GROWTH OF BILINEAR MAPS

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ABSTRACT. For a bilinear map \( * : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) of nonnegative coefficients and a vector \( s \in \mathbb{R}^d \) of positive entries, among an exponentially number of ways combining \( n \) instances of \( s \) using \( n - 1 \) applications of \( * \) for a given \( n \), we are interested in the largest entry over all the resulting vectors. An asymptotic behavior is that the \( n \)-th root of this largest entry converges to a growth rate \( \lambda \) when \( n \) tends to infinity. In this paper, we prove the existence of this limit by a special structure called linear pattern. We also pose a question on the possibility of a relation between the structure and whether \( \lambda \) is algebraic.

1. Introduction

Given a binary operation \( * \) and a fixed operand \( s \), we have a variety of ways to combine \( n \) instances of \( s \) using \( n - 1 \) applications of \( * \). The results may vary as the operation \( * \) is not necessarily commutative or associative. However, we might still expect that the “largest value” of all the combinations does not grow too arbitrarily. A problem of this type was posed in [1] by Günter Rote, where \( * \) is a bilinear map of nonnegative coefficients and \( s \) is a vector of positive entries, both in the same vector space. In this paper, the largest entry of a resulting vector will be shown to be of exponential order with a fixed growth rate.

Consider a vector \( s \in \mathbb{R}^d \) of all positive entries \( s_i \) and a bilinear map \( * : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) represented by nonnegative coefficients \( c_{i,j}^{(k)} \) in the way: If \( v = u * w \) then \( v_k = \sum_{i,j} c_{i,j}^{(k)} u_i w_j \).

Let \( A_n \) for an integer \( n \geq 1 \) be the set of vectors obtained by applying \( n - 1 \) instances of \( * \) to \( n \) instances of \( s \), that is \( A_1 = \{s\} \) and

\[
A_n = \bigcup_{1 \leq m \leq n-1} \{x * y : x, y \in A_n \times A_{n-m}\}.
\]

Let \( g(n) \) denote the largest entry over all vectors in \( A_n \), that is

\[
g(n) = \max\{x_i : x \in A_n, 1 \leq i \leq d\}.
\]

For later convenient usage we also denote by \( g_i(n) \) the largest \( i \)-th entry over all vectors in \( A_n \), that is

\[
g_i(n) = \max\{x_i : x \in A_n\}.
\]

The growth rate \( \lambda \) of the system is defined as

\[
\lambda = \lim_{n \to \infty} \sqrt[n]{g(n)}.
\]

We will prove the validity of this limit and give further discussion after introducing some definitions related to a special structure called linear pattern.

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The computation of $A_n$ can be related to the binary trees of $n$ leaves. For each binary tree of $n$ leaves, we assign an element in $A_n$ as follows: If the tree is only a single leaf then the result is $s$; otherwise, the result is $x \ast y$ where $x, y$ are the results corresponding to the left and right branches of the root, respectively. Every element of $v \in A_n$ can be computed from such a binary tree. Given a tree $T$, the corresponding vector in $A_n$ is said to be the vector obtained by $T$. Although in principle there may be more than one trees resulting in the same $v$, we assign some arbitrary tree for each $v$. Later arguments are independent of the choice.

We call a pair of a tree $T$ and a marked leaf $\ell$ of $T$ a linear pattern $P = (T, \ell)$. This definition has some interesting properties.

**Proposition 1.** Given a linear pattern $P = (T, \ell)$, if one replaces $s$ by $u$ specifically only for the leaf $\ell$, then the vector $v$ obtained by $T$ is related to $u$ by a matrix $M = M(P)$ such that

$$v = Mu.$$ 

This fact follows from a property of bilinear maps: If we fix one of the two terms of the input, the new map will be linear, that is:

$$\ast y(x) = x \ast y \text{ and } \ast x(y) = x \ast y$$

are both linear.

A sequence of trees $\{T^t\}_{t \geq 1}$ is said to be generated by a pattern $(T, \ell)$ if $T^1 = T$ and $T^t$ for $t \geq 2$ is obtained from $T$ by replacing $\ell$ by $T^{t-1}$ (see Figure 1 for example).

**Proposition 2.** For a linear pattern $P = (T, \ell)$, let $h(t)$ be the largest entry of the vector obtained from the tree $T^t$, then the so-called rate of pattern $P$

$$\lambda_P = \lim_{t \to \infty} \sqrt[h(t)]{}$$

is valid and equals to the dominant eigenvalue of the matrix $M(P)$.

This is a well known fact and that eigenvalue is often called Perron-Frobenius eigenvalue or spectral radius.

The tree $T^t$ has $t(|T| - 1) + 1$ leaves where $|T|$ is the number of leaves of $T$. While $h(t)$ is a lower bound for a subsequence of $g(n)$, the corresponding lower bound for the rate should be the $(|T| - 1)$-th root $\tilde{\lambda}_P$ of $\lambda_P$ instead.

**Proposition 3.** For every linear pattern $P = (T, \ell)$,

$$\liminf_{n \to \infty} \sqrt[g(n)]{} \geq \tilde{\lambda}_P.$$ 

Indeed, let $m = |T| - 1$, for each $n = mp + q$ ($0 \leq q < m$), consider the tree obtained from $T^p$ by replacing the marked leaf by any tree of $q$ leaves (whose evaluation can be seen to be bounded). It is not hard to see that the $n$-th root of the largest entry obtained from these trees converges to $\tilde{\lambda}_P$.

Moreover, we give the following stronger conclusion, which serves as the proof of the validity of $\lambda$.

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1 Note that we sacrifice precision for a compact notation. In fact, $M$ also depends on $\ast, s$, which are fixed from the beginning while only pattern $P$ is varied. Therefore, $M(P)$ can be clearly understood from context as $M(P, \ast, s)$

2 $\lambda_P$ is a shorten form of a more precise notation $\lambda_P^{\ast, s}$, see footnote [1]
Theorem 1. The $n$-th root of $g(n)$ converges when $n$ tends to infinity and the limit is the supremum of $\bar{\lambda}_P$ over all patterns $P$, that is

$$\lambda = \lim_{n \to \infty} \sqrt[n]{g(n)} = \sup_P \bar{\lambda}_P.$$ 

There are some cases that $\lambda > \bar{\lambda}_P$ for all $P$. The following system is one example.

Theorem 2. If $s = (1,1)$ and

$$x * y = (x_1y_1 + x_2y_2, x_2y_2),$$

then $\lambda > \bar{\lambda}_P$ for every $P$.

The optimal trees for this system are perfect binary trees (for $n$ being powers of 2), which are not recognized by any linear pattern. Actually the system in the above theorem was studied in a different formulation (see [2]) and the growth rate was shown to be

$$\lambda = \exp\left(\sum_{i \geq 1} \frac{1}{2^i} \log(1 + \frac{1}{x_i^2})\right) = 1.502836801 \ldots$$

where $x_n$ is a sequence with $x_0 = 1$ and $x_{k+1} = 1 + x_k^2$ for $k \geq 0$.

This constant has also been studied as the rates of quadratic recurrences and $x_n$ (and $g(2^n)$) is the number of binary trees of heights at most $n + 1$ (Sequence A003095). For more information on this and other sequences of the type, see [3].

The growth rate $\lambda$ in this system seems to be not an algebraic number. Since the growth rate is algebraic whenever a pattern recognizes it (and the coefficients and the entries are integers), Theorem 2 suggests the question of the other direction:

Question 1. Suppose the coefficients of $*$ and the entries of $v$ are all integers. Is it true that: If $\lambda$ is algebraic, then there exists a pattern $P$ such that $\bar{\lambda}_P = \lambda$?

It makes sense to give an example where a pattern recognizes the rate, and hence, the growth rate is algebraic.

Theorem 3. If $s = (1,1)$ and

$$x * y = (x_1y_2 + x_2y_1, x_1y_2),$$

then the growth rate $\lambda$ is the golden ratio $\phi$, which is recognized by a pattern. In particular, $g_1(n) = F_{n+1}$ and $g_2(n) = F_n$, where $F$ is the Fibonacci sequence starting with $F_1 = F_2 = 1$.

The optimal trees are binary trees where every left (or right) branch of every non-leaf vertex is just a leaf. The proof uses some inequalities involving the elements of the Fibonacci sequence, which are interesting on its own.

The readers may notice that although the two examples in Theorem 2 and Theorem 3 just slightly differ from each other, the growth rates and the patterns are of quite different natures.

A more sophisticated example can be found in [1] with growth rate $\sqrt[3]{95}$ and a complex linear pattern. It actually solves a problem on the maximal number of minimal dominating sets in a tree. One can also find a proof of the validity of $\lambda$ for that particular case there (by showing that $g(n)$ is supermultiplicative and then applying Fekete’s lemma [4]).

We give the proofs of Theorems 1, 2, 3 in Sections 2, 4, 5, respectively. Section 3 proves some lemmas used in Section 2.
Consider the dependency graph that is a directed graph whose vertices are \( \{1, \ldots, d\} \); there is a directed edge from \( k \) to \( i \) if and only if \( c_{ij}^{(k)} \) or \( c_{ji}^{(k)} \) is positive, where \( c_{ij}^{(k)} \) are the coefficients of \( * \). We say \( k \) depends on \( i \) for such an edge \( ki \). In some cases we need to say specifically that \( k \) left depends (resp. right depends) on \( i \) if \( c_{ij}^{(k)} \) (resp. \( c_{ji}^{(k)} \)) is positive.

The dependency graph can be partitioned into strongly connected components, which can be partially ordered. Component \( C_1 \) is said to be greater than Component \( C_2 \) if either there is a directed edge \( ij \) for \( i \in C_1, j \in C_2 \), or there exists another component \( C_3 \) so that \( C_1 > C_3 > C_2 \).

We give some useful lemmas, which will be proved later in Section 3.

**Lemma 1.** For every \( i \), \( g_i(n) \) is at least a constant time of \( g_i(n+1) \).

**Lemma 2.** If \( i, j \) are of the same component, then

\[
\liminf_{n \to \infty} \sqrt[n]{g_i(n)} = \liminf_{n \to \infty} \sqrt[n]{g_j(n)},
\]

\[
\limsup_{n \to \infty} \sqrt[n]{g_i(n)} = \limsup_{n \to \infty} \sqrt[n]{g_j(n)}.
\]

If \( i \in C_1, j \in C_2 \) and \( C_1 < C_2 \) then

\[
\liminf_{n \to \infty} \sqrt[n]{g_i(n)} \leq \liminf_{n \to \infty} \sqrt[n]{g_j(n)},
\]

\[
\limsup_{n \to \infty} \sqrt[n]{g_i(n)} \leq \limsup_{n \to \infty} \sqrt[n]{g_j(n)}.
\]

**Lemma 3.** Given a pattern \( P = (T, \ell) \) with its matrix \( M \). Let \( i, j \) be two vertices of the same component, then there exists a pattern \( P' = (T', \ell') \) with the difference in the number of leaves \( |T'| - |T| \) bounded and \( \lambda_{P'} \) at least a constant time of \( M_{i,j} \).

**Lemma 4.** If \( M = M(P) \) is the matrix for a pattern \( P = (T, \ell) \) with \( T \) having \( n \) leaves, then for every \( i, j \), the value \( M_{i,j} \) is at most a constant time of \( g_i(n) \) for every \( i \in C \).

**Lemma 5.** If a component \( C \) is greater than every other component, then \( g_i(n) \) is at least a constant time of \( g(n) \) for every \( i \in C \).

**Lemma 6.** For a tree of \( n > 1 \) leaves, there is a subtree of \( m \) leaves such that \( n/3 \leq m \leq 2n/3 \).

We are now ready to prove Theorem 1.

Take any component \( C \), we investigate the \( C \)-subsystem, which is the system after excluding all but the dimensions in the components smaller than or equal to \( C \). This restriction actually does not lose the generality but gives a conclusion on the convergence of \( \sqrt[n]{g_i(n)} \) for every \( i \), as we will show later.

In the \( C \)-subsystem, let \( \lambda_P^C \) and \( \bar{\lambda}_P^C \) denote the rates with respect to the \( C \)-subsystem.

It can be seen that \( \liminf_{n \to \infty} \sqrt[n]{g_i(n)} \geq \sup_{P} \lambda_P^C \) for \( i \in C \) by Proposition 3 and Lemma 5. We prove the other direction:

\[
\limsup_{n \to \infty} \sqrt[n]{g_i(n)} \leq \sup_P \bar{\lambda}_P^C.
\]
Suppose $C$ is a component $C_0$ such that
\begin{equation}
\limsup_{n \to \infty} \sqrt[n]{g_i(n)} > \limsup_{n \to \infty} \sqrt[n]{g_j(m)}
\end{equation}
for every $C' < C_0, i \in C_0, j \in C'$.

Let $i$ be a vertex in $C_0$ and denote $\theta = \limsup_{n \to \infty} \sqrt[n]{g_i(n)}$.

Then for every $\epsilon > 0$, there exist an $n_{\epsilon}$ such that for every $n > n_{\epsilon}$, $g_i(n) < (\theta + \epsilon)^n$, and for every $N$ there exists $n > N$ such that $g_i(n) > (\theta - \epsilon)^n$.

Let $k$ be a vertex and denote $\theta' = \limsup_{n \to \infty} \sqrt[n]{g_k(n)}$.

Then for every $\epsilon > 0$, there exists an $n_{\epsilon}$ such that for every $m > n_{\epsilon}$, $g_k(m) < (\theta' + \epsilon)^m$.

Fix $\epsilon$, choose $n_{\epsilon}$ that works for $i$ and every $k$, that is for all $n > n_{\epsilon}$ we have $g_i(n) < (\theta + \epsilon)^n$ and $g_k(n) < (\theta' + \epsilon)^n$. Let $N = 3n_{\epsilon}$ and take any $n > N$ such that $g_i(n) > (\theta - \epsilon)^n$.

Consider the tree $T$ corresponding to $g_i(n)$. Take a subtree $T_2$ of $m$ leaves with $n/3 \leq m \leq 2n/3$ (by Lemma 6), and combine any leaf $\ell_2$ among these $m$ leaves with $T_2$ to obtain a pattern $P_2 = (T_2, \ell_2)$. Denote by $\ell_1$ the root of $T_2$, and by $T_1$ the tree obtained from $T$ after contracting $T_2$ to $\ell_1$. We have another pattern $P_1 = (T_1, \ell_1)$. Also, consider the pattern $P = (T, \ell)$ for $\ell = \ell_2$.

Let the matrices for $P, P_1, P_2$ be $M, A, B$, respectively. Clearly, $M = AB$.

Since $g_i(n) = \sum_j M_{i,j} s_j$, there exists some $j$ such that

$$M_{i,j} \geq \text{const} g_i(n).$$

Since $M_{i,j} = \sum_k A_{i,k} B_{k,j}$, there exists $k$ such that

$$A_{i,k} B_{k,j} \geq \text{const} M_{i,j} \geq \text{const} g_i(n) \geq \text{const}(\theta - \epsilon)^n.$$

By Lemma 4 and by the definition of $\theta'$ with $m > n_{\epsilon}$,

$$B_{k,j} \leq \text{const} g_k(m) \leq \text{const}(\theta' + \epsilon)^m.$$

It means $A_{i,k}$ is at least a constant time of

$$\frac{(\theta - \epsilon)^n}{(\theta' + \epsilon)^m} \geq \left(\frac{\theta - \epsilon}{\theta' + \epsilon}\right)^m (\theta - \epsilon)^{n-m} \geq \left(\sqrt[\theta' - \epsilon]{\frac{\theta - \epsilon}{\theta' + \epsilon}}\right)^{n-m} (\theta - \epsilon)^{n-m} \geq \left(\sqrt[\theta' + \epsilon]{\frac{\theta - \epsilon}{\theta' + \epsilon}}\right)^{n-m} (\theta + \epsilon)^{n-m}.$$

The inequality step is due to $m \geq (n - m)/2$.

Suppose $k$ is of a smaller component than $C_0$, that is $\theta' < \theta$.

When $\epsilon$ is small and $n$ is large enough, the value of $A_{i,k}$ will be not bounded by a constant time of $(\theta + \epsilon)^{n-m}$ due to $(\theta - \epsilon)/(\theta' + \epsilon) > 1$ but $(\theta - \epsilon)/(\theta + \epsilon)$ tending to 1 when $\epsilon$ tends to 0. However, $A_{i,k} \leq \text{const} g_i(n-m+1) \leq \text{const} g_i(n-m) \leq \text{const}(\theta + \epsilon)^{n-m}$ by Lemma 4 and Lemma 1 a contradiction (note that $T_1$ has $n - m + 1$ leaves).
Therefore, $i$ and $k$ are of the same component, which means $B_{k,j}$ is at most a constant time of $(\theta + \epsilon)^m$. It follows that $A_{i,k}$ is at least a constant time of $(\theta - \epsilon)^n / (\theta + \epsilon)^m$.

For every $\epsilon' > 0$ there exists $\epsilon > 0$ such that

\[
\frac{(\theta - \epsilon)^n}{(\theta + \epsilon)^m} > (\theta - \epsilon')^{n-m}.
\]

By Lemma 3, the lower bound of $A_{i,k}$ shows that there exists a pattern $P'$ having $\overline{\lambda}^C_{P'}$ greater than a number arbitrarily close to $\theta$ from below when $\epsilon'$ gets smaller. In other words,

\[
\limsup_{n \to \infty} \sqrt[n]{g_i(n)} \leq \sup_{P} \overline{\lambda}_P^C.
\]

It means $\lim_{n \to \infty} \sqrt[n]{g_i(n)}$ exists for every $i \in C_0$ since the limit superior and the limit inferior are equal.

We have shown that $\sqrt[n]{g_i(n)}$ converges to a limit for every $i$ in a component satisfying the requirement $[1]$. It remains to consider components $C$ not satisfying the requirement. For such a component $C$, there is a component $C_0 < C$ satisfying that requirement and

\[
\limsup_{n \to \infty} \sqrt[n]{g_i(n)} = \limsup_{n \to \infty} \sqrt[n]{g_k(n)} \text{ for any } i \in C_0 \text{ and } k \in C.
\]

By Lemma 2,

\[
\lim_{n \to \infty} \sqrt[n]{g_k(n)} \geq \liminf_{n \to \infty} \sqrt[n]{g_i(n)} = \limsup_{n \to \infty} \sqrt[n]{g_i(n)} = \limsup_{n \to \infty} \sqrt[n]{g_k(n)}.
\]

It means $\lim_{n \to \infty} \sqrt[n]{g_k(n)}$ exists because the limit superior and limit inferior are equal. The existence of \(\lambda = \lim_{n \to \infty} \sqrt[n]{g(n)} = \max_k \lim_{n \to \infty} \sqrt[n]{g_k(n)}\) follows from the existence of $\lim_{n \to \infty} \sqrt[n]{g_k(n)}$ for every $k$.

This limit $\lambda$ equals to the supremum of $\overline{\lambda}_P$ over all patterns $P$ because for $i \in C$ satisfying $\lim_{n \to \infty} \sqrt[n]{g_i(n)} = \lambda$, we have

\[
\sup_{P} \overline{\lambda}^C_P \leq \sup_{P} \overline{\lambda}_P \leq \lim_{n \to \infty} \sqrt[n]{g(n)} = \lim_{n \to \infty} \sqrt[n]{g_i(n)} = \sup_{P} \overline{\lambda}^C_P.
\]

### 3. Proofs of the lemmas

**Proof of Lemma 1**. Let $T$ be the tree corresponding to $g_i(n + 1)$. Take any subtree $T_0$ of 2 leaves, and replace it by a leaf, denoted by $\ell$, to obtain a new tree $T'$ of $n$ leaves.

Let $v, v'$ be the vector obtained by the trees $T, T'$, respectively.

Let $M$ be the matrix for the pattern $(T', \ell)$, that is $v' = Ms$ for the vector $s$ at the leaf $\ell$. If the leaf $\ell$ is replaced by the tree $T_0$, we have the relation $v = Mu$ where $u = s * s$ is the vector obtained by $T_0$.

Since $u_i \leq g(2)$ and $s_i \geq \min_k s_k$ for every $i$,

\[
\frac{u_i}{s_i} \leq \frac{g(2)}{\min_k s_k}.
\]

We can write $u \leq (g(2) / \min_k s_k)s$. 

6
Together with $v = Mu$ and $v' = Ms$, we have
\[ \frac{v_i}{v'_i} \leq \frac{g(2)}{\min_k s_k}. \]

The conclusion follows due to $v'_i \leq g_i(n)$.

**Remark 1.** It is possible to obtain a more general conclusion by choosing $T_0$ of more than two leaves. However, we cannot guarantee the size of $T_0$ in this case but only some bound on it (as in Lemma 6). The question is: Is it true that $g(n) \leq \text{const } g(p)g(q)$ for every $p, q \geq 1, p + q = n$? The validity of $\lambda$ just follows if this is true (by Fekete’s lemma).

Before proving the remaining lemmas, we give the following useful corollary of Lemma 1.

**Corollary 1.** Given a fixed $d$, for every $i$, $g_i(n)$ is at least a constant time of $g_i(n + d)$.

**Proof of Lemma 2.** Suppose there is an edge $ki$ in the dependency graph. For each $n$, let $T_0$ be the tree corresponding to $g_i(n)$. Consider the tree $T'$ of $n + 1$ leaves where the left (resp. right) branch of the root is $T$ if $k$ left (resp. right) depends on $i$, and the other branch is just a single leaf. It can be seen from $T'$ that
\[ g_k(n + 1) \geq \text{const } g_i(n). \]

Suppose $i, j$ be two vertices so that there exists a path of length $d$ from $j$ to $i$, we have
\[ g_j(n + d) \geq \text{const } g_i(n). \]

By Corollary 1, $g_j(n + d) \leq \text{const } g_j(n)$, therefore,
\[ g_j(n) \geq \text{const } g_i(n). \]

If there is a path from $j$ to $i$, it follows from Equation (2) that
\[ \liminf_{n \to \infty} \sqrt[n]{g_i(n)} \leq \liminf_{n \to \infty} \sqrt[n]{g_j(n)}, \]
\[ \limsup_{n \to \infty} \sqrt[n]{g_i(n)} \leq \limsup_{n \to \infty} \sqrt[n]{g_j(n)}. \]

It is indeed the case when $i \in C_1, j \in C_2$ and $C_1 < C_2$.

If $i, j$ are of the same component, then there exist a path from $i$ to $j$ and also a path from $j$ to $i$. Apply the above inequalities to both $i, j$ and $j, i$, we obtain
\[ \liminf_{n \to \infty} \sqrt[n]{g_i(n)} = \liminf_{n \to \infty} \sqrt[n]{g_j(n)}, \]
\[ \limsup_{n \to \infty} \sqrt[n]{g_i(n)} = \limsup_{n \to \infty} \sqrt[n]{g_j(n)}. \]

**Proof of Lemma 3.** Let the path from $j$ to $i$ be $k_0, \ldots, k_d$ for $k_0 = j, k_d = i$ and the length of the path $d$. Construct the trees $T_0, \ldots, T_d$ such that $T_d = T$, and for $t < d$, one of the two branches of the root of $T_t$ is $T_{t+1}$ and the other is just a single leaf. If $k_t$ left (right) depends on $k_{t+1}$ then the branch of $T_{t+1}$ is on the left (right) in $T_t$.
Let \( P' \) be the pattern \((T', \ell')\) for \( T' = T_0, \ell' = \ell \), and \( M' \) the matrix of \( P' \). We can see that \(|T'| - |T|\) is bounded and \( M'_{jj} \) is at least a constant time of \( M_{ij} \). It follows that \( \lambda_{P'} \) is at least a constant time of \( M_{ij} \).

Proof of Lemma 4. Let the vector obtained from \( T \) be \( v = Ms \). Since \( v_i = \sum_j M_{ij} s_j \) and \( v_i \leq g_i(n) \), the value \( M_{ij} \) for any \( j \) is at most a constant time of \( g_i(n) \).

Proof of Lemma 5. Since there is a path from \( i \) to \( j \) for every \( i \in C \) and any other \( j \), by the same argument as in Equation (2), we have

\[
g_i(n) \geq \text{const} g_j(n).
\]

If we do not fix \( j \), we still have

\[
g_i(n) \geq \text{const} \max_j g_j(n).
\]

The conclusion follows since for each \( n \), \( g(n) = g_j(n) \) for some \( j \).

Proof of Lemma 6. Assume there is no such subtree. Consider the two branches of the root, one of them has less than \( n/3 \) leaves while the other has more than \( 2n/3 \) leaves. Consider the two sub-branches of the latter branch. Each of these sub-branches must have less than \( n/3 \) leaves. Totally, the whole tree has less than \( n/3 + n/3 + n/3 = n \) leaves, a contradiction.

4. PROOF OF THEOREM 2

We know (see [2]) that the growth rate for this system can be attained by perfect binary trees (no linear pattern recognizes them). We will show further that there is no linear pattern of the rate by verifying that given any linear pattern, we can construct another pattern of a higher rate.

Consider a pattern \( P = (T, \ell) \) with the corresponding matrix

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]

It is verifiable that \( a, b \) are always at least 1 while \( c \) is always 0 and \( d \) is always 1 (the readers can check themselves or just see it by the manipulations of patterns and matrices throughout the proof). The dominant eigenvalue of the matrix is always \( a \) since \( c = 0 \) and \( d = 1 \). The rate is therefore the \( m \)-th root of \( a \) where \( m \) is one less than the number of leaves in \( T \).

Suppose we have two patterns of \( P_1 = (T_1, \ell_1) \) and \( P_2 = (T_2, \ell_2) \) with their matrices respectively

\[
\begin{bmatrix}
a_1 & b_1 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
a_2 & b_2 \\
0 & 1
\end{bmatrix}.
\]

Their product is

\[
\begin{bmatrix}
a_1a_2 & a_1b_2 + b_1 \\
0 & 1
\end{bmatrix}.
\]

If we construct a new pattern \( P = (T, \ell) \) with \( T \) obtained from \( T_1 \) by replacing \( \ell_1 \) by \( T_2 \) and let \( \ell \) be \( \ell_2 \), then

\[
\tilde{\lambda}_P \leq \max\{\tilde{\lambda}_{P_1}, \tilde{\lambda}_{P_2}\},
\]

since the dominant eigenvalue of the product is \( a_1a_2 \). It means that we do not need to consider patterns that are decomposable into two patterns in the above way for a candidate
of the best rate. In other words, the maximal $\bar{\lambda}_P$ if exists can be found among the patterns where one branch of the root is just a leaf, which is also the marked leaf.

Let the other branch than the branch of the marked leaf has the evaluation $(a,1)$, then the matrix of the pattern

$$\begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix}$$

has the dominant eigenvalue $a$.

So, the growth rate of the system is

$$\lambda = \sup_n \sqrt[n]{g(n)}.$$

However, we do not have any $n$ so that $\sqrt[n]{g(n)} = \lambda$. Suppose the contrary, let $T$ be a tree of $n$ leaves whose first entry has value $g(n)$ attaining the maximum rate $\lambda$. Let $T'$ be a tree where each branch of the root is a copy of $T$. The first entry of the evaluation of $T'$ is $(g(n))^2 + 1$, but $T'$ has $2^n$ leaves, hence $T'$ attains a higher rate than $\lambda$, a contradiction.

So, for every linear pattern, we always obtain another pattern of a higher rate. The conclusion of Theorem 2 follows.

5. Proof of Theorem 3

We will show that the growth rate of this system is the golden ratio by showing that $g_1(n), g_2(n)$ are respectively $F_{n+1}, F_n$, where $F$ is the Fibonacci sequence with starting elements $F_0 = 0, F_1 = 1, F_2 = 1$.

It can be seen that $g_1(n) \geq F_{n+1}$ and $g_2(n) \geq F_n$ for every $n \geq 1$ since the vector $(F_{n+1}, F_n)$ is the evaluation of the tree $T^{n-1}$ for the pattern $(T,\ell)$ where $T$ is the tree of two leaves, any of them can be chosen as the marked leaf $\ell$.

In order to show that they are also the upper bounds, we prove following lemma.

Lemma 7. Let $F$ be the Fibonacci sequence with starting elements $F_0 = 0, F_1 = 1, F_2 = 1$, then the following inequalities

$$F_pF_{q-1} + F_{p-1}F_q \leq F_{p+q-1},$$

$$F_pF_q \leq F_{p+q-1}$$

hold for every $p, q \geq 1$.

Proof. The conclusion holds for any $(p,q) \in \{(1,2) \times \mathbb{N}^+ \cup (\mathbb{N}^+ \times \{1,2\}\}$, i.e. one of the four conditions $p = 1, p = 2, q = 1, q = 2$ holds.

As for the first inequality, if $p = 1$ (similarly for $q = 1$), then the inequality is equivalent to $F_{q-1} \leq F_q$. If $p = 2$ (similarly for $q = 2$), then it is equivalent to $F_{q-1} + F_q \leq F_{q+1}$.

As for the second inequality, if $p = 1$ (similarly for $q = 1$), then the inequality is equivalent to $F_q \leq F_q$. If $p = 2$ (similarly for $q = 2$), then it is equivalent to $F_q \leq F_{q+1}$.

We prove the lemma by induction. Suppose the inequalities hold for any $(p',q') \in \{p-1,p-2\} \times \{q-1,q-2\}$, we show that they also hold for $(p,q)$.
Indeed,
\[
F_p F_{q-1} + F_{p-1} F_q = (F_{p-2} + F_{p-1})(F_{q-3} + F_{q-2}) + (F_{p-3} + F_{p-2})(F_{q-2} + F_{q-1})
\]
\[
= (F_{p-2} F_{q-3} + F_{p-3} F_{q-2}) + (F_{p-2} F_{q-2} + F_{p-3} F_{q-1})
\]
\[
+ (F_{p-1} F_{q-3} + F_{p-2} F_{q-2}) + (F_{p-1} F_{q-2} + F_{p-2} F_{q-1})
\]
\[
\leq F_{p+q-5} + F_{p+q-4} + F_{p+q-4} + F_{p+q-3}
\]
\[
= F_{p+q-3} + F_{p+q-2}
\]
\[
= F_{p+q-1},
\]
and
\[
F_p F_q = (F_{p-2} + F_{p-1})(F_{q-2} + F_{q-1})
\]
\[
= F_{p-2} F_{q-2} + F_{p-2} F_{q-1} + F_{p-1} F_{q-2} + F_{p-1} F_{q-1}
\]
\[
\leq F_{p+q-5} + F_{p+q-4} + F_{p+q-4} + F_{p+q-3}
\]
\[
\leq F_{p+q-3} + F_{p+q-2}
\]
\[
\leq F_{p+q-1}.
\]
By induction, the inequalities hold for every \( p, q \geq 1 \). \(\square\)

Now the verification for the upper bounds of \( g_1(n) \) and \( g_2(n) \) becomes clear. They hold trivially for \( n = 1 \). For higher \( n \), if \( g_1(n) \) corresponds to a tree where the left branch of the root has \( p \) leaves and the right branch has \( q \) leaves \( (p + q = n) \), then the same bounds hold:
\[
g_1(n) \leq g_1(p) g_2(q) + g_2(p) g_1(q) = F_{p+1} F_q + F_p F_{q+1} \leq F_{p+q+1} = F_{n+1},
\]
and
\[
g_2(n) \leq g_1(p) g_2(q) = F_{p+1} F_q \leq F_{p+q} = F_n.
\]

Being both lower bounds and upper bounds, we have \( g_1(n) = F_{n+1} \) and \( g_2(n) = F_n \).

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