Global synchronization of partially forced Kuramoto oscillators on Networks

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Abstract

We study the synchronization of Kuramoto oscillators on networks where only a fraction of them is subjected to a periodic external force. When all oscillators receive the external drive the system always synchronize with the periodic force if its intensity is sufficiently large. Our goal is to understand the conditions for global synchronization as a function of the fraction of nodes being forced and how these conditions depend on network topology, strength of internal couplings and intensity of external forcing. Numerical simulations show that the force required to synchronize the network with the external drive increases as the inverse of the fraction of forced nodes. However, for a given coupling strength, synchronization does not occur below a critical fraction, no matter how large is the force. Network topology and properties of the forced nodes also affect the critical force for synchronization. We develop analytical calculations for the critical force for synchronization as a function of the fraction of forced oscillators and for the critical fraction as a function of coupling strength. We also describe the transition from synchronization with the external drive to spontaneous synchronization.

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I. INTRODUCTION

Coupled biological oscillators are abundant in nature and often need to work in synchrony to regulate physical activities, such as pacemaker cells in the heart [1], neurons in regions of the brain [2, 4] and fireflies flashing collectively to help females find suitable mates [5, 6]. Artificial systems, such as electrochemical oscillators [7] and coupled metronomes [8], have also been studied. There are evidences that synchronization also plays a key role in information processing in areas on the cerebral cortex [9, 10]. Even the brain rest state activity is characterized by local rhythmic synchrony that induces spatiotemporally organized spontaneous activity at the level of the entire brain [11].

The model of coupled oscillators introduced by Kuramoto [12] has become a paradigm in the study of synchronization and has been extensively explored in the last years in connection with biological systems, neural networks and the social sciences [13]. The model consists of \( N \) oscillators described by internal phases \( \theta_i \) which rotate with natural frequencies \( \omega_i \) typically selected from a symmetric distribution. In the original model all oscillators interact with each other according to the equations

\[
\dot{\theta}_i = \omega_i + \lambda \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i),
\]

where \( \lambda \) is the coupling strength and \( i = 1, \ldots, N \).

Kuramoto analyzed the system in the limit where \( N \) goes to infinity and showed that for small values of the coupling parameter the oscillators continue to move as if they were independent. However, as the coupling increases beyond a critical value, a finite fraction of oscillators start to move together, a behavior termed spontaneous synchronization. This fraction increases smoothly with the coupling, characterizing a second order phase transition in the limit of infinite oscillators. For large enough coupling the whole system oscillates on the same frequency, as if it were a single element.

Synchronization in many biological systems, however, is not spontaneous, but frequently depends on external stimuli. Information processing in the brain, for example, might be triggered by visual, auditory or olfactory inputs [2]. Different patterns of synchronized neuronal firing are observed in the mammalian visual cortex when subjected to stimuli [3]. In the sensomotor cortex synchronized oscillations appear with amplitude and spatial patterns that depend on the task being performed [3, 4]. Synchronization of brain regions not
directly related to the task in question can be associated to disorders like epilepsy, autism, schizophrenia and Alzheimer [14, 15]. In the heart, cardiac synchronization is induced by specialized cells in the sinoatrial node or by an artificial pacemaker that controls the rhythmic contractions of the whole heart [16]. The periodic electrical impulses generated by pacemakers can be seen as an external periodic force that synchronizes the heart cells. Another example of driven system is the daily light-dark cycle on the organisms [17]. In mammals, cells specialized on the sleep control exhibit intrinsic oscillatory behavior whose connectivity is still unknown [18]. The change in the light-dark cycle leads to a response in the circadian cycle mediated by these cells, which synchronize via external stimulus.

A natural extension of the Kuramoto model, therefore, is to include the influence of an external periodic force acting on the system [19–22]. In this work we consider systems where the oscillators’ interconnections form a network and where the force acts only on a fraction of the oscillators. We are interested in the conditions for global synchronization as a function of the fraction of nodes being forced and how it depends on network topology. We show that the minimum force $F_{\text{crit}}$ needed for global synchronization scales as $1/f$, where $f$ is the fraction of forced oscillators, and it is independent of the internal coupling strength $\lambda$. However, in order to reach synchronization with fraction $f$ a minimum internal strength is needed. The degree distributions of the network and the set of forced nodes modify the $1/f$ behavior in heterogeneous networks. We develop analytical approximations for $F_{\text{crit}}$ as a function of the fraction $f$ of forced oscillators and for the minimum fraction $f_{\text{crit}}$ for which synchronization occurs as a function of $\lambda$. This paper is organized as follows: in section II we describe the partially forced Kuramoto model and present the results of numerical simulations in section III. In section IV we discuss the analytical calculations for $F_{\text{crit}}(f)$ and $f_{\text{crit}}(\lambda)$ that take into account network topology and explain most of the simulations. We summarize our conclusions in section V.

II. THE FORCED KURAMOTO MODEL ON NETWORKS

Here we consider three modifications of the original Kuramoto model: first, to include the possibility that each oscillator interacts only with a subset of the other oscillators, the system will be placed on a network whose topology defines the interactions [23]; second, we include the action of an external periodic force [19, 21] and; third, we allow the external
force to act only on a subset of the oscillators, representing the 'interface' of the system that interacts with the 'outside' world, like the photo-receptor cells in the eye [3].

The system is described by the equations

\[ \dot{\theta}_i = \omega_i + F \delta_{i,C} \sin(\sigma t - \theta_i) + \frac{\lambda}{k_i} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i), \]  

(2)

where \( A_{ij} \) is the adjacency matrix defined by \( A_{ij} = 1 \) if oscillators \( i \) and \( j \) interact and zero if they do not; \( k_i \) is the degree of node \( i \), namely \( k_i = \sum_j A_{ij} \); \( F \) and \( \sigma \) are respectively the amplitude and frequency of the external force; and \( C \) is the subgroup of oscillators subjected to the external force. We have also defined \( \delta_{i,C} = 1 \) if \( i \in C \) and zero otherwise and we shall call \( N_C \) the number of nodes in the set \( C \).

Following [21] we get rid of the explicit time dependence by performing a change of coordinates to analyse the dynamics in a referential frame corotating with the driving force:

\[ \phi_i = \theta_i - \sigma t \]  

(3)

which leads to

\[ \dot{\phi}_i = \omega_i - \sigma - F \delta_{i,C} \sin \phi_i + \frac{\lambda}{k_i} \sum_{j=1}^{N} A_{ij} \sin(\phi_j - \phi_i), \]  

(4)

The behavior of the system depends now not only on the distribution of natural frequencies and coupling intensity \( \lambda \), but also on the network properties, on the intensity and frequency of the external force and on the size and properties of the set \( C \). The role of network characteristics in the absence of external forcing has been extensively studied in terms of clustering [24–26], assortativity [27] and modularity [28–30].

The behavior of the system under an external force has also been considered for very large and fully connected networks when the force acts on all nodes equally [21]. The system exhibits a rich behavior as a function of the intensity and frequency of the external force. In particular, it has been shown that if the force intensity is larger than a critical value \( F_{\text{crit}} \) the system may fully synchronize with the external frequency. Among the questions we want to answer here are how synchronization with the external force changes as we make \( N_C < N \) and how does that depend on the topology of the network and on the properties of the nodes in \( C \). In particular we are interested in studying how the critical intensity \( F_{\text{crit}} \) of the external force increases as \( N_C \) decreases and if there is a minimum number of nodes that need to be excited by \( F \) in order to trigger synchronization. In the next section
we show the results of numerical simulations considering three network topologies (random, scale-free and fully connected). Analytical calculations that describe these results will be presented next.

### III. NUMERICAL RESULTS

In order to get insight into the general behavior of the system we present a set of simulations for the following networks: (i) fully connected with \( N = 200 \) nodes (FC200), (ii) fully connected with \( N = 500 \) (FC500); (iii) random Erdos-Renyi network with \( N = 200 \) and average degree \( \langle k \rangle = 10.51 \) (ER200) and (iv) scale-free Barabasi-Albert network with \( N = 200 \) (BA200) computed starting with \( m_0 = 11 \) fully connected nodes and adding nodes with \( m = 10 \) links with preferential attachment, so that \( \langle k \rangle = 9.83 \). In all simulations we have considered a Gaussian distribution of natural frequencies \( g(\omega) \) with null mean and standard deviation \( a = 1.0 \) for the oscillators.

For the fully connected networks the critical value \( \lambda_c \) for the onset of synchronization can be estimated when \( N \to \infty \) as \( \lambda_c = 2a\sqrt{2/\pi} \approx 1.6 \). For finite networks the calculation of \( \lambda_c \) can be performed numerically (see, for example, [31]) and we have checked that \( \lambda_c = 1.6 \) is a good approximation even for \( N = 100 \) and for the other topologies we used. Full synchronization occurs only for larger values of \( \lambda \) and we define \( \lambda_f \) as the value where \( r = 0.95 \) and \( \dot{\psi} < 10^{-2} \). Here we are interested in scenarios where the system synchronizes spontaneously when \( F = 0 \) and, therefore, we set \( \lambda \) above \( \lambda_f \) to assure full spontaneous synchronization. The coupling strength \( \lambda \) has an important role in the synchronization process, as we discuss below. For each network type and fraction \( f = N_C/N \) of nodes interacting with the external force we calculate the minimum (critical) force necessary for synchronization with the external frequency.

In order to characterize the dynamics we use the usual order parameter

\[
z = re^{i\psi} = \frac{1}{N} \sum_{i=1}^{N} e^{i\phi_i},
\]

where \( r = 1 \) indicates full synchronization and \( \dot{\psi} \) the frequency of the collective motion. We note that, since we are working on a rotating frame, synchronization with \( \sigma \) will imply \( \dot{\psi} = 0 \) whereas spontaneous synchronization \( \dot{\psi} = -\sigma \).
Fig. 1. (color online) Order parameter $r$ and $\dot{\psi}$ as a function of $F$ for a fully connected network with $N = 200$, $\lambda = 20.0$ and $\sigma = 3.0$ for (a) $f = 1$ and (b) $f = 0.5$. Red dots correspond to time averaged values calculated between $t = 25$ to $t = 50$. Error bars correspond to one standard deviation. The dashed lines indicate the critical force.

Fig. 1 shows $r$ and $\dot{\psi}$ for FC200 as a function of $F$ for $\lambda = 20$ and $f = 1$ and $f = 0.5$. The system has been evolved up to $t = 50$ starting with random phases, which was enough to overcome the transient period. Because the system is finite and there are fluctuations we computed time averages and standard deviations of $r$ and $\psi$ in the interval from time 25 to 50. The system remained fully synchronized for all values of $F$, first spontaneously ($F = 0$) and later with the external frequency for $F > 3$ ($f = 1$) and for $F > 6$ ($f = 0.5$). For intermediate values of the external force, $\dot{\psi}$ oscillates and the average and standard deviations are shown. In this regime the oscillators move together ($r = 1$) but change directions constantly due to the competition between the couplings $\lambda$ and $F$. The critical force $F_{\text{crit}}$ was numerically computed as the value of $F$ where $\dot{\psi} < 10^{-2}$ and $r > 0.95$.

Fig. 2(a) shows $F_{\text{crit}}$ as a function of the fraction $f$ of excited nodes for FC200. It also shows that for a fixed value of the internal coupling $\lambda$ synchronization can only be achieved for $f$ larger than a critical value $f_{\text{crit}}(\lambda)$. For example, for $\lambda = 20$ (orange circles) synchronization is obtained only for $f > 0.22$. For $f < 0.22$ no synchronization is achieved for $\lambda = 20$, no matter how large is the external force. The value of $f_{\text{crit}}$ is shown as the last point of the corresponding symbol on the plot. Notice that the minimum value of $F$
FIG. 2. (color online) (a) Critical force $F_{\text{crit}}$ versus fraction $f$ of forced nodes for the fully connected network FC200. The continuous red curve shows the analytical calculation and the symbols are the results of numerical simulations for different values of the coupling constant $\lambda$. The point with smallest $f$ for each $\lambda$ is defined as $f_{\text{crit}}$. (b) $f_{\text{crit}}(\lambda)$ from numerical simulations (stars) and according to Eq. (15) (red curve). The dashed (blue) line was obtained from the parametric curve of Eq. (18).

for synchronization does not itself depend on $\lambda$, since the same value is obtained as long as $\lambda$ is large enough. Fig. 2(b) shows $f_{\text{crit}}$ as a function of $\lambda$. We have performed the same analysis for FC500 and both curves $F_{\text{crit}}(f)$ and $f_{\text{crit}}(\lambda)$ were essentially identical to the ones obtained for FC200, showing that these are independent of network size.

Fig. 3 shows similar results for the ER200 random network. In this case the nodes have different degrees and it matters which nodes are selected to interact with the external force. For the results in panel (a) the nodes have been ordered from high to low degree and the $fN$ first (highly connected) nodes have been selected to interact with the force. In panel (b) the nodes were chosen at random. The dependence of $f_{\text{crit}}$ on $\lambda$ is similar to the fully connected case and different values of $\lambda$ are shown with different symbols.

For the random network the differences between the two cases are not striking, since the distribution of nodes is quite homogeneous. This is not the case for the BA200 network, as shown in Fig. 4. When the external source connects with nodes of highest degree, panel (a), the critical force for synchronization is smaller than when connected randomly, panel
FIG. 3. (color online) Critical force $F_{\text{crit}}$ versus fraction $f$ of forced nodes for the random network ER200. The continuous red curve shows the analytical calculation and the symbols are the results of numerical simulations for different values of the coupling constant $\lambda$. The point with smallest $f$ for each $\lambda$ is defined as $f_{\text{crit}}$. Force is connected with nodes of (a) highest degrees; (b) random. For the red line on panel (b) we have computed the average degree $\langle k \rangle_C$ of forced set over 10 simulations to eliminate fluctuations.

(b), or with nodes of lowest degrees, panel (c), as expected. The analytical (red) curve for random connections shows an average over 10 simulations using the same network but different random choices of nodes.

IV. ANALYTICAL RESULTS

The numerical simulations show that: (i) $F_{\text{crit}}$ depends of $f$; (ii) for heterogeneous networks it depends on the properties of the set $C$; (iii) there is a critical fraction $f_{\text{crit}}$, that depends on the network type, on $C$ and on $\lambda$, below which no synchronization is possible. In this section we derive a theory for $F_{\text{crit}}(f)$ and an approximation for $f_{\text{crit}}(\lambda)$.

A. Critical Force $F_{\text{crit}}$

In order to derive an expression for $F_{\text{crit}}(f)$ we use the fact that nodes directly affect all their neighbors and, therefore, their importance should be proportional to their degree. We
FIG. 4. (color online) Critical force $F_{\text{crit}}$ versus fraction $f$ of forced nodes for the scale-free network BA200. The continuous red curve shows the analytical calculation and the symbols are the results of numerical simulations for different values of the coupling constant $\lambda$. The point with smallest $f$ for each $\lambda$ is defined as $f_{\text{crit}}$. Force is connected with nodes of (a) highest degrees; (b) random and (c) lowest degree. For the red line on panel (b) we have computed the average degree $\langle k \rangle_C$ of forced set over 10 simulations to eliminate fluctuations.

start by multiplying all terms of Eq. (4) by $k_i/\langle k \rangle$, sum over $i$ and divide by $N$ to obtain

$$\frac{d\langle\phi\rangle_k}{dt} = \langle\omega\rangle_k - \sigma - \bar{F} \langle \sin\phi \rangle_{k,C} \tag{6}$$

where

$$\langle\phi\rangle_k = \frac{1}{N} \sum_{i=1}^{N} \frac{k_i}{\langle k \rangle} \phi_i, \tag{7}$$

$$\langle\omega\rangle_k = \frac{1}{N} \sum_{i=1}^{N} \frac{k_i}{\langle k \rangle} \omega_i \tag{8}$$

and

$$\langle \sin\phi \rangle_{k,C} = \frac{1}{N_c} \sum_{i \in N_C} \frac{k_i}{\langle k \rangle} \sin \phi_i. \tag{9}$$

The term proportional to $\lambda$, containing the coupling between the oscillators, cancel out exactly. When the oscillators synchronize with the external force Eq. (9) becomes

$$\langle \sin\phi \rangle_{k,C} = \sin\langle\phi\rangle \langle\langle k \rangle \rangle_C \tag{10}$$

and we define

$$\bar{F}_k = F \langle \langle k \rangle \rangle_C F. \tag{11}$$

Since $\langle \phi \rangle$ is constant in the synchronized state Eq. (6) implies

$$\sin\langle\phi\rangle = \frac{\langle\omega\rangle_k - \sigma}{\bar{F}_k}. \tag{12}$$
Because the $\omega_i$ are randomly distributed with zero average, $\langle \omega \rangle_k$ is generally small for large networks (although not zero in a single realization of the frequency distribution). The critical force is now estimated as $\bar{F}_c = \sigma - \langle \omega \rangle_k$ and

$$F_{\text{crit}} = \frac{\sigma - \langle \omega \rangle_k}{f} \frac{\langle k \rangle}{\langle k \rangle_C} \approx \frac{\sigma}{f} \frac{\langle k \rangle}{\langle k \rangle_C}.$$  

For regular networks, in particular, where all nodes have the same degree, $\langle k \rangle = \langle k \rangle_C$, the critical force is reduced to

$$F_{\text{crit}} = \frac{\sigma}{f}.$$  

Eq. (13) shows that when nodes with high degree are being forced, $\langle k \rangle_C > \langle k \rangle$, the critical force for synchronization is smaller than the value obtained by equation (14), since the external force is directly transmitted to a large number of neighbors. On the other hand, if $\langle k \rangle_C < \langle k \rangle$ (nodes with low degree are being forced) the critical force must be higher than that estimated by (14), since these nodes have few neighbors. This agrees with the results shown in Figs. 2-4, where the continuous (red) line shows the approximation Eq.(13). For the scalefree network, in particular, when the force acts on nodes of highest degree, Fig. 4(a), $F_{\text{crit}} \approx 5$ for $f = 0.4$, whereas $F_{\text{crit}} \approx 15$ for the same value of $f$ when the force acts on the nodes with smallest degree Fig. 4(c).

B. The critical fraction $f_{\text{crit}}(\lambda)$

Eq.(6) is exact and it might appear to be completely independent of $\lambda$. This, however, is not true, since the dynamics of the angles $\phi$ are implicitly coupled by $\lambda$ and synchronization is only possible if $\lambda$ is large enough. As $f$ decreases the amplitude of the external force needed for synchronization increases and if it gets too much larger than $\lambda$ the oscillators start to move almost independently and synchronization is hindered.

An approximation for minimum value of $f$ that can lead to synchronization for a given $\lambda$ can be obtained by setting the internal coupling strength per node to the intensity of the external force, i.e., $\lambda \approx F$. Along the curve $F = F_{\text{crit}}$ we may write $\lambda \approx \sigma \langle k \rangle/(f \langle k \rangle_C)$ (see Eq.(13)) or, taking into account that complete spontaneous synchronization starts at $\lambda_f$, we propose that $f_{\text{crit}}$ can be estimated as

$$f_{\text{crit}}(\lambda) = \frac{\sigma}{\lambda - \lambda_0} \frac{\langle k \rangle}{\langle k \rangle_C},$$  

10
where $\lambda_0$ is a fit parameter. For fully connected networks $\langle k \rangle = \langle k \rangle_C$ and Eq. (15) reduces to $f_{crit}(\lambda) = \sigma/(\lambda - \lambda_0)$. For the red curve in Fig. 2(b) we obtained $\lambda_0 = 4.48 \pm 0.12$ which fits very well the numerical results (black stars). Note that the value of $\lambda$ for $f = 1$ is $\lambda_0 + \sigma = 7.48$ for which we find $r = 0.99$ for $F = 0$ although $\dot{\psi}$ is still fluctuating. Full spontaneous synchronization ($r > 0.95$ and $\dot{\psi} < 10^{-2}$) only occurs for $\lambda = 11.3$.

The heuristic approximation given by Eq. (15) can be made more precise using the bifurcation surfaces derived by Childs and Strogatz [21] for the case where the external force acts on all nodes. The derivation assumed a Lorentzian distribution for the oscillator’s natural frequencies, but is believed to be valid for a larger class of such distributions. The full bifurcation diagram is divided into five regions but is dominated by only two: one where the oscillators are locked to the same frequency as the external force and one with mutual, spontaneous, synchronization. These two main regions are separated by saddle-node bifurcations given in the $F$ versus $\sigma$ plane, for $\lambda$ fixed, by the parametric equations

$$F(\lambda, r) = \frac{\sqrt{2} r^2}{(1 - r^2)^2} \sqrt{\lambda^2 (1 - r^2)^3 + 2 \lambda (r^4 - 4 r^2 + 3) - 8}$$

(16)

$$\sigma(\lambda, r) = \frac{(1 + r^2)^{3/2}}{2(1 - r^2)^2} \sqrt{\lambda (r^2 - 1) [\lambda (r^2 - 1)^2 - 4] - 4 (r^2 + 1)}$$

(17)

where $r$ varies from approximately 0.66 to 1.0. The resulting curve $F = F(\sigma)$ can be approximated by the simple relation $F = \sigma$, as predicted by eq. (14). This approximation becomes exact as $\lambda$ goes to infinity, or when $r = 1$ and $\dot{\psi} = 0$.

Solving these equations for $F$ and $\lambda$ we obtain

$$\lambda(\sigma, r) = \frac{2}{(r^2 - 1)^2} + 2 \sqrt{\frac{r^4}{(r^2 - 1)^4} + \frac{\sigma^2 (r^2 - 1)}{(r^2 + 1)^3}}$$

(18)

and $F(\sigma, r) = F(\lambda(\sigma, r), r)$. This new set of parametric equations results in the critical curve $F = F(\lambda)$, for fixed $\sigma$. Finally, using eq. (14) $F = \sigma/f$ we can compute $f = f(\lambda)$ with the parametric functions $(\lambda(\sigma, r), \sigma/F(\sigma, r))$. This curve is shown as dashed (blue) line in Fig. 2(b) and differs from the heuristic approximation only for small values of $\lambda$.

**C. Transition from forced to mixed dynamics**

Synchronization with the external force is possible only if $F > F_{crit}$, estimated by Eq. (13). If $F < F_{crit}$ the system’s behavior is determined by the competition between spontaneous
FIG. 5. (color online) Frequency of oscillations for the fully connected network with 200 nodes for $F = 2.5$ fixed and fraction (a) $f = 100\%$; (b) $f = 90\%$; (c) $f = 80\%$ and (d) $f = 70\%$. The points show $r$ (blue triangles) and $\dot{\psi}$ (orange circles). The periods estimated from Eq. (22) are (a) $\tau = 3.8$; (b) $\tau = 3.4$; (c) $\tau = 3.1$ and (d) $\tau = 2.9$.

and forced motion. The transition between these two regimes was studied in detail in ref. [21] for the case of infinitely many oscillators, all of which coupled to the external drive. Here we present a simplified description of the transition using the analytical approach developed above.

Making the approximations $\langle \omega \rangle_k = 0$ and $\langle \sin \phi \rangle_{k,C} = \sin \langle \phi \rangle$, Eq. (6) simplifies to the
Adler equation \[32\]

\[
\frac{d\phi}{dt} = -\sigma - \bar{F}\sin \phi
\]  

(19)

where we are omitting the average symbol and considering regular networks to simplify the notation. For general networks we only need to make \( F \to \bar{F}_k \). This equation, which has been used to model fireflies \[33\] among other systems \[21\], can be solved exactly to give

\[
\sigma \tan \frac{\phi}{2} = \bar{F} + \sqrt{\bar{F}^2 - \sigma^2} \tanh \left[ \frac{1}{2} \sqrt{\bar{F}^2 - \sigma^2} (t - t_0) \right]
\]

(20)

for \( \bar{F} > \sigma \). In this case \( \phi \) converges to a constant value and the system stops (synchronizes with \( F \)). For \( \bar{F} < \sigma \), on the other hand, the solution is oscillatory,

\[
\sigma \tan \frac{\phi}{2} = \bar{F} - \sqrt{\sigma^2 - \bar{F}^2} \tan \left[ \frac{1}{2} \sqrt{\sigma^2 - \bar{F}^2} (t - t_0) \right]
\]

(21)

with period \[34\]

\[
\tau = \frac{2\pi}{\sqrt{\sigma^2 - \bar{F}^2}}.
\]

(22)

Figure 5 illustrates the frequency of oscillations for \( F < F_{\text{crit}} = 3 \) fixed and different number of nodes that receive the external drive, showing \( r \) and \( \dot{\psi} \) as a function of \( t \), for a fully connected network. Although \( r \) approaches 1 quickly (i.e., the system does synchronize), \( \dot{\psi} \) oscillates with growing periods as the number of nodes on \( C \) increases, remaining always negative. This means that \( \psi \) decreases monotonically and the order parameter \( z(t) \) oscillates, implying that a finite fraction of the oscillators has synchronized spontaneously, due to their mutual interactions and not to the drive. The approximation (22) for the periods of oscillation matches very well the results of the simulations.

D. Time to equilibrium

The time scale of dynamical processes also changes with the fraction of forced nodes. The time to equilibrium should increase when \( f \) decreases, but no simple relation seems to exist. When \( F \) is large, we can approximate Eq. (9) by

\[
\frac{d\langle \phi \rangle_k}{dt} = -F f \frac{\langle k \rangle_C}{\langle k \rangle}\sin \langle \phi \rangle_k.
\]

(23)

Defining \( t' = tF f \frac{\langle k \rangle_C}{\langle k \rangle} \) this equation becomes identical to that of a system where the force acts on all nodes. Therefore, within this crude approximation we expect that: (i) for
FIG. 6. (color online) Contour plot of time to equilibration for different values of $F$ and $f$ and fixed $\lambda = 40$. Thick lines correspond to constant times according to the approximation $F = F_0/f$ for $F_0 = 3, 5, 7$ and 9.

fixed $F$, the time to equilibration should scale as $\tau(f) = \tau_0 \langle k \rangle / [f \langle k \rangle C]$, where $\tau_0$ is the equilibration time at $f = 1$ and; (ii) along the curve $F_{\text{crit}}(f) = F \langle k \rangle / [f \langle k \rangle C]$ the time to equilibration remains constant, since the factors multiplying $F$ in Eq. (23) cancel out. Fig. (6) shows contour levels of numerically computed equilibration times in the $F \times f$ plane. Thick black lines shows predicted curves of constant times, which indeed provide a somewhat poor approximation to the computed values.

V. CONCLUSIONS

The Kuramoto model is perhaps the simplest dynamical system that allows the study of synchronization. Here we considered the problem of periodically forced oscillators where the external drive acts only on a fraction of them [21]. The problem is inspired by artificial heart pacemakers [16] and information processing in the brain induced by an external stimulus [2]. In both cases the stimulus is perceived by a subset of the system (a heart chamber of photo-receptor cells in the eye) and propagates to other parts of the network structure.

When the periodic drive acts on all oscillators, the system always synchronize with the
forced period if the force intensity is sufficiently large [21]. Using numerical simulations and analytical calculations we have shown that the force required to synchronize the entire set of oscillators increases roughly as the inverse of the fraction of forced nodes. The degree distribution of the complete network of interactions and of the set of forced nodes also affect the critical force for synchronization. Forcing oscillators with large number of links facilitates global synchronization in proportion to the average degree of the forced set to the total network.

We have also shown that below a critical fraction, that depends on $\lambda$, no synchronization occurs, no matter how large the force. We believe this is an interesting result of this study that might have consequences for adaptive systems relying on synchronization. The set of $N_C = fN$ nodes that directly receives the external drive can be interpreted as the interface of a system where the remaining $(1 - f)N$ nodes are the 'processing unit', that needs to synchronize with the external signal $F$ to perform a function. In this case it would be desirable to have $N_C$ as small as possible to increase the processing power. However, synchronization with small $f$ requires large couplings between the units, which can be costly. An example is neural network of $C. elegans$ where multiple links can connect the same two nodes and the cost of a connection is proportional to the number of such links (synaptic connections) that make it [35]. In these cases it is expected that a balance between interface size and network cost is attained, and natural systems should evolve toward this condition. The final balance will, of course, depend on the cost. If the cost is zero the system should evolve to the minimum possible interface size, given by $f = \sigma/F$ and $\lambda = \lambda_c + \sigma/f = \lambda_0 + F$ (for a fully connected network). If there is a cost it might be advantageous to work with a smaller processing unit (and larger interface) that requires smaller values of $\lambda$.

The theory developed here for $F_{crit}(f)$ considered only constant values of the coupling strength $\lambda$. In this case the term containing $\lambda$ in Eq.(4) disappears from the averaged Eq.(6). This equation, however, remains valid for arbitrary symmetric couplings $\lambda_{ij} = \lambda_{ji}$, as can be easily verified by inspection. For asymmetric couplings, $\lambda_{ij} \neq \lambda_{ji}$, this is not true and an extra term has to be included in Eq.(6). However, since this term is proportional to $\sin(\theta_j - \theta_i)$ it vanishes when the system synchronizes and Eq.(13) still holds, being, therefore, a very robust result.

As a final comment we note that here we have picked nodes for the set $C$ at random or based on their degree. Another interesting choice would be to pick them according to
their natural frequencies $\omega_i$. For finite systems the oscillator with the largest frequency determines the spontaneous synchronization of the system [31] and forcing the fastest nodes might also result in interesting dynamics.

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