A REMARK ON THE ACTIONS OF SOME GROUPS ON THE PRODUCT OF HADAMARD SPACES

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Abstract. For a product of Hadamard spaces \( X = X_1 \times X_2 \) on which some group \( G \subset Is(X_1) \times Is(X_2) \) acting, G. Link introduced the growth rate \( \delta_\theta \) of slope \( \theta \) to construct a \( G \)-invariant \((b, \theta)\)-density. She showed that \( \delta_\theta \) is upper semi-continuous in the slope \( \theta \).

First, we show that \( \delta_\theta \) is continuous in the slope \( \theta \) as the above \( G \) with some mild condition. The above result enables us to remove some assumptions of several main theorems of G. Link.

Second, we give a negative answer to a question raised by G. Link in \([3]\) in general. And further results about the question are discussed.

1. Introduction

Recall that a Hadamard space means a complete simply connected metric space with non-positive Alexandrov curvature. Let \( X = X_1 \times X_2 \) be a product of proper Hadamard spaces with the standard product metric, one can compactify \( X \), \( X_1 \) and \( X_2 \) by giving the corresponding canonical geometric boundaries \( \partial X \), \( \partial X_1 \) and \( \partial X_2 \) \((\ref{8}, \ref{11})\). A basic fact is that \( \partial_{reg} X = \partial X_1 \times \partial X_2 \times (0, \pi/2) \), \( \partial_{sing} X = \partial X_1 \cup \partial X_2 \) and \( \partial X = \partial_{reg} X \cup \partial_{sing} X \). The third component in the first term is called the slope of a point in \( \partial X_{reg} \). In particular, \( \partial X_1 \) is of slope 0 and \( \partial X_2 \) is of slope \( \pi/2 \).

Consider a group \( G \subset Is(X_1) \times Is(X_2) \) acting properly on the space \( X \), and \( G \) contains a pair of isometries which projects a pair of independent rank one elements in each factor and satisfies the regular growth condition (see the definition in the next section).

We establish the theorem as follows.

Theorem 1.1. The growth rate \( \delta_\theta \) of \( G \) with slope \( \theta \) is continuous in \( \theta \in [0, \pi/2] \), where \( G \) is as above.

By Theorem 1.1, we can get some important corollaries in the sequel.

Corollary 1.2. There exists some \( \theta^* \in [0, \pi/2] \) such that

\[
\delta_{\theta^*} = \max_{\theta \in [0, \pi/2]} \delta_{\theta}.
\]

Remark. In \([3]\), G. Link showed that such \( \theta^* \) is unique with \( \delta(G) = \delta_{\theta^*} \), where \( G \) is as above which does not necessarily satisfy the regular growth condition.

Combining some extra work with Theorem 1.1 one can propose some interesting results.

First, we can get an interesting result on \( \delta_\theta \).

Theorem 1.3. It holds that the growth rate \( \delta_\theta \) of the action \( G \rhd X = X_1 \times X_2 \) with slope \( \theta \) is positive for any \( \theta \in (0, \pi/2) \), where \( G \) is given as above.
We list some theorems of G. Link in [3], where the group $G$ does not necessarily satisfy the regular growth condition.

**Theorem 1.4** (Theorem B, [3]). If $\theta \in (0, \pi/2)$ is such that $\delta_\theta(G) > 0$, then there exists a $(b, \theta)$-density for some parameters $b = (b_1, b_2) \in \mathbb{R}^2$.

**Theorem 1.5** (Theorem C, [3]). If a $G$-invariant $(b, \theta)$-density exists for some $\theta \in (0, \pi/2)$, then $\delta_\theta \leq b_1 \cos \theta + b_2 \sin \theta$, where $b = (b_1, b_2) \in \mathbb{R}^2$.

**Theorem 1.6** (Theorem E, [3]). If $\nu$ is a $(b, \theta)$-density for $\theta \in (0, \pi/2)$ with $\delta_\theta > 0$, then a radial limit point is not a point mass for $\nu$.

Theorem 1.5 removes completely the assumption that $\delta_\theta > 0$ for some $\theta \in (0, \pi/2)$ of G. Link appeared in the above $(b, \theta)$-density existence theorems. We establish them in the following.

**Corollary 1.7.** For any $\theta \in (0, \pi/2)$, there exists a $(b, \theta)$-density with some parameters $b = (b_1, b_2) \in \mathbb{R}^2$.

**Corollary 1.8.** There exists a $(b, \theta)$-density for any $\theta \in (0, \pi/2)$ with $\delta_\theta \leq b_1 \cos \theta + b_2 \sin \theta$, where $b = (b_1, b_2) \in \mathbb{R}^2$.

**Corollary 1.9.** For any $\theta \in (0, \pi/2)$ and a $(b, \theta)$-density $\nu$, a radial limit point is never an atomic mass for $\nu$.

In [3], G. Link proposed a question as follows.

**Question 1.10.** For the unique $\theta_* \in [0, \pi/2]$ such that $\delta_{\theta_*} = \max_{\theta \in [0, \pi/2]} \delta_\theta$, does it hold that $\theta_* \in (0, \pi/2)$ not necessarily with the regular growth condition?

In the end, we give a negative answer to Question 1.10 by constructing an example such that $\theta_* = \pi/2$. But the above conclusion holds under a quite mild condition, the precise statement is the following.

**Theorem 1.11.** Assume that $(X, d) = (X_1, d_1) \times (X_2, d_2)$ is a product of Hadamard spaces on which a group $G \subset Is(X_1) \times Is(X_2)$ acts properly, $d$ is the standard product metric. If $G$ contains a pair of isometries which projects a pair of independent rank one elements in each factor and satisfies $\delta_\theta > 0$ for $\theta \in (0, \pi/2)$, with the following condition:

\[
\frac{\delta_{\pi/2}}{\delta_\theta} < \frac{1}{\sin \theta} \quad \text{and} \quad \frac{\delta_0}{\delta_{\pi/2-\beta}} < \frac{1}{\sin \beta},
\]

for some $\theta, \beta \in (0, \pi/2)$.

Then there exists some $\theta_* \in (0, \pi/2)$ such that $\delta_{\theta_*} = \max_{\theta \in [0, \pi/2]} \delta_\theta$.

Moreover, the upper bound in (1) is sharp. The interested reader is referred to see details in the last section.

## 2. Preliminaries

We say a metric space is proper if every ball of finite radius is compact. For a proper Hadamard space $Y$, one defines equivalent classes of geodesic rays by bounded Gromov-Hausdorff distance on account of giving the geometry boundary $\partial Y$ with cone topology. The point of this geometry boundary is a class of a geodesic...
ray and it is well-known that \( \hat{Y} = Y \cup \partial Y \), \( \partial Y \) are compact and \( Y \) is open in \( \hat{Y} \). The isometric action of \( G \sim X \) can be extended to a homeomorphic action of \( G \sim \partial Y \).

For a group \( G \) acting properly on a Hadamard space, an isometry \( g \in G \) is called rank one if it acts on a geodesic axis as a translation and the geodesic does not bound a half plane. Note that such \( g \) has two fixed points as the endpoints of its axis in \( \partial X \).

Given a set \( S \), we let \( |S| \) be the cardinality. Set

\[
B(o, n) = \{ g \in G : d(o, go) < n \}
\]

and

\[
A(o, n) = \{ g \in G : n - 1 \leq d(o, go) < n \}.
\]

Let

\[
\delta = \delta(G) := \limsup_{n \to \infty} \frac{\log |A(o, n)|}{n}
\]

and

\[
\hat{\delta} = \hat{\delta}(G) := \limsup_{n \to \infty} \frac{\log |B(o, n)|}{n}.
\]

We say that \( \delta \) (resp. \( \hat{\delta} \)) is the annuli (resp. balls) growth rate of the action \( G \sim X \). \( \delta \) is simply called the growth rate of the group \( G \) if the action is clear in the context.

Note that for a general group action \( G \sim X \) with \( |B(o, n)| < \infty \) for each \( n \in \mathbb{N}_{\geq 1} \), we have \( \hat{\delta} \geq \delta \) and \( \hat{\delta} \geq 0 \). However, \( \delta = -\infty \) if \( \limsup_{n \to \infty} |A(o, n)| = 0 \), and \( \delta \geq 0 \) if \( \limsup_{n \to \infty} |A(o, n)| \geq 1 \). It is straightforward that \( B(o, n) = \bigcup_{m=1}^{n} A(o, m) \). Consider the convergence radii of two formal positive series

\[
\sum_{n=1}^{\infty} |B(o, n)| z^n \quad \text{and} \quad \sum_{n=1}^{\infty} |A(o, n)| z^n, \quad z \in (0, \infty).
\]

One easily shows that \( \delta = \hat{\delta} \) if and only if \( \limsup_{n \to \infty} |A(o, n)| \geq 1 \).

An easy computation gives

\[
\delta = \inf \{ s : \sum_{g \in G} e^{-sd(o, go)} < \infty \}
\]

so one can also call \( \delta \) is the exponent of growth of \( G \).

In order to characterize the behavior of group \( G \) on the space \( X \) with slope \( \theta \in [0, \pi/2] \), one can introduce

\[
B_{\epsilon, \theta}(o, n) := \{ g \in G : d(o, go) < n, |d_2(o_2, g_2o_2) - d_1(0_1, g_1o_1) - \tan \theta| \leq \epsilon \},
\]

\[
A_{\epsilon, \theta}(o, n) := \{ g \in G : n - 1 \leq d(o, go) < n, |d_2(o_2, g_2o_2) - d_1(0_1, g_1o_1) - \tan \theta| \leq \epsilon \},
\]

where \( \epsilon > 0, n \in \mathbb{N}_{\geq 1} \). The \( \epsilon \)-close growth rate of \( G \) with slope \( \theta \) is defined as

\[
\delta_{\epsilon, \theta} = \limsup_{n \to \infty} \frac{\log |A_{\epsilon, \theta}(o, n)|}{n},
\]

and the \( \epsilon \)-close balls growth rate of \( G \) with slope \( \theta \) is defined as

\[
\hat{\delta}_{\epsilon, \theta} = \limsup_{n \to \infty} \frac{\log |B_{\epsilon, \theta}(o, n)|}{n}.
\]
Next we denote by $\delta_\theta = \liminf_{\epsilon \to 0} \delta_{\epsilon, \theta}$ the growth rate of $G$ with slope $\theta$, denote by $\hat{\delta}_\theta = \liminf_{\epsilon \to 0} \hat{\delta}_{\epsilon, \theta}$ the balls growth rate of $G$ with slope $\theta$.

Similarly, we have $\delta_\theta = \hat{\delta}_\theta$ if and only if $\limsup |A_{\epsilon, \theta}(o, n)| \geq 1$ for any $\epsilon > 0$.

**Definition 2.1.** We say that the action $G \act X$ satisfies the *regular growth condition* for $[\alpha_1, \alpha_2]$ if $\delta_\theta \geq 0$ for any $\theta \in [\alpha_1, \alpha_2] \subset [0, \pi/2]$, or equivalently $\limsup_{n \to \infty} |A_{\epsilon, \theta}(o, n)| \geq 1$ for any $\epsilon > 0$ and $\theta \in [\alpha_1, \alpha_2]$.

We specify the regular growth condition for $\alpha_1 = 0, \alpha_2 = \pi/2$.

**Remark.** If the action $G \act X$ satisfies the regular growth condition for $[\alpha_1, \alpha_2] \subset [0, \pi/2]$, it follows that $\delta_\theta = \hat{\delta}_\theta$ for $\theta \in [\alpha_1, \alpha_2]$ by the previous discussion.

G. Link defined a so-called $G$-invariant $(b, \theta)$-density for the product of Hadamard spaces $X$.

**Definition 2.2.** (Definition 1.2 [3]) Denote by $\mathcal{M}^+(\partial X)$ the cone of positive Borel measures on $\partial X$. Let $\theta \in [0, \pi/2]$ and $b = (b_1, b_2) \in \mathbb{R}^2$. A $G$-invariant $(b, \theta)$-density is a continuous map:

$$\nu : X \to \mathcal{M}^+(\partial X)$$

such that the following statements hold for any $x \in X$:

1. $\mathcal{L} \cap \partial X_{\theta} \ni \text{supp}(\nu_x) \neq \emptyset$
2. $g \nu_x = \nu_{gx}$ for any $g \in G$.
3. if $\theta \in (0, \pi/2)$, then $\frac{d\nu_x}{\alpha} = \exp\{b_1 B_{\alpha_1}(o_1, x_1) + b_2 B_{\alpha_2}(o_2, x_2)\}$
   for any $\alpha = (\alpha_1, \alpha_2, \theta) \in \text{supp}(\nu_x)$;
   if $\theta = 0$, then $b_2 = 0$ and $\frac{d\nu_x}{\alpha} = \exp\{b_1 B_{\alpha_1}(o_1, x_1)\}$
   for any $\alpha = (\alpha_1, \theta) \in \text{supp}(\nu_x)$;
   if $\theta = \pi/2$, then $b_1 = 0$ and $\frac{d\nu_x}{\alpha} = \exp\{b_2 B_{\alpha_2}(o_2, x_2)\}$
   for any $\alpha = (\alpha_2, \theta) \in \text{supp}(\nu_x)$;
   where $B_{\alpha_i}(o_i, x_i)$ is the Busemann function with respect to the geodesic $\alpha_i$ in $X_i$ for $i = 1, 2$.

We give the definition of radical limit points for $G$ acting on the product of Hadamard spaces $X$, which is different from that for a group acting on a proper hyperbolic space.

**Definition 2.3.** (Definition 1.3 [2]) We say a point $\eta \in \partial X$ is a radical limit point of the action $G \act X$ if there is a sequence $(g_n)_n = ((g_{n,1}, g_{n,2}))_n \subset G$ such that $g_n o$ converges to $\eta$ with the following:

Let $i = 1, 2$. If $\eta = (\eta_1, \eta_2, \theta) \in \partial X_{\text{reg}}$, $g_n o_i$ is in a bounded neighbourhood of one geodesic ray in the class of $\eta_i$. If $\eta = \eta_i \in \partial X_{\text{sing}}$, $g_n o_i$ is in a bounded neighbourhood of one geodesic ray in the class of $\eta_i$.

From now on we let a group $G \subset Is(X_1) \times Is(X_2)$ act properly and isometrically on a product of proper Hadamard spaces $(X, d) = (X_1, d_1) \times (X_2, d_2)$ with a fixed base point $o = (o_1, o_2)$, and $G$ contains $g = (g_1, g_2)$ and $h = (h_1, h_2)$ satisfying that $g_1, h_1$ and $g_2, h_2$ are independent rank one isometries in $Is(X_1)$ and $Is(X_2)$, respectively.
3. $\delta_\theta$ is continuous in the slope $\theta$

First we introduce a quite useful theorem due to G. Link.

**Lemma 3.1** (Theorem E, [4]). The homogeneous function $\Psi_G : \mathbb{R}_2^+ \to \mathbb{R}$ given by $x \mapsto |x|\delta_\theta(x)$, is concave, where $\theta(x) = \arctan \frac{\theta}{x_1}$, for any $x \neq 0$ and $G$ is given as before not necessarily with the regular growth condition.

Note that if $G$ is given as before satisfying regular growth condition, then $\delta_\theta(x) = \hat{\delta}_\theta(x) \geq 0$. But the above lemma does not hold by replacing $\delta_\theta(x)$ with $\hat{\delta}_\theta(x)$ for the $G$ given as before without the regular growth condition.

We illustrate it by an example.

**Example 3.2.** Let $X$ be the Cayley graph of $F_2 \times F_2$ with respect to the standard generators $(a_1, 1), (a_2, 1), (1, b_1), (1, b_2)$, $G$ be the subgroup of $Is(Cay(F_2, \{a_1, a_2\}) \times Is(Cay(F_2, \{b_1, b_2\}))$ generated by $(a_1, b_1), (a_2, b_2)$.

It is easy to see $\theta(g) = \pi/4$ for any $g \in G$. Then

$$\delta_\theta = -\infty \text{ and } \hat{\delta}_\theta = 0 \quad \text{for } \theta \in [0, \pi/2] - \{\pi/4\},$$

$$\hat{\delta}_\theta = \delta_\theta \geq 0 \quad \text{for } \theta = \pi/4.$$

The action $G \sim X$ does not satisfy the regular growth condition in Example 3.2. And Lemma 3.1 does not hold for this example by replacing $\delta_\theta(x)$ with $\hat{\delta}_\theta(x)$.

**Remark.** The regular growth condition is used to exclude the singular case $\delta_\theta = -\infty$ for $\theta \in [0, \pi/2]$. Hence $\delta_\theta = \hat{\delta}_\theta \geq 0$. It makes sense to discuss the continuity of $\delta_\theta$ in $\theta \in [0, \pi/2]$.

Now it is the time to elaborate Theorem 1.1. We present this theorem for the completeness.

**Theorem 3.3.** The growth rate $\delta_\theta$ of $G$ with slope $\theta$ is continuous in $\theta \in [0, \pi/2]$, where $G$ satisfies the regular growth condition.

**Proof.** Thanks to the regular growth condition, we have

$$\delta = \hat{\delta} \geq 0, \quad \delta_\theta = \hat{\delta}_\theta \geq 0,$$

for any $\theta \in [0, \pi/2]$.

G. Link showed $\delta_\theta$ is upper semi-continuous in the slope $\theta$, i.e. $\limsup_{\theta_j \to \theta} \delta_{\theta_j} \leq \delta_\theta$ for any sequence $(\theta_j)_j \subset [0, \pi/2]$.

Therefore, it suffices to show the lower semi-continuity of $\delta_\theta$.

We argue by contradiction. Otherwise, there are constants $\delta_1, \delta_2$ and a sequence $(\theta_j)_j \subset [0, \pi/2]$, such that $\delta_{\theta_j} < \delta_1 < \delta_2 < \delta_\theta$ and $\lim_{j \to \infty} \theta_j = \theta$.

Case 1: $\theta \in (0, \pi/2]$. Then there is a subsequence $(\theta_{k_j})_j$ such that (a) $\theta_{k_j} < \theta$, or (b) $\theta_{k_j} > \theta$.

Subcase (a). We may assume $0 \leq \beta < \theta_j < \pi/2$. Set $H_\gamma = (\cos \gamma, \sin \gamma)$ for any $\gamma \in [0, \pi/2]$. Note that

$$\Psi(H_\gamma) = \delta_\gamma.$$

For $t \in [0, 1]$, set

$$\tau(t) = \arctan \frac{t \sin \theta + (1-t) \sin \beta}{t \cos \theta + (1-t) \cos \beta}.$$
A direct computation gives
\[ \tan \tau(t) = \frac{tsin\theta + (1-t)sin\beta}{tcos\theta + (1-t)cos\beta}. \]

Taking the derivative of \( t \), we have
\[
(tan^2 \tau(t) + 1)\tau'(t) = \frac{(sin\theta - sin\beta)(tcos\theta + (1-t)cos\beta) + (cos\beta - cos\theta)(tsin\theta + (1-t)sin\beta)}{(tcos\theta + (1-t)cos\beta)^2},
\]
for all \( t \in [0,1] \).

Furthermore, we get
\[
\tau'(t) = \frac{(sin\theta - sin\beta)(tcos\theta + (1-t)cos\beta) + (cos\beta - cos\theta)(tsin\theta + (1-t)sin\beta)}{(tcos\theta + (1-t)cos\beta)^2 + (tsin\theta + (1-t)sin\beta)^2}.
\]

Nevertheless, one deduces that
\[
(tcos\theta + (1-t)cos\beta)^2 + (tsin\theta + (1-t)sin\beta)^2 \geq t^2 + (1-t)^2
\]
\[
\geq 2 \times \left( \frac{t+1-t}{2} \right)^2 = 1/2.
\]

Then one of the terms
\[
(tcos\theta + (1-t)cos\beta)( \geq 0)
\]
and
\[
(tsin\theta + (1-t)sin\beta)( \geq 0)
\]
is positive.

Hence we obtain \( \tau'(t) > 0 \) for all \( t \in [0,1] \). \( \tau(t) \) depends continuously, and is increasing on \( t \). Note that \( \tau(0) = \beta, \tau(1) = \theta \). Therefore there is a unique \( t_j \in [0,1] \) satisfying
\[
\tau(t_j) = \theta_j
\]
and
\[
\delta_{\theta_j} \geq \frac{\|\text{H}_\theta + (1-t_j)\text{H}_0\|}{\Psi_G(H\tau(t_j))} \\
= \Psi_G(t_j\text{H}_\theta + (1-t_j)\text{H}_0) \\
\geq t_j\delta_{\theta} + (1-t_j)\delta_{0}.
\]

It is clear that the right hand side tends to \( \delta_{\theta} \) as \( t_j \to 1 \).

By \( 4 \) and \( 7 \), we take sufficiently large \( j \) such that \( \delta_{\theta_j} \geq \delta_{2} \), which is a contradiction.

Subcase (b). We may assume \( \theta_j > \theta \) and set
\[
\tau(t) = \arctan \frac{tsin\theta + (1-t)sin(\pi/2)}{tcos\theta + (1-t)cos(\pi/2)}, \quad \text{for } t \in [0,1].
\]

A similar argument yields a contradiction.

If \( \theta = 0 \), it is similar to adapt this approach to show the contradiction.

Thus we complete the proof. \( \Box \)
Note that the proof of Theorem 3.3 works by replacing $[0, \pi/2]$ with any $[\alpha_1, \alpha_2] \subset [0, \pi/2]$. A straightforward consequence is:

**Corollary 3.4.** The growth rate $\delta_\theta$ of $G$ with slope $\theta$ is continuous in $\theta \in [\alpha_1, \alpha_2]$, where $G$ satisfies the regular growth condition for $[\alpha_1, \alpha_2] \subset [0, \pi/2]$.

We give an example to imply the mildness (relative to the conditions of G. Link) of the regular growth condition.

**Example 3.5.** Given $G \subset Is(X_1) \times Is(X_2)$, where $X_1, X_2$ are Hadamard spaces. Assume $(g_1, 1), (1, g_2), (h_1, h_2) \in G$ such that $g_1, h_1$ and $g_2, h_2$ are two pairs of independent rank one elements in the actions $Is(X_1) \sim X_1$ and $Is(X_2) \sim X_2$, respectively.

We claim that given any $x \in [0, \infty)$, $x$ is an accumulation point of the set $S = \{d_2(o_2, g_2^o o_2)/d_1(o_1, g_1^m o_1) : d_1(o_1, g_1^m o_1) \neq 0, m \in \mathbb{N}_{\geq 1}\}$.

**Proof.** Since $g_1, g_2$ are rank one elements in Hadamard spaces, they must be contracting elements ([1] Theorem 5.4). Then we may assume that there exists $\alpha \geq 1, c \geq 0$ such that

$\begin{align*}
1/\alpha m - c &\leq d_1(o_1, g_1^m o_1) \leq \alpha m + c, \\
1/\alpha m - c &\leq d_2(o_2, g_2^o o_2) \leq \alpha n + c.
\end{align*}$

Hence for any $\epsilon > 0$, there exists a positive integer $M$ with

$\begin{align*}
d_2(o_2, g_2^o o_2)/d_1(o_1, g_1^M o_1) &\leq \epsilon.
\end{align*}$

Note that for any integer $n \geq 0$, we have

$\begin{align*}
|d_2(o_2, g_2^{n+1} o_2)/d_1(o_1, g_1^M o_1) - d_2(o_2, g_2^n o_2)/d_1(o_1, g_1^M o_1)|
&\leq d_2(g_2^n o_2, g_2^{n+1} o_2)/d_1(o_1, g_1^M o_1) \\
&= d_2(o_2, g_2 o_2)/d_1(o_1, g_1^M o_1) \\
&\leq \epsilon.
\end{align*}$

On the other hand, by ([3], [3]) the following hold:

$\begin{align*}
\lim_{n \to \infty} d_2(o_2, g_2^n o_2)/d_1(o_1, g_1^M o_1) &= \infty, \\
\{d_2(o_2, g_2^n o_2)/d_1(o_1, g_1^M o_1) : n \in \mathbb{N}_{\geq 1}\} &\subset (0, \infty).
\end{align*}$

The claim follows from (11), (12), and (10).

Therefore, we prove that $G$ has the regular growth condition and Theorem 3.3 holds.

\[\square\]

4. SOME APPLICATIONS

In this section, we proceed to show Theorem 1.3.

**Proof.** Note that $\delta = \hat{\delta}, \delta_\theta = \hat{\delta}_\theta$ for any $\theta \in [0, \pi/2]$ by the regular growth condition. Since the images of natural projections

$\begin{align*}
p_1 : G \subset Is(X_1) \times Is(X_2) \to Is(X_1)
\end{align*}$

and

$\begin{align*}
p_2 : G \subset Is(X_1) \times Is(X_2) \to Is(X_2)
\end{align*}$
both contain a pair of independent rank one elements. Namely, $g_1,h_1$ and $g_2,h_2$ are independent rank one elements in $p_1(G)$ and $p_2(G)$, respectively.

Note that a pair of independent rank one elements acting on a proper $\text{CAT}(0)$ space is a pair of independent contracting elements (see [1], Theorem 5.4). A well-known fact is that for a proper geodesic metric $X$ on which a group $G \subset Is(X)$ acting properly with two independent contracting elements $g,h \in G$, the subgroup of $G$ generated by $g^N, h^N$ is isomorphic to a free group of rank two for sufficiently large integer $N$ (see [14], Proposition 2.7).

Therefore $\delta_{\theta_*} = \delta(G) > 0$. For any $\theta_* \neq \theta \in (0,\pi/2)$, one of the cases holds: (i) $0 < \theta < \theta_*$, or (ii) $\theta_* < \theta < \pi/2$.

In the first case (i), set

$$\tau(t) = \arctan \frac{t \sin \theta_* + (1-t) \sin \theta}{t \cos \theta_* + (1-t) \cos \theta},$$

for $t \in [0,1]$.

By the proof of main theorem and $\tau(0) = 0, \tau(1) = \theta_*$, there exists a unique $t_* \in (0,1)$ such that

$$\tau(t_*) = \theta \in (0, \theta_*).$$

Similar to [4], we get

$$\delta_0 \geq \psi(t_*,H_{\theta_*} + (1-t_*)H_0) \geq t_* \delta_0 + (1-t_*) \delta_0 \geq t_* \delta_0 > 0.$$

In the second case (ii), set

$$\tau(t) = \arctan \frac{t \sin \pi/2 + (1-t) \sin \theta_*}{t \cos \pi/2 + (1-t) \cos \theta_*},$$

for $t \in [0,1]$. By a similar argument, we show $\tau(t_*) = \theta$ for the unique $t_* \in (0,1)$ and $\delta_0 \geq (1-t_*) \delta_0 > 0$.

**Remark.** Although G. Link showed there is a unique $\theta_* \in [0,\pi/2]$ to attain the maximal value of $\delta_0$ which is positive, but she did not show $\theta_* \in (0,\pi/2)$. Thus the assumption on $\delta_0 > 0$ for $\theta \in (0,\pi/2)$ cannot be removed.

Now the previous Corollary 1.7, Corollary 1.8 and Corollary 1.9 are straightforward by Theorem 1.3.

In the end of this section, we establish another example.

**Example 4.1.** Given two free groups of rank 2, $H_1 = \langle a_1, a_2 \rangle$, $H_2 = \langle b_1, b_2 \rangle$.

Let $X = \text{Cay}(H_1, \{a_1, a_2\}) \times \text{Cay}(H_2, \{b_1, b_2\})$, and $G$ be the subgroup of

$$\text{Is}(\text{Cay}(F_2, \{a_1, a_2\})) \times \text{Is}(\text{Cay}(F_2, \{b_1, b_2\}))$$

generated by $(a_1,b_1), (a_2,b_2)$. Fix $a=(1,1) \in X$ and denote by $x=(a_1,b_1), y=(a_2,b_2)$, then any nontrivial element $g \in G$ can be expressed by a word over an alphabet $\{x, x^{-1}, y, y^{-1}\}$.

We may assume $g$ contains $n$ elements in $\{x, x^{-1}\}$ and $m$ elements in $\{y, y^{-1}\}$, hence we have

$$\tan \theta(g) = \frac{n + 2m}{n + m} = \frac{1 + 2m/n}{1 + m/n} \in [0, \infty).$$
It is not hard to show, the closure of the set of \( \theta(g) \) in \([0, \infty)\) is \([\pi/4, \arctan 2]\) for any \( g \in G \). Therefore, we deduce
\[
\hat{\delta}_\theta = -\infty, \quad \delta_\theta = 0, \quad \text{for } \theta \in [0, \pi/4) \cup (\arctan 2, \pi/2];
\]
\[
\hat{\delta}_\theta = \delta_\theta \geq 0, \quad \text{for } \theta \in [\pi/4, \arctan 2].
\]
Using our Corollary 3.4 one can obtain \( \delta_\theta \) is continuous in \( \theta \in [\pi/4, \arctan 2] \).

5. Results on Question 1.10

At present, we first give an example such that \( \delta_{\pi/2} = \max_{\theta \in [0, \pi/2]} \delta_\theta \). This implies a negative answer to Question 1.10 in general.

Example 5.1. Let the product of Hadamard spaces \( X = (\text{Cay}(F_2, T), d_1) \times (\text{Cay}(F_N, S), d_2) \), where \( \text{Cay}(F_2, T) \) (resp. \( \text{Cay}(F_N, S) \)) is the Cayley graph of the free group \( F_2 \) (resp. \( F_N \)) of rank two (resp. \( N \)) with the word metric \( d_1 \) (resp. \( d_2 \)) induced by the standard generating set \( T \) (resp. \( S \)). We may assume \( T = \{a_1, a_2\} \) and \( S = \{b_1, b_2, \ldots, b_N\} \). Let \( G \) be the subgroup in \( F_2 \times F_N \) generated by \( g_1 = (a_1, b_1), g_2 = (a_2, b_2), g_3 = (1, b_3), g_4 = (1, b_4), \ldots, g_N = (1, b_N) \). Fix the base point \( o = (1, 1) \).

We will show for any \( \theta \in [\pi/4, \pi/2] \),
\[
\delta_\theta = \log(2(N - 2) - 1) \cdot (\sin \theta) - (\log(2(N - 2) - 1) - \log 3) \cdot \cos \theta.
\]
As a consequence, for \( N \geq 4 \) we have
\[
\delta_{\pi/2} = \max_{\theta \in [0, \pi/2]} \delta_\theta.
\]

Proof. For any \( 1 \neq g \in G \), it is clear that \( g \) can be expressed uniquely in terms of \( g_1, g_2, \ldots, g_N \) by definition and the fact that \( F_N \) is free. We may assume \( g = (g_{(1)}, g_{(2)}) = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_t}^{k_t} \), where \( i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, N\} \) and \( k_1, \ldots, k_t \) are integers.

For convenience, one can assume \( i_1, i_2, \ldots, i_t \in \{1, 2\} \) and \( i_{t+1}, i_{t+2}, \ldots, i_s \in \{3, 4, \ldots, N\} \) with \( 1 \leq t \leq N \). Then we have
\[
tan \theta(g) = \frac{d_2(1, g_{(2)})}{d_1(1, g_{(1)})} = \frac{\sum_{j=1}^{N} |k_j|}{\sum_{j=1}^{t} |k_j|}.
\]

It is apparent from \( \theta \) that \( \theta(g) \in [\pi/4, \pi/2] \), \( \delta_\theta \geq 0 \) for \( \theta \in [\pi/4, \pi/2] \) and \( \delta_\theta = -\infty \) for \( \theta \in [0, \pi/4] \). For any \( g \in A_{\epsilon,\theta}(o, n) \) with \( \epsilon > 0 \), \( \theta \in [\pi/4, \pi/2] \), it is not hard to show there exists \( c(\epsilon, n) > 0 \) with \( \lim_{\epsilon \to 0} \lim_{n \to \infty} c(\epsilon, n)/n = 0 \) such that
\[
|d_2(1, g_{(2)}) - n \sin \theta| \leq c(\epsilon, n), \quad |d_1(1, g_{(1)}) - n \cos \theta| \leq c(\epsilon, n).
\]

Recall that \( g \) can be expressed uniquely in terms of \( g_1, g_2, \ldots, g_N \). If one assumes \( n_1 \) (resp. \( n_2 \)) to be the frequency of occurrence of \( \{g_1^{\pm 1}, g_2^{\pm 1}\} \) (resp. \( g_3^{\pm 1} = (1, b_3)^{\pm 1}, g_4^{\pm 1} = (1, b_4)^{\pm 1}, \ldots, g_N^{\pm 1} = (1, b_N)^{\pm 1}\)) for the expression of \( g \), via \( d_1 \) we get
\[
d_2(1, g_{(2)}) = n_1 + n_2, d_1(1, g_{(1)}) = n_1.
\]

Using \( 15, 16 \), we have
\[
|n_2 - n(\sin \theta - \cos \theta)| \leq 2c(\epsilon, n), \quad |n_1 - n \cos \theta| \leq c(\epsilon, n).
\]

On the other hand, it is well-known that the finitely generated free groups have purely exponential growth (see \( 14, 15 \)). Note that for any positive integer \( m \), we
have \( \delta(F_m) = \log(2m - 1) \) with respect to the standard generating set. Hence via [17], we have

\[
\frac{1}{c(F_{N-2})} \exp\left\{ \log(2(N - 2) - 1) \cdot (n(\sin \theta - \cos \theta) - 2c(\epsilon, n)) \right\} \\
\cdot \frac{1}{c(F_2)} \exp\{ \log 3 \cdot (ncos \theta - c(\epsilon, n)) \}
\]

\[
\leq A_{\epsilon, \theta}(o, n)
\]

\[
\leq c(F_{N-2}) \exp\left\{ \log(2(N - 2) - 1) \cdot (n(\sin \theta - \cos \theta) + 2c(\epsilon, n)) \right\} \\
\cdot c(F_2) \exp\{ \log 3 \cdot (ncos \theta + c(\epsilon, n)) \}
\]

where \( c(F_2) \) (resp. \( c(F_N) \)) are constants depending only on \( F_2 \) (resp. \( F_N \)).

Thus this yields the following for \( N \geq 4 \)

\[
\delta_\theta = \liminf_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log|A_{\epsilon, \theta}(o, n)|}{n}
\]

\[
= \log(2(N - 2) - 1) \cdot (\sin \theta - \cos \theta) + \log 3 \cdot \cos \theta
\]

\[
= \log(2(N - 2) - 1) \cdot (\sin \theta) - (\log(2(N - 2) - 1) - \log 3) \cdot \cos \theta
\]

\[
\leq \log(2(N - 2) - 1) = \delta_{\pi/2}.
\]

\[\square\]

In the next, we will present Theorem 1.11 which provides an affirmative answer to Question 1.10 under a quite mild condition.

**Theorem 5.2.** Assume that \((X, d) = (X_1, d_1) \times (X_2, d_2)\) is a product of Hadamard spaces on which a group \( G \subset Is(X_1) \times Is(X_2)\) acts properly, \( d\) is the standard product metric. If \( G\) contains a pair of isometries which projects a pair of independent rank one elements in each factor and satisfies \( \delta_\theta > 0 \) for \( \theta \in (0, \pi/2)\), with the following condition:

\[
\frac{\delta_{\pi/2}}{\delta_\theta} < \frac{1}{\sin \theta} \quad \text{and} \quad \frac{\delta_0}{\delta_{\pi/2-\beta}} < \frac{1}{\sin \beta},
\]

for some \( \theta, \beta \in (0, \pi/2)\).

Then there exists some \( \theta_* \in (0, \pi/2) \) such that \( \delta_{\theta_*} = \max_{\theta \in [0, \pi/2]} \delta_{\theta} \).

**Proof.** Since \( \delta_0 \geq \limsup_{\theta_\epsilon \to 0} \delta_{\theta_\epsilon} \geq 0 \) and \( \delta_{\pi/2} \geq \limsup_{\theta_\epsilon \to \pi/2} \delta_{\theta_\epsilon} \geq 0 \), where \( \theta_\epsilon, \theta_i \in (0, \pi/2) \).

Then \( \delta_\theta \) are nonnegative in \([0, \pi/2]\).

Recall that for any \( 0 \neq x = (x_1, x_2) \in \mathbb{R}^2_{\geq 0}, \theta(x) = \arctan \frac{x_1}{x_2} \) and set \( \theta(0) = 0 \). The key ingredient of the proof is the observation:

The following holds for any \( x, y \in \mathbb{R}^2_{\geq 0} \) and any \( t \in [0, 1] \),

\[
\|tx + (1-t)y\|\sin(\theta(tx + (1-t)y)) = t\|x\|\sin(\theta(x)) + (1-t)\|y\|\sin(\theta(y)).
\]

**Proof.** It is trivial for \( x = 0 \) or \( y = 0 \). Hence we may assume \( \|x\|, \|y\| > 0 \). One can identify any vector in \( \mathbb{R}^2_{\geq 0} \) with a point in the plane, so we let \( O \) denote \( 0 \) and \( P \in \mathbb{R}^2_{\geq 0} \) (resp. \( Q \)) denote \( x \) (resp. \( y \)). Suppose \( d \) is the standard Euclidean metric of the plane.
We may assume $0 \leq \theta(x) < \theta(y) \leq \pi/2$. Denote by $R$ the vector $tx + (1-t)y$. One can extend the segment $[O, R]$ to the point $T$ such that

$$\frac{\hat{d}(O, R)}{\hat{d}(T, R)} = \frac{\hat{d}(P, R)}{\hat{d}(Q, R)}.$$ 

Note that the triangle $\Delta TQR$ is similar to the triangle $\Delta OPR$. Thus we have

\begin{align*}
(21) & \quad \frac{\hat{d}(O, R)}{\hat{d}(T, R)} = \frac{\hat{d}(P, R)}{\hat{d}(Q, R)} = \frac{\hat{d}(O, P)}{\hat{d}(Q, P)} = \frac{1 - t}{t}.
(22) & \quad \hat{d}(O, T) = \hat{d}(O, R) + \hat{d}(R, T)
& = \left\|tx + (1-t)y\right\| + \frac{t}{1-t}\left\|tx + (1-t)y\right\|
& = \frac{\left\|tx + (1-t)y\right\|}{1-t}.
\end{align*}

If we set $\alpha = \angle TOP$, $\beta = \angle TOQ$, we have

$$\alpha + \beta = \theta(y) - \theta(x).$$

Using the Law of Sines for the triangle $\Delta TQO$, (21) and (22), we get

\begin{align*}
(24) & \quad \frac{\sin \alpha}{\sin(\pi - \alpha - \beta)} = \frac{\hat{d}(O, Q)}{\hat{d}(O, T)} = \frac{(1-t)||y||}{\left\|tx + (1-t)y\right\|},
(25) & \quad \frac{\sin \beta}{\sin(\pi - \alpha - \beta)} = \frac{\hat{d}(T, Q)}{\hat{d}(O, T)} = \frac{t||x||}{\left\|tx + (1-t)y\right\|}.
\end{align*}

Thus we have

\begin{align*}
\left\|tx + (1-t)y\right\||\sin(\theta(tx + (1-t)y))
& = t||x||\sin(\theta(x)) + (1-t)||y||\sin(\theta(y))
& \Leftrightarrow \frac{\sin \beta \sin \theta(x) + \sin \alpha \sin \theta(y)}{\sin(\theta(y) - \sin \theta(x))} = \sin(\alpha + \theta(x))
(26)
& \Leftrightarrow \sin \beta \sin \theta(x) + \sin \alpha \sin \theta(y)
& = (\sin \alpha \cos \theta(x) + \cos \alpha \sin \theta(x)) \sin(\theta(y) - \sin \theta(x))
& \Leftrightarrow \sin \alpha \sin(\theta(x) - \theta(y)) + \sin \beta \sin(\theta(x) - \theta(y)) - \cos \theta(x) \sin(\theta(y) - \theta(x))
& = (\cos \alpha \sin \theta(y) - \theta(x)) - \sin(\theta(y) - \theta(x) - \alpha) \sin \theta(x)
(28)
& \Leftrightarrow \sin \alpha \sin \theta(x) \cos(\theta(y) - \theta(x))
& = \cos \theta(y) - \theta(x) \sin \alpha \sin \theta(x),
\end{align*}

where (20) we used (21) and (25), (28) we used (23), and (21) (29) we used the basic properties of sine functions.

Recall that for any $0 \neq x, y \in \mathbb{R}_\geq 0$, we have

$$\left\|tx + (1-t)y\right\||\delta_{\theta(tx + (1-t)y)} \geq t||x||\delta_{\theta(x)} + (1-t)||y||\delta_{\theta(y)}.$$ 

Take

$$\theta(x) = \pi/2,$$

and fix $0 \neq x, y \in \mathbb{R}_\geq 2$ with $0 < \theta(y) < \pi/2$. 

Similar to (3), one can deduce that $\theta(tx + (1-t)y)$ is smooth on $t \in [0,1]$. Then this yields

$$
\lim_{t \to 1} \frac{\|tx + (1-t)y\|(1 - \sin \theta(tx + (1-t)y))}{(1-t)\|y\|} = \frac{\|x\|}{\|y\|} \lim_{t \to 1} \cos \theta(tx + (1-t)y)\theta'(tx + (1-t)y) = 0,
$$

(32)

where we applied L’Hospital’s Rule to get (32) and applied (31) to get (33).

By (33) and $\frac{\delta_{\pi/2}}{\delta_0} < \frac{1}{\sin \theta}$, for $t \in (0,1)$ such that $t$ is sufficiently close to 1 we obtain

$$
\frac{(1-t)\|y\|}{\|tx + (1-t)y\|(1 - \sin \theta(tx + (1-t)y)) + (1-t)\|y\|\sin \theta(tx + (1-t)y)} > \frac{\delta_{\pi/2}}{\delta_0(y)}.
$$

(34)

Combining (32), (31) with (20), one can have

$$
t\|x\|\delta_{\theta(x)} + (1-t)\|y\|\delta_{\theta(y)} = t\|x\|\delta_{\pi/2} + (1-t)\|y\|\delta_{\theta(y)} > \|tx + (1-t)y\|\delta_{\pi/2} = \|tx + (1-t)y\|\delta_{\theta(x)}.
$$

(35)

On the other hand, by (30) and (35) we get

$$
\|tx + (1-t)y\|\delta_{\theta(tx + (1-t)y)} > \|tx + (1-t)y\|\delta_{\pi/2}.
$$

Therefore we deduce $\delta_{\theta(tx + (1-t)y)} > \delta_{\pi/2}$ for some $t \in (0,1)$ such that $t$ is sufficiently close to 1.

If one exchange $X_1$ and $X_2$, then for $G \subset X = X_2 \times X_1$ we have $\delta_{\beta} = \delta_{\pi/2 - \beta}$. Similarly, for some $t \in (0,1)$ such that $t$ is sufficiently close to 1, we have

$$
\delta_{\pi/2 - \theta(tx + (1-t)y)} = \delta_{\theta(tx + (1-t)y)} > \delta_{\pi/2} = \delta_0.
$$

Thus we complete it.

\[ \square \]

Remark. Recall the example (5.1). If one takes $N = 4$, then $\delta_\theta = \sin \theta \cdot \delta_{\pi/2}$ by the equation (13). Hence $\frac{\delta_{\pi/2}}{\delta_\theta} = \frac{1}{\sin \theta}$. This implies the upper bound in (19) for $\frac{\delta_{\pi/2}}{\delta_\theta}$ in Theorem 5.2 is sharp.

An immediate consequence is:

**Corollary 5.3.** Assume that $(X, d) = (X_1, d_1) \times (X_2, d_2)$ is a product of Hadamard spaces on which a group $G \subset Is(X_1) \times Is(X_2)$ acts properly, $d$ is the standard product metric. If $G$ contains a pair of independent rank one elements in each factor and satisfies $\delta_\theta > 0$ for $\theta \in [0, \pi/2]$

Then there exists some $\theta^* \in (0, \pi/2)$ such that $\delta_{\theta^*} = \max_{\theta \in [0, \pi/2]} \delta_{\theta}$.

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References

[1] Bestvina M, Fujiwara K, A Characterization of Higher Rank Symmetric Spaces Via Bounded Cohomology[J]. Geom. Funct. Anal, 2009, 19(1):11-40.
[2] Dennis Sullivan, The density at infinity of a discrete group of hyperbolic motions, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 171-202.
[3] Gabriele Link, Generalized Patterson-Sullivan measures for products of Hadamard spaces, Preprint, arXiv:1107.3755v2 2011.
[4] Gabriele Link, Asymptotic geometry in products of Hadamard spaces with rank one isometries, Geom. Topol. (2010), no.2,1063-1094.
[5] Gabriele Link, Generalized conformal densities for higher products of rank one Hadamard spaces. Geom. Dedicata. (2015), 351-387
[6] Gabriele Link, Asymptotic geometry in higher products of rank one Hadamard spaces. Groups Geom. Dyn. 10 (2016), no. 3, 885-931
[7] J. F. Quint, Mesures de Patterson-Sullivan en rang supérieur, Geometric and functional analysis. 12 (2002), no. 4, 776-809.
[8] M. R. Bridson, A Haefliger, Metric spaces of non-positive curvature, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin (1999)
[9] S. J. Patterson, The limit set of a Fuchsian group, Acta Math. 136 (1976), no. 3-4, 241-273.
[10] W. Ballmann, Axial isometries of manifolds of nonpositive curvature, Math. Ann.259 (1982) 131-144
[11] W. Ballmann, Lectures on spaces of nonpositive curvature, volume 25 of DMV Seminar, Birkhäuser Verlag, Basel (1995)
[12] W. Ballmann, M Brin, Orbihedra of nonpositive curvature, Inst. Hautes Études Sci. Publ. Math. (1995) 169-209 (1996)
[13] Yang W. Patterson-Sullivan measures and growth of relatively hyperbolic groups,Preprint arXiv:1308.6326v2 2013.
[14] Yang W. Statistically convex-cocompact actions of groups with contracting elements,Preprint arXiv:1612.03648v3 2016.
[15] Yang W . Purely exponential growth of cusp-uniform actions[J]. Ergodic Theory and Dynamical Systems, 2016.

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