Stochastic Qubits

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Abstract

A new concept of qubits is given by considering entanglement of ordinary quibits with quantum measuring devices (micro-detectors). They are called stochastic qubits since they are generalized coherent states used in the stochastic (phase space) quantum theory. Entanglement is realized through the coupling of angular momenta \( l \) and \( \sigma \), where the micro-detector has \( l = 0, 1 \) and the qubit spin is \( \sigma = 1/2 \). In both cases, the stochastic qubit has total spin \( J = 1/2 \) and is entangled only when \( l = 1 \). In this case, Stochastic Bell states have been defined and teleportation has been studied. They resemble conventional ones. When the micro-detectors have only two states, Stochastic qudits have rather been used. Here, Stochastic Bell states have also been defined and teleportation is possible for special states only. In the last step of this teleportation, Bob will have to transform the qubit only, or the micro-detector only, to recover Alice state.

1 Introduction

The theory of quantum information is based on the concept of qubits. The treatment of these states depends on the representation chosen to describe them. In general, this description is realized using the Dirac formalism in which a qubit corresponds to a two-state physical system such as the electron or the photon [1]. For instance, in quantum field theory, a generalized concept of qubit (QFTbit) has been proposed as a dressed electron on account of gauge invariance [2]. Our concern is that, in many non relativistic representations, if not all of them, the measurement apparatus is not explicitly taken into account albeit quantum information processes are fundamentally based on some measurement schemes. With the ongoing miniaturization trend, the natural question of considering quantum measurement apparatuses in the formalism of quantum information arises. As is well known, these apparatuses should be entangled to the qubits in order to extract information from them. The appearance of this entanglement is a phenomenon due solely to the interaction of apparatus with the measured system [3, 4].
In this work, we develop a formalism in which the qubit is entangled to a quantum measuring apparatus (a micro-detector) and, within which, we reformulate some quantum information concepts such as the qubit itself, entanglement and teleportation [5].

Rather than basing our approach on a specific interaction, we start right away with two types of entanglement. In the first case, it is embodied in the very definition of a qubit as a generalized coherent state (GCS) of the isochronous Galilei group [6]. In the second case, we consider a Bell type entanglement between the qubit and a two-state measuring apparatus.

The reason for choosing generalized coherent states is that they provide a phase space representation of quantum mechanics which aims at solving the localization problem [7]. From the quantum informational standpoint, the importance of the generalized coherent states is that they provide positive operator valued measures (POVM) that are "informationally" complete [7]. The latter can serve as stochastic measurement operators in the sense that they yield probabilities for stochastic position, momentum, and even spin measurement outcomes [7, 8]. The stochastic values are related to the actual values of these observables through confidence distributions of the quantum measuring apparatus representing probability densities that the actual value be \( x \) when the measurement outcome is \( q \). The generalized coherent states constitute an overcomplete system, provide a resolution of the identity, and determine the aforementioned confidence functions. To be definite, we note that non orthogonality with respect to phase space variables, which is responsible of overcompleteness and which is more involved, will not be considered now but postponed to future investigation. We shall rather rely on the orthogonality with respect to spin components of the generalized coherent states. This will enable us to define stochastic qubits and to recover some conventional results with the difference that the measuring apparatus is not discarded from the formalism.

A noticeable feature in our generalization of qubits as GCS is that the measurement apparatus must have integer spin so that it cannot be a two-state system. In order to handle this case, we maintain the entanglement idea between the qubit and apparatus but not through generalized coherent states.

The paper will be organized as follows. In Sect. 2, we gather the most relevant GCS concepts which are relevant to our work and present them in the context of the stochastic quantum theory. In Sect. 3, we define the stochastic qubits as special cases of the GCS. In Sect. 4, we construct Bell states out of the stochastic qubits before using them for teleportation in Sect. 5. We reconsider these concepts for a two-state measuring apparatus in Sect. 6, and summarize all the results in the conclusion.

2 Generalized coherent states

In the spinless case, the GCS have a clear-cut physical interpretation advocated by the stochastic theory [7], namely, that of proper state vector of a quantum (imperfect) measuring apparatus. The main idea of that theory is the considera-
tion of the imprecision of the measuring apparatus in the formalism of quantum theory using a confidence measure $\mu$. The latter satisfies the following relation

$$
\mu_q(\Delta) = \int_{\Delta} \chi_q(x) dx
$$

and corresponds to the probability that the real value be in the interval $\Delta$ when the reading is $q$. The set $\{q = (q, \mu_q)\}$ constitutes the stochastic configuration space. When the value $q$ is accurate, then $\mu_q$ will be a $\delta$ measure (the characteristic function) and $\chi_q$ a $\delta$ distribution. The distribution $\chi_q$ has the form [7]

$$
\chi_q(x) = (2\pi \hbar)^{3/2} \left| \tilde{\xi}(x - q) \right|^2
$$

and is interpreted as the probability density that the real position will be $x$ when the reading is $q$. Then the function $\tilde{\xi}(x - q)$ is interpreted as the proper wave function of an apparatus located at the stochastic position $q$ [7]. The stochastic wave function is written as a scalar product in the phase space, configuration, or momentum representations with respective wave functions $\psi$, $\hat{\psi}$, and $\tilde{\psi}$ [7]

$$
\psi\xi(q,p) = \langle \xi_{q,p} | \psi \rangle = \langle \tilde{\xi}_{q,p} | \hat{\psi} \rangle = \langle \tilde{\xi}_{q,p} | \tilde{\psi} \rangle
$$

The subscript in $\psi\xi$ is a reminder that the states $| \psi\xi \rangle$ form a Hilbert subspace $L^2(\Gamma\xi)$ in the space $L^2(\Gamma)$ of square integrable functions over the phase space $\Gamma = \{(q,p)\} = \mathbb{R}^6$. They provide probabilities in the stochastic phase space

$$
\Gamma\xi = \{(\hat{q}, \hat{p}) = (\{(q,p), \mu_{q,p}\}\}
$$

and

$$
\chi_{q,p}(x,k) = \hat{\chi}_{q,p}(k)
$$

$$
\tilde{\chi}_{p}(k) = (2\pi \hbar)^{3/2} \left| \tilde{\xi}(k - p) \right|^2
$$

In the generic case, the generalized coherent states $| \eta^{JM}_{\xi_{q,p}} \rangle$, of the isochronous Galilei group, have the following form [6]:

$$
| \eta^{JM}_{\xi_{q,p}} \rangle = \frac{1}{\sqrt{(2J + 1) \sum_{j,m}}} \sum_{j,m} \langle j,m;j_z|JM \rangle | \xi_{q,p}^{lm} \rangle | jj_z \rangle
$$

$$
\tilde{\xi}_{q,p}^{lm}(k) = \frac{1}{\sqrt{(2\pi \hbar)^3}} e^{-i k \cdot q} R(||k - p||) Y^{lm}(k - p)
$$

where $\langle j,m;j_z|JM \rangle$ is a Clebsh-Gordan coefficient so that $J = |l - j|, \ldots, l + j$, $M = -J, \ldots, +J$, $m = -l, \ldots, +l$ and $j_z = -j, \ldots, +j$ with the condition $M = j_z + m$. In the spherical harmonic $Y^{lm}$, the argument $(k - p)$ stands for the spherical angle. The radial function $R$ is normalized and $| jj_z \rangle$ is the spin state canonical basis. Then, for a system with spin $J$, the state $| \psi \rangle$ and the analog of
the wave function (3) are

$$|\psi\rangle = \sum_{JM} dqdp \psi_{jmjM}(q,p) |\eta_{\xi_{q,p}}^{JM}\rangle \tag{9}$$

$$\psi_{jmjM}(q,p) = \langle \eta_{\xi_{q,p}}^{JM} | \psi \rangle \tag{10}$$

This is so because the overcomplete family of GCS provides a resolution of the identity

$$\sum_{JM} dqdp \langle \eta_{\xi_{q,p}}^{JM} | \eta_{\xi_{q,p}}^{JM} \rangle = I \tag{11}$$

From now on, $|jjz\rangle$ will represent the qubit with $j = \frac{1}{2}$ and $jz = \pm \frac{1}{2}$, and $|\xi_{q,p}^{lm}\rangle$ will correspond to the measuring apparatus.

### 3 Qubits in stochastic phase space

Let us rewrite relation (7) in the form :

$$|\eta_{\xi_{q,p}}^{JM}\rangle = \frac{1}{\sqrt{(2J+1)}} \left( \langle l, \frac{1}{2}; M - \frac{1}{2}, \frac{1}{2} | JM \rangle |0_{\xi_{q,p}}^{lm}\rangle \right)$$

$$+ \frac{1}{\sqrt{(2J+1)}} \left( \langle l, \frac{1}{2}; M + \frac{1}{2}, -\frac{1}{2} | JM \rangle |1_{\xi_{q,p}}^{lm}\rangle \right) \tag{12}$$

where the states $|0_{\xi_{q,p}}^{lm}\rangle$ and $|1_{\xi_{q,p}}^{lm}\rangle$ are

$$|0_{\xi_{q,p}}^{lm}\rangle = |\xi_{q,p}^{lm}\rangle \left| j = \frac{1}{2}, jz = \frac{1}{2} \right\rangle = |\xi_{q,p}^{lm}\rangle |0\rangle, \quad m = M - \frac{1}{2} \tag{13}$$

$$|1_{\xi_{q,p}}^{lm}\rangle = |\xi_{q,p}^{lm}\rangle \left| j = \frac{1}{2}, jz = -\frac{1}{2} \right\rangle = |\xi_{q,p}^{lm}\rangle |1\rangle, \quad m = M + \frac{1}{2} \tag{14}$$

Here, for the sake of conformity with quantum information [1], we have used the notation $|0\rangle = |j = \frac{1}{2}, jz = \frac{1}{2}\rangle$, $|1\rangle = |j = \frac{1}{2}, jz = -\frac{1}{2}\rangle$. These relations mean that the qubit states ($j = \frac{1}{2}, jz = \pm \frac{1}{2}$) are measured with the micro-detector state $|\xi_{q,p}^{lm}\rangle$. The latter has angular momentum $l$, projection $m$, and is localized at the stochastic phase space point $(q,p)$. The detection of any qubit state $|0\rangle$ or $|1\rangle$ in the measurement, involves all possible micro-detector states and generates stochastic qudit (Squdit) states (12). In the case where $J = \frac{1}{2}$, we have a Squbit. Consequently, the Squdit (or Squbit) is not defined only by the physical system, but by both the physical system and the measuring device. Moreover, we note that different detector angular momentum values generate intrinsically different Squdits. This means that one should not identify the ordinary qubit with the...
exploitable information which is encoded in the Squbit. In other words, the micro-detector is identified with the physical support of information.

We suppose that \((q, p)\) are fixed so that the state of the Squbit can be written in the following form

\[
|\Phi_{\eta_{q,p}, \frac{1}{2} lJ} \rangle = \sum_M \alpha_{\frac{1}{2} lJM} |\eta_{\xi_{q,p}}^{\frac{1}{2} lJM} \rangle
\]  

(15)

\[
\sum_M |\alpha_{\frac{1}{2} lJM}|^2 = 1
\]  

(16)

This means that we consider discrete properties of states, and defer the continuous ones to forthcoming works.

Now, we illustrate these considerations with some special cases corresponding to \(l = 0, 1\).

a) For \(l = 0\), we have \(J = j = \frac{1}{2}\) and the Squbit takes the form of a non entangled Squbit (12):

\[
\begin{array}{c|cc}
 l & J & M \\
 0 & \frac{1}{2} & 0^\frac{1}{2} l0 \eta_{\xi_{q,p}} = \frac{1}{\sqrt{3}} |0_{\xi_{q,p}}^{00}\rangle + \frac{2}{\sqrt{3}} |1_{\xi_{q,p}}^{00}\rangle \\
 & \frac{1}{2} & 0^\frac{1}{2} l1 \eta_{\xi_{q,p}} = \frac{1}{\sqrt{3}} |\xi_{q,p}^{00}\rangle \\
 & \frac{1}{2} & 1^\frac{1}{2} l0 \eta_{\xi_{q,p}} = \frac{1}{\sqrt{3}} |1_{\xi_{q,p}}^{00}\rangle \\
 & \frac{1}{2} & 1^\frac{1}{2} l1 \eta_{\xi_{q,p}} = \frac{1}{\sqrt{3}} |\xi_{q,p}^{00}\rangle \\
\end{array}
\]

b) For \(l = 1\), we have a Squbit \((J = \frac{1}{2})\) and a Squbit \((J = \frac{3}{2})\). In the first case, we have the following table:

\[
\begin{array}{c|cc}
 l & J & M \\
 1 & \frac{1}{2} & 0^\frac{1}{2} l0 \eta_{\xi_{q,p}} = -\frac{1}{\sqrt{3}} |0_{\xi_{q,p}}^{010}\rangle + \frac{2}{\sqrt{3}} |1_{\xi_{q,p}}^{010}\rangle \\
 & \frac{1}{2} & 0^\frac{1}{2} l1 \eta_{\xi_{q,p}} = -\frac{1}{\sqrt{3}} |\xi_{q,p}^{010}\rangle \\
 & \frac{1}{2} & 1^\frac{1}{2} l0 \eta_{\xi_{q,p}} = -\frac{1}{\sqrt{3}} |0_{\xi_{q,p}}^{11}\rangle + \frac{2}{\sqrt{3}} |1_{\xi_{q,p}}^{11}\rangle \\
 & \frac{1}{2} & 1^\frac{1}{2} l1 \eta_{\xi_{q,p}} = -\frac{1}{\sqrt{3}} |\xi_{q,p}^{11}\rangle \\
\end{array}
\]

These two relations show that the micro-detector is entangled with the ordinary qubit.

In the second case, we obtain the following states

\[
\begin{array}{c|cc}
 l & J & M \\
 1 & \frac{3}{2} & 0^\frac{3}{2} l0 \eta_{\xi_{q,p}} = \frac{1}{2} |0_{\xi_{q,p}}^{11}\rangle + \frac{1}{2} |1_{\xi_{q,p}}^{11}\rangle \\
 & \frac{3}{2} & 0^\frac{3}{2} l1 \eta_{\xi_{q,p}} = \frac{1}{2} (\sqrt{\frac{1}{3}} |\xi_{q,p}^{010}\rangle + \sqrt{\frac{2}{3}} |1_{\xi_{q,p}}^{010}\rangle + \sqrt{\frac{2}{3}} |\xi_{q,p}^{010}\rangle |0\rangle + \sqrt{\frac{2}{3}} |1_{\xi_{q,p}}^{010}\rangle |1\rangle \\
 & \frac{3}{2} & 1^\frac{3}{2} l0 \eta_{\xi_{q,p}} = \frac{1}{2} \left(\sqrt{\frac{1}{3}} |\xi_{q,p}^{11}\rangle + \sqrt{\frac{2}{3}} |1_{\xi_{q,p}}^{11}\rangle |0\rangle + \sqrt{\frac{2}{3}} |\xi_{q,p}^{11}\rangle |1\rangle \\
 & \frac{3}{2} & 1^\frac{3}{2} l1 \eta_{\xi_{q,p}} = \frac{1}{2} \left(\sqrt{\frac{1}{3}} |\xi_{q,p}^{11}\rangle + \sqrt{\frac{2}{3}} |1_{\xi_{q,p}}^{11}\rangle |0\rangle + \sqrt{\frac{2}{3}} |\xi_{q,p}^{11}\rangle |1\rangle \\
\end{array}
\]
Here, two Squbit states are entangled and the other two states are not.

For values of $l$ greater than one, only Squdits are obtained, some states of which are entangled.

### 4 Stochastic Bell states

We consider a bipartite state

$$|\Psi\rangle = \sum_{MM'} a_{MM'} |\eta^{JM}_{q,p}\rangle |\eta^{JM'}_{q',p'}\rangle$$

(17)

where

$$|\eta^{JM}_{q,p}\rangle |\eta^{JM'}_{q',p'}\rangle = \frac{1}{2J+1} \sum_{j=q, p} \langle l, j; m, j_z |JM\rangle \langle l, j; m, j'_z |JM'\rangle$$

$$\times |\xi_{q, p}\rangle |\xi'_{q', p'}\rangle |jj'_z\rangle$$

(18)

For simplicity, we take the same state for the two devices. The state $\xi_{q, p}$ refers to the first Squbit and $\xi_{q', p'}$ to the second one, and the canonical basis is denoted by

$$|jj_z\rangle |jj'_z\rangle = |jj'_z\rangle$$

(19)

We define the stochastic Bell states by

$$|\Psi^{\pm}_{\xi, q, r\xi', r'}\rangle = \frac{1}{\sqrt{2}} \left[ |\eta^{00}_{q,p}\rangle |\eta^{00}_{q', p'}\rangle \pm |\eta^{000}_{q,p}\rangle |\eta^{000}_{q', p'}\rangle \right]$$

(20)

$$|\Phi^{\pm}_{\xi, q, r\xi', r'}\rangle = \frac{1}{\sqrt{2}} \left[ |\eta^{00}_{q,p}\rangle |\eta^{00}_{q', p'}\rangle \pm |\eta^{000}_{q,p}\rangle |\eta^{000}_{q', p'}\rangle \right]$$

(21)

For $l = 0$, we obtain the stochastic Bell states in the form:

$$|\Psi^{\pm}_{\xi, q, r\xi', r'}\rangle = |\xi^{00}_{q,p}\rangle |\xi^{00}_{q', p'}\rangle |\Psi^{\pm}\rangle, \quad |\Phi^{\pm}_{\xi, q, r\xi', r'}\rangle = |\xi^{00}_{q,p}\rangle |\xi^{00}_{q', p'}\rangle |\Phi^{\pm}\rangle$$

(22)

The ordinary qubits are in Bell states

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle \pm |1\rangle |1\rangle \right)$$

(23)

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |1\rangle \pm |1\rangle |0\rangle \right)$$

(24)

while the micro-detectors are not entangled.

For $l = 1$, the stochastic Bell states

$$|\Psi^{\pm}_{\xi, q, r\xi', r'}\rangle = \frac{1}{\sqrt{2}} \left[ |\eta^{00}_{q,p}\rangle |\eta^{00}_{q', p'}\rangle \pm |\eta^{000}_{q,p}\rangle |\eta^{000}_{q', p'}\rangle \right]$$

(25)

$$|\Phi^{\pm}_{\xi, q, r\xi', r'}\rangle = \frac{1}{\sqrt{2}} \left[ |\eta^{00}_{q,p}\rangle |\eta^{00}_{q', p'}\rangle \pm |\eta^{000}_{q,p}\rangle |\eta^{000}_{q', p'}\rangle \right]$$

(26)
become

\[
\begin{align*}
\left| \Psi^\alpha_{q,p,q',p'} \epsilon_{1/2} \right> &= \sum_{\beta = \pm} \left( \left| \psi_{q,p,q',p'}^\alpha \right> \left| \Phi^\beta \right> + \left| \psi_{q,p,q',p'}^\alpha \right> \left| \Phi^\beta \right> \right) \\
\left| \Phi^\alpha_{q,p,q',p'} \epsilon_{1/2} \right> &= \sum_{\beta = \pm} \left( \left| \phi_{q,p,q',p'}^\alpha \right> \left| \Phi^\beta \right> + \left| \phi_{q,p,q',p'}^\alpha \right> \left| \Phi^\beta \right> \right)
\end{align*}
\]

where

\[
\begin{align*}
\left| \psi_{q,p,q',p'}^\alpha \right> &= \frac{1}{3} \left[ \delta_{\alpha,\beta} \left| \xi_{q,p}^{10} \right> \left| \xi_{q',p'}^{10} \right> + \alpha \left| \xi_{q,p}^{1-1} \right> \left| \xi_{q',p'}^{1-1} \right> + \beta \left| \xi_{q,p}^{11} \right> \left| \xi_{q',p'}^{11} \right> \right] \\
\left| \psi_{q,p,q',p'}^\alpha \right> &= -\frac{1}{3\sqrt{2}} \left[ \left| \xi_{q,p}^{10} \right> \left| \xi_{q',p'}^{10} \right> + \alpha \left| \xi_{q,p}^{1-1} \right> \left| \xi_{q',p'}^{1-1} \right> \right) + \beta \left| \xi_{q,p}^{11} \right> \left| \xi_{q',p'}^{11} \right> \right]
\end{align*}
\]

and

\[
\begin{align*}
\left| \phi_{q,p,q',p'}^\alpha \right> &= \frac{1}{3\sqrt{2}} \left[ \left| \xi_{q,p}^{10} \right> \left| \xi_{q',p'}^{1-1} \right> + \alpha \left| \xi_{q,p}^{1-1} \right> \left| \xi_{q',p'}^{10} \right> \right) + \beta \left| \xi_{q,p}^{11} \right> \left| \xi_{q',p'}^{11} \right> \right] \\
\left| \phi_{q,p,q',p'}^\alpha \right> &= \frac{1}{6} \left[ \left| \xi_{q,p}^{10} \right> \left| \xi_{q',p'}^{10} \right> + \alpha \left| \xi_{q,p}^{1-1} \right> \left| \xi_{q',p'}^{11} \right> \right) + \beta \left| \xi_{q,p}^{11} \right> \left| \xi_{q',p'}^{11} \right> \right]
\end{align*}
\]

In this case, ordinary qubits are in Bell states and the micro-detectors are in entangled states. Moreover, the ordinary qubits Bell states are entangled with the micro-detectors entangled states.

5 Teleportation

In the case \( l = 0 \), we obviously obtain the same results as for ordinary qubits multiplied by the micro-detector states. For \( l = 1 \), teleportation is also analog to the ordinary case in the following sens. In order to teleport an unknown state in Alice possession,

\[
\left| \Psi_{q,p,q',p'}^{1/2} \right> = \mu \left| \eta_{q,p,q',p'}^{1/2} \right> + \lambda \left| \eta_{q,p,q',p'}^{1/2} \right>
\]

Alice and Bob must share a stochastic Bell state, say \( \left| \Psi_{q,p,q',p'}^{1/2} \right> \). The global state can be written in the stochastic Bell basis of Alice two stochastic qubits in the usual manner.
Now, we suppose that the two ordinary qubits are in Bell state (23) or (24),

\[ \ket{\Psi^{\pm}_{\xi^{p},\eta^{p},\xi^{p'},\eta^{p'}}} = \left( \mu \ket{\frac{1}{\sqrt{2}} \xi^{p} + \frac{1}{\sqrt{2}} \eta^{p}} + \lambda \ket{\frac{1}{\sqrt{2}} \xi^{p} - \frac{1}{\sqrt{2}} \eta^{p}} \right) \ket{\Psi^{+}_{\xi^{p},\eta^{p},\xi^{p'},\eta^{p'}}} \]

and get the usual results [5, 1] except that in place of ordinary qubits, we have stochastic qubits.

### 6 Stochastic qubits with a two-state micro-detector

Rather than using (12), where \( l \) is an integer, we use entangled states between the measuring device and the qubit. For this, we suppose that the measuring device is a two-state system.

#### 6.1 Stochastic Bell states

We suppose that the qubit states are measured with the micro-detector states \( \ket{\xi_{q,p}^\pm} \). Hence, the entangled states are

\[ \ket{\Theta_{\xi_{q,p}^\pm}} = \left( \mu_{\pm} \ket{0_{\xi_{q,p}^\pm} + \lambda_{\mp} \ket{1_{\xi_{q,p}^\pm}} \right) \]

where

\[ \ket{0_{\xi_{q,p}^\pm}} = \ket{\xi_{q,p}^\pm} \ket{0} \]

\[ \ket{1_{\xi_{q,p}^\pm}} = \ket{\xi_{q,p}^\pm} \ket{1} \]

The general state of the qubit-device system is

\[ \ket{\Theta_{\xi_{q,p}}} = \mu_{+} \ket{0_{\xi_{q,p}^+} + \mu_{-} \ket{0_{\xi_{q,p}^-} + \lambda_{+} \ket{1_{\xi_{q,p}^+} + \lambda_{-} \ket{1_{\xi_{q,p}^-}}} \]

\[ 1 = \| \mu_{+} \|^2 + \| \lambda_{+} \|^2 + \| \mu_{-} \|^2 + \| \lambda_{-} \|^2 \]

Now, we suppose that the two ordinary qubits are in Bell states (23) or (24), and the two corresponding devices are also maximally entangled.
\[ |\psi_{q,p}^{\beta}\rangle_{12} = \frac{1}{\sqrt{2}} \left( \beta |\xi_{q,p}^{\beta}\rangle_{1} |\tau_{q',p'}^{\beta}\rangle_{2} + |\xi_{q,p}^{-\beta}\rangle_{1} |\tau_{q',p'}^{-\beta}\rangle_{2} \right) \]  
\[ |\varphi_{q,p}^{\beta}\rangle_{12} = \frac{1}{\sqrt{2}} \left( \beta |\xi_{q,p}^{\beta}\rangle_{1} |\tau_{q',p'}^{-\beta}\rangle_{2} + |\xi_{q,p}^{-\beta}\rangle_{1} |\tau_{q',p'}^{\beta}\rangle_{2} \right) \]

so that

\[ |\xi_{q,p}^{\beta}\rangle_{1} |\xi_{q,p}^{\beta}\rangle_{2} = \frac{1}{\sqrt{2}} \left( \beta |\psi_{q,p}^{\beta}\rangle_{12} + |\psi_{q,p}^{-\beta}\rangle_{12} \right) \]  
\[ |\xi_{q,p}^{\beta}\rangle_{1} |\tau_{q',p'}^{-\beta}\rangle_{2} = \frac{1}{\sqrt{2}} \left( \beta |\varphi_{q,p}^{\beta}\rangle_{12} + |\varphi_{q,p}^{-\beta}\rangle_{12} \right) \]

By analogy with the usual case, let's define the stochastic Bell basis as follows

\[ (\alpha, \beta = \pm) \]

\[ |\Phi_{q,q',p}^{(\alpha)\beta}\rangle_{12} = \frac{1}{\sqrt{2}} \left( |0_{q,p}^{\alpha} 1_{q',p'}^{\beta}\rangle_{12} + \alpha |1_{q,p}^{\alpha} 0_{q',p'}^{-\beta}\rangle_{12} \right) \]  
\[ |\Psi_{q,q',p}^{(\alpha)\beta}\rangle_{12} = \frac{1}{\sqrt{2}} \left( |0_{q,p}^{\alpha} 1_{q',p'}^{-\beta}\rangle_{12} + \alpha |1_{q,p}^{\alpha} 0_{q',p'}^{\beta}\rangle_{12} \right) \]  
\[ |\Phi_{q,q',p}^{(\alpha)\beta}\rangle_{12} = \frac{1}{\sqrt{2}} \left( |0_{q,p}^{\alpha} 1_{q',p'}^{-\beta}\rangle_{12} + \alpha |1_{q,p}^{\alpha} 0_{q',p'}^{\beta}\rangle_{12} \right) \]

and consequently

\[ |0_{q,p}^{\alpha} 1_{q',p'}^{\beta}\rangle_{12} = \frac{1}{\sqrt{2}} \left( |\Phi_{q,q',p}^{(\alpha)\beta}\rangle_{12} + |\Phi_{q,q',p}^{(-\alpha)\beta}\rangle_{12} \right) \]  
\[ |1_{q,p}^{\alpha} 0_{q',p'}^{\beta}\rangle_{12} = \frac{1}{\sqrt{2}} \left( |\Phi_{q,q',p}^{(\alpha)\beta}\rangle_{12} - |\Phi_{q,q',p}^{(-\alpha)\beta}\rangle_{12} \right) \]

9
The relations (44)-(47) can be rewritten in the form

$$\left| \Phi^{(\alpha)(\beta)}_{\psi_{q,p}, t_{q',p'}} \right>_{12} = \frac{1}{\sqrt{2}} \left( \beta \left| \Phi^{0\beta}_{\phi_{q,p}, t_{q',p'}} \right>_{12} + \Phi^{-\alpha\beta}_{\phi_{q,p}, t_{q',p'}} \right)_{12} \quad (56)$$

$$\left| \Phi^{(\alpha)(\beta)}_{\psi_{q,p}, t_{q',p'}} \right>_{12} = \frac{1}{\sqrt{2}} \left( \beta \left| \Phi^{0\beta}_{\phi_{q,p}, t_{q',p'}} \right>_{12} + \Phi^{-\alpha\beta}_{\phi_{q,p}, t_{q',p'}} \right)_{12} \quad (57)$$

$$\left| \Psi^{(\alpha)(\beta)}_{\psi_{q,p}, t_{q',p'}} \right>_{12} = \frac{1}{\sqrt{2}} \left( \beta \left| \Psi^{0\beta}_{\phi_{q,p}, t_{q',p'}} \right>_{12} + \Psi^{-\alpha\beta}_{\phi_{q,p}, t_{q',p'}} \right)_{12} \quad (58)$$

$$\left| \Psi^{(\alpha)(\beta)}_{\psi_{q,p}, t_{q',p'}} \right>_{12} = \frac{1}{\sqrt{2}} \left( \beta \left| \Psi^{0\beta}_{\phi_{q,p}, t_{q',p'}} \right>_{12} + \Psi^{-\alpha\beta}_{\phi_{q,p}, t_{q',p'}} \right)_{12} \quad (59)$$

where

$$\left| \Phi^{\alpha\beta}_{\psi_{q,p}, t_{q',p'}} \right>_{12} = \left| \Phi^{\alpha\beta}_{\psi_{q,p}, t_{q',p'}} \right>_{12} \quad (60)$$

and so on. Hence, we find that the stochastic Bell states are maximally entangled states of the ordinary qubits Bell states and the micro-detectors Bell states.

### 6.2 Teleportation using the stochastic Bell basis

Now, we suppose that Alice has the following stochastic qudit

$$\left| \Omega_{\xi_{q,p}} \right>_{1} = \mu^{+} \left| 0_{\xi_{q,p}} \right>_{1} + \mu^{-} \left| 0_{\xi_{q,p}} \right>_{1} + \lambda^{+} \left| 1_{\xi_{q,p}} \right>_{1} + \lambda^{-} \left| 1_{\xi_{q,p}} \right>_{1}$$

$$= \sum_{\gamma = \pm} \left( \mu_{\gamma} \left| 0_{\xi_{q,p}} \right>_{1} + \lambda_{\gamma} \left| 1_{\xi_{q,p}} \right>_{1} \right) \quad (61)$$

and shares the stochastic Bell state

$$\left| \Phi^{(\alpha)(\beta)}_{\psi_{q',p', t_{q',p'}}} \right>_{23} = \frac{1}{\sqrt{2}} \left( \left| 0_{\tau_{q',p'}} \right>_{23} - \left| 1_{\tau_{q',p'}} \right>_{23} \right) + \alpha \left| 0_{\tau_{q',p'}} \right>_{23} \quad (62)$$

with Bob. Then, the global state

$$\left| \Psi \right>_{123} = \left| \Omega_{\xi_{q,p}} \right>_{1} \otimes \left| \Phi^{(\alpha)(\beta)}_{\psi_{q',p', t_{q',p'}}} \right>_{23} \quad (63)$$

$$= \frac{1}{\sqrt{2}} \sum_{\gamma = \pm, \beta} \left( \mu_{\gamma} \left| 0_{\tau_{q',p'}} \right>_{12} \left| 0_{\tau_{q',p'}} \right>_{12} + \lambda_{\gamma} \left| 1_{\tau_{q',p'}} \right>_{12} \left| 1_{\tau_{q',p'}} \right>_{12} \right) + \mu_{\tau} \left| 0_{\tau_{q',p'}} \right>_{12} \left| 0_{\tau_{q',p'}} \right>_{12} + \lambda_{\tau} \left| 1_{\tau_{q',p'}} \right>_{12} \left| 1_{\tau_{q',p'}} \right>_{12} \right) \quad (64)$$
can be expressed in terms of Alice stochastic Bell states

\[
\left| \Psi_{123}^{(\alpha)(\beta)} \right\rangle = \frac{1}{2} \left( \lambda_{\beta} \left| 0_{\pi_{q,p}^\beta} \right\rangle_3 + \mu_{\beta} \left| 1_{\pi_{q,p}^\beta} \right\rangle_3 \right)
\]

In the second case, we have \( \mu_{\beta} = \lambda_{-\beta} = 0 \), and the global state will be

\[
\left| \tilde{\Psi}_{123}^{(\alpha)(\beta)} \right\rangle = \frac{1}{2} \left( \lambda_{\beta} \left| 0_{\pi_{q,p}^\beta} \right\rangle_3 + \mu_{\beta} \left| 1_{\pi_{q,p}^\beta} \right\rangle_3 \right)
\]

In the first case, we have \( \mu_{-\beta} = \lambda_{\beta} = 0 \), and the global state can be obtained from that of the first case by interchanging \( \psi \) and \( \varphi \), and changing the sign of \( \beta \) everywhere except in \( \pi \).

We readily see that teleportation is impossible in this case, except when the state has one of the following two special forms

\[
\left| \Theta_{\xi_{q,p}^{\pm\beta}} \right\rangle_1 = \left( \mu_{\pm\beta} \left| 0_{\xi_{q,p}^{\pm\beta}} \right\rangle_1 + \lambda_{\pm\beta} \left| 1_{\xi_{q,p}^{\pm\beta}} \right\rangle_1 \right)
\]
Then, Alice will classically inform Bob that her pair of stochastic qubits is in the state \( |\Psi^{(\alpha)(\beta)}_{\xi_{q,p},\tau_{q,p}'}\rangle_{12} \), so that Bob will know that his stochastic qubit is in the state

\[
| \Theta^{-\beta}_{\pi_{\xi_{q,p},\tau_{q,p}'}},\pi_{\xi_{q,p},\tau_{q,p}'}\rangle_3 = \left( \lambda_{-\beta} | 0_{\pi_{\xi_{q,p},\tau_{q,p}'}},\pi_{\xi_{q,p},\tau_{q,p}'}\rangle_3 + \mu_{\beta} | 1_{\pi_{\xi_{q,p},\tau_{q,p}'}},\pi_{\xi_{q,p},\tau_{q,p}'}\rangle_3 \right)
\]  

(69)

Teleportation is achieved when Bob performs a \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) gate to his qubit without changing the micro-detector state. If Alice had obtained the state \( |\Psi^{(-\alpha)(\beta)}_{\xi_{q,p},\tau_{q,p}'}\rangle_{12} \), then Bob would have performed a \( Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) gate to the qubit. However, if she had obtained \( |\Phi^{(\alpha)(\beta)}_{\xi_{q,p},\tau_{q,p}'}\rangle_{12} \) or \( |\Phi^{(-\alpha)(\beta)}_{\xi_{q,p},\tau_{q,p}'}\rangle_{12} \), he would have performed a \( X \) or \( Y \) operation to his micro-detector without changing the qubit state.

7 Conclusion

This work was mainly based on the entanglement of the qubit with the measuring device. This entanglement was realized using generalized coherent states, in which the qubit has a one-half spin and the measuring device has a zero or one angular momentum. This corresponds to the usual angular momentum coupling. A second method of entanglement was proposed. It involves measuring devices having two states.

In the case of zero angular momentum, we obtained a stochastic qubit which is not entangled with the micro-detector.

In the case where the angular momentum equals one, the stochastic qubit is an entangled state between the ordinary qubit and the micro-detector. Stochastic qudits were also obtained. These may be entangled states or not. We have defined the stochastic Bell states by analogy with the usual case and used them to study the teleportation of stochastic qubits. The results formally coincide with those of the usual teleportation.

In the last case, we have defined the stochastic Bell states in the canonical basis. We obtained a maximal entanglement between Bell states of ordinary qubits and those of the measuring devices. The stochastic Bell states have been used for teleportation of the states (61). This teleportation proved to be impossible, except when the latter states have the special form (66). We note that in order to achieve this teleportation, we had to apply the final transformation to either the ordinary qubit alone or to the measuring device alone alone.

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