Performing sequence for the thin-layer quantization scheme

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We clearly refine a fundamental framework for the thin-layer quantization scheme. Further, we definitively clarify a performing sequence in the thin-layer quantization process. The limit \(q_3 \to 0\) (\(q_3\) denotes the variable perpendicular to the curved surface) must be performed after calculating all curvilinear coordinate derivatives. In the general form, a canonical action integral, the limit \(d \to 0\) (\(d\) is the thickness of the curved surface) has to be calculated after performing all curvilinear coordinate integrations. In complete accordance with the sequence, in the presence of an electromagnetic field, for a charged particle constrained on a spatially curved surface the Lorentz gauge and the Schrödinger equation cannot be decoupled from the mean curvature of the surface simultaneously, except \(A^3 = 0\) (\(A^I\) is a contravariant component along the direction perpendicular to the curved surface of the electromagnetic field vector potential \(\vec{A}\)) with a certain gauge transformation, and the canonical action integral does not show convenience for the thin-layer quantization scheme. Additionally, we derive the modifications induced by the thickness of the surface to the geometric potential and the kinetic operator.

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I. INTRODUCTION

In 1971, the thin-layer quantization scheme was first put forward by H. Jensen and H. Koppe to research the quantum mechanics of a particle confined to move on a spatially curved surface \(^1\). Ten years later, the semiclassical method was applied by R. C. T. da Costa to investigate the effective quantum dynamics for one constrained particle \(^2\) and for \(n\) constrained particles \(^3\). The three original papers determine a fundamental framework for the thin-layer quantization scheme, but they do not explicitly define it. This probably leads to some unintentional mistakes in the performing process of the thin-layer quantization scheme.

In the presence of an externally applied electromagnetic field, the thin-layer quantization scheme was, in general, used to quantize a charged spin-less particle constrained on \(S\) \(^1\,^4\,^5\). For the electromagnetic field, the Lorentz gauge was chosen \(^6\). In the case the electromagnetic field is applied externally, it is trivial to obtain the two fundamental evidences in Ref. \(^5\). The electromagnetic field decouples from the curvature independently of the shape of the surface, of the electromagnetic field, and of the gauge. The effective Schrödinger equation can be decomposed into a surface component and a normal component analytically. In a general case of the reduced system including electric currents \(\vec{J}\) in all space directions \(^6\,^7\), the vanishing of the \(J^3\) component current is a necessary condition for the two fundamental evidences \(^6\,^8\). With the mapping of the original metric tensor \(G_{ij}\) into the confined metric tensor \(\tilde{G}_{ij}\), the coupling term \(2A^5M\) (\(M\) is the mean curvature) was canceled from the Lorentz gauge, and then the electromagnetic field was decoupled from the curvature of \(S\), and the separability of the dynamics was accomplished \(^9\).

In a general form, a canonical action integral, by using the variational approach, the quantum mechanics is re-derived \(^6\,^9\). By partially integrating, the canonical action integral is separated into a volume integral and a closed surface integral. The vanishing conditions of the closed surface integral are necessary to decouple the electromagnetic field from the curvature, and to decompose the effective quantum dynamics into a surface component and a normal component \(^6\,^7\,^9\). These necessary conditions require further discussion.

In this paper, we get inspiration from the results in Refs. \(^5\,^6\,^9\), especially the further work for the thin-layer quantization scheme opened up in Ref. \(^9\). We first explicitly refine the fundamental framework for the thin-layer quantization scheme, and then definitively indicate a sound performing sequence in the thin-layer quantization process. The limit \(q_3 \to 0\) must be performed after calculating all curvilinear coordinate derivatives. In a general form, it follows immediately that the limit \(q_3 \to 0\) should be replaced by the limit \(d \to 0\), i.e. the limit \(d \to 0\) must be calculated after
performing all curvilinear coordinate integrations. In complete agreement with the performing sequence, in the presence of an electromagnetic field we derive the quantum mechanics for a charged particle. The Lorentz gauge and the effective Schrödinger equation cannot be decoupled from the mean curvature of the surface simultaneously, except $A^3 = 0$ with a certain gauge transformation. In other words, in a general case, where electric currents $\vec{J}$ are present in all space directions, the vanishing of the $J^3$ component is a necessary condition to decouple the electromagnetic field from the curvature of the surface, and to decompose the dynamics into a surface term and a normal term in Refs. [6, 8]. In this well understood system, the canonical action integral does not supply any convenience for the thin-layer quantization scheme, and the closed surface integral plays the role of a Dirichlet boundary condition. Furthermore, we investigate the thickness effect of the thin-layer on the surface quantum dynamics by taking back the terms of first degree in $q_3$. These terms depend on $q_3$ modifying the well-known geometric potential and the kinetic operator.

II. THE FUNDAMENTAL FRAMEWORK FOR THE THIN-LAYER QUANTIZATION SCHEME

![FIG. 1:](Image)

FIG. 1: (Color online) The surface $\mathcal{S}$, the subspace $V_N$ and two auxiliary surfaces $S_1$ and $S_2$ are sketched. The surface $\mathcal{S}$ is described by $\bar{\mathcal{R}}(q_1, q_2)$. The subspace $V_N$ is enclosed by $S_1$ and $S_2$.

For the sake of convenient presentation, we first define a subspace $V_N$, which is enclosed by two parallel surfaces $S_1$ and $S_2$ with a distance $d$. The main surface $\mathcal{S}$ has a unique distance $d/2$ to $S_1$ and $S_2$. They are sketched in Fig. 1. If $\mathcal{S}$ is parametrized by $\bar{\mathcal{R}}(q_1, q_2)$, $V_N$ could be parametrized by

$$\bar{R}(q_1, q_2, q_3) = \bar{\mathcal{R}}(q_1, q_2) + q_3 \vec{n}(q_1, q_2),$$

(1) where $\vec{n}(q_1, q_2)$ is a unit vector perpendicular to $\mathcal{S}$.

From the three original papers [1][8][8], it is straightforward to learn that the scheme is to squeeze the particle on the surface $\mathcal{S}$, and to preserve the curvature effects in the surface Schrödinger equation as much as possible. In terms of the two essences, we clearly conclude the fundamental framework for the thin-layer quantization scheme. It consists of three steps:

1. A dynamical system is originally defined in the subspace $V_N$. In other words, its dynamical equation, gauge condition and all others must be defined in $V_N$.
2. In terms of the metric tensor $G_{ij}$ defined in $V_N$, one calculates all the curvilinear coordinate derivatives included in Step (1).
3. Perform the limit $q_3 \to 0$ to remove all the terms depending on $q_3$ from all expressions in Step (2), and separate the dynamical equation into surface and normal components.

The possibility of Step (3) is protected by the introduction of the squeezing potential [1]

$$V_\lambda(q_3) = \begin{cases} 0, & q_3 = 0, \\ \infty, & q_3 \neq 0, \end{cases}$$

(2)

which squeezes the particle on $\mathcal{S}$. The fundamental framework implies a performing sequence where the limit $q_3 \to 0$ must be made after calculating all curvilinear coordinate derivatives. In a general framework, a canonical action integral, the limit $q_3 \to 0$ will become the limit $d \to 0$, and the performing sequence is that the limit $d \to 0$ must be done after performing all curvilinear coordinate integrations. This performing sequence is defined by the action of the thin-layer quantization scheme, in order to preserve the curvature effects to the surface dynamics as much as possible.

III. PERFORMING THE LIMIT $q_3 \to 0$ AFTER ALL CURVILINEAR COORDINATE DERIVATIVES

In $V_N$, the metric tensor is defined by

$$G_{ij} = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad i, j = 1, 2, 3,$$

(3)

where $G_{ij} = \frac{\partial \bar{R}}{\partial q_i} \cdot \frac{\partial \bar{R}}{\partial q_j}$. Confined on $\mathcal{S}$, the metric tensor [3] is simplified in the following expression

$$\tilde{G}_{ij} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \lim_{q_3 \to 0} (G_{ij}),$$

(4)

where $g_{ab} = \frac{\partial x^a}{\partial q^b} \cdot \frac{\partial x^a}{\partial q^b}$, wherein $a, b = 1, 2$. The relation between $G = \det(G_{ij})$ and $g = \det(g_{ab}) = \det(\tilde{G}_{ij})$ is

$$G = [1 + \text{Tr}(\alpha)q_3 + \det(\alpha)q_3^2]^2 g = f^2 g,$$

(5)

where $f = 1 + \text{Tr}(\alpha)q_3 + \det(\alpha)q_3^2$, wherein the elements of the Weingarten curvature matrix $\alpha$ are

$$\alpha_{11} = \frac{1}{g} (g_{12}h_{21} - g_{22}h_{11}), \quad \alpha_{12} = \frac{1}{g} (g_{21}h_{11} - g_{11}h_{21}),$$

$$\alpha_{21} = \frac{1}{g} (g_{12}h_{22} - g_{22}h_{12}), \quad \alpha_{22} = \frac{1}{g} (g_{12}h_{21} - g_{21}h_{12}),$$

(6)
in which $h_{ab}$ are the coefficients of the second fundamental form, $h_{ab} = \hat{n} \cdot \frac{\partial^2 \hat{r}}{\partial \xi^a \partial \xi^b}$, where $\hat{n}$ is the unit vector perpendicular to $\mathcal{S}$ \cite{10},

$$\hat{n} = \frac{\frac{\partial \hat{r}}{\partial \xi^a} \times \frac{\partial \hat{r}}{\partial \xi^b}}{|\frac{\partial \hat{r}}{\partial \xi^a} \times \frac{\partial \hat{r}}{\partial \xi^b}|}. \quad (7)$$

According to the fundamental framework elucidated in Sec. II, in the presence of an electromagnetic field, a spin-less particle with a charge $-e$ constrained on $\mathcal{S}$ can be originally described by the following Schrödinger equation

$$ihD_t \psi = -\frac{\hbar^2}{2m} D_i D^i \psi + V_\lambda(q_3) \psi, \quad (8)$$

where $\psi$ is a wave function, $D_t = \partial_t + \frac{e}{\hbar} A_0, D_i = \nabla_i + \frac{e}{\hbar} A_i$, and $V_\lambda(q_3)$ is the squeezing potential \cite{2}. The covariant derivatives $\nabla_i$ are defined by $\nabla_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k$, where $v^j$ are the contravariant components of a three-dimensional (3D) vector field $\vec{v}$, $\Gamma^j_{ik}$ are the Christoffel symbols, $\Gamma^j_{ik} = \frac{1}{2} G^{ji} (\partial_k G_{il} + \partial_l G_{ik} - \partial_l G_{il})$, and $\partial_i$ are the derivatives with respect to the curvilinear coordinate variables $q_i$. Following the fundamental framework, the Lorentz gauge for the four-component potential $(A_0, \vec{A})$ must be originally defined in $V_N$. Therefore, we expand the Schrödinger equation \cite{3} and the Lorentz gauge \cite{6} in the following expressions

$$ihD_t \psi = -\frac{\hbar^2}{2m} \left\{ \frac{1}{\sqrt{G}} \partial_t (\sqrt{G} G^{ij} \partial_j \psi) \right\} + \frac{ie}{\hbar} \left\{ \frac{1}{\sqrt{G}} \partial_t (\sqrt{G} G^{ij} A_j) \right\} \psi + \frac{2ie}{\hbar} G^{ij} A_j \partial_i \psi \quad (9)$$

and

$$\vec{n} \cdot \vec{A} = \frac{1}{\sqrt{G}} \partial_a (\sqrt{G} g^{ab} A_b) + \frac{1}{\sqrt{G}} \partial_3 (\sqrt{G} A_3) = 0. \quad (10)$$

By introducing a new wave function $\chi(q_1, q_2, q_3) = \chi_s(q_1, q_2) \chi_t(q_3)$, which is related to $\psi$ by $\psi = f^{-\frac{1}{2}} \chi$, subsequently performing the limit $q_3 \to 0$, we can rewrite Eqs. \cite{3} and \cite{6} in the following forms

$$ihD_t \chi_s = -\frac{\hbar^2}{2m} \left\{ \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b \chi_s) \right\} + \frac{ie}{\hbar} \left\{ \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} A_b) \right\} \chi_s + \frac{2ie}{\hbar} g^{ab} A_a \partial_b \chi_s - \frac{\hbar^2}{2m} \partial_3 \partial^3 \chi_s + V_\lambda(q_3) \chi_s, \quad (11)$$

and

$$\frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} A_b) + \partial_3 A^3 + 2MA^3 = 0, \quad (12)$$

where $V_g$ is the well-known geometric potential \cite{11}

$$V_g = -\frac{\hbar^2}{2m} [M^2 - K], \quad (13)$$

wherein $M = \frac{1}{4} \text{Tr}(\alpha)$ is the mean curvature and $K = \det(\alpha)$ is the Gaussian curvature. In the case where sources of the electromagnetic field are neglected, it is apparently possible to perform a restricted gauge transformation to set $A^3 = 0$. Under this situation, it is obvious that the electromagnetic field can be completely decoupled from the curvature of $\mathcal{S}$ in Eqs. \cite{11} and \cite{12}, simultaneously, the effective Schrödinger equation \cite{11}

$$ihD_t \chi_t = -\frac{\hbar^2}{2m} \partial_3 \partial^3 \chi_t + V_\lambda(q_3) \chi_t. \quad (15)$$

In the general case the constrained system includes currents $\vec{J}$ in 3D, and the Schrödinger equation \cite{11} is implicitly coupled to the curvature of $\mathcal{S}$ due to the presence of the $\vec{A}^2$ term \cite{6}. It is obviously impossible to entirely remove the couplings to the mean curvature from Eqs. \cite{11} and \cite{12} simultaneously. According to the equation of motion of the electromagnetic field coupled to $\vec{J}$, $\nabla^2 \vec{A} \sim -\vec{J}$, it is apparent that the vanishing of the $J^3$ component is necessary to decouple the electromagnetic field from the curvature of $\mathcal{S}$, and to decompose
the Schrödinger equation into a surface component and a normal component [6, 8]. It is worthwhile to notice that the Lorentz gauge and the Schrödinger equation are together defined in $V_N$ to describe the discussed dynamical system. Eq. (11) represents the gauge invariance for the Schrödinger equation minimally coupled to the electromagnetic field in a curved 3D space with the metric tensor $G_{ij}$. The mapping of $G_{ij}$ into $\tilde{G}_{ij}$ preserves the gauge invariance. By fixing the gauge, the Lorentz gauge can be simplified in the following form [9]

$$\nabla \cdot \tilde{A} = \nabla_{||} \cdot \tilde{A}_{||} + \partial_3 A^3 \equiv 0. \quad (16)$$

This mapping of $G_{ij}$ into $\tilde{G}_{ij}$ changes the region of definition from $V_N$ to $S$ for the Lorentz gauge, and is identical to exchanging the performing sequence between $\lim_{q_3 \to 0}$ and $\partial_i$, which does not agree with the fundamental framework in Sec. II. It is rather obviously wrong, but it partially confirms that the limit $q_3 \to 0$ must be done after performing all the curvilinear coordinate derivatives in the Lorentz gauge.

IV. PERFORMING THE LIMIT $d \to 0$ AFTER ALL CURVILINEAR COORDINATE INTEGRALS

In a general form, for the above considered system the canonical action integral [5, 6, 9] reads

$$S = \int_{V_N} \left[ -i\hbar \psi^* D_t \psi + \frac{\hbar^2}{2m} (\tilde{D} \psi)^* \cdot (\tilde{D} \psi) + V_\lambda(q_3)|\psi|^2 \right], \quad (17)$$

where $D_t = \partial_t + \frac{i}{\hbar} A_0$, $\tilde{D} = \nabla + \frac{e}{\hbar} \tilde{A}$, $\psi$ is a wave function and $V_\lambda(q_3)$ is the squeezing potential [2]. By partially performing integration, we separate the action (17) into a volume integral

$$S_v = \int_{V_N} \left[ -i\hbar \psi^* D_t \psi - \frac{\hbar^2}{2m} \psi^* [\nabla_i \nabla^i \psi - \frac{\nabla^2}{\hbar^2} \tilde{A}^2 \psi + \frac{2ie}{\hbar} A^i \nabla_i \psi + \frac{ie}{\hbar} (\nabla_i A^i) \psi + V_\lambda(q_3)|\psi|^2 \right], \quad (18)$$

and a closed surface integral

$$S_s = \frac{\hbar^2}{2m} \int_{\partial V_N} (\psi^* \tilde{D} \psi). \quad (19)$$

The latter term plays the role of a boundary condition. By performing the limit $d \to 0$, we write the surface integral (19) in the following expression

$$S_s = A_s \lim_{d \to 0} \left[ (\psi^* D_3 \psi) \left\vert\right. \frac{4}{d} - (\psi^* D_3 \psi) \left\vert\right. \frac{-4}{d} \right], \quad (20)$$

where $D_3 = \partial_3 + \frac{i}{\hbar} A_3$, and $A_3$ denotes the area of the main surface embedded in the integral volume $V_N$. By introducing a new wave function $\chi$ with the relation $\chi = \sqrt{f} \psi$, we rewrite the expression (20) for $\chi$ as below

$$S_s = A_s \lim_{d \to 0} \left[ (\chi^* (D_3 - M) \chi) \left\vert\right. \frac{4}{d} - (\chi^* (D_3 - M) \chi) \left\vert\right. \frac{-4}{d} \right]. \quad (21)$$

The vanishing of the surface integral (19) is trivially satisfied by the smoothness of $\chi$ and $\partial_3 \chi$ passing through $S$. It is apparent that, in the presence of an electromagnetic field, an arbitrarily imposed boundary condition (including Neumann type) does not endanger the validity of the thin-layer quantization scheme [6, 9]. Therefore, the thin-layer quantization scheme can be used without restrictions. Here, it should be noted that the limit $d \to 0$ is absolutely impossible to be moved into the surface integral and replaced by $q_3 \to 0$. Taking the limit $d \to 0$ into the surface integral and replacing it by $q_3 \to 0$ is equivalent to exchanging the performing order between the limit $q_3 \to 0$ and the curvilinear coordinate derivatives. This operation conflicts with the fundamental framework in Sec. II. Therefore, we confidently indicate that the limit $d \to 0$ also has to be done after calculating all curvilinear coordinate integrals.

In the physically general case that is extended in Ref. [6], the limit $q_3 \to 0$ is probably severe and breaks with natural limits set by the uncertainty principle. On physical grounds one can argue that it is more physically consistent to assume certain variation $\delta \psi$ on $\partial V_N$. In the case that the surface integral vanishes provided $\lim_{q_3 \to 0} (\partial_3 - icA_3) \psi = 0$, a Neumann type of boundary condition imposed to the normal fluctuations of the wave function endangers the validity of the thin-layer quantization scheme, except that $S$ has a constant mean curvature. This extension is physically interesting, but it is outside the abilities of the thin-layer quantization scheme.

In complete agreement with the performing sequence, the limit $d \to 0$ cannot be brought into the volume integral and replaced by $q_3 \to 0$, and the term $\tilde{\nabla} \cdot (\psi^* \tilde{A} \psi)$ included in (19) cannot contribute a volume integral term $-\frac{i\hbar}{m} A^i \chi \tilde{\nabla} \chi$ to cancel the anomalous orbital magnetic moment in the volume integral component (18). Varying the canonical action (18), we obtain the Schrödinger equation (6). It is apparent that, in the fundamental framework, a canonical action integral does not supply convenience to the thin-layer quantization scheme.”

V. THICKNESS OF THE SURFACE MODIFIES THE SURFACE DYNAMICS

It is essentially important that the thin-layer quantization scheme causes the curvature-induced geometric potential to appear in the surface dynamical equation. This process is probably severe and breaks with natural limits set by the uncertainty principle, but it has been demonstrated that the attractive geometric potential is valid and important to some physical systems [11, 13]. On actual grounds, the thickness of the curved surface is the real existence. It is not completely clear how to introduce the thickness effect into the surface dynamics. Now we try to discuss it’s effect.

Following the fundamental framework in Sec. II, we further add a new step to the standard framework,
namely:

(4) Take the terms of first degree in $q_3$ back into the surface Schrödinger equation.

Performing Step (4), we rewrite the surface Schrödinger equation (14) in the following form

\[
\begin{align*}
\hbar D_t \chi_s &= - \frac{\hbar^2}{2m} \left\{ \frac{1}{\sqrt{g}} \partial_a [\sqrt{g} g^{ab} (\partial_b \chi_s)] - q_3 \frac{1}{\sqrt{g}} \partial_a [\sqrt{g} g^{ab} (\partial_b M) \chi_s - \sqrt{g} (g^{ab} - Mg^{ab})(\partial_b \chi_s)] \\
+ \frac{i e}{\hbar} &\left[ \partial_a (\sqrt{g} g^{ab} A_b) \right] \chi_s + q_3 \frac{i e}{\hbar} \left[ \partial_a (\sqrt{g} g^{ab} A_b) - M \partial_a (\sqrt{g} g^{ab} A_b) \right] \chi_s + \frac{2i e}{\hbar} g^{ab} A_a (\partial_b \chi_s) + q_3 \frac{2i e}{\hbar} (g^{ab}) A_a (\partial_b \chi_s) + V'_{g} \right) A_a (\partial_b \chi_s) + V'_{\chi} \right),
\end{align*}
\]

where $g^{ab}$ is included in the expression $G^{ab} \approx g^{ab} + g^{ab} q_3 + \cdots$ and $V'_g$ is a thickness-modified geometric potential in the following form

\[
V'_g = - \frac{\hbar^2}{2m_m} (M^2 - K)(1 - 4Mq_3) = V'_g (1 - 4Mq_3),
\]

wherein $M$ is the mean curvature, $K$ is the Gaussian curvature and $V'_g$ is the well-known geometric potential [13].

\[
H' = \pm \frac{\hbar^2 d}{4m} \frac{1}{\sqrt{g}} \partial_a [\sqrt{g} g^{ab} (\partial_b M) - \sqrt{g} (g^{ab} - Mg^{ab}) \partial_b] - \frac{i e}{\hbar} \left[ \partial_a (\sqrt{g} g^{ab} A_b) - M \partial_a (\sqrt{g} g^{ab} A_b) \right]
\]

\[
- \frac{2i e}{\hbar} (g^{ab} - Mg^{ab}) A_a (\partial_b) + \frac{e^2}{2} (g^{ab} - Mg^{ab}) A_a (\partial_b)
\]

These modifications can be measured by energy shifts [14].

VI. CONCLUSIONS

In this paper we conclude that the fundamental framework for the thin-layer quantization scheme consists of three steps: (1) originally define the discussed dynamics in $V_N$, (2) subsequently calculate all various curvilinear coordinate derivatives in the dynamics, (3) finally perform the limit $q_3 \rightarrow 0$ to obtain the effective surface dynamics. The fundamental framework is determined by the two essential actions of the thin-layer quantization scheme: One is to squeeze the particle on $S$, and the other is to preserve the curvature effects in the expectant surface dynamical equation as much as possible. The latter action implicitly defines that the limit $q_3 \rightarrow 0$ must be performed after calculating all curvilinear coordinate derivatives in the dynamical equation, in the necessary gauge, and in all others. In a general form, the limit $d \rightarrow 0$ must be done after performing all curvilinear coordinate integrals in the canonical action integral. In complete accordance with the fundamental framework, strictly following the performing sequence, we reconsidered a charged spin-less particle bounded on $S$ in an electromagnetic field. In the case the electromagnetic field is externally applied, it is trivial that the electromagnetic field is decoupled from the curvature of $S$, and the effective Schrödinger equation is decomposed into a surface component and a transverse component [5, 8]. In the general case of the presence of the currents $J$ in 3D, the vanishing of the $J^3$ component is necessary to the known two fundamental evidences [6, 8]. This result is ultimately determined by the fact that the region of the original definition for the Schrödinger equation and the Lorentz gauge is $V_N$ [13]. Therefore the mapping of the original metric tensor $G_{ij}$ into the confined metric tensor $\tilde{G}_{ij}$ cannot be applied to eliminate the coupling term $2A^2 M$ from the Lorentz gauge. According to the sequence, performing the limit $d \rightarrow 0$ after all curvilinear coordinate integrals, the canonical action integral does not offer any convenience for the thin-layer quantization scheme. In the fundamental framework, an imposed Neu-
mann type boundary condition does not endanger the validity of the thin-layer quantization scheme. In a general physical case of a certain variation $\delta \psi$ along the direction perpendicular to $S$, a Neumann type boundary condition invalidates the thin-layer quantization scheme except where $S$ has a constant mean curvature.

Extending the fundamental framework, we further included a step to take the terms of first degree in $q_3$ back into the expectant surface Schrödinger equation. The modifications induced by the thickness of the surface to the well-known geometric potential and to the kinetic operator were derived. Those modifications could be measured by the energy shifts \cite{14}.

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