Euclidean scalar and spinor
Green’s functions in Rindler space

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Abstract

In Rindler space, we consider the Feynman Green’s functions associated with either the Fulling-Rindler vacuum or the Minkowski vacuum. In Euclidean field theory, they become respectively the Euclidean Green’s functions $G_{\infty}$ and $G_{2\pi}$ whose we give different suitable forms. In the case of the massive spin-$\frac{1}{2}$ field, we determine also the Euclidean spinor Green’s functions $S_{\infty}$ and $S_{2\pi}$ in different suitable forms. In both cases for massless fields in four dimensions, we compute the vacuum expectation value of the energy-momentum tensor relative to the Rindler observer.

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1 Introduction

We consider a Rindler observer in an $n$-dimensional Minkowski spacetime $(n \geq 2)$ which has uniform acceleration, $g$. In the coordinate system $(\xi^0, \xi, x^i)$, $i = 1, .., n - 2$, with $\xi > 0$, Rindler space is described by the metric

$$ds^2 = -g^2\xi^2(d\xi^0)^2 + (d\xi)^2 + (dx^1)^2 + \cdots + (dx^{n-2})^2 \quad (1)$$

This coordinate system covers only a part of the Minkowski spacetime. The line given by $\xi = 1/g$ and $x^i = 0$ corresponds to the world line of the Rindler observer undergoing a uniform acceleration, $g$.

The free quantum field theory in Rindler space has been initially investigated by Fulling [1], Davies [2], Unruh [3] and also Candelas and Raine [4] and Dowker [5]. Due to the existence of the event horizon located at $\xi = 0$, the quantum field theory described in the Rindler metric is not equivalent to the usual field theory in the Minkowski metric. The Rindler observer moving in the Minkowskian vacuum sees a thermal bath of temperature $g/2\pi$. We use units in which $c = \hbar = k_B = 1$.

We recap briefly the definition of the Feynman Green’s functions in terms of Rindler metric (1). Apart from the usual Minkowski vacuum $|O_M>$, there also exists the Fulling-Rindler vacuum $|O_R>$ which is associated with the timelike Killing vector $\partial/\partial \xi^0$. Consequently, for a massive scalar field of mass $m$, we have the two corresponding Feynman Green’s functions $\Delta_R$ and $\Delta_M$, defined respectively by

$$\Delta^{(n)}_R(x, x_0; m) = <O_R | T\varphi(x_0)\varphi^+(x) | O_R> \quad (2)$$

$$\Delta^{(n)}_M(x, x_0; m) = <O_M | T\varphi(x_0)\varphi^+(x) | O_M> \quad (3)$$

where $T$ denotes the time-ordered product. We delete the factor $i$ and furthermore we choose to renormalise the vacua to one.

By performing a Wick rotation involving the acceleration $g$ given by

$$\xi^0 = -\frac{i\tau}{g} \quad (4)$$

metric (1) takes a Riemannian form

$$ds^2 = \xi^2(d\tau)^2 + (d\xi)^2 + (dx^1)^2 + \cdots + (dx^{n-2})^2 \quad (5)$$
in the coordinate system \((\tau, \xi, x^i)\), \(\xi > 0\). The properties of the Feynman Green’s functions in a static spacetime \([3]\) imply that they can be deduced by an analytic continuation from the Euclidean Green’s functions. In the metric \((5)\), we call respectively \(G_\infty\) and \(G_{2\pi}\) the corresponding Feynman Green’s functions \(\triangle_R\) and \(\triangle_M\).

These Green’s functions have already been considered in previous works but in the present paper we use the methods that we introduced in the analysis of the Euclidean Green’s functions in the spacetime describing a straight cosmic string \([4, 5]\). We can then derive a convenient form of both \(G_\infty\) and \(G_{2\pi}\) in order to straightforwardly compute the vacuum expectation value of the operators relative to the Rindler observer.

The plan of this work is as follows. In section 2, we determine the Euclidean Green’s functions \(G_{2\pi}\) and \(G_\infty\). We give different forms of the expression for \(G_\infty\) in section 3, in particular a convenient expression for \(G_{2\pi} - G_\infty\) which is valid for point \(x\) near \(x_0\) is obtained. We obtain the Euclidean spinor Green’s functions \(S_{2\pi}\) and \(S_\infty\) in section 4, in particular a convenient expression of \(S_{2\pi} - S_\infty\) valid for point \(x\) near \(x_0\). We compute in section 5 the vacuum expectation value of the energy-momentum tensor relative to the Rindler observer for a massless scalar or spin-\(\frac{1}{2}\) field in four dimensions. In section 6, we introduce the Euclidean Green’s function \(G_\beta\) and \(S_\beta\) corresponding to an arbitrary temperature \(1/\beta\).

## 2 Scalar Green’s functions

The Feynman Green’s function \(\triangle_R\) is easily constructed from the positive-frequency functions with respect to the coordinate \(\tau\) of the metric \((1)\); one finds that

\[
\Delta_R^{(n)}(x, x_0; m) = \int d^{n-2}k \frac{1}{(2\pi)^{n-2}} \exp[ik^i(x^i - x_0^i)] \times \int_0^\infty d\nu \frac{\sinh \pi \nu}{\pi^2} \exp[-i\nu(g | \xi^0 - \xi_0^0 | -i\epsilon)]K_{\nu}(\kappa_k \xi)K_{\nu}(\kappa_k \xi_0)
\]

where \(\kappa_k = \sqrt{k^i k^i + m^2}\), in the limit \(\epsilon \to 0\).

The Feynman Green’s function, \(\triangle_M\) is calculated from the development of the creation and annihilation operators associated with the Fulling-Rindler vacuum on the Minkowskian creation and annihilation operators; Spindel \([1, 2]\).
gives the following expression

\[
\Delta_m^{(n)}(x, x_0; m) = \int d^{n-2}k \frac{1}{(2\pi)^{n-2}} \exp[i k^i (x^i - x_0^i)] \times \\
\int_0^\infty d\nu \frac{1}{\pi^2} \cosh[\nu(\pi - |\xi_0^i - \xi_0^i| - \epsilon)] K_{iv}(\kappa k \xi_0^i) K_{iv}(\kappa k \xi_0^i)
\]

(7)
in the limit \(\epsilon \to 0\).

The Wick rotation (4) applied to expressions (6) and (7) yields the following expression for the Green’s function,

\[
G_\infty^{(n)}(x, x_0; m) = \int d^{n-2}k \frac{1}{(2\pi)^{n-2}} \exp[i k^i (x^i - x_0^i)] \times \\
\int_0^\infty d\nu \frac{1}{\pi^2} \sinh \frac{\pi \nu}{2} \exp[-\nu |\tau - \tau_0|] K_{iv}(\kappa k \xi_0^i) K_{iv}(\kappa k \xi_0^i)
\]

(8)

and also for the Green’s function, \(G_{2\pi}\)

\[
G_{2\pi}^{(n)}(x, x_0; m) = \int d^{n-2}k \frac{1}{(2\pi)^{n-2}} \exp[i k^i (x^i - x_0^i)] \times \\
\int_0^\infty d\nu \frac{1}{\pi^2} \cosh[\nu(\pi - |\tau - \tau_0|)] K_{iv}(\kappa k \xi_0^i) K_{iv}(\kappa k \xi_0^i)
\]

(9)

In fact, the multiple integral (9) can be explicitly calculated; one has merely

\[
G_{2\pi}^{(n)}(x, x_0; m) = \frac{m^{n/2-1}}{(2\pi)^{n/2} r_n^{n/2-1}} K_{n/2-1}(mr_n)
\]

(10)

where \(r_n = \sqrt{\xi^2 + \xi_0^2 - 2\xi \xi_0 \cos(\tau - \tau_0) + (x_1^1 - x_0^1)^2 + \cdots (x_{n-2}^2 - x_0^{n-2})^2}\). Evidently, \(G_{2\pi}\) is periodic in the coordinate \(\tau\) with period \(2\pi\).

Within Euclidean field theory, one must impose that the manifold defined by the metric (3) is regular. Hence, we require the coordinate \(\tau\) to be a periodic coordinate ranging from 0 to \(2\pi\). Thus, it coincides with the Euclidean space. The Green’s function \(G_{2\pi}\) is now the ordinary Green’s function in an Euclidean space. From expressions (8) and (9), one can prove that

\[
G_{2\pi}(\tau - \tau_0) = \sum_{n=-\infty}^{\infty} G_\infty(\tau - \tau_0 + 2\pi n)
\]

(11)
in which the dependence of the space coordinates is not indicated. In the quantum field theory at finite temperature, we can thus interpret \( G_{2\pi} \) as the thermal Euclidean Green’s function \([11, 12, 13]\) with respect to \( G_{\infty} \) which is the zero-temperature Green’s function. Considered in the spacetime \([11]\), the periodicity of \( G_{2\pi} \) in the coordinate \( \xi^0 \) is an imaginary period \( i2\pi/g \). This is the origin of the thermal character of the Minkowski vacuum perceived by the Rindler observer at temperature \( T_0 = g/2\pi \).

The vacuum expectation value of the operators relative to the Rindler observer is obtained from the Green’s function \( G_{2\pi} \), but the renormalisation is performed by removing the Green’s function \( G_{\infty} \). As in the Casimir effect, \( G_{\infty} \) plays the role of the local Green’s function and \( G_{2\pi} \) the global Green’s function \([10]\). In a symbolic manner, we write

\[
< O(x) >= O(G_{2\pi} - G_{\infty}) |_{x=x_0}
\]

(12)

where \( O \) is a differential operator in \( x \) and \( x_0 \).

### 3 Determination of \( G_{\infty} \)

We now search expressions for the Green’s function \( G_{\infty} \) which are more suitable than that given by equation (8). We start from \( n = 2 \) because we have the following recurrence relation between the Green’s functions

\[
G_{\infty}^{(n)}(\tau, \xi, x^1, \cdots, x^{n-2}, m) = \frac{1}{2\pi} \times \\
\int_{-\infty}^{\infty} dk G_{\infty}^{(n-1)}(\tau, \xi, x^1, \cdots, x^{n-3}; k^2 + m^2) \cos[k(x^{n-2} - x_0^{n-2})]
\]

(13)

for \( n \geq 3 \).

#### 3.1 Integral expression of \( G_{\infty} \)

For this purpose, we make use of the formula

\[
sinh(\pi \nu)K_{iu}(m\xi)K_{iu}(m\xi_0) = \frac{\pi}{2} \int_{\xi_2}^{\infty} du J_0[m(2\xi_0)^{1/2}(\cosh u - \cosh \xi_2)^{1/2}] \sin(\nu u)
\]

where \( \xi_2 \) is defined by

\[
\cosh \xi_2 = \frac{\xi_2^2 + \xi_0^2}{2\xi_0} \quad (\xi_2 \geq 0)
\]
By inserting this into (8) for \( n = 2 \), we have

\[
G^{(2)}_{\infty}(x, x_0; m) = \frac{1}{2\pi} \int_0^\infty d\nu \exp\left[ -\nu \mid \tau - \tau_0 \right] \times \\
\int_{\xi_2}^\infty d\nu J_0[m(2\xi_0)^{1/2}(\cosh u - \cosh \xi_2)^{1/2}] \sin(\nu u)
\]

Taking into account the formula

\[
\int_0^\infty dx \exp(-px) \sin(qx) = \frac{q}{p^2 + q^2} \quad (p > 0)
\]

we find an integral expression of \( G^{(2)}_{\infty} \) of the form

\[
G^{(2)}_{\infty}(x, x_0; m) = \frac{1}{2\pi} \times \\
\int_{\xi_2}^\infty d\nu J_0[m(2\xi_0)^{1/2}(\cosh u - \cosh \xi_2)^{1/2}] \frac{u}{u^2 + (\tau - \tau_0)^2} \quad (14)
\]

By virtue of the recurrence relation (13), we obtain in three dimensions

\[
G^{(3)}_{\infty}(x, x_0; m) = \frac{1}{2\pi^2(2\xi_0)^{1/2}} \times \\
\int_{\xi_3}^\infty d\nu \frac{\cos[m(2\xi_0)^{1/2}(\cosh u - \cosh \xi_3)^{1/2}]}{(\cosh u - \cosh \xi_3)^{1/2}} \frac{u}{u^2 + (\tau - \tau_0)^2} \quad (15)
\]

where \( \xi_3 \) is defined by

\[
\cosh \xi_3 = \frac{\xi^2 + \xi_0^2 + (x^1 - x_0^1)^2}{2\xi_0} \quad (\xi_3 \geq 0)
\]

Upon using again (13), we obtain in four dimensions

\[
G^{(4)}_{\infty}(x, x_0; m) = -\frac{1}{4\pi^2\xi_0} \int_{\xi_4}^\infty d\nu J_0[m(2\xi_0)^{1/2}(\cosh u - \cosh \xi_4)^{1/2}] \times \\
\frac{d}{du} \left[ \frac{u}{\sinh u(u^2 + (\tau - \tau_0)^2)} \right] \quad (16)
\]

where \( \xi_4 \) is defined by

\[
\cosh \xi_4 = \frac{\xi^2 + \xi_0^2 + (x^1 - x_0^1)^2 + (x^2 - x_0^2)^2}{2\xi_0} \quad (\xi_4 \geq 0)
\]
and so on as in [8]. We emphasize that (16) can be explicitly integrated in the case \( m = 0 \); we find

\[
D^{(4)}_\infty(x, x_0) = \frac{\xi_4}{4\pi^2\xi_0 \sinh \xi_4(\xi_4^2 + (\tau - \tau_0)^2)}
\]

(17)

which agrees with the result of Troost and Van Dam [14].

### 3.2 Local expression of \( G_\infty \)

As explained in section 2, the calculation of the vacuum expectation value is performed in the coincidence limit \( x = x_0 \). According to the formula (12), we must know \( G_{2\pi} - G_{\infty} \) for point \( x \) near point \( x_0 \). We anticipate and we say that the required subset of the Rindler manifold is defined by the condition

\[
| \tau - \tau_0 | < \pi
\]

(18)

satisfied of course when \( x \) tends to \( x_0 \).

For this purpose, we make use of the formula

\[
K_{\nu\nu}(m\xi)K_{\mu\mu}(m\xi_0) = \int_0^\infty du K_0[mR_2(u)] \cos(\nu u)
\]

where \( R_2(u) = \sqrt{\xi^2 + \xi_0^2 + 2\xi\xi_0 \cosh u} \). Insertion of this into (8) for \( n = 2 \) yields

\[
G^{(2)}_\infty(x, x_0; m) = \frac{1}{\pi^2} \times
\int_0^\infty d\nu \sinh(\pi\nu) \exp[-\nu | \tau - \tau_0 |] \int_0^\infty du K_0[mR_2(u)] \cos(\nu u)
\]

Due to the identity

\[
\sinh(\pi\nu) \exp(-\nu\psi) = \cosh[\nu(\psi - \pi)] - \cosh(\nu\psi) \exp(-\pi\nu)
\]

we can rewrite \( G^{(2)}_\infty \) as

\[
G^{(2)}_\infty(x, x_0; m) = G^{(2)}_{2\pi}(x, x_0; m) - \frac{1}{\pi^2} \times
\int_0^\infty du K_0[mR_2(u)] \int_0^\infty d\nu \cosh[\nu | \tau - \tau_0 |] \exp(-\pi\nu) \cos(\nu u)
\]

(19)
in which the integral converges under assumption (18). The \( \nu \)-integration in (19) can be performed with the aid of the formula

\[
\int_0^\infty d\nu \cosh(\nu \psi) \exp(-\pi \nu) \cos(\nu u) = \frac{1}{2} \left[ \frac{-\psi + \pi}{(\psi - \pi)^2 + u^2} + \frac{\psi + \pi}{(\psi + \pi)^2 + u^2} \right]
\]

Finally, we obtain the desired local expression of \( G^{(2)}_\infty \)

\[
G^{(2)}_\infty(x, x_0; m) = G^{(2)}_{2\pi}(x, x_0; m) + \frac{1}{4\pi^2} \times \int_0^\infty du K_0[mR_2(u)] F_\infty(u, \tau - \tau_0)
\]

where the function \( F_\infty(u, \psi) \), which is well defined everywhere, is given by

\[
F_\infty(u, \psi) = 2\left[ -\frac{\pi + \psi}{(\pi + \psi)^2 + u^2} + \frac{\psi - \pi}{(\psi - \pi)^2 + u^2} \right]
\]

By virtue of the recurrence relation (13), we obtain in three dimensions

\[
G^{(3)}_\infty(x, x_0; m) = G^{(3)}_{2\pi}(x, x_0; m) + \frac{1}{8\pi^2} \times \int_0^\infty du \frac{\exp[-mR_3(u)]}{R_3(u)} F_\infty(u, \tau - \tau_0)
\]

where \( R_3(u) = \sqrt{R_2^2(u) + (x^1 - x_0^1)^2} \). In four dimensions, we obtain

\[
G^{(4)}_\infty(x, x_0; m) = G^{(4)}_{2\pi}(x, x_0; m) + \frac{m}{8\pi^3} \times \int_0^\infty du \frac{K_1[mR_4(u)]}{R_4(u)} F_\infty(u, \tau - \tau_0)
\]

where \( R_4(u) = \sqrt{R_3^2(u) + (x^2 - x_0^2)^2} \) and so on as in [8], again in the subset defined by condition (18). In the case \( m = 0 \), expression (22) reduces to

\[
D^{(4)}_\infty(x, x_0) = D^{(4)}_{2\pi}(x, x_0) + \frac{1}{8\pi^3} \times \int_0^\infty du \frac{1}{[R_4(u)]^2} F_\infty(u, \tau - \tau_0)
\]
4 Spinor Green’s functions

In the quantum field theory for a spin-$\frac{1}{2}$ field in the Rindler spacetime, the
Rindler observer also sees a thermal bath of temperature $g/2\pi$ \cite{13, 16, 17}. It
is possible to use the Euclidean approach and in particular the theory at finite
temperature in which the Rindler manifold is regular. In this framework, we
must determine the Euclidean spinor Green’s functions $S_\infty$ et $S_{2\pi}$, expressed
in a vierbein well defined as that associated with the Cartesian coord inates,
which satisfy the two conditions

$$S_{2\pi}(\tau - \tau_0 + 2\pi) = -S_{2\pi}(\tau - \tau_0) \quad (24)$$

and

$$S_{2\pi}(\tau - \tau_0) = \sum_{n=-\infty}^{\infty} (-1)^n S_\infty(\tau - \tau_0 + 2\pi n) \quad (25)$$

In this interpretation, $S_{2\pi}$ represents the finite temperature spinor Green’s
function at the temperature $T_0 = g/2\pi$ and $S_\infty$ the zero-temperature Green’s
function. We point out that $S_{2\pi}$ is not the ordinary spinor Green’s function
$S_E$ in Euclidean space but it corresponds to the twisted spinor Green’s function $S_E^{(T)}$.

However in metric (3), we defines another vierbein $e^\mu_\underline{a}$ which is given by
the components

$$e^\mu_1 = (\frac{1}{\xi}, 0, ..., 0) \quad \text{et} \quad e^\mu_\underline{a} = \delta^\mu_\underline{a} \quad \underline{a} = 2, ..., n \quad (26)$$

The spinor connection $\Gamma_\mu$ then has the components

$$\Gamma_1 = -\frac{1}{4}(\gamma^2 \gamma^1 - \gamma^1 \gamma^2) \quad \text{et} \quad \Gamma_i = 0 \quad i = 2, ..., n \quad (27)$$

Taking into account the transformation laws of the spinor components, the
spinor Green’s functions can be expressed in terms of vierbein (26) in the
notations $S_\infty$ and $S_{2\pi}$. With this choice of vierbein, condition (24) takes the
following form on the spinor Green’s function $S_{2\pi}$

$$S_{2\pi}(\tau - \tau_0 + 2\pi) = S_{2\pi}(\tau - \tau_0) \quad (28)$$

and condition (25) becomes

$$S_{2\pi}(\tau - \tau_0) = \sum_{n=-\infty}^{\infty} S_\infty(\tau - \tau_0 + 2\pi n) \quad (29)$$
In \(n\) dimensions \((n \geq 2)\), we have determined in a previous work [18] the spinor Green’s function satisfying (28) in the notation \(S^{(n)}_{(T)}\) for the value \(B = 1\). We recall the result

\[
S_2(\pi) = (e^{\mu \gamma a \partial_\mu} + \frac{\gamma^2}{2\xi} - mI)\left[\text{I} \text{R} H_{2\pi} + \gamma^2 \gamma^{1} \text{I} \text{H}_{2\pi}\right]
\]

where

\[
H_{2\pi}(x, x_0; m) = \exp \left(\frac{\tau - \tau_0}{2}\right) G_{2\pi}^{(T)}(x, x_0; m)
\]

and where \(G_{2\pi}^{(T)}\) is the scalar Green’s function satisfying

\[
G_{2\pi}^{(T)}(\tau - \tau_0 + 2\pi) = -G_{2\pi}^{(T)}(\tau - \tau_0)
\]

In consequence, we obviously have

\[
G_{2\pi}^{(T)}(\tau - \tau_0) = \sum_{n=-\infty}^{\infty} (-1)^n G_{\infty}(\tau - \tau_0 + 2\pi n)
\]

where \(G_{\infty}\) is the scalar Green’s function given by expression (8) in section 3.2. Consequently, we can define the spinor Green’s function \(S_{\infty}\) by

\[
S_{\infty}(x, x_0; m) = (e^{\mu \gamma a \partial_\mu} + \frac{\gamma^2}{2\xi} - mI)\left[\text{I} \text{R} H_{\infty} + \gamma^2 \gamma^{1} \text{I} \text{H}_{\infty}\right]
\]

with

\[
H_{\infty}(x, x_0; m) = \exp \left(\frac{\tau - \tau_0}{2}\right) G_{\infty}(x, x_0; m)
\]

From relation (32), we deduce the property (29) which corresponds to the fundamental property (25).

As in the case of the scalar field, we calculate the vacuum expectation value relative to the Rindler observer by taking \(S_{2\pi} - S_{\infty}\) in the coincidence limit. In a symbolic manner, we write

\[
< O(x) >= \mathcal{O}^{(1/2)}(S_{2\pi} - S_{\infty}) \mid_{x=x_0}
\]

where \(\mathcal{O}^{(1/2)}\) is a differential operator in \(x\) and \(x_0\).

We now turn our attention to the derivation of an expression for \(S_{2\pi} - S_{\infty}\) which is valid for a point \(x\) in the neighbourhood of \(x_0\) that we can look for in the form

\[
S_{2\pi} - S_{\infty} = (S_{2\pi} - S_E) - (S_{\infty} - S_E)
\]
Taking into account the formulae \((30)\) and \((33)\), we can thus obtain the required expression by finding \(G_{2\pi}^{(T)} - G_{\infty}\) in the form
\[
G_{2\pi}^{(T)} - G_{\infty} = (G_{2\pi}^{(T)} - G_{2\pi}) - (G_{\infty} - G_{2\pi})
\]

In a previous work \([8]\), we have obtained in an arbitrary number of dimensions such an expression for \(G_{2\pi}^{(T)} - G_{2\pi}\), where \(G_{2\pi}^{(T)}\) is given under the notation \(G_{1/2}^{(n)}\) for the value \(B = 1\). On the other hand, the expression for \(G_{\infty} - G_{2\pi}\) has been given in section 3.2. So, we can determined the desired expression of \(G_{2\pi}^{(T)} - G_{\infty}\).

We confine ourselves to considering \(S_{2\pi}^{(4)}\) and \(S_{\infty}^{(4)}\) in four dimensions. We recall the expression for \(G_{2\pi}^{(4)(T)}\) subject to the condition \((18)\)
\[
G_{2\pi}^{(4)(T)}(x, x_0; m) = G_{2\pi}^{(4)}(x, x_0; m) + \frac{m}{8\pi^3} \times \int_0^{\infty} du \frac{K_1[mR_4(u)]}{R_4(u)} F^{(1/2)}_1(u, \tau - \tau_0)
\]

where the function \(F^{(1/2)}_1(u, \psi)\) is given by
\[
F^{(1/2)}_1(u, \psi) = -4 \cosh \frac{u}{2} \cos \frac{\psi}{2} \cosh u + \cos \psi
\]

Taking into account expression \((22)\), we obtain
\[
G_{2\pi}^{(4)(T)}(x, x_0; m) - G_{\infty}^{(4)}(x, x_0; m) = \frac{m}{8\pi^3} \times \int_0^{\infty} du \frac{K_1[mR_4(u)]}{R_4(u)} \times \left[ F^{(1/2)}_1(u, \tau - \tau_0) - F_{\infty}(u, \tau - \tau_0) \right]
\]

which is valid under condition \((18)\). In the case \(m = 0\), expression \((34)\) reduces to
\[
D_{2\pi}^{(4)(T)}(x, x_0) - D_{\infty}^{(4)}(x, x_0) = \frac{1}{8\pi^3} \times \int_0^{\infty} du \frac{1}{[R_4(u)]^2} \times \left[ F^{(1/2)}_1(u, \tau - \tau_0) - F_{\infty}(u, \tau - \tau_0) \right]
\]

which is, of course, valid under condition \((18)\).
5 Vacuum energy-momentum tensor \((n = 4)\)

We intend to compute the vacuum expectation value of the energy-momentum tensor relative to the Rindler observer in the coordinate system \((\tau, \xi, x^i)\) with the help of formulae (12) and (34). Since Rindler space is static we can then deduce from this the vacuum energy-momentum tensor in the coordinate system \((\xi^0, \xi, x^i)\) by using the Wick rotation (34).

We remark that we can also express these results in terms of a local temperature \(T\) given by

\[
T = \frac{T_0}{g\xi} \quad \text{or} \quad T = \frac{1}{2\pi\xi}
\]

for the value \(T_0 = g/2\pi\). It is convenient for all Rindler observers \(\xi = \text{const.}\) and \(x^i = 0\).

5.1 Case of the scalar field

At first, we calculate the mean-square field \(<\phi^2>\) using the formula

\[
<\phi^2(x)> = (G_{2\pi}^{(4)}(x, x_0; m) - G_{\infty}^{(4)}(x, x_0; m)) \big|_{x = x_0}
\]

From (22), we obtain

\[
<\phi^2(x)> = \frac{m}{4\pi^2} \int_0^\infty du \frac{K_1[\sqrt{2m}\xi \cosh(u/2)]}{\cosh(u/2)(\pi^2 + u^2)}
\]

(40)

For the case \(m = 0\), expression (34) reduces to

\[
<\phi^2(x)> = \frac{1}{8\pi^2\xi^2} \int_0^\infty du \frac{1}{\cosh(u/2)(1 + u^2)}
\]

(41)

Upon applying the formula

\[
\int_0^\infty \frac{1}{(1 + \cosh(\pi x))(1 + x^2)} = \frac{1}{12}\pi
\]

we have thereby

\[
<\phi^2(x)> = \frac{1}{48\pi^2\xi^2}
\]

(42)
The vacuum expectation value of the energy-momentum tensor is given by
\[ \langle T^{\mu \nu}(x) \rangle = T^{\mu \nu}(G_2(x, x_0; m) - G_\infty(x, x_0; m)) \bigg|_{x=x_0} \quad (43) \]
where \( T^{\mu \nu} \) is the following differential operator in \( x \) and \( x_0 \)
\[ T^{\mu \nu} = (1 - 2\Xi)\nabla_\mu \nabla^{\nu_0} - 2\Xi \nabla_\mu \nabla^{\nu} + (2\Xi - \frac{1}{2})\delta^{\nu}_{\mu}(m^2 + \nabla_\alpha \nabla^{\alpha_0}) \quad (44) \]
with \( \Xi \) the parameter of the scalar field theory under consideration. The application of formula (43) leads to well defined integrals. As an example for a conformally invariant scalar field, \( i.e. m = 0 \) and \( \Xi = 1/6 \), it is possible to integrate explicitly them. Without given details, we find
\[ \langle T^{\mu \nu}(x) \rangle = \frac{1}{1440\pi^2 \xi^4} \text{diag.}(-3, 1, 1, 1) \quad (45) \]
By the inverse of the Wick rotation (4), we deduce
\[ \langle T^{\mu_0}_{\xi_0}(x) \rangle = -\frac{1}{480\pi^2 \xi^4} \quad (46) \]
which gives a positive vacuum energy density according to the signature of metric (4).

Results (42) and (46) can be also expressed in the form
\[ \langle \phi^2(x) \rangle = \frac{1}{12}T^2 \quad \text{et} \quad \langle T^{\mu_0}_{\xi_0}(x) \rangle = -\frac{\pi^2}{30}T^4 \quad (47) \]
where the temperature \( T \) is given by (38).

5.2 Case of the spinor field

The vacuum expectation value of the energy-momentum tensor is given by
\[ \langle T^{\mu \nu}(x) \rangle = T^{(1/2)\mu \nu}(S^{(4)}(x, x_0; m) - S^{(4)}_{\infty}(x, x_0; m)) \bigg|_{x=x_0} \quad (48) \]
where \( T^{(1/2)\mu \nu} \) is the following differential operator in \( x \) and \( x_0 \)
\[ T^{(1/2)\mu \nu} = \frac{1}{4}tr[\gamma^{\mu}(e^{\nu}(\partial_\nu - \partial_{\nu_0}) + e^{\nu}(\partial_\mu - \partial_{\mu_0}))] \quad (49) \]
The application of formula (48) leads to well defined integrals. We confine ourselves to the case $m = 0$ in four dimensions. We examine firstly the component $< T^{x_1}_{x_1} >$. From (48), we find

$$< T^{x_1}_{x_1} (x) > = 4 \partial_{x_1,x_1} (H^{(4)}_{2\pi} (x, x_0) - H^{(4)}_{\infty} (x, x_0)) |_{x=x_0}$$

(50)

By substituting (37) into (50), we obtain

$$< T^{x_1}_{x_1} (x) > = \frac{1}{4\pi^3 \xi^4} \int_0^\infty du \frac{1}{(1 + \cosh u)^2} [ F^{(1/2)}_1 (u, 0) - F_\infty (u, 0) ]$$

which we can write as

$$< T^{x_1}_{x_1} (x) > = \frac{1}{\pi^3 \xi^4} \int_0^\infty du \frac{1}{(1 + \cosh u)^2} \left[ \frac{\cosh u/2}{1 + \cosh u} - \frac{\pi}{\pi^2 + u^2} \right]$$

By using the following integrals

$$\int_0^\infty dx \frac{1}{(1 + \cosh(\pi x))^2(1 + x^2)} = \frac{11}{360\pi}$$

$$\int_0^\infty dx \frac{1}{\cosh^5 x} = \frac{3}{16\pi}$$

we obtain finally

$$< T^{x_1}_{x_1} (x) > = \frac{3}{\pi^2 \xi^4} \left[ \frac{3}{64} - \frac{11}{360} \right]$$

(51)

Consequently, the vacuum energy-momentum relative to the Rindler observer has the expression

$$< T^\nu_{\mu} (x) > = \frac{47}{2880\pi^2 \xi^4} \text{diag.} (-3, 1, 1, 1)$$

(52)

since it must be traceless and conserved. Our result (52) does not agree with the one due to Candelas and Deutsch [15] and Dowker [19].

6 Fields at an arbitrary temperature

In the metric (3), the Euclidean scalar Green’s function $G_\beta$ at finite temperature $T_0$ is periodic in the coordinate $\tau$ with period $\beta$, where $\beta = 1/T_0$. 
When $\beta = 2\pi$, $G_\beta$ obviously coincides with $G_{2\pi}$. Moreover, $G_\infty$ is obtained as the limiting case of $G_\beta$ when $T_0$ tends to zero.

A transcript of our results [8] on the scalar Green’s function for a massive scalar field in the spacetime describing a straight cosmic string enable us to determine $G_\beta$. We must set $\tau = B\varphi$ and $\beta = 2\pi B$ in our formulae. We remark that, in the massless case in four dimensions, $D^{(4)}_\beta$ has been already given by Dowker [20].

We have in particular found a local form of $G_\beta$ which is convenient when one computes the vacuum expectation value of the operators. We set

$$<O(x)>_\beta = O(G_\beta(x, x_0; m) - G_\infty(x, x_0; m)) |_{x=x_0} \quad (53)$$

In the conformally invariant case, we find

$$<T^\nu_\mu(x)>_\beta = \frac{\pi^2}{90\xi^4\beta^4}\text{diag.}(-3, 1, 1, 1) \quad (54)$$

When $\beta = 2\pi$, we find again the energy-momentum tensor (45).

The Euclidean spinor Green’s function $S_\beta$ is antiperiodic in $\tau$ with period $\beta$. Likewise, a transcript of our results [18] on the twisted spinor Green’s function for a massive spin-$\frac{1}{2}$ field in the spacetime describing a straight cosmic string enable us to determine $S_\beta$, or more precisely $S_\beta$ in the choosen vierbein. We must set $\tau = B\varphi$ and $\beta = 2\pi B$ in the expression for the twisted spinor Green’s function. We also set

$$<O(x)>_\beta = O^{(1/2)}(S_\beta(x, x_0; m) - S_\infty(x, x_0; m)) |_{x=x_0} \quad (55)$$

We apply formula (53) for the energy-momentum tensor in the massless case by setting

$$S_\beta - S_\infty = (S_\beta - S_E) - (S_\infty - S_E)$$

Making use of previous calculations for $\beta > \pi$ [18, 21], we find

$$<T^\nu_\mu(x)>_\beta = \frac{1}{\xi^4}[-2w_4(\gamma) - \frac{11}{360\pi^2}]\text{diag.}(-3, 1, 1, 1) \quad (56)$$

where $w_4(\gamma)$ is the following expression

$$w_4(\gamma) = -\frac{1}{720\pi^2}\left\{11 - \frac{60\pi^2}{\beta^2}[4(\gamma - \frac{1}{2})^2 - \frac{1}{3}] + \frac{30\pi^4}{\beta^4}[16(\gamma - \frac{1}{2})^4 - 8(\gamma - \frac{1}{2})^2 + \frac{7}{15}]\right\} \quad (57)$$
where the parameter $\gamma$ is the fractional part of $1 - \beta/4\pi$ such that $0 \leq \gamma < 1$. We point out that for $\pi < \beta \leq 2\pi$

$$w_4(1 - \frac{\beta}{4\pi}) = -\frac{1}{720\pi^2} \left( -\frac{17}{8} + \frac{45\pi}{\beta} - \frac{10\pi^2}{\beta^2} - \frac{16\pi^4}{\beta^4} \right)$$  \hspace{1cm} (58)

and thus for $\beta = 2\pi$, we have

$$w_4\left(\frac{1}{2}\right) = -\frac{3}{128\pi^2}$$ \hspace{1cm} (59)

We have to take the limit of expression (57) when $\beta$ tends to infinity. We can easily prove that

$$\lim_{\beta \to \infty} w_4(\gamma) = -\frac{11}{720\pi^2}$$ \hspace{1cm} (60)

For $\beta = 2\pi$, by combining (59) and (60), we recover (52) from (56).

7 Conclusion

Within the Euclidean approach to scalar and spinor field theory, we have explicitly determined the Euclidean Green’s functions in the Rindler space-time. Our results enable us to compute straightforwardly the vacuum energy-momentum tensor relative to the Rindler observer, in particular for massless fields in four dimensions.

However, in the latter case for a spinor field, we have found an expression (52) for the vacuum energy-momentum tensor which is different to the one derived by two authors in previous papers [15, 19, 21].

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