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Abstract A geometric characterization of transition amplitudes between coherent states, or equivalently, of the hermitian scalar product of holomorphic cross sections in the associated $D_{\tilde{M}}$-module, in terms of the embedding of the coherent state manifold $\tilde{M}$ into a projective Hilbert space is proposed. Coherent state manifolds endowed with a homogeneous kaehler structure are considered. Using the coherent state approach, an effective method to find the cut loci on symmetric manifolds and generalized symmetric manifolds $\tilde{M}$ is proposed. The CW- complex structure of coherent state manifolds of flag type is discussed. Recent results of Anandan and Aharonov are commented vis-à-vis of last century constructions in projective geometry. Calculations with significance in the coherent state approach furnishe explicit proofs of the results announced by Y. C.Wong on conjugate locus in complex Grassmann manifold.

Let a quantum system with symmetry, i.e. a triplet $(K, G, \pi)$, where $\pi$ is an unitary irreducible representation of the group $G$ on the Hilbert space $K$. Let the orbit $\tilde{M} = \tilde{\pi}(G)|\psi_0>$, where $|\psi_0> \in K, \xi|\psi_0> = |\tilde{\psi}_0> \in PK$. Then $\tilde{M} \approx G/K$ (diffeomorphism), where $K$ is the stationary group of the state $|\tilde{\psi}_0>$. If $i : M \leftrightarrow PK$ is an embedding, then $\tilde{M}$ is called coherent state manifold. If $|\psi_0> \equiv |j>$ is an extremal weight vector and $G$ is compact connected simply connected, then $\tilde{M}$ is Kähler and $\pi_j$ is given by the Borel-Weil-Bott theorem. If a local section $\sigma : \tilde{M} \rightarrow S(K)$ in the unity sphere in $K$ is constructed, then the set $M = \sigma(\tilde{M})$ is named coherent vector manifold. $M$ is the holomorphic line bundle associated by a holomorphic character $\chi$ of the parabolic subgroup $P$ of the complexification $G^c$ of $G$.

The coherent states are introduced as

$$|Z, j> = \exp \sum_{\varphi \in \Delta^+_n} (Z_{\varphi}F^+_\varphi)|j>, |\tilde{Z} > = <Z|Z >^{-1/2}|Z>, \quad (*)$$

where $\Delta^+_n$ are the positive non-compact roots. If $Z = (Z_{\varphi}) \in C^d$, where $d$ is the dimension
of \( \widetilde{M} \), then \( Z \) are local coordinates in the neighborhood \( V_0 \subset \widetilde{M} \) around \( Z = 0 \). Here \( F_\varphi^+|_j \neq 0, F_\varphi^-|_j = 0, \varphi \in \Delta^+_n \). Then \( < \widetilde{Z}'|\widetilde{Z} > \) is: i) the hermitian scalar product of holomorphic sections in the line bundle \( M \) associated by \( \chi \) to the principal holomorphic bundle \( P \to G^c \to G^c/P \), or ii) the hermitian scalar product of sections with base \( \widetilde{M} \) in the \( L_{\widetilde{M}} \)-module of differentiable operators on \( \widetilde{M} \). Here \( G^c/P \) is a flag manifold.

**Problem 1:** find a geometric meaning of the transition probability on coherent state manifold.

**Proposition 1:** Let \( \widetilde{Z} \) as in (\( * \)), where \( Z \) parametrize the coherent state manifold in the \( V_0 \subset \widetilde{M} \) and let the embedding \( i : \widetilde{M} \hookrightarrow PK \). Then the angle \( \theta \equiv \arccos |< \widetilde{Z}'|\widetilde{Z} > | \),

is equal to the Cayley distance on the geodesic joining \( i(Z'), i(Z) \), where \( Z', Z \in V_0 \),

\[ \theta = d_c(i(Z'), i(Z)). \]

More generally, it is true the following relation (Cauchy formula)

\[ < \widetilde{Z}'|\widetilde{Z} > = (i(Z'), i(Z)). \]

**Proof:** The holomorphic line bundle \( M \) of coherent vectors is the pull back \( i^* \) of the hyperplan bundle of \( PK \), the dual bundle of the tautological line bundle of \( PK^* \).

Here \( (.,.) \) is the scalar product in \( K \). If \( \xi : K\{0\} \to PK, \xi : \omega \to [\omega], \) then the Cayley (1859) distance is

\[ d_c([\omega'], [\omega]) = \arccos \frac{|(\omega', \omega)|}{\|\omega'\|\|\omega\|}. \]

For the \( \infty \)-dimensional case, see Kobayashi (1959), which also gives conditions for the existence of the embedding \( i \). See also Rawnsley (1977).

**Comment:** The Cayley distance has been used in Quantum Mechanics by Wick (1967), Fivel (1973) and recently by Anandan and Aharonov (1990). The Cayley distance is useful in the geodesic approach. The (Bargmann) distance \( d_b \), used by Prevost and Vallée (1980) in the context of coherent states,

\[ d_b^2([\omega'], [\omega]) = 2(1 - \cos d_c([\omega'], [\omega])), \]

is equivalent with \( d_c : 2\sqrt{2}/\pi \leq d_b \leq d_c. \)

**Problem 2:** find those manifolds \( \widetilde{M} \) for which the angle \( \theta \equiv \arccos |< \widetilde{Z}'|\widetilde{Z} > | \) is a distance on \( \widetilde{M} \).

**Proposition 2:** Let \( \widetilde{M} \) be a coherent state manifold parametrized as in (\( * \)). Then the angle \( \theta \equiv \arccos |< \widetilde{Z}'|\widetilde{Z} > | \) is a distance on \( \widetilde{M} \) iff \( \widetilde{M} \) is a symmetric space of rank 1.

**Proof:** The problem is reduced to that of two-point homogeneous spaces, which are known (Wolf).

**Comment:** Generally, the distance \( \delta \geq \theta \), but infinitesimally, \( d\delta = d\theta \). Let us illustrate this on the complex Grassmann manifold \( G_n(C^{n+m}) \). Then

\[ \cos \theta = \cos \theta_1...\cos \theta_n, \theta = d_c(i(0), i(z)), i : G_n(C^{n+m}) \hookrightarrow \mathbb{CP} \left( \begin{array}{c} m+n \\ n \end{array} \right)^{-1}, \]
i is the Plucker embedding, \(\theta_i, i = 1, ..., n\) are the stationary angles of the two \(n\)-planes in \(\mathbb{C}^{n+m}\) (Jordan (1875)), \(\delta^2 = \sum_{i=1}^{n} \theta_i^2, (\theta_i < \pi/2, i = 1, ..., n)\) (Rosenfel’d (1941)).

**Problem 3:** find a geometric meaning of Calabi’s diastasis (1953), used by Cahen, Gutt, Rawnsley (1993) in the context of coherent states, \(D(Z', Z) = -2|n| < \overline{Z}'\overline{Z} > |\).

**Proposition 3:** The diastasis distance \(D(Z', Z)\) between \(Z', Z \in V_0 \subset M\) is related to the geodesic distance \(\theta = d_c(i(Z'), i(Z))\), where \(i : M \hookrightarrow PK\), by

\[
D(Z', Z) = ln cos^{-2}\theta.
\]

If \(\tilde{M}_n\) is noncompact and \(i' : \tilde{M}_n \hookrightarrow PK^{N-1,1} = SU(N, 1)/S(U(N) \times U(1))(i : \tilde{M}_n \hookrightarrow PK)\) and \(\delta_n(\theta_n)\) is the length of the geodesic joining \(i'(Z'), i'(Z)(\text{resp.} i(Z'), i(Z))\), then

\[
\cos\theta_n = \cosh^{-1}\delta_n = e^{-D/2}.
\]

**Proof:** cf. Prop. 1.

**Problem 4:** characterize the relationship of the number \(N\) and the manifold \(\tilde{M}\) in the embedding \(i : \tilde{M} \hookrightarrow \mathbb{C}P^{N-1}\).

**Proposition 4:** For coherent state manifolds \(\tilde{M} \approx G/K\) which have a flag manifold structure, the following 7 numbers are equal:

1) the maximal number of orthogonal coherent vectors on \(\tilde{M}\);
2) the number of holomorphic global sections in the holomorphic line bundle \(\mathcal{M}\) with base \(\tilde{M}\);
3) the dimension of the representation in the Borel-Weil-Bott theorem;
4) the minimal \(N\) appearing in the Kodaira embedding theorem, \(i : \tilde{M} \hookrightarrow \mathbb{C}P^{N-1}\);
5) the number of critical points of the energy function \(f_H\) attached to a Hamiltonian \(H\) linear in the generators of the Cartan algebra of \(G\), with inequal coefficients;
6) the Euler- Poincaré characteristic of \(\tilde{M} \approx G/K\), \(\chi(\tilde{M}) = [W_G]/[W_H]\), where \([W_G] = cardW_G\) and \(W_G\) is the Weyl group of \(G\);
7) the number of Borel-Morse cells which appear in the CW- complex decomposition of \(\tilde{M}\).

**Proof:** Use theorems 1, 2 in S.B. and A.C.Gheorghe (1987), where it is proved that \(f_H\) is a perfect Morse function, and also the Cauchy formula. Remark that \(\chi(G/K) > 0\) iff \(\text{Rank}\ G = \text{Rank}\ K\) (cf. Hopf, Samelson (1941)).

**Comment :** The Weil prequantization condition is the condition to have a Kodaira embedding, i.e. the algebraic manifold to be Hodge.

**Problem 5:** find a relationship between geodesics and coherent states.

**Proposition 5:** For a \(d\)-dimensional manifold \(G/K\) equipped with hermitian symmetric space structure, the parameters \(B_\varphi\) in formulas of coherent vectors

\[
|B, j > = \exp \sum_{\varphi \in \Delta^+_k} (B_\varphi F_\varphi^+ - B_\varphi F_\varphi^-)|j >, \tag{**}
\]

\[
|B, j > = |Z, j > = < Z, j | Z, j >^{-1/2} \exp \sum_{\varphi \in \Delta^+_k} (Z_\varphi F_\varphi^+)|j >, \tag{**}
\]

are normal coordinates in the normal neighborhood \(V_0 \approx \mathbb{C}^d\) around \(Z = 0\). The conjugate locus of the point \(Z = 0\) is obtained annullating the Jacobian of the transformation \(Z = Z(B)\).

**Proof:** See S.B. and L.Boutet de Monvel (1993).
**By-product:** The results announced by Wong (1968) on conjugate locus in $G_n(C^{n+m})$ are proved. They were contested by T. Sakai in 1977. The proof uses the Jordan angles.

**Problem 6:** characterize geometrically the polar divizor of $|0>$,

$$
\Sigma_0 = \{|\psi > ||\psi >= \text{ coherent vector}, <0|\psi >= 0\}.
$$

Let $e$ be the unity element in $G/K, \lambda : G \rightarrow G/K$ the natural projection, $g = k \oplus m$ the Lie algebra decomposition, $Exp : T_pM \rightarrow M$ the exponential mapping and $exp : g \rightarrow G$.

Let the condition

$$A1) \quad Exp|\lambda(e) = \lambda \circ exp|m,$$

i.e. the geodesics in $\tilde{M}$ are images of one-parameter subgroups of $G$.

Thimm furnishes sufficient conditions for the manifold $\tilde{M}$ to verify $A1)$, for example the reductive manifolds, and in particular the symmetric spaces verify $A1)$ (cf. Cartan).

In the next proposition, $CL_0$ denotes the cut locus of $0 \in \tilde{M}$.

**Proposition 6:** Let $\tilde{M} \approx G/K$, and suppose the parametrization $(\ast)$ around $Z = 0$ in $V_0$. Then $\tilde{M} = V_0 \cup \Sigma_0$ (disjoint union). If $A1)$ is true for the manifold $\tilde{M}$, then $\Sigma_0 = CL_0$.

Moreover, if $i$ is the embeeding $i : \tilde{M} \hookrightarrow PK$, then

$$CL_0 = \{Z \in \tilde{M}|d_c(i(0),i(z)) = \pi/2\}.$$

**Proof:** The theorems reffering to $CL$ are from Kobayashi, Nomizu Vol II. See also S.Kobayashi in ”Global differential geometry” (1989).

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References (selective list):

- A.Cayley, Phil. Trans. Royal. Soc. London 149 (1859) 61; C. Jordan, Bull. Soc. Math. France. t III (1875) 103; E.A.Rosenfel’d, Izv.Akad.Nauk SSSR, ser. Mat. 5 (1941) 353; E.Calabi, Ann.Math. 58 (1953) 1; S.Kobayashi, Trans.Amer.Math.Soc. 92 (1959) 267; G.K.Wick, On symmetry transformations, in ”Preludes in Theoretical Physics”, North Holand (1967); J.Wolf, Spaces of constant curvature (1967); Y.C.Wong, Bull.Am.Math.Soc. 74 (1968) 240; A.M.Perelomov, Commun.Math. Phys. 26 (1972) 222; D.I. Fivel, preprint Maryland 73-102 (1973); J.Rawnsley, Quart.J.Math. Oxford 28 (1977) 403; T.Sakai, Hokkaido Math. J. 6 (1977) 136; J.P.Prevost, G.ValléE, Commun.Math.Phys. 76 (1980) 289; A.Thimm, Ergod. Theory Dyn. Syst. 1 (1981) 495; S.Berceanu, A.Gheorghe, J.Math.Phys. 20 (1987) 2892; J.Anandan, Y.Aharonov, Phys.Rev.Lett. 65 (1990) 1697; S.Berceanu, L.Boutet de Monvel, J.Math.Phys.34 (1993) 2353; M.Cahen, S.Gutt, J.Rawnsley, Trans. Math. Soc. 337 (1993 ) 73.