INTRODUCTION

The study of unconventional reservoirs continues to attract increasing attention around the world due to their tremendous potential for future gas reserves and production.1-5 Permeability characterizes the fluid flow rate through the rock formation under the pressure gradient and therefore is one of the most important parameters for the evaluation and exploitation of unconventional reservoirs. Compared with conventional reservoirs, unconventional reservoirs usually have extremely low permeability, which brings difficulty to permeability measurement.6-8 Many methods have been developed to determine rock permeability in the laboratory and can be subdivided into two kinds based on their steady-state or unsteady-state nature.9,10 The steady-state methods measure the steady-state flow rate under a given pressure gradient, and the unsteady-state methods measure transient pressure variations. Generally, the unsteady-state methods are more suitable for measurements on tight rocks than the steady-state
methods, because the pressure is easier to measure than the flow rate across tight rock samples, which is often too small to be detected.\textsuperscript{11,12}

The pulse-decay method is one of the most widely used unsteady-state methods.\textsuperscript{13,14} The basic idea was pioneered by Brace et al.\textsuperscript{15} In the conventional pulse-decay measurement, a sample is placed in the core holder and two chambers are connected to its two ends. Initially, the whole system is in pressure equilibrium and then a pressure pulse is applied to one chamber. The pressure variations with time in both chambers are recorded. Unlike the steady-state method, where the permeability can be calculated directly with Darcy’s law, the pulse-decay method requires an analytical solution to evaluate the permeability from the pressure record. Brace et al.\textsuperscript{15} obtained an approximate analytical solution by assuming a constant pressure gradient within the core, which is valid when the pore volume of the sample is negligible compared with the chambers. Hsieh et al.\textsuperscript{16} and Dicker and Smits\textsuperscript{17} gave a general solution without the assumption used by Brace et al.\textsuperscript{15} The analytical solution with considerations for gas adsorption was developed by Cui et al.\textsuperscript{18} Yang et al.\textsuperscript{19} gave the solution for the transverse permeability measurements on tight core samples. Han et al.\textsuperscript{20} summarized the analytical solutions of pulse-decay methods and compared their performance under different scenarios.

In recent decades, many modified pulse-decay methods with different experimental designs have also been developed. Giot et al.\textsuperscript{21} suggested measuring the permeability by imposing a pressure pulse on a hollow cylindrical sample, and similar ideas have also been proposed by other researchers.\textsuperscript{22,23} Yang et al.\textsuperscript{24} presented a method in which one side of the sample was connected to the chamber and the other side was sealed. The radial differential pressure decay method was developed by Wu et al.\textsuperscript{10} to assess the apparent permeability with microplug samples. Metwally and Sondergeld\textsuperscript{25} proposed the pressure build-up method where the upstream pressure was kept constant during the test and the downstream pressure increase was recorded for permeability evaluation. Further research\textsuperscript{26} showed the pressure build-up method is less affected by surface defects or limited penetration fractures of the core plugs than the conventional pulse-decay method because the downstream pressure build-up is a response across the whole core plug, unlike the conventional pulse-decay response.

Despite the wide range of usage and the potential advantages, the original pressure build-up method suffers from the separation of permeability and porosity measurements. In the pulse-decay methods where chambers of finite volume are used, the porosity can be evaluated by mass conservation and Boyle’s law using the initial and final equilibrium pressure.\textsuperscript{7,27,28} However, the total mass in the pressure build-up method is not conserved because the pressure on one side is kept constant during the test, which brings difficulty to porosity measurement in the pressure build-up method.

To overcome this problem, this study proposes a modified method that can simultaneously measure the permeability and the porosity in a single test. In this method, the pressure in the downstream (or upstream) chamber is decreased (or increased) and then maintained constant. The pressure variations in the other chamber are used to evaluate the permeability and porosity of the tight rock sample. The mathematical model and the analytical solution based on this experimental design have been obtained. Using the proposed method, the permeability and the porosity of a tight sandstone sample have been measured and the accuracy of the results has been confirmed.

2 | EXPERIMENTAL PROCEDURE

The schematic of the experimental setup is shown in Figure 1. A detailed description of the experimental setup can be found in our previous studies.\textsuperscript{29,30} Since applying a negative pulse in the downstream chamber and applying a positive pulse in the upstream chamber are mathematically equivalent, we only take the former as an example in the following part of the study. To keep the downstream pressure constant during the test, the downstream volume is chosen to be much larger than the sum of the pore volume and the volume of the upstream chamber, that is, \( V_d \gg V_p + V_{wu} \) or to be open to the atmosphere. The experimental protocol is as follows:

1. The core sample was placed into the core holder and confining pressure was applied to resemble the subsurface condition. An oven was used to keep the whole setup at the desired temperature.
2. With valves 1, 2, 3 open, and valve 4 closed, the core and the two chambers were filled to the desired initial pressure. Then, valves 1, 2, and 3 were closed, and valve 4 was open. The pressure in the downstream chamber was decreased to create the initial pressure difference (<10%) and then maintained as constant.

![FIGURE 1 The schematic of the experimental setup](image-url)
Strictly speaking, the gas compressibility \( \beta \) is affected by pressure due to the poroelastic and slippage effects.31-33 However, as previously noted,15,34-36 when the initial pressure difference between the two ends of the sample in a single measurement is small, these variables can be regarded as constants and their values are taken under the mean pore pressure.

Considering the mass balance at the interface between the core sample and the upstream chamber and noting that the downstream pressure is kept constant during the test, the boundary conditions are obtained:

\[
\frac{\partial P}{\partial t} \bigg|_{x=0} = \frac{kA}{\beta \mu V_u} \frac{\partial P}{\partial x} \bigg|_{x=0}, \quad \frac{\partial P}{\partial t} \bigg|_{x=L} = 0
\]

where \( A \) is the cross-section area of the sample (m\(^2\)), \( V_u \) the upstream chamber volume (m\(^3\)), and \( L \) is the length of the sample (m).

The initial conditions are as follows:

\[
P(x, 0) = \begin{cases} 
P_u(0), & 0 \leq x < L \\
P_d(0), & x = L 
\end{cases}
\]

where \( P_u(0) \) and \( P_d(0) \) are the initial upstream and downstream pressures (Pa), respectively.

To facilitate the derivation, the following dimensionless variables with subscript \( D \) are introduced:

\[
x_D = \frac{x}{L}, \quad t_D = \frac{kt}{\beta \mu \phi L^2}, \quad P_D = \frac{P(x,t) - P_d(0)}{P_u(0) - P_d(0)}, \quad a = \frac{LA\phi}{V_u}
\]

With these dimensionless variables, the governing Equation (1) can be rewritten as:

\[
\frac{\partial P_D}{\partial t_D} = \frac{\partial^2 P_D}{\partial x_D^2}
\]

The dimensionless boundary conditions are given by:

\[
\frac{\partial P_D}{\partial t_D} \bigg|_{x_D=0} = a \frac{\partial P_D}{\partial x_D} \bigg|_{x_D=0}, \quad \frac{\partial P_D}{\partial t_D} \bigg|_{x_D=1} = 0
\]

and the dimensionless initial conditions are as follows:

\[
P_D(x_D, 0) = \begin{cases} 
1, & 0 \leq x_D < 1 \\
0, & x_D = 1
\end{cases}
\]

### 3.2 The analytical solution

The Laplace transform is used to solve the above equations (the details can be found in the appendixes), and the derived solutions are shown below.
3.2.1 | The general solution

The general solution for the dimensionless pressure distribution within the sample is:

\[ P_{D_D}(x_D, t_D) = 2 \sum_{m=1}^{\infty} \frac{2a}{(\theta_m^2 + a^2 + a) \cos \theta_m} e^{-\theta_m t_D} \]  

where \( \theta_m \) is the \( m \)th positive root of:

\[ \tan \theta_m = \frac{a}{\theta_m} \]  

The general solutions for upstream and downstream pressures can be obtained by setting \( x_D = 0 \) and \( x_D = 1 \) in Equation (8), respectively:

\[ P_{uD}(t_D) = P_D(0, t_D) = \sum_{m=1}^{\infty} \frac{2a}{(\theta_m^2 + a^2 + a) \cos \theta_m} e^{-\theta_m t_D} \]  

and

\[ P_{dD}(t_D) = P_D(1, t_D) = 0 \]

The expression for dimensionless pressure difference between two ends of the sample is given by:

\[ \Delta P_D(t_D) = P_{uD}(t_D) - P_{dD}(t_D) = \sum_{m=1}^{\infty} \frac{2a}{(\theta_m^2 + a^2 + a) \cos \theta_m} e^{-\theta_m t_D} \]  

The pressure distributions throughout the sample at different instants are depicted in Figure 3. Because the negative pulse needs time to permeate through the core sample from the downstream to the upstream, the upstream pressure remains constant rather than lowering immediately after the test begins. As time proceeds, the domain affected by the negative pulse expands and eventually reaches the upstream side of the core, after which a significant decrease in the upstream pressure can be observed.

3.2.2 | The early-time solution

The approximate solution that describes the upstream pressure variations at the beginning of the pulse-decay test is usually referred to as the early-time solution in the literature.\textsuperscript{9,16,19} The early-time solution to the proposed method is:

\[ P_{uD}(t_D) \approx 1 + 2e^{a^2 t_D} a^2 t_D \]  

which is referred to as the late-time solution in the literature.\textsuperscript{9}

The comparison between the general solution and the late-time solution for pressure difference is presented in Figure 6, and the results show that when the dimensionless
time $t_D$ is large, the two solutions agree with each other very well.

### 3.3 Permeability and porosity determination

Taking the logarithm of the late-time solution for pressure difference (Equation 14) yields:

$$\ln \Delta P_D = f - \theta_1^2 t_D = f + at$$  \hspace{1cm} (15)

where the dimensionless time is transformed into the dimensional one, and the expressions for $f$ and $\alpha$ are as follows:

$$f = \ln \frac{2a}{(\theta_1^2 + a^2 + a) \cos \theta_1}$$  \hspace{1cm} (16)

$$\alpha = -\frac{\theta_1^2 k}{\beta \mu \phi L^2}$$  \hspace{1cm} (17)

Equation (15) indicates that if the logarithmic dimensionless pressure difference $\ln \Delta P_D$ is plotted against $t_D$ or $t$, a linear relationship can be obtained in the late-time
stage, where the intercept is \( f \), and the slope is \( -\theta_1^2 \) or \( a \), respectively.

It is noted that a one-to-one correspondence between \( a \) and \( \theta_1 \), can be established through Equation (9). Therefore, by combining Equations (9) and (16), \( f \) can be solely determined if \( a \) or \( \theta_1 \) is known, and vice versa. Different values of \( a \) and the corresponding values of \( f \) are shown in Figures 7 and 8, from which a monotonic relationship between \( a \) and \( f \) is found. In other words, a one-to-one correspondence between the intercept \( f \) and the volume ratio \( a \) can be established, so the intercept \( f \) obtained from the experiment can be used to calculate the volume ratio \( a \) and therefore the porosity \( \phi \). A similar relationship can be found between the \( f \) and \( \theta_1 \) (see Figures 7 and 9).

Equations (9) and (16) are both transcendental equations, so the mappings from intercept \( f \) to \( a \) and \( \theta_1 \) are both implicit functions, that is, the expressions for \( a(f) \) and \( \theta_1(f) \) cannot be written down in an explicit manner. For ease of use, a series of values of \( (a, f) \) and \( (\theta_1, f) \) are obtained by solving Equations (9) and (16) numerically, and two analytical approximate correlations are given by fitting the series of values with polynomial and rational fractions, respectively:

\[
\begin{align*}
a &= 3091.7f^4 - 684.46f^3 + 76.014f^2 + 4.5344f \\
\theta_1 &= (4.123f^2 + 0.4846f + 3.484 \times 10^{-4}) / (f + 0.01263)
\end{align*}
\]

Equations (18) and (19) can be used to interpret the volume ratio \( a \) and \( \theta_1 \) directly from the intercept \( f \), respectively. Then, the porosity \( \phi \) can be obtained from the definition of \( a \) in Equation (4):

\[
\phi = \frac{aV_u}{LA} \quad (20)
\]

When \( \theta_1 \) and \( \phi \) are both known, the permeability can be easily determined with the rewritten form of Equation (17):

\[
k = -\frac{a\beta \mu \phi L^2}{\theta_1^2} \quad (21)
\]

4 | RESULTS AND DISCUSSION

A sandstone sample was measured with the proposed method and the pressure difference was recorded to calculate the slope and intercept. During the measurements, the downstream was directly open to the atmosphere to ensure constant downstream pressure. The permeability and porosity were evaluated with slope and intercept values. To ensure the measurement repeatability and reduce random errors, the sample was measured five times, and the standard deviation of the results was found to be smaller than 5%. Table 1 shows the basic information about the sample and the measurements. Figure 10 presents one set of experimental results on pressure decay. It is noted that the experimental data shows a platform at the beginning of the test, which is consistent with the general solution in Figure 6 and the discussions in Section 3.2.1. In the semi-log plot, it can be seen that, after a short initial period, the late-time solution fits the experimental data very well and the slope and intercept values can be easily obtained through the least-squares method.

To verify the accuracy of the proposed method, the permeability and porosity measured by the proposed method were compared with those measured by the conventional pulse-decay method and the gas expansion method, and a good agreement was found, proving that the proposed method is a reliable and accurate way to estimate the permeability and porosity of tight rocks. The comparison is depicted in Table 2.

Using the slope of the pressure decay to calculate the permeability is a common characteristic shared by the pulse-decay methods. However, using the intercept to evaluate the porosity, to the authors’ knowledge, has not been reported before. The common way to measure porosity is based on Boyle’s law and mass conservation and requires the initial and final equilibrium pressures in the measurement. However, in the proposed method, the porosity is evaluated using late-time pressure decay data and the final equilibrium is not needed. Compared with the original pressure build-up method where the permeability and porosity are measured separately, the proposed
method has two advantages: Firstly, the total test time is significantly reduced. Secondly, the proposed method ensures the permeability and porosity are measured exactly under the same loading condition.

5 | CONCLUSIONS

In this study, a modified pressure pulse-decay method is presented to measure the permeability and porosity of a low-permeable reservoir rock. In the proposed method, the pressure on one side of the sample is changed (increased/decreased) instantaneously and then kept constant. The pressure variations on the other side are recorded to evaluate the petro-physical properties of the sample. The mathematical model describing the test process was established and the analytical solution was obtained. The late-time solution was adopted for the postprocessing of the experimental data. The permeability of the sample was determined by the slope of a semilogarithmic plot of the pressure differential vs time. A one-to-one correspondence between the volume ratio and the intercept of this plot was found, and the analytical approximate correlation of this correspondence was given. Thus, the intercept can be used to calculate the porosity of the sample. A tight sandstone sample was tested with the proposed method, and the evaluated permeability and porosity agree well with the results of the conventional methods, which confirms the accuracy of the proposed method.

**TABLE 1** Parameters of the measurement

| Core sample | Sandstone          |
|-------------|--------------------|
| Testing gas | Helium             |
| Sample length (m) | 1.96 × 10⁻² |
| Sample cross section area (m²) | 1.13 × 10⁻³ |
| Confining pressure (bar) | 200 |
| Mean pore pressure (bar) | 1 |
| Temperature (°C) | 35 |

**TABLE 2** Comparison of the measured permeability and porosity

|                        | Permeability | Porosity |
|------------------------|--------------|----------|
| The proposed method    | 1.08 × 10⁻¹⁶ m² | 8.71%    |
| The other method       | 1.06 × 10⁻¹⁶ m² | 8.60%    |
| Relative error         | 2.14%        | 1.26%    |

**FIGURE 9** Relationship between the intercept \( f \) and \( \theta_1 \). The scatter points represent the numerical solution of Equations (9) and (16), and the dashed line represents the approximate correlation Equation (19).

**FIGURE 10** Experimental results for the sandstone sample. (A) Dimensionless pressure difference vs time. (B) Logarithmic differential pressure vs time. Note that the scatter points in (A) and (B) represent the experimental data, and the solid line in (B) represents the linear fitting curve.
with measuring the permeability and porosity separately, the proposed method can measure both of them in one test and under the same loading condition, which increases the accuracy and reduces the total test time.

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REFERENCES
1. Gensterblum Y, Ghanizadeh A, Cuss RJ, et al. Gas transport and storage capacity in shale gas reservoirs—a review. Part A: transport processes. J Unconvent Oil Gas Resour. 2015;12:87-122.
2. Pan H, Li J, Zhang T, Li S, Zhang L. Study on crack propagation of the CO2 presplitting blasting empty hole effect in coal seam. Energy Sci Eng. 2020;8(11):3898-3908.
3. Soeder DJ. The successful development of gas and oil resources from shales in North America. J Petrol Sci Eng. 2018;163:399-420.
4. Tao S, Chen S, Pan Z. Current status, challenges, and policy suggestions for coalbed methane industry development in China: a review. Energy Sci Eng. 2019;7(4):1059-1074.
5. Wang Z, Wang M, Chen S. Coupling of high Knudsen number and non-ideal gas effects in microporous media. J Fluid Mech. 2018;840:56-73.
6. Hannon MJ. Alternative approaches for transient-flow laboratory-scale permeametry. Transp Porous Media. 2016;114(3):719-746.
7. Jang H, Lee W, Kim J, Lee J. Novel apparatus to measure the low-permeability and porosity in tight gas reservoir. J Petrol Sci Eng. 2016;142:1-12.
8. Zhao Y, Wang C, Bi J. A method for simultaneously determining axial permeability and transverse permeability of tight reservoir cores by canister degassing test. Energy Sci Eng. 2020;8(4):1220-1230.
9. Sander R, Pan Z, Connell LD. Laboratory measurement of low permeability unconventional gas reservoir rocks: a review of experimental methods. J Nat Gas Sci Eng. 2017;37:248-279.
10. Wu T, Zhang D, Li X. A radial differential pressure decay method with micro-plug samples for determining the apparent permeability of shale matrix. J Nat Gas Sci Eng. 2020;74:103126.
11. Akkutlu IY, Fathi E. Multiscale gas transport in shales with local kerogen heterogeneities. SPE J. 2012;17(04):1002-1011.
12. Heller R, Vermelen J, Zoback M. Experimental investigation of matrix permeability of gas shales. AAPG Bull. 2014;98(5):975-995.
13. Jones S. A technique for faster pulse-decay permeability measurements in tight rocks. SPE Form Eval. 1997;12(01):19-26.
14. Mokhtari M, Tutuncu AN. Characterization of anisotropy in the permeability of organic-rich shales. J Petrol Sci Eng. 2015;133:496-506.
15. Brace WF, Walsh J, Frangos W. Permeability of granite under high pressure. J Geophys Res. 1968;73(6):2225-2236.
16. Hsieh P, Tracy J, Neuzil C, Bredehoef J, Silliman SE. A transient laboratory method for determining the hydraulic properties of ‘tight’ rocks—I. Theory. Int J Rock Mech Min Sci Geomech Abstr Elsevier. 1981;18(3):245-252.
17. Dicker A, Smits R. A practical approach for determining permeability from laboratory pressure-pulse decay measurements. In: International Meeting on Petroleum Engineering. Society of Petroleum Engineers. 1988:285-292.
18. Cui X, Bustin A, Bustin RM. Measurements of gas permeability and diffusivity of tight reservoir rocks: different approaches and their applications. Geoﬂuids. 2009;9(3):208-223.
19. Yang Z, Dong M, Zhang S, Gong H, Li Y, Long F. A method for determining transverse permeability of tight reservoir cores by radial pressure pulse decay measurement. J Geophys Res: Solid Earth. 2016;121(10):7054-7070.
20. Han G, Liu X, Huang J. Theoretical comparison of test performance of different pulse decay methods for unconventional cores. Energies. 2020;13(17):4557.
21. Girot R, Giraud A, Auvray C. Assessing the permeability in anisotropic and weakly permeable porous rocks using radial pulse tests. Oil Gas Sci Technol-Revue d’IFP Energies Nouvelles. 2014;69(7):1171-1189.
22. Cao C. Numerical interpretation of transient permeability test in tight rock. J Rock Mech Geotech Eng. 2018;10(1):32-41.
23. He B, Xie L, Zhao P, Ren L, Zhang Y. Highly efficient and simplified method for measuring the permeability of ultra-low permeability rocks based on the pulse-decay technique. Rock Mech Rock Eng. 2020;53(1):291-303.
24. Yang Z, Sang Q, Dong M, Zhang S, Li Y, Gong H. A modified pressure-pulse decay method for determining permeabilities of tight reservoir cores. J Nat Gas Sci Eng. 2015;27:236-246.
25. Metwally YM, Sondergeld CH. Measuring low permeabilities of gas-sands and shales using a pressure transmission technique. Int J Rock Mech Min Sci. 2011;48(7):1135-1144.
26. Wang Y, Chen Z, Morah V, Knabe R, Appel M. Gas-phase relative permeability characterization on tight-gas samples. Petrophysics. 2012;53(06):393-400.
27. Li Z, Feng R, Liu J, Pandey R. A simplified transient technique for porosity and permeability determination in tight formations: numerical simulation and experimental validation. Energy Sci Eng. 2021;9(3):375-389.
28. Wang C, Pan L, Zhao Y, Zhang Y, Shen W. Analysis of the pressure-pulse propagation in rock: a new approach to simultaneously determine permeability, porosity, and adsorption capacity. Rock Mech Rock Eng. 2019;52(11):4301-4317.
29. Gaus G, Amann-Hildenbrand A, Krooss BM, Fink R. Gas permeability tests on core plugs from unconventional reservoir rocks under controlled stress: a comparison of different transient methods. J Nat Gas Sci Eng. 2019;65:224-236.
30. Nolte S, Fink R, Krooss BM, et al. Experimental investigation of gas dynamic effects using nanoporous synthetic materials as tight rock analogues. Transp Porous Media. 2021;137(3):519-553.
31. Fink R, Krooss B, Amann-Hildenbrand A. Stress-dependence of porosity and permeability of the Upper Jurassic Bossier shale: an experimental study. Geol Soc Lond, Spec Publ. 2017;454(1):107-130.
32. Song W, Yao J, Li Y, et al. Apparent gas permeability in an organic-rich shale reservoir. Fuel. 2016;181:973-984.
33. Zhao Y, Wang C, Bi J. Permeability model of fractured rock with consideration of elastic-plastic deformation. Energy Sci Eng. 2020;8(2):441-451.
34. Walder J, Nur A. Permeability measurement by the pulse-decay method: effects of poroelastic phenomena and non-linear pore pressure diffusion. Int J Rock Mech Min Sci Geomech Abstr. 1986;23(3):225-232.
35. Winhausen L, Amann-Hildenbrand A, Fink R, et al. A comparative study on methods for determining the hydraulic properties of a clay shale. Geophys J Int. 2021;224(3):1523-1539.
36. Zheng J, Zheng L, Liu H-H, Ju Y. Relationships between permeability, porosity and effective stress for low-permeability sedimentary rock. Int J Rock Mech Min Sci. 2015;78:304-318.

**APPENDIX A**

The procedure for solving Equations (5)-(7) through the Laplace transform is presented in this appendix.

The Laplace transform of the dimensionless pressure \( P_D (x_D, t_D) \) is defined as:

\[
\mathcal{L} [P_D (x_D, t_D)] = \mathcal{P}_D (x_D, s) = \int_0^{+\infty} P_D (x_D, t_D) e^{-st_D} dt_D \quad (A1)
\]

where \( \mathcal{P}_D \) is the transformed counterpart of \( P_D \) and \( s \) is the transform parameter.

Applying the Laplace transform to the Equations (5)-(7), we obtain the transformed governing equation:

\[
\frac{d\mathcal{P}_D (x_D, s)}{dx_D} = s\mathcal{P}_D (x_D, s) - 1 \quad (A2)
\]

and the transformed boundary conditions:

\[
s\mathcal{P}_D (0, s) - P_D (0, 0) = a \left. \frac{d\mathcal{P}_D}{dx_D} \right|_{x_D=0}, \quad \mathcal{P}_D (1, s) = 0 \quad (A3)
\]

Note that the initial conditions have been substituted into the above transformed equations.

Combining the transformed Equations (A2) and (A3), we get the expression for \( \mathcal{P}_D (x_D, s) \):

\[
\mathcal{P}_D (x_D, s) = \frac{1}{s} \frac{\sqrt{s} \sinh (\sqrt{s} x_D)}{s \sqrt{\sinh (\sqrt{s})} + \cosh (\sqrt{s}) (A4)}
\]

Then \( P_D (x_D, t_D) \) can be obtained by applying the inverse Laplace transform to \( \mathcal{P}_D (x_D, s) \):

\[
P_D (x_D, t_D) = \mathcal{L}^{-1} [\mathcal{P}_D (x_D, s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st_D} \mathcal{P}_D (x_D, s) ds \quad (A5)
\]

where \( s \) is now a complex number, and \( c \) is a real, positive constant that is large enough that all the singularities of \( \mathcal{P}_D (x_D, s) \) lie to the left of the line \( (c - i\infty, c + i\infty) \).

The contour integral in Equation (A5) can be evaluated through Cauchy’s residue theorem:

\[
P_D (x_D, t_D) = m \sum \text{Res} \left[ e^{st_D} \mathcal{P}_D (x_D, s) \right] \quad (A6)
\]

where \( s_m \) are the poles of the integrand \( e^{st_D} \mathcal{P}_D (x_D, s) \), and \( \text{Res} [e^{st_D} \mathcal{P}_D (x_D, s)] \) are the corresponding residues.

To determine the poles of \( e^{st_D} \mathcal{P}_D (x_D, s) \), we set \( \sqrt{s_m} = i\theta_m \) and substitute it into the denominator of \( \mathcal{P}_D (x_D, s) \):

\[
\theta_m^2 \sin \theta_m - a\theta_m^2 \cos \theta_m = 0 \quad (A7)
\]

It is easy to check that \( \theta_0 = 0 \) satisfies Equation (A7), and \( \theta_m (m \geq 1) \) are the roots of the following equation:

\[
\tan \theta_m = \frac{a}{\theta_m} (m \geq 1) \quad (A8)
\]

Since Equation (A8) has infinite real roots and all of them are of the first order, the integrand has infinite real, nonpositive simple poles (recalling that \( \sqrt{s_m} = i\theta_m \)):

\[
s_0 = 0, s_m = -\theta_m^2 (m \geq 1) \quad (A9)
\]

Noting that if \( \theta_m (m \geq 1) \) is the root of Equation (A8), so will be \( -\theta_m \), and the sign of \( \theta_m \) has no effect on the value of \( s_m \). Therefore, we take \( \theta_m (m \geq 1) \) to be positive in the following derivation, without a loss of generality.

Then, we evaluate the residue of the pole \( s_0 = 0 \):

\[
\text{Res} \left[ e^{st_D} \mathcal{P}_D (x_D, s) \right] = \frac{e^{st_D} \mathcal{P}_D (x_D, s)}{s} \bigg|_{s=0} = \frac{\mathcal{P}_D (x_D, s)}{s} \bigg|_{s=0}
\]

37. Hahn DW, Özisik MN. *Heat Conduction*. John Wiley & Sons; 2012.
\[
\text{Res} \left[ e^{\alpha s} P_D (x_D, s), 0 \right] = s \lim_{s \to 0} \left[ s e^{\alpha s} P_D (x_D, s) \right] = 0 \quad (A10)
\]

and the residues of the other poles \( s_m = -\theta_m^2 \) \((m \geq 1)\) are as follows:

\[
\text{Res} \left[ e^{\alpha s} P_D (x_D, s), -\theta_m^2 \right] = s \theta_m^2 - \lim_{s \to \theta_m^2} \left[ (s + \theta_m^2) e^{\alpha s} P_D (x_D, s) \right] = 0
\]

Substituting Equations (A10) and (A11) into Equation (A6), we get the expression for \( P_D (x_D, t_D) \):

\[
P_D (x_D, t_D) = 2 \sum_{m=1}^{\infty} \frac{\alpha \cos (\theta_m x_D) - \theta_m \sin (\theta_m x_D)}{\cos (\theta_m t_D)} e^{-\theta_m^2 t_D} \quad (A12)
\]

where \( \theta_m (m \geq 1) \) are the positive roots of Equation (A8).

\[
\mathcal{L}^{-1} \left[ \frac{2a}{s \left( \sqrt{s} + a \right)} e^{\sqrt{s}} \right] = 2e^{a \sqrt{t} + \frac{1}{2 \sqrt{t}}} \text{erfc} \left( a \sqrt{t_D} + \frac{1}{2 \sqrt{t_D}} \right) - 2 \text{erfc} \left( \frac{1}{2 \sqrt{t_D}} \right) \quad (B4)
\]

and those of the remaining terms involve very complex expressions and cannot be written down in a concise manner. However, it can be proved that the inverse Laplace transform of the remaining terms is much smaller than that of the first two terms when the dimensionless time \( t_D \) is small. Therefore, by retaining the first few terms of the inverted series, an approximate expression of upstream pressure variations in a short time can be obtained:

\[
P_{ud} (t_D) \approx 1 + 2e^{a \sqrt{t} + \frac{1}{2 \sqrt{t}}} \text{erfc} \left( a \sqrt{t_D} + \frac{1}{2 \sqrt{t_D}} \right) - 2 \text{erfc} \left( \frac{1}{2 \sqrt{t_D}} \right) \quad (B6)
\]

where the hyperbolics have been converted into exponentials.

The last term in Equation (B1) can be expanded as a Taylor series:

\[
\left( 1 - \frac{\sqrt{s} - a e^{-2\sqrt{s}}}{\sqrt{s} + a} \right)^{-1} = 1 + \left( \frac{\sqrt{s} - a e^{-2\sqrt{s}}}{\sqrt{s} + a} \right) + \left( \frac{\sqrt{s} - a e^{-2\sqrt{s}}}{\sqrt{s} + a} \right)^2 + \ldots \quad (B2)
\]

\[
\mathcal{L}^{-1} \left[ \frac{1}{s} \right] = 1 \quad (B4)
\]

\[
\mathcal{L}^{-1} \left[ \frac{1}{s^2} \right] = \frac{1}{2s} \quad (B5)
\]

\[
\mathcal{L}^{-1} \left[ \frac{1}{s^3} \right] = -\frac{1}{6s^2} \quad (B5)
\]