Almost Bi-Γ-Ideals and Fuzzy Almost Bi-Γ-Ideals of Γ-Semigroups

Anusorn Simuen¹, Saleem Abdullah², Winita Yonthanthum¹, Ronnason Chinram ¹,³,*

¹ Algebra and Applications Research Unit, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand
² Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan
³ Centre of Excellence in Mathematics, Si Ayuthaya Road, Bangkok 10400, Thailand

Abstract. In this paper, we introduce the notions of almost bi-Γ-ideals and fuzzy almost bi-Γ-ideals of Γ-semigroups and give properties of them. Moreover, we investigate relationships between almost bi-Γ-ideals and fuzzy almost bi-Γ-ideals.

2020 Mathematics Subject Classifications: 20M99
Key Words and Phrases: bi-Γ-ideals, almost bi-Γ-ideals, fuzzy almost bi-Γ-ideals

1. Introduction and Preliminaries

Ideal theory in semigroups, like all other algebraic structures, plays an important role in studying them. Good and Hughes [8] introduced the notion of bi-ideals of semigroups in 1952. An introductory definition of left, right, two-sided almost ideals of semigroups was launched by Grosek and Satko [9] in 1980. They gave the characterization of these ideals when a semigroup $S$ contains no proper left, right, two-sided almost ideals in [9], and afterwards, they discovered the minimal almost ideals and the smallest almost ideals of semigroups in [10] and [11], respectively. In 1981, Bogdanovic [3] introduced the definition of almost bi-ideals in semigroups by using the definitions of almost ideals and bi-ideals in semigroups. In [5], Wattanatripop, Chinram and Changphas gave the properties of quasi-almost ideals and first defined the concept of fuzzy almost ideals in semigroups. Moreover, they provided the relationships between almost ideals and their fuzzification. Furthermore, they investigated fuzzification of almost bi-ideals in semigroups in [4]. Almost $(m,n)$-ideals and their fuzzification in semigroups were studied by Suebsung, Wattanatripop and Chinram in [23]. Moreover, the idea of almost ideals and their fuzzification were extended to $n$-ary semigroups in [21].

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v13i3.3759

Email addresses: asimuen96@gmail.com (A. Simuen), saleemabdullah@awkum.edu.pk (S. Abdullah), winita.m@psu.ac.th (W. Yonthanthum), ronnason.c@psu.ac.th (R. Chinram)
The notion of Γ-semigroups has been first studied by Sen [18] in 1981. In 1986, Sen and Saha [19] improved more general definition as follows:

**Definition 1.** ([19]) Let $M$ and $Γ$ be non-empty sets. $(M, Γ)$ is called a $Γ$-semigroup if it satisfies the following laws.

1. $aab ∈ M$ for all $a, b ∈ M$ and $α ∈ Γ$.
2. $M$ is associative under $Γ$, that is
   
   $$(aab)βc = aα(bβc)$$

   for all $a, b, c ∈ M$ and all $α, β ∈ Γ$.

Every semigroup $(S, ·)$ can be considered as a $Γ$-semigroup $S$ by choosing $Γ = \{·\}$. Then a $Γ$-semigroup is one of the generalizations of semigroups. The investigation on $Γ$-semigroups was done by certain mathematicians which are parallel to some results of semigroups, for example, one may see [6, 7, 17–19]. Similar to semigroups, ideal theory in $Γ$-semigroups plays an important role (for example, we can see in [1, 6, 7, 12–14, 20]).

Let $M$ be a $Γ$-semigroup. For nonempty subsets $A$ and $B$ of $M$, let

$$AΓB = \{aαb | a ∈ A, b ∈ B, α ∈ Γ\}.$$  

If $m ∈ M$, we let $AΓm = AΓ\{m\}$ and $mΓA = \{m\}ΓA$. If $α ∈ Γ$, we let

$$AαB = \{aαb | a ∈ A, b ∈ B\}.$$  

**Definition 2.** (see [7]) Let $M$ be a $Γ$-semigroup.

1. A nonempty subset $T$ of $M$ is called a sub $Γ$-semigroup of $M$ if $TTT ⊆ T$.

2. A sub $Γ$-semigroup $B$ of $M$ is called a bi-$Γ$-ideal of $M$ if $BΓMTB ⊆ B$.

A bi-$Γ$-ideal in $Γ$-semigroups was sometimes called a bi-ideal (see [14]). Some generalizations of this ideal were studied in [2] and [16]. Recently, Wattanatripop and Changphas first studied the concept of almost ideals in $Γ$-semigroups. In [22], they defined the definitions of left [right] almost ideals in $Γ$-semigroups. Moreover, a $Γ$-semigroup containing no proper left [right] almost ideals was characterized.

In 1965, Zadeh [24] introduced the concept of fundamental fuzzy sets. Since then, fuzzy sets have been studied in various fields. A function from a set $M$ into the closed unit interval $[0, 1]$ is called a fuzzy subset of $M$. Let $f$ and $g$ be any two fuzzy subsets of a set $M$.

1. A fuzzy subset $f ∩ g$ of $M$ is defined by
   
   $$(f ∩ g)(m) = \min\{f(m), g(m)\}$$

   for all $m ∈ M$.  

A fuzzy subset \( f \cup g \) of \( M \) is defined by
\[
(f \cup g)(m) = \max\{f(m), g(m)\}
\]
for all \( m \in M \).

If \( f(m) \leq g(m) \) for all \( m \in M \), we say that \( f \) is a subset of \( g \), and use the notation \( f \subseteq g \) and sometimes we will say that \( f \) is contained in \( g \).

For a fuzzy subset \( f \) of any set \( M \), the support of \( f \) is the set of points in \( M \) defined by
\[
\text{supp}(f) = \{m \in M \mid f(m) \neq 0\}.
\]
For a subset \( A \) of any set \( M \), the characteristic function \( \chi_A \) of \( A \) is a fuzzy subset of \( M \) defined by
\[
\chi_A(m) = \begin{cases} 1 & m \in A, \\ 0 & m \notin A. \end{cases}
\]
For any element \( m \) of any set \( M \) and \( t \in (0,1] \), a fuzzy point \( m_t \) of \( M \) is a fuzzy subset of \( M \) defined by
\[
m_t(x) = \begin{cases} t & x = m, \\ 0 & x \neq m \end{cases}
\]
(see [15]).

2. Almost bi-\( \Gamma \)-ideals

First, we define almost bi-\( \Gamma \)-ideals of \( \Gamma \)-semigroups as follows:

**Definition 3.** A non-empty subset \( B \) of a \( \Gamma \)-semigroup \( M \) is called an almost bi-\( \Gamma \)-ideal of \( S \) if
\[
B\Gamma m\Gamma B \cap B \neq \emptyset
\]
for all \( m \in M \).

**Example 1.** Let \( B \) be any bi-\( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( M \). Then \( B\Gamma m\Gamma B \subseteq B \). This implies that for any \( m \in M \), \( B\Gamma m\Gamma B \subseteq B\Gamma m\Gamma B \subseteq B \). So \( B\Gamma m\Gamma B \cap B = B\Gamma m\Gamma B \neq \emptyset \) for all \( m \in M \). Then \( B \) is an almost bi-\( \Gamma \)-ideal of \( M \).

By Example 1, we conclude that every bi-\( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( M \) is an almost bi-\( \Gamma \)-ideal of \( M \).

**Example 2.** Consider the \( \Gamma \)-semigroup \( \mathbb{Z}_8 \) with \( \Gamma = \{0,1,2\} \) under the usual addition. Let \( B = \{4,6\} \). We see that
Therefore, \( B \) is an almost bi-\( \Gamma \)-ideal of \( \mathbb{Z}_8 \). However, \( B \) is not a bi-\( \Gamma \)-ideal of \( \mathbb{Z}_8 \) because 
\[
B + \Gamma + \mathbb{Z}_8 + \Gamma + B = \mathbb{Z}_8 \not\subseteq B.
\]

From Example 2, we see that an almost bi-\( \Gamma \)-ideal of \( \Gamma \)-semigroup \( S \) need not be a bi-\( \Gamma \)-ideal of \( S \).

**Example 3.** Consider the \( \Gamma \)-semigroup \( M = \{a,b,c,d\} \) with \( \Gamma = \{\alpha, \beta\} \) and the multiplication table:

|     | a  | b  | c  | d  |
|-----|----|----|----|----|
| a   | a  | c  | c  | a  |
| b   | c  | a  | a  | c  |
| c   | c  | a  | a  | c  |
| d   | a  | c  | a  | a  |

|     | a  | b  | c  | d  |
|-----|----|----|----|----|
| a   | c  | a  | a  | c  |
| b   | a  | c  | c  | a  |
| c   | a  | c  | c  | a  |
| d   | c  | a  | a  | c  |

Let \( B = \{a, c\} \). Then

\[
B \Gamma a \Gamma B \cap B = \{a, c\} \cap \{a, c\} = \{a, c\} \neq \emptyset,
\]

\[
B \Gamma b \Gamma B \cap B = \{a, c\} \cap \{a, c\} = \{a, c\} \neq \emptyset,
\]

\[
B \Gamma c \Gamma B \cap B = \{a, c\} \cap \{a, c\} = \{a, c\} \neq \emptyset,
\]

\[
B \Gamma d \Gamma B \cap B = \{a, c\} \cap \{a, c\} = \{a, c\} \neq \emptyset.
\]

Therefore, \( B \) is an almost bi-\( \Gamma \)-ideal of \( M \).

**Theorem 1.** Assume that \( B \) is an almost bi-\( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( M \). If \( A \) is any subset of \( M \) containing \( B \), then \( A \) is also an almost bi-\( \Gamma \)-ideal of \( M \).

**Proof.** Since \( B \) is an almost bi-\( \Gamma \)-ideal of \( M \) and \( B \subseteq A \), we have \( B \Gamma m \Gamma B \cap B \neq \emptyset \) and \( B \Gamma m \Gamma B \cap B \subseteq A \Gamma m \Gamma A \cap A \) for all \( m \in M \), respectively. This implies that \( A \Gamma m \Gamma A \cap A \neq \emptyset \) for all \( m \in M \). Therefore, \( A \) is an almost bi-\( \Gamma \)-ideal of \( M \).

**Corollary 1.** The union of any two almost bi-\( \Gamma \)-ideals of a \( \Gamma \)-semigroup \( M \) is also an almost bi-\( \Gamma \)-ideal of \( M \).

**Proof.** Let \( A \) and \( B \) be any two almost bi-\( \Gamma \)-ideals of \( M \). Since \( A \subseteq A \cup B \subseteq M \), it follows from Theorem 1 that \( A \cup B \) is an almost bi-\( \Gamma \)-ideal of \( M \).
Example 4. Consider the $\Gamma$-semigroup $\mathbb{Z}_8$ with $\Gamma = \{0, 1, 2\}$ under the usual addition. Let $A = \{2, 3\}$ and $B = \{4, 5\}$. Clearly, $A$ and $B$ are almost bi-$\Gamma$-ideals of $\mathbb{Z}_8$ but $A \cap B = \emptyset$, so it is not an almost bi-$\Gamma$-ideal of $\mathbb{Z}_8$.

By Example 4, we have the following remark.

Remark 1. The intersection of any two almost bi-$\Gamma$-ideals of a $\Gamma$-semigroup $M$ need not be an almost bi-$\Gamma$-ideal of $M$.

Theorem 2. A $\Gamma$-semigroup $M$ contains a proper almost bi-$\Gamma$-ideal if and only if there exists an element $m$ of $M$ such that $M \setminus \{m\}$ is an almost bi-$\Gamma$-ideal of $M$.

Proof. Assume that a $\Gamma$-semigroup $M$ contains a proper almost bi-$\Gamma$-ideal $B$ and let $m \in M \setminus B$. Then $B \subseteq M \setminus \{m\} \subset M$. By Theorem 1, $M \setminus \{m\}$ is an almost bi-$\Gamma$-ideal of $M$.

Conversely, let $m \in M$ be such that $M \setminus \{m\}$ is an almost bi-$\Gamma$-ideal of $M$. Since $M \setminus \{m\} \subseteq M$, we get $M \setminus \{m\}$ is a proper almost bi-$\Gamma$-ideal of $M$.

Theorem 3. Let $M$ be a $\Gamma$-semigroup such that $|M| > 1$. Then $M$ has no proper almost bi-$\Gamma$-ideals if and only if for all $m \in M$ there exists $a \in M$ such that

$$(M \setminus \{m\})\Gamma a \Gamma (M \setminus \{m\}) = \{m\}.$$ 

Proof. Assume that $M$ has no proper almost bi-$\Gamma$-ideals and let $m \in M$. By Theorem 2, $M \setminus \{m\}$ is not an almost bi-$\Gamma$-ideal of $M$. Thus there exists an element $a$ of $M$ such that $(M \setminus \{m\})\Gamma a \Gamma (M \setminus \{m\}) \cap (M \setminus \{m\}) = \emptyset$. Hence, $(M \setminus \{m\})\Gamma a \Gamma (M \setminus \{m\}) = \{m\}$.

Conversely, suppose $M$ contains a proper almost bi-$\Gamma$-ideal $B$. Let $m \in M \setminus B$. By assumption, we have $(M \setminus \{m\})\Gamma a \Gamma (M \setminus \{m\}) = \{m\}$ for some element $a$ in $M$. Since $B \subseteq M \setminus \{m\} \subset M$, we get $M \setminus \{m\}$ is an almost bi-$\Gamma$-ideal of $M$ by Theorem 1. This implies that $\emptyset = \{m\} \cap (M \setminus \{m\}) = (M \setminus \{m\})\Gamma a \Gamma (M \setminus \{m\}) \cap (M \setminus \{m\}) \neq \emptyset$, which is a contradiction. Therefore, $M$ has no proper almost bi-$\Gamma$-ideals.

3. Fuzzy almost bi-$\Gamma$-ideals

For a $\Gamma$-semigroup $M$, let $\mathcal{F}(M)$ be the set of all fuzzy subsets of $M$. For each $\alpha \in \Gamma$, define a binary operation $\circ_{\alpha}$ on $\mathcal{F}(M)$ by

$$(f \circ_{\alpha} g)(m) = \begin{cases} \sup_{m' \in M} \{\min\{f(a), g(b)\}\} & \text{if } m \in M\alpha M, \\ 0 & \text{otherwise.} \end{cases}$$ 

Let $\Gamma^* := \{\circ_{\alpha} | \alpha \in \Gamma\}$. Then $(\mathcal{F}(M), \Gamma^*)$ is a $\Gamma$-semigroup.

Proposition 1. For fuzzy subsets $f$ and $g$ of a $\Gamma$-semigroup $M$ such that $f \subseteq g$ and $\alpha \in \Gamma$, if $h$ is any fuzzy subset of $M$, then $h \circ_{\alpha} f \subseteq h \circ_{\alpha} g$ and $f \circ_{\alpha} h \subseteq g \circ_{\alpha} h$. 


We define fuzzification of almost bi-Γ-ideals in Γ-semigroups as follows:

**Definition 4.** A fuzzy subset \( f \) of a Γ-semigroup \( M \) is called a fuzzy almost bi-Γ-ideal of \( M \) if for all fuzzy points \( m_t \) of \( M \), there exist \( \alpha, \beta \in \Gamma \) such that \((f \circ_\alpha m_t \circ_\beta f) \cap f \neq 0\).

**Theorem 4.** Assume that \( f \) and \( g \) are fuzzy subsets of a Γ-semigroup \( M \) such that \( f \subseteq g \). If \( f \) is a fuzzy almost bi-Γ-ideal of \( M \), then \( g \) is also a fuzzy almost bi-Γ-ideal of \( M \).

**Proof.** Since \( f \) is a fuzzy almost bi-Γ-ideal of \( M \), for each fuzzy point \( m_t \) of \( M \), there exist \( \alpha, \beta \in \Gamma \) such that \((f \circ_\alpha m_t \circ_\beta f) \cap f \neq 0\). We have that \((f \circ_\alpha m_t \circ_\beta f) \cap f \subseteq (g \circ_\alpha m_t \circ_\beta g) \cap g\), this implies that \((g \circ_\alpha m_t \circ_\beta g) \cap g \neq 0\). Hence, \( g \) is also a fuzzy almost bi-Γ-ideal of \( M \).

**Corollary 2.** If \( f \) and \( g \) are fuzzy almost bi-Γ-ideals of a Γ-semigroup \( M \), then \( f \cup g \) is also a fuzzy almost bi-Γ-ideal of \( M \).

**Proof.** It follows by Theorem 4 because of \( f \subseteq f \cup g \).

**Example 5.** Consider the Γ-semigroup \( \mathbb{Z}_5 \) where \( \Gamma = \{\overline{0}\} \) and \( \pi \gamma \delta := \pi + \gamma + \delta \). Let \( f \) and \( g \) be fuzzy subsets of \( \mathbb{Z}_5 \) defined by
\[
\begin{align*}
  f(\overline{0}) &= 0, f(\overline{1}) = 0.5, f(\overline{2}) = 0, f(\overline{3}) = 0.1, f(\overline{4}) = 0.4 \\
  g(\overline{0}) &= 0, g(\overline{1}) = 0.3, g(\overline{2}) = 0.7, g(\overline{3}) = 0, g(\overline{4}) = 0.2.
\end{align*}
\]
It is easy to check that \([(f \circ_\alpha m_t \circ_\beta f) \cap f](\overline{4}) \neq 0\) and \([(g \circ_\alpha m_t \circ_\beta g) \cap g](\overline{4}) \neq 0\) for all \( \alpha, \beta \in \Gamma, m \in \mathbb{Z}_5 \) and \( t \in (0, 1] \). So \( f \) and \( g \) are fuzzy almost bi-Γ-ideals of \( \mathbb{Z}_5 \).

From the definition of the intersection of two fuzzy subsets, we have
\[
\begin{align*}
  (f \cap g)(\overline{0}) &= 0, (f \cap g)(\overline{1}) = 0.3, (f \cap g)(\overline{2}) = 0, (f \cap g)(\overline{3}) = 0, (f \cap g)(\overline{4}) = 0.2.
\end{align*}
\]
We can easily to check that \([(f \cap g) \circ_\alpha \overline{0}_t \circ_\beta (f \cap g)](a) = 0\) for all \( \alpha, \beta \in \Gamma, t \in (0, 1] \) and \( a \in \mathbb{Z}_5 \), so \( f \cap g \) is not a fuzzy almost bi-Γ-ideal of \( \mathbb{Z}_5 \).

The following remark follows from Example 5.

**Remark 2.** The intersection of two fuzzy almost bi-Γ-ideals of a Γ-semigroup \( M \) need not be a fuzzy almost bi-Γ-ideal of \( M \).

4. Relationships between almost bi-Γ-ideals and their fuzzification

**Theorem 5.** A non-empty subset \( B \) of a Γ-semigroup \( M \) is an almost bi-Γ-ideal of \( M \) if and only if \( \chi_B \) is a fuzzy almost bi-Γ-ideal of \( M \).
Proof. Assume that $B$ is an almost bi-$\Gamma$-ideal of a $\Gamma$-semigroup $M$ and let $m_t$ be any fuzzy point of $M$. Then $B \Gamma m \cap B \neq \emptyset$. Thus there exists $b \in B$ such that $b \in B \alpha m \beta B$ for some $\alpha, \beta \in \Gamma$. This implies that $(\chi_B \circ_\alpha m_t \circ_\beta \chi_B)(b) \neq 0$ and $\chi_B(b) \neq 0$. Hence, $(\chi_B \circ_\alpha m_t \circ_\beta \chi_B) \cap \chi_B \neq \emptyset$. Therefore, $\chi_B$ is a fuzzy almost bi-$\Gamma$-ideal of $M$.

To prove the converse, we assume that $\chi_B$ is a fuzzy almost bi-$\Gamma$-ideal of $M$ and let $m \in M$. Then there exist $\alpha, \beta \in \Gamma$ such that $(\chi_B \circ_\alpha m_t \circ_\beta \chi_B) \cap \chi_B \neq 0$, so $[(\chi_B \circ_\alpha m_t \circ_\beta \chi_B)(y)](x) \neq 0$ for some $y \in M$. Hence, $y \in B$ and $y = a \alpha m \beta b$ for some $a, b \in B$ and $\alpha, \beta \in \Gamma$. Therefore, $y \in B \Gamma m \cap B$. So $B \Gamma m \cap B \neq \emptyset$. Consequently, $B$ is an almost bi-$\Gamma$-ideal of $M$.

Theorem 6. A fuzzy subset $f$ of a $\Gamma$-semigroup $M$ is a fuzzy almost bi-$\Gamma$-ideal of $M$ if and only if $\text{supp}(f)$ is an almost bi-$\Gamma$-ideal of $M$.

Proof. Assume that $f$ is a fuzzy almost bi-$\Gamma$-ideal of a $\Gamma$-semigroup $M$ and let $m \in M$ and $t \in (0, 1]$. Then there exist $\alpha, \beta \in \Gamma$ such that $(f \circ_\alpha m_t \circ_\beta f) \cap f \neq \emptyset$. Hence, $[(f \circ_\alpha m_t \circ_\beta f) \cap f](x) \neq 0$ for some $x \in M$. So there exist $y_1, y_2 \in S$ such that $x = y_1 \alpha m \beta y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. That is, $x, y_1, y_2 \in \text{supp}(f)$. Thus $[(\chi_{\text{supp}(f)} \circ_\alpha m_t \circ_\beta \chi_{\text{supp}(f)})(x)](y) \neq 0$ and $\chi_{\text{supp}(f)}(x) \neq 0$. Therefore, $(\chi_{\text{supp}(f)} \circ_\alpha m_t \circ_\beta \chi_{\text{supp}(f)}) \cap \chi_{\text{supp}(f)} \neq 0$. Hence, $\chi_{\text{supp}(f)}$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. By Theorem 5, $\text{supp}(f)$ is an almost bi-$\Gamma$-ideal of $M$.

On the other hand, we assume that $\text{supp}(f)$ is an almost bi-$\Gamma$-ideal of $M$. It follows from Theorem 5 that $\chi_{\text{supp}(f)}$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. Let $m_t$ be any fuzzy point of $M$. Thus, $(\chi_{\text{supp}(f)} \circ_\alpha m_t \circ_\beta \chi_{\text{supp}(f)}) \cap \chi_{\text{supp}(f)} \neq 0$ for some $\alpha, \beta \in \Gamma$. Then there exists an element $x \in M$ such that $[(\chi_{\text{supp}(f)} \circ_\alpha m_t \circ_\beta \chi_{\text{supp}(f)})(x)](y) \neq 0$. Therefore, $(\chi_{\text{supp}(f)} \circ_\alpha m_t \circ_\beta \chi_{\text{supp}(f)})(x) \neq 0$ and $\chi_{\text{supp}(f)}(x) \neq 0$. Then there exist $y_1, y_2 \in M$ such that $x = y_1 \alpha m \beta y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. This means that $(f \circ_\alpha m_t \circ_\beta f) \cap f \neq 0$. We conclude that $f$ is a fuzzy almost bi-$\Gamma$-ideal of $M$.

Next, we will study the minimality of fuzzy almost bi-$\Gamma$-ideals.

Definition 5. A fuzzy almost bi-$\Gamma$-ideal $f$ of a $\Gamma$-semigroup $M$ is called minimal if for all fuzzy almost bi-$\Gamma$-ideal $g$ of $M$ contained in $f$, we must have $\text{supp}(g) = \text{supp}(f)$.

Now, we provide the relationship between minimal almost bi-$\Gamma$-ideals and their fuzzification.

Theorem 7. A non-empty subset $A$ of a $\Gamma$-semigroup $M$ is a minimal almost bi-$\Gamma$-ideal of $M$ if and only if $\chi_A$ is a minimal fuzzy almost bi-$\Gamma$-ideal of $M$.

Proof. Let $A$ be a minimal almost bi-$\Gamma$-ideal of a $\Gamma$-semigroup $M$. By Theorem 5, we have that $\chi_A$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. Assume that $g$ is a fuzzy almost bi-$\Gamma$-ideal of $M$ contained in $\chi_A$. Thus, $\text{supp}(g) \subseteq \text{supp}(\chi_A) = A$. Because of $g \subseteq \chi_{\text{supp}(g)}$, we have $(g \circ_\alpha m_t \circ_\beta g) \cap g \subseteq (\chi_{\text{supp}(g)} \circ_\alpha m_t \circ_\beta \chi_{\text{supp}(g)}) \cap \chi_{\text{supp}(g)}$ for all fuzzy points $m_t$ of $M$. Thus $\chi_{\text{supp}(g)}$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. By Theorem 5, $\text{supp}(g)$ is an almost bi-$\Gamma$-ideal of $M$. Because of $A$ is a minimal, then $\text{supp}(g) = A = \text{supp}(\chi_A)$. Therefore, $\chi_A$ is minimal.

To prove the converse, assume that $\chi_A$ is a minimal fuzzy almost bi-$\Gamma$-ideal of $M$ and $B$ is an almost bi-$\Gamma$-ideal of $M$ contained in $A$. Then $\chi_B$ is a fuzzy almost bi-$\Gamma$-ideal of $M$ and $\chi_B \subseteq \chi_A$. Thus, $B = \text{supp}(\chi_B) = \text{supp}(\chi_A) = A$. We conclude that $A$ is minimal.
Corollary 3. A \( \Gamma \)-semigroup \( M \) has no proper almost bi-\( \Gamma \)-ideals if and only if for all fuzzy almost bi-\( \Gamma \)-ideal \( f \) of \( M \), \( \text{supp}(f) = M \).

Proof. Assume that \( M \) has no proper almost bi-\( \Gamma \)-ideals and let \( f \) be a fuzzy almost bi-\( \Gamma \)-ideal of \( M \). By Theorem 6, we have \( \text{supp}(f) \) is almost bi-\( \Gamma \)-ideal of \( M \). Thus \( \text{supp}(f) = M \).

To prove the converse, we let \( B \) be any almost bi-\( \Gamma \)-ideal of \( M \). Follow by Theorem 5, we have that \( \chi_B \) is a fuzzy almost bi-\( \Gamma \)-ideal of \( M \). By assumption, we get \( B = \text{supp}(\chi_B) = M \). This implies that \( M \) has no proper almost bi-\( \Gamma \)-ideals.

Definition 6. Let \( M \) be a \( \Gamma \)-semigroup and \( \alpha \in \Gamma \).

(1) An almost bi-\( \Gamma \)-ideal \( B \) of \( M \) is called \( \alpha \)-prime if

\[ x\alpha y \in B \Rightarrow x \in B \text{ or } y \in B \]

for any \( x, y \in M \).

(2) A fuzzy almost bi-\( \Gamma \)-ideal \( f \) of \( M \) is called \( \alpha \)-prime if

\[ f(x\alpha y) \leq \max\{f(x), f(y)\} \]

for any \( x, y \in M \).

Next, we investigate relationship between \( \alpha \)-prime almost bi-\( \Gamma \)-ideals and their fuzzification.

Theorem 8. A nonempty subset \( A \) of a \( \Gamma \)-semigroup \( M \) is an \( \alpha \)-prime almost bi-\( \Gamma \)-ideal of \( M \) if and only if \( \chi_A \) is an \( \alpha \)-prime fuzzy almost bi-\( \Gamma \)-ideal of \( M \).

Proof. Let \( A \) be any \( \alpha \)-prime almost bi-\( \Gamma \)-ideal of \( M \). Then \( \chi_A \) is a fuzzy almost bi-\( \Gamma \)-ideal of \( M \) by Theorem 5. Let \( x \) and \( y \) be elements in \( M \). If \( x\alpha y \in A \), then \( x \in A \) or \( y \in A \). This implies that

\[ \chi_A(x\alpha y) = 1 \leq \max\{\chi_A(x), \chi_A(y)\} \]

If \( x\alpha y \notin A \), then

\[ \chi_A(x\alpha y) = 0 \leq \max\{\chi_A(x), \chi_A(y)\} \]

We conclude that \( \chi_A(x\alpha y) \leq \max\{\chi_A(x), \chi_A(y)\} \) for all \( x, y \in M \). Therefore, \( \chi_A \) is an \( \alpha \)-prime fuzzy almost bi-\( \Gamma \)-ideal of \( M \).

To prove the converse, suppose that \( \chi_A \) is an \( \alpha \)-prime fuzzy almost bi-\( \Gamma \)-ideal of \( M \). By Theorem 5, we have that \( A \) is an almost bi-\( \Gamma \)-ideal of \( M \). Let \( x \) and \( y \) be elements in \( M \) such that \( x\alpha y \in A \). Thus, \( \chi_A(x\alpha y) = 1 \). By assumption, we have that \( \chi_A(x\alpha y) \leq \max\{\chi_A(x), \chi_A(y)\} \). Therefore, \( \max\{\chi_A(x), \chi_A(y)\} = 1 \). We can conclude that \( x \in A \) or \( y \in A \). Hence, \( A \) is an \( \alpha \)-prime almost bi-\( \Gamma \)-ideal of \( M \).
**Definition 7.** Let $M$ be a $\Gamma$-semigroup and $\alpha \in \Gamma$.

1. An almost bi-$\Gamma$-ideal $A$ of $M$ is called $\alpha$-semiprime if

$$m\alpha m \in A \Rightarrow m \in A$$

for all $m \in M$.

2. A fuzzy almost bi-$\Gamma$-ideal $f$ of $M$ is called $\alpha$-semiprime if

$$f(m\alpha m) \leq f(m)$$

for all $m \in M$.

Finally, we give relationship between $\alpha$-semiprime almost bi-$\Gamma$-ideals and their fuzzification.

**Theorem 9.** A nonempty subset $A$ of a $\Gamma$-semigroup $M$ is an $\alpha$-semiprime almost bi-$\Gamma$-ideal of $M$ if and only if $\chi_A$ is an $\alpha$-semiprime fuzzy almost bi-$\Gamma$-ideal of $M$.

**Proof.** Let $A$ be an $\alpha$-semiprime almost bi-$\Gamma$-ideal of $M$. By Theorem 5, $\chi_A$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. Let $m \in M$. If $m\alpha m \in A$, then $m \in A$. So, $\chi_A(m) = 1$. Hence, $\chi_A(m\alpha m) \leq \chi_A(m)$. If $m\alpha m \notin A$, then $\chi_A(m\alpha m) = 0 \leq \chi_A(m)$. By both cases, we conclude that $\chi_A(m\alpha m) \leq \chi_A(m)$ for all $m \in M$. Thus, $\chi_A$ is an $\alpha$-semiprime fuzzy almost bi-$\Gamma$-ideal of $M$.

Conversely, assume that $\chi_A$ is an $\alpha$-semiprime fuzzy almost bi-$\Gamma$-ideal of $M$. By Theorem 5, we have that $A$ is an almost bi-$\Gamma$-ideal of $M$. Let $m \in M$ be such that $m\alpha m \in A$. Thus $\chi_A(m\alpha m) = 1$. By assumption, we have that $\chi_A(m\alpha m) \leq \chi_A(m)$. Since $\chi_A(m\alpha m) = 1$, it follows that $\chi_A(m) = 1$. Therefore, $m \in A$. Consequently, $A$ is an $\alpha$-semiprime almost bi-$\Gamma$-ideal of $M$.

**5. Conclusion**

In this paper, we define almost bi-$\Gamma$-ideals and their fuzzification of $\Gamma$-semigroups. Every bi-$\Gamma$-ideal is an almost bi-$\Gamma$-ideal but the converse is not true in general. We show that the union of two almost bi-$\Gamma$-ideals is also an almost bi-$\Gamma$-ideal. However, it is not generally true in case the intersection. Similarly, we have that the union of two fuzzy almost bi-$\Gamma$-ideals is also a fuzzy almost bi-$\Gamma$-ideal but it is not generally true in case the intersection. Moreover, the relationships between almost bi-$\Gamma$-ideals and their fuzzification were shown in Section 4.

**Acknowledgements**

This work was supported by the Faculty of Sciences Research Fund, Prince of Songkla University, Contract no. 1-2562-02-013.

We would like to thank the reviewers for their comments and suggestions.
References

[1] M. A. Ansari. Roughness applied to generalized \( \Gamma \)-ideals of ordered LA \( \Gamma \)-ideals. *Commun. Math. Appl.*, 10:71–84, 2019.

[2] M. A. Ansari and M. R. Khan. Notes on \((m, n)\) bi-\( \Gamma \)-ideals in \( \Gamma \)-semigroups. *Rend. Circ. Mat. Palermo*, 60:31–42, 2011.

[3] S. Bogdanovic. Semigroups in which some bi-ideal is a group. *Review of Research Faculty of Science-University of Novi Sad*, 11:261–266, 1981.

[4] K. Wattanatripop; R. Chinram and T. Changphas. Fuzzy almost bi-ideals in semigroups. *Int. J. Math. Computer Sci.*, 13:51–58, 2018.

[5] K. Wattanatripop; R. Chinram and T. Changphas. Quasi-\( A \)-ideals and fuzzy \( A \)-ideals in semigroups. *J. Discrete Math. Sci. Cryptogr.*, 21:1131–1138, 2018.

[6] R. Chinram. On quasi-gamma-ideals in gamma-semigroups. *ScienceAsia*, 32:351–353, 2006.

[7] R. Chinram and C. Jirokul. On bi-\( \Gamma \)-ideal in \( \Gamma \)-semigroups. *Songklanakarin J. Sci. Techno.*, 29:231–234, 2007.

[8] R. A. Good and D. R. Hughes. Associated for a semigroup. *Bull. Amer. Math. Soc.*, 58:624–625, 1952.

[9] O. Grosek and L. Satko. A new notion in the theory of semigroups. *Semigroup Forum*, 20:233–240, 1980.

[10] O. Grosek and L. Satko. On minimal \( A \)-ideals of semigroups. *Semigroup Forum*, 20:283–295, 1981.

[11] O. Grosek and L. Satko. Smallest \( A \)-ideals in semigroups. *Semigroup Forum*, 20:297–309, 1981.

[12] H. Hedayati. Isomorphisms via congruences on \( \Gamma \)-semigroups and \( \Gamma \)-ideals. *Thai J. Math.*, 11:563–575, 2013.

[13] K. Hila. On regular, semiprime and quasi-reflexive \( \Gamma \)-semigroup and minimal quasi-ideals. *Lobachevski J. Math.*, 29:141–152, 2008.

[14] A. Iampan. Note on bi-ideals in \( \Gamma \)-Semigroups. *Int. J. Algebra*, 3:181–188, 2009.

[15] P. M. Pu and Y. M. Liu. Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. *J. Math. Anal. Appl.*, 76:571–599, 1980.

[16] M. Murali Krishna Rao. Bi-quasi ideals and fuzzy bi-ideals of \( \Gamma \)-semigroups. *Bull. Int. Math. Virtual Inst.*, 7:231–242, 2017.
REFERENCES

[17] H. Rasouli and A. R. Shabani. On gamma acts over gamma semigroups. Eur. J. Pure Appl. Math., 10:739–748, 2017.

[18] M. K. Sen. On Γ-semigroups. Lecture Notes in Pure and Appl. Math. (Algebra and its applications, New Delhi), 91:301–308, 1981.

[19] M. K. Sen and N. K. Saha. On Γ-semigroup-I. Bull. Calcutta Math. Soc., 78:180–186, 1986.

[20] M. Siripitukdet and A. Iampan. On the ideal extensions in Γ-semigroups. Kyungpook Math. J., 48:585–591, 2008.

[21] J. P. F. Solano; S. Suebsung and R. Chinram. On almost i-ideals and fuzzy almost i-ideals in n-ary semigroups. JP J. Algebra, Number Theory Appl., 40:833–842, 2018.

[22] K. Wattanatripop and T. Changphas. On left and right A-ideals of a Γ-semigroup. Thai J. Math., SI:87–96, 2018.

[23] S. Suebsung; K. Wattanatripop and R. Chinram. On almost (m,n)-ideals and fuzzy almost (m,n)-ideals in semigroups. J. Taibah Univ. Sci., 13:897–902, 2019.

[24] L. A. Zadeh. Fuzzy sets. Inf. Control, 8:338–353, 1965.