In this paper we shall describe some correlation function computations in perturbative heterotic strings that, for example, in certain circumstances can lend themselves to a heterotic generalization of quantum cohomology calculations. Ordinary quantum chiral rings reflect worldsheet instanton corrections to correlation functions involving products of elements of Dolbeault cohomology groups on the target space. The heterotic generalization described here involves computing worldsheet instanton corrections to correlation functions defined by products of elements of sheaf cohomology groups. One must not only compactify moduli spaces of rational curves, but also extend a sheaf (determined by the gauge bundle) over the compactification, and linear sigma models provide natural mechanisms for doing both. Euler classes of obstruction bundles generalize to this language in an interesting way.

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References
1 Introduction

In this paper we shall describe some correlation function computations in (large radius) heterotic nonlinear sigma models with Kähler (but not necessarily Calabi-Yau) targets, as part of a program to generalize techniques developed to understand ordinary mirror symmetry, to (0,2) mirrors. We shall describe how to generalize the rational-curve-counting of type II theories to generalizations of the $\text{27}^3$ coupling in heterotic strings. We shall also check some of the predictions made recently in [1].

Recall that a perturbative heterotic string compactification is defined by not only a manifold, but also a bundle over that manifold. We will only consider holomorphic vector bundles over complex manifolds. To fix notation, we shall denote the target space by $X$, and we shall denote the holomorphic vector bundle on the target space $X$ by $E$. We shall assume throughout this paper that $E$ possesses the following two properties:

1. $\Lambda^{top}E^\vee \simeq K_X$
2. $\text{ch}_2(E) = \text{ch}_2(TX)$

The second of these properties is the well-known anomaly cancellation condition in heterotic strings, and we shall see that it plays an important role in defining the quantum product structure. The first of these properties plays a different and equally important role. We shall see that it is required to make sense of product structures (at both the classical and quantum levels), as well as for the additive structure of the heterotic chiral ring to be well-behaved. Also, when the target space is Calabi-Yau, this condition guarantees that a left-moving $U(1)$ symmetry (precisely the $U(1)_R$ on the (2,2) locus) is nonanomalous. The left-moving $U(1)$ in question is used to build representations of the full low-energy gauge group. Hence for Calabi-Yau compactifications we preferentially consider holomorphic vector bundles with vanishing first Chern class. In any event, both of the properties above – the anomaly cancellation condition as well as the condition on first Chern classes – will play an important role in our analyses.

Technically, although we will sometimes speak of heterotic chiral ‘rings,’ following [1], in general one will not always have a well-behaved ring structure. It would be more accurate to say that we will study correlation functions between certain operators in (0,2) models,
which in certain special cases (e.g. the gauge bundle is a deformation of the tangent bundle) can be interpreted as corresponding to a ring structure. If the gauge bundle is a deformation of the tangent bundle, then this ring is a deformation of the usual $(2,2)$ chiral ring. The ring structure encodes the correlation functions in the same manner that the $(2,2)$ correlation functions are encoded in the quantum cohomology ring.

A related point concerns the possibility of loop corrections. When the target is Calabi-Yau, then our three-point correlation functions correspond to spacetime superpotential terms, and there is an old nonrenormalization theorem based on Peccei-Quinn symmetries that spacetime superpotentials do not receive loop corrections in $\alpha'$, only worldsheet instanton corrections (see e.g. [4]). However, in this paper we are also interested in massive two-dimensional theories in which the target space is not Calabi-Yau, and we do not have a suitable generalization of the nonrenormalization result to non-Calabi-Yau spaces. Thus, in this paper we compute worldsheet instanton corrections to certain correlation functions, designed to generalize mathematical computations, but sometimes, some of these correlation functions might also conceivably get loop corrections in addition. As our purpose in this note is to generalize some old mathematical computations, we will not speak of this possibility further.

Another minor technical issue revolves around the spacetime interpretation of our correlation functions. When the target is Calabi-Yau, our three-point functions compute superpotential terms. However, the spacetime superpotential is not a function, but rather a section of a line bundle over the CFT moduli space [5]. This fact is realized on the worldsheet via $U(1)_R$ rotations as one moves about on CFT moduli space [6]. The ‘gauge choice’ needed to uniquely define a superpotential term at any point in CFT moduli space therefore amounts to a choice of normalization of the vertex operators, which cannot be extended globally over CFT moduli space unless the line bundle to which the superpotential couples is trivial. We will not speak further of this issue in this paper.

We begin in section 2 with a review of the half-twisted $(0,2)$ model in which we shall be computing correlation functions, which reduces to the A model twist on the $(2,2)$ locus. In section 3 we describe the states that will enter into our correlation functions – essentially, generalizations of $27$’s. In section 4 we describe classical (no worldsheet instantons) computations of the correlation functions, and in section 5 we give a formal discussion of how to take into account worldsheet instantons. Our calculations generalize the rational curve counting of type II theories, and in particular, the obstruction sheaf story seems to generalize in an intriguing fashion. To make sense of our formal calculations, we must compactify the moduli space of worldsheet instantons and describe how to extend certain sheaves (induced by the gauge bundle) over the compactification. In section 6 we review how linear sigma models can be used to provide a natural compactification, and furthermore how they naturally define and extend the relevant sheaves, in a fashion compatible with symmetries. Finally in section 7 we apply this technology to check some predictions of [1].
We should mention some related work. For example, considerations of \((2,2)\) sigma models on supermanifolds [7] lead one to study holomorphic vector bundles on Calabi-Yau manifolds, and their extensions over moduli spaces of rational curves. Some related work in heterotic compactifications is [3, 8, 9, 10], where correlation functions of gauge singlets were studied (and shown to vanish). In this paper, we are concerned with heterotic correlation functions between vertex operators for charged states – no gauge singlets appear in our correlation functions, and so the vanishing results of [3, 8, 9, 10] are not relevant.

Finally, it should be noted throughout this paper that we will not couple our models to worldsheet gravity – our computations are performed for a fixed complex structure on the worldsheet – and furthermore, we will only consider genus zero worldsheets.

2 Half-twisted \((0,2)\) theory

In typical quantum cohomology calculations, the worldsheet theory is the A-model topological field theory. A \((0,2)\) theory cannot be twisted to give a completely topological theory, but we can do most of the twisting, and will recover most of the corresponding results.

In a standard \((0,2)\) theory, one has right-moving fermions \(\psi_+\) coupling to \(\sqrt{K_\Sigma} \otimes \phi^*TX\) and left-moving fermions \(\lambda_-\) coupling to \(\sqrt{K_\Sigma} \otimes \phi^*E\), where \(\Sigma\) denotes the worldsheet. We define the “half-twisted theory” by coupling fermions to bundles as follows:

\[
\begin{align*}
\psi^+_+ & \in \Gamma_{C^\infty} \left( \phi^*T^{1,0}X \right) \\
\psi^+_- & \in \Gamma_{C^\infty} \left( K_\Sigma \otimes \left( \phi^*T^{1,0}X \right)^\vee \right) \\
\lambda^a_- & \in \Gamma_{C^\infty} \left( K_\Sigma \otimes \left( \phi^*E \right)^\vee \right) \\
\lambda^{-a} & \in \Gamma_{C^\infty} \left( \phi^*E \right)
\end{align*}
\]

and it is this half-twisted theory that we shall be studying throughout this paper. In other words, just as in a standard topological twist, we make the worldsheet spinors into worldsheet scalars and vectors. Note that although we perform this operation on both left- and right-movers, the resulting theory will not be a topological field theory in general, since it only has \((0,2)\) worldsheet supersymmetry, and not \((2,2)\). Also note that on the \((2,2)\) locus, where \(E = TX\), the twisting above is equivalent to the A model twist. See [8] for another discussion of half-twisted theories in heterotic compactifications.

One of the reasons two-dimensional topological field theories are useful is that they simplify the computation of correlation functions (see e.g. [11]). Many three-point functions

---

\(^4\)We have used Hermitian metrics on \(X\) and \(E\) to write \(\phi^*T^{0,1}(X)\) as \((\phi^*T^{1,0}(X))^\vee\) and \(\phi^*E\) as \((\phi^*E)^\vee\). We do this to make a Riemann-Roch calculation in Section 4 more transparent.
in physical (type II) theories are the same as certain three-point functions in corresponding topological field theories. Schematically,

\[ <\psi\psi\phi>_{\text{phys}} = <\psi\psi\psi>_{\text{tft}} \]

In a nutshell, the reason for this equivalence is that the spectral flow insertion used to generate the vertex operator for the spacetime boson \( \phi \) above from the vertex operator for the corresponding spacetime spinor can also be interpreted as generating the topological field theory.

Although the mechanism for relating physical and topological correlation functions is well-known, let us pause for a moment to very briefly and schematically review it. In a nutshell, to twist the worldsheet theory means adding a term proportional to \( \int_{\Sigma} \frac{1}{2} \omega \psi \psi \) to the worldsheet action, where \( \omega \) is the worldsheet spin connection. Now, \( \psi \psi \) is proportional to the \( U(1) \) current \( J \). If we bosonize \( J \sim \partial \phi \), then the term added looks like \( \int_{\Sigma} R \phi \). By concentrating the worldsheet curvature at points, so that \( R \sim \delta^2(z-z_0) \), we see that topological twisting is essentially the same as inserting factors of the form \( \exp(\phi) \), which is spectral flow.

Analogous statements can also be made relating correlation functions in physical \((0,2)\) theories to corresponding correlation functions in the half-twisted theory described above. Recall that a three-point correlation function in a physical \((0,2)\) theory has two aspects, coming from the left- and right-movers. In a three-point function the right-movers behave just as in a type II theory, as described above. The left-movers encode gauge information, and require an analogue of spectral flow, which can also be alternately interpreted as twisting. For example, suppose we have an irreducible rank three holomorphic vector bundle, and we wish to compute a \( \mathbf{27}^3 \) coupling\(^5\). The bundle breaks \( E_8 \) to \( E_6 \), and the \( E_6 \) is built on the worldsheet from \( SO(10) \times U(1) \), the \( SO(10) \) from the left-moving fermions in the \( E_8 \) but not coupled to the bundle, and the \( U(1) \) from a left-moving symmetry\(^6\) on the fermions coupled to the bundle, under which \( \lambda^a \) and \( \lambda^\bar{a} \) have equal and opposite charges. (On the \((2,2)\) locus, this \( U(1) \) is the \( U(1)_R \) of the left-moving \( \mathcal{N} = 2 \) algebra.) Just as a physical three-point correlation function involves both spacetime spinors and a spacetime boson, one analogously uses both the \( \mathbf{16} \) and \( \mathbf{10} \) representations of \( SO(10) \) that are part of the \( \mathbf{27} \) of \( E_6 \):

\[ \mathbf{27} = \mathbf{10} + \mathbf{16} + \mathbf{1} \]

under \( SO(10) \times U(1) \). In particular, the physical \( \mathbf{27}^3 \) correlation function can be computed in the form

\[ <\psi_{\mathbf{16}}\psi_{\mathbf{10}}\phi_{\mathbf{16}}>_{\text{phys}} \]

\(^5\)On the \((2,2)\) locus, the \( \mathbf{27}^3 \) gets worldsheet instanton corrections, whereas the \( \mathbf{27}^3 \) does not. On the \((2,2)\) locus, the \( \mathbf{27}^3 \) corresponds to an A model calculation, whereas the \( \mathbf{27}^3 \) corresponds to a B model computation.

\(^6\)It is straightforward to compute that in a physical \((0,2)\) theory on a Calabi-Yau, this left-moving \( U(1) \) will be nonanomalous precisely when \( c_1(\mathcal{E}) = 0 \).
Just as in the right-movers, the vertex operator for $\phi$ could be obtained from that for $\psi$ via spectral flow, similarly the left-moving 10 can be obtained from the left-moving 16 through the action of a left-moving analogue. Just as in (2, 2) theories, where these spectral flow insertions could be interpreted as giving topological twists, in (0, 2) theories these insertions of (analogues of) spectral flow can also be interpreted as giving the half-twisted theory, hence

$$<\psi^{16}\psi^{10}\phi^{10}>_{\text{phys}} = <\psi\psi\psi>_{\text{half-twist}}.$$ 

Because the half-twisted theory simplifies computations of correlation functions as described above, in this paper we shall use it exclusively to compute correlation functions.

In computations of rational curve corrections in type II theories, the A model is often coupled to worldsheet topological gravity in the literature. In this paper, we will not couple to topological gravity. Furthermore, when computing $n$-point correlation functions on $\mathbb{P}^1$ for $n > 3$, we shall not use descendants of $n-3$ of the vertex operators; we shall compute products in the purely half-twisted (0, 2) theory.

### 3 Relevant massless states in (0,2) theories

In a type II string theory, the chiral ring consists of massless RR sector states, which (at large radius) are counted by Dolbeault cohomology $H^{p,q}(X)$ of the target space $X$, or, equivalently, elements of the sheaf cohomology groups

$$H^q(X, \Lambda^p T^* X).$$

By Hodge theory, these Dolbeault cohomology groups generate the full de Rham cohomology.

In a heterotic string compactification on a space $X$ with gauge bundle $\mathcal{E}$, recall from [12, section 3] that the charged massless RR states are counted (at large radius) by

$$H^q(X, \Lambda^p \mathcal{E}^\vee)$$

which is clearly analogous to the type II string result. In the special case that $\mathcal{E} = TX$, we duplicate the type II result.

We shall sometimes, when sensible, speak of a heterotic chiral ring associated to certain (0, 2) heterotic theories. This ring will be described additively by the sum of sheaf cohomology groups of the form above, i.e.

$$H^*_{\text{het}} \equiv \sum_{p,q} H^q(X, \Lambda^p \mathcal{E}^\vee)$$

Note that this ring is naturally bigraded. Physically, that bigrading corresponds to the distinction between left- and right-movers. The heterotic chiral ring is associated to part of
the gauge sector of the heterotic theory. The multiplication in the ring is described by the correlation functions.

In principle, there are additional massless states in a large radius heterotic compactification – there are gauge singlets that correspond to complex, Kähler, and bundle moduli, as well as additional charged matter fields, such as elements of $H^n(X, \text{End} \mathcal{E})$ come from the NS-R sector. However, we shall not consider such fields here.

Clearly we have states of the form $H^1(X, \mathcal{E}^\vee)$ in our state space, the $(0, 2)$ generalization of $H^1(T^\vee)$, corresponding to Kähler moduli on the $(2, 2)$ locus. Note that the $(0, 2)$ theory does still contain Kähler moduli, but in the $(0, 2)$ theory those states are strictly gauge neutral. A $(2, 2)$ theory contains both gauge neutral as well as gauge charged states corresponding to complex structure moduli – i.e. in addition to singlet moduli fields in the target space theory corresponding to complex moduli, there are also charged $27$’s corresponding to complex moduli. A $(0, 2)$ theory contains singlets corresponding to complex moduli, but even for a rank three $\mathcal{E}$, the $27$’s are counted by $H^1(X, \mathcal{E})$ and not $H^1(X, T)$.

In the introduction we mentioned that, in addition to the anomaly cancellation condition, throughout this paper we shall impose the constraint that

$$\Lambda^{top} \mathcal{E}^\vee \cong K_X. \tag{1}$$

We can see one motivation for this constraint already in the additive structure of our chiral states. Recall that Serre duality in $(0, 2)$ theories maps spectra back into themselves [12]. Serre duality acts as

$$H^i \left( X, \Lambda^j \mathcal{E}^\vee \right) \cong H^{n-i} \left( X, \Lambda^j \mathcal{E} \otimes K_X \right)^*$$

$$\cong H^{n-i} \left( X, \Lambda^{r-j} \mathcal{E}^\vee \otimes \Lambda^r \mathcal{E} \otimes K_X \right)^*$$

where $r$ is the rank of $\mathcal{E}$. Notice that when the line bundle

$$\Lambda^r \mathcal{E} \otimes K_X$$

on $X$ is trivial, i.e. $\Lambda^r \mathcal{E}^\vee \cong K_X$, Serre duality still closes our states back into themselves, but if this line bundle is nontrivial, Serre duality does not close the states back into themselves. Thus, just to make the additive structure well-behaved under Serre duality, we must require the constraint above.

The condition (1) also impacts correlation functions. For definiteness, let $E$ have rank $r$ and let $X$ have dimension $n$. Note that the condition $\Lambda^r \mathcal{E}^\vee \cong K_X$ implies that there is an isomorphism

$$\phi : H^n(X, \Lambda^r \mathcal{E}^\vee) \to \mathbb{C}.$$ 

This isomorphism already allows classical products of operators in the chiral ring to be evaluated, and we see that it will similarly be crucial in the evaluation of the quantum products.
For Calabi-Yau compactifications, another physical motivation for this constraint comes from the fact that our bundles must be embeddable inside $E_8$. Although $SU(n)$ has natural $E_8$ embeddings for small $n$ which cleanly correspond to worldsheet physics, realizing $U(n)$ embeddings on the worldsheet is not as well understood. In particular, for a $U(n)$ embedding there is a worldsheet anomaly in a left-moving $U(1)$ symmetry used to build vertex operators. Thus, although one can work with $(0,2)$ CFT’s with $c_1 \neq 0$, to make them useful for a compactification one typically adds extra left-movers to cancel out the $c_1$, hence as a practical matter we only consider holomorphic vector bundles with $c_1 = 0$.

4 Classical correlation functions

In a $(2,2)$ chiral ring, the classical product is obtained by wedging together enough differential forms to get a top form on the target space $X$, which can then be integrated over $X$ to produce a number. In other words, for the product of $k$ operators we have the maps

$$H^{q_1}(X, \Lambda^{p_1} T^* X) \otimes \cdots \otimes H^{q_k}(X, \Lambda^{p_k} T^* X) \longrightarrow H^n(X, \Lambda^n T^* X) \cong \mathbb{C} \quad (2)$$

where $\sum q_i = \sum p_i = n$. The first map is given by cup and wedge products, and the second map is given by integration of a top form on $X$. This mathematical analysis is backed-up physically by e.g. ghost number conservation, which implies that the only nonzero correlation functions can come from top-degree forms.

The classical correlation functions in our heterotic theory involve a product of states such that the sums of the degrees of the sheaf cohomology groups is the dimension $n$ of the target-space $X$, and that the number of $\mathcal{E}$’s in the coefficients equals the rank $r$ of $\mathcal{E}$. Indeed, wedging together representative differential forms, from such a product we get an element of

$$H^n(X, \Lambda^r \mathcal{E}^\vee) \quad (3)$$

This is the analogue of a top-degree form in the present case.

However, we can only get a number from this top-degree form in special circumstances. In the special case that $\Lambda^r \mathcal{E}^\vee \cong K_X$, then we can identify an element of (3) with a top form on $X$, which can then be integrated to get a number.

Next, let us take a moment to consider the physical situation more carefully.

In the half-twisted theory, as described in section 2, the $\psi'_+ \downarrow$ couple to $\phi^* T^{1,0} X$ and the $\psi'_+ \uparrow$ couple to $K_{\Sigma} \otimes (\phi^* T^{1,0} X)^\vee$, hence the number of $\psi'_+$ zero modes, minus the number of $\psi'_+$ zero modes, is given by Hirzebruch-Riemann-Roch as

$$\chi \left( \phi^* T^{1,0} X \right) = c_1(\phi^* TX) + n(1 - g)$$

10
This is the anomaly in the (right-moving) $U(1)_R$ symmetry. (Compare e.g. [1] above equation (154).)

This anomaly gives us a selection rule that is responsible for the top-degree-form constraint when there are no worldsheet instantons. Consider the case that our vacuum consists of constant maps $\phi$, \textit{i.e.} no worldsheet instantons. Then $\phi^*TX$ is a trivial bundle over the worldsheet, so $c_1(\phi^*TX) = 0$, hence $\chi = n(1 - g)$. Thus at string tree level there are $n$ $\psi^*$ zero modes, hence the degrees of the sheaf cohomology groups must add up to $n$ ($= \dim X$) in order to hope to get a nonzero result for the correlation function.

As outlined in section 2, to most efficiently see the product structure in the chiral ring, we also want to consider the case that the left-moving fermions are also twisted, so that $\lambda^e$ couples to $K_{X} \otimes (\phi^*E)^\vee$ and $\lambda^\pi$ couples to $\phi^*E$. Here we compute an anomaly

$$\chi(\phi^*E) = c_1(\phi^*E) + r(1 - g).$$

When we are expanding about constant $\phi$, we have $c_1(\phi^*E) = 0$. Thus at string tree level we recover the rule that for a correlation function to be nonvanishing, the number of $\lambda^e$’s appearing must equal the rank of $E$.

In any event, from counting zero modes of worldsheet fermions, we see that when there are no worldsheet instantons, for a product to be nonvanishing the sum of the degrees of the sheaf cohomology groups must match the dimension of the target, and the sum of the exterior powers of the coefficients must match the rank of $E$. In other words, for a product to be non-vanishing, the wedge product of differential forms must give us the analog of a top form, as we outlined earlier.

As noted above, in the special case that $\Lambda^eE^\vee \cong K_X$, this wedge product living in (3) can be identified with a top form and subsequently integrated to get a number. A precise choice of trivialization of $\Lambda^eE \otimes K_X^\vee$ is needed to normalize correlation functions. This is analogous to the $(2, 2)$ chiral ring, where a precise choice of nowhere-zero holomorphic top form is needed to normalize the correlation functions.

Thus, the constraint $\Lambda^eE^\vee \cong K_X$ not only makes the path integral well-defined, but is also used in order to be able to define the classical product. We will see the same constraint in defining quantum products.
5 Quantum correlation functions – formal discussion

5.1 Generalities

The selection rules derived above from anomalies no longer apply in the presence of world-sheet instantons, but rather are corrected. Previously if the sum of the degrees of the sheaf cohomology groups was greater than the dimension of the manifold, the resulting correlation function must vanish, reflecting a basic property of the (‘classical’) Yoneda product, really just a property of wedge products. In an instanton background, however, it is possible to have a nonzero correlation function even when that sum of degrees is strictly greater than the dimension of the target manifold. This is what gives rise to “quantum” products – products that are nonzero, but become zero when $\alpha' \to 0$.

Suppose that we want to compute a correlation function associated to elements of the heterotic chiral ring. Specifically, recall that the selection rule on right-movers is that the sum of the degrees of the sheaf cohomology groups must equal

$$c_1(\phi^*TX) + n(1 - g)$$

where $n$ is the dimension of the target and $g$ is the genus of the worldsheet, and the sum of the powers of the bundle $E^\vee$ appearing must be

$$c_1(\phi^*E) + r(1 - g)$$

where $r$ is the rank of $E$. In a one-instanton background, neither $c_1(\phi^*TX)$ nor $c_1(\phi^*E)$ need vanish.

There is a significant complication, not present in the $(2,2)$ case, coming from operator determinants. In a $(2,2)$ topologically-twisted theory, the operator determinants for the left- and right-moving worldsheet fermions precisely cancel against the operator determinant for the worldsheet bosons. In a physical theory, no such cancellation occurs, and certainly in a $(0,2)$ theory, twisted or not, no such cancellation is possible. (See also [9] for a more extensive discussion of this issue.) Thus, all string tree level correlation functions necessarily contain numerical factors amounting to the partition function of left-moving fermions on $S^2$.

If these numerical factors were the same for all worldsheet instanton contributions, then we could ignore them, or absorb them into redefinitions. However, these numerical factors for any one worldsheet instanton depend upon the restriction of $E$ to that rational curve, as after all they represent the partition function for left-moving fermions coupling to $E$. If two different rational curves have non-isomorphic restrictions of $E$, then the numerical factors will be different.

Far worse now is the case of a family of curves, in which the cohomology of the restriction of $E$ can jump in the family [13]. If the cohomology of the restriction of $E$ jumps, then surely
the corresponding numerical factor jumps, and so the required integral over the moduli space of zero modes can not have nearly so simple a form as appeared in [14].

We shall ignore this possibility, and work only in $g = 0$ (at fixed complex structure), so that this determinant is just a number that can be reabsorbed into other definitions. We will also ignore group theory factors associated to the representations of the unbroken gauge group needed to specify the gauge sector contributions to the operators.

In the rest of this section we shall only work formally, on smooth not necessarily compact spaces of maps of fixed degree from the worldsheet into $X$ of fixed degree. Our goal in this section is to outline the general ideas, not to fill in all details. In section 6 we shall describe how to compactify the moduli spaces and extend these constructions over those compactifications.

5.2 Integration over moduli space

In this subsection, we shall formally outline how to compute rational curve corrections in the special case that there are no excess zero modes, that there is no analogue of the obstruction sheaf. (We shall make this restriction precise momentarily.) The general case will be discussed in section 5.5. Readers unfamiliar with these computations for $(2,2)$ theories might wish to consult [15, section 3.3], [16, section 3.3] for a very readable review.

In this section we shall only work formally. We consider an idealized situation which is almost never realized: we suppose that we have a smooth and compact instanton moduli space $\mathcal{M}$ of maps from the fixed worldsheet $\Sigma$ into $X$ of fixed degree. On the other hand, it is quite common to find noncompact but smooth moduli spaces. In section 6 we shall describe how to compactify the moduli spaces and extend these constructions over those compactifications.

Recall in the $(2,2)$ case, the chiral ring is composed of $(p,q)$ forms on the target space $X$. There is a selection rule that correlation functions are nonzero when the sum of the $p$’s and $q$’s separately equals

$$(\dim X) (1 - g) + c_1(\phi^*TX)$$

(the same as the selection rule (4) on the degrees of sheaf cohomology groups). Each of those $(p,q)$ forms on $X$ is associated to a $(p,q)$ form on the moduli space $\mathcal{M}$ of worldsheet instantons of given degree, and the selection rule above becomes the statement that the wedge product of those differential forms on $\mathcal{M}$ is a formally a top form, since formally the dimension of $\mathcal{M}$ is given by

$$(\dim X) (1 - g) + c_1(\phi^*TX).$$

In the heterotic case, with a rank $r$ bundle $\mathcal{E}$ obeying $\Lambda^r \mathcal{E}^\vee \cong K_X$ as well as the anomaly
cancellation condition, elements of the sheaf cohomology groups
\[ H^p(X, \Lambda^q \mathcal{E}^\vee) \]
are mapped to sheaf cohomology groups
\[ H^p(\mathcal{M}, \Lambda^q \mathcal{F}^\vee) \]
on the moduli space \( \mathcal{M} \), where \( \mathcal{F} \) is a sheaf on \( \mathcal{M} \) of rank \( \Gamma(X, \phi^* \mathcal{E}) \). We will discuss the precise map in the next subsection, but for the purpose of outlining the general ideas, for now we merely assert such a map exists.

Let us assume that we have a universal instanton \( \alpha : \Sigma \times \mathcal{M} \to X \), where \( \Sigma \) is the worldsheet.\(^7\) We have the natural projection \( \pi : \Sigma \times \mathcal{M} \to \mathcal{M} \). Now we can define a sheaf \( \mathcal{F} \) on \( \mathcal{M} \) by
\[ \mathcal{F} = \pi_\ast \alpha^* \mathcal{E}. \]
In this section, we will assume that
\[ R^1 \pi_\ast \alpha^* \mathcal{E} = 0 = R^1 \pi_\ast \alpha^* T\Sigma. \tag{6} \]
In the case where \( \mathcal{E} = T\Sigma \), we have that \( \mathcal{F} = T\mathcal{M} \), as the fiber of \( \mathcal{F} \) at a point of \( \mathcal{M} \) corresponding to an instanton \( \phi : \Sigma \to X \) is \( H^0(\Sigma, \phi^* T\Sigma) \), the space of first-order deformations of the map \( \phi \).\(^8\) The first equality in (6) tells us that the map \( \phi \) is unobstructed. In particular \( \mathcal{M} \) is smooth. The second equality is the natural extension. By Riemann-Roch, this condition implies that the fibers of \( \mathcal{F} \) have the same dimension, so that \( \mathcal{F} \) is a vector bundle of rank given in equation (5).

Physically the assumption (6) means that there are no excess zero modes. We will discuss the more general case in section 5.5.

Under the assumption (6), we can compute \( c_1(\mathcal{F}) \) by Grothendieck-Riemann-Roch. Letting a subscript of \( k \) on a cohomology class denote its complex codimension \( k \) component, and letting \( \eta \) be the pullback to \( \Sigma \times \mathcal{M} \) of the cohomology class of a point of \( \Sigma \), we have
\[ c_1(\mathcal{F}) = \text{ch}(\mathcal{F})_1 = \pi_\ast \left( (\text{ch}(\alpha^* \mathcal{E}) \text{Td}(T\Sigma))_2 \right) = \pi_\ast \left( \alpha^* \left( c_1(\mathcal{E})^2/2 - c_2(\mathcal{E}) \right) + (1 - g) \eta \alpha^* c_1(\mathcal{E}) \right). \tag{7} \]
In (7) we have used \( \text{ch}_2(\mathcal{E}) = c_1^2(\mathcal{E})/2 - c_2(\mathcal{E}) \).

If we now apply (7) to \( \mathcal{E} = T\Sigma \) we obtain
\[ c_1(T\mathcal{M}) = \pi_\ast \left( (\text{ch}(\alpha^* T\Sigma) \text{Td}(T\Sigma))_2 \right) = \pi_\ast \left( \alpha^* \left( c_1(T\Sigma)^2/2 - c_2(T\Sigma) \right) + (1 - g) \eta \alpha^* c_1(T\Sigma) \right). \]

\(^7\)Here is where we will find it hard to enforce the compactness of \( \mathcal{M} \) in practice. In a typical situation maps from the worldsheet can degenerate to maps from the worldsheet with some 2-spheres attached.

\(^8\)Note that we are not considering stable maps here as we are not identifying maps differing by an automorphism of \( \Sigma \). As we have not coupled to worldsheet gravity, worldsheet automorphisms are irrelevant. Unlike topological \textit{string} theories, in a topological field theory, even at genus zero one can have nonzero two-point couplings.
By the anomaly condition together with (1), we conclude that \( c_1(\mathcal{F}) = c_1(TM) \), implying\(^9\) that \( \Lambda^{top}F^\vee \cong K_M \), as desired.

The selection rules (4) and (5) then imply that we can get nonvanishing correlation functions by a straightforward extension of the procedure (2), replacing \( X \) by \( \mathcal{M} \) and \( T^*X \) by \( F^\vee \):

\[
H^q(\mathcal{M}, \Lambda^p F^\vee) \otimes \cdots \otimes H^q(\mathcal{M}, \Lambda^p F^\vee) \longrightarrow H^N(\mathcal{M}, \Lambda^R F^\vee) \cong H^N(\mathcal{M}, K_M) \cong \mathbb{C}, \quad (8)
\]
given by cup and wedge product of cohomology classes followed by integration of a top form.

Technically, as mentioned in the previous section, the correlation function should be more properly described as

\[
\int_M \left( \frac{\det \bar{\partial}_{\phi^*E}}{\det \bar{\partial}_{\phi^*TX}} \right) H^{top}(\mathcal{M}, K_M)
\]
instead of merely

\[
\int_M H^{top}(\mathcal{M}, K_M).
\]
But at genus zero, the ratio of operator determinants is just a number, which we shall suppress. We shall also studiously ignore the possibility that the splitting behavior of \( \phi^*E \) changes over \( \mathcal{M} \) in such a way that this ratio changes on a set of nonzero measure on \( \mathcal{M} \). Understanding this ratio of operator determinants would be much more important for calculations at higher genus, and for properly understanding coupling to worldsheet gravity.

In (8), \( N \) is the dimension of \( \mathcal{M} \) as given by the selection rule (4) and \( R \) is the rank of \( \mathcal{F} \) as given by the selection rule (5). Also, note that the anomaly cancellation condition played a crucial role in defining this product, as it is only because of the anomaly cancellation condition that we have \( \Lambda^R F^\vee \cong K_M \).

In the next section, we will describe maps

\[
\psi_k : H^q(X, \Lambda^p E^\vee) \longrightarrow H^q(\mathcal{M}, \Lambda^p F^\vee),
\]
the subscript \( k \) corresponding to the \( k \)th insertion point. This will complete our description of the correlation functions in our ideal situation. Given cohomology classes \( \eta_i \in H^q(X, \Lambda^p E^\vee) \) satisfying the selection rules \( \sum q_i = N \) and \( \sum p_i = R \), we apply (8) to the \( \psi_i(\eta_i) \) to get the value of the desired correlation function, using algebraic geometry.

\[\text{5.3 Evaluation map}\]

In order to make sense of the calculations just described, we need to describe the maps

\[
\psi_i : H^q(X, \Lambda^p E^\vee) \rightarrow H^q(\mathcal{M}, \Lambda^p F^\vee).
\]

\(^9\)At least when \( \mathcal{M} \) is smooth, Kähler, and simply-connected, assumptions we will freely make for the purposes of the formal discussion of this section.
More precisely, by restricting the universal map $\alpha : \Sigma \times \mathcal{M} \to X$ to $p_i \in \Sigma$ (where $p_i$ is the $i^{th}$ insertion point), we have an evaluation map

$$ev_i : \mathcal{M} \to X, \quad ev_i(\phi) = \phi(p_i).$$

Note that upon identifying $\mathcal{M}$ with $\{p_i\} \times \mathcal{M}$, we see that $ev_i$ is just the restriction of $\alpha$ to $\mathcal{M}$. We need to define a map

$$\psi_i : H^q(X, \Lambda^p \mathcal{E}^\vee) \mapsto H^q(\mathcal{M}, \Lambda^p \mathcal{F}^\vee). \quad (9)$$

First, note that pulling back by $ev_i$, we get a map

$$H^q(X, \Lambda^p \mathcal{E}^\vee) \mapsto H^q(\mathcal{M}, \Lambda^p (\alpha^* \mathcal{E})^\vee |_{\{p_i\} \times \mathcal{M}}). \quad (10)$$

So to derive the map (9) we just need to find a sheaf map

$$\Lambda^p (\alpha^* \mathcal{E})^\vee |_{\{p_i\} \times \mathcal{M}} \mapsto \Lambda^p \mathcal{F}^\vee$$

yielding a corresponding map on cohomology

$$H^q(\mathcal{M}, \Lambda^p (\alpha^* \mathcal{E})^\vee |_{\{p_i\} \times \mathcal{M}}) \mapsto H^q(\mathcal{M}, \Lambda^p \mathcal{F}^\vee)$$

to compose with the pullback map (10) to arrive at $\psi_i$.

For simplicity of notation, we complete the argument in the case $p = 1$; the generalization to arbitrary $p$ is straightforward. We shall demonstrate the existence of the dual map $\mathcal{F} \to (\alpha^* \mathcal{E}) |_{\{p_i\} \times \mathcal{M}}$, and will work in terms of local sections of the corresponding sheaves. Let $U$ be an open subset of $\mathcal{M}$, then

$$\mathcal{F}(U) = (\pi_* \alpha^* \mathcal{E})(U) \quad \text{(by definition)}$$

$$= (\alpha^* \mathcal{E})(\pi^* U)$$

$$= H^0(\Sigma \times U, \alpha^* \mathcal{E})$$

so we have now expressed local sections of $\mathcal{F}$ in terms of local sections of $\alpha^* \mathcal{E}$. Next, we can restrict those sections, getting a map

$$H^0(\Sigma \times U, \alpha^* \mathcal{E}) \to H^0(U, (\alpha^* \mathcal{E}) |_{\{p_i\} \times \mathcal{M}}).$$

These are, however, precisely the local sections of the sheaf $(\alpha^* \mathcal{E}) |_{\{p_i\} \times \mathcal{M}}$ over the open set $U \subseteq \mathcal{M}$. Thus, we have a map of local sections

$$\mathcal{F}(U) \to (\alpha^* \mathcal{E}) |_{\{p_i\} \times \mathcal{M}}(U)$$

as desired, completing the description of the maps $\psi_i$. 

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Note that if $\mathcal{E} = TX$, then the composition
\[ T\mathcal{M} = \mathcal{F} \to \alpha^*\mathcal{E}|_{(p_i) \times \mathcal{M}} = ev_i^*\mathcal{E} = ev_i^*TX \]
is easily checked to be the differential of the evaluation map $ev_i : \mathcal{M} \to X$. Thus, in this case, the map $\psi_i : H^q(X, \Lambda^p\mathcal{E}^\vee) \mapsto H^q(\mathcal{M}, \Lambda^p\mathcal{F}^\vee)$ coincides with the pullback by $ev_i$ on ordinary cohomology
\[ H^{p,q}(X) \to H^{p,q}(\mathcal{M}). \]
It follows that on the $(2,2)$ locus, the correlation functions coincide with those of ordinary Gromov-Witten theory, with minor modifications due to the turning off of topological gravity.

In summary, the heterotic chiral ring is a generalization of the chiral ring of Gromov-Witten theory and quantum cohomology, and as such, deserves to be better understood mathematically.

In the next section, we relate to standard Gromov-Witten theory a little more closely.

### 5.4 Stable maps and compactifications

In this section, we describe how to modify our basic construction of the preceding sections to more realistic situations.

Suppose we want to compute a correlation function with $k$ insertions. Let $\overline{M}_{g,k}(X,d)$ be the moduli space of genus $g$ stable maps maps to $X$ of degree $d$\(^{10}\) as introduced in [17]. These are maps $f : C \to X$ of degree $d$, where $C$ is a connected algebraic curve of genus $g$ with marked points $p_1, \ldots, p_n$ in the smooth locus of $C$ modulo automorphisms of $C$ preserving $f$ and the $p_i$. The stability condition is that each genus 0 component of $C$ on which $f$ is constant has at most 3 special (nodal or marked) points. There are evaluation maps $e_i : \overline{M}_{g,k}(X,d) \to X$ defined by $e_i(f) = f(p_i)$.

There is a forgetful map $\rho : \overline{M}_{g,k}(X,d) \to \overline{M}_{g,k}$ which forgets the map and contracts any components of $C$ which have become unstable after forgetting the map. Let $\Sigma, p_1, \ldots, p_k \in \overline{M}_{g,n}$ be our worldsheet with its fixed complex structure and insertion points. For ease of notation we denote this simply by $\Sigma \in \overline{M}_{g,n}$. Then our more realistic model for the instanton moduli space is
\[ \mathcal{M} = \rho^{-1}(\Sigma). \]
Note that $\mathcal{M}$ is compact, and its elements correspond to degree $d$ maps $\phi : \Sigma \to X$, as well as maps from the union of $\Sigma$ with trees of 2 spheres to $X$ of total degree $d$.

---

\(^{10}\)More properly, we should replace $d$ with a homology class $\beta \in H_2(X, \mathbb{Z})$, but we just denote this by $d$ and call it a “degree” for ease of locution.
Now there is still a universal map. It is well known that the universal family of curves over \( \overline{M}_{g,k}(X,d) \) is given by \( \overline{M}_{g,k+1}(X,d) \). There is a map \( \pi_k : \overline{M}_{g,k+1}(X,d) \to \overline{M}_{g,k}(X,d) \) which is given by forgetting the last marked point and contracting unstable components. The fiber of \( \pi_k \) corresponds to the location of an extra point on \( C \), which explains why \( \overline{M}_{g,k+1}(X,d) \) is the universal curve over \( \overline{M}_{g,k}(X,d) \). The map \( \epsilon_{k+1} : \overline{M}_{g,k+1}(X,d) \to X \) is then clearly the universal map. There are \( k \) natural sections \( s_i : \overline{M}_{g,k}(X,d) \to \overline{M}_{g,k+1}(X,d) \) of \( \pi_k \) given by sending a stable map \( f : C \to X \) with marked points \( p_i \) to the point \( p_i \), identified with the corresponding point on the universal curve.

We now let \( C = \pi_k^{-1}(\mathcal{M}) \), and let \( \alpha \) be the restriction of \( \epsilon_{k+1} \) to \( C \). Then let \( \pi \) be the restriction of \( \pi_k \) to \( C \). This gives the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\alpha} & X \\
\downarrow \pi & & \downarrow \\
\mathcal{M} & &
\end{array}
\]

We now see that this is the proper generalization of the universal map \( \alpha : \Sigma \times \mathcal{M} \to X \) and projection \( \pi : \Sigma \times \mathcal{M} \to \mathcal{M} \) considered previously.

In particular, we can now define

\[ \mathcal{F} = \pi_s \alpha^* \mathcal{E} \]

as before. To get the generalized maps \( \psi_i : H^q(X, \Lambda^p \mathcal{E}^\vee) \to H^q(\mathcal{M}, \Lambda^p \mathcal{F}^\vee) \), we just use the sections \( s_i : \overline{M}_{g,k}(X,d) \to \overline{M}_{g,k+1}(X,d) \) in place of the embeddings \( \mathcal{M} = \{p_i\} \times \mathcal{M} \subset \Sigma \times \mathcal{M} \) and proceed as before.

The price to be paid for this general construction using well-known mathematics is that these \( \mathcal{M} \) are almost never smooth except in simple cases like projective spaces and \( g = 0 \). So to evaluate correlation functions, we would need to develop a virtual fundamental class in this context. This is very similar to Gromov-Witten theory, except that topological gravity is turned off (by restricting to the fiber of \( \rho \) and the virtual fundamental class must be modified accordingly). Furthermore, the techniques of Gromov-Witten theory do not apply in general. However, the techniques of localization can be applied if \( X \) admits a torus action and the bundle \( \mathcal{E} \) is equivariant for that torus action. Such bundles are studied in \([18,19]\) if \( X \) is a toric variety. We do not develop these techniques here since one of our goals in the present paper is to verify the claims of \([1]\) and the gauge bundles appearing there are not of this type. It would nevertheless be interesting to develop computational techniques for this equivariant situation.

In Section 6 we will describe another compactification, the linear sigma model compactification. We expect this to coincide with the nonlinear heterotic theory for simple spaces such as products of projective spaces, and, by analogy with \([20]\), to be related to the heterotic theory by a change of variables.
5.5 Generalization of obstruction sheaves

Next, let us turn to the case in which

\[ R^1\pi_*\alpha^*TX \neq 0, \quad R^1\pi_*\alpha^*E \neq 0 \]

i.e. the case that would call for obstruction sheaves on the (2, 2) locus. Our analysis in section 5.2 only makes sense physically in the case that there are \( \psi^i \) zero modes, but no \( \psi^\bar{r} \) zero modes, and \( \lambda^a \) zero modes, but no \( \lambda^b \) zero modes. The index theorems quoted in section 5.2 only specify the difference between the number of such zero modes.

We shall describe a proposal for how this case should be described mathematically, that will generalize the obstruction bundle story of the (2, 2) locus. We shall check that our proposal satisfies basic consistency tests, and reduces to the ordinary obstruction bundle story on the (2, 2) locus. However, there is a crucial mathematical statement we have not yet been able to prove.

Write the number of zero modes of the four types of worldsheet fermions as follows:

\[
\begin{align*}
\psi^i &: h^0(\Sigma, \phi^*TX) = m + p \\
\psi^{\bar{r}} &: h^1(\Sigma, \phi^*TX) = p \\
\lambda^a &: h^1(\Sigma, \phi^*E) = q \\
\lambda^{\bar{a}} &: h^0(\Sigma, \phi^*E) = r + q
\end{align*}
\]

The numbers \( m, r \) are calculated by index theory, and the numbers \( p, q \) count ‘excess’ zero modes.

We still need to make a simplifying assumption. We assume that the number of excess modes \( p \) and \( q \) are constant on the instanton moduli space \( M \), although possibly nonzero. As a consequence, \( M \) will still be smooth, of dimension \( m + p \), and \( F \) will still be a bundle, of rank \( r + q \). But now the selection rule on the correlators gives bundle-valued forms of degree \( m, r \) – so the selection rules no longer define top forms on the moduli space in the presence of excess zero modes. We will fix this problem by wedging with more bundle-valued differential forms, but before we discuss the mathematics, let us review the physics.

Just as in [14], we can soak up these excess zero modes by using the four-fermi term in the worldsheet action. Recall the four-fermi term has the form

\[
\int_\Sigma F_{\bar{a}b} \psi^i \psi^{\bar{r}} \lambda^a \lambda^{\bar{b}}
\]

(11)

where \( F \) is the curvature of the bundle \( E \). (Note, in particular, that the structure of the four-fermi term in a sigma model is asymmetric between the left- and right-movers: the curvature of the left-moving bundle \( E \) appears in the action, but there is no term representing the curvature of the right-moving bundle \( TX \).) Each time we bring down a factor of this four-fermi term, we soak up one of each type of worldsheet fermion. So long as \( p = q \), which
is the case for \((2,2)\) theories, we can use this four-fermi term to soak up all of the excess zero modes.

In general, however, \(p\) need not equal \(q\). Suppose, without loss of generality, that \(p > q\). Then we could use \(q\) factors of the four-fermi terms to soak up all of the excess \(\lambda\) zero modes, and all but \(p - q\) of the excess \(\psi\) zero modes. We could bring down an additional \(p - q\) factors of the four-fermi term, and then use \(\lambda\) propagators to contract away the excess \(\lambda\) fermions. The resulting correlation function would exhibit a dependence on the positions on the worldsheet in which the correlators were inserted. The result is not topological, but since this theory is not a topological field theory, there is no good reason to believe that all correlators under consideration can be expressed purely topologically.

As we are only interested in those correlators that can be expressed purely topologically, henceforth we shall only consider theories for which

\[
H^1 (\Sigma, \phi^* TX) = H^1 (\Sigma, \phi^* \mathcal{E})
\]

(12)

for all \(\phi\), i.e. \(p = q\) everywhere on \(\mathcal{M}\). Phrased more formally, we shall assume that

\[
\text{rank } R^1 \pi_2_* \alpha^* \mathcal{E} = \text{rank } R^1 \pi_2_* \alpha^* TX
\]

everywhere on \(\mathcal{M}\), i.e. that \(p = q\) in the notation introduced above.

Let us now return to the mathematical description of our correlators. As outlined above, the physical selection rule says that images of the vertex operators as sheaf cohomology on \(\mathcal{M}\) wedge together to form an element of

\[
H^m (\mathcal{M}, \Lambda^r \mathcal{F}^\vee)
\]

but the dimension of \(\mathcal{M}\) is \(m + p\), and the rank of \(\mathcal{F}\) is \(r + q\), so we need an additional factor if we wish the correlation functions to be expressible as integrals of top forms.

This problem has a well-known solution on the \((2,2)\) locus. In [14], bringing down factors of the four-fermi term was interpreted in terms of wedging the differential forms representing correlators (whose total degree was too small to be a top form) with a differential form representing the Euler class (coming from the four-fermi term factors), which had the effect of making the integrand a top form, so that correlation functions again naturally generate numbers.

There is an analogous phenomenon here. If we try to interpret each \((0,2)\) four-fermi term (11) as a bundle-valued differential form on the moduli space, then each such four-fermi term should be identified with an element of

\[
H^1 (\mathcal{M}, \mathcal{F}^\vee \otimes \mathcal{F}_1 \otimes \mathcal{G}_1^\vee)
\]

(13)
on symmetry grounds, where

\[ \mathcal{F}_1 \equiv R^1 \pi_* \alpha^* \mathcal{E}, \]

\[ \mathcal{G}_i \equiv R^i \pi_* \alpha^* T_X. \]

are the sheaves over the moduli space defined by the zero modes of the fermions. (The \( \psi^i \) is responsible for having degree one sheaf cohomology; the other three fermions are responsible for the coefficients.)

We should stress at this point, however, that this is an ansatz. One would like to be able to prove mathematically that the curvature \( F \) of \( \mathcal{E} \) defines an element of the sheaf cohomology group above, but we have not yet been able to do this.

Nevertheless, we can check that this ansatz is consistent. First, we shall show that with this ansatz, correlation functions can be expressed as integrals of top forms, i.e. naturally generate numbers. Previously in section 5.2, when \( R^1 \pi_* \alpha^* \mathcal{E} \) and \( R^1 \pi_* \alpha^* T_X \) both vanished, we saw that the heterotic anomaly cancellation condition and Grothendieck-Riemann-Roch implied that

\[ \Lambda^{\mathrm{top}} \mathcal{F} = K_{\mathcal{M}}^{\vee} \]

exactly as needed for our correlation functions to generate a number. However, more generally we have a slightly different statement. If we define

\[ \mathcal{F}_1 \equiv R^1 \pi_* \alpha^* \mathcal{E} \]

\[ \mathcal{G}_i \equiv R^i \pi_* \alpha^* T_X, \]

as above, then the anomaly cancellation condition and Grothendieck-Riemann-Roch imply

\[ \Lambda^{\mathrm{top}} \mathcal{F} \otimes \Lambda^{\mathrm{top}} \mathcal{F}_1 \cong \Lambda^{\mathrm{top}} \mathcal{G}_0 \otimes \Lambda^{\mathrm{top}} \mathcal{G}_1^{\vee}. \]

In particular, since \( \mathcal{G}_0 \cong T\mathcal{M}, \) this means that

\[ \Lambda^{\mathrm{top}} \mathcal{F}^{\vee} \otimes \Lambda^{\mathrm{top}} \mathcal{F}_1 \otimes \Lambda^{\mathrm{top}} \mathcal{G}_1^{\vee} \cong K_{\mathcal{M}} \]

so that we can fix up correlation functions by wedging with a representative of

\[ H^q (\mathcal{M}, \Lambda^q \mathcal{F}^{\vee} \otimes \Lambda^q \mathcal{F}_1 \otimes \Lambda^q \mathcal{G}_1^{\vee}) \] (recall \( \mathcal{F}_1 \) has rank \( q \), matching the rank of \( \mathcal{G}_1 \)). Note, however, that if we bring down enough copies of the (0, 2) four-fermi term to absorb the ‘excess’ zero modes, then from our ansatz (13), we generate precisely the factor (14) above. In other words, our interpretation of the (0, 2) four-fermi terms is precisely what we need to describe correlators as integrals of top forms.

Let us now check that the description above gives correct results on the (2, 2) locus. In this case, \( \mathcal{F}_1 \cong \mathcal{G}_1 \) so that the \( \mathcal{F}_1 \) and \( \mathcal{G}_1 \) factors in (14) cancel out. Nevertheless, the particular class of (14) used depends on \( \mathcal{G}_1 \), as we will see.
Recall that in the $(2,2)$ case, correlators are described by differential forms on $\mathcal{M}$, not sheaf cohomology groups, and the factor $(14)$ is replaced by the top Chern class of $\mathcal{G}_1$, i.e. $c_q(\mathcal{G}_1)$.

Chern classes and sheaf cohomology can be related via the Atiyah class of the bundle, which we shall briefly review. Consider expressing Chern classes in terms of the curvature of the connection on a holomorphic vector bundle $\mathcal{E}$ on a Kähler manifold $X$:

$$c_r \propto \text{Tr } F \wedge F \wedge \cdots \wedge F.$$  

Taking advantage of hermitian fiber metrics on vector bundles, we can write

$$F = F_{ij}dz^i \wedge d\overline{z}^j \wedge \lambda^a \wedge \overline{\lambda}^b.$$  

Furthermore, because of the Bianchi identity, if $F$ is a holomorphic connection (i.e. $F_{ij} = F_{i\overline{j}} = 0$), then $F$ is $\overline{\partial}$-closed. Thus, we can think of $F$ as a holomorphic $(0,1)$-form valued in $TX^\vee \otimes \mathcal{E} \otimes \mathcal{E}^\vee$, and so in particular the curvature $F$ defines an element of

$$H^1(X, TX^\vee \otimes \mathcal{E} \otimes \mathcal{E}^\vee).$$

The sheaf cohomology class above is known as the Atiyah class and is independent of the choice of connection [21]. Taking the trace defines a map

$$H^1(X, TX^\vee \otimes \mathcal{E} \otimes \mathcal{E}^\vee) \mapsto H^1(X, TX^\vee) \cong H^{1,1}(X),$$

whose image is the first Chern class of the bundle. Furthermore, note that by wedging $r$ copies of $F$ together we create an element of

$$H^r(X, \Lambda^rTX^\vee \otimes \Lambda^r\mathcal{E} \otimes \Lambda^r\mathcal{E}^\vee)$$

and the trace defines a map from the sheaf cohomology group above to

$$H^r(X, \Lambda^rTX^\vee) \cong H^{r,1}(X),$$

whose image is the $r$th Chern class.

In particular, on the $(2,2)$ locus, the factor $(14)$ is the sheaf cohomology description of the $q$th Chern class of $\mathcal{G}_1$, as claimed.

Thus, even though we do not yet understand mathematically how the Atiyah class of $\mathcal{E}$ induces an element of the sheaf cohomology group (13), our proposal satisfies reasonable consistency tests. This gives us some confidence in our interpretation.
6 Linear-sigma-model-based compactifications

6.1 Generalities

Linear sigma models were used in [20] to provide natural compactifications of moduli spaces of worldsheet instantons. Let us take a moment to review their construction. Recall that given a toric variety expressed as a GIT quotient of a set of homogeneous polynomials \( x_a \) by a set of \( \mathbb{C} \times \) actions, the linear sigma model moduli space is described as a GIT quotient of the space of zero modes of the \( x_a \). Let \( \vec{n}_a \) denote the weights of homogeneous coordinate \( x_a \) with respect to the \( \mathbb{C} \times \) actions, and let \( \vec{d} \) denote the (fixed) degree of the worldsheet instantons (expressed as a vector of weights with respect to the same \( \mathbb{C} \times \) actions). Then we can expand each \( x_a \) as

\[
x_a \in H^0 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) = x_{a0} u^{d_a} + x_{a1} u^{d_a-1} v + \ldots
\]

for \( d_a = \vec{n}_a \cdot \vec{d} \), where \( u, v \) are homogeneous coordinates on the worldsheet \( \mathbb{P}^1 \). To construct the linear sigma model moduli space, we take the space of \( x_a \)'s, and quotient by the same set of \( \mathbb{C} \times \)'s as defined the original toric variety, such that each \( x_a \) has weight vector \( \vec{n}_a \) (same as for the original \( x_a \)), after removing an exceptional set.

The original (2,2) gauged linear sigma models of [22] were generalized in [23] to describe (0,2) theories, that is, Calabi-Yau's together with bundles. The bundles are presented physically in a very special form, as the cohomology of a short complex

\[
\oplus \mathcal{O} \xrightarrow{E} \oplus \mathcal{O}(\vec{n}_a) \xrightarrow{F} \oplus \mathcal{O}(\vec{m}_i)
\]

sometimes known as a ‘monad’. The bundles in question are bundles over\(^{11}\) the ambient toric variety. If a superpotential is present to specify a Calabi-Yau subvariety, then the bundle on the subvariety is obtained by restricting the bundle on the ambient space.

The methods of [20] can be extended in a very straightforward fashion to define extensions of the sheaves \( \mathcal{F}, \mathcal{F}_1 \) over the (compact) linear sigma model moduli spaces, as we will discuss in numerous examples in the rest of this section. The basic idea is to expand each worldsheet field corresponding to a line bundle, in a basis of zero modes, and associate the coefficients to line bundles on \( \mathcal{M} \). For example, if we had a reducible bundle \( \mathcal{E} = \oplus_a \mathcal{O}(\vec{n}_a) \), described by a set of free left-moving fermions of charges \( \vec{n}_a \), then our ansatz yields the induced bundles

\[
\mathcal{F} = \oplus_a H^0 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a),
\]

\[
\mathcal{F}_1 = \oplus_a H^1 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a).
\]

\(^{11}\)In this paper, we will work with monads describing bundles over the ambient space. However, historically monads have been used to describe bundles only over a Calabi-Yau complete intersection. The distinction involves whether the composition of maps \( F \circ E \) vanishes identically (as we will assume throughout this paper) or only up to hypersurface equations (as is more typical in the older literature [2, 23]).
It can be shown\textsuperscript{12} that this ansatz matches the $R^i \pi_* \alpha^* \mathcal{E}$ on the open subset of $\mathcal{M}$ corresponding to honest maps, and furthermore that this ansatz has the desired rank for $\mathcal{F} \oplus \mathcal{F}_1$ as well as the correct determinant line bundle $\Lambda^{\text{top}} \mathcal{F}^\vee \otimes \Lambda^{\text{top}} \mathcal{F}_1$. These verifications are special cases of arguments we will present later, so we omit details.

We will describe more general cases explicitly in the rest of this section. We will also check that this ansatz for induced sheaves is compatible with needed results, \textit{e.g.} that

$$\Lambda^{\text{top}} \mathcal{F}^\vee \otimes \Lambda^{\text{top}} \mathcal{F}_1 \cong K_\mathcal{M} \otimes \Lambda^{\text{top}} \mathcal{G}_1$$

continues to hold after extending the sheaves over the compactification of the moduli space, and that in all cases, our ansatz gives sheaves that agree with the $R^i \pi_* \alpha^* \mathcal{E}$ on the open subset of $\mathcal{M}$ corresponding to honest maps.

An important technical issue is that the anomaly cancellation condition in gauged linear sigma models is slightly stronger than the mathematical statement $\text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$, as shown in [2], and can even distinguish different presentations of the same gauge bundle. We specifically require the stronger form, as described in [2], in order for our constructions to be consistent. In particular, in section 6.5 we shall see some examples which satisfy $\text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$ but fail the stronger linear sigma model anomaly cancellation condition, despite merely being alternative presentations of well-behaved linear sigma models. In these examples, our construction fails. We will discuss the linear sigma model anomaly cancellation condition in more detail later.

A related issue, also discussed in section 6.5, that the extension of the sheaves $R^i \pi_* \alpha^* \mathcal{E}$ over the compactification divisor depends upon the presentation of $\mathcal{E}$, and different presentations can give very different extensions.

### 6.2 Bundles presented as cokernels, and a check of the (2, 2) locus

#### 6.2.1 Easy example: tangent bundle of $\mathbb{P}^{N-1}$

Let us begin with an easy example: the tangent bundle of a projective space $\mathbb{P}^{N-1}$, which physically in the linear sigma model arises as a cokernel:

$$0 \longrightarrow \mathcal{O} \longrightarrow \bigoplus_{1}^{N} \mathcal{O}(1) \longrightarrow T\mathbb{P}^{N-1} \longrightarrow 0.$$ 

\textsuperscript{12}Clearly

$$R^i \pi_* \alpha^* \mathcal{E} = \bigoplus_{a} R^i \pi_* \alpha^* \mathcal{O}(\vec{n}_a)$$

for $i = 0, 1$, and these sheaves have the same ranks as the $\mathcal{F}, \mathcal{F}_1$ listed. To show that the $R^i \pi_* \alpha^* \mathcal{E}$ completely decompose into a direct sum of line bundles one uses the $T$-equivariant nature of the line bundles $\mathcal{O}(\vec{n}_a)$, as we will describe in detail in the section on bundles presented as cokernels.
Physically both the $\mathcal{O}$ and the $\oplus \mathcal{O}(1)$ correspond to worldsheet fields, the latter to $N$ (0,2) fermi multiplets of charge one, and the former to a single neutral (0,2) chiral multiplet.

Thus, when we consider a moduli space of degree $d$ maps $\mathbf{P}^1 \to \mathbf{P}^{N-1}$, we want to replace the worldsheet fields by their zero modes, coupling to bundles on the moduli space determined by the scaling properties of the original linear sigma model fields with respect to the toric action. Furthermore, those zero modes should obey relations determined by the relations in the original linear sigma model.

In other words, on the moduli space, the sheaf $\mathcal{F}$ on the moduli space corresponding to $T\mathbf{P}^{N-1}$ is given by

$$\text{coker } \left\{ H^0 \left( \mathbf{P}^1, \mathcal{O}(d \cdot 0) \right) \otimes_{\mathbb{C}} \mathcal{O} \to \bigoplus_{i=1}^{N} H^0 \left( \mathbf{P}^1, \mathcal{O}(d \cdot 1) \right) \otimes_{\mathbb{C}} \mathcal{O}(1) \right\}$$

where the maps are given by multiplication by homogeneous coordinates on the moduli space.

Let us take a moment to work through the maps in more detail. If the homogeneous coordinates on $\mathbf{P}^{N-1}$ are denoted $x_i$, and homogeneous coordinates on the worldsheet are denoted $u, v$, then for degree $d$ maps we can write

$$x_i = c_{i0}u^d + c_{i1}u^{d-1}v + \cdots + c_{id}v^d.$$ 

Similarly, arbitrary local sections of $\mathcal{O}(1)$ can be written as

$$\lambda_i = \omega_{i0}u^d + \omega_{i1}u^{d-1}v + \cdots + \omega_{id}v^d.$$ 

In the short exact sequence defining $\mathcal{F}$, the map $\mathcal{O} \to \mathcal{O}(1)^{N(d+1)}$ arising from the original map $\mathcal{O} \to \mathcal{O}(1)^N$ on $\mathbf{P}^{N-1}$ that multiplied local sections by $x_i$, can now be expressed in terms of the $u^i v^j$. In other words, at the level of zero modes, that map now multiplies 1 by $c_{ik}$ to land in the bundle whose sections are generated by $\omega_{ik}$. More generally, any map $\mathcal{O}(n) \to \mathcal{O}(m)$ on the original space induces a map

$$H^0 \left( \mathbf{P}^1, \mathcal{O}(nd) \right) \otimes_{\mathbb{C}} \mathcal{O}(n) \to H^0 \left( \mathbf{P}^1, \mathcal{O}(md) \right) \otimes_{\mathbb{C}} \mathcal{O}(m)$$

on the moduli space, given by expressing the original map (evaluated on zero modes) in terms of its $u^i v^j$ components. More explicitly, the original map is given by multiplication by a polynomial of degree $m - n$, which pulls back to a polynomial of degree $d(m - n)$ on $\mathbf{P}^1$ whose coefficients depend on the $c_{ij}$, and which in addition has degree $m - n$ in the $c_{ij}$. This is expressed in more invariant fashion as a map (15).

Since the $c_{ik}$ are homogeneous coordinates on the moduli space, it should now be clear that this sheaf is $T\mathbf{P}^{N(d+1)-1} = T\mathcal{M}$, exactly as desired.
6.2.2 More general bundles presented as cokernels

More generally, given a bundle $E$ presented in the linear sigma model as a cokernel of the form

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_a \mathcal{O}(\vec{n}_a) \longrightarrow E \longrightarrow 0$$

for worldsheet instantons of degree $d$ we have an induced long exact sequence

$$0 \longrightarrow \oplus_k H^0 \left( \mathbb{P}^1, \mathcal{O}(\vec{0} \cdot \vec{d}) \right) \otimes \mathcal{O} \longrightarrow \oplus_a H^0 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{F}$$

$$\longrightarrow \oplus_k H^1 \left( \mathbb{P}^1, \mathcal{O}(\vec{0} \cdot \vec{d}) \right) \otimes \mathcal{O} \longrightarrow \oplus_a H^1 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{F}_1$$

This simplifies to give the short exact sequence,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_a H^0 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{F} \longrightarrow 0$$

and the relation

$$\mathcal{F}_1 \cong \oplus_a H^1 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a).$$

To verify that this method is giving reasonable results, we shall next check the following things:

- First, we shall check that this gives the correct results on the $(2,2)$ locus.
- We shall argue that more generally, $\mathcal{F}$ and $\mathcal{F}_1$ agree with $R^0 \pi_+^* \mathcal{E}$, $R^1 \pi_+^* \mathcal{E}$, respectively, on the open subset of $\mathcal{M}$ corresponding to honest maps.
- Finally, we shall check that rank $(\mathcal{F} \ominus \mathcal{F}_1)$ and $\Lambda^{\text{top}} \mathcal{F}^\vee \otimes \Lambda^{\text{top}} \mathcal{F}_1$ satisfy the correct relations for our formal analysis of correlation functions to proceed.

First, let us check that our results are correct on the $(2,2)$ locus. Note that when $\mathcal{E}$ is the tangent bundle of a toric variety, $\mathcal{F}$ as described above is the tangent bundle to the linear sigma moduli space of [20].

In the case of the tangent bundle to $\mathbb{P}^{N-1}$ above, $\vec{n}_a \cdot \vec{d} = d$ for all $a$, so as long as $d > 0$, we have that $\mathcal{F}_1 = 0$ for the tangent bundle.

When $\mathcal{E} = TX$, the sheaf $\mathcal{F}_1$ is known as the obstruction bundle$^{13}$, as we shall now check. First, a couple of easy tests of this statement. Loosely, the obstruction sheaf is the sheaf over

\footnote{Technically, we shall show that $\mathcal{F}_1$ agrees with the obstruction sheaf over the open subset of the moduli space corresponding to honest maps. Strictly speaking, we are not aware of a previous general definition of obstruction sheaf over (compact) linear sigma model moduli spaces.}
the moduli space defined by the \( \psi _i \) zero modes, and that is precisely the physical meaning of \( \mathcal{F}_1 \). More concretely, since

\[
 h^1 \left( \mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) = \begin{cases} 
 -\vec{n}_a \cdot \vec{d} - 1 & \text{if } \vec{n}_a \cdot \vec{d} \leq -1 \\
 0 & \text{otherwise}
\end{cases}
\]

we see that the top Chern class of the obstruction sheaf \( \mathcal{F}_1 \) is given by

\[
 \prod_{\vec{n}_a \cdot \vec{d} \leq -1} c_1(\mathcal{O}(\vec{n}_a))^{-\vec{n}_a \cdot \vec{d} - 1},
\]

a result that precisely matches [20][equ’n (3.62)].

Next, let us check more systematically that on the open subset of the linear sigma model moduli space \( \mathcal{M} \) corresponding to honest maps, \( \mathcal{F}_1 \) really is the obstruction sheaf. Recall that on the (2,2) locus, the obstruction sheaf is \( R^1 \pi_* \alpha^* \mathcal{T} \mathcal{X} \), at least over the locus where there is a universal instanton \( \alpha : \mathbf{P}^1 \times \mathcal{M} \to \mathcal{X} \). Now if \( \mathcal{X} \) is toric, present the tangent bundle as

\[
 0 \to O \to \bigoplus_a O(\vec{n}_a) \to \mathcal{T} \mathcal{X} \to 0.
\]

Applying \( \pi_* \alpha^* \) gives rise to an exact sequence including terms

\[
 R^1 \pi_* \alpha^* \mathcal{O}(\vec{n}_a) \to \bigoplus_a R^1 \pi_* \alpha^* O(\vec{n}_a) \to R^1 \pi_* \alpha^* \mathcal{T} \mathcal{X} \to 0.
\]

The first term is zero since \( H^1(\mathbf{P}^1, \mathcal{O}) = 0 \), so we get an isomorphism

\[
 R^1 \pi_* \alpha^* \mathcal{T} \mathcal{X} \cong \bigoplus_a R^1 \pi_* \alpha^* O(\vec{n}_a).
\]

To reconcile with the computation of \( \mathcal{F}_1 \) in equation (17), we need to compare \( R^1 \pi_* \alpha^* O(\vec{n}_a) \) with

\[
 H^1 \left( \mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes_\mathbb{C} \mathcal{O}(\vec{n}_a).
\]

These sheaves both have the same ranks, since \( H^1 \left( \mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \) is the fiber of \( R^1 \pi_* \alpha^* \mathcal{O}(\vec{n}_a) \). In addition, note that the induced torus action decomposes \( R^1 \pi_* \alpha^* \mathcal{O}(\vec{n}_a) \) into eigenbundles, each of which have precisely the same torus weight as the bundle \( \mathcal{O}(\vec{n}_a) \). Since there is a single weight and \( \mathcal{M} \) is itself a toric variety, we simply get a sum of line bundles, as expected. Thus, our sheaf \( \mathcal{F}_1 \) really does match the obstruction sheaf on the (2,2) locus, as advertised.

We shall outline an example of an obstruction sheaf in section 6.2.3. Note that from equation (17), we see that the obstruction sheaf over any linear sigma model moduli space is always a direct sum of line bundles.

Now that we have checked that on the (2,2) locus the sheaves \( \mathcal{F}, \mathcal{F}_1 \) are precisely the tangent bundle and obstruction sheaf, as expected, next we shall check that \( \mathcal{F} \) and \( \mathcal{F}_1 \) match \( R^0 \pi_* \alpha^* \mathcal{E} \) and \( R^1 \pi_* \alpha^* \mathcal{E} \) on the open subset of \( \mathcal{M} \) corresponding to honest maps. The analysis
is very similar to our earlier comparison of $F_1$ on the (2, 2) locus to the obstruction sheaf. To do this, we return to the definition of $E$ as a cokernel

\[ 0 \to O^{\oplus k} \to \oplus_a O(\vec{n}_a) \to E \to 0 \]

which induces, on the open subset of $M$ corresponding to honest maps, the long exact sequence

\[ 0 \to \oplus_k R^0_\pi \alpha^* O \to \oplus_a R^0_\pi \alpha^* O(\vec{n}_a) \to R^0_\pi \alpha^* E \to 0 \]

\[ \to \oplus_k R^1_\pi \alpha^* O \to \oplus_a R^1_\pi \alpha^* O(\vec{n}_a) \to R^1_\pi \alpha^* E \]

Since $R^1_\pi \alpha^* O = 0$, this long exact sequence simplifies to become the short exact sequence,

\[ 0 \to O^{\oplus k} \to \oplus_a R^0_\pi \alpha^* O(\vec{n}_a) \to R^0_\pi \alpha^* E \to 0 \]

and the relation

\[ R^1_\pi \alpha^* E \cong \oplus_a R^1_\pi \alpha^* O(\vec{n}_a). \]

For the same reasons as in our discussion of the obstruction sheaf, the sheaves $R^i_\pi \alpha^* O(\vec{n}_a)$ all split into a direct sum of line bundles, and so we now see explicitly that when restricted to the open subset $U$ of $M$ corresponding to honest maps, $F|_U \cong R^0_\pi \alpha^* E$ and $F_1|_U \cong R^1_\pi \alpha^* E$.

Let us next check that the sheaves $F$, $F_1$ derived above do indeed possess the properties discussed in section 5.5, namely that

\[ \Lambda^{\text{top}} F^\vee \otimes \Lambda^{\text{top}} F_1 \cong K_M \otimes \Lambda^{\text{top}} G_1 \]

and that

\[ \text{rank } F - \text{rank } F_1 = \chi(\phi^* E) = c_1(\phi^* E) + \text{rank } E. \]

To perform this verification, let us present the tangent bundle to the toric variety in the form

\[ 0 \to O^{\oplus m} \to \oplus_a O(\vec{q}_n) \to TX \to 0. \]

Writing $\vec{n} = (n^i)$, and using $E$ as described above, the anomaly cancellation condition in the (0,2) gauged linear sigma model takes the form

\[ \sum_a n_a^i n_a^j = \sum_n q_n^i q_n^j \]

for each $i$, $j$, a condition which is very slightly stronger than the statement of matching second Chern characters [2]. Similarly, we shall also impose the requirement that for each $i$,

\[ \sum_a n_a^i = \sum_n q_n^i, \]

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which is a slightly stronger form of the constraint
\[ \Lambda^{\text{top}}E^\vee \cong K_X. \]

First, consider the ranks of \( \mathcal{F} \) and \( \mathcal{F}_1 \).
\[
\begin{align*}
\text{rank } \mathcal{F} &= \sum_{\vec{n}_a \cdot \vec{d} \geq 0} (\vec{n}_a \cdot \vec{d} + 1) - k \\
\text{rank } \mathcal{F}_1 &= \sum_{\vec{n}_a \cdot \vec{d} < 0} (-\vec{n}_a \cdot \vec{d} - 1)
\end{align*}
\]
from which we see that
\[
\text{rank } \mathcal{F} - \text{rank } \mathcal{F}_1 = \text{rank } \mathcal{E} + \sum_a \vec{n}_a \cdot \vec{d}
\]
precisely as desired.

Next, consider the product of top exterior powers of \( \mathcal{F} \) and \( \mathcal{F}_1 \). From their expressions above, we see that
\[
\begin{align*}
c_1(\mathcal{F}) &= \sum_{\vec{n}_a \cdot \vec{d} \geq 0} (\vec{n}_a \cdot \vec{d} + 1) n_a^i J_i \\
c_1(\mathcal{F}_1) &= \sum_{\vec{n}_a \cdot \vec{d} < 0} (-\vec{n}_a \cdot \vec{d} - 1) n_a^i J_i
\end{align*}
\]
where the \( J_i \) generate degree two cohomology of \( \mathcal{M} \) (and may be constrained by relations).
Thus,
\[
c_1(\mathcal{F} \ominus \mathcal{F}_1) = \left( \sum_a n_a^i J_i \right) + \left( \sum_a (\vec{n}_a \cdot \vec{d}) n_a^i J_i \right),
\]
where \( \ominus \) denotes the K-theoretic difference. For the tangent bundle,
\[
c_1(T\mathcal{M} \ominus \mathcal{G}_1) = \left( \sum_n q_n^i J_i \right) + \left( \sum_n (\vec{q}_n \cdot \vec{d}) q_n^i J_i \right).
\]
However, using the anomaly cancellation condition above plus the statement that \( \Lambda^{\text{top}}E^\vee \cong K_X \), we see immediately that
\[
c_1(\mathcal{F} \ominus \mathcal{F}_1) = c_1(T\mathcal{M} \ominus \mathcal{G}_1)
\]
precisely as desired.
6.2.3 Example: Obstruction sheaves and Hirzebruch surfaces

So far we have proven general properties of this class of (0,2) gauged linear sigma models, and described details of the special case of the tangent bundle of $\mathbf{P}^{N-1}$. Next, we shall consider a special case on the (2,2) locus in which the obstruction sheaf is nontrivial, to further illustrate these ideas.

Specifically, we will consider moduli spaces of maps into the curve of self-intersection $-n$ on a Hirzebruch surface $\mathbf{F}_n$. We can describe the Hirzebruch surface in terms of four homogeneous coordinates $s, t, u, v$, with weights under two $\mathbb{C}^\times$ actions as follows:

\[
\begin{array}{c|ccccc}
 & s & t & u & v \\
\lambda & 1 & 1 & n & 0 \\
\mu & 0 & 0 & 1 & 1
\end{array}
\]

The homogeneous coordinates on the $\mathbf{P}^1$ fibers are $u, v$. Maps into the $\mathbf{P}^1$ fiber have weight $(0,1)$ under $(\lambda, \mu)$, and similarly maps into the base have weight $(1,0)$. Following [24][chapter IV], maps into the curve of self-intersection $-n$ should have weight

\[(1,0) - n(0,1) = (1,-n)\]

under $(\lambda, \mu)$. For such maps, when we expand in zero modes, we find that $s$ and $t$ are sections of $\mathcal{O}_{\mathbf{P}^1}(1)$, $u$ is a section of $\mathcal{O}_{\mathbf{P}^1}$, and $v$ is a section of $\mathcal{O}_{\mathbf{P}^1}(-n)$. Let $x$ and $y$ be homogeneous coordinates on the worldsheet $\mathbf{P}^1$, then we can expand

\[
\begin{align*}
s &= ax + by \\
t &= cx + dy \\
u &= e \\
v &= 0
\end{align*}
\]

for $a, b, c, d, e$ complex numbers which we can be identified with homogeneous coordinates on the linear sigma model moduli space. The moduli space is determined by a pair of $\mathbb{C}^\times$ actions, whose actions on the homogeneous coordinates $a, b, c, d, e$ are determined by the actions on the homogeneous coordinates $s, t, u, v$ on $\mathbf{F}_n$. If we denote the $\mathbb{C}^\times$ actions on the moduli space by $(\overline{\lambda}, \overline{\mu})$, then the homogeneous coordinates $a, b, c, d, e$ have weights given by

\[
\begin{array}{c|ccccc}
 & a & b & c & d & e \\
\overline{\lambda} & 1 & 1 & 1 & 1 & n \\
\overline{\mu} & 0 & 0 & 0 & 0 & 1
\end{array}
\]

The obstruction sheaf on the moduli space, computed by the methods given earlier, is given by

\[
H^1\left(\mathbf{P}^1, \mathcal{O}(-n)\right) \otimes_{\mathbb{C}} \mathcal{O}(0,1) = \bigoplus_{i=1}^{n-1} \mathcal{O}(0,1).
\]
But $\mathcal{O}(0, 1)$ is the trivial bundle on $\mathcal{M}$ since the apparent freedom of $e$ has been removed as noted earlier. Thus the obstruction bundle is $\mathcal{O}^{n-1}$. Since the euler class of $\mathcal{O}^{n-1}$ is trivial for $n \geq 2$, all correlation functions will vanish. This was to be expected, as the virtual dimension is $E \cdot c_1(F_n) + 2 = 4 - n$, so that all three point functions vanish for $n \geq 2$ by simple dimension considerations.

### 6.3 Bundles presented as kernels

In this section, let us consider a bundle described in linear sigma model language as a kernel. We shall check that if it obeys the usual physical constraints, then the corresponding sheaf on the moduli space has the desired $c_1$.

#### 6.3.1 An example on $\mathbf{P}^{N-1}$

Let us begin by studying examples of bundles on $\mathbf{P}^{N-1}$. Consider a bundle $\mathcal{E}$ on $\mathbf{P}^{N-1}$ defined by

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_a \mathcal{O}(n_a) \xrightarrow{F_a} \bigoplus_i \mathcal{O}(m_i) \longrightarrow 0$$

where all the $n_a$ and $m_i$ are positive.

As before, the bundle $\mathcal{F}$ on the (linear sigma model) moduli space of maps of degree $d$ is given as another kernel, expressed between two bundles on the moduli space constructed from the zero modes of the worldsheet fields corresponding to the bundles $\bigoplus \mathcal{O}(n_a)$ and $\bigoplus \mathcal{O}(m_i)$. The weights under the toric action of the zero modes are the same as the weights of the original fields, hence

$$\mathcal{F} = \ker \left\{ \bigoplus_a \mathbb{H}^0 \left( \mathbf{P}^1, \mathcal{O}(n_a d) \right) \otimes \mathcal{O}(n_a) \rightarrow \bigoplus_i \mathbb{H}^0 \left( \mathbf{P}^1, \mathcal{O}(m_i d) \right) \otimes \mathcal{O}(m_i) \right\}.$$ 

The maps are determined by the maps defining $\mathcal{E}$, in the same fashion as for cokernels. More specifically, we expand out maps into homogeneous coordinates on $\mathbf{P}^{N-1}$ in terms of homogeneous coordinates $u, v$ on the worldsheet, then expand out the maps $F_i$ in terms of those $u, v$ in order to get the precise maps above. We shall assume that the resulting map is surjective, for simplicity.

First, let us check that $\mathcal{F}$, as we have defined it above, has the correct rank. On general principles, as outlined elsewhere, the virtual rank of $\mathcal{F}$ is given by

$$c_1(\phi^* \mathcal{E}) + (\text{rk } \mathcal{E})(1 - g)$$

and $g = 0$ in the present case. Our $\mathcal{F}$ above has rank

$$\sum_a h^0 \left( \mathbf{P}^1, \mathcal{O}(n_a d) \right) - \sum_i h^0 \left( \mathbf{P}^1, \mathcal{O}(m_i d) \right)$$
\[ = \sum_a (n_a d + 1) - \sum_i (m_i d + 1) \]
\[ = d \left( \sum_a n_a - \sum_i m_i \right) + \left( \sum_a 1 - \sum_i 1 \right). \]

First, note that
\[ d \left( \sum_a n_a - \sum_i m_i \right) = c_1(\phi^* \mathcal{E}) \]
and also
\[ \text{rank } \mathcal{E} = \sum_a 1 - \sum_i 1. \]

Thus, plugging back into the expression above, we find that the rank of \( \mathcal{F} \) is the same as the predicted virtual rank of \( \mathcal{F} \).

Next, we shall compute \( c_1(\mathcal{F}) \), and demonstrate that when \( \mathcal{E} \) satisfies anomaly cancellation and has \( c_1(\mathcal{E}) = c_1(T \mathbb{P}^{N-1}) \), that \( c_1(\mathcal{F}) = c_1(T \mathcal{M}) \), as desired.

If we let \( J \) denote the generator of the integral cohomology ring of the (linear sigma model) moduli space \( \mathcal{M} = \mathbb{P}^{N(d+1)-1} \), then we have that
\[ c_1(\mathcal{F}) = \sum_a \left( h^0(\mathbb{P}^1, \mathcal{O}(n_a d)) \right) n_a J - \sum_i \left( h^0(\mathbb{P}^1, \mathcal{O}(m_i d)) \right) m_i J \]
\[ = \sum_a n_a (n_a d + 1) J - \sum_i m_i (m_i d + 1) J \]
\[ = \left( \sum_a n_a - \sum_i m_i \right) J + \left( \sum_a n_a^2 - \sum_i m_i^2 \right) dJ. \]

Now, it is easy to compute that
\[ c_1(\mathcal{E}) = \left( \sum_a n_a - \sum_i m_i \right) J' \]
(where \( J' \) generates the cohomology ring of \( \mathbb{P}^{N-1} \)) so \( c_1(\mathcal{E}) = c_1(T \mathbb{P}^{N-1}) \) implies that
\[ \left( \sum_a n_a - \sum_i m_i \right) = N. \]

Similarly, if we use \( \text{ch}_2 = \frac{1}{2} c_1^2 - c_2 \), then it is straightforward to compute that
\[ \text{ch}_2(\mathcal{E}) = \frac{1}{2} \left( \sum_a n_a^2 - \sum_i m_i^2 \right) J^2, \]
so using the anomaly cancellation condition we find that
\[ \sum_a n_a^2 - \sum_i m_i^2 = N^2 - 2 \left( \binom{N}{2} \right) \]
\[ = N. \]
Plugging these results back into the expression for $c_1(F)$, we find that

$$c_1(F) = NJ + NdJ$$
$$= N(d + 1)J$$
$$= c_1(TP^{N(d+1)} - T\mathcal{M}),$$

exactly as desired.

### 6.3.2 General analysis of bundles presented as kernels

More generally, consider a bundle $E$ over a toric variety $X$ defined as the kernel of the short exact sequence

$$0 \to E \to \bigoplus_a \mathcal{O}(\bar{n}_a) \to \bigoplus_r \mathcal{O}(\bar{m}_r) \to 0.$$

Over a linear sigma model moduli space of maps of degree $\bar{d}$, the short exact sequence above induces the long exact sequence

$$0 \to F \to \bigoplus_a H^0(P^1, \mathcal{O}(\bar{n}_a \cdot \bar{d})) \otimes \mathcal{O}(\bar{n}_a) \to \bigoplus_r H^0(P^1, \mathcal{O}(\bar{m}_r \cdot \bar{d})) \otimes \mathcal{O}(\bar{m}_r)$$
$$\to F_1 \to \bigoplus_a H^1(P^1, \mathcal{O}(\bar{n}_a \cdot \bar{d})) \otimes \mathcal{O}(\bar{n}_a) \to \bigoplus_r H^1(P^1, \mathcal{O}(\bar{m}_r \cdot \bar{d})) \otimes \mathcal{O}(\bar{m}_r)$$
$$\to 0.$$

It is straightforward to check, using the same methods as for bundles presented as cokernels, that on the open subset of $\mathcal{M}$ corresponding to honest maps, $F$ and $F_1$ match $R^0\pi_*\alpha^*E$ and $R^1\pi_*\alpha^*E$, respectively.

Let us check that the sheaves $F$, $F_1$ have the expected total rank predicted in section 5.5. 

Note that

$$\text{rank } (F \oplus F_1) = \sum_{\bar{n}_a, \bar{d} \geq 0} (\bar{n}_a \cdot \bar{d} + 1) - \sum_{\bar{m}_r, \bar{d} \geq 0} (\bar{m}_r \cdot \bar{d} + 1)$$
$$- \sum_{\bar{n}_a, \bar{d} < 0} (\bar{n}_a \cdot \bar{d} - 1) + \sum_{\bar{m}_r, \bar{d} < 0} (\bar{m}_r \cdot \bar{d} - 1)$$
$$= \left( \sum_a - \sum_r \right) + \left( \sum_a \bar{n}_a \cdot \bar{d} - \sum_r \bar{m}_r \cdot \bar{d} \right)$$
$$= \text{rank } E + c_1(\phi^*E)$$
$$= \chi(\phi^*E)$$

precisely as desired.

Next, let us check that the condition

$$\Lambda^\top F^\vee \otimes \Lambda^\top F_1 \cong K_{\mathcal{M}} \otimes \Lambda^\top G^1;$$

is satisfied, essentially by an argument based on topological K-theory.
as derived formally in section 5.5, is indeed satisfied by the sheaves \( \mathcal{F}, \mathcal{F}_1 \) above. Present the tangent bundle to the toric variety in the form

\[
0 \to O^\oplus m \to \oplus_a O(\vec{q}_a) \to TX \to 0.
\]

The anomaly cancellation condition in the (0,2) gauged linear sigma model has the form

\[
\sum_a n^i_a n^j_a - \sum_r m^i_r m^j_r = \sum_n q^i_n q^j_n
\]

for all \( i, j \), and the constraint that \( \Lambda^{\text{top}} \mathcal{E}^\vee \cong K_X \) implies that

\[
\sum_a n^i_a - \sum_r m^i_r = \sum_n q^i_n
\]

for each \( i \). Then, from the long exact sequence above, we see that

\[
c_1 (\mathcal{F} \ominus \mathcal{F}_1) = \sum_a (\vec{n}_a \cdot \vec{d} + 1) n^i_a J_i - \sum_r (\vec{m}_r \cdot \vec{d} + 1) m^i_r J_i
\]

\[
- \sum_{\vec{n}_a \cdot \vec{d} < 0} (-\vec{n}_a \cdot \vec{d} - 1) n^i_a J_i + \sum_{\vec{m}_r \cdot \vec{d} < 0} (-\vec{m}_r \cdot \vec{d} - 1) m^i_r J_i
\]

\[
= \left( \sum_a n^i_a J_i \right) + \left( \sum_a (\vec{n}_a \cdot \vec{d}) n^i_a J_i \right) - \left( \sum_r m^i_r J_i \right) - \left( \sum_r (\vec{m}_r \cdot \vec{d}) m^i_r J_i \right).
\]

For the tangent bundle, we similarly obtain

\[
c_1 (TM \ominus G_1) = \left( \sum_n q^i_n J_i \right) + \left( \sum_n (\vec{q}_n \cdot \vec{d}) q^i_n J_i \right).
\]

Thus, using the anomaly cancellation condition and the statement that \( \Lambda^{\text{top}} \mathcal{E}^\vee \cong K_X \), we see that

\[
c_1 (\mathcal{F} \ominus \mathcal{F}_1) = c_1 (TM \ominus G_1),
\]

precisely as predicted on general grounds in section 5.5.

### 6.4 Bundles presented as the cohomology of a monad

The previous discussion also extends to bundles presented as monads, as we shall now briefly outline. In this case, the monad is a short complex of the form

\[
0 \to O^\oplus k \overset{E}{\to} \oplus_a O(\vec{q}_a) \overset{F}{\to} \oplus_i O(\vec{m}_i) \to 0
\]

and the bundle \( \mathcal{E} \) is defined by

\[
\mathcal{E} \equiv \ker \frac{F}{\text{im } E}.
\]
Physically, this is realized via $(0,2)$ fermi multiplets $\Lambda^a$ of charge $\vec{n}_a$, $(0,2)$ chiral multiplets $p_i$ of charge $\vec{m}_i$, and neutral $(0,2)$ chiral multiplets $\Sigma^\lambda$, together with a superpotential term realizing $F$, and susy transformations realizing $E$.

For simplicity, we break the cohomology of the complex into a pair of short exact sequences

$$\begin{align*}
0 \rightarrow \ker F &\rightarrow \oplus_a \mathcal{O}(\vec{n}_a) \xrightarrow{F} \oplus_i \mathcal{O}(\vec{m}_i) \rightarrow 0, \\
0 \rightarrow \mathcal{O}^{\oplus k} &\xrightarrow{E} \ker F \rightarrow \mathcal{E} \rightarrow 0.
\end{align*}$$

Expanding fields into their zero modes, we see that the short exact sequence (18) induces

$$\begin{align*}
0 \rightarrow (\ker F)_0 &\rightarrow \oplus_a H^0 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a) \rightarrow \oplus_i H^0 \left( \mathbb{P}^1, \mathcal{O}(\vec{m}_i \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{m}_i) \\
\rightarrow (\ker F)_1 &\rightarrow \oplus_a H^1 \left( \mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a) \rightarrow \oplus_i H^1 \left( \mathbb{P}^1, \mathcal{O}(\vec{m}_i \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{m}_i) \\
\rightarrow 0
\end{align*}$$

and the short exact sequence (19) induces

$$\begin{align*}
0 \rightarrow \mathcal{O}^{\oplus k} &\xrightarrow{E} (\ker F)_0 \rightarrow \mathcal{F} \rightarrow 0, \\
\mathcal{F}_1 &\cong (\ker F)_1.
\end{align*}$$

It is now straightforward to compute that

$$\begin{align*}
\text{rank } \mathcal{F} - \text{rank } \mathcal{F}_1 &= \text{rank } \mathcal{E} + c_1(\phi^* \mathcal{E}), \\
c_1(\mathcal{F}) - c_1(\mathcal{F}_1) &= c_1(TM) - c_1(G_1)
\end{align*}$$

exactly as desired.

### 6.5 A technical aside

A linear sigma model describes a physically natural compactification of moduli spaces of maps, and $(0,2)$ linear sigma models describe physically natural extensions of the sheaves $R^1\pi_*\alpha^*\mathcal{E}$ over the compactifications of the moduli spaces. However, we shall see in this section that the precise extensions depend upon the presentation of the gauge bundle $\mathcal{E}$, and not all presentations yield consistent linear sigma models.

In particular, it is important for our construction that the linear sigma model anomaly cancellation condition be met, which is somewhat stronger than the mathematical statement $\text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$. We shall see that the linear sigma model anomaly cancellation condition can even distinguish different presentations of the same gauge bundle. Presentations that
fail the linear sigma model anomaly cancellation condition have different extensions over the compactification divisor which do not have desirable properties.

If we schematically let charges of left-moving fermions be denoted \( \vec{n}_a \) and charges of right-moving fermions be denoted \( \vec{q}_s \), then we have imposed two conditions, namely

\[
\sum_a n_a^i = \sum_s q_s^i \quad \text{for each } i,
\]

\[
\sum_a n_a^i n_a^j = \sum_s q_s^i q_s^j \quad \text{for each } i,j.
\]

The first of these statements is the linear-sigma-model version of the requirement \( \Lambda^{top} \mathcal{E}^\vee \cong K_X \), and the second [2] is the linear-sigma-model version of the anomaly cancellation condition \( \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX) \).

Our linear-sigma-model requirements are slightly stronger than their mathematical counterparts. For example, consider the tangent bundle of \( \mathbb{P}^1 \), presented in its physically canonical form as

\[
0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^2 \longrightarrow T\mathbb{P}^1 \longrightarrow 0
\]

Suppose the gauge bundle \( \mathcal{E} \) is described by a single free fermion of charge 2. Mathematically, this is the statement that \( \mathcal{E} = \mathcal{O}(2) \), which is isomorphic to \( T\mathbb{P}^1 \). Since \( \mathcal{E} \cong T\mathbb{P}^1 \), we trivially have the statement that \( \text{ch}_2(\mathcal{E}) = \text{ch}_2(T\mathbb{P}^1) \). However, this setup no longer satisfies the linear sigma model anomaly cancellation condition:

\[
\sum_s (q_s) (q_s) = \sum_{1}^{2} (1)^2 = 2
\]

\[
\sum_a (n_a) (n_a) = \sum_{1}^{1} (2)^2 = 4
\]

and so physically, the bundle \( \mathcal{E} \) described by a single free fermion of charge 2 on \( \mathbb{P}^1 \) does not describe a consistent linear sigma model, even though mathematically \( \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX) \).

Let us apply our construction to this inconsistent linear sigma model. If we present the tangent bundle of \( \mathbb{P}^1 \) in its physically canonical form, as a pair of left-moving fermions of charge 1 with a ‘gauged fermionic symmetry,’ the resulting sheaf \( \mathcal{F} \) over the moduli space is the tangent bundle to the moduli space. However, if we present it as merely \( \mathcal{O}(2) \), i.e. as a single free fermion of charge 2, which physically is not the canonical presentation, then the resulting sheaf \( \mathcal{F} \) over the moduli space is given by

\[
H^0 \left( \mathbb{P}^1, \mathcal{O}(2d) \right) \otimes_{\mathbb{C}} \mathcal{O}(2) = \oplus_{1}^{2d+1} \mathcal{O}(2)
\]

which clearly is not only no longer the tangent bundle of the moduli space \( \mathbb{P}^{2d+1} \), but in fact does not even have the property \( \Lambda^{top} \mathcal{F}^\vee \cong K_M \). This \( \mathcal{F} \) is a very different extension of \( R^0 \pi_* \alpha^* \mathcal{E} \).
Thus, we see in this example that the gauged linear sigma model anomaly cancellation condition, which is slightly stronger than merely \( \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX) \), plays an important role in the consistency of our construction. When we applied our construction to an inconsistent linear sigma model, the induced sheaves \( \mathcal{F}, \mathcal{F}_1 \) do not have correct Chern classes. Furthermore, this also shows that the extensions across the compactification divisor of \( R^i \pi_* \alpha^* \mathcal{E} \) defined by our construction depend upon the presentation of \( \mathcal{E} \).

The reader might object that this phenomenon is too degenerate, in that \( \mathbb{P}^1 \) only has dimension one, and so can not provide good examples for subtleties revolving around second Chern classes. However, there are more examples in higher dimensions.

The tangent bundle of \( \mathbb{P}^1 \times \mathbb{P}^1 \) gives a slightly less degenerate example of this phenomenon. There are three presentations of this bundle:

1. The standard (2,2) presentation, as a cokernel

\[
0 \rightarrow \mathcal{O}^{\oplus 2} \rightarrow \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \rightarrow T(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow 0
\]

corresponding physically to four charged fermi superfields (part of the (2,2) chiral superfields describing the homogeneous coordinates) and two neutral chiral superfields (part of the (2,2) gauge superfields corresponding to the two \( U(1) \)'s).

2. A cokernel presentation for one \( T\mathbb{P}^1 \), and a free fermion presentation for the other:

\[
0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,2) \rightarrow T(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow 0.
\]

3. A free fermion presentation for both sides:

\[
T(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathcal{O}(2,0) \oplus \mathcal{O}(0,2).
\]

Only the standard (2,2) presentation satisfies the strong linear sigma model anomaly cancellation condition for a gauge bundle on \( \mathbb{P}^1 \times \mathbb{P}^1 \); the second two presentations do not satisfy this condition. As a result, each of these physical presentations of the tangent bundle gives a different induced sheaf \( \mathcal{F} \) over the compactified moduli space, and only for the (2,2) presentation does the resulting induced sheaf have desirable properties.

The first presentation yields \( \mathcal{F} \) isomorphic to the tangent bundle of the moduli space \( \mathbb{P}^{2d_1+1} \times \mathbb{P}^{2d_2+1} \).

The second presentation yields

\[
0 \rightarrow \mathcal{O} \rightarrow \oplus_1^2 \left( H^0 \left( \mathbb{P}^1, \mathcal{O}(d_1) \right) \otimes \mathcal{O}(1,0) \right) \oplus \left( H^0 \left( \mathbb{P}^1, \mathcal{O}(2d_2) \right) \otimes \mathcal{O}(0,2) \right) \rightarrow \mathcal{F} \rightarrow 0
\]

or, equivalently,

\[
\mathcal{F} \cong \pi^*_1 TP^{2d_1+1} \oplus \oplus_1^{2d_2+1} \mathcal{O}(0,2).
\]

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The third presentation yields
\[ \mathcal{F} = \left( H^0 \left( \mathbb{P}^1, \mathcal{O}(2d_1) \right) \otimes \mathcal{O}(2,0) \right) \oplus \left( H^0 \left( \mathbb{P}^1, \mathcal{O}(2d_2) \right) \otimes \mathcal{O}(0,2) \right) \]
\[ = \bigoplus_1^{2d_1+1} \mathcal{O}(2,0) \oplus \bigoplus_1^{2d_2+1} \mathcal{O}(0,2). \]

So long as \( d_1 \) and \( d_2 \) are both positive, \( \mathcal{F}_1 \equiv 0 \) in each case.

Note that for each presentation,
\[ \text{rank } \mathcal{F} = (2d_1 + 1) + (2d_2 + 1) \]
\[ = \text{rank } T \left( \mathbb{P}^1 \times \mathbb{P}^1 \right) + c_1 \left( \phi^* T \left( \mathbb{P}^1 \times \mathbb{P}^1 \right) \right). \]

However, although the ranks match, it is straightforward to check that, just as for \( TP^1 \), the first Chern classes do not match, and (except for the first presentation) do not have the desired symmetry property \( \Lambda^{\text{top}} \mathcal{F} \cong K_M \). The reason for this is the same as for \( P^1 \): the linear sigma model anomaly cancellation condition is only met for the first presentation.

Thus, we see in these examples several important technicalities of linear sigma models and our constructions:

- First, different presentations of a single, fixed gauge bundle \( \mathcal{E} \) can define different extensions of \( R^i \pi_* \alpha^* \mathcal{E} \) across the compactified moduli space.
- Second, the linear sigma model anomaly cancellation condition can distinguish different presentations of the same bundle.
- Third, if a given presentation of the gauge bundle fails the linear sigma model anomaly cancellation condition, then even if there exists an alternate presentation that satisfies that condition, the induced sheaves \( \mathcal{F}, \mathcal{F}_1 \) determined by the failing presentation have the wrong Chern classes to be useful in our calculations of correlation functions.

### 7 Verification of results of Adams-Basu-Sethi

In this section we shall give first-principles verifications of some conjectures made in [1].
7.1 Physical prediction for quantum cohomology

One of the first examples of a heterotic quantum cohomology ring\textsuperscript{14} considered in [1][section 6.2] concerns a heterotic theory on $\mathbb{P}^1 \times \mathbb{P}^1$, with gauge bundle given by a deformation of the tangent bundle. Recall that we can describe the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$ as the cokernel of the map

$$
\mathcal{O} \oplus \mathcal{O} \xrightarrow{[x_1 \ 0 \ 0 \ \tilde{x}_1 \ 0 \ \tilde{x}_2 \ 0 \ x_2]} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2
$$

where $x_1, x_2$ are homogeneous coordinates on the first $\mathbb{P}^1$, and $\tilde{x}_1, \tilde{x}_2$ are homogeneous coordinates on the second $\mathbb{P}^1$. In [1][section 6.2] a deformation of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$ is described as the cokernel of the map

$$
\mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{bmatrix} x_1 & \alpha x_1 + \alpha' x_2 \\ x_2 & \beta x_1 + \beta' x_2 \end{bmatrix}} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2
$$

where $\alpha, \alpha', \beta, \beta'$ are constants.

In the special case that $\alpha = \epsilon_1$ and $\alpha' = \epsilon_2$, and all other constants vanish, the authors of [1] obtain what can be described as a deformation of the chiral ring relations for a $(2,2)$ model on $\mathbb{P}^1 \times \mathbb{P}^1$, namely

$$
\tilde{X}^2 = \exp(it_2) \\
X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} = \exp(it_1)
$$

where $t_1, t_2$ are Kähler parameters describing the sizes of the $\mathbb{P}^1$’s, and $X, \tilde{X}$ are degree two generators. (The sign ambiguity is meaningless, because $\epsilon_1, \epsilon_2$ act as homogeneous coordinates on the deformation space.) Note in the special case of the $(2,2)$ locus, where $\epsilon_1 = \epsilon_2$, the relations above reduce precisely to the ordinary quantum chiral ring relations for $\mathbb{P}^1 \times \mathbb{P}^1$.

In terms of correlation functions, the quantum cohomology claim above allows us to compute quantum corrections in terms of classical results. We shall describe the classical computation in section 7.2. Without even doing the classical computation, it is not hard to see that consistency\textsuperscript{15} of the ring structure forces a nonzero value for $\langle X^2 \rangle$ when $\epsilon_1 \neq \epsilon_2$

\textsuperscript{14}As emphasized earlier, in a heterotic theory, one will not be able to make sense of a ring structure in general, but in special cases – such as the deformation of the tangent bundle considered here – such a structure can still be meaningful.

\textsuperscript{15}Write $\langle X^2 \rangle = g$ for some function $g$ of $\epsilon_1, \epsilon_2$, with the other classical correlation functions as listed. One can then compute $\langle X \tilde{X}^3 \rangle = \exp(it_2)$ and $\langle X^2 \tilde{X}^2 \rangle = g \exp(it_2)$. Then, plug those values into $\langle \tilde{X}^2 (X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X}) \rangle = 0$ to find that $g = \epsilon_1 - \epsilon_2$.
(i.e. off the (2, 2) locus) while the other classical correlation functions are unchanged. In any event, the classical correlation functions can be shown to be given by

\[ < \tilde{X}^2 > = < 1 > = 0, \quad < X \tilde{X} > = 1, \quad < X^2 > = \epsilon_1 - \epsilon_2. \quad (20) \]

We can formally compute a higher-order correlator by using the quantum cohomology relations, \textit{e.g.}

\[ < X \tilde{X} > = < (X \tilde{X}) \tilde{X}^2 > \quad (21) \]
\[ = < X \tilde{X} > \exp(it_2) = \exp(it_2) \quad (22) \]
\[ < X \tilde{X}^5 > = < X \tilde{X} (\tilde{X}^2)^2 > = \exp(2it_2) \quad (23) \]
\[ < X \tilde{X}^7 > = \exp(3it_2) \quad (24) \]
\[ < \tilde{X}^4 > = < 1 > \exp(2it_2) = 0 \quad (25) \]
\[ < X^2 \tilde{X}^2 > = < X^2 > \exp(it_2) = (\epsilon_1 - \epsilon_2) \exp(it_2) \quad (26) \]
\[ < X \tilde{X} (X^2 - (\epsilon_1 - \epsilon_2)X \tilde{X}) > = < X \tilde{X} > \exp(it_1) = \exp(it_1) \quad (27) \]
\[ < X^3 \tilde{X} > = \exp(it_1) + (\epsilon_1 - \epsilon_2)^2 \exp(it_2) \quad (28) \]
\[ < X^4 > = < X^2 (\exp(it_1) + (\epsilon_1 - \epsilon_2)X \tilde{X}) > \quad (29) \]
\[ = 2(\epsilon_1 - \epsilon_2)^3 \exp(it_2) \quad (30) \]

and so forth.

In section 7.2, we shall directly compute the four-point functions listed above, and we will directly confirm that they do have the form listed. In general, one would only expect to obtain the answer up to a coordinate change, of course, but in this simple example, we shall recover the advertised four-point functions.

One quick physical way to derive the quantum cohomology statement above (following [20], but slightly different from [1]) is to compute the one-loop effective superpotential

\[ \tilde{W}_{\text{eff}} = \Upsilon_1 \left( i \tilde{\tau}_1 - \frac{1}{2\pi} \log \left( \frac{\sigma_1 + \epsilon_1 \sigma_2}{\Lambda} \right) - \frac{1}{2\pi} \log \left( \frac{\sigma_1 + \epsilon_2 \sigma_2}{\Lambda} \right) \right) + \Upsilon_2 \left( i \tilde{\tau}_2 - \frac{1}{2\pi} \log \left( \frac{\sigma_2}{\Lambda} \right) - \frac{1}{2\pi} \log \left( \frac{\sigma_2}{\Lambda} \right) \right) \]

(where the \( \Upsilon \)'s are (0,2) gauge multiplets), from which setting

\[ \frac{\partial \tilde{W}_{\text{eff}}}{\partial \Upsilon_a} = 0 \]

gives the relations

\[ (\sigma_1 + \epsilon_1 \sigma_2)(\sigma_1 + \epsilon_2 \sigma_2) = q_1 \]
\[ \sigma_2^2 = q_2 \]

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which, after a change of variables, are equivalent to the relations given in [1].

In the case of interest, the gauge bundle $E$ is given by

$$0 \rightarrow \mathcal{O} \oplus \mathcal{O} \rightarrow \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \rightarrow E \rightarrow 0.$$  

(31)

For degree $(1,0)$ maps, the linear sigma model moduli space is $\mathbb{P}^3 \times \mathbb{P}^1$. If we let $\alpha_0$, $\alpha_1$, $\alpha'_0$, $\alpha'_1$ denote homogeneous coordinates on the $\mathbb{P}^3$ (obtained from expanding out the homogeneous coordinates $x_1$, $x_2$ in terms of their zero modes), and let $\beta_0$, $\beta_1$ denote the homogeneous coordinates on the $\mathbb{P}^1$, then the sheaf $F$ on the moduli space is described by

$$0 \rightarrow \mathcal{O} \oplus \mathcal{O} \rightarrow \mathcal{O}(1,0)^4 \oplus \mathcal{O}(0,1)^2 \rightarrow F \rightarrow 0.$$  

(32)

Next, we shall find polynomial representatives for the relevant sheaf cohomology groups. From (31), we derive the long exact sequence

$$0 \rightarrow H^0(X,E^\vee) \rightarrow H^0(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2) \rightarrow H^0(\mathcal{O}^2) \rightarrow H^1(X,E^\vee) \rightarrow H^1(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2) \rightarrow \cdots$$  

(33)

from which we read off that

$$H^1(X,E^\vee) = H^0(\mathcal{O}^2) = \mathbb{C}^2.$$  

(34)

In particular, the two elements of $H^1(X,E^\vee)$ can both be represented by constants. Essentially the same calculation using (32) in place of (31) reveals that

$$H^1(\mathcal{M},F^\vee) = H^0(\mathcal{M},\mathcal{O}^2) = \mathbb{C}^2.$$  

We therefore have a natural map

$$H^1(X,E^\vee) \simeq \mathbb{C}^2 \simeq H^1(\mathcal{M},F^\vee)$$

which is the linear sigma model version of the maps $\psi_i$ (9).
7.2 Computation of the correlation functions

Let us now explicitly compute the classical correlation functions listed in equation (20) and the four-point functions listed in equations (30), (28), (26), (22), and (25). The classical contributions to these four-point functions all vanish, and the only worldsheet instanton contributions can come from the $(1,0)$ and $(0,1)$ sectors.

Before we begin, we need to observe that the natural basis (34) for $H^2(P^1 \times P^1, \mathcal{E}^\vee)$ does not specialize to the usual basis for $H^2(P^1 \times P^1)$ generated by the hyperplane classes of the $P^1$ factors on the $(2,2)$ locus. Let’s denote the basis determined by (34) as $Y, \tilde{Y}$. The $4 \times 2$ matrix in (31), specialized to the $(2,2)$ locus $\epsilon_1 = \epsilon_2$, shows that $Y, \tilde{Y}$ is related to the standard quantum cohomology basis $X, \tilde{X}$ by

$$X = Y + \epsilon \tilde{Y}, \quad \tilde{X} = \tilde{Y},$$

as can be inferred from the first two rows (resp. the last 2 rows). Here we have put $\epsilon = \epsilon_1 = \epsilon_2$.

To explain the method, we compute the classical correlation functions of $Y$ and $\tilde{Y}$ in detail. We start with the dual of (31)

$$0 \to \mathcal{E}^\vee \to \mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2 \xrightarrow{\begin{bmatrix} x_1 & x_2 \\ \epsilon_1 x_1 & \epsilon_2 x_2 \end{bmatrix}} \mathcal{O}^2 \to 0. \quad (36)$$

Let us denote by $e_1, e_2, f_1, f_2$ the natural basis for $\mathcal{O}^4$, corresponding to the columns of the matrix in (36). We want to compute the cohomology classes $Y, \tilde{Y} \in H^1(\mathcal{E}^\vee)$. We do this by computing images of $(1,0)$ $(0,1) \in H^0(\mathcal{O}^2) = \mathbb{C}^2$ of the coboundary mapping in (33). We compute using the Cech cover $U_{ij} = \{ x_i \neq 0, \bar{x}_j \neq 0 \}$ of $P^1 \times P^1$, where $1 \leq i, j \leq 2$.

Let us first compute $\tilde{Y}$ as the coboundary of $(0,1)$. First we must lift $(0,1)$ to a section of $\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2$ over $U_{ij}$, and the simplest lift is $\tilde{x}_j^{-1} f_j$. Then $\tilde{Y}$ has a Cech representative

$$\tilde{Y}_{ij, \tilde{i}j'} = \tilde{x}_j^{-1} f_{j'} - \tilde{x}_j^{-1} f_j.$$  

To make sense of (37) we fix an ordering of the sets in the open cover, say $U_{11}, U_{12}, U_{21}, U_{22}$.

Similarly, we lift $(1,0)$ to $x_i^{-1} e_i - \epsilon_i \tilde{x}_j^{-1} f_j$ on $U_{ij}$, yielding the Cech representative

$$Y_{ij, \tilde{i}j'} = x_i^{-1} e_{i'} - x_i^{-1} e_i + \epsilon_i \tilde{x}_j^{-1} f_j - \epsilon_{i'} \tilde{x}_j^{-1} f_{j'}.$$  

To compute the classical correlator $\langle \tilde{Y}^2 \rangle$, we take the cup and wedge product of $\tilde{Y}$ with itself to get a Cech representative of $H^2(\Lambda^2 \mathcal{E}^\vee)$. From our constraint (1) this is identified with an element of $H^2(K) \simeq \mathbb{C}$ so we will get a number. In the present situation of $P^1 \times P^1$, we have $K \simeq \mathcal{O}(-2,-2)$. 

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To compute explicitly, note the inclusion
\[ \Lambda^2 E^\vee \cong O(-2, -2) \hookrightarrow \Lambda^2 \left( O(-1, 0)^2 \oplus O(0, -1)^2 \right) \cong O(-2, 0) \oplus O(-1, -1)^4 \oplus O(0, -2). \]
(39)

Our strategy is to first compute the cup product as a representative of \( H^2(\Lambda^2(O(-1, 0)^2 \oplus O(0, -1)^2) = H^2(O(-2, 0) \oplus O(-1, -1)^4 \oplus O(0, -2)) \) using the inclusion (39). Then we interpret this cocycle as a representative of \( H^2(\Lambda^2 E^\vee) = H^2(O(-2, -2)). \) We do this explicitly by computing the inclusion (39). So the computation proceeds from (38) and (37) by the explicit computation of (39) and the explicit computation of the cup product.

For any sheaves \( S, T, \) the cup product
\[ H^1(S) \otimes H^1(T) \to H^2(S \otimes T) \]
is given in terms of Cech representatives \( \omega \) and \( \eta \) of elements of \( H^1(S) \) and \( H^1(T) \) by
\[ (\omega \cup \eta)_{abc} = \omega_{ab} \otimes \eta_{bc}, \]
where restrictions to appropriate smaller open sets are understood and suppressed from the notation. Note that \( \omega \cup \eta \) is indeed a cocycle if \( \omega \) and \( \eta \) are:
\[ \delta (\omega \cup \eta)_{abcd} = \omega_{bc} \otimes \eta_{cd} - \omega_{ac} \otimes \eta_{cd} + \omega_{ab} \otimes \eta_{bd} - \omega_{ab} \otimes \eta_{bc} = (\omega_{bc} - \omega_{ac}) \otimes \eta_{cd} + \omega_{ab} \otimes (\eta_{bd} - \eta_{bc}) = -\omega_{ab} \otimes \eta_{cd} + \omega_{ab} \otimes \eta_{cd} = 0, \]
and similarly \( \omega \cup \eta \) is a coboundary if either \( \omega \) or \( \eta \) is a coboundary while the other is a cocycle.

To describe the map (39) we note that any map
\[ O(-2, -2) \to O(-2, 0) \oplus O(-1, -1)^4 \oplus O(0, -2) \]
is given by multiplication with an element of
\[ O(0, 2) \oplus O(1, 1)^4 \oplus O(2, 0) \]
(40)
which we now compute. From (36) we compute that on the open set where \( \tilde{x}_1 \neq 0 \), the sheaf \( E^\vee \) is the row space of
\[ \begin{pmatrix} \tilde{x}_1 x_2 & -\tilde{x}_1 x_1 & (\epsilon_2 - \epsilon_1) x_1 x_2 & 0 \\ 0 & 0 & \tilde{x}_2 & -\tilde{x}_1 \end{pmatrix} \]
with a similar expression when \( \tilde{x}_2 \neq 0 \). The maximal minors of this matrix are
\[ \begin{pmatrix} 0 & x_2 \tilde{x}_1 \tilde{x}_2 & -\tilde{x}_1^2 x_2 & -x_1 \tilde{x}_1 \tilde{x}_2 & x_1 \tilde{x}_1^2 & (\epsilon_2 - \epsilon_1) x_1 x_2 \tilde{x}_1 \end{pmatrix}. \]
Note that this is a multiple of
\[
\begin{pmatrix}
0 & x_2 \bar{x}_2 & -x_1 x_2 & -x_1 \bar{x}_2 & x_1 \bar{x}_1 & (\epsilon_2 - \epsilon_1)x_1 x_2
\end{pmatrix}.
\] (41)
We get precisely the same factor over the open set where \(x_2 \neq 0\), therefore this must be the desired element of (40). We can rewrite this as
\[
x_2 \bar{x}_2 e_1 \wedge f_1 - x_1 x_2 e_1 \wedge f_2 - x_1 \bar{x}_2 e_2 \wedge f_1 + x_1 \bar{x}_1 e_2 \wedge f_2 + (\epsilon_2 - \epsilon_1)x_1 x_2 f_1 \wedge f_2.
\] (42)
This gives a computational simplification. When we compute a cup and wedge product of elements of \(H^1(E)\), then the resulting Cech representative written as a representative of \(H^2(O(-2,0) \oplus O(-1,-1)^4 \oplus O(0,-2))\) must be a multiple of (42) on each open set. To find the multiple, hence the class in \(H^2(O(-2,-2))\), we need only compute the coefficient of one of the \(e_i \wedge f_j\), say, \(e_1 \wedge f_1\).

Let’s compute \(\langle \bar{Y}^2 \rangle\). Since there are no \(e_i\) in the Cech representatives (37) for \(\bar{Y}_{ij,i'j'}\), we cannot obtain an \(e_1 \wedge f_1\) term in the cup product. Hence \(\langle \bar{Y}^2 \rangle = 0\).

For \(\langle Y \bar{Y} \rangle\) we compute using (37) and (38), omitting terms not involving \(e_1\) or \(f_1\)
\[
\begin{align*}
(Y \cup \bar{Y})_{11,21} &= \left(\epsilon_1 \bar{x}_1^{-1} f_1\right) \wedge \left(-\bar{x}_1^{-1} f_1\right) = 0 \\
(Y \cup \bar{Y})_{12,22} &= \left(-x_1^{-1} e_1 + \left(\epsilon_1 \bar{x}_1^{-1} - \epsilon_2 \bar{x}_2^{-1}\right) f_1\right) \wedge \left(-\bar{x}_1^{-1} f_1\right) = \frac{1}{x_1 \bar{x}_1} e_1 \wedge f_1 \\
(Y \cup \bar{Y})_{11,22} &= \left(\epsilon_1 \bar{x}_1^{-1} f_1\right) \wedge 0 = 0 \\
(Y \cup \bar{Y})_{12,21} &= \left(-x_1^{-1} e_1 - \epsilon_2 \bar{x}_1^{-1} f_1\right) \wedge \left(-\bar{x}_1^{-1} f_1\right) = \frac{1}{x_1 \bar{x}_1} e_1 \wedge f_1.
\end{align*}
\]
After dividing by the coefficient \(x_2 \bar{x}_2\) of \(e_1 \wedge f_1\) from (42), we learn that the Cech representative \(\phi\) of \(Y \bar{Y}\) as an element of \(O(-2,-2)\) is given by
\[
\phi_{11,21,22} = 0, \quad \phi_{11,22,12} = (x_1 x_2 \bar{x}_1 \bar{x}_2)^{-1}, \quad \phi_{11,12,22} = 0, \quad \phi_{12,21,22} = (x_1 x_2 \bar{x}_1 \bar{x}_2)^{-1}.
\] (43)

We now make an explicit choice of isomorphism \(H^2(O(-2,-2)) \simeq \mathbb{C}\). First of all, it is not hard to see that Cech representatives of \(H^2(O(-2,-2)) \simeq \mathbb{C}\) that do not have a term involving \(x_1 x_2 \bar{x}_1 \bar{x}_2\) are coboundaries. For example, a section of the form \((x_1^2 \bar{x}_1 \bar{x}_2)^{-1}\) can be extended to \(U_{11} \cap U_{12}\) and then used to construct a coboundary. So we only need to look at coefficients of \(1/(x_1 x_2 \bar{x}_1 \bar{x}_2)\). Let \(A_{ij,i'j',i''j''}\) be these coefficients. Note that the cocycle condition implies
\[
A_{12,21,22} - A_{11,21,22} + A_{11,12,22} - A_{11,12,21} = 0.
\] (44)

Now take a section \(\rho\) on \(U_{11} \cap U_{22}\) and compute
\[
\delta \rho_{11,12,22} = \delta \rho_{11,21,22} = \rho.
\]
Similarly, if $\rho$ is a section on $U_{12} \cap U_{21}$ we compute

$$\delta \rho_{11,21} = \delta \rho_{12,21,22} = \rho.$$  

Since these generate all possible ways of getting terms $(x_1 x_2 \bar{x}_1^{-1} \bar{x}_2)^{-1}$ in coboundaries we conclude that the coboundaries satisfy

$$A_{11,12,22} = A_{11,21,22}, \quad A_{11,12,21} = A_{12,21,22}. \quad \text{(45)}$$

We therefore can define

$$\text{tr} : H^2(\mathcal{O}(-2, -2)) \to \mathbb{C}, \quad \text{tr}(\omega) = A_\omega^{11,12,22} - A_\omega^{11,21,22}, \quad \text{(46)}$$

where $A_\omega$ is the coefficient of $(x_1 x_2 \bar{x}_1^{-1} \bar{x}_2)^{-1}$ in the Cech cocycle $\omega$. Note that this is the unique (up to a multiple) linear functional on the $A$'s subject to the cocycle condition (44) which vanishes on coboundaries.

Finally, applying (46) to (43) we get $\text{tr}(\phi) = 1$. Thus $\langle Y \bar{Y} \rangle = 1$.

We similarly compute the cup product of $Y$ with itself using (38) and omitting terms not involving $e_1 \wedge f_1$

$$(Y \cup Y)_{11,12,21} = \frac{\epsilon_1}{x_1 \bar{x}_1} e_1 \wedge f_1,$$

$$(Y \cup Y)_{11,21,22} = -\frac{\epsilon_2}{x_1 \bar{x}_1} e_1 \wedge f_1,$$

$$(Y \cup Y)_{11,12,22} = \frac{\epsilon_1}{x_1 \bar{x}_1} e_1 \wedge f_1,$$

$$(Y \cup Y)_{12,21,22} = -\frac{\epsilon_2}{x_1 \bar{x}_1} e_1 \wedge f_1.$$

Thus the Cech representative $\rho$ in $H^2(\mathcal{O}(-2, -2)$ satisfies

$$A^\rho_{11,12,21} = \epsilon_1, \quad A^\rho_{11,21,22} = -\epsilon_2, \quad A^\rho_{11,12,22} = \epsilon_1, \quad A^\rho_{12,21,22} = -\epsilon_2,$$

leading to $\text{tr}(\rho) = -(\epsilon_1 + \epsilon_2)$. Hence $\langle Y^2 \rangle = -(\epsilon_1 + \epsilon_2)$.

Summarizing the classical computation, we have computed

$$\langle \bar{Y}^2 \rangle = 0, \quad \langle Y \bar{Y} \rangle = 1, \quad \langle Y^2 \rangle = -(\epsilon_1 + \epsilon_2).$$

We now make the substitution

$$X = Y + \epsilon_1 \bar{Y}, \quad \bar{X} = \bar{Y}, \quad \text{(47)}$$

which extends (35) which held on the $(2,2)$ locus. With these new variables, we once again get the usual classical correlators (20).
We now compute in the degree $(1, 0)$ instanton sector. As we have seen, there are several parts to giving a mathematical formulation of the correlation functions. First, we need to give a map

$$\psi : H^p(X, \Lambda^q \mathcal{E}^\vee) \rightarrow H^p(\mathcal{M}, \Lambda^q \mathcal{F}^\vee).$$

To compute a correlation function $\langle \phi_1, \ldots, \phi_n \rangle_{1,0}$, we start by multiplying the $\psi(\phi_i)$ (via cup and wedge products) to get an element of $H^4(\mathcal{M}, \Lambda^4 \mathcal{F}^\vee)$, so we next need to be able to compute cup and wedge products. Finally, we need to evaluate this class numerically. Since $\Lambda^4 \mathcal{F}^\vee \simeq K_{\mathcal{M}}$, we have $H^4(\mathcal{M}, \Lambda^4 \mathcal{F}^\vee) \simeq C$. As in the classical calculation, we will use Cech cohomology and a trace map

$$\text{tr} : H^4(\mathcal{M}, \Lambda^4 \mathcal{F}^\vee) \rightarrow C$$

Then the correlation functions are

$$\langle \phi_1, \ldots, \phi_n \rangle_{1,0} = \text{tr} (\psi(\phi_1) \cup \cdots \cup \psi(\phi_n)). \quad (48)$$

where the subscript emphasizes that we are looking only at instanton contributions of a fixed degree.

As we have seen before, from the dual of (31) we have $H^1(X, \mathcal{E}^\vee) = C^2$ while from the dual of (32) we have $H^1(\mathcal{M}, \mathcal{F}^\vee) = C^2$. Explicitly the exact sequence

$$0 \rightarrow \mathcal{F}^\vee \rightarrow \mathcal{O}(-1, 0)^4 \oplus \mathcal{O}(0, -1)^2 \rightarrow \mathcal{O}^2 \rightarrow 0 \quad (49)$$

whose coboundary map

$$H^0(\mathcal{M}, \mathcal{O}^2) \rightarrow H^1(\mathcal{M}, \mathcal{F}^\vee)$$

is an isomorphism by the vanishing of the cohomologies of $\mathcal{O}(-1, 0)^4 \oplus \mathcal{O}(0, -1)^2$. Thus the identification of $H^1(\mathcal{F}^\vee)$ with $C^2$ is canonical. Similarly, the isomorphism of $H^1(X, \mathcal{E}^\vee)$ with $C^2$ is canonical. Composing these canonical isomorphisms gives the desired isomorphism

$$\psi : H^0(X, \mathcal{E}^\vee) \simeq H^0(\mathcal{M}, \mathcal{F}^\vee).$$

We need to explicitly evaluate (48). For this, we have $\Lambda^4 \mathcal{F}^\vee \simeq K_{\mathcal{M}} \simeq \mathcal{O}(-4, -2)$. So the fourth exterior power of (49) gives

$$\mathcal{O}(-4, -2) \simeq \Lambda^4 \mathcal{F}^\vee \hookrightarrow \Lambda^4 \left( \mathcal{O}(-1, 0)^4 \oplus \mathcal{O}(0, -1)^2 \right) = \mathcal{O}(-4, 0) \oplus \mathcal{O}(-3, -1)^8 \oplus \mathcal{O}(-2, -2)^6.$$ 

Note that this embedding is equivalent to giving a global section of

$$\mathcal{O}(0, 2) \oplus \mathcal{O}(1, 1)^8 \oplus \mathcal{O}(2, 0)^6.$$ 

From (49) and the explicit matrices in (32), we see that over the open set where $\beta_1 \neq 0$, the rank 4 bundle $\mathcal{F}^\vee$ is spanned by the rows of the matrix

$$\begin{pmatrix}
\alpha_1 & -\alpha_0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha'_1 & -\alpha'_0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta_1 & -\beta_0 \\
-\alpha'_1 \beta_1 & \alpha_0 \beta_1 & 0 & 0 & (\epsilon_1 - \epsilon_2) \alpha_0 \alpha'_1 \\
\end{pmatrix}$$
Over the open set where $\beta_0 \neq 0$ the last row is replaced by a similar expression. The maximal minors of this matrix are computed to be

$$0, \alpha_0 \alpha_1' \beta_1^2, -\alpha_0 \alpha_0' \alpha_1' \beta_1, \ldots$$

when written as the coefficients of $e_0 \wedge e_1 \wedge e_2 \wedge e_3, e_0 \wedge e_1 \wedge e_2 \wedge f_0, \ldots$. Note that this is just $\alpha_0 \alpha_1' \beta_1$ times

$$0, \alpha_0' \beta_1, -\alpha_0' \beta_1, \ldots$$

and we would get exactly the same thing for the $\beta_0 \neq 0$ calculation. This is in fact the $(\epsilon_i$-dependent) global section of $O(0, 2) \oplus O(1, 1)^8 \oplus O(2, 0)^6$ that determines $\Lambda^4 F^\vee$.

Let $\psi(Y)$ and $\psi(\tilde{Y})$ be the natural generators of $H^1(M, F^\vee)$. Let’s start by computing $\langle \tilde{Y}^4 \rangle_{1,0}$. For simplicity, let’s put $\alpha_2 = \alpha_0'$ and $\alpha_3 = \alpha_1'$. We have the cover of $M$ given by open sets $U_{ij}$ defined by $\alpha_i \neq 0, \beta_j \neq 0$ (here $i = 0, 1, 2, 3$ and $j = 0, 1$). We have $\psi(\tilde{Y})$ as the coboundary of $(0, 1) \in H^0(O^2)$ in the exact sequence (49). On any of the open sets $U_{ij}$ we lift $(0, 1)$ to the section $\beta_j^{-1} f_j$ of $O(-1, 0)^4 \oplus O(0, -1)^2$, where $f_0, f_1$ are the natural generators of $O(0, -1)^2$. Thus $\psi(\tilde{Y}) \in H^1(F^\vee)$, represented as a Cech cocycle for this cover, is given by

$$(\psi(\tilde{Y}))_{ij, \alpha j'j} = \beta_j^{-1} f_{j'} - \beta_j^{-1} f_j.$$

Since this representation of $\psi(\tilde{Y})$ lives entirely in the rank 2 subbundle spanned by $O(0, -1)^2$, the fourth exterior powers in the cup product all vanish, and the Cech cocycle representing $\psi(\tilde{Y}) \cup \psi(\tilde{Y}) \cup \psi(\tilde{Y}) \cup \psi(\tilde{Y})$ is zero. Hence

$$\langle \tilde{Y}^4 \rangle_{1,0} = 0. \quad (51)$$

Since the third exterior powers of the representatives for $\psi(\tilde{Y})$ similarly vanish, we immediately get

$$\langle Y \tilde{Y}^3 \rangle_{1,0} = 0. \quad (52)$$

Actually the second exterior powers vanish as well, since the representatives live in the line subbundle of $O(0, -1)^2$ which is the kernel of the mapping

$$O(0, -1)^2 \to O$$

with matrix $(\beta_0, \beta_1)$. Thus

$$\langle Y^2 \tilde{Y}^2 \rangle_{1,0} = 0. \quad (53)$$

Let us next compute $\psi(Y) \in H^1(F^\vee)$. On $U_{ij}$ we can lift $(1, 0)$ to $\alpha_i^{-1} e_i - \epsilon_{ci} \beta_j^{-1} f_j$ on $U_{ij}$, where $e_0, e_1, e_2, e_3$ are local generators of $O(-1, 0)^4$ and $c(i) = (1, 1, 2, 2)$ for $i = 0, 1, 2, 3$.
respectively. This gives

\[(\psi(Y))_{ij;\epsilon'} = \alpha_i^{-1} e_{\epsilon'} - \alpha_i^{-1} e_i + \epsilon e_{(i)} \beta_j^{-1} f_j - \epsilon e_{(i')} \beta_j^{-1} f_j'\]  

(54)

It remains to compute \(\langle Y^3 \tilde{Y} \rangle_{1,0}\) and \(\langle Y^4 \rangle_{1,0}\). The straightforward way to do this is to multiply the Cech cocycles found above and then use any explicit form of the trace map.

The drawback is that there are many open sets needed to fully describe Cech 4-cocycles, so the actual computation will be somewhat tedious. Instead, we conclude by using some tricks adapted to this situation.

Since the coefficient of \(e_0 \wedge e_1 \wedge e_2 \wedge f_0\) (the second entry in (50) is nonzero), we merely need to extract the coefficient of \(e_0 \wedge e_1 \wedge e_2 \wedge f_0\) for the purpose of computing.

In computing \(\langle Y^3 \tilde{Y} \rangle_{1,0}\), the only way to get an \(e_0 \wedge e_1 \wedge e_2 \wedge f_0\) term is to use the \(e_i\) from \(\psi(\tilde{Y})\) and the \(f_0\) from \(\psi(\tilde{Y})\). Note that there is no \(\epsilon\) dependence in any of these terms. Therefore \(\langle Y^3 \tilde{Y} \rangle_{1,0}\) is independent of \(\epsilon\). Changing basis to \(X, \tilde{X}\) via (47) and using the vanishings (51), (52), and (53) we conclude that \(\langle X^3 \tilde{X} \rangle_{1,0}\) is independent of \(\epsilon\). By putting \(\epsilon_1 = \epsilon_2\), it follows from the usual quantum cohomology result that \(\langle X^3 \tilde{X} \rangle_{1,0} = 1\). Note also that each Cech term in the cup product is a scalar multiple of \((\alpha_0 \alpha_1 \alpha_2 \alpha_3 \beta_0 \beta_1)^{-1}\) times (50) so that the corresponding Cech cocycle in \(\mathcal{O}(-4, -2)\) is represented by \((\alpha_0 \alpha_1 \alpha_2 \alpha_3 \beta_0 \beta_1)^{-1}\), precisely of the form needed to represent the nonzero class in \(H^4(\mathcal{M}, \mathcal{O}(-4, -2))\).

Turning to \(\langle Y^4 \rangle_{1,0}\), a similar argument shows that the result is linear in the \(\epsilon_i\), entering via the term \(\epsilon e_{(i)} \beta_0^{-1} f_0\) in (54). But this can be evaluated up to multiple by a simple trick. Note that via the change of basis (47) and vanishings (51), (52), (53) we infer that \(\langle X^4 \rangle_{1,0}\) is linear in \(\epsilon\). Now recall that on the \((2,2)\) locus, the correlation function \(\langle X^4 \rangle_{1,0}\) vanishes. Since the \((2,2)\) locus is described by the equation \(\epsilon_1 - \epsilon_2 = 0\), we conclude that the correlation function must simply be a multiple of \(\epsilon_1 - \epsilon_2\), i.e.

\[\langle X^4 \rangle_{1,0} \propto (\epsilon_1 - \epsilon_2).\]

Summarizing, we have found

\[\langle \tilde{X}^4 \rangle_{1,0} = \langle X \tilde{X}^3 \rangle_{1,0} = \langle X^2 \tilde{X}^2 \rangle_{1,0} = 0, \langle X^3 \tilde{X} \rangle_{1,0} = 1, \langle X^4 \rangle_{1,0} \propto \epsilon_1 - \epsilon_2.\]

(55)

We can now repeat the computation for maps of degree \((0,1)\). The linear sigma model moduli space is \(\mathbb{P}^1 \times \mathbb{P}^3\). We let \(\beta_0, \beta_1\) be the natural homogeneous coordinates on the \(\mathbb{P}^1\) factor and let \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) denote the homogeneous coordinates on the \(\mathbb{P}^3\) factor. Then the bundle \(\mathcal{F}'\) on this moduli space is given by

\[
0 \rightarrow \mathcal{O} \oplus \mathcal{O} \rightarrow \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^4 \rightarrow \mathcal{F}' \rightarrow 0.
\]

(56)
There is the dual exact sequence

\[ 0 \to (F')^\vee \to \mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^4 \to \mathcal{O}^2 \to 0. \]

The section \((0,1)\) lifts to local sections \(\alpha_j^{-1}e_j\) of \(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^4\) while \((1,0)\) lifts to \(\beta_i^{-1}f_i - \epsilon_{i+1}\alpha_{i+1}^{-1}e_j\) with \(f_0, f_1, e_0, e_1, e_2, e_3\) basis vectors for \(\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^4\). Over an open set, the rank 4 bundle \((F')^\vee\) is spanned by the rows of the matrix

\[
\begin{pmatrix}
0 & 0 & \alpha_1 & -\alpha_0 & 0 \\
0 & 0 & 0 & \alpha_2 & -\alpha_1 & 0 \\
0 & 0 & 0 & 0 & \alpha_3 & -\alpha_2 \\
\alpha_0\beta_1 & -\alpha_0\beta_0 & (\epsilon_2 - \epsilon_1)\beta_0\beta_1 & 0 & 0 & 0
\end{pmatrix}
\]

An argument analogous to the \((1,0)\) case says that we must have one \(f_i\) term and 3 \(e_j\) terms in the wedge product arising from a correlation function in order to get a nonzero answer. Since there are no \(f_i\) in the lift of \((0,1)\), we immediately see that \(\langle \tilde{X}^4 \rangle_{0,1} = 0\).

Similarly, from one \(Y\) and three \(\tilde{Y}'s\), the \(f_i\) must come from \(Y\) and we see that there is no \(\epsilon\) dependence. We get that \(\langle X, \tilde{X}^3 \rangle_{0,1} = 1\) from the change of basis (47) the \((2,2)\) result.

More generally, we see by the same reasoning that each four-point function \(\langle X^{i+1}\tilde{X}^{3-i} \rangle_{0,1}>\) is a homogeneous polynomial in the \(\epsilon_i\) of degree \(i\), for \(i = 0, 1, 2, 3\). Furthermore, each of these polynomials vanish on the \((2,2)\) locus, hence is a multiple of \(\epsilon_1 - \epsilon_2\).

Summarizing, we have computed

\[
\begin{align*}
\langle \tilde{X}^4 \rangle_{0,1} &= 0, \quad \langle X\tilde{X}^3 \rangle_{0,1} = 1, \quad \langle X^2\tilde{X}^2 \rangle_{0,1} \propto \epsilon_1 - \epsilon_2, \\
\langle X^3\tilde{X} \rangle_{0,1} &= (\epsilon_1 - \epsilon_2)f_1(\epsilon), \quad \langle X^4 \rangle_{0,1} = (\epsilon_1 - \epsilon_2)f_2(\epsilon),
\end{align*}
\]

(57)

where \(f_i(\epsilon)\) is a homogeneous polynomial of degree \(i\) in \(\epsilon_1, \epsilon_2\).

Combining (55) with (57), we find agreement with (22), (25), (26), (28), and (30).

### 8 Conclusions

In this paper we have described the computation of generalizations of the \(T^8\) coupling in perturbative heterotic string compactifications. These calculations amount to a heterotic version of curve-counting, generalizing the standard A model calculations.

We spent the first third of this paper describing formally how one can calculate these correlation functions. We saw how old ideas from A model calculations generalize in the heterotic context – for example, the obstruction sheaf story seems to generalize in an interesting way.

\[\text{We could instead have argued that the representatives of the lifts live naturally in a rank 3 subbundle.}\]
Our formal methods in the first third of this paper suffered from needing not only a compactification of the moduli space of worldsheet instantons, but also extensions of induced sheaves over that compactification divisor, in a fashion that preserves certain necessary properties of the Chern classes. The second third of this paper was devoted to solving this problem using linear sigma models, which not only compactify moduli spaces, but as we described, also provide the needed sheaf extensions.

In the final part of this paper, we applied this technology to check some predictions of [1] for heterotic curve counting.

There are several open problems that need to be solved:

1. We have described how to map

\[ H^p(X, \Lambda^q E^\vee) \mapsto H^p(M, \Lambda^q F^\vee) \]

on open subsets of the moduli space, and in special cases involving linear sigma models, have used calculational tricks to extend the map over the compactification of the moduli space. However, a more general prescription for extending the map over the compactification is lacking.

2. In section 5.5, we described a proposal for generalizing obstruction sheaf constructions. From physics, we conjecture that the Atiyah class of \( E \) determines an element of

\[ H^1(M, F^\vee \otimes F_1 \otimes G_1^\vee) \]

with the property that when \( E = TX \), the element of the sheaf cohomology group above is the Atiyah class of the obstruction sheaf. With those assumptions, an easy Grothendieck-Riemann-Roch argument showed how the resulting product of sheaf cohomology groups generated a top form which can be integrated over the moduli space, and we also checked that we reproduce the usual obstruction sheaf story on the (2, 2) locus. However, although the underlying physics seems clear, these mathematical conjectures need to be checked.

There are other extensions of this work that would be interesting to pursue. For example, it would be interesting to understand how these correlation functions change when the Kähler class passes through a stability subcone wall [25], which would be the heterotic analogue of a flop.

It would also be interesting to better understand the effect of the ratio of operator determinants that we outlined earlier. For the calculations in this paper, at genus zero, that ratio is just a constant, which we have ignored. However, at higher genus, it is a nontrivial function of both the moduli of the Riemann surface and of the bundle.
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