FUNCTIONAL MODEL FOR BOUNDARY-VALUE PROBLEMS

KIRILL D. CHEREDNICHENKO, ALEXANDER V. KISELEV, AND LUIS O. SILVA

Abstract. We develop a functional model for operators arising in the study of boundary-value problems of materials science and mathematical physics. We then provide explicit formulae for the resolvents of the associated extensions of symmetric operators in terms of appropriate Dirichlet-to-Neumann maps, which can be utilised in the analysis of the properties of parameter-dependent problems, including the study of their spectra.

1. Introduction

The need to understand and quantify the behaviour of solutions to problems of mathematical physics has been central in driving the development of theoretical tools for the analysis of boundary-value problems (BVP). On the other hand, the second part of the last century witnessed several substantial advances in the abstract methods of spectral theory in Hilbert spaces, stemming from the groundbreaking achievement of John von Neumann in laying the mathematical foundations of quantum mechanics. Some of these advances have made their way into the broader context of mathematical physics [35, 21, 43]. In spite of these obvious successes of spectral theory applied to concrete problems, the operator-theoretic understanding of BVP has been lacking. However, in models of short-range interactions, the idea of replacing the original complex system by an explicitly solvable one, with a zero-radius potential (possibly with an internal structure), has proved to be highly valuable [6, 47, 14, 8, 32, 33, 56]. This facilitated an influx of methods of the theory of extensions (both self-adjoint and non-selfadjoint) of symmetric operators to problems of mathematical physics, culminating in the theory of boundary triples.

The theory of boundary triples introduced in [24, 22, 29, 30] has been successfully applied to the spectral analysis of BVP for ordinary differential operators and related setups, e.g. that of finite “quantum graphs”, where the Dirichlet-to-Neumann maps act on finite-dimensional “boundary” spaces, see [19] and references therein. However, in its original form this theory is not suited for dealing with BVP for partial differential equations (PDE), see [12, Section 7] for a relevant discussion. The key obstacle to such analysis is the lack of boundary traces \( \Gamma_0 u \) and \( \Gamma_1 u \) for functions \( u : \Omega \to \mathbb{R} \) (where \( \Omega \) is a bounded open set with a smooth boundary) in the domain of the maximal operator \( A \) corresponding to the differential expression considered (e.g. the operator \( -\Delta \) on the domain of \( L^2(\Omega) \)-functions \( u \) such that \( \Delta u \) is in \( L^2(\Omega) \)) entering the Green identity

\[
\langle Au, v \rangle_{L^2(\Omega)} - \langle u, Av \rangle_{L^2(\Omega)} = \langle \Gamma_1 u, \Gamma_0 v \rangle_{L^2(\partial \Omega)} - \langle \Gamma_0 u, \Gamma_1 v \rangle_{L^2(\partial \Omega)},
\]

in other words \( \text{dom}(A) \nsubseteq \text{dom}(\Gamma_0) \cap \text{dom}(\Gamma_1) \). Recently, when the works [26, 27, 5, 23, 52, 12] started to appear, it has transpired that, suitably modified, the boundary triples approach nevertheless admits a natural generalisation to the BVP setup, see also the seminal contributions by M. S. Birman [10], L. Boutet de Monvel [4], M. S. Birman and M. Z. Solomyak [11], G. Grubb [25], and M. Agranovich [1], which provide an analytic backbone for the related operator-theoretic constructions.

In all cases mentioned above, one can see the fundamental rôle of a certain Herglotz operator-valued analytic function, which in problems where a boundary is present (and sometimes even without an explicit boundary [2]) turns out to be a natural generalisation of the classical notion of a Dirichlet-to-Neumann map. The emergence of this object yields the possibility to apply to BVP advanced methods of complex analysis in conjunction with abstract methods of operator and spectral theory, which in turn sheds light on the...
intrinsic interplay between the mentioned abstract frameworks and concrete problems of interest in modern mathematical physics.

The present paper is a development of the recent activity [15, 16, 17, 20] aimed at implementing the above strategy in the context of problems of materials science and wave propagation in inhomogeneous media. Our recent papers [18, 19] have shown that the language of boundary triples is particularly fitting for direct and inverse scattering problems on quantum graphs, as one of the key challenges to their analysis stems from the presence of interfaces through which energy exchange between different components of the medium takes place. In the present work we continue the research initiated in these papers, adapting the technology so that BVP, especially those stemming from materials sciences, become within reach. As in [18, 19], the ideas of [36, 39] concerning the functional model allow one to efficiently incorporate into the analysis information about the mentioned energy exchange, by employing a suitable Dirichlet-to-Neumann map. In our analysis of BVP, we adopt the approach to the operator-theoretic treatment of BVP suggested by [52], which appears to be particularly convenient for obtaining sharp quantitative information about scattering properties of the medium, cf. e.g. [20], where this same approach is used as a framework for the asymptotic analysis of homogenisation problems in resonant composites.

We next outline the structure of the paper. In Section 2 we recall the main points of the abstract construction of [52] and introduce the key objects for the analysis we carry out later on, such as the dissipative operator $L$ at the centre of the functional model. In Section 3 we construct the minimal dilation of $L$, based on the ideas of [50], which in the context of extensions of symmetric operators followed the earlier foundational work [39]. Using the functional model framework thus developed, in Section 4 we construct a new version of Pavlov’s “three-component” functional model for the dilation [45] and pass to his “two-component”, or “symmetric”, model [46] (see also [39, 50]), based on the notion of the characteristic function for $L$, which is computed explicitly in terms of the $M$-operator introduced in Section 2. In Section 5 we develop formulae for the resolvents of boundary-value operators for a range of boundary conditions $\alpha \Gamma_0 u + \beta \Gamma_1 u = 0$, with $\alpha, \beta$ from a wide class of operators in $L^2(\partial \Omega)$, including those relevant to applications. The last two sections are devoted to the applications of the framework: based on the derived formulae for the resolvents, in Section 6 we establish the resolvent formulae for the operators of boundary-value problems belonging the class discussed earlier in the functional spaces stemming from the functional model, and in Section 7 we apply these formulae to obtain a description of the operators of BVPs in a class of Hilbert spaces with generating kernels.

2. Ryzhov triples for BVP

In this section we follow [52] in developing an operator framework suitable for dealing with boundary-value problems. The starting point is a self-adjoint operator $A_0$ in a separable Hilbert space $H$ with $0 \in \rho(A_0)$, where $\rho(A_0)$, as usual, denotes the resolvent set of $A_0$. Alongside $H$, we consider an auxiliary Hilbert space $E$ and a bounded operator $\Pi : E \to H$ such that

\begin{equation}
\text{dom}(A_0) \cap \text{ran}(\Pi) = \{0\} \quad \text{and} \quad \ker(\Pi) = \{0\}.
\end{equation}

Since $\Pi$ has a trivial kernel, there is a left inverse $\Pi^{-1}$, so that $\Pi^{-1}\Pi = I_E$. We define

\begin{equation}
\text{dom}(A) := \text{dom}(A_0) + \text{ran}(\Pi),
\quad A : A_0^{-1}f + \Pi \phi \mapsto f, \quad f \in H, \phi \in E,
\end{equation}

\begin{equation}
\text{dom}(\Gamma_0) := \text{dom}(A_0) + \text{ran}(\Pi),
\quad \Gamma_0 : A_0^{-1}f + \Pi \phi \mapsto \phi, \quad f \in H, \phi \in E,
\end{equation}

where neither $A$ nor $\Gamma_0$ is assumed closed or indeed closable. The operator given in (2.2) is the null extension of $A_0$, while (2.3) is the null extension of $\Pi^{-1}$. Note also that

\begin{equation}
\ker(\Gamma_0) = \text{dom}(A_0).
\end{equation}
For $z \in \rho(A_0)$, consider the abstract spectral boundary-value problem

\begin{equation}
\begin{cases}
Au = zu, \\
\Gamma_0 u = \phi, \\
\phi \in \mathcal{E},
\end{cases}
\end{equation}

where the second equation is seen as a boundary condition. As asserted in [52, Theorem 3.1], there is a unique solution $u$ of the boundary-value problem (2.5) for any $\phi \in \mathcal{E}$. Thus, there is an operator (clearly linear) which assigns to any $\phi \in \mathcal{E}$ the solution $u$ of (2.5), referred to as the solution operator\(^1\) for $A$ and denoted by $\gamma(z)$. An explicit expression for it in terms of $A_0$ and $\Pi$ can be obtained as follows. Using the fact that $A \supset A_0$, one can show (see [52, Remark 3.3]) that for all $\phi \in \mathcal{E}$ one has

\[ \Pi \phi + z(A_0 - zI)^{-1} \Pi \phi \in \ker(A - zI), \quad \Gamma_0 (\Pi \phi + z(A_0 - zI)^{-1} \Pi \phi) = \phi, \]

and therefore

\begin{equation}
\gamma(z) \phi = (I + z(A_0 - zI)^{-1}) \Pi \phi.
\end{equation}

Furthermore, note that

\begin{equation}
I + z(A_0 - zI)^{-1} = (I - zA_0^{-1})^{-1},
\end{equation}

and so (2.3), (2.6) immediately imply

\begin{equation}
\Gamma_0 \gamma(z) = I_E.
\end{equation}

By (2.6), one has $\text{ran}(\gamma(z)) \subset \ker(A - zI)$, but the inverse inclusion also holds. Indeed, taking a vector $u \in \ker(A - zI)$ and writing it in the form $u = A_0^{-1} f + \Pi \phi$, one obtains

\[ 0 = (A - zI)(A_0^{-1} f + \Pi \phi) = (I - zA_0^{-1}) f - z \Pi \phi, \]

which yields $f = z(I - zA_0^{-1})^{-1} \Pi \phi$. Thus,

\[ u = A_0^{-1} f + \Pi \phi = [zA_0^{-1}(I - zA_0^{-1})^{-1} + I] \Pi \phi = (I - zA_0^{-1})^{-1} \Pi \phi. \]

In view of (2.6), (2.7), the last expression shows that $u \in \text{ran}(\gamma(z))$. Putting together the above, one arrives at

\begin{equation}
\text{ran}(\gamma(z)) = \ker(A - zI).
\end{equation}

We remark that, since $A$ is not required to be closed, $\text{ran}(\gamma(z))$ is not necessarily a subspace. This is precisely the kind of situation that commonly occurs in the analysis of BVPs.

In what follows, we consider (abstract) BVP of the form (2.5) associated with the operator $A$, with variable boundary conditions. To this end, for a self-adjoint operator $\Lambda$ in $\mathcal{E}$, define

\begin{equation}
\text{dom}(\Gamma_1) := \text{dom}(A_0) + \Pi \text{dom}(\Lambda),
\end{equation}

\begin{equation}
\Gamma_1 : A_0^{-1} f + \Pi \phi \mapsto \Pi^* f + A \phi, \quad f \in \mathcal{H}, \phi \in \text{dom}(\Lambda).
\end{equation}

The operator $\Lambda$ can thus be seen as a parameter for the boundary operator $\Gamma_1$.

On the basis of (2.6), one obtains from (2.10) (see [52, Equation 3.7]) that

\begin{equation}
\gamma(\mathfrak{T})^* = \Gamma_1 (A_0 - zI)^{-1}, \quad z \in \rho(A_0).
\end{equation}

Also, according to [52, Theorem 3.2], the following Green’s type identity holds:

\begin{equation}
\langle Au, v \rangle_{\mathcal{H}} - \langle u, Av \rangle_{\mathcal{H}} = \langle \Gamma_1 u, \Gamma_0 v \rangle_{\mathcal{E}} - \langle \Gamma_0 u, \Gamma_1 v \rangle_{\mathcal{E}}, \quad u, v \in \text{dom}(\Gamma_1).
\end{equation}

The spectral BVP (2.5) is thus described by the triple $(A_0, \Pi, \Lambda)$, introduced by Ryzhov [52]. His setup stems from the Birman-Krein-Vishik theory [8, 32, 33, 56], rather than the theory of boundary triples [24].

**Definition 1.** For a given triple $(A_0, \Pi, \Lambda)$, define the operator-valued $M$-function associated with $A_0$ as follows: for any $z \in \rho(A_0)$, the operator $M(z)$ in $\mathcal{E}$ is defined on the domain $\text{dom}(M(z)) := \text{dom}(\Lambda)$, and its action is given by

\[ M(z) : \phi \mapsto \Gamma_1 \gamma(z) \phi, \quad \phi \in \text{dom}(M(z)). \]

\(^1\)The operator-valued function $\gamma$ is also sometimes referred to as the $\gamma$-field.
The above abstract framework is illustrated (see [52] for details) by the classical setup where $A_0$ is the Dirichlet Laplacian on a bounded domain $\Omega$ with smooth boundary $\partial \Omega$, so $A_0$ is self-adjoint on $\text{dom}(A_0) = W^2_0(\Omega) \cap W^1_2(\Omega)$. In this case $\Pi$ is simply the Poisson operator of harmonic lift, its left inverse is the operator of boundary trace for harmonic functions and $\Gamma_0$ is the null extension of the latter to $[W^2_2(\Omega) \cap W^1_2(\Omega)] + \Pi L^2(\partial \Omega)$. Furthermore, $\Lambda$ can be chosen as the Dirichlet-to-Neumann map\(^2\) which maps any function $\phi \in W^2_2(\Omega) =: \text{dom}(\Lambda)$ to $-\left( \partial u / \partial n \right)|_{\partial \Omega}$, where $u$ is the solution of the boundary-value problem
\[
\begin{align*}
\Delta u &= 0, \\
\left. u \right|_{\partial \Omega} &= \phi,
\end{align*}
\] (see e.g. [55]). Due to the choice of $\Lambda$, it follows from (2.10) that
\[
\text{dom}(\Gamma_1) = \left[ W^2_2(\Omega) \cap W^1_2(\Omega) \right] + \Pi W^2_2(\partial \Omega), \quad \Gamma_1 u = -\left. \frac{\partial u}{\partial n} \right|_{\partial \Omega}.
\] Note that (2.13) follows from the fact that $\Pi^* f = -\left( \partial u / \partial n \right)|_{\partial \Omega}$ for $u = A_0^{-1} f$. Therefore, the $M$-operator $M(z), z \in \rho(A_0)$, is the Dirichlet-to-Neumann map $\phi \mapsto -\left( \partial u / \partial n \right)|_{\partial \Omega}$ of the spectral boundary-value problem (2.5), i.e. $u \in \left[ W^2_2(\Omega) \cap W^1_2(\Omega) \right] + \Pi L^2(\partial \Omega)$ is a solution of
\[
\begin{align*}
\Delta u &= z u, \\
\left. u \right|_{\partial \Omega} &= \phi,
\end{align*}
\] where $\phi$ belongs to $L^2(\partial \Omega)$, and $M(z)$ is understood as an unbounded operator\(^3\) defined on $\text{dom}(M(z)) = W^1_2(\partial \Omega)$.

This example shows how all the classical objects of BVP appear naturally from the triple $(A_0, \Pi, \Lambda)$. In particular, it is worth noting how the energy-dependent Dirichlet-to-Neumann map $M(z)$ is “grown” from its “germ” $\Lambda$ at $z = 0$. Returning to the abstract setting and taking into account (2.10), one concludes from Definition 1 that
\[
M(z) = \Lambda + z \Pi^*(I - z A_0^{-1})^{-1} \Pi.
\] From this equality, one verifies directly that
\[
M(z) - M(w) = \Pi^* \left[ (I - z A_0^{-1})^{-1} - (I - w A_0^{-1})^{-1} \right] \Pi = (z - w) \gamma(\overline{\gamma})^\ast \gamma(w), \quad z, w \in \rho(A_0).
\] Also, due to the self-adjointness of $\Lambda$, one has
\[
M^*(z) = M(\overline{z}).
\] The properties (2.15) and (2.16) together imply that $M$ is an unbounded operator-valued Herglotz function, i.e., $M(z) - M(0)$ is analytic, and $3M(z) \geq 0$ whenever $z \in \mathbb{C}_+$. It is shown in [52, Theorem 3.3(4)] that
\[
M(z)\Gamma_0 u = \Gamma_1 u \quad \forall u \in \ker(A - z I) \cap \text{dom}(\Gamma_1).
\] In this work we consider extensions (self-adjoint and non-selfadjoint) of the “minimal” operator
\[
\widetilde{A} := A_0|_{\ker(\Gamma_1)}
\] that are restrictions of $A$. It is proven in [52, Section 5] that $\widetilde{A}$ is symmetric with equal deficiency indices. Moreover, [52, Remark 5.1] asserts that
\[
\text{dom}(\widetilde{A}) = A_0^{-1}[\text{ran}(\Pi)^\perp],
\] so $\widetilde{A}$ does not depend on the parameter operator $\Lambda$, contrary to what could be surmised from (2.17).

Still following [52], we let $\alpha$ and $\beta$ be linear operators in $\mathcal{E}$ such that $\text{dom}(\alpha) \supset \text{dom}(\Lambda)$ and $\beta$ is bounded on $\mathcal{E}$. Additionally, assume that $\alpha + \beta \Lambda$ is closable and denote its closure by $\beta$. Consider the linear set
\[
\mathcal{H}_\beta := \left\{ A_0^{-1} f + \Pi \phi : f \in \mathcal{H}, \phi \in \text{dom}(\beta) \right\}.
\]\(^2\)For convenience, we define the Dirichlet-to-Neumann map via $-\partial u / \partial n|_{\partial \Omega}$ instead of the more common $\partial u / \partial n|_{\partial \Omega}$. As a side note, we mention that this is obviously not the only choice for the operator $\Lambda$. In particular, the trivial option $\Lambda = 0$ is always possible. Our choice of $\Lambda$ is motivated by our interest in the analysis of classical boundary conditions.

\(^3\)More precisely, $M(z)$ is the sum of an unbounded self-adjoint operator and a bounded one, which will be obvious from (2.14).
The definition of $\text{dom}(L)$ implies that $\alpha \Gamma_0 + \beta \Gamma_1$ is well defined on $\text{dom}(A_0) + \Pi \text{dom}(A)$. The assumption that $\alpha + \beta A$ is closable is used to extend the domain of definition of $\alpha \Gamma_0 + \beta \Gamma_1$ to the set $(2.18)$. Moreover, one verifies that $\mathcal{H}_B$ is a Hilbert space with respect to the norm

$$
\|u\|_B^2 := \|f\|^2_{\mathcal{H}} + \|\phi\|^2_{\mathcal{H}} + \|\beta \phi\|^2_{\mathcal{H}}, \quad u = A_0^{-1}f + \Pi \phi.
$$

It follows that the constructed extension $\alpha \Gamma_0 + \beta \Gamma_1$ is a bounded operator on $\mathcal{H}_B$. According to [52, Theorem 4.1], if the operator $\alpha + \beta M(z)$ is boundedly invertible for $z \in \rho(A_0)$, the spectral boundary-value problem

$$
(A - zI)u = f, \\
(\alpha \Gamma_0 + \beta \Gamma_1)u = \phi, \quad f \in \mathcal{H}, \quad \phi \in \mathcal{E},
$$

has a unique solution $u \in \mathcal{H}_B$, where, as above, $\alpha \Gamma_0 + \beta \Gamma_1$ is a bounded operator on $\mathcal{H}_B$. Under the same hypothesis of $\alpha + \beta M(z)$ being boundedly invertible for $z \in \rho(A_0)$, it follows from [52, Theorem 5.1] that the function

$$
(A_0 - zI)^{-1} - (I - zA_0^{-1})^{-1}\Pi(\alpha + \beta M(z))^{-1}\beta \Pi^*(I - zA_0^{-1})^{-1}
$$

is the resolvent of a closed operator $A_{\alpha \beta}$ densely defined in $\mathcal{H}$. Moreover, $\tilde{A} \subset A_{\alpha \beta} \subset A$ and $\text{dom}(A_{\alpha \beta}) \subset \{u \in \mathcal{H}_B : (\alpha \Gamma_0 + \beta \Gamma_1)u = 0\}$.

Among the extensions $A_{\alpha \beta}$ of $\tilde{A}$, we single out the operator

$$
L := A_{-iI I},
$$

that is, $\alpha = -iI$ and $\beta = I$. Since in this case $\alpha$ and $\beta$ are scalar operators, and $\text{dom}(\Gamma_1) \subset \text{dom}(\Gamma_0)$, by virtue of (2.18) one has

$$
\text{dom}(L) \subset \text{dom}(\Gamma_1).
$$

The definition of $\text{dom}(L)$ implies that for all $h \in \mathcal{H}$, $z \in \mathbb{C}_-$,

$$
0 = (\Gamma_1 - i\Gamma_0)(L - zI)^{-1}h = \Gamma_1(L - zI)^{-1}h - i\Gamma_0[(L - zI)^{-1} - (A_0 - zI)^{-1}]h
$$

$$
= M(z)\Gamma_0[(L - zI)^{-1} - (A_0 - zI)^{-1}]h + \Gamma_1(A_0 - zI)^{-1}h - i\Gamma_0[(L - zI)^{-1} - (A_0 - zI)^{-1}]h
$$

$$
= M(z)\Gamma_0(L - zI)^{-1}h + \Gamma_1(A_0 - zI)^{-1}h - i\Gamma_0(L - zI)^{-1}h
$$

since, by (2.4) and the fact that $L, A_0 \subset A$, one has

$$
[(L - zI)^{-1} - (A_0 - zI)^{-1}]h \in \ker(A - zI), \quad (A_0 - zI)^{-1}h \in \ker(\Gamma_0).
$$

Thus

$$
\Gamma_0(L - zI)^{-1} = -(M(z) - iI)^{-1}\Gamma_1(A_0 - zI)^{-1}, \quad z \in \mathbb{C}_-,
$$

$$
\Gamma_0(L^* - zI)^{-1} = -(M(z) + iI)^{-1}\Gamma_1(A_0 - zI)^{-1}, \quad z \in \mathbb{C}_+,
$$

where the second equality is deduced in the same way as the first. In what follows, we will use the following relations, which are obtained by combining (2.11) and (2.23):

$$
\Gamma_0(L - zI)^{-1} = -(M(z) - iI)^{-1}\gamma(\mathcal{E})^*, \quad z \in \mathbb{C}_-,
$$

$$
\Gamma_0(L^* - zI)^{-1} = -(M(z) + iI)^{-1}\gamma(\mathcal{E})^*, \quad z \in \mathbb{C}_+.
$$

It is proven in [52, Theorem 6.1] that the operator $L$ of formula (2.21) is dissipative and boundedly invertible (hence maximal). We recall that a densely defined operator $L$ in $\mathcal{H}$ is called dissipative if

$$
\text{Im}(Lf, f) \geq 0 \quad \forall f \in \text{dom}(L).
$$

A dissipative operator $L$ is said to be maximal if $\mathbb{C}_- \subset \rho(L)$. Maximal dissipative operators are closed, and any dissipative operator admits a maximal extension.
Furthermore, the function
\begin{equation}
S(z) := (M(z) - iI)(M(z) + iI)^{-1} = I - 2i(M(z) + iI)^{-1}, \quad z \in \mathbb{C}_+,
\end{equation}
turns out to be the characteristic function of $L$, see [36, 54]. Since $M$ is a Herglotz function (see (2.16)), one has the following formula:
\begin{equation}
S^*(z) := [S(z)]^* = I + 2i(M^*(z) - iI)^{-1} = I + 2i(M(z) - iI)^{-1}, \quad z \in \mathbb{C}_-.
\end{equation}
We remark that the function $S$ is analytic in $\mathbb{C}_+$ and, for each $z \in \mathbb{C}_+$, the mapping $S(z) : \mathcal{E} \to \mathcal{E}$ is a contraction. Therefore, $S$ has nontangential limits almost everywhere on the real line in the strong operator topology [53].
Recall that a closed operator $L$ is said to be completely non-selfadjoint if there is no subspace reducing $L$ such that the part of $L$ in this subspace is self-adjoint. We refer to a completely non-selfadjoint symmetric operator as simple.

**Proposition 2.1.** If the symmetric operator $\tilde{A}$ of (2.17) is simple, then the dissipative operator $L$ is completely non-selfadjoint.

**Proof.** Suppose that $L$ has a reducing subspace $\mathcal{H}_1$ such that $L|_{\mathcal{H}_1}$ is self-adjoint. Take a nonzero $w \in \text{dom}(L) \cap \mathcal{H}_1$. Then (2.12) and (2.22) imply $\langle \Gamma w, \Gamma_0 w \rangle_{\mathcal{E}} = 0$. Since $w \in \ker(\Gamma_0 - i\Gamma_0)$, one obtains from the last equality that $\|\Gamma_0 w\| = 0$. Therefore, $w \in \ker(\Gamma_0) \cap \ker(\Gamma_1)$, which means that $w \in \text{dom}(\tilde{A})$.

The nontrivial invariant subspace $\mathcal{H}_1$ of $L$ is a nontrivial invariant subspace of its restriction $\tilde{A}$ as long as $\mathcal{H}_1 \cap \text{dom}(\tilde{A}) \neq \emptyset$. This last condition has been established above. Finally, since $\tilde{A}$ is symmetric, $\mathcal{H}_1$ is actually a reducing subspace of $\tilde{A}$. Clearly $\tilde{A}$ is self-adjoint in $\mathcal{H}_1$. □

### 3. Self-adjoint dilations for operators of BVP and a 3-component functional model

Any completely non-selfadjoint dissipative operator $L$ admits a self-adjoint dilation [53], which is unique up to a unitary transformation, under an assumption of minimality, see (3.2) below. There are numerous approaches to an explicit construction of the named dilation [13, 39, 40, 41, 45, 46, 50, 51, 54]. In applications, one is compelled to seek a realisation corresponding to a particular setup. In the present paper we develop a way of constructing dilations of dissipative operators convenient in the context of BVP for PDE.

In the formulae below, we use the subscript “±” to indicate two different versions of the same formula in which the subscripts “+” and “−” are taken individually.

Recall that for any maximal dissipative operator $L$, its dilation is defined as a self-adjoint operator $A$ in a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ with the property
\begin{equation}
P_H(A - zI)^{-1}|_{\mathcal{H}} = (L - zI)^{-1} \quad \forall z \in \mathbb{C}_-.
\end{equation}
A dilation $A$ is referred to as minimal if
\begin{equation}
\overline{\text{span}} \{ (A - zI)^{-1} \mathcal{H} \} = \mathcal{K}.
\end{equation}
We start by constructing a minimal dilation of the operator $L$ of the previous section, defined by (2.21), following a procedure similar to the one used in [44, 45]. Let
\begin{equation}
\mathcal{K} := L^2(\mathbb{R}_-, \mathcal{E}) \oplus \mathcal{H} \oplus L^2(\mathbb{R}_+, \mathcal{E}).
\end{equation}
In this Hilbert space, the operator $A$ is defined as follows. Its domain $\text{dom}(A)$ is given by
\begin{equation}
\text{dom}(A) := \{(v_-, u, v_+)^\top \in \mathcal{K} : \begin{array}{l}
v_\pm \in W^2_1(\mathbb{R}_\pm, \mathcal{E}), \ u \in \text{dom}(\Gamma_1) : \Gamma_1 u \pm i\Gamma_0 u = \sqrt{2}v_\pm(0), \end{array} \end{equation}
where $W^2_1(\mathbb{R}_+, \mathcal{E})$ and $W^2_1(\mathbb{R}_-, \mathcal{E})$ are the Sobolev spaces of functions defined on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively, and taking values in $\mathcal{E}$. We remark that the results of the previous section imply that in our case $\mathcal{H}_B = \text{dom}(\Gamma_1)$. On this domain, the operator $A$ acts according to the rule
\begin{equation}
A : \text{dom}(A) \ni (v_-, u, v_+)^\top \mapsto (iv_-', Au, iv_+')^\top \in \text{dom}(A).
\end{equation}

**Theorem 3.1.** In the dilated space $\mathcal{K}$, the operator $A$ is a self-adjoint extension of $L$. 
Proof. The fact that $A$ is an extension of $L$ follows from (2.21) and (2.22). Let us establish the self-adjointness of $A$. Abbreviating $u = (v_-, u, v_+)^\top \in \text{dom}(A)$, we have
\[
\langle Au, u \rangle - \langle u, Au \rangle = \langle iv'_-, v_- \rangle + \langle Au, u \rangle + \langle iv'_+, v_+ \rangle - \langle v_-, iv'_- \rangle - \langle u, Au \rangle - \langle v_+, iv'_+ \rangle
\]
\[
= i \int_{\mathbb{R}^-} (v'_- \nabla_- + v_- \nabla_-) + i \int_{\mathbb{R}^+} (v'_+ \nabla_+ + v_+ \nabla_+) + \langle Au, u \rangle - \langle u, Au \rangle
\]
\[
= i \|v_-(0)\|^2 - i \|v_+(0)\|^2 + \langle \Gamma_1 u, \Gamma_0 u \rangle - \langle \Gamma_0 u, \Gamma_1 u \rangle.
\]
Furthermore, taking into account the conditions defining $\text{dom}(A)$, one obtains
\[
\langle \Gamma_1 u, \Gamma_0 u \rangle - \langle \Gamma_0 u, \Gamma_1 u \rangle = \langle \sqrt{2}v_-(0) + i\Gamma_0 u, \Gamma_0 u \rangle - \langle \Gamma_0 u, \sqrt{2}v_+(0) - i\Gamma_0 u \rangle
\]
\[
= \langle \sqrt{2}v_-(0), \Gamma_0 u \rangle - \langle \Gamma_0 u, \sqrt{2}v_+(0) \rangle = i \langle v_-(0), v_+(0) - v_-(0) \rangle + i \langle v_+(0), v_-(0), v_+(0) \rangle
\]
\[
= -i \|v_-(0)\|^2 + i \|v_+(0)\|^2.
\]
It follows by combining (3.6) and (3.7) that $A$ is symmetric. To complete the proof, it suffices to show that $\text{ran}(A - zI) = \mathcal{H}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. To this end, consider the operators $\partial_{\pm}$ and $\partial_{\pm}^0$ in $L_2(\mathbb{R}_{\pm}, \mathcal{E})$ given by
\[
\text{dom}(\partial_{\pm}) := W_2^1(\mathbb{R}_{\pm}, \mathcal{E}), \quad \partial_{\pm} : y \mapsto iy'_\pm, \quad \text{dom}(\partial_{\pm}^0) := W_2^1(\mathbb{R}_{\pm}, \mathcal{E}), \quad \partial_{\pm}^0 : y \mapsto iy'_\pm.
\]
Here, $W_2^1(\mathbb{R}_{\pm}, \mathcal{E})$ is the closure in $W_2^1(\mathbb{R}_{\pm}, \mathcal{E})$ of the set of smooth functions with compact support in $\mathbb{R}_{\pm}$. The operators $\partial_{\pm}^0$ and $\partial_{\pm}^0$ are symmetric, with deficiency indices $(n_+, n_-) = (0, 1)$ and $(n_+, n_-) = (0, 1)$, respectively. Also, $\partial_{\pm}^0 = \partial_{\pm}^0$ (see [9, Chapter 4, Section 8.4]). Therefore $\rho(\partial_{\pm}) = \mathbb{C}_{\pm}$ and $\rho(\partial_{\pm}^0) = \mathbb{C}_{\mp}$.

Take any $z \in \mathbb{C}_-$ and $(h_-, h_+) \in \mathcal{H}$. It turns out that the vector $(f_-, f_+)^\top$ defined by
\[
f_- := (\partial_{-} - zI)^{-1}h_-,
\]
\[
f := (L - zI)^{-1}h + \sqrt{2}\gamma(z)(M(z) - iI)^{-1}f_- (0),
\]
\[
f_+ := (\partial_{+}^0 - zI)^{-1}h_+ + e^{-iz} \left[ i\sqrt{2}\Gamma_0(L - zI)^{-1}h + S^*(\overline{\gamma})f_- (0) \right],
\]
is an element of $\text{dom}(A)$. Indeed, clearly $f_- \in W_2^1(\mathbb{R}_-, \mathcal{E})$, and $f_+ \in W_2^1(\mathbb{R}_+, \mathcal{E})$ since
\[
f_+ = (\partial_{+}^0 - zI)^{-1}h_+ + e^{-iz} e \quad \text{for some } e \in \mathcal{E}.
\]
Also,
\[
(\Gamma_1 - i\Gamma_0) \left\{ (L - zI)^{-1}h + \sqrt{2}\gamma(z)(M(z) - iI)^{-1}f_- (0) \right\} = (\Gamma_1 - i\Gamma_0)\sqrt{2}\gamma(z)(M(z) - iI)^{-1}f_- (0)
\]
\[
= \sqrt{2}(M(z) - iI)(M(z) - iI)^{-1}f_- (0) = \sqrt{2}f_- (0),
\]
where to obtain the first equality we use (2.21), and the second equality follows from (2.8) and Definition 1. Thus
\[
(\Gamma_1 - i\Gamma_0)f = \sqrt{2}f_- (0).
\]
In addition, we have
\[
(\Gamma_1 + i\Gamma_0)f = (\Gamma_1 - i\Gamma_0)f + 2i\Gamma_0 f = \sqrt{2}f_- (0) + 2i\Gamma_0 \left\{ (L - zI)^{-1}h + \sqrt{2}\gamma(z)(M(z) - iI)^{-1}f_- (0) \right\}
\]
\[
= \sqrt{2}f_- (0) + 2i\Gamma_0(L - zI)^{-1}h + i2\sqrt{2}(M(z) - iI)^{-1}f_- (0)
\]
\[
= 2i\Gamma_0(L - zI)^{-1}h + \sqrt{2} \left[ i + 2i(M(z) - iI)^{-1} \right] f_- (0) = 2i\Gamma_0(L - zI)^{-1}h + \sqrt{2}S^*(\overline{\gamma})f_- (0),
\]
where we have used (3.10), (3.8) for the second, (2.8) for the third, and (2.26) for the fifth equality. Due to the expression for $f_+$ in (3.8), we have thus shown that
\[
(\Gamma_1 + i\Gamma_0)f = \sqrt{2}f_+ (0).
\]
The equalities (3.10) and (3.11) imply that $(f_-, f_+)^\top \in \text{dom}(A)$, see (3.4).
Next, we show that

\begin{equation}
(A - zI) \begin{pmatrix} f_- \\ f \\ f_+ \end{pmatrix} = \begin{pmatrix} h_- \\ h \\ h_+ \end{pmatrix}.
\end{equation}

On the one hand, it follows from (3.9) and the first line of (3.8) that

\begin{equation}
(3.13) \quad h_\pm = (\partial_\pm - zI)f_\pm.
\end{equation}

On the other hand, due to the fact that $L \subset A$ and the property (2.9), one has

\begin{equation}
(3.14) \quad (A - zI)[(L - zI)^{-1}h + \sqrt{2}\gamma(z)(M(z) - iI)^{-1}f_-(0)] = h.
\end{equation}

In conformity with (3.5), the identities (3.13), (3.14) yield (3.12). As $(h_-, h, h_+)^T$ is an arbitrary element in $\mathcal{H}$, we have also shown that\(\text{ran}(A - zI) = \mathcal{H}\) for $z \in \mathbb{C}_-$.

Now fix an arbitrary $z \in \mathbb{C}_+$. For any $(h_-, h, h_+)^T \in \mathcal{H}$, we redefine

\begin{equation}
(3.15) \quad f_+ := (\partial_+ - zI)^{-1}h_+,
\end{equation}

\begin{equation}
(3.16) \quad f := (L^* - zI)^{-1}h + \sqrt{2}\gamma(z)(M(z) + iI)^{-1}f_+(0),
\end{equation}

\begin{equation}
(3.17) \quad f_- := (\partial_0^0 - zI)^{-1}h_- + e^{-iz}[-i\sqrt{2}\Gamma_0(L^* - zI)^{-1}h + S(z)f_+(0)].
\end{equation}

In the same way as above, it can be shown that $(f_-, f, f_+)^T \in A$ and

\begin{equation}
(3.18) \quad (A - zI) \begin{pmatrix} f_- \\ f \\ f_+ \end{pmatrix} = \begin{pmatrix} h_- \\ h \\ h_+ \end{pmatrix},
\end{equation}

which completes the proof. 

\begin{remark}
In the proof of Theorem 3.1, we have obtained the following formulae for the resolvent of $A$:

\begin{equation}
(3.19) \quad (A - zI)^{-1} \begin{pmatrix} h_- \\ h \\ h_+ \end{pmatrix} = \begin{pmatrix} f_- \\ f \\ f_+ \end{pmatrix},
\end{equation}

where $(f_-, f, f_+)^T$ is given by (3.8) for $z \in \mathbb{C}_-$ and by (3.15) for $z \in \mathbb{C}_+$.

The following technical result will be used to prove that $A$ is a minimal dilation of $L$; at the same time, it is of a clear independent interest.

\begin{lemma}
Each of the sets

\begin{align*}
\text{span}_{u \in \mathcal{H}} \{ \Gamma_1(A_0 - zI)^{-1}u \}, \quad \text{span}_{h \in \mathcal{H}} \{ \Gamma_0(L - zI)^{-1}h \}, \quad \text{span}_{h \in \mathcal{H}} \{ \Gamma_0(L^* - zI)^{-1}h \}
\end{align*}

is dense in $\mathcal{E}$, for every $z \in \mathbb{C}_- \cup \mathbb{C}_+$, $z \in \mathbb{C}_-$, $z \in \mathbb{C}_+$, respectively.

\begin{proof}
Due to (2.23) and the fact that $\text{dom}(M(z))$ is dense in $\mathcal{E}$, it suffices to prove the assertion of the lemma about the first set.

Suppose that $\overline{v} \in \mathcal{E}$ is such that $\langle \Gamma_1(A_0 - zI)^{-1}u, \overline{v} \rangle = 0$ for all $u \in \mathcal{H}$. Using (2.11), we obtain $\langle u, \gamma(\bar{z})\overline{v} \rangle = 0 \quad \forall u \in \mathcal{H}$, and therefore $\gamma(\bar{z})\overline{v} = 0$ or, in view of (2.6), $v = -\overline{\gamma}(A_0 - \overline{z})^{-1}v$, where $v := \Pi\overline{v}$. Hence, $v \in \text{dom} A_0$ and $\overline{v} = \Gamma_0 v = 0$, as required.
\end{proof}
\end{lemma}

\begin{theorem}
The operator $A$ is a minimal self-adjoint dilation of $L$.
\end{theorem}

\begin{proof}
By Theorem 3.1, the operator $A$ is a self-adjoint extension of $L$. The property (3.1) is verified directly on the basis of Remark 1. Thus it only remains to check the minimality condition (3.2). It follows from Remark 1 that, relative to the orthogonal decomposition (3.3), one has

\begin{align*}
\text{span}_{z \in \mathbb{C} \setminus \mathbb{R}} \{ (A - zI)^{-1} \mathcal{H} \} &= \text{span}_{z \in \mathbb{C}_+} \{ e^{-iz} \Gamma_0(L^* - zI)^{-1} \mathcal{H} \} \\
&\quad \oplus \left( \text{span}_{z \in \mathbb{C}_-} \{ (L - zI)^{-1} \mathcal{H} \} + \text{span}_{z \in \mathbb{C}_+} \{ (L^* - zI)^{-1} \mathcal{H} \} \right) \oplus \text{span}_{z \in \mathbb{C}_-} \{ e^{-iz} \Gamma_0(L - zI)^{-1} \mathcal{H} \}.
\end{align*}


Since $L$ is densely defined, one clearly has
\[
\operatorname{span}\{(L - zI)^{-1}H\} + \operatorname{span}\{(L^* - zI)^{-1}H\} = \mathcal{H}.
\]

We next show that
\[
(3.16) \quad \operatorname{span}\{e^{-iz}\Gamma_0(L - zI)^{-1}H\} = L_2(\mathbb{R}_+, \mathcal{E}).
\]

Assuming that $g \in L_2(\mathbb{R}_+, \mathcal{E})$ is such that for all $z \in \mathbb{C}_-$, $h \in \mathcal{H}$, one has
\[
0 = \langle e^{-iz}\Gamma_0(L - zI)^{-1}h, g \rangle_{L_2(\mathbb{R}_+, \mathcal{E})} = \int_{\mathbb{R}_+} \langle \Gamma_0(L - zI)^{-1}h, e^{\xi z}g(\xi) \rangle_{\mathcal{E}} d\xi
\]
\[
= \left\langle \Gamma_0(L - zI)^{-1}h, \int_{\mathbb{R}_+} e^{i\xi \text{Re} z}e^{\xi \text{Im} z}g(\xi) d\xi \right\rangle_{\mathcal{E}}.
\]

By Lemma 3.2, it follows that
\[
\int_{\mathbb{R}_+} e^{i\xi \text{Re} z}e^{\xi \text{Im} z}g(\xi) d\xi = 0 \quad \forall z \in \mathbb{C}_-.
\]

Finally, fixing $\text{Im} z$ and taking the Fourier transform with respect to $\text{Re} z$ yields $g(\xi) = 0$ for a.e. $\xi \in \mathbb{R}_+$, which concludes the proof of (3.16). By a similar argument, one also shows that
\[
(3.18) \quad \operatorname{span}\{e^{-iz}\Gamma_0(L^* - zI)^{-1}H\} = L_2(\mathbb{R}_+, \mathcal{E}),
\]
which completes the proof. \qed

For convenience, we introduce the following families of sets in $\mathcal{H}$. For any $z_+ \in \mathbb{C}_+$ and $z_- \in \mathbb{C}_-$, define
\[
\mathcal{Y}(z_+, z_-) := (A - z_+ I)^{-1} \left( \begin{array}{cc} 0 & L_2(\mathbb{R}_+, \mathcal{E}) \\ 0 & 0 \end{array} \right) + (A - z_- I)^{-1} \left( \begin{array}{c} L_2(\mathbb{R}_+, \mathcal{E}) \\ 0 \end{array} \right),
\]
\[
\mathcal{G}(z_+, z_-) := \mathcal{P}_\mathcal{H}\mathcal{Y}(z_+, z_-),
\]
where $\mathcal{P}_\mathcal{H}$ is the orthogonal projection onto $\{0\} \oplus \mathcal{H} \oplus \{0\}$. Henceforth, we identify $\{0\} \oplus \mathcal{H} \oplus \{0\}$ and $\mathcal{H}$.

**Lemma 3.4.** If $\tilde{A}$ defined in (2.17) is simple, then the linear sets
\[
(3.17) \quad \operatorname{span} \mathcal{Y}(z_+, z_-), \quad \operatorname{span} \mathcal{G}(z_+, z_-)
\]
are dense in the spaces $\mathcal{H}$ and $\mathcal{H}$, respectively.

**Proof.** To simplify notation, denote by $Y$ the closure of the first set in (3.17). It follows from Remark 1 that
\[
(3.18) \quad \{0\} \oplus \{0\} \oplus L_2(\mathbb{R}_+, \mathcal{E}) \subset Y, \quad L_2(\mathbb{R}_+, \mathcal{E}) \oplus \{0\} \oplus \{0\} \subset Y.
\]

Indeed, putting $h_- = 0 \in L_2(\mathbb{R}_+, \mathcal{E})$, $h = 0 \in \mathcal{H}$ in (3.15) with $z = z_+ \in \mathbb{C}_+ \subset \rho(\partial_+)$, the first inclusion in (3.18) follows. Similarly, by putting $h_+ = 0 \in L_2(\mathbb{R}_+, \mathcal{E})$, $h = 0 \in \mathcal{H}$ in (3.8) with $z = z_- \in \mathbb{C}_- \subset \rho(\partial_-)$, the second inclusion in (3.18) follows. The inclusions (3.18) imply that the orthogonal complement of $Y$ is a subset of $\mathcal{H}$.

It remains to show that
\[
(3.19) \quad \operatorname{span} \mathcal{G}(z_+, z_-) = \mathcal{H}.
\]

Using the formulae for the resolvent of the dilation (see (3.8) for $z \in \mathbb{C}_-$, (3.15) for $z \in \mathbb{C}_+$, and Remark 1), one immediately obtains
\[
(3.20) \quad \mathcal{G}(z_+, z_-) = \gamma(z_+)(M(z_+) + iI)^{-1}\mathcal{E} + \gamma(z_-)(M(z_-) - iI)^{-1}\mathcal{E}.
\]

Suppose that $u \in \mathcal{H}$ is such that $u \perp G(z_+, z_-)$ for all $z_+ \in \mathbb{C}_+$, $z_- \in \mathbb{C}_-$. Taking into account that vectors in $\mathcal{E}$ in (3.20) can be chosen independently in the first and second summands, we obtain
\[
\langle \gamma(z_+)(M(z_+) + iI)^{-1}e, u \rangle = \langle \gamma(z_-)(M(z_-) - iI)^{-1}e, u \rangle = 0 \quad \forall z_+ \in \mathbb{C}_+, \; z_- \in \mathbb{C}_-, \; e \in \mathcal{E}.
\]
In particular, for \( z_+ \in \mathbb{C}_+ \) we have
\[
0 = \langle \gamma(z_+)(M(z_+)+iI)^{-1}e, u \rangle \\
= \langle (M(z_+)+iI)^{-1}e, \gamma(z_+)^*u \rangle = \langle (M(z_+)+iI)^{-1}e, \Gamma_1(A_0-\tau_+)^{-1}u \rangle \quad \forall e \in \mathcal{E}.
\]
Since \((M(z_+)+iI)^{-1}E = E\), it follows that \( \Gamma_1(A_0-\tau_+)^{-1}u = 0 \), and hence \((A_0-\tau_+)^{-1}u \in \text{dom}(\tilde{A})\).
Finally, noticing that \((\tilde{A} - \tau_+)E(A_0-\tau_+)^{-1}u = u\), we conclude that \( u \in \text{ran}(\tilde{A} - \tau_+) \). Similarly, we establish that \( u \in \text{ran}(\tilde{A} - \tau_-) \) for \( z_- \in \mathbb{C}_- \). Since \( z_+ \in \mathbb{C}_+, z_- \in \mathbb{C}_- \) above are arbitrary, it follows that
\[
(3.21) \quad u \in \bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \text{ran}(\tilde{A} - zI).
\]
The assumption that \( \tilde{A} \) is simple is equivalent (see [34, Section 1.3]) to the fact that the set on the right-hand side of (3.21) is trivial, and hence \( u = 0 \). This concludes the proof of (3.19). \( \square \)

Remark 2. The terms on the right-hand side of (3.20) are linearly independent. \( \square \)

Proof. Assume that \( e_1, e_2 \in \mathcal{E} \) are such that
\[
(3.22) \quad \gamma(z_+)(M(z_+)+iI)^{-1}e_1 + \gamma(z_-)(M(z_-)-iI)^{-1}e_2 = 0.
\]
Applying \( \Gamma_0 \) and \( \Gamma_1 \) to (3.22) and using the definition of \( \gamma \), we obtain
\[
(M(z_+)+iI)^{-1}e_1 + (M(z_-)-iI)^{-1}e_2 = 0,
\]
\[
(3.23) \quad e_1 - i(M(z_+)+iI)^{-1}e_1 + e_2 + i(M(z_-)-iI)^{-1}e_2 = 0,
\]
respectively. Substituting the first identity above into the second one yields
\[
e_2 + (M(z_+)-iI)(M(z_+)+iI)^{-1}e_1 = 0.
\]
Then the first equality in (3.23) becomes \((M(z_-)-iI)^{-1}(M(z_+)-iI)f_1 = f_1\), where the substitution \( f_1 := (M(z_+)+iI)^{-1}e_1 \) has been used. It follows that \( M(z_-)f_1 = M(z_+)f_1 \). Setting \( z_- = \tau_+ \), we obtain, in particular, \( \Im (M(z_+)f_1, f_1) = 0 \). Combined with the property \( \Im M(z_+) = (\Im z_+)\gamma^*(z_+)^*\gamma(z_+) \) (see (2.15)), this implies \( \|\gamma(z_+)f_1\| = 0 \), which immediately leads to \( f_1 = 0 \) due to (2.1). Finally, we infer \( e_1 = (M(z_+)+iI)f_1 = 0 \), as required. \( \square \)

4. Two-component spectral form of the functional model

Following [39], we introduce a Hilbert space in which we construct a functional model for the operator family \( A_{\alpha,\beta} \), in the spirit of Pavlov [44, 45, 46]. The functional model for completely non-selfadjoint maximal dissipative operators that can be represented as additive perturbations of self-adjoint operators was constructed in [44, 45, 46] and further developed in [39] to include non-dissipative operators. In the context of boundary triples an analogous construction was carried out in [50]. In the most general setting to date, namely the setting of adjoint operator pairs, an explicit three-component model akin to the one we presented in the previous section was constructed in [13], which however stops short of constructing a “spectral”, two-component, form of the model, which is particularly convenient for the development of a scattering theory for operator pairs.\(^4\) In this section we carry out such a construction, tailored to study operators of BVP, in the case when symbol of the operator is formally self-adjoint (but the operator itself can be non-selfadjoint due to the boundary conditions).

Next, we recall some concepts relevant to the construction of [39]. In what follows, we assume throughout that \( \tilde{A} \), see (2.17), is simple and therefore \( L \) is completely non-selfadjoint (see Proposition 2.1).

A function \( f \), analytic on \( \mathbb{C}_\pm \) and taking values in \( \mathcal{E} \), is said to be in the Hardy class \( H^2_{\pm}(\mathcal{E}) \) when
\[
\sup_{y > 0} \int_{\mathbb{R}} \| f(x \pm iy) \|^2 dx < +\infty
\]
\(^4\)We refer the reader to the paper [51], where a three-component model is constructed for a dissipative operator with at least one regular point in the upper half-plane.
and prove a fundamental regularity property for the expressions (2.24), which is crucial for our construction.

**Lemma 4.1.** Let the operators $\Gamma_0$ and $L$ be defined by (2.3) and (2.21), respectively. For all $h \in \mathcal{H}$, one has $\Gamma_0(L - \cdot)^{-1}h \in H^2_+(\mathcal{E})$ and $\Gamma_0(L^* - \cdot)^{-1}h \in H^2_+(\mathcal{E})$. Moreover,

\begin{equation}
\|\Gamma_0(L - \cdot)^{-1}h\|_{H^2_+(\mathcal{E})} \leq \sqrt{\pi} \|h\|_{\mathcal{H}}, \quad \|\Gamma_0(L^* - \cdot)^{-1}h\|_{H^2_+(\mathcal{E})} \leq \sqrt{\pi} \|h\|_{\mathcal{H}}.
\end{equation}

**Proof.** The reasoning goes along the lines of the proof of [50, Lem. 2.4] which in turn is based on the one of [39, Thm. 1].

Suppose that $z \in \mathbb{C}$. Using the Green’s identity (2.12) and the fact that $L \subset A$, we obtain, for all $h \in \mathcal{H}$,

$$2i \|\Gamma_0(L - zI)^{-1}h\|^2 = \langle i\Gamma_0(L - zI)^{-1}h, \Gamma_0(L - zI)^{-1}h \rangle - \langle \Gamma_0(L - zI)^{-1}h, i\Gamma_0(L - zI)^{-1}h \rangle$$

$$= \langle \Gamma_1(L - zI)^{-1}h, \Gamma_0(L - zI)^{-1}h \rangle - \langle \Gamma_0(L - zI)^{-1}h, \Gamma_1(L - zI)^{-1}h \rangle$$

$$= \langle (L - zI)^{-1}h, (L - zI)^{-1}h \rangle - \langle (L - zI)^{-1}h, L(L - zI)^{-1}h \rangle$$

Since $L$ is maximal dissipative, it admits a self-adjoint dilation $A$ [53]. (In the case of the operator $L$ considered here, this dilation is given explicitly by Theorem 3.3. However, we do not require this fact here.)

One concludes, by resorting to the resolvent identity, that

$$\|\Gamma_0(L - zI)^{-1}h\|^2 = \frac{1}{2i} \begin{cases} \langle [(A - zI)^{-1} - (A - zI)^{-1}]h, h \rangle + (z - \overline{z}) \|(L - zI)^{-1}h\|^2 \end{cases}$$

Denoting by $E(t)$, $t \in \mathbb{R}$, the resolution of identity [9, Chapter 6] for $A$ and setting $z = k - i\epsilon$, $k \in \mathbb{R}$, $\epsilon > 0$, one has

$$\|\Gamma_0(L - (k - i\epsilon)I)^{-1}h\|^2 = \epsilon \int_\mathbb{R} \frac{d \langle E(t)h, h \rangle}{(t - k)^2 + \epsilon^2} - \epsilon \|(L - (k - i\epsilon)I)^{-1}h\|^2.$$

Now, using Fubini’s theorem, we obtain

$$\int_\mathbb{R} \|\Gamma_0(L - (k - i\epsilon)I)^{-1}h\|^2 \, dk = \int_\mathbb{R} \left( \epsilon \int_\mathbb{R} \frac{d \langle E(t)h, h \rangle}{(t - k)^2 + \epsilon^2} \right) \, dk - \epsilon \int_\mathbb{R} \|(L - (k - i\epsilon)I)^{-1}h\|^2 \, dk$$

$$= \int_\mathbb{R} \left( \epsilon \int_\mathbb{R} \frac{dk}{(t - k)^2 + \epsilon^2} \right) d \langle E(t)h, h \rangle - \epsilon \int_\mathbb{R} \|(L - (k - i\epsilon)I)^{-1}h\|^2 \, dk$$

$$= \pi \|h\|^2 - \epsilon \int_\mathbb{R} \|(L - (k - i\epsilon)I)^{-1}h\|^2 \, dk.$$

Taking supremum with respect to $\epsilon$, it follows that

$$\|\Gamma_0(L - I)^{-1}h\|_{H^2_+(\mathcal{E})}^2 \leq \pi \|h\|^2.$$

The second inequality in (4.1) of the lemma is proven in the same way. \qed
As mentioned in Section 2, the characteristic function \( S \), given in (2.25), has nontangential limits almost everywhere on the real line in the strong topology. Thus, for a two-component vector function \( \{ \tilde{g}_n \} \in L^2(\mathbb{R}, \mathcal{E}) \), the integral

\[
\int_{\mathbb{R}} \left( \begin{pmatrix} I & S^*(s) \\ S(s) & I \end{pmatrix} \begin{pmatrix} \tilde{g}(s) \\ g(s) \end{pmatrix} \right) ds, \quad \tilde{g}, g \in L^2(\mathbb{R}, \mathcal{E}),
\]

makes sense and is nonnegative due to the contractive properties of \( S \). The space

\[
\mathcal{H} := L^2 \left( \mathcal{E} \oplus \mathcal{E}; \begin{pmatrix} I & S^* \\ S & I \end{pmatrix} \right)
\]

is the completion of the linear set of two-component vector functions \( \{ \tilde{g}_n \} : \mathbb{R} \rightarrow \mathcal{E} \oplus \mathcal{E} \) with respect to the norm (4.2), where a factorisation by vectors on which (4.2) vanishes is assumed. Naturally, not every element of the set can be identified with a pair \( \{ \tilde{g}_n \} \) of independent functions, however we keep the notation \( \{ \tilde{g}_n \} \) for the elements of this space.

Another consequence of the contractive properties of the characteristic function \( S \) is the inequalities

\[
\| \tilde{g} + S^* g \|_{L^2(\mathbb{R}, \mathcal{E})} \leq \left\| \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\|_{\mathcal{H}}, \quad \| S\tilde{g} + g \|_{L^2(\mathbb{R}, \mathcal{E})} \leq \left\| \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\|_{\mathcal{H}}, \quad \forall \tilde{g}, g \in L^2(\mathbb{R}, \mathcal{E}).
\]

They imply, in particular, that for every sequence \( \{ \tilde{g}_n, g_n \}_{n=1}^\infty \) that is Cauchy with respect to the \( \mathcal{H} \)-topology and such that \( \tilde{g}_n, g_n \in L^2(\mathbb{R}, \mathcal{E}) \) for all \( n \in \mathbb{N} \), the limits of \( \tilde{g}_n + S^* g_n \) and \( S\tilde{g}_n + g_n \) exist in \( L^2(\mathbb{R}, \mathcal{E}) \), so that the objects \( \tilde{g} + S^* g \) and \( S\tilde{g} + g \) can always be treated as \( L^2(\mathbb{R}, \mathcal{E}) \) functions.\(^6\)

Consider the following subspaces of \( \mathcal{H} \):\(^7\)

\[
\mathcal{D}_- := \begin{pmatrix} 0 \\ H^2_-(\mathcal{E}) \end{pmatrix}, \quad \mathcal{D}_+ := \begin{pmatrix} H^2_+(\mathcal{E}) \\ 0 \end{pmatrix}.
\]

It is easily seen [46] that the spaces \( \mathcal{D}_- \) and \( \mathcal{D}_+ \) are mutually orthogonal in \( \mathcal{H} \).

Define the subspace

\[
K := \mathcal{H} \ominus (\mathcal{D}_- \oplus \mathcal{D}_+),
\]

which is characterised as follows (see [44, 46]):

\[
K = \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} : \tilde{g} + S^* g \in H^2_-(\mathcal{E}), \quad S\tilde{g} + g \in H^2_+(\mathcal{E}) \right\}.
\]

The orthogonal projection \( P_K \) onto \( K \) is given by (see e.g. [38])

\[
P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+(\tilde{g} + S^* g) \\ g - P_-(S\tilde{g} + g) \end{pmatrix},
\]

where \( P_\pm \) are the orthogonal Riesz projections in \( L^2(E) \) onto \( H^2_\pm(E) \).

**Definition 2** ([50]). The mappings \( \mathcal{F}_\pm : \mathcal{H} \rightarrow L^2(\mathbb{R}, \mathcal{E}) \) are defined by

\[
\mathcal{F}_+ \begin{pmatrix} v_- \\ v \end{pmatrix} := -\frac{1}{\sqrt{\pi}} \Gamma_0(L - (-i0)I)^{-1} v + (S^* v_- + \hat{\nu}_+) (-)
\]

\(5\)This is in fact the same construction as proposed by [46] and further developed by [39]. Henceforth in this section we follow closely the analysis of the named two papers, facilitated by the fact that essentially this way to construct the functional model only relies upon the characteristic function \( S \) of the maximal dissipative operator and an estimate of the type claimed in Lemma 4.1 above. A similar argument for extensions of symmetric operators, based on the theory of boundary triples, was developed in [50], [18].

\(6\)In general, \( \tilde{g} + S^* g \) and \( S\tilde{g} + g \) are not independent of each other, see [28].

\(7\)In the language of scattering theory [35], the subspaces \( \mathcal{D}_-, \mathcal{D}_+ \) are “incoming” and “outgoing” subspaces, respectively, for the group of translations of \( \mathcal{H} \), as was first observed in [44].
and
\[
\mathcal{F}_- \begin{pmatrix} v_- \\ v_+ \end{pmatrix} := -\frac{1}{\sqrt{\pi}} \Gamma_0(L^* - (\cdot + i0)I)^{-1}v + (\tilde{v}_- + S\tilde{v}_+)(\cdot).
\]

Based on the above definition, we will now introduce a map from $\mathcal{H}$ to $\mathfrak{F}$, which will prove to be unitary. We will then show that $\mathfrak{F}$ serves as a representation space for the spectral form of the functional model discussed in Section 3. We implement this strategy in Lemmata 4.2–4.6.

**Lemma 4.2.** Fix $z_+ \in \mathbb{C}_+$, $z_- \in \mathbb{C}_-$, and consider the map
\[
\Phi : \begin{pmatrix} L_2(\mathbb{R}_-, \mathcal{E}) \\ \mathcal{G}(z_+, z_-) \end{pmatrix} \ni \begin{pmatrix} v_- \\ v_+ \end{pmatrix} \mapsto \begin{pmatrix} \tilde{v}_+(-) + \frac{i}{\sqrt{2\pi}} \left[ \frac{1}{1 - z_-} S^*(\bar{\tau}_-) w_- - \frac{1}{1 - z_+} w_+ \right] \\ \tilde{v}_-(-) - \frac{i}{\sqrt{2\pi}} \left[ \frac{1}{1 - z_-} w_- - \frac{1}{1 - z_+} S(z_+) w_+ \right] \end{pmatrix} \in \mathfrak{F},
\]
where $w_+, w_- \in \mathcal{E}$ are determined uniquely, by Remark 2, from
\[
v = \sqrt{2}\gamma(z_+)(M(z_+) + iI)^{-1}w_+ + \sqrt{2}\gamma(z_-)(M(z_-) - iI)^{-1}w_-. 
\]

The map $\Phi$ satisfies
\[
(I \quad S^*) \Phi \begin{pmatrix} v_- \\ v_+ \end{pmatrix} = \begin{pmatrix} \mathcal{F}_+ \begin{pmatrix} v_- \\ v_+ \end{pmatrix} \\ \mathcal{F}_- \begin{pmatrix} v_- \\ v_+ \end{pmatrix} \end{pmatrix} \in \begin{pmatrix} L_2(\mathbb{R}_-, \mathcal{E}) \\ \mathcal{G}(z_+, z_-) \end{pmatrix}.
\]

**Proof.** Taking into account Definition 2, one immediately verifies that (4.7) holds for $v = 0$. Since $\Phi, \mathcal{F}_\pm$ are linear, it only remains to prove the assertion when $v_- = 0 \in L_2(\mathbb{R}_- \mathcal{E}), v_+ = 0 \in L_2(\mathbb{R}_+ \mathcal{E})$. Under this assumption, consider the first row in the vector equality (4.7), where $v$ is replaced by the formula (4.6):
\[
i \sqrt{2\pi} \Gamma_0(L - (\cdot - i0)I)^{-1} \left[ \gamma(z_+) (M(z_+) + iI)^{-1}w_+ + \gamma(z_-) (M(z_-) - iI)^{-1}w_- \right]
\]

In what follows, we show that
\[
i \frac{1}{z - z_-} [S^*(\cdot) - S^*(\bar{\tau}_-)] w_- = 2\Gamma_0(L - (\cdot - i0)I)^{-1} \gamma(z_+) (M(z_-) - iI)^{-1}w_-
\]
and
\[
i \frac{1}{z - z_+} [I - S^*(\cdot)S(z_+)] w_+ = 2\Gamma_0(L - (\cdot - i0)I)^{-1} \gamma(z_+) (M(z_+) + iI)^{-1}w_+,
\]
and therefore (4.8) holds, as required. To verify (4.9) first, consider $z \in \mathbb{C}_-$. Using the second resolvent identity, it follows from (2.26) that
\[
\frac{1}{z - z_-} [S^*(\bar{\tau}) - S^*(\bar{\tau}_-)] = \frac{2i}{z - z_-} \left[ (M(z) - iI)^{-1} - (M(z_-) - iI)^{-1} \right] 
\]
\[
= \frac{2i}{z - z_-} (M(z) - iI)^{-1} [M(z) - M(z_-)] (M(z_-) - iI)^{-1}
\]
Therefore, by (2.15), (2.24), one has
\[
\frac{1}{z - z_-} [S^*(\bar{\tau}) - S^*(\bar{\tau}_-)] = 2i (M(z) - iI)^{-1} \gamma(z_-) (M(z_-) - iI)^{-1}
\]
\[
= -2i \Gamma_0 (L - zI)^{-1} \gamma(z_-) (M(z_-) - iI)^{-1}.
\]
Passing to the limit as $z$ approaches a real value, we infer that (4.9) is satisfied for all $w_+ \in \mathcal{E}$. To prove (4.10) for all $w_+ \in \mathcal{E}$, we proceed in a similar way. By straightforward calculations, one has, for $z \in \mathbb{C}_-$,

$$
\frac{1}{z - z_+} \left[ I - S^*(\overline{z})S(z+) \right] = -\frac{2i}{z - z_+} \left[ (M(z) - iI)^{-1} - (M(z_+) + iI)^{-1} - 2i(M(z) - iI)^{-1}(M(z_+) + iI)^{-1} \right] = -\frac{2i}{z - z_+} (M(z) - iI)^{-1} [(M(z_+) + iI) - (M(z) - iI) - 2iI] (M(z_+) + iI)^{-1}
$$

Proceeding in the same way as (4.12) was obtained from (4.11), one obtains

$$
\frac{1}{z - z_+} (I - S^*(\overline{z})S(z+)) = -2i \Gamma_0 (L - zI)^{-1} \gamma(z_+) (M(z_+) - iI)^{-1},
$$

which, by passing to the limit as $z$ approaches the real line, yields the required property.

The second entry of the vector equality (4.7) is proved in a similar way. \(\square\)

**Lemma 4.3.** The mapping $\Phi$, given in Lemma 4.2, is an isometry from $L_2(\mathbb{R}_-, \mathcal{E}) \oplus \{0\} \oplus L_2(\mathbb{R}_+, \mathcal{E})$ onto $\mathfrak{D}_- \oplus \mathfrak{D}_+$.

**Proof.** Clearly, for all $v \in L_2(\mathbb{R}_+, \mathcal{E})$, one has

$$
\left\| \Phi \begin{pmatrix} v_- \\ 0 \\ 0 \end{pmatrix} \right\|_{\mathfrak{D}} = \left\| \begin{pmatrix} 0 \\ \hat{v}_- \end{pmatrix} \right\|_{\mathfrak{D}}, \quad \left\| \Phi \begin{pmatrix} 0 \\ 0 \\ v_+ \end{pmatrix} \right\|_{\mathfrak{D}} = \left\| \begin{pmatrix} \hat{v}_+ \\ 0 \end{pmatrix} \right\|_{\mathfrak{D}}.
$$

Thus, taking into account that the spaces $\mathfrak{D}_-$ and $\mathfrak{D}_+$ are orthogonal (see the discussion following the formula (4.3)), one has

$$
\left\| \Phi \begin{pmatrix} v_- \\ v_+ \end{pmatrix} \right\|_{\mathfrak{D}}^2 = \left\| \begin{pmatrix} 0 \\ \hat{v}_- \end{pmatrix} \right\|_{\mathfrak{D}}^2 + \left\| \begin{pmatrix} \hat{v}_+ \\ 0 \end{pmatrix} \right\|_{\mathfrak{D}}^2.
$$

Finally note that

$$
\left\| \begin{pmatrix} 0 \\ \hat{v}_- \end{pmatrix} \right\|_{L_2(\mathbb{R}, \mathcal{E})} = \left\| v_- \right\|_{L_2(\mathbb{R}_-, \mathcal{E})}, \quad \left\| \begin{pmatrix} \hat{v}_+ \\ 0 \end{pmatrix} \right\|_{L_2(\mathbb{R}, \mathcal{E})} = \left\| v_+ \right\|_{L_2(\mathbb{R}_+, \mathcal{E})}.
$$

The surjectivity of the mapping follows from the fact that the Fourier transform is a unitary mapping between $L_2(\mathbb{R}_\pm, \mathcal{E})$ and $H^2_\pm(\mathcal{E})$, by the Paley-Wiener theorem. \(\square\)

**Lemma 4.4.** The mapping $\Phi$, given in Lemma 4.2 and extended by linearity to

$$
L_2(\mathbb{R}_- \mathcal{E}) \oplus \text{span} \ G(z_+, z_-) \oplus L_2(\mathbb{R}_+, \mathcal{E})
$$

is an isometry from the set (4.13) to $\mathfrak{D}$.

**Proof.** Due to (4.4) and Lemma 4.3, the assertion will be proved if one shows first that

$$
\Phi \begin{pmatrix} \text{span}_{z \in \mathbb{C}_\pm} G(z_+, z_-) \end{pmatrix} \subset K
$$

and, second, that for all $z \in \mathbb{C}_\pm$ and $v \in G(z_+, z_-)$ one has

$$
\left\| \Phi \begin{pmatrix} 0 \\ v \end{pmatrix} \right\|_{\mathfrak{D}} = \| v \|_{\mathcal{H}}.
$$
In view of the definition of $\Phi$, see Lemma 4.2, to establish (4.14) it suffices to verify that, for $z_\pm \in \mathbb{C}_\pm$ and $w_+, w_- \in \mathcal{E}$ chosen as in (4.6), the vectors
\[
\begin{align*}
\frac{1}{\cdot - z_-} \begin{pmatrix} S^*(\tau_-) & w_- \end{pmatrix}, \quad & \frac{1}{\cdot - z_+} \begin{pmatrix} -w_+ \end{pmatrix} \\
\end{align*}
\]
are orthogonal to $\mathcal{D}_- \oplus \mathcal{D}_+$. To this end, consider $h_\pm \in H^\pm_2(\mathcal{E})$. Taking into account the fact that
(4.16) \quad (\cdot - z_-)^{-1} w_- \in H^+_2(\mathcal{E}),
we obtain
\[
\begin{align*}
\left\langle \frac{1}{\cdot - z_-} \begin{pmatrix} S^*(\tau_-) & w_- \end{pmatrix}, \left( h_+^+ \vphantom{\begin{pmatrix} w_- \end{pmatrix}} \right) \right\rangle_{\mathcal{B}} &= \left\langle \frac{1}{\cdot - z_-} S^*(\tau_-) w_- , h_+ + S^* h_- \right\rangle_{L^2(\mathcal{E})} - \left\langle \frac{1}{\cdot - z_-} w_-, S h_+ + h_- \right\rangle_{L^2(\mathcal{E})} \\
&= \left\langle \frac{1}{\cdot - z_-} S^*(\tau_-) w_- , h_+ \right\rangle_{L^2(\mathcal{E})} - \left\langle \frac{1}{\cdot - z_-} w_-, S h_+ \right\rangle_{L^2(\mathcal{E})} \\
&= - \left\langle S^*(\cdot) - S^*(\tau_-) \frac{w_-}{\cdot - z_-} , h_+ \right\rangle_{L^2(\mathcal{E})}.
\end{align*}
\]
Now analytically continuing the function $S^*$ to the lower half-plane and using the fact that
\[
\frac{S^*(\cdot) - S^*(\tau_-)}{\cdot - z_-} w_- \in H^+_2(\mathcal{E}),
\]
we conclude that the expression (4.17) vanishes, as required.

In the same way, since
(4.18) \quad (\cdot - z_+)^{-1} w_+ \in H^+_2(\mathcal{E}),
we conclude that
\[
\left\langle \frac{1}{\cdot - z_+} \begin{pmatrix} -w_+ \end{pmatrix}, \left( h_+^+ \vphantom{\begin{pmatrix} w_- \end{pmatrix}} \right) \right\rangle_{\mathcal{B}} = - \left\langle \frac{S(\cdot) - S(z_+)}{\cdot - z_+} w_-, h_- \right\rangle_{L^2(\mathcal{E})} = 0.
\]
It remains to prove (4.15). In view of Lemma 4.2 and Definition 2, one has
\[
\left\| \Phi \begin{pmatrix} 0 \\ v \end{pmatrix} \right\|^2_{\mathcal{B}} = \left\langle \begin{pmatrix} I & S^* \\ S & I \end{pmatrix} \Phi \begin{pmatrix} 0 \\ v \end{pmatrix} \right\rangle_{L^2(\mathcal{E})} = \left\langle \begin{pmatrix} \mathcal{F}_+ & 0 \\ 0 & \mathcal{F}_- \end{pmatrix} \Phi \begin{pmatrix} 0 \\ v \end{pmatrix} \right\rangle_{L^2(\mathcal{E})} = \left\langle \begin{pmatrix} -\frac{1}{\sqrt{\pi}} \Gamma_0(L - (\cdot - i0))^{-1} v \\ \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{\cdot - z_-} S^*(\tau_-) w_- - \frac{1}{\cdot - z_+} w_+ \right] \end{pmatrix} \right\rangle.
\]
By Lemma 4.1, one has $\Gamma_0(L - \cdot)^{-1} v \in H^+_2(\mathcal{E})$ and $\Gamma_0(L^* - \cdot)^{-1} v \in H^+_2(\mathcal{E})$. Thus, in view of (4.16) and (4.18), one obtains, using the Cauchy formula for Hardy classes, that
\[
\left\| \Phi \begin{pmatrix} 0 \\ v \end{pmatrix} \right\|^2_{\mathcal{B}} = \left\langle \begin{pmatrix} \frac{1}{\sqrt{\pi}} \Gamma_0(L - (\cdot - i0))^{-1} v \\ \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{\cdot - z_+} w_+ \right] \end{pmatrix} \right\rangle_{L^2(\mathcal{E})} = \sqrt{2} \left( \langle \Gamma_0(L - \tau_- I) v, w_+ \rangle + \langle \Gamma_0(L^* - \tau_+ I) v, w_- \rangle \right)
\]
(4.19) \quad = \sqrt{2} \langle v, \gamma(z_+)(M(z_+) + iI)w_+ + \gamma(z_-)(M(z_-) - iI)w_- \rangle,
where for obtaining the last equality we use (2.24). Due to (4.6), the formula (4.19) immediately implies (4.15), as required.

Due to Lemma 3.4 and Lemma 4.4, the mapping $\Phi$ can be extended by continuity to the whole space $\mathcal{H}$, provided that the operator $\hat{A}$ is simple. We will use same notation $\Phi$ for this extension.

**Lemma 4.5.** For all $z \in \mathbb{C} \setminus \mathbb{R}$, one has $\Phi(A - zI)^{-1} = (\cdot - z)^{-1}\Phi$.

**Proof.** We prove the statement for $z \in \mathbb{C}_+$, as the case $z \in \mathbb{C}_-$ is established in a similar way.

Consider an arbitrary $(h_-, h, h_+)^T \in \mathcal{H}$, and let $(f_-, f, f_+)^T$ be the vector defined by (3.15). It follows from (3.13) that

\[(\cdot - z)\hat{f}_\pm(\cdot) = \hat{h}_\pm(\cdot) \pm \frac{i}{\sqrt{2\pi}}f_\pm(0).\]

Recall that $\hat{h}_\pm$ and $\hat{f}_\pm$ are the Fourier transforms of $h_\pm$ and $f_\pm$, respectively. According to Definition 2 and (3.15), one has

\[
\mathcal{F}_- \begin{pmatrix} f_- \\ f_0 \\ f_+ \end{pmatrix} = -\frac{1}{\sqrt{\pi}} \Gamma_0(L^* - (\cdot + i0)I)^{-1}f + \hat{f}_-(\cdot) + (\hat{S}f_+)(\cdot)
\]

\[
= -\frac{1}{\sqrt{\pi}} \Gamma_0(L^* - (\cdot + i0)I)^{-1} \left[ (L^* - zI)^{-1}h + \sqrt{2}\gamma(z)(M(z) + iI)^{-1}f_+(0) \right]
\]

\[
+ \frac{1}{\cdot - z} \left[ (\hat{h}_- + S\hat{h}_+)(\cdot) + \frac{i}{\sqrt{2\pi}}(Sf_+(0) - f_-(0)) \right]
\]

where to obtain the expression in the second square brackets we invoke (4.20). Thus, using the resolvent identity and (2.25),

\[
\mathcal{F}_- \begin{pmatrix} f_- \\ f_0 \\ f_+ \end{pmatrix} = \frac{1}{\cdot - z} \mathcal{F}_- \begin{pmatrix} h_- \\ h_0 \\ h_+ \end{pmatrix} - \frac{1}{\sqrt{\pi}} \Gamma_0(L^* - (\cdot + i0)I)^{-1}\gamma(z)(M(z) + iI)^{-1}\sqrt{2}f_+(0)
\]

\[
+ \frac{1}{\cdot - z} \sqrt{\pi} \left( \Gamma_0(L^* - zI)^{-1}h + \frac{i}{\sqrt{2}} \left[ f_+(0) - f_-(0) - 2i(M(\cdot + i0) + iI)^{-1}f_+(0) \right] \right).
\]

Consider the third term on the right-hand side of (4.21) evaluated at $\zeta \in \mathbb{C}_+$. Using the property (cf. (3.4))

\[f_+(0) - f_-(0) = \sqrt{2}i\Gamma_0f,\]

we write it as follows:

\[
\frac{1}{(\zeta - z)^{3/2}} \Gamma_0(L^* - zI)^{-1}h + \frac{i}{\sqrt{2}} \left[ f_+(0) - f_-(0) - 2i(M(\zeta) + iI)^{-1}f_+(0) \right]
\]

\[
= \frac{1}{(\zeta - z)^{1/2}} \left( \Gamma_0(L^* - zI)^{-1}h - \Gamma_0f + \sqrt{2}(M(z) + iI)^{-1}f_+(0) \right)
\]

\[
= \frac{\sqrt{2}}{(\zeta - z)^{1/2}} \left( -\Gamma_0\gamma(z)(M(z) + iI)^{-1}f_+(0) + (M(\zeta) + iI)^{-1}f_+(0) \right)
\]

\[
= \frac{\sqrt{2}}{(\zeta - z)^{1/2}} \left( (M(\zeta) + iI)^{-1} - (M(z) + iI)^{-1} \right) f_+(0)
\]

\[
= -\sqrt{\frac{2}{\pi}} (M(\zeta) + iI)^{-1} \left( \frac{M(\zeta) - M(z)}{\zeta - z} \right) (M(z) + iI)^{-1}f_+(0)
\]

\[
= -\sqrt{\frac{2}{\pi}} (M(\zeta) + iI)^{-1} \gamma(\zeta)^*(\gamma(z)(M(z) + iI)^{-1}f_+(0),
\]

\[(4.21) \quad = \sqrt{\frac{2}{\pi}} \Gamma_0(L^* - \zeta I)^{-1}\gamma(z)(M(z) + iI)^{-1}f_+(0).\]
where for the second equality $f$ is replaced by (3.15), while for the third and fourth equalities we have used (2.8) and the second resolvent identity, respectively. Furthermore, we utilise (2.15) to obtain the fifth equality. The identities (2.24) now yield the final expression (4.21).

It follows that the second and third terms on the right-hand side of (4.21) cancel each other as $\zeta$ approaches the real line. We have therefore shown that

\[(4.22) \quad \mathcal{F}_-(A - zI)^{-1} = (\cdot - z)^{-1}\mathcal{F}_-, \quad z \in \mathbb{C}_+.\]

Similarly, one proves that

\[(4.23) \quad \mathcal{F}_+(A - zI)^{-1} = (\cdot - z)^{-1}\mathcal{F}_+, \quad z \in \mathbb{C}_+.\]

Combining (4.22), (4.23), and Lemma 4.2 yields the claim. □

**Lemma 4.6.** The operator $\Phi$ maps $\mathcal{H}$ onto $\mathcal{H}$ unitarily.

**Proof.** In view of Lemma 4.4, the mapping $\Phi$ is an isometry defined in the whole space $\mathcal{H}$. It thus suffices to show that the range of $\Phi$ is dense in $\mathcal{H}$. To this end, suppose $g \in \mathcal{H}$ is such that

\[(4.24) \quad \left\langle g, \Phi \begin{pmatrix} v_- \\ v \\ v_+ \end{pmatrix} \right\rangle_{\mathcal{H}} = 0 \quad \forall \left(\begin{pmatrix} v_- \\ v \\ v_+ \end{pmatrix} \right) \in \text{span}_{z \in \mathbb{C}_+} G(z_+, z_-) \subseteq L_2(\mathbb{R}_+, \mathcal{E}) \text{ and } \text{span}_{z \in \mathbb{C}_+} G(z_+, z_-) \subseteq L_2(\mathbb{R}_+, \mathcal{E}).

By Lemma 4.3 and the definition of the subspace $K$, see (4.4), this is equivalent to the existence of a nonzero $g \in K$ such that (4.24) holds with $v_+ = 0$, $v_+ = 0$. On the other hand, since $\Phi^*g \in \mathcal{H}$, one has

\[0 = \left\langle g, \Phi \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \right\rangle_{\mathcal{H}} = \langle \Phi^*g, v \rangle_{\mathcal{H}} \quad \forall v \in \text{span}_{z \in \mathbb{C}_+} G(z_+, z_-),

which by Lemma 3.4 yields $\Phi^*g = 0$, and hence $g = 0$. □

Combining the above lemmata, we obtain the following result, concerning the representation of the dilation $A$ as the operator of multiplication in the two-component model space $\mathcal{H}$.

**Theorem 4.7.** Under the above definitions of $A$ and $\Phi$, one has

\[(L - zI)^{-1} = \Phi^* P_K (\cdot - z)^{-1} |_K \Phi, \quad z \in \mathbb{C}_-,\]

\[(A - zI)^{-1} = \Phi^* (\cdot - z)^{-1} \Phi, \quad z \in \mathbb{C} \setminus \mathbb{R},\]

where $\Phi$ is unitary from $\mathcal{H}$ to $\mathcal{H}$.

5. **Boundary traces of the resolvents of BVP**

Our aim here is to derive an explicit formula for the solution operator of the spectral boundary-value problem (2.19). To this end, consider the operator (see (2.20), (5.5), cf. [52, Section 5])

\[-(\alpha + \beta M(z))^{-1} \beta,

for all $z$ such that $0 \in \rho(\alpha + \beta M(z))$. It is convenient to assume that $\beta$ is boundedly invertible, which we do henceforth. Recall, that above (see Section 2) we have also required that $\beta$ is bounded, and $\alpha$ is such that $\text{dom}(\alpha) \supset \text{dom}(\Lambda)$ and $\alpha + \beta \Lambda$ is closable.

We note that $\tilde{M}(z) := M(z) - \Lambda$ is bounded and

\[(5.1) \quad (\alpha + \beta M(z))^{-1} \beta = (\alpha + \beta \Lambda + \beta \tilde{M}(z))^{-1} \beta = (\beta^{-1}(\alpha + \beta \Lambda) + \tilde{M}(z))^{-1}.

Furthermore, one has $\text{dom}(\Lambda) \subset \text{dom}(\alpha + \beta \Lambda)$, and

\[\beta^{-1} \alpha + \Lambda \subset \beta^{-1}(\alpha + \beta \Lambda).\]
In addition, \( \beta^{-1}(\alpha + \beta \Lambda) \) is closed, as a consequence of the general fact that whenever \( T_1 \) is bounded with a bounded inverse and \( T_2 \) is closed, the operator \( T_1 T_2 \) is closed. Therefore, \( \beta^{-1} \alpha + \Lambda \) is closable and

\[
\beta^{-1} \alpha + \Lambda = \beta^{-1} (\alpha + \beta \Lambda).
\]

Combining (5.1) and (5.2), we obtain

\[
(\alpha + \beta \mathcal{M}(z))^{-1} = (\beta^{-1} \alpha + \Lambda + \mathcal{M}(z))^{-1} = (B + M(z))^{-1}, \quad B := \beta^{-1} \alpha,
\]

and [52, Theorem 5.1] implies that

\[
Q_B := \{ z : 0 \in \rho(B + M(z)) \} \subset \rho(A_{\alpha \beta}).
\]

For convenience, henceforth we use the notation \( Q_B(\alpha \beta) := \{ z : 0 \in \rho(B + M(z)) \} \subset \rho(A_{\alpha \beta}) \).

Notice that [52, Theorem 5.1] requires \( Q_B \neq \emptyset \), which cannot be guaranteed in the most general setup. In the present article we focus on the PDE setting, where the standard choice of boundary conditions implies that \( \Lambda \) is the Dirichlet-to-Neumann map [52]. This allows us to make some reasonable assumptions that are bound to hold provided the boundary of the spatial domain in the BVP is smooth, so that [52, Theorem 5.1] is applicable and the resulting operator \( A_{\alpha \beta} \) has discrete spectrum in \( \mathbb{C}^- \cup \mathbb{C}^+ \). In what follows, we utilise the standard notation \( \mathcal{E}_\infty \) the Banach algebra of compact operators [9, Section 11] on the boundary space \( \mathcal{E} \).

Lemma 5.1. Suppose that \( \Lambda \) is the Dirichlet-to-Neumann map of a BVP problem, such that it is a self-adjoint operator with purely discrete spectrum, accumulating to \(-\infty\). Then \( M(z)^{-1} \in \mathcal{E}_\infty \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \).

Proof. Choose a finite-rank operator \( K \) such that \( \Lambda + K \) has trivial kernel and \((\Lambda + K)^{-1} \in \mathcal{E}_\infty \). Such a choice is obviously always possible. Furthermore, by the second Hilbert identity,

\[
M(z)^{-1} - (\Lambda + K)^{-1} = (\Lambda + K)^{-1} \Xi M(z)^{-1},
\]

where \( \Xi \) is a bounded operator. Hence, \( M(z)^{-1} \in \mathcal{E}_\infty \). \( \square \)

Corollary 5.2. Within the conditions of Lemma 5.1, if \( B \) is bounded, then \( BM(z)^{-1} \in \mathcal{E}_\infty \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \).

Remark 3. Note that if one drops the condition that \( B \) is bounded, it is possible for \( Q_B \) to be empty. Indeed, put \( \alpha = -\Lambda \) and \( \beta = 1 \) (as shown in [52], under these assumptions the operator \( A_{\alpha \beta} \) is the Krein extension [3] of the operator \( \bar{A} \)). Then by (2.14) one has

\[
B + M(z) = z \Pi^*(I - z A_0^{-1})^{-1} \Pi,
\]

which is shown to be compact under the assumptions of Lemma 5.1. However, the following theorem suggests that instead of the restriction that \( B \) be bounded, it suffices to assume that it is compact relative to \( M(z) \), in order to ensure that \( Q_B \) coincides with \( \mathbb{C} \setminus \mathbb{R} \) with the exception of a discrete set.

Theorem 5.3. Suppose that \( BM(z)^{-1} \in \mathcal{E}_\infty \) for at least one \( z \in \mathbb{C}^+ \) and at least one \( z \in \mathbb{C}^- \) (and hence at all \( z \in \mathbb{C} \setminus \mathbb{R} \)), where \( B \) is defined by (5.3). If \( I + BM(z)^{-1} \) is invertible for for at least one \( z \in \mathbb{C}^+ \) and at least one \( z \in \mathbb{C}^- \), then

1) The operator \( A_{\alpha \beta} \) has at most discrete spectrum in \( \mathbb{C} \setminus \mathbb{R} \) (accumulating at the real line only).

2) One has \( \rho(A_{\alpha \beta}) = Q_B \).

Proof. By the Analytic Fredholm Theorem, see [48, Theorem 8.92], the operator \( I + BM(z)^{-1} \) is invertible at all \( z \in \mathbb{C} \setminus \mathbb{R} \) with the exception of a discrete set of points. Therefore, for any \( z \) such that the inverse exists, one has

\[
(B + M(z))^{-1} = M(z)^{-1}(I + BM(z)^{-1})^{-1}.
\]

This implies that the “Krein formula”, cf. (2.20), holds at all \( z \in \mathbb{C} \setminus \mathbb{R} \) with the exception of a discrete set of points:

\[
(A_{\alpha \beta} - z)^{-1} = (A_0 - z I)^{-1} - (I - z A_0^{-1})^{-1} \Pi (B + M(z))^{-1} \Pi^*, \quad B = \beta^{-1} \alpha,
\]

and therefore \( \rho(A_{\alpha \beta}) \) is discrete in \( \mathbb{C} \setminus \mathbb{R} \), which proves the first claim.

---

8 Any BVP for a second-order elliptic PDE in a domain with smooth boundary has these properties, as follows from a straightforward argument based on the Poincaré-Wirtinger inequality and the Lax-Milgram lemma.
Furthermore, the right-hand side of (5.5) is analytic whenever its left-hand side is, i.e. on the set \( \rho(A_{\alpha\beta}) \), which immediately implies the inclusion \( \rho(A_{\alpha\beta}) \subset Q_B \). The second claim of the theorem now follows by comparing this with (5.4).

The formulae in the next lemma are analogous to [50, Eqs. (2.18), (2.22)].

**Lemma 5.4.** Assume that \( B \) defined by (5.3) is bounded. Then the following identities hold:

\[
\Gamma_0(A_{\alpha\beta} - zI)^{-1} = \Theta_B(z)^{-1} \Gamma_0(L - zI)^{-1} \quad \forall z \in \mathbb{C}_- \cap Q_B, \\
\Gamma_0(A_{\alpha\beta} - zI)^{-1} = \hat{\Theta}_B(z)^{-1} \Gamma_0(L^* - zI)^{-1} \quad \forall z \in \mathbb{C}_+ \cap Q_B,
\]

where \( \Theta_B \) and \( \hat{\Theta}_B \) are defined via their inverses:

\[
\Theta_B(z)^{-1} = 2iQ_B(z)(I - S^*(z))^{-1}, \quad z \in \mathbb{C}_- \cap Q_B, \\
\hat{\Theta}_B(z)^{-1} = 2iQ_B(z)(I - S(z))^{-1}, \quad z \in \mathbb{C}_+ \cap Q_B.
\]

**Proof.** Fix an arbitrary \( h \in \mathcal{H} \) and define

\[
g_{\alpha\beta} := (A_{\alpha\beta} - zI)^{-1}h, \quad z \in \rho(A_{\alpha\beta}),
\]

so that, in particular, \( g_{-iI}I = (L - zI)^{-1}h, z \in \mathbb{C}_+ \), and \( g_{iI}I = (L^* - zI)^{-1}h, z \in \mathbb{C}_- \).

In order to prove (5.6), suppose that \( z \in \mathbb{C}_- \cap Q_B \), so the resolvents \( (L - zI)^{-1} \) and \( (A_{\alpha\beta} - zI)^{-1} \) are defined on the whole space \( \mathcal{H} \). Clearly, the vector

\[
g := g_{-iI}I - g_{\alpha\beta} = ((L - zI)^{-1} - (A_{\alpha\beta} - zI)^{-1}) h
\]

is an element of \( \ker(A - zI) \). It follows from \( g_{-iI}I \in \text{dom}(L) \) and \( g_{\alpha\beta} \in \text{dom}(A_{\alpha\beta}) \) that \( \Gamma_1g_{-iI}I = i\Gamma_0g_{-iI}I \) and \( \beta\Gamma_1g_{\alpha\beta} = -\alpha\Gamma_0g_{\alpha\beta} \), and therefore one has

\[
0 = \beta\Gamma_1(g + g_{\alpha\beta}) - i\beta\Gamma_0(g + g_{\alpha\beta}) = \beta\Gamma_1g - i\beta\Gamma_0g + \beta\Gamma_1g_{\alpha\beta} - i\beta\Gamma_0g_{\alpha\beta}
\]

\[
= \beta M(z)\Gamma_0g - i\beta\Gamma_0g - \alpha\Gamma_0g_{\alpha\beta} - i\beta\Gamma_0g_{\alpha\beta},
\]

where in the last equality we also use the fact that \( g \in \ker(A - zI) \), together with Definition 1. Hence, by collecting the terms in the calculation (5.10), one has (cf. (5))

\[
(\alpha + \beta M(z))\Gamma_0g = (\alpha + i\beta)\Gamma_0(g + g_{\alpha\beta}) = (\alpha + i\beta)\Gamma_0g_{-iI}I,
\]

which, in turn, implies that, for \( z \in Q_B \) one has

\[
\left\{ I - \left( B + M(z) \right)^{-1}(B + iI) \right\} \Gamma_0g_{-iI}I = \Gamma_0g_{\alpha\beta}.
\]

Finally, using the second resolvent identity

\[
(B + iI)^{-1} - (B + M(z))^{-1} = (B + M(z))^{-1}(M(z) - iI)(B + iI)^{-1},
\]

we obtain

\[
I - \left( B + M(z) \right)^{-1}(B + iI) = \left( B + M(z) \right)^{-1}(M(z) - iI) = 2iQ_B(z)(I - S^*(z))^{-1},
\]

where we use the formula (2.26).

The identity (5.7) is proved by an argument similar to the above, where the vector \( g_{-iI}I \) is replaced with \( g_{iI}I \), for \( z \in \mathbb{C}_+ \), and the formula (2.25) is used instead of (2.26).

**Remark 4.** Note that the boundedness condition imposed on \( B \) in Lemma 5.4 can be relaxed. Not only can we assume that \( B \) is such that \( BM(z)^{-1} \in \mathcal{S}_\infty \), as suggested by Theorem 5.3, but the latter condition can be relaxed even further by assuming that \( B \) is bounded relative to \( M(z) \) with the bound\(^9\) less than 1 (see [31]), which clearly suffices for \( B + M(z) = B + M(z) \). In present paper, however, we limit ourselves to physically motivated applications to BVP, which renders these considerations unnecessary. For this reason in what follows we will only consider the case when the parameter \( B \) is bounded.

\(^9\)In the case when \( B \) is compact relative to \( M(z) \), the bound is zero, see [7].
6. Functional model for non-necessarily dissipative operators

In this section we obtain a useful representation for the resolvent of $A_{\alpha\beta}$ in the Hilbert space $\mathcal{H}$, i.e. in the spectral functional model representation of $L$. The results of this section generalise those of [50]. We start by proving the following lemma. Throughout we assume that the condition imposed by Lemma 5.1 holds.

**Lemma 6.1.** Suppose that $B$ defined by (5.3) is bounded, and denote

$$\chi_B^\pm \colonequals \frac{1}{2i}(\pm B + iI).$$

The following formulae hold for the functions defined in (5.8), (5.9):

$$\Theta_B(z) = S^*(\mathcal{G})\chi_B^+ + \chi_B^-, \quad z \in \mathbb{C}_- \cap \mathcal{Q}_B,$$

$$\widehat{\Theta}_B(z) = S(z)\chi_B^+ + \chi_B^-, \quad z \in \mathbb{C}_+ \cap \mathcal{Q}_B.$$

**Proof.** By the definition (5.8) and using the representation (2.26), we write, for $z \in \mathbb{C}_- \cap \mathcal{Q}_B$,

$$\Theta_B(z) = -(2i)^{-1}(I - S^*(\mathcal{G}))(B + M(z)) = -(2i)^{-1}(I - S^*(\mathcal{G}))(B - 2i(I - S^*(\mathcal{G}))^{-1} + iI)$$

$$= -(I - S^*(\mathcal{G}))\chi_B^+ + I = S^*(\mathcal{G})\chi_B^+ + \chi_B^-,$$

as claimed in (6.1). Similarly, by the definition (5.9) and using (2.25), we obtain (6.2). \hfill \square

The following is the main result of this section and is similar in form to [50, Theorem 2.5] and [39, Theorem 3]. Its proof closely follows the lines of the mentioned works.

**Theorem 6.2.** Suppose that $\beta^{-1}$ and $B = \beta^{-1}A$ are bounded. Then

(i) If $z \in \mathbb{C}_- \cap \mathcal{Q}_B$ and $(\bar{g}, g)^\top \in K$, then

$$\Phi(A_{\alpha\beta} - zI)^{-1}\Phi^* \begin{pmatrix} \bar{g} \\ g \end{pmatrix} = P_K \frac{1}{z - \chi_B} \begin{pmatrix} \bar{g} \\ g - \chi_B\Theta_B(z)^{-1}(\bar{g} + S^*g)(z) \end{pmatrix}.$$

(ii) If $z \in \mathbb{C}_+ \cap \mathcal{Q}_B$ and $(\bar{g}, g)^\top \in K$, then

$$\Phi(A_{\alpha\beta} - zI)^{-1}\Phi^* \begin{pmatrix} \bar{g} \\ g \end{pmatrix} = P_K \frac{1}{z - \chi_B} \begin{pmatrix} \bar{g} \\ g - \chi_B\widehat{\Theta}_B(z)^{-1}(\bar{g} + S^*g)(z) \end{pmatrix}.$$

Here, $(\bar{g} + S^*g)(z)$ and $(S\bar{g} + g)(z)$ denote the values at $z$ of the analytic continuations of the functions $\bar{g} + S^*g \in H_2^2(\mathcal{E})$ and $S\bar{g} + g \in H_2^2(\mathcal{E})$ into the lower half-plane and the upper half-plane, respectively.

**Proof.** We prove (i). The proof of (ii) is carried out along the same lines. For this one should establish the validity of the identities:

$$\mathcal{F}_\pm (A_{\alpha\beta} - zI)^{-1}\Phi^* \begin{pmatrix} \bar{g} \\ g \end{pmatrix} = \mathcal{F}_\pm \Phi^{-1} P_K \frac{1}{z - \chi_B} \begin{pmatrix} \bar{g} \\ g - \chi_B\Theta_B(z)^{-1}(\bar{g} + S^*g)(z) \end{pmatrix}, \quad z \in \mathbb{C}_- \cap \mathcal{Q}_B.$$ 

First we compute the left-hand-side of (6.3). It follows from Lemma 5.4 that for $z, \lambda \in \mathbb{C}_- \cap \mathcal{Q}_B$, $h \in \mathcal{H}$ one has

$$\Gamma_0(L - zI)^{-1}(A_{\alpha\beta} - \lambda I)^{-1}h = \Theta_B(z)\Gamma_0(A_{\alpha\beta} - zI)^{-1}(A_{\alpha\beta} - \lambda I)^{-1}h$$

$$= \frac{1}{z - \lambda} \Theta_B(z)\Gamma_0 \left[ (A_{\alpha\beta} - zI)^{-1} - (A_{\alpha\beta} - \lambda I)^{-1} \right] h$$

$$= \frac{1}{z - \lambda} \left[ \Gamma_0(L - zI)^{-1} - \Theta_B(z)\Gamma_0(A_{\alpha\beta} - \lambda I)^{-1} \right] h$$

$$= \frac{1}{z - \lambda} \left[ \Gamma_0(L - zI)^{-1} - \Theta_B(z)\Theta_B(\lambda)^{-1}\Gamma_0(L - \lambda I)^{-1} \right] h.$$
Letting \( z = k - \mathrm{i} \epsilon, k \in \mathbb{R} \), it follows from the above calculation that
\[
\lim_{\epsilon \to 0} \Gamma_0(L - (k - \mathrm{i} \epsilon)I)^{-1}(A_{\alpha\beta} - \lambda I)^{-1}h = \lim_{\epsilon \to 0} \frac{1}{(k - \mathrm{i} \epsilon) - \lambda} \left[ \Gamma_0(L - (k - \mathrm{i} \epsilon)I)^{-1} - \Theta_B(k - \mathrm{i} \epsilon) \Theta_B(\lambda)^{-1} \Gamma_0(L - \lambda I)^{-1} \right] h.
\]
(6.4)

Combining the expression for \( \mathcal{F}_+ \) from Definition 2 with (6.4) yields
\[
\mathcal{F}_+(A_{\alpha\beta} - \lambda I)^{-1}h = \frac{1}{-\lambda} \left[ (\mathcal{F}_+ h)(\cdot) - \Theta_B(\cdot) \Theta_B(\lambda)^{-1} \mathcal{F}_+ h(\lambda) \right].
\]
Hence, in view of the identity \( \mathcal{F}_+ h = \bar{g} + S^* g \), which follows from (4.6), we obtain
\[
\mathcal{F}_+(A_{\alpha\beta} - \lambda I)^{-1} \Phi^{-1}\left( \frac{\bar{g}}{g} \right) = \frac{1}{-\lambda} \left[ (\bar{g} + S^* g)(\cdot) - \Theta_B(\cdot) \Theta_B(\lambda)^{-1} (\bar{g} + S^* g)(\lambda) \right].
\]
(6.5)

On the basis of Lemma 5.4 and reasoning in the same fashion as was done to write (6.5), one verifies
\[
\mathcal{F}_-(A_{\alpha\beta} - \lambda I)^{-1} \Phi^{-1}\left( \frac{\bar{g}}{g} \right) = \frac{1}{-\lambda} \left[ (\bar{g} + S^* g)(\cdot) - \Theta_B(\cdot) \Theta_B(\lambda)^{-1} (\bar{g} + S^* g)(\lambda) \right].
\]
(6.6)

Let us focus on the right hand side of (6.3). Note that
\[
P_{\mathcal{F}_+} \cdot \frac{1}{z} \left( \frac{\bar{g}}{g - \chi^+_B \Theta_B(z)^{-1} (\bar{g} + S^* g)(z)} \right) = \left( \frac{1}{-\lambda} - P_+ \cdot \frac{1}{z} \left[ (\bar{g} + S^* g)(\cdot) - S^* \chi^+_B \Theta_B(z)^{-1} (\bar{g} + S^* g)(z) \right] \right)
\]
(6.7)

where (4.5) is used in the first equality and in the second the fact that if \( f \in H^2_-(\mathcal{E}) \), then, for all \( z \in \mathbb{C}_-, \)
\[
P_+ f(\cdot) = \frac{f(\cdot) + f(z) - f(z)}{-z} = P_+ \frac{f(z)}{-z} = \frac{f(z)}{-z}.
\]

Now, apply \( \mathcal{F}_+ \Phi^{-1} \) to (6.7) taking into account that \( \mathcal{F}_+ h = \bar{g} + S^* g \) once again:
\[
\mathcal{F}_+ \Phi^{-1} \cdot \frac{1}{z} \left( \frac{\bar{g} - (\bar{g} + S^* g)(z) + S^* (\bar{\gamma}) \chi^+_B \Theta_B(z)^{-1} (\bar{g} + S^* g)(z)}{g - \chi^+_B \Theta_B(z)^{-1} (\bar{g} + S^* g)(z)} \right)
\]
(6.8)

where for the last equality we have used Lemma 6.1. By combining (6.8) with (6.5), we establish the first identity in (6.3).
Finally, applying \( \mathcal{F}_- \Phi^{-1} \) to (6.7) and using the identity \( \mathcal{F}_- h = Sg + g \), we obtain
\[
\mathcal{F}_- \Phi^{-1} \cdot \frac{1}{\cdot - z} \left( \begin{array}{c}
\bar{g} - (\bar{g} + S^* g)(z) + S^* (\bar{\Theta}_B(z))^{-1}(\bar{g} + S^* g)(z) \\
g - \chi_B^+ \Theta_B(z)^{-1}(\bar{g} + S^* g)(z)
\end{array} \right)
\]
\[
= \frac{1}{\cdot - z} \left[ (S\bar{g} + g)(\cdot) - S(\bar{g} + S^* g)(z) - (I - SS^* (\bar{\Theta}))\chi_B^+ \Theta_B(z)^{-1}(\bar{g} + S^* g)(z) \right]
\]
\[
= \frac{1}{\cdot - z} \left[ (S\bar{g} + g)(\cdot) - (S\Theta_B(z) + \chi_B^+ - SS^* (\bar{\Theta})\chi_B^+) \Theta_B(z)^{-1}(\bar{g} + S^* g)(z) \right]
\]
\[
= \frac{1}{\cdot - z} \left[ (S\bar{g} + g)(\cdot) - (S\chi_B^+ + \chi_B^+) \Theta_B(z)^{-1}(\bar{g} + S^* g)(z) \right]
\]
\[
= \frac{1}{\cdot - z} \left[ (S\bar{g} + g)(\cdot) - \Theta_B(z)^{-1}(\bar{g} + S^* g)(z) \right],
\]
where in the last two equalities we use Lemma 6.1. Comparing this with (6.6), we arrive at the second identity in (6.3). \( \square \)

7. Application: a unitary equivalent model of an operator associated with BVP in a space with reproducing kernel

In the present section we demonstrate that in the setting of operators of BVP, the results of Section 4 lead to the representation of \((L^* - zI)^{-1}\) as the Toeplitz operator \(P_S(f(\cdot) - zI)^{-1})|_{K_S}\), where \(P_S\) is the orthogonal projection of \(H^2_+(\mathcal{E})\) onto \(K_S := H^2_+(\mathcal{E}) \cap SH^2_+(\mathcal{E})\). Thus this results of Section 6 can be used to represent the resolvent of \(A_{a,\beta}\) as a “triangular” perturbation of the aforementioned Toeplitz operator.

Throughout the section we assume that the condition imposed by Lemma 5.1 holds, the operator \(B\) is bounded and that the operator \(A\) is simple.

The following proposition carries over together with its proof from [28].

Proposition 7.1 ([28]). If the operator-function \(S(z)\) is inner (which implies that its boundary values \(S(k)\) are almost everywhere unitary on \(\mathbb{R}\)), then the Hilbert space \(\mathcal{H}\) is unitary equivalent to the spaces \(K_S := H^2_+(\mathcal{E}) \cap SH^2_+(\mathcal{E})\) and \(K^\dagger_S := H^2_+(\mathcal{E}) \cap S^* H^2_+(\mathcal{E})\). Moreover, the unitary transformations of \(\mathcal{H}\) to \(K_S\) and \(K^\dagger_S\) are given explicitly as the restrictions of the operators \(\mathcal{F}_-\) of Definition 2, respectively, to the spaces \((0, H, 0)\).

Remark 5. It can be verified that the characteristic function \(S\) is indeed inner if the spectrum of the operator \(L\) is discrete. The latter is satisfied by the Krein resolvent formula, provided that the conditions of Lemma 5.1 hold and the operator of the BVP with Dirichlet conditions has discrete spectrum, the latter being the case under minimal regularity conditions; however, see, e.g., the discussion in [37] and references therein.

The formula (6.5) applied to the operator \(L\) and a similar computation in relation to the operator \(L^*\) now yield the following result.

Theorem 7.2. The operator \((L - zI)^{-1}\) for \(z \in \mathbb{C}_-\) is unitary equivalent to the Toeplitz operator \(f \mapsto P_S^\dagger f(\cdot)(-z)^{-1})\) in the space \(K^\dagger_S\); the operator \((L^* - zI)^{-1}\) for \(z \in \mathbb{C}_+\) is unitary equivalent to the Toeplitz operator \(f \mapsto P_S(f(\cdot)(-z)^{-1})\) in the space \(K_S\). Here \(P^\dagger_S\) and \(P_S\) are orthogonal projections onto \(K^\dagger_S\) and \(K_S\), respectively:
\[
P_S f(\cdot)|_{\cdot - z} = P^\dagger_S f(\cdot)|_{\cdot - z} = \frac{f(\cdot) - f(z)}{\cdot - z}, \quad z \in \mathbb{C}_+,
\]
\[
P^\dagger_S f(\cdot)|_{\cdot - z} = P_S f(\cdot)|_{\cdot - z} = \frac{f(\cdot) - f(z)}{\cdot - z}, \quad z \in \mathbb{C}_-.
\]
where \(P_+, P_-\) are orthogonal projections onto Hardy classes \(H^2_+(\mathcal{E}), H^2_-(\mathcal{E})\), respectively.

For the operators of BVPs defined by different boundary conditions parameterised by the operator \(B\), including self-adjoint ones, a similar argument yields the following representation.
Theorem 7.3. The operator \((A_{\alpha\beta} - z)^{-1}\) for \(z \in \mathbb{C}_- \cap \rho(A_{\alpha\beta})\) is unitary equivalent to a “triangular” perturbation of the Toeplitz operator \(f \mapsto P_\delta^f(\cdot - z)^{-1}\) in the space \(K_\delta^S\), namely, to the operator

\[
f \mapsto P_\delta - \frac{\Theta_B(\cdot - z)^{-1}}{\Theta_B(z)^{-1}} - P_\delta f(\cdot - z).
\]

For \(z \in \mathbb{C}_- \cap \rho(A_{\alpha\beta})\) the resolvent \((A_{\alpha\beta} - z)^{-1}\) is unitary equivalent to the operator

\[
f \mapsto P_\delta P_\delta^f(\cdot - z)^{-1} P_\delta - \frac{\Theta_B(\cdot - z)^{-1}}{\Theta_B(z)^{-1}} P_\delta f(\cdot - z)
\]

in the space \(K_\delta\).

Remark 6. It is rather well-known that the spaces \(K_\delta\) and \(K_\delta^S\) are Hilbert spaces with reproducing kernels, closely linked to the corresponding de Branges spaces in the “scalar” case of \(\dim \mathcal{E} = 1\). We refer the reader to the book [42] for an in-depth survey of the subject area and of the related developments in modern complex analysis. The applications of the latter Theorem to the direct and inverse spectral problems of operators of BVPs is outside the scope of the present paper and will be dwelt upon elsewhere.

Acknowledgements

KDC is grateful for the financial support of the Engineering and Physical Sciences Research Council: Grant EP/L018802/2 “Mathematical foundations of metamaterials: homogenisation, dissipation and operator theory”. AVK has been partially supported by the Russian Federation Government megagrant 14.Y26.31.0013 and by the RFBR grant 19-01-00657-a. LOS has been partially supported by UNAM-DGAPA-PAPIIT IN110818 and SEP-CONACYT CB-2015 254062. LOS is grateful for the financial support of PASPA-DGAPA-UNAM during his sabbatical leave and thanks the University of Bath for their hospitality.

References

[1] M. S. Agranovich, Sobolev Spaces, Their Generalizations, and Elliptic Problems in Smooth and Lipschitz Domains, Springer, 2015.
[2] W. O. Amrein, D. B. Pearson, \(M\)-operators: a generalisation of Weyl-Titchmarsh theory. J. Comput. Appl. Math., 171(1-2):1–26, 2004.
[3] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, G. Teschl, A survey on the Krein-von Neumann extension, the corresponding abstract buckling problem, and Weyl-type spectral asymptotics for perturbed Krein Laplacians in nonsmooth domains. Mathematical Physics, Spectral Theory and Stochastic Analysis, 1–106, Oper. Theory Adv. Appl., 232, Birkhäuser, Basel, 2013.
[4] L. Boutet de Movel, Boundary problems for pseudo-differential operators. Acta Math., 126:11–51, 1971.
[5] J. Behrndt, M. Langer, Boundary value problems for elliptic partial differential operators on bounded domains. J. Func. Anal., 243(2):536–565, 2007.
[6] F. A. Berezin, L. D. Faddeev, Remark on the Schrödinger equation with singular potential. (Russian) Dokl. Akad. Nauk SSSR, 137:1011–1014, 1961.
[7] P. Binding, R. Hryniv, Relative boundedness and relative compactness for linear operators in Banach spaces. Proc. Amer. Math. Society, 128(8):2287–2290, 2000.
[8] M. Š. Birman, On the theory of self-adjoint extensions of positive definite operators. Mat. Sb. N.S., 38(80):431–450, 1956.
[9] M. Š. Birman, M. Z. Solomjak, Spectral theory of selfadjoint operators in Hilbert space. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.
[10] M. S. Birman, Perturbations of the continuous spectrum of a singular elliptic operator by varying the boundary and the boundary conditions, Vestn. Leningrad. Univ. 17 (1962), 22–55. English translation in: Spectral theory of differential operators, Amer. Math. Soc. Transl. Ser. 2, 225, Amer. Math. Soc., Providence, RI, 2008, pp. 19–53.
[11] M. S. Birman, M. Z. Solomjak, Asymptotics of the spectrum of variational problems on solutions of elliptic equations in unbounded domains. Funkts. Analiz Prilozhen. 14:27–35, 1980. English translation in: Funct. Anal. Appl. 14:267–274, 1981.
[12] M. Brown, M. Marletta, S. Naboko, I. Wood, Boundary triples and \(M\)-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices. J. Lond. Math. Soc. (2), 77(3):706–718, 2008.
[13] M. Brown, M. Marletta, S. Naboko, I. Wood, The functional model for maximal dissipative operators: An approach in the spirit of operator knots. Trans. Amer. Math. Soc., 373:4145–4187, 2020.
[14] J. Brüning, G. Martin, B. Pavlov, Calculation of the Kirchhoff coefficients for the Helmholtz resonator. Russ. J. Math. Phys., 16(2):188–207, 2009.
[48] M. Renardy, R. C. Rogers, An Introduction to Partial Differential Equations. Texts in Applied Mathematics 13, Springer, 2004.
[49] M. Rosenblum, J. Rovnyak, Hardy Classes and Operator Theory. Oxford University Press, 1985.
[50] V. Ryzhov, Functional model of a class of non-selfadjoint extensions of symmetric operators. Operator Theory, Analysis and Mathematical Physics, 117–158, Oper. Theory Adv. Appl., 174, Birkhäuser, Basel, 2007.
[51] V. Ryzhov, Functional model of a closed non-selfadjoint operator. Integral Equations Operator Theory 60(4):539–571, 2008.
[52] V. Ryzhov, Spectral boundary value problems and their linear operators, Analysis as a Tool in Mathematical Physics: in Memory of Boris Pavlov, 576–626, Oper. Theory Adv. Appl., 276, Birkhäuser, Basel, 2020.
[53] B. Sz.-Nagy, C. Foias, H. Bercovici, L. Kérchy, Harmonic Analysis of Operators on Hilbert Space. Springer, New York, 2010.
[54] A. V. Štraus, Functional models and generalized spectral functions of symmetric operators. St. Petersburg Math. J., 10(5):733-784, 1999.
[55] M. E. Taylor, Tools for PDE: Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials. Mathematical Surveys and Monographs 81, American Mathematical Society, Providence, Rhode Island, 2000.
[56] M. I. Višik, On general boundary problems for elliptic differential equations. Trudy Moskov. Mat. Obšč., 1:187–246, 1952.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH, CLAVERTON DOWN, BATH, BA2 7AY, UNITED KINGDOM
Email address: cherednichenkokd@gmail.com

DEPARTAMENTO DE FÍSICA MATEMÁTICA, INSTITUTO DE INVESTIGACIONES EN MATEMÁTICAS APLICADAS Y EN SISTEMAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, C.P. 04510, MÉXICO D.F. AND INTERNATIONAL RESEARCH LABORATORY “MULTISCALE MODEL REDUCTION”, AMMOSOV NORTH-EASTERN FEDERAL UNIVERSITY, YAKUTSK, RUSSIA
Email address: alexander.v.kiselev@gmail.com

DEPARTAMENTO DE FÍSICA MATEMÁTICA, INSTITUTO DE INVESTIGACIONES EN MATEMÁTICAS APLICADAS Y EN SISTEMAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, C.P. 04510, MÉXICO D.F. AND DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH, CLAVERTON DOWN, BATH, BA2 7AY, UNITED KINGDOM
Email address: smithlimas.unam.mx

25