}} & \quad \text{the top family of nonintersecting paths on } D_{\mu} \text{ for } \lambda \\
D_{\mu} & \quad \text{the set of subsets of } D_{\mu} \\
D_{\lambda,\mu} & \quad \text{the set of subsets of } D_{\mu} \text{ for } \lambda \\
D_{\lambda,\mu}^{\text{top}} & \quad \text{the top subset of } D_{\mu} \text{ for } \lambda \\
\text{SSYT}_{\lambda,\mu} & \quad \text{the set of semistandard Young tableaux on } \mu \text{ of shape } \lambda \\
\mathcal{P}_{\lambda,\mu}^{\text{top}} & \quad \text{the top semistandard Young tableaux on } \mu \text{ of shape } \lambda
\end{align*}
\]

Define \(h : \mathcal{F}_{\lambda,\mu} \rightarrow \mathcal{D}_{\lambda,\mu}\) by \(h(F) = D_{\mu} \setminus \text{Supp}(F)\). Then \(h\) maps \(\mathcal{F}_{\lambda,\mu}^{\text{top}}\) to \(\mathcal{D}_{\lambda,\mu}^{\text{top}}\) and commutes with ladder moves. Therefore it restricts to a map from \(\mathcal{F}_{\lambda,\mu}\) to \(\mathcal{D}_{\lambda,\mu}\). Injectivity of \(h\) follows immediately from Lemma 5.2. To show surjectivity, assume that \(h(F) = D\), for some \(D \in \mathcal{D}_{\lambda,\mu}\), \(F \in \mathcal{F}_{\mu}\). Suppose that a ladder move is applied to \(D\) to obtain \(D'\). Then there is a corresponding ladder move which when applied to \(F\) yields \(F'\) such that \(h(F') = D'\). Thus surjectivity of \(h : \mathcal{F}_{\lambda,\mu} \rightarrow \mathcal{D}_{\lambda,\mu}\) follows by induction on number of ladder moves.

We next give a bijection \(g\) from \(\mathcal{D}_{\lambda,\mu}\) to \(\text{SSYT}_{\lambda,\mu}\). Let \(D \in \mathcal{D}_{\lambda,\mu}\). By applying a sequence of reverse ladder moves to \(D\), \(\mathcal{D}_{\lambda,\mu}^{\text{top}}\) can be obtained. This sequence of reverse ladder moves takes the box \((x, y)\) of \(D\) to some box \((i_{x,y}, j_{x,y})\) of \(\mathcal{D}_{\lambda,\mu}^{\text{top}}\). Note that \((i_{x,y}, j_{x,y})\) depends only on \(x\) and \(y\) (and \(D\)), and not on the sequence of reverse ladder moves. Let \(g(D)\) be the tableau of shape \(\lambda\) which, for each box \((x, y)\) of \(D\), contains entry \(x\) in box \((i_{x,y}, j_{x,y})\). The definitions of ladder and reverse ladder moves on subsets of \(D_{\mu}\) imply that \(g(D)\) is semistandard. Also,
since \((x, y) \in D_\mu, y \leq \mu(x)\); thus \(y - x = j_{x,y} - i_{x,y}\) implies \(x + j_{x,y} - i_{x,y} \leq \mu(x)\). Therefore \(g(D)\) is on \(\mu\).

To show that \(g\) is bijective, we give the inverse map \(f\) from \(SSYT_{\lambda,\mu}\) to \(D_{\lambda,\mu}\). Let \(P \in SSYT_{\lambda,\mu}\), and let \(f(P)\) be the subset of \(D_\mu\) which, for each entry \(x \in P\), contains box \((x, x - r(x) + c(x))\) of \(D_\mu\) (where recall that \((r(x), c(x))\) is the box of \(x\) in \(P\)). Since \(P\) is on \(\mu\), \((x, x - r(x) + c(x)) \in D_\mu\); thus \(f(P)\) is indeed in \(D_{\mu}\). Let \(P_{\lambda,\mu}^{top}\) denote the top semistandard Young tableaux on \(\mu\) of shape \(\lambda\). Because \(f(P_{\lambda,\mu}^{top}) = D_{\lambda,\mu}^{top}\) and \(f\) commutes with ladder moves, \(f(P) \in D_{\lambda,\mu}\).

**Proof of Lemma 4.2**

Let \(\lambda = \pi(\alpha), \mu = \pi(\beta)\). Let \(\{v_{a,b} \mid (a, b) \in \beta' \times \beta\} \subseteq \mathcal{O}_\beta\) denote the basis dual to the basis of linear forms \(\{y_{a,b} \mid (a, b) \in \beta' \times \beta\} \subseteq \mathcal{O}_\beta^*\). For \(F \in \mathcal{F}_\mu\), define

\[
W_F = \text{Span}(\{v_{\beta'(z), \beta(d+1-x)} \mid (x, z) \in \text{Supp}(F)\}) \cup \{v_{a,b} \mid (a, b) \in \beta' \times \beta, a > b\}).
\]

In \([16][19][20]\), an explicit equivariant bijection (which is called the bounded RSK in \([20]\)) is constructed from \(\mathbb{C}[\bigcup_{F \in \mathcal{F}_\lambda,\mu} W_F]\) to \(\mathbb{C}[Y_{\alpha,\beta}]\). Thus in light of Lemma 5.14

\[
\text{Char}(\mathbb{C}[Y_{\alpha,\beta}]) = \text{Char}\left(\mathbb{C}\left[\bigcup_{F \in \mathcal{F}_\lambda,\mu} W_F\right]\right) = \text{Char}\left(\mathbb{C}\left[\bigcup_{F \in \mathcal{F}_\lambda,\mu} W_F\right]\right). \quad (14)
\]

For \(x \in \{1, \ldots, d\}, z \in \{1, \ldots, n - d\}\), we have that \((x, z) \in D_\alpha \iff z \leq \mu(x) \iff \beta'(z) < \beta(d+1-x)\) (see Remark 2.4). Thus \(\{(a, b) \in \beta' \times \beta \mid a < b\}\) can be expressed as \(\{(\beta'(z), \beta(d+1-x)) \mid (x, z) \in D_\mu\}\). Let \(F \in \mathcal{F}_{\lambda,\mu}\), \(D = h(F)\), and \(P = g(D)\). Since \(\text{Supp}(F)\) and \(D\) are complements in \(D_\mu\),

\[
\beta' \times \beta = \{(\beta'(z), \beta(d+1-x)) \mid (x, z) \in \text{Supp}(F)\} \cup \{(a, b) \in \beta' \times \beta \mid a > b\}
\]

\[
\cup \{(\beta'(z), \beta(d+1-x)) \mid (x, z) \in D\}.
\]

Therefore

\[
W_F = V(\{y_{\beta'(z), \beta(d+1-x)} \mid (x, z) \in D\}) = V(\{y_{\beta'(x-r(x)+c(x)), \beta(d+1-x)} \mid x \in P\}) = W_P.
\]

Consequently, \(\bigcup_{F \in \mathcal{F}_{\lambda,\mu}} W_F = \bigcup_{P \in SSYT_{\lambda,\mu}} W_P = \mathcal{W}_{\alpha,\beta}\), and thus

\[
\text{Char}\left(\mathbb{C}\left[\bigcup_{F \in \mathcal{F}_{\lambda,\mu}} W_F\right]\right) = \text{Char}\left(\mathbb{C}\left[\bigcup_{P \in SSYT_{\lambda,\mu}} W_P\right]\right) = \text{Char}(\mathcal{W}_{\alpha,\beta}). \quad (15)
\]

Combining \((14)\) and \((15)\), we obtain \(\text{Char}(\mathbb{C}[Y_{\alpha,\beta}]) = \text{Char}(\mathbb{C}[\mathcal{W}_{\alpha,\beta}])\). By \((1)\), \([Y_{\alpha,\beta}]_k = [\mathcal{W}_{\alpha,\beta}]_k\).
In this section we prove Lemma 4.4.

For a set-valued tableau $S$, we denote by $S_{i,j}$ the set of entries of $S$ which are contained in the box with row and column numbers $i$ and $j$ respectively, and we denote by $S_{i,j,1}, \ldots, S_{i,j,r}$ the entries of $S_{i,j}$, which we assume are listed in increasing order. We define $M_k(S)$ to be the $k$-element subsets of $SSYT(S)$, and $N_k(S)$ the $k$-element subsets of $SSYT(S)$ whose unions equals $S$. By definition, $N_{S,k} = |N_k(S)|$.

Define a generalized set-valued tableau to be the assignment of a possibly empty set of positive integers to each box of a Young diagram. If $S$ and $R$ are two set-valued tableaux of the same shape, then the difference $S \setminus R$ is defined to be the generalized set-valued tableau of the same shape as $S$ (or $R$) with $(S \setminus R)_{i,j} = S_{i,j} \setminus R_{i,j}$ for all $i,j$. Recall that we write $R \subset S$ to indicate that $R_{i,j} \subset S_{i,j}$ for all $i,j$. If $R \subset S$, and $S$ is clear from the context, then we also denote $S \setminus R$ by $R$. The only generalized set-valued tableaux which appear in this section in which boxes may be empty occur explicitly as differences of set-valued tableaux.

**Lemma 7.1.** $N_S = \sum_{R \subset S} (-1)^{|R|} - \sum_{R \subset S} (-1)^{|R|}$.

**Proof.** If $q_S = 0$ (i.e., $SSYT(S) = \emptyset$), then for every set-valued tableau $R \subset S$, $SSYT(R) = \emptyset$; thus the result is trivially true. Assume $q_S \neq 0$.

For $k \geq 1$, we have that $N_k(S) = M_k(S) \setminus \bigcup_{R \subset S} M_k(R)$; by the inclusion-exclusion principle,

$$|N_k(S)| = \sum_{R \subset S} (-1)^{|R|} |M_k(R)| = \sum_{R \subset S} (-1)^{|R|} \binom{|SSYT(R)|}{k} = \sum_{R \subset S} (-1)^{|R|} \binom{q_R}{k}$$

where we use the convention $\binom{a}{b} = 0$ if $a < b$. Thus,

$$N_S = \sum_{k=1}^{q_S} (-1)^{k+1} |N_k(S)|$$

$$= \sum_{k=1}^{q_S} (-1)^{k+1} \sum_{R \subset S} (-1)^{|R|} \binom{q_R}{k}$$

$$= \sum_{R \subset S} (-1)^{|R|} \sum_{k=1}^{q_S} (-1)^{k+1} \binom{q_R}{k}.$$
The following Lemma gives the value of the first summation in Lemma 7.1.

Lemma 7.2. \(\sum_{R \subset S} (-1)^{|R|} = (-1)^{|S| + \|S\|}\).

Proof.

\[
\sum_{R \subset S} (-1)^{|R|} = \sum_{R_{1,1} \subset S_{1,1}} \cdots \sum_{R_{u,v} \subset S_{u,v}} (-1)^{|S_{1,1} \setminus R_{1,1}| + \cdots + |S_{u,v} \setminus R_{u,v}|}
\]

\[
= \prod_{i,j} \sum_{R_{i,j} \subset S_{i,j}} (-1)^{|S_{i,j} \setminus R_{i,j}|}
\]

\[
= \prod_{i,j} \sum_{R^*_{i,j} \not\subseteq S_{i,j}} (-1)^{|R^*_{i,j}|}
\]

\[
= \prod_{i,j} \sum_{k=0}^{(|S_{i,j}| - 1)} (-1)^k \binom{|S_{i,j}|}{k}
\]

\[
= \prod_{i,j} (-1)^{|S_{i,j}| - 1}
\]

\[
= (-1)^{|S| + \|S\|}.
\]

If \(S\) is semistandard, then for every \(R \subset S\), \(\text{SSYT}(R) \neq \emptyset\). Thus Lemmas 7.1 and 7.2 imply Lemma 4.4(i).

The following Lemma gives the value of the second summation in Lemma 7.1 when \(S\) is not semistandard. Lemmas 7.1, 7.2, and the following Lemma imply Lemma 4.4(ii).

Lemma 7.3. If \(S\) is not semistandard, then \(\sum_{R \subset S} (-1)^{|R|} = (-1)^{|S| + \|S\|}\).

Proof. Define \(Z(S) = \{R \subset S \mid \text{SSYT}(R) = \emptyset\}\), so that

\[
\sum_{R \subset S \mid \text{SSYT}(R) = \emptyset} (-1)^{|R|} = \sum_{R \in Z(S)} (-1)^{|R|}. \quad (16)
\]

We make a series of reductions.

Let \(x\) and \(y\) be the row and column numbers of a box of \(S\) where semistandardness is violated either on the top or left, but not on the right or bottom. Define

- \(Z'(S) = \{R \in Z(S) \mid \{S_{x,y,1} = R_{x,y}\}\}
- \(Z''(S) = \{R \in Z(S) \mid \{S_{x,y,1} \not\subseteq R_{x,y}\}\}
- \(Z'''(S) = \{R \in Z(S) \mid S_{x,y,1} \not\in R_{x,y}\}\).
Then \( Z(S) = Z'(S) \cup Z''(S) \cup Z'''(S) \). Consider the bijection from \( Z''(S) \) to \( Z'''(S) \) defined by \( R \mapsto R \setminus S_{x,y,1} \). The set-valued tableaux paired under this bijection contribute opposite signs to \( (17) \). Thus

\[
\sum_{R \in Z(S)} (-1)^{|R|} = \sum_{R \in Z'(S)} (-1)^{|R|}.
\]

Let \( g = S_{x,y,1} \). Define

\[
\mathcal{Y}(S) = \{ R \in Z'(S) \mid R_{x-1,y,k} < g, R_{x,y-1,l} \leq g, \text{ some } k,l \}
\]

\[
\mathcal{Y}''(S) = \{ R \in Z'(S) \mid R_{x-1,y,k} \geq g, \text{ all } k \}
\]

\[
\mathcal{Y}'''(S) = \{ R \in Z'(S) \mid R_{x,y-1,k} > g, \text{ all } k \}.
\]

Then \( Z'(S) = \mathcal{Y}(S) \cup (\mathcal{Y}''(S) \cup \mathcal{Y}'''(S)) \). Therefore

\[
\sum_{R \in Z'(S)} (-1)^{|R|} = \sum_{R \in \mathcal{Y}(S)} (-1)^{|R|} + \sum_{R \in \mathcal{Y}''(S)} (-1)^{|R|}
\]

\[
+ \sum_{R \in \mathcal{Y}'''(S)} (-1)^{|R|}.
\]

We compute the last three summations on the right hand side of \( (17) \). Assume that \( \mathcal{Y}''(S) \neq \emptyset \). Let \( S' \) be the tableaux obtained from \( S \) by removing all entries other than \( g \) from \( S_{x,y} \) and all entries less than \( g \) from \( S_{x-1,y} \). Then \( \mathcal{Y}''(S) = \{ R \subset S' \} \). Thus

\[
\sum_{R \in \mathcal{Y}''(S)} (-1)^{|R|} = \sum_{R \subset S'} (-1)^{|S' \setminus R|}
\]

\[
= \sum_{R \subset S'} (-1)^{|S' \setminus S'| + |S' \setminus R|}
\]

\[
= (-1)^{|S' \setminus S'|} \sum_{R \subset S'} (-1)^{|S' \setminus R|}
\]

\[
= (-1)^{|S' \setminus S'|} (-1)^{|S'| + |S'|}
\]

\[
= (-1)^{|S'| + |S'|},
\]

where the second to last equality follows from Lemma 7.2 and the last equality form the fact that \(|S'| = |S'|\). In a similar manner, one shows that that

\[
\sum_{R \in \mathcal{Y}'''(S)} (-1)^{|R|} = (-1)^{|S'| + |S'|} \quad \text{if } \mathcal{Y}'''(S) \neq \emptyset, \quad \text{and}
\]

\[
\sum_{R \in \mathcal{Y}'''(S) \setminus \mathcal{Y}''(S)} (-1)^{|R|} = (-1)^{|S'| + |S'|} \quad \text{if } \mathcal{Y}''(S) \cap \mathcal{Y}'''(S) \neq \emptyset.
\]

Either \( \mathcal{Y}''(S) \) or \( \mathcal{Y}'''(S) \) must be nonempty, and if both of them are nonempty, then so must be their intersection. It follows that

\[
\sum_{R \in Z'(S)} (-1)^{|R|} = (-1)^{|S'| + |S'|} + \sum_{R \in \mathcal{Y}(S)} (-1)^{|R|}.
\]
Define 
\[ X(S) = \{ R \in \mathcal{Y}(S) \mid R_{x-1,y,k} < g, R_{x,y-l,t} \leq g, \text{ all } k, l \} \].

Let \( A = \{ a_1, \ldots, a_r \} \) be the entries of \( S_{x-1,y} \) which are greater than or equal to \( g \), let \( B = \{ b_1, \ldots, b_s \} \) be the entries of \( S_{x,y-1} \) which are greater than \( g \), and let \( t = r + s \). For each \( R \in X(S) \), define \( Y'_R(S) \) to be all the set-valued tableaux obtained by adding elements of \( A \) to \( R_{x-1,y} \) and elements of \( B \) to \( S_{x,y} \). Then
\[ Y'(S) = \bigcup_{R \in X(S)} Y'_R(S), \]
and
\[
\sum_{R \in Y'(S)} (-1)^{|R|} = \sum_{R \in X(S)} \sum_{Q \in Y'_R(S)} (-1)^{|S \setminus Q|} \\
= \sum_{R \in X(S)} (-1)^{|S \setminus R|} \left( (-1)^0 \begin{pmatrix} t \cr 0 \end{pmatrix} + (-1)^1 \begin{pmatrix} t \cr 1 \end{pmatrix} + \cdots + (-1)^t \begin{pmatrix} t \cr t \end{pmatrix} \right) \\
= 0. \tag{20}
\]
Combining (16), (17), (19), and (20), we obtain the result. \( \square \)

References

[1] N. Bergeron, *A combinatorial construction of the Schubert polynomials*, J. Combin. Theory Ser. A 60 (1992), no. 2, 168–182.

[2] N. Bergeron and S. Billey, *RC-graphs and Schubert polynomials*, Experiment. Math. 2 (1993), no. 4, 257–269.

[3] S. Billey, *Kostant polynomials and the cohomology ring for G/B*, Proc. Nat. Acad. Sci. U.S.A. 94 (1997), no. 1, 29–32.

[4] A. Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. 189 (2002), no. 1, 37–78.

[5] A. S. Buch, A. Kresch, H. Tamvakis, and A. Yong, *Grothendieck polynomials and quiver formulas*, Amer. J. Math. 127 (2005), no. 3, 551–567.

[6] S. Fomin and A. N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Proceedings of the Sixth Conference in Formal Power Series and Algebraic Combinatorics, DIMACS, 1994, pp. 183–190.

[7] ______, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. 153 (1996), no. 1-3, 123–143.

[8] S. Ghorpade and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in the symplectic Grassmannian*, preprint, arXiv:math.RT/0409338.
[9] W. Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614.

[10] S. Griffeth and A. Ram, *Affine Hecke algebras and the Schubert calculus*, European J. Combin. **25** (2004), no. 8, 1263–1283.

[11] T. Ikeda, *Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian*, preprint, arXiv:math.AG/0508110.

[12] A. Knutson, E. Miller, and A. Yong, *Grobner geometry of vertex decompositions and of flagged tableaux*, preprint, arXiv:math.AG/0502144.

[13] ———, *Tableau complexes*, preprint, arXiv:math.CO/0510487.

[14] A. Knutson and E. Miller, *Grobner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318.

[15] A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003), no. 2, 221–260.

[16] V. Kodiyalam and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in Grassmannians*, J. Algebra **270** (2003), no. 1, 28–54.

[17] C. Krattenthaler, *On multiplicities of points on Schubert varieties in Grassmannians*, Sém. Lothar. Combin. **45** (2000/01), Art. B45c, 11 pp. (electronic).

[18] ———, *On multiplicities of points on Schubert varieties in Grassmannians II*, J. Algebraic Combin. **22** (2005), 273–288.

[19] V. Kreiman, *Monomial bases and applications for Richardson and Schubert varieties in ordinary and affine Grassmannians*, Ph.D. thesis, Northeastern University, 2003.

[20] ———, *Local properties of Richardson varieties in the Grassmannian via a bounded Robinson-Schensted-Knuth correspondence*, preprint, ArXiv:math.AG/0511695.

[21] ———, *Schubert classes in the equivariant K-theory and equivariant cohomology of the Lagrangian Grassmannian*, preprint, arXiv:math.AG/0602245.

[22] V. Kreiman and V. Lakshmibai, *Multiplicities of singular points in Schubert varieties of Grassmannians*, Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000), Springer, Berlin, 2004, pp. 553–563.

[23] A. Lascoux, *Anneau de Grothendieck de la variété de drapeaux*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 1–34.
[24] A. Lascoux, B. Leclerc, and J. Thibon, *Flag varieties and the Yang-Baxter equation*, Lett. Math. Phys. 40 (1997), no. 1, 75–90.

[25] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 13, 447–450.

[26] A. Lascoux and M.-P. Schützenberger, *Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. 295 (1982), no. 11, 629–633.

[27] C. Lenart, S. Robinson, and F. Sottile, *Grothendieck polynomials via permutation patterns and chains in the Bruhat order*, preprint, arXiv:math.CO/0405539.

[28] C. Lenart, *A unified approach to combinatorial formulas for Schubert polynomials*, J. Algebraic Combin. 20 (2004), no. 3, 263–299.

[29] I. G. Macdonald, *Notes on Schubert polynomials*, Publications du LACIM, Université du Québec à Montréal, 1991.

[30] A. I. Molev and B. E. Sagan, *A Littlewood-Richardson rule for factorial Schur functions*, Trans. Amer. Math. Soc. 351 (1999), no. 11, 4429–4443.

Department of Mathematics, Virginia Tech, Blacksburg, VA 24063

Email address: vkreiman@vt.edu