Generalized Coherent States Associated with the $C_\lambda$-Extended Oscillator

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Abstract

Two new types of coherent states associated with the $C_\lambda$-extended oscillator, where $C_\lambda$ is the cyclic group of order $\lambda$, are introduced. They satisfy a unity resolution relation in the $C_\lambda$-extended oscillator Fock space (or in some subspace thereof) and give rise to Bargmann representations of the latter, wherein the generators of the $C_\lambda$-extended oscillator algebra are realized as differential-operator-valued matrices.

1 Introduction

Coherent states (CS) of the harmonic oscillator are known to have properties similar to those of the classical radiation field. In contrast, generalized CS associated with various algebras may have some nonclassical properties.

In the present communication, we shall consider some generalized CS, [1, 2] which may be associated with the recently introduced $C_\lambda$-extended oscillator. [3] The latter may be considered as a deformed oscillator with a $Z_\lambda$-graded Fock space and has proved very useful in the context of supersymmetric quantum mechanics and some of its variants. [3, 4]

The main mathematical feature of these new generalized CS is that they satisfy Klauder’s minimal set of conditions for generalized CS, [5] including the existence of a unity resolution relation, property that is not shared by generic generalized CS.
2 The $C_\lambda$-Extended Oscillator

The $C_\lambda$-extended oscillator Hamiltonian is defined by

$$H_0 = \frac{1}{2} \{a, a^\dagger\},$$

(1)

where the creation and annihilation operators $a^\dagger$, $a$ satisfy the commutation relation

$$[a, a^\dagger] = I + \sum_{\mu=1}^{\lambda-1} \kappa_\mu T^\mu = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu.$$  

(2)

Here $T = (T^\dagger)^{-1}$ denotes the generator of the cyclic group $C_\lambda = \{T, T^2, \ldots, T^{\lambda-1}, T^\lambda = I\}$ (or, more precisely, the generator of a unitary representation thereof), and $P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} \exp(-2\pi i \mu \nu/\lambda) T^\nu$, $\mu = 0, 1, \ldots, \lambda - 1$, are the projectors on the $\lambda$ inequivalent unitary irreducible matrix representations of $C_\lambda$. The parameters $\kappa_\mu$ are complex and satisfy the conditions $\kappa_\mu^* = \kappa_{\lambda-\mu}$, whereas the $\alpha_\mu$'s are real and such that $\sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0$.

The commutation rule for $T$ (or $P_\mu$) and $a^\dagger$ is given by

$$a^\dagger T = e^{-2\pi i/\lambda} T a^\dagger \quad \text{or} \quad a^\dagger P_\mu = P_{\mu+1} a^\dagger.$$  

(3)

The $C_\lambda$-extended oscillator algebra also contains a number operator $N$ such that

$$[N, a^\dagger] = a^\dagger, \quad [N, T] = 0 \quad \text{or} \quad [N, P_\mu] = 0.$$  

(4)

When $T$ (or $P_\mu$) is realized in terms of $N$, i.e., $T = \exp(2\pi i N/\lambda)$, the $C_\lambda$-extended oscillator algebra becomes a generalized deformed oscillator algebra (GDOA). In the Fock representation of the latter, there exists a vacuum state $|0\rangle$, such that $a|0\rangle = N|0\rangle = 0$, and the remaining basis states $|n\rangle$, $n = 1, 2, \ldots$, can be obtained from it by successive applications of $a^\dagger$. The Fock space $\mathcal{F}$ is $Z_\lambda$-graded: $\mathcal{F} = \sum_{\mu=0}^{\lambda-1} \mathcal{F}_\mu$, where $\mathcal{F}_\mu = \{|n\rangle = |k\lambda + \mu\rangle | k = 0, 1, 2, \ldots\}$, and $P_\mu$ projects on $\mathcal{F}_\mu$.

The $C_\lambda$-extended oscillator Hamiltonian $H_0$, defined in Eq. (1), has a spectrum made of $\lambda$ infinite sets of equally spaced levels, belonging to $\mathcal{F}_\mu$, $\mu = 0, 1, \ldots, \lambda - 1$, respectively. Its spectrum generating algebra (SGA) is generated by the operators $J_+ = (1/\lambda)(a^\dagger)^\lambda$, $J_- = (1/\lambda)a^\lambda$, $J_0 = (1/\lambda)H_0$, and is a $C_\lambda$-extended polynomial deformation of $su(1,1)$ for $\lambda > 2$, while it reduces to $su(1,1)$ for $\lambda = 2$. [1]

For $\lambda = 2$, the GDOA corresponding to the $C_\lambda$-extended oscillator algebra reduces to the Calogero-Vasiliev algebra, characterized by the commutation relation $[a, a^\dagger] = I + \alpha_0 K$, where $K = T = (-1)^N$. [1] For $\alpha_0 = p - 1$, the latter becomes the paraboson algebra of order $p$. [1]
3 Family of Coherent States $|z; \mu; \alpha\rangle$

The coherent states $|z; \mu; \alpha\rangle$, where $z \in \mathbb{C}$, $\alpha \in \{0, 1, \ldots, \lfloor \lambda/2 \rfloor\}$, $\mu \in \{0, 1, \ldots, \lambda - \alpha - 1\}$, are the solutions of the equation

$$\left[a^{\lambda-\alpha} - z \left(a^\dagger\right)^{\alpha}\right]|z; \mu; \alpha\rangle = 0.$$  \hspace{1cm} (5)

As special cases, they include (i) the annihilation-operator CS of the $C_\lambda$-extended oscillator SGA, corresponding to $\alpha = 0$ (i.e., $J_- |z; \mu; 0\rangle = (z/\lambda)|z; \mu; 0\rangle$), \cite{1} and (ii) the displacement-operator or Perelomov $su(1,1)$ CS, corresponding to $\lambda = 2$, $\mu = 0$, $\alpha = 1$ (i.e., $(a - za^\dagger)|z; 0; 1\rangle = 0$, where $|z; 0; 1\rangle \propto \exp(zJ_+)|0\rangle$). \cite{8}

The states $|z; \mu; \alpha\rangle$ belong to the subspace $F_\mu$ of $F$ and can therefore be expanded in the number states $|k\lambda + \mu\rangle$, $k = 0, 1, \ldots$, as

$$|z; \mu; \alpha\rangle = \frac{1}{\left[N^{(\alpha)}_\mu(|z|)\right]^{1/2}} \sum_{k=0}^{\infty} c_k(z, z^*; \mu; \alpha) z^k |k\lambda + \mu\rangle,$$  \hspace{1cm} (6)

where the coefficients $c_k(z, z^*; \mu; \alpha)$ are given in Ref. \cite{2} and the normalization coefficient can be expressed as

$$N^{(\alpha)}_\mu(|z|) = \alpha F_{\lambda-\alpha-1} \left(\bar{\beta}_{\mu+1}, \ldots, \bar{\beta}_{\mu+\alpha}; \bar{\beta}_1 + 1, \ldots, \bar{\beta}_\mu + 1, \bar{\beta}_{\mu+\alpha+1}, \ldots, \bar{\beta}_{\lambda-1}; y\right).$$  \hspace{1cm} (7)

In Eq. (7), the variable $y$ is defined by $y \equiv |z|^2/\lambda^{\lambda-2\alpha}$, while the constants $\bar{\beta}_\mu$ are given in terms of the parameters $\alpha_\nu$ of the $C_\lambda$-extended oscillator algebra through the relations $\bar{\beta}_\mu \equiv (\beta_\mu + \mu)/\lambda$, $\beta_\mu \equiv \sum_{\nu=0}^{m-1} \alpha_\nu$. From this, we conclude that the states $|z; \mu; \alpha\rangle$ are normalizable on the complex plane for any $\alpha \in \{0, 1, \ldots, \lfloor (\lambda - 1)/2 \rfloor\}$ and on the unit disc for $\lambda$ even, $\alpha = \lambda/2$.

In addition, the CS $|z; \mu; \alpha\rangle$ are continuous in the label $z$, i.e., $|z - z'| \to 0 \Rightarrow ||z; \mu; \alpha\rangle - |z'; \mu; \alpha\rangle|^2 \to 0$, and for appropriate values of the algebra parameters, they satisfy a unity resolution relation with a positive measure,

$$\int d^2 z h^{(\alpha)}_\mu(y)|z; \mu; \alpha\rangle(z; \mu; \alpha) = I_\mu,$$  \hspace{1cm} (8)

in $F_\mu$ (with $I_\mu$ the unit operator in $F_\mu$). Alternatively, we may write

$$\sum_{\mu=0}^{\lambda-1} \int d^2 z h^{(\alpha)}_\mu(y)|z; \mu; \alpha\rangle(z; \mu; \alpha) = I$$  \hspace{1cm} (9)
in $\mathcal{F}$. In Eqs. (8) and (9), $|z; \mu; \alpha\rangle \equiv [\mathcal{N}^{(\mu)}(|z|)]^{1/2}|z; \mu; \alpha\rangle$ denotes an unnormalized CS and the general form of $h^{(\alpha)}(y)$ is conjectured to be given in terms of a Meijer $G$-function as

$$h^{(\alpha)}(y) = A^{(\mu)} G^{\lambda-\alpha,0}_{\alpha,\lambda-\alpha} \left( \frac{\bar{\beta}_{\mu+1} - 1, \ldots, \bar{\beta}_{\mu+\alpha-1}}{0, \bar{\beta}_1, \ldots, \bar{\beta}_\mu, \bar{\beta}_{\mu+\alpha-1} - 1, \ldots, \bar{\beta}_{\lambda-1} - 1} \right),$$  

(10)

where $A^{(\mu)}$ is some numerical coefficient. Eq. (10) has actually been proved for $\alpha \neq \lambda/2$ and any $\lambda$, and for $\alpha = \lambda/2$ and $\lambda = 2, 4, 6$. [2]

Since the CS $|z; \mu; \alpha\rangle$ form a complete (in fact, an overcomplete) set in $\mathcal{F}_\mu$ for some appropriate choice of the algebra parameters if $z$ runs over the complex plane for $\alpha \leq [(\lambda - 1)/2]$ or over the unit disc for $\alpha = \lambda/2$ and $\lambda$ even, we may associate with such a set a realization of $\mathcal{F}_\mu$ as a Hilbert space $\mathcal{B}^{(\mu)}$ of entire analytic functions. This gives rise to so-called Bargmann representations [3] of the $C_\lambda$-extended oscillator, wherein the operators $X$ defined in $\mathcal{F}_\mu$ become differential operators $X^{(\mu)}$. For the SGA generators $J_+, J_-$, and $J_0$, for instance, we get

$$J^{(\mu)}_+ = \lambda \alpha^{-1} z \prod_{\nu=\mu+1}^{\mu+\alpha} \left( z \frac{\partial}{\partial z} + \bar{\beta}_\nu \right),$$  

(11)

$$J^{(\mu)}_- = \lambda \lambda^{-\alpha-1} \left( \prod_{\nu=1}^{\mu} \left( z \frac{\partial}{\partial z} + \bar{\beta}_\nu + 1 \right) \right) \left( \prod_{\nu=\mu+1}^{\mu+\alpha} \left( z \frac{\partial}{\partial z} + \bar{\beta}_\nu \right) \right) \frac{\partial}{\partial z},$$  

(12)

$$J^{(\mu)}_0 = z \frac{\partial}{\partial z} + \frac{1}{2}(\bar{\beta}_\mu + \bar{\beta}_{\mu+1}).$$  

(13)

When dealing with the whole Fock space $\mathcal{F}$, we may consider as complete set the collection of CS $|z; \mu; 0\rangle$, $\mu = 0, 1, \ldots, \lambda - 1$, and realize $\mathcal{F}$ as a Hilbert space $\mathcal{B}^{(0)} = \sum_{\mu=0}^{\lambda-1} \mathcal{B}^{(\mu)}$ of entire analytic functions. It is then convenient to introduce vector CS [10] defined as row vectors, $|z; 0\rangle \equiv (|z; 0; 0\rangle, |z; 1; 0\rangle, \ldots, |z; \lambda - 1; 0\rangle)$, the corresponding bras $(|z; 0\rangle$ being column vectors. In such a case, the operators defined in $\mathcal{F}$ are represented by some operator-valued $\lambda \times \lambda$ matrices.

4 Eigenstates $|z\rangle$ of the Annihilation Operator $a$

The eigenstates of the $C_\lambda$-extended oscillator annihilation operator $a$, defined by [2]

$$a|z\rangle = z|z\rangle,$$  

(14)

where $z \in \mathbb{C}$, include as special case the paraboson CS, [11] corresponding to $\lambda = 2$. 

4
They can be constructed as linear combinations of the CS $|\omega; \mu; 0\rangle$, $\mu = 0, 1, \ldots, \lambda - 1$, considered in Sec. 3, where one replaces $z$ by $\omega \equiv z^\lambda$,

$$|z\rangle = [\mathcal{N}(|z|)]^{-1/2} \sum_{\mu=0}^{\lambda-1} d'_\mu (z, z^*) \left( \frac{z}{\sqrt{\lambda}} \right)^\mu |\omega; \mu; 0\rangle,$$

with $d'_\mu (z, z^*) = [\mathcal{N}^{(0)}_\mu (|\omega|)/\prod_{\nu=1}^{\mu} \beta_\nu]^{1/2}$. The normalization coefficient $\mathcal{N}(|z|)$ is given by

$$\mathcal{N}(|z|) = \sum_{\mu=0}^{\lambda-1} \mathcal{O}_{\lambda-1} \left( \beta_1 + 1, \ldots, \beta_\mu + 1, \beta_{\mu+1}, \ldots, \beta_{\lambda-1}; t^\lambda \right) \frac{t^\mu}{\prod_{\nu=1}^{\mu} \beta_\nu},$$

where $t \equiv |z|^2/\lambda$. It is therefore straightforward to see that the CS $|z\rangle$ are normalizable on the complex plane and continuous in their label $z$.

They satisfy a unity resolution relation of unusual form, $[3]$

$$\sum_{\mu=0}^{\lambda-1} \int d^2 z h_\mu(t)|z_\mu\rangle (z_\mu| = \sum_{\mu=0}^{\lambda-1} \int d^2 z g_\mu(t)|z\rangle (ze^{2\pi i \mu/\lambda}| = I,$$

generalizing the known one for paraboson CS. $[11]$ Here $|z\rangle = [\mathcal{N}(|z|)]^{1/2}|z\rangle$ denotes an unnormalized CS, $|z_\mu\rangle$ its component in $\mathcal{F}_\mu$, and

$$h_\mu(t) = \lambda^\lambda \left( \prod_{\nu=1}^{\mu} \beta_\nu \right) t^{\lambda-\mu-1} h^{(0)}_\mu (t^\lambda), \quad g_\mu(t) = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{2\pi i \mu \nu/\lambda} h_\nu(t),$$

where $h^{(0)}_\mu (t^\lambda)$ is defined in Eq. (10). It can actually be shown that this nondiagonal CS resolution of unity is entirely equivalent to the diagonal one valid for the set of CS $|z; \mu; 0\rangle$, $\mu = 0, 1, \ldots, \lambda - 1$, and given in Eq. (4).

Instead of the set of CS $|z; \mu; 0\rangle$, $\mu = 0, 1, \ldots, \lambda - 1$, we may use $|z_\mu\rangle$, $\mu = 0, 1, \ldots, \lambda - 1$, as complete set in $\mathcal{F}$ and introduce a vector notation again, $||z\rangle\rangle = (|z_0\rangle, |z_1\rangle, \ldots, |z_{\lambda-1}\rangle)$. In the corresponding Bargmann space $\mathcal{B}$, the operators $N$, $a^\dagger$, $a$ of the $C_\lambda$-extended oscillator algebra take a very simple form, namely

$$\mathcal{N} = z \frac{\partial}{\partial z} \mathcal{I},$$

$$\mathcal{A}^\dagger = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ z & 0 & \cdots & 0 & 0 \\ 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} + \frac{\beta_1}{z} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\partial}{\partial z} + \frac{\beta_2}{z} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\partial}{\partial z} + \frac{\beta_{\lambda-1}}{z} \\ \frac{\partial}{\partial z} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\mathcal{I}$ denotes the $\lambda \times \lambda$ unit matrix.
5 Conclusion

In the present contribution, we presented the mathematical properties of some new CS associated with the $C_\lambda$-extended oscillator.

Such states have also some interesting physical properties. In Refs. [1, 2], we indeed investigated some of their characteristics relevant to quantum optics, such as their statistical and squeezing properties, for a wide range of parameters and from both viewpoints of real and dressed photons. Their nonclassical features for some parameter ranges were clearly demonstrated.

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