A coherent Ising machine with quantum measurement and feedback control

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A coherent Ising machine (CIM) based on a multiple-pulse degenerate optical parametric oscillator (DOPO) is proposed to solve NP-hard combinatorial optimization problems. The underlying mechanism originates from the bistable phase operation of each DOPO pulse and the inherent preference of the coupled DOPOs in selecting an oscillation mode with the minimum photon decay rate. In order to build a scalable CIM, a new quantum feedback scheme via optical homodyne detection and electronic signal processing must replace a massive parallel optical interconnections. This paper presents the basic concept, theoretical model and numerical results of the proposed scheme. Computational experiments are performed on NP-hard MAX-CUT problems with number of vertices up to 20000. The numerical results suggest the substantial speed-up of computation on a CIM with the feedback scheme against classical algorithms such as Goemans-Williamson semidefinite programming and simulated annealing.

I. INTRODUCTION

In statistical physics, the Ising model is the simplest theoretical description of magnetism and spin glass. The energy of a spin configuration \( \sigma = (\sigma_i)_{i \in V} \) on lattice sites \( V \) of size \( N \) is written by weighted two-spin interactions:

\[
\mathcal{H}(\sigma) = -\sum_{i<j} J_{ij} \sigma_i \sigma_j \quad (1 \leq i, j \leq N),
\]

where each spin takes one of two states \( \sigma_i \in \{\pm 1\} \), and matrix \( J \in \mathbb{R}^{N \times N} \) denotes a coupling constant. The Ising problem is one of the combinatorial optimization problems \( [1] \) (e.g., in computer science \( [2] \), life-science and drug discovery \( [3] \), and statistical physics \( [4] \)) to find the ground state of an Ising Hamiltonian \( [1] \). If the graph is three-dimensional, i.e., non-planar, the Ising problem is known to be non-deterministic polynomial-time (NP)-hard to find the optimal spin configurations \( [3] \). A MAX-CUT problem of finding the size of the largest cut in a given graph, where a cut is a partition of the vertices \( V \) into two sets and the size of the cut is the number of edges with one vertex on either side of the partition, is equivalent to an antiferromagnetic Ising problem in the following equation \( [6] \):

\[
C(\sigma) = -\frac{1}{2} \sum_{i<j} J_{ij} - \frac{1}{2} \mathcal{H},
\]

where the edge weight of the graph \( w_{ij} \in \mathbb{R} \) gives \( J_{ij} = -w_{ij} \) and \( \mathcal{H} \) is an Ising Hamiltonian as defined in Eq. \( [1] \).

In the optimal solution for the unweighted graph, the maximum number of spin pairs should satisfy the lower energy state of two-body interaction term in \( \mathcal{H} \). Assuming the unique games conjecture, the approximation ratio of 0.87856 achieved by the Goemans-Williamson algorithm (GW) based of semi-definite programming (SDP) is optimal.

The simulated annealing algorithm (SA) was designed by mimicking the thermal annealing procedure in metallurgy \( [7] \) to tackle NP-hard Ising problems. A quantum annealing technique was also formulated \( [8] \) and was shown to have competitive performance against simulated annealing \( [9] \). As a variant of quantum annealing, quantum adiabatic computation was devised based on the adiabatic theorem of quantum mechanics \( [10] \) with a computational power equivalent to that of a standard quantum computer based on unitary gate operation \( [11,12] \). In addition, recent algorithmic studies have reported remarkable improvement on best-known accuracy and computation time against a specific problem set \( [13] \). However, these algorithms have not yet been generally proven to be efficient for NP-hard problems.

We recently proposed a physical system, coherent Ising machine (CIM), to solve the Ising problems using lasers \( [14–16] \) and degenerate optical parametric oscillators (DOPOs) \( [17,18] \). The architecture of this machine is motivated by the principle of oscillators in which the mode with the minimum loss rate is most likely to be excited first. The energy of the Ising Hamiltonian can be mapped onto the total loss rate of a network. The desired oscillating mode in a DOPO network, which corresponds to the ground state of a given Hamiltonian, then competes with other possible oscillating modes for the available gain and reduces the gain accessible to these other modes due to the cross-gainsaturation effect.

We proposed in this paper a novel scheme of which implements the loss rate modulation via quantum-measurement feedback control. We conducted a numerical study based on this idea for a large problem size. In Section \( [19] \) we introduce the \( c \)-number Langevin equation for DOPO signal pulse amplitudes, which will be used
in the simulation of the CIM with proposed quantum measurement-feedback control scheme. We then discuss our numerical study in Sections III and IV against the two representative algorithms on current digital computers to verify improved performance in computation time. Finally, we conclude the paper in Section V.

II. MULTIPLE-PULSE DOPO WITH QUANTUM MEASUREMENT-FEEDBACK CONTROL

A. A proposed machine

We propose a new measurement-based scheme for the coherent Ising machine as shown in Fig. 1. A fiber ring resonator is composed of three components: an optical parametric amplifier and directional couplers I and II, which are used as an out-coupling port to the homodyne detectors and feedback port for providing the mutual coupling between the DOPO pulses, respectively. A second harmonic generation (SHG) pulse train at a wavelength of 0.78 µm generates multiple DOPO pulses at a wavelength of 1.56 µm inside a fiber ring cavity. The local oscillator (LO) pulse for homodyne detection and feedback pulse for the mutual coupling are directly obtained from the master laser at a wavelength of 1.56 µm.

If the round trip time of a ring resonator is properly adjusted to N times the pulse pump interval, we can simultaneously generate N independent DOPO pulses inside a cavity. Each of these pulses is used as an Ising spin. Instead of connecting these pulses with optical delay lines, we can measure the in-phase amplitude \( \tilde{c}_j \) of the \( j \)-th pulse and compute the proper feedback pulse amplitude for the \( i \)-th pulse, \( \sum_j \xi_{ij}\tilde{c}_j \), by the digital electronic circuit, where the coupling coefficient \( \xi_{ij} \) is proportional to the Ising coupling \( J_{ij} \). This electrical signal drives the intensity and phase modulators to generate a feedback pulse for the \( i \)-th pulse. Such a hybrid optoelectric coupling scheme is equivalent to the purely optical coupling scheme except for the small difference in noise penalty, which we discuss in the next section.

A clear advantage of the new scheme is that all the Ising coupling \( J_{ij} \) on the order of \( N^2 \) can be implemented by a single quantum measurement-feedback control circuit. In Sections IV B and IV C, we assume a CIM with a clock frequency of 2 GHz (pump pulse interval of 0.5 ns), fiber length of 2 km (round trip time of 10 µs) and \( 2 \times 10^4 \) DOPO pulses. Even for such a relatively small number of DOPO pulses, the above advantage is substantial in practical implementation.

B. \( \alpha \)-number Langevin equations for multi-pulse DOPO with quantum feedback control

The in-phase and quadrature-phase amplitudes of a single isolated DOPO obey the following \( \alpha \)-number Langevin equations [17]:

\[
dc = (-1 + p - c^2 - s^2)\frac{c}{2A_c} dt + \frac{1}{A_c} \sqrt{c^2 + s^2 + \frac{1}{2}} dW_1
\]

\[
ds = (-1 - p - c^2 - s^2)s \frac{ds}{2A_s} dt + \frac{1}{A_s} \sqrt{c^2 + s^2 + \frac{1}{2}} dW_2
\]

The above \( \alpha \)-number Langevin equations are derived by expanding the field density operator with the Wigner distribution function. The two approaches by the Wigner function and the generalised P-representation are equivalent for highly dissipative systems such as ours. The pump field is adiabatically eliminated in (3) and (4) by assuming that the pump photon decay rate \( \gamma_p \) is much larger than the signal photon decay rate \( \gamma_s \). The term \( A_s = (\gamma_s \gamma_p / 2 \kappa^2)^{1/2} \) is the DOPO field amplitude at a normalized pump rate \( p = F_p / F_{th} = 2 \), and \( \kappa \) is the second order nonlinear coefficient associated with the degenerate optical parametric amplification (DOPA). The term \( t = \gamma_s \tau / 2 \) is a normalized time, while \( \tau \) is a real time in seconds. The term \( F_p \) is the pump field amplitude and \( F_{th} = \gamma_s \gamma_p / 4 \kappa^2 \) is the threshold pump field amplitude. Finally, \( dW_1 \) and \( dW_2 \) are two independent Gaussian noise processes that represent the incident vacuum fluctuations from the open port of directional coupler I and the pump field fluctuation for in-phase and quadrature-phase components, respectively. The vacuum fluctuation of the signal channel contributes to the 1/2 term and the quantum noise of the pump field contributes to \( c^2 + s^2 \) in the square-root bracket in (3) and (4). We ignore the vacuum fluctuation for the open port of directional coupler II because the coupling constant is normally set to a very small value. The \( \alpha \)-number Langevin equations (3) and (4) are consistent with the master equation for the signal field density matrix and the Fokker-Planck equation for the Wigner function [18, 19].

When the \( i \)-th signal pulse is incident upon directional coupler I, the output-coupled field and remaining cavity field are written as

\[
c_{i,\text{out}} = \sqrt{T}c_i - \sqrt{1 - T}f_i / A_s
\]

\[
c_{i,\text{cavity}} = \sqrt{1 - T}c_i + \sqrt{T}f_i / A_s
\]

where \( T \) is the power transmission coefficient of directional coupler I and \( f_i \) is the incident vacuum fluctuation from the open port of that coupler. The optical balanced homodyne detection for the out-coupled-field “measures” an inferred signal amplitude

\[
\tilde{c}_i \equiv c_{i,\text{out}} / \sqrt{T} = c_i - \sqrt{1 - T}f_i / A_s
\]

Note there is no additional noise in Eq. 7 except for the incident vacuum field fluctuation \( f_i \). A balanced homodyne detector with a 3 dB coupler (50%-50% beam
FIG. 1. Multiple-pulse DOPO with quantum feedback control. Small portion of each signal pulse transmits through the directional coupler I, and its in-phase component is measured by optical balanced homodyne detector, where LO pulse is directly obtained from master laser. Two detector outputs are converted to digital signals and input into electronic digital circuit, where feedback signal for i-th signal pulse is computed. Independently obtained feedback pulse from master laser is modulated in its intensity and phase to achieve \( \sum_j \xi_{ij} \tilde{c}_j \) and coupled into i-th signal pulse by directional coupler II. Flows of optical fields and electrical signals are shown as solid and dashed lines, respectively.

The master equation for a quantum system with quantum measurement-feedback control is formulated using the Linblad operator \([20, 21]\). In the highly dissipative limit of an feedback circuit, such as in our scheme, we can use the c-number Langevin equation supplemented with the noisy feedback term. Since the transmission coefficient \( \sqrt{T'} \) of directional coupler II should be much smaller than one, in order to keep the ring cavity Q-value high enough, we do not need to consider any additional noise in the injected feedback pulse. The c-number Langevin equation \( \mathbf{8} \) can be now rewritten to include the mutual coupling terms

\[
 dc_i = [( -1 + p - c_i^2 - s_i^2 ) c_i + \sum_j \xi_{ij} \tilde{c}_j ]\,dt \\
+ \frac{1}{A_s} \sqrt{c_i^2 + s_i^2 + \frac{1}{2}} \,dW_i. \quad \text{(8)}
\]

The summation in Eq. \( \mathbf{8} \) represents the quantum measurement-feedback term including the measurement noise effect. The vacuum fluctuation coupled to the i-th pulse in directional coupler I is already taken into account in the second term of right-hand side together with the pump noise.

Note that the coupling coefficient \( |\xi_{ij}| \) can be greater than one, since we start with a high-intensity feedback pulse directly obtained from the master laser. If we do not use such a quantum measurement-feedback control scheme, we must introduce a laser amplifier with a power gain of \( G \) in the optical coupling path to achieve \( |\xi_{ij}| > 1 \). In this case, the summation in Eq. \( \mathbf{8} \) is replaced with

\[
 \sum_j \xi_{ij} (c_j - \sqrt{1 - T' T''} f_j + \sqrt{1 - T' T''} f_a), \quad \text{(9)}
\]

where \( \xi_{ij} = \sqrt{GT'} \), \( T' \) is the transmission coefficients of the two directional couplers, which transmits a part of the j-th signal pulse and adds the amplified feedback pulse to the i-th signal pulse \( \mathbf{18} \), and \( f_a \) is the internal noise of the amplifier. When \( T' \approx 1 \), the added noise power is at least doubled because a laser amplifier must add the internal noise, which is equal to or greater than the vacuum fluctuation.

We used the numerical simulation of the coupled c-number Langevin equations \( \mathbf{8} \) to evaluate the performance of a new coherent Ising machine with quantum measurement-feedback control.

### III. BENCHMARK STUDIES AGAINST SMALL INSTANCES

**A. Simple MAX-CUT-3 problem**

If each vertex in a graph has exactly three edges, the graph is called a cubic graph and the corresponding MAX-CUT problem (MAX-CUT-3) is known to be in the NP-hard class\([22]\).

The smallest simple MAX-CUT-3 problem is defined on the graph with four vertices and six edges with identical weight \( J_{ij} = -1 \). The solutions to this problem...
are the set of two-by-two cuts, which contains six degenerate ground states of the Ising Hamiltonian, i.e., \{\{↑↑↓↓\}, \{↑↓↑↓\}, \{↓↑↑↓\}, \{↓↓↑↑\}, \{↓↑↓↑\}, \{↓↓↓↑\}\}.

Figure 2 shows the time evolution of \(c_i\) (\(i = 1, \ldots, 4\)) when \(p = 1.1\) and \(\xi = -0.1\). A correct solution spontaneously emerges after several tens of roundtrips. The statistics of obtaining different states against 1000 sessions of such a numerical simulation are shown in Fig. 3 in which six correct solutions appear with almost equal probabilities with no errors found.

### B. Many-body problem

If the Ising interaction is not a standard two-body interaction but rather a four-body interaction such as

\[
\mathcal{H} = -J_{1234} \sigma_1 \sigma_2 \sigma_3 \sigma_4, \tag{10}
\]

where \(J_{1234} \in \mathbb{R}\), the coupled field into the \(i\)-th pulse is no longer given by \(\sum_j \xi_i \tilde{c}_j\) but by \(\xi \tilde{c}_i \tilde{c}_j \tilde{c}_k \tilde{c}_l\) (\(j, k, l \neq i\)).

In this case, the c-number Langevin equation (3) can be rewritten to include the four-body coupling term

\[
dc_i = \left((-1 + p - c_i^2 - s_i^2) c_i + \xi \tilde{c}_j \tilde{c}_k \tilde{c}_l\right) dt + \frac{1}{\Delta} \sqrt{c_i^2 + s_i^2 + \frac{1}{2}} dW_i. \tag{11}
\]

If the four-body coupling coefficient \(J_{1234}\) is \(-1\) (multi-body anti-ferromagnetic coupling), there are eight degenerate ground states, i.e., \{\{↑↑↓↓\}, \{↑↓↑↓\}, \{↑↑↑↓\}, \{↓↑↑↓\}, \{↓↓↑↑\}\} and the inverses. Figure 4 shows the time evolution of \(c_i\) (\(i = 1, \ldots, 4\)) when \(p = 1.1\) and \(\xi = -0.1\). One of the eight degenerate ground states emerges spontaneously after several tens of round trips. The statistics of observing different states in 1000 independent sessions of the numerical simulation of Eq. (11) are shown in Fig. 5 in which eight correct solutions are obtained with almost equal probabilities with no errors found.
IV. BENCHMARK STUDY AGAINST LARGE INSTANCES

A. Algorithm description

The Goemans-Williamson algorithm (GW) based on SDP has a 0.87856-performance guarantee for NP-hard MAX-CUT problems [23]. It achieves the optimal approximation ratio for MAX-CUT problems [23]. It achieves the optimal approximation ratio for MAX-CUT problems [23]. With this algorithm, the original MAX-CUT problem is extended to a vector-valued optimization problem as maximizing $\frac{1}{2} \sum_{i<j} w_{ij}(1-\vec{v}_i \cdot \vec{v}_j)$, $\vec{v}_i \in S^{k-1}$, where $S^{k-1}$ is a unit sphere in $\mathbb{R}^k$ and $k \leq N$ (or $\#V$: number of vertices). Since the computational complexity of this relaxation problem is P (with error $\varepsilon > 0$), there is a polynomial time algorithm to find the optimal solution, and its value is commonly called the SDP upper bound. A final solution to the original MAX-CUT problem is obtained by projecting the solution vector sets to randomly chosen one-dimensional Euclidean spaces (or dividing by random hyperplanes).

There are three types of computational complexities of the best-known algorithms for solving the SDP relaxation problem. If a graph with $N$ vertices and $m$ edges is regular, the SDP problem can be approximately solved in almost linear time as $O(m) = O(m \log N)^2 \varepsilon^{-4}$ using the matrix multiplicative weights method [23], where $\varepsilon$ represents the accuracy of the obtained solution. However, slower algorithms are required for general graphs. If the edge weights of the graph are all non-negative, the fastest algorithm runs in $O(Nm) = O(Nm(\log N)^2 \varepsilon^{-3})$ time based on the Lagrangian relaxation method [20]. For graphs with both positive and negative edge weights, the SDP problem is commonly solved using the interior-point method, which scales as $O(N^{3.5}) = O(N^{3.5} \log(1/\varepsilon))$ [21]. There are also faster general algorithms if only the matrices are sparse [28, 29].

In our computational experiments, the COPLSDP [30] was used as a MAX-CUT solver. The SDP upper bound $U_{SDP}$ and the solution $C_{SDP}$ were obtained using the following parameters: interior point method was used until the relative gap $r_{gap} = 1 - P_{obj}/D_{obj}$ reached $10^{-3}$, where $P_{obj}$ and $D_{obj}$ are the objective functions of the primal and dual of the SDP problem, respectively [31]. Random projection onto one-dimensional space was executed $N(= \#V)$ times.

The computation time of the GW algorithm generally scales in $O(N^{3.5})$, which still requires long computation time despite its complexity class of P when the number of vertices $N$ increases. Hence, various heuristic methods have been developed with a computation time proportional to the smaller power of the problem size. Metropolis et al. introduced a simple algorithm that can be used to provide an efficient simulation of a collection of atoms in equilibrium at a given temperature [32]. Kirkpatrick et al. applied the algorithm to optimization problems by replacing the energy of the atomic system to the cost function of optimization problems and using spin configurations $\sigma$, which is called the simulated annealing algorithm (SA). In each step of this algorithm, a system is given with a random spin flip and the resulting change $\Delta E$ in the energy of the cost function is computed. If $\Delta E \leq 0$, the spin flip is accepted, and the configuration with the flipped spin is used as the starting point of the next step. If $\Delta E > 0$, the spin is treated probabilistically, i.e., the probability that the new configuration is accepted is $P(\Delta E) = \exp(-\Delta E/k_B T)$ with a control parameter of system temperature $T$. This choice of $P(\Delta E)$ results in the system evolving into an equilibrium Boltzmann distribution.

These codes are written in C/C++ and ran on a Linux machine with two 6-core Intel Xeon (2.67 GHz) processors and 94 GB memory.

B. Benchmark study of accuracy

The performance of a CIM with DOPO network and quantum measurement-feedback was tested against the NP-hard MAX-CUT problem on G-set graphs [33]. These test instances were randomly constructed using a machine-independent graph generator written by G. Rinaldi, with the number of vertices ranging from 800 to 20000, edge density from 0.02% to 6%, and geometry from random, almost planar, to toroidal.

The outcomes of running the CIM, SA and GW for some of G-set graphs are summarized in Table I, where $p = 0.2$ and $\xi = -0.03$. The result of CIM is obtained in less than 1 s for each graph; (300 DOPO cavity round trips, i.e., 3 ms) $\times$ (300 trials). The result of SA is obtained in 100 s for each graph; (single cooling in 1 s) $\times$ (100 trials). For GW, the computation time ranged between 2.3 s and 1.1 $\times$ 10$^6$ s, depending on $N$. The best outputs of the CIM were about 0.8-1.79% better than GW except for some toroidal graphs (g48, g49, g50) and a disconnected random graph (g70).

As the size of optimization problems increases, the average computation accuracy will dominate practical applications. This Table I shows that for all G-set graphs, the average value in 300 trials is 0.93307, i.e., the CIM can find a CUT value larger than 0.93307 of the optimal value on average, whereas the accuracy of the GW is 0.93025 and that of the SA is 0.94290. The optimal accuracy suppressed by the probabilistically checkable proof (PCP) theorem is less than 0.94118 [34, 35].

C. Benchmark study of computation time

Computational experiments were conducted on the same machine as those discussed in the previous section and the computation time of each algorithm was measured on CPU time. Note that the input time of the problem parameters is not included and the pure computation time was evaluated. The time and temperature
TABLE I. Performance of the coherent Ising machine, simulated annealing and Goemans-Williamson SDP algorithm in solving the MAX-CUT problems on sample G-set graphs. \#V is the number \(N\) of vertices in the graph, \#E is the number \(m\) of edges, \(U_{\text{SDP}}\) is the optimal solution to the semidefinite relaxation of the MAX-CUT problem, and \(C_{\text{GW}}\) is the best solution obtained by \(N(=\#V)\) projections after SDP. \(C_{\text{SA}}\) and \(\langle C_{\text{GW}} \rangle\) are the best and average values obtained by SA in 100 trials of 1 s. \(C_{\text{CIM}}\) and \(\langle C_{\text{CIM}} \rangle\) are the best and average values in CIM 300 runs of 3 ms (= 300 DOPO cavity round trips), respectively. To make comparisons with each other, every cut value \(C\) generated from the CIM, SA, or GW algorithm is normalized according to \((C + \langle C_{\text{GW}} \rangle)/(U_{\text{SDP}} + \langle C_{\text{GW}} \rangle)\), where \(\langle C_{\text{GW}} \rangle \geq 0\) is the number of negative edges. In the bottom of this table, the average values of all 71 G-set graphs are shown.

| Graph | \#V | \#E | \(U_{\text{SDP}}\) | \(C_{\text{GW}}\) | \(C_{\text{SA}}\) | \(\langle C_{\text{SA}} \rangle\) | \(C_{\text{CIM}}\) | \(\langle C_{\text{CIM}} \rangle\) |
|-------|-----|-----|-------------------|-----------------|----------------|-----------------|----------------|----------------|
| g1    | 800 | 19176 | 0.94177 | 0.96199 | 0.95696 | 0.95796 | 0.95168 |
| g6    | 800 | 19176 | 0.94177 | 0.96663 | 0.95586 | 0.95743 | 0.95165 |
| g11   | 800 | 1600 | 0.93272 | 0.95255 | 0.94739 | 0.94263 | 0.93058 |
| g17   | 800 | 4694 | 0.93257 | 0.95386 | 0.94986 | 0.94250 |
| g22   | 2000 | 19990 | 0.91194 | 0.93485 | 0.93584 | 0.93006 |
| g27   | 2000 | 19990 | 0.91194 | 0.93770 | 0.93388 | 0.93530 | 0.92938 |
| g32   | 2000 | 4000 | 0.92717 | 0.95079 | 0.94601 | 0.93954 | 0.93141 |
| g35   | 2000 | 11778 | 0.92285 | 0.93510 | 0.93497 | 0.93422 | 0.93985 |
| g39   | 2000 | 11778 | 0.92285 | 0.93921 | 0.93350 | 0.93647 | 0.92879 |
| g43   | 1000 | 9990 | 0.93298 | 0.94411 | 0.93786 | 0.93956 | 0.93297 |
| g44   | 1000 | 9990 | 0.93298 | 0.94411 | 0.94601 | 0.94933 | 0.94633 |
| g51   | 1000 | 5909 | 0.93335 | 0.94507 | 0.94933 | 0.94983 | 0.94249 |
| g55   | 5000 | 12498 | 0.90963 | 0.91168 | 0.90642 | 0.91086 | 0.90568 |
| g57   | 5000 | 10000 | 0.93234 | 0.94800 | 0.94497 | 0.93565 | 0.91333 |
| g59   | 5000 | 29570 | 0.91483 | 0.93365 | 0.93070 | 0.92780 | 0.86414 |
| g60   | 7000 | 17148 | 0.89890 | 0.90098 | 0.90547 | 0.90908 | 0.90485 |
| g66   | 7000 | 14159 | 0.91426 | 0.93360 | 0.92975 | 0.92994 | 0.85857 |
| g67   | 10000 | 20000 | 0.92153 | 0.94566 | 0.94376 | 0.93601 | 0.93147 |
| g70   | 10000 | 9999 | 0.96130 | 0.95397 | 0.95375 | 0.95375 |
| g81   | 20000 | 40000 | 0.91947 | 0.94225 | 0.94099 | 0.93305 | 0.90535 |
| Average | | | 0.93025 | 0.94162 | 0.94120 | 0.94134 | 0.93507 |

Scheduling parameters of the SA were set as follows: Inverse temperature increased with logarithmic function. The number of spin flipping was optimized to be \(10^4\) times for some \(l \in \mathbb{N}\), which requires the minimum computation time to achieve the same accuracy as with the GW.

Since there is no guarantee that the CIM or SA can efficiently find optimal solutions for MAX-CUT problems, the GW solution has been used as the mark of sufficient accuracy (note that the actual ground state are generally not known for these size of problems). The CIM and SA then competed the computation time to reach that accuracy. Figure 5(b) shows the Ising energy in Eq. (1) as a function of real computation time when \(p = 0.2\) and \(\xi = -0.03\) (for \(N = 800\)), \(\xi = -0.003\) (for \(N = 4000\)). Both the CIM and SA ran stochastically due to quantum and thermal noise. For the CIM, the energy of all 100 runs was averaged at each round trip. For the SA, the average was calculated at each point on the time axis with an interpolated value from real time sampling. In Fig. 5(a), where \(N = 800\), the GW outputted energy equal to \(-15624\) in 22.96 s, while the CIM and SA reached the same energy in \(6.1 \times 10^{-4}\) s and in \(0.671\) s. The CIM was \(10^4\) times faster than the SA. In Fig. 5(b), the GW produced energy of \(-168160\) in 6646.25 s, while the CIM reached the same energy in \(5.5 \times 10^{-4}\) s and the SA did so in 10.1 s. The CIM was \(10^4\) times faster than the SA.

Figure 6 shows the computation time versus problem size (number of vertices). The computation time is defined as the actual time to solve a given MAX-CUT problem for GW and the actual time to reach the same accuracy achieved by GW for SA and CIM. The preparation time needed to input \(J_{ij}\) into the computing system is not included. The three types of time scaling complexity are also shown. The time complexity of the GW is consistent with previous results, \(O(N^3)\) and is dominated by the Cholesky decomposition on the interior-point method in the general Goemans Williamson algorithm. The SA seems to scale in \(O(N^2)\). This indicates that it requires the number of spin flips to be proportional to \(N\) to achieve optimal performance. Each spin flip costs computation time proportional to the vertex degree. In the case of complete graphs, this number is equal to \(N - 1\). Thus, the computation time scales as \(O(N^2)\) for the SA. The CIM exhibits a problem-size independent computation time around \(10^{-3}\) s, which is determined by the turn-on delay time of the DOPO network.

### V. SUMMARY AND DISCUSSION

To build a large-scale DOPO network as a coherent Ising machine, we proposed a quantum measurement-feedback scheme. The potential for solving NP-hard Ising problems using a CIM was numerically studied by conducting computational experiments using the MAX-CUT problems on G-set graphs and complete graphs of up to 20000. With the normalized pump rate and coupling coefficient \(p = 0.2\) and \(\xi = -0.03\), the CIM achieved a good approximation rate of 0.93307 on average and found better cut compared to the GW for 67 out of 71 graphs, but worse cut compared to the SA. The time scaling property was tested against up to 20000 complete graphs. The results imply that CIM achieves almost constant time scaling.

The choice of parameters is different from the previously reported results [17], where the success probability of finding the ground state was investigated. In contrast, we tuned the parameters so that the average accuracy for larger instances is optimized.

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FIG. 6. Performance comparison of CIM, SA, and GW in solving complete graphs (a) $K_{800}$ and (b) $K_{4000}$, where each edge was randomly weighted ±1. Each time, bundle of curves depicted average energy (solid line) ± and standard deviations (dashed line) in 100 runs. Dotted line was obtained using GW algorithm, which is shown with dot. The number of flips in SA algorithm were $10^5$ for $K_{800}$ and $10^6$ for $K_{4000}$, respectively, to optimize computation time.

FIG. 7. Computation time of coherent Ising machine, simulated annealing algorithm, and Goemans-Williamson SDP algorithm fitted to lines indicating $O(1)$, $O(N^2)$, and $O(N^{3.5})$, respectively. The computation time for CIM and SA is defined as the time to reach the same accuracy achieved by GW. Data points of CIM and SA were calculated by averaging 100 runs, except for $N = 40$ and $N = 800$. Since the optimal energy of $N = 40$ graph obtained using GW is regarded as ground state, half of the 100 runs of stochastic algorithms were post-selected to reach that value. And only for $N = 800$ case, 100 randomly ±1 weighted complete graphs are generated. The results for $N = 800$ show the average computation time for 100 graphs with error bars indicating standard deviations.

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