ON THE HODGE-NEWTON DECOMPOSITION FOR SPLIT GROUPS

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1. Introduction

The main purpose of this paper is to prove a group-theoretic generalization of a theorem of Katz on isocrystals. Along the way we reprove the group-theoretic generalization of Mazur’s inequality for isocrystals due to Rapoport–Richartz, and generalize from split groups to unramified groups a result from [KR] which determines when the affine Deligne-Lusztig subset \( X_G^\mu(b) \) of \( G(L)/G(o_L) \) is non-empty.

Let \( F \) be a finite extension of \( \mathbb{Q}_p \) with uniformizing element \( \varpi \). We write \( L \) for the completion of the maximal unramified extension of \( F \) in some algebraic closure \( \overline{F} \) of \( F \). We write \( \sigma \) for the Frobenius automorphism of \( L \) over \( F \), and we write \( o \) (respectively, \( o_L \)) for the valuation ring of \( F \) (respectively, \( L \)).

Let \( G \) be a split connected reductive group over \( o \) and let \( A \) be a split maximal torus of \( G \) over \( o \). Fix a Borel subgroup \( B = AU \) containing \( A \) with unipotent radical \( U \), as well as a parabolic subgroup \( P \) of \( G \) containing \( B \). Write \( P = MN \), where \( M \) is the unique Levi subgroup of \( P \) containing \( A \) and \( N \) is the unipotent radical of \( P \).

We write \( X_G \) for the quotient of \( X_\ast(A) \) by the coroot lattice for \( G \), and we define a homomorphism \( w_G : G(L) \to X_G \) as follows. For \( g \in G(L) \) we define \( r_B(g) \in X_\ast(A) \) to be the unique element \( \mu \in X_\ast(A) \) such that \( g \in G(o_L) \cdot \mu(\varpi) \cdot U(L) \), and we define \( w_G(g) \) to be the image of \( r_B(g) \) under the canonical surjection from \( X_\ast(A) \) to \( X_G \). (This definition of \( w_G \) suffices for the purposes of this paper; see §7 of [Kot97] for a definition that applies to groups \( G \) that are not split over \( L \).)

Applying the construction above to \( M \) rather than \( G \), we obtain \( X_M \), the quotient of \( X_\ast(A) \) by the coroot lattice for \( M \), and a homomorphism

\[
w_M : M(L) \to X_M.
\]

For \( \mu, \nu \in X_M \) we write \( \mu \leq^P \nu \) if \( \nu - \mu \) is a non-negative integral linear combination of (images in \( X_M \) of) coroots \( \alpha^\vee \), where \( \alpha \) ranges over the roots of \( A \) in \( N \).

We write \( A_P \) for the identity component of the center of \( M \); thus \( A_P \) is a split torus over \( F \). Let \( a_P \) denote the real vector space \( X_\ast(A_P) \otimes_{\mathbb{Z}} \mathbb{R} \). As usual \( P \) determines an open chamber \( a_P^+ \) in \( a_P \), defined by

\[
a_P^+ := \{ x \in a_P : \langle \alpha, x \rangle > 0 \text{ for every root } \alpha \text{ of } A_P \text{ in } N \}.
\]

The composition \( X_\ast(A_P) \to X_\ast(A) \to X_M \), when tensored with \( \mathbb{R} \), yields a canonical isomorphism \( a_P \simeq X_M \otimes_{\mathbb{Z}} \mathbb{R} \). We write \( X_M^+ \) for the subset of \( X_M \) consisting of all elements whose image in \( X_M \otimes_{\mathbb{Z}} \mathbb{R} \) lies in \( a_P^+ \).

1991 Mathematics Subject Classification. Primary 14L05; Secondary 11S25, 20G25, 14F30.
Partially supported by NSF Grant DMS-0071971.
For any coweight $\mu \in X_\iota(A)$ (usually taken to be dominant) and any $b \in G(L)$ we consider the affine Deligne-Lusztig set

$$X^G_\mu(b) := \{ x \in G(L)/G(\mathfrak{o}_L) : x^{-1}b\sigma(x) \in G(\mathfrak{o}_L)\mu(\varpi)G(\mathfrak{o}_L) \}. $$

For $b \in M(L)$ we may replace $G$ by $M$ in this definition, obtaining the affine Deligne-Lusztig set $X^M_\mu(b)$; moreover, the inclusion $M(L)/M(\mathfrak{o}_L) \hookrightarrow G(L)/G(\mathfrak{o}_L)$ induces an inclusion

$$X^M_\mu(b) \hookrightarrow X^G_\mu(b).$$

**Theorem 1.1.** Let $\mu \in X_\iota(A)$ be a dominant coweight, and let $b$ be a basic element in $M(L)$ such that $w_M(b)$ lies in the subset $X^+_M$ of $X_M$.

1. If $X^G_\mu(b)$ is non-empty, then $w_M(b) \leq \mu$. Here we are regarding $\mu$ as an element of $X_M$.
2. Suppose that $w_M(b) = \mu$ (with $\mu$ again regarded as an element of $X_M$). Then the natural injection $X^M_\mu(b) \hookrightarrow X^G_\mu(b)$ is a bijection.

See [Kot85] for the definition of basic. The first part of the theorem is a reformulation of the group-theoretic generalization of Mazur’s inequality (see Theorem 1.4.1 of [Kat79]) proved by Rapoport-Richartz [RR96]. Proposition 4.10 shows that our formulation agrees with that of [RR96]. The second part of the theorem is the group-theoretic generalization of Katz’s theorem (see Theorem 1.6.1 of [Kat79]) which was formulated in [KR] Remark 4.12 and proved there for $GL_n$ and $GSp_{2n}$ as a consequence of Katz’s theorem.

The theorem above is proved in [3]. In [4] it is generalized to the case of unramified groups (see Theorem 4.1 for a precise statement). Theorem 4.3 determines (for any unramified $G$) when the affine Deligne-Lusztig set $X^G_\mu(b)$ is non-empty, generalizing Proposition 4.6 of [KR], which treated the case of split groups.

We now direct the reader’s attention to some related work. In [ER02] (respectively, [KR]) the converse to Mazur’s inequality is proved for $GL_n$ (respectively, $GL_n$ and $GSp_{2n}$). Recently C. Leigh [Lei02] has proved the converse to Mazur’s inequality for all split classical groups. The reader who would like to know how these questions relate to the reduction modulo $p$ of Shimura varieties should consult the survey article [Rap02] by Rapoport.

It is a pleasure to thank M. Rapoport, both for the interest he has taken in this work and the helpful comments he made about a preliminary version of the paper.

2. Retractions

2.1. Notation. Let $\mathfrak{o}$ be a complete discrete valuation ring with fraction field $F$, uniformizing element $\varpi$, and residue field $k = \mathfrak{o}/\varpi\mathfrak{o}$.

Let $G$ be a split connected reductive group over $\mathfrak{o}$ and let $A$ be a split maximal torus of $G$ over $\mathfrak{o}$. We denote by $\mathcal{B} = \mathcal{B}(A)$ the set of Borel subgroups of $G$ containing $A$ (all of which are defined over $\mathfrak{o}$). For $B \in \mathcal{B}$ denote by $\mathcal{B}$ the Borel subgroup in $\mathcal{B}$ that is opposite to $B$. For $B,B_1,B_2,\ldots \in \mathcal{B}$ we denote the unipotent radical by $U,U_1,U_2,\ldots$, so that (for instance) $B = AU$. We write $K$ for $G(\mathfrak{o})$.

2.2. Definition of retractions. For $g \in G(F)$ and $B = AU$ in $\mathcal{B}$ we define $r_B(g) \in X_\iota(A)$ to be the unique element $\mu \in X_\iota(A)$ such that $g \in K \cdot \mu(\varpi) \cdot U(F)$. The family $(r_B(g))_{B \in \mathcal{B}}$ of retractions is used by Arthur to form weighted orbital integrals.
2.3. Positivity properties of families of retractions. The family \((r_B(g))_{B \in \B}\) has the following basic positivity property \cite[Lemma 3.6]{Art76}, \cite[Lemma 85]{HC66}.

Let \(B_1 = AU_1\) and \(B_2 = AU_2\) be adjacent Borel subgroups in \(\B\), and let \(\alpha\) be the unique root of \(A\) that is positive for \(B_1\) and negative for \(B_2\). Then

\[(2.1) \quad r_{B_1}(g) - r_{B_2}(g) = j \cdot \alpha^\vee,\]

where \(j\) is a non-negative integer that we will now define. (We will not recall the proof except to say that one reduces to the case of \(SL(2)\), for which a simple computation with \(2 \times 2\) matrices does the job.)

The group \(U_1\) is the semidirect product of the normal subgroup \(U_1 \cap U_2\) and the root subgroup \(U_\alpha\) determined by \(\alpha\). In particular \(U_\alpha\) is a quotient of \(U_1\), and we refer to the image of \(u_1 \in U_1\) in \(U_\alpha\) as the \(\alpha\)-component of \(u_1\). Choosing an isomorphism between \(U_\alpha\) and \(G_\alpha\) over \(\mathfrak{a}\), we may view the \(\alpha\)-component of \(u_1 \in U_1(F)\) as an element of \(F\), well-defined up to multiplication by a unit.

Now we can define \(j\). Decompose \(g = k \cdot u_1 \cdot \mu(x)\) with \(k \in K\), \(u_1 \in U_1(F)\) and \(\mu \in X_\alpha(A)\) (so that \(\mu = r_{B_2}(g)\)), and write \(x \in F\) for the \(\alpha\)-component of \(u_1\).

Then \(j\) is defined to be 0 if \(x \in \mathfrak{a}\) and is defined to be \(-\val(x)\) if \(x \notin \mathfrak{a}\).

The basic positivity property above has some obvious consequences. One is that for any \(B, B' \in \B\) the coweight \(r_B(g) - r_{B'}(g)\) is a non-negative integral linear combination of coroots \(\alpha^\vee\) that are positive for \(B_1\) and negative for \(B_2\). Thus for any \(B, B' \in \B\) we have

\[(2.2) \quad r_B(g) \leq r_{B'}(g) \leq r_B(g),\]

where \(\mu \leq \nu\) (for coweights \(\mu, \nu\)) means that \(\nu - \mu\) is a non-negative integral linear combination of coroots that are positive for \(B\).

2.4. Recognizing the subset \(K \cdot M(F)\) of \(G(F)\) using retractions. Let \(M\) be a Levi subgroup of \(G\) containing \(A\) and note that \(M\) is automatically defined over \(\mathfrak{a}\).

We write \(X_M\) for the quotient of \(X_\alpha(A)\) by the coroot lattice for \(M\). For example, when \(M = A\), we have \(X_M = X_\alpha(A)\).

**Lemma 2.1.** Let \(g \in G(F)\). Then \(g \in K \cdot M(F)\) if and only if \(r_{B_1}(g)\) and \(r_{B_2}(g)\) are equal in \(X_M\) for all \(B_1, B_2 \in \B\).

**Proof.** Assume first that \(g = km\) with \(k \in K\) and \(m \in M(F)\). We must show that all the retractions of \(g\) are equal in \(X_M\). For any \(B \in \B\) the intersection \(B \cap M\) is a Borel subgroup of \(M\), and it is clear that \(r_B(g) = r_{B \cap M}(m)\). Thus, for \(B_1, B_2 \in \B\) the coweight \(r_{B_1}(g) - r_{B_2}(g)\) is a non-negative integral linear combination of coroots \(\alpha^\vee\) for \(M\) that are positive for \(B_1\) and negative for \(B_2\), and in particular \(r_{B_2}(g) = r_{B_1}(g)\) in \(X_M\).

Now assume that all the retractions of \(g\) are equal in \(X_M\). Choose \(B = AU\) in \(\B\) and choose a minimal gallery

\[(2.3) \quad B = B_0, B_1, \ldots, B_{l-1}, B_l = \bar{B}\]

of Borel subgroups in \(\B\) joining \(B\) to \(\bar{B}\). Thus \(l\) is equal to the number of positive roots for \(B\), and the subgroups \(B_i, B_{i+1}\) are adjacent for \(0 \leq i \leq l - 1\). Write \(B_i = AU_i\) and put \(V_i := U \cap U_i\). Note that \(V_0 = U\) and \(V_l = \{1\}\). We will prove by induction on \(i\) (for \(0 \leq i \leq l\)) that \(g \in K \cdot V_i(F) \cdot M(F)\). The case \(i = 0\) is obvious and the case \(i = l\) is the statement of the lemma.
For the induction step we suppose that for some \( i \) less than \( l \) we have \( g = kum \) for \( k \in K \), \( u \in V_i(F) \), \( m \in M(F) \). The group \( V_i \) is the semidirect product of the normal subgroup \( V_i+1 \) and the root subgroup \( U_\alpha \), where \( \alpha \) is the unique root of \( A \) that is positive for \( B_i \) and negative for \( B_i+1 \). If \( \alpha \) is not a root of \( M \), then \( \alpha^\vee \) is a non-torsion element in \( X_M \), and our hypothesis that the retractions for \( B_i \) and \( B_i+1 \) are equal in \( X_M \) ensures that the \( \alpha \)-component \( u_\alpha \) of \( u \) lies in \( U_\alpha(\mathfrak{o}) \), so that we can write \( g = (k u_\alpha) \cdot u_\alpha^{-1} u \cdot m \in K \cdot V_i+1(F) \cdot M(F) \).

If \( \alpha \) is a root of \( M \), then we can write \( g = k \cdot u_{\alpha}^{-1} u \alpha m \in K \cdot V_i+1(F) \cdot M(F) \).

\[ \square \]

2.5. Review of two relations between the Iwasawa and Cartan decompositions. We now recall two results of Bruhat-Tits. Fix \( B' \in \mathcal{B} \) and a \( B \)-dominant coweight \( \mu \in X_*(A) \). Suppose that \( g \in K \cdot \mu(\varpi) \cdot K \). Then (see [BT72 4.4.4]) for all \( B' \in \mathcal{B} \)

\[(2.4) \quad r_{B'}(g)^B \leq \mu\]

and if \( r_B(g) = \mu \), then \( g \in K \cdot A(F) \).

Note that (2.2) and (2.4) together yield

\[(2.5) \quad r_B(g)^B \leq r_{B'}(g)^B \leq \mu\]

for all \( B' \in \mathcal{B} \), so that the hypothesis \( r_B(g) = \mu \) implies that all the retractions of \( g \) are equal; therefore the second result of Bruhat-Tits follows from the first together with Lemma 21. This proof of their second result (different from the one given in [BT72]) has the advantage that it generalizes immediately to parabolic subgroups, as we now check.

2.6. Variant (for parabolic subgroups) of the two results of Bruhat-Tits. Fix \( B = AU \in \mathcal{B} \) as well as a parabolic subgroup \( P \) of \( G \) containing \( B \). Write \( P = MN \), where \( M \) is the unique Levi subgroup of \( P \) containing \( A \) and \( N \) is the unipotent radical of \( P \). As before we write \( X_M \) for the quotient of \( X_*(A) \) by the coroot lattice for \( M \). For \( \mu, \nu \in X_M \), we write \( \mu \leq \nu \) if \( \nu - \mu \) is a non-negative integral linear combination of (images in \( X_M \) of) coroots \( \alpha^\vee \), where \( \alpha \) ranges over the roots of \( A \) in \( N \) (or, equivalently, \( U \)).

As in the previous section, fix a \( B \)-dominant coweight \( \mu \in X_*(A) \) and an element \( g \in K \cdot \mu(\varpi) \cdot K \). It follows immediately from (2.4) that

\[(2.6) \quad r_{B'}(g)^P \leq \mu\]

for all \( B' \in \mathcal{B} \), where the two sides of this inequality are now viewed as elements in \( X_M \).

Lemma 2.2. With \( \mu, g \) as above assume further that \( r_B(g) \) is equal to \( \mu \) in \( X_M \). Then \( g \in K \cdot M(F) \). Moreover, writing \( g = km \) for \( k \in K \) and \( m \in M(F) \), then \( m \in K_M \cdot \mu(\varpi) \cdot K_M \), where we have written \( K_M \) for \( M(\mathfrak{o}) \).

Proof. Let \( B' \in \mathcal{B} \). Since (by (2.5))

\[(2.7) \quad r_B(g)^P \leq r_{B'}(g)^P \leq \mu,\]

our hypothesis that \( r_B(g) = \mu \) in \( X_M \) implies that \( r_B(g) = r_{B'}(g) \) in \( X_M \) for all \( B' \in \mathcal{B} \). Therefore Lemma 21 implies that \( g \in K \cdot M(F) \).

Now we write \( g \) as \( km \) and verify the second statement of the lemma. By the Cartan decomposition for \( M \) there exists a unique coweight \( \nu \in X_*(A) \) that is
dominant with respect to the Borel subgroup $B \cap M$ of $M$ and is such that $m$ lies in $K_M \cdot \nu(\pi) \cdot K_M$. By the Cartan decomposition for $G$ the coweights $\mu$ and $\nu$ lie in the same orbit of the Weyl group of $A$ in $G$. Since both $\mu$ and $\nu$ are dominant for $M$, no root hyperplane for $M$ separates $\mu$ from $\nu$. Therefore $\mu - \nu$ is a non-negative integral linear combination of coroots $\alpha^\vee$, for $\alpha$ ranging through the roots of $A$ in $N$; on the other hand it is clear that $\mu$ and $\nu$ are equal in $X_M$ (since $\mu = r_B(g) = r_B \cdot m$ and $m \in K_M \nu(\pi)K_M$); therefore $\mu = \nu$, showing that $m \in K_M \cdot \mu(\pi) \cdot K_M$, as desired. □

3. Generalizations of Mazur’s inequality and Katz’s theorem

3.1. Notation. In the rest of the paper $F$ denotes a finite extension of $\mathbb{Q}_p$ and $O$ denotes the valuation ring of $F$. We write $L$ for the completion of the maximal unramified extension of $F$ in some algebraic closure $\bar{F}$ of $F$. We write $\sigma$ for the Frobenius automorphism of $L$ over $F$, and we write $O_L$ for the valuation ring of $L$.

3.2. A lemma about $\sigma$-L-spaces. Recall that a $\sigma$-L-space is a pair $(V, \Phi)$ consisting of a finite dimensional vector space $V$ over $L$ and a $\sigma$-semilinear bijection $\Phi : V \to V$. In case $F = \mathbb{Q}_p$, a $\sigma$-L-space is an isocrystal, and the theory of $\sigma$-spaces is completely parallel to that of isocrystals. In particular there are finitely many rational numbers, called slopes, attached to $(V, \Phi)$ (see §3 in [Kot85]).

Lemma 3.1. Let $(V, \Phi)$ be a $\sigma$-L-space and assume that all its slopes are strictly positive.

1. For any $v \in V$ the sequence $\Phi^n v$ approaches 0 as $n \to +\infty$.
2. Suppose that $\Lambda$ is an $O_L$-lattice in $V$ such that $\Phi \Lambda \subset \Lambda$, and suppose that $v$ is an element of $V$ such that $v - \Phi v \in \Lambda$. Then $v \in \Lambda$.

Proof. We begin by proving the first part of the lemma. Choose a positive integer $j$ such that $jr \in \mathbb{Z}$ for every slope $r$ of $(V, \Phi)$. Then, in a suitable basis for $V$ the map $\Phi^j$ can be represented by a diagonal matrix whose diagonal entries are strictly positive powers of the uniformizing parameter $\pi$ for $F$, and it is clear that $\Phi^j v^{\prime} \to 0$ as $m \to +\infty$ for every $v^{\prime} \in V$. Taking for $v^{\prime}$ the $j$ vectors $v, \Phi v, \ldots, \Phi^{j-1} v$, we see that $\Phi^n v \to 0$, as desired. Now we prove the second part of the lemma. It follows from the first part of the lemma that we may define an additive homomorphism $\Psi : V \to V$ by $\Psi = 1 + \Phi + \Phi^2 + \Phi^3 + \ldots$ and hence that the additive homomorphism $1 - \Phi$ is bijective with inverse $\Psi$. Also, it is clear from the definition of $\Psi$ that $\Psi \Lambda \subset \Lambda$.

We are given $v \in V$ such that $(1 - \Phi)v \in \Lambda$. Applying $\Psi$, we conclude that $v \in \Psi \Lambda \subset \Lambda$, as desired. □

3.3. Proof of Theorem [L.L] In the proof of Theorem [L.L] we will need the following non-abelian analog of Lemma 3.1.

Lemma 3.2. Let $\mu \in X_*(A)$ be a dominant coweight, and let $b$ be a basic element in $M(L)$ such that $w_M(b)$ lies in the subset $X_M^+$ of $X_M$. Assume further that $b \in M(O_L)\mu(\pi)M(O_L)$. Write $\Phi$ for the automorphism $n \mapsto b \sigma(n)b^{-1}$ of $N$ over $L$. Let $n \in N(L)$ and assume that $n^{-1}\Phi n \in N(O_L)$. Then $n \in N(O_L)$.

Proof. Conjugation by $M(O_L)$ preserves $N(O_L)$, and since $\mu$ is dominant, we have $\mu(\pi)N(O_L)\mu(\pi)^{-1} \subset N(O_L)$; it follows that $bN(O_L)b^{-1} \subset N(O_L)$ and hence that $\Phi N(O_L) \subset N(O_L)$. Our hypothesis that $w_M(b) \in X_M^+$ ensures that all the slopes of
Φ on Lie N(L) are strictly positive. Indeed, since b is basic in M(L), these slopes are given by \( (\alpha, w_M(b)) \), where \( \alpha \) ranges through the roots of \( A_P \) on \( \text{Lie}(N) \), and \( w_M(b) \) denotes the image of \( w_M(b) \) in \( a_P \). Thus, if \( N \) is abelian, we have only to appeal to Lemma 3.1.

In the general case we need to choose an \( M \)-stable filtration

\[ N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_r = \{1\} \]

by normal subgroups with \( N_i/N_{i+1} \) abelian for all \( i \). Each \( N_i \) is \( A \)-stable, hence is a product of root subgroups and is necessarily defined over \( \mathfrak{o} \). We will prove by induction on \( i \) (\( 0 \leq i \leq r \)) that \( n \in N_i(L) \cdot N(\mathfrak{o}_L) \). For \( i = 0 \) this statement is trivial, and for \( i = r \) it is the statement of the lemma. It remains to do the induction step. So suppose that for \( 0 \leq i < r \) we can write \( n = n_i n_{\sigma} \) for \( n_i \in N_i(L) \) and \( n_o \in N(\mathfrak{o}_L) \). Then \( n_i^{-1} \Phi(n_i) \in N_i(\mathfrak{o}_L) \). So (by Lemma 3.1) the image of \( n_i \) in \( (N_i/N_{i+1})(L) \) lies in \( (N_i/N_{i+1})(\mathfrak{o}_L) \). Since \( N_i(\mathfrak{o}_L) \) maps onto \( (N_i/N_{i+1})(\mathfrak{o}_L) \), we see that \( n_i \) can be written as \( n_{i+1} n'_{\sigma} \) with \( n_{i+1} \in N_{i+1}(L) \) and \( n'_{\sigma} \in N_i(\mathfrak{o}_L) \). Thus \( n = n_{i+1} \cdot (n'_{\sigma} n_{\sigma}) \in N_{i+1}(L) N(\mathfrak{o}_L) \), as desired. \( \Box \)

Now we are ready to prove the main theorem for split groups.

**Proof of Theorem 1.1.** Let \( g \in G(L) \) and suppose that

\[ g^{-1} b \sigma(g) \in K_{L,\mu}(\varpi) K_L, \]

where we have written \( K_L \) for \( G(\mathfrak{o}_L) \). Use the Iwasawa decomposition to write \( g \) as \( m n k \) for \( m \in M(L) \), \( n \in N(L) \) and \( k \in K_L \). It follows from (3.2) that

\[ n_1 m_1 \in K_{L,\mu}(\varpi) K_L, \]

where \( m_1 := m^{-1} b \sigma(m) \in M(L) \) and \( n_1 := n^{-1} m_1 \sigma(n) m_1^{-1} \in N(L) \). We claim that

\[ w_M(b) = r_B(n_1 m_1), \]

with the right side being regarded as an element of \( X_M \). Indeed,

\[ w_M(b) = w_M(m_1) = r_B \cdot r_B^P(m_1), \]

which in turn is equal to the image in \( X_M \) of \( r_B(n_1 m_1) \).

We conclude from (3.2), (3.3), (3.4) that \( w_M(b) \leq \mu \), which proves the first part of the theorem.

Now we prove the second part of the theorem. Under the hypothesis that \( w_M(b) = \mu \) (and with \( g, m, n, m_1, n_1 \) as above), we must prove that \( g \in M(L) \cdot K_L \). It follows from (3.2), (3.3), (3.4) and Lemma 2.2 that \( n_1 m_1 \in K_L \cdot M(L) \). Therefore \( n_1 \in K_L \cdot M(L) \), say \( n_1 = k_2 m_2 \) with \( k_2 \in K_L \) and \( m_2 \in M(L) \). Then \( n_1 m_2^{-1} \in P(\mathfrak{o}_L) \), and therefore \( n_1 \in N(\mathfrak{o}_L) \) and \( m_2 \in M(\mathfrak{o}_L) \). Since \( n_1 \in N(\mathfrak{o}_L) \), the second statement of Lemma 2.2 applies to \( n_1 m_1 \), and hence \( m_1 \in M(\mathfrak{o}_L) \mu(\varpi) M(\mathfrak{o}_L) \).

Now applying Lemma 3.2 (not to the element \( b \), but to its \( \sigma \)-conjugate \( m_1 \), which satisfies the same hypotheses as \( b \)), we see that \( n \in N(\mathfrak{o}_L) \). Therefore \( g = m \cdot n k \in M(L) K_L \), and we are done, since we have already seen that

\[ m^{-1} b \sigma(m) = m_1 \in M(\mathfrak{o}_L) \mu(\varpi) M(\mathfrak{o}_L). \]

\( \Box \)
4. Unramified groups

It is easy to generalize Theorem 4.1 from the case of split groups to that of unramified groups, in other words, quasi-split groups $G$ over $F$ that split over $L$. There is no need to generalize the results in [2] since we will apply them to the group $G$ over $L$, where it becomes split.

Continuing with unramified $G$, we will then determine precisely which affine Deligne-Lusztig sets $\mathcal{L}(4.1)$ are non-empty, generalizing Proposition 4.6 of [KR].

4.1. Notation. We will now change notation slightly, to emphasize that our maximal torus is no longer assumed to be split. We consider a Borel subgroup $B = TU$ of $G$, where $T$ is a maximal torus in $B$ and $U$ is the unipotent radical of $B$: all these subgroups are assumed to be defined over $\mathfrak{o}$. In addition we fix a parabolic subgroup $P = MN$ containing $B$, with $M$ containing $T$; again all these subgroups are assumed to be defined over $\mathfrak{o}$.

We denote by $A_P$ the maximal split torus in the center of $M$, and we write $\mathfrak{a}_P$ for $X_*(A_P) \otimes \mathbb{Z} \mathbb{R}$. In the special case $P = B$, we often write $A$ and $\mathfrak{a}$ rather than $A_B$ and $\mathfrak{a}_B$; of course $A$ is simply the maximal split torus in $T$. As usual we identify $\mathfrak{a}_P$ with a subspace of $\mathfrak{a}$.

As before we denote by $X_M$ the quotient of $X_*(T)$ by the coroot lattice for $M$. The Frobenius automorphism $\sigma$ acts on $X_M$, and we denote by $Y_M$ the coinvariants of $\sigma$ on $X_M$. Thus $Y_M$ is the quotient of $X_M$ by the image of the homomorphism $1 - \sigma : X_M \rightarrow X_M$. We introduce a partial order on $Y_M$ as follows: for $y_1, y_2 \in Y_M$ we say that $y_2 \preceq y_1$ if $y_1 - y_2$ is a non-negative integral linear combination of images in $Y_M$ of coroots $\alpha^\vee$ corresponding to simple roots $\alpha$ of $T$ in $N$.

We identify $Y_M \otimes \mathbb{Z} \mathbb{R}$ with $\mathfrak{a}_P$, and we write $Y_M^+$ for the subset of $Y_M$ consisting of those elements whose image in $\mathfrak{a}_P$ lies in the cone

\begin{equation}
\mathfrak{a}_P^+ := \{ x \in \mathfrak{a}_P : \langle \alpha, x \rangle > 0 \ \forall \ \text{root } \alpha \text{ of } A_P \text{ in } N \}.
\end{equation}

As in [Kot85] the homomorphism $\psi_M : M(L) \rightarrow X_M$ induces a map

\begin{equation}
\kappa_M : B(M) \rightarrow Y_M.
\end{equation}

Now we can generalize Theorem 4.1 to this more general context, with the affine Deligne-Lusztig sets $X_M^\mu(b)$ still defined by (4.1).

**Theorem 4.1.** Let $\mu \in X_*(T)$ be a dominant coweight, and let $b$ be a basic element in $M(L)$ such that $\kappa_M(b) \in Y_M^+$. Then

1. **[RR92]** If $X^\mu_M(b)$ is non-empty, then $\kappa_M(b) \preceq \mu$. Here we are regarding $\mu$ as an element of $Y_M$.

2. Suppose that $\kappa_M(b) = \mu$ (with $\mu$ again regarded as an element of $Y_M$). Then the natural injection $X^\mu_M(b) \hookrightarrow X^\mu_M(b)$ is a bijection.

**Proof.** The proof of part 1 is the same as the proof in the split case. The proof of part 2 is the same as in the split case, except for one new complication, which we will now explain. Suppose that $g \in G(L)$ represents an element of $X^\mu_M(b)$, so that

\begin{equation}
g^{-1}b\sigma(g) \in K_L, \mu(\varpi)K_L,
\end{equation}

where we have written $K_L$ for $G(\mathfrak{a}_L)$. As before we use the Iwasawa decomposition to write $g$ as $mnk$ for $m \in M(L), n \in N(L)$ and $k \in K_L$. It follows from (3.3) that
\[ n_1 m_1 \in K_L \mu(\varpi) K_L, \]

where \( m_1 := m^{-1} b \sigma(m) \in M(L) \) and \( n_1 := n^{-1} m_1 \sigma(n) m_1^{-1} \in N(L) \). We need to prove that \( g \in M(L) \cdot K_L \) (under the hypothesis that \( \kappa_M(b) = \mu \)). Denote by \( \nu \) the image of \( r_B(n_1 m_1) \) in \( X_M \). As in the proof of Theorem 4.1, the elements \( \nu, w_M(b) \) of \( X_M \) have the same image in \( Y_M \), and by hypothesis the image of \( w_M(b) \) in \( Y_M \) is \( \mu \). We would like to apply Lemma 2.2 to the element \( n_1 m_1 \), but for this we would need to know that \( \nu \) and \( \mu \) are equal in \( X_M \), while all we know at the moment is that they are equal in the quotient \( Y_M \) of \( X_M \). However, by (4.4) and (4.5) we also know that \( \nu \leq \mu \). Therefore Lemma 1.1 shows that \( \nu = \mu \) in \( X_M \), as desired. Thus we may apply Lemma 2.2 to the element \( n_1 m_1 \) in order to see that \( n_1 m_1 \in K_L \cdot M(L) \). The rest of the proof is exactly the same as in the split case. Of course we need to appeal to Lemma 3.2, but its statement generalizes without change to the general unramified case, and its proof stays the same too, though one should note that the subgroups \( N_i \) used in the proof need to be chosen so as to be defined over \( \sigma \).

Next we prove the lemma we just used.

**Lemma 4.2.** Let \( x \in X_M \). Suppose that \( x \geq 0 \) and suppose further that the image of \( x \) in \( Y_M \) is \( 0 \). Then \( x = 0 \).

**Proof.** Let \( X_M^G \) denote the kernel of the canonical surjection \( X_M \to X_G \), so that we get a short exact sequence
\[ 0 \to X_M^G \to X_M \to X_G \to 0. \]

Taking coinvariants for \( \sigma \), we get an exact sequence
\[ Y_M^G \to Y_M \to Y_G \to 0, \]

where \( Y_M^G \) denotes the coinvariants of \( \sigma \) on \( X_M^G \).

Clearly \( X_M^G \) is a free abelian group on the set \( S \) of coroots \( \alpha^\vee \) for simple roots \( \alpha \) of \( T \) that occur in \( N \), and \( \sigma \) permutes these basis elements. Therefore \( Y_M^G \) is a free abelian group on the set \( \hat{S} \) of orbits of \( \sigma \) on \( S \). In particular \( Y_M^G \) is torsion-free, which implies that the map \( Y_M^G \to Y_M \) is injective.

Now consider \( x \in X_M \) such that \( x \geq 0 \). In particular \( x \) lies in the subgroup \( X_M^G \). Consider the image \( y \) of \( x \) in \( Y_M \). It is clear that \( y \) lies in the subgroup \( Y_M^G \), and that its coefficients in the basis \( \hat{S} \) are given by summing over the orbits of \( \sigma \) on \( S \) the coefficients of \( x \) in the basis \( S \); since these latter coefficients are non-negative by our hypothesis that \( x \geq 0 \), we see that if \( y = 0 \), then \( x = 0 \) as well. \( \square \)

### 4.2. Non-emptiness of affine Deligne-Lusztig subsets of \( G(L)/G(\sigma_L) \)

We will now determine when the affine Deligne-Lusztig set \( X_M^G(b) \) (see (4.4)) is non-empty. The split case is treated in Proposition 4.6 of [KR], and we follow closely the method used there.

As in [KR], for \( \mu \in X_*(T) \) we put
\[ P_{\mu} := \{ \nu \in X_*(T) : \nu = \mu \text{ in } X_G, \text{ and } \nu \in \text{Conv}(W\mu) \} \]

where \( \text{Conv}(W\mu) \) denotes the convex hull of the \( W \)-orbit \( W\mu \) of \( \mu \) in \( X_*(T) \otimes \mathbb{R} \), \( W \) being the absolute Weyl group of \( T \) in \( G \). With \( P = MN \), \( Y_M \) as in (4.1) we
let $\mathcal{P}_{\mu,M}$ denote the subset of $Y_M$ obtained as the image of $\mathcal{P}_\mu$ under the canonical surjection $X_*(T) \to Y_M$.

**Theorem 4.3.** Let $b \in M(L)$ be basic, and let $\mu \in X_*(T)$. Then $X_\mu^G(b)$ is non-empty if and only if $\kappa_M(b)$ lies in $\mathcal{P}_{\mu,M}$.

Since every $\sigma$-conjugacy class in $G(L)$ contains an element that is basic in some standard Levi subgroup $M$ (see Kot85), this theorem gives a complete answer to the question of when $X_\mu^G(b)$ is non-empty. The theorem follows immediately from the following three lemmas.

**Lemma 4.4.** Let $b \in G(L)$ be basic. Then the $\sigma$-conjugacy class of $b$ meets $K_{L,\mu}(\varpi)K_L$ if and only if $\kappa_G(b)$ is equal to the image of $\mu$ in $Y_M$.

**Proof.** ($\implies$) Obvious, since the homomorphism $w_G$ is trivial on $K_L$.

($\impliedby$) We write $N(T)$ for the normalizer of $T$ in $G$. Then we have an exact sequence

$$1 \to T(\sigma_L) \to N(T)(L) \to \tilde{W} \to 1,$$

where $\tilde{W} := W \ltimes X_*(T)$ is the extended affine Weyl group for $G$ over $L$. Using this exact sequence, one sees without difficulty that the map $N(T)(L) \to \tilde{W}$ induces a bijection from $B(N(T))$ to the set of $\sigma$-conjugacy classes in the affine Weyl group $\tilde{W}$.

For $w \in W$ choose $\tilde{w} \in N(T)(\sigma_L)$ such that $\tilde{w} \mapsto w$. Associated (see Kot85) to the element $b' := \mu(\varpi)\tilde{w}$ of $N(T)(L)$ is a homomorphism $\nu : D \to N(T)$, where $D$ is the diagonalizable group with character group $Q$. It is easy to calculate $\nu$.

Indeed, choose a positive integer $r$ such that $(w\sigma)^r = \sigma^r$. We use $w$ to twist the action of $\sigma$ on $T$, obtaining a new unramified torus $T_w$ which becomes equal to $T$ over the fixed field $F_r$ of $\sigma^r$ in $L$, but for which the action of $\sigma$ is now given by $\sigma\sigma$ rather than $\sigma$. Thus $X_*(T_w)$ coincides with $X_*(T)$ as an abelian group, but $\sigma$ acts by $w\sigma$ rather than $\sigma$. From this point of view, the homomorphism $\nu : D \to N(T)$ may be regarded as an element of $\text{Hom}(D,T_w)$, or in other words as an element in $X_*(T) \otimes Q$ fixed by $w\sigma$, the explicit formula for $\nu$ being given by

$$r^{-1} \sum_{i=1}^{r} (w\sigma)^i(\mu) \in X_*(T) \otimes Q.$$  

The remarks above are valid for any $w \in W$. The well-known fact that semisimple groups over finite fields have anisotropic maximal tori translates into the fact that there exists $w \in W$ for which the torus $T_w$ is anisotropic modulo the center of $G$; for the rest of this proof we work with such an element $w$. In this case $\nu$ is forced to be central in $G$, and therefore the element $b' = \mu(\varpi)\tilde{w}$ is basic in $G(L)$.

It is obvious from the form of $b'$ that $\kappa_G(b') = \mu$. By hypothesis $b$ is basic and $\kappa_G(b) = \mu$. Therefore Kot85 $b$ is $\sigma$-conjugate in $G(L)$ to $b'$. From the form of $b'$ it is obvious that $b'$ lies in the $K_L$-double coset of $\mu(\varpi)$. This concludes the proof. □

As in KR for $\mu \in X_*(T)$ we denote by $M(\mu)$ the image of $K_{L,\mu}(\varpi)K_L \cap P(L)$ under the canonical surjection $P(L) \to M(L)$; obviously $M(\mu)$ is a union of $M(\sigma_L)$-double cosets.

**Lemma 4.5.** Let $b \in M(L)$ and let $\mu \in X_*(T)$. Then the following three conditions are equivalent:

1. $\kappa_M(b)$ lies in $\mathcal{P}_{\mu,M}$.
2. $b$ is $\sigma$-conjugate in $G(L)$ to $\kappa_G(b)$. 
3. $b$ is basic in $M(\mu)$. 

Proof. (1 $\implies$ 2) Since $b$ is basic in $M(\mu)$, it is $\sigma$-conjugate to $\mu$ by hypothesis; let $\sigma$ be such that $\sigma(\mu) = \kappa_G(b)$. Then $b = \kappa_M(b) = \kappa_M(b')$ where $b'$ is $\sigma$-conjugate to $\mu$. Therefore $b$ is $\sigma$-conjugate to $\kappa_G(b)$. 

(2 $\implies$ 3) Since $b$ is $\sigma$-conjugate to $\kappa_G(b)$, it is basic in $M(\mu)$. 

(3 $\implies$ 1) Since $b$ is basic in $M(\mu)$, it is $\sigma$-conjugate to $\mu$ by hypothesis; let $\sigma$ be such that $\sigma(\mu) = \kappa_G(b)$. Then $b = \kappa_M(b) = \kappa_M(b')$ where $b'$ is $\sigma$-conjugate to $\mu$. Therefore $b$ is $\sigma$-conjugate to $\kappa_G(b)$. This concludes the proof. □
(1) The $\sigma$-conjugacy class of $b$ in $G(L)$ meets $K_L \mu(\varpi)K_L$.
(2) The $\sigma$-conjugacy class of $b$ in $P(L)$ meets $K_L \mu(\varpi)K_L \cap P(L)$.
(3) The $\sigma$-conjugacy class of $b$ in $M(L)$ meets $M(\mu)$.

Proof. The equivalence of the first two conditions is clear from the Iwasawa decomposition $G(L) = K_L P(L)$. The equivalence of the second two conditions follows from the fact [Kot97, 3.6] that $B(P) \to B(M)$ is a bijection.

Lemma 4.6. The set $w_M(M(\mu))$ is equal to the image of $P_\mu$ under the canonical surjection $X_*(T) \to X_M$.

Proof. This is Lemma 4.5 of [KR], which can be applied since $G$ splits over $L$. Note that the assumption, made at the beginning of §4 of [KR], that the derived group of $G$ be simply connected was made merely for convenience; in particular it was not used in the proof of Lemma 4.5.

4.3. Remarks concerning the converse to Mazur’s inequality. Let $b, \mu$ be as in Theorem 4.1. Mazur’s inequality (in other words the first part of that theorem) says that if $X^G_\mu(b)$ is non-empty, then $\kappa_M(b)^P \equiv \mu$, where $\mu$ is being regarded an element of $Y_M$. Thus the converse to Mazur’s inequality is the statement, only known to be true in certain cases, that if $\kappa_M(b)^P \equiv \mu$, then $X^G_\mu(b)$ is non-empty. Since Theorem 4.3 does tell us exactly when $X^G_\mu(b)$ is non-empty, proving the converse to Mazur’s inequality is the same as proving that $\kappa_M(b)^P \equiv \mu$ is equivalent to $\kappa_M(b) \in P_{\mu,M}$. Thus, in order to prove the converse to Mazur’s inequality in general, it would be enough to answer the following question about root systems affirmatively.

Question 4.7. Let $\mu \in X_*(T)$ be a dominant coweight and let $\nu \in Y^+_M$. Are the following two conditions equivalent?

1. $\nu \equiv^P \nu \equiv \mu$
2. $\nu \in P_{\mu,M}$

It is immediate that the second condition implies the first. The challenge is to prove that the first condition implies the second; this was done for $GL_n$ and $GSp_{2n}$ in [KR] and for all split classical groups in [Le02]; non-split groups have not been examined yet.

4.4. Comparison with [RR96, 4.2]. Condition 1 in Question 4.7 looks superficially different from the one used in [RR96] (and also used to define the set $B(G, \mu)$ in [Kot97]), but in fact it is equivalent, as we now check (see Proposition 4.10 below).

This equivalent condition involves some additional notation. We write $X_R$ for the real vector space $X_*(T) \otimes \mathbb{R}$. We identify $\mathfrak{a}$ with the subspace of $\sigma$-fixed vectors in $X_R$, and we view $\mathfrak{a}$ as a direct summand of $X_R$, the projection map $X_R \to \mathfrak{a}$, denoted $x \mapsto x^\sigma$, being given by averaging over orbits of $\sigma$. We have already identified $\mathfrak{a}_P$ with a subspace of $\mathfrak{a}$. In fact we view $\mathfrak{a}_P$ as a direct summand of $\mathfrak{a}$, the projection map $pr_M : \mathfrak{a} \to \mathfrak{a}_P$ being given by averaging over the relative Weyl group $W_{M(F)}$ of $T$ in $M$. As usual we identify $W_{M(F)}$ with the fixed points of $\sigma$ in the absolute Weyl group $W_M$ of $T$ in $M$. 


The partial order \( B \leq \) on \( X_*(T) \) extends as usual to a partial order on \( X_\mathbb{R} \), which we will denote simply by \( \leq \); thus for \( x, y \in X_\mathbb{R} \) the inequality \( x \leq y \) means that 
\( y - x \) is a non-negative real linear combination of simple coroots.

Recall that we have already identified \( Y_M \otimes \mathbb{R} \) with \( \mathfrak{a}_P \). For \( \nu \in Y_M \) we denote \( \bar{\nu} \) the image of \( \nu \) in \( \mathfrak{a}_P \) (which lets us view \( \bar{\nu} \) as an element in \( X_\mathbb{R} \), as we will do in the next lemma).

Lemma 4.8. Let \( \nu \in Y_M \). Then \( \nu \overset{P}{\geq} 0 \) if and only if \( \bar{\nu} \geq 0 \) and the image of \( \nu \) in \( Y_G \) is 0.

Proof. Exercise.

Lemma 4.9. Let \( x \) be a dominant element of \( X_\mathbb{R} \), and let \( y \in \mathfrak{a}_P \). Then \( y \leq x^\flat \) if and only if \( y \leq pr_M(x^\flat) \).

Proof. \( (\Rightarrow) \) Apply \( pr_M \) to the inequality \( y \leq x^\flat \), using that \( pr_M \) preserves \( \leq \).
\( (\Leftarrow) \) This follows from the fact that \( pr_M(x^\flat) \leq x^\flat \), a consequence of the inequalities \( wx^\flat \leq x^\flat \) for \( w \in W_M(F) \), which hold since \( x^\flat \) is dominant.

Proposition 4.10. Let \( \mu \in X_*(T) \) be a dominant coweight and let \( \nu \in Y_M \). Then the following two conditions are equivalent:

1. \( \nu \overset{P}{\leq} \mu \).
2. \( \mu, \nu \) have the same image in \( Y_G \), and \( \bar{\nu} \leq \mu^\flat \).

Proof. This follows from Lemmas 4.8 and 159.

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