Basic aspects of symplectic Clifford analysis for the symplectic Dirac operator

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Abstract

In the present article we study basic aspects of the symplectic version of Clifford analysis associated to the symplectic Dirac operator. Focusing mostly on the symplectic vector space of real dimension 2, this involves the analysis of first order symmetry operators, symplectic Clifford-Fourier transform, reproducing kernel for the symplectic Fischer product and the construction of bases of symplectic monogenics for the symplectic Dirac operator.

Key words: Symplectic Dirac operator, Symmetry operators, Reproducing kernel, Fischer product, Bases of symplectic monogenics.

MSC classification: 53C27, 53D05, 81R25

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1 Introduction

Harmonic analysis is a fruitful concept built on the analysis of function spaces equipped with a Lie (finite, discrete, etc) group action. A key organizing principle in analyzing function spaces and decomposing them into simple building blocks is the notion of intertwining (differential, integral) operators. The basic example related to orthogonal
symmetry are quadratic spaces like euclidean space, sphere or hyperbolic space and the Laplace operator acting on scalar valued smooth functions. Another developed example concerns quadratic spaces, smooth functions valued in the spinor space or the Clifford algebra and the orthogonally equivariant Dirac operator (collectively known as the orthogonal Clifford analysis.)

In the present article we focus on a similar structure: we consider the symplectic symmetry instead of the orthogonal group, as for function spaces we consider the symplectic spinors (smooth functions valued in the Segal-Shale-Weil representation) and the symplectic Dirac operator as a symplectic intertwining differential operator. Namely, focusing mostly on the real dimension 2, we develop the basics of symplectic Clifford analysis including the analysis of first order symmetry operators, symplectic Clifford-Fourier transform, reproducing kernel for the symplectic Fischer product and the construction of bases for symplectic monogenics for the symplectic Dirac operator.

Let us briefly indicate the structure of our article. Section 2 contains a review of the metaplectic Howe duality, [6], a concise way of describing the space of symplectic spinors through the invariant theory of the metaplectic Lie algebra \( \mathfrak{mp}(2n, \mathbb{R}) \). Section 3 treats the concept of symmetry differential operators of the symplectic Dirac operator in general even dimensions, and then specializes to several explicit problems both in real and complex variables. In Section 4, we turn our attention to the action of the generator of the Weyl group associated to \( \mathfrak{mp}(2, \mathbb{R}) \), giving rise to the symplectic Clifford-Fourier transform. In particular, we introduce the operator of metaplectic harmonic oscillator and find its spectral decomposition into eigenspaces. Section 5 is devoted to the symplectic Fischer product and the construction of its reproducing kernel. In Section 6, we construct some explicit bases for the space of symplectic monogenics and prove several useful characterizing properties.

To summarize, our work is clearly the first attempt aiming to uncover fundamental analytical properties of the symplectic Dirac operator. The generalization of our results to a symplectic space of arbitrary dimension or a proper formulation of an analogue of the Cauchy-Kovalevskaya theorem are still missing cornerstones of symplectic Clifford analysis.

2 Representation theory and symplectic spinors

Let \((\mathbb{R}^{2n}, \omega)\) be the symplectic vector space with coordinates \(x_1, \ldots, x_n, y_1, \ldots, y_n\), and coordinate vector fields \(\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}\) or, equally, a symplectic frame \(e_1, \ldots, e_{2n}\), fulfilling

\[
\omega(e_j, e_{n+j}) = 1, \quad \omega(e_{n+j}, e_j) = -1, \quad j = 1, \ldots, n \quad (1)
\]

and zero otherwise.

The symplectic Lie algebra \( \mathfrak{sp}(2n, \mathbb{R}) \) has the matrix realization given by the span of

\[
X_{jk} = E_{j,k} - E_{n+k,n+j}, \\
Y_{jj} = E_{j,n+j}, \\
Y_{jk} = E_{j,n+k} + E_{k,n+j} \text{ for } j \neq k, \\
Z_{jj} = E_{n+j,j}, \\
Z_{jk} = E_{n+j,k} + E_{n+k,j} \text{ for } j \neq k,
\]

2
where \( j, k = 1, \ldots, n \), and \( E_{k,j} \) is the \( 2n \times 2n \) matrix with 1 on the intersection of the \( k \)-th row and the \( j \)-th column and zero otherwise. The representation of \( \mathfrak{sp}(2n, \mathbb{R}) \) on the symmetric algebra of \( \mathbb{R}^{2n} \), \( \mathcal{S}^*(\mathbb{R}^{2n}, \mathbb{C}) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \), is given by

\[
\begin{align*}
X_{jk} &= x_j \partial_{x_k} - y_k \partial_{y_j}, \\
Y_{jj} &= x_j \partial_{y_j}, \\
Y_{jk} &= x_j \partial_{y_k} + x_k \partial_{y_j} \text{ for } j \neq k, \\
Z_{jj} &= y_j \partial_{x_j}, \\
Z_{jk} &= y_j \partial_{x_k} + y_k \partial_{x_j} \text{ for } j \neq k.
\end{align*}
\]

(2)

Definition 2.1. The symplectic Clifford algebra \( \mathcal{C}_{\mathfrak{sp}}(\mathbb{R}^{2n}, \omega) \) on \( (\mathbb{R}^{2n}, \omega) \) with a basis \( \{e_1, \ldots, e_{2n}\} \) is an associative unital algebra over \( \mathbb{C} \), given by the quotient of the tensor algebra \( T(e_1, \ldots, e_{2n}) \) by a two-sided ideal \( I \subset T(e_1, \ldots, e_{2n}) \) generated by

\[
v \cdot w - w \cdot v = -i \omega(v, w)
\]

for all \( v, w \in \mathbb{R}^{2n} \) and \( i \in \mathbb{C} \) the complex unit. Namely, the relations \( e_j \cdot e_k - e_k \cdot e_j = -i \omega(e_j, e_k) \) for the basis \( \{e_1, \ldots, e_{2n}\} \) hold true.

The symplectic Clifford algebra \( \mathcal{C}_{\mathfrak{sp}}(\mathbb{R}^{2n}, \omega) \) is isomorphic to the Weyl algebra \( W_{2n} \) of complex valued algebraic differential operators on \( \mathbb{R}^n \), and the symplectic Lie algebra \( \mathfrak{sp}(2n, \mathbb{R}) \) can be realized as a subalgebra of \( W_{2n} \).

In particular, \( W_{2n} \) is an associative algebra generated by \( \{q_1, \ldots, q_n, \partial_{q_1}, \ldots, \partial_{q_n}\} \), the multiplication operator by \( q_j \) and differentiation \( \partial_{q_j}, j = 1, \ldots, n \), and the symplectic Lie algebra \( \mathfrak{sp}(2n, \mathbb{R}) \subset W_{2n} \) is

\[
\begin{align*}
X_{jk} &= q_k \partial_{q_j} + \frac{1}{2} \delta_{j,k}, \\
Y_{jj} &= -\frac{i}{2} \partial_{q_j}^2, \\
Y_{jk} &= i \partial_{q_j} \partial_{q_k} \text{ for } j \neq k, \\
Z_{jj} &= -\frac{i}{2} q_j^2, \\
Z_{jk} &= i q_j q_k \text{ for } j \neq k.
\end{align*}
\]

(3)

We denote by \( \mathcal{S}(\mathbb{R}^n) \) the space of Schwartz functions on \( \mathbb{R}^n \). The representation of \( \mathfrak{sp}(2n, \mathbb{R}) \) on the space of polynomial symplectic spinors \( \mathcal{P}(\mathbb{R}^{2n}, \mathbb{C} \otimes \mathcal{S}(\mathbb{R}^n)) \) is given by the combination of the dual representation to (2) and the representation (3),

\[
\begin{align*}
X_{jk} &= -x_j \partial_{x_k} + y_k \partial_{y_j} + q_k \partial_{q_j} + \frac{1}{2} \delta_{j,k}, \\
Y_{jj} &= -x_j \partial_{y_j} - \frac{1}{2} \partial_{q_j}^2, \\
Y_{jk} &= x_k \partial_{y_j} + x_j \partial_{y_k} + i \partial_{q_j} \partial_{q_k} \text{ for } j \neq k, \\
Z_{jj} &= -y_j \partial_{x_j} - \frac{1}{2} q_j^2, \\
Z_{jk} &= y_k \partial_{x_j} + y_j \partial_{x_k} + i q_j q_k \text{ for } j \neq k.
\end{align*}
\]

(4)

The action of the basis elements \( e_1, \ldots, e_{2n} \) on a symplectic spinor \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) is given by

\[
\begin{align*}
e_j \cdot \varphi &= i q_j \varphi, \\
e_{n+j} \cdot \varphi &= \partial_{q_j} \varphi
\end{align*}
\]

(5)
for $j = 1 \ldots , n$. The three differential operators valued in $\text{End}(\mathcal{S}(\mathbb{R}^n))$,

$$X_s = \sum_{j=1}^{n} (y_j \partial_{q_j} + ix_j \gamma_j),$$

$$D_s = \sum_{j=1}^{n} (i\gamma_j \partial_{y_j} - \partial_{x_j} \partial_{q_j}),$$

$$E = \sum_{j=1}^{n} (x_j \partial_{x_j} + y_j \partial_{y_j}),$$

(6)

are $\mathfrak{sp}(2n, \mathbb{C})$-equivariant and generate the representation of the Lie algebra $\mathfrak{sl}(2)$ on the space $\text{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$. Their commutation relations are

$$[E + n, D_s] = -D_s,$$

$$[E + n, X_s] = X_s,$$

$$[X_s, D_s] = i(E + n).$$

(7)

The metaplectic analogue of the classical theorem on the separation of variables allows to decompose polynomial symplectic spinors $\text{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ under the action of $\mathfrak{mp}(2n, \mathbb{R})$ into a direct sum of simple (irreducible) $\mathfrak{mp}(2n, \mathbb{R})$-modules, cf. [6]:

$$\text{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n) \simeq \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{\infty} X_s^j M_s^l$$

(8)

with

$$M_s^l := (\text{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)) \cap \text{Ker}(D_s),$$

which can be graphically represented as

To simplify the scheme, we used the notation $P_j$ instead of $\text{Pol}_j(\mathbb{R}^{2n}, \mathbb{C})$ and $S$ instead of $\mathcal{S}(\mathbb{R}^n)$ in the last picture. The symplectic Dirac operator $D_s$ as well as $X_s$ act horizontally in the previous picture, but in opposite directions; $E$ preserves each simple metaplectic module in the decomposition.

4
3 Symmetries of the symplectic Dirac operator

We shall start the present section with a short reminder of the notion of symmetry operators for the classical Dirac operator associated to a quadratic form, see [10] and [11], and then pass to the case of our interest: the symplectic Dirac operator.

The Clifford algebra associated to a vector space equipped with a quadratic form \( B \) is determined by the relations \( e_j \cdot e_k + e_k \cdot e_j = -2B(e_j, e_k) \), while the relations for the symplectic Clifford algebra on \( (\mathbb{R}^{2n}, \omega) \) are introduced in Definition 2.1. In the orthogonal case, the Dirac operator on \( \mathbb{R}^m \) is \( D = \sum_{j=1}^m e_j \partial_{e_j} \) and its polynomial solutions are coined spherical monogenics. The module of polynomial spherical monogenics of homogeneity \( h \) is denoted by \( M_h = \{ \text{Pol}_h(\mathbb{R}^m, \mathbb{C}) \otimes \mathbb{S} \} \cap \text{Ker}(D) \), where \( \mathbb{S} \) is the spinor space. In particular, each of the modules \( M_h, h \in \mathbb{N}_0 \), is an irreducible representation of the Lie algebra \( \mathfrak{so}(m) \), acting by the differential operators

\[
K_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j} - \frac{1}{2} e_j e_k, \quad j \neq k, \quad j, k = 1, \ldots, m.
\]

Moreover, the space \( M = \bigoplus_h M_h \) is an irreducible representation of the conformal Lie algebra \( \mathfrak{so}(m+1,1,\mathbb{R}) \), which is the linear span of \( K_{jk}, 2E + m - 1, \partial_{x_j} \) and \( T_j \), \( j, k = 1, \ldots, m \); here the operators \( T_j : M_h \rightarrow M_{h+1} \) act by

\[
T_j = X e_j + x_j (m + 2E) - |X|^2 \partial_{x_j},
\]

where

\[
X = \sum_{j=1}^m e_j x_j.
\]

Let us now turn our attention to the symplectic space \( (\mathbb{R}^{2n}, \omega) \). First of all, we find differential operators increasing the homogeneity of polynomial solutions of the symplectic Dirac operator by one. We construct them as a composition of the multiplicity by \( x_i, y_l \), \( l = 1, \ldots, n \), and projection on the kernel of the symplectic Dirac operator \( D_s \).

In the first step we check that \( D^3_s \) acts trivially on \( x_i m, y_l m \) for \( m \in M_h^s \) and coordinate functions \( x_i, y_l \) on \( \mathbb{R}^{2n} \). For \( j = 1, \ldots, n \), we have

\[
D^2_s(x_j m) = D_s(-\partial_{x_j} m) = -\sum_{k=1}^{2n} (i q_k \partial_{y_k} - \partial_{y_k} \partial_{x_k}) \partial_{q_k} m = i \partial_{y_j} m,
\]

\[
D^2_s(y_l m) = D_s(i q_l m) = \sum_{k=1}^{2n} (i q_k \partial_{y_k} - \partial_{y_k} \partial_{x_k}) i q_j m = -i \partial_{x_j} m,
\]

and so \( x_i m, y_l m \) are in the kernel of \( D^3_s \) for all \( l = 1, \ldots, n \). Denoting the identity endomorphism \( \text{Id} \), the corresponding projector of \( x_i m, y_l m \) on the homogeneity \( h + 1 \) subspace of \( \text{Ker}(D_s) \) is

\[
P_{h+1}^s = \text{Id} + c X_s D_s + d X_s^2 D_s^2
\]

for some constants \( c, d \) depending on \( h \) and \( n \). The relations (11) imply that on the spaces of homogeneous symplectic monogenics holds

\[
P_{h+1}^s m_{h+1} = m_{h+1},
\]

\[
P_{h+1}^s X_s m_h = X_s m_h + c X_s D_s X_s m_h = (1 - ic(h + n)) X_s m_h,
\]

\[
P_{h+1}^s X_s^2 m_{h-1} = X_s^2 m_{h-1} + c X_s D_s (X_s^2 m_{h-1}) + d X_s^2 D_s^2 (X_s^2 m_{h-1})
\]

\[
= X_s^2 m_{h-1} - ic X_s^2 (h - 1 + n) m_{h-1} - ic X_s (h + n) X_s m_{h-1} - d(2h + 2n - 1)(h + n - 1) X_s^2 m_{h-1}.
\]
Then the second and the third expressions in (11) are zero provided
\[ c = \frac{1}{i(h + n)}, \quad d = \frac{-1}{(h + n)(2h + 2n - 1)}, \]
hence the projector is
\[ P^s_{h+1} = \text{Id} + \frac{1}{(h + n)(2h + 2n - 1)} X_s D_s. \]  
(12)
The action of the operators \( S_l = P^s_{h+1} x_l \), \( l = 1, \ldots, n \) and \( S_{n+l} = P^s_{h+1} y_l \), \( l = 1, \ldots, n \), on \( m \in M^s_h \) is then
\[ S_j m = x_j m - c X_s \partial_{q_j} m + i d X_s^2 \partial_{y_j} m, \]
\[ S_{n+j} m = y_j m + c X_s i q_j m - i d X_s^2 \partial_{x_j} m, \]
so we can define for \( j = 1, \ldots, n \) the collection of differential operators
\[ Z_j := -i(h + n)(2h + 2n - 1) S_{n+j}, \]
\[ Z_{n+j} := i(h + n)(2h + 2n - 1) S_j. \]  
(13)

**Proposition 3.1.** Let \( n \in \mathbb{N} \). The differential operators
\[ Z_j = X_s^2 \partial_{x_j} - i y_j (E + n)(2E + 2n - 1) - i X_s q_j (2E + 2n - 1), \]
\[ Z_{n+j} = X_s^2 \partial_{y_j} + i x_j (E + n)(2E + 2n - 1) - X_s \partial_{q_j} (2E + 2n - 1) \]  
(14)
for \( j = 1, \ldots, n \) are \( \mathfrak{mp}(2n, \mathbb{R}) \)-equivariant and preserve the solution space of the symplectic Dirac operator on \( (\mathbb{R}^{2n}, \omega) \). The operators \( Z_l, l = 1, \ldots, 2n \), increase the homogeneity in the basis variables \( x_1, \ldots, x_n, y_1, \ldots, y_n \) by one:
\[ Z_l : \text{Ker}(D_s) \rightarrow \text{Ker}(D_s), \]
\[ Z_l : M_h \rightarrow M_{h+1}, \quad l = 1, \ldots, 2n, \]  
(15)
where \( M_h \) is the irreducible \( \mathfrak{mp}(2n, \mathbb{R}) \)-module of homogeneity \( h \) symplectic polynomial spinors in \( \text{Ker}(D_s) \).

**Proof:** The property of \( \mathfrak{mp}(2n, \mathbb{R}) \)-equivariance means that the vector space of dimension \( 2n \) generated by \( \{ Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{2n} \} \) transforms in the fundamental vector representation of \( \mathfrak{mp}(2n, \mathbb{R}) \) with respect to the canonical Lie algebra structure on the associative algebra \( W_{4n} \otimes C_l(\mathbb{R}^{2n}, \omega) \). Recall that \( W_{4n} \) is generated by \( x_j, y_j, \partial_{x_j}, \partial_{y_j} \) for \( j = 1, \ldots, n \). The verification of all commutation relations of the Lie algebra \( \mathfrak{mp}(2n, \mathbb{R}) \) (cf., (4)) with \( \{ Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{2n} \} \) is a straightforward but tedious computation.

The second part of the claim is a consequence of
\[ [D_s, X_s^2 \partial_{x_j}] = -i X_s \partial_{x_j} (2E + 2n - 1), \]
\[ [D_s, \omega_{j,k} \delta^{k,l} x_l (E + n)(2E + 2n - 1)] = -e_j (E + n)(2E + 2n - 1) \]
\[ + \omega_{j,k} \delta^{k,l} x_l (4E + 4n + 1) D_s, \]
\[ [D_s, X_s e_j (2E + 2n - 1)] = -i e_j (E + n)(2E + 2n - 1) \]
\[ -i X_s \partial_{x_j} (2E + 2n - 1) + 2 X_s e_j D_s, \]  
(16)
because the linear combination
\[ AX^2_\partial x_j + B\omega_{jk}\delta^{kl}_x x_l (E + n)(2E + 2n - 1) + CX^2_e x_j (2E + 2n - 1) \]
for \( A, B, C \in \mathbb{C} \) and all \( j = 1, \ldots, 2n \) commutes with \( D_s \) provided \( A = 1, B = i \) and \( C = -1 \). To shorten our notation we used \( \omega_{jk} = \omega(e_j, e_k) \), see (I).

The differential operators \( Z_j, Z_{n+j}, j = 1, \ldots, n \) are of third order, and are of second order in the base variables \( x_j, y_j \) (due to their quadratic dependence on the homogeneity operator \( E \)).

**Proposition 3.2.** The \( \mathfrak{mp}(2n, \mathbb{R}) \)-equivariant first order differential operators
\[ \partial x_j, \partial y_j, \quad j = 1, \ldots, n \]
preserve the solution space of the symplectic Dirac operator on \((\mathbb{R}^{2n}, \omega)\):
\[ \partial x_j, \partial y_j : \text{Ker}(D_s) \to \text{Ker}(D_s), \]
\[ \partial x_j, \partial y_j : M_h \mapsto M_{h-1}, \quad j = 1, \ldots, n, \]
where \( M_h \) is the irreducible \( \mathfrak{mp}(2n, \mathbb{R}) \)-module of homogeneity \( h \) symplectic polynomial spinors in \( \text{Ker}(D_s) \).

**Proof:** The property of \( \mathfrak{mp}(2n, \mathbb{R}) \)-equivariance means that the vector space of dimension \( 2n \) generated by \( \{ \partial x_1, \ldots, \partial x_n, \partial y_1, \ldots, \partial y_n \} \) transforms in the fundamental vector representation of \( \mathfrak{mp}(2n, \mathbb{R}) \) with respect to the canonical Lie algebra structure on the associative algebra \( W_{4n} \otimes \text{Cl}_4(\mathbb{R}^{2n}, \omega) \). The verification of all commutation relations of the Lie algebra \( \mathfrak{mp}(2n, \mathbb{R}) \) (cf., (I)) with \( \{ \partial x_1, \ldots, \partial x_n, \partial y_1, \ldots, \partial y_n \} \) is a straightforward computation.

The rest of the claim follows from \([\partial x_j, D_s] = 0 \) and \([\partial y_j, D_s] = 0 \) for \( j = 1, \ldots, n \). □

### 3.1 First order symmetries of the symplectic Dirac operator on \((\mathbb{R}^2, \omega)\)

The aim of the present section is to compute all first order differential operators (in both the horizontal variables \( x, y \) and the vertical variable \( q \)) which are symmetries of the symplectic Dirac operator. Here we restrict to \( n = 2 \), the case of general even dimension being notationally tedious.

We start with \((\mathbb{R}^2, \omega)\) and denote the coordinates by \( x = x_1, y = y_1 \), the coordinate vector fields by \( \partial_x, \partial_y \) and a symplectic frame is \( e_1, e_2 \) with the action on a symplectic spinor \( \varphi \in \text{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes S(\mathbb{R}) \)
\[ e_1 \cdot \varphi = i q \varphi, \quad e_2 \cdot \varphi = \partial_q \varphi. \]

Following (I), the basis elements of \( \mathfrak{mp}(2, \mathbb{R}) (\simeq \mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2)) \) act as
\[ \tilde{X} = -y \partial_x - \frac{i}{2} q^2, \]
\[ \tilde{Y} = -x \partial_y - \frac{i}{2} \partial_q^2, \]
\[ \tilde{H} = -x \partial_x + y \partial_y + q \partial_q + \frac{1}{2}, \] (19)
and satisfy the commutation relations of the Lie algebra $\mathfrak{mp}(2, \mathbb{R})$:

\[
\begin{align*}
[\hat{X}, \hat{Y}] &= \hat{H}, \\
[\hat{H}, \hat{X}] &= 2\hat{X}, \\
[\hat{H}, \hat{Y}] &= -2\hat{Y}.
\end{align*}
\]

Notice that these operators preserve homogeneity in the variables $x, y$. The three $\mathfrak{mp}(2, \mathbb{R})$-invariant operators

\[
X_s = y\partial_y + ixq, \\
D_s = iq\partial_y - \partial_x\partial_q, \\
E = x\partial_x + y\partial_y
\]  

form the Lie algebra isomorphic to $\mathfrak{sl}(2)$. The operators $X_s, D_s$ and $E$ commute with $\hat{X}, \hat{Y}$ and $\hat{H}$, i.e. they are $\mathfrak{mp}(2, \mathbb{R})$ intertwining differential operators on complex polynomials valued in the Segal-Shale-Weil representation. A consequence of Proposition 3.1 and Proposition 3.2 is

**Corollary 3.3.** The commuting operators

\[
\begin{align*}
Z_1 &= -X_s^2\partial_x + iy(E+1)(2E+1) + X_siq(2E+1), \\
Z_2 &= -X_s^2\partial_y - ix(E+1)(2E+1) + X_s\partial_q(2E+1)
\end{align*}
\]  

(21)

preserve the solution space of the symplectic Dirac operator $D_s$ and increase the homogeneity in the variables $x, y$ by one, $Z_j : M_h \mapsto M_{h+1}$, $j = 1, 2$, for $M_h$ being the irreducible $\mathfrak{mp}(2, \mathbb{R})$-module of homogeneity $h$ polynomial symplectic spinors in $\text{Ker}(D_s)$.

The commuting operators

\[
\partial_x, \partial_y
\]  

(22)

preserve the solution space of the symplectic Dirac operator $D_s$ and decrease the homogeneity in the variables $x, y$ by one.

The commutator $[Z_1, Z_2]$ is zero, and

\[
\begin{align*}
[\partial_x, Z_1] &= -2i\hat{X}(2E+1), \\
[\partial_y, Z_1] &= 2X_sD_s + i\hat{H}(2E+1) + i(2E+1)(2E+1) + \frac{i}{2}, \\
[\partial_x, Z_2] &= -2X_sD_s + i\hat{H}(2E+1) - i(2E+1)(2E+1) - \frac{i}{2}, \\
[\partial_y, Z_2] &= 2i\hat{Y}(2E+1).
\end{align*}
\]  

(23)

Moreover, we have

\[
\begin{align*}
[Z_1, \hat{H}] &= -Z_1, & [Z_2, \hat{H}] &= Z_2, \\
[Z_1, \hat{X}] &= 0, & [Z_2, \hat{X}] &= -Z_1, \\
[Z_1, \hat{Y}] &= Z_2, & [Z_2, \hat{Y}] &= 0, \\
[Z_1, E] &= -Z_1, & [Z_2, E] &= -Z_2,
\end{align*}
\]  

(24)

as well as

\[
\begin{align*}
[\partial_x, \hat{H}] &= -\partial_x, & [\partial_y, \hat{H}] &= \partial_y, \\
[\partial_x, \hat{X}] &= 0, & [\partial_y, \hat{X}] &= -\partial_x, \\
[\partial_x, \hat{Y}] &= -\partial_y, & [\partial_y, \hat{Y}] &= 0, \\
[\partial_x, E] &= \partial_x, & [\partial_y, E] &= \partial_y.
\end{align*}
\]  

(25)
Remark 3.4. The commutator of commutators \([\partial_x, Z_1] = -2i\tilde{X}(2E+1)\) and \([\partial_y, Z_2] = 2i\tilde{Y}(2E+1)\) gives

\[[-2i\tilde{X}(2E+1), 2i\tilde{Y}(2E+1)] = 4\tilde{H}(2E+1)(2E+1).\]

Then we can compute the commutator of this commutator with, for example, \([\partial_x, Z_1] = -2i\tilde{X}(2E+1)\), resulting in the third power of \((2E+1)\). In general, we can produce an arbitrarily high power of \((2E+1)\) in iterated commutators, hence the linear span of the operators \(\tilde{H}, \tilde{X}, \tilde{Y}, \partial_x, \partial_y, Z_1, Z_2\) and \(E\) is not closed under the commutator bracket.

Let us briefly mention the key concept of (generalized) differential symmetries for the symplectic Dirac operator, see [10] and references therein for an introduction. A differential operator \(A\) is a symmetry of \(D_s\) if there exists another differential operator \(B\) such that

\[D_sA = BD_s.\]  

(26)

Consequently, symmetry operators preserve the solution space of the symplectic Dirac operator.

Theorem 3.5. The first order symmetries of the symplectic Dirac operator \(D_s\) on \(\mathbb{R}^2\) are given by the linear span of differential operators \(\partial_x, \partial_y, \tilde{H}, \tilde{X}, \tilde{Y}, E\) and \(y\tilde{H} - 2x\tilde{X} + yE + \frac{3}{2}y\).

Proof: Let us consider a general first order differential operator in the variables \(x, y, q\):

\[A = F_0(x, y, q)\partial_x + F_1(x, y, q)\partial_y + F_2(x, y, q)\partial_q + F_3(x, y, q),\]

where \(F_j, j = 0, 1, 2, 3\), are convenient functions of \(x, y\) and \(q\). Then \(D_sA = AD_s + [D_s, A]\), so that (26) implies \([D_s, A] = B'D_s\) for a differential operator \(B'\). The computation of commutators gives

\[
(iq[\partial_y, F_0(x, y, q)] - \partial_q[\partial_x, F_0(x, y, q)] - [\partial_x, F_2(x, y, q)]\partial_q - \partial_q[\partial_y, F_3(x, y, q)] - \partial_q[\partial_x, F_1(x, y, q)])\partial_x
\]

\[
+ (iq[\partial_y, F_1(x, y, q)] - \partial_q[\partial_x, F_1(x, y, q)] + F_2(x, y, q)[iq, \partial_q])\partial_y
\]

\[
- [\partial_x, F_0(x, y, q)]\partial_y^2 - [\partial_q, F_1(x, y, q)]\partial_x\partial_y
\]

\[
+ iq[\partial_y, F_2(x, y, q)]\partial_q - \partial_q[\partial_x, F_2(x, y, q)]\partial_q + iq[\partial_y, F_3(x, y, q)] - \partial_y[\partial_x, F_3(x, y, q)]
\]

\[
= B'(iq\partial_q - \partial_x\partial_q).
\]

The commutator \([\partial_q, F_0(x, y, q)]\) by \(\partial_y^2\) does not depend on \(\partial_q\) and so equals to zero. Hence \(F_0(x, y, q)\) is independent of the variable \(q\), \(F_0 \equiv F_0(x, y)\). Then the commutator \([\partial_q, F_1(x, y, q)]\) by \(\partial_x\partial_y\) has to be zero as well, i.e., \(F_1(x, y)\) is independent of \(q\). Moreover, the commutator \([\partial_x, F_1(x, y)]\) by \(\partial_y[\partial_x, F_1(x, y)]\partial_q\) has to be zero, i.e., \(F_1 \equiv F_1(y)\).

We can separate the last equation into three equalities:

\[
(iq[\partial_y, F_0(x, y)] - [\partial_q, F_3(x, y, q)] - ([\partial_x, F_0(x, y)] + [\partial_y, F_2(x, y, q)])\partial_q)\partial_x
\]

\[= -B'iq\partial_q, \]  

(27)

\[
(iq[\partial_q, F_1(y)] - iF_2(x, y, q))\partial_q = B'iq\partial_q, \]  

(28)

\[
iq[\partial_y, F_2(x, y, q)]\partial_q - \partial_q[\partial_x, F_2(x, y, q)]\partial_q + iq[\partial_y, F_3(x, y, q)]
\]

\[= - \partial_q[\partial_x, F_3(x, y, q)] = 0. \]  

(29)
The equation (27) yields $iq[\partial_y, F_0(x, y)] - [\partial_q, F_3(x, y, q)] = 0$. We set

$$F_3(x, y, q) = F'_3(x, y) \frac{i}{2} q^2 + F''_3(x, y),$$

(30)

and therefore

$$[\partial_y, F_0(x, y)] = F'_3(x, y).$$

(31)

The second equality (28) implies

$$F_2(x, y, q) = F'_2(x, y) q.$$

(32)

Then $[\partial_q, F_2(x, y, q)] = F'_2(x, y)$, and equations (28) and (27) give

$$[\partial_y, F_1(y)] - F'_2(x, y) = [\partial_x, F_0(x, y)] + F_2(x, y),$$

$$[\partial_y, F_1(y)] = [\partial_x, F_0(x, y)] + 2 F'_2(x, y).$$

(33)

The equation (29) can be rewritten with the use of (30) and (32) as

$$[\partial_y, F'_2(x, y)]iq^2 \partial_q - [\partial_x, F'_2(x, y)](\partial_q + q \partial_q^2) - [\partial_y, F'_3(x, y)] \frac{i}{2} q^3$$

$$+ [\partial_y, F''_3(x, y)]iq - [\partial_x, F''_3(x, y)](iq + \frac{1}{2} iq^2 \partial_q) - [\partial_x, F''_3(x, y)] \partial_q = 0.$$

Because there is only one commutator by $q \partial_q^2$ and $q^3$, we have $F'_2 \equiv F'_2(y), F''_3 \equiv F''_3(x)$.

Then the commutators by $\partial_y$ have to be zero and $F''_3$ is independent of $x$. $F''_3 \equiv F''_3(y)$.

The commutators by $iq^2 \partial_q$ and $iq$ give the relations

$$[\partial_y, F'_3(y)] - \frac{1}{2} [\partial_x, F'_3(x)] = 0, \quad [\partial_y, F''_3(y)] - [\partial_x, F''_3(x)] = 0.$$  

(34)

The solution of (34) is $F'_3(y) = \frac{i}{2} \alpha y + \gamma$, $F''_3(x) = \alpha x + \beta$ and $F''_3 = \alpha y + \gamma$. The substitution of this solution into (31) yields $F_0(x, y) = \alpha xy + \beta y + F'_0(x)$. Substituting into (33), we get $F'_0(x) = \eta x + \zeta y$ and $F_0(y) = \alpha y^2 + (2 \gamma + \eta)y + \kappa$. Taken altogether, the functions $F_j, j = 0, 1, 2, 3$ are

$$F_0 = \alpha xy + \eta x + \beta y + \zeta,$$

$$F_1 = \alpha y^2 + (2 \gamma + \eta)y + \kappa,$$

$$F_2 = \frac{1}{2} \alpha yq + \gamma q,$$

$$F_3 = (\alpha x + \beta) \frac{i}{2} q^2 + \alpha y + \delta,$$

where $\alpha, \beta, \gamma, \delta, \eta, \zeta, \kappa \in \mathbb{C}$ are arbitrary constants. The constant $\beta$ corresponds to the operator $\hat{X}$, $\zeta$ and $\kappa$ correspond to $\partial_x$ and $\partial_y$. A combination of $\eta, \gamma$ and $\delta$ corresponds to a combination of $E$, $\hat{H}$ and the identity operator. Finally, $\alpha$ corresponds to the operator $y \hat{H} - 2x \hat{X} + yE + \frac{2}{3} y$.

We notice that $\hat{Y}$ is a second order differential operator, but it is first order in the base variables $x, y$. The operators $Z_1, Z_2$ are symmetries of $D_3$ but they are third order differential operators, second order in the base variables $x, y$. 

\square
3.2 First order symmetries in the holomorphic variable

We use the complex coordinates \( z = x + iy, \bar{z} = x - iy \), for the standard complex structure on \( \mathbb{R}^2 \), where \( \partial_x = \partial_z + \partial_{\bar{z}} \) and \( \partial_y = i(\partial_z - \partial_{\bar{z}}) \). In the complex coordinates \( z, \bar{z} \) we have

\[
X_s = \frac{i}{2}((q - \partial_q)z + (q + \partial_q)\bar{z}), \\
D_s = -(q + \partial_q)\partial_z + (q - \partial_q)\partial_{\bar{z}}, \\
E = z\partial_z + \bar{z}\partial_{\bar{z}}
\]

and

\[
Z_1 = 2X_s^2\partial_z + \bar{z}(E + 1)(2E + 1) + iX_s(\partial_q - q)(2E + 1), \\
Z_2 = 2X_s^2\partial_{\bar{z}} - z(E + 1)(2E + 1) - iX_s(\partial_q + q)(2E + 1),
\]

where \( Z_1 = \bar{Z}_1 + i\bar{Z}_2 \) and \( Z_2 = \bar{Z}_1 - i\bar{Z}_2 \), cf. \( Z_1 \) and \( Z_2 \) in Corollary 3.3.

The commutator of \( [Z_1, Z_2] \) is trivial and the commutators with (anti-)holomorphic coordinate vector fields are

\[
[Z_1, \partial_z] = 2iX_t(2E + 1), \\
[Z_1, \partial_{\bar{z}}] = 2iX_tD_s - H_t(2E + 1) - (2E + 1)(2E + 1) - \frac{1}{2}, \\
[Z_2, \partial_z] = -2iX_tD_s - H_t(2E + 1) + (2E + 1)(2E + 1) + \frac{1}{2}, \\
[Z_2, \partial_{\bar{z}}] = 2iY_t(2E + 1),
\]

where we introduced

\[
H_t = i\bar{X} - i\bar{Y}, \\
X_t = -\frac{1}{2}(\bar{X} + \bar{Y} + iH), \\
Y_t = -\frac{1}{2}(\bar{X} + \bar{Y} - iH),
\]

with \( \bar{H}, \bar{X} \) and \( \bar{Y} \) the operators \( \text{(19)} \) in the variables \( z, \bar{z} \):

\[
H_t = \bar{z}\partial_z - z\partial_{\bar{z}} + \frac{1}{2}(q^2 - \partial_q^2), \\
X_t = i\bar{z}\partial_z + \frac{i}{4}(q - \partial_q)^2, \\
Y_t = -iz\partial_{\bar{z}} + \frac{i}{4}(q + \partial_q)^2.
\]

The operators \( H_t, X_t \) and \( Y_t \) commute with \( D_s, X_s, E \), and satisfy the commutation relations of the Lie algebra \( \mathfrak{mp}(2,\mathbb{R}) \):

\[
[X_t, Y_t] = H_t, \\
[H_t, X_t] = 2X_t, \\
[H_t, Y_t] = -2Y_t.
\]

A straightforward computation reveals

\[
[Z_1, H_t] = -Z_1, \\
[Z_2, H_t] = Z_2, \\
[Z_1, X_t] = 0, \\
[Z_2, X_t] = iZ_1, \\
[Z_1, Y_t] = -iZ_2, \\
[Z_2, Y_t] = 0, \\
[Z_1, E] = -Z_1, \\
[Z_2, E] = -Z_2,
\]

(40)
\[ [\partial_z, H_t] = -\partial_{\bar{z}}, \quad [\partial_{\bar{z}}, H_t] = \partial_z, \]
\[ [\partial_z, X_t] = 0, \quad [\partial_{\bar{z}}, X_t] = i\partial_z, \]
\[ [\partial_z, Y_t] = -i\partial_{\bar{z}}, \quad [\partial_{\bar{z}}, Y_t] = 0, \]
\[ [\partial_z, E_t] = \partial_z, \quad [\partial_{\bar{z}}, E_t] = \partial_{\bar{z}}. \] (41)

4 Towards a symplectic Clifford-Fourier transform

The central role in harmonic analysis on \( \mathbb{R}^n \) is played by the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \), generated by the \( \mathfrak{so}(n, \mathbb{R}) \)-invariant Laplace operator \( \Delta \) and the norm squared \( |x|^2 \) of the vector \( x \in \mathbb{R}^n \). The classical integral Fourier transform,

\[ F(f)(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \exp^{-i(x,y)} \, dx, \quad \langle x,y \rangle = \sum_{i=1}^n x_i y_i, \] (42)

can be equivalently represented by the operator exponential that contains the generators of \( \mathfrak{sl}(2, \mathbb{C}) \):

\[ \exp^{iz} \exp^{iz^2} (\Delta - |x|^2), \] (43)

which means that the two operators have the same spectral properties. There are analogous results in the harmonic analysis for finite groups based on Dunkl operators, or Clifford analysis based on the Clifford algebra associated to a quadratic form and the Dirac operator \( D = \sum_{j=1}^n e_j \partial_{x_j} \), written in a basis \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \) with coordinates \( x_1, \ldots, x_n \), cf. [3], [4] and [5].

In the present section we discuss several basic questions in this direction, focusing on symplectic Clifford analysis and the associated symplectic Dirac operator in real dimension 2.

4.1 The eigenfunction decomposition for the operator \( D_s - cX_s \)

The symplectic Fourier transform is based on the eigenvalue equation

\[ (D_s - cX_s)f = \lambda f, \quad c \in \mathbb{R}, \lambda \in \mathbb{C}. \] (44)

As already indicated, we shall stick to the real dimension 2 and look for the solutions of this equation in terms of a linear combination of elements \( g(X_s) m_k^s \), where \( m_k^s \in M_k^s \) is a symplectic monogenic and \( g \) is a polynomial in the variable \( X_s \). We shall first focus on the problem whether for a symplectic spinor \( \varphi \) valued in \( S(\mathbb{R}) \) holds \( e^{\alpha X_s} \varphi \in S(\mathbb{R}) \) for \( \alpha \in \mathbb{C} \).

Lemma 4.1. The following identity holds,

\[ e^{\alpha X_s} e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} e^{\alpha (ix-y)(2q+\alpha y)}. \] (45)

Proof: Writing the exponential as

\[ e^{\alpha X_s} e^{-\frac{x^2}{2}} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} X_s^k e^{-\frac{x^2}{2}}, \]
we show by induction on \( k \in \mathbb{N}_0 \) that

\[
X_s^k e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(ix - y)^k - m q^k - 2m y^m}{m!(k - 2m)!2^m m}. \tag{46}
\]

Recall the notation \( \lfloor \cdot \rfloor \) for the floor function. The equation is satisfied for \( k = 0 \) and for \( k = 1 \), \( X_s e^{-\frac{x^2}{2}} \) is equal to \( e^{-\frac{x^2}{2}} q(ix - y) \). Assuming (46) holds for \( k \), we aim to prove the identity for \( k + 1 \). Let us start with odd \( k \):

\[
(i x + y \partial_y) e^{-\frac{x^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(ix - y)^k - m q^k - 2m y^m}{m!(k - 2m)!2^m m} = e^{-\frac{x^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(ix - y)^{k+1} - m q^{k+1} - 2m y^{m+1}}{m!(k - 2m)!2^m m} + e^{-\frac{x^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(ix - y)^{k-1} - m q^{k-1} - 2m y^m}{m!(k - 2m - 1)!2^m m},
\]

and the \( m \rightarrow m - 1 \) in the second sum results into

\[
e^{-\frac{x^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k + 1)!(ix - y)^{k+1} - m q^{k+1} - 2m y^{m+1}}{m!(k + 1 - 2m)!2^m m} \left( \frac{k + 1 - 2m}{k + 1} + \frac{2m}{k + 1} \right)
\]

\[
+ e^{-\frac{x^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k + 1)!(ix - y)^{k+1} - m q^{k+1} - 2m y^{m+1}}{m!(k + 1 - 2m)!2^m m} \left( \frac{k + 1}{2} \right)^2 \frac{m}{m!2^m}
\]

which proves the induction step. For \( k \) even, \( \lfloor \frac{k}{2} \rfloor = \lfloor \frac{k + 1}{2} \rfloor \) and the second expression on the last display is zero, so that

\[
e^{x \varphi} e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} \sum_{k=0}^{\infty} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \alpha^k (ix - y)^k - m q^k - 2m y^m m!/(k - 2m)!2^m m.
\]

The change of the order in the last summation while keeping \( m \) fixed gives

\[
\sum_{k=2m}^{\infty} \alpha^k (ix - y)^k - m q^k - 2m y^m m!/(k - 2m)!2^m m \]

\[
= \frac{\alpha^{2m} (ix - y)^m y^m m!2^m}{m!2^m} e^{q(ix - y)}, \tag{47}
\]

and so

\[
e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^{2m} (ix - y)^m y^m m!2^m}{m!2^m} e^{q(ix - y)} = e^{-\frac{x^2}{2}} e^{\frac{1}{2} q(x - y)(2q + 2y)}.
\]

By Lemma [14] we see that \( e^{x \varphi} \), \( \alpha \in \mathbb{C} \), is for \( \varphi = -\frac{x^2}{2} \) a Schwartz function in the variable \( q \) and a non-polynomial function in the variables \( x, y \). This property remains
true for any \( \varphi = p(x, y)e^{-\frac{x^2}{2}} \), where \( p(x, y) \in \text{Pol}(\mathbb{R}^2, \mathbb{C}) \): taking as basis elements of the Schwartz space \( q^j e^{-\frac{x^2}{2}} \in \mathcal{S}(\mathbb{R}) \), \( j \in \mathbb{N}_0 \),

\[
e^{\alpha X_s} q^j e^{-\frac{x^2}{2}} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} X^k s q^j e^{-\frac{x^2}{2}}
\]

is a Schwartz function in \( q \), because in the expansion of \( X^k s q^j e^{-\frac{x^2}{2}} \) the maximal exponent of \( q \) is just \( k + j \), cf. (46). Therefore, \( e^{\alpha X_s} q^j e^{-\frac{x^2}{2}} \) grows as \( q^j e^{-\frac{x^2}{2}} e^{\alpha q} \), \( \alpha \in \mathbb{C} \), which is a characterizing property of Schwartz function class in the variable \( q \).

It is easy to verify the following identities in the universal enveloping algebra \( U(\mathfrak{sl}(2, \mathbb{C})) \):

\[
[E + n, X^k_s] = k X^k_s,

[D_s, X^k_s] = -i(E + n)X^{k-1}_s - iX_s(E + n)X^{k-2}_s - \ldots - iX^{k-1}_s(E + n)
\]

so that for all \( \alpha \in \mathbb{C} \)

\[
[D_s, e^{\alpha X_s}] = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} [D_s, X^k_s]
\]

\[
= -i \sum_{k=1}^{\infty} \frac{\alpha^k}{(k - 1)!} X^{k-1}_s(E + n) - i\frac{\alpha^2}{2} X_s \sum_{k=2}^{\infty} \frac{\alpha^{k-2}}{(k - 2)!} X^{k-2}_s
\]

\[
= -i\alpha e^{\alpha X_s}(E + n) - i\frac{\alpha^2}{2} X_s e^{\alpha X_s}.
\]

The substitution of

\[ f = e^{\alpha X_s} g(X_s) m^s_k \]

into (41), where \( m^s_k \in M^s_k \) is a symplectic monogenic and \( g(X_s) \) is a polynomial in \( X_s \), yields

\[ D_s e^{\alpha X_s} g(X_s) m^s_k = c X_s e^{\alpha X_s} g(X_s) m^s_k = \lambda e^{\alpha X_s} g(X_s) m^s_k. \]

Because \( e^{\alpha X_s} \) is an invertible operator, we get

\[ D_s(g(X_s)m^s_k) - i\alpha (E + n)g(X_s)m^s_k - (c + i\frac{\alpha^2}{2})X_s g(X_s)m^s_k = \lambda g(X_s)m^s_k. \]

Now we set \( c = -i\frac{\alpha^2}{2} \), i.e., \( \sqrt{2c} = \alpha \) (we choose and fix one of the roots):

\[ D_s(g(X_s)m^s_k) - i\alpha (E + n)g(X_s)m^s_k = \lambda g(X_s)m^s_k \]

and substitute

\[ g(X_s) = g^j_k(X_s) = \sum_{l=0}^{j} \beta^{j,k}_l X^l_s. \]

Then (51) turns into the recursion relation

\[ \lambda \sum_{l=0}^{j} \beta^{j,k}_l X^l_s m^s_k = -i\alpha \sum_{l=0}^{j} \beta^{j,k}_l (l + n) X^l_s m^s_k + \sum_{l=0}^{j} \beta^{j,k}_l D_s(X^l_s m^s_k), \]
and noting \( D_s(X_s^l m_k^s) = -i \frac{l}{2}(2k + 2n + l - 1)X_s^{l-1}m_k^s \), see (49), we have
\[
\sum_{l=0}^{j} (\lambda + i\alpha(l + k + n))\beta_{l}^{j,k}X_s^{l}m_k^s = -i \sum_{l=0}^{j-1} \frac{l + 1}{2}(2k + 2n + l)\beta_{l+1}^{j,k}X_s^{l+1}m_k^s.
\]

Finally, we obtain the recurrence relations for \( l = 0, 1, \ldots, j - 1 \):
\[
(\lambda + i\alpha(l + k + n))\beta_{l}^{j,k} = -i \frac{l + 1}{2}(2k + 2n + l)\beta_{l+1}^{j,k},
\]
(53)
\[
(\lambda + i\alpha(l + k + n))\beta_{j}^{j,k} = 0.
\]
(54)

In order for \( g(X_s) \) to be a polynomial in \( x, y \) of degree \( j \), we need to have \( \lambda = -i\alpha(n + j + k) \) as an eigenvalue. Hence our recursion becomes
\[
\alpha(j - l)\beta_{l}^{j,k} = \frac{l + 1}{2}(2k + 2n + l)\beta_{l+1}^{j,k},
\]
which results in
\[
\beta_{l+1}^{j,k} = \frac{2\alpha(j - l)}{(l + 1)(2k + 2n + l)}\beta_{l}^{j,k} = \ldots = 2^{l+1}\alpha^{l+1}\left(\frac{j}{l + 1}\right)\frac{(2k + 2n - 1)!}{(2k + 2n - 1 + l)!}\beta_{0}^{j,k}.
\]
Therefore, we conclude that
\[
\beta_{l}^{j,k} = 2^{l}\alpha^{l}\left(\frac{j}{l}\right)\frac{(2k + 2n - 1)!}{(2k + 2n - 1 + l)!}\beta_{0}^{j,k},
\]
(55)
and we choose \( \beta_{0}^{j,k} = 1 \). Hence we have
\[
g_{j}^{k}(X_s) = \sum_{l=0}^{j} 2^{l}\alpha^{l}\left(\frac{j}{l}\right)\frac{(2k + 2n - 1)!}{(2k + 2n - 1 + l)!}X_s^{l}
\]
\[
= j!\frac{(2k + 2n - 1)!}{(2k + 2n - 1 + j)!}L_{j}^{2n+2k-1}(-2iX_s),
\]
(56)
where \( L_{j}^{\beta}(x) \) is the generalized Laguerre polynomial,
\[
L_{j}^{\beta}(x) = \sum_{l=0}^{j} (-1)^l \frac{(j + \beta)(j + \beta - 1)\ldots(j + \beta + l - j + 1)x^l}{l!}
\]
defined by the formula
\[
L_{j}^{\beta}(x) = x^{-\beta}e^x \frac{d^j}{dx^j}(x^{\beta}e^{-x}).
\]

The spectral decomposition of our operator, which can be termed the symplectic spin harmonic oscillator, is summarized in the following theorem.

**Theorem 4.2.** The operator \( H = D_s - cX_s \), \( c \in \mathbb{R} \), has a complete system of eigenfunctions (valued in the Segal-Shale-Weil representation) given by
\[
f_{j}^{k} = e^{\sqrt{2ic}X_s}L_{j}^{2n+2k-1}(-2\sqrt{2ic}X_s)m_k^s,
\]
(57)
where \( L_{j}^{\beta}(-2\sqrt{2ic}X_s) \) is the generalized Laguerre polynomial of the operator \(-2\sqrt{2ic}X_s\), \( m_k^s \in M_k^s \) is a symplectic monogenic and \( j, k \in \mathbb{N} \), with corresponding eigenvalue
\[
\lambda_{j}^{k} = \sqrt{2ic}(n + j + k).
\]
(58)

**Example 4.3.** The simplest eigenfunction for \( j = 0 \) is \( e^{\sqrt{2ic}X_s}e^{-\frac{x^2}{2}} \in M_0^s \), where \( e^{-\frac{x^2}{2}} \) is a highest weight vector of the Segal-Shale-Weil representation.
5 Fischer product and reproducing kernel on symplectic spinors

Let us briefly mention a motivation given by the classical orthogonal Fischer scalar product. For two complex polynomials valued in the Clifford algebra associated to a quadratic form, \( f \otimes a, g \otimes b \in \text{Pol}(\mathbb{R}^m, \mathbb{C}) \otimes \text{Cl}(\mathbb{R}^m) \), the Fischer scalar product is defined by

\[
\langle f \otimes a, g \otimes b \rangle = \left[ f(\partial_{x_j})g \right]_{x=0} = ab^0.
\]

Here \( f(\partial_{x_j}) \) is a differential operator, where we substitute \( \partial_{x_j} \) for the variable \( x_j \), \( j = 1, \ldots, m \), and act by the resulting differential operator on a polynomial \( g(x) \). As for the values, \([ \cdot ]_0\) denotes the zero degree part of an element in \( \text{Cl}(\mathbb{R}^m) \). The properties of scalar products are conveniently encoded in their reproducing kernels. For example, the space of homogeneous polynomials of homogeneity \( k \) satisfies

\[
\langle \langle x, y \rangle^k, g(x) \rangle = g(y)
\]

for all \( g \in \text{Pol}_k(\mathbb{R}^m, \mathbb{C}) \) and \( \langle \cdot, \cdot \rangle \) the canonical scalar product on \( \mathbb{R}^m \). Hence the reproducing kernel for homogeneity \( k \) harmonic polynomials \( \mathcal{H}_k \),

\[
Z_k(x, y) = \text{Proj}_{\mathcal{H}_k} \left( \frac{\langle x, y \rangle^k}{k!} \right),
\]

(59)
can be expressed by the use of the Gegenbauer polynomial. The interested reader can find more about this topic in, e.g., [2] and [7].

In what follows, we attempt to apply the concept of Fischer product and reproducing kernel to the space of symplectic spinors equipped with the action of the metaplectic Lie algebra. As in the previous section, after some general considerations we focus mostly on the real dimension 2.

5.1 Fischer product and reproducing kernel for \( n = 1 \)

We now aim to define the Fischer product on the space of symplectic spinors. We construct the symplectic Fischer product on \( \text{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes S(\mathbb{R}^n) \) for \( f \otimes \psi, g \otimes \phi \) with \( f, g \in \text{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \) and \( \psi, \phi \in S(\mathbb{R}^n) \), in the form

\[
\langle f \otimes \psi, g \otimes \phi \rangle = \omega(f, g) \int_{\mathbb{R}^n} \overline{\psi(q)}\phi(q) \, dq.
\]

(60)
The integral is the inner product in the fiber variables \( q_1, \ldots, q_n \) and \( \omega(f, g) \) is the evaluation of a lift of the symplectic form to symmetric tensors \( \text{Sym}_k(\mathbb{R}^{2n}) \), \( k \in \mathbb{N} \). We put

\[
\omega(v_1 \otimes \ldots \otimes v_k, w_1 \otimes \ldots \otimes w_k) = \sum_{(j_1, \ldots, j_k) \in \mathcal{S}_k} \omega(v_1, w_{j_1})\omega(v_2, w_{j_2})\ldots\omega(v_k, w_{j_k}),
\]

(61)
where \( v_j, w_j \in \mathbb{R}^{2n} \) and we sum over all even permutations of the set \( \{1, \ldots, k\} \).

As already advertised above, we now focus on the real 2-dimensional case and for a moment elaborate on the part of the inner product on \( \text{Pol}(\mathbb{R}^2, \mathbb{C}) \) given in (61). We normalize the lift of the symplectic form to be \( \omega(e_1, e_2) = 1 \) for \( v = xe_1 + ye_2 \in \mathbb{R}^2 \), and define the Fourier symplectic transformation by

\[
x \leftrightarrow \partial_y, \quad y \leftrightarrow -\partial_x.
\]
Consequently, we get for $r, s, t, u \in \mathbb{N}_0$
\[
\langle x^r y^s, x^t y^u \rangle = \omega(x^r y^s, x^t y^u) = (-1)^s \partial_y^r \partial_x^t y^u = (-1)^s u! s! \delta_{r, s} \delta_{t, u},
\]
and so we have for $f = x^r y^s, g = x^t y^u$ and $r + s = t + u$
\[
\omega(f, yg) = \langle f, yg \rangle = (-1)^s \partial_y^r \partial_x^t y^{u+1} = (-1)^s (u + 1)! s! \delta_{r, s} \delta_{t, u},
\]
\[
\omega(\partial_x f, g) = \langle \partial_x f, g \rangle = r (x^{r-1} y^s, x^t y^u) = (-1)^s r (r - 1)! s! \delta_{r, s} \delta_{t, u},
\]
\[
\omega(f, xg) = \langle f, xg \rangle = (-1)^s (t + 1)! s! \delta_{r, s} \delta_{t + 1, u},
\]
\[
\omega(\partial_y f, g) = \langle \partial_y f, g \rangle = (-1)^s - r (s - 1)! s! \delta_{r, s} \delta_{-t, u}.
\]
Hence, there are the relations
\[
\langle \partial_x f, g \rangle = \langle f, yg \rangle, \quad -\langle \partial_y f, g \rangle = \langle f, xg \rangle. \quad \tag{62}
\]
Let us now summarize our definitions and basic properties in the 2-dimensional case.

**Definition 5.1.** The symplectic Fischer product for $f(x, y) \otimes \psi, g(x, y) \otimes \phi$, with $f, g \in \text{Pol}(\mathbb{R}^2, \mathbb{C})$ and $\psi, \phi \in S(\mathbb{R})$, is given by
\[
\{f \otimes \psi, g \otimes \phi\} = [f(\partial_y, -\partial_x)g(x, y)]_{x=y=0} \int_{-\infty}^{\infty} \overline{\psi(q)} \phi(q) \, dq, \quad \tag{63}
\]
where the bar denotes the complex conjugation of a complex valued function.

**Lemma 5.2.** The bilinear form defined in **(63)** for all $a, b \in \text{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes S(\mathbb{R})$ satisfies
1. $\langle qa, b \rangle = \langle a, qb \rangle$, and $\langle iq a, b \rangle = \langle a, -iq b \rangle$.
2. $\langle \partial_q a, b \rangle = -\langle a, \partial_q b \rangle$ and $\langle i \partial_q a, b \rangle = \langle a, i \partial_q b \rangle$.
3. $\langle \partial_x a, b \rangle = \langle a, y b \rangle$.
4. $\langle \partial_y a, b \rangle = -\langle a, x b \rangle$.
5. $\langle xa, b \rangle = \langle a, \partial_y b \rangle$.
6. $\langle ya, b \rangle = -\langle a, \partial_x b \rangle$.

Now we compute the adjoints of operators $D_s, X_s$ with respect to $\langle \cdot, \cdot \rangle$.

**Lemma 5.3.** The adjoint operator for the symplectic Dirac operator $D_s$ with respect to the symplectic Fischer product is $X_s$, and vice versa. We have
\[
\langle D_s a, b \rangle = \langle a, X_s b \rangle, \quad \langle X_s a, b \rangle = \langle a, D_s b \rangle,
\]
for arbitrary $a, b \in \text{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes S(\mathbb{R})$.

**Proof:** A direct computation for $a, b \in \text{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes S(\mathbb{R})$ gives
\[
\langle D_s a, b \rangle = \langle (iq \partial_y - \partial_q \partial_x) a, b \rangle = \langle a, (iq x + \partial_q y) b \rangle,
\]
\[
\langle X_s a, b \rangle = \langle (iq x + \partial_q y) a, b \rangle = \langle a, (iq \partial_y - \partial_q \partial_x) b \rangle.
\]
\[\square\]

Consequently, we have the orthogonality relations for the symplectic Fischer decomposition,
\[
\langle X_s^j m_k^s, X_s^l m_h^s \rangle \sim \delta_{jj} \delta_{kh}, \quad \tag{64}
\]
with symplectic monogenics $m_k^s \in M_k^s, m_h^s \in M_h^s$. 

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Theorem 5.5. The projection operator $\text{Proj}^k_m$ and the reproducing kernel $Z_k$ relate to the symplectic Fischer product as follows:

1. $\text{Proj}^k_m$ is self-adjoint.
2. $Z_k(\xi_1, \xi_2, x, y, q, \partial_q) = \text{Proj}^k_m K_k(\xi_1, \xi_2, x, y)$ is the reproducing kernel for $M^*_k$.

Proof: Indeed, using this pairing, we first observe the self-adjointness property of $\text{Proj}^k_m$:

\[
\langle \text{Proj}^k_m f, g \rangle = \sum_{j=0}^{k} a^k_j \langle X^j f, D^j g \rangle = \sum_{j=0}^{k} a^k_j \langle f, X^j D^j g \rangle = \langle f, \text{Proj}^k_m g \rangle.
\]
By (69), we have for \(m_k^s \in M_k^s\)
\[
\langle Z_k(\xi_1, \xi_2, x, y, q, \partial_q), m_k^s(x, y, q) \rangle = \langle K_k(\xi_1, \xi_2, x, y), \text{Proj}_{sm}^k m_k^s \rangle
\]
\[
= m_k^s(\xi_1, \xi_2, q).
\]
and for any \(j \in \mathbb{N}\) holds
\[
\langle Z_k(\xi_1, \xi_2, x, y, q, \partial_q), X^j_s m^s_{k-j}(x, y, q) \rangle = \langle K_k(\xi_1, \xi_2, x, y), \text{Proj}_{sm}^k X^j_s m^s_{k-j} \rangle = 0.
\]

\[\square\]

**Proposition 5.6.** The reproducing kernel \(Z_k\) has the explicit form
\[
Z_k(\xi_1, \xi_2, x, y, q, \partial_q) = \sum_{j=0}^{k} \, \bar{\psi}^{\xi_j} \, \prod_{r=0}^{j-1} R_{k-j}^{k} \left( -q \xi_1 + i \partial_q \xi_2 \right) X^j_s \, \bar{\xi}_s,
\]
where \(\xi_s = -q \xi_1 + i \partial_q \xi_2\).

**Proof:** First we need an explicit formula for \(D^j_s K_k(\xi_1, \xi_2, x, y), j = 1, \ldots, k\). We obtain by the chain rule
\[
D_s K_k(\xi_1, \xi_2, x, y) = (i \partial_y - \partial_q \partial_x) \frac{1}{k!} ( -q \xi_1 + i \partial_q \xi_2 )^k
\]
\[
= K_{k-1}(\xi_1, \xi_2, x, y) (i \partial_x - \partial_q \xi_2) = i K_{k-1}(\xi_1, \xi_2, x, y) \xi_s.
\]
Therefore, \(D^j_s K_k(\xi_1, \xi_2, x, y) = \bar{\psi}^{\xi_j} K_{k-j}(\xi_1, \xi_2, x, y) \xi_s\), and so
\[
Z_k(\xi_1, \xi_2, x, y, q, \partial_q) = \sum_{j=0}^{k} \, \bar{\psi}^{\xi_j} X^j_s K_{k-j}(\xi_1, \xi_2, x, y) \xi_s
\]
\[
= \sum_{j=0}^{k} \, \bar{\psi}^{\xi_j} X^j_s \left( -q \xi_1 + i \partial_q \xi_2 \right)^{k-j} \left( \frac{1}{(k-j)!} \right) \xi_s,
\]
which proves the assertion.
\[\square\]

## 6 Explicit bases of symplectic monogenic on \((\mathbb{R}^2, \omega)\)

In the present section we construct some explicit bases for symplectic monogenics \(M^h_k\) in \(\text{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes S(\mathbb{R})\) of homogeneity \(h\), and prove several useful characterizing properties.

The first distinguished basis is written in the real coordinates \(x\) and \(y\) on \(\mathbb{R}^2\) and the (topological) basis \(q^j e^{-\frac{x^2}{2}}, j \in \mathbb{N}_0\), of the Schwartz space. The second distinguished basis for symplectic monogenics is written in the complex coordinates \(z\) and \(\bar{z}\) on \(\mathbb{R}^2 \simeq \mathbb{C}\) and the (topological) basis of Hermite functions \(\psi_j(q), j \in \mathbb{N}_0\), for \(S(\mathbb{R})\).

**Proposition 6.1.** The symplectic spinors of homogeneity \(h\) in the variables \(x, y\) and odd in the variable \(q\) for \(h, k \in \mathbb{N}_0\) with \(k \geq h\),
\[
\tilde{s}_{0,k}^h = e^{-\frac{x^2}{2}} \sum_{p=0}^{k} \, (-1)^p \frac{(2k+1)!!}{(2k-2p+1)!!} \left( \frac{h}{p} \right) q^{2k+1-2p} (x + iy)^{h-p} (iy)^p
\]
\[
= \sum_{p=0}^{k} \, (-1)^p \frac{(2k+1)!!}{(2k-2p+1)!!} \left( \frac{h}{p} \right) q^{2k+1-2p} (x + iy)^{h-p} (iy)^p
\]
and even in variable $q$ for $k \in \mathbb{N}_0$,

$$S^h_{e,k} = e^{-\frac{q^2}{2}} \sum_{p=0}^{h} (-1)^p \frac{(2k)!!}{(2k - 2p)!!} \frac{(h)}{p} q^{2k-2p}(x + iy)^{h-p}(iy)^p, \quad (71)$$

form a (topological) basis of the odd and even part of the symplectic monogenics $M^s_h$, respectively.

**Proof:** Let us consider a polynomial monogenic symplectic spinor

$$f(x, y, q) = e^{-\frac{q^2}{2}} \sum_{j=0}^{\infty} q^j p_j(x, y),$$

where $p_j(x, y)$ are polynomials in the variables $x, y$. Solving the equation $D_x f(x, y, q) = 0$, we have

$$0 = (iq\partial_y - \partial_x \partial_q) f(x, y, q)$$

$$= e^{-\frac{q^2}{2}} \sum_{j=0}^{\infty} (iq^{j+1}\partial_y p_j(x, y) + q^{j+1}\partial_x p_j(x, y) - jq^{j-1}\partial_z p_j(x, y)).$$

The Schwartz functions $e^{-\frac{q^2}{2}}q^j$, $j \in \mathbb{N}_0$ are linearly independent, hence

$$q^j ((\partial_x + i\partial_y) p_{j-1}(x, y) - (j + 1)\partial_x p_{j+1}(x, y)) = 0 \quad (72)$$

for each $j \in \mathbb{N}_0$. We get a system of recursion equations, splitting into two subsystems of odd and even homogeneity in the variable $q$ and the solution follows.

For a fixed homogeneity $h$ in the variables $x$ and $y$, the systems $s^h_{o,k}$ and $s^h_{e,k}$ contain all powers $q^j$, $j \in \mathbb{N}_0$, for appropriate $k$ and all possible combinations of $x, y$ in $\text{Pol}(\mathbb{R}^2, \mathbb{C})$ so that they are in $\text{Ker}(D_s)$. Therefore, the odd (70) and even (71) systems form a basis of $M^s_h$ because $\{q^j e^{-\frac{q^2}{2}}\}_{j \in \mathbb{N}}$ is a (topological) basis of $\mathcal{S}(\mathbb{R})$.

\[ \square \]

In the complex coordinates $z = x + iy, \bar{z} = x - iy$, with $\partial_x = \partial_z + \partial_{\bar{z}}$ and $\partial_y = i(\partial_z - \partial_{\bar{z}})$, the symplectic Dirac operator is by (35) given by

$$D_s = -(q + \partial_q)\partial_z + (q - \partial_q)\partial_{\bar{z}}. \quad (73)$$

Let us recall (see e.g. [13]) that the Hermite functions $\{\psi_k(q)\}_{k \in \mathbb{N}_0}$ form a (topological) basis of the Schwartz space $\mathcal{S}(\mathbb{R})$. The $k$-th Hermite function is

$$\psi_k(q) = \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} e^{-\frac{q^2}{2}} H_k(q) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} (q - \partial_q)^k e^{-\frac{q^2}{2}},$$

where $H_k$ is the $k$-th Hermite polynomial. The operators $(q + \partial_q)$ and $(q - \partial_q)$ act on the basis vectors by

$$(q + \partial_q)\psi_k = \sqrt{2} \sqrt{k} \psi_{k-1},$$

$$(q - \partial_q)\psi_k = \sqrt{2} \sqrt{k + 1} \psi_{k+1}, \quad (74)$$

and together with the operator acting by a multiple of identity on each $\psi_k$ form the representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We shall use the following easily verified formulas

$$\begin{aligned}
(q^2 - \partial_q^2)\psi_k &= (2k + 1)\psi_k, \\
(q - \partial_q)^2\psi_k &= 2\sqrt{(k + 1)(k + 2)}\psi_{k+2}, \\
(q + \partial_q)^2\psi_k &= 2\sqrt{k(k - 1)}\psi_{k-2}.
\end{aligned} \quad (75)$$
Proposition 6.2. The polynomial symplectic spinors of homogeneity $h$ in the variables $z, \bar{z}$ and odd in the variable $q$ for $k \in \mathbb{N}_0$,

$$s^h_{o,k} = \sum_{p=0}^{h} \sqrt{(2k + 2p)!!(2k + 2p + 1)!!} \binom{h}{p} \psi^{2k+2p+1}(q) z^{h-p} p^z, \quad (76)$$

form the basis of the odd part of the solution space of the symplectic Dirac operator $D_s$.

The polynomial symplectic spinors of homogeneity $h$ in the variables $z, \bar{z}$ and even in the variable $q$ for $k \in \mathbb{N}_0$,

$$s^h_{e,k} = \sum_{p=0}^{h} \sqrt{(2k + 2p - 1)!!(2k + 2p)!!} \binom{h}{p} \psi^{2k+2p}(q) z^{h-p} p^z, \quad (77)$$

and for $k = -1, -2, \ldots, -h$

$$s^h_{e,k} = \sum_{p=|k|}^{h} \sqrt{(2k + 2p - 1)!!(2k + 2p)!!} \binom{h}{p} \psi^{2k+2p}(q) z^{h-p} p^z, \quad (78)$$

form the basis of the even part of the solution space of the symplectic Dirac operator $D_s$.

**Proof:** Let us consider a polynomial monogenic symplectic spinor

$$f(z, \bar{z}, q) = \sum_{l=0}^{\infty} \psi_l(q) p_l(z, \bar{z}),$$

where $\psi_l(q)$ is the $l$-th Hermite function and $p_l(z, \bar{z})$ is a polynomial in the variables $z, \bar{z}$. The action of the symplectic Dirac operator is then

$$0 = D_s f(z, \bar{z}, q) = (\partial_q p_l(z, \bar{z}) - \partial_{\bar{z}} p_l(z, \bar{z})) \psi_{l+1}(q) - \partial_q p_l(z, \bar{z}) \psi_{l-1}(q) - \partial_{\bar{z}} p_l(z, \bar{z}) \psi_{l+1}(q) + \sqrt{2} \sum_{l=0}^{\infty} \sqrt{l} \psi_{l-1}(q) \partial_z p_l(z, \bar{z}) - \sqrt{l+1} \psi_{l+1}(q) \partial_{\bar{z}} p_l(z, \bar{z}).$$

The Hermite functions are linearly independent, which implies

$$\psi_l(q) (\sqrt{l+1} \partial_z p_{l+1}(z, \bar{z}) - \sqrt{l} \partial_{\bar{z}} p_{l-1}(z, \bar{z})) = 0 \quad (79)$$

for each $l \in \mathbb{N}_0$. The system of recursion equations is split into two systems with odd and even indexes in the variable $q$, each of which is easy to resolve.

For a fixed homogeneity $h$ the systems of symplectic polynomial spinors $(76), (77)$ and $(78)$ form a basis of symplectic monogenics $M^h_s$ of homogeneity $h$, and because the Hermite functions form a basis of $S(\mathbb{R})$ the above collection of symplectic monogenics is a (topological) basis of $\text{Ker}(D_s)$.

Let us now explore the properties of the symplectic Fischer product $(63)$ applied to the basis elements discussed in the Proposition 6.1 and Proposition 6.2. The motivation for this question is the existence of a basis of symplectic monogenics, which is isotropic with respect to the product $(63)$. 

□
Lemma 6.3. The basis elements \((\ref{70})\) and \((\ref{71})\) of homogeneity 2 in the symplectic Fischer product \((\ref{63})\) satisfy, for \(k, l \in \mathbb{N}, k, l \geq 2\),

\[
\langle s_{o,k}^2, s_{o,l}^2 \rangle = \frac{-3\sqrt{\pi}(2k + 2l - 5)}{2^{k+l-3}}, \\
\langle s_{o,k}^2, s_{e,l}^2 \rangle = \frac{-3\sqrt{\pi}(2k + 2l - 5)}{2^{k+l-2}}, \\
\langle s_{e,k}^2, s_{e,l}^2 \rangle = 0.
\]

Proof: Focusing just on the \(\text{Pol}(\mathbb{R}^2, \mathbb{C})\) part of the symplectic Fischer product \((\ref{63})\), the only non-zero combinations of \(x, y\) in homogeneity 2 are \(\langle x^2, y^2 \rangle = 2\) and \(\langle xy, xy \rangle = -1\). Then \(\int_{-\infty}^{\infty} e^{-q^2} q^t \, dq = \frac{\sqrt{\pi}(2t+1)}{2^t}\) for \(t \in \mathbb{N}_0\) and moreover, \(\int_{-\infty}^{\infty} e^{-q^2} q^t \, dq = 0\) for \(t\) odd.

Therefore, we see that the symplectic Fischer product \((\ref{63})\) of any two odd or even basis elements \((\ref{70}), (\ref{71})\) for \(k, l \geq 2\) is non-zero (in fact, negative) for \(k = l\). This implies that the symplectic Fischer product \((\ref{63})\) does not seem to be a convenient candidate for the scalar product on \(\text{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes S(\mathbb{R})\).

Let us rewrite the symplectic Fischer product \((\ref{63})\) in the complex variables. In the variables \(z, \bar{z}\), we have a non-trivial pairing for the pairs \(z \leftrightarrow -2i\partial_z\) and \(\bar{z} \leftrightarrow 2i\partial_{\bar{z}}\). Hence for \(f(z, \bar{z}) \otimes \psi, g(z, \bar{z}) \otimes \phi\), with \(f, g \in \text{Pol}(\mathbb{R}^2, \mathbb{C})\) and \(\psi, \phi \in S(\mathbb{R})\),

\[
\langle f \otimes \psi, g \otimes \phi \rangle = \left[\int_{-\infty}^{\infty} \psi(q) \phi(q) \, dq\right]_{z = \bar{z} = 0}.
\]

Let us look at the symplectic Fischer product \((\ref{80})\) for the low homogeneity basis elements \(s_{o,k}^3, (\ref{70})\), of odd part of the symplectic monogenics.

Example 6.4. In the homogeneity 2 and \(k, l \in \mathbb{N}\) holds

\[
\langle s_{o,k}^2, s_{o,l}^2 \rangle = -8(2k+1)!! \delta_{2k+1,2l+5} + 16(2k+2)!! \delta_{2k+2,2l+2} - 8(2k+4)!! \delta_{2k+5,2l+1},
\]

where just one of the Kronecker deltas on the previous display may be non-zero. We observe that for \(k = l\) holds \(\langle s_{o,k}^2, s_{o,k}^2 \rangle \neq 0\), because \(\delta_{2k+2,2l+2} \neq 0\).

In the homogeneity 3 and \(k, l \in \mathbb{N}\),

\[
\langle s_{o,k}^3, s_{o,l}^3 \rangle = -48i(2k+1)!! \delta_{2k+1,2l+7} - 16i(2k+2)!! \delta_{2k+2,2l+5} + \delta_{2k+3,2l+3} + 48i(2k+6)!! \delta_{2k+7,2l+1},
\]

where again just one Kronecker delta may be non-zero. For \(k = l\) the symplectic Fischer product gives zero, \(\langle s_{o,k}^3, s_{o,k}^3 \rangle = 0\), and the analogous conclusion \(\langle s_{o,k}^h, s_{o,k}^h \rangle = 0\) can be made for all odd homogeneities \(h\).

Let us now consider another skew-symmetric bilinear form on \(\text{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes S(\mathbb{R})\), which is skew-symmetric on \(S(\mathbb{R})\) and possesses several remarkable properties. We use again the complex variables on \(\mathbb{R}^2\).
Definition 6.5. Let us introduce a bilinear form $\langle \cdot, \cdot \rangle_1$ on symplectic spinors, defined on $f(z, \bar{z}) \otimes \psi, g(z, \bar{z}) \otimes \phi$ with $f, g \in \text{Pol}(\mathbb{R}^2, \mathbb{C})$ and $\psi, \phi \in S(\mathbb{R})$ by

$$
\langle f \otimes \psi, g \otimes \phi \rangle_1 = \sqrt{2} \left[ \frac{1}{h!} f(\partial_z, \partial_{\bar{z}}) g(z, \bar{z}) \right]_{z=\bar{z}=0} \int_{-\infty}^{\infty} (\partial_q \psi(q)) \phi(q) \, dq,
$$

where $h$ denotes the homogeneity of the polynomial $f(z, \bar{z})$.

In the monomial basis, we have for $r, s, t, u \in \mathbb{N}_0$

$$
\langle z^r z^s \otimes \psi, z^t z^u \otimes \phi \rangle_1 = \sqrt{2} \frac{r!s!}{(r+s)!} \delta_{r,t} \delta_{s,u} \int_{-\infty}^{\infty} (\partial_q \psi(q)) \phi(q) \, dq,
$$

where $\delta_{r,t}$ denotes the Kronecker delta. Moreover, for $a, b \in \text{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes S(\mathbb{R})$ holds

$$
\langle za,b \rangle_1 = \langle a, \partial_z b \rangle_1, \quad \langle \partial_z a, b \rangle_1 = \langle a, zb \rangle_1,
$$

Notice that the bilinear form $\langle \cdot, \cdot \rangle_1$ is not $\mathfrak{mp}(2, \mathbb{R})$-invariant on the whole space of symplectic spinors, because

$$
\langle H_i(f \otimes \psi), g \otimes \phi \rangle_1 - \langle f \otimes \psi, H_i(g \otimes \phi) \rangle_1
= [f, g] \int_{-\infty}^{\infty} q \psi(q) \phi(q) \, dq,
$$

$$
\langle X_i(f \otimes \psi), g \otimes \phi \rangle_1 - \langle f \otimes \psi, X_i(g \otimes \phi) \rangle_1
= \frac{1}{2} [f, g] \int_{-\infty}^{\infty} \left( q \psi(q) \phi(q) + 2q(\partial_q \psi(q))(\partial_q \phi(q)) \right) \, dq,
$$

$$
\langle Y_i(f \otimes \psi), g \otimes \phi \rangle_1 - \langle f \otimes \psi, Y_i(g \otimes \phi) \rangle_1
= \frac{1}{2} [f, g] \int_{-\infty}^{\infty} \left( q \psi(q) \phi(q) - 2q(\partial_q \psi(q))(\partial_q \phi(q)) \right) \, dq,
$$

with

$$
[f, g] := \sqrt{2} \left[ \frac{1}{h!} f(\partial_z, \partial_{\bar{z}}) g(z, \bar{z}) \right]_{z=\bar{z}=0}.
$$

However, $\langle \cdot, \cdot \rangle_1$ is $\mathfrak{mp}(2, \mathbb{R})$-invariant when restricted to any of the two irreducible subspaces of symplectic spinors (given by the subspaces of even and odd Schwartz functions, respectively.)

Let us now define the elements

$$
s^h_{E,l} = \frac{1}{2^h} \sum_{j=0}^{l} s^h_{e,j},
$$

which form a basis of even symplectic monogenics in the homogeneity $h$ (as well as the set $s^h_{e,l}, l \in \mathbb{N}_0$, cf. (77).)

Theorem 6.6. The basis elements $s^h_{o,k}, s^h_{E,k}, k \in \mathbb{N}_0$, of the polynomial symplectic monogenics of homogeneity $h$ in the variables $z, \bar{z}$ form two isotropic subspaces of the symplectic monogenics $M^*_h$ with respect to the form defined in (81). Namely, the basis elements satisfy

$$
\langle s^h_{o,k}, s^h_{o,l} \rangle_1 = 0, \quad \langle s^h_{o,k}, s^h_{E,l} \rangle_1 = \delta_{k,l},
$$

$$
\langle s^h_{E,k}, s^h_{E,l} \rangle_1 = 0, \quad \langle s^h_{E,k}, s^h_{o,l} \rangle_1 = -\delta_{k,l},
$$

for $k, l \in \mathbb{N}_0$ and $h \in \mathbb{N}_0$. Moreover, the form is identically zero for symplectic monogenics of different homogeneities $h, h'$.  

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Proof: Let us remind the orthonormality relation \( \int_{-\infty}^{\infty} \psi_k(q)\psi_l(q) \, dq = \delta_{k,l} \) for Hermite functions. Then the relations in the first column \((83)\) are obvious, because the derivative of a Hermite function \( \psi_k(q) \) is

\[
\partial_q \psi_k(q) = \sqrt{\frac{k}{2}} \psi_{k-1}(q) - \sqrt{\frac{k+1}{2}} \psi_{k+1}(q)
\]

and consequently, the integral in the bilinear form is zero.

As for the proof of the relation \( \langle s_{o,k}^h, s_{o,l}^h \rangle_1 = \delta_{k,l} \), we first prove

\[
\langle s_{o,k}^h, s_{o,l}^h \rangle_1 = 2^h (\delta_{k,l} - \delta_{k+1,l})
\]

for \( k, l \in \mathbb{N}_0 \). We use \((82)\) to simplify the calculation and get

\[
\langle s_{o,k}^h, s_{o,l}^h \rangle_1 = \sum_{p=0}^{h} \frac{(2k+2p+1)!!(2l+2p-1)!!}{(2k+2p+1)!!(2l+2p)!!} \frac{(h-p)!p!}{h!} \times \left( \delta_{2k+2p,2l+2p} - \delta_{2k+2p+1,2l+2p+1} \right).
\]

This is equal to \( \sum_{p=0}^{h} \binom{h}{p} = 2^h \) for \( k = l \), \(-2^h \) for \( k + 1 = l \) and zero otherwise. Then for the basis elements \( s_{E,l}^h \) we have \( \langle s_{o,k}^h, s_{E,l}^h \rangle_1 = \sum_{j=0}^{l} \delta_{k,j} - \sum_{j=0}^{l-1} \delta_{k+1,j} \), which is non-zero just for \( k = l \). The last relation in \((83)\) follows from the skew-symmetry of the integration in the variable \( q \). For different homogeneities, the statement easily follows from \((82)\).

\[
\square
\]

We remark that for \( k < 0 \), the elements \( s_{e,k}^h \) in \((78)\) satisfy

\[
\langle s_{e,k}^h, s_{e,l}^h \rangle_1 = 0, \quad \langle s_{o,k}^h, s_{e,l}^h \rangle_1 = 0, \quad \langle s_{E,j}^h, s_{e,l}^h \rangle_1 = 0, \quad \langle s_{E,j}^h, s_{e,l}^h \rangle_1 = 0,
\]

for each \( k, l \in \mathbb{Z}, -h \leq k < 0, -h \leq l \) and \( j \in \mathbb{N}_0 \).

### 6.1 The action of symmetry operators of the symplectic Dirac operator on the basis of symplectic monogenics

In the present part we determine the action of the symmetry operators introduced in Section \((82)\) on the basis of symplectic monogenics described in Proposition \((6.2)\). We remark that the action of the symmetry operators on the basis of symplectic monogenics described in Proposition \((6.1)\) is much more involved.

We shall start with the even component of the basis.

**Proposition 6.7.** The operators \( \partial_z \) and \( \partial_{\bar{z}} \) decrease the homogeneity in \( z, \bar{z} \) and preserve the elements of even basis \((84)\), \((85)\) of the kernel of the symplectic Dirac operator \( D_z \). In particular, for \( k \in \mathbb{Z}, k \leq -h, \)

\[
\partial_z s_{e,k}^h = h s_{e,k+1}^{h-1}, \\
\partial_{\bar{z}} s_{e,k}^h = h s_{e,k}^{h-1}, \quad \text{for } k \neq -h, \quad \partial_{\bar{z}} s_{e,-h}^h = 0.
\]

**Proof:** We verify the relation \( \partial_z s_{e,k}^h = h s_{e,k+1}^{h-1} \) for \( k \in \mathbb{N}_0 \), the others being analogous. We have

\[
\partial_z \sum_{p=0}^{h} \sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}} \binom{h}{p} \psi_{2k+2p}(q) \bar{z}^{h-p} z^p = \sum_{p=1}^{h} \sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}} \binom{h}{p} \psi_{2k+2p}(q) \bar{z}^{h-p} z^{p-1}.
\]
and a shift in the summation index by one gives
\[
\sum_{p=0}^{h-1} \sqrt{\frac{(2k+2p+1)!!}{(2k+2p+2)!!}} \frac{h!}{(h-p-1)!p!} \psi_{2k+2p+2}(q) z^{h-1-p} z^p = h s_{e,k+1}^h
\]
as required.

Proposition 6.8. The operators \( H_t, X_t \) and \( Y_t \), see (38), preserve the span of even elements of the basis \((77), (78)\) of the kernel of the symplectic Dirac operator, and the action on the basis elements is, for \( k \in \mathbb{Z} \), \( k \leq -h \), given by

\[
\begin{align*}
H_t s_{e,k}^h &= (h + 2k + \frac{1}{2}) s_{e,k}^h, \\
X_t s_{e,k}^h &= i(h + k + 1) s_{e,k+1}^h, \\
Y_t s_{e,k}^h &= i(k - \frac{1}{2}) s_{e,k-1}^h, \quad \text{for } k \neq -h, \quad Y_t s_{e,-h}^h = 0. \quad (87)
\end{align*}
\]

Proof: This is again a straightforward computation. For example, let us prove the relation \( X_t s_{e,k}^h = i(h + k + 1) s_{e,k+1}^h \), \( k \in \mathbb{N}_0 \):

\[
\begin{align*}
&\left(i \partial_z + \frac{i}{4}(q - \partial_q)^2\right) s_{e,k}^h \\
&= \frac{1}{2} \sum_{p=0}^{h} \sqrt{\frac{(2k+2p+1)!!}{(2k+2p+2)!!}} \frac{h!}{(h-p-1)!p!} \psi_{2k+2p+2}(q) z^{h-p} z^p \\
&= \frac{1}{2} \sum_{p=0}^{h} \sqrt{\frac{(2k+2p+1)!!}{(2k+2p+2)!!}} \frac{h!}{(h-p-1)!p!} \psi_{2k+2p+2}(q) z^{h-p+1} z^{p-1} \\
&\times 2 \sqrt{(2k+2p+1)(2k+2p+2)} \psi_{2k+2p+2}(q) z^{h-p} z^p,
\end{align*}
\]

and a shift by one in the summation index in the first sum gives

\[
\begin{align*}
&= \frac{1}{2} \sum_{p=0}^{h} \sqrt{\frac{(2k+2p+1)!!}{(2k+2p+2)!!}} \frac{h!}{(h-p-1)!p!} \psi_{2k+2p+2}(q) z^{h-p} z^p (h-p+\frac{1}{2}(2k+2p+2)) \\
&= i(h + k + 1) s_{e,k+1}^h.
\end{align*}
\]

Proposition 6.9. The operators \( Z_1 \) and \( Z_2 \), see (38), increase the homogeneity by one in the variables \( z, \bar{z} \) and preserve the even elements of the basis \((77), (78)\) of the kernel of the symplectic Dirac operator. The operators satisfy for \( k \in \mathbb{Z} \), \( k \leq -h \),

\[
\begin{align*}
Z_1 s_{e,k}^h &= 2(h + 1)(h + k + 1) s_{e,k}^{h+1}, \\
Z_2 s_{e,k}^h &= (h + 1)(2k - 1) s_{e,k-1}^{h+1}. \quad (88)
\end{align*}
\]

Proof: It follows from (38) that

\[
\begin{align*}
Z_1 &= -\frac{1}{2} \left( (q - \partial_q)^2 z^2 + 2(q^2 + \partial_q^2) z \bar{z} + (q + \partial_q)^2 \bar{z}^2 \right) \partial_z \\
&\quad + \bar{z} (E + 1)(2E + 1) + \frac{1}{2} \left( (q - \partial_q)^2 z + (q + \partial_q)(q - \partial_q) \bar{z} \right) (2E + 1) \\
Z_2 &= -\frac{1}{2} \left( (q - \partial_q)^2 z^2 + 2(q^2 + \partial_q^2) z \bar{z} + (q + \partial_q)^2 \bar{z}^2 \right) \partial_{\bar{z}} \\
&\quad - z (E + 1)(2E + 1) + \frac{1}{2} \left( (q - \partial_q)(q + \partial_q) z + (q + \partial_q)^2 \bar{z} \right) (2E + 1)
\end{align*}
\]
Now using (74), (75) and \((2E + 1)s^h_{e, k} = (2h + 1)s^h_{e, k}, (E + 1)s^h_{e, k} = (h + 1)s^h_{e, k}\), we verify the relation \(Z_2s^h_{e, k} = (h + 1)(2k - 1)s^{h+1}_{e, k-1}\) for \(k \in \mathbb{N}_0\):

\[
Z_2\left(\sum_{p=0}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \binom{h}{p} \psi_{2k+2p}(q)z^{h-p}z^p\right)
\]

is equal to the sum of three summations:

\[
-\frac{1}{2} \sum_{p=0}^{h-1} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \binom{h}{p} (h - p)
\]

\[
2\left(\sqrt{(2k + 2p + 1)(2k + 2p + 2)}\psi_{2k+2p+2}(q)z^{h-p-1}z^{p+2}
+ (4k + 4p + 1)\psi_{2k+2p}(q)z^{h-p}z^{p+1}
+ \sqrt{(2k + 2p)(2k + 2p - 1)}\psi_{2k+2p-2}(q)z^{h-p+1}z^{p}\right)
\]

\[
- \frac{1}{2} \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \binom{h}{p} (h + 1)(2h + 1)\psi_{2k+2p}(q)z^{h-p}z^{p+1}
\]

\[
+ \frac{1}{2} \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \binom{h}{p} (2h + 1)2\left((2k + 2p)\psi_{2k+2p}(q)z^{h-p}z^{p+1}
+ \sqrt{(2k + 2p)(2k + 2p - 1)}\psi_{2k+2p-2}(q)z^{h-p+1}z^{p}\right).
\]

We reorganize the sums to get the contributions to a given Hermite function,

\[
= \sum_{p=0}^{h-1} \sqrt{\frac{(2k + 2p + 1)!!}{(2k + 2p + 2)!!}} \psi_{2k+2p+2}(q)z^{h-p-1}z^{p+2} \binom{h}{p} (p - h)(2k + 2p + 2)
\]

\[
+ \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \psi_{2k+2p}(q)z^{h-p}z^{p+1} \binom{h}{p} (p - h)(4k + 4p + 1) + (2h + 1)(2k + 2p - h - 1)
\]

\[
+ \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p - 3)!!}{(2k + 2p - 2)!!}} \psi_{2k+2p-2}(q)z^{h-p+1}z^{p} \binom{h}{p} (2k + 2p - 1)(h + p + 1),
\]

and do appropriate shifts in summations and multiple expressions to produce the required combinatorial coefficients:

\[
= \sum_{p=0}^{h+1} \sqrt{\frac{(2k + 2p - 3)!!}{(2k + 2p - 2)!!}} \psi_{2k+2p-2}(q)z^{h-p+1}z^{p} \frac{(h + 1)!}{(h - p + 1)!p!}
\]

\[
\frac{1}{h - 1} (- (p - 1)p(2k + 2p - 2) + p(p - 1 - h)(4k + 4p - 3)
+ p(2h + 1)(2k + 2p - h - 3) + (h - p + 1)(2k + 2p - 1)(h + p + 1))
\]

\[
= s^h_{e, k-1}(h + 1)(2k - 1).
\]

The remaining equalities are analogous. □

The proof of the analogous statement for the odd part of the basis of the kernel of the symplectic Dirac operator is analogous to the even part in the previous proposition and so is omitted. This result is summarized in the next proposition.
Proposition 6.10.  

1. The operators $\partial_{z}$ and $\partial_{\bar{z}}$ decrease the homogeneity in the variables $z, \bar{z}$ by one and preserve odd elements of the basis (76), $k \in \mathbb{N}_0$, of the kernel of the symplectic Dirac operator $D_s$:

\[
\begin{align*}
\partial_z s_{o,k}^h &= hs_{o,k+1}^{h-1}, \\
\partial_{\bar{z}} s_{o,k}^h &= hs_{o,k}^{h-1}.
\end{align*}
\]  

(89)

2. The operators $H_t, X_t$ and $Y_t$ preserve odd elements of the basis (76), $k \in \mathbb{N}_0$, of the kernel of the symplectic Dirac operator:

\[
\begin{align*}
H_t s_{o,k}^h &= (h + 2k + \frac{3}{2})s_{o,k}, \\
X_t s_{o,k}^h &= i(h + k + \frac{3}{2})s_{o,k+1}, \\
Y_t s_{o,k}^h &= ik s_{o,k-1}, \quad \text{for } k \neq 0, \quad Y_t s_{o,0}^h = 0.
\end{align*}
\]  

(90)

3. The operators $Z_1$ and $Z_2$ increase the homogeneity by one in the variables $z, \bar{z}$, and map odd elements of the basis (76), $k \in \mathbb{N}_0$, to the elements of odd basis of the homogeneity plus one higher of the kernel of the symplectic Dirac operator,

\[
\begin{align*}
Z_1 s_{o,k}^h &= (1 + h)(2h + 2k + 3)s_{o,k}^{h+1}, \\
Z_2 s_{o,k}^h &= 2(h + 1)ks_{o,k-1}^{h+1}.
\end{align*}
\]  

(91)

Acknowledgement:  M. Holíková is supported by the Faculty of Mathematics and Physics (Charles University), grant SVV-2015-260227. P. Somberg is supported by the Grant Agency of Czech Republic, grant GA CR P201/12/G028. H. De Bie is supported by the Fund for Scientific Research-Flanders (FWO-V), grant G.0116.13N.

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