SKETCHING THE ORDER OF EVENTS

TERRY LYONS AND HARALD OBERHAUSER

ABSTRACT. We introduce features for massive data streams. These stream features can be thought of as “ordered moments” and generalize stream sketches from “moments of order one” to “ordered moments of arbitrary order”. In analogy to classic moments, they have theoretical guarantees such as universality that are important for learning algorithms.

1. Introduction

1.1. The streaming problem. A stream \( \sigma = (\sigma_i)_{i=1}^L \) is a sequence of events. An event \( \sigma_i \) is a tuple \( \sigma_i = (\lambda_i, a_i) \in \mathbb{R} \times A \) where \( A \) denotes a finite but typically very large set. Our task is to compute a summary of the stream \( \Phi(\sigma_1, \ldots, \sigma_L) \) and to update it on arrival of a new event. This summary \( \Phi(\sigma) \) should be rich enough to efficiently describe the effects of the stream \( \sigma = (\sigma_1, \ldots, \sigma_L) \), that is allow to make inference about functions \( f(\sigma) \) of the stream. We refer to \( \Phi \) as feature map and in this article we focus on so-called cash register streams, that is the space of events \( E = \mathbb{R}_{>0} \times A \) has only positive increments. We call \( \sigma_i = (\lambda_i, a_i) \in E \) the event with counter increase \( \lambda_i \geq 0 \) in the letter \( a \) and we call \( S = \bigcup_{L \geq 1} E^L \) the set of cash register streams; see [24] for more background on data streaming.

1.2. Examples. Streams taking values in large sets arise in many applications: parsing a text word-by-word or letter-by-letter (\( \lambda_i \equiv 1 \) with \( |A| \approx 2^7 \) if ASCII characters or \( |A| \approx 10^5 \) if English dictionary words are parsed); recording network traffic in a router (\( \lambda_i > 0 \) is the data volume and \( |A| \approx 10^{38} \) IP adresses); in the order book of a stock exchange (\( \lambda_i > 0 \) denoting trading volume, \( |A| \approx 10^4 \) traded assets); etc. Effects of streams are functions \( f(\sigma_1, \ldots, \sigma_L) \): the function that categorizes texts into “drama”, “comedy”, “news”, “gossip”; the function that decides if a network traffic stream contains abnormal patterns; the functions that detects trading patterns in stocks; etc. In all these examples, the order in which elements of the stream are received carries relevant information.

1.3. Features. We construct a map \( \Phi \) from the space of streams into a linear space

\[ \sigma = (\lambda_i, a_i) \mapsto \Phi(\sigma) \]

such that

(1) (Efficient algorithms) \( \Phi(\sigma) \) can be well approximated

(a) in logarithmic space complexity in \(|A|\),
(b) in “streaming fashion”: with a single pass over the stream \( \sigma = (\sigma_i) \).

(2) (Universal features) \( \Phi \) linearizes non-linear functionals \( f \) of streams, i.e.

\[ f(\sigma) \simeq \langle \ell, \Phi(\sigma) \rangle \]

where \( \ell \) is a linear functional of \( \Phi(\sigma) \) and above holds uniformly over streams \( \sigma \). This is known as “universality” in the machine learning literature and justifies the use of standard learning algorithms such as linear classifiers.
(3) **(Pattern queries)** The coordinates of \( \Phi(\sigma) \) are indexed by words build from the alphabet \( A \) and have natural interpretation as counting patterns in \( \sigma \), e.g. \( \Phi_{ai}(\sigma) = \sum_i \lambda_i 1_{a_i = a} \), \( \Phi_{ab}(\sigma) = \sum_{i<j} \lambda_i \lambda_j 1_{a_i = a, a_j = b} \). Operations on streams become algebraic operations in feature space, e.g. stream concatenation amounts to a (non-commutative!) multiplication in feature space.

(4) **(Scaling limits)** the feature map allows to understand the scaling limit, that is when the number of events in the stream becomes very large, \( L \to \infty \). Additionally, it is robust under noisy observations.

Point (1) is a central theme in the streaming community with spectacular progress in recent years [11 3 2 24]; Point (2) is a standard requirement for guarantees of most machine learning algorithms (“universality of features”); Point (3) and Point (4) are a central theme in stochastic analysis [21 11]. Providing features that address all four points is the goal of this article.

**Remark 1.** One can identify \( \sigma = (\sigma_i)_{i=1}^L \) as an element of \( \mathbb{R}^{|A|^L} \) and apply standard features for vector-valued data. This approach becomes computationally infeasible for large \( L \).

Further, methodological problems arise since streams of different lengths are mapped to different feature spaces, etc.

**Remark 2.** In analogy to the count-min sketch [3 4], our feature sketch can be modified to deal with \( \ell_2 \)-error bounds (instead of \( \ell_1 \)) to deal with turnstile instead of cash-register streams etc.; the needed modifications are analogous to the classic case, see Remark [12].

1.4. **Sketching.** Already for simple features \( \Phi \) it is not easy to address Point (1) and often it can be shown that the computational problem is NP-hard in space. A very successful approach to reduce the computational complexity are so-called sketches, that is small data structures that rely on randomized algorithms [1 3 9 24]. These algorithms compute for given \( \epsilon, \delta > 0 \) a random variable \( \hat{\Phi}(\sigma) = (\hat{\Phi}_i(\sigma)) \), such that the relative error is small in probability

\[
P\left( \frac{|\Phi_i(\sigma) - \hat{\Phi}_i(\sigma)|}{\|\Phi(\sigma)\|} < \epsilon \right) > 1 - \delta \text{ for every coordinate } \Phi_i(\sigma) \text{ of } \Phi(\sigma) = (\Phi_i(\sigma)).
\]

An important case is when the feature map are letter frequencies, that is \( \Phi(\sigma) \in \mathbb{R}^{|A|} \) with each coordinate being the frequency of an element \( \sigma \) in \( A \) and \( \|\Phi(\sigma)\| \) denotes the 1-norm. Cormode and Muthukrishnan [6] show that in this case, \( \hat{\Phi}(\sigma) \) can be constructed by sampling a random matrix \( L \in \mathbb{R}^{m \times |A|} \) with \( m \ll |A| \) and storing only the \( m \)-dimensional, random vector

\begin{equation}
L \Phi(\sigma).
\end{equation}

In contrast to compressed sensing, the entries of \( L \) are not i.i.d. but have a rich structure. This allows to construct the estimator \( \hat{\Phi}(\sigma) \) from the low-dimensional, random vector \( L \Phi(\sigma) \) without solving a constrained minimization problem. Moreover this sketch is linear in the sense that,

\[
\hat{\Phi}(a \alpha \sigma + \beta \tau) = a \hat{\Phi}(\sigma) + \beta \hat{\Phi}(\tau)
\]

for \( a, \beta \in \mathbb{R} \) where multiplication with scalars of streams is defined as \( a \sigma = (a \lambda_i, a_i) \) and \( (a \sigma, \beta \tau) \) is simply the concatenation of the streams \( a \sigma \) and \( \beta \tau \). A drawback of the such choices for

\footnote{That is, if \( \sigma = (\lambda_i, a_i)_{i=1}^L \) the coordinate for \( a \in A \) equals \( \sum_{i:a_i = a} \lambda_i \).}
Φ (such as letter frequencies, number of distinct letters, moments of frequencies, etc.) is that they are order agnostic,

\[ \Phi(\sigma) = \Phi(\sigma^\pi) \]

where \( \sigma^\pi = (\sigma_{\pi(1)}, \ldots, \sigma_{\pi(L)}) \) and \( \pi \) is a permutation of \( \{1, \ldots, L\} \). While such order agnostic features are useful and widely used in practice, they only very partially fulfill Points (2), (3), (4) and for many applications the order information carries important information, see the above examples [1, 2].

1.5. Related work and contributions. Finding efficient summaries of patterns in sequences (also called substrings, motifs, etc.) is a classic theme in computer science, data mining and machine learning [18]:

1. A standard machine learning approach are string kernels. These capture the order of events and have already been combined with sketches [29]. However, they only apply to constant counter-increases \( \lambda_i \equiv 1 \), and the behavior and universality as \( L \to \infty \) (Point 2 and Point 4) is not discussed.

2. The engineering community developed algorithms such as SPADE, APriori, Freespan, [14, 33], etc. to find patterns. Usually, properties such as universality that are relevant to machine learning tasks are not discussed (Point (2),(3),(4) above).

3. Describing a sequence as a formal power series in non-commutative variables is a well-known technique in many areas of mathematics [5, 13, 21]. For small alphabets, this was used for for various learning/statistical tasks [25, 31, 12, 17]; further, the kernelization developed in [16] covers large alphabets and string/ANOVA/time warping kernels arise as a special cases; however, the latter restricts to kernelized learning algorithms and gives no pattern queries.

4. Sketches have been used for learning and optimization tasks [10, 26, 32], see for example [15] for detection of trends in time series, but in a stream/sequential context these results are usually discussed without theoretical guarantees such as universality or scaling limits, Points (2) and (4).

To sum up: some classic constructions (such as string kernel features, frequency sketches, etc.) arise as special cases or are related to our approach. However, the central theme of our approach is to focus on the functions of stream and how sketches of features allow to linearize such functions. The key to this is the algebraic-analytic background that allows to prove properties that address Points (2), (3), (4). More precisely, is capture in the interplay of two algebras

- a graded, non-commutative algebra as the feature space (to capture order information),
- a graded, commutative algebra of linear functionals of features (that is dense among function on streams),

which can be elegantly formulated as an Hopf algebra. Another perspective that we develop in a streaming context is to identify a stream as a lattice path in the free vector space spanned by the alphabet \( A \). This elementary observation allows us to apply insights from stochastic analysis and rough path theory to the study of streams, e.g. it gives a useful topology on \( S \) and clarifies the behaviour as then number of events goes to infinity, etc. In turn, developing sketching ideas from this perspective allows for efficient computation, thus also addressing Point (1) which ultimately allows to learn nonlinear functions of massive data streams.
Remark 3. We are motivated by the count-min sketch. The underlying principle (streams as paths that are injected into the algebra of non-commutative polynomials) is not restricted to the count-min sketch and it is an interesting question how it can be applied to other classic sketching algorithms such as [11, 7], or other approaches to summarize massive streams such as sampling [8].

1.6. Applications and experiments. In Section 4 we discuss applications. These include pattern queries and building the list of patterns of heavy hitters with a single parse over the stream; \( M = 1 \) recovers the usual heavy hitter sketch. Many effects of streams are given by functions \( f(\sigma) \) that are well approximated by considering only the substream \( \sigma^H \) consisting of heavy hitters, \( f(\sigma) \approx f(\sigma^H) \). By combining universality of \( \Phi \) and our sketching result this allows to approximate

\[
 f(\sigma) \approx f(\sigma^H) \approx \langle \ell, \Phi(\sigma^H) \rangle.
\]

For example, \( f \) could assign a label to streams (normal/abnormal stream) and one can train a standard linear algorithm to find the linear functional \( \ell \).

In Section 5 we give numerical examples. Our algorithms are easy to parallelize and thus can make use of multi-threading on several CPUs or GPUs to parse high-volume streams. We implemented our algorithms in C++ and ran the following experiments on synthetic data:

1. (Speed and ground truth). We evaluated our algorithms on a deterministic stream. The dimensions of the stream was chosen such one can still calculate the ground truth for ordered moments up to \( M = 3 \). We then compared it to the error introduced by sketching and the speed.

2. (Classifying Markov chains). We sample streams from two Markov chains in a state space \( A \) consisting of \( |A| = 10^5 \) letters. Even in such a simple toy model of high-dimensional streams, order-agnostic features can become quickly useless but sketches of patterns allow to efficiently train classifiers.

2. FROM LOCAL EVENTS TO GLOBAL SUMMARIES

Fix a finite set \( A \) of letters and denote with

\[
 S = \bigcup_{L \geq 1} E^L = \bigcup_{L \geq 1} \{ \sigma = (\sigma_i)_{i=1}^L : \sigma_i \in E \}
\]

the set of turnstile streams consisting of an arbitrary number of elements, so-called events, in \( E = \mathbb{R}_{>0} \times A \). Further, denote with

\[
 A^* = \{ a_1 \cdots a_M : a_1, \ldots, a_M \in A, M \geq 0 \}
\]

the set of words; we denote the empty word with 1. Let

\[
 \mathcal{F} := \mathbb{R}\langle \langle A \rangle \rangle := \left\{ \sum_{w \in A^*} c_w w : c_w \in \mathbb{R} \right\}
\]

be the linear space of formal power series in \( |A| \) non-commuting variables. Below we show how to construct feature maps

\[
 \Phi : S \to \mathcal{F}, \quad \sigma \mapsto \Phi(\sigma) = \sum_w \Phi_w(\sigma) w
\]

by “stitching events together in feature space”.

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2.1. From local events to global descriptions. Fix a so-called event map

\[ p : E \rightarrow F \]

We evaluate \( p \) along the stream \( \sigma_1, \sigma_2, \ldots \) and update our features by (non-commutative!) multiplication in \( F \)

\[
\Phi(\emptyset) = 1, \\
\Phi(\sigma_1, \ldots, \sigma_{L+1}) = \Phi(\sigma_1, \ldots, \sigma_L) p(\sigma_{L+1})
\]

(2.1)

The non-commutativity of the multiplication in the feature space \( F \) captures the order information. A priori there are many possible choices for the event map \( p \), each choice results in different coefficients \( (\Phi_w(\sigma))_{w \in A^*} \) of \( \Phi(\sigma) = \sum \Phi_w(\sigma) w \in F \). More importantly, different choices of \( p \) give rise to different algebraic structures on the dual of \( F \).

**Example 4.** Consider the feature map

\[
p(\lambda, a) = 1 + \lambda a.
\]

With \( A = \{a, b\} \) and \( \sigma = ((1, a), (1.5, b), (1, b), (2, a)) \). Then \( A^* = \{1, a, b, a^2, ab, ba, b^2, a^3, \ldots\} \) and

\[
\Phi(\sigma) = (1 + a)(1 + 1.5b)(1 + b)(1 + 2a) = 1 + 3a + 2.5b + 2a^2 + 2.5ab + 5.5ba + 1.5b^2.
\]

(2.3)

Note that each coordinate \( \Phi_w(\sigma) \) counts how often \( w \) appears as subword in \( \sigma \) relative to the counter-increases \( \lambda_i \); the map (2.2) was introduced in [16] in the context of sequentializing kernels; e.g. for a stream with constant counter-increase \( \lambda_i = 1 \), \( \Phi(\sigma) \) recovers the classic vanilla string kernel features [19] and similarly one recovers ANOVA features etc., see [16].

**Example 5.** Let

\[
p(\lambda, a) = 1 + \lambda a + \frac{\lambda^2 a^2}{2!} + \cdots.
\]

This leads to the feature map

\[
\Phi(\sigma) = \left(1 + a + \frac{a^2}{2!} + \cdots\right) \left(1 + 1.5b + \frac{(1.5b)^2}{2!} + \cdots\right) \left(1 + b + \frac{b^2}{2!} + \cdots\right) \left(1 + 2a + \frac{(2a)^2}{2!} + \cdots\right)
\]

\[
= 1 + 3a + 2.5b + \left(2 + \frac{1}{2!} + \frac{2^2}{2!}\right) a^2 + (1.5 + 1)ab + \cdots.
\]

(2.4)

The coefficients are somewhat less intuitive than in above Example 4 but lead to a classic algebraic structure (the shuffle algebra, Appendix B). Moreover, as \( L \rightarrow \infty \) this choice of event map still makes sense for turnstile streams, whereas Example 4 leads to problems in the limit \( L \rightarrow \infty \) when turnstile streams are considered and rough paths appear, see the discussion [16] for further details in a (kernel) learning context.

**Remark 6.** The algebraic structure of Example 5 is the classic shuffle Hopf algebra, whereas the (Hopf) algebra arising from Example 4 was at least new to us, see Theorem 32 and Theorem 35 in the Appendix B.

**Remark 7.** An interesting question is how to construct a commutative product on the dual of \( F' \) for a given event map \( p \) and vice versa. Note that the non-commutative product (formal power series multiplication) in \( F \) is the same for all choices of \( p \).
2.2. (Pseudo-)Norms. For \( f = \sum_{w} c_w w \in \mathcal{F} \) define
\[
\|f\|_1 = \sum_{w} |c_w|
\]
(with the convention \( \|f\|_1 = \infty \) if the sum does not converge). For a word \( w = a_1 \cdots a_M \in A^* \) define its length \( |w| = M \) as the number of letters in \( w \). For \( M \geq 1 \) define
\[
\|f\|_{1,M} = \sum_{w; |w| \leq M} |c_w| \quad \text{and} \quad \|f\|_{1,(M)} = \sum_{w; |w| = M} |c_w|.
\]
None of these are norms on \( \mathcal{F} \) (but on appropriate sub- or quotient-spaces). However they appear naturally in our calculations.

2.3. Some algebra and feature universality. Denote with
\[
\mathcal{F}' := \mathbb{R} \langle A \rangle = \left\{ \sum_{w \in A^*} c_w w : c_w \in \mathbb{R} \text{ and } c_w = 0 \text{ for infinitely many } w \right\}
\]
the subset of \( \mathcal{F} \) that consists of finite sums. We identify elements of \( \mathcal{F}' \) as linear functionals acting on \( \mathcal{F} \) via the pairing
\[
\langle \cdot, \cdot \rangle : \mathcal{F}' \times \mathcal{F} \to \mathbb{R}, \quad \langle \ell, P \rangle := \sum_{w} c_w d_w \quad \text{where} \quad \ell = \sum_{w} d_w w, \quad P = \sum_{w} d_w w.
\]
A far reaching result is that \( \mathcal{F}' \) can be equipped with a commutative multiplication, that is for \( \ell_1, \ell_2 \in \mathcal{F}' \) there exists a \( \ell \in \mathcal{F}' \) that is given by multiplication of \( \ell_1, \ell_2 \) such that
\[
\langle \ell_1, \Phi(\sigma) \rangle \langle \ell_2, \Phi(\sigma) \rangle = \langle \ell, \Phi(\sigma) \rangle.
\]
Thus
\[
\{ \sigma \to \langle \ell, \Phi(\sigma) \rangle, \ell \in \mathcal{F}' \} \subset \mathbb{R}^S
\]
is an algebra of functions on streams. Equipping \( S \) with bounded variation topology it follows that any continuous function of the stream \( f \in C(S, \mathbb{R}) \) can be approximated by a \( \ell \in \mathcal{F}' \)
\[
f(\cdot) \approx \langle \ell, \Phi(\cdot) \rangle,
\]
uniformly over compact sets in \( S \). In the terminology of machine learning [23], the features \( (\Phi_w(\sigma))_w \) are “universal” and this allows to use linear learning algorithms; we provide the full details in the appendix A and B.

Remark 8. Algebraically it makes more sense to introduce \( \mathcal{F} \) as the dual of \( \mathcal{F}' \), see Appendix B. However, in a learning context we regard \( \Phi(\sigma) \in \mathcal{F} \) as features and learn about \( \sigma \) by linear functionals \( \ell \in \mathcal{F}' \).

Remark 9. The coefficients \( (\Phi_w(\sigma))_w \) carry redundant information: a simple calculation shows that \( (\Phi_a(\sigma), \Phi_b(\sigma), \Phi_{ab}(\sigma) - \Phi_{ba}(\sigma))_{a,b \in A} \) already completely determines \( (\Phi_w(\sigma))_{w \in A^*, |w| \leq 2} \). The reason is that \( \Phi(\sigma) \) lives in a nonlinear subset of \( \mathcal{F} \); in the case \( p(\lambda, a) = \exp(\lambda a) \) this is the free Lie group generated by \( |A| \) variables and the previous observation is simply that we can work in the Lie algebra instead of the Lie group. However, a classic result of Bourbaki is that the dimension of the Lie algebra is \( O\left(\frac{|A|^M}{M}\right) \) as \( M \to \infty \), i.e. this does not kill the exponential growth. Nevertheless, our main sketch can be immediately applied to the Lie algebra which leads to a reduction of computational cost by a constant for the price of more complicated algebraic objects.
2.4. Streams, paths and scaling limits. Given a stream $\sigma \in S$ we can identify it as a path in the free vector space spanned by the letters $A$. That is, identify the set of letters $A$ as ONB basis for $\mathbb{R}^{|A|}$ and an event $\sigma_i = (\lambda_i, a_i)$ as the vector $\lambda_i a_i \in \mathbb{R}^{|A|}$. Consequently $\sigma = (\lambda_i, a_i)_{i=1}^L$ can be seen as the continuous path $\gamma$ in the free vector space spanned by $A$ given by the Donsker embedding

$$t \mapsto \gamma(t) := \sigma_{\lfloor Lt \rfloor} (Lt - \lfloor Lt \rfloor) + \sum_{i=1}^{\lfloor Lt \rfloor} \sigma_i,$$

That is we inject $S \hookrightarrow C\left([0, 1], \mathbb{R}^{|A|}\right)$ by linear interpolation, see for example Figure (2.1).

This gives a topology on streams $S$ that captures the intuitive notion of streams as being similar if they have similar events and length. Further, it allows to study what happens if the number of events goes to infinity, $L \to \infty$, and it gives another interpretation to our features $\Phi(\sigma)$, namely

$$(2.5) \quad \Phi_{a_1 \cdots a_M}(\sigma) = \int_{0 \leq t_1 \leq \cdots \leq t_M \leq 1} dy^{a_1}(t_1) \cdots dy^{a_M}(t_M).$$

We give full details in the Appendix [B]. From this point of view, our sketch chooses a random but very structured linear map $H$ from $\mathbb{R}^{|A|}$ to $\mathbb{R}^d$ for $d \ll |A|$ to turn $\gamma \in C\left([0, 1], \mathbb{R}^{|A|}\right)$ into $\hat{\gamma} = (H\gamma(t)) \in C\left([0, 1], \mathbb{R}^d\right)$ that gives good estimates for the large coordinates of (2.5).

![Figure 2.1](image-url) The stream $\sigma = (((1,a), (1.5,b)), (1,b), (2,a))$ as a path in the free vector space spanned by the letters $a, b$.

3. Sketching the order of events

The algebraic construction of the feature map

$$\Phi : S \to \mathcal{F}, \quad \sigma = (\sigma_i)_{i=1}^L \mapsto \Phi(\sigma) = \sum_{i=1}^L p(\sigma_i)$$

based on a event map $p : \mathcal{E} \to \mathcal{F}$ gives theoretical guarantees that address Points (2), (3), (4) mentioned in the introduction, see Appendix [A] and [B] for details. In this section, we present our main result that addresses the computational aspect, Point (1): although $\mathcal{F}$ is infinite dimensional, it is graded by word length. Analogous to the classic case of polynomials in commuting variables as features, a sensible approach is to truncate at a given degree $M$, i.e. to consider

$$\Phi_M(\sigma) := \sum_{|w| \leq M} \Phi_w(\sigma) w.$$
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(in supervised learning tasks, \( M \) is a parameter that must be chosen as to minimize the generalization error, e.g. in a supervised setting by cross-validation). Unfortunately, the combinatorial explosion of coordinates is

\[
|\{ \Phi_w(\sigma) : |w| \leq M \}| = O\left(|A|^M\right)
\]

and for many streaming applications \(|A|\) is so large that already \( M = 1 \) is infeasible. Our main theorem is

**Theorem 10.** Let \( \sigma \in S \). For every \( \epsilon, \delta > 0 \) there exists a \( \mathcal{F} \)-valued random variable \( \hat{\Phi}(\sigma) \) such that

\[
\mathbb{P}\left( \frac{|\hat{\Phi}_w(\sigma) - \Phi_w(\sigma)|}{\|\Phi(\sigma)\|_{1,(M)}} > \epsilon \right) < \delta, \quad \forall w \in A^* \text{ with } |w| \leq M
\]

and we can compute the set of coordinates

\[
\{ \hat{\Phi}_w(\sigma) : |w| \leq M \}
\]

using \( O\left(\epsilon^{-M} \log \frac{1}{\delta}\right) \) memory units, \( \lceil -\log \delta \rceil \) random bits and a single pass over \( \sigma \) using Algorithm 1.

We can express the error estimate also in terms of the length of \( \sigma \), \( \|\sigma\| = \sum_{i=1}^{L} |\lambda_i| \).

**Corollary 11.** Let \( \sigma \) and \( \hat{\Phi}(\sigma) \) be as above. Then

\[
\mathbb{P}\left( |\hat{\Phi}_w(\sigma) - \Phi_w(\sigma)| > \epsilon \frac{\|\sigma\|^M}{M!} \right) < \delta
\]

for every word \( w \) of length \(|w| = M\).

Note:
- the appearance of the 1-norm \( \|\Phi(\sigma)\|_{1,(M)} \). Above is a very good estimate for “heavy hitter patterns”, i.e. for power law type distributions in the coordinates \( (\Phi_w(\sigma))_{w \in A^*} \),
- the alphabet size \(|A|\) appears only in a logarithm in the computational complexity,
- that applied with \( M = 1 \), above calculates \( \{ \hat{\Phi}_a(\sigma) : a \in A \} \) and Algorithm 1 is then simply the count-min frequency sketch of Cormode–Muthukrishnan [6].

**Remark 12.** Above can be modified for cash-register streams with simple modifications:
- the first option is to replace the coordinate-wise minimum by a median; the second option is to use a count-sketch [3, 4] giving an additional factor of \( \epsilon^{-1} \) in space complexity.
- Similarly, we can replace the 1-norm by other norms in analogy with standard sketches, \( M = 1 \). This leads to higher computational complexity, analogous to the letter frequency case, see [24].

The rest of Section 3 is devoted to the proof of Theorem 10. It is a simple but instructive exercise to run through the remainder of this section for the special case \( M = 1 \) (not patterns, just frequencies) and see how this recovers the standard proof the count-min sketch [6].

\footnote{A memory units stores a positive real number. For implementations the usual considerations and additional cost to deal with rounding errors, floating numbers, etc. apply.}
3.1. **Hashed streams.** Fix a function \( h : A \rightarrow B \), where \( B \) is finite set with \(|B| < |A|\). We study how much information we can recover about \( \Phi(\sigma) \) if we only observe the hashed stream \( \sigma^h := (\lambda_i, h(a_i)) \) of the stream \( \sigma = (\lambda_i, a_i) \). To do so, we work with three objects:

\[
\Phi(\sigma) \in \mathcal{F}, \quad \Phi(\sigma^h) \in \mathbb{R} \langle \langle B \rangle \rangle, \quad \Phi_h(\sigma) \in \mathcal{F}.
\]

the original features \( \Phi(\sigma) \), the features \( \Phi(\sigma^h) \) of the hashed stream \( \sigma^h \) and the third object, denoted \( \Phi_h(\sigma) \), is the “pull-back” of \( \Phi(\sigma^h) \) to \( \mathcal{F} = \mathbb{R} \langle \langle A \rangle \rangle \).

**Definition 13.** Let \( B \) be a finite set and \( h : A \rightarrow B \). For \( \sigma = (\sigma_i) \) with \( \sigma_i = (\lambda_i, a_i) \in \mathbb{R}_{>0} \times A \), define

\[
\sigma^h := (\sigma_i^h) \quad \text{with} \quad \sigma_i^h := (\lambda_i, h(a_i)) \in \mathbb{R}_{>0} \times B.
\]

We call \( \sigma^h \) the \( h \)-hash of \( \sigma \).

We are interested in the situation where \( B \) is much smaller than the original alphabet \( A \) such that we can afford the computational cost to calculate \( \Phi(\sigma^h) \in \mathbb{R} \langle \langle B \rangle \rangle \). Given \( \Phi(\sigma^h) \) we “pull back” these features to \( \mathcal{F} \) to get an approximation of \( \Phi(\sigma) \).

**Definition 14.** Define \( \Phi_h(\sigma) \in \mathcal{F} \) as

\[
\langle \Phi_h(\sigma), w \rangle := \left( \Phi(\sigma^h), h(w) \right) \quad \text{for} \quad w \in A^*.
\]

where \( h(w) := h(a_1) \cdots h(a_M) \in B^* \) for \( w = a_1 \cdots a_M \in A^* \).

If \(|B| < |A| \) then \( h \) is not injective, \( h(a) = h(b) \) for \( a \neq b \). These collisions are the reason \( \Phi_h(\sigma) \) overestimates \( \Phi(\sigma) \).

**Lemma 15.** For any \( h \in B^A \) we have

\[
\langle \Phi(\sigma), w \rangle \leq \langle \Phi_h(\sigma), w \rangle \quad \text{for all} \quad w \in A^*.
\]

Lemma follows directly from the formulas for \( \Phi(\sigma) \) given in Theorem 32. In fact, we get the following explicit expression for this bias.

**Proposition 16.** Let \( h \in B^A \) and \( \sigma = (\lambda_i, a_i)_{i=1}^L \in S \). Then

\[
\Phi_h(\sigma) = \Phi(\sigma) + b \quad \text{and} \quad \langle b, w \rangle = \begin{cases} 
\sum_{(1)} \frac{\lambda(i)}{1!} 1_{h(a(i)) = h(w)} & \text{if} \ p(\lambda, a) = \exp(\lambda a), \\
\sum_{(2)} \lambda(i) 1_{h(a(i)) = h(w)} & \text{if} \ p(\lambda, a) = 1 + \lambda a.
\end{cases}
\]

where the sum \( \sum_{(1)} \) is taken over all \( i = (i_1, \ldots, i_M) \) such that \( i_1 \leq \cdots \leq i_M \) and \( a(i) \neq w \) where \( a(i) := a_{i_1} \cdots a_{i_M} \); for the sum \( \sum_{(2)} \) we additionally assume \( i_1 < \cdots < i_M \).

**Proof.** If \( p(\lambda, a) = \exp(\lambda a) \) (the other case follows by similar arguments) then we know that \( \langle \Phi(\sigma), w \rangle = \sum \frac{\lambda(i)}{1!} \) where the sum is over \( i \), \( i_1 \leq \cdots \leq i_M \) such that \( a(i) = w \). This from Theorem 32 we know that

\[
\langle \Phi_h(\sigma), w \rangle = \left( \Phi(\sigma^h), h(w) \right) = \sum \frac{\lambda(i)}{1!} 1_{h(a(i)) = h(w)}
\]

with the sum over all \( i \), \( i_1 \leq \cdots \leq i_M \). The statement follows by splitting \( 1_{h(a(i)) = h(w)} = 1_{a(i) = w} + \sum_{i} \frac{\lambda(i)}{1!} 1_{h(a(i)) = h(w)} \).

**Corollary 17.** Let \( H \) be a \( B^A \)-valued random variable. For \( \sigma = (\lambda_i, a_i)_{i=1}^L \in S \) we have

\[
\mathbb{E} [\Phi_H(\sigma)] = \Phi(\sigma) + \text{bias}(\sigma),
\]

where \( \text{bias}(\sigma), w \) = \begin{cases} 
\sum_{(1)} \frac{\lambda(i)}{1!} \mathbb{P}(H(a(i)) = H(w)) & \text{if} \ p(\lambda, a) = \exp(\lambda a), \\
\sum_{(2)} \lambda(i) \mathbb{P}(H(a(i)) = H(w)) & \text{if} \ p(\lambda, a) = 1 + \lambda a.
\end{cases}
As a consequence, if $P(H(w) = H(w')) \leq q$ for words $w \neq w'$, $|w| = |w'| = M$ then

$$0 \leq \langle \text{bias}(\sigma), w \rangle \leq q \|\Phi(\sigma)\|_{1,M}$$

for $w \in A^*$ with $|w| = M$.

**Proof.** Note that $\sum_{i:t_1 \leq \cdots \leq t_M} \frac{\lambda(i)}{\lambda(t)} = \sum_{i:t_1 \leq \cdots \leq t_M} \frac{\lambda(i)}{\lambda(t)} 1_{\hat{H}(a(i)) = H(w)} = \sum_{w:|w| = M} \Phi_w(\sigma) \equiv \|\Phi(\sigma)\|_{1,M}$, hence the statement follows since $E \left[ \sum_{i:t_1 \leq \cdots \leq t_M} \frac{\lambda(i)}{\lambda(t)} 1_{\hat{H}(a(i)) = H(w)} \right] \leq q \sum_{i:t_1 \leq \cdots \leq t_M} \frac{\lambda(i)}{\lambda(t)}$. □

### 3.2. Combining independent hashes

We get tail estimates for $\Phi_H(\sigma)$ for a randomly chosen $H$.

**Proposition 18.** Let $M \in \mathbb{N}$ and $H$ be a $B^A$-valued random variable. Then

1. $\mathbb{P}(\|\Phi_H(\sigma) - \Phi(\sigma)\|_{1,M} > x) \leq \frac{\|\text{bias}(\sigma)\|_{1,M}}{x}$ for $x > \|\text{bias}(\sigma)\|_{1,M}$,
2. $\mathbb{P}(\langle \Phi(\sigma), w \rangle \in [\langle \Phi_H(\sigma), w \rangle - x, \langle \Phi_H(\sigma), w \rangle]) \geq 1 - \frac{\|\text{bias}(\sigma)\|_{1,M}}{x}$ for $w \in A^*$, $x > \langle \text{bias}(\sigma), w \rangle$.

As a consequence, if $\mathbb{P}(H(w) = H(v)) \leq q$ for words $w \neq v$, $|w| = |v| = M$ then

$$\mathbb{P}(\langle \Phi(\sigma), w \rangle \in [\langle \Phi_H(\sigma), w \rangle - 2q \|\Phi(\sigma)\|_{1,M}, \langle \Phi_H(\sigma), w \rangle]) \geq \frac{1}{2}.$$

**Proof.** We apply the Markov inequality and Corollary [17] to

$$\|\Phi_H(\sigma) - \Phi(\sigma)\|_{1,M} = \sum_{w:|w| \leq M} \langle \Phi_H(\sigma) - \Phi(\sigma), w \rangle$$

to get

$$\mathbb{P}(\|\Phi_H(\sigma) - \Phi(\sigma)\|_{1,M} > x) \leq \frac{\|\text{bias}(\sigma)\|_{1,M}}{x}.$$

Similarly, applying Markov’s inequality coordinate-wise shows

$$\mathbb{P}(\langle \Phi_H(\sigma), w \rangle - \langle \Phi(\sigma), w \rangle > x) \leq \frac{\|\text{bias}(\sigma)\|_{1,M}}{x}.$$

Hence,

$$\mathbb{P}(\langle \Phi(\sigma), w \rangle \in [\langle \Phi_H(\sigma), w \rangle - x, \langle \Phi_H(\sigma), w \rangle]) \geq 1 - \frac{\|\text{bias}(\sigma)\|_{1,M}}{x}.$$

By Corollary [17]

$$\langle \text{bias}(\sigma), w \rangle = \sum_{i} \frac{\lambda(i)}{\lambda(t)} \mathbb{P}(H(a(i)) = H(w)) \quad \text{if } p(\lambda, a) = \exp(\lambda a),$$

$$\sum_{i} \frac{\lambda(i)}{\lambda(t)} \mathbb{P}(H(a(i)) = H(w)) \quad \text{if } p(\lambda, a) = 1 + \lambda a.$$ 

and $\langle \text{bias}(\sigma), w \rangle \leq q \|\Phi(\sigma)\|_{1,M}$. This shows

$$\mathbb{P}(\langle \Phi(\sigma), w \rangle \in [\langle \Phi_H(\sigma), w \rangle - x, \langle \Phi_H(\sigma), w \rangle]) \geq 1 - q \frac{\|\Phi(\sigma)\|_{1,M}}{x}$$

for $x > q \|\Phi(\sigma)\|_{1,M}$.

The statement follows by choosing $x = 2q \|\Phi(\sigma)\|_{1,M}$. □

Since $\Phi_H$ overestimates $\Phi$, Lemma [15] we can take independent copies of $H$ and take the coordinate-wise minimum to combine these estimators.

**Proposition 19.** Let $H_1, \ldots, H_r$ be independently and identically distributed $B^A$-valued random variables. Assume there exists a $q > 0$ such that for

$$\mathbb{P}(H_j(w) = H_j(w')) \leq q \text{ for } j = 1, \ldots, r \text{ and } w, w' \in A^*, w \neq w'.$$

Then for any $w \in A^*$

1. $\mathbb{P}\left(\min_{j = 1, \ldots, r} \langle \Phi_{H_j}(\sigma), w \rangle - \langle \Phi(\sigma), w \rangle > 2q \|\Phi(\sigma)\|_{1,M}\right) \leq 2^{-r},$
We call memory. This is prohibitively expensive for our applications where functions to choose from, so specifying an element of bounds given in Proposition 23.

Choosing a random element of \( H \) from \( \Phi \) is 2-universal. Hence choosing a random element of \( H \) yields

\[
P \left( \min_{i=1,\ldots,r} \langle \Phi_H((\sigma),w),\langle \Phi(\sigma),w \rangle > x \right) = \prod_{i=1}^r P \left( \langle \Phi_H((\sigma),w),\langle \Phi(\sigma),w \rangle > x \right)
\]

\[
= \prod_{i=1}^r P \left( \langle \Phi(\sigma),w \rangle \notin \left[ \langle \Phi_H((\sigma),w) - x,\langle \Phi_H((\sigma),w) \right] \right)
\]

\[
\leq 2^{-r}.
\]

\[\square\]

Unfortunately, sampling uniformly from \( B^A \) is too expensive: there are \( |B|^{|A|} \) such functions to choose from, so specifying an element of \( B^A \) requires \( |A| \log |B| \) bits of memory. This is prohibitively expensive for our applications where \( A \) is a very large set. Fortunately, this a classic topic in computer science solved by universal hashes.

3.3. Universal hashes. We sample uniformly from a subset \( \mathcal{H} \subset B^A \). Our goal is to construct a set \( \mathcal{H} \) such that the assumptions of Proposition 19 are met, that is we need to bound \( P(H(w) \neq H(w')) \) for \( w \neq w' \). Constructing such families of functions is not trivial but a classic topic in hashing.

**Definition 20.** Fix \( \mathcal{H} \subset B^A \). Let \( H \) be a chosen uniformly at random from \( \mathcal{H} \). We call \( \mathcal{H} \) a 2-universal family of hash functions if

\[
P(H(a) = H(b)) \leq |B|^{-1}
\]

for \( a \neq b \).

Since \( P(H(a_1 \cdots a_M) = H(b_1 \cdots b_M)) \leq \sup_i P(H(a_i) = H(b_i)) \) this is enough to apply Proposition 19

**Example 21.** Let \( A = \{1,\ldots,m\}, B = \{1,\ldots,n\} \). For any prime \( p \geq m \), the set

\[
\mathcal{H} = \{ h_{a,b} \mid h_{a,b}(x) = (((ax + b) \mod p) \mod n), 1 \leq a \leq p - 1, 0 \leq b \leq p - 1 \} \subset B^A
\]

is 2-universal. Hence choosing a random element of \( \mathcal{H} \) requires \( 2 \log p \) random bits.

3.4. Sketching features. Since for every random hash, the estimator \( \Phi_H(\sigma) \) overestimates \( \Phi(\sigma) \) we simply take the minimum to get a better estimator \( \hat{\Phi}(\sigma) \) for \( \Phi(\sigma) \). The specific choice of alphabet size and number of hashes will become clear from the probabilistic bounds given in Proposition 23

**Definition 22.** For \( \epsilon, \delta > 0 \) let \( \mathcal{H} \subset B^A \) be a 2-universal family into an alphabet \( B \) of cardinality \( |B| = \lceil 2\epsilon^{-1} \rceil \). Draw \( r = \lceil \log \delta \rceil \) elements \( H_1, \ldots, H_r \) independently and uniformly from \( \mathcal{H} \) and define the random map

\[
\sigma \mapsto \hat{\Phi}(\sigma) \in \mathcal{F} \text{ by } \hat{\Phi}_w(\sigma) := \min_{i=1,\ldots,M} \langle \Phi_{H_i}(\sigma),w \rangle \text{ for } w \in A^*.
\]

We call \( \hat{\Phi}(\sigma) \) the \( (\epsilon, \delta) \)-count-min-sketch of \( \Phi(\sigma) \).

Applying the previous estimates yields
Proposition 23. Let $\hat{\Phi}(\sigma)$ be the $(\epsilon,\delta)$-count-min-sketch of $\Phi(\sigma)$. Then

$$\mathbb{P}\left(\hat{\Phi}_w(\sigma) \in \left[\Phi_w(\sigma) - \epsilon \|\Phi(\sigma)\|_1, \Phi_w(\sigma)\right]\right) \geq 1 - \delta \text{ for all } w \in A^* \text{ with } |w| = M.$$ 

Algorithm 7 computes for given $(\epsilon,\delta,M)$ the set of coordinates

$$\{\hat{\Phi}_w : |w| \leq M\}$$

with a single pass over $\sigma$ using $O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)$ memory units and $\lceil -\log \delta \rceil \log |A|$ random bits.

Proof. By universality of $\mathcal{H}$ we have $\mathbb{P}(H(w) = H(w')) \leq \sup_{a, b \in A, a \neq b} \mathbb{P}(H(a) = H(b)) \leq |B|^{-1}$. Thus the assumptions of Proposition 19 are met with $q = 2^{-1}\epsilon$. Using the 2-universal hash family, we can store $r$ hash functions in $2r \log |A|$ random bits. Each $\langle \Phi_H(\sigma), w \rangle$ for $|w| \leq M$ requires $O\left(|B|^M\right) = O(\epsilon^{-M})$ of storage, thus for $r \equiv \lceil -\log \delta \rceil$ hashes we need $O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)$ memory units. \hfill $\square$

4. Applications: Pattern queries and heavy hitter patterns

4.1. Pattern queries. Theorem 10 allows to estimate patterns in the stream $\sigma$. For the event map $p(\lambda, a) = 1 + \lambda a$, recall that the resulting features are

$$\Phi_w(\sigma) = \sum_{(i)} \lambda_{i_1} \cdots \lambda_{i_M}$$

with the sum taken over all $i_1 < \cdots < i_M$ such that $a_{i_1} \cdots a_{i_M} = w$. Thus they count how often a word $w$ appears within the stream $\sigma$, weighted by the counter increases. Algorithm 1 calculates the estimator $\hat{\Phi}^M(\sigma)$ that is given in Theorem 10. The same applies with $p(\lambda, a) = \exp(\lambda a)$ though in this case we allow for $i_1 \leq \cdots \leq i_M$ and account for this with a factorial $\frac{1}{i_1! \cdots i_M!}$.

4.2. Patterns of heavy hitters. Fix a threshold $\rho > 0$ and $M \in \mathbb{N}$. Denote

$$\mathcal{H}(\sigma) = \{a \in A : \Phi_a(\sigma) \geq \rho\}$$

where we recall the definition $\|\Phi(\sigma)\|_1 = \sum_{w : |w| = M} \Phi_w(\sigma)$. The standard (count-min) sketch produces a random set $\hat{\mathcal{H}}$ such that $\mathcal{H} \subseteq \hat{\mathcal{H}}$ and $\hat{\mathcal{H}} \setminus \mathcal{H}$ is small. Recall one of our motivations in the introduction was to learn effects/functions of streams

$$f : S \to \mathbb{R}$$

from the sketch $\hat{\Phi}(\sigma)$. A common scenario is that $f(\sigma)$ is well approximated by the stream of heavy hitters, i.e. $f(\sigma) \approx g(\sigma^\mathcal{H})$ where $g$ denotes simply the restriction of $f$ to streams in letters in $\mathcal{H} \equiv \mathcal{H}(\sigma)$, and $\sigma^\mathcal{H} = (a_i, \lambda_i)_{a_i \in \mathcal{H}}$. Combining the universality $\Phi$, Proposition 30 with the sketch of Theorem 10 yields

$$f(\sigma) \approx g(\sigma^\mathcal{H})$$

$$\approx \left\langle \ell, \Phi(\sigma^\mathcal{H}) \right\rangle$$

$$= \sum_{w \in (\mathcal{H}(\sigma))^*} \ell_w \Phi_w(\sigma^\mathcal{H}).$$
Thus we can train a linear learning algorithm with \( \left( \hat{\Phi}_w(\sigma) \right)_{w \in \mathcal{H}(\sigma)^*} \) as features to find \( \ell \in \mathcal{F}' \).

**Definition 24.** Let \( \rho > 0 \) and \( M \geq 1 \). We call
\[
\mathcal{H}(\sigma) = \{ a \in A : \Phi_a(\sigma) > \rho \}
\]
the set of heavy hitter letters of threshold \( \rho \) for the stream \( \sigma \). We call
\[
\mathcal{H}_M(\sigma) = \left\{ w \in \mathcal{H}(\sigma)^* : \Phi_w(\sigma) > \rho |w| \right\}
\]
the set of heavy patterns of length \( M \) of threshold \( \rho \) of the stream \( \sigma \).

**Remark 25.** A subtlety is that coordinates of words of different lengths scale differently: if we replace \( \sigma = (\lambda_i, a_i) \) with \( \sigma^c = (c\lambda_i, a_i) \) for some \( c > 0 \) then \( \Phi_w(\sigma^c) = c |w| \Phi_w(\sigma) \). Thus it is natural to scale the threshold accordingly as done in above definition.

To carry out the above learning method (find the heavy hitter patterns \( \mathcal{H}_M \), train a learning algorithm on \( \left( \hat{\Phi}_w(\sigma) \right)_{w \in \mathcal{H}_M} \) we need to compute the set of heavy hitter patterns \( \mathcal{H}_M \). We could use Theorem 10 and query the sketch for all \( O(|A|^M) \) coordinates but this is too expensive. Instead, as in the \( M = 1 \) case, one can approximate the list \( \mathcal{H}_M \) while parsing the stream.

**Theorem 26.** For given threshold \( \rho > 0 \), Algorithm 2 computes for a given stream \( \sigma \) a random set of words \( \hat{\mathcal{H}} \) such that
\begin{itemize}
  \item \( \mathcal{H}_M(\sigma) \subset \hat{\mathcal{H}} \),
  \item If \( w \notin \mathcal{H}_M(\sigma) \) with \( \Phi_w(\sigma) < \rho - \epsilon \| \Phi(\sigma) \|_1(M) \) then \( \mathbb{P}(w \in \hat{\mathcal{H}}) < \delta \).
\end{itemize}
Algorithm 2 uses one pass over \( \sigma \), \( O\left(\epsilon^{-M} \log \frac{1}{\delta}\right) \) memory units and \( \lceil -\log \delta \rceil \log |A| \) random bits.

**Proof.** Since the sketch of Theorem 10 overestimates,
\[
\hat{\Phi}_w(\sigma) \geq \Phi_w(\sigma) \text{ for all } w \in A^*
\]
it follows that every \( w \in A^* \) with \( \Phi_w(\sigma) > \rho \) will be an element of \( \hat{\mathcal{H}} \). On the other hand, by Theorem 10 \( \hat{\Phi}_w(\sigma) > \Phi_w(\sigma) + \epsilon \| \Phi(\sigma) \|_1(M) \) occurs with probability smaller than \( \delta \). Hence, if \( \Phi_w(\sigma) < \rho - \epsilon \| \Phi(\sigma) \|_1(M) \), the probability of erroneously including \( w \) is less than \( \delta \). The computational complexity is the same as in Theorem 10 up to a constant since we only additionally check \( \hat{\Phi}_w(\sigma) > \rho \) at every event. \( \square \)

**Remark 27.** Similar to the count-min sketch for finding the set of heavy hitter letters, choosing the threshold \( \rho \) can be an issue: if \( \rho \) is too big, the set of heavy hitters stays empty; if it is choose too small it the set \( \mathcal{H}_M(\sigma) \) becomes quickly too large. One approach is to run above heavy hitter pattern sketch simultaneous for different thresholds and stop calculating the associated heavy hitter list as soon as it becomes too big. Note that this can be done with the same sketch \( \hat{\Phi} \), thus the additional computational overhead is minor.

5. Experiments

The sketch of Theorem 10 provides an estimate for every coordinates \( \Phi_w(\sigma) \), not only the heavy hitter coordinates. In general it is an interesting question how to summarize the
quality of a sketch, that is to find a “loss function”. Since the $\ell_1$ norm naturally appears in the estimates, we record in the experiments the difference between $\Phi_w(\sigma)$ and $\hat{\Phi}_w(\sigma)$, i.e.

$$\text{Error}_M := \frac{1}{M} \sum_{m=1}^{M} \text{error}_m, \text{ where } \text{error}_m := \frac{\sum_{|w|=m} |\Phi_w(\sigma) - \hat{\Phi}_w(\sigma)|}{\|\Phi(\sigma)\|_{1,(M)}}$$

denotes the relative error on the level of the $m$-th ordered moments. Recall that the sketch always overestimates, $\hat{\Phi}_w(\sigma) \geq \Phi_w(\sigma)$, hence $\|\hat{\Phi}(\sigma)\|_{1,(M)} \geq \|\Phi(\sigma)\|_{1,(M)}$.

5.1. **Experiment 1. Stream sketch.** We apply Theorem 10 to a fixed stream $\sigma = (\sigma_i)_{i=1}^L$ consisting of $L = 10^6$ events that are drawn from an alphabet with $|A| = 100$ letters. This is a toy example in the sense that $|A| = 100$ is not sufficient for many real-world examples. However, we find it instructive since it allows to calculate and store the ground truth, i.e. $\Phi(\sigma)$ (6 hours on a multicore machine for words up to length 3) to compare it with the quality of the sketch. Table 1 and summarizes the performance of the sketch from Theorem 10.

| $|B|$ | Nr. of hashes | Events/second | memory for $\Phi(\sigma)$ | memory for $\hat{\Phi}(\sigma)$ | Error$_3$ |
|------|---------------|---------------|--------------------------|-----------------------------|----------|
| 4    | 2             | 64368         | 6012.50                  | 3923.47                     |
| 4    | 4             | 33529.3       | 3006.25                  | 3048.61                     |
| 4    | 8             | 17651.8       | 1503.13                  | 2927.01                     |
| 4    | 16            | 9120.63       | 751.56                   | 2086.38                     |
| 4    | 32            | 4620.79       | 375.78                   | 2061.50                     |
| 8    | 2             | 13231.9       | 864.81                   | 484.71                      |
| 8    | 4             | 6688.18       | 432.41                   | 364.40                      |
| 8    | 8             | 3411.47       | 216.20                   | 293.34                      |
| 8    | 16            | 1712.27       | 108                      | 268.00                      |
| 8    | 32            | 850.85        | 54.05                    | 230.30                      |
| 16   | 2             | 1567.09       | 115.63                   | 57.52                       |
| 16   | 4             | 781.82        | 57.81                    | 42.94                       |
| 16   | 8             | 390.48        | 28.91                    | 38.66                       |
| 16   | 16            | 194.98        | 14.45                    | 33.14                       |
| 16   | 32            | 97.213        | 7.23                     | 26.29                       |
| 32   | 2             | 775.96        | 1.87                     | 8.15                        |
| 32   | 4             | 388.06        | 7.47                     | 6.59                        |
| 32   | 8             | 195.25        | 3.73                     | 5.01                        |
| 32   | 16            | 97.93         | 1.87                     | 4.41                        |
| 32   | 32            | 49.21         | 0.99                     | 3.60                        |

Table 1. The stream $\sigma = (1,a_i)_{i=1}^{10^5}$ with $a_i \in A$, $|A| = 100$. There are 10 letters that make about 10 percent of the events, the rest of the events is uniformly distributed among the remaining 90 letters. Calculating the ground-truth $(\Phi_w(\sigma))_{|w|\leq 3}$ took 6 hours on a modern multicore computer (Intel Xeon, CPU E5-2690 v4, 2.60GHz, 56 CPUs with 2 threads per core). Without sketching, one needs to update for every of the $10^5$ events $10^2 + 10^4 + 10^6$ real numbers.
5.2. **Experiment 2: classifying Markov chains.** We sampled two types of random streams of constant counter increase \( \lambda_i \equiv 1 \) from an alphabet consisting of \( |A| = 10^5 \) letters. Each stream contains \( L = 10^5 \) events and two heavy hitters which we denote below wlog as \( \{1,2\} \). The first type of stream was sampled as follows: in the first 2500 events we choose at each step with probability \( p \) the letter 1 and with probability \( 1 - p \) uniformly from \( A \setminus \{1,2\} \); in the following 2500 events we do the same with 2 instead of 1; in the last 5000 events we choose with probability \( p \) uniformly from \( \{1,2\} \) and with \( 1 - p \) uniformly from \( A \setminus \{1,2\} \). The second type of streams was constructed analogous but the heavy hitters were run in reverse order (i.e. 2 as heavy hitter in first 2500 events occurring with probability \( p \), followed by 1 as heavy hitter in next 2500 events occurring with probability \( q \), uniformly from both in the last 5000 events). Note that these are Markov chains (adding time as state space to account for the change of regime). It is trivial for standard sketches to identify the heavy hitters \( \{1,2\} \), but as \( p \) approaches \( q \), it becomes harder to distinguish the streams. On the other hand, taking order information into account allows for perfect classification, see Table 2 below.

| \( q - p \) | \( q \) | Mean accuracy \( M = 1 \) | Mean accuracy \( M = 2 \) |
|---|---|---|---|
| 0.001 | 0.101 | 0.57 | 1.0 |
| 0.005 | 0.105 | 0.63 | 1.0 |
| 0.01 | 0.11 | 0.785 | 1.0 |
| 0.02 | 0.12 | 0.89 | 1.0 |
| 0.03 | 0.13 | 0.975 | 1.0 |

*Table 2.* For \( p = 0.1 \) and each of above values for \( q \) we sampled 500 streams for each type. We then ran heavy hitter pattern sketch (using 20 hash functions to an alphabet of 10 letters) for each stream and computed the corresponding features (i.e. \( \Phi_w(\sigma) : w \in \{1,2,11,12,21,22\} \)) for \( M = 2 \) resp. \( \Phi_w(\sigma) : w \in \{1,2\} \) for \( M = 1 \) and trained a logistic regression classifier (\( \ell_1 \) penalty via 5-fold cross-validation over the training set, training-test splitted as 0.8 to 0.2). Above shows the mean accuracy on the testing set. As \( |p - q| \) becomes small, \( M = 1 \) approaches the uninformed baseline of guessing the type (0.5 mean accuracy is achieved by choosing uniformly at random the type of the stream). In contrast, \( M = 2 \) uses second order information which allows for perfect classification. Calculating the heavy hitter sketch took approximately 0.15 seconds per stream.

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APPENDIX A. A topology on streams and universality

We want to regard streams that have similar length and similar events as close to each other. It is therefore natural to view them as paths in the vector space spanned by letters in A. To make this precise, we use the Donsker embedding

$$\iota: \mathcal{S} \hookrightarrow C \left( [0,1], \mathbb{R}^{[A]} \right)$$

by identifying $\sigma$ as a continuous path $\gamma_\sigma \in C \left( [0,1], \mathbb{R}^{[A]} \right)$ by using $(a_1, \ldots, a_{|A|})$ as an orthonormal basis of $\mathbb{R}^{[A]}$ and setting

$$\gamma_\sigma(t) := \left( tL - \lfloor tL \rfloor \right) \lambda_{\lfloor tL \rfloor} a_{\lfloor tL \rfloor} + \sum_{i=1}^{\lfloor tL \rfloor} \lambda_i a_i \text{ for } j = 1, \ldots, L.$$  \hfill (A.1)

In words: the path $\gamma_\sigma$ is a lattice path starts at $t = 0$ at the origin in $\mathbb{R}^{[A]}$ and upon receiving an event $(\lambda, a)$ goes with constant speed in direction proportional to $\lambda$. Denote with

$$C^{1-var}(\mathbb{R}^{[A]}), \quad \gamma \in C \left( [0,1], \mathbb{R}^{[A]} \right) : ||\gamma||_1 = \sup_{0 \leq t_1 < \cdots < t_M \leq 1} \sum_{i=1}^{M-1} |\gamma(t_{i+1}) - \gamma(t_i)| < \infty \right)$$

the set of bounded variation paths.

**Definition 29.** Denote with $\iota: \mathcal{S} \rightarrow C^{1-var}(\mathbb{R}^{[A]})$ the Donsker embedding. We equip $\mathcal{S}$ with the pullback topology of $\iota$, that is the open sets in $\mathcal{S}$ are $\iota^{-1}(U)$ where $U$ is an open set in $C^{1-var}(\mathbb{R}^{[A]})$.

**Proposition 30.** Let the event map $p: \mathcal{E} \rightarrow \mathcal{F}$ and resulting feature map $\Phi: \mathcal{S} \rightarrow \mathcal{F}$ be as in Theorem 32. For every $f \in C(\mathcal{S}, \mathbb{R})$ and compact set $K \subset \mathcal{S}$ there exists a $\ell \in \mathcal{F}'$ such that

$$\sup_{\sigma \in K} |f(\sigma) - \langle \ell, \Phi(\sigma) \rangle| < \epsilon.$$  

**Proof.** By Stone–Weierstrass we need to verify that $\{ \sigma \mapsto \langle \ell, \Phi(\sigma) \rangle, \ell \in \mathcal{F}' \}$ is a point-separating subalgebra of $C(\mathcal{S}, \mathbb{R})$. The subalgebra property follows since

$$\langle \ell, \Phi(\sigma) \rangle \langle \ell', \Phi(\sigma) \rangle = \langle m(\ell \otimes \ell'), \Phi(\sigma) \rangle$$

where $m$ denotes the shuffle (resp. infiltration product) on $\mathcal{F}'$ as detailed in Appendix A. The fact that $\sigma \mapsto \Phi(\sigma)$ is point-separating follows in the case of the event map $p(\lambda, a) = \exp \lambda a$ from classic results (injectivity of the signature for non-treeleike paths); in the case $p(\lambda, a) = 1 + \lambda a$ note that if $\sigma = (\sigma_i)_{i=1}^L$ and $\tau = (\tau_i)_{i=1}^K$ and $L < K$ then $\Phi_w(\sigma) = 0 \neq \Phi_w(\tau) > 0$ since $\Phi_w(\tau)$ is a sum over disjoint time indices. If $L = K$, and $\sigma_i = (\lambda_i, a_i)$, $\tau_i = (\rho_i, b_i)$ and there exists a $i$ such that $a_i \neq b_i$ then the result follows immediately by comparing at the coordinate $w = a_1 \cdots a_L$. Finally, if $L = K$ and $a_i = b_i$ for all $i$, we can argue as in [22].

APPENDIX B. SOME (HOPE) ALGEBRAS

We describe the interplay between feature and dual space using Hopf algebras. This is a concise way to capture the interplay between $\mathcal{F}$ and its dual space; a reader less familiar with algebra might want to skip this appendix after a brief look at Theorem 32 and Theorem 35.
B.1. **Hopf algebras.** Hopf algebras arise naturally when a linear space $\mathcal{H}$ as well as its dual $\mathcal{H}^*$ are equipped with products $m$ and $m^*$ such that $(\mathcal{H}, m)$ and $(\mathcal{H}^*, m^*)$ are algebras and $m, m^*$ are “compatible”. This is of interest to us, since the feature space $\mathcal{F}$ has the concatenation product (“a new event happens”) and we will that its dual $\mathcal{F}'$ can be equipped with commutative product. This elegantly describes the interplay between dual and feature space and allows us to address Points (2), (3) (universality of features and structure preserving) from Section 1.3.

**Example 31.** Consider a finite dimensional, linear space $\mathcal{H}$ and let $(\mathcal{H}, m)$ and $(\mathcal{H}^*, m^*)$ be algebras. The product $m$ can be written as a linear map $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ that fulfills associativity and distributivity. By duality, $m$ can be encoded as a linear map $\Delta_m : \mathcal{H}^* \to \mathcal{H}^* \otimes \mathcal{H}^*$

\[
\Delta_m : \mathcal{H}^* \to \mathcal{H}^* \otimes \mathcal{H}^*
\]

by $\langle \Delta^*(\ell), g \otimes h \rangle := \langle \ell, m(g \otimes h) \rangle$ for $f, g \in \mathcal{H}, \ell \in \mathcal{H}^*$.

Thus instead of working with two algebras $(\mathcal{H}, m)$ and $(\mathcal{H}^*, m^*)$, we can work with one space $(\mathcal{H}^*, m^*, \Delta_m)$ and two linear maps, $m^* : \mathcal{H}^* \otimes \mathcal{H}^* \to \mathcal{H}^*$, $\Delta_m : \mathcal{H}^* \to \mathcal{H}^* \otimes \mathcal{H}^*$

[or vice versa with $(\mathcal{H}, m, \Delta_m)$, $\Delta_m$ defined analogous to (B.1)]. A natural way to ensure “compatibility” of the two algebras is to require that $\Delta_m$ is an algebra morphism,

\[
m^*(\Delta_m(h) \otimes \Delta_m(g)) = \Delta(m^*(h \otimes g)) \quad \text{for } g, h \in \mathcal{H}^*.
\]

Further, an algebra also has unit $\epsilon$ which can be represented as linear map $\epsilon : \mathbb{R} \to \mathcal{H}$. Again by duality this unit $\epsilon$ translates to a linear map $\mathcal{H}^* \to \mathbb{R}$ called **counit**. More general, a not necessarily finite dimensional vector space $\mathcal{H}$ equipped with two linear maps, called **product** and **coproduct**,

\[
m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \quad \text{and} \quad \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}
\]

that fulfill natural generalizations of above properties is called a **bialgebra**. If additionally (as in our application) the space $\mathcal{H}$ is connected and graded, it is a non-trivial result that there must exist a so-called **antipode** $S : \mathcal{H} \to \mathcal{H}$

and we call $(\mathcal{H}, m, \Delta, S)$ a **Hopf algebra** (we refer to the monographs [2, 30, 28] for details). This antipode has an intuitive interpretation: it is simply the inverse map of a group structure inside the linear space $\mathcal{H}$, that is if we denote with

\[
G(\mathcal{H}) = \{ h \in \mathcal{H} : \epsilon(h) = 1, \Delta(h) = h \otimes h \}
\]

the set of group-like elements, then $(G(\mathcal{H}), m)$ is a group with inverse given by $S(g) = g^{-1}$. In many situations, the elements of $\mathcal{H}$ have a combinatorial interpretation in which case one may think of $m$ and $\Delta$ as composition and decomposition. Elements that are “simple under decomposition” with $\Delta$, are exactly $G(\mathcal{H})$. We use below that linear functionals of

\[3\text{ Takeuchi’s formula reads } S = \sum_{k \geq 0} (-1)^k m^{(k-1)}(id - \eta \epsilon)^{\otimes k} \Delta^{(k-1)} \text{ where } \eta, \epsilon \text{ are unit and counit of } \mathcal{H}.\]
group-like elements are closed under multiplication, that is if $h \in G(\mathcal{H}) \subset \mathcal{H}$ is group-like, then
\[
\langle \ell, h \rangle \langle \ell', h \rangle = \langle \ell \otimes \ell', h \otimes h \rangle = \langle \ell \otimes \ell', \Delta(h) \rangle = \langle m^*(\ell \otimes \ell'), h \rangle \text{ for } \ell, \ell' \in \mathcal{H}^*.
\]

B.2. Feature and dual space for streams. By construction of $\Phi(\sigma)$, the obvious choice for multiplication in $\mathcal{F}$ — no matter what event map $p : \mathcal{E} \to \mathcal{F}$ is used — is the non-commutative multiplication. That is we define the so-called concatenation product $m_c$

\[
m_c : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}, \quad m_c(a_1 \cdots a_M \otimes a_{M+1} \cdots a_{M+N}) := a_1 \cdots a_{M+N} \text{ for } a_1, \ldots, a_{m+n} \in \ell
\]

extended by linearity to $\mathcal{F}$. This turns $(\mathcal{F}, m_c)$ into a non-commutative algebra. To learn about $\sigma$, we apply linear functionals $\ell \in \mathcal{F}'$ to $\Phi_p(\sigma) \in \mathcal{F}$. To find a product $m_p$ that turns the dual $\mathcal{F}'$ into an algebra, it is useful to recall that our features should be group-like. Hence, this product

\[
m_p : \mathcal{F}' \otimes \mathcal{F}' \to \mathcal{F}'
\]

depends highly on the choice of event map $p$. The existence of this product is a priori not clear; a non-trivial results is that for both choices $p(\lambda, a) = 1 + \lambda a$ and $p(\lambda, a) = \exp(\lambda a)$ such a product $m_p$ exists. This turns $(\mathcal{F}', m_p)$ into a commutative algebra. The final step is to capture this structure of two algebras $(\mathcal{F}, m_c)$ and $(\mathcal{F}', m_p)$ by using one Hopf algebra.

**Theorem 32.** Define the event map $p : \mathcal{E} \to \mathcal{F}$ as either

\[
(\lambda, a) \mapsto \exp(\lambda a) \text{ or } (\lambda, a) \mapsto 1 + \lambda a,
\]

and define a feature map $\Phi : \mathcal{S} \to \mathcal{F}$ as

\[
\Phi(\sigma) = \prod_{i=1}^{L} p(\sigma_i) \text{ for streams } \sigma = (\sigma_i) \text{ of events } \sigma_i = (\lambda_i, a_i).
\]

Then

(1) The features $\Phi(\sigma) \in \mathcal{F}$ are ordered moments, that is

\[
\Phi(\sigma) = \mathbb{E}\left[\lambda_{i_1} \cdots \lambda_{i_m} a_{i_1} \cdots a_{i_m}\right],
\]

with expectation taken over the order statistics of $(i_1, \ldots, i_m)$ sampled uniformly without replacement from $\{1, \ldots, L\}$ in the case $p(\lambda, a) = 1 + \lambda a$; in the case $p(\lambda, a) = \exp(\lambda a)$ the expectation is taken over the order statistic of $(i_1, \ldots, i_L)$ sampled uniformly with replacement from $\{1, \ldots, L\}$.

(2) The coordinates $\langle \Phi(\sigma), w \rangle$ are given as

\[
\langle \Phi(\sigma), w \rangle = \begin{cases} 
\sum_{(1)} (-1)^{i} (i) & \text{if } p(\lambda, a) = \exp(\lambda a), \\
\sum_{(2)} i (i) & \text{if } p(\lambda, a) = 1 + \lambda a.
\end{cases}
\]

where the sum $\sum_{(1)}$ is taken over all $i = (i_1, \ldots, i_L) \in \mathbb{N}^L$ such that $i_1 \leq \cdots \leq i_L$ and $a_{i_1} \cdots a_{i_L} = w$; for the sum $\sum_{(2)}$ we additionally assume $i_1 < \cdots < i_L$. We denote $\lambda(i) = \lambda_{i_1} \cdots \lambda_{i_L} \in \mathbb{R}$ and $i!$ is recursively defined as $i! = i_1!$ if $L = 1$ and $(i_1, \ldots, i_L, i_{L+1})! = (i_1, \ldots, i_L)! k!$ where $k = \max_{j \geq 0; i_{L+1} < i_{L+1} = i_{L+1}} j$.

(3) There exists a linear map $m_p : \mathcal{F}' \otimes \mathcal{F}' \to \mathcal{F}'$ such that $(\mathcal{F}', m_p, \Delta_c)$ is a commutative Hopf algebra.
Proof. Points (1) and (2) follow by direct calculations. For \( p(\lambda, a) = \exp(\lambda a) \), Point (3) is a standard result in algebra and the resulting Hopf algebra is known as the shuffle Hopf algebra, see e.g. [27]. The case \( p(\lambda, a) = 1 + \lambda a \) seem to be much less known. The key is to define the commutative product on \( F' \) recursively as

\[
m_p(ab \otimes bw) = a \text{inf}(v \otimes bw) + b \text{inf}(av \otimes b) + avw1_{a=b}
\]

Note the additional third term, which is 0 in the shuffle Hopf algebra case. Then one can directly verify the properties of a Hopf algebra in a (lengthy) direct calculation. This commutative product is known as the infiltration product, see [20]. □

Remark 33.

(1) Above theorem is classic for \( p(\lambda, a) = \exp(\lambda a) \), the product \( m_p \) on \( F' \) is the “shuffle product”. If \( p(\lambda, a) = 1 + \lambda a \), the product is the less standard “infiltration product”.

(2) In the special case of constant counterincrease, \( \lambda_i \equiv 1 \), the coordinates count how often a substring \( w \) appears in \( \sigma \) and the choice of the event map \( p \) determines if an event can be counted several times. The choice \( p(\lambda, a) = 1 + \lambda a \) recovers the string kernel features from [19, 16]. The Hopf algebra structure for string kernels features seems to be new.

(3) If we replace the non-commutative product \( m_c \) in \( F \) with a commutative product, the features reduce to usual (unordered) sample moments as seen by (B.2).

(4) The event map \( (\lambda, a) \mapsto 1 + \lambda a \) and associated Hopf algebra generalizes classic string kernel features. While the coordinates are more intuitive, the resulting feature map is much less robust under scaling than \( (\lambda, a) \mapsto \exp(\lambda a) \), see the section below.

(5) We stick to the machine learning terminology and call \( F \) feature space and \( F' \) dual space. From an algebraic perspective it is the other way around: \( F = \mathbb{R}\langle\langle A \rangle\rangle \) is the algebraic dual of \( F' = \mathbb{R}\langle A \rangle \).

(6) A technical point: \( (F', m_p, \Delta_C) \) is a Hopf algebra, so we expect a Hopf algebra on feature space \( F \) by duality. However, one needs to work with a slightly smaller subset of \( F \) (called the Sweedler dual).

Example 34. As remarked in above proof, the commutative product on \( F' \) known recursively. For example

\[
m_p(ab \otimes ba) = \begin{cases} abba + 2abba + 2baab + baba & \text{if } p(\lambda, a) = \exp(\lambda a), \\ aba + bab + abab + 2abba + 2baab + baba & \text{if } p(\lambda, a) = 1 + \lambda a. \end{cases}
\]

B.3. Scaling limits and estimates. One can identify \( \sigma = (\lambda_i, a_i)_{i=1}^L \) as an element of \( \mathbb{R}|A|^L \) and apply standard features \( \varphi \) for vector-valued data. Most reasonable choice of \( \varphi \) guarantee that

\[
f(v) \approx \langle \ell, \varphi(v) \rangle \text{ for } v \in \mathbb{R}|A|^L
\]

However, this approach becomes infeasible for large \( L \) and additionally methodological problems arise if our data contains streams of different length. On the other hand, we will see that

\[
f(\sigma) \approx \langle \ell, \Phi(\sigma) \rangle
\]

and that above expression still makes sense in the scaling limit \( L \to \infty \).

Theorem 35. Let the event map \( p : E \to F \) and resulting feature map \( \Phi : S \to F \) be as in Theorem 32.
(1) Let \( (\sigma^n)_{n \geq 1} \subset S \) be a sequence of streams such that \( \gamma_{\sigma^a} \) converges in \( C^{1-\text{var}} \left( [0,1], \mathbb{R}^{|A|} \right) \) to a path \( \gamma = (\gamma^a)_{a \in A} \). Then for every word \( w = a_1 \cdots a_M \),

\[
\lim_n \langle \Phi(\sigma^n), w \rangle = \int_{0 \leq t_1 \leq \cdots \leq t_m \leq 1} d\gamma^{a_1}(t_1) \cdots d\gamma^{a_m}(t_m)
\]

and \( \lim_n \Phi(\sigma^n) \) is a group-like element of the Hopf algebra \( F \).

(2) Define \( \|\sigma\|_1 = \sum_{i=1}^L |\lambda_i| \). Then \( \|\sigma\|_1 = \|\gamma_{\sigma}\|_1 \) and

\[
\|\Phi(\sigma)\|_{1,M} \leq \frac{\|\sigma\|_1^M}{M!} \quad \text{and} \quad \|\Phi(\sigma)\|_1 \leq \exp(\|\sigma\|_1) \quad \text{for} \quad \sigma \in S.
\]

with equality if \( p(\lambda, a) = \exp(\lambda, a) \).

(3) Stream concatenation is multiplication in feature space,

\[
\Phi(\sigma, \tau) = \Phi(\sigma) \cdot \Phi(\tau) \quad \text{for} \quad \sigma, \tau \in S.
\]

Proof. Point (1) follows by a direct calculation for both event maps. For Point (2) note that under the event map \( p(\lambda, a) = \exp(\lambda, a) \) we have that

\[
\langle a_1 \cdots a_M, \Phi(\sigma) \rangle = \int_{0 < t_1 < \cdots < t_M < L} d\gamma^{a_1}_{\sigma}(t_1) \cdots d\gamma^{a_M}_{\sigma}(t_M)
\]

where \( \gamma_{\sigma} \) denotes the path \( A.1 \). Then

\[
\langle a_1 \cdots a_M, \Phi(\sigma) \rangle = \frac{1}{M!} \int_{[0,L]^M} d\gamma^{a_1}_{\sigma}(t_1) \cdots d\gamma^{a_M}_{\sigma}(t_M) = \frac{1}{M!} \prod_{m=1}^M \int_{[0,L]} d\gamma^{a_m}_{\sigma}(t) = \frac{1}{M!} \prod_{m=1}^M f_{a_m}.
\]

where denoted with \( f_a \) the frequency of letter \( a \) in \( \sigma \), i.e. \( f_a = \sum_{i:a_i=a} \lambda_i \). Hence,

\[
\left( \sum_{w:|w|=M} \langle w, \Phi(\sigma) \rangle \right) = \sum_{a_1, \ldots, a_M \in A} f_{a_1} \cdots f_{a_M} = \left( \sum_{a \in A} f_a \right)^M = \left( \sum_{i=1}^L \lambda_i \right)^M = \|\sigma\|_1^M.
\]

For the event map \( p(\lambda, a) = 1 + \lambda a \), Theorem 32 shows that the feature coordinates are smaller equal than then ones for the event map \( p(\lambda, a) = \exp(\lambda, a) \). Hence, the result follows. Point (3) follows directly from our construction of features as \( \Phi(\sigma) = \prod_{i=1}^L p(\lambda_i, a_i) \) for \( \sigma = (\lambda_i, a_i) \). \( \square \)
Algorithm 1 $(\varepsilon, \delta, M)$-sketch.

Initialize:
- Set $h \leftarrow \lceil \log \delta \rceil$ and $d \leftarrow \lceil \frac{1}{\varepsilon} \rceil$
- Sample $H_1, \ldots, H_h$ hash functions from a 2-universal hash family
- For each $H_i$ initialize $d + d^2 + \cdots + d^M$ counters to store $\Phi_i = \Phi(H_i(\sigma))$

Process:
- Fetch event $(\lambda, a)$ in stream
  - while $(\lambda, a) \neq \Box$ do
    - for $i = 1, \ldots, h$ do
      - $\Phi_i \leftarrow \Phi_i * p(\lambda, H_i(a))$
      - end for
    - Fetch next event $(\lambda, a)$
  - end while

Output: On query $w$, return $\min_{i \in \{1, \ldots, h\}} \langle \Phi_i, H_i(w) \rangle$

Algorithm 2 Heavy hitter patterns

Initialize:
- Prepare $(\varepsilon, \delta)$ sketch structure $\hat{\Phi}$ as in Algorithm 1
- $\hat{\mathcal{H}} \leftarrow \{1\}$

Process:
- Fetch event $(\lambda, a)$ in stream
  - while $(\lambda, a) \neq \Box$ do
    - for $i = 1, \ldots, h$ do
      - $\Phi_i \leftarrow \Phi_i * p(\lambda, H_i(a))$
      - end for
    - if $\min_{i \in \{1, \ldots, h\}} \langle \Phi_i, H_i(\alpha) \rangle > \rho$ then
      - $\hat{\mathcal{H}} \leftarrow \hat{\mathcal{H}} \cup \{a\}$
    - end if
    - Fetch next event $(\lambda, a)$
  - end while
  - for $w \in \hat{\mathcal{H}}^*$ do
    - if $\min_{i \in \{1, \ldots, h\}} \langle \Phi_i, H_i(w) \rangle \geq \rho^M$ then
      - $\hat{\mathcal{H}}_M \leftarrow \hat{\mathcal{H}}_M \cup \{w\}$
    - end if
  - end for

Output: Return $\hat{\mathcal{H}}_M$

E-mail address: tlyons@maths.ox.ac.uk

E-mail address: oberhauser@maths.ox.ac.uk

Mathematical Institute, University of Oxford