ARCWISE CONNECTEDNESS OF THE BOUNDARIES OF CONNECTED SELF-SIMILAR SETS

TAI-MAN TANG

Abstract. Let $T$ be the attractor of injective contractions $f_1, \ldots, f_m$ on $\mathbb{R}^2$ that satisfy the Open Set Condition. If $T$ is connected, $\partial T$ is arcwise connected. In particular, the boundary of the Lévy dragon is arcwise connected.

Key Words: self-similar sets, Lévy dragon, Lévy curve, reptiles, self-affine tiles.

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1. The theorem

Let $f_1, \ldots, f_m$ be a family of injective contractions on $\mathbb{R}^2$ satisfying the Open Set Condition: there is a nonempty bounded open set $V$ such that $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{m} f_i(V) \subset V$ (see e.g. [F]). Let $T$ be the attractor of the system. Suppose that $T$ is connected. Among other results, Luo, Rao and Tan prove that $\partial T$ is connected [LRT, Theorem 1.1]. They further ask whether $\partial T$ is arcwise connected. We answer the question in the affirmative.

Theorem 1.1. Let $f_1, \ldots, f_m$ be a family of injective contractions on $\mathbb{R}^2$ satisfying the Open Set Condition. Suppose that $T$ is connected. Then $\partial T$ is arcwise connected.

Corollary 1.2. The boundary of a connected reptile or self-affine tile is arcwise connected.

Lately there is some interest in the topology of self-similar sets, particularly for some classical reptiles and self-affine tiles (see [BKS], [BW], [LRT], [NN]). If the $f_i$ are similarities of the same contraction ratio and $T^c \neq \emptyset$, $T$ is called a reptile. A self-affine tile is defined by an expanding matrix and a digit set. The twindragon, the Heighway dragon and the Lévy dragon are classical examples in both classes. Bandt and Wang [BW] show that the twindragon is a disk. Ngai and Nguyen [NN] show that the Heighway dragon is a union of disks, each having a common point with each of its two neighboring disks. Hence
our theorem is true for the twindragon and the Heighway dragon. Notice that for these $T$, almost all points in $\partial T$ are boundary points of the components of $T^o$. The only exceptions are the two special points of the Heighway dragon, which are limit points of such components.

The non-trivial cases for our theorem are offered by those $T$ where $\partial T$ has many points that are not boundary points of the components of $T^o$, but are the limits of such components. The Lévy dragon offers an example. The Hausdorff dimension of its boundary has been calculated using different methods [DK], [SW]. Its topology is discussed by Bailey, Kim and Strichartz [BKS]. The arcwise connectedness of its boundary is an addition to the results there.

2. Preliminaries

We collect here some definitions and results from point set topology and self-similar sets.

A continuum is a compact connected set. It is non-degenerate if it has more than one point. Let $S$ be a topological space. Let $G$ be an infinite collection of subsets of $S$, not necessarily different. The set of $x \in S$ such that every neighborhood of $x$ contains points of infinitely many sets in $G$ is called the limit superior of $G$, denoted $\limsup G$. The set of $y \in S$ such that every neighborhood of $y$ contains points from all but a finite number of the sets of $G$ is called the limit inferior of $G$, written $\liminf G$. If $\liminf G = \limsup G$, then $G$ is said to be convergent, with limit $\lim G = \liminf G = \limsup G$.

A set $M$ is said to be locally connected at $p \in M$ if for every neighborhood $U$ of $p$, there exists a neighborhood $V$ of $p$ such that every point of $M \cap U$ containing $p$. Equivalently, $M$ has a local base at $p$ consisting of connected sets. $M$ is locally connected if it is locally connected at every one of its points.

**Theorem 2.1.** (a) [W, p.13] If the continuum $M \subset \mathbb{R}^2$ is not locally connected at one of its points $p$, then there is a ball $B_r(p)$ and an infinite sequence of distinct components $C$ and $C_i$ of $M \cap B_r(p)$, $i = 1, 2, \ldots$, such that $\lim \{C_i\} = C$ and $p \in C$.

(b) [W, p.14] There is a non-degenerate subcontinuum $H$ of $M$ containing $p$ such that $M$ is not locally connected at every point of $H$.

**Theorem 2.2.** [W, p.27] Every locally connected continuum is arcwise connected.

An arc is a homeomorphic image of $[0, 1]$. A simple closed curve is a homeomorphic image of a circle. A set $M$ is said to have property $S$ if for each $\epsilon > 0$, $M$ is the union of a finite number of connected sets of diameter less than $\epsilon$.

**Theorem 2.3.** [W, p.19] A continuum $M$ is locally connected if and only if $M$ has property $S$. 
Theorem 2.4. [W, p.34] If $M \subset \mathbb{R}^2$ is a locally connected continuum with no cut point, the boundary of any component of $\mathbb{R}^2 \setminus M$ is a simple closed curve.

For $\alpha = i_1 \ldots i_k \in \{1, \ldots, m\}^k$, we write $f_\alpha = f_{i_1} \circ \cdots \circ f_{i_k}$. $f_\alpha(T)$ is called a $k$th-level piece of $T$. Let
\[ c_i = \sup \left\{ \frac{|f_i(x) - f_i(y)|}{|x - y|} : x \neq y; x, y \in \mathbb{R}^2 \right\} < 1; c = \sup \{c_1, \ldots, c_m\} < 1. \]
Notice that $\operatorname{diam}(f_\alpha(T)) \leq c^k \operatorname{diam}(T) \to 0$ as $k \to \infty$. As $T = \bigcup_{\alpha \in \{1, \ldots, m\}^N} f_\alpha(T)$, and each $f_\alpha(T)$ is connected, we have part (a) of the following.

Theorem 2.5. Let $T$ be the connected attractor of injective contractions $f_i$, as in Theorem 1.1. Then

(a) $T$ has property S.

(b) $T$ is arcwise connected ([H], [K, p.33]).

(c) If $T^o \neq \emptyset$, $T = T^o$ (e.g. [LRT, p.226]).

(d) Suppose $T^o \neq \emptyset$. For different $\alpha_1, \alpha_2 \in \{1, \ldots, m\}^k$, $f_{\alpha_1}(T^o) \cap f_{\alpha_2}(T^o) = \emptyset$ (e.g. [LRT, p.226]).

3. The Proof

We prove Theorem 1.1 in this section. Under the given hypothesis, $\partial T$ is connected ([LRT, Theorem 1.1(ii)]) and hence a continuum. We will prove that it is arcwise connected.

Lemma 3.1. If $T^o = \emptyset$, $\partial T$ is arcwise connected.

Proof. In this case $\partial T = T$. The arcwise connectedness of $\partial T$ follows from that of $T$ (Theorem 2.5(b)).

Hereafter, we assume that $T^o \neq \emptyset$. Suppose $\partial T$ is not arcwise connected. We derive a contradiction in a sequence of steps.

Claim 3.2. Suppose that $\partial T$ is not arcwise connected. There is a point $p \in \partial T$, and an open ball $B_r(p)$ such that $\partial T \cap B_r(p)$ has infinitely many components $C$ and $C_i$, $i = 1, 2, \ldots$, such that $\lim \{C_i\} = C$ and $p \in C$.

Proof. $\partial T$ not arcwise connected implies that it is not locally connected (Theorem 2.2). The result follows from Theorem 2.4(a).

Claim 3.3. Let $N$ be a positive integer such that for any $N$th-level piece $f_\alpha(T)$, $\alpha \in \{1, \ldots, m\}^N$, $\operatorname{diam}(f_\alpha(T)) < r/2$. There is an $N$th-level piece of $T$, denoted $A$, that is contained in $B_r(p)$ and intersects infinitely many $C_i$.

Proof. As $\lim \{C_i\} = C$, $C_i \cap B_{r/2}(p) \neq \emptyset$ except for finitely many $i$. As $C_i \subset \partial T \subset T = \bigcup_{\alpha \in \{1, \ldots, m\}^N} f_\alpha(T)$, each of these points of intersections is
We get a contradiction by proving the following. 

Claim 3.4. There is an $N$th-level piece of a neighbor of $T$, called $B$, such that $B \subset B_r(p)$ and $B \cap A$ contains points from two of the $C_i$'s, say $C_1, C_2$. Here $A$ is as in Claim 3.3.

Proof. Choosing another $N$ if necessary, suppose that the $N$th-level pieces of $T$ and its neighbors in the blow up have diameter less than $r/2$. From Claim 3.3 $A \cap C_i \cap B_{r/2}(p) \neq \emptyset$ for infinitely many $i$. As $C_i \subset \partial T$, $A \cap C_i \cap B_{r/2}(p)$ is also contained in the neighbors of $T$. As only finitely many $N$th-level pieces of the neighbors of $T$ intersects $B_{r/2}(p)$, one such piece $B$ contains points in $A \cap C_i \cap B_{r/2}(p)$ for infinitely many $i$. As $\text{diam}(B) < r/2$, $B \subset B_r(p)$.

By renaming the $C_i$'s if necessary, suppose that $A \cap B$ contains points from $C_1, C_2$.

Let $x \in A \cap B \cap C_1, y \in A \cap B \cap C_2$. As $A$ and $B$ are arcwise connected (Theorem 2.5(b)), there are arcs $\gamma \subset A \subset T, \beta \subset B \subset T^c$ with endpoints $x, y$. We get a contradiction by proving the following.

Claim 3.5. $C_1$ and $C_2$ cannot be distinct components of $\partial T \cap \overline{B_r(p)}$.

Proof. Case 1. If $\gamma = \beta$, the arcs are in $\partial T \cap \overline{B_r(p)}$, and the claim is true.

Case 2. Suppose that $\gamma \neq \beta$, and $\gamma \cap \beta = \{x, y\}$. That is, $\gamma \cup \beta$ is a simple closed curve enclosing a region $D \subset B_r(p)$.

If $\gamma \subset \partial T$ or $\beta \subset \partial T$, then $C_1$ and $C_2$ are joined by an arc in $\partial T \cap \overline{B_r(p)}$, and the claim is true.

Suppose that $\gamma \cap T^c \neq \emptyset$. Look at the components of $T^c \cap D$ whose boundary has nonempty intersection with $\gamma$. Call them $A_i, i \in \mathbb{N}$. Notice that $\partial A_i \subset (\partial T \cap D) \cup \gamma \cup \beta$.

We claim that $\overline{A}_i$ is a locally connected continuum with no cut point. We have to prove the local connectedness of $\overline{A}_i$ at each of its points. As $T$ is a locally connected continuum (Theorem 2.3 2.5(a)), it is locally connected at each of its points. For $z \in D \cap \overline{A}_i$, local connectedness of $\overline{A}_i$ at $z$ follows from the local connectedness of $T$ at $z$.

Next consider $z \in \overline{A}_i \cap \gamma$ with the property that there is an interval $(t_1, t_2) \subset [0, 1]$ with $z \in \gamma(t_1, t_2) \subset \overline{A}_i$ (the ‘interior boundary points’). We have used the same symbol for the arc $\gamma$ and one of its parametrizations $\gamma : [0, 1] \to \mathbb{R}^2$. 

Notice that \( \text{dist}(z, \partial T \cap D) > 0 \). Suppose that \( \overline{A}_i \) is not locally connected at \( z \). Then there is a closed ball \( S \) of \( z \), such that \( S \cap \partial A_i \subset \gamma \), and \( \overline{A}_i \cap S \) has components \( C'_i, C' \) such that \( \lim \{C'_i\} = C' \) (Theorem 2.1(a)). By our choice of \( S \), \( \partial C'_i \subset \gamma \). It follows that every neighborhood of \( z \) in \( S \) intersects \( \gamma \) in infinitely many components. Hence \( \gamma \) does not have a local base of connected neighborhoods at \( z \), contradictory to the local connectedness of \( \gamma \). Hence \( \overline{A}_i \) is locally connected at \( z \). The same argument apply to the ‘interior boundary points’ on \( \overline{A}_i \cap \beta \).

It remains to establish the locally connectedness of \( \overline{A}_i \) at the ‘corner boundary points’ of \( \overline{A}_i \), the points \( z = \gamma(t) \in \gamma \) (and the similar points on \( \beta \)) with the following property. There is no interval \( (t_1, t_2) \subset [0, 1] \) containing \( t \) such that \( \gamma(t_1, t_2) \subset \overline{A}_i \). If \( \overline{A}_i \) is not locally connected at \( z \), it is not locally connected on a non-degenerate sub-continuum \( H \) of \( \overline{A}_i \) containing \( z \) (Theorem 2.1(b)). As we have established the local connectedness of \( \overline{A}_i \) at the points of \( \overline{A}_i \) in \( D \), \( H \subset \gamma \) and hence must be a non-degenerate sub-arc. But then points in \( H \) other then its two end points are the ‘interior boundary points’ discussed in the last paragraph, and \( \overline{A}_i \) is locally connected at such points. This contradicts the definition of \( H \), and proves the local connectedness of \( \overline{A}_i \) at \( z \). Hence \( \overline{A}_i \) is locally connected.

\( \overline{A}_i \) has no cut point, as for any \( z \in \overline{A}_i \), \( A_i \setminus \{z\} \) is in one component, and hence so is \( \overline{A}_i \setminus \{z\} \). This establishes our claim that \( \overline{A}_i \) is a locally connected continuum with no cut point.

By Theorem 2.4 the boundaries of the components of \( \mathbb{R}^2 \setminus \overline{A}_i \) are simple closed curves. Let \( \delta_i \) be the boundary of the unbounded component. Points on \( \delta_i \) are of three types: those in \( D \), \( \gamma \) or \( \beta \). Those in \( D \) and \( \beta \) are in \( \partial T \).

Let \( s_i := \inf \{ s : \gamma(s) \in \delta_i \} \), \( t_i := \sup \{ t : \gamma(t) \in \delta_i \} \). Then \( \delta_i \setminus \{ \gamma(s_i), \gamma(t_i) \} \) is consist of two parts, with at least one lying entirely in \( D \cup \beta \). Call one such part \( \delta'_i \). Then \( \delta'_i \subset \partial T \). Define

\[
\gamma' := (\gamma \setminus \bigcup_i \gamma(s_i, t_i)) \cup \bigcup_i \delta'_i.
\]

Then \( \gamma' \subset \partial T \). Though \( \gamma' \) may not be an arc, it is the image of a continuous curve joining \( x, y \). Therefore \( x, y \) and hence \( C_1, C_2 \) are in the same component of \( \partial T \cap B_r(p) \). This finishes the argument when \( \gamma \cap \beta = \{x, y\} \).

Case 3. Suppose that \( \gamma \neq \beta \) and \( \gamma \cap \beta \) is more than \( \{x, y\} \). Let \( (u_i, v_i), i \in \mathbb{N} \), be maximal intervals with \( \gamma(u_i, v_i) \cap \beta = \emptyset \). For each \( i \), \( \gamma(u_i), \gamma(v_i) \) bounds a segment from each of \( \gamma \) and \( \beta \). The two segments bounded a region \( D_i \). Apply the argument in case 2 to get a curve in \( \partial T \cap D_i \) joining \( \gamma(u_i) \) and \( \gamma(v_i) \). Together with the observation that \( \gamma \cap \beta \subset \partial T \cap B_r(p) \), we get that \( x, y \) and \( C_1, C_2 \) are in the same component of \( \partial T \cap B_r(p) \). \( \square \)
The contradiction obtained in Claim 3.5 proves the arcwise connectedness of $\partial T$.

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