Unitary Gate Synthesis for Continuous Variable Systems

Jaromír Fiurášek
Ecole Polytechnique, CP 165, Université Libre de Bruxelles, 1050 Brussels, Belgium and Department of Optics, Palacký University, 17. listopadu 50, 77200 Olomouc, Czech Republic

We investigate the synthesis of continuous-variable two-mode unitary gates in the setting where two modes $A$ and $B$ are coupled by a fixed quadratic Hamiltonian $H$. The gate synthesis consists of an adiabatic evolution governed by Hamiltonian $H$ interspersed by local phase shifts applied to $A$ and $B$. We concentrate on protocols that require the minimum necessary number of steps and we show how to implement the beam splitter and the two-mode squeezer in just three steps. Particular attention is paid to the Hamiltonian $x_{APB}$ that describes the effective off-resonant interaction of light with the collective atomic spin.

PACS numbers: 03.67.-a, 42.50.Dv

I. INTRODUCTION

One of the central problems of the quantum information theory is to establish what resources are sufficient for universal quantum computation. In this context, the question whether a given Hamiltonian $H$ can simulate another one has attracted considerable attention recently [1, 2, 3, 4, 5, 6, 7, 8]. In its simplest form, this problem may be formulated as follows. Consider two parties, traditionally referred to as Alice and Bob, possessing a single qubit each. The interaction between those two qubits is governed by a fixed Hamiltonian $H$, that is determined by the physical properties of the systems that represent the qubits. In addition to the interaction $H$, Alice and Bob may attach (local) ancillas to their qubits and perform arbitrary local unitary operations on their subsystems. It is usually assumed that these local operations are very fast compared to the evolution induced by the Hamiltonian $H$. The task for Alice and Bob is to simulate the evolution due to different Hamiltonian $H'$. Two kinds of simulations should be distinguished. The infinitesimal time simulation [1, 2, 3, 4, 5, 6] consists of simulating the action of the Hamiltonian $H'$ for an infinitesimally short time interval $\Delta t$. The gate synthesis [7, 8, 9, 10, 11, 12] requires the implementation of the unitary transformation $U' = \exp(-iH't)$ for some finite time $t$.

It turns out that in the two-party setting all nonlocal Hamiltonians $H$ are qualitatively equivalent. Given enough time $\tau$, Alice and Bob can, with the help of local ancillas, simulate the evolution $\exp(-iH't)$ for any $H'$ [2]. The central question, then, is what is the optimal simulation. The latter may be defined as a simulation that requires the shortest time. For the two-qubit case, this problem has been completely solved and the optimal protocols for Hamiltonian [2, 6] and unitary gate [8] simulations have been determined. The situation becomes much more complicated for higher dimensional systems and for higher number of involved parties. Simulation protocols suggested for these generic settings are rather involved and it is not known which protocols are optimal.

It should be stressed that most of the work focused on discrete variable systems: qubits or, more generally, qudits. Recently, however, Kraus et al. extended the notion of Hamiltonian simulation to continuous variable systems [13]. They assumed that Alice and Bob possess a single-mode system each and these two modes are coupled via quadratic Hamiltonian (we assume $\hbar = 1$ throughout this paper):

$$H = c_{11} x_A x_B + c_{12} x_{APB} + c_{21} p_A x_B + c_{22} p_{APB}.$$  \hspace{1cm} (1)

where $x_j$ and $p_j$ are two conjugate quadratures of the $j$th mode. Kraus et al. showed that almost every Hamiltonian [13] is capable to simulate any other Hamiltonian of the form (1) provided that Alice and Bob can apply fast local phase shifts described by single-mode Hamiltonians $H_A = x_A^2 + p_A^2$ and $H_B = x_B^2 + p_B^2$.

These results are interesting both from the theoretical and experimental points of view. In particular, the off-resonant interaction of light with the collective atomic spin [14, 15, 16, 17, 18, 19, 20] can be described by the effective unitary transformation

$$U = \exp(-itH_{AL}),$$  \hspace{1cm} (2)

where the Hamiltonian

$$H_{AL} = \kappa x_{APB}$$  \hspace{1cm} (3)

is a special instance of (1). The typical geometry of the experiments is such that a light with strong coherent field polarized along the $x$ axis propagates along the $z$ axis through the atomic sample, whose spin is also polarized along the $x$ axis. The $x$ and $p$ quadratures are defined as the properly normalized $y$- and $z$-components of the collective spin operators describing the polarization state of light and atomic ensemble, respectively [14, 16, 17]. In recent beautiful experiments it was demonstrated that the interaction [17] can be employed to squeeze the atomic spin [15], entangle two distant atomic ensembles [18], and transfer the quantum state of light into the atoms [19]. Schemes for teleportation and swapping of the quantum state of collective atomic spin have been suggested [16, 17]. These experiments and proposals in fact rely on the quantum non-demolition (QND) measurement of the atomic quadrature, possibly accompanied by a suitable feedback.
As showed by Kraus et al., the Hamiltonian (3) can simulate any Hamiltonian (1). In particular, \( H_{AL} \) can be used to implement a beam splitter and a two-mode squeezer. This is very appealing since it suggests that, for instance, the storage of the quantum state of light into atoms and the subsequent readout of the quantum memory—the transfer of quantum state of atoms onto light—can be implemented in a unitary way if the Hamiltonian (3) is used to simulate a beam splitter.

However, currently there are technical difficulties that will complicate the actual practical realization of this procedure. The effective unitary transformation (2) describes the modification of the polarization state of the light pulse after the passage through the atomic sample due to its coupling with atoms. This means that the Hamiltonian simulation requires several passages of the light pulse through the atomic sample (c.f. the detailed description of the simulation protocol in Sec. II). In currently envisaged experiments, the pulse width must be at least 1 \( \mu s \) \cite{21} which corresponds to a length 300 m. This long pulse would have to be stored somewhere (e.g. in an optical fiber) until its tail leaves the atomic sample. Only then can the pulse (with properly applied phase shifts) be fed to the atomic sample again.

These practical considerations imply that the approach relying on the infinitesimal time simulation is not very convenient from the experimental point of view. It is possible to simulate a gate by concatenating a large sequence of short-time Hamiltonian simulations but this would require a large number of manipulations and passages of the light pulse through the sample. Since, in practice, every round of the gate synthesis procedure is necessarily accompanied by some losses and other errors, the accumulation of the errors would negatively influence the simulation.

In this paper, we show how to implement several important two-mode interactions with the Hamiltonian (1) such that the number of the applications of the Hamiltonian (1) is minimized. We demonstrate that only three sequences of evolution governed by Hamiltonian \( H \), interspaced by (fast) local phase shifts on both subsystems, suffice to implement a two-mode squeezer and a beam splitter. For the specific Hamiltonian (3) we also design a single-mode squeezing gate that involves four evolution steps. These results illustrate that several important quantum information processing tasks such as entangling the light and collective atomic spin, or a transfer of the quantum state of light into atomic clouds and vice versa, can be carried out with a small number of repeated passages of the light pulse through the atomic sample.

This paper is structured as follows. In Sec. II we introduce the notation, the canonical form of the interaction Hamiltonian (1) and we describe the gate simulation protocol. In Sec. III, we consider the simple interaction Hamiltonian (3) and we show how to implement the two-mode squeezing operation, beam splitter transformation and also single-mode squeezing as a sequence of three (or four) intervals of evolution governed by Hamiltonian (3) combined with local phase shift operations. In Sec. IV we extend this analysis to the generic interaction Hamiltonians (1). Finally, Sec. V contains conclusions.

II. DESCRIPTION OF THE SIMULATION PROTOCOL

In this section we describe the simulation protocol. The gate synthesis consists of a sequence of \( N \) intervals of evolution governed by the Hamiltonian \( H \) followed by local unitary phase shift transformations. The resulting unitary gate \( G \) is given by

\[
G = V_N^1 e^{-iHt_N} V_N^2 \ldots V_1^2 e^{-iHt_2} V_1^1 e^{-iHt_1} V_1.
\]

The local phase shift operation applied to modes \( A \) and \( B \) reads

\[
V_j = e^{-i\phi_j A a_1^a} \otimes e^{-i\phi_j B b_1^b}, \tag{5}
\]

where \( a \) and \( b \) are the annihilation operators of modes \( A \) and \( B \), respectively. Note that \( V_j^1 V_j^2 \) is still of the form (4), with \( \phi_A = \phi_{j+1,A} - \phi_{j,A} \) and \( \phi_B = \phi_{j+1,B} - \phi_{j,B} \). With the help of the useful identity

\[
U^\dagger \exp(-iHt) U = \exp(-iU^\dagger H U t) \tag{6}
\]

we can rewrite Eq. (4) as

\[
G = e^{-iH_N t_N} \ldots e^{-iH_2 t_2} e^{-iH_1 t_1}, \tag{7}
\]

where \( H_j = V_j^1 HV_j \).

The Hamiltonian (1) is characterized by four parameters. However, by means of local rotations, we can always transform this Hamiltonian to a diagonal form

\[
H_c = c_1 x_A p_B + c_2 p_A x_B, \tag{8}
\]

where \( c_1 = \sigma_1 = c_2 = \sigma_2 \det|C|/\det|C| \), and \( \sigma_1 \) and \( \sigma_2 \) are the singular values of the matrix \( C \) defined as \( (C)_{ij} = c_{ij} \) \cite{13}. In close analogy to the qubit case \cite{2}, we may refer to \( H_c \) as the canonical form of \( H \). Mathematically, we have

\[
\exp(-iH_c t) = \tilde{V}^\dagger \exp(-iH t) \tilde{V}, \tag{9}
\]

where \( \tilde{V} \) is a local rotation \cite{6}. This shows that, without loss of generality, we may assume that \( H \) has the canonical form (8). In particular, it follows that \( H \) is able to simulate an arbitrary \( H' \) \cite{10} if and only if \( H_c \) is able to simulate an arbitrary canonical Hamiltonian (8).

In Eq. (9) the phase shifts \( \phi_j \) may be arbitrary. In what follows, we focus on the phase shifts that preserve the canonical form of \( H \). There are four inequivalent possibilities:

(a) \( \phi_A = 0, \phi_B = 0 \),

\[
H_1 = c_1 x_A p_B + c_2 p_A x_B. \tag{10}
\]
(b) \( \phi_A = \pi/2, \phi_B = 3\pi/2, \)

\[ H_2 = c_2x_{APB} + c_1p_{AXB}. \]  
(11)

(c) \( \phi_A = \pi, \phi_B = 0, \)

\[ H_3 = -c_1x_{APB} - c_2p_{AXB}. \]  
(12)

(d) \( \phi_A = \pi/2, \phi_B = \pi/2, \)

\[ H_4 = -c_2x_{APB} + c_1p_{AXB}. \]  
(13)

From the structure of these Hamiltonians we can deduce that two different noncommuting canonical Hamiltonians \( H_1 \) and \( H_2 \) are available. Furthermore, we can see that \( H_3 = -H_1 \) and \( H_4 = -H_2 \), hence we can implement any transformations of the form \( \exp(-iH_1t) \) and \( \exp(-iH_2t) \) where \( t \) is an arbitrary real number, positive or negative.

The two specific cases \( c_1 = c_2 \) and \( c_1 = -c_2 \) when \( H_1 = \pm H_2 \) and the simulation is not possible correspond to the Hamiltonians of a two-mode squeezer and a beam splitter, respectively.

### III. XP COUPLING

Having established the notation and described the simulation protocol, we may proceed to the unitary gate synthesis. Namely, we would like to decompose the unitary transformation \( G \) that we want to simulate into a sequence of unitary evolutions governed by Hamiltonians \( H_1 \) and \( H_2 \) that were defined in the previous section.

\[ G = e^{-iH_2t_N}e^{-iH_1t_{N-1}}...e^{-iH_2t_2}e^{-iH_1t_1}. \]  
(14)

We are particularly interested in the simulations that involve the lowest possible number of steps \( N \), because such simulations require low number of local manipulations in the eventual experimental implementation.

We note here that Eq. (14) is an example of a decomposition of a group element into a product of \( N \) other group elements. In the present case, the underlying group is the symplectic group \( Sp(4, R) \) of all linear canonical transformations of the quadratures of the two modes \( A \) and \( B \).\[ 22 \] It is worth mentioning here that the related problem of a decomposition of the symplectic transformation into a sequence of simple evolutions associated with the common passive and active linear optical elements has been studied recently. Braunstein has shown that any \( N \) mode symplectic transformation can be implemented as a sequence of an \( N \)-mode passive linear interferometer followed by \( N \) single-mode squeezers and another passive interferometer — the so-called Bloch-Messiah decomposition \[ 24 \]. The decompositions of this kind have also been applied to investigate the properties of nonlinear optical couplers \[ 25 \] \[ 26 \].

In this section, we shall consider the simplest and also the experimentally relevant coupling between the two systems described by the interaction Hamiltonian \[ 8 \].

Without loss of generality, we may assume that the coupling constant is equal to unity, hence the two relevant Hamiltonians read

\[ H_1 = x_{APB}, \quad H_2 = p_{AXB}. \]  
(15)

All the canonical Hamiltonians \[ 3 \] have the important property that the \( x \) and \( p \) quadratures are not mutually coupled when we write down the Heisenberg equations of motion for \( x_j \) and \( p_j \). This means that the evolution of the operators \( \mathbf{x} = (x_A, x_B)^T \) and \( \mathbf{p} = (p_A, p_B)^T \) is governed by the following linear canonical transformations:

\[ \mathbf{x}_{\text{out}} = S\mathbf{x}_{\text{in}}, \quad \mathbf{p}_{\text{out}} = R\mathbf{p}_{\text{in}}. \]  
(16)

This decoupling of \( x \) and \( p \) quadratures greatly simplifies the analysis. The transformation \( 10 \) must preserve the canonical commutation relations \( [x_j, p_k] = i\delta_{jk} \). From these conditions we can express the matrix \( R \) in terms of \( S \),

\[ R = (S^{-1})^T, \]  
(17)

hence the evolution of \( p \) quadratures is uniquely determined by the evolution of the \( x \) quadratures.

Our task is to implement two-mode unitary gates (symplectic transformations) as a sequence of a small number of unitary transformations generated by the Hamiltonians \[ 15 \]. The matrices \( S_1 \) and \( S_2 \) associated with the unitary evolutions \( U_1 = \exp(-iH_1t) \) and \( U_2 = \exp(-iH_2t) \) read

\[ S_1(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad S_2(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]  
(18)

The factorization \[ 14 \] can be rewritten in terms of the matrices \( S_j \) as follows,

\[ S = S_2(t_N)S_1(t_{N-1})...S_2(t_2)S_1(t_1), \]  
(19)

where \( S \) is the matrix associated with the gate \( G \). Since \( \det S_1 = \det S_2 = 1 \), we are restricted to a three parametric subgroup of transformations \( S \) such that \( \det S = 1 \). In what follows, we will discuss the implementation of three important gates: a beam splitter, a two-mode squeezer and a single-mode squeezer.

#### A. Beam splitter

The beam splitter operation is described by the matrix

\[ S_{BS}(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}. \]  
(20)

We show that this transformation can be implemented as a sequence of three evolutions \[ 13 \],

\[ S_{BS}(\phi) = S_1(\gamma)S_2(\beta)S_1(\alpha). \]  
(21)
The explicit multiplication yields
\[ S_{BS}(\phi) = \begin{pmatrix} 1 + \alpha \beta & \beta \\ \alpha + \gamma (1 + \alpha \beta) & 1 + \gamma \beta \end{pmatrix}. \] (22)

If we compare the elements of the matrices on left- and right-hand sides of Eq. (22), we obtain a set of equations for the parameters \( \alpha, \beta \), and \( \gamma \) whose solution yields:
\[ \alpha = -\tan \frac{\phi}{2}, \quad \beta = \sin \phi, \quad \gamma = \alpha. \] (23)

The parameters \( \alpha \) and \( \beta \) can be negative but this is not an obstacle as explained in the previous section since we can change the sign of the Hamiltonian \( H_1 \) or \( H_2 \) by \( \pi \) rotation of one of the systems. Two cases of particular importance are (i) the balanced beam splitter \( (\phi = \pi/4) \), that requires \( \alpha = 1 - \sqrt{2} \) and \( \beta = \sqrt{2}/2 \), and (ii) the swap \( (\phi = \pi/2) \) that exchanges the quantum states of the two systems, \( \alpha = -1, \beta = 1 \).

The swap gate is closely related to the two-step protocol for mapping the state of collective atomic spin on light that was suggested by Kuzmich and Polzik. In fact, their two-step protocol can be obtained by simply removing the last step of the present three-step swap gate. The removal of the third step means that the mapping adds some noise and the procedure is thus only approximate. A possible way how to improve its performance is to use squeezed light. For details, see [21].

**B. Two-mode squeezer**

Let us now turn our attention to the two-mode squeezer, described by the following matrix,
\[ S_{TMS}(r) = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix}. \] (24)

Similarly as in the case of the beam splitter, we attempt to implement this transformation as a sequence of three evolutions, c.f. Eq. (21). By comparison of the right-hand side of Eq. (22) with the matrix (24) we again obtain a system of nonlinear equations for the parameters \( \alpha, \beta, \) and \( \gamma \) having the solution
\[ \alpha = \tanh \frac{r}{2}, \quad \beta = \sinh r, \quad \gamma = \alpha. \] (25)

Note the similarity with the results for the beam splitter, the difference is essentially that the goniometric functions have been replaced by their hyperbolic counterparts. The parameters are finite for any finite \( r \). However, \( \beta \) grows exponentially with \( r \) and for large \( r \) we have \( \beta \propto e^r \). On the other hand, for small \( r \) we get \( \beta \approx r \). This implies that we may reduce the synthesis time if we implement the two-mode squeezing transformation as a sequence of \( n \) two-mode squeezers with \( r' = r/n \). The reduction of the time is achieved at the expense of a higher number of steps of the gate synthesis protocol. For modest values of squeezing \( r \), the sequence of three evolutions is advantageous since it involves the minimum necessary number of manipulations of the systems \( A \) and \( B \).

**C. Single-mode squeezer**

After dealing with two-mode gates, let us now focus on the single-mode gate, namely the single-mode squeezer. Since \( \det S = 1 \), the squeezing of quadrature \( x_A \) will necessarily by accompanied by anti-squeezing of quadrature \( y_A \), and vice versa,
\[ S_{SMS}(r) = \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}. \] (26)

It turns out that in this case a sequence of three transformations is insufficient and we must consider a sequence of four basic evolutions:
\[ S_{SMS}(r) = S_2(\delta)S_1(\gamma)S_2(\beta)S_1(\alpha). \] (27)

We proceed as before and derive equations for the four parameters appearing in (27):
\[
\begin{align*}
e^r &= 1 + \alpha \beta + \delta (\alpha + \gamma + \alpha \beta), \\
e^{-r} &= 1 + \gamma \beta, \\
0 &= \beta + \delta (1 + \gamma \beta), \\
0 &= \alpha + \gamma (1 + \alpha \beta).
\end{align*}
\]

This system of equations has a one-parametric class of solutions, given by
\[
\begin{align*}
\beta &= \frac{e^r - 1}{\alpha}, \\
\gamma &= -\alpha e^{-r}, \\
\delta &= \frac{e^r (1 - e^r)}{\alpha},
\end{align*}
\] (28)

and \( \alpha \) is arbitrary but nonzero. We may choose the optimal value of \( \alpha \) that minimizes the total time \( T \) needed for the implementation of the squeezing operation,
\[ T = |\alpha| + |\beta| + |\gamma| + |\delta|. \] (29)

Assuming that \( r > 0 \) we obtain by solving \( dT/d\alpha = 0 \) the optimal value,
\[ \alpha = \sqrt{\frac{e^{2r} - 1}{1 + e^{-r}}}. \] (30)

In the limit of small \( r \), all four parameters \( \alpha, \beta, \gamma, \) and \( \delta \) are proportional to \( \sqrt{r} \). This stems from the fact that the single-mode squeezing Hamiltonian \( H_{SMS} = x_{AP} \) cannot be obtained as a linear combination of the two-mode Hamiltonians \( H_1 \) and \( H_2 \) and only the terms of the order of \( O(t^2) \) or higher in \( \Omega \) may give rise to the contribution proportional to \( H_{SMS} \).

**IV. GENERIC QUADRATIC COUPLING**

In this section we shall assume that the interaction Hamiltonian has the generic canonical form. Although the mathematical analysis will be more involved we shall still be able to derive analytical formulas for the interaction times characterizing the gate synthesis. Without loss of generality, we may assume that \( c_1 = 1 \) in [3]. We
have to distinguish two classes of Hamiltonians giving rise to qualitatively different evolutions of the quadratures in the Heisenberg picture. For $c_2 > 0$ the dynamics resembles an amplifier while for $c_2 < 0$ we obtain oscillatory dynamics reminiscent that of a beam splitter. We shall discuss these two cases separately.

A. Amplifier-like Hamiltonians

Suppose first that $c_2 > 0$ and introduce a more convenient notation $c_2 = s^2$, $s > 0$, hence

$$H_1 = x_{APB} + s^2 p_{AXB}, \quad H_2 = s^2 x_{APB} + p_{AXB}.$$  \hfill (31)

It is an easy exercise to derive matrices $S_1$ and $S_2$ corresponding to the unitary evolutions governed by $H_1$ and $H_2$, respectively,

$$S_1^+(t) = \left( \begin{array}{cc} \cosh(st) & s \sinh(st) \\ \frac{1}{s} \sinh(st) & \cosh(st) \end{array} \right),$$ \hfill (32)

$$S_2^+(t) = \left( \begin{array}{cc} \cosh(st) & \frac{1}{s} \sinh(st) \\ s \sinh(st) & \cosh(st) \end{array} \right).$$ \hfill (33)

In what follows we will focus on the implementation of the beam splitter and two-mode squeezer transformations. We have seen in the previous section that these transformations could be implemented as a sequence of three basic evolutions governed by Hamiltonians $H_1$ or $H_2$ that were of the form (15). Moreover, there was an inherent symmetry in this gate synthesis; we have found that $\gamma = \alpha$. It turns out that these basic symmetry properties remain valid also for the generic Hamiltonians (31), and we can thus decompose the two-mode squeezing transformation as

$$S_{TMS}(r) = S_1^+(\alpha/s)S_2^+((\beta/s)S_1^+(\alpha/s).$$ \hfill (34)

Here the parameters $\alpha = st_1$ and $\beta = st_2$ are the rescaled interaction times. The nonlinear equations for the parameters $\alpha$ and $\beta$ are much more complicated than before. Nevertheless, analytical results can be obtained. From the condition $S_{12} = S_{21}$ we get

$$\tanh \beta = \frac{2 \sinh \alpha \cosh \alpha}{\cosh^2 \alpha - (s^{-2} + 1 + s^2) \sinh^2 \alpha}.$$ \hfill (35)

The condition $S_{11} = S_{22}$ is satisfied due to the symmetry ($\gamma = \alpha$), and the parameter $\alpha$ can be determined from the last independent equation $S_{12}/S_{11} = \tanh r$, which yields

$$\frac{2y(s + s^{-1})}{1 + y^2(s^{-2} + 1 + s^2)} = \tanh r,$$

where $y = \tanh \alpha$. This is a quadratic equation for $y$ whose solution reads

$$\tanh \alpha = \frac{s + s^{-1} - \sqrt{1 + (s^{-2} + 1 + s^2)(\cosh r)^{-2}}}{(s^{-2} + 1 + s^2) \tanh r}.$$ \hfill (36)

We have selected the root that yields the correct limit $\tanh \alpha = 0$ when $r \to 0$. In the opposite limit $r \to \infty$ we obtain

$$\tanh \alpha_{\infty} = \frac{1}{s^{-1} + 1 + s}.$$ \hfill (37)

On inserting this back into Eq. (35) we find that

$$\lim_{r \to \infty} \tanh \beta = 1.$$ \hfill (38)

It is easy to check that the equations for $\alpha$ and $\beta$ have finite solutions for any finite $r$. In the limit $r \to \infty$, $\alpha$ approaches a finite asymptotic value, cf. Eq. (37), while $\beta$ grows to infinity.

Suppose now that we want to implement the beam splitter transformation (20). The calculations of the parameters $\alpha$ and $\beta$ parallel those for the two-mode squeezer. From the condition $S_{12} = -S_{21}$ we express $\beta$ in terms of $\alpha$:

$$\tanh \beta = \frac{-2 \sinh \alpha \cosh \alpha}{\cosh^2 \alpha + (s^{-2} - 1 + s^2) \sinh^2 \alpha}.$$ \hfill (39)

Since $(s^{-2} - 1 + s^2) \geq 1$, it follows that $|\tanh \beta| \leq |\tanh(2\alpha)|$. From the condition $S_{12}/S_{11} = \tan \phi$ we obtain quadratic equation for $\tanh \alpha$ leading to,

$$\tanh \alpha = \frac{(s^{-1} - s) - \text{sign}(s^{-1} - s) \sqrt{s^{-2} - 1 + s^2} - 1}{(s^{-2} - 1 + s^2) \tan \phi}.$$ \hfill (40)

The sign function in the above formula selects the root that yields the correct limit $\alpha = 0$ when $\phi = 0$. The formula (40) is applicable in the interval $\phi \in [0, \pi/2]$. In the limiting case $\phi = \pi/2$ we have

$$\tanh \alpha_{\pi/2} = \frac{\text{sign}(s^{-1} - s)}{\sqrt{s^{-2} - 1 + s^2}},$$ \hfill (41)

which implies that $|\tanh \alpha_{\pi/2}| < 1$ iff $s \neq 1$. Furthermore, it can be shown that $\alpha$ is a monotonic function of $\phi$ in the interval $[0, \pi/2]$. We can thus conclude that with the interaction Hamiltonian of the amplifier type (31) we can implement any beam splitter transformation (20) with the mixing angles in the interval $[0, \pi/2]$, which includes the two important cases of a balanced beam splitter ($\phi = \pi/4$) and the swap ($\phi = \pi/2$). It follows from the expressions (39) and (40) that the simulation becomes more and more time consuming for Hamiltonians close to the two-mode squeezing Hamiltonian $H_{TMS} = x_{APB} + p_{AXB}$, i.e., when $s \to 1$.

B. Beam splitter-like Hamiltonians

Having derived the gate synthesis parameters for the Hamiltonians (31), we proceed to the interaction Hamiltonians leading to oscillatory dynamics,

$$H_1 = x_{APB} - s^2 p_{AXB}, \quad H_2 = -s^2 x_{APB} + p_{AXB}.$$ \hfill (42)
The $S$ matrices associated with these Hamiltonians read

$$S_1^{-}(t) = \begin{pmatrix} \cos(st) & -s \sin(st) \\ \frac{1}{2} \sin(st) & \cos(st) \end{pmatrix},$$

$$S_2^{-}(t) = \begin{pmatrix} \cos(st) & \frac{1}{2} \sin(st) \\ -s \sin(st) & \cos(st) \end{pmatrix}.$$ (43) (44)

We shall not repeat the details of the derivations of the parameters $\alpha$ and $\beta$ and we only summarize the results here. The beam splitter transformation can be accomplished by the following choice:

$$\tan \beta = \frac{-2 \sin \alpha \cos \alpha}{\cos^2 \alpha + (s^{-2} + 1 + s^2) \sin^2 \alpha}$$ (45)

and

$$\tan \alpha = \frac{s^{-1} + s - \sqrt{(s^{-2} + 1 + s^2)(\cos \phi)^{-2} + 1}}{(s^{-2} + 1 + s^2) \tan \phi}.$$ (46)

This reveals that simulation of any beam splitter with $\phi \in [0, \pi/2]$ is possible and the parameters satisfy $|\alpha| \leq \pi/2$ and $|\beta| \leq \pi/2$.

Consider now the two-mode squeezing operation $\mathcal{S}$.

After some algebra, one obtains

$$\tan \beta = \frac{2 \sin \alpha \cos \alpha}{\cos^2 \alpha - (s^{-2} - 1 + s^2) \sin^2 \alpha}$$ (47)

and

$$\tan \alpha = \frac{s^{-1} - s - \text{sign}(s^{-1} - s) \sqrt{s^{-2} - 1 + s^2}}{(s^{-2} - 1 + s^2) \tanh r}.$$ (48)

The function $\tan \alpha$ must be real which implies that the term under the square root must be non-negative. This constraint, in turn, limits the amount of two-mode squeezing that can be produced via a three-step protocol $\mathcal{S}_3$. It holds that $r \leq r_{\text{th}}$ where

$$\cosh r_{\text{th}} = \sqrt{s^{-2} - 1 + s^2}.$$ (49)

The origin of this bound lies in the fact that the dynamics governed by the Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$ is oscillatory and fully periodic with period $2\pi/s$. Squeezing above the threshold $r_{\text{th}}$ can be achieved only if we concatenate several three-step protocols. It thus appears that the amplifier-like Hamiltonians $\mathcal{H}_2$ are in certain sense more versatile than the beam-splitter like Hamiltonians $\mathcal{H}_1$, because the former allow to implement any two-mode squeezing gate and also any beam splitter with $\phi \in [0, \pi/2]$ via a three-step protocol $\mathcal{S}_3$.

V. CONCLUSIONS

In this paper we have addressed the problem of gate synthesis for continuous variable systems. We have assumed that two single-mode systems $A$ and $B$ interact via a quadratic Hamiltonian. We have studied how to implement a unitary symplectic gate $G$ with the use of the interaction Hamiltonian $H$ as a resource. The gate synthesis protocol consists of a sequence of evolutions governed by $H$ and followed by fast local phase shifts applied to the systems $A$ and $B$. We have focused on the gate simulation protocols that involve the minimal necessary number of steps, because these protocols require a low number of local control operations which is important from the experimental point of view. We have shown that a three-step protocol suffices for simulation of the two-mode squeezer as well as a beam splitter. For the specific case of the Hamiltonian $\mathcal{H}_3$ we have also established a four step implementation of a single-mode squeezer. Our results are applicable to any physical systems coupled via quadratic Hamiltonians. In particular, the gate synthesis protocols proposed in the present paper may find applications in the experiments where light interacts with atomic ensembles via a Kerr-like coupling $\mathcal{H}_2$. $\mathcal{H}_1$.

Acknowledgments

I would like to thank S. Massar, E.S. Polzik, N.J. Cerf and F. Grosshans for helpful discussions. I acknowledge financial support from the Communauté Française de Belgique under grant ARC 00/05-251, from the IUAP programme of the Belgian government under grant V-18, from the EU under projects RESQ (IST-2001-35759) and CHIC (IST-2001-32150) and from the grant LN00A015 of the Czech Ministry of Education.

[1] W. Dür, G. Vidal, J.I. Cirac, N. Linden, and S. Popescu, Phys. Rev. Lett. 87, 137901 (2001).
[2] C.H. Bennett, J.I. Cirac, M.S. Leifer, D.W. Leung, N. Linden, S. Popescu, and G. Vidal, Phys. Rev. A 66, 012305 (2002).
[3] J.L. Dodd, M.A. Nielsen, M.J. Bremner, and G. Vidal, Phys. Rev. A 66, 042301 (2002).
[4] M.A. Nielsen, M.J. Bremner, J.L. Dodd, A.M. Childs, and C.M. Dawson, Phys. Rev. A 66, 022317 (2002).
[5] P. Wocjan, M. Rötteler, D. Janzing, and T. Beth, Phys. Rev. A 65, 042309 (2002).
[6] G. Vidal and J.I. Cirac, Phys. Rev. A 66 022315 (2002).
[7] N. Khaneja, R. Brockett, and S.J. Glaser, Phys. Rev. A 63, 032308 (2001).
[8] G. Vidal, K. Hammerer, and J.I. Cirac, Phys. Rev. Lett. 88, 237902 (2002).
[9] K. Hammerer, G. Vidal, and J.I. Cirac, quant-ph/0205100.
[10] L. Masanes, G. Vidal, and J.I. Latorre, quant-ph/0202042.
[11] M.J. Bremner, C.M. Dawson, J.L. Dodd, A. Gilchrist, A.W. Harrow, D. Mortimer, M.A. Nielsen, and T.J. Osborne, Phys. Rev. Lett. 89, 247902 (2002).
[12] J. Zhang, J. Vala, S. Sastry, and K.B. Whaley, quant-ph/0212109.
[13] B. Kraus, K. Hammerer, G. Giedke, and J.I. Cirac, quant-ph/0210136.
[14] A. Kuzmich, N.P. Bigelow, and L. Mandel, Europhys. Lett. 42, 481 (1998).
[15] A. Kuzmich, L. Mandel, and N.P. Bigelow, Phys. Rev. Lett. 85, 1594 (2000).
[16] A. Kuzmich and E.S. Polzik, Phys. Rev. Lett. 85, 5639 (2000).
[17] L.M. Duan, J.I. Cirac, P. Zoller, and E.S. Polzik, Phys. Rev. Lett. 85, 5643 (2000).
[18] B. Julsgaard, A. Kozhekin, E.S. Polzik, Nature (London) 413, 400 (2001).
[19] C. Schori, B. Julsgaard, J.L. Sorensen, and E.S. Polzik, Phys. Rev. Lett. 89, 057903 (2002).
[20] A. Kuzmich and E.S. Polzik, in *Quantum Information with Continuous Variables*, edited by S.L. Braunstein and A.K. Pati (Kluwer Academic, to be published).
[21] E.S. Polzik, private communication.
[22] Arvind, B. Dutta, N. Mukunda, and R. Simon, Phys. Rev. A 49, 1567 (1994).
[23] Arvind, B. Dutta, N. Mukunda, and R. Simon, Phys. Rev. A 52, 1609 (1995).
[24] S. L. Braunstein, quant-ph/9904002.
[25] J. Fiurášek and J. Peřina, Phys. Rev. A 62, 033808 (2000).
[26] J. Řeháček, L. Míšta, Jr., J. Fiurášek, and J. Peřina, Phys. Rev. A 65, 043815 (2002).