Degree Formula for Grassmann Bundles

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Abstract. Let $X$ be a non-singular quasi-projective variety over a field, and let $E$ be a vector bundle over $X$. Let $G_X(d, E)$ be the Grassmann bundle of $E$ over $X$ parametrizing corank $d$ subbundles of $E$ with projection $\pi : G_X(d, E) \to X$, and let $Q \leftarrow \pi^*E$ be the universal quotient bundle of rank $d$, and denote by $\theta$ the Plücker class of $G_X(d, E)$, that is, the first Chern class of the Plücker line bundle, det $Q$. In this short note, a closed formula for the push-forward of powers of the Plücker class $\theta$ is given in terms of the Schur polynomials in Segre classes of $E$, which yields a degree formula for $G_X(d, E)$ with respect to $\theta$ when $X$ is projective and $\wedge^d E$ is very ample.

0. Introduction

Let $X$ be a non-singular quasi-projective variety of dimension $n$ defined over a field of arbitrary characteristic, and let $E$ be a vector bundle of rank $r$ over $X$. Let $G_X(d, E)$ be the Grassmann bundle of $E$ over $X$ parametrizing corank $d$ subbundles of $E$ with projection $\pi : G_X(d, E) \to X$, and let $Q \leftarrow \pi^*E$ be the universal quotient bundle of rank $d$ on $G_X(d, E)$. We denote by $\theta$ the first Chern class $c_1(\text{det } Q) = c_1(Q)$ of $Q$, and call $\theta$ the Plücker class of $G_X(d, E)$. Note that the determinant bundle $\text{det } Q$ is isomorphic to the pull-back of the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\wedge^d E)}(1)$ of $\mathbb{P}_X(\wedge^d E)$ by the relative Plücker embedding over $X$.

The purpose of this short note is to give a closed formula for $\pi_* \theta^N$, the push-forward of powers $\theta^N$ of the Plücker class $\theta$ to $X$ by $\pi$, in terms of the Schur polynomials in Segre classes of $E$, where $\pi_* : A^{*+d(r-d)}(G_X(d, E)) \to A^*(X)$ is the push-forward by $\pi$ between the Chow rings.

The result is

Theorem 0.1. For each integer $N \geq d(r-d)$, we have

$$
\pi_* \theta^N = \sum_{|\lambda| = N - d(r-d)} f^{\lambda+\varepsilon} \Delta_{\lambda}(s(E))
$$

in $A^{N-d(r-d)}(X)$, where $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a partition with $|\lambda| := \sum_{1 \leq i \leq d} \lambda_i$, $\Delta_{\lambda}(s(E)) := \text{det}[s_{\lambda_i+j-i}(E)]_{1 \leq i, j \leq d}$ is the Schur polynomial in Segre classes of $E$ corresponding to $\lambda$, $\varepsilon := (r-d)^d = (r-d, \ldots, r-d)$, and $f^{\lambda+\varepsilon}$ is the number of standard Young tableaux with shape $\lambda + \varepsilon$.

The Segre classes $s_i(E)$ here are the ones satisfying $s(E)c(E^\vee) = 1$ as in [1], [5], [6], where $s(E)$ and $c(E)$ denote respectively the total Segre class and the total Chern class of $E$. Note that our Segre class $s_i(E)$ differs by the sign $(-1)^i$ from the one in [2].

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Corollary 0.2 (degree formula for Grassmann bundles). If $X$ is projective and $\wedge^d \mathcal{E}$ is very ample, then $G_X(d, \mathcal{E})$ is embedded in the projective space $\mathbb{P}(H^0(X, \wedge^d \mathcal{E}))$ by the tautological line bundle $\mathcal{O}_{G_X(d, \mathcal{E})}(1)$, and its degree is given by

$$\deg G_X(d, \mathcal{E}) = \sum_{|\lambda| = n} f^{\lambda + \varepsilon} \int_X \Delta_\lambda(s(\mathcal{E})).$$

Here a vector bundle $\mathcal{F}$ over $X$ is said to be very ample if the tautological line bundle $\mathcal{O}_{\mathcal{F}}(1)$ of $\mathbb{P}(\mathcal{F})$ is very ample.

Setting $n := 0$, we recover the degree formula of Grassmann varieties, as follows:

Corollary 0.3 ([2 Example 14.7.11 (iii)]). Let $G(d, r)$ be the Grassmann variety parametrizing codimension $d$ subspaces of a vector space of dimension $r$. Then its degree with respect to the Plücker embedding is given by

$$\deg G(d, r) = \frac{(d(r - d))! \prod_{1 \leq l \leq d - 1} l!}{\prod_{1 \leq l \leq d} (r - l)!}.$$  

1. Proofs

Proof of Theorem 0.1. Let $\xi_1, \ldots, \xi_d$ be the Chern roots of the universal quotient bundle $\mathcal{Q}$. Then we can write $\theta = \xi_1 + \cdots + \xi_d$ formally. Using Pieri’s formula [3, §2.2] repeatedly, and applying Jacobi-Trudi identity [7, I, (3.4)], we obtain that

$$\theta^N = \sum_{|\mu| = N} f^\mu \Delta_\mu(\xi) = \sum_{|\mu| = N} f^\mu \Delta_\mu(s(\mathcal{Q})), $$

where $\Delta_\mu(\xi)$ is the Schur polynomial in $\xi := (\xi_1, \ldots, \xi_d)$ corresponding to a partition $\mu$. It follows from the push-forward formula of Józefiak-Lascoux-Pragacz [4, Proposition 1] that

$$\pi_* \Delta_\mu(s(\mathcal{Q})) = \Delta_{\mu - \varepsilon}(s(\mathcal{E})).$$

Therefore we obtain

$$\pi_* \theta^N = \sum_{|\mu| = N} f^\mu \Delta_{\mu - \varepsilon}(s(\mathcal{E})) = \sum_{|\lambda| = N - (r - d)} f^{\lambda + \varepsilon} \Delta_\lambda(s(\mathcal{E})),$$

where $\lambda$ is a partition, and $\varepsilon := (r - d)^d = (r - d, \ldots, r - d)$.

Proof of Corollary 0.2. By the assumption $G_X(d, \mathcal{E})$ is projective and the tautological line bundle $\mathcal{O}_{G_X(d, \mathcal{E})}(1)$ defines an embedding $\mathbb{P}(\wedge^d \mathcal{E}) \hookrightarrow \mathbb{P}(H^0(X, \wedge^d \mathcal{E}))$. Therefore $G_X(d, \mathcal{E})$ is considered to be a projective variety in $\mathbb{P}(H^0(X, \wedge^d \mathcal{E}))$ via the relative Plücker embedding $G_X(d, \mathcal{E}) \hookrightarrow \mathbb{P}(\wedge^d \mathcal{E})$ over $X$ defined by the quotient $\wedge^d \pi^* \mathcal{E} \rightarrow \wedge^d \mathcal{Q} = \det \mathcal{Q}$. Since the hyperplane section class of $G_X(d, \mathcal{E})$ is equal to the Plücker class $\theta$, we obtain the conclusion, taking $N := \dim G_X(d, \mathcal{E}) = d(r - d) + n$ in Theorem 0.1.

Proof of Corollary 0.3. The conclusion follows from Corollary 0.2 with $n := 0$, since the number $f^{\lambda + \varepsilon}$ is known to be given as follows ([31 p.53] and [33 p.54, Exercise 9]):

$$f^{\lambda + \varepsilon} = \frac{N! \prod_{1 \leq i < j \leq d} (\lambda_i - \lambda_j - i + j)}{\prod_{1 \leq i \leq d} (r + \lambda_i - i)!}.$$ 

□

Remark 1.1. Under the same assumption as in Theorem 0.1 one can prove a push-forward formula of the following form:

$$\pi_* \theta^N = \sum_{|k| = N - (r - d)} \frac{N! \prod_{1 \leq i < j \leq d} (k_i - k_j - i + j)}{\prod_{1 \leq i \leq d} (r + k_i - i)} \prod_{i=1}^d s_{k_i}(\mathcal{E})$$
in $A^{N-d(r-d)}(X) \otimes \mathbb{Q}$, where $k = (k_1, \ldots, k_d) \in \mathbb{Z}_{\geq 0}^d$ with $|k| := \sum_i k_i$, and $s_i(\mathcal{E})$ is the $i$-th Segre class of $\mathcal{E}$.

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