On the Existence of Generalized Parking Spaces for Complex Reflection Groups

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Abstract

Let $W$ be an irreducible finite complex reflection group acting on a complex vector space $V$. For a positive integer $k$, we consider a class function $\varphi_k$ given by $\varphi_k(w) = k^{\dim V^w}$ for $w \in W$, where $V^w$ is the fixed-point subspace of $w$. If $W$ is the symmetric group of $n$ letters and $k = n + 1$, then $\varphi_{n+1}$ is the permutation character on (classical) parking functions. In this paper, we give a complete answer to the question when $\varphi_k$ (resp. its $q$-analogue) is the character of a representation (resp. the graded character of a graded representation) of $W$. As a key to the proof in the symmetric group case, we find the greatest common divisors of specialized Schur functions. And we propose a unimodality conjecture of the coefficients of certain quotients of principally specialized Schur functions.

1 Introduction

1.1 Background

A (classical) parking function of length $n$ is a map $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ satisfying $\# f^{-1}(\{1, 2, \ldots, k\}) \geq k$ for each $k = 1, \ldots, n$. The notion of parking function was introduced by Pyke [15] and independently by Konheim–Weiss [8], and later becomes one of the main characters in algebraic combinatorics.

Let $PF_n$ be the set of all parking functions of length $n$. The symmetric group $S_n$ of $n$ letters $\{1, 2, \ldots, n\}$ acts on $PF_n$ by the rule $f \mapsto f \circ w$ for $f \in PF_n$ and $w \in S_n$. The corresponding permutation module is called the (classical) parking space. It is an important problem in Catalan combinatorics to generalize the parking space from $S_n$ to other complex reflection groups and from the Coxeter number to more general parameters (including the Fuss cases).

It is known ([7], Proposition 2.6.1) that the $S_n$-action on $PF_n$ is isomorphic to the $S_n$-action on $(\mathbb{Z}_{n+1})^n / \langle \mathbf{1} \rangle$, where $\mathbb{Z}_{n+1} = \mathbb{Z} / (n+1)\mathbb{Z}$ and $\langle \mathbf{1} \rangle$ is the subgroup generated by $\mathbf{1} = (1, 1, \ldots, 1) \in (\mathbb{Z}_{n+1})^n$. Hence we see that the corresponding permutation character $\chi_{PF_n}$ is given by

$$\chi_{PF_n}(w) = (n + 1)^{(\text{type}(w)) - 1} \quad (w \in S_n),$$

(1.1)

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where type($w$) is the cycle type of $w$ and $l($type($w$)) is the number of cycles in $w$. In particular, there are $(n + 1)^{n-1}$ parking functions and the number of $\mathfrak{S}_n$-orbits is equal to the $n$th Catalan number.

As a generalization of the $\mathfrak{S}_n$-set $PF_n$ to Weyl groups $W$, Haiman \[7\] studied the quotient $Q/(h + 1)Q$ of the root lattice $Q$, where $h$ is the Coxeter number of $W$. For real reflection groups $W$, Armstrong–Reiner–Rhoades \[2\] proposed two new $W$-parking spaces $\mathbb{C}[\text{Park}_{W}^{\text{NC}}]$ and $\text{Park}_{W}^{\text{alg}}$. And Rhoades \[16\] gives Fuss analogues $\mathbb{C}[\text{Park}_{W}^{\text{NC}}(p)]$ and $\text{Park}_{W}^{\text{alg}}(p)$ of the constructions in \[2\]. Note that $\text{Park}_{W}^{\text{NC}} = \text{Park}_{W}^{\text{NC}}(1)$ and $\text{Park}_{W}^{\text{alg}} = \text{Park}_{W}^{\text{alg}}(1)$. There are simple formulas for the characters of these generalized parking spaces. These character formulas lead to the following definition.

**Definition 1.1.** Let $W$ be a finite complex reflection group acting on a complex vector space $V$. For a positive integer $k$, let $\varphi_k^W$ and $\varphi_k^W$ be the class functions on $W$ defined by

$$
\varphi_k^W(w) = k^{\dim V_w} (w \in W),
$$

$$
\varphi_k^W(w) = \frac{\det_V(1-q^kw)}{\det_V(1-qw)} (w \in W),
$$

where $V_w$ is the fixed-point subspace of $w$ and $q$ is an indeterminate. We omit the superscript $W$ from $\varphi_k^W$ and $\varphi_k^W$ if there is no confusion.

Note that $\varphi_k$ is a $q$-analogue of $\varphi_k$ in the sense that $\lim_{q \to 1} \varphi_k = \varphi_k$.

For a Weyl group $W$, Sommers \[20\], Proposition 3.9 proved that, if $k$ is “very good”, in particular if $k = h + 1$, then $\varphi_k$ is the permutation character of $Q/kQ$. For a real reflection group, it is shown (or conjectured) that $\mathbb{C}[\text{Park}_{W}^{\text{NC}}(p)]$ and $\text{Park}_{W}^{\text{alg}}(p)$ have the same character $\varphi_{ph+1}$ and the graded representation $\text{Park}_{W}^{\text{alg}}(p)$ has the graded character $\varphi_{ph+1}$. Here a graded representation of $W$ is a representation $U$ of $W$ equipped with a direct sum decomposition $U = \bigoplus_{i \in \mathbb{Z}} U_i$ into $W$-stable subspaces $U_i$, and the graded character of $U$ is defined as $\sum_{i \in \mathbb{Z}} \chi_i q^i$, where $\chi_i$ is the character of $U_i$.

However, $\varphi_k$ (or $\varphi_k$) does not necessarily come from genuine (graded) representations of $W$. For example, if $W = \mathfrak{S}_2$, then $\varphi_k$ is the character of a representation if and only if $k$ is odd.

### 1.2 Main Theorem

The aim of this paper is to answer the following questions:

**Question 1.2.**

1. When is $\varphi_k^W$ the character of a representation of $W$?

2. When is $\varphi_k^W$ is the graded character of a graded representation of $W$?

We call a representation affording the character $\varphi_k$ or $\varphi_k$ a **generalized parking space**.

To state our main theorem, we introduce generalizations of $W$-Catalan numbers.

**Definition 1.3.** Let $W$ be a finite complex reflection group. The **generalized q-Catalan number** $\text{Cat}_k(W,q)$ and the **generalized dual q-Catalan number** $\text{Cat}_k^*(W,q)$ are defined by

$$
\text{Cat}_k(W,q) = \prod_{i=1}^{r} \frac{[k + d_i - 1]_q}{[d_i]_q}, \quad \text{Cat}_k^*(W,q) = q^N \prod_{i=1}^{r} \frac{[k - d_i^* - 1]_q}{[d_i]_q},
$$

where $d_i$ and $d_i^*$ are the degrees and dual degrees of the reflection $W_i$. Here $N = \dim V$. For $W = \mathfrak{S}_2$, we have $\text{Cat}_{k}(\mathfrak{S}_2,q) = \text{Cat}_{k}^{*}(\mathfrak{S}_2,q) = 1$ and $\text{Cat}_{k}(\mathfrak{S}_2,q)$ is the number of $k$-ary compositions of $r$. In particular, $\text{Cat}_{k}(\mathfrak{S}_2,q)$ is a $q$-analogue of the $k$-th Catalan number.
where \( [m]_q = (1 - q^m)/(1 - q) \) is the \( q \)-integer, \( d_1, \ldots, d_r \) (resp. \( d_1^*, \ldots, d_r^* \)) are the degrees (resp. the codegrees) of \( W \) and \( N \) is the number of reflecting hyperplanes.

For example, \( \text{Cat}_{n+1}(\mathfrak{S}_n, q) \) is MacMahon’s \( q \)-analogue of the Catalan number, and \( \text{Cat}_{h+1}(W, q) \) is a \( q \)-analogue of \( W \)-Catalan number \( \text{Cat}_{h+1}(W, 1) \).

The following is the main result of this paper.

**Theorem 1.4.** Let \( W \) be an irreducible finite complex reflection group and \( k \) a positive integer. Then we have

(A) For any irreducible complex reflection group \( W \), the following are equivalent:

(i) \( \tilde{\varphi}_k^W \) is the graded character of a graded representation of \( W \).

(ii) Both \( \text{Cat}_k(W, q) \) and \( \text{Cat}_k^*(W, q) \) are polynomials in \( q \).

(iii) \( k \) satisfies the congruence condition given in Table 1.

(B) Except for dihedral groups \( W = G(m, m, 2) = D_{2m} \), the conditions in (A) are equivalent to the following:

(iv) \( \varphi_k^W \) is the character of a representation of \( W \).

(v) \( \varphi_k^W \) is the character of a permutation representation of \( W \).

(C) If \( W = G(m, m, 2) = D_{2m} \), then the following conditions are equivalent:

(iii’) \( k = 1 \), or \( k \geq m - 1 \) and \( k \) satisfies the congruence

\[
    k^2 \equiv 1 \pmod{2m} \quad \text{if } m \text{ is even,}
    \quad \mod{m} \quad \text{if } m \text{ is odd.}
\]

(iv) \( \varphi_k^W \) is the character of a representation of \( W \).

(v) \( \varphi_k^W \) is the character of a permutation representation of \( W \).

We should remark that \( \varphi_k \) does not come from a genuine representation of \( W \) even if \( k \) is relatively prime to the Coxeter number \( h \). For example, if \( W \) is the Coxeter group of type \( H_3 \), then \( k = 3 \) is relatively prime to the Coxeter number \( h = 10 \), but \( \varphi_3 \) is not the character of a representation of \( W \).

Some partial results on the existence of generalized parking spaces can be found in the literatures. Haiman [7, Proposition 2.4.1] proved the equivalence of (i), (ii), and (iii) for the symmetric groups. Sommers [20, Proposition 3.9] proved the implication (iii) \( \implies \) (v) for Weyl groups. Note that \( k \) is “very good” in the sense of [20] if and only if \( k \) satisfies the condition in (iii). Also it is known that some of \( \tilde{\varphi}_k \), up to a power of \( q \), appear as the characters of finite dimensional representations of rational Cherednik algebras. See [3] for example.
Table 1: Condition in Theorem 1.4 (iii)

| group                        | condition on $k$                              |
|------------------------------|----------------------------------------------|
| $S_n$                        | $\gcd(k, n) = 1$                             |
| $G(m, p, n)$ (n ≥ 3 or p < m)| $k \equiv 1 \mod m$                          |
| $G(m, m, 2)$                 | $k \equiv \pm 1 \mod m$                     |
| $C_m$                        | $k \equiv 1 \mod m$                          |
| $G_4$                        | $k \equiv 1, 3 \mod 6$                       |
| $G_5$                        | $k \equiv 1 \mod 6$                          |
| $G_6$                        | $k \equiv 1, 9 \mod 12$                      |
| $G_7$                        | $k \equiv 1 \mod 12$                         |
| $G_8$                        | $k \equiv 1, 5 \mod 12$                      |
| $G_9$                        | $k \equiv 1, 17 \mod 24$                     |
| $G_{10}$                     | $k \equiv 1 \mod 12$                         |
| $G_{11}$                     | $k \equiv 1 \mod 24$                         |
| $G_{12}$                     | $k \equiv 1, 11, 17, 19 \mod 24$             |
| $G_{13}$                     | $k \equiv 1, 17 \mod 24$                     |
| $G_{14}$                     | $k \equiv 1, 19 \mod 24$                     |
| $G_{15}$                     | $k \equiv 1 \mod 24$                         |
| $G_{16}$                     | $k \equiv 1, 11 \mod 30$                     |
| $G_{17}$                     | $k \equiv 1, 41 \mod 60$                     |
| $G_{18}$                     | $k \equiv 1 \mod 30$                         |
| $G_{19}$                     | $k \equiv 1 \mod 60$                         |
| $G_{20}$                     | $k \equiv 1, 19 \mod 30$                     |
| $G_{21}$                     | $k \equiv 1, 49 \mod 60$                     |
| $G_{22}$                     | $k \equiv 1, 29, 41, 49 \mod 60$             |
| $G_{23} = W(H_3)$            | $k \equiv 1, 5, 9 \mod 10$                   |
| $G_{24}$                     | $k \equiv 1, 9, 11 \mod 14$                  |
| $G_{25}$                     | $k \equiv 1 \mod 6$                          |
| $G_{26}$                     | $k \equiv 1 \mod 6$                          |
| $G_{27}$                     | $k \equiv 1, 19, 25 \mod 30$                 |
| $G_{28} = W(F_4)$            | $k \equiv 1, 5 \mod 6$                       |
| $G_{29}$                     | $k \equiv 1, 9, 13, 17 \mod 20$              |
| $G_{30} = W(H_4)$            | $k \equiv 1, 11, 19, 29 \mod 30$            |
| $G_{31}$                     | $k \equiv 1, 13, 17, 29, 37, 41, 49, 53 \mod 60$ |
| $G_{32}$                     | $k \equiv 1, 7, 13, 19 \mod 30$              |
| $G_{33}$                     | $k \equiv 1 \mod 6$                          |
| $G_{34}$                     | $k \equiv 1, 13, 19, 25, 31, 37 \mod 42$     |
| $G_{35} = W(E_6)$            | $k \equiv 1, 5 \mod 6$                       |
| $G_{36} = W(E_7)$            | $k \equiv 1, 5 \mod 6$                       |
| $G_{37} = W(E_8)$            | $k \equiv 1, 7, 11, 13, 17, 19, 23, 29 \mod 30$ |
1.3 Strategy of proof

Our proof of Theorem 1.4 proceeds as follows. We prove the following six implications

(a) (i) $\implies$ (ii),
(b) (ii) $\implies$ (iii),
(c) (iii) $\implies$ (i),
(d) (iii) (or (iii’) for dihedral groups) $\implies$ (v),
(e) (v) $\implies$ (iv),
(f) (iv) $\implies$ (iii) (or (iii’) for dihedral groups).

Among these implications, (e) is obvious, and (a) follows from the fact that the generalized $q$-Catalan numbers $\text{Cat}_k(W, q)$ and $\text{Cat}_k^*(W, q)$ appear as multiplicities of some irreducible characters in $\tilde{\varphi}_k$ (See Proposition 2.2). The other implications will be proved in a case-by-case manner.

The implication (b) is proved by decomposing $\text{Cat}_k(W, q)$ and $\text{Cat}_k^*(W, q)$ into the cyclotomic polynomials.

For the implications (c) and (f), we write

$$\varphi_k = \sum_{\chi \in \text{Irr}(W)} m_k^\chi \chi, \quad \tilde{\varphi}_k = \sum_{\chi \in \text{Irr}(W)} \tilde{m}_k^\chi \chi,$$

where $\text{Irr}(W)$ is the set of all irreducible characters of $W$. Then the condition (iv) (resp. (i)) is equivalent to saying that $m_k^\chi$ is a nonnegative integers (resp. $\tilde{m}_k^\chi$ is a polynomials in $q$ with nonnegative integer coefficients) for all $\chi \in \text{Irr}(W)$. And the multiplicities $m_k^\chi$ and $\tilde{m}_k^\chi$ can be explicitly computed. If $W = S_n$ or $G(m, p, n)$, then these multiplicities are given in terms of specializations of Schur functions (see Proposition 3.4 and Corollary 4.3). For the exceptional groups, we can use the program GAP [17] together with CHEVIE [5] to compute these multiplicities. By using these explicit formulas for the multiplicities, we can prove (c) and (f).

In order to prove (d), we appeal to a result of Orlik–Solomon [13, 14], which provides an expression of $\varphi_k$ as a linear combination of certain permutation characters $\eta_j$:

$$\varphi_k = \sum_{j=1}^s n_j(k) \eta_j,$$

where $n_j(k)$ is a polynomial in $k$. (See Proposition 2.3) Thus we can prove (d) by showing that all $n_j(k)$’s are nonnegative integers if $k$ satisfies the condition in (iii).

In the proof for the symmetric groups, a key role is played by the following results on the greatest common divisors of specialized Schur functions, which is interesting in itself.

**Theorem 1.5.** (Theorem 3.5 below) Let $k$ and $n$ be positive integers. Then we have

$$\gcd \mathbb{Z}\{s_\lambda(1, \ldots, 1)_k : \lambda \vdash n\} = \frac{k}{\gcd(n, k)},$$

$$\gcd \mathbb{Q}[q]\{s_\lambda(1, q, \ldots, q^{k-1})_k : \lambda \vdash n\} = \frac{[k]_q}{[\gcd(n, k)]_q}.$$
1.4 Organization

The remaining of this paper is organized as follows. In Section 2, we prepare several general results which are useful in the proof of our main result (Theorem 1.4), especially for exceptional groups. Sections 3, 4, 5, and 6 are devoted to the proof of the main result in the cases where \( W = S_n, G(m, p, n), D_{2m} \) and exceptional groups, respectively. In Subsection 3.2, we prove the formulae for the greatest common divisors of specialized Schur functions. In Section 7, we conclude with some problems.

1.5 Notations

In this paper, \( \mathbb{N} \) and \( \mathbb{N}[q] \) denote the set of nonnegative integers and the set of polynomials in \( q \) with nonnegative integer coefficients respectively.

2 Preliminaries

In this section, we review two general results for complex reflection groups and present several criteria for nonnegativity and polynomiality.

2.1 \( q \)-Catalan numbers and multiplicities

We refer the readers to [11] for the theory of complex reflection groups.

Let \( V \) be a complex vector space of dimension \( r \). For an linear map \( w : V \to V \), we put 
\[
V^w = \{ v \in V : w(v) = v \}.
\]

A complex reflection on \( V \) is an invertible linear map \( w : V \to V \) satisfying \( \dim V^w = \dim V - 1 \). Such a hyperplane \( V^w \) is called a reflecting hyperplane. A complex reflection group is a subgroup \( W \) of \( \text{GL}(V) \) which is generated by complex reflections. Then \( V \) is called the reflection representation of \( W \). We say that \( W \) is irreducible if \( V \) has no \( W \)-stable subspace other than \( \{0\} \) and \( V \). Shephard–Todd [19] classified the irreducible finite complex reflection groups into three infinite families

1. the symmetric groups \( S_n \),
2. the groups \( G(m, p, n) \),
3. the cyclic groups \( C_m \),

and 34 exceptional groups \( G_4, \ldots, G_{37} \).

Let \( W \) be a complex reflection group with \( r \)-dimensional reflection representation \( V \). Given an \( n \)-dimensional \( W \)-module \( M \), the \( M \)-exponents \( m_1(M) \leq m_2(M) \leq \cdots \leq m_n(M) \) are defined by 
\[
\sum_{i=1}^{n} q^{m_i(M)} = \sum_{d \geq 0} \dim \text{Hom}_W(\mathcal{H}_d, M)q^d,
\]
where $\mathcal{H} = \bigoplus_{d \geq 0} \mathcal{H}_d$ is the space of $W$-harmonic polynomials. Then the degrees $(d_1, \ldots, d_r)$ and the codegrees $(d^*_1, \ldots, d^*_r)$ of $W$ are given in terms of the $M$-exponents for $M = V$ and $V^*$:

$$d_i = m_i(V) + 1, \quad d^*_i = m_i(V^*) - 1 \quad (1 \leq i \leq r).$$

See [11, pp. 274–275] for the data of degrees and codegrees. It is known that the number of reflecting hyperplanes are given by

$$N = \sum_{i=1}^r (d^*_i + 1). \quad (2.1)$$

It is convenient to work with the following generalization of $\varphi_k$.

**Definition 2.1.** For indeterminates $q$ and $u$, let $\widetilde{\varphi}$ be the class function on $W$ defined by

$$\widetilde{\varphi}(w) = \frac{\det_V(1 - uw)}{\det_V(1 - qw)} \quad (w \in W). \quad (2.2)$$

Note that $\widetilde{\varphi}_k$ is obtained from $\widetilde{\varphi}$ by substituting $u = q^k$. If we denote by $\text{Irr}(W)$ the set of irreducible characters of $W$, then we can write

$$\widetilde{\varphi} = \sum_{\chi \in \text{Irr}(W)} \tilde{m}^\chi \chi, \quad \widetilde{\varphi}_k = \sum_{\chi \in \text{Irr}(W)} \tilde{m}_k^\chi \chi, \quad \varphi_k = \sum_{\chi \in \text{Irr}(W)} m_k^\chi \chi, \quad (2.3)$$

where $\tilde{m}^\chi \in \mathbb{C}(q)[u]$, $\tilde{m}_k^\chi \in \mathbb{C}(q)$, and $m_k^\chi \in \mathbb{C}$. It follows from

$$\tilde{m}^\chi = \frac{1}{\#W} \sum_{w \in W} \widetilde{\varphi}(w) \chi(w),$$

that $\tilde{m}^\chi$ is a polynomial in $u$ of degree $\leq r$. Similarly we see that $m_k^\chi$ is a polynomial in $k$ of degree $\leq k$.

**Proposition 2.2.** The multiplicities of the trivial character $\text{triv}$ and the determinant character $\text{det} = \det_V$ in $\widetilde{\varphi}$ are given by

$$\tilde{m}^\text{triv} = \prod_{i=1}^r \frac{1 - uq^{d_i-1}}{1 - q^{d_i}}, \quad \tilde{m}^\text{det} = \prod_{i=1}^r \frac{1 - uq^{-d^*_i-1}}{1 - q^{d^*_i}}. \quad (2.4)$$

In particular, we have

$$\tilde{m}^\text{triv}_k = \text{Cat}_k(W, q), \quad \tilde{m}^\text{det}_k = \text{Cat}_k^*(W, q). \quad (2.5)$$

**Proof.** For a linear character $\chi$ of $W$, we have \([10, (2.3)]\)

$$\frac{1}{\#W} \sum_{w \in W} \widetilde{\varphi}(w) \chi(w) = q^{m(\mathbb{C}_\chi)} \prod_{i=1}^r \frac{1 - uq^{m_i(V \otimes \mathbb{C}_\chi) - m(\mathbb{C}_\chi)}}{1 - q^{d_i}},$$

where $\mathbb{C}_\chi$ is the one-dimensional $W$-module affording $\chi$, and $m(\mathbb{C}_\chi)$, $\{m_i(V \otimes \mathbb{C}_\chi)\}_{i=1}^r$ denote the exponents of $\mathbb{C}_\chi$ and $V \otimes \mathbb{C}_\chi$ respectively. We apply this formula to $\chi = \text{triv}$ and $\text{det}^{-1}$. By using the relations (see [10] for example)

$$m(\mathbb{C}_{\text{triv}}) = 0, \quad m(\mathbb{C}_{\text{det}^{-1}}) = N,$$

$$m_i(V \otimes \mathbb{C}_{\text{triv}}) = m_i(V), \quad m_i(V \otimes \mathbb{C}_{\text{det}^{-1}}) = N - m_{r+1-i}(V^*),$$

we obtain the formulas (2.4). \qed
2.2 Orlik–Solomon formula

Let $W$ be a complex reflection group acting on $V$, and $\mathcal{A}$ the set of reflecting hyperplanes. We denote by $L(\mathcal{A})$ the intersection lattice of $\mathcal{A}$. That is, $L(\mathcal{A})$ is the set of all intersections of hyperplanes in $\mathcal{A}$ ordered by reverse inclusion. Then $W$ acts on $L(\mathcal{A})$.

**Proposition 2.3.** (Orlik–Solomon [13, 14]) Let $X_1, \ldots, X_s$ be a complete set of representatives of $W$-orbits on $L(\mathcal{A})$. For $1 \leq j \leq s$, we put

$$W_j = \{ w \in W : X_j \subset V^w \},$$

$$\mathcal{A}_j = \{ H \cap X_j : H \in \mathcal{A}, H \not\supset X_j \},$$

and denote by $\eta_j$ the permutation character of $W$ on $W/W_j$. Then we have

$$\varphi_k = \sum_{j=1}^{s} \frac{\chi(\mathcal{A}_j, k)}{[N_W(W_j) : W_j]} \eta_j,$$

where $\chi(\mathcal{A}_j, t)$ is the characteristic polynomial of the hyperplane arrangement $\mathcal{A}_j$, and $N_W(W_j)$ is the normalizer of $W_j$ in $W$.

The polynomials $\chi(\mathcal{A}_j, t)$ are monic polynomials and the roots are known to be positive integers. For exceptional groups $W = G_{23}, \ldots, G_{37}$ of rank $\geq 3$, the $W$-orbits on $L(\mathcal{A})$ and the roots of the corresponding characteristic polynomials $\chi(\mathcal{A}_j, t)$ are listed in [13, Tables III–VIII] (Coxeter groups of type $H_3, H_4, E_6, E_7, E_8$) and [14, Tables 3–11] (other groups). For the remaining rank 2 groups $W = G_4, \ldots, G_{22}$, one can find the lists of the orbits $O_j$ with $W_j$ and their sizes $\#O_j$ in [14, Table 2], and the index is computed by

$$[N_W(W_j) : W_j] = \frac{\#W}{\#W_j \cdot \#O_j}.$$ 

For the groups $\mathfrak{S}_n$ and $G(m, 1, n)$, we can use the theory of symmetric functions to prove the corresponding results. See Propositions 3.4 (2) and 4.5.

2.3 Nonnegativity

In this subsection, we present three general lemmas. The proofs are easy, so we omit them. In order to prove the implication (iv) $\implies$ (iii) of Theorem 1.4 for exceptional groups, we need to find all integers $k$ satisfying $m_k^X \in \mathbb{N}$ for a fixed irreducible character $\chi$ of $W$. Since $m_k^X$ is a polynomial in $k$, we can appeal to the following lemma to reduce the proof to a finite amount of computation.

**Lemma 2.4.** Let $f \in \mathbb{C}[t]$ be a polynomial in $t$. Assume that there is a positive integer $L$ such that $f(t + L) - f(t)$ maps nonnegative integers to nonnegative integers. Then, for an integer $i$ with $1 \leq i \leq L$, the following are equivalent:

(i) $f(i) \in \mathbb{N}$.

(ii) $f(k) \in \mathbb{N}$ for any integer $k$ with $k \equiv i \mod L$.

The assumption in the lemma above can be checked by using the following lemma.
Lemma 2.5. Let \( g \in \mathbb{C}[t] \) be a polynomial in \( t \) of degree \( \leq r \). We define \( b_0, b_1, \ldots, b_r \) recursively by the relations

\[
b_0 = g(0), \quad b_i = g(i) - \sum_{j=0}^{i-1} b_j \binom{i}{j}.
\]

Then we have

\[
g(t) = \sum_{i=0}^{r} b_i \binom{t}{i},
\]

and, if \( b_0, \ldots, b_r \in \mathbb{N} \), then \( g(t) \) maps nonnegative integers to nonnegative integers.

This lemma can also be used in the proof of (iii) \( \implies \) (v), where we prove that \( \chi(A_j, pH + i)/[N_W(W_i) : W_i] \in \mathbb{N} \) for all \( p \in \mathbb{N} \).

The following lemma is a \( q \)-analogue of Lemma 2.5. This lemma will be used in the proof of (iii) \( \implies \) (i), where we show that \( \tilde{m}_{pH+i} = \tilde{m}^x(q^{pH+i}) \in \mathbb{N}[q] \) for all \( p \in \mathbb{N} \).

Lemma 2.6. Let \( h(u) \) be a polynomial in \( u \) of degree \( \leq r \) with coefficients in \( \mathbb{C}(q) \), and \( M \) a positive integer. Define \( c_0, c_1, \ldots, c_r \) recursively by the relations

\[
c_0 = h(1), \quad c_i = h(q^M) - \sum_{j=0}^{i-1} \left[ \begin{array}{c} i \\ j \end{array} \right]_{q^M} c_j,
\]

where \( \left[ \begin{array}{c} i \\ j \end{array} \right]_{q^M} \) is the \( q \)-binomial coefficient in base \( q^M \), i.e.,

\[
\left[ \begin{array}{c} i \\ j \end{array} \right]_{q^M} = \prod_{l=1}^{j} \frac{1 - q^{(i-l+1)M}}{1 - q^{lM}}.
\]

Then we have

\[
h(u) = \sum_{i=0}^{r} c_i \prod_{l=1}^{i} \frac{1 - uq^{(-l+1)M}}{1 - q^{lM}},
\]

and, if \( c_0, c_1, \ldots, c_r \in \mathbb{N}[q] \), then \( h(q^{pM}) \in \mathbb{N}[q] \) for all nonnegative integers \( p \).

2.4 Polynomials of \( q \)-Catalan numbers

In the proof of (ii) \( \implies \) (iii) of Theorem 1.4 we need to find all integers \( k \) such that \( \text{Cat}^*_k(W, q) \) and \( \text{Cat}^*_k(W, q) \) are both polynomials in \( q \). By using (2.1), we have

\[
\text{Cat}^*_k(W, q) = \prod_{i=1}^{r} q^{d_i+1} - q^k.
\]

Thus \( \text{Cat}^*_k(W, q) \) is a polynomial in \( q \) if and only if \( \text{Cat}^*_k(W, q) \) is a Laurent polynomial in \( q \).

Let \( \Phi_d(q) \) denotes the \( d \)-th cyclotomic polynomial in \( q \). Then \( \Phi_d(q) \in \mathbb{Z}[q] \) is an irreducible polynomial and we have

\[
1 - q^a = \prod_{d|a} \Phi_d(q), \quad (2.7)
\]
where \(d\) runs over all divisors of a positive integer \(a\). The Laurent polynomiality of \(\text{Cat}_k(W, q)\) and \(\text{Cat}^*_k(W, q)\) can be checked by using the following lemma:

**Lemma 2.7.** Given integers \(a_1, \ldots, a_r\), and positive integers \(b_1, \ldots, b_r\), we consider the rational function given by

\[
c(q) = \prod_{i=1}^r \frac{[a_i]_q}{[b_i]_q}.
\]

For a positive integer \(d\), we put

\[
N(d) = \#\{i : d \mid a_i\}, \quad D(d) = \#\{i : d \mid b_i\}.
\]

Then \(c(q)\) is a Laurent polynomial in \(q\) if and only if some of \(a_i\)'s are 0 or \(N(d) \geq D(d)\) for all divisors \(d \in T\), where

\[
T = \bigcup_{i=1}^s \{d : d \mid b_i\}.
\]

**Proof.** Note that \(c(q) = 0\) if and only if some of \(a_i\)'s are 0. So we may assume that all \(a_i\)'s are nonzero. By using the relation \([-a]_q = -q^{-a[a]_q}\) and (2.7), we have

\[
c(q) = \varepsilon q^{-S} \prod_d \Phi_d(q)^{N(d) - D(d)},
\]

where \(d\) runs over all positive integers, and \(\varepsilon \in \{1, -1\}\), \(S = \sum_{i:a_i<0} a_i\). Since \(D(d) = 0\) for \(d \notin T\), we see that \(c(q)\) is a Laurent polynomial if and only if \(N(d) - D(d) \geq 0\) for all \(d \in T\). \(\square\)

### 3 Symmetric groups

In this section, we prove the main result (Theorem 1.4) in the case where \(W = \mathfrak{S}_n\) is the symmetric group. As a key to our proof, we give formulae for the greatest common divisors of specialized Schur functions.

#### 3.1 Representations of \(\mathfrak{S}_n\) and symmetric functions

In this subsection, we review several facts concerning representations of \(\mathfrak{S}_n\) and symmetric functions. See [12, Chapter 1] or [22, Chapter 7] for details.

The symmetric group \(\mathfrak{S}_n\) of \(n\) letters is realized as an irreducible complex reflection group acting on \(V = \{x \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\}\).

A partition of a nonnegative integer \(n\) is a weakly decreasing sequence \(\lambda = (\lambda_1, \lambda_2, \ldots)\) of nonnegative integers satisfying \(\sum_{i \geq 1} \lambda_i = n\). Then we write \(|\lambda| = n\) and \(\lambda \vdash n\). The length of a partition \(\lambda\), denoted by \(l(\lambda)\), is the number of nonzero entries of \(\lambda\). Let \(\mathcal{P}_n\) be the set of partitions of \(n\). We often identify a partition \(\lambda\) with the finite sequence \((\lambda_1, \ldots, \lambda_k)\) with \(k \geq l(\lambda)\).

The conjugacy classes of \(\mathfrak{S}_n\) are parametrized by partitions of \(n\), called the cycle type. A permutation \(w \in \mathfrak{S}_n\) has the cycle type \(\mu \vdash n\), denoted by \(\text{type}(w)\), if \(w\) is decomposed into the product of disjoint cycles of lengths \(\mu_1, \mu_2, \ldots\). Then we have

\[
\det V(1 - tw) = \frac{\prod_{i \geq 1} (1 - t^{\mu_i})}{1 - t},
\]

\[10\]
and
\[ \tilde{\varphi}_k(w) = \prod_{i \geq 1} \left[ \frac{k^{q^i}}{[k]_q} \right], \quad \varphi_k(w) = k^{l(\mu) - 1}. \] (3.1)

The irreducible characters of \( S_n \) are also indexed by partitions of \( n \). Let \( \chi^\lambda \) be the irreducible character of \( S_n \) corresponding to a partition \( \lambda \), and denote by \( \chi^\mu_\lambda \) the character value at an element of cycle type \( \mu \).

Let \( R(S_n) \) be the vector space of complex-valued class functions on \( S_n \). Then the direct sum \( R(S_n) = \bigoplus_{n \geq 0} R(S_n) \) has a commutative associative graded algebra structure with respect to the product defined by
\[ f \cdot g = \text{Ind}_{S_n}^{S_{n+l}}(f \times g), \]
where \( f \in R(S_n) \) and \( g \in R(S_l) \).

Let \( \Lambda \) be the ring of symmetric functions in infinitely many variables \( x = (x_1, x_2, \ldots) \) with complex coefficients. We follow [12] for the notations of symmetric functions. For example, we denote by \( s_\lambda \) the Schur function associated to a partition \( \lambda \). The following identity is known as the Cauchy identity:

**Proposition 3.1.** For two sets of variables \( x \) and \( y \), we have
\[ \prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y) = \sum_\lambda h_\lambda(x) m_\lambda(y), \] (3.2)
where \( h_\lambda \) and \( m_\lambda \) are the complete and monomial symmetric functions respectively.

Define a linear map \( \text{ch} : R(S_n) \to \Lambda \), called the Frobenius characteristic, by
\[ \text{ch}(f) = \frac{1}{n!} \sum_{w \in S_n} f(w) p_{\text{type}(w)} \]
for \( f \in R(S_n) \). Then we have

**Proposition 3.2.**
1. The Frobenius characteristic \( \text{ch} \) is an algebra isomorphism.
2. For a partition \( \lambda \) of \( n \), the images of the irreducible character \( \chi^\lambda \) and the permutation character \( \eta^\lambda \) on \( S_n/\Sigma_\lambda \) are given by
\[ \text{ch}(\chi^\lambda) = s_\lambda, \quad \text{ch}(\eta^\lambda) = h_\lambda. \] (3.3)
3. For a partition \( \mu \) of \( n \), we have
\[ p_\mu(x) = \sum_{\lambda \vdash n} \chi^\lambda_\mu s_\lambda(x), \] (3.4)
where \( \lambda \) runs over all partitions of \( n \).

**Definition 3.3.** For a symmetric function \( f \in \Lambda \) and a positive integer \( k \), we denote by \( f(1^k) \) the specialization of \( f \) with \( x_1 = \cdots = x_k = 1 \) and \( x_{k+1} = x_{k+2} = \cdots = 0 \). And we denote by \( f(1, q, \ldots, q^{k-1}) \) the principal specialization of \( f \) with \( x_i = q^{i-1} \) (\( 1 \leq i \leq k \)) and \( x_i = 0 \) (\( i > k \)).
By using the Frobenius formula (3.4) and the Cauchy identity (3.2), we obtain the following expressions of \( \varphi_k \) and \( \tilde{\varphi}_k \).

**Proposition 3.4.** (See [24])

(1) The class function \( \tilde{\varphi}_k \) is expressed in terms of irreducible characters as

\[
\tilde{\varphi}_k = \sum_{\lambda \vdash n} s_{\lambda}(1, q, \ldots, q^k-1) \frac{[k]_q}{k} \chi^\lambda. \tag{3.5}
\]

In particular, we have

\[
\varphi_k = \sum_{\lambda \vdash n} s_{\lambda}(1^k) \frac{k}{k} \chi^\lambda. \tag{3.6}
\]

(2) The class function \( \varphi_k \) is expressed in terms of permutation characters as

\[
\varphi_k = \sum_{\lambda \vdash n} m_{\lambda}(1^k) \frac{k}{k} \eta^\lambda. \tag{3.7}
\]

**Proof.** (1) By substituting \( x_i = q^{i-1} \) \((1 \leq i \leq k)\) and \( x_i = 0 \) \((i \geq k + 1)\) in the Frobenius formula (3.3) and comparing with (3.1), we obtain (3.5).

(2) By substituting \( y_1 = \cdots = y_k = 1 \) and \( y_{k+1} = \cdots = 0 \) in the Cauchy identity (3.2), and by using (3.3), we have

\[
\text{ch}(\varphi_k) = \sum_{\lambda \vdash n} \frac{s_{\lambda}(1^k)}{k} s_{\lambda} = \sum_{\lambda \vdash n} \frac{m_{\lambda}(1^k)}{k} h_{\lambda} = \sum_{\lambda \vdash n} \frac{m_{\lambda}(1^k)}{k} \text{ch}(\eta_{\lambda}).
\]

Since ch is an isomorphism, we obtain (3.7). \( \square \)

### 3.2 Greatest common divisor of specialized Schur functions

It follows from Proposition 3.4 (1) that \( \varphi_k \) is the character of a representation of \( S_n \) if and only if \( s_\lambda(1^k) \) is divisible by \( k \) for any partitions \( \lambda \vdash n \). Similarly, \( \tilde{\varphi}_k \) is the graded character of a graded representation of \( S_n \) if and only if \( s_\lambda(1, q, \ldots, q^{k-1}) \) is divisible by \( [k]_q \) and the quotient is a polynomial with nonnegative integer coefficients for any \( \lambda \vdash n \).

So the following theorem plays an indispensable role in the proof of Theorem 1.4.

**Theorem 3.5.** Let \( k \) and \( n \) be positive integers.

(1) In the integer ring \( \mathbb{Z} \), we have

\[
\gcd_{\mathbb{Z}}\{s_{\lambda}(1^k) : \lambda \vdash n\} = \frac{k}{\gcd(n, k)}. \tag{3.8}
\]

(2) In the polynomial ring \( \mathbb{Q}[q] \), we have

\[
\gcd_{\mathbb{Q}[q]}\{s_{\lambda}(1, q, \ldots, q^{k-1}) : \lambda \vdash n\} = \frac{[k]_q}{[\gcd(n, k)]_q}. \tag{3.9}
\]

Here the greatest common divisor is taken to be a monic polynomial.
Remark 3.6. It should be noted that (1) is not an immediate consequence of its “q-analogue” (2). For example, if \( f(q) = (q^2 + 1)(q + 1)^2 \) and \( g(q) = (q + 1)^3 \), then we have \( h(q) = \gcd_{\mathbb{Q}[q]} \{ f(q), g(q) \} = (q + 1)^2 \) and \( h(1) = 4 \), while \( \gcd_{\mathbb{Z}} \{ f(1), g(1) \} = 8 \).

In the proof of Theorem 3.5, we will use the following two lemmas.

**Lemma 3.7.** (Haiman [7, Proof of Proposition 2.5.1]) Let \( d, k, r \) be positive integers such that \( d \) divides \( k \).

1. If \( d \mid r \), then no primitive \( d \)-th root of 1 is a root of \( h_r(1, q, \ldots, q^{k-1}) \).
2. If \( d \nmid r \), then any primitive \( d \)-th root of 1 is a simple root of \( h_r(1, q, \ldots, q^{k-1}) \).

**Proof.** If \( r = sd + t \) with \( s \in \mathbb{N} \) and \( 0 \leq t < d \), then we have

\[
h_r(1, q, \ldots, q^{k-1}) = \prod_{i=0}^{s-1} \prod_{j=1}^{d} \frac{1 - q^{id+j+k-1}}{1 - q^{id+j}} \prod_{j=1}^{t} \frac{1 - q^{sd+j+k-1}}{1 - q^{sd+j}},
\]

and the proof follows from this expression.

For a prime \( p \) and an integer \( x \), we denote by \( \nu_p(x) \) the highest power of \( p \) dividing \( x \).

**Lemma 3.8.** (Kummer [9, p. 116]) Let \( p \) be a prime, and \( m, r \) positive integers with \( m \geq r \). Then \( \nu_p \left( \binom{m}{r} \right) \) is equal to the number of borrow required when subtracting \( r \) from \( m \) in the base \( p \) representation.

**Proof of Theorem 3.5.** In the proof, we put \( d = \gcd(n, k) \).

First we prove (2). Let \( g(q) \) be the greatest common divisor of \( s_{\lambda}(1, q, \ldots, q^{k-1}) \)'s with \( \lambda \vdash n \). Since \( \{ h_{\lambda} : \lambda \vdash n \} \) is another \( \mathbb{Z} \)-basis of \( \Lambda_{\mathbb{Z}} = \sum_{\lambda \vdash n} \mathbb{Z} s_{\lambda} \), we have \( g(q) \) is the greatest common divisor of \( h_{\lambda}(1, q, \ldots, q^{k-1}) \)'s. Also we have

\[
\frac{[k]_q}{[d]_q} = \prod_{d \mid k, d \mid n} \Phi_d(q) = \prod_{d \mid k, d \mid n} \prod_{\zeta} (q - \zeta),
\]

where \( \Phi_d \) is the \( d \)-th cyclotomic polynomial and \( \zeta \) runs over all primitive \( d \)-th roots of unity. In order to prove \( g(q) = \frac{[k]_q}{[d]_q} \), it is enough to show the following two claims:

**Claim 1** We have

\[
\{ z \in \mathbb{C} : z \text{ is a common root of } h_{\lambda}(1, q, \ldots, q^{k-1}) \ (\lambda \vdash n) \} = \bigcup_{d \mid k, d \mid n} \{ z \in \mathbb{C} : z \text{ is a primitive } d \text{-th root of } 1 \}.
\]

**Claim 2** If \( z \) is a common root of \( h_{\lambda}(1, q, \ldots, q^{k-1}) \ (\lambda \vdash n) \), then \( z \) is a simple root of \( h_{\mu}(1, q, \ldots, q^{k-1}) \) for some \( \mu \vdash n \).
Proof of Claim 1. We put
\[
C = \{z \in \mathbb{C} : z \text{ is a common root of } h_{\lambda}(1, q, \ldots, q^{k-1}) (\lambda \vdash n)\},
\]
\[
D = \bigcup_{d | k, d \notdiv n} \{z \in \mathbb{C} : z \text{ is a primitive } d\text{-th root of 1}\}.
\]
Let \(z \in C\). Since \(z\) is a root of \(h_{\lambda}(1, q, \ldots, q^{k-1})\) and
\[
h_{\lambda}(1, q, \ldots, q^{k-1}) = \left(h_1(1, q, \ldots, q^{k-1})\right)^n = \left(\prod_{d | k, d \neq 1} \Phi_d(q)\right)^n,
\]
we see that \(z\) is a primitive \(d\)-th root of 1 for some \(d\) dividing \(k\). Since \(z\) is also a root of \(h_n(1, q, \ldots, q^{n-1})\), it follows from Lemma 3.7 that \(d\) does not divide \(n\).

In order to show the inclusion \(D \subseteq C\), let \(z \in D\) be a primitive root of 1 with \(d \div k\) and \(d \notdiv n\). Let \(\lambda\) be a partition of \(n\). Since \(\sum_i \lambda_i = n\) and \(d\) does not divide \(n\), there exists a part \(\lambda_i\) not divisible by \(d\). Then, by using Lemma 3.7, \(z\) is a root of \(h_{\lambda_i}(1, q, \ldots, q^{k-1})\), so \(z\) is a root of \(h_{\lambda}(1, q, \ldots, q^{k-1})\).

Proof of Claim 2. Suppose that \(z\) is a common root of \(h_{\lambda}(1, q, \ldots, q^{k-1})\)'s with \(\lambda \vdash n\). It follows from Claim 1 that \(z\) is a primitive \(d\)-th root of unity for some \(d\) satisfying \(d \div k\) and \(d \notdiv n\). We write \(n = sd + t\) with \(s \in \mathbb{Z}\) and \(0 \leq t < d\), and consider a partition
\[
\mu = (d, \ldots, d, t) \vdash n.
\]
Then
\[
h_{\mu}(1, q, \ldots, q^{k-1}) = h_d(1, q, \ldots, q^{k-1})^s \cdot h_t(1, q, \ldots, q^{k-1})
\]
and it follows from Lemma 3.7 that \(z\) is a simple root of \(h_{\mu}(1, q, \ldots, q^{k-1})\).

Thus we are done for the proof of (2).

Next we prove (1). Let \(g\) be the greatest common divisor of \(s_{\lambda}(1^k)\) with \(\lambda \vdash n\). Since \(\{e_{\lambda} : \lambda \vdash n\}\) is another \(\mathbb{Z}\)-basis of \(A_{\mathbb{Z}}\), the integer \(g\) is the greatest common divisor of \(e_{\lambda}(1^k)\)'s.

From (2), we already know that \([k]_q/[d]_q\) divides \(e_{\lambda}(1, q, \ldots, q^{k-1})\) for any partition \(\lambda \vdash n\). Since \([k]_q/[d]_q\) is a monic polynomial, we see that \(k/d\) divides \(e_{\lambda}(1^k)\). Hence \(k/d\) divides \(g\). We shall show that \(g\) divides \(k/d\).

We fix a prime \(p\) and show that \(\nu_p(g) \leq \nu_p(k/d)\). We put \(G = \nu_p(g)\) and \(K = \nu_p(k)\). We write \(n = s \cdot p^K + r\) with \(s \in \mathbb{N}\) and \(0 \leq r < p^K\), and consider a partition
\[
\lambda = (p^K, \ldots, p^K, r) \vdash n.
\]
Then we have
\[
p^G \mid e_{\lambda}(1^k), \quad \text{i.e.,} \quad p^G \mid \left(\binom{k}{p^K}\right)^s\left(\binom{k}{r}\right).
\]
It follows from Lemma 3.8 that \( \binom{k}{p} \) is not divisibly by \( p \), so we have

\[
p^G \bigg| \binom{k}{r}.
\]

If \( r = 0 \), then \( G = 0 \) and \( \nu_p(g) = 0 \leq \nu_p(k/d) \) as desired. So we may assume \( r > 0 \). We put \( R = \nu_p(r) \). Then it follows from Lemma 3.8 that \( \nu_p(\binom{k}{r}) = K - R \). Hence we conclude that \( K - R \geq G \). On the other hand, since \( \nu_p(n) = \nu_p(r) = R < K = \nu_p(k) \), we have \( \nu_p(d) = R \) and \( \nu_p(k/d) = K - R \). This completes the proof of Theorem 3.5.

Let \( k \) and \( n \) be positive integers and \( \lambda \) be a partition of \( n \). Then Theorem 3.5 (2) implies that \( s_\lambda(1, q, \ldots, q^{k-1}) \) is divisible by \( [k]_q/[d]_q \), where \( d = \gcd(n, k) \). Since \( [k]_q/[d]_q \) is a monic polynomial with integer coefficients, the quotient

\[
s_\lambda(1, q, \ldots, q^{k-1})/[k]_q/[d]_q
\]

is also a polynomial with integer coefficients. In fact, we can show that this quotient has nonnegative integer coefficients. Recall that a polynomial \( f(q) = \sum_{i=0}^{m} a_i q^i \) of degree \( m \) is symmetric (resp. unimodal) if \( a_i = a_{m-i} \) for all \( i \) (resp. if there is an index \( p \) such that \( a_0 \leq a_1 \leq \cdots \leq a_{p-1} \leq a_p \geq \cdots \geq a_d \)). The idea of the following lemma goes back to [1, Theorem 2].

**Lemma 3.9.** (See Guo–Krattenthaler [6] Lemma 5.1 for example.) Let \( g(q) \) be a polynomial with nonnegative integer coefficients, and let \( k \) and \( d \) be positive integers. Assume that \( g(q) \) is symmetric and unimodal and that \( [d]_q f(q)/[k]_q \) is a polynomial in \( q \). Then \( [d]_q f(q)/[k]_q \) has nonnegative integer coefficients.

**Proposition 3.10.** Let \( k \) and \( n \) be positive integers and \( d = \gcd(n, k) \). For a partition \( \lambda \) of \( n \), the quotient

\[
f_\lambda(q) = s_\lambda(1, q, \ldots, q^{k-1})/[k]_q/[d]_q
\]

is a polynomial with nonnegative integer coefficients.

This proposition is also given by Garsia–Leven–Wallach–Xin [4, Theorem 2.1].

**Proof.** It is known ([12] I.8, Example 4]) that \( s_\lambda(1, q, \ldots, q^{k-1}) \) is a symmetric unimodal polynomial with nonnegative integer coefficients. By Theorem 3.5, \( f_\lambda(q) \) is a polynomial in \( q \). Hence we can apply Lemma 3.9 to obtain the conclusion.

### 3.3 Proof of Theorem 1.4 for \( W = \mathfrak{S}_n \)

Now we are ready to prove Theorem 1.4 for \( W = \mathfrak{S}_n \). Since the implication (i) \( \implies \) (ii) follows from Proposition 2.2, it remains to show the implications (ii) \( \implies \) (iii), (iii) \( \implies \) (i), (iv) \( \implies \) (iii), and (iii) \( \implies \) (v).
Proof of (ii) \implies (iii). We prove that, if \( \text{Cat}_k(\mathcal{S}_n, q) \) is a polynomial in \( q \), then \( \gcd(n, k) = 1 \). We note that
\[
\text{Cat}_k(\mathcal{S}_n, q) = \frac{h_n(1, q, \ldots, q^{k-1})}{[k]_q} = \frac{h_n(1, q, \ldots, q^{k-1})}{\prod_{e \mid k, e \neq 1} \Phi_e(q)}.
\]
Let \( e \neq 1 \) be a divisor of \( k \). Since \( \text{Cat}_k(\mathcal{S}_n, q) \) is a polynomial, any \( e \)-th primitive root of \( 1 \) must be a root of \( h_n(1, q, \ldots, q^{k-1}) \). It follows from Lemma 3.7 that \( e \) does not divide \( n \). Hence we see that \( k \) is coprime to \( n \).

Proof of (iii) \implies (i). Suppose that \( \gcd(n, k) = 1 \). By (3.5), it is enough to show that \( s_\lambda(1, q, \ldots, q^{k-1})/k \) is a polynomial with nonnegative integer coefficients for all \( \lambda \vdash n \). This is a consequence of Proposition 3.10.

Proof of (iii) \implies (v). If \( \gcd(n, k) = 1 \), then \( m_\lambda(1^k)/k \in \mathbb{N} \) by Theorem 3.5 (2). Hence it follows from (3.7) that \( \varphi_k \) is a permutation character.

Proof of (iv) \implies (iii). If \( \varphi_k \) is a character of some representation, then it follows from (3.6) that \( s_\lambda(1^k)/k \) is an integer for all \( \lambda \vdash n \). Then \( k \) divides \( \gcd\{s_\lambda(1^k) : \lambda \vdash n \} \). Hence, by using Theorem 3.5 (1), we see that \( k \) divides \( k/\gcd(n, k) \). So we must have \( \gcd(n, k) = 1 \).

This completes the proof of Theorem 1.4 for the symmetric groups.

4 \quad G(m, p, n)

This section is devoted to the proof of the main result for the groups of the form \( W = G(m, p, n) \), except for the dihedral groups. In this section, we fix an integer \( m \geq 2 \) and put \( \zeta = e^{2\pi \sqrt{-1}/m} \).

4.1 Representations of \( G(m, 1, n) \) and symmetric functions

We review several facts concerning representations of \( G(m, 1, n) \) and their connection to symmetric functions. See [12, Chapter 1, Appendix B] for details.

Let \( \mathcal{P}^{(m)}_n \) be the set of \( m \)-tuples of partitions \( \lambda = (\lambda^{(0)}, \ldots, \lambda^{(m-1)}) \) with \( |\lambda| = \sum_{i=0}^{m-1} |\lambda^{(i)}| = n \). We put \( \mathcal{P}^{(m)} = \bigcup_{n \geq 0} \mathcal{P}^{(m)}_n \).

The group \( G(m, 1, n) \) is realized as a complex reflection group acting on \( V = \mathbb{C}^n \), which consists of all monomial matrices such that the nonzero entries are \( m \)-th roots of unity. In this subsection we write \( \mathcal{S}^{(m)}_n = G(m, 1, n) \) for short. The group \( \mathcal{S}^{(m)}_n \) is isomorphic to the semidirect product \( \mathcal{S}_n \ltimes \mathbb{C}^m \), where \( C_m \) is the cyclic group of order \( m \).

It is known that an element \( w \in \mathcal{S}^{(m)}_n \) is conjugate to the matrix of the form
\[
\bigoplus_{i=0}^{m-1} \bigoplus_{j \geq 1} C(\mu_j^{(i)}, \zeta^i),
\]
where $\mu = (\mu^{(0)}, \ldots, \mu^{(m-1)}) \in \mathcal{P}_n^{(m)}$, called the cycle type of $w$ and denoted by type($w$), and $C(l, \alpha)$ is the $l \times l$ matrix given by

$$
C(l, \alpha) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \alpha \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
$$

And two elements of $\mathfrak{S}_n^{(m)}$ are conjugate if and only if they have the same cycle types. Hence the conjugacy classes of $\mathfrak{S}_n^{(m)}$ are indexed by $\mathcal{P}_n^{(m)}$. It is easy to see that, if $w \in \mathfrak{S}_n^{(m)}$ have the cycle type $\mu$, then

$$
\det \chi(1-tw) = \prod_{i=0}^{m-1} \prod_{j \geq 0} \left(1 - \zeta^i \mu_j^{(i)} \right), \quad (4.1)
$$

The irreducible characters of $\mathfrak{S}_n^{(m)}$ are also parametrized by $\mathcal{P}_n^{(m)}$. We denote by $\chi^\lambda$ the irreducible character of $\mathfrak{S}_n^{(m)}$ corresponding to $\lambda$, and by $\chi^\lambda_\mu$ the character value of $\chi^\lambda$ at an element of cycle type $\mu$.

Let $R(\mathfrak{S}_n^{(m)})$ be the vector space of complex-valued class functions on $\mathfrak{S}_n^{(m)}$, and put $R(\mathfrak{S}_n^{(m)})^{(m)} = \bigoplus_{n \geq 0} R(\mathfrak{S}_n^{(m)})$. Then $R(\mathfrak{S}_n^{(m)})$ becomes a graded $\mathbb{C}$-algebra with respect to the product

$$
f \cdot g = \text{Ind}_{\mathfrak{S}_n^{(m)}}^{-1} \mathfrak{S}_n^{(m)} \times \mathfrak{S}_n^{(m)} (f \times g),
$$

where $f \in R(\mathfrak{S}_n^{(m)})$ and $g \in R(\mathfrak{S}_l^{(m)})$.

Let $\Lambda(x^{(i)})$ be the ring of symmetric functions in variables $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots)$, and put

$$
\Lambda^{(m)} = \Lambda(x^{(0)}) \otimes \Lambda(x^{(1)}) \otimes \cdots \otimes \Lambda(x^{(m-1)}).
$$

For $\lambda \in \mathcal{P}_n^{(m)}$, we put

$$
S_\lambda = s_{\lambda^{(0)}}(x^{(0)}) s_{\lambda^{(1)}}(x^{(1)}) \cdots s_{\lambda^{(m-1)}}(x^{(m-1)}),
$$

$$
P_\lambda = \prod_{i=0}^{m-1} \prod_{j \geq 1} \left(\sum_{r=0}^{m-1} \zeta^{ir} \mu_j^{(i)}(x^{(r)})\right).
$$

We define the Frobenius characteristic map $\text{ch}^{(m)} : R(\mathfrak{S}_n^{(m)}) \to \Lambda^{(m)}$ by

$$
\text{ch}^{(m)}(f) = \frac{1}{m^nn!} \sum_{w \in \mathfrak{S}_n^{(m)}} f(w) \overline{P_{\text{type}(w)}} \quad (f \in R(\mathfrak{S}_n^{(m)}),
$$

where

$$
\overline{P_\lambda} = \prod_{i=0}^{m-1} \prod_{j \geq 1} \left(\sum_{r=0}^{m-1} \zeta^{-ir} \mu_j^{(i)}(x^{(r)})\right).
$$

Then we have
Proposition 4.1.  (1) The Frobenius characteristic map \( \text{ch}^{(m)} \) is an algebra isomorphism.

(2) For \( \lambda \in \mathcal{P}_n^{(m)} \), we have
\[
\text{ch}^{(m)}(\chi^\lambda) = S_\lambda. \tag{4.2}
\]

(3) For \( \mu \in \mathcal{P}_n^{(m)} \), we have
\[
P_\mu = \sum_{\lambda \in \mathcal{P}_n^{(m)}} \chi^\lambda_\mu S_\lambda. \tag{4.3}
\]

We can use this setting to give a formula for the multiplicities of irreducible characters in the class function \( \hat{\varphi} \) given by (2.1).

Proposition 4.2. For \( \lambda \in \mathcal{P}_n^{(m)} \), the multiplicity \( \hat{m}^\lambda \) of \( \chi^\lambda \) in \( \hat{\varphi} \) is given by
\[
\hat{m}^\lambda = \hat{\pi}(S_\lambda),
\]
where \( \hat{\pi} : \Lambda^m \to \mathbb{C}(q, u) \) is the ring homomorphism given by
\[
\hat{\pi}(p_l(x^{(r)})) = \begin{cases} 
1 - q^{(m-1)l}u^l & \text{if } r = 0, \\
1 - q^{ml} & \text{if } 1 \leq r \leq m - 1.
\end{cases}
\]

Proof. By a direct computation, we have
\[
\frac{1 - q^{(m-1)l}u^l}{1 - q^{ml}} + \sum_{s=1}^{m-1} \zeta^s q^{sl} \frac{1 - q^{-l}u^l}{1 - q^{ml}} = 1 - \zeta^iu^l.
\]

Hence we see that
\[
\hat{\varphi}(w) = \hat{\pi}(P_{\text{type}(\mu)}) \quad (w \in G(m, 1, n)).
\]

It follows from the Frobenius formula (4.3) that
\[
\hat{\varphi}(w) = \sum_{\lambda \in \mathcal{P}_n^{(m)}} \hat{\pi}(S_\lambda) \chi^\lambda(w) \quad (w \in G(m, 1, n)).
\]

\[\square\]

Corollary 4.3.  (1) The multiplicity \( m_k^\lambda \) of \( \chi^\lambda \) in \( \varphi_k \) is given by
\[
m_k^\lambda = \prod_{x \in \lambda^{(0)}} \frac{m+k-1}{m} + c(x) \frac{m-1}{h(x)} \prod_{r=1}^{k-1} \prod_{x \in \lambda^{(r)}} \frac{k-1}{m} + c(x), \tag{4.4}
\]
where \( x \) runs over all cells in the Young diagram of \( \lambda^{(0)} \) or \( \lambda^{(r)} \), and \( h(x) \) and \( c(x) \) denote the hook length and the content of \( x \) respectively.
If \( k = pm + 1 \) with \( p \in \mathbb{Z} \), then the multiplicity \( \tilde{m}_k^\lambda \) of \( \chi^\lambda \) in \( \tilde{\varphi}_k \) is given by

\[
\tilde{m}_k^\lambda = s_{\lambda(0)}(1, q^m, \ldots, q^{pm}) \prod_{r=1}^{m-1} s_{\lambda(r)}(q^r, q^{r+m}, \ldots, q^{r+(p-1)m}), \tag{4.5}
\]

which is a polynomial in \( q \) with nonnegative integer coefficients.

**Proof.** By replacing \( u \) by \( q^k \) in Proposition 4.2, we see that the multiplicity \( \tilde{m}_k^\lambda \) in \( \tilde{\varphi}_k \) is given by

\[
\tilde{m}_k^\lambda = \bar{\pi}_k(S_\lambda),
\]

where the algebra homomorphism \( \bar{\pi}_k : \Lambda^{(m)} \rightarrow \mathbb{C}(q) \) is defined by

\[
\bar{\pi}_k(p_l(x^{(r)})) = \begin{cases} 
1 - q^{(m+k-1)l} & \text{if } r = 0, \\
1 - q^{ml} & \text{if } 1 \leq r \leq m - 1.
\end{cases}
\]

(1) By putting \( q = 1 \), we see that the multiplicity \( m_k^\lambda \) in \( \varphi_k \) is given by the specialization

\[
m_k^\lambda = \pi_k(S_\lambda),
\]

where the algebra homomorphism \( \pi_k : \Lambda^{(m)} \rightarrow \mathbb{C}(q) \) is defined by

\[
\pi_k(p_l(x^{(r)})) = \begin{cases} 
m + k - 1 & \text{if } r = 0, \\
k - 1 & \text{if } 1 \leq r \leq m - 1.
\end{cases}
\]

Hence (4.4) follows from the specialization formula ([12 I.3 Example 4])

\[
\rho_z(s_\lambda) = \prod_{x \in \lambda} \frac{z + c(x)}{h(x)},
\]

where \( \rho_z : \Lambda \rightarrow \mathbb{C}(z) \) is defined by \( \rho_z(p_l) = z \) for \( l \geq 1 \).

(2) If \( k = pm + 1 \), then the specialization \( \bar{\pi}_k \) is given by substitution

\[
x^{(r)}_j = \begin{cases} 
q^{(j-1)m} & \text{if } r = 0 \text{ and } 1 \leq j \leq p + 1, \\
q^{r+(j-1)m} & \text{if } 1 \leq r \leq m - 1 \text{ and } 1 \leq j \leq p, \\
0 & \text{otherwise}.
\end{cases}
\]

Hence we obtain (4.5). \( \square \)

Next we give an expression of \( \varphi_k \) as a linear combination of permutation characters.

**Lemma 4.4.** Let \( F : R(S_n) \rightarrow R(S_n^{(m)}) \) be the linear map defined by

\[
F(f) = \text{Ind}_{S_n}^{S_n^{(m)}} f \quad (f \in R(S_n)).
\]
Then $F$ is an graded algebra homomorphism and the following diagram commutes.

$$
\begin{array}{ccl}
R(\mathfrak{S}_*) & \xrightarrow{F} & R(\mathfrak{S}^{(m)}_*) \\
\downarrow \text{ch} & & \downarrow \text{ch}^{(m)} \\
\Lambda & \xrightarrow{\Delta} & \Lambda^{(m)}
\end{array}
$$

where $\Delta : \Lambda \to \Lambda^{\otimes m}$ is the coproduct given by the plethystic substitution

$$
\Delta(f) = f[x^{(0)} + x^{(1)} + \cdots + x^{(m-1)}].
$$

Proof. By using the definition of the products on $R(\mathfrak{S}_*)$ and $R(\mathfrak{S}^{(m)}_*)$, we see that

$$
F(f \cdot g) = \text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_l}^{\mathfrak{S}^{(m)}_n} (f \times g) = F(f) \cdot F(g)
$$

for $f \in R(\mathfrak{S}_n)$ and $g \in R(\mathfrak{S}_l)$.

If we denote by $c_\mu$ the characteristic function of the conjugacy class of $\mathfrak{S}_n$ corresponding to a partition $\mu$, then we have

$$
\text{ch } c_\mu = \frac{1}{z_\mu} p_\mu,
$$

where $z_\mu = \prod_{i \geq 1} i^{m_i} m_i!$. And $F(c_\mu)$ is the characteristic function of the conjugacy class of $\mathfrak{S}^{(m)}_n$ corresponding to $(\mu, \emptyset, \ldots, \emptyset)$, so we have

$$
\text{ch}^{(m)}(F(c_\mu)) = \frac{1}{z_\mu} \prod_{i=1}^{l(\mu)} (p_{\mu_i}(x^{(0)}) + \cdots + p_{\mu_i}(x^{(m-1)})) = \frac{1}{z_\mu} [x^{(0)} + \cdots + x^{(m-1)}].
$$

Hence the diagram commutes. \hfill \square

Proposition 4.5. If $k = pm + 1$ with $p \in \mathbb{Z}$, then we have

$$
\varphi_k^{(m)} = \sum_{r=0}^{n} \sum_{\lambda \vdash n-r} m_\lambda (1^p) \eta^{r,\lambda}.
$$

(4.6)

where $\eta^{r,\lambda}$ is the permutation character on the coset space $\mathfrak{S}^{(m)}_n / \mathfrak{S}_r^{(m)} \times \mathfrak{S}_\lambda$ by the parabolic subgroup $\mathfrak{S}_r^{(m)} \times \mathfrak{S}_\lambda = \mathfrak{S}_r^{(m)} \times \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \ldots$.

Proof. If $k = pm + 1$, then it follows from (4.5) with $q = 1$ that

$$
\text{ch}^{(m)}(\varphi_k^{(m)}) = \sum_{\lambda \in \mathcal{P}^{(m)}_{\text{even}}} s_{\lambda^{(0)}}(1^{p+1}) \prod_{i=1}^{m-1} s_{\lambda^{(i)}}(1^{p}) \prod_{i=0}^{m-1} s_{\lambda^{(i)}}(x^{(i)}).
$$

Hence, by using the Cauchy identity (3.2), we have

$$
\sum_{n \geq 0} \text{ch}(\varphi_k^{(m)}) = \sum_{\lambda \in \mathcal{P}^{(m)}} s_{\lambda^{(0)}}(1^{p+1}) \prod_{i=1}^{m-1} s_{\lambda^{(i)}}(1^{p}) \prod_{i=0}^{m-1} s_{\lambda^{(i)}}(x^{(i)}).
$$

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\[ \prod_{j} (1 - x_j^{(0)})^{-p-1} \prod_{i=1}^{m-1} \prod_{j} (1 - x_j^{(i)})^{-p} \]
\[ = \prod_{j} (1 - x_j^{(0)})^{-1} \prod_{i=0}^{m-1} \prod_{j} (1 - x_j^{(i)})^{-p} \]
\[ = \left( \sum_{r=0}^{\infty} h_r(x^{(0)}) \right) \left( \sum_{\lambda} m_\lambda (1^p) h_\lambda \left[ x^{(0)} + x^{(1)} + \cdots + x^{(r-1)} \right] \right). \]

Therefore we have
\[ \text{ch}(\varphi_k^{(m)}) = \sum_{r=0}^{n} \sum_{\lambda^r = n-r} m_\lambda (1^p) h_r(x^{(0)}) \Delta(h_\lambda). \]

Here \( h_r(x^{(0)}) = S_{(r), \emptyset, \ldots, \emptyset} \) is the Frobenius characteristic of the trivial representation of \( S_r^{\emptyset} \). And it follows from Lemma 4.4 that \( \Delta(h_\lambda) \) is the Frobenius characteristic of the permutation character on \( S_{n-r}/S_\lambda \). Hence we have
\[ \varphi_k^{(m)} = \sum_{r=0}^{n} \sum_{\lambda^r = n-r} m_\lambda (1^p) \text{Ind}^{S_r^{\emptyset}}_{S_{n-r} \times S_\lambda} \text{triv}. \]

\[ \square \]

4.2 Representation Theory of \( G(m, p, n) \)

For a divisor \( p \) of \( m \), the group \( G(m, p, n) \) is the subgroup of \( G(m, 1, n) \) consisting of the matrices in \( G(m, 1, n) \) such that the product of the nonzero entries is \( (m/p) \)-th root of unity. Then \( G(m, p, n) \) is a normal subgroup of \( G(m, 1, n) \) of index \( p \), and the quotient group \( G(m, 1, n)/G(m, p, n) \) is the cyclic group of order \( p \). Hence we can apply Clifford Theory to obtain irreducible representations of \( G(m, p, n) \) from those of \( G(m, 1, n) \). See [26, Section 6] for details.

Let \( \text{sh} : \mathcal{P}_n^{(m)} \to \mathcal{P}_n^{(m)} \) be the shift operator defined by
\[ \text{sh}(\lambda^{(0)}, \ldots, \lambda^{(m-1)}) = (\lambda^{(1)}, \ldots, \lambda^{(m-1)}, \lambda^{(0)}). \]

Then the group \( \langle \text{sh}^m/p \rangle \) acts on \( \mathcal{P}_n^{(m)} \). For \( \lambda \in \mathcal{P}_n^{(m)} \), we have
\[ \text{Res}^{G(m, 1, n)}_{G(m, p, n)} \chi^\lambda = \text{Res}^{G(m, 1, n)}_{G(m, p, n)} \chi^{\text{sh}^m/p(\lambda)}. \]

If the stabilizer of \( \chi^\lambda \) in \( \langle \text{sh}^m/p \rangle \) has the order \( t \), then \( \text{Res}^{G(m, 1, n)}_{G(m, p, n)} \chi^\lambda \) is decomposed into the sum of \( t \) distinct irreducible characters with multiplicity 1. And any irreducible character of \( G(m, p, n) \) can be obtained in this way. Since \( \varphi_k^{G(m, 1, n)} \) is the restriction of \( \varphi_k^{G(m, 1, n)} \) to \( G(m, p, n) \), we obtain the following proposition.

**Proposition 4.6.** If an irreducible character \( \chi \) of \( G(m, p, n) \) appears in the restriction \( \text{Res}^{G(m, 1, n)}_{G(m, p, n)} \chi^\lambda \) with \( \lambda \in \mathcal{P}_n^{(m)} \), then the multiplicity of \( \chi \) in \( \varphi_k^{G(m, p, n)} \) is equal to
\[ \sum_{\mu \in \langle \text{sh}^m/p \rangle \lambda} m_k^\mu \]
where the summation is taken over the $\langle sh^{m/p}\rangle$-orbit of $\lambda$ and $m_k^\mu$ is the multiplicity of $\chi^\mu$ in $\varphi_k^{G(m,1,n)}$.

4.3 Proof of Theorem 1.4 for $G(m,p,n)$ with $n \geq 3$

In this subsection, we assume $n \geq 3$.

Proof of (ii) $\Rightarrow$ (iii). We show that, if $\text{Cat}_k(W,q)$ is a polynomial, then $k \equiv 1 \text{ mod } m$.

Since the degrees $(d_1, \ldots, d_n)$ of $W = G(m,n,p)$ are given by

$$(d_1, \ldots, d_n) = (m, 2m, \ldots, (n - 1)m, mn/p),$$

we have

$$\text{Cat}_k(W,q) = \prod_{i=1}^{n-1} (k + im - 1)_q \cdot (k + mn/p - 1)_q.$$

Since $n \geq 3$, the denominator of $\text{Cat}_k(W,q)$ is divisible by $\Phi_m(q)^2$, where $\Phi_m(q)$ is the $m$-th cyclotomic polynomial. So, if $\text{Cat}_k(W,q)$ is a polynomial, then the numerator is also divisible by $\Phi_m(q)^2$. Hence at least one of the factors $[k + im - 1]_q$ with $1 \leq i \leq n - 1$ is divisible by $\Phi_m(q)$. This implies $m \mid k + im - 1$, so $k \equiv 1 \text{ mod } m$. \hfill $\square$

Proof of (iii) $\Rightarrow$ (i). Suppose that $k \equiv 1 \text{ mod } m$. Then it follows from (1.5) that $\varphi_k^{G(m,1,n)}$ is the graded character of a graded representation. By restricting to $G(m,p,n)$, we see that $\varphi_k^{G(m,p,n)}$ is also the graded character of a graded representation. \hfill $\square$

Proof of (iii) $\Rightarrow$ (v). Suppose that $k \equiv 1 \text{ mod } m$. Then it follows from (1.6) that $\varphi_k^{G(m,1,n)}$ is a permutation character of $G(m,1,n)$.

In general, by using Mackey’s Restriction Theorem (see [18, Proposition 22])

$$\text{Res}_K^G \text{ Ind}_H^G \text{ triv} = \sum_g \text{ Ind}_K^G g^{-1} g\text{ triv},$$

where $g$ runs over all double coset representatives of $K \backslash G/H$, we can see that $\varphi_k^{G(m,p,n)} = \text{Res}_{G(m,p,n)}^{G(m,1,n)} \varphi_k^{G(m,1,n)}$ is also a permutation character. \hfill $\square$

The most subtle part of the proof is the implication (iv) $\Rightarrow$ (iii).

Proof of (iv) $\Rightarrow$ (iii). Suppose that $\varphi_k^{G(m,p,n)}$ comes from a genuine representation of $G(m,p,n)$. Then, by restricting to $G(m,m,n)$, we see that $\varphi_k^{G(m,m,n)}$ also comes from a genuine representation.

Since the degrees and the codegrees of $G(m,m,n)$ are given by

$$(d_1, \ldots, d_n) = (m, 2m, \ldots, (n - 1)m, n),$$

$$(d'_1, \ldots, d'_n) = (0, m, \ldots, (n - 2)m, (n - 1)m - n),$$

we can use Proposition 2.2 to see that the multiplicities of triv and det in $\varphi_k^{G(m,m,n)}$ are given by

$$m_k^{\text{triv}} = \frac{1}{m^{n-1}n!} \prod_{i=1}^{n-1} (k + im - 1) \cdot (k + n - 1),$$

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\[ m_k^{\text{det}} = \frac{1}{m^{n-1}n!} \prod_{i=1}^{n-1} (k - im + m - 1) \cdot (k - (n-1)m + (n-1)). \]

This can be also derived from (4.4) and Proposition 4.6. We prove that, if \( m_k^{\text{triv}} \) and \( m_k^{\text{det}} \) are both integers, then \( k \equiv 1 \mod m \).

We consider two polynomials

\[
\begin{align*}
f(z) &= \frac{1}{n!} \prod_{i=1}^{n-1} (z + i) \cdot (mz + n), \\
g(z) &= \frac{1}{n!} \prod_{i=1}^{n-1} (z - i + 1) \cdot (m(z - n + 1) + n).
\end{align*}
\]

Then we have

\[
m_k^{\text{triv}} = f \left( \frac{k - 1}{m} \right), \quad m_k^{\text{det}} = g \left( \frac{k - 1}{m} \right).
\]

Hence \( z = (k - 1)/m \) is a solution of the polynomial equation

\[
\frac{n!}{2} (f(z) + g(z)) - (n - 2)! z(f(z) - g(z)) = \frac{n!}{2} (m_k^{\text{triv}} + m_k^{\text{det}}) - (n - 2)! (m_k^{\text{triv}} - m_k^{\text{det}}) z.
\]

Recall that a rational number is a rational integer if and only if it is an algebraic number. Hence it is enough to show that

\[
h(z) = \frac{n!}{2} (f(z) + g(z)) - (n - 2)! z(f(z) - g(z))
\]

is a monic polynomial of degree \( n - 1 \) with integer coefficients.

The polynomials \( f(z) \) and \( g(z) \) are written as

\[
\begin{align*}
f(z) &= f_1(z) + f_2(z), & g(z) &= g_1(z) + g_2(z),
\end{align*}
\]

where

\[
\begin{align*}
f_1(z) &= \frac{m}{n!} \prod_{i=1}^{n} (z + i - 1), & f_2(z) &= \frac{1}{(n-1)!} \prod_{i=1}^{n-1} (z + i),
\end{align*}
\]

\[
\begin{align*}
g_1(z) &= \frac{m}{n!} \prod_{i=1}^{n} (z - i + 1), & g_2(z) &= \frac{1}{(n-1)!} \prod_{i=1}^{n-1} (z - i + 1).
\end{align*}
\]

Since we have

\[
\begin{align*}
f_1(z) &= \frac{m}{n!} \left( z^n + \frac{n(n-1)}{2} z^{n-1} + \text{lower terms} \right), \\
g_1(z) &= \frac{m}{n!} \left( z^n - \frac{n(n-1)}{2} z^{n-1} + \text{lower terms} \right),
\end{align*}
\]

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we see that $h(z)$ is a monic polynomial of degree $n - 1$.

Next we show that $h(z)$ has integer coefficients. Let $c(n, j)$ be the signless Stirling number of the first kind. Then we have (see [21, 1.3.7 Proposition] for example)

\[
\prod_{i=1}^{n}(z + i - 1) = \sum_{k=0}^{n} c(n, j)z^j.
\]

By using this generating function, we see that

\[
\frac{n!}{2}(f_1(u) + g_1(u)) = m \sum_{n - j \text{ is even}} c(n, j)u^j,
\]

\[
(n - 2)!(f_1(u) - g_1(u)) = \sum_{n - j \text{ is odd}} c(n, j)u^j.
\]

Also by using the recurrence relation (see [21, 1.3.6 Lemma] for example)

\[
c(n, j) = c(n - 1, j - 1) + (n - 1)c(n - 1, j),
\]

we have

\[
\frac{n!}{2}(f_2(u) + g_2(u)) = n \sum_{n - 1 - j \text{ is even}} c(n - 1, j)u^j + \binom{n}{2} \sum_{j=1}^{n} c(n - 1, j)u^{j-1},
\]

\[
(n - 2)!(f_2(u) - g_2(u)) = n \cdot \frac{1}{\binom{n-1}{2}} \sum_{n - 1 - j \text{ is odd}} c(n - 1, j)u^j + \sum_{j=1}^{n} c(n - 1, j)u^{j-1}.
\]

Therefore it is enough to show that $c(n, j)$ is divisible by $\binom{n}{2}$ if $n - j$ is odd. The proof is reduced to the following lemma.

**Lemma 4.7.** (Stanley [25, Corollary 3.4], Zagier [27, Application 3]) If $n - j$ is odd, then $c(n, j)$ is divisible by $\binom{n}{2}$.

We will give a refinement and another proof in Appendix.

### 4.4 Proof of Theorem 1.4 for $G(m, p, n)$ with $n = 2$ and $p < m$

In this subsection, we deal with the case where $n = 2$ and $p < m$. The remaining case where $n = 2$ and $p = m$ (i.e., $W = G(m, m, 2)$ is the dihedral group of order $2m$) will be treated in the next section.

The proofs of the implications (iii) $\implies$ (i) and (iii) $\implies$ (v) are the same as in the case where $n \geq 3$. 


Proof of (ii) \implies (iii). The degrees and the codegrees of $G(m, p, 2)$ are given by
$$(d_1, d_2) = (m, mn/p), \quad (d'_1, d'_2) = (0, m),$$
so we have
$$\text{Cat}^*_k(W, q) = q^{m+2}[k-1]_q[k-m-1]_q/m_q[mn/p]_q.$$ If it is a polynomial in $q$, then the numerator is divisible by $\Phi_m(q)$, so $m \mid (k - 1)$ or $m \mid (k - m - 1)$. Hence we have $k \equiv 1 \mod m$. \hfill \square

Proof of (iv) \implies (iii). Let $\chi$ and $\eta$ be the irreducible characters of $G(m, p, 2)$ corresponding to the $(\text{sh}^{m/p})$-orbits of
$$\lambda = ((2), \emptyset, \ldots, \emptyset), \quad \text{and} \quad \mu = (\emptyset, (2), \emptyset, \ldots, \emptyset),$$
respectively. Since $p < m$, these two orbits are distinct, so $\chi \neq \eta$. Then, by using (4.4) and Proposition 4.6, the multiplicities of $\chi$ and $\eta$ in $\varphi_k$ are given by
$$m^\chi = \frac{1}{2m^2}(k + m - 1)(k + 2m - 1) + (p - 1) \cdot \frac{1}{2m^2}(k - 1)(k + m - 1)$$
$$= \frac{p}{2m^2}(k + m - 1) \left( k + \frac{2m}{p} - 1 \right),$$
$$m^\eta = \frac{p}{2m^2}(k - 1)(k + m - 1).$$
So we have
$$m^\chi - m^\eta = \frac{k - 1}{m}.$$ Thus, if both of the multiplicities $m^\chi$ and $m^\eta$ are integers, then $k \equiv 1 \mod m$. \hfill \square

5 Dihedral groups

This section is devoted to the proof of Theorem 1.3 for the dihedral groups.

5.1 Preliminaries

Let $m$ be a positive integer. The dihedral group $D_{2m}$ of order $2m$ is defined by generators and relations:
$$D_{2m} = \langle a, b \mid a^m = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$ The dihedral group $D_{2m}$ is a complex reflection group acting on $\mathbb{C}^2$ via
$$a = \begin{pmatrix} \cos 2\pi/m & -\sin 2\pi/m \\ \sin 2\pi/m & \cos 2\pi/m \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$ Then the conjugacy classes and the irreducible characters of $D_{2m}$ are described as follows. (See a standard textbook on representation theory of finite groups, e.g. [18, 5.3].)
Proposition 5.1. (1) Suppose $m$ is even. Then the $(m/2 + 3)$ elements

$$1, \ a^i \ (1 \leq i \leq m/2), \ b, \ ab$$

form a complete set of representatives of conjugacy classes of $D_{2m}$. And $D_{2m}$ has four one-dimensional irreducible characters $ξ_0, ξ_1, ξ_2, ξ_3$ and $m/2 - 1$ two-dimensional irreducible characters $χ_j \ (1 \leq j \leq m/2 - 1)$, whose character values are given by

$$ξ_0(1) = 1, \ ξ_0(a^i) = 1, \ ξ_0(b) = 1, \ ξ_0(ab) = 1,$$
$$ξ_1(1) = 1, \ ξ_1(a^i) = 1, \ ξ_1(b) = -1, \ ξ_1(ab) = -1,$$
$$ξ_2(1) = 1, \ ξ_2(a^i) = (-1)^i, \ ξ_2(b) = 1, \ ξ_2(ab) = -1,$$
$$ξ_3(1) = 1, \ ξ_3(a^i) = (-1)^i, \ ξ_3(b) = -1, \ ξ_3(ab) = 1,$$
$$χ_j(1) = 2, \ χ_j(a^i) = ζ^{ij} + ζ^{-ij}, \ χ_j(b) = 0, \ χ_j(ab) = 0.$$

(2) Suppose $m$ is odd. Then the $((m-1)/2 + 2)$ elements

$$1, \ a^i \ (1 \leq i \leq (m-1)/2), \ b$$

form a complete set of representatives of conjugacy classes of $D_{2m}$. And $D_{2m}$ has two one-dimensional irreducible characters $ξ_0, ξ_1$ and $(m-1)/2$ two-dimensional irreducible characters $χ_j \ (1 \leq j \leq (m-1)/2)$, whose character values are given by

$$ξ_0(1) = 1, \ ξ_0(a^i) = 1, \ ξ_0(b) = 1,$$
$$ξ_1(1) = 1, \ ξ_1(a^i) = 1, \ ξ_1(b) = -1,$$
$$χ_j(1) = 2, \ χ_j(a^i) = ζ^{ij} + ζ^{-ij}, \ χ_j(b) = 0.$$

Since $a^i$ is a rotation through an angle of $2π/m$ and $a^ib$ is a reflection, the values of the class function $\hat{ϕ}$ defined in [2,2] are given by

$$\hat{ϕ}(a^i) = \frac{(1 - ζ^i)(1 - ζ^{-i})}{(1 - ζ^q)(1 - ζ^{-q})}, \quad \hat{ϕ}(a^ib) = \frac{(1 - u)(1 + u)}{(1 - q)(1 + q)}.$$

The multiplicities $\hat{m}^x$ in $\hat{ϕ}$ can be computed as follows:

Proposition 5.2. (1) If $m$ is even, then we have

$$\hat{m}^{ξ_0}(u) = \frac{(1 - uq)(1 - uq^{m-1})}{(1 - q^2)(1 - q^m)},$$
$$\hat{m}^{ξ_1}(u) = q^m \frac{(1 - uq^{-1})(1 - uq^{-m+1})}{(1 - q^2)(1 - q^m)},$$
$$\hat{m}^{ξ_2}(u) = q^{m/2} \frac{(1 - uq)(1 - uq^{-1})}{(1 - q^2)(1 - q^m)},$$
$$\hat{m}^{ξ_3}(u) = q^{m/2} \frac{(1 - uq^{-1})(1 - uq^{-m+1})}{(1 - q^2)(1 - q^m)},$$
$$\hat{m}^{χ_j}(u) = (q^j + q^{m-j}) \frac{(1 - uq)(1 - uq^{-1})}{(1 - q^2)(1 - q^m)} \ (1 \leq j \leq m/2 - 1).$$
(2) If \( m \) is odd, then we have
\[
\tilde{m}^{\xi_0}(u) = \frac{(1 - uq)(1 - uq^{m-1})}{(1 - q^2)(1 - q^m)}, \\
\tilde{m}^{\xi_1}(u) = q^m \frac{(1 - uq^{-1})(1 - uq^{-m+1})}{(1 - q^2)(1 - q^m)}, \\
\tilde{m}^{\chi_j}(u) = (q^j + q^{m-j}) \frac{(1 - uq)(1 - uq^{-1})}{(1 - q^2)(1 - q^m)} \quad (1 \leq j \leq (m - 1)/2).
\]

**Proof.** It suffices to show that the right-hand sides of the above formulas satisfy the relation
\[
\sum_{\chi \in \text{Irr}(W)} \tilde{m}^{\chi}(w) = \varphi(w) \text{ for } w \in W.
\]
This can be done by a direct computation, so we leave it to the readers. \( \square \)

### 5.2 Proof of Theorem [1.4](A)

It is enough to prove (ii) \( \implies \) (iii) and (iii) \( \implies \) (i).

**Proof of (ii) \( \implies \) (iii).** We show that, if \( \text{Cat}_k(W, q) \) is a polynomial, then \( k \equiv \pm 1 \mod m \).

Since the degrees of \( D_{2m} \) are 2 and \( m \), we have
\[
\text{Cat}_k(W, q) = \frac{[k + 1]_q[k + m - 1]_q}{[2]_q[m]_q}.
\]
Suppose that \( \text{Cat}_k(W, q) \) is a polynomial in \( q \). Then \( \Phi_m(q) \) must appear in the numerator, so we have \( m \mid k + 1 \) or \( m \mid k + m - 1 \), which implies \( k \equiv -1 \mod m \) or \( k \equiv 1 \mod m \) respectively. \( \square \)

**Proof of (iii) \( \implies \) (i).** We use Lemma [2.6](b) to prove that, if \( k \equiv 1 \) or \( -1 \mod m \), then \( \tilde{m}^{\chi}_{k} = \tilde{m}^{\chi}|_{u=q} \) is a polynomial with nonnegative integer coefficients for all \( \chi \in \text{Irr}(W) \).

First consider the multiplicities of \( \xi_0 \) and \( \xi_1 \). By using the formulas given in Proposition [5.2](a), we see that
\[
\tilde{m}^{\xi_0}(qu) = 1 + \left( q^m + q^2 \frac{[2m]_q}{[2]_q} \right) \cdot \frac{1 - u}{1 - q^m} + q^{2m+2} \frac{[2m]_q}{[2]_q} \cdot \frac{(1 - u)(1 - uq^{-m})}{(1 - q^m)(1 - q^{2m})}, \\
\tilde{m}^{\xi_0}(q^{m-1}u) = \frac{[2m - 2]_q}{[2]_q} + q^m \left( \frac{[2m - 2]_q}{[2]_q} + q^{m-2} \frac{[2m]_q}{[2]_q} \right) \cdot \frac{1 - u}{1 - q^m} \\
+ q^{4m-2} \frac{[2m]_q}{[2]_q} \cdot \frac{(1 - u)(1 - uq^{-m})}{(1 - q^m)(1 - q^{2m})}, \\
\tilde{m}^{\xi_1}(qu) = q^m \cdot \frac{1 - u}{1 - q^m} + q^{m+2} \frac{[2m]_q}{[2]_q} \cdot \frac{(1 - u)(1 - uq^{-m})}{(1 - q^m)(1 - q^{2m})}, \\
\tilde{m}^{\xi_1}(q^{m-1}u) = q^m \frac{[2m - 2]_q}{[2]_q} \cdot \frac{1 - u}{1 - q^m} + q^{3m-2} \frac{[2m]_q}{[2]_q} \cdot \frac{(1 - u)(1 - uq^{-m})}{(1 - q^m)(1 - q^{2m})}.
\]

Since \( [2m]_q/[2]_q, [2m - 2]_q/[2]_q \in \mathbb{N}[q] \), we conclude that \( \tilde{m}^{\xi_0}_{pm+1}, \tilde{m}^{\xi_0}_{pm+1}, \tilde{m}^{\xi_1}_{pm+1} \) and \( \tilde{m}^{\xi_1}_{pm+m-1} \) belong to \( \mathbb{N}[q] \) for all nonnegative integers \( p \).
The remaining multiplicities have a factor

\[ M(u) = \frac{(1 - uq)(1 - uq^{-1})}{(1 - q^2)(1 - q^m)}, \]

and we see that

\[ \eta \]

denote by \( \eta \)

\[ \eta \]

even, we see that \( k \)

\[ k \]

is an integer. Also, since \( m \)

\[ m \]

are polynomials with nonnegative integer coefficients.

\[ M(qu) = \frac{[m + 2]_q [m - 2]_q}{[2]_q} \cdot \frac{1 - u}{1 - q^m} + q^{m - 2} \frac{[2m]_q}{[2]_q} \cdot \frac{(1 - u)(1 - uq^{-m})}{(1 - q^m)(1 - q^{2m})} + q^{3m - 2} \frac{[2m]_q}{[2]_q} \cdot \frac{(1 - u)(1 - uq^{-m})}{(1 - q^m)(1 - q^{2m})}. \]

If \( m \) is even, then we have \([m + 2]_q/[2]_q, [m - 2]_q/[2]_q \in \mathbb{N}[q] \) and \( M(q^{pm+1}), M(q^{pm+m-1}) \in \mathbb{N}[q] \). Hence \( \tilde{m}_{pm+1}^\chi, \tilde{m}_{pm+m-1}^\chi \in \mathbb{N}[q] \) for \( \chi = \xi_2, \xi_3 \) and \( \chi_j \). If \( m \) is odd, then we use the relations

\[ (q^j + q^{m-j}) \frac{[m + 2]_q [m - 2]_q}{[2]_q} = q^j \frac{[2m - 2j + 2]_q}{[2]_q} + q^{m-j} \frac{[2j + 2]_q}{[2]_q}, \]

\[ (q^j + q^{m-j}) \frac{[m - 2]_q}{[2]_q} = q^j \frac{[2m - 2j - 2]_q}{[2]_q} + q^{m-j} \frac{[2j - 2]_q}{[2]_q} \]

to conclude that

\[ \tilde{m}_{pm+1}^\chi = (q^j + q^{m-j})M(q^{pm+1}), \quad \tilde{m}_{pm+m-1}^\chi = (q^j + q^{m-j})M(q^{pm+m-1}). \]

are polynomials with nonnegative integer coefficients. \( \square \)

5.3 Proof of Theorem 1.4 (C)

We prove the implications (iii') \( \implies \) (v) and (iv) \( \implies \) (iii').

Proof of (iii') \( \implies \) (v). Since \( \varphi_1 \) is the trivial character, we may assume \( k \geq m - 1 \).

First we consider the case where \( m \) is even, and assume that \( k^2 \equiv 1 \mod 2m \). If we denote by \( \eta_1 \) and \( \eta_2 \) the permutation characters on \( D_{2m}/\langle b \rangle \) and \( D_{2m}/\langle ab \rangle \) respectively, then we have

\[ \varphi_k = \text{triv} + \frac{k - 1}{2} \eta_1 + \frac{k - 1}{2} \eta_2 + \frac{(k - 1)(k - m + 1)}{2m} \eta_{\text{reg}}, \quad (5.1) \]

where \( \eta_{\text{reg}} \) is the character of the regular representation. Since \( k^2 \equiv 1 \mod 2m \) and \( m \) is even, we see that \( k \) is odd and

\[ \frac{(k - 1)(k - m + 1)}{2} = \frac{k^2 - 1 - k - 1}{2m} \]

is an integer. Also, since \( k \geq m - 1 \), the coefficients in (5.1) are nonnegative.

Next we consider the case where \( m \) is odd, and assume that \( k^2 \equiv 1 \mod 2m \). If we denote \( \eta_1 \) the permutation character on \( D_{2m}/\langle b \rangle \), then we have

\[ \varphi_k = \text{triv} + (k - 1) \eta_1 + \frac{(k - 1)(k - m + 1)}{2m} \eta_{\text{reg}}, \quad (5.2) \]
Since \( k^2 \equiv 1 \mod m \) and \( m \) is odd, we see that \( (k^2 - 1)/m \) have the different parity to \( k \) and
\[
\frac{(k - 1)(k - m + 1)}{2m} = \frac{1}{2} \left( \frac{k^2 - 1}{m} - k + 1 \right)
\]
is an integer. Also, since \( k \geq m - 1 \), the coefficients in [5.2] are nonnegative.

Proof of (iv) \( \implies \) (iii'). First we consider the case where \( m \) is even. Then, by using Proposition 5.2, we have
\[
\varphi_k = \frac{(k + 1)(k + m - 1)}{2m} \xi_0 + \frac{(k - 1)(k - m + 1)}{2m} \xi_1 + \frac{(k + 1)(k - 1)}{2m} \xi_2 + \frac{(k + 1)(k - 1)}{2m} \xi_3
\]
\[
+ \sum_{j=1}^{m/2-1} \frac{(k + 1)(k - 1)}{m} \chi_j.
\]
Suppose that \( \varphi_k \) is the character of some representation of \( W \). Since the multiplicity of \( \xi_2 \) is an integer, we have \( k^2 \equiv 1 \mod 2m \). Also the multiplicity of \( \xi_1 \) is nonnegative, we have \( k \leq 1 \) or \( k \geq m - 1 \).

Similarly we can prove the case where \( m \) is odd, by using the decomposition
\[
\varphi_k = \frac{(k + 1)(k + m - 1)}{2m} \xi_0 + \frac{(k - 1)(k - m + 1)}{2m} \xi_1 + \frac{(k + 1)(k - 1)}{2m} \xi_2 + \frac{(m-1)/2}{m} \xi_3 + \sum_{j=1}^{m/2-1} \frac{(k + 1)(k - 1)}{m} \chi_j.
\]

6 Exceptional groups

In this section we explain how to prove our main result for the exceptional groups with a help of computer.

6.1 Proof of (ii) \( \implies \) (iii)

We use the criterion for polynomiality in Lemma 2.7. We can prove the following lemma by using a computer.

Lemma 6.1. Let \( W \) be an exceptional complex reflection group with degrees \((d_1, \ldots, d_r)\) and codegrees \((d_1^*, \ldots, d_r^*)\). For a positive integer \( d \) and a positive integer \( k \), we put
\[
N_k(d) = \# \{ i : d \mid (k + d_i - 1) \}, \quad N_k^*(d) = \# \{ i : d \mid (k - d_i^* - 1) \},
\]
and
\[
D(d) = \# \{ i : d \mid d_i \}, \quad T = \bigcup_{i=1}^r \{ d \mid d_i \}.
\]
Then we have

(1) The following are equivalent for a positive integer \( k \):

(i) \( N_k(d) \geq D(d) \) for all \( d \in T \).
Table 2: Condition in Lemma 6.1 (ii)

| group | condition on $k$ |
|-------|------------------|
| $W = G_{13}$ | $k \equiv 1, 5 \mod 12$ |
| $W = G_{15}$ | $k \equiv 1 \mod 12$ |

(ii) $k$ satisfies the condition in Table 1 except for $W = G_{13}$ and $G_{15}$. In these exceptions, the condition is given in Table 2.

(2) The following are equivalent for a positive integer $k$:

(i*) $N^*_k(d) \geq D(d)$ for all $d \in T$.

(ii*) $k$ satisfies the condition in Table 1.

Proof. The conditions given in (ii) and (ii*) are of the form “$k \mod H \in K$” for some integer $H$ and a subset $K \subset \{1, 2, \ldots, H\}$, where $k \mod H$ is the remainder of $k$ divided by $H$. Let $L$ be the least common multiple of $d_1, \ldots, d_r$ and $H$. Then it is easy to see that $N_k(d) = N_{L+k}(d)$ and $N^*_k(d) = N^*_{L+k}(d)$. And $k$ satisfies the condition (ii) (resp. (ii*)) if and only if $k + L$ satisfies (ii) (resp. (ii*)). Hence it is enough to show that

$$\{k \in [1, L] : N_k(d) \geq D(d) \text{ for all } d \in T\} = \{k \in [1, L] : k \text{ satisfies (ii)}\},$$
$$\{k \in [1, L] : N^*_k(d) \geq D(d) \text{ for all } d \in T\} = \{k \in [1, L] : k \text{ satisfies (ii*)}\}.$$

These equalities can be checked by using a computer. \hfill \Box

Now we are ready to prove (ii) $\implies$ (iii) of Theorem 1.4.

Proof of (ii) $\implies$ (iii). If follows from Lemmas 2.7 and 6.1 that, if Cat$_k(W, q)$ is a polynomial in $q$, then $k$ satisfies the condition (ii) of Lemma 6.1 and that, if Cat$_k(W, q)$ is a polynomial in $q$, then $k$ satisfies the condition (ii*), or $k = d^*_i + 1$ for some $i$.

If $k = d^*_i + 1$ does not satisfy the condition (ii*), then the pair $(W, k)$ is one of the following:

$$(W, k) = (G_{25}, 4), (G_{33}, 9), (G_{33}, 15), (G_{35}, 4), (G_{35}, 8).$$

In each of these cases, we can see that $k$ does not satisfy the condition (ii) of Lemma 6.3. So we conclude that, if Cat$_k(W, q)$ and Cat$_k(W, q)$ are both polynomials, then $k$ satisfies both (ii) and (ii*) of Lemma 6.1 hence the condition (iii) of Theorem 1.4. \hfill \Box

6.2 Proof of (iii) $\implies$ (i)

In order to prove (iii) $\implies$ (i), we need to compute explicitly the multiplicities $\hat{m}^\chi$ in $\hat{\varphi}$. If $\{w_1, \ldots, w_n\}$ is a complete set of representatives of conjugacy classes of $W$, then the multiplicities are given by

$$\hat{m}^\chi = \frac{1}{\#W} \sum_{w \in W} \hat{\varphi}(w)\chi(w) = \sum_{i=1}^n \frac{1}{\#Z(w_i)} \hat{\varphi}(w_i)\chi(w_i),$$

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where $Z(w_i)$ is the centralizer of $w_i$ in $W$ and $\bar{\chi(w)}$ is the complex conjugate of $\chi(w)$. By using the program GAP [17] together with CHEVIE [5], we can compute

- a complete set of representatives of the conjugacy classes of $W$,
- matrix representations of $w \in W$ in the reflection representation,
- the character table of $W$,
- the size of the centralizer of each representative.

Hence we obtain explicit formulas for $\hat{m}^{\chi}$. Then we can use Lemma 2.6 to show that $\tilde{m}_k^{\chi} = \hat{m}_k^{\chi}|_{u=q^k}$ is a polynomial with nonnegative integer coefficients provided $k$ satisfies the condition (iii) (see Table 1) of Theorem 1.4.

Remark 6.2. It is known [11, Corollary 9.39] that the coordinate ring $S = \mathbb{C}[V]$ of the reflection representation $V$ is decomposed into the tensor product of the invariant subring $S^W$ and the space of $W$-harmonic polynomials $H$. By using this decomposition, we can show that $\det_V(1 - qw)$ divides $\prod_{i=1}^r(1 - q^{d_i})$. Hence

$$\prod_{i=1}^r(1 - q^{d_i}) \cdot \hat{m}^{\chi} = \sum_{i=1}^n \frac{1}{\#Z(w_i)} \det_V(1 - uw) \cdot \frac{\prod_{i=1}^r(1 - q^{d_i})}{\det_V(1 - qw)} \cdot \chi(w_i)$$

is a polynomial in $q$ and $u$. By designing the program in terms of $\prod_{i=1}^r(1 - q^{d_i}) \cdot \hat{m}^{\chi}$ instead of $\hat{m}^{\chi}$, we can avoid rational functions to make the computation faster.

6.3 Proof of (iii) $\implies$ (v)

We use the Orlik–Solomon formula (2.6) and the data given in [13, 14]. Then, by using Lemma 2.5, we can prove that the coefficients $\chi(A_j, k)/[N_W(W_j) : W_j]$ are nonnegative integers provided $k$ satisfies the condition (iii) of Theorem 1.4.

6.4 Proof of (iv) $\implies$ (iii)

It follows from Proposition 2.2 that the Catalan numbers $\text{Cat}_k(W, 1)$ and $\text{Cat}_k^*(W, 1)$ are the multiplicities of $\text{triv}$ and $\text{det}_V$ in $\varphi_k$ respectively. Hence the proof of (iv) $\implies$ (iii) follows from the following Lemma:

Lemma 6.3. Let $W$ be an exceptional complex reflection group. Then we have

1. The following are equivalent for a positive integer $k$:
   - (i) $\text{Cat}_k(W, 1)$ is an integer.
   - (ii) $k$ satisfies the condition given in Table 1 except for $W = G_{13}, G_{15}, G_{25}, G_{33}, G_{35}$ and $G_{36}$. In these exceptions, the condition is given in Table 3.

2. The following are equivalent for a positive integer $k$:
   - (i) $\text{Cat}_k^*(W, 1)$ is an integer.
(ii) $k$ satisfies the condition given in Table 3 except for $W = G_{25}, G_{33}, G_{35},$ and $G_{36}$. In these exceptions, the condition is given in Table 4.

Proof. We note that

$$c(k) = \text{Cat}_k(W, 1) = \prod_{i=1}^{r} \frac{k + d_i - 1}{d_i}, \quad c^*(k) = \text{Cat}_k^*(W, 1) = \prod_{i=1}^{r} \frac{k - d_i \ast_i - 1}{d_i}$$

are polynomials in $k$. And the conditions in (ii) and (ii*) are of the form “$k \equiv H \in K$” for some positive integer $H$ and a subset $K \subset \{1, \ldots, H\}$. Then we can find a positive integer $L$ satisfying

(a) $L$ is a multiple of $H$, and

(b) $c(t + L) - c(t)$ and $c^*(t + L) - c^*(t)$ map nonnegative integers to nonnegative integers.

For example, we can take $L = \#W$. (We can use Lemma 2.5 to check the condition (b).) Then it is enough to show that

$$\{k \in [1, L] : c(k) = \text{Cat}_k(W, 1) \in \mathbb{N}\} = \{k \in [1, L] : k \text{ satisfies (ii)}\},$$
$$\{k \in [1, L] : c^*(k) = \text{Cat}_k^*(W, 1) \in \mathbb{N}\} = \{k \in [1, L] : k \text{ satisfies (ii*)}\}.$$

These equalities can be checked by using a computer.

This complete the proof of Theorem 1.4 for all irreducible finite complex reflection groups.
7 Concluding remarks

7.1 Uniform proof

We have proved Theorem 1.4 in a case-by-case manner. The conditions (i) and (ii) of Theorem 1.4 are stated in a uniform way (i.e., do not use the classification of irreducible complex reflection groups). So it is natural to seek a classification-free proof.

Problem 7.1. Find a uniform proof to the equivalence between (i) and (ii) of Theorem 1.4.

If $W$ is a Weyl group, then the condition (iii) is equivalent to saying that $k$ is “very good” in the sense of [20]. A positive integer $k$ is called “very good” for $W$ if $k$ is relatively prime to the coefficients of simple roots in the highest root and to the index $[Q^\vee : Q]$, where $Q$ and $Q^\vee$ is the root lattice and the coroot lattice respectively.

Problem 7.2. Find a classification-free description of the condition (iii) in Theorem 1.4.

7.2 Combinatorial/algebraic models

In Theorem 1.4 we have proved that, if $k$ satisfies the condition (iii) (or (iii’) for dihedral groups), then $\tilde{\varphi}_k$ is the graded character of a representation and $\varphi_k$ is a permutation character. However, our proof does not give any constructions of such representations. We know [20] that, if $W$ is a Weyl group, then $\mathbb{Q}/k\mathbb{Q}$ affords the character $\varphi_k$ provided $k$ is very good. Also, for a Coxeter group $W$ and $k \equiv 1 \mod h$ (Fuss cases), [2] and [10] give a combinatorial model for $\varphi_k$. So it is a natural problem to generalize their constructions to other complex reflection groups and to general $k$ satisfying the condition (iii) in Theorem 1.4.

Problem 7.3. (1) Give a combinatorial construction of a $W$-set which gives the permutation character $\varphi_k$.

(2) Give an algebraic/combinatorial construction of a graded $W$-module affording the graded character $\tilde{\varphi}_k$.

7.3 Unimodality

In Proposition 3.10 we have proved that

$$f_\lambda(q) = \frac{s_\lambda(1, q, \ldots, q^{k-1})}{[k]_q/[d]_q}$$

is a polynomial in $q$ with nonnegative integer coefficients, where $\lambda$ is a partition of $n$ and $d = \gcd(n, k)$. Recall that a finite sequence $(a_0, a_1, \ldots, a_m)$ is unimodal if there is an index $p$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{p-1} \leq a_p \geq a_{p+1} \geq \cdots \geq a_m.$$

If $n$ is a multiple of $k$, i.e., $\gcd(n, k) = k$, then it is well-known (see [12, I.8 Example 4] for example) that the coefficients of $f_\lambda(q) = s_\lambda(1, q, \ldots, q^{k-1})$ form a unimodal sequence. Also, if $\gcd(n, k) = 1$, then a certain finite dimensional module over a rational Cherednik algebra associated to $S_n$ provides a $W \times \mathfrak{sl}_2$ module with character $\tilde{\varphi}_k$. (See [3] for example.) Hence the polynomials $f_\lambda(q)$ are $\mathfrak{sl}_2$ characters. These results together with a computer experiment suggest the following conjecture.
Hence we have the partition obtained by removing \( l \) the conjugacy class \( C \).

Suppose that

\[ \text{Proof.} \]

By [23, Exercise 91 (c)], Let

Proposition A.1.

J. Bruns proved the following result, but it seems that he never published his proof.

Remark 7.5. The whole sequence \((a_0, a_1, a_2, a_3, \ldots)\) is not unimodal in general. For example, if \( n = 2, k = 3 \) and \( \lambda = (2) \), then

\[ \frac{s(2)(1, q^2)}{[3]_q} = 1 + q^2 \]

does not have unimodal coefficients.

A Divisibility of the Stirling numbers

In this appendix, we give a proof to Lemma [4.7] which asserts that the signless Stirling number \( c(n, j) \) of the first kind is divisible by \( \binom{n}{j} \) if \( n - j \) is odd. Recall that \( c(n, j) \) is equal to the number of permutations of \( n \) letters with exactly \( j \) cycles. If \( w \in S_n \) has the cycle type \( \lambda \), then we have \( n - l(\lambda) = \sum_{i \geq 1} (i - 1)m_i(\lambda) \), where \( m_i(\lambda) \) is the multiplicity of \( i \) in \( \lambda \). Hence, if \( n - l(\lambda) \) is odd, then \( \lambda \) has an even part with odd multiplicity. Now Lemma [4.7] follows from the following proposition. (According to [23], Exercise 91 (c)], J. Bruns proved the following result, but it seems that he never published his proof.)

Proposition A.1. Let \( \lambda \) be a partition of \( n \) and \( C_\lambda \) the corresponding conjugacy class of \( S_n \). If \( \lambda \) has an even part with odd multiplicity, then \( \#C_\lambda \) is divisible by \( \binom{n}{2} \).

Proof. Suppose that \( \lambda \) has an even part \( 2f \) with odd multiplicity \( l = m_{2f}(\lambda) \). The size of the conjugacy class \( C_\lambda \) is given by \( \#C_\lambda = n!/z_\lambda \), where \( z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)! \). Let \( \mu \) be the partition obtained by removing \( l \) parts equal to \( 2f \) from \( \lambda \). Then we have

\[ |\mu| = n - 2fl, \quad z_\lambda = z_\mu \cdot (2f)^l \cdot l! \]

Hence we have

\[ \frac{1}{\binom{n}{2}} \#C_\lambda = \binom{n - 2}{2fl - 2} \cdot \#C_\mu \cdot \frac{(2fl - 2)!}{f^l \cdot (2l - 2)!} \cdot \frac{(2l - 2)!}{2^{l-1} \cdot l!} \]

If \( f = 1 \), then \( (2fl - 2)!/f^l \cdot (2l - 2)! = 1 \). If \( f \geq 2 \), then \( 2l - 2 \leq fl - 2 \) and the interval \([fl - 1, 2fl - 2]\) contains \( l \) multiples of \( f \), so we see that \( (2fl - 2)!/(2l - 2)! \) is divisible by \( f^l \). It remains to show that the last factor \( (2l - 2)!/2^{l-1} \cdot l! \) is an integer.

Let \( \nu_2(x) \) denote the 2-adic valuation of a rational number \( x \). Then we have

\[ \nu_2 \left( \frac{(2l - 2)!}{l!} \right) = \sum_{i \geq 1} \left\lfloor \frac{2l - 2}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{l}{2^i} \right\rfloor, \]

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where \([x]\) denotes the largest integer not exceeding \(x\). Since \(l\) is an odd integer, we have 
\[
\left\lfloor \frac{(2l-2)/2^{i+1}}{l} \right\rfloor = \left\lfloor \frac{(l-1)/2^i}{l} \right\rfloor = \left\lfloor l/2^i \right\rfloor.
\]
Hence we have 
\[
\nu_2\left( \frac{(2l-2)!}{l!} \right) = \left\lfloor \frac{2l-2}{2} \right\rfloor = l - 1,
\]
and \((2l-2)!/l!\) is exactly divisible by \(2^{l-1}\).

\[\square\]

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