Fate of the Black String Instability

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Abstract

Gregory and Laflamme showed that certain nonextremal black strings (and $p$-branes) are unstable to linearized perturbations. It is widely believed that this instability will cause the black string horizon to classically pinch off and then quantum mechanically separate, resulting in higher dimensional black holes. We argue that this cannot happen. Under very mild assumptions, classical event horizons cannot pinch off. Instead, they settle down to new static black string solutions which are not translationally invariant along the string.
1 Introduction

It is well known that four dimensional black holes are stable \[1\]. Almost ten years ago, Gregory and Laflamme \[2, 3\] showed that this is not true for higher dimensional generalizations of black holes, such as black strings and black $p$-branes. The simplest black string solution is the product of the four dimensional Schwarzschild metric and a circle of length $L$. Gregory and Laflamme showed that this spacetime is unstable to linearized perturbations with a wavelength along the circle larger than the Schwarzschild radius of the black hole $r_0$. They also compared the total entropy of the black string with that of a five dimensional black hole with the same total mass, and found that when $L > r_0$, the black hole had greater entropy. They thus suggested that the full nonlinear evolution of the instability would result in the black string breaking up into separate black holes which would then coalesce into a single black hole. Classically, horizons cannot bifurcate, but the idea was that under classical evolution, the event horizon would pinch off and become singular. When the curvature became large enough, it was plausible that quantum effects would smooth out the transition between the black string and black holes. The same instability was found for higher dimensional black $p$-branes, and also black strings (and $p$-branes) carrying certain charges, as long as they were nonextremal.

These higher dimensional generalizations of black holes arise naturally in string theory, and the idea that a black string with $L > r_0$ will break up into black holes has been widely accepted. It has been used in many recent string discussions, e.g., descriptions of black holes in matrix theory \[4, 5, 6\], discussions of the density of states of strongly coupled field theories using the AdS/CFT correspondence \[7, 8, 9\], the relation between near extremal D2 and M2 brane configurations \[10\], discussions of black holes on brane-worlds \[11\], and various unstable D-brane configurations \[12, 13\].

We will argue that this widespread belief is incorrect: Black strings do not in fact, break up into black holes! Under very weak assumptions, we prove that an event horizon cannot pinch off in finite time. In particular, if one perturbs Schwarzschild cross a circle, an $S^2$ on the horizon cannot shrink to zero size in finite affine parameter. The basic idea is the following. The famous area theorem is based on a local result that the divergence $\theta$ of the null geodesic generators of the horizon cannot become negative. If an $S^2$ on the horizon tries to shrink to zero size, $\theta$ can stay positive only if the horizon is expanding rapidly in the circle direction. But this produces a large shear which also drives $\theta$ negative. The upshot is that the solution settles down to a new (as yet unknown) static black string
solution which is not translationally invariant along the circle.

One can view this result as an example of spontaneous symmetry breaking in general relativity. The most symmetric solution is unstable, and the stable solution has less symmetry. Unlike the usual particle physics examples where the broken symmetry is an internal one, here the broken symmetry is spatial translations.

Our arguments apply to all black $p$-branes. If the $p$-brane is charged with respect to a $p + 2$ form $F$ (so the charge is the integral of $\ast F$ over a sphere surrounding the brane), it was already known that the $p$-brane could not break up into black holes since this charge can only be carried by an object extended in $p$ directions. Since nonextremal solutions are still unstable, it was clear that there must be new static solutions with less symmetry. However, these were thought to resemble the extremal solution with a Schwarzschild black hole superposed on it. Since the extremal $p$-brane horizon is often singular, this cannot happen. There must be new solutions with nonsingular horizons. Our results also apply to black $p$-branes with “smeared” charges associated with a lower rank form. Charge conservation does not prevent these solutions from breaking up, but nevertheless, the horizon must stay connected.

2 Horizons cannot pinch off

For simplicity, we will start with the example of four dimensional Schwarzschild with radius $r_0$ cross a circle of length $L > r_0$. Since it has been suggested that cosmic censorship might be violated in the evolution of the Gregory-Laflamme instability (see below), we must be careful not to assume cosmic censorship in our analysis. We will proceed by considering the maximal Cauchy evolution of smooth initial data on a surface $\Sigma$. Consider initial data for a vacuum spacetime which looks like a static slice of the black string plus a small perturbation. Alternatively, to avoid the second asymptotically flat region in the maximal extension of Schwarzschild, one can start with initial data describing infalling matter that will produce trapped surfaces and an apparent horizon with $S^2 \times S^1$ topology. Now consider the maximal Cauchy evolution of this initial data. Since the initial data is asymptotically flat, the evolution will include at least part of null infinity $\mathcal{I}^+$. Let $\tilde{\mathcal{I}}^+ = \mathcal{I}^+ \cap D^+ [\Sigma]$ where $D^+$ denotes the future domain of dependence in the conformally completed spacetime.

\footnote{There is a slight possibility that the horizon continues to pinch off, taking an infinite time to do so. If this happened, the curvature would eventually become large and quantum effects could cause the horizon to split. We will argue that this possibility is very unlikely, but have not been able to rigorously exclude it.}

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the weak energy condition is satisfied, the trapped surfaces cannot lie in the past of $\tilde{I}^+$. Thus there must be an event horizon, defined by the boundary of $I^-[\tilde{I}^+]$, enclosing all the trapped surfaces. We want to study properties of this event horizon.

To begin, we will consider the simplest case when the spacetime has spherical symmetry, and also reflection symmetry about $z = 0$, where $z$ is the coordinate along the $S^3$. To picture the evolution, consider the metric restricted to $z = 0$. This looks like a time dependent four dimensional spherical black hole. If the horizon shrinks to zero size in finite affine parameter, the Penrose diagram would look like Fig. 1.

Figure 1: A Penrose diagram of $z = 0$ surface if horizon pinches off. We show that this cannot happen.

The Cauchy evolution stops at the dashed line. The spacetime to the future of this line assumes the horizon splits into separate black holes. Note that the spacetime looks
just like an evaporating black hole and has a naked singularity. However we are now considering classical evolution. The horizon size on the \( z = 0 \) slice can decrease because there is an effective stress energy tensor coming from the higher dimensional curvature which can have negative energy densities. However, we now show that it cannot shrink all the way to zero size in finite affine parameter.

In a neighborhood of the horizon one can introduce Gaussian null coordinates so that the metric takes the form \[14\]

\[
\begin{align*}
    ds^2 &= -f d\lambda^2 + 2 dr d\lambda + 2 \beta d\lambda dz + e^{2x} dz^2 + e^{2\psi} d\Omega \tag{2.1}
\end{align*}
\]

where \( f, \beta, \chi, \psi \) are functions of \( \lambda, r, z \). The horizon is located at \( r = 0 \) and on the horizon, \( f = 0, \partial_r f = 0, \beta = 0 \) and \( \lambda \) is an affine parameter along the null geodesic generators. Let \( \ell^\mu = (\partial/\partial \lambda)^\mu \) be tangent to these null generators, and let \( h_{\mu\nu} \) denote the metric on a \( r = 0 \), constant \( \lambda \) cross-section of the horizon\[^{[4]}\]. The divergence of the null generators is

\[
    \theta = h^{\mu\nu} \nabla_\mu \ell_\nu = \dot{\chi} + 2 \dot{\psi} \tag{2.2}
\]

where a dot denotes derivative with respect to \( \lambda \). A fundamental property of event horizons is that \( \theta \) cannot become negative. This can be proved under two different assumptions. The first is that the null geodesic generators of the horizon are complete. We do not wish to assume this since, if the horizon pinches off in finite time, the geodesics will not be complete. However, \( \theta \geq 0 \) can also be established if there are no naked singularities outside the horizon \[^{[5]}\]. Since we are working within the future domain of dependence of a spacelike surface \( \Sigma \), the spacetime is globally hyperbolic and this assumption is valid.

The shear of the null generators is defined by

\[
    \sigma_{\mu\nu} = h_{\mu}^\alpha h_{\nu}^\beta \nabla_\alpha \ell_\beta - \frac{\theta}{3} h_{\mu\nu} \tag{2.3}
\]

which reduces to

\[
    \sigma_{\mu\nu} = \frac{1}{2} \dot{h}_{\mu\nu} - \frac{\theta}{3} h_{\mu\nu} \tag{2.4}
\]

For the metric (2.1) one finds

\[
    \sigma_{\mu\nu} \sigma^{\mu\nu} = \frac{2}{3} (\dot{\chi} - \dot{\psi})^2 \tag{2.5}
\]

Since \( \theta \geq 0 \) and \( \dot{\psi} \leq 0 \), we have \( \dot{\chi} \geq -2 \dot{\psi} \) and \( \sigma_{\mu\nu} \sigma^{\mu\nu} \geq 6 \dot{\psi}^2 \). The Raychaudhuri equation in five dimensions is

\[
    \dot{\theta} = -\frac{\theta^2}{3} - \sigma_{\mu\nu} \sigma^{\mu\nu} - R_{\mu\nu} \ell^\mu \ell^{\nu} \tag{2.6}
\]

\[\text{2The indices } \mu, \nu \text{ run over all spacetime coordinates, but } h_{\mu\nu} \text{ is nonzero only when } \mu, \nu \text{ is } z \text{ or one of the angles on the spheres of spherical symmetry.}\]
Since the spacetime is Ricci flat, we have \( \dot{\theta} \leq -6\dot{\psi}^2 \). Thus if \( \theta_0 > 0 \) is the initial value of the divergence,

\[
\theta(\lambda) \leq \theta_0 - 6 \int_0^\lambda \dot{\psi}^2 \tag{2.7}
\]

Using \((\dot{\psi}+1)^2 \geq 0\), this implies \( \theta(\lambda) \leq 12\psi(\lambda) + 6\lambda + \text{constant} \). Since \( \theta \) must stay positive, \( \psi \) cannot go to minus infinity at any finite \( \lambda \). In other words, the sphere cannot shrink to zero size in finite affine parameter.

The above argument can be extended in many directions. One can consider nonspherical perturbations, higher dimensional spacetimes, horizons extended in more than one direction (black branes), and collapsing surfaces of various dimensions. Perhaps the most general result is the following theorem which states roughly that event horizons cannot have any collapsing \( S^1 \)'s.

**Theorem:** Consider the event horizon of a general \( D \) dimensional spacetime satisfying the null energy condition. Choose any \( S^1 \) in the \( \lambda = 0 \) cross-section of the horizon. Following this circle along the null generators yields a family of closed curves \( \Gamma(\lambda) \). The length of these curves cannot go to zero in finite affine parameter.

**Proof:** Choose local coordinates \( y, x^i \) \((i = 1, \cdots, D-3)\) on the horizon in a neighborhood of \( \Gamma(\lambda) \) which are constant along the null generators, such that the \( \Gamma(\lambda) \) is \( x^i = 0 \), and \( y \) is a periodic coordinate along the curves. The metric \( h_{\mu\nu} \) on each constant \( \lambda \) cross-section of the horizon can be written

\[
ds^2 = e^{2\phi}(dy + A_i dx^i)^2 + \gamma_{ij} dx^i dx^j \tag{2.8}
\]

where \( \phi, A_i \) and \( \gamma_{ij} \) depend on \( \lambda \) as well as the other coordinates. Let \( \ell^\mu \) be the null geodesic generators of the event horizon and set \( B_{\mu\nu} = h^{\alpha}_\mu h^{\beta}_\nu \nabla_\alpha \ell_\beta \). Then the Raychaudhuri equation, in any dimension, is simply

\[
\dot{\theta} = -B_{\mu\nu} B^{\mu\nu} - R_{\mu\nu} \ell^\mu \ell^\nu \tag{2.9}
\]

Decomposing \( B_{\mu\nu} \) into its trace \( \theta \) and tracefree \( \sigma_{\mu\nu} \) parts we recover \((2.6)\) when \( D = 5 \).

Using \((2.8)\), a straightforward calculation yields

\[
B_{\mu\nu} B^{\mu\nu} = \dot{\phi}^2 + \frac{1}{2} e^{2\phi} |\dot{A}_i|^2 + \frac{1}{4} |\dot{\gamma}_{ij}|^2 \tag{2.10}
\]

where the double bar means take the norm with the metric \( \gamma_{ij} \). Since the weak energy condition holds, \((2.9)\) implies

\[
\dot{\theta} \leq -\dot{\phi}^2 \tag{2.11}
\]
The previous argument now shows that $\phi$ cannot go to minus infinity in finite $\lambda$. This completes the proof.

An immediate consequence of this theorem is that cosmic censorship is not violated by the Gregory-Laflamme instability. The null geodesic generators of the horizon remain complete.

3 Discussion

Since the horizon cannot pinch off in finite affine parameter the spacetime must settle down to something at late time. The most likely possibility is that it settles down to a new static black string (or $p$-brane) solution which is not translationally invariant along the horizon. Before discussing this in more detail, we comment on the remote possibility that the horizon pinches off in infinite affine parameter. Using just the Raychaudhuri equation, it is possible for the length $L(\lambda)$ of the closed curves $\Gamma(\lambda)$ to go to zero keeping $\theta$ positive. However, since the horizon area must remain finite, $\int_{\lambda}^{\infty} \theta(\lambda) d\lambda < \infty$, and the decay rate is quite restricted. For example, it is easy to see that $L(\lambda)$ cannot decay like a simple power law or exponential. The type of decay that is not obviously forbidden is

$$L(\lambda) = e^{-(\ln \lambda)^\alpha} \quad 0 < \alpha < 1/2 \quad (3.1)$$

This seems rather unnatural. More importantly, it is a very slow decay. Since the decay is so slow, there must exist a family of new essentially static solutions. One can view the late time evolution as slow motion through this space of static solutions. But given the existence of new static solutions, there is no physical reason for the horizon to pinch off. It is much more likely that the evolution will stop at one of the static configurations.

In the case of Schwarzschild cross a circle, one can say more. Consider the metric (2.1) and assume reflection symmetry about $z = 0$. If the horizon pinches off in infinite time then $\psi(z = 0, \lambda) \to -\infty$ as $\lambda \to \infty$. Since $\theta \geq 0$ everywhere, (2.2) implies that $\chi$ must go to plus infinity. Thus it appears that the solution near $z = 0$ is evolving toward a very long thin black string, which is unstable. This not only sounds unphysical, it leads to a contradiction by looking at the effective four dimensional Einstein’s equation on the $z = 0$ surface. By using a Kaluza-Klein type reduction in the $z$ direction (which involves rescaling the effective 4D metric), the equations take the form of Einstein gravity minimally coupled to a massless scalar $\chi$. Spherically symmetric black hole solutions always have $\chi$ remain bounded on the horizon. This contradicts the fact that $\chi$ must diverge. Unfortunately
this is not sufficient to rule out the possibility that the horizon pinches off in infinite time. If $\psi, z z|_{z=0}$ grows sufficiently rapidly with $\lambda$, then the solution never resembles a long thin string. Instead it looks like a chain of spherical black holes connected by small necks. But the spacetime near one neck would be analogous to that obtained by bringing two black holes close together, and in that case it is well known that a new trapped surface forms which surrounds both black holes. This is simply because there is now double the mass within a sphere containing both black holes so the effective Schwarzschild radius moves out. Similarly, we would expect that if the apparent horizon tried to pinch off in infinite affine parameter, another apparent horizon would form outside, and the true event horizon would not pinch off.

It should be noted that the argument about Kaluza-Klein reduction in the $z$ direction also rules out the possibility that the final static solution is $z$ independent, but has $\chi(r)$ increasing with $r$ such that the length of the circle at the horizon is less than $r_0$. Since the 5D Einstein’s equation reduces to 4D gravity coupled to a scalar $\chi$, the usual no hair theorem shows that a static black hole must have constant $\chi$.

Let us now turn to the most likely alternative that the solution settles down to a new static black string solution of Einstein’s equation which is not translationally invariant along the horizon. Since we do not have the new solution explicitly, we cannot say for sure what the horizon geometry looks like. In particular, we cannot determine the minimum size of the sphere. However, since the physics is mostly determined by one scale – the initial Schwarzschild radius $r_0$ – the maximum and minimum radii are likely to be within factors of two of this scale. There is no reason for a large dimensionless number (corresponding to the ratio of these radii) to arise. Note that one cannot determine the geometry of the horizon just by examining Einstein’s equation near the horizon. Even for ordinary four dimensional black holes, if there is a static nonspherical distribution of matter far outside the black hole the horizon will be distorted. One needs to examine the field equations everywhere.

Even without the exact solutions, one can deduce certain properties of the new solutions. For example, the solution must approach the translationally invariant black string exponentially fast at infinity. This is because the asymptotic solution can be modeled by a perturbation of the translationally invariant one. Since $z$ is periodic, any $z$-dependent

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3We thank V. Hubeny for suggesting this analogy.

4The length of the circle $L$ is not likely to be important when $L \gg r_0$ since the transition between stable and unstable modes is set by $r_0$. 
perturbation satisfies a massive spin two equation and must fall off exponentially.

The translationally invariant black string is unstable for $L > L_0$ where $L_0$ is a critical length of order $r_0$. For every initial Schwarzschild radius $r_0$ and $L > L_0$, there must be at least one new static black string solution. There may even be more than one. Since the black strings with $L < L_0$ are stable, the new black string solutions probably meet the branch of solutions corresponding to the original translationally invariant solutions at $L = L_0$. This is supported by the existence of a nontrivial static perturbation to the translationally invariant black string when $L = L_0$.

If $\xi$ denotes the static Killing field, then $*d\xi$ is closed by the vacuum Einstein equation. Consider the integral of the curl of this $(D - 2)$-form over a static slice from the horizon to infinity. The surface term at infinity yields the total mass $M$, and the surface term at the horizon yields $\kappa A/4\pi$ where $\kappa$ denotes the surface gravity. Since these must be equal we obtain

$$M = \frac{\kappa A}{4\pi} \quad (3.2)$$

This is true for any static vacuum solution with a horizon. If one starts with a slightly perturbed translationally invariant black string, under evolution the mass decreases (since energy can be radiated to infinity) and the area increases, so the final surface gravity cannot be greater than the initial surface gravity. For nonvacuum spacetimes corresponding to charged branes, there is an extra term on the right hand side of (3.2) involving a volume integral of the stress energy tensor.

The final black string solution will have greater entropy than the original one but considerably less than a single higher dimensional black hole (when $L \gg r_0$). This can be seen by the following rough argument. Setting $G_5 = 1$, the original black string has mass of order $Lr_0$ and entropy of order $Lr_0^2$. Since the horizon cannot pinch off, the final black string must have an entropy less than that of a chain of five dimensional black holes of radius approximately $r_0$. Since each one has mass of order $r_0^2$ (and the total mass is $Lr_0$), there are $L/r_0$ black holes in the chain with total entropy of order $(L/r_0)r_0^2 = Lr_0^2$. So the entropy can only increase by a numerical factor. This is much less than the entropy of a single five dimensional black hole with the same mass which would be $(Lr_0)^{3/2}$. However, this does not contradict the statement that the new solutions are stable. There are many examples of stable solutions which do not maximize their entropy for given mass, e.g. static stars, tables, chairs, etc. The new black string solutions can be viewed as a local entropy maximum but not a global one. It has recently been suggested that a Gregory-Laflamme type instability should exist precisely when the entropy is not a local maximum [16, 17].
Clearly it would be very interesting to find the new black string solutions explicitly. Although an analytic solution would be preferable, numerical solutions may be needed. Explicit numerical evolution of the Gregory-Laflamme instability is now underway \cite{18}.

The implications of these new solutions for string theory remain to be investigated. Previous results which assumed black strings will break up into black holes should be reexamined. Some of these results will be unchanged, e.g., there is independent evidence for localized black holes bound to branes \cite{19, 20} so the assumptions of \cite{11} are not needed\footnote{There are also stable black strings with asymptotically AdS boundary conditions \cite{21}}. Other results may need to be modified.

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