The Tensor Quadratic Forms

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Abstract
We consider the following data perturbation model, where the covariates incur multiplicative errors. For two \( n \times m \) random matrices \( U, X \), we denote by \( U \circ X \) the Hadamard or Schur product, which is defined as \( (U \circ X)_{ij} = (U_{ij}) \cdot (X_{ij}) \). In this paper, we study the subgaussian matrix variate model, where we observe the matrix variate data \( X \) through a random mask \( U \):
\[
X = U \circ X \quad \text{where} \quad X = B^{1/2}ZA^{1/2},
\]
where \( Z \) is a random matrix with independent subgaussian entries, and \( U \) is a mask matrix with either zero or positive entries, where \( E(U_{ij}) \in [0, 1] \) and all entries are mutually independent. Subsampling in rows, or columns, or random sampling of entries of \( X \) are special cases of this model. Under the assumption of independence between \( U \) and \( X \), we introduce componentwise unbiased estimators for estimating covariance \( A \) and \( B \), and prove the concentration of measure bounds in the sense of guaranteeing the restricted eigenvalue conditions to hold on the estimator for \( B \), when columns of data matrix \( X \) are sampled with different rates. Our results provide insight for sparse recovery for relationships among people (samples, locations, items) when features (variables, time points, user ratings) are present in the observed data matrix \( X \) with heterogenous rates. Our proof techniques can certainly be extended to other scenarios.

1 Introduction

In this paper, we study the multiplicative measurement errors on matrix-variate data in the presence of missing values, and sometimes entirely missed rows or columns. Missing value problems appear in many application areas such as energy, genetics, social science and demography, and spatial statistics; see \([14, 20, 16, 24, 21, 19, 12, 7]\). For complex data arising from these application domains, missing values is a norm rather than an exception. For example, in spatio-temporal models in geoscience, it is common some locations will fail to observe certain entries, or at different time points, the number of active observation stations varies \([32, 31]\). In social science and demography, the United States Census Bureau was involved in a debate with the U.S. Congress and the U.S. Supreme Court over the handling of the undercount in the 2000 U.S. Census \([24]\). In addition to missing values, data are often contaminated with an additive source of noise on top of the multiplicative noise such as missing values \([20, 10, 9]\).

For two \( n \times m \) random matrices \( U, X \), we denote by \( U \circ X \) the Hadamard or Schur product, which is defined as \( (U \circ X)_{ij} = (U_{ij}) \cdot (X_{ij}) \). Let \( \Delta \) be a random \( n \times m \) noise matrix such that \( E\Delta_{ij} = 0 \). We consider the following data perturbation model, where the covariates incur additive and/or...
multiplicative errors; that is, instead of $X$, we observe
\[ X = U \circ X + \Delta, \]
where $U = [u^1, \ldots, u^m]$ is a random mask with either zero or positive entries and $\mathbb{E}U_{ij} \in [0, 1]$. Additive errors, subsampling in rows, or columns, or random sampling of entries of $X$ are special cases of this model.

The matrix variate normal model has a long history in psychology and social sciences, and is becoming increasingly popular in biology and genetics, econometric theory, image and signal processing and machine learning in recent years. Consider a space-time model $X(s,t)$ where $s$ denotes spatial location and $t$ denotes time. In the space-time model literature [cf. 12] a common assumption on the covariance of $X$ is separability, namely
\[ \text{Cov}(X(s_1,t_1),X(s_2,t_2)) = A_0(s_1,s_2)B_0(t_1,t_2) \]
where $A_0$ and $B_0$ are each covariance functions. Suppose we observe $X = U \circ X$, where data is randomly subsampled, and the aim is to recover the full rank covariance matrices $A_0, B_0 \succeq 0$ in the tensor-product for the matrix variate model.

The goal of this paper is to obtain operator norm-type of bounds on estimating submatrices of $B_0$ using matrix concentration of measure analyses. Specifically, we focus on deriving concentration of measure bounds for a componentwise unbiased estimator $\tilde{B}_0$ in the space-time context (2) under the observation model (1) with $\Delta = 0$. In this model, a mean zero column vector $x^j$ corresponds to values observed across $n$ spatial locations at a single time point $t_j, j = 1, 2, \ldots$. The reason we focus on $B_0$ is because of the motivation for handling streaming type of data, where for each node (row), there is a continuous data stream over time, but at each time point, we may only have a subset of the $n$ observations. That is, instead of $x^j$, we observe $x^j \circ u^j, j = 1, \ldots, m$.

An estimator that provides componentwise unbiased estimate for covariance matrix $B_0$ was introduced in [41],
\[ \tilde{B}_0 = X'X^T \odot \mathcal{M}, \quad \text{for } \mathcal{M} \text{ as defined in (4)}, \]
where $\odot$ denotes componentwise division and we use the convention of $0/0 = 0$. The matrix $\mathcal{M}$ is a linear combination of rank-one matrices $M_1, \ldots, M_m \in \mathbb{R}^{n \times n}$ as follows:
\[ \mathcal{M} := \sum_{j=1}^m a_{jj} M_j \quad \text{where } M_j = \mathbb{E}(u^j \otimes u^j) \]
and $a_{11}, a_{22}, \ldots$ are diagonal entries of matrix $A_0$. Here and in the sequel, we assume $\text{tr}(B_0) = n$ (cf. [41]). For completeness, we will also present the corresponding oracle estimator $\tilde{A}_0$ as studied in [41] in Section (2.1). In this paper, we assume that each column $u^j = (u_{1j}, \ldots, u_{nj})^T \in \{0, 1\}^n, j = 1, \ldots, m$ of the mask matrix $U$ is composed of independent Bernoulli random variables such that $\mathbb{E}u_{kj} = p_j, k = 1, \ldots, n$, and moreover, $X$ and $U$ are independent of each other.

It is difficult to obtain an accurate matrix operator norm bound from componentwise matrix maximum norm bounds, although, for diagonal matrices, these two are the same (cf. Theorem 3.1). Hence, while $\tilde{A}_0$ and $\tilde{B}_0$ provide accurate componentwise estimates for $A_0$ and $B_0$ in the sense...
of Theorems 5.1-5.4 in [41], one cannot hope to readily obtain convergence properties in terms of estimating the covariance matrix $A_0$ and $B_0$ as a whole, as they may not be positive-semidefinite.

This poses the primary question we address in this work. The theory and estimation tasks we are tackling in the present work depart significantly from the baseline model where we observe the full data matrix following a matrix variate normal model (2). In this paper, we consider this separable covariance model defined through the tensor product of $A_0, B_0$, however, now under the much more general subgaussian distribution, where we also model the sparsity in data with a random mask. This creates numerous technical challenges when analyzing the quadratic forms as there are no existing tools for handling such complex data generating models.

In this paper, we prove new concentration of measure inequalities for quadratic forms involving sparse and nearly sparse vectors; in particular, concentration of measure bounds on the following quadratic form, where $M$ is as defined in (4),

$$q^T(AX^T \odot M - E(AX^T \odot M))q \quad \text{over a class of vectors } q \in \mathbb{R}^n$$

satisfying the following cone constraint (6), where $0 < s_0 \leq n$,

$$\text{Cone}(s_0) := \{ v : |v|_1 \leq \sqrt{s_0} \|v\|_2 \}, \quad \text{where } |v|_1 = \sum_{i=1}^{n} |v_i|.$$  

Such objects arise naturally from the context of high dimensional sparse regression. Concentration of measure bounds on (5) lead to the conclusion that certain restricted eigenvalue (RE) conditions hold for design matrices with high probability, guaranteeing sparse recovery using the Lasso or Dantzig selector-type of estimators [36, 11, 8]. We state Theorem 2.7 for this purpose for the perturbation model as in Definition 2.1.

We make the following theoretical contributions: (a) We present the concentration of measure bounds on quadratic forms and certain functionals of large random matrices $AX^T$ and $X^TX$; (b) Suppose that each entry of column $j$ of data matrix $X = [x^1, \ldots, x^m]$ is observed with probability $p_j$, we introduce an estimator $\hat{M}$ for matrix $M$ as defined in (4) and show that entries in $\hat{M}$ are tightly concentrated around their mean values in $M$; (c) Combining the results we obtain on (5) and on estimating $M$ with estimator $\hat{M}$ (cf. (18)), we obtain an error bound on the quadratic form

$$\left| q^T(AX^T \odot \hat{M} - E(AX^T \odot M))q \right| \quad \text{for } q \in S^{n-1} \cap \text{Cone}(s_0)$$

Such analyses exploit and extend the concentration of measure inequalities on quadratic forms involving sparse subgaussian random vectors as studied in [41], which we generically refer to as sparse Hanson-Wright inequalities, as well as the analysis framework we have newly developed in the current work in analyzing (3) and (5). See [18, 39, 28] and references therein, for classic results and recent expositions on quadratic forms over dense vectors, and their applications.

### 1.1 Related work

RE conditions have been explored in the literature for other classes of random design matrices, see for example [5, 37, 25, 29, 22, 30]. Our analysis framework extends, with suitable adaptation, to the general distributions of $U$ with independent nonnegative elements, which are independent of the
(unobserved) matrix variate data \( X \). In particular, the mask \( \mathbf{1} \) and estimator \( \mathbf{3} \), as well as the corresponding pair for \( A_0 \) (cf. (13) and (14)), work for such general distributions of \( U \) that may not necessarily have binary entries. From these initial estimators, we can subsequently derive penalized estimators for covariance matrices and their inverses using nodewise regression \[23\] or the graphical Lasso-type estimators \[15, 3\]. It turns out that such concentration of measure properties are also essential to ensure algorithmic convergence, and hence to bound both optimization and statistical errors, for example, when approximately solving optimization problems such as the corrected Lasso using the gradient-descent type of algorithms [cf. 1, 22]. These estimators were introduced to address high dimensional errors-in-variables regression problems including the missing values; See also \[26, 27, 4\].

The problem we study here is also different from matrix completion, which focuses on recovering low-rank structures. We focus here on recovering the full rank covariance matrices \( A_0, B_0 > 0 \) in the matrix variate model, but with incomplete data, or intentionally subsampled data. In \[30\], we analyzed such quadratic forms in errors-in-variables models, where data matrix \( X \) of space-time covariance models for the observation matrix .

\( X \) is contaminated with a perturbation matrix \( \Delta \) which consists of spatially correlated subgaussian noise as column vectors; there we introduced the additive errors in the covariates, resulting in a non-separable class of space-time covariance models for the observation matrix.

When \( X \) is observed in full and free of noise, the theory is already in place on estimating matrix variate Gaussian graphical models. Under sparsity conditions, \[40\] is the first in literature to show that one can estimate the graphs, covariance and inverse covariance matrices well using only one instance from the matrix-variate normal distribution with a separable covariance structure \[2\]. See also \[2, 33, 34\], where EM based method for sparse inverse covariance matrix estimation and missing value imputation algorithms in the matrix-variate normal model were considered.

### 1.2 Definitions and notations

Let \( e_1, \ldots, e_n \) be the canonical basis of \( \mathbb{R}^n \). For a set \( J \subset \{1, \ldots, n\} \), denote \( E_J = \text{span}\{e_j : j \in J\} \). Let \( B_2^n \) and \( S^n \) be the unit Euclidean ball and the unit sphere respectively. For a symmetric matrix \( A \), let \( \lambda_{\max}(A) \) and \( \lambda_{\min}(A) \) be the largest and the smallest eigenvalue of \( A \) respectively, and its spectral radius \( \rho(A) = \{\max |\lambda| : \lambda \text{ eigenvalue of } A\} \). For a matrix \( A \), the operator norm \( \|A\|_2 \) is defined to be \( \sqrt{\lambda_{\max}(A^T A)} \). For a vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), denote by \( \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \) and \( |x|_1 := \sum_{j} |\theta_j| \). For a matrix \( A = (a_{ij}) \) of size \( m \times n \), let \( \|A\|_\infty = \max_{i} \sum_{j=1}^{n} |a_{ij}| \) denote the maximum absolute row sum and \( \|A\|_1 = \max_{j} \sum_{i=1}^{m} |a_{ij}| \) denote the maximum absolute column sum of the matrix \( A \). The matrix Frobenius norm is given by \( \|A\|_F = (\sum_{i,j} a_{ij}^2)^{1/2} \). Let \( \|A\|_{\max} = \max_{i,j} |a_{ij}| \) denote the componentwise max norm. Let \( \text{diag}(A) \) be the diagonal of \( A \). Let \( \text{offd}(A) \) be the off-diagonal of \( A \). For a given vector \( x \in \mathbb{R}^m \), \( \text{diag}(x) \) denotes the diagonal matrix whose main diagonal entries are the entries of \( x \). And we write \( D_x := \text{diag}(x) \) interchangeably. For matrix \( A \), \( r(A) \) denotes the effective rank \( \text{tr}(A)/\|A\|_2 \). Let \( \kappa(A) = \lambda_{\max}(A)/\lambda_{\min}(A) \) denote the condition number for matrix \( A \). For two numbers \( a, b \), \( a \wedge b := \min(a, b) \), and \( a \vee b := \max(a, b) \). We write \( a \asymp b \) if \( ca \leq b \leq Ca \) for some positive absolute constants \( c, C \) which are independent of \( n, m \), sparsity, and sampling parameters. We write \( f = O(g) \) or \( f \ll g \) if \( |f| \leq Cg \) for some absolute constant \( C < \infty \) and \( f = \Omega(g) \) or \( f \gg g \) if \( g = O(f) \). We write \( f = o(g) \) if \( f/g \to 0 \) as
$n \to \infty$, where the parameter $n$ will be the size of the matrix under consideration. In this paper, $C_1, c_1, c', c_0$, etc, denote various absolute positive constants which may change line by line.

**Organization of the paper.** The rest of the paper is organized as follows. In Section 2, we present our model and the main theoretical results on bounding the quadratic forms, as well as discussions on our method in this paper. We present our main technical result with respect to analyzing the random quadratic form in Section 3, where we also present the analysis framework for Theorem 2.4. We discuss our proof strategy for Theorem 3.3 in Section 4. We prove Theorems 2.4 and 2.8 in Sections 5 and 6 respectively. We place all technical proofs in the supplementary material.

### 2 Models and the main result

We need the following definitions and notation to introduce our data generative model. Let $B_0 = (b_{ij}) \in \mathbb{R}^{n \times n}$ and $A_0 = (a_{ij}) \in \mathbb{R}^{m \times m}$ be positive definite matrices. Denote by $B_0^{1/2}$ and $A_0^{1/2}$ the unique square root of $B_0$ and $A_0$ respectively. For a random variable $Z$, the subgaussian (or $\psi_2$) norm of $Z$ denoted by $\|Z\|_{\psi_2}$ is defined as

$$\|Z\|_{\psi_2} = \inf\{t > 0 : \mathbb{E}\exp(Z^2/t^2) \leq 2\}.$$

**Definition 2.1.** (Random mask sparse model) We denote by $X = [x^1 \mid x^2 \mid \ldots \mid x^m]$ the full (but not fully observed) $n \times m$ data matrix with column vectors $x^1, \ldots, x^m \in \mathbb{R}^n$ and row vectors $y^1, \ldots, y^m$. Consider the data matrix $X_{n \times m}$ generated from a subgaussian random matrix $Z_{n \times m}$:

$$X = B_0^{1/2}ZA_0^{1/2}, \quad \text{where } Z_{n \times m} = (Z_{ij}) \text{ with}$$

$$\|Z\|_{\psi_2} \leq K \quad \text{and} \quad \mathbb{E}Z_{ij}^2 = 1 \quad \forall i, j. \quad \text{Without loss of generality, we assume } K = 1. \quad \text{Suppose that we now observe}$$

$$X' = U \odot X, \quad \text{where } U \in \{0, 1\}^{n \times m} \text{ is a mask matrix and}$$

$$U = [v^1 \mid v^2 \mid \ldots \mid v^m] = [v^1 \mid v^2 \mid \ldots \mid v^m]^T \quad \text{is independent of } X,$$

with independent row vectors $v^1, \ldots, v^m \sim \mathcal{E} \in \{0, 1\}^m$ where $v$ is composed of independent Bernoulli random variables with $\mathbb{E}v_k = p_k, k = 1, \ldots, m$.

A positive definite matrix $\Sigma$ is said to be separable if it can be written as a Kronecker product of two positive definite matrices $A_0$ and $B_0$ for which we denote by $\Sigma = A_0 \otimes B_0 = (a_{ij}B_0)$, where $\otimes$ denotes the Kronecker product. Notice that we can only estimate $A_0$ and $B_0$ up to a scaled factor, as the tensor product $A_0 \otimes B_0 = A_0 \eta \otimes \frac{1}{\eta} B_0$ for any $\eta > 0$. When $Z$ in (8) is a Gaussian random ensemble with i.i.d. $\mathcal{N}(0, 1)$ entries, we say that random matrix $X$ as defined in (8) follows the matrix-variate normal distribution with a separable covariance structure $\Sigma = A_0 \otimes B_0$:

$$X_{n \times m} \sim \mathcal{N}_{n,m}(0, A_0_{m \times m} \otimes B_0_{n \times n}).$$

This is equivalent to say vec $\{X\}$ follows a multivariate normal distribution with mean 0 and covariance $\Sigma$. Here vec $\{X\}$ is formed by stacking the columns of $X$ into a vector in $\mathbb{R}^{mn}$. See [13, 17, 40] for characterization and examples on matrix-variate normal distributions.

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2.1 Estimators and problem formulations

Recall that for two \( m \times m \) matrices \( A = (a_{ij}) \) and \( W = (w_{ij}) \), we use \( W \circ A \) to denote their Hadamard product such that \( (W \circ A)_{ij} = w_{ij} \cdot a_{ij} \) while \( (W \odot A) \) denotes the Hadamard or componentwise division \( w_{ij}/a_{ij} \). We use the convention that \( 0/0 = 0 \).

**Definition 2.2.** For the rest of the paper, we assume \( B_0 \) is scaled such that \( \text{tr}(B_0) = n \) in view of (15); hence \( \|B_0\|_2 \geq 1 \). Let \( \mathcal{X} = U \circ X \). In order to estimate \( B_0 \), we define the following oracle estimator:

\[
\tilde{B}_0 = \mathcal{X} \mathcal{X}^T \odot \mathcal{M} \quad \text{where} \quad \mathcal{M}_{k\ell} = \begin{cases} \sum_{j=1}^{m} a_{jj}p_{jj} & \text{if } \ell = k; \\ \sum_{j=1}^{m} a_{jj}p_{jj}^2 & \text{if } \ell \neq k. \end{cases} \tag{12}
\]

Clearly, \( \mathbb{E}\tilde{B}_0 = B_0 \), where expectation denotes the componentwise expectation of each entry of \( \tilde{B}_0 \).

In order to estimate \( A_0 \), we consider the corresponding oracle estimator

\[
\tilde{A}_0 = \mathcal{X}^T \mathcal{X} \odot \mathcal{N} \quad \text{where} \quad \mathcal{N}_{ij} = \text{tr}(B_0)\mathbb{E}v_i^j \otimes v^i \tag{13}
\]

and \( \mathcal{N}_{ij} = \text{tr}(B_0) \begin{cases} p_i & \text{if } i = j; \\ p_ip_j & \text{if } i \neq j, \end{cases} \tag{14} \)

where \( \text{tr}(B_0) \) is the trace of matrix \( B_0 \). Similar to (11), (13) works for general distributions of \( U \), while (14) works for the model (10) under consideration. Justifications for estimators (12) and (13) appear in Section 3 and Section A in the supplementary material. Additionally, we need to estimate the population parameters in \( A_0 \) and \( \tilde{B}_0 \), as after all, we are estimating these quantities such as \( \text{tr}(B_0) \) and diagonal elements in \( A_0 \), which are identifiable only up to a scaled factor. Hence, we propose to estimate covariance matrices \( A_* \) and \( B_* \)

\[
A_* := A_0 \text{tr}(B_0)/n \quad \text{and} \quad B_* := nB_0/\text{tr}(B_0) \tag{15}
\]

using the following set of sample based plug-in estimators. Denote by

\[
p := (p_1, \ldots, p_m) \quad \text{the vector of column sampling probabilities.} \tag{16}
\]

Let \( \hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m) \) denote the estimate of sampling probabilities \( p \) (16), where \( \hat{p}_j = \frac{1}{n} \sum_{k=1}^{n} w_{kj} \) is the average number of non-zero (observed) entries for column \( j \). Let \( \hat{M} \) be as defined in (17).

Construct

\[
\hat{A}_* = \frac{1}{n} \mathcal{X}^T \mathcal{X} \odot \hat{M} \quad \text{where} \quad \hat{M}_{ij} = \begin{cases} \hat{p}_i & \text{if } i = j; \\ \hat{p}_i\hat{p}_j & \text{if } i \neq j; \end{cases} \tag{17}
\]

and

\[
\hat{B}_* = \mathcal{X} \mathcal{X}^T \odot \hat{M} \quad \text{where} \tag{18}
\]

\[
\hat{M}_{k\ell} = \begin{cases} \frac{1}{n} \text{tr}(\mathcal{X}^T \mathcal{X}) & \text{if } k = \ell, \\ \frac{1}{n(n-1)} \text{tr}(\mathcal{X}^T \mathcal{X}) & \text{if } k \neq \ell. \end{cases}
\]

Clearly \( \mathbb{E}\hat{M} = D_p + \text{offd}(p \otimes p) \), where \( D_p = \text{diag}(p) = \text{diag}(p_1, \ldots, p_m) \) denotes the diagonal matrix with entries of \( p \) (16) along its main diagonal. It is straightforward to check that \( \hat{M} \) is a componentwise unbiased estimator for \( M \) when \( \text{tr}(B_0) = n \); and hence \( \hat{B}_* \) as defined in (18) is considered as a natural plug-in estimator of \( B_* \), which is completely data-driven and does not
Throughout this paper, we use \( \psi \). Definition 2.3. Denote by \( s \) \( n \leq \) vector, where 1 \( k \leq \) \( n \). Let \( B \) be an \( n \times n \) matrix. A \( k \times k \) submatrix of \( B \) formed by deleting \( n - k \) rows of \( B \), and the same \( n - k \) columns of \( B \), is called principal submatrix of \( B \).

**Definition 2.3.** Denote by \(|B_0| = (|b_{ij}|)\) the entrywise absolute value of a matrix \( B_0 \). A vector \( q \in \mathbb{S}^{n-1} \) is \( s_0 \)-sparse, where parameter \( s_0 \leq n \), if it has at most \( s_0 \) nonzero entries. When \( q, h \) are \( s_0 \)-sparse, denote by

\[
\rho_{\text{max}}(s_0, (|b_{ij}|)) := \max_{q \in \mathbb{S}^{n-1}, s_0 \text{-sparse}} \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |q_i| |q_j| \tag{19}
\]

*Throughout this paper, we use \( \psi_B(s_0) \) to denote the following:

\[
\psi_B(s_0) := \frac{\rho_{\text{max}}(s_0, (|b_{ij}|))}{\|B_0\|_2} \tag{20}
\]

### 2.2 The main theorem

We now present our main result regarding estimator \( \overline{B}_0 \) for estimating \( B_0 = (b_{ij}) \succ 0 \) using sparse random matrix \( \mathcal{X} \) under masks. We are not optimizing over the logarithmic factors in this paper.

**Theorem 2.4. (Overall bounds for estimator \( \overline{B}_0 \))** Consider the data generating random matrices as in (9) and (10). Let \( B_0 \) be a symmetric positive definite covariance matrix and \( \overline{B}_0 \) be as defined in (12), and \( \overline{\lambda} = \overline{B}_0 - B_0 \). Suppose \( \text{tr}(B_0) = n \). Let \(|B_0| = (|b_{ij}|)\). Let \( \psi_B(s_0) = \rho_{\text{max}}(s_0, |B_0|) / \|B_0\|_2 \) be as in Definition 2.3, with a sparsity parameter \( 1 \leq s_0 < n \). Let \( 0 < \varepsilon < 1/2 \). Suppose that the sampling rates satisfy

\[
\sum_{j=1}^{n} a_{jj} \rho_j^2 \geq C_4 a_{\infty} \rho_{\text{min}}(\psi_B(2s_0 \wedge n) \lor 1)s_0 \log(n \lor m) \tag{21}
\]

where \( a_{\infty} = \max_j a_{jj} \) and \( a_{\min} = \min_j a_{jj} \) and \( C_4 > 1 \). Then with probability at least \( 1 - \frac{c'}{(n \lor m)^2} - 4 \exp(-cs_0 \log(3en/(s_0 \varepsilon))) \) for some absolute constants \( c, c' \), we have for \( \eta_A = \sqrt{\frac{a_{\infty}}{a_{\min}}} \), \( d = 2s_0 \wedge n \),

\[
\sup_{q \in \mathbb{S}^{n-1} \cap B_2} \frac{1}{\|B_0\|_2} \left| q^T (\overline{B}_0 - B_0) q \right| \ll \eta_AR_{\text{offd}}(s_0) \ell_{s_0, n}^{1/2} + r_{\text{offd}}^2(s_0) \psi_B(d)
\]

where \( B_1^n, B_2^n \) denote the unit \( \ell_1 \) and \( \ell_2 \) balls respectively,

\[
r_{\text{offd}}(s_0) \asymp \sqrt{s_0 \log \left( \frac{3en}{(s_0 \varepsilon)} \right)} \frac{\|A_0\|_2}{\sum_j a_{jj} \rho_j^2}, \text{ and } \ell_{s_0, n} = \frac{\log(n \lor m)}{\log(3en/(s_0 \varepsilon))} \tag{22}
\]
Discussions. We defer the discussion of key technical ideas for Theorem 2.4 and implications of the convergence rate to Sections 3.2 and C. We now unpack Theorem 2.4 and show how it is related to the RE conditions. Clearly, the set of vectors as defined in (6) satisfy
\[
\text{Cone}(s_0) \cap \mathbb{S}^{n-1} \subset \sqrt{s_0} B_1^n \cap B_2^n
\]

Theorem 3.1 ensures a uniform bound holds for the diagonal component of the quadratic form (29) as the operator norm is completely determined by the maximum entrywise deviation for \( \text{diag}(\tilde{B}_0 - B_0) \) (cf. (32)). In contrast, for \( \text{offd}(\tilde{B}_0 - B_0) \), the presentation of our results will necessarily focus on families of sparse vectors and vectors satisfying certain cone constraints such as (6) for chosen sparsity parameter \( s_0 \), which dominates the rate of convergence as we will show in the proof of Theorem 2.4.

In [41], it was shown that elements of \( \tilde{A}_0 \) and \( \tilde{A}_0 \) are tightly concentrated around their individual mean values when
\[
\frac{\sum_j p_j^2}{\|A_0\|_2} = \Omega \left( \frac{a_\infty \log m}{\min_i} \right) \quad \text{and} \quad p_i p_j = \Omega \left( \frac{\log m \|B_0\|_2}{\text{tr}(B)} \right) \forall i \neq j,
\]
while the diagonal components have tighter concentration than that of the off-diagonal components. To prove a uniform concentration of measure bounds for the quadratic form we pursue in Theorems 2.4 and 2.7, we drop the second condition while strengthening the first, since the second condition is only needed in order for \( \tilde{A}_0 \) to have componentwise convergence. More precisely, ignoring the logarithmic factors, an additional factor of \( \|B_0\|_2 \) is needed (cf. (21)) in order to control the quadratic form (5) over the cone, which shows sub-quadratic and perhaps superlinear dependency on \( s_0 \). We have for all \( q \in \text{Cone}(s_0) \cap \mathbb{S}^{n-1} \) and \( \delta \leq \eta \text{Ar}_{\text{offd}}(s_0) + r^2_{\text{offd}}(s_0)\psi_B(2s_0 \wedge n) \),
\[
q^T \tilde{B}_0 q \geq q^T B_0 q - \delta \|B_0\|_2 \geq \lambda_{\min}(B_0) - \delta \|B_0\|_2 > 0 \quad (23)
\]
so long as \( \|B_0\|_2 < \infty \), and we choose the lower bound on the cumulative sampling rate to be sufficiently strong in the sense of (21). It is well known that such lower bound as in (23) leads to restricted eigenvalue type of bounds to hold on \( \tilde{B}_0 \); We consider this superlinear dependency on \( s_0 \) as the price we pay for handling with such complex data model as considered in Definition 2.1. See [3, 37, 29] for background and discussions.

Crucially, as we show in Lemma 2.5, an upper (and lower) bound on \( \rho_{\max}(s_0, (|b_{ij}|)) \) which depends on \( s_0 \) rather than \( n \). Hence \( \rho_{\max}(s_0, (|b_{ij}|)) \) is upper bounded by \( \sqrt{s_0} \|B_0\|_2 \) and coincides with the operator norm of \( (|b_{ij}|) \) when \( s_0 = n \).

**Lemma 2.5.** Let \( S \subset [n] \). Let \( |B_0|_{S,S} \) denote the principal submatrix of \( |B_0| \) with rows and columns indexed by \( S \). For \( \rho_{\max}(s_0, (|b_{ij}|)) \) as defined in (19), we have for \( 1 \leq s_0 \leq n \)
\[
\rho_{\max}(s_0, (|b_{ij}|)) = \max_{S \subset [n]: |S| = s_0} \lambda_{\max}(|B_0|_{S,S}) \leq \sqrt{s_0} \|B_0\|_2
\]
On the other hand, for the lower bound, we have for \( b_\infty = \max_j b_{jj} \), \( \rho_{\max}(s_0, (|b_{ij}|)) \geq b_\infty \).

We assume \( \text{tr}(B_0) = n \); hence by definition, \( \|B_0\|_2 \geq b_{\max} = \max_j b_{jj} \geq 1 \).

We defer further discussions to Sections 3.2 and C.
2.3 Restricted eigenvalue conditions

We now state the implications of Theorem 2.4 on proving the lower and upper RE conditions (cf. Definition 2.6) in Theorem 2.7 under slightly stronger conditions on the sample size and sampling rate requirement.

Definition 2.6. (Lower-and-Upper-RE conditions) [22] The matrix $\Gamma$ satisfies a Lower-RE condition with curvature $\alpha > 0$ and tolerance $\tau > 0$ if $\theta^T \Gamma \theta \geq \alpha \| \theta \|_2^2 - \tau |\theta|_1^2 \quad \forall \theta \in \mathbb{R}^m$. The matrix $\Gamma$ satisfies an upper-RE condition with smoothness $\tilde{\alpha} > 0$ and tolerance $\tau > 0$ if $\theta^T \Gamma \theta \leq \tilde{\alpha} \| \theta \|_2^2 + \tau |\theta|_1^2 \quad \forall \theta \in \mathbb{R}^m$.

As $\alpha$ becomes smaller, or as $\tau$ becomes larger, the Lower-RE condition is easier to be satisfied. Consequently, a smaller $\tau$ implies a stronger Lower-RE condition. See [30] for comparing the Lower-RE condition to the RE condition as defined in [5]. Assuming the slightly stronger conditions on the sample size and sampling probabilities as stated in Theorem 2.7, we prove that the Lower and Upper-RE conditions hold for $\Delta = B_0 - B_0$ with suitably chosen $\alpha, \tilde{\alpha},$ and $\tau$.

Theorem 2.7. Set $1 \leq s_0 \leq n$. Suppose all conditions in Theorem 2.4 hold. Let $|B_0| = (|b_{ij}|)$. Moreover, we replace (21) with the following:

$$\sum_{i,j} a_{ij}^2 \frac{\gamma(B_0) s_0 \log(n) \eta_A(\psi_B(2s_0 \wedge n) \vee \kappa(B_0))}{\| A_0 \|_2} \gg \kappa(B_0) s_0 \log(n) \eta_A(\psi_B(2s_0 \wedge n) \vee \kappa(B_0))$$  \hspace{1cm} (24)

where $\kappa(B_0) = \| B_0 \|_2 / \lambda_{\min}(B_0)$ is the condition number of matrix $B_0$ and $\psi_B(s_0) = \frac{\rho_{\max}(s_0,|B_0|)}{\| B_0 \|_2}$ is as defined in [19]. Then with probability at least $1 - \frac{C}{(\mu \nu \eta)^2} - 4 \exp(-cs_0 \log(3en/(s_0 \epsilon)))$ for some absolute constant $C, c,$

$$\sup_{q,h \in \mathbb{R}^{n-1},s_0 \text{-sparse}} |q^T (\tilde{B}_0 - B_0) h| < \delta \leq \frac{3}{32} \lambda_{\min}(B_0);$$

and the lower and upper RE conditions hold: for all $q \in \mathbb{R}^n$,

$$q^T \tilde{B}_0 q \geq \frac{5}{8} \lambda_{\min}(B_0) \| q \|_2^2 - \frac{3 \lambda_{\min}(B_0)}{8s_0} \| q \|_1^2 \quad \text{and}$$

$$q^T \tilde{B}_0 q \leq \lambda_{\max}(B_0) + \frac{3}{8} \lambda_{\min}(B_0)) \| q \|_2^2 + \frac{3 \lambda_{\min}(B_0)}{8s_0} \| q \|_1^2.$$  \hspace{1cm} (25)

(26)

These conditions are especially convenient for analyzing optimization problems as shown in [22] under missing values; and more general forms of these conditions, namely, restricted strong convexity (RSC) and restricted smoothness (RSM) conditions are used in analyzing high-dimensional gradient descent algorithms [1]. We prove Theorem 2.7 in the supplementary material Section 1. Finally, we state Theorem 2.8 regarding our estimator $\hat{B}_*$.

Theorem 2.8. (Overall bounds with $\hat{B}_*$) Set $1 < s_0 \leq n$. Suppose all conditions in Theorem 2.4 hold. Let $\hat{B}_*$ be as defined in [18] and $B_* = nB_0 / \text{tr}(B_0) > 0$. Then with probability at least $1 - \frac{C}{(\mu \nu \eta)^2} - 4 \exp(-cs_0 \log(3en/(s_0 \epsilon)))$ for some absolute constants $C, c$, we have for $\hat{\Delta} = \hat{B}_* - B_*,$

$$\sup_{q \in \sqrt{s_0}B_n^{*} \cap B_n} \frac{1}{\| B_* \|_2} |q^T \hat{\Delta} q| \ll \eta_A r_{\text{offd}}(s_0) \ell_{s_0,n}^{1/2} + \psi_B(2s_0 \wedge n) r_{\text{offd}}(s_0)$$

where $r_{\text{offd}}(s_0)$ as defined in [22] and $\eta_A = \sqrt{a_\infty} / \sqrt{a_{\min}}$. 

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The design of $\hat{B}_*$ makes it scale-free as we divide $\mathcal{X}\mathcal{X}^T$ by the mask matrix $\hat{M}$; to see this, notice that

$$\text{tr}(\mathcal{X}^T\mathcal{X}) = \text{tr}(\mathcal{X}\mathcal{X}^T),$$

and hence by construction $\text{tr}(\hat{B}_*) = n$.

In practice, one can set $\text{tr}(\hat{B}_0) = n$ in view of discussions immediately above. We prove Theorem 2.8 in Section 6. In future work, we will also discuss jointly estimating matrix variate Gaussian graphical models corresponding to precision matrices $A_0^{-1}$ and $B_0^{-1}$. In such methods, we use sample based $\hat{A}_*$ and $\hat{B}_*$ as input to estimators that involve shrinkage, such as [15, 23]. See [40] for precise characterizations of matrix variate Gaussian graphical models and procedures for estimating such models.

### 3 Randomized quadratic forms

In this section, we present our strategy to analyze the quadratic form as defined in (5). For the random mask model (10), we are given $m$ dependent samples to estimate $B_0$, namely, $(x_j \otimes x_j)$, with each one applied an independent random mask $u_j \otimes u_j$. Let $\mathcal{M}$ be as defined in (12). Denote by $M_j = \mathbb{E} u_j \otimes u_j$. We emphasize that the only assumption we make in the following derivation is on independence of mask $U$ from data matrix $X$. Hence,

$$\Delta(B) := \mathcal{X}\mathcal{X}^T - \mathbb{E}\mathcal{X}\mathcal{X}^T = \mathcal{X}\mathcal{X}^T - \mathcal{M} \circ B_0$$

where

$$\mathcal{X}\mathcal{X}^T := (U \circ X)(U \circ X)^T = \sum_{j=1}^{m} (u_j \otimes u_j) \circ (x_j \otimes x_j)$$

and by independence of $U$ and $X$, and by definition of (4),

$$\mathbb{E}\mathcal{X}\mathcal{X}^T = \sum_{j=1}^{m} \mathbb{E}(u_j \otimes u_j) \circ \mathbb{E}(x_j \otimes x_j) = \sum_{j=1}^{m} a_{jj} M_j \circ B_0 =: \mathcal{M} \circ B_0.$$

**Reductions.** We write $\text{diag}(\mathcal{M}) = (\sum_{j=1}^{m} a_{jj} p_j) I_n$ and $\text{offd}(\mathcal{M}) = (\sum_{j=1}^{m} a_{jj} p_j^2)(11^T - I_n)$ where $1 = (1, \ldots, 1)^T \in \mathbb{R}^n$. Consider (12), we break the quadratic form into two parts: for all $q \in \mathbb{R}^n$,

$$|q^T (\tilde{B}_0 - B_0)q| = |q^T (\mathcal{X}\mathcal{X}^T \circ \mathcal{M} - \mathbb{E}(\mathcal{X}\mathcal{X}^T \circ \mathcal{M}))q|$$

$$= |q^T \Delta(B) \circ \mathcal{M}q|$$

$$\leq \frac{1}{\|\mathcal{M}\|_{\text{diag}}} |q^T \text{diag}(\mathcal{X}\mathcal{X}^T - B_0 \circ \mathcal{M})q| + \frac{1}{\|\mathcal{M}\|_{\text{offd}}} |q^T \text{offd}(\mathcal{X}\mathcal{X}^T - B_0 \circ \mathcal{M})q|,$$

where we denote the componentwise max norm for $\text{diag}(\mathcal{M})$ and $\text{offd}(\mathcal{M})$ by

$$\|\mathcal{M}\|_{\text{diag}} := \sum_{j=1}^{m} a_{jj} p_j \quad \text{and} \quad \|\mathcal{M}\|_{\text{offd}} := \sum_{j=1}^{m} a_{jj} p_j^2.$$

Hence, we have essentially reduced the problem that involves quadratic form with mask $\mathcal{M} \in \mathbb{R}^{n \times n}$ embedded inside (5) into ones not involving the masks as we pull the masks outside of the quadratic form.
forms in (30). Moreover, this decomposition enables us to treat the diagonal and the off-diagonal parts of the original quadratic form (5) separately. For diagonal matrices, we can prove a uniform bound over all quadratic forms, so long as we prove coordinate-wise concentration of measure bounds as we will state in Theorem 3.1 which follows from Theorem 5.4 [41] immediately.

For the rest of this section, we mainly focus on developing strategies for obtaining an upper bound on the off-diagonal component in (30). Denote the unique symmetric square root of the positive definite matrix $B_0^{1/2}$ by

$$B_0^{1/2} = [c_1, c_2, \ldots, c_n], \quad \text{where } c_1, c_2, \ldots, c_n \in \mathbb{R}^n$$

are the column vectors of $B_0^{1/2}$, and $\langle c_i, c_j \rangle = b_{ij}$, for all $i, j$. Denote the unique symmetric square root of the positive definite matrix $A_0^{1/2}$ by

$$A_0^{1/2} = [d_1, d_2, \ldots, d_m], \quad \text{where } d_1, d_2, \ldots, d_m \in \mathbb{R}^m$$

(31)

are the column vectors of $A_0^{1/2}$, and $\langle d_i, d_j \rangle = a_{ij}$, for all $i, j$. For each $q \in \mathbb{S}^{n-1}$, let $A_{qq}^\circ \in \mathbb{R}^{mn \times mn}$ be a random matrix that can be expressed as a quadratic form over the set of independent Bernoulli random variables $\{u_{ij}^k, k = 1, \ldots, m, j = 1, \ldots, n\}$ in the mask matrix

$$A_{qq}^\circ = \sum_k \sum_{i \neq j} u_{ij}^k u_{ij}^k q_i q_j (c_j c_i^T) \otimes (d_k d_k^T),$$

where the coefficient for a pair $u_{ij}^k u_{ij}^k$ is a tensor product $q_i q_j (c_j c_i^T) \otimes (d_k d_k^T)$ that changes with each choice of $q \in \mathbb{S}^{n-1}$. Denote by

$$Q_{\text{offd}} = \frac{1}{\|M\|_{\text{offd}}} |q^T \text{offd}(X X^T - B_0 \circ M) q| =: \frac{1}{\|M\|_{\text{offd}}} |q^T \tilde{\Delta} q|$$

the off-diagonal component in (30), which can be shown to have the following expression using the random matrix $A_{qq}^\circ$ and a subgaussian vector $Z$

$$Q_{\text{offd}} \sim \frac{1}{\|M\|_{\text{offd}}} |Z^T A_{qq}^\circ Z - \mathbb{E}(Z^T A_{qq}^\circ Z)|, \quad \text{where } Z \sim \text{vec} \{Z^T\}$$

for $Z$ as defined in (8) with $K = 1$ and $\sim$ stands that two vectors follow the same distribution. Throughout this section, we denote by $Z \in \mathbb{R}^{mn}$ a subgaussian random vector with independent components $Z_j$ that satisfy $\mathbb{E}Z_j = 0$, $\mathbb{E}Z_j^2 = 1$, and $\|Z_j\|_{\psi_2} \leq 1$.

We will take a closer look at the random matrix $A_{qq}^\circ$ appearing in the quadratic form in $Q_{\text{offd}}$. In particular, we study the operator and the Frobenius norms of this object and its cousin as defined in (39) in details, which is the focus of the paper. We will state a slightly more general result in Theorem 3.3 and discuss strategies for tackling the off-diagonal component $Q_{\text{offd}}$ in Sections 3.1 and 4. We mention in passing that the strategy we develop for dealing with $\tilde{\Delta}$ will readily apply when we deal with $\text{offd}(X^T X - A_0 \circ N)$ by symmetry of the problem, which we leave as future work.
3.1 Proof sketch for Theorem 2.4

For the diagonal part of the problem, a substantial simplification can be made as follows:

\[
\sup_{q \in \mathbb{S}^{n-1}} |q^T (\text{diag}(\mathcal{X} \mathcal{X}^T) - \mathbb{E} \text{diag}(\mathcal{X} \mathcal{X}^T)) q| \\
= \max_{q \in \{e_1, ..., e_n\}} |q^T \text{diag}(\mathcal{X} \mathcal{X}^T) q - \mathbb{E}(q^T \text{diag}(\mathcal{X} \mathcal{X}^T) q)| \\
= \|\text{diag}(\mathcal{X} \mathcal{X}^T) - \mathbb{E}(\text{diag}(\mathcal{X} \mathcal{X}^T))\|_{\max}
\]

(32)

**Theorem 3.1.** Suppose that \( \sum_{i=1}^{m} p_j = \Omega \left( \frac{a_{\infty} \log(n \vee m)}{a_{\min}} \right) \). Let \( a_{\infty} = \max_j a_{jj}, a_{\min} = \min_j a_{jj} \) and \( b_{\infty} = \max_j b_{jj} \). Denote by

\[
r_{\text{diag}} \equiv \log^{1/2}(m \vee n) \sqrt{a_{\infty} \|A_0\|_2} / a_{\min}^{1/2} \sqrt{\sum_{j=1}^{m} a_{jj} p_j}.
\]

With probability at least 1 - \( \frac{C}{(n \vee m)^d} \) for some \( d > 4 \), and \( M_{ii} = \sum_j a_{jj} p_j =: \|M\|_{\text{diag}} \),

\[
\frac{1}{\|M\|_{\text{diag}}} \|\text{diag}(\mathcal{X} \mathcal{X}^T) - \mathbb{E}(\text{diag}(\mathcal{X} \mathcal{X}^T))\|_{\max} = O(b_{\infty} r_{\text{diag}})
\]

(33)

In order to provide a bound for the quadratic form above for all \( q \in (\sqrt{s_0} B_1^n \cap B_2^m) \), we use Lemma 3.2, which follows from the proof for Lemma 37 [30]. Theorem 2.4 follows from Theorems 3.1 and 3.3 in view of Lemma 3.2. We defer the proof of Theorem 2.4 to Section 5. We then consider the class of \( s_0 \)-sparse vectors and state the main technical result in Theorem 3.3.

**Lemma 3.2.** Let \( \delta > 0 \). Set \( 0 < s_0 \leq n \). Let \( E = \bigcup_{|J| \leq s_0} E_J \). Let \( \Delta \) be an \( n \times n \) matrix such that \( |q^T \Delta h| \leq \delta, \quad \forall q, h \in \mathbb{S}^{n-1} \). Then for all \( \nu \in (\sqrt{s_0} B_1^n \cap B_2^m), |\nu^T \Delta \nu| \leq 4\delta \).

**Theorem 3.3. (Control the quadratic form over sparse vectors)** Set \( 1/2 \geq \varepsilon > 0 \). Suppose that all conditions in Theorem 2.4 hold. For a chosen sparsity parameter \( s_0 \), let \( E = \bigcup_{|J| \leq s_0} E_J \) for \( 1 < s_0 \leq n \). Let \( \eta_A = \sqrt{a_{\infty} / a_{\min}} \). Let \( \psi_B(s_0) \) be as defined in [19]. Then with probability at least 1 - \( \frac{4}{(n \vee m)^d} - 4 \exp(-c s_0 \log(3n / s_0 \varepsilon)) \) for some absolute constant \( c \), we have for \( \Delta = \text{offd}(\mathcal{X} \mathcal{X}^T - B_0 \circ \mathcal{M}) \) and \( r_{\text{offd}}(s_0) \) as defined in [22],

\[
\sup_{q, h \in \mathbb{S}^{n-1} \cap E} \left\| B_0 \right\|_2 \left\| M_{\text{offd}} \right\| \left| q^T \Delta h \right| \ll \eta_A \sqrt{s_0 \log(n \vee m)} \left\| A_0 \right\|_2 / \sum_j a_{jj} p_j + r_{\text{offd}}(s_0) \psi_B(2s_0 \land n) =: \delta_q
\]

(34)

3.2 Discussions on the main theorem

For completeness, we write Corollary 3.4 where we control the quadratic form over the entire sphere \( q \in \mathbb{S}^{n-1} \), from which we can obtain the relative error in the operator norm for estimating \( B_0 \) using \( \hat{B}_0 \). When \( s_0 = n \), we denote by \( \left\| B_0 \right\|_2 \) the operator norm of \( \left\langle |b_{ij}| \right\rangle \) and hence

\[
1 \leq \psi_B(n) := \left\| \left\langle |b_{ij}| \right\rangle \right\|_2 \leq \left\| \left\langle |b_{ij}| \right\rangle \right\|_{\infty} = \left\| B_0 \right\|_{\infty} \leq \sqrt{n}
\]

(35)
Corollary 3.4. (Operator norm bounds) Suppose all conditions in Theorem 2.4 hold with $s_0 = n$. Set $0 < \varepsilon < 1/2$. Denote by $\psi_B(n) = \frac{\|\psi_i(b_j)\|}{\|B_0\|_2}$ as in (35). Then with probability at least $1 - \frac{C}{(n \wedge m)^2} - 4\exp\left(-cn(\log(3e/\varepsilon))\right)$ for some absolute constants $c, C$, we have for $\eta_A = \sqrt{\frac{a_{\max}}{a_{\min}}}$

$$\|\tilde{B}_0 - B_0\|_2 / \|B_0\|_2 \ll \eta_A \log^{1/2}(n \wedge m) \sqrt{\frac{n \|A_0\|_2}{\sum_j a_{jj} b_j^2}} + r_{\text{offd}}(n)^2 \psi_B(n)$$

where $r_{\text{offd}}(n)$ is as defined in (22) with $s_0 = n$.

We defer the proof of Corollary 3.4 to Section E in the supplementary material. Notice that for non-negative symmetric matrix $|B_0| \geq 0$, its operator norm is the same as its spectral radius, denote by $\rho(|B_0|)$, which is lower and upper bounded by the minimum and maximum row (column) sum respectively.

We now state in Lemma 3.5 an upper bound and a more refined lower bound on $\rho(|B_0|)$ that depends on the maximum and the average row sum of $|B_0|$. In summary, we have Lemma 3.5 which follows from results about non-negative or real symmetric matrices, which we prove in Section E.

Lemma 3.5. For symmetric matrices $B_0$ and $|B_0|$, their spectral radii (and hence operator norms) must obey the following relationships:

$$\lambda_{\max}(|B_0|) = \rho(|B_0|) = \|B_0\|_2 \geq \rho(B_0) = \|B_0\|_2$$

and

$$\sqrt{n} \|B_0\|_2 \geq \|B_0\|_\infty \geq \|B_0\|_2 \geq \frac{1}{n} \sum_{i,j} |b_{ij}|.$$

Finally, if the rows of $|B_0|$ have the same sum $r$, then $\rho(|B_0|) = \|B_0\|_2 = r$.

Moreover, suppose we define a weighted graph $G = (V, E)$, where $V = [n]$ and an edge $(i, j) \in E$ exists, denoted by $i \sim j$, if $|b_{ij}| \neq 0$; moreover, we assign the edge weight to be $|b_{ij}|$. Denote by $\deg_i = \sum_{j=1}^n |b_{ij}|$. Denote by $\delta(G) = \min_{1 \leq i \leq n} \deg_i$ and $\Delta(G) = \max_{1 \leq i \leq n} \deg_i$ the minimum and maximal degrees of $G$. Then we have by Lemma 3.5

$$\frac{1}{n} \sum_{i=1}^n \deg_i \leq \lambda_{\max}(|B_0|) = \|B_0\|_2 \leq \Delta(G)$$

Hence, intuitively, when the average row or column sums of $|B_0|$ is large, or when the average node degree in the weighted graph $G$ is large, we expect to see a larger error in the tail events. In particular, when all the rows (or columns) of $|B_0|$ have the same sum, that is, when $\delta(G) = \Delta(G)$, we have $\|B_0\|_2 = \Delta(G)$. One could also define for each principal submatrix $|B_0|_{S,S}$, where $S \subset [n]$, its maximum and average column (row) sums to obtain a lower and upper bound on the largest eigenvalue $\lambda_{\max}(|B_0|_{S,S})$. Exploration of such connections is left as future work.

4 Proof strategy of Theorem 3.3

We show the proof strategy for Theorem 3.3 in this section. In proving Theorem 3.3, we make the following key technical contributions along the way: we prove Theorems 4.2 and 4.3 regarding the
spectral and Frobenius norms over a family of structured random matrices, with more discussions in the supplementary material. Such results may be of independent interests.

**Symmetrization.** Notice that vec \{ Y \} = (B_1^{1/2} \otimes A_0^{1/2})vec \{ Z^T \}, where Y = X^T = A_0^{1/2}ZB_0^{1/2} for Z as defined in (8). We write \( y^i = c_i^T \otimes A_0^{1/2}Z \), where Z ~ vec \{ Z^T \}. Then

\[
q^T \text{offd}(X^T X)h = \sum_{i \neq j} q_i h_j \langle v^i \circ y^i, v^j \circ y^j \rangle
\]

\[
= Z^T \left( \sum_{i \neq j} q_i h_j c_i c_j^T \otimes A_0^{1/2} \text{diag}(v^i \circ v^j) A_0^{1/2} \right) Z
\]

where \( (A_0^{1/2} \text{diag}(v^i \circ v^j) A_0^{1/2}) = \sum_{k=1}^m u_k^i u_k^j d_k \otimes d_k \). Clearly, we have by symmetry of the gram matrix \( X^T X \), for any \( h, q \in S^{n-1}, \)

\[
q^T \text{offd}(X^T X)h = h^T \text{offd}(X^T X)q
\]

We now symmetrize the random quadratic form (36) in view of (37), and rewrite

\[
q^T \text{offd}(X^T X)h := Z^T A^o_{qh} Z \quad \text{where}
\]

\[
A^o_{qh} = \frac{1}{2} \sum_{k=1}^m \sum_{i=1}^n \sum_{i \neq j} u_k^i u_k^j (q_i h_j + q_j h_i)(c_i c_j^T) \otimes (d_k d_k^T)
\]

is symmetric since for any index set \( (i, j, k) \), both \( (c_i c_j^T) \otimes (d_k d_k^T) \) and its transpose \( (c_j c_i^T) \otimes (d_k d_k^T) \) appear in the sum with the same coefficients. Then we have for \( q^T \Delta h \), where \( \Delta = \text{offd}(X^T X) - \mathbb{E} \text{offd}(X^T X) \),

\[
\left| q^T \Delta h \right| = \left| Z^T A^o_{qh} Z - \mathbb{E}(Z^T A^o_{qh} Z) \right|
\]

**Approximation.** To obtain a uniform large deviation bound for the quadratic form above for all sparse vectors \( q, h \in S^{n-1} \cap E \), we first consider \( q, h \in \mathcal{N} \) which is the \( \varepsilon \)-net of \( S^{n-1} \cap E \) as constructed in Lemma 4.1. We will show that under event \( F_1 \cap F_2 \) to be defined in Theorems 4.3 and 4.5,

\[
\sup_{h, q \in \mathcal{N}} \left| q^T \Delta h \right| \leq \frac{\delta_q}{\| M \|_{\text{offd}} \| B_0 \|_2} \quad \text{where } \delta_q \text{ is as defined in (33).}
\]

A standard approximation argument shows that if (41) holds, then for all \( q, h \in S^{n-1} \cap E, \)

\[
\sup_{h, q \in \mathcal{E} \cap S^{n-1}} \frac{1}{\| M \|_{\text{offd}} \| B_0 \|_2} \left| q^T \Delta h \right| \leq \frac{\delta_q}{(1 - \varepsilon)^2}.
\]

**Lemma 4.1.** Let \( 1/2 > \varepsilon > 0 \). For a set \( J \subset [n] \), denote \( E_J = \text{span} \{ e_j, j \in J \} \). For each subset \( E_J \), construct an \( \varepsilon \)-net \( \Pi_J \), which satisfies

\[
\Pi_J \subset E_J \cap S^{n-1} \quad \text{and} \quad |\Pi_J| \leq (1 + 2/\varepsilon)^{s_0}.
\]
If \( \mathcal{N} = \bigcup_{|J| = s_0} \Pi_J \), then the previous estimate implies that
\[
|\mathcal{N}| \leq (3/\varepsilon)^{s_0} \left( \frac{n}{s_0} \right)^{s_0} \leq \left( \frac{3en}{(s_0\varepsilon)} \right)^{s_0} = \exp \left( s_0 \log \left( \frac{3en}{s_0\varepsilon} \right) \right) .
\]

(43)

Clearly, when \( s_0 = n \), we have \( |\mathcal{N}| \leq (3/\varepsilon)^n \).

**Decomposition.** Let \( Z = \text{vec} \{ Z^T \} \) for \( Z \) as defined in (8). Following (40) and (27), we decompose the error into two parts: for \( \tilde{\Delta} = \text{offd}(\mathcal{X}^T - B_0 \circ \mathcal{M}) \) and \( q, h \in \mathbb{S}^{n-1} \),
\[
|q^T \tilde{\Delta} h| = |Z^T A_{qh}^0 Z - \mathbb{E}(Z^T A_{qh}^0 Z)| \leq |Z^T A_{qh}^0 Z - \mathbb{E}(Z^T A_{qh}^0 Z U)|
+ |\mathbb{E}(Z^T A_{qh}^0 Z U) - \mathbb{E}(Z^T A_{qh}^0 Z)| =: \text{I} + \text{II}
\]
To obtain a uniform bound for the quadratic form \( |q^T \tilde{\Delta} h| \) over all \( s_0 \)-sparse vectors in \( E \cap \mathbb{S}^{n-1} \), where \( 1 \leq s_0 \leq n \), we first present a uniform concentration of measure bound on Part I followed by that of Part II of (44) for all pairs \( q, h \in \mathcal{N} \), the \( \varepsilon \)-net of \( \mathbb{S}^{n-1} \cap E \). The proof of Theorem 3.3 involves checking conditions in Theorems 4.5 and 4.4 and then combining these two bounds using (44) to control the quadratic form over sparse vectors in \( \mathbb{S}^{n-1} \cap E \).

**Part I.** We first condition on \( U \) being fixed, and then the quadratic form \( q^T \text{offd}(\mathcal{X}^T h = Z^T A_{qh}^0 Z \) can be treated as a subgaussian quadratic form with \( A_{qh}^0 \) taken to be a deterministic matrix. We will prove in Theorem 4.2 that for all realizations of \( U \) and for all \( q, h \in \mathbb{S}^{n-1} \), the operator norm of \( A_{qh}^0 \) is uniformly and deterministically bounded. We then state a probabilistic uniform bound on \( \|A_{qh}^0\|_F \) in Theorem 4.3.

**Theorem 4.2.** Let \( A_{q,h}^0 \) be as defined in (39). Then for all \( q, h \in \mathbb{S}^{n-1} \),
\[
\|A_{q,h}^0\|_2 \leq \|A_0\|_2 \|B_0\|_2 .
\]

**Theorem 4.3.** Set \( 1 \leq s_0 \leq n \). Let \( N = \sum_{s=1}^{m} a_{s}^2 p_s^4 + \sum_{s \neq t} a_{s}^2 a_{t}^2 p_s^2 p_t^2 \leq \|A_0\|_2 a_{\infty} \sum_{s=1}^{m} p_s^4 \). Suppose that \( \sum_{j=1}^{m} a_{j}^2 p_j^2 \geq C_2 a_{\infty} \log(n \vee m) \) for some absolute constant \( C_2 \). Then on event \( \mathcal{F}_0^c \), where \( \mathbb{P}(\mathcal{F}_0^c) \geq 1 - \frac{4}{(nm)^r} \),
\[
\sup_{q,h \in \mathbb{S}^{n-1}, s_0 \text{-sparse}} \|A_{q,h}^0\|_F \leq W \cdot \|B_0\|_2 \|A_0\|_2^{1/2} \text{ where }
\]
\[
W \asymp \sqrt{a_{\infty} \sum_{s=1}^{m} p_s^2 + \psi_B(s_0)} \left( \|A_0\|_2^{1/2} \log^{1/2}(n \vee m) + (N \log(n \vee m))^{1/4} \right)
\]
where \( \psi_B(s_0) = \frac{d_{\text{max}}(s_0, \|B_0\|_2)}{\|B_0\|_2} \) is as defined in (19).

We prove Theorems 4.2 and 4.3 in Sections C and D in the supplementary material. The proof techniques may be of independent interests for analyzing tensor quadratic forms. Our proof for Theorem 4.2 will go through if one replaces \( u^k, k = 1, \ldots, m \) by independent Gaussian random vectors; however, the statement will be probabilistic subject to an additional logarithmic factor. Such a result may be of independent interests, as more generally, the mask matrix \( U \) may not be constrained to the family of Bernoulli random matrices. For example, one may consider \( U \) as a
matrix with arbitrary positive coefficients belonging to \([0,1]\). In particular, Lemma \(G.1\) holds for general block-diagonal matrices with bounded operator norm, which can be deterministic.

We may condition on the event that \(\mathcal{F}_0^c\) holds. Applying the Hanson-Wright inequality \(\text{[28]}\) (cf. Theorem \(\text{B.1}\)) with the preceding estimates on the operator and Frobenius norms and the union bound, we have Theorem \(\text{4.4}\). We prove Theorem \(\text{4.4}\) in Section \(\text{D}\) where we show that \(\mathcal{F}_0 \subset \mathcal{F}_1\).

**Theorem 4.4.** Let \(1/2 > \varepsilon > 0\). For a chosen sparsity \(1 \leq s_0 \leq n\), let \(E = \bigcup_{|J| \leq s_0} E_J\). Denote by \(\mathcal{N}\) the \(\varepsilon\)-net for \(S^{n-1} \cap E\) as constructed in Lemma \(\text{4.1}\). Suppose that for \(N \asymp a_\infty \|A_0\|_2 \sum_{s=1}^m P_s^4\),

\[
\sum_{j=1}^m a_{jj} P_j^2 \geq C_4 \|A_0\|_2 \log(n \lor m) a_\infty a_{\min}, \quad \text{and} \quad \sum_{j=1}^m a_{jj} P_j^2 \geq C_5 \sqrt{s_0 \log(3en/s_0 \varepsilon)} \|A_0\|_2 \psi_B(s_0)(N \log(n \lor m))^{1/4}
\]

for some absolute constants \(C_4, C_5\). Then on event \(\mathcal{F}_1^c\), which holds with probability at least \(1 - \frac{4}{(nvm)^2} - 2 \exp(-c_1 s_0 \log(3en/(s_0 \varepsilon)))\), we have

\[
\sup_{q,h \in \mathcal{N}} \left| Z^T A_{qh}^0 Z - \mathbb{E}(Z^T A_{qh}^0 Z|U) \right| \ll \eta_A r_{\text{offd}}(s_0) + r_{\text{offd}}(s_0) f_p \psi_B(s_0) = o(1)
\]

where \(r_{\text{offd}}(s_0)\) is as defined in \(\text{[22]}\), \(c_1\) is an absolute constant,

\[
f_p = \left( \log(n \lor m) \|A_0\|_2 a_\infty \sum_{s=1}^m P_s^4 \right)^{1/4} \sqrt{\sum_{j=1}^m a_{jj} P_j^2}, \quad \text{and} \quad \eta_A = \sqrt{a_\infty a_{\min}}.
\]

As we show in the proof of Theorem \(\text{3.3}\) \(\text{[21]}\) ensures that \(\text{(46)}\) and \(\text{(47)}\) hold, which in turn ensures that \(\eta_A r_{\text{offd}}(s_0) + r_{\text{offd}}(s_0) f_p \psi_B(s_0) \to 0\).

**Part II.** We now allow \(U\) to be random and obtain the following large deviation bound for Part II in \(\text{(44)}\) in Theorem \(\text{4.5}\). Denote by

\[
S_*(q,h) = \mathbb{E}(Z^T A_{qh}^0 Z|U) \mathbb{E}(Z^T A_{qh}^0 Z) = \sum_{k=1}^m \sum_{i \neq j} a_{ij}^k (u_i^k u_j^k - p_i^k)
\]

where \(a_{ij}^k\) is a shorthand for \(a_{ij}^k(q,h) = \frac{1}{m} a_{kk} b_{ij}(q_i h_j + q_j h_i)\).

**Theorem 4.5.** Set \(1/2 > \varepsilon > 0\). Denote by \(\mathcal{N}\) the \(\varepsilon\)-net for \(S^{n-1} \cap E\) as constructed in Lemma \(\text{4.1}\). Suppose for \(\psi_B(s_0)\) as defined in Theorem \(\text{2.4}\).

\[
\sum_{j=1}^m a_{jj} P_j^2 = \Omega \left( a_\infty \psi_B(2s_0 \lor n) s_0 \log(3en/(s_0 \varepsilon)) \right).
\]

Set \(\tau' = C_6 a_\infty \psi_B(2s_0 \lor n) \|B_0\|_2 s_0 \log(3en/(\varepsilon s_0))\). Then

\[
\mathbb{P}(\mathcal{F}_2) = \mathbb{P}(\exists q,h \in \mathcal{N}, |S_*(q,h)| \geq \tau') \leq 2 \exp(-c_1 s_0 (3en/(s_0 \varepsilon)))
\]

where \(C_6, c_1\) are absolute constants.
Then on event $\mathcal{F}_2$, we have
\[
\sup_{q,h \in \mathcal{N}} \frac{|S_q(h)|}{\|M\|_{\text{offd}} \|B_0\|_2} \asymp r_{\text{offd}}^2(s_0)\psi_B(2s_0 \wedge n)
\]

It is important that we separate the dependence on $s_0$ from dependence on $\rho_{\max}(s_0, (|b_{ij}|))$ as defined in (19). The result in Theorem 3.5 appears to be tight, since when we ignore the logarithmic terms, the linear dependency on $s_0$ is correct; the extra term $\rho_{\max}(s_0, |B_0|)$ is unavoidable, because of our reliance on the sparse Hanson-Wright type of moment generating function bounds as stated in Lemma B.5. We prove Theorem 3.3 in Section I in the supplementary material.

**Putting things together.** Combining the large deviation bounds using (44), we have on event $\mathcal{F}_0^c \cap \mathcal{F}_1^c \cap \mathcal{F}_2^c$, for $\Delta = \text{offd}(\mathcal{X}\mathcal{X}^T - B_0 \circ \mathcal{M})$, by Theorems 4.4 and 4.5, and a standard approximation argument in the sense of (41) and (42),
\[
\sup_{q,h \in \mathcal{N}} \frac{|q^T\Delta h|}{\|B_0\|_2 \|M\|_{\text{offd}}} \leq \sup_{q,h \in \mathcal{N}} \frac{1}{(1-\varepsilon)^2 \|B_0\|_2 \|M\|_{\text{offd}}} \leq \eta_A r_{\text{offd}}(s_0) + r_{\text{offd}}(s_0) f_p \psi_B(s_0) + r_{\text{offd}}^2(s_0) \psi_B(2s_0 \wedge n) =: \delta_{\text{overall}}
\]

(51)

The expression for the overall rate of convergence can be simplified and further depends on the lower bound on $\sum_{j=1}^n p_j^2$. Clearly, we have by (47), $r_{\text{offd}}(s_0) f_p \psi_B(s_0) = o(1)$.

**Lemma 4.6.** Let $r_{\text{offd}}(s_0)$ and $\ell_{s_0,n}$ be as defined in (22). Then using $r_{\text{offd}} := r_{\text{offd}}(s_0)$ as a shorthand, we have
\[
r_{\text{offd}} f_p \psi_B(s_0) \leq r_{\text{offd}}(\eta_A r_{\text{offd}} \psi_B(s_0))^{1/2} \ell_{s_0,n}^{1/4} \delta_{\text{overall}} \leq r_{\text{offd}} \left( \eta_A \ell_{s_0,n}^{1/2} + r_{\text{offd}} \psi_B(2s_0 \wedge n) \right)
\]

while the second term $O(r_{\text{offd}}^2 \psi_B(2s_0 \wedge n))$ also vanishes once the sample size satisfies (52) for some absolute constant $C_8$,
\[
\sum_{j=1}^m a_{jj} p_j^2 \geq C_8 \psi_B^2(2s_0 \wedge n) s_0 \log(3en/s_0 \varepsilon).
\]

(52)

Lemma 4.6 is proved in Section [E]. This completes the proof of Theorem 3.3. We defer details to Section [E] in the supplementary material.

5 Proof of Theorem 2.4

Let $E$ be as defined in Lemma 3.2. By Theorem 3.3, we have on event $\mathcal{F}_0^c \cap \mathcal{F}_1^c \cap \mathcal{F}_2^c$, for all $q, h \in E \cap \mathbb{S}^{n-1}$ and $\Delta_B = \text{offd}(\mathcal{B}_0 - B_0)/\|B_0\|_2$,
\[
|q^T\Delta_B h| = \frac{1}{\|M\|_{\text{offd}} \|B_0\|_2} |q^T \text{offd}(\mathcal{X}\mathcal{X}^T - B_0 \circ \mathcal{M}) h| \ll \delta_q < 1/5
\]
Then for all \( q \in (\sqrt{s_0}B_1^0 \cap B_2^0) \), by Lemma 3.2,
\[
|q^T \Sigma B q| = \frac{1}{\|M\|_{\text{offd}} \|B_0\|_2} |q^T \text{offd}(X^T X - B_0 \circ \mathcal{M}) q| \leq 4\delta_q < 4/5
\]

Let \( \Delta = \tilde{B}_0 - B_0 \). Hence by (30), (32) and Theorems 3.1 and 3.3, we have with probability at least
\[1 - \frac{C}{(n^\alpha m)^4} - 4 \exp(-cs_0 \log(3en/(s_0e)))\]
for some absolute constants \( C, c \), for all \( q \in (\sqrt{s_0}B_1^0 \cap B_2^0) \),
\[
|q^T \Delta q| \leq \frac{1}{\|M\|_{\text{offd}} \|B_0\|_2} |q^T \text{offd}(X^T X - B_0 \circ \mathcal{M}) q| + \frac{1}{\|M\|_{\text{diag}} \|B_0\|_2} |q^T \text{diag}(X^T X - B_0 \circ \mathcal{M}) q| \ll r_{\text{diag}} + \eta_{\text{offd}}(s_0) \ell_{s_0,n}^{1/2} + r_{\text{offd}}^2(s_0) \psi_E(2s_0 \cap n)
\]
where clearly \( r_{\text{diag}} \leq \eta_{\text{offd}}(s_0) \ell_{s_0,n}^{1/2} \times \sqrt{s_0 \log(n \lor m) \|A_0\|_2 \sum_j a_{jj} p_j^2} \).

6 Proof of Theorem 2.8

Since we are dealing with relative errors, we may assume without loss of generality that \( \text{tr}(B_0) = n \) throughout our analysis. In this case, \( B_\ast = B_0 \). Denote by the diagonal and off-diagonal components in \( \tilde{M} \) as defined in (18), which has only two unique entries,
\[
\tilde{M}_{k\ell} =: \|\tilde{M}\|_{\text{offd}} \text{ in case } k \neq \ell \text{ and } \tilde{M}_{\ell\ell} =: \|\tilde{M}\|_{\text{diag}} = \frac{1}{n} \text{tr}(X^T X) \forall \ell.
\]
In Lemma 6.1 we prove the concentration bounds for \( \|\tilde{M}\|_{\text{offd}} \) and \( \|\tilde{M}\|_{\text{diag}} \). Combining these bounds with Theorems 3.1 and 3.3, we prove Theorem 2.8 in Section 6.1. Under (24), we will be able to show that RE conditions as stated in (25) and (26) also hold for \( \tilde{B}_\ast \). Such results are omitted.

**Lemma 6.1.** Assume \( \text{tr}(B_0) = n \). Then \( \tilde{E} \tilde{M} = \mathcal{M} \), for \( \tilde{M} \) and \( \mathcal{M} \) as defined in (18) and (12) respectively. Denote by \( r(B_0) = \text{tr}(B_0)/\|B_0\|_2 \) the effective rank of \( B_0 \). Suppose that \( \sum_{s=1}^m a_{ss}^2 p_s^2 > C a_{\infty}^2 \log(m \lor n) \). Then on event \( \mathcal{F}_S \cap \mathcal{F}_0^c \), which holds with probability at least \( 1 - \frac{C}{(n^\alpha m)^4} \), for some absolute constant \( C, C' \) and \( \eta_A = \sqrt{\frac{a_{\infty}}{a_{\min}}} \),
\[
\forall k \neq \ell, \quad \delta_{\text{mask}} := \frac{|\tilde{M}_{k\ell} - \mathcal{M}_{k\ell}|}{\sum_{j=1}^m a_{jj} p_j^2} = \frac{\|\tilde{M}\|_{\text{offd}} - \|\tilde{M}\|_{\text{offd}}}{\|\mathcal{M}\|_{\text{offd}}} = O(\eta_{\text{offd}} r_{\text{offd}} / \sqrt{r(B_0)}) \text{ where } r_{\text{offd}} \leq \sqrt{\|A_0\|_2^{1/2} \log(1/2)(m \lor n) / \sum_j a_{jj} p_j^2} = o(1)
\]
is the same as \( r_{\text{offd}}(s_0) \) as defined in (22) with \( s_0 = 1 \); for \( r_{\text{diag}} \) as defined in Theorem 3.1, with probability at least \( 1 - \frac{C}{(n^\alpha m)^4} \), for all \( \ell \),
\[
\delta_{m,\text{diag}} := \frac{|\tilde{M}_{\ell\ell} - \mathcal{M}_{\ell\ell}|}{\sum_{j=1}^m a_{jj} p_j} = \frac{\|\tilde{M}\|_{\text{diag}} - \|\tilde{M}\|_{\text{diag}}}{\|\mathcal{M}\|_{\text{diag}}} \ll r_{\text{diag}} / \sqrt{r(B_0)}.
\]
6.1 Proof of Theorem 2.8

Notice that \( E(\mathcal{X}^T) = B_0 \circ \mathcal{M} \), where expectation is understood to be taken componentwise. Let \( \delta_{m,\text{diag}} \) and \( \delta_{\text{mask}} \) be as defined in Lemma 6.1. Then for all \( q \in (\sqrt{s_0}B_1^n \cap B_2^n) \), by the fact that

\[
\left| q^T \text{offd}(\hat{\mathcal{X}}_* - B_0)q \right| = \left| q^T \text{offd}(\mathcal{X}^T \circ \hat{\mathcal{X}} - E(\mathcal{X}^T \circ \mathcal{M}))q \right|
\]

\[
= \left| q^T \text{offd}(\mathcal{X}^T \circ \hat{\mathcal{M}} - (B_0 \circ \mathcal{M}) \circ \mathcal{M})q \right|
\]

\[
\leq \left| q^T \text{offd}(\mathcal{X}^T \circ \hat{\mathcal{M}} - (B_0 \circ \mathcal{M}) \circ \hat{\mathcal{M}})q \right| + \left| q^T \text{offd}(B_0 \circ \mathcal{M} \circ \hat{\mathcal{M}} - (B_0 \circ \mathcal{M}) \circ \mathcal{M})q \right| =: W_1 + W_2
\]

where we have by Theorem 3.3 and Lemma 6.1 on event \( \mathcal{F}_0^c \cap \mathcal{F}_1^c \cap \mathcal{F}_2^c \cap \mathcal{F}_3^c \), which holds with probability at least \( 1 - \frac{C}{(nvm)^2} - 4 \exp(-c_0 \log(3en/(s_0 \varepsilon))) \),

\[
W_1 = \frac{1}{\|\hat{\mathcal{M}}\|_{\text{offd}}} \left| q^T \text{offd}(\mathcal{X}^T - (B_0 \circ \mathcal{M}))q \right| \leq \frac{\|\mathcal{M}\|_{\text{offd}}}{\|\hat{\mathcal{M}}\|_{\text{offd}}} \left| q^T \text{offd}(\mathcal{X}^T - (B_0 \circ \mathcal{M}))q \right| \leq \frac{4\delta_q}{1 - \delta_{\text{mask}}} \|B_0\|_2
\]

where \( \delta_q \) is as defined in Theorem 3.3 and \( \forall q \in S^{n-1} \),

\[
W_2 = \left| q^T \text{offd}(\mathcal{M} \circ \mathcal{M} - \mathcal{M})q \right| \leq \frac{\delta_{\text{mask}}}{1 - \delta_{\text{mask}}} \|\text{offd}(\mathcal{M})\|_2.
\]

Similarly, by Theorem 3.1 and Lemma 6.1 we have for all \( q \in S^{n-1} \), with probability at least \( 1 - \frac{C}{(nvm)^2} \) and some absolute constant \( C \),

\[
\left| q^T \text{diag}(\hat{\mathcal{X}}_* - B_0)q \right| = \left| q^T \text{diag}(\mathcal{X}^T \circ \hat{\mathcal{M}} - E(\mathcal{X}^T \circ \mathcal{M}))q \right|
\]

\[
\leq \frac{\|\mathcal{M}\|_{\text{diag}}}{\|\hat{\mathcal{M}}\|_{\text{diag}}} \left| \text{diag}(\mathcal{X}^T - (B_0 \circ \mathcal{M})) \right|_2 + \frac{\|\mathcal{M}\|_{\text{diag}}}{\|\hat{\mathcal{M}}\|_{\text{diag}}} - 1 \left| \|\text{diag}(B_0 \circ \mathcal{M})\|_2 \right|
\]

\[
\leq \frac{C''}{1 - \delta_{m,\text{diag}}} (b_{\infty,\text{diag}} + \delta_{m,\text{diag}} b_{\infty}) \times b_{\infty} \approx \delta_q \|B_0\|_2
\]

Finally, combining the two bounds above using the triangle inequality, we have with probability at least \( 1 - \frac{C'}{(nvm)^2} - 4 \exp(-c_0 \log(3en/(s_0 \varepsilon))) \)

\[
\left| q^T (\hat{\mathcal{X}}_* - B_0)q \right| \leq \left| q^T \text{offd}(\hat{\mathcal{X}}_* - B_0)q \right| + \left| q^T \text{diag}(\hat{\mathcal{X}}_* - B_0)q \right|
\]

\[
\leq 8\delta_q \|B_0\|_2 (1 + \delta_{\text{mask}})
\]

where \( c, C, C', \ldots \) throughout this proof are absolute constants which may change line by line. The theorem thus holds. \( \Box \)

Lemma 6.1 is proved in Section N.1 in the supplementary material.
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A Preliminary results on estimators

First, we need the following definitions and notations. Let \( b^{(1)}, \ldots, b^{(n)} \) denote the column (row) vectors of symmetric positive-definite matrix \( B_0 \succ 0 \). Let \( u_1^s, \ldots, u_m^s, s = 1, \ldots, m \) be independent random variables with two values 0 and 1, and a polynomial \( Y = \sum_{e \in \mathbf{E}} w_e \prod_{(i,j) \in e} u_j^i \), where \( w_e \) is a weight which may have both positive and negative coefficients, and \( \mathbf{E} \) is a collection of subsets of indices \( \{(i,j), i = 1, \ldots, m, j = 1, \ldots, n\} \). If the size of the largest subset in \( \mathbf{E} \) is \( k \), \( Y \) is called a polynomial of degree \( k \). If all coefficients \( w_e \) are positive, then \( Y \) is called a positive polynomial of degree \( k \). A homogeneous polynomial is a polynomial whose nonzero terms all have the same degree.

Recall for a matrix \( A \), the effective rank \( r(A) = \text{tr}(A)/\|A\|_2 \). We use the following properties of the Hadamard product: for \( x \in \mathbb{R}^m \) and \( A \in \mathbb{R}^{m \times m} \),

\[
A \circ xx^T = D_x AD_x
\]

and

\[
\text{tr}(D_x AD_x A) = x^T(A \circ A)x
\]

from which a simple consequence is

\[
\text{tr}(D_x AD_x) = x^T(A \circ I)x = x^T\text{diag}(A)x.
\]

We use \( X \sim Y \) to denote that two random variables \( X, Y \) follow the same distribution.

A.1 Elaborations on estimators for \( M \)

Without knowing the parameters, we need to estimate \( M \) and \( N \) as used in (13) and (14). Recall \( v^i \sim v, i = 1, \ldots, n \) are independent, and

\[
\mathbb{E}v^i \otimes v^i = \begin{bmatrix} p_1 & p_1p_2 & p_1p_3 & \cdots & p_1p_m \\
p_2p_1 & p_2 & p_2p_3 & \cdots & p_2p_m \\
p_3p_1 & p_3p_2 & p_3 & \cdots & p_3p_m \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
p_mp_1 & p_mp_2 & p_mp_3 & \cdots & p_m \end{bmatrix}_{m \times m} =: M
\]

where expectation denotes the componentwise expectation of each entry of \( v^i \otimes v^i \), for all \( i \). Clearly we observe \( v^i \in \{0,1\}^m \), the vector of indicator variables for nonzero entries in the \( i \)-th row of data matrix \( X \), for all \( i = 1, \ldots, n \). That is,

\[
v^i_j = 1 \quad \text{if} \quad X_{ij} \neq 0
\]

\[
v^i_j = 0 \quad \text{if} \quad X_{ij} = 0
\]

Hence we observe \( M^i := v^i \otimes v^i, i = 1, \ldots, n \), upon which we define an estimator

\[
\hat{M} = \frac{1}{n} \sum_{i=1}^n v^i \otimes v^i = \frac{1}{n} \sum_{i=1}^n M^i
\]
Clearly $\hat{M}$ as defined in (54) is unbiased since $E\hat{M} = EM = M_0$, for $M$ as defined in (53). To justify (13) and (14), we have by independence of the mask $U$ and data matrix $X$ as defined in (8),

$$AX' = (U \circ X)^T(U \circ X) = \sum_{i=1}^{n} (v^i \otimes v^i) \circ (y^i \otimes y^i)$$

$$\mathbb{E} \frac{1}{n} AX' = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(v^i \otimes v^i) \circ \mathbb{E}(y^i \otimes y^i) = \text{tr}(B_0)/n(A_0 \circ M)$$

where $M$ is as defined in (53) and hence

$$\mathbb{E}\hat{M}_{jj} = \mathbb{E}\text{tr}(X'X)/n = \frac{\text{tr}(B_0)}{n} \sum_{j} a_{jj}p_j$$

for $\hat{M}$ as in (18),

where expectation in (56) denotes the componentwise expectation of each entry of $X'X$.

Denote by

$$S_c := \frac{n}{n-1}\text{tr}(X'X \circ \hat{M}) - \frac{1}{n-1}\text{tr}(X'X).$$

Notice that for $M^i = v^i \otimes v^i, i = 1, \ldots, n$, we have by (55) and (17), or equivalently, $\hat{M}$ as defined immediately above in (54),

$$\frac{n}{n-1}\text{tr}(X'X \circ \hat{M}) = \frac{1}{n-1}\text{tr}(X'X \circ \left(\sum_{j=1}^{n} v^j \otimes v^j\right))$$

$$= \frac{1}{n-1}\sum_{k=1}^{n}\text{tr}\left(\sum_{j=1}^{n} (v^j \otimes v^j) \circ (y^j \otimes y^j) \circ (v^k \otimes v^k)\right)$$

$$= \frac{1}{n-1}\sum_{j=1}^{n}\text{tr}\left((v^j \otimes v^j) \circ (y^j \otimes y^j)\right) + \frac{1}{n-1}\sum_{j=1}^{n}\sum_{k \neq j}^{n}\text{tr}\left(M^k \circ (y^j \otimes y^j) \circ M^j\right)$$

$$= \frac{1}{n-1}\text{tr}(X'X) + \frac{1}{n-1}\sum_{j=1}^{n}\sum_{k \neq j}^{n}\text{tr}\left(M^k \circ M^j \circ (y^j \otimes y^j)\right)$$

Rearranging we obtain for $\hat{M}$ as defined in (18),

$$\forall k \neq \ell, \hat{M}_{kk} = \frac{S_c}{n} = \frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{k \neq j}^{n}(y^j)^T\text{diag}(M^k \circ M^j)y^j$$

We will show that $\hat{M}$ is a componentwise unbiased estimator for $M$ as defined in (12) in case $\text{tr}(B_0) = n$. Clearly, $\text{diag}(\hat{M}) = \frac{1}{n}\text{tr}(X'X)I_n$ provides a componentwise unbiased estimator for $\text{diag}(M)$ as defined in (12), namely,

$$\text{diag}(M) = (\sum_{j} a_{jj}p_j)I_n \quad \text{in case} \quad \text{tr}(B_0) = n.$$
B Preliminary theoretical results

Before we go on, we first state the standard Hanson-Wright inequality \(^{28}\) and Theorem [B.2] which is the main tool to deal with the sparse quadratic forms.

**Theorem B.1.** \(^{28}\) Let \(X = (X_1, \ldots, X_m) \in \mathbb{R}^m\) be a random vector with independent components \(X_i\) which satisfy \(\mathbb{E}X_i = 0\) and \(\|X_i\|_{\psi_2} \leq K\). Let \(A\) be an \(m \times m\) matrix. Then, for every \(t > 0\),

\[
\mathbb{P}\left( |X^T AX - \mathbb{E}X^T AX| > t \right) \leq 2 \exp \left( -c \min \left( \frac{t^2}{K^4 \|A\|_F^2}, \frac{t}{K^2 \|A\|_2} \right) \right).
\]

Theorem [B.2] shows a concentration of measure bound on a quadratic form with Bernoulli random variables where an explicit dependency on \(p_i\), for all \(i\), is shown. The setting here is different from Theorem [B.1] as we deal with a quadratic form which involves non-centered Bernoulli random variables. The constants presented in Theorem [B.2] statement are entirely arbitrarily chosen. Although the proof of Theorem [B.2] shares the same line of arguments with Theorem 2.10 in \(^{41}\), we state the bound differently as we now need it to derive a concentration bound on \(S_\ast(q, h)\) as defined in \(^{39}\); we include the proof in Section [M] for self-containment.

**Theorem B.2.** Let \(\xi = (\xi_1, \ldots, \xi_m) \in \{0, 1\}^m\) be a random vector with independent Bernoulli random variables \(\xi_i\) such that \(\xi_i = 1\) with probability \(p_i\) and 0 otherwise. Let \(A = (a_{ij})\) be an \(m \times m\) matrix. Let

\[
S_\ast := \sum_{i,j} a_{ij} \xi_i \xi_j - \mathbb{E} \sum_{i,j} a_{ij} \xi_i \xi_j.
\]

Denote by \(D_{\text{max}} := \|A\|_\infty \lor \|A\|_1\). Then, for every \(|\lambda| \leq \frac{1}{16\|A\|_1 \lor \|A\|_\infty}\),

\[
\mathbb{E} \exp(\lambda S_\ast) \leq \exp \left( 32.5 \lambda^2 D_{\text{max}} e^{8|\lambda|D_{\text{max}}} \sum_{i \neq j} |a_{ij}| \sigma_i^2 \sigma_j^2 \right).
\]

\[
\exp \left( 2 \lambda^2 D_{\text{max}} e^{4|\lambda|D_{\text{max}}} \left( \sum_{i=1}^m |a_{ii}| \sigma_i^2 + 2 \sum_{i \neq j} |a_{ij}| p_j p_i \right) \right)
\]

where \(\sigma_i^2 = p_i(1 - p_i)\) and \(\mathbb{E} \sum_{i,j} a_{ij} \xi_i \xi_j = \sum_{i=1}^m a_{ii} p_i + \sum_{i \neq j} a_{ij} p_i p_j\).

Corollary [B.3] follows from Theorem [B.2] in case \(\text{diag}(A) = 0\).

**Corollary B.3.** Let \(\xi = (\xi_1, \ldots, \xi_m) \in \{0, 1\}^m\) be a random vector with independent Bernoulli random variables \(\xi_i\) such that \(\xi_i = 1\) with probability \(p_i\) and 0 otherwise. Let \(A = (a_{ij})\) be an \(m \times m\) matrix with 0s along its diagonal. Then, for every \(|\lambda| \leq \frac{1}{16\|A\|_1 \lor \|A\|_\infty}\) and \(D_{\text{max}} := \|A\|_\infty + \|A\|_1\)

\[
\mathbb{E} \exp \left( \lambda \left( \sum_{i,j} a_{ij} \xi_i \xi_j - \mathbb{E} \sum_{i,j} a_{ij} \xi_i \xi_j \right) \right)
\]

\[
\leq \exp \left( 36.5 \lambda^2 D_{\text{max}} e^{8|\lambda|D_{\text{max}}} \sum_{i \neq j} |a_{ij}| p_i p_j \right)
\]

where \(\mathbb{E} \sum_{i \neq j} a_{ij} \xi_i \xi_j = \sum_{i \neq j} a_{ij} p_i p_j\).
B.1 Event $F_1$

We first state Lemma B.4, which defines the event $F_1$.

Lemma B.4. Let $1 \leq s_0 \leq n$ and $0 < \varepsilon \leq 1/2$. Denote by $\mathcal{N}$ the $\varepsilon$-net for $S^{n-1} \cap E$ as constructed in Lemma 4.7. Suppose that under event $F_0$, (45) holds for some function $W$. Set for some absolute constants $C_1, C_2$,

$$
\tau_0 = C_1 s_0 \log \left( \frac{3en}{s_0 \varepsilon} \right) \| A_0 \|_2 \| B_0 \|_2 + C_2 \sqrt{ \frac{s_0}{s_0 \varepsilon} } \| A_0 \|_2^{1/2} \| B_0 \|_2 \cdot W
$$

Then for some absolute constant $c_1$ and $F_0$ as defined in Theorem 4.3,

$$
P(F_1) := P(\exists q, h \in \mathcal{N}, |Z^T A_{q,h}^0 Z - \mathbb{E}(Z^T A_{q,h}^0 Z[U])| > \tau_0) 
\leq 2 \exp(-c_1 s_0 \log(3en/(s_0 \varepsilon))) + P(F_0)
$$

We prove Lemma B.4 and Section H.1.

B.2 Event $F_2$

First we derive the expression for $S_* = S_*(q, h)$ as in (49). Let $\mathbb{D}(q, h)$ be the block-diagonal matrix with $k$th block along the diagonal being $\mathbb{D}_{ij}^{(k)}(q, h)$, where

$$
\mathbb{D}_{ij}^{(k)}(q, h) = \frac{1}{2} Z^T ((q_i h_j + q_j h_i) c_i c_j^T \otimes (d_k d_k^T)) Z
$$

and hence

$$
\mathbb{E}\mathbb{D}_{ij}^{(k)}(q, h) = \frac{1}{2} \text{tr}((q_i h_j + q_j h_i) c_i c_j^T \otimes (d_k d_k^T)) = \frac{1}{2} a_{kk}(q_i h_j + q_j h_i) b_{ij}
$$

Hence for $\tilde{a}_{ij}^{(k)}(q, h) = \frac{1}{2} a_{kk}(q_i h_j + q_j h_i) b_{ij}$ and $\mathbb{E}(u_i^k) = p_k$ for all $i$,

$$
\mathbb{E}(Z^T A_{q,h}^0 Z[U] - \mathbb{E}(Z^T A_{q,h}^0 Z))
= \text{vec} \{ U \}^T \mathbb{E}\mathbb{D}(q, h) \text{vec} \{ U \} - \mathbb{E}(\text{vec} \{ U \}^T \mathbb{E}\mathbb{D}(q, h) \text{vec} \{ U \})
= : \sum_{k=1}^n \sum_{i=1}^n \sum_{j \neq i}^n (u_i^k u_j^k \mathbb{E}(u_i^k u_j^k) \tilde{a}_{ij}^{(k)}(q, h) =: S_*(q, h)
$$

Lemma B.5. Let $S_*(q, h)$ be as defined in (49). Let $E = \bigcup_{|J| \leq s_0} E_J$ for $0 < s_0 \leq n$. Denote by $\bar{\rho}(s_0, |B_0|)$ the following quantity: where $|B_0| = (|b_{ij}|),$

$$
\bar{\rho}(s_0, |B_0|) := \max_{q, h \in S^{n-1}, s_0 \text{-sparse}} \sum_{i=1}^n \sum_{j=1}^n |b_{ij}| |q_i| |h_j|
$$

(59)

Denote by

$$
\tilde{\rho}(s_0, |B_0|) = \rho(s_0, |B_0|) \wedge |B_0|_2.
$$

Then for any $q, h \in E \cap S^{n-1}$, for $|\lambda| \leq 1/ (16a_{\infty} \bar{\rho}(s_0, |B_0|))$, we have

$$
\mathbb{E} \exp \left( \lambda S_*(q, h) \right) \leq \exp \left( 60\lambda^2 a_{\infty} \bar{\rho}(s_0, |B_0|) \rho(s_0, |B_0|) \sum_k a_{kk} p_k^2 \right)
$$
Hence for any \( t > 0 \), and for any \( q, h \in E \cap \mathbb{S}^{n-1} \),
\[
\mathbb{P} (|S_\tau (q, h)| > t) \leq 2 \exp \left( -c \min \left( \frac{t^2}{a_\infty \bar{\rho}(s_0, |B_0|) \bar{\rho}(s_0, |B_0|) \sum_{k=1}^m a_{kk} p_k^2}, \frac{t}{a_\infty \bar{\rho}(s_0, |B_0|)} \right) \right)
\]

We now also derive an upper bound on \( \bar{\rho}(s_0, |B_0|) \) as follows.

**Lemma B.6.** Denote by \( |q| \) the vector with absolute values of \( q_j, j = 1, \ldots, n \). Let \( E = \cup_{|j| \leq s_0} E_j \) for \( 0 < s_0 \leq n \). Let \( q, h \in E \cap \mathbb{S}^{n-1} \) be \( s_0 \)-sparse. Then for \( \bar{\rho}(s_0, |B_0|) \) as defined in (59), we have for \( |q| = (|q_1|, \ldots, |q_n|) \),
\[
\bar{\rho}(s_0, |B_0|) := \sup_{h, q \in E \cap \mathbb{S}^{n-1}} |h|^T |B_0| |q|
\leq (2 \rho_{\max}(2s_0, (|b_{ij}|)) - \rho_{\min}(s_0, B_0)) \wedge \sqrt{s_0} \|B_0\|_2.
\]

We prove Lemmas B.5 and B.6 in Section B.3. We emphasize that Lemma B.5 holds for all vectors \( q, h \in \mathbb{S}^{n-1} \), rather than for sparse vectors only; When \( q \) and \( h \) are indeed \( s_0 \)-sparse, then \( \rho_{\max}(2s_0, (|b_{ij}|)) \) is used to replace \( \sum_{i,j} |b_{ij}| |q_i| |h_j| \) appearing in the proof (cf. (62) in proof of Lemma B.6). For \( s_0 = n \), we can simplify Lemma B.5 and write the following corollary (B.7).

**Corollary B.7.** Denote by
\[
S_\tau (q) = \mathbb{E} (Z^T A^\circ q Z | U) - \mathbb{E} (Z^T A^\circ q Z) = \sum_{k=1}^m \sum_{i \neq j} a_{ij}^k (u_{i}^k u_{j}^k - p_k^2)
\]
where \( a_{ij}^k \) is a shorthand for \( a_{ij}^k(q) = a_{kk} b_{ij} q_i q_j \). Then for any \( q \in \mathbb{S}^{n-1} \) and \( |\lambda| \leq 1/(16 a_\infty \|B_0\|_2) \), we have
\[
\mathbb{E} \exp \left( \lambda S_\tau (q) \right) \leq \exp \left( 60 \lambda^2 a_\infty \|B_0\|_2 \|B_0\|_2 \sum_k a_{kk} p_k^2 \right)
\]

Hence for any \( t > 0 \), and for any \( q \in \mathbb{S}^{n-1} \),
\[
\mathbb{P} (|S_\tau (q)| > t) \leq 2 \exp \left( -c \min \left( \frac{t^2}{a_\infty \|B_0\|_2 \|B_0\|_2 \sum_{k=1}^m a_{kk} p_k^2}, \frac{t}{a_\infty \|B_0\|_2} \right) \right)
\]

**Proof.** Let \( \bar{A} = \mathbb{E} \mathbb{D}(q) = (\bar{a}_{ij}^k)_{k=1,\ldots,m} \) be the block-diagonal matrix with \( k^{th} \) block along the diagonal being \( \bar{A}^{(k)} := (\bar{a}_{ij}^k(q))_{i,j \leq n}, k = 1, \ldots, m \), where \( \bar{a}_{ij}^k(q) = a_{kk} b_{ij} q_i q_j \) for \( i \neq j \) and \( \bar{a}_{jj}^k = 0 \). Then
\[
\bar{A} = \text{diag}(A_0) \otimes \text{offd}(B_0 \circ (q \otimes q)), \tag{60}
\]
where the expectation is taken componentwise. Then for \( |\lambda| \leq \frac{1}{16 D_{\max}} \), where \( D_{\max} := \|\bar{A}\|_\infty \vee \|\bar{A}\|_1 \leq a_\infty \|B_0\|_2 \) for \( \bar{A} \) as defined in (60),
\[
\mathbb{E} \exp \left( \lambda S_\tau (q) \right) \leq \exp \left( 60 \lambda^2 D_{\max} \sum_k \sum_{i \neq j} |\bar{a}_{ij}^k(q)| p_i p_j \right)
\leq \exp \left( 60 \lambda^2 D_{\max} \|B_0\|_2 \sum_k a_{kk} p_k^2 \right)
\]

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by Theorem B.2. The rest of the proof follows that of Lemma B.5. □

B.3 Proof of Lemma B.5

Let \( \bar{A} = E(D(q, h)) = (\bar{a}_{ij}^k)_{k=1,\ldots,m} \) be the block-diagonal matrix with \( k \)th block along the diagonal being \( \bar{A}^{(k)} := (\bar{a}_{ij}^k)_{i,j \leq n}, \) for \( k = 1, \ldots, m. \) Then
\[
\bar{A} = E(D(q, h)) = \frac{1}{2} \text{diag}(A_0) \otimes \text{offd}(B_0 \circ ((q \otimes h) + (h \otimes q))),
\]
where the expectation is taken componentwise. Now we compute for all \( q, h \in \mathbb{S}^{n-1}, \) for \( \bar{a}_{ij}^k(q, h) = \frac{1}{2} a_{kk}b_{ij}(q_i h_j + q_j h_i), \) and \( \bar{A} \) as defined in (61),
\[
D_{\max} := \| \bar{A} \|_\infty \vee \| \bar{A} \|_1 \leq \max_k a_{kk} \frac{1}{2} \left( \max_i |q_i| \sum_{j \neq i} |h_j| |b_{ij}| + \max_i |h_i| \sum_{j \neq i} |q_j| |b_{ij}| \right)
\leq a_\infty \max_i \| b^{(i)} \|_2
\]
where for \( B_0 = [b^{(1)}, \ldots, b^{(n)}] \) and \( h \in \mathbb{S}^{n-1}, \)
\[
\sum_{j \neq i} |b_{ij}| |h_j| \leq \sqrt{\sum_{j \neq i} b_{ij}^2} \sqrt{\sum_{j \neq i} h_j^2} \leq \max_i \| b^{(i)} \|_2 \leq \| B_0 \|_2.
\]
On the other hand,
\[
\max_i |q_i| \sum_{j \neq i} |h_j| |b_{ij}| \leq \sum_i |q_i| \sum_{j \neq i} |h_j| |b_{ij}|,
\]
and hence
\[
\frac{1}{2} \left( \max_i |q_i| \sum_{j \neq i} |h_j| |b_{ij}| + \max_i |h_i| \sum_{j \neq i} |q_j| |b_{ij}| \right) \leq \bar{\rho}(s_0, |B_0|)
\]
Hence
\[
D_{\max} := \| \bar{A} \|_\infty \vee \| \bar{A} \|_1 \leq a_\infty (\| B_0 \|_2 \land \bar{\rho}(s_0, |B_0|)) = a_\infty \bar{\rho}(s_0, |B_0|).
\]
Then by Theorem B.2, we have for \( |\lambda| \leq \frac{1}{16a_\infty (\| B_0 \|_2 \land \bar{\rho}(s_0, |B_0|))} \leq \frac{1}{16D_{\max}}, \)
\[
\mathbb{E} \exp(\lambda S_n) \leq \prod_{k=1}^m \exp \left( 36.5 \lambda^2 D_{\max} e^{8|\lambda| D_{\max}} \sum_{i \neq j} \bar{a}_{ij}^k \mathbb{E}(u_j^k) \mathbb{E}(u_j^k) \right)
= \exp \left( 60 \lambda^2 D_{\max} \sum_{k=1}^m \sum_{i \neq j} \bar{a}_{ij}^k |p_k^2 \right) \leq \exp(\lambda^2 60 D_{\max} T)
\]
where
\[
\sum_{k=1}^m \sum_{i \neq j} \bar{a}_{ij}^k |p_k^2 = \sum_{k=1}^m a_{kk} p_k^2 \frac{1}{2} \sum_{i \neq j} |b_{ij}| (|q_i| |h_j| + |q_j| |h_i|)
\leq \bar{\rho}(s_0, |B_0|) \sum_{k=1}^m a_{kk} p_k^2 =: T
\]

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Let $t > 0$. Optimizing over $0 < \lambda < \frac{1}{\nu_{\infty}(\|B_0\|_2,\rho(s_0,|B_0|))}$, we have

$$
P(S^* > t) \leq \frac{\mathbb{E}\exp(\lambda S^*)}{e^{\lambda t}} \leq \exp(-\lambda t + \lambda^2 60D_{\max}T)
$$

$$
\leq \exp\left(-c \min\left(\frac{t^2}{\alpha_{\infty}(s_0,|B_0|)^{\rho(s_0,|B_0|)} \sum_k a_{kk}}{\rho(s_0,|B_0|)^{\rho(s_0,|B_0|)}}\right)\right) =: q^*;
$$

Repeating the same arguments, we have for $t > 0$,

$$
P(S^* < -t) = P(-S^* > t) \leq q^*
$$

hence the lemma is proved by combining these two events. □

C Proof of Lemmas 2.5 and 3.5

First, we need to define the sparse eigenvalue for matrix $B_0$.

**Definition C.1.** For $1 \leq s_0 \leq n$, we define the largest $s_0$-sparse eigenvalue of an $n \times n$ matrix $B_0 \succ 0$ to be

$$
\rho_{\max}(s_0, B_0) := \max_{v \in \mathbb{S}^{n-1}, s_0 - \text{sparse}} v^T B_0 v
$$

Clearly, as a consequence of the Rayleigh-Ritz theorem,

$$
\max_j b_{jj} =: b_\infty \leq \rho_{\max}(s_0, B_0) \leq \|B_0\|_2 \leq \|\|b_{ij}\|\|_2.
$$

**Proof of Lemma 2.5** Now since $|B_0| \geq 0$, that is, all entries $|b_{ij}|$ are either positive or zero, we have

$$
\rho_{\max}(s_0, |B_0|) := \max_{q \in \mathbb{S}^{n-1}, s_0 - \text{sparse}} \sum_{i=1}^{n} \sum_{j=1}^{n} |q_i| |q_j| |b_{ij}|
$$

$$
= \max_{S \subset [n]: |S| = s_0} \lambda_{\max}(|B_0|_S) \leq \max_{S \subset [n]: |S| = s_0} \|B_0|_S\|_{\infty}
$$

$$
= \max_{S \subset [n]: |S| = s_0} \|B_0, S, S\|_{\infty} \leq \sqrt{s_0} \|B_0\|_2
$$

On the other hand, for the lower bound, we have for $b_{\max} = \max_{j} b_{jj}$,

$$
\rho_{\max}(s_0, \{|b_{ij}|\}) := \max_{q \in \mathbb{S}^{n-1}, s_0 - \text{sparse}} \sum_{i=1}^{n} |q_i| \sum_{j=1}^{n} |b_{ij}| |q_j|
$$

$$
\geq \max_{q \in \mathbb{S}^{n-1}, s_0 - \text{sparse}} \sum_{i=1}^{n} b_{ij} q_i q_j =: \rho_{\max}(s_0, B_0) \geq b_\infty
$$

□
Proof of Lemma 3.5. For the lower bound, we have by the Rayleigh-Ritz theorem,

$$\lambda_{\max}(|B_0|) \geq \frac{\langle |B_0|, 1, 1 \rangle}{\|1\|^2_2} = \frac{1}{n} \sum_{i,j} b_{ij} = \frac{1}{n} \sum_i \deg_i$$

where $1 = (1, 1, \ldots, 1)$; By definition, the largest eigenvalue of symmetric non-negative matrix $|B_0|$ coincides with the spectral radius

$$\lambda_{\max}(|B_0|) := \max_{v \in S^{n-1}} |v^T |B_0||v|$$

$$= \max_{v \in S^{n-1}} |v^T |B_0||v| =: \rho(|B_0|);$$

Finally, $\|B_0\|_2 = \rho(B_0) \leq \rho(|B_0|) = \lambda_{\max}(|B_0|)$

$$= \|B_0\|_2 \leq ||B_0||_\infty = ||B_0||_\infty$$

in view of the previous derivations, where the upper bound follows from the fact that the matrix operator norm is upper bounded by its $\ell_\infty$ norm. \square

## D Proof of Theorem 2.7

First, we state Theorem D.1, which follows from Corollary 25 [30], adapted to our settings and allows us to use Theorems 3.1 and 3.3 to prove the lower and upper-R\E\-conditions.

**Theorem D.1.** [30] Suppose $1/8 > \delta > 0$. Let $1 \leq s_0 < n$. Let $B_0$ be a symmetric positive definite covariance matrix such that $\tr(B_0) = n$. Let $\hat{B}$ be an $n \times n$ symmetric matrix and $\hat{\Delta} = \hat{B} - B_0$. Let $E = \cup_{|j| \leq s_0} E_j$, where $E_j = \span\{e_j, j \in J\}$. Suppose that for all $q, h \in E \cap S^{n-1}$

$$|q^T \hat{\Delta} h| \leq \delta \leq \frac{3}{32} \lambda_{\min}(B_0) < \frac{1}{8}. \quad (62)$$

Then the Lower and Upper RE conditions holds: for all $q \in \mathbb{R}^m$,

$$q^T \hat{B}q \geq \frac{5}{8} \lambda_{\min}(B_0) \|q\|^2_2 - \frac{3 \lambda_{\min}(B_0)}{8s_0} |q|^2_1 \quad (63)$$

$$q^T \hat{B}q \leq (\lambda_{\max}(B_0) + \frac{3}{8} \lambda_{\min}(B_0)) \|q\|^2_2 + \frac{3 \lambda_{\min}(B_0)}{8s_0} |q|^2_1 \quad (64)$$

Let $E$ be as defined in Theorem D.1. Suppose (24) holds. Then clearly (63) holds. Then with probability at least $1 - \frac{C}{(n\sqrt{\log n})} - 4 \exp(-c s_0 \log(3en/(s_0 \varepsilon)))$ for some absolute constants $C, c$, we have by Theorems 3.1 and 3.3 for all $q, h \in E \cap S^{n-1}$,

$$\left| q^T (\hat{B}_0 - B_0) h \right| = \left| q^T ((\mathcal{A} \mathcal{X}^T + \mathcal{M}) - B_0) h \right|$$

$$\leq C \|B_0\|_2 \left( r_{\text{diag}} + \eta_A r_{\text{offd}} \epsilon_{s_0, n}^{1/2} + r_{\text{offd}}^2 \psi_B(2s_0) \right)$$

$$\leq C' \|B_0\|_2 \left( \eta_A r_{\text{offd}} \epsilon_{s_0, n}^{1/2} + r_{\text{offd}}^2 \psi_B(2s_0) \right) \leq \frac{1}{\lambda_{\min}(B_0)}$$

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where
\[ r_{offd}^2 \psi_B(2s_0) \leq \frac{\|A_0\|_2 s_0 \log(3en/(s_0 \varepsilon)) \psi_B(2s_0)}{\sum_j a_{jj}^2} \lesssim \frac{1}{\kappa(B_0)} \]
\[ \eta_A r_{offd}^{1/2} \lesssim \sqrt{\frac{\|A_0\|_2 s_0 \log(n)}{\sum_j a_{jj}^2} a_{\infty}} \lesssim \frac{1}{\kappa(B_0)} \]

Theorem 2.7 follows from Theorem D.1 in view of the uniform bound immediately above. □

E Proof of Theorem 3.3

Let \( \Delta = \text{offd}(X\mathcal{X}^T - B_0 \circ \mathcal{M}) \). Suppose event \( F_{c0} \cap F_{c1} \cap F_{c2} \) holds. Combining the large deviation bounds using (44), we have by Theorems 4.4 and 4.5, and a standard approximation argument in the sense of (41) and (42),
\[ \sup_{q,h} \frac{1}{\|A_0\|_2} \|A_{\circ} M\|_{\text{offd}} \left| Z^T A_{\circ} h Z - E Z^T A_{\circ} h Z \right| \lesssim \eta_{A} r_{offd} + r_{offd} f \psi_B(s_0) + r_{offd}^2 \psi_B(2s_0); \]

(65)

Checking the conditions. It remains to check the following conditions hold so that we can apply the results in Theorems 4.4 and 4.5. Now (21) clearly ensures that (46) and (50) hold; we now show that it also ensures that (47) holds for some constant \( C_5 \), which ensures that the relative error in (65) converges to 0 as \( n \to \infty \).

Proposition E.1. Condition (21) implies that
\[ \frac{\sum_{j=1}^m a_{jj}^2 p_j^2}{\|A_0\|_2} \geq C_4 s_0 \log(n \vee m) \left( \psi_B(2s_0 \wedge n) \wedge \frac{a_{\infty}}{a_{\text{min}}} \right) \]
\[ \geq C_6 \left( \frac{a_{\infty}}{a_{\text{min}}} \right)^{1/3} s_0^{2/3} (\psi_B(s_0))^{1/3} \log(n \vee m) \]

(66)

for suitably chosen constants \( C_4, C_6 \), given that \( \psi_B(s_0) \leq \psi_B(2s_0 \wedge n) \leq \sqrt{2s_0} \), which in turn implies that (47) holds.

Proof. First, we rewrite condition (47) as
\[ \sum_{j=1}^m a_{jj}^2 p_j^2 = \left( \sum_{j=1}^m a_{jj}^2 p_j^2 \right)^{3/4} \left( \sum_{j=1}^m a_{jj}^2 p_j^2 \right)^{1/4} \]
\[ \geq C_5 \psi_B(s_0) \sqrt{s_0 \log(n/s_0)} \|A_0\|_2^{3/4} a_{\infty}^{1/4} \left( \log(n \vee m) \sum_{i=1}^n p_i^4 \right)^{1/4} \]

It is obvious that \( \left( \sum_{j=1}^m a_{jj}^2 p_j^2 \right)^{1/4} \geq a_{\text{min}}^{1/4} (\sum_{i=1}^n p_i^4)^{1/4} \). Thus, to ensure condition (47), we require the following to hold:
\[ \left( \sum_{j=1}^m a_{jj}^2 p_j^2 \right)^{3/4} \geq C_5 \left( \frac{a_{\infty} \log(n \vee m)}{a_{\text{min}}} \right)^{1/4} \psi_B(s_0) \sqrt{s_0 \log(n/s_0)} \]
or equivalently
\[
\sum_{j=1}^{m} a_{jj} p_j^2 \geq C_0 \left( \frac{a_{\infty} \log(n \lor m)}{a_{\min}} \right)^{1/3} \psi_B(s_0)^{4/3} \left( s_0 \log \left( \frac{n}{s_0} \right) \right)^{2/3}
\]
for which (66) is a sufficient condition. \( \square \)

### E.1 Proof of Lemma 4.6

By definition, for \( r_{\text{offd}} := r_{\text{offd}}(s_0) \) and \( \ell_{s_0,n} \) as defined in (22) and \( f_p \) as defined (48) respectively,
\[
r_{\text{offd}} f_p \psi_B(s_0) \leq \frac{\|A_0\|_2^{3/4} a_{\infty}^{1/4} \psi_B(s_0) \sqrt{s_0 \log(3en/(s_0\varepsilon))} \left( \log(n \lor m) \sum_{s=1}^{m} p_s^4 \right)^{1/4}}{\sum_{j=1}^{m} a_{jj} p_j^2}
\]
\[
\leq \sqrt{\eta_A} \psi_B(s_0) \left( \frac{\log(n \lor m)}{\log(3en/(s_0\varepsilon))} \right)^{1/4} \left( \frac{\|A_0\|_2 s_0 \log(3en/(s_0\varepsilon))}{(\sum_{s=1}^{m} p_s^4)} \right)^{3/4}
\]
\[
\leq \sqrt{\eta_A r_{\text{offd}}^2(\psi_B(s_0))^{1/2}} \ell_{s_0,n}^{1/4} = r_{\text{offd}}(\eta_A r_{\text{offd}} \psi_B(s_0))^{1/2} \ell_{s_0,n}^{1/4}
\]
\[
\leq r_{\text{offd}}(\ell_{s_0,n}^{1/2} \eta_A + r_{\text{offd}} \psi_B(s_0)). \quad \square
\]

Furthermore, we have by the union bound, \( \mathbb{P} \left( \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_0^c \right) \geq 1 - \mathbb{P} (\mathcal{F}_0) - 4 \exp \left( -c_1 s_0 \log(3en/(s_0\varepsilon)) \right) \)
by Theorems 4.4 and 4.5. This completes the proof of Theorem 3.3. \( \square \)

### F Proof of Corollary 3.4

Throughout this section, we denote by \( Z \in \mathbb{R}^{mn} \) a subgaussian random vector with independent components \( Z_j \) that satisfy \( \mathbb{E} Z_j = 0, \mathbb{E} Z_j^2 = 1, \) and \( \|Z_j\|_{\psi_2} \leq 1. \) Recall
\[
A_{qq}^c = \sum_k \sum_{i \neq j} u_k^i u_j^i q_i q_j (c_j c_i^T) \otimes (d_k d_k^T),
\]
Since we are working with the sphere \( S^{n-1} \), we will bound \( Q_{\text{offd}} \) directly. Recall that we denote by \( Q_{\text{offd}} \) the off-diagonal component in (30),
\[
Q_{\text{offd}} = \frac{1}{\|M\|_{\text{offd}}} |q^T \text{offd}(\mathcal{X}_T - B_0 \circ M) q| =: \frac{1}{\|M\|_{\text{offd}}} |q^T \tilde{\Delta} q|
\]
\[
\sim \frac{1}{\|M\|_{\text{offd}}} |Z^T A_{qq}^c Z - \mathbb{E}(Z^T A_{qq}^c Z)|, \quad \text{where} \quad Z \sim \text{vec} \{ Z^T \}
\]
for \( Z \) as defined in (5) with \( K = 1. \) Following (27), we decompose the error into two parts: for \( \tilde{\Delta} = \text{offd}(\mathcal{X}_T - B_0 \circ M) \) and \( q \in S^{n-1}, \)
\[
|q^T \tilde{\Delta} q| = |Z^T A_{qq}^c Z - \mathbb{E}(Z^T A_{qq}^c Z)| \leq |Z^T A_{qq}^c Z - \mathbb{E}(Z^T A_{qq}^c Z)| + |\mathbb{E}(Z^T A_{qq}^c Z) - \mathbb{E}(Z^T A_{qq}^c Z)| =: \text{I} + \text{II}
\]
(67)
Part I Denote by $\mathcal{N}$ the $\varepsilon$-net for $S^{n-1}$ as constructed in Lemma 3.1 with $|\mathcal{N}| \leq (3/\varepsilon)^n$. By (21) with $s_0 = n$, we have for $\psi_B(n) = \|B_0\|_2 / \|B_0\|_2$, 

$$\sum_{j=1}^{m} a_{jj}p_j^2 \geq C_4 a_{\infty} \cdot \psi_B(n)n \log(n \vee m). \tag{68}$$

By Theorem 4.3 we have on event $\mathcal{F}_0^c$ and for $N$ as defined therein,

$$\sup_{q \in S^{n-1}} \|A_q\|_F \leq \|B_0\|_2 \|A_0\|_2^{1/2} W$$

where

$$W \asymp \sqrt{a_{\infty} \sum_{s=1}^{m} p_s^2 + \psi_B(n) \left( \log(n) a_{\infty} \|A_0\|_2 \sum_{j=1}^{m} p_j^4 \right)^{1/4}}. \tag{69}$$

which holds under condition (68). Hence we set

$$\tau_0 \propto n \log(3e/\varepsilon) \|A_0\|_2 \|B_0\|_2 + \sqrt{n \log(3e/\varepsilon)} \|B_0\|_2 \|A_0\|_2^{1/2} W$$

Following Lemma 3.4 we have

$$\mathbb{P} \left( \exists q \in \mathcal{N}, |Z^T A_{qq}^o Z - \mathbb{E}(Z^T A_{qq}^o Z|U) > \tau_0 \right) =: \mathbb{P}(\mathcal{F}_1) \leq \exp(-c_1 n \log(3e/\varepsilon)) + \mathbb{P}(\mathcal{F}_0)$$

Part II Now suppose that for $C_4$ large enough, set

$$\tau' = C_4 a_{\infty} \|B_0\|_2 \psi_B(n)n \log(3e/\varepsilon) \asymp a_{\infty} \|B_0\|_2 n \log(3e/\varepsilon).$$

Then by Corollary B.7 and the union bound,

$$\mathbb{P} \left( \exists q \in \mathcal{N}, |S_+(q)| > \tau' \right) =: \mathbb{P}(\mathcal{F}_2) \leq |\mathcal{N}| \exp \left( -C_5 \min \left( \frac{\psi_B(n)(n \log(3e/\varepsilon))}{\sum_{k=1}^{m} a_{kk}p_k^2} \frac{\|B_0\|_2 n \log(3e/\varepsilon)}{\|B_0\|_2} \right) \right) \leq (3/\varepsilon)^n \exp \left( -C' s_0 \log(3e/\varepsilon) \right) \leq \exp \left( -2n \log(3e/\varepsilon) \right),$$

where we have by (68) and Lemma 3.5

$$\sum_{j=1}^{m} a_{jj}p_j^2 \geq C_4 a_{\infty} \psi_B(n)n \log(3e/\varepsilon) \quad \text{and} \quad 1 \leq \psi_B(n) = O(\sqrt{n}).$$

On event $\mathcal{F}_0^c \cap \mathcal{F}_1^c \cap \mathcal{F}_2^c$, for $W$ as defined in (69), we have by the bounds immediately above,

$$\sup_{q \in \mathcal{N}} \|B_0\|_2 \frac{1}{\|B_0\|_2} \left| q^T \Delta q \right| = \sup_{q \in \mathcal{N}} \frac{1}{\|B_0\|_2} \left| Z^T A_{qq}^o Z - \mathbb{E}Z^T A_{qq}^o Z \right|$$

$$\leq \sup_{q \in \mathcal{N}} \frac{1}{\|B_0\|_2} \left( \left| Z^T A_{qq}^o Z - \mathbb{E}(Z^T A_{qq}^o Z|U) \right| + \left| \mathbb{E}(Z^T A_{qq}^o Z|U) - \mathbb{E}Z^T A_{qq}^o Z \right| \right)$$

$$\leq \frac{\tau_0 + \tau'}{\|B_0\|_2} \asymp n \log(3e/\varepsilon) \|A_0\|_2 + a_{\infty} \psi_B(n) + \sqrt{n \log(3e/\varepsilon)} \|A_0\|_2^{1/2} W$$

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Hence by a standard approximation argument, we have
\[
\sup_{q \in \mathbb{S}^{n-1}} \frac{|q^T \Delta q|}{\|B_0\|_2 \|M\|_{offd}} \leq \sup_{q \in N} \frac{1}{(1 - \epsilon)^2} \frac{|q^T \Delta q|}{\|B_0\|_2 \|M\|_{offd}} \leq \frac{r_{offd}^2(n) + \eta_A r_{offd}(n) + r_{offd}(n) f_p \psi_B(n) + r_{offd}^2(n) \psi_B(n)}{O\left(r_{offd}(n) \eta_A f_{\eta_n}^{1/2} + r_{offd}^2(n) \psi_B(n)\right)}
\]
where the expression for the overall rate of convergence can be simplified using Lemma 4.6. The rest of the proof follows from that of Theorem 2.4 using (29) and (30). This completes the proof of Corollary 3.4. \square

## G Proof of Theorem 4.2

We will prove a uniform deterministic bound on \(\|A_{q,h}^\circ\|_2\) in this section. Let \(\{u_k, k = 1, \ldots, m\}\) be the column vectors of the mask matrix \(U = [u^1 | \ldots | u^m]\). Recall that the nuclear norm or trace norm of \(d \times d\) matrix \(X\) is defined as
\[
\|X\|_* := |s(X)|_1 = \sum_{i=1}^d s_i(X) = \text{tr}(\sqrt{X^TX})
\]
where \(\sqrt{X^TX}\) represents the unique positive-semidefinite matrix \(C\) such that \(C^2 = X^TX\) and \(s(X) := (s_i(X))_{i=1}^d\) denotes the vector of singular values of \(X\). For positive-semidefinite matrix \(A \succeq 0\), clearly, \(\sqrt{A^T A} = \sqrt{A^2} = A\), and
\[
\|A\|_* := |s(A)|_1 = \text{tr}(A) = \sum_{i=1}^m \lambda_i(A)
\]
where \(\lambda_i(A)\) are eigenvalues of \(A\). First, denote by \(v_{ij}^{(k)} = c_i c_j^T \otimes (d_k d_k^T)^T \in \mathbb{R}^{mn \times mn}\). Denote by \(D_0(w)\) the block diagonal matrix such that on the \(k^{th}\) block, we have for a fixed \(w \in \mathbb{S}^{mn-1}\)
\[
D_{0,ij}^{(k)}(w) = w^T (c_i c_j^T \otimes (d_k d_k^T)) w =: w^T v_{ij}^{(k)} w \text{ and hence}
D_0^{(k)}(w) = \left(D_{0,ij}^{(k)}(w)\right) \in \mathbb{R}^{n \times n} \tag{70}
\]
As a preparation, we first state a general result in Lemma G.1 involving sum of tensor products, specialized to our settings, as well as properties of matrix \(D_0\) in Lemma G.2. We defer proofs of Lemmas G.2 and G.3 to Section G.1.

**Lemma G.1.** Denote by \(\tilde{U}\) a block-diagonal matrix with 0s along the diagonal, and on the \(k^{th}\) block, we have a symmetric matrix \(\tilde{U}^{(k)} = (\tilde{U}_{ij}^{(k)}) \in \mathbb{R}^{n \times n}\) with bounded operator norm. Consider
\[
H_0^{(k)} = \sum_{i \neq j} \tilde{U}_{ij}^{(k)} c_i c_j^T, \text{ where } \tilde{U}^{(k)} \in \mathbb{R}^{n \times n} \text{ is a symmetric matrix;}
\]
Then for $\mathbb{D}_0(w)$ as defined in (70),

$$\left\| \sum_{k=1}^{m} \mathbb{H}_0^{(k)} \otimes d_k d_k^T \right\|_2 = \sup_{w \in \mathbb{S}^{mn-1}} \left| \langle \widetilde{U}, \mathbb{D}_0(w) \rangle \right| \leq \left\| \widetilde{U} \right\|_2 \sup_{w \in \mathbb{S}^{mn-1}} \left\| \mathbb{D}_0(w) \right\|_* .$$

**Lemma G.2.** Let $\mathbb{D}_0$ be defined as in (70). Then $\mathbb{D}_0 \succeq 0$ for all $w \in \mathbb{R}^{mn}$ and $\left\| \mathbb{D}_0(w) \right\|_* = w^T (B_0 \otimes A_0) w$; Moreover,

$$\sup_{w \in \mathbb{S}^{mn-1}} \left\| \mathbb{D}_0(w) \right\|_* = \left\| B_0 \right\|_2 \left\| A_0 \right\|_2 .$$

**Lemma G.3.** Let $\widetilde{U}$ be a block-diagonal matrix with 0s along the diagonal, and $\mathbb{U}^{(k)}(q,h) \in \mathbb{R}^{n \times n}$ on the $k^{th}$ block along the diagonal such that

$$\forall k, \forall i \neq j, \quad \mathbb{U}^{(k)}_{ij}(q,h) = \frac{1}{2} (u^k_i u^k_j (q_i h_j + h_i q_j)) ;$$

Then, $\forall q,h \in \mathbb{S}^{n-1}, \forall k, \left\| \mathbb{U}^{(k)}(q,h) \right\|_2 \leq 1.$

**Proof of Theorem 4.2** Consider for arbitrary $q,h \in \mathbb{S}^{n-1}$ and $\widetilde{U} := \mathbb{U}(q,h)$ as defined in Lemma G.3

$$\mathbb{H}_0^{(k)}(q,h) = \sum_{i=1}^{n} \sum_{j \neq i} \mathbb{U}^{(k)}_{ij}(q,h) c_i c_j^T .$$

Let $\mathbb{D}_0(w)$ be as defined in (70). Now for arbitrary $q,h \in \mathbb{S}^{n-1},$

$$A^\diamond_{q,h} = \sum_{k=1}^{m} \mathbb{H}_0^{(k)}(q,h) \otimes d_k d_k^T = \sum_{k=1}^{m} \sum_{j \neq i} \mathbb{U}^{(k)}_{ij}(q,h) c_i c_j^T \otimes d_k d_k^T ,$$

we have by Lemmas G.1 and G.3

$$\left\| A^\diamond_{q,h} \right\|_2 = \left\| \sum_{k=1}^{m} \mathbb{H}_0^{(k)}(q,h) \otimes d_k d_k^T \right\|_2 = \sup_{w \in \mathbb{S}^{mn-1}} \left| \langle \widetilde{U}(q,h), \mathbb{D}_0(w) \rangle \right| \leq \left\| \widetilde{U}(q,h) \right\|_2 \sup_{w \in \mathbb{S}^{mn-1}} \left\| \mathbb{D}_0(w) \right\|_* \leq \sup_{w \in \mathbb{S}^{mn-1}} \left\| \mathbb{D}_0(w) \right\|_* ,$$

where $\forall q,h \in \mathbb{S}^{n-1},$ we have $\left\| \widetilde{U}(q,h) \right\|_2 \leq 1;$ Hence by Lemma G.2

$$\sup_{q,h \in \mathbb{S}^{n-1}} \left\| A^\diamond_{q,h} \right\|_2 \leq \sup_{w \in \mathbb{S}^{mn-1}} \left\| \mathbb{D}_0(w) \right\|_* \leq \left\| B_0 \right\|_2 \left\| A_0 \right\|_2 .$$

$\square$

**Proof of Lemma G.1** Clearly $\mathbb{H}_0^{(k)}$ is symmetric. Denote by $M = \sum_{k=1}^{m} \mathbb{H}_0^{(k)} \otimes d_k d_k^T$, then $M$ is also symmetric. Recall the operator norm of the symmetric matrix $M = \sum_{k=1}^{m} \mathbb{H}_0^{(k)} \otimes d_k d_k^T$ is the same as the spectral radius of $M$, denoted by

$$\rho(M) := \{ \max |\lambda|, \lambda \text{ eigenvalue of } M \} .$$

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Let $D_0(w)$ be as defined in (70). Hence by definition,

$$\|M\|_2 = \sup_{w \in S^{mn-1}} \left| w^T \left( \sum_{k=1}^{m} \sum_{i \neq j} U^{(k)}_{ij} c_i c_j^T \otimes d_k d_k^T \right) w \right|$$

$$= \sup_{w \in S^{mn-1}} \left| \sum_{k=1}^{m} \sum_{i \neq j} U^{(k)}_{ij} w^T (c_i c_j \otimes d_k d_k^T) w \right|$$

$$= \sup_{w \in S^{mn-1}} \left| \sum_{k=1}^{m} \langle U^{(k)}, D_0^{(k)}(w) \rangle \right|$$

$$= \sup_{w \in S^{mn-1}} \left| \langle \tilde{U}, D_0(w) \rangle \right| \leq \sup_{w \in S^{mn-1}} \left| \tilde{U} \right|_2 \|D_0(w)\|_*$$

where (71) follows since $\text{diag}(\tilde{U}) = 0$, and in (72), we use the fact that

$$\forall w \in S^{mn-1} \left| \langle \tilde{U}, D_0(w) \rangle \right| = \left| \sum_{k=1}^{m} \langle U^{(k)}, D_0^{(k)}(w) \rangle \right| \leq \left| \tilde{U} \right|_2 \|D_0(w)\|_*$$

by Hölder’s inequality. See for example Example 10.4.2 in [38]. □

G.1 Proof of Lemmas G.2 and G.3

Proof of Lemma G.2 Fix $w \in \mathbb{R}^{mn}$. We break a vector $w$ on the sphere into $n$ vectors $w^1, w^2, \ldots, w^n$, each of which has size $m$. We show that each block $(D_0^{(k)}(w))_{i,j \leq n}$ is positive semidefinite (PSD) and hence $D_0(w)$ is PSD for all $w \in \mathbb{R}^{mn}$. Indeed for any vector $h \in \mathbb{R}^{n}$

$$h^T D_0^{(k)} h = \sum_{i,j} h_i h_j \sum_{s,t} c_{i,s} \langle w_s, d_k \rangle c_{j,t} \langle w_t, d_k \rangle$$

$$= \left( \sum_{i} h_i \sum_{s} c_{i,s} \langle w_s, d_k \rangle \right) \left( \sum_{j} h_j \sum_{t} c_{j,t} \langle w_t, d_k \rangle \right) \geq 0$$

For the nuclear norm, we use the fact that for all $w \in \mathbb{R}^{mn}$, $D_0(w) \succeq 0$, and hence by definition

$$\|D_0(w)\|_* = \text{tr}(D_0(w)) = \sum_{k=1}^{m} \text{tr}(D_0^{(k)}(w))$$

$$= \sum_{k=1}^{m} \sum_{j=1}^{n} D_0^{(k)}_{i,j}(w) = \sum_{k=1}^{m} \sum_{j=1}^{n} w^T (c_j c_j^T) \otimes (d_k d_k^T) w$$

$$= w^T \left( \sum_{j=1}^{n} (c_j c_j^T) \otimes \sum_{k=1}^{m} (d_k d_k^T) \right) w = w^T (B_0 \otimes A_0) w.$$
Proof of Lemma \[\text{G.3}\] Denote by $\bar{u}^k(q) = u^k \circ q$, where $q \in \mathbb{S}^{n-1}$. Hence
\[\forall i \neq j, \quad \bar{U}^{(k)}_{ij}(q, h) = \frac{1}{2} \left( \bar{u}^k(q) \otimes \bar{u}^k(h) + \bar{u}^k(h) \otimes \bar{u}^k(q) \right)_{i,j}\]
and $\forall q, h \in \mathbb{S}^{n-1}$, and $\forall k$,
\[\left\| \bar{U}^{(k)}(q, h) \right\|_2 \leq \frac{1}{2} \left\| \text{offd}(\bar{u}^k(q) \otimes \bar{u}^k(h)) \right\|_2 + \frac{1}{2} \left\| \text{offd}(\bar{u}^k(h) \otimes \bar{u}^k(q)) \right\|_2 \leq \frac{1}{2} \left\| \text{offd}(\bar{u}^k(q) \otimes \bar{u}^k(h)) \right\|_2 + \frac{1}{2} \left\| \text{offd}(\bar{u}^k(h) \otimes \bar{u}^k(q)) \right\|_2 \leq \frac{1}{2} \left\| \text{offd}((\bar{u}^k(q) \otimes \bar{u}^k(h))^T (\bar{u}^k(q) \otimes \bar{u}^k(h))) \right\|_1 \leq 1\]
where for all $u^k, k = 1, \ldots, m$ and $q \in \mathbb{S}^{n-1}$, we have for $\left\| \bar{u}^k(q) \right\|_2 = \left\| u^k \circ q \right\|_2 \leq \left\| q \right\|_2 = 1$. \[\square\]

\section{H Proof of Theorem 4.4}

For a chosen sparsity parameter $1 \leq s_0 \leq n$, let $E = \bigcup_{|J| \leq s_0} E_J$. Denote by $\mathcal{N}$ the $\varepsilon$-net for $\mathbb{S}^{n-1} \cap E$ as constructed in Lemma \[\text{4.1}\].

By Theorem \[\text{4.2}\] we have
\[
\sup_{q,h \in \mathbb{S}^{n-1}} \left\| A^\circ_{q,h} \right\|_2 \leq \left\| A_0 \right\|_2 \left\| B_0 \right\|_2
\]
In order to apply Theorem \[\text{B.1}\] we first condition on the mask matrix $U$ being fixed, as when $U$ is fixed, the following quadratic form
\[
Z^T A^\circ_{q,h} Z =: Z^T \left( \frac{1}{2} \sum_{k=1}^{m} \sum_{i=1}^{n} \sum_{j \neq i}^{n} u^c_i u^c_j (q_i h_j + q_j h_i) (c^c_i c^c_j) \otimes (d^c_k d^c_k) \right) Z
\]
can be treated as a random subgaussian quadratic form with $A^\circ_{q,h}$ taken to be a deterministic matrix. Let
\[
V_1 := \left\| A_0 \right\|_2^{1/2} \sqrt{a_\infty \sum_{s=1}^{m} p^2_s} \quad \text{and} \quad V_2 = \left\| A_0 \right\|_2^{1/2} \psi_B(s_0) \left( N \log(n \lor m) \right)^{1/4}
\]
where $N \leq \lambda_{\max}(A_0 \circ A_0) \left\| p^2 \right\|_2^2 \leq a_\infty \left\| A_0 \right\|_2 \sum_{i=1}^{m} p^4_i$. We first prove Lemma \[\text{H.4}\] followed by Corollary \[\text{H.1}\].
Corollary H.1. Suppose that event $\mathcal{F}_0^c$ as defined in Theorem 4.3 hold. Then for some $W$ to be specified,

$$\sup_{q,h \in S_{n-1}, s_0 \text{-sparse}} \|A_{q,h}^o\|_F \leq \|B_0\|_2 \|A_0\|_2^{1/2} W$$

We have by Theorems B.1, 4.2 and 4.3, and the union bound, for any $t > 0$,

$$\mathbb{P}( \{ \exists q, h \in \mathcal{N}, |Z^T A_{q,h}^o Z - \mathbb{E}(Z^T A_{q,h}^o Z|U)| > t \} | U \in \mathcal{F}_0^c) \leq 2 |\mathcal{N}|^2 \exp \left( -c \min \left( \frac{t^2}{\|B_0\|_2^2 \|A_0\|_2 W^2 \|A_0\|_2 \|B_0\|_2} \right) \right)$$

Hence we set the large deviation to be at the order of

$$\tau_0 = C_1 s_0 \log(3en/(s_0 \varepsilon)) \|A_0\|_2 \|B_0\|_2 + C_2 s_0 \log(3en/(s_0 \varepsilon)) \|A_0\|_2^{1/2} \|B_0\|_2 \cdot W$$

Finally, we have by Theorems B.1, 4.2 and 4.3

$$\mathbb{P}( \{ \exists q, h \in \mathcal{N}, |Z^T A_{q,h}^o Z - \mathbb{E}(Z^T A_{q,h}^o Z|U)| > \tau_0 \} \cap \mathcal{F}_0^c) \leq 2 |\mathcal{N}|^2 \exp \left( -c \min \left( \frac{\tau_0^2}{\|B_0\|_2^2 \|A_0\|_2 W^2 \|B_0\|_2 \|A_0\|_2} \right) \mathbb{P}( \mathcal{F}_0^c) + \mathbb{P}( \mathcal{F}_0) \right) \leq \exp(-c_1 s_0 \log(3en/(s_0 \varepsilon))) + \mathbb{P}( \mathcal{F}_0) \quad \square$$

**Corollary H.1.** Suppose that (46) holds. Then we have

$$\sup_{q,h \in S_{n-1}, s_0 \text{-sparse}} \|A_{q,h}^o\|_F \leq W \cdot \|B_0\|_2 \|A_0\|_2^{1/2} \quad \text{where}$$

$$W \asymp \sqrt{a_{\infty} \sum_{s=1}^{m} p_{s}^2 + \psi_B(s_0) (N \log(n \vee m))^{1/4}}$$

**Proof.** By Theorem 4.3 we have on event $\mathcal{F}_0^c$,

$$\sup_{q,h \in S_{n-1}, s_0 \text{-sparse}} \|A_{q,h}^o\|_F \leq W \cdot \|B_0\|_2 \|A_0\|_2^{1/2} \quad \text{where}$$

$$W \asymp \sqrt{a_{\infty} \sum_{s=1}^{m} p_{s}^2 + \psi_B(s_0) \left( \sqrt{\|A_0\|_2 \log(n \vee m) + (N \log(n \vee m))^{1/4}} \right)}$$

where by condition (46) and the fact that $\psi_B(s_0) = O(\sqrt{s_0})$,

$$\psi_B(s_0)^2 \|A_0\|_2 \log(n \vee m) \leq s_0 \log(n \vee m) \|A_0\|_2 \leq \sum_{j} a_{jj} p_{j}^2 \leq a_{\infty} \sum_{j} p_{j}^2. \quad (73)$$

The corollary thus holds. \square
H.2 Proof of Theorem 4.4

Now by Corollary [1.1] we have on event $F_0^c$,
\[
\sup_{q,h \in S^{n-1}, s_0\text{-sparse}} \| A_{qh}^c \|_F \leq \| B_0 \|_2 \| A_0 \|_2^{1/2} W \asymp \| B_0 \|_2 (V_1 + V_2)
\]
On event $F_1^c \cap F_0^c$, we have by Lemma [B.4]
\[
\forall q, h \in \mathcal{N} \quad \left| Z^T A_{qh}^o Z - \mathbb{E}(Z^T A_{qh}^o Z | U) \right| / \| B_0 \|_2 \leq \tau_0 / \| B_0 \|_2
\]
\[
\asymp s_0 \log(3en/(s_0 \varepsilon)) \| A_0 \|_2 + \sqrt{s_0 \log(3en/(s_0 \varepsilon)) \| A_0 \|_2^{1/2} W
\]
with $W \asymp \sqrt{a_{\infty} \sum_{s=1}^m p_s^2} + \psi_B(s_0)(N \log(n \lor m))^{1/4}$; hence by (46) and (47),
\[
\sup_{q,h \in \mathcal{N}} \| \mathcal{M} \|_{\text{offd}} \| B_0 \|_2 \left| Z^T A_{qh}^o Z - \mathbb{E}(Z^T A_{qh}^o Z | U) \right|
\leq \frac{C_1 s_0 \log(3en/(s_0 \varepsilon)) \| A_0 \|_2}{\sum_{k=1}^m a_k p_k^2} + \frac{C_2 \sqrt{s_0 \log(3en/(s_0 \varepsilon)) \| A_0 \|_2^{1/2} W}{\sum_{k=1}^m a_k p_k^2} (V_1 + V_2)
\]
\[
\asymp r_{\text{offd}}^2(s_0) + r_{\text{offd}}(s_0) \eta_A + r_{\text{offd}}(s_0) f_p \psi_B(s_0) = o(1)
\]
where $r_{\text{offd}}^2(s_0) = o(r_{\text{offd}}),$
\[
\frac{s_0 \log(3en/(s_0 \varepsilon))}{\sum_{k=1}^m a_k p_k^2} V_1 \ll \frac{\| A_0 \|_2^{1/2}}{\sqrt{\sum_{k=1}^m a_k p_k^2}} \sqrt{\frac{a_{\infty}}{a_{\min}}} \asymp \eta_A r_{\text{offd}}(s_0) = o(1),
\]
and
\[
\frac{s_0 \log(n/s_0)}{\sum_{j=1}^m a_j p_j^2} V_2 = r_{\text{offd}}(N \log(n \lor m))^{1/4} \psi_B(s_0)
\]
\[
\asymp r_{\text{offd}} f_p \psi_B(s_0) = o(1).
\]
Furthermore, we show that $\mathbb{P}(F_1^c) \geq 1 - \mathbb{P}(F_0) - 2 \exp(-c_1 s_0 \log(3en/(s_0 \varepsilon)))$ in Lemma [B.4] This completes the proof of Theorem 4.4. □

I Proof of Theorem 4.5

First we consider $q, h$ being fixed. We state in Lemma [B.5] an estimate on the moment generating function of $S_\ast(q, h)$ as defined in (1.9), from which a large deviation bound immediately follows. Denote by $\mathcal{N}$ the $\varepsilon$-net for $S^{n-1} \cap E$ as constructed in Lemma 4.1.

Proof of Theorem 4.5 Now suppose $q, h \in S^{n-1}$ are $s_0$-sparse. Denote by $\bar{\rho}(s_0) = \bar{\rho}(s_0, |B_0|) = \bar{\rho}(s_0, |B_0|) \wedge \| B_0 \|_2$; we use the shorthand notation $\bar{\rho}(s_0) = \bar{\rho}(s_0, |B_0|)$ as defined in (57). Let
\[
\tau' = C_4 a_{\infty} (\| B_0 \|_2 \psi_B(s_0 \lor n)) s_0 \log(3en/(s_0 \varepsilon))
\]
\[
\asymp a_{\infty} \rho_{\max}(2s_0, |B_0|) s_0 \log(3en/(s_0 \varepsilon))
\]

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then by Lemma [B.3], we have for \( C_4 \) large enough, \( \psi_B(2s_0 \vee n) = O(\sqrt{s_0}) \), and letting \( d := 2s_0 \vee n \),

\[
\mathbb{P} \left( \exists q, h \in \mathcal{N}, |S_*(q, h)| \geq \tau' \right) =: \mathbb{P}(F_2) \leq |\mathcal{N}|^2.
\]

\[
\exp \left( -c \left( \frac{C_4 a_{\infty} \|B_0\|_2 \psi_B(d) s_0 \log \left( \frac{3en}{s_0 \varepsilon} \right)}{a_{\infty}(\bar{\rho}(s_0)) \bar{\rho}(s_0) \sum_{k=1}^{m} a_{kk} p_k^2} \right)^2 \wedge \frac{C_4 \|B_0\|_2 \psi_B(d) s_0 \log \left( \frac{3en}{s_0 \varepsilon} \right)}{\bar{\rho}(s_0)} \right)
\]

where by Lemma [B.6] we have \( \bar{\rho}(s_0, |B_0|) \leq 2\rho_{\max}(2s_0, |B_0|) \) and by condition (50),

\[
|\mathcal{N}|^2 \exp \left( -C \min \left( s_0 \log \left( \frac{3en}{s_0 \varepsilon} \right), \psi_B(2s_0) s_0 \log \left( \frac{3en}{s_0 \varepsilon} \right) \right) \right)
\]

\[
\leq \left( \frac{3}{\varepsilon} \right)^{2s_0} \left( \frac{n}{s_0} \right)^2 \exp \left( -C' s_0 \log \left( \frac{3en}{s_0 \varepsilon} \right) \right) \leq \exp \left( -C'' s_0 \log (3en/(s_0 \varepsilon)) \right).
\]

\( \square \)

It remains to prove Lemma [B.6].

**Proof of Lemma [B.6]** Note that \( \|q\| + \|h\|_2 \leq \|q\|_2 + \|h\|_2 = 2 \). Thus we have

\[
(|q| + |h|)^T |B_0| (|q| + |h|) = |q|^T |B_0| |q| + 2|h|^T |B_0| |q| + |h|^T |B_0| |h|
\]

and hence

\[
|h|^T |B_0| |q| = \frac{1}{2} \left( (|q| + |h|)^T |B_0| (|q| + |h|) - |q|^T |B_0| |q| - |h|^T |B_0| |h| \right)
\]

\[
\leq 2\rho_{\max}(2s_0, (|b_{ij}|)) - \rho_{\min}(s_0, B_0)
\]

while for general \( q, h \in \mathbb{S}^{n-1}, \|B_0\|_2 = \|(b_{ij})\|_2 \) is to be understood as the operator norm for matrix \((|b_{ij}|)\) as defined in [19]. On the other hand, \( \forall h \in \mathbb{S}^{n-1}, \forall i, \sum_{j=1}^{n} |b_{ij}| |h_j| \leq \|b^{(i)}\|_2 \|h\|_2 = \|b^{(i)}\|_2 \) and hence

\[
\max_{q, h \in \mathbb{S}^{n-1}, s_0 - \text{sparse}} \sum_{i=1}^{n} |q_i| \sum_{j=1}^{n} |b_{ij}| |h_j| \leq \max_{q \in \mathbb{S}^{n-1}, s_0 - \text{sparse}} \sum_{i=1}^{n} |q_i| \|b^{(i)}\|_2 \leq \sqrt{s_0} \|B_0\|_2
\]

where \( b^{(1)}, \ldots, b^{(n)} \) are column (row) vectors of symmetric matrix \( B_0 \succ 0 \). Thus we have

\[
\bar{\rho}(s_0, (|b_{ij}|)) \leq (2\rho_{\max}(2s_0, (|b_{ij}|)) - \rho_{\min}(s_0, B_0)) \wedge \sqrt{s_0} \|B_0\|_2
\]

\( \square \)

**J Proof sketch for Theorem 4.3**

Our analysis framework will work beyond cases considered in the present work, namely, it will work in cases where random matrix \( U \) follows other distributions; for example, one may consider \( U \) as
a matrix with positive coefficients, rather than 0,1s. First, we rewrite the off-diagonal part of the quadratic form as follows:

\[ q^T \text{offd}(XX^T)h = \sum_{i \neq j} q_i h_j \langle v^i \circ y^j, v^j \circ y^j \rangle \]

\[ = \sum_{i \neq j} q_i h_j Z^T c_i \otimes A_0^{1/2} \text{diag}(v^i \circ v^j) c_j^T \otimes A_0^{1/2} Z \]

\[ = Z^T \left( \sum_{i \neq j} q_i h_j c_i c_j^T \otimes A_0^{1/2} \text{diag}(v^i \otimes v^j) A_0^{1/2} \right) Z \]

(74)

and recall for each row vector \( y^i \) of \( X = B_0^{1/2} Z A_0^{1/2} \), for \( Z \) as defined in (38), we observe in \( X \) its sparse instance: \( \forall i = 1, \ldots, n, \)

\[ v^i \circ y^i, \text{ where } v^i_k \sim \text{Bernoulli}(p_k), \forall k = 1, \ldots, m, \] (75)

and for two vectors \( v^i, y^i \in \mathbb{R}^m \), \( v^i \circ y^i \) denote their Hadamard product such that \( (v^i \circ y^i)_k = v^i_k y^i_k = u^i_k x^i_k \). Recall the symmetric matrix \( A^o_{\text{qh}} \) as defined in (39) is the average of the asymmetric versions:

\[ A^o_{\text{qh}} = \frac{1}{2} (A^o_{\text{qh}}(\ell) + A^o_{\text{qh}}(r)) \]

where we denote by

\[ A^o_{\text{qh}}(\ell) = \sum_{i=1}^{n} \sum_{j \neq i}^{n} q_i h_j (c_i c_j^T) \otimes \left( A_0^{1/2} \text{diag}(v^i \otimes v^j) A_0^{1/2} \right) \]

and

\[ A^o_{\text{qh}}(r) = \sum_{i=1}^{n} \sum_{j \neq i}^{n} q_i h_j (c_j c_i^T) \otimes \left( A_0^{1/2} \text{diag}(v^i \otimes v^j) A_0^{1/2} \right) = (A^o_{\text{qh}}(\ell))^T \]

Recall that

\[ \| A^o_{\text{qh}}(\ell) \|_F \leq \frac{1}{2} \left( \| A^o_{\text{qh}}(\ell) \|_F + \| A^o_{\text{qh}}(r) \|_F \right) ; \]

hence

\[ \| A^o_{\text{qh}} \|_F^2 \leq \frac{1}{4} \left( 2 \| A^o_{\text{qh}}(\ell) \|_F^2 + 2 \| A^o_{\text{qh}}(r) \|_F^2 \right) = \| A^o_{\text{qh}}(r) \|_F^2 \]

First, we use the decomposition argument to express \( \| A^o_{\text{qh}}(r) \|_F^2 \) as a summation over homogeneous polynomials of degree 2, 3, 4 respectively. We then prove the concentration of measure bounds for each homogeneous polynomial respectively. For now, we have a quick summary of these random
functions and their expectations. Hence

\[
\|A_{q,h}^e(r)\|_F^2 = \text{tr} \left( \sum_{i \neq j} \sum_{k \neq \ell} q_i h_j q_k h_{\ell} \left( (c_j c_i^T) \otimes \left( A_0^{1/2} \text{diag}(v^i \circ v^j) A_0 \text{diag}(v^k \circ v^\ell) A_0^{1/2} \right) \right) \right) \\
= \sum_{i \neq j} \sum_{k \neq \ell} q_i h_j q_k h_{\ell} \text{tr} \left( (c_j c_i^T) \otimes \left( A_0^{1/2} \text{diag}(v^i \circ v^j) A_0 \text{diag}(v^k \circ v^\ell) A_0^{1/2} \right) \right) \\
= \sum_{i \neq j} \sum_{k \neq \ell} b_{ki} q_i q_k b_j h_j h_{\ell} \left( (v^i \circ v^j)^T (A_0 \circ A_0) (v^k \circ v^\ell) \right) \\
= \sum_{i,k} b_{ki} q_i q_k \sum_{j \neq i, k \neq \ell} b_j h_j h_{\ell} \left( (v^i \circ v^j)^T (A_0 \circ A_0) (v^k \circ v^\ell) \right)
\]

We now characterize the sums that involve all unique pairs, triples, and quadruples of Bernoulli random variables. It is understood that \((i, j)\) and \((k, \ell)\) are allowed to overlap in one or two vertices, but \(i \neq j\) and \(k \neq \ell\). Here and in the sequel, we use \(j \neq i \neq \ell \neq k\) to denote that \(i, j, k, \ell\) are all distinct, while \(i \neq j \neq k\) denotes that indices \(i, j, k\) are distinct, and so on. On the other hand, the conditions \(i \neq j\) and \(k \neq \ell\) do not exclude the possibility that \(k = j\) or \(k = i\), or \(i = \ell\), or \(j = \ell\), or some combination of these.

We first introduce the following definitions.

- **Fix** \(i \neq j\). In (78), when an unordered pair of indices \((k, \ell)\) is chosen to be identical with an unordered pair \((i, j)\), we add an element in \(W_2^\circ\) resulting in a homogeneous positive polynomial of degree 2

\[
W_2^\circ = \sum_{(i,j)} \binom{n}{2} w_{i,j}^e (v_i \circ v_j)^T \text{diag}(A_0 \circ A_0) (v_i \circ v_j) \quad (76)
\]

where the weight \(w_{i,j}^e \geq 0\) for all \(i \neq j\) is to be defined in (79).

- **In forming polynomial** \(W_3^\text{diag}\), it is understood that the quadruple \((i, j, k, \ell)\) in (78), where \(i \neq j\) and \(k \neq \ell\) will collapse into a triple with three distinct indices, say, \(i, j, k\). Indeed, suppose we first fix a pair of indices \((i, j)\), where \(i \neq j\) and for the second pair \((k, \ell)\), we pick a single new coordinate \(k \neq i, j\), while, without loss of generality, fixing \(\ell = i\); We then add a set of elements in \(W_3^\text{diag}\) with 4 coefficients denoted by \(\Delta_{(i,j),(i,k)}\) for \(i \neq j \neq k\):

\[
\Delta_{(i,j),(i,k)} := b_{ij} q_i^2 b_{jk} h_j h_k + b_{jk} q_j q_k b_{ij} h_i^2 \\
+ q_i h_i (b_{ik} q_k b_{ij} h_j + b_{ij} q_j b_{ik} h_k), \quad \text{and hence}
\]

\[
W_3^\text{diag} := \sum_{i=1}^n \sum_{j \neq i} \sum_{k \neq i, j} \Delta_{(i,j),(i,k)} (v_i \circ v_j)^T \text{diag}(A_0 \circ A_0) (v_i \circ v_k).
\]

In Section 12.2, we explain the counting strategy for this case and will analyze \(W_3^\circ\) in Lemma 12.3.
• In $W_4^{\text{diag}}$, we have all indices $i, j, k, \ell$ being distinct, namely,
\[
W_4^{\text{diag}} := \sum_{i \neq k} b_{ki} q_i q_k \sum_{j \neq i \neq \ell \neq k} b_{j\ell} h_j h_\ell (v^i \circ v^j) (v^k \circ v^\ell);
\]

• Finally, denote by
\[
W_4^\circ = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki} q_i q_k \sum_{j \neq i \neq \ell} b_{j\ell} h_j h_\ell (v^i \circ v^j) (v^k \circ v^\ell)
\]

Hence
\[
\|A_{q,h}(r)\|_F^2 = W_2^\circ + W_3^{\text{diag}} + W_4^{\text{diag}} + W_4^\circ.
\] (78)

In summary, we have the following bounds for the expected values of each component in Lemma J.1, which follows immediately from Lemmas K.1, K.2, K.3, and K.4. We prove Lemma J.1 in Section K.

**Lemma J.1.** Denote by $b_\infty = \max_j b_{jj}$. For all $q, h \in \mathbb{S}^{n-1}$, we have
\[
|\mathbb{E}W_2^\circ| \leq 2b_\infty^2 \sum_{i=1}^{m} a_{ii}^2 p_i^2,
\]
\[
|\mathbb{E}W_3^{\text{diag}}| \leq (2b_\infty^2 + 2b_\infty \|B_0\|_2 + 2\|B_0\|^2_2) \sum_{i=1}^{m} a_{ii}^2 p_i^3,
\]
\[
|\mathbb{E}W_4^{\text{diag}}| \leq (2b_\infty^2 + 2b_\infty \|B_0\|_2 + 6\|B_0\|^2_2) \sum_{i=1}^{m} a_{ii}^2 p_i^4,
\]
and
\[
|\mathbb{E}W_4^\circ| \leq 4\|B_0\|^2_2 \sum_{i \neq j} a_{ij}^2 p_i^2 p_j^2.
\]

In Section J.1, we prove large deviation bounds and obtain an upper bound on the following polynomial functions: $|W_2^\circ - \mathbb{E}W_2^\circ|, |W_3^{\text{diag}} - \mathbb{E}W_3^{\text{diag}}|, |W_4^{\text{diag}} - \mathbb{E}W_4^{\text{diag}}|$, and $|W_4^\circ - \mathbb{E}W_4^\circ|$ in Lemmas J.2 to J.5 respectively. It follows from the proof of Lemma J.2 that for some $0 < \varepsilon < 1/2$,
\[
(1 - \varepsilon)\mathbb{E}W_2^\circ \leq W_2^\circ \leq (1 + \varepsilon)\mathbb{E}W_2^\circ,
\]
so long as [82] holds. This is not surprising given that $W_2^\circ$ is a positive polynomial of degree 2, which is known to have strong concentration. Unfortunately, although the dominating term $W_2^\circ$ has non-negative coefficients $w_{(i,j)}$ with respect to each unique $(i,j)$ pair and their corresponding linear term $\sum_{s=1}^{m} a_{ss} a_i^s a_j^s$, the same property does not hold for others. We exploit crucially an upper bound on the sum of absolute values of coefficients (including many possibly non-positive) to derive the corresponding large deviation bounds for $W_3^{\text{diag}}$, $W_4^{\text{diag}}$ and $W_4^\circ$. In combination with the absolute value bounds on their expected values in Lemma J.1, we obtain an upper bound for each of the following terms: $W_2^\circ, |W_3^{\text{diag}}|, |W_4^{\text{diag}}|$, and $|W_4^\circ|$ using the triangle inequality, which collectively leads to a large deviation bound on the Frobenius norm as stated in Theorem 4.3.
J.1 Proof of Theorem 4.3

We first state the large deviation bounds for $W_2^\diamond$, $W_3^{\text{diag}}$ and $W_4^{\text{diag}}$, which are bounded in a similar manner. As a brief summary of the number of unique events (or unique polynomial of order 2, 3, 4) in the summation

$$\sum_{i,k} \sum_{i\neq j,k \neq \ell} w_{i,j,k,\ell} a_{i,j,k,\ell}^2 u_i^j u_j^k u_k^\ell,$$

where the labels are allowed to repeat, we end up with the following categories:

- $\#W_2^\diamond = \frac{1}{2}n(n-1)$ with coefficients $w_{i,j}^\diamond = b_{i,i}q_i^2b_{j,j}h_j^2 + b_{j,j}q_i^2b_{i,i}h_i^2 + 2b_{i,j}^2q_iq_jh_ih_j$ (79)

- $\#W_3^{\text{diag}} = \frac{n(n-1)(n-2)}{3!}$ with coefficients $b_{i,i}q_i^2b_{j,k}h_jh_k + b_{j,k}q_jq_kb_{i,i}h_i^2 + q_ih_i(b_{i,k}q_kb_{j,j}h_j + b_{j,i}b_{i,k}h_kq_j)$ and their $3!$ permutations

- $\#W_4^{\text{diag}} = \frac{(n)(n-1)(n-2)(n-3)}{4!}$ with 24 coefficients which we do not enumerate here

For a unique pair $(i,j), i \neq j$, the coefficient corresponding to the unique quadratic term $((\nu^i \circ \nu^j)^T \text{diag}(A_0 \circ A_0)(\nu^i \circ \nu^j))$ is

$$0 \leq w_{i,j}^\diamond = b_{i,i}b_{j,j}q_i^2h_j^2 + b_{j,j}b_{i,i}q_i^2h_i^2 + 2b_{i,j}^2q_iq_jh_ih_j \leq 2b_{i,i}b_{j,j}q_i^2h_j^2 + 2b_{i,j}b_{j,j}q_i^2h_i^2$$

for $B_0 \geq 0$. Exploiting symmetry, we rewrite

$$W_2^\diamond := \sum_{i \neq j} \sum_{i,j} w_{i,j}^\diamond \sum_{s=1}^m a_{i,j}^s u_i^j u_j^s \quad \text{where} \quad \sum_{i \neq j} w_{i,j}^\diamond \leq 2b_{i,j}^\diamond.$$ (80)

Thus our starting point is to obtain the large deviation bound for each such linear term $\sum_{s=1}^m a_{i,j}^s u_i^j u_j^s$, and then put together a large deviation bound for $W_2^\diamond$ from its mean using the union bound, as well as the upper bound in Lemma J.2 on the total weight $\sum_{i \neq j} w_{i,j}^\diamond \leq 2b_{i,j}^\diamond$. We prove Lemma J.2 in Section L.1.

**Lemma J.2.** ($W_2^\diamond$ bound) Let $W_2^\diamond$ be as defined in (80). Denote by

$$|S_2^\diamond| := \max_{i \neq j} |S_2^\diamond(i, j)| \quad \text{where} \quad S_2^\diamond(i, j) := \sum_{s=1}^m a_{i,j}^s (u_i^j u_j^s - p_i^s).$$ (81)

Suppose that for some absolute constant $C_a^2$,

$$\sum_{s=1}^m a_{i,j}^s p_i^s \geq 4C_a^2 \log(n \lor m).$$ (82)

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Then for \( \tau_2 = C_n a_\infty \sqrt{\log(n \vee m)} \sum_{s=1}^{m} a^2_{ss} p^2_s \),
\[
\mathbb{P}( |S^*_2| > \tau_2 ) := \mathbb{P}( \mathcal{E}_2 ) \leq \frac{1}{(n \vee m)^4}
\]

On event \( \mathcal{E}_2 \), we have for all \( q, h \in S^{n-1} \), \( \omega_{(i,j)} \geq 0 \), and
\[
|W_2^q - \mathbb{E}W_2^q| \leq \sum_{(i \neq j)} w^e_{(i,j)} |S^*(i,j)|
\]
\[
\leq 2C_n b_\infty^2 a_\infty \sqrt{\log(n \vee m)} \sum_{s=1}^{m} a^2_{ss} p^2_s \leq b_\infty^2 \sum_{s=1}^{m} a^2_{ss} p^2_s
\]

Throughout this section, it is understood that for \( |B_0| = (|b_{ij}|) \) and when \( q, h \in S^{n-1} \) are \( s_0 \)-sparse, we replace \( ||B_0||_2 = ||(|b_{ij}|)||_2 \) with its maximum \( s_0 \)-sparse eigenvalue
\[
\rho_{\max}(s_0, (|b_{ij}|)) \leq \sqrt{s_0} ||B_0||_2
\]
as defined in [19] and bounded in Lemma 2.5. Moreover, we choose constants large enough so that all probability statements hold. Recall that to extract the cubic polynomial \( W_3^{\text{diag}} \) from (78), we first allow \( v^i \) to appear on both sides of the quadratic form \( (v^i \circ v^j)^T \text{diag}(A_0 \circ A_0)(v^i \circ v^k) \); we then add an element in \( W_3^{\text{diag}} \) with weight \( \Delta(i,j),(i,k) \)
\[
\text{for } (v_i \circ v_j)^T \text{diag}(A_0 \circ A_0)(v_i \circ v_k) = \sum_{s=1}^{m} a^2_{ss} u^s_i u^s_j u^s_k,
\]
where it is understood that index pairs \((i,j)\) and \((i,k)\) on both sides of \( \text{diag}(A_0 \circ A_0) \) remain unordered, resulting in four coefficients in \( \Delta(i,j),(i,k) \) (cf. (77)).

We crucially exploit an upper bound on the sum over absolute values of coefficients corresponding to each polynomial function \( S^*_3(i,j,k) \) as stated in Lemma K.3 to derive their corresponding large deviation bounds.

**Lemma J.3.** Denote by
\[
|S^*_3| := \max_{i \neq j \neq k} |S^*_3(i,j,k)| \quad \text{where} \quad |S^*_3(i,j,k)| := \left| \sum_{s=1}^{m} a^2_{ss}(u^s_i u^s_j u^s_k - p^3_s) \right|
\]

Then for \( \tau_3 = C_2 a_\infty^2 \log(n \vee m) \sqrt{\log(n \vee m)} \sum_{j=1}^{m} a^2_{jj} p^3_j \),
\[
\mathbb{P}( |S^*_3| > \tau_3 ) := \mathbb{P}( \mathcal{E}_3 ) \leq \frac{1}{3(n \vee m)^4};
\]

Under event \( \mathcal{E}_3 \), we have for all \( q, h \in S^{n-1} \) and \( |B_0| = (|b_{ij}|) \),
\[
\left| W_3^{\text{diag}} - \mathbb{E}W_3^{\text{diag}} \right| \leq 2(||B_0||_2 b_\infty + ||B_0||_2^2) \tau_3
\]
\[
\leq ||B_0||_2^2 \sum_{s=1}^{m} a^2_{ss} p^3_s + C_3 a^2_\infty \log(n \vee m)(||B_0||_2^2 + ||B_0||_2^3)
\]

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where $C_3$ is an absolute constant; When we consider $s_0$-sparse vectors $q,h \in \mathbb{S}^{n-1}$, we replace $\|B_0\|_2 = \|(b_{ij})\|_2$ with $\rho_{max}(s_0, (|b_{ij}|))$ as defined in (19).

**Lemma J.4.** Denote by

$$|S_4^*| = \max_{(i \neq j, k \neq \ell)} |S_4^*(i, j, k, \ell)| \quad \text{where} \quad S_4^*(i, j, k, \ell) := \sum_{s=1}^{m} a_{ss}^2 (u_s^i u_s^j u_s^k u_s^\ell - p_s^4).$$

Then for $\tau_4 = C_4a_\infty(a_\infty \log(n \vee m) \vee \sqrt{\log(n \vee m) \sum_{s=1}^{m} a_{ss}^2 p_s^4})$, we have

$$\mathbb{P} (|S_4^*| > \tau_4) =: \mathbb{P} (\mathcal{E}_4) \leq \frac{1}{12(n \vee m)^4}.$$ 

Under event $\mathcal{E}_4$, we have for all $q,h \in \mathbb{S}^{n-1}$,

$$\left| W_4^{\text{diag}} - \mathbb{E} W_4^{\text{diag}} \right| \leq \text{offd}(\|B_0\|_2)^2 |S_4^*| \leq \|(b_{ij})\|_2^2 \tau_4. \quad (83)$$

When we consider $s_0$-sparse vectors $q,h \in \mathbb{S}^{n-1}$, we replace $\|(b_{ij})\|_2$ with $\rho_{max}(s_0, (|b_{ij}|))$ as defined in (19).

We next bound the sum

$$W_4^\circ - \mathbb{E} W_4^\circ = \sum_{i=1}^{n} \sum_{k=1}^{n} (b_{ik} q_i q_k) \left( \sum_{j \neq i}^{n} \sum_{\ell \neq k \neq \ell} b_{j\ell} h_j h_{\ell} \right) \mathcal{S}_4(i, j, k, \ell)
\leq \sum_{i \neq j, k \neq \ell} w_{ij}(i, j, k, \ell) (\mathcal{S}_4(i, j, k, \ell)) \quad \text{where} \quad \mathcal{S}_4(i, j, k, \ell) := \sum_{s \neq \ell} a_{st}^2 (u_s^i u_s^j u_s^k u_s^\ell - p_s^2 p_\ell^2) \quad \forall i \neq j, k \neq \ell. \quad (84)$$

**Lemma J.5.** Let $\mathcal{S}_4(i, j, k, \ell)$ be as defined in (84). Denote by

$$|S_4^\circ| = \max_{(i \neq j, k \neq \ell)} |\mathcal{S}_4(i, j, k, \ell)|$$

Denote by $\tau_5 = C_5 \|A_0\|_2 \left( \|A_0\|_2 \log(n \vee m) \vee \sqrt{\log(n \vee m) \sum_{s \neq \ell} a_{ss}^2 p_s^2 p_\ell^2} \right)$. Then

$$\mathbb{P} (|S_4^\circ| > \tau_5) =: \mathbb{P} (\mathcal{E}_5) \leq \frac{1}{2(n \vee m)^4}.$$ 

For all $q,h \in \mathbb{S}^{n-1}$, we have on event $\mathcal{E}_5$, for $|B_0| = (|b_{ij}|)$,

$$|W_4^\circ - \mathbb{E} W_4^\circ| \leq C_5 \|B_0\|_2 \left( \|A_0\|_2 \log(n \vee m) \vee \sqrt{\log(n \vee m) \sum_{s \neq \ell} a_{ss}^2 p_s^2 p_\ell^2} \right)$$

When we consider $s_0$-sparse vectors $q,h \in \mathbb{S}^{n-1}$, we replace $\|B_0\|_2 = \|(b_{ij})\|_2$ with $\rho_{max}(s_0, (|b_{ij}|))$ as defined in (19).

Lemma J.3, J.4 and J.5 are proved in Sections L.2, L.3 and L.4 respectively.
J.2 Proof of Theorem 4.3

Throughout this proof, it is understood that when we consider $s_0$-sparse vectors $q, h \in S^{n-1}$, we replace $\|B_0\|_2 = \|(b_{ij})\|_2$ with

$$\rho_{\text{max}}(s_0, (|b_{ij}|)) \leq \sqrt{s_0} \|B_0\|_2$$

as defined in (19) and shown in Lemma 2.5. First, we have by Lemma J.1, for all $J.2$, Proof of Theorem 4.3 for all $q, h \in S^{n-1}$,

$$\mathbb{E} \|A_{q,h}^o\|_F^2 = \mathbb{E} W_2^o + \mathbb{E} W_3(\text{diag}) + \mathbb{E} W_4^\text{diag} + \mathbb{E} W_4^o \leq \mathbb{E} W_2^o + \mathbb{E} W_3(\text{diag}) + \|\mathbb{E} W_4(\text{diag})\| + \|\mathbb{E} W_4^o\|$$

$$\leq 2b_\infty^2 \sum_{j=1}^{m} a_j^2 p_j^2 + 6 \|B_0\|_2^2 \left( \sum_{j=1}^{m} a_j^2 p_j^3 + \sum_{i=1}^{m} a_i^2 p_i^4 \right) + 4 \|B_0\|_2^2 N$$

where

$$N = \sum_{j=1}^{m} a_j^4 p_j^4 + \sum_{i \neq j} a_i^2 p_i^2 p_j^2 \leq \lambda_{\max}(A_0 \circ A_0) \sum_{s=1}^{m} p_s^4 \leq a_\infty \|A_0\|_2 \sum_{s=1}^{m} p_s^4$$

Let events $\mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5$ be as defined in Lemmas J.2, J.3, J.4 and J.5 respectively. We have on event $\mathcal{E}_2$, by Lemma J.2 for all $q, h \in S^{n-1}$,

$$|W_2^o - \mathbb{E} W_2^o| \leq b_\infty^2 \sum_{s=1}^{m} a_{ss}^2 p_s^2$$

Denote by

$$S_2 = \sum_{s=1}^{m} a_{ss}^2 p_s^2$$

and

$$S_4 = \sum_{s=1}^{m} a_{ss}^4 p_s^4$$

On event $\mathcal{E}_3^c$, we have by Lemma K.3 and Lemma J.3 for all $q, h \in S^{n-1}$,

$$|W_3^\text{diag}| \leq |\mathbb{E} W_3^\text{diag}| + |W_3^\text{diag} - \mathbb{E} W_3^\text{diag}| \leq 8 \|B_0\|_2 \sum_{s=1}^{m} a_{ss}^2 p_s^3 + C_3(\|B_0\|_2^2 + \|B_0\|_2^2) a_\infty^2 \log(n \lor m)$$

where $S_3 \leq (S_2 S_4)^{1/2} \leq \frac{1}{2}(S_2 + S_4)$ and $C_3 a_\infty^2 \log(n \lor m) \leq c'S_2$ by assumption on the lower bound on $S_2$. Finally, on event $\mathcal{E}_4^c \cap \mathcal{E}_5^c$, by Lemmas J.4 and J.5 we have for some absolute constants $C_1, C_1'$,

$$|W_4^\text{diag} - \mathbb{E} W_4^\text{diag}| + |W_4^o - \mathbb{E} W_4^o| \leq C_1 \log(n \lor m) \|A_0\|_2^2 \|B_0\|_2^2 + C_1' \|B_0\|_2^2 \|A_0\|_2^2 \sqrt{\log(n \lor m) N}$$
Hence, on event \( \mathcal{E}_2^c \cap \mathcal{E}_3^c \cap \mathcal{E}_4^c \cap \mathcal{E}_5^c \) by Lemmas J.1, J.2, J.4, J.3 and J.5, we have for all \( q, h \in \mathbb{S}^{n-1} \),

\[
\| A_{q,h}^c \|_F^2 \leq |W_2^c| + |W_3^{\text{diag}}| + |W_4^{\text{diag}}| + |W_4^c| \\
\leq \mathbb{E}W_2^c + |W_2^c - \mathbb{E}W_2^c| + |W_3^{\text{diag}}| + |\mathbb{E}W_4^{\text{diag}}| + |\mathbb{E}W_4^c| \\
+ |W_4^{\text{diag}} - \mathbb{E}W_4^{\text{diag}}| + |W_4^c - \mathbb{E}W_4^c| \\
\leq C_6 \|B_0\|_2^2 S_2 + C_7 \|B_0\|_2^2 a_\infty \|A_0\|_2 \sum_{s=1}^m p_s^4 + C_8 \|B_0\|_2^2 \|A_0\|_2^2 \log(n \vee m) \\
+ C_9 \|B_0\|_2^2 \|A_0\|_2 \left( \log(n \vee m) a_\infty \|A_0\|_2 \sum_{j=1}^m p_j^4 \right)
\]

for some absolute constants \( C_6, C_7, C_8, \ldots \). The theorem statement thus holds on event \( \mathcal{F}_0^c := \mathcal{E}_2^c \cap \mathcal{E}_3^c \cap \mathcal{E}_4^c \cap \mathcal{E}_5^c \), which holds with probability at least \( 1 - \frac{4}{(n \vee m)^4} \) by the union bound. \( \square \)

**K Proof of Lemma J.1**

Recall \( \{b^{(1)}, \ldots, b^{(n)}\} \) denotes the set of column (row) vectors of symmetric positive-definite matrix \( B_0 \succ 0 \).

**K.1 Case \( W_4^2 \)**

First we compute the expectation for \( W_4^2 \) for any \( q, h \in \mathbb{S}^{n-1} \). We prove Lemma K.1 in Section K.5. Recall

\[
W_4^2 = \sum_{i=1}^n \sum_{k=1}^n (b_{ik} q_i q_k) \left( \sum_{j \neq i} \sum_{\ell \neq k} (b_{ij} h_j h_\ell) \right) \sum_{s \neq t} a_{st}^2 u_s^t u_s^t u_t^t u_t^t
\]  

\( (85) \)

**Lemma K.1.** Let \( W_4^2 \) be defined in \( (85) \). Then

\[
\forall \ q, h \in \mathbb{S}^{n-1}, \quad |\mathbb{E}W_4^2| \leq 4 \|B_0\|_2^2 \sum_{i \neq j} a_{ij}^2 p_i^2 p_j^2 \text{ where}
\]

\[
\left| \sum_{i,k} b_{ik} q_i q_k \sum_{i \neq j, k \neq \ell} b_{ij} h_j h_\ell \right| \leq 4 \|B_0\|_2^2
\]  

\( (86) \)

**K.2 Case \( W_2^2 \)**

We prove Lemma K.2 in Section K.6.

**Lemma K.2.** Fix \( q, h \in \mathbb{S}^{n-1} \). Let \( w_{(i,j)}^e \) be as defined in \( (79) \). Then for all \( q, h \in \mathbb{S}^{n-1} \) and \( W_2^2 \) as defined in \( (80) \),

\[
|\mathbb{E}W_2^2| \leq 2b_\infty^2 \sum_{s=1}^a s^2 p_s^2.
\]

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K.3 Counting strategy for unique triples

We prove Lemma K.3 in Section K.7. Recall for $\Delta_{(i,j),(i,k)}$ as defined in (77),

$$W_3^{\text{diag}} := \sum_{i=1}^{n} \sum_{j \neq i}^{n} \sum_{k \neq i,j}^{n} \Delta_{(i,j),(i,k)} \cdot (v_i \circ v_j)^T \text{diag}(A_0 \circ A_0) (v_i \circ v_k).$$

**Lemma K.3.** For all $q, h \in S^{n-1},$

$$\left| \mathbb{E} W_3^{\text{diag}} \right| \leq \left( 2b_{\infty}^2 + 2 \|B_0\|_2 b_{\infty} + 2 \|B_0\|_2^2 \right) \sum_{s=1}^{m} a_{ss}^2 p_s^3,$$

where we have for $\Delta_{(i,j),(i,k)}$ as defined in (77),

$$\left| \sum_{i \neq j \neq k} \Delta_{(i,j),(i,k)} \right| \leq 2 \|B_0\|_2^2 + 2 \|B_0\|_2 b_{\infty} + 2b_{\infty}^2;$$

Moreover, for $\|B_0\|_2 = \|(b_{ij})\|_2$ and for all $q, h \in S^{n-1},$

$$\sum_{i \neq j \neq k} \left| \Delta_{(i,j),(i,k)} \right| \leq \left( 2 \|(b_{ij})\|_2 b_{\infty} + 2 \|B_0\|_2 \right)^2;$$

where it is understood that when we consider $s_0$-sparse vectors $q, h \in S^{n-1},$ we replace $\|B_0\|_2 = \|(b_{ij})\|_2$ with $\rho_{\text{max}}(s_0, (b_{ij}))$ as defined in (19) and bounded in Lemma 2.5.

K.4 $W_4^{\text{diag}}$: distinct $i, j, k, \ell$

The proof of Lemma K.4 follows the arguments in Lemma 2.1 by [6], which we defer to Section K.8.

**Lemma K.4.** Recall

$$W_4^{\text{diag}} = \sum_{i \neq k \neq j \neq \ell}^{n} b_{ki} q_i q_k b_{ij} h_j h_j (v_i \circ v_j)^T \text{diag}(A_0 \circ A_0) (v_k \circ v_{\ell}) \quad (87)$$

We have

$$\left| \mathbb{E} W_4^{\text{diag}} \right| \leq \left( 2b_{\infty}^2 + 2b_{\infty} \|B_0\|_2 + 6 \|B_0\|_2^2 \right) \sum_{i=1}^{m} a_{ii}^2 p_i^4,$$

where for any $q, h \in S^{n-1},$

$$\left| \sum_{i,j,k,\ell \text{ distinct}} b_{ki} q_i q_k b_{ij} h_j h_{\ell} \right| \leq 6 \|B_0\|_2^2 + 2b_{\infty} \|B_0\|_2 + 2b_{\infty}^2$$

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K.5 Proof of Lemma K.1

For \( i \neq j \) and \( k \neq \ell \),

\[
(v^i \circ v^j)^T \text{offd}(A_0 \circ A_0) (v^k \circ v^\ell) = \sum_{s \neq t} a^2_{st} u^s_i u^s_j u^t_k u^t_\ell;
\]

Hence by linearity of expectation,

\[
|\mathbb{E} W_4^o| = \left| \sum_{i=1}^n \sum_{k=1}^n (b_{ik} q_i q_k) \left( \sum_{j \neq i} \sum_{\ell \neq k} (b_{j\ell} h_j h_\ell) \right) \right| \mathbb{E} \left| \sum_{s \neq t} a^2_{st} u^s_i u^s_j u^t_k u^t_\ell \right|
\]

\[
= \left| \sum_{i=1}^n \sum_{k=1}^n (b_{ik} q_i q_k) \left( \sum_{j \neq i} \sum_{\ell \neq k} (b_{j\ell} h_j h_\ell) \right) \sum_{s \neq t} a^2_{st} p^2_s p^2_\ell \right|
\]

\[
\leq 4 \| B_0 \|_2^2 \sum_{i \neq j} a^2_{ij} p^2_i p^2_j
\]

To see the last step, we now examine the coefficients for \( W_4^o \): for each fixed \((i, k)\) pair,

\[
\sum_{j \neq i, \ell \neq k} b_{j\ell} h_j h_\ell = \sum_{j, \ell} b_{j\ell} h_j h_\ell - \sum_{j, \ell = k} b_{j\ell} h_j h_\ell - \sum_{j = i, \ell \neq k} b_{j\ell} h_j h_\ell + \sum_{j = i, \ell = k} b_{j\ell} h_j h_\ell
\]

\[
= \sum_{j, \ell} b_{j\ell} h_j h_\ell - h_k \sum_{j = 1}^n b_{jk} h_j - h_\ell \sum_{\ell = 1}^n b_{i\ell} h_\ell + b_{ik} h_i h_k
\]

and hence

\[
\sum_{i, k} b_{ki} q_i q_k \sum_{j \neq i, \ell \neq k} b_{j\ell} h_j h_\ell = \sum_{i, k} b_{i\ell} h_i q_i q_k \sum_{j, \ell} b_{j\ell} h_j h_\ell - \sum_{i, k} b_{ki} q_i q_k h_k \sum_{j = 1}^n b_{jk} h_j - \sum_{i, k} b_{ki} q_i q_k h_i \sum_{\ell = 1}^n b_{i\ell} h_\ell + \sum_{i, k} b_{ki} q_i q_k b_{ik} h_i h_k,
\]

where due to symmetry, we bound the middle two terms in an identical manner: denote by \( |q| = (|q_1|, \ldots, |q_n|) \),

\[
\left| \sum_{i, k} b_{ki} q_i q_k \sum_{j = 1}^n b_{jk} h_j \right| = \left| \sum_{k = 1}^n q_k h_k \sum_{i = 1}^n \sum_{j = 1}^n b_{ki} b_{kj} q_i h_j \right|
\]

\[
= \left| \sum_{k = 1}^n q_k h_k \left( q^T (b^{(k)} \otimes b^{(k)}) h \right) \right|
\]

\[
\leq \max_k \| b^{(k)} \otimes b^{(k)} \|_2 \| \langle |q|, |h| \rangle \| \leq \| B_0 \|_2^2;
\]
Similarly,
\[
\left| \sum_{i,k} b_{ki} q_i q_k h_i \sum_{\ell=1}^n b_{i\ell} h_{\ell} \right| = \left| \sum_{i=1}^n q_i h_i \sum_{k=1}^n b_{ik} q_k \sum_{\ell=1}^n b_{i\ell} h_{\ell} \right| = \left| \sum_{i=1}^n q_i h_i \left( q^T (b^{(i)} \otimes b^{(i)}) h \right) \right| \leq \|B_0\|_2^2; \]

Finally for \( q \circ h = (q_1 h_1, \ldots, q_n h_n) \)
\[
\sum_{i,k} b_{ki} q_i q_k b_{ik} h_i h_k = \sum_{i,k} b_{ki}^2 (q \circ h)_i (q \circ h)_k \leq \|B_0 \circ B_0\|_2 \leq b_\infty \|B_0\|_2. \]

Hence by (SS) and the inequalities immediately above, we have
\[
\left| \sum_{i,k} b_{ki} q_i q_k \sum_{i \neq j, k \neq \ell} b_{ji} h_j h_{\ell} \right| \leq \left| \sum_{i,k} b_{ki} q_i q_k \sum_{j, \ell} b_{ji} h_j h_{\ell} \right| + \left| \sum_{i,k} b_{ki} q_i q_k h_k \sum_{j=1}^n b_{jk} h_j \right| + \left| \sum_{i,k} b_{ki} q_i q_k h_i \sum_{\ell=1}^n b_{i\ell} h_{\ell} \right| + \left| \sum_{i,k} b_{ki} q_i q_k b_{ik} h_i h_k \right| \leq 4 \|B_0\|_2^2. \]

\( \Box \)

**K.6 Proof of Lemma [K.2]**

First we compute for all \( q, h \in S^{n-1}, \)
\[
0 \leq \sum_{i \neq j} w_{(i,j)}^e \leq \sum_{i \neq j} (2b_{ii} b_{jj} q_i^2 h_j^2 + 2b_{jj} b_{ii} q_j^2 h_i^2)
= \sum_{i=1}^n b_{ii} q_i^2 \sum_{j \neq i} b_{jj} h_j^2 + \sum_{i=1}^n b_{ii} h_i^2 \sum_{j \neq i} b_{jj} q_j^2
\leq b_\infty^2 (\sum_{i=1}^n q_i^2 \sum_{j \neq i} h_j^2 + \sum_{i=1}^n h_i^2 \sum_{j \neq i} q_j^2) \leq 2b_\infty^2
\]
where we use the fact that for \( B_0 = (b_{i,j}) > 0, \)
\[
(\sqrt{b_{ii} b_{jj}} q_i h_j)^2 + (\sqrt{b_{ii} b_{jj}} h_i q_j)^2 \geq 2b_{ii} b_{jj} |q_i q_j h_i h_j|
\geq 2b_{ij}^2 |q_i q_j h_i h_j| \quad \text{since} \quad b_{ii} b_{jj} \geq b_{ij}^2.
\]

Hence
\[
0 \leq \mathbb{E} W_2^e = \sum_{i \neq j} w_{(i,j)}^e \mathbb{E} \left( (v_i \circ v_j)^T \text{diag}(A_0 \circ A_0)(v_i \circ v_j) \right)
= \sum_{(i,j) \neq (i',j')} (b_{ii} b_{jj} q_i^2 h_j^2 + b_{jj} b_{ii} q_j^2 h_i^2 + 2b_{ij}^2 q_i q_j h_i h_j) \sum_{s=1}^2 a_{ss}^2 p_s^2 \leq 2b_\infty^2 \sum_{s=1}^2 a_{ss}^2 p_s^2.
\]
K.7 Proof of Lemma [K.3]

Throughout this proof, we assume that \( q, h \in \mathbb{S}^{n-1} \). Denote by \( b_{(i)} = (b_{i,1}, \ldots, b_{i,i-1}, b_{i,i+1}, \ldots, b_{i,n}) \). Summing over \( \Delta_{(i,j),(i,k)} \) over all unique triples \( i \neq j \neq k \), we have

\[
\sum_{i=1}^{n} \sum_{j \neq i}^{n} \sum_{k \neq j}^{n} \Delta_{(i,j),(i,k)} = \sum_{i=1}^{n} b_{ii} q_{i}^2 \sum_{j \neq k \neq i} b_{jk} h_{j} h_{k} \\
+ \sum_{i=1}^{n} b_{ii} h_{i}^2 \sum_{j \neq k \neq i} b_{jk} q_{j} q_{k} + 2 \sum_{i=1}^{n} q_{i} h_{i} \sum_{k \neq j \neq i} b_{ik} q_{k} b_{ij} h_{j}
\]

where for a fixed index \( i \) and column \( u^s \), when we sum over all \( j, k \), due to symmetry, we have

\[
\sum_{(j \neq k) \neq i} b_{ik} q_{k} b_{ij} h_{j} = \sum_{(j \neq k) \neq i} b_{ij} q_{j} b_{ik} h_{k} = q_{i}^{T} \text{offd}(b_{(i)}^{(i)} \otimes b_{(i)}^{(i)}) h_{\setminus i};
\]

Now, this leads to the following equivalent expressions for \( W_{3}^{\text{diag}} \):

\[
W_{3}^{\text{diag}} := \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq j} \Delta_{(i,j),(i,k)} \cdot (v^{i} \circ v^{j})^{T} \text{diag}(A_{0} \circ A_{0})(v^{i} \circ v^{k})
\]

(89)

\[
= \sum_{s=1}^{m} \sum_{i=1}^{n} a_{ss}^{2} u_{i}^{s} \sum_{j \neq i} \sum_{k \neq j} (b_{ii} q_{i}^2 (b_{jk} h_{j} h_{k}) + (b_{jk} q_{j} q_{k}) b_{ii} h_{i}^2) u_{j}^{s} u_{k}^{s}
\]

\[
+ \sum_{s=1}^{m} \sum_{i=1}^{n} (b_{ik} q_{k} b_{ij} h_{j}) u_{i}^{s} u_{k}^{s}
\]

where in (89), we have for each of the \( n(n-1)(n-2) \) uniquely ordered triple \( (i, j, k) \), a total weight defined by \( \Delta_{(i,j),(i,k)} \), and the second expression holds by symmetry of the quadratic form; in more details,

\[
\sum_{(j \neq k) \neq i} b_{ik} q_{k} b_{ij} h_{j} u_{i}^{s} u_{k}^{s} = \sum_{(j \neq k) \neq i} b_{ij} q_{j} b_{ik} h_{k} u_{i}^{s} u_{k}^{s}
\]

\[
= (q \circ u^{s})^{T} \text{offd}(b_{(i)}^{(i)} \otimes b_{(i)}^{(i)}) (h \circ u^{s})_{\setminus i}.
\]

Therefore, we only write \( 2b_{ik} q_{k} b_{ij} h_{j} \) rather than the sum over \( (b_{ik} q_{k} b_{ij} h_{j} + b_{ij} q_{j} b_{ik} h_{k}) \) in the summation in the sequel. Now we will show that

\[
\left| \sum_{i \neq j \neq k} \Delta_{(i,j),(i,k)} \right| \leq \sum_{i=1}^{n} b_{ii} q_{i}^2 \sum_{j \neq i} \sum_{k \neq i,j} b_{jk} h_{j} h_{k} + \sum_{i=1}^{n} b_{ii} h_{i}^2 \sum_{j \neq i} \sum_{k \neq i,j} b_{jk} q_{j} q_{k}
\]

\[
+ 2 \sum_{i=1}^{n} |q_{i}| |h_{i}| \left| \sum_{j \neq i} (b_{ij} h_{j}) (\sum_{k \neq i} b_{ik} q_{k}) \right| =: I + II + III
\]

\[
\leq 2 \| B_{0} \|_{2} b_{\infty} + 2b_{\infty}^2 + 2 \| B_{0} \|_{2}^2.
\]
Notice that
\[
\mathbb{E}(v_j \circ v_i)^T \text{diag}(A_0 \circ A_0)(v_j \circ v_i) = \sum_{s=1}^m \mathbb{E}(a_{ss}^2 u_s^8 u_k^8) = \sum_{s=1}^m a_{ss}^3.
\]
Hence by linearity of expectations, we have
\[
|\mathbb{E}W_3^{ij}| = \left| \sum_{i \neq j \neq k} \Delta_{(i,j),(i,k)} \sum_{s=1}^m a_{ss}^2 P_s^3 \right| \leq \sum_{i \neq j \neq k} \Delta_{(i,j),(i,k)} \sum_{s=1}^m a_{ss}^2 P_s^3 \leq 2 \left( \|B_0\|_2 b_\infty + b_\infty^2 + \|B_0\|_2^2 \right) \sum_{s=1}^m a_{ss}^3.
\]
It remains to show \(I, II \leq \|B_0\|_2 b_\infty + b_\infty^2\) and \(III \leq 2 \|B_0\|_2^2\). The first two terms are bounded similarly and with the same upper bound, and hence we show the first one:
\[
I = \sum_{i=1}^n b_i q_i^2 \sum_{j \neq i} \sum_{k \neq i, j} b_{jk} h_j h_k \leq \|B_0\|_2 b_\infty + b_\infty^2,
\]
where for a fixed \(i,\)
\[
\left| \sum_{j \neq k \neq k} b_{jk} h_j h_k \right| \leq \left| \sum_{k \neq i} b_{jk} h_j h_k \right| + \left| \sum_{k \neq i} b_{jj} h_j^2 \right| \leq \|B_0\|_2 + b_\infty,
\]
and similarly,
\[
II = \sum_{i=1}^n b_i h_i^2 \sum_{j \neq i} \sum_{k \neq i, j} b_{jk} h_j q_k \leq \|B_0\|_2 b_\infty + b_\infty^2.
\]
We can rewrite the sum for \(III\) as follows. Let \(|q| = (|q_1|, \ldots, |q_n|)\). Then
\[
III = 2 \sum_{i=1}^n |q_i| |h_i| \sum_{j \neq i} (b_{ij} h_j) \left( \sum_{k \neq i} b_{ik} q_k \right) \leq 2 \sum_{i=1}^n |q_i| |h_i| \sum_{j \neq i} \sum_{k \neq j \neq i} |b_{ij}| |h_j| |b_{ik}| |q_k| =: IV
\]
where for \(b_{ij}^{(i)} = (b_{1,i-1}, b_{1,i}, b_{1,i+1}, \ldots, b_{1,n}),\)
\[
IV := 2 \sum_{i=1}^n |q_i| |h_i| \sum_{j \neq i} \sum_{k \neq i} |b_{ij}| |h_j| |b_{ik}| |q_k| \leq 2 \sum_{i=1}^n |q_i| |h_i| \left\| b_{ij}^{(i)} \right\|_2^2 \leq 2 \max_i \left\| b_{ij}^{(i)} \right\|_2 \left\langle |q|, |h| \right\rangle \leq 2 \|B_0\|_2^2, \quad (90)
\]
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Similarly, we have for $|B_0| = (|b_{ij}|)$,

$$
\sum_{i \neq j \neq k} |\Delta_{(i,j), (i,k)}| \leq 2 \sum_{i=1}^{m} |q_i| |h_i| \sum_{k \neq i, j \neq k} |b_{ik}| |q_k| |b_{ij}| |h_j| + \sum_{i=1}^{n} b_{ii}q_i^2 \sum_{j \neq i, k \neq i,j} |b_{jk}| |h_j| |h_k| + \sum_{i=1}^{n} b_{ii}h_i^2 \sum_{j \neq i, k \neq i,j} |b_{jk}| |q_j| |q_k|
$$

$$=: IV + V + VI \leq 2 \|B_0\|_2^2 + 2b_\infty \|B_0\|_2,$$

where the term $IV$ is as bounded in (90), and $V + VI \leq 2b_\infty \|B_0\|_2$ since

$$V = \sum_{i=1}^{n} b_{ii}q_i^2 \sum_{k,j,k \neq j \neq i} |b_{jk}| |h_j| |h_k| \leq b_\infty \|B_0\|_2$$

where $\|B_0\|_2 := \|(|b_{ij}|)\|_2$ in the argument above is understood to denote

$$\rho_{\max}(s_0, |B_0|) = \max_{h \in \mathbb{R}^{n-1}, s_0 - \text{sparse}} \sum_{i,j} |b_{ij}| |h_i| |h_j|$$

$$\leq \sqrt{s_0} \|B_0\|_2$$

in case $h$ is $s_0 - \text{sparse};$

and $VI := \sum_{i=1}^{n} b_{ii}h_i^2 \sum_{j \neq i} \sum_{k \neq i,j} |b_{jk}| |q_j| |q_k|$ is bounded in a similar manner. \(\square\)

### K.8 Proof of Lemma [K.4]

First we have by (86),

$$\left| \sum_{k \neq i} \sum_{j \neq i, \ell \neq k} b_{k\ell} q_i q_k \sum_{j, \ell, \ell \neq k} b_{j\ell} h_j h_\ell \right| \leq 4 \|B_0\|_2^2$$

Denote by

$$E_1 = \{(i,j,k,\ell) : (i = k), i \neq j, k \neq \ell\}, \quad E_2 = \{(i,j,k,\ell) : (i = \ell), i \neq j, k \neq \ell\},$$

$$E_3 = \{(i,j,k,\ell) : (j = k), i \neq j, k \neq \ell\}, \quad E_4 = \{(i,j,k,\ell) : (j = \ell), i \neq j, k \neq \ell\}$$

Now by the inclusion-exclusion principle,

$$\sum_{i,j,k,\ell \text{ distinct}} b_{k\ell} q_i q_k b_{j\ell} h_j h_\ell = \sum_{i,k} b_{k\ell} q_i q_k \sum_{i \neq j, k \neq \ell} b_{j\ell} h_j h_\ell$$

$$- \sum_{i=k} b_{k\ell} q_i q_k \sum_{j \neq i, \ell \neq k} b_{j\ell} h_j h_\ell (E_1) - \sum_{i \neq k} b_{k\ell} q_i q_k \sum_{i \neq j, k \neq \ell = i} b_{j\ell} h_j h_\ell (E_2)$$

$$- \sum_{i \neq k} b_{k\ell} q_i q_k \sum_{j \neq i, \ell \neq j, k \neq \ell = k} b_{j\ell} h_j h_\ell (E_3) - \sum_{i,k} b_{k\ell} q_i q_k \sum_{\ell \neq k, j = \ell \neq i} b_{j\ell} h_j h_\ell (E_4)$$

$$+ \sum_{i=k} b_{k\ell} q_i q_k \sum_{i \neq j, j = \ell} b_{j\ell} h_j h_\ell (E_4 \cap E_1) + \sum_{i \neq k} b_{k\ell} q_i q_k \sum_{i \neq j, j = \ell = i} b_{j\ell} h_j h_\ell (E_2 \cap E_3)$$

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where clearly
\[ E_1 \cap E_4 = \{(i, j, k, \ell) : (i = k), (j = \ell), i \neq j\} \]
\[ E_2 \cap E_3 = \{(i, j, k, \ell) : (i = \ell), (j = k), i \neq j\} \]
while all other pairwise intersections are empty and hence all three-wise and four-wise intersections are empty. Simplifying the notation, we have
\[
\sum_{i,j,k,\ell \text{ distinct}} b_{ki}q_iq_kb_{ji}h_jh_\ell = \sum_{i,k} b_{ki}q_iq_k \sum_{i \neq j, k \neq \ell} b_{ji}h_jh_\ell \\
- \sum_i b_{ii}q_i^2 \sum_{j \neq i, \ell \neq i} b_{ji}h_jh_\ell(E_1) - \sum_{i \neq k} b_{ki}q_iq_k \sum_{i \neq j} b_{ji}h_jh_i(E_2) \\
- \sum_{i,k} b_{ki}q_iq_k \sum_{j \neq k, j \neq i} b_{jj}h_j^2(E_4) - \sum_{i \neq j} b_{ji}q_jq_j \sum_{j \neq \ell} b_{ji}h_jh_\ell(E_3) \\
+ \sum_{i=1}^n b_{ii}q_i^2 \sum_{i \neq j} b_{ji}h_j^2(E_4 \cap E_1) + \sum_{i=1}^n \sum_{j \neq i} b_{ii}q_iq_jb_{ji}h_jh_i \ (E_2 \cap E_3)
\]
where for indices \((i, j, k, \ell)\) in \(E_1\) and \(E_2\),
\[
(E_1) \quad \left| \sum_i b_{ii}q_i^2 \sum_{j \neq i, \ell \neq i} b_{ji}h_jh_\ell \right| \leq b_\infty \sum_{i=1}^n q_i^2 \sum_{j \neq i, \ell \neq i} b_{ji}h_jh_\ell \leq b_\infty \|B_0\|_2 \\
(E_2) \quad \left| \sum_{i \neq k} b_{ki}q_iq_k \sum_{j \neq i} b_{ji}h_jh_i \right| \leq \sum_{i=1}^n |q_i| |h_i| \sum_{i \neq k} b_{ki}q_k \sum_{i \neq j} b_{ji}h_j \leq \|B_0\|_2^2
\]
where for a fixed index \(i\),
\[
\left| \sum_{i \neq k} b_{ki}q_k \sum_{i \neq j} b_{ji}h_j \right| \leq \sum_{i \neq k} b_{ki}q_k \sum_{i \neq j} b_{ji}h_j \leq \left\| b^{(i)} \right\|_2^2 \leq \|B_0\|_2^2
\]
Similarly, we bound for indices \((i, j, k, \ell)\) in \(E_3\) and \(E_4\),
\[
(E_4) \quad \left| \sum_{i,k} b_{ki}q_iq_k \sum_{j \neq k, j \neq i} b_{jj}h_j^2 \right| \leq b_\infty \|B_0\|_2 \quad \text{and} \quad (E_3) \quad \left| \sum_{i \neq j} b_{ji}q_jq_j \sum_{j \neq \ell} b_{ji}h_jh_\ell \right| \leq \|B_0\|_2^2
\]
Finally, for indices \((i, j, k, \ell)\) in \((E_3 \cap E_2) \cup (E_4 \cap E_1)\), we have
\[
\sum_{i=1}^n \sum_{j \neq i} (b_{ii}q_i^2b_{jj}h_j^2 + b_{ii}q_iq_jb_{ji}h_jh_i) \leq 2b_\infty^2
\]
following Lemma [K.2] Now as we have shown in Lemma [K.1]
\[
\left| \sum_{i,k} b_{ki}q_iq_k \sum_{i \neq j, k \neq \ell} b_{ji}h_jh_\ell \right| \leq 4 \|B_0\|_2^2
\]
Thus we have by the bounds immediately above,

$$\left| \sum_{i,j,k,\ell \text{ distinct}} b_{ki} q_i q_k b_j h_j h_\ell \right| \leq 6 \|B_0\|_2^2 + 2 \|B_0\|_2 b_{\infty} + 2b_{\infty}^2$$

Finally, for $w_{i,j,k,\ell} = b_{ki} q_i q_k b_j h_j h_\ell$,

$$\left| \mathbb{E} W_4^{\text{diag}} \right| = \left| \sum_{i\neq j \neq k \neq \ell} w_{i,j,k,\ell} \sum_{s=1}^m a_{ss}^2 p_s^4 \right| \leq (8 \|B_0\|_2^2 + 2b_{\infty}^2) \sum_{s=1}^m a_{ss}^2 p_s^4.$$  

Thus the lemma holds. \(\square\)

## L Proof strategies: concentration of measure bounds

We need the following result which follows from Proposition 3.4 \(35\).

**Lemma L.1.** \((35)\) Let $A = (a_{ij})$ be an $m \times m$ matrix. Let $a_{\infty} := \max_i |a_{ii}|$. Let $\xi = (\xi_1, \ldots, \xi_m) \in \{0,1\}^m$ be a random vector with independent Bernoulli random variables $\xi_i$ such that $\xi_i = 1$ with probability $p_i$ and 0 otherwise. Then for $|\lambda| \leq \frac{1}{4a_{\infty}}$,

$$\mathbb{E} \exp \left( \lambda \sum_{i=1}^m a_{ii} (\xi_i - p_i) \right) \leq \exp \left( \frac{1}{2} \lambda^2 e^{\lambda |a_{\infty}|} \sum_{i=1}^m a_{ii}^2 \sigma_i^2 \right).$$

We are going to apply Lemma \(L.1\) and its Corollary \(L.3\) to obtain concentration of measure bounds on the diagonal components corresponding to degree-2, 3, 4 polynomials in $W_2^\circ$, $W_3^\circ(\text{diag})$, and $W_4^{\text{diag}}$ respectively, where $i \neq j \neq k \neq \ell$ are being fixed. For $W_2^\circ$, we need to derive large deviation bound on polynomial function $\overline{S}_\circ(i,j,k,\ell)$ for each quadruple $(i,j,k,\ell)$ such that $i \neq j, k \neq \ell$. More precisely, we prove Theorem \(L.2\) Denote by

$$\forall i \neq j, k \neq \ell, \quad S_\circ(i,j,k,\ell) := (v_i \circ v_j)^T \text{offd}(A_0 \circ A_0)(v_k \circ v_\ell) \quad (91)$$

$$= \sum_{s \neq \ell} a_{ss}^2 u_s^i u_j^s u_k^l u_\ell^s$$

and

$$\overline{S}_\circ(i,j,k,\ell) := S_\circ(i,j,k,\ell) - \mathbb{E} S_\circ(i,j,k,\ell) \quad (92)$$

where

$$\mathbb{E} S_\circ(i,j,k,\ell) = \sum_{s \neq \ell} a_{ss}^2 p_s^2 p_s^2 \quad (93)$$

**Theorem L.2.** Let $a_{\infty} = \max_i a_{ii}^2$. Let $\overline{S}_\circ(i,j,k,\ell)$ be as defined in \(92\). For any $t > 0$, and quadruple $(i,j,k,\ell)$ such that $i \neq j, k \neq \ell$,

$$\mathbb{P} \left( |\overline{S}_\circ(i,j,k,\ell)| > t \right) \leq 2 \exp \left( -c \min \left( \frac{t^2}{\|A_0\|_2^2 \sum_{s \neq \ell} a_{ss}^2 p_s^2 p_s^2}, \frac{t}{\|A_0\|_2^2} \right) \right)$$

where $c$ is an absolute constant.
We prove Theorem L.2 in Section L.4.2

**Corollary L.3.** Let $A_\infty = \max_i a_{ii}^2 = a_{\infty}^2$. Let $\xi = (\xi_1, \ldots, \xi_m) \in \{0, 1\}^m$ be a random vector with independent Bernoulli random variables $\xi_i$ such that $\xi_j = 1$ with probability $\mathbb{E} \xi_j$ and 0 otherwise. Let $S_* = \sum_{j=1}^m a_{ij}^2 (\xi_j - \mathbb{E} \xi_j)$. Then for $t > 0$,

$$\mathbb{P} (|S_*| > t) \leq 2 \exp \left( -c \min \left( \frac{t^2}{M}, \frac{t}{a_{\infty}^2} \right) \right)$$

where $M = a_{\infty}^2 \sum_{s=1}^m a_{ss}^2 \mathbb{E} \xi_s$.

**Proof of Corollary L.3.** Let $A = (A_0 \circ A_0) = (a_{ij})$ be an $m \times m$ matrix. Let $A_{\infty} := \max_s a_{ss}^2$. Denote by $M = A_{\infty} \sum_{s=1}^m a_{ss}^2 \mathbb{E} \xi_s$. Thus we have by Lemma L.1 for $|\lambda| \leq \frac{1}{A_{\infty}}$,

$$\mathbb{E} \exp (\lambda S_*) \leq \exp \left( \frac{1}{2} \lambda^2 e^{\lambda |A_{\infty}|} \sum_{s=1}^m a_{ss}^2 \mathbb{E} \xi_s \right) \leq \exp (\lambda^2 M)$$

where $e^{\lambda |A_{\infty}|} \leq e^{1/4} \leq 1.29$. Now for $0 < \lambda < \frac{1}{4A_{\infty}}$, we have for $t > 0$, by Markov’s inequality,

$$\mathbb{P} (S_* > t) = \mathbb{P} (\exp(\lambda S_*) > \exp(\lambda t)) \leq \mathbb{E} \exp(\lambda S_*) / \exp(\lambda) \leq \exp (-\lambda t + \lambda^2 M)$$

Optimizing over $0 < \lambda < \frac{1}{4A_{\infty}}$, we have for $t > 0$, by Markov’s inequality,

$$\mathbb{P} (S_* > t) \leq \exp \left( -c \min \left( \frac{t^2}{M}, \frac{t}{A_{\infty}} \right) \right) =: q_{\text{diag}}$$

while noting that for $0 < \lambda < \frac{1}{4A_{\infty}}$ and $t > 0$, and repeating the same arguments above, we have

$$\mathbb{P} (S_* < -t) = \mathbb{P} (\exp(-\lambda S_*) > \exp(\lambda t)) \leq \frac{\mathbb{E} \exp(-\lambda S_*)}{e^{\lambda t}} \leq \exp (-\lambda t + \lambda^2 M) \leq q_{\text{diag}}$$

The corollary is thus proved by combining these two events since $\mathbb{P} (|S_*| > t) \leq 2q_{\text{diag}}$. □

Throughout this section, it is understood that for $|B_0| = (|b_{ij}|)$ and when $q, h \in S^{n-1}$ are $s_0$-sparse, we replace $\|B_0\|_2 = \|(|b_{ij}|)\|_2$ with its maximum $s_0$-sparse eigenvalue

$$\rho_{\text{max}} (s_0, (|b_{ij}|)) \leq \sqrt{s_0} \|B_0\|_2$$

as defined in (19) and bounded in Lemma 2.5. Moreover, we choose constants large enough so that all probability statements hold.

**L.1 Proof of Lemma J.2**

Denote by $M_2 = a_{\infty}^2 \sum_{s=1}^m a_{ss}^2 p_s^2$. Notice that by assumption,

$$2C_a a_{\infty}^2 \log(n \vee m) \leq \tau_2 := C_a a_{\infty} \sqrt{\log(n \vee m) \sum_{s=1}^m a_{ss}^2 p_s^2} \leq \frac{1}{2} \sum_{s=1}^m a_{ss}^2 p_s^2.$$
By Corollary \textbf{L.3} and the union bound,
\[
\mathbb{P} (|S_2^0| > \tau_2) := \mathbb{P} \left( \max_{i \neq j} |S_2^0(i, j)| > \tau_2 \right) := \mathbb{P} (\mathcal{E}_2) \\
\leq \left( \frac{n}{2} \right) 2 \exp \left( -c \min \left( \frac{\tau_2^2}{M_2}, \frac{\tau_2^2}{a_{\infty}^2} \right) \right) \leq \frac{1}{(n \vee m)^4}
\]
which holds for \( C_2^2 \) sufficiently large. Then on event \( \mathcal{E}_c^c \), for positive weights \( w_{i,j}^e \geq 0 \), we have for all \( q, h \in \mathbb{S}^{n-1} \),
\[
|W_2^0 - \mathbb{E}W_2^0| = \left| \sum_{(i,j), i \neq j} w_{i,j}^e \left( \sum_{s=1}^{m} a_{ss}^2 u_i^s u_j^s - \sum_{s=1}^{m} a_{ss}^2 p_s^2 \right) \right| \\
\leq \left| \sum_{j \neq i} w_{i,j}^e \left( \sum_{s=1}^{m} a_{ss}^2 u_i^s u_j^s - \sum_{s=1}^{m} a_{ss}^2 p_s^2 \right) \right| \leq \sum_{j \neq i} w_{i,j}^e \tau_2 \\
\leq 2b_\infty^2 \tau_2 \leq b_\infty^2 \sum_{s=1}^{m} a_{ss}^2 p_s^2.
\]
\( \square \)

\textbf{L.2 Proof of Lemma \textbf{J.3}}

By Corollary \textbf{L.3}, the following holds for \( C_2 \) large enough and
\[
\tau_3 = C_2 a_{\infty}^2 \log(n \vee m) \vee (a_{\infty} \sqrt{\log(n \vee m) \sum_{j=1}^{m} a_{jj}^2 p_j^3}),
\]
we have by the union bound,
\[
\mathbb{P} (|S_3^0| > \tau_3) = \mathbb{P} \left( \max_{i \neq j \neq k} |S_3^0(i, j, k)| > \tau_3 \right) := \mathbb{P} (\mathcal{E}_3) \\
\leq \left( \frac{n}{3} \right) 2 \exp \left( -c \min \left( \frac{\tau_3^2}{a_{\infty}^2 \sum_{s=1}^{m} a_{ss}^2 p_s^2}, \frac{\tau_3^2}{a_{\infty}^2} \right) \right) \leq \frac{1}{3(n \vee m)^4}
\]
By \( \textbf{89} \) and Lemma \textbf{K.3} we have for all \( q, h \in \mathbb{S}^{n-1} \), on event \( \mathcal{E}_3^c \)
\[
|W_3^{\text{diag}} - \mathbb{E}W_3^{\text{diag}}| \leq \sum_{i \neq j \neq k} |\Delta(i,j), (i,k)| |S_3^0(i, j, k)| \\
\leq 2C_2 (\|B_0\|_2 b_\infty + \|B_0\|_2^2) \left( a_{\infty} \sqrt{\log(n \vee m) S_3} \vee a_{\infty}^2 \log(n \vee m) \right) \\
\leq \|B_0\|_2^2 S_3 + C_3 a_{\infty}^2 \log(n \vee m) (\|B_0\|_2^2 + \|B_0\|_2^2)
\]
where for $b_\infty \leq \|B_0\|_2 \land \|B_0\|_2$, where $B_0 = (|b_{ij}|),$

$$2C_2(\|B_0\|_2 b_\infty + \|B_0\|_2^2) a_\infty \sqrt{\log(n \lor m) S_3}$$

$$\leq 4C_2(\|B_0\|_2 \lor \|B_0\|_2) \|B_0\|_2 a_\infty \sqrt{\log(n \lor m) S_3}$$

$$\leq \|B_0\|_2^2 S_3 + 4C_2^2 a_\infty^2 \log(n \lor m) (\|B_0\|_2^2 \lor \|B_0\|_2^2)$$

while $2C_2(\|B_0\|_2 b_\infty + \|B_0\|_2^2) a_\infty \log(n \lor m)$

$$\leq 2C_2 \|B_0\|_2^2 b_\infty a_\infty^2 \log(n \lor m) + 2C_2 \|B_0\|_2^2 a_\infty^2 \log(n \lor m). \quad \Box$$

### L.3 Proof of Lemma J.4

Denote by $M_4 \asymp a_\infty^2 \sum_{s=1}^m a_s^2 p_s^4$. By Corollary L.3, we have for $\tau_4 = C_4(\sqrt{M_4 \log(n \lor m)} \lor a_\infty^2 \log(n \lor m))$, where $C_4$ is chosen large enough so that

$$\mathbb{P}(|S_4^*| > \tau_4) = \mathbb{P}(\exists \text{distinct } i, j, k, \ell : |S_4^*(i, j, k, \ell)| > \tau_4)$$

$$\leq \left(\begin{array}{c} n \end{array}\right)^2 \exp \left(-c \min \left(\frac{\tau_4^2}{M_4}, \frac{\tau_4}{a_\infty}\right)\right)$$

$$\leq \frac{1}{12} \exp \left(-4 \log(n \lor m)\right) \leq \frac{1}{12(n \lor m)^4}$$

Denote the above exception event as $E_4$. Now, we have on event $E_4^c$ and for all $q, h \in \mathbb{S}^{n-1},$

$$\left|W_4^{\text{diag}} - \mathbb{E}W_4^{\text{diag}}\right| \leq \sum_{i \neq k} |b_{ki}| |q_k| |q_i| \sum_{j \neq \ell, k \neq \ell \neq i} |b_{j\ell}| |h_j| |h_\ell| |S_4^*(i, j, k, \ell)|$$

$$\leq \|\text{offd}(|B_0|)\|_2^2 |S_4^*| \leq \|\text{offd}(|B_0|)\|_2^2 \tau_4. \quad \Box$$

### L.4 Large deviation bound on $W_4^\circ$

Our goal in this section is to prove Lemma J.5. Let $i \neq j$ and $k \neq \ell$.

#### L.4.1 Proof of Lemma J.5

We can now apply Theorem L.2 with $\tau_5 = C_5 \|A_0\|_2 \left(\|A_0\|_2 \log(n \lor m) \lor \sqrt{\log(n \lor m) \sum_{s \neq t} a_{st}^2 p_s^2 p_t^2}\right),$

$$\mathbb{P}\left(|S_5^*| > \tau_5\right) = \mathbb{P}\left(\exists i \neq j, k \neq \ell, \sum_{i,j,k,\ell} S_\circ(i, j, k, \ell) \geq t\right) =: \mathbb{P}(E_5)$$

$$\leq \left(\begin{array}{c} n \end{array}\right)^2 \exp \left(-c \min \left(\frac{\tau_5^2}{\|A_0\|_2^2 \sum_{s \neq t} a_{st}^2 p_s^2 p_t^2}, \frac{C_5 \|A_0\|_2^3 \log(n \lor m)}{\|A_0\|_2^2}\right)\right)$$

$$\leq \frac{1}{2(n \lor m)^4}$$

for $C_5$ large enough. Hence on event $E_5^c$, for $|B_0| = (|b_{ij}|)$ and all $q, h \in \mathbb{S}^{n-1},$

$$\left|W_4^\circ - \mathbb{E}W_4^\circ\right| \leq \sum_{i \neq j, k \neq \ell} |w_\circ(i, j, k, \ell)| |S_\circ(i, j, k, \ell)|$$

$$\leq \sum_{i,j,k} \sum_{j \neq i, \ell \neq k} \left(|b_{ki}| |q_k| |q_i| |b_{j\ell}| |h_j| |h_\ell| |S_\circ(i, j, k, \ell)|\right) \leq \|B_0\|_2^2 \tau_5. \quad \Box$$

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L.4.2 Proof of Theorem L.2

We obtain the following estimate on the moment generating function of \( \mathbf{S}_\circ(i,j,k,\ell) \). Although we give explicit constants here, they are by no means optimized. This result may be of independent interests. In fact, we present a proof aiming for clarity rather than optimality of the constants being involved.

Lemma L.4. For all \( |\lambda| \leq \frac{1}{32 \|A_0\|_2^2} \) and \( C_{12} = 65 e^{1/4} \), we have for all quadruple \((i,j,k,\ell)\), where \( i \neq j \) and \( k \neq \ell \),

\[
\mathbb{E} \exp(\lambda (\mathbf{S}_\circ(i,j,k,\ell) - \mathbb{E} \mathbf{S}_\circ(i,j,k,\ell))) \leq \exp \left( C_{12} \lambda^2 \|A_0\|_2^2 \sum_{s \neq t} a_{st}^2 p_{st}^2 p_{st}^2 \right)
\]

Proof of Theorem L.2. Fix \( t > 0 \). Fix \( i \neq j \) and \( k \neq \ell \). Denote by \( \mathbf{S}_\circ = \mathbf{S}_\circ(i,j,k,\ell) \). By Lemma L.4, we have for \( t > 0 \), \( 0 < \lambda \leq \frac{1}{32 \|A_0\|_2^2} \) and \( C_{12} = 65 e^{1/4} \),

\[
\mathbb{P}(\mathbf{S}_\circ \geq t) = \mathbb{P}(\lambda(\mathbf{S}_\circ) \geq \lambda t) = \mathbb{P}(\exp(\lambda(\mathbf{S}_\circ)) \geq \exp(\lambda t)) \leq \frac{\mathbb{E} \exp(\lambda(\mathbf{S}_\circ))}{e^{\lambda t}} \leq \exp \left( -\lambda t + \lambda^2 C_{12} \|A_0\|_2^2 \sum_{s \neq t} a_{st}^2 p_{st}^2 p_{st}^2 \right).
\]

Optimizing over \( \lambda \), we have for \( \mathbf{S}_\circ = \mathbf{S}_\circ(i,j,k,\ell) \),

\[
\mathbb{P}(\mathbf{S}_\circ \geq t) \leq \exp \left( -c \min \left( \frac{t^2}{\|A_0\|_2^2 \sum_{s \neq t} a_{st}^2 p_{st}^2 p_{st}^2 \|A_0\|_2^2}, \frac{t}{\|A_0\|_2^2} \right) \right) =: p_1
\]

while noting that for \( 0 < \lambda \leq \frac{1}{32 \|A_0\|_2^2} \) and \( t > 0 \), and repeating the same arguments above,

\[
\mathbb{P}(\mathbf{S}_\circ < -t) = \mathbb{P}(\exp(\lambda(-\mathbf{S}_\circ)) > \exp(\lambda t)) \leq \frac{\mathbb{E} \exp(-\lambda(\mathbf{S}_\circ))}{e^{\lambda t}} \leq p_1;
\]

Hence \( \mathbb{P}(\mathbf{S}_\circ > t) \leq 2p_1 \) and the theorem is thus proved. □

L.4.3 Proof of Lemma L.4

Denote by \( \xi = v^i \circ v^j \) and \( \xi' = v^k \circ v^\ell \) in the following steps, where \( i \neq j, k \neq \ell \). Then

\[
\mathbf{S}_\circ(i,j,k,\ell) := (v_i \circ v_j)^T \text{offd}(A_0 \circ A_0)(v_k \circ v_\ell)
\]

\[
= \sum_{s \neq t} a_{st}^2 u_s^i u_j^i u_k^t u_\ell^t = \sum_{t=1}^m \xi_t' \sum_{s \neq t} a_{st}^2 \xi_s
\]

First, we compute the expectation for

\[
\forall i \neq j, k \neq \ell, \quad \mathbb{E} \mathbf{S}_\circ(i,j,k,\ell) = \sum_{i \neq j} a_{ij}^2 p_i^2 p_j^2
\]

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Notice that the two vectors $\xi$ and $\xi'$ may not be independent, since $(i, j)$ and $(k, \ell)$ can have overlapping vertices; however, when we partition $U$ into disjoint submatrices $(U)_\Lambda := \{u^s\}_{s \in \Lambda}$ and $(U)_{\Lambda^c} := \{u^t\}_{t \in \Lambda^c}$, each formed by extracting columns of $U$ indexed by $\Lambda \subset [m]$ and its complement set $\Lambda^c$ respectively, then

$$\forall i \neq j, k \neq \ell, u^s, u^s_j, u^t_k, u^t_\ell, \ s \in \Lambda, \ t \in \Lambda^c$$

are mutually independent Bernoulli random variables; and hence each monomial $\xi'_t := u^s_k u^t_\ell, t \in \Lambda^c$ is independent of the sum $\sum_{s \in \Lambda} a^2_{st} u^s_k u^t_\ell$.

**Decoupling.** Consider independent Bernoulli random variables $\delta_i \in \{0, 1\}$ with $\mathbb{E} \delta_i = 1/2$. Let $\mathbb{E}_\delta$ denote the expectation with respect to Bernoulli random vector $\delta = (\delta_1, \ldots, \delta_m)$. Since $\mathbb{E}_\delta(1 - \delta_j) = 1/4$ for $i \neq j$ and 0 for $i = j$, we have

$$S_\circ := 4 \mathbb{E}_\delta S_{\delta} \text{ where } S_{\delta} := \sum_{s, t} \delta_s (1 - \delta_t) a^2_{st} u^s_k u^t_\ell$$

Let $\Lambda_\delta = \{i \in [m] : \delta_i = 1\}$ and $\Lambda^c_\delta$ be the complement of $\Lambda_\delta$. First notice that we can express

$$S_{\delta} := \sum_{s \in \Lambda_\delta} \sum_{t \in \Lambda^c_\delta} a^2_{st} u^s_i u^t_j u^t_k u^t_\ell = \sum_{t \in \Lambda^c_\delta} u^t_k u^t_\ell \sum_{s \in \Lambda_\delta} a^2_{st} u^s_i u^s_j$$

Hence

$$\exp(\lambda \mathbb{E} S_\circ) = \exp(4\lambda \mathbb{E}_U, S_{\delta}) =: \exp(4\lambda \mathbb{E}_U S_{\delta} | \delta) \text{ where}$$

$$\mathbb{E}(S_{\delta} | \delta) = \mathbb{E}\left(\sum_{s \in \Lambda_\delta} \sum_{t \in \Lambda^c_\delta} a^2_{st} u^s_i u^t_j u^t_k u^t_\ell | \delta\right) = \sum_{s \in \Lambda_\delta} \sum_{t \in \Lambda^c_\delta} a^2_{st} p^2_{st} p^2_{\ell t}$$

and in (95), $\mathbb{E}_\delta$ denotes the expectation with respect to the random vector $\delta = (\delta_1, \ldots, \delta_m)$, or equivalently, the random set of indices in $\Lambda_\delta$, and $\mathbb{E}_{U, \delta}$ denotes expectation with respect to both $U$ and $\delta$. Now by (94) and (95), we have for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}\exp(\lambda(S_\circ - \mathbb{E} S_{\delta})) = \mathbb{E}_U \exp(4\lambda \mathbb{E}_\delta(S_{\delta}) - \mathbb{E}_\delta \mathbb{E}(S_{\delta} | \delta))$$

$$= \mathbb{E}_U \exp \{4\lambda \mathbb{E}_\delta[S_{\delta} - \mathbb{E}(S_{\delta} | \delta)]\} =: \mathbb{E}_U \exp \{4\lambda g(U)\}$$

where the random function $g(U)$ is defined as follows:

$$g(U) = \mathbb{E}_\delta[S_{\delta} - \mathbb{E}(S_{\delta} | \delta)]$$

$$= \mathbb{E}_{\Lambda_\delta}\left(\sum_{t \in \Lambda^c_\delta} u^t_k u^t_\ell \sum_{s \in \Lambda_\delta} a^2_{st} u^s_j - \sum_{t \in \Lambda^c_\delta} \sum_{s \in \Lambda_\delta} a^2_{st} p^2_{st} |U\right)$$

while

$$\mathbb{E}_\delta S_{\delta} = \mathbb{E}_{\Lambda_\delta}\left(\sum_{t \in \Lambda^c_\delta} u^t_k u^t_\ell \sum_{s \in \Lambda_\delta} a^2_{st} u^s_j |U\right)$$

Hence we have reduced the original problem to the new problem of computing the moment generating function for $g(U)$.

**Centering.** Denote by

$$Z'_t := \xi'_t - p^2_{st} = u^t_k u^t_\ell - p^2_{st} \text{ and } Z_s := \xi_s - p^2_s = u^s_k u^s_j - p^2_s$$
Fix $\delta$. First, we express the decoupled quadratic form involving centered random variables with

$$
\sum_{s \in \Lambda_0} \sum_{t \in \Lambda_0^c} a_{st}^2 Z_s Z_t' := \sum_{s \in \Lambda_0} \sum_{t \in \Lambda_0^c} a_{st}^2 (\xi_s - \mu_s)(\xi'_t - \mu_t')
$$

$$
= \sum_{s \in \Lambda_0} \sum_{t \in \Lambda_0^c} a_{st}^2 (\xi_s\xi'_t - (\xi_s - \mu_s)\mu_{t'} - \mu_s\xi'_t - \mu_s\mu_{t'})
$$

where $\{\xi'_t - \mu_t\} \Lambda_0^c$ and $\{\xi - \mu_s\} \Lambda_0$ are each centered and mutually independent random vectors. Hence we can now express $S_\delta - \mathbb{E}(S_\delta|\delta)$ as sum of quadratic and linear forms based on centered random vectors with independent mean-zero coordinates:

$$
S_\delta - \mathbb{E}(S_\delta|\delta) = \sum_{t \in \Lambda_0^c} \xi'_t \sum_{s \in \Lambda_0} a_{st}^2 \xi_s - \sum_{t \in \Lambda_0^c} \mu_{t'} \sum_{s \in \Lambda_0} a_{st}^2 \mu_s
$$

$$
= \sum_{s \in \Lambda_0} \sum_{t \in \Lambda_0^c} a_{st}^2 (\xi_s\xi'_t - \mu_s\mu_{t'})
$$

$$
= \sum_{s \in \Lambda_0} \sum_{t \in \Lambda_0^c} a_{st}^2 Z_s Z_t' + \sum_{s \in \Lambda_0} \sum_{t \in \Lambda_0^c} p_{t'}^2 a_{st}^2 Z_s + \sum_{s \in \Lambda_0} \sum_{t \in \Lambda_0^c} a_{st}^2 Z_t'
$$

$$
= Q_1 + L_1 + L_2
$$

(99)

where for each fixed $\delta$, $Q_1, L_1, L_2$ are quadratic and linear terms involving mean-zero independent random variables in $\{Z_s\}_{s \in \Lambda_0}$ and $\{Z_t'\}_{t \in \Lambda_0^c}$, which are in turn mutually independent. Hence by (97), (98) and (99),

$$
\mathbb{E}_U \exp (4\lambda g(U)) = \mathbb{E}_U \exp (4\lambda \mathbb{E}_\delta[S_\delta - \mathbb{E}(S_\delta|\delta)])
$$

$$
= \mathbb{E}_U \exp (4\lambda \mathbb{E}_\delta(Q_1 + L_1 + L_2))
$$

$$
= \mathbb{E}_U \left( \exp (4\lambda \mathbb{E}_\delta(Q_1)) \exp (4\lambda \mathbb{E}_\delta(L_1 + L_2)) \right)
$$

$$
\leq \left( \mathbb{E}_U \mathbb{E}_\delta(Q_1) \right) 1/2 \left( \mathbb{E}_U \mathbb{E}_\delta(L_1 + L_2) \right) 1/2
$$

(100)

where $\mathbb{E}_\delta$ denotes the conditional expectation with respect to randomness in $\delta$ for fixed $U$; The second equality holds by linearity of expectations, $\mathbb{E}_\delta(Q_1 + L_1 + L_2) = \mathbb{E}_\delta(Q_1) + \mathbb{E}_\delta(L_1 + L_2)$, and (100) follows from the Cauchy-Schwartz inequality.

**Computing moment generating functions.** We have by Jensen’s inequality and Fubini theorem, for $|\lambda| \leq \frac{1}{32\|A_0\|_2}$,

$$
\mathbb{E}_U \exp (8\lambda \mathbb{E}_\delta(Q_1)) \leq \mathbb{E}_U \mathbb{E}_\delta \exp (8\lambda Q_1)
$$

$$
= \mathbb{E}_\delta \mathbb{E}_U \exp \left( 8\lambda \left( \sum_{s \in \Lambda_0} Z_s \sum_{t \in \Lambda_0^c} a_{st}^2 Z_t' \right) \right)
$$

$$
\leq \mathbb{E}_\delta \exp \left( 65\lambda^2 \|A_0\|_2^2 e^{1/4} \left( \sum_{s \in \Lambda_0} \sum_{t \in \Lambda_0^c} p_{t'}^2 a_{st}^2 p_{t'}^2 \right) \right)
$$

$$
\leq \exp \left( 65\lambda^2 \|A_0\|_2^2 e^{1/4} \left( \sum_{s \neq t} a_{st}^2 p_{s}^2 p_{t}^2 \right) \right)
$$

(101)
where (101) follows from the proof of Theorem B.2 (cf. (109) and (111)), where we replace \( A \) with \( A_0 \circ A_0 \) while adjusting constants; The second inequality holds since for all \( \delta \) and

\[
\forall t > 0, \quad \exp \left( t \sum_{s \in \Lambda_\delta} p_s^2 \sum_{t \in \Lambda_\delta^c} a_{st}^2 p_t^2 \right) \leq \exp \left( t \sum_{s \neq t} a_{st}^2 p_s^2 p_t^2 \right)
\]

Now for any fixed \( \delta \), \( L_1 \) and \( L_2 \) are independent random variables with respect to the randomness in \( U \). By Jensen’s inequality and Fubini theorem again, for all \( \lambda \in \mathbb{R} \),

\[
\mathbb{E}_U \exp (8\lambda \mathbb{E}_\delta (L_1 + L_2)) \leq \mathbb{E}_\delta \mathbb{E}_U \exp (8\lambda (L_1 + L_2)) = \mathbb{E}_\delta \left( \mathbb{E}_U \exp (8\lambda (L_1)) \right) \mathbb{E}_U \exp (8\lambda (L_2))
\]

Thus conditioned on \( \delta \), we denote by \( d_s \) and \( d_t \), the following fixed constants respectively:

\[
\forall s \in \Lambda_\delta, \quad 0 \leq d_s := \sum_{t \in \Lambda_\delta^c} a_{st}^2 p_t^2 \leq \sum_{t=1}^m a_{st}^2 p_t^2 \leq D_{\text{max}} = \| A_0 \|_2^2 \quad (102)
\]

\[
\forall t \in \Lambda_\delta^c, \quad 0 \leq d_t := \sum_{s \in \Lambda_\delta} a_{st}^2 p_s^2 \leq \sum_{s=1}^m a_{st}^2 p_s^2 \leq D_{\text{max}} = \| A_0 \|_2^2 \quad (103)
\]

where \( \mathbb{E}_s = \mathbb{E} u_s^i u_s^i = p_s^2 \) and recall that

\[
L_1 := \sum_{s \in \Lambda_\delta} \sum_{t \in \Lambda_\delta^c} p_t^2 a_{st}^2 Z_s = \sum_{s \in \Lambda_\delta} d_s (\xi_s - p_s^2)
\]

\[
L_2 := \sum_{s \in \Lambda_\delta} p_s^2 \sum_{t \in \Lambda_\delta^c} a_{st}^2 Z_t' = \sum_{t \in \Lambda_\delta^c} d_t (\xi_t' - p_t^2)
\]

Now by Lemma L.1 we have for \( \tau = 8\lambda \leq \frac{1}{4D_{\text{max}}} \) where \( D_{\text{max}} = \| A_0 \|_2^2 \),

\[
\mathbb{E}_{U|\Lambda_\delta} \exp \left( \tau \sum_{s \in \Lambda_\delta} d_s (\xi_s - p_s^2) \right) = \prod_{s \in \Lambda_\delta} \mathbb{E} \exp \left( \tau d_s ( u_s^i u_s^i - \mathbb{E} u_s^i u_s^i) \right)
\]

\[
\leq \exp \left( \frac{1}{2} \tau^2 e^{\tau |D_{\text{max}}} \sum_{s \in \Lambda_\delta} d_s^2 p_s^2 \right) \leq \exp \left( \frac{1}{2} \tau^2 D_{\text{max}} e^{\tau |D_{\text{max}}} \sum_{s \in \Lambda_\delta} \sum_{t \in \Lambda_\delta^c} a_{st}^2 p_s^2 p_t^2 \right)
\]

Similarly, for \( d_t := \sum_{s \in \Lambda_\delta} a_{st}^2 p_s^2 \) and \( \tau = 8\lambda \)

\[
\mathbb{E}_{U|\Lambda_\delta} \exp (8\lambda (L_2)) := \mathbb{E}_{U|\Lambda_\delta} \exp \left( 8\lambda \sum_{t \in \Lambda_\delta^c} d_t Z_t' \right)
\]

\[
\leq \exp \left( \frac{1}{2} \tau^2 e^{\tau |D_{\text{max}}} \sum_{t \in \Lambda_\delta^c} d_t^2 p_t^2 \right) \leq \exp \left( \frac{1}{2} \tau^2 D_{\text{max}} e^{\tau |D_{\text{max}}} \sum_{t \in \Lambda_\delta^c} \sum_{s \in \Lambda_\delta} a_{st}^2 p_s^2 p_t^2 \right)
\]

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Hence for $|\lambda| \leq \frac{1}{32\|A_0\|_2}$ and $|\tau| \leq \frac{1}{4D_{\text{max}}}$
\[
\mathbb{E}_U \exp (8\lambda\mathbb{E}_d(L_1 + L_2)) \leq \mathbb{E}_\delta (\mathbb{E}_U \exp (8\lambda(L_1)) \mathbb{E}_U \exp (8\lambda(L_2))) \\
\leq \mathbb{E}_\delta \left( \exp \left( \tau^2 D_{\text{max}} e^{\tau|D_{\text{max}}} \sum_{s \in A_\delta} \sum_{t \in A_\delta^s} a_{st}^2 p_s^2 p_t^2 \right) \right) \\
\leq \exp \left( 64e^{1/4} \lambda^2 D_{\text{max}} \sum_{s \neq t} a_{st}^2 p_s^2 p_t^2 \right) \\
\leq \exp \left( 65e^{1/4} \lambda^2 D_{\text{max}} \sum_{s \neq t} a_{st}^2 p_s^2 p_t^2 \right) \\
(104)
\]

**Putting things together.** Hence by (97), (100), (101) and (104), we have for all $|\lambda| \leq \frac{1}{32\|A_0\|_2}$,
\[
\mathbb{E} \exp(\lambda(S_0 - \mathbb{E}S_0)) = \mathbb{E}_U \exp \{4\lambda g(U)\} \\
\leq (\mathbb{E}_U \exp (8\lambda\mathbb{E}_d|Q_1|))^{1/2} (\mathbb{E}_U \exp (8\lambda\mathbb{E}_d|L_1 + L_2|))^{1/2} \\
\leq \exp \left( 65\lambda^2 \|A_0\|_2^2 e^{1/4} \left( \sum_{s \neq t} a_{st}^2 p_s^2 p_t^2 \right) \right) \\
\leq \exp \left( 65e^{1/4} \lambda^2 D_{\text{max}} \sum_{s \neq t} a_{st}^2 p_s^2 p_t^2 \right)
\]

\[\square\]

**M Proof of Theorem B.2**

We first state the following Decoupling Theorem M.1, which follows from Theorem 6.1.1 [38].

**Theorem M.1.** [38] Let $A$ be an $m \times m$ matrix. Let $X = (X_1, \ldots, X_m)$ be a random vector with independent mean zero coordinates $X_i$. Then, for every convex function $F : \mathbb{R} \mapsto \mathbb{R}$, one has
\[
\mathbb{E} F(\sum_{i \neq j} a_{ij}X_iX_j) \leq \mathbb{E} F(\sum_{i \neq j} a_{ij}X_iX'_j).
\]

where $X'$ is an independent copy of $X$.

We use the following bounds throughout our paper. For any $x \in \mathbb{R}$,
\[
e^x \leq 1 + x + \frac{1}{2} x^2 e^{x^2}.
\]

(106)

Let $Z_i := \xi_i - p_i$. Denote by $\sigma_i^2 := p_i(1 - p_i)$. For all $Z_i$, we have $|Z_i| \leq 1$, $\mathbb{E}Z_i = 0$ and
\[
\mathbb{E}Z_i^2 = (1 - p_i)^2 p_i + p_i^2 (1 - p_i) = p_i(1 - p_i) = \sigma_i^2, \\
\mathbb{E}|Z_i| = (1 - p_i)p_i + p_i(1 - p_i) = 2p_i(1 - p_i) = 2\sigma_i^2.
\]

(107) (108)

**Proof of Theorem B.2** Denote by $\tilde{a}_i := \sum_{j \neq i}(a_{ij} + a_{ji})p_j + a_{ii} \leq 2D_{\text{max}}$. We express the quadratic form as follows:
\[
\sum_{i=1}^m a_{ii}(\xi_i - p_i) + \sum_{i \neq j} a_{ij}(\xi_i \xi_j - p_i p_j) = \sum_{i \neq j} a_{ij}Z_i Z_j + \sum_{j=1}^m Z_j \tilde{a}_i := S_1 + S_2.
\]

(61)
We will show the following bounds on the moment generating functions of $S_1$ and $S_2$: for every $|\lambda| \leq \frac{1}{16\|A\|_1 \vee \|A\|_\infty}$,

\[
\mathbb{E} \exp(\lambda S_1) \leq \exp \left( 65\lambda^2 \|A\|_\infty e^{8\|A\|_\infty} \sum_{i \neq j} |a_{ij}|^2 \sigma_i^2\right) \quad \text{and} \quad \quad (109)
\]

\[
\mathbb{E} \exp(\lambda S_2) \leq \exp \left( 4\lambda^2 D_{\max} e^{4\|A\|_{\max}} \left( 2 \sum_{i \neq j} |a_{ij}| p_i p_j + \sum_{i=1}^m |a_{ii}| \sigma_i^2 \right) \right). \quad (110)
\]

The estimate on the moment generating function for $\sum_{i,j} a_{ij} \xi_i \xi_j$ then follows immediately from the Cauchy-Schwartz inequality.

**Bounding the moment generating function for $S_1$.** In order to bound the moment generating function for $S_1$, we start by a decoupling step following Theorem M.1. Let $Z'$ be an independent copy of $Z$.

**Decoupling.** Now consider random variable $S_1 := \sum_{i \neq j} a_{ij} (\xi_i - p_i)(\xi_j - p_j) = \sum_{i \neq j} a_{ij} Z_i Z_j$ and

\[S'_1 := \sum_{i \neq j} a_{ij} Z'_i Z'_j,\]

we have \(\mathbb{E} \exp(2\lambda S_1) \leq \mathbb{E} \exp(8\lambda S'_1) =: f\) by (105). Thus we have by independence of $Z_i$,

\[
f := \mathbb{E}_{Z'} \mathbb{E}_Z \exp \left( 8\lambda \sum_{i=1}^m Z_i \sum_{j \neq i} a_{ij} Z'_j \right) = \mathbb{E}_{Z'} \prod_{i=1}^m \mathbb{E} \left( \exp (8\lambda Z_i \tilde{a}_i) \right),
\]

where $Z'_j, \forall j$ satisfies \(|Z'_j| \leq 1\), and

\[
\forall i, \quad \tilde{a}_i := \sum_{j \neq i} a_{ij} Z'_j \quad \text{and hence} \quad |\tilde{a}_i| \leq \|A\|_\infty. \quad (112)
\]

First consider $Z'$ being fixed. Recall $Z_i, \forall i$ satisfies: $|Z_i| \leq 1$, $\mathbb{E}Z_i = 0$ and $\mathbb{E}Z_i^2 = \sigma_i^2$. Then by (106), for all $|\lambda| \leq \frac{1}{16\|A\|_{\max}}$ and $t_i := 8\lambda \tilde{a}_i$,

\[
\mathbb{E} \exp (8\lambda \tilde{a}_i Z_i) := \mathbb{E} \exp (t_i Z_i) \leq 1 + \frac{1}{2} t_i^2 \mathbb{E} Z_i^2 e^{t_i |Z_i|} \leq \exp \left( \frac{1}{2} t_i^2 e^{t_i |Z_i|^2} \right)
\]

\[
\leq \exp \left( 32\lambda^2 \|A\|_\infty e^{8\|A\|_\infty} |\tilde{a}_i| \sigma_i^2 \right)
\]

\[
=: \exp (\tau' |\tilde{a}_i| \sigma_i^2) \quad \text{for} \quad \tau' := 32\lambda^2 \|A\|_\infty e^{8\|A\|_\infty} \geq 0, \quad (113)
\]

where by (112)

\[
\frac{1}{2} t_i^2 e^{t_i |Z_i|} \leq \frac{1}{2} (8^2 \lambda^2 \sigma_i^2)^2 e^{8\lambda |\tilde{a}_i|} \leq 32\lambda^2 \|A\|_\infty \tilde{a}_i e^{8\|A\|_\infty}
\]

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Denote by
\[ |\bar{a}_j| := \sum_{i \neq j} |a_{ij}| \sigma_i^2 \leq \|A\|_1 / 4, \quad j = 1, \ldots, m. \quad (115) \]

Thus by (111) and (113)
\[
f \leq \mathbb{E}_{Z'} \prod_{i=1}^{m} \exp \left( \tau' |\bar{a}_i| \sigma_i^2 \right) \leq \mathbb{E}_{Z'} \exp \left( \tau' \sum_{i=1}^{m} \sigma_i^2 \sum_{j \neq i} |a_{ij}| |Z'_j| \right)
= \prod_{j=1}^{m} \mathbb{E} \exp \left( \tau' |Z'_j| \sum_{i \neq j} |a_{ij}| \sigma_i^2 \right) =: \prod_{j=1}^{m} \mathbb{E} \exp \left( \tau' |\bar{a}_j| |Z'_j| \right)
\]

where \( \mathbb{E}(Z'_j)^2 = \sigma_j^2 \) and \( \mathbb{E} |Z'_j| = 2 \sigma_j^2 \) following (107) and (108). Denote by
\[ \hat{\ell}_j := \tau' |\bar{a}_j| = 32 \lambda^2 \|A\|_{\infty} e^{8\|A\|_{\infty}} |\bar{a}_j| > 0, \]

we have by (115) and for \( |\lambda| \leq \frac{1}{16(\|A\|_1 \vee \|A\|_{\infty})} \),
\[ \hat{\ell}_j := 32 \lambda^2 \|A\|_{\infty} |\bar{a}_j| e^{8\|A\|_{\infty}} \leq \frac{\|A\|_{\infty} \|A\|_1 e^{8\|A\|_{\infty}}}{32(\|A\|_1 \vee \|A\|_{\infty})^2} \leq \frac{e^{1/2}}{32} \approx 0.052; \]

Thus we have by the elementary approximation (106), (109) holds,
\[
\mathbb{E} \exp \left( \hat{\ell}_j |Z'_j| \right) \leq 1 + \mathbb{E} \left( \hat{\ell}_j |Z'_j| \right) + \frac{1}{2} (\hat{\ell}_j)^2 \mathbb{E} (Z'_j)^2 e^{\hat{\ell}_j} \\
\leq \exp \left( 2\hat{\ell}_j \sigma_j^2 + \frac{1}{2} (\hat{\ell}_j)^2 \sigma_j^2 e^{0.052} \right) \quad \text{using the inequality of } x \leq e^x, \\
\leq \exp \left( 2\hat{\ell}_j \sigma_j^2 + 0.026\hat{\ell}_j \sigma_j^2 e^{0.052} \right) \leq \exp \left( 2.03\hat{\ell}_j \sigma_j^2 \right) \\
\leq \exp \left( 65\lambda^2 \|A\|_{\infty} e^{8\|A\|_{\infty}} \sum_{i \neq j} |a_{ij}| \sigma_i^2 \sigma_j^2 \right) \quad \text{so long as } |\lambda| \leq \frac{1}{16(\|A\|_1 \vee \|A\|_{\infty})}. 
\]

Bounding the moment generating function for \( S_2 \). Recall
\[
S_2 := \sum_{i=1}^{m} Z_i \left( \sum_{j \neq i} (a_{ij} + a_{ji}) p_j + a_{ii} \right) =: \sum_{i=1}^{m} Z_i \bar{a}_i. 
\]

Let \( a_\infty := \max_i |\bar{a}_i| \leq \|A\|_{\infty} + \|A\|_1 \leq 2D_{\max} \). Thus we have by Lemma 1.1 for all \( |\lambda| \leq \frac{1}{16(\|A\|_1 \vee \|A\|_{\infty})} \),
\[
\mathbb{E} \exp \left( 2\lambda \sum_{i=1}^{m} Z_i \bar{a}_i \right) \leq \exp \left( 2\lambda^2 e^{2|\lambda| a_\infty} \sum_{i=1}^{m} \bar{a}_i^2 \sigma_i^2 \right) \\
\leq \exp \left( 2\lambda^2 a_\infty e^{2|\lambda| a_\infty} \sum_{i=1}^{m} |\bar{a}_i| \sigma_i^2 \right) \\
\leq \exp \left( 2\lambda^2 a_\infty e^{2|\lambda| a_\infty} 2 \left( \sum_{i=1}^{m} \sum_{j \neq i} |a_{ij}| p_j + \sum_{i=1}^{m} |a_{ii}| \sigma_i^2 \right) \right)
\]

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where \( \forall i, |\tilde{a}_i| = |\sum_{j \neq i} (a_{ij} + a_{ji}) p_j + a_{ii}| \) and hence
\[
\sum_{i=1}^{m} |\tilde{a}_i| \sigma_i^2 \leq 2\sum_{i=1}^{m} \sum_{j \neq i} |a_{ij}| p_j p_j + \sum_{i=1}^{m} |a_{ii}| \sigma_i^2 \quad \text{where} \quad \sigma_i^2 = p_i(1 - p_i)
\]
Thus (110) holds for all \( |\lambda| \leq \frac{1}{16(\|A\|_\infty \vee \|A\|_1)} \). Hence by the Cauchy-Schwartz inequality, in view of (109) and (110),
\[
\mathbb{E} \exp \left( \lambda \left( \sum_{i=1}^{m} a_{ii}(\xi_i - p_i) + \sum_{i \neq j} a_{ij}(\xi_i \xi_j - p_ip_j) \right) \right) = \mathbb{E} \exp (\lambda(S_1 + S_2)) \leq (\mathbb{E} \exp(2\lambda S_1))^{1/2}(\mathbb{E} \exp(2\lambda S_2))^{1/2}
\]
for all \( |\lambda| \leq \frac{1}{16(\|A\|_\infty \vee \|A\|_1)} = \frac{1}{16D_{\max}} \).

The theorem is thus proved since \( \|A\|_1 \vee \|A\|_\infty = D_{\max} \). \( \square \)

### N Proof of Lemma 6.1

Throughout this section, let \( Y = X^T \), where \( X = B_0^{1/2}ZA_0^{1/2} \) is as defined in (8). Denote by \( D_0 = B_0^{1/2} \otimes A_0^{1/2} \). We have \( D_0^2 = B_0 \otimes A_0 \). Let \( Z = \text{vec} \{ Z^T \} \) for \( Z \) as defined in (8). Then \( Z \in \mathbb{R}^{mn} \) is a subgaussian random vector with independent components \( Z_j \) that satisfy \( \mathbb{E}Z_j = 0, \mathbb{E}Z_j^2 = 1, \text{and } \|Z_j\|_{\psi_2} \leq 1 \).

As shown in (57),
\[
\frac{1}{n}\text{tr}(X^T X) = \frac{1}{n}\mathbb{E}\text{tr}(X^T X^T) = \frac{1}{n}\mathbb{E} \sum_{i=1}^{n} \|v^i \circ y^i\|_2^2 = \frac{\text{tr}(B_0)}{n} \sum_{i=1}^{m} a_{ii}p_i.
\]
Hence \( \frac{1}{n}\text{tr}(X^T X) \) provides an unbiased estimator for entries in \( \text{diag}(M) \) for \( \text{tr}(B_0) = n \).

### N.1 Unbiased estimator for the mask matrix: off-diagonal component

Let \( \{e_1, \ldots, e_m\} \in \mathbb{R}^m \) be the canonical basis of \( \mathbb{R}^m \). Denote by
\[
\mathcal{D}_0 = \sum_{k=1}^{n} \text{diag}(e_k) \otimes D_k \quad \text{where}
\]
(116)
\[
D_k = \frac{1}{n-1} \sum_{j \neq k} \text{diag}(v^k \otimes v^j), \forall k = 1, \ldots, n
\]
(117)

Then for \( S_c \) as defined in (58), we have for \( M^i = v^i \otimes v^i \) and hence \( \text{diag}(M^i) = \text{diag}(v^i) \),
\[
S_c := \frac{1}{n-1} \sum_{j=1}^{n} \sum_{k \neq j} \text{tr} \left( M^k \circ M^j \circ (y^j \otimes y^j) \right)
\]
\[
= \frac{1}{n-1} \sum_{j=1}^{n} \sum_{k \neq j} (y^j)^T \text{diag}(v^k \circ v^i) y^j = \sum_{j=1}^{n} (y^j)^T D_j y^j = \text{vec} \{ Y \}^T \mathcal{D}_0 \text{vec} \{ Y \}
\]
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Indeed, we can compute for $E_Z$ where $D$ and $A_v := D_0 D_v D_0$, 
\[ S_c = Z^T D_0^{1/2} \otimes A_0^{1/2} \otimes D_v B_0^{1/2} \otimes A_0^{1/2} Z =: Z^T A_v Z \]
where $Z = \text{vec} \{ Z^T \}$ for $Z$ as defined in (8). It is not difficult to verify that $E S_c = \text{tr}(B_0) \sum_{i=1}^m a_{ii} p_i^2$. Indeed, we can compute for $E Z^2_1 = 1$ and $E Z_j = 0$,
\[ E S_c = E E(Z^T D_0 D_v D_0 Z | U) = \text{etr}(D_0 D_v D_0) \]
\[ = \frac{1}{n-1} \sum_{j=1}^n \sum_{i \neq j} b_{ij} \text{tr}(A_0 E \text{diag}(v^i \circ v^j)) = \text{tr}(B_0) \sum_{j=1}^m a_{jj} p_j^2 \]

**Decomposition.** We now decompose the error into two parts:
\[ |S_c - ES_c| = |Z^T A_v Z - E(Z^T A_v Z)| \leq |Z^T A_v Z - E(Z^T A_v Z | U)| + |E(Z^T A_v Z | U) - E(Z^T A_v Z)| =: I + II. \quad (118) \]

**Part I.** Since $D_v$ (116) is a block diagonal matrix with the $k^{th}$ block along the diagonal being $D_k, \forall k = 1, \ldots, m$ (117), with entries in $[0, 1]$, we have its its operator norm, row and column $\ell_1$-norms all bounded by 1 and thus
\[ \|A_v\|_2 := \|D_0 D_v D_0\|_2 \leq \|D_0\|_2 \|D_v\|_2 \leq \|A_0\|_2 \|B_0\|_2 \quad (119) \]

Lemma N.1 shows that tight concentration of measure bound on $\|A_v\|_F$ can be derived under a mild condition such as (120); Since the proof follows a similar line of arguments as in the proof of Theorem 4.3 we omit it here.

**Lemma N.1.** Suppose that (120) holds:
\[ \sum_{s=1}^m a_{ss}^2 p_s^2 \geq C a_{\infty}^2 \log(n \lor m) \quad \text{for some absolute constant } C. \quad (120) \]

Then on event $F_0^c$ as defined in Theorem 4.3, we have
\[ \|A_v\|_F^2 \leq \frac{C_1}{n-1} \|\text{diag}(B_0)\|_F^2 m \sum_{s=1}^m a_{ss}^2 p_s^2 + C_2 \|B_0\|_F^2 \sum_{s=1}^m a_{ss}^3 p_s \]
\[ + C_3 \|B_0\|_F^2 a_{\infty} \|A_0\|_2 \sum_{s=1}^m p_s^4 + C_4 \|B_0\|_F^2 \|A_0\|_2^2 \log(n \lor m) =: W_v^2 \]

Hence we set the large deviation bound to be
\[ \tau_0 = C_6 \sqrt{\log(n \lor m)} \left( \frac{2}{\sqrt{n-1}} \|\text{diag}(B_0)\|_F \sqrt{\sum_{s=1}^m a_{ss}^2 p_s^2 + \|B_0\|_F^2 \sum_{s=1}^m a_{ss}^3 p_s} \right) + \]
\[ C_7 \sqrt{\log(n \lor m)} \|B_0\|_F \left( a_{\infty} \|A_0\|_2 \sum_{s=1}^m p_s^4 + C \|A_0\|_2 \sqrt{\log(n \lor m)} \right) \]
\[ = 65 \]
By Lemma \text{N.1} and Theorem \text{B.1} for absolute constants \(c\) and \(C_6, C_7\) sufficiently large,

\[
P \left( \left| Z^T A_v Z - \mathbb{E}(Z^T A_v Z | U) \right| > \tau_0 \right) =: P(\mathcal{E}_7) = P(\mathcal{F}_0^c) + P(\mathcal{F}_0)
\]

\[
\leq 2 \exp \left( -c \min \left( \frac{\tau_0^2}{W_v^2}, \frac{\tau_0}{\|A_0\|_2 \|B_0\|_2} \right) \right) + P(\mathcal{F}_0) \leq \frac{c}{(n \vee m)^4} \tag{121}
\]

where by Theorem \text{B.1} and \text{(119)}, we have for any \(t > 0\),

\[
P \left( \left| Z^T A_v Z - \mathbb{E}(Z^T A_v Z | U) \right| > t \right| U \in \mathcal{F}_0^c \right) \leq 2 \exp \left( -c \min \left( \frac{t^2}{W_v^2}, \frac{t}{\|B_0\|_2 \|A_0\|_2} \right) \right)
\]

since on event \(\mathcal{F}_0^c\), where \(U\) is being fixed, \(\|A_v\|_F^2 \leq W_v^2\).

\textbf{Part II.} Denote by

\[
S_\star := \mathbb{E}(Z^T D_0 D_0^T D_0 | U) - \mathbb{E} \text{tr}(D_0 D_0^T)
\]

\[
:= \text{tr}(D_0 D_0^T) - \mathbb{E} \text{tr}(D_0 D_0^T)
\]

Denote by

\[
D_\star := B' \otimes \text{diag}(A_0) \quad \text{where} \quad B' := \begin{bmatrix} 0 & b_{11} & b_{12} & \ldots & b_{11} \\ b_{22} & 0 & b_{22} & \ldots & b_{22} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{nn} & b_{nn} & \ldots & \ldots & b_{nn} \\ 0 & 0 & \ldots & \ldots & 0 \end{bmatrix}_{n \times n}
\]

It is straightforward to check the following holds:

\[
\|D_\star\|_\infty \leq b_{\infty} a_{\infty} \quad \text{and} \quad \|D_\star\|_1 \leq b_{\infty} a_{\infty}.
\]

\textbf{Corollary N.2.} Let \(U\) be as defined in \text{(10)} and \(V = U^T\). Then for \(D_\star\) as defined immediately above,

\[
S_\star = \text{vec} \{ V \}^T D_\star \text{vec} \{ V \} - \text{tr}(B_0) \sum_{s=1}^m a_{ss} p_s^2
\]

Let \(A_0 = (a_{ij})\) and \(B_0 = (b_{ij})\). Let \(b_{\infty} := \max_j b_{jj}\) and \(a_{\infty} = \max_i a_{ii}\). For all \(t > 0\),

\[
P (|S_\star| > t) \leq 2 \exp \left( -c \min \left( \lambda^2 a_{\infty} b_{\infty} \text{tr}(B_0) \sum_{j=1}^m a_{jj} p_j^2, \frac{t^2}{a_{\infty} b_{\infty}} \right) \right)
\]

Set for \(C\) large enough

\[
\tau_\star = C \lambda \log^{1/2}(n \vee m) \sqrt{a_{\infty} b_{\infty} \text{tr}(B_0) \sum_{j=1}^m a_{jj} p_j^2}
\]
We have by Corollary N.2 and by assumption in Lemma 6.1, namely (120),
\[ \mathbb{P}(\lvert S^* \rvert > \tau_*) =: \mathbb{P}(E_6) \leq 2 \exp \left( -c \min \left( \frac{\tau_*^2}{\lambda^2 a^\infty b^\infty \text{tr}(B_0) \sum_{j=1}^m a_{jj} p_j^2}, \frac{\tau_*}{a^\infty b^\infty} \right) \right) \leq \frac{1}{(n \lor m)^4}. \]
Putting these two parts together, we have on event \( F_{c0} \cap E_{c6} \cap E_{c7} \),
\[ \lvert S_c - ES_c \rvert \leq \lvert S_c - E[S_c|U] \rvert + \lvert S^* \rvert \leq \tau_0 + \tau_* \] (122)

N.2 Proof of Corollary N.2

First we compute for \( B_1^{1/2} \otimes A_1^{1/2} =: D_0 \),
\[ \mathbb{E}[S_c|U] = \text{tr}(D_0 D_0 D_0) = \frac{1}{n-1} \sum_{t=1}^n b_{tt} \text{tr}(A_0 D_t) \]
\[ = \frac{1}{n-1} \sum_{t=1}^n b_{tt} \sum_{s \neq t} \text{tr}(A_0 \text{diag}(v_t \otimes v_s)) \]
\[ = \frac{1}{n-1} \sum_{t=1}^n b_{tt} \sum_{s \neq t} (v_t^T A_0 I v_s)^T = \text{vec} \{ V \}^T D_* \text{vec} \{ V \} \]
Notice that by definition of \( D_* \) we have for \( \text{E}(\text{vec} \{ V \}) =: p \in \mathbb{R}^{mn} \),
\[ \sum_{i \neq j} \lvert D_{*,ij} \rvert p_j p_i = \sum_{i=1}^n \frac{1}{n-1} \sum_{j \neq i} \lvert b_{ii} \rvert \sum_{q=1}^m \sum_{j=1}^m a_{jj} p_j^2 = \text{tr}(B_0) \sum_{j=1}^m a_{jj} p_j^2 \]
By Corollary B.3 where we substitute \( A \) with \( D_* \), and set \( D_{\text{max}} =: a^\infty b^\infty \), we have the following estimate on the moment generating function for \( S_* \). We have for \( \lvert \lambda \rvert < \frac{1}{16 a^\infty b^\infty} \),
\[ \mathbb{E}(\exp(\lambda S_*)) \leq \exp \left( 36.5 \lambda^2 D_{\text{max}} e^{1/2} \text{tr}(B_0) \sum_{j=1}^m a_{jj} p_j^2 \right) \]
where \( e^{\lvert \lambda \rvert a^\infty b^\infty} \leq e^{1/2} \leq 1.65 \). The rest of the proof follows that of Lemma B.5 and hence omitted. □

N.3 Proof of Lemma 6.1

We use the following bounds: \( \| \text{diag}(B_0) \|_F^2 \leq b^\infty \text{tr}(B_0), \text{tr}(B_0) \leq \sqrt{n} \| \text{diag}(B_0) \|_F \) and \( \| B_0 \|_F \leq \sqrt{\| B_0 \|_2^2 \text{tr}(B_0)} \).

Off-diagonal component Let \( F_0^c = \mathcal{E}_0^c \cap \mathcal{E}_7^c \). Clearly, On event \( \mathcal{E}_0^c \cap \mathcal{E}_7^c \),
\[ \forall k \neq \ell, \quad \delta_{\text{mask}} =: \frac{\widehat{M}_{k\ell} - M_{k\ell}}{M_{k\ell}} = \frac{|S_c - ES_c|}{ES_c} \leq \frac{\tau_0 + \tau_*}{ES_c} \]

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Putting things together we have on event $\mathcal{F}_n \cap \mathcal{F}_n^c$, which holds with probability at least $1 - \frac{C}{(\log n)^2}$, where $C \leq 6$, by \eqref{eq:delta_mask}, \eqref{eq:tau_0} and Corollary \ref{cor:4}.

\[
\delta_{\text{mask}} := \frac{|S_c - ES_c|}{ES_c} \leq \frac{\tau_0 + \tau_r}{\text{tr}(B_0) \sum_{j=1}^m a_{ss}p_j^2} \times \eta A_{\text{offd}} \frac{\|B_0\|_2^{1/2}}{\sqrt{\text{tr}(B_0)}} (1 + o(1)) \text{ where } r_{\text{offd}} \asymp \frac{\|A_0\|_2^{1/2} \log^{1/2}(n \lor m)}{\sqrt{\sum_j a_{jj}p_j^2}}
\]

is the same as $r_{\text{offd}}$ as defined in \cite{22} with $s_0 = 1$.

**Diagonal component** Recall $V = U^T$. Let $\xi = \text{vec} \{ V \} := (\xi_1, \ldots, \xi_{mn}) \in \{0, 1\}^{mn}$ be a random vector independent of $X$, as defined in \cite{10}. Let $A_\xi = D_0D_\xi D_0$ where $D_0 = B_0^{1/2} \otimes A_0^{1/2}$. Thus we can write for vec \{ $Y$ \} = $B_0^{1/2} \otimes A_0^{1/2}Z$ and $v := \text{vec} \{ V \}$,

\[
\text{tr}(X^T X) = \text{vec} \{ V \}^T \text{diag}(\text{vec} \{ V \} \otimes \text{vec} \{ V \}) \text{vec} \{ Y \} = Z^T B_0^{1/2} \otimes A_0^{1/2} \text{diag}(v \otimes v) B_0^{1/2} \otimes A_0^{1/2} Z =: Z^T D_0D_\xi D_0 Z
\]

Now $D_0^2 \circ D_0^2 = (B_0 \otimes A_0) \circ (B_0 \otimes A_0) = (B_0 \circ B_0) \otimes (A_0 \circ A_0)$. We can apply Theorem \ref{thm:1} \cite{41} here directly to argue that for every $t > 0$,

\[
P \left( \left| Z^T A_\xi Z - E Z^T A_\xi Z \right| > t \right) \leq 2 \exp \left( -c \min \left( \frac{t^2}{K^2 Q}, \frac{t}{K^2 \|A_0\|_2 \|B_0\|_2} \right) \right) \tag{123}
\]

where for $|p|_1 = \sum_{j=1}^m p_j$.

\[
Q = \sum_{j=1}^n b_{jj}^2 \sum_{i=1}^m a^2_{ii}p_i + E \left( \xi^T \text{offd}((B_0 \circ B_0) \otimes (A_0 \circ A_0))\xi^T \right) \leq \|B_0\|_F^2 a_\infty \|A_0\|_2 |p|_1 \text{ where } p = (p_1, p_2, \ldots, p_m)
\]

Thus by choosing for some absolute constants $C_1, C_2, c$,

\[
\tau_{\text{diag}} = C_1 \log(n \lor m) \|A_0\|_2 \|B_0\|_2 + C_2 \log^{1/2}(n \lor m) \sqrt{a_\infty \|A_0\|_2 |p|_1 \|B_0\|_F} \tag{123}
\]

We have by \eqref{eq:delta_mask}

\[
P \left( \left| Z^T A_\xi Z - E Z^T A_\xi Z \right| > \tau_{\text{diag}} \right) =: P(\mathcal{E}_8) \leq 2 \exp \left( -c \min \left( \frac{\tau_2^2}{Q}, \frac{\tau_{\text{diag}}}{\|A_0\|_2 \|B_0\|_2} \right) \right) \leq \frac{c}{(n \lor m)^4}
\]

Hence on event $\mathcal{E}_8^c$, for all $\ell$,

\[
|\overline{M}_{\ell \ell} - M_{\ell \ell}| = \frac{\|X\|_F^2 - E \|X\|_F^2}{E \|X\|_F^2} = \frac{\text{tr}(X^T X) - E \text{tr}(X^T X)}{E \text{tr}(X^T X)} \leq \frac{\tau_{\text{diag}}}{\text{tr}(B_0) \sum_{j=1}^m a_{jj}p_j} \times \frac{r_{\text{diag}} \sqrt{\|B_0\|_2}}{\sqrt{\text{tr}(B_0)}} (1 + o(1))
\]

where

\[
r_{\text{diag}} \asymp \frac{\log^{1/2} m \sqrt{a_\infty \|A_0\|_2}}{\sqrt{\sum_{j=1}^m a_{jj}p_j}} = o(1)
\]
is as defined in Theorem 3.1. The lemma is proved. □

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