3d Oscillator and Coulomb systems reduced from Kähler spaces

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To the memory of Professor Valery Ter-Antonyan

Abstract

We define the oscillator and Coulomb systems on four-dimensional spaces with $U(2)$-invariant Kähler metric and perform their Hamiltonian reduction to the three-dimensional oscillator and Coulomb systems specified by the presence of Dirac monopoles. We find the Kähler spaces with conic singularity, where the oscillator and Coulomb systems on three-dimensional sphere and two-sheet hyperboloid are originated. Then we construct the superintegrable oscillator system on three-dimensional sphere and hyperboloid, coupled to monopole, and find their four-dimensional origins. In the latter case the metric of configuration space is non-Kähler one. Finally, we extend these results to the family of Kähler spaces with conic singularities.

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1 Introduction

The oscillator and Coulomb systems play a distinguished role in theoretical and mathematical physics, due to their overcomplete symmetry group. The wide number of hidden symmetries provides these systems with unique properties, e.g., closed classical trajectories, the degenerated quantum-mechanical energy spectrum, the separability of variables in a few coordinate systems. The overcomplete symmetry allows to preserve their exact solvability even after some deformation of the potential breaking the initial symmetry, or, at least, to simplify the perturbative calculations. The reduction of these systems to low dimensions allows one to construct new integrable systems with hidden symmetries [1].

There exist nontrivial generalizations of the oscillator and Coulomb systems on the sphere and the two-sheet hyperboloid (pseudosphere) [2] given by the potentials

\[ V_{\text{Coulomb}} = -\frac{r_0^2}{r} x_{d+1}, \quad V_{\text{osc}} = \frac{\omega^2 r_0^2}{2} \frac{x^2}{x_{d+1}}. \tag{1.1} \]

Here $x, x_{d+1}$ are the (pseudo)Euclidean coordinates of the ambient space $\mathbb{R}^{d+1}(\mathbb{R}^{d+1})$: $\epsilon r^2 + x^2_{d+1} = r_0^2$, with $\epsilon = +1$ for the sphere, $\epsilon = -1$ for the hyperboloid.

The potential of oscillator has also been generalized for the complex projective space $\mathbb{C}P^N$, $N > 1$. That is defined as follows [3]

\[ V_{\text{osc}} = \omega^2 g^{ab} \partial_a K \partial_b K, \tag{1.2} \]

where $K(z, \bar{z}) = \log(1 + z\bar{z})$ is Kähler potential of $\mathbb{C}P^N$.

The generalized systems preserves the property of “maximal superintegrability” of the conventional oscillator and Coulomb systems. They have $2d - 1$ functionally independent constants of motion (here $D$ is the dimension of configuration space). The definition of the oscillator potential (1.2) tells us to define the Coulomb potential on $\mathbb{C}P^N$ as follows

\[ V_{\text{Coulomb}} = -\frac{\gamma}{\sqrt{g^{ab} \partial_a K \partial_b K}}. \tag{1.3} \]

In some cases one can establish the nontrivial relation between oscillator and Coulomb systems: the $(p+1)$–dimensional Coulomb problem can be obtained from the $2p$–dimensional oscillator by the
so-called Levi-Civita (or Bohlin) \((p = 1)\), Kustaanheimo-Stiefel \((p = 2)\) and Hurwitz transformations \((p = 4)\), when \(p = 1, 2, 4\) [4], corresponding to the reduction by the actions of \(Z_2, U(1)\) and \(SU(2)\) groups, respectively. To be more precise, these transformations connect the energy levels of oscillators with the ones parametric families of Coulomb-like systems, specified by the presence of a magnetic flux for \(p = 1\) [5]; by a Dirac monopole for \(p = 2\) (the MIC-Kepler system) [6]; and by a Yang monopole for \(p = 4\) [7] (for the review, see [8]). Among these systems most elegant (and important) one is, probably, the MIC-Kepler system, describing of the relative motion of two Dirac dyons. It is also relevant to the the scattering of two well-separated BPS monopoles and dyons. The latter system was considered in a well-known paper by Gibbons and Manton [9], where the existence of a hidden Coulomb-like symmetry was established. Nowadays the MIC-Kepler system is studied in details as well as Coulomb one [10].

To relate the four-dimensional oscillator with the MIC-Kepler system, we have to perform its Hamiltonian reduction by the action of \(U(1)\) group which leads the canonical symplectic structure on \(T^*\mathbb{C}^2\) to the twisted symplectic structure on \(T^*\mathbb{R}^3\) specified by the presence of monopole magnetic field

\[
\Omega_{can} = dz \wedge d\pi + d\bar{z} \wedge d\bar{\pi} \quad \rightarrow \quad \Omega_{red} = dx \wedge dp + s \frac{x \times dx \times dx}{|x|^3}.
\]

(1.4)

Here \(s\) denotes the value of the generator of the Hamiltonian action

\[
s = \frac{J_0}{2}, \quad J_0 = i(z\pi - \bar{z}\bar{\pi}).
\]

(1.5)

The reduced coordinates are connected with the initial one as follows

\[
x = z\sigma\bar{z}, \quad p = \frac{z\sigma\pi + \bar{\pi}\sigma\bar{z}}{2z\bar{z}},
\]

(1.6)

where \(\sigma\) denote Pauli matrices.

Upon this reduction, the energy surface of the four-dimensional oscillator yields the one of the MIC-Kepler system. Applying this reduction to the oscillator on four-dimensional sphere/hyperboloid and on the complex projective space \(\mathbb{C}P^2\) we shall get the MIC-Kepler system on three-dimensional hyperboloid [11, 3].

The appearing, in the reduced system, of the monopole field is due to Hamiltonian reduction: that corresponds to the compactification of the spatial degree of freedom in the circle, which generate the magnetic charge. So, the above reduction could be used for the construction of the three-dimensional systems with monopole from the four-dimensional systems without it. Vice versa, on can try to construct the superintegrable four-dimensional system (without monopole) by the lifting the given three-dimensional superintegrable one.

In present paper we analyze the following question: wether exist the maximally superintegrable systems on four-dimensional \(U(2)\)–invariant Kähler spaces, whose reductions yield the (superintegrable) three-dimensional oscillator and Coulomb systems with monopoles, including the systems on the configurational spaces with non-constant curvature.

For this purpose we reduce the Hamiltonian system on the four-dimensional space equipped with \(U(2)\)–invariant Kähler metric to the system on three-dimensional conformal-flat space (Section 2). We find that the the oscillator and Coulomb systems on the three-dimensional space, sphere and hyperboloid are originated on the four-dimensional oscillator and Coulomb systems on some Kähler spaces with conic singularity, so that (1.2), (1.3) give us the well-defined generalizations of the oscillator and Coulomb potentials on Kähler spaces. However, in the presence of Dirac monopole field arose due to Hamiltonian reduction, the trajectories of the three-dimensional systems becomes unclosed. Hence, in general, these systems are not superintegrable ones. On the other hand, one can define the “maximally superintegrable” generalization of the three-dimensional oscillator a with Dirac monopoles, which is originated in the four-dimensional system with non-Kähler metrics (Section 3). We also find the family of superintegrable four-dimensional oscillators, which yield the “maximally superintegrable” oscillator with monopoles, on the three-dimensional spaces with non-constant curvature (Section 4).
2 3d systems with monopoles from of $U(2)$–invariant Kähler spaces

As we mentioned in the Introduction, the Hamiltonian reduction, by the action of $U(1)$ group, of the eight-dimensional canonical symplectic structure yields the six-dimensional canonical symplectic structure twisted by the magnetic field of the Dirac monopole (1.4). Particular consequence of this reduction is the reduction of the energy surface of the oscillator (on $\mathbb{C}^2$, $S^3$, $\mathbb{CP}^2$) to the energy surface of the MIC-Kepler system (on $\mathbb{R}^3$ and $AdS_3$).

In this Section we would like to reveal which sort of system arise upon $U(1)$ Hamiltonian reduction of the four-dimensional mechanical systems on the spaces with $U(2)$–invariant Kähler metrics. Particulary, we hope do find, in this way, the four-dimensional origins of the three dimensional oscillator and Coulomb systems on Euclidean spaces, spheres and hyperboloids (which are superintegrable systems).

The Kähler potential of the $U(2)$-invariant Kähler spaces $M$, $dim_{\mathbb{Q}} M_0 = 2$ is (in the appropriate local coordinates $z^a$, $a = 1, 2$) of the form $K(z\bar{z})$. Hence, the corresponding metric reads

$$g_{\bar{a}b} = \frac{\partial^2 K(z\bar{z})}{\partial z^a \partial \bar{z}^b} = a\delta_{\bar{a}b} + a' z^a \bar{z}^b,$$

where $a = dK'(y)/dy$, $a' = a'(y)$. (2.1)

The particular cases of these spaces are the Euclidean space $\mathbb{C}^2$ (when $K = z\bar{z}$) and the complex projective space $\mathbb{CP}^2$ (when $K = \log(1 + z\bar{z})$).

The motion of particle on $M$ in the $U(2)$-invariant potential field $V$ is described by the following Hamiltonian system

$$\Omega_{can} = dz^a \land d\pi_a + d\bar{z}^a \land d\bar{\pi}_a, \quad \mathcal{H} = g^{\bar{a}b} \pi_a \bar{\pi}_b + V(z\bar{z}).$$

The Nether constants of motion corresponding to $U(2)$ symmetry are given by the generators

$$J = iz\sigma \pi - i\bar{\pi} \sigma \bar{z}, \quad J_0 = iz\pi - i\bar{\pi} \bar{z} : \quad \{J_0, J_k\} = 0, \quad \{J_k, J_l\} = 2\epsilon_{klm} J_m,$$

(2.2)

where $\sigma$ denotes standard Pauli matrices.

In order to perform the Hamiltonian reduction of this system by the action of the generator $J_0$, we have to fix its level surface,

$$J_0 = 2s,$$

(2.4)

and then factorize the level surface of by the action of vector field $\{J_0, \}$. The resulting six-dimensional phase space $T^s M_{red}$ could be parameterized by the following $U(1)$-invariant functions:

$$y = z\sigma \bar{z}, \quad \pi = \frac{z\sigma \pi + \bar{\pi} \sigma \bar{z}}{2z\bar{z}} : \quad \{y, J_0\} = \{\pi, J_0\} = 0.$$

(2.5)

In these coordinates the reduced symplectic structure and the generators of the angular momentum are given by the expressions

$$\Omega_{red} = d\pi \land dy + s \frac{y \times dy \times dy}{|y|^3}, \quad J_{red} = J/2 = \pi \times y + s \frac{y}{|y|}.$$

(2.6)

The reduced Hamiltonian is given by the expression

$$\mathcal{H}_{red} = \frac{1}{a} \left[ y\pi^2 - b(y\pi)^2 \right] + s^2 \frac{1 - by}{ay} + V(y), \quad \text{where} \quad y \equiv |y|, \quad b = \frac{a'(y)}{a + ya'(y)}.$$

(2.7)

Hence, the reduced system is specified by the presence of a Dirac monopole.

Let us perform the canonical transformation $(y, \pi) \rightarrow (x, p)$ to the coordinates, where the metric takes a conformal-flat form:
\[ x = f(y), \quad \pi = f p + \frac{df}{dy} (yp), \]  \hspace{1cm} (2.8)

where
\[ \left(1 + \frac{y f'(y)}{f}\right)^2 = 1 + \frac{ya'(y)}{a} \Rightarrow \left(\frac{d \log x}{dy}\right)^2 = \frac{d \log ya(y)}{ydy}. \]  \hspace{1cm} (2.9)

Notice, that \( x < 1 \).

In the new coordinates the Hamiltonian takes the form
\[ \mathcal{H}_{\text{red}} = \frac{x^2(y)}{ya(y)} p^2 + \frac{s^2}{y(a + ya'(y))} + V(y(x)). \]  \hspace{1cm} (2.10)

In order to express the \( y, a(y), a'(y) \) via \( x \) it is convenient to introduce the function
\[ \tilde{A}(y) \equiv \int (a + ya'(y))yf(y)dy \]  \hspace{1cm} (2.11)

and consider its Legendre transform \( A(x) \),
\[ A(x) = xa(y) - \tilde{A}(y) \]  \hspace{1cm} (2.12)

Then, we get immediately
\[ \frac{dA(x)}{dx} = a(y)y, \quad x \frac{d^2 A}{dx^2} = y\sqrt{a(a + ya'(y))}. \]  \hspace{1cm} (2.13)

By the use of these expressions, we can present the reduced Hamiltonian system as follows
\[ \mathcal{H}_{\text{red}} = \frac{x^2}{N^2} p^2 + \frac{s^2}{(2xN'(x))^2} + V(y(x)), \quad \Omega_{\text{red}} = dp \wedge dx + s \frac{x \times dx \times dx}{|x|^3}, \]  \hspace{1cm} (2.14)

where
\[ N^2(x) \equiv \frac{dA}{dx}. \]  \hspace{1cm} (2.15)

The Kähler potential of the initial system is connected with \( N \) via the equations
\[ \frac{dK}{dx} = \frac{N^3(x)}{2x^2N'(x)}, \quad \frac{d \log y}{dx} = \frac{N}{2x^2N'(x)}. \]  \hspace{1cm} (2.16)

Let us postulate, that the “oscillator potential” on the spaces under consideration by the same formula, as on the complex projective spaces, (1.2). Then, upon the reduction it will reads as follows
\[ V_{\text{osc}} = \omega^2 \partial_a K y^{ab} \partial_b K = \omega^2 \frac{ya^2(y)}{a(y) + ya'(y)} = \left(\frac{\omega N^2}{2xN'(x)}\right)^2. \]  \hspace{1cm} (2.17)

Similarly, we could choose the “Coulomb potential” (1.3), and get its reduced version
\[ V_{\text{Coulomb}} = \frac{2xN'(x)}{N^2(x)}. \]  \hspace{1cm} (2.18)

Further study we will need to consider the classical trajectories of the (reduced) system, in order to check their closeness (closeness of trajectories is the explicit indication of superintegrability).

For this purpose it is convenient to direct the \( x_3 \) axis along \( \textbf{J} \), i.e. assume, that \( J = J_3 \). Upon this choice of coordinate system one has
\[ \frac{x_3}{x} = \frac{s}{J_3}. \]  \hspace{1cm} (2.19)
Then, after obvious manipulations, we get
\[
\frac{d\phi}{dt} = \frac{2J}{N^2}, \quad \mathcal{E} = \frac{J^2 - s^2}{N^2} + \frac{J^2}{x^2 N^2} \left( \frac{dx}{d\phi} \right)^2 + \frac{s^2}{(2x N'(x))^2} + V(x),
\] (2.20)
where
\[
\phi = \arctan \frac{x_1}{x_2}
\] (2.21)
From the expression (2.20) we find,
\[
\left| \frac{\phi}{J} \right| = \int \frac{dx}{x \sqrt{(\mathcal{E} - V_{eff})N^2 - J^2 + s^2}} \quad \text{where} \quad V_{eff} = V(x) + \frac{s^2}{(2x N'/dx)^2}.
\] (2.22)

**Euclidean space**

Let us consider the simplest case, when the reduced configuration space is \( \mathbb{R}^3 \), i.e. \( N = \sqrt{2}x \). In that case the reduced Hamiltonian reads
\[
\mathcal{H}_{red} = \frac{p^2}{2} + \frac{s^2}{8x^2} + V(x).
\] (2.23)
The trajectories of the system are defined by the equations (2.19) and
\[
\left| \frac{\phi}{J} \right| = \int \frac{dx}{x \sqrt{2(\mathcal{E} - V)x^2 - J^2 + 3s^2/4}}
\] (2.24)
The Kähler potential and metric of the original four-dimensional system are of the form
\[
K = (z\bar{z})^4, \quad g_{ab} = 4(z\bar{z})^2 \left[ (z\bar{z}) \delta_{ab} + 3\bar{z}^a z^b \right].
\] (2.25)
Hence, the systems on \( \mathbb{R}^3 \) is originated on the Kähler conifold\(^1\).

Notice, that the oscillator and Coulomb potentials (1.2), (1.3) takes, on this conifold, the following form
\[
V_{Coulomb} = -\frac{\gamma}{(z\bar{z})^2}, \quad V_{osc} = \omega^2(z\bar{z})^4.
\] (2.26)

Upon reduction they yield the oscillator and Coulomb potentials on \( \mathbb{R}^3 \)!

On the other hand, for the
\[
V_{eff} = \frac{s^2}{x^2} + V(x)
\]
one has
\[
\left| \frac{\phi}{J} \right| = \int \frac{dx}{x \sqrt{2(\mathcal{E} - V)x^2 - J^2}},
\] (2.27)
so that the form of trajectory, \( \phi(x) \), is independent on “monopole number” \( s \).

Hence, the well-defined monopole generalization of the system on \( \mathbb{R}^3 \) with potential \( V(x) \) reads
\[
\mathcal{H}_s = \frac{p^2}{2} + \frac{s^2}{2x^2} + V(x), \quad \Omega_{\text{red}} = dp \wedge dx + s \frac{x \times dx \times dx}{|x|^3}.
\] (2.28)
Its four-dimensional origin is formulated as follows,
\[
\Omega_{\text{can}} = dz^a \wedge d\pi_a + d\bar{z}^a \wedge d\bar{
\pi}_a, \quad \mathcal{H} = g^{\hat{a}\hat{b}} \bar{\pi}_a \pi_b + \frac{3 f_0^2}{16(z\bar{z})^4} + V(z\bar{z}),
\] (2.29)
where \( g_{\hat{a}\hat{b}} \) is given by (2.25).

\(^1\)We thank Dmitry Fursaev for this remark.
3 Sphere and hyperboloid

In this Section we consider the particular case of our construction, when the reduced system (2.14) is formulated on the three-dimensional sphere or (two-sheet) hyperboloid.

For this purpose we choose the following value of $N$:

$$N = 2\sqrt{2r_0x}/(1 + \epsilon x^2), \quad \epsilon = 1, -1. \quad (3.1)$$

Here $\epsilon = 1$ corresponds to the sphere, $\epsilon = -1$ corresponds to the two-sheet hyperboloid.

The corresponding Hamiltonian is of the form

$$\mathcal{H}_{red} = \left(1 + \epsilon x^2\right)^2 \frac{P^2}{8r_0^2} + \frac{s^2x^2}{4x^2} + V(x) + \frac{s^2x^2}{2r_0^2(1 - \epsilon x^2)^2} + \frac{\epsilon s^2}{8r_0^2}. \quad (3.2)$$

Solving the equations (2.16) we could find the Kähler space, where the system (2.14) is originated. It is defined by the following Kähler potential and metric

$$K = \frac{\epsilon r_0^2}{2}\log(1 + 4\epsilon(z\bar{z})^4), \quad g_{a\bar{b}} = \frac{8r_0^2(z\bar{z})^2}{1 + 4\epsilon(z\bar{z})^4} \left[(z\bar{z})\delta_{a\bar{b}} + \frac{3 - 4\epsilon(z\bar{z})^4}{(1 + 4\epsilon(z\bar{z})^4)} z^a z^b\right]. \quad (3.3)$$

Hence, the systems on sphere and two-sheet hyperboloid are also originated on the Kähler conifolds.

On these conifolds the oscillator and Coulomb potentials (1.2), (1.3) read as follows

$$V_{\text{Coulomb}} = -\gamma\sqrt{2r_0(z\bar{z})}, \quad V_{osc} = 2\omega^2 r_0^2 (z\bar{z})^4. \quad (3.4)$$

Upon reduction to the sphere and hyperboloid they take the form

$$V_{\text{Coulomb}} = -\frac{\sqrt{2}\gamma}{r_0} \frac{1 - \epsilon x^2}{2x}, \quad V_{osc} = \omega^2 r_0^2 \frac{2x^2}{(1 - \epsilon x^2)^2}. \quad (3.5)$$

These potentials are precisely Coulomb and oscillator potentials on the sphere and hyperboloid (1.1) written in conformal-flat coordinates. *Hence, the oscillator/Coulomb system on the conifold (3.3), reduces, for $J_0 = 0$, to the oscillator/Coulomb system on three-dimensional sphere/hyperboloid.* Hence, the initial four-dimensional oscillator and Coulomb systems are superintegrable ones, when the constant of motion $J_0$ takes the value $J_0 = 0$.

When $J_0 \neq 0$, the relation between three- and four-dimensional systems is more complicated, and needs separate consideration of the oscillator and Coulomb cases.

Let us consider, at first, the case of oscillator. For a checking the superintegrability, let us clarify, wether trajectories of the reduced system are closed.

Substituting (3.1) and (3.5) in (2.22), we get

$$\left|\frac{\phi}{J}\right| = \int \frac{du}{\sqrt{-4r_0^2(\omega^2 r_0^2 + 2\mathcal{E}) + (8r_0^2 \mathcal{E} + l^2)u - (s^2 + l^2)u^2}} \quad (3.6)$$

where

$$l^2 = 4(J^2 - s^2), \quad 4u = (x + 1/x)^2. \quad (3.7)$$

From this expression we easily get

$$\left(\frac{x + 1}{x}\right)^2 = \frac{8r_0^2 \mathcal{E} + J^2 - s^2}{4J^2 - 3s^2} \left(1 + \sqrt{1 - 4(2r_0^2 \mathcal{E} + r_0^2 \omega^2)(4J^2 - 3s^2)} \sin 2\sqrt{1 - 3s^2/4J^2} |\phi|\right) \quad (3.8)$$

Hence, trajectories are closed only when

$$\sqrt{1 - 3s^2/4J^2} \text{ is rational number.}$$
Particularly, trajectories are closed in the “ground state”, i.e. for $s = J$. In this case they belong to “equatorial plane”, $x_3 = x$. Hence, the Hamiltonian system (2.14) on sphere/hyperboloid with the oscillator potential is not superintegrable for the arbitrary value of monopole number $s$.

However, one can get the monopole generalization of oscillator whose trajectories are closed for any $s$, choosing the potential

$$V_{osc}^s = V_{osc} + \frac{3s^2}{4x^2(dN/dx)^2}.$$  

(3.9)

In that case the trajectories are given by the expression

$$\left(x + \frac{1}{\epsilon}\right)^2 = 2\frac{r_0^2\mathcal{E} + J^2 - s^2}{2J^2}\left(1 + \sqrt{1 - 16J^2r_0^2\frac{2\mathcal{E} + \omega^2r_0^2}{(2r_0^2\mathcal{E} + J^2 - s^2)^2}\sin 2|\phi|}\right),$$  

(3.10)

i.e. they are closed for any $s$.

Hence, the superintegrable generalization of the Higgs oscillator specified by the presence of Dirac monopole is defined by the Hamiltonian

$$\mathcal{H}_{MIC-osc}^s = \frac{(1 + \epsilon x^2)^2}{8r_0^2} \left(\mathbf{p}^2 + \frac{s^2}{x^2}\right) + (\omega^2r_0^2 + \frac{s^2}{4r_0^2}) \left(1 - \epsilon x^2\right)^2, \quad \epsilon = \pm 1,$$  

(3.11)

where $\epsilon = 1$ corresponds to the sphere and $\epsilon = -1$ to the hyperboloid.

It is originated in the Hamiltonian given by the expression

$$\mathcal{H} = g^{\bar{a}b}\bar{\pi}_a\pi_b + \frac{3J_0^2}{16R(\bar{z}z)} + \omega^2K_0g^{\bar{a}b}K_b, \quad R = \frac{32r_0^2(\bar{z}z)^4}{(1 + 4\epsilon(\bar{z}z)^4)^2}. \quad (3.12)$$

There is an important difference of the above reduced oscillator from the one on the $\mathbb{R}^3$.

Namely, the four-dimensional system with “frequency” $\omega$ yields the three-dimensional oscillator with “frequency” dependent on the “monopole number” $s$

$$\omega_s = \sqrt{\omega^2 + \frac{s^2}{4r_0^2}},$$

while the frequency of the oscillator reduced to $\mathbb{R}^3$ is independent on $s$.

Now, let us consider the system with Coulomb potential (1.3) on the conifold with Kähler structure (3.3). After Hamiltonian reduction it yields the three-dimensional system with Hamiltonian (3.2) where $V = V_{\text{Coulomb}}$ (3.5).

On the level surface $s = 0$ (i.e. in the absence of Dirac monopole), the reduced system coincides with the standard Coulomb system on the sphere/hyperboloid. Therefore it is superintegrable one. On the level surface $s \neq 0$ (i.e. in the presence of monopole) the potential of the reduced system is the superposition of the Coulomb potential and of the oscillator one, proportional to $s^2/r_0^4$. So, it is not surprising, that the expression for trajectories of the reduced system, $\phi = \phi(r)$ is given by elliptic integral...

On the other hand, there are the superintegrable MIC-Kepler systems on the sphere and hyperboloid, given by the Hamiltonian

$$\mathcal{H}_{MIC}^s = \frac{(1 + \epsilon x^2)^2}{8r_0^2} \left(\mathbf{p}^2 + \frac{s^2}{x^2}\right) - \frac{\gamma}{r_0} \frac{1 - \epsilon x^2}{2x}, \quad \epsilon = \pm 1,$$  

(3.13)

where $\epsilon = 1$ corresponds to the sphere [12] and $\epsilon = -1$ to the hyperboloid [11].

To recover this system, we can try to modify the initial Coulomb system, transiting to the non-Kähler metric, as in the case of oscillator (compare with (3.12)),

$$\mathcal{H} = g^{\bar{a}b}\bar{\pi}_a\pi_b + \frac{3J_0^2}{16R(\bar{z}z)} + \frac{\gamma}{\sqrt{K_0g^{\bar{a}b}K_b}}, \quad R = \frac{32r_0^2(\bar{z}z)^4}{(1 + \epsilon(\bar{z}z)^4)^2}. \quad (3.14)$$
However, in that case the reduced Hamiltonian is given by the one of MIC-Kepler system (3.11) with additional oscillator potential. So even modified Coulomb system is not exactly solvable for any value of $J_0$.

We have found the four-dimensional oscillator and Coulomb systems on appropriate Kähler conifolds, which result, after Hamiltonian reduction, in the oscillator and Coulomb systems on three-dimensional sphere and two-sheet hyperboloid. The appearance, in the reduced system, of the Dirac monopole, breaks the superintegrability of the system. However, the superintegrability of the oscillator system, in opposite to the Coulomb one, could be restored by the transition to the non-Kähler metric.

## 4 Family of oscillator systems

The results of previous Section could be easily extended to the Kähler space whose metric is defined by the potential

$$K = \frac{\epsilon r_0^2}{2} \log(1 + 4\epsilon (z\bar{z})^n), \quad \epsilon = \pm 1, \quad n > 0. \quad (4.1)$$

For $n = 1$ the potential (4.1) defines the Fubini-Studii metric of the two-dimensional complex projective space $\mathbb{CP}^2$ (for $\epsilon = 1$) and its noncompact version, the four-dimensional Lobacewski space $L_2$ (for $\epsilon = -1$). These spaces are of the constant curvature ones, and have the the isometry group $SU(3)$ for $\epsilon = 1$ and $SU(1,2)$ for $\epsilon = -1$.

The case $n = 4$ was considered in previous Section. The system on such spaces results, after Hamiltonian reduction, in the ones on sphere($\epsilon = 1$) or two-sheet hyperboloid $\epsilon = -1$ (which have the constant curvature, and the isometry group $SO(4)$ and $SO(1,3)$ respectively). For any other $n$ both initial and reduced spaces have non-constant curvature and conic singularity.

The Hamiltonian systems on the spaces with Kähler potential (4.1) results, after reduction, in the three-dimensional systems (2.14), with

$$N^2 = 2nr_0^2 \frac{x^{\sqrt{n}}}{(1 + \epsilon x^{\sqrt{n}})^2}, \quad (4.2)$$

Hence, the metric of the reduced configuration space is given by the expression

$$ds^2 = 2nr_0^2 x^{\sqrt{n}}(dx)^2 (1 + \epsilon x^{\sqrt{n}})^2, \quad (4.3)$$

so that for $n \neq 4$ it has a conifold structure.2

The oscillator potential (2.17) looks as follows:

$$V_{osc} = 2r_0^2 \omega^2 (z\bar{z})^n. \quad (4.4)$$

It is reduces to the following form

$$V_{osc}^{red} = 2r_0^2 \omega^2 \frac{x^{\sqrt{n}}}{(1 - \epsilon x^{\sqrt{n}})^2}. \quad (4.5)$$

The trajectories of the reduced oscillator are given by the expression

$$\left| \frac{\phi}{J} \right| = \int \frac{du}{\sqrt{-nr_0^2 (r_0^2 \omega^2 + 2E) + (2nr_0^2 \omega^2 + l^2)u - (n^2 s^2 + l^2)u^2}}, \quad (4.6)$$

2In the vicinity of singularity this metric could be presented as follows $ds^2 = dR^2 + R^2 d\Omega^2$, where $R = r^{\sqrt{n}/2}/\sqrt{n}/2$, $d\Omega^2 = (n/2)^2 d\Omega^2$ where $d\Omega^2$ is a metric on $S^2$. Hence, the solid angle around singularity is equal to $n\pi$, instead of $4\pi$ (D.Fursaev).
where
\[ l^2 = 4(J^2 - s^2), \quad 4u = (x\sqrt{n}/2 + 1/x\sqrt{n}/2)^2. \] (4.7)

From this expression we easily get
\[ \left(x\sqrt{n}/2 + x^{-\sqrt{n}/2}\right)^2 = \] (4.8)
\[ \frac{mr_0^2E + 2(J^2 - s^2)}{2(J^2 - s^2(1 - 1/n))} \left(1 + \sqrt{1 - 4r_0^2(2E + r_0^2\omega^2)(J^2 - s^2(1 - 1/n))} \right) \sin 2\sqrt{1 - \frac{(n - 1)s^2}{nJ^2}}|\phi|, \] (4.9)

Hence, trajectories are closed when the following condition holds
\[ \sqrt{1 - \frac{(n - 1)s^2}{nJ^2}} \text{ is rational number.} \]

Therefore, trajectories are closed for any \( s \) only when \( n = 1 \), i.e. on the complex projective space \( \mathbb{C}P^2 \) (for \( \epsilon = 1 \)) and on its noncompact version, four-dimensional Lobachewskii space \( \mathcal{L}^2 = SU(1,2)/U(1) \times SU(2) \). In this case the potential takes quite simple form, \( V = 2\omega^2r_0^2z\bar{z}. \) The closeness of trajectories are due to the hidden symmetries of the system, given by the expressions [3]

\[ J = \frac{J_+ \sigma J_+}{2r_0^2} + 2r_0^2\omega^2z\bar{z} \quad J_+^a = \pi_a + \epsilon(\bar{z}z)\bar{z}^a, \quad J^- = J^+, \] (4.10)

were \( J_+^a \) are the translation generators. The reduced Hamiltonian is of the form
\[ H_{\text{red}} = \frac{x(1 + \epsilon x)^2p^2}{2r_0^2} + s^2 \frac{(1 + \epsilon x)^4}{2r_0^2x(1 - \epsilon x)^2} + \frac{2r_0^2\omega^2x}{(1 - \epsilon x)^2}. \] (4.11)

Fixing the energy surface \( H = E_{\text{osc}} \) of the reduced system, we can transform it in the MIC-Kepler system on hyperboloid, given by the Hamiltonian (3.11).

For the \( n \neq 1 \), we could get the superintegrable oscillator with monopole, choosing the the potential
\[ V_{\text{osc}} = V_{\text{osc}} + \frac{(n - 1)}{n}s^2x^2(dN/dx)^2. \] (4.12)

In that case the Hamiltonian of the reduced system reads
\[ H = \frac{(1 + \epsilon x\sqrt{n}/2)^2p^2}{2nr_0^2x\sqrt{n}(1 - \epsilon x\sqrt{n})^2} + s^2 \frac{(1 + \epsilon x\sqrt{n})^4}{2nr_0^2x\sqrt{n}(1 - \epsilon x\sqrt{n})^2} + \frac{2r_0^2\omega^2x}{(1 - \epsilon x\sqrt{n})^2}, \] (4.13)

while the trajectories are given by the equation
\[ (x\sqrt{n}/2 + x^{-\sqrt{n}/2})^2 = \frac{nr_0^2E + 2(J^2 - s^2)}{J^2} \left(1 + \sqrt{1 - 4r_0^2(2E + r_0^2\omega^2)(J^2 - s^2(1 - 1/n))} \sin 2|\phi| \right). \] (4.14)

This superintegrable oscillator with monopole is originated in the four-dimensional system with Hamiltonian
\[ H = g^{ab}\pi_a\pi_b + \frac{(n - 1)J^2}{4nR(z\bar{z})} + 2r_0^2\omega^2(z\bar{z})^n, \quad R = \frac{2n^2r_0^2(z\bar{z})^4}{(1 + 4\epsilon(z\bar{z})^n)^2} \] (4.15)

where \( g^{ab} \) is defined by the Kähler potential (4.1).
5 Conclusion

We considered the reduction of the mechanical systems on four-dimensional Kähler spaces with $U(2)$ isometry to the three-dimensional systems, paying special attention to the “oscillator” and “Coulomb” systems, defining their potentials by the expressions (1.2) and (1.3), respectively. From the previous study [3] it was known, that such a “oscillator” potential defines the well-defined superintegrable system on $\mathbb{C}P^n$, and is distinguished with respect to supersymmetrization as well. Since the Hamiltonian reduction by the action of $U(1)$ group generates, in the resulting three-dimensional system, the magnetic field of Dirac monopole. We hoped to find, in this way, the superintegrable generalizations of oscillator and Coulomb systems on curved spaces, specified by the presence of Dirac monopole. Particularly, we found the four-dimensional Kähler spaces (with conic singularities) where the three-dimensional oscillator and Coulomb systems on the $\mathbb{R}^3$, $S^3$, $\mathbb{H}^3$ are originated, and established, that the original oscillator and Coulomb potential are, indeed, given by (1.2) and (1.3). However, when these, four-dimensional systems results in the three-dimensional ones specified by the presence of Dirac monopoles, their trajectories become unclosed. In other words, the monopole field breaks superintegrability of the system. On the other hand, in the case of oscillator, transiting from the Kähler metric to the appropriate non-Kähler one, we can restore superintegrability of systems (both initial and reduced one), but we can’t do the same for the Coulomb system. We also extended these consideration for the some parametric family of Kähler spaces including previous ones as a particular case. We found that the unique representative of this family, where the oscillator is superintegrable, is the complex projective space $\mathbb{C}P^2$ (and its non-compact version, Lobachewski space $\mathbb{L}^2 = SU(2.1)/SU(2) \times U(1)$). The energy surface of the oscillator on this four-dimensional space leads to the energy surface of the MIC-Kepler system on three-dimensional hyperboloid. Let us notice, that while previous superintegrable generalizations of oscillator systems were formulated on constant curvature spaces, the superintegrable oscillators constructed in the present paper, could have configuration spaces with nonconstant curvature and conic singularities.

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