Mean-field theory for Heisenberg zigzag ladder: Ground state energy and spontaneous symmetry breaking

Vagharsh V Mkhitaryan\textsuperscript{1,a} and Tigran A. Sedrakyan\textsuperscript{2,b}

\textsuperscript{a} Yerevan Physics Institute, Alikhanian Br. str. 2, Yerevan 36, Armenia and The Abdus Salam ICTP, Strada Costiera 11, Trieste 34014, Italy
\textsuperscript{b} Department of Physics, University of Utah, Salt Lake City, UT 84112

Dedicated to the memory of Daniel Arnaudon

Abstract

The spin-1/2 zig-zag Heisenberg ladder ($J_1 - J_2$ model) is considered. A new representation for the model is found and a saddle point approximation over the spin-liquid order parameter \( \langle \vec{\sigma}_{n-1}(\vec{\sigma}_n \times \vec{\sigma}_{n+1}) \rangle \) is performed. Corresponding effective action is derived and analytically analyzed. We observe the presence of phase transitions at values $J_2/J_1 = 0.231$ and $J_2/J_1 = 1/2$. 

\textsuperscript{1}e-mail: vgho@mail.yerphi.am
\textsuperscript{2}e-mail: tigrans@physics.utah.edu
1 Introduction

Unconventional spin-liquid phases in frustrated spin chains attracted notable theoretical and experimental interest in recent years. The natural question is whether the frustrations in antiferromagnetic Heisenberg chains can stabilize the new phases with exotic spin excitations observed in ladder systems. The model used to analyze the effect of frustration in antiferromagnetic spin chains is the so-called spin-1/2 $J_1 - J_2$ model with the Hamiltonian

$$H = J_1 \sum_n [\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - 1] + J_2 \sum_n [\vec{\sigma}_n \cdot \vec{\sigma}_{n+2} - 1],$$

(1)

where $\vec{\sigma}_n = 2\vec{S}_n$ are Pauli matrices. The bosonization analysis of this model was performed by Haldane [11], and the phase diagram has been studied intensively by various authors [2, 3, 4, 5, 6, 7] (for a review see also [8]).

The interest in this model is not purely theoretical. There are inorganic compounds in nature, such as $\text{Cs}_2\text{CuCl}_4$ [9], $\text{CuGeO}_3$ [10], $\text{LiV}_2\text{O}_5$ [11], or $\text{SrCuO}_2$ [12], which can be described by the spin-1/2 $J_1 - J_2$ chain Hamiltonian Eq. (1).

The investigation of the spin-1/2 $J_1 - J_2$ model’s phase diagram (1) starts from the weak coupling limit when $J_2 << J_1$. Classically, in this limit, the $J_1 - J_2$ chain has a long range Neel order. The excitations are massless spin-waves, frustrated by an irrelevant perturbation. At the values $J_2/J_1 > 1/4$, the spins in the ground state are arranged in a canted configuration, in which each spin makes a fixed angle $\alpha = \arccos[-J_1/4J_2]$ with its predecessor. The classical ground state of the model is doubly degenerate since the spin configurations can turn clockwise and counterclockwise with the same energy.

In quantum field theory, it is believed that at larger values, $J_2/J_1 > 1/4$, a phase transition of the Berezinskii-Kosterlitz-Thouless (BKT) type [13] occurs, which separates the gapless spin-1/2 Heisenberg phase from a fully massive region. This phase is characterized by the two-fold degenerate dimerized ground state and a spontaneous breaking of the lattice translation symmetry. Frustration stabilizes this gapful phase. The actual value for the ratio of the coupling constants at the transition point was found numerically in [2, 3, 4, 5, 6] to be slightly lower than $J_2/J_1 = 0.241$ due to quantum fluctuations.

At the larger value (when $J_2 = 1/2, J_1 > 0$) the model coincides with the Majumdar-Ghosh (MG) model [14, 15] and is exactly integrable [16, 17]. The existence of a mass gap in the MG model has been shown rigorously in [18]. The correlation function is found to be zero at distances larger than the lattice spacing. The ground state of the phase at $J_2/J_1 > 1/2$ is found to be a condensate of dimerized singlets of pairs of neighbor spins, which is $Z_2$ degenerate. This spontaneous breaking of the $Z_2$ discrete symmetry creates a kink type of topological excitation, the tails of which end in the different $Z_2$ vacua at $\pm\infty$. The spectrum of this excitation is massive, and the gap is decreasing at $J_2/J_1 \to \infty$, when the system becomes a pair of noninteracting spin-1/2 Heisenberg chains.

In the present article, we develop an approach to analyze the model on the matter of critical behavior, based on the idea that the middle phase, $1/4 < J_2/J_1 < 1/2$, can be characterized by the spin-liquid order parameter

$$\varphi = \langle \vec{\sigma}_{n-1} \cdot (\vec{\sigma}_n \times \vec{\sigma}_{n+1}) \rangle$$

(2)

defined on the triangles of the zig-zag chain. Performing a mean-field (saddle point) approximation, we reduce the model to an extended Heisenberg chain with topological term, which
appears to be integrable. Then, by use of the technique of Thermodynamic Bethe Ansatz, we calculate the effective action (formula (34)) as a function of the spin-liquid order parameter $\phi$.

The analysis of the effective action shows that, at the points $J_2/J_1 = 0.230971$ and $J_2/J_1 = 1/2$, we indeed have phase transitions, as it was expected.

2 New representation for the $J_1 - J_2$ model

For an alternative treatment of the Hamiltonian (1), we make use of the following identity for the square of the Hermitian operator $\chi_{abc} = \vec{\sigma}_a (\vec{\sigma}_b \times \vec{\sigma}_c)$ (scalar chirality operator)

$$(\vec{\sigma}_a (\vec{\sigma}_b \times \vec{\sigma}_c))^2 = -2 \left( [\vec{\sigma}_a \vec{\sigma}_b - 1] + [\vec{\sigma}_a \vec{\sigma}_c - 1] + [\vec{\sigma}_b \vec{\sigma}_c - 1] \right).$$ (3)

On the right hand side of this identity, one may recognize a sum of three different spin exchange terms, marked by mutually non-coinciding indices $a \neq b \neq c$. The form of Eq. (3) suggests a transformation for the $J_1 - J_2$ Hamiltonian, a dual representation of which would contain the square of the above mentioned scalar chirality operator. Thus, using Eq. (3), we map the expression (1) to

$$\tilde{H} = \sum_n [\vec{\sigma}_n \vec{\sigma}_{n+1} - 1] - \frac{g}{2} \sum_n \chi_{n,n+1,n+2}, \quad g = \frac{J_2}{J_1 - 2J_2},$$ (4)

were we have rescaled the Hamiltonian $H$ by the constant factor $(J_1 - 2J_2)$, i.e.,

$$\tilde{H} = \frac{H}{J_1 - 2J_2}.$$ (5)

This will be our starting point. We would like to emphasize however, that, to the best of our knowledge, the Heisenberg zigzag ladder ($J_1 - J_2$ model) has not been represented in this form in the literature previously.

3 Mean-field theory and its Bethe Ansatz solution

Our further analysis is close in spirit to that of Affleck and Marston, applied in Ref. [19] for the solution of the Heisenberg-Hubbard model. It is based on the approximation of the partition function of the given model by an exactly integrable one. Namely, we introduce a scalar field, $\phi_n$, by a Hubbard-Stratanovich transformation, which maps the Hamiltonian $\tilde{H}$ to $\mathcal{H}$, where

$$\mathcal{H} = \tilde{H} + \frac{g}{2} \sum_n (\phi_n - \chi_{n,n+1,n+2})^2 = \sum_n [\vec{\sigma}_n \vec{\sigma}_{n+1} - 1] - g \sum_n \phi_n \chi_{n,n+1,n+2} + \frac{g}{2} \sum_n \phi_n^2.$$ (6)

This map induces a constant factor in the partition function

$$Z = Tr \left( \exp - \beta \tilde{H} \right) = \text{const} \; Tr \left( \int \exp - \beta \mathcal{H}(\{\phi_n\}) \prod d\phi_n \right),$$ (7)

which however is irrelevant. The fact that the transformation (6) leaves the dynamics of the model unchanged may be shown using the coherent state path integral representation for the
partition function, where the functional integral over the field \( \{ \phi_n \} \) can be exactly evaluated. Below, we will investigate the zero temperature limit, \( \beta \to \infty \), of the integral on the right hand side of Eq. (7) in the saddle point approximation. We are going to analyze the mean-field theory corresponding to a certain saddle point, which we believe gives the main contribution. The question of the existence of any other saddle points, however, will be left for future investigations. More precisely, we consider the saddle point equation

\[
\frac{\partial H}{\partial \phi_n} = 0.
\]

The solution of Eq. (8) with regard to the bosonic fields \( \{ \phi_n \} \) can be obtained by the substitution of Eq. (6) into Eq. (8). It has the form of a set of \( N \) (the lattice size) coupled equations

\[
\phi_n = \text{Tr} \left( \chi_{n,n+1,n+2} e^{-\beta H(\{ \phi_n \})} \right) \equiv \langle \chi_{n,n+1,n+2} \rangle,
\]

where \( n = 1 \ldots N \), and we have cyclic boundary conditions. It would be reasonable to restrict ourselves by some (quasi) translational invariant, homogeneous saddle points. Therefore, we consider the solution where the operators \( \chi_{n,n+1,n+2} \), for all \( n = 1 \ldots N \), have the same mean value. Then the set of coupled equations (9) simplifies, and for all values of \( n \) acquires the form

\[
\langle \chi_{n,n+1,n+2} \rangle = \varphi.
\]

Thus, in this way, the original problem reduces to the eigenfunction problem for the mean-field Hamiltonian

\[
\mathcal{H}_M = \sum_n \left[ \sigma_n \sigma_{n+1} - 1 \right] - \alpha \sum_n \sigma_n (\sigma_{n+1} \times \sigma_{n+2}), \quad \alpha = g \varphi.
\]

This model appears to be exactly solvable by means of Bethe Ansatz. This is because the second term in expression (11) commutes with the first term, which, in turn, is the Heisenberg Hamiltonian. Therefore, skipping the demonstration of the standard technique of Algebraic Bethe Ansatz (since it repeats the one for the XXZ model [17]), we present here only the solution. The eigenvectors can be parameterized through the set of parameters (rapidities), \( \{ x_i \} \), which satisfy the set of Bethe equations

\[
\left( \frac{x_j - i}{x_j + i} \right)^N = -\prod_{k=1}^M \frac{x_j - x_k - 2i}{x_j - x_k + 2i}.
\]

The corresponding state has a total spin projection \( S_z = N - M \) and energy

\[
E(x_1, \ldots, x_M) = -\sum_{j=1}^M \left( 1 + 2\alpha \partial_{x_j} \right) \frac{8}{x_j^2 + 1}.
\]
4 Free energy

In the present section, we present our calculations of the ground state energy of the $J_1 - J_2$ model in the mean-field approximation. Namely, we calculate the free energy in the model (1). The model (1) has been studied in Ref. [20] for a fixed value of the parameter $\alpha$. Here, we present our exact analytical calculations of the free energy of the mean-field model (11), and analyze, in detail, the free energy as a function of the parameter $\alpha$. The calculation is based on the method of Thermodynamic Bethe Ansatz, introduced in [21] (for details see also [22]). By definition, the thermodynamic limit is given by the following conditions

$$ N \to \infty, \quad M \to \infty, \quad \frac{M}{N} = \text{const}. \tag{14} $$

In the thermodynamic limit (14), the Bethe equations (12) become integral equations. In order to represent these integral equations in a convenient form, we introduce the densities $\rho_n(x)$, defined as

$$ \rho_n(x) = \left. \frac{dx}{dt} \right|_{t=t(x)} - 1. $$

Then the equations (12) reduce to the following system of $N$ integral equations for the densities

$$ a_n(x) = \rho_n(x) + \tilde{\rho}_n(x) + \sum_k T_{jk} \ast \rho_k(x). \tag{15} $$

Here we introduced the notations

$$ a_n(x) = \frac{1}{\pi} \frac{n}{x^2 + n^2}, \tag{16} $$

for $n = 1 \ldots N$. The functions $\rho_n(x)$ and $\tilde{\rho}_n(x)$, which are unknown, denote particle and hole densities, respectively. The index $n$ represents their correspondence to $n$-strings. The convolution operation, " $\ast$ ", is defined as

$$ f \ast g (x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy, \tag{17} $$

for any given pair of functions $f$ and $g$. This is the conventional definition. The functions $T_{nm}(x)$ for $n, \ m = 1 \ldots N$, involve the expressions $a_n(x)$ in their definition. They have the following form

$$ T_{nm}(x) \equiv \begin{cases} a_{|n-m|}(x) + 2a_{|n-m|+2}(x) + 2a_{|n-m|+4}(x) + \ldots + 2a_{n+m-2}(x) + a_{n+m}(x) & \text{for } n \neq m, \\ 2a_2(x) + 2a_4(x) + \ldots + 2a_{2n-2}(x) + a_{2n}(x) & \text{for } n = m. \end{cases} \tag{18} $$

As we have already mentioned, the equations (15) represent the thermodynamic limit of the Bethe Equations. These are integral equations with respect to particle and hole densities, containing all of the information about the energy spectrum. Suppose that the system is in a state characterized by densities $\rho_j(x)$ and $\tilde{\rho}_j(x)$. Then the equilibrium dynamics of the system at temperature $T$ can be extracted by minimizing the free energy, $F = E - TS$, with respect to the independent $\rho_j$. This yields the following non-linear integral equations for functions $\eta_n(x) = \tilde{\rho}_n(x)/\rho_n(x)$,
\[
\ln \eta_n = \frac{g_n}{T} + \sum_{k=1}^{\infty} T_n k^* \ln(1 + \eta_k^{-1}), \quad g_n = -8\pi (1 + 2\alpha \partial_x) a_n.
\] (19)

In order to analyze the ground state energy of the mean-field model Eq. (11) which is under our current consideration, we need to go to the zero temperature limit in Eq. (19). For this purpose, let us introduce a set of new functions, \(\epsilon_n(x)\), as \(\eta_n(x) = \exp\{\epsilon_n(x)/T\}\), \(n = 1 \ldots N\), and substitute them into Eq. (19). Then, in the zero temperature limit, \(T \to 0\), Eq. (19) acquires the following form

\[
\epsilon_1(x) = -8\pi (1 + 2\alpha \partial_x) s(x) + s * \epsilon_2^\dagger(x),
\] (20)

\[
\epsilon_n(x) = s * (\epsilon_{n-1} + \epsilon_{n+1})(x), \quad n \geq 2,
\]

where the function \(s(x)\) is defined as \(s(x) = [4 \cosh(\pi x/2)]^{-1}\). The action of the dagger (minus), \(^\dagger (-)\), in Eq. (20), leaves only the positive (negative) part of the corresponding function, \(\epsilon_n(x)\), as

\[
\epsilon_n^\dagger(x) = \begin{cases} 
\epsilon_n(x) & \text{if } \epsilon_n(x) \geq 0 \\
0 & \text{if } \epsilon_n(x) < 0,
\end{cases}
\]

\[
\epsilon_n^-(x) = \epsilon_n(x) - \epsilon_n^+(x).
\] (21)

By definition, all \(\epsilon_n(x)\) are measured in units of \(kT\) (where we set \(k = 1\)), and therefore have magnitudes of energy. Equations (20) unambiguously define the solutions for functions \(\epsilon_n(x)\) provided that

\[
\lim_{n \to \infty} \frac{\epsilon_n(x)}{n} = 2B,
\] (22)

where \(B\) is the "magnetic field", which in our case, Eq. (11), is zero. It is transparent from Eq. (20), that \(\epsilon_n(x) > 0\) for \(n = 2, 3, \ldots\), and only the function \(\epsilon_1(x)\) can be positive, as well as negative (can change its sign crossing the \(x\) axis at a certain point). The solution of the system (20) can be then expressed in terms of \(\epsilon_1(x)\) as

\[
\epsilon_n(x) = \epsilon_n^\dagger(x) = a_{n-1} * \epsilon_1^\dagger(x) + 2(n - 1)B, \quad n = 2, 3, \ldots
\]

\[
\epsilon_1(x) = -8\pi (1 + 2\alpha \partial_x) s(x) + \int_{\epsilon_1 > 0} (s * a_1)(x-y)\epsilon_1(y)dy.
\]

From the last equation we see that if there exists such a point, \(x = a\), where the function \(\epsilon_1(x)\) changes its sign (and, therefore, \(\epsilon_1(a) = 0\)), then \(\pi \alpha > 1\). Thus, one arrives at a Wiener–Hopf type integral equation for the unknown function \(\epsilon_1(x)\)

\[
\epsilon_1(x) = \epsilon_0(x) + \int_{y \geq a} R(x-y)\epsilon_1(y)dy,
\] (23)
where
\[
\epsilon_0(x) = -8\pi(1 + 2\alpha\partial_x) s(x), \quad R(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega x}}{2 \cosh \omega}.
\]

The same kind of non-linear integral equation, occurring when the so called disturbance term, \(\epsilon_0(x)\), is not an even function of \(x\) and changes its sign, appears in the solutions of staggered zigzag ladders with broken one-step translation symmetry \cite{23, 24}. Therefore, according to our experience drawn from the previous works, we assume that \(\epsilon_1(x) = 0\) for \(x < a\). Then, Eq. (23) will be valid for \(x \geq a\). This assumption does not affect the right hand side of Eq. (23) and, without loss of generality, gives the same solution. This solution can be obtained following the standard steps of the technique of Wiener–Hopf integral equations. First, we apply a Fourier transform
\[
f(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega x} \tilde{f}(\omega), \quad \tilde{f}(\omega) = \int_{-\infty}^{\infty} dx \, e^{i\omega x} f(x),
\]
to the functions \(\epsilon_0(x), \epsilon_1(x)\) and \(R(x)\). The substitution of these functions in the form of Fourier integrals into Eq. (23) yields
\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} \left\{ [1 - \tilde{R}(\omega)] \tilde{\epsilon}_1(\omega) - \tilde{\epsilon}_0(\omega) \right\} = 0, \quad x \geq a.
\] (24)

The equation (24) can be rewritten in an equivalent form, as
\[
[1 - \tilde{R}(\omega)] \tilde{\epsilon}_1(\omega) - \tilde{\epsilon}_0(\omega) = e^{i\omega a} h_-(\omega),
\] (25)

where \(h_\pm(\omega)\) are the boundary values of analytic functions which do not have poles in the upper (\(\Pi_+\)) and lower (\(\Pi_-\)) complex half-planes respectively, and have zero limiting values at corresponding infinite points. Hence, with our assumption, we will have \(\tilde{\epsilon}_1(\omega) = e^{i\omega a} \tilde{\epsilon}_-\). The kernel in Eq. (24) can be factorized. It is precisely this factorization property of the kernel which is responsible for the solvability of Eq. (24). This means that the kernel can be represented as a product
\[
[G_+^{\star}(\omega)G_-^{\star}(\omega)]^{-1} \equiv 1 - \tilde{R}(\omega) = \frac{e^{i\omega a}}{2 \cosh[\omega]},
\] (26)

where \(G_\pm(\omega)\) are the boundary values of the analytic functions which do not have zeroes or poles on \(\Pi_\pm\), respectively, and have the property \(G_+^{\star}(\infty) = G_-^{\star}(\infty) = 1\). Then, for the Fourier components \(\tilde{\epsilon}_1(\omega)\) in Eq. (23), the solution, when \(x \geq a\), will be
\[
\tilde{\epsilon}_+(\omega) = G_+^{\star}(\omega) \tilde{P}_+ [G_-^{\star}(\omega)e^{-i\omega a} \tilde{\epsilon}_0(\omega)].
\] (27)

Here, the operators \(P_\pm\) are projectors, defined as \(P_\pm[f(\omega)] = f_\pm(\omega)\), for any given function \(f(x)\). For example, the action of the projector \(P_+\) on the sum of the Fourier components of the function \(f(x)\) and a constant \(c\), yields
\[
P_+ \left[ c + \int_{-\infty}^{+\infty} dx \, e^{i\omega x} f(x) \right] = c + \int_{0}^{+\infty} dx \, e^{i\omega x} f(x).
\] (28)
The complex functions $G_+(\omega)$ and $G_-(\omega)$, from the factorization equation (26), can be calculated exactly. They have the following algebraic forms

$$G_-(\omega) = \sqrt{2\pi} \exp\{ -i\omega \pi + i\omega \ln(i\omega\pi) \} \Gamma\left(\frac{1}{2} + i\omega\pi \right), \quad G_+(\omega) = G_-(\omega).$$

Now one can derive the explicit form of the solution (27) for $a > 0$. In order to do this, one just has to make use of the above mentioned property of projectors $P_+$ and $P_-$, given by Eq. (28). Namely, upon application of Eq. (28) to the expression in brackets in the right hand side of Eq. (27), one will express the solution Eq. (27) in the form of an infinite sum

$$\tilde{\epsilon}_+(\omega) = -i4\pi G_+(\omega) \sum_{k=0}^{\infty} (-1)^k [1 - 2\pi \alpha(k + 1/2)] e^{-\pi\alpha(k+1/2)} G_-(-i\pi(k + 1/2)).$$

The first two terms of the sum in this equation have been obtained in Ref. [20]. Now, in order to calculate the free energy in the mean-field model Eq. (11), we need to find the parameter $a = a(\alpha)$, defined by the condition $\epsilon_1(a) = 0$. This condition can be rewritten as

$$0 = \epsilon_1(a) = \int \frac{d\omega}{2\pi} \tilde{\epsilon}_+(\omega) \simeq \frac{i}{2} \lim_{|\omega| \rightarrow \infty} \omega \tilde{\epsilon}_+(\omega).$$

Substituting the solution Eq. (30) for $\tilde{\epsilon}_+(\omega)$ into Eq. (31), we will represent Eq. (31) in an equivalent form as

$$\sum_{k=0}^{\infty} (-1)^k [1 - 2\pi \alpha(k + 1/2)] e^{-\pi\alpha(k+1/2)} G_-(-i\pi(k + 1/2)) = 0.$$

The free energy per site at $T = 0$, which is the ground state energy, will explicitly depend on $a$. From the definition of the free energy, we have

$$F(\alpha) = -(4\log 2 - 1) - \int_{x \in L_+} s(x) \epsilon_1(x) dx = -(4\log 2 - 1) - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega a} \tilde{\epsilon}_+(\omega)s_+(-\omega).$$
Figure 2: Vacuum energy $E_0$ versus parameter $\varphi$ for different values of $g$.

Here, the integration range of the first integral, $L_+$, is given by those values of $x$, where $\epsilon_1(x)$ is positive. Thus, substituting Eq. (30) into Eq. (31), we get the exact ground state energy of the model Eq. (11) as

$$F(\alpha) = 4 - 4 \log 2 + 2 \sum_{n,k=0}^{\infty} (-1)^{k+n}[1 - 2\pi\alpha(k + 1/2)]$$

$$\times e^{-\pi\alpha(k+n+1)}G_+(i\pi(n + 1/2))G_-(-i\pi(k + 1/2)) \frac{k + n + 1}{k + n + 1}.$$ 

5 Effective action and phase transitions

The expression (34), for the energy per site, can be considered as an effective action for the order parameter $\varphi$, defined by Eq. (2). The explicit form of the effective action, Eq. (34), allows for further investigations; in particular, with regard to the matter of critical behavior, one can analyze in details the saddle point equation corresponding to our mean-field theory. In terms of
the parameter $\alpha$, the saddle point equation \((9)\) reads

$$\partial_{\alpha} F(\alpha) = -\alpha / g, \quad \text{or, if } \alpha \neq 0, \quad -\frac{\partial_{\alpha} F(\alpha)}{\alpha} = 1 / g.$$ \hspace{1cm} (35)

For any given $g$, this equation always has a zero solution $\alpha = 0$. In order to find a non-zero solution, we performed a numerical evaluation of the function $-\partial_{\alpha} F(\alpha) / \alpha$ (where the analytical form of $F(\alpha)$ is given by Eq \((34)\)), with the condition given by Eq. \((32)\). The plot is presented in Fig.\[1\] From this picture, one can conclude that Eq. \((35)\) has a solution when and only when $g$ exceeds the critical value $g_c$, where $1 / g_c$ equals to the maximal value of $-\partial_{\alpha} f(\alpha) / \alpha$. For that value, our calculations give $g_c = 0.428646$. Then, from Eq. \((4)\), one can find the corresponding critical ratio

$$(J_2/J_1)_c = \frac{g_c}{1 + 2g_c} = 0.230791,$$

which is in a good agreement with the expected value. It is also interesting to investigate the behavior of the effective potential, Eq. \((34)\) versus the order parameter $\varphi$ for different values of $g$. The plots are presented in Fig.\[2\]. When $g$ is less than $g_c$, there is only one vacuum energy minimum at $\varphi = 0$ while, for $g \geq g_c$, two new minima appear. There exists another value of $g$, which we mark as $g_{c2}$ ($g_{c2} = 0.555083$, $[J_2/J_1]_{c2} = 0.263052$), which occur when the magnitudes of three vacuum energy minima are the same. At this point we do not have an additional phase transition, since the order parameter, $\varphi$, is smooth and finite. However, it would be interesting to understand the reason and consequences of such behavior.

Two non-zero minima become infinitely deeper and the positions of minima approach to zero from both, left and right hand sides, as $g$ further goes up to $+\infty$. This scenario corresponds to the Majumdar-Ghosh limit, $J_2/J_1 \rightarrow 1/2$, suggesting the next phase transition where the chiral order parameter $\varphi$ vanishes and another, fully dimerised phase appears. Thus, our description of the intermediate phase with non-zero chiral order parameter $\varphi$ complements the understanding of the fully dimerised phase for $J_2/J_1 \geq 1/2$.

**Concluding remarks.** We have derived an effective action for the spin-1/2 $J_1 - J_2$ model as a function of the spin-liquid order parameter $\varphi = \langle \vec{\sigma}_{n-1} \times \vec{\sigma}_{n+1} \rangle$, and have observed the presence of two phase transitions at points close to the expected values: (i) when $(J_2/J_1) < 0.230791$, we have an ordinary critical phase of isotropic Heisenberg model; (ii) in the middle phase, when $0.230791 < (J_2/J_1) < 1/2$, the ground state is $Z_2$ degenerate with two signs of order parameter. Due to this degeneracy, kink-like topological excitations are present and their condensation may characterize the third phase at $J_2/J_1 > 1/2$. Though the described picture do not coincide, but at the same time is not in contradiction with the well known description of this phase in the thermodynamic limit, when the two states, one with wave vector $k = 0$ (ground state for finite system) and another with $k = \pi$ (first exited state for finite system), collapse to each other and give rise to the dimerization pattern (two-fold degeneracy) and the breaking of one-step translational invariance.

In our opinion, the developed approach based on the chiral order parameter $\varphi$, is alternative to the known methods for description of the intermediate phase and provides promising possibility to investigate this important problem further.
6 Acknowledgment

The authors are indebted to A. G. Sedrakyan for support and illuminating discussions. It is a pleasure to thank A.A. Nersesyan and A.B. Zamolodchikov for very useful discussions. V.M. acknowledges INTAS grants 03-51-5460, YSF 05-109-5041, and Volkswagen Foundation of Germany for financial support.

References

[1] F. D. M. Haldane, Phys. Rev. B 25, 4925 (1982).
[2] R. Julien and F. D. M. Haldane, Bull. Am. Phys. Soc 28, 34 (1983).
[3] S. Eggert, Phys. Rev. B 54, R9612 (1996).
[4] K. Okamoto and K. Nomura, Phys. Lett. A 169, 433 (1992).
[5] R. Chitra, S. Pati, H. R. Krishnamurthy, D. Sen, and S. Ramasesha, Phys. Rev. B 52, 6581 (1995).
[6] S. White and I. Affleck, Phys. Rev. B 54, 9862 (1996).
[7] M. Zarea, M. Fabrizio and A. A. Nersesyan, Eur. Phys. J. B 39, 155 (2004).
[8] P. Lecheminant, Frustrated spin systems, edited by H. T. Diep, World-Scientific (2003).
[9] R. Coldea, D. A. Tennant, R. A. Cowley, D. F. McMorrow, B. Dorner and Z. Tylczynski, Phys. Rev. Lett. 79, 151 (1997).
[10] M. Hase, I. Terasaki and K. Uchinokura, Phys. Rev. Lett. 70, 3651 (1993).
[11] N. Fujiwara, H. Yasouka, M. Isobe, Y. Ueda and S. Maegawa, Phys. Rev. B 55, R11945 (1997).
[12] M. Matsuda and K. Katsumata, J. Mag. Mag. Mat. 140-144, 1671 (1995), M. Matsuda, K. Katsumata, K. M. Kojima et.al., Phys. Rev. B 55, R11953 (1997).
[13] V. L. Berezinskii, Sov. Phys. JETP 34, 610 (1972); J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973); J. M. Kosterlitz, J. Phys. C 7, 1046 (1974).
[14] C. K. Majumdar and D. K. Ghosh, J. Math. Phys. 10, 1388 (1969).
[15] B. S. Shastry and B. Sutherland, Phys. Rev. Lett. 47, 964 (1981).
[16] H. A. Bethe, Z. Physik 71, 205 (1931).
[17] L. D. Faddeev and L. Takhtajian, Zap. Nauch. Semin. LOMI, v. 109, 134 (1981).
[18] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. 59, 799 (1987); I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. 115, 477 (1988).
[19] I. Affleck and J. B. Marston, Phys. Rev. B 37, 3774 (1988).

[20] A. M. Tsvelik, Phys. Rev. B 42, 779 (1990).

[21] C. N. Yang and C. P. Yang, Phys. Rev. 150, 321 (1966);  
    C. N. Yang and C. P. Yang, Phys. Rev. 150, 327 (1966).

[22] M. Takahashi and M. Suzuki, Pro. Theor. Phys. 46, 2187 (1972).

[23] V. V. Mkhitaryan and A. G. Sedrakyan, Nucl. Phys. B 673, 455 (2003);  
    D. Arnaudon, R. Poghossian, A. Sedrakyan, P. Sorba, Nucl. Phys. B 588, 638 (2000);

[24] D. Arnaudon, A. Sedrakyan, T. Sedrakyan, Nucl. Phys. B 676, 615 (2004);  
    D. Arnaudon, A. Sedrakyan, T. Sedrakyan, Int. J. Mod. Phys. A 19, S2 16 (2004).