Magic graphs and the faces of the Birkhoff polytope

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ABSTRACT
Magic labelings of graphs are studied in great detail by Stanley in [18] and [19], and Stewart in [20] and [21]. In this article, we construct and enumerate magic labelings of graphs using Hilbert bases of polyhedral cones and Ehrhart quasi-polynomials of polytopes. We define polytopes of magic labelings of graphs and digraphs. We give a description of the faces of the Birkhoff polytope as polytopes of magic labelings of digraphs.

KEY WORDS
Magic graphs; Polyhedral cones; Birkhoff polytope.

1 Introduction.

Let $G$ be a finite graph. A labeling of $G$ is an assignment of a nonnegative integer to each edge of $G$. A magic labeling of magic sum $r$ of $G$ is a labeling such that for each vertex $v$ of $G$ the sum of the labels of all edges incident to $v$ is the magic sum $r$ (loops are counted as incident only once) [18][19][20][21]. Graphs with a magic labeling are also called magic graphs (see Figure 1 for an example of a magic labeling of the complete graph $K_6$ of magic sum 40).

We define a magic labeling of a digraph $D$ of magic sum $r$ to be an assignment of a nonnegative integer to each edge of $D$ (such that for each vertex $v_i$ of $D$, the sum of the labels of all edges with $v_i$ as the initial vertex is $r$, and the sum of the labels of all edges with $v_i$ as the terminal vertex is also $r$). Thus magic labelings of a digraph is a network flow, where the flow into and out of every vertex, is the magic sum of the labeling. If we consider the labels of the edges of $G$ as variables, the defining magic sum conditions are simply linear equations, and the set of magic labelings of $G$ becomes the set of integral points inside a pointed polyhedral cone $C_G$ [15]. Henceforth, we call $C_G$ the cone of magic labelings of $G$. A Hilbert basis of $C_G$ has the property that any magic labeling of $G$ can be expressed as a linear combination with nonnegative integer coefficients of the elements of the Hilbert basis (see [1], [2], or [15]). An irreducible magic labeling of a graph is a magic labeling that cannot be written as a sum of two other magic labelings. The minimal Hilbert basis of $C_G$ is the set of all irreducible magic labelings and is unique [15]. Henceforth, when we refer to the Hilbert basis in this article, we mean the minimal Hilbert basis.

Let $v_1, v_2, \ldots, v_n$ denote the vertices of $G$
and let $e_{i_1}, e_{i_2}, \ldots, e_{i_m}$ denote the edges of $G$ that are incident to the vertex $v_i$ of $G$. Consider the polytope

$$\mathcal{P}_G = \{ L \in C_G \subseteq \mathbb{R}^d, \sum_{j=1}^{m_i} L(e_{i_j}) = 1; i = 1, \ldots, n \}.$$ 

We will refer to $\mathcal{P}_G$ as the polytope of magic labelings of $G$ and denote $H_G(r)$ to be the number of magic labelings of $G$ of magic sum $r$. Then, $H_G(r)$ is the Ehrhart quasi-polynomial of $\mathcal{P}_G$ [16].

We define a polytope $\mathcal{P}_D$ of magic labelings of a digraph $D$ as follows. Let $e_{i_1}, \ldots, e_{i_m}$ denote the edges of $D$ that have the vertex $v_i$ as the initial vertex and let $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$ denote the edges of $D$ for which the vertex $v_i$ is the terminal vertex, then:

$$\mathcal{P}_D = \{ L \in C_D \subseteq \mathbb{R}^q, \sum_{j=1}^{m_i} L(e_{i_j}) = \sum_{j=1}^{n_i} L(f_{i_j}) = 1; i = 1, \ldots, n \}.$$ 

Let $H_D(r)$ denote the number of magic labelings of $D$ of magic sum $r$. Then like before, $H_D(r)$ is the Ehrhart quasi-polynomial of $\mathcal{P}_D$. We now connect the magic labelings of digraphs to magic labelings of bipartite graphs.

**Lemma 1.1.** For every digraph $D$, there is a bipartite graph $G_D$ such that the magic labelings of $D$ are in one-to-one correspondence with the magic labelings of $G_D$. Moreover, the magic sums of the corresponding magic labelings of $D$ and $G_D$ are also the same.

**Proof.** Denote a directed edge of a digraph $D$ with $v_i$ as the initial vertex, and $v_j$ as the terminal vertex, by $e_{ij}$. Let $L$ be a magic labeling of $D$ of magic sum $r$. Consider a bipartite graph $G_D$ in $2n$ vertices, where the vertices are partitioned into two sets $A = \{ a_1, \ldots, a_n \}$ and $B = \{ b_1, \ldots, b_n \}$, such that there is an edge between $a_i$ and $b_j$, if and only if, there is an edge $e_{ij}$ in $D$. Consider a labeling $L_{G_D}$ of $G_D$ such that the edge between the vertices $a_i$ and $b_j$ is labeled with $L(e_{ij})$. Observe that the sum of the labels of the edges incident to $a_i$ is the same as the sum of the labels of incoming edges at the vertex $v_i$ of $D$. Also, the sum of the labels of edges at a vertex $b_j$ is the sum of the labels of outgoing edges at the vertex $v_j$ of $D$. Since $L$ is a magic labeling, it follows that $L_{G_D}$ is a magic labeling of $G_D$ with magic sum $r$. Going back-wards, consider a magic labeling $L'$ of $G_D$. We label every edge $e_{ij}$ of $D$ with the label of the edge between $a_i$ and $b_j$ of $G_D$ to get a magic labeling $L_D$ of $D$. Observe that $L'$ and $L_D$ have the same magic sum. Hence, there is a one-to-one correspondence between the magic labelings of $D$ and the magic labelings of $G_D$. □

For example, the magic labelings of the Octahedral digraph with the given orientation $D_O$ in Figure 2 are in one-to-one correspondence with the magic labelings of the bipartite graph $G_{D_O}$.

A graph $G$ is called a positive graph if for any edge $e$ of $G$ there is a magic labeling $L$ of $G$ for which $L(e) > 0$ [18]. Since edges of $G$ that are always labeled zero for any magic labeling of $G$ may be ignored to study magic labelings, we will concentrate on positive graphs in general. We use the following results by Stanley from [18] and [19] to prove Theorems 1.4 and 1.5 and Corollary 1.3.1.

**Theorem 1.1 (Theorem 1.1, [19]).** Let $G$ be a finite positive graph. Then either $H_G(r)$ is the Kronecker delta $\delta_0$, or else there exist polynomials $I_G(r)$ and $J_G(r)$ such that $H_G(r) = I_G(r) + (-1)^r J_G(r)$ for all $r \in \mathbb{N}$.

**Theorem 1.2 (Theorem 1.2, [19]).** Let $G$ be a finite positive graph with at least one edge. The degree of $H_G(r)$ is $q-n+1+b$, where $q$ is the number of edges of $G$, $n$ is the number of vertices, and $b$ is
the number of connected components of $G$ which are bipartite.

**Theorem 1.3 (Theorem 1.2, [18]).** Let $G$ be a finite positive bipartite graph with at least one edge, then $H_G(r)$ is a polynomial.

We now conclude that $H_D(r)$ is a polynomial for every digraph $D$.

**Corollary 1.3.1.** If $D$ is a digraph, then $H_D(r)$ is a polynomial of degree $q - 2n + b$, where $q$ is the number of edges of $D$, $n$ is the number of vertices, and $b$ is the number of connected components of the bipartite graph $G_D$.

**Proof.** The one-to one correspondence between the magic labelings of $D$ and the magic labelings of $G_D$, implies by Theorem [13] that $H_D(r)$ is a polynomial, and by Theorem [12] that the degree of $H_D(r)$ is $q - 2n + b$, where $b$ is the number of connected components of $G_D$ that are bipartite. □

Consider the polytope $P := \{ x | Ax \leq b \}$. Let $c$ be a nonzero vector, and let $\delta = \max \{ cx | Ax \leq b \}$. The affine hyperplane $\{ x | cx = \delta \}$ is called a supporting hyperplane of $P$. A subset $F$ of $P$ is called a face of $P$ if $F = P$ or if $F$ is the intersection of $P$ with a supporting hyperplane of $P$. Alternatively, $F$ is a face of $P$ if and only if $F$ is nonempty and

$$F = \{ x \in P | A'x = b' \}$$

for some subsystem $A'x \leq b'$ of $Ax \leq b$. See [15] for basic definitions with regards to polytopes.

Therefore, a face of $P_G$ is a polytope of the form

$$\{ L \in P_G, L(e_{i_k}) = 0; e_{i_k} \in E_0 \},$$

where $E_0 = \{ e_{i_1}, \ldots, e_{i_r} \}$ is a subset of the set of edges of $G$.

**Theorem 1.4.** Let $G$ be a finite positive graph with at least one edge. Then the polytope of magic labelings of $G$, $P_G$ is a rational polytope with dimension $q - n + b$, where $q$ is the number of edges of $G$, $n$ is the number of vertices, and $b$ is the number of connected components of $G$ that are bipartite. The $d$-dimensional faces of $P_G$ are the $d$-dimensional polytopes of magic labelings of positive subgraphs of $G$ with $n$ vertices and at most $n - b + d$ edges.

Observe from Theorem [13] that there is an edge between two vertices $v_i$ and $v_j$ of $P_G$ if and only if there is a graph with at most $n - b + 1$ edges, with magic labelings $v_i$ and $v_j$. The edge graph of $P_{13}$ is given in Figure 3. Similarly, we can draw the face poset of $P_G$ (see Figure 4 for the face poset of $P_{13}$).

An $n \times n$ semi-magic square of magic sum $r$ is an $n \times n$ matrix with nonnegative integer en-
Figure 5: A magic labeling of $\Gamma_3$ and its corresponding symmetric magic square.

Figure 6: Two different graph labelings associated to a semi-magic square.

The $d$-dimensional faces of $\mathcal{P}_{\Gamma_n}$ are $d$-dimensional polytopes of magic labelings of positive graphs with $n$ vertices and at most $n + d$ edges. There are $(2^{n-1})$ faces of $\mathcal{P}_{\Gamma_{2n}}$ that are copies of the Birkhoff polytope $B_n$.

We define a digraph $D$ to be a positive digraph if the corresponding bipartite graph $G_D$ is positive.

**Theorem 1.5.** Let $D$ be a positive digraph with at least one edge. Then, $\mathcal{P}_D$ is an integral polytope with dimension $q - 2n + b$, where $q$ is the number of edges of $D$, $n$ is the number of vertices, and $b$ is the number of connected components of $G_D$ that are bipartite. The $d$-dimensional faces of $\mathcal{P}_D$ are the $d$-dimensional polytopes of magic labelings of positive subdigraphs of $D$ with $n$ vertices and at most $2n - b + d$ edges.

Let $\Pi_n$ denote the complete digraph with $n$ vertices, i.e., there is an edge from each vertex to every other, including the vertex itself (thereby creating a loop at every vertex), then $G_{\Pi_n}$ is the complete bipartite graph $K_{n,n}$. We get a one-to-one correspondence between semi-magic squares $M = [m_{ij}]$ of magic sum $r$ and magic labelings of $\Pi_n$ of the same magic sum $r$ by labeling the edges $e_{ij}$ of $\Pi_n$ with $m_{ij}$. This also implies that there is a one-to-one correspondence between semi-magic squares and magic labelings of $K_{n,n}$ (this correspondence is also mentioned in [18] and [20]). See Figure 6 for an example.

A good description of the faces of Birkhoff polytope is not known [14]. We can now give an explicit description of the faces of the Birkhoff polytope.

\begin{align*}
\mathcal{P}_{\Gamma_n} &= \left\{ L = (L(e_{ij}) \in \mathbb{R}^{n(n+1)/2} ; \\
L(e_{ij}) &\geq 0; \\
1 &\leq i, j \leq n, i \leq j, \\
\sum_{j=1}^{n} L(e_{ji}) + \sum_{j=i+1}^{n} L(e_{ij}) = 1 \\
\text{for } i &= 1, \ldots, n \right\}
\end{align*}
Corollary 1.5.1. \( P_{\Pi_n} \) is the Birkhoff polytope \( B_n \). The \( d \)-dimensional faces of \( B_n \) are polytopes of magic labelings of positive digraphs with dimension \( d \), \( n \) vertices and at most \( 2n + d - 1 \) edges. The vertices of \( P_D \), where \( D \) is a positive digraph, are permutation matrices.

See Figure 7 for the edge graph of \( B_3 \). Two faces of a polytope of magic labelings of a graph (or a digraph) are said to be isomorphic faces if the subgraphs (subdigraphs, respectively) defining the faces are isomorphic. A set of faces is said to be a generating set of \( d \)-dimensional faces if every \( d \)-dimensional face is isomorphic to one of the faces in the set. See Figures 8, 9, 10, and 11 for the generators of the edges, the two-dimensional faces, the facets, and the Birkhoff polytope \( B_3 \), respectively. The numbers in square brackets in the figures indicate the number of faces in the isomorphism class of the given face.

The proofs of Theorems 1.4 and 1.5 and Corollaries 1.4.1 and 1.5.1 are presented in Secs...
2 Polytopes of magic labelings.

A polytope \( P \) is called rational if each vertex of \( P \) has rational coordinates.

An element \( \beta \) in the semigroup \( S_{CG} \) is said to be completely fundamental, if for any positive integer \( n \) and \( \alpha, \alpha' \in S_{CG} \), \( n\beta = \alpha + \alpha' \) implies \( \alpha = i\beta \) and \( \alpha' = (n-i)\beta \), for some positive integer \( i \), such that \( 0 \leq i \leq n \) (see [17]).

**Lemma 2.1.** \( P_G \) is a rational polytope.

*Proof.* Proposition 4.6.10 of Chapter 4 in [17] states that the set of extreme rays of a cone and the set of completely fundamental solutions are identical. Proposition 2.7 in [18] states that every completely fundamental magic labeling of a graph \( G \) has magic sum 1 or 2. Thus, the extreme rays of the cone of magic labelings of a graph \( G \) are irreducible 2-matches of \( G \). We get a vertex of \( P_G \) by dividing the entries of an extreme ray by its magic sum. Thus, \( P_G \) is a rational polytope. \( \square \)

**Lemma 2.2.** The dimension of \( P_G \) is \( q - n + b \), where \( q \) is the number of edges of \( G \), \( n \) is the number of vertices, and \( b \) is the number of connected components that are bipartite.

*Proof.* Ehrhart’s theorem states that the degree of \( H_G(r) \) is the dimension of \( P_G \) [6]. The degree of \( H_G(r) \) is \( q - n + b \) by Theorem 1.2. Therefore, the dimension of \( P_G \) is \( q - n + b \). \( \square \)

**Lemma 2.3.** The \( d \)-dimensional faces of \( P_G \) are the \( d \)-dimensional polytopes of magic labelings of positive subgraphs of \( G \) with \( n \) vertices and at most \( n - b + d \) edges.

*Proof.* An edge \( e \) labeled with a zero in a magic labeling \( L \) of \( G \) does not contribute to the magic sum, therefore, we can consider \( L \) as a magic labeling of a subgraph of \( G \) with the edge \( e \) deleted. Since a face of \( P_G \) is the set of magic labelings of \( G \) where some edges are always labeled zero, it follows that the face is also the set of all the magic labelings of a subgraph of \( G \) with these edges deleted. Similarly, every magic labeling of a subgraph \( H \) with \( n \) vertices corresponds to a magic labeling of \( G \), where the missing edges of \( G \) in \( H \) are labeled with 0. Now, let \( H \) be a subgraph such that the edges \( e_{r1}, \ldots, e_{rm} \) are labeled zero for every magic labeling of \( H \). Then the face defined by \( H \) is the same as the face defined by the positive graph we get from \( H \) after deleting the edges \( e_{r1}, \ldots, e_{rm} \). Therefore, the faces of \( P_G \) are polytopes of magic labelings of positive subgraphs.

By Lemma 2.2 the dimension of \( P_G \) is \( q - n + b \). Therefore, to get a \( d \)-dimensional polytope, we need to label at least \( q - n + b - d \) edges always 0. This implies that the \( d \)-dimensional face is the set of magic labelings of a positive subgraph of \( G \) with \( n \) vertices and at most \( n - b + d \) edges. \( \square \)

The proof of Theorem 1.4 follows from Lemmas 2.1, 2.2, and 2.3. We can now prove Corollary 1.4.1.

*Proof of Corollary 1.4.1.* It is clear from the one-to-one correspondence between magic labelings of \( \Gamma_n \) and symmetric magic squares that \( P_{\Gamma_n} \) has the given description. Since the graph \( \Gamma_n \) has \( \frac{n(n+1)}{2} \) edges and \( n \) vertices, and every graph is a subgraph of \( \Gamma_n \), it follows from Theorem 1.4 that the dimension of \( P_{\Gamma_n} \) is \( \frac{n(n-1)}{2} \); the \( d \)-dimensional faces of \( P_{\Gamma_n} \) are \( d \)-dimensional polytopes of magic labelings of positive graphs with \( n \) vertices and at most \( n + d \) edges.

We can partition the vertices of \( \Gamma_{2n} \) into two equal sets \( A \) and \( B \) in \( \binom{2n-1}{n} \) ways: Fix the vertex \( v_1 \) to be in the set \( A \), then we can choose the \( n \) vertices for the set \( B \) in \( \binom{2n-1}{n} \) ways, and the remaining \( n - 1 \) vertices will belong to the set \( A \). By adding the required edges, we get a complete bipartite graph for every such partition of the vertices of \( \Gamma_{2n} \). Thus, the number of subgraphs of \( \Gamma_{2n} \) that are isomorphic to \( K_{n,n} \) is \( \binom{2n-1}{n} \). Therefore, there are \( \binom{2n-1}{n} \) faces of \( P_{\Gamma_{2n}} \) that are Birkhoff polytopes because every isomorphic copy of \( K_{n,n} \) contributes to a face of \( P_{\Gamma_{2n}} \). \( \square \)

**Lemma 2.4.** Let \( G \) be a graph with \( n \) vertices. A labeling \( L \) of \( G \) with magic sum \( s \) can be lifted to
Proof. Since $G$ is a subgraph of $\Gamma_n$, every labeling $L$ of $G$ can be lifted to a labeling $L'$ of $\Gamma_n$, where

$$L'(e_{ij}) = \begin{cases} L(e_{ij}) & \text{if } e_{ij} \text{ is also an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Since the edges with nonzero labels are the same for both $L$ and $L'$, it follows that the magic sums are also the same. □

Lemma 2.5. Let $G$ be a graph with $n$ vertices. The minimal Hilbert basis of $C_G$ can be lifted to a subset of the minimal Hilbert basis of $C_{\Gamma_n}$.

Proof. If $L$ is an irreducible magic labeling of $G$, then clearly it lifts to an irreducible magic labeling $L'$ of $\Gamma_n$. Since the minimal Hilbert basis is the set of all irreducible magic labelings, we get that the minimal Hilbert basis of $C_G$ corresponds to a subset of the minimal Hilbert basis of $C_{\Gamma_n}$. □

Similarly, we can prove:

Lemma 2.6. For a digraph $D$ with $n$ vertices, a magic labeling $L$ with magic sum $s$ can be lifted to a magic labeling $L'$ of $\Pi_n$ with the same magic sum $s$. The minimal Hilbert basis of $C_D$ can be lifted to a subset of the minimal Hilbert basis of $C_{\Pi_n}$.

Lemma 2.7. Let $D$ be a digraph with $n$ vertices. All the elements of the minimal Hilbert basis of $C_D$ have magic sum 1.

Proof. It is well-known that the minimal Hilbert basis of semi-magic squares are the permutation matrices (see [15]) and therefore have magic sum 1. The one-to-one correspondence between magic labelings of $\Pi_n$ and semi-magic squares implies that the minimal Hilbert basis elements of $C_{\Pi_n}$ have magic sum 1. It follows by Lemma 2.6 that all the elements of the minimal Hilbert basis of $C_D$ have magic sum 1. □

We now prove our results about polytope of magic digraphs.

Proof of Theorem 1.4. By Lemma 2.7 all the elements of the Hilbert basis of $C_D$ have magic sum 1. Since the extreme rays are a subset of the Hilbert basis elements, it follows that the vertices of $P_D$ are integral. Since $P_D = P_{G_D}$, it follows by Theorem 1.4 that the dimension of $P_D$ is $q - 2n + b$; the $d$-dimensional faces of $P_D$ are the $d$-dimensional polytopes of magic labelings of positive subdigraphs of $D$ with $n$ vertices and at most $2n - b + d$ edges. □

We derive our results about the faces of the Birkhoff polytope as a consequence.

Proof of Corollary 1.5.1. The one-to-one correspondence between semi-magic squares and magic labelings of $\Pi_n$ gives us that $P_{\Pi_n} = B_n$. Since every digraph with $n$ vertices is a subgraph of $\Pi_n$, by Theorem 1.5 it follows that its $d$-dimensional faces are $d$-dimensional polytopes of magic labelings of positive digraphs with $n$ vertices and at most $2n - 1 + d$ edges. Since the vertex set of a face of $B_n$ is a subset of the vertex set of $B_n$ it follows that the vertices of $P_D$, where $D$ is a positive digraph, are permutation matrices. □

3 Applications.

In this section, we present some examples and applications of magic graphs. Interesting examples of magic digraphs are Cayley digraphs of finite groups. Let $G$ be a finite group $\{g_1, g_2, \ldots, g_n = 1\}$. The Cayley group digraph of $G$ is a graphical representation of $G$: every element $g_i$ of the group $G$ corresponds to a vertex $v_i$ ($i = 1, 2, \ldots, n$) and every pair of distinct vertices $v_i, v_j$ is joined by an edge labeled with $\alpha$ where $g_{\alpha} = g_i g_i^{-1}$ [12]. For example, the Cayley digraph for the permutation group

$$S_3 = \{g_1 = (123), g_2 = (132), g_3 = (23),
\quad g_4 = (12), g_5 = (13), g_6 = 1\}$$

is given in Figure 12.

Proposition 3.1. The Cayley digraph of a group of order $n$ is a magic digraph with magic sum $n(n-1)/2$.

Proof. Let $e_{ij}$ denote an edge between the vertex $v_i$ and $v_j$ of the Cayley digraph such that $v_i$ is the
initia vertex and $v_j$ is the terminal vertex. Let $v_i$ be a vertex of the Cayley digraph, and let $\alpha$ be an integer in the set $\{1, 2, \ldots, n-1\}$. Let $g_p = g_\alpha g_t$ and let $g_q = g_\alpha g_a$. Then, the edges $e_{ip}$ and $e_{qj}$ are labeled by $\alpha$. Also, $g_jg_i^{-1} = g_n = I$ if and only if $i = j$. Hence, a Cayley group digraph is a magic digraph with magic sum $1 + 2 + \cdots + (n - 1) = \frac{n(n-1)}{2}$ (see also Chapter 8, Section 5 in [12]). □

A digraph is called Eulerian if for each vertex $v$ the indegree and the outdegree of $v$ is the same. Therefore, Eulerian digraphs can also be studied as magic digraphs where all the edges are labeled by 1 (see [4] for the applications of Eulerian digraphs to digraph colorings). An $n$-matching of $G$ is a magic labeling of $G$ with magic sum at most $n$ and the labels are from the set $\{1, 2, \ldots, n\}$ (see [13], chapter 6). A perfect matching of $G$ is a 1-matching of $G$ with magic sum 1.

**Proposition 3.2.** The perfect matchings of $G$ are the minimal Hilbert basis elements of $C_G$ of magic sum 1 and the number of perfect matchings of $G$ is $H_G(1)$. □

Hilbert bases can also be used to study factorizations of labeled graphs. We define Factors of a graph $G$ with a labeling $L$ to be labelings $L_i, i = 1, \ldots, r$ of $G$ such that $L(G) = \sum_{i=1}^{r} L_i(G)$, and if $L_i(e_k) \neq 0$ for some edge $e_k$ of $G$, then $L_j(e_k) = 0$ for all $j \neq i$. A decomposition of $L$ into factors is called a factorization of $G$. An example of a graph factorization is given in Figure 13. See Chapters 11 and 12 of [12] for a detailed study of graph factorizations.

Our results enable us to reprove some known facts about the Birkhoff polytope as well. For example, Theorem 1.5 gives us that the dimension of $B_n$ is $(n-1)^2$. The leading coefficient of the Ehrhart polynomial of $B_n$ is the volume of $B_n$. This number has been computed for $n = 1, 2, \ldots, 9$ (see [5] and [9]).

The software 4ti2 [11] can be used to find the Hilbert bases of $C_G$, and the software LattE [10] can be used to compute the generating functions of $H_G(r)$ effectively. See [13] for other results about magic labelings of graphs. See [22] for a study of other types of magic graphs.

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