Unitary dynamics
of spherical null gravitating shells

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Abstract

The dynamics of a thin spherically symmetric shell of zero-rest-mass matter in its own gravitational field is studied. A form of action principle is used that enables the reformulation of the dynamics as motion on a fixed background manifold. A self-adjoint extension of the Hamiltonian is obtained via the group quantization method. Operators of position and of direction of motion are constructed. The shell is shown to avoid the singularity, to bounce and to re-expand to that asymptotic region from which it contracted; the dynamics is, therefore, truly unitary. If a wave packet is sufficiently narrow and/or energetic then an essential part of it can be concentrated under its Schwarzschild radius near the bounce point but no black hole forms. The quantum Schwarzschild horizon is a linear combination of a black and white hole apparent horizons rather than an event horizon.

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1 Introduction

According to general relativity, all parts of a massive object definitely disappear if the object falls through its Schwarzschild radius. The problem to be tackled in the present paper is whether also a quantum system is, or is not, irretrievably lost if it falls under its Schwarzschild radius.

We limit ourselves to a sufficiently simple model so that no approximations are needed and the quantum theory can be constructed without problems. In this way, an important question about the validity of approximative methods such as WKB expansion can also be touched. The simplest system that can ever be invented for these aims seems to be a thin shell with its own gravitational field made of light-like material, everything spherically symmetric. A Hamiltonian action principle [1] for this system has been transformed to a form suitable for quantization in [2] (foregoing paper); this will be used as a starting point. Most of the results of the present paper have already been published in a short review [3]; here, all derivations and calculations will be described in sufficient detail, some new results will be added, and a new interpretation of the results will be given.

The plan of the paper is as follows. In Sec. 2, the starting assumptions and equations are collected. The action for the system from Ref. [1] is written down because we shall need the form of the constraints. The same action after the transformation to a set of embedding variables and Dirac observables is then given, the key notion of background manifold is introduced, and the meaning of the new variables is discussed. A construction of quantum mechanics including the position and the direction-of-motion operators is contained in Sec. 3. The so-called group-theoretical quantization method is used, which is well adapted to the problems such as the limited ranges of spectra and the construction of a unitary dynamics. The quantum mechanics is formulated as a dynamics of the shell on the background manifold; this enables straightforward and unique interpretations. In Sec. 4, motion of wave packets is investigated. It turns out that no shell reaches the zero radius if it starts away from it and so the singularity is avoided. The wave packets contract, bounce and then expand, reaching the asymptotic region from which they have been sent in, so the dynamics is unitary from the point of view of one family of observers. Some of the packets can be sufficiently concentrated near their bouncing point so that an essential part of them comes under the corresponding Schwarzschild radius, but no event horizon forms.

In Sec. 5, we consider the seemingly contradictory claims that the quantum shell can cross its Schwarzschild radius and still re-expand. The solution of the paradox is that if the matter creates a Schwarzschild (apparent) horizon outside then the horizon can be, even in the classical version of the theory, of two types: white or black, that is, corresponding to the white or black hole horizon in the Schwarzschild
spacetime. The “colour” of the apparent horizon in a Cauchy surface depends on the direction of motion of the shell: the horizon is black if the shell is contracting and it is white if the shell is expanding. The quantum horizon is a linear combination of both because the motion of the shell is. The quantum horizon is “grey”, changing from mostly black to mostly white.

The semi-classical approximation fails blatantly near the bouncing point of the quantum shell because every classical shell reaches its Schwarzschild radius, forms a black hole and falls into the singularity. A cautious discussion of this point is given in Sec. 6. In particular, the reason is explained why our results do not prevent massive quantum systems from collapsing to black-hole-like objects.

2 Canonical formalism

In this section, we shall summarize the formulae derived in Refs. [1] (abbreviated as LWF further on) and [2] that are needed to start the present paper.

In LWF, the spherically symmetric metric outside the shell is written in the form

\[ ds^2 = -N^2 d\tau^2 + \Lambda^2 (d\rho + N^r d\tau)^2 + R^2 d\Omega^2, \]

and the shell is described by its radial coordinate \( \rho = r \). The LWF action reads

\[ S_0 = \int d\tau \left[ p\dot{\rho} + \int_0^\infty d\rho (P_\Lambda \dot{\Lambda} + P_R \dot{R} - H_0) \right], \]

and the LWF Hamiltonian is

\[ H_0 = N\mathcal{H} + N^\rho \mathcal{H}_\rho + N_\infty M_\infty, \]

where \( N_\infty := \lim_{\rho \to \infty} N(\rho) \), \( M_\infty \) is the ADM energy, \( \mathcal{H} \) and \( \mathcal{H}_\rho \) are the constraints,

\begin{align*}
\mathcal{H} &= \frac{\Lambda P_\Lambda^2}{2R} - \frac{P_\Lambda P_R}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R^2}{2\Lambda} - \frac{\Lambda}{2} + \frac{\eta P}{\Lambda} \delta(\rho - r), \\
\mathcal{H}_\rho &= P_R R' - P_\Lambda' \Lambda - p \delta(\rho - r); \quad (1)
\end{align*}

the prime denotes the derivative with respect to \( \rho \) and the dot that with respect to \( \tau \).

In Ref. [2], the variables \( \eta, r, p, \Lambda, P_\Lambda, R \) and \( P_R \) have been transformed to the embedding variables \( U(\rho) \) and \( V(\rho) \), their canonical conjugates \( P_U(\rho) \) and \( P_V(\rho) \), and the shell variables \( u, v, p_u \) and \( p_v \). The pair \( (U(\rho), V(\rho)) \) defines an embedding of the half-axis into the so-called background manifold \( \mathcal{M} \) that is covered by the coordinates \( U \) and \( V \) with the ranges

\[ \frac{U + V}{2} \in (-\infty, \infty), \quad \frac{-U + V}{2} \in (0, \infty). \]
The transformation to embedding variables is determined by a gauge condition, and there has been a definite condition used in Ref. [2], where all details are given. The background manifold carries then a set of metrics, one representative for each geometry. The variables $u$ and $v$ are the coordinates of the shell trajectory in the background manifold:

$$U = u(\tau), \quad V = v(\tau).$$

The full action that results has the form of the so-called \textit{Kuchař decomposition}

$$S = \int d\tau (p_u \dot{u} + p_v \dot{v} - np_u p_v) + \int d\tau \int_0^\infty d\rho (P_U \dot{U} + P_V \dot{V} - H),$$

(3)

where $H = N^U P_U + N^V P_V$; $N^U (\rho)$ and $N^V (\rho)$ are Lagrange multipliers.

The variables $u$, $v$, $p_u$ and $p_v$ span an extended phase space of the shell. They contain all true degrees of freedom of the system. The phase space has non-trivial boundaries:

$$p_u \leq 0, \quad p_v \leq 0, \quad -u + v^2 \geq 0.$$  \hfill (4)

The constraint surface of the extended action of the shell consists of two components: outgoing shells for $p_v = 0$ and in-going shells for $p_u = 0$.

### 3 Group quantization

To quantize the system defined by the action (3), we apply the so-called group-theoretical quantization method [4]. There are three reasons for this choice. First, the method as modified for the generally covariant systems by Rovelli [5] (see also [6] and [7]) is based on the algebra of Dirac observables of the system; dependent degrees of freedom don’t influence the definition of Hilbert space. Second, the group method has, in fact, been invented to cope with restrictions such as Eq. (4). Finally, the method automatically leads to self-adjoint operators representing all observables.

In particular, a unique self-adjoint extension of the Hamiltonian is obtained in this way, and this is the reason that the dynamics is unitary. The uniqueness of the self-adjoint extension of the Hamiltonian is truly a result of the group quantization in the sense that the Hamiltonian operator itself, as calculated from the constraint, possesses a one-dimensional family of such extensions.

To begin with, we have to find a complete system of Dirac observables. Let us choose the functions $p_u$, $p_v$, $D_u := up_u$ and $D_v := vp_v$. Observe that $u$ alone is constant only along outgoing shell trajectories ($p_u \neq 0$), and $v$ only along in-going
ones \((p_v \neq 0)\), but \(u p_u\) and \(v p_v\) are always constant. The only non vanishing Poisson
brackets are

\[
\{D_u, p_u\} = p_u, \quad \{D_v, p_v\} = p_v.
\]

This Lie algebra generates a group \(G_0\) of symplectic transformations of the phase
space that preserve the boundaries \(p_u = 0\) and \(p_v = 0\). \(G_0\) is the Cartesian product
of two copies of the two-dimensional affine group \(A\).

The group \(A\) generated by \(p_u\) and \(D_u\) has three irreducible unitary representa-
tions. In the first one, the spectrum of the operator \(\hat{p}_u\) is \([0, \infty)\), in the second, \(\hat{p}_u\)
is the zero operator, and in the third, the spectrum is \((-\infty, 0]\), see Ref. [8]. Thus, we
must choose the third representation; this can be described as follows (details are
given in Ref. [8]).

The Hilbert space is constructed from complex functions \(\psi_u(p)\) of \(p \in [0, \infty)\); the
scalar product is defined by

\[
(\psi_u, \phi_u) := \int_0^\infty \frac{dp}{p} \psi_u^*(p) \phi_u(p),
\]

and the action of the generators \(\hat{p}_u\) and \(\hat{D}_u\) on smooth functions is

\[
(\hat{p}_u \psi_u)(p) = -p \psi_u(p), \quad (\hat{D}_u \psi_u)(p) = -ip \frac{d\psi_u(p)}{dp}.
\]

Similarly, the group generated by \(p_v\) and \(D_v\) is represented on functions \(\psi_v(p)\); the
group \(G_0\) can, therefore, be represented on pairs \((\psi_u(p), \psi_v(p))\) of functions:

\[
\hat{p}_u \left(\psi_u(p), \psi_v(p)\right) = (-p \psi_u(p), 0),
\hat{p}_v \left(\psi_u(p), \psi_v(p)\right) = (0, -p \psi_v(p)),
\hat{D}_u \left(\psi_u(p), \psi_v(p)\right) = (-ip \frac{d\psi_u(p)}{dp}, 0),
\hat{D}_v \left(\psi_u(p), \psi_v(p)\right) = (0, -ip \frac{d\psi_v(p)}{dp}).
\]

This choice guarantees that the Casimir operator \(\hat{p}_u \hat{p}_v\) is the zero operator on this
Hilbert space, and so the constraint is satisfied.

Handling the last inequality (4) is facilitated by the canonical transformation:

\[
t = (u + v)/2, \quad r = (-u + v)/2, \quad (5)
\]
\[
p_t = p_u + p_v, \quad p_r = -p_u + p_v. \quad (6)
\]

The constraint function then becomes \(p_u p_v = (p_t^2 - p_r^2)/4\).
The positivity of \( r \) is simply due to its role as the radius of the shell: it is defined as a square root of a sum of squares of coordinates with the range \( \mathbb{R}^3 \). This suggests the following trick. Let us extend the phase space so that \( r \in (-\infty, +\infty) \) and let us define a symplectic map \( I \) on this extended space by \( I(t, r, p_t, p_r) = (t, -r, p_t, -p_r) \). The quotient of the extended space by \( I \) is isomorphic to the original space, and we adopt it as our phase space.

Clearly, only those functions on the extended space that are invariant with respect to \( I \) will define functions on the quotient. Dirac observables of this kind are, e.g., \( p_t, p_r^2 \), the “dilation” \( D := tp_t + rp_r = up_u + vp_v \) and the square of the “boost” \( J^2 := (tp_r + rp_t)^2 = (-up_u + vp_v)^2 \). The “action” of the map \( \hat{I} \) on the functions \( p_u \), \( p_v \), \( D_u \) and \( D_v \) is:

\[
I p_u I = p_v, \quad ID_u I = D_v, \quad Ip_v I = p_u, \quad ID_v I = D_u.
\]

There are only two choices for \( \hat{I} \) that preserve these relations in the quantum theory:

\[
\hat{I} \left( \psi_u(p), \psi_v(p) \right) = \left( \pm \psi_v(p), \pm \psi_u(p) \right).
\]

We choose the plus sign; it is easy to see that the other choice leads to an equivalent theory. Observe that the resulting representation of the group \( G := G_0 \otimes (\text{id}, I) \) is irreducible.

There are two eigenspaces of \( \hat{I} \): one to the eigenvalue +1, consisting of the pairs with \( \psi_u(p) = \psi_v(p) \), the other to the eigenvalue -1, containing the pairs with \( \psi_u(p) = -\psi_v(p) \). If we choose one of these eigenspaces as our final Hilbert space, we obtain a representation of the classical algebra on the quotient space. Again, the two possible choices give equivalent theories. The final result can easily be brought to the following form. The states are determined by complex functions \( \varphi(p) \) on \( \mathbb{R}_+ \); the scalar product \( \langle \varphi, \psi \rangle \) is

\[
\langle \varphi, \psi \rangle = \int_0^{\infty} \frac{dp}{p} \varphi^*(p)\psi(p);
\]

let us denote the corresponding Hilbert space by \( \mathcal{K} \). The representatives of the above algebra are

\[
\langle \hat{p}_t \varphi \rangle(p) = -p \varphi(p),
\]

\[
\langle \hat{p}_r^2 \varphi \rangle(p) = p^2 \varphi(p),
\]

\[
\langle \hat{D} \varphi \rangle(p) = -ip \frac{d\varphi(p)}{dp},
\]

\[
\langle \hat{J}^2 \varphi \rangle(p) = -p \frac{d\varphi(p)}{dp} - p^2 \frac{d^2 \varphi(p)}{dp^2}.
\]

The next question is that of time evolution. Time evolution of a generally covariant system described by Dirac observables may seem self-contradictory or gauge
dependent. Here, we apply the approach that has been worked out in [3] and [4] using the symmetry group of time shifts found in Sec. 2 of Ref. [2], which is generated by the function \( p_t \). The operator \( -\hat{p}_t \) has the meaning of the total energy \( M \) of the system. We observe that it is a self-adjoint operator with a positive spectrum and that it is diagonal in our representation. The parameter \( t \) of the unitary group \( \hat{U}(t) \) that is generated by \( -\hat{p}_t \) is easy to interpret: \( t \) represents the quantity that is conjugated to \( p_t \) in the classical theory and this is given by Eq. (5). Hence, \( \hat{U}(t) \) describes the evolution of the shell states between the levels of the function \( (U + V)/2 \) on \( \mathcal{M} \).

The missing piece of information of where the shell is on \( \mathcal{M} \) is carried by the quantity \( r \) of Eq. (5). We try to define the corresponding position operator in three steps.

First, we observe that \( r \) itself is not a Dirac observable, but the boost \( J \) is, and that the value of \( J \) at the surface \( t = 0 \) coincides with \( rp_t \). It follows that the meaning of the Dirac observable \( Jp_t^{-1} \) is the position at the time \( t = 0 \). This is in a nice correspondence with the Newton-Wigner construction on one hand, and with the so-called evolving constants of motion by Rovelli [10] on the other.

Second, we try to make \( Jp_t^{-1} \) into a symmetric operator on our Hilbert space. As it is odd with respect to \( I \), we have to square it. Let us then chose the following factor ordering:

\[
\hat{r}^2 := \frac{1}{\sqrt{p}} \int \frac{dp}{p} \frac{1}{\sqrt{p}} \frac{1}{\sqrt{p}} = -\sqrt{p} \frac{d^2}{dp^2} \frac{1}{\sqrt{p}}.
\] (7)

Other choices are possible; the above one makes \( \hat{r}^2 \) essentially a Laplacian and this simplifies the subsequent mathematics. Indeed, we can map \( \mathcal{K} \) unitarily to \( L^2(\mathbb{R}_+) \) by sending each function \( \psi(p) \in \mathcal{K} \) to \( \tilde{\psi}(p) \in L^2(\mathbb{R}_+) \) as follows:

\[
\tilde{\psi}(p) = \frac{1}{\sqrt{p}} \psi(p).
\]

Then, the operator of squared position \( \hat{r}^2 \) on \( L^2(\mathbb{R}_+) \) corresponding to \( \hat{r}^2 \) is

\[
\hat{r}^2 = \frac{1}{\sqrt{p}} \hat{r}^2 \left( \sqrt{p} \tilde{\psi}(p) \right) = -\frac{d^2}{dp^2} \tilde{\psi}(p) = -\tilde{\Delta} \tilde{\psi}(p).
\]

Third, we have to extend the operator \( \hat{r}^2 \) to a self-adjoint one. The Laplacian on the half-axis possesses a one-dimensional family of such extensions [11]. The parameter is \( \alpha \in [0, \pi) \) and the domain of \( \tilde{\Delta}_\alpha \) is defined by the boundary condition at zero:

\[
\tilde{\psi}(0) \sin \alpha + \tilde{\psi}'(0) \cos \alpha = 0.
\]
The complete system of normalized eigenfunctions of $\tilde{\Delta}_\alpha$ is given by:

$$\tilde{\psi}_{\alpha}(r,p) = \sqrt{\frac{2}{\pi}} \frac{r \cos \alpha \cos rp - \sin \alpha \sin rp}{\sqrt{r^2 \cos^2 \alpha + \sin^2 \alpha}};$$

if $\alpha \in (0, \pi/2)$, there is one additional bound state,

$$\tilde{\psi}_{\alpha}(b,p) = \frac{1}{\sqrt{2 \tan \alpha}} \exp(-p \tan \alpha),$$

so that

$$-\tilde{\Delta}_\alpha \tilde{\psi}_{\alpha}(r,p) = r^2 \tilde{\psi}_{\alpha}(r,p),$$

$$-\tilde{\Delta}_\alpha \tilde{\psi}_{\alpha}(b,p) = -\tan^2 \alpha \tilde{\psi}_{\alpha}(r,p).$$

The corresponding eigenfunctions $\psi_{\alpha}$ of the operator $\hat{r}_\alpha^2$ are:

$$\psi_{\alpha}(r,p) = \sqrt{\frac{2p}{\pi}} \frac{r \cos \alpha \cos rp - \sin \alpha \sin rp}{\sqrt{r^2 \cos^2 \alpha + \sin^2 \alpha}},$$

and we restrict ourselves to $\alpha \in [\pi/2, \pi]$, so that there are no bound states and the operator $\hat{r}$ is self-adjoint.

To restrict the choice, we apply the idea of Newton and Wigner. First, the subgroup of $G_0$ that preserves the surface $t = 0$ is to be found. This is, in our case, $U_D(\lambda)$ generated by the dilatation $D$. Then, in the quantum theory, the eigenfunctions of the position at $t = 0$ are to transform properly under this group; this means that the eigenfunction for the eigenvalue $r$ is to be transformed to that for the eigenvalue $U_D(\lambda)r$, for each $\lambda$. The dilatation group generated by $\hat{D}$ acts on a wave function $\psi(p)$ as follows:

$$\psi(p) \rightarrow U_D(\lambda)\psi(p) = \psi(e^{-\lambda}p),$$

where $U_D(\lambda)$ is an element of the group parameterized by $\lambda$. Applying $U_D(\lambda)$ to $\psi_{\alpha}(r,p)$ yields

$$U_D(\lambda)\psi_{\alpha}(r,p) = e^{-\lambda/2} \sqrt{\frac{2p}{\pi}} \frac{r \cos \alpha \cos(e^{-\lambda}rp) - \sin \alpha \sin(e^{-\lambda}rp)}{\sqrt{r^2 \cos^2 \alpha + \sin^2 \alpha}}.$$

The factor $e^{-\lambda/2}$ in the resulting functions of $p$ keeps the system $\delta$-normalized.

Let $\alpha = \pi/2$; then

$$U_D(\lambda)\psi_{\pi/2}(r,p) = e^{-\lambda/2}\psi_{\pi/2}(e^{-\lambda}r,p).$$

Similarly, for $\alpha = \pi$,

$$U_D(\lambda)\psi_{\pi}(r,p) = e^{-\lambda/2}\psi_{\pi}(e^{-\lambda}r,p),$$

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but such relation can hold for no other $\alpha$ from the interval $[\pi/2, \pi]$, because of the form of the eigenfunction dependence on $r$. Now, Newton and Wigner require that

$$U_D(\lambda)\psi(r, p) = e^{-\lambda/2}\psi(e^{-\lambda}r, p).$$

Then all values of $\alpha$ except for $\alpha = \pi/2$ and $\alpha = \pi$ are excluded.

We have, therefore, only two choices for the self-adjoint extension of $\hat{r}^2$:

$$\psi(r, p) := \sqrt{\frac{2p}{\pi}} \sin rp, \quad r \geq 0, \quad (8)$$

and

$$\psi(r, p) := \sqrt{\frac{2p}{\pi}} \cos rp, \quad r \geq 0.$$  

Let us select the first set, Eq. (8); by that, the construction of a position operator is finished.

The construction contains a lot of choice: the large factor-ordering freedom, and the freedom of choosing the self-adjoint extension. One can react to this ambiguity in two ways.

The first is to ask how the different choices influence the results. It seems plausible that the qualitative, rough properties of the quantum system will be the same for all possible choices. We hope (provisionally) that this is true.

The second question to ask is how the position is, in fact, measured in praxis. This question hits the crux of the problem. Indeed, the Newton-Wigner construction may be formally elegant but, to my knowledge, nobody managed to describe the corresponding measurement. If we search for methods of how the position of various constituents in a microscopic system is measured, we find the scattering method to dominate. For that it is necessary to use a particular coupling the system under study, a crystal, say, has with another agent, X-rays, say. One has to send the X-rays onto the crystal and to view what comes out.

It seems, therefore, that the following approach would be more reliable than attempts at a formal definition of a position operator of the shell. One can try, for example, to couple the shell to some field, the quanta of which could be emitted by the shell on its way down and up. The quanta will, or will not reach the asymptotic observers and their properties at infinity might reveal something of what is going on with the shell. This is a future project because it will be mathematically more difficult than our provisional attempt with the position operator.

Another observable that we shall need is $\hat{\eta}$; this is to tell us the direction of motion of the shell at the time zero, having the eigenvalues $+1$ for all purely outgoing shell states, and $-1$ for the in-going ones. In fact, in the classical theory, $\eta = -\text{sgn} pr$,.
but \( p_r \) does not act as an operator on the Hilbert space \( K \), only \( p_r^2 \). Hence, we need the following trick.

Consider the classical dilatation generator \( D = tp_t + rp_r \). It is a Dirac observable; at \( t = 0 \), its value is \( rp_r \). Thus, for positive \( r \), the sign of \(-D\) at \( t = 0 \) has the required value. On the quotient space, the values at negative \( r \) correspond to the \( I \)-mapped states with positive \( r \), and, as \( D \) is \( I \)-invariant, the relation of the sign to the direction of motion is again valid. Hence, we have the relation:

\[
\text{sgn}D = -\eta_{t=0}.
\]

The normalized eigenfunctions \( \psi_a(p) \) of the operator \( \hat{D} \) are solutions of the differential equation:

\[
\hat{D}\psi_a(p) = a\psi_a(p).
\]

The corresponding normalized system is given by

\[
\psi_a(p) = \frac{1}{\sqrt{2\pi}} e^{ia}. 
\]

Hence, the kernels \( P_\pm(p,p') \) of the projectors \( \hat{P}_\pm \) on the purely out- or in-going states are:

\[
P_+(p,p') = \int_{-\infty}^{0} da \psi_a(p) \frac{\psi^*_a(p')}{p'}, \quad P_-(p,p') = \int_{0}^{\infty} da \psi_a(p) \frac{\psi^*_a(p')}{p'}
\]

so that

\[
(\hat{\eta}\psi)(p) = \int_{0}^{\infty} dp'[P_+(p,p') - P_-(p,p')]\psi(p').
\]

This finishes our construction of the shell quantum mechanics.

### 4 Motion of wave packets

We shall work with the family of wave packets on the energy half-axis that are defined by

\[
\psi_{\kappa \lambda}(p) := \frac{(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} p^{\kappa+1/2} e^{-\lambda p},
\]

where \( \kappa \) is a positive integer and \( \lambda \) is a positive number with dimension of length. Using the formula

\[
\int_{0}^{\infty} dp p^n e^{-\nu p} = \frac{n!}{\nu^{n+1}}, \tag{9}
\]
which is valid for all non-negative integers $n$ and for all complex $\nu$ that have a positive real part, we easily show that the wave packets are normalized,

$$\int_0^\infty \frac{dp}{p} \psi^2_{\kappa\lambda}(p) = 1.$$ 

The expected energy,

$$\overline{M}_{\kappa\lambda} := \int_0^\infty \frac{dp}{p} p \psi^2_{\kappa\lambda}(p),$$

of the packet can be calculated by the same formula with the simple result

$$\overline{M}_{\kappa\lambda} = \frac{\kappa + 1/2}{\lambda}.$$ 

The (energy) width of the packet can be represented by the mean quadratic deviation, $\Delta \overline{M}_{\kappa\lambda}$, which is

$$\Delta \overline{M}_{\kappa\lambda} = \frac{\sqrt{2\kappa + 1}}{2\lambda}.$$ 

Hence, by choosing $\kappa$ and $\lambda$ suitably, we can approximate any required energy and width arbitrarily closely.

The time evolution of the packet is generated by $-\hat{p}_t$:

$$\psi_{\kappa\lambda}(t, p) = \psi_{\kappa\lambda}(p)e^{-ipt}.$$ 

Let us calculate the corresponding wave function $\Psi_{\kappa\lambda}(r, t)$ in the $r$-representation,

$$\Psi_{\kappa\lambda}(t, r) := \int_0^\infty \frac{dp}{p} \psi_{\kappa\lambda}(t, p)\psi(r, p),$$

where the functions $\psi(r, p)$ are defined by Eq. (8). Formula (9) then yields:

$$\Psi_{\kappa\lambda}(t, r) = \frac{1}{\sqrt{2\pi}} \frac{\kappa!(2\lambda)^{\kappa+1/2}}{(2\kappa)!} \left[ \frac{i}{(\lambda + it + ir)^{\kappa+1}} - \frac{i}{(\lambda + it - ir)^{\kappa+1}} \right]. \quad (10)$$

It follows immediately that

$$\lim_{r \to 0} |\Psi_{\kappa\lambda}(t, 0)|^2 = 0.$$ 

The scalar product measure for the $r$-representation is just $dr$ because the eigenfunctions (8) are normalized, so the probability to find the shell between $r$ and $r + dr$ is $|\Psi_{\kappa\lambda}(t, r)|^2 dr$.

Our first important result is, therefore, that the wave packets start away from the center $r = 0$ and then are keeping away from it during the whole evolution. This
can be interpreted as the *absence of singularity* in the quantum theory: no part of the packet is squeezed up to a point, unlike the shell in the classical theory.

Observe that the equation $\Psi_{\kappa\lambda}(t,0) = 0$ is *not* a result of a boundary condition imposed on the wave function. It is a result of the unitary dynamics. The nature of the question that we are studying requires that the wave packets start in the asymptotic region so that their wave function vanishes at $r = 0$ for $t \to -\infty$; this is the only condition put in by hand. The fact that the dynamics preserves this equation is the property of the unique self-adjoint extension of the Hamiltonian operator.

A more tedious calculation is needed to obtain the time dependence $\bar{r}_{\kappa\lambda}(t)$ of the expected radius of the shell,

$$\bar{r}_{\kappa\lambda}(t) := \int_0^\infty dr \, r |\Psi_{\kappa\lambda}(t,r)|^2. \tag{11}$$

Let first $\kappa = 0$. The wave function of the packet then is

$$\Psi_{0\lambda}(t,r) = 2\sqrt{\frac{\lambda}{\pi}} \frac{r}{r^2 + (\lambda + it)^2},$$

so the expectation value of $\hat{r}$ is

$$\bar{r}_{0\lambda}(t) = 4\lambda \sqrt{\frac{1}{\pi}} \int_0^\infty dr \, \frac{r^3}{(r^2 + \lambda^2 - t^2)^2 + 4\lambda^2 t^2}.$$

This integral diverges logarithmically, so

$$\bar{r}_{0\lambda}(t) = \infty.$$

Let $\kappa \neq 0$. The substitution of Eq. (10) into (11) leads to:

$$\bar{r}_{\kappa\lambda}(t) = \frac{1}{2\pi} \frac{(\kappa!)^2(2\lambda)^{2\kappa+1}}{(2\kappa)!} \left( I_{\kappa\lambda}(t) - J_{\kappa\lambda}(t) \right),$$

where

$$I_{\kappa\lambda}(t) = \int_0^\infty r \, dr \left\{ \frac{1}{[\kappa + 1]} + \frac{1}{[\kappa - 1]} \right\},$$

$$J_{\kappa\lambda}(t) = \int_0^\infty r \, dr \left\{ \frac{1}{[\kappa + 1]} + \frac{1}{[\kappa - 1]} \right\}.$$

The first integral can be brought by elementary methods to the following form:

$$I_{\kappa\lambda}(t) = \frac{1}{\kappa(t^2 + \lambda^2)^\kappa} + t \int_{-t}^t \frac{ds}{(s^2 + \lambda^2)^{\kappa+1}}.$$
Let us calculate the second integral. We obtain after a simple rearrangement:

\[
J_{\kappa\lambda}(t) = (-1)^{\kappa+1} \int_0^\infty dr \left\{ \frac{r + i\lambda}{[(r + i\lambda)^2 - t^2]^{\kappa+1}} - \frac{i\lambda}{[(r + i\lambda)^2 - t^2]^{\kappa+1}} \right. \\
+ \left. \frac{r - i\lambda}{[(r - i\lambda)^2 - t^2]^{\kappa+1}} + \frac{i\lambda}{[(r - i\lambda)^2 - t^2]^{\kappa+1}} \right\}.
\]

This suggests the introduction of integration contours \( C_1 \) defined in the complex plane by \( z = r + i\lambda \) for \( r \in (0, \infty) \), and \( C_2 \) by \( z = r - i\lambda \), \( r \in (0, \infty) \). Then \( J_{\kappa\lambda}(t) \) can be written as follows:

\[
J_{\kappa\lambda}(t) = (-1)^{\kappa+1} \int_{C_1} dz \left[ \frac{z}{(z^2 - t^2)^{\kappa+1}} - \frac{i\lambda}{(z^2 - t^2)^{\kappa+1}} \right] - (-1)^{\kappa+1} \int_{C_2} dz \left[ \frac{z}{(z^2 - t^2)^{\kappa+1}} + \frac{i\lambda}{(z^2 - t^2)^{\kappa+1}} \right].
\]

The integrals of the first terms in the square brackets can be done immediately:

\[
J_{\kappa\lambda}(t) = -\frac{1}{\kappa} \frac{1}{(\lambda^2 + t^2)^{\kappa}} + (-1)^{\kappa+1} i\lambda \int_{-C_1+C_2} \frac{dz}{(z^2 - t^2)^{\kappa+1}}. \tag{12}
\]

We obtain as the final result:

\[
\bar{r}_{\kappa\lambda}(t) = \frac{1}{2\pi} \frac{(\kappa!)^2 (2\lambda)^{2\kappa+1}}{(2\kappa)!} \left[ \frac{2}{\kappa} \frac{1}{(\lambda^2 + t^2)^{\kappa}} + t \int_{-t}^t \frac{dx}{(x^2 + \lambda^2)^{\kappa+1}} \right]
+ i\lambda (-1)^{\kappa+1} \int_{C_1} \frac{dz}{(z^2 - t^2)^{\kappa+1}} - i\lambda (-1)^{\kappa+1} \int_{C_2} \frac{dz}{(z^2 - t^2)^{\kappa+1}}. \tag{13}
\]

In fact, the R. H. side diverges for \( \kappa = 0 \) so, in this sense, this formula can be considered as completely general, i.e., valid for all \( \kappa \) and \( t \).

Let us study some properties of the function \( \bar{r}_{\kappa\lambda}(t) \). Eq. (13) implies that

\[
\bar{r}_{\kappa\lambda}(t) = \bar{r}_{\kappa\lambda}(-t),
\]

so the average motion of the packet is symmetric under time reversal. Eq. (13) is also suitable for the calculation of the expansions about the points \( t = 0 \) and \( t = \pm\infty \). Consider first the point \( t = 0 \). Expanding the first term in the square bracket is easy:

\[
\frac{2}{\kappa} \frac{1}{(\lambda^2 + t^2)^{\kappa}} = \frac{2}{\kappa \lambda^{2\kappa}} \sum_{k=0}^\infty (-1)^k \binom{\kappa + k - 1}{k} \left( \frac{t}{\lambda} \right)^{2k}.
\]

The series on the R. H. side converges for \( |t| < \lambda \).
To expand the next term, we expand the integrand in the powers of \(x/\lambda\); the series converges for \(|t| < \lambda\). Integrating term by term yields:

\[
t \int_{-t}^{t} \frac{dx}{(x^2 + \lambda^2)^{\kappa+1}} = \frac{2}{\lambda^{2\kappa}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \binom{\kappa+k}{k} \left(\frac{t}{\lambda}\right)^{2k+2}.
\]

Again, this series converges for \(|t| < \lambda\).

A similar method can be applied to the remaining integrals:

\[
\frac{1}{(z^2 - t^2)^{\kappa+1}} = \frac{1}{z^{2\kappa+2}} \sum_{k=0}^{\infty} \binom{\kappa+k}{k} \left(\frac{t}{z}\right)^{2k}.
\]

The convergence is granted for \(|t| < |z|\). As the minimal \(|z|\) along both contours is \(\lambda\), the expansion is always valid for \(|t| < \lambda\). Then

\[
i\lambda(-1)^{\kappa+1} \int_{C_1-C_2} \frac{dz}{(z^2 - t^2)^{\kappa+1}} = -\frac{2}{\lambda^{2\kappa}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2\kappa + 2k + 1} \binom{\kappa+k}{k} \left(\frac{t}{\lambda}\right)^{2k}.
\]

Collecting all terms, we obtain the expansion around \(t = 0\),

\[
\bar{r}_{\kappa\lambda}(t) = \frac{\lambda}{\pi} \frac{(\kappa!)^2 2^{2\kappa+1}}{(2\kappa)!} \left[\frac{\kappa+1}{\kappa(2\kappa+1)} \right]^{\frac{1}{2}} (\kappa+1) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2\kappa + 2k + 1)k(2k-1)} \binom{\kappa+k-1}{k} \left(\frac{t}{\lambda}\right)^{2k},
\]

and the equation holds for \(|t| < \lambda\). As the \(k = 1\) term in Eq. (14) is positive, there is a minimal expected radius \(\bar{r}_{\kappa\lambda}(0)\) at \(t = 0\),

\[
\bar{r}_{\kappa\lambda}(0) = \frac{1}{\pi} \frac{2^{2\kappa}(\kappa!)^2}{(2\kappa)!} \frac{\kappa+1}{\kappa+1/2} > 0.
\]

For a large \(\kappa\), the minimum at \(t = 0\) may be only local and/or the curve may oscillate for \(t \in (-\lambda, \lambda)\).

Let us turn to the asymptotics \(t \to \pm \infty\). It is sufficient to consider the case \(t \to \infty\) because the other one is obtained by \(t \mapsto -t\). The first term in the square brackets in Eq. (13) is clearly of the order \(O(t^{-2\kappa})\) and it is not difficult to convince oneself that the last two terms are both of the order \(O(t^{-2\kappa-1})\).

The second term need more care. First, we use the relation

\[
\int_{-t}^{t} \frac{dx}{(x^2 + \lambda^2)^{\kappa+1}} = \frac{1}{\kappa!} \left(\frac{1}{2\lambda} \frac{d}{d\lambda}\right)^\kappa \int_{-t}^{t} \frac{dx}{x^2 + \lambda^2}
\]

so that we obtain

\[
t \int_{-t}^{t} \frac{dx}{(x^2 + \lambda^2)^{\kappa+1}} = \frac{2}{\kappa!} \left(\frac{1}{2\lambda} \frac{d}{d\lambda}\right)^\kappa \left(\frac{t}{\lambda} \arctan \frac{t}{\lambda}\right).
\]
For $t/\lambda > 0$, the following formula holds:

$$\arctan \frac{t}{\lambda} = \frac{\pi}{2} - \arctan \frac{\lambda}{t}.$$ 

Using it and expanding the function $\arctan(\lambda/t)$ around zero leads to

$$t \int_{-t}^{t} \frac{dx}{x^2 + \lambda^2} = \frac{\pi t}{2^\kappa \kappa!} \left( -\frac{1}{\lambda} \frac{d}{d\lambda} \right)^\kappa \frac{\lambda}{x} - 2(-1)^\kappa \left( \frac{d}{d(\lambda)^2} \right)^\kappa \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left( \frac{\lambda}{t} \right)^{2k}.$$ 

The series converges for $t > \lambda$ and can be differentiated term by term in this interval. The first non-zero term comes only from $k = \kappa$ and it has the value

$$-\frac{2\kappa!}{2\kappa + 1} \frac{1}{t^{2\kappa}}.$$ 

It holds also

$$\left( -\frac{1}{\lambda} \frac{d}{d\lambda} \right)^\kappa \frac{\lambda}{x} = \frac{(2\kappa - 1)!!}{\lambda^{2\kappa+1}}.$$ 

Hence,

$$t \int_{-t}^{t} \frac{dx}{x^2 + \lambda^2} = \frac{\pi t}{2^\kappa \lambda^{2\kappa+1} \kappa!} \frac{(2\kappa - 1)!!}{2^\kappa \lambda^{2\kappa+1} \kappa!} + O(t^{-2\kappa}).$$ 

Substituting this into Eq. (13) and using the symmetry $t \mapsto -t$, we obtain for both cases $t \to \pm \infty$:

$$\bar{r}_{\kappa \lambda}(t) \approx |t| + O(t^{-2\kappa}). \quad (16)$$

A further interesting question about the motion of the packets is about the portion of a given packet that moves in—is purely in-going—at a given time $t$. The portion is given by $\|P_- \psi_{\kappa \lambda}\|^2$, where $P_-$ is the projector defined in Sec. 3. Let us calculate this quantity.

If we write out the projector kernel and make some simple rearrangements in the expression of the norm, we obtain:

$$\|P_- \psi_{\kappa \lambda}\|^2 = \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dq'' \left( \int_{0}^{\infty} da \psi_a^*(e^{q'}) \psi_a(e^{q''}) \right) \psi_{\kappa \lambda}^*(t, e^{q'}) \psi_{\kappa \lambda}(t, e^{q''}),$$

where the transformation of integration variables $p'$ and $p''$ to $e^{q'}$ and $e^{q''}$ in the projector kernels has been performed.

The integral in the parenthesis,

$$\int_{0}^{\infty} da \psi_a^*(e^{q'}) \psi_a(e^{q''}) = \frac{1}{2\pi} \int_{0}^{\infty} da e^{ia(q''-q')},$$
is a kernel in an integral that is exponentially damped at the infinities. Thus, we can calculate it as a limit,

\[
\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_0^\infty da \, e^{ia(q''-q')-i\epsilon a} = \frac{i}{2\pi} \lim_{\epsilon \to 0} \frac{1}{(q'' - q') + i\epsilon} = \frac{i}{2\pi} \mathcal{P} \frac{1}{q'' - q'} + \frac{1}{2}\delta(q'' - q'),
\]

where \(\mathcal{P}\) denotes the principal value.

Doing the integral over the \(\delta\)-function gives the simple result:

\[
\frac{1}{2}(\psi_{\kappa\lambda}, \psi_{\kappa\lambda}) = \frac{1}{2}.
\]

The rest can be written as follows:

\[
\|\hat{P}_-\psi_{\kappa\lambda}\|^2 = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dq'' \psi_{\kappa\lambda}(e^{q'})\psi_{\kappa\lambda}(e^{q''}) \frac{\cos t(q'' - q')}{(q'' - q')}.
\]

The integrand is a sum of a symmetric and an anti-symmetric functions of the variables \(q'\) and \(q''\). The principal value integral annihilates the anti-symmetric part. The integral from the symmetric part is already regular, and we can write the final formula:

\[
\|\hat{P}_-\psi_{\kappa\lambda}\|^2 = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dq'' \psi_{\kappa\lambda}(e^{q'})\psi_{\kappa\lambda}(e^{q''}) \frac{\sin t(q'' - q')}{(q'' - q')}.
\]

Let us calculate the in-going portion for some simple values of \(t\). Thus, for \(t = 0\), we obtain immediately:

\[
\|\hat{P}_-\psi_{\kappa\lambda}\|^2_{t=0} = \frac{1}{2}.
\]

At the time zero, the probabilities to catch the shell going in or out are equal.

The limit \(t \to \pm \infty\) can be obtained, if we use the formula:

\[
\lim_{t \to \pm \infty} \frac{\sin tx}{x} = \pm \pi \delta(x).
\]

Hence,

\[
\lim_{t \to \pm \infty} \frac{\sin t(e^{q''} - e^{q'})}{q'' - q'} = \pm \pi \frac{e^{q''} - e^{q'}}{q'' - q'} \delta(e^{q''} - e^{q'}).
\]

Substituting this into the integral of Eq. (17) and returning back to the variables \(p'\) and \(p''\) results in:

\[
\|\hat{P}_-\psi_{\kappa\lambda}\|^2_{t=\pm\infty} = \frac{1}{2} + \frac{1}{2} \int_0^\infty \int_0^\infty \frac{dp'}{p'} \frac{dp''}{p''} \psi_{\kappa\lambda}(p')\psi_{\kappa\lambda}(p'') \frac{p'' - p'}{\log p'' - \log p'} \delta(p'' - p').
\]
The expression
\[ \frac{p'' - p'}{\log p'' - \log p'} \]
is smooth and equal to \( p' \) at \( p'' = p' \). Hence, finally
\[ \| \hat{P}_{\psi_{\kappa\lambda}^t} \|_{t \to -\infty}^2 = 1, \quad \| \hat{P}_{\psi_{\kappa\lambda}^t} \|_{t \to \infty}^2 = 0, \]
and we have only in-going, or only outgoing shells at the infinity.

The obvious interpretation of these formulae is that quantum shell always bounces at the center and re-expands. We can, however, ask further questions. For example, what is the time delay of the re-expansion as compared, say, with the same trajectory in the background manifold \( \mathcal{M} \) that carries the flat metric
\[ ds^2 = -dUdV + \frac{1}{4}(U + V)^2 d\Omega^2 \]
in our coordinates \( U \) and \( V \)? To find this time delay, the “true” metric with respect to these coordinates had to be calculated. The metric is determined by the quantum state in a similar way as the position and the colour of the horizon are (see the next section). However, unlike the points and the metric inside \( \mathcal{M} \), the points and metric in the asymptotic region are gauge invariant quantities. The method by which we should calculate the asymptotic metric ought to make the gauge invariance of the result transparent. Such a method has first to be developed.

The result that the quantum shell bounces and re-expands is clearly at variance with the classical idea of black hole forming in the collapse and preventing anything that falls into it from re-emerging. It is, therefore, natural to ask, if the packet is squeezed enough so that an important part of it comes under its Schwarzschild radius. We can try to answer this question by comparing the minimal expected radius \( \bar{r}_{\kappa\lambda}(0) \) with the expected Schwarzschild radius \( \bar{r}_{\kappa\lambda H} \) of the wave packet. The Schwarzschild radius is given by
\[ \bar{r}_{\kappa\lambda H} = 2G\bar{M}_{\kappa\lambda} = 2\frac{\bar{M}_{\kappa\lambda}}{M_P^2}, \]
where \( M_P \) is the Planck energy. Now, the values of \( \kappa \) and \( \lambda \) for which a large part of the packet gets under its Schwarzschild radius clearly satisfy the inequality
\[ \bar{r}_{\kappa\lambda}(0) < \bar{r}_{\kappa\lambda H}, \]
or
\[ (\lambda M_P)^2 < 2\pi \frac{\kappa(\kappa + 1/2)^2}{\kappa + 1} \frac{(2\kappa)!}{2^{2\kappa}(\kappa!)^2}. \]
Interpreting $\lambda$ roughly as the spatial width of the packet, we have $\lambda M_P \gg 1$ for reasonably broad packets. Then the right-hand side can be estimated by the Stirling formula:

$$2\pi \frac{\kappa (\kappa + 1/2)^2}{\kappa + 1} \frac{(2\kappa)!}{2^{2\kappa} (\kappa!)^2} \approx \sqrt{2\pi} \kappa.$$ 

Substituting this into the inequality (19) yields

$$\bar{M}_{\kappa \lambda} > \frac{\lambda M_P}{\sqrt{2\pi}} M_P,$$

which implies that the threshold energy for squeezing the packet under its Schwarzschild radius is much larger than the Planck energy. For narrow wave packets, we have that $\lambda M_P \approx 1$, so the inequality (19) is satisfied, and the threshold energy is about one Planck energy. The inequality (20) expresses, therefore, always the desired property. To summarize: Reasonably narrow packets can, in principle, get under their Schwarzschild radius; their energy must be much larger than Planck energy. Even in such a case, the shell bounces and re-expands.

This apparent paradox will be explained in the next section.

5 Grey horizons

In this section, we try to explain the apparently contradictory result that the quantum shell can cross its Schwarzschild radius in both directions. The first possible idea that comes to mind is simply to disregard everything that our model says about Planck regime. This may be justified, because the model can hardly be considered as adequate for this regime. However, the model is mathematically consistent, simple and solvable; it must, therefore, provide some mechanism to make the horizon leaky. We shall study this mechanism in the hope that it can work in more realistic situations, too.

To begin with, we have to recall that the Schwarzschild radius is the radius of a non-diverging null hyper-surface; anything moving to the future can cross such a hyper-surface only in one direction. The local geometry is that of an apparent horizon. (Whether or not an event horizon forms, that can also depend on the geometry near the singularity [12]). However, as Einstein’s equations are invariant under time reversal, there are two types of Schwarzschild radius: that associated with a black hole and that associated with a white hole. Let us call these Schwarzschild radii themselves black and white. The explanation of the paradox that follows from the model is that quantum states can contain a linear combination of black and white horizons, and that no event horizon forms. We call such a combination a grey horizon.
The existence of grey horizons can be shown as follows. The position and the “colour” of a Schwarzschild radius outside the shell is determined by the spacetime metric. For our model, this metric is a combination of purely gauge and purely dependent degrees of freedom, and so it is determined, within the classical version of the theory, by the physical degrees of freedom through the constraints.

To explain the idea in more detail, let us start with the general case in the ADM formalism. There are 16 canonical variables, the 6 components of the three-metric \( q_{kl} \), the 6 components of the conjugate momentum \( \pi_{kl} \), 1 lapse and three shift functions. These can be decomposed (non-uniquely) into physical, gauge, and dependent variables. Fixing the gauge variables by hand (this also includes some boundary conditions in non-compact cases) means that a particular space-like surface \( \Sigma \) is chosen, and a particular coordinate system \( x^k \), \( k = 1, 2, 3 \), is lain onto this surface. Then the constraints turn into differential equations determining the dependent part of \( q_{kl} \) and \( \pi_{kl} \) in terms of the physical one and so the tensor fields \( q_{kl} \) and \( \pi_{kl} \) are determined uniquely along \( \Sigma \) in the coordinates \( x^k \) by the physical degrees of freedom. By this, the full spacetime metric \( g_{\mu\nu} \) and all its first derivatives are known at each point of \( \Sigma \). Indeed, if we choose the Gaussian coordinates \( x^0, x^k \), adapted to \( \Sigma \), then the four-metric at \( \Sigma \) is

\[
d s^2 = -(dx^0)^2 + q_{kl}dx^kdx^l,
\]

and the derivatives of this metric with respect to the coordinates \( x^0 \) and \( x^k \) are given by

\[
\frac{\partial g_{00}}{\partial x^0} = 0, \quad \frac{\partial g_{00}}{\partial x^k} = 0, \quad \frac{\partial g_{0k}}{\partial x^0} = 0,
\]

\[
\frac{\partial g_{kl}}{\partial x^0} = -2K_{kl}, \quad \frac{\partial g_{kl}}{\partial x^m} = \frac{\partial q_{kl}}{\partial x^m}, \quad \frac{\partial g_{0k}}{\partial x^l} = 0,
\]

where \( K_{kl} := (\det q_{kl})^{-1/2}(1/2 q^{mn}\pi_{mn}q_{kl} - \pi_{kl}) \) is the second fundamental form of the surface \( \Sigma \). Observe that the choice of Gaussian coordinates is equivalent to specifying the lapse and shift at \( \Sigma \) by hand. The lapse and shift could also be fixed by the condition that the gauge is preserved by the evolution.

Let \( S \) be a closed two-surface on \( \Sigma \). We can calculate the Gaussian coordinates adapted to \( S \) in \( \Sigma \) from the metric \( q_{kl} \) of \( \Sigma \); let they be \( x^A \) and \( x^3 \), \( A = 1, 2 \), so that \( S \) is given by \( x^3 = 0 \) and \( x^3 \) increases in the outside direction. Let the corresponding components of the tensor fields be \( q'_{kl} \) and \( K'_{kl} \). Then the induced two-metric on \( S \) is \( q'_{AB} \) and the full three-metric on \( \Sigma \) is

\[
d s^2 = (dx^3)^2 + q'_{AB}dx^Adx^B.
\]

Now, let \( l^\mu \) and \( n^\mu \) be null vectors orthogonal to \( S \), \( l^\mu \) being the outgoing and \( n^\mu \) the in-going one. Their component in terms of the coordinates \( x^0 := x^0 \) and \( x^k \) are

\[
l^\mu = (1, 0, 0, 1), \quad n^\mu = (1, 0, 0, -1).
\]
Then
\[
\frac{\partial g'_{AB}}{\partial x'^\mu} = \frac{\partial g'_{AB}}{\partial x'^0} + \frac{\partial g'_{AB}}{\partial x'^3} = -2K'_{AB} + \frac{\partial q'_{AB}}{\partial x'^3},
\]
and, similarly,
\[
\frac{\partial g'_{AB}}{\partial x'^\mu} = -2K'_{AB} - \frac{\partial q'_{AB}}{\partial x'^3},
\]
We can, therefore, check, whether or not the following equation holds
\[
g'_{AB} \left( -2K'_{AB} \pm \frac{\partial q'_{AB}}{\partial x'^3} \right) = 0,
\]
and so can find, if \( S \) is an out- (in-)going apparent horizon—Eq. (21) is then valid with the above (lower) sign.

Let us look to see how this algorithm works for the shell model. The constraints are \( P_U(\rho) = 0 \) and \( P_V(\rho) = 0 \). If the transformation from the original variables \( \Lambda(\rho), R(\rho), P_\lambda(\rho), P_R(\rho), \eta, r \) and \( p \) to \( U(\rho), V(\rho) \), \( P_U(\rho) P_V(\rho) \), \( \eta, u, v, p_u \) and \( p_v \) were known, it would provide the functionals:
\[
P_U(\rho) = P_U[\lambda, R, P_\lambda, P_R, \eta, r, p; \rho),
\]
and
\[
P_V(\rho) = P_V[\lambda, R, P_\lambda, P_R, \eta, r, p; \rho).
\]
The transformation is not known explicitly, but we know that the constraint equations
\[
P_U[\lambda, R, P_\lambda, P_R, \eta, r, p; \rho) = 0, \quad P_V[\lambda, R, P_\lambda, P_R, \eta, r, p; \rho) = 0
\]
are equivalent to the original constraints, Eqs. (1) and (2). Hence, our first trick is to work with Eqs. (1) and (2), instead of Eqs. (22).

The constraints (1) and (2) contain the physical variables \( \eta, M u \) and \( v \) also through \( r \) and \( p \). We can choose the gauge variables to be \( R(\rho) \) and \( \Lambda(\rho) \). A fixed function \( R(\rho) \) determines \( \rho \) in terms of the geometrical quantity \( R \) and so it fixes a radial coordinate along \( \Sigma \). \( \Lambda(\rho) \) contains derivatives of the embedding functions, so it determines the slope of the embedding at each \( \rho \). Integrating the slope gives a family of surfaces; a suitable boundary condition at infinity selects one of them.

In order to obtain a suitable surface \( \Sigma \) the functions \( R(\rho) \) and \( \Lambda(\rho) \) have to satisfy some further boundary conditions at the infinity, at the shell and at the regular center. The condition at the infinity, \( \rho \to \infty \), is to guarantee that \( \Sigma \) is asymptotically flat. That at the shell is necessary in order that \( \Sigma \) is smooth across
the shell. Finally, at $\rho = 0$, we require that $\Sigma$ cut the regular center rather than the singularity and that it be a smooth surface at this point. Similarly, $P_R$ and $P_\Lambda$ have to satisfy suitable boundary conditions at $\rho = 0$, $\rho = r$ and $\rho \to \infty$. The explicit form of these boundary conditions are carefully discussed in [1].

Then the constraints become equations for the two functions $P_R(\rho)$ and $P_\Lambda(\rho)$. Eq. (1) is an algebraic equation for $P_R$; solving it and inserting the solution into Eq. (2) gives an ordinary differential equation for $P_\Lambda$. The differential equation, together with the boundary conditions, determines $P_\Lambda(\rho)$ uniquely, and this, in turn, together with Eq. (1), gives $P_R(\rho)$. The solution is unique. From the known functions $R(\rho)$, $\Lambda(\rho)$, $P_R(\rho)$ and $P_\Lambda(\rho)$, we can determine $q_{kl}$ and $K_{kl}$ along $\Sigma$ and check Eq. (21).

One can try to solve Eqs. (1) and (2) for $P_R(\rho)$ and $P_\Lambda(\rho)$ explicitly, by choosing the functions $R(\rho)$ and $\Lambda(\rho)$ in some way that simplifies the equations. Instead, we use the uniqueness of the solution in the following simple trick. Any solution of the constraint equations in the spherically symmetric case defines an initial data and surface for a solution to Einstein’s equations that is itself spherically symmetric. Hence, every such solution of constraints forms a space-like surface that can be embedded in some Schwarzschild spacetime. There will always be the Schwarzschild solution of mass zero inside the shell, and the Schwarzschild solution of mass $M$ outside it.

In this way, we find by inspection from the Kruskal diagram: If the shell is ingoing, $\eta = -1$, then it is contracting and any space-like surface containing such a shell can at most intersect an outgoing apparent horizon at the radius $R = 2GM$, independently of which of the two infinities the surface is connecting the shell with. Analogous result holds for $\eta = +1$, where the shell is expanding. The corresponding $\rho_H$ is determined by the equation $R(\rho_H) = 2GM$, and the horizon will cut $\Sigma$ if and only if $\rho_H > r$. We can assign the value $+1 (-1)$ to the horizon that is out-(in-)going and denote the quantity by $c$ (colour: black or white hole). Then $c = -\eta$. In particular, if we choose the gauge so that $R(\rho) = \rho$, then $r$ is just $r(t)$, where $t$ is the value of the parameter $t$ at which the shell intersect $\Sigma$, and we have:

1. The condition that an apparent horizon intersects $\Sigma$ is $r_t < 2GM$.
2. The position of the horizon at $\Sigma$ is $\rho_H = 2GM$.
3. The value of $c$ is $c = -\eta$.

In this way, questions about the existence and colour of an apparent horizon outside the shell are reduced to equations containing dynamical variables of the shell. In particular, the result that $c = -\eta$ can be expressed by saying that the shell always creates a horizon outside that cannot block its motion. All that matters is that the shell can bounce at the singularity (which it cannot within the classical theory).
These results can be carried over to quantum mechanics after quantities such as $2G\bar{M} - r$ and $\bar{\eta}$ are expressed in terms of the operators describing the shell. Then we obtain a “quantum horizon” with the “expected radius” $2G\bar{M}$ and with the “expected colour” $-\bar{\eta}$ to be mostly black at the time when the expected radius of the shell crosses the horizons inwards, neutrally grey at the time of the bounce and mostly white when the shell crosses it outwards.

This proof has, however, two weak points. First, the spacetime metric on the background manifold is not a gauge invariant quantity; although all gauge invariant geometrical properties can be extracted from it within the classical version of the theory, this does not seem to be possible in the quantum theory. Second, calculating the quantum spacetime geometry along hyper-surfaces of a foliation on a given background manifold is foliation dependent. For example, one can easily imagine two hyper-surfaces $\Sigma$ and $\Sigma'$ belonging to different foliations, that intersect each other at a sphere outside the shell and such that $\Sigma$ intersects the shell in its in-going and $\Sigma'$ in its outgoing state. Observe that the need for a foliation is only due to our insistence on calculating the quantum metric.

The essence of these problems is the gauge dependence of the results of the calculation. However, it seems that this dependence concerns only details such as the distribution of different hues of grey along the horizon, not the qualitative fact that the horizon exists and changes colour from almost black to almost white. Still, a more reliable method to establish the existence and properties of grey horizons would require another material system to be coupled to our model; this could probe the spacetime geometry around the shell in a gauge-invariant way.

It may still seem difficult to imagine any spacetime that contains an apparent horizon of mixed colours. Nevertheless, examples of such space-times can readily be constructed if the assumption of differentiability is abandoned. A continuous, piece-wise differentiable spacetime can make sense as a history within the path integral method.

The simplest construction of this kind is based on the existence of the time reversal isometry $T$ as defined in the foregoing paper that maps an in-going shell spacetime onto an outgoing one.

Let us choose a space-like hyper-surface $\Sigma_1$ crossing the shell before this hits the singularity in a $(-1, M, u)$-spacetime $\mathcal{M}$, and find the corresponding surface $T\Sigma_1$ in the spacetime $T\mathcal{M}$ with the parameters $(1, M, v)$. Then we cut away the part of $\mathcal{M}$ that lies in the future of $\Sigma_1$ and the part of $T\mathcal{M}$ in the past of $T\Sigma_1$. As their boundaries are isometric to each other, the remaining halfs can be stuck together in a continuous way. In the resulting spacetime, the shell contracts from the infinity until it reaches $\Sigma_1$ at the radius $r_1$; then, it turns its motion abruptly to expand towards infinity again. There is no singularity and the spacetime is flat everywhere.
inside the shell. If \( r_1 < 2GM \), then there is an apparent horizon at \( R = 2GM \). It comes into being where the in-going shell crosses the radius \( r = 2GM \) and is outgoing (black) until it reaches \( \Sigma_1 \). Then, it changes its colour abruptly to white (in-going) and lasts only until the outgoing shell crosses it again.

The space-like hyper-surface \( \Sigma_1 \) can be chosen arbitrarily in \( \mathcal{M} \). The construction can, therefore, be repeated in the future of \( T\Sigma_1 \) in an analogous way so that we obtain a spacetime with two “pleats”; the shell contracts, then expands, then contracts again and hits the singularity. The horizon starts as a black ring, then changes to a white one, and then it becomes black for all times. This history is, however, not continuous. Clearly, one can repeat the construction arbitrary many times; this leads to a “pleated” spacetime with a zig-zag motion of the shell and alternating horizon rings of white and black colour. If the spacetime is to be singularity free, however, there must be an odd number of pleats and an even number of rings, beginning with the black ring and ending with a white one.

The conditions that the surface \( \Sigma_1 \) cuts the trajectory of the shell at some small value of the Schwarzschild radial coordinate \( R \), is smooth and space-like everywhere and hits the space-like infinity \( i^0 \) for large values of \( R \) allow a considerable freedom. We can require in addition that \( \Sigma_1 \) joins smoothly to the surface \( T = T_1 \), where \( T \) is the Schwarzschild time coordinate and \( T_1 \) some constant so that \( \Sigma_1 \) coincides with \( T = T_1 \) for all values of \( R \) larger than, say, \( R_1 \). It is clear from the Penrose diagram that such a \( \Sigma_1 \) can ran arbitrarily close to the incoming shell trajectory and can be joined to \( T = T_1 \) for arbitrary low value of \( T_1 \in (-\infty, \infty) \), if \( R_1 \) is chosen sufficiently large. On the other hand, for \( R_1 = 2GM + \epsilon \), \( \Sigma_1 \) can join \( T = T_1 \) for arbitrarily large \( T_1 \in (-\infty, \infty) \), just if \( \epsilon > 0 \).

Consider now an observer at the fixed value \( R_0 \) of the Schwarzschild radius in each shell spacetime. We shall choose \( \Sigma_1 \) in such a way that \( R_1 < R_0 \). With this choice, the observer trajectory \( R = R_0 \) remains smooth at \( \Sigma_1 \). Then, the lower bound on the possible values of \( T_1 \) is \( T_c \), which is the Schwarzschild time of the point at which the observer crosses the shell. There is, however, no upper bound on \( T_1 \). Hence, we can construct a one-pleat spacetime for each value of \( T_1 \) from the interval \( (T_c, \infty) \) with a smooth trajectory of the observer. For each value of \( T_1 \), the observer will measure the proper time

\[
\Delta \tau = 2\sqrt{\left(1 - \frac{2GM}{R_0}\right)}(T_1 - T_c) \in (0, \infty)
\]

between his two encounters with the shell. Thus, the time delay can be made arbitrarily small or large. (Of course, all such histories and many others must be integrated with some suitable measure in a path integral in order to obtain a reasonable value of the delay).
Let us choose a gauge in each spacetime constructed above such that the trajectory of the shell is $V = u$ for the in-going part and $U = u$ for the outgoing one. Then, the metric in the asymptotic region, where the observer is, will read

$$ds^2 = -A(U,V)dUdV + R^2(U,V)dR^2,$$

and it is clear that the functions $A(U,V)$ and $R(U,V)$ must have different forms for different values of $T_1$, or else the proper time $\Delta \tau$ measured by the observer will be independent of $T_1$. In most cases, the asymptotic behaviour of the metric in this gauge will be different from (18). On the other hand, in each such spacetime, there will be double null coordinates $U_1$ and $V_1$, say, in which the metric will have the asymptotic behaviour (18). However, the trajectory of the shell will then be given, with respect to the coordinates $U_1$ and $V_1$, by different equations for different values of $T_1$.

6 Concluding remarks

Comparison of the motion of wave packets of Sec. 4 with the classical dynamics of the shell as described in Sec. 3 of [2] shows a marked difference. Whereas all classical shells cross their Schwarzschild radius and reach the singularity in some stage of their evolution, the quantum wave packets never reach the singularity, but always bounce and re-expand; few of them manage to cross their Schwarzschild radius during their motion. This behaviour is far from being a small perturbation around a classical solution if the classical spacetime is considered as a whole. Even locally, the semi-classical approximation is not valid near the bouncing point. It is surely valid in the whole asymptotic region, where narrow wave packets follow more or less the classical trajectories of the shell.

The most important question, however, concerns the validity of the semi-classical approximation near the Schwarzschild radius. We have seen that the geometry near the radius can resemble the classical black hole geometry in the neighbourhood of the point where the shell is crossing the Schwarzschild radius inwards. Then, the radius changes its colour gradually and the geometry becomes very different from the classical one. Finally, near the point where the shell crosses the Schwarzschild radius outwards, the radius is predominantly white and the quantum geometry can be again similar to the classical geometry, this time of a white hole horizon.

If the change of colour is very slow then the neighbourhood of the inward crossing where the classical geometry is a good approximation can be large. It seems that sufficiently large time delays would allow for arbitrarily slow change of colour. We cannot exclude, therefore, that the quantum spacetime contains an extended region with the geometry resembling its classical counterpart near a black hole horizon,
at least locally. This can be true even if the quantum spacetime as a whole differs strongly from any typical classical collapse solution.

One can even imagine the following scenario (which needs a more realistic model than a single thin shell). A quantum system with a large energy collapses and re-expands with huge time delay. The black hole horizon phase is so long, that Hawking evaporation becomes significant and must be taken into account in the calculation. It does then influence the time delay and the period of validity of the black hole approximation. The black hole becomes very small and only then the change of horizon colour becomes significant. The white hole stage is quite short and it is only the small remnant of the system that, finally, re-expands. The whole process can still preserve unitarity. In fact, this is a scenario for the issue of Hawking evaporation process. At least, it is not excluded by the results of the present paper.

The calculations of this paper are valid only for null shells. Similar calculations have been performed in [20]. There has been re-expansion and unitarity for massive shells if the rest mass has been smaller than the Planck mass ($10^{-5}$ g). It is very plausible that the interpretation of these results is similar to that given in the present paper. Thus, we can expect the results valid at least for all “light” shells. There is, in any case, a long way to any astrophysically significant system and a lot of work is to be done before we can claim some understanding of the collapse problem.

Our method of dealing with the problem employs simplified models and a kind of effective theory of gravity; it does not worry about the final form of a full-fledged theory of quantum gravity. This need not be completely unreasonable approach. Even if the ultimate quantum gravity theory were known, most calculations would still be performed within the approximation of some effective theory and for simplified models (compare the situation in the QCD). The method can give useful hints also because of the fact that the black hole geometry is “made up” from purely dependent degrees of freedom of the gravitational field, and these degrees of freedom have no proper quantum character of their own.

To summarize: We have demonstrated, at least for light shells, that quantum theory can smoothly unify two states of motion, one being the time reversal of the other, into one history. In this way, geometry containing a piece of a black hole horizon can be followed by geometry containing a piece of a white hole horizon—just the opposite to the situation we know from the Kruskal diagram of the classical general relativity. In this way, the quantum evolution can stay unitary and the question posed at the beginning of the paper can be answered as follows: A quantum system is not always lost if it falls under its Schwarzschild radius.
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