CHOW GROUP OF 1-CYCLES OF THE MODULI OF PARABOLIC BUNDLES OF RANK 2 OVER A CURVE

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ABSTRACT. Let $X$ be a nonsingular projective curve of genus $g \geq 3$ over $\mathbb{C}$, and choose a point $x \in X$. Let $M$ denote the moduli space of isomorphism classes of stable vector bundles of rank 2 and fixed determinant $O_X(x)$ over $X$. Let us moreover fix $n$ distinct closed points $S = \{p_1, p_2, ..., p_n\}$ over $X$, and weights $(\alpha) := 0 \leq \alpha_1 < \alpha_2 < 1$ over the parabolic points. We also assume that the weights are generic. Let $M_\alpha$ denote the moduli space of $S$-equivalence classes of parabolic stable vector bundles of rank 2 over $X$ of fixed determinant $O_X(x)$. I. Choe and J. Hwang in their paper [CH, Main Theorem] showed that there is a canonical isomorphism of the following Chow groups with $\mathbb{Q}$-coefficients: $CH^1_0(M) \cong CH^0_0(X)$. Here our aim is to show that $CH^1_0(M_\alpha) \cong \mathbb{Q}^n \oplus CH^0_0(X)$ for generic weights $\alpha$.

1. INTRODUCTION

Let us fix a nonsingular projective curve $X$ of genus $g \geq 3$ over $\mathbb{C}$, and choose a point $x \in X$. Let $M$ denote the moduli space of isomorphism classes of stable vector bundles of rank 2 and fixed determinant $O_X(x)$ over $X$. Let us moreover fix $n$ distinct closed points $S = \{p_1, p_2, ..., p_n\}$ over $X$, referred to as parabolic points, and parabolic weights $(\alpha) := 0 \leq \alpha_1 < \alpha_2 < 1$ over the parabolic points. We also assume that the weights are generic. Let $M_\alpha$ denote the moduli space of $S$-equivalence classes of parabolic stable vector bundles of rank 2 over $X$ of fixed determinant $O_X(x)$.

The Chow groups of these moduli spaces are interesting objects to study. I. Choe and J. Hwang in their paper [CH, Main Theorem] showed that there is a canonical isomorphism $CH^1_0(M) \cong CH^0_0(X)$. Here our aim is to show that $CH^1_0(M_\alpha) \cong \mathbb{Q}^n \oplus CH^0_0(X)$ for generic weights $\alpha$.

Here is a brief outline of this paper:
In section 2, we briefly recall the notions necessary for our discussions, like (parabolic) semistability and stability of (parabolic) vector bundles, their moduli spaces, Chow groups and so on. In section 3, we study the relations between the chow groups of 1-cycles of parabolic bundles over $X$ for different generic weights. The main result of section 3 is proven in theorem 3.10:

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Theorem 1.1. For any two generic weights $\alpha$ and $\beta$, there exists a canonical isomorphism
\[ \text{CH}^0_1(\mathcal{M}_\alpha) \cong \text{CH}^0_1(\mathcal{M}_\beta). \]

Finally, in section 4, we restrict to rank 2 and determinant $\mathcal{O}_X(x)$-case and prove the following result in theorem 4.4:

Theorem 1.2. In case of rank 2 and determinant $\mathcal{O}_X(x)$, for any generic weight $\alpha$, we have
\[ \text{CH}^0_1(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \text{CH}^0_0(\mathcal{X}), \]
where $n = |S|$.

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2. Preliminaries

2.1. Semistability and stability of vector bundles. Let $X$ be a nonsingular projective curve over $\mathbb{C}$. Let $E$ be a holomorphic vector bundle of rank $r$ over $X$. Here onwards, by a variety we will always mean an irreducible quasi-projective variety.

Definition 2.1 (Degree and slope). The degree of $E$, denoted $\text{deg}(E)$, is defined as the degree of the line bundle $\text{det}(E) := \wedge^r E$. The slope of $E$, denoted $\mu(E)$, is defined as
\[ \mu(E) := \frac{\text{deg}(E)}{r}. \]

Definition 2.2 (Semistability and stability). $E$ is called semistable (resp. stable), if for any sub-bundle $F \hookrightarrow E$, $0 < \text{rank}(F) < r$, we have
\[ \mu(F) \leq \mu(E). \]

It is easy to check that if $\gcd(r, \text{deg}(E)) = 1$, then the notion of semistability and stability coincide for a vector bundle $E$.

2.2. Moduli space of vector bundles. We briefly recall the notion of the moduli space of vector bundles over $X$. If $E$ is a semistable bundle of rank $r$, then there exists a Jordan-Hölder filtration for $E$ given by
\[ E = E_k \supset E_{k-1} \supset \cdots \supset E_1 \supset 0 \]

The filtration is not unique, but the associated graded object $\text{gr}(E) := \bigoplus_{i=1}^k E_i/E_{i-1}$ is unique up to isomorphism. Two vector bundles $E$ and $E'$ are called $S$-equivalent if $\text{gr}(E) \cong \text{gr}(E')$. When $E$, $E'$ are stable, being $S$-equivalent is same as being isomorphic as vector bundles over $X$.

The moduli space of $S$-equivalence classes of vector bundles of rank $r$ and determinant $\mathcal{L}$ on $X$, denoted $\mathcal{M}_{r,\mathcal{L}}$, is a normal projective variety of dimension $(r^2 - 1)(g - 1)$; its singular locus is given by the strictly semistable bundles.
In the case when \( \gcd(r, \deg(L)) = 1 \), \( M_{r,L} \) is the isomorphism class of stable vector bundles on \( X \), is a nonsingular projective variety; moreover, it is a fine moduli space.

When \( r, L \) are fixed, we shall denote the moduli space by \( M \), when there is no scope for confusion.

### 2.3. Parabolic bundles and stability.

**Definition 2.3 (Parabolic bundles).** Let us fix a set \( S \) of \( n \) distinct closed points on \( X \). A parabolic vector bundle of rank \( r \) on \( X \) is a holomorphic vector bundle \( E \) on \( X \) with a parabolic structure along points of \( S \). By this, we mean a collection weighted flags of the fibers of \( E \) over each point \( p \in S \):

\[
E_p = E_{p,1} \supsetneq E_{p,2} \supsetneq \cdots \supsetneq E_{p,s_p} \supsetneq E_{p,s_p+1} = 0,
\]

(2.1)

\[
0 \leq \alpha_{p,1} < \alpha_{p,2} < \cdots < \alpha_{p,s_p} < 1,
\]

(2.2)

where \( s_p \) is an integer between 1 and \( r \). The real number \( \alpha_{p,i} \) is called the weight attached to the subspace \( E_{p,i} \). The multiplicity of the weight \( \alpha_{p,i} \) is the integer \( m_{p,i} := \dim(E_{p,i}) - \dim(E_{p,i-1}) \). Thus \( \sum_i m_{p,i} = r \). We call the flag to be full if \( s_p = r \), or equivalently \( m_{p,i} = 1 \forall i \).

Let \( \alpha := \{ (\alpha_{p,1}, \alpha_{p,2}, \ldots, \alpha_{p,s_p}) | p \in S \} \) and \( m := \{ (m_{p,1}, \ldots, m_{p,s_p}) | p \in S \} \). We call the tuple \( (r, L, m, \alpha) \) as the parabolic data for the parabolic bundle \( E \), where \( L := \det(E) \). Usually we denote the parabolic bundle as \( E_* \) to distinguish from the underlying vector bundle \( E \).

**Definition 2.4 (Parabolic degree and slope).** The degree of a parabolic bundle \( E_* \) is defined as \( \deg(E) \), \( E \) being the underlying vector bundle. The Parabolic degree of \( E_* \), denoted \( \text{Pardeg}(E) \), is defined as

\[
\text{Pardeg}(E_*) := \deg(E) + \sum_{p \in S} \sum_{i=1}^{s_p} m_{p,i} \alpha_{p,i}.
\]

The parabolic slope of \( E_* \) is defined as

\[
\text{Par}\mu(E_*) := \frac{\text{Pardeg}(E_*)}{\text{rank}(E)}.
\]

**Definition 2.5 (Parabolic semistability and stability).** Any vector sub-bundle \( F \hookrightarrow E \) obtains a parabolic structure in a canonical way: For each \( p \in S \), the flag at \( F_p \) is obtained intersecting \( F_p \) with the flag at \( E_p \), and the weight attached to the subspace \( F_{p,j} \) is \( \alpha_k \), where \( k \) is the largest integer such that \( F_{p,j} \subseteq E_{p,k} \). (for more details see [MS, Definition 1.7].) We call the resulting parabolic bundle to be a parabolic sub-bundle, and denote it by \( F_* \).

A parabolic bundle \( E_* \) is called parabolic semistable (resp. parabolic stable), if for every proper sub-bundle \( F \hookrightarrow E \) we have

\[
\text{Par}\mu(F_*) \leq \text{Par}\mu(E_*). \quad \text{(resp.} < \text{)}
\]

\[
\text{Par}\mu(F_*) \leq \text{Par}\mu(E_*).
\]

\[
\text{Par}\mu(F_*) < \text{Par}\mu(E_*).
\]
2.4. **Generic weights and walls.** We briefly recall the notion of generic weights and walls. For more details we refer to [BH, BY].

Fix a set $S$, rank $r$, line bundle $L$ on $X$ and multiplicities $m$ as defined above. Let $\Delta^r := \{(a_1, ..., a_r) | 0 \leq a_1 \leq ... \leq a_r < 1 \}$, and define $W := \{\alpha : S \to \Delta^r\}$. Note that the elements of $W$ determine both weights and the multiplicities at the parabolic points, and hence a parabolic data. Conversely, given any parabolic data $(r, d, m, \alpha)$ as defined above, we can associate a map $S \to \Delta^r$, by repeating each weight $\alpha_{p,i}$ according to its multiplicity $m_{p,i}$. This leads to a natural notion of when a given weight $\alpha$ is compatible with the multiplicity $m$. The set of all weights compatible with $m$ is a product of $|S|$-many simplices. We denote by $V_m$ the set of all weights compatible with $m$.

Let $\alpha \in V_m$. If a parabolic bundle $E_*$ with data $(r, d, m, \alpha)$ is parabolic semistable but not parabolic stable, then it would contain a parabolic sub-bundle with same parabolic slope. It is easy to see that this gives a linear condition on $V_m$, i.e. such weights belong to the intersection of a hyperplane with $V_m$.

There can be only finitely many such hyperplanes (see [BY, BH]); call them $H_1, ..., H_l$.

**Definition 2.6.** (Walls and generic weights) We call each of the intersections $H_i \cap V_m$ a wall in $V_m$. There are only finitely many such walls.

We call the connected components of $V_m \setminus \cup_{1 \leq i \leq l} H_i$ as chambers, and weights belonging to these chambers are called generic.

Clearly, for weights in $V_m \setminus \cup_{1 \leq i \leq l} H_i$, a parabolic bundle is parabolic semistable iff it is parabolic stable.

2.5. **Moduli of parabolic bundles.** Again, we briefly recall the notion of moduli space of parabolic semistable bundles over $X$. The construction is analogous to section 2.2; for details we refer to [MS].

For a parabolic semistable bundle $E_*$ with fixed parabolic data $(r, L, m, \alpha)$, there exists a Jordan-Holder filtration, and an associated graded object $gr_\alpha(E_*)$ analogous to section 2.2. Again, we call two parabolic semistable bundles to be $S$-equivalent if their associated graded objects are isomorphic. Let $\mathcal{M}(r, L, m, \alpha)$ denote the moduli space of $S$-equivalence classes of parabolic semistable bundles over $X$ with parabolic data $(r, L, m, \alpha)$. It is a normal projective variety, with singular locus given by the strictly semistable bundles. When $r, L, m$ are fixed, we will denote the moduli space by $\mathcal{M}_\alpha$ if no confusion occurs.

For generic weight $\alpha$, $\mathcal{M}_\alpha = \text{moduli space of isomorphism classes of parabolic stable bundles on } X$, is a nonsingular projective variety; moreover, it is a fine moduli space ([BY, Proposition 3.2]).

2.6. **Chow groups.** For a variety $Y$ over $\mathbb{C}$, let $Z_k(Y)$ denote the free abelian group generated by the irreducible $k$-dimensional closed subvarieties of $Y$. The *Chow group of*
$k$-cycles, denoted $\text{CH}_k(Y)$, is given by

$$\text{CH}_k(Y) := \frac{Z_k(Y)}{\sim},$$

where $\sim$ denotes "rational equivalence". We refer to [Voi, Section 9] and [Ful] for the details regarding Chow groups and the related notions (proper pushforward and flat pullback of cycles, intersection product, Chern class of vector bundles etc.)

Let $\text{CH}_k^\mathbb{Q}(Y) := \text{CH}_k(Y) \otimes \mathbb{Q}$; this is a $\mathbb{Q}$-vector space. By a slight abuse of notation, throughout the rest of the discussion, we will address $\text{CH}_k^\mathbb{Q}(Y)$ as 'Chow group' as well, since no confusion will arise.

We recall a few results from [Ful] which we will require in section 3:

**Theorem 2.7** ([Ful, Theorem 3.3]). Let $E$ be a vector bundle of rank $r$ on $Y$, with projection $\pi : E \to Y$. The flat pull-back

$$\pi^* : \text{CH}_{k-r}(Y) \to \text{CH}_k(E)$$

is an isomorphism for all $k$.

**Definition 2.8** ([Ful, Definition 3.3]). Let $s$ denote the zero section of the bundle $E$ above. Hence $\pi \circ s = \text{Id}_Y$. Then there exist Gysin homomorphisms:

$$s^* : \text{CH}_k(E) \to \text{CH}_{k-r}(Y),$$

$$s^*(W) := (\pi^*)^{-1}(W)$$

where $r = \text{rank}(E)$.

(We make a small remark that $s^* \neq \pi^*$).

**Lemma 2.9** ([Ful, Example 3.3.2]). If $s$ is the zero section of a vector bundle $E$ of rank $r$ on $Y$, then

$$s^*s_*(Z) = c_r(E) \cap Z \text{ for all } Z \in \text{CH}_s(Y).$$

**Proposition 2.10** ([Ful, Proposition 6.7(a)]). Let $Y$ be a nonsingular variety, and $X \hookrightarrow Y$ be a nonsingular closed subvariety of codimension $d$, with normal bundle $N$. Let $\widetilde{Y}$ be the blow-up of $Y$ along $X$, and $\widetilde{X}$ be its exceptional divisor. We have a fiber square:

$$\begin{array}{ccc}
\widetilde{X} & \xrightarrow{j} & \widetilde{Y} \\
\downarrow{g} & & \downarrow{f} \\
X & \xleftarrow{i} & Y
\end{array}$$

Let $E := g^*(N)/\mathcal{O}_N(-1)$ be the Excess normal bundle. Then for all $Z \in \text{CH}_k(X)$,

$$f^*i_*(Z) = j_*(c_{d-1}(E) \cap g^*(Z)).$$
3. Relation between Chow groups of 1-cycles of moduli of parabolic bundles for arbitrary generic weights

Fix a set $S$ of parabolic points, rank $r$ and determinant $L$. We assume that we are working with full flags, i.e. $m_{p,i} = 1 \forall p, i$. Consider $V_m$, the set of weights compatible with $m$, as in section 2.4. Recall that $V_m$ is cut out by finitely many walls. Moreover, as the flags are full, $V_m$ contains a generic weight by [BY, Proposition 3.2].

Let $\alpha, \beta \in V_m$ be two generic weights in adjacent chambers separated by a single wall. Let $H$ be the hyperplane separating $\alpha$ and $\beta$. Let $\gamma$ be the weight lying on $H$ and the line joining $\alpha$ and $\beta$. Then $M_\alpha$ and $M_\beta$ are nonsingular projective varieties, while $M_\gamma$ is normal projective variety, with the singular locus $\Sigma_\gamma \subset M_\gamma$ given by the class of strictly semistable bundles. Note that since $\gamma$ lies on only one hyperplane in $W$, $\Sigma_\gamma$ is nonsingular ([BH, Section 3.1]).

Let us recall the following theorem:

**Theorem 3.1** (BY, Theorem 3.1). There are canonical projective morphisms

\[
\begin{array}{ccc}
M_\alpha & \leftarrow & M_\gamma \\
\downarrow \phi_\alpha & & \downarrow \phi_

\end{array}
\]

\[
\begin{array}{ccc}
M_\gamma & \rightarrow & M_\beta \\
\downarrow \phi_\beta & & \downarrow \phi_

\end{array}
\]

so that: a) $\phi_\alpha$ and $\phi_\beta$ are isomorphisms along $M_\gamma \setminus \Sigma_\gamma$,

b) along $\Sigma_\gamma$, $\phi_\alpha$ and $\phi_\beta$ are $\mathbb{P}^{n_\alpha}$ and $\mathbb{P}^{n_\beta}$-bundles respectively,

and c) $\text{codim} \Sigma_\gamma = 1 + n_\alpha + n_\beta$.

Since $\Sigma_\gamma$ is nonsingular and $\phi_\alpha^{-1}(\Sigma_\gamma), \phi_\beta^{-1}(\Sigma_\gamma)$ are projective bundles, they are nonsingular closed subvarieties of $M_\alpha, M_\beta$ respectively.

Let $N := M_\alpha \times M_\beta$. Let $\psi_\alpha$ and $\psi_\beta$ denote the natural maps from $N$ to $M_\alpha$ and $M_\beta$ respectively. Then according to the discussion in the end of section 1 in [BY], $N$ is the common blowdown along $\phi_\alpha^{-1}(\Sigma_\gamma)$ and $\phi_\beta^{-1}(\Sigma_\gamma)$, and hence $N$ is the common blow-up with exceptional divisor a $(\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta})$-bundle over $\Sigma_\gamma$.

Call the exceptional divisor $E$, with $j : E \hookrightarrow N$ the inclusion.

We have the following diagram:
Here $E$ is a $\mathbb{P}^{n_{\beta}}$-bundle over $\phi_{\beta}^{-1}(\Sigma_{\gamma})$ via $\psi_{\alpha}|_E$, and a $\mathbb{P}^{n_{\alpha}}$-bundle over $\phi_{\beta}^{-1}(\Sigma_{\gamma})$ via $\psi_{\beta}|_E$.

**Remark 3.2.** From the diagram above, we note that $E \cong \phi_{\alpha}^{-1}(\Sigma_{\gamma}) \times \phi_{\beta}^{-1}(\Sigma_{\gamma})$, since

$$E = \mathcal{N} \times \phi_{\beta}^{-1}(\Sigma_{\gamma}) = (\mathcal{M}_{\alpha} \times \mathcal{M}_{\beta}) \times \phi_{\beta}^{-1}(\Sigma_{\gamma})$$

$$\cong \mathcal{M}_{\alpha} \times \phi_{\beta}^{-1}(\Sigma_{\gamma})$$

$$\cong \phi_{\alpha}^{-1}(\Sigma_{\gamma}) \times \phi_{\beta}^{-1}(\Sigma_{\gamma}).$$

[\because \phi_{\beta}^{-1}(\Sigma_{\gamma}) \text{ maps to } \Sigma_{\gamma}]

**Lemma 3.3.** $\phi_{\alpha}^{-1}(\Sigma_{\gamma})$ and $\phi_{\beta}^{-1}(\Sigma_{\gamma})$ are rational varieties (i.e. birational to $\mathbb{P}^n$ for some $n$); hence $\text{CH}_0(\phi_{\alpha}^{-1}(\Sigma_{\gamma})) \cong \mathbb{Q} \cong \text{CH}_0(\phi_{\beta}^{-1}(\Sigma_{\gamma}))$.

**Proof.** By equation (5) in [BH], $\Sigma_{\gamma}$ is the product of two smaller dimensional moduli, which are rational (by [BY, Theorem 6.1]), so $\Sigma_{\gamma}$ is itself rational.

Since $\phi_{\alpha}^{-1}(\Sigma_{\gamma})$ and $\phi_{\beta}^{-1}(\Sigma_{\gamma})$ are projective bundles over $\Sigma_{\gamma}$, they are also rational. This proves the first assertion.

Moreover, by [Ful, Example 16.1.11], the Chow groups of 0-cycles is a birational invariant; and $\text{CH}_0(\mathbb{P}^n) \cong \mathbb{Z} \forall n$, so we get the second assertion as well. \hfill \Box

Recall the fiber diagram from Remark 3.2:

\[
\begin{array}{ccc}
E & \xrightarrow{\psi_{\beta}|_E} & \phi_{\beta}^{-1}(\Sigma_{\gamma}) \\
\downarrow{\psi_{\alpha}|_E} & & \downarrow{\phi_{\beta}} \\
\phi_{\alpha}^{-1}(\Sigma_{\gamma}) & \xrightarrow{\phi_{\alpha}} & \Sigma_{\gamma}
\end{array}
\]

\[
\therefore \text{ If we choose a point } p \in \Sigma_{\gamma}, \text{ then by base changing to } \{p\}, \text{ the diagram above transforms to}
\]

\[
\begin{array}{ccc}
\mathbb{P}^{n_{\alpha}} \times \mathbb{P}^{n_{\beta}} & \xrightarrow{p_2} & \mathbb{P}^{n_{\beta}} \cong \phi_{\beta}^{-1}(p) \\
p_1 & & \downarrow \\
\mathbb{P}^{n_{\alpha}} & \cong \phi_{\alpha}^{-1}(p) & \xrightarrow{} \{p\}
\end{array}
\]

where $p_1, p_2$ denote the first and second projections respectively.
Let $\phi^{-1}_\alpha(p) \cong \mathbb{P}^{n_\alpha} \overset{i_\alpha}{\hookrightarrow} \phi^{-1}_\alpha(\Sigma_\gamma)$, $\psi^{-1}_\alpha(\phi^{-1}_\alpha(p)) \cong \mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta} \overset{\tilde{\iota}}{\hookrightarrow} E$ denote the inclusions. We have the fiber diagram

\[
\begin{array}{ccc}
\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta} & \xrightarrow{\cong} & \psi^{-1}_\alpha(\phi^{-1}_\alpha(p)) \xrightarrow{\tilde{\iota}} E \\
p_1 \downarrow & & \downarrow \psi_{\alpha|E} \\
\mathbb{P}^{n_\alpha} & \xrightarrow{\cong} & \phi^{-1}_\alpha(\phi^{-1}_\alpha(p)) \xrightarrow{\iota_\alpha} \phi^{-1}_\alpha(\Sigma_\gamma)
\end{array}
\]

Choose a point $x \in \mathbb{P}^{n_\alpha} \cong \phi^{-1}_\alpha(\Sigma_\gamma)$. In the following, under slight abuse of notation, we will think of the element $[x] \in \text{CH}_0^Q(\mathbb{P}^{n_\alpha})$ as an element of $\text{CH}_0^Q(\phi^{-1}_\alpha(p))$, and we will think of the element $\{x\} \times \mathbb{P}^{n_\beta} \in \text{CH}_{n_\beta}^Q(\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta})$ as an element of $\text{CH}_{n_\beta}^Q(\phi^{-1}_\alpha(\phi^{-1}_\alpha(p)))$.

**Lemma 3.4.** \((\psi_{\alpha|E})^*((\iota_\alpha)_*[x]) = \tilde{\iota}_*[\{x\} \times \mathbb{P}^{n_\beta}]\)

**Proof.** This follows from [Ful, Proposition 1.7], since $\psi_{\alpha|E}$ is flat, being a projective bundle map, and $\iota_\alpha$ is proper, being a closed immersion. \qed

Now, since $\mathcal{N}$ is the blow-up over $\mathcal{M}_\alpha$ along $\phi^{-1}_\alpha(\Sigma_\gamma)$, hence by [Voi, Theorem 9.27] there is an isomorphism of Chow groups:

\[
\text{CH}_0^Q(\phi^{-1}_\alpha(\Sigma_\gamma)) \oplus \text{CH}_1^Q(\mathcal{M}_\alpha) \xrightarrow{g_\alpha} \text{CH}_1^Q(\mathcal{N}) \tag{3.1}
\]

given by \((W_0, W_1) \mapsto j_*(c_1(h_\alpha)^{n_\beta-1} \cap (\psi_{\alpha|E})^*(W_0)) + \psi_{\alpha|E}^*(W_1) \tag{3.2}\)

where $h_\alpha := O_E(1)$ (E thought of as a $\mathbb{P}^{n_\beta}$-bundle over $\phi^{-1}_\alpha(\Sigma_\gamma)$), and $\cap$ denotes the intersection product.

Similarly, there exists an isomorphism defined similarly to $g_\alpha$ above:

\[
\text{CH}_0^Q(\phi^{-1}_\beta(\Sigma_\gamma)) \oplus \text{CH}_1^Q(\mathcal{M}_\beta) \xrightarrow{g_\beta} \text{CH}_1^Q(\mathcal{N}) \tag{3.3}
\]

given by \((Z_0, Z_1) \mapsto j_*(c_1(h_\beta)^{n_\alpha-1} \cap (\psi_{\beta|E})^*(Z_0)) + \psi_{\beta|E}^*(Z_1) \tag{3.4}\)

where $h_\beta := O_E(1)$ (E thought of as a $\mathbb{P}^{n_\alpha}$-bundle over $\phi^{-1}_\beta(\Sigma_\gamma)$), and $\cap$ denotes the intersection product.

**Remark 3.5.** Again, identifying $\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta}$ and $\psi^{-1}_\alpha(\phi^{-1}_\alpha(p))$, it is easy to see that the pull-back bundle $\tilde{\iota}^*(h_\alpha) \cong O_{\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta}}(1)$.

By lemma 3.3 \(\text{CH}_0^Q(\phi^{-1}_\alpha(\Sigma_\gamma)) \cong \mathbb{Q}\), hence the class $(\iota_\alpha)_*([x])$ will be a $\mathbb{Q}$-basis.
Lemma 3.6. \(g_\alpha((\iota_\alpha)_*(|[x]\times l)) = j_*(\tilde{\iota}_*[{x}\times \mathbb{P}^n])\), where \(l\) is a line in \(\mathbb{P}^n\).

Proof. By (3.2),
\[
g_\alpha((\iota_\alpha)_*([x])) = j_*(c_1(h_\alpha)^{n_\beta-1}\cap (\psi_\alpha|_E)^*((\iota_\alpha)_*([x])))
= j_*(c_1(h_\alpha)^{n_\beta-1}\cap (\tilde{\iota}_*([x]\times \mathbb{P}^n]))\quad \text{[Lemma 3.4]} \tag{3.5}
\]

By Remark 3.5 and projection formula applied to \(\tilde{\iota}\) (cf. [Ful, Proposition 2.5]),
\[
c_1(h_\alpha)^{n_\beta-1}\cap (\tilde{\iota}_*([x]\times \mathbb{P}^n)) = \tilde{\iota}_*(c_1(\tilde{\iota}^*(h_\alpha))^{n_\beta-1}\cap [x]\times \mathbb{P}^n)]
= \tilde{\iota}_*(c_1(\mathcal{O}_{\mathbb{P}^n_\alpha\times \mathbb{P}^n_\beta}(1))^{n_\beta-1}\cap [x]\times \mathbb{P}^n]) \tag{3.6}
\]

But \(\mathcal{O}_{\mathbb{P}^n_\alpha\times \mathbb{P}^n_\beta}(1)|_{[x]\times \mathbb{P}^n_\beta} = \mathcal{O}_{[x]\times \mathbb{P}^n_\beta}(1)\), which corresponds to the divisor of a hyperplane section \(H\) (say), and so by definition of intersection product,
\[
c_1(\mathcal{O}_{\mathbb{P}^n_\alpha\times \mathbb{P}^n_\beta}(1))\cap [x]\times \mathbb{P}^n = [x]\times H.
\]

Repeating this \(n_\beta - 1\) times, we get
\[
c_1(\mathcal{O}_{\mathbb{P}^n_\alpha\times \mathbb{P}^n_\beta}(1))^{n_\beta-1}\cap [x]\times \mathbb{P}^n = [x]\times l,
\]
where \(l\) is a line in \(\mathbb{P}^n\). Hence from (3.5) and (3.6) we finally get \(g_\alpha((\iota_\alpha)_*([x])) = j_*(\tilde{\iota}_*[x]\times \mathbb{P}^n])\), as claimed. \(\square\)

Let \([x]' := (\iota_\alpha)_*([x])\) and \([x]\times l' := \tilde{\iota}_*[x]\times l\).

Proposition 3.7. Let \(Z := g_\alpha([x]) \in \text{CH}_1^G(\mathcal{N})\), then \(Z \neq (\psi_\beta^*\circ \psi_\beta)(Z)\).

Proof. Let \(j_\beta : \phi_\beta^{-1}(\Sigma_\gamma) \hookrightarrow \mathcal{M}_\beta\) be the inclusion, so that we have the following blow-up diagram:

\[
\begin{array}{cccc}
E & \xrightarrow{j} & \mathcal{N} \\
\downarrow{\psi_\beta|_E} & & \downarrow{\psi_\beta} \\
\mathbb{P}^n_\alpha & \xrightarrow{\phi_\beta^{-1}(\Sigma_\gamma)} & \mathcal{M}_\beta \\
\end{array}
\]

If \(E = \mathbb{P}(N)\), where \(N\) denotes the normal bundle of the embedding \(j_\beta\), Let \(Q := (\psi_\beta|_E)^*(N) / \mathcal{O}_E(-1)\) be the Excess normal bundle of rank \(n_\alpha\), as defined in [Ful, §6.7].

Let \(\pi : E \to \mathcal{N}\) be the vector bundle projection, and let \(c_{n_\alpha}(Q)\) denote the top chern class of the bundle \(Q\).

By lemma 3.6, we want to show that
\[
(\psi_\beta^*\circ \psi_\beta)(j_*([x]\times l')) \neq j_*([x]\times l').
\]

To the contrary, suppose they are equal. We have:
\[ \text{LHS} = (\psi_\beta \circ \psi_\beta)(j_\ast \{x\} \times l') \]
\[ = \underbrace{\psi_\beta \circ (j_\ast (\psi_\beta | E)_\ast \{x\} \times l')}_{\therefore \psi_\beta \circ j = j_\beta \circ (\psi_\beta | E)} \]
\[ = j_\ast (c_{n_\alpha}(Q) \cap (\psi_\beta | E)_\ast (\psi_\beta | E)_\ast (\{x\} \times l')), \quad [\text{Proposition 2.10}] \]

So, we would get \( j_\ast (c_{n_\alpha}(Q) \cap (\psi_\beta | E)_\ast (\psi_\beta | E)_\ast (\{x\} \times l')) = j_\ast \{x\} \times l' \).

\[ \therefore \text{denoting } Z := c_{n_\alpha}(Q) \cap (\psi_\beta | E)_\ast (\psi_\beta | E)_\ast ((\{x\} \times l')) - \{x\} \times l', \text{ we have } j_\ast (Z) = 0; \]

\[ \therefore Z = 0 \text{ by } [\text{Ful, Proposition 6.7(c)}], \text{ i.e.} \]
\[ \{x\} \times l' = c_{n_\alpha}(Q) \cap (\psi_\beta | E)_\ast (\psi_\beta | E)_\ast ((\{x\} \times l')). \quad (3.7) \]

Moreover, by lemma 2.9,
\[ c_{n_\alpha}(Q) \cap (\psi_\beta | E)_\ast (\psi_\beta | E)_\ast ((\{x\} \times l')) = s_\ast s_\ast ((\psi_\beta | E)_\ast (\psi_\beta | E)_\ast ((\{x\} \times l')), \]

where \( s : E \to Q \) denotes the zero section of the bundle map \( Q \xrightarrow{\psi} E \), and \( s_\ast \) is defined as in definition 2.8.

\[ \therefore \text{from (3.7) we would finally get:} \]
\[ \{x\} \times l' = s_\ast s_\ast ((\psi_\beta | E)_\ast (\psi_\beta | E)_\ast ((\{x\} \times l')). \quad (3.8) \]

Let us write down the following square:
\[
\begin{array}{ccccc}
\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta} & \xrightarrow{\cong} & \psi_\beta^{-1}(\phi_\beta^{-1}(p)) & \xleftarrow{\tilde{\pi}} & E \\
p_2 & & \downarrow & & \downarrow \\
\mathbb{P}^{n_\beta} & \xrightarrow{\cong} & \phi_\beta^{-1}(p) & \xrightarrow{s \text{ flat}} & \phi_\beta^{-1}(\Sigma_\gamma) \\
\end{array}
\]

Recalling \( (\{x\} \times l') := \tilde{\pi}_\ast \{x\} \times l \), we get from the diagram above:
\[ (\psi_\beta | E)_\ast (\psi_\beta | E)_\ast ((\{x\} \times l')) = (\psi_\beta | E)_\ast (\psi_\beta | E)_\ast (\tilde{\pi}_\ast \{x\} \times l) \]
\[ = (\psi_\beta | E)_\ast (\psi_\beta | E)_\ast (\tilde{\pi}_\ast (p_2 \circ p_2 \ast \{x\} \times l)) \]
\[ = (\tilde{\pi}_\ast (p_2 \circ p_2 \ast \{x\} \times l)) \]
\[ = (\tilde{\pi}_\ast (p_2 \circ p_2 \ast \{l\})) \]
\[ = (\tilde{\pi}_\ast (\mathbb{P}^{n_\alpha} \times l)) \quad (3.9) \]

\[ \therefore (3.8) \text{ becomes} \]
\[ \{x\} \times l' = s_\ast s_\ast (\tilde{\pi}_\ast (\mathbb{P}^{n_\alpha} \times l)) \quad (3.10) \]
\[ \implies \pi_\ast (\{x\} \times l') = s_\ast (\tilde{\pi}_\ast (\mathbb{P}^{n_\alpha} \times l)) \quad [\therefore s_\ast = (\pi_\ast)^{-1}] \quad (3.11) \]

But applying \( \pi_\ast \) to both sides, we see that \( \pi_\ast \pi_\ast (\{x\} \times l') = 0 \), since clearly \( \pi_\ast \circ \pi_\ast = 0 \) as taking inverse image under a bundle map increases the dimension and then taking
image decreases the dimension.

On the other hand, since $\pi \circ s = \text{Id}_E$,

$$
\pi_\ast \circ s_\ast(\overline{\mathbb{P}^n \times l}) = \overline{\mathbb{P}^n \times l}
$$

\[\therefore\] we would get $\overline{\mathbb{P}^n \times l} = 0$. But from (3.10) we would get $[\{x\} \times l'] = 0$, which would give, by lemma 3.6, that $g_\alpha([x']) = 0$, which is a contradiction since $g_\alpha$ is an isomorphism. Hence the claim is proved. \[\square\]

3.1. Proof of the main theorem. Before coming to the main theorem, let us make an useful remark:

**Remark 3.8.** We note a fact about vector spaces. Let $V, W$ be two $\mathbb{Q}$-vector spaces with an isomorphism $\varphi : \mathbb{Q}\langle e \rangle \oplus V \sim \mathbb{Q}\langle f \rangle \oplus W$, where $e$ and $f$ are any two basis elements. Consider the composite map

$$
\mathbb{Q}\langle e \rangle \hookrightarrow \mathbb{Q}\langle e \rangle \oplus V \xrightarrow{\varphi} \mathbb{Q}\langle f \rangle \oplus W
$$

and moreover assume that $\psi(e) \neq 0$. In other words, the composition

$$
\mathbb{Q}\langle e \rangle \hookrightarrow \mathbb{Q}\langle e \rangle \oplus V \xrightarrow{\varphi} \mathbb{Q}\langle f \rangle \oplus W \xrightarrow{p_1} \mathbb{Q}\langle f \rangle
$$

is non-zero.

**claim:** $\varphi|_V$ induces an isomorphism $V \cong W$.

**proof:** Since $\psi : \mathbb{Q}\langle e \rangle \to \mathbb{Q}\langle f \rangle$ is nonzero, $\psi$ is an isomorphism of $\mathbb{Q}$-vector spaces. By assumption, we can write $\psi(e) = r \cdot f$ for some $0 \neq r \in \mathbb{Q} \implies e = \psi^{-1}(r \cdot f)$.

\[\therefore\] $\varphi|_{\mathbb{Q}\langle e \rangle}$ looks like $xe \mapsto (x \cdot rf, x \cdot \phi(\psi^{-1}(rf))) = r \cdot (xf, x \cdot g(f)) \forall x \in \mathbb{Q}$, where $g := \phi \psi^{-1} : \mathbb{Q}\langle f \rangle \to W$.

\[\therefore\] ignoring the multiplication by $r$, we get that under $\varphi$, $\text{image}(\mathbb{Q}\langle e \rangle) = \text{graph}(g)$.

\[\therefore\] $V \cong \frac{\mathbb{Q}\langle e \rangle \oplus V}{\mathbb{Q}\langle e \rangle} \xrightarrow{\varphi|_{\mathbb{Q}\langle e \rangle}} \frac{\mathbb{Q}\langle f \rangle \oplus W}{\text{graph}(g)} \cong W$.

The last isomorphism follows, since the map

$$
\mathbb{Q}\langle f \rangle \oplus W \to W, \ (v, w) \mapsto g(v) - w
$$

is surjective, with kernel $\text{graph}(g)$. (claim proved)

**Proposition 3.9.** For generic weights $\alpha, \beta$ in adjacent chambers, the map $g_\beta^{-1} \circ g_\alpha$, when restricted to $\text{CH}_1^\mathbb{Q}(\mathcal{M}_\alpha)$, induces isomorphism

$$
\text{CH}_1^\mathbb{Q}(\mathcal{M}_\alpha) \xrightarrow{g_\beta^{-1} \circ g_\alpha} \sim \text{CH}_1^\mathbb{Q}(\mathcal{M}_\beta).
$$
Proof. Using lemma 3.3, let us write \( \text{CH}_0^Q(\phi_\alpha^{-1}(\Sigma_\gamma)) = \mathbb{Q}\langle e \rangle \) and \( \text{CH}_0^Q(\phi_\beta^{-1}(\Sigma_\gamma)) = \mathbb{Q}\langle f \rangle \), where \( e, f \) are some basis elements. Recall the maps \( g_\alpha, g_\beta \) from (3.1) and (3.3).

Consider the composition
\[
\mathbb{Q}\langle e \rangle \twoheadrightarrow \mathbb{Q}\langle e \rangle \oplus \text{CH}_1^Q(\mathcal{M}_\alpha) \xrightarrow{g_\beta^{-1} \circ g_\alpha} \mathbb{Q}\langle f \rangle \oplus \text{CH}_1^Q(\mathcal{M}_\beta) \xrightarrow{p_1} \mathbb{Q}\langle f \rangle \tag{3.12}
\]
where \( p_1 \) is the first projection.

According to remark 3.8, we will be done if we can show that the composition in (3.12) is nonzero. Consider the first projection \( p_1 \circ g_\beta^{-1} : \text{CH}_1^Q(\mathcal{N}) \twoheadrightarrow \mathbb{Q}\langle f \rangle \) with respect to \( g_\beta \). This map can be described as follows:

We note that the other projection with respect to \( g_\beta \), namely \( p_2 \circ g_\beta^{-1} : \text{CH}_1^Q(\mathcal{N}) \twoheadrightarrow \text{CH}_1^Q(\mathcal{M}_\beta) \) is given by \( (\psi_\beta)_* \), since by \([\text{Voi}, \text{Corollary 9.15}]\) \( \psi_\beta_* \circ \psi_\beta^* = \text{Id}_{\mathcal{M}_\beta} \), and \( \psi_\beta_* \) sends the terms coming from \( \text{CH}_0^Q(\phi_\beta^{-1}(\Sigma_\gamma)) \) to 0, since their image under \( \psi_\beta \) has strictly smaller dimension than the source.

\[ \forall Z \in \text{CH}_1^Q(\mathcal{N}), \ Z = (Z - (\psi_\beta^* \circ \psi_\beta_*)(Z)) + \psi_\beta^*(\psi_\beta_*(Z)), \] and by description of \( g_\beta \) in (3.4), we get that \( Z - (\psi_\beta^* \circ \psi_\beta_*)(Z) \) is the first projection with respect to \( g_\beta \), i.e.

\[ (p_1 \circ g_\beta^{-1})(Z) = Z - (\psi_\beta^* \circ \psi_\beta_*)(Z). \]

∴ from proposition 3.7 we get that

\[ (p_1 \circ g_\beta^{-1})(Z) \neq 0, \text{ where } Z = g_\alpha([x']), \]

also, \([x']\) is a basis for \( \text{CH}_0^Q(\phi_\alpha^{-1}(\Sigma_\gamma)) = \mathbb{Q}\langle e \rangle \); in other words, the composite map in (3.12) is nonzero. Hence we are done by remark 3.8, as mentioned before.

\[ \square \]

**Theorem 3.10.** For any two generic weights \( \alpha \) and \( \beta \), there exists a canonical isomorphism

\[ \text{CH}_1^Q(\mathcal{M}_\alpha) \cong \text{CH}_1^Q(\mathcal{M}_\beta). \]

**Proof.** By [BH, Lemma 2.7 and Remark 2.9], the moduli spaces corresponding to weights in the same chamber are isomorphic. Moreover, we can order the finitely many chambers in such a way that any two consecutive chambers are separated by a single wall. Combining this fact with proposition 3.9, we get our claim.

\[ \square \]

4. **Chow group of 1-cycles of rank 2 parabolic bundles**

In this section, we look at rank 2 parabolic semistable bundles of fixed determinant \( \mathcal{L} \) of degree 1. We also assume that all the flags at the parabolic points are full, i.e. \( m_{p,i} = 1 \ \forall i, p \in S \). Since we are in rank 2 case, this amounts to giving a 1-dimensional subspace of each fiber over the parabolic points.
4.1. The case of small weights. We recall the following proposition from [BY]:

**Proposition 4.1** (BY, Proposition 5.2). Suppose $E$ be a vector bundle of rank $r$ and degree $d$ on $X$. Define the following quantities:

$$
\epsilon_\pm(d, r) = \inf \{ \pm \left( \frac{d}{r} - \frac{d'}{r'} \right) \mid d', r' \in \mathbb{Z}, 1 \leq r' < r, \text{and} \pm \left( \frac{d}{r} - \frac{d'}{r'} \right) > 0 \} \\
\epsilon(d, r) = \min \{ \epsilon_\pm(d, k) \mid k = 1, ..., r \}
$$

Furthermore, suppose $\sum_{p \in S} \sum_{i=1}^{s_p} m_{p,i} \alpha_{p,i} < \epsilon(d, r)/2$.

(i) If $E$ is stable as a regular bundle, then $E_\ast$ is parabolic stable.

(ii) If $E_\ast$ is parabolic stable, then $E$ is semistable as regular bundle.

**Lemma 4.2.** Let rank $=2$, and choose a generic weight $\alpha$ as in proposition 4.1. There exists a canonical morphism $M_\alpha \to M$ making $M_\alpha$ into a $(\mathbb{P}^1)^n$-bundle over $M$, where $n = |S|$. 

*Proof.* Since $\alpha$ is generic, $E_\ast$ is in fact parabolic stable, so by proposition 4.1 (ii), $E$ is regular semistable bundle (hence stable) as well. Hence there is a map

$$g : M_\alpha \to M$$

by forgetting the parabolic structure.

For simplicity, first let $n = 1$, i.e. only one parabolic point. Recall that $M$ is a fine moduli space, since $\deg(L) = 1$. ([BY, Proposition 3.2]). Consider the the universal (or Poincare) bundle over $X \times M$, whose fiber over each $(p, [E])$ is given by $E_p$. Restrict the bundle over $\{p\} \times M$. Call the resulting bundle $E$. For each $[E] \in M$, the fiber of $\mathbb{P}(E)$ over $(p, [E])$ is $\mathbb{P}(E_p)$, i.e. lines in $E_p$. Hence the fiber of $\mathbb{P}(E)$ over $(p, [E])$ gives the set of all possible full flags at $E_p$. Moreover, by proposition 4.1 (i), the parabolic bundle $E_\ast$ resulting from the weight $\alpha$ and parabolic point $p$ will be automatically parabolic stable. In other words, for each $[E] \in M$, each point in the fiber of $\mathbb{P}(E)$ over $(p, [E])$ corresponds to a unique point $[E_\ast] \in M_\alpha$. This way we get a map $\psi : \mathbb{P}(E) \to M_\alpha$.

On the other hand, by [Har, Proposition 7.12], giving a morphism $M_\alpha \to \mathbb{P}(E)$ over $M$ is equivalent to giving a line bundle $L$ on $M_\alpha$ and a surjective map of sheaves $f : g^*E \to L$; we can choose $L := g^*E$ and $f = 1_{g^*E}$ as our candidate. This gives a morphism $\varphi : M_\alpha \to \mathbb{P}(E)$ in the opposite direction. It is easy to check that $\varphi$ and $\psi$ are inverses of each other, giving $\mathbb{P}(E) \cong M_\alpha$.

In general, if the parabolic data consists of $n$ distinct set of closed points $S = \{p_1, \cdots, p_n\}$ and generic weight $\alpha$ as in proposition 4.1, for each $i = 1, ..., n$ let $E_i$ denote the restriction of the universal bundle over $X \times M$ to $\{p_i\} \times M$. Then an analogous argument as above shows that $M_\alpha$ is isomorphic to the fiber product of $\mathbb{P}(E_i)$’s over $M$, i.e.

$$M_\alpha \cong \mathbb{P}(E_1) \times_M \mathbb{P}(E_2) \times_M \cdots \times_M \mathbb{P}(E_n)$$

\[\square\]
Proposition 4.3. For rank=2 and generic weights \( \alpha \) as in 4.1,
\[
\CH_Q^1(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \CH_Q^1(\mathcal{M}), \quad \text{where } n = |S|.
\]

Proof. For each \( 1 \leq i \leq n \), let \( \mathcal{F}_i := \mathbb{P}(\mathcal{E}_1) \times_{\mathcal{M}} \mathbb{P}(\mathcal{E}_2) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathbb{P}(\mathcal{E}_i) \).
By 4.2 we have \( \mathcal{M}_\alpha \cong \mathcal{F}_n \), so we have the following fiber diagram:

\[
\begin{array}{c}
\mathcal{M}_\alpha \\
\downarrow \\
\mathcal{F}_{n-1} \end{array} \longrightarrow \begin{array}{c}
\mathbb{P}(\mathcal{E}_n) \\
\downarrow \\
\mathcal{M} \end{array}
\]

The left and right vertical arrows above are \( \mathbb{P}^1 \)-bundles, and hence by [Voi, Theorem 9.25], there exist isomorphisms of Chow groups:
\[
\CH_Q^1(\mathbb{P}(\mathcal{E}_n)) \cong \CH_Q^0(\mathcal{M}) \oplus \CH_Q^1(\mathcal{M})
\]
and
\[
\CH_Q^1(\mathcal{M}_\alpha) \cong \CH_Q^1(\mathcal{F}_n) \cong \CH_Q^0(\mathcal{F}_{n-1}) \oplus \CH_Q^1(\mathcal{F}_{n-1})
\]
Iterating the same for \( \mathcal{F}_{n-1}, \mathcal{F}_{n-2}, \) and so on, we get from (4.1):
\[
\CH_Q^1(\mathcal{M}_\alpha) \cong \CH_Q^1(\mathcal{F}_n) \cong \CH_Q^0(\mathcal{F}_{n-1}) \oplus (\CH_Q^0(\mathcal{F}_{n-2}) \oplus \CH_Q^1(\mathcal{F}_{n-2}))
\]
\[
\vdots
\]
\[
\cong \bigoplus_{i=1}^{n-1} \CH_Q^0(\mathcal{F}_i) \oplus \CH_Q^1(\mathcal{F}_1)
\]
\[
\cong \bigoplus_{i=1}^{n-1} \CH_Q^0(\mathcal{F}_i) \oplus \CH_Q^0(\mathcal{M}) \oplus \CH_Q^1(\mathcal{M})
\]
Now, by [KS, Theorem 1.2] \( \mathcal{M} \) is rational, and hence any projective bundle over it must also be rational; so each \( \mathcal{F}_i \) must be rational. By [Ful, Example 16.1.11], the Chow group of 0-cycles is a birational invariant, hence it follows that \( \CH_Q^0(\mathcal{M}) \cong \mathbb{Q} \), and \( \CH_Q^0(\mathcal{F}_i) \cong \mathbb{Q} \) \( \forall i \).

Hence we conclude that
\[
\CH_Q^1(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \CH_Q^1(\mathcal{M}).
\]
\( \square \)

4.2. The case of determinant \( \mathcal{O}_X(x) \). Let us fix determinant \( \mathcal{O}_X(x) \), where \( x \in X \) is any closed point. In this case, we have the following result due to I. Choe and J. H. Hwang:

Theorem 4.4 ([CH, Main theorem]). There is a canonical isomorphism \( \CH_Q^1(\mathcal{M}) \cong \CH_Q^0(\mathcal{M}) \).
We are now able to extend this result for the moduli of parabolic bundles.
**Theorem 4.5.** In case of rank 2 and determinant $O_X(x)$, for any generic weight $\alpha$, we have

$$\text{CH}_1^Q(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \text{CH}_{0}^Q(X),$$

where $n = |S|$.

**Proof.** Combining proposition 4.3 and theorem 4.4, we get $\text{CH}_1^Q(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \text{CH}_{0}^Q(X)$ for weight $\alpha$ small enough as in proposition 4.1. But using theorem 3.10, we can conclude that the same result holds true for arbitrary generic weights as well. □

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