A new topological aspect of the arbitrary dimensional topological defects

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Abstract

We present a new generalized topological current in terms of the order parameter field $\vec{\phi}$ to describe the arbitrary dimensional topological defects. By virtue of the $\phi$-mapping method, we show that the topological defects are generated from the zero points of the order parameter field $\vec{\phi}$, and the topological charges of these topological defects are topological quantized in terms of the Hopf indices and Brouwer degrees of $\phi$-mapping under the condition that the Jacobian $J(\frac{\phi}{\nu}) \neq 0$. When $J(\frac{\phi}{\nu}) = 0$, it is shown that there exist the crucial case of branch process. Based on the implicit function theorem and the Taylor expansion, we detail the bifurcation of generalized topological current and find different directions of the bifurcation. The arbitrary dimensional topological defects are found splitting or merging at the degenerate

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point of field function $\bar{\phi}$ but the total charge of the topological defects is still unchanged.

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I. INTRODUCTION

The world of topological defects is amazingly rich and have been the focus of much attention in many areas of contemporary physics [1–3]. The importance of the role of defects in understanding a variety of problems in physics is clear [1–7]. Whenever we have a field theory with a set of vacua given by a non-connected space there is the possibility of having different regions in space living on different vacuum sectors. Two such regions will meet at what is generally named a topological defect, i.e. a thin hypersurface where the field rapidly evolves from one vacuum to the other [8].

As evidence of cosmological phase transitions in the early Universe, topological defects, such as cosmic string, domain wall, monopole and texture, remain somewhere in our Universe, and can help to resolve some long standing puzzles such as the origin of structure formation [3]. The physics of spacetimes containing the defects has been investigated extensively [10]. The existence of topological defects [2,10] with a non-trivial core phase structure has recently been demonstrated for embedded global domain walls and vortices [11]. A solution for a Schwarzschild particle with global monopole charge has been obtained by Barriola and Vilenkin [2]. The texture model of structure formation in the universe [12] and the one–texture universe [13] have been studied by many researchers [14]. \(p\)-branes [15], which have been found to play important roles in \(M\)-theory [16], are also proved to be topological defects in gauge theory [17]. Recently, some physicists noticed [18] that the topological defects are closely related to the spontaneously broken of \(O(m)\) symmetry group to \(O(m-1)\) by \(m\)-component order parameter field \(\vec{\phi}\) and pointed out that for \(m = 1\), one has domain walls, \(m = 2\), strings and \(m = 3\), monopoles, for \(m = 4\), there are textures. And \(O(m)\) symmetric vector field theories are a class of models describing the critical behavior of an great variety of important physical systems [19–20]. But for the lack of a powerful method, the topological properties of these systems are not very clear, some important topological informations have been lost, also the unified theory of describing the topological properties of all these defect objects is not established yet.
In this paper, in the light of $\phi$–mapping topological current theory [21], a useful method which plays a important role in studying the topological invariants [22,23] and the topological structures of physical systems [24,25], we will investigate the topological quantization and the branch process of arbitrary dimensional topological defects. We will show that the topological defects are generated from the zero points of the order parameter field $\vec{\phi}$, and their topological charges are quantized in terms of the Hopf indices and Brouwer degrees of $\phi$–mapping under the condition that the zero points of field $\vec{\phi}$ are regular points. While at the critical points of the order parameter field $\vec{\phi}$, i.e. the limit points and bifurcation points, there exist branch processes, the topological current of defect bifurcates and the topological defects split or merge at such point, this means that the topological defects system is unstable at these points.

This paper is organized as follows. In section 2, we investigate the topological quantization of these topological defect and point out that the topological charges of these defects are the Winding numbers which are determined by the Hopf indices and the Brouwer degrees of the $\phi$–mapping. In section 3, we study the branch process of the defect topological current at the limit points, bifurcation points and higher degenerated points systematically by virtue of the $\phi$–mapping theory and the implicit function theorem.

II. TOPOLOGICAL QUANTIZATION OF TOPOLOGICAL DEFECTS

In our previous papers [21,25–28], we have studied the topological properties of point like defects and string like defects systematically via the $\phi$–mapping topological current theory and rank-2 topological current theory, respectively. In this paper, in order to study the topological properties of arbitrary dimensional topological defects, we will extend the concept to present an arbitrary dimensional generalized topological current.

It is well known that the $m$–component vector order parameter field $\vec{\phi}(x) = (\phi^1(x), ..., \phi^m(x))$ determines the properties of the topological defect system, and it can be looked upon as a smooth mapping between the $n$–dimensional Riemannian spacetime $G$
(with the metric tensor $g_{\mu\nu}$ and local coordinates $x^\mu$ ($\mu, \nu = 1, \ldots, n$)) and a $m$–dimensional Euclidean space $R^m$ as $\phi : G \rightarrow R^m$. By analogy with the discussion in our previous work \cite{25–28}, from this $\phi$–mapping, one can deduce a topological tensor current as

$$j^{\mu_1 \cdots \mu_k} = \frac{1}{A(S^{m-1})(m-1)!} \sqrt{g_x} \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} \epsilon_{a_1 \cdots a_m} \partial_{\mu_{k+1}} n^{a_1} \partial_{\mu_{k+2}} n^{a_2} \cdots \partial_{\mu_n} n^{a_m}. \quad (1)$$

to describe the system of the topological defects, where $k = n - m$. In this expression, $\partial_\mu$ stands for $\partial/\partial x^\mu$, $A(S^{m-1}) = 2\pi^{m/2}/\Gamma(m/2)$ is the area of $(m-1)$–dimensional unit sphere $S^{m-1}$ and $n^a(x)$ is the direction field of the $m$–component order parameter field $\vec{\phi}$

$$n^a(x) = \frac{\phi^a(x)}{||\phi(x)||}, \quad ||\phi(x)|| = \sqrt{\phi^a(x)\phi^a(x)} \quad (2)$$

with $n^a(x)n^a(x) = 1$. It is obviously that $n^a(x)$ is a section of the sphere bundle $S(G)$ \cite{21} and it can be looked upon as a map of $G$ onto a $(m-1)$–dimensional unit sphere $S^{m-1}$ in order parameter space. Clearly, the zero points of the order parameter field $\vec{\phi}(x)$ are just the singular points of the unit vector $n^a(x)$. It is easy to see that $j^{\mu_1 \cdots \mu_k}$ are completely antisymmetric, and from the formulas above, we conclude that there exists a conservative equation of the topological tensor current in (1)

$$\nabla_\mu j^{\mu_1 \cdots \mu_k} = 0, \quad i = 1, \ldots, k.$$

In the following, we will investigate the intrinsic structure of the generalized topological current $j^{\mu_1 \cdots \mu_k}$ by making use of the $\phi$–mapping method. From \cite{2}, we have

$$\partial_\mu n^a = \frac{1}{||\phi||} \partial_\mu \phi^a + \phi^a \partial_\mu (\frac{1}{||\phi||}), \quad \frac{\partial}{\partial \phi^a} (\frac{1}{||\phi||}) = - \frac{\phi^a}{||\phi||^3}$$

which should be looked upon as generalized functions \cite{29}. Due to these expressions the generalized topological current (1) can be rewritten as

$$j^{\mu_1 \cdots \mu_k} = C_m \frac{1}{\sqrt{g_x}} \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} \epsilon_{a_1 \cdots a_m} \cdot \partial_{\mu_{k+1}} \phi^a \cdots \partial_{\mu_n} \phi^{a_m} \frac{\partial}{\partial \phi^{a_1}} \frac{\partial}{\partial \phi^{a_1}} (G_m(||\phi||)), \quad (3)$$

where $C_m$ is a constant.
\[ C_m = \begin{cases} -\frac{1}{4A(S^{m-1})(m-2)(m-1)!}, & m > 2, \\ \frac{1}{2\pi}, & m = 2, \end{cases} \]

and \( G_m(||\phi||) \) is a Green function

\[ G_m(||\phi||) = \begin{cases} \frac{1}{||\phi||^{m-2}}, & m > 2, \\ \ln ||\phi||, & m = 2. \end{cases} \]

Defining general Jacobians \( J_{\mu_1\cdots\mu_k}(\frac{\phi}{x}) \) as following

\[ \epsilon^{a_1\cdots a_m}J_{\mu_1\cdots\mu_k}(\frac{\phi}{x}) = \epsilon^{\mu_1\cdots\mu_k\mu_{k+1}\cdots\mu_n} \partial_{\mu_{k+1}}\phi^{a_1} \partial_{\mu_{k+2}}\phi^{a_2} \cdots \partial_{\mu_n}\phi^{a_m} \]

and by making use of the \( m \)-dimensional Laplacian Green function relation in \( \phi \)-space [21]

\[ \Delta_\phi(G_m(||\phi||)) = -\frac{4\pi^{m/2}}{\Gamma\left(\frac{m}{2} - 1\right)} \delta(\tilde{\phi}) \]

where \( \Delta_\phi = \left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}\right) \) is the \( m \)-dimensional Laplacian operator in \( \phi \)-space, we do obtain the \( \delta \)-function structure of the defect topological current rigorously

\[ j_{\mu_1\cdots\mu_k} = \frac{1}{\sqrt{g_x}}\delta(\tilde{\phi})J_{\mu_1\cdots\mu_k}(\frac{\phi}{x}). \quad (4) \]

This expression involves the total defect information of the system and it indicates that all the defects are located at the zero points of the order parameter field \( \tilde{\phi}(x) \). It must be pointed out that, comparing to similar expressions in other papers, the results in (4) is gotten theoretically in a natural way. We find that \( j_{\mu_1\cdots\mu_k} \neq 0 \) only when \( \tilde{\phi} = 0 \), which is just the singularity of \( j_{\mu_1\cdots\mu_k} \). In detail, the Kernel of the \( \phi \)-mapping is the singularities of the topological tensor current \( j_{\mu_1\cdots\mu_k} \) in \( G \), i.e. the inner structure of the topological tensor current is labelled by the zeroes of \( \phi \)-mapping. We think that this is the essential of the topological tensor current theory and \( \phi \)-mapping is the key to study this theory.

From the above discussions, we see that the kernel of \( \phi \)-mapping plays an important role in the topological tensor current theory, so we are focussed on the zero points of \( \tilde{\phi} \) and will search for the solutions of the equations \( \phi^a(x) = 0 \) \( (a = 1, \ldots, m) \) by means of the implicit function theorem. These points are topological singularities in the orientation of the order parameter field \( \phi(x) \).
Suppose that the vector field $\vec{\phi}(x)$ possesses $l$ zeroes, according to the implicit function theorem [30], when the zeroes are regular points of $\phi$–mapping at which the rank of the Jacobian matrix $[\partial_\mu \phi^a]$ is $m$, the solutions of $\vec{\phi} = 0$ can be expressed parameterizedly by

$$x^\mu = z_i^\mu(u^1, \cdots, u^k), \quad i = 1, \ldots, l,$$

where the subscript $i$ represents the $i$–th solution and the parameters $u^I (I = 1, \ldots, k)$ span a $k$–dimensional submanifold with the metric tensor $g_{IJ} = g_{\mu\nu} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\nu}{\partial u^J}$ which is called the $i$–th singular submanifold $N_i$ in the Riemannian manifold $G$ corresponding to the $\phi$–mapping. These singular submanifolds $N_i$ are just the world volumes of the topological defects. For each singular manifold $N_i$, we can define a normal submanifold $M_i$ in $G$ which is spanned by the parameters $v^A$ with the metric tensor $g_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial v^A} \frac{\partial x^\nu}{\partial v^B}$ ($A, B = 1, \ldots, m$), and the intersection point of $M_i$ and $N_i$ is denoted by $p_i$ which can be expressed parameterizedly by $v^A = p_i^A$. In fact, in the words of differential topology, $M_i$ is transversal to $N_i$ at the point $p_i$. By virtue of the implicit function theorem at the regular point $p_i$, it should be held true that the Jacobian matrices $J(\vec{\phi}_v)$ satisfies

$$J(\vec{\phi}_v) = \frac{D(\phi^1, \cdots, \phi^m)}{D(v^1, \cdots, v^m)} \neq 0.$$  \hspace{1cm} (6)

In the following, we will investigate the topological charges of the topological defects and their quantization. Let $\Sigma_i$ be a neighborhood of $p_i$ on $M_i$ with boundary $\partial \Sigma_i$ satisfying $p_i \notin \partial \Sigma_i$, $\Sigma_i \cap \Sigma_j = \emptyset$. Then the generalized winding number $W_i$ of $n^a(x)$ at $p_i$ [31] can be defined by the Gauss map $n : \partial \Sigma_i \rightarrow S^{m-1}$

$$W_i = \frac{1}{A(S^{m-1})(m-1)!} \int_{\partial \Sigma_i} n^* (\epsilon_{a_1 \cdots a_m} n^{a_1} d n^{a_2} \wedge \cdots \wedge d n^{a_m})$$  \hspace{1cm} (7)

where $n^*$ denotes the pull back of map $n$. The generalized winding numbers is a topological invariant and is also called the degree of Gauss map [32]. It means that, when the point $v^A$ covers $\partial \Sigma_i$ once, the unit vector $n^a$ will cover a region $n[\partial \Sigma_i]$ whose area is $W_i$ times of $A(S^{m-1})$, i.e. the unit vector $n^a$ will cover the unit sphere $S^{m-1}$ for $W_i$ times. Using the Stokes’ theorem in exterior differential form and duplicating the above process, we get the compact form of $W_i$.
\[ W_i = \int_{\Sigma_i} \delta(\vec{\phi}) J(\frac{\vec{\phi}}{v}) d^m v. \]  

(8)

By analogy with the procedure of deducing \( \delta(f(x)) \), since

\[
\delta(\vec{\phi}) = \begin{cases} 
+\infty, & \text{for } \vec{\phi}(x) = 0 \\
0, & \text{for } \vec{\phi}(x) \neq 0 
\end{cases} = \begin{cases} 
+\infty, & \text{for } x \in N_i \\
0, & \text{for } x \notin N_i 
\end{cases},
\]

(9)

we can expand the \( \delta \)-function \( \delta(\vec{\phi}) \) as

\[ \delta(\vec{\phi}) = \sum_{i=1}^{l} c_i \delta(N_i), \]

(10)

where the coefficients \( c_i \) must be positive, i.e. \( c_i = |c_i| \). \( \delta(N_i) \) is the \( \delta \)-function in space-time \( G \) on a submanifold \( N_i \) \( [29] \)

\[ \delta(N_i) = \int_{N_i} 1 \sqrt{g_x} \delta^m(x - \vec{z}_i(u^1, \ldots, u^k)) \sqrt{g_u} d^k u, \]

(11)

where \( g_x = \det(g_{\mu\nu}), g_u = \det(g_{IJ}) \). Substituting (10) into (8), and calculating the integral, we get the expression of \( c_i \)

\[ c_i = \frac{\beta_i \sqrt{g_v}}{|J(\frac{\vec{\phi}}{v})|_{p_i}} = \frac{\beta_i \eta_i \sqrt{g_v}}{J(\frac{\vec{\phi}}{v})|_{p_i}}, \quad g_v = \det(g_{AB}), \]

(12)

where \( \beta_i = |W_i| \) is a positive integer called the Hopf index \( [32] \) of \( \phi \)-mapping on \( M_i \), it means that when the point \( v \) covers the neighborhood of the zero point \( p_i \) once, the function \( \vec{\phi} \) covers the corresponding region in \( \vec{\phi} \)-space \( \beta_i \) times, and \( \eta_i = \text{sign} J(\frac{\vec{\phi}}{v})|_{p_i} = \pm 1 \) is the Brouwer degree of \( \phi \)-mapping \( [32] \). Substituting this expression of \( c_i \) and (10) into (4), we gain the total expansion of the rank-\( k \) topological current

\[ j^{\mu_1 \cdots \mu_k} = \frac{1}{\sqrt{g_x}} \sum_{i=1}^{l} \frac{\beta_i \eta_i \sqrt{g_v}}{J(\frac{\vec{\phi}}{v})|_{p_i}} \delta(N_i) J^{\mu_1 \cdots \mu_k}(\frac{\vec{\phi}}{x}). \]

or in terms of parameters \( y^{A'} = (v^1, \ldots, v^m, u^1, \ldots, u^k) \)

\[ j^{A'_1 \cdots A'_k} = \frac{1}{\sqrt{g_y}} \sum_{i=1}^{l} \frac{\beta_i \eta_i \sqrt{g_v}}{J(\frac{\vec{\phi}}{v})|_{p_i}} \delta(N_i) J^{A'_1 \cdots A'_k}(\frac{\vec{\phi}}{x}). \]

(13)

From the above equation, we conclude that the inner structure of \( j^{\mu_1 \cdots \mu_k} \) or \( j^{A'_1 \cdots A'_k} \) is labelled by the total expansion of \( \delta(\vec{\phi}) \), which includes the topological information \( \beta_i \) and \( \eta_i \).
Taking $u^I$ and $u^J$ ($I = 2, \ldots, k$) be time-like evolution parameter and space-like parameters, respectively, the inner structure of the generalized topological current just represents $l$ $(k - 1)$–dimensional topological defects with topological charges $g_i = \beta_i \eta_i$ moving in the $n$–dimensional Riemann manifold $G$. The $k$-dimensional singular submanifolds $N_i$ ($i = 1, \ldots, l$) are their world sheets in the space-time. Mazenko [19] and Halperin [20] also got similar results for the case of point-like defects and line defects, but unfortunately, they did not consider the case $\beta_i \neq 1$. In fact, what they lost sight of is just the most important topological information for the charge of topological defects. In detail, the Hopf indices $\beta_i$ characterize the absolute values of the topological charges of these defects and the Brouwer degrees $\eta_i = +1$ correspond to defects while $\eta_i = -1$ to antidefects. Furthermore, they did not discuss what will happen when $\eta_i$ is indefinite, which we will study in detail in section 3.

Corresponding to the rank–$k$ topological tensor currents $j^{\mu_1 \cdots \mu_k}$, it is easy to see that the Lagrangian of many defects is just

$$ L = \sqrt{\frac{1}{k!} g_{\mu_1 \nu_1} \cdots g_{\mu_k \nu_k} j^{\mu_1 \cdots \mu_k} j^{\nu_1 \cdots \nu_k} = \delta(\vec{\phi}) } $$

which includes the total information of arbitrary dimensional topological defects in $G$ and is the generalization of Nielsen’s Lagrangian [33]. The action in $G$ is expressed by

$$ S = \int_G L \sqrt{g_2} \sqrt{dx} = \sum_{i=1}^{l} \beta_i \eta_i \int_{N_i} \sqrt{g_2} d^k u = \sum_{i=1}^{l} \beta_i \eta_i S_i $$

where $S_i$ is the area of the singular manifold $N_i$. It must be pointed out here that the Nambu–Goto action [34], which is the basis of many works on defect theory, is derived naturally from our theory. From the principle of least action, we obtain the evolution equations of many defect objects

$$ \frac{1}{\sqrt{g_2}} \frac{\partial}{\partial u^I} \left( \sqrt{g_2} g^{IJ} \frac{\partial x^\nu}{\partial u^J} \right) + g^{IJ} \Gamma^\nu_{\mu \lambda} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\lambda}{\partial u^J} = 0, \quad I, J = 1, \ldots, k. \quad (14) $$

As a matter of fact, this is just the equation of harmonic map [35].
III. THE BRANCH PROCESSES OF THE TOPOLOGICAL DEFECTS

With the discussion mentioned above, we know that the results in the above section are obtained straightly from the topological viewpoint under the condition

$$J(\phi/v)|_{p_i} \neq 0,$$

i.e., at the regular points of the order parameter field $\phi$. When the condition fails, i.e., the Brouwer degree $\eta_i$ are indefinite, what will happen? In what follows, we will study the case when $J(\phi/v)|_{p_i} = 0$. It often happens when the zero points of field $\phi$ include some branch points, which lead to the bifurcation of the topological current.

In this section, we will discuss the branch processes of these topological defects. In order to simplify our study, we select the parameter $u^1$ as the time-like evolution parameter $t$, and let the space-like parameters $u^I = \sigma^I (I = 2, \ldots, k)$ be fixed. In this case, the Jacobian matrices $J^{A_1 \cdots A_k}(\phi/y)$ are reduced to

$$J^{A_1 \cdots I_{k-1}}(\phi/y) \equiv J^A(\phi/y), \quad J^{A_1 \cdots I_{k-2}}(\phi/y) = 0, \quad J^{(m+1) \cdots n}(\phi/y) = J(\phi/v),$$

$$A, B = 1, \ldots, (m + 1), \quad I_j = m + 2, \ldots, n,$$

for $y^A = v^A (A \leq m)$, $y^{m+1} = t$, $y^{m+I} = \sigma^I (I \geq 2)$. The branch points are determined by the $m + 1$ equations

$$\phi^a(v^1, \ldots, v^m, t, \vec{\sigma}) = 0, \quad a = 1, \ldots, m$$  \hspace{1cm} (16)

and

$$\phi^{m+1}(v^1, \ldots, v^m, t, \vec{\sigma}) \equiv J(\phi/v) = 0$$  \hspace{1cm} (17)

for the fixed $\vec{\sigma}$. and they are denoted as $(t^*, p_i)$. In $\phi$–mapping theory usually there are two kinds of branch points, namely the limit points and bifurcation points [30], satisfying

$$J^1(\phi/y)|_{(t^*,p_i)} \neq 0$$  \hspace{1cm} (18)

and
respectively. In the following, we assume that the branch points \((t^*, p_i)\) of \(\phi\)-mapping have been found.

**A. Branch process at the limit point**

We first discuss the branch process at the limit point satisfying the condition (18). In order to use the theorem of implicit function to study the branch process of topological defects at the limit point, we use the Jacobian \(J^1(\frac{\phi}{y})\) instead of \(J(\frac{\phi}{v})\) to discuss the problem. In fact, this means that we have replaced the parameter \(t\) by \(v^1\). For clarity we rewrite the problem as

\[
\phi^a(t, v^2, \ldots, v^m, v^1, \vec{\sigma}) = 0, \quad a = 1, \ldots, m. \tag{20}
\]

Then, taking account of the condition (18) and using the implicit function theorem, we have an unique solution of the equations (20) in the neighborhood of the limit point \((t^*, p_i)\)

\[
t = t(v^1, \vec{\sigma}), \quad v^i = v^i(v^1, \vec{\sigma}), \quad i = 2, 3, \ldots, m \tag{21}
\]

with \(t^* = t(p_i^1, \vec{\sigma})\). In order to show the behavior of the defects at the limit points, we will investigate the Taylor expansion of (21) in the neighborhood of \((t^*, p_i)\). In the present case, from (18) and (17), we get

\[
\frac{dv^1}{dt}\bigg|_{(t^*, p_i)} = \frac{J^1(\frac{\phi}{y})}{J(\frac{\phi}{y})}\bigg|_{(t^*, p_i)} = \infty,
\]

i.e.

\[
\frac{dt}{dv^1}\bigg|_{(t^*, p_i)} = 0.
\]

Then we have the Taylor expansion of (21) at the point \((t^*, p_i)\)

\[
t = t(p_i, \vec{\sigma}) + \frac{dt}{dv^1}\bigg|_{(t^*, p_i)}(v^1 - p_i^1) + \frac{1}{2}\frac{d^2t}{(dv^1)^2}\bigg|_{(t^*, p_i)}(x^1 - p_i^1)^2
\]
\[ t^* + \frac{1}{2} \frac{d^2 t}{(dv^1)^2} |(v^1, p_i)(v^1 - p_i^1)^2. \]

Therefore

\[ t - t^* = \frac{1}{2} \frac{d^2 t}{(dv^1)^2} |(v^1, p_i)(v^1 - p_i^1)^2 \] (22)

which is a parabola in the \( v^1 - t \) plane. From (22), we can obtain the two solutions \( v^1(1)(t, \vec{\sigma}) \) and \( v^1(2)(t, \vec{\sigma}) \), which give the branch solutions of the system (16) at the limit point. If \( \frac{d^2 t}{(dv^1)^2} |(v^1, p_i) > 0 \), we have the branch solutions for \( t > t^* \) (Fig 1(a)), otherwise, we have the branch solutions for \( t < t^* \) (Fig 1(b)). The former is related to the creation of defect and antidefect in pair at the limit points, and the latter to the annihilation of the topological defects, since the topological current of topological defects is identically conserved, the topological quantum numbers of these two generated topological defects must be opposite at the limit point, i.e. \( \beta_1 \eta_1 + \beta_2 \eta_2 = 0 \).

### B. Branch process at the bifurcation point

In the following, let us consider the case (19), in which the restrictions of the system (16) at the bifurcation point \((t^*, p_i)\) are

\[ J(\phi_v)|_{(t^*, p_i)} = 0, \quad J(\phi_y)|_{(t^*, p_i)} = 0. \] (23)

These two restrictive conditions will lead to an important fact that the dependency relationship between \( t \) and \( v^1 \) is not unique in the neighborhood of the bifurcation point \((t^*, p_i)\). In fact, we have

\[ \frac{dv^1}{dt}|_{(t^*, p_i)} = \frac{J^1(\phi_y)}{J(\phi_y)}|_{(t^*, p_i)} \] (24)

which under the restraint (23) directly shows that the tangential direction of the integral curve of equation (24) is indefinite at the point \((t^*, p_i)\). Hence, (24) does not satisfy the conditions of the existence and uniqueness theorem of the solution of a differential equation. This is why the very point \((t^*, \vec{z}_i)\) is called the bifurcation point of the system (16).
In the following, we will find a simple way to search for the different directions of all branch curves at the bifurcation point. As assumed that the bifurcation point \((t^*, p_i)\) has been found from (16) and (17), the following calculations are all conducted at the value \((t^*, p_i)\). As we have mentioned above, at the bifurcation point \((t^*, p_i)\), the rank of the Jacobian matrix \(\frac{\partial \phi}{\partial v}\) is smaller than \(m\). In order to derive the calculating method, we consider the rank of the Jacobian matrix \(\frac{\partial \phi}{\partial v}\) is \(m - 1\). The case of a more smaller rank will be discussed in next subsection. Suppose that one of the \((m - 1) \times (m - 1)\) submatrix \(J_1(\phi_v)\) of the Jacobian matrix \(\frac{\partial \phi}{\partial v}\) is

\[
J_1(\phi_v) = \begin{pmatrix}
\phi_1^1 & \phi_1^2 & \cdots & \phi_1^m \\
\phi_2^1 & \phi_2^2 & \cdots & \phi_2^m \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{m-1}^1 & \phi_{m-1}^2 & \cdots & \phi_{m-1}^m
\end{pmatrix}
\]  

(25)

and its determinant \(\text{det } J_1(\phi_v)\) does not vanish at the point \((t^*, p_i)\) (otherwise, we have to rearrange the equations of (16)), where \(\phi^A_a\) stands for \((\partial \phi^a / \partial v^A)\) \((a = 1, ..., m - 1; \ A = 2, ..., m)\). By means of the implicit function theorem we obtain one and only one functional relationship in the neighborhood of the bifurcation point \((t^*, p_i)\)

\[
v^A = f^A(v^1, t, \sigma^2, \ldots, \sigma^k), \quad A = 2, 3, ..., n
\]  

(26)

with the partial derivatives

\[
f^A_1 = \frac{\partial v^A}{\partial v^1}, \quad f^A_t = \frac{\partial v^A}{\partial t}, \quad A = 2, 3, ..., n.
\]

Then, for \(a = 1, ..., m - 1\) we have

\[
\phi^a = \phi^a(v^1, f^2(v^1, t, \bar{\sigma}), ..., f^m(v^1, t, \bar{\sigma}), t, \bar{\sigma}) \equiv 0
\]

which gives

\[
\sum_{A=2}^{m} \frac{\partial \phi^a}{\partial v^A} f^A_1 = - \frac{\partial \phi^a}{\partial v^1}, \quad a = 1, ..., m - 1
\]  

(27)
\[ \sum_{A=2}^{m} \frac{\partial \phi^a}{\partial v^A} f^A_t = - \frac{\partial \phi^a}{\partial t}, \quad a = 1, \ldots, m - 1 \]  

(28)

from which we can calculate the first order derivatives of \( f^A \): \( f^A_1 \) and \( f^A_t \). Denoting the second order partial derivatives as

\[ f^A_{11} = \frac{\partial^2 v^A}{(\partial v^1)^2}, \quad f^A_{1t} = \frac{\partial^2 v^A}{\partial v^1 \partial t}, \quad f^A_{tt} = \frac{\partial^2 v^A}{\partial t^2} \]

and differentiating (27) with respect to \( v^1 \) and \( t \) respectively, we get

\[ \sum_{A=2}^{m} \phi^a_A f^A_{11} = - \sum_{A=2}^{m} \left[ 2 \phi^a_A f^A_1 + \sum_{B=2}^{m} (\phi^a_{AB} f^B_t) f^A_1 \right] - \phi^a_{11}, \quad a = 1, 2, \ldots, m - 1 \]  

(29)

\[ \sum_{A=2}^{m} \phi^a_A f^A_{1t} = - \sum_{A=2}^{m} \left[ \phi^a_{A1} f^A_1 + \phi^a_A f^A_t + \sum_{B=2}^{m} (\phi^a_{AB} f^B_t) f^A_1 \right] - \phi^a_{1t}, \quad a = 1, 2, \ldots, m - 1. \]  

(30)

And the differentiation of (28) with respect to \( v^1 \) gives

\[ \sum_{A=2}^{m} \phi^a_A f^A_{tt} = - \sum_{A=2}^{m} \left[ 2 \phi^a_A f^A_1 + \sum_{B=2}^{m} (\phi^a_{AB} f^B_t) f^A_1 \right] - \phi^a_{tt}, \quad a = 1, 2, \ldots, m - 1 \]  

(31)

where

\[ \phi^a_{AB} = \frac{\partial^2 \phi^a}{\partial v^A \partial v^B}, \quad \phi^a_{At} = \frac{\partial^2 \phi^a}{\partial v^A \partial t} \]

The differentiation of (28) with respect to \( v^1 \) gives the same expression as (30). If we use the Gaussian elimination method to the three vectors at the right hands of the formulas (29), (30) and (31), we can obtain the three partial derivatives \( f^A_{11} \), \( f^A_{1t} \) and \( f^A_{tt} \). Notice that the three equations (29), (30) and (31) have the same coefficient matrix \( J_1(\phi_f) \), which are assumed to be nonzero, and we should substitute the values of the partial derivatives \( f^A_1 \) and \( f^A_t \), which have been calculated out in the former, into the right hands of the three equations.

The above discussions do not matter to the last component \( \phi^m(v^1, \ldots, v^m, t, \bar{\sigma}) \). In order to find the different values of \( dv^1/dt \) at the bifurcation point, let us investigate the Taylor expansion of \( \phi^m(v^1, \ldots, v^m, t, \bar{\sigma}) \) in the neighborhood of \( (t^*, p_i) \). Substituting the existing, but unknown, dependency relationship (26) into \( \phi^m(v^1, \ldots, v^m, t, \bar{\sigma}) \), we get the function of two variables \( v^1 \) and \( t \)
\[ F(t, v^1, \vec{\sigma}) = \phi^m(v^1, f^2(v^1, t, \vec{\sigma}), ..., f^m(v^1, t, \vec{\sigma}), t, \vec{\sigma}) \] (32)

which according to (16) must vanish at the bifurcation point

\[ F(t^*, p_i) = 0. \] (33)

From (32), we can calculate the first order partial derivatives of \( F(t, v^1, \vec{\sigma}) \) with respect to \( v^1 \) and \( t \) respectively at the bifurcation point \((t^*, p_i)\)

\[
\frac{\partial F}{\partial v^1} = \phi_1^m + \sum_{A=2}^{m} \phi_A^m f_A^1, \quad \frac{\partial F}{\partial t} = \phi_1^m + \sum_{A=2}^{m} \phi_A^m f_A^t. \] (34)

Using (27) and (28), the first equation of (23) is expressed by

\[
J(\phi_v)|_{(t^*, p_i)} = \begin{vmatrix}
- \sum_{A=2}^{m} \phi_A^1 f_A^1 & \phi_1^1 & \cdots & \phi_m^1 \\
- \sum_{A=2}^{m} \phi_A^2 f_A^1 & \phi_1^2 & \cdots & \phi_m^2 \\
\vdots & \vdots & \ddots & \vdots \\
- \sum_{A=2}^{m} \phi_A^{m-1} f_A^1 & \phi_1^{m-1} & \cdots & \phi_m^{m-1} \\
\phi_A^m & \phi_2^m & \cdots & \phi_m^m
\end{vmatrix}_{(t^*, p_i)} = 0
\]

which, by Cramer’s rule, (25) and (34), can be rewritten as

\[
J(\phi_v)|_{(t^*, p_i)} = \begin{vmatrix}
0 & \phi_1^2 & \cdots & \phi_m^1 \\
0 & \phi_2^2 & \cdots & \phi_m^2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \phi_1^{m-1} & \cdots & \phi_m^{m-1} \\
\phi_1^m + \sum_{A=2}^{m} \phi_A^m f_A^1 & \phi_2^m & \cdots & \phi_m^m
\end{vmatrix}_{(t^*, p_i)} = \frac{\partial F}{\partial v^1} \det J_1(\phi_v)|_{(t^*, p_i)} = 0.
\]

Since

\[
\det J_1(\phi_v)|_{(t^*, p_i)} \neq 0
\]

which is our assumption, the above equation leads to
\[ \frac{\partial F}{\partial u} |_{(t^*, p_i)} = 0. \] (35)

With the same reasons, we can prove that

\[ \frac{\partial F}{\partial t} |_{(t^*, p_i)} = 0. \] (36)

The second order partial derivatives of the function \( F(t, v^1, \vec{\sigma}) \) are easily to find out to be

\[
\frac{\partial^2 F}{(\partial v^1)^2} = \phi_{11}^m + \sum_{A=2}^m [2\phi_{1A}^m f_1^A + \phi_A^m f_{11}^A + \sum_{B=2}^m (\phi_{AB}^m f_B^A)]
\]

\[
\frac{\partial^2 F}{\partial v^1 \partial t} = \phi_{1t}^m + \sum_{A=2}^m [\phi_{1A}^m f_t^A + \phi_A^m f_{1t}^A + \sum_{B=2}^m (\phi_{AB}^m f_B^A)]
\]

\[
\frac{\partial^2 F}{\partial t^2} = \phi_{tt}^m + \sum_{A=2}^m [2\phi_{At}^m f_t^A + \phi_A^m f_{tt}^A + \sum_{B=2}^m (\phi_{AB}^m f_B^A)]
\]

which at \((t^*, p_i)\) are denoted by

\[
A = \frac{\partial^2 F}{(\partial v^1)^2} |_{(t^*, p_i)}, \quad B = \frac{\partial^2 F}{\partial v^1 \partial t} |_{(t^*, p_i)}, \quad C = \frac{\partial^2 F}{\partial t^2} |_{(t^*, p_i)}. \] (37)

Then, by virtue of (33), (35), (36) and (37), the Taylor expansion of \( F(t, v^1, \vec{\sigma}) \) in the neighborhood of the bifurcation point \((t^*, p_i)\) can be expressed as

\[
F(t, v^1, \vec{\sigma}) = \frac{1}{2} A(v^1 - p_i^1)^2 + B(v^1 - p_i^1)(t - t^*) + \frac{1}{2} C(t - t^*)^2 \] (38)

which is the expression of \( \phi^m(v^1, \ldots, v^m, t, \vec{\sigma}) \) in the neighborhood of \((t^*, p_i)\). The expression (38) shows that at the bifurcation point \((t^*, p_i)\)

\[
A(v^1 - p_i^1)^2 + 2B(v^1 - p_i^1)(t - t^*) + C(t - t^*)^2 = 0. \] (39)

Dividing (39) by \((v^1 - p_i^1)^2\) or \((t - t^*)^2\), and taking the limit \( t \to t^* \) as well as \( v^1 \to p_i^1 \) respectively, we get two equations

\[ C\left(\frac{dt}{dv^1}\right)^2 + 2B \frac{dt}{dv^1} + A = 0. \] (40)

and
\[ A\left(\frac{dv_1}{dt}\right)^2 + 2B\frac{dv_1}{dt} + C = 0. \] (41)

So we get the different directions of the branch curves at the bifurcation point from the solutions of (40) or (41). There are four possible cases:

Firstly, \( A \neq 0, \Delta = 4(B^2 - AC) > 0, \) from Eq. (40) we get two different solutions: \( \frac{dv_1}{dt} \bigg|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}, \) which is shown in Fig. 2, where two topological defects meet and then depart at the bifurcation point. Secondly, \( A \neq 0, \Delta = 4(B^2 - AC) = 0, \) there is only one solution: \( \frac{dv_1}{dt} = \frac{-B}{A}, \) which includes three important cases: (a) two topological defects tangentially collide at the bifurcation point (Fig 3(a)); (b) two topological defects merge into one topological defect at the bifurcation point (Fig 3(b)); (c) one topological defect splits into two topological defects at the bifurcation point (Fig 3(c)). Thirdly, \( A = 0, C \neq 0, \Delta = 4(B^2 - AC) > 0, \) from Eq. (41) we have \( \frac{dt}{dv_1} = 0 \) and \( -2B/C. \) There are two important cases: (i) One topological defect splits into three topological defects at the bifurcation point (Fig 4(a)); (ii) Three topological defects merge into one at the bifurcation point (Fig 4(b)). Finally, \( A = C = 0, \) Eqs. (40) and (41) give respectively \( \frac{dv_1}{dt} = 0 \) and \( \frac{dt}{dv_1} = 0. \) This case is obvious as in Fig. 5, which is similar to the third situation.

In order to determine the branches directions of the remainder variables, we will use the relations simply

\[ dv^A = f_1^A dv_1 + f_t^A dt, \quad A = 2, 3, ..., n \]

where the partial derivative coefficients \( f_1^A \) and \( f_t^A \) have given in (27) and (28). Then, respectively

\[ \frac{dv^A}{dv_1} = f_1^A + f_t^A \frac{dt}{dv_1} \]

or

\[ \frac{dv^A}{dt} = f_1^A \frac{dv_1}{dt} + f_t^A. \] (42)

where partial derivative coefficients \( f_1^A \) and \( f_t^A \) are given by (27) and (28). From this relations we find that the values of \( \frac{dv^A}{dt} \) at the bifurcation point \( (t^*, z_i) \) are also possibly different because (41) may give different values of \( \frac{dv_1}{dt}. \)
C. Branch process at a higher degenerated point

In the following, let us discuss the branch process at a higher degenerated point. In the above subsection, we have analysed the case that the rank of the Jacobian matrix \( \frac{\partial \phi}{\partial v} \) of the equation (17) is \( m - 1 \). In this section, we consider the case that the rank of the Jacobian matrix is \( m - 2 \) (for the case that the rank of the matrix \( \frac{\partial \phi}{\partial v} \) is lower than \( m - 2 \), the discussion is in the same way). Let the \((m - 2) \times (m - 2)\) submatrix \( J_2(\frac{\phi}{v}) \) of the Jacobian matrix \( \frac{\partial \phi}{\partial v} \) be

\[
J_2(\frac{\phi}{v}) = \begin{pmatrix}
\phi_1^1 & \phi_2^1 & \cdots & \phi_m^1 \\
\phi_1^2 & \phi_2^2 & \cdots & \phi_m^2 \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1^{m-2} & \phi_2^{m-2} & \cdots & \phi_m^{m-2}
\end{pmatrix}
\]

and suppose that \( \det J_2(\frac{\phi}{v})|_{(t^*,p_i)} \neq 0 \). With the same reasons of obtaining (26), we can have the function relations

\[
v^A = f^A(v^1, v^2, t, \bar{\sigma}), \quad A = 3, 4, ..., m. \quad (43)
\]

For the partial derivatives \( f_1^A, f_2^A \) and \( f_i^A \), we can easily derive the system similar to the equations (27) and (28), in which the three terms at the right hand of can be figured out at the same time. In order to determine the 2–order partial derivatives \( f_{11}^A, f_{12}^A, f_{1i}^A, f_{22}^A, f_{2i}^A \) and \( f_{ii}^A \), we can use the equations similar to (29), (30) and (31). Substituting the relations (43) into the last two equations of the system (16), we have the following two equations with respect to the arguments \( v^1, v^2, t, \bar{\sigma} \)

\[
\begin{align*}
F_1(v^1, v^2, t, \bar{\sigma}) &= \phi^{m-1}(v^1, v^2, f^3(v^1, v^2, t, \bar{\sigma}), \cdots, f^m(v^1, v^2, t, \bar{\sigma}), t, \bar{\sigma}) = 0 \\
F_2(v^1, v^2, t, \bar{\sigma}) &= \phi^{m}(v^1, v^2, f^3(v^1, v^2, t, \bar{\sigma}), \cdots, f^m(v^1, v^2, t, \bar{\sigma}), t, \bar{\sigma}) = 0.
\end{align*} \quad (44)
\]

Calculating the partial derivatives of the function \( F_1 \) and \( F_2 \) with respect to \( v^1, v^2 \) and \( t \), taking notice of (13) and using six similar expressions to (35) and (36), i.e.

\[
\left. \frac{\partial F_j}{\partial v^1} \right|_{(t^*,p_i)} = 0, \quad \left. \frac{\partial F_j}{\partial v^2} \right|_{(t^*,p_i)} = 0, \quad \left. \frac{\partial F_j}{\partial t} \right|_{(t^*,p_i)} = 0, \quad j = 1, 2, \quad (45)
\]
we have the following forms of Taylor expressions of $F_1$ and $F_2$ in the neighborhood of $(t^*, p_i)$

$$
F_j(v^1, v^2, t, \sigma) \approx A_{j1}(v^1 - p^1_i)^2 + A_{j2}(v^1 - p^1_i)(v^2 - p^2_i) + A_{j3}(v^1 - p^1_i)
$$

$$(t - t^*) + A_{j4}(v^2 - p^2_i)^2 + A_{j5}(v^2 - p^2_i)(t - t^*) + A_{j6}(t - t^*)^2 = 0$$

$$j = 1, 2. \quad (46)$$

In the case of $A_{j1} \neq 0, A_{j4} \neq 0$, by dividing (46) by $(t - t^*)^2$ and taking the limit $t \to t^*$, we obtain two quadratic equations of $\frac{dv^1}{dt}$ and $\frac{dv^2}{dt}$

$$A_{j1}\left(\frac{dv^1}{dt}\right)^2 + A_{j2}\frac{dv^1}{dt}\frac{dv^2}{dt} + A_{j3}\frac{dv^1}{dt} + A_{j4}\left(\frac{dv^2}{dt}\right)^2 + A_{j5}\frac{dv^2}{dt} + A_{j6} = 0 \quad (47)$$

$$j = 1, 2.$$

Eliminating the variable $dv^1/dt$, we obtain a equation of $dv^2/dt$ in the form of a determinant

$$\begin{vmatrix}
A_{11} & A_{12}Q + A_{23} & A_{14}Q^2 + A_{15}Q + A_{16} & 0 \\
0 & A_{11} & A_{12}Q + A_{13} & A_{14}Q^2 + A_{15}Q + A_{16} \\
A_{21} & A_{22}Q + A_{23} & A_{24}Q^2 + A_{25}Q + A_{26} & 0 \\
0 & A_{21} & A_{22}Q + A_{23} & A_{24}Q^2 + A_{25}Q + A_{26}
\end{vmatrix} = 0 \quad (48)$$

where $Q = dv^2/dt$, which is a 4th order equation of $dv^2/dt$

$$a_0\left(\frac{dv^2}{dt}\right)^4 + a_1\left(\frac{dv^2}{dt}\right)^3 + a_2\left(\frac{dv^2}{dt}\right)^2 + a_3\left(\frac{dv^2}{dt}\right) + a_4 = 0. \quad (49)$$

Therefore we get different directions at the bifurcation point corresponding to different branch curves. The number of different branch curves is four at most. If the degree of degeneracy of the matrix $\frac{\partial \phi}{\partial v}$ is more higher, i.e. the rank of the matrix $\frac{\partial \phi}{\partial v}$ is more lower than the present $(m - 2)$ case, the procedure of deduction will be more complicate. In general supposing the rank of the matrix $\frac{\partial \phi}{\partial v}$ be $(m - s)$, the number of the possible different directions of the branch curves is $2^s$ at most.
At the end of this section, we conclude that there exist crucial cases of branch processes in our topological defect theory. This means that a topological defect, at the bifurcation point, may split into several (for instance $s$) topological defects along different branch curves with different charges. Since the topological current is a conserved current, the total quantum number of the splitting topological defects must precisely equal to the topological charge of the original defect i.e.

$$
\sum_{j=1}^{s} \beta_{ij} \eta_{ij} = \beta_{i} \eta_{i}
$$

for fixed $i$. This can be looked upon as the topological reason of the defect splitting. Here we should point out that such splitting is a stochastic process, the sole restriction of this process is just the conservation of the topological charge of the topological defects during this splitting process. Of course, the topological charge of each splitting defects is an integer.

In summary, we have studied the topological property of the arbitrary dimensional topological defects in general case by making use of the $\phi$–mapping topological current theory and the implicit function theorem. We would like to point out that all the results in this paper are gained from the viewpoint of topology without any particular models or hypothesis.
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FIGURES’ CAPTIONS

Fig. 1. (a) The creation of two topological defects. (b) Two topological defects annihilate in collision at the limit point.

Fig. 2. Two topological defects collide with different directions of motion at the bifurcation point.

Fig. 3. Topological defects have the same direction of motion. (a) Two topological defects tangentially collide at the bifurcation point. (b) Two topological defects merge into one topological defect at the bifurcation point. (c) One topological defect splits into two topological defects at the bifurcation point.

Fig. 4. (a) One topological defect splits into three topological defects at the bifurcation point. (b) Three topological defects merge into one topological defect at the bifurcation point.

Fig. 5. This case is similar to Fig. 4. (a) Three topological defects merge into one topological defect at the bifurcation point. (b) One topological defect splits into three topological defects at the bifurcation point.
Fig. 1(b)
Fig. 3(a)
Fig. 3(b)
Fig. 3(c)
Fig. 4(a)
Fig. 4(b)
Fig. 5(a)
Fig. 5(b)