Average contra-rotation and co-rotation of line segments for flow field analysis

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Abstract. The earlier concept of the average co-rotation of infinitesimal radial line segments near a point is extended to the case of contra-rotation. The tensor of the contra-rotation is introduced and averaged over “all planar cross sections” going through the examined point. Both the average contra-rotation and co-rotation, representing shear-free quantities, are applied to describe a complex flow structure.

1. Introduction
The co-rotational approach is based on the analysis of the relative motion near a point which is usually described in terms of the velocity-gradient tensor \([1,2]\). The relevant quantities under consideration are the vorticity tensor (expressing an average angular velocity of fluid elements) and the strain-rate tensor (dealing with deformation of fluid elements). These quantities determine the local flow kinematics and, in the present context, the local co-rotation and contra-rotation of line segments.

The notion of co-rotation of instantaneous infinitesimal radial line segments near a point has been recently introduced in connection with local vortex-identification schemes [3]. At an arbitrary point, the vector of local co-rotation is defined in a planar cross section (of a general 3D flow field) with the orientation of a unit normal vector of the plane. Let us briefly characterize an arbitrary cutting plane at a given point at a given instant of time. The vorticity vector, associated with the cutting plane and aligned with its unit normal vector, represents an average angular velocity of all radial line segments near a point. The vector of local co-rotation is the shear-free collinear part of the vorticity vector given by the least-absolute-value angular velocity of all radial line segments, according to figure 1.

The least-absolute-value angular velocity in figure 1 is a measure of the local rigid-body rotation of a fluid element. The rigid-body rotation is obtained after eliminating the local shearing motion, characterized by the shear vorticity, and quantified by the residual vorticity. This part of vorticity is determined through the corresponding decomposition of the velocity-gradient tensor \(\nabla \mathbf{u}\) proposed in the frame of the triple decomposition of the relative motion near a point [4]. From the vortex-identification viewpoint, the residual vorticity represents a local vortex intensity.

There exists a number of pointwise vortex-identification methods based on the velocity-gradient tensor. The well-known criteria are \(Q\) (Hunt et al. [5]), \(\Delta\) (Chong et al. [6]), \(\lambda_2\) (Jeong and Hussain [7]), and \(\lambda_{ci}\) (Zhou et al. [8]). These criteria are briefly reviewed, for example, in Chakraborty et al. [9].

Though the co-rotational approach discussed below is closely related to vortex identification as well, the main difference lies in the ability to characterize also a vortex counterpart, the shear-free deformation, and even shearing effect. A similar but not the same duality property is, from the popular...
schemes, inherent only to the quantity $Q$ which is just the difference of vorticity and strain-rate tensor magnitudes. The co-rotational approach coincides with the triple decomposition of the relative motion [4] in 2D, distinguishing three elementary motions as indicated in figure 1: rigid-body rotation, shearing motion, and irrotational straining (elongation and/or contraction). However, it differs in 3D.

In 3D, the triple-decomposition method (TDM) becomes computationally expensive. On the other hand, the vortex-identification method based in 3D on the average co-rotation vector [3] provides very similar identification results as the TDM at relatively low computational cost [10]. The aim of the present contribution is to show the algorithm for the determination of (i) the (shear-free) average contra-rotation and (ii) average shear, which is less evident than the algorithm for the determination of the average co-rotation. The new approach motivates an advanced comparison with the TDM.

The application to a wake flow shows the complex flow structure in terms of isosurfaces of magnitudes of the average co-rotation, contra-rotation, and average shear, in direct comparison with the TDM results taken for reference.

Figure 1. Co-rotation and contra-rotation of radial line segments near a point and elementary motions.

2. Contra-rotation of material line segments and its tensor representation
Recall that, at a given point, the vector of average co-rotation of line segments is defined as the average of the instantaneous local rigid-body rotation over “all planar cross sections” going through
the examined point [3]. The procedure consists of two consecutive steps: Firstly, the vector of local co-rotation of material line segments at a point is defined in a plane. This planar concept is directly related to the residual vorticity according to the TDM applied in 2D. Secondly, a proper averaging process is applied to all planar cross sections going through the given point. The averaging procedure is performed as surface integration over a unit sphere, which is approximated using numerical quadrature. Analogous steps — definition in a plane followed by averaging over all planar cross sections — need to be performed to determine the tensor of local contra-rotation and average contra-rotation.

The necessary condition for the local contra-rotation of instantaneous infinitesimal radial line segments in a plane is that the deviatoric strain rate is greater than vorticity (figure 1). While the co-rotation is, according to the TDM, quantified by the residual vorticity, the contra-rotation is quantified by the residual strain rate \( s_{\text{RES}} \). The quantity \( s_{\text{RES}} \) reads for a 2D velocity gradient [4] (the symbol \( s_{\text{SH}} \) stands for the shear strain rate associated with the separated shearing motion)

\[
\begin{align*}
\text{\( s_{\text{RES}} = s_{D} - s_{\text{SH}} = (\text{sgn} \ s_{D}) \left| s_{D} \right| - \left| \omega \right| \text{ for } \left| s_{D} \right| \geq \left| \omega \right| \)} \quad (1a) \\
\text{\( s_{\text{RES}} = s_{D} - s_{\text{SH}} = 0 \text{ for } \left| s_{D} \right| \leq \left| \omega \right| \)} \quad (1b)
\end{align*}
\]

where vorticity \( \omega \) and deviatoric principal strain rate \( s_{D} \) are given by (see figure 1)

\[
\left| s_{D} \right| = \left( \left( s_{x} - s_{y} \right)^2 + \left( s_{y} + s_{x} \right)^2 \right)^{1/2} / 2 .
\]

(3)

For the co-rotation case (figure 1, \( \left| \omega \right| < \left| s_{D} \right| \)), \( s_{D} \) can be expressed in terms of angular velocities as follows

\[
\left| s_{D} \right| = \left( \left| \Omega_{\text{HIGH}} \right| - \left| \Omega_{\text{LOW}} \right| \right) / 2 .
\]

(4)

Alternatively, for the contra-rotation case (figure 1, \( \left| \omega \right| > \left| s_{D} \right| \)), \( s_{D} \) can be expressed as

\[
\left| s_{D} \right| = \left( \left| \Omega_{\text{HIGH}} \right| + \left| \Omega_{\text{LOW}} \right| \right) / 2 .
\]

(5)

Finally, for the shear (\( \left| \omega \right| = \left| s_{D} \right| \)), \( s_{D} \) is given by

\[
\left| s_{D} \right| = \left| \Omega_{\text{HIGH}} \right| / 2 .
\]

(6)

In view of the averaging procedure over a unit sphere and to respect all contributing cross sections we have to construct a tensor representation of the residual strain rate — as a measure of the local contra-rotation — for an arbitrary cross section. This tensor input can then be averaged in the same manner as the vector input in the case of co-rotation [3].

For general angles of spherical coordinates \( \phi \in [0,2\pi] \) and \( \theta \in [0,\pi] \), the sequence of two rotational transformations leads to the final 3x3 rotation matrix (orthogonal linear transformation) [4]

\[
Q = \begin{bmatrix}
\cos \theta & \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\
-\sin \theta & \cos \theta \cos \phi & \cos \theta \sin \phi & \sin \phi \\
\sin \phi \cos \theta & -\sin \phi \sin \theta & \cos \theta & 0
\end{bmatrix} .
\]

(7)

Consider pointwise data input as the general, not necessarily divergence-free, velocity-gradient tensor \( \nabla \mathbf{u} \). The transformation (7) of the original coordinate frame results in the rotated tensor components determined by \( \mathbf{Q} (\nabla \mathbf{u}) \mathbf{Q}^T \). An arbitrarily rotated plane can be identified with the \((x,y)\)-coordinate plane of the rotated frame. The corresponding leading 2x2 submatrix can be extracted by application of the 2x3 restriction matrix \( \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \) on both sides,

\[
\begin{bmatrix}
u_x \\ v_x \\
u_y \\ v_y
\end{bmatrix} = \mathbf{R} \mathbf{Q} (\nabla \mathbf{u}) \mathbf{Q}^T \mathbf{R}^T .
\]

(8)
Let us now focus on the \((x,y)\)-coordinate plane. Assuming transformation of the 2x2 velocity gradient \(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\) to the frame of principal axes of the planar strain rate and subtracting half of the trace from the obtained diagonal entries, we find the deviatoric planar strain rate and residual strain rate in the simple 2x2 diagonal matrix form as

\[
\begin{pmatrix}
  s_D & 0 \\
  0 & -s_D \\
\end{pmatrix} \Rightarrow \begin{pmatrix}
  s_{\text{RES}} & 0 \\
  0 & -s_{\text{RES}} \\
\end{pmatrix}
\]

(9)

where the value of \(s_{\text{RES}}\) is given by \((1a, b)\). Denote the 2x2 matrix of eigenvectors of (8), expressed in the original coordinates \((x,y)\), as \(W_2\). This leads to a planar representation of the contra-rotation as

\[
S_{2,\text{RES}} = W_2 \begin{pmatrix}
  s_{\text{RES}} & 0 \\
  0 & -s_{\text{RES}} \\
\end{pmatrix} W_2^T.
\]

(10)

Here the subscript 2 denotes a 2x2 matrix.

The next step is the search for a proper 3x3 tensor representation of the contra-rotation valid for the original 3D coordinate frame. To this end, the reverse transformation by the matrices \(R\) and \(Q\) is used. In particular, the desired 3x3 contra-rotation tensor \(S_{\text{RES}}\) reads

\[
S_{\text{RES}}(x_0, n) = Q^T R^T \left( S_{2,\text{RES}} \right) R Q.
\]

(11)

### 3. Determination of the average contra-rotation and the scaling factor

We follow the averaging procedure introduced in [3] and apply it to all components of the matrix \(S_{\text{RES}}\). For a given position vector \(x_0\) the averaging procedure is applied over all planar cross sections. The infinite set of all admissible planes can be defined by the infinite set of unit normal vectors which is the same as for a unit sphere \(\sigma(0,1)\). Keeping in mind the key role of a normal \(\mathbf{n}\), the present averaging process translates to an integration over the unit sphere \(\sigma(0,1)\). Consequently, the average contra-rotation tensor at \(x_0\) is defined as

\[
S_{\text{RAVG}}(x_0) = \beta \cdot \int_{\sigma(0,1)} \int_{\sigma(0,1)} S_{\text{RES}}(x_0, n) d\sigma = \frac{\beta}{4\pi} \int_{\sigma(0,1)} \int_{\sigma(0,1)} S_{\text{RES}}(x_0, n) d\sigma
\]

(12)

where \(\beta\) is a scaling factor. A natural choice \(\beta = 5/2\) is derived below. It should be noted that equation (12) represents nine independent surface integrals of scalar functions, one for each component of the average contra-rotation.

A considerable amount of research has been devoted to efficient numerical evaluation of surface integrals over the unit sphere such as equation (12), see the review paper [11]. The approach based on Fibonacci integration from [12] is employed in our computations.

Let us pay attention to the scaling factor \(\beta\). Suppose a given divergence-free strain-rate tensor \(S_D\) should be reproduced by the given averaging procedure. The coefficient is determined from the requirement that for such input \(S_{\text{RAVG}} = S_D\). The substitution from equations (10), (11) and from the degenerate relation \(s_{\text{RES}} = s_D\) (valid due to the purely irrotational input \(S_D\)) into the component-wise integration (12) gives the result \(S_{\text{RAVG}} = 2\beta S_D / 5\) requiring \(\beta = 5/2\) for the reproduction of \(S_D\). A diagonalized input of \(S_D\) makes this evaluation easier.

The average contra-rotation tensor, similarly as the average co-rotation vector, is not affected by a non-zero divergence in the case of compressibility. This fact can be seen from relations (1a, b) dealing with the 2D deviatoric principal strain rate \(s_D\) which is free from any impact of a 3D uniform dilatation. Consequently, without loss of generality and applicability to compressible flows, a non-zero divergence can be removed prior to the analysis of \(\nabla u\); only the deviatoric part of \(\nabla u\) is decisive.
Figure 2. Comparison of isosurfaces for the TDM (top, isovalue from left to right: 1.2, 5.0, 1.2) and for the co-rotation scheme (bottom, isovalue: 1.78, 4.16, 1.26) at $Re=300$.

Figure 3. Comparison of isosurfaces for the TDM (top, isovalue from left to right: 1.2, 8.0, 1.2) and for the co-rotation scheme (bottom, isovalue: 1.88, 7.42, 1.28) at $Re=1200$.

4. Average shear
Let us recall that the average co-rotation vector was introduced in [3] as
\[ \omega_{\text{RAVG}}(x_0) = \frac{3}{4\pi} \int_{\sigma(0,1)} \omega_{\text{RES}}(x_0, \mathbf{n}) \, d\sigma \]  

(13)

where \( \omega_{\text{RES}}(x_0, \mathbf{n}) \) denotes the co-rotation vector. The corresponding average co-rotation tensor reads

\[ \Omega_{\text{RAVG}}(x_0) \equiv \Omega_{\text{RAVG},ij} = -\varepsilon_{ijk} \omega_{\text{RAVG},k} / 2. \]  

(14)

A divergence-free analogue of the TDM [4], \( \nabla \mathbf{u} = \mathbf{S}_{\text{RES}} + \Omega_{\text{RES}} + (\nabla \mathbf{u})_{\text{SH}} \), which separates the shearing effect and the residual vorticity and strain rate, takes in the present context the form

\[ \nabla \mathbf{u} = \mathbf{S}_{\text{RAVG}} + \Omega_{\text{RAVG}} + (\nabla \mathbf{u})_{\text{SAVG}}. \]  

(15)

The first two terms on the right-hand side of equation (15) are already known, so the average shear \((\nabla \mathbf{u})_{\text{SAVG}}\) can be determined in a direct manner. However, the quantity \((\nabla \mathbf{u})_{\text{SAVG}}\) is also obtainable by an analogous averaging procedure applied to both its symmetric and antisymmetric parts.

5. Application of the average co-rotation, contra-rotation and shear

The application deals with an impulsively started incompressible flow around a flat plate (aspect ratio 2) at an angle of attack of 30 deg solved numerically for \( Re=300 \) and \( Re=1200. \) In figures 2 and 3, isosurfaces of the quantities obtained by the TDM are shown for representative isovalues. Here, the symbol \( \mathbf{n} \) denotes the standard Frobenius norm, defined for a tensor \( \mathbf{G} = (G_{ij}) \) as \( \| \mathbf{G} \| = \left[ \text{tr}(\mathbf{GG}^T) \right]^{1/2} \).

For comparison, isosurfaces of their averaged counterparts are presented, and their isovalues are found by the volumetric matching procedure from [3]. A relatively high level of shear (behind the plate edges) is reflected by a high isovalues adopted for both \( Re \).

6. Conclusions

The earlier concept of the average co-rotation of line segments near a point is extended to the case of contra-rotation. The tensor representation of the contra-rotation is introduced and averaged over all planar cross sections going through the examined point. Note that (i) the third elementary motion, shearing, is also “visualized” in terms of the (shear-free) average co-rotation and contra-rotation of line segments, (ii) the volumetric change near a point (compression or expansion) does not affect the co-rotational analysis what results in direct applicability to compressible flows, similarly as the TDM, and (iii) the results obtained for the co-rotational scheme are remarkably similar to those by the TDM.

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