Quantitative rigidity of differential inclusions in two dimensions

Xavier Lamy∗ Andrew Lorent† Guanying Peng‡

Abstract

For any compact connected one-dimensional submanifold $K \subset \mathbb{R}^2 \times \mathbb{R}^2$ which has no rank-one connection and is elliptic, we prove the quantitative rigidity estimate

$$\inf_{M \in K} \int_{B_{1/2}} |Du - M|^2 \, dx \leq C \int_{B_1} \text{dist}^2(Du, K) \, dx, \quad \forall u \in H^1(B_1; \mathbb{R}^2).$$

This is an optimal generalization, for compact connected submanifolds of $\mathbb{R}^2 \times \mathbb{R}^2$, of the celebrated quantitative rigidity estimate of Friesecke, James and Müller for the approximate differential inclusion into $SO(n)$. The proof relies on the special properties of elliptic subsets $K \subset \mathbb{R}^2 \times \mathbb{R}^2$ with respect to conformal-anticonformal decomposition, which provide a quasilinear elliptic PDE satisfied by solutions of the exact differential inclusion $Du \in K$. We also give an example showing that no analogous result can hold true in $\mathbb{R}^n \times \mathbb{R}^n$ for $n \geq 3$.

1 Introduction

In 1850, Liouville [16] proved that, given a domain $\Omega \subset \mathbb{R}^3$, any smooth map $u: \Omega \to \mathbb{R}^3$ satisfying the differential inclusion $Du(x) \in \mathbb{R}^+O(n)$ for all $x \in \Omega$ must be either affine or a Möbius transform. A corollary to Liouville’s Theorem is that a $C^3$ function whose gradient belongs everywhere to $SO(n)$ is an affine mapping. This phenomenon of being able to globally control a map satisfying a certain differential inclusion $Du \in K$ is known as “rigidity”.

In [8] Friesecke, James and Müller solved a long standing open problem by proving an optimal quantitative rigidity estimate for $K = SO(n)$. Specifically, they showed that for every bounded open connected Lipschitz domain $U \subset \mathbb{R}^n$, $n \geq 2$, there exists a constant $C(U)$ such that, for $K = SO(n)$,

$$\inf_{R \in K} \|Du - R\|_{L^2(U)} \leq C(U) \|\text{dist}(Du, K)\|_{L^2(U)}, \quad \forall v \in H^1(U; \mathbb{R}^n). \quad (1)$$

Here and below, $\text{dist}(M, K)$ denotes the distance from a matrix $M \in \mathbb{R}^{n \times n}$ to a subset $K \subset \mathbb{R}^{n \times n}$ measured in the Euclidean norm. This result strengthened earlier work of a series of authors, including John [12], Reshetnyak [18], and Kohn [14], and it has had a number of important applications, in particular to thin film limits of elastic structures [8, 9]. One of the

∗Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Xavier.Lamy@math.univ-toulouse.fr
†Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, USA. Email: lorentaw@uc.edu
‡Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA. Email: gpeng@wpi.edu
remarkable things about this result is that it is a striking fact about classical mathematical objects that could be understood by the mathematicians of hundreds of years ago.

A number of works have extended the above result \[1\] to cover various larger classes of matrices than \( K = SO(n) \). Chaudhuri and Müller \[3\] and later De Lellis and Szekelyhidi \[5\] considered a set of the form \( K = SO(n)A \cup SO(n)B \) where \( A \) and \( B \) are strongly incompatible in the sense of Matos \[17\]. Faraco and Zhong \[7\] proved an analogous quantitative rigidity result with \( K = m \cdot SO(n) \) where \( m \subset (0, +\infty) \) is compact. There the infimum in the left-hand side of \[1\] also needs to include the gradients of Möbius transforms, and the integral is over a smaller subset \( U' \subset U \).

Our main result is an optimal generalization of the quantitative rigidity estimate of \[8\] in the context of compact connected submanifolds \( K \subset \mathbb{R}^{2 \times 2} \) without boundary.

**Theorem 1.1.** Let \( K \subset \mathbb{R}^{2 \times 2} \) be a smooth, compact and connected 1-manifold without rank-one connections, and elliptic: there exists \( C_* > 0 \) such that

\[
|M - M'|^2 \leq C_* \det(M - M') \quad \forall M, M' \in K.
\]

Then for any \( u \in H^1(B_1; \mathbb{R}^2) \) we have

\[
\inf_{M \in K} \int_{B_{1/2}^1} |Du - M|^2 \, dx \leq C \int_{B_1} \text{dist}^2(Du, K) \, dx,
\]

for some constant \( C = C(K) > 0 \).

**Remark 1.2.** A covering argument as in \[7\] shows that the estimate \[3\] in the balls \( B_{1/2} \subset B_1 \) automatically improves to

\[
\inf_{M \in K} \int_{\Omega'} |Du - M|^2 \, dx \leq C \int_{\Omega} \text{dist}^2(Du, K) \, dx, \quad \forall u \in H^1(\Omega; \mathbb{R}^2),
\]

for open bounded sets \( \Omega' \subset \subset \Omega \subset \mathbb{R}^2 \) and a constant \( C = C(K, \Omega', \Omega) > 0 \).

This result is optimal among compact connected submanifolds \( K \subset \mathbb{R}^{2 \times 2} \) without boundary for the following reasons:

- First, it is classical that the no-rank-one-connections assumption is necessary for the rigidity of the exact differential inclusion (see e.g. \[13\]).

- Second, ellipticity is necessary for the validity of the linearized version of \[3\] (see Remark \[2,3\]), because non-ellipticity would imply that the tangent space \( T_M K \) has a rank-one connection for some \( M \in K \).

- Third, the two previous conditions (no rank-one connections and ellipticity) imply that the submanifold \( K \) must be of dimension 1. \[22\] Corollary 3.5 & 3.6.

Moreover, we provide in Section \[5\] an example showing that the two-dimensional setting is also optimal: there exists an elliptic 1-submanifold \( K \subset \mathbb{R}^{3 \times 3} \) without rank-one connection but which contains a so-called \( T_4 \) configuration, a well-known obstruction to compactness of sequences \( \{u_k\} \subset H^1 \) satisfying \( \text{dist}(Du_k, K) \to 0 \) in \( L^2 \) \[2\], and therefore to any type of quantitative rigidity estimate.
One of our motivations for studying differential inclusion into general submanifolds $K \subset \mathbb{R}^{2 \times 2}$ is our previous work \cite{15} where we obtained a rigidity result for a non-elliptic differential inclusion related to the so-called Aviles-Giga functional, and pointed out the nice consequences that a corresponding quantitative rigidity estimate would have. Theorem 1.1 is not valid for non-elliptic differential inclusions, but the ideas in the present work should be relevant to attain that goal.

While the statements of the quantitative rigidity results of \cite{8, 3, 7} are elementary, their proofs are not. Their starting point, in addition to rigidity of the exact differential inclusion, is a linearized version of (1) for the differential inclusion $Du \in T_{M_0}K$ into a tangent space $T_{M_0}K$. For $K = SO(n)$ and $M_0 = I$, this is Korn’s inequality. A natural linearization procedure then provides a quantitative rigidity estimate, but in terms of the $L^\infty$ norm of $\text{dist}(Du, K)$, rather than $L^2$. Strengthening the $L^2$ bound on $\text{dist}(Du, K)$ into an $L^\infty$ bound constitutes therefore the main difficulty. A key idea, introduced in \cite{8}, is to use the regularity of an elliptic PDE satisfied by solutions of the exact differential inclusion: for $K = SO(n)$ the exact differential inclusion $Du \in SO(n)$ implies that the coordinate functions $u_k$ are harmonic. For $K \subset \mathbb{R}_+ \times SO(n)$ the coordinate functions satisfy the $(n-2)$-Laplace equation $\text{div}(\nabla u_k |^{n-2} \nabla u_k) = 0$. Such PDE follows from the universal identity $\text{div} \ \text{cof}(Du) = 0$ (where cof denotes the cofactor matrix), together with identities satisfied by matrices in the specific set $K$. It is satisfied by solutions of the exact differential inclusion, and for a general map $u$ the error from solving that PDE can be controlled in terms of the right-hand side of (1). This allows to reduce the proof of (1) to maps solving that PDE. Elliptic regularity then provides, via a compactness argument, a uniform bound on $\text{dist}(Du, K)$ and the linearization can be performed.

Following this scheme, the main ingredient to prove Theorem 1.1 is to embed $K$ into the graph of a uniformly monotone vector field: this will be enough to turn the identity $\text{div} \ \text{cof}(Du) = 0$ into a quasilinear elliptic equation for the exact differential inclusion $Du \in K$.

**Proposition 1.3.** Let $K$ be as in Theorem 1.1. There exist $G_1, G_2 : \mathbb{R}^2 \to \mathbb{R}^2$ smooth, globally Lipschitz vector fields such that

$$K \subset \left\{ \begin{pmatrix} A \\ iG_1(A) \end{pmatrix} : A \in \mathbb{R}^2 \right\} \cap \left\{ \begin{pmatrix} -iG_2(B) \\ B \end{pmatrix} : B \in \mathbb{R}^2 \right\},$$

and $G_1, G_2$ are uniformly monotone, that is

$$(G_j(X) - G_j(X')) \cdot (X - X') \geq \lambda |X - X'|^2 \quad \forall X, X' \in \mathbb{R}^2,$$

for some constant $\lambda > 0$ depending only on $K$.

Proposition 1.3 relies on remarkable properties of elliptic subsets of $\mathbb{R}^{2 \times 2}$ with respect to the decomposition into conformal and anticonformal parts, discovered in \cite{22} and exploited in a striking manner in \cite{6} (see also \cite{13}). (It is also related to the classical link between two-dimensional elliptic PDEs of second order and complex Beltrami equations, see e.g. the introduction of \cite{11} and references therein.) Then the proof of Theorem 1.1 follows the scheme outlined above.

The article is organized as follows. In Section 2 we establish the two basic prerequisites to Theorem 1, rigidity for the exact differential inclusion and the linearized estimate. In
Section 3 we give the proof of Proposition 1.3. In Section 4 we gather these ingredients to prove Theorem 1.1. In Section 5 we describe the counterexample in $\mathbb{R}^{3\times 3}$.

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2 Basic ingredients: rigidity of the exact inclusion and linearized estimate

In this section we prove the following two Lemmas.

Lemma 2.1. If $u \in H^1(B_1; \mathbb{R}^2)$ is such that $Du \in K$ a.e., then $Du \equiv M$ for some $M \in K$.

Lemma 2.2. For all $M \in K$ and $u \in H^1(B_1; \mathbb{R}^2)$ we have

$$\inf_{X \in T_M K} \int_{B_1} |Du - X|^2 \, dx \leq C \int_{B_1} \text{dist}^2(Du, T_M K) \, dx$$

for some constant $C = C(K) > 0$, where $T_M K$ denotes the linear tangent space to $K$ at $M$.

Remark 2.3. The linearized estimate (4), or rather its weaker interior version

$$\inf_{X \in T_M K} \int_{B_{1/2}} |Du - X|^2 \, dx \leq C \int_{B_{1/2}} \text{dist}^2(Du, T_M K) \, dx,$$

is a necessary condition for (3) to be valid. Assume indeed that (3) is verified, fix $u \in C^1(B_1; \mathbb{R}^2)$, and apply (3) to $v_\epsilon(x) = Mx + \epsilon u(x)$ for $\epsilon \ll 1$. There exists $M_\epsilon \in K$ such that

$$\int_{B_{1/2}} |M - M_\epsilon - \epsilon Du|^2 \, dx \leq C \int_{B_{1/2}} \text{dist}^2(M + \epsilon Du, K) \, dx \leq C\epsilon^2 \int_{B_{1/2}} \text{dist}^2(Du, T_M K) \, dx + o(\epsilon^2).$$

Hence, letting $X_\epsilon = \epsilon^{-1}(M - M_\epsilon)$, we have

$$\int_{B_{1/2}} |X_\epsilon - Du|^2 \leq C \int_{B_{1}} \text{dist}^2(Du, T_M K) \, dx + o(1).$$

In particular $X_\epsilon$ is bounded, and extracting a converging subsequence we obtain $X \in T_M K$ showing the validity of (5) for $u \in C^1(\overline{B_1}; \mathbb{R}^2)$, and then by density for $u \in H^1(\overline{B_1}; \mathbb{R}^2)$.

Proof of Lemma 2.1. Let $\ell = |K|$ and $M: \mathbb{R}/\ell \mathbb{Z} \to K$ be an arc-length parametrization of $K$, the ellipticity assumption ensures that $M'(t)$ is invertible for all $t \in \mathbb{R}$. Let $u \in H^1(B_1; \mathbb{R}^2)$ such that $Du \in K$ a.e., then $u$ is smooth by [21], and since $B_1$ is simply connected there exists a smooth lifting $\theta: B_1 \to \mathbb{R}$ such that $Du = M(\theta)$. Using that $\text{div} \, \text{cof}(Du) = 0$, where $\text{cof}$ denotes the cofactor matrix, we find $\text{cof}(M'(\theta))\nabla \theta = 0$, hence $\nabla \theta = 0$ since $\text{cof}(M'(\theta))$ is invertible. Therefore $Du$ is constant. \qed
Proof of Lemma 2.2. For $M \in K$ we denote by $P_M: \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$ the orthogonal projection onto $(T_M K)^\perp$, so that
\[\text{dist}^2(X, T_M K) = |P_M X|^2.\]

We denote by $P_M^{\alpha\beta jk} \in \mathbb{R}$ the coefficients of $P_M$, that is,
\[(P_M X)_{\alpha\beta} = \sum_{jk} P_M^{\alpha\beta jk} X_{jk} \quad \forall X = (X_{jk}) \in \mathbb{R}^{2 \times 2}.

Define $\mathbb{P}_M(i\xi) \in \mathcal{L}(\mathbb{C}^2, \mathbb{C}^{2 \times 2})$ by
\[(\mathbb{P}_M(i\xi) v)_{\alpha\beta} = \sum_{jk} P_M^{\alpha\beta jk} v_j \xi_k \quad \forall \xi, v \in \mathbb{C}^2,

so the differential operator $u \mapsto P_M Du$ has symbol $\mathbb{P}_M(i\xi)$, i.e.,
\[(P_M (D f))_{\alpha\beta} = \frac{1}{2\pi} \int (\mathbb{P}_M (i\xi) \hat{f}(\xi))_{\alpha\beta} e^{ix \cdot \xi} d\xi.

We claim that $\mathbb{P}_M(\xi)$ has trivial kernel for all non-zero $\xi \in \mathbb{C}^2$. Let indeed $v \in \mathbb{C}^2$ such that $\mathbb{P}_M(\xi)v = 0$. This implies that $P_M \text{Re}(v \otimes \xi) = P_M \text{Im}(v \otimes \xi) = 0$, because the coefficients $P_M^{\alpha\beta jk}$ are real-valued. In other words, the real and imaginary parts of $v \otimes \xi$ both belong to $\ker P_M = T_M K$. Since $T_M K$ is a one-dimensional subspace of $\mathbb{R}^{2 \times 2}$ which doesn’t contain any rank-one matrix, we have $T_M K = \mathbb{R} X_0$ for some invertible matrix $X_0$. Hence we deduce that $v \otimes \xi = \lambda X_0$ for some $\lambda \in \mathbb{C}$ and an invertible matrix $X_0 \in T_M K$. But $v \otimes \xi$ has zero determinant, so $\lambda = 0$ and we must have $v = 0$. This proves that $\mathbb{P}_M(\xi)$ has trivial kernel. Therefore we have the representation formula [20 Theorem 4.1] and the coercive inequality [20 Theorem 8.15] that follows from it,

\[
\int_{B_1} |Du|^2 \, dx \leq C \int_{B_1} |P_M Du|^2 \, dx + C \int_{B_1} |u|^2 \, dx,
\]

for all $u \in H^1(B_1; \mathbb{R}^2)$. (In the notation of [20], $N = 4$, $M = 2$, and the index set $\{1, 2, 3, 4\}$ for $j$ is in our case given by $\{1, 2\}^2$, and we can take $m_j = 1$ for $j \in \{1, 2\}$ and $l_i = 0$ for $i \in \{1, 2\}$.) The constant $C > 0$ in (6) depends a priori on the fixed matrix $M \in K$. Denote by $C(M)$ the best possible constant in (6). Then for any $M, M' \in K$ we have
\[
\int_{B_1} |Du|^2 \, dx \leq 2C(M) \int_{B_1} |P_{M'} Du|^2 \, dx + C(M) \int_{B_1} |u|^2 \, dx
\]
\[+ 2C(M)\|P_M - P_{M'}\|^2 \int_{B_1} |Du|^2 \, dx.
\]

For all $M \in K$, there exists $\delta(M) > 0$ sufficiently small such that for all $M' \in K \cap B_{\delta(M)}(M)$, we have $2C(M)\|P_M - P_{M'}\|^2 < 1/2$. It follows that
\[
C(M') \leq \frac{2C(M)}{1 - 2C(M)\|P_M - P_{M'}\|^2} < 4C(M) \quad \forall M' \in K \cap B_{\delta(M)}(M).
\]
By compactness this implies that $M \mapsto C(M)$ is bounded on $K$, so we can take the constant $C$ in (6) to depend only on $K$.

Moreover, if $u \in H^1(B_1; \mathbb{R}^2)$ satisfies $P_M Du = 0$ a.e., then $Du = \lambda X_0$ for some $\lambda \in L^2(B_1; \mathbb{R})$, and the distributional identity $0 = \text{div} \, \text{cof}(Du) = \text{cof}(X_0) \nabla \lambda$ implies that $\lambda$ is constant, hence $Du = X$ for some $X \in T_M K$.

Therefore (4) follows from (6) via a compactness argument: assume by contradiction the existence of sequences $M_k \in K$, and $u^k \in H^1(B_1; \mathbb{R}^2)$ such that

$$\inf_{X \in T_{M_k} K} \int_{B_1} |Du^k - X|^2 \, dx = 1, \quad \int_{B_1} |P_{M_k} Du^k|^2 \, dx \to 0.$$ 

Subtracting from $u^k$ its average and $X_k x$ for the matrix $X_k$ at which the infimum in the left-hand side is attained, we may in fact assume

$$\int_{B_1} u^k \, dx = \int_{B_1} Du^k \cdot X \, dx = 0 \quad \forall X \in T_{M_k} K,$$

and

$$\int_{B_1} |Du^k|^2 \, dx = 1, \quad \int_{B_1} |P_{M_k} Du^k|^2 \, dx \to 0.$$ 

Thus we may extract subsequences (not relabeled) $u^k \to u$ weakly in $H^1(B_1; \mathbb{R}^2)$ and strongly in $L^2(B_1; \mathbb{R}^2)$, and $M_k \to M \in K$. It follows that $P_{M_k} Du^k \to P_M Du$ in $L^2(B_1; \mathbb{R}^{2 \times 2})$, and thus by lower semicontinuity of the $L^2$ norm under weak convergence, we have $P_M Du = 0$ a.e., which implies $Du \equiv X$ for some $X \in T_M K$. Approximating $X$ by a sequence $X^k \in T_{M_k} K$ and using $Du^k \rightharpoonup Du$ in $L^2(B_1; \mathbb{R}^{2 \times 2})$, we deduce that $0 = \int_{B_1} Du \cdot X \, dx = |B_1||X|^2$, and thus $Du \equiv X = 0$. Further $u$ satisfies $\int_{B_1} u \, dx = 0$, which implies $u \equiv 0$. Plugging $u^k$ into (6) gives

$$1 = \int_{B_1} |Du^k|^2 \, dx \leq C \int_{B_1} |P_{M_k} Du^k|^2 \, dx + C \int_{B_1} |u|^2 \, dx.$$ 

Passing to the limit as $k \to \infty$ and using the strong $L^2$ convergence, we have $1 \leq C \int_{B_1} |u|^2 \, dx = 0$, which gives a contradiction. 

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### 3 Proof of Proposition 1.3

We only prove the existence of $G_1$, the existence of $G_2$ is obtained by the same arguments. The proof relies on the properties of the conformal and anticonformal projections of $K$ uncovered in [22, 6]. For any $z_+, z_- \in \mathbb{C}$, we denote by

$$[z_+, z_-] = \begin{pmatrix} \text{Re} \, z_+ & -\text{Im} \, z_+ \\ \text{Im} \, z_+ & \text{Re} \, z_+ \end{pmatrix} + \begin{pmatrix} \text{Re} \, z_- & \text{Im} \, z_- \\ \text{Im} \, z_- & -\text{Re} \, z_- \end{pmatrix} \in \mathbb{R}^{2 \times 2},$$

the $2 \times 2$ matrix whose conformal, respectively anticonformal, part is represented by $z_+$, respectively $z_-$. For any $A \in \mathbb{R}^{2 \times 2}$, the decomposition $A = [z_+, z_-]$ is unique, and we have the identities

$$\det A = |z_+|^2 - |z_-|^2, \quad |A|^2 = 2|z_+|^2 + 2|z_-|^2, \quad ||A|| = |z_+| + |z_-|,$$
where $|A|$ and $\|A\|$ denote the Hilbert-Schmidt and the operator norms of $A$, respectively. We denote by $p_+: [z_+, z_-] \mapsto z_+$ the projection onto the conformal part.

Using these notations, the ellipticity assumption (2) is equivalent to

$$2|z_+ - z_+'|^2 + 2|z_- - z_-'|^2 \leq C_\ast (|z_+ - z_+|^2 + |z_- - z_-'|^2),$$

for all $[z_+, z_-], [z_+', z_-'] \in K$. This implies that the curve $\hat{K}$ is $C_\ast$-elliptic in the sense of [6, Def. 1], and it follows from the analysis in [22] (see also the proofs of Lemmas 1 and 2 in [6]) that $p_+(K)$ is a Jordan curve and

$$K = \{[z, H(z)]: z \in p_+(K)\},$$

for some $k$-Lipschitz function $H: p_+(K) \to \mathbb{C}$ with $0 \leq k = (C_\ast - 1)/(C_\ast + 1) < 1$. Further the explicit formula $p_+(A) = (a_{11} + a_{22})/2 - i(a_{12} - a_{21})/2$ for $A = (a_{ij})$ and the smoothness of $K$ imply that $p_+(K) \subset \mathbb{C}$ is smooth.

**Lemma 3.1.** The function $H$ admits a smooth extension $H: \mathbb{C} \to \mathbb{C}$ which is $k$-Lipschitz for some (possibly larger) $0 \leq k < 1$.

**Proof of Lemma 3.1.** We first fix, thanks to Kirszbraun’s theorem, a $k$-Lipschitz extension $\tilde{H}: \mathbb{C} \to \mathbb{C}$. In the rest of the proof we modify $\tilde{H}$ to make it smooth while still agreeing with $H$ on $p_+(K)$, at the cost of slightly increasing its Lipschitz constant.

Let $[z_+(t), z_-(t)], t \in \mathbb{R}/\ell\mathbb{Z}$, denote a smooth arc-length parametrization of $K$, so that $2|\dot{z}_+|^2 + 2|\dot{z}_-|^2 = 1$, where $\dot{z}_\pm = \frac{d}{dt}z_\pm$. As $z_- = H(z_+)$ and $\tilde{H}$ is $k$-Lipschitz, it follows that $|\dot{z}_-| \leq k|\dot{z}_+|$, so $|\dot{z}_+|^2 \geq \frac{1}{2(1+k^2)} > 0$. Therefore we may reparametrize and consider

$$K = \{[z(s), H(z(s))]: s \in \mathbb{R}/\ell\mathbb{Z}\},$$

with $z(s)$ an arc-length parametrization of $p_+(K)$ and $\ell$ its length, and the map $s \mapsto H(z(s))$ is smooth by smoothness of $K$.

For small enough $\delta > 0$, the map

$$\varphi: \mathbb{R}/\ell\mathbb{Z} \times (-2\delta, 2\delta) \to \mathcal{U}_{2\delta} = \{z \in \mathbb{C}: \text{dist}(z, p_+(K)) < 2\delta\},$$

$$(s, r) \mapsto z(s) + ri\dot{z}(s),$$

is a smooth diffeomorphism. We first modify $\tilde{H}$ by setting

$$\tilde{H} = \tilde{H} \circ \Phi, \quad \Phi(Z) = \begin{cases} z(s) + \lambda(r)i\dot{z}(s) & \text{if } Z = \varphi(s, r) \in \mathcal{U}_{2\delta}, \\ Z & \text{otherwise}, \end{cases}$$

where $\lambda$ is the odd $(1 - \delta)^{-1}$-Lipschitz function given for $r > 0$ by

$$\lambda(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq \delta^2, \\ -\frac{\delta^2}{1-\delta} + \frac{r}{1-\delta} & \text{for } \delta^2 < r \leq \delta, \\ r & \text{for } r > \delta. \end{cases}$$

In particular we have

$$\tilde{H}(Z) = H(z(s)) \quad \forall Z = \varphi(s, r) \in \mathcal{U}_{2\delta},$$
so \( \tilde{H} \) is smooth in \( U_{\delta^2} \) (by smoothness of \( s \mapsto H(z(s)) \) and \( \varphi^{-1} \)) and agrees with \( H \) on \( p_+(K) \). Note that by definition of \( \varphi \) and \( \lambda \) we have \( \Phi(Z) = Z \) in \( \mathbb{C} \setminus U_\delta \) and therefore \( \Phi \) is Lipschitz in \( \mathbb{C} \). Since \( D\varphi(s, 0) \in SO(2) \) for all \( s \in \mathbb{R}/\ell + \mathbb{Z} \), we have \( \|D\varphi\| \leq 1 + C\delta \) on \( \mathbb{R}/\ell + \mathbb{Z} \times (-2\delta, 2\delta) \). Further, we have \( \|D(\varphi^{-1})\| \leq 1 + C\delta \) on \( U_{\delta^2} \). Denoting by \( \psi \) the \( (1 - \delta)^{-1} \)-Lipschitz map \( (s, r) \mapsto (s, \lambda(r)) \), we write \( \Phi(Z) = \varphi(\psi(\varphi^{-1}(Z))) \) and deduce that \( \|D\Phi\| \leq 1 + C\delta \) a.e. in \( U_{\delta^2} \). This inequality is also true in the rest of \( \mathbb{C} \) by definition of \( \Phi \), so we conclude that \( \tilde{H} \) is \( k \)-Lipschitz in \( \mathbb{C} \), with \( k = (1 + C\delta)k < 1 \) for small enough \( \delta > 0 \). Now \( \delta \) is fixed and we define, for \( \epsilon \in (0, \delta^2/4) \),

\[
H_\epsilon(z) = \int_\mathbb{C} \tilde{H}(z + \epsilon \chi(z)y)\rho(y) \, dy,
\]

for a smooth kernel \( \rho \geq 0 \) with support in \( B_1 \) and \( \int \rho(y) \, dy = 1 \), and some smooth cut-off function \( \chi \) with \( 1_{U_{\delta^2/4}} \leq 1 - \chi \leq 1_{U_{\delta^2/2}} \). In \( U_{3\delta^2/4} \), the map \( H_\epsilon \) is smooth thanks to the smoothness of \( \tilde{H} \) in \( U_{\delta^2} \). In \( \mathbb{C} \setminus U_{\delta^2/2} \), we have \( H_\epsilon(z) = \int_\mathbb{C} \tilde{H}(z + \epsilon y)\rho(y) \, dy \) is also smooth. Therefore the map \( H_\epsilon \) is smooth in \( \mathbb{C} \). Further, \( H_\epsilon(z) = \tilde{H}(z) \) for \( z \in U_{\delta^2/4} \), and thus agrees with \( H \) on \( p_+(K) \). Finally, denoting by \( L \) the Lipschitz constant of \( \chi \), we have

\[
|H_\epsilon(z) - H_\epsilon(z')| \leq \int_{B_1} |\tilde{H}(z + \epsilon \chi(z)y) - \tilde{H}(z' + \epsilon \chi(z')y)|\rho(y) \, dy
\leq \int_{B_1} \tilde{k}(1 + \epsilon L |y|)|z - z'|\rho(y) \, dy \leq \tilde{k}(1 + \epsilon L)|z - z'|.
\]

So \( H_\epsilon \) is \( k_\epsilon \)-Lipschitz with \( k_\epsilon \leq \tilde{k}(1 + \epsilon L) < 1 \) for small enough \( \epsilon \).

\[\blacklozenge\]

**Lemma 3.2.** The map \( F: z \mapsto \tilde{z} + H(z) \) is a smooth diffeomorphism from \( \mathbb{C} \) onto \( \mathbb{C} \). Moreover \( F \) is \( (1 + k) \)-Lipschitz and \( F^{-1} \) is \( (1 - k)^{-1} \)-Lipschitz.

**Proof of Lemma 3.2.** For any \( w \in \mathbb{C} \) the equation

\[
w = \tilde{z} + H(z) \iff z = \bar{w} - \tilde{H}(z),
\]

admits a unique solution \( z \in \mathbb{C} \) thanks to the fixed point theorem, since \( z \mapsto \bar{w} - \tilde{H}(z) \) is \( k \)-Lipschitz and \( 0 \leq k < 1 \). This shows that \( F \) is bijective. The inequalities

\[
(1 - k)|z - z'| \leq |F(z) - F(z')| \leq (1 + k)|z - z'|,
\]

follow directly from the fact that \( H \) is \( k \)-Lipschitz and imply the announced Lipschitz constants of \( F \) and \( F^{-1} \). The inverse \( F^{-1} \) is smooth thanks to the Inverse Function Theorem, since \( DF(z): h \mapsto \tilde{h} + D\tilde{H}(z)h \) is invertible for all \( z \in \mathbb{C} \) because \( \|DH\| < 1 \).

\[\blacklozenge\]

**Proof of Proposition 1.3 completed.** Then, identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \), we define

\[
G_1(A) = F^{-1}(A) - H(F^{-1}(A)),
\]

so that a short calculation shows that

\[
[z, H(z)] = \begin{pmatrix} A \\ iG_1(A) \end{pmatrix} \quad \text{for } A = F(z).
\]
The map $G_1$ is smooth and globally Lipschitz with Lipschitz constant $\Lambda = (1 + k)/(1 - k)$. Moreover, for all $A = F(z)$, $A' = F(z')$, we have
\[
(G_1(A) - G_1(A')) \cdot (A - A') = \det \left( \begin{array}{c} A - A' \\ i(G_1(A) - G_1(A')) \end{array} \right) \\
= \det([z - z', H(z) - H(z')]) \\
= |z - z'|^2 - |H(z) - H(z')|^2 \\
\geq (1 - k^2)|z - z'|^2.
\]
Since $F$ is $(1 + k)$-Lipschitz, this implies
\[
(G_1(A) - G_1(A')) \cdot (A - A') \geq \lambda |A - A'|^2,
\]
for $\lambda = (1 - k^2)/(1 + k)^2 = (1 - k)/(1 + k) > 0$. This concludes the proof of Proposition 1.3. □

4 Proof of Theorem 1.1

Step 1. We may assume that $u$ is Lipschitz, thanks to the truncation argument of [8, Proposition A.1]. Let $K \subset B_R \subset \mathbb{R}^{2 \times 2}$ for some $R > 0$. Then for all $X \in \mathbb{R}^{2 \times 2}$ with $|X| > 2R$, we have $|X| \leq 2 \text{dist}(X, K)$. An application of [8, Proposition A.1] with $\lambda = 2R$ gives $v : B_1 \rightarrow \mathbb{R}^2$ satisfying
\[
\|Dv\|_{L^\infty(B_1)} \leq 2C_0 R, \\
\int_{B_1} |Du - Dv|^2 \, dx \leq C_0 \int_{\{x \in B_1 : |Du| > 2R\}} |Du|^2 \, dx \\
\leq 4C_0 \int_{B_1} \text{dist}^2(Du, K) \, dx,
\]
for some constant $C_0$ (depending only on $B_1$). If there exists $M \in K$ such that
\[
\int_{B_{1/2}} |Dv - M|^2 \, dx \leq \tilde{C}(K) \int_{B_1} \text{dist}^2(Dv, K) \, dx,
\]
then repeated applications of the triangle inequality give
\[
\int_{B_{1/2}} |Du - M|^2 \, dx \leq \left(16\tilde{C}(K)C_0 + 4\tilde{C}(K) + 8C_0\right) \int_{B_1} \text{dist}^2(Du, K) \, dx.
\]
Thus if Theorem 1.1 holds for all Lipschitz mappings $v$ for some constant $\tilde{C}(K)$, then it is also valid for all $H^1$ mappings with $C(K) = 16\tilde{C}(K)C_0 + 4\tilde{C}(K) + 8C_0$.

Step 2. We may assume in addition that $u \in C^2(B_1; \mathbb{R}^2)$ solves
\[
\text{div} G_1(Du_1) = \text{div} G_2(Du_2) = 0 \quad \text{in} \ B_1.
\]
Consider indeed $w \in C^2(B_1)$ such that $w = u$ on $\partial B_1$ and
\[
\text{div} G_1(Dw_1) = \text{div} G_2(Dw_2) = 0 \quad \text{in} \ B_1.
\]
The existence of such \( w \) is guaranteed by the ellipticity of the equation \( 0 = \text{div} G_j(Dw_j) = \text{tr}(DG_j(Dw_j)D^2w_j) = 0 \), invoking e.g. \cite{10} Theorem 12.5: the inequality \( \lambda|\xi|^2 \leq DG_j(A)\xi \cdot \xi \leq A|\xi|^2 \), valid for all \( A, \xi \in \mathbb{R}^2 \) thanks to Proposition \ref{prop:1.3} ensures that the eigenvalues of the symmetric part \([DG_j(A)]_s = (DG_j(A) + DG_j(A)^T)/2 \) of \( DG_j(A) \) are bounded above and below (since \( DG_j(A)\xi \cdot \xi = [DG_j(A)]_s\xi \cdot \xi \) for all \( \xi \in \mathbb{R}^2 \)) and in particular condition (ii) in \cite{10} Theorem 12.5 is satisfied. Letting \( v = u - w \) and using the uniform monotonicity of \( G_1 \) we find

\[
\lambda \int_{B_1} |Dv| dx \leq \int_{B_1} (G_1(Du) - G_1(Dw)) \cdot Dv dx.
\]

Since \( \text{div} G_1(Dw_1) = 0 \) and \( \text{div}(iDu_2) = 0 \) we rewrite this as

\[
\lambda \int_{B_1} |Dv|^2 dx \leq \frac{1}{2\lambda} \int_{B_1} |G_1(Du_1) + iDu_2|^2 dx + \frac{\lambda}{2} \int_{B_1} |Dv|^2 dx,
\]

and infer

\[
\int_{B_1} |Dv|^2 dx \leq \frac{1}{\lambda^2} \int_{B_1} |G_1(Du_1) + iDu_2|^2 dx.
\]

According to Proposition \ref{prop:1.3} the function \( M \mapsto G_1(A) + iB \), where \( A, B \) denote the first and second row of the matrix \( M \), vanishes on \( K \). Since that function is Lipschitz we deduce that \( |G_1(Du_1) + iDu_2| \leq C \text{dist}(Du,K) \), and therefore

\[
\int_{B_1} |Dv|^2 dx \leq C \int_{B_1} \text{dist}^2(Du,K) dx.
\]

Applying a similar argument to \( v_2 \) we obtain

\[
\int_{B_1} |Dv|^2 dx \leq C \int_{B_1} \text{dist}^2(Du,K) dx.
\]

Recalling that \( v = u - w \) and using the triangle inequality we deduce

\[
\int_{B_1} \text{dist}^2(Dw,K) dx \leq C \int_{B_1} \text{dist}^2(Du,K) dx,
\]

\[
\int_{B_{1/2}} |Du - M|^2 dx \leq 2 \int_{B_{1/2}} |Dw - M|^2 dx + C \int_{B_1} \text{dist}^2(Du,K) dx.
\]

As a consequence, if Theorem \ref{thm:1.1} is valid for \( w \) then we obtain it for \( u \). This proves Step 2.

**Step 3.** As \( u_j \in C^2(B_1) \) satisfies \( (7) \), it is a weak solution of

\[
\text{div} (\partial_i(G_j(Du_j))) = \text{div} (DG_j(Du_j)D(\partial_iu_j)) = 0 \quad \text{for } i = 1, 2.
\]

Invoking the De Giorgi-Nash estimates \cite{4} (see e.g. \cite{11} Theorem 4.11] for the precise statement we use here) for \( \partial_iu_j \), \( i, j \in \{1, 2\} \), we obtain

\[
\|Du\|_{C^0(\overline{B}_{1/2})} \leq C\|Du\|_{L^2(B_1)}.
\]
for some $\alpha > 0$ and some constant $C = C(K)$. Thanks to this estimate and the exact rigidity obtained in Lemma 2.1, we may argue exactly as in [8, Lemma 4.5] to deduce that
\[
\inf_{M \in K} \| Du - M \|_{L^\infty(B_{1/2})} \leq \rho \left( \int_{B_1} \text{dist}^2(Du, K) \, dx \right),
\]  
for some function $\rho$ depending only on $K$ and satisfying $\rho(\epsilon) \to 0$ as $\epsilon \to 0$.

**Step 4.** We finally combine Step 3 with the linearized estimate of Lemma 2.2, to obtain our main estimate (3). The basic idea, as in [8, 7], is to linearize dist

Step 4. We finally combine Step 3 with the linearized estimate of Lemma 2.2, to obtain our main estimate (3). The basic idea, as in [8, 7], is to linearize dist^2(\cdot, K) around $M_0 \in K$ such that $|Du - M_0|$ is uniformly small. When doing so, (3) formally turns into the linearized estimate (4) of Lemma 2.2, and it remains to control the error terms. Due to the modification in the left-hand side of the linearized estimate (4), it is not directly obvious that the error terms are negligible. In [8] this problem is absent because their equivalent of (8) comes with an explicit $\rho(\epsilon) = C\epsilon^1$. In [7] it is taken care of via a topological degree argument [7, Proposition 4.7] (see also [19]) which allows to avoid the translation. Here we present an alternative argument relying on elementary estimates.

We assume without loss of generality that
\[
\int_{B_1} \text{dist}^2(Du, K) \, dx = \epsilon \leq \epsilon_0,
\]  
where $\epsilon_0 = \epsilon_0(K)$ is to be chosen in the course of the proof. If (9) is not valid then (3) is automatically satisfied for a large enough constant $C$ because the left-hand side of (3) is bounded thanks to Step 1.

We fix $\delta_0 > 0$ depending only on $K$, such that the nearest-point projection $\Pi_K$ onto $K$ is uniquely defined and smooth in the neighborhood $N_{2\delta_0}(K)$. We first choose $\epsilon_0$ small enough that $\rho(\epsilon_0) \leq \delta_0$, so thanks to (8) the projection $\Pi_K(Du)$ is well-defined.

We claim that, for every $M \in K$, there exists $Y_M \in K$ such that
\[
\int_{B_{1/2}} |Du - Y_M|^2 \, dx \leq C \int_{B_{1/2}} \text{dist}^2(Du, K) \, dx + C \int_{B_{1/2}} |Du - M|^4 \, dx,
\]  
and
\[
|Y_M - M|^2 \leq C \int_{B_{1/2}} |Du - M|^2 \, dx.
\]

Here and in the rest of this proof we denote by $C > 0$ a generic constant depending only on $K$.

To prove (10), we first invoke Lemma 2.2 according to which we have
\[
\inf_{X \in T_M K} \int_{B_{1/2}} |Du - M - X|^2 \, dx \leq C \int_{B_{1/2}} \text{dist}^2(Du - M, T_M K) \, dx.
\]

Choosing $X = X_M \in T_M K$ attaining the infimum in the left-hand side, we obtain
\[
\int_{B_{1/2}} |Du - M - X_M|^2 \, dx \leq C \int_{B_{1/2}} \text{dist}^2(Du - M, T_M K) \, dx.
\]

Moreover the minimizing property of $X_M$ implies that $\int_{B_{1/2}} (Du - M - X_M) \, dx$ is orthogonal to $X_M$. We deduce
\[
|X_M|^2 = \int_{B_{1/2}} X_M \cdot (Du - M) \, dx \leq \frac{1}{2} |X_M|^2 + C \int_{B_{1/2}} |Du - M|^2 \, dx,
\]
\[ |X_M|^2 \leq C \int_{B_{1/2}} |Du - M|^2 \, dx. \]  

(12)

Recalling from the proof of Lemma 2.2 that \( P_M \) denotes the orthogonal projection onto \((T_M K)^\perp\), we estimate the integrand in the right-hand side of (11) using

\[
\text{dist}(Du - M, T_M K) = |P_M(Du - M)|
\]

\[
\leq |Du - \Pi_K(Du)| + \Pi_K(Du) - M - (I - P_M)(Du - M)|
\]

\[
\leq |Du - \Pi_K(Du)| + C|Du - M|^2.
\]

The last inequality follows from the fact that \( I - P_M = D\Pi_K(M) \). Since \( |Du - \Pi_K(Du)| = \text{dist}(Du, K) \), plugging this into (11) we find

\[
\int_{B_{1/2}} |Du - M - X_M|^2 \, dx \leq C \int_{B_{1/2}} \text{dist}^2(Du, K) \, dx + C \int_{B_{1/2}} |Du - M|^4 \, dx
\]

(13)

Now we may choose \( Y_M \in K \) such that

\[
|M + X_M - Y_M| \leq C|X_M|^2.
\]

(14)

Indeed, if \(|X_M| \leq \delta_0\) then one can simply take \( Y_M = \Pi_K(M + X_M) \) and use the fact that \( D\Pi_K(M)X_M = (I - P_M)X_M = X_M \), and if \(|X_M| \geq \delta_0\) one may take \( Y_M = M \). From (13) and (14) we infer

\[
\int_{B_{1/2}} |Du - Y_M|^2 \, dx \leq C \int_{B_{1/2}} \text{dist}^2(Du, K) \, dx + C \int_{B_{1/2}} |Du - M|^4 \, dx + C|X_M|^4.
\]

Using (12) and Cauchy-Schwarz to estimate the last term, we deduce the first inequality in (10), and the estimate on \(|M - Y_M|\) in (10) follows from (14) and (12) (taking into account that \(|X_M| \leq C\) thanks to (12) and Step 1).

Our goal is to find \( M \in K \) for which we can discard the last term in (10). To that end, we apply (10) to define by induction a sequence \((M_j) \subset K\) satisfying

\[
M_{j+1} = Y_{M_j} \quad \forall j \geq 0,
\]

and \( M_0 \) to be chosen later. According to (10) we have

\[
\int_{B_{1/2}} |Du - M_{j+1}|^2 \, dx \leq C \int_{B_{1/2}} \text{dist}^2(Du, K) \, dx + C \int_{B_{1/2}} |Du - M_j|^4 \, dx,
\]

and

\[
|M_j - M_{j+1}|^2 \leq C \int_{B_{1/2}} |Du - M_j|^2 \, dx,
\]

Therefore, setting

\[
q_j = \int_{B_{1/2}} |Du - M_j|^2 \quad \text{and} \quad s_j = \sup_{B_{1/2}} |Du - M_j|,
\]

(15)
and recalling the definition of $\epsilon = \int_{B_{1/2}} \text{dist}^2(Du, K) \, dx$ [9], we find
\begin{equation}
q_{j+1} \leq C \epsilon + Cs_j^2 q_j \quad \text{and} \quad s_{j+1} \leq s_j + Cq_j^\frac{1}{2}.
\end{equation}
We wish to find $j_0 \geq 0$ such that
\begin{equation}
q_{j_0} \leq K \epsilon,
\end{equation}
for some constant $K > 0$ depending only on $K$. Recalling the definitions of $q_j$ [15] and $\epsilon$ [9], this concludes the proof of [3]. We prove by induction on $j_0$ that, if $s_0 \leq 1/(2K)$ and
\begin{equation}
q_j > K \epsilon \quad \text{for } j = 0, \ldots, j_0,
\end{equation}
then
\begin{equation}
q_j \leq (Ks_0)^{2j} s_0^2 \quad \text{for } j = 0, \ldots, j_0.
\end{equation}
The constant $K > 0$ depends only on $K$ and will be adjusted in the course of the proof. Initialization at $j_0 = 0$ is obvious since $|B_{1/2}| = \pi/4 \leq 1$. Next we assume [18]-[19] and prove that either [17] or [19] is satisfied at $j_0 + 1$. Combining [19] with the second inequality in [16], we find
\begin{align*}
s_{j_0} \leq s_0 + Cs_0 \sum_{j=0}^{j_0-1} (Ks_0)^j \leq (1 + C)s_0,
\end{align*}
provided $2Ks_0 \leq 1$. Therefore the first inequality in [16] gives
\begin{align*}
q_{j_0+1} \leq C \epsilon + C(1 + C)^2 s_0^2 q_{j_0},
\end{align*}
and using [19] we obtain
\begin{align*}
q_{j_0+1} \leq C \epsilon + C(1 + C)^2 s_0^4 (Ks_0)^{2j_0} \\
= C \epsilon + \frac{C(1 + C)^2}{K^2} (Ks_0)^{2j_0 + 2} s_0^2.
\end{align*}
Therefore we have
\begin{align*}
\text{either } q_{j_0+1} \leq 2C \epsilon, \quad \text{or } q_{j_0+1} \leq 2 \frac{C(1 + C)^2}{K^2} (Ks_0)^{2j_0 + 2} s_0^2.
\end{align*}
In the first case [17] is satisfied for $j = j_0 + 1$ provided $K \geq 2C$, and in the second case [19] is satisfied at $j_0 + 1$, provided $K \geq (2C)^{1/2}(1 + C)$. This concludes the proof of [19] by induction.

As a consequence, assuming by contradiction that there is no $j_0 \geq 0$ satisfying [17], we would have
\begin{align*}
q_j > K \epsilon \quad \text{and} \quad q_j \leq (Ks_0)^{2j} s_0^2 \quad \forall j \geq 0.
\end{align*}
As $s_0 \leq 1/(2K)$, this implies $q_j \to 0$ and, passing to the limit in $q_j > K \epsilon$ we deduce $\epsilon = 0$. But in that case by Lemma 2.1 we have $Du \equiv M_0$ and this choice of $M_0$ gives $q_0 = 0$.

We conclude that there exists $j_0 = j_0(\epsilon)$ satisfying [17], if we can choose $M_0$ in such a way that $s_0 \leq 1/(2K)$. But thanks to [8] we do have $s_0 \leq \rho(\epsilon) \leq 1/(2K)$ provided $\epsilon_0$ is small enough.
5 A 3 × 3 counter-example

In this section we prove that the two-dimensional setting of Theorem 1.1 is optimal in the following sense: a connected 1-submanifold of $\mathbb{R}^{3 \times 3}$ which has no rank-one connection and is elliptic may not satisfy Sverak’s compactness result [21], and even less a quantitative rigidity estimate.

**Proposition 5.1.** There exists a closed compact $1$-submanifold $\Pi \subset \mathbb{R}^{3 \times 3}$ which is elliptic and has no rank-one connection, but contains a $T_4$ configuration.

By a known construction, see e.g. [2, Theorem 3.1], Proposition 5.1 implies the existence of a sequence of maps $u_k : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$
\int_{B_1} \text{dist}^2(Du_k, \Pi) \, dx \to 0 \quad \text{as } k \to 0,
$$

but $(Du_k)$ is not precompact in $L^2(B_{1/2})$. In particular, one certainly cannot hope for a quantitative estimate

$$
\inf_{M \in \Pi} \int_{B_{1/2}} |Du_k - M|^2 \, dx \leq \rho \left( \int_{B_1} \text{dist}^2(Du_k, \Pi) \, dx \right),
$$

for any function $\rho (\epsilon) \to 0$ as $\epsilon \to 0$.

Proposition 5.1 is a consequence of the construction below. Let $a > 0$ and define matrices $T_1, T_2, T_3, T_4$ by

$$
T_1 = -T_3 = \begin{pmatrix} 1 + a & 0 & 0 \\ 0 & 1 & 0 \\ 1 + a & 0 & 0 \end{pmatrix}, \quad T_2 = -T_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 + a & 0 \\ -1 & 0 & 0 \end{pmatrix},
$$

and $C_1, C_2, C_3, C_4$ by

$$
C_1 = -C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_2 = -C_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
$$

We have that $T_k - C_k$ is rank-one for $k = 1, 2, 3, 4$, and (with the convention that $C_5 = C_1$)

$$
T_k - \frac{2 + a}{a} (T_k - C_k) = C_{k+1} \quad \text{for } k = 1, 2, 3, 4,
$$

so $\{T_1, T_2, T_3, T_4\}$ forms a $T_4$ configuration. Next we construct a curve $\Pi = \Pi_\theta$ as in Proposition 5.1 which contains this $T_4$ configuration. Let $\theta_a = \arctan (1/(1 + a))$, so

$$
\cos \theta_a = \frac{1 + a}{r_a}, \quad \sin \theta_a = \frac{1}{r_a}, \quad r_a = \sqrt{1 + (1 + a)^2}.
$$

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth monotonically increasing function to be determined later that satisfies

- $\rho (\theta + 2\pi) = \rho (\theta) + 2\pi$ for $\theta \in \mathbb{R}$,
\[ \rho \left( \hat{\theta}_k \right) = \hat{\theta}_k = \theta_a + (k - 1)\frac{\pi}{2}, \quad k = 1, 2, 3, 4. \]

Define \( \Pi_a = \Gamma_a(\mathbb{R}/2\pi \mathbb{Z}) \), with

\[ \Gamma_a(\theta) := \begin{pmatrix} r_a \cos(\theta) & -\sin(8\theta - 8\theta_a) & \sin(6\rho(\theta) - 6\theta_a) \\ \sin(6\theta - 6\theta_a) & r_a \sin(\theta) & \sin(8\rho(\theta) - 8\theta_a) \\ r_a \cos(\theta) & \sin(8\theta - 8\theta_a) & \sin(6\rho(\theta) - 6\theta_a) \end{pmatrix} \]  \( (20) \)

Then we have

\[ T_k = \Gamma_a(\hat{\theta}_k) \quad \text{for} \quad k = 1, 2, 3, 4, \]

so \( \Pi_a \) contains the \( T_4 \) configuration \( \{ T_1, T_2, T_3, T_4 \} \). Next we adjust the parameter \( a > 0 \) and the function \( \rho \) in order to ensure that \( \Pi_a \) has no rank-one connection and is elliptic.

**Notation.** With the \( M_{i_1i_2,j_1j_2} \) minor we mean the determinant of the \( 2 \times 2 \) submatrix corresponding to the rows \( i_1, i_2 \) and columns \( j_1, j_2 \).

**Lemma 5.2.** If \( a > 0 \) is such that \( \theta_a / \notin \frac{\pi}{48} \mathbb{Z} \), the curve \( \Pi_a \) is elliptic, i.e. \( \text{Rank} \Gamma_a'(\theta) > 1 \) for all \( \theta \in \mathbb{R} \).

**Proof.** The derivative \( \Gamma_a' \) is given by

\[ \Gamma_a'(\theta) := \begin{pmatrix} -r_a \sin(\theta) & -8 \cos(8\theta - 8\theta_a) & 6\rho'(\theta) \cos(6\rho(\theta) - 6\theta_a) \\ 6 \cos(6\theta - 6\theta_a) & r_a \cos(\theta) & 8\rho'(\theta) \cos(8\rho(\theta) - 8\theta_a) \\ -r_a \sin(\theta) & 8 \cos(8\theta - 8\theta_a) & 6\rho'(\theta) \cos(6\rho(\theta) - 6\theta_a) \end{pmatrix} \]

Assume \( \text{Rank} \Gamma_a'(\theta) \leq 1 \) for some \( \theta \in \mathbb{R} \). Then calculating the \( M_{12,12} \) minor we have

\[ -r_a^2 \sin(\theta) \cos(\theta) + 48 \cos(8\theta - 8\theta_a) \cos(6\theta - 6\theta_a) = 0 \]

and calculating the \( M_{23,12} \) minor we have

\[ 48 \cos(8\theta - 8\theta_a) \cos(6\theta - 6\theta_a) + r_a^2 \sin(\theta) \cos(\theta) = 0. \]

Adding and subtracting these two equations we obtain that

\[ \sin(\theta) \cos(\theta) = 0 \quad \text{and} \quad \cos(8\theta - 8\theta_a) \cos(6\theta - 6\theta_a) = 0. \]

The first equality implies \( \theta \in \frac{\pi}{4} \mathbb{Z} \), and then the second equality becomes

\[ \cos(8\theta_a) \cos(6\theta_a) = 0, \]

which is impossible for \( \theta_a / \notin \frac{\pi}{48} \mathbb{Z} \).

\[ \square \]

**Lemma 5.3.** For any \( a > 0 \) such that \( \theta_a / \notin \frac{\pi}{24} \mathbb{Z} \), for any \( \epsilon > 0 \), there exists a smooth monotonic function \( \rho: \mathbb{R} \to \mathbb{R} \) with the following properties

- \( \rho(\theta + 2\pi) = \rho(\theta) + 2\pi \) for \( \theta \in \mathbb{R} \),
\[
\rho \left( \hat{t}_k \right) = \hat{t}_k \text{ for } t_k = \hat{t}_a + (k-1)\frac{\pi}{2}, \ k = 1, 2, 3, 4.
\]

- For any \( \theta, \theta' \in [0, 2\pi) \cap \frac{\pi}{24} \mathbb{Z} \), we have \( \rho(\theta) - \rho(\theta') \notin \frac{\pi}{12} \mathbb{Z} \).
- \( \sup_{x \in \mathbb{R}} |\rho(x) - x| < \epsilon \)

**Proof.** Let \( \delta = \frac{1}{2} \text{dist}(\theta_a, \frac{\pi}{24} \mathbb{Z}) = \frac{1}{2} \text{dist}(\{\hat{t}_k\}, \frac{\pi}{24} \mathbb{Z}) > 0 \) and fix a smooth function \( \varphi \) such that \( \text{supp} \varphi \subset (-\delta, \delta) \), \( 0 \leq \varphi \leq \varphi(0) = 1 \) and \( |\varphi'| \leq 2/\delta \). Define \( \rho \) on \([0, 2\pi)\) by setting

\[
\rho(\theta) = \theta + \sum_{j=1}^{47} t_j \varphi \left( \theta - j \frac{\pi}{24} \right) \quad \text{for } \theta \in [0, 2\pi),
\]

where \( t_1, \ldots, t_{47} \in (-\eta, \eta) \) are to be fixed later and \( \eta = \min(\epsilon, \delta/2) \). The choice of \( \delta > 0 \) ensures that \( \rho(\hat{t}_k) = \hat{t}_k \), also since \( |t_j| < \epsilon \) we have \( |\rho - \text{id}| < \epsilon \), and finally since \( |t_j| < \delta/2 \) we have \( \rho' > 0 \) on \([0, 2\pi)\). Moreover the function \( \rho - \text{id} \) is identically zero near 0 and \( 2\pi \), so it can be extended to a smooth \( 2\pi \) periodic function, thus yielding a smooth monotonic extension \( \rho: \mathbb{R} \to \mathbb{R} \) satisfying \( \rho(\theta + 2\pi) = \rho(\theta) + 2\pi \) for \( \theta \in \mathbb{R} \). It remains to argue that we can pick \( t_1, \ldots, t_{47} \in [0, \eta] \) to ensure that the third condition in Lemma 5.3 is satisfied.

Denote \( t_0 = 0 \). By induction, we may for each \( j = 1, \ldots, 47 \) choose \( t_j \in (-\eta, \eta) \) to ensure that

\[
\rho(j \pi/24) - \rho(\ell \pi/24) = t_j - t_\ell + (j - \ell)\pi/24 \notin \frac{\pi}{12} \mathbb{Z}
\]

for all \( \ell \in \{0, \ldots, j - 1\} \). This is possible because at each step there is only a discrete set of values of \( t_j \) to avoid. \( \square \)

**Lemma 5.4.** If \( a > 0 \) is such that \( \theta_a \notin \frac{\pi}{48} \mathbb{Z} \), \( \epsilon > 0 \) is small enough and \( \rho \) is as in Lemma 5.3, then the curve \( \Pi_a \subset \mathbb{R}^{3 \times 3} \) does not contain Rank-1 connections.

**Proof.** Note as in Lemma 5.2 that the assumption \( \theta_a \notin \frac{\pi}{48} \mathbb{Z} \) implies

\[
\cos(8\theta_a) \neq 0 \quad \text{and} \quad \cos(6\theta_a) \neq 0.
\]

We assume that there exist \( \theta \neq \theta' \in \mathbb{R}/2\pi \mathbb{Z} \) such that

\[
\text{Rank} \ (\Gamma_a(\theta) - \Gamma_a(\theta')) = 1,
\]

and we obtain a contradiction. We do this in several steps.

**Step 1.** We have

\[
\theta + \theta' \in \pi \mathbb{Z} \quad \text{and} \quad \theta - \theta' \in \frac{\pi}{3} \mathbb{Z} \cup \frac{\pi}{4} \mathbb{Z}.
\]

**Proof of Step 1.** From (20) calculating the \( M_{12, 12} \) minor we have

\[
0 = r_a^2 \left( \cos(\theta) - \cos(\theta') \right) \left( \sin(\theta) - \sin(\theta') \right)
\]

\[
+ \left( \sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a) \right) \left( \sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a) \right)
\]

\[
= -4r_a^2 \sin \left( \frac{\theta + \theta'}{2} \right) \cos \left( \frac{\theta + \theta'}{2} \right) \sin^2 \left( \frac{\theta - \theta'}{2} \right)
\]

\[
+ 4 \sin \left( 3(\theta - \theta') \right) \sin \left( 4(\theta - \theta') \right) \cos \left( 3(\theta + \theta') - 6\theta_a \right) \cos \left( 4(\theta + \theta') - 8\theta_a \right).
\]

(23)
And calculating the $M_{23,12}$ minor we have

$$0 = -r_a^2 \left( \cos(\theta) - \cos(\theta') \right) \left( \sin(\theta) - \sin(\theta') \right) + \left( \sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a) \right) \left( \sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a) \right)$$

$$= 4r_a^2 \sin \left( \frac{\theta + \theta'}{2} \right) \cos \left( 
\frac{\theta + \theta'}{2} \sin^2 \left( \frac{\theta - \theta'}{2} \right) 
\right) + 4 \sin \left( 3(\theta - \theta') \right) \sin \left( 4(\theta - \theta') \right) \cos \left( 3(\theta + \theta') - 6\theta_a \right) \cos \left( 4(\theta + \theta') - 8\theta_a \right). \quad (24)$$

Adding and subtracting $23$ and $24$, we obtain the equations

$$0 = \sin \left( 3(\theta - \theta') \right) \sin \left( 4(\theta - \theta') \right) \cos \left( 3(\theta + \theta') - 6\theta_a \right) \cos \left( 4(\theta + \theta') - 8\theta_a \right) \quad (25)$$

and

$$0 = \sin \left( \frac{\theta + \theta'}{2} \right) \cos \left( \frac{\theta + \theta'}{2} \sin^2 \left( \frac{\theta - \theta'}{2} \right) \right). \quad (26)$$

Since $\theta \neq \theta'$ in $\mathbb{R}/2\pi\mathbb{Z}$, the last factor of $26$ is nonzero, so either the first or the second must be zero. This implies $\theta + \theta' \in \pi\mathbb{Z}$. As a consequence, the last two factors in $25$ are equal to $\pm \cos(6\theta_a)$ and $\cos(8\theta_a)$ and are nonzero by our choice of $a$. So one of the first two factors of $25$ must vanish, that is, $\theta - \theta' \in \frac{\pi}{3}\mathbb{Z} \cup \frac{\pi}{4}\mathbb{Z}$.

**Step 2.** We have $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$.

**Proof of Step 2.** Considering the $M_{13,23}$ minor of $\Gamma_a(\theta) - \Gamma_a(\theta')$ we obtain the equation

$$0 = \left( \sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a) \right) \times \left( \sin(6\rho(\theta) - 6\theta_a) - \sin(6\rho(\theta') - 6\theta_a) \right)$$

$$= 4 \sin \left( 4(\theta - \theta') \right) \cos \left( 4(\theta + \theta') - 8\theta_a \right) \times \sin \left( 3(\rho(\theta) - \rho(\theta')) \right) \cos \left( 3(\rho(\theta) + \rho(\theta')) - 6\theta_a \right).$$

From Step 1 we have $\theta + \theta' \in \pi\mathbb{Z}$ so the second factor is $\cos(6\theta_a) \neq 0$. The last factor is arbitrarily close to $\pm \cos(6\theta_a) \neq 0$ since $|\rho - id| \leq \epsilon$. The third factor is nonzero by construction of $\rho$ because $\theta, \theta' \in \frac{\pi}{2\pi}\mathbb{Z}$ by $22$. So we must have $\sin(4(\theta - \theta')) = 0$ hence $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$.

**Step 3.** We have $\theta - \theta' \in \pi\mathbb{Z}$.

**Proof of Step 3.** Considering the $M_{12,13}$ minor of $\Gamma_a(\theta) - \Gamma_a(\theta')$ we obtain the equation

$$0 = r_a \left( \cos(\theta) - \cos(\theta') \right) \left( \sin(8\rho(\theta) - 8\theta_a) - \sin(8\rho(\theta') - 8\theta_a) \right)$$

$$- \left( \sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a) \right)$$

$$\times \left( \sin(6\rho(\theta) - 6\theta_a) - \sin(6\rho(\theta') - 6\theta_a) \right),$$

which we rewrite as

$$- 4r_a \sin \left( \frac{\theta + \theta'}{2} \right) \sin \left( \frac{\theta - \theta'}{2} \right)$$

$$\times \sin \left( 4(\rho(\theta) - \rho(\theta')) \right) \cos \left( 4(\rho(\theta) + \rho(\theta')) - 8\theta_a \right)$$

$$= 4 \sin \left( 3(\theta - \theta') \right) \cos \left( 3(\theta + \theta') - 6\theta_a \right)$$

$$\times \sin \left( 3(\rho(\theta) - \rho(\theta')) \right) \cos \left( 3(\rho(\theta) + \rho(\theta') - 6\theta_a) \right). \quad (27)$$
Since $\theta - \theta' \in \frac{\pi}{4} \mathbb{Z}$ and $|\rho - id| \leq \epsilon$, the third factor in the left-hand side of (27) has absolute value $\leq 4\epsilon$. Since $\theta + \theta' \in \pi \mathbb{Z}$, the second factor in the right-hand side of (27) has absolute value equal to $|\cos(6\theta_a)| > 0$, and for small enough $\epsilon$ the absolute value of the last factor is $\geq |\cos(6\theta_a)|/2 > 0$. Taking also into account that the first and third factor in the right-hand side of (27) differ from each other by an error $\leq 3\epsilon$, we must have

$$\sin^2(3(\theta - \theta')) \leq c_a \epsilon,$$

for some $c_a > 0$ depending only on $a$. Because $\theta - \theta' \in \frac{\pi}{4} \mathbb{Z}$ by Step 2, provided $\epsilon$ is chosen small enough, this implies $\theta - \theta' \in \frac{\pi}{2} \mathbb{Z} \cap \frac{3\pi}{4} \mathbb{Z} = \pi \mathbb{Z}$.

**Step 4: Conclusion.** From Step 1 and Step 3 we have $\theta + \theta', \theta - \theta' \in \pi \mathbb{Z}$, so $\theta, \theta' \in \pi \mathbb{Z}$. Since me may without loss of generality exchange the roles of $\theta$ and $\theta'$ and choose arbitrary representants in $\mathbb{R}/2\pi \mathbb{Z}$, this amounts to $\theta = 0$ and $\theta' = \pi$. Considering the $M_{12,13}$ minor of $\Gamma_a(0) - \Gamma_a(\pi)$ as in (27) we deduce

$$\sin(4(\rho(\pi) - \rho(0)) \cos(4(\rho(\pi) + \rho(0)) + 8\theta_a) = 0.$$

Using again that $|\rho - id| \leq \epsilon$, the second factor has absolute value $\geq |\cos(8\theta_a)|/2$ provided $\epsilon$ is small enough, and the first factor is nonzero by construction of $\rho$, so we conclude that (21) is not possible for $\theta \neq \theta' \in \mathbb{R}/2\pi \mathbb{Z}$.

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