The multicomponent 2D Toda hierarchy: generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann–Hilbert problems

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Abstract
We consider the relation of the multicomponent 2D Toda hierarchy with the matrix orthogonal and biorthogonal polynomials. The multi-graded Hankel reduction of this hierarchy is considered and the corresponding generalized matrix orthogonal polynomials are studied. In particular, for these polynomials we consider the recursion relations, and for rank 1 weights its relation with multiple orthogonal polynomials of mixed type with a type II normalization and the corresponding link with a Riemann–Hilbert problem.

1. Introduction
Recently, multiple orthogonal polynomials, the related Riemann–Hilbert problems and their application to different areas, for example Brownian motions, have deserved much attention [1]. The field of matrix orthogonal polynomials has also been a growing area of research with some similarities with the scalar case but much more richness, see [2]. The relation of multiple orthogonal polynomials with multicomponent KP hierarchies has been noted in [3] and the string equation formalism of integrable systems has been applied in [4]. Ten years ago Adler and van Moerbeke [5]—in the context of the so-called discrete KP hierarchy—introduced what they named as generalized orthogonal polynomials, and what they claimed to be the corresponding Riemann–Hilbert problem, and later on they studied the related Darboux transformations [6]. Recently, and for the Toeplitz case Cafasso [7] extended this work in order to consider block matrices within the non-Abelian Ablowitz–Ladick lattice.

Following Ueno and Takasaki [8] and the seminal paper of Mulase [9] in our paper [10] we gave a description of the infinite multicomponent 2D Toda lattice hierarchy in terms of a Gaussian factorization (also known as Borel factorization) in an infinite-dimensional Lie group, while later on [11] we analyzed the dispersionless limit of the hierarchy with the
aid of the factorization problem. (See [12] for a discussion on different cases where this
factorization makes sense.) Following [5] we could argue as follows: (i) on the one hand, the
multicomponent Toda hierarchy may be viewed as an LU factorization of certain deformed
infinite-dimensional matrix and (ii) on the other hand, the same matrix could be thought as
a moment matrix and the corresponding LU factorization should give us the corresponding
generalized matrix orthogonal polynomials. In this manner, we would be able to build a
bridge between multicomponent Toda hierarchy and matrix orthogonal polynomials. This is
the main idea developed in this paper. First, we connect matrix orthogonal and biorthogonal
polynomials with the multicomponent 2D Toda lattice hierarchy, focusing in particular on the
Hankel reduction. Second, we generalize the band condition of [5] to the multicomponent
case and consider what we refer to as multi-graded Hankel. This leads to a multicomponent
extension of generalized orthogonal polynomials which in some cases can be described in
terms of multiple orthogonal polynomials of mixed type with a type II normalization. This
connection allows us to give an appropriate Riemann–Hilbert problem for these generalized
orthogonal polynomials (note that the one discussed in [5] is not correct).

The layout of the paper is as follows. In this introduction we give an overview of
the Gaussian factorization and the semi-infinite multicomponent 2D Toda lattice hierarchy.
Then, in section 2 we discuss how matrix orthogonal polynomials and the multicomponent
2D Toda lattice hierarchy are connected. Finally, in section 3 we consider the multi-graded
Hankel reduction, extended generalized orthogonal polynomials, mixed multiple orthogonal
polynomials and corresponding Riemann–Hilbert problems.

1.1. Gaussian factorization and the semi-infinite multicomponent 2D Toda lattice hierarchy

For the construction of a Lie group theoretical setting we denote by \( \Lambda \) the shift operator
for matrix-valued sequences. The associative algebra of linear operators on these sequences
can be identified with the associative algebra of semi-infinite matrices with entries taking
values in \( \mathbb{C}^{N \times N} \), the set of \( N \times N \) complex matrices. With the usual commutator for linear
operators this algebra is also a Lie algebra denoted by \( g \) whose Lie group \( G \) is the group
of invertible linear operators in \( g \). Let us take \( g \in G \) and consider the following Gaussian
factorization problem
\[
g = S^{-1} \bar{S}
\]
where \( S \) is a block lower triangular matrix, with \( S_{ii} = I_N \), being \( I_N \in \mathbb{C}^{N \times N} \) the identity matrix, and \( \bar{S} \) is a block upper triangular matrix. In [12]
it is proven that the Borel decomposition holds if all the principal minors do not vanish.
Thus, the factorization holds under ‘small’ continuous deformations and we can consider
the factorization \( g(t) = S(t)^{-1} \bar{S}(t) \) where \( t \) stands for a set of complex variables. As was
discussed in [10] this factorization problem leads to an integrable hierarchy of nonlinear PDE
known as multicomponent 2D Toda lattice hierarchy. Let us discuss these issues in more depth.

Observe that the matrix associated with the shift operator is the block matrix
\( \Lambda_{i,j} \) and \( \Lambda' \) is the operator associated with the transposed matrix \( \Lambda_{i,j} \). If \( E_{ab} \), \( a, b = 1, \ldots, N \),
is the canonical basis of \( \mathbb{C}^{N \times N} \) and \( t = (t_{ja}, \bar{t}_{ja}) \), \( j = 1, 2, \ldots \) and \( a = 1, \ldots, N \), is a
collection of complex parameters we introduce
\[
W_0 := \sum_{a=1}^{N} E_{a a} \exp \left( \sum_{j=1}^{\infty} t_{ja} \Lambda' \right),
\]
\[
\bar{W}_0 := \sum_{a=1}^{N} E_{a a} \exp \left( \sum_{j=1}^{\infty} \bar{t}_{ja} (\Lambda')' \right),
\]
and consider Gaussian factorization of \( g(t) := W_0(t)g\bar{W}_0(t)^{-1} \).
Following [10] we define the Lax operators

\[ L := S \Lambda S^{-1} = \Lambda + u_0 + u_1 \Lambda^t + u_2 (\Lambda^t)^2 + \cdots, \]

\[ C_a := S E_a S^{-1} = E_a + c_{a_1} \Lambda^t + \cdots, \]

\[ \bar{L} := \bar{S} \Lambda \bar{S}^{-1} = e^\phi \Lambda^t + \bar{u}_0 + \bar{u}_1 \Lambda + \cdots, \]

\[ \bar{C}_a := \bar{S} E_a \bar{S}^{-1} = \bar{c}_{a_0} + \bar{c}_{a_1} \Lambda + \cdots, \]

where all the coefficients in the \( \Lambda \)-expansions belong to \( \mathbb{C}^{N \times N} \). The multicomponent 2D Toda hierarchy has the following Lax representation:

\[
\frac{\partial L}{\partial t_{ja}} = [(L^j C_a)^+, L], \quad \frac{\partial C_b}{\partial t_{ja}} = [(L^j C_a)^+, C_b], \\
\frac{\partial \bar{L}}{\partial t_{ja}} = [(\bar{L}^j \bar{C}_a)^-, \bar{L}], \quad \frac{\partial \bar{C}_b}{\partial t_{ja}} = [(\bar{L}^j \bar{C}_a)^-, \bar{C}_b], \\
\frac{\partial \bar{L}}{\partial \bar{t}_{ja}} = [(\bar{L}^j \bar{C}_a)^-, \bar{L}], \quad \frac{\partial \bar{C}_b}{\partial \bar{t}_{ja}} = [(\bar{L}^j \bar{C}_a)^-, \bar{C}_b],
\]

where the sub-indices + and − denote the block upper triangular, strictly block lower triangular projections, respectively.

2. Matrix orthogonal polynomials and the multicomponent 2D Toda lattice hierarchy

Following Adler and van Moerbeke [5] we construct families of matrix orthogonal and bi-orthogonal polynomials associated with the 2D Toda lattice hierarchy.

In the first place, we define the following families of (time-dependent) matrix polynomials:

\[ p(z) \equiv \{p_i(z)\}_{i \geq 0} := S \chi(z), \quad \bar{p}(z) \equiv \{\bar{p}_i(z)\}_{i \geq 0} := (\bar{S}^{-1})^\dagger \chi(z), \]

where \( \chi(z) := (I_N, z I_N, z^2 I_N, \ldots)^t \) and the symbol \( ^\dagger \) denotes Hermitian conjugation. Next, we consider a matrix-valued bilinear pairing between matrix polynomials. Given matrix polynomials \( P(z) = \sum_{k=0}^i P_k z^k \) and \( Q(z) = \sum_{l=0}^j Q_l z^l \) (of degrees \( i, j \), respectively) we have

\[ \langle P(z), Q(z) \rangle = \sum_{k=1 \ldots i} \sum_{l=1 \ldots j} P_k (z^k | I_N, z^l I_N) Q_l^\dagger, \]

where \( (z^k | I_N, z^l I_N) \) denotes the matrix for the bilinear pairing in the canonical basis and for each \( (k, l) \) is an \( N \times N \) complex matrix.

This pairing has the following properties.

(1) It is linear in the first component:

\[ \langle c_1 P_1(z) + c_2 P_2(z), Q(z) \rangle = c_1 \langle P_1(z), Q(z) \rangle + c_2 \langle P_2(z), Q(z) \rangle, \quad \forall c_1, c_2 \in \mathbb{C}^{N \times N}. \]

(2) It is skew-linear in the second component:

\[ \langle P(z), c_1 Q_1(z) + c_2 Q_2(z) \rangle = \langle P(z), Q_1(z) \rangle c_1^\dagger + \langle P(z), Q_2(z) \rangle c_2^\dagger, \quad \forall c_1, c_2 \in \mathbb{C}^{N \times N} \]
Proposition 1.

(1) If \( \langle z^i I_N, z^j I_N \rangle = g_{ij} \) where \( g_{ij} \) is the \( C_N \times C_N \) block in the position \( (i, j) \), then the families \( p(z) \) and \( \bar{p}(z) \) are biorthogonal matrix polynomials for the linear pairing, i.e.
\[
\langle p_i(z), \bar{p}_j(z) \rangle = \delta_{ij} I_N.
\]
Moreover,
\[
\langle p_i(z), z^l I_N \rangle = 0, \quad l = 0, \ldots, i - 1, \quad \langle z^l I_N, \bar{p}_j(z) \rangle = 0, \quad l = 0, \ldots, j - 1.
\]

(2) In addition, if \( g(t) \) is Hermitian for all \( t \), then \( p(z) \) and \( \bar{p}(z) \) are two families of matrix orthogonal polynomials. Moreover, the two families are proportional.

Proof.

(1) With the previous definitions for \( p(z) \) and \( \bar{p}(z) \) we have
\[
p_i(z) = \sum_{k=0}^{i} S_{ik} z^k, \quad \bar{p}_j(z) = \sum_{l=0}^{j} (\bar{S}^{-1}_{lj})^\dagger z^l,
\]
where \( S_{ik} \) and \( \bar{S}^{-1}_{lj} \) are the blocks \((i, k)\) and \((l, j)\) for \( S \) and \( \bar{S}^{-1} \), respectively. Hence,
\[
\langle p_i(z), \bar{p}_j(z) \rangle = \sum_{k,l} S_{ik} \langle z^k I_N, z^l I_N \rangle (\bar{S}^{-1}_{lj})^\dagger = \sum_{k,l} S_{ik} \langle z^k I_N, z^l I_N \rangle (S^{-1})^\dagger_{lj} = (Sg(t)S^{-1})_{ij} = \delta_{ij} I_N,
\]
as desired. Finally, (2) is proven by induction: first we have that \( \langle p_i(z), p_0(z) \rangle = 0 \), but \( p_0(z) = (\bar{S}^{-1}_{00})^\dagger \) is invertible and therefore we conclude \( \langle p_i(z), I_N \rangle = 0 \). Now, \( \langle p_i(z), \bar{p}_1(z) \rangle = 0 \), but \( \bar{p}_1(z) = (\bar{S}^{-1}_{11})^\dagger z + (\bar{S}^{-1}_{10})^\dagger \), and using the skew-linearity, the previous result and the fact that \( (\bar{S}^{-1}_{ij})^\dagger \) is invertible we deduce that \( \langle p_i(z), z I_N \rangle = 0 \), and so forth.

(2) Let us study the conditions under which \( p(z) \) is a family of matrix orthogonal polynomials. If we take two polynomials in the family, such as \( p_i(z) = \sum_{k=1}^{i} S_{ik} z^k \) and \( p_j(z) = \sum_{l=1}^{j} S_{lj} z^l \), we have
\[
\langle p_i(z), p_j(z) \rangle = \sum_{k,l=1}^{i,j} S_{ik} \langle z^k I, z^l I \rangle (S_{lj})^\dagger = \sum_{k,l=0}^{\infty} S_{ik} \langle z^k I_N, z^l I_N \rangle (S^\dagger)_{lj} = (Sg(t)S^\dagger)_{ij} = (\bar{S} S^\dagger)_{ij}.
\]
Observe that \( \bar{S} S^\dagger \) is clearly block upper diagonal with its Hermitian conjugate given by
\[
(\bar{S} S^\dagger)^\dagger = \bar{S} S^\dagger = S g(t)(g(t)^{-1})^\dagger \bar{S}^\dagger = \bar{S} g(t)^{-1}\bar{S}^\dagger = \bar{S} S^\dagger.
\]
Therefore, \( \bar{S} S^\dagger \) is Hermitian and block upper diagonal, which implies that \( \bar{S} S^\dagger \) is a block diagonal matrix and the blocks in the diagonal are \( C_N \times C_N \) Hermitian matrices. We conclude that \( \langle p_i(z), p_j(z) \rangle = \delta_{ij} (h_i)^{-1} \), where \( h_i \) is a Hermitian matrix. Note also that as a consequence \( \bar{p}(z) = h p(z) \) where \( h = \text{diag}(h_1, h_2, \ldots) \). □
2.1. The Hankel case

We choose $g$ to be a block Hankel matrix so that $\Lambda g = g \Lambda^t$ or

$$(\Lambda g)_{ij} = g_{i+1,j} = g_{i,j+1} = (g \Lambda^t)_{ij}.$$ 

In this case we have for the blocks of the moment matrix $g$

$$g_{ij} = \gamma^{(i+j)},$$

for some matrices $\gamma^{(j)} \in \mathbb{C}^{N \times N}$. From

$$\Lambda g(t) = W_0 \Lambda g W_0^{-1} = W_0 \Lambda g \Lambda^t W_0^{-1} = W_0 \Lambda g \Lambda^t W_0^{-1} = g(t) \Lambda^t,$$

we easily deduce that $g(t)$ is block Hankel if $g$ is. We have

**Proposition 2.** Assume that $g$ is block Hankel with $\gamma^{(j)}$ a block moment matrix, i.e. $\gamma^{(j)} = \int_{\mathbb{R}} x^j \rho(x) \, dx$. Then, the pairing can be viewed as a scalar product in the real line

$$\langle P(x), Q(x) \rangle = \int_{\mathbb{R}} P(x) \rho(x) Q(x)^\dagger \, dx.$$ 

**Proof.** On one hand, we have $\langle P(x), Q(x) \rangle = \sum_{ij} P_i \gamma^{(i+j)} Q_j^\dagger$. Using the previous definition $\int_{\mathbb{R}} x^j \rho(x) \, dx = \langle x^j 1_N, x^k 1_N \rangle = \gamma^{(j+k)}$ (the Hankel symmetry ensures that there is only dependence in $j+k$) we have

$$\langle P(x), Q(x) \rangle = \sum_{ij} P_i \gamma^{(i+j)} Q_j^\dagger = \sum_{ij} P_i \int_{\mathbb{R}} x^j x^k \rho(x) \, dx Q_j^\dagger = \int_{\mathbb{R}} P(x) \rho(x) Q(x)^\dagger \, dx.$$ 

In general arbitrary continuous deformations do not preserve the Hermitian character of $g$. If we look for families of matrix orthogonal polynomials on the real line we should make restricted deformations. Let us make this point clear. In the following $z^*$ denotes the complex conjugate of $z \in \mathbb{C}$.

**Proposition 3.** If the matrix $g$ is block Hankel and the matrices $\gamma^{(j)}$ are Hermitian, then

(1) the families $p(z)$ and $\bar{p}(z)$ are proportional, being each of them a family of matrix orthogonal polynomials in the real line.

(2) Moreover, if the continuous deformation parameters satisfy one of the two following conditions,

(a) $t_{ja}, \bar{t}_{ja} \in \mathbb{R}$ and satisfy $t_{ja} = \bar{t}_{ja} = \bar{t}_j, a = 1, \ldots, N.$

(b) $t_{ja}, \bar{t}_{ja}$ satisfy $t_{ja} + \bar{t}_{ja} = 0,$

the result holds for the time-dependent moment matrix.

**Proof.**

(1) If the matrix $g$ is block Hankel and the blocks are Hermitian the matrix $g$ is itself Hermitian. Given $i, j$ pair of indices for an element of $g$, there exist four integer indices $(k, l), (m, n)$ with $k, l \geq 0, m, n = 1, \ldots, N$ that satisfy $a_{ij} = (A_{kl})_{mn} = (A_{kl})_{nm} = (A_{kl})_{nm} = a_{ji}$ (we use $A \in \mathbb{C}^{N \times N}$ for a block of $g$) as a consequence $g = g^\dagger$ and by proposition 1 the first part of the result holds.
Proof. From the definition of taking the definition \(1\) we conclude

The polynomials \(\sigma_j(t_1, \ldots, t_j)\) are defined by the following generating relation \(\exp(\sum_{j=1}^{\infty} t_j z^j) = \sum_{j=1}^{\infty} \sigma_j(t_1, t_2, \ldots, t_j) z^j\), and therefore \(\sigma_j = \sum_{i=1}^{j} \sum_{j_i+j_2+\cdots+j_r=j} t_{j_1} t_{j_2} \cdots t_{j_r}\). Given a partition \(\vec{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_+^r\) we have the the Schur function \(s_\vec{n}(t) = \det(\sigma_{n_i-j}(t))_{1 \leq i, j \leq r}\). For more on the relation of these Schur functions and those in [13] see [14].

Consequently and taking \(s_j := \sum_{a=1}^{N} E_{aa}s_j^{(a)}\)

\[
W_0 = \sum_{j=0}^{\infty} \sum_{k=1}^{N} E_{aa}s_j^{(a)} \Lambda^j = \sum_{j=0}^{\infty} s_j \Lambda^j,
\]

so it is straightforward to see the block structure of \(W_0\), whose blocks are given by \((W_0)_{ij} = s_{j-i}\) if \(j-i \geq 0\) and \(0\) otherwise.

A similar argument for \(W_0^{-1}\) leads to \((W_0^{-1})_{ij} = s_{j-i}\) if \(j-i \geq 0\) and \(0\) otherwise (here \(s_j := \sum_{a=1}^{N} E_{aa}s_j^{(a)}\) and \(s_j^{(a)}\) is the \(j\)th Schur polynomial for the component \(a\) but now in the variables \(-t_{ja}\)).

We are now ready to compute \(g(t)_{ij} = \sum_{k,l \geq 0} (W_0)^{k+l} \sigma_{k+l}(W_0^{-1})_{ij} = \sum_{k,l \geq 0} s_{k-i} \gamma^{(k+l)} s_{l-j} = \sum_{k,l \geq 0} s_k \gamma^{(i+j+k+l)} s_l\).

Then,

(a) if the first condition holds, then all matrices \(s_j\) and \(s_j\) are real and scalar so

\[
g(t)_{ij} = \sum_{k,l \geq 0} s_k \gamma^{(i+j+k+l)} s_l = \sum_{k,l \geq 0} s_k \gamma^{(i+j+k+l)} s_l
\]

(b) if the second condition holds, \(s_j = \bar{s}_{j}\) for all \(j\) so

\[
g(t)_{ij} = \sum_{k,l \geq 0} \bar{s}_k \gamma^{(i+j+k+l)} \bar{s}_l = \sum_{k,l \geq 0} s_k \gamma^{(i+j+k+l)} s_l = g(t)_{ij}.
\]

Under any of the two conditions \(g(t)\) is block-Hankel with Hermitian blocks, so is itself Hermitian. That proves the second part of the proposition.

We now discuss the recursion formulas. Using the Hankel condition \(\Lambda_R = g\Lambda^T\) and taking the definition \(1\) we conclude \(L = \bar{L}\) and hence we have \(L = \Lambda + u(n) + v(n)\Lambda^T\).

Proposition 4. The polynomials \(p(z), \bar{p}(z)\) satisfy a three-term recurrence law given by \(p_{n+1}(z) = zp_n(z) - u(n)p_n(z) - v(n)p_{n-1}(z)\).

Proof. From the definition of \(L\) we have \(Lp(z) = LS\chi(z) = zp(z) = (\Lambda + u(n) + v(n)\Lambda^T)p(z)\). If we take the sequence terms \(\{p_n(z)\}_n\) we conclude \(zp_n(z) = p_{n+1}(z) + u(n)p_n(z) + v(n)p_{n-1}\). For the polynomials \(\bar{p}(z)\) we have \(\bar{L}\bar{p}(z) = z\bar{p}(z) = (\Lambda v^T(n) + u^T(n) + \Lambda^T)\bar{p}(z)\), and using the sequence \(\{\bar{p}_n(z)\}_n\) we obtain another recurrence law.
3. Multi-graded Hankel reduction, generalized orthogonality, multiple orthogonal polynomials and Riemann–Hilbert problems

In this section we will study generalized Hankel-type conditions. Given a multi-index \( \tilde{n} = (n_1, \ldots, n_N) \) with \( n_a \) non-negative integers we define for \( A \in g \) the power \( A^{\tilde{n}} = \sum_{a=1}^N A^{n_a} E_{aa} \).

For two multi-indices \( \tilde{n} \) and \( \tilde{m} \) a matrix \( g \) is said to be a \( (\tilde{n}, \tilde{m}) \)-multi-graded Hankel if

\[
\Lambda_1 g = g(\Lambda_1)^{\tilde{m}}. \tag{3}
\]

If as before \( g_{ij} \in \mathbb{C}^{N \times N} \) denotes a block in \( g \), then we can write \( g_{ij} = (g_{ij,ab})_{1 \leq a, b \leq N} \) and the multi-graded Hankel condition reads \( g_{i+n,a,j+ab} = g_{ij+m,ab} \). An ample family of multi-graded Hankel matrices can be constructed in terms of weights \( \rho_{j,ab} \) as the moments

\[
g_{ij,ab} = \int_{\mathbb{R}} x^j \rho_{j,ab}(x) \, dx, \tag{4}
\]

where the weights satisfy a generalized periodicity condition of the form

\[
\rho_{j+m,a,ab}(x) = x^{n_a} \rho_{j,ab}(x). \tag{5}
\]

Thus, given the weights \( \rho_{0,ab}, \ldots, \rho_{m_{i-1},ab} \), all the others are fixed by (5). From now on, we concentrate only on these cases of multi-graded Hankel matrices. We note that for the 1-component case and for the \( n_1 = m_1 \) case these moment matrices were studied in [5, 6].

**Proposition 5.** For multi-graded Hankel matrices the matrix polynomials \( p_i \) satisfy the following generalized orthogonality conditions:

\[
\int_{\mathbb{R}} p_i(x) \rho_j(x) \, dx = 0, \quad j = 0, \ldots, i - 1, \quad \rho_j := (\rho_{j,ab}) \in \mathbb{C}^{N \times N}. \tag{6}
\]

**Proof.** From \( Sg(t) S^{-1} = 1_G \) we get \( \sum_{j=1}^{m_c} P_{i,j,ab} g_{jl,bc} = 0 \) for \( a, c = 1, \ldots, N \) and \( l = 0, \ldots, i - 1 \). Now recalling (4) we get the result. \( \square \)

Using the Euclidean division \( i = \theta_c m_c + \sigma_c \), with \( \theta_c \geq 0 \), \( 0 \leq \sigma_c < m_c \) we get a better insight of the orthogonality relations (6), for \( a, c = 1, \ldots, N \),

\[
\sum_{b=1}^{N} \int_{\mathbb{R}} p_{i,ab}(x) \rho_{j,bc}(x)(x^{n_a})^l \, dx = 0, \quad j = 0, \ldots, m_c - 1, \quad l = 0, \ldots, \theta_c - 1, \tag{7}
\]

\[
\sum_{b=1}^{N} \int_{\mathbb{R}} p_{i,ab}(x) \rho_{j,bc}(x)(x^{n_a})^\theta \, dx = 0, \quad j = 0, \ldots, \sigma_c - 1.
\]

3.1. Evolution

From (3) we conclude that \( g(t) \) is of \( (\tilde{n}, \tilde{m}) \)-multi-graded Hankel type if \( g \) is. In general, the evolution of \( g \) is given in terms of elementary Schur polynomials by

\[
g_{ij}(t) = \sum_{k,l \geq 0} s_k g_{i+k,j+l}
\]

which recalling (4) leads to the following evolution of the weights:

\[
\rho_{j}(t) = \sum_{k,l \geq 0} x^k s_k \rho_{j+l} = \exp(t(x)) \sum_{l \geq 0} \rho_{j+l} \tag{8}
\]
where \( t(x) = \sum_{a=1}^{N} t_a(x) E_{a1} \) and \( t_a(x) := \sum_{j \geq 1} t_{ja} x^j \). It can be easily checked that \( \rho_j(t) \) satisfies the periodicity condition (5) if \( \rho_j \) does. From (8) we infer that

\[
\begin{pmatrix}
\rho_{0,ab}(t) \\
\vdots \\
\rho_{m_{b-1},ab}(t)
\end{pmatrix} = \exp(t_a(x))
\begin{pmatrix}
\mathcal{J}_{0,ab} & \mathcal{J}_{1,ab} & \mathcal{J}_{2,ab} & \cdots & \mathcal{J}_{m_{b-1},ab} \\
\mathcal{J}_{0,ab} & \mathcal{J}_{1,ab} & \mathcal{J}_{2,ab} & \cdots & \mathcal{J}_{m_{b-1},ab} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathcal{J}_{0,ab} & \mathcal{J}_{1,ab} & \mathcal{J}_{2,ab} & \cdots & \mathcal{J}_{m_{b-1},ab}
\end{pmatrix}
\begin{pmatrix}
\rho_{0,ab} \\
\vdots \\
\rho_{m_{b-1},ab}
\end{pmatrix}
\] (9)

where

\[
\mathcal{J}_{i,ab} = \sum_{j \geq 0} \xi_{i+m_j}(x^{n_j}) = \frac{1}{m_b x^{n_a/m_b}} \sum_{k=0}^{m_{b-1}} \epsilon_{b}^{ik} x^{j+m_{b}i}, \quad \epsilon_{b}^{m_{b}} = 1.
\]

If we denote

\[
\tilde{t}_a^{(l)}(x) = \sum_{j \geq 0} \tilde{t}_{j+m_{b}l,a} x^{j+m_{b}l}, \quad l = 0, 1, \ldots, m_{b} - 1,
\]

we have

\[
\tilde{t}_a(\epsilon_{b}^{ik} x) = \tilde{t}_a^{(0)}(x) + \tilde{t}_a^{(l)}(x),
\]

\[
\tilde{t}_a^{(l)}(x) = \sum_{l=1}^{m_{b} - 1} \tilde{t}_{j+m_{b}l,a} x^{j+m_{b}l},
\]

and therefore

\[
\mathcal{J}_{i,ab} = \frac{1}{m_b x^{n_a/m_b}} \exp(-\tilde{t}_a^{(0)}(x^{n_a/m_b}) \sum_{k=0}^{m_{b}-1} \epsilon_{b}^{ik} x^{j+m_{b}i}).
\] (10)

Finally, we deduce

\[
\rho_{j,ab}(t) = \exp \left( t_a(x) - \tilde{t}_a^{(0)}(x^{n_a/m_b}) \sum_{k=0}^{m_{b}-1} \epsilon_{b}^{ik} x^{j+m_{b}i} \right) \rho_{j,ab},
\] (11)

where we have used the discrete Fourier transform of the weights

\[
\tilde{\rho}_{j,ab}^{(k)} := \frac{1}{m_b} \sum_{i=1}^{m_{b}-1} \epsilon_{b}^{ik} x^{-n_a/m_b} \rho_{j+i,ab}.
\]

### 3.2. Recursion relations and symmetries

In terms of Lax operators the multi-graded Hankel reduction reads [10]

\[
\mathcal{L} := \sum_{a=1}^{N} L^{a1} C_a = \sum_{b=1}^{N} L^{m_{b}} C_b,
\] (12)

Within this subsection we assume that

\[
n_1 \geq \cdots \geq n_N \geq 1, \quad m_1 \geq \cdots \geq m_N \geq 1,
\]
and suppose that \( n_1 = \cdots = n_r \) and \( n_r > n_{r+1} \). Given (1), from (12) we deduce that

\[
\mathcal{L} = (E_{11} + \cdots + E_{rr}) \Lambda^{n_1} + \mathcal{L}_{r-1} \Lambda^{n_{r-1}} + \cdots + \mathcal{L}_0 + \mathcal{L}_{-1} \Lambda^{2} + \mathcal{L}_{-2} \Lambda^{1} + \cdots,
\]

while we also have \( \mathcal{L} = \sum_{b=1}^N L^b b \) and therefore

\[
\mathcal{L} = \mathcal{L}_{-m_1} (\Lambda^{m_1}) + \mathcal{L}_{-m_1+1} (\Lambda^{m_1-1}) + \cdots + \mathcal{L}_0 + \mathcal{L}_1 \Lambda + \mathcal{L}_2 \Lambda^2 + \cdots.
\]

We conclude the block band structure

\[
\mathcal{L} = (E_{11} + \cdots + E_{rr}) \Lambda^{n_1} + \mathcal{L}_{r-1} \Lambda^{n_{r-1}} + \cdots + \mathcal{L}_{-m_1} (\Lambda^{m_1}) + \mathcal{L}_{-m_1+1} (\Lambda^{m_1-1}) + \mathcal{L}_{-m_1+2} (\Lambda^{m_1-2}) + \cdots.
\]

(13)

**Proposition 6.** The polynomials \( p_i(z) \) are subject to

\[
(E_{11} + \cdots + E_{rr}) p_{n_1}(z) + \cdots + \mathcal{L}_{-m_1} p_{-m_1}(z) = p_i(z) \left( \sum_{a=1}^n z^{a^1} E_{aa} \right)
\]

**Proof.** We only have to show that \( \mathcal{L} p = p(\sum_{a=1}^N z^{a^1} E_{aa}) \). But this follows from

\[
\mathcal{L} p = S \left( \sum_{a=1}^N \Lambda^{a^1} E_{aa} \right) S^t S \mathcal{L} = S \left( \sum_{a=1}^N \Lambda^{a^1} E_{aa} \right) = p(z) \left( \sum_{a=1}^N z^{a^1} E_{aa} \right).
\]

Similarly, from \( \mathcal{L}^i p(z) = \tilde{p}(z) \left( \sum_{a=1}^N z^{a^1} E_{aa} \right) \) a recursion relation follows for \( p_i \).

Finally we note that in this case the following symmetry conditions hold [10]:

\[
\left( \sum_{a=1}^N \frac{\partial}{\partial h_{ma,a}} + \sum_{a=1}^N \frac{\partial}{\partial h_{ma,a}} \right) L = \left( \sum_{a=1}^N \frac{\partial}{\partial h_{ma,a}} + \sum_{a=1}^N \frac{\partial}{\partial h_{ma,a}} \right) \tilde{L} = 0,
\]

\[
\left( \sum_{a=1}^N \frac{\partial}{\partial h_{ma,a}} + \sum_{a=1}^N \frac{\partial}{\partial h_{ma,a}} \right) C_i = \left( \sum_{a=1}^N \frac{\partial}{\partial h_{ma,a}} + \sum_{a=1}^N \frac{\partial}{\partial h_{ma,a}} \right) \tilde{C}_b = 0,
\]

for \( i \geq 1 \) and \( b = 1, \ldots, N \).

**3.3. Relation with multiple orthogonal polynomials**

We will see that when the weights \( \rho_j \) are particular rank 1 matrices there is a good correspondence with multiple orthogonal polynomials of mixed type with a normalization of type II. Following [1] we take two sets of non-negative multi-indices \( \bar{\nu} = (\nu_1, \ldots, \nu_p) \), \( \bar{\mu} = (\mu_1, \ldots, \mu_q) \) and write \( |\nu| = \sum_{j=1}^p \nu_j \) and \( |\mu| = \sum_{k=1}^q \mu_k \). We also take weights \( \{w_{1j}\}_{j=1}^p \) and \( \{w_{2k}\}_{k=1}^q \) which are assumed to be non-negative functions on the real line. For a fixed pair \( \bar{\nu}, \bar{\mu} \) we say that \( \{A_{\bar{\nu},\bar{\mu},j}\}_{j=1}^p \) is a set of multiple orthogonal polynomials of mixed type if deg \( A_{\bar{\nu},\bar{\mu},j} \leq \nu_j - 1 \) and the following orthogonality relations

\[
\int_{\mathbb{R}} \sum_{j=1}^p A_{\bar{\nu},\bar{\mu},j}(x) w_{1j}(x) w_{2k}(x) x^\alpha dx = 0, \quad \alpha = 0, \ldots, \mu_k - 1, \quad K = 1, \ldots, q,
\]

(14)

are satisfied. Alternatively defining the following linear forms \( Q_{\bar{\nu},\bar{\mu}}(x) := \sum_{j=1}^p A_{\bar{\nu},\bar{\mu},j}(x) w_{1j}(x) \) the orthogonality relations can be written in the following way:

\[
\int_{\mathbb{R}} Q_{\bar{\nu},\bar{\mu}}(x) w_{2k}(x) x^\alpha dx = 0, \quad \alpha = 0, \ldots, \mu_k - 1, \quad K = 1, \ldots, q.
\]

For multiple orthogonal polynomials as described above we will take \( |\nu| = |\mu| \) and assume that the following conditions hold.
(1) For each \( K = 1, \ldots, p \) the orthogonality relations for the multi-indices \( \bar{\nu} + e_K \), \( \bar{\mu} \) have a unique solution with \( A_{\bar{\nu}, \bar{\mu}, K} \) monic and \( \deg A_{\bar{\nu}, \bar{\mu}, K} = v_K - 1 \), with \( \deg A_{\bar{\nu}, \bar{\mu}, J} \leq v_J - 1 \) if \( J \neq K \). We will call it the type II normalization to the \( K \)th component and write that normalized solution as \( \{ A_{(II, K)}^{\bar{\nu}, \bar{\mu}, J} \}_{J = 1}^{\ldots, p} \).

(2) For each \( K = 1, \ldots, q \) the orthogonality relations for the multi-indices \( \bar{\nu}, \bar{\mu} - e_K \) have a unique solution with the following normalization: \( \deg A_{\bar{\nu}, \bar{\mu}, J} \leq \nu_J - 1 \) and
\[
\int_{\mathbb{R}} Q_{\nu, \mu}(x) w_{2K} x^{\mu_K} dx = 1.
\]
We will call it the normalization of type I to the \( K \)th component and write that normalized solution as \( \{ A_{(I, K)}^{\bar{\nu}, \bar{\mu}, J} \}_{J = 1}^{\ldots, p} \).

In order to connect (7) with multiple orthogonal polynomials we consider the Euclidean division \( i = q a n_a + r_a \), with \( q a \geq 0 \) and \( 0 \leq r_a < n_a \), and write
\[
 p_{i, ab}(z) = n_a^{-1} \sum_{j=1}^{n_a} z^{-1} \Pi_{ij, ab}(z^{n_a}),
\]
where \( \Pi_{ij, ab}(z^{n_a}) \) are the polynomials in \( z^{n_a} \) such that
\[
\deg \Pi_{ij, ab} \leq \begin{cases} q_b, & j \leq r_b \\
 b - 1, & j > r_b + 1 \end{cases} \quad \text{and} \quad b = a,
\]
\[
\begin{cases} q_b - 1, & j = r_b + 1 \quad \text{and} \quad b = a, \\
r_b - 1, & j > r_b + 1 \quad \text{and} \quad b \neq a.
\end{cases}
\]
Note that the monic character of \( p_i \) gives the normalization of \( \Pi_{i, ra+1, aa} \) which happens to be a monic polynomial with \( \deg \Pi_{i, ra+1, aa} = q_a \).

The inversion formula for (15) can be deduced as follows. If we denote by \( \epsilon_b := \exp(2\pi i/n_b) \), a primitive \( n_b \)th root of the unity, and evaluate at \( \epsilon_k^b z \), \( k = 0, \ldots, n_b - 1 \), we get the following system of equations:
\[
 p_{i, ab}(\epsilon_k^b z) = n_a^{-1} \sum_{j=1}^{n_a} \epsilon_k^b - j \epsilon_b^{-j} p_{i, ab}(\epsilon_k^b z), \quad k = 0, \ldots, n_b - 1,
\]
that we solve in order to obtain the polynomials \( \Pi_{i, k, ab} \) in terms of a discrete Fourier transform of the polynomial \( p_i \) through the formula
\[
\Pi_{i, k+1, ab}(z^{n_a}) = \frac{1}{n_a z^k} \sum_{j=0}^{n_a-1} \epsilon_b^{-jk} p_{i, ab}(\epsilon_b^j z).
\]

Then, (7) can be written as
\[
\int_{\mathbb{R}} \sum_{b=1}^{N} \sum_{j=1}^{n_b} \Pi_{ij, ab}(x^{n_a}) x^{l-1} p_{k, bc}(x)(x^{n_b})^l dx = 0,
\]
\[
k = 0, \ldots, m_c - 1, \quad l = 0, \ldots, \sigma_c - 1,
\]
\[
\int_{\mathbb{R}} \sum_{b=1}^{N} \sum_{j=1}^{n_b} \Pi_{ij, ab}(x^{n_a}) x^{l-1} p_{k, bc}(x)(x^{n_b})^l dx = 0, \quad k = 0, \ldots, \sigma_c - 1.
\]

These equations strongly suggest to perform a change of variables in each integrand of the type \( y = x^{n_b} \). For that aim, it is relevant that when \( n_b \) is an even number \( \text{supp} (\rho_{j, bc}) \subset \mathbb{R}^+ \), otherwise the change of variables is ill-defined. In fact, for these even cases, one easily sees that the weights must be supported on the positive axis or uniqueness of the orthogonal polynomials is not ensured. Moreover, for \( n_b \) odd it is also necessary to assume that the weight is supported either only in the positive real numbers or only in the negative real line; this requirement comes from the positivity condition on the weights and the use of (19).
Hereon we will assume that all weights are supported on the positive real semi-line. When the mentioned change of variable is performed in (16) we get
\[ \int_{\mathbb{R}} \sum_{b=1}^{N} \sum_{j=1}^{n_b} \Pi_{ij,ab}(y) \bar{\rho}_{jk,bc}(y) y^l \, dy = 0, \quad k = 0, \ldots, m_c - 1, \quad l = 0, \ldots, \theta_c - 1, \]
\[ \int_{\mathbb{R}} \sum_{b=1}^{N} \sum_{j=1}^{n_b} \Pi_{ij,ab}(y) \bar{\rho}_{jk,bc}(y) y^\sigma \, dy = 0, \quad k = 0, \ldots, \sigma_c - 1, \]
with
\[ \bar{\rho}_{jk,bc}(y) = \frac{1}{n_b} y^{\frac{1}{\sigma}} \rho_{jk,bc}(y^{\frac{1}{\sigma}}). \]

Now, if the matrix weights \( \rho_k \) are rank 1 matrices of the following particular form:
\[ \rho_{k,bc}(x) = v_1,b(x)w_{2,kc}(x^{\sigma_b}), \]
we get
\[ \int_{\mathbb{R}} \sum_{b=1}^{N} \sum_{j=1}^{n_b} \Pi_{ij,ab}(y) w_{1,jb}(y) w_{2,kc}(y) y^l \, dy = 0, \quad k = 0, \ldots, m_c - 1, \quad l = 0, \ldots, \theta_c - 1, \]
\[ \int_{\mathbb{R}} \sum_{b=1}^{N} \sum_{j=1}^{n_b} \Pi_{ij,ab}(y) w_{1,jb}(y) w_{2,kc}(y) y^\sigma \, dy = 0, \quad k = 0, \ldots, \sigma_c - 1, \]
where
\[ w_{1,jb}(y) = \frac{1}{n_b} y^{\frac{1}{\sigma}} v_1,b(y^{\frac{1}{\sigma}}). \]

We are ready to describe the relation among multiple orthogonal polynomials of type II and generalized matrix polynomials. First, given \( (a, j) \) with \( a = 1, \ldots, N \) and \( j \in \mathbb{N} \) we make the definitions \( N(a, j) := n_1 + \cdots + n_{a-1} + j \) and \( M(a, j) := m_1 + \cdots + n_{a-1} + j + 1 \).

**Proposition 7.** Relations (18) are particular cases of (14) with
\begin{enumerate}
\item \( J = N(b, j) \) for \( b = 1, \ldots, N \) and \( j = 1, \ldots, n_b; \) and \( K = M(c, k) \) for \( c = 1, \ldots, N \) and \( k = 0, \ldots, m_c - 1. \) We have, therefore, the identification \( J = n_1 + \cdots + n_{a-1} + j \) and \( K = m_1 + \cdots + m_{c-1} + k + 1. \)
\item \( p = |\bar{n}| = n_1 + \cdots + n_N \) and \( q = |\bar{m}| = m_1 + \cdots + m_N. \)
\item \( v_{N(b,j)} = \begin{cases} q_b + 1 & \text{if } j \leq r_b \text{ and } b = a, \\ q_b, & \text{if } j > r_b \text{ and } b = a, \\ q_b, & \text{if } j > r_b + 1, \end{cases} \)
where \( q_b = \begin{cases} 1 & \text{if } j = r_b + 1 \text{ and } b = a, \\ 0 & \text{otherwise.} \end{cases} \)
\item \( \mu_{M(c,k)} = \begin{cases} 0 & \text{if } k = 0 \text{ and } c = 1, \\ \theta_c, & \text{if } k > 0 \text{ and } c = 1, \\ \sigma_c & \text{if } k \leq m_c - 1, \end{cases} \)
\end{enumerate}
The reader should note that the evolution of the weights given through (8) gives the following evolution of \( w_{1,jb} \) and \( w_{2,kc} \):

\[
w_{1,jb}(t) = \exp \left( \sum_{k \geq 1} y^k/n_k t_k b \right) w_{1,jb}, \quad w_{2,kc}(t) = \sum_{l \geq 0} w_{2,k+l,c}^{(c)},
\]

Using (11) we may write

\[
w_{1,jb}(t) = \exp \left( \sum_{k \geq 1} y^k/n_k t_k b \right) w_{1,jb}, \quad w_{2,kc}(t) = \exp \left( -\bar{t}_b(y^{1/m_c}) \right) \sum_{l \geq 0} \hat{w}_{2,k+c}^{(l)} \exp \left( -\bar{t}_b(y^{1/m_c}) \right).
\]

These evolved weights fulfill the positivity condition (recall that their support is included in the positive real line) when we have \( t_{jb}, \bar{t}_{jm_c} \in \mathbb{R} \) and \( \bar{t}_{jc} \leq 0 \) for \( j \neq 0 \) mod \( (m_c) \).

### 3.4. Riemann–Hilbert problems

As we have just shown, the generalized matrix polynomials are connected with a family of multiple orthogonal polynomials for a particular rank 1 moment matrix. In [1] the Riemann–Hilbert problem for multiple orthogonal polynomials of mixed type was presented, see also [15]. We will discuss its relation with the generalized orthogonal polynomials \( p_1(z) \).

Let us recall the reader that the Cauchy transform is defined by

\[
\hat{\mathcal{Q}}_{\nu',\mu'}(z) := -\frac{1}{2\pi i} \int_{\mathbb{R}} \mathcal{Q}_{\nu',\mu'}(x) \frac{1}{z-x} w_{2k}(x) \, dx.
\]

Now we can make the following definition for the \((p + q) \times (p + q)\) complex-valued matrix \( Y(z) \)

\[
Y_{K,J} := A_{\nu',\mu',J}^{(I,K)}, \quad J = 1, \ldots, p \quad K = 1, \ldots, p
\]

\[
Y_{K,J+P} := A_{\nu',\mu',J}^{(I,K)}, \quad J = 1, \ldots, q \quad K = 1, \ldots, p
\]

\[
Y_{K+P,J} := -2\pi i A_{\nu',\mu',J}^{(I,K)}, \quad J = 1, \ldots, p \quad K = 1, \ldots, q
\]

\[
Y_{K+P,J+P} := -2\pi i A_{\nu',\mu',J}^{(I,K)}, \quad J = 1, \ldots, q \quad K = 1, \ldots, q.
\]

We will also use the following real-valued \((p + q) \times (p + q)\) matrix \( D(x) \) defined by blocks

\[
D(x) := \begin{pmatrix}
I_p & W(x) \\
0_{q \times p} & I_q
\end{pmatrix}
\]

where \( W_{JK}(x) = w_{1,J}(x) w_{2,K}(x) \).

We adapt to the present situation a result of [1] taking into account the support of the weights.

**Theorem 1.** Let \( \nu', \mu' \) be two multi-indices such that \(|\nu'| = |\mu'|\) and suppose that the normality conditions hold. Let \( \{w_{1,J}\}_{J=1}^{p} \) and \( \{w_{2,K}\}_{K=1}^{q} \) also be two sets of weight functions such that for every \( J, K \), \( w_{1,J}(x) w_{2,K}(x) \) are differentiable a.e. in \( \mathbb{R}_+ \) and \( x^J w_{1,J}, x^K w_{2,K} \in H^1(\mathbb{R}_+) \), \( j = 0, \ldots, \nu'_K - 1 \). In \( x = 0 \) we require the weight functions to be bounded. Then, the matrix \( Y(z) \) is the only solution of the following Riemann–Hilbert problem.
(1) \( Y(z) \) is analytic in \( \mathbb{C} \setminus \mathbb{R}_+ \).

(2) \( Y(x)_a = Y(x)_b \cdot D(x) \) for all \( x > 0 \).

(3) \( Y(z) \text{diag}(z^{-v_1}, \ldots, z^{-v_p}, z^{\mu_1}, \ldots, z^{\mu_q}) = \|p_{ab}\| + O(z^{-1}) \) for \( z \to \infty \).

(4) \( Y(z) = O \left( \begin{array}{ccc} 1 & 1 & \cdots & 1 & \log |z| & \log |z| & \cdots & \log |z| \\ 1 & 1 & \cdots & 1 & \log |z| & \log |z| & \cdots & \log |z| \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & \log |z| & \log |z| & \cdots & \log |z| \end{array} \right) \) when \( z \to 0 \).

**Proof.** First we see uniqueness of the solutions. First we see that as \( \det D(x) = 1 \) \( \det Y(z) \) is analytical across the integration contour; then the only possible singularity of \( \det Y(z) \) is in \( z = 0 \). As \( z \) \( \det Y(z) \to 0 \) when \( z \to 0 \) the isolated singularity must be removable and \( \det Y(z) \) is an entire function. The asymptotic behavior and Liouville’s theorem gives \( \det Y(z) = 1 \). Consequently \( Y(z)^{-1} \) exists. Given two possible solutions for the problem \( Y(z) \) and \( \tilde{Y}(z) \), the matrix \( Y(z)\tilde{Y}(z)^{-1} \) can be singular only in \( z = 0 \). As before the singularity must be removable so \( Y(z)\tilde{Y}(z)^{-1} \) is entire and bounded, and hence \( Y(z)\tilde{Y}(z)^{-1} = \|p_{ab}\| \).

Now we prove that the matrix \( Y(z) \) is a solution of the R–H problem. Condition (i) follows from the fact that polynomials are entire functions and general theory of Cauchy integrals [16] gives analytic behavior outside the integration contour. Now, observe that condition (ii) is a jumping condition on the positive real axis and is a consequence of Plemelj formulas. For condition (iii) we note that \( Y(z)_{ii} \) is a monic polynomial with degree \( v_i \) for \( i = 1, \ldots, p \). We also see that the leading term of each \( Y(z)_{ii} \) is \( z^{-v_i} \) for \( i = p, \ldots, (p + q) \) due to the orthogonality relations and the type I normalization. Consequently the diagonal elements of \( Y(z) \text{diag}(z^{-v_1}, \ldots, z^{-v_p}, z^{\mu_1}, \ldots, z^{\mu_q}) \) are equal to 1 and the rest of them vanish like \( \frac{1}{z} \) asymptotically. Finally, condition (iv) is a consequence of the behavior of the polynomials and the Cauchy transforms. The bounness condition of the weights at \( z = 0 \) makes the Cauchy integrals to have at much log-type singularities at \( z = 0 \).

The theorem applies to our situation giving

**Proposition 8.** If \( v' = \bar{v} = \epsilon_{N(a',r'+1)} \) and \( \bar{\mu}' = \bar{\mu} \), with \( \bar{v} \) and \( \bar{\mu} \) as in proposition 7, we have \( \Pi_{ij,ab} = \Pi_{N(a,r+1),N(b,j)} \) and

\[
p_{i,ab}(z) = \sum_{j=1}^{n_b} z^{j-1} Y_{N(a,r+1),N(b,j)}(z^n),
\]

\[
Y_{N(a,r+1),N(b,j)}(z^n) = \frac{1}{n_b z^{j-1}} \sum_{l=0}^{n_b-1} \epsilon_{b/l}^{-(j-l-1)} p_{i,ab}(\epsilon_{b/l}^n).
\]
4. Conclusions and perspectives

In this paper we have studied further the relation of the Gauss–Borel factorization with integrable hierarchies, on the one hand, and with orthogonality and bi-orthogonality, on the other hand. We have seen how this basic idea can be used in the context of matrix orthogonal polynomials and also with generalized orthogonality. For some finite-rank constraints the connection with a Riemann–Hilbert problem has been proven. It is well known that multiple orthogonality and the corresponding Riemann–Hilbert problem founds applications in the theory of Brownian motion, and also in random matrix theory, and therefore our work might have applications in these two subjects. However, let us point out that another possible application, following the work of Karlin and McGregor [17], is the introduction of memory in death–birth processes. In such processes the transition probability functions satisfy differential equations with operators that have \( n \)-diagonal matrix representations \((n > 3)\), just as in our case. Moreover, the existence of Riemann–Hilbert problems for some of our examples indicates that probably the needed strong asymptotic behavior of the multi-orthogonal polynomials of mixed type, considered in this paper, could be determined. These ideas constitute the kernel of a work in progress.

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