Research Article

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A dynamical approach to the variational inequality on modified elastic graphs

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Abstract: We consider the variational inequality on modified elastic graphs. Since the variational inequality is derived from the minimization problem for the modified elastic energy defined on graphs with the unilateral constraint, a solution to the variational inequality can be constructed by the direct method of calculus of variations. In this paper we prove the existence of solutions to the variational inequality via a dynamical approach. More precisely, we construct an $L^2$-type gradient flow corresponding to the variational inequality and prove the existence of solutions to the variational inequality via the study on the limit of the flow.

Keywords: obstacle problem, elastic flow on graphs, minimizing movements

MSC: 35K25, 53C44, 35K86, 49J40

1 Introduction

In this paper we are interested in a dynamical approach to the obstacle problem for modified elastic graphs: find $u : [0, 1] \rightarrow \mathbb{R}$ such that

$$\min_{v \in K_\psi} \mathcal{E}_\lambda(v)$$

(1.1)

with $\mathcal{E}_\lambda(v) := \lambda \mathcal{L}(v) + \mathcal{W}(v)$, where $\lambda$ is a given nonnegative constant,

$$\mathcal{L}(v) := \int_0^1 \ell(v'(x)) \, dx = \int_0^1 \sqrt{1 + v'(x)^2} \, dx,$$

(1.2)

$$\mathcal{W}(v) := \int_0^1 \kappa \gamma(x)^2 \ell(v'(x)) \, dx = \int_0^1 \frac{v''(x)^2}{(1 + v'(x)^2)^{5/2}} \, dx,$$

(1.3)

$$K_\psi := \{ v \in \mathcal{H}(0, 1) \mid v \geq \psi \text{ in } [0, 1] \}, \quad \mathcal{H}(0, 1) := H^2(0, 1) \cap H^1_0(0, 1),$$

(1.4)

and $\psi : [0, 1] \rightarrow \mathbb{R}$ is a given obstacle function satisfying the following:

$$\psi(0) < 0, \quad \psi(1) < 0 \quad \text{and} \quad \psi \in C([0, 1]).$$

(1.5)

The functional $\mathcal{W}$ is the squared integral of the curvature $\kappa_\gamma$ of the curve $\gamma(x) := (x, v(x))$ with respect to the arc length parameter of $\gamma$ and is called the bending energy, the elastic energy, or the one-dimensional Willmore functional. One of strategy to find a solution to (1.1) is the direct method of calculus of variations. Indeed, [3, 13] proved the existence of minimizer of (1.1) with $\lambda = 0$. If $u$ is a minimizer of (1.1), then $u$ satisfies

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Moreover, the solution of the following variational inequality:

\[ \int_0^1 \left[ \frac{2}{v(u)} (v'' - u)'' - \left( \frac{5}{v(u)} \frac{|u''|^2}{v(u)'} - \lambda \right) \frac{u'(v - u)'}{v(u)'} \right] dx \geq 0 \quad \text{for} \quad v \in K_\psi. \]  

(1.6)

The purpose of this paper is to prove the existence of solutions to (1.6) via a dynamical approach. Although problem (1.1) can be solved by the direct method of calculus of variations, it is significant to prove the existence of solutions to (1.6) via the other strategy. Because, it is not clear whether the set of all minimizers of (1.1) is equivalent to the set of all solutions to (1.6) or not, due to the lack of the convexity of the functional \( \lambda \mathcal{L} + \mathcal{W} \).

Recently M. Müller [14] gave a dynamical approach to the existence of solutions to (1.6) with \( \lambda = 0 \). In [14] the \( H^2 \)-gradient flow for the minimization problem (1.1) with \( \lambda = 0 \) was constructed, and it was proved that a solution of the flow subconverges to a solution of (1.6) as \( t \to \infty \). We are interested in an \( L^2 \)-type gradient flow for the minimization problem (1.1). As the Willmore flow, which is the \( L^2 \)-gradient flow for the Willmore functional, was inspired by the Willmore conjecture (e.g., see [11, 12, 20]), generally it is natural to construct \( L^2 \)-gradient flows corresponding to variational problems on the elastic energy (e.g., see [5, 7, 16, 21]).

To this end, we consider the following variational inequality of parabolic type:

\[ \int_0^T \int_0^1 \left[ \psi_t (v-u) + 2 \psi'' (v-u)'' - \left( \frac{5}{\psi' v} \frac{|u''|^2}{\psi'} - \lambda \right) \frac{u'(v-u)'}{\psi'} \right] dx dt \geq 0 \]

\[ \text{for} \quad v \in K_T := \{ v \in L^\infty (0, T; \mathcal{H}(0, 1)) \cap H^1 (0, T; L^2 (0, 1)) \}

\[ |v|_{\kappa} \text{ in } (0, 1) \times [0, T], \quad v|_{t=0} = u_0 \text{ in } (0, 1). \]

Following the motivation as we stated above, we prove the following:

(i) problem (P) possesses a unique local-in-time solution with an \( L^2 \)-gradient structure for \( \lambda \mathcal{L} + \mathcal{W} \);

(ii) problem (P) possesses a unique global-in-time solution for some initial data \( u_0 \) and the solution converges to a solution of (1.6) as \( t \to \infty \).

The first main result of this paper is concerned with (i):

**Theorem 1.1.** Let \( \psi : [0, 1] \to \mathbb{R} \) satisfy (1.5). Then for each \( u_0 \in K_\psi \) there exists \( T = T(u_0) > 0 \) such that (P) possesses a unique solution \( u \in K_T \). Moreover, \( \mathcal{E}_A(u(t)) \) is well-defined for all \( t \in [0, T] \) and satisfies

\[ \mathcal{E}_A(u(t_2)) - \mathcal{E}_A(u(t_1)) \leq \frac{1}{2} \int_{t_1}^{t_2} \left| \psi_t u \right|^2 dx dt \quad \text{for} \quad 0 \leq t_1 \leq t_2 \leq T. \]

(1.7)

We infer from Theorem 1.1 that problem (P) is solvable for all natural initial data, and the energy \( \mathcal{E}_A \) is non-increasing along the orbit of the solution to (P). Moreover, we obtain the regularity properties of the solutions to (P) (see Theorem 5.1).

In order to prove (ii), we need a certain restriction on \( \lambda, \psi \) and \( u_0 \). Indeed, even if \( \lambda = 0 \), problem (1.6) has no solution for some \( \psi \) satisfying (1.5) (e.g., see [3, 13]). Let

\[ c_0 := \int_{-\infty}^{\infty} \frac{d\tau}{(1 + \tau^2)^{3/4}}. \]

(1.8)

Then the second main result of this paper is stated as follows:

**Theorem 1.2.** Let \( 0 < \lambda < \frac{c_0^2}{4} \) and assume that \( \psi \) satisfies (1.5) and

\[ \inf_{v \in K_\psi} \mathcal{E}_A(v) \leq \frac{c_0^2}{4}. \]

(1.9)

Then for each \( u_0 \in K_\psi \) satisfying \( \mathcal{E}_A(u_0) \leq \frac{c_0^2}{4} \), problem (P) possesses a unique global-in-time solution \( u \in K_{\infty} \). Moreover, the solution \( u \) subconverges to a solution \( u^* \in K_\phi \) of (1.6) as \( t \to \infty \).
If the obstacle $\psi$ is small enough, then it is verified that the assumption (1.9) holds true (see Remark 6.5). Theorem 1.2 means that, for $\psi$ satisfying (1.9), a solution of (1.6) can be constructed from a limit of a solution to (P). For the case $\lambda = 0$, we can obtain the same assertion as Theorem 1.2 under an assumption which is stronger than assumption (1.9) (see Remark 6.6).

We also prove that a solution to problem (P) may blow up (see Theorem 6.2 and Lemma 6.3). It will be interesting to study the aspect of blow up solutions to problem (P). Moreover, if the uniqueness of solutions to problem (1.6) does not hold, it may be interesting to study a stability notion of solutions to (1.6) via parabolic problem (P).

This paper is organized as follows: In Section 2, we collect notations and inequalities which are used in this paper. In Section 3, we prove the uniqueness of solutions to (P). In Section 4, we discuss a family of approximate solutions via minimizing movements: existence, regularity and convergence. We prove Theorems 1.1 and 1.2 in Sections 5 and 6, respectively.

2 Preliminary

From now on, we denote by $I$ the open interval $(0, 1)$. We denote by $\mathcal{H}(I)$ the Hilbert space $H^2(I) \cap H_0^1(I)$ equipped with the scalar product

$$(u, v)_{\mathcal{H}} := \int_0^1 u'v'' \, dx \quad \text{for} \quad u, v \in \mathcal{H}(I).$$

In this paper we employ the norm $\| \cdot \|_{\mathcal{H}(I)}$ on $\mathcal{H}(I)$ as

$$\|u\|_{\mathcal{H}} := (u, u)^{1/2} \quad \text{for} \quad u \in \mathcal{H}(I),$$

which is equivalent to $\| \cdot \|_{H^2(I)}$ (see e.g. [9, Theorem 2.31]), i.e., there exists a positive constant $c_H$ such that

$$c_H \|u\|_{H^2(I)} \leq \|u\|_{\mathcal{H}} \leq \|u\|_{H^2(I)}. \quad (2.1)$$

We collect interpolation inequalities used in this paper.

**Lemma 2.1.**

$$\|\varphi'\|_{L^p(I)} \leq \sqrt{2} \|\varphi\|_{L^q(I)}^2 \|\varphi\|_{L^\infty(I)}^{1/2} \quad \text{for} \quad \varphi \in \mathcal{H}(I).$$

**Proof.** Since $\varphi(0) = \varphi(1) = 0$, we deduce from integration by part and Hölder’s inequality that

$$\|\varphi'\|_{L^2(I)}^2 = -\int_0^1 \varphi \cdot \varphi'' \, dx \leq \|\varphi\|_{L^1(I)} \|\varphi''\|_{L^1(I)}. \quad (2.2)$$

Using $\varphi(0) = \varphi(1) = 0$ again, we find $x_0 \in I$ such that $\varphi'(x_0) = 0$, and then

$$|\varphi'(x)|^2 = \int_{x_0}^x (|\varphi'(\xi)|^2) \, d\xi = \int_{x_0}^x 2\varphi' \varphi'' \, d\xi \leq 2 \|\varphi'\|_{L^1(I)} \|\varphi''\|_{L^1(I)}$$

for $x \in I$. This together with (2.2) gives us the conclusion. \qed

We also introduce the following interpolation inequality (see [1], [8, Theorem 6.4]).

**Proposition 2.2.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set satisfying the cone condition. Let $k, l, m$ and $m$ be integers such that $0 \leq k \leq l \leq m$. Let $1 \leq p < q < \infty$ if $(m - l)p \geq N$, and let $1 \leq p < q < \infty$ if $(m - l)p > N$. Then, there exists $C > 0$ such that

$$\|D^lf\|_{L^q(\Omega)} \leq C(\|D^mf\|_{L^p(\Omega)}^\theta \|D^lf\|_{L^p(\Omega)}^{1-\theta} + \|D^lf\|_{L^p(\Omega)}).$$
for all $f \in W^{m,p}(\Omega)$, where

$$\theta := \frac{1}{m-k} \left( \frac{N}{p} - \frac{N}{q} + 1 - k \right).$$

As in (1.2), we let $\ell(s) := (1 + s^2)^{1/2}$. By the mean value theorem we have

$$|\ell(s)^m - \ell(t)^m| \leq C|s - t|$$

(2.3)

for $s, t \in \mathbb{R}$ and $m > 0$, where the constant $C$ depends only on $m$.

From now on, the letter $C$ denotes generic positive constants and it may have different values also within the same line.

### 3 Uniqueness

In this section we prove the uniqueness of solutions to (P).

**Lemma 3.1.** Let $u \in \mathcal{K}_T$ be a solution to (P). Then

$$\int_{\tau_1}^{\tau_2} \left[ \int_{\tau_0}^{1} \left[ \partial_t u(v-u) + 2 \frac{u''(v-u)}{\ell(u)^3} - \left( 5 \frac{|u''|^2}{\ell(u)^5} - \lambda \right) \frac{u'(v-u)}{\ell(u)} \right] dx \right] dt \geq 0$$

(3.1)

for $0 \leq \tau_1 \leq \tau_2 \leq T$ and $v \in \mathcal{K}_T$.

**Proof.** Assume that (3.1) does not hold. Then there exist $0 < \tau_1 < \tau_2 < T$ and $v \in \mathcal{K}_T$ such that

$$J := \int_{\tau_1}^{\tau_2} \left[ \int_{\tau_0}^{1} \left[ \partial_t u(v-u) + 2 \frac{u''(v-u)}{\ell(u)^3} + \left( 5 \frac{|u''|^2}{\ell(u)^5} + \lambda \right) \frac{u'(v-u)}{\ell(u)} \right] dx \right] dt < 0.$$

We first consider the case of $0 < \tau_1 < \tau_2 < T$. For $0 < \varepsilon < 1$, we define $\eta_\varepsilon \in H^1(0, T)$ by

$$\eta_\varepsilon(t) := \begin{cases} \frac{1}{\varepsilon}(t - \tau_1) + 1 & \text{if } \tau_1 - \varepsilon \leq t < \tau_1, \\ 1 & \text{if } \tau_1 \leq t \leq \tau_2, \\ -\frac{1}{\varepsilon}(t - \tau_2) + 1 & \text{if } \tau_2 < t \leq \tau_2 + \varepsilon, \\ 0 & \text{otherwise}, \end{cases}$$

and set $v_\varepsilon := (1 - \eta_\varepsilon)u + \eta_\varepsilon v$. Since $v_\varepsilon \in \mathcal{K}_T$ for $\varepsilon > 0$, taking $v_\varepsilon$ as the test function in (P), we have

$$J + J_1 + J_2 \geq 0,$$

(3.2)

where

$$J_1 := \int_{\tau_1}^{\tau_1 - \varepsilon} \int_{\tau_0}^{1} \left[ \partial_t u(v-u) + 2 \frac{u''(v-u)}{\ell(u)^3} - \left( 5 \frac{|u''|^2}{\ell(u)^5} - \lambda \right) \frac{u'(v-u)}{\ell(u)} \right] dx \, dt,$$

$$J_2 := \int_{\tau_2}^{\tau_2 + \varepsilon} \int_{\tau_0}^{1} \left[ \partial_t u(v-u) + 2 \frac{u''(v-u)}{\ell(u)^3} - \left( 5 \frac{|u''|^2}{\ell(u)^5} + \lambda \right) \frac{u'(v-u)}{\ell(u)} \right] dx \, dt.$$
Theorem 3.2. Let

\begin{equation*}
\varepsilon \leq C \sqrt{\varepsilon} \left\| \partial_2 u \right\|_{L^2([0,T];L^1(I))} \leq C \sqrt{\varepsilon},
\end{equation*}

where we used the fact that \( u, v \in C([0, T]; W^{1,\infty}(I)) \). Similarly, we have

\begin{align*}
\left| \int_{\tau_2}^{\tau_2+\varepsilon} \int_{0}^{1} |u''(v - u)| \ell(u')^{-\gamma} \, dx \, dt \right| &\leq \int_{\tau_2}^{\tau_2+\varepsilon} \int_{0}^{1} \left| u'' \right| \left( v - u \right) \ell(u')^{-\gamma} \, dx \, dt \leq C, \\
\left| \int_{\tau_2}^{\tau_2+\varepsilon} \int_{0}^{1} \left| u' \right| \left( v - u \right) \ell(u')^{-\gamma} \, dx \, dt \right| &\leq C.
\end{align*}

Thus we obtain \( J_2 \leq C \sqrt{\varepsilon} \). Along the same line as above, we have \( J_1 \leq C \sqrt{\varepsilon} \). Letting \( \varepsilon > 0 \) small enough, we see that (3.2) contradicts \( J < 0 \). Similarly, we also lead a contradiction if \( \tau_1 = 0 \) or \( \tau_2 = T \). Therefore Lemma 3.1 follows. \( \square \)

**Theorem 3.2.** Let \( u_1, u_2 \in \mathcal{K}_T \) be solutions to (P) for some \( T > 0 \) and assume that \( u_1|_{\varepsilon=0} = u_2|_{\varepsilon=0} \) in \( I \). Then \( u_1 = u_2 \) in \( L^\infty(0, T; \mathcal{C}(I)) \cap H^1(0, T; L^2(I)) \).

**Proof.** Fix \( \tau \in [0, T] \) arbitrarily. It follows from Lemma 3.1 that

\begin{equation}
\int_{0}^{1} \int_{0}^{\tau} \left[ \partial_1 u_i(v - u_i) + 2 \frac{u''}{\ell(u')^3} \right] (v - u_i)'' \, dx \, dt \geq 0 \quad (3.3)
\end{equation}

for \( v \in \mathcal{K}_T \) and \( i = 1, 2 \). Setting \( v = u_2 \) in (3.3) for \( u_1 \) and \( v = u_1 \) in (3.3) for \( u_2 \), and adding them, we infer from Fubini’s Theorem that

\begin{align*}
\frac{1}{2} \| u_1(\tau) - u_2(\tau) \|_{L^2(I)}^2 &\leq 2 \int_{0}^{1} \int_{0}^{\tau} \left( \frac{u''_1}{\ell(u')^3} - \frac{u''_2}{\ell(u')^3} \right) (u_2 - u_1)'' \, dx \, dt \\
&\quad - 5 \int_{0}^{1} \int_{0}^{\tau} \left( \frac{u''_1}{\ell(u')^3} - \frac{u''_2}{\ell(u')^3} \right) (u_2 - u_1)' \, dx \, dt \\
&\quad + \lambda \int_{0}^{1} \int_{0}^{\tau} \left( \frac{u''_1}{\ell(u')^3} - \frac{u''_2}{\ell(u')^3} \right) (u_2 - u_1) \, dx \, dt
\end{align*}

(3.4)

where we used \( u_1|_{\varepsilon=0} = u_2|_{\varepsilon=0} \) in \( I \). We deduce from (2.3) that

\begin{align*}
J_1 &\leq -2 \int_{0}^{1} \int_{0}^{\tau} \frac{(u''_2 - u''_1)^2}{\ell(u')^3} \, dx \, dt + C \int_{0}^{1} \int_{0}^{\tau} \left| u_2'' \right| \left| u_1' - u_2' \right| \left| u_1'' - u_2'' \right| \, dx \, dt.
\end{align*}

Since it follows from Hölder’s inequality and Lemma 2.1 that

\begin{align*}
\int_{0}^{1} \int_{0}^{\tau} \left| u_2'' \right| \left| u_1' - u_2' \right| \left| u_1'' - u_2'' \right| \, dx \, dt &\leq \int_{0}^{1} \left| u_2'' \right| \left| u_1' - u_2' \right| \left\| u_1'' - u_2'' \right\|_{L^2(I)} \, dt \\
&\leq C \int_{0}^{1} \left| u_1 - u_2 \right| \left| u_1' \right| \left| u_2'' \right| \, dt.
\end{align*}
we deduce from Young’s inequality that
\[
J_1 \leq -2 \int_0^1 \int_0^r \frac{(u''_1 - u''_2)^2}{\epsilon(u'_1)^2} \, dx \, dt + \int_0^r \left[ \epsilon \|u'_1 - u'_2\|_{L^2(I)}^2 + C \|u_1 - u_2\|_{L^2(I)}^2 \right] \, dt
\]
for \( \epsilon > 0 \). Next we turn to \( J_2 \). We reduce \( J_2 \) into
\[
J_2 = 5 \int_0^r \left[ \|u''_1\|^2 (u'_1 - u'_2)^2 \epsilon(u'_1)^{-7} + ((u''_1)^2 - (u''_2)^2) u'_2 (u'_1 - u'_2) \epsilon(u'_1)^{-7}
\right.
\]
\[
\left. + (\epsilon(u'_1)^{-7} - \epsilon(u'_2)^{-7}) |u''_1|^2 |u'_2 (u'_1 - u'_2)| \right] \, dx \, dt.
\]
Since \( \ell(s) \geq 1 \) and \( u_1, u_2 \in C([0, T]; W^{1,\infty}(I)) \), along the same line as in the estimate on \( J_1 \), we have
\[
J_2 \leq \int_0^r \left[ \epsilon \|u''_1 - u''_2\|_{L^2(I)}^2 + C \|u_1 - u_2\|_{L^2(I)}^2 \right] \, dt
\]
for \( \epsilon > 0 \). Similarly, we obtain
\[
J_3 \leq C \int_0^r \|u'_1 - u'_2\|_{L^2(I)}^2 \, dt \leq \int_0^r \left[ \epsilon \|u''_1 - u''_2\|_{L^2(I)}^2 + C \|u_1 - u_2\|_{L^2(I)}^2 \right] \, dt
\]
for \( \epsilon > 0 \). Since \( \ell(u'_1) \leq C \), letting \( \epsilon > 0 \) small enough, we deduce from (3.4) that
\[
\|u_1(\tau) - u_2(\tau)\|_{L^2(I)}^2 + C \int_0^\tau \|u''_1 - u''_2\|_{L^2(I)}^2 \, dt \leq C \int_0^\tau \|u_1 - u_2\|_{L^2(I)}^2 \, dt
\]
for \( \tau > 0 \). Then Gronwall’s inequality implies that
\[
\|u_1(\tau) - u_2(\tau)\|_{L^2(I)} = 0 \quad \text{for} \quad \tau \in [0, T].
\]
Combining (3.5) with (3.6) we also see that
\[
\int_0^\tau \|u''_1 - u''_2\|_{L^2(I)}^2 \, dt = 0 \quad \text{for} \quad \tau \in [0, T].
\]
Thus \( u_1 = u_2 \) in \( L^\infty(0, T; \mathcal{H}(I)) \). Moreover, since \( u_1 = u_2 \) in \( L^2(0, T; L^2(I)) \), we deduce from the uniqueness of weak derivatives that \( u_1 = u_2 \) in \( H^1(0, T; L^2(I)) \). We complete the proof.

\[\square\]

4 Approximate solutions

In this section we construct a family of approximate solutions of a solution to (P) via De Giorgi’s minimizing movements (e.g., see [2]). We divide Section 4 into three subsections, existence and uniform estimates, regularity, and convergence.

4.1 Existence and uniform estimates

We construct a family of approximate solutions of a local-in-time solution to (P). For this purpose, we first define a constant \( T > 0 \) as follows:
Let \( u_0 \in \mathcal{H}(I) \) satisfy (1.5). Let \( M_0 > 0 \) be a constant such that
\[
\max_{x \in I} |u_0'(x)| \leq M_0.
\]
Let \( L = L(u_0, \lambda) > 0 \) be
\[
L := \max \{ c_{\ast}^{-1} \ell(2M_0)^{\frac{5}{2}}, \sqrt{2} \mathcal{E}_0(u_0)^{\frac{1}{2}} \}.
\]
We define \( T > 0 \) by a constant small enough to satisfy
\[
2\sqrt{2}LT^{\frac{1}{2}} \leq \frac{1}{2} M_0.
\tag{4.1}
\]

Under the above setting, we define the discrete approximate solutions. For \( T \) defined by (4.1) and \( n \in \mathbb{N} \) we set
\[
\tau_n := \frac{T}{n}.
\]

We define inductively \( \{ u_{i,n} \} := \{ u_{i,n} \}_{i=0}^n \) as follows: Let \( u_{0,n}(x) := u_0(x) \). For each \( i = 1, \ldots, n \), we define \( u_{i,n} \) by a solution of the minimization problem
\[
\min_{v \in K_{\phi}} G_{i,n}(v),
\tag{M_{i,n}}
\]
where
\[
G_{i,n}(v) := \mathcal{E}_0(v) + P_{i,n}(v), \quad P_{i,n}(v) := \frac{1}{2\tau_n} \int_0^1 |v - u_{i-1,n}|^2 \, dx,
\]
and
\[
K_{\psi} := \{ v \in \mathcal{H}(I) \mid v \in \mathcal{H}, \max_{x \in I} |v'(x)| \leq 2M_0 \}.
\]

**Lemma 4.1.** For any \( n \in \mathbb{N} \) and each \( i = 1, \ldots, n \), problem \( (M_{i,n}) \) possesses a solution \( u_{i,n} \).

**Proof.** Let \( \{ v_j \}_{j \in \mathbb{N}} \subset K_{\phi} \) be a minimizing sequence on problem \( (M_{i,n}) \), that is,
\[
\lim_{j \to \infty} G_{i,n}(v_j) = \inf_{v \in K_{\phi}} G_{i,n}(v).
\tag{4.2}
\]

Since we may assume that \( G_{i,n}(v_j) < C \) for \( j \in \mathbb{N} \), we deduce from \( v_j \in K_{\phi} \) that
\[
el(2M_0)^{-\frac{5}{2}} \int_0^1 |v_j''|^2 \, dx \leq \int_0^1 |v_j''|^2 e^{\ell(2M_0)^{-\frac{5}{2}}} \, dx = W(v_j) \leq G_{i,n}(v_j) \leq C,
\]
i.e.,
\[
\|v_j\|_{C^1}^2 \leq C \ell(2M_0)^{\frac{5}{2}} \quad \text{for} \quad j \in \mathbb{N}.
\]

Recalling (2.1), we see that \( \{ v_j \} \) is uniformly bounded in \( H^2(I) \). Thus, extracting a subsequence, we find \( \tilde{u} \in H^2(I) \) such that
\[
v_j \rightharpoonup \tilde{u} \quad \text{weakly in} \quad H^2(I).
\tag{4.3}
\]

Moreover, this together with the Rellich-Kondrachov compactness theorem implies that
\[
v_j \to \tilde{u} \quad \text{in} \quad C^1(I).
\tag{4.4}
\]

This clearly implies that \( \tilde{u}(0) = \tilde{u}(1) = 0 \), \( \tilde{u} \geq \psi \) in \( I \) and \( \max_{x \in I} |\tilde{u}'(x)| \leq 2M_0 \). Thus we have \( \tilde{u} \in K_{\phi} \).

We prove that \( \tilde{u} \) is a minimizer of \( G_{i,n} \) in \( K_{\phi} \). It follows from (4.4) that
\[
\mathcal{L}(\tilde{u}) = \lim_{j \to \infty} \mathcal{L}(v_j), \quad P_{i,n}(\tilde{u}) = \lim_{j \to \infty} P_{i,n}(v_j).
\]
Moreover, we deduce from (4.3) and (4.4) that
\[
\mathcal{W}(\tilde{u}) = \lim_{j \to \infty} \frac{1}{\ell (\tilde{u}''(0))^3} \int_0^1 \frac{\tilde{u}'''}{\ell (\tilde{u}'')^{5/2}} \, dx = \lim_{j \to \infty} \frac{1}{\ell (\tilde{u}'')^{5/2}} \int_0^1 \tilde{u}'' \, dx \leq \liminf_{j \to \infty} \mathcal{W}(v_j) \frac{\mathcal{W}(\tilde{u})}{2},
\]
where the last inequality followed from Hölder’s inequality. Thus we obtain
\[
\mathcal{W}(\tilde{u}) \leq \liminf_{j \to \infty} \mathcal{W}(v_j).
\]
Recalling (4.2) and \( G_{i,n}(v) = \lambda \mathcal{L}(v) + \mathcal{W}(v) + P_{i,n}(v) \), we see that
\[
G_{i,n}(\tilde{u}) \leq \liminf_{j \to \infty} \left[ \mathcal{W}(v_j) + \lambda \mathcal{L}(v_j) + P_{i,n}(v_j) \right] = \inf_{v \in \bar{K}_\phi} G_{i,n}(v). \tag{4.5}
\]
This completes the proof.

In the following, we define \( V_{i,n} : \bar{I} \to \mathbb{R} \) by
\[
V_{i,n}(x) := \frac{u_{i,n}(x) - u_{i-1,n}(x)}{\tau_n},
\]
for each \( x \in \bar{I} \). We define the piecewise linear interpolation of \( \{u_{i,n}\} \) as follows:

**Definition 4.2.** We define \( u_n(x, t) : \bar{I} \times [0, T] \to \mathbb{R} \) by
\[
u_n(x, t) := u_{i-1,n}(x) + (t - (i-1)\tau_n) V_{i,n}(x),
\]
if \( (x, t) \in \bar{I} \times [(i-1)\tau_n, i\tau_n] \) for each \( i = 1, \ldots, n \).

We also make use of the piecewise constant interpolation of \( \{u_{i,n}\} \) and \( \{V_{i,n}\} \):

**Definition 4.3.** We define \( \tilde{u}_n(x, t) : \bar{I} \times (0, T] \to \mathbb{R} \) and \( V_n : \bar{I} \times (0, T] \to \mathbb{R} \) by
\[
\tilde{u}_n(x, t) := u_{i,n}(x), \quad V_n(x, t) := V_{i,n}(x),
\]
if \( (x, t) \in \bar{I} \times [(i-1)\tau_n, i\tau_n] \) for each \( i = 1, \ldots, n \).

We derive the following uniform estimates on \( \{u_{i,n}\} \) and \( V_n \):

**Lemma 4.4.** For any \( n \in \mathbb{N} \), it holds that
\[
\max \left\{ \sup_{0 \leq t \leq \tau_n} \|u_{i,n}\|_{H^1(I)}, \|V_n\|_{L^2(0,T;L^2(I))} \right\} \leq L. \tag{4.6}
\]

**Proof.** It suffices to consider the case of \( 1 \leq i \leq n \). By the minimality of \( u_{i,n} \), we have
\[
\mathcal{E}_\lambda(u_{i,n}) + P_{i,n}(u_{i,n}) = G_{i,n}(u_{i,n}) \leq G_{i,n}(u_{i-1,n}) = \mathcal{E}_\lambda(u_{i-1,n}) \tag{4.7}
\]
for each \( i = 1, \ldots, n \), where we used the fact that \( u_{i-1,n} \in \bar{K}_\phi \). This clearly implies that \( \mathcal{E}_\lambda(u_{i,n}) \leq \mathcal{E}_\lambda(u_{0}) \) for \( i = 1, \ldots, n \). Thus we obtain
\[
\ell (2M_0)^{-5} \|u_{i,n}\|_{H^2(I)}^2 \leq \int_0^1 |u_{i,n}''|^2 \ell (\tilde{u}_{i,n}'')^{-5} \, dx = \mathcal{W}(u_{i,n}) \leq \mathcal{E}_\lambda(u_{i,n}) \leq \mathcal{E}_\lambda(u_{0}),
\]
for \( i = 1, \ldots, n \). This together with (2.1) implies that
\[
\sup_{0 \leq t \leq \tau_n} \|u_{i,n}\|_{H^2(I)}^2 \leq c_\bar{u}^2 (2M_0)^5 \mathcal{E}_\lambda(u_{0}). \tag{4.8}
\]
We turn to the estimate on \( V_n \). Since it follows from (4.7) that
\[
P_{i,n}(u_{i,n}) \leq E'_A(u_{i-1,n}) - E'_A(u_{i,n}),
\]
we have
\[
\int_0^T \int_0^1 |V_n|^2 \, dx \, dt = \sum_{i=1}^{n} \int_0^{\tau_n} \int_0^1 |V_{n,i}|^2 \, dx \, dt = 2 \sum_{i=1}^{n} P_{i,n}(u_{i,n}) \leq 2 \sum_{i=1}^{n} \left[ E'_A(u_{i-1,n}) - E'_A(u_{i,n}) \right] \leq 2 E'_A(u_0).
\] (4.9)

Thus Lemma 4.4 follows.

Due to the artificial condition \( \|V\|_{L^\infty(I)} \leq 2M_0 \) in \( \tilde{k}_\phi \), we need to control \( \|u_{i,n}\|_{L^\infty(I)} \) for each \( i = 1, \ldots, n \).

**Lemma 4.5.** For any \( n \in \mathbb{N} \) and each \( i = 1, \ldots, n \), it holds that
\[
\|u_{i,n}\|_{L^\infty(I)} \leq \frac{3}{2} M_0.
\]

**Proof.** Fix \( 0 \leq t_1 < t_2 \leq T \) arbitrarily. By Lemma 2.1 we have
\[
\|u_n(t_2) - u_n(t_1)\|_{L^\infty(I)} \leq \sqrt{2} \|u(t_2) - u(t_1)\|_{L^3(I)}^{1/3} \|u_n(t_2) - u_n(t_1)\|_{L^3(I)}^{1/3}.
\] (4.10)

It follows from (2.1) and Lemma 4.4 that
\[
\sup_{t \in [0,T]} \|u_n(t)\|_{H^3(I)} \leq \sup_{t \in [0,T]} \|u_i(t)\|_{H^3(I)} \leq L.
\] (4.11)

Since \( \partial u_n / \partial t = V_n \), we observe from Fubini’s theorem and Lemma 4.4 that
\[
\|u_n(t_2) - u_n(t_1)\|_{L^2(I)}^2 = \int_0^{t_2} \int_0^{t_1} \frac{\partial u_n}{\partial t} \, dx \, dt \leq (t_2 - t_1) \|V_n\|_{L^2(I)}^2 \leq L^2(t_2 - t_1).
\] (4.12)

Plugging (4.11) and (4.12) into (4.10), we have
\[
\|u_n(t_2) - u_n(t_1)\|_{L^\infty(I)} \leq 2 \sqrt{2} L(t_2 - t_1)^{1/3}.
\]

This together with (4.1) implies that
\[
\|u_{i,n}(t_2) - u_{i,n}(t_1)\|_{L^\infty(I)} \leq \frac{1}{2} M_0.
\] (4.13)

In particular, taking \( t_2 = i \tau_n \) and \( t_1 = 0 \) in (4.13), we have
\[
\|u_{i,n}\|_{L^\infty(I)} \leq \|u_{i,n,i \tau_n}\|_{L^\infty(I)} = \|u_0\|_{L^\infty(I)} + \frac{1}{2} M_0 + M_0 = \frac{3}{2} M_0
\]
for \( i = 1, \ldots, n \). Thus Lemma 4.5 follows.

**4.2 Regularity**

In this subsection we discuss the regularity of approximate solutions with respect to the space variable. Fix a nonnegative function \( \phi \in C^\infty_c(I) \) arbitrarily. Then it follows from Lemma 4.5 that \( u_{i,n} + \varepsilon \phi \in \tilde{k}_\phi \) for \( i = 1, \ldots, n \) and \( \varepsilon > 0 \) small enough. Hence, by the minimality of \( u_{i,n} \) we have
\[
\frac{d}{d \varepsilon} G_{i,n}(u_{i,n} + \varepsilon \phi) \bigg|_{\varepsilon = 0} \geq 0.
\]
The inequality is reduced into
\[
L_{i,n}(\varphi) := \int_{0}^{1} \left[ \frac{1}{\ell(u''_{i,n})} \left( \frac{2}{\|u''_{i,n}\|} \right) V_{i,n} \varphi \right] dx \geq 0 \tag{4.14}
\]
for \( \varphi \in C_{c}^{\infty}(I) \) with \( \varphi \geq 0 \). We observe from (4.14) that \( L_{i,n} \) is a nonnegative distribution on \( C_{c}^{\infty}(I) \). Therefore, thanks to the Riesz representation theorem (e.g., see [18, 2.14 Theorem]), we find a nonnegative Radon measure \( \mu_{i,n} \) such that
\[
L_{i,n}(\varphi) = \int_{0}^{1} \varphi \, d\mu_{i,n} \tag{4.15}
\]
for all \( \varphi \in C_{c}^{\infty}(I) \) and \( i = 1, \ldots, n \).

**Definition 4.6.** For each \( i = 1, \ldots, n \), we define \( \mathcal{N}_{i,n} \subset I \) by
\[
\mathcal{N}_{i,n} := \{ x \in I \mid u_{i,n}(x) > \psi(x) \}. \tag{4.17}
\]
We note that, since \( u_{i,n}, \psi \in C(I) \), the set \( \mathcal{N}_{i,n} \) is open. For any \( n \in \mathbb{N} \) and each \( i = 1, \ldots, n \), the measure \( \mu_{i,n} \) has a support on \( I \setminus \mathcal{N}_{i,n} \), that is,
\[
\mu_{i,n}(\mathcal{N}_{i,n}) = 0. \tag{4.16}
\]
Indeed, for any \( \varphi \in C_{c}^{\infty}(I) \) with \( \text{supp} \varphi \subset \mathcal{N}_{i,n} \), we have \( u_{i,n} \pm \varepsilon \varphi \geq \psi \) in \( I \) for \( \varepsilon > 0 \) small enough. Thus we have \( L_{i,n}(\varphi) = 0 \), and then (4.15) yields
\[
\int_{0}^{1} \varphi \, d\mu_{i,n} = 0 \quad \text{for} \quad \varphi \in C_{c}^{\infty}(I) \quad \text{with} \quad \text{supp} \varphi \subset \mathcal{N}_{i,n}. \tag{4.17}
\]
Thus (4.16) follows from (4.17) (see e.g. [10, Chapter II, Theorem 6.9]).

**Lemma 4.7.** There exist \( 0 < a < b < 1 \) such that
\[
(0, a) \cup (b, 1) \subset \mathcal{N}_{i,n} \tag{4.18}
\]
for \( n \in \mathbb{N} \) and \( i = 1, \ldots, n \).

**Proof.** We prove the existence of the constant \( a > 0 \). Since \( \psi \in C(I) \) and \( \psi(0) < 0 \), there exists \( \delta \in I \) such that
\[
\psi(x) < \frac{3}{4} \psi(0) \quad \text{for} \quad x \in [0, \delta]. \tag{4.18}
\]
Moreover, by Sobolev’s embedding theorem and Lemma 4.4 we have
\[
\|u_{i,n}\|_{W^{1,\infty}(I)} \leq C\|u_{i,n}\|_{H^{1}(I)} \leq CL \tag{4.19}
\]
for \( n \in \mathbb{N} \) and \( i = 1, \ldots, n \). Since \( u_{i,n}(0) = 0 \), we observe from (4.19) that
\[
|u_{i,n}(x)| \leq CL|x| \quad \text{for} \quad x \in I. \tag{4.19}
\]
Hence, taking \( \delta' > 0 \) small enough to satisfy \( \delta' < -\psi(0)/4CL \), we obtain
\[
u_{i,n}(x) \geq \frac{1}{4} \psi(0) \quad \text{for} \quad x \in [0, \delta']. \tag{4.20}
\]
Thus, by (4.18) and (4.20) we see that \( a := \min\{\delta, \delta'\} \) satisfies \( (0, a) \subset \mathcal{N}_{i,n} \). We note that (4.19) implies that \( a \) is independent of \( i \) and \( n \). Along the same line as above, the existence of \( b \) follows. We complete the proof. \( \Box \)
Lemma 4.8. There exists a constant $M > 0$ being independent of $n$ such that
\[ \tau_n \sum_{i=1}^{n} \mu_{i,n}(I)^2 \leq M \quad \text{for} \quad n \in \mathbb{N}. \]

Proof. Letting $0 < a < b < 1$ be the constants obtained by Lemma 4.7, we decompose $I$ into $(0, a) \cup [a, b] \cup (b, 1)$. Combining (4.16) with Lemma 4.7, we have
\[ \mu_{i,n}(I) = \mu_{i,n}(0, a) + \mu_{i,n}([a, b]) + \mu_{i,n}(b, 1) = \mu_{i,n}([a, b]). \]

Thus it suffices to estimate $\mu_{i,n}([a, b])$. Fix $\zeta \in C_c^\infty(I)$ with $\zeta \equiv 1$ in $[a, b]$ and $0 \leq \zeta \leq 1$ in $I$. We deduce from (4.15) and Hölder's inequality that
\[
\mu_{i,n}([a, b]) \leq \int_0^1 \zeta(x) \, d\mu_{i,n} = L_{i,n}(\zeta)
\leq 2\|u_{i,n}\|_{L^\infty(\zeta)} + 5\|u_{i,n}\|_{L^3(\zeta)}\|u'_{i,n}\|_{L^1(\zeta)}\|\zeta''\|_{L^1(\zeta)} \\
+ \lambda\|u'_{i,n}\|_{L^2(\zeta)}\|\zeta\|_{L^2(\zeta)} + \|V_{i,n}\|_{L^1(\zeta)}\|\zeta\|_{L^1(\zeta)}.
\]

Since $\|u'_{i,n}\|_{L^\infty(\zeta)} \leq 2M_0$, by Lemma 4.4 we reduce (4.21) into
\[ \mu_{i,n}([a, b]) \leq C(1 + \|V_{i,n}\|_{L^1(\zeta)}). \]

Thus we infer from Lemma 4.4 that
\[
\tau_n \sum_{i=1}^{n} \mu_{i,n}([a, b])^2 \leq CT + C \int_0^T \int_0^1 |V_n|^2 \, dx \, dt \leq CT + CL^2.
\]

Therefore Lemma 4.8 follows. \hfill \Box

In order to study the regularity of the approximate solution $\tilde{u}_n$, we adopt the idea used in [3, Proposition 3.2] and [4, Theorem 3.9]. They made use of the following test functions:

Lemma 4.9. Fix $\eta \in C_c^\infty(I)$ and set
\[
\varphi_1(x) := \int_0^x \int_0^y \eta(s) \, ds \, dy + \alpha x^2 + \beta x^3,
\]
\[
\varphi_2(x) := \int_0^x \eta(y) \, dy + \left( \int_0^1 \eta(y) \, dy \right) \left( -3x^2 + 2x^3 \right),
\]

for $x \in \bar{I}$, where
\[
\alpha := \int_0^1 \eta(y) \, dy - 3 \int_0^y \eta(s) \, ds \, dy, \quad \beta := -\alpha - \int_0^1 \eta(s) \, ds \, dy.
\]

Then, $\varphi_1, \varphi_2 \in H^2_0(I)$ and there exists $C > 0$ such that
\[
\|\varphi_1\|_{C^1(\bar{I})}, \quad |\alpha|, \quad |\beta| \leq C\|\eta\|_{L^1(\zeta)},
\]
\[
\|\varphi_2\|_{L^\infty(\zeta)} \leq C\|\eta\|_{L^1(\zeta)}, \quad \|\varphi_2''\|_{L^p(\zeta)} \leq C\|\eta\|_{L^p(\zeta)} \quad \text{for} \quad p \in [1, \infty).
\]

The proof of Lemma 4.9 follows from a direct modification of the arguments in [3, 4].

Lemma 4.10. Let $\tilde{u}_n$ be the piecewise constant interpolation of $\{u_{i,n}\}$. Then there exists $C > 0$ such that
\[
\|\tilde{u}_n''\|_{L^p(0, T; L^2(\bar{I}))} \leq C(T + 1)
\]
(4.22)
for \( p \in (1, 2) \), and
\[
\|u''_n\|_{L^1(0,T;L^p(I))} \leq C(T + 1),
\] (4.23)
for \( n \in \mathbb{N} \).

**Proof.** For the measure \( \mu_{i,n} \) obtained in (4.15), we define a function \( m_{i,n}(x) \) as
\[
m_{i,n}(x) := \mu_{i,n}(0, x) \quad \text{for} \quad x \in I \text{ and } i = 1, \ldots, n.
\]
Then \( m_{i,n} \) is of bounded variation on \( I \). Thus we can consider the Lebesgue-Stieltjes integral induced by \( m_{i,n} \).

Moreover, using integration by parts for Lebesgue-Stieltjes integrals ([19, Chapter III, Theorem 14.1]), we have
\[
\int_0^1 \phi \, d\mu_{i,n}(x) = -\int_0^1 m_{i,n}(x)\phi'(x) \, dx
\] (4.24)
for \( \phi \in C^\infty_c(I) \). Combining (4.15) with (4.24), we obtain
\[
2 \int_0^1 \frac{u''_{i,n}}{\ell(u'_{i,n})} \phi'' \, dx = 5 \int_0^1 \frac{u''_{i,n}^2 u'_{i,n}}{\ell(u'_{i,n})^3} \phi' \, dx - \lambda \int_0^1 \frac{u''_{i,n}}{\ell(u'_{i,n})^2} \phi' \, dx - \int V_{i,n} \phi \, dx - \int m_{i,n} \phi' \, dx
\] (4.25)
for \( \phi \in C^\infty_c(I) \). By a density argument we see that (4.25) also holds for \( \phi \in H^1_0(I) \).

Fix \( \eta \in C^\infty_c(I) \) arbitrarily and define \( \varphi_1 \) as in Lemma 4.9. Since \( \varphi''_1 = \eta + 2\alpha + 6\beta x \), taking \( \varphi_1 \) as \( \varphi \) in (4.25), we have
\[
2 \int_0^1 \frac{u''_{i,n}}{\ell(u'_{i,n})} \eta \, dx = -4 \int_0^1 \frac{u''_{i,n}}{\ell(u'_{i,n})} (\alpha + 3\beta x) \, dx + 5 \int_0^1 \frac{u''_{i,n}^2 u'_{i,n}}{\ell(u'_{i,n})^3} \varphi_1' \, dx
\]
\[-\lambda \int_0^1 \frac{u''_{i,n}}{\ell(u'_{i,n})} \varphi_1' \, dx - \int V_{i,n} \varphi_1 \, dx - \int m_{i,n} \varphi_1' \, dx
\]
\[=: J_1 + J_2 + J_3 + J_4 + J_5.
\]
Since \( u_{i,n} \in \tilde{\mathcal{K}}_\eta \), by Lemmas 4.4 and 4.9 we obtain
\[
\begin{align*}
|J_1| &\leq (4|\alpha| + 12|\beta|)\|u''_{i,n}\|_{L^1(I)} \leq C \|\eta\|_{L^1(I)}, \\
|J_2| &\leq 5 \|u''_{i,n}\|_{L^1(I)}^2 \|\varphi_1\|_{C^1(I)} \leq CL^2 \|\eta\|_{L^1(I)}, \\
|J_3| &\leq \lambda \|\varphi_1\|_{C^1(I)}, \\
|J_4| &\leq \|V_{i,n}\|_{L^1(I)} \|\varphi_1\|_{L^\infty(I)} \leq C \|V_{i,n}\|_{L^1(I)} \|\eta\|_{L^1(I)}, \\
|J_5| &\leq \sup_{x \in I} m_{i,n}(x) \|\varphi_1\|_{C^1(I)} \leq C \mu_{i,n}(I) \|\eta\|_{L^1(I)},
\end{align*}
\]
where we used \( |u'_{i,n}| \leq \ell(u'_{i,n}) \) in \( I \). Thus (4.26) is reduced into
\[
\left| \int_0^1 \frac{u''_{i,n}}{\ell(u'_{i,n})} \eta \, dx \right| \leq C (1 + \|V_{i,n}\|_{L^1(I)} + \mu_{i,n}(I)) \|\eta\|_{L^1(I)} := C_{i,n} \|\eta\|_{L^1(I)}
\] (4.27)
for \( \eta \in C^\infty_c(I) \). Then (4.27) implies that
\[
\|u''_{i,n} \ell(u'_{i,n})^{-3}\|_{L^\infty(I)} \leq C_{i,n}.
\] (4.28)
Moreover, recalling that \( \|u'_{i,n}\|_{L^\infty(I)} \leq 2M_0 \), we deduce from (4.28) that
\[
\|u''_{i,n}\|_{L^\infty(I)} \leq \ell(2M_0)^3 C_{i,n}.
\] (4.29)
We turn to the uniform $H^3$-estimate on $u_{i,n}$. Fix $\eta \in C_c^\infty(I)$ arbitrarily and define $\varphi_2$ as in Lemma 4.9. Since
\[
\varphi''_2(x) = \eta'(x) + (-6 + 12x) \int_0^1 \eta(y)dy,
\]
taking $\varphi_2$ as $\varphi$ in (4.25), we reduce (4.25) into
\[
\int_0^1 \frac{u''_{i,n}}{\ell(u''_{i,n})^5} \eta' \ dx = \int_0^1 \frac{u''_{i,n}}{\ell(u''_{i,n})^5} (1 - 2x) \int_0^1 \eta(y)dy \ dx + \frac{5}{2} \int_0^1 \frac{|u''_{i,n}|^2}{\ell(u''_{i,n})} \varphi'_2 \ dx - \frac{\lambda}{2} \int_0^1 \frac{u''_{i,n}}{\ell(u''_{i,n})} \ varphi'_2 \ dx - \frac{1}{2} \int_0^1 V_{i,n} \varphi_2 \ dx - \frac{1}{2} \int_0^1 m_{i,n}(x) \varphi_2' \ dx.
\]
(4.30)
where we used (4.27). This clearly implies that
\[
\int_0^1 \left(\frac{u''_{i,n}}{\ell(u''_{i,n})^5} \eta' \right) \ dx \leq \frac{5}{2} \|u''_{i,n}\|_{L^\infty(I)} \|\varphi'_2\|_{L^1(I)} \leq C \|u''_{i,n}\|_{L^\infty(I)} \|\varphi'_2\|_{L^1(I)}
\]
for $p \in [1, 2]$. Similarily we have
\[
|\varphi_2| \leq C \|\varphi_2\|_{L^1(I)},
\]
\[
|\varphi_4| \leq \|V_{i,n}\|_{L^1(I)} \|\varphi_2\|_{L^\infty(I)} \leq C \|V_{i,n}\|_{L^1(I)} \|\varphi_2\|_{L^1(I)}.
\]
Hence we reduce (4.30) into
\[
\int_0^1 \frac{u''_{i,n}}{\ell(u''_{i,n})^5} \eta' \ dx \leq C \|\varphi_2\|_{L^1(I)} \|\varphi_2\|_{L^1(I)}
\]
for $\eta \in C_c^\infty(I)$ and $p \in [1, 2]$, where we used (4.27). This clearly implies that
\[
\|u''_{i,n}(\ell(u''_{i,n})^{-5})\|_{L^{p,r}(I)} \leq C \|u''_{i,n}\|_{L^\infty(I)}
\]
for $p \in (1, 2]$. Since $\|u''_{i,n}\|_{L^\infty(I)} \leq 2M_0$, this together with (4.29) and Lemma 4.4 implies that
\[
\|u''_{i,n}\|_{L^{p,r}(I)} \leq C \|u''_{i,n}\|_{L^\infty(I)}
\]
(4.31)
It follows from (4.31) and Jensen's inequality that
\[
\int_0^\tau \|u''_{i,n}\|_{L^{p,r}(I)} \ dx = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|u''_{i,n}\|_{L^{p,r}(I)} \ dx 
\]
(4.32)
\[
\leq CT + C \int_0^\tau |V_n|^2 \ dx dt + C \tau_n \sum_{i=1}^n \mu_{i,n}(I)^2.
\]
Plugging Lemmas 4.4 and 4.8 into (4.32), we obtain (4.22). Moreover, for the case $p \geq 1$, similarly we obtain (4.23). Thus Lemma 4.10 follows. □
4.3 Convergence

In this subsection we prove the convergence of approximate solutions. To begin with, along the standard argument in minimizing movements, we have:

Lemma 4.11. Let \( \{u_{i,n}\} \) be the family of functions obtained by Lemma 4.1. Let \( u_n \) and \( \tilde{u}_n \) be the interpolations of \( \{u_{i,n}\} \) defined in Definitions 4.2 and 4.3, respectively. Then there exists \( u \in L^\infty(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \) such that

\[
\begin{align*}
    u_n & \to u \quad \text{weakly}^* \quad \text{in} \quad L^\infty(0, T; H^2(\Omega)), \\
    V_n & \to \partial_t u \quad \text{weakly} \quad \text{in} \quad L^2(0, T; L^2(\Omega)), \\
    \tilde{u}_n & \to u \quad \text{weakly}^* \quad \text{in} \quad L^\infty(0, T; H^2(\Omega)), \\
    u_n & \to u \quad \text{weakly} \quad \text{in} \quad H^1(0, T; L^2(\Omega)), \\
    \tilde{u}_n & \to u \quad \text{in} \quad L^2(0, T; L^2(\Omega)),
\end{align*}
\]

as \( n \to \infty \) up to a subsequence.

Proof. Thanks to the uniform estimates on \( u_n \) and \( \tilde{u}_n \) obtained by Lemma 4.4, we complete the proof along the same line as in the proof of [15, Section 4] or [17, Lemma 3.5, Theorem 3.1].

Lemma 4.10 allows us to discuss an extra regularity on the limit \( u \) obtained by Lemma 4.11:

Lemma 4.12. Let \( \{u_{i,n}\} \) be the family of functions obtained by Lemma 4.1, and let \( \tilde{u}_n \) be the piecewise constant interpolation of \( \{u_{i,n}\} \). Then for each \( p \in (1, 2] \)

\[
\begin{align*}
    \tilde{u}_n & \to \tilde{u} \quad \text{weakly} \quad \text{in} \quad L^p(0, T; W^{3, \frac{p}{p-1}}(\Omega)), \\
    \tilde{u}'_n & \to u' \quad \text{in} \quad L^3(0, T; L^\infty(\Omega)), \\
    \tilde{u}''_n & \to u'' \quad \text{in} \quad L^2(0, T; L^\infty(\Omega)),
\end{align*}
\]

as \( n \to \infty \) up to a subsequence, where \( u \) is the function obtained by Lemma 4.11.

Proof. We show (4.38). Fix \( p \in (1, 2] \) arbitrarily. By Lemmas 4.4 and 4.10 we find \( \tilde{u} \in L^p(0, T; W^{3, \frac{p}{p-1}}(\Omega)) \) such that

\[
\tilde{u}_n \to \tilde{u} \quad \text{weakly} \quad \text{in} \quad L^p(0, T; W^{3, \frac{p}{p-1}}(\Omega)) \quad (4.41)
\]

as \( n \to \infty \) up to a subsequence. Thanks to (4.36), combining (4.41) with the uniqueness of weak limit, we see that \( \tilde{u} = u \) in \( L^p(0, T; L^3(\Omega)) \). Hence, by \( \tilde{u} \in L^p(0, T; W^{3, \frac{p}{p-1}}(\Omega)) \) we obtain (4.38).

We turn to (4.39). By Lemma 2.1 we have

\[
\|\tilde{u}_n(t) - u'(t)\|_{L^\infty(\Omega)} \leq \sqrt{2}\|\tilde{u}_n(t) - u(t)\|_{L^2(\Omega)}^{1/4}\|\tilde{u}_n(t) - u(t)\|_{L^2(\Omega)}^{3/4} \quad (4.42)
\]

for a.e. \( t \in (0, T) \). It follows from (2.1), (4.6), (4.37) and (4.42) that

\[
\|\tilde{u}_n - u'\|_{L^8(0,T;L^\infty(\Omega))}^8 \leq 2^4 (2L)^6 \|\tilde{u}_n - u\|_{L^2(0,T;L^2(\Omega))}^2 \to 0 \quad \text{as} \quad n \to \infty,
\]

which implies (4.39).

Finally we show (4.40). We deduce from Proposition 2.2 that

\[
\|\tilde{u}_n(t) - u''(t)\|_{L^\infty(\Omega)} \leq C\|\tilde{u}_n(t) - u''(t)\|_{L^2(\Omega)}^{1/4}\|\tilde{u}_n(t) - u'(t)\|_{L^2(\Omega)}^{1/4} + C\|\tilde{u}_n(t) - u'(t)\|_{L^2(\Omega)} \quad (4.43)
\]

for a.e. \( t \in (0, T) \). We observe from (4.38) that

\[
\|u\|_{L^2(0,T;H^1(\Omega))} \leq \liminf_{n \to \infty} \|\tilde{u}_n\|_{L^2(0,T;H^1(\Omega))} \leq C(T + 1), \quad (4.44)
\]
where the last inequality followed from Lemmas 4.4 and 4.10. Plugging (4.44) into (4.43), we deduce from (4.39) that
\[ \| \tilde{u}''_n - u'' \|_{L^2(0, T; L^2(I))} \leq C(T + 1)(\| \tilde{u}'_n - u' \|_{L^2(0, T; L^2(I))} + \| \tilde{u}'_n - u' \|_{L^2(0, T; L^2(I))}) \to 0 \]
as \( n \to \infty \). Thus (4.40) follows.

Along the same argument as in the proof of [15, Theorem 4.2], we have:

**Lemma 4.13.** Let \( \{ u_{i,n} \} \) be the family of functions obtained by Lemma 4.1, and let \( u_n \) be a piecewise linear interpolation of \( \{ u_i, n \} \). Then
\[ u_n \to u \quad \text{in} \quad C^{0, \beta}([0, T]; C^{1, \alpha}(I)) \quad \text{as} \quad n \to \infty \]
for \( \alpha \in (0, 1/2) \) and \( \beta = (1 - 2\alpha)/8 \), up to a subsequence, where \( u \) is the function obtained by Lemma 4.11.

### 5 Proof of Theorem 1.1

To begin with, we prove the existence of local-in-time solutions to (P) and its regularity properties:

**Theorem 5.1.** Let \( \psi : I \to \mathbb{R} \) satisfy (1.5). Then for each \( u_0 \in K_{\vartheta} \), there exists \( T = T(u_0) > 0 \) such that (P) possesses a unique solution \( u \in \mathcal{X}_T \). Moreover, the solution \( u \) satisfies the following:

(i) For any \( p \in (1, 2) \), \( \alpha \in (0, 1/2) \) and \( \beta = (1 - 2\alpha)/8 \),
\[ u \in L^p(0, T; W^{3,\frac{p}{2}}(I)) \cap C^{0, \beta}([0, T]; C^{1, \alpha}(I)). \] (5.1)

(ii) For a.e. \( t \in (0, T) \),
\[ u''(\cdot, t) \in BV(I) \] (5.2)
holds and the distribution
\[ \mu_t := \partial_t u(\cdot, t) + 2 \left( \frac{u''(\cdot, t)}{\ell(u'(\cdot, t))} \right)'' + \left[ \frac{1}{2} \left( \frac{|u''(\cdot, t)|^2}{\ell(u'(\cdot, t))} - \lambda \right) \frac{u'(\cdot, t)}{\ell(u'(\cdot, t))} \right]' \] (5.3)
defines a Radon measure on \( I \) and satisfies
\[ \text{supp} \ \mu_t \subset \{ x \in I \mid u(x, t) = \psi(x) \}. \] (5.4)

Moreover, there exists \( C = C(u_0, \psi) > 0 \) such that
\[ \int_0^T \mu_t(I)^2 \ dt \leq C. \] (5.5)

**Proof.** We divide the proof into three steps. From now on, we denote by \( u \) the limit obtained by Lemma 4.11.

**Step 1.** We prove \( u \in \mathcal{X}_T \). By Lemma 4.11 we have \( u \in L^\infty(0, T; \mathcal{H}(I)) \cap H^1(0, T; L^2(I)) \). Moreover, since \( u_n(\cdot, 0) = u_0(\cdot) \) and \( u_n \uparrow \psi \) in \( L^1[I \times [0, T]] \), it follows from Lemma 4.13 that \( u(\cdot, 0) = u_0(\cdot) \) and \( u \uparrow \psi \) in \( L^1[I \times [0, T]] \). Thus we have \( u \in \mathcal{X}_T \).

**Step 2.** We prove that \( u \) satisfies (P) for \( \nu \in \mathcal{X}_T \). Fix \( \nu \in \mathcal{X}_T \) arbitrarily. Since \( \nu \uparrow \psi \), for any \( n \in \mathbb{N} \), \( i = 1, \ldots, n \) and \( \varepsilon \in (0, 1) \) we have
\[ u_{i,n}(x) + \varepsilon [v(x, t) - u_{i,n}(x)] \geq \psi(x) \quad \text{for} \quad (x, t) \in I \times [0, T]. \]
Moreover, since \( v(t) \in \mathcal{H}(I) \) a.e. \( t \in (0, T) \), it follows from Lemma 4.5 that
\[ \| u_{i,n}^\prime(\cdot) + \varepsilon [v^\prime(\cdot, t) - u_{i,n}^\prime(\cdot)] \|_{L^\infty(I)} \leq (1 - \varepsilon)^{3/2}M_0 + \varepsilon \| v^\prime(\cdot, t) \|_{L^\infty(I)} \leq 2M_0. \]
for $\varepsilon > 0$ small enough. Thus $u_{i,n}(\cdot) + \varepsilon (v(\cdot, t) - u_{i,n}(\cdot)) \in \hat{K}_\phi$ for a.e. $t \in (0, T)$. Since $u_{i,n}$ is a solution to $(M_{i,n})$, we obtain

$$\left. \frac{d}{d\varepsilon} G_{i,n}(u_{i,n}(\cdot) + \varepsilon (v(\cdot, t) - u_{i,n}(\cdot))) \right|_{\varepsilon = 0} \geq 0$$

(5.6)

for a.e. $t \in ((i-1)r_n, ir_n)$ and $i = 1, \ldots, n$. Integrating (5.6) with respect to $t$ on $((i-1)r_n, ir_n)$ and summing over $i = 1, \ldots, n$, we have

$$\int_0^1 \left[ 2 \tilde{u}_n''(v - \tilde{u}_n)' - \left( 5 \frac{\tilde{u}_n''}{\ell(\tilde{u}_n)^5} - \lambda \right) \frac{\tilde{u}_n''(v - \tilde{u}_n)'}{\ell(\tilde{u}_n)^6} + V_n(v - \tilde{u}_n) \right] dx dt \geq 0$$

(5.7)

for $v \in \mathcal{K}_T$. It follows from (4.34) and (4.37) that

$$\int_0^1 V_n(v - \tilde{u}_n) dx dt \to \int_0^1 \partial_t u(v - u) dx dt \quad \text{as} \quad n \to \infty$$

(5.8)

up to a subsequence. Thanks to Lemma 4.12, extracting a subsequence, we have

$$\int_0^1 \lambda \tilde{u}_n'' \tilde{u}_n'(v - \tilde{u}_n)' dx dt \to \int_0^1 \lambda u'(u)'^T(v - u)' dx dt$$

(5.9)

as $n \to \infty$. Moreover, combining (4.40) with Lemma 4.12, we observe that

$$\int_0^1 |\tilde{u}_n''|^2 \tilde{u}_n'(v - \tilde{u}_n)' dx dt \to \int_0^1 |u''|^2 u'(u)'^T(v - u)' dx dt,$$

(5.10)

$$\int_0^1 2 \tilde{u}_n'' \tilde{u}_n''(v - \tilde{u}_n)'' dx dt \to \int_0^1 2 u'' u'(v - u)'' dx dt,$$

(5.11)

as $n \to \infty$ up to a subsequence. Plugging (5.8), (5.9), (5.10) and (5.11) into (5.7), we see that $u$ satisfies (P) for $v \in \mathcal{K}_T$.

**Step 3. We discuss on (i) and (ii).** Since Lemmas 4.12 and 4.13 clearly imply (5.1), the property (i) follows. We turn to the property (ii). We first prove that the distribution (5.3) defines a measure on $I$. Fix $0 \leq \varphi \in C_0^\infty(I)$ arbitrarily and set

$$\langle \mu_t, \varphi \rangle = \int_0^1 \left[ \frac{u''(\cdot, t)\varphi''}{\ell(u'(\cdot, t))^5} - \left( 5 \frac{u''(\cdot, t)}{\ell(u'(\cdot, t))^6} - \lambda \right) \frac{u'(\cdot, t)\varphi'(\cdot, t)}{\ell(u'(\cdot, t))} + \partial_t u(\cdot, t)\varphi(\cdot, t) \right] dx$$

for a.e. $t \in (0, T)$. Then it is easy to check that $\langle \mu_t, \varphi \rangle \in L^1(0, T)$. Fix $\tau \in (0, T)$ arbitrarily. For sufficiently small $\varepsilon > 0$, let $\eta \in C_0^\infty(0, T)$ be such that

$$\eta(t) = \begin{cases} 1 & \text{in } [\tau, \tau + \varepsilon], \\ 0 & \text{in } [0, \tau - \varepsilon] \cup [\tau + 2\varepsilon, T], \end{cases}$$

where $0 \leq \eta(t) \leq 1$ in $[\tau - \varepsilon, \tau] \cup (\tau + \varepsilon, \tau + 2\varepsilon)$. Since $u + \varphi \eta \in \mathcal{K}_T$, taking $u + \varphi \eta$ as $v$ in Lemma 3.1, we have

$$\int_\tau^{\tau+\varepsilon} \langle \mu_t, \varphi \rangle dt = \int_\tau^{\tau+\varepsilon} \int_0^1 \left[ \frac{u''(\cdot, t)\varphi''}{\ell(u'(\cdot, t))^5} - \left( 5 \frac{u''(\cdot, t)}{\ell(u'(\cdot, t))^6} - \lambda \right) \frac{u'(\cdot, t)\varphi'(\cdot, t)}{\ell(u'(\cdot, t))} + \partial_t u \varphi(\cdot, t) \right] dx dt \geq 0.$$
Thus \( \mu_t \) is a nonnegative distribution on \( C_c^\infty(I) \) for a.e. \( t \in (0, T) \). Then it follows from Riesz’s theorem that \( \mu_t \) defines a Radon measure on \( I \). Thanks to Lemma 4.13, we see that

\[
\mathcal{N}_t := \left\{ x \in I \mid u(x, t) > \psi(x) \right\}
\]

is an open set in \( I \) for \( t \in [0, T] \). Hence, similarly to (4.16), we obtain (5.4).

Next we prove (5.5). By Lemma 4.13 we have \( u \in C^{0,\beta}([0, T]; C^{1,\alpha}(I)) \). Thus, similarly to Lemma 4.7, we find \( 0 < \tilde{a} < \tilde{b} < 1 \) being independent of \( t \) such that

\[
(0, \tilde{a}) \cup (\tilde{b}, 1) \subset \mathcal{N}_t,
\]

and then, \( \mu_t(I) = \mu_t([\tilde{a}, \tilde{b}]) \). For \( \zeta \in C_c^\infty(I) \) with \( 0 \leq \zeta \leq 1 \) and \( \zeta \equiv 1 \) on \( [\tilde{a}, \tilde{b}] \), we have

\[
\mu_t([\tilde{a}, \tilde{b}]) \leq \int_0^1 \zeta \, d\mu_t = \int_0^1 \left[ \frac{2}{t} \frac{u''(\cdot, t) \zeta''}{(u''(\cdot, t))^2} - \left( \frac{5 |u''(\cdot, t)|^2}{t(u''(\cdot, t))} + \lambda \right) \frac{u'(\cdot, t) \zeta'}{t(u''(\cdot, t))} + \partial_t u(\cdot, t) \zeta \right] \, dx.
\]

(5.13)

It follows from Lemmas 4.5 and 4.13 that

\[
\|u'(t)\|_{L^\infty(I)} \leq \frac{3}{2} M_0 \quad \text{for} \quad t \in [0, T].
\]

Thus we reduce (5.13) into

\[
\mu_t([\tilde{a}, \tilde{b}]) \leq C \left( 1 + \|u(t)\|_{L^\infty(I)} + \|\partial_t u(t)\|_{L^2(I)} \right),
\]

(5.14)

where the constant \( C \) is independent of \( t \). By Lemmas 4.4 and 4.11 we have

\[
\|u\|_{L^\infty(0, T; C(I))} \leq \liminf_{n \to \infty} \|\bar{u}_n\|_{L^\infty(0, T; C(I))} \leq L.
\]

This together with (5.14) implies that

\[
\mu_t([\tilde{a}, \tilde{b}]) \leq C \left( 1 + \|\partial_t u(t)\|_{L^2(I)} \right).
\]

Using Lemmas 4.4 and 4.11 again, we obtain

\[
\int_0^T \mu_t(I)^2 \, dt \leq C \int_0^T \left[ 1 + \|\partial_t u(t)\|_{L^2(I)}^2 \right] \, dt \leq C(T + L^2).
\]

It remains to show (5.2). Similarly to (4.24), it follows from (5.12) that

\[
\langle \mu_t, \varphi \rangle = -\int_I m_t(x) \varphi'(x) \, dx \quad \text{for a.e. } t \in (0, T) \text{ and } \varphi \in C_c^\infty(I),
\]

(5.15)

where \( m_t(x) := \mu_t(0, x) \). By (5.1) we notice that \( u(\cdot, t) \in H^3(I) \) for a.e. \( t \in (0, T) \) and hence we obtain

\[
\int_0^1 \left[ -2 \left( \frac{u''(\cdot, t)}{t(u''(\cdot, t))} \right)' + \left( \frac{5 |u''(\cdot, t)|^2}{t(u''(\cdot, t))} + \lambda \right) \frac{u'(\cdot, t)}{t(u''(\cdot, t))} - U(\cdot, t) \right] \varphi' \, dx = -\int_0^1 m_t(x) \varphi'(x) \, dx
\]

for a.e. \( t \in (0, T) \) and \( \varphi \in C_c^\infty(I) \), where

\[
U(x, t) := \int_0^x \partial_t u(y, t) \, dy.
\]

We note that, since \( \partial_t u(\cdot, t) \in L^1(I) \) for a.e. \( t \in (0, T) \), \( U(x, t) \) is absolutely continuous on \( I \) for a.e. \( t \in (0, T) \). Hence there exists \( c \in \mathbb{R} \) such that

\[
-2 \left( \frac{u''(\cdot, t)}{t(u''(\cdot, t))} \right)' + \left( \frac{5 |u''(\cdot, t)|^2}{t(u''(\cdot, t))} + \lambda \right) \frac{u'(\cdot, t)}{t(u''(\cdot, t))} - U(\cdot, t) + m_t(x) = c,
\]

which implies \( u''(\cdot, t) \in BV(I) \) since \( U(\cdot, t) \) is absolutely continuous and \( m_t(x) \) is of bounded variation. Thus (5.2) follows and we complete the proof of Theorem 5.1. \( \Box \)
In the rest of this section we derive the $L^2$-gradient structure of $\mathcal{E}_A(u(t))$ in a weak sense, where $u$ denotes the weak solution to (P) obtained by Theorem 5.1.

**Lemma 5.2.** Let $u$ be the solution to (P) obtained by Theorem 5.1 for some $T > 0$. Then

$$u(\cdot, t) \in H^2(I) \quad \text{for all} \quad t \in [0, T].$$

**Proof.** Fix $t \in [0, T]$ arbitrarily. By (4.11) we find $v_t \in H^2(I)$ such that

$$u_n(\cdot, t) \rightharpoonup v_t \quad \text{weakly in} \quad H^2(I)$$

up to a subsequence. Moreover, it follows from the Rellich-Kondrachov compactness theorem that $u_n(\cdot, t) \rightharpoonup v_t(\cdot)$ in $C^{0,1}(I)$. This together with Lemma 4.13 implies that $v_t(\cdot) = u(\cdot, t)$ in $C^{0,1}(I)$ for $t \in [0, T)$. Since $v_t \in H^2(I)$, we have

$$u(\cdot, t) = v_t \quad \text{in} \quad H^2(I) \quad \text{for} \quad t \in [0, T].$$

Thus Lemma 5.2 follows. \hfill $\square$

**Lemma 5.3.** Let $u$ be the solution to (P) obtained by Theorem 5.1 for some $T > 0$. Then

$$\mathcal{E}_A(u(T)) - \mathcal{E}_A(u_0) \leq -\frac{1}{2} \int_0^T \int_I |\partial_t u(x, t)|^2 \, dx \, dt. \quad (5.16)$$

**Proof.** Since $u_{0,n}(\cdot) = u_0(\cdot)$ and $u_n(\cdot, T) = u_{n,n}(\cdot)$, we deduce from (4.9) that

$$\frac{1}{2} \int_0^T \int_I |V_n(x, t)|^2 \, dx \, dt \leq \mathcal{E}_A(u_{0,n}) - \mathcal{E}_A(u_{n,n}) = \mathcal{E}_A(u_0) - \mathcal{E}_A(u_n(T)). \quad (5.17)$$

By Lemmas 4.11 and 4.13 we find a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that

$$u_{n_k} \rightharpoonup u \quad \text{in} \quad C^{0,\bar{\alpha}}([0, T]; C^{1,\alpha}(I)), \quad (5.18)$$

$$V_{n_k} \rightharpoonup \partial_t u \quad \text{weakly in} \quad L^2(0, T; L^2(I)). \quad (5.19)$$

On the other hand, thanks to (4.11), we also find $\{u_{n_k}^{\prime}\} \subset \{u_{n_k}\}$ and $v_T \in H^2(I)$ such that

$$u_{n_k}^{\prime}(T) \rightharpoonup v_T \quad \text{weakly in} \quad H^2(I).$$

This together with (5.18) implies that $u(T) = v_T$ in $C^{1,\alpha}(I)$. Then, along the same line as in the proof of Lemma 5.2, we have $u(T) = v_T$ in $H^2(I)$, that is,

$$u_{n_k}^{\prime}(T) \rightharpoonup u(T) \quad \text{weakly in} \quad H^2(I). \quad (5.20)$$

Similarly to (4.5), we deduce from (5.18) and (5.20) that

$$\mathcal{E}_A(u(T)) \leq \liminf_{n_k \to \infty} \mathcal{E}_A(u_{n_k}^{\prime}(T)). \quad (5.21)$$

Thus, combining (5.17), (5.19) and (5.21), we obtain (5.16). \hfill $\square$

As a generalization of Lemma 5.3, we have:

**Lemma 5.4.** Let $u$ be the solution to (P) obtained by Theorem 5.1 for some $T > 0$. Then

$$\mathcal{E}_A(u(s)) - \mathcal{E}_A(u_0) \leq -\frac{1}{2} \int_0^s \int_I |\partial_t u(x, t)|^2 \, dx \, dt \quad \text{for} \quad 0 \leq s \leq T. \quad (5.22)$$
Proof. Fix $s \in (0, T)$ arbitrarily. Similarly to Theorem 5.1, we can construct the solution $\tilde{u}: I \times [0, s] \rightarrow \mathbb{R}$ to (P) with the initial datum $u_0$. Then it follows from Lemma 5.3 that

$$E_A(\tilde{u}(s)) - E_A(u_0) \leq -\frac{1}{2} \int_0^s \int_0^1 |\partial_t \tilde{u}(x, t)|^2 \, dxdt.$$ 

On the other hand, by Lemma 3.1 we see that $u|_{[0,s]}$ also satisfies (P). Thus we observe from Theorem 3.2 that $\tilde{u} = u|_{[0,s]}$ in $L^\infty(0, s; \mathcal{H}(I)) \cap H^1(0, s; L^2(I))$. Since $\tilde{u}(\cdot, s) = u(\cdot, s)$ in $L^2(I)$, along the same line as in the proof of Lemma 5.2, we have $\hat{u}(s) = u(s)$ in $H^2(I)$, which implies that $E_A(u(s)) = \tilde{E}_A(\tilde{u}(s))$. Thus we obtain (5.22). □

We are in a position to prove Theorem 1.1:

Proof of Theorem 1.1. Thanks to Theorem 5.1, it suffices to prove the gradient structure (1.7). Let $u: I \times [0, T] \rightarrow \mathbb{R}$ be the solution to (P) obtained by Theorem 5.1. Fix $0 \leq s_1 \leq s_2 \leq T$ arbitrarily.

To begin with, we prove the following:

For any $\tau \in [s_1, s_2]$, there exists $\rho = \rho(s_1, s_2) > 0$ being independent of $\tau$ such that 

(P) with initial data $u(\tau)$ has a unique weak solution $w_\tau: I \times [0, \rho] \rightarrow \mathbb{R}$. \hspace{1cm} (5.23)

Set $\hat{M} := 2 \sup_{s_1 \leq s_2 \leq T} ||u'(\cdot, t)||_{L^\infty(I)}$ and define $\hat{L}$ by

$$\hat{L} := \max\{c_H^{-1}((2\hat{M})^\frac{1}{2}, \sqrt{2}) E_A(u_0)^\frac{1}{2}\}.$$ 

We define $\rho > 0$ by a constant small enough to satisfy

$$2\sqrt{2} \hat{L} \rho^\frac{1}{2} \leq \frac{1}{2} \hat{M}.$$ 

Let $n \in \mathbb{N}$ and set $\rho_n = \rho/n$. We define a family of functions $\{w_{i,n}\}$ inductively. Let $w_{0,n} := u(\tau)$. For $i = 1, \ldots, n$, we define $w_{i,n}$ by argmin$_{v \in \hat{K}_\psi}$ $\hat{G}_{i,n}(v)$ with

$$\hat{G}_{i,n}(v) := E_A(v) + \hat{P}_{i,n}(v),$$ 

where

$$\hat{P}_{i,n}(v) := \frac{1}{2\rho_n} \int_0^1 |v - w_{i-1,n}|^2 \, dx,$$

$$\hat{K}_\psi := \left\{v \in \mathcal{H}(I) \mid v \neq \psi \text{ in } I, \max_{x \in I} |v'(x)| \leq 2\hat{M}\right\}.$$ 

By the minimality of $w_{i,n}$ we have

$$E_A(w_{i,n}) + \hat{P}_{i,n}(w_{i,n}) = \hat{G}_{i,n}(w_{i,n}) \leq \hat{G}_{i,n}(w_{i-1,n}) = E_A(w_{i-1,n}),$$

and then $E_A(w_{i,n}) \leq E_A(w_{0,n}) = E_A(u(\tau))$ for $i = 1, \ldots, n$. This together with Lemma 5.4 implies that $E_A(w_{i,n}) \leq E_A(u_0)$ for $i = 0, 1, \ldots, n$. Along the same argument as in (4.8), we have

$$\sup_{0 \leq s \leq n} ||w_{i,n}||_{H^2(I)} \leq \hat{L}.$$ 

Then, similarly to the proof of Theorem 5.1, we obtain the desired solution $w_\tau$.

Let $w_\tau$ be the solution to (P) obtained by (5.23). Thanks to Lemma 5.2, we see that $E_A(w_\tau(s))$ is well-defined for $s \in [0, \rho]$. Since $w_\tau(\cdot, 0) = u(\cdot, \tau)$, similarly to Lemma 5.4, we have

$$E_A(w_\tau(s)) - E_A(w_\tau(0)) \leq -\frac{1}{2} \int_0^s \int_0^1 |\partial_t w_\tau(x, t)|^2 \, dxdt \quad \text{for} \quad s \in [0, \rho].$$ \hspace{1cm} (5.24)
Along the same line as in the proof of Lemma 5.4, we obtain
\[ w_s(\cdot, s) = u(\cdot, s_1 + s), \quad E_A(w_s(s)) = E_A(u(s + s_1)), \quad \text{for} \quad s \in [0, \rho]. \]
This together with (5.24) implies that
\[ E_A(u(s)) - E_A(u(s_1)) \leq -\frac{1}{2} \int_{s_1}^{s} \int_{0}^{1} |\partial_t u(x, t)|^2 \, dx \, dt \quad \text{for} \quad s \in [s_1, s_1 + \rho]. \]  
(5.25)
If \( s_1 + \rho \geq s_2 \), then we obtain the conclusion. If not, we repeat the argument. Indeed, if \( s_1 + \rho < s_2 \), then we have
\[ w_{s_1+\rho}(\cdot, s) = u(\cdot, s_1 + \rho + s), \quad E_A(w_{s_1+\rho}(s)) = E_A(u(s + s_1 + \rho)), \quad \text{for} \quad s \in [0, \rho]. \]
This together with (5.24) implies that
\[ E_A(u(s)) - E_A(u(s_1)) \leq -\frac{1}{2} \int_{s_1+\rho}^{s} \int_{0}^{1} |\partial_t u(x, t)|^2 \, dx \, dt \]  
for \( s \in [s_1 + \rho, s_1 + 2\rho] \). Combining (5.25) with (5.26), we obtain
\[ E_A(u(s)) - E_A(u(s_1)) \leq -\frac{1}{2} \int_{s_1}^{s} \int_{0}^{1} |\partial_t u(x, t)|^2 \, dx \, dt \quad \text{for} \quad s \in [s_1, s_1 + 2\rho]. \]
By induction we complete the proof. \( \square \)

6 Proof of Theorem 1.2

By Theorem 1.1 we prove the existence of local-in-time solutions to (P). In this section we first give a characterization of the maximal existence time of solutions to (P), which is defined as follows:

Definition 6.1. Let \( u \) be the solution to (P) with initial data \( u_0 \). We define the maximal existence time \( T_M(u_0) \) of \( u \) as follows:
\[ T_M = T_M(u_0) := \sup \{ \tau > 0 \mid u \text{ can be uniquely extended to a solution with } u(0) = u_0 \text{ to } (P) \text{ in } I \times [0, \tau]. \} \]
We give a characterization of the maximal existence time:

Theorem 6.2. Let \( \psi : [0, 1] \rightarrow \mathbb{R} \) satisfy (1.5). Assume that \( u_0 \in K_\psi \) satisfies \( T_M(u_0) < \infty \). Then
\[ \liminf_{t \uparrow T_M(u_0)} \|u'(t)\|_{L^\infty(I)} = \infty, \]
where \( u \) is the solution to (P) starting from \( u_0 \).

Proof. Let \( T_M(u_0) < \infty \). Assume that
\[ \gamma := \liminf_{t \uparrow T_M} \|u'(t)\|_{L^\infty(I)} < \infty. \]
Then we find a sequence \( \{t_l\}_{l \in \mathbb{N}} \subset [0, T_M) \) such that \( t_l \rightarrow T_M \) as \( l \rightarrow \infty \) and
\[ \|u'(t_l)\|_{L^\infty(I)} < 2\gamma \quad \text{for} \quad l \in \mathbb{N}. \]  
(6.1)
By Lemma 5.4 we have
\[ E_A(u(t_l)) \leq E_A(u_0) \quad \text{for} \quad l \in \mathbb{N}. \]  
(6.2)
Combining (6.1) with (6.2), we find $\rho > 0$ being independent of $l$ such that $u$ can be uniquely extended to the solution of (P) in $I \times [0, t_1 + \rho]$ for $l \in \mathbb{N}$. Indeed, similarly to the proof of (5.23), setting
\[ L_\gamma := \max\{c^l l(2\gamma)^{\frac{1}{2}}, \sqrt{2} \mathcal{E}_A(u_0)^{\frac{1}{2}}\}, \]
and taking $\rho > 0$ small enough such that
\[ 2\sqrt{2}L_\gamma \rho^{\frac{1}{2}} \leq \frac{1}{2} \gamma, \]
we observe from the argument in the proof of (5.23) that (P) with initial data $u(t_l)$ has a unique solution in $I \times [t_l, t_1 + \rho]$ for $l \in \mathbb{N}$. On the other hand, we can take $N \in \mathbb{N}$ large enough to satisfy $t_N + \rho > T_M$. However, this contradicts the definition of $T_M$. 

Similarly to [14, Theorem 5.10], we prove that global-in-time solutions to (P) have two alternatives.

**Lemma 6.3.** Let $\psi$ satisfy (1.5) and assume that $u_0 \in K_{\psi}$ satisfies $T_M(u_0) = \infty$. Then one of the following holds:

(a) There exists $\{t_l\} \in \mathbb{N}$ with $t_l \to \infty$ such that $\|u(t_l)\|_{L^\infty} \to \infty$ as $j \to \infty$;

(b) There exist $\{t_l\} \in \mathbb{N}$ with $t_l \to \infty$ and a solution $u_\ast \in K_{\psi} \cap H^3(I)$ to (1.6) such that $u(t_l) \rightharpoonup u_\ast$ weakly in $H^3(I)$ as $j \to \infty$.

**Proof.** Assume that (a) does not hold. Then there exists $\gamma > 0$ such that
\[ \|u'(t)\|_{L^\infty} \leq \gamma \text{ for all } t \in [0, \infty). \] (6.3)

We define $M_\ast$ and $L_\ast$ by
\[ M_\ast := 2\gamma \quad \text{and} \quad L_\ast := \max\{c^l l(2M_\ast)^{\frac{1}{2}}, \sqrt{2} \mathcal{E}_A(u_0)^{\frac{1}{2}}\}, \]
respectively. We take $T_\ast > 0$ small enough such that
\[ 2\sqrt{2}L_\ast T_\ast^{\frac{1}{2}} \leq \frac{1}{2} M_\ast. \]

Fix $m \in \mathbb{N}$ arbitrarily. Using the same argument as in (5.23), we can identify $u|_{[mT_\ast,(m+1)T_\ast]}$ with the solution in $I \times [0, T_\ast]$ to (P) with initial data $u(mT_\ast)$. In particular, we have
\[ \int_{mT_\ast}^{(m+1)T_\ast} \int_{0}^{1} \left[ \partial_t u(v - u) + 2 \frac{u''(v - u)v'' - \left(5 \frac{|u''|^2}{l(u')^5} - \lambda \right) u'(v - u)}{l(u')} \right] dx dt \geq 0 \] (6.4)
for $v \in K_{\psi}$. Moreover, along the same line as in Lemma 5.3, we have
\[ \frac{1}{2} \int_{mT_\ast}^{(m+1)T_\ast} \|\partial_t u(t)\|_{L^2}^2 dt \leq \mathcal{E}_A(u(mT_\ast)) - \mathcal{E}_A(u((m + 1)T_\ast)). \] (6.5)

Since $m \in \mathbb{N}$ is arbitrary, it follows from (6.4) and the Lebesgue differential Theorem that
\[ \int_{0}^{1} \left[ \partial_t u(v - u) + 2 \frac{u''(v - u)v'' - \left(5 \frac{|u''|^2}{l(u')^5} - \lambda \right) u'(v - u)}{l(u')} \right] dx \geq 0 \] (6.6)
for $v \in K_{\psi}$ and a.e. $t \in (0, \infty)$. Furthermore, thanks to (6.5), we find a sequence $\{t_k\}$ with $t_k \to \infty$ such that
\[ \partial_t u(t_k) \to 0 \text{ in } L^2(I) \text{ as } k \to \infty. \] (6.7)

Since (6.6) holds for a.e. $t \in (0, \infty)$, we may assume that (6.6) holds true for $t \in \{t_k\}_{k \in \mathbb{N}}$. On the other hand, by (6.3) and Theorem 1.1 we have
\[ \ell(\gamma)^{-\frac{5}{2}} \|u(t_k)\|_{L^\infty}^2 \leq \mathcal{W}(u(t_k)) \leq \mathcal{E}_A(u(t_k)) \leq \mathcal{E}_A(u_0) \] (6.8)
for \( k \in \mathbb{N} \). Therefore, by the same argument as in Lemma 4.7 we find constants \( 0 < a^* < b^* < 1 \) being independent of \( k \in \mathbb{N} \) such that \( \mu_k(I) = \mu_k((a^*,b^*)]. \) Thus, along the same line as in the derivation of (5.14), we have

\[
\mu_k(I) \leq C(1 + ||u(t_k)||_{L^p}^p + ||\partial_t u(t_k)||_{L^q(I)}) \quad \text{for} \quad k \in \mathbb{N}. \tag{6.9}
\]

Plugging (6.7) and (6.8) into (6.9), and extracting a subsequence, we find a constant \( C > 0 \) such that

\[
\mu_k(I) \leq C \quad \text{for} \quad k \in \mathbb{N}. \tag{6.10}
\]

Similarly to the proof of Lemma 4.10, we deduce from (6.8) and (6.10) that

\[
||u''(t_k)||_{L^p(I)} \leq \mu_k(I) \leq C \quad \text{for} \quad k \in \mathbb{N}.
\]

Therefore, extracting a subsequence, we find \( u^* \in H^3(I) \) such that

\[
u(t_k) \rightharpoonup u^* \quad \text{weakly in} \quad H^3(I).
\]

This together with the Rellich-Kondrachov compactness theorem implies that

\[
u(t_k) \rightarrow u^* \quad \text{in} \quad H^2(I).
\]

Since \( \nu \in \mathcal{H}_T \) for all \( T > 0 \), we see that

\[
u(0,t_k) = u(1,t_k) = 0, \quad u(x,t_k) \geq \psi(x) \quad \text{in} \quad 1,
\]

for \( k \in \mathbb{N} \). Plugging (6.11) into (6.12), we obtain

\[
u(0) = u(1) = 0, \quad u^*(x) \geq \psi(x) \quad \text{in} \quad 1.
\]

Thus \( u^* \in K_\psi \). Moreover, we infer from (6.6), (6.7) and (6.11) that \( u^* \) satisfies (1.6). Therefore Lemma 6.3 follows.

\[
\square
\]

In order to prove the existence of global-in-time solutions to (P), we introduce the function \( G : (-c_0/2, c_0/2) \rightarrow \mathbb{R} \) defined by

\[
G(s) := \int_0^s \frac{d\tau}{(1+\tau^2)^{3/2}},
\]

where \( c_0 \) is defined by (1.8).

**Lemma 6.4.** Assume that \( v \in \mathcal{H}(I) \) satisfies \( \mathcal{W}(v) < c_0^2/4 \). Then

\[
||v'||_{L^\infty(I)} \leq G^{-1} \left( \mathcal{W}(v)^{\frac{1}{2}} \right) < \infty. \tag{6.13}
\]

**Proof.** Since \( v(0) = v(1) = 0 \), we find \( x_0 \in [0,1] \) such that \( v'(x_0) = 0 \). Fix \( x \in [0, x_0] \) arbitrarily. Then it follows from the definition of \( G \) that

\[
\mathcal{W}(v) \geq \int_x^{x_0} v''(y)^2 G'(v(y))^2 \, dy.
\]

This together with the Schwartz inequality implies that

\[
|G(v'(x)) - G(v'(x_0))| = \left| \int_x^{x_0} G'(v(y))v''(y) \, dy \right| \leq \sqrt{x_0 - x} \mathcal{W}(v)^{\frac{1}{2}}.
\]

Since \( G \) is an odd function, we have

\[
G(|v'(x)|) = |G(v'(x))| \leq \mathcal{W}(v)^{\frac{1}{2}} \quad \text{for} \quad x \in [0, x_0].
\]

The same estimate holds for \( x \in (x_0, 1] \). Then it follows from \( \mathcal{W}(v) < c_0^2/4 \) that (6.13) holds.

\[
\square
\]
We are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** By the assumption (1.9) we can find $u_0 \in K_\psi$ such that $\mathcal{E}_\lambda(u_0) \leq \frac{c_0^2}{4}$. Then it follows from $0 < \lambda < \frac{c_0^2}{4}$ that

$$\|u'(t)\|_{L^\infty(t)} \leq G^{-1}\left(\frac{c_0^2}{4} - \lambda\right). \quad (6.14)$$

Indeed, since $\mathcal{L}(v) \geq 1$ for $v \in K_\psi$, we have

$$\mathcal{W}(u_0) \leq \frac{c_0^2}{4} - \lambda.$$

This together with Lemma 6.4 implies (6.14).

By Theorem 1.1 we have a unique local-in-time solution $u$ of problem (P) such that

$$\mathcal{E}_\lambda(u(t)) \leq \mathcal{E}_\lambda(u_0) \leq \frac{c_0^2}{4} \quad \text{for} \quad t \in [0, T_M(u_0)). \quad (6.15)$$

Assume that $T_M(u_0) < \infty$. Then it follows from Theorem 6.2 that

$$\liminf_{t \uparrow T_M(u_0)} \|u'(t)\|_{L^\infty} = \infty. \quad (6.16)$$

On the other hand, similarly to (6.14), we infer from (6.15) that

$$\|u'(t)\|_{L^\infty(t)} \leq G^{-1}\left(\frac{c_0^2}{4} - \lambda\right) \quad \text{for} \quad t \in [0, T_M(u_0)).$$

This clearly contradicts (6.16). Thus the solution $u$ can be extended globally in time. In particular, we obtain

$$\mathcal{E}_\lambda(u(t)) \leq \mathcal{E}(u_0) \leq \frac{c_0^2}{4} \quad \text{for} \quad t \in [0, \infty).$$

Then, similarly to (6.14) again, we have

$$\|u'(t)\|_{L^\infty(t)} \leq G^{-1}\left(\frac{c_0^2}{4} - \lambda\right) \quad \text{for} \quad t \in [0, \infty).$$

This together with Lemma 6.3 implies that the solution $u$ subconverges to a solution $u_* \in K_\psi$ of (1.6). Therefore Theorem 1.2 follows.

\[\square\]

**Remark 6.5.** We give an example of the obstacle $\psi$ satisfying the assumption (1.9). To this end, we introduce a function $u_c : [0, 1] \rightarrow \mathbb{R}$ defined by

$$u_c(x) := \frac{2}{c(1 + G^{-1}(\frac{c}{2} - cx)^2)^{1/4}} - \frac{2}{c(1 + G^{-1}\left(\frac{c}{2}\right)^2)^{1/4}},$$

where $c \in (-c_0/2, c_0/2) \setminus \{0\}$. The function $u_c$ is given by [6, Lemma 4]. In [6] it is proved that $u_c$ satisfies

$$u_c(0) = u_c(1) = 0, \quad \mathcal{W}(u_c) = c^2,$$

for $c \in (-c_0/2, c_0/2) \setminus \{0\}$. From now on we let $c \in (0, c_0/2)$ for the simplicity. Since $\mathcal{L}(u_c)$ is monotone with respect to $c \in (0, c_0/2)$, recalling that $\lambda \in (0, \frac{c_0^2}{4})$, we find a unique constant $c^* \in (0, c_0/2)$ such that

$$\mathcal{E}_\lambda(u_c) = \lambda \mathcal{L}(u_c) + \mathcal{W}(u_c) \leq \frac{c_0^2}{4} \quad \text{for} \quad c \in (0, c^*]. \quad (6.17)$$

Fix $c \in (0, c^*)$ arbitrarily and assume that $\psi$ satisfies (1.5) and

$$\psi(x) \leq u_c(x) \quad \text{for} \quad x \in [0, 1].$$

Then $u_c \in K_\psi$. Thus we deduce from (6.17) that the assumption (1.9) holds.

**Remark 6.6.** For the case $\lambda = 0$, we assume that $\psi$ satisfies (1.5) and

$$\inf_{v \in K_\psi} \mathcal{E}_\lambda(v) < \frac{c_0^2}{4},$$

instead of (1.9).

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