Reducibility of Parameter Ideals in Low Powers of the Maximal Ideal

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To Roger and Sylvia Wiegand in celebration of their combined 150th birthday.

Abstract. A commutative noetherian local ring \((R, m)\) is Gorenstein if and only if every parameter ideal of \(R\) is irreducible. Although irreducible parameter ideals may exist in non-Gorenstein rings, Marley, Rogers, and Sakurai show there exists an integer \(\ell\) (depending on \(R\)) such that \(R\) is Gorenstein if and only if there exists an irreducible parameter ideal contained in \(m^\ell\). We give upper bounds for \(\ell\) that depend primarily on the existence of certain systems of parameters in low powers of the maximal ideal.

1. Introduction

Let \((R, m, k)\) be a commutative noetherian local ring of dimension \(\dim R = d\). It is known that \(R\) is Gorenstein if and only if every parameter ideal is irreducible, but we cannot characterize Gorenstein rings by the existence of an irreducible parameter ideal. For example, the non-Gorenstein ring \(\mathbb{Q}[x, y]/(x^2, xy)\) has an irreducible parameter ideal \((y)\), although \((y^j)\) is reducible for \(j \geq 2\). Marley, Rogers, and Sakurai show [8], however, that the existence of a parameter ideal in a sufficiently high power of the maximal ideal does characterize Gorenstein rings:

Theorem A (Marley, Rogers, and Sakurai). There exists an integer \(\ell\), depending on \(R\), such that \(R\) is Gorenstein if and only if some parameter ideal contained in \(m^\ell\) is irreducible.

The integer \(\ell\) in Theorem 1, considered previously by Goto and Sakurai [6, Lemma 3.12] may be taken to be the least integer \(i\) such that the canonical map

\[
\Ext^d_R(R/m^i, R) \rightarrow \lim_{\to \gamma} \Ext^d_R(R/m^j, R) \cong H^d_m(R)
\]

(1.1)

becomes surjective after applying the socle functor \(\Hom_R(R/m, -)\). The existence of such an integer is guaranteed as the socle \(\Hom_R(R/m, H^d_m(R))\) is finitely generated, but determining an upper bound for \(\ell\) seems to be somewhat subtle. Indeed, we show in Example 2.7 that for each integer \(i \geq 1\), there exists a ring which requires \(\ell > i\). To understand how deep in the maximal ideal one must go before detecting whether \(R\) is Gorenstein in terms of reducibility of parameter ideals, we consider the problem, posed to the authors by Marley, of finding an upper bound for the integer \(\ell\) in Theorem 1.
For rings of dimension one, we take a direct approach to determine surjectivity of the maps in (1.1) after applying $\text{Hom}_R(R/m, -)$. We show (see Theorem 2.4):

**Theorem B.** Assume $\dim R = 1$ and $k$ is infinite. If $n$ is the least integer such that $m^n = (x)m^{n-1}$ for some parameter $x$ and $m^{n-1} \cap \Gamma_m(R) = 0$, then for $i \geq n$ the canonical map (1.1) becomes surjective after applying $\text{Hom}_R(R/m, -)$.

Thus, in this setting, $R$ is Gorenstein if and only if some parameter ideal contained in $m^n$ is irreducible; see Corollary 2.6. This consequence of Theorem B can also be deduced from work of Rogers [10] and Marley, Rogers, and Sakura [8].

For a ring $R$ with arbitrary dimension $d$ and system of parameters $x_1, ..., x_d$, we instead consider—in place of (1.1)—the least integer $i$ such that the canonical map

$$R/(x_1^i, ..., x_d^i) \xrightarrow{\lim} R/(x_1^i, ..., x_d^i) \cong H^i_m(R)$$

becomes surjective after applying $\text{Hom}_R(R/m, -)$. We focus on the case where $x_1, ..., x_d$ is a $p_s$-standard system of parameters for an integer $s \geq 1$, a variant of the $p$-standard systems of parameters considered by Cuong [3]. These systems of parameters (both $p_s$-standard and $p$-standard) are chosen in a way as to annihilate certain local cohomology modules (see Section 3).

We show in Proposition 3.6 that if $s \geq 2$ and $x_1, ..., x_d$ is a $p_s$-standard system of parameters, then for $i \geq s$ the canonical map (1.2) is a surjection after applying $\text{Hom}_R(R/m, -)$. In particular, we obtain (see Theorem 3.8 and Corollary 3.9):

**Theorem C.** Assume $R$ has a dualizing complex. If $n$ is the least integer such that $m^n \subseteq (x_1^s, ..., x_d^s)$ for a $p_s$-standard system of parameters $x_1, ..., x_d$ with $s \geq 2$, then $R$ is Gorenstein if and only if some parameter ideal contained in $m^n$ is irreducible.

The conditions in Theorems B and C ensure that the bounds given are finite: For the former, $k$ being infinite guarantees that there exists a parameter $x$ such that $m^n = (x)m^{n-1}$. For the latter, the existence of a dualizing complex implies the existence of a $p_s$-standard system of parameters.

* * *

Throughout this paper, let $(R, m, k)$ be a commutative noetherian local ring. Let $\dim R = d$ be the Krull dimension of $R$. We briefly recall a few facts and notation; for additional background, we refer to [1, 2, 7].

For an $R$-module $M$, submodule $N \subseteq M$, and ideal $a \subseteq R$, we consider the submodule $(N :_M a) = \{ y \in M \mid ay \subseteq N \}$ of $M$. If $a = (x)$, just write $(N :_M x)$. Also, $(0 :_M m) \cong \text{Soc} M$, where $\text{Soc} M = \text{Hom}_R(R/m, M)$ is the socle of $M$.

A system of parameters of $R$ is a sequence $x_1, ..., x_d$ such that $m^i \subseteq (x_1, ..., x_d)$ for some integer $i$; an ideal generated by a system of parameters is a parameter ideal. If $M$ is an $R$-module with $\dim M = t$, then a sequence $x_1, ..., x_t$ in $R$ is a system of parameters of $M$ if $M/(x_1, ..., x_t)M$ has finite length; an element $x \in R$ such that $\dim M/xM < \dim M$ is referred to as a parameter of $M$.

For an ideal $a$ of $R$, denote by $\Gamma_a(-)$ the $a$-torsion functor; its right derived functors yield the usual local cohomology functors $H^i_a(-)$ for $i \geq 0$.

2. A BOUND IN DIMENSION ONE

Assume in this section that the ring $(R, m, k)$ has an infinite residue field $k$ and $\dim(R) = 1$. Moreover, we fix the next two invariants throughout this section; the
first is finite because $k$ is infinite\(^1\) [2, Corollary 4.6.10], the second is finite because $\Gamma_m(R)$ has finite length [1, Theorem 7.1.3]:

\[
c = \inf \{ i \mid \text{there exists a parameter } x \text{ such that } m^{i+1} = (x)m^i \};
\]

\[g = \inf \{ i \mid m^i \cap \Gamma_m(R) = 0 \}.
\]

These invariants have been considered elsewhere; $c$ is the reduction number of $m$, and the bound we consider below, $\max\{c, g\} + 1$, is used by Rogers [10, Theorem 2.3]. We begin with two elementary lemmas involving these invariants:

**Lemma 2.2.** Let $x \in R$ be a parameter and $y \in m^g$. If $x^iy = 0$ for some $i \geq 1$, then $y = 0$.

**Proof.** As $(x)$ is a parameter ideal, there exists an integer $j$ such that $m^j \subseteq (x)$ hence $m^{ij} \subseteq (x^i)$ for $i \geq 1$. If $x^iy = 0$, then $m^{ij}y = 0$. It thus follows that $y \in m^g \cap (0 :_R m^j) \subseteq m^g \cap \Gamma_m(R) = 0$, hence $y = 0$. \hfill $\square$

**Lemma 2.3.** Set $n = \max\{c, g\}$. If $\{y_1, \ldots, y_e\}$ is a minimal generating set of $m^n$, then $\{x^iy_1, \ldots, x^iy_e\}$ is a minimal generating set of $m^{n+i}$ for each $i \geq 1$.

**Proof.** Let $\{y_1, \ldots, y_e\}$ be a minimal generating set of $m^n$. As $n \geq c$, the equality $(x)m^n = m^{n+1}$ implies $(x^i)m^n = m^{n+i}$ by induction. Thus $(x^iy_1, \ldots, x^iy_e) = m^{n+i}$. If there exists $r_i \in R$ such that $x^iy_j = \sum_{q \neq j} r_qx^jy_q$ for some $j$, then we have $x^iy_j - \sum_{q \neq j} r_qy_q = 0$. Since $y_j - \sum_{q \neq j} r_qy_q \in m^g$, we have $y_j - \sum_{q \neq j} r_qy_q = 0$ by Lemma 2.2; this contradicts the fact that $y_1, \ldots, y_e$ is a minimal generating set, hence we must have $\{x^iy_1, \ldots, x^iy_e\}$ is a minimal generating set for $m^{n+i}$. \hfill $\square$

**Theorem 2.4.** Assume $k$ is infinite and $\dim(R) = 1$. For $i \geq \max\{c, g\} + 1$, the canonical map

\[
\varphi_i : \Ext^1_R(R/m^i, R) \longrightarrow \lim_{\rightarrow j} \Ext^1_R(R/m^{j}, R) \cong H^1_m(R)
\]

becomes surjective after applying $\text{Soc}(-) = \text{Hom}_R(R/m, -)$.

**Proof.** Let $x$ be a parameter such that $(x)m^c = m^{c+1}$ and set $n = \max\{c, g\}$, and let $\{u_1, \ldots, u_e\}$ be a minimal generating set for $m^n$. By Lemma 2.3, we know $\{x^iu_1, \ldots, x^iu_e\}$ is a minimal generating set for $m^{n+i}$ for $i \geq 0$. We will consider $\Soc\Ext^1_m(R/m^{n+i}, R)$ by examining a projective resolution of $R/m^{n+i}$.

We first show that, for $i \geq 0$, one may choose free resolutions of $R/m^{n+i}$ which agree starting in degree 1. Let $\bar{u} = [u_1 \cdots u_e] : R^e \rightarrow R$. The containment $\ker(\bar{u}) \subseteq \ker(x^i\bar{u})$ is clear, and equality holds because if $\bar{r} \in \ker(x^i\bar{u})$, then $\bar{u}\bar{r} = 0$ by Lemma 2.2. Thus, for $i \geq 0$, there is a commutative diagram with exact rows:

\[
\begin{array}{c}
R^f \xrightarrow{A} R^c \xrightarrow{x} R^e \xrightarrow{x^i\bar{u}} R \longrightarrow R/m^{n+i} \longrightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
R^f \xrightarrow{A} R^c \xrightarrow{x} R^e \xrightarrow{x^i\bar{u}} R \longrightarrow R/m^{n+i} \longrightarrow 0
\end{array}
\]

\(^1\)For the purposes of this paper, the assumption of $k$ being an infinite field may be replaced with the assumption that a parameter $x$ exists so that $m^{i+1} = (x)m^i$ for some integer $i$.\]
Applying $\text{Hom}_R(-, R)$ to this diagram yields a commutative diagram:

\[
\begin{array}{ccc}
R & \xrightarrow{x^{i+1} \hat{u}^T} & R^e \\
\downarrow{=} & \downarrow{x} & \downarrow{A^T} \\
R & \xrightarrow{x^i \hat{u}^T} & R^e \\
\end{array}
\]

Taking cohomology, we obtain that $\text{Ext}^1_R(R/m^{n+i}, R) \cong \ker(A^T)/\text{im}(x^i \hat{u}^T)$ for $i \geq 0$. Set $K = \ker(A^T) \subseteq R^e$, $I_i = \text{im}(x^i \hat{u}^T) \subseteq K$, and identify $\text{Soc} \text{Ext}^1_R(R/m^{n+i}, R) \cong \text{Soc}(K/I_i)$ for $i \geq 0$. Moreover, for $i \geq 0$, the map $\text{Ext}^1_R(R/m^{n+i}, R) \to \text{Ext}^1_R(R/m^{n+i+1}, R)$ is induced by multiplication by $x$, see (2.5), as well as is the induced map after applying $\text{Soc}(-)$. Indeed, for $j \geq 1$, the map $x^j : \text{Soc}(K/I_i) \to \text{Soc}(K/I_{i+j})$ is defined by $\hat{z} + I_j \to x^j \hat{z} + I_{i+j}$.

In order to show that $\text{Soc} \varphi_i$ is surjective for $i \geq n + 1$, where $\varphi_i$ is as in the statement, it will be enough to show that $\text{Soc} \varphi_{n+1}$ is surjective (this follows from the definition of direct systems). Note that for $\bar{v} + I_i \in \text{Soc} K/I_i$, the function $\text{Soc} \varphi_i$ is induced by $\varphi_i$ and hence $(\text{Soc} \varphi_i)(\bar{v} + I_i) = \varphi_i(\bar{v} + I_i)$. Hereafter, we use the latter notation.

Let $\sigma \in \text{lim}_{i \to j} \text{Soc} K/I_j$. As $\text{lim}_{i \to j} \text{Soc} K/I_j \cong \text{Soc} H^d_m(R)$ is finitely generated, $\text{Soc} \varphi_i$ is surjective for $i \gg 0$, thus $\sigma = \varphi_{n+p}(\bar{v} + I_p)$ for some $\bar{v} + I_p \in \text{Soc} K/I_p$ for some $p \geq 1$. If $p = 1$, then $\sigma \in \text{im} \text{Soc} \varphi_{n+1}$ as desired, so assume $p > 1$.

We proceed by descending induction: that is, we aim to show there is an element $\bar{w} + I_{p-1} \in \text{Soc} K/I_{p-1}$ such that

\[
x^j(\bar{v} + I_p) = x^{i+1}(\bar{w} + I_{p-1})
\]

for some $i \geq 1$, and hence $\varphi_{n+p-1}(\bar{w} + I_{p-1}) = \varphi_{n+p}(\bar{v} + I_p)$.

We consider the element $x^g(\bar{v} + I_p) = x^g \bar{v} + I_{p+g}$, recalling that $g$ is the least integer such that $m^g \cap \Gamma_m(R) = 0$. As $x^g \bar{v} + I_{p+g}$ is a socle element, we have:

\[
x(x^g \bar{v} + I_{p+g}) = 0 + I_{p+g}
\]

\[
\implies x^{g+1} \bar{v} \in I_{p+g}
\]

\[
\implies x^{g+1} \bar{v} = ax^{p+g} \bar{u}^T \text{ for some } a \in R, \text{ recalling } I_{p+g} = \text{im}(x^{p+g} \bar{u}^T),
\]

\[
\implies x(x^g \bar{v} - ax^{p+g-1} \bar{u}^T) = 0
\]

\[
\implies x^g \bar{v} = ax^{p+g-1} \bar{u}^T, \text{ by Lemma 2.2.}
\]

Since $p \geq 2$, we may set $\bar{w} = ax^{p+2} \bar{u}^T$, and notice that $x^g \bar{v} = x^{g+1} \bar{w}$.

We claim $\bar{w} + I_{p-1} \in \text{Soc}(K/I_{p-1})$. First, $\bar{v} \in K$ implies that $A^T \bar{v} = 0$, hence $0 = x^g A^T \bar{v} = x^{g+1} A^T \bar{w}$. As the entries of $\bar{w}$ are contained in $m^g$, so are the entries of $A^T \bar{w}$. Lemma 2.2 yields $A^T \bar{w} = 0$, hence $\bar{w} \in K$. Next, for any $z \in m$, we have

\[
xz \bar{w} = xz \bar{v} = bx^{p+g} \bar{u}^T, \text{ for some } b \in R,
\]

since $x^g \bar{v} + I_{p+g}$ is a socle element in $K/I_{p+g}$. Thus $x^{g+1}(z \bar{w} - bx^{p-1} \bar{u}^T) = 0$. The entries of $z \bar{w} - bx^{p-1} \bar{u}^T$ are all in $m^g$, so Lemma 2.2 implies that $z \bar{v} = bx^{p-1} \bar{u}^T$. 
Therefore $\bar{\varphi} + I_p^{-1} \in \text{Soc}(K/I_p^{-1})$, hence
\[
\varphi_{n+p}(\bar{\varphi} + I_p) = \varphi_{n+p+g}(x^g \bar{\varphi} + I_g + p) \\
= \varphi_{n+p+g}(x^{g+1} \bar{\varphi} + I_g + p) \\
= \varphi_{n+p-1}(\bar{\varphi} + I_{p-1}).
\]

By descending induction, there exists $\bar{\varphi}' + I_1 \in \text{Soc} K/I_1$ such that
\[
\varphi_{n+p}(\bar{\varphi} + I_p) = \varphi_{n+1}(\bar{\varphi}' + I_1).
\]
The desired map $\text{Soc} \varphi_i$ is therefore surjective for $i \geq n + 1 = \max\{c, g\} + 1$. □

The following consequence allows us to characterize Gorenstein rings in terms of the existence of irreducible parameter ideals in $m^n$ for $n = \max\{c, g\} + 1$; it can also be obtained using Rogers’ [10, Theorem 2.3] in place of Theorem 2.4. Recall that an ideal $q$ of $R$ is reducible if $q = b \cap c$ for two strictly larger ideals $b$ and $c$ of $R$; if such a decomposition is not possible, then $q$ is irreducible.

**Corollary 2.6.** Assume $k$ is infinite and $\dim(R) = 1$. Set $n = \max\{c, g\} + 1$. The ring $R$ is Gorenstein if and only if some parameter ideal in $m^n$ is irreducible.

**Proof.** This follows from Theorem 2.4 and the [8, Theorem 2.7]. □

The least integer $\ell$ required to determine whether $R$ is Gorenstein in terms of the existence of an irreducible parameter ideal in $m^\ell$ depends on $R$ and is thus at most $\max\{c, g\} + 1$ in the case of a dimension 1 local ring with an infinite residue field. The next example shows that given an integer $i$, there exists a ring with $\ell < \ell \leq 2i + 1$.

**Example 2.7.** For an infinite field $k$ and $i \geq 1$, set $Q = k[x, y]/(x^{i+1}, xy^i)$. Let $m = (x, y)$ be the maximal ideal of $Q$. The ring $Q$ has dimension 1 and depth 0, hence is non-Gorenstein. We first show that the parameter ideal $(y^i)$ is irreducible, thus $\ell > i$. To see this, it is enough to show that any ideal of $Q$ properly containing $(y^i)$ also contains the nonzero element $x^iy^{i-1}$. Let $b \subseteq Q$ be an ideal that properly contains $(y^i)$ and fix $\beta \in b \setminus (y^i)$. We may write $\beta = \sum_{s, t \geq 0} a_{s, t}x^sy^t$, with $a_{s, t} \in k$. Because $\beta \notin (y^i)$, the set $\Lambda = \{(s, t) \mid a_{s, t} \neq 0, s \leq i, \text{ and } t \leq i - 1\}$ is nonempty. Choose $(s_0, t_0) \in \Lambda$ with $s_0 + t_0 \leq s + t$ for all $(s, t) \in \Lambda$. Noting that we have $x^{s_0}y^i^{i-1-t_0} \beta \in m^{2i-1} = (x^iy^{i-1}, y^{2i-1})$, it follows that $b$ contains $a_{s_0, t_0}^{-1}x^{i-s_0}y^i^{i-1-t_0} \beta = x^iy^{i-1} + \varepsilon$ with $\varepsilon \in (y^{2i-1}) \subseteq (y^i)$. Thus $x^iy^{i-1} \in b$, showing that $(y^i)$ is irreducible. On the other hand, this ring has $c \leq i$ and $g = 2i$, thus $\max\{c, g\} + 1 = 2i + 1$. Corollary 2.6 now implies that every parameter ideal in $m^{2i+1}$ is reducible, that is, $\ell \leq 2i + 1$.

3. A bound in higher dimensions

For rings of higher dimension, the problem of determining surjectivity of the socle of (1.1) becomes more subtle, with obstructions similar to those noted by Fouli and Huneke [5, Discussion 4.5]. In particular, it is not clear to us whether the same type of “lifting” technique employed in Theorem 2.4 can be used to show surjectivity of the socle of (1.1) if $\dim R > 1$. Our solution here is to instead consider surjectivity of the socle of (1.2) for $p_a$-standard systems of parameters (defined below) by comparing it to a composition of connecting homomorphisms in local cohomology.
For this section, \((R, m, k)\) is a commutative noetherian local ring with \(\dim R = d\). Let \(M\) be a finitely generated \(R\)-module with \(\dim M = t\). As in \([11]\), denote the annihilator of \(H_m^t(M)\) by \(a_i(M) = \text{ann}_R H_m^{i+1}(M)\), and put \(a(M) = a_0(M) \cdots a_{t-1}(M)\).

In particular, if \(r \in a(M)\) then \(r H_m^t(M) = 0\) for \(i = 0, \ldots, t - 1\).

A system of parameters \(x_1, \ldots, x_t\) of \(M\) is called a \(p\)-standard system of parameters if \(x_t \in a(M)\) and \(x_i \in a(M/(x_{i+1}, \ldots, x_t)M)\) for \(i = 1, \ldots, t - 1\). Such systems were defined at this level of generality by Cuong \([3]\), who noted that a result of Schenzel \([11, \text{Korollar 2.2.4}]\) implies that every finitely generated \(R\)-module has a \(p\)-standard system of parameters provided \(R\) has a dualizing complex. In detail, if \(R\) has a dualizing complex, then \(\dim R/a(M) < t\) by \([11, \text{Korollar 2.2.4}]\) and so prime avoidance provides an element \(x_t \in a(M)\) that is a parameter of \(M\). Inductively, this shows the existence of \(p\)-standard systems of parameters, as well as the existence of the following variant\(^2\), provided \(R\) has a dualizing complex.

**Definition 3.1.** Let \(M\) be a finitely generated \(R\)-module with \(\dim M = t\). For an integer \(s \geq 1\), a system of parameters \(x_1, \ldots, x_t\) of \(M\) is called a \(p_s\)-standard system of parameters if \(x_t \in a(M)\) and \(x_i \in a(M/(x_{i+1}, \ldots, x_t)M)\) for \(i = 1, \ldots, t - 1\).

Evidently, \(p_1\)-standard and \(p\)-standard systems of parameters are the same.

Let \(x_1, \ldots, x_d\) be a system of parameters of \(R\), fix a positive integer \(t\), and set \(\mathfrak{m} = R/(x_{t+1}, \ldots, x_d)\). The exact sequence \(0 \to \mathfrak{m}/(0: \mathfrak{m} x_t) \overset{0: \mathfrak{m} x_t}{\to} \mathfrak{m}/(x_t \mathfrak{m}) \to \mathfrak{m}/x_t \mathfrak{m} \to 0\) induces a canonical connecting homomorphism \(H_{\mathfrak{m}}^{t-1}(R/(x_t \mathfrak{m})) \to H_{\mathfrak{m}}^t(\mathfrak{m}/(0: \mathfrak{m} x_t))\).

The containment \(x_t \mathfrak{m} \subseteq (0: \mathfrak{m} (0: \mathfrak{m} x_t))\) implies \(\dim(0: \mathfrak{m} x_t) \leq \dim \mathfrak{m}/x_t \mathfrak{m} < t\), hence the surjection \(\mathfrak{m}/x_t \mathfrak{m} \to \mathfrak{m}/(0: \mathfrak{m} x_t)\) yields an isomorphism
\[(3.2)\quad H_{\mathfrak{m}}^t(\mathfrak{m}/x_t \mathfrak{m}) \cong H_{\mathfrak{m}}^t(\mathfrak{m}/(0: \mathfrak{m} x_t)).\]

Combined with the connecting homomorphism, this yields a homomorphism
\[(3.3)\quad H_{\mathfrak{m}}^{t-1}(R/x_t \mathfrak{m}) \longrightarrow H_{\mathfrak{m}}^t(\mathfrak{m}).\]

In light of the isomorphism (3.2), the next result essentially follows from \([4, \text{Proposition 2.1}]\), but we spell out some of the details in order to keep track of the map inducing the surjection, which we will need later.

**Lemma 3.4.** Let \(x_1, \ldots, x_d\) be a system of parameters of \(R\). Fix \(t \geq 1, s \geq 2\), and set \(\mathfrak{m} = R/(x_{s+1}, \ldots, x_d)\). If \(x_t \in a_{s-1}(\mathfrak{m})\), then the map \(\delta_t : H_{\mathfrak{m}}^{t-1}(R/x_t \mathfrak{m}) \to H_{\mathfrak{m}}^t(\mathfrak{m})\) defined in (3.3) for the system of parameters \(x_1, \ldots, x_d\), induces a split surjection
\[\text{Soc} \delta_t : \text{Soc} H_{\mathfrak{m}}^{t-1}(R/x_t \mathfrak{m}) \longrightarrow \text{Soc} H_{\mathfrak{m}}^t(\mathfrak{m}).\]

**Proof.** The inclusion \((0 : \mathfrak{m} x_t) \subseteq (0 : \mathfrak{m} x_t^s)\) induces the left vertical map in the next commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathfrak{m}/(0: \mathfrak{m} x_t^s) & \longrightarrow & \mathfrak{m}/x_t \mathfrak{m} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{m}/(0: \mathfrak{m} x_t) & \longrightarrow & \mathfrak{m}/x_t \mathfrak{m} & \longrightarrow & 0
\end{array}
\]

\(^2\)In fact, one may find by \([11, \text{Korollar 2.2.4}]\) an element in \(a_{s-1}(M)\) that is a parameter of \(M\) provided \(R\) has a dualizing complex. Hence for our purposes, one may instead find a system of parameters \(x_1, \ldots, x_t\) of \(M\) satisfying \(x_t \in a_{s-1}(M/(x_{s+1}, \ldots, x_t)M)\) for \(i = 1, \ldots, t\). The notion considered in Definition 3.1 is chosen to be reminiscent of \(p\)-standard systems.
From (3.2), using also that $x_1^t, \ldots, x_d^t$ and $x_1, \ldots, x_t, x_{t+1}^t, \ldots, x_d^t$ are systems of parameters, we have $H^t_m(R/(0 : x_t^t R)) \cong H^t_m(R) \cong H^t_m(R/(0 : x_t^t R))$. We thus obtain a commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
H^t_{m-1}(R) & \xrightarrow{x_1} & H^t_{m-1}(R) & \longrightarrow & H^t_{m-1}(R/x_t R) & \longrightarrow & H^t_{m-1}(R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^t_{m-1}(R) & \xrightarrow{x_1^t} & H^t_{m-1}(R) & \longrightarrow & H^t_{m-1}(R/x_t R) & \longrightarrow & H^t_{m-1}(R) \\
\end{array}
\]

Since $x_t H^t_{m-1}(R) = 0$, this yields the next commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^t_{m-1}(R) & \longrightarrow & H^t_{m-1}(R/x_t R) & \longrightarrow & (0 : H^t_{m-1}(R) x_t) \\
\downarrow & & \downarrow & & \downarrow \iota & & \downarrow \\
0 & \longrightarrow & H^t_{m-1}(R) & \longrightarrow & H^t_{m-1}(R/x_t R) & \longrightarrow & (0 : H^t_{m-1}(R) x_t) \\
\end{array}
\]

Following the argument in [4, proof of Proposition 2.1], note that the middle vertical map induces $\varepsilon : (0 : H^t_{m-1}(R) x_t) \to H^t_{m-1}(R/x_t R)$ such that $\delta_t \varepsilon = \iota$. Since $\iota$ is the natural inclusion, and $\text{Soc}(0 : H^t_{m-1}(R) x_t) = \text{Soc}(0 : H^t_{m-1}(R) x_t^t) = \text{Soc} H^t_m(R)$, we see that $\text{Soc} \iota = 1_{\text{Soc}(H^t_{m-1}(R))}$ and thus $(\text{Soc} \delta_t)(\text{Soc} \varepsilon) = 1_{\text{Soc}(H^t_{m-1}(R))}$. It follows that $\text{Soc} \delta_t$ is a split surjection.

Given a system of parameters $x_1, \ldots, x_d$ of $R$, the canonical map (1.2) to the direct limit $R/(x_1^t, \ldots, x_d^t) \xrightarrow{\lim_{x_t}} R/(x_1^t, \ldots, x_d^t)$ is induced by the direct system

\[
R/(x_1^t, \ldots, x_d^t) \xrightarrow{x_1^t} R/(x_1^t, x_2^t, \ldots, x_d^t) \xrightarrow{x_1^t} \cdots.
\]

Moreover, there is a unique isomorphism, see [1, Theorem 5.2.9]:

\[
\lim_{x_t} R/(x_1^t, \ldots, x_d^t) \xrightarrow{\cong} H^d_m(R).
\]

**Lemma 3.5.** If $x_1, \ldots, x_d$ is a system of parameters of $R$, then the canonical map $R/(x_1, \ldots, x_d) \xrightarrow{\lim_{x_t}} R/(x_1^t, \ldots, x_d^t)$ equals the composition

\[
H^0_m(R/(x_1, \ldots, x_d)) \to H^0_m(R/(x_2, \ldots, x_d)) \to \cdots \to H^d_m(R) \to H^d_m(R),
\]

where these maps are the connecting homomorphisms as defined in (3.3).

**Proof.** First note that $R/(x_1, \ldots, x_d) = H^0_m(R/(x_1, \ldots, x_d))$. Further, the canonical map $R/(x_1, \ldots, x_d) \xrightarrow{\lim_{x_t}} R/(x_1^t, \ldots, x_d^t)$ can be decomposed as the composition of the following canonical maps:

\[
\begin{array}{ccccccc}
R & \xrightarrow{\lim_{x_t}} & R & \xrightarrow{\lim_{x_t}} & R & \xrightarrow{\lim_{x_t}} & R \\
(x_1, \ldots, x_d) & \xrightarrow{R} & (x_1, \ldots, x_d) & \xrightarrow{R} & (x_1, \ldots, x_d) & \xrightarrow{R} & (x_1, \ldots, x_d) \\
\end{array}
\]

Fix $0 \leq t \leq d$. It is therefore sufficient to show that the next two maps agree up to isomorphism; indeed, by [1, Theorem 5.2.9] there is a unique isomorphism between the domains, and another between the codomains, of the next two maps:

\[
\alpha_t : \lim_{x_t} R/(x_1, \ldots, x_d) \xrightarrow{R} \lim_{x_t} R/(x_1, \ldots, x_d), \text{ and}
\]

\[
\delta_t : H^t_m(R/(x_1, \ldots, x_d)) \xrightarrow{H^t_m R} H^t_m(R/(x_1, \ldots, x_d)).
\]
where $\alpha_t$ is the canonical map to the direct limit and $\delta_t$ is defined by (3.3) from the exact sequence $0 \to R/(x_{t+1}, \ldots, x_d) \to R/(x_t, \ldots, x_d) \to R/(x_{t+1}, \ldots, x_d) \to 0$.

For $0 \leq u \leq d - 1$, there is by [1, Theorem 5.2.9] a natural equivalence of functors $\lim_{\to} (R/\langle x_{i_{u+1}}, \ldots, x_i \rangle) \otimes_R - \to H^0_{x_{i_{u+1}}, \ldots, x_i}(-)$. This provides the isomorphisms and commutativity in the following diagram:

$$
\begin{array}{ccc}
\lim_{\to} (R/\langle x_{i_{u+1}}, \ldots, x_i \rangle) & \cong & H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle) \\
\downarrow \alpha_t & & \downarrow \beta_t \\
\lim_{\to} (R/\langle x_{i_{u+1}}, \ldots, x_i \rangle) & \cong & H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle),
\end{array}
$$

where $\beta_t$ is the canonical map to the direct limit and the modules on the right have also utilized the fact that $H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle) \cong H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t}^{j+1}, x_{t+1}, \ldots, x_d \rangle)$ for each $j \geq 1$; see [1, Example 4.2.2]. It remains only to show that $\beta_t$ is isomorphic to $\delta_t$. To see this, consider the short exact sequence of direct systems:

$$
\begin{array}{cccccc}
0 & \to & R/(x_{t+1}, \ldots, x_d) & \to & R/(x_{t+1}, \ldots, x_d) & \to & 0 \\
\downarrow & & \downarrow x_t & & \downarrow x_t & & \downarrow 0 \\
0 & \to & R/(x_{t+1}, \ldots, x_d) & \to & R/(x_{t+1}, \ldots, x_d) & \to & 0 \\
\downarrow & & \downarrow x_t & & \downarrow x_t & & \downarrow 0 \\
& & \vdots & & \vdots & & \vdots
\end{array}
$$

Each row of this diagram yields a long exact sequence in local cohomology; along with (3.2) and (3.3), this gives the following diagram with exact rows:

$$
\begin{array}{cccccccc}
H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle) & \to & H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t}, \ldots, x_d \rangle) & \to & H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle) & \to & H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle) & \to & 0 \\
\downarrow x_t & & \downarrow x_t & & \downarrow x_t & & \downarrow x_t & & \downarrow 0 \\
H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle) & \to & H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t}, \ldots, x_d \rangle) & \to & H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle) & \to & H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle) & \to & 0 \\
\downarrow x_t & & \downarrow x_t & & \downarrow x_t & & \downarrow x_t & & \downarrow 0 \\
& & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

The map from the second term in the first row, $H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle)$, to the direct limit of the second column is $\beta_t$. The direct limits of the left and right columns are the localizations $(H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle))_{x_t}$ and $(H^0_{x_{i_{u+1}}, \ldots, x_i}(R/\langle x_{t+1}, \ldots, x_d \rangle))_{x_t}$, respectively. These are zero since they are both $m$-torsion [1, 2.1.3] and multiplication by $x_t \in m$ is invertible on either module. Hence the middle map becomes an isomorphism upon taking the direct limit, showing that $\beta_t$ and $\delta_t$ are isomorphic. □

The main distinction between the next result and [8, Proposition 2.5] or [6, Lemma 3.12] is that here we have some control for the point at which the induced maps on socles are surjective.
Proposition 3.6. If there is a \( p_s \)-standard system of parameters \( x_1, \ldots, x_d \) of \( R \) for some \( s \geq 2 \), then the canonical map \( R/(x_1^s, \ldots, x_d^s) \to \lim_{\rightarrow} R/(x_j^s, x_i^s) \cong H^d_m(R) \) induces a split surjection

\[
\text{Soc } R/(x_1^s, \ldots, x_d^s) \longrightarrow \text{Soc } H^d_m(R).
\]

Proof. Fix \( s \geq 2 \) and let \( x_1, \ldots, x_d \) be a \( p_s \)-standard system of parameters of \( R \). By definition, for each \( t \leq d \) we have \( x_t \in \mathfrak{a}(R/(x_{t+1}^s, \ldots, x_d^s)) \subseteq \mathfrak{a}_{t-1}(R/(x_{t+1}^s, \ldots, x_d^s)) \), and so Lemma 3.4 yields that the induced map

\[
\text{Soc } H_m^{t-1}(R/(x_1^s, x_{t+1}^s, \ldots, x_d^s)) \longrightarrow \text{Soc } H_m^d(R/(x_1^s, x_{t+1}^s, \ldots, x_d^s))
\]

is a split surjection. Thus the composition

\[
\begin{align*}
H_m^0(R/(x_1^s, \ldots, x_d^s)) & \longrightarrow H_m^1(R/(x_2^s, \ldots, x_d^s)) \longrightarrow \cdots \longrightarrow H_m^{d-1}(R/(x_d^s)) \longrightarrow H_m^d(R),
\end{align*}
\]

which is equal to the canonical map \( R/(x_1^s, \ldots, x_d^s) \to \lim_{\rightarrow} R/(x_j^s, x_i^s) \cong H^d_m(R) \) by Lemma 3.5, induces the desired split surjection on socles.

Remark 3.7. Suppose \( y_1, \ldots, y_d \) and \( x_1, \ldots, x_d \) are systems of parameters such that \( (y_1, \ldots, y_d) \subseteq (x_1, \ldots, x_d) \). The following hold by [5, p. 2681 and Corollary 2.5]:

1. If \( A = (a_{ij}) \) is any matrix such that \( y_i = \sum_{j=1}^d a_{ij}x_j \), then multiplication by \( \det A \) induces a well-defined map \( R/(x_1, \ldots, x_d) \to R/(y_1, \ldots, y_d) \).

2. If \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are matrices with \( y_i = \sum_{j=1}^d a_{ij}x_j = \sum_{j=1}^d b_{ij}x_j \), then multiplication by \( \det A \) and \( \det B \) determine the same map.

After a reduction in order to apply Proposition 3.6 in place of [8, Proposition 2.5], the proof of the next result is similar to that of [8, Theorem 2.7]. Recall that a \( p_s \)-standard system of parameters of \( R \) exists if \( R \) has a dualizing complex (for example, if \( R \) is complete).

Theorem 3.8. Suppose there exists a \( p_s \)-standard system of parameters \( x_1, \ldots, x_d \) of \( R \) for some \( s \geq 2 \). The ring \( R \) is Gorenstein if and only if some parameter ideal contained in \( (x_1^s, \ldots, x_d^s) \) is irreducible.

Proof. Suppose \( x_1, \ldots, x_d \) is a \( p_s \)-standard system of parameters of \( R \) for some \( s \geq 2 \). All parameter ideals in a Gorenstein ring are irreducible [9, Theorem 18.1], so it is sufficient to prove the converse.

Assume \( y_1, \ldots, y_d \) is a system of parameters such that \( (y_1, \ldots, y_d) \) is irreducible and contained in \( (x_1^s, \ldots, x_d^s) \). We first claim that, for the direct system

\[
R/(y_1, \ldots, y_d) \overset{y_1 \cdots y_d}{\longrightarrow} R/(y_1^2, \ldots, y_d^2) \overset{y_1 \cdots y_d}{\longrightarrow} \cdots,
\]

the canonical map \( R/(y_1, \ldots, y_d) \to \lim_{\rightarrow} R/(y_1^j, y_d^j) \cong H^d_m(R) \) is surjective when restricted to socles. As \( x_1^s, \ldots, x_d^s \) and \( y_1^j, \ldots, y_d^j \) are systems of parameters for all \( j \geq 1 \), there exist families of positive integers \( \{1 = t_1 < t_2 < \cdots\} \), \( \{u_1 < u_2 < \cdots\} \), and \( \{v_1 < v_2 < \cdots\} \) such that for \( i \geq 1 \) we have containments:

\[
(x_1^{u_i}, \ldots, x_d^{u_i}) \supseteq (y_1^{v_i}, \ldots, y_d^{v_i}) \supseteq (x_1^{u_i}, \ldots, x_d^{u_i}) \supseteq (y_1^{v_i}, \ldots, y_d^{v_i}).
\]
By Remark 3.7, we obtain natural maps (coming from determinants of matrices) making the following commutative diagram of direct systems:

\[
\begin{array}{cccccc}
R/(x_1^s, \ldots, x_d^s) & \xrightarrow{\sigma_s} & R/(y_1, \ldots, y_d) & \xrightarrow{\tau_s} & R/(x_1^{u_1}, \ldots, x_d^{u_1}) & \xrightarrow{\rho_s} & R/(y_1^{v_1}, \ldots, y_d^{v_1}) \\
R/(x_1^{2s}, \ldots, x_d^{2s}) & \xrightarrow{\sigma_{2s}} & R/(y_1^{2s}, \ldots, y_d^{2s}) & \xrightarrow{\tau_{2s}} & R/(x_1^{u_2}, \ldots, x_d^{u_2}) & \xrightarrow{\rho_{2s}} & R/(y_1^{v_2}, \ldots, y_d^{v_2}) \\
R/(x_1^s, \ldots, x_d^s) & \xrightarrow{\sigma_s} & R/(y_1, \ldots, y_d) & \xrightarrow{\tau_s} & R/(x_1^{u_1}, \ldots, x_d^{u_1}) & \xrightarrow{\rho_s} & R/(y_1^{v_1}, \ldots, y_d^{v_1}) \\
R/(x_1^{2s}, \ldots, x_d^{2s}) & \xrightarrow{\sigma_{2s}} & R/(y_1^{2s}, \ldots, y_d^{2s}) & \xrightarrow{\tau_{2s}} & R/(x_1^{u_2}, \ldots, x_d^{u_2}) & \xrightarrow{\rho_{2s}} & R/(y_1^{v_2}, \ldots, y_d^{v_2}) \\
\end{array}
\]

Moreover, Remark 3.7 also yields that the compositions of horizontal maps are the familiar ones: \(\tau_{is}\sigma_{is} = x_1^{u_{is}} \cdots x_d^{u_{is}}\) and \(\rho_{is}\tau_{is} = y_1^{v_{is}} \cdots y_d^{v_{is}}\), for \(i \geq 1\). The direct limits of all four columns are isomorphic to \(H^d_m(R)\), and it thus follows from the universal property of direct limits that the induced maps on these direct limits are isomorphisms. Hence we obtain a commutative diagram of canonical maps:

\[
\begin{array}{ccc}
\text{Soc } R/(x_1^s, \ldots, x_d^s) & \xrightarrow{\text{Soc } \sigma_s} & \text{Soc } R/(y_1, \ldots, y_d) \\
\downarrow & & \downarrow \\
\text{Soc } H^d_m(R) & \cong & \text{Soc } H^d_m(R)
\end{array}
\]

The left vertical map is a surjection by Proposition 3.6, hence it follows that the right vertical map is a surjection as well.

Let \(\phi : R/(y_1, \ldots, y_d) \to \lim \nrightarrow \to R/(y_1, \ldots, y_d) \cong H^d_m(R)\) be the canonical map and proceed as in the proof of [8, Theorem 2.7]: Recall that the limit closure of \(y_1, \ldots, y_d\) is defined as \(J = \{y_1, \ldots, y_d\}^\lim = \bigcup_{n \geq 0}((y_1^{n+1}, \ldots, y_d^{n+1}) R (y_1 \cdots y_d)\). By [8, Remark 2.2], \(\ker(\phi) = \{y_1, \ldots, y_d\}^\lim R / (y_1, \ldots, y_d)\). Applying \(\text{Soc}(-)\) to the canonical maps, we obtain the next exact sequence:

\[0 \to \text{Soc } \ker(\phi) \to \text{Soc } R/(y_1, \ldots, y_d) \to \text{Soc } H^d_m(R) \to 0.\]

Irreducibility of \((y_1, \ldots, y_d)\) yields \(\dim_{H^d_m(R)} \text{Soc } R/(y_1, \ldots, y_d) = 1\). As \(H^d_m(R) \neq 0\) we obtain \(\text{Soc } \ker(\phi) = 0\). Applying \(\text{Soc}(-)\) to the canonical maps, we obtain the next exact sequence:

\[0 \to \text{Soc } \ker(\phi) \to \text{Soc } R/(y_1, \ldots, y_d) \to \text{Soc } H^d_m(R) \to 0.\]

Thus \(R\) is Cohen-Macaulay with \(\dim_{H^d_m(R)} = 1\), hence Gorenstein.

**Corollary 3.9.** Assume \(R\) has a dualizing complex. If \(n\) is the least integer with \(m^n \subseteq (x_1^n, \ldots, x_d^n)\) for a \(p_2\)-standard system of parameters \(x_1, \ldots, x_d\), then \(R\) is Gorenstein if and only if some parameter ideal contained in \(m^n\) is irreducible.

**Proof.** Immediate by Theorem 3.8. \(\square\)

We end by noting that if \(R\) has finite local cohomologies, in which case there is an integer \(n_0\) such that \(m^{n_0} H^i_m(R) = 0\) for \(i < d\) (this is not assumed in the results above), a similar bound can be obtained by using [4, Corollary 4.3] in conjunction with [8, Theorem 2.7]; in particular, it follows from these that \(R\) is Gorenstein if and only if some parameter ideal contained in \(m^{2n_0}\) is irreducible. In particular, if \(R\) is Buchsbaum so that we have \(m H^i_m(R) = 0\) for \(i < d\), then \(R\) is Gorenstein if and only if some parameter ideal contained in \(m^2\) is irreducible.
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