On the error exponents of binary state discrimination with composite hypotheses

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Abstract

The trade-off between the two types of errors in binary state discrimination may be quantified in the asymptotics by various error exponents. In the case of simple i.i.d. hypotheses, each of these exponents is equal to a divergence (pseudo-distance) of the two states. In the case of composite hypotheses, represented by sets of states $R, S$, one always has the inequality $e(R\|S) \leq E(R\|S)$, where $e$ is the exponent, $E$ is the corresponding divergence, and the question is whether equality holds. This has been shown to be the case for various general settings in classical state discrimination, and for the Stein exponent with simple alternative hypothesis for finite-dimensional quantum state discrimination. On the other hand, examples with strict inequality have been shown for the Stein and the Chernoff exponents for discriminating a single finite-dimensional quantum state from a set of states with continuum cardinality representing the alternative hypothesis. These results suggest that the relation between the composite exponents and the worst pairwise exponents may be influenced by a number of factors: the type of exponents considered; whether the problem is classical or quantum; the cardinality and the geometric properties of the sets representing the hypotheses; and, on top of the above, possibly whether the underlying Hilbert space is finite- or infinite-dimensional.

Our main contribution in this paper is clarifying this landscape considerably: We exhibit explicit examples for hitherto unstudied cases where the above inequality fails to hold with equality, while we also prove equality for various general classes of state discrimination problems. In particular, we show that equality may fail for any of the error exponents even in the classical case, if the system is allowed to be infinite-dimensional, and the alternative hypothesis contains countably infinitely many states. Moreover, we show that in the quantum case strict inequality is the generic behavior in the sense that, starting from any pair of non-commuting density operators of any dimension, and for any of the exponents, it is possible to construct an example with a simple null-hypothesis and an alternative hypothesis consisting of only two states, such that strict inequality holds for the given exponent.

I. INTRODUCTION

State discrimination is one of the fundamental problems in statistics, with applications in many information-theoretic problems [21, 22, 53]. In a typical binary i.i.d. quantum state discrimination problem, an experimenter is presented with several identically prepared quantum systems, all in the same state that either belongs to a set $R \subseteq S(\mathcal{H})$ (null-hypothesis $H_0$), or to another set $S \subseteq S(\mathcal{H})$ (alternative hypothesis $H_1$), where $S(\mathcal{H})$ is the set of all density operators on the system’s Hilbert space $\mathcal{H}$. The experimenter’s task is to guess which hypothesis is correct, in the sense that, starting from any pair of non-commuting density operators of any dimension, and for any of the hypotheses, it is possible to construct an example with a simple null-hypothesis and an alternative hypothesis consisting of only two states, such that strict inequality holds for the given exponent.

The experimenter makes an erroneous decision by rejecting the null-hypothesis when it is true (type I error), or by accepting it when it is false (type II error). The worst-case probabilities of these events are given by

$$\alpha_n(R|T_n) := \sup_{\sigma \in R} \text{Tr} \sigma^\otimes n (I - T_n), \quad \beta_n(S|T_n) := \sup_{\sigma \in S} \text{Tr} \sigma^\otimes n T_n,$$

(type I),

(type II).

Clearly, there is a trade-off between the two types of error probabilities, which can be quantified in the asymptotics $n \to +\infty$ by various error exponents, depending on the way the two types of errors are optimized with respect to each other. The most often studied ones are the Chernoff exponent $c(R\|S)$ of symmetric state discrimination, the Stein exponent $s(R\|S)$ of asymmetric state discrimination, and the one-parameter family of direct exponents $d_r(R\|S)$, $0 < r < s(R\|S)$, describing the whole trade-off curve when both errors vanish asymptotically; we will give precise definitions of these notions in Section II C.

Let us denote by $e(R\|S)$ any of these exponents. It is easy to see that the exponents for the state discrimination problem $R$ vs. $S$ cannot be better than the worst pairwise exponents for the state discrimination problems $\rho$ vs. $\sigma$, 

$$
\alpha_n(R|T_n) := \sup_{\rho \in R} \text{Tr} \rho^\otimes n (I - T_n), \quad \beta_n(S|T_n) := \sup_{\sigma \in S} \text{Tr} \sigma^\otimes n T_n,
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(type I),

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$\rho \in R, \sigma \in S$, i.e.,
\[ e(R\|S) \leq \inf_{\rho \in R, \sigma \in S} e(\rho\|\sigma), \tag{I.3} \]
where we write $e(\rho|\sigma)$ instead of $e(\{\rho\}||\{\sigma\})$. Moreover, the error exponents for simple hypotheses ($R = \{\rho\}$ vs. $S = \{\sigma\}$) can be explicitly expressed as certain divergences of the two states [4, 21, 28, 44, 49, 52], and therefore if (I.3) holds as an equality, these results provide expressions for the exponents of the composite hypothesis testing problem in terms of divergence distances of the two sets $R$ and $S$. Therefore, it is of fundamental importance to know when the inequality in (I.3) holds as an equality.

Such equalities have been established, or follow easily from known results, in the following cases (the Hilbert space dimension is assumed to be finite):

- for the Stein and the Chernoff exponents when all states commute (classical case), and both $R$ and $S$ are compact convex sets [11];
- for the Stein exponent when $|S| = 1$ (Stein’s lemma with composite null-hypothesis) [8, 10, 38, 48];
- for the Chernoff exponent when $|R|, |S| < +\infty$, and all states commute, or $R$ consists of a single pure state [3].

In fact, the first result was established in [11] in a strictly stronger form, for the non-i.i.d. problem of classical adversarial hypothesis testing, and the equality for the Stein exponent was established in [48] also for the non-i.i.d. problem of quantum arbitrarily varying state discrimination. We note that the statement in [10] is somewhat weaker than how we formulated the problem above, as in [10] the type I error is not shown to converge to zero uniformly over all candidate states in the null hypotheses, but only individually. This is a subtle but important difference, which will also appear at various places in our analysis of the error exponents; we will refer to the corresponding error exponents as relaxed exponents. On the other hand, it was shown recently in [8] that (I.3) may hold as a strict inequality for the Stein exponent, in a finite-dimensional setting, with a simple null-hypothesis and a composite alternative hypothesis of continuum cardinality.

Instead of the worst-case error probabilities (I.1), it is also very natural to work with some weighted averages of the error probabilities, defined as
\[ \alpha_n(p|T_n) := \int_R dp(q) \text{Tr} \rho^\otimes n (I - T_n), \quad \text{(type I)}, \tag{I.4} \]
\[ \beta_n(q|T_n) := \int_S dq(\sigma) \text{Tr} \sigma^\otimes n T_n, \quad \text{(type II)}, \tag{I.5} \]
where $p, q$ are probability measures on the Borel sets of the state space, reflecting some prior knowledge about the likeliness of the candidate states, or about the relative severity of misidentifying them. We call this the mixed i.i.d. setting. It is easy to see that
\[ e(\text{supp} p\| \text{supp} q) \leq e(p\|q) \leq \inf_{\rho \in \text{supp} p, \sigma \in \text{supp} q} e(\rho\|\sigma), \tag{I.6} \]
and hence we get a refinement of the equality problem in (I.3): the two inequalities in (I.6) can be analyzed separately, and if both of them hold with equality then so does the inequality in (I.3) (with $R = \text{supp} p$ and $S = \text{supp} q$). It is easy to see that the first equality in (I.6) holds as an equality whenever $\text{supp} p$ and $\text{supp} q$ are finite, and hence the analysis of this inequality becomes relevant when at least one of the probability distributions is supported on a set of infinite cardinality. The state discrimination problem with the weighted error probabilities may also be interpreted as a binary state discrimination problem with simple, but non-i.i.d. hypothesis, given as $H_0 : (\int dp(\rho) \rho^\otimes n)_{n \in \mathbb{N}}$ vs. $H_1 : (\int dq(\sigma) \sigma^\otimes n)_{n \in \mathbb{N}}$, a special case of which is i.i.d. state discrimination with group covariant measurements [25]. In this setting, an explicit example demonstrating strict inequality for the Chernoff exponent and for the direct exponents was given in [25, Example 6.2], with with $q$ being a Dirac measure (simple i.i.d. alternative hypothesis) and $p$ supported on a set of continuum cardinality.

The above results suggest that the relation between the composite exponents and the worst pairwise exponents may be influenced by a number of factors: the type of exponents considered; whether the problem is classical or quantum; the cardinality and the geometric properties of the sets representing the hypotheses; and, on top of the above, possibly whether the underlying Hilbert space is finite- or infinite-dimensional. Our main contribution in this paper is clarifying this landscape considerably: We exhibit explicit examples for hitherto unstudied cases where the above inequalities fail to hold with equality, while we also prove equality for various general classes of state discrimination problems.

The structure of the paper is as follows. In Section II we give the necessary preliminaries and prove some simple general results about the error exponents and the related divergences between sets of states, which will be used throughout the paper.
Section III is devoted to classical hypothesis testing. In Section III A we show that equality holds in (I.3) for all the error exponents (Stein, Chernoff, direct exponents) in the most general classical setting (represented by commutative von Neumann algebras) whenever both sets $R$ and $S$ are finite. In fact, we show this more generally for countable sets and the “relaxed” exponents, where the errors are not required to follow the given asymptotics uniformly over all states in the two hypotheses, but only individually. On the other hand, in Section III B we show that this is no longer true for any of the “strong” exponents (defined from the worst-case errors (I.1)), at least if we allow the system to be infinite-dimensional; we construct an explicit example with probability distributions on the $[0,1]$ interval, where the null-hypothesis is simple (the uniform distribution), the alternative hypothesis consists of countably infinitely many states, and both inequalities in (I.6) are strict for all the error exponents. In Section III C we study the finite-dimensional case, and show that equality holds in (I.3) for all the error exponents provided that both $R$ and $S$ are compact convex sets. In fact, we prove the stronger statement that all the error exponents for arbitrary $R$ and $S$ coincide with the respective worst pairwise error exponents over the closed convex hulls of $R$ and $S$ in the arbitrarily varying and in the adversarial settings. This extends previous results in [11], where the cases of the Stein and the Chernoff exponents were proved in the adversarial setting for closed convex sets, and in [19, 20], where the cases of the Stein and the direct exponents were proved for finite sets.

In Section IV we consider the one-parameter family of strong converse exponents $sc_r(R\|S)$, $r > s(R\|S)$. In this case the trivial relation is opposite to (I.3),

$$sc_r(R\|S) \geq \sup_{\rho \in R, \sigma \in S} sc_r(\rho\|\sigma),$$

and the question is again when equality holds. It was shown recently in [12] that equality holds when the alternative hypothesis is simple and the null-hypothesis consists of finitely many states. As we show in Section IV A, this can be easily extended to the relaxed version of the strong converse exponent for a null-hypothesis of arbitrary cardinality, provided that it contains a countable set that is dense with respect to the max-relative entropy pseudo-distance; this is satisfied, for instance, in the classical case, or if the states all have the same support. The opposite case, with a simple null-hypothesis and a composite alternative hypothesis, turns out to behave very differently. Indeed, in Section IV C 1 we give an explicit 2-dimensional classical example with a simple null-hypothesis and two alternative hypotheses where (I.7) holds with strict equality. This is in sharp contrast also with the behavior of the other error exponents discussed above, where equality holds in the classical case for finitely many hypotheses. It turns out, however, that equality still holds in the finite-dimensional classical case when the alternative hypothesis is given by a closed convex set. More generally, we show in Section IV C 2 that in the finite-dimensional classical case, the strong converse exponents for arbitrary $R$ and $S$ coincide with the respective worst pairwise strong converse exponents over the closed convex hulls of $R$ and $S$ in the arbitrarily varying and in the adversarial setting. When combined with the result of Section IV A, this yields equality in (I.7) for the relaxed strong converse exponent for an arbitrary null-hypothesis and a closed convex alternative hypothesis. Finally, in Section IV C 3 we show equality in (I.7) for an arbitrary null-hypothesis and a simple alternative hypothesis under some technical conditions and a restricted, but non-trivial range of parameters.

II. BACKGROUND AND PRELIMINARY RESULTS

A. Mathematical background

We will consider the binary state discrimination problem in a slightly more general setting than in the Introduction, on the one hand allowing the quantum systems to be modeled by general von Neumann algebras (to incorporate infinite-dimensional classical systems in the formalism), and on the other hand considering a more general notion of hypotheses. Therefore, below we collect some basic notions of such models of quantum systems. Note, however, that we only use this formalism for convenience of presentation, and we will only actually work with the quantum and classical models familiar to any quantum information theorists. Therefore, no familiarity with the theory of von
Neumann algebras is required to follow the paper, apart from the basic notions and terminology that we collect in this section.

For a Hilbert space \( \mathcal{H} \), let \( \mathcal{B}(\mathcal{H}) \) denote the set of all linear operators on \( \mathcal{H} \). We say that \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \) is a von Neumann algebra on \( \mathcal{H} \) if it is a linear subspace of \( \mathcal{B}(\mathcal{H}) \) that is closed under the operator product, the adjoint, contains the identity operator, and is closed in the weak operator topology. (The last condition is automatically satisfied when the underlying Hilbert space is finite-dimensional.)

In this paper we will only consider the two simplest cases of von Neumann algebras: The simple quantum case where \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \), and the classical case, where \( \mathcal{M} = \{ \mathcal{M}_f : f \in L^\infty(\mathcal{X}, \mathcal{F}, \mu) \} \), with \( (\mathcal{X}, \mathcal{F}, \mu) \) a measure space, and \( \mathcal{M}_f \) being the multiplication operator \( L^2(\mathcal{X}, \mathcal{F}, \mu) \ni g \mapsto fg \). In the classical case, we may naturally identify \( \mathcal{M} \) with the function algebra \( L^\infty(\mathcal{X}, \mathcal{F}, \mu) \), in which the algebraic operations are the usual point-wise operations on functions. In particular, when \( \mathcal{X} \) is finite, we will choose \( \mu \) to be the counting measure \( \mu(\{x\}) = 1, x \in \mathcal{X} \), on the full power set of \( \mathcal{X} \), in which case the function algebra \( L^\infty(\mathcal{X}, \mathcal{F}, \mu) \) is simply \( \mathbb{C}^\mathcal{X} = \{ f : \mathcal{X} \to \mathbb{C} \} \), and the corresponding operator algebra is the collection of all operators on \( \mathbb{C}^\mathcal{X} \) that are diagonal in the canonical orthonormal basis \( (|x\rangle)_{x \in \mathcal{X}} \) of \( l^2(\mathcal{X}) \), i.e., can be written as \( \sum_{x \in \mathcal{X}} a(x)|x\rangle\langle x| \) with some \( a \in \mathbb{C}^\mathcal{X} \). The only infinite-dimensional von Neumann algebra that we will actually use in this paper is the classical algebra \( L^\infty([0,1]) \), where the measure is the Lebesgue measure on the Lebesgue measurable subsets of \([0,1]\).

For a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \), we will use the notations \( \mathcal{M}_{sa}, \mathcal{M}_{\geq 0}, \mathcal{M}_{\geq 0}, \) and \( \mathcal{M}_{> 0} \) for the set of self-adjoint, positive semi-definite (PSD), non-zero positive semi-definite, and positive definite operators in \( \mathcal{M} \), respectively, and

\[
\mathcal{M}_{[0,1]} := \{ T \in \mathcal{M} : 0 \leq T \leq I \}
\]

for the set of tests in \( \mathcal{M} \). A test \( T \) is projective if \( T^2 = T \). In the classical case \( \mathcal{M} = L^\infty(\mathcal{X}, \mathcal{F}, \mu) \), projective tests may be identified with measurable subsets of \( \mathcal{X} \), up to the equivalence \( A \sim B \) if \( \mu((A \setminus B) \cup (B \setminus A)) = 0 \); the test corresponding to a subset is the multiplication operator by the characteristic function of the given subset.

We denote by \( \mathcal{M}_* \), the predual of \( \mathcal{M} \), i.e., the space of normal linear functionals on \( \mathcal{M} \). The \( w^* \)-topology on \( \mathcal{M}_* \) is the weak topology induced by \( \mathcal{M}_* \) on \( \mathcal{M} \). The following is an immediate consequence of the Banach-Alaoglu theorem:

**Lemma II.1** \( \mathcal{M}_{[0,1]} \) is \( w^* \)-compact.

We denote by \( \mathcal{M}_*^+ \) the set of normal positive functionals on \( \mathcal{M} \). A state on \( \mathcal{M} \) is a \( \varphi \in \mathcal{M}_*^+ \) that is normalized, i.e., \( \varphi(I) = 1 \). We denote the set of states on \( \mathcal{M} \) by \( \mathcal{S}(\mathcal{M}) \). In the simple quantum case, every \( \varphi \in \mathcal{B}(\mathcal{H})_*^+ \) is uniquely determined by a positive trace-class operator \( \hat{\varphi} \) on \( \mathcal{H} \) via \( \varphi(.) = \text{Tr} \hat{\varphi}(.) \), and \( \varphi \) is a state if and only if \( \hat{\varphi} \) is a density operator, i.e., \( \text{Tr} \hat{\varphi} = 1 \). In this case we are going to identify normal positive functionals with their density operators, and denote the set of density operators (states) on \( \mathcal{H} \) by \( \mathcal{S}(\mathcal{H}) \).

In the classical case, every non-negative function \( \varphi \in L^1(\mathcal{X}, \mathcal{F}, \mu) \) defines a positive normal functional via \( \varphi(f) := \int_X f(x)\varphi(x)\,d\mu(x) \), and it is a state if and only if \( \hat{\varphi} \) is a probability density function, i.e., \( \int_X \hat{\varphi}(x)\,d\mu(x) = 1 \). When no confusion arises, we will use the same notation \( \varphi \) and terminology (state) for both the function and the corresponding functional on the algebra. For a measurable subset \( A \in \mathcal{F} \), we will also use the notation

\[
\varphi(A) := \varphi(1_A) = \int_A \hat{\varphi}(x)\,d\mu(x).
\]

In particular, in a finite classical model, states of the system are simply probability density functions on the finite set \( \mathcal{X} \), or equivalently, in the operator formalism, density operators on the finite-dimensional Hilbert space \( l^2(\mathcal{X}) \) that are diagonal in the canonical basis. In this case, we will denote the set of states by \( \mathcal{S}(\mathcal{X}) \).

For topological notions (e.g., compactness of subsets of \( \mathcal{M}_* \), (semi-)continuity of functions on \( \mathcal{M}_* \)) we will always consider the weak topology induced by \( \mathcal{M} \) on \( \mathcal{M}_* \). In the finite-dimensional case this is just the usual topology induced by any norm on \( \mathcal{M}_* \).

For any \( \varphi \in \mathcal{M}_*^+ \), we will denote its support projection by \( \varphi^0 \).

**B. Quantum divergences for state discrimination**

The error exponents of binary i.i.d. state discrimination may be expressed as certain divergences (distance-like quantities) of the two states representing the two hypotheses. The properties of these divergences are well-studied in the literature for pairs of states; the main purpose of this section is to extend some of this analysis to pairs of subsets of states, which we will need in our study of the error exponents of composite hypothesis testing.
For a density operator $\varrho$ and a PSD operator $\sigma$ on a finite-dimensional Hilbert space $\mathcal{H}$, their (Petz-type) Rényi $\alpha$-divergence $D_\alpha(\varrho\|\sigma)$ is defined as

$$D_\alpha(\varrho\|\sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \varrho^\alpha \sigma^{1-\alpha}$$

for any $\alpha \in [0, 1)$. The limit $\alpha \to 1$ gives the Umegaki relative entropy

$$D_1(\varrho\|\sigma) := \lim_{\alpha \to 1} D_\alpha(\varrho\|\sigma) = D(\varrho\|\sigma) := \begin{cases} \text{Tr} \varrho (\log \varrho - \log \sigma); & \varrho^0 \leq \sigma^0, \\ +\infty, & \text{otherwise}, \end{cases}$$

where log stands for the natural logarithm, and $\widehat{\log}$ is its extension by $\widehat{\log} 0 := 0$.

The above formulas already define the Rényi divergences and the relative entropy for a state and a positive functional on a finite-dimensional classical system, represented as diagonal operators in the same ONB. In the general commutative case $M = L^\infty(\mathcal{X}, \mathcal{F}, \mu)$, and non-negative functions $\varrho, \sigma \in L^1(\mathcal{X}, \mathcal{F}, \mu)$ with $\int_\mathcal{X} \varrho(x) \, d\mu(x) = 1$, the above divergences are defined as

$$D_\alpha(\varrho\|\sigma) := \frac{1}{\alpha - 1} \log \int_\mathcal{X} \varrho(x)^\alpha \sigma(x)^{1-\alpha} \, d\mu(x),$$

and

$$D_1(\varrho\|\sigma) := D(\varrho\|\sigma) := \begin{cases} \int_\mathcal{X} \varrho(x)(\widehat{\log} \varrho(x) - \widehat{\log} \sigma(x)) \, d\mu(x); & \varrho^0 \leq \sigma^0, \\ +\infty, & \text{otherwise}. \end{cases}$$

Moreover, the notions of relative entropy and Rényi divergences may be extended to pairs of positive normal functionals on an arbitrary von Neumann algebra $[2, 24, 54]$.

The Chernoff divergence of $\varrho \in \mathcal{S}(M)$ and $\sigma \in M_+^1$ is defined as

$$C(\varrho\|\sigma) := \sup_{\alpha \in (0,1)} (1-\alpha) D_\alpha(\varrho\|\sigma),$$

and their Hoeffding divergence with parameter $r \geq 0$ as

$$H_r(\varrho\|\sigma) := \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} \left[ r - D_\alpha(\varrho\|\sigma) \right]. \quad (\text{II.8})$$

It is immediate from their definitions that the Rényi- and the Hoeffding divergences satisfy the following scaling laws:

$$D_\alpha(t\varrho\|s\sigma) = D_\alpha(\varrho\|\sigma) - \log s + \frac{\alpha}{\alpha - 1} \log t, \quad H_r(t\varrho\|s\sigma) = H_{r+\log s}(\varrho\|\sigma) - \log t \quad (\text{II.9})$$

for any $\varrho \in \mathcal{S}(M)$, $\sigma \in M_+^1$, $t, s \in (0, +\infty)$, $\alpha \in [0, 1)$, and $r \geq 0$.

The above divergences may be extended to pairs of subsets $R \subseteq \mathcal{S}(M)$, $S \subseteq M_+^1$ by

$$\Delta(R\|S) := \inf_{\varrho \in R, \sigma \in S} \Delta(\varrho\|\sigma),$$

where $\Delta$ stands for any of the above divergences.

**Lemma II.2** If $\Delta$ is any of the above divergences, then the map $\mathcal{S}(M) \times M_+^1 \ni (\varrho, \sigma) \mapsto \Delta(\varrho\|\sigma)$ is convex and lower semi-continuous. Moreover, the map $[0, +\infty) \times \mathcal{S}(M) \times M_+^1 \ni (r, \varrho, \sigma) \mapsto H_r(\varrho\|\sigma)$ is convex.

**Proof** These statements are well-known for the relative entropy and the Rényi divergences; see, for instance, [24]. Thus, the Chernoff divergence and any Hoeffding divergence for a fixed $r$ are suprema of convex and lower semi-continuous functions, and therefore they are also convex and lower semi-continuous. By the above, for any fixed $\alpha \in (0, 1)$, $\frac{\alpha - 1}{\alpha} [r - D_\alpha(\varrho\|\sigma)]$ is a convex function of the triple $(r, \varrho, \sigma) \in [0, +\infty) \times \mathcal{S}(M) \times M_+^1$, and hence the same holds for its supremum over $\alpha$, i.e., $H_r(\varrho\|\sigma)$. \qed

**Corollary II.3** For any convex subsets $R \subseteq \mathcal{S}(M)$ and $S \subseteq M_+^1$, $H_r(R\|S)$ is a convex function of $r \in [0, +\infty)$. 
Proof Immediate from Lemma II.2 and the fact that taking the infimum of a jointly convex function in some of its variables yields a convex function.

Let us introduce the notation
\[
\psi(R||S|\alpha) := (\alpha - 1)D_{\alpha}(R||S) = \sup_{\varphi \in R, \sigma \in S} (\alpha - 1)D_{\alpha}(\varphi||\sigma), \quad \alpha \in [0, 1), \quad \psi(R||S|1) := \lim_{\alpha \to 1} \psi(R||S|\alpha).
\]
(The limit exists due to the convexity of \(\psi\); see Lemma II.6.) With this, the definition of the Hoeffding divergences (II.8) can be rewritten as
\[
H_r(\varphi||\sigma) = \sup_{\alpha \in (0, 1]} \left[ \frac{\alpha - 1}{\alpha}r - \frac{1}{\alpha}\psi(\varphi||\sigma|\alpha) \right] = \sup_{u \in (-\infty, \infty)} \left[ ur - (1 - u)\psi \left( \frac{\varphi||\sigma}{1 - u} \right) \right].
\]

Remark II.4 It is known [24] that for \(\varrho \in S(M), \sigma \in M^+_X\),
\[
\psi(\varrho||\sigma|1) = \log \varrho(\sigma^0) \leq 0, \quad \text{and hence} \quad \psi(R||S|1) \leq 0
\]
for any \(R \subseteq S(M), S \subseteq M^+_X\).

For proving the \(\alpha \to 1\) limit of the Rényi divergences in Lemma II.6, we will need the following minimax theorem from [39, Corollary A.2].

Lemma II.5 Let \(X\) be a compact topological space, \(Y\) be an ordered set, and let \(f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}\) be a function. Assume that
(i) \(f(., y)\) is lower semicontinuous for every \(y \in Y\) and
(ii) \(f(x, .)\) is monotonic increasing for every \(x \in X\), or \(f(x, .)\) is monotonic decreasing for every \(x \in X\).
Then
\[
\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y), \quad \text{(II.11)}
\]
and the infima in (II.11) can be replaced by minima.

Lemma II.6 For any \(R \subseteq S(M), S \subseteq M^+_X\), \(\alpha \to \psi(R||S|\alpha)\) is a convex function on \([0, 1]\), and
\[
\forall \alpha \in [0, 1] : \psi(R||S|\alpha) > -\infty \iff \exists \varrho \in R, \sigma \in S : \varrho^0 \not\perp \sigma^0. \quad \text{(II.12)}
\]
The function \(\alpha \to D_{\alpha}(R||S)\) is monotone increasing with
\[
D_0(R||S) = \lim_{\alpha \to 0} D_{\alpha}(R||S) \leq \lim_{\alpha \to 1} D_{\alpha}(R||S) =: D_1^{-}(R||S) \leq D(R||S). \quad \text{(II.13)}
\]
If, moreover, \(R\) and \(S\) are both compact then we also have \(D_1^{-}(R||S) = D(R||S)\).

Proof The above are well-known when \(|R| = |S| = 1\) [24], from which all the assertions follow immediately, except for the last one. Assume that \(R\) and \(S\) are both compact; then
\[
\lim_{\alpha \to 1} D_{\alpha}(R||S) = \sup_{\alpha \in (0, 1)} \inf_{(\varrho, \sigma) \in R \times S} D_{\alpha}(\varrho||\sigma) = \inf_{(\varrho, \sigma) \in R \times S} \sup_{\alpha \in (0, 1)} D_{\alpha}(\varrho||\sigma) = D(R||S),
\]
where the second equality follows by Lemmas II.5 and II.2. This proves the last assertion.

Corollary II.7 For any fixed \(R, S \subseteq S(M)\),
\[
\tilde{\psi}(R||S|u) := (1 - u)\psi \left( \frac{R||S}{1 - u} \right) = (1 - u) \sup_{\varrho \in R, \sigma \in S} \psi \left( \frac{\varrho||\sigma}{1 - u} \right)
\]
is a non-positive convex function of \(u\) on \((-\infty, 0)\), and it is finite at every \(u \in (-\infty, 0)\) if and only if the support condition in (II.12) holds.
Proof By Lemma II.6, $\alpha \mapsto \psi(\varrho|\sigma|\alpha)$ is convex, and therefore also continuous, on $(0, 1)$. Hence, it can be written as $\psi(\varrho|\sigma|\alpha) = \sup_{i \in I} \{c_i \alpha + d_i\}$ for some index set $I$, and $c_i, d_i \in \mathbb{R}$. Thus,

$$\psi(\varrho|\sigma|u) = \frac{1}{1-u} \psi\left(\varrho|\sigma|\frac{1}{1-u}\right) = \frac{1}{1-u} \sup_{i \in I} \left\{ c_i \frac{1}{1-u} + d_i \right\} = \sup_{i \in I} \left\{ c_i + d_i (1-u) \right\},$$

which, as the supremum of convex functions on $(-\infty, 0)$, is itself convex. Taking the supremum over $\varrho \in R, \sigma \in S$ yields the convexity of $u \mapsto \psi(R|S|u)$, and the rest of the assertions are obvious.

Lemma II.8 For any fixed $R \subseteq S(M), S \subseteq M_+^*$,

$$H_r(R|S) \geq \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} [r - D_\alpha(R|S)]. \tag{II.14}$$

If, moreover, both $R$ and $S$ are compact and convex then the above inequality holds as an equality.

Proof By (II.10), we have

$$H_r(R|S) = \inf_{\varrho \in R, \sigma \in S} \sup_{u \in (-\infty, 0)} \left[ ur - \psi(\varrho|\sigma|u) \right] \geq \sup_{u \in (-\infty, 0)} \inf_{\varrho \in R, \sigma \in S} \left[ ur - \psi(\varrho|\sigma|u) \right] = \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} [r - D_\alpha(R|S)].$$

If both $R$ and $S$ are compact and convex then the inequality above holds as an equality due to the minimax theorem in [17, Theorem 5.2], and Lemma II.2 and Corollary II.7.

Lemma II.9 For any fixed $R \subseteq S(M), S \subseteq M_+^*$, the function $r \mapsto H_r(R|S)$ is monotone non-increasing on $[0, +\infty)$,

$$r < D_0(R|S) \quad \Rightarrow \quad H_r(R|S) = +\infty \quad \Rightarrow \quad r \leq D_0(R|S), \tag{II.15}$$

and

$$0 \leq -\psi(R|S|1) \leq H_r(R|S), \quad r \in [0, +\infty). \tag{II.16}$$

Moreover,

$$r > D(R|S) \quad \Rightarrow \quad H_r(R|S) = 0 \quad \Rightarrow \quad r \geq D_1-(R|S), \tag{II.17}$$

and if $R, S$ are both compact and convex then

$$H_r(R|S) = 0 \quad \iff \quad r \geq D(R|S). \tag{II.18}$$

Proof It follows by its definition (II.8) that $r \mapsto H_r(R|S)$ is monotone non-increasing.

The first implication in (II.15) follows as $\lim_{\alpha \searrow 0} (r - D_\alpha(R|S)) = r - D_0(R|S)$ by Lemma II.6, and $\lim_{\alpha \searrow 0} \frac{\alpha - 1}{\alpha} = -\infty$. To see the second implication, assume that $r > D_0(R|S)$, and hence there exist $\varrho \in R, \sigma \in S$ such that $r > D_0(\varrho|\sigma)$. Then $r > D_\alpha(\varrho|\sigma)$ for all small enough $\alpha > 0$, according to (II.13), and hence $\lim_{\alpha \searrow 0} \frac{\alpha - 1}{\alpha} [r - D_\alpha(\varrho|\sigma)] = -\infty$. On the other hand, $\lim_{\alpha \searrow 1} \frac{\alpha - 1}{\alpha} [r - D_\alpha(\varrho|\sigma)] = -\psi(\varrho|\sigma|1) < +\infty$, where the inequality follows from the fact that $r > D_0(\varrho|\sigma)$ implies $\varrho^0 \not\subseteq \sigma^0$, and hence $\psi(\varrho|\sigma|\alpha)$ is a finite-valued function on $[0, 1]$, according to Lemma II.6. Since $\alpha \mapsto \frac{\alpha - 1}{\alpha} [r - D_\alpha(\varrho|\sigma)]$ is continuous on $(0, 1)$, we obtain that its supremum is finite, i.e., $+\infty > H_r(\varrho|\sigma) \geq H_r(R|S)$, where the last inequality is by definition.

The inequalities in (II.16) follow from (II.14) and Remark II.4 as

$$\lim_{\alpha \searrow 1} \frac{\alpha - 1}{\alpha} [r - D_\alpha(R|S)] = -\psi(R|S|1).$$

If $r > D(R|S)$ then there exist $\varrho \in R, \sigma \in S$ such that $r > D(\varrho|\sigma) \geq D_\alpha(\varrho|\sigma), \alpha \in (0, 1)$, where the second inequality is due to Lemma II.6. This implies immediately that $0 = H_r(\varrho|\sigma) \geq H_r(R|S) \geq 0$, where the first inequality is by definition, and the second one is by (II.16). This proves the first implication in (II.17).

If $r < D_1-(R|S)$ then $r < D_\alpha(R|S)$ for some large enough $\alpha < 1$ by Lemma II.6, and $0 < \frac{\alpha - 1}{\alpha} [r - D_\alpha(R|S)] \leq H_r(R|S)$, where the second inequality is by (II.14). This proves (II.18).

Finally, if $R, S$ are both convex and compact then $D(R|S) = D_1-(R|S) \geq D_\alpha(R|S), \alpha \in (0, 1)$, according to Lemma II.6. Hence, if $r = D(R|S)$ then $\frac{\alpha - 1}{\alpha} [r - D_\alpha(R|S)] \leq 0, \alpha \in (0, 1)$, which implies $H_r(R|S) = \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} [r - D_\alpha(R|S)] \leq 0$, where the equality is due to Lemma II.8.
In Sections III C and V C, we will need the following more detailed knowledge about the Hoeffding divergence for classical states \( \varrho, \sigma \in \mathcal{S}(\mathcal{X}) \) for some finite set \( \mathcal{X} \). The statements below are well-known and easy to prove, but we collect them here for ease of reference in later parts of the paper, and because it does not seem to be easy to find a single reference for all the formulas that we will need.

We will assume that \( \mathcal{X}_{\varrho, \sigma} := (\text{supp } \varrho) \cap (\text{supp } \sigma) \neq \emptyset \). For every \( \alpha \in \mathbb{R} \) and \( u \in (\infty, 1) \), let

\[
\psi(\alpha) := \psi(\varrho \| \sigma | \alpha) := \log \sum_{x \in \mathcal{X}_{\varrho, \sigma}} \varrho(x)^{\alpha} \sigma^{1-\alpha}, \quad \hat{\psi}(u) := \hat{\psi}(\varrho \| \sigma | u) := (1-u)\psi((1-u)^{-1}). \tag{II.19}
\]

These coincide with \( \psi(\varrho \| \sigma | \alpha) \) and \( \hat{\psi}(\varrho \| \sigma | u) \) defined previously for \( \alpha \in [0, 1] \) and \( u \in (\infty, 0) \), respectively. A straightforward computation shows that

\[
\psi'(\alpha) = \mathbb{E}_{\mu_{\alpha}} \left( \log \varrho - \log \sigma \right), \quad \psi''(\alpha) = \mathbb{E}_{\mu_{\alpha}} \left( (\log \varrho - \log \sigma)^2 \right) - \left( \mathbb{E}_{\mu_{\alpha}} (\log \varrho - \log \sigma) \right)^2, \tag{II.20}
\]

where

\[
\mu_{\alpha, \varrho, \sigma}(x) := \mu_{\alpha}(x) := \frac{\varrho(x)^{\alpha} \sigma(x)^{1-\alpha}}{\sum_{y \in \mathcal{X}_{\varrho, \sigma}} \varrho(y)^{\alpha} \sigma(y)^{1-\alpha}} 1_{\mathcal{X}_{\varrho, \sigma}}(x), \quad x \in \mathcal{X}. \tag{II.21}
\]

(The family of probability distributions \( \mu_{\alpha, \varrho, \sigma} \), \( \alpha \in [0, 1] \) is called the Hellinger arc, and it connects \( \frac{\varrho}{\sigma} 1_{\mathcal{X}_{\varrho, \sigma}} \) and \( \frac{\sigma}{\varrho} 1_{\mathcal{X}_{\varrho, \sigma}} \).) Moreover,

\[
\hat{\psi}'(u) = -\psi \left( \frac{1}{1-u} \right) + \frac{1}{1-u} \psi' \left( \frac{1}{1-u} \right), \quad \hat{\psi}''(u) = \left( \frac{1}{1-u} \right)^3 \psi'' \left( \frac{1}{1-u} \right). \tag{II.22}
\]

This immediately yields the following (see also [26, Lemma 3.2] for a generalization to quantum states):

**Lemma II.10** In the above setting, \( \psi \) and \( \hat{\psi} \) are convex, and the following are equivalent:

(i) \( \psi'(\alpha) = 0 \) for some \( \alpha \in \mathbb{R} \);

(ii) \( \log \varrho - \log \sigma \) is constant on \( \text{supp } \varrho \cap \text{supp } \sigma \);

(iii) there exists a \( \kappa > 0 \) such that \( \varrho(x) = \kappa \sigma(x), \) \( x \in \text{supp } \varrho \cap \text{supp } \sigma \);

(iv) \( \psi(\alpha) = (\alpha - 1) \log \varrho(\text{supp } \varrho) + \log (\text{supp } \varrho \cap \text{supp } \sigma), \) \( \alpha \in \mathbb{R} \); in particular, \( \psi \) is affine;

(v) \( \hat{\psi}(u) = \log (\text{supp } \varrho \cap \text{supp } \sigma) - u \log (\text{supp } \varrho \cap \text{supp } \sigma) \), \( u \in (-\infty, 1) \);

(vi) \( \hat{\psi}''(u) = 0 \) for all \( u \in (-\infty, 1) \);

(vii) \( \hat{\psi}''(u) = 0 \) for some \( u \in (-\infty, 1) \).

If, moreover, \( \varrho^0 \leq \sigma^0 \) then the above is further equivalent to

(viii) \( \alpha \mapsto \frac{\psi(\alpha)}{\alpha-1} \) is constant on any/some non-trivial subinterval of \((0, \infty)\),

in which case the constant is equal to \(- \log (\text{supp } \varrho) = D_0(\varrho \| \sigma) \).

Let

\[
\Psi(c) := \sup_{\alpha \in (0, +\infty)} \{c\alpha - \psi(\alpha)\}, \quad \bar{\Psi}(c) := \sup_{u \in (-\infty, 1)} \{cu - \hat{\psi}(u)\}, \quad c \in \mathbb{R},
\]

be the Legendre-Fenchel transforms of \( \psi \) and \( \hat{\psi} \), respectively. A straightforward computation, using (II.20) and (II.22), yields that for every \( \alpha \in (0, +\infty) \) and \( u = (\alpha - 1)/\alpha \),

\[
D(\mu_{\alpha} \| \sigma) = \alpha \psi'(\alpha) - \psi(\alpha) = \Psi(\psi'(\alpha)) = \hat{\psi}'(u), \tag{II.23}
\]

\[
D(\mu_{\alpha} \| \varrho) = (\alpha - 1) \psi'(\alpha) - \psi(\alpha) = \Psi(\psi'(\alpha)) - \psi'(\alpha) = u \hat{\psi}'(u) - \hat{\psi}'(u) = \bar{\Psi}(\hat{\psi}'(u)). \tag{II.24}
\]

Assume for the rest that \( \varrho^0 \leq \sigma^0 \). Let
\[
D^*_\infty(\varrho\|\sigma) := \log \max \left\{ \frac{\varrho(x)}{\sigma(x)} : x \in \mathcal{X} \right\}, \quad \mathcal{X}_{\infty} := \left\{ x \in \mathcal{X} : \log \frac{\varrho(x)}{\sigma(x)} = D^*_\infty(\varrho\|\sigma) \right\},
\]
where \( D^*_\infty(\varrho\|\sigma) \) is the max-relative entropy of \( \varrho \) and \( \sigma \); see Section V B for details. We have
\[
\lim_{\alpha \to 1} \mu_\alpha(x) = \frac{\sigma(x)}{\sigma(\text{supp } \varrho)} \mathbf{1}_{\text{supp } \varrho}(x), \quad \mu_1(x) = \varrho(x), \quad \lim_{\alpha \to +\infty} \mu_\alpha(x) = \frac{\sigma(x)}{\sigma(\mathcal{X}_{\infty})} \mathbf{1}_{\mathcal{X}_{\infty}}(x), \quad x \in \mathcal{X},
\]
and hence, by (II.20) and (II.23)–(II.24),
\[
\lim_{u \to -\infty} \bar{\psi}'(u) = -\psi(0) = -\log \sigma(\text{supp } \varrho) = D_0(\varrho\|\sigma), \quad \psi'(1) = \bar{\psi}'(0) = D(\varrho\|\sigma), \tag{II.25}
\]
\[
\lim_{\alpha \to +\infty} \psi'(\alpha) = D^*_\infty(\varrho\|\sigma), \quad \lim_{u \to 1} \bar{\psi}'(u) = -\log \sigma(\mathcal{X}_{\infty}) = r_\infty(\varrho\|\sigma). \tag{II.26}
\]

**Remark II.11** If \( \varrho^0 \leq \sigma^0 \) and \( \psi \) is affine then, by Lemma II.10 and (II.25)–(II.26),
\[
-\log \sigma(\text{supp } \varrho) = D_0(\varrho\|\sigma) = D(\varrho\|\sigma) = D^*_\infty(\varrho\|\sigma) = r_\infty(\varrho\|\sigma).
\]
The above observations yield immediately the following:

**Lemma II.12** Let \( \mathcal{X} \) be a finite set, and \( \varrho, \sigma \in \mathcal{S}(\mathcal{X}) \) be such that \( \varrho^0 \leq \sigma^0 \). For every \( r \in (D_0(\varrho\|\sigma), r_\infty(\varrho\|\sigma)) \) there exists a unique \( u_r \in (-\infty, 1) \) and corresponding \( \alpha_r := 1/(1-u_r) \in (0, +\infty) \) such that
\[
r = \bar{\psi}'(u_r) = \alpha_r \psi'(\alpha_r) - \psi(\alpha_r) = D(\mu_{\alpha_r, \varrho, \sigma}\|\sigma), \tag{II.27}
\]
and for this \( u_r \) and \( \alpha_r \),
\[
\bar{\Psi}(r) = u_r r - \bar{\psi}(u_r) = \frac{\alpha_r - 1}{\alpha_r} - \frac{1}{\alpha_r} \psi(\alpha_r) = (\alpha_r - 1) \psi'(\alpha_r) - \psi(\alpha_r) = D(\mu_{\alpha_r, \varrho, \sigma}\|\sigma). \tag{II.28}
\]
Moreover, \( r \in (D_0(\varrho\|\sigma), D(\varrho\|\sigma)) \iff u_r \in (-\infty, 0) \iff \alpha_r \in (0, 1) \), in which case
\[
\bar{\Psi}(r) = H_r(\varrho\|\sigma), \tag{II.29}
\]
and \( r \in (D(\varrho\|\sigma), r_\infty(\varrho\|\sigma)) \iff u_r \in (0, 1) \iff \alpha_r \in (1, +\infty) \), in which case
\[
\bar{\Psi}(r) = H^*_r(\varrho\|\sigma) = r - \bar{\psi}(1) = r - D_\infty(\varrho\|\sigma). \tag{II.30}
\]
where \( H^*_r(\varrho\|\sigma) := \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} [r - D_\alpha(\varrho\|\sigma)] \) is the Hoeffding anti-divergence; see Section V B. Finally, for \( r \geq r_\infty(\varrho\|\sigma) \),
\[
\bar{\Psi}(r) = H^*_r(\varrho\|\sigma) = r - \bar{\psi}(1) = r - D_\infty(\varrho\|\sigma). \tag{II.31}
\]

**Proof** If \( \psi \) is affine then \( -\log \sigma(\text{supp } \varrho) = r_\infty(\varrho\|\sigma) \), and (II.31) holds trivially. Otherwise \( \psi \) is strictly convex, according to Lemma II.10, and all the statements follow easily from the observations after Lemma II.10. \( \square \)

Finally, let us briefly comment on different notions of Rényi divergences for positive definite operators \( \varrho, \sigma \) on a finite-dimensional Hilbert space. For any \( \alpha \in (0, 1) \), the *log-Euclidean Rényi \( \alpha \)-divergence* of \( \varrho \) and \( \sigma \) [5, 42, 51, 52] is defined as
\[
D^*_\alpha(\varrho\|\sigma) := \frac{1}{\alpha - 1} \log \text{Tr } e^{\alpha \log \varrho + (1-\alpha) \log \sigma} - \frac{1}{\alpha - 1} \log \text{Tr } \varrho,
\]
and their *maximal Rényi \( \alpha \)-divergence* [37] as
\[
D^*_\alpha(\varrho\|\sigma) := \frac{1}{\alpha - 1} \log \text{Tr } \sigma^\#_{\alpha \varrho} - \frac{1}{\alpha - 1} \log \text{Tr } \varrho,
\]
where
\[
\sigma^\#_{\alpha \varrho} := \sigma^{1/2} \left( \sigma^{-1/2} \varrho \sigma^{-1/2} \right)^\alpha \sigma^{1/2} \tag{II.32}
\]
is the Kubo-Ando \( \alpha \)-geometric mean [35]. The Petz-type Rényi \( \alpha \)-divergence is also defined with the extra term \( -\frac{1}{\alpha - 1} \log \text{Tr } \varrho \) when \( \text{Tr } \varrho \neq 1 \). The following is an immediate consequence of Theorems 2.1 and 3.1 in [23], which we will use in Section IV A.
Lemma II.13 For any \( \alpha \in (0, 1) \),
\[
D_\alpha(q||\sigma) \leq D_\alpha^\prime(q||\sigma) \leq D_\alpha^{\text{max}}(q||\sigma).
\]  
(II.33)
If \( q \neq \sigma \) then all the inequalities in (II.33) are strict, and hence, in particular,
\[
\log \sigma \#_\alpha q \neq \alpha \log q + (1 - \alpha) \log \sigma.
\]

We will also use a further family of quantum Rényi divergences, the sandwiched Rényi divergences [43, 60], in Section V B.

C. Asymptotic binary quantum state discrimination

We consider various generalizations of the binary state discrimination problem presented in the Introduction. The first level of generalization unifies the composite i.i.d. and the (simple) mixed i.i.d. settings discussed in the Introduction. In the unified setting, an experimenter is presented with several identically prepared quantum systems with von Neumann algebra \( \mathcal{M} \), and the knowledge that the state of the system was either chosen according to a probability distribution \( p \) belonging to a set of probability distributions \( \mathcal{P} \) on the Borel sets of \( S(\mathcal{M}) \) (null-hypothesis \( H_0 \)), or according to a probability distribution \( q \) belonging to a set of probability distributions \( \mathcal{Q} \) on the Borel sets of \( S(\mathcal{M}) \) (alternative hypothesis \( H_1 \)).

The experimenter’s test on \( n \) copies of the system is represented by an operator \( T_n \in \mathcal{M}^\otimes n \), \( 0 \leq T_n \leq I \), and the two types of error probabilities are given by
\[
\alpha_n(\mathcal{P}|T_n) := \sup_{p \in \mathcal{P}} \frac{\int_{S(\mathcal{M})} dp(p) \; p^\otimes n(I - T_n)}{\int_{S(\mathcal{M})} dp(p) \; p^\otimes n}, \quad \text{(type I)}
\]
\[
\beta_n(\mathcal{Q}|T_n) := \sup_{q \in \mathcal{Q}} \frac{\int_{S(\mathcal{M})} dq(q) \; q^\otimes n(T_n)}{\int_{S(\mathcal{M})} dq(q) \; q^\otimes n}, \quad \text{(type II)}
\]
(Recall that here \( p^\otimes n(I - T_n) \) is the functional \( p^\otimes n \) evaluated on the operator \( (I - T_n) \).) The composite i.i.d. setting discussed in the Introduction is obtained as a special case by choosing \( \mathcal{P} = \{ \delta_{\omega} \}_{\omega \in R} \) and \( \mathcal{Q} = \{ \delta_{\sigma} \}_{\sigma \in S} \), where \( \delta_{\omega} \) is the Dirac measure concentrated on \( \omega \). In this case, we will write \( R \) and \( S \) instead of \( \mathcal{P} \) and \( \mathcal{Q} \), respectively, in the error probabilities and in the error exponents introduced below. The mixed i.i.d. setting, also discussed in the Introduction, may be obtained as the special case \( \mathcal{P} = \{ p \} \) and \( \mathcal{Q} = \{ q \} \), where each hypothesis is represented by a single probability distribution.

The above setting can be further generalized by dropping the i.i.d. assumption. In the most general setting, an asymptotic binary state discrimination problem is specified by a sequence \( (\mathcal{M}_n)_{n \in \mathbb{N}} \) of von Neumann algebras, and two sequences of sets of states \( (R_n)_{n \in \mathbb{N}} \) and \( (S_n)_{n \in \mathbb{N}} \), representing the null hypothesis \( H_0 \) and the alternative hypothesis \( H_1 \), respectively. We denote the two types of error probabilities corresponding to a test \( T_n \in (\mathcal{M}_n)_{[0,1]} \) in this case by
\[
\alpha_n(R_n|T_n) := \sup_{q \in R_n} q_n(I - T_n), \quad \beta_n(S_n|T_n) := \sup_{\sigma \in S_n} \sigma_n(T_n).
\]
This level of generality is the usual setting in investigations using the information spectrum method; see, e.g., [45] for the case of binary quantum state discrimination with simple hypotheses. In this paper we will consider the non-i.i.d. problems of arbitrarily varying and of adversarial classical state discrimination; see Section III C.

A summary of the important special cases (including the ones discussed above) is given in order of decreasing generality as follows:

- **Consistent:** It is specified by a sequence of von Neumann algebras \( (\mathcal{M}_n)_{n \in \mathbb{N}} \), two index sets \( I, J \), and two functions \( \varrho : [N \times I] \rightarrow S(\mathcal{M}_n) \), \( \sigma : [N \times J] \rightarrow S(\mathcal{M}_n) \). The null and the alternative hypotheses are given by \( H_0 : (R_n = (\varrho_{n,i})_{i \in I})_{n \in \mathbb{N}} \) and \( H_1 : (S_n = (\sigma_{n,j})_{j \in J})_{n \in \mathbb{N}} \). (This is the natural setting, for instance, to consider the discrimination of correlated states on an infinite spin chain, where \( \varrho_{n,i} \) is the \( i \)-th site restriction of some state \( \varrho_{\infty,i} \) on the infinite chain, and similarly for \( \sigma_{n,j} \); see, e.g., [26, 41].)

- **Mixed i.i.d.:** \( \mathcal{M}_n = \mathcal{M}^\otimes n, \; n \in \mathbb{N} \), and the null and the alternative hypotheses are specified by two sets of probability distributions \( \mathcal{P}, \mathcal{Q} \) on the Borel sets of \( S(\mathcal{M}) \), with \( R_n = \{ \int_{S(\mathcal{M}^\otimes n)} dp(p) \varrho^\otimes n \}_{p \in \mathcal{P}} \) and \( S_n = \{ \int_{S(\mathcal{M}^\otimes n)} dq(q) \sigma^\otimes n \}_{q \in \mathcal{Q}} \).

- **I.i.d.:** In the above, \( \mathcal{P} = \{ \delta_{\varrho} \}_{\varrho \in R} \) and \( \mathcal{Q} = \{ \delta_{\sigma} \}_{\sigma \in S} \) for some sets \( R, S \subseteq S(\mathcal{M}) \).
Note that once the sequence of von Neumann algebras is fixed (typically the tensor powers of a given algebra), we may consider mixtures of the above cases by either requiring only one of the hypotheses to fall in one of the above classes, or by requiring one of the hypotheses to fall in one class, and the other in a possibly different class; e.g., we may consider an i.i.d. vs. mixed i.i.d. state discrimination problem. On top of the above, we say that the null hypothesis is simple if \(|R_n| = 1, n \in \mathbb{N}\), and composite otherwise, and similarly for the alternative hypothesis. The state discrimination problem is simple if both the null and the alternative hypotheses are simple, and composite otherwise. We will use the notations \(\alpha_n(\varrho|T_n), \alpha_n(R|T_n), \alpha_n(p|T_n), \alpha_n(P|T_n), \alpha_n(R_n|T_n)\) for the type I, and similarly for the type II error probabilities, as well as for the error exponents defined later, depending on the precise setting that we consider. If some concept or statement is independent of the setting, we will also use the notations \(\alpha_n(H_0|T_n)\) and \(\beta_n(H_1|T_n)\).

It is known that in the case of simple binary i.i.d. hypothesis testing (i.e., \(|R| = |S| = 1\), if \(R \neq S\) then both error probabilities can be made to vanish asymptotically with an exponential speed by a suitably chosen sequence \((T_n)_{n \in \mathbb{N}}\) of tests, and the same holds in most of the more general composite settings, too. Clearly, the faster the error probabilities decay with \(n\) the better, and hence it is a natural question to ask what exponents are achievable in the following sense:

**Definition II.14** We say that \((r, r') \in [0, +\infty]^2\) is an achievable exponent pair if there exists a sequence of tests \(T_n \in \mathcal{M}_n[0,1], n \in \mathbb{N}\), such that

\[
\frac{1}{n} \log \beta_n(H_1|T_n), \quad r' \leq \liminf_{n \to +\infty} \frac{1}{n} \log \alpha_n(H_0|T_n).
\]

We denote the set of achievable exponent pairs by \(\mathbb{K}(H_0||H_1)\).

In the case where the null hypothesis is consistent, given by \((R_n = \{\varrho_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\), we may introduce the relaxed set of achievable exponent pairs \(\mathbb{K}^{(0)}\) \(\left((\varrho_{n,i})_{i \in I}||H_0\right)\) as the set of all pairs \((r, r') \in [0, +\infty]^2\) for which there exists a sequence of tests \(T_n \in \mathcal{M}_n[0,1], n \in \mathbb{N}\), such that

\[
r \leq \liminf_{n \to +\infty} \frac{1}{n} \log \beta_n(H_1|T_n), \quad \forall i \in I : \quad r' \leq \liminf_{n \to +\infty} \frac{1}{n} \log \alpha_n(\varrho_{n,i}|T_n).
\]

The set of achievable exponent pairs \(\mathbb{K}^{(1)}\) \(\left((\sigma_{n,j})_{j \in J}||H_0\right)\) and \(\mathbb{K}^{(0,1)}\) \(\left((\sigma_{n,j})_{j \in J}||\left((\varrho_{n,i})_{i \in I}||H_0\right)\right)\) are defined analogously when the alternative hypothesis or both hypotheses are consistent, respectively.

Note that if the index set \(I\) is finite then \(\mathbb{K}^{(0)}\) \(\left((\varrho_{n,i})_{i \in \mathbb{N}}||H_0\right) = \mathbb{K}(\left((\varrho_{n,i})_{i \in \mathbb{N}}||H_0\right)\), and similarly for the other variants, and hence these only play a role for infinite index set(s).

We will use \(\mathbb{K}^t\) \(\left(H_0||H_1\right)\) to denote any of the above sets of achievable exponent pairs, where \(t = \emptyset, \{0\}, \{1\}, \{0,1\}\).

**Remark II.15** The notation \(\mathbb{K}^{(0)}\) \(\left(H_0||H_1\right)\) implicitly implies that the null hypothesis is consistent, and we use the analogous conventions for \(t = \{1\}\) and \(\{0,1\}\), as well as for the error exponents defined below.

**Remark II.16** It is clear from the definition that

\[
(r, r') \in \mathbb{K}^t \left((R_n)_{n \in \mathbb{N}} || (S_n)_{n \in \mathbb{N}}\right) \iff (r', r) \in \mathbb{K}^{\bar{t}} \left((S_n)_{n \in \mathbb{N}} || (R_n)_{n \in \mathbb{N}}\right),
\]

where \(\bar{t} := \{1 - b : b \in t\}\).

The following is immediate from the definition:

**Lemma II.17** If \((r, r') \in \mathbb{K}^t(H_0||H_1)\) then also \((\bar{r}, \bar{r}') \in \mathbb{K}^t(H_0||H_1)\) for all \(0 \leq \bar{r} \leq r\) and \(0 \leq \bar{r}' \leq r'\).

**Lemma II.18** \(\mathbb{K}(H_0||H_1)\) is closed, and the same holds for \(\mathbb{K}^t(H_0||H_1)\) for \(t = \{0\})/\{1\}/\{0,1\}\), provided that \(H_0/H_1/\) both is/are consistent with (a) countable set(s).

**Proof** The proof is standard; we give the details for readers’ convenience in the case \(t = \{0,1\}\) and \(I = J = \mathbb{N}\). Let \((r_m, r'_m) \in \mathbb{K}^{(0,1)} \left((\varrho_{n,i})_{i \in \mathbb{N}} || (\sigma_{n,j})_{j \in J} || \mathbb{N}\right)\) be such that \(r_m \to r, r'_m \to r'\). We assume that \(r, r' < +\infty\), in which case we may also assume without loss of generality that \(r_m, r'_m < +\infty\); the other cases follow in a similar way. For every \(m \in \mathbb{N}\), there exists an \(N_m \in \mathbb{N}\) such that for every \(n \geq N_m\), and every \(i, j \in [m], \alpha_n(\varrho_{n,i}|T_{m,n}) < e^{-n(r_m - 1/m)}\), \(\beta_n(\sigma_{n,j}|T_{m,n}) < e^{-n(r'_m - 1/m)}\) for some test \(T_{m,n}\). We may also assume that \(N_1 < N_2 < \ldots\). Therefore, for every \(n \in \mathbb{N}\), there exists a unique \(m_n \in \mathbb{N}\) such that \(N_{m_n} \leq n < N_{m_n+1}\). Then with \(T_n := T_{m_n,n}, n \in \mathbb{N}\), we have \(\liminf_{n \to +\infty} \frac{1}{n} \log \alpha_n(\varrho_{n,i}|T_n) \geq r', \liminf_{n \to +\infty} \frac{1}{n} \log \beta_n(\sigma_{n,j}|T_n) \geq r, \) for every \(i, j \in \mathbb{N}\). \(\square\)
Definition II.19 The direct exponents with type II rate $r > 0$ of testing $H_0$ against $H_1$ are
\[
d_r^1(H_0 \| H_1) := \sup \{ r' : (r, r') \in A^1(H_0 \| H_1) \}. \tag{II.34} \]

Definition II.20 The Chernoff exponents of testing $H_0$ against $H_1$ are
\[
c_r^1(H_0 \| H_1) := \sup \{ r : (r, r) \in A^1(H_0 \| H_1) \}. \]

Remark II.21 Note that $d_r^1(H_0 \| H_1)$ is the length of the line segment obtained by intersecting $A^1(H_0 \| H_1)$ with $\{r\} \times \mathbb{R}$, while $c_r^1(H_0 \| H_1)$ is $2^{-1/2}$ times the length of the line segment obtained by intersecting $A^1(H_0 \| H_1)$ with the diagonal line $D := \{(r, r) : r \in \mathbb{R}\}$. That is,
\[
d_r^1(H_0 \| H_1) = \lambda_1 (\{(r) \times \mathbb{R}\} \cap A^1(H_0 \| H_1)), \quad c_r^1(H_0 \| H_1) = 2^{-1/2} \lambda_1 (D \cap A^1(H_0 \| H_1)),
\]
where $\lambda_1$ is the one-dimensional Lebesgue measure (length) of the line segment in its argument, and that the intersections are indeed line segments follows from Lemma II.17.

While these formulations of the error exponents are slightly different from how these quantities are usually defined, the simple geometric picture behind them is very convenient in the analysis of the error exponents of composite state discrimination problems; see, e.g., Lemma II.27. It is straightforward to rewrite the direct exponent in the familiar way, as
\[
d_r(H_0 \| H_1) = \sup \left\{ \liminf_{n \to +\infty} - \frac{1}{n} \log \alpha_n(H_0|T_n) : \liminf_{n \to +\infty} - \frac{1}{n} \log \beta_n(H_1|T_n) \geq r \right\},
\]
where the supremum is over all sequences of tests satisfying the indicated constraint. The relaxed variants for consistent null- and/or alternative hypotheses can be expressed similarly, e.g., as
\[
d_r^{01}(H_0 \| H_1) = \sup \left\{ \inf \liminf_{i \to +\infty} - \frac{1}{n} \log \alpha_n(\varrho_{n,i}|T_n) : \liminf_{n \to +\infty} - \frac{1}{n} \log \beta_n(H_1|T_n) \geq r \right\}.
\]

For the Chernoff exponent, we have the following:

Lemma II.22 The Chernoff exponent $c(H_0 \| H_1)$ can be expressed as
\[
c(H_0 \| H_1) = \liminf_{n \to +\infty} - \frac{1}{n} \log \min_{T_n \in (\mathcal{M}_n)_{[0, t]}} \{\alpha_n(H_0|T_n) + \beta_n(H_1|T_n)\}. \tag{II.35} \]

Proof If $(r, r)$ is achievable by a test sequence $(T_n)_{n \in \mathbb{N}}$ then
\[
r \leq \liminf_{n \to +\infty} - \frac{1}{n} \log (\alpha_n(H_0|T_n) + \beta_n(H_1|T_n)) \leq \liminf_{n \to +\infty} - \frac{1}{n} \log \min_{T_n \in (\mathcal{M}_n)_{[0, t]}} \{\alpha_n(H_0|T_n) + \beta_n(H_1|T_n)\},
\]
and taking the supremum over all such $r$ yields LHS$\leq$RHS in (II.35). On the other hand, if LHS$<$RHS there exists a $\delta > 0$ and a sequence of tests $(T_n)_{n \in \mathbb{N}}$ such that for all large enough $n$, $\alpha_n(H_0|T_n) < e^{-n(c(H_0 \| H_1) + \delta)}$, $\beta_n(H_1|T_n) < e^{-n(c(H_0 \| H_1) + \delta)}$, and hence $(c(H_0 \| H_1) + \delta, c(H_0 \| H_1) + \delta)$ is achievable, a contradiction. \hfill $\square$

Definition II.23 The Stein exponent of testing $H_0$ against $H_1$ is
\[
s_r(H_0 \| H_1) := \sup \left\{ \liminf_{n \to +\infty} - \frac{1}{n} \log \beta_n(H_1|T_n) : \liminf_{n \to +\infty} \alpha_n(H_0|T_n) = 0 \right\} \tag{II.36} \]
\[
= \sup \left\{ r \geq 0 : \exists T_n \in (\mathcal{M}_n)_{[0, t]}, n \in \mathbb{N}, \liminf_{n \to +\infty} - \frac{1}{n} \log \beta_n(H_1|T_n) \geq r, \lim_{n \to +\infty} \alpha_n(H_0|T_n) = 0 \right\}. \tag{II.37} \]
where the supremum in (II.36) is over all sequences of tests $T_n \in (\mathcal{M}_n)_{[0, t]}$, $n \in \mathbb{N}$, satisfying the constraint. The variants $s_r^{01}(H_0 \| H_1)$, etc., can be defined by a straightforward modification of (II.37).

It is well-known that in finite-copy state discrimination, general tests may provide an advantage over projective tests (i.e., where $T_n^* = T_n$); see, e.g., [30, Example 2.27]. This is not the case, however, in the asymptotic study of the error exponents. Indeed, we have the following:
Lemma II.24 All error exponents defined above remain unchanged if in their definitions we restrict the tests to be projections.

Proof Let $f : [0, 1] \rightarrow \{0, 1\}$ be the function defined by the formula

$$f(x) = \begin{cases} 0 & \text{if } x < 1/2, \\ 1 & \text{if } x \geq 1/2. \end{cases}$$

(II.38)

For any test $T_n \in (\mathcal{M}_n)[0,1]$, $Q_n := f(T_n)$ is a projection, $0 \leq Q_n \leq 2T_n$ and $0 \leq I - Q_n \leq 2(I - T_n)$ since $f(x) \leq 2x$ and $1 - f(x) \leq 2(1 - x)$ for all $x \in [0,1]$. Thus, $\alpha_n(H_0|Q_n) \leq 2\alpha_n(H_0|T_n)$ and $\beta_n(H_1|Q_n) \leq 2\beta_n(H_1|T_n)$, from which the assertion follows immediately. \hfill \square

In the case of i.i.d. state discrimination with simple hypotheses, i.e., when $R = \{\varrho\}$, $S = \{\sigma\}$ for some states $\varrho, \sigma \in S(\mathcal{M})$, each of the above discussed exponents is known to be equal to a certain divergence of the two states:

Lemma II.25 For any von Neumann algebra $\mathcal{M}$, and any states $\varrho, \sigma \in S(\mathcal{M})$,

$$s(\varrho\|\sigma) = D(\varrho\|\sigma), \quad \text{(Stein’s lemma)}$$

$$e(\varrho\|\sigma) = C(\varrho\|\sigma), \quad \text{(Chernoff bound)}$$

$$d_r(\varrho\|\sigma) = H_r(\varrho\|\sigma), \quad r > 0. \quad \text{(Hoeffding bound)}$$

See [28, 52] for the Stein’s lemma, [4, 49] for the Chernoff bound, and [21, 44] for the direct exponents when $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with a finite-dimensional Hilbert space $\mathcal{H}$, and [31] for all the error exponents in the case of a general von Neumann algebra. The importance of the above results is twofold: on the one hand, they give single-copy expressions for the error exponents (i.e., not involving a limit $n \rightarrow +\infty$), while on the other hand, they provide operational interpretations to the divergences appearing in them.

In this paper we are mainly interested in the error exponents of binary i.i.d. state discrimination, and their relations to the worst pairwise exponents. Let us start with a few simple observations. First, in the most general case, it is intuitively clear that “larger” hypotheses are less distinguishable; formally, if $\alpha_n(\tilde{R}_n|T_n) \geq \alpha_n(R_n|T_n)$, $\beta_n(\tilde{S}_n|T_n) \geq \beta_n(S_n|T_n)$, $T_n \in (\mathcal{M}_n)[0,1]$, $n \in \mathbb{N}$, then

$$\alpha_n(\tilde{R}_n|T_n) \geq \alpha_n(R_n|T_n), \quad \beta_n(\tilde{S}_n|T_n) \geq \beta_n(S_n|T_n), \quad T_n \in (\mathcal{M}_n)[0,1], \quad n \in \mathbb{N},$$

and hence,

$$e(\tilde{R}_n)_{n \in \mathbb{N}} \parallel (\tilde{S}_n)_{n \in \mathbb{N}} \leq e(\tilde{R}_n)_{n \in \mathbb{N}} \parallel (S_n)_{n \in \mathbb{N}}, \quad \text{(II.39)}$$

where $e$ may be any of the Stein-, Chernoff, or the direct exponents. In the consistent case we further have

$$e(\tilde{R}_n)_{n \in \mathbb{N}} \parallel (\tilde{S}_n)_{n \in \mathbb{N}} \leq e(\tilde{R}_n)_{n \in \mathbb{N}} \parallel (\tilde{S}_n)_{n \in \mathbb{N}} \leq \inf_{e \in E, s \in S} e(\varrho, \sigma) = E(\|S\|), \quad \text{(II.42)}$$

where $x = 0$ or $x = 1$, and analogous inequalities may be obtained by a straightforward modification when only one of the hypotheses is assumed to be/is treated as consistent. Specializing to the i.i.d. case yields

$$e(\|S\|) \leq \left\{ \begin{array}{l} e(0)(\|S\|) \\ e(1)(\|S\|) \end{array} \right\} \leq \inf_{e \in E, s \in S} e(\|S\|) = E(\|S\|), \quad \text{(II.42)}$$

where $E$ is the divergence corresponding to the exponent $e$ as in Lemma II.25.

Second, in the mixed i.i.d. case,

$$\alpha_n(\mathcal{P}|T_n) = \sup_{p \in \mathcal{P}} \int_{S(\mathcal{M})} dp(\varrho) \varrho^{\otimes n}(I - T_n) \leq \sup_{\varrho \in \text{supp } \mathcal{P}} \varrho^{\otimes n}(I - T_n) = \alpha_n(\text{supp } \mathcal{P}|T_n),$$

and similarly, $\beta_n(\mathcal{Q}|T_n) \leq \beta_n(\text{supp } \mathcal{Q}|T_n)$, where $\text{supp } \mathcal{P} := \cup_{p \in \mathcal{P}} \text{supp } p$, and $\text{supp } \mathcal{Q} := \cup_{q \in \mathcal{Q}} \text{supp } q$. Combining the above, we get that in the mixed i.i.d. case, with hypotheses $\mathcal{P}, \mathcal{Q}$,

$$e(\text{supp } \mathcal{P}) \parallel \text{supp } \mathcal{Q} \leq \left\{ \begin{array}{l} e(\|\mathcal{P}\|) \parallel \text{supp } \mathcal{Q} \\ e(\|\mathcal{P}\|) \parallel \text{supp } \mathcal{Q} \end{array} \right\} \leq e(\|\mathcal{Q}\|) \leq \inf_{p \in \mathcal{P}, q \in \mathcal{Q}} e(p\|q). \quad \text{(II.43)}$$
In particular, for any state \( q \in \mathcal{S}(\mathcal{M}) \), and any probability distribution \( p \) on the Borel sets of \( \mathcal{S}(\mathcal{M}) \), (II.42) and (II.43) give the incomparable bounds

\[
e(\|q\| \supp q) \leq \begin{cases} \inf_{\sigma \in \supp q} e(\|q\|\sigma), \\ e(\|q\|q). \end{cases} \tag{II.44}
\]

**Remark II.26** It is easy to see that for finitely supported probability distributions \( p, q \) on \( \mathcal{S}(\mathcal{M}) \), we have

\[
e(\supp p \| \supp q) = e(p\|q).
\]

In particular, for a finitely supported \( q \), (II.44) reduces to

\[
e(\|q\| \supp q) = e(\|q\|) \leq \inf_{\sigma \in \supp q} e(\|q\|\sigma). \tag{II.45}
\]

Our main goal in this paper is to give sufficient conditions for (some of) the above inequalities to hold as equalities, and exhibit explicit examples demonstrating strict inequality when these conditions are not satisfied. Note that equality between the first and the last term in (II.42) is equivalent to

\[
e(R\|S) = E(R\|S).
\]

In particular, such an equality gives an operational interpretation of the divergence of two sets of states, extending that in Lemma II.25 for pairs of single states.

We start our investigation with some simple but very useful necessary and sufficient conditions in terms of the set of achievable exponent pairs. In particular, Lemmas II.27 and II.29 below show that to prove equality for the Stein and the Chernoff exponents, it is sufficient to prove equality for the direct exponents. We will utilize this in Theorems III.2 and III.8.

**Lemma II.27** Let \( H_0 : (R_n)_{n \in \mathbb{N}}, H_1 : (S_n)_{n \in \mathbb{N}} \) specify a binary state discrimination problem, and for every \( k \) in some index set \( K \), let \( H_0^{(k)} : (R_n^{(k)})_{n \in \mathbb{N}}, H_1^{(k)} : (S_n^{(k)})_{n \in \mathbb{N}} \) also specify binary state discrimination problems (on possibly different sequences of von Neumann algebras). Moreover, let \( t, k \in \{0, \{1\}, \{0, 1\}, k \in K \). Then

\[
\mathbb{A}^t(H_0\|H_1) \subseteq \bigcap_{k \in K} \mathbb{A}^t_k(H_0^{(k)}\|H_1^{(k)}) \quad \implies \quad d_t^r(H_0\|H_1) \leq \inf_{k \in K} d_k^r(H_0^{(k)}\|H_1^{(k)}), \quad r > 0, \tag{II.46}
\]

\[
\implies \quad \overline{A}^t(H_0\|H_1) \subseteq \bigcap_{k \in K} \overline{A}^t_k(H_0^{(k)}\|H_1^{(k)}) \tag{II.47}
\]

\[
\implies \quad e^t(H_0\|H_1) \leq \inf_{k \in K} e^{t_k}(H_0^{(k)}\|H_1^{(k)}), \tag{II.48}
\]

where \( \overline{A}^t, \overline{A}^t_k \) stand for the closures of the achievable sets, and

\[
\mathbb{A}^t(H_0\|H_1) \supseteq \bigcap_{k \in K} \mathbb{A}^t_k(H_0^{(k)}\|H_1^{(k)}) \quad \implies \quad d_t^r(H_0\|H_1) \geq \inf_{k \in K} d_k^r(H_0^{(k)}\|H_1^{(k)}), \quad r > 0, \tag{II.49}
\]

\[
\implies \quad \overline{A}^t(H_0\|H_1) \supseteq \bigcap_{k \in K} \overline{A}^t_k(H_0^{(k)}\|H_1^{(k)}) \tag{II.50}
\]

\[
\implies \quad e^t(H_0\|H_1) \geq \inf_{k \in K} e^{t_k}(H_0^{(k)}\|H_1^{(k)}). \tag{II.51}
\]
Proof We only prove (II.46)–(II.48), as (II.49)–(II.51) follow the same way. We have
\[ \mathcal{A}^t(H_0||H_1) \subseteq \bigcap_{k \in K} \mathcal{A}^{t_k}(H_0^{(k)}||H_1^{(k)}) \]

\[ \implies \forall r > 0 : \quad (\{r\} \times \mathbb{R}) \cap \mathcal{A}^t(H_0||H_1) \subseteq \left( \{r\} \times \mathbb{R} \right) \cap \left( \bigcap_{k \in K} \mathcal{A}^{t_k}(H_0^{(k)}||H_1^{(k)}) \right) \]

\[ = \bigcap_{k \in K} \left( (\{r\} \times \mathbb{R}) \cap \mathcal{A}^{t_k}(H_0^{(k)}||H_1^{(k)}) \right) \]

\[ \implies \forall r > 0 : \quad \lambda_1 \left( \left( (\{r\} \times \mathbb{R}) \cap \mathcal{A}^t(H_0||H_1) \right) \right) \leq \inf_{k \in K} \lambda_1 \left( \left( (\{r\} \times \mathbb{R}) \cap \mathcal{A}^{t_k}(H_0^{(k)}||H_1^{(k)}) \right) \right) \]

\[ = \bigcap_{k \in K} \left( (\{r\} \times \mathbb{R}) \cap \mathcal{A}^{t_k}(H_0^{(k)}||H_1^{(k)}) \right) \]

(The intersection \((\{0\} \times \mathbb{R}) \cap \mathcal{A}^t(H_0||H_1)\) is \(\{0\} \times [0, +\infty)\) for any \(H_0, H_1\), and hence it is enough to consider \(r > 0\) above.) This proves (II.46)–(II.47), and (II.48) follows in the same way as
\[ c^t(H_0||H_1) = \lambda_1 \left( D \cap \mathcal{A}^t(H_0||H_1) \right) \leq \lambda_1 \left( \bigcap_{k \in K} \left( D \cap \mathcal{A}^{t_k}(H_0^{(k)}||H_1^{(k)}) \right) \right) = \inf_{k \in K} \lambda_1 \left( D \cap \mathcal{A}^{t_k}(H_0^{(k)}||H_1^{(k)}) \right), \]

where \(D := \{(r, r) : r \in \mathbb{R}\}\), and the inequality follows from (II.47). \(\square\)

Corollary II.28 In the case of binary mixed i.i.d. hypothesis testing with null-hypothesis represented by \(\mathcal{P}\), and the alternative hypothesis by \(\mathcal{Q}\), we have
\[ \mathcal{A}(\mathcal{P}||\mathcal{Q}) \supseteq \bigcap_{p \in \mathcal{P}, q \in \mathcal{Q}} \mathcal{A}(p||q) \iff d_r(\mathcal{P}||\mathcal{Q}) = \inf_{p \in \mathcal{P}, q \in \mathcal{Q}} d_r(p||q), \quad r > 0 \quad (II.52) \]
\[ \implies c(\mathcal{P}||\mathcal{Q}) = \inf_{p \in \mathcal{P}, q \in \mathcal{Q}} c(p||q). \quad (II.53) \]

Proof Immediate from (II.43), Lemma II.27, and Lemma II.18. \(\square\)

Lemma II.29 Let \(H_0 : (R_n)_{n \in \mathbb{N}}\) and \(H_1 : (S_n)_{n \in \mathbb{N}}\) be the null and the alternative hypotheses of a general binary state discrimination problem, and let \(R, S \subseteq \mathcal{S}(\mathcal{M})\) be compact sets for some von Neumann algebra \(\mathcal{M}\).

If \(d_r(H_0||H_1) \geq \inf_{\varrho \in R, \sigma \in S} d_r(\varrho||\sigma), \quad r > 0\), then \(s^t(H_0||H_1) \geq \inf_{\varrho \in R, \sigma \in S} s(\varrho||\sigma). \quad (II.54)\)

Proof Assume that \(d_r^t(H_0||H_1) \geq \inf_{\varrho \in R, \sigma \in S} d_r(\varrho||\sigma), \quad r > 0\). Then
\[ s^t(H_0||H_1) \geq \sup \{r > 0 : d_r^t(H_0||H_1) > 0\} \geq \sup \{r > 0 : \inf_{\varrho \in R, \sigma \in S} d_r(\varrho||\sigma) > 0\} \]
\[ = D(R||S) = \inf_{\varrho \in R, \sigma \in S} s(\varrho||\sigma), \]

where the inequality in the first step is obvious, the second inequality follows by assumption, the first equality is due to Lemma II.25 (Hoeffding bound) and Lemma II.9, and the last equality follows again by Lemma II.25 (Stein’s lemma). \(\square\)

Finally, we note that the standard techniques used for the achievability parts (i.e., \(c(\varrho||\sigma) \geq E(\varrho||\sigma)\)) of the equalities in Lemma II.25 also yield achievability bounds in the most general case; however, the resulting lower bounds are given in terms of regularized divergences, which are not feasible to evaluate in general. We give these bounds below with detailed proofs for completeness. We remark that an improved lower bound for the Stein exponent in the composite i.i.d. case was given recently in [8].
Proposition II.30 For the general binary state discrimination problem $H_0 : (R_n)_{n \in \mathbb{N}}$ vs. $H_1 : (S_n)_{n \in \mathbb{N}}$, we have

$$c((R_n)_{n \in \mathbb{N}})(S_n)_{n \in \mathbb{N}} \geq \liminf_{n \to +\infty} \frac{1}{n} \sup_{\alpha \in (0,1)} (1 - \alpha)D_\alpha(\co(R_n)\|\co(S_n))$$

$$\geq \liminf_{n \to +\infty} \frac{1}{n} C(\co(R_n)\|\co(S_n)), \quad (\text{II.55})$$

$$d_r((R_n)_{n \in \mathbb{N}})(S_n)_{n \in \mathbb{N}} \geq \liminf_{n \to +\infty} \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} \left[ r - \frac{1}{n} D_\alpha(\co(R_n)\|\co(S_n)) \right]$$

$$\geq \liminf_{n \to +\infty} \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} \left[ r - \frac{1}{n} D_\alpha(\co(R_n)\|\co(S_n)) \right]$$

$$= \liminf_{n \to +\infty} \frac{1}{n} H_{r,\alpha}(\co(R_n)\|\co(S_n)), \quad r > 0, \quad (\text{II.56})$$

$$s((R_n)_{n \in \mathbb{N}})(S_n)_{n \in \mathbb{N}} \geq \sup_{\alpha \in (0,1)} \liminf_{n \to +\infty} \frac{1}{n} D_\alpha(\co(R_n)\|\co(S_n)). \quad (\text{II.60})$$

**Proof** The following argument is quite standard, but we explain it in detail for readers’ convenience. For any $c \in \mathbb{R}$, we have

$$\inf_{T \in (\mathcal{M}_n)_{0}} (\alpha_n(R_n|T) + e^{nc} \beta_n(S_n|T)) = \inf_{T \in (\mathcal{M}_n)_{0}} (\alpha_n(\co(R_n)|T) + e^{nc} \beta_n(\co(S_n)|T))$$

$$= \inf_{T \in (\mathcal{M}_n)_{0}, \rho \in \co(R_n), \sigma \in \co(S_n)} (\rho(I - T) + e^{nc} \sigma(T))$$

$$= \min_{T \in (\mathcal{M}_n)_{0}, \rho \in \co(R_n), \sigma \in \co(S_n)} (\rho(I - T) + e^{nc} \sigma(T))$$

$$= \sup_{\rho \in \co(R_n), \sigma \in \co(S_n)} \min_{T \in (\mathcal{M}_n)_{0}} (\rho(I - T) + e^{nc} \sigma(T))$$

$$\leq \sup_{\rho \in \co(R_n), \sigma \in \co(S_n)} e^{nc(1 - \alpha)}(\alpha - 1)D_\alpha(\rho\|\sigma), \quad \alpha \in (0,1),$$

$$= e^{nc(1 - \alpha)}(\alpha - 1)D_\alpha(\rho\|\sigma), \quad \alpha \in (0,1).$$

The first equality above follows as $(\rho, \sigma) \mapsto \alpha_n(R_n|T) + e^{nc} \beta_n(S_n|T)$ is affine, and the second equality is by definition. Note that $T \mapsto \rho(I - T) + e^{nc} \sigma(T)$ is continuous in the $w^*$-topology, and hence its supremum over $\co(R_n) \times \co(S_n)$ is lower semi-continuous, and therefore it attains its infimum on the $w^*$-compact set $(\mathcal{M}_n)_{0}$ (see Lemma II.1). This proves the third equality. The fourth equality follows by the Kneser-Fan minimax theorem [15, 34]. The inequality is due to Audenaert’s inequality [4], and its extension to von Neumann algebras [31, 50], and the last equality is obvious.

Taking $c = 0$, we obtain the existence of a test $T_n$ such that

$$\max \{ \alpha_n(R_n|T_n), \beta_n(S_n|T_n) \} \leq \alpha_n(R_n|T_n) + \beta_n(S_n|T_n) \leq e^{\inf_{\alpha \in (0,1)} (\alpha - 1)D_\alpha(\co(R_n)\|\co(S_n))},$$

from which (II.55) follows immediately. Moreover, we have

$$\inf_{\alpha \in (0,1)} (\alpha - 1)D_\alpha(\co(R_n)\|\co(S_n)) \leq \inf_{\alpha \in (0,1)} (\alpha - 1)D_\alpha(\co(R_n)\|\co(S_n))$$

$$= \inf_{\alpha \in (0,1)} \sup_{\rho \in \co(R_n), \sigma \in \co(S_n)} (\alpha - 1)D_\alpha(\rho\|\sigma)$$

$$= \sup_{\rho \in \co(R_n), \sigma \in \co(S_n)} \inf_{\alpha \in (0,1)} (\alpha - 1)D_\alpha(\rho\|\sigma)$$

$$= -C(\co(S)\|\co(S)),$$

where the second equality follows again from the Kneser-Fan minimax theorem, since $\psi(\rho\|\sigma|\alpha)$ is convex in $\alpha$, and concave in $(\rho, \sigma)$, according to Lemmas II.2 and II.6. This yields (II.56).

Let us now return to the case of a general $c$. By the above argument, for any $c \in \mathbb{R}$ and any $\alpha \in (0,1)$, there exists a test $T_{n,c,\alpha}$ such that

$$\alpha_n(R|T_{n,c,\alpha}) \leq e^{nc(1 - \alpha) + (\alpha - 1)D_\alpha(\co(R_n)\|\co(S_n))}, \quad \beta_n(R|T_{n,c,\alpha}) \leq e^{nc(1 - \alpha) + (\alpha - 1)D_\alpha(\co(R_n)\|\co(S_n))}.$$
Choosing \( c(r, \alpha) := \frac{e^{-(1-\alpha)D_\alpha(\|co(R_n)\|_\sigma, co(S_n))}}{\alpha} \) yields, with \( T_{n,\alpha} := T_{n,c(r,\alpha),\alpha} \)

\[
\alpha_n(R|T_{n,\alpha}) \leq e^{-n \frac{\alpha}{\alpha-1} \left[r - \frac{1}{n}D_\alpha(\|co(R_n)\|_\sigma, co(S_n))\right]}, \quad \beta_n(R|T_{n,\alpha}) \leq e^{-nr}.
\]

The above argument fails only if \( D_\alpha(\|co(R_n)\|_\sigma, co(S_n)) = +\infty \); however, in that case \( g_n \perp \sigma_n \) for all \( g_n \in co(R_n) \) and \( \sigma_n \in co(S_n) \), and hence we may choose \( T_{n,\alpha} \) to be any test that perfectly distinguishes \( co(R_n) \) and \( co(S_n) \).

Now, for any \( \gamma \) smaller than the RHS of (II.57), and every large enough \( n \), we have

\[
\sup_{\alpha \in (0,1)} \frac{\alpha-1}{\alpha} \left[r - \frac{1}{n}D_\alpha(\|co(R_n)\|_\sigma, co(S_n))\right] > \gamma,
\]

and hence there exists an \( \alpha_{\gamma,n} \in (0,1) \) such that \( \frac{\alpha_{\gamma,n}-1}{\alpha_{\gamma,n}} \left[r - \frac{1}{n}D_{\alpha_{\gamma,n}}(\|co(R_n)\|_\sigma, co(S_n))\right] > \gamma \). Thus, with the test sequence \( T_n := T_{n,\alpha_{\gamma,n}} \), we get

\[
\liminf_{n \to +\infty} -\frac{1}{n} \log \beta_n(S|T_n) \geq r, \quad \liminf_{n \to +\infty} -\frac{1}{n} \log \alpha_n(R|T_n) \geq \gamma, \quad \text{whence} \quad d_r(R|S) \geq \gamma.
\]

Taking the supremum over all \( \gamma \) as above yields the inequality in (II.57). Finally, the inequality in (II.58) is trivial, and the equality in (II.59) follows by Lemma II.8.

Let us finally consider (II.60). If the RHS of (II.60) is zero then there is nothing to prove, and hence for the rest we assume that it is strictly positive. Consider any \( 0 < r < \sup_{\alpha \in (0,1)} \liminf_{n \to +\infty} \frac{1}{n}D_\alpha(\|co(R_n)\|_\sigma, co(S_n)) \). Then

\[
d_r((R_n)_{n \in \mathbb{N}}||(S_n)_{n \in \mathbb{N}}) \geq \liminf_{n \to +\infty} \sup_{\alpha \in (0,1)} \frac{\alpha-1}{\alpha} \left[r - \frac{1}{n}D_\alpha(\|co(R_n)\|_\sigma, co(S_n))\right]
\geq \sup_{\alpha \in (0,1)} \frac{\alpha-1}{\alpha} \left[r - \liminf_{n \to +\infty} \frac{1}{n}D_\alpha(\|co(R_n)\|_\sigma, co(S_n))\right] > 0,
\]

where the first inequality is by (II.57), the second inequality is obvious, and the third inequality is immediate from our assumption. Thus, \( s((R_n)_{n \in \mathbb{N}}||(S_n)_{n \in \mathbb{N}}) > r \) for any \( r \) as above, and taking the supremum over every such \( r \) yields (II.60).

\[ \square \]

### III. Classical State Discrimination

#### A. Countably many hypotheses: Equality holds for the relaxed exponents

**Lemma III.1** Let \( \mathcal{M} \) be a commutative von Neumann algebra, and \( T_{i,j} \in \mathcal{M}_{[0,1]} \) be tests for every \( i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, m\} \). Then there exists a test \( T \in \mathcal{M}_{[0,1]} \) such that

\[
\forall i: I - T \leq \sum_{j=1}^{m} (I - T_{i,j}), \quad \text{and} \quad \forall j: T \leq \sum_{i=1}^{k} T_{i,j}.
\]

**Proof** Without loss of generality we may assume that \( \mathcal{M} = L^\infty(\mathcal{X}, \mathcal{F}, \mu) \) for some measure space \( (\mathcal{X}, \mathcal{F}, \mu) \). Then \( T_{i,j} \) are nonnegative functions on \( \mathcal{X} \) and

\[
T(x) := \max_j \min_i T_{i,j}(x), \quad x \in \mathcal{X},
\]

defines a test \( T \in \mathcal{M}_{[0,1]} \). For any \( i \in \{1, \ldots, k\} \) and \( x \in \mathcal{X} \) we have \( T(x) \geq \min_j T_{i,j}(x) \), and hence

\[
1 - T(x) \leq 1 - \min_j T_{i,j}(x) = \max_j (1 - T_{i,j}(x)) \leq \sum_{j=1}^{m} (1 - T_{i,j}(x)),
\]

implying that \( I - T \leq \sum_{j=1}^{m} (I - T_{i,j}) \). The other inequality can be shown in a similar manner. \[ \square \]
Theorem III.2 Let \((M_n)_{n \in \mathbb{N}}\) be a sequence of commutative von Neumann algebras, \(I, J\) be countable index sets, and for every \(n \in \mathbb{N}\), \(R_n, S_n, \{\varrho_{n,i}\}_{i \in I}, \{\sigma_{n,j}\}_{j \in J} \subseteq \mathcal{S}(M_n)\). Then

\[
\begin{align*}
e^{(0)} \left( \left\{ (\varrho_{n,i})_{i \in I} \right\}_{n \in \mathbb{N}} \right) &= \inf_i \left\{ \left((\varrho_{n,i})_{n \in \mathbb{N}} \right) \right\}, \\
e^{(1)} \left( \left\{ (\varrho_{n,i})_{i \in I} \right\}_{n \in \mathbb{N}} \right) &= \inf_j \left\{ \left((\sigma_{n,j})_{n \in \mathbb{N}} \right) \right\}, \tag{III.62} \\
e^{(0,1)} \left( \left\{ (\varrho_{n,i})_{i \in I} \right\}_{n \in \mathbb{N}} \right) &= \inf_{i,j} \left\{ \left((\varrho_{n,i})_{n \in \mathbb{N}} \right) \right\}, \tag{III.63}
\end{align*}
\]

where \(e\) may be any of the error exponents \(e = s, c, \text{ or } d_r, \ r > 0\). Moreover, the exponents \(e^{(0)}/e^{(1)}/e^{(0,1)}\) on the left hand sides above coincide with \(e\) if \(I\) is finite/\(J\) is finite/both \(I\) and \(J\) are finite, respectively.

Proof Again, we may assume that for every \(n \in \mathbb{N}\), \(M_n = L^\infty(\mathcal{X}_n, \mathcal{F}_n, \mu_n)\) for some measure space \((\mathcal{X}_n, \mathcal{F}_n, \mu_n)\). Let us prove (III.63) under the assumption that \(I = J = \mathbb{N}\); the rest of the statements follow in a similar way. Note that we have LHS\(\leq\)RHS in (III.63), and hence we only need to prove the converse inequality. Moreover, it is enough to prove the converse inequality for \(e = d_r, \ r \in (0, +\infty)\), according to Lemmas II.27 and II.29.

Thus, let \(e = d_r\), for some \(r \in (0, +\infty)\), and let \(r'\) be the RHS of (III.63). By assumption, for every \(n > 0\) and every \(i, j \in \mathbb{N} \times \mathbb{N}\), there exists a sequence of tests \((T_{\delta,i,j,n})_{n \in \mathbb{N}}\) such that

\[
\liminf_{n \to +\infty} -\frac{1}{n} \log \varrho_{n,i}(I - T_{\delta,i,j,n}) > r' - \delta, \quad \liminf_{n \to +\infty} -\frac{1}{n} \log \sigma_{n,i}(T_{\delta,i,j,n} > r - \delta).
\]

For every \(n \in \mathbb{N}\), let \(k_{\delta,n} := \max\{k \in [n] : \varrho_{m,i}(I - T_{\delta,i,j,m}) < e^{-m(r' - \delta)}, \sigma_{m,j}(T_{\delta,i,j,m}) < e^{-m(r - \delta)}, m \geq n, i, j \in [k]\}\), and define \(T_{\delta,n} := \max_{i \in [k_{\delta,n}]} \min_{j \in [k_{\delta,n}]} T_{\delta,i,j,n}(x)\). As shown in the proof of Lemma III.1,

\[
\forall i \in [k_{\delta,n}] : \quad I - T_{\delta,n} \leq \sum_{k=1}^{k_{\delta,n}} (I - T_{\delta,i,k,n}), \quad \forall j \in [k_{\delta,n}] : \quad T_{\delta,n} \leq \sum_{k=1}^{k_{\delta,n}} T_{\delta,k,j,n}. \tag{III.64}
\]

By assumption, \(\lim_{n \to +\infty} k_{\delta,n} = +\infty\), and hence for every \((i, j) \in \mathbb{N} \times \mathbb{N}\), there exists an \(N_{i,j}\) such that for all \(n \geq N_{i,j}, i, j \in [k_{\delta,n}]\), and thus, by (III.64),

\[
\varrho_{n,i}(I - T_{\delta,n}) \leq n e^{-n(r' - \delta)}, \quad \sigma_{n,j}(T_{\delta,n}) \leq n e^{-n(r - \delta)}.
\]

This shows that \((r - \delta, r' - \delta) \in A^{(0,1)} \left( \left\{ (\varrho_{n,i})_{i \in I} \right\}_{n \in \mathbb{N}} \right) \left( \left\{ (\sigma_{n,j})_{i \in I} \right\}_{n \in \mathbb{N}} \right)\). Since this holds for every \(\delta > 0\), Lemma II.18 yields that \((r, r') \in A^{(0,1)} \left( \left\{ (\varrho_{n,i})_{i \in I} \right\}_{n \in \mathbb{N}} \right) \left( \left\{ (\sigma_{n,j})_{i \in I} \right\}_{n \in \mathbb{N}} \right)\), and therefore \(A^{(0,1)} \left( \left\{ (\varrho_{n,i})_{i \in I} \right\}_{n \in \mathbb{N}} \right) \left( \left\{ (\sigma_{n,j})_{i \in I} \right\}_{n \in \mathbb{N}} \right) \geq r'\).

B. Infinite dimension with countably infinitely many alternative hypotheses: Equality may not hold

In this section we show that the composite hypothesis testing error exponents may be strictly smaller than the worst pairwise error exponents even in the classical case, provided that the system is allowed to be infinite-dimensional, and the alternative hypothesis of infinite cardinality. More precisely, we prove the following:

Theorem III.3 There exists a probability density function \(q\) on \([0, 1]\) (w.r.t. the Lebesque measure), and a probability distribution \(q\) supported on a countably infinite set of probability density functions on \([0, 1]\), such that

\[
e(q\|q) \leq e(q\|q) < \inf_{\sigma \in \text{supp } q} e(q\|\sigma) \tag{III.65}
\]

for any of the Stein-, Chernoff-, and the direct exponents. Moreover, the first inequality above is also strict for the Stein-, Chernoff-, and the direct exponents \(d_r\), with \(0 < r < D(q\|q)\).

Remark III.4 Recall that probability density functions on \([0, 1]\) represent states on the commutative infinite-dimensional von Neumann algebra \(L^\infty([0, 1])\); see Section II A.
We start with describing our example first; Theorem III.3 will then follow from the more precise statement in Proposition III.5.

Assume that we receive \( n \) real numbers drawn independently from the interval \([0,1]\), all with the same distribution. Our null hypothesis is that this distribution is the uniform one. The alternative hypothesis is that first a random natural number \( N \) was generated with probabilities

\[
\text{"prob. of } N = k \text{"} = \frac{6}{\pi^2} \frac{1}{k^2} =: q_k,
\]

and then the \( n \) real numbers were drawn independently from the interval \([0,1]\) with distribution \( \mu_k \), where \( k \) is the obtained value of \( N \). The measure \( \mu_k \) is given by the density function (w.r.t. the Lebesgue measure, i.e. the uniform distribution) \( 2 \cdot \mathbf{1}_{H_k} \), where \( H_1 = [0,\frac{1}{2}], H_2 = [0,\frac{1}{4}] \cup \left[ \frac{1}{2}, \frac{3}{4} \right] \), and in general

\[
H_k = \bigcup_{j=0}^{2^{k-1}-1} \left[ \frac{j}{2^{k-1}}, \frac{j+1}{2^{k-1}} + \frac{1}{2^k} \right],
\]

where \( \mathbf{1}_{H_k} \) stands for the characteristic function (or indicator function) of \( H_k \). Note that the factor \( 6/\pi^2 \) is there to make the sum of probabilities \( \sum_{k=1}^{\infty} q_k = 1 \).

Using the notations and conventions explained in the preliminaries, for a fixed number of copies \( n \), our null hypothesis is represented by the single state \( \varrho^\otimes n = \mathbf{1}^\otimes_{[0,1]} = \mathbf{1}_{[0,1]^n} \), whereas our alternative hypothesis may be represented by the convex combination of i.i.d. states

\[
\sigma^{(n)} := \sum_{k=1}^{\infty} q_k \sigma_k^{\otimes n} = \sum_{k=1}^{\infty} q_k (2 \cdot \mathbf{1}_{H_k})^{\otimes n} = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{2^n}{k^2} \mathbf{1}_{H_k}^n
\]

where \( H_k^n = H_k \times \ldots \times H_k \) is the \( n \)-fold Cartesian product of \( H_k \) with itself. Our aim is to contrast the error-exponents of this problem with the worst one we would get if our null and alternate hypotheses were just \( \varrho^\otimes n \) and \( \sigma_k^{\otimes n} \) for some \( k \in \mathbb{N} \). For this, we obtain the following:

**Proposition III.5** In the above example, we have

(i) \( s(\varrho||q) = \log 2 < \infty = s(\varrho||\sigma_k) \);

(ii) \( d_r(\varrho||q) = \max\{\log 2 - r, 0\} < \log 2 = d_r(\varrho||\sigma_k), \quad r > 0 \);

(iii) \( c(\varrho||q) = \frac{1}{2} \log 2 < \log 2 = c(\varrho||\sigma_k) \);

for every \( k \in \mathbb{N} \).

**Proof** Let us begin by noting that \( \lambda(H_k) \) – i.e., the Lebesgue measure of the set \( H_k \) – is always \( 1/2 \), regardless of the value of \( k \). Since \( \varrho = \lambda \) is the uniform distribution, this immediately implies that the “\( \varrho^\otimes n \) versus \( \sigma_k^{\otimes n} \)” cases are all equivalent in the sense that none of the exponents depend on \( k \). Indeed, we have

\[
\psi(\varrho||\sigma_k|\alpha) = \log \int_{[0,1]} \left( \mathbf{1}_{[0,1]}(x) \right)^\alpha (2 \mathbf{1}_{H_k}(x))^{1-\alpha} d\lambda(x) = \log \left( 2^{1-\alpha} \int_{[0,1]} \mathbf{1}_{H_k}(x) d\lambda(x) \right) = -\alpha \log 2,
\]

and Lemma II.25 yields

\[
d_r(\varrho||\sigma_k) = c(\varrho||\sigma_k) = \log(2), \quad r > 0, \quad k \in \mathbb{N},
\]

while \( \text{supp } \varrho \not\subseteq \text{supp } \sigma_k \) implies

\[
s(\varrho||\sigma_k) = +\infty, \quad k \in \mathbb{N}.
\]

We remark that any of these exponents remain unchanged if we replace \( \varrho \) and \( \sigma_k \) by the \( 2 \times 2 \) diagonal density matrices in the following manner:

\[
\varrho = \mathbf{1}_{[0,1]} \iff \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \text{and} \quad \sigma_k = 2 \cdot \mathbf{1}_{H_k} \iff 2 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.
\]
The real challenge, of course, is not the computation of the error exponents of the “one-versus-one” cases, but rather, the estimation of the “one versus a convex combination” case. As a first step, observe that, by construction

$$
\lambda(H_k \cap H_j) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \lambda(H_k) \lambda(H_j)
$$

for all $k \neq j$. Thus, the subsets $H_1, H_2, \ldots$ (as events) are all independent with respect to the uniform distribution. It follows that

$$
\lambda \left( \bigcup_{k=1}^{m} H_k \right) = 1 - \lambda \left( \bigcap_{k=1}^{m} (H_k^c) \right) = 1 - \prod_{k=1}^{m} (1 - \lambda(H_k^c)) = 1 - \left( 1 - \frac{1}{2^n} \right)^m
$$

(III.66)

where, with a slight abuse of notation, we use $\lambda$ for the Lebesgue measure both on the interval $[0,1]$ as well as on the hypercube $[0,1]^n$, and $(H_k^c)$ denotes the complement $[0,1]^n \setminus H_k$.  

According to Lemma II.24, in the study of the error exponents, it is sufficient to consider projective tests only. Let $T_n$ be a projective test in $L^\infty([0,1])^\otimes n \equiv L^\infty([0,1]^n)$; that is, $T_n = 1_{K_n}$ where $K_n$ is a (Lebesgue) measurable subset of $[0,1]^n$. Then the type I error is

$$
\alpha_n(T_n) = \int_{[0,1]^n} \sigma^\otimes n (1 - 1_{K_n}) d\lambda = 1 - \lambda(K_n) = \lambda(K_n^c)
$$

(III.67)

is simply the volume of the complement of $K_n$. The type II error is

$$
\beta_n(T_n) = \int_{[0,1]^n} \sum_{k=1}^{\infty} q_k \sigma_k^\otimes n 1_{K_n} \ d\lambda = 2^n \sum_{k=1}^{\infty} q_k \lambda(H_k^c \cap K_n),
$$

(III.68)

which we shall lower bound by 1) replacing the summation over $k$ from 1 to $\infty$ by a summation over $k$ from 1 to a certain value $m$, 2) by replacing the individual coefficients $q_k$ ($k = 1, \ldots, m$) by $\min\{q_k|k=1,\ldots,m\} = q_m$, and 3) by replacing a sum of volumes by the volume of the union:

$$
\beta_n(T_n) \geq 2^n \sum_{k=1}^{m} q_k \lambda(H_k^c \cap K_n) \geq 2^n q_m \sum_{k=1}^{m} \lambda(H_k^c \cap K_n) \geq 2^n q_m \lambda \left( \bigcup_{k=1}^{m} (H_k^c \cap K_n) \right)
$$

(III.69)

Since $\bigcup_{k=1}^{m} (H_k^c \cap K_n) = (\bigcup_{k=1}^{m} H_k) \setminus K_n^c$, we can further bound the volume of the union in question as follows:

$$
\lambda \left( \bigcup_{k=1}^{m} (H_k^c \cap K_n) \right) = \lambda \left( \left( \bigcup_{k=1}^{m} H_k \right) \setminus K_n^c \right) \geq \lambda \left( \bigcup_{k=1}^{m} H_k \right) - \lambda(K_n^c).
$$

Substituting the above into (III.69) together with the actual value of the coefficient $q_m$, the volume of $\cup_{k=1}^{m} H_k$ given in III.66, and the equality $\lambda(K_n^c) = \alpha_n(T_n)$, we obtain that

$$
\beta_n(T_n) \geq 2^n \frac{6}{\pi^2} \frac{1}{m^2} \left( 1 - \left( 1 - \frac{1}{2^n} \right)^m - \alpha_n(T_n) \right)
$$

(III.70)

for any $m \in \mathbb{N}$. We may view this as a sequence of trade-off relations: for any given value of $m$, it says that a certain $m$-dependent weighted combination of the errors must be larger than some ($m$-dependent) value.

The sequence $k \mapsto (1 - 1/k)^k$ is monotone increasing and converges to $1/e$. Moreover, using elementary calculus one can easily check that $1 - e^{-x} \geq x/2$ on the interval $x \in [0,1]$. So, for $m \leq 2^n$, the term $1 - (1 - 1/2^n)^m$ in (III.70) may be bounded from below as follows:

$$
1 - \left( 1 - \frac{1}{2^n} \right)^m = 1 - \left( 1 - \frac{1}{2^n} \right)^{2^m} \geq 1 - e^{-m^{2^{-n}}} \geq \frac{1}{2} m^{2^{-n}}.
$$

Substituting this into (III.70), after some reordering we get that

$$
2^{(n+1)} \alpha_n(T_n) + \frac{\pi^2}{3} m^2 \beta_n(T_n) \geq m
$$

(III.71)
for any $m \leq 2^n$. The choice $m := 2^n$ yields

$$2\alpha_n(T_n) + \frac{\pi^2}{3} 2^n \beta_n(T_n) \geq 1. \quad \text{(III.72)}$$

In particular, if $\alpha_n(T_n) < \frac{1}{3}$, then $\beta_n(T_n) \geq \frac{1}{2\pi} 2^{-n}$, implying that $\log 2 \geq s(q\|q)$. Since $s(q\|q) \geq \sup\{r > 0 : d_r(q\|q) > 0\}$, we also get that for $r > \log 2$, $d_r(q\|q) = 0 = \max\{\log 2 - r, 0\}$.

Consider now a fixed $0 < r < \log 2$, and assume that $(T_n)_{n \in \mathbb{N}}$ is a sequence of tests such that

$$\liminf_{n \to +\infty} -\frac{1}{n} \log \beta_n(T_n) \geq r.$$ 

By (III.72), if $\beta_n(T_n) \leq \frac{1}{2\pi} 2^{-n}$ for infinitely many $n$ then $\alpha_n(T_n) \geq 5/12$ for the same indices, and hence

$$\liminf_{n \to +\infty} -\frac{1}{n} \log \alpha_n(T_n) = 0.$$ 

Hence, for the rest we assume that $\beta_n(T_n) > \frac{1}{2\pi} 2^{-n}$, and hence

$$m := \left\lceil \frac{1}{3\pi^2 \beta_n(T_n)} \right\rceil$$

is a positive integer smaller than $2^n$, for all sufficiently large $n$. Therefore, we can apply (III.71) to obtain

$$2^{(n+1)} \alpha_n(T_n) \beta_n(T_n) + \frac{\pi^2}{3} (m \beta_n(T_n))^2 \geq m \beta_n(T_n) \quad \implies \quad \alpha_n(T_n) \beta_n(T_n) \geq \pi^{-2} 2^{-(n+3)}.$$ 

Thus,

$$r + \limsup_{n \to +\infty} -\frac{1}{n} \log \alpha_n(T_n) \leq \limsup_{n \to +\infty} -\frac{1}{n} \log (\alpha_n(T_n) \beta_n(T_n)) \leq \log 2.$$ 

This proves that $d_r(q\|q) \leq \log 2 - r$. Monotonicity of $r \mapsto d_r(q\|q)$ yields that $d_r(q\|q) \leq \log 2 - r = 0$ for $r = \log 2$.

So far we have bounded the error exponents from above. To be able to deduce their precise value, we also need to bound them from below which we shall do by presenting an actual series of tests. Consider the projective tests $T_n := 1_{K_n}$, where $K_n$ is the complement of $\cup_{k \leq m_n} H_k^n$ in $[0,1]^n$ and $n \mapsto m_n \in \mathbb{N}$ is some function of $n$. Using that $K_n \cap H_k^n = \emptyset$ for $k \leq m_n$ and the volume formula (III.66), for our specific test, the type I and II errors given by (III.67) and (III.68) simplify to

$$\alpha_n(T_n) = 1 - \left(1 - \frac{1}{2^n}\right)^{m_n}, \quad \text{and} \quad \beta_n(T_n) = 2^n \sum_{k=m_n+1}^{\infty} q_k \lambda(K_n \cap H_k^n).$$

This time we bound the errors from above. Since $\lambda(K_n \cap H_k^n) \leq \lambda(H_k^n) = 2^{-n}$, the type II error can be estimated as follows:

$$\beta_n(T_n) \leq \sum_{k=m_n+1}^{\infty} q_k = \frac{6}{\pi^2} \sum_{k=m_n+1}^{\infty} \frac{1}{k^2} \leq \frac{6}{\pi^2} \int_{m_n}^{\infty} \frac{1}{x^2} dx = \frac{6}{\pi^2 m_n}.$$ 

We now specifically choose $m_n$ to be the upper integer part of $e^{nr}$ for some $r > 0$. Then, by the derived bound on $\beta_n(T_n)$, we find that

$$\liminf_{n \to +\infty} -\frac{1}{n} \log \beta_n(T_n) \geq r. \quad \text{(III.73)}$$

On the other hand, for the type I error we have

$$\alpha_n(T_n) = 1 - (1 - 2^{-n})^{[e^{nr}]} = 2^{-n} \sum_{k=0}^{[e^{nr}] - 1} (1 - 2^{-n})^k \leq [e^{nr}] 2^{-n},$$ 

and hence

$$\liminf_{n \to +\infty} -\frac{1}{n} \log \alpha_n(T_n) \geq \log 2 - r. \quad \text{(III.74)}$$

By (III.73) and (III.74), $d_r(q\|q) \geq \log 2 - r$ for all $r > 0$. Combining this with the previous upper bound and taking account of the nonnegativity of $d_r$, yields $d_r(q\|q) = \max\{\log 2 - r, 0\}$ for all $r > 0$. Using this expression and the derived bound on the Stein exponent, we can now also confirm the claimed values for the Stein and Chernoff exponents (see Remark II.21).
Proposition III.6 In the above example, we have \( e(\varrho|\text{supp}\varrho) = 0 \) for any of the exponents \( \epsilon = s, c, d_r \) (\( r > 0 \)).

Proof Assume again that \( T_n = 1_K_n \) for some measurable subset \( K_n \subseteq [0,1]^n \). As in the previous proof, we have that \( \alpha_n(T_n) = 1 - \lambda(K_n) = \lambda(K_n^c) \). What changes is the type two error; this time
\[
\beta_n(T_n) = \sup_k \int_{[0,1]^n} \sigma_k \mathbf{1}_{K_n} \, d\lambda = \sup_k 2^n \lambda(H_k^c \cap K_n).
\]

We shall lower bound this by 1) replacing the supremum by the average of the first \( 2^n \) terms and 2) replacing the sum of volumes by the volume of the union:
\[
\beta_n(T_n) \geq \frac{1}{2^n} \sum_{k=1}^{2^n} 2^n \lambda(H_k^c \cap K_n) \geq \lambda \left( \bigcup_{k=1}^{2^n} (H_k^c \cap K_n) \right) = \lambda \left( \left( \bigcup_{k=1}^{2^n} H_k \right) \setminus K_n^c \right).
\]
Thus for the sum of the two types of errors we have
\[
\alpha_n(T_n) + \beta_n(T_n) \geq \lambda(K_n^c) + \lambda \left( \left( \bigcup_{k=1}^{2^n} H_k \right) \setminus K_n^c \right) \geq \lambda \left( \bigcup_{k=1}^{2^n} H_k \right) = 1 - \left( 1 - \frac{1}{2^n} \right)^2.
\]
Since the expression on the right tends to \( 1 - 1/e \) as \( n \to \infty \), it follows that the two types of errors cannot both converge to zero, from where our claim follows immediately. \( \square \)

C. Finite dimension: Equality under convexity

Consider now the composite i.i.d. hypothesis testing problem with the null- and the alternative hypotheses represented by \( R, S \subseteq S(\mathcal{M}) \). For a fixed number of copies \( n \), the two hypotheses are represented by
\[
H_0 : \ R^{\otimes n} := \{ \otimes_{k=1}^n \varrho : \varrho \in R \} \quad H_1 : \ S^{\otimes n} := \{ \otimes_{k=1}^n \sigma : \sigma \in S \}.
\]
The *arbitrarily varying* version of the above problem is when the hypotheses are represented by
\[
H_0 : \ R^{\otimes n}_{av} := \{ \otimes_{k=1}^n \varrho_k : \varrho_k \in R, k \in [n] \} \quad H_1 : \ S^{\otimes n}_{av} := \{ \otimes_{k=1}^n \sigma_k : \sigma_k \in S, k \in [n] \}.
\]
That is, the problem is uniquely determined by the sets of states \( R, S \) on the single-copy algebra, and the state of the \( n \)-copy system is still a product state (independence), but the state of the \( k \)-th system can be an arbitrary member of the state set representing the given hypothesis. We denote the corresponding error probabilities and error exponents with a superscript \( \text{av} \).

In the finite-dimensional classical case, where the elements of \( R, S \) are probability density functions on some finite set \( \mathcal{X} \), one may also define the *adversarial* version [11], given by
\[
H_0 : \ R^{(n)}_{adv} := \{ \varrho \in S(\mathcal{X}^n) : \varrho_1 \in R, \forall k = 1, \ldots, n-1, \forall x \in \mathcal{X}^k : \varrho_{k+1}(\cdot|x_1 \ldots x_k) \in R \},
H_1 : \ S^{(n)}_{adv} := \{ \sigma \in S(\mathcal{X}^n) : \sigma_1 \in S, \forall k = 1, \ldots, n-1, \forall x \in \mathcal{X}^k : \sigma_{k+1}(\cdot|x_1 \ldots x_k) \in S \},
\]
where for a probability distribution \( \omega \) on \( \mathcal{X}^n \), \( \omega(x) := \omega(\{x\} \times \mathcal{X} \times \ldots \times \mathcal{X}) \), and for any \( k \in \{1, \ldots, n-1\} \) and \( x_1, \ldots, x_k \in \mathcal{X} \), \( \omega_{k+1}(\cdot|x_1 \ldots x_k) \) is the conditional distribution
\[
\omega_{k+1}(\cdot|x_1 \ldots x_k) := \frac{\omega(\{x_1\} \times \ldots \times \{x_k\} \times \{x\} \times \mathcal{X} \times \ldots \times \mathcal{X})}{\omega(\{x_1\} \times \ldots \times \{x_k\} \times \mathcal{X} \times \ldots \times \mathcal{X})}.
\]

For a subset \( A \subseteq S(\mathcal{X}) \) of states, let \( \text{co}(A) \) denote the convex hull of \( A \), and \( \overline{\text{co}}(A) \) the closure of the convex hull.

Theorem III.7 Let \( \mathcal{X} \) be a finite set and \( R, S \subseteq S(\mathcal{X}) \). Let \( \epsilon \) be any of the Stein-, Chernoff-, or the direct exponents, and \( E \) be the corresponding divergence as in Lemma II.25. Then
\[
E(\overline{\text{co}}(R)||\overline{\text{co}}(S)) = \inf_{\varrho_{\overline{\text{co}}(R)},\sigma_{\overline{\text{co}}(S)}} e(\varrho||\sigma) \tag{III.75}
\]
\[
\leq e^{\text{adv}}(\overline{\text{co}}(R)||\overline{\text{co}}(S)) \leq e^{\text{adv}}(R||S) \tag{III.76}
\]
\[
\leq e^{\text{av}}(R||S) = e^{\text{av}}(\overline{\text{co}}(R)||\overline{\text{co}}(S)) \tag{III.77}
\]
\[
\leq e(\overline{\text{co}}(R)||\overline{\text{co}}(S)) \leq \left\{ \begin{array}{l} e^{(0)}(\overline{\text{co}}(R)||\overline{\text{co}}(S)) \\ e^{(1)}(\overline{\text{co}}(R)||\overline{\text{co}}(S)) \end{array} \right\} \leq e^{(01)}(\overline{\text{co}}(R)||\overline{\text{co}}(S)) \tag{III.78}
\]
\[
\leq \inf_{\varrho_{\overline{\text{co}}(R)},\sigma_{\overline{\text{co}}(S)}} e(\varrho||\sigma), \tag{III.79}
\]
and hence all the inequalities above hold as equalities.
Proof All the inequalities except for the first one in (III.76) are obvious from (II.39)–(II.42), and the equality in (III.75) is due to Lemma II.25. The equality in (III.77) is easily seen to follow as for any $R_n, S_n \subseteq S(\mathcal{X}^n)$, and any test $T_n$, 
\[ \alpha_n(\mathcal{C}(R_n)|T_n) = \alpha_n(R_n|T_n), \quad \beta_n(\mathcal{C}(S_n)|T_n) = \beta_n(S_n|T_n), \]
and \((\mathcal{C}(R))_{av} \subseteq \mathcal{C}(R_{av}^\otimes n) \subseteq \mathcal{C}((\mathcal{C}(R))_{av}^\otimes n)\) implies, by taking the closed convex hull of each term, that \(\mathcal{C}((\mathcal{C}(R))_{av}^\otimes n) = \mathcal{C}(R_{av}^\otimes n)\).

Hence, the only thing left to be proved is the first inequality in (III.76), which we establish separately below. \(\square\)

We note that the Stein and the direct exponents were determined for the arbitrarily varying setting in [19] and [20], respectively, for finite $R$ and $S$. Theorem III.7 gives an extension of these results, on the one hand by treating $R$ and $S$ of arbitrary cardinality, and on the other hand by also determining the Chernoff exponent. We note that our proof is different from the proofs in [19] and [20].

Regarding the first inequality in (III.76), we note that the cases of the Stein and the Chernoff exponent were already treated in [11], and our proof is a straightforward adaptation of the proof for the Chernoff exponent in [11]. On the other hand, we not only extend the results of [11] to the case of the whole range of direct exponents, but the proof presented below is also a simplification of that in [11], where the statements for the Stein and the Chernoff exponents were proved separately, whereas we give a proof for the direct exponent $d_r$ for every $r > 0$, and obtain the cases of the Stein and the Chernoff exponents as immediate consequences. Moreover, we also utilize this method to show equality for the strong converse exponent, albeit only in the case of the null hypothesis i.i.d.; see Section V C 2.

The main idea of the proof is simple: For any $0 < r < D(R||S)$, one takes the pair $\vartheta_r \in R$, $\sigma_r \in S$, that minimizes the distance between $R$ and $S$ in the $H_r$ divergence, and uses the optimal Neyman-Pearson test for discriminating the i.i.d. extensions of these two states; the challenge is to show that this test is in fact universally good, and that it is so even in the adversarial setting.

**Theorem III.8** The first inequality in (III.76) holds.

**Proof** Our aim is to prove the first inequality in (III.76), and hence we may assume without loss of generality that $R$ and $S$ are closed and convex. Hence, our aim is to prove
\[ \inf_{\vartheta \in R, \sigma \in S} e(\vartheta||\sigma) \leq e_{adv}(R||S), \] (III.80)
where $e$ may be the Stein-, the Chernoff-, or any of the direct exponents. Moreover, it is sufficient to prove (III.80) for the direct exponents, from which the inequalities for the Chernoff and the Stein exponents follow immediately by Lemma II.27 and Lemma II.29, respectively.

Let $r > 0$ be a fixed type II rate, and $e = d_r$ be the corresponding direct exponent, so that, by Lemma II.25,
\[ \inf_{\vartheta \in R, \sigma \in S} e(\vartheta||\sigma) = H_r(R||S). \]
We assume that $r$ is such that $H_r(R||S) > 0$, since otherwise the statement is trivial. By Lemma II.9, this is equivalent to assuming that $r < D(R||S)$. In particular, $D(R||S) > 0$, which is equivalent to $R \cap S = \emptyset$.

To avoid potential difficulties arising from differing supports of the probability distributions, we introduce the “smoothed” sets
\[ R_\vartheta := \{(1 - \vartheta)\vartheta + \vartheta \frac{1}{|\mathcal{X}|} : \vartheta \in R\}, \quad S_\vartheta := \{(1 - \vartheta)\sigma + \vartheta \frac{1}{|\mathcal{X}|} : \sigma \in R\}, \] (III.81)
for every $\vartheta \in [0,1)$. Note that, by definition, for any $\vartheta_{[n]} \in R_{adv}\otimes n$ there exists a $\tilde{\vartheta}_{[n]} \in (R_{\vartheta})_{adv}^{(n)}$ such that
\[ \tilde{\vartheta}_1(x) = (1 - \vartheta)\vartheta_1(x) + \vartheta \frac{1}{|\mathcal{X}|}, \quad x \in \mathcal{X}, \]
\[ \tilde{\vartheta}_{k+1}(\cdot |x_1, \ldots, x_k) = (1 - \vartheta)\vartheta_{k+1}(\cdot |x_1, \ldots, x_k) + \vartheta \frac{1}{|\mathcal{X}|}, \quad k = 2, \ldots, n - 1, x_1, \ldots, x_k \in \mathcal{X}. \]

In particular,
\[ \tilde{\vartheta}_{[n]}(x_1, \ldots, x_n) \geq (1 - \vartheta)^n \vartheta_{[n]}(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in \mathcal{X}, \] (III.82)
and we have analogous inequalities for $S$ in place of $R$. This implies that for any test $T_n$,
\[ \alpha_n(T_n|R^{(n)}_{\text{adv}}) \leq (1 - \theta)^{-n} \alpha_n(T_n|(R_0)^{(n)}_{\text{adv}}), \quad \beta_n(T_n|S^{(n)}_{\text{adv}}) \leq (1 - \theta)^{-n} \beta_n(T_n|(S_0)^{(n)}_{\text{adv}}). \]

Clearly, $R_0$ and $S_0$ are again closed convex sets, and $R_0 \cap S_0 = \emptyset$. Since the relative entropy is jointly lower semi-continuous, for every $\theta$ there exist $\tilde{\theta}_0 \in R_0$, $\tilde{\sigma}_0 \in S_0$ such that $D(\tilde{\theta}_0||\tilde{\sigma}_0) = D(R_0||S_0)$. Using compactness of $R$ and $S$, we may choose a sequence $\tilde{\theta}_k \to 0$ such that $\tilde{\theta}_k \to 0 \in R$, $\tilde{\sigma}_k \to \sigma_0 \in S$ as $k \to +\infty$. Thus,
\[ r < D(R||S) \leq D(\tilde{\theta}_0||\tilde{\sigma}_0) \leq \lim \inf_{k \to +\infty} D(\tilde{\theta}_k||\tilde{\sigma}_k) = \lim \inf_{k \to +\infty} D(R_{\theta_k}||S_{\theta_k}), \tag{III.83} \]

where the first inequality is by assumption, the second one is by definition, and the third one is due to the lower semi-continuity of the relative entropy. Hence, $r < D(R_{\theta_k}||S_{\theta_k})$ for all large enough $k$, and for the rest we assume that this is satisfied. Since $D_0(\varrho||\sigma) = 0$ for all $\varrho \in R_0$, $\sigma \in S_0$, Lemma II.9 implies that
\[ 0 < H_r(\varrho||\sigma) < +\infty, \quad \varrho \in R_{\theta_k}, \sigma \in S_{\theta_k}. \]

Let us fix a large enough $k$ as above. By Lemma II.2, $H_r$ is lower semi-continuous on the compact set $R_{\theta_k} \times S_{\theta_k}$, and therefore there exists a pair of states $(\tilde{\varrho}_r, \tilde{\sigma}_r) \in R_{\theta_k} \times S_{\theta_k}$ such that $H_r(R_{\theta_k}||S_{\theta_k}) = H_r(\tilde{\varrho}_r||\tilde{\sigma}_r) := H_r$. Let $\psi(\alpha) := \psi(\tilde{\varrho}_r||\tilde{\sigma}_r||\alpha)$, $\tilde{\psi}(u) := \tilde{\psi}(\tilde{\varrho}_r||\tilde{\sigma}_r||u) = (1 - u)\tilde{\psi}(1 - u)^{-1}$, and let $\alpha_r$ and $u_r$ be as in Lemma II.12. For $\varrho \in R_{\theta_k}$, $\sigma \in S_{\theta_k}$, there exists a $\delta > 0$ such that $\varrho(t) := (1 - t)\varrho_r + t\varrho$ and $\sigma(t) := (1 - t)\sigma_r + t\sigma$ are probability density functions for every $t \in (-\delta, 1)$. Hence, for all such $t$, we may define
\[
 f(t, u) := ur - \tilde{\psi}(\varrho(t)||\sigma_r) = ur - (1 - u)\log \sum_x \varrho(t)(x) \frac{1}{\varrho_r(x)} \sigma_r(x)^{1 - \frac{1}{\varrho_r(x)}}, \\
g(t, u) := ur - \tilde{\psi}(\varrho_r||\sigma(t)) = ur - (1 - u)\log \sum_x \varrho_r(x) \frac{1}{\varrho_r(x)} \sigma(t)^{1 - \frac{1}{\varrho_r(x)}}.
\]

Then
\[ H_r(\varrho(t)||\sigma_r) = \max_{u \in (-\infty, 0)} f(t, u) = f(t, u_1(t)), \quad H_r(\varrho_r||\sigma(t)) = \max_{u \in (-\infty, 0)} g(t, u) = g(t, u_2(t)), \]

where $u_1(t)$ and $u_2(t)$ are the unique maximizers. Since $u_1(t)$ is the unique solution of $\partial_2 f(t, u) = 0$, and $\partial_2^2 f(t, u) > 0$ at every $u \in (-\infty, 0)$, according to Lemma II.12, the function $t \mapsto u_1(t)$ is differentiable due to the implicit function theorem, and similarly for $t \mapsto u_2(t)$. Thus,
\[
 \frac{d}{dt} H_r(\varrho(t)||\sigma_r) = \partial_1 f(t, u_1(t)) + \partial_2 f(t, u_1(t)) \frac{d}{dt} u_1(t) = \partial_1 f(t, u_1(t)) \\
= \sum_x \varrho(t)(x) \frac{1 - \varrho_r(x)}{\varrho_r(x)} \sigma_r(x)^{1 - \frac{1}{\varrho_r(x)}} (\varrho_r(x) - \varrho(t)) \\
= \sum_x \varrho(t)(x) \frac{1 - \varrho_r(x)}{\varrho_r(x)} \sigma_r(x)^{1 - \frac{1}{\varrho_r(x)}} (\sigma(t) - \sigma_r(x)).
\]

By the minimizing property of $\varrho_r$, we have
\[ 0 \leq \frac{d}{dt} H_r(\varrho(t)||\sigma_r) \bigg|_{t=0} \implies \sum_x \varrho(x) \left( \frac{\sigma_r(x)}{\varrho_r(x)} \right)^{1 - \alpha_r} \leq \sum_x \varrho(x) \left( \frac{\sigma_r(x)}{\varrho_r(x)} \right)^{1 - \alpha_r}, \tag{III.84} \]

where we used that $u_1(0) = u_2(0) = u_r = (\alpha_r - 1)/\alpha_r$. A similar computation yields
\[
 \frac{d}{dt} H_r(\varrho_r||\sigma(t)) = \partial_1 g(t, u_2(t)) + \partial_2 g(t, u_2(t)) \frac{d}{dt} u_2(t) = \partial_1 g(t, u_2(t)) \\
= \frac{u_2(t) \sum_x \varrho_r(x) \frac{1 - u_2(t)}{\varrho_r(x)} \sigma(t)^{1 - \frac{1}{\varrho_r(x)}} (\sigma(t) - \sigma_r(x))}{\sum_x \varrho_r(x) \frac{1 - u_2(t)}{\varrho_r(x)} \sigma(t)^{1 - \frac{1}{\varrho_r(x)}}}, \tag{III.85}
\]

and
\[
 0 \leq \frac{d}{dt} H_r(\varrho_r||\sigma(t)) \bigg|_{t=0} \implies \sum_x \sigma(x) \left( \frac{\varrho_r(x)}{\sigma(x)} \right)^{\alpha_r} \leq \sum_x \sigma(x) \left( \frac{\varrho_r(x)}{\sigma(x)} \right)^{\alpha_r}. \tag{III.86}
\]
Define $c_r := \psi'(\alpha_r)$, and let

$$T_{n,r} := \left\{ x \in \mathcal{X}^n : \frac{\varphi_r^{(n)}(x)}{\varphi_r^{(n)}(e)} \geq e^{nc_r} \right\}, \text{ so that } \mathcal{X}^n \setminus T_{n,r} = \left\{ x \in \mathcal{X}^n : \frac{\varphi_r^{(n)}(x)}{\varphi_r^{(n)}(e)} > e^{-nc_r} \right\}. \tag{III.87}$$

As is usual in the classical case, we identify the subset $T_{n,r} \subseteq \mathcal{X}^n$ with a projective test (in the algebraic formalism, this projective test is the multiplication operator corresponding to the characteristic function of $T_{n,r}$). Then

$$\beta_n(T_{n,r}) = \sup_{\sigma_n \in S_{adv}^{(n)}} \sum_{x \in T_{n,r}} \sigma_n(x) \geq (1 - \vartheta_k)^{-n} \rho \sum_{x \in T_{n,r}} \sigma_n(x) \geq (1 - \vartheta_k)^{-n} \rho \sum_{x \in \mathcal{X}^n} \sigma_n(x) \alpha_r \leq (1 - \vartheta_k)^{-n} e^{-nc_r\alpha_r} \times$$

$$\sum_{x \in \mathcal{X}^n} \sigma_n(x) \left( \frac{\varphi_r(x)}{\varphi_r(e)} \right)^{\alpha_r} \leq \sum_{x \in \mathcal{X}} \sigma_n(x) \left( \frac{\varphi_r(x)}{\varphi_r(e)} \right)^{\alpha_r}, \tag{III.88}$$

where the first inequality follows from (III.82), the second inequality is due to the Markov inequality, and the third one is due to (III.86). Iterating the above, we get

$$\beta_n(T_{n,r}) \leq (1 - \vartheta_k)^{-n} e^{-nc_r\alpha_r} \left( \sum_{x \in \mathcal{X}} \sigma_n(x) \left( \frac{\varphi_r(x)}{\varphi_r(e)} \right)^{\alpha_r} \right)^n = (1 - \vartheta_k)^{-n} e^{-n(c_r, \alpha_r, \psi(\alpha_r))} \leq e^{-n(r + \log(1 - \vartheta_k))}, \tag{III.89}$$

where the last equality is by (II.27). An exactly analogous argument yields

$$\alpha_n(T_{n,r}) \leq (1 - \vartheta_k)^{-n} e^{-c_r(1 - \alpha_r)} \left( \sum_{x \in \mathcal{X}} \varphi_r(x) \left( \frac{\varphi_r(x)}{\varphi_r(e)} \right)^{1 - \alpha_r} \right)^n = (1 - \vartheta_k)^{-n} e^{n(c_r(1 - \alpha_r) + \psi(\alpha_r))}$$

where the last equality is by (II.28)–(II.29). Thus, the pair of exponents

$$(r + \log(1 - \vartheta_k), H_r(R_{\theta_k} || S_{\theta_k} + \log(1 - \vartheta_k))) \text{ is achievable} \tag{III.90}$$

for any large enough $k$.

Note that the optimal states $\varrho_r, \sigma_r$ above depend on $\vartheta_k$, and for the rest we indicate this by denoting them as $\varrho_{\theta_k, r}, \sigma_{\theta_k, r}$. Again by the compactness of $R$ and $S$, and by passing to a subsequence if necessary, we may assume that $\varrho_{\theta_k, r} \to \varrho_{0, r} \in R, \sigma_{\theta_k, r} \to \sigma_{0, r} \in S$ as $k \to +\infty$. Thus,

$$H_r(R || S) \leq H_r(\varrho_{0, r} || \sigma_{0, r}) \leq \lim_{k \to +\infty} \inf H_r(\varrho_{\theta_k, r} || \sigma_{\theta_k, r}),$$

by the same argument as in (III.83), using the lower semi-continuity of the Hoeffding divergence (Lemma II.2). By (III.90) and Lemmas II.18 and II.17, the pair of exponents

$$r = \lim_{k \to +\infty} (r + \log(1 - \vartheta_k)) \text{ and } H_r(R || S) \leq \lim_{k \to +\infty} \inf (H_r(R_{\theta_k} || S_{\theta_k} + \log(1 - \vartheta_k)))$$

is achievable, and hence

$$d_r^{adv}(R || S) \geq H_r(R || S) = \inf_{\varrho \in R, \sigma \in S} d_r(\varrho || \sigma), \tag{III.91}$$

proving (III.80).
IV. QUANTUM STATE DISCRIMINATION

A. Finite dimension with two alternative hypotheses: Equality may not hold

In this section, for any of the error exponents \( e = s, c, d_r \), we shall construct three density operators \( \varrho, \sigma_1, \sigma_2 \) on a finite-dimensional Hilbert space satisfying the strict inequality
\[
\epsilon(\varrho \parallel (\sigma_1, \sigma_2)) < \min\{ \epsilon(\varrho \parallel \sigma_1), \epsilon(\varrho \parallel \sigma_2) \}.
\]
As it turns out, in some sense, such triplets are fairly common. So the real problem is not finding (or constructing) such triplets of density operators, but proving that we have a strict inequality between the worst-case pairwise error exponent and the composite one. Of course, the pairwise error exponents are easy to compute due to the explicit formulas given in Lemma II.25; the hard part is estimating the error exponents from above in the composite case, which amounts to estimating the two types of errors from below.

Throughout this section we shall work on finite-dimensional Hilbert spaces (so all operators appearing in our constructions, even if not explicitly mentioned, are acting on some finite-dimensional space). Our idea is to employ the well-known operator form of the inequality between the arithmetic and the geometric means to bound the type II error. Recall that the geometric mean of two positive definite operators is defined as
\[
\sigma_{\text{geometric}} := \sqrt{\sigma_1 \sigma_2}
\]
(corresponding to \( \alpha = 1/2 \) in (II.32)), and it can be extended to pairs of positive semi-definite operators by taking decreasing limits; see, e.g., [35]. The geometric-arithmetic mean inequality says that
\[
\epsilon(\varrho, \sigma_1 \# \sigma_2) \leq \epsilon(\varrho, \sigma_{\text{geometric}}),
\]
for any density operators \( \sigma_1, \sigma_2 \), and test \( T_n \). The point of replacing the arithmetic mean with the geometric mean is to replace the composite i.i.d. problem with a simple i.i.d. problem. This is achieved because the geometric mean is easily seen to have the following multiplicativity property:
\[
(A_1 \# B_1) \otimes (A_2 \# B_2) = (A_1 \otimes A_2) \# (B_1 \otimes B_2),
\]
implying that the term \( \sigma_1 \# \sigma_2 \otimes T_n \) appearing in our estimation can be rewritten as \( (\sigma_1 \# \sigma_2) \otimes T_n \).

We remark that for states \( \sigma_1, \sigma_2 \in S(\mathcal{H}) \), their geometric mean \( \sigma_1 \# \sigma_2 \) is in general a subnormalized state. Indeed, we have
\[
\text{Tr}(\sigma_1 \# \sigma_2) \leq \left( \frac{\sigma_1^{1/2} \sigma_2^{1/2}}{2} \right)^{1/2} = F(\sigma_1, \sigma_2) \leq 1,
\]
where \( F \) is the fidelity [46, 58], the first inequality was shown in [37], and the second one is well-known [46, Chapter 9]. This also implies that
\[
\text{Tr}(\sigma_1 \# \sigma_2) = 1 \iff F(\sigma_1, \sigma_2) = 1 \iff \sigma_1 = \sigma_2,
\]
where the second equivalence is again well-known [46, Chapter 9]. Moreover, for any state \( \varrho \in S(\mathcal{H}) \),
\[
D(\varrho \parallel \sigma_1 \# \sigma_2) = D(\varrho \parallel \sigma_1 \# \sigma_2) - \log \text{Tr} \sigma_1 \# \sigma_2 \geq -\log \text{Tr} \sigma_1 \# \sigma_2 \geq 0,
\]
and \( D(\varrho \parallel \sigma_1 \# \sigma_2) = 0 \) if and only if \( \varrho = \sigma_1 = \sigma_2 \).

**Lemma IV.1** For any density operators \( \sigma_1, \sigma_2 \in S(\mathcal{H}) \), and any set of density operators \( R \subseteq S(\mathcal{H}) \), we have
\[
s(R \parallel \{\sigma_1, \sigma_2\}) \leq D(R \parallel \sigma_1 \# \sigma_2) \quad \text{and} \quad d_r(R \parallel \{\sigma_1, \sigma_2\}) \leq H_r(R \parallel \sigma_1 \# \sigma_2), \quad \forall r > 0.
\]
Proof If $\sigma_1 \# \sigma_2 = 0$ then $D(\varrho\|\sigma_1 \# \sigma_2) = H_\varrho(\varrho|\sigma_1 \# \sigma_2) = +\infty$, and there is nothing to prove. Hence, we may assume that $\sigma_1 \# \sigma_2 \neq 0$ in which case $\lambda := \text{Tr}(\sigma_1 \# \sigma_2) > 0$ and $\tilde{\sigma} := (\sigma_1 \# \sigma_2)/\lambda$ is a density operator. By (IV.93),
\[
\beta_n(\{\sigma_1, \sigma_2\}|T_n) \geq \text{Tr}((\sigma_1 \# \sigma_2)\otimes^n T_n) = \lambda^n \beta_n(\tilde{\sigma}|T_n),
\]
and hence
\[
\liminf_{n \to +\infty} -\frac{1}{n} \log \beta_n(\{\sigma_1, \sigma_2\}|T_n) \leq -\log \lambda + \liminf_{n \to +\infty} -\frac{1}{n} \log \beta_n(\tilde{\sigma}|T_n). \tag{IV.94}
\]
Thus, by the definition of the error exponents in question, we get
\[
s(R||\{\sigma_1, \sigma_2\}) \leq -\log \lambda + s(R||\tilde{\sigma}) \leq -\log \lambda + \inf_{\varrho \in R} s(\varrho||\tilde{\sigma}) = -\log \lambda + D(R||\tilde{\sigma}) = D(R||\sigma_1 \# \sigma_2),
\]
\[
d_r(R||\{\sigma_1, \sigma_2\}) \leq d_r + \log \lambda(R||\tilde{\sigma}) \leq \inf_{\varrho \in R} d_r + \log \lambda(R||\tilde{\sigma}) = H_r + \log \lambda(R||\tilde{\sigma}) = H_r(R||\sigma_1 \# \sigma_2),
\]
where the first inequalities follow from (IV.94), the second inequalities from (II.42), the first equalities from Lemma II.25, and the last equalities from the scaling laws (II.9).

By the above lemma, if for some density operators $\varrho, \sigma_1, \sigma_2$, $D(\varrho||\sigma_1 \# \sigma_2) < \min\{D(\varrho||\sigma_1), D(\varrho||\sigma_2)\}$, then $s(\varrho||\{\sigma_1, \sigma_2\}) \leq D(\varrho||\sigma_1 \# \sigma_2) < \min\{D(\varrho||\sigma_1), D(\varrho||\sigma_2)\} = \min\{s(\varrho||\sigma_1), s(\varrho||\sigma_2)\}$, i.e. we can achieve a strict inequality for the Stein exponents. In Theorem IV.3, we will give a systematic way to construct such triplets, corresponding to any pair of non-commuting density operators. Before that, however, we will need to establish a fact regarding the relation between the geometric and the exponential mean.

For two positive definite operators $A, B$ let
\[
\Lambda(A, B) = \log(A\#B) - \frac{\log(A) + \log(B)}{2}.
\]

By the above lemma, if for some density operators $\varrho, \sigma_1, \sigma_2$, $D(\varrho||\sigma_1 \# \sigma_2) < \min\{D(\varrho||\sigma_1), D(\varrho||\sigma_2)\}$, then $s(\varrho||\{\sigma_1, \sigma_2\}) \leq D(\varrho||\sigma_1 \# \sigma_2) < \min\{D(\varrho||\sigma_1), D(\varrho||\sigma_2)\} = \min\{s(\varrho||\sigma_1), s(\varrho||\sigma_2)\}$, i.e. we can achieve a strict inequality for the Stein exponents. In Theorem IV.3, we will give a systematic way to construct such triplets, corresponding to any pair of non-commuting density operators. Before that, however, we will need to establish a fact regarding the relation between the geometric and the exponential mean.

Lemma IV.2 Let $A, B$ be two positive definite operators. Then $\text{Tr} \Lambda(A, B) = 0$. In particular, if $AB \neq BA$ then there exists a density operator $\varrho$ such that $\text{Tr}(\varrho \Lambda(A, B)) = \delta_{A,B} := \lambda_{\text{max}}(A, B) > 0$, and there exists an invertible density operator $\varrho$ such that $\text{Tr}(\varrho \Lambda(A, B)) = \delta_{A,B}/2$.

Proof For a positive definite operator $X$ one has $\text{Tr}(\log(X)) = \log(\det(X))$. Thus, $\text{Tr}(\Lambda(A, B) = 0$ can be justified by a straightforward reordering relying on (IV.92) and the multiplicative property of the determinant. By the above, $AB \neq BA$ implies $\Lambda(A, B) \neq 0$, which, together with $\text{Tr}(\Lambda(A, B) = 0$, yields that $\lambda_{\text{max}}(\Lambda(A, B)) > 0$. Choosing $\varrho = |\psi\rangle\langle\psi|$ with an eigen-vector of $\Lambda(A, B)$ corresponding to $\lambda_{\text{max}}(\Lambda(A, B))$ yields $\text{Tr}(\varrho \Lambda(A, B)) = \lambda_{\text{max}}(\Lambda(A, B))$. The existence of an invertible $\varrho$ with the given property then follows from the fact that $\{\varrho \in S(\mathcal{H}) : \text{Tr}(\varrho \Lambda(A, B)) > 0\}$ is an open subset of $S(\mathcal{H})$.

After the above preparation, it is easy to construct triplets of density operators with strict inequality for the Stein exponent.

Theorem IV.3 For any density operators $\varrho, \sigma_1, \sigma_2 \in S(\mathcal{H})$ such that $\sigma_1, \sigma_2$ are invertible,
\[
\min_{i=1,2} s(\hat{\varrho}||\hat{\sigma}_i) \leq s(\hat{\varrho}||\{\hat{\sigma}_1, \hat{\sigma}_2\}) = s(\hat{\varrho}||\tilde{\sigma}_j) \leq D(\hat{\varrho}||\tilde{\sigma}_j) = 2 \text{Tr} \varrho \Lambda(\sigma_1, \sigma_2), \quad j = 1, 2, \tag{IV.95}
\]
where
\[
\hat{\varrho} := \frac{1}{2} \left( \begin{array}{cc} 0 & \varrho \\ \varrho & 0 \end{array} \right), \quad \hat{\sigma}_1 := \frac{1}{2} \left( \begin{array}{cc} \sigma_1 & 0 \\ 0 & \sigma_2 \end{array} \right), \quad \hat{\sigma}_2 := \frac{1}{2} \left( \begin{array}{cc} \sigma_2 & 0 \\ 0 & \sigma_1 \end{array} \right).
\]

In particular, if $\sigma_1, \sigma_2$ are non-commuting then there exists an invertible density operator $\varrho$ such that
\[
\min_{i=1,2} s(\hat{\varrho}||\hat{\sigma}_i) \leq s(\hat{\varrho}||\{\hat{\sigma}_1, \hat{\sigma}_2\}) \geq D(\hat{\varrho}||\tilde{\sigma}_j) = 2 \text{Tr} \varrho \Lambda(\sigma_1, \sigma_2) = \delta_{\sigma_1, \sigma_2} > 0.
\]
Proof The inequality in (IV.95) is immediate from Lemma II.25 and Lemma IV.1, and the equality in (IV.95) follows by a straightforward computation using that $\text{Tr} \hat{\varrho} \log \hat{\sigma}_1 = \text{Tr} \hat{\varrho} \log \hat{\sigma}_2$. The last assertion follows immediately from Lemma IV.2.

Remark IV.4 Note that (IV.95) gives an explicitly computable lower bound for the gap between the composite Stein exponent and the worst pairwise Stein exponent.

We will use triplets of density operators as in Theorem IV.3 to construct further triplets exhibiting strict inequality also for the Chernoff and the direct exponents. To this end, let us introduce, for any density operators $\varrho, \sigma_1, \sigma_2 \in S(\mathcal{H})$, and any $\lambda, \eta, \mu, \nu \in [0, 1]$, the density operators

$$\varrho_{\lambda, \eta} := \frac{\eta \lambda}{2} \varrho + \frac{\eta (1 - \lambda)}{2} \lambda \varrho + \frac{(1 - \eta)}{2} \varrho,$$

$$\sigma_{j, \mu, \nu} := \frac{\nu \mu}{2} \sigma_{j} + \frac{\nu (1 - \mu)}{2} \sigma_{j - 1} \oplus \nu (1 - \mu) \oplus \nu = (\nu \mu) \sigma_{j} + \nu (1 - \mu) \oplus \nu, \quad j = 1, 2,$$

on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{C} \oplus \mathcal{C}$. Note that $\varrho_{1, 1} = \hat{\varrho} \oplus 0 \oplus 0$, $\sigma_{j, 1, 1} = \hat{\sigma}_j \oplus 0 \oplus 0$ with the notations of Theorem IV.3. Clearly,

$$\sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu} = (\sigma_1 \# \sigma_2)_{\mu, \nu} = \frac{\nu \mu}{2} \sigma_1 \# \sigma_2 \oplus \frac{\nu (1 - \mu)}{2} \sigma_1 \# \sigma_2 \oplus \nu (1 - \mu) \oplus \nu = (\nu \mu) \sigma_1 \# \sigma_2 \oplus \nu (1 - \mu) \oplus \nu,$$

(IV.96)

and a straightforward computation shows that

$$D(\varrho_{\lambda, \eta} || \sigma_{j, \mu, \nu}) = \lambda \eta D(\hat{\varrho} || \hat{\sigma}_j) + \eta d_2(\lambda || \mu) + d_2(\eta || \nu),$$

(IV.97)

$$D(\varrho_{\lambda, \eta} || \sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu}) = \lambda \eta D(\hat{\varrho} || \hat{\sigma}_1 \# \hat{\sigma}_2) + \eta d_2(\lambda || \mu) + d_2(\eta || \nu),$$

(IV.98)

where

$$d_2(\alpha, \beta) := D((\alpha, 1 - \alpha) || (\beta, 1 - \beta)), \quad \alpha, \beta \in [0, 1].$$

We start with the following refined version of Theorem IV.3.

Proposition IV.5 Let $\varrho, \sigma_1, \sigma_2 \in S(\mathcal{H})$ be as in Theorem IV.3. For every $r > 0$ and every $0 < \lambda < \min\{1, r/D(\hat{\varrho} || \hat{\sigma}_1 \# \hat{\sigma}_2)\}$, there exists a $\mu \in (0, 1)$ such that

$$s(\varrho_{\lambda, 1} || \{\sigma_{1, \mu, 1}, \sigma_{2, \mu, 1}\}) \leq r = D(\varrho_{\lambda, 1} || \sigma_{1, \mu, 1} \# \sigma_{2, \mu, 1})$$

$$= D(\varrho_{\lambda, 1} || \sigma_{j, \mu, 1}) - \lambda \delta_{\sigma_1, \sigma_2}, \quad j = 1, 2.$$

Proof By Theorem IV.3,

$$r_0 := D(\hat{\varrho} || \hat{\sigma}_1 \# \hat{\sigma}_2) < r_0 + \delta_{\sigma_1, \sigma_2} = D(\hat{\varrho} || \hat{\sigma}_j), \quad j = 1, 2.$$

Let $\lambda$ be as in the statement, so that $\lambda \in (0, 1)$ and $r - \lambda r_0 > 0$, and let $\mu \in (0, 1)$ be such that $r - \lambda r_0 = d_2(\lambda || \mu)$. Such a $\mu$ exists, since $\mu \mapsto d_2(\lambda || \mu)$ is convex, it is finite-valued on $(0, 1)$, $d_2(\lambda || \lambda) = 0$, and $\lim_{\mu \to 0} d_2(\lambda || \mu) = \lim_{\mu \to 1} d_2(\lambda || \mu) = +\infty$. By (IV.97)–(IV.98),

$$D(\varrho_{\lambda, 1} || \sigma_{j, \mu, 1}) = \lambda D(\hat{\varrho} || \hat{\sigma}_j) + d_2(\lambda || \mu) = r + \lambda \delta_{\sigma_1, \sigma_2}, \quad j = 1, 2;$$

(IV.99)

$$D(\varrho_{\lambda, 1} || \sigma_{1, \mu, 1} \# \sigma_{2, \mu, 1}) = \lambda D(\hat{\varrho} || \hat{\sigma}_1 \# \hat{\sigma}_2) + d_2(\lambda || \mu) = r.$$

(IV.100)

Thus,

$$s(\varrho_{\lambda, 1} || \{\sigma_{1, \mu, 1}, \sigma_{2, \mu, 1}\}) \leq D(\varrho_{\lambda, 1} || \sigma_{1, \mu, 1} \# \sigma_{2, \mu, 1}) = r = D(\varrho_{\lambda, 1} || \sigma_{j, \mu, 1}) - \lambda \delta_{\sigma_1, \sigma_2} = s(\varrho_{\lambda, 1} || \sigma_{j, \mu, 1}) - \lambda \delta_{\sigma_1, \sigma_2}, \quad j = 1, 2,$$

where the first inequality is due to Lemma IV.1, and the last equality is by Lemma II.25. This proves the assertion. }

Remark IV.6 Note that in Proposition IV.5,

$$\text{supp} \varrho_{\lambda, 1} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{C} \oplus 0 = \text{supp} \sigma_{j, \mu, 1}.$$
Using the freedom in the choice of the parameters $\lambda, \eta, \mu, \nu$, we can also get examples with strict inequality for the direct exponents. Our construction below also provides examples with strict inequality for the direct exponents in the setting where the null-hypothesis is composite (consisting of two hypotheses), and the alternative hypothesis is simple.

**Theorem IV.7** Let $\varrho, \sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{H})$ be as in Theorem IV.3. For every $r, t > 0$, there exist $\lambda, \eta, \mu, \nu \in (0, 1)$ and $\gamma > 0$ such that for every $r', r'' \in (r - \gamma, r + \gamma)$,

$$d_r(\varrho_{\lambda, \eta} || \{\sigma_1, \mu, \nu, \sigma_2, \mu, \nu\}) < t < \min_{i \in \{1, 2\}} d_{r''}(\varrho_{\lambda, \eta} || \sigma_i, \mu, \nu, \sigma_3, \nu), \quad j = 1, 2. \quad \text{(IV.101)}$$

Moreover, if $r' < r''$ then

$$d_t(\{\sigma_1, \mu, \nu, \sigma_2, \mu, \nu\} || \varrho_{\lambda, \eta}) < r' < r'' \leq \min_{i \in \{1, 2\}} d_t(\sigma_i, \mu, \nu || \varrho_{\lambda, \eta}), \quad j = 1, 2. \quad \text{(IV.102)}$$

**Proof** Let us fix $r$, and choose $\lambda, \mu$ as in Proposition IV.5. Note that

$$\varrho_{\lambda, \eta} = \eta(\lambda \hat{\sigma} \oplus (1 - \lambda) \oplus 0) + 0 \oplus 0 \oplus 0 \oplus (1 - \eta), \quad \sigma_{j, \mu, \nu} = \nu(\mu \hat{\sigma}_j \oplus (1 - \mu) \oplus 0) + 0 \oplus 0 \oplus 0 \oplus (1 - \nu).$$

Thus, for any $\eta \in (0, 1]$,

$$H_r(\varrho_{\lambda, \eta} || \sigma_{1, \mu, 1} \# \sigma_{2, \mu, 1}) = -\log \eta + H_r(\varrho_{\lambda, 1} || \sigma_{1, \mu, 1} \# \sigma_{2, \mu, 1}) = -\log \eta - \log \eta + H_r(\varrho_{\lambda, 1} || \sigma_{1, \mu, 1}) = H_r(\varrho_{\lambda, \eta} || \sigma_{j, \mu, 1}), \quad j = 1, 2. \quad \text{(IV.103)}$$

where the first and the last equalities follow by straightforward computations using the scaling laws (II.9). The second equality follows by Lemma II.9, since $r = D(\varrho_{\lambda, 1} || \sigma_{1, \mu, 1} \# \sigma_{2, \mu, 1})$ according to (IV.100), and the inequality follows again by Lemma II.9, since $r < D(\varrho_{\lambda, 1} || \sigma_{j, \mu, 1})$ according to (IV.99).

Let $s \in (0, 1/3)$ be such that $sk < t$. Set $\eta := e^{sk - t} \in (0, 1)$. By (IV.104), we have

$$H_r(\varrho_{\lambda, \eta} || \sigma_{1, \mu, 1} \# \sigma_{2, \mu, 1}) = t - sk < t, \quad H_r(\varrho_{\lambda, \eta} || \sigma_{j, \mu, 1}) = -\log \eta + \kappa = t + \kappa(1 - s) > t + 2\kappa/3. \quad \text{(IV.105)}$$

By Lemma II.2, $\nu \mapsto H_r(\varrho_{\lambda, \eta} || \sigma_{j, \mu, \nu})$ is a convex lower semi-continuous function on $[0, 1]$. Moreover, it is finite-valued; for $\nu < 1$ this follows from

$$\supp \varrho_{\lambda, \eta} = \mathcal{H} \ominus \mathcal{H} \oplus \mathbb{C} \oplus \mathbb{C} = \supp \sigma_{j, \mu, \nu}, \quad \text{(IV.106)}$$

while for $\nu = 1$ it follows from (IV.104). By (IV.96), $\sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu}$ is affine in $\nu$, and hence, again by Lemma II.2, $\nu \mapsto H_r(\varrho_{\lambda, \eta} || \sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu})$ is convex and lower semi-continuous, and it is also finite-valued, by the same argument as above. Hence, both functions are continuous in $\nu$, and thus there exists a $\nu_0 \in [0, 1)$ such that for all $\nu \in (\nu_0, 1)$,

$$H_r(\varrho_{\lambda, \eta} || \sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu}) < t, \quad H_r(\varrho_{\lambda, \eta} || \sigma_{j, \mu, \nu}) > t + 2\kappa/3, \quad \text{(IV.107)}$$

according to (IV.105). Let us fix any such $\nu$.

By (IV.106), $D_0(\varrho_{\lambda, \eta} || \sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu}) = 0 = D_0(\varrho_{\lambda, \eta} || \sigma_{j, \mu, \nu})$, whence the functions $r \mapsto H_r(\varrho_{\lambda, \eta} || \sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu})$ and $r \mapsto H_r(\varrho_{\lambda, \eta} || \sigma_{j, \mu, \nu})$ are finite-valued on $(0, +\infty)$ according to Lemma II.9, and, by Corollary II.3, they are also convex, and hence continuous. Thus, by (IV.107), there exists a $\gamma \in (0, r)$ such that

$$H_r(\varrho_{\lambda, \eta} || \sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu}) < t, \quad H_r(\varrho_{\lambda, \eta} || \sigma_{j, \mu, \nu}) > t + 2\kappa/3, \quad \text{for any } r' \in (r - \gamma, +\infty) \text{ and } r'' \in (0, r + \gamma), \quad \text{where we also took into account that } H_r \text{ is monotone non-increasing in } r. \quad \text{(IV.108)}$$

For any $r' \in (r - \gamma, +\infty)$ and $r'' \in (0, r + \gamma)$, where we also took into account that $H_r$ is monotone non-increasing in $r$. Finally, for any $r', r'' \in (r - \gamma, r + \gamma)$,

$$d_r(\varrho_{\lambda, \eta} || \{\sigma_1, \mu, \nu, \sigma_2, \mu, \nu\}) \leq H_r(\varrho_{\lambda, \eta} || \sigma_{1, \mu, \nu} \# \sigma_{2, \mu, \nu}) < t < H_r(\varrho_{\lambda, \eta} || \sigma_{j, \mu, \nu}) = d_{r''}(\varrho_{\lambda, \eta} || \sigma_{j, \mu, \nu}), \quad j = 1, 2. \quad \text{(IV.108)}$$

where the first inequality is due to Lemma IV.1, and the equality is due to Lemma II.25. This proves (IV.101).

For $r'$ as above, we have

$$d_r(\varrho_{\lambda, \eta} || \{\sigma_1, \mu, \nu, \sigma_2, \mu, \nu\}) < t \iff (r', t) \notin A(\varrho_{\lambda, \eta} || \{\sigma_1, \mu, \nu, \sigma_2, \mu, \nu\}) \iff (t, r') \notin A(\{\sigma_1, \mu, \nu, \sigma_2, \mu, \nu\} || \varrho_{\lambda, \eta}) \iff d_t(\{\sigma_1, \mu, \nu, \sigma_2, \mu, \nu\} || \varrho_{\lambda, \eta}) < r', \quad \text{(IV.109)}$$

Theorem IV.7
where the first inequality is by (IV.101), the first and the last equivalences are by definition, and the second equivalence is by Remark II.16. Similarly, for \( r'' \) as above,

\[
  t < d_{r'', (\rho, \eta, \mu, \nu)}(\sigma_{j,\mu,\nu}) \quad \iff \quad (r'', t) \in \mathcal{K}(\rho, \eta, \mu, \nu)
  \iff \quad (t, r'') \in \mathcal{K}(\sigma_{j,\mu,\nu} \| \rho, \eta, \mu, \nu)
  \iff \quad r'' \leq d_{t}(\sigma_{j,\mu,\nu} \| \rho, \eta, \mu, \nu).
\]  

(IV.110)

Finally, if \( r' < r'' \) then (IV.109) and (IV.110) yield (IV.102). □

Theorem IV.7 yields immediately a construction with strict inequality for the Chernoff exponent.

**Theorem IV.8** Let \( \rho, \sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{H}) \) be as in Theorem IV.3. For every \( r > 0 \), there exist \( \lambda, \eta, \mu, \nu \in (0, 1) \) such that

\[
c(\rho, \lambda, \eta, \mu, \nu) \{ \sigma_{1,\mu,\nu}, \sigma_{2,\mu,\nu} \} \leq r < \min_{i \in \{1, 2\}} c(\rho, \lambda, \eta, \mu, \nu) = c(\rho, \lambda, \eta, \mu, \nu), \quad j = 1, 2.
\]

(IV.111)

**Proof** Let \( t := r \), and choose \( \lambda, \eta, \mu, \nu \) as in Theorem IV.7. By (IV.101), \( d_{t}(\rho, \lambda, \eta, \mu, \nu) \{ \sigma_{1,\mu,\nu}, \sigma_{2,\mu,\nu} \} \leq r \), and hence \( c(\rho, \lambda, \eta, \mu, \nu) \{ \sigma_{1,\mu,\nu}, \sigma_{2,\mu,\nu} \} \leq r \).

On the other hand, let \( r'' > r \) be such that (IV.101) holds, and let \( \hat{r} := d_{r''}(\rho, \lambda, \eta, \mu, \nu) \). Then \( \hat{r} > r \), according to (IV.101). Finally, with \( \hat{r} := \min\{r'', \hat{r}\} \), we have \( d_{\hat{r}}(\rho, \lambda, \eta, \mu, \nu) \geq d_{r''}(\rho, \lambda, \eta, \mu, \nu) = \hat{r} \geq \hat{r} \), whence \( c(\rho, \lambda, \eta, \mu, \nu) \geq \hat{r} > r \), proving (IV.111). □

**Remark IV.9** Note that the unitaries \( U_0 := I, U_1 : X \oplus Y \oplus t \oplus s \rightarrow Y \oplus X \oplus t \oplus s \) give a unitary representation of \( \mathbb{Z}_2 \) on \( \mathcal{H} \oplus \mathcal{H} \oplus \mathbb{C} \oplus \mathbb{C} \), and we have

\[
  \rho, \lambda, \eta, \sigma_{1,\mu,\nu}, \sigma_{2,\mu,\nu} = U_k \rho, \lambda, \eta, U^*_k, \quad k = 0, 1, \quad U_1 \sigma_{1,\mu,\nu}, U^*_1 = \sigma_{2,\mu,\nu}.
\]

Thus, the examples in Theorem IV.3, Proposition IV.5, and Theorems IV.7–IV.8 fit into the setting of the group symmetric state discrimination problem considered in [25], and they refute the conjecture formulated in [25, Section VII], which, if true, would have implied the equality \( c(\sigma_{1,\mu,\nu}, \sigma_{2,\mu,\nu}) \{ \rho, \lambda, \eta \} = c(\sigma, \lambda, \eta, \mu, \nu) \), \( j = 1, 2 \), for the Chernoff and the direct exponents.

**Remark IV.10** Our example for the strict inequality in the direct exponents works for a non-trivial parameter range, which depends on the states constructed. It would be interesting to find an example of similarly simple structure, for which strict inequality holds simultaneously over the whole trade-off curve. An example with this property, but with continuum many null-hypotheses, was given in [25, Example 6.2].

**Remark IV.11** The above constructed examples are minimal in the sense that their null-hypothesis is always a singleton while their alternative hypothesis contain only two elements. However, regarding the dimension of the Hilbert space, they are not necessarily minimal. For example, to have a strict inequality for the asymmetric error exponent, our construction uses matrices of size at least \( 4 \times 4 \). This can in fact be improved: by an explicit computation of geometric means and relative entropies, one can verify that the \( 2 \times 2 \) matrices

\[
  \Omega := \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \sigma_1 := \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \quad \sigma_2 := \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}
\]

also give an example for the strict inequality \( s(\Omega \| \{ \sigma_1, \sigma_2 \}) < \min\{s(\Omega \| \sigma_1), s(\Omega \| \sigma_2)\} \), thereby providing an example that is also minimal in dimension. On the other hand, for the strict inequality in the symmetric case, we were unable to find an example using \( 2 \times 2 \) matrices; numerical attempts to compute the asymptotic behaviour using the first \( \sim 40 \) tensor powers[47] seemed to indicate that for that, one needs at least \( 3 \times 3 \) matrices.

**B. Equality cases in quantum state discrimination**

In this section we consider the case where both the null and the alternative hypotheses are given by finite sets of density operators \( R = \{ \sigma_1, \ldots, \sigma_k \} \) and \( S = \{ \sigma_1, \ldots, \sigma_m \} \) on some (possibly infinite-dimensional) Hilbert space \( \mathcal{H} \).

As we have seen in Section IV A, the optimal error exponents need not coincide with the worst pairwise error exponents even in this simple case, and therefore it becomes important to identify classes of states for which such an equality can nevertheless be established, for some or for all of the error exponents discussed in Section II C. As we have already mentioned in Section II C, equality holds in the classical case, i.e., when all states commute with each
other. Our first result, in Section IV B 1, shows that this condition can be relaxed, and equality holds even in the semi-classical case, where we only require that $\rho \varphi = \sigma \varphi$ for all $\rho \in R$ and $\sigma \in S$. In Section IV B 2 we show that equality holds also when all states $\rho, \sigma$ are pure.

Note that the above two special cases are disjoint in the sense that if $R$ and $S$ satisfy both conditions then for any $\rho \in R$ and $\sigma \in S$, they are either equal or orthogonal to each other, a trivial situation.

We mention that, on top of the equality cases discussed in this section and the ones mentioned in the Introduction, further special cases of equality are given in [57] for the Chernoff exponent. Moreover, it is easy to see that the main result of [36] on symmetric multi-state discrimination implies that we also have equality for the Chernoff exponent when the Hilbert space is finite-dimensional and the null and alternative hypotheses $R, S$ are such that

$$\min_{\rho \in R, \sigma \in S} C(\rho \| \sigma) = \min_{\tau_1, \tau_2 \in R \cup S, \tau_1 \neq \tau_2} C(\tau_1 \| \tau_2).$$

1. Semi-classical case

For a PSD operator $A \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and a test $T \in \mathcal{B}(\mathcal{H})_{[0,1]}$, let

$$\alpha(A|T) := \text{Tr} A(I - T), \quad \beta(A|T) := \text{Tr} AT.$$

**Lemma IV.12** Let $A_1, \ldots, A_k, B \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be PSD trace-class operators, and $T_1, \ldots, T_k \in \mathcal{B}(\mathcal{H})_{[0,1]}$ be tests. If $[A_i, B] = 0$ for every $i$, then there exists a projective test $Q$ such that

$$\max_{1 \leq i \leq k} \alpha(A_i|Q) \leq 2 \sum_{i=1}^{k} \alpha(A_i|T_i), \quad \text{and} \quad \beta(B|Q) \leq 2 \sum_{i=1}^{k} \beta(B|T_i).$$

**Proof** Note that replacing each $T_i$ with its diagonal in an orthonormal basis in which both $A_i$ and $B$ are diagonal, does not change the $\alpha$ and $\beta$ quantities, and therefore we may assume that $T_i$ commutes with both $A_i$ and $B$. Let $Q_i := f(T_i)$ with the function $f$ in (II.38). Then $Q_i$ is a projective test that also commutes with $A_i$ and $B$, and, by Lemma II.24,

$$\alpha(A_i|Q_i) \leq 2 \alpha(A_i|T_i), \quad \text{and} \quad \beta(B|Q_i) \leq 2 \beta(B|T_i).$$

Let $Q := \overline{Q}$ be the projection onto the support of $\overline{Q} := \sum_{i=1}^{k} Q_i$. Then

$$\max_{1 \leq i \leq k} \alpha(A_i|Q) \leq \max_{1 \leq i \leq k} \alpha(A_i|Q_i) \leq \max_{1 \leq i \leq k} 2 \alpha(A_i|T_i) \leq 2 \sum_{i=1}^{k} \alpha(A_i|T_i),$$

where the first inequality follows from $Q \geq Q_i, i \in [k]$. This proves the first inequality in (IV.112).

Let $P_\lambda$ denote the spectral projection of $B$ corresponding to $\lambda \in \text{spec}(B)$, and for an arbitrary PSD operator $X \in \mathcal{B}(\mathcal{H})_{\geq 0}$, let $E^X(H)$ denote its spectral projection corresponding to a Borel set $H \subseteq \mathbb{R}$. Note that $B$ commutes with $\overline{Q}$, and hence all $P_\lambda$ commute with the spectral projections $E^\overline{Q}(\cdot)$ of $\overline{Q}$. In particular, $E^P_\lambda \overline{Q}(H) = P_\lambda E^\overline{Q}(H)$ for any Borel set $H \subseteq (0, +\infty)$, and

$$(P_\lambda \overline{Q})^0 = \int_{(0, +\infty)} E^{P_\lambda \overline{Q}}(dt) = \int_{(0, +\infty)} P_\lambda E^\overline{Q}(dt) = P_\lambda \int_{(0, +\infty)} E^\overline{Q}(dt) = P_\lambda \overline{Q}^0 = P_\lambda Q.$$

Thus,

$$\text{Tr}(P_\lambda Q) = \text{Tr} \left( \sum_{i=1}^{k} P_\lambda Q_i \right)^0 = \dim (\text{span} \cup_{i=1}^{k} (\text{ran} P_\lambda \cap \text{ran} Q_i)) \leq \sum_{i=1}^{k} \dim (\text{ran} P_\lambda \cap \text{ran} Q_i) = \sum_{i=1}^{k} \text{Tr} P_\lambda Q_i.$$

Finally,

$$\beta(B|Q) = \sum_{\lambda \in \text{spec}(B) \setminus \{0\}} \lambda \text{Tr}(P_\lambda Q) \leq \sum_{i=1}^{k} \sum_{\lambda \in \text{spec}(B) \setminus \{0\}} \lambda \text{Tr}(P_\lambda Q_i) = \sum_{i=1}^{k} \beta(B|Q_i) \leq 2 \sum_{i=1}^{k} \beta(B|T_i),$$

proving the second inequality in (IV.112).
Lemma IV.13 Let \( A_i, \ldots, A_k, B_1, \ldots, B_m \in \mathcal{B}(\mathcal{H})_0 \) be PSD trace-class operators, and \( T_{i,j} \in \mathcal{B}(\mathcal{H})_{[i,j]}, i \in [k], j \in [m] \), be tests. If \( [A_i, B_j] = 0 \) for all \( i, j \), then there exists a projective test \( Q \) such that

\[
\max_{1 \leq i \leq k} \alpha(A_i | Q) \leq 4k \sum_{i=1}^{k} \alpha(A_i | T_{i,j}), \quad \text{and} \quad \max_{1 \leq j \leq m} \beta(B_j | Q) \leq 4 \sum_{j=1}^{m} \beta(B_j | T_{i,j}). \tag{IV.113}
\]

Proof The proof goes by a double application of Lemma IV.12. First, we fix \( B := B_j \) and \( T_i := T_{i,j} \) to obtain a projective test \( Q_j \) such that

\[
\max_{1 \leq i \leq k} \alpha(A_i | Q_j) \leq 2 \sum_{i=1}^{k} \alpha(A_i | T_{i,j}), \quad \text{and} \quad \beta(B_j | Q_j) \leq 2 \sum_{i=1}^{k} \beta(B_j | T_{i,j}). \tag{IV.114}
\]

Next, we apply Lemma IV.12 with \( \tilde{A}_j := B_j \), \( \tilde{B} := \tilde{A} := \sum_{i=1}^{k} A_i \), and \( \tilde{T}_j := I - Q_j \), to obtain a projective test \( \tilde{Q} \) such that

\[
\max_{1 \leq j \leq m} \alpha(\tilde{A}_j | \tilde{Q}) \leq 2 \sum_{j=1}^{m} \alpha(\tilde{A}_j | \tilde{T}_j) = 2 \sum_{j=1}^{m} \beta(B_j | Q_j) \leq 2 \sum_{j=1}^{m} 2 \sum_{i=1}^{k} \beta(B_j | T_{i,j}), \quad \text{and} \quad \beta(\tilde{B} | \tilde{Q}) \leq 2 \sum_{j=1}^{m} \beta(\tilde{B} | \tilde{T}_j) = 2 \sum_{j=1}^{m} \sum_{i=1}^{k} \alpha(A_i | Q_j) \leq 2 \sum_{j=1}^{m} \sum_{i=1}^{k} \sum_{i=1}^{k} \alpha(A_i | T_{i,j}). \tag{IV.115}
\]

The first inequality in (IV.115) is due to Lemma IV.12, and the second one follows from the second inequality in (IV.114). Similarly, the first inequality in (IV.116) is due to Lemma IV.12, and the second inequality follows from the first inequality in (IV.114). Defining \( Q := I - \tilde{Q} \), the LHS in (IV.115) becomes \( \max_{1 \leq j \leq m} \beta(B_j | Q) \), and we obtain the second inequality in (IV.113) from (IV.115), while the LHS in (IV.116) becomes

\[
\beta(\tilde{B} | \tilde{Q}) = \sum_{i=1}^{k} \text{Tr} A_i (I - Q) = \sum_{i=1}^{k} \text{Tr} A_i (I - Q) = \sum_{i=1}^{k} \alpha(A_i | Q) \geq \max_{1 \leq i \leq k} \alpha(A_i | Q),
\]

and combining it with the rest of (IV.116) yields the first inequality in (IV.113). \( \square \)

Corollary IV.14 Consider the composite hypothesis testing problem given by the finite sets of density operators \( \mathcal{R} = \{ q_1, \ldots, q_k \} \) (null-hypothesis) and \( \mathcal{S} = \{ \sigma_1, \ldots, \sigma_m \} \) (alternative hypothesis). If \( [q_i, \sigma_j] = 0 \) for all \( i, j \), then \( e(\mathcal{R} \Vert \mathcal{S}) = \min_{i,j} e(q_i \Vert \sigma_j) \) for any of the exponents \( e = s, c, \) and \( d_r \) with \( r > 0 \).

Proof By (II.42), it is enough to show that \( e(\mathcal{R} \Vert \mathcal{S}) \geq \min_{i,j} e(q_i \Vert \sigma_j) \), which follows immediately from the inequalities in Lemma IV.13. \( \square \)

2. Pure state case

Theorem IV.15 Let \( \mathcal{H} \) be a Hilbert space, and \( \mathcal{R}, \mathcal{S} \subseteq \mathcal{S}(\mathcal{H}) \). If both \( \mathcal{R} \) and \( \mathcal{S} \) are finite collections of rank-one projections (pure states), then

\[
e(\mathcal{R} \Vert \mathcal{S}) = \min_{q \in \mathcal{R}, \sigma \in \mathcal{S}} e(q \Vert \sigma)
\]

for any of the error exponents \( e = s, c, \) and \( d_r \), \( r > 0 \).

Proof Let \( \mathcal{R} = \{ q_1, \ldots, q_k \} \) and \( \mathcal{S} = \{ \sigma_1, \ldots, \sigma_m \} \) where \( q_i = | \Psi_i \rangle \langle \Psi_i | \) and \( \sigma_j = | \Phi_j \rangle \langle \Phi_j | \) are rank-one projections for every \( i, j \). We may assume without loss of generality that \( q_i \neq q_j \), \( i \neq j \), and \( \sigma_i \neq \sigma_j \), \( i \neq j \). Note that for every \( i \in [k], j \in [m] \),

\[
\psi(q_i \Vert \sigma_j | \alpha) = \log \text{Tr} q_i \sigma_j = -C(q_i \Vert \sigma_j) = -C_{i,j}
\]
for every $\alpha \in (0, 1)$, and thus
\[
d_r(\varphi_i | \sigma_j) = H_r(\varphi_i | \sigma_j) = \sup_{\alpha \in (0, 1)} \frac{\alpha - 1}{\alpha} \left[ r - \frac{1}{\alpha - 1} \log \text{Tr} \varphi_i \sigma_j \right] = r - \inf_{\alpha \in (0, 1)} \frac{1}{\alpha} | r - C_{i,j} | = \begin{cases} +\infty, & r < C_{i,j}, \\ C_{i,j}, & r \geq C_{i,j}, \end{cases}
\]
where the first equality is by Lemma II.25, and the rest are immediate from the definition of $H_r$. Hence,
\[
A_r(\varphi_i | \sigma_j) = [0, +\infty]^2 \setminus (C_{i,j}, +\infty)^2,
\]
and thus $\bigcap_{i,j} A_r(\varphi_i | \sigma_j) = [0, +\infty]^2 \setminus (C_{\text{min}}, +\infty)^2$,

where $C_{\text{min}} := \min_{i,j} C_{i,j}$. By Corollary II.28 and Lemma II.29, the assertion will follow for all the exponents if we can show that $[0, +\infty]^2 \setminus (C_{\text{min}}, +\infty)^2 \subset A(R|S)$. For this it is sufficient to show that $(C_{\text{min}}, \infty)$ and $(\infty, C_{\text{min}})$ are both in $A(R|S)$, according to Lemma II.17. Due to the symmetry of the problem with respect to interchanging $R$ and $S$, it is sufficient to show the achievability of the first pair. To this end, let us define
\[
K_n := \sum_{i=1}^{k} \varphi_i^{\otimes n}, \quad T_n := K_n^0, \quad n \in \mathbb{N},
\]
so that
\[
\alpha_n(R|T_n) = \max_{1 \leq i \leq k} \text{Tr} \varphi_i^{\otimes n} (I - T_n) = 0, \quad n \in \mathbb{N} \implies \liminf_{n \to +\infty} -\frac{1}{n} \log \alpha_n(R|T_n) = +\infty.
\]

Note that supp $K_n = \text{span}\{\Psi_i^{\otimes n} : i \in [k]\}$ is a $k$-dimensional subspace in $H^{\otimes n}$, and the nonzero eigenvalues of $K_n$ coincide with those of the $k \times k$ Gram matrix $G^{(n)}$ with entries
\[
G^{(n)}_{i,j} = \langle \Psi_i^{\otimes n}, \Psi_j^{\otimes n} \rangle = \langle \Psi_i, \Psi_j \rangle^n.
\]

Clearly, $G^{(n)}$ converges to the $k \times k$ identity matrix as $n \to \infty$. Hence, for every large enough $n$, the smallest non-zero eigenvalue of $K_n$ is larger than $1/2$, and thus $2K_n \geq K_n^0 = T_n$. Therefore,
\[
\beta(R|T_n) = \max_{1 \leq i \leq m} \text{Tr} (\sigma_j^{\otimes n} T_n) \leq 2 \max_{1 \leq j \leq m} \text{Tr} (\sigma_j^{\otimes n} K_n) = 2 \max_{1 \leq j \leq m} \sum_{i=1}^{k} \text{Tr} \sigma_j^{\otimes n} \varphi_i^{\otimes n} \leq 2 \left( \max_{1 \leq j \leq m} \max_{1 \leq i \leq k} \text{Tr}(\sigma_j \varphi_i) \right)^n.
\]

Hence,
\[
\liminf_{n \to +\infty} -\frac{1}{n} \log \beta(R|T_n) \geq -\max_{i,j} \log \text{Tr}(\varphi_i \sigma_j) = C_{\text{min}},
\]
completing the proof. □

\section{The Strong Converse Exponent}

By the definition of the Stein exponent $s(H_0||H_1)$, if the type II error exponent $r > s(H_0||H_1)$ then the type I error cannot converge to 0. In typical scenarios it even converges to 1 (this is called the strong converse property), and hence in this case the exponent of interest is the following quantity:

\textbf{Definition V.1} The strong converse exponent of the hypothesis testing problem $H_0$ vs. $H_1$ with type II error exponent $r$ is defined as
\[
sc_r(H_0||H_1) := \inf \left\{ \limsup_{n \to +\infty} -\frac{1}{n} \log (1 - \alpha_n(H_0|T_n)) : \liminf_{n \to +\infty} -\frac{1}{n} \log \beta_n(H_1|T_n) \geq r \right\},
\]
where the infimum is taken along all test sequences $(T_n)_{n \in \mathbb{N}}$ satisfying the indicated condition. The relaxed exponent $sc_r^t(H_0||H_1)$ for $t = \{0\}, \{1\},$ and $\{0, 1\}$ are defined analogously to Definition II.14.
Clearly, we have
\[
sc_r^i(H_0\|H_1) = 0, \quad r \leq s(H_0\|H_1),
\]
and hence we will be interested in the strong converse exponent only for type II error exponents \(r > s(H_0\|H_1)\).

Similarly to the case of the direct exponents, it follows immediately from the definitions that in the case of a consistent null hypothesis,
\[
sc_r((\{q_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\|\{S_n\}_{n \in \mathbb{N}}) \geq \sup_{i \in I} \min_{\epsilon} sc_r^i((\{q_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\|\{S_n\}_{n \in \mathbb{N}}),
\]
and similar inequalities hold when the alternative hypothesis is consistent, and when both hypotheses are consistent.

In this section we analyze cases where the inequalities in (V.117) do/do not hold as equalities. Of course, \(I\) is finite \(\implies\) \(sc_r((\{q_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\|\{S_n\}_{n \in \mathbb{N}}) = \max_{i \in I} sc_r^i((\{q_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\|\{S_n\}_{n \in \mathbb{N}}),\) and therefore in this case there is only one inequality to consider.

A. Composite null hypothesis and arbitrary alternative hypothesis

It was shown in [12] that \(sc_r((\{q_i\}_{i \in I})\|\sigma) = \max_{i \in I} sc_r(q_i\|\sigma)\) in the composite i.i.d. vs. simple i.i.d. setting if the index set \(I\) is finite. The same proof method gives the equality in the more general setting of Proposition V.2 below. We give a proof for completeness, and as a preparation for the proof of Proposition V.4.

**Proposition V.2** Let \(H_0 : (\{q_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\) vs. \(H_1 : (S_n)_{n \in \mathbb{N}}\) be a binary state discrimination problem with a consistent null hypothesis. If \(I\) is finite then
\[
sc_r((\{q_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\|H_1) = \max_{i \in I} sc_r^i((\{q_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\|H_1).
\]

**Proof** We have LHS \(\geq\) RHS in (V.118) due to the general inequalities in (V.117), and hence we only have to prove the converse inequality. Let \(s > \max_{i \in I} sc_r((\{q_{n,i}\}_{n \in \mathbb{N}}\|H_1)\), so that for every \(i \in I\) there exists a sequence of tests \(T_{n,i}, n \in \mathbb{N}\), satisfying
\[
\lim_{n \to +\infty} \frac{1}{n} \log(1 - \alpha_n(q_{n,i}|T_{n,i})) < s, \quad \liminf_{n \to +\infty} \frac{1}{n} \log(\beta(R_n|T_{n,i})) \geq r.
\]
Therefore there exists an \(N \in \mathbb{N}\) such that \(\min_{i} q_{n,i}(T_n) > e^{-ns}, n \geq N\). Let \(T_n := |I|^{-1} \sum_{i \in I} T_{n,i}, n \in \mathbb{N}\). Then
\[
\min_{i \in I} q_{n,i}(T_n) > |I|^{-1} e^{-ns}, \quad \sup_{\sigma_n \in S_n} \sigma_n(T_n) \leq |I|^{-1} \sum_{\sigma_n \in S_n} \sup_{\sigma_n \in S_n} \sigma_n(T_n), \quad n \geq N,
\]
from which the assertion follows immediately.

Next, we give some general converses to the inequalities in (V.117). For this, we will need the following:

**Definition V.3** For \(q, \sigma \in S(M)\), their max-relative entropy is defined as \([14, 56]\)
\[
D^*_\infty(q\|\sigma) := \inf \{\lambda \in \mathbb{R} : q \leq e^\lambda \sigma\}.
\]

The following proposition gives a generalization of Proposition V.2. The main idea of the proof is due to [59].

**Proposition V.4** Let \(H_0 : (\{q_{n,i}\}_{i \in I})_{n \in \mathbb{N}}\) vs. \(H_1 : (S_n)_{n \in \mathbb{N}}\) be a binary state discrimination problem with a consistent null hypothesis, and assume that there exists a countable max-relative entropy dense set for the null hypothesis in the following sense: there exists a countable \(\tilde{I} \subseteq I\) such that for every \(i \in I\) and every \(\epsilon > 0\) there exists a \(k_{i,\epsilon} \in \tilde{I}\) and \(N_{i,\epsilon}\) such that \(D^*_\infty(q_{n,k_{i,\epsilon}}\|q_{n,i}) \leq \epsilon\) for every \(n \geq N_{i,\epsilon}\). Then
\[
\inf_{r' > r} \sup_{i \in I} sc_r((\{q_{n,i}\}_{n \in \mathbb{N}}\|H_1) \geq \sup_{i \in I} sc_r^i((\{q_{n,i}\}_{n \in \mathbb{N}}\|H_1) (V.119)
\]

Clearly, the sequence (\(\tilde{n}\)) assumption. Then for every \(M\) such that \(\tilde{\tilde{n}}\) is of countably infinite cardinality, and hence we may assume without loss of generality that \(M = N\).

Clearly, (V.119) is equivalent to

\[
\text{sc}_r^{(0)}((\{q_{n,i}\}_{i \in I})_{n \in N} \| H_1) \geq \sup_{i \in I} \text{sc}_r((q_{n,i})_{n \in N} \| H_1) \geq \sup_{i \in I} \text{sc}_r((q_{n,i})_{n \in N} \| H_1) \geq \sup_{r' < r} \text{sc}_{r'}^{(0)}((\{q_{n,i}\}_{i \in I})_{n \in N} \| H_1),
\]

(V.120)

and we will prove the statement in this form. The inequalities in (V.120) are immediate from the general inequalities in (V.117), and hence we only have to prove the inequality in (V.121).

Let \(s > \sup_{i \in I} \text{sc}_r((q_{n,i})_{n \in N} \| H_1)\). Then for every \(k \in I\) there exists a sequence of tests \(T_{n,k}, n \in N\), such that

\[
\lim_{n \to +\infty} -\frac{1}{n} \log(1 - \alpha_n(q_{n,k} \| T_{n,k})) < s, \quad \liminf_{n \to +\infty} -\frac{1}{n} \log \beta_n(R_n | T_{n,k}) \geq r.
\]

(V.122)

Let us fix a probability distribution \((q_k)_{k \in \mathbb{N}}\) with all \(q_k > 0\). For every \(\delta > 0\) and every \(n \in \mathbb{N}\), let

\[
\hat{I}_{n,\delta} := \left\{ k \in I \cap [n] : q_{m,k}(T_{m,k}) > e^{-m\alpha}, \sup_{\sigma_m \in S_m} \sigma_m(T_{m,k}) < e^{-m(r - \delta)}, \ m \geq n \right\}, \quad T_n := \sum_{k \in \hat{I}_n} q_k T_{n,k}.
\]

Clearly, the sequence \((\hat{I}_{n,\delta})_{n \in \mathbb{N}}\) is increasing, and by (V.122), \(\cup_{n \in \mathbb{N}} \hat{I}_{n,\delta} = \hat{I}\), i.e., for every \(k \in \hat{I}\) there exists an \(M_{k,\delta} \in \mathbb{N}\) such that \(k \in \hat{I}_{n,\delta}\) for every \(n \geq M_{k,\delta}\). For arbitrary \(i \in I\) and \(\epsilon > 0\), let \(k_{i,\epsilon}\) and \(N_{i,\epsilon}\) be as in the assumption. Then for every \(n \geq \max\{N_{i,\epsilon}, M_{k_{i,\epsilon},\delta}\},
\[
q_{n,i}(T_n) \geq e^{-n\epsilon} q_{n,k_{i,\epsilon}}(T_{n,k_{i,\epsilon}}) > e^{-n\epsilon} q_{n,k_{i,\epsilon}}(T_{n,k_{i,\epsilon}}) > \epsilon \sup_{\sigma_n \in \mathcal{S}_n} \sigma_n(T_n) < e^{-n(r - \delta)}.
\]

This shows that \(\text{sc}_r^{(0)}((\{q_{n,i}\}_{i \in I})_{n \in N} \| H_1) \leq s + \epsilon\) for all \(\epsilon > 0\) and \(s\) as above. Taking the infimum in \(\epsilon\) and \(s\), and then in \(\delta\), yields (V.121).

Let us now consider the case where the null hypothesis is composite i.i.d., given by a subset \(R \subseteq \mathcal{S}(\mathcal{M})\). We say that a subset \(\hat{R} \subseteq R\) is max-relative entropy dense in \(R\), if for every \(\varrho \in R\) and every \(\epsilon > 0\) there exists a \(\varrho_{\epsilon} \in \hat{R}\) such that \(D^*_{\varrho}(\varrho_{\epsilon} \| \varrho) < \epsilon\), or equivalently, \(e^{-\epsilon} \varrho_{\epsilon} \leq \varrho\). Clearly then \(D^*_{\varrho}(\varrho_{\epsilon}^{\otimes n} \| \varrho^{\otimes n}) < n\epsilon\) for every \(n \in \mathbb{N}\). Hence, if a countable \(R\) exists with the above properties then the sufficient condition in Proposition V.4 is satisfied, and therefore the inequalities in (V.119) hold. Hence, our next goal is to find sufficient conditions for the existence of a countable max-relative entropy dense set.

**Lemma V.5** Let \(\varrho, \sigma\) be non-zero PSD operators on a finite-dimensional Hilbert space \(\mathcal{H}\). If \(\varrho^0 \leq \sigma^0\) then

\[
\frac{1}{2} \| \varrho - \sigma \|_1^2 \leq D^*_{\varrho}(\varrho \| \sigma) \leq \log \left( 1 + \frac{1}{\lambda_{\min}(\sigma)} \| \varrho - \sigma \|_1 \right) \leq \frac{1}{\lambda_{\min}(\sigma)} \| \varrho - \sigma \|_1,
\]

(V.123)

where \(\lambda_{\min}(\sigma)\) is the smallest non-zero eigen-value of \(\sigma\).

**Proof** The first inequality is an immediate consequence of the quantum Csizásr-Pinsker inequality [27]. The duality of linear programming yields that

\[
D^*_{\varrho}(\varrho \| \sigma) = \log \max \left\{ \frac{\text{Tr} \varrho T}{\text{Tr} \sigma T} : T \in \mathcal{B}(\mathcal{H})|_{[0,1]}, T^0 \leq \sigma^0 \right\} = \log \max \left\{ \frac{\text{Tr} \varrho \tau}{\text{Tr} \sigma \tau} : \tau \in \mathcal{S}(\mathcal{H}), \tau^0 \leq \sigma^0 \right\}.
\]

By the Hölder inequality, \(|\text{Tr} \varrho \tau - \text{Tr} \sigma \tau| \leq \| \varrho - \sigma \|_1\), and hence

\[
\left| \frac{\text{Tr} \varrho \tau}{\text{Tr} \sigma \tau} - 1 \right| \leq \frac{1}{\text{Tr} \sigma \tau} \| \varrho - \sigma \|_1 \leq \frac{1}{\lambda_{\min}(\sigma)} \| \varrho - \sigma \|_1,
\]

from which the statement follows. \(\square\)
Lemma V.6 Let $\mathcal{H}$ be a finite-dimensional Hilbert space and $R \subseteq S(\mathcal{H})$. If only countably many of the sets
\[
R_P := \{ \varrho \in R : \varrho^0 = P \}, \quad P \text{ projection on } \mathcal{H},
\]
are non-empty then there exists a countable max-relative entropy dense set in $R$. This is satisfied, for instance, if $R$ is a convex set and (i) $R \setminus \text{relint } R$ is countable, or (ii) $R$ is classical, i.e., any two elements in $R$ commute.

Proof Finite-dimensionality implies that for each $P$, there exists a countable dense set $\hat{R}_P$ in $R_P$ with respect to the trace-norm, and therefore the main claim follows immediately from Lemma V.5 due to $R = \bigcup_{P \text{ projection}} R_P$.

Assume now that $R$ is convex. It is easy to see that any two $\varrho, \sigma \in \text{relint } R$ have the same support, from which the assertion in case (i) follows. In case (ii), we may represent the elements on $R$ as probability density functions on a finite set $X$. Hence, there are only finitely many possible support sets, from which the assertion in case (ii) follows. □

Corollary V.7 Consider a binary state discrimination problem with composite i.i.d. null-hypothesis $H_0 : (R^{\otimes n})_{n \in \mathbb{N}}$ vs. $H_1 : (S_n)_{n \in \mathbb{N}}$. If $R$ is as in Lemma V.6 then (V.119) holds (with $I = R$).

We will utilize the above corollary in the proof of Corollary V.23.

Remark V.8 Let $R \subseteq S(\mathbb{C}^2)$ be a set obtained by slicing off a part of the Bloch ball by a hyperplane, such that $|R| > 1$. Then $R$ does not contain a countable max-relative entropy dense set, because for every pure state $|\psi\rangle\langle\psi| \in R$ and every $|\psi\rangle\langle\psi| \neq \varrho \in R$, $D_\infty^*(\varrho\| |\psi\rangle\langle\psi|) = +\infty$. This shows that max-relative entropy density is a strictly stronger notion than density in any norm, and also demonstrates how the sufficient conditions in Lemma V.6 might fail.

B. Divergences for the strong converse exponent

In the case of finite-dimensional simple i.i.d. quantum state discrimination, the list of equalities in Lemma II.25 can be continued with $s_{cr}(\varrho\|\sigma) = H_\alpha^*(\varrho\|\sigma)$, where the quantity on right-hand side is the Hoeffding anti-divergence [40]. In this section we discuss some important properties of the extension of these quantities to pairs of sets of states. While we will only use these later in the classical case, in Section V B 1 we consider the more general quantum case, as this does not make the discussion more complicated.

Similarly to the Hoeffding divergence, the Hoeffding anti-divergence is also a certain transform of a function defined from a family of Rényi $\alpha$-divergences (denoted by $\psi$ and $\psi^*$ in the respective cases), which is convex in $\alpha$ for pairs of states. However, for a pair of sets of states $R,S$, the function is given by $\psi(R\|S|\alpha) = \sup_{\varrho \in R, \sigma \in S} \psi(\varrho\|\sigma|\alpha)$ in the first case, and hence it is still convex in $\alpha$, while in the second case it is $\psi^*(R\|S|\alpha) = \inf_{\varrho \in R, \sigma \in S} \psi^*(\varrho\|\sigma|\alpha)$, and it is not clear whether convexity in $\alpha$ still holds. For this reason, establishing some of the relevant properties is more complicated for the Hoeffding anti-divergences, and in Section V B 2 we give an analysis of these properties under minimal assumptions on the properties of $\psi$.

1. The Hoeffding anti-divergence

For a density operator $\varrho \in S(\mathcal{H})$, a non-zero PSD operator $\sigma \in B(\mathcal{H}) \geq 0$, and $\alpha \in (0, +\infty)$, let
\[
Q^*_\alpha(\varrho\|\sigma) := \begin{cases} 
\text{Tr} \left( \varrho^{1/2} \sigma^{1/2} \varrho^{1/2} \right)^\alpha, & \alpha \in (0,1) \text{ or } \varrho^0 \leq \sigma^0, \\
+\infty, & \text{otherwise},
\end{cases}
\]
and
\[
\psi^*(\varrho\|\sigma|\alpha) := \log Q^*_\alpha(\varrho\|\sigma).
\]
The sandwiched Rényi $\alpha$-divergences of $\varrho$ and $\sigma$ are defined as [43, 60]
\[
D^*_\alpha(\varrho\|\sigma) := \frac{1}{\alpha - 1} \psi^*(\varrho\|\sigma|\alpha), \quad \alpha \in (0, +\infty) \setminus \{1\}.
\]
These notions have also been extended to the setting of general von Neumann algebras in [9, 32, 33], but here we only consider the finite-dimensional case. It is known that $\alpha \mapsto D^*_\alpha(\varrho\|\sigma)$ is monotone non-decreasing, with
\[
D^*_1(\varrho\|\sigma) := \lim_{\alpha \to 1} D^*_\alpha(\varrho\|\sigma) = D(\varrho\|\sigma), \quad \lim_{\alpha \to +\infty} D^*_\alpha(\varrho\|\sigma) = D^*_\infty(\varrho\|\sigma) := \inf\{\lambda \in \mathbb{R} : \varrho \leq e^\lambda \sigma\}; \quad (V.124)
\]
see [43].

In the above setting, let us introduce the **Hoeffding anti-divergence** of \( \varrho \) and \( \sigma \) as

\[
H^*_r(\varrho \| \sigma) := \max_{u \in [0,1]} \left[ ur - \tilde{\psi}^*(\varrho \| \sigma | u) \right], 
\]

(V.125)

where \( r > 0 \) is a parameter, and

\[
\tilde{\psi}^*(\varrho \| \sigma | u) := (1-u)\psi^*(\varrho \| \sigma | (1-u)^{-1}).
\]

(The existence of the max follows from properties of the \( \psi^* \) function that we establish below.) In the above, we use the conventions

\[
\frac{+\infty - 1}{+\infty} := \lim_{\alpha \to +\infty} \frac{\alpha - 1}{\alpha} = 1, \quad 0 \cdot (\pm \infty) := 0, \quad \tilde{\psi}^*(\varrho \| \sigma | 1) := \lim_{u \to 1} \tilde{\psi}^*(\varrho \| \sigma | u) = D^*_\infty(\varrho \| \sigma).
\]

The equality in (V.125) follows by the simple change of variables \( u := (\alpha - 1)/\alpha \).

**Remark V.9** Note that for commuting states \( \varrho, \sigma \) (which can be assumed to be probability distributions on a finite set \( \mathcal{X} \)), \( \psi^*(\varrho \| \sigma | \alpha) = \psi(\varrho \| \sigma | \alpha) \) and \( \tilde{\psi}^*(\varrho \| \sigma | u) = \psi(\varrho \| \sigma | u) \), where the latter were defined in (II.19). However, \( H_r(\varrho \| \sigma) \neq H^*_r(\varrho \| \sigma) \) in general, even if the states commute, because in their definitions not only the types of Rényi divergences differ, but also the ranges of optimization.

**Remark V.10** \( H^*_r \) is called an anti-divergence because it is monotone non-decreasing under CPTP maps; this is immediate from the monotonicity of \( D^*_\alpha \) under CPTP maps for \( \alpha \geq 1 \) [7, 18].

The importance of the Hoeffding anti-divergence stems from the following fact [40]:

**Lemma V.11** In the simple binary i.i.d. state discrimination problem \( \varrho \) vs. \( \sigma \),

\[
sc_r(\varrho \| \sigma) = H^*_r(\varrho \| \sigma), \quad r > 0.
\]

For two sets \( R \subseteq S(\mathcal{H}) \) and \( S \subseteq B(\mathcal{H})_{\geq 0} \), let their Hoeffding anti-divergence be defined as

\[
H^*_r(R \| S) := \sup_{\varrho \in R, \sigma \in S} H^*_r(\varrho \| \sigma) = \sup_{\alpha \in [1, +\infty]} \frac{\alpha - 1}{\alpha} \left[ r - \inf_{\varrho \in R, \sigma \in S} D^*_\alpha(\varrho \| \sigma) \right] = \sup_{\alpha \in [1, +\infty]} \frac{\alpha - 1}{\alpha} \left[ r - D^*_\alpha(R \| S) \right].
\]

Note that

\[
\lim_{\alpha \to 1} D^*_\alpha(R \| S) = \inf_{\alpha > 1} D^*_\alpha(R \| S) = \inf_{\alpha > 1} \inf_{\varrho \in R, \sigma \in S} D^*_\alpha(\varrho \| \sigma) = \inf_{\varrho \in R, \sigma \in S} \inf_{\alpha > 1} D^*_\alpha(\varrho \| \sigma) = \inf_{\varrho \in R, \sigma \in S} D(\varrho \| \sigma) = D(R \| S),
\]

(V.126)

and thus

\[
H^*_r(R \| S) \geq 0, \quad \text{and} \quad H^*_r(R \| S) = 0 \iff r \leq D(R \| S).
\]

(V.127)

**Remark V.12** Note that

\[
D^*_\infty(R \| S) = \inf_{\varrho \in R, \sigma \in S} D^*_\infty(\varrho \| \sigma) = \inf_{\varrho \in R, \sigma \in S} \sup_{\alpha > 1} D^*_\alpha(\varrho \| \sigma)
\]

\[
\geq \sup_{\alpha > 1} \inf_{\varrho \in R, \sigma \in S} D^*_\alpha(\varrho \| \sigma) = \sup_{\alpha > 1} D^*_\alpha(R \| S) = \lim_{\alpha \to +\infty} D^*_\alpha(R \| S).
\]

If \( R \) and \( S \) are compact then the inequality holds with equality; this follows from Lemma II.5 and the lower semi-continuity of the sandwiched Rényi divergences; see Lemma V.15 below.

The following is immediate from the definition:

**Lemma V.13** For any \( R, S \subseteq S(\mathcal{H}) \), the function \( (0, +\infty) \ni r \mapsto H^*_r(R \| S) \) is finite-valued, monotone increasing, and convex; in particular, it is continuous.
Proof Monotonicity is trivial from the definition, and convexity follows as \( H_r^*(R\|S) \) is the supremum of convex functions. For \( r \leq D(R\|S) \), \( H_r(R\|S) = 0 \). For \( r > D(R\|S) \), we have
\[
\frac{\alpha - 1}{\alpha} \left[ r - D_{\alpha r}^*(R\|S) \right] \leq r - D(R\|S)
\] (V.128)
for any \( \alpha \geq 1 \), and hence \( H_r^*(R\|S) \leq r - D(R\|S) \), showing finiteness. Continuity then follows from convexity and finiteness.

We will need certain further continuity properties of the above quantities. Recall that an extended real-valued function \( f \) on a topological space \( X \) is upper semi-continuous, if \( \{ x \in X : f(x) \geq c \} \) is closed for every \( c \in \mathbb{R} \). The following is easy to show from the definition:

**Lemma V.14** Let \( X \) be a topological space, \( Y \) be an arbitrary set, and \( f : X \times Y \to \mathbb{R} \cup \{ \pm \infty \} \) be a function.

(i) If \( f(., y) \) is upper semi-continuous for every \( y \in Y \) then \( \inf_{y \in Y} f(., y) \) is upper semi-continuous.

(ii) If \( Y \) is a compact topological space, and \( f \) is upper semi-continuous on \( X \times Y \) w.r.t. the product topology, then \( \sup_{y \in Y} f(., y) \) is upper semi-continuous.

Proof For \( \alpha \in (1, +\infty) \), the map \((0, +\infty) \ni \varepsilon \mapsto (\sigma + \varepsilon)^{\frac{\alpha}{\sigma + \varepsilon}} \) is monotone non-increasing, and hence so is \( \varepsilon \mapsto Q_\alpha^*(\varrho\|\sigma + \varepsilon) \). Moreover, it is easy to verify that
\[
Q_\alpha^*(\varrho\|\sigma) = \sup_{\varepsilon > 0} Q_\alpha^*(\varrho\|\sigma + \varepsilon), \quad \psi^*(\varrho\|\sigma|\alpha) = \sup_{\varepsilon > 0} \psi^*(\varrho\|\sigma + \varepsilon|\alpha), \quad D_\alpha^*(\varrho\|\sigma) = \sup_{\varepsilon > 0} D_\alpha^*(\varrho\|\sigma + \varepsilon),
\] (V.129)
where the last equality also holds for \( \alpha = 1 \).

**Lemma V.15** For any fixed \( r > 0 \), the map
\[
[1, +\infty] \times \mathcal{S}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})_{\geq 0} \ni (\alpha, \varrho, \sigma) \mapsto \frac{\alpha - 1}{\alpha} [r - D_\alpha^*(\varrho\|\sigma)]
\]
is upper semi-continuous.

Proof For any fixed \( \varepsilon > 0 \), the map
\[
[1, +\infty] \times \mathcal{S}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})_{\geq 0} \ni (\alpha, \varrho, \sigma) \mapsto \frac{\alpha - 1}{\alpha} [r - D_\alpha^*(\varrho\|\sigma + \varepsilon)]
\]
is continuous, (where \((+\infty - 1)/ + \infty := 1\)), and hence
\[
[1, +\infty] \times \mathcal{S}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})_{\geq 0} \ni (\alpha, \varrho, \sigma) \mapsto \frac{\alpha - 1}{\alpha} [r - D_\alpha^*(\varrho\|\sigma)] = \inf_{\varepsilon > 0} \frac{\alpha - 1}{\alpha} [r - D_\alpha^*(\varrho\|\sigma + \varepsilon)],
\]
where the equality is due to (V.129), is upper semi-continuous according to (i) of Lemma V.14.

**Corollary V.16** For any fixed \( r > 0 \), the map \( \mathcal{S}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})_{\geq 0} \ni (\varrho, \sigma) \mapsto H_r^*(\varrho\|\sigma) \) is upper semi-continuous.

Proof Immediate from Lemma V.15 and (ii) of Lemma V.14.

**Corollary V.17** For any fixed \( r > 0 \) and compact sets \( R \subseteq \mathcal{S}(\mathcal{H}) \), \( S \subseteq \mathcal{B}(\mathcal{H})_{\geq 0} \), there exist \( \varrho_r \in R \), \( \sigma_r \in S \), and \( \alpha_r \in [1, +\infty) \) such that
\[
H_{\alpha_r}^*(R\|S) = \frac{\alpha_r - 1}{\alpha_r} [r - D_{\alpha_r}^*(R\|S)] = \frac{\alpha_r - 1}{\alpha_r} [r - D_{\alpha_r}^*(\varrho_r\|\sigma_r)] = H_{\alpha_r}^*(\varrho_r\|\sigma_r).
\]
In particular, \( D_{\alpha_r}^*(\varrho_r\|\sigma_r) = D_{\alpha_r}^*(R\|S) \).

Proof Immediate from Lemma V.15 and the fact that an upper semi-continuous function attains its supremum on a compact set.

Some of the important properties of the Hoeffding anti-divergence, which we will need in Section V.C.2, can be obtained from very general properties of the function \( \psi(\alpha) := \inf_{\varrho \in R, \sigma \in S} \psi^*(\varrho\|\sigma|\alpha) \). We discuss these in the following separate section, which essentially gives a generalization of Lemmas 15 and 16 in [41] to the case where \( \psi \) is not assumed to be convex or differentiable.
2. Legendre-Fenchel representation of the Hoeffding anti-divergence

Let \( \psi : [1, +\infty) \to \mathbb{R} \) be a function such that

\[
\psi(1) = 0, \quad \text{and} \quad \alpha \mapsto \frac{\psi(\alpha)}{\alpha - 1} \text{ is monotone increasing.}
\]

This implies that for the function \( \tilde{\psi}(u) := (1 - u)\psi\left(\frac{1}{1-u}\right), \ u \in [0, 1), \)

\[
\tilde{\psi}(0) = 0, \quad \text{and} \quad u \mapsto \frac{\tilde{\psi}(u)}{u} \text{ is monotone increasing.}
\]

(Note that \( \tilde{\psi} \) is the so-called transpose function of \( \psi \) evaluated at \( 1 - u \).) The monotonicity assumption yields the existence of the half-sided limits

\[
\psi(\alpha^-) := \lim_{\beta \nearrow \alpha} \psi(\beta), \quad \psi(\alpha^+) := \lim_{\beta \searrow \alpha} \psi(\beta), \quad \tilde{\psi}(u^-) := \lim_{\nu \searrow u} \tilde{\psi}(\nu), \quad \tilde{\psi}(u^+) := \lim_{\nu \nearrow u} \tilde{\psi}(\nu)
\]

for every \( \alpha \in (1, +\infty) \) and \( u \in (0, 1) \), and the existence of

\[
D_1^+ := \Psi^+(1) := \lim_{\alpha \searrow 1} \frac{\psi(\alpha)}{\alpha - 1} = \lim_{u \searrow 0} \frac{\tilde{\psi}(u)}{u} =: \Psi^+(0) \quad (V.130)
\]

\[
D_\infty := \lim_{\alpha \to +\infty} \frac{\psi(\alpha)}{\alpha - 1} = \lim_{u \nearrow 1} \frac{\tilde{\psi}(u)}{u} =: \Psi^+(1). \quad (V.131)
\]

(It is easy to see that if \( \psi \) is convex then \( D_\infty = \lim_{\alpha \to +\infty} \partial^- \psi(\alpha) = \lim_{\alpha \to +\infty} \partial^+ \psi(\alpha) \).) For every \( c \in \mathbb{R} \), let

\[
\Psi(c) := \sup_{1 \leq \alpha < +\infty} \{\alpha c - \psi(\alpha)\}, \quad \Psi^-(c) := \sup_{1 \leq \alpha < +\infty} \{c(\alpha - 1) - \psi(\alpha)\} = \Psi(c) - c,
\]

where \( \Psi \) is the Legendre-Fenchel transform of \( \psi \).

**Lemma V.18**

(i) \( \Psi \) and \( \Psi^- \) are convex, increasing, and lower semi-continuous on \( \mathbb{R} \).

(ii) For every \( c \in \mathbb{R} \), \( \Psi(c) \geq c \), \( \Psi^-(c) \geq 0 \), and the inequalities are strict if and only if \( c > D_1^+ \).

(iii) \( \Psi(c) = \Psi^-(c) = +\infty \) for \( c > D_\infty \).

Assume for the rest that \( D_1^+ > -\infty \).

(iv) \( \Psi(c) \) and \( \Psi^-(c) \) are finite for \( c < D_\infty \).

(v) \( \Psi \) and \( \Psi^- \) are continuous on \( (-\infty, D_\infty] \), and \( \Psi \) is strictly increasing on \( (-\infty, D_\infty] \).

(vi) For every \( c \in (D_1^+, D_\infty) \) there exist \( 1 < \alpha_{c,\min} < \alpha_c \leq \alpha_{c,\max} < +\infty \) such that

\[
\sup_{\alpha \in [1, +\infty) \setminus [\alpha_{c,\min}, \alpha_{c,\max}]} \{(\alpha - 1)c - \psi(\alpha)\} < \frac{\Psi^-(c)}{2} < \Psi^-(c), \quad \text{and} \quad \Psi^-(c) = c(\alpha_c - 1) - \psi(\alpha_c^+).
\]

(vii) \( \Psi^- \) is strictly increasing on \( (D_1^+, D_\infty] \).

**Proof** The properties listed in (i) are obvious from the definitions of \( \Psi \) and \( \Psi^- \).

The inequalities in (ii) are obvious from taking \( \alpha = 1 \), for which \( c(\alpha - 1) - \psi(\alpha) = 0 \). We have

\[
\Psi^-(c) = \max \left\{0, \sup_{\alpha > 1} \left\{(\alpha - 1) \left[ c - \frac{\psi(\alpha)}{\alpha - 1}\right]\right\}\right\} > 0 \iff \exists \alpha > 1: c - \frac{\psi(\alpha)}{\alpha - 1} > 0 \iff c > \inf_{\alpha > 1} \frac{\psi(\alpha)}{\alpha - 1} = D_1^+,
\]
proving the assertion about the strict inequalities in (ii). If $D_\infty < +\infty$ then
\[
\Psi^-(c) \geq \limsup_{\alpha \to +\infty} (\alpha - 1) \left[ c - D_\infty + D_\infty - \frac{\psi(\alpha)}{\alpha - 1} \right] \geq \lim_{\alpha \to +\infty} (\alpha - 1) [c - D_\infty],
\]
and the last expression is equal to $+\infty$ for $c > D_\infty$, proving (iii).

Assume for the rest that $D_1^+ > -\infty$. If $c < D_\infty$ then $(\alpha - 1) \left[ c - \frac{\psi(\alpha)}{\alpha - 1} \right] \leq 0$ for all $\alpha > \alpha_{c,\text{max}} := \sup \left\{ \alpha > 1 : \frac{\psi(\alpha)}{\alpha - 1} < c \right\} < +\infty$. As a consequence, $\Psi^-(c) \leq (\alpha_{c,\text{max}} - 1) [c - D_1^+]$, proving (iv).

Since $\Psi$ and $\Psi^-$ are convex and lower semi-continuous, they are also continuous on the closure of the interval where they take finite values; by (iii) and (iv), this is $(-\infty, D_\infty]$. Since $\Psi^-$ is increasing on $(-\infty, D_\infty]$, and $\Psi(c) = \Psi^-(c) + c$, it is clear that $\Psi$ is strictly increasing on the same interval. This proves (v).

$D_1^+ > -\infty$ implies that $\lim_{\alpha \to +\infty} (\alpha - 1) \left[ c - \frac{\psi(\alpha)}{\alpha - 1} \right] = 0$ for every $c \in \mathbb{R}$. If $c > D_1^+$ then $\Psi^-(c) > 0$ by (ii), and by the observation in the previous sentence, $(\alpha - 1) c - \psi(\alpha) < \Psi^-(c)/2$ for $c \in [1, \alpha_{c,\text{min}})$ with some $\alpha_{c,\text{min}} > 1$. Hence, if $c \in (D_1^+, D_\infty)$ then the inequalities in (V.132) hold. Therefore,
\[
\Psi^-(c) = \sup_{\alpha_{c,\text{min}} \leq \alpha \leq \alpha_{c,\text{max}}} \left\{ (\alpha - 1) \left[ c - \frac{\psi(\alpha)}{\alpha - 1} \right] \right\}, \tag{V.133}
\]
and therefore both inequalities are equalities. If the sequence $(\alpha_{c,k(n)})_{n \in \mathbb{N}}$ is monotone increasing then the first limit in (V.134) is $c(\alpha_{c,1}) - \psi(\alpha_{c,1})$, as required. If $(\alpha_{c,k(n)})_{n \in \mathbb{N}}$ is monotone decreasing then the first limit in (V.134) is $c(\alpha_{c,1}) - \psi(\alpha_{c,1})$. This shows that $\psi(\alpha_{c,1}) = \psi(\alpha_{c,1})$, since otherwise $\Psi^-(c) = (\alpha_{c,1} - 1) \left[ c - \frac{\psi(\alpha_{c,1})}{\alpha_{c,1} - 1} \right] \leq -\Psi^-(c)$, a contradiction. This proves (vi).

Let $c \in (D_1^+, D_\infty)$, and $\alpha_c$ be as above. For every $\alpha > \alpha_c$,
\[
\Psi^-(c') \geq \lim_{\alpha \to \alpha_c} \left\{ (\alpha - 1) - \psi(\alpha) \right\} = c'(\alpha_{c,1}) - \psi(\alpha_{c,1}) = c(\alpha_{c,1}) - \psi(\alpha_{c,1}) + (c' - c)(\alpha_{c,1} - 1) - \Psi^-(c),
\]
showing the strict monotonicity of $\Psi^-$ on $(D_1^+, D_\infty)$, as stated in (vii).

Consider now the Legendre-Fenchel transform of $\tilde{\psi}$,
\[
\tilde{\Psi}(r) := \sup_{0 \leq u \leq 1} \{ ur - \tilde{\psi}(u) \} = \sup_{0 \leq u < 1} \{ ur - \tilde{\psi}(u) \} = \sup_{1 \leq u} \left\{ \frac{\alpha - 1}{\alpha} r - \frac{\psi(\alpha)}{\alpha} \right\}, \tag{V.135}
\]
where the first equality follows as $r - \tilde{\psi}(1) = \lim_{u \to 1} \{ ur - \tilde{\psi}(u) \}$, according to (V.131), and the second equality follows by the change of variables $u = (\alpha - 1)/\alpha$.

By the monotonicity of $\Psi$, (i) of Lemma V.18, the limits
\[
r_1^+ := \lim_{c \to D_1^+} \Psi(c), \quad r_\infty := \lim_{c \to D_\infty} \Psi(c),
\]
exist. If $-\infty < D_1^+$, then (i) and (ii) of Lemma V.18 imply
\[
r_1^+ = \Psi(D_1^+) = D_1^+ = \partial^+ \tilde{\psi}(0), \tag{V.137}
\]
If $D_\infty < +\infty$ then
\[
r_\infty = \Psi(D_\infty) \geq \lim_{\alpha \to +\infty} \alpha D_{\alpha} - \psi(\alpha) = \lim_{u \to 1} \frac{\tilde{\psi}(u) - D_{\infty}}{u - 1} = \partial^- \tilde{\psi}(1). \tag{V.138}
\]
It is easy to see that if $\tilde{\psi}$ is convex then the inequality in (V.138) is an equality, i.e., $r_\infty = \partial^- \tilde{\psi}(1)$, in complete analogy with $r_1^+ = \partial^+ \tilde{\psi}(0)$ in (V.137); see also (II.26).
Proposition V.19 In the above setting assume that $D_1^+ > -\infty$.

(i) $\tilde{\Psi}$ is a finite-valued, monotone increasing, convex, and continuous function on $\mathbb{R}$.

(ii) $\tilde{\Psi}(r) \geq 0$, and $\tilde{\Psi}(r) = 0 \iff r \leq r_1^+$.

(iii) The Legendre-Fenchel transform of $\tilde{\psi}$ can be expressed as

\[
\tilde{\Psi}(r) = u_r r - \tilde{\psi}(u_r) = \begin{cases} 0, & r \leq r_1^+, \\ r - \Psi^{-1}(r) = \Psi^-(\Psi^{-1}(r)), & r \in (r_1^+, r_\infty), \\ -D_\infty, & r \geq r_\infty, \end{cases}
\]  

(V.139)

where $u_r := \alpha r - \tilde{\psi}(u_r) \in (0, 1)$ with $\alpha r - \tilde{\psi}(u_r) \in (1, +\infty)$ as in (vi) of Lemma V.18 when $r \in (r_1^+, r_\infty)$, and $u_r := 1$ when $r \geq r_\infty$. This proves (ii) and the first case in (V.139).

(iv) For every $r \in (r_1^+, r_\infty)$ there exist $0 < u_{r,\min} < u_{r,\max} < 1$ such that

\[
\sup_{u \in [0,1] \setminus [u_{r,\min}, u_{r,\max}]} \{ ur - \tilde{\psi}(u) \} < \tilde{\Psi}(r).
\]  

(V.140)

(v) For every $r > r_\infty$, and every $u \in (0, 1)$, $ur - \tilde{\psi}(u) < \tilde{\Psi}(r)$.

Proof The assertion in (i) is immediate from the definition.

Clearly, $\tilde{\Psi}(r) \geq 0 \cdot r - \tilde{\psi}(0) = 0$. For any $u \in [0, 1)$, $ur - \tilde{\psi}(u) = u(r - \tilde{\psi}(u)/u) \leq 0$ if $r \leq r_1^+ = D_1^+ = \inf_{u \in [0, 1]} \frac{r - \tilde{\psi}(u)}{u}$, showing that $\tilde{\Psi}(r) = 0$. By the same reasoning, $ur - \tilde{\psi}(u) > 0$ for some $u \in (0, 1]$ if $r > r_1^+$, showing that $\tilde{\Psi}(r) > 0$. This proves (ii) and the first case in (V.139).

Assume next that $r \in (r_1^+, r_\infty)$, and let $c_r := \Psi^{-1}(r)$. By the assumption on $r$, $c_r \in (D_1^+, D_\infty)$. Let $\alpha_{c_r}$ be as in (vi) of Lemma V.18. For any $\alpha \geq 1$,

\[
\Psi^-(c_r) - \left[ \frac{\alpha - 1}{\alpha} r - \frac{\psi(\alpha)}{\alpha} \right] = \frac{1}{\alpha} \left[ \Psi^-(c_r) - (c_r (\alpha - 1) - \psi(\alpha)) \right] \geq 0,
\]  

(V.141)

where the equality follows by substituting $r = \Psi(c_r) = \Psi^-(c_r) + c_r$. Taking the limit $\alpha \nearrow \alpha_{c_r}$ yields

\[
\Psi^-(c_r) - \left[ \frac{\alpha_{c_r} - 1}{\alpha_{c_r}} r - \frac{\psi(\alpha_{c_r})}{\alpha_{c_r}} \right] = \frac{1}{\alpha_{c_r}} \left[ \Psi^-(c_r) - (c_r (\alpha_{c_r} - 1) - \psi(\alpha_{c_r})) \right] = 0,
\]

where the second equality is due to (vi) of Lemma V.18. This shows that

\[
\Psi^-(c_r) = \sup_{0 \leq u < 1} \left[ \frac{\alpha - 1}{\alpha} r - \frac{\psi(\alpha)}{\alpha} \right] = \frac{1}{\alpha_{c_r}} \left[ \Psi^-(c_r) - (c_r (\alpha_{c_r} - 1) - \psi(\alpha_{c_r})) \right] \geq \frac{\Psi^-(c_r)}{2\alpha_{c_r,\min}} > 0.
\]

(V.142)

On the other hand, let $\delta > 0$ be such that $c_r + \delta < D_\infty$, and define $\alpha_{c_r,\delta} := \sup \{ \alpha \geq 1 : \frac{\psi(\alpha)}{\alpha - 1} \leq c_r + \delta \}$, $u_{r,\max} := (\alpha_{c_r,\delta} - 1)/\alpha_{c_r,\delta}$. Then $1 < \alpha_{c_r,\min} < \alpha_{c_r,\delta} < +\infty$ and thus $0 < u_{r,\min} < u_{r,\max} < 1$. By (V.141), for any $u > u_{r,\max}$, and corresponding $\alpha = 1/(1 - u) > \alpha_{c_r,\delta}$,

\[
\tilde{\Psi}(r) - (ur - \tilde{\psi}(u)) = \Psi^-(c_r) - \left[ \frac{\alpha - 1}{\alpha} r - \frac{\psi(\alpha)}{\alpha} \right] > \frac{1}{\alpha} \left[ \Psi^-(c_r) - (c_r (\alpha - 1) - \psi(\alpha)) \right] > \frac{1}{\alpha} \Psi^-(c_r) + \frac{\alpha - 1}{\alpha} \delta > \left( 1 - \frac{1}{\alpha_{c_r,\delta}} \right) \delta > 0.
\]
This proves (V.140).

The third case in (V.139) is only interesting when \( r_\infty < +\infty \) (since otherwise there is nothing to prove), in which case we have \( D_\infty = \lim_{r \to D_\infty} c \leq \lim_{r \to D_\infty} \Psi(c) = r_\infty < +\infty \), and thus \( r_\infty = \Psi(D_\infty) \), according to Lemma V.18. This yields that for any \( \alpha \geq 1 \), \( r \geq \Psi(D_\infty) \geq \alpha D_\infty - \psi(\alpha) \), whence \( D_\infty \leq (r + \psi(\alpha))/\alpha \), and thus,

\[
r - D_\infty \geq \sup_{\alpha \geq 1} \left( \frac{\alpha - 1}{\alpha} r - \frac{\psi(\alpha)}{\alpha} \right) = \sup_{0 < \alpha < 1} \{ur - \tilde{\psi}(u)\}.
\]

Obviously, \( r - D_\infty = \lim_{u \to \infty} \{ur - \tilde{\psi}(u)\} \leq \sup_{0 < \alpha < 1} \{ur - \tilde{\psi}(u)\} \), completing the proof in the third case in (V.139).

When \( r > r_\infty \) then we have the strict inequality \( r > \Psi(D_\infty) \geq \alpha D_\infty - \psi(\alpha) \), which yields the strict inequality \( r - D_\infty > \frac{\alpha - 1}{\alpha} r - \frac{\psi(\alpha)}{\alpha} \) for any \( \alpha \geq 1 \), proving (v).

\[\square\]

**Corollary V.20** Let \( \mathcal{H} \) be a finite-dimensional Hilbert space, and \( R, S \subseteq S(\mathcal{H}) \) be compact subsets such that there exist \( \varrho \in R \) and \( \sigma \in S \) for which \( \varrho^0 \leq \sigma^0 \). For every \( \alpha \in [1, +\infty) \) and \( c \in \mathbb{R} \), let

\[
\psi(\alpha) := \inf_{\varrho \in R, \sigma \in S} \psi^*(\varrho\|\sigma|\alpha), \quad \Psi(c) := \sup_{\alpha \in [1, +\infty)} \{c\alpha - \psi(\alpha)\}.
\]

The following hold:

(i) \( \Psi \) is an increasing convex and continuous function, which is strictly increasing on \((-\infty, D_\infty^*(R\|S)]\).

(ii) The Hoeffding anti-divergence \( r \mapsto H_\alpha^+(R\|S) \) is convex, continuous and strictly increasing on \([D(R\|S), +\infty)\), and it can be expressed as

\[
H_\alpha^+(R\|S) = u_r r - \tilde{\psi}(\varrho_r\|\sigma_r|ur_r),
\]

\[
= \begin{cases} 0, & r \leq D(R\|S), \\ r - \Psi^{-1}(r) = \Psi^{-1}(\Psi^{-1}(r)), & r \in (D(R\|S), r_\infty(R\|S)), \\ r - D_\infty^*(R\|S), & r \geq r_\infty(R\|S), \end{cases}
\]

for some \( \varrho_r \in R, \sigma_r \in S \) and \( u_r \in [0, 1] \), where \( r_\infty(R\|S) := \Psi(D_\infty^*(R\|S)) \).

(iii) For any \( u_r \) satisfying the first equality in (V.142),

\[
r \leq D(R\|S) \Rightarrow u_r = 0, \quad r \in (D(R\|S), r_\infty(R\|S)) \Rightarrow u_r \in (0, 1), \quad r > r_\infty(R\|S) \Rightarrow u_r = 1.
\]

**Proof** It is clear that \( \psi \) is finite-valued, \( \psi(0) = 0 \), and \( \alpha \mapsto \frac{\psi(\alpha)}{\alpha - 1} = \inf_{\varrho \in R, \sigma \in S} \frac{\psi(\varrho\|\sigma|\alpha)}{\alpha - 1} = D_\alpha^+(R\|S) \) is monotone increasing. We have

\[
D_1^+ = \inf_{\alpha > 1} \frac{\psi(\alpha)}{\alpha - 1} = D(R\|S) \geq 0, \quad D_\infty = \sup_{\alpha > 1} \frac{\psi(\alpha)}{\alpha - 1} = D_\infty^*(R\|S) < +\infty,
\]

according to (V.126) and Remark V.12, and hence the conditions in Proposition V.19 are satisfied. The existence of \( \varrho_r, \sigma_r \) and \( u_r \) with the given properties is immediate from Corollary V.17, and the rest of the statements follow from Proposition V.19.

\[\square\]

**C. Classical systems**

1. **Two alternative hypotheses in a 2-dimensional classical system: Equality might fail**

   In this section we demonstrate by an explicit example that the equality \( sc_r(\varrho\|S) = \sup_{\sigma \in S} sc_r(\varrho\|\sigma) \) may fail to hold in the simplest possible setting, where the system is classical and 2-dimensional, and \( S \) has only two elements. The example may be considered as a hypothesis testing between a fair coin and two biased coins, given by

\[
\varrho, \sigma_1, \sigma_2 : \{h = \text{heads}, t = \text{tails}\} \to [0, 1], \quad \varrho(h) := \frac{1}{2}, \quad \sigma_1(h) := \frac{1}{4}, \quad \sigma_2(h) := \frac{3}{4}.
\]

One might suspect that the failure of equality is a consequence of the fact that \( \varrho \in \co(\{\sigma_1, \sigma_2\}) \). However, as we show below, equality fails even if we replace all states with the same states on some fixed tensor power \( k \), while \( \varrho^{\otimes k} \notin \co(\{\sigma_1^{\otimes k}, \sigma_2^{\otimes k}\}) \) for any \( k \geq 2 \), as one can easily verify.
Proposition V.21 For $\varrho, \sigma_1, \sigma_2$ as above, and arbitrary $k \in \mathbb{N}$,

$$\text{sc}_r(\varrho^\otimes k||\{\sigma_1^\otimes k, \sigma_2^\otimes k\}) - \max_{j \in \{1,2\}} \text{sc}_r(\varrho^\otimes k||\sigma_j^\otimes k) \begin{cases} > 0, & r > D(\varrho^\otimes k||\sigma_j^\otimes k) = k \log \frac{2}{\sqrt{3}}; \\ \geq k \log \sqrt{3}, & r \geq k \log 4. \end{cases}$$

Proof Let $\hat{\varrho} := \varrho^\otimes k$, $\hat{\sigma}_1 := \sigma_1^\otimes k$, $\hat{\sigma}_2 := \sigma_2^\otimes k$. By Lemma II.10, $\psi(\hat{\varrho}||\hat{\sigma}_j|\cdot)$ is strictly convex, and hence

$$k \log \frac{2}{\sqrt{3}} = D(\hat{\varrho}||\hat{\sigma}_j) < D^*_n(\hat{\varrho}||\hat{\sigma}_j) < D^*_\infty(\hat{\varrho}||\hat{\sigma}_j) = k \log 2 < r_\infty(\hat{\varrho}||\hat{\sigma}_j) = k \log 4,$$

for $j = 1, 2$, and any $\alpha \in (1, +\infty)$, where the explicit values follow by a straightforward computation (using (II.26) for $r_\infty$). In particular,

$$\text{sc}_r(\hat{\varrho}||\hat{\sigma}_j) = H^*_r(\hat{\varrho}||\hat{\sigma}_j) = \max_{\alpha \in [1, +\infty]} \frac{\alpha - 1}{\alpha} [r - D^*_n(\hat{\varrho}||\hat{\sigma}_j)] < r - D(\hat{\varrho}||\hat{\sigma}_j) = r - k \log \frac{2}{\sqrt{3}} \quad (V.143)$$

for any $r > D(\hat{\varrho}||\hat{\sigma}_j)$, where the first equality is due to (V.11), and the strict inequality is straightforward to verify.

For any sequence of tests $(T_n)_{n \in \mathbb{N}}$, we have

$$1 - \alpha_n(H_0|T_n) = \sum_{x \in \{h, t\}^k} \hat{\varrho}^\otimes n(x) T_n(x) \leq \max_{\mathcal{H} \in \{h, t\}^k} \left( \frac{\hat{\varrho}^\otimes n(x)}{\hat{\sigma}_1^\otimes n(x) + \hat{\sigma}_2^\otimes n(x)} \right) \sum_{\mathcal{H}} \left( \hat{\varrho}^\otimes n(x) + \hat{\sigma}_2^\otimes n(x) \right) T_n(x) \leq \frac{1}{2} \left( \frac{2}{\sqrt{3}} \right)^k 2\beta_n(H_1|T_n), \quad (V.144)$$

where the first two inequalities are obvious, and the last inequality follows by $\hat{\varrho}^\otimes n(x) = 1/2^kn$, $x \in \{h, t\}^k$, and an application of the inequality between the arithmetic and geometric means, as

$$\hat{\sigma}_1^\otimes n(x) + \hat{\sigma}_2^\otimes n(x) = \left( \frac{1}{4} \right)^m \left( \frac{3}{4} \right)^{kn-m} + \left( \frac{3}{4} \right)^m \left( \frac{1}{4} \right)^{kn-m} \geq 2 \sqrt{\left( \frac{1}{4} \right)^m \left( \frac{3}{4} \right)^{kn-m} \left( \frac{3}{4} \right)^m \left( \frac{1}{4} \right)^{kn-m}} = 2 \left( \frac{\sqrt{3}}{4} \right)^k,$$

where $m := |\{i : x_i = h\}|$. Hence, if $\liminf_{n \to +\infty} -\frac{1}{n} \log \beta_n(H_1|T_n) \geq r$ then (V.144) yields

$$\liminf_n -\frac{1}{n} \log(1 - \alpha_n(H_0|T_n)) \geq k \log \frac{\sqrt{3}}{2} + \liminf_n -\frac{1}{n} \log \beta_n(H_1|T_n) \geq k \log \frac{\sqrt{3}}{2} + r = r - D(\hat{\varrho}||\hat{\sigma}_j). \quad (V.145)$$

Hence,

$$\text{sc}_r(\hat{\varrho}||\{\hat{\sigma}_1, \hat{\sigma}_2\}) \geq r - D(\hat{\varrho}||\hat{\sigma}_j) > H^*_r(\hat{\varrho}||\hat{\sigma}_j) = \text{sc}_r(\hat{\varrho}||\hat{\sigma}_j),$$

where the second inequality holds for $r > D(\hat{\varrho}||\hat{\sigma}_j)$, according to (V.143). Finally, when $r \geq k \log 4 = r_\infty(\hat{\varrho}||\{\hat{\sigma}_1, \hat{\sigma}_2\})$, we have

$$\text{sc}_r(\hat{\varrho}||\{\hat{\sigma}_1, \hat{\sigma}_2\}) - \max_{j \in \{1,2\}} \text{sc}_r(\hat{\varrho}||\hat{\sigma}_j) \geq r - D(\hat{\varrho}||\hat{\sigma}_j) - (r - D^*_\infty(\hat{\varrho}||\hat{\sigma}_j)) = D^*_\infty(\hat{\varrho}||\hat{\sigma}_j) - D(\hat{\varrho}||\hat{\sigma}_j) = k \log \sqrt{3},$$

where the inequality is again due to (V.145), and we used Corollary V.20.

2. Composite iid vs adversarial: Equality for the relaxed exponent

Below we consider classical state discrimination problems where the alternative hypothesis may be given in the composite i.i.d., the arbitrarily varying, or in the adversarial setting, and use the notations $\text{sc}_r$, $\text{sc}_{rav}$, $\text{sc}_{rav}$ for the strong converse exponents, similarly to the notations introduced in Section III.C for the direct exponents. Theorem V.22 below is an analogy of Theorem III.7 in the case where the null hypothesis is simple.
Theorem V.22 Let $X$ be a finite set, $\varrho \in S(X)$, and $S \subseteq S(X)$. Assume that $D(\varrho || S) < D^*_\infty (\varrho || S)$. Then for any $r > 0$,

$$H^*_r (\varrho || S) = \sup_{\sigma \in \overline{S}} \, sc_r (\varrho || \sigma)$$

(V.146)

$$\geq \, sc^{adv}_r (\varrho || \overline{S}) \geq sc^{adv}_r (\varrho || S)$$

(V.147)

$$\geq \, sc^\sigma_r (\varrho || S) = sc^\sigma_r (\varrho || \overline{S})$$

(V.148)

$$\geq \, sc_r (\varrho || \overline{S}) \geq \hat{s}^{\{1\}_r} (\varrho || \overline{S})$$

(V.149)

$$\geq \sup_{\sigma \in \overline{S}} \, sc_r (\varrho || \sigma),$$

(V.150)

and hence all the inequalities above hold as equalities.

Proof All the inequalities are obvious, except for the first one in (V.147), the equality in (V.148) holds for the same reason as in Theorem III.7, and the equality in (V.146) is due to Lemma V.11. Hence, we only have to prove the first inequality in (V.147), and therefore we may assume that $S$ is convex and compact, so that $\overline{S} = S$, to simplify notation. Also, we may assume that $r > D(\varrho || S)$, since otherwise $sc^{adv}_r (\varrho || S) = 0$, according to Theorem III.7. Finally, we may assume that $\varrho \notin S$, since otherwise $H^*_r (\varrho || S)$ is easily seen to be equal to $r$, which is indeed an upper bound to $sc^{adv}_r (\varrho || \overline{S})$, as demonstrated by the test sequence $T_n := e^{-nr} I$, $n \in \mathbb{N}$.

Consider first the case $r \in (D(\varrho || S), r_\infty (\varrho || S))$, where we use the notations of Corollary V.20. Let $\varrho_r = \varrho, \sigma_r$, and $u_r \in (0,1)$ be as in Corollary V.17 and Corollary V.20. $r > D(\varrho || S)$ implies that $0 < H^*_r (\varrho || \sigma_r) = H^*_r (\varrho || \sigma_r)$, and hence, in turn, $D(\varrho || \sigma_r) < r$, according to (V.127). In particular, $\varrho^0 \leq \sigma^0_r$. Moreover, $u_r < 1$ implies that $r \leq r_\infty (\varrho || \sigma_r)$, according to Corollary V.20 applied to the sets $\{\varrho\}$ and $\{\sigma_r\}$. Thus, $D(\varrho || \sigma_r) < r \leq r_\infty (\varrho || \sigma_r)$. Therefore, by Lemma II.10 and Remark II.11, $\hat{\psi} (\varrho || |\sigma_r|)$ is strictly convex, and thus $\hat{\psi} (\varrho || |\sigma_r|) : = (t - r) - \hat{\psi} (\varrho || |\sigma_r|)$ is strictly concave. This implies that $u_r$ is the unique maximizer of $\hat{\psi} (\varrho || |\sigma_r|)$, and that for any small enough $\varepsilon > 0$, there exist $0 < u_{r,-} < u_r < u_{r,+} < 1$ such that $\hat{\psi} (\varrho || |\sigma_r| u_{r,-}) = H^*_r (\varrho || \sigma_r) - \varepsilon = \hat{\psi} (\varrho || |\sigma_r| u_{r,+})$, and

$$\hat{\psi} (\varrho || |\sigma_r| u) < H^*_r (\varrho || \sigma_r) - \varepsilon, \quad u \in [0,1] \setminus \{u_{r,-}, u_{r,+}\}. \tag{V.151}$$

For any fixed $\sigma \in S$ and $t \in [0,1]$, let $\sigma^{(t)} := (1 - t)\sigma + t \sigma \in S$. Applying Corollary V.20 to the sets $\{\varrho\}$ and $\{\sigma^{(t)}\}$, we get the existence of some $t(\varrho) \in [0,1]$ such that $H^*_r (\varrho || |\sigma^{(t)}|) = r u(t) - \hat{\psi} (\varrho || |\sigma^{(t)}| u(t)) = \max_{u \in [0,1]} \hat{\psi} (\varrho || |\sigma^{(t)}| u)$, where $\hat{\psi} (\varrho || |\sigma^{(t)}| u) := u r - \hat{\psi} (\varrho || |\sigma^{(t)}| u)$. As we have seen above, $\varrho^0 \leq \sigma^0_r$, and hence $\lim_{t \to 0} \hat{\psi} (\varrho || |\sigma^{(t)}| u) = \hat{\psi} (\varrho || |\sigma_r| u)$ for every $u \in [0,1]$. However, since all $\hat{\psi} (\varrho || |\sigma^{(t)}| u)$ are concave, the convergence is actually uniform in $u$, i.e., for every $\varepsilon > 0$ there exists a $t_{\varepsilon} > 0$ such that for all $t \in [0, t_{\varepsilon})$,

$$\max_{u \in [0,1]} \left| \hat{\psi} (\varrho || |\sigma^{(t)}| u) - \hat{\psi} (\varrho || |\sigma_r| u) \right| < \varepsilon. \tag{V.152}$$

Combining (V.151) and (V.152) yields

$$\sup_{u \in [0,1] \setminus \{u_{r,-}, u_{r,+}\}} \hat{\psi} (\varrho || |\sigma^{(t)}| u) \leq \sup_{u \in [0,1] \setminus \{u_{r,-}, u_{r,+}\}} \hat{\psi} (\varrho || |\sigma_r| u) + \varepsilon/3 \leq H^*_r (\varrho || \sigma_r) - 2\varepsilon/3,$n

$$\hat{\psi} (\varrho || |\sigma^{(t)}| u) \geq \hat{\psi} (\varrho || |\sigma_r| u) - \varepsilon/3 = H^*_r (\varrho || \sigma_r) - \varepsilon/3.$$

This implies that $u(t) \in [u_{r,-}, u_{r,+}]$ for every $t < t_{\varepsilon}$. Since $\lim_{t \to 0} u_{r,-} = \lim_{t \to 0} u_{r,+} = u_r$, we finally obtain that

$$\lim_{t \to 0} u(t) = u_r.$$n

This also implies that for every small enough $t$, $\hat{\psi} (\varrho || |\sigma^{(t)}| u) < H^*_r (\varrho || |\sigma^{(t)}|)$, and hence $r \leq r_\infty (\varrho || |\sigma^{(t)}|)$, according to Corollary V.20. Using again that $\varrho^0 \leq \sigma^0_r$, we see that $\lim_{t \to 0} D(\varrho || |\sigma^{(t)}|) = D(\varrho || \sigma_r) < r$. Thus, for every small enough $t$, $D(\varrho || |\sigma^{(t)}|) < r \leq r_\infty (\varrho || |\sigma^{(t)}|)$, and therefore $\frac{d}{du} \hat{\psi} (\varrho || |\sigma^{(t)}| u) < 0$ for all $u \in (0,1)$, according to Lemma II.10 and Remark II.11.

Since $u(t)$ is the unique solution of $\frac{d}{du} \hat{\psi} (\varrho || |\sigma^{(t)}| u) = 0$, the implicit function theorem implies that $t \mapsto u(t)$ is differentiable on $(0,t_0)$ for some $t_0 \in (0,1)$, and hence so is $t \mapsto \hat{\psi} (\varrho || |\sigma^{(t)}| u(t))$. Therefore, the same computation as
in the proof of Theorem III.8 yields that

\[
0 \geq \lim_{t \searrow 0} \frac{H^*_r(q(\sigma(t))) - H^*_r(q(\sigma_r))}{t} = \lim_{t \searrow 0} \frac{\dot{\psi}(q(\sigma(t))|u(t)) - \dot{\psi}(q(\sigma_r)|u_r)}{t} = \lim_{t \searrow 0} \frac{d}{dt} \dot{\psi}(q(\sigma(t))|u(t)) = u(t) \sum_x q(x)^{1-\nu}(\sigma(t)(x)^{-1/\nu}(\sigma(x) - \sigma_r(x))) \sum_x q(x)^{1-\nu} \sigma_r(x)^{-1/\nu}(\sigma(x) - \sigma_r(x)),
\]

where in the last step we used that \(\lim_{t \searrow 0} u(t) = u_r\), as shown above. Rearranging, we get

\[
\sum_x \sigma(x) \left( \frac{q(x)}{\sigma_r(x)} \right)^{\alpha_r} \leq \sum_x \sigma_r(x) \left( \frac{q(x)}{\sigma_r(x)} \right)^{\alpha_r},
\]

where \(\alpha_r = 1/(1-u_r)\). Since this holds for every \(\sigma \in S\), the same argument as in (III.88)–(III.89) yields that

\[
\beta_n(T_{n,r}) \leq e^{-nr},
\]  
(V.153)

where the test \(T_{n,r}\) is defined as in (III.87), with \(\vartheta_r = \vartheta\).

To estimate the type I success probability from below, note that the logarithmic moment generating function of

\[
\log \frac{\hat{\vartheta}(\vartheta \sigma_r)}{\sigma_r}\text{ w.r.t. } \vartheta
\]

is defined as in (III.90), with \(\vartheta \sigma_r = \vartheta_r\). The Legendre-Fenchel transform of \(\Lambda\) at \(c_r = (\vartheta)^{(\alpha_r)}\) is

\[
\Lambda^{\circ}(c_r) := \sup_{\alpha \in \mathbb{R}} \{c_r \alpha - \psi(\alpha + 1)\} = \sup_{\alpha \in \mathbb{R}} \{c_r \alpha - \psi(\alpha)\} = \Psi(c_r) - c_r = H^*_r(q(\sigma_r)) = H^*_r(q(S)),
\]

according to (V.142) and Lemma II.12. Hence, by Cramér’s large deviation theorem [1], we get that

\[
\limsup_{n \to +\infty} -\frac{1}{n} \log (1 - \alpha_n(T_{n,r})) \leq \limsup_{n \to +\infty} -\frac{1}{n} \log \hat{\vartheta}^{\circ n}(T_{n,r}) \leq \Psi(c_r) - c_r = H^*_r(q(S)).
\]  
(V.154)

Putting together (V.153) and (V.154) yields

\[
\text{sc}_{\vartheta}^{\text{adv}}(q(S)) \leq H^*_r(q(S)).
\]

Assume next that \(r \geq r_\infty(q(S))\). In this case the Neyman-Pearson tests may not give the optimal exponent, and a different approach is needed, which was originally introduced in [45, Theorem 4]; see [40, Theorem 4.10] for details. Let \(\psi(\alpha) := \inf_{\sigma \in S} \psi(\vartheta \sigma_r | \sigma_r)\), and \(\Psi(c) := \sup_{\alpha \geq 1} \{c \alpha - \psi(\alpha)\}\), \(\alpha, c \in \mathbb{R}\). For every \(c \in (D(q(S)), D_\infty(q(S)))\), let \(r_c := \Psi(c)\), and \(T_{n,r_c}\) be the test as above. By the strict monotonicity of \(\Psi\) (Corollary V.20), \(\Psi(c) < \Psi(D_\infty(q(S))) = r_\infty(q(S)) \leq r\), and hence

\[
T_{n}(r,c) := e^{-n(r-\Psi(c))} T_{n,r_c}
\]

is a test. By (V.153) and (V.154),

\[
\beta_n(T_{n}(r,c)) = e^{-n(r-\Psi(c))} \beta_n(T_{n,r_c}) \leq e^{-n(r-\Psi(c))} e^{-nr_c} = e^{-nr}, \quad n \in \mathbb{N},
\]

\[
\lim_{n \to +\infty} -\frac{1}{n} \log (1 - \alpha_n(T_{n}(r,c))) = r - \Psi(c) + \lim_{n \to +\infty} -\frac{1}{n} \log (1 - \alpha_n(T_{n,r_c}))
\]

\[
\leq r - \Psi(c) + (\Psi(c) - c) = r - c.
\]

Hence,

\[
\text{sc}_{\vartheta}^{\text{adv}}(q(S)) \leq \inf_{c \in (D(q(\sigma_r)), D_\infty(q(\sigma_r)))} \{r - c\} = r - D^*_\infty(q(S)) = H^*_r(q(S)),
\]

where the equality is due to Corollary V.20.

Theorem V.22 and Corollary V.7 yield immediately the following:
Corollary V.23 Let $\mathcal{X}$ be a finite set, and $R, S \subseteq \mathcal{S}(\mathcal{X})$ be such that $D(\varrho\|S) < D^*_\varrho(S)$ for every $\varrho \in R$. Then for any $r > 0$,

$$H^*_r(R\|\mathcal{C}(S)) = \sup_{\varrho \in R, \sigma \in \mathcal{C}(S)} \text{sc}_r(\varrho\|\sigma)$$

(V.155)

$$\geq \text{sc}_{r(0)}(R\left\|\mathcal{C}(S)_{\text{adv}}(n)\right\|_{n \in \mathbb{N}}) \geq \text{sc}_{r(0)}(R\left\|\mathcal{C}(S)_{\text{adv}}(n)\right\|_{n \in \mathbb{N}})$$

(V.156)

$$\geq \text{sc}_{r(0)}(R\left\|\mathcal{C}(S)_{\text{adv}}(n)\right\|_{n \in \mathbb{N}}) = \text{sc}_{r(0)}(R\left\|\mathcal{C}(S)_{\text{adv}}(n)\right\|_{n \in \mathbb{N}})$$

(V.157)

$$\geq \text{sc}_{r(0)}(R\left\|\mathcal{C}(S)\right\|_{n \in \mathbb{N}}) = \text{sc}_{r(0)}(R\left\|\mathcal{C}(S)\right\|_{n \in \mathbb{N}})$$

(V.158)

$$\geq \sup_{\varrho \in R, \sigma \in \mathcal{C}(S)} \text{sc}_r(\varrho\|\sigma),$$

(V.159)

and hence all the inequalities above hold as equalities.

Proof The equality in (V.155) is due to Lemma V.11, and the equality in (V.157) follows the same way as in Theorem III.7. All the inequalities are obvious except for the first one in (V.156), for which we have

$$\text{sc}_{r(0)}(R\left\|\mathcal{C}(S)\right\|_{n \in \mathbb{N}}) \leq \inf_{r' > r} \sup_{\varrho \in R} \text{sc}_{r'}(\varrho\left\|\mathcal{C}(S)\right\|_{n \in \mathbb{N}}) \leq \inf_{r' > r} \sup_{\varrho \in R} H^*_r(\varrho\|\mathcal{C}(S)) = \inf_{r' > r} H^*_r(R\|\mathcal{C}(S)) = H^*_r(R\|\mathcal{C}(S)).$$

The first inequality above is due to Corollary V.7, the second inequality follows from Theorem V.22, the first equality is by definition, and the last equality follows from the continuity of $r \mapsto H^*_r(R\|\mathcal{C}(S))$; see Corollary V.20. □

3. Composite i.i.d. versus simple i.i.d.: Equality for a restricted parameter range

For a sequence $x \in \mathcal{X}^n$, let $P_x$ denote its empirical distribution, or type, defined by $P_x(y) := \frac{1}{n} \{i : x_i = y\}, y \in \mathcal{X}$. For any $n \in \mathbb{N}$, let $P_n := \{P_x : x \in \mathcal{X}^n\}$ be the set of $n$-types. It is clear that any probability distribution on $\mathcal{X}$ can be arbitrarily well approximated by $n$-types for sufficiently large $n$. We will need the following refined statement, whose proof follows by a simple modification of that of Lemma A.2 in [29]. We give a detailed proof in Appendix A for readers’ convenience.

Lemma V.24 Let $\varrho \in \mathcal{S}(\mathcal{X}) \cap A_{v,c}$, where $A_{v,c} := \{f \in \mathbb{R}^{\mathcal{X}} : \sum_{x \in \mathcal{X}} v(x)f(x) \geq c\}$ is a half-space, and let $r := |\text{supp}\varrho|$. For every $n \geq r(r-1)$, there exists a $\varrho_n \in P_n \cap A_{v,c}$ such that $\varrho_n^0 \leq \varrho^0$ and $\|\varrho - \varrho_n\|_1 \leq \frac{2(r-1)}{n}$. We will also need the following continuity bound for the relative entropy, which is a simple consequence of an analogous continuity bound for the entropy [6, 16, 55]:

Lemma V.25 Let $\varrho_1, \varrho_2, \sigma \in \mathcal{S}(\mathcal{X})$ be such that $\text{supp}\varrho_1 \cup \text{supp}\varrho_2 \subseteq \text{supp}\sigma$. Then

$$|D(\varrho_1\|\sigma) - D(\varrho_2\|\sigma)| \leq \frac{1}{2} \|\varrho_1 - \varrho_2\|_1 \log(|\mathcal{X}| - 1) + h_2\left(\frac{1}{2} \|\varrho_1 - \varrho_2\|_1\right) + \|\varrho_1 - \varrho_2\|_1 \left(\log \min_{x \in \text{supp}\varrho_1 \cup \text{supp}\varrho_2} \sigma(x)\right),$$

where $h_2(x) := -x \log x - (1-x) \log(1-x), x \in [0, 1]$.

Theorem V.26 Consider the classical i.i.d state discrimination problem with composite null hypothesis $R \subseteq \mathcal{S}(\mathcal{X})$ and simple alternative hypothesis $\sigma \in \mathcal{S}(\mathcal{X})$. Assume that

$$\delta_{\text{min}}(R) := \sup_{\varrho \in R} (\varrho - \log \min_{x \in \text{supp}\varrho} \varrho(x)) < +\infty.$$

Then, for any $D(R\|\sigma) < r < \inf_{\varrho \in R} \varrho_{\infty}(\varrho\|\sigma)$,

$$\text{sc}_r(R\|\sigma) = H^*_r(R\|\sigma) = \sup_{\varrho \in R} \text{sc}_r(\varrho\|\sigma).$$

(V.160)
\textbf{Proof} The second equality in (V.160) is immediate from Lemma V.11, and hence we only need to prove the first one. For a fixed $r$ as in the statement, and $n \in \mathbb{N}$, consider the test
\[ B_{n,r} := \{ x \in X^n : D(P_x \| \sigma) \geq r \}. \]
Then, by a standard type estimate (see, e.g., Eq. (2.1.12) in [1]),
\[ \beta_n(B_{n,r}) \leq (n + 1)^{\|X\|} e^{-n \inf_{\omega \in P_n} D(\omega \| \sigma) \geq r} D(\omega \| \sigma) \leq (n + 1)^{\|X\|} e^{-nr}. \] (V.161)

Assume that $\rho \in R$ is such that $D(\rho \| \sigma) \geq r$. Since \{ $\omega \in \mathcal{S}(X) : D(\omega \| \sigma) \leq r$ \} is a convex set, it can be separated from $\rho$ by a hyperplane, i.e., $\rho \in A_{n,c}$ for some $v \in \mathbb{R}^X$ and $c \in \mathbb{R}$, and \{ $\omega \in \mathcal{S}(X) : D(\omega \| \sigma) \leq r$ \} $\subseteq A_{-v,-c}$. For every $n \geq |X|^2$, let $\rho_n \in \mathcal{S}(X) \cap A_{n,c}$ as in Lemma V.24. Then $D(\rho_n \| \sigma) \geq r$, and thus
\[ \varrho_n(B_{n,r}) \geq \varrho_n(\{ x \in X^n : P_x = \rho_n \}) \geq (n + 1)^{\|X\|} e^{-n D(\rho_n \| \sigma)}, \] (V.162)
where the last inequality follows by another well-known type estimate [13, Lemma 2.6]. By lemma V.25,
\[ D(\rho_n \| \sigma) \leq D(\varrho \| \sigma) = \frac{|X|^2}{n} + h_2 \left( \frac{|X|}{n} \right) + \frac{2|X| \delta_{\min}(R)}{n}, \] (V.163)
where we used (V.127). Combining (V.162) and (V.163), we get
\[ \frac{1}{n} \log \varrho_n(B_{n,r}) \leq H^*_r(\varrho \| \sigma) + |X| \log(n + 1) + |X|^2 \frac{|X|}{n} + h_2 \left( \frac{|X|}{n} \right) + \frac{2|X| \delta_{\min}(R)}{n}. \] (V.164)

Consider now a $\rho \in R$ such that $D(\rho \| \sigma) < r$. By assumption, we also have $r < r_\infty(\rho \| \sigma)$, and hence, by Lemma II.12, there exists a unique $\alpha \in (1, +\infty)$ such that $D(\mu_{\alpha,\rho,\sigma} \| \sigma) = r$, $D(\mu_{\alpha,\rho,\sigma} \| \rho) = H^*_r(\varrho \| \sigma)$. Again, $\mu_{\alpha,\rho,\sigma}$ can be separated by a hyperplane from \{ $\omega \in \mathcal{S}(X) : D(\omega \| \sigma) \leq r$ \}, and Lemma V.24 yields the existence of a $\mu_n \in P_n$ such that $D(\mu_n \| \sigma) \geq r$ and $\|\mu_n - \mu_{\alpha,\rho,\sigma}\|_1 \leq \frac{2|X|^2}{n}$, whenever $n \geq |X|^2$. By the same argument as above,
\[ \varrho_n(B_{n,r}) \geq \varrho_n(\{ x \in X^n : P_x = \mu_n \}) \geq (n + 1)^{\|X\|} e^{-n D(\mu_n \| \sigma)}, \] and

\[ D(\mu_n \| \rho) \leq D(\mu_{\alpha,\rho,\sigma} \| \rho) = H^*_r(\varrho \| \sigma) + |X| \log(n + 1) + |X|^2 \frac{|X|}{n} + h_2 \left( \frac{|X|}{n} \right) + \frac{2|X| (-\log \sigma_{\min})}{n}, \]
where $\sigma_{\min} := \min_{x \in \text{supp} \sigma} \sigma(x)$. Thus,
\[ \frac{1}{n} \log \varrho_n(B_{n,r}) \leq H^*_r(\varrho \| \sigma) + |X| \log(n + 1) + |X|^2 \frac{|X|}{n} + h_2 \left( \frac{|X|}{n} \right) + \frac{2|X| (-\log \sigma_{\min})}{n}. \] (V.165)

Combining (V.164) and (V.165) yields
\[ \frac{1}{n} \log \alpha_n(B_{n,r} \| R) = \sup_{\rho \in R} \frac{1}{n} \log \varrho_n(B_{n,r}) \leq H^*_r(R \| \sigma) + \frac{c}{n} + |X| \log(n + 1) + h_2 \left( \frac{|X|}{n} \right). \] (V.166)
for some positive constant $c$. Finally, (V.161) and (V.166) together yield $sc_\alpha(R \| \sigma) \leq H^*_r(R \| \sigma)$, as required. \qed

\textbf{Remark V.27} It is clear from its definition in (II.26) that $r_\infty(\rho \| \sigma) \geq D^*_\infty(\rho \| \sigma)$, and thus Theorem V.26 is valid, in particular, for every $r \in (D(R \| \sigma), D^*_\infty(R \| \sigma))$. This is typically a non-degenerate interval; an easily verifiable case where this holds is when $R$ is compact and $\alpha \mapsto \psi(\rho \| \sigma | \alpha)$ is not affine for a $\rho \in R$ such that $D^*_\infty(\rho \| \sigma) = D^*_\infty(R \| \sigma)$; see Lemma II.10.
VI. CONCLUSION

While the error exponents of simple binary i.i.d. state discrimination are by now completely characterized in terms of divergences of the two states representing the two hypotheses, the composite case seems to present far more open problems than definitive solutions so far. The results presented in this paper clarify the picture to some extent, but of course there are many problems still left open. The most obvious one is to find tractable expressions for the error exponents in cases where the equality \( E(R|S) = E(R||S) \) does not hold, or at least find some possibly sub-optimal but non-trivial and universally applicable lower bounds. Though a complete characterization of the Stein exponent in the finite-dimensional quantum case was given in [8], and general lower bounds are available as in Proposition II.30, these involve regularizations that are practically not feasible to solve explicitly, and call for the search of better computable lower bounds. One such bound was given in [25], where it was shown that \( c(q||S) \geq 1/2 C(q||S) \) when \( S \) is finite. It would be interesting to find similar bounds for more general settings and for other exponents, too.

It is interesting that even the finite-dimensional classical case seems to present some open problems. We are not aware of the equality problem \( E(R||S) = E(R|S) \) having been completely solved for the Stein, Chernoff, and the direct exponents without some constraint (finiteness, convexity) on the sets \( R, S \). On the other hand, it also seems to be open whether closedness and convexity of the sets always guarantees equality; in our classical counterexamples to equality in Sections III B and V C 1 the composite sets are non-convex. On the other hand, convexity is certainly not sufficient for equality in the quantum setting: an example with a closed convex alternative hypothesis and a simple null-hypothesis \( q \) was shown in [8] for which \( s(q||S) \neq D(q||S) \). It is quite likely that this example is also a counterexample to equality for the strong converse exponents with rates \( s(q||S) < r < D(q||S) \), since \( H_r^*(q||S) = 0 \) in this range; the missing part is whether \( s(c(q||S) > 0 \) when \( r > s(q||S) \), i.e., the exponential strong converse part of Stein’s lemma with composite alternative hypothesis and a simple null-hypothesis.

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Appendix A: Proof of Lemma V.24

**Proof** Let \( x_1, \ldots, x_r \) be an ordering of \( \text{supp } q \) and let \( p_i := q(x_i), v_i := v(x_i) \). If \( r = 1 \) then there is nothing to prove, and hence for the rest we assume the contrary. Obviously, there exists an \( i \) such that \( p_i \geq 1/r \); assume without loss of generality that \( p_r \geq 1/r \). For each \( i < r \), let

\[
 k_i := \begin{cases} 
 [np_i], & v_i > v_r, \\
 [np_i], & v_i \leq v_r,
\end{cases}
\]

so that \( (v_i - v_r)(k_i - np_i) \geq 0 \). (A.1)

Define \( k_r := n - (k_1 + \ldots + k_{r-1}) \). By the above,

\[
k_r \geq n - (np_1 + 1) - \ldots - (np_{r-1} + 1) = np_r - (r - 1) \geq \frac{n}{r} - (r - 1) = \frac{n - r(r - 1)}{r},
\]

and therefore \( k_r \geq 0 \) if \( n \geq r(r - 1) \). Thus, \( g_n(x_i) := k_i^n, i = 1, \ldots, r, g_n(x) := 0, x \in X \setminus \{x_1, \ldots, x_r\} \), defines an element of \( P_n \), and it is clear from the definition that \( g_n^0 \leq \theta^0 \). We have

\[
|k_r - np_r| = |n - k_1 - \ldots - k_{r-1} - n(1 - p_1 - \ldots - p_{r-1})| \leq \sum_{k=1}^{r-1} |k_i - np_i| \leq r - 1,
\]

and therefore

\[
\|g_n - \theta\|_1 = \sum_{k=1}^{r} |k_i/n - p_i| \leq \frac{2(r - 1)}{n}.
\]
Finally,

\[
0 \leq \sum_{i=1}^{r-1} (v_i - v_r)(k_i/n - p_i) = \sum_{i=1}^{r-1} v_i(k_i/n - p_i) - v_r \sum_{i=1}^{r-1} (k_i/n - p_i) = \sum_{i=1}^{r-1} v_i(k_i/n - p_i) - \sum_{i=1}^{r-1} v_i p_i \leq \sum_{x} v(x) \varrho_n(x) - c,
\]

where the first inequality is due to (A.1). Thus, \( \varrho_n \in A_{v,c} \). \( \square \)
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