BILINEAR ENDPOINT ESTIMATES FOR CALDERÓN COMMUTATOR WITH ROUGH KERNEL

XUDONG LAI

Abstract. In this paper, we establish some bilinear endpoint estimates of Calderón commutator $C[\nabla A, f](x)$ with a homogeneous kernel when $\Omega \in L \log^+ L(S^{d-1})$. More precisely, we prove that $C[\nabla A, f]$ maps $L^q(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ to $L^{r, \infty}(\mathbb{R}^d)$ if $q > d$ which improves previous result essentially. If $q = d$, we show that Calderón commutator maps $L^{d,1}(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ to $L^{r, \infty}(\mathbb{R}^d)$ which is new even if the kernel is smooth. The novelty in the paper is that we prove a new endpoint estimate of the Mary Weiss maximal function which may have its own interest in the theory of singular integral.

1. Introduction

The purpose of this paper is to study some bilinear endpoint estimates which are unsolved in the previous work of A. P. Calderón [2], C. P. Calderón [4, 5]. Before stating our results, we give some notation and background.

In 1965, A. P. Calderón introduced the commutator defined by

$$C[\nabla A, f](x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x - y) \left(A(x) - A(y)\right)}{|x - y|^d} \cdot f(y)dy,$$

which is called the Calderón commutator, here $\Omega$ is a function defined on $\mathbb{R}^d \setminus \{0\}$ which satisfies:

(1.2) $\Omega(r\theta) = \Omega(\theta)$ for all $r > 0$ and $\theta \in S^{d-1}$;

(1.3) $\int_{S^{d-1}} \Omega(\theta)\theta^\alpha \, d\sigma(\theta) = 0$ for multi-index $|\alpha| = 1$;

and $\Omega \in L^1(S^{d-1})$. $S^{d-1}$ is the unit surface in $\mathbb{R}^d$ and $d\sigma$ denotes the surface measure on $S^{d-1}$. It is easy to see that $C[\nabla A, f](x)$ is well defined if $A$ and $f$ are smooth functions with compact supports. Calderón commutator $C[\nabla A, f](x)$ is a typical example of non-convolution Calderón-Zygmund...

Date: October 25, 2017.
2010 Mathematics Subject Classification. 42B20.
Key words and phrases. Bilinear estimate, Endpoint, Calderón commutator, rough kernel.
singular integral. We can regard $C[\nabla A, f](x)$ as a generalization of
\[ [A, S]f(x) = A(x)S(f)(x) - S(Af)(x) \]
\[ = -\text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-y} \frac{A(x) - A(y)}{x-y} f(y) dy \]
where $S = \frac{d}{dx} \circ H$ and $H$ denotes the Hilbert transform. It is well known that
the commutator $[A, S]$ is a fundamental operator in harmonic analysis and
plays an important role in the theory of the Cauchy integral along Lipschitz
curve in $\mathbb{C}$, the boundary value problem of elliptic equation on non-smooth
domain, and the Kato square root problem on $\mathbb{R}$ (see e.g. [2], [3], [9], [13],
[14] for the details). Recently, there has been a renewed interest into the
commutator $[A, S]$ and $d$-commutator introduced by Christ-Journé (see [7])
since it has application in the mixing flow problem (see e.g. [15], [12]).

In this paper, we are interested in the following strong bilinear estimate
(or weak type estimate)
\begin{equation}
(1.4) \quad \|C[\nabla A, f]\| \leq C\|\nabla A\| \|f\|_{L^p(\mathbb{R}^d)},
\end{equation}
with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$. Let us recall some historic literature about the above
inequality. We divide them into three cases (see also the complete picture
of $(1/p, 1/q)$ in Figure 1 in Theorem 1.1).

Case $r \geq 1$. A. P. Calderón [2] showed that if $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$ with $1 < r < \infty$,
$1 < q \leq \infty$, $1 < p < \infty$, then $(1.4)$ holds when $\Omega$ satisfies the condition $(1.2)$
and either the property (p-i) or (p-ii) defined as follows:

- (p-1). $\Omega$ is even and $\Omega \in L^1(S^{d-1})$;
- (p-2). $\Omega \in L \log^+ L(S^{d-1})$ is odd and satisfies $(1.3)$.

Later C. P. Calderón [4] proved $(1.4)$ is still true in the case $r = 1$, $1 < p < \infty$, $1 = \frac{1}{q} + \frac{1}{p}$, and also the case $1 < r = q < \infty$, $p = \infty$ where $\Omega$ satisfies
the condition $(1.2)$ and either the property (p-i) or (p-ii). For the endpoint
case $(q, p) = (\infty, 1)$, Y. Ding and the author [8] recently proved that if $\Omega$ satisfies
$(1.2)$, $(1.3)$ and $\Omega \in L \log^+ L(S^{d-1})$, then $C[\nabla A, f](x)$ is bounded
from $L^\infty(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$, the weak $L^1$ space. The study of this
topic in this case is quite related to weak $(1,1)$ bound of rough singular
integral (see e.g. [6], [11]).

Case $r < d/(d+1)$. C. P. Calderón [5] gave an example shows that if $r < d/(d+1)$, $q \geq 1$ and $p \geq 1$, $C[\nabla A, f](x)$ is unbounded on a ball in $\mathbb{R}^d$
for some functions $f, A$ satisfying $f \in L^p(\mathbb{R}^d)$ and $\nabla A \in L^q(\mathbb{R}^d)$.

Case $d/(d+1) \leq r < 1$. In the same paper [5], C. P. Calderón proved that if $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$ with $\frac{d}{d+1} \leq r < 1$, $1 \leq q < d$, $1 < p \leq \infty$, $C[\nabla A, f]$ maps
$L^q(\mathbb{R}^d) \times L^p(\mathbb{R}^d)$ to $L^{r,\infty}(\mathbb{R}^d)$ when $\Omega$ satisfies the condition $(1.2)$
and either the property (p-i) or (p-ii). Specially in this case if $1 < q < d$, C. P. Calderón
pointed out that $L^{r,\infty}(\mathbb{R}^d)$ space can be replaced by $L^r(\mathbb{R}^d)$ by using the
interpolation theorem developed by himself in [4]. If $d/(d+1) \leq r < 1$,
$q > d$, $p \geq 1$, C. P. Calderón got the following result.

\begin{equation}
(1.5) \quad \|C[\nabla A, f]\| \leq C\|\nabla A\| \|f\|_{L^p(\mathbb{R}^d)},
\end{equation}
Theorem A (see Theorem D in [4]). Suppose that $\Omega$ satisfies (1.2), (1.3), $\Omega \in L^1(S^{d-1})$ and the Hörmander condition
\begin{equation}
\int_{|x| \geq 2|y|} \frac{|\Omega(x-y) - \Omega(y)|}{|x-y|^d} dy < +\infty.
\end{equation}
Then $\mathcal{C}[\nabla A, f](x)$ is bounded from $L^q(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ to $L^{r, \infty}(\mathbb{R}^d)$ where $\frac{1}{r} = \frac{1}{q} + 1$ with $q > d$. Moreover we have $\mathcal{C}[\nabla A, f](x)$ is bounded from $L^q(\mathbb{R}^d) \times L^p(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ where $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$ with $q > d$ and $p > 1$.

Based on the previous theory of rough singular integral, now a natural question is that whether the conclusions in Theorem A hold if $\Omega$ is a rough kernel. Also notice that there is a case $r = d/(d+1)$, $p = 1$ and $q = d$ which is not developed even if the kernel satisfies the Hörmander condition (1.5). In this case is it possible to establish some kind of estimate like (1.4) or weak type estimate? Well, the present paper will give confirm answers to those questions. Our main results are as follows.

Theorem 1.1. Let $\mathcal{C}[\cdot, \cdot]$ be defined in (1.1). Suppose $\Omega$ satisfies (1.2), (1.3) and $\Omega \in L \log^+ L(S^{d-1})$ for $d \geq 2$. Then we have the following conclusions:

(i). For any $\lambda > 0$, there exists a finite constant $C_{\Omega, d} > 0$ such that
\begin{equation}
\lambda^r |\{x \in \mathbb{R}^d : |\mathcal{C}[\nabla A, f](x)| > \lambda\}| \leq C_{\Omega, d} \|\nabla A\|_{L^r(\mathbb{R}^d)} \|f\|_{L^1(\mathbb{R}^d)},
\end{equation}
where $\frac{1}{r} = \frac{1}{q} + 1$ and $q > d$.

(ii). Let $\frac{1}{r} = \frac{1}{d} + 1$. Then for any $\lambda > 0$, there exists a finite constant $C_{\Omega, d}$ such that
\begin{equation}
\lambda^r |\{x \in \mathbb{R}^d : |\mathcal{C}[\nabla A, f](x)| > \lambda\}| \leq C_{\Omega, d} \|\nabla A\|_{L^{d,1}(\mathbb{R}^d)} \|f\|_{L^{1}(\mathbb{R}^d)},
\end{equation}
here $L^{d,1}(\mathbb{R}^d)$ is the standard Lorentz space (see [17]).

Combining these results of A. P. Calderón [2], C. P. Calderón [4, 5], Y. Ding and the author [8], we may conclude all possible $(1/p, 1/q)$ in the following figure:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Our main results in Theorem 1.1 are corresponding to the case $0 < \frac{1}{q} \leq \frac{1}{d}$ and $p = 1$, see the blue line and point.}
\end{figure}
Remark 1.2. Here we should point that the strong estimate in Theorem A (i.e \( C[\nabla A, f] \) maps \( L^q(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \) to \( L^r(\mathbb{R}^d) \) where \( \frac{1}{r} = \frac{1}{q} + \frac{1}{p} \) with \( q > d \) and \( p > 1 \)) just follows from the multilinear interpolation theorem (see [11], Theorem 7.2.2). The proof is standard. So we do not include this result in our main theorem.

Remark 1.3. In (i) of Theorem [11] we only require \( \Omega \in L \log^+ L(S^{d-1}) \) which is strictly weaker than the regularity condition [1.5]. Hence we improve Theorem A essentially. To the best knowledge of the author, the estimate in (ii) of Theorem [1.1] is new even when the kernel is smooth.

In [4], C. P. Calderón used the method by making a Calderón-Zygmund decomposition of \( f \) so that \( f \) can be written as a good function and a bad function. It is easy to deal with the good function. However when estimating those terms related to the bad function, the Hörmander condition [1.5] is crucial. In this paper, the method here is different and relies on our recent results [8] that \( C(\nabla A, f)(x) \) is weak type (1,1) bounded if \( A \) is a Lipchitz function and \( \Omega \in L \log^+ L(S^{d-1}) \) (see Lemma 2.3 below). Roughly speaking, for a function \( A \) satisfying \( \nabla A \in L^q(\mathbb{R}^d) \), we will construct an except set which satisfies required weak type estimate. And on the complementary of except set the function \( A \) is a Lipchitz function. Therefore the weak type (1,1) boundedness of \( C[\nabla A, f](x) \) could be applied there.

Notice that the estimate \( \|\nabla A\|_{L^q(\mathbb{R}^d)} \) is related to the Sobolev space \( W^{1,q}(\mathbb{R}^d) \). When \( 1 \leq q < d \), Sobolev space \( W^{1,q}(\mathbb{R}^d) \) is embedded into \( L^{q^*}(\mathbb{R}^d) \) with \( q^* = dq/(d - q) \). This property is crucial in [5]. When \( q > d \), except set can be constructed by using the Mary Weiss maximal operator \( M \) (see section 2 for its definition), which maps \( L^q(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \) (or \( L^{q,\infty}(\mathbb{R}^d) \)) only when \( q > d \). But when \( q = d \), Sobolev \( W^{1,d}(\mathbb{R}^d) \) may be imbedded into an Orlicz space (see [1]) which may be not useful to us. This forces us to study the Mary Weiss maximal operator on \( L^d(\mathbb{R}^d) \), which is quite difficult. Fortunately, we find a substitute that \( M \) maps the Lorentz space \( L^{d,1}(\mathbb{R}^d) \) to \( L^{d,\infty}(\mathbb{R}^d) \) which is enough to construct except set.

Throughout this paper, we only consider the dimension \( d \geq 2 \) and the letter \( C \) stands for a positive finite constant which is independent of the essential variables and not necessarily the same one in each occurrence. \( A \lesssim B \) means \( A \leq CB \) for some constant \( C \). Sometimes we write \( C_\varepsilon \) means that it depends on the parameter \( \varepsilon \). \( A \approx B \) means that \( A \lesssim B \) and \( B \lesssim A \). For a set \( E \subset \mathbb{R}^d \), we denote by \( |E| \) or \( m(E) \) the Lebesgue measure of \( E \). \( \nabla A \) will stand for the vector \( (\partial_1 A, \cdots, \partial_d A) \) where \( \partial_i A(x) = \partial A(x)/\partial x_i \). Define

\[
\|\nabla A\|_X = \left\| \left( \sum_{i=1}^d |\partial_i A|^2 \right)^{1/2} \right\|_X
\]

for \( X = L^p(\mathbb{R}^d) \) or \( X = L^{d,1}(\mathbb{R}^d) \).
2.2. Proof of Theorem 1.1

2.1. Some Lemmas.

Before giving the proof of Theorem 1.1, we introduce some lemmas which play a key role in the proof of Theorem 1.1. For those readers who are not familiar with the theory of the Lorentz space $L^{p,q}(\mathbb{R}^d)$, we refer to see [17], Chapter V.3. We will use the theory of the Lorentz space $L^{p,q}(\mathbb{R}^d)$ in Lemma 2.2. Now we begin by some properties of a special maximal function which was introduced by Mary Weiss (see [4]). It is defined as

$$M(\nabla A)(x) = \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{|A(x+h) - A(x)|}{|h|}.$$ 

**Lemma 2.1.** Let $\nabla A \in L^p(\mathbb{R}^d)$ with $p > d$. Then $M$ is bounded on $L^p(\mathbb{R}^d)$, that is

$$\|M(\nabla A)\|_{L^p(\mathbb{R}^d)} \leq C \|\nabla A\|_{L^p(\mathbb{R}^d)},$$

where the constant $C$ is independent of $A$.

**Proof.** By using a standard limiting argument, we only need to consider $A$ as a $C^\infty$ function with compact support. Then the lemma just follows from the inequality

$$\frac{|A(x) - A(y)|}{|x-y|} \lesssim \left( \frac{1}{|x-y|^d} \int_{|z-x| \leq 2|x-y|} |\nabla A(z)|^q \, dz \right)^{\frac{1}{q}},$$

which holds for any $q > d$ (see Lemma 1.4 in [4]) and the fact that the Hardy Littlewood maximal operator is of strong $(p,p)$ for $p > 1$.

**Lemma 2.2.** Let $\nabla A \in L^{d,1}(\mathbb{R}^d)$. Then for any $\lambda > 0$, there exist a finite constant $C$ independent of $A$ such that

$$\lambda^d |\{x \in \mathbb{R}^d : M(\nabla A)(x) > \lambda\}| \leq C \|\nabla A\|_{L^{d,1}(\mathbb{R}^d)}^d.$$

**Proof.** It suffices to consider $A$ as a smooth function with compact support. By the formula given in [16], page 125, (17), we may write

$$A(x) = C_n \sum_{i=1}^d \int_{\mathbb{R}^d} \frac{x_i - y_i}{|x-y|^d} \partial_i A(y) \, dy = K * f(x)$$

where $K(x) = 1/|x|^{d-1}$, $f = C_n \sum_{j=1}^d R_j(\partial_j A)$ with $R_j$ the Riesz transforms. By using the fact the Riesz transform $R_j$ maps $L^{d,1}(\mathbb{R}^d)$ to itself which follows from the general form of the Marcinkiewicz interpolation theorem (see [17], Theorem 3.15 in page 197), one can easily get that

$$\|f\|_{L^{d,1}(\mathbb{R}^d)} \lesssim \|\nabla A\|_{L^{d,1}(\mathbb{R}^d)}.$$

Hence to prove the lemma, it is enough to show that

$$\lambda^d |\{x \in \mathbb{R}^d : M(\nabla A)(x) > \lambda\}| \lesssim \|f\|_{L^{d,1}(\mathbb{R}^d)}.$$
with \( A = K \ast f \). In the following our goal is to prove that for any \( x \in \mathbb{R}^d \), the estimate

\[
|A(x + h) - A(x)| \lesssim |h| T(f)(x)
\]

holds uniformly for \( h \in \mathbb{R}^d \setminus \{0\} \) with \( T \) an operator maps \( L^{d,1}(\mathbb{R}^d) \) to \( L^{d,\infty}(\mathbb{R}^d) \). Once we prove this, we get (2.1.3) and hence complete the proof of Lemma 2.2. We write

\[
\Lambda(x + h) - \Lambda(x)
\]

\[
= \int_{|x - y| \leq 2|h|} |x + h - y|^{-d+1} f(y)dy - \int_{|x - y| \leq 2|h|} |x - y|^{-d+1} f(y)dy
\]

\[
+ \int_{|x - y| > 2|h|} (|x + h - y|^{-d+1} - |x - y|^{-d+1}) f(y)dy
\]

\[
= I + II + III.
\]

Let us first consider \( I \). By an elementary calculation, one may get \( K \in L^{d,\infty}(\mathbb{R}^d) \) where \( d' = d/(d - 1) \). Set \( B(x, r) = \{ y \in \mathbb{R}^d : |x - y| \leq r \} \). Using the rearrangement inequality (see [10], page 74, Exercise 1.4.1), we have

\[
|I| \leq \int_{\mathbb{R}^d} K(x + h - y)(f \chi_{B(x,2|h|)})(y)dy \leq \int_0^\infty K^*(s)(f \chi_{B(x,2|h|)})^*(s)ds
\]

\[
\leq \|f \chi_{B(x,2|h|)}\|_{L^{d,1}(\mathbb{R}^d)} \|K\|_{L^{d',\infty}(\mathbb{R}^d)}.
\]

here \( f^* \) represents the decreasing rearrangement of \( f \). By an elementary calculation, one may get \( \| \chi_E \|_{L^{d,1}(\mathbb{R}^d)} = \| \chi_E \|_{L^{d}(\mathbb{R}^d)} \) holds for any characteristic function \( \chi_E \) of set \( E \) of finite Lebesgue measure, thus \( \| \chi_{B(x,2|h|)}\|_{L^{d,1}(\mathbb{R}^d)} = C_d|h| \). Therefore we get

\[
|I| \lesssim |h| \Lambda(f)(x), \text{ where } \Lambda(f)(x) = \sup_{r > 0} \|f \chi_{B(x,r)}\|_{L^{d,1}(\mathbb{R}^d)}.
\]

Below we need to show that the operator \( \Lambda \) maps \( L^{d,1}(\mathbb{R}^d) \) to \( L^{d,\infty}(\mathbb{R}^d) \). Since \( L^{d,1}(\mathbb{R}^d) \) is a Banach space (see [17], page 204, Theorem 3.22), it is sufficient to show that \( \Lambda \) maps the characteristic function \( \chi_E \in L^{d,1}(\mathbb{R}^d) \) to \( L^{d,\infty}(\mathbb{R}^d) \) (see [10], page 62, Lemma 1.4.20). However in this case, it is equivalent to show that

\[
\lambda |\{ x \in \mathbb{R}^d : M(\chi_E)(x) > \lambda \}| \lesssim \| \chi_E \|_{L^1(\mathbb{R}^d)},
\]

where \( M \) is the Hardy-Littlewood maximal operator. It is well known that \( M \) is of weak type \((1,1)\), hence we have shown that \( \Lambda \) maps \( L^{d,1}(\mathbb{R}^d) \) to \( L^{d,\infty}(\mathbb{R}^d) \).

Next we consider \( II \). This estimate is quite simple. Since the kernel

\[
k(x) = \varepsilon^{-d} |x|^{-d+1} \chi_{\{|x| \leq \varepsilon\}}
\]

is a radial non-increasing function and \( L^1 \) integrable in \( \mathbb{R}^d \), we get

\[
|II| \lesssim \|k\|_{L^1(\mathbb{R}^d)} |h| M(f)(x).
\]
Notice that $L^{p,1}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ and $M$ is of strong type $(p, p)$, $1 < p < \infty$, of course those imply that $M$ maps $L^{d,1}(\mathbb{R}^d)$ to $L^{d,\infty}(\mathbb{R}^d)$.

Finally we give an estimate of $\text{III}$. Notice that we only consider $|x - y| > 2|h|$. Then by the Taylor expansion of $|x - y + h|^{-d+1}$, one may have

$$
(2.2) \quad \frac{1}{|x - y + h|^{d-1}} - \frac{1}{|x - y|^{d-1}} = (-d + 1) \sum_{j=1}^{d} h_j \frac{x_j - y_j}{|x - y|^{d+1}} + R(x, y, h)
$$

where the Taylor expansion’s remainder term $R(x, y, h)$ satisfies

$$
|R(x, y, h)| \leq C|h|^2 |x - y|^{-d-1}, \quad |x - y| > 2|h|.
$$

Inserting (2.2) into the term $\text{III}$ with the above estimate of $R(x, y, h)$, we conclude that

$$
|\text{III}| \lesssim |h| \sum_{j=1}^{d} R_j^*(f)(x) + |h|^2 \int_{|x-y|>2|h|} |x - y|^{-d-1} |f(y)|dy
$$

where $R_j^*$ is the maximal Riesz transform which is defined by

$$
R_j^*(f)(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x - y|^{d+1}} f(y)dy \right|.
$$

Since $R_j^*$ is bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, one immediately gets that $R_j^*$ maps $L^{d,1}(\mathbb{R}^d)$ to $L^{d,\infty}(\mathbb{R}^d)$. The second term which controls $\text{III}$ can be dealt with the same as we do in the estimate of $\text{II}$ once we notice that the function $\varepsilon |x|^{-d-1} \chi_{\{|x|>\varepsilon\}}$ is radial non-increasing and $L^1$ integrable. \hfill $\Box$

**Lemma 2.3** (see Theorem 5.2 in [8]). Let $f \in L^1(\mathbb{R}^d)$ and $A$ be a Lipschitz function. Then for any $\lambda > 0$, we have

$$
\lambda |\{x \in \mathbb{R}^d : |C[\nabla A, f](x)| > \lambda\} | \leq C_{\Omega, d}\|\nabla A\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^1(\mathbb{R}^d)}.
$$

### 2.2. Proof of (i) in Theorem 1.1

We start to prove (i) of Theorem 1.1. Let $\frac{1}{r} = \frac{1}{q} + 1$ and $q > d$. By using a standard limiting argument, we only need to show that when $A$ and $f$ are $C^\infty$ functions with compact supports, the following inequality

$$
|\{x \in \mathbb{R}^d : |C[\nabla A, f](x)| > \lambda\} | \leq C_{\Omega, d}\lambda^{-r}\|\nabla A\|_{L^r(\mathbb{R}^d)} \|f\|_{L^1(\mathbb{R}^d)}
$$

holds for any $\lambda > 0$ with the constant $C_{\Omega, d}$ independent of $\lambda$, $A$ and $f$. By a simple scaling argument, we may assume that

$$
\|f\|_{L^1(\mathbb{R}^d)} = \|\nabla A\|_{L^q(\mathbb{R}^d)} = \|\Omega\|_{L^{\log^+ L}(\sigma^{d-1})} = 1.
$$

Now fix $\lambda > 0$. For convenience set $E_\lambda = \{x \in \mathbb{R}^d : |C[\nabla A, f](x)| > \lambda\}$ and define the except set

$$
J_\lambda = \{x \in \mathbb{R}^d : \mathcal{M}(\nabla A)(x) > \lambda^\frac{q}{2}\}.
$$
We need to show $|E_\lambda| \leq \lambda^{-r}$. From Lemma 2.1, $\mathcal{M}$ is bounded on $L^q(\mathbb{R}^d)$ with $q > d$. Hence $\mathcal{M}$ maps $L^q(\mathbb{R}^d)$ to $L^{q, \infty}(\mathbb{R}^d)$, i.e.

$$
|J_\lambda| \lesssim \lambda^{-r}\|
abla A\|_{L^q(\mathbb{R}^d)}^q = \lambda^{-r}.
$$

Choose an open set $G_\lambda$ which satisfies the following conditions: (1) $J_\lambda \subset G_\lambda$; (2) $m(G_\lambda) \leq 2|J_\lambda|$. By the property (2.3) of $J_\lambda$, we see that $m(G_\lambda) \lesssim \lambda^{-r}$. Next making a Whitney decomposition of $G_\lambda$ (see [10]), we may get a family of disjoint dyadic cubes $\{Q_k\}_k$ such that

(i) $G_\lambda = \bigcup_{k=1}^{\infty} Q_k$;

(ii) $\sqrt{d} \cdot l(Q_k) \leq dist(Q_k, (G_\lambda)^c) \leq 4\sqrt{d} \cdot l(Q_k)$.

With those properties (i) and (ii), for each $Q_k$, we could construct a larger cube $Q_k^*$ so that $Q_k \subset Q_k^*$, $Q_k^*$ is centered at $y_k$ and $y_k \in (G_\lambda)^c$, $|Q_k^*| \leq C|Q_k|$. The constant $C$ here is only dependent on the dimension. By the property (ii) above, the distance between $Q_k$ and $(G_\lambda)^c$ equals to $Cl(Q_k)$. Therefore by the construction of $Q_k^*$ and $y_k$, we get

$$
dist(y_k, Q_k) \approx l(Q_k).
$$

Now we return to give an estimate of $E_\lambda$. Split $f$ into two parts $f = f_1 + f_2$ where $f_1(x) = f(x)\chi_{(G_\lambda)^c}(x)$ and $f_2(x) = f(x)\chi_{G_\lambda}(x)$. By the definition of $J_\lambda$, when restricted on $(G_\lambda)^c$, $A$ is a Lipschitz function with $\|
abla A\|_{L^{\infty}((G_\lambda)^c)} \leq \lambda^{\frac{r}{\omega}}$. Let $\tilde{A}$ stand for the Lipschitz extension of $A$ from $(G_\lambda)^c$ to $\mathbb{R}^d$ (see [16], page 174, Theorem 3) so that

$$
\tilde{A}(y) = A(y) \quad \text{if } y \in (G_\lambda)^c;
$$

$$
|\tilde{A}(x) - \tilde{A}(y)| \leq \lambda^{\frac{r}{\omega}}|x - y| \quad \text{for all } x, y \in \mathbb{R}^d.
$$

Since the operator $C[\cdot, \cdot]$ is bilinear, we split $E_\lambda$ as three terms

$$
|\{x \in \mathbb{R}^d : |C[\nabla A, f](x)| > \lambda\}| \\
\leq |10G_\lambda| + |\{x \in (10G_\lambda)^c : |C[\nabla A, f_1](x)| > \lambda/2\}| \\
+ |\{x \in (10G_\lambda)^c : |C[\nabla A, f_2](x)| > \lambda/2\}|.
$$

The first term above satisfies $|10G_\lambda| \lesssim \lambda^{-r}$, which is the required bound. In the following, we only consider $x \in (10G_\lambda)^c$. By the definition of $f_1$, we see that $C[\nabla A, f_1](x) = C[\nabla \tilde{A}, f_1](x)$. With this equality in hand, Lemma 2.3 implies

$$
|\{x \in (10G_\lambda)^c : |C[\nabla A, f_1](x)| > \lambda/2\}| \\
= |\{x \in (10G_\lambda)^c : |C[\nabla \tilde{A}, f_1](x)| > \lambda/2\}| \\
\leq \lambda^{-1}C_{\Omega, \omega}\|
abla \tilde{A}\|_{L^{\infty}(\mathbb{R}^d)}\|f_1\|_{L^1(\mathbb{R}^d)} \lesssim \lambda^{-1 + \frac{r}{\omega}} = \lambda^{-r}.
$$

Let us turn to $C[\nabla A, f_2](x)$, which can be rewritten as

$$
C[\nabla A, f_2](x) = C[\nabla \tilde{A}, f_2](x) + \int_{\mathbb{R}^d} \frac{\Omega(x - y)\tilde{A}(y) - A(y)}{|x - y|^d} f_2(y)dy.
$$
Using the similar method of dealing with $C[\nabla \tilde{A}, f_1]$, we may get
\[
|\{x \in (10G_\lambda)^c : |C[\nabla \tilde{A}, f_2](x)| > \lambda/4\}| \lesssim \lambda^{-r}.
\]
Therefore it remains to consider the second term in (2.5). Using the notation in the Whitney decomposition of $G_\lambda$, we may write
\[
\int_{\mathbb{R}^d} \frac{\Omega(x-y) \tilde{A}(y) - A(y)}{|x-y|^d} f_2(y)dy = \sum_k \int_{Q_k} \frac{\Omega(x-y) \tilde{A}(y) - A(y)}{|x-y|^d} f(y)dy
\]
\[
=: H(\tilde{A}, f)(x) - H(A, f)(x),
\]
where
\[
H(A, f)(x) = \sum_k \int_{Q_k} \frac{\Omega(x-y) A(y) - A(y_k)}{|x-y|^d} f(y)dy.
\]
By using the Chebyshev inequality, the Fubini theorem and $\tilde{A}$ is a Lipschitz function with Lipschitz bound $\lambda^{r/q}$, we conclude that
\[
|\{x \in (10G_\lambda)^c : |H(\tilde{A}, f)(x)| > \lambda/8\}|
\]
\[
\lesssim \lambda^{-1} \int_{(10G_\lambda)^c} \sum_k \int_{Q_k} \frac{|\Omega(x-y)|}{|x-y|^d} \frac{||\tilde{A}(y) - A(y_k)||}{|x-y|} |f(y)|dy
\]
\[
\lesssim \lambda^{-1+\frac{2}{q}} \sum_k \int_{Q_k} |y_k - y| \left( \int_{(10G_\lambda)^c} \frac{|\Omega(x-y)|}{|x-y|^d+1} dx \right) |f(y)|dy
\]
\[
\lesssim \lambda^{-r} \|\Omega\|_{L^1(S^{d-1})} \sum_k \int_{Q_k} |f(y)|dy \lesssim \lambda^{-r},
\]
where in the second inequality we use the fact: $|x-y| \geq l(Q_k) \approx |y-y_k|$, since $x \in (10G_\lambda)^c$ and (2.4); and in the last inequality we use $\|\Omega\|_{L^1(S^{d-1})} \leq \|\Omega\|_{L^0_{\log} + L^0_{S^{d-1}}} = 1$.

Notice that by the construction of $y_k$, we have $y_k \in (G_\lambda)^c$. It follows that $\mathcal{M}(\nabla A)(y_k) \lesssim \lambda^{\frac{2}{q}}$. By using the Chebyshev inequality, the Fubini theorem and the above fact, we get
\[
|\{x \in (10G_\lambda)^c : |H(A, f)(x)| > \lambda\}|
\]
\[
\lesssim \lambda^{-1} \int_{(10G_\lambda)^c} \sum_k \int_{Q_k} \frac{|\Omega(x-y)|}{|x-y|^d} \frac{|A(y) - A(y_k)|}{|x-y|} |f(y)|dy
\]
\[
\lesssim \lambda^{-1} \sum_k \mathcal{M}(\nabla A)(y_k) |y - y_k| \int_{(10G_\lambda)^c} \frac{|\Omega(x-y)|}{|x-y|^d+1} dx |f(y)|dy
\]
\[
\lesssim \lambda^{-1+\frac{2}{q}} \|\Omega\|_{L^1(S^{d-1})} \sum_k \int_{Q_k} |f(y)|dy \lesssim \lambda^{-r},
\]
where the third inequality follows from $|x-y| \geq l(Q_k) \approx |y-y_k|$. Therefore we complete the proof of (i) in Theorem 1.1. □
2.3. **Proof of (ii) in Theorem [1.1]**

The proof of (ii) is similar to that of (i) in Theorem [1.1] once we choose $q = d$. The only difference is that when we give an estimate of except set $J_\lambda$, we will use Lemma 2.2 instead of Lemma 2.1. Proceeding the rest proof as we do in the proof of (i), we may obtain the result of (ii).

\[\square\]

**Acknowledgement.** The author would like to thanks A. Seeger for suggesting to consider the Lorentz space $L^{d,1}(\mathbb{R}^d)$ endpoint estimate of the Mary Weiss maximal operator $\mathcal{M}$.

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Xudong Lai: Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin, 150001, People’s Republic of China
E-mail address: xudonglai@hit.edu.cn xudonglai@mail.bnu.edu.cn