Numerical Investigation of a Bifurcation Problem with Free Boundaries Arising from the Physics of Josephson Junctions

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Abstract

A direct method for calculating the minimal length of “one-dimensional” Josephson junctions is proposed, in which the specific distribution of the magnetic flux retains its stability. Since the length of the junctions is a variable quantity, the corresponding nonlinear spectral problem as a problem with free boundaries is interpreted.

The obtained results give us warranty to consider as “long”, every Josephson junction in which there exists at least one nontrivial stable distribution of the magnetic flux for fixed values of all other parameters.
I. Posing the Problem

It is known that the stationary distributions of the magnetic flux $\varphi(x)$ in “long” (one-dimensional) Josephson junctions (JJ) are solutions of the nonlinear boundary value problem (BVP)

$$- \varphi_{xx} + j_D(x) \sin \varphi \pm \gamma = 0, \quad x \in (-R, R),$$

$$\varphi_x(\pm R) = h_B,$$  

where $h_B$ is the external magnetic field alongside the axis $y$ on the junction plane (see Fig.1a).

We note that the kind of boundary conditions (2), either in presence or absence of current $\gamma$ in the right side of eq.(1), are determined by the geometry of the junction. Here we consider simple junctions with overlap geometry\cite{4} in which the current $\gamma$ can be approximately considered as a constant. The generalization of our results in cases of any other geometry, for example in-line geometry, does not require big efforts.

We suppose that the given continuous function $0 \leq j_D(x) \leq 1$ describes the variations of the Josephson current amplitude, caused by the possible local inhomogeneities of the dielectric layer thickness. When $j_D(x) \equiv 1$ the junction is homogeneous. Otherwise when the junction is inhomogeneous the function $j_D(x)$ is usually modelled by Dirac $\delta$- function\cite{2,9} or its continuous approximations, for example hyperbolic functions\cite{1} splines\cite{3} etc. At the present work as is in\cite{1} we use an isosceles trapezium with base $\mu$ (see Fig.1b) as more suitable in physical sense model of inhomogeneity.

Every solution of the equation (1) is simultaneously a stationary solution of the perturbed Sin-Gordon equation (SGE)

$$\varphi_{tt} + \alpha \varphi_t - \varphi_{xx} + j_D(x) \sin \varphi \pm \gamma = 0,$$  

where $\alpha$ is coefficient of a resistance. When $\alpha > 0$ the second term in the above equation is dissipative and hence arbitrary distribution $\varphi(x,t)$ of the magnetic flux as result of an energy loss can be “attracted” by some steady distribution $\varphi(x)$.

In order to study the stability of some concrete solution $\varphi(x)$ of the BVP (1),(2) we consider the following Sturm-Liouville problem (SLP)

$$- \psi_{xx} + q(x) \psi = \lambda \psi, \quad x \in (-R, R),$$

where $q(x) = j_D(x) \cos \varphi(x)$ is a potential, originated by the solution $\varphi(x)$, and boundary conditions of Neumann’s type

$$\psi_x(\pm R) = 0.$$
\[ <\psi,\psi> = \int_{-R}^{R} \psi^2(x) \, dx = 1. \quad (6) \]

If the minimal eigenvalue \( \lambda_{\text{min}} > 0 \), the respective solution \( \varphi(x) \) of BVP (1), (2) is stable with respect to small time-space perturbations. If the minimal eigenvalue \( \lambda_{\text{min}} < 0 \), this solution is unstable. The eigenvalue \( \lambda_{\text{min}} = 0 \) is a bifurcation point, in which the stable solutions of eq. (1) go to unstable ones and vice versa (for details see\textsuperscript{9}).

Apart from the space coordinate \( x \), the virtual solutions of the nonlinear BVP (1), (2) depend also on the physical parameters \( h_B, \gamma \) and “technological” ones \( \mu \) and \( R \), i.e., \( \varphi = \varphi(x, p) \), where we simply substitute \( p \equiv \{h_B, \gamma, \mu, R\} \). The varying of every of those parameters causes a variation of the distribution \( \varphi(x, p) \) and therefore subsequent variations of the potential \( q(x, p) \), the eigenvalues \( \lambda(p) \) and the respective eigenfunctions \( \psi(x, p) \). Thus we can conclude that every solution of BVP (1), (2) has an area where it remains stable with regard to the variations of the parameters \( p \). The equation

\[ \lambda_{\text{min}}(p) = 0 \quad (7) \]

determinates in the parametric space a hypersurface which points appear to be bifurcation points corresponding to the solution under consideration (1). The intersections of the bifurcation hypersurface (7), when there are fixed pairs of the parameters \( p \), we call bifurcation curves and respective values of the parameters - bifurcation (critical) parameters. The most interesting from the physical viewpoint seem to be bifurcation curves “external current - magnetic flux” \( \lambda_{\text{min}}(h_B, \gamma) = 0 \), when the geometrical parameters \( \mu \) and \( R \) are fixed. That curves could be relatively easy obtained experimentally.\textsuperscript{9}

From the technological point of view, however, it is worth investigating the bifurcation curves as functions with respect to at least one of geometrical parameters

\[ \lambda_{\text{min}}(h_B, R) = 0, \quad \lambda_{\text{min}}(\gamma, R) = 0, \quad \text{or} \quad \lambda_{\text{min}}(\mu, R) = 0. \]

Such kind of problems can be connected with the optimization of sizes of devices, containing Josephson junctions.

Formally the numerical modelling of the bifurcation curves as function of some concrete parameter \( p_0 \in p \) (at the paper\textsuperscript{4} it is chosen \( p_0 = h_B \)) can be schemed as follows. We find some solution of the BVP (1), (2). After that we check his stability with respect to small perturbations solving the corresponding SLP (4) - (6). If \( \lambda_{\text{min}} > 0 \) we set a new value \( p_0 := p_0 + \Delta p \), where \( \Delta p \) is given increment, and solve the BVP (1), (2) again using the obtained solution as initial approximation. We repeat this iteration while at the “current stage” \( p_0 \) the equation \( \lambda_{\text{min}} = 0 \) is satisfied numerically. Then one point at the searched bifurcation curve is calculated.

At Fig. 2 the typical relationships \( \lambda_{\text{min}}(\Delta) \) (here \( \Delta = 2R \) is the junction length) corresponding to the so called “main fluxon”\textsuperscript{5} (see Fig. 3) in inhomogeneous junction containing one resistance inhomogeneity placed at the point \( x = 0 \) when \( \mu = 0.5 \) and \( \mu = 1 \) respectively are shown. The points \( B_0 \) and \( B_1 \) appear to be bifurcation points of

\textsuperscript{1}In infinite JJ with one \( \delta \)-shaped microinhomogeneity at the point \( x = 0 \) “main” fluxon/antifluxon is represented by the exact solution \( \varphi(x) = 4 \arctan \exp(\pm x) \).

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corresponding relationships $\lambda_{\text{min}}(\Delta) = 0$. According to the above mentioned reasonings these points determine the minimum length of JJ, for which main fluxon is still stable.

Obviously such kind of algorithm is quite hard. Therefore it is quite natural to put the question how to calculate directly the bifurcation curves. A general approach study for the posed problem is proposed at the articles. We consider the equations (1), (2), (4) - (6) as a closed nonlinear system with respect to the functions $\varphi(x), \psi(x)$ and one of parameters $p$ (for example, $h_B$ or $\gamma$), while the other 3 parameters, and $\lambda$ also, are given. Then fixing $\lambda$ to be small enough, (for example $\lambda = 0.01$), every solution of the above system with a priori prescribed accuracy (the derivative $\frac{\partial \lambda}{\partial p_0} \to \infty$ when $p_0$ approaches the its critical value) belongs to the small vicinity of the searched bifurcation curve.

The calculation of the critical half-length $R$ of homogeneous or inhomogeneous JJ, corresponding to a concrete nontrivial distribution of the magnetic flux, is an important practical problem. The shortcoming in this connection ensues from the fact that the equations (1), (2), (4) - (6) are implicit with respect to the quantity $R$. At the present work we propose how to overcome the mentioned imperfection.

II. Method of Solution

For given values of parameters $\lambda, \mu, h_B$ and $\gamma$ we consider the system (1), (2), (4) - (6) as a nonlinear eigenvalue problem with respect to the eigenfunctions $\varphi(x), \psi(x)$ and to the eigenvalue $R$. As the parameter $R$ does not occur explicitly, it is convenient to use the Landau transformation $\xi = \frac{x}{f(R)}$. Choosing simply $f(R) \equiv R$ we map the original interval $[-R, R]$ to the interval $[-1, 1]$. Taking into account that $\frac{d}{dx} = \frac{1}{R} \frac{d}{d\xi}$ we obtain that the above system renders to the system:

\[-\bar{\varphi}_{\xi\xi} + R^2 [\bar{j}_D(\xi) \sin \bar{\varphi} + \gamma] = 0,\]
\[\bar{\varphi}(\pm 1) - R h_B = 0,\]
\[-\bar{\psi}_{\xi\xi} + R^2 [\bar{j}_D(\xi) \cos \bar{\varphi} - \lambda] \bar{\psi} = 0,\]
\[\bar{\psi}(\pm 1) = 0,\]
\[<\bar{\psi}, \bar{\psi}> - 1 \equiv R \int_{-1}^{1} \bar{\psi}^2(\xi) \, d\xi - 1 = 0,\]

where $(\cdot)$ denotes the function of $\xi$. Without fear of confusion the bars will be omitted henceforth. Let us consider the left-hand side of the system (8)-(12) as a functional vector $F(\varphi, \psi, R)$. Then we have $F(\varphi, \psi, R) = 0$.

The nonlinear system under consideration will be solved using the continuous analog of Newton’s method. Following it, we introduce a continuous parameter (“time”) $t \in [0, \infty)$ and suppose the quantities $\varphi, \psi$ and $R$ depend on $t$, i.e., $\varphi(t, \xi), \psi(t, \xi)$ and $R(t)$. The following abstract differential equation is used

\[\frac{dF}{dt} + F \equiv F_{\varphi} \dot{\varphi} + F_{\psi} \dot{\psi} + F_R \dot{R} + F = 0.\]
Here

\[ F'_\varphi u = \frac{d}{d\epsilon} F(\varphi + \epsilon u) \bigg|_{\epsilon=0}, \quad F'_\psi v = \frac{d}{d\epsilon} F(\psi + \epsilon v) \bigg|_{\epsilon=0}, \quad F'_R \rho = \frac{d}{d\epsilon} F(R + \epsilon \rho) \bigg|_{\epsilon=0} \]

are Frechet’s derivatives of \( F \), and

\[ u = \dot{\varphi}, \quad v = \dot{\psi}, \quad \rho = \dot{R} \quad (13) \]

are the “time” derivatives of the functions \( \varphi, \psi \) and \( R \), respectively. Simplifying above system we obtain

\[ -u_{\xi\xi} + R^2 j_D(\xi) \cos \varphi u + \{2R \ [j_D(\xi) \sin \varphi + \gamma] + R^2 j_D,\xi(\xi) \sin \varphi\} \rho = 0, \]
\[ -\varphi_{\xi\xi} + R^2 [j_D(\xi) \sin \varphi + \gamma] = 0, \]  
\[ u_{\xi}(\pm1) - \rho h_B + \varphi_{\xi}(\pm1) - R h_B = 0, \]  
\[ -v_{\xi\xi} + R^2 [q(\xi) - \lambda] \psi + \{2R \ [q(\xi) - \lambda] \psi + R^2 j_D,\xi(\xi) \psi \cos \varphi\} \rho = 0, \]
\[ -R^2 j_D(\xi) \psi \sin \varphi u + \psi_{\xi\xi} + R^2 [q(\xi) - \lambda] \psi = 0, \]
\[ v_{\xi}(\pm1) + \psi_{\xi}(\pm1) = 0, \]  
\[ R < \psi, v > + \rho < \psi, \psi > + R < \psi, \psi > -1 = 0, \]  

This system can be solved using the following decomposition:

\[ u = u_1 + \rho u_2, \quad v = v_1 + \rho v_2, \]

where \( u_1(\xi), u_2(\xi), v_1(\xi), v_2(\xi) \) are new unknown functions of \( \xi \). That assumption yields four linear two-point boundary problems with respect to the new introduced functions:

\[ -u_{1\xi\xi} + R^2 q(\xi) u_1 = \varphi_{\xi\xi}(\xi) - R^2 [j_D(\xi) \sin \varphi(\xi) + \gamma] \]
\[ u_{1\xi}(\pm1) = R h_B - \varphi_{\xi}(\pm1), \quad (19) \]
\[ -u_{2\xi\xi} + R^2 q(\xi) u_2 = -2R \ [j_D(\xi) \sin \varphi(\xi) + \gamma] - R^2 j_D,\xi(\xi) \xi \sin \varphi(\xi) \]
\[ u_{2\xi}(\pm1) = h_B, \quad (20) \]
\[ -v_{1\xi\xi} + R^2 [q(\xi) - \lambda] v_1 = \psi_{\xi\xi}(\xi) - R^2 [j_D(\xi) \cos \varphi(\xi) - \lambda] \psi(\xi) + R^2 j_D,\xi(\xi) \psi(\xi) \sin \varphi(\xi) u_1(\xi) \]
\[ v_{1\xi}(\pm1) = -\psi_{\xi}(\pm1), \quad (21) \]
\[-v_{2\xi} + R^2 [q(\xi) - \lambda] v_2 = R^2 j_D(\xi) \psi(\xi) \sin \varphi(\xi) u_2(\xi) - 2R [q(\xi) - \lambda] \psi(\xi) - R^2 j_D,\xi(\xi) \xi \psi(\xi) \cos \varphi(\xi)\]

\[v_{2\xi}(\pm1) = 0.\] (22)

At last the norm condition renders to

\[\rho = \frac{1 - R < \psi, \psi > - 2R < \psi, v_1 >}{< \psi, \psi > + 2R < \psi, v_2 >}.\] (23)

For numerical computation the time derivatives \(u, v\) and \(\rho\) at (13) are discretized by Euler’s method (see details in [3]). For given iteration \(k\), the approximation at the next stage \(k + 1\) is obtained as follows

\[R^{k+1} = R^k + \tau^k \rho^k, \quad \varphi^{k+1} = \varphi^k + \tau^k u^k, \quad \psi^{k+1} = \psi^k + \tau^k v^k.\]

Here \(\tau^k \in (0, 1]\) is a parameter (“time” step) which can be chosen satisfying the condition the residual to be minimal (see [6]).

Let us note that every one of BVP (19) -(22) can be presented simply in the form

\[-y_{\xi\xi} + p(\xi) y = r(\xi),\]

\[y_\xi(\pm1) = y\pm,\] (24)

where \(p(\xi), r(\xi)\) are given functions, and \(y\pm\) are given constants. Further we define an uniform set at the interval \([-1, 1]\) namely \(\xi_j = -1 + jh, \quad j = 0, \ldots, N\), where \(N\) is number of knots, \(h = \frac{2}{N}\) is the step of set. Let \(S(\xi)\) is a cubic spline interpolating the function \(y(\xi)\) over the \(\xi\)-mesh. We assume that \(M(\xi) =: S_{\xi\xi}(\xi)\). Then taking into account the continuity condition concerning the first moments of the spline \(S(\xi)\) and BVP (24), we obtain a three-diagonal algebraic system (see [3]).

The iteration process starts from the initial conditions \(\varphi^0_j = \varphi^0_j, \psi^0_j = \psi^0_j, \quad j = 1, \ldots, N\) and \(R^0 = R^k\), where \(k\) denotes the number of the iteration. We solve consequently the two-point BVP (19) -(22). Thus the values for the grid functions \(u_{1j}^k, u_{2j}^k, v_{1j}^k, v_{2j}^k\) are obtained. Then using (23) we calculate the increment \(\rho\). Hereupon we calculate the increments \(u\) and \(v\) and obtain the predictions for the junction length \(R^{k+1}\) and the grid functions \(\varphi_j^{k+1}, \psi_j^{k+1}\) at the new stage \(k + 1\). The criterion for terminating the iteration is

\[\max_j (\delta \varphi, \delta \psi, \delta R) \leq \varepsilon,
\]

where \(\delta \varphi, \delta \psi\) and \(\delta R\) are the corresponding residuals. We use the norm estimation \(\varepsilon \sim 10^{-8} \div 10^{-12}\).
The numerical correctness of the used scheme is verified through appropriate numerical experiments. We used different meshes with sizes \( N = 256, 512 \) and \( 1024 \). The relative differences for the magnetic flux \( \varphi(x) \), first eigenfunction \( \psi(x) \) do not exceed 0.004% and 0.03%, respectively. We estimated the order of approximation of the obtained solution using the Runge method. The calculations carried out on meshes with spacings \( h = \frac{1}{128}, \frac{1}{256}, \frac{1}{512} \) are presented on the table. It is easy to show the approximate relationship

\[
\frac{z_h - z_{\frac{h}{2}}}{z_{\frac{h}{2}} - z_{\frac{h}{4}}} \approx 2^2
\]

holds at the knots \( 0, \frac{N}{2}, N \). Here \( z = \{ \varphi, \psi, R \} \). On the ground of that comparison we conclude the second-order approximation for the functions \( \varphi(x) \), \( \psi(x) \) and the junction half-length \( R \) is satisfied.

| \( N \) | \( h \) | \( \varphi_0 \equiv \varphi(-R) \) | \( \varphi_{\frac{N}{2}} \equiv \varphi(0) \) | \( \varphi_N \equiv \varphi(R) \) | \( \psi_0 \equiv \psi_{\frac{N}{2}} \equiv \psi(\pm R) \) | \( \psi_{\frac{N}{4}} \equiv \psi(0) \) |
|---|---|---|---|---|---|---|
| 256 | \( \frac{1}{128} \) | 1.1770473 | 3.1415927 | 5.1061380 | 0.4203551 | 0.5178962 |
| 512 | \( \frac{1}{256} \) | 1.1770277 | 3.1415927 | 5.1061576 | 0.4202415 | 0.5177725 |
| 1024 | \( \frac{1}{512} \) | 1.1770244 | 3.1415927 | 5.1061609 | 0.4202198 | 0.5177485 |

All results, stated below, are related to the solutions of kind “main fluxon” (see Fig. 3).

The upper curve on the Fig.4 presents the initial distribution of the magnetic field \( \varphi_x(x) \) alongside the junction. This distribution is chosen to be a solution of BVP (1), (2) for \( R = 5 \), \( h_B = 0 \) and \( \gamma = 0 \). At the same figure the lower curve presents the calculated distribution of the magnetic field corresponding to the minimal eigenvalue \( \lambda_{\text{min}} = 0.01 \). Hence this is the searched bifurcation distribution and in the framework of this model the value of the spectral parameter \( R_{\text{min}} \approx 2.11 \) is the minimal half-length of the junction providing a stable main fluxon. In this sense, junctions, whose length lies below the critical \( R_{\text{min}} \), it is necessary to be considered as “short” for distributions of the kind “main fluxon”. As it was shown\(^2\) for such short length, there is a unique stable distribution of the magnetic flux in the junction - Meissner’s solution\(^3\).

On Fig.5 the distributions of magnetic field \( \varphi_x(x) \) alongside the junction in absence of external current \( (\gamma = 0) \) depending on its boundary values \( h_B = 0 \) and \( h_B = 1 \) respectively, are represented. It is seen that two solutions could be approximated by means of two-degree polynomials. Furthermore they seem to be geometrically similar. Namely the curve \( \varphi_x(x) \) corresponding to magnetic field \( h_B > 0 \) may be considered as obtained from the other curve doing a homothety alongside axis \( x \) with respect to the pole \( x = 0 \) and a translation alongside the axis \( \varphi_x(x) \).

On Fig.6 the obtained relationship \( \lambda_{\text{min}}(h_B, \gamma) = 0 \) is drawn, depending on the length of the trapezium base \( \mu \). It is well noticeable the stabilizing influence of the

\(^2\)This is the trivial solution \( \varphi(x) = 0 \) (stable) and \( \varphi(x) = \pi \) (unstable) for \( h_B = 0 \) and \( \gamma = 0 \).
boundary magnetic field upon the critical length of the junction, i.e., increasing the magnitude of above mentioned quantity one decreases the critical length of junction $2R_{\text{min}}$. Thus the calculated critical half-length of the Josephson junction corresponding to the so called main soliton ($h_B = 0, \gamma = 0$) is $R_{\text{min}} \approx 2.11$ while the same quantity when $h_B = 1$ and $\gamma = 0$ is $R_{\text{min}} \approx 1.35$.

IV. Concluding Remarks

A direct iterative method for obtaining the minimal half-length $R_{\text{min}}$, corresponding to fixed distribution of the magnetic flux in a long Josephson junction is developed. The mere existence of the minimal length is a warranty enough for us to name “long”, every JJ in which there exists at least one nontrivial stable distribution of the magnetic flux for given values of other parameters.

An appropriate linearization based on the continuous analog of Newton’s method renders the original nonlinear spectral problem to four two-point linear BVP. A spline-difference scheme in second order of approximation for solving of these BVP is used.

This method may be applied for solving more general nonlinear problems.

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VI. FIGURE CAPTIONS

Figure 1: a) Geometrical sketch of an inhomogeneous JJ; b) Geometrical model of the amplitude $j_D(x)$ of Josephson current.

Figure 2: The minimal eigenvalue $\lambda_{min}$ as function of the junction length $\Delta = 2R$. 
Figure 3: The magnetic flux $\varphi(x)$, magnetic field $\varphi_x(x)$ and first eigenfunction $\psi(x)$ as functions of the junction length $\Delta = 2R$.

Figure 4: The magnetic field $\varphi_x(x)$ alongside the junction as function of its length $2R$: "∇" - initial distribution; "○" - final (bifurcation) distribution, corresponding to the minimal length.

Figure 5: The magnetic field $\varphi_x(x)$ alongside the junction as function of its length $2R$ for length of the trapezium base $\mu = 1$, bias current $\gamma = 0$ and different values of the boundary magnetic field $h_B$: "○" - $h_B = 0$; "⋄" - $h_B = 1$.

Figure 6: The minimal (bifurcation) eigenvalue $\lambda_{\text{min}}$ as a function of the boundary magnetic field $h_B$ for different lengths of the trapezium base $\mu$: "∇" - $\mu = 0.5$; "△" - $\mu = 1$. 
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