PSEUDOSPECTRUM FOR OSEEN VORTICES OPERATORS

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Abstract. In this paper, we give resolvent estimates for the linearized operator of the Navier-Stokes equation in \( \mathbb{R}^2 \) around the Oseen vortices, in the fast rotating limit \( \alpha \to +\infty \).

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1. INTRODUCTION

1.1. The origin of the problem. Consider the motion of a viscous incompressible fluid in the whole plane, which is described by the Navier-Stokes equation in \( \mathbb{R}^2 \). In two dimensions where the vorticity is a scalar, it is more convenient to study the evolution of the vorticity which is given by

\[
\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \nu \Delta \omega, \quad x \in \mathbb{R}^2, \quad t \geq 0,
\]

where \( \nu \) is the kinematic viscosity, \( \omega(x, t) \in \mathbb{R} \) is the vorticity of the fluid, \( v(x, t) \in \mathbb{R}^2 \) is the divergence-free velocity field reconstructed from \( \omega \) by the Biot-Savart law

\[
v(x, t) = (K_{BS} \ast \omega)(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)\perp}{|x - y|^2} \omega(y, t) dy,
\]

where we denote \( x\perp = (-x_2, x_1) \) for \( x = (x_1, x_2) \in \mathbb{R}^2 \). The equation \( (1.1) \) is globally well-posed in \( L^1(\mathbb{R}^2) \) (\cite{1}, \cite{13}), i.e. for any initial data \( \omega_0 \in L^1(\mathbb{R}^2) \), \( (1.1) \) has a
unique global solution \( \omega \in C^0([0, +\infty); L^1(\mathbb{R}^2)) \) such that \( \omega(0) = \omega_0 \). The total circulation of the velocity field

\[
\int_{\mathbb{R}^2} \omega(x, t) \, dx = \lim_{R \to +\infty} \oint_{|x| = R} v(x, t) \cdot dl
\]
is a quantity conserved by the semi-flow defined by (1.1) in \( L^1(\mathbb{R}^2) \). It is well-known that the equation (1.1) has a family of explicit self-similar solutions, called Oseen vortices, which is given by

\[
\omega(x, t) = \frac{\alpha}{\nu t} G\left(\frac{x}{\sqrt{\nu t}}\right), \quad v(x, t) = \frac{\alpha}{\sqrt{\nu t}} v^G\left(\frac{x}{\sqrt{\nu t}}\right),
\]
where

\[
G(x) = \frac{1}{4\pi} e^{-|x|^2/4}, \quad v^G(x) = \frac{1}{2\pi} \frac{x}{|x|^2} \left(1 - e^{-|x|^2/4}\right), \quad x \in \mathbb{R}^2,
\]
and the parameter \( \alpha \in \mathbb{R} \) is referred to as the circulation Reynolds number. In fact these solutions are trivial in the sense that \( v(x, t) \cdot \nabla \omega(x, t) \equiv 0 \) so that (1.1) reduces to the linear heat equation, and the Oseen vortices are the only self-similar solutions to the Navier-Stokes equations in \( \mathbb{R}^2 \) whose vorticity is integrable. Moreover, it is proved by T. Gallay and C.E. Wayne in [10] that if the initial vorticity \( \omega_0 \) is in \( L^1(\mathbb{R}^2) \), then the solution \( \omega(x, t) \) of (1.1) satisfies

\[
\lim_{t \to +\infty} \|\omega(\cdot, t) - \frac{\alpha}{\nu t} G\left(\frac{\cdot}{\sqrt{\nu t}}\right)\|_{L^1(\mathbb{R}^2)} = 0,
\]
where \( \alpha = \int_{\mathbb{R}^2} \omega_0(x) \, dx \). In physical terms, this means that the Oseen vortices are globally stable for any value of the circulation Reynolds number \( \alpha \). In contrast to many situations in hydrodynamics, such as the Poiseuille or the Taylor-Couette flows, increasing the Reynolds number does not produce any instability.

In order to investigate the stability of the Oseen vortices, we introduce the self-similar variables \( \tilde{x} = x/\sqrt{\nu t}, \ t = \log(t/T) \) and we set

\[
\omega(x, t) = \frac{1}{t} \tilde{\omega}\left(\frac{x}{\sqrt{\nu t}}, \log \frac{t}{T}\right), \quad v(x, t) = \sqrt{\frac{\nu}{t}} \tilde{v}\left(\frac{x}{\sqrt{\nu t}}, \log \frac{t}{T}\right).
\]
Then the rescaled system reads (replacing \( \tilde{x} \) by \( x \), \( \tilde{\omega} \) by \( \omega \) and so on)

\[
\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \Delta \omega + \frac{1}{2} x \cdot \nabla \omega + \omega, \quad x \in \mathbb{R}^2, \ t \geq 0,
\]
where \( \omega(x, t) \in \mathbb{R} \) is the rescaled vorticity, \( v(x, t) \in \mathbb{R}^2 \) is the rescaled velocity field again given by the Biot-Savart law (1.2). Then for any \( \alpha \in \mathbb{R} \), the Oseen vortex \( \omega = \alpha G \) is a stationary solution of (1.1). Linearizing the equation (1.1) at \( \alpha G \), we get a linear evolution equation

\[
\frac{\partial \omega}{\partial t} = -(\mathcal{L} + \alpha \Lambda) \omega,
\]
where

\[
\mathcal{L} \omega = -\Delta \omega - \frac{1}{2} x \cdot \nabla \omega - \omega, \quad \Lambda \omega = v^G \cdot \omega + (K_{BS} \ast \omega) \cdot \nabla G.
\]
It turns out that the operator \( \mathcal{L} \) is self-adjoint, non-negative on the weighted space \( L^2(\mathbb{R}^2; G^{-1} \, dx) \) and \( \Lambda \) is a relatively compact perturbation of \( \mathcal{L} \), which is the sum of two skew-adjoint operators on \( L^2(\mathbb{R}^2; G^{-1} \, dx) \). The spectrum of \( \mathcal{L} + \alpha \Lambda \) is a
sequence of eigenvalues by classical perturbation theory ([14]). Introducing the
following subspaces of \( Y = L^2(\mathbb{R}^2; G^{-1}dx) \):

\[
Y_0 = \{ \omega \in Y; \int_{\mathbb{R}^2} \omega(x)dx = 0 \} = \{ G \}^\perp,
\]
\[
Y_1 = \{ \omega \in Y_0; \int_{\mathbb{R}^2} x_j \omega(x)dx = 0 \text{ for } j = 1, 2 \} = \{ G; \partial_1 G; \partial_2 G \}^\perp,
\]
\[
Y_2 = \{ \omega \in Y_1; \int_{\mathbb{R}^2} |x|^2 \omega(x)dx = 0 \} = \{ G; \partial_1 G; \partial_2 G; \Delta G \}^\perp,
\]

which are invariant spaces for \( \mathcal{L} \) and \( \Lambda \), the following spectral bounds for \( \mathcal{L} + \alpha \Lambda \) are proved in [10],

\[
\text{Spec}(\mathcal{L} + \alpha \Lambda) \subset \{ z \in \mathbb{C}; \ \text{Re}(z) \geq 0 \} \quad \text{in } Y,
\]
\[
\text{Spec}(\mathcal{L} + \alpha \Lambda) \subset \{ z \in \mathbb{C}; \ \text{Re}(z) \geq \frac{1}{2} \} \quad \text{in } Y_0,
\]
\[
\text{Spec}(\mathcal{L} + \alpha \Lambda) \subset \{ z \in \mathbb{C}; \ \text{Re}(z) \geq 1 \} \quad \text{in } Y_1,
\]
\[
\text{Spec}(\mathcal{L} + \alpha \Lambda) \subset \{ z \in \mathbb{C}; \ \text{Re}(z) > 1 \} \quad \text{in } Y_2, \text{ if } \alpha \neq 0.
\]

These spectral bounds allow us to obtain estimates on the semigroup associated to \( \mathcal{L} + \alpha \Lambda \), which can be used to show that Oseen vortex \( \alpha G \) is a stable stationary solution of (1.7) for any \( \alpha \in \mathbb{R} \). However, these bounds are not precise. The eigenvalues that do not move are those which correspond to eigenvectors in the kernel of \( \Lambda \). All eigenvalues of \( \mathcal{L} + \alpha \Lambda \) which correspond to eigenvectors in the orthogonal complement of \( \text{ker}(\Lambda) \), have a real part that goes to \(+\infty\) as \(|\alpha| \to \infty\), observed numerically by A. Prochazka and D. Pullin [18] and recently proved by Y. Maekawa [16].

In this paper, we are interested in pseudospectral properties of this linearized operator. We conjugate the linear operators \( \mathcal{L} \) and \( \Lambda \) with \( G^{1/2} \), then we obtain two operators on \( L^2(\mathbb{R}^2; dx) \)

\[
L \omega = G^{-1/2} \mathcal{L} G^{1/2} \omega = -\Delta \omega + \frac{|x|^2}{16} \omega - \frac{1}{2} \omega,
\]
\[
M \omega = G^{-1/2} \Lambda G^{1/2} \omega = v^G \cdot \nabla \omega - \frac{1}{2} G^{1/2} x \cdot (K_{BS} * (G^{1/2} \omega)).
\]

Up to some numerical constants, \( L \) is the two-dimensional harmonic oscillator, which is self-adjoint and non-negative on \( L^2(\mathbb{R}^2; dx) \). On the other hand, both terms in \( M \) are separately skew-adjoint on \( L^2(\mathbb{R}^2; dx) \). Letting

\[
\mathcal{H}_\alpha \omega = L \omega + \alpha M \omega, \quad \omega \in L^2(\mathbb{R}^2; dx)
\]
\[
= \left( -\Delta \omega + \frac{|x|^2}{16} \omega - \frac{1}{2} \omega \right) + \alpha \left[ v^G \cdot \nabla \omega - \frac{1}{2} G^{1/2} x \cdot (K_{BS} * (G^{1/2} \omega)) \right],
\]

our aim is to give estimates for the resolvent of the non-self-adjoint operator \( \mathcal{H}_\alpha \) along the imaginary axis, in the fast rotating limit \( \alpha \to +\infty \).

1.2. About non-self-adjoint operators. In many problems originated from mathematical physics, one encounters a linear evolution equation with a non-self-adjoint generator, of the form \( H = A + iB \), where \( A \) is self-adjoint, non-negative and \( iB \) is skew-adjoint such that \( A, B \) do not commute. \( A \) is usually called the dissipative term and \( iB \) the conservative term. The conservative term can affect and sometimes
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enhance the dissipative effects or the regularizing properties of the whole system. When a large skew-adjoint term $iB$ is present, the spectrum and the pseudospectrum of the whole operator $H$ may be strongly stabilized. In particular, the norm of the resolvent $\| (H - z)^{-1} \|$ may tend to 0 quickly.

In the paper [7], a one-dimensional analogue of $H_\alpha$ is studied by I. Gallagher, T. Gallay and F. Nier

$$H_\epsilon = -\partial_x^2 + x^2 + \frac{i}{\epsilon} f(x), \quad x \in \mathbb{R},$$

where $\epsilon > 0$ is a small parameter, $f: \mathbb{R} \to \mathbb{R}$ is a bounded smooth function. Here the limit $\epsilon \to 0$ corresponds to the fast rotating limit $\alpha \to +\infty$. They studied the asymptotics of two quantities related to the spectral and pseudospectral properties in the limit $\epsilon \to 0$. More precisely, they define $\Sigma(\epsilon)$ as the infimum of the real part of the spectrum of $H_\epsilon$ and

$$\Psi(\epsilon)^{-1} = \sup_{\lambda \in \mathbb{R}} \| (H_\epsilon - i\lambda)^{-1} \|$$
as the supremum of the norm of the resolvent of $H_\epsilon$ along the imaginary axis. Under some appropriate conditions on $f$, both quantities $\Sigma(\epsilon)$, $\Psi(\epsilon)$ tend to infinity as $\epsilon \to 0$ and lower bounds are given by using the so-called hypocoercive method. Furthermore, they focused on Morse functions of $C^3(\mathbb{R}; \mathbb{R})$ which are bounded together with their derivatives up to the third order, and which behave like $|x|^{-k}$ as $|x| \to \infty$ (Hypothesis 1.6 in [7]). For functions verifying these hypotheses, some precise and optimal estimates on $\Psi(\epsilon)$ are proved (Theorem 1.8 in [7]): there exists $M \geq 1$ such that for any $\epsilon \in (0, 1]$, \[
\frac{1}{M\epsilon^\nu} \leq \Psi(\epsilon) \leq \frac{M}{\epsilon^\nu}, \quad \text{with } \nu = \frac{2}{k+4}.
\]

Their proof is based on the localization techniques and some semiclassical subelliptic estimates.

In our recent work [5], a two-dimensional non-self-adjoint operator is considered

$$\mathcal{L}_\alpha = -\Delta + |x|^2 + \alpha \sigma(|x|) \partial_\theta, \quad x \in \mathbb{R}^2,$$

where $\sigma(r) = r^{-2}(1 - e^{-r^2})$, $\partial_\theta = x_1 \partial_2 - x_2 \partial_1$ and $\alpha$ is a positive parameter tending to infinity. Note that up to some numerical constants, the differential operator $\mathcal{L}_\alpha$ is equal to the operator $H_\alpha$ given in (1.11), by neglecting the second member in the skew-adjoint part $\alpha M$, which is a non-local, lower-order term. In that paper, we gave a complete study of the resolvent of $\mathcal{L}_\alpha$ along the imaginary axis in the limit $\alpha \to +\infty$ and proved an estimate of type (Theorem 2.2 in [5])

$$\sup_{\lambda \in \mathbb{R}} \| (\mathcal{L}_\alpha - i\lambda)^{-1} \|_{\mathcal{L}(L(\mathbb{R}^2))} \leq C\alpha^{-1/3},$$

which is optimal. The result is established by using a multiplier method, metrics on the phase space and localization techniques.

The present paper is devoted to proving resolvent estimates similar to (1.14) for the whole linearized operator $\mathcal{H}_\alpha$ in (1.11).

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invitation of the summer school “Spectral analysis of non-selfadjoint operators and applications”, held in University Rennes I, June 2011, where the notes [8] were taken.

2. Statement of the result

2.1. The theorem. Using the notations in Section 1.1, we consider the operator on $L^2(\mathbb{R}^2; dx)$

$$\mathcal{H}_\alpha \omega = -\Delta \omega + \frac{|x|^2}{16} \omega - \frac{1}{2} \omega$$

where $G_v$ is given by [1.5], $K_{BS}$ is given in (1.2) and $\alpha \geq 1$ is a large parameter. The real part of $\mathcal{H}_\alpha$ is the two-dimensional harmonic oscillator and the imaginary part of $\mathcal{H}_\alpha$ is the sum of a divergence-free vector field and a non-local integral operator, multiplied by the circulation Reynolds number $\alpha$.

The skew-adjoint part of $\mathcal{H}_\alpha$ vanishes on radial functions and in particular the function $e^{-|x|^2/8}$ is an eigenfunction of $\mathcal{H}_\alpha$ corresponding to the eigenvalue 0, for any $\alpha \in \mathbb{R}$, which implies that the ground state of the two-dimensional harmonic-oscillator does not move under the large skew-adjoint perturbation. Moreover, one can also check that the skew-adjoint part of $\mathcal{H}_\alpha$ vanishes on the functions $x_1 e^{-|x|^2/8}$, $x_2 e^{-|x|^2/8}$. Thus we shall work in some subspaces of $L^2(\mathbb{R}^2; dx)$, defined below.

Using polar coordinates in $\mathbb{R}^2$, for $k_0 \geq 1$, we define the subspace of $L^2(\mathbb{R}^2; dx)$

$$X_{k_0} = \left\{ \omega \in L^2(\mathbb{R}^2; dx); \omega(r \cos \theta, r \sin \theta) = \sum_{|k| \geq k_0} \omega_k(r)e^{ik\theta} \right\},$$

which is a Hilbert space equipped with the norm $\| \cdot \|_{L^2(\mathbb{R}^2)}$ and which is an invariant space for $\mathcal{H}_\alpha$.

Definition 2.1 (Domain of $\mathcal{H}_\alpha$). Let $D = \{ \omega \in L^2(\mathbb{R}^2); \omega \in H^2(\mathbb{R}^2), |x|^2 \omega \in L^2(\mathbb{R}^2) \}.$

Then $(\mathcal{H}_\alpha, D)$ is a closed operator on $L^2(\mathbb{R}^2)$. Moreover, for any $k_0 \geq 1$, $\mathcal{H}_\alpha$ is a closed operator on $X_{k_0}$ with dense domain $D \cap X_{k_0}$ and its numerical range defined by

$$\Theta(\mathcal{H}_\alpha; X_{k_0}) = \{ (\mathcal{H}_\alpha \omega, \omega)_{L^2(\mathbb{R}^2)} \in \mathbb{C}; \omega \in D \cap X_{k_0}, \| \omega \|_{L^2(\mathbb{R}^2)} = 1 \}$$

is included in the set $\{ z \in \mathbb{C}; \text{Re} z \geq k_0/2 \}$, so that its spectrum is also contained in $\{ z \in \mathbb{C}; \text{Re} z \geq k_0/2 \}$.

Now let us state our main result.

Theorem 2.2. There exist constants $C > 0$, $k_0 \geq 3$, $\alpha_0 \geq 8\pi$ such that for all $\alpha \geq \alpha_0$, $\lambda \in \mathbb{R}$, for all $\omega \in C_0^\infty(\mathbb{R}^2) \cap X_{k_0}$, we have

$$\| (\mathcal{H}_\alpha - i\lambda) \omega \|_{L^2(\mathbb{R}^2)} \geq C \alpha^{1/3} \| D\theta \|_{L^2(\mathbb{R}^2)},$$

where $|D\theta|^{1/3} \omega = \sum_k |k|^{1/3} \omega_k(r)e^{ik\theta},$ for $\omega = \sum_k \omega_k(r)e^{ik\theta}$. In particular, we have

$$\| (\mathcal{H}_\alpha - i\lambda)^{-1} \|_{L(X_{k_0})} \leq C^{-1} \alpha^{-1/3} k_0^{-1/3}.$$
The resolvent estimate \((2.4)\) gives information about the pseudospectrum of the family of operators \(\{\mathcal{H}_\alpha\}_{\alpha \geq 1}\).

**Definition 2.3.** For the operators \(\{\mathcal{H}_\alpha\}_{\alpha \geq 1}\) on \(X_{k_0}\), we define the pseudospectrum of \(\{\mathcal{H}_\alpha\}_{\alpha \geq 1}\) as the complement of the set of \(z \in \mathbb{C}\) such that
\[
\exists N_0 \in \mathbb{N}, \quad \sup_{\alpha \geq 1} \| (\mathcal{H}_\alpha - z)^{-1} \|_{\mathcal{L}(X_{k_0})} \alpha^{-N_0} < +\infty.
\]

**Corollary 2.4.** The pseudospectrum of \(\{\mathcal{H}_\alpha\}_{\alpha \geq 1}\) is included in the set
\[
\{ z \in \mathbb{C}; \quad \text{Re} z \geq C\alpha^{1/3}k_0^{1/3} \}.
\]

Indeed, if \(\text{Re} z \leq 0\), then for \(\omega \in D \cap X_{k_0}\) with \(\| \omega \|_{L^2(\mathbb{R}^2)} = 1\),
\[
\| (\mathcal{H}_\alpha - z)^{-1} \|_{\mathcal{L}(X_{k_0})} \leq 2k_0^{-1},
\]
implicating \(\| (\mathcal{H}_\alpha - z)^{-1} \|_{\mathcal{L}(X_{k_0})} < +\infty\).

Let \(\kappa \in (0, 1)\). For \(z = \mu + i\lambda\) with \(0 < \mu \leq \kappa C\alpha^{1/3}k_0^{1/3}\) and \(\lambda \in \mathbb{R}\), we infer from the resolvent formula
\[
(\mathcal{H}_\alpha - \mu - i\lambda)^{-1} - (\mathcal{H}_\alpha - i\lambda)^{-1} = \mu(\mathcal{H}_\alpha - i\lambda)^{-1}(\mathcal{H}_\alpha - \mu - i\lambda)^{-1}
\]
and the resolvent estimate \((2.4)\) that
\[
\| (\mathcal{H}_\alpha - \mu - i\lambda)^{-1} \|_{\mathcal{L}(X_{k_0})} \leq \frac{\| (\mathcal{H}_\alpha - i\lambda)^{-1} \|_{\mathcal{L}(X_{k_0})}}{1 - \mu \| (\mathcal{H}_\alpha - i\lambda)^{-1} \|_{\mathcal{L}(X_{k_0})}} \leq \frac{C^{-1}\alpha^{-1/3}k_0^{-1/3}}{1 - \kappa}.
\]
As a result, the set \(\{ z \in \mathbb{C}; \quad \text{Re} z < C\alpha^{1/3}k_0^{1/3} \}\) is included in the complement of the pseudospectrum of \(\{\mathcal{H}_\alpha\}_{\alpha \geq 1}\), so that the corollary is proved.

2.2. **Comments.**

2.2.1. **The nonlocal term.** The term
\[
\frac{\alpha}{2} G^{1/2} x \cdot (K_{BS} * (G^{1/2} \omega))
\]
is an integral operator which is non-local and skew-adjoint. This term should be carefully treated as it has a large coefficient \(\alpha\).

2.2.2. **A weight.** We shall reduce the two-dimensional operator \(\mathcal{H}_\alpha - i\lambda\) to a family of one-dimensional operators acting on the positive-half real line \(\mathbb{R}_+\) by using polar coordinates and expanding the angular variable \(\theta\) in Fourier series, indexed by the Fourier mode parameter \(k \in \mathbb{Z}\). Then we transform the problem onto the whole real line \(\mathbb{R}\) by making a change of variable \(r = e^t\) and multiplying by a weight \(e^{2t}\). After these transformations, the properties of self-adjointness and skew-adjointness are preserved (see Section 3.1), and the non-local term turns out to be a skew-adjoint pseudodifferential operator with \(\mathcal{L}(L^2(\mathbb{R}; dt))\)-norm bounded above by \(\alpha|k|^{-1}\). The discussion is divided into different cases according to a change-of-sign situation.

2.2.3. **Multiplier method.** The proof relies on a classical multiplier method. For the non-trivial cases where the change-of-sign takes place (see Section 3.3), we shall construct a multiplier bounded on \(L^2(\mathbb{R}; dt)\), which is a pseudodifferential operator associated to a Hörmander-type metric. The non-local term will be treated as a perturbation and will be absorbed by the main term letting \(|k| \geq k_0\), with \(k_0\) a constant independent of the circulation parameter \(\alpha\).
2.2.4. The value of $k_0$. Given $\epsilon_0, \epsilon_1 \in (0, 1)$, we shall discuss 4 different cases given in (3.21). Then $k_0$ can be expressed as a function of $(\epsilon_0, \epsilon_1)$. For example, if we take $\epsilon_0 \approx 0.462, \epsilon_1 \approx 0.426$, then Theorem 2.2 holds with $k_0 = 84$, see [4]. (In fact, we can obtain $k_0 = 51$ if we do some improvements.)

3. The proof

3.1. First reductions. The operator $H_\alpha$ in (2.1) is invariant under rotations with respect to the origin in $\mathbb{R}^2$. We can reduce the problem to a family of one-dimensional operators by using polar coordinates and expanding the angular variable $\theta$ in Fourier series.

3.1.1. Polar coordinates. We can write for $\omega \in L^2(\mathbb{R}^2)$ and $v = K_{BS} \ast \omega$ given by (1.2) as

$$\omega(r \cos \theta, r \sin \theta) = \sum_{k \in \mathbb{Z}} \omega_k(r)e^{ik\theta},$$

$$v(r \cos \theta, r \sin \theta) = \sum_{k \in \mathbb{Z}} \left( \frac{u_k(r)}{r}e_r + \frac{w_k(r)}{r}e_\theta \right)e^{ik\theta},$$

where $e_r = (\cos \theta, \sin \theta)$ and $e_\theta = (-\sin \theta, \cos \theta)$. The relations $\partial_1 v_1 + \partial_2 v_2 = 0, \partial_1 v_2 - \partial_2 v_1 = \omega$ become

$$u'_k + \frac{ik}{r}w_k = 0, \quad w'_k - \frac{ik}{r}u_k = r\omega_k,$$

so that $-\Delta_k u_k = ik\omega_k$, where

$$-\Delta_k = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2}.$$ 

If $k \neq 0$, the Poisson equation $-\Delta_k \Omega = f$ has the explicit solution $\Omega = K_k[f]$, where

$$K_k[f](r) = \frac{1}{2|k|} \int_0^{\infty} \left( \frac{r}{s} \right)^{|k|} H(r)H(s - r) + \left( \frac{r}{s} \right)^{|k|} H(s)H(r - s) \right)f(s)sds,$$

where $H(r)$ is the Heaviside function. We thus have

$$u_k = ikK_k[\omega_k] \quad \text{and} \quad w_k = -rK_k[\omega_k]'$$

if $k \neq 0$. For $k = 0$, we find $u_0 = 0$ and $w_0' = r\omega_0$, hence $w_0(r) = \int_0^r s\omega(s)ds$.

By using the following notations:

$$\sigma(r) = \frac{1 - e^{-r^2/4}}{r^2/4}, \quad g(r) = e^{-r^2/8}, \quad \text{for } r > 0,$$

and observing that $v^G = \frac{1}{8\pi}r\sigma(r)e_\theta$, we rewrite the skew-adjoint part of $H_\alpha$ in polar coordinates as

$$\alpha v^G \cdot \nabla \omega = \sum_{k \neq 0} \frac{i\alpha k}{8\pi} \sigma(r)\omega_k(r)e^{ik\theta},$$

$$\frac{\alpha}{2}(G^{1/2} \cdot (K_{BS} \ast (G^{1/2} \omega))) = \sum_{k \neq 0} \frac{i\alpha k}{8\pi} g(r)K_k[g\omega_k](r)e^{ik\theta}.$$ 

Thus we find that for $H_\alpha$ given by (2.1), $\lambda \in \mathbb{R}$ and for $\omega = \sum_{k \in \mathbb{Z}} \omega_k(r)e^{ik\theta}$,

$$((H_\alpha - i\lambda)\omega)(r \cos \theta, r \sin \theta) = \sum_{k \in \mathbb{Z}} (H_{\alpha,k}\lambda\omega_k)(r)e^{ik\theta},$$

(3.3)
where $\mathcal{H}_{\alpha,k,\lambda}$ acts on $L^2(\mathbb{R}_+; r dr)$ and is given by

$$
\mathcal{H}_{\alpha,k,\lambda} v = -\partial_t^2 v - \frac{1}{r} \partial_r v + \frac{k^2}{r^2} v + \frac{\nu^2}{16} v - \frac{1}{2} v + \frac{i k \alpha}{8\pi} \left( \sigma(r) v - g K_k [g v] \right) - i \lambda v.
$$

Introducing two new notations

$$
\beta_k = \frac{\alpha k}{8\pi}, \quad \lambda = \beta_k \nu_k, \quad \nu_k \in \mathbb{R},
$$

we are led to study the resolvent of the one-dimensional operator $\mathcal{H}_{\alpha,k,\lambda}$ on $L^2(\mathbb{R}_+; r dr)$ for $|\beta_k| \to +\infty$, where (we omit the indices $\alpha, \lambda$ in $\mathcal{H}_{\alpha,k,\lambda}$)

$$
\mathcal{H}_k v = -\partial_t^2 v - \frac{1}{r} \partial_r v + \frac{k^2}{r^2} v + \frac{\nu^2}{16} v - \frac{1}{2} v
$$

self-adjoint and non-negative on $L^2(\mathbb{R}_+; r dr)$

$$
+i \beta_k (\sigma(r) - \nu_k) v - i \beta_k g K_k [g v].
$$

skew-adjoint on $L^2(\mathbb{R}_+; r dr)$

Note that the non-local term is transformed to $i \beta_k g K_k g$ with $K_k$ given by (3.1).

Moreover, $C_0^\infty((0, +\infty))$ is a core for the closed operator $\mathcal{H}_k$ with domain

$$
D(\mathcal{H}_k) = \{ v \in L^2(\mathbb{R}_+; r dr); \partial_t^2 v, \frac{1}{r} \partial_r v, \frac{1}{r^2} v, r^2 v \in L^2(\mathbb{R}_+; r dr) \}, \quad \text{if } |k| \geq 2,
$$

and

$$
D(\mathcal{H}_k) = \{ v \in L^2(\mathbb{R}_+; r dr); \partial_t^2 v, \partial_r \left( \frac{v}{r} \right), r^2 v \in L^2(\mathbb{R}_+; r dr) \}, \quad \text{if } |k| = 1.
$$

3.1.2. Change of variables. We wish to transform the operator $\mathcal{H}_k$ in (3.6) acting on the positive half-line into an operator acting on the whole real line, by making the change of variables $r = e^t$. A simple but key observation is

**Lemma 3.1.** For $v \in L^2(\mathbb{R}_+; r dr)$, define $u(t) = v(e^t)$. Then

$$
||e^t u||_{L^2_1(\mathbb{R}; dt)} = \int_\mathbb{R} |e^t u(t)|^2 dt = \int_0^{+\infty} |v(r)|^2 r dr = ||v||_{L^2(\mathbb{R}_+; r dr)}.
$$

Moreover, for $v \in C_0^\infty((0, +\infty))$, multiplying $(\mathcal{H}_k v)(e^t)$ by the weight $e^{2t}$, we have

$$
e^{2t} (\mathcal{H}_k v)(e^t) = (\mathcal{L}_k u)(t), \quad \text{for } u(t) = v(e^t),$$

where

$$
\mathcal{L}_k = -\partial_t^2 + k^2 + \frac{1}{16} e^{4t} - \frac{1}{2} e^{2t}
$$

$$
+i \beta_k e^{2t} (\sigma(e^t) - \nu_k) - i \beta_k e^{2t} g(e^t)(k^2 + D_t^2)^{-1} e^{2t} g(e^t).
$$

**Proof.** Indeed, we have

$$
r^2 (\partial_t^2 + r^{-1} \partial_r) = (r \partial_t)^2 = \partial_t^2 \quad \text{for } r = e^t.
$$
On the other hand, by the definition (3.1) of $K_k$, we have for $v \in C_0^\infty((0, +\infty))$,
\[
e^{2t}(gK_k[gv])(e^t) = e^{2t}g(e^t)K_k[gv](e^t) = e^{2t}g(e^t)h(s - e^t)
= \frac{1}{2|k|} \int_0^{\infty} e^{2t}g(e^t) \left[ \left( \frac{e^t}{s} \right)^{|k|} H(e^t)H(s - e^t) + \left( \frac{s}{e^t} \right)^{|k|} H(s)H(e^t - s) \right] g(s)v(s)sds
= \frac{1}{2|k|} \int_0^{\infty} e^{2t}g(e^t) \left[ \left( \frac{e^t}{s} \right)^{|k|} H(e^s - e^t) + \left( \frac{s}{e^t} \right)^{|k|} H(e^t - e^s) \right] g(e^s)v(e^s)e^{2s}ds
= \frac{1}{2|k|} \int_0^{\infty} e^{2t}g(e^t)e^{-|k|(t-s)}H(s) + e^{-|k|(t-s)}H(t - s) \right] e^{2s}g(e^s)v(e^s)ds
= \frac{1}{2|k|} \int_0^{\infty} e^{2t}g(e^t)e^{-|k|(t-s)}e^{2s}g(e^s)u(s)ds.
\]
For $k \neq 0$, we have
\[
\frac{1}{2|k|} \int_0^{\infty} e^{-|k||\tau|}e^{i\tau}dt = \frac{1}{k^2 + \tau^2},
\]
(see Lemma 4.4) so that the non-local term $gK_kg$ becomes
\[
e^{2t}(gK_k[gv])(e^t) = (e^{2t}g(e^t)(k^2 + D^2)^{-1}e^{2t}g(e^t)u)(t), \quad \text{for } u(t) = v(e^t),
\]
which is a self-adjoint, positive (non-local) pseudodifferential operator on $L^2(\mathbb{R}; dt)$. The proof of the lemma is complete.

When $\tilde{H}_k$ given by (3.9) is viewed as an operator on $L^2(\mathbb{R}; dt)$, we see that
\[
\tilde{H}_k = -\partial_t^2 + k^2 + \frac{1}{16}e^t - \frac{1}{2}e^{2t} + i\beta_ke^{2t}(\sigma(e^t) - \nu_k) - i\beta_ke^{2t}g(e^t)(k^2 + D^2)^{-1}e^{2t}g(e^t).
\]
After the change of variables $r = e^t$ and the multiplication by the weight $e^{2t}$, the self-adjoint (resp. skew-adjoint) part of $H_k$ in (3.6) does not lose its self-adjointness (resp. skew-adjointness), and in particular, the non-local term $i\beta_ke^{2t}g\tilde{K}_k$ stays skew-adjoint. Moreover, the power 2 in the weight is the only power to keep these properties unchanged.

In view of (3.7) and (3.8) in Lemma 3.1, the problem is reduced to prove estimates for the operator $\tilde{H}_k$ in (3.9) of type
\[
\|e^{-t}\tilde{H}_k u\|_{L^2(\mathbb{R}; dt)} \geq C|\beta_k|^a\|e^t u\|_{L^2(\mathbb{R}; dt)}
\]
for some $a > 0$, which correspond to the estimates for the operator $H_k$ given in (3.6)
\[
\|H_k v\|_{L^2(\mathbb{R}^+; rdr)} \geq C|\beta_k|^a\|v\|_{L^2(\mathbb{R}^+; rdr)},
\]
where $u(t) = v(e^t)$, since we have exactly
\[
\|e^{-t}\tilde{H}_k u\|_{L^2(\mathbb{R}; dt)} = \|H_k v\|_{L^2(\mathbb{R}^+; rdr)}; \quad \|e^t u\|_{L^2(\mathbb{R}; dt)} = \|v\|_{L^2(\mathbb{R}^+; rdr)}.
\]
Furthermore, we need only to prove estimates (3.10) for $u \in C_0^\infty(\mathbb{R})$, since it is enough to get (3.11) for $u \in C_0^\infty((0, +\infty))$.

As in [5], we divide our discussion into different cases, according to the change-of-sign situation of $\sigma(e^t) - \nu_k$, where the function $\sigma$ is given in (3.2). Note that $\sigma(e^t)$ is a decreasing function of the variable $t$ and has range $(0, 1)$. When $\sigma(e^t) - \nu_k$ does
not change sign, it is easy to deal with by using the multipliers \( \text{Id}, \pm i\text{Id} \) (see Section \( \ref{section:multipliers} \)). If \( \sigma(e^t) - \nu_k \) changes sign at one point, it is more complicated (see Section \( \ref{section:complicated} \)). In this case, we will construct a multiplier well-adapted to this change-of-sign situation, which is a pseudodifferential operator depending on a Hörmander metric on the phase space. Compared with the method in \( \ref{section:method} \), the multiplier that we shall construct is a global one, because of the existence of the non-local term, which possesses a large coefficient and would produce a commutator of size \( |\beta_k| \) if we just used a partition of unity on \( \mathbb{R}_t \) as done in \( \ref{section:method} \).

3.1.3. Notations. In Section \( \ref{section:notations} \), \( \ref{section:notations2} \) and \( \ref{section:notations3} \), we shall always assume that \( k \geq 1 \) hence \( \beta_k > 0 \), and we denote by \( \| \cdot \|, \langle \cdot, \cdot \rangle \) the \( L^2(\mathbb{R}; dt) \)-norm, inner-product respectively. We shall also be able to neglect the term \( -\frac{1}{2} e^{2t} \) in the real part of \( \widehat{\mathcal{L}}_k \) and by introducing two notations,

\[
\langle D_k \rangle^{-2} = (D_t^2 + k^2)^{-1}, \quad \gamma(t) = e^{2t}g(e^t) = e^{2t}e^{-e^{2t}/8},
\]

we shall study

\[
\mathcal{L}_k = D_t^2 + k^2 + \frac{1}{16} e^{4t} + i\beta_k e^{2t}(\sigma(e^t) - \nu_k) - i\beta_k \gamma(t) \langle D_k \rangle^{-2} \gamma(t).
\]

In fact, as soon as we prove \((\ref{section:inequalities})\) for \( \mathcal{L}_k \) in \((\ref{section:multipliers})\) with \( a > 0 \), we have for the operator \( \widehat{\mathcal{L}}_k \) given in \((\ref{section:transformed})\)

\[
\|e^{-t} \widehat{\mathcal{L}}_k u\|_{L^2(\mathbb{R}; dt)} = \|e^{-t}(\mathcal{L}_k - \frac{1}{2} e^{2t}) u\|_{L^2(\mathbb{R}; dt)} \geq C\beta_k \|e^t \|_{L^2(\mathbb{R}; dt)} - \frac{1}{2}\|e^t \|_{L^2(\mathbb{R}; dt)},
\]

so that it suffices to let \( \alpha \) large enough since \( k \geq 1 \), \( \beta_k \geq \alpha/8\pi \).

We present in Appendix \( \ref{section:appendix} \) some inequalities concerning the functions \( \sigma \) and \( g \) given in \((\ref{section:inequalities})\) that will be used in the proof. We have for all \( u \in C_0^\infty(\mathbb{R}) \),

\[
\text{Re} \langle \mathcal{L}_k u, u \rangle_{L^2(\mathbb{R}; dt)} = \langle (D_t^2 + k^2 + \frac{1}{16} e^{4t}) u, u \rangle_{L^2(\mathbb{R}; dt)},
\]

\[
0 \leq \langle \gamma \langle D_k \rangle^{-2} \gamma u, u \rangle_{L^2(\mathbb{R}; dt)} \leq k^{-2}\|\gamma u\|_{L^2(\mathbb{R}; dt)}^2, \quad \text{by Lemma } \ref{section:lemma},
\]

where \( \mathcal{L}_k \) is given in \((\ref{section:multipliers})\) and \( \gamma, \langle D_k \rangle^{-2} \) are given in \((\ref{section:transformed})\).

3.2. Easy cases. In this section, we study the cases where \( \sigma(e^t) - \nu_k \) does not change sign, that is \( \nu_k \geq 1 \) or \( \nu_k \leq 0 \).

Lemma 3.2. Suppose \( \nu_k \geq 1 \). There exists \( C > 0 \) such that for all \( k \geq 1, \alpha \geq 8\pi \) and for \( u \in C_0^\infty(\mathbb{R}) \),

\[
\|e^{-t} \mathcal{L}_k u\| \geq C\beta_k^{1/2}\|e^t u\|,
\]

where \( \mathcal{L}_k \) is given in \((\ref{section:multipliers})\) and \( \beta_k \) is given in \((\ref{section:beta})\).

Proof. If \( \nu_k \geq 1 \), then \( \sigma(e^t) - \nu_k \) is non-positive. Using the multiplier \(-i\text{Id} \) and by \((\ref{section:inequalities})\), we have

\[
\text{Re} \langle \mathcal{L}_k u, -iu \rangle = \beta_k \langle e^{2t}(\nu_k - \sigma(e^t)) u, u \rangle + \beta_k \langle \gamma \langle D_k \rangle^{-2} \gamma u, u \rangle \geq \beta_k \langle e^{2t}(1 - \sigma(e^t)) u, u \rangle.
\]
Adding (3.14), (3.17) together, we obtain
\[
\text{Re}(\mathcal{L}_ku, (1-i)u) \geq \left( k^2 e^{-2t} + \beta_k (1 - \sigma(e^t)) \right) e^2tu, u).
\]
Then using the second inequality in (4.17), we get
\[
\text{Re}(e^{-t}\mathcal{L}_ku, e^t(1-i)u) \geq C\beta_k^{1/2} \langle e^{2tu}, u \rangle = C\beta_k^{1/2} \|e^u\|^2.
\]
By Cauchy-Schwarz inequality, the estimate (3.16) is proved.

\[\square\]

**Lemma 3.3.** Suppose \(\nu_k \leq 0\). There exists \(C > 0\) such that for all \(k \geq 2\), \(\alpha \geq 8\pi\) and for \(u \in C_0^\infty(\mathbb{R})\),
\[(3.18) \quad \|e^{-t}\mathcal{L}_ku\| \geq C\beta_k^{1/2}\|e^u\|,
\]
where \(\mathcal{L}_k\) is given in (3.13) and \(\beta_k\) in (3.5).

**Proof.** If \(\nu_k \leq 0\), then \(\sigma(e^t) - \nu_k\) is non-negative. Using the multiplier \(i\text{Id}\) and by (3.15), we have
\[
\text{Re}(\mathcal{L}_ku, iu) = \beta_k \langle e^{2t}(\sigma(e^t) - \nu_k)u, u \rangle - \beta_k \langle \gamma(D_k)^{-2} \gamma u, u \rangle \\
\geq \beta_k \left( e^{2t}\sigma(e^t)u, u \right) - \beta_k \|e^{2t}g(e^t)u\|^2.
\]
Using (4.12), we get for \(k \geq 2\),
\[(3.19) \quad \text{Re}(\mathcal{L}_ku, iu) \geq (1 - (4\delta)^{-1}) \beta_k \langle e^{2t}\sigma(e^t)u, u \rangle,
\]
with \(1 - (4\delta)^{-1} > 0\). Adding (3.14), (3.19) together we obtain
\[
\text{Re}(\mathcal{L}_ku, (1+i)u) \geq \left( \frac{1}{16} e^{2t} + (1 - (4\delta)^{-1}) \beta_k \sigma(e^t) \right) e^{2tu}, u), \quad k \geq 2.
\]
Using the first inequality in (4.17), we get
\[
\text{Re}(e^{-t}\mathcal{L}_ku, e^t(1+i)u) \geq C\beta_k^{1/2}\langle e^{2tu}, u \rangle = C\beta_k^{1/2}\|e^u\|^2, \quad k \geq 2.
\]
By Cauchy-Schwarz inequality, the estimate (3.18) is proved. \[\square\]

**Remark 3.4.** When \(k = 1\) and \(\nu_k = 0\), the imaginary part of \(\mathcal{H}_1\) vanishes on the function \(v(r) = rg(r) \in D(\mathcal{H}_1)\), i.e. we have \(gK_1|gv| = \sigma v\). Consequently, when \(\nu_k = 0\), the imaginary part of \(\mathcal{L}_1\) vanishes on the function \(u(t) = e^tu\).

### 3.3 Nontrivial cases

We turn to study the cases where the change-of-sign of \(\sigma(e^t) - \nu_k\) takes place, that is \(\nu_k \in (0, 1)\). We have thus \(\nu_k = \sigma(e^{t_k})\) for some \(t_k \in \mathbb{R}\). Then the operator \(\mathcal{L}_k\) can be written as
\[(3.20) \quad \mathcal{L}_k = D_t^2 + k^2 + \frac{1}{16} e^{2t} + i\beta_k e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) - i\beta_k \gamma(t)\langle D_k \rangle^{-2} \gamma(t).
\]
Suppose \(\epsilon_0, \epsilon_1 \in (0, 1)\). We discuss four cases according to the behavior of the function \(\sigma\) near the point \(e^{t_k}\):
\[(3.21) \quad e^{t_k} > \epsilon_0^{-1} \quad \text{or} \quad e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}] \quad \text{or} \quad e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1) \quad \text{or} \quad e^{t_k} \leq \beta_k^{-1/4}.
\]
Before going through the proofs for each case, let us first choose some functions that will be used to construct the multipliers. Suppose that \(c_0 \in (0, 1)\) is the constant
chosen in Proposition 4.7. Let $\chi_j \in C^\infty(\mathbb{R}; [0,1]), j = 0, 1, -1$, satisfying that

$$\begin{cases}
\text{supp}\chi_0 \subset [-c_0, c_0], \\
\chi_+ = 1 \text{ on } [c_0, +\infty), \quad \text{supp}\chi_+ \subset \left(\frac{c_0}{2}, +\infty\right), \\
\chi_- = 1 \text{ on } (-\infty, -c_0], \quad \text{supp}\chi_- \subset (-\infty, -\frac{c_0}{2}], \\
\chi_0(\theta)^2 + \chi_+(\theta)^2 + \chi_-(\theta)^2 = 1, \quad \forall \theta \in \mathbb{R}.
\end{cases}$$

(3.22)

See Figure 1. Choose a function $\tilde{\chi}_0 \in C^\infty_0(\mathbb{R}; [0,1])$ such that

$$\tilde{\chi}_0 = 1 \text{ on } [-2c_0, 2c_0], \quad \text{supp}\tilde{\chi}_0 \subset [-3c_0, 3c_0].$$

(3.23)

Take a decreasing function $\psi \in C^\infty(\mathbb{R}; [-1,1])$ such that

$$\begin{cases}
\psi = 1 \text{ on } (-\infty, -2], \quad \psi = -1 \text{ on } [2, +\infty), \quad \psi' = -\frac{1}{2} \text{ on } [-1,1].
\end{cases}$$

(3.24)

We can assume that $\psi$ has a factorization

$$\psi(\theta) = -e(\theta)\theta,$$

(3.25)

where $e \in C^\infty_0(\mathbb{R}; [0,1])$ satisfies\footnote{We denote by $C^\infty_0(\mathbb{R}; [0,1])$ the set of smooth functions defined on $\mathbb{R}$ with values in $[0,1]$ such that all their derivatives are bounded.} that

$$e(\theta) = \frac{1}{2} \text{ for } \theta \in [-1,1], \quad e(\theta) = |\theta|^{-1} \text{ for } |\theta| \geq 2.$$

3.3.1. Plan of the paragraph. The sections 3.3, 3.4 are organized as follows. Recall the four cases given in (3.21) and we give in Proposition 4.7 inequalities about the function $\sigma$ that will be used in the proof for the first three cases.

Section 3.3.2 is devoted to the proof for Case 1 where $e^{t_k} > \epsilon_0^{-1}$. We shall construct a multiplier adapted to the change-of-sign situation. Moreover, there is a special localization effect in this case (see Remark 3.13).

In Section 3.3.3 we prove estimates for Case 2 where $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$. The multiplier to be used in this case is the same as that in Case 1.
In Section 3.4.1, we prove estimates for Case 3 where \( e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1) \). The multiplier will be different from that in the previous cases and the condition \( e^{t_k} > \beta_k^{-1/4} \) is required such that the metric verifies the uncertainty principle.

Finally, Section 3.4.2 is devoted to proving estimates for the last case where \( e^{t_k} \leq \beta_k^{-1/4} \) and estimates are easily obtained by using the multipliers \( \text{Id}, -i\text{Id} \).

3.3.2. Case 1: \( e^{t_k} > \epsilon_0^{-1} \). We present in Proposition 4.7, (1) some inequalities about the function \( \sigma \) that will be used in this case.

**Theorem 3.5.** Suppose \( e^{t_k} > \epsilon_0^{-1} \). There exist \( C > 0, k_0 \geq 1 \) such that for all \( k \geq k_0, \alpha \geq 8\pi, u \in C_0^\infty(\mathbb{R}) \),

\[
\| e^{-t} \mathcal{L}_k u \| \geq C \beta_k^{1/3} \| e^t u \|,
\]

where \( \mathcal{L}_k \) is given in (3.20) and \( \beta_k \) is given in (3.5).

a. Definition of the multiplier. We first give the definition of the Hörmander-type metric that we shall work with (see Appendix 4.1).

**Definition 3.6.** Define a metric on the phase space \( \mathbb{R}_t \times \mathbb{R}_\tau \)

\[
\Gamma = |dt|^2 + \frac{|d\tau|^2}{\tau^2 + \beta_k^{2/3}},
\]

which is admissible with

\[
\lambda_\Gamma = (\tau^2 + \beta_k^{2/3})^{1/2} \geq \beta_k^{1/3} \geq (\frac{\alpha}{8\pi})^{1/3} \geq 1, \quad \text{provided } \alpha \geq 8\pi.
\]

**Remark 3.7.** We give a proof for the uniform admissibility (w.r.t. \( k \geq 1, \alpha \geq 8\pi \)) of the metric \( \Gamma \) in Lemma 4.1. Moreover, the function \( f(\beta_k^{-1/3} \tau) \) belongs to \( S(1, \Gamma) \) whenever \( f \in S(1, \frac{|dt|^2}{1+\beta^2}) \), since for any \( n \in \mathbb{N} \),

\[
| \frac{\partial^n}{\partial \tau^n} (f(\beta_k^{-1/3} \tau)) | = | f^{(n)}(\beta_k^{-1/3} \tau) \beta_k^{-n/3} |
\leq C_n (1 + |\beta_k^{-1/3} \tau|^2)^{-n/2} \beta_k^{-n/3} = C_n (\beta_k^{2/3} + \tau^2)^{-n/2}.
\]

Now we can construct the multiplier, using the functions that we have chosen in (3.22), (3.24).

**Definition 3.8.**

\[
M_k = m_{0,k}^w + m_{+,k}^w + m_{-,k}^w,
\]

where

\[
m_{0,k}(t, \tau) = \chi_0(t - t_k) \sharp \psi(\beta_k^{-1/3} \tau) \sharp \chi_0(t - t_k),
\]

\[
m_{+,k}(t, \tau) = -i\beta_k^{-1/3} \chi_+ (t - t_k)^2,
\]

\[
m_{-,k}(t, \tau) = i\beta_k^{-1/3} \chi_- (t - t_k)^2,
\]

where \( a^w \) stands for the Weyl quantization for the symbol \( a \) and \( \sharp \) denotes the composition law in Weyl calculus. (See Appendix 4.1 for Weyl calculus.)
Remark 3.9. The functions $\chi_0(t-t_k)$, $\chi_\pm(t-t_k)$, $\psi(\beta_k^{-1/3}r)$ are real-valued symbols in $S(1,\Gamma)$. Then $M_k$ given in Definition 3.8 is a bounded operator on $L^2(\mathbb{R}; dt)$. Moreover, we see that

\begin{equation}
(3.29) \quad m_{0,k}^w = \chi_0(t-t_k)\psi(\beta_k^{-1/3}D_t)\chi_0(t-t_k)
\end{equation}

and $M_k$ can be written as

\begin{equation}
M_k = \chi_0(t-t_k)\psi(\beta_k^{-1/3}D_t)\chi_0(t-t_k) - i\beta_k^{-1/3}\chi_+(t-t_k)^2 + i\beta_k^{-1/3}\chi_-(t-t_k)^2.
\end{equation}

Furthermore, the operator $e^t M_k^w e^{-t}$ is bounded on $L^2(\mathbb{R}; dt)$, since

\begin{equation}
(3.30) \quad e^t m_{0,k}^w e^{-t} = [e^{t-t_k} \chi_0(t-t_k)] \psi(\beta_k^{-1/3}D_t) \left[ \chi_0(t-t_k) e^{-(t-t_k)} \right],
\end{equation}

and $|e^{t-t_k} \chi_0(t-t_k)| \leq e^{c_0}$.

The three parts in $M_k$ are used to handle different zones in the phase space. We use $m_{0,k}^w$ to localize near the point $t_k$, where the change-of-sign of $\sigma(e^t) - \sigma(e^{t_k})$ happens. The Fourier multiplier $\psi(\beta_k^{-1/3}D_t)$ allows us to obtain some subelliptic estimate in this zone, acting with the skew-adjoint part of $L_k$. As we shall see in the computations, it is important to put the cutoff function $\chi_0(t-t_k)$ on both sides of $\psi(\beta_k^{-1/3}D_t)$, so that we are able to do symbolic calculus with the exponential functions since they are all localized near $t_k$.

The other two multipliers $m_{+,-}^w$ are used for dealing with the zones where there is no change-of-sign of $\sigma(e^t) - \sigma(e^{t_k})$, that is $t$ away from the point $t_k$, and the sign of $m_{+,-}$ corresponds exactly to the sign of $\sigma(e^t) - \sigma(e^{t_k})$ on their supports. If the non-local term $i\beta_k \gamma(D_k)^{-2} \gamma$ were not present, then we could remove the factor $\beta_k^{-1/3}$ to get better estimates in these zones, as we have already done in [5]. However, we see that the non-local term has a large coefficient $\beta_k$ and it does not commute with $\chi_\pm(t-t_k)$, so that we would obtain a commutator of size $\beta_k$ that we would not know how to control. Our strategy is to weaken the multiplier in these regions by multiplying a factor $\beta_k^{-1/3}$.

The method that we use here is perturbative: the non-local term is treated as a perturbation with respect to the main term $\sigma(e^t) - \sigma(e^{t_k})$. Thanks to the operator $(D_k)^{-2}$ and the nice function $\gamma(t)$ (see (3.12)), this perturbation is controlled by the main term with an extra factor $k^{-2}$. Letting $k \geq k_0$, with $k_0 \geq 1$ a constant independent of the parameter $\alpha$, we can get the desired result.

However, it is of course impossible to consider the non-local term as a “global” perturbation, i.e. to absorb it by a term controlled by $\|e^{-t} L_k u\|_{L^2(\mathbb{R}, dt)}$, where $L_k$ is the unperturbed part of $L_k$: in fact the size of that perturbation is $\beta_k$ and the best estimate we can hope is controlling a factor $\beta_k^{1/3}$. We have instead to follow our multiplier method to check the effect of the perturbation.

b. Computations. Now let us compute $2 \text{Re} \langle L_k u, M_k u \rangle$.

Proposition 3.10. Suppose $e^{t_k} > e^{-1}$. There exist $c, C > 0$ such that for all $k \geq 1$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,

\begin{equation}
2 \text{Re} \langle L_k u, M_k u \rangle \geq c_0 \beta_k^{2/3} \langle \rho(t,t_k) u, u \rangle - C \beta_k^{2/3} k^{-2} \kappa(e^{t_k}) \| e^{2t} g(e^t) \chi u \|_2^2
\end{equation}

\begin{equation}
- 2 \beta_k^{2/3} k^{-2} \| e^{2t} g(e^t) \chi u \|_2^2 - C \| D_t u \|_2^2 - C k \| u \|_2^2 - C e^{2t} \| u \|_2^2,
\end{equation}
where $L_k$ is given in (3.20), $M_k$ in Definition 3.8, $\chi_0, \chi_{\pm}$ in (3.22), $\sigma, g$ in (3.2), $\beta_k$ in (3.5),

$$\rho(t, t_k) = \chi_0(t - t_k)^2 + e^{2t}\sigma(e^{t_k})\chi_+(t - t_k)^2 + e^{2t}\sigma(e^{t})\chi_-(t - t_k)^2$$

(3.32) and

$$\kappa(e^{t_k}) = g(e^{t_k})^{1/2} \max \left(1, \sqrt{\frac{3}{4}} e^{2t_k}, \frac{1}{\sqrt{16}} e^{4t_k}\right)$$

(3.33)

**Proof of Proposition 3.10.** First recall that for all $u \in C_0^\infty(\mathbb{R})$,

$$\text{Re}(\mathcal{L}_k u, u) = \|Du\|^2 + k^2\|u\|^2 + \frac{1}{16}\|e^{2t}u\|^2.$$  

(3.34)

In the following computations, we omit the dependence of $\chi_{\beta}(t - t_k)$ on $t - t_k$ for the sake of brevity.

**Estimates for $2\text{Re}(\mathcal{L}_k u, m_{w_{0,k}} u)$.

$$A := 2\text{Re}(\mathcal{L}_k u, m_{w_{0,k}} u) = 2\text{Re}(i\beta_k e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, m_{w_{0,k}} u)$$

$$- 2\text{Re}(i\beta_k (D_k)^{-2} \gamma u, m_{w_{0,k}} u)$$

$$+ 2\text{Re}( (D_k^2 + k^2 + \frac{1}{16}e^{4t}) u, m_{w_{0,k}} u)$$

(3.35)

$$=: A_1 + A_2 + A_3.$$  

Noticing $\chi_0 \tilde{\chi}_0 = \chi_0$ and (3.29), we have

$$A_1 = 2\text{Re}(i\beta_k e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, \chi_0 \psi(\beta_k^{-1/3} D_t) \chi_0 u)$$

$$= 2\text{Re}(i\beta_k \tilde{\chi}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \chi_0 u, \psi(\beta_k^{-1/3} D_t) \chi_0 u)$$

$$= \langle \psi(\beta_k^{-1/3} D_t), i\beta_k \tilde{\chi}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \rangle \chi_0 u, \chi_0 u \rangle.$$  

By (4.19), we know that the symbol $\beta_k \tilde{\chi}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))$ belongs to $S(\beta_k, \Gamma)$ and we get

$$\left[ \psi(\beta_k^{-1/3} D_t), i\beta_k \tilde{\chi}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] = b_1^w + \sigma_1^w,$$

where $b_1 \in S(\beta_k \lambda_1^{-1}, \Gamma)$ is a Poisson bracket and $r_1 \in S(\beta_k \lambda_1^{-3}, \Gamma) \subset S(1, \Gamma)$, with $\lambda_1$ given in (3.27) (see (4.11)). More precisely, we have

$$b_1(t, \tau) = \frac{1}{i} \left\{ \psi(\beta_k^{-1/3} \tau), i\beta_k \tilde{\chi}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right\}$$

$$= \beta_k^{2/3} \psi'(\beta_k^{-1/3} \tau) \frac{d}{dt} \left( \tilde{\chi}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right) \in S(\beta_k^{2/3}, \Gamma).$$

By (3.23), (3.24) and (4.18), we have in the zone $\{|t - t_k| \leq 2\sigma_0, |\tau| \leq \beta_k^{1/3}\}$

$$b_1(t, \tau) = \beta_k^{2/3} \psi'(\beta_k^{-1/3} \tau) \frac{d}{dt} \left( \tilde{\chi}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right) \geq \frac{C_1}{2} \beta_k^{2/3}.$$  

This implies for all $t, \tau \in \mathbb{R}$,

$$\frac{C_1}{2} \beta_k^{2/3} \leq b_1(t, \tau) + \frac{C_1}{2} \tau^2 + \tilde{C}_1 \beta_k^{2/3} \left( 1 - \tilde{\chi}_0(2(t - t_k)) \right) \in S(\lambda_1^2, \Gamma),$$

where $\tilde{C}_1 = 2\|b_1\|_{0,S(\beta_k^{2/3}, \Gamma)}$. Indeed, the function

$$b_1(t, \tau) + \tilde{C}_1 \beta_k^{2/3} \left( 1 - \tilde{\chi}_0(2(t - t_k)) \right) \geq \frac{C_1}{2} \beta_k^{2/3} \text{ for all } t \in \mathbb{R} \text{ and } |\tau| \leq \beta_k^{1/3},$$  

(3.36)
and it is non-negative for all \( t, \tau \in \mathbb{R} ; \) if \( |\tau| \geq \beta_k^{1/3} \), then \( \tau^2 \geq \beta_k^{2/3} \), which proves the inequality in \((3.36)\). Moreover, each term in the right hand side of \((3.36)\) is in \( S(\lambda_t^2, \Gamma) \). The Fefferman-Phong inequality (Proposition 4.2) implies
\[ b_1(t, \tau)^w + \frac{C_1}{2} D_t^2 + \tilde{C}_1 \beta_k^{2/3} \left( 1 - \tilde{\omega}(2(t - t_k)) \right) \geq \frac{C_1}{2} \beta_k^{2/3} - C'. \]
Applying to \( \chi_0 u \) and noting \( \chi_0(\cdot) \tilde{\chi}_0(2\cdot) = \chi_0(\cdot) \), we obtain
\[ A_1 + \frac{C_1}{2} \langle D_t^2 \chi_0 u, \chi_0 u \rangle = \langle \left( \frac{C_1}{2} D_t^2 + \tilde{b}^w \right) \chi_0 u, \chi_0 u \rangle \geq \frac{C_1}{2} \beta_k^{2/3} \| \chi_0 u \|^2 - C'' \| \chi_0 u \|^2. \]
On the other hand, we have
\[ \langle D_t^2 \chi_0 u, \chi_0 u \rangle = \| D_t \chi_0 u \|^2 \leq 2 \| \chi_0 D_t u \|^2 + 2 \| \chi_0 u \|^2 \leq C \| D_t u \|^2 + C \| u \|^2, \]
which gives
\[ A_1 \geq \frac{C_1}{2} \beta_k^{2/3} \| \chi_0 u \|^2 - C \| D_t u \|^2 - C \| u \|^2. \]
For the term \( A_2 \) defined in \((3.35)\), we have
\[ A_2 = -2 \text{Re} \langle i \beta_k \gamma (D_k)^{-2} \gamma u, m_{0,k}^w u \rangle = -2 \text{Re} \langle i \beta_k (D_k)^{-2} \gamma u, m_{0,k}^w u \rangle = -2 \text{Re} \langle i \beta_k (D_k)^{-2} \gamma u, m_{0,k}^w \gamma u \rangle - 2 \text{Re} \langle i \beta_k (D_k)^{-2} \gamma u, [\gamma, m_{0,k}^w] u \rangle = A_{21} + A_{22}. \]
For \( A_{21} \) in \((3.38)\), since \( \langle D_k \rangle^{-2} \) is skew-adjoint and \( m_{0,k}^w \) is self-adjoint, we have
\[ A_{21} = i \beta_k \langle \left[ (D_k)^{-2}, m_{0,k}^w \right] \gamma u, \gamma u \rangle. \]
Recalling \((3.29)\) and noting that \( \langle D_k \rangle^{-2} \) commutes with \( \psi(\beta_k^{-1/3} D_t) \), we get
\[ \left[ (D_k)^{-2}, m_{0,k}^w \right] = \left[ (D_k)^{-2}, \chi_0 \right] \psi(\beta_k^{-1/3} D_t) \chi_0 + \chi_0 \psi(\beta_k^{-1/3} D_t) \left[ (D_k)^{-2}, \chi_0 \right]. \]
We compute the commutator as follows
\[ \left[ (D_k)^{-2}, \chi_0 \right] = (D_k)^{-2} \chi_0 - \chi_0 (D_k)^{-2} = (D_k)^{-2} \left( \chi_0 (D_t^2 + k^2) - (D_t^2 + k^2) \chi_0 \right) (D_k)^{-2} = (D_k)^{-2} \left[ [\chi_0, D_t] D_t + D_t [\chi_0, D_t] \right] (D_k)^{-2} = i (D_k)^{-2} \chi_0 D_t (D_k)^{-2} + i (D_k)^{-2} D_t \chi_0 (D_k)^{-2}, \]
which implies
\[ \left[ (D_k)^{-2}, \chi_0 \right] D_t = i \sum_{k \leq 2} \chi_0 D_t (D_k)^{-2} D_t + i \sum_{k \leq 1} \chi_0 (D_k)^{-2} D_t \chi_0 (D_k)^{-2} D_t, \]
and
\[ \| \left[ (D_k)^{-2}, \chi_0 \right] D_t \|_{L^2(B(\mathbb{R}, dt))} \leq \frac{5}{4} \| \chi_0 \|_{L^\infty k^{-2}}. \]
The factorization \((3.25)\) of \( \psi \) gives
\[ \psi(\beta_k^{-1/3} D_t) = -\beta_k^{-1/3} D_t e(\beta_k^{-1/3} D_t) = -\beta_k^{-1/3} e(\beta_k^{-1/3} D_t) D_t, \]
so that
\[
\left[ (D_k)^{-2}, \chi_0 \right] \psi(\beta_k^{-1/3} D_t) \chi_0 = -\beta_k^{-1/3} \left( (D_k)^{-2}, \chi_0 \right) D_t e(\beta_k^{-1/3} D_t) \chi_0,
\]
has norm $\leq C k^{-2}$ bounded

and
\[
\left| \langle (D_k)^{-2}, \chi_0 \rangle \psi(\beta_k^{-1/3} D_t) \chi_0, \gamma u \rangle \right| \leq C \beta_k^{-1/3} k^{-2} \| \chi_0 \gamma u \| \| \gamma u \|.
\]
Similarly we can get
\[
\left| \langle \chi_0 \psi(\beta_k^{-1/3} D_t) (D_k)^{-2}, \chi_0 \rangle, \gamma u \rangle \right| \leq C \beta_k^{-1/3} k^{-2} \| \chi_0 \gamma u \| \| \gamma u \|.
\]
By \((3.39)\) and \((3.40)\), we obtain
\[
(3.41) \quad |A_{21}| \leq C \beta_k^{2/3} k^{-2} \| \chi_0 \gamma u \| \| \gamma u \|.
\]
For the term $A_{22}$ defined in \((3.38)\), we have, using \((3.29)\)
\[
A_{22} = -2 \text{Re} \langle i \beta_k (D_k)^{-2} \gamma u, [\gamma, m_{0,k}^w] u \rangle
\]
\[
= -2 \text{Re} \langle i \beta_k (D_k)^{-2} \gamma u, \chi_0 [\gamma, \psi(\beta_k^{-1/3} D_t)] \chi_0 u \rangle,
\]
so that we should compute the commutator $[\gamma, \psi(\beta_k^{-1/3} D_t)]$, for which we will do some symbolic calculus with the metric $\Gamma$ given in Definition \((3.6)\). The symbol $\gamma(t)$ is in $S(1, \Gamma)$ since $\gamma(t) = e^{2t} e^{-2t/8}$ belongs to $C_{\infty}^0(\mathbb{R})$, and we get
\[
[\gamma, \psi(\beta_k^{-1/3} D_t)] = b_2^w + r_2^w,
\]
where $b_2 \in S(\lambda_1^{-1}, \Gamma)$ is a Poisson bracket and $r_2 \in S(\lambda_1^{-3}, \Gamma) \subset S(\beta_1^{-1}, \Gamma)$, with $\lambda_1$ given in \((3.27)\) (see \((4.11)\)). By direct computation, we have
\[
b_2 = \frac{1}{i} \left\{ \gamma(t), \psi(\beta_k^{-1/3} \tau) \right\}
\]
\[
= -\frac{1}{i} \beta_k^{-1/3} \psi'(\beta_k^{-1/3} \tau) \gamma'(t) \in S(\beta_k^{-1/3}, \Gamma)
\]
\[
= -\frac{1}{i} \beta_k^{-1/3} \psi'(\beta_k^{-1/3} \tau) \gamma'(t) + b_3 + r_3,
\]
where $b_3 \in S(\beta_k^{-1/3}, \lambda_1^{-1}, \Gamma)$ is again a Poisson bracket and $r_3$ belongs to $S(\beta_k^{-1/3}, \Gamma)$ thus to $S(\beta_1^{-1}, \Gamma)$, since $\lambda_1 \geq \beta_k^{1/3}$. We continue to expand $b_3$
\[
b_3 = -\frac{1}{2i} \left\{ -\frac{1}{i} \beta_k^{-1/3} \psi'(\beta_k^{-1/3} \tau), \gamma'(t) \right\}
\]
\[
= -\frac{1}{2} \beta_k^{-2/3} \psi''(\beta_k^{-1/3} \tau) \gamma''(t) \in S(\beta_k^{-2/3}, \Gamma)
\]
\[
= -\frac{1}{2} \beta_k^{-2/3} \psi''(\beta_k^{-1/3} \tau) \gamma''(t) + r_4,
\]
with $r_4 \in S(\beta_k^{-2/3}, \lambda_1^{-1}, \Gamma) \subset S(\beta_1^{-1}, \Gamma)$. Thus we get for $w \in C_{0}^\infty(\mathbb{R})$,
\[
[\gamma, \psi(\beta_k^{-1/3} D_t)] w = -\frac{1}{i} \beta_k^{-1/3} \psi'(\beta_k^{-1/3} D_t) \gamma'(t) w - \frac{1}{2} \beta_k^{-2/3} \psi''(\beta_k^{-1/3} D_t) \gamma''(t) w
\]
\[
+ (r_2^w + r_3^w + r_4^w) w,
\]
where $r_2, r_3, r_4 \in S(\beta_1^{-1}, \Gamma)$. Since $\psi'(\beta_k^{-1/3} D_t)$ and $\psi''(\beta_k^{-1/3} D_t)$ are bounded on $L^2(\mathbb{R}; dt)$, we deduce that for $w \in C_{0}^\infty(\mathbb{R})$,
\[
\left\| \left[ \gamma(t), \psi(\beta_k^{-1/3} D_t) \right] w \right\| \leq C \beta_k^{-1/3} \| \gamma' w \| + C \beta_k^{-2/3} \| \gamma'' w \| + C \beta_k^{-1} \| w \|.
\]
Applying the above inequality to $\chi_0 u$, we get the estimate for $A_{22}$ defined in (3.38):

$$|A_{22}| \leq 2\beta_k \|\langle D_k \rangle^{-2} \gamma u \| \|\chi_0 \gamma \xi (\beta_k^{-1/3} D_t) \chi_0 u\| \leq 2\beta_k k^{-2} \|\gamma u\| \times (C\beta_k^{-1/3} \|\gamma u\| + C\beta_k^{-2/3} \|\gamma'' u\| + C\beta_k^{-1} \|\chi_0 u\|).$$

(3.42)

It follows from (3.38), (3.41) and (3.42) that

$$|A_2| \leq C\beta_k^{3/2} k^{-2} \|\gamma u\| \times (\|\chi_0 \gamma u\| + \|\chi_0 \gamma' u\|) + C\beta_k^{1/3} k^{-2} \|\gamma u\| \|\chi_0 u\|.$$

(3.43)

Recall $\gamma(t) = e^{2t} g(e^t) = e^{2t} e^{-e^{2t}} \leq 8/e$ and $g(e^t) = e^{-e^{2t}}$, then

$$\gamma'(t) = e^{2t} g(e^t) \left(2 - \frac{1}{4} e^{2t}\right), \quad \gamma''(t) = e^{2t} g(e^t) \left(4 - \frac{3}{2} e^{2t} + \frac{1}{16} e^{4t}\right).$$

(3.44)

Letting

$$\kappa(r) = g(r)^{1/2} \max \left(1, \left|2 - \frac{1}{4} r^2\right|, \left|4 - \frac{3}{2} r^2 + \frac{1}{16} r^4\right|\right),$$

we deduce from (3.43) that

$$|A_2| \leq C\beta_k^{3/2} k^{-2} \kappa(e^t) \|e^{2t} g(e^t)^{1/2} u\|^2 + Ck^{-2} \|u\|^2.$$

(3.46)

**Remark 3.11.** When $e^{t_k}$ is taken large (and we do not need $k$ large), $\kappa(e^{t_k})$ is very small. In particular, if $\epsilon_0$ is small, since $e^{t_k} > \epsilon_0^{-1}$, $\kappa(e^{t_k})$ is bounded above by $\kappa(\epsilon_0^{-1})$.

For the term $A_3$ defined in (3.35), we have

$$A_3 = 2\text{Re} \langle D^2_t u, m^w_{0,k} u \rangle + 2\text{Re} \langle k^2 u, m^w_{0,k} u \rangle + \frac{1}{8} \text{Re} \langle e^{4t} u, m^w_{0,k} u \rangle \quad \text{defined in (3.35)}.$$

(3.47)

For $A_{31}$ in (3.47),

$$A_{31} = 2\text{Re} \langle D_t u, D_t \Gamma^w_{0,k} u \rangle = 2\text{Re} \langle D_t u, m^w_{0,k} D_t u \rangle + 2\text{Re} \langle D_t u, [D_t, m^w_{0,k}] u \rangle = 2\text{Re} \langle D_t u, m^w_{0,k} D_t u \rangle + 2\text{Re} \langle [D_t, D_t, m^w_{0,k}] u \rangle.$$

Since $m_{0,k} \in S(1, \Gamma)$ and $\tau \in S(\lambda, \Gamma)$, the double commutator $[D_t, D_t, m^w_{0,k}]$ has a symbol in $S(1, \Gamma)$. We get

$$|A_{31}| \leq C \|D_t u\|^2 + C \|u\|^2.$$  

(3.48)

Using the $L^2(\mathbb{R};dt)$-boundedness of $m^w_{0,k}$, we get for $A_{32}$ defined in (3.47)

$$|A_{32}| \leq C k^2 \|u\|^2.$$  

(3.49)

For $A_{33}$ in (3.47), we have by (3.30)

$$8A_{33} = \text{Re} \langle e^{4t} u, \chi_0 \psi (\beta_k^{-1/3} D_t) \chi_0 u \rangle = \text{Re} \langle e^{2t} u, e^{2t} \chi_0 \psi (\beta_k^{-1/3} D_t) \chi_0 e^{-2t} e^{2t} u \rangle.$$

Hence

$$|A_{33}| \leq C \|e^{2t} u\|^2.$$  

(3.50)
By \(3.47\), \(3.48\), \(3.49\) and \(3.50\) we get
\[
|A_3| \leq C\|Du\|^2 + Ck^2\|u\|^2 + C\|e^{2t}u\|^2. \tag{3.51}
\]
We deduce from \(3.35\), \(3.37\), \(3.36\) and \(3.51\) that
\[
A \geq \frac{C}{2}\beta_k^{2/3}\|\chi_0u\|^2 - C\beta_k^{2/3}k^{-2}\kappa(e^{t_k})\|e^{2t}g(e^t)^{1/2}u\|^2
- C\|Du\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2, \tag{3.52}
\]
with \(\kappa(e^{t_k})\) given by \(3.45\).

Estimates for \(2\text{Re}(\mathcal{L}_k u, m_{+,k}^u)\). Recall that \(m_{+,k}^u = -i\beta_k^{-1/3}\chi_+(t - t_k)^2\).
\[
B^+ := 2\text{Re}(\mathcal{L}_k u, m_{+,k}^u) = 2\text{Re}(\mathcal{L}_k u, -i\beta_k^{-1/3}\chi_+^2) = 2\text{Re}(i\beta_k e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, -i\beta_k^{-1/3}\chi_+^2)
- 2\text{Re}(i\beta_k (D_k)^{-2}\gamma u, -i\beta_k^{-1/3}\chi_+^2)
+ 2\text{Re}(D_k^2 u, -i\beta_k^{-1/3}\chi_+^2). \tag{3.53}
\]
The support of \(\chi_+(t - t_k)\) is included in the set \(\{t - t_k \geq c_0/2\}\). By \(4.20\) we have
\[
B^+_1 = 2\beta_k^{2/3}(e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))u, \chi_+^2) \geq 2c_1\beta_k^{2/3}(e^{2t}(\sigma(e^{t_k})\chi_+^2, u, u). \tag{3.54}
\]
For \(B^+_2\) in \(3.53\) we have
\[
B^+_2 = \beta_k^{2/3}2\text{Re}(\chi_+(D_k)^{-2}\gamma u, \chi_+^2)
= \beta_k^{2/3}2\text{Re}(\chi_+(D_k^{-2})\gamma u, \chi_+^2) + \beta_k^{2/3}2\text{Re}(D_k^{-2}\gamma u, \chi_+^2)
= \beta_k^{2/3}(\chi_+(D_k^{-2})\gamma u, \chi_+^2) + \beta_k^{2/3}2\text{Re}(D_k^{-2}\gamma u, \chi_+^2). \tag{3.55}
\]
The kernel of \([\chi_+, [\chi_+, (D_k)^{-2}]]\) is
\[
\frac{1}{2k}e^{-k|t-s|}[\chi_+(t - t_k) - \chi_+(s - t_k)]^2,
\]
v vanishing if \(\max(t, s) \leq t_k + c_0/2\) and also if \(\min(t, s) \geq t_k + c_0\). Then we have
\[
\|([\chi_+, [\chi_+, (D_k)^{-2}]]\gamma u, \gamma u)\|
= \left| \int_{t - t_k \geq c_0/2} \frac{1}{2k}e^{-k|t-s|}[\chi_+(t - t_k) - \chi_+(s - t_k)]^2(\gamma u)(s)(\gamma u)(t)dt ds
+ \int_{t - t_k \leq c_0/2} \frac{1}{2k}e^{-k|t-s|}[\chi_+(t - t_k) - \chi_+(s - t_k)]^2(\gamma u)(s)(\gamma u)(t)dt ds \right|
\leq g(e^{t_k})^{1/2} \int_{t - t_k \geq c_0/2} \frac{1}{2k}e^{-k|t-s|}(e^{2s}g(e^s)|u(s)|)(e^{2t}g(e^t)^{1/2}|u(t)|)dt ds
+ g(e^{t_k})^{1/2} \int_{t - t_k \leq c_0/2} \frac{1}{2k}e^{-k|t-s|}(e^{2s}g(e^s)^{1/2}|u(s)|)(e^{2t}g(e^t)|u(t)|)dt ds
\leq g(e^{t_k})^{1/2}k^{-2}\|e^{2t}g(e^t)^{1/2}u\|^2, \tag{3.56}
\]
so that we obtain

\begin{equation}
B_2^+ \geq -\beta^{2/3}_k k^{-2} \kappa(e^k)\|e^{2t}g(e^t)^{1/2}u\|^2,
\end{equation}

where \(\kappa(e^k)\) is given in (3.45). For \(B_3^+\) in (3.53), we have

\[
B_3^+ = \beta^{1/3}_k \langle \{D^2_t, -i\chi_+^2 \} u, u \rangle
= i\beta^{1/3}_k \langle \{2\chi_+ \chi''^2 + 2\chi^2, u, u \rangle + i\beta^{1/3}_k \langle 4\chi_+ \chi', \partial_t u, u \rangle,
\]

which implies

\begin{equation}
|B_3^+| \leq C\beta^{-1/3}_k \|u\|^2 + C\beta^{-1/3}_k \|\partial_t u\| \|u\| \leq C\|D_t u\|^2 + C\|u\|^2.
\end{equation}

We get from (3.53), (3.54), (3.55) and (3.56) that

\begin{equation}
B^+ \geq 2c_1\beta^{2/3}_k \langle e^{2t} \sigma(e^t) \chi^2_+ u, u \rangle
- \beta^{2/3}_k k^{-2} \kappa(e^k)\|e^{2t}g(e^t)^{1/2}u\|^2 - C\|u\|^2 - C\|D_t u\|^2.
\end{equation}

**Estimates for \(2\text{Re}(\mathcal{L}_k u, m^w_{-k} u)\).** Recall that \(m^w_{-k} = i\beta^{-1/3}_k \chi_-(t - t_k)^2\).

\[
B^- := 2\text{Re}(\mathcal{L}_k u, m^w_{-k} u) = 2\text{Re}(\mathcal{L}_k u, i\beta^{1/3}_k \chi^2_- u)
= 2\text{Re}(i\beta e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, i\beta^{1/3}_k \chi^2_- u) - 2\text{Re}(i\beta \gamma(D_k)^{-2}u, i\beta^{1/3}_k \chi^2_- u) + 2\text{Re}(D^2_t u, i\beta^{1/3}_k \chi^2_- u)
= B_1^- + B_2^- + B_3^-.
\]

Recall (4.20) and note that the support of \(\chi_-(t - t_k)\) is included in \(\{t - t_k \leq -c_0/2\}\), then we get

\begin{equation}
B_1^- \geq 2c_1\beta^{2/3}_k \langle e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, \chi^2_- u \rangle
\end{equation}

For \(B_2^-\) in (3.58), we have

\[
B_2^- = -2\beta^{2/3}_k \text{Re}(\chi_- (D_k)^{-2}\gamma u, \chi_- u)
= -2\beta^{2/3}_k \text{Re}(\langle [\chi_-, (D_k)^{-2}] \gamma u, \chi_- u \rangle - 2\beta^{2/3}_k \text{Re}(\langle D_k)^{-2}\chi_- \gamma u, \chi_- u \rangle
= -\beta^{2/3}_k \langle [\chi_-, [\chi_-, (D_k)^{-2}] \gamma u, \gamma u \rangle - 2\beta^{2/3}_k \text{Re}(\langle (D_k)^{-2}\chi_- \gamma u, \chi_- u \rangle
=: B_{21}^- + B_{22}^-.
\]

For \(B_{21}^-\), we have

\[
0 \geq B_{22}^- = -2\beta^{2/3}_k \text{Re}(\langle (D_k)^{-2}\chi_- \gamma u, \chi_- u \rangle \geq -2\beta^{2/3}_k k^{-2}\|\chi_- u\|^2.
\]

By using the method that is used to estimate the double commutator in \(B_2^+\), we find

\[
|B_{21}^-| \leq C\beta^{2/3}_k k^{-2} \kappa(e^k)\|e^{2t}g(e^t)^{1/2}u\|^2,
\]

where \(\kappa(e^k)\) is given in (3.45), so that

\begin{equation}
B_2^- \geq -2\beta^{2/3}_k k^{-2}\|\chi_- u\|^2 - C\beta^{2/3}_k k^{-2} \kappa(e^k)\|e^{2t}g(e^t)^{1/2}u\|^2.
\end{equation}
For $B_3^-$ in (3.58), we have
\[ B_3^− = β_k^{−1/3} (\{ D^2_r, iμ^2 \} u, u) \]
\[ = −iβ_k^{−1/3} (\{ 2μ_− + 2μ^2 \} u, u) − iβ_k^{−1/3} (4μ_− μ_0, u) , \]
which implies
\[ |B_3^-| \leq Cβ_k^{−1/3} ||u||^2 + Cβ_k^{−1/3} ||μ_0|| ||u|| \leq C||u||^2 + C||D_t u||^2. \]
We get from (3.60) and (3.61) that
\[ B^- \geq 2c_1e_2^{−2/3} (e_2^{−2/3} μ_0^2 u, u) − C||u||^2 - C||D_t u||^2 \]
\[ - 2β_k^{−2/3} e_2^{−2} ||μ_0|| ||e_2^t||^2 - β_k^{−2/3} e_2^{−2} μ_0|| ||e_2^t||^2 , \]
where $e_2^t$ is given in (3.45). This completes the proof of (3.31) in Proposition 3.10. □

Recall the definition (3.32) of $μ(t, t_k)$, then (3.31) implies
\[ 2Re{Luku, M_k u} \geq \beta_k^{−2/3} \{ \left( cμ(t, t_k) − Ck^{−2} e_2^{−2} μ_0^2 \right) μ_0^2 u, u \} \]
\[ C||D_t u||^2 - Ck^{−2} ||u||^2 \}
where $k(μ_0)$ is bounded above by a constant depending on $μ_0$ (see Remark 3.11). We have the following two estimates for $μ(t, t_k)$.

**Lemma 3.12.** There exist $C_4, C_5 > 0$ such that for $e_k > μ_0^−1$, $t \in R$, $α \geq 8π$, $k \geq 1$,
\[ μ(t, t_k) \geq C_4 e^{−2} μ_0^2 , \]
(3.65)
\[ β_k^{−2/3} μ(t, t_k) + e_2^{−2} \geq C_5 β_k^{1/3} e_2^{1/3} μ_0^2 , \]
where $μ(t, t_k)$ is given in (3.32), $e_2$ is given in (3.2) and $β_k$ is given in (3.5).

**Proof of Lemma 3.12.** Suppose $e_k > μ_0^−1$, then $μ(μ_0) \geq δe_2^{−2} μ_0^2$ for some $δ > 0$. Note also that the function $r^2 g(r)$ is bounded. If $t$ is in the support of $μ_0(· − t_k)$, i.e. $|t − t_k| ≤ c_0$, we have
\[ e_2^{−2} μ_0^2 (μ_0^−1) ≤ C e_2^{−2} μ_0^2 (μ_0^−1) ≤ C, \]
\[ β_k^{−2/3} e_2^{−2} + e_2^{−2} = \{ β_k^{−2/3} e_2^{−2} + e_2^{−2} \} e_2^{−2} \geq β_k^{1/3} e_2^{2} . \]
When $t$ is in the support of $μ_0(· − t_k)$, i.e. $t \geq t_k + c_0/2$, we have
\[ e_2^{−2} μ_0^2 (μ_0^−1) \geq δe_2^{−2} μ_0^2 \geq δ e_2^{−2} μ_0^2 \geq C e_2^{−2} μ_0^2 (μ_0^−1) , \]
\[ β_k^{−2/3} e_2^{−2} μ_0^2 (μ_0^−1) + e_2^{−2} \geq \{ δβ_k^{−2/3} e_2^{−2} μ_0^2 + e_2^{−2} \} e_2^{−2} \geq δ^{1/2} β_k^{1/3} e_2^{2} . \]
When \( t \) is in the support of \( \chi_-(\cdot - t_k) \), i.e. \( t \leq t_k - c_0/2 \), we have, using (4.13) and the first inequality in (4.17)
\[
e^{4t}g(e^t) \leq 16e^{2t}\sigma(e^t),
\]
\[
\beta_k^{2/3}e^{2t}\sigma(e^t) + e^{4t} \geq \left(\beta_k^{2/3}\sigma(e^t) + e^{2t}\right)e^{2t} \geq (2\log 2)^{-1}\beta_k^{1/3}e^{2t}.
\]
This completes the proof of (3.64) and (3.65). \( \square \)

**Proof of Theorem 3.5.** The estimates (3.63) and (3.64) imply that there exists \( k_0 \geq 1 \), for all \( k \geq k_0 \),
\[
2\Re\langle \mathcal{L}_k u, M_k u \rangle \geq \frac{c}{2}\beta_k^{2/3}\rho(t, t_k)u - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t} u\|^2.
\]
Together with (3.34), by choosing \( C_6 > 0 \) large enough, we have for \( k \geq k_0 \),
\[
(3.66) \quad \Re\langle \mathcal{L}_k u, (C_6 + 2M_k)u \rangle \geq \frac{c}{2}\left(\beta_k^{2/3}\rho(t, t_k) + D_t^2 + k^2 + e^{4t}\right)u, u).\]
It follows from (3.66) and (3.65) that for \( k \geq k_0 \),
\[
(3.67) \quad \Re\langle \mathcal{L}_k u, (C_6 + 2M_k)u \rangle \geq C\beta_k^{1/3}e^{2t} u, u).\]
Noticing that \( e^t(C_6 + 2M_k)e^{-t} \) is bounded on \( L^2(\mathbb{R}; dt) \) by (3.30) and that
\[
\langle \mathcal{L}_k u, (C_6 + 2M_k)u \rangle = \langle e^{-t}\mathcal{L}_k u, (e^t(C_6 + 2M_k)e^{-t})(e^t u) \rangle,
\]
we deduce from (3.67) and Cauchy-Schwarz inequality that
\[
\|e^{-t}\mathcal{L}_k u\|\|e^t u\| \geq C\beta_k^{1/3}\|e^t u\|^2, \quad \text{for } k \geq k_0,
\]
which is
\[
\|e^{-t}\mathcal{L}_k u\| \geq C\beta_k^{1/3}\|e^t u\|, \quad k \geq k_0,
\]
completing the proof of Theorem 3.5. \( \square \)

**Remark 3.13.** There is a localization effect taking place in this case. We see in (3.31) of Proposition 3.10 that the coefficient of the term \( \|e^{2t}g(e^t)\|/2 \) has a factor \( \kappa(e^t) \), which is small if \( e^{t_k} \) is taken very large (see Remark 3.11). As a result, if we suppose \( e^{t_k} \) large enough, this term is negligible, and the only bad term coming from the nonlocal operator that we need to control is
\[
2\beta_k^{2/3}k^{-2}\|e^{2t}g(e^t)\|\chi_- u\|^2.
\]
On the other hand, we can prove that there exists \( \epsilon_2 > 0 \) such that for all \( e^{t_k} > \epsilon_2^{-1} \),
\[
\forall k \geq 2, \forall t \leq t_k - \frac{c_0}{2}, \quad 2e^{2t}\left(\sigma(e^t) - \sigma(e^{t_k})\right) > 2k^{-2}e^{4t}g(e^t)^2.
\]
This implies that it suffices to take \( k \geq 2 \) to absorb the remainders and thus Theorem 3.5 holds for \( k_0 = 2 \) and \( e^{t_k} > \epsilon_2^{-1} \). Furthermore, we shall see that this localization effect does not present in Case 2 and Case 3.
3.3.3. Case 2: $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$.

Theorem 3.14. Suppose $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$. Then there exist $C > 0$, $k_0 \geq 1$ such that for all $k \geq k_0$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$, 

\[(3.68) \quad \|e^{-t_L}L_k u\| \geq C\gamma k^{2/3}\|e^t u\|,\]

where $L_k$ is given in (3.20) and $\gamma_k$ is given in (3.5).

We present some inequalities concerning $\sigma$ that will be used in Case 2 in Proposition 4.7 (2). Note that they are similar to those in the Case 1 (given in Proposition 4.7 (1)).

We use the metric $\Gamma$ and the multiplier $M_k$ in Definition 3.6, 3.8 and use the notations $A_1, A_2, A_3, B^+, B^-$ in (3.35), (3.53), (3.58). The estimate (3.37) and $A_1$ is valid with constant $C_1$ replaced by $C_2$ (which is given in (4.21)) and the estimate (3.51) for $A_3$ holds in Case 2. For $A_2$, the estimate (3.43) remains true:

\[(3.69) \quad |A_2| \leq C\gamma k^{2/3}\|\gamma u\|^2 + C\gamma k^{-2}\|u\|^2.\]

For the terms $B^+, B^-$, we have by (4.23),

\[B^+ = 2\text{Re}\langle L_k u, m_{\pm,k} w \rangle \geq 2c_2\gamma k^{2/3}\langle e^{t} \sigma(e^{t})\chi u \rangle u, u \rangle - C\|u\|^2 - C\|D_t u\|^2 - 2\gamma k^{2/3}\|\gamma u\|^2,\]

\[B^- = 2\text{Re}\langle L_k u, m_{-k} w \rangle \geq 2c_2\gamma k^{2/3}\langle e^{t} \sigma(e^{t})\chi u \rangle u, u \rangle - C\|u\|^2 - C\|D_t u\|^2 - 2\gamma k^{2/3}\|\gamma u\|^2.\]

Summarizing, we get that for all $k \geq 1$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,

\[2\text{Re}\langle L_k u, M_k u \rangle = A + B^+ + B^- \geq \frac{C_2}2 \gamma k^{2/3}\|\chi u\|^2 + 2c_2\gamma k^{2/3}\langle e^{t} \sigma(e^{t})\chi u \rangle u, u \rangle - C\|D_t u\|^2 - C\|e^t u\|^2,\]

so that the following proposition is proved:

Proposition 3.15. Suppose $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$. There exist $c > 0, C > 0$ such that for $k \geq 1$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,

\[(3.70) \quad 2\text{Re}\langle L_k u, M_k u \rangle \geq \gamma k^{2/3}\langle (c\rho(t_k) - Ck^{-2}e^t g(e^{t_k})) u, u \rangle - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^t u\|^2,\]

where $L_k$ is given in (3.20), $M_k$ in Definition 3.8, $g$ in (3.2), $\beta_k$ in (3.5) and

\[(3.71) \quad \rho(t, t_k) = \chi_0(t - t_k) + e^u \sigma(e^{t_k})\chi_+(t - t_k)^2 + e^u \sigma(e^{t})\chi_-(t - t_k)^2,\]
with \( \chi_0, \chi_\pm \) defined in (3.22) and \( \sigma \) in (3.2).

We have the following estimates for \( \rho(t, t_k) \).

**Lemma 3.16.** There exist \( C_7, C_8 > 0 \) such that for all \( e^t \in [\epsilon_1, \epsilon_0^{-1}] \), \( t \in \mathbb{R} \),

\[
(3.72) \quad \rho(t, t_k) \geq C_7 e^{\frac{1}{4}} g(e^t)^2,
\]

\[
(3.73) \quad \rho(t, t_k) \geq C_8 e^{2t},
\]

where \( \rho(t, t_k) \) is given in (3.71) and \( g \) is given in (3.2).

**Proof of Lemma 3.16.** Indeed, we have for \( e^t_k \in [\epsilon_1, \epsilon_0^{-1}] \), \( \sigma(e^t_k) \geq \delta \). If \( t \) is in the support of \( \chi_0(\cdot - t_k) \), we have \( |t - t_k| \leq c_0 \), then

\[
e^{4t} g(e^t)^2 \leq C, \quad e^{2t} \leq \epsilon_0^{-2} e^{2c_0}.
\]

If \( t \) is in the support of \( \chi_+^\pm(\cdot - t_k) \), we have \( t \geq t_k + c_0/2 \),

\[
e^{4t} g(e^t)^2 \leq C e^{2t} \sigma(e^t_k), \quad e^{2t} \leq \delta^{-1} e^{2t} \sigma(e^t_k)
\]

If \( t \) is in the support of \( \chi_-^\pm(\cdot - t_k) \), we have \( t \leq t_k - c_0/2 \), then \( \sigma(e^t) \geq \delta \),

\[
e^{4t} g(e^t)^2 \leq 3 e^{2t} \sigma(e^t) \) (by (4.12)), \quad e^{2t} \leq \delta^{-1} e^{2t} \sigma(e^t).
\]

Thus (3.72), (3.73) are proved.

**Proof of Theorem 3.14.** (3.70) and (3.72) imply that there exists \( k_0 \geq 1 \), for \( k \geq k_0 \),

\[
(3.74) \quad 2 \text{Re}(\mathcal{L}_k u, M_k u) \geq \frac{c}{2} \beta_k^{2/3} \langle \rho(t, t_k) u, u \rangle - C \| D_t u \|^2 - C k^2 \| u \|^2 - C \| e^{2t} u \|^2.
\]

Hence by choosing \( C_9 > 0 \) large enough, we get for \( k \geq k_0 \)

\[
\text{Re}(\mathcal{L}_k u, (C_9 + 2M_k) u) \geq \frac{c}{2} \left( \beta_k^{2/3} \rho(t, t_k) + D_t^2 + k^2 + e^{4t} \right) u, u.
\]

and in particular by (3.73)

\[
\text{Re}(\mathcal{L}_k u, (C_9 + 2M_k) u) \geq C \beta_k^{2/3} e^{2t} u, u, \quad k \geq k_0.
\]

Finally we obtain the inequality (3.68) by using the \( L^2(\mathbb{R}; dt) \)-boundedness of the operator \( e^t(C_9 + M_k)e^{-t} \) and Cauchy-Schwarz inequality. \( \Box \)

### 3.4. Nontrivial cases, continued.

**3.4.1. Case 3: \( \beta_k^{-1/4} < e^t < \epsilon_1 \).** We present in Proposition 4.7 (3) the inequalities about the function \( \sigma \) to be used in this case. Moreover, we assume \( \alpha_0 \geq 8\pi \) such that the interval \( (\beta_k^{-1/4}, \epsilon_1) \) is not empty for any \( k \geq 1 \), \( \alpha \geq \alpha_0 \).

**Theorem 3.17.** Suppose \( e^t_k \in (\beta_k^{-1/4}, \epsilon_1) \). Then there exist \( C > 0 \), \( k_0 \geq 3 \) such that for all \( k \geq k_0 \), \( \alpha \geq \alpha_0 \), \( u \in C_0^\infty(\mathbb{R}) \),

\[
(3.75) \quad \| e^{-t} \mathcal{L}_k u \| \geq C \beta_k^{1/2} \| e^t u \|,
\]

where \( \mathcal{L}_k \) is given in (3.20) and \( \beta_k \) is given in (3.5).

We shall modify the metric \( \Gamma \) and the multiplier \( M_k \) as follows.
Definition 3.18.
(3.76) \[ \Gamma = |dt|^2 + \frac{|d\tau|^2}{\tau^2 + (\beta_k e^{4t_k})^{2/3}}, \quad (t, \tau) \in \mathbb{R}_t \times \mathbb{R}_\tau, \]
(3.77) \[ M_k = m_{0,k}^w + m_{+,k}^w + m_{-,k}^w, \]
where
\[ m_{0,k}(t, \tau) = \chi_0(t-t_k)^2 \psi((\beta_k e^{4t_k})^{-1/3} \tau)^2 \chi_0(t-t_k), \]
\[ m_{+,k}(t, \tau) = -i(\beta_k e^{4t_k})^{-1/3} \chi_+(t-t_k)^2, \]
\[ m_{-,k}(t, \tau) = i(\beta_k e^{4t_k})^{-1/3} \chi_-(t-t_k)^2, \]
with \( \chi_0, \chi_\pm, \psi \) given in (3.22), (3.24).

Remark 3.19. Since we are in the region \( e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1) \), we have
(3.78) \[ \lambda_\Gamma = (\tau^2 + (\beta_k e^{4t_k})^{2/3})^{1/2} \geq (\beta_k e^{4t_k})^{1/3} \geq 1, \]
so that the metric \( \Gamma \) verifies the uncertainty principle and moreover, \( \Gamma \) is uniformly admissible (see Lemma 4.1). Furthermore, the operator \( M_k \) is bounded on \( L^2(\mathbb{R}; dt) \).

Proposition 3.20. Suppose \( e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1) \). There exist \( c, C > 0 \) such that for all \( k \geq 3, \alpha \geq \alpha_0, u \in C^\infty_0(\mathbb{R}) \),
\begin{align*}
2\text{Re}\langle \mathcal{L}_k u, M_k u \rangle & \geq \beta_k(\beta_k e^{4t_k})^{-1/3}
\left( \left( c \tilde{\rho}(t, t_k) - Ck^{-2}e^{4t}g(e^t)^2 \right) u, u \right) \\
& \quad - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2,
\end{align*}
(3.79) where \( \mathcal{L}_k \) is given in (3.20), \( M_k \) in Definition 3.18, \( g \) in (3.2), \( \beta_k \) in (3.5) and (3.80) \( \tilde{\rho}(t, t_k) = e^{4t_k} \chi_0(t-t_k)^2 + e^{2t}(1-\sigma(e^t)) \chi_+(t-t_k)^2 + e^{2t}(1-\sigma(e^t_k)) \chi_-(t-t_k)^2, \)
with \( \chi_0, \chi_\pm \) defined in (3.22) and \( \sigma \) given in (3.2).

Proof of Proposition 3.20. Estimates for \( 2\text{Re}\langle \mathcal{L}_k u, m_{0,k}^w u \rangle \).
\[ A := 2\text{Re}\langle \mathcal{L}_k u, m_{0,k}^w u \rangle = 2\text{Re}\langle i\beta_k e^{2t} (\sigma(e^t) - \sigma(e^t_k)) u, m_{0,k}^w u \rangle \\
- 2\text{Re}\langle i\beta_k \gamma(D_k)^{-2} \gamma u, m_{0,k}^w u \rangle \\
+ 2\text{Re}\langle (D_t^2 + k^2 + \frac{1}{16} e^{4t}) u, m_{0,k}^w u \rangle \]
(3.81) \[ =: A_1 + A_2 + A_3. \]

For \( A_1 \) in (3.81), we get a commutator
\[ A_1 = 2\text{Re}\langle i\beta_k e^{2t} (\sigma(e^t) - \sigma(e^t_k)) u, \chi_0 \tilde{\rho}((\beta_k e^{4t_k})^{-1/3} D_t) \chi_0 u \rangle \]
\[ = \langle \left[ \psi((\beta_k e^{4t_k})^{-1/3} D_t), i\beta_k \chi_0 e^{2t} (\sigma(e^t) - \sigma(e^t_k)) \right] \chi_0 u, \chi_0 u \rangle, \]
where \( \tilde{\chi}_0 \) is given in (3.23). We know that, with \( \Gamma \) given in Definition 3.18
\[ \left[ \psi((\beta_k e^{4t_k})^{-1/3} D_t), i\beta_k \tilde{\chi}_0 e^{2t} (\sigma(e^t) - \sigma(e^t_k)) \right] = b_1^w + r_1^w, \]
\[ \in S(\beta_k e^{4t_k} \Gamma) \] by (4.25).
where $b_1$ is a Poisson bracket and $r_1 \in S(\beta_3 e^{4t_k} \lambda_{t_k}^{-3}, \Gamma) \subset S(1, \Gamma)$, with $\lambda_{t_k}$ given in (3.78) (see (4.11)). More precisely,

$$b_1(t,\tau) = \frac{1}{i} \{ \psi((\beta_k e^{4t_k})^{-1/3} t), i\beta_k \bar{z}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \}$$

$$= \beta_k (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3} t) \frac{d}{dt} \left( \bar{z}_0 e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right)$$

$$\in S((\beta_k e^{4t_k})^{2/3}, \Gamma) \subset S(\lambda_{t_k}^2, \Gamma).$$

By (3.23), (3.24) and (4.24), we have in the zone $\{|t - t_k| \leq 2e_0, \tau \leq (\beta_k e^{4t_k})^{1/3}\}$

$$b_1(t,\tau) = \beta_k (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3} t) \frac{d}{dt} \left( e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right)$$

$$\geq \beta_k (\beta_k e^{4t_k})^{-1/3} \times \frac{1}{2} C_3 e^{4t_k} \geq \frac{C_3}{2} (\beta_k e^{4t_k})^{2/3}. \quad (3.82)$$

This implies for all $t, \tau \in \mathbb{R}$,

$$C_3 \left( \frac{1}{2} (\beta_k e^{4t_k})^{2/3} \leq b_1(t,\tau) + \frac{C_3}{2} e^2 + \tilde{C}_3 (\beta_k e^{4t_k})^{2/3} \left( 1 - \bar{z}_0 (2(t - t_k)) \right) \right) \in S(\lambda_{t_k}^2, \Gamma),$$

where $\tilde{C}_3 = 2\|b_1\|_{0,S((\beta_k e^{4t_k})^{2/3}, \Gamma)}$. Indeed, the function

$$b_1(t,\tau) + \tilde{C}_3 (\beta_k e^{4t_k})^{2/3} \left( 1 - \bar{z}_0 (2(t - t_k)) \right) \geq \frac{C_3}{2} (\beta_k e^{4t_k})^{2/3}$$

for all $t \in \mathbb{R}$ and $|\tau| \leq (\beta_k e^{4t_k})^{1/3}$, and it is non-negative for all $t, \tau \in \mathbb{R}$; if $|\tau| \geq (\beta_k e^{4t_k})^{1/3}$, then $\tau^2 \geq (\beta_k e^{4t_k})^{2/3}$, which proves the inequality in (3.83). Moreover, each term in the right hand side of (3.83) is in $S(\lambda_{t_k}^2, \Gamma)$. The Fefferman-Phong inequality (Proposition 4.2) implies

$$b_1(t,\tau)^w + \frac{C_3}{2} D_t^2 + \tilde{C}_3 (\beta_k e^{4t_k})^{2/3} \left( 1 - \bar{z}_0 (2(t - t_k)) \right) \geq \frac{C_3}{2} (\beta_k e^{4t_k})^{2/3} - C'w.$$ 

Applying to $\chi_0 u$, we get

$$A_1 + \frac{C_3}{2} \langle D_t^2 \chi_0 u, \chi_0 u \rangle = \langle \frac{C_3}{2} D_t^2 + b_i^w \rangle \chi_0 u, \chi_0 u \rangle + \langle r_1^w \chi_0 u, \chi_0 u \rangle$$

$$\geq \frac{C_3}{2} (\beta_k e^{4t_k})^{2/3} \|\chi_0 u\|^2 - C' \|\chi_0 u\|^2.$$

Hence we get the estimate for $A_1$:

$$A_1 \geq \frac{C_3}{2} (\beta_k e^{4t_k})^{2/3} \|\chi_0 u\|^2 - C \|D_t u\|^2 - C \|u\|^2. \quad (3.84)$$

For $A_2$ defined in (3.81), we have

$$A_2 = -2 \text{Re}(i\beta_k \gamma \langle D_k \rangle^{-2} \gamma u, m_{w,0}^w u)$$

$$= -2 \text{Re}(i\beta_k \langle D_k \rangle^{-2} \gamma u, m_{w,0}^w \gamma u) - 2 \text{Re}(i\beta_k \langle D_k \rangle^{-2} \gamma u, [\gamma, m_{w,0}^w] u)$$

$$= A_{21} + A_{22}. \quad (3.85)$$

For $A_{21}$ in (3.85), since $i\langle D_k \rangle^{-2}$ is skew-adjoint and $m_{0,k}^w$ is self-adjoint, we get

$$A_{21} = i\beta_k \langle \langle D_k \rangle^{-2}, m_{0,k}^w \rangle \gamma u, \gamma u \rangle.$$
Noting that $\langle D_k \rangle^{-2}$ commutes with $\psi((\beta_k e^{4t_k})^{-1/3} D_t)$, we have
\[
\begin{align*}
\langle (D_k)^{-2}, m^{\omega}_{0,k} \rangle &= \langle (D_k)^{-2}, \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) \chi_0 \rangle \\
&= \langle (D_k)^{-2}, \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) \chi_0 + \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) \rangle \langle (D_k)^{-2}, \chi_0 \rangle.
\end{align*}
\]
By using the method that is used in Case 1, we can get
\[
\begin{align*}
|\langle (D_k)^{-2}, \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) \gamma_0 u, \gamma u \rangle| &\leq C(\beta_k e^{4t_k})^{-1/3} k^{-2} \|\chi_0 \gamma u\| \|\gamma u\|, \\
|\langle \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) \rangle \langle (D_k)^{-2}, \chi_0 \rangle \gamma_0 u, \gamma u \rangle| &\leq C(\beta_k e^{4t_k})^{-1/3} k^{-2} \|\gamma u\| \|\chi_0 \gamma u\|,
\end{align*}
\]
so that
\[
(3.86) \quad |A_{21}| \leq C(\beta_k e^{4t_k})^{-1/3} k^{-2} \|\chi_0 \gamma u\| \|\gamma u\|.
\]
For $A_{22}$ in (3.85), we have
\[
A_{22} = -2 \text{Re}(i \beta_k \langle D_k \rangle^{-2} \gamma u, [\gamma, m^{\omega}_{0,k}] u)
\]
\[
= -2 \text{Re}(i \beta_k \langle D_k \rangle^{-2} \gamma u, \chi_0 \left[\bar{x}_0 \gamma, \psi((\beta_k e^{4t_k})^{-1/3} D_t) \right] \chi_0 u),
\]
where $\bar{x}_0$ is given in (3.23). Since $\bar{x}_{0,\gamma} = \bar{x}_0 (t - t_k) e^{2t} g(e^t) \in S(e^{2t_k}, \Gamma)$, we get
\[
\left[\bar{x}_0 \gamma, \psi((\beta_k e^{4t_k})^{-1/3} D_t) \right] = b^{\omega}_2 + r^{\omega}_2,
\]
where $b_2 \in S(e^{2t_k} \lambda^{-1}, \Gamma)$ is a Poisson bracket and $r_2$ belongs to $S(e^{2t_k} \lambda^{-3}, \Gamma) \subset S(e^{2t_k} (\beta_k e^{4t_k})^{-1}, \Gamma)$, with $\lambda$ given in (3.78) (see (4.11)). We compute $b_2$ as follows
\[
b_2 = \frac{1}{i} \left\{ \bar{x}_0 \gamma, \psi((\beta_k e^{4t_k})^{-1/3} \right\}
\]
\[
= -\frac{1}{i} (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3} \frac{\bar{x}_0 \gamma}{\gamma})' = S(e^{2t_k} (\beta_k e^{4t_k})^{-1/3}, \Gamma)
\]
\[
= -\frac{1}{i} (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3} \frac{\bar{x}_0 \gamma}{\gamma})' \frac{\bar{x}_0 \gamma}{\gamma} + b_3 + r_3,
\]
where $b_3 \in S(e^{2t_k} (\beta_k e^{4t_k})^{-2/3}, \Gamma)$ is a Poisson bracket and $r_3 \in S(e^{2t_k} (\beta_k e^{4t_k})^{-1}, \Gamma)$. We continue to expand $b_3$
\[
b_3 = -\frac{1}{2i} \left\{ -\frac{1}{i} (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3} \frac{\bar{x}_0 \gamma}{\gamma})' \right\}
\]
\[
= -\frac{1}{2} (\beta_k e^{4t_k})^{-2/3} \psi''((\beta_k e^{4t_k})^{-1/3} \frac{\bar{x}_0 \gamma}{\gamma}'' \frac{\bar{x}_0 \gamma}{\gamma}' + r_4
\]
where $r_4 \in S(e^{2t_k} (\beta_k e^{4t_k})^{-1}, \Gamma)$. Thus we get for $w \in C^\infty(\mathbb{R})$,
\[
\left[\bar{x}_0 \gamma, \psi((\beta_k e^{4t_k})^{-1/3} D_t) \right] w = -\frac{1}{i} (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3} D_t) (\bar{x}_0 \gamma)'(t) w
\]
\[
-\frac{1}{2} (\beta_k e^{4t_k})^{-2/3} \psi''((\beta_k e^{4t_k})^{-1/3} D_t) (\bar{x}_0 \gamma)''(t) w + r_2 w + r_3 + r_4 w,
\]
where $r_2, r_3, r_4 \in S(e^{2t_k} (\beta_k e^{4t_k})^{-1}, \Gamma)$. Using the boundedness of $\psi'$ and $\psi''$, we obtain for $w \in C^\infty(\mathbb{R})$,
\[
\left[\bar{x}_0 \gamma, \psi((\beta_k e^{4t_k})^{-1/3} D_t) \right] w \leq C(\beta_k e^{4t_k})^{-1/3} \|\bar{x}_0 \gamma\| w
\]
\[
+ C(\beta_k e^{4t_k})^{-2/3} \|\bar{x}_0 \gamma\||w| + C e^{2t_k} (\beta_k e^{4t_k})^{-1} |w|.
\]
Now the term $A_{22}$ defined in (3.85) can be estimated as follows:

\[ |A_{22}| = |2\text{Re}(i\beta_k\chi_0(D_k)^{-2}\gamma u, [\chi_0\gamma, \psi((\beta_k e^{4t_k})^{-1/3}D_t)]\chi_0 u)| \]

\[ \leq 2\beta_k\|\chi_0(D_k)^{-2}\gamma u\| \|\chi_0\gamma, \psi((\beta_k e^{4t_k})^{-1/3}D_t)\chi_0 u\| \]

\[ \leq 2\beta_k\|\chi_0(D_k)^{-2}\gamma u\| \times \left( C(\beta_k e^{4t_k})^{-1/3}\|(\chi_0\gamma)'\chi_0 u\| \right. \]

\[ + C(\beta_k e^{4t_k})^{-2/3}\|(\chi_0\gamma)''\chi_0 u\| + C e^{2t_k}(\beta_k e^{4t_k})^{-1}\chi_0 u\| \right) \]

\[ \leq C\beta_k\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|\|\gamma'\chi_0 u\| + C\beta_k(\beta_k e^{4t_k})^{-2/3}k^{-2}\|\gamma u\|\|\gamma''\chi_0 u\| \]

(3.87)

where in the last inequality we use the following estimate for $\|\chi_0(D_k)^{-2}\gamma u\| = \|\chi_0 e^{2t_k} e^{-2t_k}(D_k)^{-2}e^{2t_k} g'(e) u\| \leq C e^{2t_k} k^{-2}\|g(e') u\|$

(see Lemma 4.5).

It follows from (3.85), (3.86) and (3.87) that

\[ |A_2| \leq C\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|\|\chi_0\gamma u\| + C\beta_k(\beta_k e^{4t_k})^{-2/3}k^{-2}\|\chi_0\gamma'' u\| + C k^{-2}\|u\|^2. \]

(3.88)

From (3.44) we deduce that for $e^{t_k} < \epsilon_1$,

\[ |\chi_0(t - t_k)\gamma'(t)| \leq C\gamma(t), \quad |\chi_0(t - t_k)\gamma''(t)| \leq C\gamma(t), \]

with $C$ depending only on $\epsilon_1$, so that

\[ |A_2| \leq C\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2 + C k^{-2}\|u\|^2. \]

(3.89)

The estimate for $A_3$ defined in (3.81) is the same as that in Case 1

\[ |A_3| \leq C\|D_t u\|^2 + C\|u\|^2 + C\|e^{2t} u\|^2. \]

(3.90)

We deduce from (3.81), (3.84), (3.89) and (3.90) that

\[ A \geq C \frac{3}{2}(\beta_k e^{4t_k})^{2/3}\|\chi_0 u\|^2 - C\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2 - C k^{-2}\|u\|^2 \]

(3.91)

Estimates for $2\text{Re}(\mathcal{L}_k u, m^{w}_{t+k}, u)$. Recall $m^{w}_{t+k} = -i(\beta_k e^{4t_k})^{-1/3}\chi_{t+k}^{2}(t - t_k)^2$.

\[ B^+ := 2\text{Re}(\mathcal{L}_k u, m^{w}_{t+k} u) = 2\text{Re}(\mathcal{L}_k u, -i(\beta_k e^{4t_k})^{-1/3}\chi_{t+k}^{2} u) \]

\[ = 2\text{Re}(i\beta_k e^{2t}(\sigma(e') - \sigma(e')) u, -i(\beta_k e^{4t_k})^{-1/3}\chi_{t+k}^{2} u) \]

\[ - 2\text{Re}(i\beta_k \gamma(D_k)^{-2}\gamma u, -i(\beta_k e^{4t_k})^{-1/3}\chi_{t+k}^{2} u) \]

\[ + 2\text{Re}(D_t^2 u, -i(\beta_k e^{4t_k})^{-1/3}\chi_{t+k}^{2} u) \]

(3.92)

\[ = B^+_1 + B^+_2 + B^+_3. \]

Recall that the support of $\chi_{t+k}^{2}(t - t_k)$ is included in $\{t - t_k \geq c_0/2\}$ and (4.26). Thus

\[ B^+_1 = 2\beta_k(\beta_k e^{4t_k})^{-1/3}(e^{2t}(\sigma(e') - \sigma(e')) u, \chi_t^2 u) \]

\[ \geq 2c_3\beta_k(\beta_k e^{4t_k})^{-1/3}(e^{2t}(1 - \sigma(e')) \chi_t^2 u, u). \]

(3.93)
For $B_2^+$ in (3.92) we get
\[
|B_2^+| = |2\beta_k(\beta_ke^{4t_k})^{-1/3}\text{Re}\langle D_k \rangle^{-2}\gamma u, \chi_+^{2}u| \\
\leq 2\beta_k(\beta_ke^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2.
\] (3.94)

For $B_3^+$ in (3.92) we have
\[
|B_3^+| = \left| (\beta_ke^{4t_k})^{-1/3}\left[ D_t^2, -i\chi_+^{2} \right] u, u \right| \\
\leq C(\beta_ke^{4t_k})^{-1/3}(\|u\|^2 + \|\partial_t u\|\|u\|) \\
\leq C\|D_t u\|^2 + C\|u\|^2.
\] (3.95)

We get from (3.92), (3.93), (3.94) and (3.95) that
\[
\begin{align*}
B^+ &\geq 2c_3\beta_k(\beta_ke^{4t_k})^{-1/3}(e^{2t}(1 - \sigma(e^t))\chi_+^{2}u, u) \\
&\quad - C\|u\|^2 - C\|D_t u\|^2 - 2\beta_k(\beta_ke^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2.
\end{align*}
\] (3.96)

Estimates for $2\text{Re}\langle \mathcal{L}_k u, m_{w-k} u \rangle$. Recall that $m_{w-k} = i(\beta_ke^{4t_k})^{-1/3}\chi_-(t + t_k)^2$.

\[
\begin{align*}
B^- := 2\text{Re}\langle \mathcal{L}_k u, m_{w-k} u \rangle &= 2\text{Re}\langle \mathcal{L}_k u, i(\beta_ke^{4t_k})^{-1/3}\chi_+^{2} u \rangle \\
&= 2\text{Re}\langle i\beta_ke^{2t}(\sigma(e^t) - \sigma(e^t_k))u, i(\beta_ke^{4t_k})^{-1/3}\chi_+^{2} u \rangle \\
&\quad - 2\text{Re}\langle i\beta_k\gamma(D_k)^{-2}\gamma u, i(\beta_ke^{4t_k})^{-1/3}\chi_+^{2} u \rangle \\
&\quad + 2\text{Re}\langle D_t^2 u, i(\beta_ke^{4t_k})^{-1/3}\chi_+^{2} u \rangle \\
&= B_1^- + B_2^- + B_3^-.
\end{align*}
\] (3.97)

Recall that the support of $\chi_-(t + t_k)$ is included in $\{t - t_k \leq -c_0/2\}$ and (4.26). Thus
\[
B_1^- = 2\beta_k(\beta_ke^{4t_k})^{-1/3}(e^{2t}(\sigma(e^t) - \sigma(e^t_k))u, \chi_+^{2} u) \\
\geq 2c_3\beta_k(\beta_ke^{4t_k})^{-1/3}(e^{2t}(1 - \sigma(e^t_k))\chi_+^{2}u, u). 
\] (3.98)

For $B_2^-$ in (3.97) we have
\[
|B_2^-| = |2\beta_k(\beta_ke^{4t_k})^{-1/3}\text{Re}\langle D_k \rangle^{-2}\gamma u, \chi_+^{2} u| \\
\leq 2\beta_k(\beta_ke^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2.
\] (3.99)

For $B_3^-$ in (3.97), we have
\[
|B_3^-| = \left| (\beta_ke^{4t_k})^{-1/3}\left[ D_t^2, i\chi_+^{2} \right] u, u \right| \\
\leq C(\beta_ke^{4t_k})^{-1/3}(\|u\|^2 + \|\partial_t u\|\|u\|) \\
\leq C\|u\|^2 + C\|D_t u\|^2.
\] (3.100)

We get from (3.97), (3.98), (3.99) and (3.100) that
\[
\begin{align*}
B^- &\geq 2c_3\beta_k(\beta_ke^{4t_k})^{-1/3}(e^{2t}(1 - \sigma(e^t_k))\chi_+^{2}u, u) \\
&\quad - C\|u\|^2 - C\|D_t u\|^2 - 2\beta_k(\beta_ke^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2.
\end{align*}
\] (3.101)
End of the proof of Proposition 3.20. By (3.91), (3.96), (3.101) and the definition of the proof of Proposition 3.20, we get

$$2 \text{Re}(\mathcal{L}u, M_k u) = A + B^+ + B^- \geq \frac{C_3}{2} (\beta_k e^{4t_k})^{2/3} \| \chi_0 u \|^2 + 2c_3\beta_k (\beta_k e^{4t_k})^{-1/3} (e^{2t_k} (1 - \sigma(e^t)) \chi^2_k u, u) + 2c_3\beta_k (\beta_k e^{4t_k})^{-1/3} (e^{2t} (1 - \sigma(e^t_k)) \chi^2_k u, u) - C \beta_k (\beta_k e^{4t_k})^{-1/3} k^{-2} \| u \|^2 - C \| D_t u \|^2 - C k^2 \| u \|^2 - C e^{2t} \| u \|^2$$

$$\geq \beta_k (\beta_k e^{4t_k})^{-1/3} \left( (c \rho(t, t_k) - C k^2 e^{4t_k} g(e^t))^2 \right) u, u.$$

where $\rho(t, t_k)$ is given in (3.80). This completes the proof of (3.79) in Proposition 3.20.

We have the following estimates for $\rho(t, t_k)$.

Lemma 3.21. There exist $C_{10}, C_{11} > 0$ such that for all $e^t < \epsilon_1$, $t \in \mathbb{R}$, $\alpha \geq \alpha_0$, $k \geq 1$,

$$\rho(t, t_k) \geq C_{10} e^{4t_k} g(e^t)^2,$$

(3.102)

$$\beta_k (\beta_k e^{4t_k})^{-1/3} \rho(t, t_k) + k^2 \geq C_{11} \beta_k^{1/2} e^{2t},$$

(3.103)

where $\rho$ is given in (3.80), $g$ is given in (3.2) and $\beta_k$ is given in (3.5).

Proof of Lemma 3.21. Suppose $e^t < \epsilon_1$, then $1 - \sigma(e^t_k) \geq \delta e^{2t_k}$. If $t$ is in the support of $\chi_0(-t_k)$, i.e. $|t - t_k| \leq c_0$, we have

$$e^{4t} g(e^t)^2 \leq e^{4t} \leq e^{4c_0} e^{4t_k},$$

$$\beta_k (\beta_k e^{4t_k})^{-1/3} e^{4t_k} + k^2 \geq \left( (\beta_k e^{4t_k})^{2/3} \right)^{3/4} (k^2)^{1/4} = \beta_k^{1/2} e^{2t_k} k^{1/2} \geq e^{-2c_0} \beta_k^{1/2} e^{2t}.$$

If $t$ is in the support of $\chi_-(\cdot - t_k)$, i.e. $t \leq t_k - c_0/2$, we have

$$e^{4t} g(e^t)^2 \leq e^{2t} e^{2t_k} \leq \delta^{-1} e^{2t} \left( 1 - \sigma(e^t_k) e^{-2t_k} \right),$$

$$\beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t_k} \left( 1 - \sigma(e^t_k) e^{-2t_k} \right) + k^2$$

$$\geq \left( \delta \beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t_k} + k^2 e^{-2t_k} \right) e^{2t}$$

$$\geq C \left( \beta_k^{2/3} e^{2t_k} \right)^{3/4} (k^2 e^{-2t_k})^{1/4} e^{2t} = C \beta_k^{1/2} k^{1/2} e^{2t}.$$

If $t$ is in the support of $\chi_+(\cdot - t_k)$, i.e. $t \geq t_k + c_0/2$, we have, by (4.13)

$$e^{4t} g(e^t)^2 \leq 8e^{2t} \left( 1 - \sigma(e^t) \right).$$

Suppose $t \geq t_k + c_0/2$, if $e^t \leq 2$, we have $1 - \sigma(e^t) \geq e^{2t/16}$ and then

$$\beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t_k} \left( 1 - \sigma(e^t) \right) + k^2 \geq \left( \beta_k^{2/3} e^{4t_k} / 16 + k^2 e^{-2t_k} \right) e^{2t}$$

$$\geq \left( \frac{1}{16} \beta_k^{2/3} e^{2t_k} \right)^{3/4} (k^2 e^{-2t_k})^{1/4} e^{2t} = \frac{1}{8} \beta_k^{1/2} k^{1/2} e^{2t}.$$
if \( e^t \geq 2 \), we have \( 1 - \sigma(e^t) \geq 1 - \sigma(2) = e^{-1} \) and then
\[
\beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t (1 - \sigma(e^t))} \geq e^{-1} \beta_k^{2/3} e^{-4t_k/3} e^{2t} \geq e^{-1} \epsilon_1^{-4/3} \beta_k^{2/3} e^{2t} \geq e^{-1} \epsilon_1^{-4/3} \beta_k^{1/2} e^{2t}.
\]
This completes the proof of (3.102), (3.103).

Proof of Theorem 3.17. The estimates (3.79) and (3.102) imply that there exists \( k_0 \geq 3 \), for all \( k \geq k_0 \),
\[
2 \text{Re} \langle \mathcal{L}_k u, M_k u \rangle \geq \frac{c}{2} \beta_k (\beta_k e^{4t_k})^{-1/3} (\rho(t, t_k) u, u)
\]
\[
- C\|D_t u\|^2 - Ck^2 \|u\|^2 - C\|e^{2t} u\|^2.
\]
Choosing \( C_{12} > 0 \) large enough, we have, using (3.34)
\[
2 \text{Re} \langle \mathcal{L}_k u, (C_{12} + M_k) u \rangle \geq \left( \beta_k (\beta_k e^{4t_k})^{-1/3} \rho(t, t_k) + D_t^2 + k^2 + e^{4t} \right) u, u).
\]
We deduce from (3.103) and (3.104) that for \( k \geq k_0 \),
\[
2 \text{Re} \langle \mathcal{L}_k u, (C_{12} + M_k) u \rangle \geq C \beta_k^{1/2} e^{2t} u, u).
\]
Using that \( e^t (C_{12} + M_k) e^{-t} \) is bounded on \( L^2(\mathbb{R}; dt) \) and Cauchy-Schwarz inequality, we get
\[
\|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|, \quad \forall k \geq k_0,
\]
completing the proof of Theorem 3.17. \( \square \)

3.4.2. Case 4: \( e^{t_k} \leq \beta_k^{-1/4} \). If \( e^{t_k} \leq \beta_k^{-1/4} \), we can get estimate by using the multipliers \( \text{Id} \) and \( i\text{Id} \).

Lemma 3.22. Suppose \( e^{t_k} \leq \beta_k^{-1/4} \). There exists \( C > 0 \) such that for all \( k \geq 1 \), \( \alpha \geq 8\pi \), \( u \in C_0^\infty(\mathbb{R}) \),
\[
(3.105) \quad \|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|
\]
where \( \mathcal{L}_k \) is given in (3.20) and \( \beta_k \) is given in (3.5).

Proof. At first note that
\[
\forall r > 0, \quad 1 - \sigma(r) \leq \frac{1}{8} r^2.
\]
We have for \( e^{t_k} \leq \beta_k^{-1/4} \),
\[
\text{Re} \langle \mathcal{L}_k u, -iu \rangle = \beta_k \langle e^{2t} (\sigma(e^{t_k}) - \sigma(e^t)) u, u \rangle + \beta_k \langle \gamma(D_k)^{-2} \gamma u, u \rangle
\]
\[
\geq \beta_k \langle e^{2t} \left( (1 - \sigma(e^t)) - (1 - \sigma(e^{t_k})) \right) u, u \rangle
\]
\[
\geq \beta_k \langle e^{2t} \left( (1 - \sigma(e^t)) - \frac{1}{8} \beta_k^{-1/2} \right) u, u \rangle,
\]
so that with (3.34) we get,
\[
\text{Re} \langle \mathcal{L}_k u, (1 - i) u \rangle \geq \left( k^2 e^{-2t} + \beta_k \left( 1 - \sigma(e^{t_k}) \right) - \frac{1}{8} \beta_k^{1/2} \right) e^{2t} u, u \rangle,
\]
\[
\geq e^{-1} \beta_k^{1/2}, \quad \text{by (3.17)}
\]
thus
\[
\text{Re} \langle \mathcal{L}_k u, (1 - i) u \rangle \geq \left( \frac{1}{c} - \frac{1}{8} \right) \beta_k^{1/2} (e^{2t} u, u).\]
By Cauchy-Schwarz inequality, we complete the proof of (3.105). \( \square \)
3.5. End of the proof of Theorem 2.2. Summarizing the estimates in Lemma 3.2, 3.3, Theorem 3.5, 3.14, 3.17 and Lemma 3.22, we have proved the estimate for the operator $L_k$ given in (3.13): There exist $C > 0$, $k_0 \geq 3$, $\alpha_0 \geq 1$ such that for all $|k| \geq k_0$, $\alpha \geq \alpha_0$, $u \in C_0^\infty(\mathbb{R})$,

\begin{equation}
\|e^{-tL_k}u\|_{L^2(\mathbb{R};dt)} \geq C|\beta_k|^{1/3}\|e^tu\|_{L^2(\mathbb{R};dt)},
\end{equation}

and an estimate of the same type for $\tilde{L}_k$ given in (3.9) (with different constants $C, \alpha_0$). This corresponds to the following estimate for the operator $\mathcal{H}_{\alpha,k,\lambda} = \mathcal{H}_k$ given in (3.4), (3.6) for $v \in C_0^\infty((0, +\infty))$, by the equivalence of (3.10) and (3.11)

\begin{equation}
\|\mathcal{H}_{k,\alpha,\lambda}v\|_{L^2(\mathbb{R}^+;vdr)} \geq C|\beta_k|^{1/3}\|v\|_{L^2(\mathbb{R}^+;vdr)},
\end{equation}

Then noticing (3.3), we get for $\omega = \sum_{|k| \geq k_0} \omega_k(r)e^{ik\theta} \in C_0^\infty(\mathbb{R}^2) \cap X_{k_0}$,

\begin{equation}
\|(\mathcal{H}_\alpha - i\lambda)\omega\|_{L^2(\mathbb{R}^2)}^2 = \sum_{|k| \geq k_0} \|\mathcal{H}_{k,\alpha,\lambda}\omega_k\|_{L^2(\mathbb{R}^+;vdr)}^2 \geq \sum_{|k| \geq k_0} C^2|\beta_k|^{2/3}\|\omega_k\|_{L^2(\mathbb{R}^+;vdr)}^2 = C^2\alpha^{2/3}\|D\omega\|_{L^2(\mathbb{R}^2)}^2,
\end{equation}

since $\|D\omega\|_{L^2(\mathbb{R}^2)} = \sum_{|k| \geq k_0} |k|^{1/3}\|\omega_k\|_{L^2(\mathbb{R}^+;vdr)}$.

Thus (2.3) is proved. Since $k_0 \geq 3$, we know that the imaginary axis does not intersect with the spectrum of $\mathcal{H}_\alpha$ viewed as an operator acting on $X_{k_0}$, which gives (2.4). The proof of Theorem 2.2 is complete.

4. Appendix

4.1. Weyl calculus. We present some facts about the Weyl calculus, which can be found in [12, Chapter 18] as well as in [15, Chapter 2]. The Weyl quantization associates to a symbol $a$ the operator $a^w$ defined by

\begin{equation}
(a^wu)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x+y/2, \xi) u(y) dy d\xi.
\end{equation}

Consider the symplectic space $\mathbb{R}^{2n}$ equipped with the symplectic form $\sigma = \sum_{i=1}^n d\xi^i \wedge dx^i$. Given a positive definite quadratic form $\Gamma$ on $\mathbb{R}^{2n}$, we define

\begin{equation}
\Gamma^\sigma(T) = \sup_{\Gamma(Y) = 1} \sigma(T, Y)^2,
\end{equation}

which is also a positive quadratic form. We say that $\Gamma$ is an admissible metric if there exist $C_0$, $C$, $\tilde{N}_0$ such that for all $X, Y \in \mathbb{R}^{2n}$,

\begin{equation}
\begin{cases}
\text{uncertainty principle:} & \Gamma_X \leq \Gamma_X^0, \\
\text{slowness:} & \Gamma_X(X-Y) \leq C_0^{-1} \implies (\Gamma_Y/\Gamma_X)^{\alpha_1} \leq C_0, \\
\text{temperance:} & \Gamma_X \leq \tilde{C}_0 \Gamma_Y(1 + \Gamma_X^\sigma(X-Y))^{\tilde{N}_0}.
\end{cases}
\end{equation}

$C_0, \tilde{C}_0, \tilde{N}_0$ in (4.2) are called structure constants of the metric $\Gamma$. An admissible weight is a positive function $m$ on the phase space $\mathbb{R}^{2n}$, such that there exist $C_0^*, \tilde{C}_0^*$,
\[ \tilde{N}_0 > 0 \] so that for all \( X, Y \in \mathbb{R}^{2n} \),

\[ \frac{\dot{N}_0}{N_0} \geq \frac{X}{Y} \geq \frac{C_0}{N_0} \implies \frac{m(Y) - m(X)}{m(X)} \leq C_0. \tag{4.3} \]

\( \tilde{C}_0, \tilde{N}_0, \tilde{N}_1 \) in (4.3) are called structure constants of the weight \( m \). In particular, the function defined by

\[ \lambda_T = \inf_{T, \sigma} \left( \frac{\Gamma(X)}{\Gamma(T)} \right)^{1/2} \]

is an admissible weight for \( \Gamma \) and its structure constants depend only on the structure constants of \( \Gamma \) (see [6]). The uncertainty principle is equivalent to \( \lambda_T \geq 1 \).

We prove the uniform admissibility of a special type of metrics, including those we have used in the proof, given in Definition [3,6].

**Lemma 4.1.** For \( \gamma \geq 1 \), the metric on the phase space \( \mathbb{R}^t \times \mathbb{R}^x \) given by

\[ \Gamma = |dt|^2 + \frac{|d\tau|^2}{\tau^2 + \gamma^2} \]

is admissible. Moreover, the structure constants of \( \Gamma \) defined in (4.2) are bounded above independently of \( \gamma \).

**Proof.** First we notice that

\[ \lambda_T = (\tau^2 + \gamma^2)^{1/2} \geq \gamma \geq 1, \]

so that \( \Gamma \) satisfies the uncertainty principle.

**Slowness.** It suffices to prove for \( X = (x, \xi), Y = (y, \eta) \), \( T = (t, \tau) \), \( \Gamma_X(X - Y) \leq s^2 \) implies \( \Gamma_Y \leq C_0 \Gamma_X \). Indeed, if \( \Gamma_X(X - Y) \leq s \) then \( \gamma \leq s^2(\xi^2 + \gamma^2) \), and we obtain

\[ \xi^2 \leq 2(\xi - \eta)^2 + 2\eta^2 \leq 2s^2(\xi^2 + \gamma^2) + 2\eta^2, \]

thus \( (1 - 2s^2)(\xi^2 + \gamma^2) \leq 2(\eta^2 + \gamma^2) \).

By choosing \( 0 < s < 1/\sqrt{2} \) and \( C_0 = 2(1 - 2s^2)^{-1} > 1 \), we get

\[ \xi^2 + \gamma^2 \leq C_0(\eta^2 + \gamma^2). \]

Then \( \Gamma_Y(T) = t^2 + \frac{\tau^2}{\eta^2 + \gamma^2} \leq t^2 + \frac{C_0\tau^2}{\xi^2 + \gamma^2} \leq C_0 \Gamma_X(T) \).

**Temperance.** We have

\[ \Gamma_X = (\xi^2 + \gamma^2)|dt|^2 + |d\tau|^2, \]

\[ \frac{\Gamma_X(T)}{\Gamma_Y(T)} \leq \max \left( 1, \frac{\eta^2 + \gamma^2}{\xi^2 + \gamma^2} \right). \]

If \( |\eta| \leq 2|\xi| \) or \( |\eta| \leq \gamma \), the right-hand side of the last inequality is bounded from above by 4. If \( |\eta| > 2|\xi| \) and \( |\eta| \geq \gamma \), then \( |\xi - \eta| \geq 1/2|\eta| \), which implies that \( \Gamma_X(X - Y) \geq (\xi - \eta)^2 \geq \frac{1}{4}\eta^2 \); on the other hand, we have

\[ \frac{\eta^2 + \gamma^2}{\xi^2 + \gamma^2} \leq \frac{\eta^2 + \gamma^2}{\gamma^2} = 1 + \gamma^{-2}\eta^2, \]

since \( \gamma \geq 1 \), we have

\[ \frac{\Gamma_X(T)}{\Gamma_Y(T)} \leq 1 + 4\Gamma_X(X - Y). \]
So the inequality $\Gamma_X(T)/\Gamma_Y(T) \leq 4(1 + \Gamma_X(X - Y))$ holds for any $X,Y,T$. As a result, we have proved that $\Gamma$ is admissible. From the proof above, we see that the structure constants are independent of $\gamma$, and this ends the proof of lemma.

The space of symbols $S(m, \Gamma)$ is defined as the set of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that the following semi-norms for all $k \in \mathbb{N}$

$$
(4.5) \quad \sup_{\Gamma_X(T_j) \leq 1} \left| a^{(k)}(X)(T_1, \ldots, T_k) \right|m(X)^{-1} < +\infty.
$$

The composition law is defined by $a^w b^w = (a^w b)^w$ and we have

$$
(4.6) \quad (a^w b)(X) = \exp \left( \frac{i}{2} \sigma(D_X, D_Y) \right) a(X) b(Y)_{Y=X}.
$$

For $a \in S(m_1, \Gamma)$, $b \in S(m_2, \Gamma)$, we have the asymptotic expansion

$$
(4.7) \quad (a^w b)(x, \xi) = \sum_{0 \leq k < N} w_k(a, b) + r_N(a, b),
$$

with $w_k(a, b) = 2^{-k} \sum_{|\alpha|+|\beta| = k} \frac{(-1)^{|eta|}}{\alpha! \beta!} D^\alpha_x \partial^\beta_x a D^\beta_x \partial^\alpha_x b \in S(m_1 m_2 \lambda_1^{-k}, \Gamma),$

$$
(4.9) \quad r_N(a, b)(X) = R_N(a(X) \otimes b(Y))_{Y=X} \in S(m_1 m_2 \lambda_1^{-N}, \Gamma),
$$

$$
(4.10) \quad R_N = \int_0^1 \left( 1 - \theta \right)^{N-1} \frac{(N-1)!}{2\pi} \left[ \partial_X, \partial_Y \right] d\theta \left( \frac{1}{2\pi} [\partial_X, \partial_Y] \right)^N.
$$

We use here the notation $D = i^{-1} \partial$. The $w_k(a, b)$ with $k$ even are symmetric in $a, b$ and skew-symmetric for $k$ odd. In particular, we have

$$
(4.11) \quad a^w b - b^w a = \frac{1}{k} \{ a, b \} + \tilde{r}, \quad \tilde{r} \in S(m_1 m_2 \lambda_1^{-3}, \Gamma),
$$

where $\{ , \}$ is the Poisson bracket, implying that $[a^w, b^w] = \frac{1}{k} \{ a, b \}^w + \tilde{r}^w$.

The symbols in $S(1, \Gamma)$ are quantified in bounded operators on $L^2(\mathbb{R}^n)$, with operator norm depending on the structure constants of $\Gamma$ defined in (4.2) and a semi-norm of the symbol in $S(1, \Gamma)$, whose order depends only on the dimension $n$ and the structure constants of $\Gamma$. See [6].

**Proposition 4.2** (Fefferman-Phong inequality). If $a \in S(\lambda_1^2, \Gamma)$ and $a \geq 0$, then $a^w$ is bounded from below by a constant depending on the structure constants of $\Gamma$ given in (4.2) and a semi-norm of the symbol $a$ in $S(\lambda_1^2, \Gamma)$, whose order depends only on the dimension $n$ and the structure constants of $\Gamma$.

4.2. For the operator $(k^2 + D_t^2)^{-1}$.

**Lemma 4.3.** For $k \geq 1$, we have

$$
(\tau^2 + k^2)^{-1} \in S((\tau^2 + k^2)^{-1}, \frac{|d\tau|^2}{\tau^2 + k^2}) \subset S(k^{-2}, \frac{|d\tau|^2}{\tau^2 + k^2}),
$$

with semi-norms bounded above independently of $k$. Moreover, the Fourier multiplier $(D_k)^{-2} = (k^2 + D_t^2)^{-1}$ is bounded on $L^2(\mathbb{R}; dt)$ with $L(L^2(\mathbb{R}, dt))$-norm bounded by $k^{-2}$.
Proof. We see that
\[
\frac{1}{\tau^2 + k^2} = \frac{k^{-2}}{(k^{-1} \tau)^2 + 1}.
\]
Then for any \( m \geq 0 \),
\[
\left| \frac{d^m}{d\tau^m} \left( \frac{1}{\tau^2 + k^2} \right) \right| \leq C_m k^{-2} ((k^{-1} \tau)^2 + 1)^{-1-m/2} k^{-m} = C_m (k^2 + \tau^2)^{-1-m/2},
\]
where \( C_m \) is a positive constant depending only on \( m \). This completes the proof of the lemma.

We can also compute the kernel of the operator \((k^2 + D_t^2)^{-1}\).

**Lemma 4.4.** For \( k \geq 1 \), we have
\[
\frac{1}{2k} \int_{\mathbb{R}} e^{-k|t|} e^{it\tau} dt = \frac{1}{k^2 + \tau^2}.
\]
As a consequence, \((D_k)^{-2}\) is just the convolution operator with the function \((2k)^{-1} e^{-k|\cdot|}\).

Proof. We have
\[
\frac{1}{2k} \int_{\mathbb{R}} e^{-k|t|} e^{it\tau} dt = \frac{1}{k} \int_0^{+\infty} e^{-kt} \cos(t\tau) dt = \frac{1}{k} \text{Re} \int_0^{+\infty} e^{-t(k+i\tau)} = \frac{1}{k^2 + \tau^2}.
\]

As a corollary of the lemma, the following result is used in the proof of Case 3.

**Lemma 4.5.** For \( k \geq 3 \), the operator \( e^{-2t} (D_k)^{-2} e^{2t} \) is bounded on \( L^2(\mathbb{R}; dt) \) with \( \mathcal{L}(L^2(\mathbb{R}; dt)) \)-norm bounded above by \( 3k^{-2} \).

Proof. We deduce from the previous lemma that the operator \( e^{-2t} (D_k)^{-2} e^{2t} \) has kernel
\[
T_k(t, s) = \frac{1}{2k} e^{-2t} e^{-k|t-s|} e^{2s} = \frac{1}{2k} e^{-k|t-s|-2(t-s)}.
\]
We have
\[
|T_k(t, s)| \leq \frac{1}{2k} e^{-(k-2)|t-s|}.
\]
If \( k \geq 3 \), then the convolution with \( \frac{1}{2k} e^{-(k-2)|\cdot|} \) is bounded on \( L^2(\mathbb{R}; dt) \) with norm
\[
\frac{1}{2k} \| e^{-(k-2)|\cdot|} \|_{L^1(\mathbb{R}; dt)} = \frac{1}{k(k - 2)},
\]
which is smaller than \( 3k^{-2} \) since \( k \geq 3 \). This completes the proof of the lemma.

4.3. **Some inequalities.** We present some inequalities that we have used in the proof. Recall the functions \( \sigma, g \) given in (3.2)
\[
\sigma(r) = \frac{1 - e^{-r^2/4}}{r^2/4}, \quad g(r) = e^{-r^2/8}, \quad r > 0.
\]
Firstly, a calculation shows that
\[
\inf_{\theta > 0} \theta^{-2}(e^\theta - 1) \simeq 1.54414\ldots
\]
so that
(4.12) \( \forall r > 0, \quad \delta r^2 g(r)^2 \leq \sigma(r), \) with \( \delta \simeq \frac{1}{4} \times 1.54414. \)

We verify easily
(4.13) \( \forall r > 0, \quad r^2 g(r) \leq 16 \sigma(r), \quad r^2 g(r)^2 \leq 8(1 - \sigma(r)). \)
We can get by induction on $n \in \mathbb{N}$ that
\begin{equation}
\sigma^{(n)}(r) = (-1)^n 4 r^{-n-2} ((n + 1)! - p_n(r) e^{-r^2/4}),
\end{equation}
where $p_n$ is a polynomial of degree $2n$. In particular, we have
\begin{equation}
\forall r > 0, \quad \sigma'(r) = -\frac{8}{r^4} (1 - e^{-r^2/4} - \frac{r^2}{4} e^{-r^2/4}) < 0,
\end{equation}
so that $\sigma$ is decreasing. We have the Taylor expansion of $\sigma$ near 0
\begin{equation}
\sigma(r) = 1 - \frac{1}{2} \cdot \frac{r^2}{4} + \frac{1}{3!} \left(\frac{r^2}{4}\right)^2 - \cdots + \frac{(-1)^l}{(l+1)!} \left(\frac{r^2}{4}\right)^l + O(r^{2l+2}), \quad \text{as } r \to 0.
\end{equation}

**Lemma 4.6.** For all $k \geq 1$, $\alpha \geq 8\pi$, $r > 0$, we have
\begin{equation}
\begin{cases}
r^2 + \beta_k \sigma(r) \geq (2 \log 2)^{-1} \beta_k^{1/2}, \\
\frac{k^2}{r^2} + \beta_k (1 - \sigma(r)) \geq e^{-1} \beta_k^{1/2},
\end{cases}
\end{equation}
where $\beta_k = \alpha k / (8\pi)$ is given in (3.5) and $\sigma(r)$ is given in (3.2).

**Proof.** By the definition of $\sigma$, we have
\begin{align*}
&\text{if } r \leq 2(\log 2)^{1/2}, \quad \text{then } \sigma(r) \geq \sigma(2(\log 2)^{1/2}) = (2 \log 2)^{-1}, \\
&\text{if } r > 2(\log 2)^{1/2}, \quad \text{then } e^{-r^2/4} < \frac{1}{2}, \text{ implying } \sigma(r) > 2r^{-2}.
\end{align*}
Therefore, we get
\begin{align*}
&\text{if } r \leq 2(\log 2)^{1/2}, \quad \text{then } r^2 + \beta_k \sigma(r) \geq (2 \log 2)^{-1} \beta_k, \\
&\text{if } r > 2(\log 2)^{1/2}, \quad \text{then } r^2 + \beta_k \sigma(r) \geq r^2 + 2\beta_k r^{-2} \geq 2\sqrt{2} \beta_k^{1/2},
\end{align*}
which implies for any $r \geq 0$,
\[r^2 + \beta_k \sigma(r) \geq \min \left( (2 \log 2)^{-1} \beta_k, 2\sqrt{2} \beta_k^{1/2} \right) \geq (2 \log 2)^{-1} \beta_k^{1/2}, \quad \text{if } \alpha \geq 8\pi,
\]
proving the first inequality in (4.17). Noting that
\[\forall 0 \leq \theta \leq 1, \quad e^{-\theta} - 1 + \theta \geq \frac{\theta^2}{4},
\]
we have
\[\forall 0 < r \leq 2, \quad 1 - \sigma(r) = \frac{4}{r^2} \left( e^{-r^2/4} - 1 + \frac{r^2}{4} \right) \geq \frac{r^2}{16}.
\]
Then we obtain
\begin{align*}
&\text{if } r \leq 2, \quad \frac{k^2}{r^2} + \beta_k (1 - \sigma(r)) \geq \frac{k^2}{r^2} + \beta_k \frac{r^2}{16} \geq \frac{1}{2} k \beta_k^{1/2}, \\
&\text{if } r > 2, \quad \frac{k^2}{r^2} + \beta_k (1 - \sigma(r)) \geq \beta_k (1 - \sigma(2)) = e^{-1} \beta_k.
\end{align*}
Therefore for any $r \geq 0$, $k \geq 1$,
\[\frac{k^2}{r^2} + \beta_k (1 - \sigma(r)) \geq \min \left( \frac{1}{2} k \beta_k^{1/2}, e^{-1} \beta_k \right) \geq e^{-1} \beta_k^{1/2}, \quad \text{if } \alpha \geq 8\pi,
\]
which proves the second inequality in (4.17). \(\square\)

Now we prove inequalities about the function $\sigma$ that are used in the proof of Case 1, 2 and 3.
Proposition 4.7. Given $\epsilon_0, \epsilon_1 \in (0, 1)$, there exist $c_0 \in (0, 1)$, $C_1, C_2, C_3, \tilde{C}_n > 0$, $c_1, c_2, c_3 \in (0, 1)$ satisfying the following. Recall that $\sigma$ is given in (3.2).

1. For $e^{t_k} > \epsilon_0^{-1}$, we have

\begin{equation}
\forall |t - t_k| \leq 2c_0, \quad \frac{d}{dt} \left[ e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] \leq -C_1,
\end{equation}

\begin{equation}
\forall |t - t_k| \leq 3c_0, \forall n \in \mathbb{N}, \quad \left| \frac{d^n}{dt^n} \left[ e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] \right| \leq \tilde{C}_n,
\end{equation}

\begin{equation}
\text{and } \quad \left\{ \begin{array}{ll}
\sigma(e^t) - \sigma(e^{t_k}) \leq -c_1 \sigma(e^{t_k}), & \text{for } t - t_k \geq c_0/2, \\
\sigma(e^t) - \sigma(e^{t_k}) \geq c_1 \sigma(e^t), & \text{for } t - t_k \leq -c_0/2.
\end{array} \right.
\end{equation}

2. For $\epsilon_1 \leq e^{t_k} \leq \epsilon_0^{-1}$, we have

\begin{equation}
\forall |t - t_k| \leq 2c_0, \quad \frac{d}{dt} \left[ e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] \leq -C_2,
\end{equation}

\begin{equation}
\forall |t - t_k| \leq 3c_0, \forall n \in \mathbb{N}, \quad \left| \frac{d^n}{dt^n} \left[ e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] \right| \leq \tilde{C}_n,
\end{equation}

\begin{equation}
\text{and } \quad \left\{ \begin{array}{ll}
\sigma(e^t) - \sigma(e^{t_k}) \leq -c_2 \sigma(e^{t_k}), & \text{for } t - t_k \geq c_0/2, \\
\sigma(e^t) - \sigma(e^{t_k}) \geq c_2 \sigma(e^t), & \text{for } t - t_k \leq -c_0/2.
\end{array} \right.
\end{equation}

3. For $e^{t_k} < \epsilon_1$, we have

\begin{equation}
\forall |t - t_k| \leq 2c_0, \quad \frac{d}{dt} \left[ e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] \leq -C_3 e^{4t_k},
\end{equation}

\begin{equation}
\forall |t - t_k| \leq 3c_0, \forall n \in \mathbb{N}, \quad \left| \frac{d^n}{dt^n} \left[ e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] \right| \leq \tilde{C}_n e^{4t_k},
\end{equation}

\begin{equation}
\text{and } \quad \left\{ \begin{array}{ll}
\sigma(e^t) - \sigma(e^{t_k}) \leq -c_3 (1 - \sigma(e^t)), & \text{for } t - t_k \geq c_0/2, \\
\sigma(e^t) - \sigma(e^{t_k}) \geq c_3 (1 - \sigma(e^t)), & \text{for } t - t_k \leq -c_0/2.
\end{array} \right.
\end{equation}

Proof. Step 1. The essential step is to choose $c_0 \in (0, 1)$ such that (4.18), (4.21) and (4.24) hold. By (4.15), given $\epsilon_0, \epsilon_1 \in (0, 1)$, there exist $\mu_1 > \mu_2 > 0$ such that

\begin{align*}
\forall r > e^{-2} \epsilon_0^{-1}, & \quad -\mu_1 r^{-3} \leq \sigma'(r) \leq -\mu_2 r^{-3}, \\
\forall r \in [e^{-2} \epsilon_1, e^2 \epsilon_0^{-1}], & \quad -\mu_1 \leq \sigma'(r) \leq -\mu_2,
\end{align*}

\begin{equation}
\forall r < e^2 \epsilon_1, \quad -\mu_1 r \leq \sigma'(r) \leq -\mu_2 r.
\end{equation}

Let us denote

\begin{displaymath}
f(t, t_k) := \frac{d}{dt} \left[ e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] = e^{2t} \sigma'(e^t) + 2e^{2t} (\sigma(e^t) - \sigma(e^{t_k})).
\end{displaymath}

The Taylor’s formula gives

\begin{displaymath}
\sigma(e^t) - \sigma(e^{t_k}) = \int_0^1 \sigma'(e^{t_k + \theta(t-t_k)}) e^{t_k + \theta(t-t_k)} (t - t_k) d\theta.
\end{displaymath}

Suppose $|t - t_k| \leq 2c_0$ with $c_0 < 1$. Using (4.27), we get the following
• if \( e^t > e_0^{-1} \), then \( e^t > e^{-2}e_0^{-1} \) and
\[
 f(t, t_k) \leq -\mu_2 + 4\mu_1e^{4c_0}c_0;
\]
• if \( e^t \in [e_1, e_0^{-1}] \), then \( e^t \in [e^{-2}e_1, e^2e_0^{-1}] \) and
\[
 f(t, t_k) \leq -\mu_2e^{3t} + 4\mu_1e^{3t}e^{2c_0}c_0 = -(\mu_2 - 4\mu_1e^{2c_0})e^{3t};
\]
• if \( e^t < e_1 \), then \( e^t < e^2e_1 \) and
\[
 f(t, t_k) \leq -\mu_2e^{4t} + 4\mu_1e^{4t}e^{4c_0}c_0 = -(\mu_2 - 4\mu_1e^{4c_0})e^{4t}.
\]

Let \( c_0 \in (0, 1) \) satisfying
\[
 4c_0e^{4c_0} \leq \frac{\mu_2}{2\mu_1},
\]
then we get (4.18), (4.21), (4.24) with
\[
 C_1 = \mu_2/2, \quad C_2 = \mu_2e^{-6e_1}/2 \quad \text{and} \quad C_3 = \mu_2e^{-8c_0}/2.
\]

**Step 2.** The inequalities (4.19), (4.22) and (4.25) are consequences of (4.14), noticing that for \( |t - t_k| \leq 3c_0 \),
\[
 |\sigma(e^t) - \sigma(e^{t_k})| \leq \begin{cases} 
 Ce^{-2t_k}, & \text{if } e^t > e_0^{-1}, \\
 C, & \text{if } e^t \in [e_1, e_0^{-1}], \\
 Ce^{2t_k}, & \text{if } e^t < e_1.
\end{cases}
\]

**Step 3.** It remains to prove (4.20), (4.23) and (4.26). Denoting \( r_k = e^{t_k} \) and \( r = e^t \), then (4.20), (4.23) are equivalent to the following
\[
 \forall r_k > e_0^{-1}, \quad \begin{cases} 
 \sigma(r) \leq (1 - c_1)\sigma(r_k), & \text{for } r \geq r_ke^{c_o/2}, \\
 \sigma(r_k) \leq (1 - c_1)\sigma(r), & \text{for } r \leq r_ke^{-c_o/2}.
\end{cases}
\]

(4.29)
\[
 \forall r_k \in [e_1, e_0^{-1}], \quad \begin{cases} 
 \sigma(r) \leq (1 - c_2)\sigma(r_k), & \text{for } r \geq r_ke^{c_o/2}, \\
 \sigma(r_k) \leq (1 - c_2)\sigma(r), & \text{for } r \leq r_ke^{-c_o/2}.
\end{cases}
\]

The function \( \sigma \) is decreasing, so that in order to prove (4.28) and (4.29), it suffices to prove the following
\[
 \exists c \in (0, 1), \forall r \geq e^{c_o/2}, \quad \sigma(re^{c_o/2}) \leq (1 - c)\sigma(r).
\]

We know that for any \( \lambda > 1 \), the function
\[
 [0, 1] \ni \theta \mapsto f_1(\theta; \lambda) = \frac{1 - \theta^\lambda}{\lambda(1 - \theta)}
\]
is strictly increasing in \((0, 1)\) and \( f_1(1; \lambda) = 1 \). Hence for all \( \lambda > 1 \), there exists \( 0 < \delta_1(\lambda) < 1 \) such that
\[
 \forall 0 < \theta \leq \exp(-\epsilon_1^2e^{c_o}/4), \quad f_1(\theta; \lambda) \leq \delta_1(\lambda).
\]

Then we have for \( r \geq e_1e^{c_o/2} \),
\[
 \frac{\sigma(re^{c_o/2})}{\sigma(r)} = \frac{1 - (e^{-r^2/4})^{c_o}}{e^{c_o}(1 - e^{-r^2/4})} = f_1(e^{-r^2/4}; e^{c_o}) \leq \delta_1(e^{c_o}), \quad \text{since } e^{c_o} > 1,
\]
which proves (4.30) with \( c = 1 - \delta_1(e^{c_o}) \). Thus (4.20) and (4.23) are proved.
Now we turn to prove (4.26), which is equivalent to the following
\begin{equation}
\forall r_k < \epsilon_1, \quad \begin{cases}
\sigma(r) - \sigma(r_k) \leq -c_3(1 - \sigma(r)), & \text{for } r \geq r_ke^{c_2/2}, \\
\sigma(r) - \sigma(r_k) \geq c_3(1 - \sigma(r_k)), & \text{for } r \leq r_ke^{-c_2/2}.
\end{cases}
\end{equation}

Since $1 - \sigma(r)$ is increasing, we need only to prove
\begin{equation}
\exists c_3 \in (0, 1), \forall r < \epsilon_1, \quad 1 - \sigma(r) \leq (1 - c_3)(1 - \sigma(re^{c_2/2})).
\end{equation}

By direct computation, we find that for any $\lambda > 1$, the function
\[
(0, +\infty) \ni x \mapsto f_2(x; \lambda) = \frac{\lambda(e^{-x} - 1 + x)}{e^{-\lambda x} - 1 + \lambda x}
\]
is continuous on $[0, +\infty)$, $f_2(0; \lambda) = \lambda^{-1}$ and $f_2(x; \lambda) < 1$ for all $x > 0$. Hence for any $\lambda > 1$, there exists $0 < \delta_2(\lambda) < 1$ such that
\[
f_2(x; \lambda) < \delta_2(\lambda), \quad \forall 0 < x < 1/4.
\]
We get for all $r < \epsilon_1 < 1$,
\[
\frac{1 - \sigma(r)}{1 - \sigma(re^{c_2/2})} = e^{c_2} \frac{e^{-r^2/4} - 1 + r^2/4}{e^{-c_2r^2/4} - 1 + e^{c_2r^2/4}} = f_2(r^2/4; e^{c_2}) < \delta_2(e^{c_2}),
\]
which proves (4.32) with $c_3 = 1 - \delta_2(e^{c_2})$ thus (4.26) is proved. The proof of Proposition 4.7 is now complete.\hfill \Box

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