Numerical ranges of $C_0(N)$ contractions

Chafiq Benhida, Pamela Gorkin and Dan Timotin

Mathematics Subject Classification (2010). 47A12, 47A20.
Keywords. Contraction, unitary dilation, numerical range.

Abstract. A conjecture of Halmos proved by Choi and Li states that the closure of the numerical range of a contraction on a Hilbert space is the intersection of the closure of the numerical ranges of all its unitary dilations. We show that for $C_0(N)$ contractions one can restrict the intersection to a smaller family of dilations. This generalizes a finite dimensional result of Gau and Wu.

1. Introduction

Suppose $\mathcal{H}, \mathcal{H}'$ are separable Hilbert spaces; we will denote by $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ the space of bounded linear operators $T : \mathcal{H} \to \mathcal{H}'$ and $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$. The numerical range of an operator $T \in \mathcal{L}(\mathcal{H})$ is the set

$$W(T) := \{ \langle Tx, x \rangle : \|x\| = 1 \}.$$ 

Much is known about this set; for example, it is convex, in the finite-dimensional case it is compact, and if $T$ is normal, the closure of $W(T)$ is the convex hull of the spectrum of $T$. In general, however, the numerical range is difficult to compute. In this paper, we study new ways of obtaining the numerical range of a contraction $T$ from the numerical ranges of certain unitary dilations of $T$.

If there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an operator $\hat{T} \in \mathcal{L}(\mathcal{K})$ such that $T = P_\mathcal{H} \hat{T}|\mathcal{H}$, where $P_\mathcal{H}$ denotes the orthogonal projection onto $\mathcal{H}$, the operator $T$ is said to dilate to the operator $\hat{T}$. (We note that we are considering the so-called weak dilations here, and not power dilations treated in Sz.-Nagy dilation theory.) The operator $\hat{T}$ is said to be a dilation of $T$; more precisely, if $\dim(\mathcal{K} \ominus \mathcal{H}) = k$, then $\hat{T}$ is called a $k$-dilation.

We will be interested in unitary dilations. A result of Halmos [14 Problem 222(a)] shows that every contraction $T$ has unitary dilations. It is easy to see that

$$\overline{W(T)} \subseteq \cap \{ \overline{W(U)} : U \text{ is a unitary dilation of } T \}.$$
Choi and Li showed that, in fact,
\[ W(T) = \bigcap \{ W(U) : U \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } T \}, \]
answering a question raised by Halmos (see, for example, [13]). We note that in the case that \( \mathcal{H} \) is \( n \)-dimensional, these unitary dilations are \( n \)-dilations; that is, the dilations are of size \( 2^n \times 2^n \).

Before Choi and Li’s work was completed, Gau and Wu [9] studied the so-called compressions of the shift on finite-dimensional spaces and their numerical ranges. If \( SS_n \) is the class of all completely nonunitary contractions \( T \) (that is, \( \|T\| \leq 1 \) and \( T \) has no eigenvalue of modulus one) on an \( n \)-dimensional space with \( \text{rank}(I - T^*T) = 1 \), Gau and Wu [9, Corollary 2.8] showed that, in fact, if \( T \in SS_n \), then
\[ W(T) = \bigcap \{ W(U) : U \text{ is an } (n+1)\text{-dimensional unitary dilation of } T \}. \]

(There is no need to take the closure in the case of finite-dimensional spaces.) Thus, the unitary dilations may be chosen to be 1-dilations when \( \text{rank}(I - T^*T) = 1 \). An extension of this result can be found in [8]: namely, if \( T \) is an \( n \times n \) contraction with \( \text{rank}(I - T^*T) = k \), then
\[ W(T) = \bigcap \{ W(U) : U \in M_{n+k} \text{ is a unitary } k\text{-dilation of } T \}. \]  

(1.1)

It is easy to see that if \( \text{rank}(I - T^*T) = k \), then \( T \) has no unitary \( \ell \)-dilations for \( \ell < k \), which explains why Gau, Li and Wu refer to (1.1) in [8] as the most “economical” solution to the Halmos problem. We also refer the reader to the papers [10], [11], [12], and [20] for work related to this discussion. These authors, as well as others, (in particular, [16], [17], and [5]) have studied this problem from a geometric point of view.

The analogue of \( SS_n \) on a space of infinite dimension is the class of contractions with \( \text{rank}(I - T^*T) = \text{rank}(I - TT^*) = 1 \) for which \( T^n \) and \( T^{*n} \) tend strongly to 0. It is well known (see, for instance, [19]) that such a \( T \) is unitarily equivalent to some model operator \( S_\theta \) defined as follows: Suppose \( S \) is the unilateral shift on \( H^2 \). For \( \theta \) an inner function on the unit disc \( \mathbb{D} \), define \( K_\theta = H^2 \ominus \theta H^2 \) and \( S_\theta = P_{K_\theta}S|_{K_\theta} \). (The operator \( S_\theta \) is often called a compression of the shift.) Noting that when \( \theta(0) = 0 \) all unitary 1-dilations of \( S_\theta \) are equivalent to rank-1 perturbations of \( S_{z\theta} \), the authors of [3] show that when \( \theta = B \) is a Blaschke product we have
\[ W(S_B) = \bigcap \{ W(U) : U \text{ a rank-1 perturbation of } S_{zB} \}. \]

Our goal in this paper is to extend these results to operator-valued inner functions. After two preliminary sections, the main results appear in Section 4, where we show that the closure of the numerical range of \( S_\Theta \), where \( \Theta \) is an inner function in \( H^2(\mathbb{C}^N) \), is the intersection of the closures of the numerical ranges of an appropriate family of unitary dilations of \( S_\Theta \) (see Corollary 4.7). In Theorem 4.8 this result is extended to a larger class of contractions, called \( C_0(N) \) (see Definition 2.3). In Section 5 we describe the spectrum of the unitary dilations, obtaining a generalization of the scalar
case. We conclude the paper with a brief discussion of a conjecture about the numerical ranges of contractions with finite defect index.

2. Preliminaries

2.1. Matrix–valued analytic functions

The basic reference that we will use for matrix–valued analytic functions (or, equivalently, functions with values in $\mathcal{L}(\mathbb{C}^N)$) is [15]; our definitions are simpler since we will consider only bounded (in the operator norm) analytic functions $F : \mathbb{D} \to \mathcal{M}_N$ (the set of $N \times N$ matrices). These share certain factorization properties similar to those of scalar analytic functions.

A bounded analytic matrix-valued function $F : \mathbb{D} \to \mathcal{M}_N$ is called outer if $\det F(z)$ is outer, and inner if the boundary values (which can be defined as radial limits almost everywhere) are isometries for almost all $e^{it} \in \mathbb{T}$.

It is known [15, Theorem 5.4] that any analytic bounded $F$ can be factorized as

$$F = \Theta E$$

(2.1)

where $\Theta$ is inner and $E$ is outer, and, if $F = \hat{\Theta} \hat{E}$, then $\hat{\Theta} = \Theta V$, $\hat{E} = V^* E$ for some constant unitary $V$.

The inner function appearing in (2.1) can be further factorized in two parts. Recall that a Blaschke–Potapov factor $b(P, \lambda)(z)$ determined by a point $\lambda \in \mathbb{D}$ and an orthogonal projection $P$ on $\mathbb{C}^N$ is an inner function given by the formulas

$$b(P, \lambda)(z) = \frac{|\lambda|}{\lambda} \frac{\lambda - z}{1 - \lambda z} P + (I - P) \text{ for } \lambda \neq 0, \quad b(P, 0) = zP + (I - P).$$

A finite Blaschke–Potapov product is a product

$$B_n(z) = b(P_1, \lambda_1)(z) \cdots b(P_n, \lambda_n)(z),$$

for some $\lambda_j, P_j, j = 1, \ldots, n$. If $(\lambda_j)$ is a Blaschke sequence in $\mathbb{D}$ (that is, $\sum_j (1 - |\lambda_j|) < \infty$), while $P_j$ is an arbitrary sequence of projections on $\mathbb{C}^N$, then the sequence $B_n(z)$ converges at each point $z \in \mathbb{D}$ to $B(z)$, where $B$ is an inner function denoted by $\prod_j b(P_j, \lambda_j)$. A function that can be written as $B(z)V$, where $V$ is a constant unitary, is called an (infinite) Blaschke–Potapov product. The convergence is uniform on all compact subsets of $\mathbb{D}$. (Note that such a function is sometimes called a left Blaschke–Potapov product; we will not have the occasion to use right Blaschke–Potapov products.) Finally, an inner function $\Theta$ is called singular if $\det \Theta(z) \neq 0$ for all $z \in \mathbb{D}$.

With these definitions, Theorem 4.1 in [15] states that any inner function $\Theta$ decomposes as $\Theta = BS$, where $B$ is a (finite or infinite) Blaschke–Potapov product and $S$ is singular. As in the case of inner–outer factorization, the decomposition is unique up to a unitary constant; more precisely, if we also have $\Theta = \hat{B} \hat{S}$ with $\hat{B}$ a Blaschke–Potapov product and $\hat{S}$ singular, then $\hat{B} = BV$ and $\hat{S} = V^* S$ for some constant unitary $V$.

The next lemma is a Frostman-type theorem that follows from [15].
Lemma 2.1. Every inner function $\Theta$ in $H^2(\mathbb{C}^N)$ is a uniform limit of infinite Blaschke–Potapov products.

Proof. For $\lambda \in \mathbb{D}$, $(\Theta - \lambda I)(I - \bar{\lambda} \Theta)^{-1}$ is inner and $I - \lambda \Theta$ is outer; thus

$$\Theta - \lambda I = ((\Theta - \lambda I)(I - \bar{\lambda} \Theta)^{-1})(I - \bar{\lambda} \Theta)$$

is the inner–outer factorization of $\Theta - \lambda I$. But Corollary 6.1 from [15] says that for a dense set of $\lambda \in \mathbb{D}$ the inner factor of $\Theta - \lambda I$ is a Blaschke–Potapov product. If we take a sequence $\lambda_n \to 0$ with this property and we denote the corresponding Blaschke–Potapov product by $B^{(n)}$, then

$$B^{(n)} = (\Theta - \lambda_n I)(I - \bar{\lambda}_n \Theta)^{-1},$$

whence

$$\Theta = \lambda_n I + B^{(n)}(I - \bar{\lambda}_n \Theta) = \lim_{n \to \infty} B^{(n)}.$$

□

2.2. Model spaces

Let $\mathcal{E}, \mathcal{E}^*$ be Hilbert spaces. Suppose we are given an operator-valued inner function $\Theta(z) : \mathcal{E} \to \mathcal{E}^*$. The model space associated to it is

$$K_\Theta := H^2(\mathcal{E}^*) \ominus \Theta H^2(\mathcal{E}),$$

The operator $T_\Theta : H^2(\mathcal{E}) \to H^2(\mathcal{E}^*)$ defined by $T_\Theta f = \Theta f$ is an isometry, and we have

$$P_{K_\Theta} = I - T_\Theta T_\Theta^*.$$  \hspace{1cm} (2.2)

In particular, $\Theta(z) = zI_{\mathcal{E}^*} : \mathcal{E} \to \mathcal{E}^*$ is inner; we will denote the corresponding $T_z$ simply by $T_z$. The model operator $S_\Theta$ is the compression of $T_z$ to $K_\Theta$; that is, $S_\Theta = P_{K_\Theta} T_z P_{K_\Theta}|K_\Theta$.

An inner function $\Theta$ is called pure if it has no constant unitary direct summand; this is equivalent to assuming $\|\Theta(0)x\| < \|x\|$ for all $x \neq 0$. A general inner function is the direct sum of a pure inner function and a unitary constant; from the point of view of model spaces and operators we may consider only pure inner functions. Thus, from now on, we assume that $\Theta$ is a pure inner function.

Recall that the defect operators and spaces of a contraction $T$ are defined by $D_T = (I - T^*T)^{1/2}$ and $D_T^* = \overline{\text{ran}D_T}$. The next lemma shows how one can identify the defect spaces of $S_\Theta$; a good reference is [7, Section 1].

Lemma 2.2. Suppose $\Theta(z) : \mathcal{E} \to \mathcal{E}^*$ is a pure inner function; in particular, $D_{\Theta(0)}$ and $D_{\Theta(0)^*}$ have dense ranges. Define the maps $\iota : \mathcal{E} \to H^2(\mathcal{E}^*)$, $\iota_* : \mathcal{E}^* \to H^2(\mathcal{E}^*)$ (on dense domains) by

$$\iota(D_{\Theta(0)}\xi) = \frac{1}{z}(\Theta(z) - \Theta(0))\xi, \quad \xi \in \mathcal{E};$$

$$\iota_*(D_{\Theta(0)^*}\xi^*) = (I - \Theta(z)\Theta(0)^*)\xi^*, \quad \xi^* \in \mathcal{E}^*.\hspace{1cm} (2.3)$$
Numerical ranges of $C_0(N)$ contractions

Then $\iota$ and $\iota_*$ are isometries with ranges $D_{S_\Theta}$ and $D_{S_\Theta^*}$ respectively, and the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\iota} & D_{S_\Theta} \\
\downarrow_{-\Theta(0)} & & \downarrow_{S_\Theta} \\
\mathcal{E}_* & \xrightarrow{\iota_*} & D_{S_\Theta^*}
\end{array}
$$

(2.4)

In particular, $\dim D_{S_\Theta} = \dim \mathcal{E}$ and $\dim D_{S_\Theta^*} = \dim \mathcal{E}_*$.

We will occasionally write $\iota_\Theta$ and $\iota_*^\Theta$ to indicate the dependence on $\Theta$.

From the Sz-Nagy–Foias theory it follows that any $C_0$ contraction $T$ (that is, a contraction such that the powers of the adjoint tend strongly to 0) is unitarily equivalent to some $S_\Theta$, where we can take $\mathcal{E} = D_T$ and $\mathcal{E}_* = D_{T^*}$.

We are actually interested in the particular case when $\dim D_T = \dim D_T^* = N < \infty$. The following definition appears in [19].

**Definition 2.3.** A contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be of class $C_0(N)$ if $\dim D_T = \dim D_T^* = N < \infty$ and $T^n, T^*n$ tend strongly to 0.

If $T \in C_0(N)$, then $T$ is unitarily equivalent to $S_\Theta$, with $\Theta(z) : \mathbb{C}^N \to \mathbb{C}^N$. In this case $\Theta(0)$ is a strict contraction, and formulas (2.3) are defined on all of $\mathbb{C}^N$. The next lemma collects a few facts that we shall use.

**Lemma 2.4.** Suppose $\Theta(z) : \mathbb{C}^N \to \mathbb{C}^N$ is an inner function.

(i) $K_\Theta$ is finite dimensional if and only if $\Theta$ is a finite Blaschke–Potapov product.

(ii) If $\Theta_n \to \Theta$ in the uniform norm, then $P_{K_{\Theta_n}} \to P_{K_\Theta}$ uniformly.

(iii) Let $b_j = b(P_j, \lambda_j)$. If $B = \prod b_k$ is an infinite Blaschke–Potapov product and $B_n = b_1 \cdots b_n$, then

(a) $K_B = \bigcup_n K_{B_n}$.

(b) $B_n \xi \to B \xi$ in $H^2(\mathbb{C}^N)$, for any $\xi \in \mathbb{C}^N$.

**Proof.** Statement (i) can be found, for instance, in [15, Ch.2, Lemma 5.1], while (ii) follows from (2.2).

As for (iii), a standard normal family argument shows that $BH^2(\mathbb{C}^N) = \bigcap_n B_n H^2(\mathbb{C}^N)$, and therefore (a) follows by passing to orthogonal complements.

For (b), write $B = B_n \tilde{B}_n$, where $\tilde{B}_n$ is also an infinite Blaschke–Potapov product. If $B(0)$ is invertible, the pointwise convergence of $B_n$ to $B$ implies that $\tilde{B}_n(0) \to I_{\mathbb{C}^N}$, whence (taking norms and scalar products in $H^2(\mathbb{C}^N)$)

$$
\|B_n \xi - B \xi\|^2 = 2\|\xi\|^2 - 2R(B_n \xi, B \xi) = 2\|\xi\|^2 - 2R(\xi, \tilde{B}_n \xi) \\
= 2\|\xi\|^2 - 2R(\xi, \tilde{B}_n(0) \xi) \to 0.
$$

In the general case, write $B_n = CD_n$, where $C$ contains the Blaschke–Potapov factors $b(P, \lambda)$ corresponding to $\lambda = 0$. We have then $B = CD$ (with $D$ an infinite Blaschke–Potapov product), while the previous argument shows that
$D_n \xi \to D \xi$ in $H^2(\mathbb{C}^N)$. Multiplying with the inner function $C$ yields the result.

\section{Unitary $N$-dilations}

The next result is folklore; we give a short proof.

\textbf{Proposition 3.1.} Suppose $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$. If $U \in \mathcal{L}(\mathcal{H} \oplus \mathcal{E})$ is a unitary $N$-dilation of $T$, then there exist unitary operators $\omega : \mathcal{E} \to \mathcal{D}_T, \omega_* : \mathcal{E} \to \mathcal{D}_{T^*}$, such that

$$U = \begin{pmatrix} T & D_{T^*} \omega \\ \omega_* D_T & -\omega_* T^* \omega \end{pmatrix}.$$  \hspace{1cm} (3.1)

Conversely, any choice of $\omega : \mathcal{E} \to \mathcal{D}_T, \omega_* : \mathcal{E} \to \mathcal{D}_{T^*}$ yields, through formula \[(3.1),\] a unitary $N$-dilation of $T$.

\textbf{Proof.} Theorem 1.3 of [1] says that if $(T_{12} T_{21} T_{22}) : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H} \oplus \mathcal{E}$ is a contraction, then there exist contractions $\Gamma_1 : \mathcal{E} \to \mathcal{D}_{T^*}, \Gamma_2 : \mathcal{D}_T \to \mathcal{E}$ and $\Gamma : \mathcal{D}_{\Gamma_1} \to \mathcal{D}_{\Gamma_2^*}$ such that $T_{12} = D_{T^*} \Gamma_1, T_{21} = \Gamma_2 D_T$ and $T_{22} = -\Gamma_2 T^* \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1}$. We apply this result to $U$.

Since

$$\|U(x \oplus 0)\|^2 = \|Tx\|^2 + \|\Gamma_2 D_T x\|^2 \leq \|Tx\|^2 + \|D_T x\|^2 = \|x\|^2,$$

and the first column of $U$ is an isometry, the last term is equal to the first; so the middle inequality is an equality. This means that $\Gamma_2$ acts isometrically on the image of $D_T$; but this is precisely $\mathcal{D}_T$, whence $\Gamma_2$ has to be an isometry.

In fact, $\Gamma_2$ is unitary, since it acts between spaces of the same dimension $N$. Similarly, we obtain that $\Gamma_1$ is unitary, which implies $\Gamma$ acts between 0 spaces. The result follows if we let $\omega = \Gamma_1$ and $\omega_* = \Gamma_2^*$.

The converse is immediate. \hspace{1cm} $\square$

We can write \[(3.1)\] as

$$U = \begin{pmatrix} I & 0 \\ 0 & \omega_* \end{pmatrix} \begin{pmatrix} T & D_{T^*} \\ \omega_* D_T & -T^* \omega \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \omega \end{pmatrix}.$$  \hspace{1cm} (3.2)

The next corollary follows immediately from this formula.

\textbf{Corollary 3.2.} Suppose $T \in \mathcal{L}(\mathcal{H})$ is a contraction with $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = N < \infty$ and

$$u = \begin{pmatrix} T \\ u_{21} \\ u_{22} \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{E})$$

is a unitary $N$-dilation of $T$. Then any unitary $N$-dilation $U$ of $T$ on $\mathcal{H} \oplus \mathcal{E}'$ (for some Hilbert space $\mathcal{E}'$ with $\dim \mathcal{E}' = N$) is given by the formula

$$U = \begin{pmatrix} I & 0 \\ 0 & \Omega_* \end{pmatrix} u \begin{pmatrix} I & 0 \\ 0 & \Omega \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \omega \end{pmatrix}.$$  \hspace{1cm} (3.2)
where \( \Omega, \Omega^* : \mathcal{E}' \to \mathcal{E} \) are unitaries.

We are interested in consequences for \( S_\Theta \). The notation below refers to that of Lemma 2.2.

**Lemma 3.3.** All unitary \( N \)-dilations of \( S_\Theta \) to \( K_\Theta \oplus \mathbb{C}^N \) can be indexed by unitaries \( \Omega, \Omega^* : \mathbb{C}^N \to \mathbb{C}^N \), according to the formula

\[
U_{\Omega, \Omega^*} = \left( \begin{array}{c}
S_\Theta & \ell_\ast D_{\Theta(0)} \ast \Omega \\
\Omega^*_\ast D_{\Theta(0)} \ast \ell^* & \Omega^*_\ast \Theta(0) \ast \Omega
\end{array} \right).
\]

(3.3)

**Proof.** Let us apply Proposition 3.1 (the part stated as a converse) to the case \( T = S_\Theta, \mathcal{E} = \mathbb{C}^N, \omega = \ell_\ast, \omega_\ast = \ell \). We obtain the following \( N \)-dilation of \( T \) to \( \mathcal{H} \oplus \mathbb{C}^N \):

\[
V = \left( \begin{array}{c}
S_\Theta & D_{\Theta(0)} \ast \ell^* \\
\ell^* D_{S_\Theta} & -\ell^* S_\Theta \ast \ell_\ast
\end{array} \right).
\]

The commutative diagram (2.4) yields the relations

\[
\begin{align*}
S_\Theta \ast \ell = -\ell_\ast \Theta(0), & \quad \ell^* S_\Theta^* = -\Theta(0) \ast \ell_\ast. 
\end{align*}
\]

(3.4)

It follows immediately that \( -\ell_\ast S_\Theta^* \ell_\ast = \Theta(0)^\ast \).

From (3.4) we have \( \ell^* S_\Theta \ast \ell = \Theta(0)^\ast \Theta(0) \), whence \( \ell^* I_{\mathcal{H}} - S_\Theta^* S_\Theta \ell = I_{\mathbb{C}^N} - \Theta(0)^\ast \Theta(0) \) and thus \( \ell^* D_{S_\Theta} \ell = D_{\Theta(0)} \). Multiplying the last relation with \( \ell^* \) on the right, and taking into account that \( \ell^* = P_{D_{S_\Theta}} \) and \( D_{S_\Theta} P_{D_{S_\Theta}} = D_{S_\Theta} \), we obtain

\[
\ell^* D_{S_\Theta} = D_{\Theta(0)} \ell^*.
\]

A similar computation yields

\[
D_{S_\Theta} \ast \ell = \ell_\ast D_{\Theta(0)} \ast \ell
\]

and thus

\[
V = \left( \begin{array}{c}
S_\Theta & \ell_\ast D_{\Theta(0)} \ast \ell^* \\
D_{\Theta(0)} \ast \ell^* & \Theta(0)^\ast
\end{array} \right).
\]

(3.5)

Applying Corollary 3.2 to \( \mathcal{E} = \mathcal{E}' = \mathbb{C}^N \) and \( \textbf{v} = V \) finishes the proof.

We have thus parametrized all unitary \( N \)-dilations of \( S_\Theta \) to \( K_\Theta \oplus \mathbb{C}^N \) by pairs of unitaries on \( \mathbb{C}^N \). If we are interested only in classes of unitary equivalence, we may take a single unitary as parameter, since \( U_{\Omega, \Omega^*}, U_{\Omega^* \ast \ell, \ell} \), and \( U_{I, \Omega, \Omega^*} \) are all unitarily equivalent, and therefore have the same numerical range. In the sequel, we let

\[
U_\Theta := \left( \begin{array}{c}
S_\Theta & \ell_\ast D_{\Theta(0)} \ast \Omega \\
D_{\Theta(0)} \ast \ell^* & \Theta(0)^\ast \Omega
\end{array} \right).
\]

(3.5)

**4. The main result**

Let \( \mathfrak{K} \) denote the complete metric space of all nonempty compact subsets of \( \mathbb{C} \), endowed with the Hausdorff distance \( \mathfrak{d} \). Suppose that \( A \in \mathfrak{K} \) and \( \tau : X \to \mathfrak{K} \) is a continuous mapping defined on some compact space \( X \). We will say that \( \tau \) *wraps* \( A \) if for each open half-plane \( \mathbb{H} \) in \( \mathbb{C} \) that contains \( A \) there exists \( x \in X \) such that \( \tau(x) \subset \mathbb{H} \).
Lemma 4.1. Let $A_n, A \in \mathcal{R}$ with $A_n \to A$. Let $X$ be a compact space, and $\tau_n, \tau : X \to \mathcal{R}$ be continuous mappings such that $\tau_n \to \tau$ uniformly on $X$. Suppose that for each $n$, $\tau_n$ wraps $A_n$. Then $\tau$ wraps $A$.

Proof. If $\mathbb{H}$ is an open half-plane and $A \subset \mathbb{H}$, let $\mathbb{H}'$ be a slight translate of $\mathbb{H}$ towards $A$ such that we still have $A \subset \mathbb{H}'$. For $n$ sufficiently large $A_n \subset \mathbb{H}'$. It follows then from the assumption that for each $n$ sufficiently large there exists $x_n \in X$ such that $\tau_n(x_n) \subset \mathbb{H}'$. Letting $x$ be a limit point of $x_n$ in $X$, a simple $\epsilon/2$ argument shows that $\tau(x) \subset \mathbb{H}$.

Remark 4.2. Suppose $A \subset \tau(x)$ for all $x$, and $A$ and $\tau(x)$ are convex for all $x$. If $\tau$ wraps $A$, then $A = \bigcap_{x \in X} \tau(x)$. The converse is not true, as can easily be seen by considering $A$ to be the intersection of two line segments. However, the result that we quote below (in Theorem 4.6 Step 1) from [8] actually yields a wrapping property of $A$, not only intersection.

The following simple lemma will be used in Section 6.

Lemma 4.3. Suppose $A \in \mathcal{R}$ and $\tau : X \to \mathcal{R}$ wraps $A$. If $B \in \mathcal{R}$, $\tilde{A} := \text{co}(A, B)$, $\tilde{\tau}(x) = \text{co}(\tau(x), B)$, then $\tilde{\tau}$ wraps $\tilde{A}$.

Proof. Take a half-plane $\mathbb{H}$ that contains $\tilde{A}$. Then it contains $A$ and $B$. By hypothesis, there exists $x \in X$ such that $\tau(x) \subset \mathbb{H}$. Since $\mathbb{H}$ is convex, it follows that $\tilde{\tau}(x) \subset \mathbb{H}$, which proves the lemma.

The elements of $\mathcal{R}$ that we will consider are closures of numerical ranges. The next lemma states some continuity properties for these sets.

Lemma 4.4. (i) Let $T, S \in \mathcal{L}(H)$. Then $d(W(T), W(S)) \leq \|T - S\|$.

(ii) If $H_n \subset H_{n+1} \subset \cdots \subset H$ and $\bigcup_n H_n = H$, then for all $T \in \mathcal{L}(H)$,

$$W(T) = \bigcup_n W(P_{H_n}TP_{H_n}|H_n).$$

In particular, $d(W(P_{H_n}TP_{H_n}|H_n), W(T)) \to 0$.

(iii) Suppose $T \in \mathcal{L}(H)$, and $P, Q$ are orthogonal projections on $H$, with $\|P - Q\| < 1$. Then

$$d(W(PTP|PH), W(QTQ|QH)) \leq \|T\| \cdot \|P - Q\| \left[1 + \frac{2}{(1 - \|P - Q\|)^2}\right].$$

In particular, if $P_n, P$ are orthogonal projections and $P_n \to P$ uniformly, then $d(W(P_nTP_n|P_nH), W(PTP|PH)) \to 0$.

In the sequel we will let $X$ denote the space of unitary operators on $\mathbb{C}^N$ and we define $\tau^\Theta(\Omega) = W(U^\Theta_{\Omega})$, where $U^\Theta_{\Omega}$ is given by (3.5).

The next lemma singles out a technical argument that will be used twice in the proof of Theorem 4.6.

Lemma 4.5. Suppose that $\Theta, \Theta_n : \mathbb{D} \to \mathcal{L}(\mathbb{C}^N)$ are inner functions, such that

(a) $\Theta_n \xi \to \Theta \xi$ in $H^2(\mathbb{C}^N)$, for any $\xi \in \mathbb{C}^N$;

(b) $d(W(S\Theta_n), W(S\Theta)) \to 0$;
(c) if we define
\[ V_n,\Omega = \begin{pmatrix} S_{\Theta_n} & \Pi_{\Theta_n}^* D_{\Theta(0)^*}\Omega \\ D_{\Theta(0)^*} \Pi_{\Theta_n} & \Theta(0)^*\Omega \end{pmatrix}, \]
then \( \partial(W(V_n,\Omega), W(U_{\Omega}^{\Theta_n})) \to 0 \) uniformly in \( \Omega \).

If \( \tau^{\Theta_n} \) wraps \( W(S_{\Theta_n}) \) for all \( n \), then \( \tau^\Theta \) wraps \( W(S_\Theta) \).

Proof. Condition (a) implies, by formulas \( \eqref{2.3} \), that \( \iota^{\Theta_n} \to \iota^\Theta \) and \( \iota^*_{\Theta_n} \to \iota^*_{\Theta} \); whence \( \| V_n,\Omega - U_{\Omega}^{\Theta_n} \| \to 0 \) uniformly in \( \Omega \). Therefore \( \partial(W(V_n,\Omega), W(U_{\Omega}^{\Theta_n})) \to 0 \) uniformly in \( \Omega \), which, together with (c), yields \( \partial(W(U_{\Omega}^{\Theta}), W(U_{\Omega}^{\Theta_n})) \to 0 \) uniformly in \( \Omega \).

We may then apply Lemma \( \ref{4.1} \) with \( A_n = W(S_{\Theta_n}) \), \( A = W(S_\Theta) \), \( \tau_n = \tau^{\Theta_n} \), and \( \tau = \tau^\Theta \). With these notations, the assumption of Lemma \( \ref{4.5} \) becomes that \( \tau_n \) wraps \( A_n \), and it follows that \( \tau \) wraps \( A \). \( \square \)

**Theorem 4.6.** For any inner function \( \Theta : \mathbb{D} \to \mathcal{L}(\mathbb{C}^N) \), the map \( \tau^\Theta \) wraps \( W(S_\Theta) \).

Proof. The proof will be done in three steps.

**Step 1.** In case \( \Theta \) is a finite Blaschke–Potapov product, the space \( K_\Theta \) is finite dimensional and the statement is a consequence of \( \cite{8} \) Theorem 1.2 (see Remark \( \ref{4.2} \)).

**Step 2.** To pass to infinite Blaschke–Potapov products, suppose that \( \Theta = B \) and \( \Theta_n = B_n \) (where the notation is as in Lemma \( \ref{2.4} \) (iii)). We want to use Lemma \( \ref{4.5} \) Condition (a) therein is satisfied by Lemma \( \ref{2.4} \) (iii)(b). Applying Lemma \( \ref{4.4} \) (ii) to \( T = S_B, H = K_B, H_n = K_{B_n} \), we obtain \( \partial(W(S_{\Theta_n}), W(S_\Theta)) \to 0 \), and therefore (b) is also satisfied. Finally, to obtain (c), we apply Lemma \( \ref{4.4} \) (ii) again, this time to \( T = U_{\Omega}^{\Theta_n}, H = K_B \oplus \mathbb{C}^N, H_n = K_{B_n} \oplus \mathbb{C}^N \). By Step 1 we know that \( \tau^{B_n} \) wraps \( W(S_{B_n}) \) for all \( n \), and Lemma \( \ref{4.5} \) implies that \( \tau^B \) wraps \( W(S_B) \).

**Step 3.** According to Lemma \( \ref{2.1} \) we take a sequence of Blaschke–Potapov products \( \Theta_n \) that tend uniformly to an arbitrary inner function \( \Theta \). Condition (a) in Lemma \( \ref{4.3} \) is obviously satisfied. By Lemma \( \ref{2.4} \) (ii), we have \( P_{K_{\Theta_n}} \to P_{K_\Theta} \) uniformly. Since \( S_{\Theta_n} = P_{K_{\Theta_n}} T_z P_{K_{\Theta_n}} | K_\Theta \) and \( S_\Theta = P_{K_\Theta} T_z P_{K_\Theta} | K_\Theta \), Lemma \( \ref{4.3} \) (iii), applied to \( H = H^2(\mathbb{C}^N), T = T_z, P_n = P_{K_{\Theta_n}}, \) and \( P = P_{K_\Theta}, \) yields condition (b) in Lemma \( \ref{4.5} \).

To obtain (c), apply Lemma \( \ref{4.4} \) (iii) again, this time to \( H = H^2(\mathbb{C}^N) \oplus \mathbb{C}^N, P_n = P_{K_{\Theta_n}} \oplus \mathbb{C}^N, P = P_{K_\Theta} \oplus \mathbb{C}^N, \) and
\[
T = \begin{pmatrix} T_z & \iota_{\Theta} D_{\Theta(0)^*} \Theta(0)^*\Omega \\ D_{\Theta(0)^*} \iota^* & \Theta(0)^*\Omega \end{pmatrix}.
\]
Once again, we use the fact that \( P_{K_{\Theta_n}} \to P_{K_\Theta} \) uniformly to conclude that (c) is also satisfied.

By Step 2 we know that \( \tau^{\Theta_n} \) wraps \( W(S_{\Theta_n}) \) for all \( n \), and Lemma \( \ref{4.5} \) implies that \( \tau^\Theta \) wraps \( W(S_\Theta) \). The proof of the theorem is finished. \( \square \)
Corollary 4.7. Suppose \( \Theta : \mathbb{D} \to \mathcal{L}(\mathbb{C}^N) \) is an inner function. Then
\[
W(S_\Theta) = \bigcap_\Omega W(U^\Theta_\Omega),
\]
where \( U^\Theta_\Omega \) is defined by (3.3), while the intersection is taken with respect to all unitary operators \( \Omega \) on \( \mathbb{C}^N \).

Since \( C_0(\mathbb{N}) \) contractions are unitarily equivalent to model operators \( S_\Theta \), with \( \Theta : \mathbb{D} \to \mathcal{L}(\mathbb{C}^N) \) inner, we may extend the result to this class.

Theorem 4.8. Suppose \( T \in \mathcal{L}(\mathcal{H}) \) is a contraction of class \( C_0(\mathbb{N}) \), \( \mathcal{U} \) is the set of unitary \( N \)-dilations of \( T \) to \( \mathcal{H} \oplus \mathbb{C}^N \), and \( \tau : \mathcal{U} \to \mathcal{K} \) is defined by \( \tau(U) = W(U) \). Then \( \tau \) wraps \( W(T) \). In particular,
\[
W(T) = \bigcap_{U \in \mathcal{U}} W(U).
\]

5. Spectrum and numerical range of \( N \)-dilations

In the case that \( \Theta \) is a finite (scalar) Blaschke product, the spectrum of the extensions \( U^\Theta_\Omega \) can be identified precisely. Since \( U^\Theta_\Omega \) is a unitary operator, the numerical range is the (closed) convex hull of the spectrum, and we obtain a complete description of \( W(U^\Theta_\Omega) \), (see, for example, [3], [5], and [9]). The same can be done in the case of a general matrix-valued inner function, by relating these functions to perturbations of a "slightly larger" model operator. We need some preliminary material, for which the reference is [7].

If \( T \in \mathcal{L}(\mathcal{H}) \) is a contraction, then \( T(D_T) \subset D_{T^{*}} \), \( T(D^\perp_T) \subset D^\perp_{T^{*}} \), and \( T \) acts unitarily from \( D^\perp_T \) onto \( D^\perp_{T^{*}} \). For \( A : D_T \to D_{T^{*}} \), define \( T[A] \in \mathcal{L}(\mathcal{H}) \) by the formula
\[
T[A]x = \begin{cases} 
Ax & \text{if } x \in D_T, \\
Tx & \text{if } x \in D^\perp_T.
\end{cases}
\]

It is easy to see that \( T[A] \) is a contraction (respectively isometry, coisometry, unitary) if and only if \( A \) is a contraction (respectively isometry, coisometry, unitary).

We will be interested in the particular situation when \( T = S_\Xi \), with \( \Xi(z) = z\Theta(z) \), with \( \Theta : \mathbb{D} \to \mathcal{L}(\mathbb{C}^N) \) an inner function and \( A \) unitary. According to formulas (2.3), we have then \( D_{S_\Xi} = \Theta \mathbb{C}^N, D^\perp_{S_\Xi} = \mathbb{C}^N \). Thus a unitary mapping \( A : D_T \to D_{T^{*}} \) is given by \( A\Theta(z)\xi = \omega\xi \), with \( \omega : \mathbb{C}^N \to \mathbb{C}^N \) unitary. We will write \( A = A_\omega \). Since
\[
K_\Xi = K_\Theta \oplus \Theta \mathbb{C}^N = zK_\Theta \oplus \mathbb{C}^N,
\]
we have unitary operators \( J, J^*: K_\Theta \oplus \mathbb{C}^N \to K_\Xi \) defined by
\[
J(f \oplus \xi) = f + \Theta\xi, \quad J^*(f \oplus \xi) = zf + \xi.
\]

We will write
\[
Z_\Xi(\omega) := J^*S_\Xi[A_\omega]J \in \mathcal{L}(K_\Theta \oplus \mathbb{C}^N).
\]
With these notations, the lemma below follows from [7, Theorem 3.6]. For a more general result, see [2, Theorem 4.5].

**Lemma 5.1.** With the above assumptions, the spectrum of $Z_\Xi(\omega)$ is the union of the sets of points $\zeta \in \mathbb{T}$ at which $\Xi$ has no analytic continuation and the set of points $\zeta \in \mathbb{T}$ at which $\Xi$ has an analytic continuation but $\Xi(\zeta) - \omega$ is not invertible.

In particular, if $\Xi$ is a finite Blaschke–Potapov product, then

$$\sigma(Z_\Xi(\omega)) = \{\zeta \in \mathbb{T} : \det(\Xi(\zeta) - \omega) = 0\}.$$  

The relation with $N$-dilations is given by the next proposition.

**Proposition 5.2.** Suppose $\Theta : D \to \mathcal{L}(\mathbb{C}^N)$ is an inner function. Define $\Xi(z) = z\Theta(z)$. Then $U_\Omega^O = Z_\Xi(\Omega)$.

**Proof.** We have

$$Z_\Xi(\Omega) = J^*S_\Xi[A_\Omega]J = (J^*J_*)(J^*S_\Xi[A_\Omega]J). \quad (5.2)$$

Since

$$S_\Xi[A_\Omega]J(f \oplus \xi) = S_\Xi[A_\Omega](f + \Theta\xi) = zf + \Omega\xi = J_*(f \oplus \Omega\xi),$$

it follows that

$$(J^*_S\Xi[A_\Omega]J)(f \oplus \xi) = f \oplus \Omega\xi. \quad (5.3)$$

To compute $J^*_S\Xi : K_\Theta \oplus \mathbb{C}^N \to K_\Theta \oplus \mathbb{C}^N$, denote the corresponding matrix by $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, and let $P_i$ denote the projection on the $i$-th component in $K_\Theta \oplus \mathbb{C}^N$. We have

$$A_{11}(f) = P_1(J^*_S\Xi(J_*(f \oplus 0))) = P_1(J^*zf) = P_{K_\Theta}zf = S_\Theta f.$$ 

Further, $J_*(0 \oplus \xi) = \xi$, viewed as a constant function in $K_\Xi$. This decomposes with respect to $K_\Xi = K_\Theta \oplus \Theta\mathbb{C}^N$ as

$$\xi = (1 - \Theta(0)^*)\xi + \Theta(0)^*\xi = \iota_*D_{\Theta(0)^*}\xi + \Theta(0)^*\xi.$$ 

It follows that

$$J^*J_*(0 \oplus \xi) = \iota_*D_{\Theta(0)^*}\xi \oplus \Theta(0)^*\xi,$$

and thus

$$A_{12} = \iota_*D_{\Theta(0)^*}, \quad A_{22} = \Theta(0)^*.$$ 

To obtain $A_{21}$, we work now with the adjoint map $J^*_SJ$. We have $J(0 \oplus \xi) = \Theta\xi$, and the last function decomposes with respect to $K_\Xi = zK_\Theta \oplus \mathbb{C}^N$ as

$$\Theta\xi = z \left( \frac{\Theta - \Theta(0)}{z} \right)\xi + \Theta(0)\xi = z\iota D_{\Theta(0)}\xi + \Theta(0)\xi,$$

whence

$$J^*_SJ(0 \oplus \xi) = \iota D_{\Theta(0)}\xi \oplus \Theta(0)\xi.$$ 

Therefore

$$A_{21}^* = \iota D_{\Theta(0)}, \quad A_{21} = D_{\Theta(0)}\iota^*.$$
Finally,
\[ J^* J = \begin{pmatrix} S_\Theta & \iota_* D_{\Theta(0)*} \\ D_{\Theta(0)} t^* & \Theta(0)* \end{pmatrix} \quad (5.4) \]

Now the proof follows by comparing equations (5.2), (5.3), and (5.4) with (3.5).

From Lemma 5.1 and Proposition 5.2, the final result about spectrum and numerical range of \( N \)-dilations follows.

**Theorem 5.3.** With the above notations, the spectrum \( \sigma(U^\Theta_{\Omega}) \) is the union of the sets of points \( \zeta \in \mathbb{T} \) at which \( \Theta \) has no analytic continuation and the set of points \( \zeta \in \mathbb{T} \) at which \( \Theta \) has an analytic continuation but \( \zeta \Theta(\zeta) - \Omega \) is not invertible, while \( \overline{W(U^\Theta_{\Omega})} \) is the closed convex hull of \( \sigma(U^\Theta_{\Omega}) \).

In particular, if \( \Theta \) is a finite Blaschke–Potapov product, then
\[ \sigma(U^\Theta_{\Omega}) = \{ \zeta \in \mathbb{T} : \det(\zeta \Theta(\zeta) - \Omega) = 0 \} \]
and \( \overline{W(U^\Theta_{\Omega})} \) is the closed convex hull of the zeros of the polynomial \( \det(\zeta \Theta(\zeta) - \Omega) \).

The scalar case of Theorem 5.3 is contained in [11, Theorem 6.3].

6. Final remarks

It seems natural to formulate the following conjecture, which would complement Choi and Li’s answer to Halmos’ question.

**Conjecture 6.1.** Suppose \( T \in \mathcal{L}(\mathcal{H}) \) is a contraction with \( \dim \mathcal{D}_T = \dim \mathcal{D}_{T*} = N < \infty \), \( \mathcal{U} \) is the set of unitary \( N \)-dilations of \( T \) to \( \mathcal{H} \oplus \mathbb{C}^N \), and \( \tau : \mathcal{U} \to \mathbb{R} \) is defined by \( \tau(U) = \overline{W(U)} \). Then \( \tau \) wraps \( \overline{W(T)} \). In particular,
\[ \overline{W(T)} = \bigcap_{U \in \mathcal{U}} \overline{W(U)} \quad (6.1) \]

Note that the conjecture is open even for \( N = 1 \). The main points that have been settled are presented below. In the sequel \( T \in \mathcal{L}(\mathcal{H}) \) will be a contraction with \( \dim \mathcal{D}_T = \dim \mathcal{D}_{T*} = N < \infty \).

6.1. Theorem 4.8 shows that the conjecture is true for \( C_0(N) \) contractions.

6.2. As we show below, if we add a unitary operator to one for which the conjecture holds, the conjecture will still hold.

**Lemma 6.2.** If \( T_1 = T \oplus V \), where \( V \) is unitary, and Conjecture 6.1 is true for \( T \), then it is true for \( T_1 \).

**Proof.** The unitary \( N \)-dilations of \( T \oplus V \), for \( V \) unitary, are exactly \( U \oplus V \), with \( U \) a unitary \( N \)-dilation of \( T \). Since the numerical range of a direct sum is the convex hull of the numerical ranges of the components, the statement follows from Lemma 4.3. \( \square \)
Again by [19], it is known that an arbitrary contraction is the direct sum of a completely nonunitary contraction and a unitary; it follows then from Lemma 6.2 that it is enough to prove Conjecture 6.1 for a completely nonunitary $T$.

6.3. We now specialize to the case $N = 1$. Suppose $T$ is a completely nonunitary contraction with scalar characteristic function $\theta$ [19]; now $\theta$ is an arbitrary function in the unit ball of $H^\infty$. Then $T$ is unitarily equivalent to the model operator $T_\theta \in \mathcal{L}(K_\theta)$, where

$$K_\theta = (H^2 \oplus L^2(\Delta)) \ominus \{ \theta f \oplus (1 - |\theta|^2)^{1/2} f : f \in H^2 \},$$

with $\Delta = \{ \zeta \in \mathbb{T} : |\theta(\zeta)| < 1 \}$, while $T_\theta(f \oplus g) = P_{K_\theta}(zf \oplus \zeta g)$.

If $\theta$ is inner, then $\Delta = \emptyset$ and we are back in the $C_0(1)$ case discussed in 6.1.

On the other hand, the spectrum of $T_\theta$ may be precisely identified in terms of the characteristic function: $\sigma(T_\theta)$ is the union of the zeros of $\theta$ inside $\mathbb{D}$ and the complement of the open arcs of $\mathbb{T}$ on which $|\theta(\zeta)| = 1$ and through which $\theta$ has an analytic extension outside the unit disk (see again [19] for a general statement; in the scalar case it was known earlier and is usually called the Livsic–Moeller theorem).

In particular, it follows that Conjecture 6.1 can be settled for a situation at the opposite extreme of the case in which $\theta$ is inner. Namely, if $|\theta(\zeta)| < 1$ almost everywhere on $\mathbb{T}$, then $\sigma(T_\theta) \supset T$. In this case, Conjecture 6.1 is trivially true: $\overline{W(T)}$ as well as every $\overline{W(U)}$ must equal $\mathbb{D}$.

A final remark: the case $\dim \mathcal{D}_T = \dim \mathcal{D}_T^* = N < \infty$ is the only one in which we can hope to obtain the numerical range of $T$ by using “economical” unitary dilations. If $\dim \mathcal{D}_T \neq \dim \mathcal{D}_T^*$, or if both dimensions are infinite, then it is easy to see that for any unitary dilation $U \in \mathcal{L}(\mathcal{K})$ of $T \in \mathcal{L}(\mathcal{H})$ one must have $\dim(\mathcal{K} \ominus \mathcal{H}) = \infty$.

Acknowledgements

We thank the referee for useful remarks and, in particular, for pointing out the need for a slight change to the proof of Theorem 4.6.

References

[1] Arsene, Gr., Gheondea, A., Completing matrix contractions, J. Operator Theory 7 (1982), 179–189.

[2] Ball, J.A., Lubin, A., On a class of contractive perturbations of restricted shifts, Pacific J. Math. 63 (1976), 309–323.

[3] Chalendar, I., Gorkin, P., Partington, J. R., Numerical ranges of restricted shifts and unitary dilations, Oper. Matrices 3 (2009), 271–281.

[4] Choi M.-D., Li C.-K., Constrained unitary dilations and numerical ranges, J. Operator Theory 46 (2001), 435–447.
5. Daep, U., Gorkin, P., Voss, K., Poncelet’s Theorem, Sendov’s Conjecture and Blaschke Products, *J. Math. Anal. Appl.* **365** (2010), 93-102.

6. Durszt, E., On the numerical range of normal operators, *Acta Sci. Math. (Szeged)* **25** (1964), 262–265.

7. Fuhrmann, P.A., On a class of finite dimensional contractive perturbations of restricted shifts of finite multiplicity, *Israel J., Math.* **16** (1973), 162–175.

8. Gau, H.-L., Li, C.-K, Wu, P.Y., Higher-rank numerical ranges and dilations, *J. Operator Theory* **63** (2010), 181–189.

9. Gau, H.-L., Wu, P. Y., Numerical range of \( S(\phi) \), *Linear and Multilinear Algebra* **45** (1998), no. 1, 49–73.

10. Gau, H.-L., Wu, P. Y., Numerical range circumscribed by two polygons, *Linear Algebra Appl.* **382** (2004), 155–170.

11. Gau, H.-L., Wu, P. Y., Numerical range and Poncelet property, *Taiwanese J. Math.* **7** (2003), no. 2, 173–193.

12. Gau, H.-L., Wu, P. Y., Dilation to inflations of \( S(\phi) \), *Linear and Multilinear Algebra* **45** (1998), no. 2-3, 109–123.

13. Halamb, P.R., Numerical ranges and normal dilations, *Acta Sci. Math. (Szeged)* **25** (1964), 1–5.

14. Halamb, P. R., *A Hilbert space problem book*. Second edition. Graduate Texts in Mathematics, 19. Encyclopedia of Mathematics and its Applications, 17. Springer-Verlag, New York-Berlin, 1982.

15. Katsnelson, V. E., Kirstein, B., On the theory of matrix-valued functions belonging to the Smirnov class. *Topics in interpolation theory* 299–350, Oper. Theory Adv. Appl., 95, Birkhäuser, Basel, 1997.

16. Mirman, B., Borovikov, V., Ladyzhensky, L., Vinograd, R., Numerical ranges, Poncelet curves, invariant measures, *Linear Algebra Appl.* **329** (2001), no. 1-3, 61–75.

17. Mirman, B., Numerical ranges and Poncelet curves, *Linear Algebra Appl.* **281** (1998), no. 1-3, 59–85.

18. Peller, V.V., *Hankel Operators and their Applications*, Springer Verlag, 2003.

19. Sz-Nagy, B., Foias, C., *Harmonic Analysis of Operators on Hilbert Space*, North-Holland Publishing, 1970.

20. Wu, P.Y., Polygons and numerical ranges, *Amer. Math. Monthly*, **107** (2000), no. 6, 528–540.

Chaqiq Benhida
UFR de Mathématiques, Université des Sciences et Technologies de Lille, F-59655 Villeneuve D’Ascq Cedex, France
e-mail: Chafiq.Benhida@math.univ-lille1.fr

Pamela Gorkin
Department of Mathematics, Bucknell University, Lewisburg, PA 17837, U.S.A.
e-mail: pgorkin@bucknell.edu

Dan Timotin
Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 014700, Romania
e-mail: Dan.Timotin@imar.ro