GAUGED LOCALLY SUPERSYMMETRIC

$D = 3$ NONLINEAR SIGMA MODELS

Bernard de Wit, Ivan Herger and Henning Samtleben

Institute for Theoretical Physics & Spinoza Institute,
Utrecht University, Postbus 80.195, 3508 TD Utrecht, The Netherlands

B.deWit@phys.uu.nl, I.Herger@phys.uu.nl, H.Samtleben@phys.uu.nl

Abstract

We construct supersymmetric deformations of general, locally supersymmetric, nonlinear sigma models in three spacetime dimensions, by extending the pure supergravity theory with a Chern-Simons term and gauging a subgroup of the sigma model isometries, possibly augmented with R-symmetry transformations. This class of models is shown to include theories with standard Yang-Mills Lagrangians, with optional moment interactions and topological mass terms. The results constitute a general classification of three-dimensional gauged supergravities.
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1 Introduction

Supergravity theories with vector gauge fields can usually be deformed by introducing gauge charges for the various fields. These charges generate a corresponding gauge group. Supersymmetry then necessitates the presence of additional interactions consisting of masslike terms and a scalar potential, beyond the standard gauge interactions; the possible gauge groups are often severely restricted. For theories with a high degree of supersymmetry, gaugings constitute the only known supersymmetric deformations.

In three spacetime dimensions the situation is special in two respects. First of all, pure extended supergravity is topological. Non-topological theories are constructed by coupling supergravity to matter. In three dimensions the obvious matter supermultiplets are scalar multiplets, so that the resulting Lagrangians take the form of a nonlinear sigma model coupled to supergravity. These theories have been constructed and classified in [1]. Supersymmetry leads to stringent conditions on the target space, which can only be satisfied when the number of supersymmetries is restricted to \( N \leq 16 \), implying that there are at most 32 supercharges. Beyond \( N = 4 \) the target space has to be homogeneous. There exists no theory with \( N = 7 \) supersymmetry and beyond \( N = 8 \) there are only four possible theories. They are \( N = 9, 10, 12 \) and 16 supersymmetric, and their target spaces are unique and equal to the symmetric spaces \( F_4(-20)/\text{SO}(9) \), \( E_6(-14)/(\text{SO}(10) \times \text{SO}(2)) \), \( E_7(-5)/(\text{SO}(12) \times \text{SO}(3)) \) and \( E_8(8)/\text{SO}(16) \), respectively.

Secondly, the gauging of these theories seems impossible at first sight, because of the lack of vector gauge fields. However, one can introduce a Chern-Simons term in three dimensions, which is topological just as pure supergravity itself, and the corresponding gauge fields can be coupled to the nonlinear sigma model by gauging a subgroup of the target space isometries. Such gaugings have been constructed in [2, 3, 4] for \( N = 16, 8 \), and in [5, 6] for some abelian gauge groups for the case of \( N = 2 \). The gauging is defined by the gauge group embedding in the isometry group, which in this case is defined in terms of a symmetric embedding tensor. The latter defines the so-called \( T \)-tensors. The viability of the gauging depends in a subtle manner on the properties of these \( T \)-tensors and the gauged supergravity models have an elegant mathematical structure (see [7] for the corresponding analysis of the maximal supergravities in higher dimensions). In this paper we exhibit this structure and derive the precise conditions for having a consistent gauging. For \( N > 3 \) these conditions amount to the absence of \( T \)-tensor components transforming in a particular irreducible \( \text{SO}(N) \) representation.

The gauged supergravities come with a scalar potential that allows for supersymmetric anti-de Sitter groundstates. Therefore these theories can be connected to two-dimensional superconformal theories that live on the boundary of the anti-de Sitter space. The three-
The dimensional setting may offer advantages when studying the adS/CFT correspondence, because the supergravity theory is more amenable to nonperturbative studies, while at the same time the large variety of two-dimensional superconformal theories has been studied extensively in the literature. The theories constructed in this paper include the effective theories that arise in the compactification of high-dimensional supergravities, as we will show below. These include the compactifications on spheres [8, 9], compactifications with nontrivial fluxes [10, 11, 12, 13], as well as the theories whose existence has been inferred from computing the Kaluza-Klein spectra in the context of the adS-CFT correspondence [14, 15, 16, 17, 18].

The results of this paper constitute a complete classification of gauged supergravities in three dimensions which can be regarded as an extension of the classification of ungauged supergravities presented in [1]. In both cases the matter supermultiplets comprise scalar and spinor fields. Because vector fields can always be converted to scalar fields by a suitable duality transformation, the restriction to such scalar multiplets does not seem relevant. However, the presence of gauge charges often poses an obstacle for performing duality transformations. As it turns out, no such obstacle arises in the three-dimensional context. Below we will indicate how, by introducing compensating fields, every Yang-Mills Lagrangian can be converted into a Lagrangian that belongs to the class of Lagrangians discussed in this paper. Therefore the classification of the gauged supergravities presented here, is on a par with the classification of the ungauged supergravities given in [1], albeit that it is not quite possible to present an exhaustive classification of all possible gauge groups.

We briefly indicate how the conversion of three-dimensional Yang-Mills Lagrangians can be done. In this conversion every gauge field is replaced by two gauge fields and a new scalar field, which together describe the same number of dynamic degrees of freedom as the original gauge field. Our presentation is a further elaboration of the results of [13, 19] and is completely general. Consider the Lagrangian, in three spacetime dimensions, with Yang-Mills term quadratic in the field strengths, and moment interactions proportional to a gauge covariant operator $O^A_{\mu\nu}$,

$$\mathcal{L} = -\frac{1}{4} \sqrt{g} \left( F^A_{\mu\nu}(A) + O^A_{\mu\nu}(A, \Phi) \right) M_{AB}(\Phi) \left( F^{B\mu\nu}(A) + O^{B\mu\nu}(A, \Phi) \right) + \mathcal{L}'(A, \Phi). \quad (1.1)$$

Here $A^A_{\mu}$ and $F^A_{\mu\nu}(A)$ denote the nonabelian gauge fields and corresponding field strengths, labeled by indices $A, B, \ldots$, and $\Phi$ generically denotes possible matter fields transforming according to certain representations of the gauge group $G_{YM}$. The structure constants of this group are denoted by $f_{ABC}^A$, so that the field strengths read,

$$F^A_{\mu\nu}(A) = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f^{A}_{BC} A^B_\mu A^C_\nu.$$
The symmetric matrix $M_{AB}(\Phi)$ may depend on the matter fields and transforms covariantly under $G_{YM}$. The last term, $\mathcal{L}'(A, \Phi)$, in the Lagrangian is separately gauge invariant and its dependence on the gauge fields is exclusively contained in covariant derivatives of the matter fields or in topological mass terms (i.e. Chern-Simons terms). The Bianchi identities and vector field equations take the form,

$$D_\mu \tilde{F}^A\mu(A) = 0, \quad D_{[\mu} \left( M_{AB}(\Phi) (\tilde{F}^B_{\nu]}(A) + \tilde{O}^B_{\nu]}(A, \Phi)) \right) - J_{A\mu\nu}(A, \Phi) = 0, \quad (1.2)$$

where we use the definitions

$$\tilde{F}^A\mu(A) = \frac{i}{2} \sqrt{g} \varepsilon_{\mu\nu\rho} F^{A\nu\rho}(A), \quad J_{A\mu\nu}(A, \Phi) = \frac{i}{2} \sqrt{g} \varepsilon_{\mu\nu\rho} \frac{\partial \mathcal{L}'(A, \Phi)}{\partial A^A\rho},$$

$$\tilde{O}^A\mu(A, \Phi) = \frac{i}{2} \sqrt{g} \varepsilon_{\mu\nu\rho} O^{A\nu\rho}(A, \Phi).$$

Usually the duality is effected by regarding the field strength as an independent field on which the Bianchi identity is imposed by means of a Lagrange multiplier. Because the Lagrangian (1.1) depends explicitly on both the field strengths and on the gauge fields, we proceed differently and write the field strength in terms of new vector fields $B_{A\mu}$ and the derivative of compensating scalar fields $\phi_A$, all transforming in the adjoint representation of the gauge group. The explicit expression,

$$\tilde{F}^A\mu(A) + \tilde{O}^A\mu(A) = M^{AB}(B_{B\mu} - D_{\mu}\phi_B), \quad (1.3)$$

where $M^{AC} M_{CB} = \delta^B_A$, should be regarded as a field equation that follows from the new Lagrangian that we are about to present. The structure of (1.3) implies that we are dealing with additional gauge transformations as its right-hand side is invariant under the combined transformations,

$$\delta B_{A\mu} = D_{\mu} \Lambda_A, \quad \delta \phi_A = \Lambda_A, \quad (1.4)$$

under which all other fields remain invariant. The corresponding abelian gauge group, $T$, has nilpotent generators transforming in the adjoint representation of $G_{YM}$. Obviously, the $\phi_A$ act as compensating fields with respect to $T$. The combined gauge group is now a semidirect product of $G_{YM}$ and $T$ and its dimension is twice the dimension of the original gauge group $G_{YM}$. The covariant field strengths belonging to the new gauge group are $F_{\mu\nu}^A(A)$ and $F_{A\mu\nu}(B, A) = 2 D_{[\mu} B_{A\nu]}$, and transform under $T$ according to $\delta F_{\mu\nu}^A = 0$ and $\delta F_{A\mu\nu} = - \Lambda_C f_{ABC} F_{\mu\nu}^C$. The fully gauge covariant derivative of $\phi_A$ equals

$$\hat{D}_\mu \phi_A \equiv D_\mu \phi_A - B_{A\mu} = \partial_\mu \phi_A - f_{ABC} A^B_\mu \phi_C - B_{A\mu}, \quad (1.5)$$
and is invariant under $\mathcal{T}$ transformations.

The field equations corresponding to the new Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\sqrt{g} \hat{D}_\mu \phi_A M^{AB}(\Phi) \hat{D}_\mu \phi_B + \frac{1}{2} i \varepsilon^{\mu\nu\rho} (F_A^{\mu\nu} B_{A\rho} - O_A^{\mu\nu} \hat{D}_\rho \phi_A) + \mathcal{L}'(A, \Phi), \quad (1.6)$$

lead to (1.3) and (1.2), as well as to the same field equations as before for the matter fields $\Phi$. Observe that the Lagrangian is fully gauge invariant up to a total derivative. Hence, the Yang-Mills Lagrangian has now been converted to a Chern-Simons Lagrangian, with a different gauge group and a different scalar field content. To obtain the original Lagrangian (1.1), one simply imposes the gauge $\phi_A = 0$ and integrates out the fields $B_{A\mu}$.

This paper is organized as follows. In section 2 we briefly summarize the results of [1] for the ungauged theories. Subsequently we analyze the possible invariances of the Lagrangian possibly related to target space isometries. Then, in section 3, we discuss the gauging of possible subgroups of the isometry group. We determine the potential and the masslike terms in the general case and derive the extra conditions that must be satisfied in order to preserve supersymmetry. This is the central result of this paper. In section 4 we analyze these restrictions in detail for $N \leq 4$. For $N > 4$ the target spaces are homogeneous, and their consistent gaugings are discussed in section 5. We present some concluding remarks in section 6. Some technical details are relegated to two appendices.

## 2 Nonlinear sigma models coupled to supergravity

In this section we summarize and elaborate on the construction of three-dimensional nonlinear sigma models coupled to supergravity. For the derivation and conventions we refer to [1]. The fields of the nonlinear sigma model are scalar fields $\phi^i$ and spinor fields $\chi^i$, with $i = 1, \ldots, d$; the supergravity fields are the dreibein $e_\mu^a$, the spin-connection field $\omega^{ab}_\mu$ and $N$ gravitini fields $\psi^I_\mu$ with $I = 1, \ldots, N$. The gravitini transform under the $R$-symmetry group $\text{SO}(N)$, which is not necessarily a symmetry group of the Lagrangian. The scalar fields parametrize a target space endowed with a Riemannian metric $g_{ij}(\phi)$.

### 2.1 Target-space geometry

Pure supergravity is topological in three dimensions and exists for an arbitrary number $N$ of supercharges and corresponding gravitini [20]. Its coupling to a nonlinear sigma model requires the existence of $N - 1$ almost complex structures $f^{P_{ij}}(\phi)$, labeled by
\( P = 2, \ldots, N, \) which are hermitean,

\[ g_{ij} f^{Pj}_k + g_{kj} f^{Pj}_i = 0, \]  

(2.1)

and generate a Clifford algebra,

\[ f^{Pj}_k f^{Qj}_k + f^{Qj}_k f^{Pj}_k = -2 \delta^{PQ} \delta_j^i. \]  

(2.2)

From the \( f^P \) one constructs \( \frac{1}{2} N(N-1) \) tensors

\[ f_{ij}^{IJ} = -f_{ij}^{JI} = -f_{ji}^{IJ} \]  

that can act as generators for the group \( \text{SO}(N) \),

\[ f^{PQ} = f^{[P} f^{Q]}, \quad f^{1P} = -f^{P1} = f^P, \]  

(2.3)

where, here and henceforth, \( I, J = 1, \ldots, N \). The tensors \( f^{IJ} \) satisfy (in obvious matrix notation),

\[ f^{IJ} f^{KL} = f^{[IJ} f^{KL]} - 4 \delta^{[I[K} f^{L]\mathcal{]}J} - 2 \delta^{I[K} \delta^{L]J} 1, \]

\[ f^{IJ ij} f^{KL}_{ij} = 2d \delta^{I[K \delta^{L]J]} - \delta_{N,A} \varepsilon^{IJKL} \text{Tr}(J). \]  

(2.4)

Only for \( N = 4 \), the tensor \( J^{ij} \) is relevant; it is defined by \( J = \frac{1}{6} \varepsilon_{PQR} f^P f^Q f^R \), so that

\[ f^P f^Q = -\delta^{PQ} 1 - \varepsilon^{PQR} J f^R. \]  

(2.5)

Furthermore, \( J \) satisfies,

\[ J f^P = f^P J, \quad J^2 = 1, \quad J_{ij} = J_{ji}, \quad J = \frac{1}{24} \varepsilon^{IJKL} f^{IJ} f^{KL}, \]  

(2.6)

and has eigenvalues equal to \( \pm 1 \). It turns out that \( J \) is also covariantly constant, which implies that the target space is locally the product of two separate Riemannian spaces of dimension \( d_{\pm} \), where \( d_+ + d_- = d \) and \( d_{\pm} \) are both multiples of 4. These two spaces correspond to inequivalent \( N = 4 \) supermultiplets. Hence the case \( N = 4 \) is rather special, and the last identity (2.4) can be written as

\[ f^{IJ ij} f^{KL}_{ij} = 4 \left( d_+ \mathbb{P}_{+}^{IJ, KL} + d_- \mathbb{P}_{-}^{IJ, KL} \right), \]  

(2.7)

with projectors,

\[ \mathbb{P}_{\pm}^{IJ, KL} = \frac{1}{2} \delta^{I[K} \delta^{L]J} \mp \frac{1}{4} \varepsilon^{IJKL}. \]  

(2.8)
For rigidly supersymmetric nonlinear sigma models, the number of supersymmetries is equal to \( N = 1, 2 \) or 4, and the Lagrangians are manifestly invariant under SO\((N)\) R-symmetry transformations acting exclusively on the fermion fields through multiplication with the complex structures. The case \( N = 3 \) is not distinct from \( N = 4 \), because the existence of two complex structures necessarily implies the existence of a third one. In case of \( N = 4 \), the Lagrangian is a sum of two separate Lagrangians corresponding to the \( d_\perp \)-dimensional target spaces. For \( N = 3, 4 \) the target spaces are hyperkähler.

When coupling to supergravity the Lagrangian and supersymmetry transformations depend on SO\((N)\) target-space connections denoted by \( Q^I_J(\phi) \). These connections are nontrivial in view of

\[
R^I_J(Q) \equiv \partial_i Q^I_J - \partial_J Q^I_i + 2Q^K_i Q^J_K = \frac{1}{2} f^I_J. \tag{2.9}
\]

For local supersymmetry \( N \) can take the values \( N = 1, \ldots, 6 \) and \( 8, 9, 10, 12 \) or 16. The situation regarding SO\((N)\) symmetry is more subtle in this case, as we shall discuss in due course. The \( N = 3 \) theory is no longer equivalent to an \( N = 4 \) theory, as it has only three gravitini. In view of the three almost complex structures, the target space is a quaternionic space. For \( N = 4 \) the target space decomposes locally into a product of two quaternionic spaces of dimension \( d_\perp \). The \( f^I_J \) are covariantly constant, both with respect to the Christoffel and the SO\((N)\) connections, \( \Gamma^k_{ij} \) and \( Q^I_J \), respectively,

\[
D_i (\Gamma, Q) f^I_J \equiv \partial_i f^I_J - 2 \Gamma^I_i [f^J]_k + 2 Q^K_i [f^J]_K = 0. \tag{2.10}
\]

For \( N > 2 \) we are thus dealing with almost complex, rather than with complex, structures. This implies an integrability condition for the target-space Riemann tensor \( R_{ijkl} \),

\[
R_{ijmn} f^{IJM} - R_{ijmd} f^{IJM} = - f^I_i f^J_m f^K_k = 0, \tag{2.11}
\]

where we made use of (2.9). Contracting (2.11) with \( f^{MNkl} \) gives, for general \( N > 2 \),

\[
R_{ijkl} f^{IJK} = \frac{1}{4} d^I f^J, \tag{2.12}
\]

so that the target space has nontrivial SO\((N)\) holonomy, while contracting (2.11) with \( g^I \), using the cyclicity of the Riemann tensor and the above result, yields (for \( N > 2 \))

\[
R_{ij} \equiv R_{ikjl} g^{kl} = c g_{ij}, \tag{2.13}
\]
with $c = N - 2 + \frac{1}{8}d > 0$. Hence the target space must be an Einstein space.\footnote{For $N = 3$ this is in accord with the fact that quaternionic spaces of $d > 4$ are always Einstein \cite{21}. In the case at hand, the result also holds true for a $d = 4$ target space. For $N = 4$ the equations (2.12) and (2.13) read

$R_{ijkl} f^{f_{ij}}_{kl} = \frac{1}{2} \left( d_+ P^{IJ, KL} + d_- P^{IJ, KL}_{-} \right) f_{ij}^{KL}$, \\
$R_{ij} = (2 + \frac{1}{8}d) g_{ij} + \frac{1}{8}(d_+ - d_-) h_{ij}$, \\
and we have a product space of two quaternionic manifolds, which are both Einstein. For $N = 2$ the target space is Kähler and $f^{f_{ij}}$ is a complex structure. The SO(2) holonomy is undetermined.}

Following \cite{1} we introduce a complete set of linearly independent, antisymmetric, tensors $h^{\alpha}_{ij}$, labeled by indices $\alpha$, that commute with the complex structures, \textit{i.e.},

$$h^{\alpha}_{ik} f^{f}_{IJ, j} - h^{\alpha}_{ik} f^{f}_{IJ, j} = 0 . \quad (2.14)$$

For $N = 2$, there is only one tensor $f^{f_{ij}}$ which commutes with itself, so that this decomposition is not meaningful. For $N > 2$ we must have $h^{\alpha}_{ij} f^{f}_{ij} = 0$. The tensors $h^{\alpha}_{ij}$ generate a subgroup $H' \subset SO(d)$ that commutes with the group $SO(N)$ generated by the tensors $f^{f_{ij}}$. They can be normalized according to

$$h^{\alpha}_{ij} h^{\beta}_{jk} \propto \delta^{\alpha\beta} \text{ and are covariantly constant with respect to the Christoffel connection and a new connection } \Omega^{\alpha\beta}_{i}.$$ 

The Riemann tensor can be written as ($N > 2$)

$$R_{ijkl} f^{f}_{ij} f^{f}_{kl} = \frac{1}{8} \left( d_+ P^{IJ, KL} + d_- P^{IJ, KL}_{-} \right) f_{ij}^{KL}, \quad (2.16)$$

where $C_{\alpha\beta}(\phi)$ is a symmetric tensor. This result implies that the holonomy group is contained in $SO(N) \times H' \subset SO(d)$ which must act irreducibly on the target space. For $N = 4$ this result is modified because of the product structure. Observe that the Bianchi identities on the curvature tensor are not manifest for the expression on the right-hand side of (2.16), something that plays an important role in the analysis of \cite{1}. The $H'$ curvatures can now be shown to take the form,

$$2(\partial_{[i} \Omega^{\alpha\beta}_{j]} - \Omega^{\alpha\gamma}_{[i} \Omega^{\beta\gamma}_{j]} = \frac{1}{8} f^{f}_{\alpha\beta} C^{\gamma\delta}_{\alpha\beta} h^{\delta}_{ij} , \quad (2.17)$$

where we have made use of the structure constants of the group $H'$ defined by,

$$h^{\alpha}_{i} h^{\beta}_{j} - h^{\beta}_{i} h^{\alpha}_{j} = f^{f}_{\alpha\beta} C^{\gamma}_{\alpha\beta} h^{\gamma}_{ij}. \quad (2.18)$$

The above result shows that the connections $\Omega^{\alpha\beta}_{i}$ can be restricted to the form $\Omega^{\alpha\beta}_{i} \propto f^{f}_{\alpha\beta} Q^{\gamma}_{\alpha\beta}$.
2.2 Lagrangian and invariances

Let us now turn to the Lagrangian and supersymmetry transformations. In the following it is convenient to adopt an SO($N$) covariant notation which allows to select the $N-1$ almost complex structures from the $f^{IJ}$ tensors by specifying some arbitrary unit $N$-vector $\alpha_I$ and identifying the complex structures with $\alpha_J f^{JI}$. By extending the fermion fields $\chi^i$ to an overcomplete set $\chi^{iI}$, defined by

$$\chi^{iI} = (\chi^i, f^{P i} \chi^j),$$

we can write the Lagrangian and transformation rules in a way that does no longer depend explicitly on the almost-complex structures. The fact that we have only $d$ fermion fields, rather than $dN$, can be expressed by the SO($N$) covariant constraint,

$$\chi^{iI} = \mathbb{P}_{iJ}^{I} \chi^{J} \equiv \frac{1}{N} \left( \delta^{IJ} \delta^i_j - f^{IJi} \right) \chi^{J}.$$  (2.20)

The trace of this projector equals $\mathbb{P}^{I}_{iI} = d$, which confirms that the total number of fermion fields is not altered. We should stress here, that the introduction of $\chi^{iI}$ is a purely notational exercise and we do not aim at implementing the constraint (2.20) at the Lagrangian level. At every step in the computation one may change back to the original notation by choosing $\chi^i = \alpha_I \chi^{iI}$. The covariant notation does not imply that the theory is SO($N$) invariant, but the covariant setting allows us to treat the $N$ supersymmetries and the corresponding gravitini on equal footing and it facilitates the various derivations in later sections.

The supersymmetry transformations read

$$\delta \epsilon^a = \frac{1}{2} \epsilon^I \gamma^a \psi^I, \quad \delta \psi^I = D_\mu \epsilon^I - \frac{1}{8} g_{ij} \chi^{iI} \gamma^\mu \epsilon^j - \delta \phi^j Q^{IJi} \psi^J, \quad \delta \phi^i = \frac{1}{2} \epsilon^I \chi^{iI}, \quad \delta \chi^{iI} = \frac{1}{2} (\delta^{IJ} 1 - f^{IJ}) \tilde{\partial} \phi^j \epsilon^j - \delta \phi^j (\Gamma^i_{jk} \chi^{kI} + Q^{IJ} \chi^{iJ}),$$

where the supercovariant derivative $\tilde{\partial} \phi^j$ and the covariant derivative $D_\mu (\omega, Q) \epsilon^I$ are defined by

$$\tilde{\partial} \phi^j = \partial_\mu \phi^j - \frac{1}{2} \bar{\psi}_\mu \chi^{iI}, \quad D_\mu \epsilon^I = (\partial_\mu + \frac{1}{2} \omega^a_{\mu} \gamma^a) \epsilon^I + \partial_\mu \phi^j Q^{IJ} \epsilon^J.$$  (2.21)

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Observe that the terms proportional to $\delta \phi$ in $\delta \chi^I$ do not satisfy the same constraint as $\chi^I$ itself, because the projection operator $P_{IJ}$ itself transforms under supersymmetry. As in [1], we use the Pauli-Källén metric with hermitean gamma matrices $\gamma^a$, satisfying $\gamma^a \gamma^b = \delta^{ab} + i \varepsilon^{abc} \gamma^c$.

Let us now turn to the Lagrangian, which reads

\[ L_0 = -\frac{1}{2} i \varepsilon^{\mu \nu} \left( \epsilon_{\mu}^a R_{\nu \rho a} + \psi^I_{\mu} D_{\nu} \psi^I_{\rho} \right) - \frac{1}{2} g_{ij} \left( g^{\mu \nu} \partial_{\mu} \phi^i \partial_{\nu} \phi^j + N^{-1} \chi^{iI} D \chi^{jI} \right) \]

\[ + \frac{1}{4} g_{ij} \chi^{iI} \gamma^\mu \gamma^N \psi^I_{\mu} \left( \partial_{\nu} \phi^j + \tilde{\partial}_{\nu} \phi^j \right) - \frac{1}{24} N^{-2} R_{ijkl} \chi^{iI} \gamma^j \chi^{jI} \gamma^a \chi^{kJ} \]

\[ + \frac{1}{24} N^{-2} \left( 3 (g_{ij} \chi^{iI} \chi^{jI})^2 - 2 (N - 2) (g_{ij} \chi^{iI} \gamma^a \chi^{jI})^2 \right), \tag{2.23} \]

Here we used the covariant derivatives

\[ D_{\mu} \psi^I_{\nu} = \left( \partial_{\mu} + \frac{1}{2} \omega_{\mu}^a \gamma_a \right) \psi^I_{\nu} + \partial_{\mu} \phi^j Q^{IJ}_{\nu} \psi^J_{\nu}, \]

\[ D_{\mu} \chi^{iI} = \left( \partial_{\mu} + \frac{1}{2} \omega_{\mu}^a \gamma_a \right) \chi^{iI} + \partial_{\mu} \phi^j \left( \Gamma_{ijk} \chi^{kI} + Q^{IJ}_{\mu} \chi^{iJ} \right). \tag{2.24} \]

We emphasize that the above results coincide with the results of [1], written in a different form. The conversion makes use of (2.11). The Lagrangian and transformation rules are consistent with target-space diffeomorphisms and field-dependent SO($N$) R-symmetry rotations, acting on $\psi^I_{\mu}$, $\chi^{iI}$ and $Q^{IJ}$. According to

\[ \delta \psi^I_{\mu} = \Lambda^{IJ}_{\mu}(\phi) \psi^J_{\nu}, \quad \delta \chi^{iI} = \Lambda^{IJ}_{iI}(\phi) \chi^{jI}, \quad \delta Q^{IJ}_{\mu} = -D_i \Lambda^{IJ}_{\mu}(\phi). \tag{2.25} \]

Combining the last result with (2.23), one concludes that the $f^{IJ}$ should also be rotated,

\[ \delta f^{IJ} = 2 \Lambda^{IK} \left( f^{JK} \right). \tag{2.26} \]

The target-space diffeomorphisms and field-dependent SO($N$) R-symmetry rotations correspond to reparametrizations within certain equivalence classes, but do not, in general, constitute an invariance.

In the remainder of this section we discuss the invariances of these models, other than supersymmetry, spacetime diffeomorphisms and local Lorentz transformations. The target space may have isometries, generated by Killing vector fields $X^i(\phi)$. Some of them can be extended to invariances of the full Lagrangian, possibly after including a field-dependent SO($N$) transformation according to (2.25) and (2.26). Hence, we combine an isometry characterized by a Killing vector field $X^i$ with a special SO($N$) transformation whose parameters depend on $X^i(\phi)$ and on the scalar fields. Denoting the infinitesimal SO($N$) transformations by $S^{IJ}(X, \phi)$, we require invariance of the target-space metric, the
SO($N$) connections and the almost complex structures, up to a uniform SO($N$) rotation, i.e.,

$$\mathcal{L}_X g_{ij} = 0,$$
$$\mathcal{L}_X Q_i^{IJ} + D_i S^{IJ}(\phi, X) = 0,$$
$$\mathcal{L}_X f_{ij}^{IJ} - 2 S^{K[I}(\phi, X) f_{ij}^{J]K} = 0.$$  \hfill (2.27)

The Lagrangian (2.23) is then invariant under the combined transformations,

$$\delta \phi^i = X^i(\phi), \quad \delta \psi^I_\mu = S^{IJ}(\phi, X) \psi^J_\mu, \quad \delta \chi^i_I = \chi^j_I \partial_j X^i + S^{IJ}(\phi, X) \chi^i_J.$$  \hfill (2.28)

The fermion transformations can be rewritten covariantly,

$$\delta \psi^I_\mu = V^{IJ}(\phi, X) \psi^J_\mu - \delta \phi^i Q_i^{IJ} \psi^J_\mu,$$
$$\delta \chi^i_I = D_j X^i \chi^j_I + V^{IJ}(\phi, X) \chi^{ij} - \delta \phi^j (\Gamma^i_{jk} \chi^{kj} + Q_j^{IJ} \chi^j_I),$$  \hfill (2.29)

where $V^{IJ}(\phi, X) \equiv X^i Q_i^{IJ}(\phi) + S^{IJ}(\phi, X)$. The significance of this result will become apparent in a sequel. Using (2.9) and (2.10), one verifies that the second and third equation of (2.27) can be written as,

$$D_i V^{IJ}(\phi, X) = \frac{1}{2} f_{ij}^{IJ}(\phi) X^j(\phi),$$
$$f^{IK}[i(\phi) D_j X_k(\phi) = f_{ij}^{K[I}(\phi) V^{J]K}(\phi, X).$$  \hfill (2.30)

The first equation in (2.30) shows that $V^{IJ}(\phi, X)$ can be regarded as the moment map associated with the isometry $X^i$. The second equation is just the integrability condition of the first equation, so it is not independent. After contraction with $f^{MN} ij$, it leads to

$$f^{IJ} ij D_i X_j = \left\{ \begin{array}{ll}
\frac{1}{2} d V^{IJ}, & \text{for } N \neq 2, 4 \\
(d_+ P_+^{IJKL} + d_- P_-^{IJKL}) V^{KL}, & \text{for } N = 4
\end{array} \right.$$  \hfill (2.31)

From combining the above equations it follows that $\Delta V^{IJ} = \frac{1}{4} d V^{IJ}$, where $\Delta$ denotes the SO($N$) covariant Laplacian. This result applies to $N > 2$, with obvious modifications for $N = 4$. The above analysis shows that there are no restrictions for $N > 2$ to extend an isometry to a symmetry of the Lagrangian. For $N = 2$ this is different, as the isometry should be holomorphic, i.e., it should leave the complex structure invariant. In this case $V^{IJ}$ is determined by (2.30) up to an integration constant. This constant is related to an invariance under constant SO(2) transformations of the fermions.

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For $N = 4$ we note that the complex structures $P_{\pm \pm}^{IJKL} f^{KL}$ live in the two separate quaternionic subspaces. The same is true for $P_{\pm}^{IJKL} V^{KL}$, which according to (2.30) depends only on the corresponding subspace coordinates. Note, however, that when one of the subspaces is trivial, say when $d_- = 0$, then $P_{\pm}^{IJKL} V^{KL}$ corresponds to a triplet of arbitrary constants. This is a consequence of the fact that the model in this case has a rigid SO(3) invariance which acts exclusively on the fermions.

The supersymmetry transformations do not commute with the isometries, as one can verify most easily on the fields $\phi^i$, where one derives

$$\left[ \delta_Q(\epsilon), \delta_G(X) \right] = \delta_Q(\epsilon') , \quad \text{with} \quad (\epsilon')^I = S^{IJ}(\phi, X) \epsilon^J . \quad (2.32)$$

The isometries that can be extended to an invariance of the Lagrangian, generate an algebra $g$. Denoting $\{X^M\}$ as a basis of generators, we have

$$X^M \partial_i X^N - X^N \partial_i X^M = f^{MN} \kappa X^\kappa , \quad (2.33)$$

with structure constants $f^{MN} \kappa$. Closure of the algebra implies that the corresponding induced SO($N$) rotations, $S^{MI} \equiv S^{IJ}(\phi, X^M)$, satisfy,

$$[S^M, S^N]^{IJ} = -f^{MN} \kappa S^{K}{}^{IJ} + (X^M \partial_i S^N{}^{IJ} - X^N \partial_i S^{M}{}^{IJ}) . \quad (2.34)$$

Using (2.33) and the second equation (2.27), this can be rewritten as

$$[\mathcal{V}^M, \mathcal{V}^N]^{IJ} = -f^{MN} \kappa \mathcal{V}^K{}^{IJ} + \frac{1}{2} f_{ij}^{IJ} X^M X^N , \quad (2.35)$$

with $\mathcal{V}^{MI} \equiv V^{IJ}(\phi, X^M)$. In the case of $N = 2$ and of $N = 4$ with $d_+d_- = 0$, the R-symmetry, which acts only on the fermions, is realized as a separate invariance. Obviously, R-symmetry commutes with the isometry group; for $N = 2$ the R-symmetry may define a central extension of the isometry group, while for $N = 4$ with $d_+d_- = 0$, the invariance group takes the form of a direct product of the isometry group with an SO(3) factor of the R-symmetry group. This implies that there are generators for which the Killing vector $X^M$ vanishes and $S^{MI}$ is constant. We return to this in our discussion of the specific cases in section 4.

We now note that the second equation of (2.30) implies that $D_i X_j - \frac{1}{4} f_{ij}^{MN} \mathcal{V}^{MN}$ commutes with the almost complex structures, so that it can be decomposed in terms of the antisymmetric tensors $h_{ij}^\alpha$ that were introduced in (2.14),

$$D_i X_j^M - \frac{1}{4} f_{ij}^{IJ} \mathcal{V}^{MI} \equiv h_{ij}^\alpha \mathcal{V}^M_\alpha . \quad (2.36)$$
This result and the results given in the remainder of this section apply only to \( N > 2 \).

Using the general result for Killing vectors, \( D_i D_j X_k = R_{jkl} X^l \), we can evaluate the derivative of \( \mathcal{V}^M \alpha \). Introducing furthermore the notation \( \mathcal{V}^M i \equiv X^M i \), we establish the following system of linear differential equations,

\[
\begin{align*}
D_i \mathcal{V}^M I J &= \frac{1}{2} f^{I J i} \mathcal{V}^M j, \\
D_i \mathcal{V}^M j &= \frac{1}{4} f^{I J i} \mathcal{V}^M I J + h_{i j}^{\alpha} \mathcal{V}^M \alpha, \\
D_i \mathcal{V}^M \alpha &= \frac{1}{8} C_{\alpha \beta} h_{i j}^{\beta} \mathcal{V}^M j, \\
\end{align*}
\]

(2.37)

where the covariant derivative contains the Christoffel connection as well as the \( \text{SO}(N) \times H' \) connections. One can prove that \( C_{\alpha \beta} \) is covariant under the isometry, i.e., \( \mathcal{V}^M i D_i C_{\alpha \beta} = 2 \mathcal{V}^M \gamma f^{\beta \gamma (\alpha} C_{\beta) \gamma} \). Other than that the integrability of the above equations is guaranteed by previous results. Furthermore, by substituting the second identity of (2.37) into (2.33), and by taking the derivative of (2.33) and exploiting previous identities, we derive the following two equations,

\[
\begin{align*}
f^M N K \mathcal{V}^K i &= \frac{1}{4} f^{I J i} (\mathcal{V}^M I J \mathcal{V}^N j - \mathcal{V}^N I J \mathcal{V}^M j) + h_{i j}^{\alpha} (\mathcal{V}^M \alpha \mathcal{V}^N j - \mathcal{V}^N \alpha \mathcal{V}^M j), \\
f^M N K \mathcal{V}^K \alpha &= f^{\beta \gamma} \alpha \mathcal{V}^M \beta \mathcal{V}^N \gamma + \frac{1}{8} C_{\alpha \beta} h_{i j}^{\beta} \mathcal{V}^M i \mathcal{V}^N j. \\
\end{align*}
\]

(2.38)

The quantities \( \mathcal{V}^M I J, \mathcal{V}^M i \) and \( \mathcal{V}^M \alpha \) transform according to the adjoint representation of the invariance group, up to field-dependent \( \text{SO}(N) \times H' \) transformations, as is shown by (note that for \( \mathcal{V}^M i \) this already follows from (2.33)),

\[
\begin{align*}
\mathcal{V}^N \alpha D_i \mathcal{V}^M I J &= -f^M N K \mathcal{V}^K \alpha + [\mathcal{V}^N, \mathcal{V}^M] I J, \\
\mathcal{V}^N \alpha D_i \mathcal{V}^M \alpha &= -f^M N K \mathcal{V}^K \alpha + f^{\beta \gamma} \alpha \mathcal{V}^N \gamma \mathcal{V}^M \beta. \\
\end{align*}
\]

(2.39)

We close this section with a few observations regarding the structure of the last equations (2.38) and (2.39). Let us first note that the following operators

\[
\mathcal{D}^M = \delta^i_j \mathcal{V}^M k D_k + \frac{1}{4} f^{I J i j} \mathcal{V}^M I J + h^{\alpha i j} \mathcal{V}^M \alpha, \\
\]

(2.40)

acting in the space of target-space tensors, provide a realization of the Lie algebra \( \mathfrak{g} \) associated with the invariance group, according to

\[
[\mathcal{D}^M, \mathcal{D}^N] = f^M N K \mathcal{D}^K. \\
\]

(2.41)
The equations (2.38) can also be encoded in the following algebraic structure. Define
the algebra $a \equiv \{t^A\} \equiv \{t^{IJ}, t^\alpha, t^i\}$, as an extension of $\mathfrak{so}(N) \oplus \mathfrak{h}'$ with commutation relations,

$$[t^{IJ}, t^{KL}] = -4 \delta^{I[K} t^{L]J}, \quad [t^\alpha, t^\beta] = f^\alpha\beta_\gamma t^\gamma, \quad [t^{IJ}, t^i] = \frac{1}{2} f^{IJ}_{\ \ j} i^j,$$

$$[t^i, t^j] = \frac{1}{4} f_{ij}^{\ \ \ K} t^{IJ} + \frac{1}{8} C_{\alpha\beta} h^{ij} t^\alpha. \quad (2.42)$$

Unless $C_{\alpha\beta}$ is an $H'$-invariant tensor, this algebra will be nonassociative (or may alternatively be realized as a soft associative algebra upon imposing $[t^\alpha, C_{\beta\gamma}] = -2 f_{\alpha(\beta C_{\gamma)}}. Equations (2.35) and (2.38) then imply that the map,

$$\mathcal{V} : \mathfrak{g} \rightarrow a, \quad \mathcal{V}(X^M) := \mathcal{V}^{\alpha}_{\ \ \ A} t^A = \frac{1}{2} \mathcal{V}^{\alpha}_{\ \ IJ} t^{IJ} + \mathcal{V}^{\alpha}_{\ \ \ i} t^i + \mathcal{V}^{\alpha}_{\ \ \ i} t^i , \quad (2.43)$$

defines a Lie algebra homomorphism, i.e. $\mathcal{V}([X^M, X^N]) = [\mathcal{V}(X^M), \mathcal{V}(X^N)]$. In particular, the image of $\mathfrak{g}$ under $\mathcal{V}$ is an associative subalgebra of $a$. Furthermore, (2.37) takes the simple form,

$$D_i \mathcal{V}(X^M) = [g_{ij} t^j, \mathcal{V}(X^M)]. \quad (2.44)$$

When the inverse $C^{\alpha\beta}$ exists, one can prove that

$$\mathcal{V}^{\alpha}_{\ \ IJ} \mathcal{V}^{\beta}_{\ \ \ i} + \mathcal{V}^{\alpha}_{\ \ \ i} \mathcal{V}^{\beta}_{\ \ \ i} - 8 C^{\alpha\beta} \mathcal{V}^{\alpha}_{\ \ i} \mathcal{V}^{\beta}_{\ \ j} , \quad (2.45)$$

equals a constant.

3 Gauged isometries

In this section we elevate a subgroup of the isometries to a local symmetry by making the parameters spacetime dependent. With increasing $N$, supersymmetry then poses severe constraints on the possible choices of gauge groups.

3.1 Gauge group and embedding tensor

A subgroup of isometries (possibly extended with R-symmetry transformations for $N = 2, 4$) can be encoded in an embedding tensor $\Theta_{MN}$ which defines the Killing vectors that generate the gauge group by

$$X^i = g \Theta_{MN} \Lambda^M(x) X^N i , \quad (3.1)$$
with spacetime dependent parameters \( \Lambda^{\mathcal{N}}(x) \) and a gauge coupling constant \( g \). Unless the gauge group coincides with the full group of isometries, the embedding tensor acts as a projector which reduces the number of independent parameters. In order that this subset of Killing vectors generates a group, \( \Theta_{\mathcal{M}\mathcal{N}} \) must satisfy the following condition,

\[
\Theta_{\mathcal{M}\mathcal{P}} \Theta_{\mathcal{N}\mathcal{Q}} f_{\mathcal{R}}^{\mathcal{P}\mathcal{Q}} = \hat{\Theta}_{\mathcal{M}\mathcal{N}} f_{\mathcal{P}\mathcal{R}},
\]

for certain constants \( \hat{\Theta}_{\mathcal{M}\mathcal{N}} \), which are subsequently identified as the structure constants of the gauge group. One can verify that the validity of the Jacobi identity for the gauge group structure constants follows directly from the Jacobi identity associated with the full group of isometries. For a semi-simple gauge group the embedding tensor is simply the sum of the Cartan-Killing forms of the different group factors with different and a priori unrelated coupling constants. The embedding tensors of non-semisimple gauge groups may take more complicated forms \[22\].

The next step is to introduce gauge fields \( A_{\mu}^{\mathcal{M}} \) corresponding to the gauge group parameters \( \Lambda^{\mathcal{M}}(x) \) and include them into the definition of the covariant derivatives. For example, we have

\[
D_{\mu} \phi^i = \partial_{\mu} \phi^i + g \Theta_{\mathcal{M}\mathcal{N}} A_{\mu}^{\mathcal{M}} X^{\mathcal{N}i},
\]

which transforms under local isometries according to

\[
D_{\mu} \phi^i \rightarrow D_{\mu} \phi^i + g \Theta_{\mathcal{M}\mathcal{N}} \Lambda^{\mathcal{M}} \partial_{j} X^{\mathcal{N}i} D_{\mu} \phi^j,
\]

provided we assume the gauge fields transformations,

\[
\Theta_{\mathcal{M}\mathcal{N}} \delta A_{\mu}^{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}} \left( -\partial_{\mu} \Lambda^{\mathcal{M}} + g \hat{\Theta}_{\mathcal{P}\mathcal{Q}}^{\mathcal{M}} A_{\mu}^{\mathcal{P}} \Lambda^{\mathcal{Q}} \right).
\]

The covariant field strengths follow from the commutator of two covariant derivatives, e.g.,

\[
[D_{\mu}, D_{\nu}] \phi^i = g \Theta_{\mathcal{M}\mathcal{N}} F_{\mu\nu}^{\mathcal{M}} X^{\mathcal{N}i},
\]

and take the form

\[
\Theta_{\mathcal{M}\mathcal{N}} F_{\mu\nu}^{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}} \left( \partial_{\mu} A_{\nu}^{\mathcal{M}} - \partial_{\nu} A_{\mu}^{\mathcal{M}} - g \hat{\Theta}_{\mathcal{P}\mathcal{Q}}^{\mathcal{M}} A_{\mu}^{\mathcal{P}} A_{\nu}^{\mathcal{Q}} \right).
\]
The gauge transformations on the fermion fields follow from (2.29) upon substitution of (3.1). From this we derive the covariant derivatives for the spinor fields,

\[
\mathcal{D}_\mu \psi^I = (\partial_\mu + \frac{1}{2} \omega^a_\mu \gamma_a) \psi^I + \partial_\mu \phi^i Q^I_i \psi^J + g \Theta_{MN} A^M_\mu \gamma^N \psi^J,
\]

\[
\mathcal{D}_\mu \chi^{iI} = (\partial_\mu + \frac{1}{2} \omega^a_\mu \gamma_a) \chi^{iI} + \partial_\mu \phi^j (\Gamma^i_{jk} \chi^{kI} + Q^I_j \chi^{iJ}) + g \Theta_{MN} A^M_\mu (\delta^i_j \gamma^N \psi^J - \delta^i_j g^{ik} D_k \gamma^N \chi^j).
\]

In view of the commutator (2.32), the covariant derivative on the supersymmetry parameter acquires also an additional covariantization,

\[
\mathcal{D}_\mu \epsilon^I = (\partial_\mu + \frac{1}{2} \omega^a_\mu \gamma_a) \epsilon^I + \partial_\mu \phi^i Q^I_i \epsilon^j + g \Theta_{MN} A^M_\mu \gamma^N \epsilon^J.
\]

In this section we only make use of the previous results (2.30) that apply to arbitrary \( N > 0 \).

The extra minimal couplings (3.3) induce modifications of the supersymmetry variations and of the Lagrangian. As long as we are dealing with first derivatives, these new couplings do not lead to complications as they are controlled by gauge covariance. However, commutators of the new covariant derivatives lead to new (covariant) terms proportional to the field strength (3.7). These terms, which cause a violation of supersymmetry, take the form,

\[
\delta \mathcal{L} = -\frac{1}{2} i g \Theta_{MN} F^N_{\mu \nu} \epsilon^{\mu \nu} \left( \gamma^N \psi^J_\mu \epsilon^J + \frac{1}{2} \gamma^N \chi^{iI} \gamma^I \gamma^I \epsilon^I \right).
\]

They are cancelled by introducing a Chern-Simons term for the vector fields,

\[
\mathcal{L}_{CS} = \frac{1}{4} i g \epsilon^{\mu \nu \rho} A^M_\mu \Theta_{MN} \left( \partial_\nu A^N_\rho - \frac{1}{3} g \hat{\phi}_{\rho S} A^S_\rho A^Q_\rho \right),
\]

provided the embedding tensor \( \Theta_{MN} \) is symmetric and provided we assume the following supersymmetry transformations,

\[
\Theta_{MN} \delta A^M_\mu = \Theta_{MN} \left[ 2 \gamma^N \psi^J_\mu \epsilon^J + \gamma^M \chi^{iI} \gamma^I \epsilon^I \right].
\]

At this point one can also derive the field equation for the vector fields, which reads,

\[
\Theta_{MN} \left[ \hat{F}^M_{\mu \nu} + 2 i e \epsilon_{\mu \nu \rho} \hat{D}^\rho \phi^i \gamma^i \chi^{iI} + \frac{1}{12} \chi^{iI} \gamma^i \chi^{iJ} (g_{ij} \gamma^M \chi^J - \delta^i_J D_i \gamma^M \chi^J) \right] = 0,
\]

where \( \hat{F}^M_{\mu \nu} \) denotes the supercovariant curvature.
The embedding tensor is a gauge group invariant element of $\text{Sym}(g \otimes g)$ and therefore satisfies $\hat{f}_{MN}^Q \Theta_{QP} + \hat{f}_{NP}^Q \Theta_{QN} = 0$, which implies

$$\Theta_{PL} (f^{KL} M \Theta_{N K} + f^{KL} N \Theta_{MK}) = 0.$$  

(3.13)

Consequently, the structure constants of the gauge group can be expressed as $\hat{f}_{MN}^P = \Theta_{MQ} f_{PN}^Q$, as they satisfy (3.2) by virtue of (3.13).

We define the so-called $T$-tensor (originally introduced in higher-dimensional supergravity [23]) as the image of $\Theta$ under the map $\Psi$ from (2.43), i.e.

$$T_{ij}^{KL} \equiv \Psi^M_{ij} (f_{KL}^M \Theta_{MN}) \Psi_{N}^{KL}, \quad T_{ij}^{IJ} \equiv \Psi^M_{ij} (f_{IJ}^M \Theta_{MN}) \Psi_{N}^{IJ}, \quad T_{ij} \equiv \Psi^M_{ij} (f_{i}^M \Theta_{MN}) \Psi_{N}^{ij}, \quad T_{\alpha i} \equiv \Psi^M_{\alpha i} (f_{i}^M \Theta_{MN}) \Psi_{N}^{i}, \quad T_{\alpha \beta} \equiv \Psi^M_{\alpha \beta} (f_{i}^M \Theta_{MN}) \Psi_{N}^{i}, \quad T_{IJ \alpha} \equiv \Psi^M_{IJ \alpha} (f_{i}^M \Theta_{MN}) \Psi_{N}^{i}.$$  

(3.14)

The $T$-tensor components that carry indices $\alpha, \beta$ do not play an important role, as they do not appear directly in the Lagrangian and transformation rules. For the case of $N = 2$ these components are not defined. From (3.13), (2.39) and (2.33), it readily follows that the $T$-tensor transforms covariantly under the gauged isometries. Furthermore, we note the following identities,

$$D_{(i} T_{jk)} = 0,$$

$$D_{i} T_{Ij}^{JK} = \frac{1}{2} T_{k(i} f_{j)k}^{IJK},$$

$$D_{i} T_{IJKL}^{IJ} = \frac{1}{2} f_{ij}^{IJKL} T_{KL}^{IJK} + \frac{1}{2} f_{ij}^{IJKL} T_{KL}^{IJK}.$$  

(3.15)

The covariance under the gauged isometries also allows the derivation of identities quadratic in the $T$-tensors. Two examples of such identities are,

$$T^{MN}_{\phantom{MN}ij} T^{KL}_{\phantom{KL}ij} f_{ij}^{IJ} + T^{MN}_{\phantom{MN}ij} T^{IJ}_{\phantom{IJ}ij} f_{ij}^{KL} = 4 T^{MN, P[ I} T^{J]} P KL + 4 T^{MN, P[ K} T^{L]} P I J,$$

$$T^{ki}_{\phantom{ki}ij} T^{KL}_{\phantom{KL}ij} f_{ij}^{IJKL} + T^{ki}_{\phantom{ki}ij} T^{IJKL} f_{ij}^{IJ} = 4 T^{P[ I} T^{J]} P KL + 4 T^{P[ K} T^{K} T^{L]} P I J.$$  

(3.16)

### 3.2 Constraints from supersymmetry

The supersymmetry variations of the vector fields in (3.3), (3.8) cause additional supersymmetry variations of order $g$. The variations linear in the spinor fields are

$$\delta \mathcal{L} = -e g \Theta_{MN} \left(2 \Psi^M_{ij} \bar{\chi}^I_{i} \bar{\epsilon}^J + \Psi^M_{ij} \bar{\chi}^I_{i} \epsilon^I \right) \Psi^N_{j} D^\mu \phi^\mu.$$  

(3.17)
They should be cancelled by introducing mass-like terms,

\[ \mathcal{L}_g = e g \left\{ \frac{1}{2} A_{1}^{IJ} \bar{\psi}_{I}^{I} \gamma_{\mu} \psi_{J}^{J} \gamma_{\mu} + A_{2}^{IJ} \bar{\psi}_{I}^{I} \gamma_{\mu} j_{\mu}^{J} + \frac{1}{2} A_{3}^{IJ} \bar{\psi}_{I}^{I} \chi_{\mu}^{J} \right\}, \] (3.18)

accompanied by additional modifications of the supersymmetry transformation rules

\[ \delta_{g} \psi_{I}^{I} = g A_{1}^{IJ} \gamma_{\mu} \epsilon^{J}, \quad \delta_{g} \chi_{I}^{I} = -g N A_{2}^{ij} \epsilon^{J}. \] (3.19)

Obviously the tensors \( A_{1} \) and \( A_{3} \) are symmetric, \( A_{1}^{II} = A_{1}^{IJ} \), \( A_{3}^{II} = A_{3}^{ij} \). Furthermore, in view of (2.20), \( A_{2} \) and \( A_{3} \) are subject to the constraints,

\[ \mathbb{P}_{I}^{i} A_{2}^{ij} = A_{2}^{ji}, \quad \mathbb{P}_{I}^{i} A_{3}^{ik} = A_{3}^{ik}. \] (3.20)

The variations of (3.18) and the additional variations (3.19) of the original Lagrangian together cancel the terms (3.17), provided that \( A_{2}^{ij} \) and \( A_{3}^{ij} \) take the following form,

\[
\begin{align*}
A_{2}^{ij} &= \frac{1}{N} \left\{ D_{i} A_{1}^{IJ} + 2 T_{i}^{IJ} \right\}, \\
A_{3}^{ij} &= \frac{1}{N^{2}} \left\{ -2 D_{(i} D_{j)} A_{1}^{IJ} + g_{ij} A_{1}^{IJ} + A_{1}^{K[i} f_{j]K}^{J} \\
&\quad + 2 T_{ij} \delta^{IJ} - 4 D_{(i} T_{j)I} - 2 T_{ij} f_{K[i}^{J]}. \right\}
\end{align*}
\] (3.21)

Here \( A_{3} \) has the required symmetry structure. For the convenience of the reader we also give the (dependent) result,

\[ D_{i} A_{2}^{ij} = \frac{1}{2} g_{ik} A_{1}^{iK} \mathbb{P}_{j}^{K} - \frac{1}{2} N A_{3}^{ij} + T_{ik} \mathbb{P}_{j}^{K}. \] (3.22)

In addition, we need to ensure that both \( A_{2} \) and \( A_{3} \) as defined in (3.21) satisfy the projection constraints (3.20). In view of (3.22), it is sufficient to impose this constraint on \( A_{2} \), which implies the following two equations,

\[
\begin{align*}
f^{K(i}_{j} D_{j} A_{1}^{iK} + (N - 1) D_{i} A_{1}^{IJ} + 2 D_{i} T^{IK,JK} &= 0, \\
2 f^{K(i}_{j} (D_{j} A_{1}^{iK} + 2 T^{J]K}) - 2 (N - 1) T^{IJ}_{i} &= 0.
\end{align*}
\] (3.23)

These have several consequences. By iterating the first equation (i.e. by resubstituting the result for \( D_{i} A_{1} \)), we derive

\[
\begin{align*}
f^{K(i}_{j} D_{j} (4 T^{J]L,KL} + (N - 2) A_{1}^{iK}) - D_{i} (4 T^{J]L,JL} + (N - 2) A_{1}^{iJ}) &= f^{IJ}_{i} D_{j} A_{1}^{iK} + 2 T^{J]K}_{j} - 2 T^{iJ}_{i} A_{1}^{iK}.
\end{align*}
\] (3.24)
This result can be combined with the second equation (3.23) to eliminate $A_{1}^{J}$ and to find a linear constraint for the components $T^{I,J}_{i}$ of the $T$-tensor

$$(N - 4) T^{I,J}_{i} + 2 f^{K[I}_{ij} T^{J]K}_{j} - \frac{1}{N - 1} (f^{I,J}_{ij} f^{K,L}_{ij}) T^{K,L}_{ij} = 0.$$  (3.25)

Applying this constraint to the combination $f^{I,J} T^{K,L} + f^{K,L} T^{I,J}$, the resulting equation may be integrated to

$$(N - 2) (T^{I,J,K,L} - T^{[I,J,K,L]} - 4 \delta^{I[K}_{L]T^{M]J,M}} + \frac{2}{N - 1} \delta^{I[K}_{L}\delta^{L]J} T^{M,N,M,N} = 0.$$  (3.26)

A priori, this equation holds up to a covariantly constant term. Because of (2.9), covariantly constant terms cannot exist, unless they are SO($N$) invariant and therefore constant. However, the above equation does not contain a singlet contribution so that it is in fact exact for any $N$.

Vice versa, one can show that the covariant derivative of (3.26) implies (3.25), such that these two equations are in fact equivalent. It is not straightforward to disentangle various independent equations, due to the nontrivial properties of the $1/2 N(N - 1)$ almost-complex structures $f^{I,J}$. For example, the following equation is not independent,

$$(N - 8) D_{i}T^{[I,J,K,L]} - \frac{1}{N - 1} (f^{I,J}_{ij} f^{K,L}_{ij}) D_{j}T^{M,N,M,N} + 5 (f^{M,I}_{ij} f^{J,K}_{ij}) T^{L,M,J} = 0.$$  (3.27)

One can systematize this analysis by employing a set of SO($N$) projection operators, as we briefly sketch in appendix A.

Now we use (3.25) to rewrite the $f^{K,I} D T^{I,L,K,L}$ term in (3.24). Combining the result with (3.23) to remove the $f^{K,I} DA_{1,I}$ terms, we may integrate the resulting equation to obtain

$$4 T^{I,L,J,L} + (N - 2) A_{1}^{I,J} - \frac{2}{N - 1} T^{M,N,M,N} \delta^{I,J} = \mu (N - 2) \delta^{I,J},$$  (3.28)

with an as yet undetermined constant $\mu$. Putting things together, we have shown that supersymmetry at linear order in $g$ determines the tensors $A_{1}, A_{2}, A_{3}$ according to (3.21), (3.28) in terms of the $T$-tensor (3.14) while the latter satisfies the (equivalent) constraints (3.25), (3.26).

Before proceeding to the remaining terms in the action and transformation rules, we take a brief look at the supersymmetry algebra. The supersymmetry commutator leads to a covariantized translation, and a supersymmetry and Lorentz transformation with
parameters proportional to $\chi^2$. When switching on the gauge coupling, there is an extra Lorentz transformation, but more importantly, also a local isometry with parameter given by

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_G (2 \mathcal{V}^{ij} \epsilon^i_1 \epsilon^j_2) + \cdots. \quad (3.29)$$

The supersymmetry established so far guarantees the closure of the algebra to that order, except for the gauge fields which appear multiplied by a coupling constant. Their closure (up to the field equations (3.12)) implies another constraint, namely,

$$\Theta_{MN}(2 \mathcal{V}^N_{\mathcal{K}i} A^M_i)^K + \mathcal{V}^{N\mathcal{I}} D_i A^{ij} = 0. \quad (3.30)$$

It implies that the function $A_1$ is gauge covariant; in particular, its trace is invariant, i.e.

$$\Theta_{MN} V_M i D_i A_{II} = 0.$$

This is in agreement with equation (3.28) since we have already proven that the $T$-tensors are gauge covariant. Moreover, equations (3.21) show that the tensors $A_2$ and $A_3$ are covariant as well, because they depend on the $T$-tensors and $A_1$ and covariant derivatives thereof. Again we can derive certain identities from (3.30) that involve some of the $T$-tensors and $A_1$, such as

$$T^{ij} D_i A_1^{KL} + 2 T^{i,j,\mathcal{M}(K} A^L_{i} \mathcal{M)} = 0,$$

$$T^{ij} D_j A_1^{KL} + 2 T^{M(K} A^L_{i} \mathcal{M)} = 0. \quad (3.31)$$

In order to preserve supersymmetry to order $g^2$ one determines the corresponding variations linear in $\psi^I_\mu$ and $\chi^{ij}$. They reveal the need for a (gauge invariant) scalar potential in the Lagrangian,

$$\mathcal{L}_{g^2} = -e V \equiv \frac{4 \epsilon g^2}{N} (A_1^{IJ} A_1^{IJ} - \frac{1}{2} N g^{ij} A_2^{IJ} A_2^{IJ})$$

$$= \frac{4 \epsilon g^2}{N^2} (N A_1^{IJ} A_1^{IJ} - \frac{1}{2} g^{ij} D_i A_1^{IJ} D_j A_1^{IJ} - 2 g^{ij} T^{i}_i T^{j}_j). \quad (3.32)$$

We note that the variation of the scalar potential is given by

$$\partial_i \mathcal{L}_{g^2} = e g^2 \left(3 A_1^{IJ} A_2^{IJ} + N A_3^{IJ} A_2^{IJ} \right), \quad (3.33)$$

by virtue of (3.21), (3.22) and (3.30). In order that all supersymmetry variations of the potential cancel, the following two quadratic equations must be satisfied,

$$2 A_1^{IK} A_1^{KJ} - N A_2^{IK} A_2^{KJ} = \frac{1}{N} \delta^{IJ} \left(2 A_1^{KL} A_1^{KL} - N A_2^{KiL} A_2^{KiL} \right),$$

$$3 A_1^{IK} A_2^{KJ} + N g^{kl} A_2^{IK} A_3^{KJ} = \mathcal{P}^{ij} \left(3 A_1^{KL} A_2^{KL} + N g^{kl} A_2^{KL} A_3^{KL} \right). \quad (3.34)$$
It can be shown after some computation that these relations are a direct consequence of \((3.31)\) upon using \((3.21)\) and \((3.28)\) and require the integration constant \(\mu\) in the latter equation to vanish.

What remains is to analyze the supersymmetry variations cubic in the fermion fields. Their cancellation depends almost entirely on the results presented so far. E.g. supersymmetry variations proportional to \(\psi^3\) cancel provided that

\[
\delta \left[ K \right]_{A_1}^J = -T^{IJ,KL} + T^{[IJ,KL]}, \tag{3.35}
\]

which is in agreement with equation \((3.28)\). The variations that are proportional to \(\chi \psi^2\) cancel by virtue of \((3.21)\). We have not verified the cancellation of the supersymmetry variations proportional to \(\chi^2 \psi\) and \(\chi^3\), but we expect that all these terms vanish by means of the constraints derived so far. In the case of the maximal \(N = 16\) theory this has been verified explicitly \[3\].

The central result of this paper, is that a gauge group \(G_0\) with a gauge invariant embedding tensor \(\Theta_{MN}\) describing the minimal couplings according to \((3.3)\), is consistent with supersymmetry if and only if the associated \(T\)-tensor \((3.14)\) satisfies the constraint,

\[
T^{IJ,KL} = T^{[IJ,KL]} - \frac{4}{N-2} \delta \left[ K \right]_{L}^{[M,MJ]} - \frac{2 \delta \left[ K \right]_{L}^{[J]} \delta \left[ L \right]^{[J]}}{(N-1)(N-2)} T^{MN,MN}. \tag{3.36}
\]

From this constraint all further consistency conditions can be derived. The Lagrangian is modified by a Chern-Simons term \((3.10)\), mass-like fermionic terms \((3.18)\) and a scalar potential \((3.32)\) and the fermions have additional supersymmetry variations \((3.19)\). Note that the constraint \((3.36)\) is well-defined even for \(N = 1\) and \(N = 2\), but degenerates into an identity. The consistency constraint \((3.36)\) has a simple group theoretical meaning in \(SO(N)\): denoting the irreducible parts of \(T^{IJ,KL}\) under \(SO(N)\) by

\[
\begin{array}{c}
\ \ \ 1 \times \text{sym} \ \ = \ 1 + \begin{array}{c}
\ 1 \ \ 2 \ \ N-1 \\
0 \ \ 1 \ \ 2 \\
1 \ \ 2 \ \ N-1
\end{array} + \begin{array}{c}
\ 1 \ \ 2 \ \ N-1 \\
0 \ \ 1 \ \ 2 \\
1 \ \ 2 \ \ N-1
\end{array} + \begin{array}{c}
\ 1 \ \ 2 \ \ N-1 \\
0 \ \ 1 \ \ 2 \\
1 \ \ 2 \ \ N-1
\end{array},
\end{array} \tag{3.37}
\]

with each box representing a vector representation of \(SO(N)\),\(^2\) equation \((3.36)\) expresses that

\[
\begin{array}{c}
\ \ \ 1 \times \text{sym} \ \ = \ 0 .
\end{array} \tag{3.38}
\]

\(^2\)We use the standard Young tableaux for the orthogonal groups; \(i.e.\) the four representations in the decomposition \((3.37)\) are of dimension 1, \(1\), \(N(N+1)\) – 1, \(\frac{1}{12} N(N-3)(N+1)(N+2)\), and \(\binom{N}{4}\), respectively. For \(N = 8\), the last representation is reducible, but this does not affect the argument here.
4 The theories with $N \leq 4$

4.1 $N = 1$

In this case, the target space is a Riemannian manifold of arbitrary dimension $d$. The consistency conditions for the gauged theory simplify considerably; in particular the quadratic constraints (3.34) become identities.

The tensor $A_1$ has just one component, which defines a function $F$ on the target space. According to (3.30) $A_1$ is gauge invariant, and so is $F$,

$$\Theta_{\mathcal{M}\mathcal{N}} X^\mathcal{M} \partial_i F = 0. \hspace{1cm} (4.1)$$

Reading off the values for $A_2$ and $A_3$ from (3.21), we obtain

$$A_1 = F, \hspace{0.5cm} A_{2i} = \partial_i F, \hspace{0.5cm} A_{3ij} = g_{ij} F - 2 D_i \partial_j F + 2 T_{ij}, \hspace{1cm} (4.2)$$

with $T_{ij} = X^\mathcal{M}_i \Theta_{\mathcal{M}\mathcal{N}} X^\mathcal{N}_j$.

As a consequence of (4.1), any subgroup of isometries can be gauged (for example, by choosing a constant function $F$). The gravitino $\psi_\mu$ is never charged under the gauge group, as can be seen directly from (3.8), and the gauging is restricted to the matter sector. The scalar potential $V$ (3.32) is given by

$$V = -g^2 \left(4 F^2 - 2 g^{ij} \partial_i F \partial_j F \right), \hspace{1cm} (4.3)$$

i.e. the function $F$ serves as the real superpotential. Supersymmetry (in a maximally symmetric spacetime) requires the vanishing of $A_2$ (c.f. (3.5) below), so that the stationary points of $F$ define (anti-de Sitter) supersymmetric ground states.

There exist deformations of the original theory that are not induced by gaugings. They correspond to $\Theta_{\mathcal{M}\mathcal{N}} = 0$ and $F \neq 0$ and are described by the Lagrangian (2.23), together with the mass-like terms (3.18), subject to (4.2), and with the scalar potential (4.3).

4.2 $N = 2$

The target space is now a Kähler manifold. Some (partial) results for abelian gaugings have already been obtained in [5, 6]. As it turns out, any subgroup of the invariance group can be gauged. These gaugings share some features with the $N = 1$ gaugings of four-dimensional supergravity [24, 25, 26].
Many of the quantities introduced above simplify considerably. It is therefore convenient to introduce the notation
\[ f = f^{12}, \quad Q_i = Q_i^{12}, \quad V = V^{12}, \]
\[ T_i = T_i^{12}, \quad T = T^{I,J,J} = 2T^{12,12}. \] (4.4)

To avoid confusion, we keep using the notation $\Lambda^{12}$ and $S^{12}$ for the parameters of the SO(2) transformations. Further, we have
\[ \partial_i Q_j - \partial_j Q_i = \frac{1}{2} f_{ij}, \quad D_i(\Gamma)f^k_j = 0, \] (4.5)
where $\Gamma_{ij}^k$ is the Christoffel connection. For a Kähler target space it is convenient to decompose the $d$ real fields into $d/2$ complex ones and their complex conjugates, $\phi^i \rightarrow (\phi^i, \bar{\phi}^j)$ in a basis where $f_{ij} = i \delta^i_j$, $f_{ij} = -i \delta^j_i$. From the fact that $f$ is hermitian, it follows that only the components $g_{ij} = g_{ij}$ are non-zero, and therefore $f_{ij} = ig_{ij} = -f_{ji}$. The fact that $f$ is covariantly constant then leads to
\[ \partial_i g_{jk} = \partial_j g_{ik}, \] (4.6)
which implies that the metric can locally be written as $g_{ij} = \partial_i \partial_j K$, where $K(\phi, \bar{\phi})$ is the Kähler potential. Furthermore (3.15) implies,
\[ T_i = \frac{1}{2} i \partial_i T, \quad T_{ij} = -D_i \partial_j T. \] (4.7)

The projectors defined in (2.20), decompose into a holomorphic and an anti-holomorphic component,
\[ P^{Ii}_{jj} = \frac{1}{2} \delta^i_j (\delta^{IJ} + i \epsilon^{IJ}), \quad P^{Ii}_{jj} = \frac{1}{2} \delta^i_j (\delta^{IJ} - i \epsilon^{IJ}). \] (4.8)

The Kähler potential $K$ is defined up to Kähler transformations,
\[ K(\phi, \bar{\phi}) \rightarrow K(\phi, \bar{\phi}) + \Lambda(\phi) + \bar{\Lambda}(\bar{\phi}). \] (4.9)

A solution of (4.5) is provided by
\[ Q_i = -\frac{1}{4} i \partial_i K, \quad Q_i = \frac{1}{4} i \partial_i K. \] (4.10)

This solution is not unique as it is subject to field-dependent gauge transformations. By adopting (4.10) we have removed this gauge freedom, but the Kähler transformations now act on $Q$ in the form of a field-dependent SO(2) gauge transformation with parameter
\[ \Lambda^{12}(\phi, \bar{\phi}) = \frac{1}{4} i (\Lambda(\phi) - \bar{\Lambda}(\bar{\phi})). \] (4.11)
Consequently, all quantities transforming nontrivially under SO(2) become now subject to Kähler transformations induced by (4.11). Note that transformations where Λ equals an imaginary constant, correspond to SO(2) transformations acting exclusively on the fermions and not on the Kähler potential. These transformations constitute an invariance group of the ungauged Lagrangian and they are in the center of the full group of combined isometries and SO(2) transformations of the fermions.

According to (2.27) only holomorphic isometries of the target space can be extended to symmetries of the Lagrangian. Such isometries, parameterized by Killing vector fields \((X^i, X^\bar{i})\), preserve both the metric and the complex structure, i.e.,

\[ \mathcal{L}_X g = \mathcal{L}_X f = 0. \]

The invariance of the complex structure implies that \(X^i\) and \(X^\bar{i}\) must be holomorphic and anti-holomorphic, respectively. The invariance of the metric gives rise to the Killing equations

\[ D_i X_j + D_j X_i = 0, \quad D_i X_j + D_j X_i = 0. \]

Because of the holomorphicity of \(X^i\), the second condition is automatically satisfied, whereas the first condition implies that the Kähler potential remains invariant under the isometry up to a Kähler transformation. We write this special Kähler transformation in terms of a holomorphic function \(S(\phi)\), i.e.,

\[ \delta K(\phi, \bar{\phi}) = -X^i \partial_i K - X^\bar{i} \partial_{\bar{i}} K = 4i (S - \bar{S}). \] (4.12)

According to (2.30), the function \(V\), defined by

\[ V = X^i Q_i + X^\bar{i} Q_\bar{i} + S^{12} = -\frac{1}{4}i X^i \partial_i K + \frac{1}{4}i X^\bar{i} \partial_{\bar{i}} K + S^{12}, \]

must satisfy the equation,

\[ \partial_i V = \frac{1}{2}i g_{ij} X^j. \]

As the right-hand side can be written as a derivative, this equation can now be solved and we obtain

\[ S^{12}(\phi, \bar{\phi}) = S(\phi) + \bar{S}(\bar{\phi}). \] (4.13)
Consequently we have,

\[ V = -\frac{1}{4} i (X^i \partial_i K - X_i^\bar{i} \partial_{\bar{i}} K) + S + \bar{S} = -\frac{1}{2} i X^i \partial_i K + 2 \mathcal{S}. \]

For every generator \( X^M \) of the isometry group we may thus identify a holomorphic function \( \mathcal{S}^M \), determined by (4.12) up to a real constant. The particular transformation (which we denote with the extra label \( \mathcal{M} = 0 \))

\[ X^0 i = 0, \quad \mathcal{S}^0 = \frac{1}{2}, \quad \mathcal{V}^0 = 1, \] (4.14)

constitutes a central extension of the isometry group and generates the SO(2) R-symmetry group that acts exclusively on the fermions. The closure relation (2.34) yields,

\[ X^M i \partial_i \mathcal{S}^N - X^N i \partial_i \mathcal{S}^M = \sum_{K>0} f^{\mathcal{M} N \mathcal{K}} \mathcal{S}^\mathcal{K} + f^{\mathcal{M} N 0}. \] (4.15)

For a semi-simple isometry group, the \( f^{\mathcal{M} N 0} \) can be absorbed by suitable constant shifts into the functions \( \mathcal{S}^M \) (or, equivalently, into \( \mathcal{V}^M \)).

It is always possible to gauge the R-symmetry group. In that case we have a gauge field \( A_\mu^0 \) associated with SO(2) transformations of the fermions and the \( T \)-tensor must be manifestly invariant under this group. When the structure constants \( f^{\mathcal{M} N 0} \) do not vanish when projected onto the gauged subgroup of the isometries, and cannot be absorbed by suitable shifts, the R-symmetry group must be gauged. In that case it is more practical to choose a given set of functions \( \mathcal{V}^M \) whose algebra will exhibit a certain central charge, without trying to modify the structure constants by constants shifts. Instead one can then vary the embedding tensor \( \Theta_{MN} \). The reason is that the Lagrangian is invariant under a change of basis in the Lie algebra, according to \( \mathcal{V}^M \rightarrow \mathcal{V}^M + c^M, A_\mu^M \rightarrow A_\mu^M + c^M A_\mu^0, \) and

\[
\Theta_{\mathcal{M}0} \rightarrow \Theta_{\mathcal{M}0} - c^N \Theta_{N\mathcal{M}}, \quad (\mathcal{M} \neq 0);
\Theta_{00} \rightarrow \Theta_{00} - 2 c^M \Theta_{M0} + c^M \Theta_{MN} c^N, \] (4.16)

where the \( c^M \) are arbitrary constants with \( c^0 = 0 \). To see this one observes that the combinations \( \mathcal{V}^M \Theta_{MN} \mathcal{V}^N, \mathcal{V}^M i \Theta_{MN} \mathcal{V}^N \) and \( \mathcal{V}^M i \Theta_{MN} A_\mu^N \) remain invariant under the combined substitutions.

Let us further note that, given a gauge group \( G_0 \) whose embedding tensor \( \Theta \) satisfies (3.13), another solution to (3.13) can always be obtained by the deformation

\[ \Theta_{\mathcal{M}0} \rightarrow \Theta_{\mathcal{M}0} + \delta_{\mathcal{M}}, \] (4.17)
provided the generator \( \delta_M X^M \) commutes with the gauge group \( G_0 \). This is the three-dimensional analogue of the local version of the Fayet-Iliopoulos mechanism in four-dimensional \( N = 1 \) supergravity [25].

Let us now determine the various quantities involved. It is convenient to decompose the tensor \( A^I_J \) in terms of a singlet part \( A^{11}_{\bar{1}} + A^{22}_{\bar{1}} \) and a complex quantity

\[
e^{K/2} W \equiv \frac{1}{2}(A^{22}_{\bar{1}} - A^{11}_{\bar{1}}) + iA^{12}_{\bar{1}}.
\]

Kähler transformations are induced by the SO(2) transformations (4.11),

\[
\delta(e^{K/2} W) = 2i \Lambda^{12}(e^{K/2} W).
\]

This implies that \( W \) transforms under Kähler transformations according to

\[
\delta W = -\Lambda(\phi) W.
\]

Imposing the equations (3.23) then leads directly to the following result,

\[
\partial_i(A^{11}_{\bar{1}} + A^{22}_{\bar{1}} + 2T) = 0, \quad \partial_i\overline{W} = \partial_i W = 0.
\]

The function \( W \) can be identified as the holomorphic superpotential. Gauge covariance of \( A_1 \) imposes the additional relation

\[
\Theta_{MN}(X^N D_iW + 4i \mathcal{V}^N W) = \Theta_{MN}(X^N \partial_i W + 4i \mathcal{S}^N W) = 0,
\]

with the Kähler covariant derivative \( D_iW \equiv \partial_i W + \partial_i K W \), implied by (4.18). For nonvanishing \( W \) the momentum maps associated with the gauge group generators can be expressed in terms of the superpotential and are proportional to the corresponding Killing vectors. As a consequence, one may verify from (4.15) that the structure constants \( f^{MN}_{\bar{0}} \) vanish when projected onto the gauge group, i.e., \( \Theta_{MN} \Theta_{0K} f^{K\bar{0}} = 0 \). The gauging of the SO(2) R-symmetry group requires \( W \) to vanish (because \( W \) transforms nontrivially under SO(2)). Therefore nonzero \( W \) implies that we have \( \Theta_{N0} = 0 = \Theta_{00} \).

The integration constant in \( A^{11}_{\bar{1}} + A^{22}_{\bar{1}} \) is finally determined by the quadratic constraints (3.31) provided that \( W \) is nonvanishing. The tensor \( A^I_J \) is then given by

\[
A^{11}_{\bar{1}} = -T - e^{K/2} \Re W, \quad A^{22}_{\bar{1}} = -T + e^{K/2} \Re W, \quad A^{12}_{\bar{1}} = A^{21}_{\bar{1}} = e^{K/2} \Im W.
\]
The tensor $A_2$ can be derived from (3.21) and its components are given by

$$A_{2i}^{1} = -iA_{2i}^{12} = -\frac{1}{2}(\partial_i T + e^{K/2}D_i W),$$
$$A_{2i}^{22} = iA_{2i}^{21} = -\frac{1}{2}(\partial_i T - e^{K/2}D_i W).$$

(4.21)

Finally, the tensor $A_3$ can be evaluated from (3.21); this leads to, for example,

$$A_{3i}^{11} = \frac{1}{4}e^{K/2}D_i D_j W,$$
$$A_{3i}^{11} = \frac{1}{2}T_{ij} - \frac{1}{4}g_{ij} T + \frac{1}{2}\partial_i \partial_j T.$$

(4.22)

One can verify the consistency of these results by inserting $A_1$, $A_2$ and $A_3$ into the quadratic constraints (3.34). Indeed, these cancel by virtue of (4.19). When $W = 0$ the vanishing of the integration constant in $A_{11}^{11} + A_{22}^{22}$ cannot be derived from the quadratic constraints, but follows instead from the identity (3.35). We note in passing that pure $N = 2$ supergravity (without gauging) can have a cosmological constant corresponding to a constant $W$ and vanishing $T$. This implies that the gravitino mass matrix is traceless. An alternative way to generate a cosmological term in pure supergravity makes use of gauging the R-symmetry group. In that case, $T$ equals a nonzero constant and $W = 0$; the gravitino mass matrix is then proportional to the identity.

The scalar potential (3.32) of the gauged theory is given by

$$V = -g^2 \left( 4T^2 - 4g^{ij}\partial_i T \partial_j T + 4e^K |W|^2 - g^{i\bar{j}} e^K D_i W D_{\bar{j}} W \right).$$

(4.23)

Note that in three dimensions, the scalar potential is quartic in the moment map $\mathcal{V}$, since the $T$-tensor is quadratic in $\mathcal{V}$. This is in contrast with, e.g., four dimensions, where the scalar potential is quadratic in $\mathcal{V}$.

Analogous to the $N = 1$ case, there are two kinds of supersymmetric deformations of the original theory. On the one hand, there are the gaugings, which are completely characterized by an embedding tensor $\Theta_{MN}$. The above analysis shows that there is no restriction on the $T$-tensor, and therefore any subgroup of the invariance group of the theory is an admissible gauge group, as long as its embedding tensor satisfies (3.13). On the other hand there are the deformations described by the holomorphic superpotential $W$, which are not induced by a gauging. In case both deformations are simultaneously present, their compatibility requires (4.19).
4.3 $N = 3$

In this case the target space is a quaternionic manifold. The condition \( (3.36) \), from which we have derived all other consistency constraints, is identically satisfied, so that each subgroup of isometries can be consistently gauged. The gauging follows uniquely from the embedding tensor and there are no other deformations. In particular, the scalar tensors \( A_1, A_2, A_3 \) are defined by \( (3.21), (3.28) \), and the scalar potential is given by \( (3.32) \).

4.4 $N = 4$

For \( N = 4 \) the target space is locally a product of two quaternionic manifolds of dimension \( d_+ \), associated with the positive and negative eigenvalues of the tensor \( J \), whose real coordinates we denote by \( \phi^i, \bar{\phi}^\bar{i} \), respectively. Because the almost-complex structures commute with \( J \), they decompose into two sets of three almost-complex structures \( f^\pm \); the only nonvanishing components are \( f^+_ij^P, f^-_i\bar{j}^P \), where \( \pm \) denotes the split according to \( (P, P = 1, 2, 3) \)

\[
\begin{align*}
f^+_P &\equiv \frac{1}{2} (J + 1) f^P = \frac{1}{2} f^P - \frac{1}{4} \epsilon^{PQR} f^QR, \\
f^-_P &\equiv \frac{1}{2} (J - 1) f^P = - \frac{1}{2} f^P - \frac{1}{4} \epsilon^{PQR} f^QR. 
\end{align*}
\]

(4.24)

The two sets of almost-complex structures satisfy the multiplication rule,

\[
\begin{align*}
f^+_P f^+_Q &= - \mathbf{1} \delta^{PQ} - \epsilon^{PQR} f^+_R, \\
f^-_P f^-_Q &= - \mathbf{1} \delta^{PQ} - \epsilon^{PQR} f^-_R.
\end{align*}
\]

(4.25)

There is a corresponding decomposition of the SO(4) R-symmetry group,

\[ \text{SO}(4) = \text{SO}(3)^+ \times \text{SO}(3)^- \].

(4.26)

Obviously the isometry group splits into the product of the isometry groups of the two subspaces whose generators we label by \( X^M \) and \( X^\bar{M} \), respectively. These \( N = 4 \) three-dimensional gaugings share some similarities with the \( N = 2 \) gaugings of four-dimensional supergravity [28, 29, 30, 31], although the precise relation remains to be understood.

Upon reduction to three spacetime dimensions the special Kähler and the quaternionic manifolds that describe the interactions of the four-dimensional vector multiplets and hypermultiplets, respectively, give rise to the two quaternionic spaces that span the target space manifold [32, 33].

Let us first consider the nondegenerate case \( d_4 d_- \neq 0 \). According to the discussion of equation \( (2.31) \), the quantities \( \mathbf{\mathcal{V}^M}^{IJ}(\phi, X) \) decompose into two triplets denoted by
\( V^M P \) and \( V^{\bar{M}} \bar{P} \), which live in each of the corresponding subspaces. The triplets \( V \) are the momentum maps associated with the isometries of the quaternionic spaces \([27]\).

A priori, the embedding tensor \( \Theta_{MN} \) decomposes into diagonal components, \( \Theta_{MN} \) and \( \Theta_{\bar{M}\bar{N}} \), and off-diagonal components \( \Theta_{M\bar{N}} \). The latter are, however, severely constrained by the invariance condition \((3.13)\),

\[
\Theta_{PM} f^{MN}_K \Theta_{\bar{N}\bar{L}} = \Theta_{\bar{P}\bar{N}} f^{\bar{M}\bar{N}}_K \Theta_{\bar{M}\bar{L}},
\]

\((4.27)\)

Under \((4.26)\), the \( T^{IJ,KL} \) component of the \( T \)-tensor \((3.14)\) takes values in the representations,

\[
T^{IJ,KL} : (1,1) + (1,1) + (5,1) + (1,5) + (3,3).
\]

\((4.28)\)

The constraint \((3.36)\), which is necessary and sufficient for the existence of a supersymmetric gauging, implies the absence of the \((1,5) + (5,1)\) representation in this decomposition; in the basis \((4.24)\) it implies

\[
T^{PQ} = \frac{1}{3} \delta^{PQ} T^{RR}, \quad \text{where} \quad T^{PQ} = \mathcal{V}^M P \Theta_{MN} \mathcal{V}^{N Q},
\]

\((4.29)\)

and correspondingly for \( T^{\bar{P}\bar{Q}} \). The off-diagonal components, \( T^{PQ} \), which are proportional to \( \Theta_{MN} \), remain unconstrained. Unlike the cases \( N < 4 \), it is no longer possible to gauge any subgroup of the isometry group; the consistency of the gauged theory depends on the condition \((4.29)\) for the momentum maps \( \mathcal{V}^M P \) and \( \mathcal{V}^{\bar{M}} \bar{P} \) and the gauge group invariant diagonal components \( \Theta_{MN} \) and \( \Theta_{\bar{M}\bar{N}} \) of the embedding tensor, together with the compatibility relation \((4.27)\) for the off-diagonal components \( \Theta_{MN} \) of the embedding tensor. For symmetric quaternionic spaces there are convenient techniques for finding admissible gauge groups, as we will discuss in the next section. However, for non-symmetric \([34, 35, 36]\) or even non-homogeneous quaternionic spaces it remains to directly analyze equation \((4.29)\) in order to determine the possible solutions for \( \Theta_{MN} \).

Let us finally discuss the degenerate case in which one of the two quaternionic manifolds vanishes, i.e. let us assume that \( d_- = 0 \). The Lagrangian \((2.23)\) then admits an additional global symmetry \( SO(3)^- \) acting exclusively on the fermions. Similar to \((4.14)\) above we can conveniently incorporate these invariances into our framework by defining three extra generators with label \( M = \bar{P} = 1, 2, 3 \), satisfying

\[
X^P = 0, \quad [S^P, S^{\bar{Q}}] = \varepsilon^{PQR} S^R,
\]

\((4.30)\)
so that the four-by-four matrices $S^\bar{P}$ generate the $\text{SO}(3)^- \text{SO}(3)^-$ group on the fermions. With these definitions, (4.29) implies

$$\Theta_{\bar{P}\bar{Q}} = \theta \delta_{\bar{P}\bar{Q}} \iff T^{\bar{P}\bar{Q}} = \theta \delta^{\bar{P}\bar{Q}}.$$  \hspace{1cm} (4.31)

Assuming that $\theta \neq 0$, the $\text{SO}(3)^- \text{SO}(3)^-$ gauge transformations on the spinors can combine with possible target space isometries, provided that the isometry group contains an $\text{SO}(3)$ subgroup. The gauge group decomposes into a direct product $G_0 \times \text{SO}(3)$, where the gauge symmetries associated with $G_0$ correspond to the embedding tensor,

$$\Theta_{MN} - \Theta_{\bar{M}\bar{P}} \Theta^{\bar{P}\bar{Q}} \Theta_{\bar{Q}N},$$  \hspace{1cm} (4.32)

with $\Theta^{\bar{P}\bar{Q}}$ the inverse of $\Theta_{\bar{P}\bar{Q}}$.

When $\Theta^{\bar{P}\bar{Q}} = 0$, the gauge group can still include the $\text{SO}(3)^- \text{SO}(3)^-$ R-symmetry group (or an $\text{SO}(2)$ subgroup thereof) through the mixed components of the embedding tensor. Hence we distinguish three $\text{SO}(3)$ generators labeled by $P = 1, 2, 3$,

$$T_P = \Theta_{P\bar{M}} X^M + \Theta_{P\bar{P}} S^{\bar{P}},$$  \hspace{1cm} (4.33)

where $\Theta_{P\bar{M}} X^M$ denote possible corresponding $\text{SO}(3)$ Killing vectors. The presence of the mixed components $\Theta_{P\bar{P}}$ induces another set of gauge group generators,

$$T_P = \Theta_{P\bar{Q}} X^Q,$$  \hspace{1cm} (4.34)

where the $\Theta_{P\bar{Q}} X^Q$ denote three more Killing vectors. From (4.27) it then follows that the generators $T_P$ are mutually commuting and transform as a vector under the $\text{SO}(3)$ isometries (4.33). The $\text{SO}(3)$ Killing vectors $\Theta_{P\bar{M}} X^M$ must be nonvanishing. Hence the gauge group consists of a semidirect product of $\text{SO}(3)$ with the three-dimensional abelian group $\mathcal{T}$ generated by the $T_P$, possibly multiplied with another subgroup of the isometry group. These are the type of theories one obtains upon dimensional reduction of four-dimensional $N = 2$ supergravity without hypermultiplets and a gauged $\text{SU}(2)$ subgroup of the R-symmetry group [37]. It is possible to restrict the embedding tensor, such that we gauge only an $\text{SO}(2)$ subgroup of the $\text{SO}(3)^- \text{SO}(3)^-$ R-symmetry group. By similar reasoning as above, it follows that one must at least have an $\text{SO}(2) \times \text{SO}(2)$ gauge group.

5 Symmetric spaces

For $N > 4$, it has been shown in [1] that the target spaces are symmetric homogeneous spaces $G/H$ such that $d = \dim G - \dim H$. A list of these spaces is given in table [1]. In
this section, we show that the underlying group structure allows to translate the consistency condition for admissible gauge groups into a projection equation for the embedding tensor $\Theta$. This provides an efficient way of classifying and constructing solutions to this equation which has been applied to the gaugings of maximal supergravity in [2, 3].

For a homogeneous target space manifold $G/H$, the scalar fields are described by means of a $G$-valued matrix $L$, on which the rigid action of $G$ is realized by left multiplication, while $H$ acts as a local symmetry by multiplication from the right. The latter gauge freedom may be used to eliminate the spurious degrees of freedom in $L$ and obtain a coset representative so that $L = L(\phi^i)$ is directly parametrized by the $d$ scalar fields $\phi^i$.

In the case at hand, the group $H$ is a maximal compact subgroup of $G$ and given by $SO(N) \times H'$. The generators of the group $G$ constitute a Lie algebra $\mathfrak{g}$, which decomposes into $\{t^M\} = \{X^{IJ}, X^\alpha, Y^A\}$. The $X^{IJ}$ generate $SO(N)$ and the $X^\alpha$ generate the group $H'$; together they span the subalgebra $\mathfrak{h}$ while the remaining (noncompact) generators $Y^A$ transform in a (not necessarily irreducible) spinor representation of $SO(N)$. The relevant representations are collected in appendix B. Expanding the dependence of $L(\phi^i)$ on $\phi^i$ defines

$$L^{-1}\partial_i L = \frac{1}{2} Q_i^{IJ} X^{IJ} + Q_i^\alpha X^\alpha + \epsilon_i^A Y^A .$$

(5.1)

We note the connections $Q_i$ (the ones associated with the $X^\alpha$ were introduced at the end of subsection 2.1). The vielbein $e_i^A$ may be used to convert curved target-space indices into flat $SO(N)$ spinor indices. The target-space metric $g_{ij}$ and the antisymmetric tensors $f_{ij}^{IJ}$ are realized by

$$g_{ij} = e_i^A e_j^B \delta_{AB} , \quad f_{ij}^{IJ} = - \Gamma_{AB}^{IJ} e_i^A e_j^B ,$$

(5.2)

with $SO(N)$ $\Gamma$-matrices $\Gamma_{AB}^{IJ}$. The matter fermion fields are redefined by converting their target-space indices into indices associated with the conjugate spinor representation of $SO(N)$,

$$\chi^\dot{A} = \frac{1}{N} \epsilon_i^A \Gamma_A^{\dot{A}} \chi^i .$$

(5.3)

$^3$Only for $N = 9$ and $N = 16$ the $Y^A$ transform in an irreducible spinor representation of $SO(N)$. Generically, the $Y^A$ comprise a reducible representation of $SO(N) \times H'$ (c.f. appendix B for a complete list). Correspondingly, the $\Gamma$ matrices $\Gamma_{A\dot{A}}^{I}$, $\Gamma_{AB}^{IJ}$ are understood unambiguously as acting separately on the different subspaces and as identity on each $H'$-representation factor. Moreover, they define what we will refer to as the conjugate spinor representation with associated indices $\dot{A}$. E.g. for $N = 10$, we have $Y^A = 16^+ + 16^-$, and the fermions transform in the conjugate representation $16^+ + 16^-$.  

32
All the general formulae obtained above may be conveniently translated, noting that the projector $P$ from (2.20) factorizes according to
\[ g_{ik} P_{jk} = \frac{1}{N} \left( e^A_i \Gamma^I_{A\bar{A}} \left( \Gamma^J_{B\bar{A}} e^B_j \right) \right). \] (5.4)

The isometries are generated by the left action of $G$ on $L(\phi)$, accompanied by a compensating $H$-transformation to remain in the coset representative,
\[ X^M \partial_L = t^M L - L S^M(\phi^i) \quad \text{for all components belonging to } \mathfrak{h}. \] (5.5)

Here $S^M$ decomposes into $S^{M_{IJ}}$ (these quantities were already introduced in a more general context in section 2.2) and $S^{M_{\alpha}}$, belonging to $\text{SO}(N)$ and $\text{H}'$, respectively. Subsequently one forms the combinations $V^M = S^M + X^M Q_i$ for all components belonging to $\mathfrak{h}$. For any coset space one can show [38] that these $V^M$, together with the $V^{M_i} = X^{M_i}$ are subject to a system of linear first-order differential equations, which includes the generators of $H$ and the curvatures of the connections $Q_i$. For the case at hand the resulting equations coincide precisely with (2.37). The $V^M_i$ can also generally be expressed in terms of the coset representatives $L$, and the combined expression for all the $V$ takes the following form,
\[ L^{-1} t^M L \equiv V^M_A t^A = \frac{1}{2} V^{M_{IJ}} X^{IJ} + V^{M_{\alpha}} X^\alpha + V^{M_A} Y_A. \] (5.6)

where $V^{M_i} = g^{ij} e_j^A V^M_A$. Hence the $V$ span an element in the Lie algebra $\mathfrak{g}$, which coincides with the algebra of the generators of the isometries. At this point we can make direct contact with the map (2.43), which now defines an isomorphism, corresponding to the field-dependent conjugation (5.6). In particular, the $T$-tensor (3.14), given by
\[ T_{AB} = V^M_A \Theta_{MN} V^N_B, \] (5.7)

where $A = \{IJ, \alpha, A\}$, contains the embedding tensor of the gauge group as $\Theta = T|_{V=I}$.

5.1 Lifting the consistency constraints

We recall that the consistency condition for a supersymmetric gauging takes the form of a single equation [3.38] for the $T$-tensor, and dictates the absence of the $\text{SO}(N)$ representation $[38]$ in $T^{IJKL}$. In order to satisfy this equation on the entire scalar manifold $G/H$, the structure (5.7) of the $T$-tensor shows that the entire G-orbit of the $\text{SO}(N)$
representation \( \mathbb{H} \) must vanish. Consider now the decomposition of the \( T \)-tensor under \( G \), to

\[
R_{\text{adj}} \times \text{sym} R_{\text{adj}} = 1 \oplus \left( \bigoplus_i R_i \right), \quad \implies \quad T_{AB} = \theta \eta_{AB} + \sum_i P_{R_i} T_{AB}, \quad (5.8)
\]

where \( 1 \) and \( R_{\text{adj}} \) are the trivial and the adjoint representation of \( G \), respectively; \( \eta_{AB} \) is the Cartan-Killing form of \( G \), and "\times \text{sym}" denotes the symmetrized tensor product. By \( P_{R_i} \) we denote the \( G \)-invariant projector onto the representation \( R_i \). From the \( \text{SO}(N) \) composition of the generators, it is clear that there is a unique irreducible representation \( R_0 \) of \( G \) appearing in the sum in (5.8) that branches under \( \text{SO}(N) \) such that it contains the representation \( \mathbb{H} \). The condition (3.38) is thus equivalent to

\[
P_{R_0} T_{AB} = 0. \quad (5.9)
\]

The other \( \text{SO}(N) \) representations contained in this equation can be obtained explicitly by successively taking derivatives of (3.38). Because (5.9) is a \( G \)-covariant condition, it is also equivalent to

\[
P_{R_0} \Theta_{MN} = 0. \quad (5.10)
\]

The underlying coset structure thus allows to translate the field-dependent form (3.36) of the consistency condition into a single condition for the constant embedding tensor of the gauge group \( G_0 \). After identifying the representation \( R_0 \), the condition for the embedding tensor corresponding to a consistent gauging can thus be given in explicit form. In table II we have collected the decompositions (5.8) and the representations \( R_0 \) for all theories with \( N > 4 \). Given a subgroup \( G_0 \subset G \) with a corresponding embedding tensor \( \Theta_{MN} \), equation (5.10) provides a simple and efficient criterion for checking whether \( G_0 \) can be consistently gauged while preserving all supersymmetries. The solutions of (5.10) will be referred to as admissible gauge groups \( G_0 \). In the following two sections, we discuss some of these solutions, case by case for the different \( N \). We close this section with some general remarks on the solutions of (5.10).

A direct consequence of the projection equation (5.10) is that the Cartan-Killing form of \( G \) is a solution to this equation as it corresponds to the singlet in the decomposition of (5.8). Therefore the full isometry group \( G \) is always an admissible gauge group. The potential of the corresponding gauged theory reduces to a cosmological constant because the dependence on the scalars disappears as a result of the \( G \)-invariance of the potential.

\[\text{We used the LiE package [39] for computing the decompositions of tensor products and the branching of representations; throughout this paper we use the corresponding conventions for the Dynkin weights.}\]
Table I: Symmetric spaces for \( N > 4 \). The representation \( R_0 \) from (5.10) is underlined in the decomposition (5.8). Dots ‘…’ represent zero weights.

| \( N \) | \( G/H \)             | \( d \)     | \( R_{\text{adj}} \)                  | \( R_{\text{adj}} \times \text{sym} R_{\text{adj}} \) |
|-------|----------------------|------------|--------------------------------------|--------------------------------------------------|
| 5     | \( \text{Sp}(2,k) \) | \( 8k \)   | (2, 0, ...)                          | (0, ...) + (0, 1, ...) + (0, 2, ...) + (4, 0, ...) |
| 6     | \( \text{SU}(4,k) \times \text{Sp}(k) \) | \( 8k \)   | (1, 0, 1)                            | (0, 0, 0) + (0, 1, 0, 1) + (2, 0, 0, 0) |
| 8     | \( \text{SO}(8,k) \times \text{SO}(k) \) | \( 8k \)   | (0, 0, ...)                          | (0, 0, 0, 1, 0, 0) + (2, 0, 0, 0) |
| 9     | \( F_4(-20) \) \( \text{SO}(9) \) | 16         | 52                                   | 1 + 324 + 1053                                    |
| 10    | \( E_6(2) \) \( \text{SO}(10) \times \text{U}(1) \) | 32         | 78                                   | 1 + 650 + 2430                                    |
| 12    | \( E_{7(-5)} \) \( \text{SO}(12) \times \text{Sp}(1) \) | 64         | 133                                  | 1 + 1539 + 7371                                   |
| 16    | \( E_8(8) \) \( \text{SO}(16) \) | 128        | 248                                  | 1 + 3875 + 27000                                 |

In fact all scalars fields may simply be gauged away by means of the gauged isometries. Apart from this trivial solution, one may distinguish different classes of solutions of (5.10): (i) compact gauge groups, of which in general there are very few, (ii) semisimple non-compact gauge groups, (iii) non-semisimple gauge groups, and (iv) complex gauge groups embedded in the real group \( G \) \[22\]. In the following, we restrict the discussion to some semisimple solutions of (5.10); non-semisimple gauge groups may generically be obtained by boosting the embedding tensors of their semisimple cousins, (see \[22\] for a detailed discussion in the \( N = 16 \) theory).

For a compact gauge group, the components \( \Theta^I_{JA} \), \( \Theta^I_{J\alpha} \), and \( \Theta^A_B \) of the embedding tensor vanish. Its \( \text{SO}(N) \) part \( \Theta^I_{JKL} \) must satisfy (3.36), i.e. it must be of the form

\[
\Theta^I_{JKL} = \theta \delta^K_L + \delta^K_J \Xi^I_{L} + \Xi^I_{JKL},
\]

with a traceless symmetric tensor \( \Xi^I_{J} \) and a completely antisymmetric tensor \( \Xi^I_{JKL} \). Explicit inspection of (5.10) shows that for \( N > 5 \) the embedding tensor must moreover satisfy \( \Gamma^I_{AB} \Xi^I_{JKL} \equiv 0 \), which implies \( \Xi^I_{JKL} \equiv 0 \), except for \( N = 8 \) where the fourfold antisymmetric product of vector representations becomes reducible.

Let us therefore consider in some more detail compact gauge groups with embedding tensor of the form

\[
\Theta^I_{JKL} = \theta \delta^K_L + \delta^K_J \Xi^I_{L}.
\]

It is straightforward to verify, that the choice

\[
\Xi^I_{J} = \begin{cases} 
2(1 - \frac{I}{N}) \delta^I_J & \text{for } I \leq p \\
-2 \frac{p}{N} \delta^I_J & \text{for } I > p
\end{cases}, \quad \theta = \frac{2p - N}{N},
\]

\[35\]
describes the embedding of $\text{SO}(p) \times \text{SO}(N-p) \subset \text{SO}(N)$ as $\Theta = \Theta^{\text{SO}(p)} - \Theta^{\text{SO}(N-p)}$, i.e. with opposite coupling constant. This ratio is fixed by the requirement that the embedding tensor takes the form (5.12). Note that $\theta = 0$ can only be achieved for even $N = 2p$. Likewise, one can check that no product $\text{SO}(p_1) \times \ldots \times \text{SO}(p_n)$ with more than two factors can be embedded into $\text{SO}(N)$ with an embedding tensor of the form (5.12). This severely restricts the possible choices of compact gauge groups.

5.2 The theories with $8 < N \leq 16$

For the theories with $N = 9, 10, 12, 16$, the physical fields form a single supermultiplet, out of which the scalars parametrize the exceptional coset spaces $F_4(−20)/\text{SO}(9)$, $E_6(−14)/(\text{SO}(10) \times \text{U}(1))$, $E_7(−5)/(\text{SO}(12) \times \text{SU}(2))$, and $E_8(8)/\text{SO}(16)$, respectively. The decomposition (5.8) for all these groups contains three irreducible representations only (c.f. Table I), so that the embedding tensor of an admissible gauge group (5.10) is entirely contained in the union of a single $G$-representation $R_1$ and the singlet

$$\Theta_{MN} = \theta \eta_{MN} + \mathbb{P}_{R_1} \Theta_{MN}. \quad (5.14)$$

This enables one to identify solutions of (5.10) by purely group-theoretical reasoning as we shall summarize in the following observations.

- Let $G_0 \subset G$ be a semisimple subgroup of $G$ such that the decomposition of $R_0$ from (5.10) under $G_0$ does not contain a singlet. Then $G_0$ is an admissible gauge group.

- Let $G_0 = G^{(1)} \times G^{(2)} \subset G$ be a semisimple subgroup of $G$ such that the decomposition of $R_0$ from (5.10) under $G_0$ contains precisely one singlet. Then $G_0$ is an admissible gauge group, provided a fixed ratio of coupling constants of $G^{(1)}$ and $G^{(2)}$ [3]. We denote its embedding tensor by

$$\Theta_{MN} = g_1 \Theta_{MN}^{(1)} + g_2 \Theta_{MN}^{(2)}, \quad (5.15)$$

where $\Theta_{MN}^{(1)}$ and $\Theta_{MN}^{(2)}$ are the restrictions of the Cartan-Killing form $\eta_{MN}$ of $G$ onto $G^{(1)}$ and $G^{(2)}$, respectively.

- Let $G_0 = G^{(1)} \times G^{(2)} \subset G$ be a semisimple subgroup of $G$ satisfying the above assumptions with embedding tensor (5.15). Let moreover $G' \subset G$ be a group such that the decomposition of $R_1$ in (5.14) under $G'$ contains no singlet. Then the ratio of coupling constants in (5.15) is given by

$$\frac{g_1}{g_2} = \frac{\dim G' \dim G^{(2)} - \dim G \dim (G^{(2)} \cap G')} {\dim G' \dim G^{(1)} - \dim G \dim (G^{(1)} \cap G')} . \quad (5.16)$$
This is shown by contracting \((5.15)\) over \(G_N\) and over the full group \(G\).

Using these facts, we now give a brief case by case discussion of some of the semisimple admissible gauge groups for the theories with \(N > 8\).

\(N = 16\): The gaugings of the maximal three-dimensional supergravity have been constructed and discussed in great detail in [2, 3]: we include some of the results here for completeness. The embedding tensor of an admissible compact gauge groups must take the form \((5.12)\). However, the explicit decomposition of the \(T\)-tensor \((B.1)\) shows that the two singlets in \((B.1)\) are linearly related (specifically \(8 \Theta_{IJ,IJ} = -15 \Theta_{AA}\)). Since \(\Theta_{AB} = 0\) for a compact gauge group, this requires \(\theta = 0\). From \((5.13)\) it then follows that the only compact admissible gauge group is the product \(SO(8) \times SO(8)\) with opposite gauge coupling constants. The noncompact admissible gauge groups include the \(SO(p, 8-p) \times SO(8)\) but also the exceptional groups \(F_4(-20) \times \text{G}_2(-14)\), \(E_6(-14) \times \text{SU}(3)\), \(E_7(-5) \times \text{SU}(2)\), and different real forms thereof (see [3] for a detailed list). They all satisfy the assumptions leading to \((5.15)\). The ratios of coupling constants for these groups are straightforwardly derived from \((5.16)\) with \(G_N = SO(16)\).

\(N = 12\): There are three singlets in the decomposition \((B.2)\) related by a single linear condition. This leads to a larger variety of compact gauge groups. Roughly speaking, a nonvanishing \(\theta\) in \((5.12)\) may be compensated by switching on the extra \(SU(2)\) factor from \(H = SO(12) \times SU(2)\) in the gauge group, such that the noncompact part \(\Theta_{AB}\) still remains zero. For example, the decomposition of \((B.3)\) under \(H\) shows that this subgroup itself satisfies the assumptions leading to \((5.15)\); unlike in the maximal case, \(H\) is itself an admissible gauge group with a fixed ratio of coupling constants. This ratio can be derived from \((5.16)\) (using for example \(G_N = SU(6, 2)\)) and gives \(\Theta = \Theta_{SO(12)} - 3 \Theta_{SU(2)}\). In general, the admissible compact gauge groups are given by the products \(SO(p) \times SO(12-p) \times SU(2)\) with embedding tensor

\[
\Theta = \Theta_{SO(p)} - \Theta_{SO(12-p)} + \frac{1}{2}(6-p) \Theta_{SU(2)}. \tag{5.17}
\]

The relative coupling constant between the two first factors is determined by \((5.13)\), while the relative factor in front of the last term stems from the relation \(3 \Theta_{I,I,I} = -22 \Theta_{aa}\) for compact gauge groups (c.f. \((B.2)\)) whose relative coefficient may be fixed from the case \(p = 12\) given above. Note that (only) for \(p = 6\), the gauge group lies entirely in \(SO(12)\) and the \(SU(2)\) factor is not gauged. Among the noncompact admissible gauge groups there are \(E_6(2) \times U(1)\), \(F_4(-20) \times SU(2)\), \(G_2(2) \times Sp(3)\), all of which are maximal subgroups of \(E_7(-5)\).
\[ N = 10: \] Similar to the above, the admissible compact gauge groups in this case are the products \( \text{SO}(p) \times \text{SO}(10-p) \times \text{U}(1) \) with embedding tensor

\[
\Theta = \Theta_{\text{SO}(p)} - \Theta_{\text{SO}(10-p)} + \frac{1}{3}(5-p)\Theta_{\text{U}(1)}. \tag{5.18}
\]

The relative coupling constants are fixed as in (5.17), using for example \( G_{N} = \text{Sp}(2, 2) \).

For \( p = 5 \), the gauge group lies entirely in the \( \text{SO}(10) \) and the \( \text{U}(1) \) factor is not gauged. Among the noncompact admissible gauge groups there are \( \text{SU}(4, 2) \times \text{SU}(2), \) \( G_{2(-14)} \times \text{SU}(2, 1) \), as well as the simple group \( F_{4(-20)} \). All these gauge groups are maximal subgroups in \( E_{6(-14)} \).

\[ N = 9: \] In this case there is no additional factor in \( H \); however, the explicit decomposition (B.4) shows that the two singlets appearing are independent. Therefore, again a compact gauge group does not necessarily require vanishing \( \theta \) in (5.12). In particular, the group \( H = \text{SO}(9) \) itself is an admissible gauge group. More generally, the admissible compact gauge groups are the products \( \text{SO}(p) \times \text{SO}(9-p) \) with embedding tensor

\[
\Theta = \Theta_{\text{SO}(p)} - \Theta_{\text{SO}(9-p)}. \tag{5.19}
\]

Among the noncompact admissible gauge groups there are \( G_{2(-14)} \times \text{SL}(2) \) and \( \text{Sp}(2, 1) \times \text{SU}(2) \) which are maximal subgroups of \( F_{4(-20)} \).

5.3 The theories with \( 4 < N \leq 8 \)

For \( N = 5, 6, 8 \), the field content of the ungauged theories is given by an arbitrary number \( k \) of supermultiplets whose scalars fields parametrize the coset spaces \( \text{Sp}(2, k)/(\text{Sp}(2) \times \text{Sp}(k)), \text{SU}(4, k)/(\text{SU}(2) \times \text{U}(k)), \) and \( \text{SO}(8, k)/(\text{SO}(8) \times \text{SO}(k)) \), respectively.

\[ N = 8: \] The \( N = 8 \) ungauged theories have been constructed in [40], their gaugings were discussed in [4].\(^5\) Consistency of the gauging is again encoded in (5.10) with \( R_{0} = (0, 2, 0, 0, \ldots) \).\(^6\) Just as in the previous examples, one may consistently gauge the entire compact subgroup \( H = \text{SO}(8) \times \text{SO}(k) \) with a fixed ratio between the two coupling constants. More interesting are the admissible gauge groups that lie entirely in the group \( \text{SO}(8) \). Note that for \( N = 8 \), the condition \( \Gamma_{AB}^{IJKL}\Theta_{IJKL} = 0 \), no longer forces the

\(^{5}\) Note, that our conventions here differ from those used in [40, 4] by a triality rotation \( 8_{v} \leftrightarrow 8_{s} \) of \( \text{SO}(8) \) in order to fit into the general scheme.

\(^{6}\) Just as in [4], a slightly stronger consistency condition had been given, namely simultaneous absence of the \((2, 0, 0, 0, \ldots) \). This is in general not necessary. Restricting to compact gauge groups \( G_{0} \subset \text{SO}(8) \) however, the two conditions turn out to be equivalent.
entire antisymmetric part \( \Theta_{IJ,KL} \) of the embedding tensor to vanish, rather the embedding tensor of a compact gauge group \( G_0 \subset SO(8) \) takes the general form

\[
\Theta_{IJ,KL} = \delta_{I[K} \Xi_{J]L} + \Xi_{IJKL},
\]

(5.20)

with a traceless symmetric tensor \( \Xi_{IJ} \) and selfdual \( \Xi_{IJKL} = \frac{1}{24} \varepsilon_{IJKLMNPQ} \Xi_{PQR} \), relaxing (5.12). In the standard way (5.13), the group \( SO(4) \times SO(4) \) may be embedded with relative coupling constant of \(-1\). A nonvanishing \( \Xi_{IJKL} \) in (5.20) furthermore allows to introduce an arbitrary relative coupling constant \( \alpha \) between the two factors inside of each \( SO(4) \) (see [4] for details). This corresponds to the existence of the one-parameter family \( D^1(2,1;\alpha) \) of \( N = 4 \) superextensions of the AdS group \( SO(2,2) \), which appear as spacetime isometries.

\( N = 6 \): Closer inspection of the decomposition (B.6) of \( \Theta \) shows that the maximal compact subgroup \( H = SU(4) \times SU(k) \times U(1) \) is among the admissible gauge groups with an embedding tensor which forms a linear combination of the corresponding singlets in (B.6). Since a compact gauge group requires \( \Theta_{AB} = 0 \), and the four singlets appearing in (B.6) are linearly related, there are only two independent coupling constants. Other compact gauge groups are obtained by replacing the SU(4) factor by one of its subgroups \( SO(p) \times SO(6-p) \), the embedding tensor taking the form (5.13). The total embedding tensor is given by

\[
\Theta = \Theta_{SO(p)} - \Theta_{SO(6-p)} + \alpha \Theta_{SU(k)} - \frac{4\alpha(k-1) + k(p-3)}{4+k} \Theta_{U(1)},
\]

(5.21)

with a free parameter \( \alpha \). The relative coefficients are obtained in a similar way as in (5.17), (5.18), generalizing (5.16) to products of more factors and using \( G_N = SO(4,k) \). The only compact admissible gauge group which lies entirely in \( SO(6) \) is its subgroup \( SO(3) \times SO(3) \).

\( N = 5 \): The explicit decomposition of the \( T \)-tensor (B.7) shows that the group \( Sp(2) \sim SO(5) \) as well as its product with the entire \( Sp(k) \) and independent coupling constants, are admissible gauge groups. The embedding tensors are given by a linear combination of the two corresponding singlets in the decomposition (B.7). Instead of \( Sp(2) \) one may also gauge any of its subgroups \( SO(4) \), or \( SO(2) \times SO(3) \), the embedding tensor taking the form (5.13). From (B.7) it follows moreover that for \( N = 5 \) even the completely antisymmetrized tensor \( \Theta_{[IJ,KL]} \) may be nonvanishing for a compact gauge group, since the two \( (5,0,0,0,\ldots) \) representations in (B.7) may be chosen independently.
\begin{tabular}{|c|c|c|c|}
\hline
G/H & d & $R_{\text{adj}}$ & $R_{\text{adj}} \times_{\text{sym}} R_{\text{adj}}$ \\
\hline
Sp(m,1) & 4m & (2,0,\ldots) & (0,0,\ldots) + (0,1,\ldots) + (0,2,\ldots) + (4,0,\ldots) \\
Sp(m) \times Sp(1) & 4m & (1,\ldots,1) & (0,\ldots,0) + (0,1,\ldots,1,0) + (1,\ldots,1) + (2,\ldots,2) \\
SU(m,2) & 4m & (0,1,\ldots) & (0,0,\ldots) + (0,0,0,1,\ldots) + (2,0,\ldots) + (0,2,\ldots) \\
SU(m) \times U(2) & 4m & & \\
SO(m,4) & 4m & & \\
SO(m) \times SO(4) & 4m & & \\
\hline
G_2(2) & 8 & 14 & 1 + 27 + 77 \\
Sp(3) \times Sp(1) & 28 & 52 & 1 + 324 + 1053 \\
E_6(2) & 40 & 78 & 1 + 650 + 2430 \\
SU(6) \times Sp(1) & 64 & 133 & 1 + 1539 + 7371 \\
SO(12) \times Sp(1) & 112 & 248 & 1 + 3875 + 27000 \\
\hline
\end{tabular}

Table II: Symmetric spaces for $N = 4$. The representation $R_{0}$ from \[5.10\] is underlined.

### 5.4 $N = 4$: the symmetric spaces

Recall, that for $N = 4$ the target space manifold is a product of two quaternionic spaces and consistency of the gauged theory is encoded in equation \[4.29\] for the $T$-tensor to be satisfied on the entire target space manifold. In case, the quaternionic spaces are symmetric, one can apply the techniques described in this section to lift equation \[4.29\] to an algebraic projection equation on the embedding tensor, and exploit the known decomposition of the isometry group under SO(4) to perform a similar analysis of the admissible gauge groups. To this end, we list all quaternionic symmetric spaces \[41\] together with the decompositions \[5.8\] and the representations $R_{0}$ defining \[5.10\] in table \[4\] We leave the further study of these gaugings and the admissible gauge groups to the reader.

### 6 Concluding remarks

We have constructed the general $N$-extended gauged supergravity theories in three space-time dimensions. The gaugings constitute supersymmetric deformations of the ungauged theories of \[1\], and are entirely characterized by a symmetric embedding tensor $\Theta_{MN}$ that specifies the gauge group as a subgroup of the full invariance group of the ungauged theories. This invariance group consists of the target-space isometries, and (for $N = 2,4$) possible R-symmetry transformations. The embedding tensor must generate a proper subgroup and must be invariant under the gauge group. This is expressed by the conditions \[3.13\]. Supersymmetry imposes additional conditions on the embedding tensor, which are expressed in terms of constraints on the so-called $T$-tensor. For $N > 2$ these
constraints are encoded in (3.36). We have analyzed these constraints for the different values of \( N \). Any subgroup can be gauged for \( N \leq 3 \). For \( N = 1, 2 \) there exist supersymmetric deformations that are not induced by a gauging; for \( N = 2 \) their presence may form an obstacle to certain gauge groups. For \( N \geq 4 \) there are restrictions on the embedding tensor and thus on the corresponding subgroups that can be gauged. For \( N > 4 \) all target spaces are symmetric. For symmetric target spaces with \( N \geq 4 \) the restriction on the gauge group can conveniently be formulated in terms of the algebraic projection equation (5.10) on the embedding tensor \( \Theta \).

The gaugings require the usual masslike terms and a scalar potential in the Lagrangian, parametrized by three tensors \( A_1, A_2 \) and \( A_3 \),

\[
\mathcal{L} = \mathcal{L}_0 + \varepsilon g \left\{ \frac{1}{2} A_1^{IJ} \bar{\psi}_I^\mu \gamma^{\mu \nu} \psi_J^\nu + A_2^{IJ} \bar{\psi}_I^\mu \gamma^{\mu} \chi^{iJ} + \frac{1}{2} A_3^{IJ} \bar{\chi}^i_I \chi^{jJ} \right\} + \frac{4 \varepsilon g^2}{N} \left( A_1^{IJ} A_1^{IJ} - \frac{1}{2} N g^{ij} A_2^{IJ} A_2^{IJ} \right),
\]

(6.1)

where \( \mathcal{L}_0 \) denotes the ungauged Lagrangian (2.23) with the spacetime derivatives extended by extra covariantizations associated with the gauging, as specified in (3.3), (3.8) and (3.9).

The supersymmetry transformations of the fermion fields acquire extra terms proportional to \( A_1 \) and \( A_2 \). They read

\[
\delta \psi_I^\mu = \mathcal{D}_\mu \epsilon^I - \frac{1}{8} g_{ij} \bar{\chi}^I_i \gamma^\nu \chi^{jJ} \gamma_{\mu \nu} \epsilon^J - \delta \phi^i Q_i^{1J} \psi_I^\mu + g A_1^{IJ} \gamma_{\mu} \epsilon^J,
\]

\[
\delta \chi^{iJ} = \frac{1}{2} \left( \delta^{IJ}_1 - f^{IJ}_1 \right) j_i \bar{\mathcal{D}}_j \phi^i \epsilon^J - \delta \phi^j \left( \Gamma^i_{jk} \chi^{kI} + Q_i^{1J} \chi^{iJ} \right) - g N A_2^{1I} \epsilon^J.
\]

(6.2)

The transformation rules for \( e_{\mu}^a \) and \( \phi^i \) remain as given in (2.21), for the vector fields \( A_\mu^M \) they have been given in (3.11).

For \( N > 2 \), the tensors \( A_1, A_2 \) and \( A_3 \) are uniquely given in terms of the \( T \)-tensor (c.f. (3.14)) by means of (3.21) and (3.28),

\[
A_1^{IJ} = -\frac{4}{N-2} T^{IJ,JM} + \frac{2}{(N-1)(N-2)} \delta^{IJ} T^{MN,MM},
\]

\[
A_2^{IJ} = \frac{2}{N} T^{IJ} + \frac{4}{N(N-2)} f^{M(I}_j T^{J)M} + \frac{2 \delta^{IJ}}{N(N-1)(N-2)} f^{KL}_j m T^{KL} m,
\]

\[
A_3^{IJ} = \frac{1}{N^2} \left\{ -2 D_{(i} D_{j)} A_1^{IJ} + g_{ij} A_1^{IJ} + A_1^{[I} f^{J]K} + 2 T_{ij} \delta^{IJ} - 4 D_{[i} T^{IJ}_{j]} - 2 T_{k[i} f^{IJk}_{j]} \right\}.
\]

(6.3)

The cases \( N = 1, 2 \) require a separate analysis, which was presented in section 4. For symmetric spaces with \( N \geq 4 \), these results simplify considerably.
All gauged theories exhibit a potential (3.32) for the scalar fields \( \phi^i \). In certain cases this potential is constant and simply constitutes a cosmological term. In applications one is often interested in stationary points of this potential which give rise to anti-de Sitter or Minkowski solutions with residual supersymmetries, or in de Sitter solutions. Extremal points in the maximal \( N = 16 \) theory have been analyzed in some detail in \[3, 42, 43\]. Here we give a generalization to arbitrary \( N \) of some essential formulae that may enable the reader to carry out a similar analysis for the theories presented in this paper.

Stationary points of the scalar potential are characterized by (3.33), which together with the second equation of (3.34) implies the following relation at the stationary point,\(^7\)

\[
3 A_1^{IK} A_2^{KJ} + N g^{kl} A_2^{IK} A_3^{KJ} = 0 .
\]  

(6.4)

The residual supersymmetry of the corresponding solution (assuming maximally symmetric spacetimes) is parametrized by spinors \( \epsilon^I \) satisfying the condition,

\[
A_2^{JI} \epsilon^J = 0 ,
\]  

(6.5)

which ensures that the fermions \( \chi^{iI} \) remain invariant in a bosonic background. Full unbroken supersymmetry implies that \( A_2 \) vanishes at the stationary point. From the gravitino variations one derives the condition,

\[
A_1^{IK} A_1^{KJ} \epsilon^J = - \frac{V_0}{4 g^2} \epsilon^I = \frac{1}{N} \left( A_1^{IJ} A_1^{IJ} - \frac{1}{2} N g^{ij} A_2^{IJ} A_2^{IJ} \right) \epsilon^I ,
\]  

(6.6)

where \( V_0 \) is the potential taken at the stationary point. Let us emphasize that the two conditions (6.5) and (6.6) are in fact equivalent by virtue of the first equation of (3.34), so that the condition (6.6) suffices for establishing the (residual) supersymmetry. Obviously the potential must be non-positive at the stationary point, so that the maximally symmetric spacetime must be a Minkowski or an anti-de Sitter spacetime.

From the above results it follows that residual supersymmetries are associated with eigenvalues of \( A_1^{IJ} \) equal to \( \pm \sqrt{-V_0/4 g^2} \). The massive gravitini that may arise are associated with different eigenvalues and span an orthogonal subspace. We will distinguish the indices associated to this orthogonal subspace by \( \hat{I}, \hat{J}, \ldots \). The massive gravitini can be identified by the linear combinations,

\[
\psi^I_{\mu} \propto \left( A_1^{IK} A_1^{KJ} + \frac{V_0}{4 g^2} \delta^{IJ} \right) \psi^J_{\mu} + \frac{1}{2 g} A_2^{IJ} \partial_\mu \chi^{jJ} + \frac{i}{2} A_1^{IK} A_2^{KJ} \gamma_\mu \chi^{jJ} ,
\]  

(6.7)

\(^7\)In the remainder of this section the tensors \( A_1, A_2 \) and \( A_3 \) are constant and equal to their values at the stationary point.
which are explicitly restricted to the orthogonal subspace. Imposing the unitary gauge by the condition
\[ A_{2j}^{IJ} \chi^{ij} = 0 , \] (6.8)
we can simply determine the fermionic mass matrices by projection,
\[ M_{\text{gravitini}} = g A_{1i}^{ij} , \]
\[ M_{\text{fermions}} = g A_{3ij} + 6 g A_{2i}^{\hat{K}l} \left[ \frac{g^2 A_1}{4g^2 A_1^2 - 1} V_{0j} \right]_{KL} A_{2j}^{ij}. \] (6.9)
Observe that the restriction to the indices associated with the orthogonal subspace is crucial as otherwise the second mass matrix would diverge. To show that this matrix is orthogonal to the condition (6.8), one makes use of the identity (6.4).

It remains to evaluate the mass matrices for the bosons, which can simply be read off from the Lagrangian and are equal to,
\[ M_{\text{vectors}} = g^2 \Theta_{MN} V^{\nu_i} V^{\nu_j} \]
\[ M_{\text{scalars}}^{2} = g^2 D_i \partial_j V . \] (6.10)
The derivatives of the potential may be explicitly evaluated using the various identities derived previously. Moreover, the form of the vector mass matrix together with the corresponding kinetic Chern-Simons term shows that the physical vector masses are encoded in the matrix \( g T_{ij} = g \Theta_{MN} V^{\nu_i} V^{\nu_j} \), so that eventually all mass matrices can be expressed in terms of the \( T \)-tensors. This is a common feature of all gauged supergravity theories. In case of residual supersymmetry the spectrum (6.9), (6.10) decomposes into representations of the appropriate superextension of the AdS3 isometry group \( \text{SO}(2, 2) = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \).

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A About SO(N) projectors

For a symmetric target space, the consistency conditions on the \( T \)-tensor combine into the G-invariant form (5.9). In the general case, these consistency conditions take an SO(N)
covariant form, c.f. (3.25), (3.26), etc. Whereas the SO($N$) representation content of (3.26) is obvious (c.f. (3.38)), it is complicated to disentangle the independent parts of the consistency conditions (3.25) for the components $T^{IJ}_{i}$, derived from (3.20), (3.21). In this appendix, we present a systematic approach to handle these equations by means of SO($N$) covariant projectors.

The tensors $f^{IJ}$ from (2.9) define a $d$-dimensional representation of SO($N$). Consider the space of tensors $\Xi^{IJ}_{ij}$ of dimension $dN^2$. Using the $f^{IJ}$, we can build the following SO($N$) covariant maps on this space

$$
\text{Id}^{KL}_{IJ} \equiv \delta_{i}^{K} \delta_{j}^{L} \delta_{n}^{n}, \quad P^{KL}_{IJ} \equiv \frac{1}{N} \delta_{i}^{K} (\delta_{j}^{L} \delta_{n}^{n} - f^{JL}_{Jn}),
$$

$$
\Pi^{KL}_{IJ} \equiv \delta_{L}^{K} \delta_{J}^{i} \delta_{n}^{n}, \quad P_{0}^{KL}_{IJ} \equiv \frac{1}{N} \delta_{I,J} \delta_{KL} \delta_{n}^{n}.
$$

(A.1)

They satisfy the following set of relations

$$
\Pi^{2} = \text{Id}, \quad P^{2} = P, \quad P^{2} = \Pi P_{0} = P_{0} \Pi = P_{0}, \quad P_{0} P \Pi = \frac{2}{N} P_{0} - P_{0} P, \quad \Pi P P_{0} = \frac{2}{N} P_{0} - P P_{0}, \quad P_{0} P P_{0} = \frac{1}{N} P_{0}, \quad P \Pi P = \frac{2}{N} P - N PP_{0} P,
$$

(A.2)

which follow from (2.4). An equivalent form of the last relation is

$$
\Pi^{Km}_{Ii} \Pi^{Pj}_{Lm} + \Pi^{Lm}_{Ii} \Pi^{Pj}_{Km} = \frac{2}{N} \delta^{KL} \Pi^{Pj}_{Ii},
$$

(A.3)

which proves to be useful in checking the absence of several cubic fermion terms in the supersymmetry variation of the Lagrangian.

For a tensor $\Xi^{IJ}_{ij}$, consider the following inhomogeneous system of linear equations

$$
P \Xi = \Xi, \quad (\text{Id} - \Pi) \Xi = 2 Z.
$$

(A.4)

This system admits the unique solution $\Xi = T Z$ if and only if $Z$ satisfies the projection equation

$$
Z = \mathcal{P} Z,
$$

(A.5)

where $\mathcal{T}$ and the projector $\mathcal{P}$ are given by

$$
\mathcal{P} \equiv \frac{N (P - P \Pi - \Pi P + \Pi P \Pi)}{2(N-2)} - \frac{N (P_{0} - N (PP_{0} + P_{0} P) + N^{2} PP_{0} P)}{(N-1)(N-2)},
$$

$$
\mathcal{T} \equiv \frac{N}{N-2} (P - P \Pi) + \frac{N^{2}}{(N-1)(N-2)} (PP_{0} - NPP_{0} P),
$$

44
with $\mathcal{P} \mathcal{P} = \mathcal{P}$, and $\text{tr} \, \mathcal{P} = dN$. The necessity of the projection condition (A.3) follows from $2Z = (\text{Id} - \Pi) \mathcal{P} \Xi$ and from using the relation

\[(\text{Id} - \Pi) \mathcal{P} = \mathcal{P} (\text{Id} - \Pi) \mathcal{P} , \quad (A.6)\]

which follows straightforwardly from (A.2). Likewise, one may verify the relations

\[(\text{Id} - \Pi) \mathcal{T} = 2\mathcal{P} , \quad \mathcal{P} \mathcal{T} = \mathcal{T} = \mathcal{T} \mathcal{P} , \quad (A.7)\]

which ensure that $\Xi = \mathcal{T} Z$ together with (A.5) solves (A.4).

This algebra can be applied to perform a systematic analysis of the constraint equations in section 3. As an example, note that the antisymmetric part of the first equation in (3.21) together with (3.20) constitutes a system of type (A.4) with $Z = \frac{2}{N} \mathcal{T}$. Its solubility thus implies the consistency relation

\[T = \mathcal{P} \mathcal{T} , \quad (A.8)\]

on $T^I_J$, which precisely agrees with (3.25). Since $\mathcal{P}$ is a projector, this shows that equation (3.25) indeed describes a closed set of consistency relations with nontrivial solution. The tensor $A_2$ is given as $A_2 = \frac{2}{N} \mathcal{T} \mathcal{T}$ which agrees with (3.21) upon eliminating $A_1$ by means of (3.28). The proof again makes use of the constraint (A.8) on $T^I_J$.

\section*{B Explicit decompositions of the $T$-tensor}

The representation content of the $T$-tensor under the group $G$ for the various values of $N > 4$ has been given in table I. In this appendix, we give the explicit decomposition of the $T$-tensor under the compact group $H = \text{SO}(N) \times H'$. Since the embedding tensor of the gauge group is obtained as $\Theta = T_{\nu=1}$, it satisfies the same decomposition. This has been used in the main text to further analyze the admissible compact gauge groups in sections 5.2 and 5.3.

\textbf{$N = 16$}: Under $\text{SO}(16)$, the adjoint representation of $E_{8(8)}$ decomposes into

\[X^{IJ} : \quad 120 , \quad Y^A : \quad 128 , \]

implying that the $T$-tensor of the gauged theory consists of

\[T^{IJKL} = 1 + 135 + 1820 , \quad T^{AB} = 1 + 1820 , \quad T^{IJA} = 1920 , \quad (B.1)\]
where the two singlets 1 and the two representations 1820 coincide.

\[ N = 12 : \] Under SO(12) \( \times \) SU(2), the adjoint representation of E\(_7(-5)\) decomposes into

\[
X^{IJ} : (66, 1) , \quad X^\alpha : (1, 3) , \quad Y^A : (32, 2) ,
\]

implying that the \( T \)-tensor consists of

\[
T^{IJ,KL} = (1, 1) + (77, 1) + (495, 1) , \quad T^{\alpha \beta} = (1, 1) , \quad T^{IJ\alpha} = (66, 3) ,
\]

\[
T^{AB} = (1, 1) + (495, 1) + (66, 3) ,
\]

\[
T^{IJ,A} = (32, 2) + (352, 2) ,
\]

\[
T^{\alpha A} = (32, 2) , \quad (B.2)
\]

with a linear relation between the three singlets \((1, 1)\), and where the two representations in the \((66, 3)\), \((32, 2)\), and \((495, 1)\), respectively, coincide.

\[ N = 10 : \] Under SO(10) \( \times \) U(1), the adjoint representation of E\(_6(-14)\) decomposes into

\[
X^{IJ} : 45^0 , \quad X^\alpha : 1^0 , \quad Y^A : 16^+ + \overline{16}^- ,
\]

implying that the \( T \)-tensor consists of

\[
T^{IJ,KL} = 1^0 + 54^0 + 210^0 , \quad T^{\alpha \beta} = 1^0 , \quad T^{IJ\alpha} = 45^0 ,
\]

\[
T^{AB} = 1^0 + 10^+ + 10^- + 45^0 + 210^0 ,
\]

\[
T^{IJ,A} = 16^+ + \overline{16}^- + \overline{144}^+ + 144^- ,
\]

\[
T^{\alpha A} = 16^+ + \overline{16}^- , \quad (B.3)
\]

with a linear relation between the three singlets \(1^0\), and where the two representations in the \(16^+, \overline{16}^-, 45^0, \) and \(210^0\), respectively, coincide.

\[ N = 9 : \] Under SO(9), the adjoint representation of F\(_4(-20)\) decomposes into

\[
X^{IJ} : 36 , \quad Y^A : 16 ,
\]
implying that the $T$-tensor consists of

\[
T^{IJ,KL} = 1 + 44 + 126 ,
T^{AB} = 1 + 9 + 126 ,
T^{IJ,A} = 16 + 128 ,
\]

where the two $126$ representations coincide.

$N = 8$: Under $\text{SO}(8) \times \text{SO}(k)$, the adjoint representation of $\text{SO}(8,k)$ decomposes into

\[
X^{IJ} : (28, (0,0,0,0,...)) , \quad X^\alpha : (1, (0,1,0,0,...)) ,
Y^A : (8_s, (1,0,0,0,...)) ,
\]

implying that the $T$-tensor consists of

\[
T^{IJ,KL} = (1 + 35_s + 35_s + 35_c , (0,0,0,0,...) )
T^{\alpha\beta} = (1 , (0,0,0,0,...) + (2,0,0,0,...) + (0,0,0,1,...)) ,
T^{IJA} = (28 , (0,1,0,0,...))
T^{AB} = (1 , (0,0,0,0,...) + (2,0,0,0,...)) + (35_s , (0,0,0,0,...))
\quad + (28 , (0,1,0,0,...)) ,
T^{IJ,A} = (8_s + 56_s , (1,0,0,0,...)) ,
T^{\alpha A} = (8_s , (1,0,0,0,...) + (0,0,1,0,...)) ,
\]

with a linear relation between the three singlets $(1 , (0,0,0,0,...))$, and where the two representations in the $(1 , (2,0,0,0,...)) , (35_s , (0,0,0,0,...)) , (8_s , (1,0,0,0,...))$, and $(28 , (0,1,0,0,...))$, respectively, coincide.

$N = 6$: Under $\text{SU}(4) \times \text{SU}(k)$, the adjoint representation of $\text{SU}(4,k)$ decomposes into

\[
X^{IJ} : (15 , (0,0,...,0,0)) , \quad X^\alpha : (1 , (0,0,...,0,0) + (1,0,...,0,1)) ,
Y^A : (\bar{4} , (1,0,...,0,0) + (4 , (0,0,...,0,1)) ,
\]

implying that the $T$-tensor consists of

\[
T^{IJ,KL} = (1 + 15 + 20' , (0,0,...,0,0)) ,
\]

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\( T^{\alpha\beta} = (1, 2 \cdot (0, 0, \ldots, 0, 0) + 2 \cdot (1, 0, \ldots, 0, 1) + (0, 1, \ldots, 1, 0)) \),
\( T^{IJ\alpha} = (15, (0, 0, \ldots, 0, 0) + (1, 0, \ldots, 0, 1)) \),
\( T^{AB} = (1 + 15, (0, 0, \ldots, 0, 0) + (1, 0, \ldots, 0, 1)) \\
+ (6, (0, 1, \ldots, 0, 0) + (0, 0, \ldots, 1, 0)) \),
\( T^{IJ,A} = (\mathbf{10} + 20, (1, 0, \ldots, 0, 0)) + (\mathbf{4} + 2\mathbf{10}, (0, 0, \ldots, 0, 1)) \),
\( T^{\alpha A} = (4, (0, 0, \ldots, 1, 0) + 2 \cdot (0, 0, \ldots, 0, 1)) \\
+ (\mathbf{4}, (0, 1, \ldots, 0, 1) + 2 \cdot (1, 0, \ldots, 0, 0)) \),

(B.6)

with a linear relation between the four singlets, a linear relation between the three representations in the \((1, (1, 0, \ldots, 0, 1)), (\mathbf{4}, (0, 0, \ldots, 0, 1)), (\mathbf{4}, (1, 0, \ldots, 0, 0)), \) and \((15, (0, 0, \ldots, 0, 0)), \) respectively, and where the two representations \((15, (1, 0, \ldots, 0, 1)) \) coincide.

\( N = 5: \) Under \( \text{Sp}(2) \times \text{Sp}(k) \), the adjoint representation of \( \text{Sp}(2, k) \) decomposes into

\( X^{IJ}: (\mathbf{10}, (0, 0, 0, \ldots)) \), \( X^\alpha: (1, (2, 0, 0, \ldots)) \), \( Y^A: (\mathbf{4}, (1, 0, 0, \ldots)) \),

implying that the \( T \)-tensor consists of

\( T^{IJ,KL} = (1 + 5 + 14, (0, 0, 0, \ldots)) \),
\( T^{\alpha\beta} = (1, (0, 0, 0, \ldots) + (0, 1, 0, \ldots) + (0, 2, 0, \ldots)) \),
\( T^{IJ\alpha} = (\mathbf{10}, (2, 0, 0, \ldots)) \),
\( T^{AB} = (1 + 5, (0, 0, 0, \ldots) + (0, 1, 0, \ldots)) + (\mathbf{10}, (2, 0, 0, \ldots)) \),
\( T^{IJ,A} = (\mathbf{4} + 16, (1, 0, 0, \ldots)) \),
\( T^{\alpha A} = (\mathbf{4}, (1, 0, 0, \ldots) + (1, 1, 0, \ldots)) \),

(B.7)

where the two representations \((\mathbf{10}, (2, 0, 0, \ldots)) \) in \( T^{IJ\alpha} \) and \( T^{AB} \) coincide.

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