Locally integrable non-Liouville analytic geodesic flows

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Contents

1 Introduction and statement of the results 1

2 An iterative procedure
   2.1 The first and second steps 4
   2.2 The 3-rd step 6

3 The s-th step
   3.1 Formal expansion 12
   3.2 Bounds 16

4 The p-dependent case 19

5 Chaotic Behavior 19

6 Final remarks 21

1 Introduction and statement of the results

Let $\mathcal{M}$ be a $n$-dimensional smooth manifold with Riemanninan metric $\bar{g} = (\bar{g}_{ij})$. Using the standard transformation $p_j = \bar{g}_{ij}\dot{q}_i$ the geodesic flow associated with $\bar{g}$ can be regarded as an hamiltonian system

\[
\begin{aligned}
\dot{p} &= \partial_q \bar{H} \\
\dot{q} &= -\partial_p \bar{H}
\end{aligned}
\] (1.1)
The geodesic flow (1.1) is called completely Liouville integrable if it admits $n$ smooth independent functions $H_1(q, p), \ldots, H_n(q, p)$ such that

- each $H_i$ is an integral of the geodesic flow, i.e. it is constant along each geodesic line $(q(t), p(t))$.
- the functions $H_i$ Poisson-commute on $T^*\mathcal{M}$, i.e. \( \{H_i, H_j\} := \partial_p H_i \cdot \partial_q H_j - \partial_q H_i \cdot \partial_p H_j = 0 \)

Let us focus on the case $\mathcal{M} = T^2$ and use coordinates $q = (q_1, q_2) \in T^2$. If the Hamiltonian $\tilde{H}$ is of the form $\tilde{H}(q, p) = (g_1(q_1) + g_2(q_2))(p_1^2 + p_2^2)$ then the corresponding metric is said to be separable or Liouville: it is well known (see for instance $[3]$) that every surface of revolution admits a Liouville metric.

Suppose that the metric is “diagonal”, namely the corresponding Hamiltonian is of the form:

$$\tilde{H}(q, p) = V(q)(p_1^2 + p_2^2).$$

We can rewrite $\tilde{g}$ as a Jacobi metric $g := (e - f(q))\tilde{g}$ with

$$e := \max_{q \in T^2} V(q), \quad f(q) := e - V(q)$$

so that the geodesic flow corresponds to the Hamiltonian flow associated with the mechanical Hamiltonian

$$H(q, p) = \frac{1}{2}|p|^2 + f(q)$$

and the metric is separable if the Hamiltonian above is separable, namely if the potential $f$ can be written as the sum of a function of $q_1$ only plus a function of $q_2$ only; we refer the reader to $[6]$ for a recent review on the topic and some related open questions.

In $[1]$ the Authors prove that if a metric on $T^2$ is such that the geodesic flow admits an integral which is quadratic in the momenta then the metric is Liouville; in $[5]$ the Author says that Liouville metrics are the largest known class of integrable metrics.

**A Forklore Conjecture** If a metric on $T^2$ is integrable, then it is Liouville.

The present paper provides a counterexample to the conjecture above; let us now state our result precisely.

Denote by $\mathcal{P}$ the cone in the action space

$$\mathcal{P} := \{p \in \mathbb{R}^2 : -\sqrt{2}p_2 < p_1 < \sqrt{2}p_2, \, p_1 > 0\}. \quad (1.2)$$
The choice of the aperture of the cone is made so that the boundaries are Diophantine directions. It will be useful to use polar coordinates to describe the $p$-space (see Remark 2.5) i.e. we may write
\[ P = \{ p = (\varphi_p, r_p) : -\arctan(\frac{1}{\sqrt{2}}) < \varphi_p < \arctan(\frac{1}{\sqrt{2}}), \ r_p > 0\}. \] (1.3)

Denote by $| \cdot |$ the euclidean norm of a two dimensional vector. Our main result is the following.

**Theorem 1.1.** There exists a real-on-real analytic mechanical Hamiltonian
\[ H_\varepsilon(q, p) = \frac{|p|^2}{2} + f(q, \varepsilon) = \frac{|p|^2}{2} + \sum_{s \geq 1} \varepsilon^s f_s(q), \] (1.4)

with a nonzero potential $f(q, \varepsilon)$ and an analytic change of variables $\Phi$ such that $H_\varepsilon \circ \Phi = |p|^2/2$ on the energy surface $\{H_\varepsilon = 1/2\}$ and $p \in P$.

**Corollary 1.2.** There is a non-Liouville analytic metric on $T^2$ which is integrable in an open set of the energy surface $\{H_\varepsilon = 1/2\}$.

Of course if one wants an example on $T^n$, one can for instance decompose $T^n = T^2 \times T^{n-2}$ and consider a metric which is the product of the metric provided by Theorem 1.1 for $T^2$ and any integrable metric for $T^{n-2}$. However our construction strongly depend on the dimension; see Section 6 for additional comments.

On the other hand, if we allow $f$ to depend also on $p$, it is much easier to find non-separable Hamiltonian which is integrable on a whole domain of the phase space, and moreover the construction holds in any dimension. Actually, denoting by $P$ any domain in the action space, one has the following result.

**Theorem 1.3.** There exists a real-on-real analytic Hamiltonian
\[ H_\varepsilon(q, p) = \frac{|p|^2}{2} + f(q, p, \varepsilon) = \frac{|p|^2}{2} + \sum_{s \geq 1} \varepsilon^s f_s(q, p), \] (1.5)

with a nonzero $f(q, p, \varepsilon)$ and an analytic change of variables $\Phi$ such that $H_\varepsilon \circ \Phi = |p|^2/2$ on a domain $T^n \times P$.

The proof of Theorem 1.1 is an explicit iterative construction of the potential $f$ as convergent power series. In Section 2 we perform by hand the first 3 steps in order to understand the general picture: as it is quite common in perturbation theory, from the 4th step on the construction “stabilizes” and in Section 3 we describe the generic step $s$ and show the convergence of the series. Then in Section 4 we prove Theorem 1.3 which is rather straightforward. Finally in Section 6 we make some further comment about the results and their proofs.

**Acknowledgements.** Part of this research was performed during a period when L.C. was supported by a CRC Postdoctoral Fellowship at McMaster University. V.K. acknowledges a partial support of the NSF grant DMS-1402164. L.C. acknowledges a partial support of the NSF grant DMS-1500943.
2 An iterative procedure

Our aim is to explicitly construct a real-on-real potential

\[ f(q; \varepsilon) = \sum_{s \geq 1} \varepsilon^s f_s(q) \quad (2.1) \]

and a change of variables \( \Phi \) such that \( H_\varepsilon \circ \Phi = |p|^2/2 + h(p; \varepsilon) \) on the level surface \( \{ H_\varepsilon = 1/2 \} \), for some

\[ h(p; \varepsilon) = \sum_{s \geq 1} \varepsilon^s h_s(p) . \quad (2.2) \]

As usual in KAM-like problems, the change of variables \( \Phi \) will be the time-1 map generated by an Hamiltonian

\[ G(p, q) = G(p, q; \varepsilon) = \sum_{s \geq 1} \varepsilon^s G_s(p, q) , \quad (2.3) \]

so that

\[ H \circ \Phi = \sum_{n \geq 0} \frac{1}{n!} \{ H, G \}^{(n)} , \quad (2.4) \]

where we used the notation

\[ \{ H, G \}^{(0)} := H , \]
\[ \{ H, G \}^{(n)} := \{ \{ H, G \}^{(n-1)}, G \} ; \quad (2.5) \]

see for instance [2].

We shall construct \( f \) and \( \Phi \) via an iterative procedure: at each step we suitably fix \( f_s \) and \( G_s \), and we provide appropriate bounds from which we eventually infer the convergence of both the series \( (2.1) \) and \( (2.3) \).

Let \( \mathcal{K} \) be the dual cone

\[ \mathcal{K} := \{ k \in \mathbb{Z}^2 \setminus \{ 0 \} : |k \cdot p| \geq \frac{1}{2} |p| |k|, \text{ for all } p \in \mathcal{P} \} . \quad (2.6) \]

We start our procedure by choosing any non-separable \( f_1(q) \) whose Fourier modes are supported in \( \mathcal{K} \), i.e.

\[ f_1(q) = \sum_{k \in \mathcal{K}} f_{1,k} e^{i k \cdot q} . \quad (2.7) \]

Of course, by analyticity we have

\[ |f_{1,k}| \leq M e^{-\xi_0 |k|} \quad (2.8) \]

for some given positive constants \( M, \xi_0 \). Moreover, in order for \( f_1 \) to be real-on-real, we need to require

\[ f_{1,-k} = \overline{f_{1,k}} , \quad (2.9) \]
where for a complex number $z$ we denoted by $\overline{z}$ its complex conjugate.

Collecting together the same orders in $\varepsilon$ and denoting $H_0 = H_0(p) := |p|^2/2$ we have

$$H \circ \Phi = H_0(p) + \sum_{s \geq 1} \varepsilon^s \left( f_s + \sum_{m \geq 1} \frac{1}{m!} \sum_{n_0 + \ldots + n_m = s, n_0 \geq 0, n_1, \ldots, n_m \geq 1} \{\ldots \{f_{n_0}, G_{n_1}\}, G_{n_2}\} \ldots \}, G_{n_m}\} \right). \tag{2.10}$$

For all $n \geq 1$, we denote

$$H_0^{(n)} = H_0(p) + \sum_{s = 1}^{n} \varepsilon^s f_s(q), \tag{2.11}$$

with $f_1(q)$ given in (2.7) while $f_s(q)$ for $s \geq 2$ are still to be found.

We now compute explicitly the first three orders in order to understand the general behavior.

### 2.1 The first and second steps

At order $\varepsilon$, the r.h.s. of (2.10) reads

$$f_1 + \{H_0, G_1\} = f_1 + p \cdot \partial q G_1, \tag{2.12}$$

so that, defining formally

$$G_1(p, q) = -\sum_{k \in K} \frac{f_{1,k}}{ip \cdot k} e^{ik \cdot q} \tag{2.13}$$

we have

$$\{H_0, G_1\} = -f_1. \tag{2.14}$$

Note that, due to (2.9), $G_1$ is real-on-real as well.

For $1/2 < |p| < 2$ we have the bound

$$|G_{1,k}| \leq \frac{2M}{|k|} e^{-\xi_0 |k|}, \tag{2.15}$$

so that the function $G_1(p, q)$ in (2.13) is well defined on $(\{1/2 < |p| < 2\} \cap \mathcal{P}) \times T^2$. Moreover we can take $h_1(p) \equiv 0$.

At order $\varepsilon^2$ the r.h.s. of (2.10) is

$$f_2 + \{f_1, G_1\} + \{H_0, G_2\} + \frac{1}{2} \{\{f_0, G_1\}, G_1\} \stackrel{2.10}{=} f_2 + \frac{1}{2} \{f_1, G_1\} + \{H_0, G_2\}. \tag{2.16}$$

We start by imposing that on the energy surface $\{H_0 = 1/2\}$ we have

$$\{H_0, G_2\} = -(f_2 + \frac{1}{2} \{f_1, G_1\}) + h_2, \tag{2.17}$$
which in Fourier reads
\[(ip \cdot k)G_{2,k} = -(f_{2,k} + \left(\frac{1}{2}(f_{1,G_1})_k\right)), \quad k \neq 0 \quad (2.18a)\]
\[h_2 = f_{2,0} + \left(\frac{1}{2}(f_{1,G_1})_0\right), \quad k = 0. \quad (2.18b)\]

Note that (2.18b) leaves us \(f_{2,0}\) as a free parameter while we need to define \(f_{2,k}\) is such a way that the r.h.s. of (2.18a) is zero when \(|p \cdot k|\) is \(O(\varepsilon)\) small. We can’t do it in a uniform way, so we solve it with a precision \(O(\varepsilon)\).

For \(k \neq 0\), denote
\[p_k^0 := k^\perp/|k|, \quad (2.19)\]
and note that the Poisson bracket appearing in (2.18) is non-zero only for \(k \in K_2\) defined as
\[K_2 := \{k \in \mathbb{Z}^2 : k = k_1 + k_2, \text{ for some } k_1, k_2 \in K\}. \quad (2.20)\]

We distinguish two subset of \(K_2\), namely
\[K_2^{\text{big}} := \{k \in K_2 : p_k^0 \notin \mathcal{P}\} \quad (2.21)\]
and
\[K_2^{\text{small}} := \{k \in K_2 : p_k^0 \in \mathcal{P}\}, \quad (2.22)\]
and we analyze the Fourier modes separately.

**case 1.** \(k \in K_2^{\text{big}}\). In this case the l.h.s. of (2.18) cannot vanish. Of course, although \(p_k^0 \notin \mathcal{P}\) there might be \(p \in \mathcal{P}\) such that \(p \cdot k\) is “too small”: this might happen if \(p_k^0\) is “close” to the boundary of \(\mathcal{P}\), so we set
\[f_{2,k} := -\frac{1}{2} \sum_{k_1 + k_2 = k} \frac{k_1 f_{1,k_1} k_2 f_{1,k_2}}{(p_k^0 \cdot k_2)^2}, \quad (2.23)\]
where \(\overline{p}_k\) is the minimizer of \(|p - p_k^0|\) for \(p\) varying in the closure of \(\mathcal{P} \cap \{|p| = 1\}\), i.e. it is either \((\sqrt{2}/3, 1/3)\) or \((\sqrt{2}/3, -1/3)\). Note that in both cases \(\overline{p}_k\) is a Diophantine vector so we do not have to worry about the smallness of denominators of the form \(\overline{p}_k \cdot k'\) for any \(k' \in \mathbb{Z}^2 \setminus \{0\}\).

**case 2.** \(k \in K_2^{\text{small}}\). In this case the l.h.s. of (2.18) vanishes when \(p = p_k^0\) so that we first need to impose
\[f_{2,k} := -\frac{1}{2} \sum_{k_1 + k_2 = k} \frac{k_1 f_{1,k_1} k_2 f_{1,k_2}}{(p_k^0 \cdot k_2)^2}. \quad (2.24)\]

In both cases 1,2 we can define, at least formally,
\[G_{2,k} = -\frac{1}{ip \cdot k} \left(f_{2,k} + \frac{1}{2} \sum_{k_1 + k_2 = k} \frac{-k_1 f_{1,k_1} f_{1,k_2}}{(p \cdot k_2)^2}\right), \quad (2.25)\]
and
\[G_{2,k}^0 = -\frac{1}{ip \cdot k} \left(f_{2,k} + \frac{|p|^2}{2} \sum_{k_1 + k_2 = k} \frac{-k_1 f_{1,k_1} f_{1,k_2}}{(p \cdot k_2)^2}\right). \quad (2.26)\]
Remark 2.1. When $|p|^2 - 1 = O(\varepsilon)$, $G_2$ and $G_2^0$ differ by $O(\varepsilon)$ and the mismatch goes to the next order. In particular, $G_2$ will turn out to be not well defined in this neighbourhood, but $p \cdot \partial_q G_2$ is well defined there.

Remark 2.2. Note that in the sum appearing in (2.24) in principle the term with $k_1 = k_2$ might be the source of a problem. Indeed if $k$ is such that there exists $k_1$ so that $2k_1 = k$, then the denominator in (2.24) satisfies

$$p_0^k \cdot k_2 = p_0^k \cdot k_1 = \frac{1}{2} p_0^k \cdot k = 0.$$  

However this cannot happen because in this case $k \parallel k_1$, which implies $k \in K \subseteq K_{\text{big}}$ so that $f_{2,k}$ is given by (2.23) and not by (2.24).

Remark 2.3. Note that, because of (2.9), in both cases 1 and 2 one has

$$f_{2,-k} = -\frac{1}{2} \sum_{k_1 + k_2 = -k} \frac{k_1 f_{1,k_1} k_2 f_{1,k_2}}{(p \cdot k_2)^2} \quad (2.28)$$

$$= -\frac{1}{2} \sum_{k_1 + k_2 = k} \frac{(-k_1) f_{1,-k_1} (-k_2) f_{1,-k_2}}{(-\vec{p} \cdot k_2)^2} \quad (2.29)$$

$$= -\frac{1}{2} \sum_{k_1 + k_2 = k} \frac{k_1 \vec{f}_{1,k_1} k_2 \vec{f}_{1,k_2}}{(-\vec{p} \cdot k_2)^2} = \vec{f}_{2,k}$$

where $\vec{p}$ is equal to $\vec{p}_k$ in case 1 and to $p_0^k$ in case 2.

Regarding the bounds, first of all we see that in both cases 1 and 2 one has

$$|f_{2,k}| \leq \frac{M^2}{2} \sum_{k_1 + k_2 = k} \frac{|k_1|}{|k_2|} e^{-\xi_0(|k_1| + |k_2|)} \leq M^2 e^{-\xi_0 |k|/2}, \quad (2.26)$$

then we notice that, setting

$$F_{2,k}(p) := \frac{|p|^2}{2} \sum_{k_1 + k_2 = k} \frac{-k_1 f_{1,k_1} k_2 f_{1,k_2}}{(p \cdot k_2)^2}, \quad (2.27)$$

in case 1 one has

$$f_{2,k} + \frac{1}{2} \{f_1, G_1\}_k = F_{2,k}(\vec{p}) - F_{2,k}(p) \quad (2.28)$$

whereas in case 2 we have

$$f_{2,k} + \frac{1}{2} \{f_1, G_1\}_k = F_{2,k}(p_0^k) - F_{2,k}(p). \quad (2.29)$$

Note that clearly, as in Remark 2.3 one has

$$F_{2,-k} = \vec{F}_{2,k}. \quad (2.30)$$
Remark 2.4. One has
\[ p \cdot \partial_p F_{2,k}(p) = |p|^2 \sum_{k_1 + k_2 = k} \frac{-k_1 f_{1,k_1} k_2 f_{1,k_2}}{(p \cdot k_2)^2} + |p|^2 \sum_{k_1 + k_2 = k} \frac{k_1 f_{1,k_1} k_2 f_{1,k_2}}{(p \cdot k_2)^2} = 0. \]

Given two vectors \( u, v \in \mathbb{R}^2 \) we denote by \( \varphi(u, v) \) the smaller angle between the two vectors, so with this notation we can rewrite
\[ F_{2,k}(p) := \frac{1}{2} \sum_{k_1 + k_2 = k} \frac{-k_1 f_{1,k_1} k_2 f_{1,k_2}}{|k_2|^2 \cos^2(\varphi(p, k_2))}. \]

Remark 2.5. Note that if \( p, p' \) are parallel, then \( F_{2,k}(p) - F_{2,k}(p') = 0 \). This is the reason why it is convenient to describe the p-variables in polar coordinates as in \([13]\); indeed with that notation we have \( F_{2,k}(p) = F_{2,k}(\varphi_p) \), i.e. it is a function of the angular variable only.

Using the notation \([2.31]\) and setting
\[ \tilde{p}_k := \begin{cases} \bar{p}_k & k \in K_2^{big} \\ 0 & \bar{p}_k \in K_2^{small} \end{cases}, \quad \hat{p}_k := \begin{cases} 0 & k \in K_2^{big} \\ \bar{p}_k & \bar{p}_k \in K_2^{small} \end{cases} \]
we see that
\[ G_{2,k}^0 = \frac{F_{2,k}(p) - F_{2,k}(\tilde{p}_k)}{i(p - \hat{p}_k) \cdot k} \]
and hence
\[ G_{2,-k}^0 = \overline{G_{2,k}^0}. \]
Moreover we can bound
\[ |G_{2,k}^0| = \frac{|F_{2,k}(p) - F_{2,k}(\tilde{p}_k)|}{|p - \bar{p}_k| \cdot k} \]
\[ \leq \frac{1}{2} \sum_{k_1 + k_2 = k} \frac{|k_1 f_{1,k_1} k_2 f_{1,k_2}|}{|k_2|^2 \cos^2(\varphi(p, k_2)) \cos^2(\varphi(\tilde{p}_k, k_2))} \quad \frac{|\cos^2(\varphi(p, k_2)) - \cos^2(\varphi(\tilde{p}_k, k_2))|}{|(p - \bar{p}_k) \cdot k|}. \]

Now if \( |(p - \bar{p}_k) \cdot k| = \delta \), then \( \cos(\varphi(p, k)) = (|k|r_p)^{-1} \delta \) which in turn implies
\[ |\varphi(p, k) - \varphi(\tilde{p}_k, k)| \leq \frac{2\delta}{|k|r_p}, \]
and hence
\[ |\cos^2(\varphi(p, k)) - \cos^2(\varphi(\tilde{p}_k, k))| \leq 4 \left| \sin \left( \frac{\varphi(p, k_2) - \varphi(\tilde{p}_k, k_2)}{2} \right) \right| \leq \frac{2\delta}{|k|r_p}, \]
from which we deduce
\[ |G_{2,k}^0| \leq \sum_{k_1 + k_2 = k} \frac{2|k_1 f_{1,k_1} k_2 f_{1,k_2}|}{|k_1|^2 r_p^2 |k_2|^2 \cos^2(\varphi(p, k_2)) \cos^2(\varphi(\tilde{p}_k, k_2))} \leq CM^2 e^{-3\delta_0|k|/4}, \]

for some positive constant $C$, where in the last inequality we used (2.26) and the fact that $r_p^2 = |p|^2 = 1 + O(\varepsilon) \leq 2$. Note that we are left with the average $G_{2,0}^0$ as a further free parameter. Note also that $G_{2,0}^0$ is analytic as function of $p$ in a $O(\sqrt{\varepsilon})$-neighbourhood of $\{H_0 = 1/2\}$.

In conclusion $G_{2,0}^0$ is a real-on-real function, well defined and analytic in a $O(\sqrt{\varepsilon})$-neighbourhood of $\{H_0 = 1/2\}$ and we have the uniform bound (2.34) for its Fourier coefficients; on the other hand $G_2$ is well defined and its Fourier coefficients admit the bound (2.34) only on the surface $\{H_0 = 1/2\}$, being equal to $G_{2,0}^0$ on such surface. Moreover the averages $f_{2,0}$ and $G_{2,0}^0$ are still free parameters.

### 2.2 The 3-rd step

At the previous step we obtained
\[ f_2(q) = \sum_{k \in \mathbb{Z}^d} f_{2,k} e^{ik \cdot q} \quad (2.37) \]
with $f_{2,0}$ a free parameter and $f_{2,k}$ given by (2.23) or (2.24) according on $k$, as the restriction of $\{f_1, G_1\}$ to the surface $\{H_0 = 1/2\}$.

We now refine this definition in order to fit the restriction onto the energy surface $\{H_1(\varepsilon) = 1/2\}$. In other words we want the r.h.s. of (2.17) to vanish on $\{H_1(\varepsilon) = 1/2\}$. Let $p_k^1(q, \varepsilon)$ be either $p_k$ for $k \in K_2^{big}$ or the unique point in $P$ such that
\[ p \cdot k = 0, \quad \frac{1}{2}|p|^2 + \varepsilon f_1(q) = \frac{1}{2}, \quad (2.38) \]
for $k \in K_2^{small}$. Notice that for $k \in K_2^{small}$, $p_k^1(q, \varepsilon)$ can be written in the form
\[ p_k^1(q, \varepsilon) = p_k^0(1 - 2\varepsilon f_1(q))^{1/2}; \]

Now set
\[ f_{2,k}^1(q, \varepsilon) := -\frac{1}{2} \sum_{k_1+k_2=k} \frac{k_1 f_{1,k_1} f_{1,k_2} f_{2,k}^1}{(p_k^1(q, \varepsilon) \cdot k_2)^2}, \quad (2.39) \]
and note that
\[ f_{2,-k}^1(q, \varepsilon) = \overline{f_{2,k}^1(q, \varepsilon)}. \]

Define also
\[ f_3^2(q) := \lim_{\varepsilon \to 0} \sum_k \left( \frac{f_{2,k}^1(q, \varepsilon) - f_{2,k}}{\varepsilon} - \frac{\{H_0, G_{2,0}^0\}_{p=p_k^1(q, \varepsilon)}}{\varepsilon} \right) e^{ik \cdot q} \]
\[ = \lim_{\varepsilon \to 0} \sum_k \left( \frac{f_{2,k}^1(q, \varepsilon) - f_{2,k}}{\varepsilon} - ip \cdot k G_{2,k}^0(p_k^1(q, \varepsilon)) \right) e^{ik \cdot q} \quad (2.40) \]
\[ = f_1(q) \left( f_2(q) + \sum_k p_k^0 \cdot \partial_p F_{2,k}(p_k^0) e^{ik \cdot q} \right) \quad \text{Rmk[2.4]} \]
\[ = f_1(q) f_2(q); \]

9
where we used also the fact that
\[
\lim_{\varepsilon \to 0} p \cdot \partial_q G_2^0(q, p_k^0(q, \varepsilon)) = p \cdot \partial_q G_{2,k}(q, p_k^0) = 0
\]
by construction. In particular we see that \(f_3^3(q)\) is also real-on-real.

Using the formal definitions of \(G_1, G_2\), the equation for \(G_3\) formally reads
\[
p \cdot \partial_q G_3 = h_3 - \left( f_3 + f_3^* + \{f_1, G_2\} + \{f_2, G_1\} \right)
+ \frac{1}{2} (\{\{H_0, G_2\}, G_1\} + \{\{H_0, G_1\}, G_2\}) + \frac{1}{6} \{\{H_0, G_1\}, G_1\}, G_1\}
\]
(2.41)
\[
= h_3 - \left( f_3 + f_3^* + \frac{1}{2} \{f_1, G_2\} + \frac{1}{2} \{f_2, G_1\} + \frac{1}{2} \{\{f_1, G_1\}, G_1\} \right)
\]
namely in Fourier, again formally,
\[
(ip \cdot k)G_{3,k} = - \left( f_{3,k} + f_{3,k}^* + \frac{1}{2} \{f_1, G_2\}_k + \frac{1}{2} \{f_2, G_1\}_k + \frac{1}{12} \{\{f_1, G_1\}, G_1\}_k \right), \quad k \neq 0
\]
(2.42a)
\[
h_3 = \left( f_{3,0} + f_{3,0}^* + \frac{1}{2} \{f_1, G_2\}_0 + \frac{1}{2} \{f_2, G_1\}_0 + \frac{1}{12} \{\{f_1, G_1\}, G_1\}_0 \right)
\]
(2.42b)
where \(\{\cdots, \cdots\}_k\) is the \(k\)-th Fourier coefficient of the underlying function. Notice that we have “energy reduction” correction term coming from the difference between restricting \(\{f_1, G_1\}\) to \(\{H_0 = 1/2\}\) and to \(\{H_0^{(l)} = 1/2\}\).

As before we note that the Poisson brackets above are non-zero only for
\[
k \in \mathcal{K}_3 := \{k \in \mathbb{Z}^2 : k = k_1 + k_2 + k_3 \text{ for some } k_1, k_2, k_3 \in \mathcal{K}\},
\]
(2.43)
and again we distinguish the two subsets
\[
\mathcal{K}^{big}_3 := \{k \in \mathcal{K}_3 : p_k^0 \notin \mathcal{P}\}
\]
(2.44)
and
\[
\mathcal{K}^{small}_3 := \{k \in \mathcal{K}_3 : p_k^0 \in \mathcal{P}\},
\]
(2.45)
and as before we define the \(k \neq 0\) Fourier coefficients \(f_{3,k}\) differently for modes in \(\mathcal{K}^{big}_3\) or in \(\mathcal{K}^{small}_3\).

**case 1.** \(k \in \mathcal{K}^{big}_3\). In this case we set (recall that \(p_k\) is Diophantine)
\[
f_{3,k} := -f_{3,k} - \frac{1}{2} \{f_1, G_2^0\}_k |_{p = \overline{p}_k} - \frac{1}{2} \{f_2, G_1\}_k |_{p = \overline{p}_k} - \frac{1}{12} \{\{f_1, G_1\}, G_1\}_k |_{p = \overline{p}_k}
\]
(2.46)

**case 2.** \(k \in \mathcal{K}^{small}_3\). In this case we set
\[
f_{3,k} := -f_{3,k} - \frac{1}{2} \{f_1, G_2^0\}_k |_{p = p_k^0} - \frac{1}{2} \{f_2, G_1\}_k |_{p = p_k^0} - \frac{1}{12} \{\{f_1, G_1\}, G_1\}_k |_{p = p_k^0}
\]
(2.47)
Remark 2.6. As for the second step, we introduced the factors $|p|^a$, $a = 2, 4$, in order to make sure that $F_{3,k}$ depend on $p$ only through $\varphi_p$, and this is the reason behind the choice of the exponents; see also Remark 2.5. Indeed from \[2.48\] we see that

$$G_{3,k}^0(p) = \frac{-1}{i p \cdot k} \left( f_{3,k} + p \cdot k \right) + \frac{|p|^2}{2} \{ f_1, G_2^0 \} + \frac{|p|^2}{2} \{ f_2, G_1 \} + \frac{|p|^4}{12} \{ \{ f_1, G_1 \}, G_1 \}.$$ \[2.48\]

Again by definition one has

$$G_{3,-k}^0(p) = \overline{G_{3,k}^0(p)}.$$

Notice that in \[2.48\] we use $G_2^0$, which is well defined and uniformly bounded in a $O(\sqrt{\varepsilon})$-neighbourhood of $\{H_0 = 1/2\}$. As done for $G_3^0$ we obtain a uniform bound for $G_{3,k}^0$ in a $O(\varepsilon)$-neighbourhood of $\{H_{1/2} = 1/2\}$. Indeed setting

$$F_{3,k}(p) := \frac{|p|^2}{2} \{ f_1, G_2^0 \} + \frac{|p|^2}{2} \{ f_2, G_1 \} + \frac{|p|^4}{12} \{ \{ f_1, G_1 \}, G_1 \},$$ \[2.49\]

we see that

$$|G_{3,k}^0(p)| = \frac{|F_{3,k}(p) - F_{3,k}(\overline{p})|}{|p - \overline{p}|},$$

where we are using the notation \[2.32\] with $\mathcal{K}_2 \sim \mathcal{K}_3$. Hence we can reason exactly as in \[2.34\] to get $|G_{3,k}^0(p)| \leq CM^3 e^{-5\xi_0 |k|/8}$.

**Remark 2.6.** As for the second step, we introduced the factors $|p|^a$, $a = 2, 4$, in order to make sure that $F_{3,k}$ depend on $p$ only through $\varphi_p$, and this is the reason behind the choice of the exponents; see also Remark 2.5. Indeed from \[2.33\] we see that $G_{2,k}^0$ is of the form

$$G_{2,k}^0 = \frac{1}{p \cdot k} g_k(\varphi_p)$$

for some function $g_k$ depending on $p$ only through $\varphi_p$, and thus by explicit computation one sees that $|p|^2 \{ f_1, G_2^0 \}$ depends on $p$ only through $\varphi_p$; the same type of argument apply for the other terms. In other words, the normalization factors $|p|^a$ in \[2.49\] are made so that $F_{3,k}(p) = \overline{F_{3,k}(\varphi_p)}$.

Let $G^{(3)} := \varepsilon G_1 + \varepsilon^2 G_2 + \varepsilon^3 G_3$ which is real-on-real by construction, and $\Phi^{(3)}$ be the time-$1$ map of $G^{(3)}$. By construction, fixing $f_s(q)$ as above for $s = 1, 2, 3$ while $f_s(q)$ are still arbitrary for $s \geq 4$, we get

$$H \circ \Phi^{(3)} = \frac{p^2}{2} + \varepsilon h_1(p) + \varepsilon^2 h_2(p) + \varepsilon^3 h_3(p) + \varepsilon^4 R_4,$$
where $R_4$ is a suitable remainder; precisely

$$
\varepsilon^4 R_4 = \frac{1}{2}\left(\\{\{H, \varepsilon G_1^0\}, \varepsilon^2 G_2^0\} + \\{\{H, \varepsilon^2 G_2^0\}, \varepsilon^2 G_2^0 + \varepsilon^3 G_3^0\} + \\{\{H, \varepsilon^3 G_3^0\}, G^{(3)}\}\right) \\
+ \frac{1}{3!} \left(\\{\{H, \varepsilon G_1^0\}, \varepsilon G_1^0\}, \varepsilon^2 G_2^0 + \varepsilon^3 G_3^0\} + \\{\{H, \varepsilon G_1^0\}, \varepsilon^2 G_2^0 + \varepsilon^3 G_3^0\}, G^{(3)}\}\right) \\
+ \sum_{n \geq 4} \frac{1}{n!} \{H, G^{(3)}\}^{(n)}.\tag{2.50}
$$

Note that each term in (2.50) above has at least a factor $\varepsilon^4$. Note also that $G^{(3)}$ is analytic as function of $p$.

3 The $s$-th step

We now describe the procedure at the $s$-th step: as we shall see we first need to perform formal computations and then provide suitable modifications that allow us to obtain the bounds needed for the convergence of the algorithm.

3.1 Formal expansion

Consider the formal truncation of $H \circ \Phi$ at order $s - 1$, namely

$$
H_0(p) + \varepsilon (f_1 + \{H_0, G_1^0\}) + \varepsilon^2 (f_2 + \{H_0, G_2^0\} + \{f_1, G_1^0\}) + \\
+ \sum_{n=3}^{s-1} \varepsilon^n \left( \sum_{m=1}^{n} \frac{1}{m!} \sum_{n_0 + \cdots + n_m = n} \{\cdots \{f_{n_0}, G_{n_1}^0, \cdots, G_{n_m}^0\} \cdots\}, G_{n_m}^0\right),
$$

where we denoted $G_0^1 = G_1$. Recall that $f_2$ as in (2.24) annihilate the $O(\varepsilon^2)$ term, when restricted to the energy surface $\{H_0 = 1/2\}$. Then using (2.39) we produced a correction term $f_3^*$ so that $f_2 + \varepsilon f_3^*$ annihilate the $O(\varepsilon^2)$ term to the order $O(\varepsilon^4)$, when restricted to the energy surface $\{H_\varepsilon^1 = 1/2\}$ and so on; eventually we achieve the $O(\varepsilon^{s+1})$ order cancellation on $\{H_\varepsilon^{(s-1)} = 1/2\}$ we need to cancel the $\varepsilon^2(\cdots)$ bracket to $O(\varepsilon^{s+1})$, the $\varepsilon^3(\cdots)$ bracket to $O(\varepsilon^{s+2})$, and so on the $\varepsilon^{s-1}(\cdots)$ bracket to $O(\varepsilon^2)$.

Assume recursively that for all $n = 1, \ldots, s - 1$ the functions $G_{n,k}^0$ are analytic for $p$ in a neighborhood of $\{H_\varepsilon^{(s-1)} = 1/2\} \cap \mathcal{P}$, and one has

$$
f_{n,-k} = \overline{f}_{n,k}, \quad f_{n,-k}^* = \overline{f}_{n,k}^*, \quad G_{n,-k}^0 = \overline{G}_{n,k}^0.\tag{3.51}
$$

Of course we have to prove that (3.51) above is satisfied also at step $s$. 

12
For all $j = 1, \ldots, s$ set
\[
\mathcal{K}_j = \{ k \in \mathbb{Z}^2 : k = \sum_{l=1}^{j} k_l \text{ for some } k_1, \ldots, k_j \in \mathcal{K} \},
\] (3.52)
distinguish the two subsets
\[
\mathcal{K}_j^{\text{big}} := \{ k \in \mathcal{K}_j : p_k^0 \notin \mathcal{P} \} \quad \text{(3.53)}
\]
and
\[
\mathcal{K}_j^{\text{small}} := \{ k \in \mathcal{K}_j : p_k^0 \in \mathcal{P} \}, \quad \text{(3.54)}
\]
and for all $j = 1, \ldots, s - 1$ let $p_j^j(q, \varepsilon)$ either $p_k^0$ for $k \in \mathcal{K}_{s-1}^{\text{big}}$ or the unique point in $\mathcal{P}$ such that
\[
p \cdot k = 0
\]
\[
\frac{1}{2} |p|^2 + \sum_{k=1}^{j} \varepsilon^k f_k(q) = \frac{1}{2}.
\]
Notice that for $k \in \mathcal{K}_j^{\text{small}}$ such points can be written in the form
\[
p_j^j(q, \varepsilon) = p_k^0(1 - \sum_{l=1}^{j} \varepsilon^l f_l(q))^{1/2},
\] (3.55)
while we recall that for $k \in \mathcal{K}_j^{\text{big}}$ one has that $\overline{p}_k$ is Diophantine.

We define the correction to $f_s(q) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \sum_{m=1}^{n} \frac{1}{m!} \sum_{n_0 + \ldots + n_m = n} \sum_{n_0 \geq 0} \{\ldots \{f_{n_0}, G_{n_1}^0, G_{n_2}^0 \ldots, G_{n_m}^0\}\ldots\}, G_{n_m}^0\} \right)$
\[
\sum_{n=1}^{n} \frac{1}{m!} \sum_{n_0 + \ldots + n_m = n} \sum_{n_0 \geq 0} \{\ldots \{f_{n_0} + f_{n_0}^*, G_{n_1}^0, G_{n_2}^0 \ldots, G_{n_m}^0\}\ldots\}, G_{n_m}^0\} \bigg|_{p = p_k^{s-1}(q, \varepsilon)}
\]
\[
- \{\ldots \{f_{n_0} + f_{n_0}^*, G_{n_1}^0, G_{n_2}^0 \ldots, G_{n_m}^0\}\ldots\}, G_{n_m}^0\} \bigg|_{p = p_k^{s-2}(q, \varepsilon)}
\]
\[
\bigg), \quad \text{(3.56)}
\]
where we denoted $f_1^* = f_2^* \equiv 0$ and $f_0 = H_0$.

Set
\[
f_s(q) := \sum_{n=1}^{s-2} f_n^s(q) = \sum_{k} f_{s,k} e^{i k q},
\] (3.57)
and
\[ p_k := \begin{cases} p_k & k \in \mathcal{K}_s^{\text{big}}, \\ p_k^0 & k \in \mathcal{K}_s^{\text{small}}. \end{cases} \tag{3.58} \]

We define the correction term \( f_s(q) \) by setting its Fourier coefficients as
\[ f_{s,k} := -\left( f_{s,k}^* + \sum_{m=1}^{s} \frac{1}{m!} \sum_{\substack{n_0+\ldots+n_m=s \\ n_m \geq 0 \\ n_1,\ldots,n_m \geq 1}} \{\ldots\{f_{n_0} + f_{n_0}^*, G_{n_1}^0\}, \ldots, G_{n_m}^0\}\right) \tag{3.59} \]

Note that \( f_s(q) \) is well defined since the functions \( G_{n,k}^0 \) are analytic in \( p \).

**Remark 3.7.** Note that by construction \( f_{s,-k} = \overline{f}_{s,k}^* \) so that the first two recursive assumptions in (3.51) are satisfied also at step \( s \).

**Remark 3.8.** Both in (3.56) and (3.59) we need to use \( G_{n,k}^0 \) because the functions \( G_{n,k} \) are not well defined in an open neighborhood of \( |p| = 1 \) for \( n_1 \geq 2 \). However, their derivatives are formally equal.

The formal equation for \( G_s \) is, therefore,
\[ p \cdot \partial_q G_s = -\left( f_s + f_s^* + \sum_{m=1}^{s} \frac{1}{m!} \sum_{\substack{n_0+\ldots+n_m=s \\ n_m \geq 0 \\ n_1,\ldots,n_m \geq 1}} \{\ldots\{f_{n_0} + f_{n_0}^*, G_{n_1}^1\}, \ldots, G_{n_m}^1\}\right) \tag{3.60} \]

but actually there are cancellations allowing to get rid of the case \( n_0 = 0 \) simply changing the combinatorial factors, as the following result shows.

**Lemma 3.9.** The formal equation (3.60) is equivalent to
\[ p \cdot \partial_q G_s = -\left( f_s + f_s^* + \sum_{m=1}^{s-1} c_m \sum_{\substack{n_0+\ldots+n_m=s \\ n_i \geq 1}} \{\ldots\{f_{n_0} + f_{n_0}^*, G_{n_1}^1\}, \ldots, G_{n_m}^1\}\right) \tag{3.61} \]

where \( f_1^* = f_2^* \equiv 0 \) and
\[ c_0 := 1, \quad c_m := \frac{1}{m!} - \sum_{j=1}^{m} \frac{1}{(j+1)!} c_{m-j}. \tag{3.62} \]
Proof. We prove the result by double induction on \( s \geq 2, m \geq 1 \). The case \( s = 2, m = 1 \) is the explicit computation in (2.16). Assume inductively the statement to be true up to order \( s \) with the coefficients \( c_1, \ldots, c_{s-1} \) given by (3.62). At order \( s + 1 \) the equation (3.60) reads

\[
p \cdot \partial_q G_{s+1} = - \left( f_{s+1} + f_{s+1}^* + \sum_{m \geq 1} \frac{1}{m!} \sum_{n_0 + \ldots + n_m = s} \{ \ldots \{ f_{n_0} + f_{n_0}^*, G_{n_1} \}, G_{n_2} \ldots \}, G_{n_m} \right)
\]

\[
= - \left( f_{s+1} + f_{s+1}^* + \sum_{m \geq 1} \frac{1}{m!} \sum_{n_0 + \ldots + n_m = s} \{ \ldots \{ f_{n_0} + f_{n_0}^*, G_{n_1} \}, G_{n_2} \ldots \}, G_{n_m} \right)
\]

\[
+ \sum_{m \geq 2} \frac{1}{m!} \sum_{n_0 + \ldots + n_m = s} \{ \ldots \{ H_0, G_{n_1} \}, G_{n_2} \ldots \}, G_{n_m} \right) \right)
\]

(3.63)

By the inductive hypothesis we have

\[
\{ H_0, G_{n_0} \} = - \left( f_{n_0} + f_{n_0}^* + \sum_{l=1}^{n_0-1} c_l \sum_{h_0 + \ldots + h_l = n_0} \{ \ldots \{ f_{h_0} + f_{h_0}^*, G_{h_1} \}, G_{h_2} \ldots \}, G_{h_l} \right) , (3.64)
\]

which inserted into (3.63) gives

\[
p \cdot \partial_q G_{s+1} = - \left( f_{s+1} + f_{s+1}^* + \sum_{m \geq 1} \frac{1}{m!} \sum_{n_0 + \ldots + n_m = s} \{ \ldots \{ f_{n_0} + f_{n_0}^*, G_{n_1} \}, G_{n_2} \ldots \}, G_{n_m} \right)
\]

(3.65)

In equation (3.65) above all the indices are strictly positive, and in order to prove (3.62) we need to collect together the terms with the same number of Poisson brackets, since it is clear
that their coefficient do not depend on the indices $n_i$ or $h_i$. In the first line the terms with $m$ Poisson brackets appear with a coefficient $1/m!$, in the second line the terms with $m$ Poisson brackets appear with coefficient $-1/(m+1)!$ and finally in the last line the terms with $m$ Poisson brackets appear with coefficient
$$-\sum_{j+l=m}^{j+l=m} \frac{1}{(j+1)!^{c_l}},$$
therefore the assertion follows.

**Remark 3.10.** The cancellation provided by Lemma 3.9 is needed because it allows us to count the number of summand appearing in (3.61). Indeed since all the indices $n_i$ in (3.61) are strictly positive, the number of summand is equal to the partition function $p(s)$ of the natural number $s$ which, be the Hardy-Ramanujan asymptotic formula [4] grows like
$$p(s) \sim \frac{1}{4s^\sqrt{3\pi}} \cdot e^{2\sqrt{s/\sqrt{3}}} \quad \text{as} \quad s \to \infty. \quad (3.66)$$

### 3.2 Bounds

We are now ready to modify the previous construction in order to get the bounds needed for the convergence.

For a function $F = F(q, p)$ and fixed $\sigma > 1$ let us introduce the scale of analytic norms
$$\|F\|_{\xi} = \|F\|_{\xi,\sigma} := \sum_{k \in \mathbb{Z}^2} \langle k \rangle^{2\sigma} e^{2\xi |k|} \sup_{p \in P} |F_k(p)|^2, \quad \xi > 0 \quad (3.67)$$
where we used the standard notation for the Japanese symbol $\langle k \rangle := \max\{1, |k|\}$; recall that since $\sigma > 1$ we have the algebra property
$$\|FG\|_{\xi} \leq C_0 \|F\|_{\xi} \|G\|_{\xi}, \quad (3.68)$$
where $C_0$ is some positive constant (depending on $\sigma$). We omit the index $\sigma$ in the norm because it is fixed once and for all.

Set
$$\xi_s := \xi_0 \left(1 - \frac{1}{2} \sum_{j=1}^{s} 2^{-j}\right), \quad (3.69)$$
and note that
$$\xi_s > \xi_{s+1} \rightarrow \frac{\xi_0}{2}. \quad (3.70)$$

Define
$$F_{s,k}(p) := \sum_{m=1}^{s-1} c_m |p|^{2m} \sum_{\substack{n_0 + \ldots + n_m = s \\
n_i \geq 1}} \left\{\ldots \left\{ f_{n_0} + f_{n_0}^*, G_{n_1}^{0}, G_{n_2}^{0}, \ldots \right\}, G_{n_{m}}^{0} \right\}; \quad (3.71)$$
note that the normalization exponents appearing in (3.71) are chosen so that \( F_{s,k}(p) = F_{s,k}(p') \) whenever \( p, p' \) are parallel, namely \( F_{s,k}(p) = F_{s,k}(\varphi_p) \), where we are using the polar coordinates \((\varphi_p, r_p)\). Note also that one has
\[
F_{s,-k}(p) = F_{s,k}(p) .
\]

**Remark 3.11.** The functions \( F_{s,k}(p) \) defined in (3.71) are well defined (and actually analytic) for all \( p \) in a neighborhood of \( \{ H_{\frac{s-1}{2}} \} \cap P \).

Let us set
\[
G_{s,k}^{0}(p) := \frac{-1}{ip \cdot k} \left( f_{s,k} + f_{s,k}^{*} + F_{s,k}(p) \right) = \frac{F_{s,k}(p) - F_{s,k}(\tilde{p}_k)}{i(p - \tilde{p}_k) \cdot k} ,
\]
and note that
\[
G_{s,-k}^{0}(p) = G_{s,k}^{0}(p)
\]
so that also the third recursive assumption in (3.51) is satisfied.

In particular, if we set \( G(s) = \sum_{m=1}^{s} \varepsilon^m G_{m}^{0} \) and \( \Phi(s) \) is the time-1 map generated by \( G(s) \), by definition we formally get
\[
H \circ \Phi(s) = \frac{p^2}{2} + \varepsilon h_1(p) + \ldots + \varepsilon^s h_s(p) + \varepsilon^{s+1} R_{s+1} ,
\]
where
\[
\varepsilon^{s+1} R_{s+1} := \sum_{n=2}^{s+1} \frac{1}{n!} \left\{ \ldots \left\{ H, \sum_{m_1=1}^{s} \varepsilon^{m_1} G_{m_1}^{0} \right\}, \sum_{m_2=1}^{s} \varepsilon^{m_2} G_{m_2}^{0} \right\}, \ldots, \sum_{m=\max\{0, s+1-m_1-\ldots-m_{n-1}\}}^{s} \varepsilon^{m} G_{m}^{0} \right\}.
\]

(3.75)

Note that each summand in (3.75) above has indeed at least a factor \( \varepsilon^{s+1} \).

By construction we have the following result.

**Lemma 3.12.** There exist a constant \( C > 0 \) (depending on \( \sigma \)) such that
\[
\| F_s \|_{\xi_s} \leq \left( C \| f_1 \|_{\xi_0} \right)^s ,
\]
\[
\| f_s \|_{\xi_s}, \| \partial^h f_s \|_{\xi_{s+1}} \leq \left( C \| f_1 \|_{\xi_0} \right)^s ,
\]
\[
\| f_s^{*} \|_{\xi_s}, \| \partial^h f_s^{*} \|_{\xi_{s+1}} \leq \left( C \| f_1 \|_{\xi_0} \right)^s ,
\]
\[
\| G_{s}^{0} \|_{\xi_{s+1}}, \| \partial^h G_{s}^{0} \|_{\xi_{s+1}} \leq \left( C \| f_1 \|_{\xi_0} \right)^s ,
\]
where we denoted
\[
\partial^h = \partial_{p_1}^{\alpha_1} \partial_{p_2}^{\alpha_2} \partial_{q_1}^{\beta_1} \partial_{q_2}^{\beta_2}, \quad \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = h .
\]
Proof. We prove the result by induction on $s$; the case $s = 2$ have been considered explicitly in Section 2. Assume inductively the bounds (3.76) up to $s$. The bound (3.76a) follows directly by the definition (3.71) and Remark 3.11 in particular the Fourier coefficients will have at most a factor growing polynomially in $k$ and we can control such growth with a shrink of the analyticity strip, which provides an exponential decay $e^{-\xi_{s+1}|k|/2}$. Then from the Hardy-Ramanujan formula (3.66) we can bound the number of the summands in (3.71) and deduce the bound (3.76a), which in turn gives the bound for $f_{s+1}$, $f^*_{s+1}$ and their derivatives. Finally, to get the bound for $G_0^{s+1}$ we can reason as done for $G_0^{s+1}$. Indeed from the definition (3.73) we get

$$|G_{s+1,k}^0(p)| = \left|\frac{F_{s,k}(p) - F_{s,k}(\tilde{p}_k)}{(p - \tilde{p}_k) \cdot k}\right|$$

so that (3.76d) follows.

From Lemma 3.12 above we deduce the convergence of the algorithm.

**Lemma 3.13.** One has $\|R_{s+1}\|_{\xi_{s+1}} \leq e^{C\|f_1\|_{\xi_0}}$.

**Proof.** Let us factor out the common factor $\varepsilon_{s+1}$ in (3.75). We use the algebra property (3.68) and Lemma 3.12 above to deduce that each summand in (3.75) can be bounded by $(C\|f_1\|_{\xi_0})^n$, so that the assertion follows.

Finally setting

$$G^{(\infty)} = \sum_{s=1}^{\infty} \varepsilon^s G_s^0,$$

$$f = \sum_{s=1}^{\infty} \varepsilon^s f_s,$$

$$f^* = \sum_{s=1}^{\infty} \varepsilon^s f^*_s,$$

we have the following result.

**Lemma 3.14.** For all $|\varepsilon| < (C\|f_1\|_{\xi_0})^{-1}$ one has

$$\|f(\cdot; \varepsilon)\|_{\xi_0/2}, \|f^*(\cdot; \varepsilon)\|_{\xi_0/2}, \|G^{(\infty)}(\cdot, \cdot; \varepsilon)\|_{\xi_0/2} < \infty.$$  \hspace{1cm} (3.78)

**Proof.** One has

$$\|f(\cdot; \varepsilon)\|_{\xi_0/2} \leq \sum_{s \geq 1} \varepsilon^s (\|f_s\|_{\xi_0/2})^s \leq \sum_{s \geq 1} \varepsilon^s (\|f_s\|_{\xi_0})^s < \infty.$$  \hspace{1cm} (3.76b)

Same for $f^*$ and $G^{(\infty)}$.

Combining Lemmata 3.13 and 3.14 we conclude the proof of Theorem 1.1.
4 The $p$-dependent case

In this section we show that if we allow the correction $f$ to depend also on $p$ then the construction is much more easy and allows to find local integrability in a whole domain of the phase space.

Now our aim is to explicitly construct a correction

$$f(q, p; \varepsilon) = \sum_{s \geq 1} \varepsilon^s f_s(q, p)$$

(4.1)

and a Hamiltonian $G(q, p; \varepsilon)$ such that $H_\varepsilon \circ \Phi_G = |p|^2/2$ on an open subset of the phase space, where $\Phi_G$ is the time-1 flow generated by $G$.

We start by choosing $f_1(q, p)$ so that $f_1, k(p) = (i p \cdot k) g_1, k(p)$, with $g_2, k(p)$ any function satisfying $|g_2, k(p)| \leq C M e^{-\xi_0 |k|}$, so that (2.14) is easily solved. Clearly such this makes $f_1$ real-on-real.

At order $\varepsilon^2$ we still need to solve (2.17) but now we are on the whole domain $P$ and we allow $f_2$ to depend on $p$. Passing to the Fourier side, we have to solve (2.18); for all $k$, if we define

$$f_{2, k}(p) := -\frac{1}{2} \sum_{k_1 + k_2 = k} \frac{k_1 f_{1, k_1} k_2 f_{1, k_2}}{(p \cdot k)^2} + (i p \cdot k) g_{2, k}(p),$$

(4.2)

where $g_{2, k}(p)$ is any function satisfying $|g_{2, k}(p)| \leq C M^2 e^{-3\xi_0 |k|/4}$, then no small divisors appear in (2.18), and one has simply

$$G_{2, k}(p) = g_{2, k}(p).$$

Clearly one can reason exactly in the same way to all orders, by setting

$$f_{s, k}(p) := \left( \sum_{m=1}^{s} \frac{1}{m!} \sum_{n_0 + \ldots + n_m = s, n_0 \geq 0, n_1, \ldots, n_m \geq 1} \{ \ldots \{ f_{n_0} + f^*_{n_0}, G_{n_1}^0, \ldots, G_{n_m}^0 \} \ldots \} + (i p \cdot k) g_{s, k}(p) \right).$$

(4.3)

with $g_{s, k}(p)$ any function satisfying $|g_{s, k}(p)| \leq C M^* e^{-\xi_s |k|}$; in this way the solution of the homological equation is simply

$$G_{s, k}(p) = g_{s, k}(p).$$

Moreover the functions $f_s$ obtained are clearly real-on-real.

This concludes the proof of Theorem 1.3.

5 Chaotic Behavior

We now prove that it is possible to choose the potential $f(q, \varepsilon)$ in (1.4) so that in the region $\{H_\varepsilon = 1/2\} \cap \mathcal{P}^C$ there is at least one orbit that exhibit chaotic behavior. In order to do so of
course we need to be close to a resonance in $p$-space: for instance we may want to be close to $\overline{p} = (0, 1)$.

First of all let $n$ be large enough (to be chosen?) so that

$$f_{1,(n,0)} = f_{1,(-n,0)} = \frac{1}{2}$$

and suppose that

$$f_1(q) = \cos(nq_1) + \sum_{k \in \mathbb{K}, |k| < n} f_{1,k} e^{ik \cdot q}.$$  

(5.2)

Of course the high order coefficients of the potential have to be constructed starting from $f_1$ as done in the previous sections.

Let us consider the truncated Hamiltonian

$$H_1(q, p) = \frac{|p|^2}{2} + \varepsilon f_1(q)$$

(5.3)

so that the equation of motion are

$$\begin{cases}
\dot{q}_1 = p_1 \\
\dot{p}_1 = \varepsilon n \sin(nq_1) + \varepsilon \sum_{k \in \mathbb{K}, |k| < n} ik_1 f_{1,k} e^{ik \cdot q} \\
\dot{q}_2 = p_2 \\
\dot{p}_2 = \varepsilon \sum_{k \in \mathbb{K}, |k| < n} ik_2 f_{1,k} e^{ik \cdot q}.
\end{cases}$$

(5.4)

Let us now introduce the rescaling

$$p_1 = \sqrt{\varepsilon} y_1 \quad \tau = \sqrt{\varepsilon} t$$

(5.5)

so that (5.4) can be rewritten as

$$\begin{cases}
q'_1 = y_1 \\
y'_1 = n \sin(nq_1) + \sum_{k \in \mathbb{K}, |k| < n} ik_1 f_{1,k} e^{ik \cdot q} \\
q'_2 = \frac{1}{\sqrt{\varepsilon}} p_2 \\
p'_2 = \sqrt{\varepsilon} \sum_{k \in \mathbb{K}, |k| < n} ik_2 f_{1,k} e^{ik \cdot q}.
\end{cases}$$

(5.6)

where ' denotes the derivative w.r.t. $\tau$.  

20
6 Final remarks

The proof of Theorem 1.1 heavily relies on the dimension; precisely we are able to perform our construction because at each step \( s \) there is a unique point in \( \mathcal{P} \) such that

\[
p \cdot k = 0
\]

\[
\frac{1}{2} |p|^2 + \sum_{k=1}^{s-2} \varepsilon^k f_k(q) = \frac{1}{2}.
\] (6.1)

One already sees the problem at the second step when, in order to solve the homological equation (2.18), one needs to make sure that when \( p \cdot k = 0 \) also the l.h.s. vanishes.

The existence of a unique point in \( \mathcal{P} \) solving (6.1) allow us to define \( f \) as a function of \( q \) only and this is clearly not possible in higher dimension. Indeed if \( p \in \mathbb{R}^n \) with \( n \geq 3 \) there is a whole curve \( \Gamma \) in \( \mathcal{P} \) solving (6.1), and thus one seem to be forced to define \( f_{2,k} \) as a function of \( p \), at least for \( p \in \Gamma \): of course the same type of argument can be carried at each step.

On the other hand, if we allow \( f \) to depend also on \( p \), the construction is way easier, as the proof of Theorem 1.3 shows, but of course in that case the Hamiltonian (1.5) is not associated with a metric.

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