On integrable Newton Cartan strings with fluxes

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Abstract
We map 2D sigma models corresponding to torsional Newton Cartan (TNC) strings on \( R \times S^2 \) into 1D deformed Neumann-Rosochatius integrable models in the presence of background NS-NS fluxes. We propose the nonrelativistic analogue of Uhlenbeck integrals (those are in involution) and explore the associated Hamiltonian constrained structure by introducing appropriate Dirac brackets. Furthermore, we notice that the tensionless/ large \( c \) limit of the sigma model trivially reduces to deformed Rosochatius like integrable models thereby ensuring the underlying integrable structure in the nonrelativistic sector. In both the cases, we obtain the corresponding dispersion relations and compute one loop stringy corrections to it.

1 Overview and Motivation

Over the past one decade, a considerable amount of attention has been paid towards the formulation of nonrelativistic (NR) string sigma models [1]-[3] on curved manifolds with local (centrally extended) Galilean invariance. Manifolds of such kind could in general be classified into two different categories. For example, in the absence of any torsion, gauging the centrally extended Galilean algebra leads to what is known as string Newton-Cartan (SNG) geometry [5]-[10]. When torsion is included the resulting target space is known as torsional Newton-Cartan (TNC) geometry [11]-[20].

Studying string theory over SNC/TNC geometries could in principle have several motivations among which two are centrally important. One of the possible inspirations comes from the quest for a UV complete theory of non relativistic quantum gravity that has found directions very recently in the context of bosonic sigma models [10], [10]. The other question is related to a much deeper understanding of gauge/string duality or the so called holographic principle in the context of non relativistic string theory. The present paper is actually an effort to gain some insights along the second line of thought.

The standard framework behind the celebrated AdS/CFT correspondence is primarily based on the principle of matching the spectra between \( AdS_5 \times S^5 \) (super) strings in 10D and that of the \( \mathcal{N} = 4 \) SYM in 4D. It is the underlying integrable structure (existing on both sides of the duality) that has finally made it possible to go for some remarkable as well as striking tests of the conjectured duality and in particular in the limit of large quantum

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numbers [21]-[25]. It is therefore quite natural to ask similar questions in the context of non relativistic sigma model/gauge theory correspondence and to test the corresponding framework of non relativistic holography in the limit of large $R$ charge.

The question that we are primarily interested in is whether NR strings are integrable or not. This question has one of the standard/traditional answers [26]-[36] in AdS$_5 \times$ S$^5$ (super)string theory is by reducing the corresponding 2D sigma model to 1D Neumann-Rosochatius integrable models [37]-[41]. In other words, the question that we are interested in is whether similar reduction is possible in case of NR sigma models and what are the possible consequences of that from the perspective of gauge/string duality. At the moment this sounds like a potentially interesting question (in the context of non relativistic gauge/string duality) whose many details are in fact missing in the existing literature.

We address this question explicitly by constructing 2D sigma models corresponding to rotating (semi)classical string states over torsional Newton Cartan (TNC) geometry with $R \times S^2$ topology. Our analysis reveals that TNC strings allow a natural deformation of the Neumann-Rosochatius integrable models (irrespective of any background fluxes),

$$\mathcal{L}_{TNC} = \mathcal{L}_{NR} + \Delta(\varpi)$$

where $\mathcal{L}_{NR}$ stands for the standard Neumann-Rosochatius oscillator model [33] along with the deformation piece [21] that turns out to be proportional to the angular frequency $\varpi(\ll 1)$ of the corresponding rotating string configuration. We further figure out the NR generalization of Ulhenbeck integral(s) [38] in the context of TNC strings which results in a vanishing Dirac bracket with the primary Hamiltonian of the system. Interestingly enough, we notice that the large $c$/tensionless limit of the 2D sigma model yields almost identical results except for the fact that it is now mapped to “$\varpi$ deformed” Rosochatius model thereby preserving the integrability in the corresponding NR sector.

As a further continuation of our analysis, we construct class of NR rotating string configurations with constant radii [27] and obtain the corresponding dispersion relation in the semi-classical limit. The dispersion relation essentially expresses the classical string energy ($E_S$) as a function of t’Hooft coupling ($\lambda$),

$$E_S = J_\varphi \left( \frac{a_1}{\sqrt{\lambda}} + a_2 \sqrt{\lambda} + \ldots \right) = J_\varphi f(\tilde{\lambda})$$

where we identify, $J_\varphi$ as the angular momentum of the string and $\tilde{\lambda} = \frac{\lambda}{J^2} \sim 1$ as the effective expansion parameter in the semi-classical limit. Our analysis reveals that unlike the relativistic case [27], the leading order contribution to $f(\tilde{\lambda}) \sim (\sqrt{\tilde{\lambda}})^{-1}$ which also persists in the corresponding large $c$/tensionless limit. To complete the spectrum, we further take into account quantum corrections over constant radii solutions and provide a detailed formalism that estimates the excited string states in the context of deformed/extended Neumann-Rosochatius integrable models. Interestingly enough, we notice that none of the qualitative features are changed in the large $c$/tensionless limit of the sigma model. It remains to be an open question how to interpret this spectrum from the perspective of a dual gauge theory. In the tensionless limit [11], the matching between stringy excitations and operator spectrum could be done identifying appropriate sector in the dual Spin-Matrix theory (SMT) [14] and computing anomalous dimension with large $R$ charge.
The organization for the rest of the paper is as follows. We start our analysis in Section 2 with building up TNC sigma models over \( R \times S^2 \) in the presence of background NS-NS fluxes. In the embedding coordinate formalism this naturally reduces to deformed 1D Neumann-Rosochatius integrable models. We provide a detailed Hamiltonian formulation of such deformed models in Section 3 where we introduce appropriate Dirac brackets and identify the generalized Uhlenbeck integrals of motion. Section 4 is devoted towards computing the spectrum knowing the solutions in the dimensionally reduced model. For simplicity, we take into account solutions with fixed radii and obtain the corresponding spectra in the semi-classical limit. As a further illustration of our analysis, we also consider quantum/stringy corrections to these solutions and provide a general algorithm that solves the spectrum. In Section 5, we address similar issues in the large \( c/ \) tensionless limit of the sigma model that is dual to the decoupled \( (\lambda \to 0) \) sector of the \( \mathcal{N} = 4 \) SYM in 4D \([11]\). We build up the corresponding Hamiltonian formulation of the theory and obtain the generalized Uhlenbeck integrals. In Section 6, we explore the stringy spectrum corresponding to constant radii solutions in the reduced integrable model and calculate one loop quantum corrections to it. Finally, we conclude in Section 7.

## 2 TNC strings on \( R \times S^2 \)

We start by writing down the sigma model on TNC geometry with \( R \times S^2 \) topology \([17]\),

\[
ds_{\text{TNC}}^2 = 2\tau(du - m) + h_{\mu\nu}dX^\mu dX^\nu
\]

where we identify each of the individual one forms above in \([3]\) as \([13]\),

\[
\begin{align*}
\tau &= dt + \frac{1}{2}d\psi - \frac{1}{2}\cos\theta d\varphi \\
m &= \frac{1}{4}\cos\theta d\varphi
\end{align*}
\]

together with the metric on the two sphere,

\[
ds_{S^2}^2 = h_{\mu\nu}dX^\mu dX^\nu = d\theta^2 + \sin^2\theta d\varphi^2.
\]

As a next step, we set \( X^0 = t = \kappa\tau \) together with the so-called embedding coordinates,

\[
\begin{align*}
Z_1 &= X^1 + iX^2 = \sin\theta e^{i\varphi(\tau,\sigma)} = \vartheta_1(\sigma)e^{i\varphi(\tau,\sigma)} \\
Z_2 &= X^3 = \cos\theta = \vartheta_2(\sigma)
\end{align*}
\]

such that, \( ds_{S^2}^2 = dZ_id\bar{Z}_i \) \((i = 1, 2)\) together with the constraint, \( \vartheta_1^2 = \vartheta_2^2 + \vartheta_2^2 = 1 \). In terms of embedding coordinates \((X^m \ (m = 1, 2, 3))\) the metric on two sphere has an equivalent representation, \( ds_{S^2}^2 = h_{mn}dX^mdX^n \) where \( h_{mn} = diag(1, 1, 1) \).

Closed strings propagating over \([3]\) in the presence of background NS-NS two form \([1]\) \((B_{\mu\nu})\) are described by the 2D sigma model of the following form \([18]\),

\[
S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \mathcal{L}_{\text{TNC}}
\]

\footnote{For our present computation we choose to work with \( B_{\theta\varphi} = B\sin\theta \).}
where the corresponding sigma model Lagrangian could be formally expressed as:

\[ L_{TNC} = \left( \sqrt{-\gamma} \gamma^{\alpha\beta} h_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu} \right) \partial_\alpha X^\mu \partial_\beta X^\nu - \sqrt{-\gamma} \gamma^{\alpha\beta} (\tau_\alpha m_\beta + \tau_\beta m_\alpha) + 2 \left( \sqrt{-\gamma} \gamma^{\alpha\beta} \tau_\beta + \varepsilon^{\alpha\beta} (b_\beta + \partial_\beta \zeta) \right) A_\alpha. \]  

(10)

Here we identify, \( \gamma^{\alpha\beta} = \eta^{ab} \varepsilon^a e^b \) \((a = 0, 1)\) as being 2D Lorentzian metric on the world-sheet together with the determinant \( \sqrt{-\gamma} = |\varepsilon| \). It turns out that the sigma model (10) allows d.o.f.s namely the abelian one form,

\[ A_\alpha = m_\alpha + \frac{1}{2} (\lambda_+ - \lambda_-) e_\alpha^0 + \frac{1}{2} (\lambda_+ + \lambda_-) e_\alpha^1 \]  

(11)

together with a world-sheet scalar \( \zeta(\sigma) \) that essentially plays the role of an additional compact dimension associated with the null reduced target space geometry. A non zero winding of string along \( \zeta \) is what guarantees conserved momentum of string along the null isometry direction of the TNC geometry [18].

Substituting (11) into (10) we find,

\[ L_{TNC} = \left( \sqrt{-\gamma} \gamma^{\alpha\beta} h_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu} \right) \partial_\alpha X^\mu \partial_\beta X^\nu + 2 \varepsilon^{\alpha\beta} m_\alpha (b_\beta + \partial_\beta \zeta) + \lambda_+ \varepsilon^{\alpha\beta} e_\alpha^0 (\tau_\beta + b_\beta + \partial_\beta \zeta) + \lambda_- \varepsilon^{\alpha\beta} e_\alpha^1 (\tau_\beta - b_\beta - \partial_\beta \zeta) \]  

(12)

where we identify, \( e_\alpha^\pm = e_\alpha^0 \pm e_\alpha^1 \). The equations of motion corresponding to Lagrange multipliers impose the following set of constraint equations,

\[ \varepsilon^{\alpha\beta} e_\alpha^\pm (\tau_\beta \pm b_\beta \pm \partial_\beta \zeta) = 0 \]  

(13)

which has a natural solution of the form,

\[ e_\alpha^\pm = (\tau_\alpha \pm b_\alpha \pm \partial_\alpha \zeta). \]  

(14)

Substituting (14) into (12) we arrive at the following sigma model Lagrangian,

\[ L_{TNC} = \frac{\varepsilon^{\alpha\bar{\alpha}} \varepsilon^{\beta\bar{\beta}}}{\varepsilon^{\alpha\beta} e_\alpha^0 e_\beta^1} (e_\bar{\alpha}^0 e_\bar{\beta}^0 - e_\alpha^1 e_\beta^1) h_{\mu\nu} \partial_\alpha X^m \partial_\beta X^n + \varepsilon^{\alpha\beta} B_{MN} \partial_\alpha X^M \partial_\beta X^N + 2 \varepsilon^{\alpha\beta} m_\alpha (b_\beta + \partial_\beta \zeta) \]  

(15)

where for the present configuration we note down,

\[ e_\alpha^0 = \tau_\alpha = \partial_\alpha t + \frac{1}{2} \partial_\alpha \psi - \frac{1}{2} \cos \theta \partial_\alpha \varphi \]  

(16)

\[ e_\alpha^1 = b_\alpha + \partial_\alpha \zeta \]  

(17)

together with the 2D Levi-Civita convention, \( \varepsilon^{01} = -\varepsilon_{01} = +1 \) both for the world-sheet as well as tangent space indices. Here, \( X^m \)s \((m = 1, 2, 3)\) are the embedding coordinates as introduced earlier in (7) and (8).

To proceed further, we choose to work with the following ansatz,

\[ \varphi(\tau, \sigma) = \omega \tau + \xi(\sigma) \; ; \; t = \kappa \tau \; ; \; \psi = \text{const.} \; ; \; \zeta = \sigma. \]  

(18)

\[ ^2 \text{We set [18], } B_{\xi\mu} = 0 \text{ and } b_\zeta = 1. \text{ Here, } X^M = \{ \zeta, X^m \} \text{ are the target space coordinates together with, } \mu = \{ t, \theta, \varphi \}. \text{ In these coordinates the background NS-NS fields could be further decomposed as, } B_{MN} \equiv B_{\mu\nu}, B_{\xi\mu} = b_\mu \text{ and } b_M = \{ b_\mu, b_\zeta \} [18]. \]
In the following, we will be working with closed string configurations which eventually impose constraints of the following form,

\[ \vartheta_i(\sigma + 2\pi) = \vartheta_i(\sigma) ; \xi(\sigma + 2\pi) = \xi(\sigma) + 2\pi n ; \quad n \in \mathbb{R}. \]  

(19)

Using (18) we finally obtain,

\[ L_{TNC} = \frac{1}{2} \left( \kappa - \frac{\varpi}{2} \vartheta_2 \right) \left( \vartheta_1'^2 + \vartheta_2'^2 \xi_{\delta_{11}}^2 \right) + \frac{1}{8} \frac{\xi^2 \varpi^2 \vartheta_1' \vartheta_2'^2}{(\kappa - \varpi \vartheta_2)} \]

\[ + \frac{\varpi}{2} \vartheta_2 \vartheta_2'^2 \xi_{\vartheta_1}^2 + \frac{2 \vartheta_1^2 \varpi}{(\kappa - \varpi \vartheta_2)} - \frac{2 \vartheta_1^2 \varpi^2}{(\kappa - \varpi \vartheta_2)} + \varpi B \vartheta_2' + \varpi \vartheta_2 - \Lambda(\vartheta_1^2 + \vartheta_2^2 - 1) \]

(20)

where, \( \Lambda \) is the so called Lagrange multiplier of the dynamical system.

To proceed further we consider the small frequency limit namely, \( |\varpi| \ll 1 \ll \kappa \) which eventually leads towards deformed Neumann-Rosochatius model (with \( \kappa = 2 \)),

\[ L_{TNC} \approx \vartheta_1'^2 + (\xi^2 \varpi^2 - \vartheta_1^2 \varpi^2) \delta_{11} - \Lambda(\vartheta_1^2 - 1) + \Delta(\varpi) \]  

(21)

together with the deformation piece,

\[ \Delta(\varpi) \approx -\frac{\varpi}{4} \vartheta_2 \vartheta_2'^2 + \frac{\varpi \xi^2 \vartheta_1' \vartheta_2'^2}{4} \left( 1 + \frac{\varpi \vartheta_2}{4} \right) + \varpi B \vartheta_2' + \varpi \vartheta_2 + O(\varpi^2 / \kappa^2). \]  

(22)

The above Lagrangian (21) essentially is the starting point of our subsequent analysis. It effectively describes a one \( (i = 1) \) dimensional harmonic oscillator that is constrained to remain on a unit two sphere that is subjected to the constraint as mentioned above (9).

### 3 Deformed integrable models

The integrability of the reduced 1D model (21) could be anticipated from a naive counting of the integrals of motion associated with the dynamical phase space under consideration.

Given a dynamical phase space configuration of dimension \( 2N \) that is subjected to set of (secondary) constraints \( \Psi_i (i = 1, \ldots, n) \) where, \( n < N \) is said to be Liouville integrable if it possess \( I_a (a = N - \frac{n}{2}) \) conserved charges those are in involution.

Given the present dynamical configuration, it is trivial to notice that \( N = 3 \). Therefore, the only task remains is to uncover the underlying constraint structure associated with the dynamical phase space configuration which we focus next.

#### 3.1 Hamiltonian formulation

Given the deformed Lagrangian (21), the first step is to find out the corresponding Hamiltonian dynamics and in particular classify the underlying constraint structure associated to the deformed 1D Neumann-Rosochatius model under consideration. In order to do so, we first note down the canonical momenta,

\[ \pi_1 = 2 \vartheta_1' \left( 1 - \frac{\varpi}{4} \vartheta_2 \right) \]  

(23)

\[ \pi_2 = 2 \vartheta_2' \left( 1 - \frac{\varpi}{4} \vartheta_2 \right) + \varpi B \]  

(24)

\[ \pi_\xi = 2 \xi' \vartheta_1^2 \left( 1 + \frac{\varpi \vartheta_2}{4} + \frac{\varpi^2 \vartheta_2^2}{16} \right) \equiv C \]  

(25)
where in the last equation (25), we have used (50) which stems from the fact that the Lagrangian (21) is cyclic in the variable \( \xi \).

Using (23)-(25), the corresponding canonical Hamiltonian density,

\[
H_c = \pi_1 \vartheta_1' + \pi_2 \vartheta_2' + \pi_\xi \xi' - \mathcal{L}_{TNC}
\]

(26)
could be formally expressed as

\[
H_c \approx \pi_2^i i + \left( \varpi_2^2 \vartheta_2^i + \frac{\pi_\xi^2}{\vartheta_1^2} \right) \delta_{i1} + \Lambda (\vartheta_1^2 - 1) + \Delta \mathcal{H} + \mathcal{O}(\varpi^{-3}) ;
\]

(27)

where we identify the deformation,

\[
\Delta \mathcal{H} \approx \frac{\pi_2^2}{4} \varpi \vartheta_2^2 \left( 1 + \frac{\varpi \vartheta_2^2}{4} \right) - \frac{\varpi^2}{4} \left( \frac{\vartheta_2}{\vartheta_1} \right) - \varpi \vartheta_2^i \left( 1 + \frac{\varpi \vartheta_2^2}{4} \right)
\]

(28)

Given the canonical Hamiltonian (27), our next task is to define appropriate Dirac brackets between different dynamical variables in the system. The first step towards this direction is to construct the primary Hamiltonian for the system,

\[
H_P = H_c + \lambda \Phi_P
\]

(29)

where we introduce,

\[
\Phi_P = \pi_\Lambda = \frac{\partial \mathcal{L}_{TNC}}{\partial \Lambda'} \approx 0
\]

(30)
as the primary constraint associated with the dynamical system under consideration.

### 3.2 Constraint structure

With (29) in hand, we are now in a position to find out all the secondary constraints associated with the dynamical configuration. Requiring that the variation of (30) vanishes, we arrive at the secondary constraint of the following form,

\[
\Psi_1 \approx \{ \Phi_P, H_P \}_P = \vartheta_1^2 - 1 \approx 0.
\]

(31)

Requiring further that the variation of \( \Psi_1 \) vanishes namely,

\[
\Psi_1' \approx \{ \Psi_1, H_P \}_P \approx \frac{2(\pi_i \vartheta_i - \varpi \vartheta_1^2)}{1 - \frac{\varpi \vartheta_2^2}{4}}
\]

(32)

we further unveil new constraint structure of the following form,

\[
\Psi_2 \approx \pi_i \vartheta_i - \varpi \vartheta_1^2 \approx 0.
\]

(33)

\footnote{Here we re-scale the canonical momenta as, \( \pi_a \rightarrow \tilde{\pi}_a = \frac{\pi_a}{2} \ (a = i, \xi) \) and do not use tildes for simplicity.}
A further variation of $\Psi_2$ eventually gives us back the canonical Hamiltonian structure which thereby concludes the chain of secondary constraints in the theory. We identify $\Psi_{1,2}$ as being the only independent secondary constraints associated with the dynamical system under consideration. The number of conserved charges and/or the integrals of motion associated with the phase space configuration is therefore two. One of which we identify as being the conjugate momenta $(\pi_\xi)$ corresponding to cyclic coordinate $\xi$. The other integral of motion which we identify as being the generalization of the Uhlenbeck constant in the context of NR sigma models is what we construct below.

### 3.3 Integrals of motion

Before we get into the integral(s) of motion it is customary first to introduce Dirac brackets in the context of TNC sigma models. The non trivial fact to be noticed here is,

$$\{\Psi_1, \Psi_2\}_PB = \Gamma_{12} = 2 = -\{\Psi_2, \Psi_1\}_PB = -\Gamma_{21}$$

which defines what is known as second class constraints. Given (34), we define Dirac bracket between observables in the phase space as,

$$\{A_1, A_2\}_DB = \{A_1, A_2\}_PB - \{A_1, \Psi_i\}(\Gamma^{-1})_{ij}\{\Psi_j, A_2\}.$$  (35)

Using (35), the Dirac bracket between basic phase space variables could be estimated as follows,

$$\{\pi_i, \pi_j\}_DB = \pi_i\partial_j - \pi_j\partial_i + \varpi B(\partial_i\delta_{2j} - \partial_j\delta_{2i})$$  (36)

$$\{\pi_i, \vartheta_j\}_DB = -\delta_{ij} + \vartheta_i\vartheta_j$$  (37)

$$\{\vartheta_i, \vartheta_j\}_DB = 0.$$  (38)

In order to find the second integral of motion, we propose the following NR generalization of the Uhlenbeck integral,

$$I_{NR} = \vartheta_2^2 + 1 \varpi B(\partial_i^2 + \pi_2^2\vartheta_2) + \varphi_{NR}$$  (39)

where we identify,

$$\varphi_{NR} = B^2\vartheta_2^2 + \varpi^{-2}\Delta H$$  (40)

as NR modifications to the Uhlenbeck integral together with, $J_{ij} = \partial_i\pi_j - \partial_j\pi_i$. At this stage, it is indeed trivial to notice that,

$$H_c = \varpi^2 I_{NR} + \pi_2^2$$  (41)

subjected to the evaluation of the constraints and (33). The above identity therefore ensures that the generalized integral (39) is in involution namely it satisfies,

$$\{I_{NR}, H_P\}_DB = 0 ; \{I_{NR}, \pi_\xi\}_DB = 0.$$  (42)

The second identity in (42) is trivial to satisfy as $\pi_\xi$ is a constant of motion. On the other hand, a straightforward computation reveals that the first Dirac bracket in (42) vanishes identically following the original prescription of (35).
Finally, looking back at the relativistic counterpart [26], [33] it would be indeed worthwhile to define the analogue of Neumann and Rosochatius integrable models in the context of TNC strings. The Neumann model is introduced in the limit, $\pi_\xi \to 0$ which thereby yields the associated Hamiltonian structure of the form,

$$
H_N = \pi_i^2 + \varpi^2 \varphi_1^2 + \Delta H_N
$$

subjected to the constraint conditions (31) and (33). Here, we define $\Delta H_N = \Delta H|_{\pi_\xi=0}$. The corresponding generalized Uhlenbeck integral (39) reduces to,

$$
\mathcal{F}_N = \varphi_1^2 + \frac{J_{ij}^2}{2\varpi^2} + B^2 \varphi_2^2 + \varpi^{-2} \Delta H_N
$$

such that the relation,

$$
H_N = \varpi^2 \mathcal{F}_N
$$

is trivially satisfied. Finally, we take the limit, $\varpi \to 0$ and note down the corresponding Hamiltonian density,

$$
H_R = \pi_i^2 + \frac{\pi_\xi^2}{\varphi_1^2}
$$

which we identify as being the analogue of Rosochatius model [26] in the context of NR string sigma model. The corresponding integral of motion we note down to be,

$$
\mathcal{F}_R \approx \frac{J_{ij}^2}{2} + \frac{\pi_\xi^2 \varphi_2^2}{\varphi_1^2} + O(\varpi) \approx \lim_{\varpi \to 0} \varpi^2 \mathcal{I}_{NR}
$$

which trivially implies,

$$
H_R \approx \mathcal{F}_R + \pi_\xi^2.
$$

4 The TNC spectrum

The purpose of this Section is to obtain stringy spectrum corresponding to deformed Neumann-Rosochatius model as described above. Typically here we look for a dispersion and/or energy momentum relation of the form, $E = f(\lambda) J_\varphi$ where, $f(\lambda)$ is considered to be a polynomial in the t’Hooft coupling ($\lambda$).

To start with, we notice that the Lagrangian (21) is cyclic in the variable $\xi$ which thereby naturally yields the corresponding conjugate momenta as

$$
\xi'(\sigma) = \frac{C}{2\varphi_1^2 \left(1 + \frac{\varphi_2^2}{4} + \frac{\varphi_2^2 \varphi_2^2}{16}\right)}
$$

\footnote{Here $C$ is the constant of integration}
together with equations of motion corresponding to radial functions,

\[
\begin{align*}
\vartheta_1'' &= \frac{1}{(1 - \varpi \vartheta_2^2)} \left( \frac{\varpi}{4} \vartheta_1'' - (\varpi^2 + \Lambda) \vartheta_1 + \frac{C \xi'}{2 \vartheta_1} \right) \\
\vartheta_2'' &= \frac{\varpi (\vartheta_2^2 - \vartheta_1^2)}{8 (1 - \varpi \vartheta_2^2)} + \frac{\varpi \xi'^2 \vartheta_1^2 (1 + \varpi \vartheta_2)}{8 (1 - \varpi \vartheta_2^2)} + \frac{\varpi - 2 \Lambda \vartheta_2}{2 (1 - \varpi \vartheta_2^2)}.
\end{align*}
\]

Next, we note down conserved charges associated with the sigma model in its most generic form namely,

\[
P_a = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \, p_a^{(r)}
\]

where, we identify the corresponding charge density associated with the sigma model (15),

\[
p_a^{(r)} = \frac{\delta L_{TNC}}{\delta (\partial_\tau X^a)} ; \quad a = t, \varphi.
\]

A straightforward computation finally reveals (with \(\kappa = 2\)),

\[
\begin{align*}
p_t^{(r)} &= \mathcal{E} \approx \frac{1}{2} (\vartheta_1'^2 + \vartheta_2'^2 + \vartheta_1^2 \xi'^2) \\
p_\varphi^{(r)} &= \mathcal{J}_\varphi \approx -\frac{\vartheta_1'}{4} (\vartheta_1'^2 + \vartheta_2'^2) - 2\varpi \vartheta_1^2 + \frac{\vartheta_1^2 \vartheta_2^2 \xi'^2}{4} \left(1 + \frac{\varpi \vartheta_2}{2}\right) + B \vartheta_2 + \vartheta_2
\end{align*}
\]

where we drop terms \(\sim O(\varpi^2/\kappa^2)\). Given (55) and (56) the total energy and momentum could therefore be expressed as,

\[
\begin{align*}
E_S &= \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \mathcal{E} ; \\
J_\varphi &= \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \mathcal{J}_\varphi.
\end{align*}
\]

Our next task would be to evaluate the above integrals in (57) knowing the functions \(\vartheta_1(\sigma)\) and \(\vartheta_2(\sigma)\). This will be done in two steps. First, we would like to go for the simplest possible solution namely solving the equations of motion (51)-(52) considering derivatives of both the functions to be zero. Next, we would like to go for more generic solutions by considering fluctuations over the space of these constant radii solutions.

### 4.1 Constant radii solutions

In order to compute the spectrum in the sigma model, we choose to work with constant radii solutions [27] of the following form,

\[
\begin{align*}
\vartheta_1 &= \alpha_1 ; \\
\vartheta_2 &= \alpha_2
\end{align*}
\]

which is supplemented by the following ansatz [29],

\[
\xi(\sigma) = b \sigma.
\]

\[\text{From periodicity condition (19) it is trivial to see that } b \text{ is an integer namely, } b = n.\]
Substituting (59) into (50) we find the winding number,

\[ b = \frac{C}{2a_1^2 \left( 1 + \frac{\varpi a_2}{4} + \frac{\varpi^2 a_2^2}{16} \right)}. \]  

(60)

On the other hand, given the ansatz (58) the equations of motion (51)-(52) further simplify as,

\[ \frac{Cb}{2a_1^2} - \varpi^2 - \Lambda = 0 \]  

(61)

\[ \frac{\varpi b^2 a_1^2}{4} \left( 1 + \frac{\varpi a_2}{2} \right) + \varpi - 2\Lambda a_2 = 0. \]  

(62)

The obvious next step would be use (61)-(62) in order to express radii \( a_i \) \((i = 1, 2)\) as a function frequency \( \varpi \). This would certainly be subjected to the constraint, \( a_i^2 = 1 \).

Notice that, here \( |a_i| < 1 \) therefore one can always find an approximate solution keeping terms upto quadratic order in the variables namely, \( \varpi \) and \( a_i \). After certain trivial steps one essentially finds,

\[ a_2 \approx \frac{(b^2 + 4) \varpi}{4bC} + O(\varpi^3) \]  

(63)

\[ a_1 = \sqrt{1 - a_2^2} \approx 1 - \frac{(b^2 + 4)^2 \varpi^2}{32b^2C^2} + O(\varpi^4). \]  

(64)

Using (63)-(64), from (57) we obtain the following polynomial for \( \varpi \) namely \[ -\frac{32bC}{\sqrt{\lambda}} + \varpi \left( (b^2 + 4)^2 - 32bC \right) + O(\varpi^3) \approx 0 \]  

(65)

which therefore has a solution of the following form,

\[ \varpi = \frac{32bC}{\sqrt{\lambda} ((b^2 + 4)^2 - 32bC)}. \]  

(66)

Using (66), we finally obtain the dispersion relation corresponding to rotating TNC strings as,

\[ |E_S| = \frac{b^2}{4} J_\varphi \left( \frac{1}{\sqrt{\lambda} \left( (b^2 + 4)^2 - 32bC \right)} \right). \]  

(67)

4.2 One loop correction

4.2.1 Dynamics of fluctuations

We now go for studying one loop stringy corrections to classical dispersion relation (67). This would be achieved considering fluctuations over classical space of solutions (63)-(64). Instead of solving fluctuations explicitly, our purpose would be to outline solutions

\[ \lambda = \frac{1}{\lambda}, \] is the effective coupling in the dual gauge theory such that \( |\lambda| \lesssim 1 \) in the limit when both \( \lambda(\gg 1) \) and \( J_\varphi \) are large.
in a schematic fashion. In order to do that, we first introduce fluctuations as, \( \vartheta_i = \alpha_i + \lambda^{-1/4} \omega_{(i)}(\tau, \sigma) \) which essentially leads to,

\[
Z_i = \bar{Z}_i + \lambda^{-1/4} \delta Z_i
\]  

(68)

where we identify, \( \bar{Z}_i \) as constant radii solutions discussed in the previous Section together with \( \delta Z_i(\tau, \sigma) \approx \omega_{(i)} e^{j\varphi}(j = \sqrt{-1}) \) as (stringy) fluctuations over the space of solutions \( (63)-(64) \). The corresponding constraint that one must satisfy is of the following form,

\[
\bar{Z}_i \delta Z_i^* + \delta Z_i \bar{Z}_i^* \approx 0
\]  

(69)

which is valid only in the limit, \(|\delta Z_i| \ll 1\). This finally leads to the quadratic Lagrangian of the following from,

\[
\mathcal{L}_{TNC}^{(2)} \approx \mathcal{G}^{\alpha\beta}(\bar{X}^p) \partial_\alpha \delta X_m \partial_\beta \delta X^m + \varepsilon^{\alpha\beta} B_{MN}(\bar{X}^m) \partial_\alpha \delta X^M \partial_\beta \delta X^N
\]

\[
+ \varepsilon^{\alpha\beta} \delta X^m \delta^m_\alpha \partial_\alpha \varphi \partial_\beta \bar{\zeta} + \left( \frac{\delta \mathcal{G}^{\alpha\beta}}{\delta X^p} \right) \delta X^p (\partial_\alpha \bar{X}_m \partial_\beta \delta X^m + \partial_\alpha \delta X^m \partial_\beta \bar{X}_m)
\]

\[
+ 2 \varepsilon^{\alpha\beta} \left( \frac{\delta B_{MN}}{\delta X^n} \right) \delta X^n \partial_\alpha \delta X^M \partial_\beta \bar{X}^N - \Lambda (\delta X^m)^2
\]  

(70)

subjected to the fact that embedding coordinates \( (X^m) \) can be mapped to target space coordinates \( X^M(= X^M(X^m)) \) and vice versa. Here we introduce short hand notation for the metric \( \mathcal{G}^{\alpha\beta}(X^p) = \varepsilon^{\alpha\beta} \varepsilon_{\alpha\beta} \delta X_m (\delta X_m - \varepsilon_{\alpha\beta} \varepsilon_{\beta\gamma}) \). Also notice that the function, \( \mathcal{G}^{\alpha\beta}(X^m) \) when evaluated over constant radii solutions \( (\bar{X}^m) \) yields a trivial constant. Finally, it is worthwhile to notice that \( \bar{\zeta}(\tau, \sigma) \) has been introduced as fluctuations associated with the dual target space coordinate \( (\zeta) \) along which the string wraps.

Equations of motion that readily follow from \( (70) \) could be formally expressed as,

\[
\Box^{(2)} \delta X^m + \partial_\alpha (\varepsilon^{\alpha\beta} B_{MN} \partial_\beta \delta X^N) \left( \frac{\delta X^M}{\delta X^m} \right) - \frac{\varepsilon^{\alpha\beta}}{2} \delta^m_\alpha \partial_\alpha \varphi \partial_\beta \bar{\zeta} + \Lambda \delta X^m
\]

\[
- \frac{1}{2} \left( \frac{\delta \mathcal{G}^{\alpha\beta}}{\delta X^m} \right) (\partial_\alpha \bar{X}_p \partial_\beta \delta X^p + \partial_\alpha \delta X^p \partial_\beta \bar{X}_p) - \varepsilon^{\alpha\beta} \left( \frac{\delta B_{MN}}{\delta X^m} \right) \partial_\alpha \delta X^M \partial_\beta \bar{X}^N
\]

\[
+ \frac{1}{2} \left[ \partial_\alpha \left( \frac{\delta \mathcal{G}^{\alpha\beta}}{\delta X^m} \delta X^p \partial_\beta \bar{X}_p \right) + (\alpha \leftrightarrow \beta) \right] + \varepsilon^{\alpha\beta} \partial_\alpha \left( \frac{\delta B_{MN}}{\delta X^m} \delta X^n \partial_\beta \bar{X}^N \right) \left( \frac{\delta X^M}{\delta X^m} \right) = 0.
\]  

(71)

The above \( (71) \) is a second order inhomogeneous linear differential equation which therefore allows its most generic form of solution in terms of two dimensional Green’s function \( \mathcal{G}^{(2)}(\sigma^\alpha - \sigma^\alpha) \). In order to do so, we first rewrite \( (71) \) as,

\[
(\Box^{(2)} + 2\Lambda) \delta X^m = \mathcal{J}^m(\sigma^\alpha)
\]  

(72)

\[\text{Notice that here we include all four spacetime coordinates into one symbol namely, } X^p \equiv \{X^0, \zeta, X^m\} \text{ subjected to the constraint,} \]

\[
\sum_{m=1}^{3} (X^m)^2 = 1.
\]
where we identify the corresponding source density,

\[ j^m(\sigma^\alpha) = \frac{\varepsilon^{\alpha\beta}}{2} \delta^m_\alpha \partial_\beta \hat{\zeta} - \partial_\alpha (\varepsilon^{\alpha\beta} B_{MN} \partial_\beta \delta X^N) \left( \frac{\delta X^M}{\delta X^m} \right) \]

\[ + \frac{1}{2} \left( \frac{\delta G^{\alpha\beta}}{\delta X^m} \right) (\partial_\alpha \bar{X}_p \partial_\beta \delta X^p + \partial_\alpha \delta X^p + \partial_\beta \bar{X}_p) + \varepsilon^{\alpha\beta} \left( \frac{\delta B_{MN}}{\delta X^m} \right) \partial_\alpha \delta X^M \partial_\beta \bar{X}^N \]

\[ - \frac{1}{2} \left[ \partial_\alpha \left( \frac{\delta G^{\alpha\beta}}{\delta X^p} \delta X^p \partial_\beta \bar{X}^m \right) + (\alpha \leftrightarrow \beta) \right] - \varepsilon^{\alpha\beta} \partial_\alpha \left( \frac{\delta B_{MN}}{\delta X^n} \delta X^n \partial_\beta \bar{X}^N \right) \left( \frac{\delta X^M}{\delta X^m} \right) \]  

(73)

together with the identification of the 2D Laplacian, \( \Box^{(2)} = G^{\alpha\beta}(\bar{X}^p) \partial_\alpha \partial_\beta \). The most generic solution corresponding to (72) could therefore be expressed as\(^8\)

\[ \delta X^m(\sigma^\alpha) = - \int d^2 \tilde{\sigma} j^m(\tilde{\sigma}^\alpha) \Theta^{(2)}(\sigma^\alpha - \tilde{\sigma}^\alpha) \]  

(74)

subjected to the fact that,

\[ (\Box^{(2)} + 2\Lambda) \Theta^{(2)}(\sigma^\alpha - \tilde{\sigma}^\alpha) = -\delta^{(2)}(\sigma^\alpha - \tilde{\sigma}^\alpha). \]  

(75)

Using (74), one can finally estimate the (one loop) sigma model correction to the classical stringy spectrum (67) as,

\[ \delta E_S^{(1)} \approx \left( \frac{\sqrt{\lambda}}{4\pi} \right)^{1/2} \int_0^{2\pi} d\sigma \partial_\sigma \bar{X}^m \int d^2 \tilde{\sigma} j(\tilde{\sigma}^\alpha) \partial_\sigma \Theta^{(2)}(\sigma^\alpha - \tilde{\sigma}^\alpha). \]  

(76)

### 4.2.2 Solving the Green’s function

Below we briefly outline general solution to (75) by introducing the Fourier transform,

\[ \Theta^{(2)}(\sigma^\alpha - \tilde{\sigma}^\alpha) = \int d^2 \hat{K} e^{-i\hat{K}_a(\sigma^\alpha - \tilde{\sigma}^\alpha)} \Theta^{(2)}(\mathbf{w}, \hat{K}). \]  

(77)

Substituting (77) into (75) we find,

\[ (\Box^{(2)} + 2\Lambda) \Theta^{(2)}(\sigma^\alpha - \tilde{\sigma}^\alpha) = \int d^2 \hat{K} \left( \beta \mathbf{w}^2 + \gamma \mathbf{w} \hat{\mathbf{K}} + \tilde{\beta} \hat{\mathbf{K}}^2 - 2\Lambda \right) \times e^{-i\hat{K}_a(\sigma^\alpha - \tilde{\sigma}^\alpha)} \Theta^{(2)}(\mathbf{w}, \hat{K}) \]  

(78)

which finally yields,

\[ \Theta^{(2)}(\mathbf{w}, \hat{K}) = \frac{1}{\beta \mathbf{w}^2 + \gamma \mathbf{w} \hat{\mathbf{K}} + \tilde{\beta} \hat{\mathbf{K}}^2 - 2\Lambda} \]  

(79)

where we identify \( \mathbf{w} \) and \( \hat{K} \) respectively as the frequency and momentum in the Fourier space. Moreover, here \( \beta, \gamma \) and \( \tilde{\beta} \) are constant coefficients which are essentially determined by the elements of \( 2 \times 2 \) matrix \( \Theta^{\alpha\beta}(\bar{X}^p) \).

---

\(^8\)The solution (74) is subjected to the periodicity condition (19). This implies that the Green’s function must satisfy, \( \Theta^{(2)}(\tau, \sigma + 2\pi) = \Theta^{(2)}(\tau, \sigma) \).
5 Large $c$ limit

We are now going to explore the integrability criteria in the tensionless limit \(^{[11]}\) of the TNC sigma model \((15)\). This limit is achieved by taking simultaneous large $c$ limit of the associated world-sheet d.o.f. and is indeed an important limit to consider for the following reasons. Taking a second scaling limit on the world-sheet fields results in the so called $U(1)$ Galilean geometry \(^{[11]}\) (with $R \times S^2$ topology) as the target space over which strings are propagating. In case of type IIB (super)strings the corresponding dual gauge theory (also known as the Spin-Matrix Theory (SMT)\(^{[14]}\)) has been identified as some sort of a decoupling ($\lambda \to 0$) limit of $N = 4$ SYM where only states close to the near BPS bound survive \(^{[11]}\). This therefore opens up new possibilities for better understanding as well as test the duality conjecture at least in the near BPS sector in semi-classical limit.

To start with, we consider the following scaling limit \(^{[17]}-^{[18]}\) associated with the world-sheet fields,

\[
\begin{align*}
\lambda &= \frac{g}{c^2}; \\
t &= c^2 t; \\
\psi &= \psi; \\
\theta &= \theta; \\
\varphi &= \varphi; \\
\zeta &= c \tilde{\zeta}; \\
B_{MN} &= c \tilde{B}_{MN}; \\
b_\mu &= c \tilde{b}_\mu
\end{align*}
\]

which upon substitution into the original action \((15)\) yields the NR action, \(\tilde{S} = \frac{\sqrt{g}}{4\pi} \int d^2 \sigma \mathcal{L}_{NG}\) together with the NR Lagrangian,

\[
\mathcal{L}_{NG} \approx \frac{\epsilon^{\alpha \alpha'} \epsilon^{\beta \beta'}}{\epsilon^{\alpha \beta}} \partial_\alpha t \partial_\beta t \ h_{mn} \partial_\alpha X^m \partial_\beta X^n + \epsilon^{\alpha \beta} \tilde{B}_{MN} \partial_\alpha X^M \partial_\beta X^N + 2\epsilon^{\alpha \beta} m_\alpha (\tilde{b}_\beta + \partial_\beta \tilde{\zeta}) + \mathcal{O}(c^{-2}).
\]  

(81)

We choose to work with the following ansatz,

\[
\varphi(\tau, \sigma) = \varpi \tau + \xi(\sigma); \quad t = 2 \tau; \quad \tilde{\zeta} = \sigma; \quad \theta = \theta(\sigma)
\]

(82)

which upon substitution into \((81)\) yields,

\[
\mathcal{L}_{NG} = \varpi^2 \vartheta_1^2 + \xi^2 \vartheta_1^2 \delta_{11} + \omega \mathcal{B} \vartheta_2' + \varpi \vartheta_2 - \Lambda(\vartheta_1^2 - 1).
\]

(83)

5.1 Deformed Rosochatius model

Given \((83)\), the corresponding canonical momenta could be formally expressed as,

\[
\begin{align*}
\Pi_1 &= 2\vartheta_1' \\
\Pi_2 &= 2\vartheta_2' + \omega \mathcal{B} \\
\Pi_\xi &= 2\xi' \vartheta_1^2 = const.
\end{align*}
\]

(84)

(85)

(86)

which yields the corresponding canonical Hamiltonian density as,

\[
\mathcal{H}_c = \Pi_1^2 + \frac{\Pi_2^2}{\vartheta_1^2} + \tilde{\Lambda}(\vartheta_1^2 - 1) + \Delta \mathcal{H}
\]

(87)

where we have re-scaled the Hamiltonian (as well as the Lagrange multiplier, $\Lambda \to \tilde{\Lambda} = 4\Lambda$) by an overall factor of 4 and identify the corresponding deformation piece,

\[
\Delta \mathcal{H} = \varpi^2 \mathcal{B}^2 - 2\varpi \mathcal{B} \Pi_2 - 4\varpi \vartheta_2.
\]

(88)
Notice that, the deformation $\Delta h$ vanishes as we set the limit, $\varpi \to 0$ and therefore the corresponding Hamiltonian system \(^{87}\) reduces to the standard Rosochatius like integrable models as discussed in the previous Section.

Finally, we note down the primary Hamiltonian,

$$ \mathfrak{H}_P = \mathfrak{H}_c + \tilde{\lambda} \phi_P $$ \hspace{1cm} (89)

where we identify the associated primary constraint,

$$ \phi_P = \Pi \tilde{\Lambda} = \frac{\partial L_{NG}}{\partial \Lambda'} \approx 0. \hspace{1cm} (90) $$

Given the structure \(^{89}\), we next categorize the class of constraints associated with the dynamical system under consideration. Setting the variation of $\phi_P$ equal to zero we find,

$$ \psi_1 \approx \{ \phi_P, \mathfrak{H}_P \}_{PB} = \vartheta_i^2 - 1 \approx 0. \hspace{1cm} (91) $$

The zero variation of $\psi_1$ results in the secondary constraint of the following form,

$$ \psi_2 \approx \{ \psi_1, \mathfrak{H}_P \}_{PB} = \Pi_i \vartheta_i - \varpi B \vartheta_2 \approx 0. \hspace{1cm} (92) $$

Any further variation of $\psi_2$ gives back the canonical structure \(^{87}\) which thereby concludes the chain of constraints in the theory. Like in the previous example, we are therefore left with two integrals of motion one of which is the canonical momenta $(\Pi_\xi)$ conjugate to $\xi$. The other integral of motion is what we identify below as the generalized conserved charge associated with deformed Rosochatius model constructed above.

### 5.2 Integrals of motion

Below we propose the generalized integral of motion,

$$ \mathfrak{F}_R = \frac{1}{2} (\Pi_i \vartheta_j - \Pi_j \vartheta_i)^2 + \Pi_i^2 \frac{\partial f}{\partial \vartheta_i^2} + f(\Pi_i, \vartheta_i) $$ \hspace{1cm} (93)

subjected to the evaluation of the constraints \(^{91}\) and \(^{92}\). Below we set,

$$ f(\Pi_i, \vartheta_i) = \varpi^2 B^2 (1 + \vartheta_2^2) - 2 \varpi B \Pi_2 - 4 \varpi \vartheta_2 $$ \hspace{1cm} (94)

which thereby yields,

$$ \mathfrak{F}_R = \mathfrak{H}_c - \Pi_\xi^2. \hspace{1cm} (95) $$

Notice that, $f(\Pi_i, \vartheta_i) \to 0$ in the limit $\varpi \to 0$ and thereby one recovers the original Rosochatius model. As a further crosscheck we notice that,

$$ \{ \mathfrak{F}_R, \mathfrak{H}_c \}_{DB} = \{ \mathfrak{H}_c, \mathfrak{H}_c \}_{DB} = 0 $$ \hspace{1cm} (96)

together with the fact, $\{ \mathfrak{F}_R, \Pi_\xi \}_{DB} = 0$. Which thereby ensures that all the charges are in involution.
6 NR spectrum

Like in the previous example for TNC strings, we now proceed towards finding the corresponding dispersion relation for NR strings rotating over $R \times S^2$. The equations of motion that readily follow from (83) could be formally expressed as,

$$\vartheta_1'' = (b^2 - \Lambda) \vartheta_1$$  \hspace{1cm} (97)

$$\vartheta_2'' = \frac{\varpi}{2} - \Lambda \vartheta_2.$$  \hspace{1cm} (98)

Next, we note down the conserved charges associated with the sigma model,

$$P_a = \frac{\sqrt{g_4}}{4\pi} \int_0^{2\pi} d\sigma \Theta_a^{(r)}$$  \hspace{1cm} (99)

where we define the momentum density,

$$\Theta_a^{(r)} = \frac{\delta \mathcal{L}_{NG}}{\delta (\partial_\tau X^a)} ; \ a = t, \varphi.$$  \hspace{1cm} (100)

An explicit computation reveals the energy and angular momentum of the NR string,

$$E_S = \frac{\sqrt{g}}{8\pi} \int_0^{2\pi} d\sigma (\vartheta_1'^2 + \vartheta_2'^2 + \vartheta_2^2 \xi^2.)$$  \hspace{1cm} (101)

$$J_\varphi = \frac{\sqrt{g}}{4\pi} \int_0^{2\pi} d\sigma (B\vartheta_2' + \vartheta_2).$$  \hspace{1cm} (102)

6.1 Constant radii solutions

The solutions to the above set of equations (97)-(98) are subjected to the constraint (91). In order to obtain solutions, like before we set the radii to be constant namely,

$$\vartheta_2(\sigma) = c_2 : \vartheta_1(\sigma) = c_1$$  \hspace{1cm} (103)

which thereby sets the Lagrange multiplier, $\Lambda = b^2$. From (98), the solution corresponding to $c_2$ follows immediately,

$$c_2 = \frac{\varpi}{2b^2}$$  \hspace{1cm} (104)

and thereby the solution corresponding to,

$$c_1 = \sqrt{1 - \frac{\varpi^2}{4b^4}} \approx 1 - \frac{\varpi^2}{8b^4} \mathcal{O}(\varpi^4).$$  \hspace{1cm} (105)

Substituting constant radii solutions into (102) we find,

$$J_\varphi = \frac{\sqrt{g} \varpi}{4b^2}.$$  \hspace{1cm} (106)

Using (106), we finally arrive at the dispersion relation of the following form,

$$|E_S| = \frac{b^2}{4} J_\varphi \left( \frac{4}{\sqrt{g}} - \sqrt{\hat{g}} \right)$$  \hspace{1cm} (107)

where, like in the case for TNC strings, we introduce an effective coupling $\hat{g}(= \frac{g}{5^2})$ in the dual SMT theory such that, $|\hat{g}| \lesssim 1$ in the limit of large R charge.
6.2 Adding fluctuations

In order to introduce fluctuations, as in the previous case, we consider following expansion
of the world-sheet fields,

\[ X^m = \bar{X}^m + g^{-1/4}\delta x^m(\sigma^\alpha) \]  

(108)

together with fluctuations along other two coordinates namely, \( \delta X^0 \sim \tilde{\sigma}^\alpha \) and \( \delta \tilde{\zeta} \sim \frac{\tilde{\sigma}^\alpha}{g^{1/4}} \). Moreover, here \( \bar{X}^m \)s correspond to classical background solutions (103) and \( \delta \bar{x}^m \)s are the corresponding stringy excitations. Using (108), the corresponding quadratic (in fluctuations) Lagrangian density turns out to be,

\[ \mathcal{L}^{(2)}_{NG} \approx \partial_\sigma \delta \bar{x}_m \partial_\sigma \delta x^m + \varepsilon^{\alpha\beta} \mathcal{B}_{MN}(\bar{X}^m) \partial_\alpha \delta X^M \partial_\beta \delta X^N + \varepsilon^{\alpha\beta} \partial_\sigma \delta \bar{x}_m \partial_\alpha \delta x^m \partial_\beta \delta x^m - \frac{\varepsilon^{\alpha\beta}}{4\pi} \int d^2 \tilde{\sigma} \tilde{j}(\tilde{\sigma}^\alpha) \partial_\alpha \tilde{\sigma} \mathcal{G}^{(2)}(\sigma^\alpha - \tilde{\sigma}^\alpha) \]  

(109)

which is therefore structurally quite similar to that of (70) except for the fact that we identify the new function, \( \tilde{G}^{\alpha\beta}(X^0, \tilde{\zeta}) = \frac{\varepsilon^{\alpha\beta}}{e_{\alpha\beta} t_{b \alpha} t_{b \beta} + \varepsilon_{\alpha\beta} t_{b \alpha} t_{b \beta}} \partial_\alpha \tilde{t} \partial_\beta \tilde{t} \). The rest of the analysis therefore follows quite trivially and one essentially lands up in an expression that corresponds to first order correction to the spectrum,

\[ \delta \mathcal{E}^{(1)}_S \approx \frac{(\sqrt{g})^{1/2}}{4\pi} \int_0^{2\pi} d\sigma \partial_\sigma X^m \int d^2 \tilde{\sigma} \tilde{j}(\tilde{\sigma}^\alpha) \partial_\alpha \tilde{\sigma} \mathcal{G}^{(2)}(\sigma^\alpha - \tilde{\sigma}^\alpha) \]  

(110)

where the expression for the corresponding source density \( \tilde{j}(\tilde{\sigma}^\alpha) \) remains quite identical to the previous expression in (73) except for the fact that the old function \( \mathcal{G}^{\alpha\beta}(X^p) \) is now replaced by the new one namely, \( \tilde{G}^{\alpha\beta}(X^0, \tilde{\zeta}) \). The solution to the Green’s function could be obtained by going to the Fourier space as in the earlier case. A straightforward computation finally reveals the momentum space Green’s function as,

\[ \mathcal{G}^{(2)}(w, \kappa) \sim \frac{1}{\kappa^2 - 2\Lambda}. \]  

(111)

7 Summary and final remarks

We conclude our paper with a brief summary of the key results and figuring out some of the possible future applications. We reduce 2D sigma models corresponding to torsional Newton Cartan strings on \( R \times S^2 \) into (spin)deformed 1D Neumann-Rosochatius integrable models and thereby show the underlying integrable structure of the theory. Taking large \( c \) limit of the world-sheet d.o.f. we further map the sigma model into deformed 1D Rosochatius model. In both cases we obtain the (semi)classical string spectrum and unlike the relativistic case [27], the leading order correction appears to be \( \sim (\sqrt{\lambda})^{-1} \). The question that remains to be explored is how to reproduce the nonrelativistic (NR) stringy spectrum from a dual gauge theory perspective where the operators in the dual (gauge) theory are supposed to be labelled with large quantum numbers.
There are some further issues that also remain to be explored. For example, the present analysis only concerns about a particular class of solutions and fluctuations around it. However, it could be extended further to construct the most general class of solutions by introducing so called ellipsoidal coordinates \[26\]. The other direction that could possibly be interesting is to construct the nonrelativistic (NR) analogue of the 2D dual version of the rotating string ansatz \[7\]-\[8\] by exchanging the role of world-sheet coordinates \[27\]. This leads to what is known as NR pulsating string configurations \[17\] with time dependent radial functions. We hope to address some of these issues in the near future.

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