Limit distributions for the discretization error of stochastic Volterra equations

Masaaki Fukasawa and Takuto Ugai

Graduate School of Engineering Science, Osaka University

Abstract

Our study aims to specify the asymptotic error distribution in the discretization of a stochastic Volterra equation with a fractional kernel. It is well-known that for a standard stochastic differential equation, the discretization error, normalized with its rate of convergence $1/\sqrt{n}$, converges in law to the solution of a certain linear equation. Similarly to this, we show that a suitably normalized discretization error of the Volterra equation converges in law to the solution of a certain linear Volterra equation with the same fractional kernel.

1 Introduction

The discretization of the stochastic differential equations (SDE) has been studied by many researchers for many years. Let $T > 0$ and consider a standard $d$-dimensional SDE and its discretization:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad t \in [0, T],$$

$$\hat{X}_t = X_0 + \int_0^t b(\hat{X}_s)ds + \int_0^t \sigma(\hat{X}_s)dW_s, \quad t \in [0, T].$$

Back in the 1990s, the limit distribution of the scaled error $U^n = \sqrt{n}(X - \hat{X})$ was studied by Kurtz and Protter [15] and Jacod and Protter [13]. They proved that $U^n$ stably converges in law to $U$ as $n$ tends to infinity, where $U = (U^1, \ldots, U^d)$ is the solution of the following SDE:

$$U^n_i = \sum_{k=1}^d \int_0^t U^n_i \left[ \partial_b b^i(X_s)ds + \sum_{j=1}^m \partial_k \sigma^j(X_s)dW^j_s \right] - \frac{1}{\sqrt{n}} \sum_{j=1}^m \sum_{k=1}^d \sum_{l=1}^d \int_0^t \partial_k \sigma^j(X_s)\sigma^l_j(X_s)dB^l_s, \quad (1.1)$$

where $B$ is an $m^2$-dimensional standard Brownian motion independent of the $m$-dimensional standard Brownian motion $W$. Recent developments include extensions to stochastic time partitions [7] and SDEs driven by a fractional Brownian motion [11, 17, 2]. Applications include the optimal choice of tuning parameters for the Multi-Level Monte Carlo method [3].

The aim of our study is to extend their result to $d$-dimensional stochastic Volterra equations (SVE) of the form:

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s, \quad t \in [0, T], \quad (1.2)$$

where $K(t) = \frac{t^{H-1/2}}{\Gamma(H)}, H \in (0, 1/2]$. Because of the singularity at the origin of the kernel function $K$, a sample path of the solution $X$ with $H < 1/2$ has lower Hölder regularity than that of a SDE has. In particular, the solution $X$ is not a semimartingale. Recently such a SVE has attracted attention in mathematical finance in the context of rough volatility modeling; see Abi Jaber et al. [1] and references therein. Applications in financial practice, such as pricing path-dependent options, require numerical methods to simulate the solution of the SVE. The discretization of the SVE is then the most natural step. An estimation of the associated numerical error is therefore of practical importance.

As in the case of SDEs, let $\hat{X}$ be the solution of the discretized SVE of (1.2), that is,

$$\hat{X}_t = X_0 + \int_0^t K(t-s)b(\hat{X}_s)ds + \int_0^t K(t-s)\sigma(\hat{X}_s)dW_s, \quad t \in [0, T]. \quad (1.3)$$
The solution $\hat{X}_t$ of (1.3) is constructed in an inductive way. Indeed, we have

$$\hat{X}_t = X_0 + b(X_0) \int_0^t K(\frac{t}{n} - s)ds + \sigma(X_0) \int_0^t K(\frac{t}{n} - s)dW_s,$$

$$\hat{X}_t = X_0 + b(X_0) \int_0^t K(\frac{t}{n} - s)ds + b(\hat{X}_k) \int_0^t K(\frac{t}{n} - s)ds + \sigma(X_0) \int_0^t K(\frac{t}{n} - s)dW_s + \sigma(\hat{X}_k) \int_0^t K(\frac{t}{n} - s)dW_s.$$

Iterating this procedure, we consequently obtain the solution:

$$\hat{X}_t = X_0 + \sum_{j=0}^m b(\hat{X}_j) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K(t-s)ds + \sum_{j=0}^m \sigma(\hat{X}_j) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K(t-s)dW_s, \quad (1.4)$$

In particular, we can construct $(\hat{X}_1, \ldots, \hat{X}_n)$ on the regular grid $t_k = \frac{k}{n}$ by generating the independent sequence of Gaussian vectors (matrices when the dimension of $W$ is larger than two)

$$\left( \int_{t_{k-1}}^{t_k} K(t-s)dW_s, \int_{t_{k-1}}^{t_k} K(t_{k+1}-s)dW_s, \ldots, \int_{t_{k-1}}^{t_k} K(t_n-s)dW_s \right), \quad k = 1, 2, \ldots, n \quad (1.5)$$

using the Cholesky decomposition of the identical covariance matrix

$$\Sigma = [\Sigma_{ij}], \quad \Sigma_{ij} = \int_0^1 K\left(\frac{i}{n} - s\right)K\left(\frac{j}{n} - s\right)ds.$$ 

Therefore (1.3) describes a feasible numerical scheme. Further, as noted by Fukasawa and Hirano [6], the matrix $\Sigma$ is nearly degenerate, meaning that the projection to a low (but more than three when $H \neq 1/2$) dimensional Gaussian vector gives an accurate and efficient approximation to (1.5). The extreme approximation is the one dimensional projection that essentially corresponds to the Euler scheme for SVE, for which Richard et al. [20] gave the rate of convergence. The discretization scheme (1.3) is expected to be more accurate than the Euler scheme in the sense that it gives a lower mean squared error (with the same rate of convergence). We determine the asymptotic error distribution for the scheme (1.3) in this paper. The corresponding analysis for the Euler scheme is however remained for future research.

The Jacod [12] theory of stable convergence for semimartingales played a key role in the study of the discretization error in the SDE case. Since a solution of a SVE with a singular fractional kernel is not a semimartingale any more, the argument of Jacod and Protter [13] is not directly extended to the case of SVE. To overcome this technical difficulty, we exploit the fact that the convolution with respect to the fractional kernel $K$ is a continuous map between Hölder spaces. This fact is utilized recently by Horvath et al. [10] to show the convergence of random walk approximations to rough volatility models.

We therefore formulate our limit theorem in terms of the weak convergence of laws on Hölder spaces. A tightness criterion for a separable subset of the space of Hölder functions is studied by Hamadouche [9]. We give, similar but not the same, a useful criterion for the tightness in the full space of the Hölder functions in Appendix.

We give our main result in Section 2. The proofs for some lemmas are deferred to Section 4, after some preliminaries are given in Section 3.

2 The main result

2.1 The statement

Let $(\Omega, F, P, (\mathcal{F}_t)_{t \geq 0})$ be a filtered probability space satisfying the usual conditions, and $W$ be an $m$-dimensional standard Brownian motion defined on this space. Assume the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ in (1.2) and (1.3) are continuously differentiable functions with the derivatives being bounded and uniformly continuous.\(^1\)

\(^1\)We remark here that the uniform continuity of the derivatives would be relaxed to the simple continuity assumption by a localization argument: stopping the processes at the time when the process is going out of some compact set. Similarly, if we know a priori that $X$ stays in a domain of $\mathbb{R}^d$, then the regularity conditions on $\mathbb{R}^d$ would be relaxed to those on the domain.
We denote by $C_0$ the set of the $\mathbb{R}^d$-valued continuous functions on $[0,T]$ vanishing at $t=0$ and by $C_0^\lambda$ the set of the $\mathbb{R}^d$-valued $\lambda$-Hölder continuous functions with the same property. Also, $\|\cdot\|_\infty,\|\cdot\|_{C_0},\|\cdot\|_{L_p}$ denote the supremum norm on $[0,T]$, the Hölder norm on $[0,T]$, and the $L_p$ norm with respect to $\mathbb{P}$, respectively. Throughout the paper, we always view elements of $\mathbb{R}^d$ as column vectors, and denote by $A^\top$ the transpose of $A$ for any matrix $A$. We also denote by $C$ a constant which may differ from one place to another one.

The following theorem is our main result.

**Theorem 2.1.** Let $c \in (0,H)$. Then the process $U^n = n^H(X - \hat{X})$ stably converges in law in $C_0^{H-c}$ to a process $U = (U^1, \ldots, U^d)$ which is the unique continuous solution of the SVE

$$U^i_t = \sum_{k=1}^{d} \frac{1}{\sqrt{(2H+2)\sin \pi H}} \int_0^t K(t-s) \partial_k \sigma^i_j(X_s) dW^j_s \quad \frac{1}{\sqrt{1(2H+2)\sin \pi H}} \sum_{j=1}^{m} \sum_{k=1}^{d} \int_0^t K(t-s) \partial_k \sigma^i_j(X_s) dW^j_s, \quad t \in [0,T], \quad i = 1, \ldots, d, \quad (2.1)$$

where $B$ is an $m^2$-dimensional standard Brownian motion, independent of $\mathcal{F}$ and defined on some extension of $(\Omega, \mathcal{F}, \mathbb{P})$.

Note that a stable convergence in law is stronger than a convergence in law and weaker than a convergence in probability. See Jacod and Protter [14] and Haehuser and Luschgy [8] for the details. Note also that, since the inclusion $C_0^{H-c} \to C_0$ is continuous, the continuous mapping theorem implies the convergence in law of $U^n$ to $U$ in $C_0$. When $H = 1/2$, we recover the classical result (1.1).

An interesting observation from Theorem 2.1 is that when $H$ is small, the discretization suffers from not only a small rate of convergence but also a large limit variance due to the factor $\sin \pi H$ in the denominator of (2.1). As in the classical case (1.1), the limit law $U_t$ is Gaussian conditionally on $\mathcal{F}$.

### 2.2 Lemmas

Here we list key steps to prove Theorem 2.1 as lemmas, for which the proofs are deferred to Section 4. We start with observing the following decomposition of $U^n = (U^{n,1}, \ldots, U^{n,d})$:

$$U^{n,j}_t \approx \int_0^t K(t-s) \left( \nabla \hat{V}^j(\hat{X}^{0,j}_s) \hat{X}^{0,j}_s + \sum_{j=1}^{m} \nabla \sigma^j(X^{0,j}_s) \right) dW^j_s + \int_0^t K(t-s) \nabla \hat{V}^j(\hat{X}^{0,j}_s) \hat{X}^{0,j}_s + \sum_{j=1}^{m} \sum_{k=1}^{d} \int_0^t K(t-s) \partial_k \sigma^j(X^{0,j}_s) dW^k_s, \quad (2.2)$$

where $\hat{X} = (\hat{X}^1, \ldots, \hat{X}^d)$ is the solution of (1.3) and $V^n = (V^{n,k,j})$ is defined as

$$V^{n,k,j} = n^H \int_0^t (\hat{X}^k_s - \hat{X}^{0,k}_s) dW^j_s, \quad 1 \leq k \leq d, \quad 1 \leq j \leq m.$$ 

The first lemma gives the limits of the quadratic variation and covariation of $V^n$ and $W$.

**Lemma 2.2.** For all $t \in [0,T], (k_1, k_2) \in \{1, \ldots, d\}^2, 1 \leq j \leq m$,

(i) $$\lim_{n \to \infty} \frac{1}{n^H} \sum_{j=1}^{m} \int_0^t \sigma^j_k(X_s) \sigma^j_k(X_s) dW^j_s = 0,$$

(ii) $$\lim_{n \to \infty} \frac{1}{n^H} \sum_{j=1}^{m} \int_0^t \sigma^j_k(X_s) \sigma^j_l(X_s) dW^j_s = 0.$$

Note that $\langle V^{n,k,j}, V^{n,k,j} \rangle = 0$ and $\langle V^{n,k,j}, W_l \rangle = 0$ for $(k_1, k_2) \in \{1, \ldots, d\}^2, (i, j) \in \{1, \ldots, m\}^2, i \neq j$. Then Lemma 2.2 and the results of Jacod [12] lead us to specify the limit distribution of $V^n$ in Lemma 2.3.
Lemma 2.3. The process $V^n$ stably converges in law in $C_0$ to a continuous process $V = \{V^k\}$ of the following form:

$$V^k = \frac{1}{\sqrt{\Gamma(2H + 2) \sin \pi H}} \sum_{i=1}^{m} \int_0^1 a_i'(X_s) dB^i_s, \quad 1 \leq k \leq d, \ 1 \leq j \leq m.$$ 

where $B$ is an $m^2$-dimensional standard Brownian motion, independent of $\mathcal{F}$ and defined on some extension of $(\Omega, \mathcal{F}, P)$.

We also show that the second integral term of (2.2) vanishes in $C_0^{H-\epsilon}$ as $n$ goes to infinity.

Lemma 2.4. For all $i \in \{1, \ldots, d\}$,

$$\int_0^t K(t-s) n^{H} \nabla b_i'(\hat{X}_s)\nabla (\hat{X}_s - \hat{X}_m) ds \xrightarrow{n \to \infty} 0.$$ 

The difference between both sides of (2.2) converges to zero in $C_0^{H-\epsilon}$ for any $\epsilon \in (0, H)$ as $n$ goes to infinity as we prove in Lemma 2.5.

Lemma 2.5. The $\gamma$-Hölder norm of the difference between both sides of (2.2) tends to zero in $L_p$ for any $\gamma \in (0, H)$ and $p \geq 1$.

Denote by $D_e$ the space of the cadlag functions on $[0, T]$ taking values in $\mathbb{R}^d$ equipped with the Skorokhod topology. Define $\varphi_n : [0, T] \to [0, T]$ by $\varphi_n(t) = \lfloor nt \rfloor/n$. The above lemmas are used in the proof of Lemma 2.6.

Lemma 2.6. If the sequence

$$(U^n, V^n, \{\nabla b_i'(\hat{X} \circ \varphi_n)\}_{i}, \{\partial_k a_j'(\hat{X} \circ \varphi_n)\}_{i,j,k})$$

converges in law in $C_0^{H-\epsilon} \times C_0 \times D_e \times D_{e,m}$ to

$$(U, V, \{\nabla b_i'(X)\}_{i}, \{\partial_k a_j'(X)\}_{i,j,k}),$$

then $U$ is the solution of (2.1).

We will additionally show the following lemma.

Lemma 2.7. The sequence $U^n$ is tight in $C_0^{H-\epsilon}$ for any $\epsilon \in (0, H)$.

We will use the uniqueness in law of the solution of (2.1).

Lemma 2.8. If there is a strong solution for (2.1), it is in $L_p$, continuous, and unique in law.

2.3 Proof of Theorem 2.1

Using the above lemmas, we now prove the theorem.

Proof of Theorem 2.1. By Lemmas 2.7 and B.1, $X \to U$ in probability in the uniform topology. Therefore,

$$((\nabla b_i'(\hat{X} \circ \varphi_n))_{i}, \{\partial_k a_j'(\hat{X} \circ \varphi_n)\}_{i,j,k}) \to ((\nabla b_i'(X))_{i}, \{\partial_k a_j'(X)\}_{i,j,k})$$

in probability in the uniform topology as well. Together with Lemmas 2.3 and 2.7, we conclude that

$$(U^n, V^n, \{\nabla b_i'(\hat{X} \circ \varphi_n)\}_{i}, \{\partial_k a_j'(\hat{X} \circ \varphi_n)\}_{i,j,k}, Y)$$

is tight in $C_0^{H-\epsilon} \times C_0 \times D_e \times D_{e,m} \times \mathbb{R}$ for any random variable $Y$ on $(\Omega, \mathcal{F}, P)$. For any subsequence of this tight sequence, there exists a further subsequence which converges by Prokhorov’s theorem (see e.g., Theorem 5.1 of Billingsley [4] for a nonseparable case). Lemmas 2.6 and 2.8 imply the uniqueness of the limit. Therefore the original sequence itself has to converge. The limit $U$ of $U^n$ is characterized by (2.1) again by Lemma 2.6. The convergence of $U^n$ is stable because $Y$ is arbitrary. □
3 Preliminaries

The fractional kernel satisfies the following condition.

\[
K \in L_\beta(0, T) \text{ for some } \beta \in (2, \frac{5}{2}],
\]

\[
\int_0^T K(t)dt = O(h^{H+1/2}), \quad \int_0^T (K(t + h) - K(t))dt = O(h^{H+1/2}),
\]

\[
\left( \int_0^h K(t)^2 dt \right)^{\frac{1}{2}} = O(h^{H}) \text{ and } \left( \int_0^T (K(t + h) - K(t))^2 dt \right)^{\frac{1}{2}} = O(h^H).
\]

(3.1)

We fix here \( \beta \in (2, \frac{5}{2}] \) and denote by \( \beta^* \) the conjugate index of \( \beta/2 \), namely, \( \beta^* = \beta/(\beta - 2) \), for the technical purposes.

Before discussing the moments and Hölder continuities, we remark the existence and uniqueness for the solutions of (1.2) and (1.3).

Remark 3.1. The existence and uniqueness of the strong continuous solution \( X \) for (1.2) are guaranteed by Abi Jabar et al. (Theorem 3.3 [1]). The strong solution \( X \) uniquely exists by (1.4).

We introduce the following lemma which is used several times to evaluate integrals with convolution kernel.

Lemma 3.2. The following inequalities hold for any adapted \( \mathbb{R}^d \)-valued process \( Y \) and \( \mathbb{R}^{d\times m} \)-valued process \( Z \):

1. for \( p \geq 2 \),

\[
E \left[ \int_0^\tau |K(t-s)Y_s|^{p} \right] \leq C \int_0^\tau E \left[ |Y_t|^{p} \right] ds,
\]

2. for \( p > 2\beta^* \),

\[
E \left[ \int_0^\tau |K(t-s)Z_s|^{p} \right] \leq C \int_0^\tau E \left[ |Z_t|^{p} \right] ds,
\]

3. for \( p \geq 1 \),

\[
E \left[ \int_0^\tau (K(t+h-s) - K(t-s))Y_s ds \right]^{p} \leq C \int_0^\tau E \left[ |Y_s|^{p} \right] ds,
\]

4. for \( p \geq 2 \),

\[
E \left[ \int_0^\tau (K(t+h-s) - K(t-s))Z_s dW_s \right]^{p} \leq C \int_0^\tau E \left[ |Z_s|^{p} \right] ds,
\]

where \( C \) depends only on any of \( K, \beta, p, \) and \( T \).

Proof of Lemma 3.2. Let us show (1), (3) first. Take \( p \geq 2 \). Minkowski’s integral inequality, the Cauchy-Schwarz inequality and Hölder’s inequality show that

\[
E \left[ \int_0^\tau |K(t-s)|E \left[ |Y_s|^{p} \right]^{\frac{1}{2}} ds \right]^{p} \leq \left( \int_0^\tau |K(t-s)|^{p} ds \right)^{\frac{1}{2}} \left( \int_0^\tau E \left[ |Y_s|^{p} \right]^{\frac{1}{2}} ds \right)^{\frac{1}{2}} \leq C \int_0^\tau E \left[ |Y_s|^{p} \right] ds.
\]

For (3), we observe

\[
E \left[ \int_0^\tau (K(t+h-s) - K(t-s))Y_s ds \right]^{p} + E \left[ \int_t^{t+h} K(t+h-s)Y_s ds \right]^{p} \leq \sup_{r \in [0,T]} E \left[ |Y_r|^{p} \right] \left( \int_0^\tau (K(s+h) - K(s))ds \right)^{\frac{1}{p}} + \left( \int_0^\tau |K(t+h-s)|E \left[ |Y_s|^{p} \right]^{\frac{1}{p}} ds \right)^{p} \leq C \int_0^\tau E \left[ |Y_s|^{p} \right]^{\frac{1}{p}} ds \leq C \int_0^\tau E \left[ |Y_s|^{p} \right] ds.
\]

\[
\text{Proof of Lemma 3.2.}
\]
by Minkowski’s integral inequality and (3.1). We next show (2), (4). For (2), let \( p > 2\beta \). Then, using the Burkholder-Davis-Gundy (BDG for short) inequality, Hölder’s inequality and Fubini’s theorem properly yields that

\[
E \left[ \left( \int_0^t |K(t-s)Z_s| \sigma_s \, ds \right)^p \right] \leq C_p E \left[ \left( \int_0^t |K(t-s)|^2 |Z_s|^p \, ds \right)^{\frac{p}{2}} \right] \\
\leq \left( \int_0^t |K(t-s)|^2 \, ds \right)^{\frac{p}{2}} E \left[ \left( \int_0^t |Z_s|^p \, ds \right)^{\frac{p}{2}} \right] \\
\leq C \int_0^t E \left[ |Z_s|^p \right] \, ds.
\]

For (4), take \( p \geq 2 \). By the BDG inequality and Minkowski’s inequality, we obtain

\[
E \left[ \left( \int_0^t (K(t+h-s) - K(t-s))Z_s \sigma_s \, ds \right)^p \right] + E \left[ \left( \int_t^{t+h} K(t+h-s)Z_s \sigma_s \, ds \right)^p \right] \\
\leq C_p E \left[ \left( \int_0^t |K(t+h-s) - K(t-s)|^2 |Y_s|^2 \, ds \right)^{\frac{p}{2}} \right] + C_p E \left[ \left( \int_t^{t+h} |K(t+h-s)|^2 |Y_s|^2 \, ds \right)^{\frac{p}{2}} \right] \\
\leq C_p \left( \int_0^t (K(t+h-s) - K(t-s))^2 \, ds \right)^{\frac{p}{2}} + C_p \left( \int_t^{t+h} K(t+h-s)^2 \, ds \right)^{\frac{p}{2}} E \left[ |Y_s|^p \right] \\
\leq Ch^{2p} \sup_{r \leq [0, t]} E \left[ |Y_r|^p \right].
\]

This completes the proof. \( \square \)

The \( p \)th moment and the Hölder continuity of the solution of the standard SVE, have already been studied. The following two results are corollaries of Abi Jaber et al. (Lemmas 3.1 and 2.4 [1]).

**Lemma 3.3.** Let \( p \geq 1 \), Then,

\[
\sup_{r \leq [0, t]} E \left[ |X_r|^p \right] \leq C,
\]

where \( C \) is a constant that only depends on \( |X_0|, |\beta(0)|, |\sigma(0)|, K, p \) and \( T \).

**Lemma 3.4.** Let \( p > H^{-1} \). Then

\[
E \left[ |X_t - X_s|^p \right] \leq C |t - s|^{|p|}, \quad t, s \in [0, T]
\]

and \( X \) admits a version which is Hölder continuous on \([0, T]\) of any order \( \alpha < H - p^{-1} \). Denoting this version again by \( X \), one has

\[
E \left[ \sup_{0 \leq s \leq T} \left| \frac{|X_t - X_s|^p}{|t - s|^\alpha} \right| \right] \leq C \alpha
\]

for all \( \alpha \in (0, H - p^{-1}) \), where \( C_\alpha \) is a constant. As a consequence, we can regard \( X \) as a \( C^\alpha \) valued random variable for any \( \alpha < H \).

The following two lemmas are analogues of the above lemmas.

**Lemma 3.5.** Let \( p \geq 1 \), Then,

\[
\sup_{r \leq [0, T]} E \left[ |X_r|^p \right] \leq C,
\]

where \( C \) is a constant that only depends on \( |X_0|, |\beta(0)|, |\sigma(0)|, K, p, \beta, \) and \( T \).
Proof. We will prove only in the case \( p > 2\beta' \), which is sufficient for \( p \geq 1 \). Let \( \tau_m = \inf \{ t \geq 0 \mid \hat{X}_t \leq m \} \land T \) and observe that

\[
|\hat{X}_t|^p 1_{[t < \tau_m]} \leq |X_0| + \int_0^t K(t-s)b(\hat{X}_{\lfloor m \rfloor 1_{[s < \tau_m]}})ds + \int_0^t K(t-s)\sigma(\hat{X}_{\lfloor m \rfloor 1_{[s < \tau_m]}})dW_s. \tag{3.2}
\]

Indeed, for \( t \geq \tau_m \) the left-hand side is zero while the right-hand side is nonnegative. For \( t < \tau_m \) the local property of the stochastic integral implies that the right-hand side is equal to

\[
|X_0| + \int_0^t K(t-s)b(\hat{X}_{\lfloor m \rfloor 1_{[s < \tau_m]}})ds + \int_0^t K(t-s)\sigma(\hat{X}_{\lfloor m \rfloor 1_{[s < \tau_m]}})dW_s. \tag{3.2}
\]

Then by (3.2), we have

\[
E \left[ |\hat{X}_t|^p 1_{[t < \tau_m]} \right] \leq 3\beta' |X_0|^p + |X_0|^p \int_0^t E \left[ K(t-s)b(\hat{X}_{\lfloor m \rfloor 1_{[s < \tau_m]}})ds \right]^{(p-1)} + 3\beta' E \left[ \int_0^t K(t-s)\sigma(\hat{X}_{\lfloor m \rfloor 1_{[s < \tau_m]}})dW_s \right]^{(p-1)}.
\]

Therefore, by Lemma 3.2-(1),(2) and the Lipschitz condition of \( \sigma \), we see

\[
E \left[ |\hat{X}_t|^p 1_{[t < \tau_m]} \right] \leq C_1 + C_2 \int_0^t E \left[ \left( b(\hat{X}_{\lfloor m \rfloor})^p + |\sigma(\hat{X}_{\lfloor m \rfloor})|^p \right) 1_{[s < \tau_m]} \right] ds
\leq C_1 + C_2 \int_0^t E \left[ |\hat{X}_{\lfloor m \rfloor}|^p 1_{[s < \tau_m]} \right] ds
\leq C_1 + C_2 \int_0^t \sup_{t \in [0,1]} E \left[ |\hat{X}_t|^p 1_{[t < \tau_m]} \right] ds,
\]

where \( C_1, C_2 \geq 0 \) are some constants independent of \( n \) and \( m \) (remark that \( \{ s : t < \tau_m \} \subset \{ \frac{m}{n} < \tau_n \} \) for all \( n \in (0, \bar{T}) \)). Putting \( f_m(t) = \sup_{t \in [0,1]} E \left[ |\hat{X}_t|^p 1_{[t < \tau_m]} \right] \), we see

\[
f_m(t) \leq C_1 + C_2 \int_0^t \sup_{s \in [0,1]} E \left[ |\hat{X}_t|^p 1_{[t < \tau_m]} \right] ds = C_1 + C_2 \int_0^t f_m(s)ds.
\]

Since \( f_m(t) \) is bounded and, therefore, integrable on \([0, \bar{T}]\), we can apply Gronwall’s lemma to obtain

\[
f_m(t) \leq C_1 e^{C_2 t} \leq C_1 e^{C_2 \bar{T}}.
\]

By Fatou’s lemma, we have

\[
E \left[ |\hat{X}_t|^p \right] = E \left[ \lim \inf_{m \to \infty} |\hat{X}_t|^p 1_{[t < \tau_m]} \right] \leq \lim \inf_{m \to \infty} E \left[ |\hat{X}_t|^p 1_{[t < \tau_m]} \right] \leq \lim \inf_{m \to \infty} f_m(t) \leq C_1 e^{C_2 \bar{T}}
\]

for all \( t \in [0, \bar{T}] \), which completes the proof. \( \square \)

Lemma 3.6. Let \( p > H^{-1} \). Then

\[
E \left[ |\hat{X}_t - \hat{X}_s|^p \right] \leq C|t - s|^p, \quad t, s \in [0, \bar{T}]
\]

and \( \hat{X} \) admits a version which is Hölder continuous on \([0, \bar{T}]\) of any order \( \alpha < H - p^{-1} \). Denoting this version again by \( \hat{X} \), one has

\[
E \left[ \left( \sup_{0 \leq s \leq \bar{T}} |\hat{X}_t - \hat{X}_s|^p \right)^1 \right] \leq C_\alpha
\]

for all \( \alpha \in (0, H - p^{-1}) \), where \( C_\alpha \) is a constant that does not depend on \( n \). As a consequence, we can regard \( \hat{X} \) as a \( C_\alpha \) valued random variable for any order \( \alpha < H \) for all \( n \).
Proof. Since
\[ 
\sup_{t \in [0, T]} \mathbb{E} \left[ |b(X_t)|^p \right] + \sup_{t \in [0, T]} \mathbb{E} \left[ |\sigma(X_t)|^p \right] \leq C
\]
with C independent of n by Lemma 3.5, applying Lemma 2.4 in [1] yields
\[ 
\mathbb{E} \left[ |\hat{X}_t - \hat{X}_s|^p \right] \leq C|t - s|^{\frac{4p}{p+1}}, \quad t, s \in [0, T],
\]
and thus, the result follows. \(\square\)

Lemma 3.7. Let \(p \geq 1\). Then the process \(X_t - \hat{X}_t\) uniformly converges to zero in \(L_p\) with the rate \(n^{-Hp}\) as \(n\) goes to infinity, that is,
\[ 
\sup_{t \in [0, T]} \mathbb{E} \left[ |X_t - \hat{X}_t|^p \right] \leq Cn^{-H_p},
\]
where \(C\) is a positive constant which does not depend on \(n\).

Proof. We start with the case \(p > 2q^*\). First, we decompose the error as the following:
\[ 
|X_t - \hat{X}_t|^p = \left| \int_0^t K(t - s)(b(X_s) - b(\hat{X}_s))ds + \int_0^t K(t - s)(\sigma(X_s) - \sigma(\hat{X}_s))dW_s \right|^p
\]
\[ 
\leq 4^{p-1} \left( \left| \int_0^t K(t - s)(b(X_s) - b(\hat{X}_s))ds \right|^p + \left| \int_0^t K(t - s)(\sigma(X_s) - \sigma(\hat{X}_s))dW_s \right|^p \right)
\]
\[ 
+ 4^{p-1} \left( \left| \int_0^t K(t - s)(\sigma(X_s) - \sigma(\hat{X}_s))dW_s \right|^p \right)
\]
\[ 
=: 4^{p-1} (i) + (ii) + (iii) + (iv)
\]
For (i) and (iii), Lemma 3.2-(1),(2), and the Lipschitz condition yield that
\[ 
\mathbb{E} \left[ (i) + (iii) \right] \leq C \int_0^t \mathbb{E} \left[ |b(X_s) - b(\hat{X}_s)|^p \right] \mathbb{E} \left[ |\sigma(X_s) - \sigma(\hat{X}_s)|^p \right] ds
\]
\[ 
\leq C \int_0^t \mathbb{E} \left[ |X_s - \hat{X}_s|^p \right] ds.
\]
By Lemmas 3.2-(1),(2) and 3.6, we have for (ii) and (iv),
\[ 
\mathbb{E} \left[ (ii) + (iv) \right] \leq C \int_0^t \mathbb{E} \left[ |b(X_s) - b(\hat{X}_s)|^p \right] + \mathbb{E} \left[ |\sigma(X_s) - \sigma(\hat{X}_s)|^p \right] ds
\]
\[ 
\leq C \int_0^t \mathbb{E} \left[ |X_s - \hat{X}_s|^p \right] ds \leq Cn^{-H_p}.
\]
Hence, putting \(f(t) = \mathbb{E} \left[ |X_t - \hat{X}_t|^p \right]\), we obtain
\[ 
\int_0^t f(s)ds, \quad where C_1, C_2 are some positive constants independent of n and t. By Lemmas 3.4 and 3.6, X and \(\hat{X}\) are both continuous, so \(f\) is continuous. Therefore, Gronwall’s lemma yields that
\[ 
f(t) \leq C_1 n^{-H_p} + C_2 \int_0^t f(s)ds,
\]
if \(1 \leq p \leq 2q^*\), it follows from the concavity that
\[ 
\mathbb{E} \left[ |X_t - \hat{X}_t|^p \right] \leq C \mathbb{E} \left[ |X_t - \hat{X}_t|^q \right] \leq Cn^{-H_p q} = Cn^{-H_p}
\]
for some \(q > 2q^*\). This completes the proof. \(\square\)

From these estimations, an application of the Garsia-Rodemich-Rumsey inequality gives the following result as in Section 4.3.2 of Richard et al. [20].

Lemma 3.8. For all \(p \geq 1\) and \(\epsilon \in (0, H)\), there exists a constant \(C > 0\) which does not depend on \(n\) such that
\[ 
\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t - \hat{X}_t|^p \right] \leq Cn^{-p(H-\epsilon)}.
\]
4 Proofs for Lemmas 2.2-8

We first show the following auxiliary lemma which plays a key role to prove Lemma 2.2.

Lemma 4.1. For \( v < s \),

\[
n^{2H} \int_0^{n^{\delta}} \left( (s-u)^{H-1/2} - \left( \frac{m^H_n}{n} - u \right)^{H-1/2} \right) \left( (v-u)^{H-1/2} - \left( \frac{m^H_n}{n} - u \right)^{H-1/2} \right) du \to 0.
\]

Proof. The claim is clear when \( H = 1/2 \). Therefore we only need to consider the case \( H < 1/2 \). Put \( \delta_{(n,v)} = v - \frac{m^H_n}{n} \). Then, it follows

\[
n^{2H} \int_0^{n^{\delta}} \left( (s-u)^{H-1/2} - \left( \frac{m^H_n}{n} - u \right)^{H-1/2} \right) \left( (v-u)^{H-1/2} - \left( \frac{m^H_n}{n} - u \right)^{H-1/2} \right) du \leq n^{2H} \int_0^{n^{\delta}} \left( (y+s - \frac{m^H_n}{n})^{H-1/2} - (y + \frac{m^H_n}{n})^{H-1/2} \right) \left( (y + \delta_{(n,v)})^{H-1/2} - y^{H-1/2} \right) dy
\]

\[
= (n\delta_{(n,v)})^{2H} \int_0^{n^{\delta}} \eta_n(z)dz \leq \int_0^{\infty} \left( 1_{[0,\frac{m^H_n}{n})}(z)\eta_n(z)dz, \right.
\]

where

\[
\eta_n(z) = \left( z + \frac{s - [nv]/n}{v - [nv]/n} \right)^{H-1/2} - \left( z + \frac{[ns]/n - [nv]/n}{v - [nv]/n} \right)^{H-1/2} \left( (z + 1)^{H-1/2} - z^{H-1/2} \right)
\]

when \( \delta_{(n,v)} > 0 \) and \( \eta_n(z) = 0 \) otherwise. By the triangle inequality,

\[
\left| \left( z + \frac{s - [nv]/n}{v - [nv]/n} \right)^{H-1/2} - \left( z + \frac{[ns]/n - [nv]/n}{v - [nv]/n} \right)^{H-1/2} \right| \leq 2 \left( z + \frac{[ns]/n - [nv]/n}{v - [nv]/n} \right)^{H-1/2} \leq 2z^{H-1/2}
\]

holds for all \( z \in (0, \infty) \) when \( \delta_{(n,v)} > 0 \), so we have

\[
\left| 1_{[0,\frac{m^H_n}{n})}(z)\eta_n(z) \right| \leq 2z^{H-1/2} \left( (z + 1)^{H-1/2} - z^{H-1/2} \right).
\]

Then, the evaluation

\[
\int_0^{\infty} z^{H-1/2} \left( (z + 1)^{H-1/2} - z^{H-1/2} \right) dz \leq \int_0^{1} 4z^{2H-1/2} dz + \int_1^{\infty} 2z^{H-1/2}z^{-3/2}dz < \infty
\]

enables us to applying the dominated convergence theorem (DCT for short) that leads to

\[
\lim_{n \to \infty} \int_0^{\infty} 1_{[0,\frac{m^H_n}{n})}(z)\eta_n(z)dz = 0 \quad \text{(4.1)}
\]

since \( \eta_n(z) \to (0 - 0)((z + 1)^{H-1/2} - z^{H-1/2}) = 0 \) for \( v < s \).

\( \square \)

4.1 Proof of Lemma 2.2-(i)

Here we compute the limit of \( \langle V^{m_{k_i}}, V^{m_{k_j}} \rangle \). Remind that

\[
\hat{X}_s^k - \hat{X}_u^k = \int_0^{n^H} \left( K(s-u) - K(\frac{m^H_n}{n} - u) \right)b^k(\hat{X}_s^{\hat{t}})du + b^k(\hat{X}_s^{\hat{t}}) \int_{n^H}^{\infty} K(s-u) du
\]

\[
+ \sum_{j=1}^{m^H} \int_0^{n^H} \left( K(s-u) - K(\frac{m^H_n}{n} - u) \right) \sigma_j^k(\hat{X}_s^{\hat{t}})dW_s^j + \sum_{j=1}^{m^H} \int_{n^H}^{\infty} K(s-u) \sigma_j^k(\hat{X}_s^{\hat{t}})dW_s^j
\]

and put the sums of the former two and latter two terms of the right-hand as \( \psi_{1,s}^{n,k}, \psi_{2,s}^{n,k} \) respectively. Then we have

\[
\langle V^{m_{k_i}}, V^{m_{k_j}} \rangle = \int_0^{n^{2H}} (\hat{X}_s^k - \hat{X}_u^k)(\hat{X}_s^{\hat{t}} - \hat{X}_u^{\hat{t}})ds
\]

\[
= \int_0^{n^{2H}} (\psi_{1,s}^{n,k}(\hat{X}_s^k - \hat{X}_u^k) + \psi_{2,s}^{n,k}(\hat{X}_s^k - \hat{X}_u^k))ds.
\]
Define \( \delta_{(n,\varepsilon)} = s - \frac{|m|}{n} \). We first introduce the following evaluations:

\[
\int_0^\infty K(s-u)^2 du = \int_0^{s-|m|} \frac{2^{2H-1}}{G} du = \frac{1}{2HG} \delta^{2H}_{(n,\varepsilon)} \leq Cn^{-2H},
\]

\[
\int_0^{s-|m|} (K(s-u) - K(\frac{|m|}{n} - u))^2 du = C \int_0^{s-|m|} |\mu(u,s - \frac{|m|}{n})|^2 du
\]

which implies \( \delta^{2H}_{(n,\varepsilon)} \int_0^{s-|m|} |\mu(r,1)|^2 dr \leq C \delta^{2H}_{(n,\varepsilon)} \leq Cn^{-2H}, \)

where \( G = \Gamma(H+1/2)^2 \) and \( \mu(r,y) = (r+y)^{H-1/2} - r^{H-1/2} \). We now show \( n^H \psi_1^{nH} \) vanishes in \( L_2 \) for \( \kappa \in \{k_1, k_2\} \) and any \( s \). By Minkowski’s integral inequality, (3.1), and Lemma 3.5, we have

\[
E \left[ \psi_1^{nH,\varepsilon} \right] \leq 2E \left[ \int_0^{s-|m|} (K(s-u) - K(\frac{|m|}{n} - u)) \psi(\hat{X}_n) du \right]^2 + 2E \left[ \psi(\hat{X}_n) \int_0^\infty K(s-u) du \right]^2
\]

\[
\leq 2 \left\{ \left( \int_0^{s-|m|} (K(s-u) - K(\frac{|m|}{n} - u)) \right) E \left[ \psi(\hat{X}_n) \right]^2 du \right\} + 2E \left[ \psi(\hat{X}_n) \right]^2 \left( \int_0^\infty K(s-u) du \right)^2
\]

\[
\leq 2 \sup_{r \in [0,1]} E \left[ \psi(\hat{X}_r) \right]^2 \left\{ \left( \int_0^\infty (K(u + s - \frac{|m|}{n}) - K(u)) du \right)^2 + \left( \int_0^{s-|m|} K(u) du \right)^2 \right\}
\]

\[
\leq C n^{-2H+1},
\]

which implies \( n^H \psi_1^{nH} \rightarrow 0 \) in \( L_2 \) uniformly in \( s \). Doing the same as the proof of Lemma 3.2-(4) and using Lemma 3.5, we have

\[
E \left[ n^{2H} \psi_2^{nH} \right] \leq 2n^{2H} E \left[ \int_0^{s-|m|} (K(s-u) - K(\frac{|m|}{n} - u)) \sigma(\hat{X}_n) du \left\| \int_0^\infty K(s-u) \sigma(\hat{X}_n) du \right\|^2 \right]
\]

\[
\leq C n^{2H} \delta^{2H}_{(n,\varepsilon)} \sup_{r \in [0,1]} E \left[ \sigma(\hat{X}_r) \right]^2 \leq C
\]

with \( C \) being independent of \( n \) and \( s \), so by the Cauchy-Schwarz inequality and the bounded convergence theorem (BCT for short), we observe

\[
E \left[ n^{2H} \int_0^\infty (\psi_1^{nH} \hat{X}_n + \psi_2^{nH} \psi_1^{nH}) ds \right] \leq \int_0^\infty E \left[ n^{2H} \psi_1^{nH} \right]^2 + E \left[ n^{2H} \hat{X}_n^2 \right]^2 ds + \int_0^\infty E \left[ n^{2H} \psi_2^{nH} \right]^2 ds \rightarrow 0
\]

It becomes sufficient that we only consider the last term on the right-hand side of (4.3). We start with
decomposing as follows:

\[
\begin{align*}
&n^{2H} \int_0^t \psi_{1,2}^{n,k_1} \psi_{1,2}^{n,k_2} \, ds \\
&= \sum_{j,l} n^{2H} \int_0^t \left( \int_0^u f_{k_l}(s,u) \, dW_u^j \right) \left( \int_0^u f_{k_j}(s,u) \, dW_u^l \right) \, ds \\
&\quad + \sum_{j,l} n^{2H} \int_0^t \sigma_j^k(\tilde{X}_u^{(j)}) \left( \int_0^u f_{k_l}(s,u) \, dW_u^j \right) \left( \int_0^u K(s-u) \, dW_u^l \right) \, ds \\
&\quad + \sum_{j,l} n^{2H} \int_0^t \sigma_l^k(\tilde{X}_u^{(j)}) \left( \int_0^u f_{k_l}(s,u) \, dW_u^j \right) \left( \int_0^u K(s-u) \, dW_u^l \right) \, ds \\
&\quad + \sum_{j,l} n^{2H} \int_0^t \sigma_j^k(\tilde{X}_u^{(j)}) \sigma_l^k(\tilde{X}_u^{(l)}) \left( \int_0^u K(s-u) \, dW_u^j \right) \left( \int_0^u K(s-u) \, dW_u^l \right) \, ds \\
&=: \sum_{j=1}^m \sum_{l=1}^m (I_{j,l} + II_{j,l} + III_{j,l} + IV_{j,l}),
\end{align*}
\]

where

\[f_{k_l}(s,u) = \left(K(s-u) - K\left(\frac{\ln u}{\ln n}\right)\right) \sigma_l^k(\tilde{X}_u^{(j)}).\]

We use the following equality which is derived from Itô's formula: for any progressively measurable square integrable function \(h_1, h_2,\)

\[
\int_s^t h_1(u) \, dW_u^j \int_s^t h_2(u) \, dW_u^l = \int_s^t \left( \int_s^u h_1(r) \, dW_r^j \right) h_2(u) \, dW_u^l + \int_s^t \left( \int_s^u h_2(r) \, dW_r^l \right) h_1(u) \, dW_u^j + \int_s^t h_1(u) h_2(u) \, d\langle W^j, W^l \rangle_u.
\]

For \(I_{j,l}.\) According to (4.6), \(I_{j,l}\) can be written as

\[
I_{j,l} = n^{2H} \int_0^t \left[ \int_0^u \left( \int_0^u f_{k_l}(s,r) \, dW_r^j \right) f_{k_j}(s,u) \, dW_u^l \right] \\
\quad \quad \quad + \int_0^u \left( \int_0^u f_{k_l}(s,r) \, dW_r^j \right) f_{k_j}(s,u) \, dW_u^l + \int_0^u f_{k_l}(s,u) f_{k_j}(s,u) \, d\langle W^j, W^l \rangle_u \, ds,
\]

For the last term, it vanishes if \(j \neq l,\) and if \(j = l,\) it can be transformed as

\[
\begin{align*}
&n^{2H} \int_0^u \left( \int_0^u f_{k_l}(s,u) f_{k_j}(s,u) \, ds \right) \\
&= n^{2H} \int_0^u \left( \int_0^u \left( K(r+\delta_{(n,s)}) - K(r) \right)^2 \sigma_j^k(\tilde{X}_r^{(j)}) \sigma_l^k(\tilde{X}_r^{(l)}) \right) dr \\
&= \int_0^u \left( n\delta_{(n,s)} \right)^{2H} \frac{1}{\mathcal{G}_0} \int_0^u \left( \int_0^u \left( K(r+\delta_{(n,s)}) - K(r) \right)^2 \sigma_j^k(\tilde{X}_r^{(j)}) \sigma_l^k(\tilde{X}_r^{(l)}) \right) dr.
\end{align*}
\]

We will apply Lemma C.2 here, so it must be checked that the hypothesis is satisfied. We are to show that

\[
\int_0^{\frac{\ln u}{\ln n}} |\mu(r,1)|^2 \prod_{q \in [k_1,k_2]} \sigma_j^q(\tilde{X}_{ns+\delta_{(n,s)}}) \, dr \xrightarrow{L_2(ds \, dP)} \sigma_j^k(X_s) \sigma_l^k(X_s) \int_0^\infty |\mu(r,1)|^2 \, dr.
\]

We note that the limit on the right is certainly a continuous function of \(s.\) It follows from Fubini’s
theorem and Minkowski’s integral inequality that

\[
\mathbb{E} \left[ \int_0^t \left( \mu(r, 1)^2 \left( \int_{\{0 < t \wedge \tau_n \leq r \}} \sigma_j^q(X_t) \sigma_j^k(X_s) \right) dr \right)^2 ds \right]
\]

\[
\leq \int_0^t \left( \int_0^\infty \left( \mu(r, 1)^2 \left( \int_{\{0 < t \wedge \tau_n \leq r \}} \sigma_j^q(X_t) \sigma_j^k(X_s) \right) dr \right)^2 \right) ds.
\]

(4.8)

Remark that \( \| \cdot \|_{L_p} \) simply means the \( L_p \) norm with respect to \( P \). Then, by Minkowski’s inequality,

\[
\left\| \prod_{i=1}^k \sigma_j^k(X_t) - \sigma_j^k(X_s) \right\|_{L_2} \leq \left\| \prod_{i=1}^k (\hat{X}_{n|n-1|\cdots|i} - \hat{X}_{n|n-1|\cdots|i}) \right\|_{L_2} + \left\| \prod_{i=1}^k (\hat{X}_{n|n-1|\cdots|i} - \hat{X}_{n|n-1|\cdots|i}) \right\|_{L_2}.
\]

(4.9)

Therefore, since

\[
\frac{[ns]}{n^H} \to \infty, \quad \frac{[ns] + [-n\delta(n,s)]}{n} \to s,
\]

we get

\[
\left\| \prod_{i=1}^k (\hat{X}_{n|n-1|\cdots|i} - \hat{X}_{n|n-1|\cdots|i}) \right\|_{L_2} \to 0.
\]

Consequently, the right-hand side of (4.8) tends to zero by applying the DCT for the integral of \( ds \) and \( dr \) respectively, and then, Lemma C.2 gives the evaluation for (4.7) as follows:

\[
n^{2H} \int_0^t \int_0^1 f_{k_1}(s, u) f_{k_2}(s, u) du ds = \int_0^t (ns - [ns])^{2H} \frac{1}{G} \int_0^\infty \left( \int_{\{0 < t \wedge \tau_n \leq r \}} \mu(r, 1)^2 \left( \int_{\{0 < t \wedge \tau_n \leq r \}} \sigma_j^q(X_t) \sigma_j^k(X_s) \right) dr \right) ds.
\]

\[
\to \frac{1}{(2H + 1)G} \int_0^\infty \left( \int_0^t \mu(r, 1)^2 dr \right) \int_0^\infty \sigma_j^k(X_s) \sigma_j^k(X_s) ds.
\]

We next show that the remainder term of \( I_{\beta} \),

\[
I_{\beta}^{I_1} := n^{2H} \int_0^1 \int_0^u \left( \int_0^u f_{k_1}(s, r) dW_{r} \right) f_{k_2}(s, u) dW_{u} ds,
\]

converges to zero in \( L_2 \) as \( n \to \infty \). Set

\[
D_{\beta}^{I_1} := n^{2H} \int_0^1 \int_0^u \left( \int_0^u f_{k_1}(s, r) dW_{r} \right) f_{k_2}(s, u) dW_{u} ds.
\]
and observe that
\[
E \left[ \mu_t^2 \right] = E \left[ \int_0^t \int_0^s D_{t,s}^\mu D_{s,t}^\mu ds \right] = 2 \int_0^t \int_0^s E \left[ D_{t,s}^\mu D_{s,t}^\mu \right] ds
\]
by Fubini’s theorem. Then, by (4.6) and Fubini’s theorem, we have
\[
E \left[ D_{t,s}^\mu D_{s,t}^\mu \right] = n^{4H} \int_0^{\min(t,s)} \left( \int_0^\infty f_{k_1}(s, r) dW_r^k \right) \left( \int_0^\infty f_{k_2}(v, r) dW_r^\nu \right) f_{k_j}(s, u) f_{k_j}(v, u) dW_s^k W_s^\nu
\]
for \( j \neq l \),
\[
= \begin{cases} \frac{0}{n^{4H} \int_0^{\min(t,s)}} (K(s - u) - K(\frac{\ln n}{n} - u))(K(v - u) - K(\frac{\ln n}{n} - u)) \tilde{f}(u) du, & \text{if } j = l, \end{cases}
\]
where
\[
\tilde{f}(u) := E \left[ \mu_{t,j}^2 (\hat{X}_{\mu,nu})^2 \right] \left( \int_0^\infty f_{k_j}(s, r) dW_r^j \right) \left( \int_0^\infty f_{k_j}(s, r) dW_r^j \right)
\]
Again by (4.6), it follows that for \( u \in [0, \frac{\ln n}{n}] \),
\[
|f(u)| \leq E \left[ \mu_{t,j}^2 (\hat{X}_{\mu,nu})^2 \int_0^u \left( \int_0^\infty f_{k_j}(s, y) dW_y^j \right) f_{k_j}(v, r) dW_r^j \right]
+ E \left[ \mu_{t,j}^2 (\hat{X}_{\mu,nu})^2 \int_0^u \left( \int_0^\infty f_{k_j}(s, y) dW_y^j \right) f_{k_j}(v, r) dW_r^j \right]
+ E \left[ \mu_{t,j}^2 (\hat{X}_{\mu,nu})^2 \int_0^u f_{k_j}(s, r) f_{k_j}(v, y) dr \right]
\]
For the first term on the right-hand side, by using the Cauchy-Schwarz inequality, the Itô isometry, Fubini’s theorem, Lipschitz continuity of \( \sigma \), and Lemma 3.5 properly, we see
\[
E \left[ \mu_{t,j}^2 (\hat{X}_{\mu,nu})^2 \int_0^u \left( \int_0^\infty f_{k_j}(s, y) dW_y^j \right) f_{k_j}(v, r) dW_r^j \right]
\leq E \left[ \mu_{t,j}^2 (\hat{X}_{\mu,nu})^4 \right] \frac{1}{2} E \left[ \left( \int_0^u \left( \int_0^\infty f_{k_j}(s, y) dW_y^j \right)^2 \right)^{\frac{1}{2}} \left( \int_0^u \left( f_{k_j}(v, r)^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]
\leq C \left( \int_0^u (K(v - r) - K(\frac{\ln n}{n} - r))^2 E \left[ \left( \int_0^u \left( f_{k_j}(s, y) dW_y^j \right)^2 \mu_{t,j}^2 (\hat{X}_{\mu,nu})^2 \right)^{\frac{1}{2}} \right] dr \right)
\]
for some \( C \) which does not depend on \( n \). Applying Hölder’s inequality, the BDG inequality, Minkowski’s integral inequality, Lemma 3.5, and (4.4) properly, we have
\[
E \left[ \left( \int_0^u f_{k_1}(s, y) dW_y^j \right)^2 \mu_{t,j}^2 (\hat{X}_{\mu,nu})^2 \right] \leq E \left[ \left( \int_0^u f_{k_1}(s, y) dW_y^j \right)^{2} \right]^{\frac{1}{2}} \left( \frac{1}{2} \right)^{\frac{1}{2}} E \left[ \mu_{t,j}^2 (\hat{X}_{\mu,nu})^2 \right]^{\frac{1}{2}}
\leq C E \left[ \left( \int_0^u \left| f_{k_1}(s, y) \right|^2 dy \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}
\leq C \left( \int_0^u \left( K(s - y) - K(\frac{\ln n}{n} - y) \right)^2 dW_s^j W_s^\nu \right)
\leq C \left( \int_0^u \left( K(s - y) - K(\frac{\ln n}{n} - y) \right)^2 dy \right) \leq C n^{-2H}.
Hence, (4.12) is evaluated as

$$
\left| \mathbb{E} \left[ |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 \int_0^\infty \left( \int_0^y f_{k_1}(s, y)dW_s \right) f_{k_1}(v, r)dW_r \right] \right| 
\leq Cn^{-H} \left( \int_0^\infty \left( K(v - r - K(\frac{\ln^2}{n}) - r) \right) ^2 dr \right) ^{\frac{1}{2}} \leq Cn^{-2H}.
$$

(4.13)

Similarly, we have the same evaluation as (4.12)-(4.13) for the second term on the right-hand side of (4.11). For the last term remained in (4.11), by Fubini’s theorem we have

$$
\left| \mathbb{E} \left[ |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 \int_0^\infty f_{k_1}(s, r)f_{k_1}(v, y)d\mathbf{r} \right] \right| 
= \mathbb{E} \left[ |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 \int_0^\infty \left( K(v - r - K(\frac{\ln^2}{n}) - r) \right) \left( K(s - r - K(\frac{\ln^2}{n}) - r) \right) |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 dr \right] 
= \int_0^\infty \left( K(v - r - K(\frac{\ln^2}{n}) - r) \right) \left( K(s - r - K(\frac{\ln^2}{n}) - r) \right) \mathbb{E} \left[ |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 \right] dr.
$$

Since the evaluation

$$
\mathbb{E} \left[ |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 \right] \leq \mathbb{E} \left[ |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^4 \right]^{\frac{1}{2}} \mathbb{E} \left[ |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^4 \right]^{\frac{1}{2}} \leq C < \infty
$$

is derived from the Cauchy-Schwarz inequality and Lemma 3.5, it follows from the same inequality and (4.4) that

$$
\left| \mathbb{E} \left[ |\alpha_j^x(\hat{\mathbf{x}}_{\mathbf{m}})|^2 \int_0^\infty f_{k_1}(s, r)f_{k_1}(v, y)d\mathbf{r} \right] \right| 
\leq C \left( \int_0^{\frac{\ln^2}{n}} \left( K(v - r - K(\frac{\ln^2}{n}) - r) \right) ^2 dr \right) ^{\frac{1}{2}} \left( \int_0^{\frac{\ln^2}{n}} \left( K(s - r - K(\frac{\ln^2}{n}) - r) \right) ^2 dr \right) ^{\frac{1}{2}} 
\leq Cn^{-2H}.
$$

Therefore, $|\tilde{f}(u)| \leq Cn^{-2H}$ is obtained with $C$ being independent of $n$, and it is estimated that for (4.10),

$$
\left| \mathbb{E} \left[ D_{l,s}^\beta D_{s,t}^\beta \right] \right| \leq n^4 H^4 \int_0^{\frac{\ln^2}{n}} \left( K(s - u - K(\frac{\ln^2}{n}) - u) \right) \left( K(v - u - K(\frac{\ln^2}{n}) - u) \right) |\tilde{f}_n(u)| du 
\leq Cn^{2H} \int_0^{\frac{\ln^2}{n}} \left( K(s - u - K(\frac{\ln^2}{n}) - u) \right) \left( K(v - u - K(\frac{\ln^2}{n}) - u) \right) du.
$$

Therefore, Lemma 4.1 leads to $|\mathbb{E}[D_{l,s}^\beta D_{s,t}^\beta]| \to 0$. Applying the BCT with respect to the integration of $d\mathbf{v} \otimes ds$, we have $I_{1,2}^\beta \to 0$ in $L_2$, and thus,

$$
I_{1,2}^\beta \to 0 \text{ in } L_2 \left( \frac{\ln^2}{n}, 0 \right) \int \mu(r, 1)^2 dr \int \sigma_j^0(X_t, X_s) d\mu(X_t, X_s), \text{ if } j = l, \\
0, \text{ if } j \neq l.
$$

For $I_{2,2}$. Write $I_{2,2} = \int \sigma_j^0(X_t, X_s) D_{2,2}^\beta ds$, where the process $D_{2,2}^\beta$ is the following:

$$
D_{2,2}^\beta = n^{2H} \left( \int_0^{\frac{\ln^2}{n}} f_{k_1}(s, u)dW_s \right) \left( \int_0^{\frac{\ln^2}{n}} K(s - u)dW_u \right).
$$
We are to show $I_{j,l} \to 0$ for all $j, l$. Fubini’s theorem yields that
\[
E[|I_{j,l}|^2] = E \left[ \int_0^\infty \int_0^\infty \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \,dvds \right] \\
= \int_0^\infty \int_0^\infty E \left[ \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \right] \,dvds \\
= 2 \int_0^\infty \int_0^\infty E \left[ \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \right] \,dvds \\
+ 2 \int_0^\infty \int_0^\infty E \left[ \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \right] \,dvds \\
= 2 \left( I_{2,1}^\beta + I_{2,2}^\beta \right).
\]

Then, it follows that for all $n$,
\[
I_{2,1}^\beta = \int_0^\infty \int_0^\infty E \left[ \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) E \left[ D_{2,\sigma}^\beta \mathcal{F}_\infty \right] \right] \,dvds \\
= \int_0^\infty \int_0^\infty E \left[ \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) n^{2H} \int_0^\infty f_{k,j}(s, u)dW_u \right] \,dvds \\
\leq 0.
\]

We next show $I_{2,2}^\beta$ tends to zero. By the Cauchy-Schwarz inequality,
\[
E \left[ \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \right] \leq E \left[ |\sigma_{i,l}^2(\tilde{X}_\infty)|^2 \right] E \left[ |D_{2,\sigma}^\beta|^2 \right]^{\frac{1}{2}}
\]
follows, and by again the same inequality, along with Lemma 3.5, the BDG inequality, and (4.4), we have the following evaluation:
\[
E \left[ |\sigma_{i,l}^2(\tilde{X}_\infty)|^2 \right] \leq E \left[ |\sigma_{i,l}^2(\tilde{X}_\infty)|^2 \right] E \left[ |D_{2,\sigma}^\beta|^2 \right]^{\frac{1}{2}} \\
\leq C n^{4H} \left[ \int_0^\infty f_{k,j}(s, u)dW_u \right] \left[ \int_0^\infty K(s, u)dW_u \right]^{\frac{1}{2}} \\
\leq C n^{4H} \left[ \int_0^\infty |f_{k,j}(s, u)|^2 du \right] \left[ \int_0^\infty |K(s, u)|^2 du \right]^{\frac{1}{2}} \\
\leq C n^{4H} \int_0^\infty (K(s, u) - K(\frac{\lfloor n \rfloor}{n} - u))^2 du \left[ \int_0^\infty |\sigma_{i,l}^2(\tilde{X}_\infty)|^2 du \right] \int_0^\infty K(s, u)^2 du \\
\leq C \sup_{u \in [0, T]} \left[ |\sigma_{i,l}^2(\tilde{X}_\infty)|^2 \right] n^{2H} \int_0^\infty (K(s, u) - K(\frac{\lfloor n \rfloor}{n} - u))^2 du \\
\leq C
\]
with $C$ being independent of $n$ and $s$. Hence,
\[
E \left[ \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \right] \leq C
\]
holds, and therefore, the BCT yields
\[
\lim_{n \to \infty} I_{2,2}^\beta = \int_0^\infty \int_0^\infty \lim_{n \to \infty} 1_{(\frac{\lfloor n \rfloor}{n}, \frac{\lfloor n \rfloor}{n})}(u) \left[ \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \sigma_{i,l}^2(\tilde{X}_\infty) D_{2,\sigma}^\beta \right] \,dvds = 0,
\]
which concludes that $\lim_{n \to \infty} E[|I_{j,l}|^2] = 0$, namely, $I_{j,l} \to 0$ in $L_2$.

For $III_{j,l}$. Similarly to $II_{j,l}$, it holds that $III_{j,l} \to 0$ in $L_2$. 

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For $\mathbf{I}_V$. According to (4.6), we have

$$\mathbf{I}_V = n^{2H} \int_0^1 E_s \left[ \int_{\mathbb{1}_n} K(s-r) \, dW^i_r \right] K(s-u) \, dW^i_u + \int_{\mathbb{1}_n} K(s-r) \, dW^i_r \right] K(s-u) \, dW^i_u + \int_{\mathbb{1}_n} |K(s-u)|^2 \, d(W^1, W^j)_s \right] ds, \quad (4.14)$$

where $E_s = \sigma^k_j (X(s)) \sigma^k_j (X(s))$. Then, for the last term, it vanishes if $j \neq l$. Otherwise,

$$n^{2H} \int_0^1 E_s^{jj} \int_{\mathbb{1}_n} |K(s-u)|^2 \, du \, ds = \int_0^1 E_s^{jj} n^{2H} S_{2H}^j(s) \, ds$$

follows from (4.4), and therefore, Lemma C.2 yields

$$\int_0^1 E_s^{jj} (ns - [ns])^{2H} \, ds \xrightarrow{\text{var} \to \infty} \frac{1}{2HG(2H + 1)} \int_0^1 \sigma^k_j (X(s)) \sigma^k_j (X(s)) \, ds$$

since

$$E \left[ \int_0^1 |\sigma^k_j (X(s)) \sigma^k_j (X(s)) - \sigma^k_j (X(s)) \sigma^k_j (X(s))|^2 \, ds \right] \leq C n^{-H} \to 0$$

is given by a similar calculation to (4.9). We now consider the remaining first two terms of (4.14). It suffices to show that one of them vanishes as $n \to \infty$ since they are symmetry. Define $I_4^\beta$ as

$$I_4^\beta = \int_0^1 E_s^{jj} \, ds,$$

where

$$D_{4,s}^\beta = n^{2H} \int_{\mathbb{1}_n} g_j(s, u) \, dW^i_u, \quad g_j(s, u) = K(s-u) \int_{\mathbb{1}_n} K(s-r) \, dW^i_r.$$
Finally, we arrive at

$$\mathbb{E} \left[ D_{4, s}^\beta D_{4, v}^\beta \mid \mathcal{F}_{4, s} \right] = n^{4H} \mathbb{E} \left[ \int_0^s g_j(s, u) dW^t_u \int_0^u \int_0^s (u) g_j(v, u) dW^t_u \mid \mathcal{F}_{4, s} \right]$$

The last equal follows from Fubini’s theorem. Substituting $g_j$, we have

$$\mathbb{E} \left[ g_j(s, u) g_j(v, u) \mid \mathcal{F}_{4, s} \right]$$

$$= K(s-u)K(v-u) \mathbb{E} \left[ \left( \int_0^u K(s-r) dW^t_r \right) \left( \int_0^u K(v-r) dW^t_r \right) \mid \mathcal{F}_{4, s} \right]$$

$$= K(s-u)K(v-u) \int_0^u K(s-r)K(v-r) dr.$$ 

Since $K$ is decreasing, using (4.4), we obtain

$$\left| \mathbb{E} \left[ D_{4, s}^\beta D_{4, v}^\beta \mid \mathcal{F}_{4, s} \right] \right| = n^{4H} \left| \int_0^u K(s-u)K(v-u) \int_0^u K(s-r)K(v-r) drdu \right|$$

$$\leq n^{4H} \int_0^u K(v-u)^2 \int_0^u K(v-r)^2 drdu$$

$$\leq \left( n^{2H} \int_0^u K(v-u)^2 du \right)^2 \leq C,$$

and hence, we have

$$\left| \int_0^u 1_{\{s > u\}}(v) \mathbb{E} \left[ |E^\beta_{v, s} D_{4, s}^\beta E^{\beta}_{v, s} D_{4, v}^\beta | \right] \right| \leq C \mathbb{E} |E^\beta_{v, s}|^2 \leq C,$$

with $C$ being independent of $s, v,$ and $u$. Therefore, by the BCT, we get

$$\int_0^u |E^\beta_{v, s}| \mathbb{E} \left[ |E^\beta_{v, s} D_{4, s}^\beta E^{\beta}_{v, s} D_{4, v}^\beta | \right] dv \xrightarrow{n \rightarrow \infty} 0.$$ 

By the above evaluations, $l_{4, v}^\beta$ converges to zero in $L_2$, and thus,

$$IV_{\beta_{n \rightarrow \infty}} \xrightarrow{\text{in } L_2} \begin{cases} \frac{1}{2H(2H+1)} \int_0^1 \psi_j^k (X_s) \sigma_j^k (X_s) ds & \text{if } j = l, \\ 0, & \text{if } j \neq l. \end{cases}$$ 

Finally, we arrive at

$$n^{2H} \int_0^1 \psi_j^{(k_1)} \psi_j^{(k_2)} ds \to \sum_{j=1}^m \frac{1}{2H(2H+1)} G \left( 2H \int_0^1 |\mu(r, 1)|^2 dr + 1 \right) \int_0^1 \sigma_j^k (X_s) \sigma_j^k (X_s) ds.$$ 

Since

$$2H \int_0^1 |\mu(r, 1)|^2 dr + 1 = \frac{G}{(2H) \sin \pi H}$$

holds by Mishura (Theorem 1.3.1 and Lemma A.0.1 in [18]), by taking (4.3) and (4.5) into account, we obtain for all $i \in [1, \ldots, m]$ and $(k_1, k_2) \in \{1, \ldots, d\}^2$,

$$\langle V^{n, k_1, i}, V^{n, k_2, j} \rangle \xrightarrow{n \rightarrow \infty} \sum_{j=1}^m \frac{1}{2H(2H+2) \sin \pi H} \int_0^1 \sigma_j^k (X_s) \sigma_j^k (X_s) ds.$$ 

$\square$
4.2 Proof of Lemma 2.2-(ii)

Here we compute the limit of

\[ \langle V_{n,i}^{m,j}, W^i \rangle_t = \int_0^t n^{H} (\hat{X}_s^k - \hat{X}_s^k) \, ds \]

for \( i \in \{1, \ldots, m\} \). Write \( \Delta \hat{X}_s = \hat{X}_s - \hat{X}_{s-} \). Then, it follows from Fubini’s theorem that

\[
E \left[ |\langle V_{n,i}^{m,j}, W^i \rangle_t |^2 \right] = E \left[ n^{2H} \int_0^t \int_0^t \Delta \hat{X}_s^k \Delta \hat{X}_v^k \, dv \, ds \right] \\
= \int_0^t 2 \int_0^t n^{2H} E \left[ \Delta \hat{X}_s^k \Delta \hat{X}_v^k \right] \, dv \, ds \\
= 2 \int_0^t \int_0^t n^{2H} E \left[ \Delta \hat{X}_s^k \Delta \hat{X}_v^k \right] \, dv \, ds.
\]

Write \( h_\ell(s, u) = K(s-u) - K(\frac{|m|}{n} - u)b^\ell(\hat{X}_{\frac{|m|}{n}}) \). Using the same notation \( f_{j,\ell} \) as in the above proof, by the tower property and (4.2), we observe that for \( v \in (0, \frac{|m|}{n}) \), \( s \in (0, t) \),

\[
E \left[ \Delta \hat{X}_s^k \Delta \hat{X}_v^k \right] = E \left[ \Delta \hat{X}_s^k \mathbb{E} \left[ \Delta \hat{X}_v^k \left| \mathcal{F}_{\frac{|m|}{n}} \right. \right] \right] \\
= E \left[ \Delta \hat{X}_s^k \left( \int_0^{\frac{|m|}{n}} h_\ell(s,u) \, du + \int_{\frac{|m|}{n}}^t K(s-u)b^\ell(\hat{X}_{\frac{|m|}{n}}) \, du \right) \right] \\
+ \sum_{j=1}^m E \left[ \Delta \hat{X}_s^k \int_0^{\frac{|m|}{n}} f_{j,\ell}(s,u) \, dW^j_u \right] \\
= E \left[ \Delta \hat{X}_s^k \left( \int_0^{\frac{|m|}{n}} h_\ell(s,u) \, du + \int_{\frac{|m|}{n}}^t K(s-u)b^\ell(\hat{X}_{\frac{|m|}{n}}) \, du \right) \right] \\
+ \sum_{j=1}^m E \left[ \Delta \hat{X}_s^k \int_0^{\frac{|m|}{n}} f_{j,\ell}(s,u) \, dW^j_u \left( \int_0^{\frac{|m|}{n}} h_\ell(v,u) \, du + \int_{\frac{|m|}{n}}^t K(v-u)b^\ell(\hat{X}_{\frac{|m|}{n}}) \, du \right) \right] \\
+ \sum_{j=1}^m \sum_{l=1}^n E \left[ \Delta \hat{X}_s^k \int_0^{\frac{|m|}{n}} f_{l,\ell}(s,u) \, dW^l_u \left( \int_0^{\frac{|m|}{n}} f_{l,\ell}(v,u) \, du + \int_{\frac{|m|}{n}}^t K(v-u)c^\ell_l(\hat{X}_{\frac{|m|}{n}}) \, du \right) \right]
\]

\( =: (I) + \sum_{j=1}^m (\text{II})_j + \sum_{j=1}^m \sum_{l=1}^n (\text{III})_{jl} \)

According to (3.1), (I) is calculated as

\[
|I| \leq E \left[ \int_0^{\frac{|m|}{n}} \left| \Delta \hat{X}_s^k h_\ell(s,u) \right| \, du + \int_{\frac{|m|}{n}}^t K(s-u) \left| \Delta \hat{X}_v^k b^\ell(\hat{X}_{\frac{|m|}{n}}) \right| \, du \right] \\
\leq \int_0^{\frac{|m|}{n}} (K(\frac{|m|}{n} - u) - K(s-u)) E \left[ \left| \Delta \hat{X}_s^k b^\ell(\hat{X}_{\frac{|m|}{n}}) \right| \right] \, du + \int_{\frac{|m|}{n}}^t K(s-u) E \left[ \left| \Delta \hat{X}_v^k b^\ell(\hat{X}_{\frac{|m|}{n}}) \right| \right] \, du \\
\leq Cn^{-H} \left( \int_0^{\frac{|m|}{n}} (K(\frac{|m|}{n} - u) - K(s-u)) \, du + \int_{\frac{|m|}{n}}^t K(s-u) \, du \right) \leq Cn^{-2H+1/2}
\]

since

\[
E \left[ \left| \Delta \hat{X}_s^k b^\ell(\hat{X}_{\frac{|m|}{n}}) \right| \right] \leq E \left[ \left| \Delta \hat{X}_s^k \right|^2 \right] E \left[ b^\ell(\hat{X}_{\frac{|m|}{n}})^2 \right] \leq Cn^{-H}
\]

is true by Lemmas 3.5 and 3.6 (remark C is independent of \( u, v, \) and \( s \)). Hence, this term vanishes as \( n \to \infty \) even if it is multiplied by \( n^{2H} \). For (II)_j, we can derive the same evaluation as (I) since

\[
E[\int_0^{\frac{|m|}{n}} |f_{j,\ell}(s,u)|^2 \, du] \leq \int_0^{\frac{|m|}{n}} E[|f_{j,\ell}(s,u)|^2] \, du \leq Cn^{-H}
\]
holds by the Itô isometry, so $n^{2H}(\text{II})_l$ vanishes as $n \to \infty$. What remained to be considered is $(\text{III})_j$. For the former part of $(\text{III})_j$, it follows from (4.6) that

$$\left| E \left[ \int_0^{\frac{m}{n}} f_k(s, u)dW_u \int_0^{\frac{m}{n}} 1_{(0, \frac{m}{n})}(u)f_k(v, u)dW_v \right] \right| = \int_0^{\frac{m}{n}} \left( K(s - u) - K\left( \frac{m}{n} - u \right) \right) \left( K(v - u) - K\left( \frac{m}{n} - u \right) \right) E \left[ \sigma_j^2(X_{\frac{m}{n}}) \sigma_j^2(X_{\frac{m}{n}}) \right] d(W', W').$$

If $j \neq l$, it is zero. If $j = l$, since $\sup_{t \in [0, \frac{T}{n}]} E \left[ |\sigma_j^2(X_t)|^2 \right] < C$ for some finite $C$ which does not depend on $n$ by Lemma 3.5, it follows from Lemma 4.1 that

$$n^{2H}E \left[ \int_0^{\frac{m}{n}} f_k(s, u)dW_u \int_0^{\frac{m}{n}} 1_{(0, \frac{m}{n})}(u)f_k(v, u)dW_v \right] \to 0.$$ 

For the latter part of $(\text{III})_j$, it holds by (4.6) that

$$E \left[ \int_0^{\frac{m}{n}} f_k(s, u)dW_u \int_0^{\frac{m}{n}} 1_{(0, \frac{m}{n})}(u)f_k(v - u)\sigma_j^2(X_{\frac{m}{n}})dW_v \right] = E \left[ \int_0^{\frac{m}{n}} f_k(s, u)K(v - u)\sigma_j^2(X_{\frac{m}{n}})d(W', W')_u \right].$$

Then, it becomes zero if $j \neq l$, and if $j = l$, again by Lemma 3.5,

$$\left| E \left[ \int_0^{\frac{m}{n}} f_k(s, u)K(v - u)\sigma_j^2(X_{\frac{m}{n}})du \right] \right| = E \left[ \sigma_j^2(X_{\frac{m}{n}}) \right] \int_0^{\frac{m}{n}} \left( K(s - u) - K\left( \frac{m}{n} - u \right) \right) K(v - u)du \leq C \int_0^{\frac{m}{n}} \left( (r + s - v)^{H-1/2} - (r + \frac{m}{n} - v)^{H-1/2} \right) r^{H-1/2} du \leq Cn^{-2H} \int_0^{\frac{m}{n}} \left( (r + \frac{s - v}{n^{H-1/2}})^{H-1/2} - (r + \frac{m}{n} - v)^{H-1/2} \right) r^{H-1/2} du$$

holds. Doing similarly to (4.1) yields

$$\int_0^{\frac{m}{n}} \left( (r + \frac{s - v}{n^{H-1/2}})^{H-1/2} - (r + \frac{m}{n} - v)^{H-1/2} \right) r^{H-1/2} du \to 0,$$

and thus, it holds

$$n^{2H}E \left[ \int_0^{\frac{m}{n}} f_k(s, u)K(v - u)\sigma_j^2(X_{\frac{m}{n}})du \right] \to 0,$$

which concludes that $n^{2H}(\text{III})_j \to 0$. Therefore,

$$n^{2H}E \left[ \Delta X_{\frac{m}{n}}^k \Delta X_{\frac{m}{n}}^k \right] \to 0, \quad 0 < v < \frac{\lfloor m \rfloor}{n}, 0 < s < t.$$ 

On the other hand, for $v \in (\frac{\lfloor m \rfloor}{n}, s), s \in (0, t)$, from Lemma 3.6, together with the Cauchy-Schwarz inequality,

$$E \left[ [\Delta X_{\frac{m}{n}}^k]^2 \right] \leq E \left[ [\Delta X_{\frac{m}{n}}^k]^2 \right] \leq C \left( s - \frac{\lfloor m \rfloor}{n} \right)^H \left( v - \frac{\lfloor m \rfloor}{n} \right)^H \leq Cn^{-2H}$$

follows and

$$\int_0^{\frac{m}{n}} \int_0^{\frac{m}{n}} 1_{(0, \frac{m}{n})} n^{2H}E \left[ [\Delta X_{\frac{m}{n}}^k]^2 \right] \to 0,$$

holds by the BCT. Since (4.15) is valid for all $s, v \in [0, t]$, applying the BCT with respect to the integral of $d\nu \otimes ds$ and concluding above discussions, we obtain for all $n \in \mathbb{N}, k \in \{1, \ldots, d\}, i \in \{1, \ldots, m\}$,

$$E \left[ (\nu^m)^{\frac{k}{i}}(W)^{\frac{i}{k}} \right] \to 0.$$
4.3 Proof of Lemma 2.3

By Lemma 2.2, we can apply Theorem 4-1 of Jacod [12] to see that $V^n$ stably converges in law in $C_0$ to a conditionally Gaussian martingale $V = [V^k]$ with

$$\langle V^{k_1,j_1}, V^{k_2,j_2} \rangle_t = \begin{cases} \sum_{j=1}^d \frac{1}{\Gamma(2H+2) \sin \pi H} \int_0^t \sigma_j^k(X_s) \sigma_j^k(X_s) ds, & \text{if } j_1 = j_2, \\ 0, & \text{if } j_1 \neq j_2, \end{cases}$$

for all predictable processes almost surely bounded by one. By Lemma 2.2, a process $F$ is zero in $L^0$ if $\langle F \rangle_t = 0$. Here we show the convergence to 0 of $V_t$. Thus, $\hat{V}_1 - \hat{V}_2 \equiv 0$ for all $t \geq 0$ in probability in $C_0$. This concludes the proof. 

4.4 Proof of Lemma 2.4

Here we show the convergence to 0 of the process $V_t = \sum_{i=1}^m \frac{1}{\sqrt{\Gamma(2H+2) \sin \pi H}} \int_0^t \sigma_i^j(X_s) dB_i^j$. Since $\langle V^{k,j} \rangle_t$ tends to zero in $L^1$ for all $t \in [0, T]$ by Lemma 2.2, we will obtain the result by using Theorem 7.10 of Kurtz and Protter [16] and the continuity of $\mathcal{F}^n$ given in Lemma A.3. We start with showing the tightness in $C_0$ of $\langle V^{k,j} \rangle_{n \in \mathbb{N}}$. By Minkowski’s integral inequality and Lemma 3.6, we have for $0 \leq s < t \leq T$,

$$E \left[ \left| \langle V^{k,j} \rangle_t - \langle V^{k,j} \rangle_s \right|^p \right] \leq n^{H_p} \left( \frac{\int_s^t (\hat{X}_n - \hat{X}_n) ds}{s/t} \right)^p \leq Cn^{H_p} \left( \frac{\int_s^t u^{-1/p} du}{s/t} \right)^p \leq C \left| t - s \right|^p.$$

Therefore, Kolmogorov’s continuity theorem implies $E[\|V^{k,j}\|_{C_0^p}] \leq C_p$ for any $p \in (0, 1)$ with $C_p$ being independent of $n$, and then, $\langle V^{k,j} \rangle_{n \in \mathbb{N}}$ is tight in $C_0^p$ for any $p \in (0, 1)$ by Theorem B.2. Assume an arbitrary subsequence $\{\langle V^{k,j} \rangle_{n \in \mathbb{N}} \} \subset \{\langle V^{k,j} \rangle_{n \in \mathbb{N}} \}$ of $\{\langle V^{k,j} \rangle_{n \in \mathbb{N}} \}$ and a process $F \in C_0^p$ such that $\langle V^{k,j} \rangle_{n \in \mathbb{N}}$ weakly tends to $F$ in $C_0^p$. However, $\langle V^{k,j} \rangle_t \to 0$ in $L^1$ by Lemma 2.2 for all $t$, so $F_t = 0$ for all $t$. This implies $\langle V^{k,j} \rangle_{n \in \mathbb{N}}$ tends to zero also in probability in $C_0$. Let $\{\hat{V}^k \}_{n \in \mathbb{N}}$ be the set of simple predictable processes almost surely bounded by one. By Lemma 3.6, we have

$$E \left[ \int_0^t Y_n d\langle V^{k,j} \rangle_s \right] \leq \int_0^t n^{H_p} E \left| \hat{X}_n^{k,j} - \hat{X}_n^{k,j} \right| ds \leq C,$$

which implies $\langle V^{k,j} \rangle_t$ is uniform tight in the sense of Definition 7.4 of Kurtz and Protter [16]. Noting also that $\hat{X} \to X$ in probability in $C_0$ by Lemma 3.8, Theorem 7.10 of Kurtz and Protter [16] yields

$$(\partial_k b^i(\hat{X}_n)), \langle V^{k,j} \rangle, \partial_k b^i(\hat{X}_n) \cdot \langle V^{k,j} \rangle \xrightarrow{n \to \infty} (\partial_k b^i(X), 0, 0).$$
This implies \( \int_0^s \partial_k b(\mathring{X}_{tu}) d(V^{n,k,i}, W^j)_u \) converges in probability in \( C_0 \) to zero process. We have also for \( 0 \leq s < t \leq T \),

\[
\begin{align*}
E \left[ \left| \sum_{k=1}^d \left( \int_0^t \partial_k b(\mathring{X}_{tu}) d(V^{n,k,i}, W^j)_u - \int_0^s \partial_k b(\mathring{X}_{tu}) d(V^{n,k,i}, W^j)_u \right) \right|^p \right] \\
\leq \left( \int_s^t n^H \sum_{k=1}^d E \left[ \left| \partial_k b(\mathring{X}_{tu}) (\mathring{X}_t^k - \mathring{X}_{tu}^k) \right|^p \right] \, du \right)^{\frac{p}{p'}} \\
\leq C \left( \int_s^t n^H (u - \frac{\ln u}{n})^H \, du \right)^p \leq C|t - s|^p
\end{align*}
\]

by Minkowski’s integral inequality, Lemma 3.6, and the boundedness of the derivatives of \( b \). Thus, Kolmogorov’s continuity theorem yields

\[
E \left[ \left\| \sum_{k=1}^d \int_0^t \partial_k b(\mathring{X}_{tu}) d(V^{n,k,i}, W^j)_u \right\|_{C^{\rho}_b} \right] \leq C
\]

with \( C \) being independent of \( n \) for any \( \rho \in (0, 1) \). Consequently, by Corollary B.3, we have the process \( \sum_k \int \partial_k b(\mathring{X}_{tu}) d(V^{n,k,i}, W^j)_u \) converges in \( C^{1/2-\varepsilon} \) to zero process, and the desired result is obtained by Lemma A.3.

\[
\Box
\]

### 4.5 Proof of Lemma 2.5

Here we show the convergence to 0 of \( \Delta^{n,i,j} \) defined by

\[
\Delta^{n,i,j} = U^{n,i,j} - \int_0^t K(t-s) \left( \nabla b(\mathring{X}_{tu})^T U_t^i + \sum_{j=1}^m \nabla \sigma_j(\mathring{X}_{tu})^T U_t^j dW_t^j \right)
\]

\[
+ \int_0^t K(t-s) n^H \nabla b(\mathring{X}_{tu})^T (\mathring{X}_s - \mathring{X}_{tu}) \, ds + \sum_{j=1}^m \sum_{k=1}^d \int_0^t K(t-s) \partial_j \sigma_j(\mathring{X}_{tu}) dV^{n,k,i}_s.
\]

By Taylor’s theorem, one has the following identity for \( \sigma \):

\[
\sigma(x + h) - \sigma(x) - h \sigma'(x) = h \int_0^1 (\sigma'(x + rh) - \sigma'(x)) \, dr.
\]

Using this identity, we can rewrite \( \Delta^{n,i,j} \) as

\[
\Delta^{n,i,j} = \sum_{k=1}^d \int_0^t K(t-s) n^H (X^k_s - \mathring{X}^k_{tu}) \left( \int_0^1 \partial_k b(\mathring{X}_s + r(X_s - \mathring{X}_s)) - \partial_k b(\mathring{X}_s) \right) \, ds
\]

\[
+ \sum_{k=1}^d \int_0^t K(t-s) n^H (X^k_s - \mathring{X}^k_{tu}) \left( \int_0^1 \partial_k b(\mathring{X}_{tu} + r(\mathring{X}_s - \mathring{X}_{tu})) - \partial_k b(\mathring{X}_{tu}) \right) \, ds
\]

\[
+ \sum_{j=1}^m \sum_{k=1}^d \int_0^t K(t-s) n^H (X^k_s - \mathring{X}^k_{tu}) \left( \int_0^1 \partial_j \sigma_j(\mathring{X}_{tu} + r(\mathring{X}_s - \mathring{X}_{tu})) - \partial_j \sigma_j(\mathring{X}_{tu}) \right) \, dW^j_s
\]

\[
+ \sum_{j=1}^m \sum_{k=1}^d \int_0^t K(t-s) n^H (X^k_s - \mathring{X}^k_{tu}) \left( \int_0^1 \partial_j \sigma_j(\mathring{X}_{tu} + r(\mathring{X}_s - \mathring{X}_{tu})) - \partial_j \sigma_j(\mathring{X}_{tu}) \right) \, dW^j_s
\]

\[
= \sum_{k=1}^d \tilde{\Delta}^{n,k,i}(X, \mathring{X}) + \sum_{k=1}^d \tilde{\Delta}^{n,k,i}(\mathring{X}, \mathring{X}_{tu}) + \sum_{j=1}^m \sum_{k=1}^d \tilde{\Delta}^{n,i,j,k}(X, \mathring{X}) + \sum_{j=1}^m \sum_{k=1}^d \tilde{\Delta}^{n,i,j,k}(\mathring{X}, \mathring{X}_{tu}).
\]
where

\[ \tilde{\Lambda}_{i,j}^{(n,k)}(x, y) = \int_0^t (K(t - s) - K(t - s)) n^H \psi^{(k)}_n(s, x, y)ds, \quad \Lambda_{i,j}^{(n,k)}(x, y) = \int_0^t K(t - s) n^H \psi^{(k)}_n(s, x, y)dB_s, \]

\[ \psi^{(k)}_n(s, x, y) = (x^k - \tilde{y}^k_s) \int_0^t (\partial_s a(y_s + r(x_s - y_s)) - \partial_y a(y_s))dr, \quad a \in [b^i, a^j], \]

for any adapted processes \( x, y \). From Lemma 3.2-(3),(4), it follows that for \( t + h, t \in [0, T], h > 0, p > 2\beta^* \) and any adapted processes \( x, y \),

\[ \mathbb{E} \left[ \left| \tilde{\Lambda}_{i,t+h}^{(n,k)}(x, y) - \Lambda_{i,t}^{(n,k)}(x, y) \right|^p \right] \leq 2^{p-1} \mathbb{E} \left[ \left| \int_0^h (K(t - s) - K(t - s)) n^H \psi^{(k)}_n(s, x, y)ds \right|^p \right] + 2\beta^* \mathbb{E} \left[ \left| \int_0^{h} K(t - s) n^H \psi^{(k)}_n(s, x, y)dB_s \right|^p \right] \]

\[ \leq C h^{p-1} \mathbb{E} \left[ \left| \psi^{(k)}_n(s, x, y) \right|^p \right] \]

and

\[ \mathbb{E} \left[ \left| \Lambda_{i,t+h}^{(n,k)}(x, y) - \Lambda_{i,t}^{(n,k)}(x, y) \right|^p \right] \leq 2^{p-1} \mathbb{E} \left[ \left| \int_0^h (K(t + h - s) - K(t - s)) n^H \psi^{(k)}_n(s, x, y)ds \right|^p \right] + 2\beta^* \mathbb{E} \left[ \left| \int_0^{h} K(t + h - s) n^H \psi^{(k)}_n(s, x, y)dB_s \right|^p \right] \]

\[ \leq C h^{p-1} \mathbb{E} \left[ \left| \psi^{(k)}_n(s, x, y) \right|^p \right]. \]

We here define the modulus of continuity of a continuous function \( a \) as follows:

\[ m(a, \delta) = \sup_{|u - v| \leq \delta} |a(u) - a(v)|. \]

Using this notation, by the Cauchy-Schwarz inequality, we have

\[ \mathbb{E} \left[ \left| \psi^{(k)}_n(s, x, y) \right|^p \right] = \mathbb{E} \left[ \left| x^k_s - \tilde{y}^k_s \right|^p \right] + \left[ \int_0^1 (\partial_s a(y_s + r(x_s - y_s)) - \partial_y a(y_s))dr \right]^2 \]

\[ \leq \mathbb{E} \left[ \left| x^k_s - \tilde{y}^k_s \right|^2 \right] \mathbb{E} \left[ \left| \int_0^1 m(\partial_s a, r(x_s - y_s))dr \right|^2 \right] \]

\[ \leq \mathbb{E} \left[ \left| x^k_s - \tilde{y}^k_s \right|^2 \right] \mathbb{E} \left[ \left| m(\partial_s a, ||x - y||_\infty) \right|^2 \right]. \]

By the hypothesis, the derivatives of \( b \) and \( \sigma \) are all bounded, and therefore, if \( ||x - y||_\infty \to 0 \) in probability as \( n \to \infty \), it follows from the BCT and the property of the modulus of continuity that

\[ \lim_{n \to \infty} \mathbb{E} \left[ \left| m(\partial_s a, ||x - y||_\infty) \right|^2 \right] = \mathbb{E} \left[ \lim_{n \to \infty} \left| m(\partial_s a, ||x - y||_\infty) \right|^2 \right] = 0 \]

for \( a = b^i \) and \( a = a^j \).
Substituting \((x, y) = (X, \hat{X})\) and \((x, y) = (\hat{X}, \hat{X}_{\|\cdot\|_\infty})\) and using Lemmas 3.7 and 3.6 respectively, we have

\[
\mathbb{E} \left[ |\hat{\Delta}_{t+h}^{n,j} - \Delta_{t+h}^{n,j}|^p \right] 
\leq C \sum_{k=1}^{d} \mathbb{E} \left[ |\hat{\Delta}_{t+h}^{n,j,k}(X, \hat{X}) - \hat{\Delta}_{t,j}^{n,j,k}(X, \hat{X})|^p \right] + C \sum_{k=1}^{d} \mathbb{E} \left[ |\hat{\Delta}_{t+h}^{n,j,k}(\hat{X}, \hat{X}_{\|\cdot\|_\infty}) - \hat{\Delta}_{t,j}^{n,j,k}(\hat{X}, \hat{X}_{\|\cdot\|_\infty})|^p \right]
\]

where \(C\) is independent of \(n, t\) and \(h\). Therefore, since \(\Delta_{t,j}^{n,j} = 0\) and both \(\|X - \hat{X}\|_\infty\) and \(\|\hat{X} - \hat{X}_{\|\cdot\|_\infty}\|_\infty\) tend to zero in probability, the desired result is obtained by

\[
\mathbb{E} \left[ |\hat{\Delta}_{t}^{n,j}|^p \right] \leq C \sum_{k=1}^{d} \mathbb{E} \left[ |\hat{\Delta}_{t}^{n,j,k}(\hat{X} - \hat{X}_{\|\cdot\|_\infty})|^p \right] + C \sum_{k=1}^{d} \mathbb{E} \left[ |\hat{\Delta}_{t}^{n,j,k}(\hat{X} - \hat{X}_{\|\cdot\|_\infty})|^p \right] \to 0, \quad \forall \gamma \in (0, H),
\]

which is derived from Kolmogorov’s continuity theorem; see Revus and Yor (Theorem 1.2.1 [19]).

### 4.6 Proof of Lemma 2.6

First we show that \(V^n\) is uniformly tight in the sense of Definition 7.4 of Kurtz and Protter [16]. Let \(\{Y^n\}_{n \in \mathbb{N}}\) be the set of simple predictable processes almost surely bounded by one. Then for all \(t \in [0, T]\), it follows from the Itô isometry,

\[
\mathbb{E} \left[ \int_0^t Y^n_{s+} dV^n_{s+} \right] = \mathbb{E} \left[ \int_0^t |Y^n_{s+}|^2 d\langle V^n \rangle_s \right] \leq \mathbb{E} \left[ \langle V^n \rangle_t \right] < \infty,
\]

where the bound is uniform in \(n\) since \(\{\langle V^n \rangle_t\}_{n \in \mathbb{N}}\) is the convergent sequence in \(L_1\) and, therefore, bounded sequence in \(L_2\) by Lemma 2.2.

Now, define \(\Phi^n = (\Phi^n)^{1}, \ldots, (\Phi^n)^{m}\) by

\[
\Phi^n = \sum_{k=1}^{m} \int_0^t \partial_k b'(\hat{X}_{\|\cdot\|_\infty}) U^n_{s+} ds + \sum_{k=1}^{m} \int_0^t \partial_k \sigma_j'(\hat{X}_{\|\cdot\|_\infty}) U^n_{s+} dW^j_s + \sum_{k=1}^{m} \int_0^t \partial_k \sigma_j'(X) dV^n_{s+}.
\]

By the uniform tightness of \(V^n\), Theorem 7.10 of Kurtz and Protter [16] implies \((U^n, \Phi^n) \to (U, \Phi)\) in law, where \(\Phi = (\Phi^1, \ldots, \Phi^m)\) are defined by

\[
\Phi = \sum_{k=1}^{m} \int_0^t \partial_k b'(X) U^k_{s+} ds + \sum_{k=1}^{m} \int_0^t \partial_k \sigma_j'(X) U^k_{s+} dW^j_s + \sum_{k=1}^{m} \int_0^t \partial_k \sigma_j'(X) dV^k_{s+}.
\]

We can also show \(\Phi^n\) is tight as a \(C^{1/2-\epsilon}_{0}\) -valued sequence for any \(\epsilon \in (0, 1/2)\). Indeed, denoting by

\[
\hat{V}^{n,j} = \sum_{j=1}^{m} \int_0^t \partial_k \sigma_j'(\hat{X}_{\|\cdot\|_\infty}) dV^n_{s+}
\]

(4.16)
we have for any $p > 2$ and $0 \leq s < t \leq T$,

$$
E\left[ \| \tilde{V}_{t+s}^{n,i} - \tilde{V}_{t}^{n,i} \|^p \right] \leq C_{p,d,m} \sum_{j=1}^{m} \sum_{k=1}^{d} E\left[ \left( \int_{s}^{t} |\partial_k \sigma_j'(\tilde{X}_{s}^{n,i}) h^{ij}(\tilde{X}_{s}^{n,i} - \tilde{X}_{s}^{n,i}) \right)^p dt \right] \\
\leq C \sum_{j=1}^{m} \sum_{k=1}^{d} \left( \int_{s}^{t} n^{2H} |\partial_k \sigma_j'(\tilde{X}_{s}^{n,i})|^p \| \tilde{X}_{s}^{n,i} - \tilde{X}_{s}^{n,i} \|^p dt \right)^{\frac{p}{2}} \\
\leq C \sum_{j=1}^{m} \sum_{k=1}^{d} \left( \int_{s}^{t} n^{2H} (t - [\lceil n \rceil])^{2H} dt \right)^{\frac{p}{2}} \\
\leq C \sum_{j=1}^{m} \sum_{k=1}^{d} \left( \int_{s}^{t} n^{2H} (t - [\lceil n \rceil])^{2H} dt \right)^{\frac{p}{2}}
$$

and so, for $t + h$, $t \in [0, T]$, $h > 0$,

$$
\| \Phi_{t+h}^{n,i} - \Phi_{t}^{n,i} \|_{L^p_v} \leq \frac{d}{\sum_{k=1}^{m}} \left( \int_{t}^{t+h} \| \partial_k h'(\tilde{X}_{s}^{n,i}) U_{s}^{n,i} \| dt + \sum_{j,k} \left( \int_{t}^{t+h} |\partial_k \partial_j'(\tilde{X}_{s}^{n,i}) U_{s}^{n,i} \| dt \right) \right) + C_1 \sum_{j,k} \left( \int_{t}^{t+h} \| \partial_k \partial_j'(\tilde{X}_{s}^{n,i}) U_{s}^{n,i} \| dt \right)^{\frac{p}{2}} + C_2 h^{1/2} \\
\leq C h^{1/2}
$$

by the BDG inequality, Minkowski's integral inequality, Lemmas 3.6 and 3.7, and the boundedness of the derivatives of $\sigma$. Hence, Kolmogorov’s continuity theorem yields $E\left[ \| \Phi_{t}^{n,i} \|_{C^{0,\beta}} \right] \leq C$ uniformly in $n$, and therefore, by Corollary B.3, $\Phi^0$ converges in law in $C^{0,\beta}$ to $\Phi$.

Let $a = \frac{1}{2} - H$ and $\gamma = \frac{1}{2} - \varepsilon$. By Lemma A.3, the operator $F^a$ is continuous from $C^\gamma$ into $C^{\gamma - a}$. Since $(U^p, \Phi^p)$ converges in law to $(U, \Phi)$ in $C^\gamma \times C^\gamma$, the continuous mapping theorem implies that $U^p - F^a \Phi^p$ converges in law to $U - F^a \Phi$ in $C^{\gamma - a} = C^\gamma$. On the other hand, Lemmas 2.4 and 2.5 imply $U^p - F^a \Phi^p$ converges in law to zero. Consequently $U - F^a \Phi = 0$, which is equivalent to (2.1).  \( \square \)

### 4.7 Proof of Lemma 2.7

Set $\tilde{U}^{n,i}$ as

$$
\tilde{U}^{n,i} = \int_{0}^{T} K(t-s) \nabla b'(\tilde{X}_{s}^{n,i}) \nabla u_{s}^{n,i} ds + \sum_{j=1}^{d} \sum_{k=1}^{m} \int_{0}^{t} K(t-s) u_{s}^{n,i} \partial_k \sigma_j'(\tilde{X}_{s}^{n,i}) dW_{s}^{j}.
$$

We first show the tightness of $\{\tilde{U}^{n,i}\}$. By Theorem B.2, it suffices to show that there exists a uniform bound for $E[\|\tilde{U}^{n,i}\|_{C^{0,\beta}}]$ for $\beta < \varepsilon$. To show the Hölder continuity of $\tilde{U}^{n,i}$, we start with decomposing the amount of its change from $t$ to $t + h$ for some $h > 0$ and $p \geq 2$ as the following:

$$
\| \tilde{U}^{n,i}_{t+h} - \tilde{U}^{n,i}_{t} \|^p \leq 4^{p-1} \left( \int_{0}^{t} (K(t+h-s) - K(t-s)) \nabla b'(\tilde{X}_{s}^{n,i}) \nabla u_{s}^{n,i} ds \right)^p \\
+ \left( \int_{t}^{t+h} K(t+s) \nabla b'(\tilde{X}_{s}^{n,i}) \nabla u_{s}^{n,i} ds \right)^p \\
+ \sum_{j=1}^{d} \sum_{k=1}^{m} \left( \int_{0}^{t} (K(t+h-s) - K(t-s)) \partial_k \sigma_j'(\tilde{X}_{s}^{n,i}) u_{s}^{n,i} dW_{s}^{j} \right)^p \\
+ \sum_{j=1}^{d} \sum_{k=1}^{m} \int_{0}^{t+h} K(t+h-s) \partial_k \sigma_j'(\tilde{X}_{s}^{n,i}) u_{s}^{n,i} dW_{s}^{j}. \quad (4.18)
$$

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Then, by Lemma 3.2-(3), we have

$$E \left[ \left| \tilde{U}_{t+h}^{n,i} - \tilde{U}_t^{n,i} \right|^p \right] \leq C h^p \sup_{t \in [0,T]} E \left[ |U_t^{n,i}|^p \right] = C h^p \sup_{t \in [0,T]} E \left[ |U_t^{n,i}|^p \right] \leq C h^p,$$

where C is independent of $t$ and $n$, and hence, Kolmogorov’s continuity theorem and Theorem B.2 leads to the tightness in $C_{0}^{1,p} \cap C_{T}^{1,p}$ of $\{ U_t^{n,i} \}$.

In light of Lemmas 2.4 and 2.5, it is only remained to show that $\{ \int_0^t K(t-s)d\tilde{V}_{s}^{n,i} \}_{i \in \mathbb{N}}$ is tight, where $\tilde{V}_{s}^{n,i}$ is defined by (4.16). Similarly to the above, for any $t + h, t \in [0,T], h > 0, p > 2$,

$$E \left[ \left| \int_0^{t+h} K(t-h-s)d\tilde{V}_{s}^{n,i} - \int_0^t K(t-s)d\tilde{V}_{s}^{n,i} \right|^p \right] \leq 2^{p-1} E \left[ \left| \int_0^{t} (K(t-h-s) - K(t-s))d\tilde{V}_{s}^{n,i} \right|^p \right] + \left| \int_t^{t+h} K(t-h-s)d\tilde{V}_{s}^{n,i} \right|^p \right] \leq 2^{p-1} E \left[ \left| \int_0^{t} (K(t-h-s) - K(t-s)) \sum_{jk} \partial_k \partial_j \tilde{X}_{s} |n^H (\tilde{X}_{s} - \tilde{X}_{s}^k) dW_{s}^j \right|^p \right] + 2^{p-1} E \left[ \left| \int_t^{t+h} K(t-h-s) \sum_{jk} \partial_k \partial_j \tilde{X}_{s} |n^H (\tilde{X}_{s} - \tilde{X}_{s}^k) dW_{s}^j \right|^p \right] \leq C h^p \sum_{k=1}^{d} \sup_{t \in [0,T]} E \left[ |\tilde{X}_{s}^k - \tilde{X}_{s}^k|^p \right] \leq C h^p$$

follows from Lemmas 3.2-(4), 3.6, and the boundedness for the derivative of $\sigma$. Then, by Kolmogorov’s continuity theorem, together with Theorem B.2, we obtain the tightness in $C_{0}^{1,p} \cap C_{T}^{1,p}$ of $\{ \int_0^t K(t-s)d\tilde{V}_{s}^{n,i} \}$. From these arguments, the tightness in $C_{0}^{1,p} \cap C_{T}^{1,p}$ of $\{ U_t^{n,i} \}$ is verified.

### 4.8 Proof of Lemma 2.8

We first show the $L_p$ integrability with the similar way to the proof of Lemma 3.5. The SVE (2.1) is transformed as

$$U_t^{i} = \int_0^t K(t-s)\nabla b_i (X_s) \cdot U_s ds + \sum_{j=1}^{d} \sum_{k=1}^{m} \sum_{l=1}^{m} \int_0^t K(t-s) \partial_k \partial_j \tilde{X}_s |U_s dW_s^j + C_H \partial_l \tilde{X}_s |B_s^j,$$

where $C_H = \frac{1}{\sqrt{(12H+2) \sinh 2H}}$. Let $\tau_m = \inf \{ t \mid |U_t| \geq m \}$. Using the local property, Lemmas 3.2-(1),(2), 3.3 and the boundedness of the derivative of $\sigma$, we observe that

$$E \left[ |U_t^{i}|^p 1_{|U_t^{i}| < \tau_m} \right] \leq 2^{p-1} E \left[ \left| \int_0^t K(t-s) \nabla b_i (X_s) \cdot U_s ds \right|^p \right]$$

$$+ C_{p,m,d} \sum_{j,k,l} E \left[ \left| \int_0^t K(t-s) \partial_k \partial_j \tilde{X}_s |U_s dW_s^j + C_H \partial_l \tilde{X}_s |B_s^j \right|^p \right]$$

$$\leq C_1 + C_2 \int_0^t E \left[ \left| \nabla b_i (X_s) \right|^p + \left| \partial_k \partial_j \tilde{X}_s \right|^p |U_s dW_s^j + \left| \partial_l \tilde{X}_s \right|^p |C_H \partial_l \tilde{X}_s |B_s^j \right]$$

$$\leq C_1 + C_2 \int_0^t E \left[ |U_s|^{p} 1_{|U_s| < \tau_m} \right] ds.$$

for some $C_1, C_2 > 0$, independent of $t$. Therefore, Gronwall’s lemma yields

$$E \left[ |U_t^{i}|^p 1_{|U_t^{i}| < \tau_m} \right] \leq C_1 e^{C_1 t} \leq C_1 e^{C_1 T},$$

and thus, by Fatou’s lemma, we see

$$E \left[ |U_t^{i}|^p \right] = E \left[ \liminf_{m \to \infty} |U_t^{i}|^p 1_{|U_t^{i}| < \tau_m} \right] \leq \liminf_{m \to \infty} E \left[ |U_t^{i}|^p 1_{|U_t^{i}| < \tau_m} \right] \leq C_1 e^{C_1 T},$$

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that is, \( U_t \) is in \( L_p \) for any \( t \in [0, T] \).

Next we discuss the continuity. It is sufficient to verify the continuity of the stochastic integral part. Similarly to (4.18), we see for any \( t, t + h \in [0, T], h > 0 \),

\[
E \left[ |U_{t+h} - U_t|^p \right] \leq C h^p E \left[ |U_t|^p + \sum_{k} |\partial_k \sigma^j(X_s)C_{ih} \sigma^j(X_s)|^p \right] + C h^p.
\]

Then, Kolmogorov’s continuity theorem ensure the continuity of \( U_t \).

Finally, we show the uniqueness in law of the solution. Assume there exist two continuous solutions \( \nu^1, \nu^2 \) satisfying (2.1). Then

\[
\nu^1_t - \nu^2_t = \int_0^t K(t-s)\sigma'(X_s)(\nu^1_s - \nu^2_s)dW_s
\]

follows, and we will evaluate this difference in \( L_p, p > 2\beta^* \). Write \( \nu = \nu^1 - \nu^2 \). By Lemma 3.2-(1) and the boundedness of \( \sigma' \), we have

\[
E \left[ |\nu_t|^p \right] \leq C \int_0^t E \left[ |\sigma'(X_s)|^p|\nu_s|^p \right] ds
\]

\[
\leq C \int_0^t E \left[ |\nu_s|^p \right] ds.
\]

Since \( \nu \) is continuous, by Gronwall’s lemma, we see

\[
E \left[ |\nu_t|^2 \right] = 0,
\]

which implies that for all \( t, \nu^1_t = \nu^2_t \) almost surely. Taking the continuities of \( \nu^1 \) and \( \nu^2 \) into account, we can verify these processes are indistinguishable. As a consequence, the uniqueness in law is derived from the Yamada-Watanabe argument. \( \square \)

## A Stochastic integrals as the fractional derivatives

We introduce the representation of stochastic integral as the fractional derivatives; see also Horvath et al. [10]. We let \( \lambda \in (0, 1) \) and \( K(t) = C_t t^{-\lambda}, \alpha \in (0, \lambda) \) in this section.

**Definition A.1.** Let \( f \in C_0^1 \). We define the integral operator \( \mathcal{J}^\alpha \) as

\[
(\mathcal{J}^\alpha f)(t) := K(t)f(t) - \int_0^t K'(t-s)(f(t) - f(s))ds,
\]

where \( K' = dK/dt \).

**Proposition A.2.** Let \( Y \) be a continuous semimartingale taking values in \( C_0^1 \) such that the stochastic integral \( \int_0^t K(t-s)dY_s \) is well defined for all \( t \in [0, T] \). Then the integral is almost surely represented by \( \mathcal{J}^\alpha \) as

\[
\int_0^t K(t-s)dY_s = (\mathcal{J}^\alpha Y)(t)
\]

for all \( t \in [0, T] \).

**Proof.** Let \( \epsilon > 0 \) be a sufficiently small number. Itô’s integration by parts yields

\[
\int_0^{\epsilon-c} K'(t-s)(Y_t - Y_s)ds = \left( \int_0^{\epsilon-c} K'(t-s)ds \right)Y_t - \int_0^{\epsilon-c} K'(t-s)Y_sds
\]

\[
= \left( \int_{\epsilon-c} K'(s)ds \right)Y_t + [K(t-s)Y_s]_{s=0}^{s=\epsilon-c} - \int_0^{\epsilon-c} K(t-s)Y_sds
\]

\[
= (K(t) - K(\epsilon))Y_t - K(\epsilon)Y_{t-c} - \int_0^{\epsilon-c} K(t-s)Y_sds
\]

\[
= K(t)Y_t - \int_0^{\epsilon-c} K(t-s)Y_s + K(\epsilon)(Y_{t-c} - Y_t).
\]

(A.1)
Since $V$ satisfies the Hölder condition of an order $\lambda > \alpha$, the last term vanishes as $\epsilon \to 0$. Indeed, letting $A(\omega)$ be the Hölder constant of $Y(\omega)$, we see

$$|K(\epsilon)(Y_{t-\epsilon}(\omega) - Y_t(\omega))| \leq e^{-\alpha} C_K A(\omega) \epsilon^1 = C_K A(\omega) \epsilon^{1-\alpha},$$

which vanishes as $\epsilon$ tends to zero uniformly in $t$. The second term of on the right-hand side of (A.1) obviously tends to the original stochastic integral. On the other hand, the integral on the left-hand side of (A.1) converges to the integration on $[0, t]$ by the DCT since

$$K'(t-s)|Y_t(\omega) - Y_s(\omega)| \leq C_K A(\omega) |t-s|^{1-\alpha-1}$$

for all $s \in [0, t]$ and

$$\int_0^t C_K A(\omega) |t-s|^{1-\alpha-1}ds \leq \frac{C_K}{\lambda - \alpha} A(\omega) t^{1-\alpha}.$$

Therefore, taking the limit of both sides of (A.1) with $\epsilon \to 0$, we have

$$\int_0^t K(t-s)dY_s = K(t)Y_t - \int_0^t K'(t-s)(Y_t - Y_s)ds = (\mathcal{J}^\alpha Y)(t), \quad (A.2)$$

which is the desired result. $\Box$

The operator $\mathcal{J}^\alpha$ is one of the fractional derivative operators, called the Marchaud fractional derivative, which coincides with the Riemann-Liouville fractional derivative on $C_0^1$.

**Lemma A.3** (Samko et al. [21]). The operator $\mathcal{J}^\alpha$ is bounded (continuous) from $C_0^1$ into $C_0^{1-\alpha}$.

### B Tightness criterion for a space of Hölder continuous processes

We will discuss the tightness of Hölder continuous processes. We first introduce a compact set in a space of Hölder continuous functions.

**Lemma B.1.** A set of $\alpha$-Hölder continuous functions

$$K_\delta = \{ f \in C_0^{1} \mid \|f\|_{C_0^{1}} \leq \delta \}$$

is compact in $C_0^{1}$ for any $\delta > 0$ and $\beta \in (0, \alpha)$.

**Proof.** We first prove $K_\delta$ is compact in $C_0$. Take any sequence $\{f_n\}_{n \in \mathbb{N}} \subset K_\delta$ such that $\|f_n - f\|_{\infty} \to 0$ for some $f$. We have for all $t, s \in [0, T], s \neq t$,

$$\frac{|f_n(t) - f_n(s)|}{|t-s|^\alpha} \leq \delta - \|f_n\|_{\infty}.$$

Letting $n \to \infty$, we obtain

$$\frac{|f(t) - f(s)|}{|t-s|^\alpha} \leq \delta - \|f\|_{\infty},$$

since $f_n(t)$ converges to $f(t)$ for all $t \in [0, T]$. Then, we obtain $\|f\|_{C_0^{1}} \leq \delta$, and thus, $f \in K_\delta$. Hence, $K_\delta$ is closed in $C_0$, and along with the Arzelà-Ascoli theorem, we observe that $K_\delta$ is compact in $C_0$.

We now prove the compactness of $K_\delta$ in $C_0^1$. By the compactness in $C_0$, for an arbitrary sequence $\{g_n\}_{n \in \mathbb{N}} \subset K_\delta$ there exist a subsequence $\{g_{n_k}\}_{k \in \mathbb{N}}$ and $g \in K_\delta$ such that $\|g_{n_k} - g\|_{\infty} \xrightarrow{k \to \infty} 0$. Since $g_{n_k}$ and $g$ belong to $K_\delta$,

$$|g_{n_k}(t) - g(t) - (g_{n_k}(s) - g(s))| \leq 2\delta|t-s|^\alpha, \quad \forall k \in \mathbb{N}, \forall t, s \in [0, T]$$

follows, and it also holds

$$|g_{n_k}(t) - g(t) - (g_{n_k}(s) - g(s))| \leq |g_{n_k}(t) - g(t)| + |g_{n_k}(s) - g(s)| \leq 2\|g_{n_k} - g\|_{\infty}, \quad \forall k \in \mathbb{N}, \forall t, s \in [0, T].$$

Hence, the inequality

$$a \wedge b \leq a^b \theta^{-b}, \quad a, b > 0, \theta \in [0, 1]$$

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leads to
\[ |g_{n}(t) - g(t) - (g_{n}(s) - g(s))| \leq (2\delta |t - s|^\beta) \wedge (2\|g_{n} - g\|_{\infty}) \]
\[ \leq 2\delta^{\beta} |t - s|^\beta \|g_{n} - g\|_{\infty}^{1 - \frac{\beta}{2}} \]
for all \( t, s \), where \( 0 < \beta < \alpha \). Thus, we have
\[ \|g_{n} - g\|_{C^0} = \|g_{n} - g\|_{\infty} + \sup_{t, s \in [0, T]} \frac{|g_{n}(t) - g(t) - (g_{n}(s) - g(s))|}{|t - s|^\beta} \]
\[ = \|g_{n} - g\|_{\infty} + C\|g_{n} - g\|_{\infty}^{1 - \frac{\beta}{2}} \overset{n \to \infty}{\to} 0, \]
which implies \( K_0 \) is compact in \( C^0_\theta \). \( \square \)

**Theorem B.2.** Let \( \{Y^n\}_{n \in \mathbb{N}} \) be a sequence of \( C^0_\theta \)-valued random variables. If \( \mathbb{E}[\|Y^n\|_{C^0_\theta}] \) is bounded uniformly in \( n \), the sequence \( \{Y^n\}_{n \in \mathbb{N}} \) is tight in \( C^0_\theta \) for \( 0 < \beta < \alpha \).

**Proof.** We are to show that for any \( \varepsilon > 0 \) there exists a compact set \( K \) in \( C^0_\theta \) such that \( \mathbb{P}[Y^n \notin K] < \varepsilon \). Let \( \varepsilon > 0 \) and assume \( K_0 \) be the same set as in Lemma B.1. Since \( \mathbb{E}[\|Y^n\|_{C^0_\theta}] < C \), taking \( \delta = C\varepsilon^{-1} + 1 \) and Markov’s inequality yield that
\[ \mathbb{P}[Y^n \notin K_0] = \mathbb{P}[\|Y^n\|_{C^0_\theta} > \delta] \leq \frac{\mathbb{E}[\|Y^n\|_{C^0_\theta}]}{\delta} \leq \frac{C}{\delta} < \varepsilon. \]
The result follows from the compactness of \( K_0 \) verified in Lemma B.1. \( \square \)

**Corollary B.3.** Let \( Y^n \) be a stochastic process which converges to a process \( Y \) weakly in \( C_0 \) as \( n \) goes to infinity. If \( Y^n \) satisfies \( \mathbb{E}[\|Y^n\|_{C^0_\theta}] \leq C \) for some \( C \) uniformly in \( n \), it converges to \( Y \) weakly in \( C^0_\theta \) for any positive \( \beta < \alpha \).

**Proof.** Let \( 0 < \beta < \alpha \) and let \( \{Y^m\}_{m \in \mathbb{N}} \) be an arbitrary subsequence of \( \{Y^n\}_{n \in \mathbb{N}} \). Then, by Theorem B.2, together with Prokhorov’s theorem, we see that there exist a subsequence \( \{Y^n_i\}_{i \in \mathbb{N}} \) of \( \{Y^n\}_{n \in \mathbb{N}} \) and \( Y \in C^0_\theta \) such that \( Y^n \) tends to \( Y \) weakly in \( C^0_\theta \) and, in particular, in \( C_0 \). However, by the assumption, \( Y^n \) tends to \( Y \) weakly in \( C^0_\theta \). Therefore, it must holds \( \tilde{Y} = Y \). Since each subsequence of \( \{Y^n\}_{n \in \mathbb{N}} \) includes a subsequence which converges to \( Y \) weakly in \( C^0_\theta \), we see that the original sequence converges to \( Y \) weakly in \( C^0_\theta \) too. This completes the proof. \( \square \)

## C Limit distribution of the integral of fractional parts

The following lemma is analogous to that of Delattre and Jacod (Lemma 6.1 [5]) and Tukey [22].

**Lemma C.1.** Let \( g \in L_1(0, 1) \) be either nonnegative or nonpositive and let \( \{Y_n\}_{n \in \mathbb{N}} \) be a sequence of random variables on \([0, t]\) whose density functions are each
\[ f_{Y_n}(s) = C_{n,t}g(ns - [ns]), \quad 0 < s < t, \]
where
\[ C_{n,t} = \left( \frac{[nt]}{n} \right) \int_0^1 g(r)dr + \frac{1}{n} \int_{[nt]-[nt]}^1 g(r)dr \]
is the normalizing constant. Then \( Y_n \) converges in law to the uniform distribution on \([0, t]\) as \( n \) goes to infinity.

**Proof.** Firstly, we confirm that \( f_{Y_n}(y) \) is certainly a probability density function. It is easily checked by the simple calculation:
\[ \int_0^1 g(ns - [ns])ds = \sum_{j=0}^{[nt]-1} \int_j^1 g(ns - j)ds + \int_{[nt]}^1 g(ns - [nt])ds \]
\[ = \frac{1}{n} \left( \sum_{j=0}^{[nt]-1} \int_j^1 g(r)dr + \int_{[nt]-[nt]}^1 g(r)dr \right) \]
\[ = \frac{[nt]}{n} \int_0^1 g(r)dr + \frac{1}{n} \int_{[nt]-[nt]}^1 g(r)dr \]
\[ = (C_{n,t})^{-1}. \]
We now show the convergence of the characteristic function of $Y_n$. For all $x \in \mathbb{R}$ and $t = \sqrt{-1}$, we have
\[
\int_0^1 \overline{e^{ix}f_x(s)} ds = \int_0^1 \overline{e^{ix}C_{n,t}g(ns - \lfloor ns \rfloor)ds}
\]
\[
= C_{n,t} \sum_{k=0}^{[nt]-1} \int_0^{\frac{k+1}{n}} e^{ix}g(ns - k)ds + C_{n,t} \int_{\frac{k}{n}}^{\frac{k+1}{n}} e^{ix}g(ns - \lfloor nt \rfloor)ds
\]
\[
= \frac{C_{n,t}}{n} \sum_{k=0}^{[nt]-1} \int_0^1 e^{i\frac{xk}{n}}g(r)dr + \frac{C_{n,t}}{n} \int_0^{nt-[nt]} e^{i\frac{x}{n}}g(r)dr.
\]
Since $0 < nt - [nt] < 1$,
\[
\left| \frac{1}{n} \int_0^{nt-[nt]} g(r)dr \right| \leq \frac{1}{n} \int_0^1 |g(r)|dr \to 0,
\]
and therefore, by the convergence
\[
C_{n,t} = \left( \frac{[nt]}{n} \int_0^1 g(r)dr + \frac{1}{n} \int_0^{nt-[nt]} g(r)dr \right)^{-1} \to \frac{1}{\int_0^1 g(r)dr},
\]
we see
\[
\left| \frac{C_{n,t}}{n} \int_0^{nt-[nt]} e^{i\frac{x}{n}}g(r)dr \right| \leq \frac{C_{n,t}}{n} \int_0^1 |g(r)|dr \to 0.
\]
By the triangle inequality, it holds
\[
\left| \frac{1}{n} \sum_{k=0}^{[nt]-1} e^{i\frac{xk}{n}} - \int_0^1 e^{ix}ds \right| \leq \left| \frac{1}{n} \sum_{k=0}^{[nt]-1} e^{i\frac{xk}{n}} - \int_0^\frac{[nt]}{n} e^{ix}ds \right| + \int_\frac{[nt]}{n}^1 e^{ix}ds.
\]
The last term on the right-hand side vanishes as $n$ goes to infinity. The other term on the right-hand side is evaluated as
\[
\left| \frac{1}{n} \sum_{k=0}^{[nt]-1} e^{i\frac{xk}{n}} - \int_0^\frac{[nt]}{n} e^{ix}ds \right| = \left| \frac{1}{n} \sum_{k=0}^{[nt]-1} \left( e^{i\frac{xk}{n}} - n \int_0^{\frac{k+1}{n}} e^{ix}ds \right) \right|
\]
\[
= \left| \frac{1}{n} \sum_{k=0}^{[nt]-1} \left( e^{i\frac{xk}{n}} - \int_0^{1} e^{ix}ds \right) \right|
\]
\[
\leq \left| 1 - \int_0^{1} e^{i\frac{x}{n}}ds \right| \cdot \frac{1}{n} \sum_{k=0}^{[nt]-1} \left| e^{i\frac{k}{n}} \right|
\]
\[
\leq \int_0^{1} \left| 1 - e^{i\frac{x}{n}} \right| ds \cdot \frac{[nt]}{n} \to 0.
\]
Hence, we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{[nt]-1} e^{i\frac{k}{n}} = \int_0^1 e^{ix}ds,
\]
and then, along with the DCT, it is implied that
\[
C_{n,t} \sum_{k=0}^{[nt]-1} \int_0^1 e^{i\frac{xk}{n}}g(r)dr = C_{n,t} \left( \frac{1}{n} \sum_{k=0}^{[nt]-1} e^{i\frac{k}{n}} \right) \int_0^1 e^{i\frac{x}{n}}g(r)dr
\]
\[
= \left( t \int_0^1 g(r)dr \right)^{-1} \int_0^1 e^{ix}ds \int_0^t g(r)dr
\]
\[
= \int_0^t \frac{1}{t} e^{ix}ds.
\]
Indeed, a dominating function is derived as
\[
\left| \frac{1}{n} \sum_{k=0}^{[nt]-1} e^{ixk} g(r) \right| \leq \frac{[nt]}{n} |g(r)| \leq T |g(r)|.
\]
Thus,
\[
\int_0^t e^{ixs} f_{\gamma_n}(s) ds \to \int_0^t \frac{1}{t} e^{ixs} ds
\]
holds. Since the function \( s \mapsto 1/t \) is the density function of the uniform distribution on [0, t], this means the convergence of the characteristic functions, which concludes the proof. \( \square \)

By this lemma, for any continuous function \( k \), we have
\[
\int_0^1 k(s) g(ns - [ns]) ds = (C_n,t)^{-1} \int_0^1 k(s) f_{\gamma_n}(s) ds
\]
(C.2)
by the property of convergence in law. We will apply this result to stochastic processes.

Lemma C.2. Assume further \( g \in L_2(0, 1) \). Let \( H^{(n)} \) and \( H \) be stochastic processes on \([0, T]\) such that
\[
E \left[ \int_0^T |H^{(n)}_s - H_s|^2 ds \right] \to 0
\]
with \( H \) being almost surely continuous. Then, for all \( t \in [0, T] \),
\[
\int_0^t H^{(n)}_s g(ns - [ns]) ds \xrightarrow{n \to \infty} \int_0^t g(r) dr \int_0^t H_s ds.
\]

Proof. The following evaluation is derived in a similar way to (C.1) and we use it several times in this proof:
\[
\int_0^t |g(ns - [ns])|^2 ds = \frac{[nt]}{n} \int_0^1 |g(r)|^2 dr + \int_0^{[nt]-[nt]} |g(r)|^2 dr \leq C,
\]
(C.3)
where \( C \) does not depend on \( n \). It follows from Minkowski’s inequality,
\[
\left\| \int_0^t H^{(n)}_s g(ns - [ns]) ds - \int_0^t g(r) dr \int_0^t H_s ds \right\|_{L_2} \\
\leq \left\| \int_0^t (H^{(n)}_s - H_s) g(ns - [ns]) ds \right\|_{L_2} + \left\| \int_0^t H_s \left( g(ns - [ns]) - \int_0^1 g(r) dr \right) ds \right\|_{L_2}.
\]
By the Cauchy-Schwarz inequality and (C.3), the first term on the right-hand side satisfies
\[
E \left[ \left( \int_0^t (H^{(n)}_s - H_s) g(ns - [ns]) ds \right)^2 \right] \leq \left( \int_0^t |g(ns - [ns])|^2 ds \right) E \left[ \int_0^t |H^{(n)}_s - H_s|^2 ds \right] \\
\leq CE \left[ \int_0^t |H^{(n)}_s - H_s|^2 ds \right] \to 0.
\]
Since \( H \) is continuous, according to (C.2),
\[
\left\| \int_0^t H_s \left( g(ns - [ns]) - \int_0^1 g(r) dr \right) ds \right\|_{L_2}^2 \to 0
\]
holds almost surely. We have also
\[
\left\| \int_0^t H_s \left( g(ns - [ns]) - \int_0^1 g(r) dr \right) ds \right\|_{L_2}^2 \leq \int_0^t H_s^2 ds \int_0^t \left( g(ns - [ns]) - \int_0^1 g(r) dr \right)^2 ds
\]

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by the Cauchy-Schwarz inequality. The hypothesis $\mathbb{E}\left[ \int_0^t H_s^2 ds \right] < \infty$ and the evaluation
\[
\int_0^t \left( g(ns - [ns]) - \int_0^1 g(r) dr \right)^2 ds \leq \int_0^t |g(ns - [ns])|^2 dr + 2T \left( \int_0^1 g(r) dr \right)^2 < \infty,
\]
which is derived from (C.3), enable us to apply the DCT with respect to the integration of $P$ to obtain
\[
\mathbb{E}\left[ \left( \int_0^t H_s \left( g(ns - [ns]) - \int_0^1 g(r) dr \right) ds \right)^2 \right] \to 0,
\]
which concludes the proof. □

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