EMBEDDINGS OF 3–MANIFOLDS VIA OPEN BOOKS

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Abstract. In this note, we discuss embeddings of 3–manifolds via open books. First, we show that every 3-dimensional orientable closed manifold open book embeds in the standard open book of $S^2 \times S^3$. We then use open book embeddings to reprove that every closed orientable 3–manifold embeds in $S^5$.

1. Introduction

Let $M$ be a manifold and $B$ a co-dimension 2–submanifold of $M$. We say that $M$ has an open book decomposition with binding $B$ provided $M \setminus B$ fibers over $S^1$ such that the fibration in a neighborhood of $B$ looks like the trivial fibration of $B \times D^2 \setminus B \to S^1$ sending $(x, r, \theta)$ to $\theta$. Open book decompositions of simply connected manifolds were studied by Winkelnkemper in [Wi], where he proved the existence of such decompositions on simply connected manifolds of dimension bigger than 6 provided the dimension of the manifold is not divisible by 4. He also established that if the dimension of the manifold is divisible by 4 then it admits an open book decomposition if and only if its index is zero. Winkelnkemper’s results were then extended by J. Lawson [La], F. Quinn [Qu] and I. Tamura [Ta]. Due to their works, the conditions under which a manifold admits an open book decomposition is now well known. These conditions are generally very mild and hence a very large class of manifolds satisfy them. A particular class of manifold that will be of interest to us consists of odd dimensional orientable manifolds. It can be easily deduced from [Wi] that every orientable odd dimensional manifold admits an open book decomposition. See also [Qu] for more details.

In recent times the study of open book decomposition of manifolds has become very prominent due to the connection – discovered by Giroux [Gi] – between open book decompositions and contact structures. Let $M$ be an odd dimensional manifold. Recall that a contact structure $\xi$ on $M$ is a nowhere integrable codimension one distribution. Giroux in his seminal work [Gi] showed that there is a one to one correspondence between certain kinds of open book decompositions up to positive stabilizations and contact structures on manifolds up to isotopy. We refer to [Gi] and [Ko] for more details. This correspondence has been extremely useful in understanding contact structures on manifolds.

In this article, we first study open book embeddings of 3–manifolds. Roughly speaking, we say that $M$ admits an open book embedding in $N$ provided there is an embedding of $M$ in $N$ such that an open book decomposition of $M$ restricts to an open book decomposition of $N$. A more precise definition of open book embedding is given in section 3.2. Open book embeddings were used by A.Mori [Mr] to prove that every contact 3-manifold open book embeds in $S^7$. His result was generalized by David Martienz-Torres in [Mr] to show that there is an open book embedding of any contact manifold of dimension $2n + 1$ in $S^{4n+3}$. In addition, the article by Etnyre and Lekili [EL] uses open book embeddings to produce contact embeddings. Apart from these there are no known results dealing with open book embeddings. In particular, it is not known, if every 3–manifold admits an open book embedding in $S^5$. Our first theorem is regarding open book embeddings of orientable 3–manifolds in $S^3 \times S^2$. We now state this:

Theorem 1. Let $M$ be an orientable 3–dimensional manifold together with an open book decomposition $(B, \pi)$. The open book $(B, \pi)$ admits a smooth open book embedding in the standard open book associated to $S^3 \times S^2$. 

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For a precise definition of standard open book of $S^3 \times S^2$, see the Remark[2] given in the Section 2.

Using methods used in establishing the Theorem[1] we establish the following:

**Theorem 2.** Every closed orientable 3–manifold admits a smooth embedding in $S^5$.

This theorem was first discovered by M. Hirsch in [Hi]. Embeddings of manifolds in Euclidean spaces have a long history starting from the seminal work of H. Whitney establishing that every closed $n$–manifold admits an embedding in $\mathbb{R}^{2n}$. In fact, a general result of A. Haefliger and M. Hirsch [HH] implies that every odd dimensional closed orientable manifold embeds in $\mathbb{R}^{2n-1}$. There are other proofs of the Theorem[2] See, for example, the article [HLM] for a proof using what is now known as braided embeddings. We refer to [Wa] and [Ro] for embeddings of non-orientable 3–manifold in $\mathbb{R}^5$.

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## 2. Preliminaries

In this section, we quickly review notions necessary for this article pertaining mapping class groups, open books, contact structures and relationships among them.

### 2.1. Mapping class group.

Let us begin by recalling the definition of a mapping class groups as in [BM].

**Definition 3** (Mapping class group). Let $\Sigma$ be an orientable manifold. By the mapping class group of $\Sigma$, we mean the group of orientation preserving self diffeomorphisms of $\Sigma$ up to isotopy. In case, $\Sigma$ has a non-empty boundary $\partial \Sigma$, then we always assume that diffeomorphisms are the identity in a collar neighborhood of the boundary.

We denote the mapping class group of a surface $\Sigma$ by $\text{MCG}(\Sigma)$. In case, $\Sigma$ has a non-empty boundary and we want to emphasis this fact, we will denote the mapping class group by $\text{MCG}(\Sigma, \partial \Sigma)$. In this article, unless specified otherwise, we say that two diffeomorphisms $f$ and $g$ of a manifold $(M, \partial M)$ are equal provided they represent the same element in $\text{MCG}(M, \partial M)$.

Lickorish in [Li] showed that every element of the mapping class group of an orientable surface is a product of Dehn twists. Recall that, by definition, a Dehn twist is a self-diffeomorphism of $S^1 \times I$ given by $(\theta, t) \mapsto (\theta + 2\pi t, t)$. Note that the Dehn twist fixes both the boundary components of $S^1 \times I$. Hence, given an embedded $S^1$ in an orientable surface $\Sigma$, we can define the Dehn twist of a tubular neighborhood of this $S^1$ which is the identity when restricted to the boundary of this tubular neighborhood. Clearly, we can extend this diffeomorphism by the identity in the complement of the tubular neighborhood to produce a self-diffeomorphism of $\Sigma$. This diffeomorphism is called a Dehn twist along the embedded $S^1$ in the surface $\Sigma$. For more details, refer [BM].

In fact, it was later established by Lickorish that every element of the mapping class group of a closed orientable surface of genus $g$ is a product of Dehn twists along curves depicted in the Figure[1]. For an orientable surface with connected boundary component, it was established by D.L. Johnson [Jo] that the same generators that generate the mapping class group of the closed surface obtained by attaching a disk along the boundary of the surface are sufficient to generate the mapping class group. In case, the boundary of the surface has more than one connected components then we need – additionally – Dehn twists along certain simple closed curves which separate boundary components. See [BM] p. 114 for a picture depicting generators in this case. We will always call curves necessary to generate the mapping class group of a surface as *Lickorish curves* and Dehn twists along these Lickorish curves as *Lickorish generators*. 
2.2. Open books.

Let us review few results related to open book decompositions of manifolds. We first recall the following:

**Definition 4** (Open book decomposition). An open book decomposition of a manifold $M$ consists of a co-dimension 2 submanifold $B$ with trivial normal bundle and a fibration $\pi : M \setminus B \to S^1$ such that $\pi^{-1}(\theta)$ is an interior of co-dimension 1 submanifold $N_\theta$ such that $\partial N_\theta = B$ for all $\theta$. The submanifold $B$ is called the binding and $N_\theta$ is called a page of the open book. We denote the open book decomposition of $M$ by $(M, \text{Ob}(B, \pi))$ or sometimes simply by $\text{Ob}(B, \pi)$.

Next, we discuss the notion of an abstract open book decomposition. To begin with, let us recall the following:

**Definition 5** (Mapping torus). Let $\Sigma$ be a manifold with non-empty boundary $\partial \Sigma$. Let $\phi$ be an element of the mapping class group of $\Sigma$. By the mapping torus $\mathcal{M}T(\Sigma, \partial \Sigma, \phi)$, we mean $\Sigma \times [0,1] / \sim$ where $\sim$ is the equivalence relation identifying $(x,0)$ with $(\phi(x),1)$.

We are now in a position to define an abstract open book decomposition.

**Definition 6.** Let $\Sigma$ and $\phi$ as in the previous definition. An abstract open book decomposition of $M$ is pair $(\Sigma, \phi)$ such that $M$ is diffeomorphic to $\mathcal{M}T(\Sigma, \partial \Sigma, \phi) \cup_{\text{id}} \partial \Sigma \times D^2$

where $\text{id}$ denotes the identity mapping of $\partial \Sigma \times S^1$

The map $\phi$ is called the monodromy of the open book. The manifold obtained by identifying the boundary of $\mathcal{M}T(\Sigma, \phi)$ with the boundary of $\partial \Sigma \times D^2$ as described in the Definition 6 will be denoted by $\text{Aob}(\Sigma, \phi)$. A manifold $M$ together with a given abstract open book decomposition will be denoted by $(M, \text{Aob}(\Sigma, \phi))$.

One can easily see that an abstract open book decomposition of $M$ gives an open book decomposition of $M$ up to diffeomorphism and vice versa. Hence, we will not generally distinguish between open books and abstract open books.

**Remark 7.** (1) Notice that $S^n$ admits an open book decomposition with pages $D^{n-1}$ and the monodromy the identity map of $D^{n-1}$. We call this open book the trivial open book. For more details regarding open books, refer the lecture notes [Et] and [Gi] [chpt-4.4.2].

(2) $S^1 \times S^2$ admits an open book decomposition with pages $T^1 S^2$ and the monodromy identity. We call this open book decomposition of $S^1 \times S^2$ as the standard open book decomposition of $S^3 \times S^2$.

**Definition 8.** Given two manifolds abstract open books $\text{Aob}(\Sigma_1, \phi_1)$ and $\text{Aob}(\Sigma_2, \phi_2)$ we can make the band connected sum of pages of $\text{Aob}(\Sigma_1, \phi_1)$ and $\text{Aob}(\Sigma_1, \phi_2)$. There clearly exists a diffeomorphism — denoted
by \( \phi_1 \# \phi_2 \) – which acts on \( \Sigma_1 \subset \Sigma_1 \# \Sigma_2 \) by \( \phi_1 \) and on \( \Sigma_2 \subset \Sigma_1 \# \Sigma_2 \) by \( \phi_2 \). The abstract open book \( \text{Aob}(\Sigma_1 \# \Sigma_2, \phi_1 \# \phi_2) \) is then a connected sum of \( \text{Aob}(\Sigma_1, \phi_1) \) and \( \text{Aob}(\Sigma_2, \phi_2) \). This connected sum will be called the connected sum of abstract open books.

For more details about this construction, refer [Ko].

3. Open book embeddings of 3-manifolds in \( S^3 \times S^2 \) and

In this section, we produce open book embeddings of orientable 3-manifolds in the standard open book of \( S^3 \times S^2 \). We begin by reviewing quickly some well known results about embedded Hopf band in \( S^3 \). See, for example, [Et] for more details.

3.1. Hopf band in \( S^3 \) and the mapping class group of an annulus.

To begin with, we go through a proof of the following well known result.

**Lemma 9.** Let \( A \) be an annulus and let \( \phi \) be an element of the mapping class group \( \text{MCG}(A) \) of \( A \). Then, there exists an embedding \( f \) of \( A \) in \( S^3 \) that satisfies the following:

1. \( f(A) \) is a Hopf band in \( S^3 \).
2. There exists a diffeomorphism of \( \Psi_1 \) of \( S^3 \), isotopic to the identity via an isotopy \( \Psi_1 \) such that \( f^{-1} \circ \Psi_1 \circ f = \phi \).
3. The isotopy \( \Psi_1 \) fixes the boundary of \( A \) point wise for all \( t \).

**Proof.** We know that \( S^3 \) admits an open book decomposition with pages a Hopf annulus and the monodromy the Dehn twist around its center circle. This, in particular, implies that there exists a flow \( \Phi_1 \) on \( S^3 \) whose time 1 map \( \Phi_1 \) maps Hopf annulus – say \( A \) – to itself and \( \Phi_1 \) restricted to \( A \) is a Dehn twist along the center circle on \( A \). The lemma is now a straight forward consequence of the fact that every element of the mapping class group of an annulus is just a power of the Dehn twist along its center circle. \( \square \)

Since \( S^3 \times [0, 1] \) can be regarded as a collar of \( \partial D^4 \) in \( D^4 \) with \( \partial D^4 = S^3 \times 1 \). Since \( \Psi \) constructed in the Lemma is isotopic to the identity, we have the following:

**Corollary 10.** There exists a proper embedding \( f \) of \( A \) in \( (D^4, \partial D^4) \) which satisfies the property that for every element \( \phi \in \text{MCG} \), there exists a diffeomorphism \( \Psi_{\phi} \) of \( (D^4, \partial D^4) \) isotopic to the identity such that \( \phi = f^{-1} \circ \Psi_{\phi} \circ f \).

**Proof.** We begin by considering the embedding of \( A \) in \( S^3 \times [0, 1] \) as the union of a Hopf band in \( S^3 \times \{0\} \) and boundary Hopf link times \([0, 1] \). We perturb this embedding – if necessary – to make it into a smooth embedding.

Now let \( \Psi_t \) be the isotopy of \( S^3 \) such that \( \Psi_1 \) realizes the given element of \( \text{MCG}(B) \). Using the isotopy \( \Psi_t \), we construct a diffeomorphism \( \Gamma_1 \) of \( S^3 \times [-1, 1] \) that satisfy the following:

1. \( \Gamma_1 \) is isotopic to the identity via a family of diffeomorphisms \( \Gamma_t \).
2. \( \Gamma_1 \) restricted to \( S^3 \times \{0\} \) is \( \Psi_1 \).

This diffeomorphism is defined as follows:

\[
\Gamma_1(x, t) = \begin{cases} 
\Psi_{1-t}(x) & \text{if } t \geq 0 \\
\Psi_{t+1} & \text{if } t \leq 0
\end{cases}
\]

Since \( S^3 \times [-1, 1] \) can be regarded as a collar of \( \partial D^4 \) in \( (D^4, \partial D^4) \), we are through as \( \Gamma_1 \) clearly extends to a diffeomorphism of \( (D^4, \partial D^4) \) by the identity in the complement of the collar. \( \square \)

3.2. Open book embeddings.

In this section, we review the notion of open book embeddings. More concretely, we will make precise the notion of abstract open book embeddings. However, as earlier remarked, since open book decomposition and abstract open book decompositions are closely related, we will often not distinguish between abstract open book embeddings and open book embeddings.
Definition 11 (Open book embedding). Let $M^k$ be a manifold with open book decomposition $(B_1, \pi_1)$ and $N^l$ be another manifold with open book decomposition $(B_2, \pi_2)$. We say an embedding $f : M \to N$ is an open book embedding of $(M, \text{Ob}(B_1, \pi_1))$ in $(N, \text{Ob}(B_2, \pi_2))$ provided $f$ embeds $B_1$ in $B_2$ and the following diagram commute:

\[
\begin{array}{ccc}
M \setminus B_1 & \xrightarrow{f} & N \setminus B_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
S^1 & \xrightarrow{\text{Id}} & S^1
\end{array}
\]

Just as an abstract open book is defined, we can define an abstract open book embedding as follows:

Definition 12 (Abstract open book embeddings). Let $M = \text{Ob}(\Sigma_1, \phi_1)$ and $N = \text{Ob}(\Sigma_2, \phi_2)$ be two abstract open books. We say that there exists an abstract open book embedding of $M$ in $N$ provided there exists a proper embedding $f$ of $\Sigma_1$ in $\Sigma_2$ such that $\phi_2 = f^{-1} \circ \phi_1 \circ f$.

It is clear from the definition that an abstract open book embedding produces an embedding for the associated open book and vice versa.

There are some obvious examples of open book embeddings.

Examples 13. (1) Each sphere $S^n$ embeds in $S^{n+k}$ with $k > 0$ via obvious inclusion such that the trivial open book of $S^{n+k}$ restricts to the trivial open book of $S^n$.

(2) Notice that since we can embed $S^{n-1} \times I$ in $D^{n+1}$ properly, $S^1 \times S^n$ admits an open book embedding in $S^{n+2}$.

(3) We showed in previous section that $S^3$ with open book decomposition having pages an annulus and the monodromy a Dehn twist admits an open book embedding in $S^5$.

3.3. Open book embeddings of 3-manifolds in $S^3 \times S^2$.

In this subsection, we establish Theorem 1. Recall that we need to show that every 3-dimensional manifold with given open book decomposition open book embeds in $S^3 \times S^2$ with its standard open book having pages $T^*S^2$ and its monodromy the identity. In fact, we will establish a slightly general result in order to achieve this.

We begin by introducing few terminologies. We refer to [GS] for more details regarding these. We know that when we add a 2-handle to a 4-ball $B^4$ along an unknot on the boundary with framing $n, n \in \mathbb{Z}$ we produce a disk bundle with Euler characteristic $n$. Let us denote this disk bundle by $DE(n)$.

Next we establish a lemma. The techniques used in the proof of this lemma is adopted from techniques developed by Hirose and Yasuhara in [HY] to establish flexible embeddings of closed surfaces in certain manifolds. Hirose and Yasuhara called an embedding $f$ of a surface $\Sigma$ in a 4-manifold $M$ flexible provided for every element $\phi$ of the mapping class group of $\Sigma$ there exists a diffeomorphism $\Psi$ of $M$, isotopic to the identity, which maps $f(\Sigma)$ to itself and $f^{-1} \circ \Psi_1 \circ f = \phi$.

Lemma 14. Let $(\Sigma, \partial \Sigma)$ be a surface with non-empty boundary. There exists an embedding $f$ of $\Sigma$ in $DE(n)$ which satisfies the following:

(1) The embedding is proper.

(2) Given any diffeomorphism $\phi$ of $(\Sigma, \partial \Sigma)$, there exists a family $\Psi_t$ of diffeomorphisms of $DE(n)$ with $\Psi_0 = \text{Id}$ such that $\Psi_1$ maps $\Sigma$ to itself and satisfies the property that $f^{-1} \circ \Psi_1 \circ f$ is isotopic to the given diffeomorphism $\phi$ of $(\Sigma, \partial \Sigma)$.

Proof. We know that $DE(n)$ is obtained by attaching a 2-handle attached to $B^4$ along an unknot. This implies we can regard it as a union of $B^4$ with $D^2 \times D^2$. We first describe an embedding of $(\Sigma, \partial \Sigma)$ in $S^3 = \partial B^4$ that we will need in order to establish the Lemma. Let us assume that $\partial \Sigma$ has $n \in \mathbb{N}$ boundary components. Let us denote by $\Sigma$ the closed surface obtained from $(\Sigma, \partial \Sigma)$ after attaching disks to each boundary component of $\partial \Sigma$. First embed $\Sigma$ in $S^3$ such that it bounds the standard unknotted handlebody as shown in the Figure.
Now observe that by removing the disks $D_1, D_2, \ldots, D_n$ as shown in Figure 2, we get an embedding of $(\Sigma, \partial \Sigma)$ in $S^3$ such that each boundary component of $(\Sigma, \partial \Sigma)$ is the boundary of $D_i$ for some $i$.

Next, we attach a band with one full-twist around a properly embedded arc in the disk $D_1$ to the surface $\Sigma$ as shown in Figure 3. This produces an embedded surface $S$ with $(n + 1)$ boundary components in $S^3$. Notice that out of these $n + 1$ boundary components, $n - 1$ boundary components correspond to boundaries of the disks $D_i, i = 2, \ldots, n$. The remaining two boundary components form a Hopf link as depicted in Figure 3. We denote these boundary components by $H_1$ and $H_2$. We use this embedding of the surface $S$ with $n + 1$ boundary components to properly embed the surface $(\Sigma, \partial \Sigma)$ in $DE(n)$ in the following way:

![Figure 2](image1.png)

**Figure 2.** Embedding of $\Sigma$ together with disks $D_1, \ldots, D_n$

![Figure 3](image2.png)

**Figure 3.** Embedding of the surface $S$ with $n + 1$ boundary components which contains a Hopf band $H$ as a subsurface

![Figure 4](image3.png)

**Figure 4.** Embedding of the surface $S$ with $n + 1$ boundary components. The boundary component with dashed line bounds a properly embedded disk in $DE(n)$
Observe that by construction, $S$ admits an embedding of a Hopf band $H$ with boundary components $H_1$ and $H_2$ as shown in Figure 4. Now, consider $S^3$ being embedded as $S^3 \times \{\frac{1}{2}\}$ in $S^3 \times [0, 1]$, where we regard $S^3 \times [0, 1]$ as a collar of $\partial B^4$. We now observe that we can attach a 2-handle along one of the boundary components of the Hopf band in such a way that we obtain $DE(n)$ from $B^4$. More precisely, consider one of the boundary component – say $H_1$ – of the Hopf band and consider the cylinder $H_1 \times \{\frac{1}{2}\}$ and assume that $H_1 \times \{1\}$ is the unknot along which the 2–handle with framing $n$ is attached. In Figure 3, the boundary component $H_1$ is denoted by dashed circle. Thus, $H_1$ bounds a disk $D$ in $DE(n)$. We attach this disc to the surface $S$ to get a new embedding – say $\tilde{f} - f$ of $(\Sigma, \partial \Sigma)$ in $DE(n)$ with its $n$ boundary components. Let us denote these boundary components by $\partial D_1, \partial D_2, \cdots, \partial D_n$ as shown in Figure 4.

Consider $n$ cylinders $\partial D_i \times [\frac{1}{2}, 1]$ for $i = 1$ to $n$. Using these cylinders, we now modify the embedding $\tilde{f}$ to get a proper embedding $f$ of $(\Sigma, \partial \Sigma)$ in $DE(n)$. This we do by considering the union $\tilde{f}(\Sigma) \cup \partial D_1 \times [\frac{1}{2}, 1] \cup \cdots \cup \partial D_n \times [\frac{1}{2}, 1]$.

The embedding described above then clearly gives a proper embedding of $(\Sigma, \partial \Sigma)$ in $DE(n)$. We perturb this embedding – if necessary – to make it into a smooth and proper embedding. By slight abuse of notation, let us again denote this embedding of $(\Sigma, \partial \Sigma)$ by $f$.

We now observe that the embedding $f$ satisfies the property that any simple closed curve $C$ and its ambient band connected sum with the center curve $C_H$ (depicted by dark cure in the Figure 3) of the Hopf band $H$, are ambiently isotopic. This is because, $C_H$ is isotopic the boundary component of $H_1$ which bounds the disk $D$. Hence $C_H$ can be shrunk to a point in the interior of $f(\Sigma)$. This implies that we can isolate one to the other using the disk $D$.

Note that the regular neighborhood of the curve $C \# b C_H$ is a Hopf annulus. We claim that there is an isotopy – say $\Phi_t$ – of $DE(n)$ which is fixed near the boundary of $DE(n)$ and which induces a Dehn twist along $C \# b C_H$. In fact, the isotopy can be assumed to be the identity when restricted to the 2–handle as well. This can be done as follows:

To begin with, recall that the whole surface $\Sigma$ except the 2–disk $D$ coming for the attached 2–handle is still embedded in $B^4$. In fact, we would like to point out that, everything except the cylinders $\partial D_i \times [\frac{1}{2}, 1]$ are still embedded in the level $S^3 \times \{\frac{1}{2}\}$ of the collar $S^3 \times [0, 1]$ of $\partial B^4$. In particular, a fixed neighborhood $N(C \# b C_H)$ is contained in $S^3 \times \{\frac{1}{2}\}$. In order to get the isotopy $\Phi_t$, as claimed, we push the neighborhood slightly towards $S^3 \times \{0\}$ in the collar in such a way that at a fixed level between 0 and $\frac{1}{2}$ the intersection of this pushed neighborhood is a Hopf annulus and this Hopf annulus contains the curve $C \# b C_H$ as its center curve. Let us denote this level by $S^3 \times \{s_0\}$. We now perform an isotopy to induce a Dehn twist along the pushed $C \# b C_H$ in such a way that this isotopy is supported in a small neighborhood of $S^3 \times \{s_0\}$ not intersecting $S^3 \times \frac{1}{2}$. After performing this isotopy, we further isotope the pushed neighborhood $N(C \# C_H)$ back to its original place in $S^3 \times \{\frac{1}{2}\}$. Clearly, the effect of successive compositions of these isotopies is an isotopy $\Phi_t$ as claimed.

We are almost done. We now recall that the mapping class group of $(\Sigma, \partial \Sigma)$ is generated by Dehn twists along Lickorish curves as shown in Figure 1. Since on each Lickorish curve it is possible to perform a Dehn twist via an ambient isotopy of $DE(n)$, the lemma follows.

\begin{remark}
Notice that the Lemma 14 above shows that in $DE(n)$ there exists a proper flexible embedding.
\end{remark}

\begin{proof}[Proof of Theorem 7]
Notice that $S^3 \times S^2$ admits an open book decomposition with pages the unit disk bundle of $T^* S^2 = DE(-2)$ and the monodromy any diffeomorphism $\psi$ of $DE(-2)$ which is the identity near a collar of the boundary and is isotopic to the identity. Observe that Lemma 14 implies there is an abstract open book embedding of $Aob(\Sigma, \phi)$ in $Aob(DE(-2), \psi)$, where $\psi$ is isotopic to the identity. Hence, $(M, Ob(B, \pi))$ admits an open book embedding in $S^3 \times S^2$ with standard open book as claimed.
\end{proof}

4. Embeddings of 3–manifolds in $S^5$

In this section we use techniques developed to establish the Lemma 14 to reprove the theorem that every orientable 3–manifold embeds in $S^5$.}


Figure 5. In the left side of the above figure, we depict the Kirby diagram of \( DE(1) \). The unknot \( K \) with framing +1 is the attaching circle for the 2-handle of \( DE(1) \). While in the right side, we depict the unknot \( K \) together with the unknot \( K' \) which is the boundary of a slightly pushed copy of the core of the attaching handle. The blue knot is the unknot \( U \) linking both \( K \) and \( K' \) once. The knot \( K' \) is assumed to be on the boundary of the attaching region which is a solid torus around \( K \).

**Proof of Theorem 2.** In order to produce an embedding of a 3–manifold \( M \) in \( S^3 \), we first notice that it is sufficient to embed \( M \) in \( S^3 \times \mathbb{R}^2 \). Hence, in what follows, we show how to embed \( M \) in \( S^3 \times \mathbb{R}^2 \).

To begin with, we fix and review some notations. We parametrize a collar of \( \partial B^4 = S^3 \times \{0, 1\} \) such that this region is contained in \( S^3 \times \{1\} \). The unknot \( K \), which is the attaching circle of the 2-handle is then contained in \( S^3 \times \{1\} \). This is depicted in the left of the Figure 5 by black circle with framing +1. Let us denote the zero section of the bundle \( DE(1) \) by \( S \). We can regard \( S \) as the sphere obtained by considering the union of attaching disc of the 2-handle with \( K \times [0, 1] \) and the obvious disk \( K \times \{0\} \) bounds in \( S^3 \times \{0\} \). Next, we denote by \( \mathcal{N}(K) \) a tubular neighborhood of \( K \) which is the attaching region of the 2–handle \( H_2 = D^2 \times D^2 \). If \( p \) is a point on the boundary of \( D^2 \) then the disk \( D^2 \times \{p\} \) embedded in the 2–handle \( H_2 \) intersects the boundary \( S^3 \times \{1\} \) in a curve \( K' \) which links \( K \) once. This is depicted by the red curve in the Figure 5. Notice that \( K' \) lies on the boundary of the attaching region \( \mathcal{N}(K) \).

We now describe how to embed a surface with one boundary component which is disjoint from the zero section \( S \) of \( DE(1) \) and is flexible in \( DE(1) \). Consider an unknot \( U \) which links the attaching region \( \mathcal{N}(K) \) as depicted in the right of the Figure 5. Consider the two circles \( U \times \{\frac{1}{2}\} \) and \( K' \times \{\frac{1}{2}\} \) in the sphere \( S^3 \times \{\frac{1}{2}\} \). Notice that the complement of \( \mathcal{N}(K) \times \{\frac{1}{2}\} \) in \( S^3 \times \{\frac{1}{2}\} \) is a solid torus \( S^1 \times D^2 \). The circle \( U \times \{\frac{1}{2}\} \) is the center circle \( S^1 \times \{0\} \) of this solid torus while \( K' \times \{\frac{1}{2}\} \) is a curve going once around the longitude and once around the meridian of the solid torus. This implies circles \( U \times \{\frac{1}{2}\} \) and \( K' \times \{\frac{1}{2}\} \) bound a Hopf annulus in \( S^3 \times \{\frac{1}{2}\} \) which is disjoint for \( K \times \{\frac{1}{2}\} \) as it lies inside the solid torus \( S^1 \times D^2 \). We call this Hopf annulus \( \mathcal{A} \). Now observe that the boundary component of the annulus \( \mathcal{A} \) corresponding to \( K' \times \{\frac{1}{2}\} \) bounds a disk – say \( D \) – in \( DE(1) \) by construction. By attaching the annulus \( U \times \{\frac{1}{2}, 1\} \) to \( D \) along its boundary \( U \times \{\frac{1}{2}\} \), we produce a properly embedded disc in \( DE(1) \).

Next, let \( \Sigma \) be a standardly embedded handle-body contained in the solid torus \( S^1 \times D^2 \) which is disjoint from the Hopf annulus \( \mathcal{A} \). Let \( \Sigma \) be the boundary of this handle-body. We can perform an ambient connected sum of \( \Sigma \) with \( \mathcal{A} \) in \( S^3 \times \{\frac{1}{2}\} \) such that the surface obtained after ambient connected sum is still contained in the complement of \( K \times \{\frac{1}{2}\} \). Notice that since \( \mathcal{A} \) is an annulus in a properly embedded disc described in the previous paragraph. This connected sum operation produces a properly embedded surface with one boundary component. By a slight abuse of notation let us continue to denote this surface by \( \Sigma \).

Observe that an argument similar to the one used in the proof of the Lemma 14 implies that \( \Sigma \) is a properly embedded flexible surface in \( DE(1) \). This is because, we can isotope every generator of the mapping class group of \( \Sigma \) in such a way that it admits a neighborhood which is Hopf annulus embedded in \( S^3 \times \{\frac{1}{2}\} \). Furthermore, notice that \( \Sigma \) does not intersect the zero section \( S \) of the bundle \( DE(1) \) as it does not intersect the core disk of the 2–handle as well as the annulus \( K \times \{\frac{1}{2}, 1\} \).
Now, let \( M \) be any orientable 3–manifold. Clearly, we can regard \( M \) as \( \mathcal{A} \text{ob}(\Sigma, \phi) \) for some orientable surface \( \Sigma \) with one boundary component. Since \( \Sigma \) admits a flexible embedding in \( \mathcal{D} \mathcal{E}(1) \) disjoint form \( S \), there exists an embedding of \( M \) in \( S^2 \times S^3 = \mathcal{A} \text{ob}(\mathcal{D} \mathcal{E}(1), \text{Id}) \).

We now notice that since \( \Sigma \) is disjoint from \( S \), the mapping tours associated to \( M = \mathcal{A} \text{ob}(\Sigma, \phi) \) is in fact, properly embedded in a manifold diffeomorphic to \( S^1 \times S^3 \times \mathbb{R} \). This follows from the fact that the complement of the zero section in \( \mathcal{D} \mathcal{E}(1) \) is \( S^3 \times \mathbb{R} \), see, for example, [GS, p. 119]. But this clearly implies that the embedding of \( M \) in \( S^2 \times S^3 \) is contained in a manifold diffeomorphic to \( S^3 \times \mathbb{R}^2 \) as required. This completes our argument. 

\[ \square \]

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