Spiral chains in wavenumber space of two dimensional turbulence

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Self-similar, fractal nature of turbulence is discussed in the context of two dimensional turbulence, by considering the fractal structure of the wave-number domain using spirals. In loose analogy with phyllotaxis in plants, each step of the cascade can be represented by a rotation and a scaling of the interacting triad. Using a constant divergence angle and a constant scaling factor, one obtains a family of such fractals depending on the distance of interactions. Scaling factors in such sequences are given by the square roots of known ratios such as the plastic ratio, the supergolden ratio or some small Pisot numbers. While spiral chains can represent mono-fractal models of self-similar cascade, which can span a very large range in wave-number domain with good angular coverage, it is also possible that spiral chains or chains of consecutive triads play an important role in the cascade. As numerical models, the spiral chain models based on decimated Fourier coefficients have problems such as the dual cascade being overwhelmed by statistical chain equipartition due to almost stochastic evolution of the complex phases. A generic spiral chain model based on evolution of energy is proposed, which is shown to recover the dual cascade behavior in two-dimensional turbulence.

I. INTRODUCTION

Spiral patterns emerge in many nonlinear problems in nature, from galaxy formation to crystal growth, from plants to animals and from atmospheric cyclones to small scale turbulence, they appear at very different scales and in very different problems. They are a fundamental element of phyllotaxis -the dynamical phenomenon of arrangement of seeds or petals of a plant (sometimes in the form of flowers) as it grows[1]. Mathematically, the particularity of the spiral form is that it keeps certain quantities (such as the angle between two consecutive elements) invariant as the structure is scaled and rotated. This provides a natural self-similar framework with which the some physical systems operate. One of the key aspect of phyllotaxis is how a discrete structure that grows through iteration manages optimal packing, leading to the observed fractal pattern[2, 3]. Similar concepts apply to reaction-diffusion systems where spiral patterns arise in a continuum of deformations[4]. Incidentally, spiral patterns also occur in turbulence[5], especially in two dimensions[6, 7], mainly as a result of self shearing of smaller scale structures by large scale flows, and the resulting self-similarity of the turbulent flow, where the structure remains the same as it scales and turns. In fact the basic motion of scale and rotate (i.e. “swirl”), associated with a turbulent flow naturally implies a spiral-like pattern.

Spirals in wave-vector space are also potentially interesting for the study of turbulent dynamics. Common sense suggests that nonlinear interactions tend to scale and rotate real space structures, and hence they would do the same to the wave-vectors as well. For instance, if we have a particular direction of anisotropy, at a given scale, nonlinearity tends to generate a “next” scale in the hierarchchy, which is anisotropic in a direction that is “at a certain angle” (maybe perpendicular) to the original direction of anisotropy. Thus, when there is a large scale source of anisotropy, going towards smaller scales the direction of anisotropy at each scale keeps changing, which results in a virtually isotropic spectrum in statistical sense.

Energy (and enstrophy for two dimensions), gets transferred via triadic interactions in turbulent flows[8]. In general for a given scale, there are many such triads that can transfer energy or enstrophy in either directions to other scales. If, for some reason, one of these triads is “dominant” -for example due to the fact that it maximizes the interaction coefficient-, it is natural that this triad will take more of the energy or enstrophy along. Then, at the next scale the energy goes, the “same triad” (now rotated and scaled), will likely win again for the same reason that it won at the first scale, transferring the energy to the next one along a chain of such dominant triads. It is unclear if the small differences among nearby triads in terms of their capacity to transfer energy and enstrophy justifies a reduction of the turbulent transfer to picture of transfer along a single chain of scaled and rotated triads that arrange naturally into a spiral. Nonetheless the picture of turbulent energy transfer as taking place along chains of spirals (instead of the naive and incorrect picture of a “radial” flux in k-space) that compete with and couple to one another is instructive.

Various kinds of reduced models have been proposed in the past, in order to study both the nonlinear cascade and the direction of anisotropy in turbulent flows from shell models[9, 10], to differential approximation models[11, 12] to closure based models[13, 14] to tree models[15, 16] to reduced wave-number representations[17]. Here we propose a reduction of two dimensional turbulence based on spiral chains, which are chains of wave-numbers that

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are obtained by scaling and rotating a single triad such that the smaller wave-number of the triad, after scaling and rotation (or after a few scalings and rotations), becomes, first the middle wave-number and then the larger wave-number. In principle a number of such spiral chains can be used, instead of a single one, in order to span the \( k \)-space more completely.

The rest of the paper is organized as follows. In section II, the problem of a single triad is revisited and the concept of triad chains or consecutive triads by which the energy is transferred is discussed. In section III regular spiral chain models for certain chains with relatively local interactions are introduced. The general case of arbitrarily distant interactions is also covered in this section where a list of possible solutions are discussed in Section IIIA conservation of energy and enstrophy for spiral chains is formulated in Section IIIB and zero flux solutions are investigated in Section IIIC. In Section IV a spiral chain model formulated for chain energy \( E_n \) is introduced. Re-interpreting this model as a model for shell energy, with the assumption of isotropy, which allows the interactions to be infinitesimally local, the continuum limit is computed and found to be the usual differential approximation model form for the two dimensional Euler turbulence in section IV A. A four spiral chain model with good angular coverage is introduced in IVB. Numerical results for a subset of these spiral chain models are given in V. Section VI is conclusion.

II. DYNAMICS OF A SINGLE TRIAD

Two dimensional turbulence, as represented by an equation of advection of vorticity\cite{nawa1}, or more generally, of potential vorticity\cite{nawa2}, can be relevant as a simplified limiting case of many physical problems from rotating turbulence in laboratory experiments\cite{lumley}, to geostrophic turbulence in planetary atmospheres\cite{gold}, to drift wave turbulence in tokamak plasmas\cite{torok}. Consider the two dimensional Euler equation

\[
\partial_t \nabla^2 \Phi + \mathbf{z} \times \nabla \Phi \cdot \nabla \nabla^2 \Phi = 0 ,
\]

(1)

to which viscosity or hyper-viscosity can be added for dissipation of energy and enstrophy. Its Fourier transform can be written in general as

\[
\partial_t \Phi_k = \sum_{p+q=-k} \frac{\mathbf{z} \times \mathbf{p} \cdot \mathbf{q}}{k^2} \frac{(q^2 - p^2)}{k^2} \Phi^*_p \Phi^*_q
\]

with the convention that \( \sum_{p+q=-k} \) represents a sum over \( p \) and \( q \) such that \( k + p + q = 0 \). Now consider a single triad consisting of \( k, p, q \) such that \( k < p < q \). If \( \eta \equiv \frac{\ln(q/k)}{\ln(p/k)} \in \mathbb{Q} \) (i.e. is rational) we can write \( p = kg^\ell \) and \( q = kg^m \) (i.e. \( \eta = m/\ell \)). Obviously not all triangles satisfy the condition \( \eta \in \mathbb{Q} \). However there is usually an approximately equivalent triangle from a physics or numerics perspective, which does. If one is restricted to low order rationals for \( \eta \), it is only a particular class of triangles, which can be represented as \( p = kg^\ell \) and \( q = kg^m \) with \( \ell \) and \( m \) integers and \( g > 1 \) (i.e. \( g \in \mathbb{R} \)).

For those triangles, we can write the interaction as:

\[
\partial_t \Phi_k = k^2 \sin \alpha_{qp} g^{m+\ell} (g^{2m} - g^{2\ell}) \Phi^*_p \Phi^*_q
\]

\[
\partial_t \Phi_p = k^2 \sin \alpha_{qp} g^{\ell-m} (g^{2\ell} - 1) \Phi^*_q \Phi^*_k
\]

\[
\partial_t \Phi_q = k^2 \sin \alpha_{qp} g^{\ell-m} (g^{2\ell} - 1) \Phi^*_p \Phi^*_k
\]

where we have used \( (\mathbf{z} \times \mathbf{p} \cdot \mathbf{q}) = \sin \alpha_{qp} = \sin(\theta_q - \theta_p) \). Since \( g > 1 \) and the middle leg of the triad (i.e. \( p \)) is unstable as long as \( m > \ell \) (which we have assumed by assuming \( q > p \)) and gives its energy to the other two wave-numbers.

The energy evolves according to

\[
\partial_t E_k = (g^{2m} - g^{2\ell}) t_{kpq}
\]

\[
\partial_t E_p = (1 - g^{2m}) t_{kpq}
\]

\[
\partial_t E_q = (g^{2\ell} - 1) t_{kpq}
\]

where

\[
t_{kpq} = g^{m+\ell} k^4 \sin \alpha_{qp} \Phi^*_p \Phi^*_q \Phi^*_k
\]

It is easy to see that the total energy of the triad is conserved. Following the reasoning discussed in Ref.\cite{lavender}, the instability assumption implies \( t_{kpq} > 0 \) since \( E_p \) should decrease in time, where the overbar implies statistical ensemble average, which can be replaced by time average in most cases.

The energy that is transferred from \( p \) to \( k \) is \( g^{2m} t_{kpq} \), while the energy that is transferred from \( q \) to \( p \) is simply \( t_{kpq} \). On the other hand there is energy that is transferred from \( k \) to \( q \) (from the smallest to the largest wave-number), which is \( g^{2\ell} t_{kpq} \). Since \( g^{2m} > g^{2\ell} \), \( E_k \) gets more energy than it looses. However since \( g^{2\ell} > 1 \), \( E_q \) also gets more energy than it looses. This means the energy is transferred from the middle wave-number to the larger and smaller wave-numbers. If the sign of \( t_{kpq} \) changes, then the flow will be towards the middle wave-number, in fact the system will naturally undergo such oscillations as the energy of the middle wave-number gets depleted.

A. Consecutive triads:

Imagine the triad \( k, p, q \) discussed above. If we scale it by \( g^{-\ell} \) and rotate by \( -\theta_p \), we obtain a second triad
where \( \mathbf{k} \) becomes the middle wave-number instead of the smallest one (we call the other two wave-numbers as \( p' \) and \( q' \) with \( p' < k < q' \)) and if we scale it by \( g^{-m} \), and rotate by \( -\theta_q \), \( \mathbf{k} \) becomes the largest wave-number (with \( p'' \) and \( q'' \) such that \( p'' < q'' < k \)). Note that \( p' = kg^{-t}, \ q' = kg^{-m-t}, \ p'' = kg^{-m}, \ q'' = kg^{-m-t} \). By defining \( k \rightarrow k_n \), and assuming that those three triads exist, we can write the evolution equation for \( \Phi_{k_n} \to \Phi_n \) as

\[
\partial_t \Phi_n = k_n^2 \sin \alpha_{qp} \left[ g^{m+\ell} \left(g^{2m} - g^{2\ell}\right) \Phi_{n+\ell} \Phi_{n+m}^* \right.
+ g^{m-3\ell} \left(1 - g^{2m}\right) \Phi_{n-\ell}^* \Phi_{n-m+\ell}^* 
+ g^{\ell-3m} \left(g^{2\ell} - 1\right) \Phi_{n-m}^* \Phi_{n-m+\ell}^* \left. \right].
\]

The three terms on the right hand side of (2) are the contributions from \( (p, q), (p', q') \) and \( (p'', q'') \) respectively or to the three triangles from the largest to the smallest. Note that for a given triangle shape, the three terms in (2) appear naturally representing the three different size triangles (but of the same shape), where \( \mathbf{k} \) play the role of the smallest, the middle and the largest wave-numbers consecutively. In fact one can also imagine adding a sum over different shapes of triangles in order to provide a complete description.

If we call the triangles from the smallest to the largest as \( \triangle_1, \triangle_2 \) and \( \triangle_3 \) respectively, we obtain \( \triangle_2 \) by scaling \( \triangle_1 \) by \( g^{m-\ell} \) and rotating it by \( \alpha_{qp} = \theta_q - \theta_p \), and \( \triangle_3 \), by scaling \( \triangle_2 \) by \( g^\ell \) and rotating it by \( \theta_p \). Obviously we can repeat the procedure of rotating and scaling in order to cover a whole range of \( k \) vectors in the wave-number domain. However while the scaling is regular (i.e. we can define a \( k_n = k_0g^n \) such that scaled wave-numbers always have the form \( k_n \) with \( n \in \mathbb{Z} \), in general the angles are not perfectly regular.

Consider for example the triangle with \( g = \sqrt{\varphi} \) where \( \varphi = (1 + \sqrt{5})/2 \) is the golden ratio so that \( k = 1, \ p = q \) and \( g = g^2 \). The angle between \( k \) and \( p \) is a right angle (since \( \sqrt{1+g^2} = g^2 \) with \( g = \sqrt{\varphi} \)), while the one between \( p \) and \( q \) can be computed from the law of cosines as \( \cos \alpha_{qp} = \frac{1-g^2-q^2}{2qp} = \frac{1-g^2-g^4}{2g}, \) which gives an angle about \( \alpha_{pq} = 141.83^\circ \) (note that \( \alpha_{pq} \) is the angle between the two vectors, which is \( \pi/2 \) minus the angle between the two edges of the triangle). This corresponds to the triangle defined by \( \ell = 1, \ m = 2 \) and \( g = \sqrt{\varphi} \). Scaling this triangle \( \triangle_1 \) by \( g \) and rotating by \( \pi/2 \), we obtain triangle \( \triangle_2 \), obtained \( \triangle_3 \) by \( g \) and rotating by \( 141.83^\circ \) we obtain \( \triangle_3 \). We can construct a chain of such triads that are connected to one another by the common wave-number as shown in figure 1 for which the equation of motion will still be (2). However the grid that is generated by the triad chain is, in general, irregular.

However, it is obvious from this emerging picture that if we had \( \alpha_{qp} = ma_{pk} \) where \( m \) is some integer, we could write the whole thing as a regular spiral, with \( k_n = k_0g^n \) and \( \theta_n = na \). It is also obvious that the class of triangles that would result in such a regular spiral, are a very special class: Each wave-number involved in such a system is a rotated and scaled version of the wave-number before it in a regular fashion.

### III. Spiral Chain Models

Let us introduce the symbol \( \gamma_{l,m}^{s_l,s_m} \) to refer to a basic spiral chain consisting of the triad \( k_n + s_l k_{n+l} + s_m k_{n+m} = 0 \), where \( k_n = k_0 g^n \) and \( \theta_n = an \) (or using the equivalence between two dimensional vectors and complex numbers, \( k_n^* = k_0 \left( ge^{i\alpha} \right)^n \) with \( k_n = \text{Re} \left( k_n^* \right) x + \text{Im} \left( k_n^* \right) y \)). Note that \( g \) and \( \alpha \) follows from \( \ell, m, s_l \) and \( s_m \), and therefore need not be stated explicitly. Here \( s_l \) and \( s_m \) are the signs in front of the wave-numbers in order to satisfy the triad condition.

Considering \( \ell = 2, m = 3 \) in (2), with \( \theta_n = na \), so that \( a_{pk} = 2a \), \( \alpha_{qp} = \alpha \) and \( \alpha_{qk} = 3a \), and all possible interaction forms (i.e. \( k \pm p \pm q = 0 \)), we find that the law of cosines for the different cases give

\[
\cos \alpha_{pk} = \pm \left(\frac{g^2-k^2-p^2}{2kp}\right) = \pm \left(\frac{g^6-g^4-1}{2g^2}\right) = \cos 2a
\]
\[
\cos \alpha = \frac{1}{\sqrt{2}} \left( \sqrt{2} \left[ \left( 1 - \sqrt{23} \right)^{1/3} + \left( 1 + \sqrt{23} \right)^{1/3} \right] \right)^{1/3}
\]

1. Chain \( C_{2,3}^- \)

For this basic chain, which can be denoted by \( C_{2,3}^- \) a basic evolution equation can be written as follows:

\[
\partial_t \Phi_n = k_n^2 \sin \alpha \left[ g^{-7} \left( g^4 - 1 \right) \Phi_{n-3} - g^{-3} \left( g^6 - 1 \right) \Phi_{n-2} + g^3 \left( g^2 - 1 \right) \Phi_{n+3} \right] + P_n - D_n \tag{3}
\]

with \( \Phi_n = \hat{\Phi}(k_n) \) as the Fourier coefficient of \( \Phi \), with the wavevector \( k_n = k_n \left( \cos \alpha_n, \sin \alpha_n \right) \), where \( k_n = k_0 g^n \) and \( \alpha_n = \alpha n, g = \sqrt{\rho} \) being the logarithmic scaling factor and \( \alpha = \arccos \left( -g^3 / 2 \right) \), being the divergence angle. \( P_n \) and \( D_n \) are energy injection and dissipation respectively (i.e. \( D_n = \nu k_n^2 \Phi_n \) for a usual kinematic viscosity and \( P_n = \gamma_n \Phi_n \) for an internal instability drive).

Note that using the relations \( g^6 - 1 = g^2, g^4 - 1 = g^2 \) and \( g^2 - 1 = g^{-8} \), possible due to the choice \( g = \sqrt{\rho} \), we can write \( \Phi \) also as:

\[
\partial_t \Phi_n = k_n^2 \sin \alpha \left[ g^{-9} \Phi_{n-3} - g^{-1} \Phi_{n-2} + g^3 \Phi_{n+3} \right] + P_n - D_n \tag{4}
\]

While \( \Phi \) conserves energy and enstrophy for \( g = \sqrt{\rho} \), \( \Phi \) does so for arbitrary \( g \), which makes it somewhat more useful even though the two equations are identical for the given value of \( g \).

2. Chain \( C_{2,3}^{+,+} \)

It is clear that there are many similar chains, such as the one with \( \alpha = \pi - \arccos \left( -g^3 / 2 \right) = \arccos \left( g^3 / 2 \right) \), which gives a similar model, but with a different conjugation structure:

\[
\partial_t \Phi_n = k_n^2 \sin \alpha \left[ g^{-7} \left( g^4 - 1 \right) \Phi_{n-3} + g^{-3} \left( g^6 - 1 \right) \Phi_{n+1} + g^3 \left( g^2 - 1 \right) \Phi_{n+3} \right] + P_n - D_n \tag{5}
\]

and a different sampling of wave-vector directions.

3. Chain \( C_{1,3}^{+,+} \) (or \( C_{1,3}^{++,+} \))

We can obtain another chain by choosing \( \ell = -1, m = 2 \), which gives \( \alpha_{pk} = -\alpha, \alpha_{ap} = 3\alpha \) and \( \alpha_{aq} = 2\alpha \). Using the law of cosines and the relations between \( \cos \alpha \), \( \cos 3\alpha \) and \( \cos \alpha \), we obtain \( g \approx 1.21061 \), or \( g = \sqrt{\bar{\rho}} \) where:

\[
\psi = \frac{1}{3} \left[ 1 + \frac{1}{2^{1/3}} \left( \left( 29 + 3\sqrt{93} \right)^{1/3} + \left( 29 - 3\sqrt{93} \right)^{1/3} \right) \right]
\]

Figure 2. The spiral chain \( \ell = 2, m = 3 \) with \( q = \sqrt{\rho} \). The counter clockwise primary spiral chain is shown in black dashed lines while the clockwise secondary spirals are shown in blue dashed lines. Note that as the energy travels along the primary chain, it gets exchanged between the 5 secondary chains. Finally an interacting triad with \( k = k_n \) (black arrow), \( p = k_{n-2} \) (red arrow) and \( q = k_{n+1} \) (blue arrow) is shown (i.e. \( k + q - p = 0 \)).
is the so-called super golden ratio, and $\alpha = \arccos \left( \frac{g^3}{2} \right)$ for the form $k + p + q = 0$, and thus an evolution equation of the form:

$$
\partial_t \Phi_n = k^2_n \sin \alpha \left[ g^{-11} (g^2 - 1) \Phi^*_{n-2} \Phi^*_{n-3} 
- g^{-3} (g^6 - 1) \Phi^*_{n-1} \Phi^*_{n+2} 
+ g^3 (g^4 - 1) \Phi^*_{n+3} \Phi_{n+1} \right] + P_n - D_n \tag{6}
$$

where we have used the fact that for this particular value of $\alpha$, we have $\sin 3\alpha = -g^{-27}\sin \alpha$.

4. Chain $C^{-1,+}_{-1,2}$ (or $C^{-1,-}_{1,3}$)

A similar case to $C^{+,+}_{-1,2}$ exists with $g = \sqrt[3]{\psi}$ and $\alpha = \arccos \left( -\frac{g^3}{2} \right) = \pi - \arccos \left( \frac{g^3}{2} \right)$, which corresponds to $k + q - p = 0$ and the evolution equation of the form:

$$
\partial_t \Phi_n = k^2_n \sin \alpha \left[ g^{-11} (g^2 - 1) \Phi^*_{n-2} \Phi^*_{n-3} 
- g^{-3} (g^6 - 1) \Phi^*_{n-1} \Phi^*_{n+2} 
+ g^3 (g^4 - 1) \Phi^*_{n+3} \Phi_{n+1} \right] + P_n - D_n
$$

The chain denoted by $\ell = 1$, $m = 3$ corresponds to the same chain as the one denoted by $\ell = -1$, $m = 2$. (since we can obtain one from the other by exchanging $k$ and $p$). This means we can write $C^{+,+}_{1,2} = C^{+,+}_{-1,3}$ and $C^{-,-}_{1,2} = C^{-,-}_{-1,3}$ or in general $C^{s_l,s_m}_{\ell,m} = C^{s_l,s_m}_{-\ell,-m-\ell}$. This means that it is sufficient to consider the case $m > \ell > 0$.

5. Chains $C^{+,-}_{2,3} + C^{+,-}_{1,5}$

Remarkably, the case $\ell = 1$ and $m = 5$ gives $g = \sqrt[3]{\psi}$ and $\alpha = \arccos \left( -\frac{g^3}{2} \right)$ exactly as in the case $\ell = 2$ and $m = 3$. This means that in fact these two spiral chains are inseparable since a choice of $g$ and $\alpha$, will lead to an evolution equation of the form:

$$
\partial_t \Phi_n = k^2_n \sin \alpha \left[ -g^{-19} (g^2 - 1) \Phi^*_{n-5} \Phi^*_{n-4} 
+ g^{-7} (g^4 - 1) \Phi^*_{n-3} \Phi^*_{n+1} 
- g^{-3} (g^6 - 1) \Phi^*_{n-1} \Phi^*_{n+2} 
+ g^3 (g^{10} - 1) \Phi^*_{n-1} \Phi^*_{n+4} 
- g^9 (g^2 - 1) \Phi^*_{n+2} \Phi_{n+3} \right] + P_n - D_n
$$

It is easy to show that these are in fact all the interactions that take place among the points of this particular spiral (i.e. defined by $g$ and $\alpha$). Similarly there is another double chain of the form $C^{+,+}_{2,3} + C^{+,+}_{1,5}$ as well.

6. Supplementary chains

Consider the two chains represented by $C^{+,+}_{1,3}$ and $C^{-,-}_{1,3}$ discussed above. The two chains have the same $g$'s but supplementary angles. This means that while the $++$ chain has the angles $\theta_n = n\alpha$, the supplementary chain has the angles $\theta_n = n(\pi - \alpha)$. However since both $\Phi_n$ and $\Phi^*_n$ are considered for a given $k_n$, adding or subtracting $\pi$ to an angle is equivalent to taking the complex conjugate or replacing $k_n \rightarrow -k_n$. Therefore we can instead use $\theta_n = -n\alpha$, and note that it corresponds to the spiral that rotates in the opposite direction to the original spiral. But with $k_n + k_{n+1} + k_{n+3} = 0$, since the signs of $k_{n+\ell}$ for odd $\ell$ change direction.

7. Other Chains:

If we consider other $\ell$ and $m$ values, it is clear that $\ell = 4$, $m = 6$ gives $g_{4,6} = (g_{2,3})^{1/2}$ and $\alpha_{4,6} = \alpha_{2,3}/2$ etc. These are not unique chains but simply the same chains that are repeated twice [or $n$ times to get $g_{2n,3n} = (g_{2,3})^{n}$, and $\alpha_{2n,3n} = \alpha_{2,3}/n$]. In contrast, for a unique chain, we have to compute $g$ and $\alpha$. In general, for any $\ell$ and $m$ such that $k_n + s\ell k_{n+\ell} + s_m k_{n+m} = 0$, we can write

$$
\cos \ell \alpha = s\ell \frac{(g^{2\ell} - g^{2\ell - 1})}{2g^{2\ell}}
$$

$$
\cos m\alpha = s_m \frac{(g^{2m} - g^{2m - 1})}{2g^{2m}}
$$

$$
\cos (m - \ell) \alpha = s_m s_{\ell} \frac{(1 - g^{2m} - g^{2\ell})}{2g^{2(m+\ell)}}
$$

Consistency requires that:

$$
\frac{1}{\ell} \arccos \left[ s\ell \frac{(g^{2m} - g^{2\ell - 1})}{2g^{2\ell}} \right] = \frac{1}{m} \arccos \left[ s_m \frac{(g^{2\ell} - g^{2m - 1})}{2g^{2m}} \right] = \frac{1}{m - \ell} \arccos \left[ s\ell s_m \frac{(1 - g^{2m} - g^{2\ell})}{2g^{2(m+\ell)}} \right]
$$

where the arccos function is considered as multi-valued. These equations can be solved numerically in order to obtain spiral chains for any $\ell$ and $m$ values. In general for a given $\ell$ and $m$, one may have multiple solutions of $g$ because of the multivaluedness of the arccosine functions. Note that the combination of $s\ell$ and $s_m$ and $g$ define a unique angle $\alpha$. See table [1] for the list of all possible chains up to $m = 9$. Note that for each chain that is
represented in table \[4\] there is also the supplementary chain with \( \alpha' = \pi - \alpha \) and \( s'_\ell = \begin{cases} s_\ell & \ell : \text{even} \\ -s_\ell & \ell : \text{odd} \end{cases} \) and \( s'_m = \begin{cases} s_m & m : \text{even} \\ -s_m & m : \text{odd} \end{cases} \).

A. Power law steady state solutions

Substituting \( \Phi_n \rightarrow Ak_n^\alpha \) in (2), the nonlinear term vanishes when:

\[
g^{(\alpha+3)m+(\alpha+1)\ell} - g^{m(\alpha+1)+(\alpha+3)\ell} + g^{(\alpha+1)m-(2\alpha+3)m} - g^{(\alpha+3)m-(2\alpha+3)\ell} + g^{(\alpha+3)\ell-(2\alpha+3)m} - g^{(\alpha+1)\ell-(2\alpha+3)m} = 0
\]

which can be satisfied if a) \( (\alpha + 1) = -(2\alpha + 3) \) (i.e. \( \alpha = -4/3 \)) independent of the value of \( \ell \) and \( m \), in which case the first term cancels the fourth one, the second term cancels the fifth and the third term cancels last one, or b) \( (\alpha + 3) = -(2\alpha + 3) \) (i.e. \( \alpha = -2 \)), where the first term cancels the last one, second term cancels the third one and the fourth term cancels the fifth one. These correspond to the usual Kraichnan-Kolmogorov spectra \( E(k) \propto \{k^{-3}, k^{-5/3}\} \) since \( E(k_n) \equiv \Phi_n^2k_n^{2.4} \). Note that these self-similar power law solutions on any spiral chain \( C_{k,m}^{\ell,m} \) may be anisotropic in the sense that \( \Phi_{k_0,m} \neq \Phi_{0,k_0} \) for a given scale, are isotropic in the sense that if we average over a few consecutive scales we get a solution that is independent of the direction of \( k \).

However, numerical integration of the model with energy injected roughly in the middle of the spiral does not seem to converge to these solutions (see sections [13] and [14]). Instead it seems that the \( \Phi_n \) act as “random” variables and the system goes to a chain equipartition solution expected from statistical equilibrium such that

\[
P(\Phi_n) = e^{-\frac{\beta_1k_n^4|\Phi_n|^2 + \beta_2k_n^2|\Phi_n|^4}{2}},
\]

which gives (i.e. \( T_1 = \beta_1^{-1} \) and \( T_2 = \beta_2^{-1} \)):

\[
\langle |\Phi_n|^2 \rangle = \frac{T_1}{k_n^4 + \frac{T_1}{T_2}k_n^2}.
\]

and thus a spectral energy density scaling of the form \( E(k) \propto \{k^{-3}, k^{-1}\} \). In general which of these solutions will be observed depends on various factors from numerical details to the way the system is driven.

B. Energy and Enstrophy

Multiplying (2) by \( \Phi_n^*k_n^2 \) and taking the real part, we can write the evolution of energy:

\[
\partial_t E_n = \left[ (g^{2m} - g^{2\ell}) t_{n+\ell} + (1 - g^{2m}) t_{n\ell} + (g^{2\ell} - 1) t_{n-m+\ell} \right] + P_n^E - D_n^E
\]

where \( E_n = k_n^2 |\Phi_n|^2 \)

\[
t_{n\ell} \equiv \Re \left[ g^{m-3\ell}k_n^4 \sin \alpha q \Phi_{n-\ell+m}^* \Phi_{\ell}^* \right]
\]

or multiplying (2) by \( \Phi_n^*k_n^4 \),

\[
\partial_t W_n = \left[ (g^{2(m-\ell)} - 1) t_{n\ell}^W + (1 - g^{2m}) t_{n\ell}^W + (g^{2m} - g^{2(\ell-\ell)}) t_{n-m+\ell}^W \right] + P_n^W - D_n^W
\]

where \( W_n = k_n^4 |\Phi_n|^2 \), and

\[
t_{n\ell}^W \equiv \Re \left[ g^{m-3\ell}k_n^6 \sin \alpha q \Phi_{n-\ell+m}^* \Phi_{\ell}^* \Phi_{\ell}^* \right].
\]

It is easy to see that total energy \( E = \sum_n E_n \) and total enstrophy \( W = \sum_n W_n \) are conserved since \( t_{n\ell}^E \) cancel each other at different orders. This is basically due to the fact that each triad conserves energy and enstrophy, and thus each chain of triads represented by the spiral chain conserves energy and enstrophy independently. Considering mid scale, well localized drive (say around the wave-number \( k_f \)), with both large scale and small scale dissipations. If we sum over (9) from \( n = 0 \) up to an \( n \) such that \( k_n < k_f \), in the inertial range for energy, we get:

\[
\partial_t \sum_{n'=0}^n E_{n'} + \Pi_n^E = -\varepsilon \ell
\]

where \( \varepsilon \ell \) is the total large scale energy dissipation and

\[
\Pi_n^E = -\frac{\left( g^{2m} - g^{2\ell} \right) \sum_{j=1}^m t_{n-m+\ell+j}^E + (1 - g^{2m}) \sum_{j=1}^{m-\ell} t_{n-m+\ell+j}^E}{\sum_{j=1}^m t_{n-m+\ell+j}^E}
\]

A statistical steady state may imply:

\[
\Pi_n^E = -\varepsilon \ell
\]

and if \( t_{n\ell}^E \) is independent of \( n \) for an inertial range, we can write

\[
\Pi_n^E = -\lambda E t_{n\ell}^E
\]

where \( \lambda = \left[ (1 - g^{2\ell}) m - (1 - g^{2m}) \ell \right] \). Note that for \( g = 1 + \varepsilon \), so that \( g^{2\ell} = 1 + 2\varepsilon + (2\ell^2 - \ell) \), finally
Similarly by computing the sum over (11) from $\ell,m g \alpha s$ have exactly the same $\partial W_{\ell,m g \alpha s} / \partial t$, resulting in an inverse cascade of energy (i.e. $1 - \ell - m - \epsilon$)

$$\ell, m g \alpha s$$

$\{g, \rho, s\}$

$$\sum_{j=1}^{m} t_{n-m+\ell+j}^{W}$$

$\Pi_{n}^{W} = \lambda W_{n}^{W}$

where $\varepsilon_{s}$ is the total small scale dissipation and

$$\Pi_{n}^{W} \equiv \left( g^{2(m-\ell)} - 1 \right) \sum_{j=1}^{m} t_{n-m+\ell+j}^{W} + \left( 1 - g^{2m} \right) \sum_{j=1}^{m} t_{n-m+\ell+j}^{W}$$

is the $k$-space flux of enstrophy. Assuming that in the inertial range $t_{n}^{W}$ remain independent of $n$, we get:

$$\Pi_{n}^{W} = \lambda W_{n}^{W}$$

where $\varepsilon_{s}$ is the total small scale dissipation and

$$\Pi_{n}^{W} \equiv \left( g^{2(m-\ell)} - 1 \right) \sum_{j=1}^{m} t_{n-m+\ell+j}^{W} + \left( 1 - g^{2m} \right) \sum_{j=1}^{m} t_{n-m+\ell+j}^{W}$$

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$$\Pi_{n}^{W} \equiv \left( g^{2(m-\ell)} - 1 \right) \sum_{j=1}^{m} t_{n-m+\ell+j}^{W} + \left( 1 - g^{2m} \right) \sum_{j=1}^{m} t_{n-m+\ell+j}^{W}$$

is the $k$-space flux of enstrophy. Assuming that in the inertial range $t_{n}^{W}$ remain independent of $n$, we get:

$$\Pi_{n}^{W} = \lambda W_{n}^{W}$$
where
\[ \lambda_W \equiv (1 - g^{2m}) (m - \ell) - (1 - g^{2(m-\ell)}) m > 0 \]
which can be seen from the fact that \( \lambda_W \) has the same form as \( \lambda_E \) but \( \ell \) replaced by \( m - \ell \), and \( m - \ell < m \). The instability assumption for a single triad suggests \( t_n^W > 0 \), so we get a forward cascade of enstrophy.

C. Zero flux solutions

A zero flux solution for the energy can be obtained more easily for specific values of \( \ell \) and \( m \). We therefore consider the case \( \ell = 1 \), \( m = 3 \) first, with \( \Pi^E_n = 0 \), which gives
\[
\Pi_n^E \equiv - \left[ (g^6 - g^2) \left( t_{n-1}^E + t_n^E + t_{n+1}^E \right) + (1 - g^6) \left( t_{n-1}^E + t_n^E \right) \right] = 0 .
\]
Assuming \( t_{n+1} = g^\mu t_n \) we get:
\[ g^{2\mu} g^2 (g^2 + 1) - g^\mu - 1 = 0 , \]
whose solution is \( \mu = \ln \left[ \frac{1}{2g^2(g^2+1) \sqrt{1 + 4g^2(g^2+1)}} \right] / \ln g = -2 \), for \( g = \sqrt{2} \).

Since \( t_n \sim k_n^{-2} \), and \( t_n \propto k_n^4 |\Phi_n|^3 \), one obtains a spectral energy density of the form \( E(k_n) = |\Phi_n|^2 k_n \propto k_n^{-3} \).

Similarly, the zero enstrophy flux solution for \( \ell = 1 \), \( m = 3 \) gives \( \mu = 2 \), which means \( t_n^W \sim k_n^2 \) and therefore \( E(k_n) \propto k_n^{-5/3} \). The fact that the zero flux solution for energy gives the same scaling as the forward enstrophy cascade solution (i.e. \( E(k) \propto k^{-3} \)) and the zero flux solution for enstrophy gives the same scaling as the inverse energy cascade solution (i.e. \( E(k) \propto k^{-5/3} \)), is a nice feature of the spiral chain structure.

In order to see if this works in the general case, we can substitute \( t_n \propto k_n^{-2} \) into the general expressions for the energy flux:
\[
\Pi^E_n \equiv - \left[ (g^{2m} - g^{2\ell}) \sum_{j=0}^{m-1} g^{j\mu} + (1 - g^{2m}) \sum_{j=0}^{m-\ell-1} g^{j\mu} \right] t_{n-m+\ell+j}^E = 0 .
\]

Using the relation
\[ \sum_{j=0}^{m-1} g^{j\mu} = \frac{(1 - g^{m\mu})}{(1 - g^\mu)} \]
one can see that the energy flux vanishes if
\[ \left( g^{2m} - g^{2\ell} \right) \frac{(1 - g^{m\mu})}{(1 - g^\mu)} + (1 - g^{2m}) \frac{(1 - g^{m-\ell\mu})}{(1 - g^\mu)} = 0 . \]

We find that \( \mu = -2 \) is a solution of this, since if we substitute it into the above expression, we get
\[ \left( g^{2m} - g^{2\ell} \right) (1 - g^{-2m}) + (1 - g^{2m}) (1 - g^{-2(m-\ell)}) = 0 \]
This means that for any combination of \( \ell, m \) and \( g \), \( t_n^E \propto k_n^{-2} \), gives a zero flux solution of energy with \( E(k) \propto k^{-3} \).

Similarly it is easy to see that \( \mu = 2 \) is a solution of the general relation for the vanishing enstrophy flux
\[ \Pi^W_n \equiv \left[ (g^{2(m-\ell)} - 1) \frac{(1 - g^{m\mu})}{(1 - g^\mu)} + (1 - g^{2m}) \frac{(1 - g^{(m-\ell)\mu})}{(1 - g^\mu)} \right] t_n^W_{n-m+\ell+j} = 0 \]
resulting in \( t_n^W \propto k_n^2 \) and therefore \( E(k) \propto k^{-5/3} \).

IV. THE MODEL FOR \( E_n \)

The general model for the evolution of turbulent energy on the spiral chain can be formulated as
\[ \partial_t E_n = \left[ (g^{2m} - g^{2\ell}) t_{n+\ell}^E + (1 - g^{2m}) t_n^E \right] + (1 - g^{2\ell}) t_n^E + P_n^E - D_n^E \]
\[ \Pi^E_n = \left[ (g^{2(m-\ell)} - 1) \frac{(1 - g^{m\mu})}{(1 - g^\mu)} + (1 - g^{2m}) \frac{(1 - g^{(m-\ell)\mu})}{(1 - g^\mu)} \right] t_n^W_{n-m+\ell+j} = 0 \]
\[ t_n^E = g^{-\ell} k_n \sin [(m-\ell) \alpha] E_n^{3/2} \]
Note that \( E(k_n) = E_n k_n^{-3} \) and that \( E_n > 0 \) and \( P_n^E > 0 \) to assure realizability. The model still conserves energy and enstrophy, and results in a clean dual cascade solution. And the difference from a model that solves the complex amplitudes \( \Phi_n \) is mainly in the definition \( \Pi^E_n \) vs. \( \Pi^W_n \).

The two models would become “equivalent” if the sums of the complex phases would vanish at each scale (for example for \( \ell = 2 \), \( m = 3 \), this would mean \( \phi_n + \phi_{n+1} - \phi_{n-2} = 0 \), where \( \phi_n \) are the complex phases). The condition is nontrivial and is not satisfied in the nonlinear stage by a complex chain model for \( \Phi_n \). Hence the complex chain fails to describe the cascade but instead evolves towards statistical chain equipartition.

Model in \( \Pi^W_n \), works for any \( \ell \) and \( m \) combination given in Table \( \Pi^W_n \), but one should pay attention to the fact that as \( \ell \) and \( m \) change, \( g \), and therefore the range of wave-numbers that are covered by the model changes, which means that the dissipation and the boundary terms should also be modified accordingly. Note finally that the assumption of \( t_n^E \propto k_n E_n^{3/2} \) corresponds to the Kovasznay’s form.
forcing \( \Phi_n \), whereas the black line is the model for \( E_n \). While the model for \( E_n \) is driven with constant forcing \( P_n = 2.5 \times 10^{-4} \), the model for \( \Phi_n \) is driven with random forcing such that \( \langle P_n \rangle = 2.5 \times 10^{-4} \). The spectrum for the \( \Phi_n \) model is averaged over a long stationary phase, where the spectrum for the \( E_n \) model is averaged over \( t = [5000, 10000] \), whereas the spectrum for the \( E_n \) model is averaged over \( t = [190, 200] \). (in fact the instantaneous solution is not that different from the averaged result).

\[ \langle \Phi_n \rangle = \langle k_n \rangle k_n, \text{ which is integrated up to } t = 10000 \text{ and the average is computed over } t = [5000, 10000], \] whereas the spectrum for the \( E_n \) model is averaged over \( t = [190, 200] \).

A. Continuum limit

It is also possible to interpret (13) as a shell model by disregarding the information on angles and therefore lifting the restriction on \( g \) values. In this case the resulting model is a simple discrete formulation of a general model where any value of \( g \) is allowed and an arbitrary factor [instead of the \( (m - \ell) \alpha \)] multiplies the nonlinear term, as in shell models. This interpretation allows us to transform the problem into a differential approximation model by considering the continuum limit of (13), with \( \ell = 1, m = 2 \), by considering \( g \to 1 + \epsilon \). Defining \( E(k) = E_n k_n^{-1} \) and \( F(k) = k^{3/2} E(k)^{3/2} \), so that \( k_{n+1} = k(1 + \epsilon) \text{ and } k_{n-1}(1 - \epsilon + \epsilon^2) \), so that

\[ F(k_{n+1}) \approx \left( F + k \epsilon \frac{dF}{dk} + \frac{1}{2} \epsilon^2 k^2 \frac{d^2 F}{dk^2} \right) \]

\[ F(k_{n-1}) \approx \left( F(k) - k \epsilon - \epsilon^2 \right) \frac{dF}{dk} + \frac{1}{2} \epsilon^2 k^2 \frac{d^2 F}{dk^2} \]

and

\[ g^2 k_n^{-1} \epsilon + (1 + g^2) k_{n-1}^{-1} \epsilon_n + k_n^{-1} \epsilon_{n-1} \]

\[ \approx 3 \epsilon^2 F + 5 \epsilon^2 k \frac{dF}{dk} + \epsilon^2 k^2 \frac{d^2 F}{dk^2}. \]

This finally gives:

\[ \partial_t E - C \frac{\partial}{\partial k} \left( k^{-1} \frac{\partial}{\partial k} \left( k^{3/2} E^{3/2} \right) \right) = P_E(k) - D_E(k) \]

as a differential approximation model\[11\]. It is clear that the two solutions \( E(k) \propto k^{-5/3} \) and \( E(k) \propto k^{-3} \) both cause the nonlinear term to vanish. In fact the way the flux is approximated, it works nicely that \( k^{-5/3} \) gives a constant and negative energy flux. In fact the constant flux solution of the above equation is \( E(k) = (\frac{2k}{\pi})^{2/3} k^{-5/3} \), which is helpful for picking the value of \( C \) in order to normalize the model properly. The continuum limit as discussed above results in an isotropic model, since its derivation starts from a shell-model with no regards to angles.

B. 4-Spiral Chain Model

Considering the model in (7) and using 4 such spiral chains that are basically rotated by \( \delta \alpha = j \alpha / 4 \) and scaled by \( g^{1/4} \) where \( j = 1, 2, 3 \) with respect to the original spiral (together with the original spiral itself, see fig. [5]) gives us a 4-spiral chain model, where the each spiral chain is coupled with itself but not with the other three. The advantage of the existence of the other chains is therefore a better coverage of the k-space but not a better description of the nonlinear interaction (i.e. the number of triads in the 4 spiral chain model is basically 4 times the single
spiral chain one). Such a model can be formulated alternatively by defining \( g = \rho^{1/8} \) and \( \alpha = \frac{1}{4} \arccos \left( -\frac{2^{12}}{7} \right) \) and using \( k_g = k_n (\cos \alpha_n, \sin \alpha_n) \), where \( k_n = k_0 g^n \) and \( \alpha_n = \alpha n \) as usual (note that \( g \) here is obviously different from the earlier one). The evolution for \( \Phi \) can then be written as

\[
\partial_t E_n = k_n \sin \alpha \left[ g^{16} (g^8 - 1) E_{n+8}^{3/2} + (g^{32} - 1) g^{-12} E_{n+4}^{3/2} + (g^{-8} - 2g^{16} + g^{-24}) E_n^{3/2} + (g^{16} - 1) g^{-12} E_{n-4}^{3/2} + (g^8 - 1) g^{-40} E_{n-16}^{3/2} + P_n^E - D_n^E \right] \quad (16)
\]

Please note the simplicity of the nonlinear couplings in this model. Albeit the fact that the model considers two kinds of triangles and spans roughly about 10 different directions for a given “scale” it represents these nonlinear interactions with only 5 terms.

The spiral grid corresponding to the 4-spiral chain, and its reflection with respect to the origin is also shown in figure 5. The grid provides an alternative way of looking at the spiral chain as a partition of the \( k \)-space. The surface element for a given cell, can then be written as:

\[
S_n = \frac{\pi (g^1 - g^{-1}) (g^5 - g^{-5}) k_n^2}{20 \ln (g)} \approx 0.03534 \times \pi k_n^2
\]

which is basically a small percentage of the area of the circle with that same radius. One obvious problem with this perspective is the “hole” that it leaves at the center. One can remedy this either by computing the actual shape of the leftover region and adding it as a partition cell, or alternatively adding a circular cell around the origin and reducing the surface elements of the first few cells of the partition by subtracting the part of the circular region that intersects with the cell that is left for the circular element defined at the origin. While rather promising, spiral partitioning of \( k \)-space is not the focus of this paper. Thus we leave it for future studies to resolve its particular issues.

V. NUMERICAL RESULTS

Existence of all possible triads enabled by neatly matching grid points of a regular mesh allows important advantages such as good statistical behavior, mathematical clarity and use of efficient numerical methods such as fast fourier transforms. The models that we present in this paper are not likely to replace direct numerical simulation schemes such as pseudo-spectral methods even when very large wave-number ranges are needed. Instead, they may be used as models of cascade that can provide a mathematical framework for understanding the detailed structure of the cascade process through self-similar triad interactions.

Various models introduced in this paper, can be considered as sets of ordinary differential equations that can be solved numerically in the presence of well-localized forcing and dissipation in the hope of establishing nu-
merical inertial range cascade behavior. However, note that the primary goal of this paper is to introduce the framework of spiral chains and not to perform a detailed numerical study of these models.

The results for the basic chain model for complex amplitudes $\mathbf{\Phi}_n$‘s for the chain $\ell = 1$, $m = 3$, driven with stochastic forcing, with dissipation of the form $D_n = (\nu k^4 + \nu_L k^8) \Phi_k$, can be seen in figure 3 and Figure 8 along with the model for $E_n$ for comparison. Even though the evolution of the complex phase is due to nonlinear couplings, the phases rapidly become “random” in practice, causing the fluxes to oscillate (both in time and along the chain), resulting in a statistical chain equipartition solution, which overwhelms the cascade process. In contrast the results for the chain model for $E_n$ for $\ell = 1$, $m = 3$ show a clear dual cascade and thus a distinct Kraichnan-Kolmogorov spectrum. Here we used a simple python solver [27], based on scipy ode solver [28].

The four chain model introduced in section IVB has a good coverage of the $k$-space both in radial and in angular directions. Here, we present the two dimensional wave number spectrum that we obtain from this model, with $N = 440$, $\nu = 10^{-24}$, $\nu_L = 10$, and anisotropic forcing $P_n = 2.5 \times 10^{-4}$ for the 4 wave-numbers closest to $k_x = 0$, $k_y = \pm 2 \times 10^3$ in Figure 6. Even though the drive is anisotropic, the resulting spectrum is isotropic since the flux along the spiral chain results naturally in isotropization of the spectrum. The time evolution of the wave-number spectrum is shown in Figure 7 and the fluxes are shown in Figure 8. Finally no intermittency has been observed in any of the models for $E_n$, since

VI. CONCLUSION

The geometry of the self-similar dual cascade in two dimensions as the energy or enstrophy is transferred from one wave-vector to another through triadic interactions are considered. The resulting picture is that of a chain of triangles that are rotated and scaled, such that the smallest wave-number of one triangle becomes the middle and largest wave-numbers of the consecutive triads. A particular class of triangles, make it such that one can form a regular logarithmic spiral grid out of the wave-numbers $k_n = k_0 (ge^{\alpha n})^n$, where the complex number is interpreted as a two-dimensional vector so that the real and imaginary parts are the $x$ and $y$ components, with $g$ and $\alpha$ being the scaling factor and the divergence angle respectively. Nonlinear interactions take place among the wave vectors $k_n$, $k_{n+\ell}$ and $k_{n+m}$ on such a spiral, where the values of $\ell$ and $m$ define (not necessarily uniquely) particular values of $g$ and $\alpha$. There is in fact a large number of such triangles, some of which are listed explicitly in table II. It is argued that the self-similar cascade takes place along triad chains, and therefore the concept of spiral chains can give us further insight into this mechanism, without the explicit assumption of isotropy.

In order to demonstrate the usefulness of the concept, a series of spiral chain models both for the complex amplitudes $\mathbf{\Phi}_n$ as well as energy $E_n$ have been developed. It is shown that analytical solutions of these models with constant or zero flux cases agree with the
Kraichnan-Kolmogorov phenomenology of isotropic cascade. While the complex models, that are basically “shell models” with elongated triads can not numerically reproduce the dual cascade (because the nonlinear evolution of the phases, lead to oscillatory solutions for the fluxes of conserved quantities), and instead converge to unphysical chain equipartition solutions. The model for $E_n$ in \[13\] can reproduce the dual cascade results numerically for any $\ell$ and $m$.

In particular, a 4-spiral chain model for $E_n$ is introduced in \[16\], which has good angular coverage and has two kinds of triads thanks to the choice of $g$ and $\alpha$ to include $\ell = 2$, $m = 3$ and $\ell = 1$, $m = 5$ simultaneously. While a simple test of anisotropic energy injection leads to the usual isotropic dual cascade result, the model can be developed for self-consistent drive or other similar cases for more complex problems such as two dimensional plasmas or geophysical fluids.

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