Deep learning Profit & Loss

Pietro Rossi*  Flavio Cocco†  Giacomo Bormetti‡

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Abstract

Building the future profit and loss (P&L) distribution of a portfolio holding, among other assets, highly non-linear and path-dependent derivatives is a challenging task. We provide a simple machinery where more and more assets could be accounted for in a simple and semi-automatic fashion. We resort to a variation of the Least Square Monte Carlo algorithm where interpolation of the continuation value of the portfolio is done with a feed forward neural network. This approach has several appealing features not all of them will be fully discussed in the paper. Neural networks are extremely flexible regressors. We do not need to worry about the fact that for multi assets payoff, the exercise surface could be non connected. Neither we have to search for smart regressors. The idea is to use, regardless of the complexity of the payoff, only the underlying processes. Neural networks with many outputs can interpolate every single assets in the portfolio generated by a single Monte Carlo simulation. This is an essential feature to account for the P&L distribution of the whole portfolio when the dependence structure between the different assets is very strong like the case where one has contingent claims written on the same underlying.

Keywords: Feed-forward neural networks; profit & loss distribution; non-linear portfolios

*Prometeia S.p.A. and University of Bologna, Bologna, Italy. Corresponding author: pietro.rossi@prometeia.it
†Prometeia S.p.A. and University of Bologna, Bologna, Italy.
‡University of Bologna, Italy
1 Introduction

The computation of the future profit & loss distribution of a portfolio is the key step to manage risk and to set aside regulatory capital. To perform it, the financial industry devised different strategies, ranging from historical full evaluation to parametric modeling, linear and quadratic mapping on risk factors, and closed-form analytic approximations (McNeil et al., 2015). As it is well known, the problem has a direct and viable solution if we consider only assets whose future price has a fast analytic solution. In this case, we just generate enough trajectories to the desired time horizon and, for each generated trajectory we have to compute the present value of the portfolio. To build an accurate distribution we need several trajectories of the underlying and, if the portfolio has a large number of assets, the computation can still be formidable, but it can be easily parallelised. If we are not within this context, and the only way to compute the present value of an asset is via Monte Carlo simulation, we are in the unpleasant situation that for each scenario we must perform a new Monte Carlo simulation. The situation is even worse if we are dealing with strongly path-dependent derivatives – such as American or Bermuda options – and highly non-linear payoff in multiple dimensions. The problem can easily turn out to be not tractable.

One way out of this deadlock is to resort to techniques inspired by the Least Square Monte Carlo (LSM) (Longstaff and Schwartz, 2001) to estimate, via back propagation, the continuation value of the portfolio produced by the optimal strategy (Karlin and Taylor, 1975). This method, that usually is performed with polynomial interpolation of the continuation value, has proven effective in many situations. The major drawback is that for each asset in the portfolio the ideal strategy has to be tailored to the asset and, the polynomial interpolation, when used with many variables, shows a marked preference for over fitting.

The approach we follow is along the lines of the LSM method, but with some significant variation. Interpolation is done using feed forward neural networks (FFNN) (Bishop et al., 1995; Ferguson and Green, 2018). The impact on the entire procedure is striking. This
cures the curse of dimensionality associated with the polynomial interpolators. FFNN flexibility and ability to learn (fit) the price of the portfolio is remarkable. Rather than using just one trajectory to propagate back the continuation value, from each time horizon we perform a short Monte Carlo with very few trajectories and, at each time horizon, we train the network to learn the continuation value. Once we have the coefficients of the trained network at each $t$ we launch a large simulation and compute the desired distribution using our trained networks as proxies of the continuation value. The key idea behind, that is the main contribution of this work, is that even with relatively few trajectories in the training phase, we can still obtain an unbiased set of FFNN capable to produce the correct P&L distribution. It is worth to remark again that the procedure we detail is capable to deal with whole portfolios, rather than just single assets, and it seems to be the only feasible approach to the proper reconstruction of the P&L distribution in presence of highly non-linear effects and strong cross-dependence among portfolio components. As a final comment, our work contributes to the spurring stream of quantitative research approaching the hedging problem by means of machine learning techniques (Cao et al., 2019; Buehler et al., 2019).

2 The problem at hand

When dealing with market risk what we are mainly interested in is the P&L distribution. Given the value $V_t$ for our portfolio at a future time $t$ the profit or loss is defined as the difference

$$D(0,t)V_t - V_0,$$

with $D(0,t)$ the discount factor. This way we compare two quantities originating at the same time. The quantity $D(0,t)V_t$ is a random variable defined as

$$D(0,t)V_t = \mathbb{E} [D(0,T)V_T | \mathcal{F}_t] = D(0,t)\mathbb{E} [D(t,T)V_T | \mathcal{F}_t],$$

that is the expected value, conditional to the information known in $t$. 

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The steps needed to estimate the P&L distribution are:

1. sample the event space described by $\mathcal{F}_t$

2. for each sampled event, compute $D(0,t)V_t$

3. build the cumulative distribution function (CDF) for $D(0,t)V_t - V_0$.

Once we take control of the CDF for our portfolio, we can tackle issues like the computation of risk measures, Value-at-Risk (VaR), Expected Shortfall, expectiles, or the expected positive/negative exposure, if we are interested in CVA or DVA contributions.

Computation of $V_t$, possibly for a large set of values of $t$ can be a daunting process. If we do not have available a fast way to perform it, we must resort to Monte Carlo simulation, but a Monte Carlo nested inside another Monte Carlo is most of the time just too expensive from the computational point of view. The strategy we pursue in this work is to produce a fast device to compute $V_t$ for every $t$ we are interested in and every possible event in $\mathcal{F}_t$. Then, we will use it to produce the desired CDF. The fast device we are hinting at is an FFNN taking in input, as regressors, all of the underlyings entering the problem and producing a multivariate fit, one value for each asset making up our portfolio.

3 An optimal stopping problem

In a financial world – to fix ideas we can think about handling a portfolio – we assume decisions based on what we know. These decisions will have consequences according to what will happen in the future but we do not know yet. The main consequence is that our decision will generate a cash flow $i$ (positive, negative or null ) and will have some impact on future cash flows.

To make the reasoning more formal, we look at a time horizon $T$, and break it up into $N$ intervals

$$\text{now} = t_0 < t_1 < \cdots < t_{N-1} < t_N = T = \text{portfolio horizon}.$$
Let $S_n$ the set of variables describing the situation in $t_n$ and, in loose speech, the filtration $\mathcal{F}_n$ can be seen as:

$$\mathcal{F}_n = \bigcup_{i=0}^{n} S_i,$$

all of the information available up to $t_n$. The stochastic nature of the world is described by a transition probability $p(S_n,t_n \mid S_{n-1}, t_{n-1})$ to go from the state $S_{n-1}$ in $t_{n-1}$ to the state $S_n$ in $t_n$.

A strategy $\phi$ is an adapted function to the filtration. In other words, it is a decision one takes based solely on past and present information. The decision we assume in $t_n$ is based on $\mathcal{F}_n$, and has immediate consequences depending on $S_n$ and future (unknown) consequences. The immediate consequence is the generation of a cash flow $i_n$. Think of an American Put option: based on current knowledge, we decide to exercise and the immediate consequence, is the payoff that we pocket. The future, unknown consequence is that we forgo the possibility to exercise at a later time, possibly in a more advantageous condition.

The following discussion, purely a review, relies heavily on material from (Karlin and Taylor, 1975; Longstaff and Schwartz, 2001) and is a standard issue in the optimal stopping time literature. Let’s call $a_n$ the decision we take in $t_n$, $\phi$ the strategy emerging from the $N$ choices we will make and $D(0,t_n)$ the discount factor to be applied to a cash flow in $t_n$.

The value of the $\phi$ strategy we decide to enact will be:

$$I(\phi, S_0) = \mathbb{E} \left[ \sum_{n=1}^{N} D(0,t_n) i(S_n,a_n) \right]. \quad (1)$$

1. evolution occurs in the interval $[t_{n-1}, t_n)$, and we make our decision in $t_n$ having full knowledge of everything that happened up to that point;

2. after we have made our choice, $S_n$ will evolve to $S_{n+1}$ with a probability described by the matrix $p(S_{n+1},t_{n+1} \mid S_n, t_n)$.

Next step is to build a function that is an upper bound for the value described in eq. (1). Let $V(S_N)$ the payoff at maturity of the contract in exam. It is well known that

\[1\text{In the American Put option example, } a_n \text{ can take one of either two values } \{h = \text{hold}, s = \text{stop}\}.\]
the optimal solution to this problem is given by the recursive equation described by:

\[ f_N(S_N) = i(S_N, a_N) \overset{def}{=} V(S_N), \quad \forall S_N \]

\[ f_{n-1}(S_{n-1}) = \max_{a_{n-1}} (i(S_{n-1}, a_{n-1}), \mathbb{E}[D(t_{n-1}, t_n) f_n(S_n) | \mathcal{F}_{n-1}]), \quad \text{for } n = 1, \ldots, N - 1. \]

The previous solution was first derived in (Bellman, 1952) as the optimality condition in dynamic programming and it is experiencing a new life as the key formula in Reinforcement Learning (Cao et al., 2019).

4 Working on a portfolio

Different assets, say \( K \), in a portfolio are characterized by the fact that they have different cash flow structure \( i_k^n \) and the previous result can be easily generalized as:

\[ f^k_N(S_N) = i^k(S_N, a^k_N) \overset{def}{=} V^k(S_N), \quad \forall S_N \]

\[ f^k_{n-1}(S_{n-1}) = \max_{a^k_{n-1}} (i^k(S_{n-1}, a^k_{n-1}), \mathbb{E}[D(t_{n-1}, t_n) f^k_n(S_n) | \mathcal{F}_{n-1}]), \quad \text{for } n = 1, \ldots, N - 1. \]

From the above section we conclude that what we need is a reliable estimate of the transition probability

\[ p(S_{n+1}, t_{n+1} | S_n, t_n). \]

Even though modeling the right process is the major concern when pricing a portfolio, in this paper we are more focused on the methodological aspects. We will make our life simpler and assume that all of the assets undergo a log-normal process. More complex dynamics can be readily dealt with, provided they belong to the class of Markov processes, possibly of order higher than one.

Let \( S \) denote the vector of underlying prices \( S_1, \ldots, S_K \). Starting from \( t_{N-1} \), for each trajectory \( j \) we have a value \( S(j, t_{N-1}) \). From each \( S(j, t_{N-1}) \) we launch \( M \) one-step trajectories from \( t_{N-1} \) to \( t_N \) according to the law \( p(S_N, t_N | S_{N-1}, t_{N-1}) \). The small set of trajectories originating from \( S(j, t_{N-1}) \) will be denoted as \( S^m(j, t_N) \) for \( m = 1, \ldots, M \). In
$t_N$, we know what is the payoff for each possible triplet $S^m(j, t_N)$ so we can easily compute the corresponding payoffs $V^k(S^m(j, t_N))$ for each asset $k = 1, \ldots, K$. The price in $t_{N-1}$ conditioned on $S(j, t_{N-1})$ is given by

$$V^k(t_{N-1} \mid S(j, t_{N-1})) = \mathbb{E}[D(t_{N-1}, t_N) V^k(S(j, t_N)) \mid S(j, t_{N-1})]$$

and the right hand side of the above equation can be estimated by the quantity

$$\frac{1}{M} \sum_{m=1}^{M} D(t_{N-1}, t_N) V^k(S^m(j, t_N)).$$

Now we have $V^k(t_{N-1} \mid S(j, t_{N-1}))$ for $k = 1, \ldots, K$ at each point $S(j, t_{N-1})$ and we use a neural network to fit it using as input variables $S(j, t_{N-1})$. We name the interpolating function $C_{int}^{N-1}$. The symbol $C$ has been chosen as a mnemonic for “Continuation” and is basically a vector value function:

$$C_{int}^{N-1}: S(j, t_{N-1}) \rightarrow V(t_{N-1} \mid S(j, t_{N-1})),$$

where $V(t_{N-1} \mid S(j, t_{N-1})$ is the vector of prices subject to the fact that we have not exercised our option till $t_{N-1}$. Components of $V$, defined in eq. (2), are the price processes of every single asset making up the portfolio. At this point we repeat the process at $t_{N-2}$. We launch again a small set of one-step trajectories from each $S(j, t_{N-2})$ producing $S^m(j, t_{N-1})$. The difference this time is that in $t_{N-1}$ instead of using the known payoff to compute prices for the one-step trajectory, we will use the interpolating function $C_{int}^{N-1}$ and compute the price of the trajectory as

$$\max(i^k(S^m(j, t_{N-1})), C_{int}^{N-1}(S^m(j, t_{N-1})))$$

where $i^k$ is the cash flow obtained by exercising the option. The recursive structure of this scheme is rather evident and we can build interpolating functions all the way down to the first exercise date $t_1$. At each time slice, we have calibrated an interpolating function.
that we can use to compute continuation values. Now, generating P&L distributions is a straightforward matter:

1. generate $L$ trajectories from $t_0$ to $t_n$;

2. for each trajectory use the interpolator $C^n_{int}$ to compute $V^j(t_n)$;

3. the sample distribution of $D(0, t_n)V^j(t_n) - V(t_0)$ will provide the wanted CDF of the portfolio P&L.

As a non negligible bonus, we obtain all the marginal distributions for each asset in the portfolio.

The continuation value, propagated backward, is what we use to build the P&L distribution and it could be used as a proxy for the price itself. Unfortunately, it is not a reliable neither an accurate quantity. For once, the numbers obtained can hardly be considered independent. Therefore we cannot produce at all an estimate for the statistical error. Besides, as is well known in literature (Longstaff and Schwartz, 2001) such a procedure produces a value biased towards higher values. For accurate pricing, it is a much better procedure to use our interpolator as our policy maker. We generate a brand new set of trajectories and for each trajectory we use our FFNN-based continuation value as the strategy maker. For each time horizon we decide to exercise if the payoff is higher than the continuation value, otherwise we check the next time slice. It is worth pointing out that this way of proceeding establishes a strategy that is not looking forward, therefore it can be, at best, as good as the optimal. The price computed this way is always a lower bound for the real price. It is a simple consequence of the fact that any legitimate strategy is at most as good as the optimal strategy.
5 Numerical Experiments

In all the numerical experiments performed, the stochastic process of the underlying is defined by:

\[ S_i(t + \Delta t) = S_i(t) \exp \left( (r - \delta_i)\Delta t - \frac{\sigma_i^2}{2}\Delta t + \sigma_i (W_i(t + \Delta t) - W_i(t)) \right), \]

with \( r \) the instantaneously short rate, \( \delta_i \) the continuously compounded dividend yield, \( \sigma_i \) the volatility of the Wiener process \( W_i^t \). All Wiener processes are non correlated. Furthermore in modeling the portfolios used as an example, we have made some technically simplified assumption, namely all of the assets within a given portfolio share the same exercise schedule and maturity.

5.1 Portfolio Composition

The portfolio we study is made up of three assets:

1. an American Put written on \( S_x \), whose payoff reads

   \[ \text{payoff}_{AM} = (\kappa_{am} - S_x(T))^+, \]

2. a European Call on Min written on \( S_x \) and \( S_y \), where

   \[ \text{payoff}_{Cm} = (\min(S_x(T), S_y(T)) - \kappa_{cm})^+, \]

3. a Bermuda Call on Max written on \( S_x \), \( S_y \), and \( S_z \), whose payoff is given by

   \[ \text{payoff}_{bCM} = (\max(S_x(T), S_y(T), S_z(T)) - \kappa_{bcm})^+. \]

In section 5.4 we look at a one year maturity portfolio with monthly exercise schedule both for the American Put and the Bermuda option, while in section 5.5 we look at the same portfolio on a three years horizon and four months exercise schedule.
5.2 The Interpolator

The network used for the interpolation is a very simple feed forward network with two hidden layers, with 10 nodes each. Its topology is detailed in fig.(1). The activation functions are sigmoidals on both layers. We have tried with different activation functions without any visible benefit.

![Network Diagram](image)

Figure 1: The topology of the FFNN used as interpolator at each exercise date. All the nodes in the hidden layers are connected even though, for graphical reason, this is not properly represented. For the same reason, the bias node at the input layer and each of the hidden layers, is not represented. The output nodes will produce, as described in the text, the continuation value of each of the assets in the portfolio. The legenda, from top down is as follow: American Put, European Call on Min, Bermuda Call on Max.
5.3 Experiments and checks

The portfolio described above has been simulated in two different contexts: the first one for a one-year maturity and monthly exercise schedule, the second for a three-year maturity and triannual exercise schedule. In sections (5.4, 5.5) we show results for the price of the portfolio in these two scenarios and the quantiles for each individual assets as well as the whole portfolio. Given our definition of P&L negative quantiles correspond to losses, then VaR is the negative of the quantile. To check for correctness of the results is not very simple. We elected to perform the following checks. As far as the individual assets are concerned, we compared results, both for prices and quantiles, obtained while handling the whole portfolio in one simulation, with results obtained simulating each asset individually. Results show no difference in the two cases.

For some of the assets considered there are in the literature benchmark results, namely the Bermuda Call on Max and the American Put can be compared with high precision values coming from PDE approaches. Results form both of these checks are presented in section 6. Figures (2) and (3) come from the three-year portfolio. They compare the P&L distribution four months after contract start and two years after contract start. The comparison is done for each single asset individually as well as the whole portfolio. The filled gray curve corresponds to the P&L CDF of the whole portfolio. The bold, dashed, and dashed-dotted lines represent the P&L CDF of the American Put, European Call on Min, and Bermudan Call on Max, respectively. It is important to stress that the CDF of each asset was computed concurrently with the P&L distribution of the entire portfolio. For ease of readability, both figures report the horizontal lines corresponding to the 1, 10, 50, 90, and 99 percent probability levels. The Tables (5) and (6) detail the associated quantile values for the portfolio and for each single component.

The procedure is capable to provide accurate results and the ability to deal with whole portfolios, rather than just single assets allows for a rather accurate estimate of the hedging effects. In this toy example we have clearly the American Put hedging against the two calls, and the results of the VaR show markedly this effect. It is worth pointing out that the VaR
of the whole portfolio could not have been estimated by any approach that would consider each asset separately. The portfolio was build with assets hedging each other and the only way to account for this effect is to compute, for each scenario, the contribution to the P&L distribution of the whole portfolio. This hedging effect is quite noticeable both in table 2 than in table 3. The portfolio VaR is significantly different than the sum of VaR for each asset.

5.4 One-year maturity

The maturity is one year after contract start, exercise dates are monthly, both for the American Put and the Bermuda Call on Max. Parameter values are as follows

\[ S_x = 1.0, \quad S_y = 1.0, \quad S_z = 1.0, \]

\[ \kappa_{am} = 1.0, \quad \kappa_{cm} = 0.9, \quad \kappa_{bCM} = 1.0, \]

\[ \sigma_x = \sigma_y = \sigma_z = 0.2, \quad r = 5.0\%, \quad \delta_x = \delta_y = \delta_z = 3.0\%. \]

5.4.1 Summary of Results

| Asset | Price       |
|-------|-------------|
| AM    | 6.943 ± 0.003 |
| Cm    | 5.876 ± 0.003 |
| bCM   | 19.518 ± 0.005 |

Table 1: The price of each asset within the portfolio. The maturity of every asset is one year and the exercise schedule is monthly.

5.5 Three-year maturity

The maturity is three years after contract start, exercise dates are every four months, both for the American Put and the Bermuda Call on Max. Parameter values are as follows
Table 2: Quantiles of the P&L distribution one months after contract start for a portfolio with one-year maturity and monthly exercise schedule. Quantiles are given for the whole portfolio as well as each single asset.

| Asset/Quantile | .01  | .10  | .50  | .90  | .99  |
|----------------|------|------|------|------|------|
| portfolio      | -8.08| -4.79| 0.07 | 5.12 | 8.80 |
| AM             | -4.01| -2.77| .27  | 3.49 | 7.17 |
| Cm             | -3.63| 2.37 | .15  | 2.47 | 3.78 |
| bCM            | -7.39| -4.26| .11  | 4.40 | 7.24 |

Table 3: Quantiles of the P&L distribution six months after contract start for a portfolio with one-year maturity and monthly exercise schedule. Quantiles are given for the whole portfolio as well as each single asset.

| Asset/Quantile | .01  | .10  | .50  | .90  | .99  |
|----------------|------|------|------|------|------|
| portfolio      | -15.36| -10.18| -.50 | 13.51 | 26.34 |
| AM             | -5.92 | -5.21 | -1.00 | 7.40  | 16.25 |
| Cm             | -5.47 | -4.35 | -.86  | 5.94  | 12.75 |
| bCM            | -13.81| -8.92 | -.29  | 10.93 | 20.30 |

\[ S_x = 1.0, \quad S_y = 1.0, \quad S_z = 1.0, \]
\[ \kappa_{am} = 1.0, \quad \kappa_{cm} = 1.0, \quad \kappa_{bCM} = 1.0, \]
\[ \sigma_x = \sigma_y = \sigma_z = 0.2, \quad r = 5.0\%, \quad \delta_x = \delta_y = \delta_z = 10.0\%. \]

5.5.1 Summary of Results

| Asset | Price          |
|-------|----------------|
| AM    | 18.020 ± 0.005 |
| Cm    | 0.844 ± 0.002  |
| bCM   | 18.666 ± 0.006 |

Table 4: The price of each asset within the portfolio. The maturity of every asset is three years and triannual exercise schedule. All assets have been computed concurrently using an interpolating network with three output units.
| Asset/Quantile | .01 | .10 | .50 | .90 | .99 |
|---------------|-----|-----|-----|-----|-----|
| portfolio     | -8.32 | -5.55 | -0.03 | 8.61 | 18.35 |
| AM            | -10.37 | -6.26 | 0.07 | 7.22 | 13.43 |
| Cm            | -0.75 | -0.58 | -0.13 | 0.87 | 2.38 |
| bCM           | -10.56 | -6.56 | -0.12 | 8.54 | 17.61 |

Table 5: Quantiles of the P&L distribution four months after contract start for a portfolio with three-year maturity and triannual exercise schedule. Quantiles are given for the whole portfolio as well as for each asset. All assets have been computed concurrently using an interpolating network with three output units.

| Asset/Quantile | .01 | .10 | .50 | .90 | .99 |
|---------------|-----|-----|-----|-----|-----|
| portfolio     | -21.23 | -17.93 | -2.14 | 28.74 | 69.53 |
| AM            | -17.94 | -15.64 | 0.01 | 22.41 | 36.83 |
| Cm            | -0.93 | -0.85 | -0.76 | 1.58 | 12.92 |
| bCM           | -18.49 | -16.81 | -5.13 | 25.51 | 64.65 |

Table 6: Quantiles of the P&L distribution two years after contract start for a portfolio with three-year maturity and triannual exercise schedule. Quantiles are given for the whole portfolio as well as for each asset. All assets have been computed concurrently using an interpolating network with three output units.
Figure 2: Three years portfolio, four months after contract start. The filled gray curve corresponds to the P&L CDF of the whole portfolio. The bold, dashed, and dashed-dotted lines represent the P&L CDF of the American Put, European Call on Min, and Bermudan Call on Max, respectively. The CDF of each single asset was computed while processing the whole portfolio.

6 Comparison with existing results

There are no results listed in the literature for the quantiles of the future P&L distribution but, for some of the parameters used above, namely for the three-year exercise schedule, some results are provided in (Becker et al., 2019; Goudenège et al., 2020), concerning the Bermuda Call on Max.

Table (7) compares our results with those in (Becker et al., 2019) where the price and the 95% confidence interval (CI) are provided. These results have been obtained using the interpolated continuation value as a proxy for the exercise surface. FFNN-based prices are fully consistent with the benchmark 95% CI.

Furthermore, we can perform accurate checks for the American Put option. In table (8) we give the corresponding results. The column labeled $\Delta t$ is the interval between exercise dates, and the value $\Delta t = 0$ is the result of a linear regression on the non zero $\Delta t$. The
Figure 3: Three years portfolio, two years after contract start. Legend as in caption of figure (2).

| Nr Asset | $S_0$ | price ± 95% MC error | results from (Becker et al., 2019) | 95% CI          |
|----------|------|----------------------|----------------------------------|----------------|
| 2        | 90   | 8.071 ± 0.005        |                                  | 8.074 [8.060, 8.081] |
| 2        | 100  | 13.901 ± 0.007       |                                  | 13.899 [13.880, 13.910] |
| 2        | 110  | 21.345 ± 0.006       |                                  | 21.349 [21.336, 21.354] |
| 3        | 90   | 11.275 ± 0.007       |                                  | 11.287 [11.276, 11.290] |
| 3        | 100  | 18.683 ± 0.008       |                                  | 18.690 [18.673, 18.699] |
| 3        | 110  | 27.562 ± 0.007       |                                  | 27.573 [27.545, 27.591] |

Table 7: Bermuda Call on Max prices and Monte Carlo errors from FFNN compared with results from (Becker et al., 2019).

column labeled PDE provides the result coming from a method based on partial differential equations. The PDE result has been obtained solving the equation with a semi-implicit method, both with an SOR and an FFT based preconditioning. Some results with dividends are shown in table (9). The quality of the agreement between the FFNN-based values and the PDE results is remarkable.

7 Conclusion

We presented a general and simple machinery to compute the future P&L distribution of a portfolio including non-linear, path-dependent, and strongly correlated derivatives. The
Table 8: $r = 5\%, \sigma = 20\%, T = 1$ year, Strike= 100, $\delta = 0$.

| $\Delta t$ | $S_0$ | Price  | Err  | PDE |
|------------|-------|--------|------|-----|
| 2M         | 90    | 11.340 | $\pm$ 0.003 |      |
| 1M         | 90    | 11.416 | $\pm$ 0.003 |      |
| 2W         | 90    | 11.454 | $\pm$ 0.003 |      |
| 1W         | 90    | 11.472 | $\pm$ 0.003 |      |
| 0          | 90    | 11.489 |  | 11.490 |

| 2M         | 100   | 5.997  | $\pm$ 0.003 |      |
| 1M         | 100   | 6.041  | $\pm$ 0.002 |      |
| 2W         | 100   | 6.066  | $\pm$ 0.003 |      |
| 1W         | 100   | 6.075  | $\pm$ 0.003 |      |
| 0          | 100   | 6.086  |  | 6.089 |

| 2M         | 110   | 2.936  | $\pm$ 0.003 |      |
| 1M         | 110   | 2.958  | $\pm$ 0.002 |      |
| 2W         | 110   | 2.972  | $\pm$ 0.003 |      |
| 1W         | 110   | 2.978  | $\pm$ 0.003 |      |
| 0          | 110   | 2.983  |  | 2.985 |

Table 9: $r = 5\%, \sigma = 20\%, T = 1$ year, Strike= 100, $\delta = 3\%$.

| $\Delta t$ | $S_0$ | Price  | Err  | PDE |
|------------|-------|--------|------|-----|
| 2M         | 90    | 12.309 | $\pm$ 0.003 |      |
| 1M         | 90    | 12.348 | $\pm$ 0.006 |      |
| 2W         | 90    | 12.370 | $\pm$ 0.004 |      |
| 1W         | 90    | 12.377 | $\pm$ 0.004 |      |
| 0          | 90    | 12.387 |  | 12.384 |

idea was to leverage the flexibility of feed forward neural networks as universal approximators and to employ them in a Least-Square Monte Carlo approach to price American and Bermuda derivatives. The advantages of the approach are manifold: i) the machinery can easily manage the inclusion of new instruments. ii) The FFNNs cure the drawbacks related with the curse of dimensionality, which are inherently a problem with polynomial approximating functions. iii) The neural networks are better designed to deal with non differentiable payoffs. iv) It is the unique viable approach for concurrently pricing instruments with different exercise style and sensitive to the same risk factors avoiding the computationally burden nested Monte Carlo procedure. In this respect, it is worth stressing once more that our approach jointly recovers the portfolio P&L distribution and the single instruments’ marginals.

This paper details the results from a toy experiment where we considered a portfolio
composed by three strongly dependent derivatives – an American Put, a European Call on Min, and a Bermuda Call on Max – whose underlying assets follow a simple dynamics. As a future perspective, we plan to investigate more realistic dynamics in a higher dimensional setting. Specifically, we want to assess to which extent the approach can be extended to manage non-Markov dynamics. This case is of particular relevance given the flourishing of stochastic models where the volatility is driven by a fractional dynamics. The interplay among the long-memory features and the high-dimensional nature of a portfolio may result in a mixture whose complexity can be cured only by resorting to a neural network approach.

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