General Classes of Lower Bounds on the Probability of Error in Multiple Hypothesis Testing

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Abstract

In this paper, two new classes of lower bounds on the probability of error for m-ary hypothesis testing are proposed. Computation of the minimum probability of error which is attained by the maximum a-posteriori probability (MAP) criterion, is usually not tractable. The new classes are derived using Hölder’s inequality and reverse Hölder’s inequality. The bounds in these classes provide good prediction of the minimum probability of error in multiple hypothesis testing. The new classes generalize and extend existing bounds and their relation to some existing upper bounds is presented. It is shown that the tightest bounds in these classes asymptotically coincide with the optimum probability of error provided by the MAP criterion for binary or multiple hypothesis testing problem. These bounds are compared with other existing lower bounds in several typical detection and classification problems in terms of tightness and computational complexity.

Keywords

maximum a-posteriori probability (MAP), Ziv-Zakai lower bound (ZZLB), detection, lower bounds, hypothesis testing, probability of error, performance lower bounds

I. Introduction

Lower bounds on the probability of error are of great importance in system design and performance analysis in many applications, such as signal detection, communications [1], classification, and pattern recognition [2]. It is well known that the minimum probability of error is attained by the maximum a-posteriori probability
(MAP) criterion, however, its probability of error is often difficult to calculate and usually not tractable. In such cases, lower bounds on the probability of error are useful for performance analysis, feasibility study and system design. These bounds can be useful also for derivation of analytical expressions for the family of Ziv-Zakai lower bounds (ZZLB) for parameter estimation [3]. One of the difficulties in computation of the ZZLB is that they involve an expression for the minimum probability of error of a binary hypothesis problem. Analytic expressions for lower bounds on the probability of error may be useful to simplify the calculation of the bound. Another application of these bounds is a sphere packing lower bound on probability of error under MAP of the ensemble of random codes [4].

Several lower bounds on the probability of error have been presented in the literature, for specific problems, such as signals in white Gaussian noise [5], [6], and for general statistical models. The general bounds can be divided into bounds for binary hypothesis problems [7,11] and bounds for multiple-hypothesis problems [4], [12,21]. Several lower and upper bounds utilize distance measures between statistical distributions, like Bhattacharyya distance [7, 8], Chernoff [9], Bayesian distance [17], Matusita distance [19], and the general mean distance [20, 21]. Two classical lower bounds on the multiple-hypothesis error probability that have been used in proving coding theorems are the Shannon [12] and Fano [13] inequalities. The relations between entropy and error probability have been used to derive the bounds in [4,14,15]. The bound in [16] has been derived using proofs of converse channel coding theorems in information theory. In addition, there are several ad-hoc binary hypothesis testing bounds that directly bound the minimum function on the \textit{a-posteriori} probabilities. This class includes the “Gaussian-Sinusoidal” upper and lower bounds [11] and the exponential bound [10], that are found to be useful in some specific cases. A brief review of some existing lower bounds on the probability of error is presented in Appendix B and in [22].

Practical and useful lower bounds on the probability of error are expected to be computationally simple, tight, and appropriate for general multi-hypothesis problems. In this paper, two new classes of lower bounds with the aforementioned desired properties are derived using Hölder’s inequality and reverse Hölder’s inequality. The bounds in these classes provide good prediction of the minimum probability of error in multiple hypothesis testing and are often easier to evaluate than the MAP probability of error. These bounds are
compared with other existing lower bounds. In addition, it is shown that the new classes generalize some existing lower bounds \cite{4, 17, 21, 23, 24}. The tightest lower bound under each class of bounds is derived and it is shown that the tightest bound asymptotically coincides with the optimum probability of error provided by the MAP criterion.

The paper is organized as follows. The new classes of bounds are derived in Section II and the bounds properties are presented in Section III. In Section IV simple versions of the ZZLB for parameter estimation are derived using the proposed classes of bounds. The performances of the proposed bounds for various examples is evaluated in Section V. Finally, our conclusions appear in Section VI.

II. General classes of bounds on probability of error

A. Problem statement

Consider an $M$-ary hypothesis testing problem, in which the hypotheses are $\theta_i, \quad i = 1, \ldots, M$ with the corresponding a-priori probabilities $P(\theta_i), \quad i = 1, \ldots, M$. Let $P(\theta_i|x), \quad i = 1, \ldots, M$ denote the conditional probability of $\theta_i$ given the random observation vector, $x$. The probability of error of the decision problem is denoted by $P_e$. It is well known that the minimum probability of error obtained by the MAP criterion, is given by \cite{14}

$$P_e^{(\text{min})} = 1 - \mathbb{E} \left[ \max_{i=1,\ldots,M} P(\theta_i|x) \right]$$

(1)

where the MAP detector is

$$\hat{\theta}_{\text{MAP}} = \arg \max_{\theta \in \{\theta_1, \ldots, \theta_M\}} P(\theta|x) .$$

However, the minimum probability of error in (1) is often difficult to calculate and usually not tractable. Therefore, computable and tight lower and upper bounds on the probability of error are useful for performance analysis and system design.

B. Derivation of the general classes of bounds

Consider the above $M$-ary hypothesis testing problem with detector $\hat{\theta} = \hat{\theta}(x)$. The detector $\hat{\theta}(x)$ is assumed to be an arbitrary detector that decides on one of the hypotheses with positive (non-zero) a-posteriori
probabilities. That is, in the case where \( P(\theta_j | x) = 0 \) we assume that \( \hat{\theta}(x) \neq \theta_j \) with probability 1 (w.p.1).

Let
\[
u(x, \theta) \triangleq 1_{\hat{\theta} \neq \theta} = \begin{cases} 1 & \text{if } \hat{\theta} \neq \theta \\ 0 & \text{if } \hat{\theta} = \theta \end{cases}
\] (2)

where \( \theta \) is the true hypothesis. It can be verified that
\[
Pe = E[u(x, \theta)] = E[|u(x, \theta)|^p]
\] (3)

and
\[
Pe = 1 - E[1 - u(x, \theta)] = 1 - E[|1 - u(x, \theta)|^p]
\] (4)

for every \( p > 0 \), where \( Pe \) is the probability of error of the detector \( \hat{\theta} \). Then, according to Hölder’s inequality and reverse Hölder’s inequality \[22\], \[25\]:
\[
E^\frac{1}{p} [|u(x, \theta)|^p] E^{-\frac{1}{p}} \left[ |v_1(x, \theta)|^{\frac{1}{p-1}} \right] \geq E [|u(x, \theta)v_1(x, \theta)|], \quad p > 1
\] (5)

and
\[
E \left[ |(1 - u(x, \theta))v_2(x, \theta)| \right] \geq E^p \left[ |1 - u(x, \theta)|^{\frac{1}{p}} \right] E^{1-p} \left[ |v_2(x, \theta)|^{\frac{1}{1-p}} \right], \quad p > 1
\] (6)

for arbitrary scalar functions \( v_1(x, \theta) \) and \( v_2(x, \theta) \).

By substituting of (3) and (4) into (5) and (6), respectively, one obtains the following lower bounds on the probability of error:
\[
Pe \geq E^p [|u(x, \theta)v_1(x, \theta)|] E^{1-p} \left[ |v_1(x, \theta)|^{\frac{1}{p-1}} \right], \quad p > 1
\] (7)

\[
Pe \geq 1 - E^\frac{1}{p} \left[ |(1 - u(x, \theta))v_2(x, \theta)| \right] E^{-\frac{1}{p}} \left[ |v_2(x, \theta)|^{\frac{1}{1-p}} \right], \quad p > 1.
\] (8)

By substituting different functions \( v_1(x, \theta), \ v_2(x, \theta) \) in (7)-(8), one obtains different lower bounds on the probability of error. In general, this bound is a function of the detector via \( u(x, \theta) \). The following theorem states the condition to obtain valid bounds which are independent of the estimator.
Theorem 1: Under the assumption that $P(\theta_i|x) > 0 \ \forall x \in \chi$ and $\theta_i$, $i = 1, ..., M$, a necessary and sufficient condition to obtain a valid bound on the probability of error which is independent of the detector $\hat{\theta}$, is that the functions $v_1(x, \theta)$ and $v_2(x, \theta)$ should be structured as follows

$$v_k(x, \theta_i) = \frac{\zeta_k(x)}{P(\theta_i|x)}, \ \ k = 1, 2, \ \ i = 1, \ldots, M$$ \hspace{1cm} (9)

where $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ are arbitrary functions of the observations $x$ and with no loss of generality they should be chosen to be non-negative.

Proof: In Appendix A.

Using (9) it is shown in Appendix A that using $\{v_k(x, \theta_i)\}_{k=1,2}$ defined in (9)

\[
\mathbb{E}[|u(x, \theta)v_1(x, \theta)|] = (M - 1)\mathbb{E}[\zeta_1(x)] , \hspace{1cm} (10)
\]

\[
\mathbb{E}[|(1 - u(x, \theta))v_2(x, \theta)|] = \mathbb{E}[\zeta_2(x)] , \hspace{1cm} (11)
\]

and

\[
\mathbb{E}[|v_1(x, \theta)|^{\frac{p}{1-p}}] = \mathbb{E}\left[\zeta_1^\frac{p}{1-p}(x) \sum_{i=1}^{M} P_1^{\frac{1}{1-p}}(\theta_i|x)\right] , \hspace{1cm} (12)
\]

\[
\mathbb{E}[|v_2(x, \theta)|^{\frac{1}{1-p}}] = \mathbb{E}\left[\zeta_2^{\frac{1}{1-p}}(x) \sum_{i=1}^{M} P_i^{\frac{p}{1-p}}(\theta_i|x)\right] . \hspace{1cm} (13)
\]

By substitution of (10), (12) into (7), and (11), (13) into (8), the new classes of lower bounds can be rewritten as:

\[
P_e \geq (M - 1)^p\mathbb{E}[\zeta_1(x)] \mathbb{E}^{\frac{p}{1-p}}\left[\zeta_1^{\frac{p}{1-p}}(x) \sum_{i=1}^{M} P_1^{\frac{1}{1-p}}(\theta_i|x)\right] , \ \ p > 1 \hspace{1cm} (14)
\]

\[
P_e \geq 1 - \mathbb{E}[\zeta_2(x)] \mathbb{E}^{\frac{p-1}{p}}\left[\zeta_2^{\frac{1}{1-p}}(x) \sum_{i=1}^{M} P_i^{\frac{p}{1-p}}(\theta_i|x)\right] , \ \ p > 1 . \hspace{1cm} (15)
\]
C. The tightest subclasses in the proposed classes of lower bounds

According to Hölder’s inequality [25]

\[
E_p [\zeta_1(x)]^1 - p \left[ \zeta_1^{\frac{p}{p-1}}(x) \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i | x) \right] \geq E \left[ \zeta_1^p(x) \left( \zeta_1^{\frac{p}{p-1}}(x) \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i | x) \right)^{1-p} \right] = E \left[ \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i | x) \right]^{1-p} \tag{16}
\]

and it becomes an equality iff

\[
\zeta_1(x) = c_1 \zeta_1^{\frac{p}{p-1}}(x) \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i | x) \tag{17}
\]

where \(c_1\) denotes a constant independent of \(x\) and \(\theta_i, i = 1, \ldots, M\). In similar,

\[
E_p [\zeta_2(x)]^1 - p \left[ \zeta_2^{\frac{1}{p-1}}(x) \sum_{i=1}^{M} P^p (\theta_i | x) \right] \geq E \left[ \zeta_2^p(x) \left( \zeta_2^{\frac{1}{p-1}}(x) \sum_{i=1}^{M} P^p (\theta_i | x) \right)^{\frac{p}{p-1}} \right] = E \left[ \sum_{i=1}^{M} P^p (\theta_i | x) \right]^{\frac{p}{p-1}} \tag{18}
\]

and it becomes an equality iff

\[
\zeta_2(x) = c_2 \zeta_2^{\frac{1}{p-1}}(x) \sum_{i=1}^{M} P^p (\theta_i | x) \tag{19}
\]

where \(c_2\) denotes a constant independent of \(x\) and \(\theta_i, i = 1, \ldots, M\). Thus, the tightest subclasses of bounds in the two classes are:

\[
P_e \geq B_p^{(1)} \triangleq (M - 1)^p E \left[ \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i | x) \right]^{1-p}, \quad \forall p > 1 \tag{20}
\]

\[
P_e \geq B_p^{(2)} \triangleq 1 - E \left[ \sum_{i=1}^{M} P^p (\theta_i | x) \right]^{\frac{p-1}{p}}, \quad \forall p > 1 \tag{21}
\]

obtained by substituting (17) and (19) in (14) and (15), respectively.

D. Simplifications of the bound

The bounds in (20) and (21) can be simplified using Jensen’s inequality [26]. Let \(\varepsilon_1 = \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i | x) > 0\), then for \(p > 1\) \(B_p^{(1)} = \varepsilon_1^{1-p}\) is a convex function of \(\varepsilon_1 > 0\). According to Jensen’s inequality for convex
functions

\[ B_p^{(1)} = (M - 1)^p E \left[ \left( \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i|x) \right)^{1-p} \right] \geq JB_p^{(1)} \triangleq (M - 1)^p E \left[ \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i|x) \right], \quad \forall p > 1. \quad (22) \]

In similar, \( B_p^{(2)} = \varepsilon_2^{\frac{p-1}{p}} \) is a concave function of the positive term \( \varepsilon_2 = \sum_{i=1}^{M} P^{\frac{1}{p-1}} (\theta_i|x) \). According to Jensen’s inequality for concave functions

\[ B_p^{(2)} = 1 - E \left[ \left( \sum_{i=1}^{M} P^{\frac{1}{p-1}} (\theta_i|x) \right)^{\frac{p-1}{p}} \right] \geq JB_p^{(2)} \triangleq 1 - E \left[ \sum_{i=1}^{M} P^{\frac{1}{p-1}} (\theta_i|x) \right], \quad \forall p > 1. \quad (23) \]

**III. Properties of the proposed classes of bounds**

**A. Asymptotic properties**

According to [25] (Theorem 19, page 28), for any sequence of nonnegative numbers, \( a_1, \ldots, a_M \)

\[ \left( \sum_{i=1}^{M} a_i^s \right)^{\frac{t}{s}} \geq \left( \sum_{i=1}^{M} a_i^t \right)^{\frac{t}{s}}, \quad \forall 0 < s < t \quad (24) \]

and thus, the term within the expectation in (20)

\[ \left( \sum_{i=1}^{M} P^{\frac{1}{1-p}} (\theta_i|x) \right)^{1-p} = \frac{1}{\left( \sum_{i=1}^{M} P^{\frac{1}{p-1}} (\theta_i|x) \right)^{\frac{1}{p-1}}} \]

is a decreasing function of \( p \) for all \( p > 1 \). Therefore, in the binary case, the bound in (20) satisfies

\[ B_p^{(1)} \geq B_r^{(1)}, \quad \forall 1 < p \leq r, \quad M = 2. \quad (25) \]

In particular, for \( p \to 1^+ \), the bound in (20) becomes

\[ B_1^{(1)} = \lim_{p \to 1^+} B_p^{(1)} = E \left[ \min_{i=1,2} P (\theta_i|x) \right] = 1 - E \left[ \max_{i=1,2} P (\theta_i|x) \right], \quad (26) \]

which is the tightest lower bound on the probability of error in the first proposed class of lower bounds for \( M = 2 \). Thus, for the binary hypothesis testing the bound in (20) with \( p \to 1^+ \) is tight and attains the minimum probability of error, presented in (1).

In similar, using (24) the term \( \left( \sum_{i=1}^{M} P^{\frac{1}{p-1}} (\theta_i|x) \right)^{\frac{p-1}{p}} \) from (21), which is the \( \frac{p}{p-1} \) norm of \( \{ P(\theta_i|x) \}_{i=1,\ldots,M} \), is a decreasing function of \( p \) for all \( p > 1 \). Therefore, in the general case

\[ B_p^{(2)} \geq B_r^{(2)}, \quad \forall 1 < p \leq r, \quad \forall M. \quad (27) \]
In particular, for \( p \to 1^+ \), the bound in (21) becomes

\[
B^{(2)}_\infty = \lim_{p \to 1^+} B^{(2)}_p = 1 - \mathbb{E} \left[ \max_{i=1,\ldots,M} P(\theta_i|\mathbf{x}) \right]
\]

which is the minimum probability of error, obtained by the MAP criterion. Thus, for the M-hypothesis testing, the bound in (21) with \( p \to 1^+ \) is tight and attains the minimum probability of error presented in (1).

### B. Generalization of existing bounds

In this section, we show that the proposed classes of bounds generalize some existing bounds. In particular, the lower bounds in [4], [17], and [21] can be interpreted as special cases of the proposed general M-hypotheses bounds, presented in (20) and (21). In the binary hypothesis testing, the bound in (20) with \( p = 2 \) can be written by the following simple version:

\[
P_e \geq \mathbb{E} \left[ P(\theta_1|\mathbf{x}) P(\theta_2|\mathbf{x}) \right]
\]

which is identical to the harmonic lower bound [4] and to the Vajdas quadratic entropy bound [23], [24] with \( M = 2 \).

In the multiple hypothesis testing the bound in (21) with \( p = 2 \) can be written in the following simple version:

\[
P_e \geq 1 - \mathbb{E} \left[ \sqrt{\sum_{i=1}^{M} P^2(\theta_i|\mathbf{x})} \right]
\]

which is identical to the Bayesian lower bound [4], [17], \( B^{(Bayes3)} \), described in Appendix B. The bound \( JB^{(2)}_p \) in (23) with \( p = 2, \ M = 2 \) is

\[
P_e \geq JB^{(2)}_2 = 1 - \mathbb{E} \left[ \sum_{i=1}^{M} P^2(\theta_i|\mathbf{x}) \right]
\]

is identical to the Bayesian lower bound [17], \( B^{(Bayes2)} \), described in Appendix B. In addition, in the multiple hypothesis testing, the subclass of bounds in (21) is a “general mean distance” class of bounds presented in Appendix B in (62).
C. Relation to upper bounds on minimum probability of error

In [4], a class of upper bounds on the MAP probability of error for binary hypothesis testing is derived using the negative power mean inequalities:

\[
P_e^{(\text{min})} \leq 2^{(p-1)} E_x \left[ \left( \sum_{i=1}^{2} P_1 (\theta_i|\mathbf{x}) \right)^{1/p} \right]^{1-p}
\]

for any \( p > 1 \). It can be seen that this class of upper bounds is proportional to the proposed tightest subclass of lower bound in (20) with a factor of \( 2^{p-1} \). This factor controls the tightness between upper and lower bounds in the probability of error for binary hypothesis testing. This upper bound coincides with the proposed lower bound \( B_p^{(1)} \) in the limit of \( p \rightarrow 1^+ \).

In [21], the “general mean distance” is used in order to derive upper bounds on the MAP probability of error. One subclass of upper bounds presented in this reference is

\[
P_e^{(\text{min})} \leq B_p^{(\text{GMD3})} = 1 - M^{1/p} E_x \left[ \sum_{i=1}^{M} P_{\theta_i|\mathbf{x}} (\theta_i|\mathbf{x}) \right]^{1/p}, \quad p > 1.
\]

It can be seen that \( 1 - B_p^{(\text{GMD3})} = M^{1/p} \left( 1 - B_p^{(2)} \right) \). Thus, the computations of (33) for specific hypothesis testing problems can be utilized to compute the lower bounds in (20) for the same problems.

IV. Application: simple versions of the Ziv-Zakai lower bound

The new classes of lower bounds on the probability of error can be used to derive simple closed forms of the ZZLB for Bayesian parameter estimation. A critical factor in implementing the extended ZZLB is the evaluation of the probability of error in a binary detection problem. The bounds are useful only if the probability of error is known or can be tightly lower bounded. Thus, the new classes of lower bounds in (14), (15) can be used in order to derive lower bounds on the ZZLB, providing less tighter MSE bounds which may be easier to compute. Note that the derivation in this section is performed under the assumption that \( \mathbf{x} \) is continuous random variable. Extension to any random variable \( \mathbf{x} \) with \( E[\mathbf{x}^2] < \infty \) is straightforward.

Consider the estimation of a continuous scalar random variable \( \phi \in \Phi \), with \textit{a-priori} probability density function (pdf) \( f_\phi(\phi) \), based on an observation vector \( \mathbf{x} \in \chi \). The pdf’s \( f_{\phi|\mathbf{x}}(\cdot|\mathbf{x}) \) denotes the conditional pdf
of $\phi$ given $x$. For any estimator $\hat{\phi}(x)$ with estimation error $\epsilon = \hat{\phi}(x) - \phi$, the mean-square-error (MSE) is defined as $E \left[ \left( \hat{\phi}(x) - \phi \right)^2 \right]$. The extended ZZLB is \cite{27}:

$$E \left[ \left( \hat{\phi}(x) - \phi \right)^2 \right] \geq \text{ZZLB} = \frac{1}{2} \int_0^\infty V \left\{ \int_{-\infty}^\infty (f_\phi(\varphi) + f_\phi(\varphi + h)) P_{\min}(\varphi, \varphi + h) \, d\varphi \right\} \, dh$$

\hspace{1cm} (34)

where $P_{\min}(\varphi, \varphi + h)$ is the minimum probability of error for the following detection problem:

$$H_0 : f_{x|H_0}(x) = f_{x|\phi}(x|\varphi)$$

$$H_1 : f_{x|H_1}(x) = f_{x|\phi}(x|\varphi + h)$$

with prior probabilities

$$P(H_0) = \frac{f_\phi(\varphi)}{f_\phi(\varphi) + f_\phi(\varphi + h)}, \quad P(H_1) = 1 - P(H_0).$$

\hspace{1cm} (36)

The operator $V$ returns a nonincreasing function by filling in any valleys in the input function

$$Vf(h) = \max_{\xi \geq 0} f(h + \xi), \quad h \in \mathbb{R}.$$  

\hspace{1cm} (37)

Since $f_\phi(\varphi)$, $f_\phi(\varphi + h)$, and $P_{\min}(\varphi, \varphi + h)$ are non-negative terms, the inner integral term in (34) can be lower bounded by bounding $P_{\min}(\varphi, \varphi + h)$. Thus,

$$E \left[ \left( \hat{\phi}(x) - \phi \right)^2 \right] \geq \text{ZZLB} \geq C_p \triangleq \frac{1}{2} \int_0^\infty V \left\{ \int_{-\infty}^\infty (f_\phi(\varphi) + f_\phi(\varphi + h)) LB(\varphi, h) \, d\varphi \right\} \, dh$$

\hspace{1cm} (38)

where $LB(\varphi, h)$ is any lower bound on the minimum probability of error of the detection problem stated in (35). By substituting the lower bound on the probability of outage error from (14) and (15), respectively, in (38) with $M = 2$ and using arbitrary non-negative functions $\zeta_1(x)$, $\zeta_2(x)$ one obtains different MSE lower bounds. In particular, by substituting $LB = B_p^{(1)}$ and $LB = B_p^{(2)}$ from (20) and (21), respectively in (38), one obtains the tightest classes of MSE lower bounds

$$E \left[ \left( \hat{\phi}(x) - \phi \right)^2 \right] \geq C_p^{(1)} \triangleq \frac{1}{2} \int_0^\infty V \left\{ \int_{-\infty}^\infty \left[ \left( f_{\phi|x}^{1-p}(\varphi|x) + f_{\phi|x}^{1-p}(\varphi + h|x) \right)^{1-p} \right] \, d\varphi \right\} \, dh, \quad \forall p > 1$$

\hspace{1cm} (39)

and

$$E \left[ \left( \hat{\phi}(x) - \phi \right)^2 \right] \geq C_p^{(2)} \triangleq \frac{1}{2} \int_0^\infty V \left\{ 2 - \int_{-\infty}^\infty \left[ \left( f_{\phi|x}^{p}(\varphi|x) + f_{\phi|x}^{p}(\varphi + h|x) \right)^{\frac{p-1}{p}} \right] \, d\varphi \right\} \, dh$$

\hspace{1cm} (40)
∀p > 1. For p → 1⁺, the bounds in (39) and (40) become

\[
E \left[ \left| \phi(x) - \phi \right|^2 \right] \geq \frac{1}{2} \int_0^\infty V \left\{ \int_{-\infty}^\infty E \left[ \min (f_{\phi|x} (\varphi|x), f_{\phi|x} (\varphi + h|x)) \right] \, d\varphi \right\} \, h \, dh, \tag{41}
\]

which coincides with the ZZLB as presented in [28].

V. EXAMPLES

A. Bounds comparison

Fig. 1 depicts the lower bounds \(B_p^{(1)}\) and \(B_p^{(2)}\), presented in (20) and (21), for the binary hypothesis problem against the conditional probability \(P(\theta_1|x)\), for different values of the parameter \(p\) and given \(x\). It can be seen that the bounds in (20) and (21) become tighter as \(p\) decreases and that for given \(p\), \(B_p^{(2)}\) is always tighter than \(B_p^{(1)}\).

Fig. 2 depicts the lower bound \(B_p^{(2)}\), presented in (21), for the binary hypothesis problem against the conditional probability \(P(\theta_1|x)\), for different values of the parameter \(p\) and given \(x\). The new bound is compared to the bounds \(B^{(Gauss-sin)}\) and \(B^{(ATLB)}\) with \(\alpha = 5\), presented in Appendix B. It can be seen that \(B_p^{(2)}\) becomes tighter as \(p\) decreases, and that for \(p = 1.1\), the new bound is tighter than the other lower bounds almost everywhere.

B. Example: Binary hypothesis problem

Consider the following binary hypothesis testing problem:

\[
\begin{align*}
\theta_1 : f(x|\theta_1) &= \lambda_1 e^{-\lambda_1 x} u(x) \\
\theta_2 : f(x|\theta_2) &= \lambda_2 e^{-\lambda_2 x} u(x)
\end{align*}
\tag{42}
\]

where \(u(\cdot)\) denotes the unit step function, \(P(\theta_1) = P(\theta_2) = \frac{1}{2}\), and \(\lambda_1 = \frac{1}{2}, \lambda_2 > \lambda_1\). For this problem, the bounds in (20) with \(p = 2\) and \(p = 1.5\) are

\[
\begin{align*}
B_2^{(1)} &= \frac{1}{2} \left(\frac{\lambda_2}{\lambda_1 - \lambda_2}, 1; \frac{\lambda_1 - 2\lambda_2}{\lambda_1 - \lambda_2}; -\frac{\lambda_2}{\lambda_1}\right) \\
B_{1.5}^{(1)} &= \frac{1}{2} \left(\frac{\lambda_2}{2(\lambda_1 - \lambda_2)}, \frac{1}{2}; 1 - \frac{\lambda_2}{2(\lambda_1 - \lambda_2)}; -\frac{\lambda_2^2}{\lambda_1}\right)
\end{align*}
\]
and the bounds in [24] are

\[
B_{q}^{(2)} = 1 - \frac{1}{2} {}_2F_1 \left( -\frac{\lambda_1}{q(\lambda_1 - \lambda_2)}; -\frac{1}{q}; 1 - \frac{\lambda_1}{q(\lambda_1 - \lambda_2)}; \frac{\lambda_2^q}{\lambda_1^q} \right) \quad \forall q = 2, 3, \ldots
\]

where \( {}_2F_1 \) is the hypergeometric function [29]. Several bounds on the probability of error and the minimum probability of error obtained by the MAP detector are presented in Fig. 3 as a function of the distribution parameter, \( \lambda_2 \). The bounds depicted in this figure are: \( B^{(BLB1)} \), \( B^{(BLB2)} \), \( B^{(Bayes1)} \) in addition to the proposed lower bounds \( B_p^{(1)} \) with \( p = 1.5, 2 \) and \( B_p^{(2)} \) with \( p = 1.11, 1.5, 2 \). It can be seen that \( B^{(BLB1)} \) is lower than any proposed lower bound. In addition, for \( \lambda_2 \geq 0.65 \) the proposed bound \( B_{1.11}^{(2)} \) is tighter than all the other bounds and it is close to the minimum probability of error obtained by the MAP decision rule. For \( \lambda_2 \geq 0.8 \) \( B_{1.5}^{(1)} \) is tighter than the Bhattacharyya lower bounds and \( B_2^{(1)} \) and \( B_2^{(2)} \) are tighter than the \( B^{(BLB1)} \)
Fig. 2

The proposed lower bounds, \( B_p^{(1)} \) and \( B_p^{(2)} \) with \( p = 1.25, 1.1 \) compared to other existing bounds as a function of the conditional probability \( P(\theta_1|x) \) for binary hypothesis testing.

everywhere and tighter than other bounds in some specific regions. Fig. 2 presents the proposed lower bounds \( B_p^{(1)} \) with \( p = 1.5, 2 \) as a function of \( \lambda_2 \) compared to the upper bounds on the MAP probability of error [4], given by (32). It can be seen that this class of upper bounds is proportional to the proposed tightest subclass of lower bounds in [20] with a factor of \( 2^{p-1} \).
C. Example: Multiple hypothesis problem

Consider the following multiple hypothesis testing problem:

\[ \theta_1 : f(x|\theta_1) = \frac{3}{2} \cos^2(x/2)e^{-|x|} \]
\[ \theta_2 : f(x|\theta_2) = 2 \sin^2(x/2)e^{-|x|} \]  
\[ \theta_3 : f(x|\theta_3) = \frac{5}{2} \sin^2(x)e^{-|x|} \]  

(43)
with $P(\theta_1) = \frac{15}{28}$, $P(\theta_2) = \frac{5}{28}$, and $P(\theta_3) = \frac{8}{28}$. In this problem, the exact probability of error of the MAP detector is difficult to compute. The bounds $B^{(\text{Bayes1})}$, $B^{(\text{Bayes2})}$, $B^{(\text{Bayes3})}$, and $B^{(\text{quad})}$ are not tractable. The proposed bound with $q = 2$ is computable and is equal to

$$B_q^{(1)} = \frac{40}{14} \int_0^\infty \frac{e^{-x}}{\cos^2(x/2)} + \frac{1}{\sin^2(x/2)} + \frac{1}{\sin^2(x)} \, dx =$$

$$= \frac{2}{35} e^{-x} (\cos(2x) - 2\sin(2x) - 5) \bigg|_0^\infty = 0.2286.$$
This example demonstrates the simplicity of the proposed bound with $q = 2$, while the other bounds are intractable.

VI. Conclusion

In this paper, new classes of lower bounds on the probability of error in multiple hypothesis testing were presented. The proposed classes depend on a parameter, $p$, which at the limit of $p \to 1^+$ approach the minimum attainable probability of error provided by the MAP detector. It is shown that these classes of bounds generalize some existing bounds for binary and multiple hypothesis testing. New variations using the proposed classes. It was shown via examples that the proposed bounds outperform other existing bounds in terms of tightness and simplicity of calculation.

Appendix

A. Necessary and sufficient condition for independency of (7) and (8) on $\hat{\theta}$

In this appendix, it is shown that the expectation $E[|u(x, \theta)v_k(x, \theta)|]$, $k = 1, 2$ is independent of the detector $\hat{\theta}$ iff

$$v_k(x, \theta_i) = \frac{\zeta_k(x)}{P(\theta_i|x)}, \quad k = 1, 2, \quad i = 1, \ldots, M$$

where $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ are arbitrary functions of the observations $x$.

Sufficient condition: By substituting the function $v_k(x, \theta_i) = \frac{\zeta_k(x)}{P(\theta_i|x)}$ for almost every $x$ in $E[|u(x, \theta)v_k(x, \theta)|]$, one obtains

$$E[|u(x, \theta)v_k(x, \theta)|] = E \left[ \sum_{i=1}^{M} u(x, \theta_i) \zeta_k(x) \right] = E \left[ \sum_{i=1}^{M} \zeta_k(x) \right] = (M - 1)E[\zeta_k(x)]$$

which is independent of the detector $\hat{\theta}$. Substitution of (45) in (7) and (8) results in bounds that are independent of the detector $\hat{\theta}$.

Necessary condition: Let $E[|u(x, \theta)v_k(x, \theta)|]$ be independent of the detector $\hat{\theta}$ and define the following
sequence of two-hypothesis detectors

\[ \hat{\theta}_{A,j,l}(x) = \theta_j 1_{x \in A} + \theta_l 1_{x \in A^c} = \begin{cases} 
\theta_j & \text{if } x \in A \\
\theta_l & \text{if } x \in A^c
\end{cases}, \quad j, l = 1, \ldots, M \tag{46} \]

where \( x \) is a random observation vector with positive probability measure and \( A^c \) is the complementary event of \( A \). For each detector

\[
E[|u(x, \theta)v_k(x, \theta)|] =
\]

\[
= E \left[ \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) \mathbf{1}_{\theta \neq \theta_i} \mathbf{1}_{x \in A} \right] P(x \in A) + E \left[ \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) \mathbf{1}_{\theta \neq \theta_i} \mathbf{1}_{x \in A^c} \right] P(x \in A^c). \tag{47}
\]

Using (46) and the law of total probability, one obtains

\[
E[|u(x, \theta)v_k(x, \theta)|] =
\]

\[
= E \left[ \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) \mathbf{1}_{x \in A} \right] P(x \in A) + E \left[ \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) \mathbf{1}_{x \in A^c} \right] P(x \in A^c)
\]

\[
= E \left[ \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) - \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) \right] P(x \in A) + E \left[ \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) \right] P(x \in A^c)
\]

\[
= E \left[ \left( \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) - \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) \right) \mathbf{1}_{x \in A} \right] P(x \in A) + E \left[ \sum_{i=1}^{M} P(\theta_i|x)v_k(x, \theta_i) \right] P(x \in A^c). \tag{48}
\]

Under the assumption that \( E[|u(x, \theta)v_k(x, \theta)|] \) is independent of the detector \( \hat{\theta} \), in particular, \( (47) \) is identical for all \( j, l = 1, \ldots, M \), that is this term is independent of \( A \), \( \theta_j \), and \( \theta_l \). Thus, for given \( \theta_l \), the term

\[
E \left[ (P(\theta_l|x)v_k(x, \theta_l) - P(\theta_j|x)v_k(x, \theta_j)) \mathbf{1}_{x \in A} \right] P(x \in A)
\]

is identical for every \( A \) and \( \theta_j \). In particular, by setting \( A = \emptyset \) where \( \emptyset \) is the empty set, one obtains

\[
E \left[ (P(\theta_l|x)v_k(x, \theta_l) - P(\theta_j|x)v_k(x, \theta_j)) \mathbf{1}_{x \in \emptyset} \right] P(x \in \emptyset) = 0, \quad j = 1, \ldots, M
\]
and therefore
\[
E[P(\theta_l|x)v_k(x, \theta_l) - P(\theta_j|x)v_k(x, \theta_j) | x \in A] P(x \in A) = 0, \quad \forall A, j = 1, \ldots, M
\]
which is possible only if
\[
P(\theta_l|x)v_k(x, \theta_l) - P(\theta_j|x)v_k(x, \theta_j) = 0, \quad \forall j = 1, \ldots, M.
\]
Because \(l\) is arbitrarily chosen, one obtains
\[
P(\theta_l|x)v_k(x, \theta_l) = P(\theta_j|x)v_k(x, \theta_j) = \zeta_k(x), \quad \forall j, l = 1, \ldots, M
\]
where \(\zeta_k(x)\) does not depend on the hypothesis.

B. Review of Existing Lower Bounds

In this appendix, some existing lower bounds on the minimum probability of error are presented. Part of these bounds are presented also in the review in [30].

**Binary hypothesis testing bounds**

Most of the binary hypothesis testing bounds are based on divergence measures of the difference between two probability distributions, known as \(f\)-divergences or Ali-Silvey distances [31]. In [7], the divergence and two Bhattacharyya-based lower bounds were proposed. The divergence lower bound is

\[
P_e \geq B^{(\text{div})} = \frac{1}{8} e^{-J/2}
\]

where \(J = E[\log L(x)|\theta_1] - E[\log L(x)|\theta_2]\) and \(L(x) = \frac{P(\theta_1|x)P(\theta_2)}{P(\theta_2|x)P(\theta_1)}\) is the likelihood ratio function. A simple Bhattacharyya-based lower bound is

\[
P_e \geq B^{(\text{BLB1})} = \frac{E^2 \left[ \sqrt{P(\theta_1|x)P(\theta_2|x)} \right]}{8P(\theta_1)P(\theta_2)}.
\]

This bound is always tighter than the divergence lower bound [7]. The second Bhattacharyya-based bound on \(P_e\) is

\[
P_e \geq B^{(\text{BLB2})} = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4E^2 \left[ \sqrt{P(\theta_1|x)P(\theta_2|x)} \right]}.
\]
Another $f$-divergence bound is proposed in [8]:

\[
P_e \geq B^{(f)} = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \mathbb{E}[(4P(\theta_1|x) P(\theta_2|x))^{L}]} \tag{52}
\]

where $L \geq 1$. For $L = 1$ this bound can be obtained also by applying Jensen’s inequality on the MAP probability of error. The harmonic lower bound was proposed in [4]:

\[
P_e \geq B^{(HLB)} = \mathbb{E}[P(\theta_1|x)P(\theta_2|x)] . \tag{53}
\]

The pairwise separability measure, $J_{\alpha}(\theta|x) = \mathbb{E}[|P(\theta_1|x) - P(\theta_2|x)|^{\alpha}]$, is used to derive the following binary bound [32]

\[
P_e \geq B^{(J_{\alpha})} = \frac{1}{2} - \frac{1}{2} J_{\alpha}^{\frac{1}{\alpha}}(\theta|x), \quad 1 \geq \alpha . \tag{54}
\]

The “Gaussian-Sinusoidal” lower bound [11] is given by

\[
P_e \geq B^{(Gauss-sin)} = 0.395\mathbb{E}\left[\sin(\pi P(\theta_1|x))e^{-\alpha(P(\theta_1|x)-\frac{1}{2})^2}\right] \tag{55}
\]

where $\alpha = 1.8063$. Although this bound is tight, it is usually not tractable. An arbitrarily tight lower bound [10] is given by

\[
P_e \geq B^{(ATLB)} = \frac{1}{\alpha} \mathbb{E}\left[\log \frac{1 + e^{-\alpha}}{e^{-\alpha P(\theta_1|x)} + e^{-\alpha P(\theta_2|x)}}\right] \tag{56}
\]

for any $\alpha > 0$. By selecting high enough values for $\alpha$, this lower bound can be made arbitrarily close to $P_e^{(min)}$. However, in general this bound is difficult to evaluate.

**Multiple hypothesis testing bounds**

For multiple hypothesis testing problems, the following lower bounds have been proposed. In [17], Devijver derived the following bounds using the conditional Bayesian distance:

\[
P_e \geq B^{(Bayes1)} = \frac{M - 1}{M} \left(1 - \sqrt{\frac{M \times \mathbb{E}\left[\sum_{i=1}^{M} P^{2}(\theta_i|x)\right] - 1}{M - 1}}\right) \tag{57}
\]

and

\[
P_e \geq B^{(Bayes2)} = 1 - \sqrt{\mathbb{E}\left[\sum_{i=1}^{M} P^{2}(\theta_i|x)\right]} \tag{58}
\]
where \( \text{E} \left[ \sum_{i=1}^{M} P^2 (\theta_i | x) \right] \) is the conditional Bayesian distance. The bound in (57) with \( M = 2 \) is identical to (54) with \( \alpha = 2 \). In [17], it is analytically shown that for the binary case the Bayesian distance lower bound in (57) is always tighter than the Bhattacharyya bound in (51). Using Jensen’s inequality, the following bound is tighter than the bound in (58) [4], [17]

\[
P_e \geq B^{(Bayes3)} = 1 - E \left[ \sqrt[2]{\sum_{i=1}^{M} P^2 (\theta_i | x)} \right].
\] (59)

The bound

\[
P_e \geq B^{(quad)} = \frac{1}{2} - \frac{1}{2} E \left[ \sum_{i=1}^{M} P^2 (\theta_i | x) \right]
\] (60)

was proposed in [23] and [24] in the context of Vajdas quadratic entropy and the quadratic mutual information, respectively. Note that the bound \( B^{(quad)} \) can be interpreted as an \( M \)-ary extension to the harmonic mean bound, presented in [53]. In [17], it is claimed that \( B^{(quad)} \leq B^{(Bayes2)} \leq B^{(Bayes1)} \). The affinity measure of information relevant to the discrimination among the \( M \) hypothesis is defined as lower bound on \( P_e \) [19]

\[
P_e \geq B^{(MLB)} = \frac{M - 1}{M^{M-1}} \left( E \left[ \prod_{i=1}^{M} P^{1/\beta} (\theta_i | x) \right] \right)^M.
\] (61)

The “general mean distance” between the \( M \) hypotheses is \( G_{\alpha,\beta} = E \left[ \left( \sum_{i=1}^{M} P^{\beta} (\theta_i | x) \right)^\alpha \right] \) [20], [21]. Many lower bounds on \( P_e \) and upper bounds on \( P_e^{(\min)} \) can be obtained from this distance [21]. For example, the binary bound in (62) and the following classes of bounds:

\[
P_e \geq B^{(GMD1)} = 1 - G^{1/\alpha}_{\alpha,\beta}, \quad 0 < \alpha, \quad 1 < \beta, \quad \frac{1}{\alpha} \leq \beta
\] (62)

\[
P_e \geq B^{(GMD2)} = 1 - G_{\alpha,\beta}, \quad 0 < \alpha, \quad 1 < \beta \leq \frac{1}{\alpha}
\] (63)

It can be seen that by substituting \( \beta = 2, \alpha = 1 \) in (62) we obtained the lower bounds in [55]. By substituting \( 0 < \alpha < 1 \) and \( \frac{1}{\alpha} = \beta \) in (62) or in (63), one obtains the lower bound

\[
P_e \geq 1 - E \left[ \left( \sum_{i=1}^{M} P^{\beta} (\theta_i | x) \right)^{1/2} \right].
\] (64)

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