Spin $\frac{1}{2}$ Field Theory in the de Sitter space-time

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Abstract

A covariant quantization of the free spinor fields ($s = \frac{1}{2}$) in 4-dimensional de Sitter (dS) space-time based on analyticity in the complexified pseudo-Riemannian manifold is presented. We define the Wightman two-point function $W(x, y)$, which satisfies the conditions of: a) positivity, b) locality, c) covariance, and d) normal analyticity. Then the Hilbert space structure and the field operators $\psi(f)$ are defined. A coordinate-independent formula for the unsmeared field operator $\psi(x)$ is also given.

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1 Introduction

In general on curved space-time, no true spectral condition can be satisfied by Quantum Field Theory (QFT) and no unique vacum state exists. In the case of de Sitter space-time, it has been discovered that the Hadamard condition selects an unique vacuum state [1]. The Hadamard condition is related to normal analyticity, that is to say, the two-point function is the boundary value of an analytic function. One can replace the usual spectral condition by a certain geodesic spectral condition (or KMS condition), and one can consider the generalized free fields on dS space-time. The generalized free fields can be defined entirely in terms of Wightman two-point function. In this context we present local free spinor fields ($s=\frac{1}{2}$) in 4-dimensional dS space-time based on analyticity in the complexified pseudo-Riemannian manifold. First we derive the dS-Dirac field equation as an eigenvalue equation for the Casimir operator and we find the solutions in terms of coordinate-independent dS plane-waves in tube domains. We define the two-point function $W(z_1, z_2)$ in terms of spinor dS plane-waves in their tube domains. Normal analyticity allows one to define Wightman two-point function $W(x, y)$ as the boundary value of $W(z_1, z_2)$ from the tube domains. Then the Hilbert space structure and the field operators $\psi(f)$

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are defined. Finally, the unsmeared field operator $\psi(x)$ in terms of a coordinate-independent dS plane waves is also defined. This work is in the continuation of the previous ones concerning the scalar case [2, 3].

2 Notation

De Sitter space-time is visualized as the hyperboloid with equation:

$$X_R = \{ x^\alpha \in \mathbb{R}^5 : x.x = \eta_{\alpha\beta}x^\alpha x^\beta = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = -R^2 \}$$

$$\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1); \quad \alpha, \beta = 0, 1, ..., 4. \quad (1)$$

Let us define the punctured half-con with:

$$C = \{ \xi^\alpha \in \mathbb{R}^5 : \eta_{\alpha\beta}\xi^\alpha \xi^\beta = 0 \}.$$ The kinematical group of the de Sitter space-time is $G_R = SO_0(1, 4)$ and the double (and universal) covering group of $G_R$ is given in a quaternionic realisation by [4]

$$Sp(2, 2) = \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{det}g = 1, \quad \gamma^0 \tilde{g}^t \gamma^0 = g^{-1}; \quad a, b, c, d \in \mathbb{H} \}, \quad (2)$$

$g^t$ denotes the $2 \times 2$ transpose of $g$, $\tilde{g}$ denotes its quaternionic conjugate, and $\text{det}g$ is the determinant of $g$ viewed as a $4 \times 4$ matrix with complex entries. Now we need five $\gamma$ matrices instead of the usual four ones in Minkowski space-time. They are defined by the Clifford algebra:

$$\{\gamma^\alpha, \gamma^\beta\} = \gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta}, \quad \gamma^\dagger = \gamma^0 \gamma^\alpha \gamma^0.$$ The quaternion representation of the $\gamma$ matrices is given by

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} \quad (3)$$

in terms of the $2 \times 2$ unit $\mathbb{I}$ and Pauli matrices $\sigma^i$. Casimir operator and infinitesimal generators read

$$Q = -\frac{1}{2}L_{\alpha\beta}L^{\alpha\beta}, \quad L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta} = -i(x_\alpha \tilde{\partial}_\beta - x_\beta \tilde{\partial}_\alpha) - \frac{i}{4}[\gamma_\alpha, \gamma_\beta], \quad (4)$$

where $\tilde{\partial}$ is the tangential derivative: $\tilde{\partial}_\beta = \partial_\beta + H^2 x_\beta x_\partial$.

3 dS-Dirac field equation and plane-waves solution

Starting from the Casimir operator and using the infinitesimal generators and the Casimir eigenvalue equation $Q\psi(x) = (\nu^2 + \frac{3}{2})\psi(x)$ [3] give

$$Q\psi(x) = \left\{ (\frac{1}{2}\gamma_\alpha \gamma_\beta M_{\alpha\beta} + 2i)^2 + \frac{3}{2} \right\} \psi(x) = (\nu^2 + \frac{3}{2})\psi(x), \quad (5)$$

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where $\psi(x)$ is the 4-component spinor wave function and $\nu \in \mathbb{R}$. A possible spinorial solution $\psi(x)$ to the above equation is afforded by the first-order equation \[7\]:

$$(-i \slashed{\partial} + 2i - \nu)\psi(x) = 0, \quad \text{(dS Dirac field equation)}$$

where $\slashed{\partial} = x_{\gamma}$ in usual notations. For large $R$ behaviour ($R \to \infty$) we obtain the Dirac field equation in Minkowskian space. We know that any field quantity living on de Sitter space-time $X_R$ can be viewed as an homogenous function of the $\mathbb{R}^5$-variable $x^\alpha$ with some arbitrarily chosen degree $\sigma$. Let us choose a solution to the dS-Dirac equation of the type \[7\]

$$\psi(x) = (\frac{1}{2} \gamma^\alpha \gamma^\beta M_{\alpha\beta} + i + \nu)\phi(x) U_T,$$

where $\phi(x)$ is a scalar field with degree $\sigma$ and $U_T$ is an arbitrary four-component spinor. $T$ denotes the “orbital basis” of $C^+$ with respect to a subgroup $L_e$ of $G_R$ which is the stabilizer of an unit vector $e = (1, 0, 0, 0)$ in $\mathbb{R}^5$ \[3\]. We have two solutions for $\psi(x)$ \[7\]:

$$\psi_1(x) = (\frac{x_{\xi}}{R})^{-2+iv} \frac{\slashed{\partial} \xi}{\sqrt{2(\xi^0 + 1)}} U_T(\xi^0) = (\frac{x_{\xi}}{R})^{-2+iv} V(x, \xi),$$

$$\psi_2(x) = (\frac{x_{\xi}}{R})^{-2-iv} \frac{\xi \gamma^4}{\sqrt{2(\xi^0 + 1)}} U_T(\xi^0) = (\frac{x_{\xi}}{R})^{-2-iv} U(\xi).$$

We now consider the transformation of the Dirac free field $\psi(x)$ in such a way that the transformed $\psi'(x')$ obeys the same dS-Dirac equation in the new frame. This lead to the simple relation $\psi'(x') = g \psi(\Lambda^{-1}(g)x)$ where $g \in Sp(2, 2)$ is viewed as a $4 \times 4$ matrix when acting on the 4-spinor $\psi(x)$ and $\Lambda \in SO(1, 4)$. We have

$$\psi_1(x') = g \psi_1(x) \Lambda = \psi_2(x) \Lambda.$$

We see how the dS action is transfered onto the "reciprocal space" $C^+$ to which the parameter $\xi$ belongs, and the 4-component arbitrary spinor $U_T$. The plane-wave solutions to the free Dirac equation in Minkowskian space are given by the large-R behavior of the plane-wave solutions to the dS-Dirac field equation \[7\]. For obtaining general field solutions, we consider the solution in the complexified dS space-time $X_R^{(c)}$

$$X_R^{(c)} = \{ z = x + iy \in \mathbb{C}^5; \quad \eta_{\alpha\beta} z^\alpha z^\beta = (z^0)^2 - \bar{z} \cdot z - (z^4)^2 = -R^2 \}$$

$$= \{(x, y) \in \mathbb{R}^5 \times \mathbb{R}^5; \quad x^2 - y^2 = -R^2, \quad x \cdot y = 0 \}.$$  \[10\]

Let $T^\pm = \mathbb{R}^5 + iV^\pm$ be the forward and backward tubes in $\mathbb{C}^5$. $V^+$ (resp. $V^-$) stems from the causal structure on $X_R$, $V^\pm = \{ x \in \mathbb{R}^5; \quad x^0 > \sqrt{\| \bar{x} \|^2 + (x^4)^2} \}$. \[11\]

We then introduce their respective intersections with $X_R^{(c)}$, $T^\pm = T^\pm \cap X_R^{(c)}$, which will be called forward and backward tubes in the complex dS space-time $X_R^{(c)}$. Finally we define the set $T_{12} = \{(z_1, z_2); \quad z_1 \in T^+, \quad z_2 \in T^- \}$.
as a tube above \( X_R \times X_R \) in \( X_R^{(c)} \times X_R^{(c)} \). Details are given in [3]. When \( z \) varies in \( T^+ \) (or \( T^- \)) and \( \xi \) lies in the positive cone \( C^+ \), the plane wave solutions \( \psi_{\xi,\nu}(z) = (z_\xi R)^{\nu}U(z, \xi) \), \( \sigma \in \mathbb{C} \) are globally defined because the imaginary part of \((z,\xi)\) has a fixed sign. Now we can define the Wightman two-point function.

## 4 Two point function

Let us briefly recall the conditions we require on the Wightman two-point function

\[
\mathcal{W}(x, y) = \langle \Omega, \psi(x)\bar{\psi}(y)\Omega \rangle,
\]

where \( x, y \in X_R \) and \( \bar{\psi} = \psi^\dagger \gamma^0 \gamma^4 \), is the spinor field conjugate to \( \psi \). This function is \( 4 \times 4 \) matrix-valued in the present case, and has to satisfy the following requirements:

a) **Positivity**

for any test function \( f \in \mathcal{D}(X_R) \) with values in \( \mathbb{C}^4 \)

\[
\int_{X_R \times X_R} \bar{f}(x)\mathcal{W}(x, y)f(y)d\sigma(x)d\sigma(y) \geq 0, \tag{12}
\]

where \( d\sigma(x) \) denotes the dS-invariant measure on \( X_R \).

b) **Locality**

for every space-like separated pair \((x, y)\), i.e. \( x \cdot y > -R^2 \),

\[
\mathcal{W}_{ij}(x, y) = -\mathcal{W}_{ji}(y, x), \tag{13}
\]

where \( \mathcal{W}_{ji}(y, x) = \langle \Omega, \bar{\psi}_j(y)\psi_i(x)\Omega \rangle \).

c) **Covariance**

\[
g\mathcal{W}(\Lambda^{-1}(g)x, \Lambda^{-1}(g)y)i(g^{-1}) = \mathcal{W}(x, y), \tag{14}
\]

where \( i(g^{-1}) = -\gamma^4g^{-1}\gamma^4 \)[3],

d) **Normal analyticity**

\( \mathcal{W}(x, y) \) is the boundary value (in the sense of distributions) of a function \( W(z_1, z_2) \) which is analytic in the domain \( T_{12} \).

The two-point \( W^\nu(z_1, z_2) \), labelled by the principal-series parameter \( \nu \), is given by the following class of integral representations [3]

\[
W^\nu_{ij}(z_1, z_2) = c_{\nu} \int_T (z_1, \xi)^{-2-4\nu}(z_2, \xi)^{-2+4\nu} \sum_{a=1,2} U^a_i(\xi)\bar{U}^a_j(\xi)d\mu_T(\xi),
\]

and the two-point function \( W^\nu_{ji}(z_2, z_1) \) is given by the following class of integral representations

\[
W^\nu_{ji}(z_2, z_1) = H^2c_{\nu} \int_T (z_1, \xi)^{-2+4\nu}(z_2, \xi)^{-2-4\nu} \sum_{a=1,2} V^a_i(z_1, \xi)\bar{V}^a_j(z_2, \xi)d\mu_T(\xi).
\]
$d\mu_T(\xi)$ is a measure invariant under $L_e$. We can write

$$W^\nu(z_1, z_2) = D(z_2)\gamma^A\mathcal{N}(z_1, z_2), \quad D(z_2) = \frac{1}{\nu + i}(-i \not{\partial} z_2 \partial z_2 + i + \nu).$$

$\mathcal{N}(z_1, z_2)$ is a scalar two-point function

$$\mathcal{N}(z_1, z_2) = c_\nu \int_T (z_1, \xi)^{-2- i\nu} (z_2, \xi)^{-1+ i\nu} d\mu_T(\xi). \tag{15}$$

In terms of the generalized Legendre function of the first kind we have [5, 6]

$$W^\nu_{ij}(z_1, z_2) = C_\nu (D(z_2)\gamma^4)_{ij} D^{(5)}_{-1+ i\nu} \frac{(z_1, z_2)}{R^2} = 4C_\nu (D(z_2)\gamma^4)_{ij} (1 - \frac{z_1, z_2}{R^2})^{-\frac{1}{2}} \mathcal{P}_{i\nu}^{-1}\frac{(z_1, z_2)}{R^2} \tag{16}$$

The boundary value of $W^\nu_{ij}(z_1, z_2)$ provides us with the following representation for the Wightman two-point function [7]:

$$\mathcal{W}(x, y) = c_\nu \int_T [(x, \xi)^{-2+i\nu} + e^{i\pi(-2-i\nu)}(x, \xi)^{-2-i\nu}$$

$$[(y, \xi)^{-2+i\nu} + e^{-i\pi(-2+i\nu)}(y, \xi)^{-2-i\nu}] \tilde{\xi} \gamma^A d\mu_T. \tag{17}$$

This function satisfies the conditions of: a) positivity, b) locality, c) covariance, and d) normal analyticity [7]. The existence of $\mathcal{W}$ allows one to make the QF formalism work [8]. The spinor fields $\psi(x)$ are expected to be operator-valued distributions on $X_R$ acting on a Hilbert space $\mathcal{H}$. The Hilbert space $\mathcal{H}$ of the representation can be described as the Hilbertian sum

$$\mathcal{H} = \mathcal{H}_0 \bigoplus_{n=1}^{\infty} A\mathcal{H}_1 \otimes^n.$$

$A$ denotes the antisymmetrisation operation and $\mathcal{H}_0 = \{\lambda \Omega, \quad \lambda \in \mathbb{C}\}$ where the vector $\Omega$, cyclic for the polynomial algebra of field operators and invariant under the representation of $G_R$, is “the vacuum”. $\mathcal{H}_1$ is defined by the scalar product

$$(h_1, h_2) = \int_{X_R \times X_R} \bar{h}_1(x) \mathcal{W}(x, y) h_2(y) d\sigma(x) d\sigma(y) \geq 0, \tag{18}$$

where $h \in \mathcal{D}(X_{\mathcal{H}})$ with values in $\mathbb{C}^4$. Each field operator $\psi(f)$ can be defined in terms of annihilation and creation operators by [8]:

$$\psi(f) h^{(n)}(i_1, x_1; i_2, x_2; ...; i_n, x_n) = \sqrt{n+1} \int_{X_R \times X_R} f^i(x) \mathcal{W}_{ji}(y, x) h^{(n+1)}(i, y; i_1, x_1; ...; i_n, x_n) d\sigma(x) d\sigma(y)$$

$$+ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-1)^{k+1} f_{ik}(x_k) h^{(n-1)}(i_1, x_1; ...; \hat{i}_k, x_k; ...; i_n, x_n), \tag{19}$$
where the $i_k = 1, 2, 3, 4$, are spin indices. Here $\hat{\psi}_k$ means omit it. By using the Fourier-Bros transformation on $X_R$, we can write the unsmeared field operators $\psi(x)$

$$
\psi(x) = \int \sum_{a=1,2} \{ a_a(\xi, \nu) U^a(\xi)[(x,\xi)_+^{2-i\nu} + e^{i\pi(2-i\nu)}(x,\xi)_-^{2-i\nu}] \\
+ d^a_a(\xi, \nu)|\nu^a(x, \xi)|(x,\xi)_+^{2+i\nu} + e^{-i\pi(2+i\nu)}(x,\xi)_-^{2+i\nu}] \} d\mu_T(\xi),
$$

(20)

where $a_a(\xi, \nu)$ and $d_a(\xi, \nu)$ are defined by $a_a(\xi, \nu) |\Omega> = 0 = d_a(\xi, \nu) |\Omega>$. The field anticommutator is given by

$$
\{\psi_i(x), \bar{\psi}_j(y)\} = W_{ij}(x, y) + W_{ji}(y, x).
$$

It can be easily checked that for space-like separated points $(x, y)$ we have $\{\psi_i(x), \bar{\psi}_j(y)\} = 0$.

The integral representation for the two-point function is defined on the space which carries the principal series of the dS group $Sp(2, 2)/\mathbb{Z}_2 \approx SO_0(1, 4)$. In the limit $R \to \infty$, we obtained

$$
\lim_{R \to \infty} \{\psi(x), \bar{\psi}(y)\} = \frac{1}{2(2\pi)^3} \int \{e^{-ik.(X-Y)}(k\gamma^4 + m) + e^{ik.(X-Y)}(k\gamma^4 - m)\} \frac{d^3k}{k^0},
$$

(21)

which is of the same form of the Minkowskian space.

### 5 Conclusion and outlook

Using the dS-plane waves in tube domains for spinor field, one can construct an analytic function $W(z_1, z_2)$ that its boundary value is the Wightman two-point function $W(x, y)$. Then the dS-plane waves allow us to construct the quantum field on dS space in the same way as the quantum field on Minkowski space. In the case of the massless spinor field, we must replace $\nu$ with 0 in the Wightman two-point function as well as in the field operator $\psi(x)$. In this case the corresponding UIR is known as the first term of the spinor discrete series of representation, which is written as $\Pi_{1/2}^{\pm}$ in [4]. The $\pm$ define the helicity of the massless spinor field. We now intend to use these methods to construct a covariant quantum fields with spin-1 and spin-2. In the case of the massive field the procedure is the same as spin-$1/2$. In the case of the massless field, we must use the Gupta-Bleuler quantization for obtaining a fully covariant theory. At this moment, this kind of work is in progress on the ”massless” representations of dS group and the related QFT.

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