Effect of suction on the MHD fluid flow past a non-linearly stretching/shrinking sheet: Dual solutions

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Abstract. This paper investigated the effects of suction for incompressible magnetohydrodynamics (MHD) fluid over a non-linearly stretching/shrinking sheet. A proper similarity trans-formation has been used to reduce the system of partial differential equations to a system of ordinary differential equations, which was then solved numerically by using the bvp4c function in the MATLAB software. Dual solutions are found for certain values of the suction parameter when the sheet is shrinking. The generated results were presented and discussed in the relevance of the governing parameters.

1. Introduction

The boundary layer flow past a continuous moving surface has many industrial and engineering applications, such as the extrusion of a polymer sheet from a dye, hot rolling, metallic plates cooling and continuous casting. The study was initiated by Sakiadis \cite{1} in 1961, then followed by Crane \cite{2} where he discovered the closed analytic solution of two-dimensional Navier-Stokes equations by considering the linear relationship between the flow velocity and the distances from a fixed point. Erickson et al. \cite{3} extended \cite{1} by considering the case of non-zero transverse velocity. Further, the three-dimensional unsteady fluid flow along a continuously stretching surface has been presented by Lakshmisha et al. \cite{4} by considering the effects of the magnetic field, rate of heat and mass transfer.

Several years later, Miklavcic and Wang \cite{5} studied the steady three-dimensional boundary layer flow due to a shrinking surface, and they discovered that an adequate suction is required to sustain the flow past the shrinking sheet. This type of flow is essentially a backward flow, as discussed by Goldstein \cite{6} and it shows physical phenomena quite distinct from the forward stretching flow. Besides, they have also discovered the non-unique/multiple solutions at a specific
rate of suction. This discovery has been discussed by Davey [7] where he mentioned that a unique solution indicates the nodal point flow, non-uniqueness solutions reflect a saddle point flow, and no solution means the equations become insoluble. The analysis done in [5] was also extended in various directions for different fluids by many researchers (see [8–15]).

The papers discussed in all the above are related to the case of linearly stretching/shrinking surface. There exists another physical phenomenon in which the surface is stretched or shrunk in a nonlinear fashion. In 2001, Vajravelu [16] obtained the numerical solution of flow and heat transfer in a viscous fluid over a nonlinearly stretching sheet by using a fourth-order Runge-Kutta integration scheme and he showed that the heat flow is always from the sheet to the fluid. Later, some works on the boundary layer flow along with the nonlinearly stretching or shrinking sheet. Shit and Haldar [17] investigated the thermal radiation effects on the magnetohydrodynamic (MHD) flow and heat transfer over a nonlinear shrinking porous sheet. Recently, Khan et al. [18] studied the nonlinear radiation effects on MHD flow of nanofluid over a nonlinearly stretching/shrinking wedge, while Jusoh and Nazar [19] discussed the magnetohydrodynamic (MHD) stagnation point flow and heat transfer of an electrically conducting nanofluid over a nonlinear stretching/shrinking sheet.

The present study considers the numerical solutions of the magnetohydrodynamic two-dimensional boundary layer flow past a non-linearly stretching/shrinking sheet with suction effect, which is an extension from [16]. The present work also attempted to obtain the second solution, which has not been done in [16]. This study is essential in industrial applications such as automotive plastic fuel tanks, heating pipe and rigid packing for entrees [20]. The numerical results presented in Section 5 are generated by using the bvp4c function in the MATLAB software.

2. Problem Formulation

We consider the steady two dimensional and incompressible free conventional boundary layer flow of a Newtonian fluid over a stretching or shrinking sheet, as shown in figures 1 and 2. The pressure gradient and external force are assumed to be neglected in this problem. The flow is generated by the nonlinear stretching/shrinking sheet along the $x$-axis where $x$ is the coordinate measured along the sheet. We also assume that the variable magnetic field $\beta$ is applied normal to the sheet and the induced magnetic field is neglected. Using boundary layer approximations, the governing equations of this problem can be expressed as the following

\[ u \alpha (x) \]

\[ v \]

\[ w \]

\[ \lambda \]

\[ u \]

\[ \beta \]

\[ v_0 \]

\[ \alpha \]

\[ \beta \]

\[ x \]

\[ y \]

Figure 1. Sketch of stretching sheet.

Figure 2. Sketch of shrinking sheet.
while the corresponding boundary conditions for the current problem can be written as

\[ u = \lambda C x^n, \quad v = v_w \text{ at } y = 0, \]
\[ u \to 0 \text{ as } y \to \infty, \]

where \( u \) and \( v \) are the velocity components in \( x \) and \( y \) directions, respectively, while \( \nu \) is the kinematic viscosity, \( \rho \) is the density, \( \beta \) is the strength of the applied magnetic field, \( \sigma \) is the electrical conductivity of the fluid, \( v_w \) is the suction (\( v_w < 0 \)) and injection (\( v_w > 0 \)) parameter, and \( \lambda \) is the stretching or shrinking parameter with \( \lambda > 0 \) is for stretching surface and \( \lambda < 0 \) is for shrinking surface. Furthermore, \( m \) is the nonlinear parameter with \( m = 1 \) for the linear case and \( m \neq 1 \) is for the nonlinear case.

3. Similarity Transformation

Following [16], we look now for a similarity solution of equations (1) and (2) subjected to the initial and boundary condition (3) of the following form

\[ u = C x^n f'(\eta), \quad v = -\sqrt{\frac{C \nu (m+1)}{2}} x^{\frac{m-1}{2}} \left[ f(\eta) + \left( \frac{m-1}{m+1} \right) \eta f'(\eta) \right], \]
\[ \eta = y \sqrt{\frac{C (m+1)}{2 \nu}} x^{\frac{m-1}{2}}, \]

where primes denote derivatives with respect to \( \eta \). Furthermore, we assume

\[ s = -\frac{v_w}{\sqrt{\frac{C \nu (m+1)}{2}} x^{\frac{m-1}{2}}}, \]

where \( s > 0 \) corresponds to suction while \( s < 0 \) corresponds to the fluid withdrawal, respectively.

Using (4), equation (1) is satisfied, while equation (2) is transformed into the following ordinary differential equation

\[ f''' + ff'' - \frac{2m}{m+1} f'^2 - M f' = 0, \]

along with the corresponding boundary conditions

\[ f(0) = s, \quad f'(0) = \lambda, \quad f'(\eta) \to 0 \text{ as } \eta \to \infty, \]

where \( M = \frac{2\sigma \beta^2}{\mu^2 (m+1)} \) is the magnetic parameter, while \( s \) is the mass transfer parameter with \( s > 0 \) denotes suction, while \( s < 0 \) refers to injection and \( s = 0 \) is for nonpermeable surface.

The main physical quantity of interest is the reduced skin friction coefficient in \( x \)-direction which is defined as

\[ Cf_x = -2 \sqrt{\frac{m+1}{2}} R e_x^{-\frac{1}{2}} f''(0), \]

where \( R e_x = \frac{U_x \nu}{\nu} \).
4. Application of the bvp4c function Method

As mentioned in the previous section, we implement the bvp4c function to solve the resulting non-linear ordinary differential equation (6) with boundary condition (7). The MATLAB solver bvp4c function uses the collocation method and effective in solving the boundary value problem [21]. To achieve this, we first write equation (6) as a system of first order equation in the following form

\[\begin{align*}
  f(\eta) &= y_1, \\
  f'(\eta) &= y_2, \\
  f''(\eta) &= y_3, \\
  f'''(\eta) &= \frac{2m}{m+1} y_2^2 + My_2 - y_1y_3,
\end{align*}\]

while the boundary condition (7) is transformed to

\[\begin{align*}
  f(0) &= ya_1, \\
  f'(0) &= ya_2, \\
  f'(\infty) &= yb_2, \\
  ya_1 &= s, \\
  ya_2 &= \lambda, \\
  yb_2 &= 0.
\end{align*}\]

Aside from equations (9) and (10), a good initial guess is necessary to obtain the dual solutions. Dual solutions comprise of upper branch solution and lower branch solution. The upper branch solution is a solution which converged asymptotically with a thin boundary layer, whereas the lower branch solution converged asymptotically with the thicker boundary layer. The relative tolerance has been fixed to $1 \times 10^{-10}$ throughout the computation process.

5. Results and Discussion

The nonlinear ordinary differential equation (6) and its boundary condition (7) are solved numerically using the bvp4c function in MATLAB, as stated in the previous section. The validity of the method is measured by comparing the present numerical results with the previous study. The following Table 1 shows an excellent comparison of the present study with the previous results obtained in [16], which proves that the present method is accurate and the results are correct.

| $m$ | Vajravelu [16] | Current |
|-----|---------------|---------|
| 0   | -             | -0.6276 |
| 1   | -1            | -1      |
| 5   | -1.1945       | -1.1945 |
| 7   | -             | -1.2169 |
| 9   | -             | -1.2301 |
| 10  | -1.2348       | -1.2349 |

Figure 3 shows the effects of the magnetic parameter $M$ towards the skin friction coefficient which is denoted by $f''(0)$ along with a stretching/shrinking sheet. The upper branch solution of $f''(0)$ increases as $M$ increases, while the lower branch solution of $f''(0)$ decreases as $M$ increases. An increment in $M$ that increases the electromagnetic forces creates a stronger Lorentz force which induces the flow to be more resistant. As $M$ increases, the flow is decelerated, yet the boundary layer is thinner (see figure 4) because the sheet is permeable. Hence, the fluid velocity is enhanced (see figure 4), and the values of $f''(0)$ increase along with the increment in $M$. However, the lower branch solution exhibits the decrement of $f''(0)$ as $M$ increases along with a penetrable shrinking sheet. An increment in $M$ reduces the fluid velocity as in figure 4, because the sheet...
may exceed the limit of its permeability. Hence, the fluid velocity decreases and reduces the value of $f''(0)$ when $M$ increases. The opposite trend which been presented by the lower branch solution reflects the flow with separation, and it is noticed that the higher values of $M$ increase the availability of the lower branch solution when the sheet is stretching.

Figure 3. Variations of $f''(0)$ with $\lambda$ for some values of $M$

Figure 4. Velocity profiles $f'(\eta)$ for different values of $M$

Figure 5 displays the effects of the velocity power index $m$ on $f''(0)$. The upper branch solution conveys the increment in $m$ depreciates the value of $f''(0)$ as the sheet is shrinking. The nonlinearity of the shrinking sheet reduces the fluid velocity as in figure 6, and $f''(0)$ decreases. However, the lower branch solution of $f''(0)$ increases as $m$ increases and the velocity profiles as in figure 6 support the trend as the fluid velocity increases as $m$ increases.
Figure 5. Variations of $f''(0)$ with $\lambda$ for some values of $m$

Figure 6. Velocity profiles $f'(|\eta|)$ for different values of $m$

Next, figure 7 unveils the variations of the mass transfer parameter $s$ towards $f''(0)$. The present study considers the fluid flow with influences of the suction ($s > 0$). The upper branch solution of $f''(0)$ shows an increment as the value of $s$ increases from 2.5 to 3.5. The higher intensity of suction along with a shrinking sheet essentially increases the fluid velocity since the state of suction traps the slow moving molecules, and results in the enhancement in $f''(0)$. The velocity profiles as in figure 8 evident this as the velocity of the upper branch solution increases as $s$ increases. The lower branch solution, by contrast, shows the decrement of $f''(0)$ as $s$ increases, and it is proved by the velocity profiles of the lower branch solution as in figure 8, where the fluid velocity decreases as $s$ increases. It is possible for the lower branch solution to reflect such a trend due to the state of shrinking which limits the permeability of the sheet.
Figure 7. Variations of $f''(0)$ with $\lambda$ for some values of $s$

Figure 8. Velocity profiles $f'(\eta)$ for different values of $s$

On the other hand, figure 9 views the effects of $M$ with respect to $f''(0)$ and $s$. As the effect of suction gets stronger along a sheet that is shrinking at the limit of $-2$, an increment of $M$ reports the upper branch solution of $f''(0)$ gradually increases. Although a stronger effect of $M$ forms the retarded flow, the state of suction found to improve the fluid velocity and increases the value of $f''(0)$. It is also perceived that the lower branch solution explains that $f''(0)$ decreases as $M$ increases. The state of a nonlinearly shrinking sheet may be causing the shear stress along with the sheet decreases and lead to the formation of backward flow.
Figure 9. Variations of $f''(0)$ with $s$ for some values of $M$

Figure 10 illustrates the increment of the upper branch solution of $f''(0)$ as $m$ decreases along with a permeable shrinking sheet. When the intensity of suction gets stronger, $f''(0)$ found to be increasing along with a nonlinear shrinking sheet. Meanwhile, the lower branch solutions when $s > 2.5$ as in figure 10 depicts the uneven trend of $f''(0)$ as $m$ varies.

Figure 10. Variations of $f''(0)$ with $s$ for some values of $m$

Figure 11 expresses the variation of $\lambda$ with respect to $f''(0)$ and $s$. It is found that as the sheet is shrinking, the value of $f''(0)$ increases. Both solutions show the same trend. As been mentioned before, the state of shrinking initiates the backward flow and increases the skin friction along with the sheet. The velocity profiles as in figure 12 clarifies this by showing an increment in the fluid velocity as the sheet is shrinking.
Figure 11. Variations of $f''(0)$ with $s$ for some values of $\lambda$

Figure 12. Velocity profiles $f'(\eta)$ for different values of $\lambda$

6. Conclusions
The present study was devoted to solving boundary layer flow problem over a permeable stretching/shrinking sheet under the influence of the magnetohydrodynamics (MHD). Interestingly, this paper identified the dual solutions, namely upper branch solution and lower branch solution and explained its behaviour physically as the governing parameters vary. When the magnetic parameter $M$ and the mass transfer parameter $s$ increases, the upper branch solution shows increment while the lower branch solution conveys decrement. The upper branch solution decreases while the lower branch solution increases when the velocity power index $m$ increases.

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