Instantons in $G_2$ manifolds from $J$-holomorphic curves in coassociative submanifolds

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Abstract

In $G_2$ manifolds, 3-dimensional associative submanifolds (instantons) play a role similar to $J$-holomorphic curves in symplectic geometry. In [21], instantons in $G_2$ manifolds were constructed from regular $J$-holomorphic curves in coassociative submanifolds. In this exposition paper, after reviewing the background of $G_2$ geometry, we explain the main ingredients in the proofs in [21]. We also construct new examples of instantons.

1 Introduction

A $G_2$-manifold $M$ is a Riemannian manifold of dimension seven equipped with a nontrivial parallel 2-fold vector cross product (VCP) $\times$. If the VCP is 1-fold instead, namely a Hermitian complex structure, then the manifold is a Kähler manifold $X$. Note that every 2-fold VCP comes from the restriction of the algebra product on the octonions $\mathbb{O}$ or quaternions $\mathbb{H}$ to its imaginary part. Thus the only submanifolds in $M$ preserved by $\times$ are of dimension three and they are called instantons or associative submanifolds, which are analogs to holomorphic curves in Kähler manifolds.

In Physics, $G_2$-manifolds are internal spaces for compactification in M-theory in eleven dimensional spacetimes, similar to the role of Calabi-Yau threefolds $X$ in string theory in ten dimensional spacetimes. Instantons in string theory are holomorphic curves $\Sigma$ in $X$ with the natural boundary condition being $\partial \Sigma$ lies inside a Lagrangian submanifold $L$ in $X$. Very roughly speaking, the Fukaya category $\text{Fuk}(X)$ is defined by counting holomorphic disks with Lagrangian boundary condition. The homological mirror symmetry (HMS) conjecture of Kontsevich says that $\text{Fuk}(X)$ is equivalent to the derived category $D^b(X^\vee)$ of coherent sheaves of the mirror manifold $X^\vee$. The proof of this conjecture and its generalizations in many specific cases relies on the work of Fukaya-Oh [8] when $X = T^*L$ and the Lagrangian $L_1$ is the graph of an exact one form $df$ scaled

\footnote{Throughout this paper $M$ is a $G_2$-manifold and $X$ is a symplectic manifold possibly with extra structures, like Kähler or CY.}
by small \( t \in \mathbb{R} \). They showed that holomorphic disks with boundary in \( L \cup L_t \) one to one correspond to gradient flow lines of \( f \), which are in fact instantons in quantum mechanics according to Witten’s Morse theory [31]. (Fukaya-Oh [8] actually proved the case of \( k \) Lagrangians \( L_j = t df_j \) \((j = 1, 2, \cdots, k)\), where holomorphic polygons bounding on these Lagrangians correspond to gradient flow trees of the Morse functions \( \{f_j\}_{j=1}^k \).

In [21] we proved a corresponding result for instantons in any \( G_2 \)-manifold with boundary in the coassociative submanifold \( C \cup C_t \). Here the family \( C_t \) is constructed by a self-dual harmonic two form \( \omega \) on the four manifold \( C \). We assume that \( \omega \) is non-degenerate, thus defining an almost complex structure \( J \) on \( C \). Our main result Theorem 13 gives a correspondence between such instantons in \( M \) and \( J \)-holomorphic curves \( \Sigma \) in \( C \). Thus the number of instantons in \( M \) is related to the Seiberg-Witten invariant of \( C \) by the celebrated work of Taubes on \( GW = SW \) ([26], [28], [27]). We suspect this holds true without the need of the non-degeneracy of \( \omega \).

Our result is similar to the \( k = 2 \) case of Fukaya-Oh [8], however the analysis involved in the proof is essentially different from theirs in the following 3 aspects.

1. The instanton equation is on 3-dimensional domains, and there is no analogous way of finding associative submanifold by constructing associative maps as was done in constructing \( J \)-holomorphic curves thanks to the conformality of Cauchy-Riemann equation. So we have to deform submanifolds rather than maps as in the Lagrangian Floer theory. A good choice of normal frames of submanifolds turns out to be essential.

2. The instanton equation is more nonlinear than the Cauchy-Riemann equation for \( J \)-holomorphic curves. It is a first order PDE system involving cubic terms \( \frac{\partial V_i}{\partial x_1} \frac{\partial V_j}{\partial x_2} \frac{\partial V_k}{\partial x_3} \), while the Cauchy-Riemann operator \( \frac{\partial u}{\partial \tau} + J(u) \frac{\partial u}{\partial t} \) has no product of derivative terms. Consequently, the needed quadratic estimate appears to be unavailable in the \( W^{1,p} \) setting, as for \( V^i, V^j, V^k \in W^{1,p}, \frac{\partial V_i}{\partial x_1} \frac{\partial V_j}{\partial x_2} \frac{\partial V_k}{\partial x_3} \notin L^p \) in general. So instead we use the Schauder \((C^{1,\alpha})\) setting, which creates new complications (Subsection 3.2.4).

3. The linearized instanton equation is more weakly coupled than the Cauchy-Riemann equation. It is a Dirac type equation for spinor \((u, v) \in S^+ \oplus S^-\) where \( u \) and \( v \) play the role of the real and imaginary parts in Cauchy-Riemann equations, but the interrelation between \( \nabla u \) and \( \nabla v \) becomes weaker. This causes several difficulties in the \( C^{1,\alpha} \) estimates (See comments below (14)).

Besides above difficulties, our domains are \([0, \varepsilon] \times \Sigma\) for compact Riemannian surfaces \( \Sigma \), and they collapse to \( \Sigma \) as \( \varepsilon \to 0 \), causing lack of uniform ellipticity, which in turn creates difficulty to obtain a uniform right inverse bound needed in gluing arguments. This also occurs in Proposition 6.1 of Fukaya-Oh [8] in the \( W^{1,p} \) setting, but in our \( C^{1,\alpha} \) setting the boundary estimates become more subtle.
To deal with these difficulties, our paper \cite{21} becomes rather technical. Therefore in this article we give an outline of the main arguments. The organization is as follows. In Section \ref{sec:background} we review the background of $G_2$ geometry, instantons, coassociative boundary condition and give the motivations of counting instantons. In Section \ref{sec:main_theorem} we state the main theorem in \cite{21} and explain the main ingredients in the proof. In Section \ref{sec:examples} we apply the theorem to construct new examples of instantons and discuss possible generalizations.

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## 2 Review on $G_2$-geometry

### 2.1 $G_2$ manifolds

$G_2$ manifolds are 7-dimensional Riemannian manifolds $(M, g)$ with a parallel vector cross product (VCP) $\times$, i.e. for $u, v \in T_pM$ there is a VCP $\times$ such that

1. $u \times v$ is perpendicular to both $u$ and $v$.
2. $|u \times v| = |u \wedge v| = \text{Area of the parallelogram spanned by } u \text{ and } v$,

and $\times$ is invariant under the parallel transport of the Levi-Civita connection $\nabla$.

In the octonion algebra $\mathbb{O}$, we can construct a VCP $\times$ on $\text{Im} \mathbb{O}$ as follows:

$$u \times v = \text{Im} (\overline{v} u),$$

where $\overline{v}$ is the conjugate of $v$. The same formula for $\mathbb{H}$ gives another VCP on $\text{Im} \mathbb{H}$, which is indeed the useful vector product on $\mathbb{R}^3$. Together, they form the complete list of VCP because the normed algebra structures can be recovered from the VCP structures. In particular,

$$\text{Aut} \left( \mathbb{R}^7, \times \right) = \text{Aut}_{\text{alg}} (\mathbb{O}).$$

Hence a $G_2$ manifold is simply a 7-dimensional Riemannian manifold $(M, g)$ with holonomy group inside the exceptional Lie group $G_2 = \text{Aut}_{\text{alg}} (\mathbb{O})$.

Equivalently, $G_2$ manifolds are 7-dimensional Riemannian manifolds $(M, g)$ with a nondegenerate 3-form $\Omega$ such that $\nabla \Omega = 0$. The relation between $\Omega$ and $\times$ is

$$\Omega (u, v, w) = g (u \times v, w), \quad \text{for } u, v, w \in T_pM.$$

**Example 1 (Linear case)** The $G_2$ manifold $\text{Im} \mathbb{O} \simeq \mathbb{R}^7$: Let

$$\mathbb{R}^7 \simeq \text{Im} \mathbb{O} \simeq \text{Im} \mathbb{H} \oplus \mathbb{H} = \{(x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}, x_4 + x_5 \mathbf{i} + x_6 \mathbf{j} + x_7 \mathbf{k})\},$$
and the standard basis $e_i = \frac{\partial}{\partial x^i}$, $i = 1, 2, \cdots, 7$. The vector cross product $\times$ for $u, v \in \text{Im} \mathcal{O}$ is defined by $\mathbf{(1)}$. The standard $G_2$ 3-form $\Omega = \Omega_0$ is

$$\Omega_0 = \omega^{123} - \omega^{167} - \omega^{527} - \omega^{563} - \omega^{154} - \omega^{264} - \omega^{374},$$

where $\omega^{ijk} = dx^i \wedge dx^j \wedge dx^k$.

**Example 2** (Product case) $M = X \times S^1$ is a $G_2$ manifold if and only if $X$ is a Calabi-Yau threefold. The $G_2$ 3-form $\Omega$ of $X$ is related to the holomorphic volume form $\Omega_X$ and the Kähler form $\omega_X$ of $X$ as follow

$$\Omega = \text{Re} \Omega_X + \omega_X \wedge d\theta,$$

where $d\theta$ is the standard angular-form on $S^1$.

So far all compact irreducible $G_2$ manifolds are constructed by solving non-linear PDE of the $G_2$ metric using the implicit function theorem, including (i) resolving orbifold singularities by Joyce [17], [18] and (ii) twisted connected sum by Kovalev [7] and Corti-Haskins-Nordstrom-Pancini [19].

We remark a useful construction of local $G_2$ frames.

**Remark 3** (Cayley-Dickson construction) A convenient basis in $\text{Im} \mathcal{O}$ can be constructed inductively (Lemma A.15 in [13]): Given orthogonal unit vectors $v_1$ and $v_2$, we let $v_3 := v_1 \times v_2$; Then we take any unit vector $v_4 \perp \text{span}\{v_1, v_2, v_3\}$, and let $v_i+4 := v_i \times v_4$ for $i = 1, 2, 3$. Then $\{v_i\}_{i=1}^7$ is an orthonormal basis of $\text{Im} \mathcal{O}$ satisfying the same $\times$ relation as the standard basis $\{e_i\}_{i=1}^7$ in $\text{Im} \mathcal{O}$.

This basis $\{v_i\}_{i=1}^7$ is useful in many local calculations in [21] (see also [3]), where $\{v_1, v_2\}$ span the tangent spaces of a $J$-holomorphic curve.

### 2.2 Instantons (associative submanifolds)

A 3-dimensional submanifold $A$ in a $G_2$ manifold $M$ is called an **associative submanifold** (or an **instanton**) if its tangent space $TA$ is close under $\times$. This notion was introduced by Harvey-Lawson ([13], see also [20]). The following are two other equivalent conditions for $A$ to be an associative submanifold: (1) $A$ is calibrated by $\Omega$, namely $\Omega|_A = dv_A$; (2) $\tau|_A = 0 \in \Omega^3(A, T_M|_A)$ where $\tau \in \Omega^3(M, TM)$ is defined by the following equation

$$g(\tau(u,v,w),z) = (\ast \Omega)(u,w,w,z)$$

for any $u, v, w$ and $z \in T_p M$, where $\ast \Omega \in \Omega^4(M)$ is the Hodge-$\ast$ of $\Omega$. This measurement $\tau|_A$ of associativity is important for perturbing almost instantons to an instanton.
Example 4 In $\text{Im } \mathbb{O} \cong \text{Im } \mathbb{H} \oplus \mathbb{H}$, the subspace $\text{Im } \mathbb{H} \oplus \{0\}$ is an instanton. Explicitly $\tau$ is \((5.4)\) in [24]

$$\tau = (\omega^{256} - \omega^{247} + \omega^{346} - \omega^{357}) \partial_1 + (\omega^{156} - \omega^{147} - \omega^{345} + \omega^{367}) \partial_2$$

$$+ (\omega^{245} - \omega^{267} - \omega^{146} - \omega^{157}) \partial_3 + (\omega^{567} - \omega^{127} + \omega^{136} - \omega^{235}) \partial_4$$

$$+ (\omega^{126} - \omega^{456} - \omega^{137} + \omega^{234}) \partial_5 + (\omega^{457} - \omega^{125} - \omega^{134} + \omega^{237}) \partial_6$$

$$+ (\omega^{124} - \omega^{456} - \omega^{135} - \omega^{236}) \partial_7,$$

where $\partial_i = \frac{\partial}{\partial x^i}$. For a map $V = (V^4, V^5, V^6, V^7) : \text{Im } \mathbb{H} \to \mathbb{H}$, we let

$$A = \text{graph}(V) = \{(x_1, x_2, x_3, V(x_1, x_2, x_3)) | (x_1, x_2, x_3) \in \text{Im } \mathbb{H}\}.$$

By (3), the condition $\tau|_A = 0$ becomes $V^*\tau = 0$ in coordinates $(x_1, x_2, x_3)$, and we see the instanton equation for $A$ is a first order cubic PDE system involving product terms $\frac{\partial V^i}{\partial x^3} \frac{\partial V^j}{\partial x^2} \frac{\partial V^k}{\partial x^1}$.

Example 5 For any holomorphic curve $\Sigma$ in a Calabi-Yau manifold $X$, $\Sigma \times S^1$ is an instanton in the $G_2$ manifold $X \times S^1$.

The deformations of an instanton $A$ in a $G_2$ manifold $M$ are governed by a twisted Dirac operator on its normal bundle $N_{A/M}$, regarded as a Clifford module over $\mathcal{T}A$ with $\times$ as the Clifford multiplication.

Theorem 6 ([23]) Infinitesimal deformations of instantons at $A$ are parameterized by the space of harmonic twisted spinors on $A$, i.e. kernel of the twisted Dirac operator.

Example 7 For the instanton $\text{Im } \mathbb{H} \oplus \{0\} \subset \text{Im } \mathbb{O}$ in Example 4 regarding sections $V = V^4 + jV^6 + kV^7$ in its normal bundle as $\mathbb{H}$-valued functions, then the twisted Dirac operator is $\mathcal{D} = i\nabla_1 + j\nabla_2 + k\nabla_3$, and

$$\mathcal{D}V = -(V_1^5 + V_2^5 + V_3^7) + i(V_1^4 + V_3^5 - V_1^7)$$

$$+ j(V_2^4 - V_3^5 + V_1^7) + k(V_3^4 + V_2^5 - V_1^7).$$

Remark 8 Constructing instantons in $G_2$ manifolds in general is difficult, partly because the deformation theory can be obstructed. Our main theorem in [24] provides a construction of instantons from $J$-holomorphic curves in coassociative submanifolds. Other constructions are in [7].

2.3 Boundary value problem on coassociative submanifolds

Coassociative submanifolds are analogues of Lagrangian submanifolds in symplectic geometry.
Definition 9 A 4-dimensional submanifold $C$ in a $G_2$ manifold $(M, g, \Omega)$ is called coassociative if $\Omega|_C = 0$. Equivalently, $C$ is calibrated by $\ast \Omega$.

Example 10 When $M = X \times S^1$ is a product with $X$ a Calabi-Yau threefold, then (i) $C = L \times S^1$ is coassociative if and only if $L \subset X$ is a special Lagrangian submanifold $L$ of phase $\pi/2$ and (ii) $C = S \times \{ \theta \}$ is coassociative if and only if $S \subset X$ is a complex surface.

Deformations of a coassociative submanifold $C$ is studied by McLean \[24\]: Infinitesimally, they are parameterized by $H^2_+ (C)$, the space of self-dual harmonic 2-forms on $C$, and they are always unobstructed.

Finding instantons with boundaries on coassociative submanifolds is a elliptic problem, similar to finding $J$-holomorphic curves with boundaries on Lagrangian submanifolds in symplectic geometry.

Theorem 11 (Theorem 4.2, \[10\]) The linearization of the instanton equation on an instanton $A$ with boundaries $\Sigma$ lying on a coassociative submanifold $C$ is an elliptic Fredholm operator, with the Fredholm index given by

$$\int_{\Sigma} c_1 (N_{\Sigma/C}) + 1 - g(\Sigma).$$

We have the following orthogonal decomposition of $TM$ along $\Sigma$,

$$TM|_{\Sigma} = \mathbb{R} \langle n \rangle \oplus T\Sigma \oplus N_{\Sigma/C} \oplus N_{(C,n)/M}|_{\Sigma},$$

where $n$ be the unit inward normal vector field of $\Sigma$ in $A$. We call $N_{\Sigma/C}$ the “intrinsic” normal bundle of $\Sigma$ in $C$ and $N_{(C,n)/M}$ the “extrinsic” normal bundle defined as the orthogonal complement of $TC \oplus \langle n \rangle$ in $TM$. In \[10\], Gayet-Witt showed that (1) $T\Sigma, N_{\Sigma/C}$ and $N_{(C,n)/M}|_{\Sigma}$ are complex line bundles with respect to the almost complex structure $J_n := n \times$ and (2) $N_{\Sigma/C} \otimes C^{0,1}(T^*\Sigma) \simeq N_{(C,n)/M}^*$ as complex line bundles over $\Sigma$ which is obtained by changing the tensor product $\otimes$ to the vector cross product $\times$, and used metric to identify conjugate bundle with dual bundle. This relationship between intrinsic and extrinsic geometry of $\Sigma$ in $C$ is needed \[21\] in order to show two natural Dirac operators on $\Sigma$ agree which is needed in section 3.2.5 for the proof of our main theorem.

The following table gives an interesting comparison between $G_2$-geometry and symplectic geometry.
2.4 Intersection theory?

Intersection theory of Lagrangian submanifolds is an essential part of symplectic geometry. By counting the number of holomorphic disks bounding intersecting Lagrangian submanifolds, Floer and others defined the celebrated Floer homology theory. It plays an important role in mirror symmetry for Calabi-Yau manifolds and string theory in physics.

In M-theory, Calabi-Yau threefolds are replaced by $G_2$-manifolds. The analogs of holomorphic disks (resp. Lagrangian submanifolds) are instantons or associative submanifolds (resp. coassociative submanifolds or branes) in $M^{[20]}$. The problem of counting instantons has been discussed by many physicists. For example Harvey and Moore $^{[14]}$ discussed the mirror symmetry aspects of it; Aganagic and Vafa $^{[2]}$ related it to the open Gromov-Witten invariants for local Calabi-Yau threefolds; Beasley and Witten $^{[4]}$ argued that one should count instantons using the Euler characteristic of their moduli spaces.

On the mathematical side, the compactness issues of the moduli of instantons is a very challenging problem because the bubbling-off phenomena of (3-dimensional) instantons is not well understood. This makes it very difficult to define an honest invariant by counting instantons. Note the Fredholm theory for instantons with coassociative boundary conditions has been set up $^{[10]}$.

In symplectic geometry, Fukaya and Oh $^{[8]}$ considered two nearby Lagrangian submanifolds $L$ and $L_t$, where $L_t$ is the graph of a closed 1-form $\alpha$ on $L$ scaled by a small $t$, then holomorphic disks bounding them with small volume is closely related to gradient flow lines of $\alpha$. They actually deal with $J$-holomorphic polygons bounding $k$ Lagrangians and need to smooth vertex singularities on gradient flow trees when $k \geq 3$. For simplicity we only state their result for the $k = 2$ case here.

Given a closed 1-form $\alpha$ on $L$, $L_t := \text{graph}(t\alpha)$ is a Lagrangian submanifold in $T^*L$ near the zero section $L$. By Weinstein’s neighborhood theorem, a small tubular neighborhood of a Lagrangian $L$ in a symplectic manifold can be identified as a tubular neighborhood of $L$ in $T^*L$.

**Theorem 12** ($^{[8]}$) For any compact Riemannian manifold $(L, g)$, $T^*L$ has a natural almost Kähler structure $(\omega, J_g)$. Let $L_t := \text{graph}(t\alpha)$ be a Lagrangian
submanifold in $T^*L$ for a closed 1-form $\alpha$ on $L$. There is a constant $\varepsilon_0 > 0$, such that for any $t \in (0, \varepsilon_0]$, there is a 1-1 correspondence between $J_n$-holomorphic curves bounding $L \cup L_t$ and gradient flow lines of $\alpha$ on $L$.

We want to build the following analogue: instantons $A$ bounding $C \cup C_t$ have 1-1 correspondence to $J_n$-holomorphic curves $\Sigma$ on $C$.

The meaning of $J_n, C_t$ and the precise statement of our result will be explained in the next section.

### 3 Main Theorem: instantons from $J$-holomorphic curves

#### 3.1 Statement of the main theorem

Let $C = \bigcup_{0 \leq t \leq C_t}$ be a family of coassociative manifolds $C_t$ in a $G_2$-manifold $M$, regarded as a deformation of $C = C_0$ along the normal vector field $n := \frac{dC_t}{dt}|_{t=0}$. Then $\iota_n \Omega$ is a self-dual harmonic 2-form on $C$ by McLean’s Theorem on deformations of coassociative submanifolds (Section 2.3). In particular $\omega_n := \iota_n \Omega$ defines a symplectic structure on $(C \setminus n^{-1}(0), \omega_n)$ as $\omega_n \wedge \omega_n = \star \omega_n \wedge \omega_n = |\omega_n|^2 dv_C$ is nonzero outside $(n = 0)$. Furthermore $J_n := \frac{n}{|n|} \times$ defines a compatible almost complex structure on $(C \setminus n^{-1}(0), \omega_n)$. When $n$ has no zero, we have the following main theorem in [21]

**Theorem 13** Suppose that $(M, \Omega)$ is a $G_2$-manifold and $\{C_t\}$ is an one-parameter smooth family of coassociative submanifolds in $M$. When $\iota_n \Omega \in \Omega^2_+(C_0)$ is non-vanishing, then

1. (Proposition 6) If $\{A_t\}$ is any one-parameter family of associative submanifolds (i.e. instantons) in $M$ satisfying

   $\partial A_t \subset C_t \cup C_0, \quad \lim_{t \to 0} A_t \cap C_0 = \Sigma_0$ in the $C^1$-topology,

   then $\Sigma_0$ is a $J_n$-holomorphic curve in $C_0$.

2. (Theorem 24) Conversely, every regular $J_n$-holomorphic curve $\Sigma_0$ (namely those for which the linearization of $\overline{\partial} J_n$ on $\Sigma_0$ is surjective) in $C_0$ is the limit of a family of associative submanifolds $A_t$’s as described above.

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\(^2\)Here $\iota_n$ is the contraction of a form by the vector field $n$. 

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Our results are similar to those in Fukaya-Oh [8] and the proofs also share some similarities: relating the Fredholm regular property of higher dimensional linearized instanton equations to lower dimensional ones; necessity to deal with the lack of uniform ellipticity as the domain collapses when \( \epsilon \to 0 \); using the periodic reflection technique to “thicken” the collapsing domain to achieve a uniform right inverse estimate in the \( W^{1,p} \) setting.

However the proof in our case has more difficulties than those needed in the \( k = 2 \) case of [8], as explained in the introduction. For \( k \geq 3 \), [8] contains difficulties we have not encountered here: to find the local models of the singularities of degenerating \( J \)-holomorphic polygons and resolve them.

Remark 14: Given any Riemann surface \( \Sigma \subset M \), it can always be thickened to a instanton by the Cartan-Kähler theory ([13], [10]). However its boundary may not lie inside any coassociative submanifold (see [10]). In our case, we produce an instanton by the Cartan-Kähler theory ([13], [10]). However its boundary may not lie inside any coassociative submanifold (see [10]). In our case, we produce an instanton \( A_\epsilon \) with boundary in the coassociative submanifold \( C \cup C_\epsilon \), but \( \partial A_\epsilon \cap C \) is only close but not equal to \( \Sigma \).

3.2 Main ingredients of the proof

3.2.1 Formulating the instanton equation near an almost instanton

We first produce an almost instanton with boundaries on \( C_0 \cup C_\epsilon \). Let \( \phi : [0, \epsilon] \times C \to M \) be a parametrization of the family of coassociative submanifolds \( \{C_t\}_{0 \leq t \leq \epsilon} \). Under the assumptions that \( n = \frac{dC_0}{dt}|_{t=0} \) is nonvanishing and \( \Sigma \) is a \( J_n \)-holomorphic curve in \( C = C_0 \), we define

\[
A_\epsilon := [0, \epsilon] \times \Sigma, \quad \text{and} \quad A_\epsilon' = \phi(A_\epsilon) \subset M,
\]

then \( A_\epsilon' \) is an almost instanton with \( \partial A_\epsilon' \subset C_0 \cup C_\epsilon \) in the sense that \( |\tau|_{A_\epsilon'}|_{C_0} \) is small. Recall that a 3-dimensional submanifold \( A \subset M \) is an instanton if and only if \( |\tau|_A = 0 \), where \( \tau \in \Omega^3(M, TM) \) is defined in (2). The reason for the smallness is that, at \( p \in \phi(\{0\} \times \Sigma) \subset A_\epsilon' \), \( T_pA_\epsilon' \) is associative by the \( J_n \)-holomorphic property of \( \Sigma \) so \( \tau|_{T_pA_\epsilon'} = 0 \), and any point \( q \) on \( A_\epsilon' \) has \( \varepsilon \)-order distance to \( \phi(\{0\} \times \Sigma) \) while \( \tau|_{T_qA_\epsilon'} \) smoothly depends on \( q \).

Next we formulate \( \tau|_A \) as a nonlinear map \( F_\epsilon \) on the space \( \Gamma(A_\epsilon', N_{A_\epsilon'/M}) \) of sections of the normal bundle \( N_{A_\epsilon'/M} \) of \( A_\epsilon' \), in particular \( F_\epsilon(V) = 0 \) if and only if

\[
\kappa_\epsilon(V) := (\exp V)(k_\epsilon') \quad (5)
\]

is an instanton where \( \exp V : k_\epsilon' \to M \).

To do this, we let

\[
C^\alpha(A_\epsilon', N_{k_\epsilon'/M}) = \{ V \in \Gamma(A_\epsilon', N_{k_\epsilon'/M}) \mid V \in C^\alpha \},
\]

\[
C^{1,\alpha}_{-}(A_\epsilon', N_{k_\epsilon'/M}) = \{ V \in \Gamma(A_\epsilon', N_{k_\epsilon'/M}) \mid V \in C^{1,\alpha}, V|_{\partial k_\epsilon'} \subset TC_0 \cup TC_\epsilon \},
\]

(\( "-" \) in \( C^{1,\alpha}_{-} \) is for the coassociative boundary condition), and

\[
F_\epsilon : C^{1,\alpha}_{-}(A_\epsilon', N_{k_\epsilon'/M}) \to C^\alpha(A_\epsilon', N_{k_\epsilon'/M})
\]

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\[ F_\varepsilon (V) = *_{A'} \circ \bot_{A'} \circ (T_V \circ (\exp V)^* \tau), \]

where

1. \((\exp V)^* \) pulls back the differential form part of the tensor \(\tau\)
2. \(T_V : T_{\exp_p (tV)} M \to T_p M\) pulls back the vector part of \(\tau\) by parallel transport along the geodesic \(\exp_p (tV)\)
3. \(\bot_{A'} : TM |_{A'} \to N_{A'}/M\) is the orthogonal projection with respect to \(g\)
4. \(*_{A'} : \Omega^3 (A') \to \Omega^0 (A')\) is the quotient by the volume form \(d\text{vol}_{A'}\)

If \(A'\) is an almost instanton, then a \(G_2\)-linear algebra argument shows that when \(\|V\|_{C^1,\alpha} (A', N_{A'}/M)\) is small we have

\[ F_\varepsilon (V) = 0 \iff A_\varepsilon (V) \text{ is an instanton.} \]

To ensure that \(A_\varepsilon (V)\) satisfies the boundary condition, in the definition of \(\exp V\) we actually need to modify the metric \(g\) near \(C_0 \cup C_\varepsilon\) to make them totally geodesic, but we will keep the original metric in \(T_V\), \(\bot_{A'}\) and \(*_{A'}\). This modification will not change the expression of \(F'_\varepsilon (0)\) \(\tag{6}\) (see Remark 10 (1) in \cite{21}). So our estimate for \(\|F'_\varepsilon (0)^{-1}\|\) is still valid in the new metric.

### 3.2.2 Linearizing the instanton equation using a good frame

We can make \(F_\varepsilon (V)\) more explicit by using a good local frame field \(\{W_\alpha\}_{\alpha=1}^7\) near \(p \in A'_\varepsilon = A_\varepsilon (0)\). Since \(\tau\) is a vector-valued 3-form, following Einstein’s summation convention, we can write

\[ \tau = \omega^\alpha \otimes W_\alpha \in \Omega^3 (M, TM), \]

where local 3-forms \(\{\omega^\alpha\}_{\alpha=1}^7\) are determined by \(\{W_\alpha\}_{\alpha=1}^7\). For any \(q = \exp_p V \in A_\varepsilon (V)\), we let

\[ F (V) (p) = (\exp V)^* \omega^\alpha (p) \otimes T_V W_\alpha (q). \]

\(F (V)\) is the essential part of \(F_\varepsilon (V)\) as \(*_{A'} \circ \bot_{A'}\) is only a projection. It is independent on the choice of frame \(\{W_\alpha\}_{\alpha=1}^7\). To compute \(F' (0)\), we have the following generalized Cartan formula \(\tag{6}\):

\[ F' (0) V (p) = L_V \omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha \]
\[ = d (i_V \omega^\alpha) \otimes W_\alpha + i_V d\omega^\alpha (p) \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha (p). \]
\[ \text{symbol part} \quad \text{0-th order part} \]
\[ = d (i_V \omega^\alpha) \otimes W_\alpha (p) + B_\alpha (p) V \otimes W_\alpha + \omega^\alpha \otimes C_\alpha (p) V \]
\[ = D_V \otimes \text{vol}_{A'_\varepsilon} + E (p) (V) \tag{7} \]
where $B\alpha$, $C\alpha$ and $E$ are certain matrix-valued functions.

We require a “good” frame field $\{W_\alpha\}^7_{\alpha=1}$ to satisfy the following conditions in any small $\varepsilon$-ball around $p$:

1. $B\alpha$, $C\alpha$ and $E$ are of $\varepsilon$-order in $C^1$ norm,
2. $\omega^\alpha$'s are $\varepsilon$-close to $\omega^{ijk}$'s in (3) for $\mathbb{R}^7$ in $C^1$ norm, in the sense that $\omega^{ijk}$ are replaced by $W_\alpha^* \wedge W_\beta^* \wedge W_\gamma^*$, where $W_\alpha^*$ is the dual vector of $W_\alpha$,
3. $DV(p)$ is $\varepsilon$-close to the twisted Dirac operator (4) for $\mathbb{R}^7$ in $C^1$ norm, in the sense that $\frac{\partial}{\partial x_i}$ in (4) are replaced by $\nabla_{W_i}$.

Condition 1 holds when $\{W_\alpha\}^7_{\alpha=1}$ is parallel along the normal bundle directions. Conditions 2 holds if whenever $e_\alpha = e_\alpha \times e_{\beta}$ in $\text{Im} \Sigma$ we have $\|W_\alpha - W_\alpha \times W_\beta\|_{C^1} = O(\varepsilon)$. Condition 3 holds if we further have the normal covariant derivatives $\|\nabla_{W_i} W_k\|_{C^1} = O(\varepsilon)$ assuming that $W_i$'s span $TA'_\varepsilon$ and $W_k$’s span $NA'/M$.

Such a good frame $\{W_\alpha\}^7_{\alpha=1}$ can be constructed by the Cayley-Dickson construction as explained in Remark 3. The principal part of the linearized instanton equation $F'(0)V$ on $A'_\varepsilon$ is the term $DV$ in (4), which is a first order differential operator with a nice geometric meaning (see the next subsection).

### 3.2.3 A simplified model: Dirac operators on thin manifolds

We temporarily leave $G_2$ geometry and consider a (“thin” when $\varepsilon$ is small) 3-manifold

$$A_\varepsilon := [0, \varepsilon] \times \Sigma = \{(x_1, z := x_2 + ix_3)\},$$

with the warped product metric

$$g_{A_\varepsilon, h} = h(z) dx_1^2 + g_\Sigma,$$

where $h(z) = |n|^2 > 0$ and $n = \frac{dC_\Sigma}{dt}|_{t=0}$ is the nonvanishing normal vector field on $C_0$. This is a first order approximation of the induced metric on $A'_\varepsilon \subset M$.

We first consider the geometry of a $J_n$-holomorphic curve $\Sigma$ in $C$. Let $L = N_{\Sigma/C}$ be the normal bundle of $\Sigma$ in $C$, then $L$ is a Hermitian $J_n$-complex line bundle over $\Sigma$ (see Proposition 24). Let $\tilde{\bar{\partial}} = \left(\bar{\partial}, \bar{\partial}\right)$ be the Dirac operator on the Dolbeault complex $\Omega^0_\Sigma(L) \oplus \Omega^{0,1}_\Sigma(L)$ of the spinor bundle of $\Sigma$

$$S_\Sigma = S_\Sigma^+ \oplus S_\Sigma^- = L \oplus \Lambda^{0,1}_\Sigma(L)$$

such that

$$\Omega^0_\Sigma(L) \oplus \Omega^{0,1}_\Sigma(L) \xrightarrow{\tilde{\bar{\partial}}, \bar{\partial}} \Omega^{0,1}_\Sigma(L) \oplus \Omega^0_\Sigma(L),$$

(8)
holomorphic curve $\Sigma$ in $\mathbb{C}$ we use the notation $\partial = \overline{\partial} - \partial$ for the Dirac Dolbeault operator on $A$ of $V$, the sections of $S$, ... tangent vector field on $A$ where $\bar{\partial}$ acts on $S$ (in the sense of [5]):

$$\partial^+ = -\bar{\partial} \partial$$ and $\partial^- = \bar{\partial} \partial$.

To describe the spinor bundle $S$ over the 3-manifold $A$, we pullback $S_\Sigma = S_\Sigma^+ \oplus S_\Sigma^-$ to $A_\varepsilon = [0, \varepsilon] \times \Sigma$, and denote it as $S = S^+ \oplus S^-$. Let $e_1$ be the unit tangent vector field on $A_\varepsilon$ along $x_1$-direction. For $S$ to be the spinor bundle of $A_\varepsilon$, the Clifford multiplication with $e_1$ should be $\pm i$ on $S^\pm$. To describe the Dirac operator on $S$, we define

$$\mathcal{D} = e_1 \cdot h^{-\frac{1}{2}}(z) \frac{\partial}{\partial x_1} + \bar{\partial},$$

where $\bar{\partial} = \left( \partial, \bar{\partial} \right)$ is the Dirac Dolbeault operator in equation (S). $\mathcal{D}$ acts on the sections $V = (u, v)$ of $S = S^+ \oplus S^-$ over $A_\varepsilon$ with local elliptic boundary condition (in the sense of [5]):

$$v|_{\partial A_\varepsilon} = 0.$$

We can write a local expression of $\mathcal{D}$. Consider the section $V = (u, v)$ of $S$ with $u = V^4 + iV^5 \in S^+$ and $v = V^6 + iV^7 \in S^-$ (for $S^\pm$ are complex line bundles). Then $e_1 \cdot = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and

$$DV = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} h^{-\frac{1}{2}}(z) \frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & i\partial_z \\ i\bar{\partial}_z & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

$$= \left(h^{-\frac{1}{2}}(z) \frac{\partial u}{\partial x_1} - \partial_z v\right) + \left(-h^{-\frac{1}{2}}(z) \frac{\partial v}{\partial x_1} + i\bar{\partial}_z u\right) \cdot j \quad (9)$$

where $\partial_z := \nabla_2 + i\nabla_3$ and $\bar{\partial}_z := \nabla_2 - i\nabla_3$.

On can check that $\mathcal{D}$ agrees with the linearized instanton equation on $\{0\} \times \Sigma$, and on $A_\varepsilon$, they are very close (Subsection 3.2.4). This is why we can use $\mathcal{D}$ of the linear model to study deformations of instantons in $G_2$ manifolds. The precise comparison is in Subsection 3.2.5 and Subsection 3.2.6.

### 3.2.4 Key estimates of $D^{-1}$ of the linear model

The most difficult part of [21] is to derive an explicit $\varepsilon$-dependent bound of the operator norm of $D^{-1} : C^\alpha(A_\varepsilon, S) \to C^{1,\alpha}(A_\varepsilon, S)$. This is nontrivial because we lose uniform ellipticity as $A_\varepsilon = [0, \varepsilon] \times \Sigma$ collapses to $\Sigma$ and consequently $\|D^{-1}\|$ may blow up. A good control of $\|D^{-1}\|$ is crucial for singular perturbation problems in general. We could achieve this in our case roughly because of

1. The Fredholm regularity property of the $J_n$-holomorphic curve $\Sigma$, which supplies the transversality for $\mathcal{D}$,

2. The coassociative boundary condition on $\partial A_\varepsilon$, which enables us to periodically reflect $A_\varepsilon$ to a bigger domain $A_{k(\varepsilon)}$, with integer $k(\varepsilon)$ such that $1/2 \leq k(\varepsilon) \varepsilon \leq 3/2$, thus restoring the uniform ellipticity.
This is an over-simplified description as sections on $A_\xi$ may become discontinuous after periodical reflection, so condition 2 only helps in the $W^{1,p}$ setting to get a uniform bound of $\|D^{-1}\|$ as in [8]. More effort is needed to estimate $\|D^{-1}\|$ in the Schauder $C^{1,\alpha}$ setting.

Recall in Subsection 3.2.1 we have formulated instantons nearby $A_\xi'$ as solutions of the nonlinear equation $F_\varepsilon (V) = 0$, and it turns out $\|F_\varepsilon (0)\|_{C^{\alpha}} \leq C_\varepsilon^{1-\alpha}$. In Subsection 3.2.2 we have computed the linearization $F'_\varepsilon (0)$. To apply the implicit function theorem to perturb $A_\xi'$ to a true instanton, we need to estimate $\|F'_\varepsilon (0)^{-1}\|$, and we will see in Subsection 3.2.6 it is comparable to $\|D^{-1}\|$ (Proposition 23). So a key estimate that we need is the following

**Theorem 15** ($\varepsilon$-dependent bound) Suppose that the first eigenvalues for $\overline{\partial}\partial^* \overline{\partial}$ and $\overline{\omega}^* \overline{\partial}$ are bounded below by $\lambda > 0$. Then for any $0 < \alpha < 1$ and $p > 3$ there is a positive constant $C = C (\alpha, p, \lambda, h)$ such that for any $V \in C^{1,\alpha}_{\pm} (A_\xi, S)$, we have

$$\|V\|_{C^{1,\alpha}} \leq C \varepsilon^{-\left(\frac{2+2\alpha}{\alpha}\right)} \|DV\|_{C^{\alpha}}. \tag{10}$$

Here the notations $C^{1,\alpha}_{\pm} (A_\xi, S)$ are chosen to indicate the local elliptic boundary conditions for sections $V = (u, v)$ of $S = S^+ \oplus S^-:

$$C^{1,\alpha}_{+} (A_\xi, S) = \{ V \in C^{1,\alpha} (A_\xi, S) : u|_{\partial_+^\varepsilon} = 0 \}$$

$$C^{1,\alpha}_{-} (A_\xi, S) = \{ V \in C^{1,\alpha} (A_\xi, S) : u|_{\partial_-^\varepsilon} = 0 \}.$$

Similarly $W^{k,p}_{\pm} (A_\xi, S)$ will be used to indicate these boundary conditions.

To prove this theorem we first recall (for simplicity, we assume $h \equiv 1$),

$$DV = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & -i \partial \overline{\partial} \end{bmatrix} \\ i \overline{\partial} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{11}$$

Along $\Sigma$ we have the following

**Lemma 16** Assume the $J_n$-holomorphic curve $\Sigma \subset C$ is Fredholm regular, then

$$\lambda_{\overline{\partial}} > 0$$

where $\lambda_{\overline{\partial}}$ is the first eigenvalue of $\Delta_\Sigma = \overline{\partial}\overline{\partial}$ on $W^{1,2} (\Sigma, S^+)$.

**Proof.** This follows from the fact that $\overline{\partial}$ is the normal Cauchy-Riemann operator on $N_{\Sigma/C}$, and its adjoint is $\overline{\partial}^*$. So the Fredholm regular property of $\Sigma$ is equivalent to $\ker \overline{\partial} = \{0\}$, i.e. $\lambda_{\overline{\partial}} > 0$. ■

**$L^2$ estimate**

The operator $D = D_{\pm} : W^{1,2}_{\pm} (A_\xi, S) \to L^2 (A_\xi, S)$ is self-adjoint by the boundary condition, thus $\text{coker} D_{\pm} = \ker D_{\pm}$. By the Rayleigh quotient method, we know

$$\lambda_{D_{\pm}} := \inf_{0 \neq V \in W^{1,2}_{\pm} (A_\xi)} \frac{\|DV\|^2_{L^2 (A_\xi)}}{\|V\|^2_{L^2 (A_\xi)}} \tag{12}$$

is the first eigenvalue of the Laplacian $D_{\pm} D_{\pm}$. 

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Theorem 17 (First eigenvalue estimate) For \( D_{\pm} : W_{1,2}^{1,2}(A_{\varepsilon}, \Sigma) \rightarrow L^2(A_{\varepsilon}, \Sigma) \), we have
\[
\lambda_{D_-} \geq \min \left\{ \lambda_{\partial} \frac{2}{\varepsilon^2} \right\} \quad \text{and} \quad \lambda_{D_+} \geq \min \left\{ \lambda_{\partial}^* \frac{2}{\varepsilon^2} \right\}.
\]

**Remark 18** Theorem 17 enables us to control the first eigenvalue of \( D_{\pm} \) on 3-dimensional \( A_{\varepsilon} \) by that of \( \partial \) on 2-dimensional \( \Sigma \), when \( \varepsilon \) is small. The control is due to the boundary condition of \( D_{\pm} \), as will be clear from the following proof.

**Proof.** We prove the estimate for \( \lambda_{D_-} \) (\( \lambda_{D_+} \) is similar). By the boundary condition \( v|_{\partial A_{\varepsilon}} = 0 \), we have
\[
\langle DV, DV \rangle_{L^2} = \int_{[0, \varepsilon] \times \Sigma} \left( \left| \frac{\partial V}{\partial x_1} \right|^2 + \| \bar{\partial} v \|^2 + \| \overline{\partial u} \|^2 \right)
\geq \int_{[0, \varepsilon] \times \Sigma} \left( \| \bar{\partial} u \|^2 + \| v_{x_1} \|^2 \right).
\]
Then use the Rayleigh quotient for \( \bar{\partial} \) and notice that \( v|_{\partial A_{\varepsilon}} = 0 \).

**Corollary 19** If \( \Sigma \subset C \) is Fredholm regular, then for small enough \( \varepsilon > 0 \), \( D : W_{1,2}^{1,2}(A_{\varepsilon}, \Sigma) \rightarrow L^2(A_{\varepsilon}, \Sigma) \) is surjective.

**Proof.** By Lemma 16, the Fredholm regular property of \( \Sigma \) implies that \( \lambda_{\bar{\partial}} > 0 \). So by Theorem 17, \( \lambda_{D_+} > 0 \), i.e. \( \ker D_+ = \{ 0 \} \). By the self-adjoint property of \( D \), \( \text{coker} D_- = \ker D_+ = \{ 0 \} \).

Now, from the definition of \( \lambda_{D_-} \) we obtain the \( L^2 \) estimate

**Corollary 20** (\( L^2 \)-estimate) For any \( V \in W_{1,2}^{1,2}(A_{\varepsilon}, \Sigma) \) and small enough \( \varepsilon \), we have
\[
\| V \|_{W_{1,2}^{1,2}(A_{\varepsilon}, \Sigma)} \leq C(\lambda) \| DV \|_{L^2(A_{\varepsilon}, \Sigma)},
\]
where the constant \( C(\lambda) \) only depends on \( \lambda_{\bar{\partial}} \).

In the following we will derive a \( C^0 \)-estimate of \( V \). For this purpose let us examine the equation \( D_- V = W \) more closely.

For \( V = (u, v) \), \( W = (w_1, w_2) \), the equation \( D_- V = W \) (assuming \( h = 1 \)) is equivalent to the following system
\[
\begin{align*}
ux_{x_1} - i \bar{\partial} v & = w_1 \\
v_{x_1} + i \bar{\partial} u & = w_2
\end{align*}
\quad \text{and} \quad v|_{\partial A_{\varepsilon}} = 0.
\]

This is similar to the Cauchy-Riemann equation, but the relation between \( u \) and \( v \) is weaker, since \( \nabla u \) can only control \( \bar{\partial} v \) and \( v_{x_1} \), which are only half of partial derivatives of \( v \); the same applies to \( \nabla v \). This issue makes it more difficult to obtain the \( C^0 \)-estimate of \( u \) than in the Cauchy-Riemann type equations.
$C^0$ estimate

The $C^0$ estimate of $V$ is derived from a $W^{1,p}$ estimate of $V$ and Sobolev embedding. To obtain a uniform estimate for $\|D^{-1}\|$ in the $L^p$ setting, we use the periodic reflection technique.

**Theorem 21** ($L^p$-estimate, $p > 3$) For any $V \in W^{-1,p}(A_\varepsilon, S)$, we have

$$\|V\|_{W^{-1,p}(A_\varepsilon, S)} \leq C_p(\lambda) \|DV\|_{L^p(A_\varepsilon, S)}.$$  

where the constant $C_p(\lambda)$ only depends on $\lambda\Sigma$, $Vol(\Sigma)$ and $p$.

**Proof.** Given any $\varepsilon > 0$, we choose an integer $k(\varepsilon)$ so that $1/2 \leq k(\varepsilon) \varepsilon \leq 3/2$. In the following we will simply write $k(\varepsilon)$ as $k$. We reflect $A_\varepsilon$ to $A_{k\varepsilon}$ periodically and extend $(u, v)$ along the boundaries $\varepsilon \mathbb{Z} \times \Sigma$ such that

$$v(x, z) = \begin{cases} -v((2j + 2)\varepsilon - x, z) & x \in [(2j + 1)\varepsilon, (2j + 2)\varepsilon] \\ v(x - 2j\varepsilon, z) & x \in [2j\varepsilon, (2j + 1)\varepsilon] \end{cases},$$

$$u(x, z) = \begin{cases} u((2j + 2)\varepsilon - x, z) & x \in [(2j + 1)\varepsilon, (2j + 2)\varepsilon] \\ u(x - 2j\varepsilon, z) & x \in [2j\varepsilon, (2j + 1)\varepsilon] \end{cases},$$

(Notice that a $W^{k,p}$ section will remain so after reflections), i.e. we do odd extensions of $v$ and even extensions of $u$ along the boundaries. By (13), this induces odd extensions of $w_1$ and even extensions of $w_2$ along the boundaries. Since the shape of $A_{k\varepsilon}$ is uniformly bounded, we have the elliptic estimate

$$\tilde{C}(p)\|V\|_{W^{-1,p}(A_{k\varepsilon}, S)} \leq \|DV\|_{L^p(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S)} + \|V\|_{L^p(A_{2k\varepsilon} \cup A_{-2k\varepsilon}, S)}.$$ with a uniform constant $\tilde{C}(p)$ for all $\varepsilon$. Then we use the interpolation inequality

$$\|V\|_{L^{p\prime}_1(A_{k\varepsilon} \cup A_{-2k\varepsilon}, S)} \leq C\left(\delta^{p\prime - p}\|V\|_{W^{1,p}(A_{k\varepsilon} \cup A_{-2k\varepsilon}, S)} + \delta^{-p}\|V\|_{L^{p\prime}_1(A_{k\varepsilon} \cup A_{-2k\varepsilon}, S)}\right)$$

(16)

To pass from the $L^2$ estimate to the $L^p$ estimate

$$\|V\|_{W^{-1,p}(A_{k\varepsilon}, S)} \leq C_p(\lambda) \|DV\|_{L^p(A_{k\varepsilon}, S)}.$$  

(17)

This is because the boundary condition of $V \in W^{-1,p}(A_\varepsilon, S)$ and (13) yields

$$\|V\|_{L^{p\prime}_1(A_{k\varepsilon} \cup A_{-2k\varepsilon}, S)} \leq 2C(\lambda) \|DV\|_{L^{p\prime}(A_{k\varepsilon}, S)} \leq C_p(\lambda) \|DV\|_{L^p(A_{k\varepsilon}, S)},$$

where $C_p(\lambda) = 2C(\lambda)\left(\frac{2}{3}Vol(\Sigma)\right)^{\frac{3-p}{p}}$. Last we obtain the inequality (15) on $A_\varepsilon$ from (17) by the periodicity of the $L^p$ integrals on the reflected domains. □
**Theorem 23**

$$\|V\|_{C^0(A, \mathcal{S})} \leq C \varepsilon^{1-\frac{2}{p}} \|\nabla V\|_{C^0(A, \mathcal{S})}, \tag{18}$$

**Proof.** We have

$$\|V\|_{C^0(A, \mathcal{S})} \leq C \varepsilon^{1-\frac{2}{p}} \|V\|_{C^1(A, \mathcal{S})} \leq C \varepsilon^{1-\frac{2}{p}} \|V\|_{W^{1,p}(A, \mathcal{S})} \leq C \varepsilon^{1-\frac{2}{p}} \|\nabla V\|_{C^0(A, \mathcal{S})},$$

where in the first inequality we have used the condition \(u(x, z) = 0 = v(x', z)\) and the definition of the Schauder \(C^{1-\frac{2}{p}}\) norm. Then we notice the periodicity on \(A_{k\varepsilon}\) so we get \([18] \). \(\blacksquare\)

**Corollary 22** (*C^0* estimate): If for each \(z \in \Sigma\) there exist \(x, x' \in [0, \varepsilon]\) such that \(u(x, z) = 0\) and \(v(x', z) = 0\), then

$$\|V\|_{C^0(A_{k\varepsilon}, \mathcal{S})} \leq C \varepsilon^{1-\frac{2}{p}} \|\nabla V\|_{C^0(A_{k\varepsilon}, \mathcal{S})}. \tag{19}$$

**Proof.** We can change \(\mathcal{D}_- V = W\) into a system of second order elliptic equations

\[
\begin{align*}
  u_{x_1x_1} - \partial_i \partial_i u &= i \partial_1 w_2 + \partial_{x_1} w_1 \\
  u_{x_1x_1} - \partial_i \partial_i v &= -i \partial_1 w_1 + \partial_{x_1} w_2 \quad \text{and} \quad v|_{\partial A_k} = 0, \tag{19}
\end{align*}
\]

where \(V = (u, v)\) and \(W = (w_1, w_2)\).

To get the uniform estimate of \(\|\mathcal{D}_- V\|\) in the \(C^{1,\alpha}\) setting, we can not just rely on periodic reflections, because if \(w_1 \neq 0\) then the extension of \(\mathcal{D} V\) is no longer continuous on \(A_{k\varepsilon}\), not to mention in \(C^0\) \((A_{k\varepsilon}, \mathcal{S})\).

**Theorem 23** (*C^{1,\alpha}* estimate) For any \(V \in C^{1,\alpha}_{-}(A_{\varepsilon}, \mathcal{S})\), we have

$$C \varepsilon^{\frac{3+2\alpha}{p}} \|V\|_{C^{1,\alpha}_{-}(A_{\varepsilon}, \mathcal{S})} \leq \|\nabla V\|_{C^0(A_{\varepsilon}, \mathcal{S})}. \tag{20}$$

**Proof.** We can decouple \(W = (w_1, w_2)\) into dealing with \((0, w_2)\) and \((w_1, 0)\) cases separately because \(\mathcal{D}\) is linear and surjective from Corollary \(19\).

When \(w_1 = 0\) along \(\partial A_k\) we have \(u_{x_1} = \partial_1 v = 0\) (for \(v|_{\partial A_k} = 0\)), so we can reflect \(u\) by even extension and it is still in \(C^{1,\alpha}\). We can extend \(W\) in \(C^0(A_{\varepsilon}, \mathcal{S})\) as well, for \(w_1 = 0\). In this case we can restore uniform ellipticity in the Schauder setting by periodic reflection.

We deal with the more difficult case when \(w_1 \neq 0\) but \(w_2 = 0\). The \(v\) component is easier, for it satisfies the Dirichlet boundary condition \(v|_{\partial A_k} = 0\). Standard Schauder estimate on half balls \((\Pi)\) implies

$$C \varepsilon^{1+\alpha} \|v\|_{C^{1,\alpha}_{-}(A_{\varepsilon}, \mathcal{S})} \leq \varepsilon \|w_1\|_{C^{0}(A_{\varepsilon}, \mathcal{S}^+)_{-}} + \|v\|_{C^{0}_{-}(A_{\varepsilon}, \mathcal{S})}. \tag{20}$$

Plugging the \(C^0\) estimate \([18]\) in above inequality we get

$$\|v\|_{C^{1,\alpha}_{-}(A_{\varepsilon}, \mathcal{S})} \leq C \varepsilon^{-\left(\frac{3+2\alpha}{p}\right)} \|W\|_{C^0(A_{\varepsilon}, \mathcal{S})}. \tag{20}$$
The $u$ component is much harder, since $u|_{\partial A_0} \neq 0$ in general. We carry out the following steps:

(a) We homogenize $u$ by introducing
\[
\tilde{u} = u - \rho \left( \frac{x_1}{\varepsilon} \right) u(\varepsilon, z) - \left( 1 - \rho \left( \frac{x_1}{\varepsilon} \right) \right) u(0, z),
\]
where $\rho : [0, 1] \to [0, 1], \rho(0) = 0, \rho(1) = 1$ is a smooth cut-off function such that $\|\rho\|_{C^{2,\alpha}[0,1]} \leq C$. Then we get
\[
\tilde{u}_{x_1} - \partial^+ \partial^\varepsilon \tilde{u} = \partial_{x_1} \tilde{w}_1 + g \quad \text{and} \quad \tilde{u}|_{\partial A_0} = 0,
\]
where the functions $\tilde{w}_1$ and $g$ involve $w_1$ and the boundary values of $u$. Applying elliptic regularity for this Dirichlet problem of $\tilde{u}$ and then writing back in $u$, we have
\[
\varepsilon \|w_1\|_{C^\alpha} + \varepsilon^{1+\alpha} \|u\|_{C^{\alpha}} + \|v\|_{C^{\alpha}} \geq C\varepsilon^{1+2\alpha} \|u\|_{C^{1,\alpha}}.
\] (21)

This is not quite easy, since we cannot use $\nabla v$ to completely control $\nabla u$ from (14). This was done by carefully analyzing the $\varepsilon$-orders behaviors of the derivatives of the cut-off function $\rho \left( \frac{x_1}{\varepsilon} \right)$ and elliptic estimates of $\overline{\partial}$ and $\overline{\partial^\varepsilon}$ on slices $\{x\} \times \Sigma$.

(b) We show on each segment $[0, \varepsilon] \times \{z\}$ there exists $x$ with $u(x, z) = 0$. Consider the average of $u$ along each segment $[0, \varepsilon] \times \{z\}$:
\[
\overline{u}(z) = \frac{1}{\varepsilon} \int_0^\varepsilon u(x_1, z) \, dx_1.
\]
From (14) we have (note $w_2 = 0$)
\[
\overline{\nabla u}(z) = - \frac{1}{\varepsilon} \partial_{x_1} v(x_1, z) \, dx_1 = v(0, z) - v(\varepsilon, z) = 0.
\]
But from the assumption that $\overline{\partial^\varepsilon \overline{\partial^\varepsilon}}$ has trivial kernel on $\Sigma$ we get that $\overline{u}(z) \equiv 0$.

Then by the intermediate value theorem there must exist some $x \in [0, \varepsilon]$ such that $u(x, z) = 0$. So by Corollary 22 we get the desired $C^0$ estimate of $u$. Similarly for $v$.

(c) Putting the $C^0$ estimate of $V = (u, v)$ back to (21), we finally get
\[
\|V\|_{C^{1,\alpha}(\mathbb{A}, \mathbb{S})} \leq C\varepsilon^{-\left( \frac{1}{2} + 2\alpha \right)} \|DV\|_{C^{\alpha}(\mathbb{A}, \mathbb{S})}.
\]

Theorem 15 is proved. $\blacksquare$

3.2.5 Geometry of $J_n$-holomorphic curves in $G_2$ manifolds: agreement of two Dirac operators

Recall given a family $\mathcal{C} = \bigcup_{0 \leq t \leq \varepsilon} C_t$ of coassociative manifolds $C_t$ in $M$, the (nonvanishing) deformation vector field $n := \frac{dC_t}{dt}|_{t=0}$ defines an almost complex
structure $J_n$ on $C = C_0$. For any $J_n$-holomorphic curve $\Sigma \subset C$, the $G_2$-structure on $M$ gives a natural identification between $N_{\Sigma/C}$ and $N_{C/M}|\Sigma$ \[10\]. Further calculations \[21\] established the following close relation between the “intrinsic” and “extrinsic” Dirac operators.

**Proposition 24** (Proposition 16,17, \[21\]) Let $\Sigma \subset C \subset C \subset M$ be as above. We have an orthogonal decomposition

$$TM|_\Sigma = T\Sigma \oplus N_{\Sigma/C} \oplus N_{C/M}|_\Sigma \oplus N_{C/C}|_\Sigma,$$

1. For $L = T\Sigma$, $N_{\Sigma/C}$ or $N_{C/M}|_\Sigma$, the induced connection $\nabla^L$ from the Levi-Civita connection on $M$ is Hermitian, i.e. $\nabla^L J_n = 0$.

2. The spinor bundle $S_\Sigma$ over $\Sigma$ is identified with $N_{\Sigma/C} \oplus N_{C/M}|_\Sigma$ in such a way that the Clifford multiplication is given by the $G_2$ multiplication $\times$ and the spinor connection equals $\nabla^N_{\Sigma/C} \oplus N_{C/M}|_\Sigma$.

3. The Dirac operator on $S_\Sigma = N_{\Sigma/C} \oplus N_{C/M}|_\Sigma$ agrees with the Dolbeault operator on $N_{\Sigma/C} \oplus \wedge^{0,1}(N_{\Sigma/C})$.

The agreement of the two Dirac type operators on $\Sigma$ is the geometric reason that the Fredholm regularity property of the “intrinsic” Cauchy-Riemann operator on $N_{\Sigma/C}$ gives control of the “extrinsic” linearized instanton operator.

### 3.2.6 Comparison of $D$ and linearized instanton equation

When we move from $\Sigma$ to the interior of the almost instanton $A'_\epsilon$, the nice agreement of the two Dirac operators no longer holds. We need to control the difference between the Dirac operator $D$ on $S$ and the linearized instanton operator $F'_\epsilon(0)$ on $N_{A'_\epsilon/M}$. In order to compare them, we need a good identification between $S$ and $N_{A'_\epsilon/M}$.

For this purpose we defined an exponential-like map $\exp : S \to M$ such that $\exp(A_\epsilon) = A'_\epsilon$ and its differential $d\exp|_{A_\epsilon}$ has the following properties on $\{0\} \times \Sigma$ (see Appendix of \[21\]):

1. On fiber directions of $S$, $d\exp|_{\{0\} \times \Sigma} = (id, f) : N_{\Sigma/C} \oplus \wedge^{0,1}(N_{\Sigma/C}) \to N_{\Sigma/C} \oplus N_{C/M}|_\Sigma$, where $f$ is the isomorphism between the second summands given in Lemma 3.2 of \[10\];

2. On base directions of $S$, $d\exp|_{\{0\} \times \Sigma} = id : T\Sigma \to T\Sigma$ and $d\exp|_{\{0\} \times \Sigma} : \frac{\partial}{\partial z} \to n(z)$;

3. On the boundary $\{0, \epsilon\} \times \Sigma$ of $A_\epsilon$, $d\exp|_{\{0\} \times \Sigma} (S^+ \oplus 0) \subset C_0$, $d\exp|_{\{\epsilon\} \times \Sigma} (S^+ \oplus 0) \subset C_\epsilon$.

Using $d\exp|_{A_\epsilon}$, we can relate the spin bundle $S \to A_\epsilon$ to the normal bundle $N_{A'_\epsilon/M} \to A'_\epsilon$. Because of the compatibilities given in 1-3 above, using Proposition \[24\] we have the following comparison result.
Proposition 25 (Proposition 18, [21]) For any $V_1 \in C^{1,\alpha}(A_\varepsilon, S)$, we have
\[
\|F'_\varepsilon(0) (d\exp \cdot V_1) - (d\exp) \circ D V_1\|_{C^\alpha(A'_\varepsilon, N_{A'_\varepsilon}/M)} \leq C \varepsilon^{1-\alpha} \|V_1\|_{C^{1,\alpha}(A_\varepsilon, S)}.
\]

Then we get the uniform inverse estimate of $F'_\varepsilon(0)$ from $\|D^{-1}\|$ by the following diagram, using that $d\exp$ and $(d\exp)^{-1}$ are both smooth with a uniform $C^2$ bound for all $\varepsilon$:

\[
\begin{array}{c}
C^{1,\alpha}_-(A_\varepsilon, S) \xrightarrow{\exp} C^{\alpha}(A_\varepsilon, S) \\
\downarrow d\exp \\
C^{1,\alpha}_-(N_{A'_\varepsilon}/M) \xrightarrow{F'_\varepsilon(0)} C^{\alpha}(N_{A'_\varepsilon}/M)
\end{array}
\]

3.2.7 Quadratic estimate

Using any local frame field $\{W_\alpha\}_{\alpha=1}^7$, we could compare the linearizations of $F(V)$ at two different almost instantons, up to a curvature term $B$ as follows:

\[
F'(V_0) |_{A_\varepsilon(0)} V(p) = \left( (\exp V_0)^* \otimes T_{V_0} \circ [F'(0) |_{A_\varepsilon(0)} V_1(q)] + B^\alpha(V_0, V) W_\alpha(p) \right),
\]

where $V_1 = (d\exp V_0) |_{A_\varepsilon(0)} V$. This means the derivative $F'(V_0)$ on $A_\varepsilon(0)$ can be expressed by the derivative $F'(0)$ on $A_\varepsilon(V_0)$ via the transform $(\exp V_0)^* \otimes T_{V_0}$, up to the curvature term $B^\alpha(V_0, V) W_\alpha(p)$. Using (22) we get

\[
F'(V_0) V(p) - F'(0)(V_0) V(p) = (I) + (II)
\]

with

\[
(I) = \left( (\exp V_0)^* \cdot d (i_{V_1} \omega^\alpha) - d (i_{V_1} \omega^\alpha) \right) |_{A_\varepsilon(0)} \otimes W_\alpha(p),
\]

\[
(II) = \left( (\exp V_0)^* \otimes T_{V_0} \circ [i_{V_1} d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_{V_1} W_\alpha] \right)(p) + B^\alpha(V_0, V) W_\alpha(p).
\]

Here (I) consists of 1st order terms whose $C^\alpha$-norm are bounded by $\|V_0\|_{C^{1,\alpha}} \|V\|_{C^{1,\alpha}}$ because $d$, the pull back operator, the parallel transport and various exponential maps are Frechet smooth with respect to variations of $V$. (II) consists of 0th order terms and it is easier to bound. Thus we get

Proposition 26 (Quadratic estimate) For any $V_0, V \in C^{1,\alpha}(A'_\varepsilon, N_{A'_\varepsilon}/M)$ with small $\|V_0\|_{C^{1,\alpha}}$, we have
\[
\|F'_\varepsilon(V_0) V - F'_\varepsilon(0) V\|_{C^\alpha} \leq C \|V_0\|_{C^{1,\alpha}} \|V\|_{C^{1,\alpha}},
\]

where the constant $C$ is independent on $\varepsilon$.

Remark 27 We also have some pointwise estimates tied to the feature that $\tau$ is a cubic-form. In [21] we derived that

\[
|F'(V_0) V - F'(0) V| (p) \leq \left[ C_7 (|d\varphi| + |\nabla V_0|)^2 (|V_0| |V| + |\nabla V_0| |V| + |V_0| |\nabla V|) + C_5 |V_0| |V| \right] (p),
\]

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where $\varphi : A_\varepsilon \to M$ is the embedding we used to define $A'_\varepsilon$. Because of the cubic terms, the following quadratic estimate needed in the implicit function theorem in $W^{1,p}$ setting

$$\|F'(V_0) V - F'(0) V\|_{L^p} \leq C \|V_0\|_{W^{1,p}} \|V\|_{W^{1,p}}.$$  

appears unavailable, for $\left(|\nabla V_0|^3\right)^p \notin L^p$ in general. In contrast, the Cauchy-Riemann operator of $J$-holomorphic curves is more linear in the $L^p$ setting: it has the quadratic estimate (Proposition 3.5.3, [23])

$$\|F'(V_0) V - F'(0) V\|_{L^p(\Sigma)} \leq C \|V_0\|_{W^{1,p}(\Sigma)} \|V\|_{W^{1,p}(\Sigma)}.$$  

This is one of the key reasons that we use the Schauder setting.

By our construction of the almost instanton $A'_\varepsilon = \varphi (A_\varepsilon)$ and the smoothness of $\varphi$, it is not hard to get the error estimate

$$\|F_\varepsilon (0)\|_{C^\alpha (A'_\varepsilon \times N A'_\varepsilon /M)} \leq C\varepsilon^{1-\alpha}.$$  

Combining (10), (23), by the implicit function theorem (e.g. Proposition A.3.4, [23]) we can solve $F_\varepsilon (V) = 0$, thus our main theorem 13 is proved. QED.

4 Applications and further discussions

4.1 New examples of instantons

Our main theorem can be used to construct new examples of instantons.

Let $X$ be a Calabi-Yau threefold containing a complex surface $S \subset X$ which contains a smooth curve $\Sigma \subset S$ which satisfy (i) $H^0 (S, K_S) \neq 0$ (i.e. $p_g (S) \neq 0$) and (ii) $H^0 (\Sigma, K_S|\Sigma) = 0$. Condition (i) implies that $S$ can be deformed inside $X$ and condition (ii) is equivalent to $H^1 (\Sigma, N_{\Sigma/S}) = 0$, namely $\Sigma \subset S$ is Fredholm regular.

Let $\{S_t\}_{0 \leq t \leq \varepsilon}$ be a smooth family of deformations of $S$ inside $X$ and let $v = \frac{dS_t}{dt}|_{t=0}$ be the normal vector field on $S = S_0$. By our assumption $v$ is nontrivial, and after possible rescaling of the parameter $t$ for the family $S_t$, we may assume $v$ is very small. Let $M = X \times S^1$ be the $G_2$ manifold as in Example 2 and $C_t := S_t \times \{t\} \subset M$ $(0 \leq t \leq \varepsilon)$ be the family of coassociative submanifolds. They are disjoint since their second components $t$ are different.

Let $n_0 := (0, \frac{\partial}{\partial \theta})$ be the normal vector field on $C_0 = S \times \{0\}$, then the original complex structure $J_0$ on $S$ is induced from $n_0$, i.e. $J_0 = \frac{\partial}{\partial \theta}$. 

Let $n = \frac{dC_t}{dt}|_{t=0} = (v, \frac{\partial}{\partial \theta})$ be the other normal vector field on $C_0$. Then $n$ is nonvanishing for $\frac{\partial}{\partial \theta} / n$ is nonvanishing on $S^1$. The almost complex structure $J_n = \frac{\partial}{\partial \theta} \times$ is close to but not equal to the original complex structure $J_0$ on $S$, because $n$ is close to but not equal to $n_0$. 

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There must exist a Fredholm regular $J_n$-holomorphic curve $\Sigma_n \subset S$ near the original $J_0$-holomorphic curve $\Sigma$, because $\Sigma \subset S$ is Fredholm regular and will persist after small perturbations of $J_0$ on $S$.

Applying our main theorem to $\Sigma_n \subset C_0$, we get an instanton $A \subset M$ with boundaries on $C_0 \cup C_\varepsilon$. It is not the trivial instanton $\Sigma \times [0, \varepsilon]$, which has upper boundary lying on $S \times \{\varepsilon\}$, not on $C_\varepsilon = S(\varepsilon) \times \{\varepsilon\}$.

4.2 Further remarks

A few remarks of our main theorem are in order: First, counting such thin instantons is basically a problem in four manifold theory because of Bryant’s result [6] which says that the zero section $C$ in $\Lambda^2_+ (C)$ is always a coassociative submanifold for an incomplete $G_2$-metric on its neighborhood provided that the bundle $\Lambda^2_+ (C)$ is topologically trivial.

Second, when the normal vector field $n = \frac{dC}{dt}_{t=0}$ has zeros, our main Theorem 13 should still hold true (work in progress [22]). However it would require a possible change of the Fredholm set-up of the current method and a good understanding of the Seiberg-Witten theory on any four manifold with a degenerated symplectic form as in Taubes program (29, 30). In the special case when $\Sigma$ is disjoint from $\{n = 0\}$, our theorem 13 is obviously true as our analysis only involves the local geometry of $\Sigma$ in $M$.

Third, if we do not restrict to instantons of small volume, then we have to take into account of bubbling phenomenon as in the pseudo-holomorphic curves case, and gluing of instantons of big and small volumes similar to [25] in Floer trajectory case. Nevertheless, one does not expect bubbling can occur when volumes of instantons are small, thus they would converge to a $J_n$-holomorphic curve in $C$ as $\varepsilon \to 0$.

Last, we expect our result still holds in the almost $G_2$ setting, namely $\Omega$ is only a closed form rather than a parallel form. This is because our gluing analysis relies mainly on the Fredholm regularity property of the linearized instanton equation. Such a flexibility could be useful for finding regular holomorphic curves $\Sigma \subset C$ in order to apply our theorem.

References

[1] B. S. Acharya, B. Spence, Supersymmetry and M theory on 7-manifolds, [hep-th/0007213].

[2] M. Aganagic, C. Vafa, Mirror Symmetry and a $G_2$ Flop, J. High Energy Phys. 2003, no. 5, 061, 16 pp.

[3] S. Akbulut, S. Salur, Deformations in $G_2$ manifolds, Advances in Mathematics Volume 217, Issue 5, 20 March 2008, p2130-2140.

[4] C. Beasley, E. Witten, A Note on Fluxes and Superpotentials in M-theory Compactifications on Manifolds of $G_2$ Holonomy, Journal of High Energy Physics, Volume 2002.
[5] B. Booss-Bavnbek and K. Wojciechowski, *Elliptic Boundary Problems For Dirac Operators*, Birkhäuser (1993).

[6] R. Bryant, *Calibrated embeddings in the special Lagrangian and coassociative cases*. Special issue in memory of Alfred Gray (1939–1998). Ann. Global Anal. Geom. 18 (2000), no. 3-4, 405–435.

[7] A. Corti, M. Haskins, J. Nordstrom, T. Pancini, *$G_2$ manifolds and associative submanifolds via semi-Fano 3-folds*, arXiv: 1207.4470.

[8] K. Fukaya, Y.G. Oh, *Zero-loop open strings in the cotangent bundle and Morse homotopy*, Asian J. Math. 1 (1997), no. 1, 96–180.

[9] K. Fukaya, Y.G. Oh, H. Ohta, K. Ono, *Lagrangian intersection Floer theory - anomaly and obstruction*, AMS/IP Studies in Advanced Mathematics, International Press.

[10] D. Gayet, F. Witt, *Deformations of associative submanifolds with boundary*, Adv. Math. 226 (2011), 2351-2370.

[11] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer Verlag (1983).

[12] R. Gopakumar, C. Vafa, *M-theory and topological strings - II*, hep-th/9812127

[13] R. Harvey, B. Lawson, *Calibrated geometries*, Acta Math. 148 (1982), 47-157.

[14] H.A. Harvey, G. Moore, *Superpotentials and membrane instantons*, hep-th/9907026.

[15] N. Hitchin, *The geometry of three forms in 6 and 7 dimensions*, J. Differential Geom. 55 (2000), no. 3, 547–576.

[16] N. Hitchin, *Stable forms and special metrics*, Contemporary Math. 288 (2001), 70-89.

[17] D. D. Joyce, *Compact Riemannian 7-manifolds with holonomy $G_2$. I, II*, J. Diff. Geom. 43 (1996), no. 2, 291-328, 329-375.

[18] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[19] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. 565 (2003), 125-160.

[20] J.H. Lee and N.C. Leung, *Instantons and Branes in Manifolds with Vector Cross Product*, Asian Journal of Math. 2009.
[21] N.C. Leung, X. Wang, K. Zhu, *Thin instantons in G₂-manifolds and Seiberg-Witten invariants*, to appear in J. Diff. Geom., also available at arXive: math/SG/1107.1947.

[22] N.C. Leung, X. Wang, K. Zhu, *Thin instantons in G₂-manifolds with boundaries on intersecting coassociative submanifolds*, in preparation.

[23] D. McDuff, D. Salamon, *J-holomorphic Curves and Symplectic Topology*, Colloquium Publications, vol 52, AMS, Providence RI, 2004.

[24] R.C. McLean, *Deformations of Calibrated Submanifolds*, Commun. Analy. Geom. 6 (1998) 705-747.

[25] Y. Oh, K. Zhu, *Thick-thin decomposition of Floer trajectories and adiabatic gluing*, arXive: 1103.3525.

[26] C. Taubes, *Gr=>SW From pseudo-holomorphic curves to Seiberg-Witten solutions*, J. Differential Geom. 51 (1999), no. 2, 203–334.

[27] C. Taubes, *SW=>Gr: From the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc, Vol 9, no. 3, July 1996, 845-918.

[28] C. Taubes, *Gr=SW: Counting curves and connections*, J. Differential Geom. 52 (1999), no. 3, 453–609.

[29] C. Taubes, *The geometry of the Seiberg-Witten invariants*. Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 493–504.

[30] C. Taubes, *Seiberg-Witten invariants and pseudo-holomorphic subvarieties for self-dual, harmonic 2-forms*. Geom. Topol. 3 (1999), 167–210.

[31] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. Volume 17, Number 4 (1982), 661-692.

[32] S.-T. Yau *On Ricci curvature of a compact Kähler manifold and complex Monge-Ampère equation I*. Comm. Pure and App. Math. 31, pp. 339–411, 1978.