Universal off-equilibrium scaling of critical cumulants in the QCD phase diagram

Swagato Mukherjee,1 Raju Venugopalan,1,2 and Yi Yin1

1 Department of Physics, Brookhaven National Laboratory, Upton, New York 11973-5000
2 Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany

(Dated: November 30, 2016)

Exploiting the universality between the QCD critical point and the three dimensional Ising model, closed form expressions derived [1] for non-equilibrium critical cumulants on the crossover side of the critical point reveal that they can differ both in magnitude and sign from equilibrium expectations. We demonstrate here that key elements of the Kibble-Zurek framework of non-equilibrium phase transitions can be employed to describe the dynamics of these critical cumulants. Our results suggest that observables sensitive to critical dynamics in heavy-ion collisions should be expressible as universal scaling functions, thereby providing powerful model independent guidance in searches for the QCD critical point.

Theoretical work on the phase diagram of Quantum Chromodynamics (QCD) [2–4] in the temperature $T$ and baryon chemical potential $\mu_B$ plane suggests the existence of a critical end point (CEP), the end point of a line of first-order phase transitions, that separates, in the chiral limit, a chirally symmetric quark-gluon plasma (QGP) phase from a hadron matter phase. This CEP is widely believed to lie in the static universality class of the three-dimensional Ising model [5, 6]. A definitive characterization of the phase diagram is hindered by the sign problem in lattice QCD at finite $\mu_B$; nevertheless, significant progress has been made in extending lattice thermodynamics from the finite temperature $T \neq 0$, $\mu_B = 0$ axis into the domain of finite $\mu_B$ [7, 8].

A parallel intensive experimental effort is underway to locate and characterize this critical point through a beam energy scan (BES) of heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC), from the highest center of mass energies ($\sqrt{s} = 200$ GeV/nucleon) down to energies per nucleon a few times the nucleon mass [9–12]. The fireballs created in such collisions traverse trajectories in the $T$--$\mu_B$ plane as they expand and cool before freezing out in a shower of hadrons. If the initial conditions are propitious, their dynamics can be expressed in terms of “protocols”–classes of trajectories in the relevant parameter space with differing sensitivity to critical fluctuations of the universal Ising order parameter. In the BES, protocols on the crossover side of the CEP are most likely, and we will restrict our attention to these.

Crossover protocols are subject to the critical slowing down of the relaxation rate of critical fluctuations. According to the theory of dynamical critical phenomena [13], the relaxation time for critical modes is related to their equilibrium correlation length as $\tau_{\text{eff}} \sim \xi^{\alpha}_{\text{eq}}$, where the dynamic scaling exponent $\alpha = 3$ for QCD [14–17] lies in the model $H$ universality class. Recently, employing the Fokker-Planck master equation describing the non-equilibrium dynamics of critical fluctuations [13], we derived closed form expressions for the temporal evolution of the first four cumulants $\kappa_n = 1, 2, 3, 4$ of the zero mode of the critical field [1]. This work significantly extended prior work on Gaussian fluctuations [18] and showed that both the magnitude and the sign of off-equilibrium non-Gaussian cumulants could differ from their equilibrium counterparts [19–22].

While memory effects persisting from critical slowing down could thus be detectable in the BES analyses, our results were sensitive to a number of non-universal inputs governing the protocols that include i) the mapping of the Ising variables – the reduced temperature $r = (T - T_c)/T_c$ (with $T_c$ denoting the Ising critical temperature) and the rescaled magnetic field $h$ – to the QCD thermodynamic variables $T, \mu_B$, ii) the details of trajectories in QCD phase diagram and iii) the relaxation rate of the critical mode $\tau_{\text{eff}}$. We shall henceforth collectively label these non-universal inputs with the symbol $\Gamma$. Uncertainties in $\Gamma$ can be reduced by careful modeling of the hydrodynamical evolution of the fireball and by further developments in lattice QCD studies at finite $\mu_B$. However, our prior results suggest that the model dependence of the critical cumulants will survive.

In this letter, we will show that significant progress towards model independent results for $\kappa_n$ can be achieved by employing the Kibble-Zurek (KZ) framework of non-equilibrium phase transitions to express the critical cumulants $\kappa_n$ for diverse protocols in terms of universal scaling functions. The KZ framework was initiated by Kibble to describe the formation and evolution of topological defects in cosmological phase transitions [23]. It was generalized to describe critical phenomena in a variety of contexts by Zurek [24, 25]; a fruitful application is in the description of Quantum Phase Transitions [26]. Experimental observations of KZ scaling in various condensed matter systems have also been reported; for a recent example, see Ref. [27]. Our work is inspired by a study of KZ dynamics in terms of the universal scaling of correlation functions [28]. For further discussion, employing powerful holographic techniques, see Ref. [29] and references within.

We begin by noting that a reduction in the number of parameters is seen already for equilibrium Ising critical cumulants which can be expressed as $\kappa_n^{\text{eq}} = \xi^{\alpha} f_n^{\text{eq}}(\theta)$, where $\xi^{\text{eq}}(r, h)$ depends universally on $r$, $h$, and $\theta$ is related to the product $(r^{-5/3} h)$ [30]. To ad-
dressed the possibility of an analogous off-equilibrium scaling. Consider a system undergoing a slow quench, where initially $\tau_{\text{eff}}$ of the critical mode is much smaller than the quench times

$$
\tau_{\text{q}}^\xi = \left| \frac{\xi_{\text{eq}}(\tau)}{\partial_{\tau} \xi_{\text{eq}}(\tau)} \right|, \quad \tau_{\text{q}}^\theta = \left| \frac{\theta(\tau)}{\partial_{\tau} \theta(\tau)} \right|, \quad (1)
$$

governing the rate of change of equilibrium cumulants as the system cools. Consider further, two distinct protocols. In the first, of type A, trajectories are very close to the Ising critical point at $r, h = 0$, corresponding to $T_c, \mu_B^0$ in the QCD phase diagram. In the Ising model, $\xi_{\text{eq}} \sim |h|^{-2/5}$ and $\theta(\tau) \sim \text{sgn}(\tilde{\tau})$, for $\tilde{\tau} = (\tau - \tau_c)$, with $\tau_c$ the proper time at which a trajectory crosses the crossover line at $h = 0$. Near $\tau_c$, one can expand $h(\tilde{\tau}) \approx (\tilde{\tau}/\tau_Q)^\alpha$, where $\tau_Q$ controls the rate of change of $h$ and $\alpha$ is positive definite since $h(\tau_c) = 0$. Hence $\xi_{\text{eq}}(\tau) \sim |\tilde{\tau}/\tau_Q|^{-2(\alpha+1)/5}$ and $\tau_{\text{q}}^\xi$ defined in (1) will go to zero as $\tilde{\tau} \to 0$. In contrast, $\tau_{\text{q}}^\theta$ remains finite. Thus due to critical slowing down, for protocol A, $\tau_{\text{q}}^\xi \ll \tau_{\text{eff}}$ very rapidly.

We can also identify a novel protocol B for the Ising universality class. This protocol corresponds to trajectories on the crossover side ($\mu_B \leq \mu_B^0$) of the QCD phase diagram that are only weakly sensitive to critical slowing down, with $\xi_{\text{eq}}(\tilde{\tau})$ reaching a maximal value at the crossover line. This implies that $\tau_{\text{q}}^\xi$ is large. However since $\theta$ flips sign across the crossover line, $\theta \propto \tilde{\tau}$ and $\tau_{\text{q}}^\theta$ goes to zero. Hence even though $\tau_{\text{q}}^\theta \gg \tau_{\text{eff}}$ for protocol B trajectories, one can have $\tau_{\text{q}}^\theta \ll \tau_{\text{eff}}$. Representative trajectories in protocols A and B are shown in Fig. 1.

The qualitative change in behavior of the quench rates relative to the relaxation rate is at the heart of the KZ dynamics. It allows us to define a proper time, denoted by $\tau^*$, at which $\tau_{\text{eff}}(\tau = \tau^*) = \tau_{\text{q}}(\tau = \tau^*)$, giving rise to an emergent time scale $\tau_{\text{KZ}}$, defined through the condition,

$$
\tau_{\text{KZ}} = \tau_{\text{eff}}(\tau^*) = \tau_{\text{q}}(\tau^*). \quad (2)
$$

with $\tau_{\text{q}} = \min\left(\tau_{\text{q}}^\xi, \tau_{\text{q}}^\theta\right)$. One can equivalently define an emergent length scale and magnetization angle respectively to be

$$
l_{\text{KZ}} = \xi_{\text{eq}}(\tau^*), \quad \theta_{\text{KZ}} = \theta(\tau^*). \quad (3)
$$

Because critical fluctuations freeze out after $\tau^*$, the system retains memory of these emergent scales at later times. The equilibrium scaling of critical cumulants suggests the following ansatz:

$$
\kappa_n(\tau; \Gamma) \sim \tau_{\text{KZ}}^{-\Gamma} \xi_{\text{eq}}^{(n-1)} \xi_{\text{q}}(\tau; \Gamma),
$$

with $\Gamma = \tilde{\tau}/\tau_{\text{KZ}}$ and I labels different protocol classes. While $\tau_{\text{KZ}}, l_{\text{KZ}}$ and $\theta_{\text{KZ}}$ depend non-universally on $\Gamma$, the functions $\xi_{\text{q}}^{(n)}$ are universal for all the trajectories characterizing a given protocol. A possible regime of protocol B where such scaling may hold is sketched in Fig. 1.

In Fig. 2(a), we plot the temporal evolution of $\tau_{\text{q}}^\xi$ for a characteristic quench scale $\tau_{\text{q}}$ for $\mu_B = \mu_B^0$ and compare it to $\tau_{\text{eff}} = \tau_{\text{rel}}(\xi/\xi_{\text{min}})^3$, where $\tau_{\text{rel}}$ and $\xi_{\text{min}}$ are the relaxation time of the critical mode and equilibrium correlation length respectively at the boundary of the critical regime. For the two different $\tau_{\text{rel}}$ along a trajectory in protocol A, we obtain distinct values of $\tau_{\text{KZ}}$ when $\tau_{\text{eff}}$ crosses $\tau_{\text{q}}^\xi$; one can also straightforwardly extract $l_{\text{KZ}}$. For protocols B, Fig. 2(b) shows that one similarly obtains a $\tau_{\text{KZ}}$ that corresponds to a novel KZ magnetization angle $\theta_{\text{KZ}}$. Note that $\tau_{\text{rel}}, \xi_{\text{min}}$ are non-universal parameters that are part of $\Gamma$ and $\tau_{\text{KZ}}, l_{\text{KZ}}, \theta_{\text{KZ}}$ depend on $\Gamma$.

In Appendix A, we present analytical arguments that justify the scaling form in Eq. (4) for both protocols. However, one can use the closed form expressions [1] for $\xi_{\text{q}}^{(n)}$ to check numerically the existence and domain of validity of the scaling. Towards this end, we will adopt a widely used but non-universal map [18, 31] between the Ising and QCD parameters, wherein $(T - T_c)/\Delta T = h$ and $(\mu_B - \mu_B^0)/\Delta \mu_B = -r$, with $\Delta T, \Delta \mu$ denoting the width of the critical regime in the QCD phase diagram. (The normalization of $r, h$ are fixed by the conditions $\xi(r = 1, h = 0) = \xi(r = 0, h = 1) = \xi_{\text{min}}$.) For the fireball in heavy ion collisions, we will use $T = T_c[\tau/\tau_c]^{-3\alpha/2}$, with the temperature evolution of the three dimensional isentropic expansion [32] determined by the speed of sound $c_s$.

In Fig. 3(a), we plot the non-equilibrium correlation length $\xi$ over $\xi_{\text{min}}$ for different choices of $\tau_{\text{rel}}$ in protocol A. The trajectory for each such choice is clearly non-universal and varies significantly with $\tau_{\text{rel}}$. Now using Eq. (4) and constructing $\tau_{\text{KZ}}$ as specified, we plot the function $f_{\text{KZ}}$ as a function of $t$. As anticipated by our
FIG. 3. (a): the evolution of the non-equilibrium effective correlation length \( \xi(\bar{t})/\xi_{\text{min}} \) for protocol A. The corresponding equilibrium value is plotted in dotted curve. (b): the rescaled function \( \bar{f}_2^A(t) \) vs the rescaled time \( t = \bar{t}/\tau_{\text{eff}} \). Results with \( \tau_{\text{rel}}/\tau_c = 0.02, 0.06, 0.1, 0.14 \) are shown in red, dashed blue, dotted green and dot-dashed orange curves respectively.

scaling ansatz, it scales beautifully; the different curves in Fig. 3(b), obtained by solving the cumulant equation in Ref. [1] for \( \kappa_2 \), collapse onto a single nearly universal curve. Equally impressive scaling is seen for the magnetization \( (\kappa_1) \), skewness \( \kappa_3 \) and kurtosis \( \kappa_4 \). The equivalent protocol A plots for these are respectively shown in Figs. 6, 7 and 8 of Appendix B.

Turning now to protocol B, we will examine the behavior of the four trajectories shown in Fig. 1. We tune \( \tau_{\text{rel}} \) in such a way that \( \theta_{\text{KZ}} \) is identical \( (\theta_{\text{KZ}} = -0.1) \) for the evolution along each trajectory. In Figs. 4(a) and 5(a), we show the corresponding cumulants \( \kappa_3 \) and \( \kappa_4 \) obtained from solving cumulant equation in Ref. [1]. Following the same procedure as for protocol A, we plot the functions \( \bar{f}_2^B, \bar{f}_4^B \) as a function of \( t \) in in Figs. 4(b) and 5(b). Very good scaling is observed in both cases, confirming the validity of our hypothesis. One naively expects the non-equilibrium scaling hypothesis to only apply in the regime \( |\bar{t}| < \tau_{\text{KZ}} \) (or \(|t| < 1\)). This is because the critical cumulants will approach their corresponding equilibrium values outside the KZ regime. Our numerical results for both protocols demonstrate that the KZ scaling solution persists for much longer, suggesting that the KZ scaling functions are attractor solutions. For a discussion of the latter, see Ref. [33].

We will now consider what these findings imply for the BES search for the CEP in the QCD phase diagram. An immediate consequence is that if BES trajectories are sensitive to the critical point in some window of \( \sqrt{s} \) (center of mass), the centrality (degree of overlap), and rapidity in the collisions, cumulants of hadron multiplicity distributions sensitive to the critical modes [10, 34, 35] should be expressible in the scaling form suggested by Eq. (4). In particular, if the KZ scaling regime is probed by the freeze-out curve of hadrons emitted at proper time \( \bar{t}_f \) from the BES fireballs, the critical cumulants, after rescaling with the appropriate powers of \( l_{\text{KZ}} \), will only depend on \( \theta_{\text{KZ}} \) and \( t_f \equiv \bar{t}_f/\tau_{\text{KZ}} \) for trajectories in the same protocol. How can the search for KZ scaling be achieved in practice?
FIG. 4. (a): Nonequilibrium evolution of $\kappa_3(\tilde{\tau})$ (normalized by its initial equilibrium value) for representative trajectories in protocol B. The corresponding equilibrium values are plotted in dotted curves. (b): the rescaled function $\bar{f}_B^4(t)$ versus the rescaled time $t = \tau / \tau_{KZ}$. The red, blue dashed, green dotted and orange dot-dashed curves correspond to those shown in Fig. 1.

3. Compute rescaled cumulant data of observables sensitive to critical dynamics as $\bar{f}_{data}^n \equiv \kappa_{data}^n / l_{KZ}^{1/2} + 2^{(n-1)}$. One can than establish a mapping between $\kappa_{data}^n$ to a point in $(\bar{f}_{data}^n, t_f, \theta_{KZ})$ space. We note from the previous step that this mapping depends on $\Gamma_{crit}$.

4. Repeat the above steps for windows in $\sqrt{s}$, centrality and rapidity that are sensitive to critical dynamics. Data on the corresponding cumulants mapped to $(\bar{f}_{data}^n, t_f, \theta_{KZ})$ space should collapse onto a single surface by suitably adjusting $\Gamma_{crit}$. This surface will be described by the scaling functions $\bar{f}_n(t, \theta_{KZ})$.

5. In parallel to the previous steps, compute the universal scaling functions by solving the cumulant equations along one representative trajectory of each protocol. Compare the theoretically computed $\bar{f}_n(t_f, \theta_{KZ})$ with rescaled data to further confirm the scaling hypothesis.
If such theory-data comparisons are successful, they would provide unambiguous evidence for the existence of the QCD CEP [37]. The analysis sketched above should also allow us to extract $\Gamma_{\text{crit}}$, which encodes important properties of QCD matter near the CEP. The procedure outlined, with examples including mock BES data, will be pursued in future work. It can also be explored in models that explicitly couple critical and bulk dynamics—along the lines of previous work [38].

There are a number of features of our results that are of broader interest. The non-equilibrium scaling of non-Gaussian cumulants has received little attention in the literature on the KZ dynamics. A noteworthy exception is an approach based on the reparametrization invariance [39] of the stochastic master equations representing the mathematical content of different dynamical universality classes [13]. This approach has much in common with our analytical discussion of the structure of cumulant classes [13].

This approach has much in common with our analytical discussion of the structure of cumulants [13]. The non-equilibrium scaling of non-equilibrium models that explicitly couple critical and bulk dynamics—along the lines of previous work [38].

In this appendix, we will show analytically the existence of our results that are of broader interest. The non-equilibrium scaling of non-Gaussian cumulants has received little attention in the literature on the KZ dynamics. A noteworthy exception is an approach based on the reparametrization invariance [39] of the stochastic master equations representing the mathematical content of different dynamical universality classes [13]. This approach has much in common with our analytical discussion of the structure of cumulants [13].

This approach has much in common with our analytical discussion of the structure of cumulants [13]. The non-equilibrium scaling of non-equilibrium models that explicitly couple critical and bulk dynamics—along the lines of previous work [38].

In this appendix, we will show analytically the existence of our results that are of broader interest. The non-equilibrium scaling of non-Gaussian cumulants has received little attention in the literature on the KZ dynamics. A noteworthy exception is an approach based on the reparametrization invariance [39] of the stochastic master equations representing the mathematical content of different dynamical universality classes [13]. This approach has much in common with our analytical discussion of the structure of cumulants [13].

This approach has much in common with our analytical discussion of the structure of cumulants [13]. The non-equilibrium scaling of non-equilibrium models that explicitly couple critical and bulk dynamics—along the lines of previous work [38].

In this appendix, we will show analytically the existence of our results that are of broader interest. The non-equilibrium scaling of non-Gaussian cumulants has received little attention in the literature on the KZ dynamics. A noteworthy exception is an approach based on the reparametrization invariance [39] of the stochastic master equations representing the mathematical content of different dynamical universality classes [13]. This approach has much in common with our analytical discussion of the structure of cumulants [13].

This approach has much in common with our analytical discussion of the structure of cumulants [13]. The non-equilibrium scaling of non-equilibrium models that explicitly couple critical and bulk dynamics—along the lines of previous work [38].

In this appendix, we will show analytically the existence of our results that are of broader interest. The non-equilibrium scaling of non-Gaussian cumulants has received little attention in the literature on the KZ dynamics. A noteworthy exception is an approach based on the reparametrization invariance [39] of the stochastic master equations representing the mathematical content of different dynamical universality classes [13]. This approach has much in common with our analytical discussion of the structure of cumulants [13].

This approach has much in common with our analytical discussion of the structure of cumulants [13]. The non-equilibrium scaling of non-equilibrium models that explicitly couple critical and bulk dynamics—along the lines of previous work [38].
$\frac{5}{2}|\tau|$ and the condition in Eq. (2) to determine $\tau_{KZ}$ becomes

$$\tau_{rel} \left| \frac{\tau^*}{T_0} \right|^{-\frac{3}{2}} = \frac{5}{2} |\tau^*|,$$  \hspace{1cm} (A6)

where we have used $\tau_{eff} = \tau_{rel} \left( \frac{\kappa}{\xi_{eq}} \right)^3$. Likewise, $l_{KZ}$ can be determined from Eq. (3) and one can check that

$$\tilde{\xi}_{eq}(t) \approx \frac{5}{2} t^{-2/5}, \quad \tau_{eff} \approx \tilde{\xi}_{eq}^3,$$  \hspace{1cm} (A7a)

for evolution near $T_c$. Turning now to $\theta(\tau)$, we found from $\theta(\tau) \sim \text{sgn}(\tilde{\tau})$ and the definition of $\theta_{KZ}$ in Eq. (3) that

$$\theta(\tau) \approx \theta_{KZ} \text{sgn}(t).$$  \hspace{1cm} (A7b)

This concludes our proof for protocol A that Eq. (A4) has a scaling solution of the form $\bar{f}_n^A(t; \theta_{KZ})$ near $T_c$.

We next consider protocol B. Since for protocol B, $\xi_{eq}$ reaches its maximum when crossing the crossover line, we have $\xi_{eq} \sim l_{KZ}$ and $\tau_{eff} \sim \tau_{KZ}$ for evolution in the vicinity of the crossover line. Thus $\tilde{\xi}_{eq} \approx 1, \tilde{\tau}_{eff} \approx 1$. On the other hand, since $\theta \propto \tilde{\tau} \equiv \tau - \tau_c$, we will have from Eq. (3),

$$\theta(\tau) \approx \theta_{KZ} t,$$  \hspace{1cm} (A8)

for protocol B. We therefore conclude that Eq. (A4) has a scaling solution of the form $f_n^B(t; \theta_{KZ})$. Since $\tau_{eff}(t), \xi_{eq}(t), \theta(t)$ take different forms for protocol A and B, $f_n^A$ and $f_n^B$ correspond to distinct universal scaling functions.

We conclude this section by collecting explicit expressions for the right hand side of Eq. (A1) and Eq. (A4) for $n = 1, 2, 3, 4$. Following Ref. [1], $\tilde{F}_n$ in (A1) reads

$$\tilde{F}_1[\theta, b; \kappa_1] = \delta \tilde{M} \left[ 1 + \tilde{\lambda}_3(\theta)(\delta \tilde{M}) + \tilde{\lambda}_4(\theta)(\delta \tilde{M}) \right],$$

$$\tilde{F}_2[\theta, b; \kappa_1, \kappa_2] = (b^3) \left[ \frac{\kappa_2}{b} F_2(\delta \tilde{M}; \theta) - 1 \right],$$

$$\tilde{F}_3[\theta, b; \kappa_1, \kappa_2, \kappa_3] = -\left( \frac{b^3}{\kappa_3} \right) \left[ \frac{\kappa_3}{\kappa_2 b^3} F_2(\delta \tilde{M}; \theta) + \frac{\kappa_2}{b^3} \right] F_3(\delta \tilde{M}; \theta) + 6 \frac{\kappa_2}{b^3} \tilde{\lambda}_4(\theta),$$

As in Ref. [1], $\delta \tilde{M}, F_{2,3}$ are given by

$$\delta \tilde{M}[\theta, b; \kappa_1] = \left( \frac{b}{\kappa_1} \right) \kappa_1 - C_0^{-1} \tilde{\sigma}(\theta),$$

$$F_2(\delta \tilde{M}) = 1 + 2 \tilde{\lambda}_3(\theta)(\delta \tilde{M}) + 3 \tilde{\lambda}_4(\theta)(\delta \tilde{M})^2,$$

$$F_3(\delta \tilde{M}) = 2 \left[ \tilde{\lambda}_3(\theta) + 3 \tilde{\lambda}_4(\theta)(\delta \tilde{M}) \right].$$  \hspace{1cm} (A9)

and $\tilde{\sigma}(\theta), \tilde{\lambda}_3(\theta), \tilde{\lambda}_4(\theta)$ are determined from a linear parametrization model of the Ising equation of state [42, 43]:

$$\tilde{\sigma}(\theta) = \frac{5^{1/4} \theta}{(3 + 2\theta^2)^{1/4}},$$  \hspace{1cm} (A11)

$$\tilde{\lambda}_3(\theta) = \frac{1}{5^{1/4}} \frac{2\theta(9 + \theta^2)}{(3 - \theta^2)(3 + 2\theta^2)^{3/4}},$$  \hspace{1cm} (A12)

$$\tilde{\lambda}_4(\theta) = \frac{1}{\sqrt{5}} \frac{2 (27 + 45\theta^2 - 31\theta^4 - \theta^6)}{(3 - \theta^2)^3 (3 + 2\theta^2)^{1/2}}.$$  \hspace{1cm} (A13)

Here the dimensionless quantity $C_0$ is non-universal.

We next consider $\tilde{G}_n$ which appears in (A4). By
FIG. 7. The evolution of $\kappa_3(\bar{\tau})$ (a) and $\tilde{f}_3^4$ (b) for a representative protocol A trajectory along the lines described in the caption for Fig. 3.

FIG. 8. The evolution of $\kappa_4(\bar{\tau})$ (a) and $\tilde{f}_4^4$ (b) for a representative protocol A trajectory along the lines described in the caption for Fig. 3.

straightforward calculation, we have:

$$\tilde{G}_1 \left[ \xi_{eq}, \theta; f_1 \right] = G_1 \left[ f_1; \xi_{eq}, \theta \right] G_0 \left[ f_1; \xi_{eq}, \theta \right],$$

$$\tilde{G}_2 \left[ \xi_{eq}, \theta; f_1, f_2 \right] = \xi_{eq} \left\{ G_2 \left[ f_1; \xi_{eq}, \theta \right] f_2 - \xi_{eq} \right\},$$

$$\tilde{G}_3 \left[ \xi_{eq}, \theta; f_1, f_2, f_3 \right] = \xi_{eq} \left\{ G_2 \left[ f_1; \xi_{eq}, \theta \right] f_3 \right. + 2 G_2 \left[ f_1; \xi_{eq}, \theta \right] f_3 \left[ f_2^2 \right],$$

$$\tilde{G}_4 \left[ \xi_{eq}, \theta; f_1, f_2, f_3 \right] = \xi_{eq} \left\{ G_2 \left[ f_1; \xi_{eq}, \theta \right] f_4 \right. + 6 G_3 \left[ f_1; \xi_{eq}, \theta \right] f_2 f_3 \right. + 6 \lambda_4(\theta) f_2^2 \left. \right\}. \quad \text{(A14)}$$

where

$$G_0 \left[ f_1; \xi_{eq}, \theta \right] = \left[ f_1 - \xi_{eq}^{-1/2} \tilde{\sigma}(\theta) \right],$$

$$G_1 \left[ f_1; \xi_{eq}, \theta \right] = \left[ \xi_{eq}^{-1} + \xi_{eq}^{-1/2} \tilde{\lambda}_3(\theta) G_0 + \tilde{\lambda}_4(\theta) \right] (G_0)^2,$$

$$G_2 \left[ f_1; \xi_{eq}, \theta \right] = \left[ \xi_{eq}^{-1} + 2 \xi_{eq}^{-1/2} \tilde{\lambda}_3(\theta) G_0 + 3 \tilde{\lambda}_4(\theta) \right] (G_0)^2,$$

$$G_3 \left[ f_1; \xi_{eq}, \theta \right] = \left[ \xi_{eq}^{-1/2} \tilde{\lambda}_3(\theta) + 3 \tilde{\lambda}_4(\theta) G_0 \right]. \quad \text{(A15)}$$

Appendix B: More detailed numerical results for trajectories A and B

We now present further detailed numerical tests of the non-equilibrium scaling hypothesis. To solve evolution equation Eq. (A1) along a trajectory on the crossover side of the critical regime, we need to specify 1) the trajectory in the Ising phase diagram, 2) the mapping between the Ising variables $r-h$ and the QCD variables $T-\mu_B$, 3) the evolution of QCD variables along the trajectory.

Throughout this work, we will consider trajectories in
which $r$ and $h$ are related by

$$ r = r_c - a_h h^2, \quad (B1) $$

where $r_c$ is the value of $r$ on the cross-over line. As we shall see later, by changing $r_c$ and $a_h$, we will obtain trajectories which lie in protocol A or protocol B. As mentioned previously, we will use the linear map $(T - T_c)/\Delta T = h$ and $(\mu_B - \mu_B^c) = -r$. We will also employ a simple model of the medium that mimics the expanding fireball formed in heavy ion collisions. Specifically, we consider the evolution of temperature to be of the form

$$ T(\tau) = T_c \left[ \frac{\tau}{\tau_c} \right]^{-3\xi^2}, \quad (B2) $$

and we will use $\xi^2 = 0.1$.

To confirm the scaling hypothesis numerically, we first consider a representative trajectory in protocol A. In particular, we will consider a trajectory with fixed $r$: $a_h = 0$ and thus $r = r_c$ in Eq. (B1). From the definition of protocol A, this trajectory will pass the crossover line in the vicinity of the critical point. Therefore $r_c \ll 1$.

We will present below numerical results with $r_c = 0.02$.

in Figs. 6, 3, 7, 8. They correspond to solutions with $\tau_{rel}/\tau_c = 0.02, 0.06, 0.1, 0.14$. We have also verified the scaling behavior for other choices of $r_c \ll 1$. In producing Fig. 3(a), we have defined the non-equilibrium correlation length as $\xi \equiv \sqrt{\kappa V_c/T_c}$.

We now turn to protocol B. The four trajectories representing this protocol in Fig. 1 correspond to $r_c = 0.9, 0.8, 0.7, 0.6$ (from left to right). We fix $a_h$ in Eq. (B1) such that the trajectories approach the equal-$\xi_{eq}$ contour in the vicinity of the crossover line. This reflects the character of protocol B that the quench of the equilibrium correlation length $\xi_{eq}$ is very slow near the crossover line. The evolution equations were solved numerically along these trajectories. To test the scaling hypothesis, we tuned $\tau_{rel}$ to ensure $\theta_{KZ} = -0.1$ for all these trajectories. The prediction based on the scaling hypothesis is that the rescaled functions $f_{1,2,3,4}(\tau/\tau_{KZ})$ are independent of the choice of trajectories. Figs. 9, 10, 4, 5 demonstrate that there is indeed a large time window around crossover line where the scaling hypothesis works.
[1] S. Mukherjee, R. Venugopalan, and Y. Yin, Phys. Rev. C92, 034912 (2015), arXiv:1506.00645 [hep-ph].
[2] M. A. Stephanov, Prog.Theor.Phys.Suppl. 153, 139 (2004), arXiv:hep-ph/0402115 [hep-ph].
[3] M. Stephanov, PoS LAT2006, 024 (2006), arXiv:hep-lat/0701002 [hep-lat].
[4] K. Fukushima and T. Hatsuda, Rept.Prog.Phys. 74, 014001 (2011), arXiv:1005.4814 [hep-ph].
[5] H. Fujii and M. Ohtani, Phys. Rev. D73, 056001 (2006), arXiv:hep-ph/0605079 [hep-ph].
[6] H.-T. Ding, F. Karsch, and S. Mukherjee, Int. J. Mod. Phys. E24, 1530007 (2015), arXiv:1504.05274 [hep-lat].
[7] J. Berges and K. Rajagopal, Nucl.Phys. B538, 215 (1999), arXiv:hep-ph/9804233 [hep-ph].
[8] D. Son and M. Stephanov, Phys.Rev.D61, 064304 (2000), arXiv:hep-th/0002271 [hep-th].
[9] P. Vranas, Ann. Rev. Nucl. Part. Sci. 58, 317 (2008).