CLASSIFICATION OF MINIMAL ACTIONS OF A COMPACT KAC ALGEBRA WITH AMENABLE DUAL ON INJECTIVE FACTORS OF TYPE III

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ABSTRACT. We classify a certain class of minimal actions of a compact Kac algebra with amenable dual on injective factors of type III. Our main technical tools are the structural analysis of type III factors and the theory of canonical extension of endomorphisms introduced by Izumi.

1. Introduction

The purpose of this paper is to extend the classification result of \cite{18} to type III case, that is, to show uniqueness of certain minimal actions of a compact Kac algebra with amenable dual on injective type III factors.

On compact group actions on type III factors, there are some preceding works relevant with our work. The complete classification for compact abelian groups was obtained by Y. Kawahigashi and M. Takesaki \cite{15}. The recent result due to M. Izumi \cite{12} is remarkable. Among other things, he showed the conjugacy result for certain minimal actions of compact groups. More precisely, if minimal actions of a compact group on type III\(_0\) factors are faithful on the flow of weights and have the common Connes-Takesaki modules \cite{5}, then they are conjugate.

In this paper, we classify minimal actions whose dual actions are approximately inner and centrally free, which can be regarded as the generalization of classification results for trivial invariant case in \cite{15}. One should notice that these objects are different from Izumi’s ones because minimal actions studied in \cite{12} are duals of free and centrally trivial actions. Our strategy is on the whole same as \cite{18}, that is, we mainly handle actions of an amenable discrete Kac algebra \(\hat{G}\) instead of a compact Kac algebra \(G\), and obtain our main theorem through a duality argument \cite{6}. It seems, however, difficult to generalize the argument in \cite{18} to type III McDuff factors because of the lack of traces.

We present a different approach for injective type III factors starting from the classification for type II\(_1\) case \cite{18}. More precisely, we extend given actions of \(\hat{G}\) on type III factors to larger von Neumann algebras, which are the crossed products by abelian group actions. Then we classify the composed actions of the extended actions of \(\hat{G}\) and the dual actions. Splitting the dual actions and taking the partial crossed products, we show that all approximately inner and centrally free actions come from a free action on the injective type II\(_1\) factor. In these processes,
what play crucial roles are the structure analysis of type III factors \[5, 26\], Izumi’s theory on canonical extension of endomorphisms \[12\] and the characterization of approximate innerness and central triviality of endomorphisms \[19\].

This paper is organized as follows.

In §2, our main results and their applications are stated.

In §3, we prove the necessary results for the study in the later sections. In particular, the relative Rohlin theorem proved in §3.3 plays an important role for our model action splitting argument.

In §4, type III\(_{\lambda}\) case \((0 < \lambda < 1)\) is studied. Considering the discrete decomposition, we can reduce our problem to classifying actions of direct product of \(\hat{G}\) and the integer group \(\mathbb{Z}\) on the injective type II\(_\infty\) factor. Here, the \(\mathbb{Z}\)-action has non-trivial Connes-Takesaki module, and the main theorem of \[18\] is not immediately applicable. However, we can show the model action splitting as in \[2\] that enables us to cancel the Connes-Takesaki module and to use the main theorem of \[18\].

In §5, type III\(_0\) case is studied. We make use of the continuous decomposition, and represent a flow of weights as a flow built under a ceiling function. Then all things are reduced to the type II case as in \[23, 24\]. We classify actions of the direct product of \(\hat{G}\) and an AF ergodic groupoid on the injective type II\(_\infty\) factor by using \[18\] and Krieger’s cohomology lemma \[13\].

In §6, type III\(_1\) case is studied. Following the line of Connes and Haagerup’s theory of classification of injective factors of type III\(_1\) \[4, 9\], we consider the discrete decomposition of the type III\(_\lambda\) factor by the type III\(_1\) factor. Then we classify actions of the direct product of \(\hat{G}\) and the torus coming from the dual action by showing the model action splitting in §6.3 and 6.4. The key point here is approximate innerness of modular automorphisms.

In §7, we prove some basic results on the canonical extension in order that readers can smoothly shift from theory of endomorphisms to that of actions of discrete Kac algebras. Most of them directly follow from \[19\] by making use of the notion of a Hilbert space in a von Neumann algebra \[21\].

Acknowledgments.

A part of this work was done while the first named author stayed at Fields Institute and the second named author stayed at Katholieke Universiteit Leuven. They express gratitude for their warm hospitality.

2. Notations and Main theorem

Throughout this paper, we treat only separable von Neumann algebras, except for ultraproduct von Neumann algebras. We freely use the notations in \[18\]. For example, \(\hat{G} = (L^\infty(\hat{G}), \Delta, \varphi)\) denotes a discrete Kac algebra. Although some of our results are applicable to a general discrete Kac algebra, we always assume the amenability of \(\hat{G}\) before \[7\] See \[18\] and the references therein for the definition of a discrete (or compact) Kac algebra and its amenability.
For a von Neumann algebra $M$, we denote by $U(M)$ the set of unitary elements in $M$. By $W(M)$, we denote the set of faithful normal semifinite weights on $M$.

By [3,16,4,9], injective type III factors are determined by their flow of weights. We denote by $\mathcal{R}_0$, $\mathcal{R}_{0,1}$, $\mathcal{R}_\lambda$ and $\mathcal{R}_\infty$ the injective factor of type $\Pi_1$, type $\Pi_\infty$, type $\Pi_\lambda$ ($0 < \lambda < 1$) and type $\Pi_1$, respectively.

Let $\mathcal{M}$ be a factor. For a finite dimensional Hilbert space $K$, let $\text{Mor}_0(\mathcal{M}, \mathcal{M} \otimes B(K))$ be a set of all homomorphisms with finite index. When $\mathcal{M}$ is properly infinite, we can identify $\text{Mor}_0(\mathcal{M}, \mathcal{M} \otimes B(K))$ with $\text{End}_0(\mathcal{M})$, the set of endomorphisms of $\mathcal{M}$ with finite index. (See [7]) By $\text{Tr}_K$ and $\text{tr}_K$, we denote the non-normalized trace and the normalized trace on $B(K)$, respectively.

2.1. Actions and cocycle actions

We recall some definitions and notations used in [18] for readers’ convenience. Let $\mathcal{M}$ be a von Neumann algebra, $\alpha \in \text{Mor}(\mathcal{M}, \mathcal{M} \otimes L^\infty(\hat{\mathbb{G}}))$ and $u \in \text{U}(\mathcal{M} \otimes L^\infty(\hat{\mathbb{G}}) \otimes L^\infty(\hat{\mathbb{G}}))$. The pair $(\alpha, u)$ (or simply $\alpha$) is called a cocycle action of $\hat{\mathbb{G}}$ on $\mathcal{M}$ if the following holds:

1. $(\alpha \otimes \text{id}) \circ \alpha = \text{Ad} u \circ (\text{id} \otimes \Delta) \circ \alpha$;
2. $u \otimes 1 (\text{id} \otimes \Delta \otimes \text{id})(u) = \alpha(u)(\text{id} \otimes \text{id} \otimes \Delta)(u)$;
3. $u \cdot 1 = u \cdot 1 = 1$.

Here, $\alpha(u) := (\alpha \otimes \text{id} \otimes \text{id})(u)$, which is one of our conventions frequently used in our paper, that is, we will omit $\text{id}$ when the place where $\alpha$ acts is apparent. If $u = 1$, $\alpha$ is called an action. We introduce a left inverse $\Phi^\alpha_\pi : \mathcal{M} \otimes B(H_\pi) \to \mathcal{M}$ of $\alpha_\pi$ for each $\pi \in \text{Irr}(\mathbb{G})$ as follows:

$$
\Phi^\alpha_\pi(x) = (1 \otimes T^*_\pi \alpha)u^*_\pi \alpha(x)u^*_\pi(1 \otimes T^*_\pi) \quad \text{for} \quad x \in \mathcal{M} \otimes B(H_\pi).
$$

Then $\Phi^\alpha_\pi$ is a faithful normal unital completely positive map with $\Phi^\alpha_\pi \circ \alpha_\pi = \text{id}_\mathcal{M}$ [18] Lemma 2.4]. In general, a left inverse of $\alpha_\pi$ is not uniquely determined, but we only use the left inverse above. If $\mathcal{M}$ is a factor, then $\Phi^\alpha_\pi$ is standard, that is, the conditional expectation $\alpha_\pi \circ \Phi^\alpha_\pi : \mathcal{M} \otimes B(H_\pi) \to \alpha_\pi(\mathcal{M})$ is minimal (see Proposition[7,10]). The other easy but useful remark is the fact that $u$ is evaluated in $(M^\alpha)' \cap \mathcal{M}$, where $M^\alpha := \{ x \in \mathcal{M} \mid \alpha(x) = x \otimes 1 \}$ is a the fixed point algebra. This means that $(\alpha|_{(M^\alpha)' \cap \mathcal{M}}, u)$ is a cocycle action on $(M^\alpha)' \cap \mathcal{M}$.

2.2. Approximate innerness and central freeness

We collect basic notions of homomorphisms and actions from [18].

Definition 2.1. Let $\mathcal{M}$ be a von Neumann algebra and $\alpha \in \text{Mor}_0(\mathcal{M}, \mathcal{M} \otimes B(K))$ with the standard left inverse $\Phi^\alpha$. We say that

1. $\alpha$ is properly outer if there exists no non-zero element $a \in \mathcal{M} \otimes B(K)$ such that $a(x \otimes 1) = \alpha(x)a$ for any $x \in \mathcal{M}$;
2. $\alpha$ is approximately inner if there exists a sequence $\{ U^\nu \}_\nu \subset \text{U}(\mathcal{M} \otimes B(K))$ such that
   $$\lim_{\nu \to \infty} \|(\varphi \otimes \text{tr}_K) \circ \text{Ad}(U^\nu)^* - \varphi \circ \Phi^\alpha\| = 0 \quad \text{for all} \quad \varphi \in M_*;$$
3. $\alpha$ is centrally trivial if $\alpha^\omega(x) = x \otimes 1$ for all $x \in M_\omega$;
4. $\alpha$ is centrally non-trivial if $\alpha$ is not centrally trivial;
Lemma 8.3. Hence a free action \( \alpha \) then central non-triviality is equivalent to properly central non-triviality \([18, \text{Lemma } 2.8]\). If \( \alpha \) is irreducible, then central non-triviality is equivalent to properly central non-triviality \([18, \text{Lemma } 8.3]\). Hence a free action \( \alpha \) on a factor is centrally free if and only if \( \alpha \) is centrally non-trivial for each \( \pi \in \operatorname{Irr}(G) \setminus \{1\} \).

2.3. Main theorem

We recall the notion of the cocycle conjugacy for two (cocycle) actions.

**Definition 2.3.** Let \( M \) and \( N \) be von Neumann algebras. Let \( \alpha \in \operatorname{Mor}(M, M \otimes L^\infty(G)) \) and \( \beta \in \operatorname{Mor}(N, N \otimes L^\infty(G)) \) be cocycle actions of \( G \) with 2-cocycles \( u \) and \( v \), respectively.

1. \( \alpha \) and \( \beta \) are said to be conjugate if there exists an isomorphism \( \theta : M \to N \) such that
   - \( \alpha = (\theta^{-1} \otimes \text{id}) \circ \beta \circ \theta \);
   - \( u = (\theta^{-1} \otimes \text{id} \otimes \text{id})(v) \).

We say \( \alpha \approx \beta \) if \( \alpha \) and \( \beta \) are conjugate.

2. \( \alpha \) and \( \beta \) are said to be cocycle conjugate if there exist an isomorphism \( \theta : M \to N \) and \( w \in U(M \otimes L^\infty(G)) \) such that
   - \( \text{Ad} \ w \circ \alpha = (\theta^{-1} \otimes \text{id}) \circ \beta \circ \theta \);
   - \( w \alpha(w)u(\text{id} \otimes \Delta)(w^*) = (\theta^{-1} \otimes \text{id} \otimes \text{id})(v) \).

We write \( \alpha \sim \beta \) if \( \alpha \) and \( \beta \) are cocycle conjugate.

When \( \alpha \) is an action, \( v \in U(M \otimes L^\infty(G)) \) is called an \( \alpha \)-cocycle if \( (v \otimes 1) \alpha(v) = (\text{id} \otimes \Delta)(v) \). The following is the main theorem of this paper which asserts the uniqueness of approximate inner and centrally free action.

**Theorem 2.4.** Let \( G \) be a compact Kac algebra with amenable dual, \( M \) an injective factor, \( \alpha \) an approximately inner and centrally free action of \( \hat{G} \) on \( M \), and \( \alpha^{(0)} \) a free action of \( \hat{G} \) on \( \mathcal{R}_0 \). Then \( \alpha \) is cocycle conjugate to \( \text{id}_M \otimes \alpha^{(0)} \).

This implies the following in view of the duality theorem \([3, \text{Theorem IV.3}]\).

**Theorem 2.5.** Let \( G \) be a compact Kac algebra with amenable dual, \( M \) an injective factor, \( \alpha \) a minimal action of \( G \) on \( M \), and \( \alpha^{(0)} \) a minimal action of \( G \) on \( \mathcal{R}_0 \). If the dual action of \( \alpha \) is approximately inner and centrally free, then \( \alpha \) is cocycle conjugate to \( \text{id}_M \otimes \alpha^{(0)} \). If \( \alpha \) is a dual action, then \( \alpha \) and \( \text{id}_M \otimes \alpha^{(0)} \) are conjugate.
As a corollary, we obtain the following classification of minimal actions of compact Lie groups on $\mathcal{R}_\infty$.

**Corollary 2.6.** Let $G$ be a semisimple connected compact Lie group. Then any two minimal actions of $G$ on $\mathcal{R}_\infty$ are conjugate to each other.

**Proof.** This follows from Theorem 2.5, [19, Theorem 3.15, 4.12] and [12, Corollary 5.14]. \qed

Our main purpose is to prove Theorem 2.4. In [18, Theorem 7.1], we have proved that in type II$_1$ case. The remaining cases are type II$_\infty$, III$_\lambda$ ($0 < \lambda < 1$), III$_0$ and III$_1$. Type II$_\infty$ case is easily shown as follows.

- **Proof of Theorem 2.4** for $\mathbb{R}_{0,1}$.

Let $\alpha$ be an approximately inner and centrally free action of $\hat{\mathcal{G}}$ on $\mathcal{R}_{0,1}$. Let $\tau$ be a normal trace on $\mathcal{R}_{0,1}$. Since $\alpha$ is approximately inner, we have $\tau \circ \Phi_\pi = \tau \otimes \text{tr}_\pi$ for $\pi \in \text{Irr}(\hat{\mathcal{G}})$ by Corollary 7.7. Hence $\tau$ is invariant under $\alpha$.

Let $\{e_{i,j}\}_{i,j=1}^{\infty} \subset \mathbb{R}_{0,1}$ be a system of matrix units with a finite projection $e_{11}$. Since $(\tau \otimes \text{tr}_\pi)(e_{11} \otimes 1) = (\tau \otimes \text{tr}_\pi)(\alpha_{\pi}(e_{11}))$ for each $\pi \in \text{Irr}(\hat{\mathcal{G}})$, we can take $v \in \mathbb{R}_{0,1} \otimes \mathcal{L}\infty(\hat{\mathcal{G}})$ such that $vv^* = e_{11} \otimes 1$ and $v^*v = \alpha(e_{11})$. Set a unitary $V = \sum_{i=1}^{\infty}(e_{i1} \otimes 1)v\alpha(e_{i1})$. Then the perturbed cocycle action $\text{Ad}V \circ \alpha$ fixes the type I factor $B := \{e_{i,j}\}_{i,j}^\infty$. Therefore $\text{Ad}V \circ \alpha|_{B' \cap \mathcal{R}_{0,1}}$ is an approximately inner and centrally free cocycle action on the injective type II$_1$ factor $B' \cap \mathcal{R}_{0,1}$. By [18, Theorem 6.2], we can perturb $\text{Ad}V \circ \alpha|_{B' \cap \mathcal{R}_{0,1}}$ to be an action. Then this action is cocycle conjugate to the model action $\alpha^{(0)}$. Therefore we have $\alpha \sim \text{id}_{B(\ell_2)} \otimes \alpha^{(0)}$. Using $\alpha^{(0)} \sim \text{id}_{\mathcal{R}_0} \otimes \alpha^{(0)}$, we obtain $\alpha \sim \text{id}_{\mathcal{R}_{0,1}} \otimes \alpha^{(0)}$. \qed

By Theorem 2.4 any two approximately inner and centrally free actions $\alpha$ and $\beta$ on an injective factor $M$ are cocycle conjugate. This can be more precisely stated as [18, Theorem 7.1].

**Theorem 2.7.** Let $M$ be an injective factor and $\hat{\mathcal{G}}$ an amenable discrete Kac algebra. Let $\alpha$ and $\beta$ be approximately inner and centrally free actions of $\hat{\mathcal{G}}$ on $M$. Then there exists $\theta \in \text{Int}(M)$ and an $\alpha$-cocycle $v \in M \otimes \mathcal{L}\infty(\hat{\mathcal{G}})$ such that

$$\text{Ad}v \circ \alpha = (\theta^{-1} \otimes \text{id}) \circ \beta \circ \theta.$$ 

**Proof.** Since $M$ is injective, $M$ is isomorphic to $\mathcal{R}_0 \otimes M$. Fix an isomorphism $\Psi : M \rightarrow M \otimes \mathcal{R}_0$. Let $\alpha^{(0)}$ be a free action of $\hat{\mathcal{G}}$ on $\mathcal{R}_0$. Set $\gamma := (\Psi^{-1} \otimes \text{id}) \circ (\text{id}_M \otimes \alpha^{(0)}) \circ \Psi$, which is an approximately inner and centrally free action on $M$. By Theorem 2.4 we can take $\theta_0 \in \text{Aut}(M)$ and an $\alpha$-cocycle $v$ such that $\text{Ad}v \circ \alpha = (\theta_0^{-1} \otimes \text{id}) \circ \gamma \circ \theta_0$. To prove the theorem, it suffices to show the statement for $\beta = \gamma$.

Set $\theta_1 := \Psi \circ \theta_0 \circ \Psi^{-1} \in \text{Aut}(M \otimes \mathcal{R}_0)$. Note that the core of $M \otimes \mathcal{R}_0$ canonically coincides with $M \otimes \mathcal{R}_0$. Since the module map $\text{mod} : \text{Aut}(M) \rightarrow \text{Aut}(\mathcal{Z}(\hat{M}))$ is surjective by [25], there exists $\theta_2 \in \text{Aut}(M)$ such that $\text{mod}(\theta_1) = \text{mod}(\theta_2 \otimes \text{id}_{\mathcal{R}_0})$. Set $\theta_3 := \Psi^{-1} \circ (\theta_2 \otimes \text{id}_{\mathcal{R}_0}) \circ \Psi \in \text{Aut}(M)$. Then $\theta_3^{-1} \theta_0 = \Psi^{-1} \circ (\theta_2^{-1} \otimes \text{id}_{\mathcal{R}_0}) \theta_1 \circ \Psi$. \qed
implies $\text{mod}(\theta_3^{-1}\theta_0) = \text{id}$. Putting $\theta := \theta_3^{-1}\theta_0$, we have $\text{Ad} v \circ \alpha = (\theta^{-1} \otimes \text{id}) \circ \gamma \circ \theta$ because $\theta_3$ commutes with $\gamma$. Moreover $\theta \in \text{Int}(M)$ by [14, Theorem 1(1)]. □

3. Preliminaries

The results in this section are frequently used in the later sections. One of the most important results is the relative Rohlin theorem (Theorem 3.13).

3.1. Basic results on cocycle conjugacy

**Lemma 3.1.** Let $(\alpha, u)$ be a cocycle action of $\hat{G}$ on a properly infinite von Neumann algebra $M$. Let $H$ be a Hilbert space. Then $(\alpha, u)$ and $(\text{id}_{B(H)} \otimes \alpha, 1 \otimes u)$ are cocycle conjugate.

**Proof.** Take a Hilbert space $\mathcal{H} \subset M$ with support 1 and the same dimension $d \leq \infty$ as $H$ [21]. Let $\{\xi_i\}_{i=1}^d$ be an orthonormal basis of $\mathcal{H}$. Then we have the isomorphism $\Psi : B(H) \otimes M \to M$ such that $\Psi(e_{ij} \otimes x) = \xi_i x \xi^*_j$ for all $x \in M$ and $i, j \in \mathbb{N}$, where $\{e_{ij}\}_{ij}$ is a canonical system of matrix units of $B(H)$.

Define the unitary $v := \sum_{i=1}^d (\xi_i \otimes 1) \alpha(\xi^*_i)$. We check that $\Psi$ and $v$ satisfy the statement. For $x \in M$ and $i, j \in N$, we have

$$\text{Ad} v \circ \alpha \circ \Psi(e_{ij} \otimes x) = v \alpha(\xi_i x \xi^*_j) v^* = (\xi_i \otimes 1) \alpha(x) (\xi^*_j \otimes 1) = (\Psi \otimes \text{id}) \circ (\text{id} \otimes \alpha) (e_{ij} \otimes x).$$

Hence (1) holds. On (2), we have

$$(v \otimes 1) \alpha(v) u (\text{id} \otimes \Delta)(v^*)$$

$$= \sum_{i,j=1}^d (\xi_i \otimes 1 \otimes 1) (\alpha(\xi^*_i) \otimes 1) \cdot (\alpha(\xi_j) \otimes 1) \alpha(\alpha(\xi^*_j)) \cdot u (\text{id} \otimes \Delta)(v^*)$$

$$= \sum_{i=1}^d (\xi_i \otimes 1 \otimes 1) u (\text{id} \otimes \Delta)(\alpha(\xi^*_i)) (\text{id} \otimes \Delta)(v^*)$$

$$= \sum_{i=1}^d (\xi_i \otimes 1 \otimes 1) u (\xi^*_i \otimes 1 \otimes 1) = (\Psi \otimes \text{id} \otimes \text{id})(1 \otimes u).$$

□

**Lemma 3.2.** Let $(\alpha, u)$ be a cocycle action on a properly infinite von Neumann algebra $M$. Then $u$ is a coboundary.

**Proof.** By the previous lemma, it suffices to prove that $(\text{id}_{B(L^2(\hat{G}))} \otimes \alpha, 1 \otimes u)$ can be perturbed to an action. Write $\overline{\alpha} = \text{id}_{B(L^2(\hat{G}))} \otimes \alpha$ and $\overline{u} = 1 \otimes u$. Then we set a unitary $v := W_{31} u_{231} \in B(L^2(\hat{G})) \otimes M \otimes L^\infty(\hat{G})$, where $W \in L^\infty(\hat{G}) \otimes L^\infty(G)$ is the multiplicative unitary defined in [18, Section 2]. Using the 2-cocycle relation
of $u$ and $\Delta(x) = W^* (1 \otimes x) W$ for $x \in L^\infty(\hat{\mathcal{G}})$, we have

$$v_{123}^{-1}(v) \overline{w}((\text{id} \otimes \text{id} \otimes \Delta)(v^*))$$

$$= W_{31} u_{231}^* \cdot W_{41} (\alpha(u^*) \cdot (u \otimes 1) \cdot (\text{id} \otimes \Delta \otimes \text{id}))(u)_{2341} (\Delta \otimes \text{id})(W^*)_{341}$$

$$= W_{31} u_{231}^* \cdot W_{41} ((\text{id} \otimes \text{id} \otimes \Delta)(u))_{2341} (\Delta \otimes \text{id})(W^*)_{341}$$

$$= W_{31} u_{231}^* \cdot W_{41} (W_{34} u_{124} W_{34})_{2341} (\Delta \otimes \text{id})(W^*)_{341}$$

$$= W_{31} u_{231}^* \cdot W_{41} \cdot W_{41} u_{231} W_{41} \cdot (\Delta \otimes \text{id})(W^*)_{341}$$

$$= W_{31} W_{41} \cdot (\Delta \otimes \text{id})(W^*)_{341} = 1.$$

$\square$

Next we discuss the cocycle conjugacy of extended actions. For definition of the canonical extension of a cocycle action, readers are referred to [12] and [17].

**Lemma 3.3.** Let $\alpha$ be an action of $\hat{\mathcal{G}}$ on a properly infinite von Neumann algebra $M$. Then the second canonical extension $\tilde{\alpha}$ on $\tilde{M} \rtimes_g \mathbb{R}$ is cocycle conjugate to $\alpha$.

**Proof.** This is immediately obtained from Lemma 3.1 and Corollary 7.15. $\square$

We close this subsection with the following lemma.

**Lemma 3.4.** Let $\hat{\mathcal{G}}^i$ be a discrete Kac algebra for each $i = 1, 2$. Let $\alpha^i$ and $\beta^i$ be actions of $\hat{\mathcal{G}}^i$ on von Neumann algebras $M$ and $N$, respectively. Assume the following:

- $\alpha^1$ and $\alpha^2$ commute;
- $\beta^1$ and $\beta^2$ commute;
- The $\hat{\mathcal{G}}^1 \times \hat{\mathcal{G}}^2$ actions $\alpha := (\alpha^1 \otimes \text{id}) \circ \alpha^2$ and $\beta := (\beta^1 \otimes \text{id}) \circ \beta^2$ are cocycle conjugate.

Then the action $\alpha^1$ (resp. $\beta^1$) extends to the action $\overline{\alpha}^1$ on $M \rtimes_{\alpha^2} \hat{\mathcal{G}}^2$ (resp. $\overline{\beta}^1$ on $M \rtimes_{\beta^2} \hat{\mathcal{G}}^2$). Moreover, $\overline{\alpha}^1$ and $\overline{\beta}^1$ are cocycle conjugate.

**Proof.** Let $v$ be an $\alpha$-cocycle and $\Psi : M \to N$ be an isomorphism such that $\text{Ad} v \circ \alpha = (\Psi^{-1} \otimes \text{id}) \circ \beta \circ \Psi$. Set unitaries $v^\ell := v \otimes 1 \in M \otimes L^\infty(\hat{\mathcal{G}}^1)$ and $v^r := v_{1\otimes} \in M \otimes L^\infty(\hat{\mathcal{G}}^2)$, which are $\alpha^1$-cocycle and $\alpha^2$-cocycle, respectively. Then we define an isomorphism $\Theta : M \rtimes_{\alpha^2} \hat{\mathcal{G}}^1 \to N \rtimes_{\beta^2} \hat{\mathcal{G}}^2$ by $\Theta(x) = (\Psi \otimes \text{id})(v^r x (v^\ell)^*)$. We set a unitary $u := (\alpha^2 \otimes \text{id})(v^\ell) \in (M \rtimes_{\alpha^2} \hat{\mathcal{G}}^2) \otimes L^\infty(\hat{\mathcal{G}}^1)$. Then $u$ is an $\overline{\alpha}^1$-cocycle. By direct calculation, we have $\text{Ad} u \circ \overline{\alpha}^1 = (\Theta^{-1} \otimes \text{id}) \circ \overline{\beta}^1 \circ \Theta$. $\square$

3.2. Rohlin property

See [18] Section 3 for notions of ultraproduct algebras and actions on them. First we recall the following definition [18] Definition 3.4, 3.13].

**Definition 3.5.** Let $\gamma \in \text{Mor}(M^\omega, M^\omega \otimes L^\infty(\hat{\mathcal{G}}))$ be an action of $\hat{\mathcal{G}}$. We say that
(1) $\gamma$ is strongly free when for any $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$ and any countably generated von Neumann subalgebra $S \subset M^\omega$, there exists no non-zero $a \in M^\omega \otimes B(H_\pi)$ such that $\gamma_\pi(x)a = a(x \otimes 1)$ for all $x \in S' \cap M^\omega$.

(2) $\gamma$ is semi-liftable when for any $\pi \in \text{Irr}(\mathbb{G})$, there exist elements $\beta^\nu, \beta \in \text{Mor}_0(M, M \otimes B(H_\pi))$, $\nu \in \mathbb{N}$, such that $\beta^\nu$ converges to $\beta$ and $\gamma_\pi((x^\nu)^\nu) = \beta^\nu(x^\nu \nu)$ for all $(x^\nu)^\nu \in M^\omega$.

Note that a cocycle action $\alpha \in \text{Mor}(M, M \otimes L^\infty(\mathbb{G}))$ is centrally free if and only if $\alpha^\omega$ is strongly free [18, Lemma 8.2]. For $(x^\nu)^\nu \in M^\omega$, we set $\tau^\omega(x) := \lim_{\nu \to \omega} x^\nu$. Then $\tau^\omega : M^\omega \to M$ is a faithful normal conditional expectation.

**Definition 3.6.** Let $\mathbb{G}$ be an amenable discrete Kac algebra and $\gamma \in \text{Mor}(M^\omega, M^\omega \otimes L^\infty(\mathbb{G}))$ an action. We say that $\gamma$ has the Rohlin property when for any central $F \in \text{Projf}(L^\infty(\mathbb{G}))$, $\delta > 0$, $(F, \delta)$-invariant central $K \in \text{Projf}(L^\infty(\mathbb{G}))$ with $K \geq e_1$, any countable subset $S \subset M^\omega$ and any faithful state $\phi \in M_*$, there exists a projection $E \in M^\omega \otimes L^\infty(\mathbb{G})$ such that

(R1) $E$ is supported over $K$, that is, $E = E(1 \otimes K)$;

(R2) $E$ almost intertwines $\gamma$ and $\Delta$ in the following sense:

$$|\gamma_F(E) - (\text{id} \otimes_F \Delta)(E)||_{\phi \otimes_{\psi} \phi \otimes \varphi} \leq 5\delta^{1/2}|F|_\phi;$$

(R3) $E$ gives a copy of $L^\infty(\mathbb{G})K$, that is, if we decompose $E$ as

$$E = \sum_{\pi \in \text{supp}(K)} \sum_{i,j \in I_\pi} d(\pi)^{-1} E_{\pi,i,j} \otimes e_{\pi,i,j},$$

then, for all $i, j \in I_\pi$, $k, \ell \in I_\rho$ and $\pi, \rho \in \text{supp}(K)$, we have

$$E_{\pi,i,j} E_{\rho,k,\ell} = \delta_{\pi,\rho} \delta_{j,k} E_{\pi,i,\ell};$$

(R4) $(\text{id} \otimes \varphi_{\pi})(E) \in S' \cap M_\omega$ for any $\pi \in \text{supp}(K)$;

(R5) $E$ gives a partition of unity of $S' \cap M_\omega$, that is, $(\text{id} \otimes \varphi)(E) = 1$.

The above projection $E$ is called a Rohlin projection.

**Definition 3.7.** Let $\gamma \in \text{Mor}(M^\omega, M^\omega \otimes L^\infty(\mathbb{G}))$ be an action. Assume that $M^\omega$ is globally invariant under $\gamma$ and $\gamma|M_\omega$ has the Rohlin property. We say that $\gamma$ has the joint Rohlin property when for any $F \in \text{Projf}(L^\infty(\mathbb{G}))$, $\delta > 0$, $(F, \delta)$-invariant central $K \in \text{Projf}(L^\infty(\mathbb{G}))$ with $K \geq e_1$, any countable family of $\gamma$-cocycles $\mathcal{C}$ which are evaluated in $M^\omega$, there exists a projection $E \in M^\omega \otimes L^\infty(\mathbb{G})$ such that

(S1) $E$ satisfies (R1), (R2), (R3), (R4) and (R5);

(S2) For any $v \in \mathcal{C}$, a projection $vEv^*$ also satisfies (R3);

(S3) For any $v \in \mathcal{C}$ and $\pi \in \text{supp}(K)$, we have $(\text{id} \otimes \varphi_{\pi})(vEv^*) = (\text{id} \otimes \varphi_{\pi})(E)$.

**Lemma 3.8.** If $\gamma$ has the joint Rohlin property and $E$ is a projection as above, then the element $(\text{id} \otimes \varphi)(vE)$ is a unitary for all $v \in \mathcal{C}$.
Proof. Set $\mu := (\id \otimes \varphi)(vE)$ and $E' := vEv^*$. Then,

$$
\mu = \sum_{\pi \in \Irr(G)} \sum_{i,j \in I_\pi} \varphi_{\pi,i,j} E_{\pi,i} = \sum_{\pi \in \Irr(G)} \sum_{i,j \in I_\pi} E'_{\pi,i} \varphi_{\pi,i,j}.
$$

Using (R3) for $E$ and $E'$, we can check $\mu \mu^* = 1 = \mu^* \mu$ as follows,

$$
\mu \mu^* = \sum_{\pi \in \Irr(G)} \sum_{i,j,k,\ell \in I_\pi} \varphi_{\pi,i,j} E_{\pi,i}^* \varphi_{\pi,k,\ell} E_{\pi,k,\ell}^* = \sum_{\pi \in \Irr(G)} \sum_{i,j,k,\ell \in I_\pi} v_{\pi,i,j} E_{\pi,k,\ell} v_{\pi,k,\ell}^* = (\id \otimes \varphi)(E') = 1,
$$

and

$$
\mu^* \mu = \sum_{\pi \in \Irr(G)} \sum_{i,j,k,\ell \in I_\pi} \varphi_{\pi,i,j}^* (E')_{\pi,i,j} \varphi_{\pi,k,\ell} E_{\pi,k,\ell} = \sum_{\pi \in \Irr(G)} \sum_{i,j,k,\ell \in I_\pi} v_{\pi,i,j}^* (E')_{\pi,i,j} v_{\pi,k,\ell} E_{\pi,k,\ell} = (\id \otimes \varphi)(v^* E' v) = (\id \otimes \varphi)(E) = 1.
$$

Such an element $(\id \otimes \varphi)(vE)$ is called a Shapiro unitary.

Let $\hat{G}_i := G = (L^\infty(\hat{G}), \Delta), \hat{G}_2 = (L^\infty(\hat{G}^2), \Delta^2)$ be amenable discrete Kac algebras with the invariant weights $\varphi^1 := \varphi$ and $\varphi^2$, respectively. The product Kac algebra $G \times G^2$ is naturally constructed. The invariant weight and the coproduct are denoted by $\tilde{\varphi} = \varphi_{G \times G^2}$ and $\tilde{\Delta} = \Delta_{G \times G^2}$, respectively.

**Lemma 3.9.** Take $(F_i, \delta_i)$-invariant central projection $K_i \in L^\infty(\hat{G}_i)$ for $i = 1, 2$, respectively. Then $K_1 \otimes K_2$ is $(F_1 \otimes F_2, \delta_1 + \delta_2)$-invariant.

**Proof.**

$$
|\langle (F_1 \otimes F_2 \otimes 1 \otimes 1) \Delta(K_1 \otimes K_2) - F_1 \otimes F_2 \otimes K_1 \otimes K_2 | \tilde{\varphi} \otimes \tilde{\varphi} \rangle
\leq |\langle (F_1 \otimes F_2 \otimes 1 \otimes 1) \Delta(K_1 \otimes K_2) - (F_1 \otimes F_2 \otimes K_1 \otimes 1) \Delta^2(K_2)_{24} | \varphi \otimes \varphi \rangle
\ + |\langle (F_1 \otimes F_2 \otimes K_1 \otimes 1) \Delta^2(K_2)_{24} - F_1 \otimes F_2 \otimes K_1 \otimes K_2 | \tilde{\varphi} \otimes \tilde{\varphi} \rangle
\ = |\langle (F_1 \otimes 1) \Delta(K_1) - (F_1 \otimes K_1) | \varphi_1 \otimes \varphi_2 | \langle (F_2 \otimes 1) \Delta^2(K_2)_{24} | \varphi_2 \otimes \varphi_2 \rangle
\ + |\langle (F_2 \otimes 1) \Delta^2(K_2)_{24} - (F_2 \otimes K_2) | \varphi_2 \otimes \varphi_2 | F_1 \otimes K_1 | \varphi_1 \otimes \varphi_1 \rangle
\ < (\delta_1 + \delta_2) |F_1 \otimes F_2 | \tilde{\varphi} |K_1 \otimes K_2 | \tilde{\varphi}.
$$

The following elementary lemma is useful.

**Lemma 3.10.** Let $P, Q$ be von Neumann algebras. Let $\phi \in P_*$ and $\psi \in Q_*$ be faithful positive functionals, respectively. Let $X, Y \in P \otimes Q$ be given. If $X \in (P \otimes Q)_{\phi \otimes \psi}$, then one has

$$
|\langle (\id \otimes \psi)(YX) | \phi \rangle | \leq \|Y\| \|X\|_{\phi \otimes \psi}.
$$
Proof. Let \((\text{id} \otimes \psi)(YX) = w|[(\text{id} \otimes \psi)(YX)]|\) be the polar decomposition. Since \(X\) commutes with \(\phi \otimes \psi\), we have

\[
|(\text{id} \otimes \psi)(YX)|_{\phi} = \phi(w^*(\text{id} \otimes \psi)(YX)) = (\phi \otimes \psi)((w^* \otimes 1)YX) \\
\leq \|w^* \otimes 1\|_Y Y \|X\|_{\phi \otimes \psi} \leq \|Y\|_Y \|X\|_{\phi \otimes \psi}.
\]

\(\Box\)

3.3. Relative Rohlin theorem

Throughout this subsection, we are given the following:

(A1) A von Neumann algebra \(M\);
(A2) A \(\hat{G}\)-action \(\gamma^1\) on \(M^\omega\) and a \(\hat{G}^2\)-action \(\gamma^2\) on \(M^\omega\), and they are commuting;
(A3) The \(\hat{G} \times \hat{G}^2\)-action \(\gamma := (\gamma^1 \otimes \text{id}) \circ \gamma^2\) is strongly free and semi-liftable;
(A4) \(M\) is globally invariant under \(\gamma\);
(A5) \(\tau^\omega \circ \Phi_{(\pi, \rho)}^\gamma = \tau^\omega \otimes \text{tr}_\pi \otimes \text{tr}_\rho\) on \(M_\omega \otimes B(H_{(\pi, \rho)})\) for all \((\pi, \rho) \in \text{Irr}(\hat{G}) \times \text{Irr}(\hat{G}^2)\).

The assumption (A3) restricts not only \(\gamma\) but also \(M_{\omega}\). For example, \(M_\omega = \mathbb{C}\) is excluded. When \(M\) is a factor, (A5) automatically holds. Indeed by semi-liftable, we can take \((\beta^\nu)_\nu\), a sequence of homomorphisms on \(M\) converging to \(\beta\) and defining \(\gamma((\pi, \rho))\), that is, \(\gamma((\pi, \rho))(x) = (\beta^\nu(x))_\nu\) for \(x = (x^\nu)_\nu \in M^\omega\). Then by [18 Lemma 3.3], we obtain \(\tau^\omega \circ \Phi_{(\pi, \rho)}^\gamma = \Phi^\beta \circ (\tau^\omega \otimes \text{id})\) on \(M^\omega \otimes B(H_{(\pi, \rho)})\). Since \(M\) is a factor, \(\tau^\omega|_{M_{\omega}}\) is a trace. Hence for \(x \in M_\omega\) and \(y \in B(H_{(\pi, \rho)})\), we have

\[
\tau^\omega \circ \Phi_{(\pi, \rho)}^\gamma(x \otimes y) = \Phi^\beta \circ (\tau^\omega \otimes \text{id})(x \otimes y) = \tau^\omega(x) \Phi^\beta(1 \otimes y) \\
= \tau^\omega(x) \tau^\omega(\Phi_{(\pi, \rho)}^\gamma(1 \otimes y)) = \tau^\omega(x) (\text{tr}_\pi \otimes \text{tr}_\rho)(y).
\]

This shows \(\tau^\omega \circ \Phi_{(\pi, \rho)}^\gamma = \tau^\omega \otimes \text{tr}_\pi \otimes \text{tr}_\rho\) as desired.

Our aim is to prove the relative Rohlin theorem which assures that a Rohlin projection for \(\gamma^1\) can be evaluated in \(M^\omega_\gamma\).

Let us take \(F_i, K_i\) and \(\delta_i\) for \(i = 1, 2\) as in Lemma 3.9. We may assume that \(K_i \geq e_i\) for each \(i\) (see [18 §2.3]). Set \(F = F_1 \otimes F_2\), \(\delta = \delta_1 + \delta_2\) and \(K = K_1 \otimes K_2\). Set \(K_i = \text{supp}(K_i)\) for each \(i\) and \(K = \text{supp}(K) = K_1 \times K_2\). We fix a faithful state \(\phi \in M_\omega\) and set \(\psi := \phi \circ \tau^\omega\). Let \(\mathcal{C}\) be a countable family of \(\gamma^1\)-cocycles. Let \(S \subset M^\omega\) and \(T \subset (M^\omega)^\gamma\) a countably generated von Neumann subalgebras. For a projection \(E \in M^\omega \otimes L^\infty(\hat{G} \times \hat{G}^2)\), we denote by \(\hat{E}\) the sliced element \((\text{id} \otimes \text{id} \otimes \varphi^2)(E)\).

Define the set \(\mathcal{J}\) consisting of projections in \((T' \cap M_\omega) \otimes L^\infty(\hat{G} \times \hat{G}^2)K\) such that \(E \in \mathcal{J}\) if and only if \(E\) satisfies (R1), (R3) and (R4) and, in addition, \(\hat{E}\) satisfies (S2) and (S3) for \(\mathcal{C}\). Since \(0 \in \mathcal{J}\), \(\mathcal{J}\) is non-empty. Define the following
functions $a$, $b$, $c$, and $d$ from $\mathcal{J}$ to $\mathbb{R}_+$:

$$
\begin{align*}
    a_E &= |F|^{-1}_\varphi \gamma_F(E) - (\text{id} \otimes_F \Delta)(E)|_{\psi \otimes \varphi \otimes \tilde{\psi}}; \\
    b_E &= |E|_{\psi \otimes \tilde{\psi}}; \\
    c_E &= |F_2|^{-1}_\varphi \gamma_{F_2}(E)_{132} - \hat{E} \otimes F_2|_{\psi \otimes \varphi^{3} \otimes \varphi^{1}}; \\
    d_E &= |F_1|^{-1}_\varphi \gamma_{F_1}(E) - (\text{id} \otimes_{F_1} \Delta_{\tilde{\psi}} \otimes \text{id})(E)|_{\psi \otimes \varphi^{1} \otimes \tilde{\psi}}.
\end{align*}
$$

**Lemma 3.11.** Let $E \in \mathcal{J}$. Assume that $b_E < 1 - \delta^{1/2}$. Then there exists $E' \in \mathcal{J}$ such that

1. $a_{E'} - a_E \leq 3\delta^{1/2}(b_{E'} - b_E)$;
2. $0 < (\delta^{1/2}/2)|E' - E|_{\psi \otimes \varphi} \leq b_{E'} - b_E$;
3. $c_{E'} - c_E \leq 4\delta^{1/2}(b_{E'} - b_E)$;
4. $d_{E'} - d_E \leq 3\delta^{1/2}(b_{E'} - b_E)$.

**Proof.** Our proof is similar to the one presented in [20]. We may assume that $S$ contains $T$ and the matrix entries of all $v \in \mathcal{C}$, and that $S$ is globally $\gamma$-invariant. We add the matrix entries of $E$ to $S$ and denote the new countably generated von Neumann algebra by $\tilde{S}$. Again we may and do assume that $\tilde{S}$ is globally $\gamma$-invariant. Take $\delta_{\tilde{S}} > 0$ such that $b_{E'} < (1 - \delta_{\tilde{S}})(1 - \delta^{1/2})$.

Recall our assumptions (A3) and (A5). Then by [18, Lemma 5.3], there is a partition of unity $\{e_i\}_{i=0}^q \subset \tilde{S}' \cap M_\omega$ such that

1. $|e_0|_\psi < \delta_{\tilde{S}}$;
2. $(e_i \otimes 1_\pi \otimes 1_\rho)\gamma_{(\pi, \rho)}(e_i) = 0$ for all $1 \leq i \leq q$ and $(\pi, \rho) \in \mathcal{K} \times \mathcal{K} \setminus \{1\}$.

Set a projection $f_i := (\text{id} \otimes \varphi)(\gamma_K(e_i)) \in \tilde{S}' \cap M_\omega$. We claim that at least one $i$ with $1 \leq i \leq q$ satisfies $|E(f_i \otimes 1 \otimes 1)|_{\psi \otimes \tilde{\psi}} < (1 - \delta^{1/2})|f_i|_\psi$. Since

$$
\begin{align*}
    |E(f_i \otimes 1 \otimes 1)|_{\psi \otimes \tilde{\psi}} &= (\psi \otimes \tilde{\phi})(E(f_i \otimes 1)) \\
    &= \psi((\text{id} \otimes \varphi)(E)(\text{id} \otimes \tilde{\phi}))(\gamma_K(e_i)) \\
    &= \sum_{(\pi, \rho) \in \mathcal{K}} d(\pi)^2 d(\rho)^2 \psi((\text{id} \otimes \tilde{\phi})(E)\Phi_{(\pi, \rho)}(e_i \otimes 1_\pi \otimes 1_\rho)) \\
    &= \sum_{(\pi, \rho) \in \mathcal{K}} d(\pi)^2 d(\rho)^2 \psi(\Phi_{(\pi, \rho)}(\gamma_{(\pi, 1)}((\text{id} \otimes \tilde{\phi})(E))(e_i \otimes 1_\pi \otimes 1_\rho))) \\
    &= \sum_{(\pi, \rho) \in \mathcal{K}} d(\pi)^2 d(\rho)^2 \psi((\text{id} \otimes \text{tr}_{\pi} \otimes \text{tr}_{\rho})\gamma_{(\pi, 1)}((\text{id} \otimes \tilde{\phi})(E))(e_i \otimes 1_\pi \otimes 1_\rho))) \\
    &= (\psi \otimes \varphi)(\gamma_K((\text{id} \otimes \varphi)(E))(e_i \otimes 1 \otimes 1)),
\end{align*}
$$

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we have the following:

\[
\sum_{i=1}^{q} |E(f_i \otimes 1 \otimes 1)|_{\psi \otimes \varphi} = (\psi \otimes \varphi)(\gamma_\Pi((\text{id} \otimes \bar{\varphi})(E))(e_0^1 \otimes 1 \otimes 1))
\]

\[
\leq (\psi \otimes \varphi)(\gamma_\Pi((\text{id} \otimes \bar{\varphi})(E))) = |K|_\varphi(\psi \otimes \varphi)(E)
\]

\[
b_E |K|_\varphi < (1 - \delta_3)(1 - \delta^{1/2})|K|_\varphi.
\]

If \( |E(f_i \otimes 1 \otimes 1)|_{\psi \otimes \varphi} \geq (1 - \delta^{1/2})|f_i|_\psi = (1 - \delta^{1/2})|e_i|_\psi|K|_\varphi \) for all \( 1 \leq i \leq q \), then we have

\[
(1 - \delta^{1/2})|e_0^1|_\psi|K|_\varphi < (1 - \delta_3)(1 - \delta^{1/2})|K|_\varphi.
\]

This is a contradiction with \( |e_0|_\psi < \delta_3 \). Hence there exists \( f_i \) such that \( |E(f_i \otimes 1 \otimes 1)|_{\psi \otimes \varphi} < (1 - \delta^{1/2})|f_i|_\psi \). Set \( e := e_i \) and \( f := (\text{id} \otimes \bar{\varphi})(\gamma_K(e)) \in \bar{S}' \cap M_\omega \).

Define the projection \( E' \in M_\omega \otimes L^\infty(\hat{\mathbb{G}} \times \hat{\mathbb{G}}^2) \) by

\[
E' = E(f^1 \otimes 1 \otimes 1) + \gamma_K(e).
\]

Since \( T \subset (M^\omega)_{\varphi} \) and \( e \in \bar{S}' \cap M_\omega \), \( E' \in (T' \cap M_\omega) \otimes L^\infty(\hat{\mathbb{G}} \times \hat{\mathbb{G}}^2)K \). Then \( E' \) satisfies (R1), (R3) and (R4) by [18, Lemma 5.7]. We have to check \( \hat{E}' \) satisfies (S2) and (S3). Set a projection \( e' = (\text{id} \otimes \varphi^2)(\gamma^2_{K^2}(e)) \in \bar{S}' \cap M_\omega \). If we show \( (e' \otimes 1)\gamma^1_\pi(e') = 0 \) for each \( \pi \in \mathcal{K}^1 \cdot \mathcal{K}^1 \setminus \{1\} \), then we are immediately done in view of [18, Lemma 5.7]. This is verified as follows. First we compute the following: for \( \pi \in \mathcal{K}^1 \cdot \mathcal{K}^1 \setminus \{1\} \) and \( \rho \in \mathcal{K}^2 \),

\[
(e \otimes 1 \otimes 1)\gamma^2_\pi^2(\gamma^1_\pi(e')) = (e \otimes 1 \otimes 1)\gamma^2_\pi^2((\text{id} \otimes \varphi^2)(\gamma^2_{K^2}(e)))
\]

\[
= \sum_{\sigma \in \mathcal{K}^2} (\text{id} \otimes \text{id} \otimes \text{id} \otimes \varphi^2_{\sigma})(e \otimes 1 \otimes 1 \otimes 1)\gamma^2_{\pi^2}(\gamma^1_{\pi^2}(\gamma^2_{\sigma}(e)))
\]

\[
= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \varphi^2_{\rho})(e \otimes 1 \otimes 1 \otimes 1)\gamma^2_{\pi^2}(\gamma^1_{\pi^2}(\gamma^2_{\rho}(e)))
\]

\[
= 0,
\]

where we have used the starting condition for \( e \). Using this, we get

\[
(e' \otimes 1)\gamma^1_\pi(e') = ((\text{id} \otimes \varphi^2)(\gamma^2_{K^2}(e)) \otimes 1)\gamma^1_\pi(e')
\]

\[
= \sum_{\rho \in \mathcal{K}^2} d(\rho)^2(\Phi^2_{\pi^2}(e \otimes 1) \otimes 1)\gamma^1_{\pi^2}(e')
\]

\[
= \sum_{\rho \in \mathcal{K}^2} d(\rho)^2(\Phi^2_{\pi^2} \otimes \text{id})(e \otimes 1 \otimes 1)\gamma^2_{\pi^2}(\gamma^1_{\pi^2}(e'))
\]

\[
= 0.
\]

Therefore \( \hat{E}' \) satisfies (S2) and (S3), which means \( E' \in \mathcal{J} \).

Next we estimate \( b_{E'} \) as follows:

\[
b_{E'} - b_E = (\psi \otimes \varphi)(E' - E)
\]

\[
= (\psi \otimes \varphi)(-E(f \otimes 1 \otimes 1) + \gamma_K(e))
\]

\[
> -(1 - \delta^{1/2})|f|_\psi + |f|_\psi = \delta^{1/2}|f|_\psi.
\]
Hence
\[ \delta^{1/2}|f|_\psi < b_{E'} - b_E. \]  
(3.1)

We check the inequalities in the statements. The first, the second and the fourth ones are derived in a similar way to the proof of [18, Lemma 5.11]. We only present a proof for the third one. Since

\[ \gamma^2_F(\hat{E}')_{132} - \hat{E}' \otimes F_2 \]
\[ = \gamma^2_F(\hat{E})_{132}(\gamma^2_F(f)_{13} - f^\perp \otimes 1 \otimes F_2) \]
\[ + (\gamma^2_F(\hat{E})_{132} - (\hat{E} \otimes F_2))(f^\perp \otimes 1 \otimes F_2) \]
\[ + \gamma^1_{K_1}((\text{id} \otimes \text{id} \otimes \varphi^2)((\text{id} \otimes \Delta^2)(\gamma^2_{K_2}(e))(1 \otimes F_2 \otimes K_2))) \]
\[ - \gamma^1_{K_1}((\text{id} \otimes \text{id} \otimes \varphi^2)((\text{id} \otimes \Delta^2)(\gamma^2_{K_2}(e))(1 \otimes F_2 \otimes 1)), \]
we have

\[ |\gamma^2_F(\hat{E}')_{132} - \hat{E}' \otimes F_2|_{\psi \otimes \hat{\varphi}} \]
\[ \leq |\gamma^2_F(\hat{E})_{132}(\gamma^2_F(f)_{13} - f^\perp \otimes 1 \otimes F_2)|_{\psi \otimes \hat{\varphi}} \]
\[ + |(\gamma^2_F(\hat{E})_{132} - (\hat{E} \otimes F_2))(f^\perp \otimes 1 \otimes F_2)|_{\psi \otimes \hat{\varphi}} \]
\[ + |\gamma^1_{K_1}((\text{id} \otimes \text{id} \otimes \varphi^2)((\text{id} \otimes \Delta^2)(\gamma^2_{K_2}(e))(1 \otimes F_2 \otimes K_2)))|_{\psi \otimes \hat{\varphi}} \]
\[ + |\gamma^1_{K_1}((\text{id} \otimes \text{id} \otimes \varphi^2)((\text{id} \otimes \Delta^2)(\gamma^2_{K_2}(e))(1 \otimes F_2 \otimes K_2^1))|_{\psi \otimes \hat{\varphi}}. \]

Then we have the following estimates:

\[ (3.3) \leq c_E, \text{ and } (3.4), \ (3.5) < \delta_2|F_2|\varphi|f|_\psi. \]

On (3.2), we have

\[ (3.2) = (\psi \otimes \hat{\varphi})(\gamma^2_F(\hat{E})_{132}|\gamma^2_F(f)_{13} - f \otimes 1 \otimes F_2|) \]
\[ = (\psi \otimes \varphi^2)(\gamma^2_F((\text{id} \otimes \hat{\varphi})(E)|\gamma^2_F(f) - f \otimes F_2|) \]
\[ \leq (\psi \otimes \varphi^2)(|\gamma^2_F(f) - f \otimes F_2|) \]
\[ \leq (\psi \otimes \varphi^2)((|\text{id} \otimes \varphi^1 \otimes \hat{\varphi} \otimes \varphi^2) \circ \gamma^1_{K_1}((|\text{id} \otimes F_2 \Delta^2_{K_2}(\gamma^2_{K_2}(e))) | \]
\[ + (\psi \otimes \varphi^2)((|\text{id} \otimes \varphi^1 \otimes \hat{\varphi} \otimes \varphi^2) \circ \gamma^1_{K_1}((|\text{id} \otimes F_2 \Delta^2_{K_2}(\gamma^2_{K_2}(e)) | \]
\[ \leq \delta_2|K_1|\varphi^1|F_2|\varphi^2|K_2|\varphi^2|e|_\psi + \delta_2|K_1|\varphi^1|F_2|\varphi^2|K_2|\varphi^2|e|_\psi \]
\[ = 2\delta_2|F_2|\varphi^2|f|_\psi. \]

By using (3.1), we have

\[ c_{E'} \leq c_E + 4\delta_2|f|_\psi < c_E + 4\delta_2^{1/2}(b_{E'} - b_E). \]

Thus we obtain the following as [18, Theorem 5.9].

**Lemma 3.12.** Let \( \gamma = (\gamma^1 \otimes \text{id}) \circ \gamma^2 \), \( F, K, S, T \) and \( C \) be as before. Then the following statements hold:

1. \( \gamma \) has the Rohlin property;
Then we have
\[ \gamma_{\hat{E}}^{2}(\hat{E})_{132} - \hat{E} \otimes F_{2}|_{\psi \otimes \varphi^{1}} < 5\delta_{2}^{1/2}|F_{2}|_{\varphi^{2}}; \]
\[ |\gamma_{F_{1}}^{1}(E) - (\text{id} \otimes F_{1}\Delta \otimes \text{id})(E)|_{\psi \otimes \varphi^{1}\otimes \varphi} < 5\delta_{1}^{1/2}|F_{1}|_{\varphi^{1}}. \]

Our main theorem in this subsection is the following.

**Theorem 3.13** (Relative Rohlin theorem). Let \( M \) be a von Neumann algebra and \( \gamma = (\gamma^{1} \otimes \text{id}) \circ \gamma^{2} \) an action of \( \hat{G} \times \hat{G}^{2} \) on \( M^{\omega} \) such that
- The \( \hat{G} \)-action \( \gamma^{1} \) on \( M^{\omega} \) commutes with the \( \hat{G}^{2} \)-action \( \gamma^{2} \) on \( M^{\omega} \);
- The \( \hat{G} \times \hat{G}^{2} \)-action \( \gamma := (\gamma^{1} \otimes \text{id}) \circ \gamma^{2} \) is strongly free and semi-liftable;
- \( M_{\omega} \) is globally invariant under \( \gamma \);
- \( \tau^{\omega} \circ \Phi_{(\pi,\rho)}^{\gamma} = \tau^{\omega} \otimes \text{tr}_{\pi} \otimes \text{tr}_{\rho} \) on \( M_{\omega} \otimes B(H_{(\pi,\rho)}) \) for all \( (\pi, \rho) \in \text{Irr}(\hat{G}) \times \text{Irr}(\hat{G}^{2}) \).

Then \( \gamma^{1} \) has the joint Rohlin property. Moreover, for any countably generated von Neumann subalgebra \( T \subset (M^{\omega})^{\gamma} \), \( \gamma^{1} \) has a Rohlin projection \( E \in (T' \cap M_{\omega}^{\gamma}) \otimes L^{\infty}(\hat{G}) \) satisfying (S1), (S2) and (S3).

**Proof.** Let \( F_{i}, K_{i} \) and \( \delta_{i} \) \((i = 1, 2)\) be given as before. Take a Rohlin projection \( E \in (T' \cap M_{\omega}^{\gamma}) \otimes L^{\infty}(\hat{G} \times \hat{G}^{2}) \) supported over \( K_{1} \otimes K_{2} \) as in the previous lemma. Then we have
\[ |\gamma_{F_{2}}^{2}(\hat{E})_{132} - \hat{E} \otimes F_{2}|_{\psi \otimes \varphi^{1}} \leq 5\delta_{2}^{1/2}|F_{2}|_{\varphi^{2}}; \quad (3.6) \]
\[ |\gamma_{F_{1}}^{1}(E) - (\text{id} \otimes F_{1}\Delta \otimes \text{id})(E)|_{\psi \otimes \varphi^{1} \otimes \varphi} \leq 5\delta_{1}^{1/2}|F_{1}|_{\varphi^{1}}. \quad (3.7) \]

We set \( \hat{E} = (\text{id} \otimes \text{id} \otimes \varphi^{2})(E) \). By (R3), \( \hat{E} \) gives a partition of unity by matrix elements along with \( K_{1} \). We estimate the equivariance of \( \hat{E} \) with respect to \( \gamma^{1} \).
\[ |\gamma_{F_{1}}^{1}(\hat{E}) - (\text{id} \otimes F_{1}\Delta)(\hat{E})|_{\psi \otimes \varphi^{1} \otimes \varphi} \]
\[ = |(\text{id} \otimes \text{id} \otimes \varphi^{2})(\gamma_{F_{1}}^{1}(E) - (\text{id} \otimes F_{1}\Delta \otimes \text{id})(E))|_{\psi \otimes \varphi^{1} \otimes \varphi} \]
\[ \leq |\gamma_{F_{1}}^{1}(E) - (\text{id} \otimes F_{1}\Delta \otimes \text{id})(E)|_{\psi \otimes \varphi^{1} \otimes \varphi} \quad \text{(by Lemma 3.10)} \]
\[ \leq 5\delta_{1}^{1/2}|F_{1}|_{\varphi^{1}} \quad \text{(by (3.7))}. \]

Take an increasing sequence \( \{ F_{2}(n) \}_{n=1}^{\infty} \subset \text{Proj}(Z(L^{\infty}(\hat{G}^{2}))) \) with \( F_{2}(n) \to 1 \) strongly as \( n \to \infty \). Next we take \( \delta_{2}(n) > 0 \) such that \( \delta_{2}(n)^{1/2}|F_{2}(n)|_{\varphi^{2}} \to 0 \) as \( n \to \infty \). Take a sequence of Rohlin projections \( \{ E(n) \}_{n} \) satisfying the above inequalities (3.6) and (3.7) for \( F_{2}(n) \) and \( \delta_{2}(n) \). By using the index selection trick [18, Lemma 3.11] for \( (\hat{E}(n))_{n} \in \ell^{\infty}(\mathbb{N}, M_{\omega}) \), we obtain a Rohlin projection \( E_{1} \in (T' \cap M_{\omega}^{\gamma}) \otimes L^{\infty}(\hat{G}) \) supported on \( K_{1} \) such that
\[ |\gamma_{F_{1}}^{1}(E_{1}) - (\text{id} \otimes F_{1}\Delta)(E_{1})|_{\psi \otimes \varphi^{1} \otimes \varphi} \leq 5\delta_{1}^{1/2}. \]

\[ \square \]
Corollary 3.14 (Rohlin theorem). Let $M$ be a von Neumann algebra and $\gamma$ an action of $\widehat{G}$ on $M^\omega$. Assume the following:

- $\gamma$ is strongly free and semifinite;
- $M_\omega$ is globally invariant under $\gamma$;
- $\tau^\omega \circ \Phi^\gamma_\pi = \tau^\omega \otimes \text{tr}_\pi$ on $M_\omega \otimes B(H_\pi)$ for all $\pi \in \text{Irr}(G)$.

Then

1. $\gamma$ has the joint Rohlin property;
2. For any countably generated von Neumann subalgebra $T \subset (M^\omega)^\gamma$, $\gamma$ has a Rohlin projection $E \in (T' \cap M_\omega) \otimes L^\infty(\widehat{G})$ satisfying (S1), (S2) and (S3);
3. $\gamma$ is stable on $M^\omega$, that is, for any $\gamma$-cocycle $v \in M^\omega \otimes L^\infty(\widehat{G})$, there exists a unitary $\mu \in M^\omega$ such that $v = (\mu \otimes 1)\gamma(\mu^*)$. If $v \in M_\omega \otimes L^\infty(\widehat{G})$, then $\mu$ can be taken from $M_\omega$.

Proof. In the previous theorem, we put $\widehat{G}^2 = \{1\}$. Then (1) and (2) hold.

(3) Let $\{F_\nu\}_{\nu \in N} \subset L^\infty(\widehat{G})$ be an increasing family of finitely supported central projections such that $F_\nu \rightarrow 1$ strongly as $\nu \rightarrow \infty$. For each $\nu$, take an $(F_\nu, 1/\nu)$-invariant finite projection $K_\nu \in L^\infty(\widehat{G})$ with $\ell_1 \leq K_\nu$. Let $E^\nu \in M_\omega \otimes L^\infty(\widehat{G})$ be a Rohlin projection satisfying (S1), (S2) and (S3) for a faithful state $\psi = \phi \circ \tau^\omega \in (M^\omega)^\gamma$, $F_\nu$, $K_\nu$ and $1/\nu^2$. Then we get the Shapiro unitary $\nu^\gamma = (id \otimes \phi)(vE^\nu)$, and we have

$$v\gamma_{F_\nu}(\mu^\nu) - \mu^\nu \otimes F_\nu = (id \otimes id \otimes \phi)((v \otimes 1)\gamma_{F_\nu}(vE^\nu)) - (id \otimes id \otimes \phi)((id \otimes F_\nu \Delta)(vE^\nu))$$
$$= (id \otimes id \otimes \phi)((id \otimes F_\nu \Delta)(v)(\gamma_{F_\nu}(E^\nu)) - (id \otimes id \otimes \phi)((id \otimes F_\nu \Delta)(vE^\nu))$$
$$= (id \otimes id \otimes \phi)((id \otimes F_\nu \Delta)(v)(\nu\gamma_{F_\nu}(E^\nu) - (id \otimes F_\nu \Delta)(E^\nu))).$$

Since the element $\nu\gamma_{F_\nu}(E^\nu) - (id \otimes F_\nu \Delta)(E^\nu))$ is in the centralizer of $\psi \otimes \phi \otimes \phi$, we can use Lemma 3.10 and we get

$$|v\gamma_{F_\nu}(\mu^\nu) - \mu^\nu \otimes F_\nu|_{\psi \otimes \phi \otimes \phi} \leq |\nu\gamma_{F_\nu}(E^\nu) - (id \otimes F_\nu \Delta)(E^\nu)|_{\psi \otimes \phi \otimes \phi} \leq 5/\nu.$$

By using the index selection map for $(\mu^\nu)_\nu \in \ell^\infty(N, M^\omega)$, we get $\mu \in M^\omega$ such that $v\gamma(\mu) = \mu \otimes 1$. When $v$ is evaluated in $M_\omega$, each $\mu^\nu$ is in $M_\omega$, and so is $\mu$ by the property of the index selection map.

Corollary 3.15. Let $M$ be a von Neumann algebra, $\gamma$ an action of $\widehat{G}$ on $M^\omega$ and $\theta \in \text{Aut}(M^\omega)$. Regard $\theta$ as an action of $\mathbb{Z}$ on $M^\omega$. Assume the following:

- $\theta$ commutes with $\gamma$;
- The $\widehat{G} \times \mathbb{Z}$-action $\gamma \circ \theta$ is strongly free and semi-liftable;
- $M_\omega$ is globally invariant under $\gamma \circ \theta$;
- $\tau^\omega \circ \theta = \tau^\omega$ on $M_\omega$ and $\tau^\omega \circ \Phi^\gamma_\pi = \tau^\omega \otimes \text{tr}_\pi$ on $M_\omega \otimes B(H_\pi)$ for all $\pi \in \text{Irr}(G)$.
Then for any $n > 0$ and any countably generated von Neumann subalgebra $T \subset (M^\omega)^\otimes$, there exists a partition of unity $\{E_i\}_{i=0}^{n-1} \subset T' \cap M_\omega^\gamma$ such that $\theta(E_i) = E_{i+1}$, where $E_n = E_0$.

**Proof.** For $m > 0$, set $\delta_m = 2/nm$ and $K_m := \{0, 1, 2, nm - 1\}$. Then $K_m$ is $(\{1\}, \delta_m)$-invariant subset of $\mathbb{Z}$. By Theorem 3.13 we have a partition of unity $\{F_i\}_{i \in K_m}$ in $M_\omega^\gamma$ such that $\sum_{i=0}^{nm} |\theta(F_i) - F_{i+1}| \psi \leq 5\delta_m^\frac{1}{n}$. For $0 \leq i \leq n - 1$, set $E_i := \sum_{k=0}^{m-1} F_{kn+i}$. Then for $0 \leq i \leq n - 2$, we have

$$|\theta(E_i) - E_{i+1}| \psi \leq \sum_{k=0}^{m-1} |\theta(F_{kn+i}) - F_{kn+i+1}| \psi \leq 5\delta_m^\frac{1}{n}.$$

By applying the index selection trick to $\{E_i\}_{i=0}^{n-1}$, $0 \leq i \leq n - 1$, we get $\theta(E_i) = E_{i+1}$ for $0 \leq i \leq n - 2$. Then $\theta(E_{n-1}) = E_0$ follows automatically. \(\square\)

Recall the following result [18] Lemma 4.3. The statement is slightly strengthened here, but the same proof is applicable if we replace $M_\omega$ with $A' \cap M_\omega$. Note that $A' \cap M_\omega$ is of type II$_1$ for any countably generated von Neumann subalgebra $A \subset M^\omega$ when $M_\omega$ is of type II$_1$.

**Theorem 3.16** (2-cohomology vanishing). Let $M$ be a von Neumann algebra such that $M_\omega$ is of type II$_1$. Let $A \subset M^\omega$ be a countably generated von Neumann subalgebra. Let $(\gamma, w)$ be a cocycle action of $\hat{G}$ on $M^\omega$. Assume the following:

- $A' \cap M_\omega$ is globally invariant under $\gamma$;
- $w \in (A' \cap M_\omega) \otimes L^\infty(\hat{G}) \otimes L^\infty(\hat{G})$;
- $\gamma$ is of the form $\gamma = \text{Ad} U \circ \beta$, where $U \in U(M^\omega \otimes L^\infty(\hat{G}))$ and $\beta \in \text{Mor}(M^\omega, M^\omega \otimes L^\infty(\hat{G}))$ is semi-liftable.

Then the 2-cocycle $w$ is a coboundary in $A' \cap M_\omega$.

**Corollary 3.17.** Let $\gamma$ be a strongly free and semi-liftable action of $\hat{G}$ on $M_\omega$. Let $S \subset (M^\omega)^\gamma$ is countably generated von Neumann subalgebra. If $M_\omega$ is of type II$_1$, then the von Neumann algebra $S' \cap M_\omega^\gamma$ is also of type II$_1$.

**Proof.** Let $I$ be a finite index set. Since $S' \cap M_\omega$ is of type II$_1$, we can take a system of matrix units $\{e_{i,j}\}_{i,j \in I}$ in $S' \cap M_\omega$. Let $\pi \in \text{Irr}(G)$. Let $Q$ be a finite dimensional subfactor generated by $\{e_{i,j}\}_{i,j \in I}$. Take an index $i_0 \in I$. Since $\{\gamma_i(e_{i,j})\}_{i,j \in I}$ and $\{e_{i,j} \otimes 1_\pi\}_{i,j \in I}$ are systems of matrix units in $(S' \cap M_\omega) \otimes B(H_\pi)$, $e_{i_0,i_0} \otimes 1_\pi$ and $\gamma_i(e_{i_0,i_0})$ are equivalent. Hence there exists $v_\pi \in (S' \cap M_\omega) \otimes B(H_\pi)$ such that $e_{i_0,i_0} \otimes 1_\pi = v_\pi v_\pi^*$ and $\gamma_i(e_{i_0,i_0}) = v_\pi^* v_\pi$. Set the unitary

$$\tilde{v}_\pi = \sum_{i \in I} (e_{i,i_0} \otimes 1_\pi) v_\pi \gamma_i(e_{i_0,i}).$$

Then $\tilde{v}_\pi$ is satisfying $\tilde{v}_\pi \gamma_i(e_{i,j}) \tilde{v}_\pi^* = e_{i,j} \otimes 1$. Setting $\tilde{v} = (\tilde{v}_\pi)_\pi \in M_\omega \otimes L^\infty(\hat{G})$, we have

$$\tilde{v}_\pi(x) \tilde{v}_\pi^* = x \otimes 1 \quad \text{for all } x \in Q.$$
Hence the map $\text{Ad} \, \tilde{v} \circ \gamma$ is a cocycle on $Q' \cap (S' \cap M_\omega)$. Using the previous 2-cohomology vanishing result for $Q' \cap (S' \cap M_\omega)$, we obtain a unitary $w \in (Q' \cap (S' \cap M_\omega)) \otimes L^\infty(\hat{G})$ such that $w \tilde{v}$ is an $\gamma$-cocycle. Now we have

$$w \tilde{v} \gamma(x) \tilde{v}^* w^* = x \otimes 1 \quad \text{for all } x \in Q.$$

Since $\gamma$ has the joint Rohlin property, the action $\gamma|_{M_\omega}$ is stable by Corollary 3.14. Hence the $M_\omega$-valued $\gamma$-cocycle $w \tilde{v}$ is of the form $w \tilde{v} = (\nu^* \otimes 1)\gamma(\nu)$ where $\nu \in U(S' \cap M_\omega)$. This implies that a subfactor $\nu Q \nu^*$ is fixed by $\gamma$. Hence $S' \cap M_\omega^\gamma$ contains a subfactor with arbitrary finite dimension, and it is of type $\Pi_1$. □

3.4. Approximately inner actions

Let $M$ be a von Neumann algebra, $\hat{G}$ an amenable discrete Kac algebra and $\Gamma$ a discrete amenable group with the neutral element $e$. In this subsection, we study the following situation:

- We are given two actions $\alpha \in \text{Mor}(M, M \otimes L^\infty(\hat{G}))$, $\theta : \Gamma \to \text{Aut}(M)$ and unitaries $(v_g)_{g \in \Gamma} \in U(M \otimes L^\infty(\hat{G}))$ such that
  $$\theta_g \otimes \text{id}) \circ \alpha \circ \theta_g^{-1} = \text{Ad} v_g^* \circ \alpha;$$

- $M_\omega$ is of type $\Pi_1$ and $Z(M) \subset M^\theta$;

- $(v_g)_{g \in \Gamma}$ is a $(\theta \otimes \text{id})$-cocycle;

- $v_g^*$ is an $\alpha$-cocycle for each $g \in \Gamma$;

- $\alpha$ is approximately inner;

- $\alpha_g \theta_g$ is properly centrally non-trivial for each $(\pi, g) \in \text{Irr}(\hat{G}) \times \Gamma \setminus \{1, e\}$.

Take $U^\nu_\pi \in U(M \otimes B(H_\pi))$ for $\nu \in \mathbb{N}$ such that $\text{Ad} U^\nu_\pi$ converges to $\alpha_\pi$ for each $\pi \in \text{Irr}(\hat{G})$. Set $U := (U^\nu_\pi)_{\pi \in M^\omega \otimes L^\infty(\hat{G})}$ where $U_\pi := (U^\nu_\pi)_\nu \in M^\omega \otimes B(H_\pi)$. Then $\alpha = \text{Ad} U$ on $M$. Our first task is to replace $U$ with a new one which well behaves to the action $\theta^\omega$.

**Lemma 3.18.** For each $\pi \in \text{Irr}(\hat{G})$ and $g \in \Gamma$, the sequence $(v_g(\theta_g \otimes \text{id})(U^\nu_\pi))_\nu$ approximates $\alpha_\pi$. In particular, $U^\nu v_g(\theta_g^* \otimes \text{id})(U) \in M_\omega \otimes L^\infty(\hat{G})$.

**Proof.** Take $\phi \in M_\omega$. We verify that $(\phi \otimes \text{tr}_\pi) \circ \text{Ad}(\theta_g \otimes \text{id})((U^\nu_\pi)^*) v_g^*$ converges to $\phi \circ \Phi^\alpha_\pi$ as follows:

$$
\begin{align*}
\lim_{\nu \to \infty} (\phi \otimes \text{tr}_\pi) \circ \text{Ad}(\theta_g \otimes \text{id})((U^\nu_\pi)^*) v_g^* & = \lim_{\nu \to \infty} ((\theta_g^{-1} \otimes \text{id})(v_g) U^\nu_\pi (\phi \circ \theta_g \otimes \text{tr}_\pi)(U^\nu_\pi)^*(\theta_g^{-1} \otimes \text{id})(v_g^*)) \circ (\theta_g^{-1} \otimes \text{id}) \\
& = ((\theta_g^{-1} \otimes \text{id})(v_g) (\phi \circ \theta_g \circ \Phi^\alpha_\pi)(\theta^{-1} \otimes \text{id})(v_g^*)) \circ (\theta_g^{-1} \otimes \text{id}) \\
& = v_g (\phi \circ \theta_g \circ \Phi^\alpha_\pi \circ (\theta^{-1} \otimes \text{id})) v_g^* = v_g (\phi \circ \Phi^\alpha_\pi \circ \text{Ad} v_g^*) v_g^* \\
& = v_g (\phi \circ \Phi^\alpha_\pi \circ \text{Ad} v_g^*) v_g^*.
\end{align*}
$$

The latter statement follows from [18, Lemma 3.6]. □

**Lemma 3.19.** There exists $u \in U(M_\omega \otimes L^\infty(\hat{G}))$ such that $v_g(\theta_g^* \otimes \text{id})(U) = U u$. 17
Proof. Since the $\Gamma$-action $\theta^\omega$ is strongly free, it has the joint Rohlin property. Let $S \subset M^\omega$ be a von Neumann subalgebra generated by all matrix entries of $(\theta^\omega \otimes \text{id})(U)$ and $v_g$ for all $g \in \Gamma$. Let $F \subset \Gamma$ be a finite subset and $\delta > 0$. Since $\Gamma$ is amenable, there exists a finite subset $K \subset \Gamma$ such that $\sum_{g \in F} |gK\Delta K| < \delta |F| |K|$. Fix a faithful state $\phi \in \mathcal{M}_\ast$ and set $\psi := \phi \circ \tau^\omega$. Take a Rohlin projection $(E_g)_{g \in \Gamma} \subset (S' \cap \mathcal{M}_\omega)$ such that $E_g = 0$ for $g \notin K$ and $\sum_{\ell \in \Gamma} |\theta^\omega(\ell \epsilon) - E_g| \leq 5\delta^{1/2}$. Define $u \in M^\omega \otimes \mathcal{L}_{\infty}(\hat{\mathcal{G}})$ by $u = \sum_{k \in K} U^* v_k (\theta^\omega_k \otimes \text{id})(U)(E_k \otimes 1)$. By the previous lemma, $u$ is in $\mathcal{M}_\omega \otimes \mathcal{L}_{\infty}(\hat{\mathcal{G}})$. Then it is easy to see that $u$ is a unitary element, and for $g \in F$ we have

$$U^* v_g (\theta^\omega_g \otimes \text{id})(U) \cdot (\theta^\omega_g \otimes \text{id})(u) = U^* v_g (\theta^\omega_g \otimes \text{id})(U) \sum_{k \in K} (\theta^\omega_g \otimes \text{id})(U^*)(\theta^\omega_g \otimes \text{id})(v_k)(\theta^\omega_{gk} \otimes \text{id})(U)(\theta^\omega_g(E_k) \otimes 1)$$

$$= \sum_{k \in K} U^* v_{gk} (\theta^\omega_{gk} \otimes \text{id})(U)(\theta^\omega_g(E_k) \otimes 1)$$

$$= \sum_{k \in K} U^* v_{gk} (\theta^\omega_{gk} \otimes \text{id})(U)((\theta^\omega_g(E_k) - E_{gk}) \otimes 1) + \sum_{\ell \in \Gamma} U^* v_{\ell} (\theta^\omega_{\ell} \otimes \text{id})(U)(E_{\ell} \otimes 1).$$

Take a partial isometry $w_g \in M^\omega \otimes \mathcal{L}_{\infty}(\hat{\mathcal{G}})$ such that

$$|U^* v_g (\theta^\omega_g \otimes \text{id})(U) \cdot (\theta^\omega_g \otimes \text{id})(u) - u| = w_g^* (U^* v_g (\theta^\omega_g \otimes \text{id})(U)(\theta^\omega_g \otimes \text{id})(u) - u).$$

Let $\chi \in \mathcal{L}_{\infty}(\hat{\mathcal{G}})_\ast$ be a faithful state. Then we have

$$|U^* v_g (\theta^\omega_g \otimes \text{id})(U) \cdot (\theta^\omega_g \otimes \text{id})(u) - u|_{\psi \otimes \chi} = \sum_{k \in K} (\psi \otimes \chi)(w_g^* U^* v_{gk} (\theta^\omega_{gk} \otimes \text{id})(U)((\theta^\omega_g(E_k) - E_{gk}) \otimes 1))$$

$$- \sum_{\ell \in \Gamma \setminus gK} (\psi \otimes \chi)(w_g^* U^* v_{\ell} (\theta^\omega_{\ell} \otimes \text{id})(U)(E_{\ell} \otimes 1)).$$

(3.8)

Since $E_k \in (\mathcal{M}_\omega)_\psi$, we can use Lemma 3.10 and we have

$$|U^* v_g (\theta^\omega_g \otimes \text{id})(U) \cdot (\theta^\omega_g \otimes \text{id})(u) - u|_{\psi \otimes \chi} \leq 5\delta^{1/2}.$$

By the assumption of $K$, we have

$$|U^* v_g (\theta^\omega_g \otimes \text{id})(U) \cdot (\theta^\omega_g \otimes \text{id})(u) - u|_{\psi \otimes \chi} \leq 10\delta^{1/2}. \quad (3.10)$$
For each $\nu \in \mathbb{N}$, take $u_\nu \in M_\omega \otimes L^\infty(\hat{G})$ satisfying (3.10) for $\delta = 1/\nu$. Take an increasing sequence $F_\nu \in \Gamma$ with $\bigcup_{\nu = 1}^\infty F_\nu = \Gamma$. Applying the index selection trick to $(\nu^\nu)_\nu$, we get $u \in M_\omega \otimes L^\infty(\hat{G})$ with $U^*v_\nu(\theta_g^\nu \otimes \text{id})(Uu) = u$ for all $g \in \Gamma$. □

Replacing $U$ with $Uu$, we may assume that $U = (U^\nu)_\nu$ also satisfies

$$v_\nu(\theta_g^\nu \otimes \text{id})(U) = U.$$ 

As in [18], we consider the cocycle actions $\gamma^{-1} = \text{Ad}U^* \circ \alpha^\omega$ and $\gamma^0 = \text{Ad}U^*$ on $M_\omega$. Their 2-cocycles $w^{-1}$ and $w^0$ are given by

$$w^{-1} = (U^* \otimes 1)\alpha^\omega(U^*)(\text{id} \otimes \Delta)(U), \quad w^0 = (U^* \otimes 1)U^*_\nu(\text{id} \otimes \Delta^{\text{opp}})(U).$$

Here note that $\gamma^{-1}$ and $\gamma^0$ are cocycle actions of $\hat{G}$ and $\hat{G}^{\text{opp}}$, respectively.

**Lemma 3.20.** In the above setting, $\gamma^{-1}$ and $\gamma^0$ are cocycle actions on $M_\omega^\theta$.

**Proof.** First, we show that $\gamma^{-1}$ and $\gamma^0$ commute with $\theta_\omega$. Using $v_\nu(\theta_g^\nu \otimes \text{id})(U) = U$, we have $(\theta_g^\nu \otimes \text{id})\circ \gamma^{-1} = \gamma^{-1} \circ \theta_g^\nu \nu$. This equality holds on $M^\omega$, and in particular, $\gamma^{-1}$ commutes with $\theta_\omega$ on $M_\omega$.

Let $x \in M_\omega$. Since $v_\nu$ commutes with $\theta_g^\nu(x) \otimes 1$, we have

$$(\theta_g^\nu \otimes \text{id})(\gamma^0(x)) = U^*v_\nu(\theta_g^\nu(x) \otimes 1)v_\nu^*U = U^*(\theta_g^\nu(x) \otimes 1)U = \gamma^0(\theta_g^\nu(x)).$$

Hence $\gamma^0$ also commutes with $\theta^\omega$.

Second, we check that the 2-cocycles $w^{-1}$ and $w^0$ are evaluated in $M_\omega^\theta$. Since $v_\nu^*$ is an $\alpha$-cocycle, we have

$$(\theta_g^\nu \otimes \text{id} \otimes \text{id})(w^{-1}) = ((\theta_g^\nu \otimes \text{id})(U^*) \otimes 1) \cdot (\theta_g^\nu \otimes \text{id} \otimes \text{id})(\alpha^\omega(U^*)) \cdot (\theta_g^\nu \otimes \Delta)(U)$$

$$= (U^* \otimes 1)(v_\nu \otimes 1) \cdot (v_\nu^* \otimes 1)(\alpha^\omega((\theta_g \otimes \text{id})(U^*))(v_\nu \otimes 1)$$

$$\cdot (\text{id} \otimes \Delta)(v_\nu^*U)$$

$$= (U^* \otimes 1)\alpha^\omega(U^*v_\nu)(v_\nu \otimes 1)(\text{id} \otimes \Delta)(v_\nu^*)(\text{id} \otimes \Delta)(U)$$

$$= w^{-1},$$

and

$$(\theta_g^\nu \otimes \text{id} \otimes \text{id})(w^0) = (\theta_g^\nu \otimes \text{id})(U^* \otimes 1) \cdot (\theta_g^\nu \otimes \text{id})(U^*) \cdot (\theta_g^\nu \otimes \Delta^{\text{opp}})(U)$$

$$= (U^* \otimes 1)(v_\nu \otimes 1) \cdot U^*_\nu(v_\nu) \cdot (\text{id} \otimes \Delta^{\text{opp}})(v_\nu^*U)$$

$$= (U^* \otimes 1)U^*_\nu\alpha(v_\nu)(\alpha^\nu(U^*_\nu)(\alpha^\omega(U^*)v_\nu)(\text{id} \otimes \Delta)(v_\nu^*)(\text{id} \otimes \Delta^{\text{opp}})(U)$$

$$= w^0.$$  

Define the cocycle action $\gamma$ of $\hat{G} \times \hat{G}^{\text{opp}}$ on $M_\omega$ by $\gamma := (\gamma^{-1} \otimes \text{id}) \circ \gamma^0$. Its 2-cocycle $w$ is given by

$$w := U^*\alpha^\omega(U^*)_{123}\alpha^\omega(U^*_2 \alpha^\omega(U^*))_{1245}(\text{id} \otimes \Delta\hat{G} \times \hat{G}^{\text{opp}})(\alpha^\omega(U)U_{12}).$$

By direct computation, we have

$$w = \gamma^{-1}(w_{123}(w_{12}^{-1}))^*_{1234}w_{124}^{-1}(\text{id} \otimes \Delta \otimes \text{id} \otimes \text{id})(\gamma^{-1}(w^0))_{12345}.$$ 

Hence $w$ is evaluated in $M_\omega^\theta$, that is, $\gamma$ is a cocycle action on $M_\omega^\theta$. 

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Then we apply Theorem 3.16 to $\gamma$ and get $c \in M^\omega_\omega \otimes L^\infty(\hat{G} \times \hat{G}^{opp})$ such that
\[
c_{123}\gamma(c)w(id \otimes \Delta_{G\times\hat{G}^{opp}})(c^*) = 1.
\]
Here we note that the proof of [18, Lemma 4.3] works in our case by replacing $M_\omega$ with $M^\omega_\omega$. Also note that $M^\omega_\omega$ is of type II$_1$.

Set the unitaries $c^0 := c_{\cdot1}$ and $c^* := c_{1\cdot}$ in $M^\omega_\omega \otimes L^\infty(\hat{G})$. Then the proof similar to that of [18, Lemma 4.6] shows that
- $c^0U^*$ is an $\alpha^\omega$-cocycle;
- $U(c^0)^*$ is a unitary representation of $\hat{G}$;
- $U(c^*)^*$ is fixed by the perturbed action $\text{Ad}(c^0U^*) \circ \alpha^\omega$.

Exchanging $U$ with $U(c^*)^*$, we obtain the following.

**Lemma 3.21.** Let $\alpha, \theta$ and $(v_g)_{g \in \Gamma}$ be as before. Then there exists $U \in U(M^\omega \otimes L^\infty(\hat{G}))$ and $c \in U(M_\omega \otimes L^\infty(\hat{G}))$ such that
- (1) $(\text{Ad} U_\omega^\nu)_\nu$ approximates $\alpha_\pi$ for all $\pi \in \text{Irr}(G)$;
- (2) $U$ is a unitary representation of $\hat{G}$, that is, we have $(id \otimes \Delta)(U) = U_{12}U_{13}$;
- (3) $cU^*$ is an $\alpha^\omega$-cocycle;
- (4) $U$ is fixed by the perturbed action $\text{Ad} cU^* \circ \alpha^\omega$;
- (5) $v_g(\theta^\omega_{\hat{G}} \otimes \text{id})(U) = U$ and $(\theta^\omega_{\hat{G}} \otimes \text{id})(c) = c$ for all $g \in \Gamma$.

Now we set the following maps on $M^\omega$ as before:
\[
\gamma^1 := \text{Ad} cU^* \circ \alpha^\omega, \quad \gamma^2 := \text{Ad} U^*(\cdot \otimes 1),
\]
which are actions of $\hat{G}$ and $\hat{G}^{opp}$, respectively. They preserve $M_\omega$ and $M^\omega_\omega$.

**Lemma 3.22.** In the above settings, one has the following:
- (1) $v_g^*U$ is a unitary representation of $\hat{G}$;
- (2) For all $\pi \in \text{Irr}(G)$ and $X \in M^\omega \otimes B(H_\pi)$, \( \Phi_{\pi}^\omega(X) = (id \otimes \text{tr}_\pi)(UXU^*) \).

**Proof.** (1) Since $v_g^*$ is an $\alpha$-cocycle, we have
\[
(v_g^*U)_{12}(v_g^*U)_{13} = (v_g^*)_{12}(\alpha(v_g^*)U_{12}U_{13}) = (id \otimes \Delta)(v_g^*U).
\]

(2) Let $S_{\pi,\pi}$ be an isometric intertwiner from $1$ into $\pi \otimes \pi$ for $\hat{G}^{opp}$. For $X \in M^\omega \otimes B(H_\pi)$, we have
\[
\Phi_{\pi}^\omega(X) = (1 \otimes S_{\pi,\pi}^\omega)(U^*)_{12}X_{13}(U)_{12}(1 \otimes S_{\pi,\pi})
\]
\[
= (1 \otimes S_{\pi,\pi}^\omega)(id \otimes \Delta^{opp})(U^*)U_{13}X_{13}U^{*13}
\]
\[
= (1 \otimes S_{\pi,\pi}^\omega)(id \otimes \text{tr}_\pi)(UXU^*).
\]

\[\square\]

Our next aim is to replace $U$ with a new one such that we can retake $c = 1$.

**Lemma 3.23.** There exists $z \in U(M_\omega^\omega)$ such that $UcU^* = (z \otimes 1)\alpha^\omega(z^*)$. 

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Proof. By definition of $\gamma^1$, we have $\Phi^{\gamma^1_\theta} = \theta^\omega \circ \Phi^{\gamma^1_\pi} \circ \text{Ad} U c^*$. Since $\theta^\omega \circ \tau^\omega = \tau^\omega$ on $M_\omega$, we get $\tau^\omega \circ \Phi^{\gamma^1_\theta} = \tau^\omega \otimes \text{tr}_\pi$ on $M_\omega \otimes B(H_\pi)$ for all $(\pi, g) \in \text{Irr}(G) \times \Gamma$.

By Lemma 3.20, $\gamma^1 \circ \theta^\omega$ is a $\widehat{G} \times \Gamma$-action. It is easy to see that $\gamma^1 \circ \theta^\omega$ is strongly free. Since $\text{Ad}(c^\omega U^{\mu^\omega}) \circ \alpha$ converges to the trivial action, $\gamma^1$ is semiliftable. Hence $\gamma^1 \circ \theta^\omega$ has the joint Rohlin property.

Now we have two $\gamma^1$-cocycles $U c^*$ and $U$. Let $K \in \text{Proj}(Z(L^\infty(\widehat{G})))$ be an $(F, \delta)$-invariant projection with $K \geq e_1$. By Theorem 3.13 we can take a Rohlin projection $E \in M_{\omega}^\theta \otimes L^\infty(\widehat{G})K$ for $C = \{U, U c^*\}$. Set the Shapiro unitaries $\mu^\delta := (\text{id} \otimes \varphi)(UE)$ and $\nu^\delta := (\text{id} \otimes \varphi)(U c^* E)$. Then we claim the following:

**Claim 1.**

$$
\mu^\delta \nu^\delta = (\text{id} \otimes \varphi)(U Ec^*) \in M_{\omega}^\theta.
$$

Indeed, the first equality is shown by using (R3) and $\varphi = \oplus_{\pi \in \text{Irr}(G)} d(\pi) \text{Tr}_\pi$. Next we show that $\mu^\delta \nu^\delta \in M_{\omega}^\theta$. By Lemma 3.22 we have

$$
\mu^\delta \nu^\delta = (\text{id} \otimes \varphi)(U Ec^*) = \sum_{\pi \in \text{Irr}(G)} d(\pi)^2 (\text{id} \otimes \text{tr}_\pi)(U Ec^*)
$$

$$
= \sum_{\pi \in \text{Irr}(G)} d(\pi)^2 \Phi^{\gamma^2_\pi}(Ec).
$$

Since $Ec \in (M_\omega)^{\theta^\omega} \otimes B(H_\pi)$, $\mu^\delta \nu^\delta$ is in $M_{\omega}^\theta$. Using the commutativity of $\gamma^2 |_{M_\omega}$ and $\theta^\omega$, we have

$$
\theta^\omega_g(\mu^\delta \nu^\delta) = \sum_{\pi \in \text{Irr}(G)} d(\pi)^2 \theta^\omega_g(\Phi^{\gamma^2_\pi}(Ec)) = \sum_{\pi \in \text{Irr}(G)} d(\pi)^2 \Phi^{\gamma^2_\pi}((\theta^\omega_g \otimes \text{id})(Ec))
$$

$$
= \sum_{\pi \in \text{Irr}(G)} d(\pi)^2 \Phi^{\gamma^2_\pi}(Ec) = \mu^\delta \nu^\delta.
$$

Next we claim the following:

**Claim 2.**

$$
|U \gamma^1_F(\mu^\delta) - \mu^\delta \otimes F|_{\psi \otimes \varphi} \leq 5\delta^{1/2}; \quad (3.11)
$$

$$
|U c^* \gamma^1_F(\nu^\delta) - \nu^\delta \otimes F|_{\psi \otimes \varphi} \leq 5\delta^{1/2}. \quad (3.12)
$$

Let $U \gamma^1_F(\mu^\delta) - \mu^\delta \otimes F = v|U \gamma^1_F(\mu^\delta) - \mu^\delta \otimes F|$ be the polar decomposition. Then we have

$$
|U \gamma^1_F(\mu^\delta) - \mu^\delta \otimes F| = v^*(U \gamma^1_F(\mu^\delta) - \mu^\delta \otimes F)
$$

$$
= v^*(\text{id} \otimes \text{id} \otimes \varphi)(U_{12} U_{13} \gamma^1(1)(E))
$$

$$
- v^*(\text{id} \otimes \text{id} \otimes \varphi)(U_{12} U_{13}(\text{id} \otimes F \Delta)(E))
$$

$$
= v^*(\text{id} \otimes \text{id} \otimes \varphi)(U_{12} U_{13}(\gamma^1(1)(E) - (\text{id} \otimes F \Delta)(E))).
$$
Using Lemma 3.10, we have
\[ |U^{γ_1}(μ) - μ F|_{φ φ} = |(id ⊗ id ⊗ φ)(v^*_{12}U_{12}U_{13}(γ_1(E) - (id ⊗ F Δ)(E)))|_{φ φ} \]
\[ \leq |γ_1(E) - (id ⊗ F Δ)(E)|_{φ φ} \leq 5δ^{1/2}. \]

Similarly we can prove (3.12).

Now we use the index selection trick. For decreasing \( δ_n = 1/n \rightarrow 0 \) and increasing finite rank central projections \( F_n \rightarrow 1 \) in \( L^∞(G) \) as \( N \ni n \rightarrow ∞ \), we take \( μ(n) := μ_{1/n} \) and \( ν(n) := ν_{1/n} \) in \( U(M^ω) \) for \( n ∈ N \). Set \( μ = (μ(n))_n \) and \( ν = (ν(n))_n \). From them, we construct \( μ \) and \( ν \) in \( U(M^ω) \) by index selection. Since \( μ(n) ν(n)^* ∈ M^{θω}_ω, μν^* ∈ M^{θω}_ω \). By definition of an index selection map (i.e. it commutes with \( γ^1 \)), we have \( Uγ^1(μ) = μ ⊗ 1 \) and \( Uc^*γ^1(ν) = ν ⊗ 1 \). These imply
\[ α^ω(νμ^*) = Uc^*γ^1(νμ^*)cU^* = (νμ^* ⊗ 1)UcU^*. \]
Therefore, \( z := μν^* \) is a desired solution. \( □ \)

By the previous lemma, we get \( z ∈ U(M^{θω}_ω) \) such that \( UcU^* = (z ⊗ 1)α^ω(z^*) \).

Then we consider \( V = (z^* ⊗ 1)U(z ⊗ 1) \), which is a representation of \( G \) in \( M^ω \).

By the previous lemma, we have
\[ V^* = (z^* ⊗ 1)cU^* \cdot Uc^*U^*(z ⊗ 1) = (z^* ⊗ 1)cU^*α^ω(z). \]
Since \( cU^* \) is an \( α^ω \)-cocycle, so is \( V^* \). Moreover we have
\[ v_θ(θ ⊗ id)(V) = v_θ(z^* ⊗ 1)v_θ^*U(z ⊗ 1) = (z^* ⊗ 1)U(z ⊗ 1) = V. \]

Finally we again replace \( U \) with \( V = (z^* ⊗ 1)U(z ⊗ 1) \), and we get the following.

**Theorem 3.24.** Let \( M \) be a von Neumann algebra. Assume the following:
- We are given two actions \( α \in Ω(M,M ⊗ L^∞(G)), θ : Γ → Ω(M) \) and unitaries \( (ν_g)_{g ∈ G} ∈ U(M ⊗ L^∞(G)) \) such that
  \[ (θ ⊗ id) α θ^{-1}_g = Ad ν_g α; \]
- \( M^ω \) is of type \( II_1 \) and \( Z(M) ⊂ M^θ; \)
- \( (ν_g)_{g ∈ G} \) is an \( (θ ⊗ id) \)-cocycle;
- \( ν_g^* \) is an \( α \)-cocycle for each \( g ∈ Γ; \)
- \( α \) is approximately inner;
- \( α_θ \) is properly centrally non-trivial for each \( (π,g) ∈ Ω(G) × Γ \setminus (1,e). \)

Then there exists \( U = (U^γ)^ν ∈ U(M^ω ⊗ L^∞(G)) \) such that
1. \( (Ad U^ν)_ν \) converges to \( α_π \) for all \( π ∈ Ω(G); \)
2. \( U \) is a representation of \( G \) that is, \( (id ⊗ Δ)(U) = U_{12}U_{13}; \)
3. \( U^* \) is a \( α^ω \)-cocycle, that is, \( U^*_{12}α^ω(U^*) = (id ⊗ Δ)(U^*); \)
4. \( ν_g(θ^ω ⊗ id)(U) = U \) for all \( g ∈ Γ. \)

**Corollary 3.25.** Let \( w ∈ M^ω ⊗ L^∞(G) \) be an \( α^ω \)-cocycle. Take \( U ∈ U(M^ω ⊗ L^∞(G)) \) as in the previous theorem. If \( U^*wU \) is in \( M^{θω}_ω ⊗ L^∞(G) \), then there exists \( z ∈ U(M^{θω}_ω) \) such that \( w = (z ⊗ 1)α^ω(z^*). \)
Proof. The proof is similar to that of Lemma 3.23. Let $\gamma^1 = \text{Ad} U^* \circ \alpha^\omega$, $\gamma^2 = \text{Ad} U^* (\cdot \otimes 1)$ and $\gamma = (\gamma^1 \otimes 1) \circ \gamma^2$ as before.

Now we have two $\gamma^i$-cocycles $U$ and $wU$. Let $K \in \text{Projf}(L^\infty(\hat{G}))$ be an $(F, \delta)$-invariant central projection. By Theorem 3.13, we can take a Rohlin projection $E \in M_\omega^\omega \otimes L^\infty(\hat{G})K$ as in Definition 3.13 for $\mathcal{C} = \{U, wU\}$. Set the Shapiro unitaries $\mu^\delta := (\text{id} \otimes \varphi)(UE)$ and $\nu^\delta := (\text{id} \otimes \varphi)(wUE)$. Then we have

$$\mu^\delta \nu^\delta = (\text{id} \otimes \varphi)(UEU^* w^*), \quad \mu^\delta \nu^\delta \in M_\omega^\omega.$$  

Next we show that $\mu^\delta \nu^\delta \in M_\omega^\omega$. By Lemma 3.22, we have

$$\mu^\delta \nu^\delta = (\text{id} \otimes \varphi)(UEU^* w^*) = \sum_{\pi \in \text{Irr}(G)} d(\pi)^2 (\text{id} \otimes \pi)(UEU^* w^*)$$

$$= \sum_{\pi \in \text{Irr}(G)} d(\pi)^2 \Phi^*_\pi(UEU^* w^*) \cdot \text{tr}_\pi.$$  

Since $EU^* wU \in (M_\omega^\omega \otimes B(H))$ by our assumption on $w$, $\mu^\delta \nu^\delta$ is in $M_\omega$. Using the commutativity of $\gamma^2|_{M_\omega}$ and $\theta_\omega$, we have $\theta_\omega^* (\mu^\delta \nu^\delta) = \mu^\delta \nu^\delta$. Now we get $\mu$ and $\nu$ in $U(M^\omega)$ by the index selection as before. Then $\mu\nu^* \in M_\omega^\omega$. By definition of an index selection map (i.e. it commutes with $\gamma^1$), we have $U\gamma^1(\mu) = \mu \otimes 1$ and $wU\gamma^1(\nu) = \nu \otimes 1$. These imply $\nu\omega^\omega(\nu\mu^*) = wU\gamma^1(\nu\mu^*)U^* = (\nu\mu^*) \otimes 1$. Therefore, $z := \nu\mu^*$ is a desired solution.

The previous result yields the following, which can be also proved by using [17] Theorem 7.2.

Corollary 3.26. Let $M$ be an injective factor and $\alpha$ an approximately inner and centrally free cocycle action of $\hat{G}$ on $M$. Let $\varphi \in W(M)$ and $T > 0$. Then there exists a sequence $\{w_n\}_n \subset U(M)$ such that

- $\sigma^\varphi_T = \lim_{n \to \infty} \text{Ad} w_n$ in Aut($M$);
- $[D\varphi \circ \Phi_\pi : D\varphi \otimes \text{tr}_\pi]_T = \lim_{n \to \infty} \alpha_\pi(w_n)(w_n^* \otimes 1)$ for all $\pi \in \text{Irr}(G)$,

where the latter limit is taken in the strong* topology.

Proof. By [18] Theorem 6.2 and Lemma 3.2, we can perturb $\alpha$ to be an action. Considering the chain rule of Connes’ cocycles, we may and do assume that $\alpha$ is an action. Applying the previous theorem to $\alpha$ and $\Gamma = \{\epsilon\}$, we can take a unitary $U = (U^\nu)_\nu$ in $M^\omega \otimes L^\infty(\hat{G})$ such that $\text{Ad} U^\nu$ approximates $\alpha$ and $U^*$ is an $\alpha^\omega$-cocycle.

Take a sequence of unitaries $\{v^\nu\}_\nu \subset M$ such that $\sigma^\nu_T = \lim_{\nu \to \infty} \text{Ad} v^\nu$. This is possible because $\sigma^\nu_T$ is approximately inner [11]. We set $\nu := (v^\nu)_\nu \in M^\omega$.

For $\pi \in \text{Irr}(G)$, we set a unitary $w^\nu_\pi := ((v^\nu)^* \otimes 1)[D\varphi \circ \Phi^\nu_\pi : D\varphi \otimes \text{tr}_\pi]_\alpha_\pi(v^\nu)$ in $M \otimes B(H_\pi)$, and also set $w^\nu := (w^\nu_\pi)_\pi \in M \otimes L^\infty(\hat{G})$ and $w = (w^\nu)_\nu \in M^\omega \otimes L^\infty(\hat{G})$. Then by Lemma 7.12, we see that $w$ is an $\alpha^\omega$-cocycle. We will check that $U^* wU \in M_\omega \otimes L^\infty(\hat{G})$. Take any $\pi \in \text{Irr}(G)$ and $\psi \in M_\pi$. Recall the notation $\phi^\nu \sim \psi^\nu$ for sequences $(\phi^\nu)_\nu, (\psi^\nu)_\nu \subset (M \otimes B(H))$, with $\lim_{\nu \to \omega} \|\phi^\nu - \psi^\nu\| = 0$.  

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Using $\Phi_\pi^\alpha \circ \sigma_T^\nu \Phi_\pi^\nu = \sigma_T^\nu \circ \Phi_\pi^\alpha$ (see [19, §3.2]), we have

\[(U_\pi^\nu)^* w_\pi^\nu U_\pi^\nu \cdot (\psi \otimes \text{tr}_\pi) \cdot (U_\pi^\nu)^* (w_\pi^\nu)^* U_\pi^\nu \sim (U_\pi^\nu)^* (w_\pi^\nu \cdot (\psi \circ \Phi_\pi^\nu) \cdot (w_\pi^\nu)^* U_\pi^\nu
\]

\[= (U_\pi^\nu)^* (v_\nu^\nu \otimes 1) [D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]_T^* \cdot \alpha_\pi (v_\nu^\nu) \cdot (\psi \circ \Phi_\pi^\alpha)
\]

\[\cdot \alpha_\pi((v_\nu^\nu)') [D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]_T (v_\nu^\nu \otimes 1) U_\pi^\nu
\]

\[= (U_\pi^\nu)^* (v_\nu^\nu \otimes 1) [D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]_T^* \cdot ((v_\nu^\nu \cdot (\psi \circ (v_\nu^\nu)' \circ \Phi_\pi^\alpha).
\]

\[\cdot [D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]_T (v_\nu^\nu \otimes 1) U_\pi^\nu
\]

\[= (U_\pi^\nu)^* \cdot \left( (D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]^* T \cdot ((\psi \circ \sigma_T^\nu) \circ \Phi_\pi^\alpha) \cdot [D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]_T
\]

\[\circ (\sigma_T^\nu \otimes \text{tr}_\pi) \right) \cdot U_\pi^\nu
\]

\[= (U_\pi^\nu)^* \cdot ((\psi \circ \sigma_T^\nu) \circ \Phi_\pi^\alpha \circ \sigma_T^\nu \circ \Phi_\pi^\nu \circ U_\pi^\nu
\]

\[= (U_\pi^\nu)^* \cdot (\psi \circ \Phi_\pi^\alpha) \cdot U_\pi^\nu
\]

\[\sim \psi \otimes \text{tr}_\pi.
\]

By [18, Lemma 3.6], we see that $U w U$ is in $M_\nu \otimes L^\infty(\hat{G})$. Using Corollary 3.25, we can take a unitary $z \in M_\nu$ such that $w = (z \otimes 1)\alpha^\nu(z^*)$, that is, $[D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]_T = (vz \otimes 1)\alpha^\nu(z^*)$. Then a representing sequence of $vz$ satisfies the desired properties.

### 4. Classification for type $\text{III}_\lambda$ case

#### 4.1. Canonical extension to discrete cores and the main result

As explained in Introduction, our idea in type $\text{III}_\lambda$ case is that we reduce the classification problem to type $\text{II}_\infty$ case by using the discrete decomposition. For this purpose, we have to consider the canonical extension of endomorphisms of a type $\text{III}_\lambda$ factor to its discrete core. This is possible for endomorphisms with trivial Connes-Takesaki modules as follows [12, Proposition 4.5]. Readers are referred to [?] for relations between the results of [12] and [19].

Let $R$ be a type $\text{III}_\lambda$ factor, $0 < \lambda < 1$, and $\phi$ a generalized trace, that is, $\phi(1) = \infty$ and $\sigma_T^\phi = \text{id}$, $T = -2\pi/\log \lambda$, hold. Then $R \rtimes \phi \mathbb{T}$ is called the discrete core. We denote by $\lambda^\phi(t)$ the unitary implementing $\sigma_T^\phi$ for $t \in \mathbb{T}$.

**Definition 4.1.** Let $R$ be a type $\text{III}_\lambda$ factor and $K$ a finite dimensional Hilbert space. For $\beta \in \text{Mor}_0(R, R \otimes B(K))$ with the standard left inverse $\Phi$ and $\text{mod}(\beta) = \text{id}$, we define the canonical extension $\tilde{\beta} \in \text{Mor}(R \rtimes \phi \mathbb{T}, (R \rtimes \phi \mathbb{T}) \otimes B(K))$ by

1. $\tilde{\beta}(x) = \beta(x)$ for all $x \in R$;
2. $\tilde{\beta}(\lambda^\phi(t)) = [D\phi \circ \Phi : D\phi \otimes \text{tr}_K]_t(\lambda^\phi(t) \otimes 1)$ for all $t \in \mathbb{R}/T\mathbb{Z}$.

For a cocycle action $\alpha \in \text{Mor}(R, R \otimes L^\infty(\hat{G}))$, we can prove that $\tilde{\alpha} := (\tilde{\alpha}_x)_x$ is a cocycle action in a similar way to the proof of Theorem 7.13.
Lemma 4.2. If $\beta \in \text{Mor}(\mathcal{R}_\lambda, \mathcal{R}_\lambda \otimes L^\infty(\hat{G}))$ is an approximately inner and centrally free cocycle action, then $\tilde{\beta}$ is also approximately inner and centrally free.

Proof. We check $\text{mod}(\tilde{\beta}_\pi) = \text{id}$ for each $\pi \in \text{Irr}(G)$. Let $\hat{\phi}$ be the dual weight on $M$. Then $\sigma_t^\hat{\phi} = \text{Ad} \lambda^\phi(t)$ for $t \in \mathbb{T}$. Take a positive operator $h$ such that $\lambda^\phi(t) = h^{-it}$ for $t \in \mathbb{T}$. Then $\hat{\phi}_h$ is a trace on $M := \mathcal{R}_\lambda \rtimes_{\sigma_h} \mathbb{T}$. Note that $\Phi^\beta_\pi$ commutes with the dual action $\theta$. Let $T_\theta: M \to \mathcal{R}_\lambda$ be the operator valued weight obtained by averaging the $\mathbb{Z}$-action $\theta$. Using $\hat{\phi} \circ \Phi^\beta_\pi = \hat{\phi} \circ \Phi^\beta_\pi \circ (T_\theta \otimes \text{id})$, we can compute as follows:

$$[D\hat{\phi}_h \circ \Phi^\beta_\pi : D\hat{\phi}_h \otimes \text{tr}_\pi]_t = [D\hat{\phi}_h \circ \Phi^\beta_\pi : D\hat{\phi} \circ \Phi^\beta_\pi]\cdot[D\hat{\phi} \circ \text{tr}_\pi : D\hat{\phi}_h \otimes \text{tr}_\pi]_t$$

$$= \tilde{\beta}_\pi(h^it)[D\hat{\phi} \circ \Phi^\beta_\pi \circ (T_\theta \otimes \text{id}) : D\hat{\phi} \circ T_\theta \otimes \text{tr}_\pi](h^{-it} \otimes 1)$$

$$= \tilde{\beta}_\pi(\lambda^\phi(t)^*)[D\hat{\phi} \circ \Phi^\beta_\pi : D\hat{\phi} \circ \text{tr}_\pi]_t(\lambda^\phi(t) \otimes 1) = 1.$$ 

By Corollary 4.1, $\tilde{\beta}$ is approximately inner.

Next we check the freeness of $\tilde{\beta}$. If $\tilde{\beta}_\pi$ is not properly outer for some $\pi \neq 1$, then $\tilde{\beta}_\pi$ is actually implemented by a unitary. This fact is proved as in the proof of [12, Proposition 3.4] because of the irreducibility of $\beta_\pi$ [18, Lemma 2.8]. Also note Lemma 4.1. Using $(\mathcal{R}_\lambda)_\omega \subset M_\omega$ (see the proof of [19, Lemma 4.11]), we see that $\beta_\pi$ is centrally trivial, and this is a contradiction.

We show that $\tilde{\beta}$ is centrally free action. The second canonical extension $\tilde{\beta}$ is cocycle conjugate to $\beta$ by Lemma 3.3. Hence $\tilde{\beta}$ is centrally free on $M \rtimes_{\theta} \mathbb{Z}$, and $(\tilde{\beta}_\pi)\omega$ is non-trivial on $(M \rtimes_{\theta} \mathbb{Z})_\omega$ for any $\pi \neq 1$. Since $(M \rtimes_{\theta} \mathbb{Z})_\omega$ is naturally isomorphic to $M^\theta_\omega$ and $(\tilde{\beta})\omega|_{M_\omega} = (\tilde{\beta}_\pi)\omega|_{M_\omega}$, $(\tilde{\beta}_\pi)\omega$ is non-trivial on $M^\theta_\omega$ for any $\pi \neq 1$. In particular, $\tilde{\beta}$ is a centrally free action because $\tilde{\beta}$ is free.

Though the action $\tilde{\beta}$ is unique up to cocycle conjugacy, we need to consider the $\mathbb{Z}$-action $\theta$ to obtain the uniqueness of the original $\beta$. Our aim is to classify the $\hat{G} \times \mathbb{Z}$-action $\tilde{\beta}\theta$ on $\mathcal{R}_{0,1}$. The following is our main theorem in this section.

Theorem 4.3. Let $M \cong \mathcal{R}_{0,1}$ with a trace $\tau$, $\theta \in \text{Aut}(M)$, $\alpha$ be an action of $\hat{G}$ on $M$, $\beta$ an action of $\hat{G}$ on $\mathcal{R}_\lambda$. Assume the following:

- $\theta \in \text{Aut}(M)$ satisfies $\tau \circ \theta = \lambda \tau$, $0 < \lambda < 1$;
- $\alpha$ is approximately inner and centrally free;
- $\alpha$ and $\theta$ commute;
- $\beta$ is free.

Then $\alpha\theta$ is cocycle conjugate to $\theta \otimes \beta$.

Once proving the above theorem, we can show Theorem 2.4 for $\mathcal{R}_\lambda$ as follows.

- **Proof of Theorem 2.4** for $\mathcal{R}_\lambda$, $0 < \lambda < 1$. 

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Let \( \varphi \) be a generalized trace on \( \mathcal{R}_\lambda \), and \( M := \mathcal{R}_\lambda \rtimes_{\sigma^\varphi} \mathbb{T} \). Then \( M \) is isomorphic to \( \mathcal{R}_{0,1} \). Let \( \theta \) be a dual action by \( \mathbb{Z} \) on \( M \), and \( \tilde{\alpha} \) the canonical extension of \( \alpha \). Then \( \tilde{\alpha} \) is approximately inner and centrally free by Lemma 4.2. Applying the previous theorem to \( \tilde{\alpha}\theta \), we get \( \tilde{\alpha}\theta \sim \theta \otimes \beta \).

By Lemma 3.3, the second extension \( \tilde{\alpha} \) on \( \mathcal{R}_{0,1} \mathbb{Z} \) is cocycle conjugate to \( \text{id} \otimes \beta \) on \( \mathcal{R}_{0,1} \mathbb{Z} \otimes \mathcal{R} \). By Lemma 3.3, \( \tilde{\alpha} \) is cocycle conjugate to \( \alpha \). Hence \( \alpha \) is cocycle conjugate to \( \text{id}_{\mathcal{R}_\lambda} \otimes \beta \). □

4.2. Model action splitting

The rest of this section is devoted to prove Theorem 4.3. Let \( M \), \( \tau \), \( \alpha \) and \( \theta \) be as in that theorem. We also take a faithful normal state \( \phi \) on \( M \). We fix their notations from here. Since \( \theta \) scales the trace, the \( \hat{G} \times \mathbb{Z} \)-action \( \alpha\theta \) is not approximately inner.

**Lemma 4.4.** The \( \hat{G} \times \mathbb{Z} \)-action \( \alpha\theta \) is centrally free.

**Proof.** Since \( \tau \circ \theta = \lambda \tau \) with \( \lambda \neq 1 \), \( \theta \) is centrally free. Assume that \( \alpha_n\theta^n \) is centrally trivial for some \( \pi \in \text{Irr}(G) \) and \( n \in \mathbb{Z} \). Then the map \( \alpha_n\theta^n \) is implemented by a unitary by Corollary 7.7, but we have \( \text{mod}(\alpha_n\theta^n) = \text{mod}(\theta^n) \) because \( \text{mod}(\alpha_n) = \text{id} \). Hence \( n = 0 \), and \( \pi = 1 \) by central freeness of \( \alpha \). □

Take \( U \in U(M^\omega \otimes L^\infty(\hat{G})) \) as in Theorem 3.24 with \( \Gamma = \mathbb{Z} \) and \( v_g = 1 \). Define the \( \hat{G} \times \hat{G}^\text{opp} \)-action \( \gamma = (\gamma^1 \otimes \text{id}) \circ \gamma^2 \) as before, where

\[\gamma^1(x) = U^*\alpha\omega(x)U, \quad \gamma^2(x) = U^*(x \otimes 1)U \quad \text{for} \quad x \in M^\omega.\]

Since \( U \) is fixed by \( \theta^\omega \), \( \gamma \) commutes with \( \theta^\omega \) on \( M^\omega \). Hence \( \gamma\theta^\omega \) is a \( \hat{G} \times \hat{G}^\text{opp} \times \mathbb{Z} \)-action on \( M^\omega \). Applying Corollary 3.15 to the strongly free and semi-liftable action \( \gamma^1 \circ \theta \) and the set \( T = \{ U_{\pi, i, j} \} \), we have the following result. Note that \( T' \cap (M^\omega)^\gamma = (M^\omega)^\gamma \).

**Lemma 4.5.** For any \( n \in \mathbb{N} \), there exists a partition of unity \( \{ E_i \}_{i=0}^{n-1} \) in \( M^\gamma \) such that \( \theta^\omega(E_i) = E_{i+1} \) for \( 0 \leq i \leq n - 1 \) (\( E_0 := E_0 \)).

As in [2], we obtain the following stability result by using the above lemma.

**Lemma 4.6.** The \( \mathbb{Z} \)-action \( \theta^\omega \) on \( M^\gamma \) is stable, that is, for any \( u \in U(M^\gamma) \), there exists \( w \in U(M^\gamma) \) such that \( u = w\theta^\omega(w^*) \).

**Lemma 4.7.** For any \( n \in \mathbb{N} \), there exists a system of matrix units \( \{ f_{ij} \}_{i,j=0}^{n-1} \subset M^\gamma \) with \( \theta^\omega(f_{ij}) = \mu_i^\gamma - j f_{ij} \), where \( \mu = e^{2\pi \sqrt{-1}/n} \).

**Proof.** By Corollary 3.17 for \( \gamma^1 \theta^\omega \), we see that \( (T' \cap M^\omega)^\gamma \theta^\omega = M^\gamma \theta^\omega \) is of type \( \Pi_1 \). Hence we can take a system of matrix units \( \{ e_{ij} \}_{i,j=0}^{n-1} \subset M^\gamma \theta^\omega \). Set \( u := \sum_{i=0}^{n-1} \mu^i e_{ii} \), and by Lemma 4.6, we have \( w \in U(M^\gamma) \) such that \( u = w\theta^\omega(w^*) \). Set \( f_{ij} := w^* e_{ij} w \in M^\gamma \). Then we have

\[\theta^\omega(f_{ij}) = \theta^\omega(w^*) e_{ij} \theta^\omega(w) = w^* u e_{ij} u^* w = \mu_i^\gamma - j f_{ij}.\]

□
Recall the following result \cite[Proposition 7.1]{20}.

**Lemma 4.8.** Let $e, f$ be projections in $M^\omega$ with $v^*v = e$, $vv^* = f$ for an element $v \in M^\omega$. Let $e = (e(\nu))_\nu$ and $f = (f(\nu))_\nu$ be representing sequences such that $e(\nu)$ and $f(\nu)$ are equivalent for each $\nu \in \mathbb{N}$. Then we can choose a representing sequence of $v$, $v = (v(\nu))_\nu$ so that $v^*(\nu)v(\nu) = e(\nu)$ and $v(\nu)v(\nu)^* = f(\nu)$ for each $\nu \in \mathbb{N}$.

**Lemma 4.9.** Let $n \in \mathbb{N}$ and $\mu = e^{2\pi \sqrt{-1}/n}$. Then for any $F \in \text{Irr}(\mathbb{G})$, $\Psi \in (M_*)_+$, and $\epsilon > 0$, there exists a unitary $u \in M \otimes L^\infty(\mathbb{G})$, a unitary $w \in M$ and a system of matrix units $\{f_{ij}\}_{i,j=0}^{n-1}$ in $M$ such that

(i) $\|u_\pi - 1\|_{\phi \otimes \text{tr}_\pi}^\# < \epsilon$ for all $\pi \in F$;

(ii) $\|w - 1\|_\phi^\# < \epsilon$;

(iii) $\|[f_{ij}, \psi]\| < \epsilon$ for all $\psi \in \Psi$ and $0 \leq i, j \leq n - 1$;

(iv) $\text{Ad} u \circ \alpha_{ij}(f_{ij}) = f_{ij} \otimes 1$ for all $0 \leq i, j \leq n - 1$;

(v) $\text{Ad} w \circ \theta(f_{ij}) = \mu^{i-j}f_{ij}$ for all $0 \leq i, j \leq n - 1$.

**Proof.** Let $\{f_{ij}\}_{i,j=0}^{n-1}$ be a system of matrix units in $M_\omega^n$ as in Lemma 4.7. Then $\gamma(f_{ij}) = f_{ij} \otimes 1$ implies $\alpha_{ij}(f_{ij}) = f_{ij} \otimes 1$. Take a representing sequence of $f_{ij}$, $(\alpha_{ij}(f_{ij}))_\nu$ such that $(\alpha_{ij}(f_{ij}))_{ij=0}^{n-1}$ is a system of matrix units in $M$ for all $\nu$.

By Lemma 4.8, for each $\pi \in \text{Irr}(\mathbb{G})$, there exists $v_\pi(\nu) \in M \otimes B(H_\pi)$ such that $v(\nu)^*v(\nu) = f_{00}(\nu) \otimes 1$, $v(\nu)^*v(\nu) = \alpha_{ij}(f_{00}(\nu))$ and $(v(\nu))_\nu = f_{00} \otimes 1$. Set a unitary $u(\nu) := \sum_{i=0}^{n-1}(f_{i0}(\nu) \otimes 1)v(\nu)^*\alpha_{ij}(f_{00}(\nu))$. Then $\text{Ad} u(\nu) \circ \alpha_{ij}(f_{ij}(\nu)) = f_{ij}(\nu) \otimes 1$ holds. We have $(u(\nu))_\nu = 1$ in $M^{\omega} \otimes B(H_\pi)$. Indeed,

$$(u(\nu))_\nu = \sum_{i=0}^{n-1}(f_{i0}(\nu) \otimes 1, v(\nu)^* \alpha_{ij}(f_{00}(\nu)))_\nu = \sum_{i=0}^{n-1}(f_{i0} \otimes 1)(f_{00} \otimes 1)(f_{00} \otimes 1) = 1.$$

Set a unitary $u(\nu) = (u(\nu))_\pi$ in $M \otimes L^\infty(\mathbb{G})$.

Next we construct $w$. Applying Lemma 4.8 to $\theta_{ij}(f_{00}) = f_{00}$, there exists $v(\nu) \in M$ such that $v(\nu)^*v(\nu) = f_{00}$, $v(\nu)^*v(\nu) = \theta(f_{00}(\nu))$ and $(v(\nu))_\nu = f_{00}$. Set a unitary $w(\nu) := \sum_{i=0}^{n-1}\mu^{i}f_{i0}(\nu)v(\nu)$. Then $\text{Ad} w(\nu) \circ \theta(f_{ij}(\nu)) = \mu^{i-j}f_{ij}(\nu)$ holds for all $0 \leq i, j \leq n - 1$ and $\nu \in \mathbb{N}$. We can show $w(\nu) \to 1$ strongly* as $\nu \to \infty$ as above.

Hence we can choose $\nu \in \mathbb{N}$ such that $u = u(\nu)$, $w = w(\nu)$ and $f_{ij} \nu$ satisfy the desired conditions. $\square$

Let $\Psi_n \in M_*$ be an increasing subset such that $\Psi = \bigcup_{n=1}^{\infty} \Psi_n$ is total in $M_*$. We recall the following result due to Connes \cite[Lemma 2.3.6]{2}.

**Lemma 4.10.** Let $M_1, M_2, \cdots, M_n \subset M$ be mutually commuting finite dimensional subfactors. Denote $\sqrt[n-1]{M_k} := N$. If $\sum_{k=1}^{\infty} \|\psi \circ E_{M_k \cap M} - \psi\| < \infty$ for all $\psi \in \Psi$, then $N$ is a hyperfinite subfactor of type $II_1$ and we have $M = N \vee N^\prime \cap M \cong N \otimes N^\prime \cap M$. Here $E_{M_k \cap M} = \text{tr}_{M_k} \otimes \text{id}_{M_k \cap M}$.

Let $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ be such that any $n \in \mathbb{N}$ appears infinitely many times. Set $\mu_k := e^{2\pi \sqrt{-1}/n_k}$. For a system of $n_k \times n_k$ matrix units $\{e_{ij}\}_{i,j=0}^{n_k-1}$, set $u_{n_k} :=}$
Proof. We set \( \bar{\omega} := \text{Ad} \bar{\theta} \). Set \( \omega \in \Psi_k \) and \( k \in \mathbb{N} \).

We set \( M_k := (\{f^k_{ij}\}_{i,j=0}^{n_k-1})^\circ \) and \( E_m := V^m_{k=1} M_k \).

Recall that we have fixed a faithful normal state \( \phi \in M_* \). We will construct the following families:

1. Matrix units, \( \{f^k_{ij}\}_{i,j=0}^{n_k-1} \subset M \) for \( k \in \mathbb{N} \) such that they are mutually commuting for \( k \) and satisfy \( \|\psi, f^k_{ij}\| \leq \epsilon_k/n_k \) for all \( 0 \leq i, j \leq n_k, \psi \in \Psi_k \) and \( k \in \mathbb{N} \).

We set \( M_n := (\{\bar{f}^k_{ij}\}_{i,j=0}^{n_k-1})^\circ \) and \( E_m := V^m_{k=1} M_k \).

2. Unitaries \( u^m \in (E'_{m-1} \cap M) \otimes L^\infty(\hat{G}) \) and \( w^m \in E'_{m-1} \cap M \) satisfying the following for each \( m \in \mathbb{N} \):

\begin{itemize}
  \item \( \|u^m - 1\|_{\phi \otimes \text{tr}_\pi} < \epsilon_m \) and \( \|w^m - 1\|_{\phi} < \epsilon_m \) for all \( \pi \in \mathcal{F}_m \);
  \item We set \( \bar{u}^m := u^m u^{m-1} \cdots u^1 \) and \( \bar{w}^m := w^m w^{m-1} \cdots w^1 \). Then we have, for all \( 0 \leq i, j \leq n_k - 1 \) and \( 1 \leq k \leq m \):
    \begin{align*}
    \text{Ad} \bar{u}^m \circ \alpha(f^k_{ij}) &= f^k_{ij} \otimes 1; \\
    \text{Ad} \bar{w}^m \circ \theta(f^k_{ij}) &= \mu_k^{\langle i - j \rangle} f^k_{ij}.
    \end{align*}
\end{itemize}

Assume we have constructed up to \( k = m \). Set \( \alpha_m := \text{Ad} \bar{u}^m \circ \alpha \), and \( \theta(m) := \text{Ad} \bar{w}^m \circ \theta \). Since \( \alpha_m \) fixes \( E_m \), \( \alpha_m \) is a cocycle action on \( E'_m \cap M \).

Let \( \{\bar{e}^i\} \) a basis for \( E'_m \). Let us decompose \( \psi \in \Psi_{m+1} \) as \( \psi = \sum_{i=1}^{\dim(E_m)} \bar{e}^i \otimes \psi_i \), \( \psi_i \in (E'_m \cap M)_* \), and denote by \( \hat{\Psi}_{m+1} \) the set of all such \( \psi_i \). Fix \( \delta_m + 1 > 0 \) so that \( \delta_{m+1} \leq \epsilon_m (n_{m+1} \dim E_m)^{-1} \).

Claim. There exist the following elements:

1. A system of matrix units \( \{f^m_{ij}\}_{i,j=0}^{n_m-1} \subset E'_m \cap M \) such that \( \|\psi, f^m_{ij}\| \leq \delta_{m+1} \) for \( \psi \in \hat{\Psi}_{m+1} \).

Set \( M_{m+1} := (\{f^m_{ij}\}_{i,j=0}^{n_m-1})^\circ \) and \( E_{m+1} = E_m \cap M_{m+1} \).

2. Unitaries \( u^{m+1} \in (E'_m \cap M) \otimes L^\infty(\hat{G}) \) and \( w^{m+1} \in E'_m \cap M \) satisfying the following:

\begin{itemize}
  \item \( \|u^{m+1} - 1\|_{\phi \otimes \text{tr}_\pi} < \epsilon_{m+1} \) and \( \|w^{m+1} - 1\|_{\phi} < \epsilon_{m+1} \) for all \( \pi \in \mathcal{F}_{m+1} \);
  \item \( \text{Ad} u^{m+1} \circ \alpha_m(f^m_{ij}) = f^m_{ij} \) for all \( 0 \leq i, j \leq n_{m+1} - 1 \); \\
    \( \text{Ad} w^{m+1} \circ \theta_m(f^{m+1}_{ij}) = \mu_{m+1}^{\langle i - j \rangle} f^{m+1}_{ij} \) for all \( 0 \leq i, j \leq n_{m+1} - 1 \).
\end{itemize}

Indeed, we can prove this as follows. By the natural isomorphism \( (E'_m \cap M)_\omega = E'_m \cap M^\omega \), we have

\[
((E'_m \cap M)_\omega \subset (E'_m \cap M)^\omega) = (M_\omega \subset E'_m \cap M^\omega).
\] (4.1)

On \( E'_m \cap M \), we have a \( \hat{G} \)-cocycle action \( \alpha_m \) and a \( \mathbb{Z} \)-action \( \theta_m \). Using Lemma 4.7, we take a system of matrix units \( \{f^m_{ij}\}_{i,j=0}^{n_m-1} \subset M^\omega \) such that \( \theta_m(f^m_{ij}) = \mu_{m+1}^{\langle i - j \rangle} f^m_{ij} \) for \( 0 \leq i, j \leq n_{m+1} - 1 \). Then we get \( \theta_m(f^m_{ij}) = w^{m+1} \theta_m(f^m_{ij})(w^{m+1})^* = \mu_{m+1}^{\langle i - j \rangle} f^m_{ij} \).
Hence $\alpha^w(f_{ij}) = f_{ij} \otimes 1$ as before. Hence we have $(\alpha^m)^w(f_{ij}) = u^m(f_{ij} \otimes 1)(\tilde{u}^m)^* = f_{ij} \otimes 1$. By using (14), we can represent $\{f_{ij}\}_{i,j=0}^{n_{m+1}-1}$ as sequences $\{(f_{ij}(\nu))_\nu\}_{i,j=0}^{n_{m+1}-1}$ in $E'_m \cap M$. Then we can take a desired elements in the Claim as in Lemma 4.9.

Now the condition (1) in the Claim implies $\|[\psi, f_{ij}^{m+1}]\| \leq \epsilon_m/n_m$ for $\psi \in \Psi_{m+1}$. Thus we complete induction. We have constructed families $u^m$, $w^m$ and $E^m$ for $m \in \mathbb{N}$. Since for $\psi \in \Psi_k$ we have

$$\|\psi \circ E_{M'_m \cap M} - \psi\| = \left\| \frac{1}{n_k} \sum_{ij=0}^{n_k-1} f_{ij}^k \psi f_{ji}^k - \psi \right\| \leq \frac{1}{n_k} \sum_{ij=0}^{n_k-1} \|\psi, f_{ji}^k\| \leq \frac{1}{n_k} \cdot n_k^2 \cdot \epsilon_k = \epsilon_k,$$

we can check $\sum_{k=1}^{\infty} \|\psi \circ E_{M'_m \cap M} - \psi\| < \infty$ for $\psi \in \Psi$. Then Lemma 4.10 implies that $E := \bigvee_{k=1}^{\infty} E_k$ is isomorphic to $\mathcal{R}_0$ and yields a tensor product splitting $M = E \vee (E' \cap M) \cong E \otimes (E' \cap M)$.

Step 2. From the condition (2), the strong* limits $\overline{w}^\infty = \lim_{m \to \infty} \overline{w}^m$ and $\overline{\pi}^\infty = \lim_{m \to \infty} \overline{\pi}^m$ exist, and together with (1), we have $\text{Ad} \overline{w}^\infty \circ \alpha(x) = x \otimes 1$ and $\text{Ad} \overline{\pi}^\infty \circ \theta(x) = \sigma(x)$ for $x \in E$. Extend $\overline{\pi}^\infty$ to the $\theta$-cocycle naturally and denote it also by $\overline{w}^\infty \in M \otimes \ell^\infty(\mathbb{Z})$. Then we get the perturbation from the $\hat{G} \times \mathbb{Z}$-action $\alpha \theta$ to the $\hat{G} \times \mathbb{Z}$-cocycle action $(\text{Ad} \overline{w}^\infty \alpha(\overline{w}^\infty) \circ \alpha \theta, v)$. Set $\beta := \text{Ad} \overline{w}^\infty \alpha(\overline{w}^\infty) \circ \alpha \theta$. Then $\beta$ is of the form $\sigma \otimes \beta'$ on $E \otimes (E' \cap M)$, where $\beta' = \beta|_{E' \cap M}$. We claim that $v$ is evaluated in $E' \cap M$, and $(\beta', v)$ is a cocycle action.

By definition of $v$, $(\beta \otimes \text{id}) \circ \beta = \text{Ad} v \circ (\text{id} \otimes \Delta_{\hat{G} \times \mathbb{Z}}) \circ \beta$. Let $k \in \mathbb{N}$ and $0 \leq i, j \leq n_k - 1$. Then we have the following:

$$(\beta(\pi, \ell) \otimes \text{id})(\beta(\rho, m)(f_{ij}^k)) = \mu_k^{m(i-j)}(f_{ij}^k \otimes 1_\rho) = \mu_k^{(\ell+m)(i-j)}(f_{ij}^k \otimes 1_\rho)$$

and

$$(\text{id} \otimes \Delta_{\hat{G} \times \mathbb{Z}})(\beta(\pi, \ell) \otimes \rho, m) = (\text{id} \otimes \Delta)(\beta(f_{ij}^k) \otimes (\cdot, \ell+m)) = \mu_k^m(i-j)(f_{ij}^k \otimes 1_\rho)$$

Hence $v$ is evaluated in $M'_k \cap M$ for any $k \in \mathbb{N}$, and hence in $E' \cap M$.

We have shown that $\alpha \theta$ is cocycle conjugate to the cocycle action $\sigma \otimes \beta'$. Since $E' \cap M$ is type III, we can perturb $(\beta', v)$ to a $\hat{G} \times \mathbb{Z}$-action $\beta''$ by Lemma 3.2. Hence $\alpha \theta \sim \sigma \otimes \beta''$. Since $\sigma \otimes \sigma \sim \sigma$, we get $\alpha \theta \sim \sigma \otimes \sigma \otimes \beta'' \sim \sigma \otimes \alpha \theta$. \qed

Remark 4.12. We can use the Jones-Ocneanu cocycle argument in [20, Lemma 2.4] to obtain cocycle conjugacy $\sigma \sim \sigma \otimes \alpha \theta$ in Step 2 above. We set $\nu := \overline{w}^\infty \alpha(\overline{w}^\infty)$. Then we have $\text{Ad} \nu \circ \alpha \theta = \beta = \sigma \otimes \beta'$. Since $\sigma$ is conjugate to $\sigma \otimes \sigma$, there exists an isomorphism $\gamma$ from $E \otimes E$ onto $E$ with $\gamma^{-1} \circ \sigma \circ \gamma = \sigma \otimes \sigma$. So
we have
\[
(\gamma^{-1} \otimes \text{id} \otimes \text{id}_{L^\infty(\hat{G} \times \mathbb{Z})}) \circ \text{Ad} \nu \circ \alpha \theta \circ (\gamma \otimes \text{id}) = \gamma^{-1} \circ \sigma \circ \gamma \otimes \beta' = \sigma \otimes \sigma \otimes \beta'
\]
\[
= \sigma \otimes \text{Ad} \nu \circ \alpha \theta.
\]

Then the following holds:
\[
\text{Ad}(\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \nu^*) \nu \circ \alpha \theta = (\gamma \otimes \text{id} \otimes \text{id}) \circ (\sigma \circ \alpha \theta) \circ (\gamma^{-1} \otimes \text{id}).
\]
We will verify that \((\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \nu^*) \nu\) is an \(\alpha \theta\)-cocycle. Here note that \((\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \nu) = \nu\) holds because the 2-cocycle \(\nu\) is evaluated in \(E' \cap M\). Then the following holds:
\[
((\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \nu^*) \nu \otimes 1) \cdot \alpha \theta((\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \nu^*) \nu)
\]
\[
= ((\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \nu^*) \otimes 1) \cdot (\sigma \otimes \beta')(((\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \nu^*))((\nu \otimes 1)\alpha \theta)(\nu)
\]
\[
= (\gamma \otimes \text{id} \otimes \text{id})((1 \otimes \nu^* \otimes 1)(\sigma \otimes \sigma \otimes \beta')(1 \otimes \nu^*))\nu(id \otimes \Delta)(\nu)
\]
\[
= (\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \alpha \theta(\nu^*)\nu)(id \otimes \Delta)(\nu)
\]
\[
= (\gamma \otimes \text{id} \otimes \text{id})(1 \otimes (id \otimes \Delta)(\nu^*)\nu)(id \otimes \Delta)(w)
\]
\[
= (id \otimes \Delta)((\gamma \otimes \text{id} \otimes \text{id})(1 \otimes \nu^*)\nu).
\]

Hence \(\alpha \theta\) and \(\alpha \theta \otimes \sigma\) are cocycle conjugate.

- **Proof of Theorem 4.3.**

Note that \(\theta \otimes \theta^{-1}\) is cocycle conjugate to \(\text{id}_{B(\ell_2)} \otimes \sigma\) by Connes [2]. Then the following holds:
\[
\alpha \theta \sim \text{id}_{B(\ell_2)} \otimes \alpha \theta \quad \text{(by Lemma 3.1)}
\]
\[
\sim \text{id}_{B(\ell_2)} \otimes \sigma \otimes \alpha \theta \quad \text{(by Lemma 4.1)}
\]
\[
\sim \theta \otimes \theta^{-1} \otimes \alpha \theta.
\]

Since the action \(\theta^{-1} \otimes \alpha \theta\) preserves the trace of \(R_{\ell_1}\), it is approximately inner. The central freeness is clear. Then \(\theta^{-1} \otimes \alpha \theta\) is cocycle conjugate to \(\text{id}_{B(\ell_2)} \otimes \sigma \otimes \beta\) by Theorem 2.4 for type \(\Pi_\infty\) case, and the following holds:
\[
\theta \otimes \theta^{-1} \otimes \alpha \theta \sim \theta \otimes \text{id}_{B(\ell_2)} \otimes \sigma \otimes \beta
\]
\[
\sim \theta \otimes \sigma \otimes \beta
\]
\[
\sim \theta \otimes \beta.
\]

Therefore we get \(\alpha \theta \sim \theta \otimes \beta\). \(\Box\)

We close this section with the following lemma which is used in Section 6.

**Lemma 4.13.** Let \(N\) be a type \(\Pi_\lambda\) factor with \(0 < \lambda < 1\) and \(\alpha\) an approximately inner action of \(\hat{G}\) on \(N\). Let \(\psi\) be a generalized trace on \(N\). Then there exists a \(\hat{G}\)-action \(\beta\) on \(N\) such that
- \(\beta \sim \alpha\);
- \(\psi \circ \Phi^\beta_{\pi} = \psi \otimes \text{tr}_{\pi}\) for all \(\pi \in \text{Irr}(G)\).
Proof. Since $\alpha$ is approximately inner, we see that $\psi \circ \Phi^\alpha$ is a generalized trace for all $\pi \in \text{Irr}(\mathbb{G})$. Hence there exists a unitary $v_\pi \in N \otimes B(H_\pi)$ such that $\psi \circ \Phi^\alpha = (\psi \otimes \tau_\pi) \circ \text{Ad} v_\pi$. Set $v = (v_\pi)_\pi \in N \otimes L^\infty(\hat{\mathbb{G}})$, and consider the cocycle action $\delta := \text{Ad} v \circ \alpha$, whose 2-cocycle is given by $u := (v \otimes 1) \alpha(v)(\text{id} \otimes \Delta)(v^*)$.

Then we have $\psi \circ \Phi^\delta = \psi \otimes \text{tr}_\pi$, and $\sigma^\psi$ and $\delta$ commute in particular.

We check that $u$ is evaluated in $N_\psi$ as follows: for $\pi, \rho \in \text{Irr}(\mathbb{G})$,

$$u^*(\psi \otimes \text{tr}_\pi \otimes \text{tr}_\rho)u = u^* \cdot (\psi \circ \Phi^\delta \circ (\Phi^\delta \otimes \text{id})) \cdot u$$

$$= (\text{id} \otimes \Delta)\alpha_\pi(v^*_\rho) (v^*_\rho \otimes 1) \cdot (\psi \circ \Phi^\delta \circ (\Phi^\delta \otimes \text{id})) \cdot (v_\pi \otimes 1) \alpha_\pi(v_\rho)(\text{id} \otimes \Delta)(v^*)$$

$$= (\text{id} \otimes \Delta)\alpha_\pi(v^*_\rho) \cdot (\psi \circ \Phi^\delta \circ (\Phi^\delta \otimes \text{id})) \cdot \alpha_\pi(v_\rho)(\text{id} \otimes \Delta)(v^*)$$

$$= (\text{id} \otimes \Delta)(v) \cdot (\psi \circ \Phi^\delta \circ (\Phi^\delta \otimes \text{id})) \cdot (\text{id} \otimes \Delta)(v^*)$$

$$= (\text{id} \otimes \Delta)(v) \cdot (\psi \circ \Phi^\alpha \circ (\Phi^\alpha \otimes \text{id})) \cdot (\text{id} \otimes \Delta)(v^*)$$

$$= \sum_{\sigma < \pi \otimes \rho} \sum_{S \in \text{ONB}(\sigma, \pi \otimes \rho)} \frac{d(\sigma)}{d(\pi)d(\rho)} (1 \otimes S)v_\sigma \cdot (\psi \circ \Phi^\alpha_\sigma) \cdot v^*_\sigma (1 \otimes S^*) (\text{id} \otimes \Delta)(v^*)$$

Hence $u \in N_\psi \otimes L^\infty(\hat{\mathbb{G}}) \otimes L^\infty(\hat{\mathbb{G}})$, and $(\delta|_{N_\psi}, u)$ is a cocycle action on the type II$_\infty$ factor $N_\psi$. By Lemma 3.2 there exists a unitary $w \in N_\psi \otimes L^\infty(\hat{\mathbb{G}})$ perturbing $(\delta, u)$ to the action $(\text{Ad} w \circ \delta, 1)$. Then $wv$ is an $\alpha$-cocycle and we set $\beta := \text{Ad} wv \circ \alpha$. We check that $\beta$ satisfies the second condition. Since $\Phi^\delta = \Phi^\alpha \circ \text{Ad} w^*_\pi$ and $w \in N_\psi \otimes L^\infty(\hat{\mathbb{G}})$, we have

$$\psi \circ \Phi^\beta = \psi \circ \Phi^\alpha \circ \text{Ad} w^*_\pi = (\psi \otimes \text{tr}_\pi) \circ \text{Ad} w^*_\pi = \psi \otimes \text{tr}_\pi .$$

5. GROUPOID ACTIONS AND TYPE III$_0$ CASE

Let $M$ be an injective factor of type III$_0$ and $\{\tilde{M}, \theta, \tau_M\}$ the canonical core of $M$. Let $(X, \nu, \mathcal{F}_t)$ be the flow of weights for $M$, that is, $Z(M) = L^\infty(X, \nu)$, $\theta_t(f)(x) = f(\mathcal{F}_t x)$ and $\nu$ is a measure on $X$. We represent $(X, \nu, \mathcal{F}_t)$ as a flow built under the ceiling function, that is, there exist a measure space $(Y, \mu)$, $f \in L^\infty(Y, \mu)$ with $f(x) \geq R$ for some $R > 0$, and a nonsingular transformation $T$ on $(Y, \mu)$ such that $X$ is identified with $\{(y, t) \mid y \in Y, 0 \leq t < f(y)\}$, $\nu = \mu \times dt$. 

and \( F_t(y, s) = (y, t + s) \) where we identify \((y, f(y))\) and \((Ty, 0)\). Then we have two kinds of measured groupoids, \( \mathcal{G} := \mathbb{R} \ltimes \mathcal{F} X \) and \( \mathcal{G} := \mathbb{Z} \ltimes_T Y \). In fact, \( \mathcal{G} \) is characterized as \( \mathcal{G} = \{ \gamma \in \mathcal{G} \mid s(\gamma), r(\gamma) \in Y \} \). Here for a \( \Gamma \)-space \( Z \), the groupoid \( \Gamma \ltimes Z \) is defined as \((g, h, s, r)(x) = (gh, x)\) for \( g, h \in \Gamma \) and \( x \in Z \). The source map \( s \) and the range map \( r \) are defined by \( s(g, x) = x \) and \( r(g, x) = gx \), respectively.

Let \( \alpha \) be an approximately inner action of \( \tilde{\mathcal{G}} \) on \( M \). Then \( \text{mod}(\alpha) = \text{id} \) by Theorem 7.6, that is, the canonical extension \( \tilde{\alpha} \) fixes \( L^\infty(X, \nu) \). We first discuss the reduction of the study of \( \hat{\mathcal{G}} \ltimes R \) actions of \( \mathcal{G} \ltimes R \) actions.

Let \( \tilde{M} = \int_X \tilde{M}(x) dx \) be the central decomposition. Since \( \tilde{M}(x) \) are injective for almost every \( x \in X \), \( M(x) \cong R_{0,1} \) holds for almost every \( x \in X \). As in [24], we obtain a family of actions \( \{ \tilde{\alpha}_x \}_{x \in X} \) of \( \tilde{\mathcal{G}} \) on \( \tilde{M}(x) \) determined by
\[
\tilde{\alpha}(a) = \int_X \tilde{\alpha}(a)(x) d\mu(x) = \int_X \tilde{\alpha}_x(a(x)) d\mu(x),
\]
and an action \( \{ \theta_{\gamma} \}_{\gamma \in \mathcal{G}} \) of \( \mathcal{G} \) by
\[
\theta_{\gamma}(a) = \int_X \theta_{\gamma}(a)(x) d\mu(x) = \int_X \theta_{\gamma}(a(F_{-t}x)) d\mu(x),
\]
where \( \gamma = (t, F_{-t}x) \). Of course \( \theta_{\gamma} \) is a isomorphism from \( \tilde{\mathcal{M}}(s(\gamma)) \) onto \( \tilde{\mathcal{M}}(r(\gamma)) \). Then \( \theta_{\gamma} \) and \( \tilde{\alpha}_x \) commute in the following sense: \( \tilde{\alpha}_{r(\gamma)} \circ \theta_{\gamma} = (\theta_{\gamma} \otimes \text{id}) \circ \tilde{\alpha}_{s(\gamma)} \).

Since \( \tilde{\alpha} \) preserves \( \tau_{\tilde{\mathcal{M}}} \) by Lemma 7.14, each \( \tilde{\alpha}_x \) preserves \( \tau_x \), a trace on \( \tilde{\mathcal{M}}(x) \). We denote the \( \pi \)-component of \( \alpha_x \) by \( \alpha_{\pi,x} \). We introduce the notion of a \( \tilde{\mathcal{G}} \ltimes \mathcal{G} \)-action.

**Definition 5.1.** Let \( R \) be a von Neumann algebra, \( \tilde{\mathcal{G}} \) a discrete Kac algebra, and \( \mathcal{G} \) a groupoid.

1. Let \( \{ \alpha_x \}_{x \in \mathcal{G}^{(0)}} \) be a family of actions of \( \tilde{\mathcal{G}} \) and \( \{ \alpha_{\gamma} \}_{\gamma \in \mathcal{G}} \) an action of \( \mathcal{G} \) on \( R \). We say that \( \alpha \) is a \( \tilde{\mathcal{G}} \ltimes \mathcal{G} \)-action if \( \alpha_{r(\gamma)} \circ \alpha_{\gamma} = (\alpha_{\gamma} \otimes \text{id}) \circ \alpha_{s(\gamma)} \) for all \( \gamma \). We denote \( \alpha_{r(\gamma)} \alpha_{\gamma} \) and \( \alpha_{\pi,r(\gamma)} \alpha_{\gamma} \) by \( \alpha_{\gamma} \) and \( \alpha_{\pi,\gamma} \) for simplicity. We call \( \{ \alpha_x \}_{x \in \mathcal{G}^{(0)}} \) and \( \{ \alpha_{\gamma} \}_{\gamma \in \mathcal{G}} \) the \( \mathcal{G} \)-part and the \( \mathcal{G} \)-part of \( \alpha \), respectively.

2. For two \( \tilde{\mathcal{G}} \ltimes \mathcal{G} \) actions \( \alpha \) and \( \beta \) on \( R \), we say that \( \alpha \) and \( \beta \) are \( \mathcal{G} \)-cocycle conjugate if there exist a Borel function \( \sigma: X \to \text{Aut}(R) \), a \( \beta_x \)-cocycle \( u_x \) for \( x \in X \) and a \( \beta_\gamma \)-cocycle \( u_\gamma \) for \( \gamma \in \mathcal{G} \) satisfying, for all \( x \in X \) and \( \gamma \in \mathcal{G} \),
\[
(\sigma_{r(\gamma)} \otimes \text{id}) \circ \alpha_{\gamma} \circ \sigma_{s(\gamma)}^{-1} = \text{Ad}(u^{r(\gamma)}_{r(\gamma)})(u_\gamma) \circ \beta_{r(\gamma)}
\]
and
\[
u^{r(\gamma)}_{r(\gamma)}(u_\gamma) = (u_\gamma \otimes 1)(\beta_{\gamma} \otimes \text{id})(u^{s(\gamma)}).
\]

In this case, we simply say that \( u^{r(\gamma)}_{r(\gamma)}(u_\gamma) \) is a \( \beta \)-cocycle.

The following can be shown as [24] p.430.\[32\]

**Lemma 5.2.** Let \( \alpha, \beta \) be actions of \( \tilde{\mathcal{G}} \ltimes \mathcal{G} \) on a type III\(0 \) injective factor \( M \). Suppose that \( \text{mod}(\alpha) = \text{id} = \text{mod}(\beta) \).
(1) The $\tilde{\mathbb{G}} \times \mathbb{R}$-actions $\tilde{\alpha} \theta$ and $\tilde{\beta} \theta$ on $\tilde{M}$ are cocycle conjugate if and only if the $\tilde{\mathbb{G}} \times \mathbb{G}$-actions $\tilde{\alpha}_{r(\gamma)} \theta_\gamma$ and $\tilde{\beta}_{r(\gamma)} \theta_\gamma$ on $\mathcal{R}_{0,1}$ are cocycle conjugate.

(2) If the $\tilde{\mathbb{G}} \times \mathbb{G}$-actions $\tilde{\alpha}_{r(\gamma)} \theta_\gamma$ and $\tilde{\beta}_{r(\gamma)} \theta_\gamma$ on $\mathcal{R}_{0,1}$ are cocycle conjugate, then they are also cocycle conjugate as the $\tilde{\mathbb{G}} \times \mathbb{G}$-actions.

Hence we only have to classify two $\tilde{\mathbb{G}} \times \mathbb{G}$-actions $\tilde{\alpha}_{r(\gamma)} \theta_\gamma$ and $\tilde{\beta}_{r(\gamma)} \theta_\gamma$ on $\mathcal{R}_{0,1}$. Here the $\tilde{\mathbb{G}}$-parts preserve the trace, and the $\mathbb{G}$-parts come from $\theta$, which are independent from $\alpha$ and $\beta$. Now we consider the following situation:

- We are given two $\tilde{\mathbb{G}} \times \mathbb{G}$-actions $\alpha$ and $\beta$ on $\mathcal{R}_{0,1}$;
- The $\tilde{\mathbb{G}}$-parts of $\alpha$ and $\beta$ are free actions;
- The $\mathbb{G}$-parts of $\alpha$ and $\beta$ preserve the trace on $\mathcal{R}_{0,1}$;
- $\text{mod}(\alpha_\gamma) = \text{mod}(\beta_\gamma)$ for $\gamma \in \mathbb{G}$.

Note that $\mathbb{G}$ is an ergodic, approximately finite (AF), orbitally discrete principal groupoid, and the following Krieger’s cohomology lemma provides a powerful tool for study of actions of such groupoids [10]. (Also see [13] Appendix.)

**Theorem 5.3.** Let $G$ be a Polish group, and $N$ a normal subgroup. Let $\mathbb{G}$ be an ergodic AF orbitally discrete principal groupoid. Let $\theta_1$ and $\theta_2$ be homomorphisms from $\mathbb{G}$ to $G$ with $\theta_1^0 \equiv \theta_2^0 \mod N$. Then there exist Borel maps $\sigma: \mathbb{G}^{(0)} \to N$ and $\nu: \mathbb{G} \to N$ such that $\sigma_{r(\gamma)} \theta_1^0 \sigma_{s(\gamma)}^{-1} = \nu_\gamma^\nu$.

We need some preparations as in [13, 23]. Let $\sigma$ be a trace preserving free action of $\tilde{\mathbb{G}}$ on $\mathcal{R}_{0,1}$. Let $C^{(1)}_\sigma$ be the set of pairs $(\theta, v)$, where $\theta \in \text{Int}(\mathcal{R}_{0,1})$ and $v$ is a $\sigma$-cocycle such that $\text{Ad} v \circ \sigma = (\theta \otimes \text{id}) \circ \sigma \circ \theta^{-1}$. We define the multiplication on $C^{(1)}_\sigma$ by $(\theta_1, v_1)(\theta_2, v_2) := (\theta_1 \theta_2, (\theta_1 \otimes \text{id})(v_2)v_1)$. Let $\text{Aut}_\sigma(\mathcal{R}_{0,1} \rtimes_\sigma \tilde{\mathbb{G}})$ be the set of all automorphisms which commute with the dual action of $\mathbb{G}$. Then we have $C^{(1)}_\sigma \subset \text{Aut}_\sigma(\mathcal{R}_{0,1} \rtimes_\sigma \tilde{\mathbb{G}})$ in a canonical way, and $C^{(1)}_\sigma$ is a Polish group. In fact, $C^{(1)}_{\sigma_0} = \text{Aut}_{\sigma_0}(\mathcal{R}_{0,1} \rtimes_\sigma \tilde{\mathbb{G}}) \cap \text{Ker}(\text{mod})$ holds. Let $C^{(0)}_{\sigma_0} := \{(\text{Ad} v, (v \otimes 1)\sigma(v^*)) \mid v \in U(\mathcal{R}_{0,1})\}$. Then $C^{(0)}_{\sigma_0}$ is a normal subgroup of $C^{(1)}_{\sigma_0}$.

**Lemma 5.4.** $C^{(0)}_{\sigma_0}$ is dense in $C^{(1)}_{\sigma_0}$.

**Proof.** Since $\sigma$ is trace preserving and free, $\sigma$ is approximately inner and centrally free by Corollary [7, 4]. Then we can take a unitary $U = (U^\nu)_{\nu} \in \mathcal{R}_{0,1}^\omega \otimes L^\infty(\tilde{\mathbb{G}})$ as in Theorem [3.24] with $\Gamma = \{e\}$.

Take $(\theta, v) \in C^{(1)}_{\sigma_0}$ and choose $\{v^\nu\}_\nu \subset U(\mathcal{R}_{0,1})$ with $\theta = \lim_{\nu \to \infty} \text{Ad} v^\nu$. Then

$$\text{Ad} v \circ \sigma = (\theta \otimes \text{id}) \circ \sigma \circ \theta^{-1} = \lim_{\nu \to \infty} \text{Ad}(v^\nu \otimes 1) \circ \sigma \circ \text{Ad}(v^\nu)^* = \lim_{\nu \to \infty} \text{Ad}(v^\nu \otimes 1)\sigma((v^\nu)^*) \circ \sigma.$$

Set $V := (v^\nu)_{\nu} \in \mathcal{R}_{0,1}^\omega$. Then $w := v^* (V \otimes 1)\sigma^\omega(V^*)$ is a $\sigma^\omega$-cocycle, and $U^* w U \in (\mathcal{R}_{0,1})^\omega \otimes L^\infty(\tilde{\mathbb{G}})$. By Corollary [3.25] there exists $z \in (\mathcal{R}_{0,1})^\omega$ such that $w = (z \otimes 1)\sigma^\omega(z^*)$. This implies $(z^* V \otimes 1)\sigma^\omega(V^* z) = v$. Let $(\mu^\nu)_{\nu}$ be a representing sequence of $z^* V$. Then $\theta = \lim_{\nu \to \omega} \text{Ad} \mu^\nu$ and $v = \lim_{\nu \to \omega} (\mu^\nu \otimes 1)\sigma((\mu^\nu)^*)$. \(\square\)
Lemma 5.5. Suppose that $\beta_x$ is constant, that is, $\beta_x = \beta_{x_0}$ for some $x_0 \in \mathcal{G}(0)$. Then there exist Borel families of automorphisms $\{\sigma_x\}_{x \in \mathcal{G}(0)} \subset \text{Int}(\mathcal{R}_{0,1})$ and $\beta_x$-cocycles $\{w^x\}_{x \in \mathcal{G}(0)} \subset U(\mathcal{R}_{0,1} \otimes L^\infty(\widehat{\mathcal{G}}))$ such that $(\sigma_x \otimes \text{id}) \circ \alpha_x \circ \sigma_x^{-1} = \text{Ad} w^x \circ \beta_x$.

Proof. Set $N := \mathcal{R}_{0,1} \rtimes_x \widehat{\mathcal{G}}$ and $N(x) := \mathcal{R}_{0,1} \rtimes_{x_0} \widehat{\mathcal{G}}$ for each $x \in X$. Note that $N$ and $N(x)$ act on the common Hilbert space $L^2(\mathcal{R}_{0,1}) \otimes L^2(\widehat{\mathcal{G}})$.

Let $B_x$ be the set of pairs $(\sigma, v)$, where $\sigma \in \text{Aut}(\mathcal{R}_{0,1})$ and $v$ is a 1-cocycle for $\alpha_x$ such that $(\sigma^{-1} \otimes \text{id}) \circ \text{Ad} v \circ \alpha_x \circ \sigma = \beta_{x_0}$. Then $B_x$ is non-empty because of Theorem 2.4 for $\mathcal{R}_{0,1}$ and it is identified with the set of isomorphisms from $N$ onto $N(x)$ preserving $\mathcal{R}_{0,1}$. Moreover, $B_x$ is a Polish group because it is identified with a closed subset of unitary maps $L^2(N)$ onto $L^2(N(x))$ which intertwine $N$ and $N(x)$, preserve positive cones and $L^2(\mathcal{R}_{0,1})$ and commute with modular conjugation [8]. Then thanks to the measurable cross section theorem [27, Theorem A.16, vol.I], we can choose a Borel family $(\sigma_x, v^x) \in B_x$ as in the proof of [27, Theorem IV.8.28, Proposition IV.8.29].

Theorem 5.6. Let $\alpha$ and $\beta$ be $\mathcal{G}$-$\mathcal{G}$-actions on $\mathcal{R}_{0,1}$ as before. Assume that $\beta_x$ is constant. Then $\alpha$ and $\beta$ are cocycle conjugate as $\mathcal{G}$-$\mathcal{G}$ actions.

Proof. By the previous lemma, we can take Borel families $\{\sigma_x\}_{x \in \mathcal{G}(0)} \subset \text{Int}(\mathcal{R}_{0,1})$ and $\{w^x\}_{x \in \mathcal{G}(0)} \subset U(\mathcal{R}_{0,1} \otimes L^\infty(\widehat{\mathcal{G}}))$, $\beta_x$-cocycles such that $(\sigma_x \otimes \text{id}) \circ \alpha_x \circ \sigma_x^{-1} = \text{Ad} w^x \circ \beta_x$. By replacing $\alpha_{r(\gamma)} \gamma$ with $(\sigma_{r(\gamma)} \otimes \text{id}) \circ \alpha_{r(\gamma)} \gamma \circ \sigma_{s(\gamma)}^{-1}$, we may assume $\alpha_x = \text{Ad} w^x \circ \beta_x$ and mod$(\alpha_\gamma) = \text{mod}(\beta_\gamma)$. Since $(\beta_\gamma \otimes \text{id})\beta_{s(\gamma)} = \beta_{r(\gamma)} \beta_\gamma$, we can regard $\beta_\gamma$ as a homomorphism from $\mathcal{G}$ to $\text{Aut}_\sigma(\mathcal{R}_{0,1} \rtimes_{\beta_{x_0}} \widehat{\mathcal{G}})$ by $\gamma \mapsto (\beta_{\gamma}, 1)$. We also have

$$(\alpha_\gamma \otimes \text{id})\beta_{s(\gamma)} = (\alpha_\gamma \otimes \text{id}) \circ \text{Ad} w^{s(\gamma)*} \circ \alpha_{s(\gamma)}$$

$$= \text{Ad}(\alpha_\gamma \otimes \text{id})(w^{s(\gamma)*}) \circ (\alpha_\gamma \otimes \text{id})\alpha_{s(\gamma)}$$

$$= \text{Ad}(\alpha_\gamma \otimes \text{id})(w^{s(\gamma)*}) \circ \alpha_{r(\gamma)} \alpha_\gamma$$

$$= \text{Ad}(\alpha_\gamma \otimes \text{id})(w^{s(\gamma)*}) w^{r(\gamma)} \circ \beta_{r(\gamma)} \alpha_\gamma,$$

where $(\alpha_\gamma \otimes \text{id})(w^{s(\gamma)*}) w^{r(\gamma)}$ is a $\beta_{r(\gamma)}$-cocycle. So we can regard $\alpha$ as a homomorphism from $\mathcal{G}$ to $\text{Aut}_\sigma(\mathcal{R}_{0,1} \rtimes_{\beta_{x_0}} \widehat{\mathcal{G}})$ by $\gamma \mapsto (\alpha_\gamma, (\alpha_\gamma \otimes \text{id})(w^{s(\gamma)*}) w^{r(\gamma)})$. Here note that $C_{\beta_x}^{(1)} = C_{\beta_{x_0}}^{(1)}$ because $\beta_x$ is constant.

We next show that $\alpha_\gamma \equiv \beta_\gamma \mod(C_{\beta_{r(\gamma)}}^{(1)})$. Since mod$(\alpha_\gamma) = \text{mod}(\beta_\gamma)$, it is clear that $\alpha_\gamma \beta_\gamma^{-1} \in \text{Int}(\mathcal{R}_{0,1})$. By the above computation, we also have the following:

$$(\alpha_\gamma \beta_\gamma^{-1} \otimes \text{id}) \circ \beta_{r(\gamma)} = (\alpha_\gamma \otimes \text{id}) \circ \beta_{s(\gamma)} \beta_\gamma^{-1}$$

$$= \text{Ad}(\alpha_\gamma \otimes \text{id})(w^{s(\gamma)*}) w^{r(\gamma)} \circ \beta_{r(\gamma)} \alpha_\gamma \beta_\gamma^{-1}.$$
u: \mathcal{G} \ni \gamma \mapsto u_\gamma \in U(\mathcal{R}_{0,1}) such that

\begin{align*}
(\text{Ad } u_\gamma, u_\gamma \beta_{\gamma}(u_\gamma^*) \cdot (\beta_\gamma, 1) \\
= (\sigma_{\gamma}(\gamma), v^r(\gamma) \cdot (\alpha_\gamma, (\alpha_\gamma \otimes \text{id})(w^{s(\gamma)*}w^r(\gamma)) \cdot (\sigma_{s(\gamma)}^{-1}, \sigma_{s(\gamma)}^{-1}(v^{s(\gamma)*}))).
\end{align*}

The left hand side is equal to (Ad \ u_\gamma \circ \beta_\gamma, u_\gamma \beta_{\gamma}(u_\gamma^*)). We compute the right hand side. For simplicity we write \(\alpha_\gamma\) for \(\alpha_\gamma \otimes \text{id}\) and so on.

\begin{align*}
(\sigma_{\gamma}(\gamma), v^r(\gamma) \cdot (\alpha_\gamma, (\alpha_\gamma(w^{s(\gamma)*})w^r(\gamma)) \cdot (\sigma_{s(\gamma)}^{-1}, \sigma_{s(\gamma)}^{-1}(v^{s(\gamma)*}))) \\
= (\sigma_{\gamma}(\gamma), (\alpha_\gamma(w^{s(\gamma)*})w^r(\gamma)) v^r(\gamma) \cdot \left(\sigma_{s(\gamma)}^{-1}, \sigma_{s(\gamma)}^{-1}(v^{s(\gamma)*})\right) \\
= (\sigma_{\gamma}(\gamma), \sigma_{s(\gamma)}^{-1}, \sigma_{s(\gamma)}^{-1}(v^{s(\gamma)*})\sigma_{s(\gamma)}(\alpha_\gamma(w^{s(\gamma)*})w^r(\gamma))) v^r(\gamma).
\end{align*}

By comparing the first component, we have Ad \ u_\gamma \circ \beta_\gamma = \sigma_{\gamma}(\gamma) \circ \alpha_\gamma \circ \sigma_{s(\gamma)}^{-1}. Since \(\mathcal{G}\) is generated by a single transformation, we may assume that \(u_\gamma\) is a \(\beta\)-cocycle.

The second component is computed as follows:

\begin{align*}
\sigma_{s(\gamma)}^{-1}(v^{s(\gamma)*})\sigma_{\gamma}(\alpha_\gamma(w^{s(\gamma)*})w^r(\gamma)) v^r(\gamma) \\
= \text{Ad } u_\gamma \beta_{\gamma}(v^{s(\gamma)*}) \cdot \sigma_{\gamma}(\alpha_\gamma(w^{s(\gamma)*})w^r(\gamma)) v^r(\gamma) \\
= \text{Ad } u_\gamma \beta_{\gamma}(v^{s(\gamma)*}) \cdot \sigma_{\gamma}(\alpha_\gamma(w^{s(\gamma)*})w^r(\gamma)) v^r(\gamma).
\end{align*}

Set \(u^x := \sigma_x(w^x)v^x\). By using \(\sigma_x \circ \beta_x \circ \sigma_x^{-1} = \text{Ad } v^x \circ \beta_x\), it follows that \(u^x\) is a \(\beta_x\)-cocycle and \(\sigma_x \circ \alpha_x \circ \sigma_x^{-1} = \text{Ad } u^x \circ \beta_x\). By comparing the second component, we have \(\beta_\gamma(u_\gamma^*) = \beta_\gamma(u^{s(\gamma)*})u_\gamma^*u^r(\gamma)\), and equivalently \(u^r(\gamma)\beta_\gamma(u_\gamma) = u_\gamma \beta_\gamma(u^{s(\gamma)}\gamma)\). This shows that \(u_\gamma\) is a \(\beta\)-cocycle, and \(\alpha_\gamma \circ \alpha_\gamma \circ \alpha_{s(\gamma)} = \text{Ad } u_\gamma \circ \beta_\gamma\). Thus \(\alpha\) and \(\beta\) are cocycle conjugate.

\begin{itemize}
\item \textbf{Proof of Theorem 2.4 for type III}_0 factors. Let \(M\), \(\alpha\) and \(\alpha(0)\) be as in Theorem 2.4 Then \(\tilde{\alpha}\) and \(\text{id}_M \otimes \alpha(0) = \text{id}_{\tilde{M}} \otimes \alpha(0)\) act on \(Z(\tilde{M})\) trivially and free on \(\tilde{M}\) by Theorem 7.6. By using an isomorphism \(\mathcal{R}_{0,1} \cong \mathcal{R}_{0,1} \otimes \mathcal{R}_0\), we see that \((\tilde{\alpha}_x)_{x \in X}\) and \((\text{id}_{\tilde{M}(x)} \otimes \alpha(0))_{x \in X}\) are satisfying the condition of Theorem 5.6. Then the two \(\tilde{G}\)-\(\mathcal{G}\)-actions on \(\mathcal{R}_{0,1}\) arising from \(\tilde{\alpha}\theta\) and \(\theta \otimes \alpha(0)\) are cocycle conjugate. This implies the cocycle conjugacy of the \(\tilde{G} \times \mathbb{R}\)-actions \(\tilde{\alpha}\theta\) and \((\theta \otimes \alpha(0))\) by Lemma 5.2. Considering the partial crossed product by \(\theta\), we get \(\tilde{\alpha} \sim \text{id}_{\tilde{M}} \cong \tilde{\alpha}(0)\) as in the proof of Lemma 3.4. Thus \(\alpha\) and \(\text{id}_M \otimes \alpha(0)\) are cocycle conjugate by Lemma 3.3.
\end{itemize}

\begin{remark}
In general, there may appear some obstructions in combining the \(\tilde{G}\)-part and the \(\mathcal{G}\)-part. In [13, 23, 15], model actions absorbing obstructions are constructed. In our case, however, we are treating only free actions, and no obstructions appear. Hence we do not need such model actions.
\end{remark}
6. Classification for type III₁ case

6.1. Basic results on canonical extensions

In [4] we obtained the classification of approximately inner and centrally free actions of an amenable discrete Kac algebra on the injective factor of type IIIₙ. Using this result together with ideas of [4, 9] (also see [17]), we classify actions on the injective factor of type III₁.

Let \( M \cong \mathbb{R}_\infty \) and \( \varphi \) be a faithful normal state on \( M \). Fix \( T > 0 \). Set \( N := M \rtimes_{\sigma_T^\varphi} \mathbb{Z} \), which is an injective factor of type IIIₙ, \( \lambda := e^{-\frac{\pi}{T}} \), and let \( U \in N \) be the unitary implementing \( \sigma_T^\varphi \). The dual action of the torus \( T = \mathbb{R}/2\pi\mathbb{Z} \) is denoted by \( \theta \), which acts on \( U \) by \( \theta_t(U) = e^{-\sqrt{-1} t} U \) for \( t \in T \). Using the averaging expectation \( E_{\theta}: N \to M \) by \( \theta \), we extend \( \varphi \) to \( \hat{\varphi} := \varphi \circ E_{\theta} \). Throughout this section, we keep these notations.

Now we introduce the extension \( \hat{\rho}: \text{End}_0(M) \to \text{End}_0(N) \) defined by

\[
\hat{\rho}(x) = \rho(x) \quad \text{for all } x \in M;
\]

\[
\hat{\rho}(U) = d(\rho)^iT[D\varphi \circ \phi_\rho : D\varphi]_TU.
\]

Note that \( \hat{\rho} \) is one of the variants of the canonical extension. Indeed, regarding \( N \subset \tilde{M} \) by \( U = \lambda^\varphi(T) \), we see that \( \hat{\rho} = \rho|_N \).

Lemma 6.1. For any \( \rho \in \text{End}_0(M) \), \( \text{mod}(\hat{\rho}) = \text{id} \).

Proof. Since \( \sigma_T^\rho = \text{Ad}U \). We can take a positive operator \( h \) affiliated with \( N_\varphi \) such that \( U = h^{iT} \). We set \( \psi := \hat{\varphi}_{h^{-1}} \), whose modular automorphism has the period \( T \). Note that \( E_{\theta} \circ \phi_\rho = \phi_\rho \circ E_\theta = \phi_\rho \circ E_\rho \) because \( \phi_\rho|_M = \phi_\varphi \) (see Theorem 6.3 (2)). Then we can compute \([D\varphi \circ \phi_\rho : D\varphi]_T \) as follows:

\[
[D\varphi \circ \phi_\rho : D\varphi]_T = [D\varphi \circ \phi_\rho : D\varphi \circ \phi_\rho]_T[D\varphi \circ \phi_\rho : D\varphi]_T[D\varphi : D\varphi]_T \\
= \hat{\rho}([D\varphi : D\varphi]_T)[D\varphi \circ \phi_\rho : D\varphi]_T[D\varphi : D\varphi]_T \\
= \hat{\rho}(U^*)[D\varphi \circ \phi_\rho : D\varphi]_TU \\
= \hat{\rho}(U^*)[D\varphi \circ \phi_\rho : D\varphi]_TU = d(\rho)^{-iT}.
\]

By [11, Theorem 2.8] \( d(\rho) = d(\hat{\rho}) \), so the above equality means \( \text{mod}(\hat{\rho}) = \text{id} \). \( \square \)

We denote by \( \text{End}^f_0(N) \) the set of endomorphisms with finite indices on \( N \) which commute with \( \theta \), and by \( \text{Ker(mod)} \) the set of endomorphisms with finite indices in \( \text{End}(N)_{\text{CT}} \) with trivial Connes-Takesaki modules. Note that \( \hat{\rho} \in \text{End}^f_0(N) \) for all \( \rho \in \text{End}_0(N) \). We will analyze the relative commutant \( \hat{\rho}(N)' \cap N \), which admits the torus action \( \theta \). Define the following linear space for each \( n \in \mathbb{Z} \):

\[
I_n := \{ a \in \hat{\rho}(N)' \cap N \mid \theta_t(a) = e^{\sqrt{-1} nt} a \text{ for all } t \in T \}.
\]

Lemma 6.2. For each \( n \in \mathbb{Z} \), one has \( I_n = U^{-n}(\rho, \sigma_{nT}^\rho) \).

Proof. Take \( a \in I_n \). Then \( \theta_t(U^na) = U^na \) for \( t \in T \), and \( b := U^na \in M \). We check \( b \in (\rho, \sigma_{nT}^\rho) \) as follows: for \( x \in M \),

\[
bp(x) = U^na \rho(x)a = U^na \rho(x)U^na = \sigma_{nT}^\rho(\rho(x))b.
\]
Hence $I_n \subset U^{-n}(\rho, \sigma_{nT}^\varphi \rho)$.

Next we show the converse inclusion. Set a unitary $u := d(\rho)^{iT} [D\varphi \circ \phi_\rho : d\varphi]_T$. Take $b \in (\rho, \sigma_{nT}^\varphi \rho)$. By direct computation, we see that $U^{-n}b \in I_n$ if and only if $b = \sigma_{nT}^\varphi(u)\sigma_{nT}^\varphi(b)u^* \in (\rho, \sigma_{nT}^\varphi \rho)$. Then $\mu$ is a well-defined unitary, here the inner product is given by $\langle a, b \rangle = \phi_\rho(b^*a)$ for $a, b \in (\rho, \sigma_{nT}^\varphi \rho)$. Hence it suffices to prove that $\mu$ is actually an identity map. Since $(\rho, \sigma_{nT}^\varphi \rho)$ is finite dimensional, it is spanned by eigenvectors of $\mu$. Let $b$ be an eigenvector $\mu(b) = e^{\sqrt{i\pi}sb}$ for some $s \in [0, 2\pi)$. We claim that $U^{-n}b \in (\theta_n \hat{\rho}, \hat{\rho})$. For $x \in M$, we have the following.

$$U^{-n}b\theta_n(\hat{\rho}(x)) = U^{-n}\sigma_{nT}^\varphi(\rho(x))b = \rho(x)U^{-n}b = \hat{\rho}(x)U^{-n}b.$$ 

We also have the following.

$$U^{-n}b\hat{\theta}_n(\hat{\rho}(U)) = U^{-n}b\theta_n(uU) = U^{-n}b \cdot e^{\sqrt{i\pi}sb}uU = U^{-n}\mu(b)uU = uU^{-n}\sigma_{nT}^\varphi(b)U = uU^{1-n}b = \hat{\rho}(U)U^{-n}b.$$ 

Thus we have verified the claim. By the Frobenius reciprocity, $\dim(\theta_n \hat{\rho}, \hat{\rho}) = \dim(\theta_n, \hat{\rho})$, and hence $\hat{\rho}\hat{\rho}$ contains $\theta_n$ as an irreducible component. However by the previous lemma, $\hat{\rho}$ has trivial Connes-Takesaki module, and $\text{mod}(\theta_n) = \text{mod}(\hat{\rho}\hat{\rho}) = 0$. This is possible only if $s = 0$. Therefore $\mu = \text{id}$. \hfill $\Box$

**Theorem 6.3.** Let $\rho \in \text{End}_0(M)$. Then one has the following:

1. $\hat{\rho}$ is irreducible if and only if $\rho$ is irreducible. In this case, the inclusion $\rho(M) \subset N$ is irreducible;
2. The standard left inverse $\phi_{\hat{\rho}}$ is given by:
   $\phi_{\hat{\rho}}(xU^n) = d(\rho)^{\text{in}T}\phi_\rho(x[D\varphi \circ \phi_\rho : D\varphi]^*_n)U^n$ for all $x \in M, n \in \mathbb{Z}$;
3. The extension $\hat{\rho}$ is a bijection from $\text{End}_0(M)$ onto $\text{End}_0^0(N) \cap \text{Ker(mod)}$;
4. $\hat{\rho} \in \text{Cnd}(N)$ if and only if $\rho \in \text{Cnd}(M)$.

**Proof.** (1) If $\hat{\rho}$ is irreducible, then $I_0 = \mathbb{C}$, and $(\rho, \rho) = \mathbb{C}$ follows from the previous lemma. Conversely if $\rho$ is irreducible, then $\hat{\rho}\hat{\rho}$ contains nontrivial modular automorphisms because the $T$-set $T(M)$ is trivial. This means $(\rho, \rho) = \mathbb{C}$, and $(\sigma_{nT}^\varphi \rho, \rho) = 0$ for $n \neq 0$. Hence $I_0 = \mathbb{C}$, and $I_n = 0$ for $n \neq 0$. Since $\hat{\rho}(N^\vee \cap N$ is densely spanned by $\{I_n\}_{n \in \mathbb{Z}}$, $\hat{\rho}$ is irreducible.

We prove the latter statement in (1). Take $x \in \rho(M)^\vee \cap N$ and let $x = \sum_{n \in \mathbb{Z}} x_n U^n$ be the formal decomposition. Then for each $n \in \mathbb{Z}$, $x_n = \rho, \sigma_{nT}^\varphi \rho)$. From the above argument, $x_0 \in \mathbb{C}$ and $x_n = 0$ for $n \neq 0$. Hence $\rho(M)^\vee \cap N = \mathbb{C}$.

(2) By [19] Lemma 3.5, the map $\phi_{\hat{\rho}}$ is well-defined. By [11] Theorem 2.8, $\hat{\rho}$ is the minimal conditional expectation, and it follows that $\phi_{\hat{\rho}}$ is standard.

(3) Let $\psi$ be a periodic weight constructed as in the proof of Lemma 6.1. By Lemma 6.1, we see that $\hat{\rho} \in \text{End}_0^0(N) \cap \text{Ker(mod)}$. So, the given map is well-defined. We show that the map is a bijection. Clearly it is injective, and it suffices to show the surjectivity. Let $\sigma \in \text{End}_0^0(N) \cap \text{Ker(mod)}$. Since $\text{mod}(\sigma) = \text{id}$, we have $d(\sigma)^{iT}[D\psi \circ \phi_\sigma : D\varphi]_T = 1$. This is equivalent to

$$\sigma(U) = d(\sigma)^{iT}[D\varphi \circ \phi_\sigma : D\varphi]_T U. \quad (6.1)$$
Set $\rho = \sigma|_M$. The action $\theta$ of $\mathbb{T}$ on $\sigma(N)$ is dominant, and $d(\sigma) = d(\rho)$ follows from [11, Theorem 2.8 (2)]. In the proof of [11, Theorem 2.8 (2)], it is also shown that $\sigma \circ \phi_\sigma|_M$ is the minimal expectation from $M$ onto $\rho(M)$. Hence $\phi_\rho = \phi_\sigma|_M$. Then the equality (6.1) yields $\sigma = \sigma|_M$.

(4) Let $\rho \in \text{Cnd}(M)$. We may and do assume that $\rho$ is irreducible. Then by [19, Theorem 4.12], there exists $t \in \mathbb{R}$ such that $[\rho] = [\sigma^t]$. Then $[\hat{\rho}] = [\sigma^t]$, and $\hat{\rho} \in \text{Cnd}(N)$.

Conversely we assume that $\hat{\rho} \in \text{Cnd}(N)$. Thanks to (1), we may and do assume that $\hat{\rho}$ is irreducible. By [19, Theorem 4.12], there exist $t \in \mathbb{R}$ and $u \in U(N)$ such that $\hat{\rho} = \text{Ad} u \circ \sigma^t$. Considering the formal decomposition of $u$, we see that $(\sigma_{nT+t}^u, \rho) \neq 0$ for some $n \in \mathbb{Z}$. Since $\rho$ is irreducible by (1), this means $[\rho] = [\sigma_{nT+t}]$, and $\rho \in \text{Cnd}(M)$.

Let $K$ be a finite dimensional Hilbert space. Following the procedure introduced in [7], we define the canonical extension $\overline{\beta} \in \text{Mor}(N, N \otimes B(K))$ for $\beta \in \text{Mor}_0(M, M \otimes B(K))$ by

$$\overline{\beta}(x) = \beta(x) \quad \text{for all } x \in M;$$
$$\overline{\beta}(U) = d(\beta)^T [D\varphi \circ \Phi^\beta : D\varphi \otimes \text{Tr}_K]_T(U \otimes 1).$$

By $\text{Mor}_0^\theta(N, N \otimes B(K))$, we denote the set of homomorphisms in $\text{Mor}_0(N, N \otimes B(K))$ commuting with $\theta$. The following is a direct consequence of the previous theorem. The fourth statement follows from the third one and Theorem [7,6].

**Lemma 6.4.** Let $K$ be a finite dimensional Hilbert space. Then one has the following:

1. Let $\beta \in \text{Mor}_0(M, M \otimes B(K))$. Then $\overline{\beta}$ is irreducible if and only if $\beta$ is irreducible. In this case, the inclusion $\beta(M) \subset N \otimes B(K)$ is irreducible;
2. Let $\beta \in \text{Mor}_0(M, M \otimes B(K))$. Then $d(\overline{\beta}) = d(\beta)$ and the standard left inverse $\Phi^\beta$ is given by the following equality: for $x \in M \otimes B(K)$ and $n \in \mathbb{Z}$, $\Phi^\beta(x(U^n \otimes 1)) = d(\beta)^{-inT} \Phi^\beta(x[D\varphi \otimes \text{Tr}_K : D\varphi \circ \Phi^\beta]_{nT})U^n$;
3. The extension $\overline{\tau}$ is a bijection from $\text{Mor}_0(M, M \otimes B(K))$ onto $\text{Mor}_0^\theta(N, N \otimes B(K)) \cap \text{Ker}(\text{mod})$;
4. Let $\beta \in \text{Mor}_0(M, M \otimes B(K))$. If $d(\beta) = \dim(K)$, then $\overline{\beta} \in \text{Int}(N, N \otimes B(K))$;
5. Let $\beta \in \text{Mor}_0(M, M \otimes B(K))$. Then $\beta \in \text{Cnt}(N, N \otimes B(K))$ if and only if $\overline{\beta} \in \text{Cnt}(N, N \otimes B(K))$.

6.2. Reduction to the classification of actions on $\mathcal{R}_\lambda$

Let $\alpha$ a centrally free cocycle action of $\hat{G}$ on $M \cong \mathcal{R}_\infty$. Then $\alpha$ is automatically approximately inner from Corollary [7,4]. For each $\pi \in \text{Irr}(\hat{G})$, we consider the canonical extension $\overline{\pi}_\sigma \in \text{Mor}_0(N, N \otimes B(K))$ as before. Then $\overline{\pi}$ is a cocycle action on $N$ with the same 2-cocycle.
Proposition 6.5. Let $\alpha$ be a centrally free cocycle action of $\hat{G}$ on $M$. Then $\pi$ is an approximately inner and centrally free cocycle action of $\hat{G}$ on $N$.

Proof. For each $\pi \in \text{Irr}(G) \setminus \{1\}$, $\pi_\alpha$ is approximately inner and centrally non-trivial by Lemma 6.4 (4) and (5). Since $\pi_\alpha$ is properly outer, $\alpha_\pi$ is irreducible \cite[Lemma 2.8]{18}. Hence so is $\pi_\alpha$ by Lemma 6.4 (1). Then by \cite[Lemma 8.3]{18}, $\pi_\alpha$ is properly centrally non-trivial. Thus the cocycle action $\pi$ is centrally free. \hfill \Box

Our main theorem of this section is the following:

Theorem 6.6. Let $\alpha$ be a centrally free action of $\hat{G}$ on $M$. Then the $\hat{G} \times T$-action $\pi \theta$ on $N$ is cocycle conjugate to $\theta \otimes \alpha^{(0)}$, where $\alpha^{(0)}$ is a free action of $\hat{G}$ on $R_\alpha$.

- Proof of Theorem 2.4 for $R_\infty$.

Since the natural extension of $\pi$ to $N \rtimes_\theta T$ is cocycle conjugate to $\alpha$ by Takesaki duality, we see that Theorem 6.6 implies Theorem 2.4 considering the partial crossed product by $\theta$ as before. \hfill \Box

The rest of this section is devoted to show Theorem 6.6. The essential part of our proof is the model action splitting result Proposition 6.10. The following lemma shows that the canonical extension well behaves to cocycle perturbations.

Lemma 6.7. For $i = 1, 2$, let $M^i$ be a type $\text{III}_1$ factor, $\varphi^i \in W(M^i)$ and $(\alpha^i, u^i)$ be a cocycle action of $\hat{G}$ on $M^i$. We set $N^i := M^i \rtimes_{\sigma^i} \mathbb{Z}$ and the dual action $\theta^i := \sigma^i$. If $(\alpha^1, u^1)$ is cocycle conjugate to $(\alpha^2, u^2)$, then there exists an isomorphism $\Psi : N^1 \to N^2$ and a unitary $v \in M^2 \otimes L^\infty(\hat{G})$ such that

- $\Psi \circ \theta^1_t = \theta^2_t \circ \Psi$ for all $t \in T$;
- $(\Psi \otimes \text{id}) \circ \pi_1 \circ \Psi^{-1} = \text{Ad} v \circ \pi_2$;
- $(\Psi \otimes \text{id} \otimes \text{id})(u^1) = (v \otimes 1)\alpha^2(v)u^2(id \otimes \Delta)(v^*)$.

In particular, the $\hat{G} \times T$-cocycle action $\pi \varphi^1$ is cocycle conjugate to $\pi \varphi^2$.

Proof. Since $(\alpha^1, u^1) \sim (\alpha^2, u^2)$, there exists an isomorphism $\Psi_0 : M^1 \to M^2$ and $v \in M^2 \otimes L^\infty(\hat{G})$ such that

- $(\Psi_0 \otimes \text{id}) \circ \alpha^1 \circ \Psi_0^{-1} = \text{Ad} v \circ \alpha^2$;
- $(\Psi_0 \otimes \text{id} \otimes \text{id})(u^1) = (v \otimes 1)\alpha^2(v)u^2(id \otimes \Delta)(v^*)$.

We set $\psi^2 := \varphi^1 \circ \Psi_0^{-1} \in W(M^2)$. Then there exists an isomorphism $\Psi : N^1 \to M^2 \rtimes_{\sigma^2} \mathbb{Z}$ such that $\Psi(xU^\varphi^1) = \Psi_0(x)U^\psi^2$, where $U^\varphi^1$ and $U^\psi^2$ are the implementing unitaries for $\sigma^\varphi^1_T$ and $\sigma^\psi^2_T$, respectively. Then $\Psi$ intertwines the dual actions. Regard $M^2 \rtimes_{\sigma^2} \mathbb{Z} = N^2$ in the core $\hat{M}^2$. It suffices to show the second equality holds on the implementing unitary $U^\psi^2$. This is checked as follows: for
\[ \pi \in \text{Irr}(\mathbb{G}), \text{ we have} \]
\[ (\Psi \otimes \text{id}) \circ \overline{\alpha}_\pi^1 \circ \Psi^{-1}(U^{\psi^2}) \]
\[ = (\Psi \otimes \text{id})(\overline{\alpha}_\pi^1(U^{\psi^1})) \]
\[ = (\Psi \otimes \text{id})[(D\varphi^1 \circ \Phi_\pi^\alpha : D\varphi^1 \otimes \text{tr}_\pi]T(U^{\psi^1} \otimes 1)) \]
\[ = [D\varphi^1 \circ \Phi_\pi^\alpha \circ (\Psi_0^{-1} \otimes \text{id}) : D\varphi^1 \circ \Psi_0^{-1} \otimes \text{tr}_\pi]T(U^{\psi^2} \otimes 1) \]
\[ = [D\psi^2 \circ \Psi_0 \circ \Phi_\pi^\alpha \circ (\Psi_0^{-1} \otimes \text{id}) : D\psi^2 \otimes \text{tr}_\pi]T(U^{\psi^2} \otimes 1) \]
\[ = [D\psi^2 \circ \Phi_\pi^{\varphi(0) \otimes \alpha^1 \circ \Psi_0^{-1}} : D\psi^2 \otimes \text{tr}_\pi]T(U^{\psi^2} \otimes 1) \]
\[ = [D\psi^2 \circ \Phi_\pi^{\text{Ad} \circ \alpha^2} : D\psi^2 \otimes \text{tr}_\pi]T(U^{\psi^2} \otimes 1) \]
\[ = [D\psi^2 \circ \Phi_\pi^{\alpha^2} \circ \text{Ad} \pi^{\ast} : D\psi^2 \otimes \text{tr}_\pi]T(U^{\psi^2} \otimes 1) \]
\[ = v_\pi \sigma_T^{\psi \otimes \varphi^\pi} (v_\pi^*) [D\psi^2 \circ \Phi_\pi^{\alpha^2} : D\psi^2 \otimes \text{tr}_\pi]T(U^{\psi^2} \otimes 1) \]
\[ = v_\pi [D\psi^2 \circ \Phi_\pi^{\alpha^2} : D\psi^2 \otimes \text{tr}_\pi]T v_\pi^{\ast \otimes \text{tr}_\pi}(v_\pi^*) (U^{\psi^2} \otimes 1) \]
\[ = v_\pi [D\psi^2 \circ \Phi_\pi^{\alpha^2} : D\psi^2 \otimes \text{tr}_\pi]T (U^{\psi^2} \otimes 1) v_\pi^{\ast \otimes \text{tr}_\pi} \]
\[ = \text{Ad} \pi^{\ast} \circ \overline{\varphi}(U^{\psi^2}). \]
\[ \square \]

The following lemma is an equivariant version of [4, Lemma I.2]. Recall that \( \alpha^{(0)} \) is a free action of \( \widehat{\mathbb{G}} \) on \( \mathcal{R}_0 \).

**Lemma 6.8.** One has the following:

1. Let \( \delta \) be an action of \( \widehat{\mathbb{G}} \) on \( \mathcal{R}_\lambda \) and \( \gamma \in \text{Aut}(\mathcal{R}_\lambda) \) such that
   - \( \delta \) commutes with \( \gamma \);
   - \( \mathcal{R}_\lambda \rtimes \gamma \mathbb{Z} \cong \mathcal{R}_\lambda \);
   - The natural extension \( \overline{\delta} \) of \( \delta \) to \( \mathcal{R}_\lambda \rtimes \gamma \mathbb{Z} \) is approximately inner and centrally free;
   - The \( \widehat{\mathbb{G}} \times \mathbb{T} \)-action \( \overline{\delta} \gamma \) is centrally free on \( \mathcal{R}_\lambda \rtimes \gamma \mathbb{Z} \).

Then \( \widehat{\mathbb{G}} \times \mathbb{Z} \)-action \( \delta \gamma \) on \( \mathcal{R}_\lambda \) is cocycle conjugate to \( \text{id}_{\mathcal{R}_\lambda} \otimes \gamma^{(0)} \otimes \alpha^{(0)} \), where \( \gamma^{(0)} \) is an aperiodic automorphism on \( \mathcal{R}_0 \).

2. Let \( \delta \) be an action of \( \widehat{\mathbb{G}} \) on \( \mathcal{R}_\lambda \), and \( \beta \) an action of \( \mathbb{T} \) on \( \mathcal{R}_\lambda \) such that
   - \( \delta \) is approximately inner and centrally free;
   - \( \delta \) commutes with \( \beta \);
   - The \( \widehat{\mathbb{G}} \times \mathbb{T} \)-action \( \delta \beta \) is centrally free on \( \mathcal{R}_\lambda \);
   - \( \mathcal{R}_\lambda \rtimes \beta \mathbb{T} \cong \mathcal{R}_\lambda \).

Then the \( \widehat{\mathbb{G}} \times \mathbb{T} \)-action \( \delta \beta \) is cocycle conjugate to \( \text{id}_{\mathcal{R}_\lambda} \otimes \gamma^{(0)} \otimes \alpha^{(0)} \).

**Proof.** (1) Set \( \mathcal{R} := \mathcal{R}_\lambda \) which admits the \( \widehat{\mathbb{G}} \times \mathbb{Z} \)-action \( \delta \gamma \). Let \( W \in \mathcal{R} \rtimes \gamma \mathbb{Z} \) be the unitary implementing \( \gamma \).

**Step 1.** We show that \( \gamma \) is approximately inner and centrally free.

This follows from [4, Lemma I.2]. Also see [27, Lemma XVIII.4.18].

**Step 2.** We show that the \( \widehat{\mathbb{G}} \times \mathbb{Z} \)-action \( \delta \gamma \) is approximately inner.
It is known that $R$ and $R \rtimes_{\gamma} \mathbb{Z}$ have the common flow of weights \[15, 22\]. Since $\delta$ is approximately inner on $R \rtimes_{\gamma} \mathbb{Z}$, $\text{mod}(\delta) = \text{mod}(\delta) = \text{id}$. Hence $\delta$ is approximately inner on $R$ by Theorem \[7.6\] and so is $\delta \gamma$.

**Step 3.** We show that the $\hat{G} \times \mathbb{Z}$-action $\delta \gamma$ is centrally free.

Fix a generalized trace $\psi$ on $R$. Note that our assumption of (1) is satisfied for any perturbed actions of $\delta \gamma$. By Lemma \[4.13\] we may and do assume that $\psi$ is invariant by $\delta \gamma$.

For each $\pi \in \text{Irr}(G)$, we set $Q_\pi := \delta_\pi(R)' \cap (R \rtimes_{\gamma} \mathbb{Z} \otimes B(H_\pi))$. We can show that $Q_\pi$ is finite dimensional in a similar way to the proof of Theorem \[6.3\] (1), where the freeness of $\gamma$ is crucial. Also we can show that $\text{Ad}(W \otimes 1)$ ergodically acts on $Q_\pi$, and the torus action $\hat{\gamma}$ preserves $Q_\pi$. Therefore, there exist atoms $\{p_i\}_{i=1}^m \subset Q_\pi$ such that $p_i \in R \otimes B(H_\pi)$, $\gamma(p_i) = p_{i+1}$ for $1 \leq i \leq m-1$ and

$$Q_\pi = \delta_\pi(R)' \cap (R \otimes B(H_\pi)) = \mathbb{C}p_1 + \cdots + \mathbb{C}p_m.$$ \[6.2\]

Take an isometry $V_i \in N \otimes B(H_\pi)$ such that $V_i V_i^* = p_i$. Set $V_i := (\gamma^{-1} \otimes \text{id})(V_1)$ for $1 \leq i \leq m$. Then we have $V_i V_i^* = p_i$.

Now assume that $\delta_\pi \gamma^n$ is not properly centrally non-trivial on $R$ for some $\pi \in \text{Irr}(G)$ and $n \in \mathbb{Z}$. Set $\beta_i := V_i^* \delta_\pi(\gamma^n(\cdot))V_i$ for each $i$. Then $\beta_i \in \text{Mor}_0(R, R \otimes B(H_\pi))$ is irreducible and $\delta_\pi \gamma^n = \sum_{i=1}^m V_i(\beta_i(\cdot))V_i^*$. Then $\beta_i$ is not properly centrally non-trivial for some $i$. We may and do assume $i = 1$. Since $\beta_1$ is irreducible, $\beta_1$ is centrally trivial \[18\ Lemma 8.3\]. Then by Corollary \[7.7\], we see that $\beta_1 = \text{Ad} u \circ \sigma_{t_0}^\psi$ for some $u \in U(R)$ and $t_0 \in \mathbb{R}$.

So we have $\delta_\pi \gamma^n(x) = V_i u \sigma_{t_0}^\psi(x)$ for $x \in R$. Applying $\gamma^{-1}$ to the both sides, we have $\delta_\pi \gamma^n(\gamma^{-1}(x)) V_i \gamma^{-1}(u) = V_i \gamma^{-1}(u) \sigma_{t_0}^\psi(\gamma^{-1}(x))$ for $x \in R$, where we have used the fact that $\gamma$ commutes with $\sigma^\psi$. By definition of $\beta_i$, we obtain $\beta_i(\gamma^{-1}(x)) \gamma^{-1}(u) = \gamma^{-1}(u) \sigma_{t_0}^\psi(\gamma^{-1}(x))$, that is, $\beta_i = \text{Ad} \gamma^{-1}(u) \circ \sigma_{t_0}^\psi$. Hence $\{\beta_i\}_{i=1}^m$ define the equivalent sectors. By \[6.2\], this is possible when $m = 1$, that is, $\delta_\pi \gamma^n$ is irreducible. Hence we may assume that $\delta_\pi \gamma^n = \text{Ad} u \circ \sigma_{t_0}^\psi$.

Since $\psi$ is invariant under $\delta \gamma$, $u \in R_\psi$, and $\gamma(u) = e^{\sqrt{-1} s_0 u}$ for some $s_0 \in \mathbb{R}$. We can check that $\delta_\pi \circ \text{Ad} W^n = \text{Ad} u \circ \sigma_{t_0}^\psi \circ \hat{\gamma}_{-s_0}$ holds on $R \rtimes_{\gamma} \mathbb{Z}$. Hence $\delta_\pi \gamma^n$ is centrally trivial, and the assumption (1) yields $\pi = 1$ and $s_0 = 0$, and $\gamma^n = \text{Ad} u \circ \sigma_{t_0}^\psi$. Then we get $n = 0$ from central freeness of $\gamma$.

**Step 4.** We use the classification result for actions on $\mathcal{R}_\lambda$.

The $\hat{G} \times \mathbb{Z}$-action $\delta \gamma$ on $\mathcal{R}_\lambda$ is an approximately inner and centrally free. So $\delta \gamma$ is cocycle conjugate to $\text{id}_N \otimes \gamma(0) \otimes \alpha(0)$ by Theorem \[2.4\] for $\mathcal{R}_\lambda$.

(2) Let $N = \mathcal{R}_\lambda \rtimes_{\beta} \mathbb{T}$ and $\gamma = \overline{\beta}$. Extend the action $\delta$ to $N$, which is also denoted by $\delta$. Using the Takesaki duality \[26\], we see that all the assumptions of (1) are fulfilled. Then we get $\delta \overline{\beta} \sim \text{id}_{\mathcal{R}_\lambda} \otimes \gamma(0) \otimes \alpha(0)$. Comparing the crossed products by $\overline{\beta}$ and $\gamma(0)$, we obtain $\delta \beta \sim \text{id}_{\mathcal{R}_\lambda} \otimes \overline{\gamma(0)} \otimes \alpha(0)$.

**Lemma 6.9.** Let $M \cong \mathcal{R}_\infty$, $N = M \rtimes_{\gamma} \mathbb{Z}$ as before, and $\alpha$ a centrally free action of $\hat{G}$ on $M$. Then the $\hat{G} \times \mathbb{T}$-action $\theta_t \otimes \overline{\sigma_t}$ on $N \otimes N$ is cocycle conjugate to $\text{id}_N \otimes \overline{\gamma(0)} \otimes \alpha(0)$. 

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Proof. We can identify \((N \otimes N) \rtimes_{\theta \otimes \theta} \mathbb{T}\) with \((M \otimes M) \rtimes_{\sigma_{t} \otimes \sigma_{t}} \mathbb{Z}\) [4, Lemma 1 (b)]. Hence \((N \otimes N) \rtimes_{\theta \otimes \theta} \mathbb{T}\) is a factor of type III\(_{\lambda}\). By Proposition 6.5, \(\pi\) is approximately inner and centrally free, hence so is \(\text{id} \otimes \pi\). It is obvious that \(\theta_{t} \otimes \pi \theta_{t}\) is a centrally free action. Then the previous lemma can be applied. □

**Proposition 6.10.** Let \(M, N, \alpha, \theta\) be as above. Let \(\beta_{t}\) be a product type action of \(\mathbb{T}\) on \(R_{0} \cong \bigotimes_{i=1}^{\infty} M_{2}(\mathbb{C})\) given by \(\beta_{t} = \bigotimes_{i=1}^{\infty} \text{Ad} \left( \begin{bmatrix} 1 & 0 \\ 0 & e^{\sqrt{-1}t} \end{bmatrix} \right)\) for \(t \in \mathbb{R}\). Then \(\alpha \theta_{t}\) is cocycle conjugate to \(\text{id}_{\mathcal{R}_{\lambda}} \otimes \beta_{t} \otimes \alpha \theta_{t}\).

The proof of Proposition 6.10 will be presented in the sequel subsections. Here we prove Theorem 6.6 assuming Proposition 6.10.

**Proof of Theorem 6.6.**

Note that \(\beta_{t}\) is a minimal action of \(\mathbb{T}\), hence is dual, and conjugate to \(\hat{\gamma}(0)\).

Since \(\theta_{t} \otimes \theta_{t}\) (resp. \(\theta_{t} \otimes \theta_{t} \otimes \text{id}_{\mathcal{R}_{\lambda}}\)) is cocycle conjugate to \(\text{id}_{\mathcal{R}_{\lambda}} \otimes \beta_{t} \otimes \alpha(0)\) (resp. \(\theta_{t} \otimes \beta_{t} \otimes \text{id}_{\mathcal{R}_{\lambda}}\)) by the theory of Connes [4, Lemma 5] and Haagerup [9], we have

\[
\overline{\pi} \theta_{t} \sim \text{id}_{\mathcal{R}_{\lambda}} \otimes \beta_{t} \otimes \overline{\pi} \theta_{t} \quad \text{(by Proposition 6.10)}
\sim \theta_{t} \otimes \theta_{t} \otimes \overline{\pi} \theta_{t}
\sim \theta_{t} \otimes \beta_{t} \otimes \text{id}_{\mathcal{R}_{\lambda}} \otimes \alpha(0) \quad \text{(by Lemma 6.9)}
\sim \theta_{t} \otimes \alpha(0).
\]

Hence \(\overline{\pi} \theta_{t}\) is cocycle conjugate to \(\theta_{t} \otimes \alpha(0)\). Taking the crossed product by \(\theta\), we see that \(\alpha\) is cocycle conjugate to \(\text{id}_{\mathcal{R}_{\lambda}} \otimes \alpha(0)\).

Therefore the proof of Theorem 2.4 has been reduced to proving Proposition 6.10. We will show that \(\overline{\pi} \theta \sim \text{id}_{\mathcal{R}_{\lambda}} \otimes \alpha\) in Corollary 6.15, and \(\overline{\pi} \theta \sim \beta \otimes \overline{\pi} \theta\) in Theorem 6.17, and complete the proof of Proposition 6.10.

### 6.3. \(\lambda\)-stability

As an analogue of the property \(L^{\prime}_{a}\) in [1], we introduce the following notion.

**Definition 6.11.** Let \(\hat{G}\) be a discrete Kac algebra, \(P\) a factor, and \(\alpha\) a cocycle action of \(\hat{G}\) on \(P\). For \(0 < \lambda < 1\), set \(a = \frac{\lambda}{1+\lambda}\). We say that \((P, \alpha)\) satisfies the property \(L^{\prime}_{a}\) if we have the following:

For any \(\varepsilon > 0\), any finite sets \(\mathcal{F} \subseteq \text{Irr}(\hat{G})\) and \(\Psi_{\pi} \subseteq (P \otimes B(H_{\pi}))\), for \(\pi \in \mathcal{F}\), there exists a partial isometry \(u \in P\) such that for \(\psi \in \Psi_{\pi}, \pi \in \mathcal{F}\),

\[
\begin{align*}
u \nu^{*} + \nu^{*} \nu &= 1, \quad \nu^{2} = 0; \\
\| (u \otimes 1) \cdot \psi - \lambda \psi \cdot (u \otimes 1) \| &< \varepsilon; \\
\| (u \otimes 1 - \alpha_{\pi}(u)) \cdot \psi \| &< \varepsilon; \\
\| \psi \cdot (u \otimes 1 - \alpha_{\pi}(u)) \| &< \varepsilon.
\end{align*}
\]

Note that the property \(L^{\prime}_{a}\) is stable under perturbations of a cocycle action.

**Lemma 6.12.** Let \(\alpha\) be a centrally free cocycle action of \(\hat{G}\) on \(\mathcal{R}_{\infty}\). Then \((\mathcal{R}_{\infty}, \alpha)\) has the property \(L^{\prime}_{a}\), \(a = \frac{\lambda}{1+\lambda}\), for any \(0 < \lambda < 1\).
Proof. Since $M := R_\infty$ is properly infinite, we may and do assume that $\alpha$ is an action. Take $\pi \in \text{Irr}(G)$, and set $\pi' := d(\pi)1 \oplus \pi$, a direct sum representation of $G$. Consider an inclusion $\alpha_{\pi'}(M) \subset M \otimes B(H_{\pi'})$. We can identify $M \otimes B(H_{\pi'})$ with $M_2(M \otimes B(H_{\pi}))$ and

$$\alpha_{\pi'}(M) = \left\{ \begin{pmatrix} x \otimes 1_{\pi} & 0_{\pi} \\ 0_{\pi} & \alpha_{\pi}(x) \end{pmatrix} \mid x \in M \right\}.$$  

Then $\alpha_{\pi'}(M) \subset M \otimes B(H_{\pi'})$ is an inclusion of injective factors of type III$_1$ with the minimal index $4d(\pi)^2$. The minimal expectation $E^\pi$ is given by

$$E^\pi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \frac{1}{2}\alpha_{\pi'}((\text{id} \otimes \text{tr}_\pi)(a) + \Phi_\pi(d)).$$

For a fixed $0 < \lambda < 1$, we construct the type III$_\lambda$ factor $N := M \rtimes_{\sigma_{\pi}^\lambda} Z \subset \widetilde{M}$, $T = -2\pi/\log \lambda$ as before. The implementing unitary is denoted by $U^\varphi = \lambda^\varphi(T)$. Set $\gamma := \sigma_{T}^{\varphi \circ \Phi_\pi^\alpha}$, where $\Phi_\pi^\alpha = \alpha_{\pi'}^{-1} \circ E^\pi$. Then $\gamma$ globally preserves the inclusion $\alpha_{\pi'}(M) \subset M \otimes B(H_{\pi'})$.

Claim 1. We show that the inclusion $\alpha_{\pi'}(M) \rtimes_{\gamma} Z \subset (M \otimes B(H_{\pi'})) \rtimes_{\gamma} Z$ is isomorphic to $\overline{\alpha_{\pi'}}(N) \subset N \otimes B(H_{\pi'})$.

We identify $(M \otimes B(H_{\pi'})) \rtimes_{\gamma} Z$ with $(M \otimes B(H_{\pi'})) \rtimes_{\varphi \otimes \text{tr}_{\pi'}} Z$ in the core algebra $Q$ of $M \otimes B(H_{\pi'})$. Then

$$\alpha_{\pi'}(M) \rtimes_{\gamma} Z = \alpha_{\pi'}(M) \vee \{\lambda^{\circ \Phi_\pi^\alpha}(T)\}'' = \alpha_{\pi'}(M) \vee \{[D\varphi \circ \Phi_{\pi}^\alpha : D\varphi \otimes \text{tr}_{\pi'}]_{T} \lambda^{\circ \text{tr}_{\pi'}}(T)\}''.$$ 

The canonical isomorphism $\Psi : Q \rightarrow \widetilde{M} \otimes B(H_{\pi'})$ satisfies $\Psi|_{M \otimes B(H_{\pi'})} = \text{id}$ and $\Psi(\lambda^{\circ \text{tr}_{\pi'}}(T)) = \lambda^T(1) \otimes 1$. Hence

$$\Psi(\lambda^{\circ \Phi_\pi^\alpha}(T)) = [D\varphi \circ \Phi_{\pi}^\alpha : D\varphi \otimes \text{tr}_{\pi'}]_{T}(\lambda^T(1) \otimes 1) = \overline{\alpha_{\pi'}}(\lambda^T(T)).$$

Then we have $\Psi(\alpha_{\pi'}(M) \rtimes_{\gamma} Z) = \overline{\alpha_{\pi'}}(N)$ and $\Psi((M \otimes B(H_{\pi'})) \rtimes_{\gamma} Z) = N \otimes B(H_{\pi'})$.

Claim 2. We show that the inclusion $\overline{\alpha_{\pi'}}(N) \subset N \otimes B(H_{\pi'})$ is relatively $\lambda$-stable.

Since $\alpha$ is approximately inner and centrally free on $N$ by Proposition 6.5, $\alpha$ is cocycle conjugate to $\text{id}_R \otimes \alpha$ by Theorem 2.3 for type III$_\lambda$ case. Hence the inclusion $\overline{\alpha_{\pi'}}(N) \subset N \otimes B(H_{\pi'})$ is relatively $\lambda$-stable in the sense that $\overline{\alpha_{\pi'}}(N) \subset N \otimes B(H_{\pi'}) \cong R_\lambda \rtimes \overline{\alpha_{\pi'}}(N) \subset (R_\lambda \otimes N) \otimes B(H_{\pi'})$.

Claim 3. We show that $\gamma$ is an approximately inner automorphism on the subfactor $\alpha_{\pi'}(M) \subset M \otimes B(H_{\pi'})$.

By Corollary 3.26 we can choose $\{w_n\}_n \subset U(M)$ such that $\sigma^\varphi_{T} = \lim_{n \rightarrow \infty} \text{Ad} w_n$ and $[D\varphi \circ \Phi_{\pi} : D\varphi \otimes \text{tr}_{\pi}]_{T} = \lim_{n \rightarrow \infty} \alpha_{\pi}(w_n)(w^*_n \otimes 1)$ for all $\pi \in \text{Irr}(G)$. Since $2\varphi \circ \alpha_{\pi'}^{-1} \circ E^\pi$ is nothing but a balanced functional $\varphi \otimes \text{tr}_{\pi} \oplus \varphi \circ \Phi_{\pi}$,

$$\gamma = \text{Ad} \left( \begin{pmatrix} 1_{\pi} & 0_{\pi} \\ 0_{\pi} & \alpha_{\pi} \end{pmatrix} \right) \circ (\sigma^\varphi_{T} \otimes \text{id}_{\pi}).$$
Thus \( \gamma = \lim_{n \to \infty} \Ad \alpha_{\pi'}(w_n) \), and \( \gamma \) is approximately inner in a subfactor sense.

By the previous three claims, we can show that the inclusion \( \alpha_{\pi'}(M) \subset M \otimes B(H_{\pi'}) \) is relatively \( \lambda \)-stable. Indeed, the proof is similar to that of [12, Corollary II.3]. (Also see [17, Theorem 3.6].) Hence for any \( \varepsilon > 0 \) and any \( \{ \psi_i \}^n_{i=1} \subset (M \otimes B(H_{\pi'}))^* \), there exists \( u \in M \) such that \( u^2 = 0 \), \( uu^* + u^*u = 1 \) and

\[
\| \alpha_{\pi'}(u) \cdot \psi_i - \lambda \psi_i \cdot \alpha_{\pi'}(u) \| < \varepsilon, \quad \text{for all } 1 \leq i \leq n.
\]

For \( \psi \in (M \otimes B(H_{\pi'}))^* \), define \( \psi_{ij} \in (M \otimes B(H_{\pi'})) \), by \( \psi_{ij}(a) = \psi(a_{ij}) \) via identification of \( M \otimes B(H_{\pi'}) \) with \( M_2(M \otimes B(H_{\pi})) \) and \( \alpha_{\pi'}(x) = \text{diag}(x \otimes 1_{\pi}, \alpha_{\pi}(x)) \) for \( x \in M \). Assume we have chosen \( u \) so that

\[
\| \alpha_{\pi'}(u) \cdot \psi_{ij} - \lambda \psi_{ij} \cdot \alpha_{\pi'}(u) \| < \varepsilon \quad \text{for all } i, j = 1, 2.
\]

Then we obtain the following four inequalities.

\[
\|(u \otimes 1_{\pi}) \cdot \psi - \lambda \psi \cdot (u \otimes 1_{\pi})\| < \varepsilon, \quad \|(u \otimes 1_{\pi}) \cdot \psi - \lambda \psi \cdot \alpha_{\pi}(u)\| < \varepsilon,
\]

\[
\|\alpha_{\pi}(u) \cdot \psi - \lambda \psi \cdot (u \otimes 1_{\pi})\| < \varepsilon, \quad \|\alpha_{\pi}(u) \cdot \psi - \lambda \psi \cdot \alpha_{\pi}(u)\| < \varepsilon.
\]

It is easy to deduce that \( u \) satisfies the condition in Definition [6,11] for \( \psi \). So far, we have considered a single element \( \pi \in \text{Irr}(G) \). For a finite subset \( \mathcal{F} \subset \text{Irr}(G) \), define \( \Pi := \bigoplus_{\pi \in \mathcal{F}} \pi' \), and consider the similarly defined inclusion \( \alpha_{\Pi}(M) \subset M \otimes B(H_{\Pi}) \). Then the same argument is applicable.

**Lemma 6.13.** Let \( P \) be a properly infinite factor, \( H \) a finite dimensional Hilbert space, \( \alpha \in \text{Mor}_0(P, P \otimes B(H)) \) and \( \Phi \in (P \otimes B(H))^* \) a finite set of faithful states. Let \( 0 < \varepsilon < 1 \) and \( 0 < \lambda \leq 1 \). Assume that there exists \( u \in P \) such that \( uu^* + u^*u = 1 \), \( u^2 = 0 \) and for all \( \varphi \in \Phi \),

\[
\|(u \otimes 1_{\pi}) \cdot \varphi - \lambda \varphi \cdot (u \otimes 1_{\pi})\| \leq \lambda \varepsilon, \quad \|\varphi \cdot (u \otimes 1_{\pi}) - \lambda^{-1}(u \otimes 1_{\pi}) \cdot \varphi\| \leq \lambda \varepsilon;
\]

\[
\|(u \otimes 1_{\pi} - \alpha(u)) \cdot \varphi\| \leq \lambda \varepsilon, \quad \|\varphi \cdot (u \otimes 1 - \alpha(u))\| \leq \lambda \varepsilon.
\]

Then there exists a unitary \( v \in P \otimes B(H) \) such that \( \text{Ad} v \circ \alpha = \text{id} \) on the type \( I_2 \) subfactor \{\( u'\)^{\#}\} and \( \|v - 1\|_{\Phi}^2 < 2 \sqrt{2} \varepsilon \) for all \( \varphi \in \Phi \).

**Proof.** In the following, we frequently use the inequalities \( \|x\|^2_{\varphi} \leq \|x\| \|x \cdot \varphi\| \), \( \|x \cdot \varphi\| \leq \sqrt{\|\varphi\|} \|x\|_{\varphi} \). First we show \( uu^* \otimes 1 - \alpha(uu^*) \) are close as follows:

\[
\|(uu^* \otimes 1 - \alpha(uu^*)) \cdot \varphi\| \leq \|uu^* \varphi - \lambda^{-1} \alpha(u) \cdot \varphi \cdot (u \otimes 1)\| + \|\lambda^{-1}(u) \cdot \alpha(u) \cdot \varphi \cdot (u \otimes 1)\|
\]

\[
= \|uu^* \varphi - \lambda^{-1} (u \otimes 1) \cdot \varphi \cdot (u^* \otimes 1)\| + \|\lambda^{-1} (u \otimes 1) \cdot \varphi \cdot (u^* \otimes 1) - \lambda^{-1} \alpha(u) \cdot \varphi \cdot (u^* \otimes 1)\|
\]

\[
+ \|\lambda^{-1} \alpha(u) \cdot (u^* \otimes 1) - (u \otimes 1) \cdot \varphi\| + \|uu^* \varphi - \alpha(u^*) \cdot \varphi\| \leq 4 \varepsilon.
\]

Since \( \|x\|^2_{\varphi} \leq \|x \cdot \varphi\| \|x\|_{\varphi} \), we have \( \|uu^* \otimes 1 - \alpha(uu^*)\|^2_{\varphi} \leq 8 \varepsilon \). In the same way, we have \( \|u^*u \otimes 1 - \alpha(u^*u)\|^2_{\varphi} \leq 8 \varepsilon \). Hence we have \( \|uu^* \otimes 1 - \alpha(uu^*)\|^2_{\varphi} \leq 2 \sqrt{2} \varepsilon \) and \( \|u^*u \otimes 1 - \alpha(u^*u)\|^2_{\varphi} \leq 2 \sqrt{2} \varepsilon \).
By [2, Lemma 1.1.4] and [20, Lemma 8.1.1], there exists a partial isometry $w \in P \otimes B(H)$ with $wu^* = uu^* \otimes 1$, $w^*w = \alpha(uu^*)$, $\|w - uu^* \otimes 1\|_{\#}^2 \leq 7\|uu^* \otimes 1 - \alpha(uu^*)\|_{\#}$ for $\varphi \in \Phi$. Hence we have $\|w - uu^* \otimes 1\|_{\#}^2 \leq 14\sqrt{2}\varepsilon$.

Set $v := (uu^* \otimes 1)\omega(uu^*) + (u^* \otimes 1)\omega(u)$. It is standard to see $\text{Ad} v \circ \alpha(x) = x \otimes 1$ for $x \in \{u\}''$. We estimate $\|(v - 1) \cdot \varphi\|$ and $\|\varphi \cdot (v - 1)\|$. Since $\|x\|_{\#} \leq \sqrt{2}\|x\|_{\#}^2$, and $\|x\varphi\| \leq \sqrt{\|\varphi\|}\|x\|_{\#}$, we have

$$\|(w - uu^* \otimes 1) \cdot \varphi\| \leq \|w - uu^* \otimes 1\|_{\#} \leq \sqrt{2}\|w - uu^* \otimes 1\|_{\#} \leq 28\sqrt{\varepsilon}.$$ 

Since $\|[uu^* \otimes 1, \varphi]\| \leq 2\varepsilon$, we get

$$\|(w uu^* \otimes 1) w\alpha(uu^*) - uu^* \otimes 1) \cdot \varphi\| \leq \|(w uu^* - uu^* \otimes 1) \cdot \varphi\| \leq \|((w uu^* - uu^* \otimes 1) \cdot \varphi\| + \|w uu^* \otimes 1) \cdot \varphi\| \leq 4\varepsilon + \|w uu^* \otimes 1) \cdot \varphi\| \leq 4\varepsilon + 4\varepsilon + 28\sqrt{\varepsilon} \leq 36\sqrt{\varepsilon}$$

and

$$\|(w uu^* \otimes 1) w\alpha(u) - uu^* \otimes 1) \cdot \varphi\| \leq \|w uu^* \otimes 1) \cdot \varphi\| \leq \|w uu^* \otimes 1) \cdot \varphi\| \leq \|w uu^* \otimes 1) \cdot \varphi\| \leq 4\varepsilon + 4\varepsilon + 28\sqrt{\varepsilon} \leq 31\sqrt{\varepsilon}.$$ 

Hence $\|(v - 1) \cdot \varphi\| \leq 36\sqrt{\varepsilon} + 31\sqrt{\varepsilon} = 67\sqrt{\varepsilon}$, and $\|v - 1\|_{\#}^2 \leq 134\sqrt{\varepsilon}$ holds. Next we estimate $\|\varphi \cdot (v - 1)\|$ as follows:

$$\|\varphi \cdot ((uu^* \otimes 1) w\alpha(uu^*) - uu^* \otimes 1)\| \leq \|\varphi \cdot ((uu^* \otimes 1) (w\alpha(uu^*) - uu^* \otimes 1))\| \leq \|[\varphi, uu^* \otimes 1] (w\alpha(uu^*) - uu^* \otimes 1)\| + \|((uu^* \otimes 1) \cdot \varphi \cdot (w\alpha(uu^*) - uu^* \otimes 1))\| \leq 4\varepsilon + \|[uu^* \otimes 1) \cdot \varphi \cdot (w\alpha(uu^*) - uu^* \otimes 1)\| \leq 4\varepsilon + 28\sqrt{\varepsilon} + \|\varphi \cdot (uu^* \otimes 1) \cdot (\alpha(uu^*) - uu^* \otimes 1)\| \leq 32\sqrt{\varepsilon} + 4\varepsilon + \|\varphi \cdot (\alpha(uu^*) - uu^* \otimes 1)\| \leq 32\sqrt{\varepsilon} + 4\varepsilon + 4\varepsilon \leq 40\sqrt{\varepsilon}$$
and
\[ \| \varphi \cdot (u^* \otimes 1) w \alpha (u) - u^* u \otimes 1 \| \]
\[ \leq \| ((\varphi \cdot (u^* \otimes 1) - \lambda (u^* \otimes 1) \cdot \varphi) \cdot (w \alpha (u) - u \otimes 1) \| \]
\[ + \| \lambda u^* \varphi \cdot (w \alpha (u) - u \otimes 1) \| \]
\[ \leq 2 \varepsilon + \| \varphi \cdot (w - uu^* \otimes 1) \alpha (u) \| + \| \varphi \cdot ((uu^* \otimes 1) \alpha (u) - u \otimes 1) \| \]
\[ \leq 2 \varepsilon + 28 \sqrt{\varepsilon} + \| \varphi \cdot (uu^* \otimes 1) \cdot (\alpha (u) - u \otimes 1) \| \]
\[ \leq 30 \sqrt{\varepsilon} + 4 \varepsilon + \| \varphi \cdot (\alpha (u) - u \otimes 1) \| \leq 35 \sqrt{\varepsilon} . \]
Hence \( \| \varphi \cdot (v - 1) \| \leq 75 \sqrt{\varepsilon} \), and \( \| v^* - 1 \|_{\varphi}^2 \leq 150 \sqrt{\varepsilon} \) holds. This implies that \( \| v - 1 \|_{\varphi}^2 = \frac{1}{2} (\| v - 1 \|_{\varphi}^2 + \| v^* - 1 \|_{\varphi}^2) \leq 142 \sqrt{\varepsilon} \), and \( \| v - 1 \|_{\varphi}^2 \leq 12 \sqrt{\varepsilon} \). \( \square \)

**Theorem 6.14.** Let \( \alpha \) be a centrally free action of \( \hat{G} \) on \( R_\infty \). Then \( \alpha \) is cocycle conjugate to \( id_{R_\alpha} \otimes \alpha \) for all \( 0 < \lambda < 1 \).

**Proof.** Set \( M := \mathcal{R}_\infty, \varepsilon_n := 16^{-n} \). Let \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \) be an increasing sequence of finite sets of \( \text{Irr}(G) \) with \( \bigcup_{n=1}^{\infty} \mathcal{F}_n = \text{Irr}(G) \). Let \( \{ \psi_n \}_{n=1}^{\infty} \subset (M_+) \) be a countable dense subset such that \( \psi_1 \) is a faithful state. For each \( k \in \mathbb{N} \), we will construct a mutually commuting sequence of \( 2 \times 2 \)-matrix units \( \{ e_{ij}(k) \}_{i,j=1}^{2} \), and unitaries \( v^k, \tilde{v}^k \in M \otimes L^\infty(\hat{G}) \) with the following five conditions:

\[ \tilde{v}^n = v^n v^{n-1} \ldots v^1; \]
\[ \text{Ad} \tilde{v}^n \circ \alpha_\pi (e_{ij}(k)) = e_{ij}(k) \otimes 1_\pi, \quad i, j = 1, 2, \quad 1 \leq k \leq n, \quad \pi \in \mathcal{F}_n; \]
\[ \| v^n_\pi - 1 \|_{\psi_1 \otimes \tau} \leq 12 \sqrt{\varepsilon_n}, \quad \pi \in \mathcal{F}_n; \]
\[ \| v^n_\pi - 1 \|_{(\psi_1 \otimes \tau) \alpha_\pi \circ \text{Ad} \tilde{v}^{n-1}_\pi} \leq 12 \sqrt{\varepsilon_n}, \quad \pi \in \mathcal{F}_n; \]
\[ \| \psi_k \cdot e_{ij}(n) - \lambda^{ij} e_{ij}(n) \cdot \psi_k \| < 2 \varepsilon_n, \quad 1 \leq k \leq n. \]

Since \((M, \alpha)\) has the property \( L'_a \) for any \( 0 < a < 1/2 \) by Lemma 6.12, we can choose \( u \in M \) such that

\[ uu^* + u^* u = 1, \quad u^2 = 0; \]
\[ \| (u \otimes 1 - \alpha_\pi (u)) \cdot (\psi_1 \otimes \tau) \| < \lambda \varepsilon_1, \quad \pi \in \mathcal{F}_1; \]
\[ \| (\psi_1 \otimes \tau) \cdot (u \otimes 1 - \alpha_\pi (u)) \| < \lambda \varepsilon_1, \quad \pi \in \mathcal{F}_1; \]
\[ \| u \cdot \psi_1 - \lambda \psi_1 \cdot u \| < \lambda^2 \varepsilon_1. \]

Then by Lemma 6.13, there exists a unitary \( v^n_\pi \) such that \( \| v^n_\pi - 1 \|_{\psi_1 \otimes \tau} \leq 12 \sqrt{\varepsilon_1}, \pi \in \mathcal{F}_1, \) and \( \text{Ad} v^n_\pi \circ \alpha_\pi (u) = u \otimes 1 \). We define \( v^1_\rho, \rho \notin \mathcal{F}_1, \) in a similar way to the proof of Lemma 6.13. Set \( \{ e_{ij}(1), e_{i2}(1), e_{21}(1), e_{22}(1) \} := \{ uu^*, u, u^*, u^* u \} \). Note that \( \| e_{ij}(1), \psi_1 \| < 2 \varepsilon_1 \), so the first step is complete.

Suppose we have done up to the \( n \)-th step. Set \( E_n := \bigvee_{k=1}^{n} \{ e_{ij}(k) \}_{i,j=1}^{2} \), \( \alpha^{n+1} := \text{Ad} \tilde{v}^n \circ \alpha \), and \( M_{n+1} := E'_n \cap M \). Then \( \alpha^{n+1} \) is a centrally free cocycle action on \( M_{n+1} \cong \mathcal{R}_\infty \). Hence \((M_{n+1}, \alpha^{n+1})\) has the property \( L'_a \) by Lemma 6.12. Let \( \{ w_\ell \}_{\ell=1}^{n} \) be a basis for \( E'_n \) with \( \| w_\ell \| \leq 1 \), and decompose \( \psi_k = \sum_{\ell=1}^{n} w_\ell \otimes \psi_{k \ell} \).
Take $u \in M_{n+1}$ satisfying $uu^* + u^*u = 1$, $u^2 = 0$ and the following conditions: for any $\pi \in \mathcal{F}_{n+1}$,

\[
\| (u \otimes 1 - \alpha_{\pi}^{n+1}(u)) \cdot (\psi_{1} \otimes \text{tr}_{\pi}) \| < \lambda \varepsilon_{n+1};
\]
\[
\| (\psi_{1} \otimes \text{tr}_{\pi}) \cdot (u \otimes 1 - \alpha_{\pi}^{n+1}(u)) \| < \lambda \varepsilon_{n+1};
\]
\[
\| (u \otimes 1 - \alpha_{\pi}^{n+1}(u)) \cdot ((\psi_{1} \otimes \text{tr}_{\pi}) \circ \text{Ad} \bar{\psi}_{\pi}^{n*}) \| < \lambda \varepsilon_{n+1};
\]
\[
\| ((\psi_{1} \otimes \text{tr}_{\pi}) \circ \text{Ad} \bar{\psi}_{\pi}^{n*}) \cdot (u \otimes 1 - \alpha_{\pi}^{n+1}(u)) \| < \lambda \varepsilon_{n+1};
\]
\[
\| (u \otimes 1) \cdot ((\psi_{1} \otimes \text{tr}_{\pi}) \circ \text{Ad} \bar{\psi}_{\pi}^{n*}) - \lambda ((\psi_{1} \otimes \text{tr}_{\pi}) \circ \text{Ad} \bar{\psi}_{\pi}^{n*}) \cdot (u \otimes 1) \| < \lambda^2 \varepsilon_{n+1};
\]
\[
\| u \cdot \psi_{k\ell} - \lambda \psi_{k\ell} \cdot u \| < 4^{-n} \lambda^2 \varepsilon_{n+1}, \quad 1 \leq k \leq n + 1, \ 1 \leq \ell \leq 4^n.
\]

Here we have regarded $\psi_{1}$ and $(\psi_{1} \otimes \text{tr}_{\pi}) \circ \text{Ad} \bar{\psi}_{\pi}^{n*}$ as states on $M_{n+1}$ and $M_{n+1} \otimes B(H_{\pi})$, respectively. The last inequality yields $\| u \cdot \psi_{k} \cdot \lambda \psi_{k} \cdot u \| \leq \lambda^2 \varepsilon_{n+1}$, and in particular, $\| (u \otimes 1) \cdot (\psi_{1} \otimes \text{tr}) - \lambda (\psi_{1} \otimes \text{tr}) \cdot (u \otimes 1) \| \leq \lambda^2 \varepsilon_{n+1}$. By Lemma 6.13 there exists a unitary $v_{\pi}^{n+1} \in M_{n+1} \otimes B(H_{\pi})$ for $\pi \in \mathcal{F}_{n+1}$ such that

\[
\text{Ad} v_{\pi}^{n+1} \circ \alpha_{\pi}^{n}(u) = u \otimes 1, \quad \pi \in \mathcal{F}_{n+1};
\]
\[
\| v_{\pi}^{n+1} - 1 \|_{\psi_{1} \otimes \text{tr}_{\pi}}^{\#, \#} < 12 \sqrt{2} \varepsilon_{n+1}, \quad \pi \in \mathcal{F}_{n+1};
\]
\[
\| v_{\pi}^{n+1} - 1 \|_{(\psi_{1} \otimes \text{tr}_{\pi}) \circ \text{Ad} \bar{\psi}_{\pi}^{n*}}^{\#, \#} < 12 \sqrt{2} \varepsilon_{n+1}, \quad \pi \in \mathcal{F}_{n+1}.
\]

Set $\{e_{11}(n+1), e_{12}(n+1), e_{21}(n+1), e_{22}(n+1)\} := \{uu^*, u, u^*, uu\}$. Then $\| \psi_{k} \cdot e_{ij}(n+1) - \lambda^{i-j} e_{ij}(n+1) \cdot \psi_{k} \| < 2 \varepsilon_{n+1}$ holds for $1 \leq k \leq n + 1$. Define $v_{\pi}^{n+1}$ by extending $v_{\pi}^{n+1}$, $\pi \in \mathcal{F}_{n+1}$, as before. Thus we have finished the $(n+1)$-st step, and this completes our induction.

Define $E_{\infty} := \bigvee_{k=1}^{\infty} \{e_{ij}(k)\}_{i,j=1,2}$. Since $\sum_{k=1}^{\infty} \| \psi_{n} \cdot e_{ij}(k) - \lambda^{i-j} e_{ij}(k) \cdot \psi_{n} \| < \infty$ for all $n \in \mathbb{N}$, $E_{\infty}$ is an injective factor of type III$_{\lambda}$, and we have the factorization

$M = E_{\infty} \vee E'_{\infty} \cap M \cong E_{\infty} \otimes E'_{\infty} \cap M$ by [11, Theorem 1.3]. (Also see [27, Lemma XVIII.4.5] ) Next we show the convergence of $\{v_{\pi}^{n}\}_{n=1}^{\infty}$. If $\pi \in \mathcal{F}_{n}$, we have

\[
\| v_{\pi}^{n+1} - \bar{\psi}_{\pi}^{n*} \|_{\psi_{1} \otimes \text{tr}_{\pi}} = \| (v_{\pi}^{n+1} - 1)^{\#} \|_{\psi_{1} \otimes \text{tr}_{\pi}} = \| v_{\pi}^{n+1} - 1 \|_{(\psi_{1} \otimes \text{tr}_{\pi}) \circ \text{Ad} \bar{\psi}_{\pi}^{n*}} < 12 \sqrt{2} \sqrt{\varepsilon_{n+1}}
\]

and

\[
\| (v_{\pi}^{n+1} - \bar{\psi}_{\pi}^{n*})^{\#} \|_{\psi_{1} \otimes \text{tr}_{\pi}} = \| (v_{\pi}^{n+1} - 1)^{*} \|_{\psi_{1} \otimes \text{tr}_{\pi}} < 12 \sqrt{2} \sqrt{\varepsilon_{n+1}}.
\]

Hence for each $\pi \in \text{Irr}(\mathcal{G})$, $\{v_{\pi}^{n}\}_{n=1}^{\infty}$ is a Cauchy sequence in the strong* topology, and set $\bar{\psi} := \lim_{n \to \infty} v_{\pi}^{n}$. Set $\bar{v} = (\bar{\psi})_{\pi} \in M \otimes L^{\infty}(\mathcal{G})$. By the choice of $v_{\pi}^{n}$, $\alpha' := \text{Ad} \bar{v} \circ \alpha$ acts trivially on $E_{\infty}$. Hence $\alpha'$ is a cocycle action on $E'_{\infty} \cap M$ with a 2-cocycle $u = (\bar{v}(12)(\alpha \otimes \text{id})(\bar{v})(\text{id} \otimes \Delta)(\bar{v}^{*}))$. Since $E'_{\infty} \cap M$ is of type III, $u$ is a coboundary by Lemma 3.2. Hence $\alpha'$ is cocycle conjugate to $\text{id}_{E_{\infty}} \otimes \beta$ for some action $\beta$ of $\mathcal{G}$ on $E'_{\infty} \cap M$. Since $E_{\infty} \otimes E_{\infty} \cong E_{\infty} \cong \mathcal{R}_{\lambda}$, $\alpha \sim \text{id}_{E_{\infty}} \otimes \beta \approx \text{id}_{E_{\infty}} \otimes \text{id}_{E_{\infty}} \otimes \beta \sim \text{id}_{\mathcal{R}_{\lambda}} \otimes \alpha$. \hfill \Box
Corollary 6.15. Let $M \cong R_\infty$, $N = M \rtimes_{\sigma_\beta} \mathbb{Z}$, $T = -2\pi/\log \lambda$ and $\theta$ be as before. Let $\alpha$ be a centrally free action of $\hat{G}$ on $R_\infty$. Then the $\hat{G} \times \mathbb{T}$-action $\tilde{\alpha} \theta$ is cocycle conjugate to $id \otimes \alpha \theta$.

Proof. This is immediate from Lemma 6.7 and Theorem 6.14 when we consider the state of the form $\varphi \otimes \varphi$ on $R_\lambda \otimes M$, where $\varphi$ is a periodic state on $R_\lambda$. □

6.4. Model action splitting

Lemma 6.16. Let $\alpha$ be a centrally free cocycle action of $\hat{G}$ on $M \cong R_\infty$. Then there exists a centralizing sequence of partial isometries $\{u_n\}_{n \in \mathbb{N}} \subset N$ with $u_n u_n^* + u_n^* u_n = 1$, $u_n^2 = 0$, $\theta_t(u_n) = e^{\sqrt{-1}t} u_n$ for all $t \in \mathbb{R}$ and $\lim_{n \to \infty} \overline{\alpha}(u_n) - u_n \otimes 1 = 0$ in the $\sigma$-strong* topology.

Proof. Since $M$ is properly infinite, $\alpha$ is cocycle conjugate to an action $\alpha'$. Then $\alpha' \sim id_{R_\lambda} \otimes \alpha'$ by Theorem 6.14 and $R_\lambda \cong R_0 \otimes R_\lambda$, $\alpha'$ is cocycle conjugate to $id_{R_0} \otimes \alpha'$ via an isomorphism $R_0 \otimes M \cong M$. By Lemma 6.7 it suffices to show the statements for $id_{R_0} \otimes \alpha$ and $N = (R_0 \otimes M) \rtimes_{\sigma_\beta} \mathbb{Z}$ assuming that $\alpha$ is an action. We denote by $U$ the implementing unitary.

Let $\{v_n\}_{n=1}^\infty \subset R_0 \otimes \mathcal{C} \subset R_0 \otimes M$ be a centralizing sequence of partial isometries with $v_n v_n^* + v_n^* v_n = 1$, $v_n^2 = 0$, and $(id_{R_0} \otimes \alpha)(v_n) = v_n \otimes 1$. Let $\{w_n\}_{n=1}^\infty \subset \mathcal{C} \otimes M$ as in Corollary 3.26. Set $u_n := U^* w_n v_n^*$ for each $n \in \mathbb{N}$. Since $[w_n, v_n] = 0$ and $U v_n U^* = \sigma_T^{tr} \otimes \varphi(v_n) = v_n$, we have $u_n^* u_n = v_n v_n^*$, $u_n^* v_n = v_n^* v_n \in M$, $u_n u_n^* + u_n^* u_n = 1$ and $u_n^2 = 0$. Since $(U^* w_n)$ is centralizing, $\{u_n\}_{n=1}^\infty$ is a centralizing sequence in $N$, and $\theta_t(u_n) = e^{\sqrt{-1}t} u_n$ for all $t \in \mathbb{T}$. Take a faithful normal state $\psi$ on $N \otimes B(H_\pi)$. Then we have

$$|| \psi \cdot ((id \otimes \overline{\alpha}_\pi)(u_n) - u_n \otimes 1) ||$$

$$= || \psi \cdot (U^* \otimes 1) \left( [D_\varphi \circ \Phi_\pi : D_\varphi \otimes \text{tr}_\pi]^* (id \otimes \alpha_\pi)(w_n)(v_n^* \otimes 1) - (w_n v_n^* \otimes 1) \right) ||$$

$$= || \psi \cdot (U^* \otimes 1) \left( [D_\varphi \circ \Phi_\pi : D_\varphi \otimes \text{tr}_\pi]^* - (w_n \otimes 1)(id \otimes \alpha_\pi)(w_n) \right) ||$$

$$\to 0$$

as $n \to \infty$. In a similar way, we get $\lim_{n \to \infty} ||(id \otimes \overline{\alpha}_\pi)(u_n) - u_n \otimes 1) \cdot \psi || = 0$. These implies that $\overline{\alpha}_\pi(u_n) - u_n \otimes 1$ converges to $0$ $\sigma$-strongly*.

Theorem 6.17. Let $M$, $N$, $\theta$ be as before. Let $\alpha$ be a centrally free action of $\hat{G}$ on $M$. Let $\beta$ be the infinite tensor product type action of $\mathbb{T}$ on $R_\lambda$ given in Proposition 6.10. Then the $\hat{G} \times \mathbb{T}$-action $\tilde{\alpha} \theta$ is cocycle conjugate to $\beta \otimes \overline{\pi} \theta$.

Proof. The proof is similar to that of Theorem 6.14. Set $\varepsilon_n := 16^{-n}$. Let $\{\psi_n\}_{n=1}^\infty \subset (M_\ast)_\varepsilon$ be a countable dense subset such that $\psi_1$ is a faithful state. For each $k \in \mathbb{N}$, we will construct a mutually commuting sequence of $2 \times 2$-matrix
units \{e_{ij}(k)\}_{i,j=1}^2 \subset N, and unitaries \upsilon^k, \tilde{\upsilon}^k \in M \otimes B(H_\pi) with the following:
\[\tilde{\upsilon}^k = \upsilon^k \upsilon^{k-1} \ldots \upsilon^1;\]
\[\text{Ad} \tilde{\upsilon}^n \circ \overline{\pi}_\pi(e_{ij}(k)) = e_{ij}(k) \otimes 1 \quad \text{for all } i, j = 1, 2, 1 \leq k \leq n, \pi \in \mathcal{F}_n;\]
\[\|\upsilon_\pi^n - 1\|_{\psi_1 \otimes \text{tr}_\pi} < 12 \sqrt{\varepsilon_n} \quad \text{for all } \pi \in \mathcal{F}_n;\]
\[\|\upsilon_\pi^n - 1\|_{(\psi_1 \otimes \text{tr}_\pi) \circ \text{Ad} \cdot \psi_1^{-1}} < 12 \sqrt{\varepsilon_n} \quad \text{for all } \pi \in \mathcal{F}_n;\]
\[\theta_t(e_{ij}(k)) = \text{Ad} \left( \begin{array}{cc} 1 & 0 \\ 0 & e^{\sqrt{-t}} \end{array} \right) (e_{ij}(k)) \quad \text{for all } t \in \mathbb{R}, i, j = 1, 2, k \in \mathbb{N};\]
\[\|\psi_k \cdot e_{ij}(n) - e_{ij}(n) \cdot \psi_k\| < 2\varepsilon_n \quad \text{for all } i, j = 1, 2, 1 \leq k \leq n.\]

By Lemma 6.16 there exists a partial isometry \(u \in N\) such that \(u^2 = 0, uu^* + u^*u = 1, \theta_t(u) = e^{\sqrt{-u}}u, uu^*, u^*u \in M\) and
\[\|(u \otimes 1 - \overline{\pi}_\pi(u)) \cdot (\psi_1 \otimes \text{tr}_\pi)\| < \varepsilon_1 \quad \text{for all } \pi \in \mathcal{F}_1;\]
\[\|(\psi_1 \otimes \text{tr}_\pi) \cdot (u \otimes 1 - \overline{\pi}_\pi(u))\| < \varepsilon_1 \quad \text{for all } \pi \in \mathcal{F}_1;\]
\[\|u \cdot \psi_1 - \psi_1 \cdot u\| < \varepsilon_1.\]

Since \(uu^* \otimes 1, \overline{\pi}_\pi(uu^*) = \alpha_\pi(uu^*) \in M \otimes B(H_\pi)\), we can take \(w\) from \(M\) in the proof of Lemma 6.13. Then \(v_\pi^1\) constructed in Lemma 6.13 is in \(M \otimes B(H_\pi)\), and we have
\[\|v_\pi^n - 1\|_{\psi_1} < 12 \sqrt{\varepsilon_1} \quad \text{for all } \pi \in \mathcal{F}_1;\]
\[\text{Ad} v_\pi^1 \circ \overline{\pi}_\pi(u) = u \otimes 1 \quad \text{for all } \pi \in \mathcal{F}_1.\]

Set \(\{e_{11}(1), e_{12}(1), e_{21}(1), e_{22}(1)\} := \{u^*u, u^*, u, uu^*\}\). Define \(v_1^1 \in M \otimes L^\infty(\widehat{G})\) by extending \(v_1^1, \pi \in \mathcal{F}_1, \) as before. Note that \(\|[e_{ij}(1), \psi_1]\| < 2\varepsilon_1\) for \(i, j = 1, 2\). So the first step is complete.

Set \(\overline{\pi}^2 := \text{Ad} v^1 \circ \overline{\pi}, \) and \(N_2 := \{u\}' \cap N\). Take \(w \in M\) an isometry with \(ww^* = e_{11}(1)\). Set \(s_1 := e_{11}(1)w\) and \(s_2 := e_{21}(1)w\). Then \(s_is_j^* = e_{ij}(1), \theta_t(s_1) = s_1\) and \(\theta_t(s_2) = e^{-it}s_2\) hold. Let \(\rho(x) := \sum_{i=1}^2 s_is_j^*\). Then \(\rho\) is an isomorphism between \(N\) and \(N_2\) which intertwines \(\theta\). Then \(\theta' := N_2^\theta = \rho(M)\) is the injective factor of type \(\text{III}_1\), and \(\theta\) is the dual action for \(\sigma_1^\theta\), \(\varphi' := \varphi \circ \rho^{-1} \in (M_2)^*\). Since \(\theta\) commutes with \(\overline{\pi}^2\) because of \(v_1^1 \in M \otimes B(H_\pi), \overline{\pi}^2\) preserves \(M_2\). Note that \(v_1^1(\pi^1)(\text{id} \otimes \Delta)(v_1^1)^*\), a 2-cocycle of \(\overline{\pi}^2\) is in \(N_2\) and fixed by \(\theta\), and it is indeed in \(M_2\). This means that \(\overline{\pi}^2_{|M_2}\) is a cocycle action. Obviously we have \(Z(\widehat{N}) = Z(\widehat{N}_2)\).

Hence \(\overline{\pi}^2\) has trivial Connes-Takesaki module, and \(\overline{\pi}^2\) is approximately inner. By Lemma 6.4 \(\overline{\pi}^2\) is the canonical extension of \(\overline{\pi} := \overline{\pi}^2_{|M_2}\). Since \(\overline{\pi}\) is centrally free, \(\overline{\pi}^2\) is centrally free, and \(\overline{\pi}^2\) is centrally free on \(M_2\) by Lemma 6.4.

Then we can apply Lemma 6.16 to \(M_2, \overline{\pi}^2, \) and \(\theta\). The rest of the proof is same as that of Theorem 6.14.

7. Appendix

We discuss relations between the canonical extension of endomorphisms and homomorphisms. In this section, we do not assume the amenability of \(\widehat{G}\).
7.1. Canonical extension of homomorphisms

Let $M$ be a properly infinite factor and $H$ a finite dimensional Hilbert space with $\dim H = n$. Let $\tilde{M}$ be the canonical core of $M$ [7, Definition 2.5]. We denote by $\text{Tr}_H$ and $\text{tr}_H$ the non-normalized and the normalized traces on $B(H)$, respectively. Then we can introduce an isomorphism between the inclusions $M \subset \tilde{M}$ and $M \otimes B(H) \subset \tilde{M} \otimes B(H)$ as follows. Fix isometries $\{v_i\}_{i=1}^n \subset M$ with orthogonal ranges and $\sum_{i=1}^n v_i v_i^* = 1$. Define $\sigma \in \text{Mor}(\tilde{M} \otimes B(H), \tilde{M})$ by

$$\sigma(x) = \sum_{i,j=1}^n v_i x_{ij} v_j^*.$$ 

It is easy to see that $\sigma$ is an isomorphism with $\sigma^{-1}(x) = \sum_{i,j=1}^n v^*_i x v_j \otimes e_{ij}$. The map $\sigma$ derives the following bijection:

$$\sigma_* : \text{Mor}_0(M, M \otimes B(K)) \to \text{End}_0(M), \quad \alpha \mapsto \sigma \circ \alpha.$$ 

We can check that $d(\alpha) = d(\sigma \circ \alpha)$ and the standard left inverse of $\rho := \sigma \circ \alpha$ is given by $\phi_\rho = \Phi_\alpha \circ \sigma^{-1}$. Hence $\Phi_\alpha(x) = \phi_\rho \circ \sigma(x) = \sum_{i,j=1}^n \phi_\rho(v_i x_{ij} v_j^*)$ holds.

Recall the topology on $\text{End}_0(M)$ introduced in [19, Definition 2.1]. We also introduce a topology on $\text{Mor}_0(M, M \otimes B(H))$ similarly.

Lemma 7.1. The map $\sigma_*$ is a homeomorphism.

Proof. Take any $\varphi \in M_*$. Assume that $\alpha^\nu \to \alpha$ in $\text{Mor}_0(M, M \otimes B(H))$ as $\nu \to \infty$, that is, we have the norm convergence $\varphi \circ \Phi^\alpha \to \varphi \circ \Phi^\alpha$ in $(M \otimes B(H))_*$. Write $\rho^\nu = \sigma_*(\alpha^\nu)$ and $\rho = \sigma_*(\alpha)$. Using $\phi_{\rho^\nu} = \Phi^\alpha \circ \sigma^{-1}$ and $\phi_{\rho} = \Phi^\alpha \circ \sigma^{-1}$, we have the norm convergence $\varphi \circ \phi_{\rho^\nu} \to \varphi \circ \phi_{\rho}$, that is, $\rho^\nu \to \rho$ as $\nu \to \infty$. Hence $\sigma_*$ is continuous. Similarly we can prove that $\sigma_*^{-1}$ is continuous. \qed

Lemma 7.2. Let $\varphi$ be a faithful normal state on $M$. Then one has

$$[D\varphi \circ \Phi^\alpha : D\varphi \otimes \text{Tr}]_t = \sum_{i,j=1}^n v_i^*[D\varphi \circ \phi_{\sigma_*(\alpha)} : D\varphi]_t \sigma^\varphi_t(v_j) \otimes e_{ij} \quad \text{for all } t \in \mathbb{R}.$$ 

Proof. Set $\rho := \sigma_*(\alpha)$ and a unitary $u_t := \sum_{i,j=1}^n v_i^*[D\varphi \circ \phi_{\rho} : D\varphi]_t \sigma^\rho_t(v_j) \otimes e_{ij}$. Then $u_t$ is a $\sigma^\rho \otimes \text{Tr}$-cocycle. We verify that $u_t$ satisfies the relative modular condition. Let $D := \{z \in \mathbb{C} \mid 0 < \text{Im}(z) < 1\}$, and

$$A(D) := \{f(z) \mid f(z) \text{ is analytic on } D, \text{ bounded, continuous on } \overline{D}\}.$$ 

Take $x, y \in M \otimes B(H)$. By the relative modular condition for $[D\varphi \circ \Phi^\alpha : D\varphi]_t$ and $\sum_{k=1}^n v_k x_k t$ and $\sum_{k=1}^n y_{kj} v_j^*$, we can choose $F_t(z) \in A(D)$ such that

$$F_t(t) = \sum_{j,k=1}^n \varphi \circ \phi_{\rho} ([D\varphi \circ \phi_{\rho} : D\varphi]_t \sigma^\rho_t(v_{kx_k t}) y_{kj} v_j^*) \quad \text{for all } t \in \mathbb{R}$$

and

$$F_t(t + \sqrt{-1}) = \sum_{j,k=1}^n \varphi (y_{kj} v_j^*[D\varphi \circ \phi_{\rho} : D\varphi]_t \sigma^\rho_t(v_{kx_k}) \text{ for all } t \in \mathbb{R}.$$
Set $F(z) := \sum_{\ell=1}^n F_\ell(z) \in \mathcal{A}(D)$. Then we have

$$\varphi \circ \Phi^\sigma(u_t \sigma^\sigma_{t}^{\operatorname{Tr}}(x)y) = \sum_{i,j=1}^n \varphi \circ \phi_{\rho} \left( v_i (u_t \sigma^\sigma_{t}^{\operatorname{Tr}}(x)y)_{ij} v_j^* \right)$$

$$= \sum_{i,j,k,\ell=1}^n \varphi \circ \phi_{\rho} \left( v_i u_{t,ik} \sigma^\sigma_{t}(x_{k\ell}) y_{\ell j} v_j^* \right)$$

$$= \sum_{j,k,\ell=1}^n \varphi \circ \phi_{\rho} \left( \left[ D\varphi \circ \phi_{\rho} : D\varphi \right]_t \sigma^\sigma_{t}(x_{k\ell}) y_{\ell j} v_j^* \right)$$

$$= F(t),$$

and

$$(\varphi \otimes \operatorname{Tr})(yu_t \sigma^\sigma_{t}^{\operatorname{Tr}}(x)) = \sum_{\ell=1}^n \varphi \left( (yu_t \sigma^\sigma_{t}^{\operatorname{Tr}}(x))_{\ell \ell} \right) = \sum_{j,k,\ell=1}^n \varphi (y_{\ell j} u_{t,jk} \sigma^\sigma_{t}(x_{k\ell}))$$

$$= \sum_{j,k,\ell=1}^n \varphi (y_{\ell j} v_j^* \left[ D\varphi \circ \phi_{\rho} : D\varphi \right]_t \sigma^\sigma_{t}(x_{k\ell}) y_{\ell j} v_j^*)$$

$$= F(t + \sqrt{-1}).$$

This shows that $u_t$ satisfies the relative modular condition. $\square$

Let $\sim : \text{End}(M)_0 \to \text{End}(\widetilde{M})$ be the canonical extension [12, Theorem 2.4]. We define the map $\sim : \text{Mor}_0(M, M \otimes B(K)) \to \text{Mor}(\widetilde{M}, \widetilde{M} \otimes B(K))$ by

$$\widetilde{\alpha} = \sigma^{-1} \circ \phi \circ \alpha$$

for all $\alpha \in \text{Mor}_0(M, M \otimes B(K))$.

In fact, $\widetilde{\alpha}$ does not depend on $\sigma$ as follows.

**Theorem 7.3.** One has the following:

1. $\widetilde{\alpha}(x) = \alpha(x)$ for all $x \in M$;
2. $\widetilde{\alpha}(\lambda^\varphi(t)) = d(\alpha)^{it}[D\varphi \circ \Phi^\sigma : D\varphi \otimes \operatorname{Tr}_K]_t(\lambda^\varphi(t) \otimes 1)$ for all $t \in \mathbb{R}$.

**Proof.** Set $\rho := \sigma_\ast(\alpha)$. Then by definition, we have

$$\widetilde{\rho}(x) = \rho(x) \quad \text{for all } x \in M,$$

$$\widetilde{\rho}(\lambda^\varphi(t)) = d(\rho)^{it}[D\varphi \circ \phi_{\rho} : D\varphi]_t \lambda^\varphi(t) \quad \text{for all } t \in \mathbb{R}.$$
Since $\sigma^{-1} \circ \rho = \alpha$, (1) follows. On (2), we have
\[
\tilde{\alpha}(\lambda^\varphi(t)) = \sigma^{-1}(\tilde{\rho}(\lambda^\varphi(t))) = \sum_{k,\ell=1}^{n} v_k^* \tilde{\rho}(\lambda^\varphi(t)) v_{\ell} \otimes e_{k\ell}
\]
\[
= \sum_{k,\ell=1}^{n} d(\rho)^{it} v_k^*[D\varphi \circ \phi_\rho : D\varphi]_t \lambda^\varphi(t) v_{\ell} \otimes e_{k\ell}
\]
\[
= \sum_{k,\ell=1}^{n} d(\rho)^{it} (v_k^*[D\varphi \circ \phi_\rho : D\varphi]_t \lambda^\varphi(t) \otimes e_{k\ell})(\lambda^\varphi(t) \otimes 1)
\]
\[
= d(\alpha)^{it}[D\varphi \circ \Phi^\alpha : D\varphi \otimes \text{Tr}_K]_t(\lambda^\varphi(t) \otimes 1). \quad \text{(by Lemma 7.2)}
\]

We say that $\alpha \in \text{Mor}_0(M, M \otimes B(K))$ is inner if there exists a unitary $U \in M \otimes B(K)$ such that $\alpha = U(\cdot \otimes 1)U^*$. Denote by $\text{Int}(M, M \otimes B(K))$, $\overline{\text{Int}}(M, M \otimes B(K))$ and $\text{Cnt}(M, M \otimes B(K))$ the set of the inner homomorphisms, the approximately inner homomorphisms and the centrally trivial homomorphisms in $\text{Mor}_0(M, M \otimes B(K))$, respectively. (See Definition 2.1.) Then we have the following bijective correspondence. See [19] for the notations used here.

**Lemma 7.4.** The bijection $\sigma_* : \text{Mor}_0(M, M \otimes B(K)) \to \text{End}_0(M)$ yields the following bijective maps:

1. $\sigma_* : \text{Int}(M, M \otimes B(K)) \to \text{Int}_{\text{dim}(K)}(M)$;
2. $\sigma_* : \overline{\text{Int}}(M, M \otimes B(K)) \to \overline{\text{Int}}_{\text{dim}(K)}(M)$;
3. $\sigma_* : \text{Cnt}(M, M \otimes B(K)) \to \text{Cnd}(M)$.

**Proof.** (1) Assume that $\alpha = \text{Ad} U(\cdot \otimes 1)$ for some unitary $U \in M \otimes B(K)$. Set $\rho := \sigma_*(\alpha)$ and a Hilbert space $\mathcal{H} \subset M$ which is spanned by $w_k := \sum_{i=1}^{n} v_i U_{ik}$, $k = 1, \ldots, n$. Then for $x \in M$, we have
\[
\rho(x) = \sigma(\alpha(x)) = \sigma(U(x \otimes 1)U^*) = \sum_{i,j,k=1}^{n} \sigma(U_{ik} x U_{jk}^* \otimes e_{ij})
\]
\[
= \sum_{i,j,k=1}^{n} v_i U_{ik} x U_{jk}^* v_j = \sum_{k=1}^{n} w_k x w_k^* = \rho_{\mathcal{H}}(x).
\]
Hence $\rho = \rho_{\mathcal{H}} \in \text{Int}_{\text{dim}(K)}(M)$. Conversely if we have $\rho = \rho_{\mathcal{H}}$ with $\text{dim} \mathcal{H} = n$, then setting $U_{ik} := v_i^* w_k$ for some orthonormal basis $\{w_k\}_{k=1}^{n} \subset \mathcal{H}$, we have $\sigma^{-1} \circ \rho = \text{Ad} U(\cdot \otimes 1)$.

(2) This follows from (1) and Lemma 7.1.

(3) Assume that $\alpha \in \text{Cnt}(M, M \otimes B(K))$. Set $\rho := \sigma_*(\alpha)$. Take an $\omega$-centralizing sequence $(x^\nu)_\nu$ in $M$. Then $\alpha(x^\nu) - x^\nu \to 0$ strongly as $\nu \to \omega$. Hence $\rho(x^\nu) - \sigma(x^\nu \otimes 1) \to 0$. Since $\sigma(x^\nu \otimes 1) = \sum_{i,j=1}^{n} v_i x^\nu v_i^*$, we see that $\rho(x^\nu) - x^\nu \to 0$, that is, $\rho \in \text{Cnd}(M)$. The converse can be proved similarly. \qed

We define the following set:
\[
\text{Mor}_{\text{CT}}(M, M \otimes B(K)) = \{ \alpha \in \text{Mor}_0(M, M \otimes B(K)) \mid \sigma_*(\alpha) \in \text{End}(M)_{\text{CT}} \}.
\]
The following lemma shows that this set does not depend on \( \sigma \).

**Lemma 7.5.** Let \( \alpha \in \text{Mor}_0(M, M \otimes B(K)) \). Then the following are equivalent:

1. \( \alpha \in \text{Mor}_{CT}(M, M \otimes B(K)) \);
2. There exists \( \gamma \in \text{Auto}_\theta(Z(M)) \) such that \( \tilde{\alpha}(z) = \gamma(z) \otimes 1 \) for all \( z \in Z(\widetilde{M}) \).

**Proof.** Assume that \( \alpha \in \text{Mor}_{CT}(M, M \otimes B(K)) \). Set \( \rho := \sigma_*(\alpha) \). Then \( \rho \) has Connes-Takesaki module \( \text{mod}(\rho) \). By definition, \( \sigma^{-1}(z) = z \otimes 1 \) for \( z \in Z(\widetilde{M}) \). For \( z \in Z(\widetilde{M}) \), we have

\[
\tilde{\alpha}(z) = \sigma^{-1}(\rho(z)) = \sigma^{-1}(\text{mod}(\rho)(z)) = \text{mod}(\rho)(z) \otimes 1.
\]

Conversely, assume that such \( \gamma \) exists. Then we have

\[
\rho(z) = \sigma(\tilde{\alpha}(z)) = \sigma(\gamma(z) \otimes 1) = \gamma(z).
\]

Hence \( \rho \) has the Connes-Takesaki module \( \gamma \), that is, \( \alpha \in \text{Mor}_{CT}(M, M \otimes B(K)) \).

\( \square \)

In this situation, we say that \( \alpha \) has the *Connes-Takesaki module* \( \text{mod}(\alpha) := \gamma \).

**Theorem 7.6.** Let \( M \) be a properly infinite injective factor. Then one has the following:

1. \( \alpha \in \overline{\text{Int}}(M, M \otimes B(K)) \)
   \( \iff \alpha \in \text{Mor}_{CT}(M, M \otimes B(K)) \) with \( \text{mod}(\alpha) = \theta_{\log(\dim(K)/d(\alpha))} \);
2. \( \alpha \in \text{Cnt}(M, M \otimes B(K)) \)
   \( \iff \) There exists a unitary \( U \in M_{d(\alpha), \dim(K)}(\widetilde{M}) \) such that \( \tilde{\alpha} = U(\cdot \otimes 1)U^* \).

**Proof.** This follows from [18, Theorem 3.15, 4.12]. Note that if \( \alpha \in \text{Cnt}(M, M \otimes B(K)) \), then \( d(\alpha) \) is integer [12, Theorem 3.3 (5)].

\( \square \)

We obtain the following corollary.

**Corollary 7.7.** The following statements hold:

1. If \( M = \mathcal{R}_{0,1} \), then
   - \( \alpha \in \overline{\text{Int}}(M, M \otimes B(K)) \)
     \( \iff \tau \circ \Phi^\alpha = \tau \otimes \text{tr}_K \) where \( \tau \) is a trace on \( M \);
   - \( \alpha \in \text{Cnt}(M, M \otimes B(K)) \)
     \( \iff \) there exist \( n \in \mathbb{N} \) and a unitary \( U \in M \otimes M_{\dim(K), n}(\mathbb{C}) \) such that \( \alpha(x) = U(x \otimes 1)U^* \) for all \( x \in M \).
2. If \( M = \mathcal{R}_\lambda \) with \( 0 < \lambda < 1 \), then
   - \( \alpha \in \overline{\text{Int}}(M, M \otimes B(K)) \)
     \( \iff [D\varphi \circ \Phi^\alpha : D\varphi \otimes \text{tr}_K]_T = 1 \), where \( \varphi \) is a generalized trace on \( M \) and \( T = -2\pi/\log \lambda \);
   - \( \alpha \in \text{Cnt}(M, M \otimes B(K)) \)
     \( \iff \) there exist \( n \in \mathbb{N} \), a unitary \( U \in M \otimes M_{\dim(K), n}(\mathbb{C}) \) and \( \{s_i\}_{i=1}^n \subset \mathbb{R} \) such that
     \[
     \alpha(x) = U \text{diag}(\sigma_{s_1}^\varphi(x), \ldots, \sigma_{s_n}^\varphi(x))U^* \quad \text{for all } x \in M.
     \]
(3) If $M = \mathcal{R}_\infty$, then
- $\text{Int}(M, M \otimes B(K)) = \text{Mor}_0(M, M \otimes B(K))$;
- $\alpha \in \text{Cn}(M, M \otimes B(K))$
  \( \iff \) there exist $n \in \mathbb{N}$, a unitary $U \in M \otimes M_{\dim(K),n}(\mathbb{C})$ and \( \{s_i\}_{i=1}^n \subset \mathbb{R} \) such that
  \[ \alpha(x) = U \text{diag}(\sigma_{s_1}^\varepsilon(x), \ldots, \sigma_{s_n}^\varepsilon(x))U^* \quad \text{for all} \ x \in M. \]

7.2. Canonical extension of cocycle actions

We discuss canonical extension of cocycle actions. Let \((G, \pi, \alpha)\) be a system of \(\mathcal{L}(B)\) actions. For \(\pi \in \text{Irr}(\mathcal{G})\), we define the left inverse \(\Phi_\pi^\alpha\) for \(\alpha\) by
\[
\Phi_\pi^\alpha(x) = (1 \otimes T_{\pi,\pi}^\alpha)_{\pi,\pi}(\alpha_{\pi} \otimes \text{id})(x)_{\pi,\pi}(1 \otimes T_{\pi,\pi}^\alpha) \quad \text{for all} \ x \in M \otimes B(H_\pi),
\]
where \(T_{\pi,\pi}^\alpha\) is an isometry intertwining \(1\) and \(\pi \otimes \pi\) [18, p.491]. Then \(\alpha \circ \Phi_\pi^\alpha\) is a faithful normal conditional expectation from \(M \otimes B(H_\pi)\) onto \(\alpha(M)\). Set \(d(\pi) := \dim(H_\pi)\). Recall the diagonal operator \(a \in M \otimes L_\infty(\hat{\mathcal{G}})\) of \(u\) [18, Definition 5.5]:
\[
(a \otimes 1)(1 \otimes \Delta(e_1)) = u(1 \otimes \Delta(e_1)).
\]

Lemma 7.8. One has \(\text{Ind}(\alpha \circ \Phi_\pi^\alpha) = d(\pi)^2\) for all \(\pi \in \text{Irr}(\mathcal{G})\).

Proof. Set \(E_\pi := \alpha \circ \Phi_\pi^\alpha\) and \(d(\pi) = \dim H_\pi\). Let \(\{e_{\pi,ij}\}_{i,j=1}^{d(\pi)}\) be a system of \(\mathcal{G}\) matrix units of \(B(H_\pi)\). We show that \(\{d(\pi)^{1/2}(1 \otimes e_{\pi,ij})a_{\pi}^*\}_{i,j=1}^{d(\pi)}\) is a quasi basis for \(E_\pi\) [28, Definition 1.2.2]. For any \(y \in M\) and \(1 \leq k, \ell \leq d(\pi)\), we have
\[
\begin{align*}
\Phi_\pi^\alpha(a_{\pi}(1 \otimes e_{\pi,ij})(y \otimes e_{\pi,\ell})) &= \delta_{ik}(1 \otimes T_{\pi,\pi}^\alpha)_{\pi,\pi}(\alpha_{\pi}(a_{\pi}(y)) \otimes e_{\pi,\ell})(1 \otimes T_{\pi,\pi}^\alpha) \\
&= \delta_{ik}(1 \otimes T_{\pi,\pi}^\alpha)_{\pi,\pi}(a_{\pi}^* \otimes 1)(1 \otimes T_{\pi,\pi}^\alpha) \\
&= \delta_{ik}(1 \otimes T_{\pi,\pi}^\alpha)_{\pi,\pi}(a_{\pi}^*(y) \otimes e_{\pi,\ell})(1 \otimes T_{\pi,\pi}^\alpha) \\
&= d(\pi)^{-1}\delta_{ik}(\alpha_{\pi}(y)a_{\pi})_{\pi,\ell}.
\end{align*}
\]

Using this equality, we have
\[
\begin{align*}
&\sum_{i,j=1}^{d(\pi)} d(\pi)(1 \otimes e_{\pi,ij})a_{\pi}^*E_\pi(a_{\pi}(1 \otimes e_{\pi,ij})(y \otimes e_{\pi,\ell})) \\
&= \sum_{i,j=1}^{d(\pi)} \delta_{ik}(1 \otimes e_{\pi,ij})a_{\pi}^*\alpha_{\pi}((\alpha_{\pi}(y)a_{\pi})_{\pi,\ell}) \\
&= \sum_{j=1}^{d(\pi)} (1 \otimes \varepsilon_{\pi,k})(1 \otimes \varepsilon_{\pi,j}^* \otimes \varepsilon_{\pi,\ell}^*)a_{\pi}^*(1 \otimes 1 \otimes \varepsilon_{\pi,\ell}) \\
&= \sum_{j=1}^{d(\pi)} (1 \otimes \varepsilon_{\pi,k})(1 \otimes \varepsilon_{\pi,j}^* \otimes \varepsilon_{\pi,\ell}^*)u_{\pi,\pi}^\alpha(\alpha_{\pi}(y)a_{\pi})(1 \otimes 1 \otimes \varepsilon_{\pi,\ell}).
\end{align*}
\]
\[ \frac{d(\pi)}{d(\pi)} = \sum_{j=1}^{d(\pi)} (1 \otimes \varepsilon_{\pi_j})(1 \otimes \varepsilon^*_{\pi_j} \otimes \varepsilon^*_{\pi_j}) (\alpha(y)u^*_{\pi_j}\alpha(\pi)(1 \otimes 1 \otimes \varepsilon_{\pi_i}) \]
\[ = \sum_{j=1}^{d(\pi)} (y \otimes \varepsilon_{\pi_j})(1 \otimes \varepsilon^*_{\pi_j} \otimes \varepsilon^*_{\pi_j}) u^*_{\pi_j} \alpha(\pi)(1 \otimes 1 \otimes \varepsilon_{\pi_i}) \]
\[ = \sum_{j=1}^{d(\pi)} (y \otimes \varepsilon_{\pi_j})(1 \otimes \varepsilon^*_{\pi_j} \otimes \varepsilon^*_{\pi_j})(a^*_{\pi_j} \otimes 1) \alpha(\pi)(1 \otimes 1 \otimes \varepsilon_{\pi_i}) \]
\[ = \sum_{j=1}^{d(\pi)} (y \otimes \varepsilon_{\pi_j})(1 \otimes \varepsilon^*_{\pi_j} \otimes \varepsilon^*_{\pi_j})(1 \otimes 1 \otimes \varepsilon_{\pi_i}) = y \otimes e_{\pi_k}. \]

Hence \( \{d(\pi)^{1/2}(1 \otimes e_{\pi_{ij}})a^*_{\pi} \}_{i,j=1}^{d(\pi)} \) is a quasi basis for \( E_\pi \), and we have
\[
\text{Ind}(E_\pi) = \sum_{i,j=1}^{d(\pi)} d(\pi)^{1/2}(1 \otimes e_{\pi_{ij}})a^*_{\pi} \cdot d(\pi)^{1/2}a_{\pi}(1 \otimes e^*_{\pi_{ij}}) = d(\pi)^2(\text{id} \otimes \text{tr}_\pi)(a^*_{\pi}a_{\pi}) = d(\pi)^2. \text{ (by [18] Lemma 5.6(2))}
\]

\[ \square \]

**Definition 7.9.** We say that a cocycle action \( \alpha \in \text{Mor}(M, M \otimes L^\infty(\hat{G})) \) is **standard** when the left inverse \( \Phi^\alpha_\pi \) is standard for each \( \pi \in \text{Irr}(G) \).

**Proposition 7.10.** Let \( \alpha \in \text{Mor}(M, M \otimes L^\infty(\hat{G})) \) be a cocycle action. Then the following hold:

1. If \( \alpha \) is cocycle conjugate to a standard cocycle action \( \beta \in \text{Mor}(M^2, M^2 \otimes L^\infty(\hat{G})) \), then \( \alpha \) is standard.
2. If \( \alpha \) is free, then \( \alpha \) is standard;
3. If \( \hat{G} \) is amenable, then \( \alpha \) is standard.

**Proof.**
1. Let \( \pi \in \text{Irr}(G) \). Since the inclusion \( \beta_\pi(M^2) \subset M^2 \otimes B(H_\pi) \) is isomorphic to \( \alpha_\pi(M) \subset M \otimes B(H_\pi) \), \( [M \otimes B(H_\pi) : \alpha_\pi(M)]_0 = [\beta_\pi(M^2) : M^2 \otimes B(H_\pi)]_0 = d(\pi)^2 \). Hence \( \alpha \) is standard.
2. For any \( \pi \in \text{Irr}(G) \), the expectation \( \alpha_\pi \circ \Phi^\alpha_\pi \) is minimal because of the irreducibility of the inclusion \( \alpha_\pi(M) \subset M \otimes B(H_\pi) \) [18], Lemma 2.8]. Hence \( \alpha \) is standard.
3. Since \( [B(\ell_2) \otimes M \otimes B(H_\pi) : B(\ell_2) \otimes \alpha_\pi(M)]_0 = [M \otimes B(H_\pi) : \alpha_\pi(M)]_0 \), we may and do assume that \( M \) is properly infinite by considering \( \text{id} \otimes \alpha \). Then \( \alpha \) is cocycle conjugate to an action \( \beta \) on \( M \) by Lemma [5,2]. By (1), it suffices to show that \( \beta \) is standard. We check \( E_\pi^{-1} = d(\pi)^2E_\pi \) on \( Q_\pi := \beta_\pi(M)^t \cap (M \otimes B(H_\pi)) \) to use [11] Theorem 1(2)]. Take \( x \in Q_\pi \). Then by [28], p.62 Remark, we have
\[ E_\pi^{-1}(x) = \sum_{i,j=1}^{d(\pi)} d(\pi)^{1/2}(1 \otimes e_{\pi_{ij}})xd(\pi)^{1/2}(1 \otimes e^*_{\pi_{ij}}) = d(\pi)^2((\text{id} \otimes \text{tr}_\pi)(x) \otimes 1). \]

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So, $E_\pi$ is minimal if and only if the following holds:

$$\text{id} \otimes \text{tr}_\pi(x) = \Phi_\pi^\beta(x) \in \mathbb{C}. \quad (7.1)$$

If we can find a $\beta$-invariant state $\psi \in M^*$, the proof is finished. Indeed, applying $\psi$ to $\Phi_\pi^\beta$, we have

$$\psi(\Phi_\pi^\beta(x)) = T_{\pi,\pi}(\psi \otimes \text{id} \otimes \text{id})(\beta(x))T_{\pi,\pi} = T_{\pi,\pi}(1_\pi \otimes (\psi \otimes \text{id})(x))T_{\pi,\pi}$$

$$= (\psi \otimes \text{tr}_\pi(x))$$

Hence $(7.1)$ holds. Such a state $\psi$ is constructed by using an invariant mean $m \in L^\infty(\hat{\mathcal{G}})^*$. Take a state $\varphi$ on $M^\alpha$ and set $\psi := m((\varphi \otimes \text{id})(\beta(x)))$. Then we have $(\psi \otimes \text{id})(\beta(x)) = \psi(x)1$ for all $x \in M$, that is, $\psi$ is invariant under $\beta$. □

**Problem 7.11.** Is any cocycle action of $\mathcal{G}$ on a factor standard?

Let $\alpha \in \text{Mor}(M, M \otimes L^\infty(\hat{\mathcal{G}}))$ be a standard cocycle action with a 2-cocycle $u$. Now for $\pi \in \text{Irr}(\mathcal{G})$, we consider the canonical extension $\tilde{\alpha}_\pi \in \text{Mor}(\tilde{\mathcal{M}}, \tilde{\mathcal{M}} \otimes B(H_\pi))$. Collecting $(\tilde{\alpha}_\pi)_\pi$, we obtain a map $\tilde{\alpha} \in \text{Mor}(\tilde{\mathcal{M}}, \tilde{\mathcal{M}} \otimes L^\infty(\hat{\mathcal{G}}))$, which is called the canonical extension of the action $\alpha$. We have the following equalities:

$$\tilde{\alpha}_\pi(x) = \alpha_\pi(x) \quad \text{for all } x \in M;$$

$$\tilde{\alpha}_\pi(\lambda_\varphi(t)) = [D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]_t(\lambda_\varphi(t) \otimes 1) \quad \text{for all } t \in \mathbb{R}, \varphi \in W(M).$$

The following two results even for actions of general Kac algebras are obtained in [29], where operator valued weight theory is fully used, but we can directly prove them for the discrete $\mathcal{G}$. We present their proofs for readers’ convenience.

Take $\varphi \in W(M)$. For $t \in \mathbb{R}$, we define $w_t = (w_{t,\pi})_\pi \in U(M \otimes L^\infty(\hat{\mathcal{G}}))$ by

$$w_{t,\pi} = [D\varphi \circ \Phi_\pi^\alpha : D\varphi \otimes \text{tr}_\pi]_t^*.$$

**Lemma 7.12.** The unitary $w_t$ satisfies the following:

$$(w_t \otimes 1)\alpha(w_t)u(\text{id} \otimes \Delta)(w_t^*) = (\sigma_t^\varphi \otimes \text{id})(u).$$

**Proof.** By the chain rule of Connes’ cocycles, we may and do assume that $\varphi$ is a state. Let $\pi, \rho \in \text{Irr}(\mathcal{G})$. Using the isomorphism $\alpha_\pi^{-1} : \alpha_\pi(M) \to M$, we have

$$\alpha_\pi(w_{t,\rho}) = [D\varphi \circ \Phi_\rho^\alpha : D\varphi \otimes \text{tr}_\rho]_t^*.$$

Since $E_\pi := \alpha_\pi \circ \Phi_\pi^\alpha : M \otimes B(H_\pi) \to \alpha_\pi(M)$ is a conditional expectation, we have

$$\alpha_\pi(w_{t,\rho}) = [D\varphi \circ \Phi_\rho^\alpha : D\varphi \otimes \text{tr}_\rho]_t^*$$

$$= [D\varphi \circ \Phi_\rho^\alpha : D\varphi \otimes \text{tr}_\rho]_t^*$$

Then we have

$$\text{(3.1)}$$

$$= [D\varphi \otimes \text{tr}_\rho : D\varphi \circ \Phi_\rho^\alpha \otimes \text{tr}_\rho]_t [D\varphi \circ \Phi_\rho^\alpha \otimes \text{tr}_\rho]_t [D\varphi \circ \Phi_\rho^\alpha \otimes \text{tr}_\rho]_t$$

$$= [D\varphi \otimes \text{tr}_\rho : D\varphi \circ \Phi_\rho^\alpha \otimes \text{tr}_\rho]_t.$$
Applying \((\sigma^\rho_t \otimes \text{id} \otimes \text{id})(u^*_{\pi, \rho})\) and \(u_{\pi, \rho}\) to the both sides, we have
\[
(\sigma^\rho_t \otimes \text{id} \otimes \text{id})(u^*_{\pi, \rho})(w_t \otimes 1_\rho)\alpha_\pi(w_{t, \rho})u_{\pi, \rho} = (\sigma^\rho_t \otimes \text{id} \otimes \text{id})(u^*_{\pi, \rho})u_{\pi, \rho} \\
\cdot [D\varphi \otimes tr_\pi \otimes tr_\rho : D\varphi \circ \Phi^\alpha_\rho \circ (\Phi^\alpha_\pi \otimes \text{id}) \circ \text{Ad } u_{\pi, \rho}]_t \\
= [D\varphi \otimes tr_\pi \otimes tr_\rho : D\varphi \circ \Phi^\alpha_\rho \circ (\Phi^\alpha_\pi \otimes \text{id}) \circ \text{Ad } u_{\pi, \rho}]_t.
\]
(7.2)

Recall the following formula [18 Lemma 2.5]: for \(X \in M \otimes B(H_\pi) \otimes B(H_\rho)\),
\[
\Phi^\alpha_\rho \circ (\Phi^\alpha_\pi \otimes \text{id})(u_{\pi, \rho}Xu^*_{\pi, \rho}) = \sum_{\sigma < \pi, \rho} \sum_{S \in \text{ONB}(\sigma, \pi, \rho)} \frac{d(\sigma)}{d(\pi)d(\rho)} \Phi^\alpha_\sigma((1 \otimes S^*)X(1 \otimes S)).
\]
Hence for \(S \in \text{ONB}(\sigma, \pi \cdot \rho)\), we have
\[
\Phi^\alpha_\rho((\Phi^\alpha_\pi \otimes \text{id})(u_{\pi, \rho}(1 \otimes SS^*)Xu^*_{\pi, \rho})) = \frac{d(\sigma)}{d(\pi)d(\rho)} \Phi^\alpha_\sigma((1 \otimes S^*)X(1 \otimes S)) \\
= \Phi^\alpha_\rho((\Phi^\alpha_\pi \otimes \text{id})(u_{\pi, \rho}X(1 \otimes SS^*)u^*_{\pi, \rho})).
\]
In particular, \(1 \otimes SS^*\) is in the centralizer of \(\varphi \otimes \Phi^\rho_\pi \circ (\Phi^\alpha_\pi \otimes \text{id}) \circ \text{Ad } u_{\pi, \rho}\). Trivially, it is also in the centralizer of \(\varphi \otimes tr_\pi \otimes tr_\rho\). Hence we see that the both sides of (7.2) commutes with \(1 \otimes SS^*\), and we have
\[
(7.2) = \sum_{\sigma < \pi, \rho} \sum_{S \in \text{ONB}(\sigma, \pi, \rho)} [D\varphi \otimes tr_\pi \otimes tr_\rho : D\varphi \circ \Phi^\alpha_\rho \circ (\Phi^\alpha_\pi \otimes \text{id})](1 \otimes SS^*) \\
= \sum_{\sigma < \pi, \rho} \sum_{S \in \text{ONB}(\sigma, \pi, \rho)} [D(\varphi \otimes tr_\pi \otimes tr_\rho)_{1 \otimes SS^*} : D(\varphi \circ \Phi^\alpha_\rho \circ (\Phi^\alpha_\pi \otimes \text{id}) \circ \text{Ad } u_{\pi, \rho})_{1 \otimes SS^*}]_t,
\]
(7.3)

where the last cocycles are evaluated in \((M \otimes B(H_\pi) \otimes B(H_\rho))_{1 \otimes SS^*}\).

Let \(\Theta_S : B(H_\sigma) \rightarrow (B(H_\pi) \otimes B(H_\rho))_{SS^*}\) be the isomorphism defined by \(\Theta_S(x) = SS^*\) for \(x \in B(H_\sigma)\). Using
\[
(tr_\pi \otimes tr_\rho)_{SS^*} = \frac{d(\sigma)}{d(\pi)d(\rho)} tr_\sigma \circ \Theta^{-1}_S \\
(\varphi \circ \Phi^\rho_\pi \circ (\Phi^\alpha_\pi \otimes \text{id}) \circ \text{Ad } u_{\pi, \rho})_{1 \otimes SS^*} = \frac{d(\sigma)}{d(\pi)d(\rho)} \varphi \circ \Phi^\rho_\sigma \circ (\text{id} \otimes \Theta^{-1}_S),
\]
we have
\[
(7.3) = \sum_{\sigma < \pi, \rho} \sum_{S \in \text{ONB}(\sigma, \pi, \rho)} [D\varphi \otimes tr_\sigma \circ \Theta^{-1}_S : D\varphi \circ \Phi^\alpha_\sigma \circ (\text{id} \otimes \Theta^{-1}_S)]_t \\
= \sum_{\sigma < \pi, \rho} \sum_{S \in \text{ONB}(\sigma, \pi, \rho)} (\text{id} \otimes \Theta_S)([D\varphi \otimes tr_\sigma : D\varphi \circ \Phi^\alpha_\sigma])_t \\
= \sum_{\sigma < \pi, \rho} \sum_{S \in \text{ONB}(\sigma, \pi, \rho)} (\text{id} \otimes \Delta)(w_t)(1 \otimes SS^*) = (\text{id} \otimes \Delta_\rho)(w_t).
\]
Thus we get
\[(\sigma_t\otimes\text{id}\otimes\text{id})(u_{\pi,\rho}^*)(w_{t,\pi}\otimes1_{\rho})\alpha_{\pi}(w_{t,\rho})u_{\pi,\rho} = (\text{id}\otimes\Delta\rho)(w_t)\].

\[\square\]

**Theorem 7.13.** Let \((\alpha, u)\) be a standard cocycle action of \(\hat{G}\) on a factor \(M\). Then the canonical extension \((\tilde{\alpha}, u)\) is a cocycle action on \(\tilde{M}\).

**Proof.** We will check \((\tilde{\alpha}\otimes\text{id})\circ\tilde{\alpha} = \text{Ad} u \circ (\text{id}\otimes\Delta)\circ\tilde{\alpha}\). We have \(\tilde{\alpha} = \alpha\) on \(M\), and that is trivial. For \(t \in \mathbb{R}\), \(\alpha(\lambda^\varphi(t)) = w_t^*(\lambda^\varphi(t) \otimes 1)\). The previous lemma yields

\[(\tilde{\alpha}\otimes\text{id})(\tilde{\alpha}(\lambda^\varphi(t))) = (\tilde{\alpha}\otimes\text{id})(w_t^*(\lambda^\varphi(t) \otimes 1))\]
\[= (\alpha\otimes\text{id})(w_t^*)(w_t^* \otimes 1)(\lambda^\varphi(t) \otimes 1 \otimes 1)\]
\[= u(\text{id}\otimes\Delta)(w_t^*)(\lambda^\varphi(t))(\text{id}\otimes\Delta)(\tilde{\alpha}(\lambda^\varphi(t)))u^*\].

\[\square\]

**Lemma 7.14.** Let \((\alpha, u)\) be a standard cocycle action of \(\hat{G}\) on \(M\). The canonical trace \(\tau\) on \(\tilde{M}\) is invariant under \(\tilde{\alpha}\), that is, \(\tau \circ \Phi^\alpha = \tau \otimes \text{tr}_\pi\) for all \(\pi \in \text{Irr}(G)\).

**Proof.** Let \(\varphi \in W(M)\). Take a positive operator \(h\) affiliated in \(\tilde{M}_\varphi\) such that \(h^{it} = \lambda^\varphi(t)\). Then the canonical trace is given by \(\tau := \hat{\varphi}_{h^{-1}}\), which does not depend on the choice of the weight \(\varphi\). Let \(T_\theta : \tilde{M} \rightarrow M\) be the averaging operator valued weight for \(\theta\). Then \(\hat{\varphi} = \varphi \circ T_\theta\). Since \(\theta\) commutes with \(\tilde{\alpha}\), we have

\[[D\hat{\varphi} \circ \Phi^\alpha : D\text{tr}_\pi]\t = [D\varphi \circ \Phi^\alpha \circ (T_\theta \otimes \text{id}) : D\varphi T_\theta \otimes \text{tr}_\pi]\t = [D\varphi \circ \Phi^\alpha : D\varphi \otimes \text{tr}_\pi]\t.

This implies

\[[D\tau \circ \Phi^\alpha : D\tau \otimes \text{tr}_\pi]\t = [D\tau \circ \Phi^\alpha : D\hat{\varphi} \circ \Phi^\alpha]\t[D\hat{\varphi} \circ \Phi^\alpha : D\varphi \otimes \text{tr}_\pi][D\tau \otimes \text{tr}_\pi]_t\]
\[= \tilde{\alpha}_\pi(h^{-it})[D\varphi \circ \Phi^\alpha : D\varphi \otimes \text{tr}_\pi]\t(h^{it} \otimes 1) = 1\].

\[\square\]

Since \(\tilde{\alpha}\) commutes with \(\theta\), \(\tilde{\alpha}\) extends to an action on \(\tilde{M} \rtimes_\theta \mathbb{R}\). We call it the second canonical extension and denote that by \(\tilde{\alpha}\).

**Corollary 7.15.** Let \(\alpha \in \text{Mor}(M, M \otimes L^\infty(\hat{G}))\) be a standard action. The second canonical extension \(\tilde{\alpha}\) is cocycle conjugate to \(\text{id}_{B(L^2(\mathbb{R}))} \otimes \alpha\).

**Proof.** Let \(\varphi\) be a faithful normal semifinite weight on \(M\). We regard \(\tilde{M} = M \rtimes_{\sigma_\varphi} \mathbb{R}\). Define \(w(\cdot) \in U(L^\infty(\mathbb{R}) \otimes M \otimes L^\infty(\hat{G}))\) by \(w(t) = w_{-t}\) for \(t \in \mathbb{R}\). Then \(w\) is an idempotent in \(\tilde{M} \rtimes_\theta \mathbb{R} \cong B(L^2(\mathbb{R})) \otimes M\) intertwining the actions \(\tilde{\alpha}\) and \(\text{Ad} u \circ (\text{id} \otimes \alpha)\).
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