JACOBI TRANSFORM OF \((\nu, \gamma, p)\)-JACOBI–LIPSCHITZ FUNCTIONS IN THE SPACE \(L^p(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)\)

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Abstract: Using a generalized translation operator, we obtain an analog of Younis' theorem [Theorem 5.2, Younis M.S. Fourier transforms of Dini–Lipschitz functions, Int. J. Math. Math. Sci., 1986] for the Jacobi transform for functions from the \((\nu, \gamma, p)\)-Jacobi–Lipschitz class in the space \(L^p(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)\).

Keywords: Jacobi operator, Jacobi transform, Generalized translation operator.

1. Introduction and preliminaries

Younis [8, Theorem 5.2] characterized the set of functions in \(L^2(\mathbb{R})\) satisfying the Dini–Lipschitz condition by means of an asymptotic estimate of the growth of the norm of their Fourier transforms.

\textbf{Theorem 1.} [8, Theorem 5.2] Let \(f \in L^2(\mathbb{R})\). Then the following conditions are equivalent:

\begin{enumerate}
  \item \(\|f(\cdot + h) - f(\cdot)\|_{L^2(\mathbb{R})} = O\left(\frac{h^\alpha}{(\log 1/h)^\beta}\right)\) as \(h \to 0\), \(0 \leq \alpha < 1\), \(\beta > 0\),
  \item \(\int_{|\lambda| \geq r} |\mathcal{F}(f)(\lambda)|^2 d\lambda = O\left(r^{-2\alpha}(\log r)^{-2\beta}\right)\) as \(r \to +\infty\),
\end{enumerate}

where \(\mathcal{F}\) stands for the Fourier transform of \(f\).

The main aim of this paper is to establish an analog of Theorem 1 for the Jacobi transform in the space \(L^p(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)\). For this purpose, we use a generalized translation operator which was defined by Flensted–Jensen and Koornwinder [5].

In order to confirm the basic and standard notation, we briefly overview the theory of Jacobi operators and related harmonic analysis. The main references are \([1, 4, 6]\).

Let \(\lambda \in \mathbb{C}\), \(\alpha \geq \beta \geq -1/2\), and \(\alpha \neq 0\). The Jacobi function \(\phi_\lambda\) of order \((\alpha, \beta)\) is the unique even \(C^\infty\)-solution of the differential equation

\[(D_{\alpha,\beta} + \lambda^2 + \rho^2)u = 0, \quad u(0) = 1, \quad u'(0) = 0,
\]

where \(\rho = \alpha + \beta + 1\), \(D_{\alpha,\beta}\) is the Jacobi differential operator defined as

\[D_{\alpha,\beta} = \frac{d^2}{dx^2} + \left(\frac{\Delta'_{(\alpha,\beta)}(x)}{\Delta_{(\alpha,\beta)}(x)}\right) \frac{d}{dx}\]

\(^1\)Dedicated to Professor Radouan Daher for his 61’s birthday.
\[
\Delta_{(\alpha, \beta)}(x) = (2 \sinh x)^{2\alpha + 1}(2 \cosh x)^{2\beta + 1},
\]
and \(\Delta'_{(\alpha, \beta)}(x)\) is the derivative of \(\Delta_{(\alpha, \beta)}(x)\).

The Jacobi functions \(\phi_\lambda\) can be expressed in terms of Gaussian hypergeometric functions as
\[
\phi_\lambda(x) = \phi^{(\alpha, \beta)}_\lambda(x) = F\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha + 1, -\sinh^2 x\right),
\]
where the Gaussian hypergeometric function is defined as
\[
F(a, b, c, z) = \sum_{m=0}^{\infty} \frac{a_m b_m}{c_m m!} z^m, \quad |z| < 1,
\]
with \(a, b, z \in \mathbb{C}, c \notin -\mathbb{N}, a_0 = 1, \) and \(a_m = a(a+1)\cdots(a+m-1)\).

The function \(z \rightarrow F(a, b, c, z)\) is the unique solution of the differential equation
\[
z(1-z)u''(z) + (c - (a+b+1)z)u'(z) - abu(z) = 0,
\]
which is regular at 0 and equals 1 there.

From [7, Lemmas 3.1–3.3], we obtain the following statement.

**Lemma 1.** The following inequalities are valid for a Jacobi function \(\phi_\lambda(t) \ (\lambda, t \in \mathbb{R}^+):\)

(1) \(|\phi_\lambda(t)| \leq 1;\)

(2) \(|1 - \phi_\lambda(t)| \leq t^2(\lambda^2 + \rho^2);\)

(3) there is a constant \(d > 0\) such that
\[
1 - \phi_\lambda(t) \geq d \quad \text{for} \quad \lambda t \geq 1.
\]

Let \(L^p_{\alpha, \beta}([0, \infty)) = L^p([0, \infty), \Delta_{(\alpha, \beta)}(t) dt), 1 \leq p \leq 2,\) be the space of \(p\)-power integrable functions on \([0, \infty)\) endowed with the norm
\[
\|f\|_p = \left(\int_0^\infty |f(x)|^p \Delta_{(\alpha, \beta)}(x) dx\right)^{1/p} < \infty.
\]

Let \(L^p_{\mu}([0, \infty)) = L^p([0, \infty), d\mu(\lambda)/2\pi), 1 \leq p \leq 2,\) be the space of measurable functions \(f\) on \([0, \infty)\) such that
\[
\|f\|_p_{\mu} = \left(\frac{1}{2\pi} \int_0^{\infty} |f(x)|^p d\mu(\lambda)\right)^{1/p},
\]
where \(d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda\) and the \(c\)-function \(c(\lambda)\) is defined as
\[
c(\lambda) = \frac{2\rho^{-i\lambda} \Gamma(\alpha + 1) \Gamma(i\lambda)}{\Gamma(1/2 \cdot (i\lambda + \alpha + \beta + 1)) \Gamma(1/2 \cdot (i\lambda + \alpha - \beta + 1))}.
\]

Now, we define the Jacobi transform
\[
\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x) \Delta_{(\alpha, \beta)}(x) dx,
\]
for all functions \(f\) on \([0, \infty)\) and complex numbers \(\lambda\) for which the right-hand side is well defined.

The Jacobi transform reduces to the Fourier transform when \(\alpha = \beta = -1/2.\)

We have the following inversion formula [6].
Theorem 2. If $f \in L^p_{a, \beta}(\mathbb{R}^+)$, then
\[
f(x) = \frac{1}{2\pi} \int_0^\infty \hat{f}(\lambda) \phi_\lambda(x) d\mu(\lambda).
\]

From [3], we have the Hausdorff–Young inequality
\[
\|\hat{f}\|_{q,\mu} \leq C_2 \|f\|_p \quad \text{for all } f \in L^p_{a, \beta}(\mathbb{R}^+),
\]
where $1/p + 1/q = 1$ and $C_2$ is a positive constant.

The generalized translation operator $T_h$ of a function $f \in L^p_{a, \beta}(\mathbb{R}^+)$ is defined as
\[
T_h f(x) = \int_0^\infty f(z) K(x, h, z) \Delta_{(\alpha, \beta)}(z) dz,
\]
where $K$ is an explicitly known kernel function such that
\[
K(x, y, z) = 2^{-2\nu} \frac{\Gamma(\alpha + 1)(\cosh x \cosh y \cosh z)^{\alpha - \beta - 1}}{\Gamma(1/2)\Gamma(\alpha + 1/2)(\sinh x \sinh y \sinh z)^{2\alpha}} (1 - B^2)^{\alpha - 1/2}
\times F \left( \alpha + \beta, \alpha - \beta, \alpha + \frac{1}{2}, \frac{1}{2}(1 - B) \right) \quad \text{for } |x - y| < z < x + y,
\]
and $K(x, y, z) = 0$ elsewhere and
\[
B = \frac{\cosh^2 x + \cosh^2 y + \cosh^2 z - 1}{2 \cosh x \cosh y \cosh z}.
\]

From [2], we have
\[
(\hat{T}_h f)(\lambda) = \phi_\lambda(h) \hat{f}(\lambda).
\]

2. Main results

In this section, we give the main result of this paper. We need first to define the $(\nu, \gamma, p)$-Jacobi–Lipschitz class.

Definition 1. Let $\nu, \gamma > 0$. A function $f \in L^p_{a, \beta}(\mathbb{R}^+)$ is said to be in the $(\nu, \gamma, p)$-Jacobi–Lipschitz class, denoted by $\text{Lip}(\nu, \gamma, p)$, if
\[
\|T_h f(x) - f(x)\|_p = O \left( \frac{h^\nu}{(\log 1/h)^\gamma} \right) \quad \text{as } h \to 0.
\]

Theorem 3. Let $f$ belong to $\text{Lip}(\nu, \gamma, p)$. Then
\[
\int_N^{+\infty} |\hat{f}(\lambda)|^q d\mu(\lambda) = O(N^{-\nu q}(\log N)^{-\gamma q}) \quad \text{as } N \to +\infty.
\]

Proof. Let $f \in \text{Lip}(\nu, \gamma, p)$. Then we have
\[
\|T_h f(x) - f(x)\|_p = O \left( \frac{h^\nu}{(\log 1/h)^\gamma} \right) \quad \text{as } h \to 0.
\]

Therefore,
\[
\int_0^{+\infty} |1 - \phi_\lambda(h)|^q |\hat{f}(\lambda)|^q d\mu(\lambda) \leq C_2^q \|T_h f(x) - f(x)\|_p^q.
\]
If \( \lambda \in [1/h, 2/h] \), then \( \lambda h \geq 1 \) and inequality (3) of Lemma 1 implies that

\[
1 \leq \frac{1}{d^{qk}} |1 - \phi_{\lambda}(h)|^{qk}.
\]

Then

\[
\int_{1/h}^{2/h} |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda) \leq \frac{1}{d^{qk}} \int_{1/h}^{2/h} |1 - \phi_{\lambda}(h)|^{qk} |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda)
\]

\[
\leq \frac{1}{d^{qk}} \int_{0}^{+\infty} |1 - \phi_{\lambda}(h)|^{qk} |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda) \leq \frac{1}{d^{qk}} C_{2}^{q} \|T_{h} f(x) - f(x)\|_{p}^{p} = O\left(\frac{h^{q\nu}}{(\log 1/h)^{q\gamma}}\right).
\]

Then

\[
\int_{N}^{2N} |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda) = O\left(N^{-q\nu}(\log N)^{-q\gamma}\right) \quad \text{as} \quad N \to +\infty.
\]

Thus, there exists \( C_{4} \) such that

\[
\int_{N}^{2N} |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda) \leq C_{4} N^{-q\nu}(\log N)^{-q\gamma}.
\]

Furthermore, we have

\[
\int_{N}^{+\infty} |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda) = \left[\int_{N}^{2N} + \int_{2N}^{4N} + \int_{4N}^{8N} + \ldots\right] |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda)
\]

\[
\leq C_{4} N^{-q\nu}(\log N)^{-q\gamma} + C_{4}(2N)^{-q\nu}(\log 2N)^{-q\gamma} + C_{4}(4N)^{-q\nu}(\log 4N)^{-q\gamma} + \ldots
\]

\[
\leq C_{4} N^{-q\nu}(\log N)^{-q\gamma}(1 + 2^{-q\nu} + (2^{-q\nu})^{2} + (2^{-q\nu})^{3} + \ldots)
\]

\[
\leq C_{k} N^{-q\nu}(\log N)^{-q\gamma},
\]

where \( C_{k} = (1 - 2^{-q\nu})^{-1} \) since \( 2^{-q\nu} < 1 \).

This proves that

\[
\int_{N}^{+\infty} |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda) = O\left(N^{-q\nu}(\log N)^{-q\gamma}\right) \quad \text{as} \quad N \to +\infty,
\]

and this completes the proof. \( \square \)

**Definition 2.** A function \( f \in L^{p}_{\alpha, \beta}(\mathbb{R}^{+}) \) is said to be in the \((\psi, p)\)-Jacobi–Lipschitz class, denoted by \( \text{Lip}(\psi, p) \), if

\[
\|T_{h} f(x) - f(x)\|_{p} = O\left(\frac{\psi(h)}{(\log 1/h)^{\gamma}}\right), \quad \gamma > 0, \quad \text{as} \quad h \to 0,
\]

where

1. \( \psi(t) \) is a continuous increasing function on \([0, \infty)\);
2. \( \psi(0) = 0 \);
3. \( \psi(ts) \leq \psi(t)\psi(s) \) for all \( s, t \in [0, \infty) \).

**Theorem 4.** Let \( f \in L^{p}_{\alpha, \beta}(\mathbb{R}^{+}) \), \( \psi \) be a fixed function satisfying the conditions of Definition 2, and let \( f(x) \) belong to \( \text{Lip}(\psi, p) \). Then

\[
\int_{N}^{+\infty} |\hat{\lambda}(\lambda)|^{q} d\mu(\lambda) = O(\psi(N^{-q})(\log N)^{-q\gamma}) \quad \text{as} \quad r \to +\infty.
\]
Proof. Let \( f \in \text{Lip}(\psi, p) \). Then we have
\[
\|T_h f(x) - f(x)\|_p = O \left( \frac{\psi(h)}{(\log 1/h)^\gamma} \right) \quad \text{as} \quad h \to 0
\]
and
\[
\int_0^{+\infty} |1 - \phi_\lambda(h)|^q |\hat{f}(\lambda)|^q d\mu(\lambda) \leq C_2^q \|T_h f(x) - f(x)\|_p^q.
\]

If \( \lambda \in [1/h, 2/h] \), then \( \lambda h \geq 1 \) and, similarly to the proof of Theorem 3, by inequality (3) of Lemma 1, we obtain
\[
1 \leq \frac{1}{d^q k} |1 - \phi_\lambda(h)|^{q k}.
\]
Then
\[
\int_{1/h}^{2/h} |\hat{f}(\lambda)|^q d\mu(\lambda) \leq \frac{1}{d^q k} \int_{1/h}^{2/h} |1 - \phi_\lambda(h)|^{q k} |\hat{f}(\lambda)|^q d\mu(\lambda)
\leq \frac{1}{d^q k} C_2^q \|T_h f(x) - f(x)\|_p^q = O \left( \frac{\psi(h^q)}{(\log 1/h)^{\gamma q}} \right).
\]

There exists a positive constant \( C_5 \) such that
\[
\int_{N}^{2N} |\hat{f}(\lambda)|^q d\mu(\lambda) \leq C_5 \frac{\psi(N^{-q})}{(\log N)^{\gamma q}}.
\]
Thus,
\[
\int_{N}^{+\infty} |\hat{f}(\lambda)|^q d\mu(\lambda) = \left[ \int_{N}^{2N} + \int_{2N}^{4N} + \int_{4N}^{8N} + \ldots \right] |\hat{f}(\lambda)|^q d\mu(\lambda)
\leq C_5 \frac{\psi(N^{-q})}{(\log N)^{\gamma q}} + C_5 \frac{\psi((2N)^{-q})}{(\log 2N)^{\gamma q}} + C_5 \frac{\psi((4N)^{-q})}{(\log 4N)^{\gamma q}} + \ldots
\leq C_5 \frac{\psi(N^{-q})}{(\log N)^{\gamma q}} + C_5 \frac{\psi((2N)^{-q})}{(\log 2N)^{\gamma q}} + C_5 \frac{\psi((4N)^{-q})}{(\log 4N)^{\gamma q}} + \ldots
\leq C_5 \frac{\psi(N^{-q})}{(\log N)^{\gamma q}} (1 + \psi(2^{-q}) + (\psi(2^{-q}))^2 + (\psi(2^{-q}))^3 + \ldots
\leq C_5 K_1 \frac{\psi(N^{-q})}{(\log N)^{\gamma q}},
\]
where \( K_1 = (1 - \psi(2^{-q}))^{-1} \) since (1) and (3) from Definition 2 imply that \( \psi(2^{-q}) < 1 \).

This proves that
\[
\int_{N}^{+\infty} |\hat{f}(\lambda)|^q d\mu(\lambda) = O \left( \psi(N^{-q})(\log N)^{-\gamma q} \right) \quad \text{as} \quad N \to +\infty,
\]
and this completes the proof.

Acknowledgements

The authors would like to thank the referee for his valuable comments and suggestions.
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