Solutions to Master Equations of Quantum Brownian Motion in a General Environment with External Force

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Abstract

We revisit the model of a system made up of a Brownian quantum oscillator under the influence of an external classical force and linearly coupled to an environment made up of many quantum oscillators at zero or finite temperature. We show that the HPZ master equation for the reduced density matrix derived earlier [B.L. Hu, J.P. Paz, Y. Zhang, Phys. Rev. D 45, 2843 (1992)] with coefficients obtained from solutions of integro-differential equations can assume closed functional forms for a fairly general class of spectral densities of the environment at arbitrary temperature and coupling strength. As an illustration of these new results we solve the corresponding master equation and calculate, among other physical quantities, the uncertainty function whose late time behavior can be obtained fully. This produces a formula for investigating the standard quantum limit which is central to addressing many theoretical issues in macroscopic quantum phenomena and experimental concerns related to low temperature precision measurements. We find that any initial state always settles down to a Gaussian density matrix whose covariance is determined by the thermal reservoir and whose mean is determined by the external force. For more general spectra we show that the solution of the master equation can be reduced to solving for the motion of a classical parametric oscillator with parametric frequency determined by the unsolved for master equation coefficients. States in these systems experience evolution that is parametrically similar to the simpler evolution explicitly determined for in the case of Laurent-series spectra.
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In this paper we continue the lineage of work on quantum Brownian motion via the influence functional path-integral method of Feynman and Vernon[1] used by Caldeira and Leggett[2] to derive a master equation for a high-temperature ohmic environment, which corresponds to the Markovian regime. Following this, Caldeira, Cerdeira and Ramaswamy (CCR)[4] derived the Markovian master equation for the system with weak coupling to an ohmic bath, that was claimed to be valid at arbitrary temperature (Sec. IID). At the same time Unruh and Zurek[5] derived a more complete and general master equation that incorporated a colored noise at finite temperature. Finally Hu, Paz and Zhang (HPZ) [12] derived an exact master equation for a general environment (arbitrary temperature and spectral density) which preserves the positive definiteness of the reduced density matrix, an
important property missed out in many earlier derivations.

For many calculations, physicists often invoke Markovian master equations as they are easier to solve owing to their having compact, functional representations. But one runs into trouble if one is interested in low temperatures, short times or non-ohmic baths, which likely fall under the non-Markovian regime (see Ref. [21] for a discussion). The HPZ equation is capable of dealing with the full range of parameters for a general environment but its form is somewhat involved. For example the coefficients are represented by solutions to integro-differential equations and multiple integrals over Green’s functions. In this paper we present the correct (cf. for instance CCR) late-time non-Markovian master equation in compact functional representation for arbitrary temperature and coupling strength (Sec. II C 2). Additionally we obtain the coefficients for a large class of spectral densities and their time dependent form at times before the oscillator and reservoir have equilibrated.

Solutions to the HPZ equation have been attempted before [22] and the role of the master equation coefficients (renormalized frequency, dissipation, diffusion, and anomalous diffusion coefficients) has been discussed in the past. Here we present complete solutions of the master equation. We consider a quantum oscillator under the influence of an external classical force that is linearly coupled to a thermal reservoir of quantum oscillators. The particle begins decoupled from the reservoir and uncorrelated with it, with a short switch-on time \( \tau \) for the coupling. The master equation coefficients are derived for spectral functions which are Laurent series in the frequency with ultraviolet (UV) cut-off \( \Lambda \) much larger than the system frequencies, and infrared (IR) cut-off \( \lambda \) much smaller than the system frequencies. These coefficients are calculated in detail for arbitrary temperature and arbitrary time after the switch-on time. The master equation is solved explicitly for this class of spectra, and the general solution for arbitrary coefficients is reduced to solving the classical equation of motion of a parametric oscillator.

It is clearly shown how each coefficient enters into the solution and how the state evolves in time. The anomalous diffusion coefficient is actually an “anti-diffusion” term that keeps the position uncertainty finite even when the UV cut-off tends to infinity (in contrast with the momentum uncertainty). Having the solution gives us a plethora of information such as the late time thermal covariance [Eq. (IV.21)] and uncertainty function (see Sec. IV B 2). These results generalize the work of Anastopoulos and Halliwell [15], who already found the late time state to be a Gaussian and placed lower bounds on the uncertainty function.
which we discuss in Sec. IV B 2. This also crystallizes the work of Hu and Zhang [13] on the
generalized uncertainty function for Gaussian states.

Finally, we derive the master equation that includes the influence of a classical external
force and solve the master equation in that case as well.

B. Systematic Overview

The paper is organized as follows. We begin our derivation of the master equation
in Sec. II with a quantum oscillator linearly coupled to a thermal reservoir of quantum
oscillators. The spectral density here is assumed to be ohmic with high-frequency cut-off Λ.
The particle begins both decoupled from and uncorrelated with the reservoir. The coupling
is turned off at the initial time \( t = 0 \) and is switched on within a short time scale \( τ \). At
moderate times, \( t \gg τ \) (but not necessarily \( t > Ω^{-1}, γ^{-1}_0 \) ), we have obtained the master
equation coefficients as an expansion from zero temperature (Sec. II B 1), as an expansion
from high temperature (Sec. II B 2), and in closed form as an approximation that is exact at
both zero and extreme temperatures (Sec. II B 3). Exact closed form solutions are possible in
terms of various special functions, but they do not straightforwardly reveal the behavior seen
in our expansions and approximations. Most importantly, we have closed form solutions for
the master equation coefficients at late times, \( t \gg γ^{-1}_0 \), and arbitrary temperature (Sec. II B).

In Sec. III we extend these results first to analytic spectral functions (Sec. III B) and
then to Laurent-series spectra (Sec. III C). With the inclusion of subohmic terms we must
introduce a low frequency cut-off \( λ \). We take both the UV and IR cut-offs to be very
large and very small respectively and we only consider spectra that give contributions to the
master equation coefficients no more divergent than in the ohmic case (which has logarithmic
dependence on the UV cut-off). This limits our study to equations of motion for the system
trajectories with the same form as in the ohmic case and whose contributions to the master
equation coefficients are not markedly different either. Spectral densities of arbitrary powers
have been studied numerically in the past, but we do not consider them as they correspond
to classical paths that involve fractional calculus (\( i.e. \), integro-differential equations) and
would be difficult to solve analytically.

In Sec. IV we solve the master equation for this class of spectra. It is seen that the
initial solution undergoes damped oscillations while evolving into a Gaussian state of thermal
equilibrium. All the cumulants of the Wigner distribution are easily determined as functions of time (Sec. IV B 1). In particular we can provide the exact late-time uncertainty function (Sec. IV B 2). In Sec. IV C we extend this solution to more general spectra, provided that one knows the form of the master equation coefficients. The solution to this master equation is reduced to solving for the motion of a classical parametric oscillator. It is seen that simple damped oscillations may now be parametric damped oscillations, otherwise the form of the solution is the same.

Finally in Sec. V we extend the method of Calzetta, Roura, and Verdaguer [18] to include the influence of a classical force acting upon the oscillator. The master equation turns out to be what one would naively guess. We solve this master equation in Sec. V B and it is seen that the external force drives the mean around while the evolution of the variance remains unchanged and determined by the reservoir.

In the last section we conclude with a list of our findings and suggestions for their applications.

II. MASTER EQUATION COEFFICIENTS FOR AN OHMIC SPECTRUM

A. Ohmic Spectrum Master Equations

The Lagrangian of a system consisting of a quantum Brownian oscillator with mass \( M \), natural frequency \( \Omega \) and coordinate \( x \) coupled with coupling constants \( c_n \) to an environment at temperature \( T \) made up of \( n \) oscillators with mass \( m_n \), natural frequency \( \omega_n \) and coordinates \( x_n \) is given by

\[
L = \frac{1}{2} M \left( \dot{x}^2 - \Omega^2 x^2 \right) + \sum_n \frac{1}{2} m_n \left( \dot{x}_n^2 - \omega_n^2 x_n^2 \right) - \theta_\tau(t) \sum_n c_n x x_n, \tag{II.1}
\]

where \( \theta_\tau(t) \approx e^{-\left(\frac{t}{\tau}\right)^2} \) is a switch-on function with a very short characteristic time-scale \( \tau \). In this section we work with an ohmic bath so that the spectral density function is given by

\[
I(\omega) = \frac{2}{\pi} \gamma_0 M \omega \quad (0 < \omega < \Lambda), \tag{II.2}
\]

where \( \Lambda \) is a high-frequency cut-off of the bath and \( \gamma_0 \) is the dissipation frequency. We will be using this hard cut-off for all our calculations. Gaussian and exponential decay cut-off functions, of the form \( e^{-\left(\omega/\Lambda\right)^2} \) and \( e^{-\omega/\Lambda} \) respectively, are also common choices.
The system and environment begin decoupled at $t = 0$, but are fully engaged in a time $\tau$ which is assumed to be very short but non-zero. This time-dependent coupling is easily dealt with following the work of Hu and Matacz[14] on the time-dependent problems where all parameters of the system and bath oscillators and their couplings are allowed to be time-dependent. When only the system-environment couplings are time-dependent, as in our case, the dissipation and noise kernels are given respectively by

$$
\mu(t, s) = 2\gamma_0 M \frac{d}{dt} \delta_\Lambda(t - s) \theta_\tau(t) \theta_\tau(s), \tag{II.3}
$$

$$
\nu(t, s) = \int_0^\infty d\omega \cos[\omega(t - s)] \coth\left(\frac{\omega}{2T}\right) I'(\omega) \theta_\tau(t) \theta_\tau(s), \tag{II.4}
$$

where

$$
\delta_\Lambda(t) = \frac{1}{\pi} \int_0^\Lambda d\omega \cos(\omega t) = \frac{\sin(\Lambda t)}{\pi t}, \tag{II.5}
$$

and it tends to a Dirac delta distribution in the limit $\Lambda \to \infty$. Had we chosen a Gaussian high-frequency cut-off function instead of a hard cut-off, we would have found the dissipation kernel (II.3) to be the derivative of a Gaussian distribution that corresponds to an alternative representation of the delta distribution in the infinite cut-off limit. On the other hand, the integral for the noise kernel (II.4) is more complicated, even without the cut-off. Although it can also be expressed in terms of distributions, for the purposes of this paper it is best left to the last.

1. Initial Time Divergences

In HPZ[12] the system-environment coupling was taken to be constant in time, yet the system and environment were initially uncorrelated. This gives rise to initial “jolts” in the normal diffusion coefficient of the master equation with a characteristic time-scale of order $\Lambda^{-1}$ and an amplitude proportional to $\Lambda$, which diverges in the limit $\Lambda \to \infty$. It also gives rise to a delta-like term at the initial time in the equation of motion:

$$
\ddot{u}(t) + 2\gamma_0 \dot{u}(t) + \Omega_\tau^2 u(t) = -4\gamma_0 \delta_\Lambda(t) u(0). \tag{II.6}
$$
In the limit of infinite cut-off this sudden frequency change gives an initial kick to the homogeneous solutions of the equation of motion, which can be seen as follows:

\[
\dot{u} = v \\
\Delta u = \lim_{\epsilon \to 0} \int_{0}^{\epsilon} dt \dot{u} = 0 \quad (\text{II.7})
\]

\[
\dot{v} = -2\gamma_0(t)v - \Omega_r^2(t)u \\
\Delta v = \lim_{\epsilon \to 0} \int_{0}^{\epsilon} dt \dot{v} = -2\gamma_0 u(0) \quad (\text{II.8})
\]

\[
\Omega_r^2(t) = \Omega_r^2 + 4\gamma_0 \delta(t) \quad (\text{II.9})
\]

Thus, the classical paths experience a finite velocity change within an infinitesimal time. Following the approach in Ref. [18] one can easily see that this kick translates into a distortion of the Wigner distribution from the bare initial state to a shifted one

\[
W_{\text{bare}}(x, p) \rightarrow W_{\text{ren}}(x, p) = W_{\text{bare}}(x, p + 2M\gamma_0 x). \quad (\text{II.10})
\]

One expects that this result, which corresponds to replacing \( \delta_{\Lambda}(t) \) with a Dirac delta in Eq. (II.6), is a good approximation whenever all the relevant time-scales (\( \Omega_r^{-1}, \gamma_0 \) and \( t \)) are much larger than \( \Lambda^{-1} \).

The physical origin of the jolts in the coefficients of the master equation as well as other initial time divergences (such as the divergent contributions to correlation functions of system observables that are due to divergent boundary terms at the initial time (see Appendix D in Ref. [17]) can be understood as follows. In general when a system couples to an environment with an infinite number of modes, well-behaved states exhibit correlations with arbitrarily high-frequency modes. In contrast, states that are uncorrelated for sufficiently high frequencies (and hence completely factorizable states in particular) are pathological. For instance, they have infinite energy and their Hilbert space is even unitarily inequivalent to the space of physical states, spanned by the basis of energy eigenvectors of the whole system Hamiltonian including the system-environment interaction. (Of course for a finite UV cut-off there are no divergences, but the potentially divergent terms are very sensitive to changes in the value of the cut-off.) Physically acceptable initial states that correspond to the thermal equilibrium state for the whole system can be obtained using Euclidean path integrals [3]. However, the instantaneous preparation functions employed in Ref. [3] to produce other states in addition to the thermal equilibrium state still give rise to initial divergences, as explained in Ref. [6]. In order to obtain finite results, one needs to prepare the new initial state within a non-vanishing time [7], which corresponds to a physically more realistic situation. The
alternative approach that we follow here is to switch on the interaction smoothly within a
time \( \tau \) much longer than \( \Lambda^{-1} \) but shorter than any other relevant time-scale of the prob-
lem. In this way the factorized initial state, which is perfectly acceptable in the uncoupled
case, becomes adequately correlated with the arbitrarily high-frequency modes in a regular
fashion.

When adding the short time switch-on function (II.1) to the spectral density to turn on
the interaction gradually, the initial jolt is no longer present in the results for the master
equation coefficients. Moreover, the equation of motion exhibits a smooth transition between
the decoupled and the coupled system without the term proportional to \( \delta_\Lambda(t) \) on the right-
hand side:

\[
\ddot{u}(t) + \Omega^2 u(t) = 0, \quad t \ll \tau \tag{II.11}
\]
\[
\ddot{u}(t) + 2\gamma_0 \dot{u}(t) + \Omega_r^2 u(t) = 0, \quad \tau \ll t. \tag{II.12}
\]

However, since the constant bare frequency \( \Omega^2 \) is of order \( \Lambda \) so as to cancel out the divergent
contribution that arises when integrating out the environment and give a finite value of the
renormalized frequency at late times (larger than the switch-on time), the frequency will
change significantly in a short period of time of order \( \tau \). As long as \( \tau \) is much shorter than
all the other relevant time-scales in the problem (except for \( \Lambda^{-1} \)), the dynamics for \( t \gg \tau \)
(a)
can be obtained by approximating the time-dependent frequency by a delta function. More
specifically, since the renormalized frequency changes within a characteristic time-scale of
order \( \tau \) and with an amplitude of order \( \Lambda \), it can be written as

\[
\Omega_r^2(t) = \Omega_r^2 - 2c\Lambda \tau \delta_\tau(t), \tag{II.13}
\]

where \( \Omega_r^2 \) is the asymptotic constant value for times much larger than \( \tau \), \( c \) is a constant of
order one, and \( \delta_\tau(t) \) is a function peaked at \( t = 0 \) with amplitude of order \( \tau^{-1} \) and width of
order \( \tau \) that becomes a Dirac delta in the limit \( \tau \to 0 \). Therefore, for \( t \gg \tau \) the dynamics
is governed by

\[
\ddot{u}(t) + 2\gamma_0 \dot{u}(t) + \Omega_r^2 u(t) = 2c\Lambda \tau \delta_\tau(t)u(0). \tag{II.14}
\]

Note that \( \Lambda \tau \gg 1 \) if the switch-on time is sufficiently large to cure the initial divergences
discussed above.

The term on the right-hand side of Eq. (II.14) has the same form as in Eq. (II.6). It will,
therefore, give the same kind of initial kick to the solutions of the equation and generate the
same kind of transformation of the reduced Wigner function:

\[ W_{\text{bare}}(x, p) \to W_{\text{ren}}(x, p) = W_{\text{bare}}(x, p - c \Lambda \tau M x). \quad (\text{II.15}) \]

This phase-space transformation has a Jacobian with determinant equal to unity:

\[ L = \begin{pmatrix} 1 & 0 \\ -c \Lambda \tau M & 1 \end{pmatrix} \quad \text{det } L = 1. \quad (\text{II.16}) \]

Therefore, it is simple to calculate renormalized expectation values in terms of bare expectation values and vice versa:

\[
\langle A(x, p) \rangle_{\text{ren}} = \iint dxdp A(x, p) W_{\text{ren}}(x, p), \\
\langle A(x, p) \rangle_{\text{ren}} = \langle A(x, p + c \Lambda \tau M x) \rangle_{\text{bare}}.
\]

(II.17)  
(II.18)

We can immediately see that the normalization, linear entropy (see Sec. IV B 3) and state overlap are all unchanged by the kick. We can also check that the Heisenberg uncertainty relation is also preserved as follows. First, we start with the covariance matrix for \( x \) and \( p \) corresponding to the Wigner distribution

\[ \sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xp} \\ \sigma_{px} & \sigma_{pp} \end{pmatrix}, \quad (\text{II.19}) \]

with \( \sigma_{xx} = \langle xx \rangle_{\text{ren}} \), \( \sigma_{xp} = \sigma_{px} = \langle xp \rangle_{\text{ren}} \) and \( \sigma_{pp} = \sigma_{pp} = \langle pp \rangle_{\text{ren}} \), and which transforms in the following way under linear phase space transformations:

\[ \sigma \to L^T \sigma L. \quad (\text{II.20}) \]

Hence, from Eq. (II.16) we have

\[ \text{det } \sigma_{\text{bare}} = \text{det } \sigma_{\text{ren}}. \quad (\text{II.21}) \]

Finally, one takes into account that

\[ (\text{det } \sigma) \geq \frac{\hbar^2}{4}, \quad (\text{II.22}) \]

corresponds to the formulation in terms of the Wigner function of the generalized Heisenberg uncertainty relation due to Schrödinger [8, 9]:

\[ (\Delta x)^2(\Delta p)^2 - \langle \{ \hat{x} - \langle \hat{x} \rangle, \hat{p} - \langle \hat{p} \rangle \} \rangle^2 \geq \frac{\hbar^2}{4}, \quad (\text{II.23}) \]
where \( \{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A} \). In addition to being real and normalized, a phase space distribution corresponds to the Wigner function of a physical state only when the corresponding density matrix (undoing the Wigner transformation) satisfies the condition \( \rho^2 \leq \rho \). For Gaussian distributions this is guaranteed if and only if Eq. (II.22) is satisfied \([10]\). Thus, the kick does not change whether a given Gaussian distribution corresponds to a physical state. On the other hand, for non-Gaussian distributions Eq. (II.22) is no longer a sufficient condition for that, but one can use the following argument. For a finite switch-on time, the frequency change simply corresponds to unitary evolution associated with a time-dependent Hamiltonian, which leaves the eigenvalues of the density matrix non-negative. By continuity, the limit of a very short switch-on time cannot make these eigenvalues negative.

If one is interested in studying the evolution of a certain state of the system properly correlated with the environment (such as the global equilibrium states considered in Ref. \([3]\) or states prepared from those in a finite time) one can always consider the reduced Wigner function associated with that state and invert Eq. (II.22) to obtain the corresponding initial Wigner function before the interaction was switched on. (Note that for times larger than the switch-on time the result is then essentially equivalent to having introduced a time-dependent counterterm for the frequency so that the renormalized frequency was constant in time.)

2. Expressions for the Master Equation Coefficients

The HPZ master equations for the reduced density matrix \( \rho_r \) and the reduced Wigner function are given respectively by

\[
\begin{align*}
\dot{\hat{\rho}}_r & = [\hat{H}_r, \hat{\rho}_r] + \Gamma [\hat{x}, \{\hat{p}, \hat{\rho}_r\}] + \imath D_{xp} ([\hat{x}, [\hat{p}, \hat{\rho}_r]] + [\hat{p}, [\hat{x}, \hat{\rho}_r]]) - \imath D_{pp} [\hat{x}, [\hat{x}, \hat{\rho}_r]], \\
\frac{\partial}{\partial t} W_r & = [H_r, W_r] + 2\Gamma \frac{\partial}{\partial p} p W_r - 2D_{xp} \frac{\partial^2}{\partial x \partial p} W_r + D_{pp} \frac{\partial^2}{\partial p^2} W_r,
\end{align*}
\]

(II.24) (II.25)

where the detailed form of the dissipation function \( \Gamma(t) \) and the diffusion functions \( D_{xp}(t) \) and \( D_{pp}(t) \) can be found in \([12]\). Following the derivation of Calzetta, Roura, and Verdaguer \([18]\) the coefficients in the HPZ master equation for an ohmic bath with a high cut-off \( \Lambda \) and
evaluated at times larger than the switch-on time are given by:

\[ \Omega_{\text{ren}}^{2} = \Omega_{r}^{2}, \]  

\[ \Gamma = \gamma_{0}, \]  

\[ D_{xp} = -\frac{1}{2} \int_{0}^{t} ds \, \nu(t, s) G_{\text{ret}}(t, s), \]  

\[ D_{pp} = M \int_{0}^{t} ds \, \nu(t, s) \frac{\partial}{\partial t} G_{\text{ret}}(t, s), \]  

\[ G_{\text{ret}}(t, s) = \frac{1}{M \tilde{\Omega}} \sin \tilde{\Omega}(t - s) e^{-\gamma_{0}(t-s)}, \]  

where \( \tilde{\Omega} = \sqrt{\Omega_{r}^{2} - \gamma_{0}^{2}} \) and \( G_{\text{ret}}(t, s) \) is the retarded Green function associated with the differential equation that results from multiplying Eq. (II.12) by \( M \). According to the discussion in the previous subsection, when considering times much larger than the switch-on time, its effect on the equation of motion can be approximated by a delta function, as seen in Eq. (II.14). Since \( G_{\text{ret}}(t, s) \) for \( t \leq 0 \), the contribution from the delta term on the right-hand side of Eq. (II.14) vanishes when solving for the retarded propagator. Therefore, it is indeed independent of the initial kick and given by Eq. (II.14). There is still some dependence on the switch-on function \( \theta_{\tau}(s) \) in the noise kernel for \( s < \tau \), but for \( t \gg \tau \) the contribution from \( s < \tau \) to the time integrals in Eqs. (II.28)-(II.29) is negligible provided that the behavior during the switch-on time is sufficiently regular, which is indeed the case. Similarly, the behavior of the master equation coefficients during switch-on time, which we do not calculate here, will be sufficiently regular so that its contribution to the evolution of the reduced Wigner function (or density matrix) is also negligible when considering times larger than \( \tau \) and one only needs to include the effect of the kick on the initial Wigner function, given by Eq. (II.15).

Some clarification is in order here about the use of a finite frequency cut-off \( \Lambda \) for the environment spectrum. Having a finite cut-off in the frequency integral for the noise kernel, given by Eq. (II.4), is necessary to obtain a finite result for the coefficients of the master equation because, as it will be seen below, it gives rise to contributions to the diffusion coefficients that become logarithmically divergent in the limit of large \( \Lambda \). In contrast, the use of a finite cut-off in the frequency integral of the dissipation kernel is not required to

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1 This method has the nice property that the master equation coefficients after the switch-on time are explicitly independent of any reasonable behavior during the initial switch-on.
obtain a finite result. In fact, for computational convenience we are going to take Λ → ∞ in this case because then the dissipation kernel becomes local and we just have to deal with an ordinary differential equation rather than an integro-differential one. One may object that treating the noise and dissipation kernels on a different footing could lead to inconsistencies (e.g. the fluctuation-dissipation relation will not hold exactly any more, as pointed out by HPZ [12] with regard to the treatment of this problem in Ref. [5]). However, our results are consistent when properly understood. If one considered an expansion of the exact master equation coefficients in terms Λ, one would have leading terms of order log Λ, terms of order one and subdominant terms involving inverse powers of Λ. Our approach gives the right result for the log Λ and order-one terms: using the local approximation for the dissipation kernel only alters the result for the subdominant terms. This can be seen as follows. When integrating δ_Λ(t − s), given by Eq. (II.5), with a sufficiently well-behaved function f(s), one has

$$
\int_{-\infty}^{\infty} ds \delta_\Lambda(t − s)f(s) = f(t) + f'(t)\Lambda^{-1} + O(\Lambda^{-2}).
$$

(II.31)

This condition will indeed be satisfied in our case since all the other relevant time-scales in the problem are much larger than Λ^{-1} and all the functions that δ_Λ(t − s) will be integrated with are regular and extremely uniform over that scale. Therefore, the discrepancy between using the Dirac delta function and using δ_Λ(t − s) corresponds to terms of order Λ^{-1} or higher inverse powers, which give contributions to the master equation coefficients of order Λ^{-1} log Λ or higher.

In order to compute the coefficients of the master equation, we first perform the relatively simple time integrals in Eqs. (II.28)-(II.29). On the other hand, the frequency integral in the noise kernel, as given by Eq. (II.4), is particularly difficult due to the 1/sinh(ω/2T) in coth(ω/2T). It is this integral that we save for last. Others[3, 5] have also seen the utility or at least the simplicity of reducing the master equation to a collection of one-dimensional frequency integrals. We are able to obtain four sets of new results: (a) exact results at late time for arbitrary temperature, (b) asymptotic expansions for high and low temperatures, (c) approximate general results for arbitrary temperature and all times (larger than the switch-on time), (d) time-dependent diffusion functions in closed form at high temperatures for all times (larger than the switch-on time).

After computing the time integrals in Eqs. (II.28) and (II.29), the diffusion coefficients
become

\[ D_{xp}(t) = + \frac{\gamma_0}{\pi} (\text{FI}_3 - \Omega_r^2 \text{FI}_1) \]
\[ - \frac{\gamma_0}{\pi} \cos(\bar{\Omega}t) e^{-\gamma_0 t} \left( \text{FC}_3(t) - \Omega_r^2 \text{FC}_1(t) + 2\gamma_0 \text{FS}_2(t) \right) \]
\[ + \frac{\gamma_0}{\pi \Omega} \sin(\bar{\Omega}t) e^{-\gamma_0 t} \left( \gamma_0 \left( \text{FC}_3(t) + \Omega_r^2 \text{FC}_1(t) \right) + \left( \bar{\Omega}^2 - \gamma_0^2 \right) \text{FS}_2(t) - \text{FS}_4(t) \right), \quad (\text{II.32}) \]

\[ D_{pp}(t) = + \frac{4M\gamma_0^2}{\pi} \text{FI}_3 \]
\[ - \frac{2M\gamma_0}{\pi} \cos(\bar{\Omega}t) e^{-\gamma_0 t} \left( 2\gamma_0 \text{FC}_3(t) + \Omega_r^2 \text{FS}_2(t) - \text{FS}_4(t) \right) \]
\[ - \frac{2M\gamma_0}{\pi \Omega} \sin(\bar{\Omega}t) e^{-\gamma_0 t} \left( -\Omega_r^4 \text{FC}_1(t) + \left( \bar{\Omega}^2 - \gamma_0^2 \right) \text{FC}_3(t) + \gamma_0 \left( \Omega_r^2 \text{FS}_2(t) + \text{FS}_4(t) \right) \right), \quad (\text{II.33}) \]

in terms of a single family of frequency integrals \( \text{FC}_N, \text{FS}_N, \text{FI}_N \):

\[ \text{FI}_N = \int_0^\Lambda \frac{\omega^N \coth \left( \frac{\omega}{2T} \right)}{(\omega^2 - \Omega_r^2)^2 + 4\gamma_0^2 \omega^2} d\omega, \quad (\text{II.34}) \]
\[ \text{FC}_N(t) = \int_0^\Lambda \frac{\omega^N \cos(\omega t) \coth \left( \frac{\omega}{2T} \right)}{(\omega^2 - \Omega_r^2)^2 + 4\gamma_0^2 \omega^2} d\omega, \quad (\text{II.35}) \]
\[ \text{FS}_N(t) = \int_0^\Lambda \frac{\omega^N \sin(\omega t) \coth \left( \frac{\omega}{2T} \right)}{(\omega^2 - \Omega_r^2)^2 + 4\gamma_0^2 \omega^2} d\omega, \quad (\text{II.36}) \]
\[ \text{FI}_N = \text{FC}_N(0), \quad (\text{II.37}) \]
\[ \text{FS}_N(t) = -\frac{d}{dt} \text{FC}_{N-1}(t), \quad (\text{II.38}) \]
\[ \text{FC}_N(t) = +\frac{d}{dt} \text{FS}_{N-1}(t). \quad (\text{II.39}) \]

Fortunately the only integral that needs to be computed is \( \text{FC}_1(t) \). All other integrals can be generated from this one integral. In general the integrals cannot be solved directly, therefore it is necessary to expand the hyperbolic cotangent into a series of simpler functions. This will be done with low and high-temperature expansions.

When computing the time integrals that led to Eqs. (II.32)-(II.33), we just took the switch-on functions equal to one, so that sufficiently simple analytical results could be obtained. This means that, within our approach, the results in Eqs. (II.32)-(II.33) are only valid for \( t \gg \tau \). In fact, for \( t \lesssim \tau \), they essentially coincide with those of Ref. [12] for an ohmic environment. For instance, the term \( \text{FS}_4(t) \) in Eq. (II.33) gives rise to the same initial jolt, with a width of order \( \Lambda^{-1} \) and an amplitude proportional to \( \Lambda \), found in Ref. [12]. However, as we discussed above, the results should be valid for times larger than the switch-
on time because the contribution from the switch-on period to both the master equation coefficients and the evolution of the density matrix at \( t \gg \tau \) is negligible.

It is interesting to note that the coefficients \( D_{xp}(t) \) and \( D_{pp}(t) \) both exhibit logarithmic divergences in the limit \( \Lambda \to \infty \) (for times larger than the switch-on time and thus larger than \( \Lambda^{-1} \)) due to the term proportional to \( \text{FI}_3 \). This has been pointed out for \( D_{xp}(t) \) in Ref. [11], where the coefficients of the master equation were calculated perturbatively to second order in the system-environment coupling constants (linear order in \( \gamma_0 \)). The fact that there is also a logarithmic divergence in \( D_{pp}(t) \) was not seen in that reference because it is quartic in the system-environment coupling constants (quadratic in \( \gamma_0 \)), as it can be seen in Eq. (II.33). Moreover, such kinds of perturbative calculations cannot be employed to study the long time behavior since they are only valid for \( \gamma_0 t \ll 1 \) and they miss for instance the exponential decay of the second and third terms on the right-hand side of Eqs. (II.32)-(II.33).

**B. Exact Late-Time Behavior and Approximate High and Low-Temperature Results**

For the late-time diffusion coefficients, only two integrals need to be performed and they can be expressed in closed form with reasonably intuitive functions:

\[
\text{FI}_1 = \frac{\pi T}{2\gamma_0 \Omega^2} + \frac{1}{2\gamma_0 \Omega} \text{Im} \left[ H \left( \frac{\gamma_0 + i\hat{\Omega}}{2\pi T} \right) \right],
\]

(II.40)

\[
\text{FI}_3 = \frac{\pi T}{2\gamma_0} + \frac{\hat{\Omega}^2 - \gamma_0^2}{2\gamma_0 \Omega} \text{Im} \left[ H \left( \frac{\gamma_0 + i\hat{\Omega}}{2\pi T} \right) \right] + \text{Re} \left[ H \left( \frac{\Lambda}{2\pi T} \right) - H \left( \frac{\gamma_0 + i\hat{\Omega}}{2\pi T} \right) \right],
\]

(II.41)

where terms involving negative powers of the cut-off have been neglected and \( H(z) \) is the harmonic number function defined in Appendix A 1. It behaves like \( \log (z) \) except near the

---

2 As it is well-known and can be checked from Eq. (II.34), the term logarithmic in \( \Lambda \) is not present in \( \text{FI}_3 \) when considering a fixed finite cut-off \( \Lambda \) and temperatures much higher than \( \Lambda \).

3 Many of the expressions derived throughout this paper assume underdamping with an extreme cut-off, i.e., \( \gamma_0 < \Omega_r < \Lambda \) with \( \hat{\Omega} = \sqrt{\Omega_r^2 - \gamma_0^2} \). They can be used for the overdamping regime by making the following analytical continuation: \( \hat{\Omega} \to i\hat{\gamma} \) with \( \hat{\gamma} = \sqrt{\gamma_0^2 - \Omega_r^2} \) real.

Therefore, Eqs. (II.40)-(II.41) can be applied to the overdamping case if the Im and Re terms are first expanded assuming \( \hat{\Omega} \) is real, and then the analytical continuation \( \hat{\Omega} \to i\hat{\gamma} \) is made.
origin, where it does not diverge and actually vanishes. The Re[· · ·] term in FI is effectively the \( \log(\Lambda/\Omega_r) \) divergence. It disappears at extreme temperature (higher than the cut-off).

1. Expansion from Zero Temperature

A low-temperature expansion of the master equation coefficients can be obtained using the following expansion for the hyperbolic cotangent in the corresponding frequency integrals:

\[
\coth\left(\frac{\omega}{2T}\right) = 1 + 2 \sum_{k=1}^{\infty} e^{-k\frac{\pi}{2T}}.
\] 

Moreover, since the results for all the relevant integrals can be obtained from FC\(_1(t)\) simply by differentiating with respect to \( t \), we will concentrate on computing Eq. (II.35) with \( N = 1 \). In order to do that, one first decomposes the integrand in Eq. (II.35) into partial fractions and splits the integral \( \int_0^\Lambda d\omega \) into \( \int_0^{\infty} d\omega - \int_\Lambda^{\infty} d\omega \). Finally, using Eq. (II.42) they become a linear combination of exponential integrals. The result can be written as

\[
FC_1(t) = \frac{1}{4\gamma_0 \tilde{\Omega}} \text{Im} \left[ \frac{E_1\left(+(\gamma_0 - \imath \tilde{\Omega})t\right)}{e^{-(\gamma_0 - \imath \tilde{\Omega})t}} + \frac{E_1\left(-(\gamma_0 - \imath \tilde{\Omega})t\right)}{e^{+(\gamma_0 - \imath \tilde{\Omega})t}} \right]
+ \frac{\pi}{4\gamma_0 \tilde{\Omega}} \cos(\tilde{\Omega}t)e^{-\gamma_0 t} + \Delta FC^{\text{PT}}_1(t) + \Delta FC^{\text{LA}}_1(t),
\]

where \( E_1(z) \) is the exponential integral detailed in Appendix A2. The first three terms come from the \( \int_0^{\infty} d\omega \) integral of the constant term in Eq. (II.42). On the other hand, \( \Delta FC^{\text{PT}}_1(t) \), which vanishes at zero temperature, is the contribution from the \( \int_0^{\infty} d\omega \) integral of the remaining terms in Eq. (II.42):

\[
\Delta FC^{\text{PT}}_1(t) = 2 \sum_{k=1}^{\infty} \int_0^\infty \frac{\omega \cos(\omega t)e^{-k\frac{\pi}{2T}}}{(\omega^2 - \Omega_r^2)^2 + 4\gamma_0^2 \omega^2} d\omega.
\]

Finally, the term \( \Delta FC^{\text{LA}}_1(t) \) contains all the dependence on the cut-off \( \Lambda \), which arises from all the \( \int_\Lambda^{\infty} d\omega \) integrals. In particular, its value for zero temperature, which corresponds to taking only the first term in Eq. (II.42), can be calculated explicitly:

\[
\Delta FC^{\text{ZA}}_1(t) = -\frac{1}{8\gamma_0 \tilde{\Omega}} \text{Im} \left[ \frac{E_1\left(+(\gamma_0 - \imath (\tilde{\Omega} + \Lambda))t\right)}{e^{-(\gamma_0 - \imath \tilde{\Omega})t}} + \frac{E_1\left(-(\gamma_0 - \imath (\tilde{\Omega} + \Lambda))t\right)}{e^{+(\gamma_0 - \imath \tilde{\Omega})t}} \right]
+ \frac{E_1\left(+(\gamma_0 - \imath (\tilde{\Omega} - \Lambda))t\right)}{e^{-(\gamma_0 - \imath \tilde{\Omega})t}} + \frac{E_1\left(-(\gamma_0 - \imath (\tilde{\Omega} - \Lambda))t\right)}{e^{+(\gamma_0 - \imath \tilde{\Omega})t}} \right].
\]
Note that, since $FC_1(t)$ is finite in the limit $\Lambda \to \infty$, the contribution from $\Delta FC_1^{LA}(t)$ will be very small for sufficiently large $\Lambda$. However, when differentiating several times with respect to $t$, this term gives rise to the divergences that $FC_3(t)$ and $FS_4(t)$ exhibit during a short initial period of order $\Lambda^{-1}$, which are responsible for the initial jolt of the master equation coefficients found in Ref. [12]. Moreover, even in those cases it would be enough to consider just $\Delta FC_1^{Z\Lambda}(t)$ rather than $\Delta FC_1^{LA}(t)$ because the integral in Eq. (II.44) is convergent for arbitrary positive powers of the frequency due to the exponential factor.

In fact, given that the divergent behavior for short times is irrelevant when considering a switch-on time sufficiently longer than $\Lambda^{-1}$, it turns out that in our case we can completely neglect the contribution from the cut-off dependent terms for sufficiently large $\Lambda$. It would only be necessary when obtaining $FI_3$ by evaluating $FC_3(t)$ at $t = 0$, but it is much simpler and more accurate in general to use the exact result for $FI_1$ and $FI_3$ computed earlier and given by Eqs. (II.40)-(II.41).

To sum up, taking into account the remarks in the previous paragraph together with the fact that the contribution from the third term in Eq. (II.43) cancels out when adding all the time-dependent terms in Eqs. (II.32)-(II.33), all that one needs [in addition to Eqs. (II.40)-(II.40)] to calculate the master equation coefficients is

$$\Delta FC_1(t) = \frac{1}{4\gamma_0 \tilde{\Omega}} \text{Im} \left[ \frac{E_1 \left( + \left( \gamma_0 - i\tilde{\Omega} \right) t \right)}{e^{-\left( \gamma_0 - i\tilde{\Omega} \right) t}} + \frac{E_1 \left( - \left( \gamma_0 - i\tilde{\Omega} \right) t \right)}{e^{\left( \gamma_0 - i\tilde{\Omega} \right) t}} \right]$$

$$+ 2 \sum_{k=1}^{\infty} \int_{0}^{\infty} \frac{\omega \cos(\omega t) e^{-k\frac{\omega}{2}}}{\left( \omega^2 - \Omega_r^2 \right)^2 + 4\gamma_0^2 \omega^2} d\omega. \quad (II.46)$$

Given the asymptotic behavior of $E_1(z)$ for large $|z|$, one can see that the Im$[\cdots]$ terms decay only weakly in time like $1/(\Omega_r t)^2$; they both have complicated $\tilde{\Omega}$ frequency oscillations. The integral in the last term, which vanishes when the temperature tends to zero, can be performed explicitly, but the sum becomes particularly complicated.

2. **Asymptotic Expansion from High Temperature**

A high-temperature expansion can be obtained by making use of the following expression for the hyperbolic cotangent:

$$\coth \left( \frac{\omega}{2T} \right) = \frac{2T}{\omega} + \sum_{k=1}^{\infty} \frac{4T \omega}{\omega^2 + (2\pi Tk)^2}, \quad (II.47)$$
which will yield a series solution that is best in the high temperature, late time regime $2\pi T t \gg 0$. $\text{FC}_1(t)$ can then be calculated in several steps.

First, one calculates the $\int_0^\infty d\omega$ integral corresponding to Eq. (II.35) considering only the first term in Eq. (II.47), decomposes the resulting integrand into partial fractions and ends up with a linear combination of Exponential Integrals. After simplification, the result can be written as

$$\text{FC}_1^{\text{HT}}(t) = \frac{\pi T}{2\tilde{\Omega}^2 \gamma_0} \left( \tilde{\Omega} \cos(\tilde{\Omega} t) + \gamma_0 \sin(\tilde{\Omega} t) \right) e^{-\gamma_0 t}. \quad (II.48)$$

Next, we perform the $\int_0^\infty d\omega$ integral corresponding to Eq. (II.35) with the general term in the sum of Eq. (II.47) replacing the hyperbolic cotangent. After decomposing into partial fractions we end up again with a linear combination of Exponential Integrals, and the final result after simplification is

$$\text{FC}_1^{\text{LT}}(t) = \sum_{k=1}^{\infty} \frac{\pi T e^{-\gamma_0 t}}{\tilde{\Omega} \gamma_0} \left( \tilde{\Omega}^4 + 2 \left( 4k^2 \pi^2 T^2 + \gamma_0^2 \right) \tilde{\Omega}^2 + (\gamma_0^2 - 4k^2 \pi^2 T^2) \right) \left[ -4k \pi T \tilde{\Omega} \gamma_0 e^{(\gamma_0 - 2k \pi T) t} 
+ \gamma_0 \left( -4k^2 \pi^2 T^2 + \tilde{\Omega}^2 + \gamma_0^2 \right) \sin(\tilde{\Omega} t) + \tilde{\Omega} \left( 4k^2 \pi^2 T^2 + \tilde{\Omega}^2 + \gamma_0^2 \right) \cos(\tilde{\Omega} t) \right]. \quad (II.49)$$

The sum over $k$ can be performed explicitly for the terms multiplying the sine and the cosine, so that we get

$$\text{FC}_1^{\text{LT}}(t) = \frac{\pi}{4 \gamma_0 \tilde{\Omega}} \frac{\sinh \left( \frac{\tilde{\Omega}}{T} \right) \cos(\tilde{\Omega} t) + \sin \left( \frac{\gamma_0}{T} \right) \sin(\tilde{\Omega} t)}{\cosh \left( \frac{\tilde{\Omega}}{T} \right) - \cos \left( \frac{\gamma_0}{T} \right)} e^{-\gamma_0 t} + \Delta \text{FE}_1^{\text{LT}}(t), \quad (II.50)$$

with

$$\Delta \text{FE}_N^{\text{LT}}(t) = \frac{1}{(-2\pi T)^2} \sum_{k=1}^{\infty} \frac{k e^{-2\pi T tk}}{(k^2 + \left( \frac{\gamma_0}{2\pi T} \right)^2) - 4 \left( \frac{\gamma_0}{2\pi T} \right)^2 k^2}, \quad (II.51)$$

which satisfies the following property:

$$\frac{d}{dt} \Delta \text{FE}_N^{\text{LT}}(t) = -(2\pi T) \Delta \text{FE}_{N+1}^{\text{LT}}(t). \quad (II.52)$$

Putting everything together, we have

$$\text{FC}_1(t) = \frac{\pi}{4 \gamma_0 \tilde{\Omega}} \frac{\sinh \left( \frac{\tilde{\Omega}}{T} \right) \cos(\tilde{\Omega} t) + \sin \left( \frac{\gamma_0}{T} \right) \sin(\tilde{\Omega} t)}{\cosh \left( \frac{\tilde{\Omega}}{T} \right) - \cos \left( \frac{\gamma_0}{T} \right)} e^{-\gamma_0 t} + \Delta \text{FE}_1^{\text{LT}}(t) + \Delta \text{FC}_1^{\text{HA}}(t), \quad (II.53)$$

where $\Delta \text{FC}_1^{\text{HA}}(t)$ contains all the dependence on the cut-off $\Lambda$, and comes from all the $\int_\Lambda^\infty d\omega$ integrals. Nevertheless, for the same reasons given in the previous subsection, in our case one
can neglect this term for sufficiently large values of $\Lambda$. Furthermore, since the contribution from the first term in Eq. (II.53) cancels out when adding all the time-dependent terms in Eqs. (II.32)-(II.33), and provided that we use Eqs. (II.40)-(II.41) to compute $F_{I1}$ and $F_{I3}$, it is sufficient to consider

$$\Delta F_{C1}(t) = \frac{1}{(-2\pi t)^2} \sum_{k=1}^{\infty} \left( k e^{-2\pi T t k} \frac{k e^{-2\pi T t k}}{(k^2 + (\frac{\Omega T}{2\pi T})^2) - 4 \left(\frac{\gamma_0}{2\pi T}\right)^2 k^2} \right),$$

in order to calculate the master equation coefficients.

All terms on the right-hand side of Eq. (II.54) decay with temperature-dependent timescales. Since it cannot be expressed in closed form with any intuitive functions, the sum is not explicitly performed. Although useful for numerical evaluation, this form does not immediately reveal the $\gamma_0 t$ and $\tilde{\Omega} t$ behavior. Therefore, it is convenient to approximate the sum by an integral that can be computed explicitly, as follows:

$$\Delta F_{C1}(t) \approx \frac{1}{4\gamma_0 \tilde{\Omega}} \text{Im} \left[ \frac{\text{Ei} \left( (\gamma_0 - i\tilde{\Omega} - 2\pi T)t \right)}{e^{(\gamma_0 - i\tilde{\Omega})t}} + \frac{\text{Ei} \left( (-\gamma_0 + i\tilde{\Omega} - 2\pi T)t \right)}{e^{(-\gamma_0 + i\tilde{\Omega})t}} \right].$$

(II.55)

From this expression one can see that the most lingering terms decay exponentially as $e^{-2\pi T t}$ $O(1/t)$. There are also oscillations with frequency $\tilde{\Omega}$.

3. **Approximate General Solution**

By inspecting the corresponding terms in the high (II.55) and low-temperature (II.46) expansions, one can get a good idea of the general solution:

$$F_{C1}(t) \approx \frac{\pi}{4\gamma_0 \tilde{\Omega}} \frac{\sinh \left( \frac{\tilde{\Omega}}{T} \right) \cos \left( \tilde{\Omega} t \right) + \sin \left( \frac{\gamma_0}{T} \right) \sin \left( \tilde{\Omega} t \right)}{\cosh \left( \frac{\tilde{\Omega}}{T} \right) - \cos \left( \frac{\gamma_0}{T} \right)} e^{-\gamma_0 t}$$

$$+ \frac{1}{4\gamma_0 \tilde{\Omega}} \text{Im} \left[ \frac{\text{Ei} \left( (-\gamma_0 + i\tilde{\Omega} + 2\pi T)t \right)}{e^{(\gamma_0 - i\tilde{\Omega})t}} + \frac{\text{Ei} \left( (\gamma_0 - i\tilde{\Omega} + 2\pi T)t \right)}{e^{(-\gamma_0 + i\tilde{\Omega})t}} \right]$$

$$+ \Delta F_{C1}^{\text{HA}}(t).$$

(II.56)

This expression is exact at both zero temperature and high temperature, and it has only minor discrepancies within a small range of intermediate temperatures. Moreover, as explained
above, all that one really needs to calculate the master equation coefficients is

\[
\Delta FC_1(t) \approx \frac{1}{4\gamma_0\tilde{\Omega}} \text{Im} \left[ \frac{E_1 \left( -\gamma_0 + i\tilde{\Omega} + 2\pi T \right) t}{e^{(\gamma_0 - \tilde{\Omega})t}} + \frac{E_1 \left( \gamma_0 - i\tilde{\Omega} + 2\pi T \right) t}{e^{(-\gamma_0 + \tilde{\Omega})t}} \right].
\] (II.57)

4. Comparison with Numerical Results

From our analysis of the integrals it can be seen that there is only one fluctuation frequency \(\tilde{\Omega}\) and it is always damped. There are only two damping rates, \(\gamma_0\) and \(2\pi T\). All temperature-damped terms also have coupling damping, so one can say that \(t \gg \gamma_0^{-1}\) is always late time, although the temperature can hasten this.

Since we have solved the integrals in the different temperature regimes, we will now plot the lowest order terms of each expansion in the different temperature regimes. The coupling will be set at a moderate level to enhance their differences.

![Graph](image)

**FIG. 1:** The case of a moderate temperature. FC\(_1\) calculated \(-\text{numerically}\), to first order in the \(\cdots\) low-temperature expansion, to first order in the \(\cdots\) high-temperature expansion, and with the \(-\text{approximate general solution}\). All of the approximate solutions are fairly close with some minor discrepancy at short times. The low-temperature expansion will continue to be slightly off into late times.

Note that the high-temperature expansion (including the first order correction) gives very good agreement even at much lower temperatures \(T \sim \Omega_r\) (much lower than the regime \(T > \Lambda\) where the high-temperature expansion is clearly expected to be very accurate).
FIG. 2: The case of a high temperature. The approximate general solution and the first order high-temperature solution stick very well to the numeric solution, while the first order low-temperature solution has amplitude and phase discrepancies.

FIG. 3: The case of a low temperature. The approximate general solution and the first order low temperature solution stick very well to the numeric solution, while the first order high temperature solution has some amplitude and phase discrepancy initially.

When we say \( n \)th order in the high-temperature expansion, we more accurately mean up to the \( e^{-2\pi Ttn} \) term in the high-temperature expansion. Strictly speaking this is not a temperature expansion but an expansion in \( e^{-2\pi Tt} \), the only terms which decay at a temperature-dependent rate, so it is really a high-temperature, late-time expansion. Similarly, when we say \( n \)th order in the low-temperature expansion, we mean keeping the \( n \)th order term in the
sum appearing in the last term of Eq. (II.46).

C. Some Closed Form Results

1. Extreme Temperature, Arbitrary Time

Considering just the first term on the right-hand side of Eq. (II.47) for the hyperbolic cotangent, one gets the following result for temperatures much higher than the cut-off:

\[ T \gg \Lambda, \]
\[ D_{xp} = 0, \quad \text{(II.58)} \]
\[ D_{pp} = +2M\gamma_0 T. \quad \text{(II.59)} \]

This is the common, but questionable high-temperature limit. There is no first order term for the anomalous diffusion coefficient, but there are lower order terms that vanish only for infinite temperature. They include the term that gives rise to the log \( \Lambda \) divergence when the temperature becomes much smaller than \( \Lambda \).

2. Arbitrary Temperature, Late Time

Neglecting the terms multiplied by the factor \( e^{-\gamma_0 t} \) in Eqs. (II.32)-(II.33), we obtain the following result, which is valid for times much larger than the relaxation time (\( t \gg \gamma_0^{-1} \)):

\[ t \gg \frac{1}{\gamma_0}, \]
\[ D_{xp} = -\frac{\gamma_0^2}{\pi \Omega} \text{Im} \left[ H \left( \frac{\gamma_0 + i\tilde{\Omega}}{2\pi T} \right) \right] \]
\[ + \frac{\gamma_0}{\pi} \text{Re} \left[ H \left( \frac{\Lambda}{2\pi T} \right) - H \left( \frac{\gamma_0 + i\tilde{\Omega}}{2\pi T} \right) \right], \quad \text{(II.60)} \]
\[ D_{pp} = +2M\gamma_0 T + \frac{2M\gamma_0}{\pi \Omega} (\Omega^2 - \gamma_0^2) \text{Im} \left[ H \left( \frac{\gamma_0 + i\tilde{\Omega}}{2\pi T} \right) \right] \]
\[ + \frac{4M\gamma_0^2}{\pi} \text{Re} \left[ H \left( \frac{\Lambda}{2\pi T} \right) - H \left( \frac{\gamma_0 + i\tilde{\Omega}}{2\pi T} \right) \right]. \quad \text{(II.61)} \]

The \( \text{Re} \left[ \cdots \right] \) terms become \( \log(\Lambda/\Omega_r) \) at zero temperature and vanish in the infinite temperature limit. On the other hand, the value of the \( \text{Im} \left[ \cdots \right] \) terms is \( \cos^{-1}(\gamma_0/\Omega_r) \) at zero temperature, whereas it also vanishes in the infinite-temperature limit.
D. Comparison with Caldeira et al.

We now compare our results to CCR’s weak coupling master equation [4], which differs from the HPZ equation and has the following diffusion coefficients:

\[ D_{xp}^C = 0, \]  
\[ D_{pp}^C = \gamma_0 M \Omega_r \coth \left( \frac{\Omega_r}{2T} \right). \]  

The CCR master equation is frequently used for its simplicity and is believed to be accurate at late times for weak coupling and arbitrary temperature. One can see from Fig. 4 that, ignoring the contribution from the cut-off, the CCR approximation matches extremely well with our exact results for the normal diffusion coefficient at weak coupling. The CCR error from neglecting the cut-off dependence is determined by the order of magnitude of the cut-off scale since it is logarithmic in \( \Lambda \). In any case, the CCR approximation underestimates the magnitude of the diffusion coefficient.

Unlike the normal diffusion coefficient, the difference is much more severe for the anomalous diffusion coefficient, which is completely absent in CCR. The largest contribution (in the weak coupling regime) to the anomalous diffusion coefficient comes from the cut-off and it does not vanish at finite temperature (see Fig. 5). Moreover, it cannot be regarded as vanishing at weak coupling because it is only proportional to one power of the coupling constant, which is the order to which CCR’s master equation should be valid.
FIG. 5: Late time $D_{xp}$ for — high temperature or equivalently Caldeira, — HPZ at $\Lambda = 10^3\Omega_r$, and ⋯ HPZ at $\Lambda = 10^9\Omega_r$.

The final point that should be made is that CCR’s arbitrary temperature master equation is only valid in the weak coupling regime for late times and, unfortunately, late time always means $t \gg \gamma_0^{-1}$ (according to our exact results for the HPZ master equation). Therefore, the weaker the coupling, the longer one must wait for CCR’s master equation to be valid.

III. MASTER EQUATION COEFFICIENTS WITH LAURENT SERIES SPECTRA

A. Derivation of The General Integrals

We start by considering the independent effect of power-law terms in the spectral function with ultraviolet cut-off $\Lambda$ and infrared cut-off $\lambda$ whenever necessary. We take these cut-offs to be extremely large and extremely small respectively, not because that should necessarily be the case for any relevant physical situation, but to restrict the model to something that we can solve explicitly. Hence, we take a spectral function of the form

$$I_N(\omega) = \frac{2}{\pi} M \gamma_{N-1} \omega \left( \frac{\omega}{\omega_c} \right)^{N-1} (\lambda < \omega < \Lambda),$$

(III.1)

where $N$ is an integer number and $\omega_c^{[N-1]}$ is a product of characteristic frequencies chosen so that well-behaved results are obtained for the master equation coefficients in the limits $\Lambda \to \infty$ and $\lambda \to 0$. Traditionally $\omega_c$ has been chosen to be $\Lambda$ whenever a UV cut-off was
necessary; we will end up making a similar choice here.

1. The Classical Trajectories

Given a general spectral function the equation of motion for the system trajectories corresponds to an integro-differential equation:

\[ 0 = \ddot{u}(s) + \Omega^2 u(s) + \frac{2}{M} \int_0^s \mu(s - s') u(s') ds'. \]  

(III.2)

It is thus convenient to perform a Laplace transform

\[ \hat{f}(\zeta) = \int_0^\infty e^{-\zeta s} f(s) ds, \]  

(III.3)

under which the equation becomes purely algebraic, since the integral involving the dissipation kernel is merely a convolution. We then have

\[ \zeta u_0 + \dot{u}_0 = \left( \zeta^2 + \frac{2}{M} \hat{\mu}(\zeta) + \Omega^2 \right) \hat{u}(\zeta), \]  

(III.4)

or, equivalently,

\[ \hat{u}(\zeta) = \frac{\zeta u_0 + \dot{u}_0}{\zeta^2 + \frac{2}{M} \hat{\mu}(\zeta) + \Omega^2}, \]  

(III.5)

where \( u_0, \dot{u}_0 \) are the initial conditions for \( u(s) \). Given the expression of the dissipation kernel

\[ \mu(s) = -\int_{\Lambda}^\lambda I_N(\omega) \sin(\omega s) d\omega, \]  

(III.6)

one can easily compute its Laplace transform:

\[ \hat{\mu}(\zeta) = \frac{2}{\pi} M \gamma_{N-1} \int_{\Lambda}^\lambda \frac{\omega^2}{\omega^2 + \zeta^2} \left( \frac{\omega}{\omega_c} \right)^{N-1} d\omega. \]  

(III.7)

The results for different integer values of \( N \) are listed in Table I. The column labeled with \( \hat{\mu}_c(\zeta) \) corresponds to the outcome of the integral in Eq. (III.7) after taking the limits \( \Lambda \to \infty \) and \( \lambda \to 0 \) wherever that gave a finite result. The values of \( \omega_c \) were chosen so that the integrals that appear when calculating the diffusion coefficients were finite (more details can be found in the next two subsections). Finally, the column labeled with \( \hat{\mu}_c(\zeta) \) follows from substituting \( \omega_c \) in \( \hat{\mu}_c(\zeta) \) with the values of the previous column and taking the limits \( \Lambda \to \infty \) and \( \lambda \to 0 \) whenever they give a finite result.

Had we chosen fractional powers of the frequency in the spectrum or even a different functional dependence such as a logarithm, this would have resulted in an integro-differential
TABLE I: Laplace transform of the dissipation kernels for various power-law spectral functions after discarding $O(\lambda)$ and $O(1/\Lambda)$ terms. $N$ is the power of the spectrum: $N = 1$ is ohmic, $N > 1$ is supraohmic and $N < 1$ is subohmic. $\hat{\mu}_c(\zeta)$ is the Laplace transform of the dissipation kernel before a reasonable choice of characteristic frequencies $\omega_c|_{N-1}$ is made, whereas $\hat{\mu}(\zeta)$ is the chosen physical dissipation kernel. Constant terms are renormalizations to the bare frequency.

| $N - 1$ | $\hat{\mu}_c(\zeta)$ | $\omega_c|^{N-1}$ | $\hat{\mu}(\zeta)$ |
|---------|-----------------------|-------------------|-------------------|
| $-n$    | ...                   | $\lambda^{n-1}\omega_c$ | 0                |
| $-4$    | $M\gamma_{-4}\left(-\frac{2}{\pi}\frac{\omega^3}{\zeta^2} + \frac{\omega^4}{\zeta^4}\right)$ | $\lambda^3\omega_c$ | 0                |
| $-3$    | $\frac{2}{\pi}M\gamma_{-3}\left(-\frac{\omega^3}{\zeta^2} \log \frac{\Lambda}{\zeta}\right)$ | $\lambda^2\omega_c$ | 0                |
| $-2$    | $M\gamma_{-2}\left(-\frac{\omega^2}{\zeta}\right)$ | $\lambda\omega_c$ | 0                |
| $-1$    | $\frac{2}{\pi}M\gamma_{-1}\left(-\omega_c \log \frac{\Lambda}{\zeta}\right)$ | $\omega_c$ | $\frac{2}{\pi}M\gamma_{-1}\left(-\omega_c \log \frac{\Lambda}{\zeta}\right)$ |
| 0       | $M\gamma_0 \left(\zeta - \frac{2}{\pi}\lambda\right)$ | 1 | $M\gamma_0 \left(\zeta - \frac{2}{\pi}\lambda\right)$ |
| 1       | $\frac{2}{\pi}M\gamma_1 \left(\frac{\zeta^2}{\omega_c} \log \frac{\Lambda}{\zeta} - \frac{\Lambda^2}{2\omega_c}\right)$ | $\Lambda$ | $-\frac{2}{\pi}M\gamma_1 \frac{\Lambda}{2}$ |
| 2       | $M\gamma_2 \left(-\frac{\zeta^4}{\omega_c^2} + \frac{2}{\pi}\frac{\Lambda^2}{\omega_c^4} \zeta^2 - \frac{2}{\pi}\frac{\Lambda^3}{\omega_c^6} \zeta^2\right)$ | $\Lambda^2$ | $-\frac{2}{\pi}M\gamma_2 \frac{\Lambda^2}{4}$ |
| 3       | $\frac{2}{\pi}M\gamma_3 \left(-\frac{\zeta^4}{\omega_c^2} \log \frac{\Lambda}{\zeta} + \frac{\Lambda^2}{2\omega_c^2} \zeta^2 - \frac{\Lambda^4}{4\omega_c^6}\right)$ | $\Lambda^3$ | $-\frac{2}{\pi}M\gamma_3 \frac{\Lambda^3}{16}$ |
| $n$     | ...                   | $\Lambda^n$ | $-\frac{2}{\pi}M\gamma_n \frac{\Lambda}{n+1}$ |

equation for the classical system trajectories. In addition to being less classical, they would also be much more difficult to solve.

Integer subohmic spectra, (corresponding to $N - 1 < 0$ in Table I) all exhibit a similar behavior, except for the $N = 0$ case. For $N < 0$ it is never possible to change the classical ordinary differential equation (ODE) while keeping the diffusion coefficients finite in the limit $\lambda \to 0$. Indeed, the only way to guarantee that all the frequency integrals involving the noise kernel that appear when calculating the diffusion coefficients are finite is by taking $\omega_c^n \sim \lambda^{n-1}$ (higher powers of $\lambda$ are allowed, but would give vanishing results for the diffusion coefficients). In that case, there are no additional contributions to the equation of motion left. In contrast, for $N = 0$ the diffusion coefficients only exhibit a logarithmic IR divergence (proportional to log $\lambda$) when considering a finite non-vanishing value of $\omega_c$ independent of $\lambda$. Since we tolerated logarithmic dependences on the UV cut-off $\Lambda$ in the ohmic case, it would be natural to allow a similar situation with the IR cut-off $\lambda$. However, in that case the Laplace transform of the dissipation kernel exhibits a logarithmic term that makes the
equation of motion for the system trajectories (which corresponds to an integro-differential equation) much more difficult to solve. Therefore, we will not consider the $N = 0$ case, for the same reason why we did not consider fractional powers of the frequency in the spectral function or even a non-logarithmic dependence on the frequency.

On the other hand, integer *supraohmic* spectra (corresponding to $N - 1 > 0$ in Table I) all exhibit a similar behavior. None of them are capable of changing the nature of the classical ODE while keeping the diffusion coefficients finite in the limit $\Lambda \to \infty$. Indeed, the only way to guarantee that all the frequency integrals involving the noise kernel that appear when calculating the diffusion coefficients are finite is by taking $\omega_n^\gamma \sim \Lambda^n$ (higher powers of $\Lambda$ are allowed, but would give vanishing results for the diffusion coefficients). In that case, the only contributions to the equation of motion that are left are frequency renormalization terms.

### B. Analytic Spectra Master Equation Coefficients

In this subsection we consider spectral functions that are analytic in the frequency (and vanish in the limit of zero frequency). They have the following form:

$$I_{\{\gamma\}}(\omega) = \frac{2}{\pi} M \omega \gamma \left( \frac{\omega}{\Lambda} \right) \quad (\omega < \Lambda), \quad (\text{III.8})$$

where $\gamma$ is an analytic function. Eq. (III.8) can be written as

$$I_{\{\gamma\}}(\omega) = \frac{2}{\pi} M \omega \sum_{n=0}^{\infty} \gamma_n \left( \frac{\omega}{\Lambda} \right)^n \quad (\omega < \Lambda), \quad (\text{III.9})$$

which corresponds to a linear combination of terms with $N \geq 1$ among those considered in the previous subsection.

#### 1. The Classical Trajectories

The classical equation of motion associated with these spectra is an ODE that differs trivially from the ohmic case:

$$\zeta \dot{u}_0 + \dot{u}_0 = \zeta^2 \ddot{u}(\zeta) + \frac{2}{M} \left( M \gamma_0 \zeta - \frac{2}{\pi} M \sum_{n=0}^{\infty} \frac{\gamma_n}{n+1} \Lambda \right) \ddot{u}(\zeta) + \Omega^2 \dot{u}(\zeta), \quad (\text{III.10})$$
which can be written in the more compact form

\[ \zeta u_0 + \dot{u}_0 = \left( \zeta^2 + 2\gamma_0 \zeta + \Omega^2 \right) \dot{u}(\zeta), \quad (\text{III.11}) \]

by introducing the renormalized frequency

\[ \Omega^2_r = \Omega^2 - \frac{4}{\pi} \Lambda \sum_{n=0}^{\infty} \frac{\gamma_n}{n+1}. \quad (\text{III.12}) \]

We can see that only the \( \omega^1 \) power (from the ohmic-like term) gives rise to the dissipation term\(^4\), whereas all the remaining supraohmic terms merely contribute to an additional renormalization of the bare frequency. In fact, one could rig supraohmic anticoupling terms to eradicate the divergent difference between the bare and renormalized frequency, although there is no physical motivation to do so.

2. Master Equation Coefficients

For times longer than the switch-on time, when our explicit expressions for the diffusion coefficients can be used, the only difference between the master equation for an analytic spectrum and the ohmic spectrum lies in a modification of the contribution from the \( \text{FI}_3 \) integrals to the diffusion coefficients. This can be seen as follows. When considering a term proportional to \( (\omega/\omega_c)^n \) in the spectral function, instead of the \( \text{FI}_3 \) integral of the ohmic case, one gets

\[ \frac{\text{FI}_{3+n}}{\omega_c^n} = \frac{1}{\omega_c^n} \int_0^{\Lambda} \frac{\omega^{3+n} + \mathcal{O}(\omega^{2+n})}{\omega^4 + \mathcal{O}(\omega^3)} d\omega \quad (\text{III.13}) \]

\[ = \frac{1}{\omega_c^n} \int_0^{\Lambda} \left( \omega^{n-1} + \mathcal{O}(\omega^{n-2}) \right) d\omega \quad (\text{III.14}) \]

\[ = \frac{1}{\omega_c^n} \left( \frac{\Lambda^n}{n} + \mathcal{O}(\Lambda^{n-1}) \right) \quad (\text{III.15}) \]

and the only way to get a finite non-vanishing contribution is by taking \( \omega_c \) proportional to \( \Lambda \) (which was the choice already made in Eqs. (III.8)-(III.9)). Therefore, one just needs to

\(^4\) Note that, since an additional term \( u_0 \) is missing, the inverse Laplace transform of \( 2\gamma_0 \zeta \dot{u}(\zeta) \) gives not only the local dissipation term \( 2\gamma_0 \dot{u}(t) \), but also a term proportional to \( \delta(t) u(0) \), which corresponds to the term on the right-hand side of Eq. (II.6) and is responsible for the initial kick when no smooth switching-on function is present.
introduce the following simple substitution in the expressions for the diffusion coefficients of the ohmic case:

\[
\text{Ohmic} \rightarrow \text{Analytic} \\
\text{FI}_3 \rightarrow \text{FI}_3 + \sum_{n=1}^{\infty} \frac{\gamma_n \text{FI}_{3+n}}{\Lambda^n} = \text{FI}_3 + \sum_{n=1}^{\infty} \frac{\gamma_n}{n\gamma_0},
\]

(III.16)

where we took \( \omega_c = \Lambda \) and discarded terms involving negative powers of \( \Lambda \). Note that all other integrals contributing to the diffusion coefficients will exhibit lower powers of the frequency in the numerator so that their result will be proportional to inverse powers of \( \Lambda \) and can be neglected. The integrals that correspond to \( \text{FC}_{3}(t) \) and \( \text{FS}_{4}(t) \) in the ohmic case also exhibit divergences that would lead to a non-vanishing result (and even a divergent one for \( \text{FS}_{4}(t) \)) when dividing by \( \Lambda^n \), but that is only for short times of order \( \Lambda^{-1} \) after the initial time. This is the regime where our explicit expressions are not accurate if the interaction is gradually switched on with a much longer characteristic time-scale, in which case the contribution to the diffusion coefficients from that period is very small.

We close this subsection by pointing out that, since \( \text{FI}_3 \) is where the \( \log \Lambda \) divergence arises for the ohmic case, one could rig an infinite number of finite supraohmic anticoupling terms to cancel the divergence, though there is no physical motivation to do so. One could even ask whether we can both renormalize the diffusion coefficients and keep the bare frequency finite and unmodified, which amounts to requiring the following two conditions to hold simultaneously:

\[
\sum_{n=1}^{\infty} \frac{\gamma_n}{n+1} = -\gamma_0,
\]

(III.17)

\[
\sum_{n=1}^{\infty} \frac{\gamma_n}{n} \approx -\gamma_0 \log \frac{\Lambda}{\Omega_r}.
\]

(III.18)

The answer is obviously in the negative, at least not with finite couplings.

C. Laurent Series Spectra Master Equation Coefficients

In this subsection we extend the form of the spectral function considered in the previous subsection to that of a Laurent series. This is done by adding an analytic function of \( \omega^{-1} \) to the analytic function of \( \omega \) already considered there, so that we have

\[
I_{(\gamma,\varphi)}(\omega) = I_{(\gamma)}(\omega) - \frac{2}{\pi} M \frac{\lambda}{\omega} \varphi \left( \frac{\lambda}{\omega} \right) \left( \lambda < \omega \right),
\]

(III.19)
where \( \varphi \) is an analytic function. Eq. (III.8) can be written as

\[
I_{\{\gamma, \varphi\}}(\omega) = I_{\{\gamma\}}(\omega) - \frac{2}{\pi} M \sum_{n=0}^{\infty} \varphi_n \left( \frac{\lambda}{\omega} \right)^{n+1} (\lambda < \omega),
\]

which corresponds to a linear combination of terms with \( N \leq 0 \) among those considered in Sec. III A.

As we saw in Sec. III A, in the limit \( \lambda \to 0 \) there is no modification of the equations of motion due to the terms in Eq. (III.20) with negative powers of \( \omega \). Hence, the dissipation and renormalized frequency terms, \( \Gamma(t), \Omega_{\text{ren}}^2(t) \), in the master equation are exactly the same as with ohmic and, more generally, analytic spectra: \( \Gamma(t) = \gamma_0, \Omega_{\text{ren}}^2 = \Omega_r^2 \). However, there will be some non-trivial contribution to the diffusion coefficients that we analyze below.

For times longer than the switch-on time the only difference between the master equation for the Laurent-series spectrum and the analytic spectrum lies in a modification of the contribution from the terms most sensitive to IR divergences, namely FI\(_1\) and FC\(_1\)(\(t\)). This can be seen as follows. When considering a term proportional to \( (\omega_c/\omega)^n \) in the spectral function, instead of the FC\(_1\)(\(t\)) integral of the ohmic case, one gets

\[
\omega_n^c FC_{1-n}(t) = \omega_n^c \int_{\lambda}^{\infty} d\omega \frac{1}{\omega^n} \left( \frac{2T}{\omega} + O(\omega^0) \right) \cos(\omega t) = \frac{2T}{\Omega_r} \omega_n^c \int_{\lambda}^{\infty} d\omega \left( \omega^{-(n+1)} + O(\omega^{-(n-1)}) \right) \cos(\omega t) = \frac{2T}{\Omega_r^2} \omega_n^c \left[ \frac{1}{n\lambda^n} + O(\lambda^{-(n-2)}) \right] \cos(\lambda t),
\]

where we used the expansion \( \coth(\omega/2T) = (2T/\omega) + O(\omega^0) \) in the first equality. The only way to get a finite non-vanishing contribution is by taking \( \omega_c \) proportional to \( \lambda \) (which was the choice already made in Eqs. (III.8)-(III.9)). While FI\(_1\) can be obtained by evaluating FC\(_1\)(\(t\)) at \( t = 0 \), all the remaining integrals appearing in the expressions for the diffusion coefficients will involve a less negative power of the frequency and, when dividing them by \( \lambda^n \), will give a result corresponding to positive powers of \( \lambda \), which can be neglected in the limit \( \lambda \to 0 \).

Thus, in order to obtain the diffusion coefficients for a Laurent-series spectral function of the type considered in this subsection, one just needs to introduce these simple substitutions.
in the expressions for the diffusion coefficients of the ohmic case:

\[
\text{Ohmic} \rightarrow \text{Laurent}
\]

\[
\gamma_0 F I_3 \rightarrow \gamma_0 F I_3 + \ell \\
\ell = \sum_{n=1}^{\infty} \frac{\gamma_n}{n}, \tag{III.22}
\]

\[
\gamma_0 F C_1(t) \rightarrow \gamma_0 F C_1(t) - \frac{2T}{\Omega_r^4} \cos(\lambda t) \phi \\
\phi = \sum_{n=0}^{\infty} \frac{\varphi_n}{n+1}, \tag{III.23}
\]

\[
\gamma_0 F I_1 \rightarrow \gamma_0 F I_1 - \frac{2T}{\Omega_r^4} \phi.
\]

In summary, the supraohmic terms shift the bare frequency and FI\(_3\) integrals, and can be used to shift their values in the ohmic theory, whereas the subohmic terms shift the FC\(_1\) and FI\(_1\) integrals.

**IV. SOLUTIONS TO THE MASTER EQUATION**

**A. Solutions to the Laurent-Series Spectra Master Equation**

1. *Matrix Representation*

First, we express the master equation

\[
\frac{\partial}{\partial t} W_r = \left( -\frac{p}{M} \frac{\partial}{\partial x} + M \Omega_r^2 x \frac{\partial}{\partial p} + 2\gamma_0 \frac{\partial}{\partial p} p - 2D_{xp} \frac{\partial^2}{\partial x \partial p} + D_{pp} \frac{\partial^2}{\partial p^2} \right) W_r, \tag{IV.1}
\]

in a more compact form:

\[
\frac{\partial}{\partial t} W_r = (\nabla_q^t D \nabla_q + \nabla_q^t H q) W_r, \tag{IV.2}
\]

with

\[
q = \begin{bmatrix} x \\ p \end{bmatrix}, \quad \nabla_q = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial p} \end{bmatrix}, \tag{IV.3}
\]

\[
H = \begin{bmatrix} 0 & -\frac{1}{M} \\ M \Omega_r^2 & 2\gamma_0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -D_{xp} \\ -D_{xp} & D_{pp} \end{bmatrix}. \tag{IV.4}
\]

This is a hyperbolic second order partial differential equation (PDE). As \(D\) is a function of time, in general the equation is not separable in time. It is not separable in phase space either.
2. Phase-Space Fourier Transform: The Characteristic Function

The nature of the PDE suggests a Fourier transform of the phase-space variables since derivatives are more complicated than algebraic parameters. Furthermore, not only does a Fourier transform reduce the PDE to first order, but the computation of expectation values also becomes trivial since we are then working with the characteristic function of the distribution.

The Fourier transform is defined as

\[ \mathcal{F}\{f\}(k) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \ e^{-ik \cdot q} f(q), \quad (IV.5) \]

and it exhibits the usual properties:

\[ i^n \frac{\partial^n \mathcal{F}\{f\}}{\partial k^n}(0) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \ q^n_j f(q). \quad (IV.6) \]

The master equation then becomes

\[ \frac{\partial}{\partial t} \mathcal{W}_r = (i k^T D k + i k^T H_t \nabla_k) \mathcal{W}_r, \quad (IV.7) \]

where \( \mathcal{W}_r = \mathcal{F}\{W_r\} \) and the normalization of \( W_r(t,q) \) implies \( W_r(t,0) = 1 \). Finally, it is convenient to group all the derivatives on the left-hand side, so that we have

\[ \left( \frac{\partial}{\partial t} + k^T H \nabla_k \right) \mathcal{W}_r = -k^T D k \ \mathcal{W}_r. \quad (IV.8) \]

3. Method of Characteristic Curves

The method of characteristic curves involves looking for parameterized curves in the domain \((t,k)\) along which the first order PDE becomes a set of first order ODEs. For each one of those curves we have

\[ \mathcal{W}_r(t,k) = \mathcal{W}_r(t(s), k(s)), \quad (IV.9) \]

\[ \frac{d}{ds} \mathcal{W}_r = \frac{dt}{ds} \frac{\partial}{\partial t} \mathcal{W}_r + \frac{dk^T}{ds} \nabla_k \mathcal{W}_r, \quad (IV.10) \]

Next, we attempt to match the right-hand side of Eq. (IV.10) to the left-hand side of Eq. (IV.8). This results in a system of ODEs in the parameter \( s \). We will look for curves that synchronize with the initial time so that \( t(0) = 0, \ k(0) = k_0 \). The time solution is simple:

\[ \frac{dt}{ds} = 1 \Rightarrow t(s) = s. \quad (IV.11) \]
On the other hand, the solution for the Fourier transform of the phase-space variables is a bit more involved:

\[
\frac{dk^T}{ds} = k^T H \Rightarrow k^T = k_0^T e^{sH}. \tag{IV.12}
\]

In order to calculate the exponential of the \( H \) matrix we diagonalize it, so that

\[
k^T H q_\pm = h_\pm k^T q_\pm. \tag{IV.13}
\]

The eigenvalues, eigenvectors, and exponential matrix are given by

\[
h_\pm = \gamma_0 \pm i\tilde{\Omega} \quad q_\pm = \begin{bmatrix} 1 \\ -Mh_\pm \end{bmatrix}, \tag{IV.14}
\]

\[
e^{-tH} = \begin{pmatrix} 2M\gamma_0 + M(\partial/\partial t) & 1 \\ -M^2\Omega^2_r & M(\partial/\partial t) \end{pmatrix} G_{ret}(t, 0), \tag{IV.15}
\]

where \( G_{ret}(t, 0) \) is the retarded Green function in Eq. (II.30). We now have the rules for transforming back and forth between the domain coordinates \((t, k)\) and the characteristic curve coordinates \((s, k_0)\). \(k_0\) does not change along the characteristic curve, but for a given \(s\) it uniquely specifies a particular curve (except at the origin, where all curves intersect).

Using these results, we can immediately apply the method of characteristic curves to solving Eq. (IV.8) as follows:

\[
\frac{d}{ds} W_r(t(s), k(s)) = -k^T D(t) k W_r(t(s), k(s)), \tag{IV.16}
\]

\[
\frac{d}{ds} W_r(s, e^{sH^f} k_0) = -k_0^T e^{sH} D(s) e^{sH^f} k_0 W_r(s, e^{sH^f} k_0). \tag{IV.17}
\]

The last equation is a linear ODE whose solution can easily be found to be

\[
W_r(s, e^{sH^f} k_0) = W_r(0, k_0) e^{-\int_0^s ds' e^{s'H} D(s') e^{s'H^f} k_0}, \tag{IV.18}
\]

where \( W_r(0, k_0) \) is the initial reduced Wigner function at \( t = 0 \). We can now express the solution back in terms of \( k \) and get the final result

\[
W_r(t, k) = W_r(0, e^{-tH^f} k) e^{-\frac{1}{2}k^T \sigma_T(t) k}, \tag{IV.19}
\]

with

\[
\sigma_T(t) = 2 \int_0^t e^{(s-t)H} D(s) e^{(s-t)H^f} ds. \tag{IV.20}
\]
One can check that this result agrees with those in Ref. [18] if one considers equal times in the two-point correlation functions there.

From Eqs. (IV.15) and (II.30) one can see that all the integrals in Eq. (IV.20) involve a factor $e^{2(s-t)\gamma_0}$ times some oscillatory factor. Taking into account that the diffusion coefficients tend to constant asymptotic values $D_{xp}$ and $D_{pp}$ for sufficiently late times, one can see that for $t \gg \gamma_0$ the integral is dominated by large values of $s$. Therefore, the asymptotic value of $\sigma_T(t)$ can be calculated using the constant asymptotic values of the diffusion coefficients in Eq. (IV.20) and taking the limit $t \to \infty$, with the following result:

$$
\sigma_T^\infty = \begin{pmatrix}
\frac{1}{(M\Omega_r)^2} \left( \frac{1}{2\gamma_0} D_{pp}^\infty - 2MD_{xp}^\infty \right) & 0 \\
0 & \frac{1}{2\gamma_0} D_{pp}^\infty
\end{pmatrix}.
$$

The solution (IV.19) has two factors. The first one tends to one in the long time limit and encodes the disappearance of the initial state (we will call it the death factor). The second factor describes the appearance of a Gaussian state that evolves in time and tends asymptotically to a state that corresponds to thermal equilibrium (we will refer to this as the birth factor). All initial distributions evolve towards this final Gaussian state, whose covariance matrix is given by Eq. (IV.21). This state does not look like the thermal state of a free harmonic oscillator because of the coupling to the environment. It results from considering the thermal equilibrium state for the whole system (system plus environment) including the system-environment interaction, which gives rise to a non-trivial correlation between them, and tracing out the environment.

The death factor contains the information on the initial conditions, describes the gradual disappearance of the initial distribution and it is always temperature independent. The initial distribution undergoes damped oscillations with characteristic time-scales $\gamma_0, \tilde{\Omega}$. The higher cumulants of the distribution, discussed in the next subsection, oscillate and decay more rapidly. This is responsible for the inspirals in phase space of the evolution of Gaussians plotted by Unruh and Zurek [5], which are calculated in the next subsection and plotted in Fig. 6. Of course this is all assuming a non-vanishing ohmic term in the spectral function. If the spectral function is purely non-ohmic (i.e., with $\gamma_0 = 0$) then this factor will describe the initial state oscillating with a renormalized frequency, but which never decays away.

The birth factor describes the complicated birth of and settlement into a state of thermal equilibrium. This factor is always Gaussian with a covariance matrix given by Eq. (IV.20),
which involves a convolution of the diffusion matrix with propagators that reflect the natural oscillatory decay of the system. This covariance matrix vanishes at the initial time and tends at late times to the equilibrium covariance matrix (IV.21), with the diffusion coefficients being their asymptotic constant values given by Eqs. (II.60)-(II.61). This covariance matrix is positive definite for all reasonable diffusion constants. Moreover, the anomalous diffusion coefficient actually acts as an “anti-diffusion” term that makes $\sigma_{xx}$ (and the uncertainty in $x$) free of the $\log(\Lambda/\Omega_r)$ divergence, as it will be discussed in the next subsection.

B. Analysis of the Solutions

1. Trajectories of the Cumulants

As we have already mentioned, the Fourier transform of the reduced Wigner function corresponds to its characteristic function, from which the correlation functions for the phase-space variables can be easily derived using Eq. (IV.6). The general expressions for the cumulants can be obtained straightforwardly from the logarithm of the reduced Wigner function in Fourier space as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n!} \kappa^{(n)}_{i_1 \ldots i_n} (t) \prod_{l=1}^{n} i k^{i_l} = \log \mathcal{W}_r (t, \kappa), \quad (IV.22)$$

where $k^{i_l}$ denotes the components of the vector $k$ and we used the Einstein summation convention for pairs of repeated indices (i.e., it is implicitly understood that a sum $\sum_{i_l=1}^{2}$ should be preformed over each pair of repeated indices $i_l$). $\kappa^{(n)}$ is the $n^{th}$ cumulant and acts as a tensor of order $n$ contracted with $n$ copies of $k$. Using the result for $\mathcal{W}_r (t, \kappa)$ from Eq. (IV.19) we have

$$\sum_{n=1}^{\infty} \frac{1}{n!} \kappa^{(n)}_{i_1 \ldots i_n} (t) \prod_{l=1}^{n} i k^{i_l} = \sum_{n=1}^{\infty} \frac{1}{n!} \kappa^{(n)}_{i_1 \ldots i_n} (0) \prod_{l=1}^{n} i \left( e^{-i\mathbf{H}_T^\tau} k \right)^{i_l} - \frac{1}{2} k^t \sigma_T (t) k, \quad (IV.23)$$

where $\kappa^{(n)}_{j_1 \ldots j_n} (0)$ are the cumulants associated with the initial distribution. Eq. (IV.23) implies

$$\kappa^{(n)}_{i_1 \ldots i_n} (t) = \kappa^{(n)}_{j_1 \ldots j_n} (0) \prod_{l=1}^{n} i \left( e^{-i\mathbf{H}_T^\tau} \right)^{j_i} + \delta_{n2} \sigma_T^{j_1 i_2} (t). \quad (IV.24)$$

We can see that the only cumulant with a non-vanishing asymptotic value, which is a consequence of the thermal fluctuations, is the covariance matrix (with $n = 2$). The closely
related second momenta of the distribution are given by

\[ \langle q q^I \rangle(t) = e^{-iHt} \langle q q^I \rangle_0 e^{-iHt} + \sigma_T(t), \]  

(IV.25)

where \( \langle \cdots \rangle_0 \) denotes expectation value with respect to the reduced Wigner function at the initial time. All other cumulants experience oscillatory decay with time scales of \( n\gamma_0, n\tilde{\Omega} \) with \( n \) being the order of the cumulant. In particular, the expectation value

\[ \langle q \rangle(t) = e^{-iHt} \langle q \rangle_0, \]  

(IV.26)

follows a trajectory plotted in Fig. 6, where one can see that the trajectory of the expectation values \( \langle x \rangle, \langle p \rangle \) for any initial distribution inspiral into the origin.

\[ \frac{\langle p \rangle}{M \Omega_x} \]

\[ \langle x \rangle \]

FIG. 6: The trajectory of the expectation values \( \langle x \rangle, \langle p \rangle \).

2. Late Time Uncertainty Function

As we have seen above, any specific features of the initial distribution decay away (assuming a non-vanishing ohmic term in the spectral function) and at late times the state tends generically to a Gaussian with a covariance matrix given by Eq. (IV.21). Therefore, from Eq. (IV.25) it follows that at late times \( (\Delta x)^2 = (\sigma^\infty_{xx})_x \) and \( (\Delta p)^2 = (\sigma^\infty_{pp})_p \).

From Eq. (IV.21) and using Eqs. (II.32)-(II.33) we can express the position and momentum uncertainties at late times as

\[ (\Delta x)^2 = \frac{2}{\pi} \frac{\gamma_0}{M} \text{FI}_1, \]  

(IV.27)

\[ (\Delta p)^2 = \frac{2}{\pi} \frac{\gamma_0}{M} \text{FI}_3. \]  

(IV.28)
The product of the two uncertainties

\[(\Delta x)^2(\Delta p)^2 = \left(\frac{2\gamma_0}{\pi}\right)^2 F_1 F_1,\]

(IV.29)
can be expanded for high temperatures as

\[(\Delta x)^2(\Delta p)^2 = \left(\frac{T}{\Omega_r}\right)^2 + (\cdots)T + (\cdots)T^0.\]

(IV.30)
Inspecting the terms in powers of \(T\) immediately reveals the high-temperature result of classical statistical mechanics for the case of an ohmic spectrum. Integer subohmic terms would strictly decrease this amount. One can also see that at weak coupling the uncertainty function agrees with the weak coupling approximation for moderate values of the cut-off scale, as shown in Fig. 7.

![Graph showing \(\Delta x\Delta p\) vs. \(T/\Omega_r\)](image)

FIG. 7: Late time \(\Delta x\Delta p\) for – high temperature, classical statistical mechanics, · · · weak coupling approximation \(\frac{1}{2} \coth \frac{\Omega_r}{2T}\), – HPZ at \(\Lambda = 10^3\Omega_r\), and · · · HPZ at \(\Lambda = 10^9\Omega_r\).

Had one naively tried to have finite diffusion coefficients in the limit \(\Lambda \to \infty\) subtracting by hand the \(\log(\Lambda/\Omega_r)\) term, one would find a violation of the Heisenberg uncertainty principle at low temperature and strong coupling (see Fig.8), which renders the theory unphysical. Of course this does not happen with the unsubtracted theory, as seen in Fig. 9. It is thus clear that the logarithmic dependence on the ultraviolet cut-off that appears in the diffusion coefficients is a physically important parameter and not something that can be subtracted away.

From Eqs. (II.40)-(II.41) [or alternatively from Eq. (II.34)] one can see that mentioned earlier, only the momentum uncertainty contains a logarithmically divergent cut-off depen-
Subtracted Theory

\[ \Delta x \Delta p \]

FIG. 8: Late time \( \Delta x \Delta p \) for the subtracted theory.

Unsubtracted Theory

\[ \Lambda = 10^3 \Omega_r \]

FIG. 9: Late time \( \Delta x \Delta p \) for the unsubtracted theory.

dence. In contrast, the position uncertainty is much smaller and finite in the limit \( \Lambda \to \infty \) (this had already been noticed for Gaussian wave-packets in Ref. [5]). In fact, if the Brownian particle is coupled to a reservoir at arbitrarily low temperature and arbitrarily strong
coupling, the uncertainty in position will be arbitrarily small:

$$\lim_{\gamma_0 \to \infty} \lim_{T \to 0} \Delta x = 0,$$

$$\lim_{\gamma_0 \to \infty} \lim_{T \to 0} \Delta p = \infty.$$ (IV.31) (IV.32)

Hence, strong coupling to a low temperature reservoir tends to localize the Wigner distribution in position. From this result, one would qualitatively expect strong coupling to suppress quantum tunneling while high temperature would aid both tunneling and classical escape (thermal activation).

3. Linear Entropy

In this subsection we investigate the linear entropy [19], which can be easily obtained from the Wigner distribution as follows:

$$S_L = 1 - \text{Tr}(\hat{\rho}^2) = 1 - 2\pi \int d^2q W_r^2(q).$$ (IV.33)

In Fourier space it becomes

$$S_L = 1 - \frac{1}{2\pi} \int d^2k |\mathcal{W}_r(k)|^2,$$ (IV.34)

and using the result in Eq. (IV.19) we finally get

$$S_L = 1 - \frac{1}{2\pi} \int d^2k |\mathcal{W}_r(0,e^{-iHT}k)|^2 e^{-k^T\sigma_T(t)k}. \quad \text{(IV.35)}$$

At the initial time the linear entropy is that of the initial state, and at late times it tends to $$S_L = 1 - (1/2)(\text{det} \sigma_T^\infty)^{-1/2}.$$

Alternatively, one can express the linear entropy in terms of an integral of the Fourier-transformed reduced Wigner function at the initial time by introducing the change of variables $$k_0 = e^{-iHT} k$$. Eq. (IV.35) can then be written as

$$S_L = 1 - \frac{1}{2\pi} \int d^2k_0 \text{det} \left( e^{iHT} \right) |\mathcal{W}_r(0,k_0)|^2 e^{-k_0^T e^{-iHT} \sigma_T(t)e^{iHT}k_0}$$

$$= 1 - \frac{1}{2 \text{det}^{1/2}[\sigma_T(t)]} \int d^2k_0 |\mathcal{W}_r(0,k_0)|^2 N\left(0,e^{iHT}\sigma_T(t)e^{iHT};k_0\right), \quad \text{(IV.36)}$$

In this limit one needs to use the expressions for the overdamping regime, as explained in footnote 3. Moreover, one needs to keep $$\Lambda > \gamma_0$$, which means that $$\Lambda$$ also tends to infinity as $$\gamma \to \infty$$. Alternatively, one should use Eqs. (II.40)-(II.41) plus the terms involving negative powers of $$\Lambda$$ which were neglected there, so that the restriction $$\Lambda > \gamma_0$$ does not apply. In both cases one gets the result quoted in Eqs. (IV.31)-(IV.32).
where $N(\mu, \sigma; k_0)$ is a normalized Gaussian distribution for the variable $k_0$ with mean $\mu$ and covariance $\sigma$. For small times this integral is similar to that for the initial state, whereas for long times the normalized Gaussian distribution becomes increasingly close to a delta function.

For a Gaussian initial state $W_r(0, k_0) = \exp[-k_0^T \sigma_T(t) k_0 - i k_0 \langle q \rangle_0]$ the integral in Eq. (IV.35) can be explicitly computed:

$$S_L = 1 - \frac{1}{2\pi} \int d^2 k e^{-k^T \{e^{-iH} \sigma_0 e^{-iH^T} + \sigma_T(t)\} k} = 1 - \frac{1}{2 \det^{1/2} \left[ e^{-iH} \sigma_0 e^{-iH^T} + \sigma_T(t) \right]}. \tag{IV.37}$$

For these Gaussian states, reasonable linear entropy is synonymous with reasonable uncertainty functions (i.e., the linear entropy will be positive if and only if the Heisenberg uncertainty principle is satisfied). We have already found that the late time uncertainty is well behaved. The uncertainty at the initial and intermediate times should not violate the Heisenberg uncertainty principle either. As a particular example, let us consider an initial state that corresponds to the ground state associated with the bare frequency. At the initial time the determinant is completely specified by the determinant of $\sigma_0$, which equals $1/4$ as a consequence of the cancellation between the large $(\Delta p)^2$ factor (of the order of $M \Omega_{\text{bare}}$ with $\Omega_{\text{bare}} \sim \sqrt{\Lambda}$) and the small $(\Delta x)^2$ factor (of order $1/M \Omega_{\text{bare}}$). However, after a very short time, when $\sigma_T(t)$ starts acquiring non-vanishing values, it will dominate the total contribution to $(\Delta x)^2$ and the linear entropy will increase suddenly (this kind of behavior for the entropy was found in Ref. [20]). At later times it will relax to its thermal equilibrium values.

C. Solutions of the General Master Equation

In situations more general than the Laurent series spectrum and where the classical equation of motion involves fractional calculus (i.e., it becomes an integro-differential equation rather than an ODE), the master equation can be of a slightly more general nature, with additional time dependence in the frequency and the dissipation coefficients:

$$\frac{\partial}{\partial \tau} W_r = \left( -\frac{p}{M} \frac{\partial}{\partial x} + M \Omega_{r}^2(t)x \frac{\partial}{\partial p} + 2\Gamma(t) \frac{\partial}{\partial p} p + \nabla_q D(t) \nabla_q \right) W_r. \tag{IV.38}$$
The method for finding its solutions will be almost the same, except that solving for the characteristic curves of the Fourier-transformed phase-space variables is now less straightforward due to the time dependence of $H$, and the procedure followed to solve Eq. (IV.12) needs to be generalized. The system of ODEs that corresponds to

$$\frac{dk^p}{ds} = k^p H(s), \quad (IV.39)$$

can be decoupled through a process of differentiation and substitution that leads to the equations for a pair of independent parametric oscillators where solving one effectively solves the other up to an integral. The second order ODE satisfied by $k_p$ is

$$0 = \ddot{k}_p - 2\Gamma\dot{k}_p + \left(\Omega_r^2 - 2\dot{\Gamma}\right)k_p. \quad (IV.40)$$

After solving it, $k_x$ can be simply obtained as

$$k_x = k_x^0 + M \int_0^s ds' \Omega_r^2(s')k_p(s'). \quad (IV.41)$$

Alternatively, one can proceed in the reverse order as follows:

$$0 = \ddot{k}_x - \left(2\Gamma + \frac{\dot{\Omega}_r^2}{\Omega_r^2}\right)\dot{k}_x + \Omega_r^2k_x, \quad (IV.42)$$

$$k_p = \left(k_p^0 - \frac{1}{M} \int_0^s ds' e^{-2\int_0^{s'} \Gamma(s'')ds''} k_x(s')\right) e^{+2\int_0^s \Gamma(s')ds'}. \quad (IV.43)$$

Solving for $k_p$ first appears to be simpler since everything is then linear in the coefficients. It is convenient to factor out the exponential growth, which reduces the ODE to its undamped form:

$$k_p = e^{\int_0^s \Gamma(s')ds'} j_p, \quad (IV.44)$$

$$0 = \ddot{j}_p + \left(\Omega_r^2 - \Gamma^2 - \dot{\Gamma}\right)j_p. \quad (IV.45)$$

If one is able to solve this differential equation, which for simple enough functions could be performed with Floquet analysis and variation of parameters, then the characteristic curves can be expressed in terms of the following matrix equation:

$$k = \Phi(s)k_0, \quad (IV.46)$$

where $\Phi(s)$ is the so-called transition matrix, which has many of the properties of an exponential and contains the parametric oscillatory behavior of the characteristic curves. If $\Gamma(t)$
and $\Omega_r(t)$ tend to some asymptotic values at large times, then the behavior of $\Phi(s)$ tends to that of a damped harmonic oscillator.

Once the transition matrix is available one can apply exactly the same approach as above in order to solve the master equation. One starts by writing the master equation as

$$
\frac{d}{ds} W_r(t(s), \Phi(s)k_0) = -k_0^2 \Phi^T(s) D(s) \Phi(s)k_0 \ W_r(t(s), \Phi(s)k_0).
$$

(IV.47)

Next, one solves this linear ODE in $s$ and, reexpressing the result in terms $t$ and $k$, one can finally write the solution of the master equation as

$$
W_r(t, k) = W_r(0, \Phi^{-1}(t)k) e^{-\frac{1}{2}k^2 \sigma_T(t)k},
$$

(IV.48)

with

$$
\sigma_T(t) = 2 \left( \int_0^t \left[ \Phi^{-1}(t) \Phi^T(s) D(s) \Phi(s) \Phi^{-1}(t) \right] ds \right).
$$

(IV.49)

For ohmic-like coefficients, the qualitative behavior of the solution (IV.48) is very similar to that of the ohmic solution. The moments of the initial distribution will experience oscillatory decay and the solution will tend to a thermal equilibrium state at late times. In purely supraohmic regimes where $\Gamma$ vanishes, the initial state would remain with renormalized frequency. In purely subohmic regimes where $\Gamma$ grows quickly, the initial state would vanish quickly.

V. INFLUENCE OF A CLASSICAL FORCE

In this section we consider the case of a classical force $F(t)$ acting on the quantum oscillator. This is done by introducing a time-dependent potential $-F(t)x$:

$$
L_x = \frac{1}{2} M \left( \dot{x}^2 - \Omega^2 x^2 \right) + F(t)x.
$$

(V.1)

A. The Master Equation Coefficients

To derive the form of the master equation we follow the method of Calzetta, Roura, and Verdaguer [18] as it is of a very general nature and can be adapted quite straightforwardly. In the presence of an external force the system action takes the following form in terms of the “center of mass” and “relative” coordinates, $X = (x + x')/2$ and $\Delta = x - x'$:

$$
S(x) - S(x') = M \int_0^t \left( M ds \left[ \dot{X}(s) \dot{\Delta}(s) - \Omega^2 X(s) \Delta(s) \right] + F(s) \Delta(s) \right).
$$

(V.2)
The reduced Wigner function can still be expressed as an average over a stochastic process and a distribution for the initial conditions as follows:

\[
W_r(x, p, t) = \left\langle \delta(X(t) - x) \delta(M\dot{X}(t) - p) \right\rangle_{X_0, p_0}, \quad \text{(V.3)}
\]

\[
\langle \ldots \rangle_{\xi} \equiv \int D\xi \ldots e^{-\frac{1}{2} \xi \cdot \nu - \frac{1}{2} \cdot \xi}, \quad \text{(V.4)}
\]

where we used the notation \(\cdot \equiv \int_{t_0}^t\) in the last equation, and \(X(t)\) is now a solution to the Langevin equation with the external force:

\[
M \left(\frac{d^2}{dt^2} + 2\gamma_0 \frac{d}{dt} + \Omega_r^2\right) X(t) = F(t) + \xi(t). \quad \text{(V.5)}
\]

Differentiating with respect to time reveals the usual dissipation, renormalized harmonic potential and the classical force potential \(-F(t)x\):

\[
\frac{\partial}{\partial t} W_r = \left(\nabla^T q Q - F(t) \frac{\partial}{\partial p}\right) W_r - \frac{\partial}{\partial p} \left(\langle \xi(t) \delta(X(t) - x) \delta(M\dot{X}(t) - p) \rangle_{\xi} \right)_{X_0, p_0}. \quad \text{(V.6)}
\]

After a functional integration by parts, the last term gives rise to the diffusion terms, with the following coefficients:

\[
D_{xp} = -\frac{1}{2} \int_{t_0}^t ds \nu(t, s) \frac{\delta X(t)}{\delta \xi(s)}, \quad \text{(V.7)}
\]

\[
D_{pp} = M \int_{t_0}^t ds \nu(t, s) \frac{\partial}{\partial t} \frac{\delta X(t)}{\delta \xi(s)}. \quad \text{(V.8)}
\]

Eq. (V.5) can be solved in the same way as in the case without external force to obtain \(X(t)\). It is simply given by a homogeneous solution that contains all the information on the initial conditions, plus a convolution of the external force with the retarded propagator associated with the homogeneous part of the equation, plus a convolution of the stochastic source with the same retarded propagator. By the linearity of the functional derivative, \(\frac{\delta X(t)}{\delta \xi(s)}\) only depends on the retarded propagator, which is the same as in the case of no external force. Therefore, the master equation has exactly the same form, except for the addition of the classical force in the potential derivative term of the Poisson bracket:

\[
\frac{\partial}{\partial t} W_r = \left(\nabla^T q Dq + \nabla^T q Q - F(t) \frac{\partial}{\partial p}\right) W_r. \quad \text{(V.9)}
\]

**B. Solutions of the Master Equation**

Fourier transforming the phase-space variables, the master equation becomes

\[
\left(\frac{\partial}{\partial t} + k^T H \nabla_k\right) W_r = -(k^T Dk + iF(t)k_p) W_r. \quad \text{(V.10)}
\]
This equation can also be solved via the method of characteristic curves, following the same method of the previous section:

\[
\frac{d}{ds} W_r(t(s), k(s)) = -(k^T D_k + i F(s) k_p) W_r(t(s), k(s)),
\]

\[
\frac{d}{ds} W_r(s, e^{sH^r} k_0) = -\left( k_0^T e^{sH} D(s) e^{sH^r} k_0 + i F(s) \hat{k}_p e^{sH^r} k_0 \right) W_r(s, e^{sH^r} k_0),
\]

(V.11)

(V.12)

with the solution

\[
W_r(s, e^{sH^r} k_0) = W_r(0, k_0) e^{-\int_0^s ds' \left( k_0^T e^{s'H} D(s') e^{s'H^r} k_0 + i F(s') \hat{k}_p e^{s'H^r} k_0 \right) },
\]

(V.13)

which can be finally written as

\[
W_r(t, k) = W_r(0, k_0) e^{-\frac{1}{2} k^T \sigma_T(t) k - i \langle k \rangle F(t) \hat{k}_p},
\]

(V.14)

where

\[
\langle k \rangle_F(t) = \int_0^t ds F(s) e^{(s-t)H^r} \hat{k}_p.
\]

(V.15)

One can see that just as all temperature dependence only appears in the second cumulant, or covariance, the external force only affects the first cumulant, or mean. The change in the evolution of the mean, given by Eq. (V.15), simply corresponds to adding the response to the external force. As with an ordinary driven oscillator, the response is a convolution of the retarded propagator, which exhibits the natural oscillatory decay of the system, and the driving force.

VI. SUMMARY OF RESULTS

Quantum Brownian motion of an oscillator coupled to a thermal reservoir of quantum oscillators has been the canonical model for studying the environmental effects on a quantum system, even of macroscopic scale, such as quantum dissipation, diffusion, decoherence and entanglement. It also provides important information on quantum measurement, such as noise, fluctuations, correlations, uncertainty relation and standard quantum limit in mesoscopic systems. Many experiments have been carried out for testing these processes. Fifteen years ago an exact master equation \[12\] for the reduced density matrix of the system were derived for a general environment of arbitrary spectral density and temperature. There are claims that exact solutions have been found \[22\]. In this paper we report on solutions to
this equation for a fairly general set of physical conditions and analyze its salient features. We expect these solutions to be useful in realistic settings for the analysis of many problems which can be described by this model.

The first question we addressed concerns the initial time divergences. These divergences result from the sudden coupling of an initially uncorrelated system and environment. This situation is unphysical and can be resolved by switching on the system-environment interaction smoothly over a time-scale longer than the inverse of the characteristic UV cut-off scale of the environment. Having switched on the coupling, the initial frequency of the oscillator, as it quickly evolves from the bare to the renormalized frequency, is the only remaining initial singular behavior. But this merely implies a transition from the bare initial state to a “renormalized” version of that state, which can be taken as the effective initial state for the subsequent regular evolution.

We then solved the master equation coefficients for a general class of bilinear system-environment couplings, which includes any combination of ohmic and integer supraohmic and subohmic spectral densities with extreme cut-offs. Although it is by no means a complete class of spectral densities, it is a privileged class in that the classical trajectories are determined by ordinary differential equations and not integro-differential or fractional differential equations. In that sense, their dynamics is more classical.

For these system-environment couplings, we have solved the master equation coefficients for all temperature ranges, beyond weak coupling, and at all times after the short initial time-scale during which the coupling is switched on. This has enabled us to clarify the validity and shortcomings of previously obtained approximations and provide more general results. Perhaps the most useful are our simple expressions for the master equation coefficients at late time [Eq. (II.60)-(II.61)]. They clearly reveal the existence of logarithmic divergences in the limit of infinite UV cut-off not only in the anomalous diffusion coefficient, but also in the normal one. These divergences cannot be consistently subtracted and condition the possibility of obtaining meaningful results upon the existence of a physically well-motivated cut-off.

For the system-environment couplings under discussion, which all have local dissipation, we have also solved the master equation in its entirety. All initial states evolve into a given Gaussian state corresponding to thermal equilibrium. We have obtained the covariance matrix for this state [Eq. (IV.21)], and the corresponding uncertainty function (Sec. IV B 2).
Interestingly, arbitrarily strong coupling to a zero temperature reservoir will tend to localize the position with arbitrary precision.

For more general systems with non-local dissipation, we have drastically reduced the task of solving the master equation to that of solving a one-dimensional classical parametric oscillator problem. The master equation solutions are parametrically similar to the case of local dissipation. The same kind of terms arise, including a thermal covariance, but it is not guaranteed that the system will relax to a thermal state. That of course depends upon the behavior of the classical trajectories.

Finally, we extended the model of the quantum oscillator linearly coupled to a thermal reservoir of oscillators by including a classical driving force. This modifies the dynamics by driving the mean position and momentum around just as with a classical driven system. In this model we found that the force has no effect upon the width of the wave-packet or any cumulant other than the mean. These results may be useful for the study of low-temperature measurements of forced oscillators, which are relevant for experiments with nanomechanical resonators [23, 24]. They also play a crucial role in future schemes for the detection of gravitational waves with high-intensity laser interferometers, where the radiation pressure effects on the cavity mirrors are important [25, 26].

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APPENDIX

1. Harmonic Number

The Harmonic Number $H(n)$ is a function similar to a logarithm. Its analytical continuation to the complex plane is similar too.

\[ H(n) = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{Z}^+ \]  
(A.1)

\[ H(0) = 0, \]  
(A.2)

\[ \gamma_E = \lim_{n \to \infty} (H(n) - \log(n)), \]  
(A.3)

\[ H(z) = \gamma_E + \psi(z+1), \quad z \in \mathbb{C} \]  
(A.4)

\[ \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \]  
(A.5)

\[ \psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \cdots \quad \text{if} \quad |\arg(z)| < \pi, \]  
(A.6)

where Eq. (A.6) is the asymptotic expansion for $|z| \to \infty$.

2. Exponential Integrals

The following definition and properties of the exponential integral are used in this paper:

\[ \text{Ei}(x) = -\text{P.V.} \int_{-x}^{\infty} dt \frac{e^{-t}}{t} \quad \text{if} \quad x > 0, \]  
(A.7)

\[ E_1(z) = \int_{z}^{\infty} dt \frac{e^{-t}}{t} \quad \text{if} \quad |\arg(z)| < \pi, \]  
(A.8)

\[ E_1(z) \sim \frac{e^{-z}}{z} \left(1 - \frac{1}{z} + \frac{2}{z^2} + \cdots \right), \]  
(A.9)

where P.V. denotes the Cauchy principal value, corresponding to the singularity at $t = 0$ of the integrand in Eq. (A.7), and Eq. (A.9) is an asymptotic expansion for $|z| \to \infty$.

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