\( \gamma_5 \) in Dimensional Regularization: a Novel Approach

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Abstract: A new Dimensional Regularization of \( \gamma_5 \) is proposed. Cyclicity and Lorentz covariance are enforced. The extension to generic dimension is based on the integral representation of the trace of gamma's, presented in a previous paper.

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1 Introduction

In a previous paper [1] we have introduced an integral representation in $D$ dimensions of the trace of $N$ gamma’s and one $\gamma_\chi$ ($= \gamma_5$ for integer $D = 4$)

$$\text{Tr}(\hat{p}_1 \hat{p}_2 \ldots \hat{p}_{N-1} \hat{p}_N) = \int d^N \bar{c} \exp \left( \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \right)$$

$$\text{Tr}(\hat{p}_1 \hat{p}_2 \ldots \hat{p}_{N-1} \hat{p}_N \gamma_\chi) = \int d^D \xi d^D \bar{c} \exp \left( \sum_{i=1}^{N} \bar{c}_i (p_i)_{\mu} \xi_{\mu} \right) + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j$$

where $\mu = 1 \cdots D$ and $i, j = 1 \cdots N$ and $(p_i p_j) = \sum_{\mu=1}^{D} (p_i)_{\mu} (p_j)_{\mu}$. $\bar{c}_i$ and $\xi_{\mu}$ are real Grassmannian variables [2]; i.e.

$$\int d\xi_{\mu} = 0, \quad \int d\xi_{\mu} \xi_{\nu} = \delta_{\mu\nu}, \quad \int d\bar{c}_i = 0, \quad \int d\bar{c}_i \bar{c}_j = \delta_{ij}$$

{\{p_1, \ldots, p_N\}} are generic vectors (e.g. momenta and polarization vectors) in $D$ dimensions. Finally the normalization factor is chosen to be

$$\gamma_\chi^\dagger = \gamma_\chi, \quad \gamma_\chi^2 = 1$$

for integer $D$. More references on $\gamma_5$ and its use in Dimensional Renormalization can be found in [1].

Simple examples (for integer $D$) of eqs. [1] and [2] can be easily given

$$\text{Tr}(\hat{p}_1 \hat{p}_2) = \int d\bar{c}_2 \ d\bar{c}_1 \exp \left( \bar{c}_1 (p_1, p_2) \bar{c}_2 \right) = (p_1, p_2)$$

$$\Rightarrow \text{Tr}(\mathcal{I}) = 1$$

and for $D = 4$

$$\text{Tr}(\hat{p}_1 \hat{p}_2 \hat{p}_3 \hat{p}_4 \gamma_\chi) = i \frac{D(D-1)}{2} \int d^D \xi d^D \bar{c} \exp \left( \sum_{j=1}^{D} c_j (p_j)_{\mu} \xi_{\mu} \right) + \sum_{i<j=1}^{4} \bar{c}_i (p_i p_j) \bar{c}_j \bigg|_{D=4}$$

$$= \int d^D \xi \left( (p_1)_{\mu} \xi_{\mu} (p_2)_{\nu} \xi_{\nu} (p_3)_{\rho} \xi_{\rho} (p_4)_{\sigma} \xi_{\sigma} \right) \left( - \frac{D(D-1)}{2} \right) \bigg|_{D=4}$$

$$= -\epsilon_{\mu\nu\rho\sigma} (p_1)_{\mu} (p_2)_{\nu} (p_3)_{\rho} (p_4)_{\sigma} \cdot$$

(6)
See Ref. [3] for some early work on the relation between the trace of Dirac matrices and the Pfaffian. For the Pfaffian written as an integral over Grassmannian variables see Ref. [4].

The formulae in eqs. (1) and (2) are nice formal interpolations on different values of the space-time dimensions $D$. The very existence of this integral representation is a hint to search for a consistent management of $\gamma_\chi$ in generic $D$.

In the usual matrix representation to move from $D = 3$ to $D = 4$ one has to redefine $\gamma_5$ i.e. from $\gamma_5 = -i\gamma_1\gamma_2\gamma_3$ to $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$, which are unique for the chosen space dimension. While in eq. (2) the completely antisymmetric tensor emerges from the generic expression of the integration over $d^D\xi$, by taking $D = 3$ or $D = 4$. The integral representation looks as the perfect tool to tackle such a problem of continuation in $D$.

Part I (integer $D$) is devoted to the generalization of eq. (2) to the case of multiple $\gamma_\chi$ factors. Lorentz covariance and Cyclicity are protected in the procedure. Then we consider the mechanism of pairing, i.e. we integrate over the Grassmann variables pertinent to a pair of $\gamma_\chi$. After the mechanism of pairing has removed all the pairs, the trace contains zero or at most one $\gamma_\chi$. Finally we bring the last $\gamma_\chi$ to the far right of the trace: the canonical form.

The procedure will be cast in a set of very simple rules. In particular the algebra for integer values of $D$ is

$$\gamma_\chi\gamma_\mu = -(-)^D\gamma_\mu\gamma_\chi, \quad D \in \mathcal{N},$$

(7)

i.e. the algebra of the standard matrix representation.

Unfortunately this very simple result cannot be extended to generic values of $D$. In fact the algebra

$$\gamma_\chi\gamma_\mu = q\gamma_\mu\gamma_\chi$$

(8)

implies

$$\gamma_\chi\gamma_\mu\gamma_\mu = q^2\gamma_\mu\gamma_\mu\gamma_\chi,$$

(9)

i.e.

$$q^2 = 1,$$

(10)
which forbids to continue eq. (7) to complex $D$.

This negative result is mitigated in some explicit calculations by the fact that it is not necessary to know the explicit form of

$$\left\{ \gamma_\chi, \gamma_\mu \right\};$$

instead one can use

$$\text{Tr}\left( \left\{ \gamma_\chi, \gamma_\mu \right\} \gamma_{\mu_1} \cdots \gamma_{\mu_k} \right) = \text{Tr}\left( \gamma_\chi \left\{ \gamma_\mu, \gamma_{\mu_1} \cdots \gamma_{\mu_k} \right\} \right).$$

On the basis of this assumption (i.e. the existence of an expansion in powers of $(D-4)$) we have tested Dimensional Regularization by explicit calculations: i) in [1] the ABJ anomaly [5] [6] and the invariance of the path integral functional (Local Functional Equation [7] [10]); ii) in [11] the isoscalar anomaly in a non-abelian $SU(2)$ gauge theory [12] [13]; iii) in [14] the Chern-Simons term and photon self-energy in QED with a CPT- and Lorentz-violating action term [15] [16].

In all the listed calculations the technique has been very successful: no ambiguity emerges and the results coincide with those present in the literature, obtained via gauge invariant regulators (Pauli-Villars).

However in all the cases mentioned above the Feynman amplitude is finite for $(D=4)$, i.e. the pole in $D=4$ is cancelled by the zero emerging from the $\gamma$'s algebraic manipulations based on

$$\left\{ \gamma_\mu, \gamma_\nu \right\} = 2\delta_{\mu\nu}. \quad (13)$$

It would be very interesting to extend the representation of $\gamma_5$ to generic $D$ in any situation. For instance in the case of a single (divergent) graph it is very helpful to have a consistent Dimensional Regularization method. This would allow formal algebraic manipulations. A similar situation occurs in the calculation of a divergent Feynman amplitude, where a renormalization procedure is required, possibly via pole subtraction. The aim of the paper is to provide this tool.

In Part II (non-integer $D$) we suggest a novel procedure of Dimensional Renormalization: the pole removal must be performed before the final integration over $\xi$ (any generic Grassmannian variable that generates $\gamma_\chi$ when integrated over). In this way we get rid of the completely antisymmetric tensor in the subtraction procedure and of the problem of its analytic continuation. Moreover the algebra in eq. [17] is implemented when the limit of
integer $D$ is taken. The mechanism of pairing can also be implemented with some simple modifications. Thus the procedure is straightforward down to the final step, when the completely antisymmetric tensor is recovered by the integration over $\xi$.

This procedure can be formalized by introducing the GT for which most of the properties of the conventional trace are valid or require minor adjustments.

In conclusion of the paper the rules will be tested on the example developed in [1] for ABJ anomaly.

## 2 Standard Identities for Integer $D$

We recollect some standard properties of $\gamma_5$.

Let $D$ be integer and odd. Thus we can consider the product

$$\gamma_5 \equiv \prod_{\mu=1}^{D} \gamma_\mu.$$  \hfill (14)

By using the gamma’s algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \forall \mu, \nu = 1 \ldots D$$  \hfill (15)

one gets

$$[\gamma_5, \gamma_\mu] = 0, \quad \forall \mu = 1 \ldots D.$$  \hfill (16)

From Schur’s lemma we get that $\gamma_5$ is a number.

Let $D$ be integer and even. Thus we can consider the product

$$\gamma_5 \equiv \prod_{\mu=1}^{D} \gamma_\mu$$  \hfill (17)

and we get

$$\{\gamma_5, \gamma_\mu\} = 0, \quad \forall \mu = 1 \ldots D.$$  \hfill (18)

Now we recall some further identities obtained from eq. (15)

$$\gamma_\mu \gamma_\rho \gamma_\mu = (-D + 2)\gamma_\rho$$

$$\gamma_\mu \gamma_\rho \gamma_\sigma \gamma_\mu = (D - 4)\gamma_\rho \gamma_\sigma + 4\delta_{\rho\sigma}$$

$$\gamma_\mu \gamma_\mu_1 \cdots \gamma_\mu_k \gamma_\mu = (-)^k(D - 2k)\gamma_\mu_1 \cdots \gamma_\mu_k$$

$$+4(-)^k \sum_{\{ij\}} \delta_{\mu_1 \mu_j} \gamma_{\mu_1} \cdots \hat{\gamma}_{\mu_i} \cdots \hat{\gamma}_{\mu_j} \cdots \gamma_{\mu_k}.$$  \hfill (19)
where ` means omitted and $\delta P$ the parity of the permutations to take them in front.
Part I

**Integer $D$**

3 The Use of Cyclicity

When $\gamma_\chi$ is absent, as in eq. (1), Cyclicity is a property of the trace. The proof is given in Ref. [1].

If $\gamma_\chi$ is introduced as in eq. (2) then Cyclicity can not be formulated, since its position is fixed by its very definition. Instead, if we require this property, then we can extend the expression in eq. (2) to the cases, where the position of $\gamma_\chi$ is generic in the trace expression.

To illustrate this fact let us consider the identity

$$\frac{i^{D(D-1)}}{2} \int d^D \chi d^N \bar{c} \left(\bar{c}_k(p_k p_N)\bar{c}_N\right) \exp \left(\sum_{i=1}^{N} \bar{c}_i(p_i)\mu \chi_\mu\right)$$

$$+ \sum_{i<j} \bar{c}_i(p_ip_j)\bar{c}_j = \delta_P Tr(\hat{\phi}_1 \ldots \{ \hat{\phi}_k, \hat{\phi}_N \} \ldots \hat{\phi}_{N-1} \gamma_\chi), \quad (20)$$

where $\delta_P = (-)^{N-k-1}$ is the parity of the permutations necessary to order the factors in the form:

$$d\bar{c}_N \bar{c}_N d\bar{c}_k \bar{c}_k. \quad (21)$$

Now we use the identity

$$\hat{\phi}_N \hat{\phi}_1 \ldots \hat{\phi}_{N-1} = \sum_{k=1}^{N-1} \hat{\phi}_1 \ldots \{ \hat{\phi}_k, \hat{\phi}_N \} \ldots \hat{\phi}_{N-1} (-)^{k-1}$$

$$+ (-)^{N-1} \hat{\phi}_1 \ldots \hat{\phi}_{N-1} \hat{\phi}_N \quad (22)$$

and sum over $k$ in eq. (20)

$$\frac{i^{D(D-1)}}{2} \int d^D \chi d^N \bar{c} \sum_{k=1}^{N-1} \left[\bar{c}_k(p_k p_N)\bar{c}_N\right] \exp \left(\sum_{i=1}^{N} \bar{c}_i(p_i)\mu \chi_\mu\right)$$

$$+ \sum_{i<j} \bar{c}_i(p_ip_j)\bar{c}_j = (-)^N Tr \left( \hat{\phi}_N \hat{\phi}_1 \ldots \hat{\phi}_{N-1} \gamma_\chi \right)$$

$$+ Tr \left( \hat{\phi}_1 \ldots \hat{\phi}_{N-1} \hat{\phi}_N \gamma_\chi \right). \quad (23)$$
By using Cyclicity one gets
\[
\frac{i^{D(D-1)}}{2} \int d^D \chi d^N \bar{c} \exp \left( \sum_{i=1}^{N-1} \bar{c}_i(p_i) \chi \right) \exp \left( \sum_{i=1}^{N} \bar{c}_i(p_i) \right) + \sum_{i<j} \bar{c}_i(p_i p_j) \bar{c}_j = Tr\left( \phi_1 \ldots \phi_{N-1} \phi_N \gamma_X \right) + (-)^N Tr\left( \phi_1 \ldots \phi_{N-1} \gamma_X \phi_N \right).
\]

Eq. (24) suggests how to represent the trace when $\gamma_X$ is in second position. To illustrate this we consider

\[
\frac{i^{D(D-1)}}{2} \int d\bar{c}_N d^D \chi d^{(N-1)} \bar{c} \exp \left( \sum_{i=1}^{N-1} \bar{c}_i(p_i) \right) + \sum_{i<j} \bar{c}_i(p_i p_j) \bar{c}_j = \int d\bar{c}_N d^D \chi d^{(N-1)} \bar{c} \exp \left( \sum_{i=1}^{N} \bar{c}_i(p_i) \right) + \sum_{i<j} \bar{c}_i(p_i p_j) \bar{c}_j.
\]

Now we use eq. (24) in eq. (25)

\[
\frac{i^{D(D-1)}}{2} \int d\bar{c}_N d^D \chi d^{(N-1)} \bar{c} \exp \left( \sum_{i=1}^{N-1} \bar{c}_i(p_i) \right) + \sum_{i<j} \bar{c}_i(p_i p_j) \bar{c}_j = (-)^D \frac{i^{D(D-1)}}{2} \int d^D \chi d^{N-1} \bar{c} \left( 1 - 2 \sum_{j=1}^{N-1} \bar{c}_j(p_j p_N) \bar{c}_N \right) \exp \left( \sum_{i=1}^{N} \bar{c}_i(p_i) \right) + \sum_{i<j} \bar{c}_i(p_i p_j) \bar{c}_j) = (-)^D \frac{i^{D(D-1)}}{2} \int d^D \chi d^{N-1} \bar{c} \left( 1 - 2 \sum_{j=1}^{N-1} \bar{c}_j(p_j p_N) \bar{c}_N \right) \exp \left( \sum_{i=1}^{N} \bar{c}_i(p_i) \right) + \sum_{i<j} \bar{c}_i(p_i p_j) \bar{c}_j.
\]

Notice that $(-)^{D-N}$ is always equal one. This shows that the integral representation in eq. (2) is correct provided the order of the gamma’s is reproduced in RHS of the equation.
4 More $\gamma_\chi$’s

The generalization of eq. (2) to more than one $\gamma_\chi$ is achieved by following the suggestion of the result in eq. (26) and by the method of unfolding $\gamma_\chi$ into a product of $D$ all different $\gamma$’s for integer dimensions:

$$\gamma_\chi = (i)^{(D-1)D/2} \gamma_1 \cdots \gamma_D. \quad (27)$$

Thus we can write the trace as it were with no $\gamma_\chi$. See Section 6 in Ref. [1] for details.

**Assumption 1:** Multiple $\gamma_\chi$ trace is represented by integration over Grassmannian variables. Each $\not{p}^j$ is associated to the integration variable $d\bar{c}^j$ while $\gamma_\chi$’s are represented by integration over $d^D \chi \cdots d^D \eta \cdots d^D \xi \cdots$. The order in the trace is faithfully reproduced in the order of integration and in the terms entering in the exponential. A factor

$$ (i)^{(D-1)D/2} \quad (28)$$

is introduced for each $\gamma_\chi$.

5 Example with Two $\gamma_\chi$

We illustrate the algorithm in the case of two $\gamma_\chi$.

\[
\begin{align*}
\text{Tr}(\not{p}_1 \cdots \not{p}_k \gamma_\chi \cdots \not{p}_N \gamma_\chi) &= (-)^{(D-1)D/2} \int d^D \eta \ d^{(N-k)} \bar{c} \ d^D \xi d^k \bar{c} \\
& \exp \left( \sum_{i=1}^{N} \bar{c}_i (p_i)_\mu \eta_\mu + \xi_\mu \eta_\mu + \xi_\mu \sum_{i=k+1}^{N} \bar{c}_i (p_i)_\mu + \sum_{i=1}^{k} \bar{c}_i (p_i)_\mu \xi_\mu + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \right) \\
&= (-)^{(D-1)D/2} \int d^D \eta \ d^{(N-k)} \bar{c} \ d^D \xi d^k \bar{c} \\
& \exp \left( \sum_{i=1}^{N} \bar{c}_i (p_i)_\mu \eta_\mu + \xi_\mu \left[ \eta_\mu + \sum_{i=k+1}^{N} \bar{c}_i (p_i)_\mu - \sum_{i=1}^{k} \bar{c}_i (p_i)_\mu \right] + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \right). \quad (29)
\end{align*}
\]

Now we perform the integration over both $\xi, \eta$. The procedure is to force the integration over $\xi, \eta$ by an isolated term $\xi \eta$ in the exponential. Only the $\xi_\mu$
in \( \xi \eta \) can saturate the integration over \( \xi_\mu \) which has a factor \( \eta_\mu \). Therefore also the integration over \( \eta \) is constrained

\[
\int d^D\eta d^D\xi \, e^{\xi \eta} = \int d^D\eta d^D\xi \prod_\mu (1 + \xi_\mu \eta_\mu)
= \int d^D\eta d^D\xi \prod_\mu \xi_\mu \eta_\mu = (-)^{\frac{D(D-1)}{2}}.
\] (30)

In order to employ the procedure of eq. (30) we replace in eq. (29)

\[
\eta \to \eta - \sum_{i=k+1}^{N} \tilde{c}_i(p_i) + \sum_{i=1}^{k} \tilde{c}_i(p_i)
\] (31)

and we get

\[
\text{Tr}(\not{p}_1 \cdots \not{p}_k \gamma_\chi \cdots \not{p}_N \gamma_\chi) = (-)^{\frac{D(D-1)}{2}} \int d^D\eta \, d^D\xi \prod_\mu (\xi_\mu \eta_\mu)
\]

\[
\exp \left( \sum_{i=1}^{N} \tilde{c}_i(p_i) \eta_\mu - 2 \sum_{i=1}^{k} \tilde{c}_i(p_i) \right) \left( \sum_{i=k+1}^{N} \tilde{c}_i(p_i) \right)
+ \xi \eta + \sum_{i<j=1}^{N} \tilde{c}_i(p_i p_j) \tilde{c}_j
\] (32)

Now the integration over \( \xi \eta \) can be performed

\[
\text{Tr}(\not{p}_1 \cdots \not{p}_k \gamma_\chi \cdots \not{p}_N \gamma_\chi) = (-)^{(D-1)(N-k)} (-)^{(D(N-k))} (-)^{\frac{D(D-1)}{2}}
\]

\[
\int d^N \tilde{c} \exp \left( - \left[ \sum_{i=1}^{k} \tilde{c}_i(p_i) \right] \left( \sum_{i=k+1}^{N} \tilde{c}_i(p_i) \right) \right)
+ \sum_{i<j=1}^{k} \tilde{c}_i(p_i p_j) \tilde{c}_j + \sum_{i<j=k+1}^{N} \tilde{c}_i(p_i p_j) \tilde{c}_j
\] (33)

Finally we change sign to \( \tilde{c}_j \), \( j = k+1, \ldots, N \).

\[
\text{Tr}(\not{p}_1 \cdots \not{p}_k \gamma_\chi \cdots \not{p}_N \gamma_\chi) = (-)^{(D-1)(N-k)}
\]

\[
\int d^N \tilde{c} \exp \left( \left[ \sum_{i=1}^{k} \tilde{c}_i(p_i) \right] \left( \sum_{i=k+1}^{N} \tilde{c}_i(p_i) \right) + \sum_{i<j=1}^{k} \tilde{c}_i(p_i p_j) \tilde{c}_j + \sum_{i<j=k+1}^{N} \tilde{c}_i(p_i p_j) \tilde{c}_j \right)
= (-)^{(D-1)(N-k)} \text{Tr}(\not{p}_1 \cdots \not{p}_N \gamma_\chi^2). \] (34)

In particular for \( k = N \) we get

\[
\text{Tr}(\not{p}_1 \cdots \not{p}_N \gamma_\chi^2) = \text{Tr}(\not{p}_1 \cdots \not{p}_N). \] (35)
Notice that the result in eqs. (34) and (35) coincides with the standard algebra of the gamma matrices (integer dimensions).

This integration over a couple of variables describing a pair of \( \gamma \) will be denoted as pairing.

We shall provide more examples involving three \( \gamma \).

6 Example with three \( \gamma \chi \)

It is very instructive to consider a case with an odd number of \( \gamma \). After the use of the tool of pairing, only one \( \gamma \) is left over at the end.

We consider the explicit example where three \( \gamma \) are present in the trace. The order among the Grassmannian variable follows faithfully the order inside the trace as in eq. (29)

\[
\text{Tr}(\not{p}_1 \not{p}_2 \cdots \not{p}_{N-2} \not{p}_{N-1} \gamma \chi \not{p}_N) = (-i)^{(D-1)} \int d^D \chi d^D \xi d^D \bar{\xi} d^D \eta d^D \bar{\eta} \exp \left\{ \sum_{i=1}^{N} \bar{\xi}_i(p_i) \right\}.
\]

The question arises on the relations among the different paths of pairings. Thus we explore some possibilities. The calculation is straightforward and similar to the previous leading to eq. (34).

6.1 Integration over \( \chi_\mu \) and \( \eta_\mu \)

If we want to perform the pairing by integration over \( \chi_\mu \) and \( \eta_\mu \), it is easier to isolate the factor \( \sum_\mu \eta_\mu \chi_\mu \) so that we get rid of all the integration \( d^D \chi d^D \eta \). This is achieved by the substitution \( \eta_\mu \rightarrow \eta_\mu - \xi_\mu - \sum_{i=1}^{N} \bar{\xi}_i(p_i)_\mu \). Then one can show that

\[
\text{Tr}(\not{p}_1 \not{p}_2 \cdots \not{p}_{N-2} \not{p}_{N-1} \gamma \chi \not{p}_N) = (-1)^{(D-1)} \text{Tr}(\not{p}_1 \not{p}_2 \cdots \not{p}_{N-2} \not{p}_{N-1} \gamma \chi \not{p}_N).
\]
6.2 Integration over $\chi_{\mu}$ and $\xi_{\mu}$

For the integration over $d^D\chi d^D\xi$ we proceed as in the previous case: we isolate a term $\xi\chi$ by a substitution $\xi \rightarrow \xi - \eta - \sum_{i=1}^{N} \bar{c}_i p_i$. Explicit calculation yields

$$Tr(\hat{p}_1 \hat{p}_2 \cdots \hat{p}_{N-2} \hat{p}_{N-1} \gamma_{\chi} \hat{p}_{N} \gamma_{\chi}) = (-)^{D-1} Tr(\hat{p}_1 \hat{p}_2 \cdots \hat{p}_{N-2} \hat{p}_{N-1} \hat{p}_{N} \gamma_{\chi}).$$ (38)

6.3 Integration over $\xi \eta$

Finally we consider the last pairing in eq. (36) by integrating over $d^D\xi d^D\eta$. We isolate a common factor $\exp \eta\chi$ by using the substitution $\eta_{\mu} \rightarrow \eta_{\mu} + \chi_{\mu} + \bar{c}_N (p_N)_{\mu} - \sum_{i=1}^{N-1} \bar{c}_i (p_i)_{\mu}$. One gets

$$Tr(\hat{p}_1 \hat{p}_2 \cdots \gamma_{\chi} \hat{p}_{N-2} \hat{p}_{N-1} \gamma_{\chi} \hat{p}_{N} \gamma_{\chi}) = (-)^{2(D-1)} Tr(\hat{p}_1 \hat{p}_2 \cdots \hat{p}_{N-2} \hat{p}_{N-1} \hat{p}_{N} \gamma_{\chi}).$$ (39)

We see that, in all the example presented, pairing is obtained by using the naive algebra

$$\gamma_{\chi} \gamma_{\mu} = (-)^{D-1} \gamma_{\mu} \gamma_{\chi}$$

$$\gamma_{\chi}^2 = 1.$$ (40)

The final result of the pairing process is independent from the chosen sequence, as one can verify by using eqs. (37), (38), (39) and (40). One can prove that the properties in eq. (40) are true for any set of $\gamma$ and $\gamma_{\chi}$ for integer dimensions $D$. Moreover under the same conditions one can prove Cyclicity.

7 Algebra and Cyclicity in General for Integer $D$

One can prove that the properties in eq. (40) are true for any set of $\gamma_{\mu}$ and $\gamma_{\chi}$ for integer dimensions $D$. Moreover under the same conditions one can prove Cyclicity

$$Tr(\hat{p}_1 A) = Tr(A \hat{p}_1),$$ (41)

where $A$ is any product of $\hat{p}$ and $\gamma_{\chi}$.

Similarly one has

$$Tr(\gamma_{\chi} A) = Tr(A \gamma_{\chi}).$$ (42)
Part II

Generalized Trace (GT)

In the Introduction it was argued that in the equations $D$ cannot be continued to complex values. Moreover the completely antisymmetric tensor is identified by the number of the dimensions. Thus it is a quantity that cannot depend smoothly on $D$. It is the insurmountable obstacle for any continuation in $D$.

In this paper we suggest a way out to this impasse. We take full advantage of the integral representation of the trace in eq. (2) and its generalizations with more than one $\gamma_{\chi}$ discussed in Part I.

We separate the integration over the variables generating the $\gamma_{\chi}$ from those producing the trace without $\gamma_{\chi}$ denoted by $\bar{c}$. We take careful booking of the Jacobian $\delta R$ generated by this rearrangement of the differentials. Typically a factor $(-1)$ to some power depending on $D$ and possibly on $N$.

Thus we can identify a new object: the GT denoted by $R$ defined by the integral over $d^N \bar{c}$. $R$ depends on the momenta and polarization vectors and the Grassmann variables $\xi, \ldots$ associated to the $\gamma_{\chi}$’s.

The new renormalization procedure is described by the following assumption

Assumption 2: The Dimensional subtractions have to be performed on the GT, given by the expression in eq. (43), i.e. before the final integration over $d^D \xi$ . . . is performed.

For one $\gamma_{\chi}$ the GT is

$$R(p_1 \ldots p_{N-1} \ p_N|\xi) = \int d^N \bar{c} \exp \left( \sum_{i=1}^{N} \bar{c}_i (p_i)_\mu \xi^\mu \right)$$

$$+ \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j$$

(43)

i.e. we drop the integration over $d^D \xi$ and consequently the GT becomes function of the Grassmannian real variable $\xi$. In presence of two $\gamma_{\chi}$’s one has a similar expression

$$R(p_1 \ldots p_{N-2}|\eta) \ p_{N-1} \ p_N|\xi) = \int d^N \bar{c}$$
\[
\exp \left( \sum_{i=1}^{N} \bar{c}_i p_i \xi + \eta \xi + \sum_{i=1}^{N} \bar{c}_i (p_i p_i) \right). \tag{44}
\]

According to the prescription, after all pole subtractions have been performed the limit to the required integer is performed by the final integration on \((i) \frac{D(d-1)}{2} d^D \xi\) in eq. (43) or \((-1)^{\frac{D(d-1)}{2}} d^D \xi (-\xi)^{2D} d^D \eta\) in eq. (45).

### 8 Properties of the GT \(R\)

The rules (3) of Grassmann integration requires that the single \(\bar{c}_j\) appears only once in the integrands of eqs. (43) and (44). Then the GT is a linear function of \(p_j\), for each \(j = 1, \ldots, N\).

#### 8.1 Expansion in Powers of \(\xi\)

Let us develop the formalism with a limited number of \(\gamma\)'s present in the trace.

The GT has no explicit dependence on \(D\). Thus we can expand it in powers of \(\xi\) by using

\[
\exp \left( \sum_{i=1}^{N} \bar{c}_i (p_i) \xi \right) = \prod_{i=1}^{N} e^{\bar{c}_i (p_i) \xi} = \prod_{i=1}^{N} (1 + \bar{c}_i (p_i) \xi) \\
= 1 + \sum_{i=1}^{N} \bar{c}_i (p_i) \xi + \sum_{i<j}^{N} \bar{c}_i (p_i) \bar{c}_j (p_j) \xi + \sum_{i<j<k}^{N} \bar{c}_i (p_i) \bar{c}_j (p_j) \bar{c}_k (p_k) \xi + \ldots. \tag{45}
\]

The integration on \(d^N \bar{c}\) is partly on the chosen terms of the expansion (45) and the rest on the Taylor expansion of the other exponential factor in eq. (43). These last terms are bilinear in \(\bar{c}\), thus the power of \(\xi\) is even or odd depending on the value of \(\text{mod}(N, 2) = 0\) or \(\text{mod}(N, 2) = 1\).

#### 8.2 Fundamental Formula

After the integration on \(\bar{c}\) present in the eq. (43), a typical expansion in terms of powers of \(\xi\) is given by

\[
R(\mathbf{\hat{p}}_1 \mathbf{\hat{p}}_2 \cdots \mathbf{\hat{p}}_{N-1} \mathbf{\hat{p}}_N | \xi) = \sum_{\mathcal{P}} \delta_p \text{Tr}(\mathbf{\hat{p}}_{i_1} \cdots \mathbf{\hat{p}}_{i_{N-K}} | p_{j_1}, \xi) \cdots (p_{j_K}, \xi) \tag{46}
\]
where the sum is over all partitions $\mathcal{P}$ of the $N$ integers in two mutually disjoint ordered sets $(i_1, \ldots, i_{N-K})$ and $(j_1, \ldots, j_K)$. The parity $\delta_P$ counts the permutations needed to perform the integrations over $d\tilde{c}_{j_1}, \ldots, d\tilde{c}_{j_K}$.

The quantity in eq. (46) is the perfect tool to be continued in $D$.

### 8.3 The Algebra Represented in $R$

It is worth checking how the algebra $\{\hat{p}_i, \hat{p}_j\} = 2(p_i, p_j)$ is implemented on the GT. By following the same argument in Ref. [1] Sections 2 and 3 one can show that the gamma’s Clifford algebra is represented on the GT by

$$R(\{\hat{p}_1, \hat{p}_2\} \hat{p}_3 \ldots \hat{p}_{N-1} \hat{p}_N | \xi) = 2(p_1, p_2)R(\hat{p}_3 \ldots \hat{p}_{N-1} \hat{p}_N | \xi).$$

(47)

It is interesting to study the above equation when projected on a definite partition of the integer $3, \ldots, N$. For each partition there is a term composed of factors of the form $(p_j, \xi)$, as shown in eq. (46). Thus eq. (47) tells that the coefficients of the left- and right-hand side must be equal. Finally we conclude that the relation expressing the Clifford algebra is valid for the conventional trace and the generalized one.

### 8.4 Algebra of the $\gamma$’s on $R$: Cyclicitiy

Consider again

$$R(\hat{p}_1 \hat{p}_2 \ldots \hat{p}_{N-2} \hat{p}_{N-1} | \xi, \hat{p}_N) = \int d^N \tilde{c}$$

$$\exp \left( \xi p_N \tilde{c}_N + \left( \sum_{i=1}^{N-1} \tilde{c}_i p_i \right) \xi + \sum_{i<j=1}^{N} \tilde{c}_i (p_i p_j) \tilde{c}_j \right)$$

(48)

and rename

$$\tilde{c}_N \to -\tilde{c}_1$$

$$\tilde{c}_j \to \tilde{c}_{j+1}, \quad j = 1, \ldots, N-1.$$

(49)

Then

$$\sum_{i<j=1}^{N} \tilde{c}_i (p_i p_j) \tilde{c}_j = \sum_{i<j=1}^{N-1} \tilde{c}_i (p_i p_j) \tilde{c}_j + \sum_{i=1}^{N-1} \tilde{c}_i (p_i p_N) \tilde{c}_N$$

$$\to \sum_{i<j=1}^{N-1} \tilde{c}_{i+1} (p_i p_j) \tilde{c}_{j+1} - \sum_{i=1}^{N-1} \tilde{c}_{i+1} (p_i p_N) \tilde{c}_1$$

$$d^N \tilde{c} \to (-)^N d^N \tilde{c}.$$

(50)
We get

\[
R(\phi_1 \phi_2 \ldots \phi_{N-2} \phi_{N-1} | \xi | \phi_N) = (-)^N \int d^N \bar{c} \\
\exp \left( \bar{c}_1 p N \xi + \left( \sum_{i=2}^{N} \bar{c}_i p_{i-1} \right) \xi + \sum_{i<j=2}^{N} \bar{c}_i (p_{i-1} p_{j-1}) \bar{c}_j + \sum_{i=2}^{N} \bar{c}_i (p_i p_{i-1}) \bar{c}_i \right)
= (-)^N R(\phi_N \phi_1 \phi_2 \ldots \phi_{N-2} \phi_{N-1} | \xi)
\]

(51)

\underline{Anti-Cyclicity for N odd.}

Cyclicity is working also on \( \xi \). Let us consider

\[
R(\xi | \phi_1 \phi_2 \ldots \phi_{N-2} \phi_{N-1} \phi_N)
= \int d^N \bar{c} \exp \left( \xi \left( \sum_{i=1}^{N} \bar{c}_i p_i \right) + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \right)
= (-)^N \int d^N \bar{c} \exp \left( -\xi \left( \sum_{i=1}^{N} \bar{c}_i p_i \right) + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \right)
= (-)^N R(\phi_1 \phi_2 \ldots \phi_{N-2} \phi_{N-1} \phi_N | \xi),
\]

(52)

where we have performed the change of variable \( \bar{c}_j \rightarrow -\bar{c}_j, \quad j = 1, \ldots, N \).

Or, by keeping the sign of \( \xi \) (no change of variable on \( \xi \) to avoid the Jacobian!)

\[
R(\xi | \phi_1 \phi_2 \ldots \phi_{N-2} \phi_{N-1} \phi_N)
= \int d^N \bar{c} \exp \left( \xi \left( \sum_{i=1}^{N} \bar{c}_i p_i \right) + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \right)
= \int d^N \bar{c} \exp \left( \left( \sum_{i=1}^{N} \bar{c}_i p_i \right) (-\xi) + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \right)
= R(\phi_1 \phi_2 \ldots \phi_{N-2} \phi_{N-1} \phi_N | -\xi),
\]

(53)

In a more complicated situation one has

\[
R(\xi | \phi_1 \phi_2 \ldots \phi_{N-2} \phi_{N-1} | \eta | \phi_N)
= \int d^N \bar{c} \exp \left( \xi \left( \sum_{i=1}^{N} \bar{c}_i p_i \right) + \eta \bar{c}_N p_N + \left( \sum_{i=1}^{N-1} \bar{c}_i p_i \right) \eta + \sum_{i<j=1}^{N-1} \bar{c}_i (p_i p_j) \bar{c}_j \right)
= \int d^N \bar{c} \exp \left( \left( \sum_{i=1}^{N} \bar{c}_i p_i \right) (-\xi) + \eta (-\xi) + \eta \bar{c}_N p_N + \left( \sum_{i=1}^{N-1} \bar{c}_i p_i \right) \eta + \sum_{i<j=1}^{N-1} \bar{c}_i (p_i p_j) \bar{c}_j \right)
= R(\phi_1 \phi_2 \ldots \phi_{N-2} \phi_{N-1} | \eta | \phi_N | -\xi).
\]

(54)

This result can be proven valid for general cases.
8.5 Pairings on $R$: $\gamma_x$ & $\gamma_x$

We now investigate on the possibility of using the pairing mechanism on the GT. We consider

$$
Tr(\not p_1 \not p_2 ... \not p_{N-2} \not p_{N-1} \gamma_x \not p_N \gamma_x) = (-)^{D(D-1)/2} \int d^D \xi d^D \eta d^{N-1} \bar{c} \exp \left( \sum_{j=1}^{N} \bar{c}_j p_j \xi + \eta \xi + \eta \bar{c}_N p_N \right)
$$

$$+
\sum_{j=1}^{N-1} \bar{c}_j p_j \eta + \sum_{i<j} \bar{c}_i(p_i,p_j) \bar{c}_j \right). \quad (55)
$$

Now we separate the $\bar{c}$ integration form those over $\xi$ and $\eta$. Thus we isolate the $R$ trace

$$
Tr(\not p_1 \not p_2 ... \not p_{N-2} \not p_{N-1} \gamma_x \not p_N \gamma_x) = (-)^D (-)^D \int d^D \xi d^D \eta \int d^N \bar{c} \exp \left( \sum_{j=1}^{N} \bar{c}_j p_j \xi + \eta \xi + \eta \bar{c}_N p_N \right)
$$

$$+
\sum_{j=1}^{N-1} \bar{c}_j p_j \eta + \sum_{i<j} \bar{c}_i(p_i,p_j) \bar{c}_j \right). \quad (56)
$$

Thus we consider

$$
R(\not p_1 \not p_2 ... \not p_{N-2} \not p_{N-1} | \not p_N | \gamma_x) \equiv \int d^N \bar{c} \exp \left( \sum_{j=1}^{N} \bar{c}_j p_j \xi + \eta \xi + \eta \bar{c}_N p_N \right)
$$

$$+
\sum_{j=1}^{N-1} \bar{c}_j p_j \eta + \sum_{i<j} \bar{c}_i(p_i,p_j) \bar{c}_j \right) \quad (57)
$$

for pairing. We have

$$
R(\not p_1 \not p_2 ... \not p_{N-2} \not p_{N-1} | \not p_N | \gamma_x) \equiv \int d^N \bar{c} \exp \left( \sum_{j=1}^{N} \bar{c}_j p_j + \eta \right)
$$

$$\left[ \xi + \bar{c}_N p_N - \sum_{j=1}^{N-1} \bar{c}_j p_j \right] - 2 \sum_{j=1}^{N-1} \bar{c}_j p_j \bar{c}_N p_N + \sum_{i<j} \bar{c}_i(p_i,p_j) \bar{c}_j \right) \quad (58)
$$

By the substitution principle we get

$$
R(\not p_1 \not p_2 ... \not p_{N-2} \not p_{N-1} | \not p_N | \gamma_x) = e^{\eta \xi} \int d^N \bar{c} \exp \left( -2 \sum_{j=1}^{N-1} \bar{c}_j p_j \bar{c}_N p_N + \sum_{i<j} \bar{c}_i(p_i,p_j) \bar{c}_j \right) \quad (59)
$$
Finally, with \( p_N \to -p_N \) we get

\[
R(\phi_1, \ldots, \phi_{N-1}|\eta, \phi_N|\xi) = e^{\eta \xi} Tr(\phi_1, \ldots, \phi_{N-1}, -\phi_N). \tag{60}
\]

### 8.6 Pairings on \( R \): \( \gamma^x \& \gamma^2_x \)

\[
Tr(\phi_1, \phi_2, \ldots, \phi_{N-2}, \phi_{N-1}\gamma_x, \phi_N\gamma^2_x)
\]

\[
= (-i) \frac{D(D-1)}{2} \int d^D \eta d^D c d\bar{c}_N d^D \xi d^{N-1} \bar{c} \exp \left( \left( \sum_{i<j} \bar{c}_j p_j + \xi \right) \eta + \chi \right)
\]

\[
+ \chi \eta + \xi \bar{c}_N p_N + \sum_{i<j} \bar{c}_j p_j + \sum_{i<j} \bar{c}_i (p_i, p_j) \bar{c}_j \right).
\]

We isolate the factor \( e^{\eta \chi} \)

\[
Tr(\phi_1, \phi_2, \ldots, \phi_{N-2}, \phi_{N-1}\gamma_x, \phi_N\gamma^2_x)
\]

\[
= (-i) \frac{D(D-1)}{2} \int d^D \eta d^D c d\bar{c}_N d^D \xi d^{N-1} \bar{c} \exp \left( \left( \sum_{i<j} \bar{c}_j p_j + \xi \right) \eta + \chi \right)
\]

\[
[\eta - \sum_{i<j} \bar{c}_j p_j - \xi] + \xi \bar{c}_N p_N + \sum_{i<j} \bar{c}_j p_j + \sum_{i<j} \bar{c}_i (p_i, p_j) \bar{c}_j \right)
\]

\[
= (-i) \frac{D(D-1)}{2} (-1)^D \int d^D \eta d^D c d^D \xi d^{N-1} \bar{c} \exp \left( \xi \bar{c}_N p_N + \sum_{i<j} \bar{c}_j p_j + \sum_{i<j} \bar{c}_i (p_i, p_j) \bar{c}_j \right).
\]

The GT is then

\[
R(\phi_1, \phi_2, \ldots, \phi_{N-2}, \phi_{N-1}|\gamma|\phi_N|\chi|\eta) = e^{\eta \chi} \int d^N \bar{c}
\]

\[
\exp \left( \xi \bar{c}_N p_N + \sum_{i<j} \bar{c}_j p_j + \sum_{i<j} \bar{c}_i (p_i, p_j) \bar{c}_j \right)
\]

\[
= e^{\eta \chi} R(\phi_1, \phi_2, \ldots, \phi_{N-2}, \phi_{N-1}|\gamma|\phi_N). \tag{63}
\]

If we want to factor out \( e^{\xi \chi} \), from eq. \( \text{(61)} \)

\[
Tr(\phi_1, \phi_2, \ldots, \phi_{N-2}, \phi_{N-1}\gamma_x, \phi_N\gamma^2_x)
\]

\[
= (-i) \frac{D(D-1)}{2} \int d^D \eta d^D c d\bar{c}_N d^D \xi d^{N-1} \bar{c} \exp \left( \left( \sum_{i<j} \bar{c}_j p_j - \eta + \xi \right)
\]

\[
\exp \left( \xi \bar{c}_N p_N + \sum_{i<j} \bar{c}_j p_j + \sum_{i<j} \bar{c}_i (p_i, p_j) \bar{c}_j \right)
\]

\[
= e^{\eta \chi} R(\phi_1, \phi_2, \ldots, \phi_{N-2}, \phi_{N-1}|\gamma|\phi_N).
\]
\[
\chi + \eta + \bar{c}_N p_N - \sum_{1}^{N-1} \bar{c}_j p_j - 2\left( \sum_{1}^{N-1} \bar{c}_j p_j - \eta \right) \bar{c}_N p_N + \sum_{1}^{N} \bar{c}_j p_j \eta + \sum_{i<j}^{N-1} \bar{c}_i (p_i, p_j) \bar{c}_j
\]

\[
= (-i)^{\frac{D(D-1)}{2}} (-)^D \int d^D \eta d^D \chi d^D \xi \exp \left( - \sum_{1}^{N-1} \bar{c}_j p_j \bar{c}_N p_N - \bar{c}_N p_N \eta + \sum_{1}^{N-1} \bar{c}_j p_j \eta + \sum_{i<j}^{N-1} \bar{c}_i (p_i, p_j) \bar{c}_j \right). \quad (64)
\]

The GT is then

\[
R(p_1, p_2, \ldots, p_{N-2}, p_{N-1} | \xi, \eta) = e^{\xi \chi} \int d^N \bar{c} \exp \left( - \sum_{1}^{N-1} \bar{c}_j p_j \bar{c}_N p_N - \bar{c}_N p_N \eta + \sum_{1}^{N-1} \bar{c}_j p_j \eta + \sum_{i<j}^{N-1} \bar{c}_i (p_i, p_j) \bar{c}_j \right).
\]

\[
= e^{\xi \chi} R(p_1, p_2, \ldots, p_{N-2}, p_{N-1} | -, \eta) . \quad (65)
\]

### 8.7 Formula of Pairing in GT

We consider the most general pairing setup. We evaluate

\[
Tr\left( A \gamma \chi B \gamma \chi \right) = (i)^{(D-1)D} \int d^D \xi d^D \eta d^A \exp \left( (B + \eta + A) \xi + B * B + (\eta + A) B + A \eta + A * A \right) \quad (66)
\]

where \( A = \{a_1, \ldots, a_K\} \) and \( B = \{b_1, \ldots, b_M\} \) are sets of elements \( \bar{c}_i (p_i) \mu \) and \( \xi_\mu \) all mutually different. The traces and products are defined by

\[
Tr\left( A \right) = Tr\left( \ldots, \gamma_j, \ldots \right)
\]

\[
A * A = \sum_{i<j=1}^{K} (a_i, a_j)
\]

\[
A \xi = \sum_{i=1}^{K} (a_i, \xi).
\]

The differentials \( dA \) are defined by the product of \( d\bar{c}_i \) and \( d^D \xi \) according to the order in \( A \). Additional factors (as \( (i)^{(D-1)D} \) to \( d^D \xi \) for extra insertions of \( \gamma \chi \) will be resumed when the GT is extracted from the expression of the conventional trace.

Now we proceed to factor out the exponential \( e^{\eta \chi} \)

\[
Tr\left( A \gamma \chi B \gamma \chi \right) = (-i)^{(D-1)D} \int d^D \xi d^D \eta d^A
\]
\[
\exp \left( \left[ B + A + \eta \right] \left[ \xi + B - A \right] - 2AB + B * B + AB + A * A \right) \tag{68}
\]

Then we use the substitution principle and get

\[
Tr \left( A \gamma_\chi B \gamma_\chi \right) = (i)^{(D-1)D} \int d^D \xi d^D \eta dA e^{\eta \xi} \exp \left( B * B + BA + A * A \right), \tag{69}
\]

by changing the sign of all elements of \( B \). In terms of GT we have

\[
R \left( A|\eta|B|\xi \right) \simeq e^{\eta \xi} R \left( A(-B) \right) \tag{70}
\]

where the sign \( \simeq \) means that equality will be achieved after the integration over \( d^D \xi d^D \eta \). Eq. (70) shows that the elements of \( B \) encapsulated between two \( \gamma_\chi \) change sign in the process of pairing. The identity (70) should be carefully extracted from (69), in particular the same Jacobian and the same differentials in the same order have to be used in both sides of eq. (69). Moreover no explicit dependence on \( D \) should appear in eq. (70).

The result in eq. (70) is very surprising and it allows to remove all \( \gamma_\chi \)'s from the trace: only one or none is left over at the end of the process of pairing.

### 9 Ward Identity

It is very interesting to see what is the destiny of Ward identities in the present formalism. We consider the case (tree level \( D = 4 \))

\[
(p - \bar{p})_\mu \gamma_\mu \gamma_5 = (\bar{p} - m) \gamma_5 + \gamma_5 (\bar{p} - m) + 2m \gamma_5. \tag{71}
\]

Then we have to compare traces like

\[
Tr \left( A p \gamma_\chi \right) \tag{72}
\]
\[
Tr \left( A \gamma_\chi \bar{p} \right) \tag{73}
\]

in the framework of the GT's. We can always consider the situation where \( A \) has no \( \gamma_\chi \) inside. Let us move to GT's. We can write the expression in
eq. (72)

\[
\text{Tr}(A \not{p} \gamma \chi) = (i)^{(D-1)D/2} \int d^D \xi \, d\bar{c} \, dA \exp \left( (A + \bar{c} p) \xi + A \bar{c} p + A \star A \right).
\]

(74)

The corresponding GT is then

\[
R(A \not{p} \xi) = \int d\bar{c} \, dA \exp \left( (A + \bar{c} p) \xi + A \bar{c} p + A \star A \right).
\]

(75)

Similarly we can write the expression in eq. (73)

\[
\text{Tr}(A \not{p} \chi) = (i)^{(D-1)D/2} \int d^D \xi \, dA \exp \left( (A + \bar{c} \not{p}) \xi + A \bar{c} \not{p} + A \star A \right),
\]

(76)

The corresponding GT is

\[
R(A | \xi | \not{p}) = \int d\bar{c} \, dA \exp \left( (A + \bar{c}) \xi + A \bar{c} + A \star A \right).
\]

(77)

In order establish the relation between the traces in eqs. (72) and (73) we introduce a partition of unity

\[
\text{Tr}(A \not{p} \gamma \chi) = \text{Tr}(A \gamma \chi \not{p})
\]

\[
= (i)^{(D-1)D/2} (-)^{(D-1)D} \int d^D \xi \, d\bar{c} \, d^D \chi \, d^D \eta \, dA \exp \left( (A + \bar{c} + \chi + \eta) \xi + (A + \chi + \eta) \bar{c} + (A + \eta) \chi + (A + \eta) \chi + (A + \eta) \star A \right)
\]

\[
= (i)^{(D-1)D/2} (-)^{(D-1)D} \int d^D \xi \, d\bar{c} \, d^D \chi \, d^D \eta \, dA
\]
\[
\exp \left( [\mathcal{A} - \bar{c} - \xi + \eta][\chi + \xi + \bar{c} - \mathcal{A}] + (\mathcal{A} + \bar{c})\xi + \mathcal{A}\bar{c} + \mathcal{A} \ast \mathcal{A} \right) \\
= (i)^{(D-1)D} \int d^D \xi \ d\bar{c} \ d\mathcal{A} \exp \left( (\mathcal{A} + \bar{c})\xi + \mathcal{A}\bar{c} + \mathcal{A} \ast \mathcal{A} \right). \tag{78}
\]

Similarly we remove the integration over \( d^D \xi d^D \chi \)

\[
Tr \left( \mathcal{A} \gamma^\dagger \chi \right) = Tr \left( \mathcal{A} \gamma^2 \chi \right) \\
= (i)^{(D-1)D} \frac{(-1)^{(D-1)D}}{2} \int d^D \xi \ d\bar{c} \ d^D \mathcal{A} \ d^D \eta \ d\mathcal{A} \\
\exp \left( (\mathcal{A} + \bar{c} + \chi + \eta)\xi + (\mathcal{A} + \chi + \eta)\bar{c} + (\mathcal{A} + \eta)\chi + (\mathcal{A} + \mathcal{A} \ast \mathcal{A}) \right) \\
= (i)^{(D-1)D} \frac{(-1)^{(D-1)D}}{2} \int d^D \xi \ d\bar{c} \ d^D \mathcal{A} \ d^D \eta \ d\mathcal{A} \\
\exp \left( [\mathcal{A} + \bar{c} + \eta + \chi][\xi + \bar{c} - \mathcal{A} - \eta] + 2\bar{c}(\mathcal{A} + \eta) + (\mathcal{A} + \eta)\bar{c} + +\mathcal{A}\eta + \mathcal{A} \ast \mathcal{A} \right) \tag{79}
\]

We use the substitution property \( \chi \rightarrow \chi - \mathcal{A} - \bar{c} - \eta \) and \( \xi \rightarrow \xi - \bar{c} + \mathcal{A} + \eta \)

and integrate

\[
Tr \left( \mathcal{A} \gamma^\dagger \chi \right) = Tr \left( \mathcal{A} \gamma^2 \chi \right) \\
= (-)^D (i)^{(D-1)D} \int d\bar{c} \ d^D \eta \ d\mathcal{A} \exp \left( -(\mathcal{A} + \mathcal{A} \ast \mathcal{A}) \right) \tag{80}
\]

Change sign to \( \bar{c} \) and get

\[
Tr \left( \mathcal{A} \gamma^\dagger \chi \right) = Tr \left( \mathcal{A} \gamma^2 \chi \right) \\
= -(D)(i)^{(D-1)D} \int d\bar{c} \ d^D \eta \ d\mathcal{A} \exp \left( (\mathcal{A} \ast \mathcal{A}) \right) \\
= -(D)Tr \left( \mathcal{A} \gamma \ chi \right) \tag{81}
\]

Finally, by using eqs. \(75\), \(76\) and \(81\) we get

\[
R(\mathcal{A}|\xi|\bar{p}) = -R(\mathcal{A}p|\xi). \tag{82}
\]

We want to stress that the relation in eq. \(82\) between the GT’s is \( D \)-independent.
The Ward identity in eq. (71) can now be written with the help of (81)
\[ \text{Tr} \left( A(p-\bar{p})\gamma_\chi \right) \]
\[ = \text{Tr} \left( A \bar{p}\gamma_\chi \right) - \text{Tr} \left( A \bar{p}\gamma_\chi \right) \]
\[ = \text{Tr} \left( A \bar{p}\gamma_\chi \right) + (-)^D \text{Tr} \left( A\gamma_\chi \bar{p} \right), \] (83)
where \( D \) is supposed to take an integer value and all the poles in the Feynman amplitudes have been removed at the level of GT’s (Dimensional Renormalization).

10 Conclusions: the Rules

In Part I of the present paper we have derived the algebra for the integral representation of the trace. We obtained the standard algebra of the matrix representation of the gamma’s for generic integer \( D \) dimensions. \( \gamma_\chi \) obeys a consistent algebra
\[ \gamma_\chi\gamma_\mu + (-)^D\gamma_\mu\gamma_\chi = 0 \]
\[ \gamma_\chi^2 = 1. \] (84)

The above algebra allows the use of pairing technique: all pairs of \( \gamma_\chi \) can be removed inside the trace.

Lorentz covariance and Cyclicity are properties of the trace in the integral representation. The Clifford algebra of the gamma’s is also implemented.

However this algebra cannot be continued to complex \( D \) since eq. (84) requires \( [-(-)^D]^2 = 1 \).

In Part II we have presented a way to remove this obstacle to the continuation in \( D \). We have introduced a GT obtained from the integral representation of the usual trace deprived of the last integrations on Grassmannian variables \( \xi \) generating \( \gamma_\chi \). With this new tool we define a new strategy for the pole subtraction in Dimensional Regularization. The GT is function of the momenta \( p_j, j = 1, \ldots, N \) and of the Grassmannian real variables \( \xi_\mu, \eta_\mu, \ldots \) which generate the \( \gamma_\chi \)’s upon integration (no completely antisymmetric tensor is present). The GT has all the correct properties as the conventional trace with appropriate modifications.
The pole subtraction is performed on the GT. First we expand the GT in a sum of monomials
\[(p_{j_1}, \xi) \ldots (p_{j_K}, \xi)\] (85)
for every \(K \leq D\) and any partition \(j_1, \ldots, j_K\) of \(1, \ldots, N\). In this way we get rid of the completely antisymmetric tensor and only powers of the momenta are present. Thus Feynman amplitudes can be evaluated in generic \(D\) dimensions and the poles can be subtracted.

After the amplitudes are properly defined for the required integer \(D\) dimensions (by poles subtraction), the relevant integration over the variables \(\xi\) restores the completely antisymmetric tensor.

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### A Example: ABJ Anomaly

Now we can try to use eq. (46). Our procedure of pole subtraction removes any ambiguity in Dimensional Renormalization. In particular, even in the case where the sum of all graphs at given loop order is finite thanks to the presence of a \((D - 4)\) factor removing the pole, the divergent single graphs can be manipulated in a safe way under the protection of the Regularization.

We consider the ABJ anomaly. The relevant term to be evaluated is the divergent part of the Feynman amplitude in Ref. [1]. In particular we start from eq. (65) containing the trace factor
\[Tr\left(\gamma_\mu \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \gamma_\chi\right)b_\mu(q + r - k)_\alpha \epsilon_1(\beta + r)\beta \epsilon_2(q + p)\iota.\] (86)
where \(k, p\) are incoming momenta, \(\epsilon_{1,2}\) the abelian field polarizations, \(b_\mu\) an external source and \(r = yk - xp + yp\) for the Feynman parameters \(x, y\). Three terms are divergent
\[Tr\left(\gamma_\mu \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \gamma_\chi\right)b_\mu(q)\alpha \epsilon_1(q)\beta \epsilon_2(r + p)\iota.\] (87)
\[
Tr \left( \gamma_\mu \gamma_\rho \gamma_\sigma \gamma_\chi \right) b_\mu \left( r - k \right)_\alpha \epsilon_{1\rho} (q)_\beta \epsilon_{2\sigma} (q)_\iota \tag{88}
\]
\[
Tr \left( \gamma_\mu \gamma_\rho \gamma_\sigma \gamma_\chi \right) b_\mu (q)_\alpha \epsilon_{1\rho} (r)_\beta \epsilon_{2\sigma} (q)_\iota. \tag{89}
\]

Now we have to expand all three expressions of eq. (89) according to eq. (46). Of the many terms only those with two \( q \) in the \( \xi \) factor are zero by symmetry. We can avoid this lengthy procedure by using the symmetric integration in \( q \) before we use the expansion of eq. (46).

\[
\gamma_\alpha \gamma_\rho \gamma_\alpha = (2 - D) \gamma_\rho
\]
\[
\gamma_\alpha \gamma_\rho \gamma_\sigma \gamma_\alpha = (D - 4) \gamma_\rho \gamma_\sigma + 4 \delta_\rho_\sigma
\]
\[
\gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\alpha = (6 - D) \gamma_\rho \gamma_\beta \gamma_\sigma
\]
\[
-4 (\delta_\rho_\beta \gamma_\sigma - \delta_\rho_\sigma \gamma_\beta + \delta_\sigma_\beta \gamma_\rho). \tag{90}
\]

Thus after symmetrization the \( \xi \) fourth power term is unique. While lower powers than 4 yield zero under integration over \( \int d^4 \xi \). The terms in eq. (89) yield

\[
\bigg( (4 - D - 2)(r + p)_\iota + (4 - D - 2)(r - k)_\iota - (4 - D + 2)(r)_\iota \bigg)
\]
\[
\frac{q^2}{D} b_\mu \epsilon_{1\rho} \epsilon_{2\sigma} \xi_\mu \xi_\rho \xi_\sigma \xi_\iota \nonumber
\]
\[
= \left( (4 - D - 6)r_\iota + (4 - D - 2)(p - k)_\iota \right) \frac{q^2}{D} b_\mu \epsilon_{1\rho} \epsilon_{2\sigma} \xi_\mu \xi_\rho \xi_\sigma \xi_\iota. \tag{91}
\]

Finally

\[
\frac{q^2}{D} b_\mu \epsilon_{1\rho} \epsilon_{2\sigma} \xi_\mu \xi_\rho \xi_\sigma \xi_\iota
\]
\[
= \left( (4 - D - 6)r_\iota + (4 - D - 2)(p - k)_\iota \right) \frac{q^2}{D} b_\mu \epsilon_{1\rho} \epsilon_{2\sigma} \xi_\mu \xi_\rho \xi_\sigma \xi_\iota. \tag{93}
\]

We insert the value of \( r = yk - xp + yp \) and integrate \( \int_0^1 dx \int_0^x dy \)

\[
\left( (4 - D - 6)r_\iota + (4 - D - 2)(p - k)_\iota \right) \frac{q^2}{D} b_\mu \epsilon_{1\rho} \epsilon_{2\sigma} \xi_\mu \xi_\rho \xi_\sigma \xi_\iota
\]
\[
= \left( \frac{1}{6} (D + 2) + \frac{1}{2} (2 - D) \right)(p - k)_\iota \frac{q^2}{D} b_\mu \epsilon_{1\rho} \epsilon_{2\sigma} \xi_\mu \xi_\rho \xi_\sigma \xi_\iota
\]
\[
= \frac{1}{3} (4 - D)(p - k)_\iota \frac{q^2}{D} b_\mu \epsilon_{1\rho} \epsilon_{2\sigma} \xi_\mu \xi_\rho \xi_\sigma \xi_\iota. \tag{94}
\]

which agrees with the derivation of the same expression in eq. (71) of Ref. [1], after integration over \( \int d^4 \xi \).
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