A differentiable measure of pointwise shared information

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Abstract

Partial information decomposition (PID) of the multivariate mutual information describes the distinct ways in which a set of source variables contains information about a target variable. The groundbreaking work of Williams and Beer has shown that this decomposition cannot be determined from classic information theory without making additional assumptions, and several candidate measures have been proposed, often drawing on principles from related fields such as decision theory. None of these measures is differentiable with respect to the underlying probability mass function. We here present a novel measure that draws only on principles linking the local mutual information to exclusion of probability mass. This principle is foundational to the original definition of the mutual information by Fano. We reuse this principle to define measure of shared information that is differentiable and is well-defined for individual realizations of the random variables. We show that the measure can be interpreted as a local mutual information with the help of an auxiliary variable. We also show that it has a meaningful Moebius inversion on a redundancy lattice and obeys a target chain rule. We give an operational interpretation of the measure based on the decisions an agent should take if given only the shared information.

Keywords: pointwise information theory; mutual information; partial information decomposition; multivariate statistical dependency; synergy; redundancy; unique information; redundant information; neural networks

1 Introduction

What are the distinct ways in which a set of source variables may contain information about a target variable? How much information do input variables provide uniquely about the output, such that this information about the output variable cannot be obtained by any other input variable, or collections thereof. How much information is provided in a shared way, i.e. redundantly, by multiple input variables, or even multiple collections of these, about the output, and how much information about the output can only be obtained by considering many or all input variables together? Answering questions of this nature is the scope of partial information decomposition (PID). The groundbreaking study of Williams and Beer [18] provided first insights by establishing that information theory is lacking axioms to answer these questions, which as a consequence have to be chosen in a way that satisfies our intuition about how these question should be answered (at least in simple corner cases). However, further studies by [1, 8] quickly revealed that not all intuitively desirable properties, like positivity, zero redundant information for statistically independent input, a chain rule for composite output variables, etc. were compatible, and the initial measure proposed by Williams and Beer was rejected on the grounds of not fulfilling certain desiderata favored in the community. Nevertheless, the work of Williams and Beer clarified that indeed an axiomatic approach is necessary and also highlighted the possibility that the higher
order terms (or questions) that arose when considering more than two input variables could be elegantly organized into contributions on the lattice of antichains (see more below).

Subsequently, multiple PID frameworks have been proposed, and each of them has merit in the application case indicated by its operational interpretation (Bertschinger and colleagues for example, justify their measure of unique information in a decision theoretic setting [2]). For the setting of interest to us, i.e. distributed computation, learning neural networks and problems from the domain of physics, however, all measures lacked the property to be well defined on individual realizations of inputs and outputs (localizability), continuity and differentiability in the underlying joint probability distribution. While the first two properties have very recently been provided by the pointwise partial information decomposition (PPID) of Finn and Lizier [5], differentiability is still missing, as is the extension of most measures to continuous variables. Differentiability, however, seems pivotal to exploit PID measures for learning in neural networks as suggested for example in [17], and also in physics problems.

Here, we therefore set out to enhance the definition of Finn and Lizier [5] in order to define a novel PID measure of shared mutual information that is localizable and also differentiable (and that does not have obvious limitations with respect to a future extension to continuous variables). In doing so, we also aim for a measure that adheres as closely as possible to the original definition of (local) mutual information based on the exclusion of probability mass [4] — in the hope that our measure will inherit most of the operational interpretation of a local mutual information. We also seek to avoid invoking assumptions or desiderata from outside the scope of information theory, e.g. we explicitly seek to avoid invoking desiderata from decision or game theory. We do so in the attempt to keep the interpretation of our PID measures as close to measures of mutual information as possible. We note that adhering as closely as possible to information theoretic concepts should simplify finding localizable and differentiable measures.

These goals suggest that we have to abandon positivity for the parts (or ‘atoms’ [18]) of the decomposition, simply because the local MI can be already negative. With respect to a negative shared information in the PID we aim to preserve the interpretation of negative terms as being misinformative, in the sense that obtaining the negative information will make a rational agent more likely to make the wrong prediction about the value of a target variable. Our goals also strongly suggest to avoid computing the minimum (or maximum) of multiple information expressions anywhere in the definition of the measure. This is because taking a minimum or maximum would almost certainly collide with differentiability and also a later extension to continuous variables.

1.1 Glossary and Notation

Throughout the text, we will use the following notations:

- $(\Omega, \mathcal{A}, \mathbb{P})$ for the underlying probability space;
- $\mathcal{A}_R$, for the alphabet of any random variable $R$;
- $S_1, ..., S_n, T$ for the source and target random variables on $\Omega$;
- $s_1, ..., s_n, t$ for the realized values of the source and target random variables;
- $\{n\}$ for the index set of the source realizations, i.e. $\{n\} = \{1, ..., n\}$;
- $\langle \mathcal{L}, \preceq \rangle$ for a lattice $\mathcal{L}$ equipped with the partial order $\preceq$;
- $a$ for the event $\bigcap_{i \in a} s_i$ associated with the set of indices $a$;
- $\alpha$ for a set of collections of source realization indices i.e. $\alpha = \{a_1, \ldots, a_m\}$;
- $p_T(\cdot), p_T, p_T(S(\cdot), \cdot), p_T|S(\cdot|\cdot)$ for the (joint or conditional) probability mass function of the corresponding random variables;
- $p(\cdot), p(\cdot, \cdot), p(\cdot|\cdot)$ shorthand notation where the random variable is evident from the context;
2 Conceptual definition of shared information based on Boolean logic

To define the (local) shared information that two or more variables $S_1, \ldots, S_n$ (or collections thereof) have about a target $T$, we first look for a statement about the realizations of source variables that is true when these variables take on realizations $S_1 = s_1, \ldots, S_n = s_n$, and, crucially, whose truth can be verified knowing the realization of a single source variable, no matter which (i.e. knowing that $S_i = s_i$ for at least one $i$). We will then investigate what the truth of such a statement provides in information about the target realization $T = t$. Such a statement $W$ about the source variables is: $W = (S_1 = s_1 \lor \ldots \lor S_n = s_n)$, i.e. the inclusive OR of the statements that each source variable has taken on its specific realization. While this seems to tell us very little about the source variables, it can be already very informative about the realization of the target variable $T$. To see this, consider a system where the source variables are two independent random bits with uniform probability distributions and where the target variable is the logical AND of these two sources. In this case the statement $S_1 = 0 \lor S_2 = 0$ contains very little about the relation of the two source variables but fully determines the realizations of the target variable to be $T = 0$.

Formally, we can thus define the local shared information that the sources have about the target by:

$$i_{t;W}^S(t : s_1; \ldots; s_n) = \log \frac{p(t | W = \text{true})}{p(t)}.$$

(1)

In the following we show how to turn this formal definition into a proper measure of information that can be evaluated from the joint probability distribution $P(T, S_1, \ldots, S_n)$. For this purpose we will draw on a definition of mutual information based on the exclusion of probability mass, as it was introduced originally by [4,19] and recently also used in relation to PID by [5,6].

It is important to note here already that any operational interpretation of our definition of shared information in a decision or game theoretic context needs to take into account the situation that an agent is only given the information that the statement $W$ is true, but does not have access to any of the source variables in full. Also, and crucially, the agent is not given the (meta-) information that this statement reflects shared information, nor that its truth could be evaluated from each source variable. This means that an agent will have to base its actions solely on the truth of $W$; this is different from other frameworks considering agents that have access to source variables in full, and where the PID-frameworks then derive measures of shared information from the rational actions that such agents can take in some form of competitive or comparative setting [2,5].

3 Mutual information from exclusions of probability mass

In this section we will (re-)derive the local mutual information, and the joint mutual information based on exclusions of probability mass to set the stage for our definition of the shared information in the next section. To this end, consider the discrete random variables $T$, $S$ (and later $S_1$, $S_2$), defined on a probability space. Furthermore, we consider these random variables to have taken on some realizations $T = t$, $S = s$, $S_1 = s_1$, $S_2 = s_2$. We note that the probability mass of

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4 We note already here that the use of this logical statement to formulate "sharedness" instead of any measure related to the mutual information between source variables is different from other frameworks of PID and has the potential to easily explain how the so-called mechanistic shard information arises, i.e. how shared information can exist without mutual information between the source variables.

5 Giving either of the two pieces of information would enable the exploitation of more than just the shared information.
a realization, i.e. \( p(s) = \mathbb{P}(\{S = s\}) \) is the measure of the corresponding event (set) in the probability space. This means we can also use set notation when calculating probabilities in the following way:

\[
p(s) = \mathbb{P}(s) \quad (2a)
\]

\[
p(s_1, s_2) = \mathbb{P}(s_1 \cap s_2) \quad (2b)
\]

where \( s, s_1, s_2 \) are sets in the \( \sigma \)-Algebra on the sample space – corresponding to the realizations \( s, s_1, s_2 \) of the random variables. The reader may use equations (2a) to translate forth and back between both notations. Any further more rigorous treatment of the measure theoretic foundations is referred to the appendix B.

We start with the standard definition of the local mutual information [4] obtained from a realization \((t, s)\) of two random variables \( T, S \) as:

\[
i(t : s) = \log \frac{p(t|s)}{p(t)} \quad (3)
\]

and rewrite this using the standard laws of probability and the set notation from above in the following way:

\[
i(t : s) = \log \frac{p(t) - \sum_{u \in A \backslash \{s\}} p(t, u)}{1 - \sum_{u \in A \backslash \{s\}} p(u)} - \log p(t) \quad (4a)
\]

\[
= \log \frac{\mathbb{P}(t) - \mathbb{P}(t \cap \bar{s})}{1 - \mathbb{P}(\bar{s})} - \log \mathbb{P}(t) \quad (4b)
\]

where we made use of the fact that the realizations \( s, t \) of the random variables in probability theory correspond to sets on measure spaces and where \( \bar{s} \) is the set complement of \( s \) and \( A_S \) is the alphabet of \( S \).

We now see that the local mutual information from equation (3) compares the initial probability on a logarithmic scale to another probability, \( p(t|s) \), that is valid after observation of \( s \). The procedure to obtain \( p(t|s) \) can be interpreted\(^3\) as follows (see Figure 1 for a graphical representation; see appendix B for a rigorous axiomatic treatment):

1. "Removing" all events from the initial sample space \( \Omega \) that are incompatible with the observation of a specific \( s \) (i.e. "removing" \( \bar{s} \)). This is visible for example in the term \( -\mathbb{P}(t \cap \bar{s}) \).
2. Rescaling the probability measure to again have properly normalized probabilities. This step is visible in the division by \( 1 - \mathbb{P}(\bar{s}) \).

If we now extend our consideration to the case of three random variables \( T, S_1, S_2 \), and their realizations \( t, s_1, s_2 \), then we obtain the joint local mutual information as:

\[
i(t : s_1, s_2) = \log \frac{\mathbb{P}(t) - \mathbb{P}(t \cap (\bar{s}_1 \cup \bar{s}_2))}{1 - \mathbb{P}((\bar{s}_1 \cup \bar{s}_2))} - \log \mathbb{P}(t) \quad (5)
\]

because \( \bar{s} = (\bar{s}_1 \cup \bar{s}_2) = \bar{s}_1 \cup \bar{s}_2 \).

The key two principles here are that (i) the mutual information is always induced by exclusion of the probability mass related to events that are impossible after the observation of \( s_1, \ldots, s_n \), expressed via the sets \( \bar{s}_1, \ldots, \bar{s}_n \), and (ii) that the probabilities are rescaled by taking into account

\(^3\)In a rigorous description of the process, the 'excluded' events are simply assigned the probability mass zero in a new measure on the sample space. The 'rescaling' then is a consequence of the measure being properly normalized.

\(^4\)Removing more precisely means giving it a new measure of size zero.
Figure 1: Graphical depiction of the how local mutual information $i(t : s)$ arises from the exclusion of the probability mass of the event $\bar{s}$ that become impossible after the observation of the event $s$. (A) Two event partition $t, \bar{t}$ in the sample space $\Omega$. (B) Two event partition $s, \bar{s}$ of the source variable $S$ in the sample space $\Omega$. The occurrence of $s$ renders $\bar{s}$ impossible (red stripes). (C) $t$ may intersect with $s$ and $\bar{s}$ (red hashed region). The relative size of the two intersections is important to determine whether we obtain information or misinformation, i.e. whether $t$ becomes relatively more likely after considering $s$, or not (D), considering the necessary rescaling of the probability measure (E). The formula in (F) gives the reason why the misinformative exclusion $P(t \cap \bar{s})$ (intersection of red hashes with green region) cannot be cleanly separated from the informative exclusion, $P(\bar{t} \cap \bar{s})$ (blue framed region), as stated already in [6]. This is because these overlaps appear together in a sum inside the logarithm, but this logarithm in turn guarantees the additivity of information terms. Thus the additivity of (mutual) information terms is incompatible with an additive separation of informative and misinformative exclusions inside the logarithms of the information measures.

these very same exclusions. We will stick to these core information theoretic principles [6] in our definition of the shared mutual information, below.

Note that, since probability mass may be excluded by observation of either realization $s_1$ OR $s_2$ OR ... OR $s_m$, it is the union of their complements $\bar{s}_i$ that matters to obtain the total or joint mutual information.

To reiterate, measuring the joint mutual information that multiple source variables have about a target variable relies on taking the union of all sets of events excluded by the observation of specific realizations of the source variables and on using the excluded events in a sequence of two
operations: (1) the removal from the sample space of events incompatible with the observation of the sources, (2) a change (rescaling) of the probabilities.

In our approach to shared information we suggest to keep this central information theoretic principle that binds the exclusion of probability mass to mutual information fully intact. Therefore, in our definition of shared information we only additionally require that the probability mass to be excluded is derived from events that are excluded redundantly by the realizations of all source variables. That is, any subset \( \epsilon \) of \( \Omega \) is only to be excluded for the computation of shared information if it is in in all complements \( \bar{\epsilon} \). This shared exclusion encodes the knowledge that we gain from knowing that the statement \( W \) from section 2 is true.

## 4 Shared mutual information from shared exclusions of probability mass

### 4.1 Definition of \( \xi^{\text{sx}}_{(i)} \)

For a subset \( \epsilon \in \Omega \) to contribute to the shared mutual information via its exclusion we require that \( \epsilon \) is redundantly excluded by all observations \( s_1, s_2, \ldots, s_n \), i.e. we require that it is contained in the intersection, instead of the union, of \( \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n \). In other words, instead of removing and rescaling by the union of the complements \( \bar{\epsilon}_i \), we use the intersection of them. This yields:

\[
\xi^{\text{sx}}_{(i)}(t : s_1 ; s_2 ; \ldots ; s_n) := \log \frac{P(t) - P(t \cap (\bar{s}_1 \cap \bar{s}_2 \cap \ldots \cap \bar{s}_n))}{1 - P(\bar{s}_1 \cap \bar{s}_2 \cap \ldots \cap \bar{s}_n)} - \log P(t),
\]

where the superscript ‘sx’ in \( \xi^{\text{sx}}_{(i)} \) stands for ‘shared exclusions’.

In a PID setting one also has to consider the case that multiple variables are considered jointly, i.e. the case where collections of variables \( A_i := \{ \ell \mid \ell \in a_i \} \), with corresponding collection of source realizations indices \( a_i := \{ j_1, \ldots, j_{i,k_i} \} \), and corresponding events \( a_i := \cap_{\ell \in a_i} \bar{s}_\ell \) enter the set of sources of which the shared information is to be determined. In this case the complements \( \bar{s}_\ell \) of all source events of \( a_i \) are considered as a union because we know that the \( s_\ell \) in collection \( a_i \) occur together. These unions of complements for each \( a_i \) are then intersected with the unions of complements formed by the other sets \( a_{k \neq i} \) as before:

\[
\xi^{\text{sx}}_{(i)}(t : a_1 ; a_2 ; \ldots ; a_m) := \log \frac{P(t) - P(t \cap (\bar{a}_1 \cap \bar{a}_2 \cap \ldots \cap \bar{a}_m))}{1 - P(\bar{a}_1 \cap \bar{a}_2 \cap \ldots \cap \bar{a}_m)} - \log P(t),
\]

where \( P(\bar{a}_i) = P(\cup_{\ell \in a_i} s_\ell) \).

### 4.2 Interpretation of \( \xi^{\text{sx}}_{(i)} \) as a regular local mutual information with an auxiliary variable

We will show here that \( \xi^{\text{sx}}_{(i)} \) can be rewritten as a regular local mutual information using an auxiliary variable that depends on the current realizations of sources and target. This will show how \( \xi^{\text{sx}}_{(i)} \) fulfills the formal definition of the shared information from section 2.

We start by introducing the random variable \( W_{a_1, \ldots, a_m} \) that takes the value \( w_{a_1, \ldots, a_m}^{(1)} \) if the event \( \bar{a}_1 \cap \bar{a}_2 \cap \ldots \cap \bar{a}_m \) \( \Leftrightarrow \) \( a_1 \cup a_2 \cup \ldots \cup a_m \) occurred, and the value \( w_{a_1, \ldots, a_m}^{(0)} \) otherwise. Thus, \( W_{a_1, \ldots, a_m} \) indicates whether none of the complement events \( a_i \) has occurred or equivalently at least one of the events \( a_i \) occurred. \( W_{a_1, \ldots, a_m} \) can also be considered as the Boolean variable indicating whether the statement \( W \) from section 2 is true.
Figure 2: Exclusions of elementary events by three source events. (Upper left) A sample space with three events $s_1$, $s_2$, $s_3$ from three source variables (the complements of each of these events are depicted under (4)). The target event $t$ is not depicted for clarity, but may arbitrarily intersect with any intersections/unions of source events. The remainder of the figure shows the induced exclusions for all nodes of the distributive lattice of exclusions. Although not crucial, we have organized the lattice via the following color scheme: An intersection exclusion is colored by the mix of the individual colors, e.g., the $\{1\} \{2\}$ exclusion is $\bar{s}_1 \cap \bar{s}_2$ and mixes red and blue to purple. The union exclusion is colored by a pattern of the individual colors, e.g., the $\{1\} \{2\}$ exclusion is $\bar{s}_1 \cup \bar{s}_2$ and takes a light red-blue pattern. For the levels (1)-(3) we demonstrate how the relevant exclusions are generated from intersections of the complements of source events (or unions of source events for joint variables), such that the sets brought to intersection are on the left, while the intersections (shared exclusions) are on the right. For level (4)-(7) we only show shared exclusions, e.g., the $\{1\} \{2\} \{3\}$ exclusion is $\bar{s}_1 \cup (\bar{s}_2 \cap \bar{s}_3)$ and takes a red-olive pattern.
We then rewrite equation (7) as:

\[ i^{\infty}_{\sum} (t : a_1; a_2; \ldots; a_m) = \log \frac{P(t) - P(t \cap (a_1 \cap a_2 \cap \ldots \cap a_m))}{1 - P(a_1 \cap a_2 \cap \ldots \cap a_m)} - \log P(t) \]

\[ = \log \frac{P(t \cap (a_1 \cap a_2 \cap \ldots \cap a_m))}{P(a_1 \cap a_2 \cap \ldots \cap a_m)} - \log P(t) \]

\[ = \log \frac{P(t \cap (a_1 \cup a_2 \cup \ldots \cup a_m))}{P(a_1 \cup a_2 \cup \ldots \cup a_m)} - \log P(t) \]

\[ = \log \frac{p(t, w_{a_1}, \ldots, a_m)}{p(w_{a_1}, \ldots, a_m)} - \log p(t) \]

\[ = i(t : w_{a_1}, \ldots, a_m) \]

\[ = i(t : W = true). \]

Thus, we see that \( i^{\infty}_{\sum} \) is just a local mutual information between the target realization \( t \) and the realization \( w_{a_1}, \ldots, a_m \) of auxiliary variable \( W_{a_1}, \ldots, a_m \), indicating that the event we’re considering is somewhere in the union of all source events. We see that the way in which \( i^{\infty}_{\sum} \) represents shared information can be interpreted also in the following way: Given two source events \( a_1, a_2 \), we can know trivially from either source event alone that the final joint event \( (t, a_1, a_2) \) lies in the union of \( a_1 \) and \( a_2 \) (while instead knowing both source events lets us know that it lies in the intersection of \( a_1 \) and \( a_2 \).

It is important to note here that the auxiliary variable \( W_{a_1}, \ldots, a_m \) in equation (8) is a different random variable for each realization of source and target. Therefore, the use of auxiliary variables and the connection to Boolean logic can only be expressed using the concept of the local mutual information [6].

Also, the reader may wonder whether the introduction of the auxiliary binary variable \( W_{a_1}, \ldots, a_m \) limits the local, and also the average, shared information to one bit at most. This is not the case however, because the local entropy \( h(w_{a_1}, \ldots, a_m) = \log \frac{1}{p(w_{a_1}, \ldots, a_m)} \) is unbounded. Since each possible realization of the original variables comes with its own auxiliary \( W_{a_1}, \ldots, a_m \), and since it is always only the specific realization \( w_{a_1}, \ldots, a_m \) that is evaluated (see equation (8)), the local entropy \( h(w_{a_1}, \ldots, a_m) \) can exceed one bit for every realization, and the global shared information \( i^{\infty}_{\sum} \) can also exceed one bit.

### 4.2.1 Consequences of \( i^{\infty}_{\sum} \) being a local mutual information

The fact that \( i^{\infty}_{\sum} \) can be written as a regular local mutual information has several interesting consequences.

First, we can assess the self-shared information of a collection of variables:

\[ i^{SS}_{\sum} (a_1; a_2; \ldots; a_m : a_1; a_2; \ldots; a_m) = i(w_{a_1}^{(1)}, \ldots, a_m) : w_{a_1}^{(1)}, \ldots, a_m) = h(w_{a_1}^{(1)}, \ldots, a_m). \]

This quantity is greater or equal to zero and is the upper bound of shared information that the source variables can have about any realization \( u \) of any target variable \( U \), i.e.:

\[ i^{SS}_{\sum} (a_1; a_2; \ldots; a_m : a_1; a_2; \ldots; a_m) \geq i^{SS}_{\sum} (a_1; a_2; \ldots; a_m : u) \quad \forall u \in A_U. \]

This upper bound has conceptual links to maximum extractable shared information from [12]. Moreover, this upper bound may be non-zero even for independent sources.

Second, if we include the variable \( t \) in the set of sources, we get a shared (self-) information that is a fully symmetric shared information in all (collections of) variables, i.e. \( i^{SS}_{\sum} (t; a_1; a_2; \ldots; a_m : t; a_1; a_2; \ldots; a_m) \). Third, a target chain rule can be derived, see section 5, below.
4.2.2 Interpretation of negative shared information in XOR

Using our formulation of $i_{SN}$ results in negative local shared information for the classic XOR example. To see this, assume that $S_1, S_2$ are independent, uniformly distributed random bits, and $T = XOR(S_1, S_2)$, and assume for example the realization $s_1 = 1, s_2 = 1, t = 0$. From equation (6) we get:

$$i_{SN}^X(t = 0 : s_1 = 1; s_2 = 1) = \log \frac{1/2 - 1/4}{1 - 1/4} + \log \frac{1}{1/2}$$

$$= - \log 3 + \log 2 < 0$$

We argue that this result reflects that an agent receiving the shared information is misinformed (see e.g. [6] for the concept of misinformation) about $t$. To understand the source of this misinformation, consider that the agent is only provided with the shared information, i.e. the agent knows only that $W$ is true. This will let the agent predict that the joint realization is one out of three realizations with equal probability: $(s_1 = 0, s_2 = 0, t = 0)$, $(s_1 = 0, s_2 = 1, t = 1)$, $(s_1 = 1, s_2 = 0, t = 1)$ (see Figure 3). Of these three realizations, only one realization points to the correct target realization $t = 0$, the other two point to the “wrong” realization $t = 1$ — while it was equal probabilities for $t = 0$ and $t = 1$ before the agent received the shared information from the sources. As a consequence, the local shared information becomes negative.

5 Target chain rule

Our measure of shared information $i_{SN}$ is just the local mutual information between the realization $T = t$ and the realization of the auxiliary variable $W_{a_1,\ldots,a_m} = w_{a_1,\ldots,a_m}$. This fact immediately yields a target chain rule for a composite target variable $T = \{t_1, t_2\}$:

$$i_{SN}^X(t_1, t_2 : a_1; a_2;\ldots; a_m) = i_{SN}^X(t_1 : a_1; a_2;\ldots; a_m) + i_{SN}^X(t_2 : a_1; a_2;\ldots; a_m | t_1),$$

where

$$i_{SN}^X(t_2 : a_1; a_2;\ldots; a_m | t_1) = \log \frac{P(t_2 | t_1) - P(t_2, a_1, a_2,\ldots, a_m | t_1)}{1 - P(a_1, a_2,\ldots, a_m | t_1)} - \log P(t_2 | t_1).$$

We can obtain a corresponding target chain rule for the average quantities easily from the linearity of the averaging:

$$I_{SN}^X := \sum_{t,s_1,\ldots,s_n} p(t,s_1,\ldots,s_n)i_{SN}^X(t : a_1;\ldots; a_m)$$

$$= \sum_{t,s_1,\ldots,s_n} p(t,s_1,\ldots,s_n)i(t : w_{a_1,\ldots,a_m}^{(1)}),$$

where probabilities related to the auxiliary variable $W_{a_1,\ldots,a_m}$ have to be recomputed for each possible combination of source and target events. Note that in equation (13b) the averaging still runs over all combinations of $t, s_1,\ldots, s_n$, and the weights are still given by $p(t,s_1,\ldots,s_n)$, not $p(t,w_{a_1,\ldots,a_m}^{(1)})$. Having different variables in the averaging weights and the local mutual information terms makes the average shared information structurally different from a mutual

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5 This means the agent is being told: “One of the two sources has outcome 1, the other one we don’t know.”

6 Due to $i(t : s_i) = 0$ for $i = 1, 2$ in the XOR example, this negative shared information is then compensated by positive unique information and the synergy is reduced from 1 bit to 1 minus once this unique information. This is the price to be paid for operating with negative quantities as if they were areas.
Figure 3: Worked example of $\alpha^\pi_{\pi^\pi}$ for the two source-variable case, where the source variables $S_1$ and $S_2$ are both independent uniformly distributed random bits, and $T = \text{XOR}(S_1, S_2)$. We consider the realization $(s_1, s_2, t) = (1, 1, 0)$. (A) Sample space $\Omega$ with the four possible events. (B) The realized event in $\Omega$, marked in blue. (C) The exclusion of events induced by learning that $S_1 = 1$, or correspondingly, $\bar{s}_1 = \{0\}$, marked in grey. (D) Same for $\bar{s}_2 = \{0\}$. (E) Union of exclusions fully determines the joint event $(1, 1, 0)$ and gives rise to 1 bit of joint local mutual information $i(t = 0 : s_1 = 1, s_2 = 1)$. (F) The exclusions that are shared by $\bar{s}_1 = \{0\}$ and $\bar{s}_2 = \{0\}$, i.e., $\bar{s}_1 \cap \bar{s}_2$ exclude contain only the event $(0, 0, 0)$. This is a misinformative exclusion, as it raises the probability of events that did not happen, i.e. $t = 1$, relative to those that did happen, i.e. $t = 0$, compared to the case of complete ignorance. (G) Learning about one full variable, i.e. obtaining the statement that $\bar{s}_1 = \{0\}$ adds additional probability mass to the exclusion (marked in green). The shared exclusion (red) and the additional unique exclusion (green) induced by $s_1$ create an exclusion that is uninformative, i.e. the probabilities for $t = 0$ and $t = 1$ remain unchanged by learning $s_1 = 1$. At the level of the $\pi^\pi$-atoms, the shared and the unique information atom cancel each other. (H) Lattice with $\alpha^\pi_{\pi^\pi}$ and $\pi^\pi$-terms for this realization. Other realization are equivalent by the symmetry of XOR, thus, the averages yield the same numbers. Note that the necessity to cancel the negative shared information twice to obtain both, $i(t = 0 : s_1 = 1) = 0$, and $i(t = 0 : s_2 = 1) = 0$, results in a synergy < 1 bit. Also note that while adding the shared exclusion from (F) and the unique exclusions for $s_1$ and $s_2$ results in the full exclusion from (E), information atoms add differently due to the non-linear transformation of excluded probability mass into information via $-\log p(\cdot)$ – compare (H).
Figure 4: Worked example of $i^x_{\cap}$ for the four source source-variable case, evaluating the shared information $i^x_{\cap}(t : a_1; a_2)$ with $a_1 = \{1, 2\}, a_2 = \{3, 4\}, s = (s_1, s_2, s_3, s_4) = (0, 0, 1, 0),$ and $t = \text{Parity}(s) = 1$. (A) Sample space – the relevant event is marked by the blue outline. (B) exclusions induced by the two collections of source realization indices $a_1$ (yellow), $a_2$ (light blue), and the shared exclusion relevant for $i^x_{\cap}$ (grey). After removing and rescaling, the probability for the target event that was actually realized, i.e. $t = 1$, is reduced from $1/2$ to $3/7$. Hence the shared exclusion leads to negative shared information. The corresponding $\pi$ atom for this lattice node is: $-0.0145$ bit.
information. Thus, the local shared information may be expressed as a local mutual information with an auxiliary variable constructed for that purpose, and multiple such variables have to be constructed for a definition of a global shared information.

6 Lattice structure

It is well known that the local mutual information can be positive or negative. Hence, also the decomposition implied by equation (7) will potentially contain a mix of positive and negative \( I_{\gamma}^{\pm}(t : a_1; a_2; \ldots; a_m) \), preventing the construction of a partial ordering (lattice) and the desired Moebius inversion to obtain the partial information decomposition “atoms” \( \pi^k \) (see [18]). As already realized by Lizier, Ince and others a useful strategy is to decompose the (local) shared information further into all positive components and construct lattices for these components individually.

We, therefore, suggest to decompose \( I_{\gamma}^{\pm}(t : a_1; a_2; \ldots; a_m) \) further into the following two parts that are both of the form of a local Shannon information and individually fulfil the Williams and Beer axioms. The informative-misinformative decomposition is as follows:

\[
I_{\gamma}^{\pm}(t : a_1; a_2; \ldots; a_m) = \log \frac{1}{1 - \mathbb{P}(a_1 \cap a_2 \cap \ldots \cap a_m)} - \log \frac{\mathbb{P}(t)}{\mathbb{P}(t) - \mathbb{P}(t \cap (a_1 \cap a_2 \cap \ldots \cap a_m))},
\]

(14a)

\[
= \log \frac{1}{\mathbb{P}(a_1 \cup a_2 \cup \ldots \cup a_m)} - \log \frac{\mathbb{P}(t)}{\mathbb{P}(t \cap (a_1 \cup a_2 \cup \ldots \cup a_m))}
\]

(14b)

\[
= I_{\gamma}^{+}(t : a_1; a_2; \ldots; a_m) - I_{\gamma}^{-}(t : a_1; a_2; \ldots; a_m).
\]

(14c)

Here, the first term in (14b) is considered to be the informative part as it is what can be inferred from the sources (\( a_i \) are collection of sources). The second term in (14b) quantifies the (misinformative) relative loss of \( p(t) \), the probability mass of the event \( t \) (which actually happened) when excluding the mass of \( a_1 \cap a_2 \cap \ldots \cap a_n \).

Similar to the decomposition into positive and negative terms from [5] our decomposition does not separate the \( t \cap (a_1 \cup a_2 \cup \ldots \cup a_n) \) from \( t \cap (a_1 \cap a_2 \cap \ldots \cap a_n) \), i.e. the informative and misinformative exclusions in the original local mutual information (see Fig. 1 for more details on the general problem of separating such contributions).

We now proceed to the construction of the informative and misinformative lattices for the positive and negative part of \( I_{\gamma}^{\pm} \) in (14c) respectively. In order to construct these lattices, the informative and misinformative shared information measures \( I_{\gamma}^{+} \) and \( I_{\gamma}^{-} \) must individually fulfill a certain set of the Williams and Beer axioms [18].

Axiom 6.1 (Symmetry). \( I_{\gamma}^{+} \) and \( I_{\gamma}^{-} \) are invariant under any permutation \( \sigma \) of collections of source events,

\[
I_{\gamma}^{+}(t : a_1; a_2; \ldots; a_m) = I_{\gamma}^{+}(t : \sigma(a_1); \sigma(a_2); \ldots; \sigma(a_m)),
\]

\[
I_{\gamma}^{-}(t : a_1; a_2; \ldots; a_m) = I_{\gamma}^{-}(t : \sigma(a_1); \sigma(a_2); \ldots; \sigma(a_m)).
\]

Axiom 6.2 (Monotonicity). \( I_{\gamma}^{+} \) and \( I_{\gamma}^{-} \) decreases monotonically as more source events are included,

\[
I_{\gamma}^{+}(t : a_1; \ldots; a_m; a_{m+1}) \leq I_{\gamma}^{+}(t : a_1; \ldots; a_m),
\]

\[
I_{\gamma}^{-}(t : a_1; \ldots; a_m; a_{m+1}) \leq I_{\gamma}^{-}(t : a_1; \ldots; a_m).
\]

(15)

(16)

with equality iff there exists \( i \in \{1, \ldots, m\} \) such that \( a_{m+1} \subseteq a_i \).

Note that, in Axiom 6.2, the equality condition is conversed but not negated compared to the original Williams and Beer Monotonicity axiom [18].

---

As was to be expected from the difficulties encountered in the past trying to define measures of shared information.
Axiom 6.3 (Self-redundancy). $i^+_i$ and $i^-_i$ for a single source event $a_i$ equals the specificity and ambiguity respectively

$$i^+_i(t:a_i) = h(a_i), \quad (17)$$

$$i^-_i(t:a_i) = h(a_i | t). \quad (18)$$

Theorem 6.1. $i^\pm_{i1}$ and $i^\pm_{i1}$ satisfy Axioms 6.1, 6.2, and 6.3.

The proof is deferred to Appendix A. Theorem 6.1, as we will show in Appendix A, leads to the construction of informative and misinformative lattices. Since $i^\pm_{i1}$ are pointwise, then for every realization $s = (s_1, \ldots, s_n)$, there are two lattices, informative and misinformative. These lattices are in fact two copies of the redundancy lattice $(s', \preceq')$ (defined in Appendix A) where $i^\pm_{i1}$ and $i^\pm_{i1}$ are the cumulative information measure for each lattice respectively.

The terms of informative and misinformative pointwise partial information decomposition will be evaluated by $\pi^\pm_{i1}$ and $\pi^\pm_{i1}$ respectively. In Appendix A, we will show how to obtain these terms recursively from $i^\pm_{i1}$. We conclude by the following theorems that imply the existence of informative and misinformative pointwise partial information decomposition based on the informative and misinformative shared information $i^\pm_{i1}$.

Theorem 6.2. $i^\pm_{i1}$ increase monotonically on the lattice $(s', \preceq')$.

Theorem 6.3. The atoms $\pi^\pm_{i1}$ of the informative and misinformative lattice $(s', \preceq')$, are non-negative.

In Appendix A, we will provide the necessary tools to prove the above theorems, in particular, Theorem 6.3.

7 Differentiability of $i^\pm_{i1}$ and $\pi^\pm_{i1}$

In this section, we will discuss the differentiability of the PPID using $i^\pm_{i1}$ as the redundancy measure. Formally, let $\Delta_P$ be the probability simplex of the joint distributions of $(T, S_1, \ldots, S_n)$ and $\mathcal{A}(s)$ be the redundancy lattice (see Section A) for any realization $s = (s_1, \ldots, s_n)$. Then, for any $\alpha \in \mathcal{A}(s)$, $i^\pm_{i1}$ and $\pi^\pm_{i1}$ are differentiable over the interior of $\Delta_P$.

Recall that $\log$ is continuously differentiable over the open domain $R_+$. Thus, using the definition in (14c), $i^\pm_{i1}$ and $i^\pm_{i1}$ are both continuously differentiable over the interior of $\Delta_P$. From Theorem A.2 and Proposition A.3, we know that for any $\alpha \in \mathcal{A}(s)$,

$$\pi^\pm_{i1}(t:a) = \sum_{\gamma \in \mathcal{P}(\alpha \setminus \{\gamma_1\})} (-1)^{\gamma_1} \log \left( \frac{p(\gamma) + d_1}{p(\gamma)} \right),$$

where $\alpha^- = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ are the children of $\alpha$ ordered increasingly w.r.t. their probability mass. Note that the set of children $\alpha^-$ of a node $\alpha$ is defined as $\{\beta \in \mathcal{A}(s) | \beta \triangleleft \alpha, \beta \preceq \gamma \prec \alpha \Rightarrow \beta = \gamma\}$.

Then, $\pi^\pm_{i1}$ is continuously differentiable over the interior of $\Delta_P$ since the function $x + d_1/x$ and its inverse are continuously differentiable over the open domain $R_+$. Similarly, $\pi^\pm_{i1}$ is continuously differentiable over the interior $\Delta_P$.

8 Discussion

8.1 Operational interpretation of $i^\pm_{i1}$

As laid out in the introduction our aim here was to design a measure of local shared information that is defined as close as possible to the way local mutual information is defined based on exclusions of the probability mass of events in a measure space. As a consequence, our measure derived this way turned out to locally be a mutual information with respect to a specifically constructed auxiliary variable $W$. This has the advantage that all the standard interpretations
of the local mutual information carry over to our measure. It also ensures that our measure is differentiable, and that an extension to continuous variables appears possible. Last, it turned out that the auxiliary variable $W_{s_1,\ldots,s_n}$ reflected the truth value of a simple logical statement with respect to the realizations $s_1,\ldots, s_n$ of the source variables. The operational meaning here is that the truth value of $W$ can be asserted by inspecting any of the source variables. The statement $W$ whose truth value $W$ represents can always be formulated 'in plain english'. We can thus derive an operational interpretation of our measure of shared mutual information by simply asking what actions or predictions a rational agent should make upon receiving the information that statement $W$ is true.

One important aspect of this operational interpretation that may be easily overlooked is the fact that the agent only receives the shared information. In other words, we never assume our agent has access to any full source variable (in contrast to some other definitions of shared information, see below). Also, the agent does not receive the meta-information that $W$ is a statement that provides shared information, nor that the truth of $W$ can be asserted by each source.

A negative value of the shared information indicates that an agent who is only in possession of the shared information is more likely to mispredict the outcome of the target (see Figures 3, 4 for examples) than without the shared information, a positive value means that the shared information makes the agent more likely to choose the correct outcome. The unsigned magnitude of the shared information informs us about how certain the agent should be about his prediction by comparing $p(t|w_{s_1,\ldots,s_n}^{(1)})$ with $\sum_{u \in \mathcal{A}_T \setminus \{t\}} p(u|w_{s_1,\ldots,s_n}^{(1)})$.

The setting of our operational interpretation contrasts with that of other approaches to PID that take the perspective of multiple agents having full access to individual source variables (or collections thereof), and that then design measures of unique and redundant information based on actions these agents can take or rewards they obtain in decision- or game-theoretic settings based on their access to full source variables (e.g. in [2,5]). While certainly useful in the scenarios invoked in [2,5], we feel that these operational interpretations may almost inevitably mix inference problems (i.e. information theory proper) with decision theory. Also, they typically bring with them the use of minimization or maximization operations to satisfy the competitive settings of decision or game theory. This, in turn, renders it difficult to obtain a differentiable measure of local shared information.

In sum, we feel that the question of how to decompose the information provided by multiple source variables about a target variable may indeed not be a single question, but multiple question in disguise. The most useful answer will therefore depend on the scenario where the question arose. Our answer seems to be useful in communication settings, and where quantitative statements about dependencies between variables are important (e.g. the field of statistical inference, where the PID enumerates all possible types of dependencies of the dependent (target) variable on the independent (source) variables).

8.1.1 Evaluation of $I_{sx}^{\infty}$ on $P$ and on optimization distributions obtained in other frameworks $Q_0(P) \in \Delta P$

Since our approach to PID relies only on the original joint distribution $P$ it can be applied to other PID frameworks where optimized distributions $Q(P)$ are derived from the original $P$ of the problem – e.g. via optimization procedures, as it is done for example in [2]. This yields some additional insights into the operational interpretation of our approach compared to others, by highlighting how the optimization from $P$ to $Q(P)$ shifts information between PID atoms in our framework.

8.2 Number of PID atoms vs alphabet size of the joint distribution

Given the rapid rise in the number of lattice nodes with increasing numbers of sources that may outgrow the joint probability distribution one may ask about the independence of the atoms on the lattice in those cases (remember that the atoms were introduced in order to have the ‘independent’ information contributions of respective variable-configurations at the lattice nodes). As shown in
Fig. 5 and 6 our framework reveals multiple additional constraints at the level of exclusions via the family of mappings from Proposition A.3. This explains mechanistically why not all atoms are independent in cases where the number of atoms is larger than the number of symbols in the joint distribution.

8.3 Key applications

Due to the fact that PID solves a basic information theoretic problem, its applications seem to cover almost all fields where information theory can be applied. Below, we highlight a few that we deem of particular interest.

8.3.1 Learning neural goal functions

In [17] we had argued that information theory, and in particular the PID framework, lends itself to unify various neural goal functions, e.g. like infomax and others. We had also shown how to apply this to learning in neural networks via the coherent infomax framework of Kay and Phillips. Yet, this framework was restricted to goal functions expressible using combinations, albeit complex ones, of terms from classic information theory, due to the lack of a differentiable PID measure. Goal functions that were only expressible using PID proper could not be learned in the Kay and Phillips framework, and in those cases PID would only serve to assess the approximation loss. Our new measure now removes this obstacle and neural networks or even individual neurons can now be devised to learn pure PID goal functions. A possible key application are hierarchical neural networks with a hierarchy of modules, where each module contains two populations of neurons. These two populations represent supra- and infragranular neurons and coarsely mimic their different functional roles. One population represents so called layer 5 pyramidal cells. It serves to send the shared information between their bottom-up (e.g. sensory) inputs and their top down (contextual) inputs downwards in the hierarchy; the other population represents layer 3 pyramidal cells and sends the synergy between the bottom-up inputs and the top down inputs upwards in the hierarchy. For the first population the extraction of shared information between higher and lower levels in the hierarchy can be roughly equated to learning an internal model, while for the second population the extraction of synergy is akin to computing a generalized error. Thus, a hierarchical network of this kind can perform an elementary type of predictive coding. The full details of this application scenario are the topic of another study, however.

8.3.2 Information modification in distributed computation in complex systems

If one desires to frame distributed computation in complex system in terms of the elementary operations on information performed by a Turing-machine, i.e. the storage, transfer and modification of information, information theoretic measures for each of these component operations are required. For storage and transfer well established measures are available, i.e. the active information storage [11] and the transfer entropy [10, 14, 16]. For modification, in contrast, no established measures exists, yet an appropriate measure of synergistic mutual information from a partial information decomposition has been proposed as a candidate measure of information modification. An appropriate measure in his context has to be localizable (i.e. it must be possible to evaluate the measure for a single event) to serve an analysis of computation locally in space and time, and it has to be continuous in terms of the underlying probability distribution. Both of these conditions were already met for the PPID measure of Finn and Lizier [5]; our novel measure here mostly adds the possibility to differentiate the measure on the interior of the probability simplex, which makes it even more like a classic information measure.

8.3.3 Statistical modeling

In statistical modeling, we ask about the dependency structure between a so-called 'dependent' target variable and the 'independent' source variables. The most general way to describe such dependencies is to quantify the deviation from independence via a Kullback-Leibler divergence,
i.e. to compute the relevant (conditional) mutual information terms. In this setting a PID allows a fine grained assessment of the dependence structure. At a conceptual level, for example, the PID reveals that a statistical model with four source variables and a target variable should in principle accommodate for the 166 possible ways (lattice nodes) in which sets of source variables could contribute to the mutual information with the target. In practice, of course, the available data will hardly ever allow for such complex full models, but it may be useful to remember the inherent complexity of the inference problem, when interpreting results of statistical models, even when they have few variables.

8.3.4 Application to Neural Coding

Since the local mutual information $i(t : s_1, \ldots, s_n)$ is symmetric the perspective of which variable(s) are sources and target can be reversed and the PID formalism can be applied to scenarios where information flows from a single variable to multiple receiving variables. This is useful for example to answer the question how multiple neurons encode a stimulus variable. Specifically, one may analyze whether the stimulus information is encoded by the neurons mostly in unique form, i.e. each neuron coding for a different, independent aspects of the stimulus, or whether it is encoded redundantly, such that all neurons have shared information about the stimulus, or whether all neurons code jointly for the stimulus, such that knowledge about the activity of all of them is necessary to decode the stimulus information. This inverted perspective with respect to sources and targets is taken for example by Bertschinger et al. in [2] (also described in [15, Section 4]).

8.3.5 Other Applications in Coding and Communication

Due to its operational interpretation as a local mutual information we consider our novel measure as useful also in scenarios from classic information theory, i.e. coding and communication problems, where invoking arguments from decision or even game theory could be considered ‘extraneous’. Network coding may be a possible scenario from this category (see chapter 2 of [20]).

9 Examples

In this section, we present the PID provided by our $i_{sx}^{\cap}$ measure for some exemplary probability distributions. Most of the distributions are chosen from Finn and Lizier [5] and previous examples in the PID literature.

9.1 Probability Distribution PwUnq

We start by the point-wise unique distribution (PwUNQ) introduced by Finn and Lizier [5]. The PwUNQ distribution is constructed such that for each realization, only one of the sources holds complete information about the target while the other holds no information. The aim was to structure a distribution where at no point (realization) the two sources give the same information about the target. Hence, Finn and Lizier argue that, for such a distribution, there should be no shared (redundant) information.

Since in all of the realizations, the shared exclusion does not alter the likelihood of any of the target events compared to the case of total ignorance, our measure will indeed give zero redundant information. Thus, the PID terms resulting from $i_{sx}^{\cap}$ are the same as the those resultant from the Finn-Lizier measure (see Table 2).

In contrast, recall the Assumption ($*$) of Bertschinger et al. [2] which states that the unique and shared information should only depend on the marginal distributions $P(S_1, T)$ and $P(S_2, T)$. Finn and Lizier in [5] showed that all measures which satisfy the Assumption ($*$) will yield a zero unique information, i.e. a nonzero redundant information whenever $P(S_1, T)$ the marginal distribution of $S_1$ and $T$ is isomorphic to that of $S_2$ and $T$; this includes the above PwUNQ distribution. For example, $I_{\text{min}}$ [18], $I_{\text{red}}$ [8], $\bar{U}$ [2], and $S_{VK}$ [7] are all measures which suffer from this problem.
Altogether, the underlying cause for this is that Assumption (∗) does not take into consideration the pointwise nature of information.

### 9.2 Probability Distribution RndErr

Recall RND, the redundant probability distribution, where both sources are fully informative about the target and exhibit the same information. More precisely, \( s_1 = s_2 = t = 0 \) and \( s_1 = s_2 = t = 1 \) are the only two realizations that occur with equal chances and we denote them by the redundant realizations. Derived from RND, the RNDErr is a noisy redundant distribution of two sources where one source occasionally misinforms about the target while the other remains fully informative about the target. For example, if \( S_2 \) occasionally misinforms about the target, then realizations \( s_2 \neq s_1 = t = 0 \) and \( s_2 \neq s_1 = t = 1 \) are possible and equally likely but occur less frequently than the redundant realizations. In our example of RNDErr here (Table 3), we stick to the probability masses given by Finn and Lizier [5] which are \( 3/8 \) for the redundant realizations and \( 1/8 \) for the “faulty” realizations. Thus, we suggest that \( S_2 \) will hold misinformative (negative) unique information about \( T \).

For this distribution, our measure results in the following PID: misinformative unique information held by \( S_2 \), informative unique information held by \( S_1 \), informative shared information, and informative synergistic information that balances the misinformation held by \( S_2 \). For detailed results, see Table 3.

| Realization | \( \pi^\text{sx}_+ \) | \( \pi^\text{sx}_- \) |
|-------------|-----------------|-----------------|
| \( p \)     | \( s_1 \ s_2 \ t \) | \( \{1\} \{2\} \{1\} \{2\} \{12\} \{1\} \{2\} \{1\} \{2\} \{12\} \) |
| 3/8         | 0 0 0           | \( \lg(5/6) \ \lg(5/6) \ \lg(5/6) \ \lg(16/15) \) |
| 3/8         | 1 1 1           | \( \lg(5/6) \ \lg(5/6) \ \lg(5/6) \ \lg(16/15) \) |
| 1/8         | 0 1 0           | \( \lg(5/7) \ \lg(7/4) \ \lg(7/4) \ \lg(16/7) \) |
| 1/8         | 1 1 0           | \( \lg(5/7) \ \lg(7/4) \ \lg(7/4) \ \lg(16/7) \) |

Average Values: \( 0.557 \ 0.443 \ 0.443 \ 0.367 \ 0.0 \ 0.811 \ 0 \)

\( \Pi^\text{sx}(T : \{1\} \{2\}) = 0.557 \text{ bit} \quad \Pi^\text{sx}(T : \{1\}) = 0.443 \text{ bit} \quad \Pi^\text{sx}(T : \{2\}) = -0.367 \text{ bit} \quad \Pi^\text{sx}(T : \{12\}) = 0.367 \text{ bit} \)

Table 3: Example RNDErr. (Left): probability mass diagrams for each realisation. (Right): the pointwise partial information decomposition for the informative and misinformative. (Bottom): the average partial information decomposition.
9.3 Probability Distribution 3-bit Parity

Let $S_1, S_2, S_3$ be independent, uniformly distributed random bits, and $T = \sum_{i=1}^{3} S_i \mod 2$. This distribution is the 3-bit parity, where $T$ indicates the parity of the total number of 1-bits in $(S_1, S_2, S_3)$ is even and 1-bit otherwise. Note that all possible realizations occur with probability $\frac{1}{8}$ and result in the same PPID as well as the average PID due to the symmetry of the variables. Table 4 shows the informative, misinformation, and their difference for any realization.

\begin{table}[h]
\centering
\begin{tabular}{ccc}
\hline
\{123\} & \{12\} & \{13\} & \{23\} \\
0.2451 & 0.1699 & 0.1699 & 0.1699 \\
\{12\} & \{13\} & \{23\} & \{12\}\{13\}\{23\} \\
0.0931 & 0.0931 & 0.0931 & 0.0182 \\
\{1\} & \{2\} & \{3\} & \{1\}\{2\}\{3\} \\
0.3219 & 0.3219 & 0.3219 & 0.1699 \\
\{1\}\{2\} & \{1\}\{3\} & \{2\}\{3\} & \{1\}\{2\}\{3\} \\
0.2224 & 0.2224 & 0.2224 & 0.3219 \\
\{1\}\{2\}\{3\} & \{1\} & \{2\} & \{3\} \\
0.1926 & 0.3219 & 0.3219 & 0.3219 \\
\hline
\end{tabular}
\end{table}

Table 4: Example 3-bit Parity. (Left): the average informative partial information decomposition is evaluated. (Right): the average misinformative partial information decomposition is evaluated. (Center): the average partial information decomposition is evaluated.

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separating ambiguities and specificities. We also note that a similar idea to separate positive and negative contributions has been independently suggested by Robin Ince [9].

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A Lattice Structure: Supporting Proofs and Further Details

In this appendix, we will provide the proofs for the construction of informative and misinformativellattices equipped with the positive and negative pointwise shared information $i\text{sx}^\pm$ and $i\text{sx}^-\cap$ respectively. Subsequently, we will show the nonnegativity of the Moebuis inversion over each of the lattices that yield the pointwise partial information decomposition terms.

A.1 Informative and Misininformative Lattices

Finn and Lizier [5] adapted the Williams and Beer axioms to the pointwise framework, denoted as PPID axioms, in order to construct the specificity and ambiguity lattices. Similarly, for the construction of the informative and misinformative lattices for $i\text{sx}^\pm$ we used the PPID axioms but with a slight twist in the Monotonicity axiom (Section 6). Despite this alteration of the Monotonicity axiom, the lattices are constructed via the same underlying structure, namely, antichain that was used to build the specificity and ambiguity lattices [5]. We start by showing that $i\text{sx}^\pm$ satisfies the PPID axioms stated in Section 6 to then further construct the lattices.

**Theorem A.1.** $i\text{sx}^\pm$ satisfy Axioms 6.1, 6.2, and 6.3.

**Proof.** By the symmetry of intersection, $i\text{sx}^\pm$ defined in (14) satisfy the symmetry Axiom 6.1. For any collection $\mathbf{a}$, using (14), the informative and misinformative shared information are

$$i\text{sx}^+(t : \mathbf{a}) = \log \frac{1}{p(\mathbf{a})} = h(\mathbf{a})$$

$$i\text{sx}^-(t : \mathbf{a}) = \log \frac{p(t)}{p(t, \mathbf{a})} = h(\mathbf{a} | t).$$

and so they satisfy Axiom 6.3. For Axiom 6.2, note that

$$P(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m, \bar{a}_{m+1}) \leq P(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m)$$

This implies that $i\text{sx}^\pm$ decrease monotonically if joint source realizations are added, where the equality

$$P(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m, \bar{a}_{m+1}) = P(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m)$$

holds iff there exists $i \in \{1, \ldots, m\}$ such that $\bar{a}_{m+1} \supseteq \bar{a}_i$, i.e., iff there exists $i \in \{1, \ldots, m\}$ such that $\mathbf{a}_{m+1} \subseteq \mathbf{a}_i$. \qed
Let \( s = (s_1, \ldots, s_n) \) be any source realization and \([n] = \{1, \ldots, n\}\) be the index set of \( s \). Now define the Hasse lattice \( (\mathcal{P}(s), \subseteq) \) over the power set \( \mathcal{P}([n]) \) equipped with the ordering \( \subseteq \). So the elements \( a \in \mathcal{P}(s) \setminus \{\emptyset\} \) (denoted by \( \mathcal{P}_1(s) \)) are the index sets of the source events of any collection of sources. For example, the index set \( \{1, 2\} \in \mathcal{P}_1(s) \) represents the source event \( s_1 \cap s_2 \) and the realization \((s_1, s_2)\).

Williams and Beer relied on all combinations of collections \( a \) of sources in order to realise all the information contributions by the sources about the target, i.e., PID. So, \( a \in \mathcal{P}_1(\mathcal{P}_1(s)) \) are sets of collections of indices of sources events. For example, the set \( \{(1, 2), (1, 3)\} \in \mathcal{P}_1(\mathcal{P}_1(s)) \) represents the source event \((s_1 \cap s_2) \cup (s_1 \cap s_3)\) and the collection of realizations \((s_1, s_2) \) and \((s_1, s_3)\).

Therefore, the lattice \((\mathcal{P}_1(\mathcal{P}_1(s)), \subseteq)\) of all combinations of collections \( a \) will be used to represent all the information contributions of the sources about the target.

In order to quantify all the information contributions by the sources about the target, we will endow the lattice \((\mathcal{P}_1(\mathcal{P}_1(s)), \subseteq)\) with the function \( i_{\pi}^{\pm} \). So, for any \( a \in (\mathcal{P}_1(\mathcal{P}_1(s)), \subseteq) \), define the following

\[
\begin{align*}
\mathcal{P}(a) & = \mathcal{P}\left( \bigcup_{a \in \mathcal{P}_1(\mathcal{P}_1(s))} \mathcal{P}(a) \right) \\
\mathcal{P}(t, a) & = \mathcal{P}\left( \bigcup_{a \in \mathcal{P}_1(\mathcal{P}_1(s))} \mathcal{P}(t \cap a) \right) \\
i_{\pi}^{\pm}(t : a) & = \log \frac{1}{\mathcal{P}(a)} - \log \frac{\mathcal{P}(t)}{\mathcal{P}(t \cap a)} \\
& = i_{\pi}^{\pm +}(t : a) - i_{\pi}^{\pm -}(t : a).
\end{align*}
\]

In fact, as discussed by Finn-Lizier for a pointwise PID [5], we are going to endow two copies of \((\mathcal{P}_1(\mathcal{P}_1(s)), \subseteq)\) with \( i_{\pi}^{\pm +} \) and \( i_{\pi}^{\pm -} \) to get the informative (specificity) information contributions and the misinformative (ambiguity) ones respectively. Nonetheless, the equality condition of Axiom 6.2 satisfied by \( i_{\pi}^{\pm} \) implies that instead of using \((\mathcal{P}_1(\mathcal{P}_1(s)), \subseteq)\) as an underlying structure, we can restrict ourselves to \( \mathcal{A}(s) \), the lattice of the antichains of \( \mathcal{P}_1(s) \) — sets of subsets in \( \mathcal{P}_1(s) \) which are not comparable under inclusion — if such a lattice can be constructed. Fortunately, Crampton et al. [3] showed that there exists a lattice \((\mathcal{A}(s), \subseteq)\) where

\[
\alpha \preceq \beta \iff \forall \ b \in \beta, \exists \ a \in \alpha \ | \ a \subseteq b \ \forall \, \alpha, \beta \in \mathcal{A}(s).
\]

Therefore, following the footsteps of Finn-Lizier, namely, the specificity (informative) and ambiguity (misinformative) lattices for local partial information decomposition, we assume that \( i_{\pi}^{\pm} \) at each node \( \alpha \) are cumulative functions that aggregate all the nodes \( \beta \) in the down set of \( \alpha \), i.e.,

\[
i_{\pi}^{\pm}(t : \alpha) = \sum_{\beta \preceq \alpha} i_{\pi}^{\pm}(t : \beta) \ \forall \, \alpha, \beta \in \mathcal{A}(s).
\]  

(19)

Each atom \( \pi_{\pm}^{\mathcal{A}} \) is a term of the pointwise partial information decomposition (PPID). Thus, we derive below a closed form of the Moebius inversion of (19) that is used to obtain each term of the PPID and in the next section to prove that the informative and misinformative PPID terms are nonnegative.

**Theorem A.2.** Let \( i_{\pi}^{\pm} \) be measures on the redundancy lattice, then we have the following closed form for each atom \( \pi_{\pm}^{\mathcal{A}} \)

\[
\pi_{\pm}^{\mathcal{A}}(t : \alpha) = i_{\pi}^{\pm}(t : \alpha) - \sum_{\emptyset \neq \mathcal{B} \subseteq \alpha} (-1)^{|\mathcal{B}| - 1} i_{\pi}^{\pm}(t : \bigwedge \mathcal{B}).
\]

(20)

The proof of the above theorem follows from that of [5, Theorem A1].

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A.2 Nonnegativity of $\pi^{\pm}_{\gamma}$

In order for our information decomposition to be interpretative, the informative and misinformative atoms, $\pi^{\pm}_{\gamma}$, must be nonnegative. As a start, it is easy to see that the informative and misinformative redundancies, $\pi_{\gamma}^{\pm}(t : \{1\} \{2\} \ldots \{n\})$, are nonnegative since $\pi^{\pm}_{\gamma}$ are nonnegative.

Proposition A.1. $\pi^{\pm}_{\gamma}$ are nonnegative.

Proof. $\pi^{\pm}_{\gamma}(t : a_{1} : a_{2} : \ldots : a_{m}) = \log \frac{1}{P(a_{1} \cup a_{2} \cup \ldots \cup a_{m})} \geq 0$. Similarly, $\pi^{\pm}_{\gamma}(t : a_{1} : a_{2} : \ldots : a_{m}) = \log \frac{1}{P(\bigcap(t \in (\cap(t \in a_{1} \cup a_{2} \cup \ldots \cup a_{m}) \cup \ldots \cup \cap(t \in a_{m})))} \geq 0$. □

To prove the nonnegativity of $\pi^{\pm}_{\gamma}(t : \alpha)$ for any $\alpha \in \mathcal{A}(s)$, we need a couple of auxiliary theorems. We start by showing that $\pi^{\pm}_{\gamma}$ is monotonically increasing over $\mathcal{A}(s)$ (Theorem 6.2, Section 6).

Theorem A.3. $\pi^{\pm}_{\gamma}$ increase monotonically on the lattice $\langle \mathcal{A}(s), \preceq \rangle$.

Proof. Let $\alpha, \beta \in \mathcal{A}(s)$ and $\alpha \preceq \beta$. Then $\alpha$ and $\beta$ are of the form $\alpha = \{a_{1}, \ldots, a_{m}\}$ and $\beta = \{b_{1}, \ldots, b_{k}\}$. Because $\alpha \preceq \beta$ there is a function $f : \beta \rightarrow \alpha$ such that $f(b) \subseteq b$ \(^8\). Now we have for all $b \in \beta$

$$\bigcap_{i \in b} g_{i} \subseteq \bigcap_{i \in f(b)} g_{i}$$

Hence,

$$P(\beta) = P \left( \bigcup_{b \in \beta, i \in b} g_{i} \right) \leq P \left( \bigcup_{b \in \beta, i \in f(b)} g_{i} \right) \leq P \left( \bigcup_{a \in \alpha, i \in a} g_{i} \right) = P(\alpha)$$

(21)

The last inequality is true because the term on its L.H.S. is the probability of a union of intersections related to collections $\alpha \in \mathcal{A}$ (the $f(b)$), i.e. it is the probability of a union of events of the type $\bigcap_{i \in \alpha} g_{i}$. The probability of such a union can only get bigger if we take it over all events of this type. Using (21), it immediately follows that $\pi^{\pm}_{\gamma}(t : \alpha) \leq \pi^{\pm}_{\gamma}(t : \beta)$ and $\pi^{\pm}_{\gamma}$ is monotonically increasing. Using the same argument, $\pi^{\pm}_{\gamma}$ is monotonically increasing. □

Since $\pi^{\pm}_{\gamma}$ are convex over the probability mass of $\mathcal{A}(s)$, we state the first order convexity condition that will play a key role in proving the nonnegativity of the Moebuis inversion of (19).

Theorem A.4 (Theorem 2.67 [13]). Let $f : \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a continuously differentiable function. Then, $f$ is convex if and only if for all $x$ and $y$

$$f(y) \geq f(x) + \nabla f(x)(y - x).$$

Using Theorem A.3, the convex functions of interest, $\pi^{\pm}_{\gamma}$, are monotonically decreasing as functions of the probability mass over $\mathcal{A}(s)$ and thus satisfy the following property.

Proposition A.2. $f : \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a continuously differentiable convex function. If $f$ is monotonically decreasing, then for any $x_{0} \leq y_{0}$

$$-\sum_{i} \frac{\partial h}{\partial x_{i}}(y_{0}) \leq -\sum_{i} \frac{\partial h}{\partial x_{i}}(x_{0}).$$

such that $(y_{0})_{i} - (x_{0})_{i} = c \geq 0$ for all $i$.

\(^8\)This function does not have to be surjective: Suppose $\alpha = \{\{1\}, \{2\}, \{3\}\}$ and $\beta = \{\{1, 2, 3, 4\}\}$. Then necessarily two sets in $\alpha$ will not be in the image of $f$. It also does not have to be injective. Consider $\alpha = \{1\}$ and $\beta = \{\{1\}, \{3\}\}$. Then both elements of $\beta$ have to be mapped to the only element of $\alpha$.
Proof. For any \( x_0, y_0 \in \mathbb{R}^n \), using Theorem A.4 by interchanging the roles of \( x_0 \) and \( y_0 \),
\[
-\nabla^T h(y_0) (y_0 - x_0) \leq h(x_0) - h(y_0) \leq -\nabla^T h(x_0) (y_0 - x_0).
\]
Now if \( x_0 \leq y_0 \) and \( (y_0)_i - (x_0)_i = c \geq 0 \), then
\[
-e\nabla^T h(y_0)^{1} \leq -e\nabla^T h(x_0)^{1} - \sum_i \frac{\partial h}{\partial x_i}(y_0) \leq -\sum_i \frac{\partial h}{\partial x_i}(x_0).
\]
\[
\square
\]

Before proceeding to show that \( \pi^{\text{sx}}(t : \alpha) \) are nonnegative given that \( \pi^{\text{sx}} \) is the redundancy measure, we construct a family of mappings from \( \mathcal{P}(\alpha^-) \) where \( \alpha^- \) is the set of children of \( \alpha \) to the \( \mathcal{A}(s) \), e.g., Figure 5 presents this family of mappings for \( \alpha = \{123\} \). This family of mappings will be used to exploit (20) and thereupon the proof of the nonnegativity we are aiming for.

Proposition A.3. Let \( \alpha \in \mathcal{A}(s) \) and \( \alpha^- = \{\gamma_1, \ldots, \gamma_k\} \) ordered increasingly w.r.t. the probability mass be the set of children of \( \alpha \) on \( \langle \mathcal{A}(s), \leq \rangle \). Then, for any \( 1 \leq i \leq k \)
\[
f_i : \mathcal{P}(\alpha^- \setminus \{\gamma_i\}) \cup \{\alpha\} \rightarrow \mathcal{A}(s) \quad \mapsto \quad \bigwedge_{\beta \in B} \beta \land \gamma_i
\]
is a mapping such that \( \mathbb{P}(f_i(B)) = \mathbb{P}(\bigwedge_{\beta \in B} \beta) + d_i \) where \( d_i = \mathbb{P}(\gamma_i) - \mathbb{P}(\alpha) \) and the complement is taken w.r.t. \( \mathcal{P}(\alpha^-) \).

Proof. Since \( \gamma_i \in \alpha^- \) and \( \beta \in \alpha^- \) for any \( \beta \in B \), then \( (\bigwedge_{\beta \in B} \beta) \lor \gamma_i = \alpha \). Now, for any \( B \in \mathcal{P}(\alpha^- \setminus \{\gamma_i\}) \) using the inclusion-exclusion, \( \beta \land \gamma_i = \beta \cup \gamma_i \) and \( \beta \lor \gamma_i = \uparrow \beta \lor \gamma_i \),
\[
\mathbb{P}(f_i(B)) = \mathbb{P}(\bigwedge_{\beta \in B} \beta \land \gamma_i) = \mathbb{P}(\bigwedge_{\beta \in B} \beta) + \mathbb{P}(\gamma_i) - \mathbb{P}(\bigwedge_{\beta \in B} \beta \lor \gamma_i) = \mathbb{P}(\beta) + \mathbb{P}(\gamma_i) - \mathbb{P}(\alpha).
\]
\[
\square
\]

The following lemma shows that for any node \( \alpha \in \mathcal{A}(s) \), the recursive equation (20) should be nonnegative. This lemma will be used in the proof of Theorem 6.3 to conclude the nonnegativity of the informative and misinformative atoms of \( \langle \mathcal{A}(s), \preceq \rangle \) endowed with \( \mathbb{P}^\alpha \).

Lemma A.1. Let \( \alpha \in \mathcal{A}(s) \), then
\[
-\log \mathbb{P}(\alpha) + \sum_{B \subseteq \alpha^-} (\mathcal{B})^{-1} \log \mathbb{P}(\bigwedge B) \geq 0.
\]

Proof. Suppose that \( |\alpha^-| = k \) and w.l.o.g. that \( \alpha^- = \{\gamma_1, \ldots, \gamma_k\} \) is ordered increasingly w.r.t. the probability mass. The proof will follow by induction over \( k = |\alpha^-| \). We will demonstrate the inequality (22) for \( k = 3, 4 \) to show the induction basis.

For \( k = 3 \), the L.H.S. of (22) can be written as
\[
\log \frac{\mathbb{P}(\gamma_1) \mathbb{P}(\gamma_2) \mathbb{P}(\gamma_3) \mathbb{P}(\gamma_1 \land \gamma_2 \land \gamma_3)}{\mathbb{P}(\alpha) \mathbb{P}(\gamma_1 \land \gamma_2) \mathbb{P}(\gamma_1 \land \gamma_3) \mathbb{P}(\gamma_2 \land \gamma_3)} = \log \frac{\mathbb{P}(\gamma_1) \mathbb{P}(\gamma_2)}{\mathbb{P}(\alpha) \mathbb{P}(\gamma_1 \land \gamma_2)} - \log \frac{\mathbb{P}(\gamma_1 \land \gamma_3) \mathbb{P}(\gamma_2 \land \gamma_3)}{\mathbb{P}(\alpha \land \gamma_3) \mathbb{P}(\gamma_1 \land \gamma_2 \land \gamma_3)}
\]
\[
= \log \frac{\mathbb{P}(\alpha) + d_1}{\mathbb{P}(\alpha) + d_2} - \log \frac{\mathbb{P}(\alpha) + d_3 + d_4}{\mathbb{P}(\alpha)} = [h_3(\mathbb{P}(\alpha)) - h_3(\mathbb{P}(\alpha) + d_2)]
\]
\[
- [h_3(\mathbb{P}(\alpha) + d_3) - h_3(\mathbb{P}(\alpha) + d_4 + d_2)],
\]
\[
23
\]
Figure 5: The family of mappings from $P_1(\alpha - \gamma_i) \cup \{\alpha\}$ to $A(s)$, introduced in Proposition A.3, that preserve the probability mass difference where $\alpha$ is the synergy of the trivariate lattice. The orange region is $\alpha^-$, the set of children of $\alpha$. Each color depicts one of the mappings in the family based on some $\gamma \in \alpha^-$. The blue mapping is based on $\gamma_1$, the green mapping is based on $\gamma_2$ and the reddish brown mapping is based on $\gamma_3$.

Figure 6: Depiction of set differences corresponding to the probability mass difference introduced in Proposition A.3 and shown in Fig. 5, for the sets from Fig. 2.

where $h_3(x) = \log(1 + d_1/x)$, $d_i := P(\gamma_i) - P(\alpha)$ for $i \in \{1, 2, 3\}$, and $d_3 \geq d_2 \geq d_1 \geq 0$. Note that $h_3$ is a continuously differentiable convex function that is monotonically decreasing.


\[ x = \mathbb{P}(\alpha) \text{ and } y = \mathbb{P}(\alpha) + d_3, \text{ then} \]

\[
\begin{align*}
  h_3(\mathbb{P}(\alpha)) - h_3(\mathbb{P}(\alpha) + d_2) \\
  \geq -d_2 h'_3(\mathbb{P}(\alpha) + d_2) \\
  \geq -d_2 h'_3(\mathbb{P}(\alpha) + d_3) \\
  \geq h_3(\mathbb{P}(\alpha) + d_3) - h_3(\mathbb{P}(\alpha) + d_3 + d_2)
\end{align*}
\]

where the first and third inequalities hold using Theorem A.4 and the second inequality holds using Proposition A.2 and so the inequality (22) holds when \( k = 3 \).

For \( k = 4 \), we have \( \alpha^- = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \) ordered increasingly w.r.t. the probability mass. Using Proposition A.3 on \( \alpha^- \), the L.H.S. of (22) can be written as

\[
\begin{align*}
\left[ h_3(\mathbb{P}(\alpha)) - h_3(\mathbb{P}(\alpha) + d_2) - \left( h_3(\mathbb{P}(\alpha) + d_3) - h_3(\mathbb{P}(\alpha) + d_3 + d_2) \right) \right] \\
- \left[ h_3(\mathbb{P}(\alpha) + d_4) - h_3(\mathbb{P}(\alpha) + d_4 + d_2) - \left( h_3(\mathbb{P}(\alpha) + d_4 + d_3) - h_3(\mathbb{P}(\alpha) + d_4 + d_3 + d_2) \right) \right] \\
= \left[ h_4(\mathbb{P}(\alpha), \mathbb{P}(\alpha) + d_2) - h_4(\mathbb{P}(\alpha) + d_3, \mathbb{P}(\alpha) + d_3 + d_2) \right] \\
- \left[ h_4(\mathbb{P}(\alpha) + d_4, \mathbb{P}(\alpha) + d_4 + d_2) - h_4(\mathbb{P}(\alpha) + d_4 + d_3, \mathbb{P}(\alpha) + d_4 + d_3 + d_2) \right],
\end{align*}
\]

where \( d_i := \mathbb{P}(\gamma_i) - \mathbb{P}(\alpha) \) for \( i \in \{2, 3, 4\} \), \( d_4 \geq d_3 \geq d_2 \geq 0 \), and \( h_4(x_1, x_2) = \log(1 + \delta x_2 + x_1 + \delta d_4) \). Note that on the set \( H_4 := \{ x \in \mathbb{R}_{++}^2 \mid x_2 = x_1 + \delta \text{ and } \delta \geq 0 \} \), the function \( h_4 \) is monotonically decreasing on \( H_4 \) since the inequality (22) holds when \( k = 3 \). In addition, \( h_4 \) is convex since for any \( x, y \in H_4 \) and \( \theta \in [0, 1] \)

\[
\theta h_4(x) + (1 - \theta) h_4(y) - h_4(\theta x + (1 - \theta)y)
\]

is nonnegative as \( h_3 \) is convex and monotonically decreasing. Now, take \( x = (\mathbb{P}(\alpha), \mathbb{P}(\alpha) + d_2) \) and \( y = (\mathbb{P}(\alpha) + d_4, \mathbb{P}(\alpha) + d_4 + d_2) \), then

\[
\begin{align*}
  h_4(\mathbb{P}(\alpha), \mathbb{P}(\alpha) + d_2) - h_3(\mathbb{P}(\alpha) + d_3, \mathbb{P}(\alpha) + d_3 + d_2) \\
  \geq -\nabla h_4(\mathbb{P}(\alpha) + d_3, \mathbb{P}(\alpha) + d_3 + d_2) (d_3, d_3) \\
  \geq -\nabla h_4(\mathbb{P}(\alpha) + d_4, \mathbb{P}(\alpha) + d_4 + d_2) (d_3, d_3) \\
  \geq h_4(\mathbb{P}(\alpha) + d_4, \mathbb{P}(\alpha) + d_4 + d_2) - h_4(\mathbb{P}(\alpha) + d_4 + d_3, \mathbb{P}(\alpha) + d_4 + d_3 + d_2),
\end{align*}
\]

where the first and third inequalities hold using Theorem A.4 and the second inequality holds using Proposition A.2 and so the the inequality (22) holds.

Suppose that the inequality holds for \( k \) and let us proof it for \( k+1 \). Here \( \alpha^- = \{\gamma_1, \gamma_2, \ldots, \gamma_{k+1}\} \) and using Proposition A.3, the L.H.S. of (22) can be written as

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where \( a_{k-2} := (P(\alpha), \ldots, P(\alpha) + \sum_{i=2}^{k-2} d_i) \in \mathbb{R}^{2^{k-2}} \), \( d_i := P(\gamma_i) - P(\alpha) \) for \( i \in \{2, \ldots, k+1\} \), \( d_{k+1} \geq \cdots \geq d_2 \geq 0 \), and \( h_{k+1}(x_1, \ldots, x_{2^{k-1}}) = h_k(x_1, \ldots, x_{2^{k-1}}) - h_k(x_2, x_3, \ldots, x_{2^{k-1}}) \).

Note that on the set \( H_{k+1} := \{ x \in \mathbb{R}^{2^{k-1}} \mid x_i = x_j \pm \delta, i \equiv j \mod 2^{k-2}, \delta \geq 0 \} \) the function \( h_{k+1} \) is monotonically decreasing because the inequality (22) holds for \( k \). In addition, \( h_{k+1} \) is convex since for any \( x, y \in H_{k+1} \) and \( \theta \in [0,1] \)

\[
\begin{align*}
\theta h_{k+1}(x_1, \ldots, x_{2^{k-1}}) + (1-\theta)h_{k+1}(y_1, \ldots, y_{2^{k-1}}) - h_{k+1}(\theta x_1 + (1-\theta)y_1, \ldots, \theta x_{2^{k-1}} + (1-\theta)y_{2^{k-1}}) \\
= \left[ \theta h_k(x_1, \ldots, x_{2^{k-1}}) + (1-\theta)h_k(y_1, \ldots, y_{2^{k-1}}) - h_k(\theta x_1 + (1-\theta)y_1, \ldots, \theta x_{2^{k-1}} + (1-\theta)y_{2^{k-1}}) \right] \\
- h_k(\theta x_1 + (1-\theta)y_1 + \delta, \ldots, \theta x_{2^{k-1}} + (1-\theta)y_{2^{k-1}} + \delta).
\end{align*}
\]

is nonnegative. Now, take \( x = (a_{k-2}, a_{k-2} + d_{k-1} \mathbf{1}_{k-2}) \) and \( y = (a_{k-2} + d_{k+1} \mathbf{1}_{k-2}, a_{k-2} + (d_{k+1} + d_{k-1}) \mathbf{1}_{k-2}) \), then

\[
\begin{align*}
h_{k+1}(a_{k-2}, a_{k-2} + d_{k-1} \mathbf{1}_{k-2}) - h_{k+1}(a_{k-2} + d_k \mathbf{1}_{k-2}, a_{k-2} + (d_k + d_{k-1}) \mathbf{1}_{k-2}) \\
\geq -d_k \nabla^T h_{k+1}(a_{k-2} + d_k \mathbf{1}_{k-2}, a_{k-2} + (d_k + d_{k-1}) \mathbf{1}_{k-2}) \mathbf{1}_{k-1} \\
\geq -d_k \nabla^T h_{k+1}(a_{k-2} + d_{k+1} \mathbf{1}_{k-2}, a_{k-2} + (d_{k+1} + d_{k-1}) \mathbf{1}_{k-2}) \mathbf{1}_{k-1} \\
\geq h_{k+1}(a_{k-2} + d_{k+1} \mathbf{1}_{k-2}, a_{k-2} + (d_{k+1} + d_{k-1}) \mathbf{1}_{k-2}) - h_{k+1}(a_{k-2} + d_{k-1} \mathbf{1}_{k-2}, a_{k-2} + (d_{k-1} + d_{k-1}) \mathbf{1}_{k-2}),
\end{align*}
\]

where the first and third inequalities hold using Theorem A.4 and the second inequality holds using Proposition A.2 and so the inequality (22) holds for \( k+1 \).

Now we restate Theorem 6.3 and prove it using Lemma A.1. This theorem shows the existence of nonnegative pointwise partial information decomposition atoms \( \pi_{\pm}^{\text{XX}} \) when the redundancy lattice \( \mathcal{A}(s) \) is endowed with \( \tilde{\gamma}^{\pm}_{\pm} \) as the pointwise shared information measures.

**Theorem A.5.** The atoms \( \pi_{\pm}^{\text{XX}} \) of the positive and negative lattice \( (\mathcal{A}(s), \subseteq) \), are non-negative.

**Proof.** For any \( \alpha \in \mathcal{A}(s) \),

\[
\pi_{\pm}^{\text{XX}}(t : \alpha) = \tilde{\gamma}^{\pm}_{\pm}(t : \alpha) - \sum_{\emptyset \neq B \subseteq \alpha} (-1)^{|B| - 1} \pi_{\pm}^{\text{XX}}(t : \bigwedge B) \\
= - \log P(\alpha) + \sum_{\emptyset \neq B \subseteq \alpha} (-1)^{|B|-1} \log P(\bigwedge B).
\]

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So, by Lemma A.1 $\pi^+(t : \alpha) \geq 0$. Similarly, $\pi^-(t : \alpha) \geq 0$ since

$$\pi_t^\pm(t : \alpha) = - \log P(t \cap \alpha) + \sum_{\emptyset \neq B \subseteq \alpha} (-1)^{|B| - 1} \log P(t \cap B).$$

and intersecting with $t$ has no affect on the nonnegativity shown in Lemma A.1.

\[ \square \]

**B Definition of $i_{\cap}^{sx}$ starting from a general probability space**

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $S_1, ..., S_n, T$ be discrete and finite random variables on that space, i.e.

\[ S_i : \Omega \rightarrow \mathcal{A}_{S_i}, \quad (\mathcal{A}, \mathcal{P}(\mathcal{A}_{S_i})) \text{ measurable} \]

\[ T : \Omega \rightarrow \mathcal{A}_T, \quad (\mathcal{A}, \mathcal{P}(\mathcal{A}_T)) \text{ measurable}, \]

where $\mathcal{A}_{S_i}$ and $\mathcal{A}_T$ are the finite alphabets of the corresponding random variables and $\mathcal{P}(\mathcal{A}_{S_i})$ and $\mathcal{P}(\mathcal{A}_T)$ are the power sets of these alphabets. Given a subset of source realization indices $a \subseteq \{1, ..., n\}$ the local mutual information of source realizations $(s_i)_{i \in a}$ about the target realization $t$ is defined as

$$i(t : (s_i)_{i \in a}) = i(t : a) = \log \frac{P(t \cap \bigcup_{i \in a} s_i)}{P(t)}.$$

The local shared information of an antichain $\alpha = \{a_1, ..., a_m\}$ (representing a set of collections of source realizations) about the target realization $t \in \mathcal{A}_T$ is defined in terms of the original probability measure $P$ as a function $i_{\cap}^{sx} : \mathcal{A}_T \times \mathcal{A}(s) \rightarrow \mathbb{R}$ with

$$i_{\cap}^{sx}(t : \alpha) = i_{\cap}^{sx}(t : a_1; \ldots; a_m) := \log \frac{P(t \cap \bigcup_{i=1}^m a_i)}{P(t)}.$$

An special case of this quantity is the local shared information of a complete sequence of source realizations $(s_1, ..., s_n)$ about the target realization $t$. This is obtained by setting $a_i = \{i\}$ and $m = n$:

$$i_{\cap}^{sx}(t : \{1; \ldots; n\}) = \log \frac{P(t \cap \bigcup_{i=1}^n s_i)}{P(t)}.$$

In contrast to other shared information terms, this is an atomic quantity corresponding the very bottom of the lattice of antichains. Rewriting $i_{\cap}^{sx}$ allows us to decompose it into the difference of two positive parts:

$$i_{\cap}^{sx}(t : a_1, ..., a_m) = \log \frac{P(t \cap \bigcup_{i=1}^m a_i)}{P(t)P(\bigcup_{i=1}^m a_i)} = \log \frac{1}{P(\bigcup_{i=1}^m a_i)} - \log \frac{P(t)}{P(t \cap \bigcup_{i=1}^m a_i)},$$

using standard rules for the logarithm. We call

$$i_{\cap}^{sx}^+(t : a_1, ..., a_m) := \log \frac{1}{P(\bigcup_{i=1}^m a_i)}$$

the informative local shared information and

$$i_{\cap}^{sx}^-(t : a_1, ..., a_m) := \log \frac{P(t)}{P(t \cap \bigcup_{i=1}^m a_i)}$$

the misinformative local shared information.