The quantum adjacency algebra and subconstituent algebra of a graph

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Abstract
Let Γ denote a finite, undirected, connected graph, with vertex set X. Fix a vertex x ∈ X. Associated with x is a certain subalgebra T = T(x) of MatX(C), called the subconstituent algebra. The algebra T is semisimple. Hora and Obata introduced a certain subalgebra Q ⊆ T, called the quantum adjacency algebra. The algebra Q is semisimple. In this paper we investigate how Q and T are related. In many cases Q = T, but this is not true in general. To clarify this issue, we introduce the notion of quasi-isomorphic irreducible T-modules. We show that the following are equivalent:
(i) Q ≠ T; (ii) there exists a pair of quasi-isomorphic irreducible T-modules that have different endpoints. To illustrate this result we consider two examples. The first example concerns the Hamming graphs. The second example concerns the bipartite dual polar graphs. We show that for the first example Q = T, and for the second example Q ≠ T.

Keywords: subconstituent algebra, Terwilliger algebra, quantum adjacency algebra, quasi-isomorphism, quantum decomposition.

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1 Introduction
In [20] the subconstituent algebra was introduced, and used to investigate commutative association schemes. Since then the subconstituent algebra has received considerable attention; some notable papers are [3, 5, 7, 8, 10, 11, 12, 13, 15, 16, 17, 18, 19, 23, 25].

In the present paper we consider the subconstituent algebra of a graph. Let Γ denote a finite, undirected, connected graph, with vertex set X and path-length distance function
Let $A \in \text{Mat}_X(\mathbb{C})$ denote the adjacency matrix of $\Gamma$. Fix $x \in X$ and define $D = \max\{\partial(x, y) \mid y \in X\}$. For $0 \leq i \leq D$ let $E_i^x$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y, y)$-entry
\[
(E_i^x)_{yy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X).
\]
Let $T$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A$ and $\{E_i^x\}_{i=0}^D$. We call $T$ the subconstituent algebra (or Terwilliger algebra) of $\Gamma$ with respect to $x$ \cite[p. 380]{20}. The algebra $T$ is semisimple (see Section 2).

In \cite{9} A. Hora and N. Obata introduced the quantum adjacency algebra, and used it to investigate quantum probability. This algebra is described as follows. Define
\[
L = \sum_{i=1}^{D} E_{i-1}^x AE_i^x, \quad F = \sum_{i=0}^{D} E_i^x AE_i^x, \quad R = \sum_{i=0}^{D-1} E_{i+1}^x AE_i^x.
\]
Observe that $L, F, R \in T$. The equation $A = L + F + R$ is called the quantum decomposition of $A$ with respect to $x$ \cite[Definition 2.24]{9}. Let $Q$ denote the subalgebra of $T$ generated by $L, F, R$. We call $Q$ the quantum adjacency algebra of $\Gamma$ with respect to $x$ \cite[p. 78]{9}. The algebra $Q$ is semisimple (see Lemma \cite[4.9]{9}).

In this paper we investigate how $Q$ and $T$ are related. In many cases $Q = T$, but it turns out that this is not always true. To clarify this issue, we compare the $Q$-modules and $T$-modules. Let $W$ denote a $T$-module. For the $T$-module $W$ the restriction of the $T$-action to $Q$ turns $W$ into a $Q$-module. Assume that the $T$-module $W$ is irreducible. We show that the $Q$-module $W$ is irreducible. Let $U$ and $W$ denote irreducible $T$-modules. Then the $Q$-modules $U, W$ are irreducible. Assume that the $T$-modules $U, W$ are isomorphic. By construction the $Q$-modules $U, W$ are isomorphic. Next assume that the $T$-modules $U, W$ are not isomorphic. We find a necessary and sufficient condition for the $Q$-modules $U, W$ to be isomorphic. To describe this condition, we introduce the notion of quasi-isomorphic irreducible $T$-modules. The main result of the paper is that the following are equivalent: (i) $Q \neq T$; (ii) there exists a pair of quasi-isomorphic irreducible $T$-modules that have different endpoints. To illustrate the main result we consider two examples. The first example concerns the Hamming graphs \cite[p. 261]{11}. The second example concerns the bipartite dual polar graphs \cite[p. 274]{11}. We show that for the first example $Q = T$, and for the second example $Q \neq T$.

The paper is organized as follows. In Section 2 we recall some background concerning semisimple algebras and their modules. In Section 3 we recall the subconstituent algebra $T$. In Section 4 we consider the quantum adjacency algebra $Q$ and its relationship to $T$. In Section 5 we describe a $\mathbb{Z}$-grading for $Q$ and $T$. In Sections 6 and 7 we compare the irreducible $Q$-modules and $T$-modules. In Section 8 we introduce the notion of quasi-isomorphic irreducible $T$-modules. In Section 9 we compare the algebras $Q$ and $T$; Theorem \cite[9.1]{9} is the main result of the paper. In Section 10 we consider a type of irreducible $T$-module, said to be thin. In Section 11 we discuss the Hamming graphs and the bipartite dual polar graphs.

2 Preliminaries

In this section we review some basic facts concerning algebras and their modules.

Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let
$I \in \text{Mat}_X(\mathbb{C})$ denote the identity matrix. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of the column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. Observe that $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We endow $V$ with the Hermitean inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^t \overline{v}$ for $u, v \in V$. Here $t$ denotes transpose and $\overline{\cdot}$ denotes complex conjugation. Let $W, W'$ denote nonempty subsets of $V$. These subsets are said to be orthogonal whenever $\langle w, w' \rangle = 0$ for all $w \in W$ and $w' \in W'$.

Let $S$ denote a subalgebra of $\text{Mat}_X(\mathbb{C})$. By an $S$-module we mean a subspace $W$ of $V$ such that $SW \subseteq W$. We refer to $V$ itself as the standard module for $S$. An $S$-module $W$ is said to be irreducible whenever $W$ is nonzero and contains no $S$-module other than 0 and $W$.

Let $U$ and $W$ denote $S$-modules. By an isomorphism of $S$-modules from $U$ to $W$ we mean a vector space isomorphism $\sigma : U \to W$ such that $(\sigma s - s \sigma)U = 0$ for all $s \in S$. The $S$-modules $U$ and $W$ are called isomorphic whenever there exists an isomorphism of $S$-modules from $U$ to $W$.

For the rest of this section, assume that $S$ is closed under the conjugate-transpose map. Then $S$ is semisimple [20, Lemma 3.4]. Let $W$ denote an $S$-module. Then its orthogonal complement $W^\perp$ is an $S$-module. It follows that every $S$-module is an orthogonal direct sum of irreducible $S$-modules. In particular, $V$ is an orthogonal direct sum of irreducible $S$-modules.

Let $S^\vee$ denote the set of isomorphism classes of irreducible $S$-modules. The elements of $S^\vee$ are called types. For $\lambda \in S^\vee$ let $V_\lambda$ denote the subspace of $V$ spanned by the irreducible $S$-modules of type $\lambda$. Observe that $V_\lambda$ is an $S$-module. We have

$$V = \sum_{\lambda \in S^\vee} V_\lambda \quad \text{(orthogonal direct sum)}.$$  

(1) For $\lambda \in S^\vee$ let $d_\lambda$ denote the dimension of an irreducible $S$-module that has type $\lambda$.

**Proposition 2.1.** [24, Proposition 2.2] Let $S$ denote a subalgebra of $\text{Mat}_X(\mathbb{C})$ that is closed under the conjugate-transpose map. Then

$$\dim S = \sum_{\lambda \in S^\vee} d_\lambda^2.$$  

The following notation will be useful.

**Notation 2.2.** Given subspaces $Y, Z$ of the vector space $\text{Mat}_X(\mathbb{C})$ define

$$YZ = \text{Span}\{yz \mid y \in Y, z \in Z\}.$$  

### 3 The subconstituent algebra

In this section we recall the subconstituent algebra; see [20] for more background information.

Throughout the paper $\Gamma$ denotes a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$ and path-length distance function $\partial$. Denote by $A$ the matrix in $\text{Mat}_X(\mathbb{C})$ with $(x, y)$-entry

$$A_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = 1, \\ 0 & \text{if } \partial(x, y) \neq 1 \end{cases} \quad (x, y \in X).$$  

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We call $A$ the adjacency matrix of $\Gamma$. Note that $A$ is real and symmetric. Let $M$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A$. We call $M$ the adjacency algebra of $\Gamma$.

We now recall the dual adjacency algebras of $\Gamma$ [20, p. 378]. For the rest of this paper, fix $x \in X$. Define $D = D(x)$ by $D = \max\{\partial(x, y) \mid y \in X\}$. We call $D$ the diameter of $\Gamma$ with respect to $x$. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y, y)$-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X).$$

We call $E_i^*$ the $i$-th dual idempotent of $\Gamma$ with respect to $x$. The following definition is for notational convenience. For $i \in \mathbb{Z}$,

$E_i^* = 0$ unless $0 \leq i \leq D$. 

Observe that

$$I = \sum_{i=0}^{D} E_i^*$$

and

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad (0 \leq i, j \leq D).$$

The matrices $\{E_i^*\}_{i=0}^{D}$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{C})$. We call $M^*$ the dual adjacency algebra of $\Gamma$ with respect to $x$. Note that

$$V = \sum_{i=0}^{D} E_i^* V \quad (\text{orthogonal direct sum}).$$

We now recall how $M$ and $M^*$ are related. By [2] and the definition of $A$,

$$E_i^* A E_j^* = 0 \quad \text{if } |i - j| > 1, \quad (0 \leq i, j \leq D).$$

Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$. We call $T$ the subconstituent algebra of $\Gamma$ with respect to $x$ [20, p. 380]. The algebra $T$ is often called the Terwilliger algebra [2, 5].

Observe that $T$ is generated by $A$ and $\{E_i^*\}_{i=0}^{D}$. These generators are real and symmetric, so $T$ is closed under the conjugate-transpose map. Therefore our discussion in Section 2 applies to $T$. By Proposition 2.1,

$$\dim T = \sum_{\lambda \in T^*} d_{\lambda}^2.$$ 

Let $W$ denote an irreducible $T$-module. Observe that $W$ is an orthogonal direct sum of the nonzero subspaces among $E_0^* W, \ldots, E_D^* W$. Define the endpoint $r = r(W)$ by

$$r = \min\{i \mid 0 \leq i \leq D, E_i^* W \neq 0\}.$$ 

Define the diameter $d = d(W)$ by

$$d = \left| \{i \mid 0 \leq i \leq D, E_i^* W \neq 0\} \right| - 1.$$ 

Using the idea from [20, Lemma 3.9(ii)] we have $E_i^* W \neq 0$ if and only if $r \leq i \leq r + d$ $(0 \leq i \leq D)$. By the above comments,

$$W = \sum_{i=0}^{d} E_{r+i}^* W \quad (\text{orthogonal direct sum}).$$

Note that isomorphic irreducible $T$-modules have the same endpoint and the same diameter.
4 The quantum adjacency algebra

In this section we recall from [9] the quantum adjacency algebra.

Definition 4.1. [9] p. 10] Define
\[ L = \sum_{i=1}^{D} E_{i-1}^* AE_i^*, \quad F = \sum_{i=0}^{D} E_i^* AE_i^*, \quad R = \sum_{i=0}^{D-1} E_{i+1}^* AE_i^*. \]

Lemma 4.2. The matrices \( L, F, R \) are contained in \( T \).
Proof. By Definition 4.1.

Lemma 4.3. The matrices \( L, F, R \) are real. Moreover \( L^t = R \) and \( F^t = F \).
Proof. The matrix \( A \) is real and symmetric. For \( 0 \leq i \leq D \) the matrix \( E_i^* \) is real and diagonal. The results follow by Definition 4.1.

Lemma 4.4. For \( 0 \leq i \leq D \) the matrices \( L, F, R \) act on \( E_i^* V \) in the following way:
\[ LE_i^* V \subseteq E_{i-1}^* V, \quad FE_i^* V \subseteq E_i^* V, \quad RE_i^* V \subseteq E_{i+1}^* V. \]

Proof. By (5) and Definition 4.1.

Lemma 4.5. [6] p. 10] We have
\[ A = L + F + R. \] (10)
Proof. Multiply \( A \) on the left and right by \( I \). Evaluate the result using (1) and (7). Line (10) follows by Definition 4.1.

Note 4.6. In [9] the matrices \( L, F, R \) are called \( A^-\), \( A^0\), \( A^+ \) respectively. In [9] Definition 2.24] the equation (10) is called the quantum decomposition of \( A \) with respect to \( x \).

Definition 4.7. [9] p. 78]. Let \( Q = Q(x) \) denote the subalgebra of \( T \) generated by \( L, F, R \). We call \( Q \) the quantum adjacency algebra of \( \Gamma \) with respect to \( x \).

Note 4.8. The algebra \( Q \) is called \( \tilde{A} \) in [9].

Lemma 4.9. The algebra \( Q \) is closed under the conjugate-transpose map.
Proof. By Lemma 4.3 and Definition 4.7.

Lemma 4.10. We have
\[ \dim Q = \sum_{\mu \in Q^c} d^2_{\mu}. \] (11)
Proof. By Proposition 2.1.

As we will see, in some cases \( Q = T \), and in other cases \( Q \neq T \). We now consider how \( Q \) is related to \( T \) in general.

Lemma 4.11. We have
\[ LE_i^* = E_{i-1}^* L \quad (1 \leq i \leq D), \quad LE_0^* = 0, \quad E_D^* L = 0, \]
\[ FE_i^* = E_i^* F \quad (0 \leq i \leq D), \quad RE_D^* = 0, \quad E_0^* R = 0. \]
Proof. By (5) and Definition 4.1.

Corollary 4.12. We have

\[ LM^* = M^*L, \quad FM^* = M^*F, \quad RM^* = M^*R. \]

Proof. By Lemma 4.11 and since \( M^* \) is spanned by \( \{ E_i^* \}_{i=0}^D \).

Proposition 4.13. We have

\[ QM^* = T = M^*Q. \]

Proof. By Corollary 4.12 and since \( Q \) is generated by \( L, F, R \), we see that \( QM^* = M^*Q \). This common value is a subalgebra of \( T \) that contains \( M^* \) and \( Q \). The algebra \( T \) is generated by \( M^*, Q \). The result follows.

5 A \( \mathbb{Z} \)-grading for \( Q \) and \( T \)

In this section we describe a \( \mathbb{Z} \)-grading for \( Q \) and \( T \). These \( \mathbb{Z} \)-gradings will be used to compare the \( Q \)-modules and \( T \)-modules.

Lemma 5.1. The following is a direct sum of vector spaces:

\[ T = \sum_{i=0}^D \sum_{j=0}^D E_i^* T E_j^*. \] (12)

Proof. Multiply \( T \) on the left and right by \( I \), and use (4) to obtain (12). The sum (12) is direct by (4) and (5).

Definition 5.2. For \( n \in \mathbb{Z} \) define a subspace \( T_n \subseteq T \) by

\[ T_n = \sum_{i \in \mathbb{Z}} E_{i+n}^* T E_i^*. \]

Lemma 5.3. The following is a direct sum of vector spaces:

\[ T = \sum_{n \in \mathbb{Z}} T_n. \] (13)

Proof. Combine Lemma 5.1 and Definition 5.2.

Lemma 5.4. For \( n \in \mathbb{Z} \) we have \( T_n = 0 \) unless \( -D \leq n \leq D \).

Proof. By (3) and Definition 5.2.

Lemma 5.5. For \( n \in \mathbb{Z} \) and \( S \in T \) the following are equivalent:

(i) \( S \in T_n; \)

(ii) \( SE_i^* V \subseteq E_{i+n}^* V \) for \( i \in \mathbb{Z} \).

Proof. (i) \( \Rightarrow \) (ii) By Definition 5.2 and (3).

(ii) \( \Rightarrow \) (i) By Lemma 5.3 and (5), (6).
**Lemma 5.6.** We have $M^* \subseteq T_0$. Moreover

$$L \in T_{-1}, \quad F \in T_0, \quad R \in T_1. \quad (14)$$

*Proof.* By (5) and Definition 5.2, $T_0$ contains $E_j^*$ for $0 \leq j \leq D$. Therefore $T_0$ contains $M^*$. Line (14) follows from Lemmas 4.4 and 5.5. \hfill \square

**Lemma 5.7.** For $n \in \mathbb{Z}$ and $S \in T$ the following are equivalent:

1. $S \in T_n$;
2. $SE_i^* = E_{i+n}^* S$ for $i \in \mathbb{Z}$.

Suppose (i), (ii) hold. Then $SE_i^* = E_{i+n}^* S = E_{i+n}^* S$ for $i \in \mathbb{Z}$.

*Proof.* (i) $\Rightarrow$ (ii) By Definition 5.2 and (5).
(ii) $\Rightarrow$ (i) By Lemma 5.1, (5), and Definition 5.2.
Suppose (i), (ii) hold. Then the last assertion holds by Definition 5.2 and (5). \hfill \square

**Lemma 5.8.** We have

$$T_n T_m \subseteq T_{n+m} \quad (n, m \in \mathbb{Z}). \quad (15)$$

*Proof.* Use Definition 5.2 and (5). \hfill \square

**Note 5.9.** By Lemmas 5.3 and 5.8, the sequence $\{T_n\}_{n \in \mathbb{Z}}$ is a $\mathbb{Z}$-grading of $T$.

**Lemma 5.10.** The subspace $T_0$ is a subalgebra of $T$.

*Proof.* In (13) set $m = n = 0$. \hfill \square

**Definition 5.11.** For $n \in \mathbb{Z}$ define $Q_n = Q \cap T_n$.

**Lemma 5.12.** For $n \in \mathbb{Z}$ we have $Q_n = 0$ unless $-D \leq n \leq D$.

*Proof.* By Lemma 5.4 and Definition 5.11. \hfill \square

**Lemma 5.13.** For $n \in \mathbb{Z}$ and $S \in Q$ the following are equivalent:

1. $S \in Q_n$;
2. $SE_i^* V \subseteq E_{i+n}^* V$ for $i \in \mathbb{Z}$.

*Proof.* By Lemma 5.5 and Definition 5.11. \hfill \square

**Lemma 5.14.** We have

$$I \in Q_0, \quad L \in Q_{-1}, \quad F \in Q_0, \quad R \in Q_1.$$  

*Proof.* By Lemma 5.6 and Definition 5.11. \hfill \square

**Proposition 5.15.** The following is a direct sum of vector spaces:

$$Q = \sum_{n \in \mathbb{Z}} Q_n. \quad (16)$$

Moreover,

$$Q_n Q_m \subseteq Q_{n+m} \quad (n, m \in \mathbb{Z}). \quad (17)$$

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Proof. Line (16) follows from Lemmas 5.3, 5.6 and Definition 5.11. Line (17) follows from (15) and Definition 5.11. 

Note 5.16. By Proposition 5.15, the sequence \( \{ Q_n \}_{n \in \mathbb{Z}} \) is a \( \mathbb{Z} \)-grading of \( Q \).

Lemma 5.17. The subspace \( Q_0 \) is a subalgebra of \( Q \).

Proof. In (17) set \( m = n = 0 \).

In general the \( \{ E^*_i \}_{i=0}^D \) are not contained in \( Q \). Nevertheless we have the following.

Lemma 5.18. For \( n \in \mathbb{Z} \) and \( S \in Q \) the following are equivalent:

(i) \( S \in Q_n \);

(ii) \( SE^*_i = E^*_{i+n}S \) for \( i \in \mathbb{Z} \).

Suppose (i), (ii) hold. Then \( SE^*_i = E^*_{i+n}SE^*_i = E^*_{i+n}S \) for \( i \in \mathbb{Z} \).

Proof. By Lemma 5.7 and Definition 5.11.

6 Irreducible \( T \)-modules and \( Q \)-modules

Recall that \( Q \) is a subalgebra of \( T \). Let \( W \) denote a \( T \)-module. For the \( T \)-module \( W \) the restriction of the \( T \)-action to \( Q \) turns \( W \) into a \( Q \)-module. Assume that the \( T \)-module \( W \) is irreducible. In this section we show that the \( Q \)-module \( W \) is irreducible.

Lemma 6.1. Let \( W \) denote an irreducible \( T \)-module. Then for \( 0 \neq v \in W \) we have \( Tv = W \).

Proof. Since \( Tv \) is a nonzero \( T \)-submodule of \( W \).

Lemma 6.2. Let \( W \) denote an irreducible \( T \)-module. Pick a nonzero \( v \in W \) that is a common eigenvector for \( M^* \). Then \( W = Qv \).

Proof. By assumption \( M^*v = Cv \). Using Proposition 4.13 and Lemma 6.1 we obtain \( W = Tv = QM^*v = Qv \).

Proposition 6.3. Let \( W \) denote an irreducible \( T \)-module. Then the \( Q \)-module \( W \) is irreducible.

Proof. Let \( r \) and \( d \) denote the endpoint and diameter of \( W \), respectively. Let \( W' \) denote a nonzero \( Q \)-submodule of \( W \). We show that \( W = W' \). Pick \( 0 \neq z \in W' \). By (14) we have \( z = \sum_{i=0}^{d} E^*_r z \). Define \( j = \max\{i|0 \leq i \leq d, E^*_r z \neq 0\} \). Thus \( z = \sum_{i=0}^{j} E^*_r z \) and \( E^*_r z \neq 0 \). We have

\[
Q_{-j} z = Q_{-j} \sum_{i=0}^{j} E^*_r z = Q_{-j} E^*_r z = E^*_r Q_{-j} E^*_r z = E^*_r Q E^*_r z = E^*_r W. \tag{18}
\]

In the above line, the second equality holds by Lemma 5.13, the definition of \( r \), and since \( W \) is a \( Q \)-module. The third equality holds by Lemma 5.18. The fourth equality holds by \( \{5\}, \{16\}, \) and Definition 5.11. The last equality follows by Lemma 6.2 and since \( E^*_r z \) is a common eigenvector for \( M^* \). By Lemma 6.2 we have \( W = Q E^*_r W \). By this and (18),

\[
W = Q E^*_r W = QQ_{-j} z \subseteq Qz \subseteq W'.
\]

Therefore \( W = W' \).
Let $U$ and $W$ denote irreducible $T$-modules. We just saw that the $Q$-modules $U, W$ are irreducible. Assume for the moment that the $T$-modules $U, W$ are isomorphic. Then the $Q$-modules $U, W$ are isomorphic. Assume for the moment that the $T$-modules $U, W$ are not isomorphic. It is possible that the $Q$-modules $U, W$ are isomorphic. We now explain this point in more detail.

By (1) the following sums are direct:

$$V = \sum_{\lambda \in T^\vee} V_\lambda, \quad V = \sum_{\mu \in Q^\vee} V_\mu.$$  

By this and Proposition 6.3, the inclusion map $Q \to T$ induces a surjective map $\psi : T^\vee \to Q^\vee$ such that for $\mu \in Q^\vee$,

$$V_\mu = \sum_{\lambda \in T^\vee : \psi(\lambda) = \mu} V_\lambda.$$  

For $\mu \in Q^\vee$ define

$$m_\mu = \left| \{ \lambda \in T^\vee | \psi(\lambda) = \mu \} \right|. \quad (19)$$

Note that $m_\mu$ is a positive integer.

Lemma 6.4. We have

$$\dim T = \sum_{\mu \in Q^\vee} m_\mu d_\mu. \quad (20)$$

Proof. For $\lambda \in T^\vee$ and $\mu \in Q^\vee$ such that $\psi(\lambda) = \mu$, we have $d_\lambda = d_\mu$. Using this fact and (19), we evaluate equation (8). The result follows.

## 7 Irreducible $T$-modules and $Q$-modules, cont.

In this section we describe how an irreducible $T$-module looks when viewed as a $Q$-module.

Lemma 7.1. Let $W$ denote an irreducible $T$-module with endpoint $r$ and diameter $d$. For $i \in \mathbb{Z}$ and $r \leq j \leq r + d$,

$$Q_i E_j^* W = E_{i+j}^* W. \quad (21)$$

Proof. By Lemma 6.2, $W = Q E_j^* W$. By this and Lemma 5.18,

$$E_{i+j}^* W = E_{i+j}^* Q E_j^* W = Q_i E_j^* W.$$  

Lemma 7.2. Let $W$ denote an irreducible $T$-module with endpoint $r$ and diameter $d$. Then for $0 \leq i \leq d$,

$$Q_i W = \sum_{\ell=i}^{d} E_{r+\ell}^* W, \quad Q_{-i} W = \sum_{\ell=0}^{d-i} E_{r+\ell}^* W.$$  

Moreover for $i \geq d + 1$,

$$Q_i W = 0, \quad Q_{-i} W = 0.$$  

Proof. Use (21) and Lemma 7.1.

Corollary 7.3. Let $W$ denote an irreducible $T$-module with endpoint $r$ and diameter $d$. Then

$$Q_d W = E_{r+d}^* W, \quad Q_{-d} W = E_r^* W.$$  


Proof. Set $i = d$ in Lemma 7.2.

Corollary 7.4. Let $W$ denote an irreducible $T$-module. Then the diameter $d$ of $W$ is given by

$$d = \max\{i \mid 0 \leq i \leq D, Q_i W \neq 0\}.$$  

Proof. By Corollary 7.3 and the last assertion of Lemma 7.2.

Lemma 7.5. Let $U$ and $W$ denote irreducible $T$-modules that are isomorphic as $Q$-modules. Then they have the same diameter.

Proof. By Corollary 7.4.

Proposition 7.6. Let $U$ and $W$ denote irreducible $T$-modules. Assume that there exists an isomorphism of $Q$-modules $\sigma : U \to W$. Then $\sigma E_{i+1}^* U = E_{i+1}^* W$ for $0 \leq i \leq d$. Here $d = d(U) = d(W)$ and $r = r(U)$, $r' = r(W)$.

Proof. Observe that

$$\sigma E_{i+1}^* U = \sigma Q_i E_{i+1}^* U = \sigma Q_i Q_{-d} U = Q_i Q_{-d} \sigma U = Q_i Q_{-d} W = Q_i E_{i+1}^* W = E_{i+1}^* W.$$ 

8 Quasi-isomorphisms of $T$-modules

In this section we compare $Q$-module isomorphisms and $T$-module isomorphisms. To do this, we introduce the notion of a quasi-isomorphism of $T$-modules.

Definition 8.1. Let $U$ and $W$ denote irreducible $T$-modules. By a quasi-isomorphism of $T$-modules from $U$ to $W$ we mean a $C$-linear bijection $\sigma : U \to W$ such that on $U$,

$$\sigma L = L \sigma, \quad \sigma F = F \sigma, \quad \sigma R = R \sigma$$  

and

$$\sigma E_i^* = E_{i+n}^* \sigma \quad i \in \mathbb{Z},$$  

where $n = r(W) - r(U)$.

Lemma 8.2. Let $U$ and $W$ denote irreducible $T$-modules. For a $C$-linear map $\sigma : U \to W$, the following are equivalent:

(i) $\sigma$ is a quasi-isomorphism of $T$-modules from $U$ to $W$;

(ii) $\sigma^{-1}$ is a quasi-isomorphism of $T$-modules from $W$ to $U$.

Proof. Use Definition 8.1.

Definition 8.3. Irreducible $T$-modules $U$ and $W$ are called quasi-isomorphic whenever there exists a quasi-isomorphism of $T$-modules from $U$ to $W$.

We make two observations.

Lemma 8.4. Let $U$ and $W$ denote irreducible $T$-modules with the same endpoint. Then for a $C$-linear map $\sigma : U \to W$ the following are equivalent:
(i) \( \sigma \) is a quasi-isomorphism of \( T \)-modules from \( U \) to \( W \);

(ii) \( \sigma \) is an isomorphism of \( T \)-modules from \( U \) to \( W \).

**Proof.** Set \( n = 0 \) in Definition 8.1.

**Corollary 8.5.** For irreducible \( T \)-modules \( U, W \) the following are equivalent:

(i) the \( T \)-modules \( U, W \) are quasi-isomorphic and have the same endpoint;

(ii) the \( T \)-modules \( U, W \) are isomorphic.

**Proof.** By Lemma 8.4 and since isomorphic irreducible \( T \)-modules have the same endpoint.

**Proposition 8.6.** Let \( U \) and \( W \) denote irreducible \( T \)-modules. Then for a \( \mathbb{C} \)-linear map \( \sigma : U \to W \) the following are equivalent:

(i) \( \sigma \) is an isomorphism of \( Q \)-modules from \( U \) to \( W \);

(ii) \( \sigma \) is a quasi-isomorphism of \( T \)-modules from \( U \) to \( W \).

**Proof.** (i) \( \Rightarrow \) (ii) We check that \( \sigma \) satisfies the conditions in Definition 8.1. The map \( \sigma \) satisfies (22) since \( \sigma \) is an isomorphism of \( Q \)-modules. The map \( \sigma \) satisfies (23) by Proposition 7.6.

(ii) \( \Rightarrow \) (i) Use (22).

**Corollary 8.7.** For irreducible \( T \)-modules \( U, W \) the following are equivalent:

(i) the \( Q \)-modules \( U, W \) are isomorphic;

(ii) the \( T \)-modules \( U, W \) are quasi-isomorphic.

**Proof.** By Proposition 8.6.

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**9 Comparing \( Q \) and \( T \)**

Recall that \( Q \) is a subalgebra of \( T \). In this section we consider when are these two algebras equal.

**Theorem 9.1.** The following (i)–(iv) are equivalent:

(i) \( Q \neq T \);

(ii) \( Q \subsetneq T \);

(iii) there exists a pair of nonisomorphic irreducible \( T \)-modules that are isomorphic as \( Q \)-modules;

(iv) there exists a pair of quasi-isomorphic irreducible \( T \)-modules that have different endpoints.

**Proof.** (i) \( \Leftrightarrow \) (ii) By construction \( Q \subseteq T \).

(ii) \( \Leftrightarrow \) (iii) Assertion (iii) means that there exists \( \mu \in Q^\vee \) such that \( m_\mu > 1 \). In this light, compare (11) and (20).

(iii) \( \Leftrightarrow \) (iv) By Corollaries 8.5 and 8.7.
10 Thin irreducible $T$-modules

In Sections 6–8 we considered the irreducible $T$-modules. In this section we consider a special type of irreducible $T$-module, said to be thin.

Definition 10.1. An irreducible $T$-module $W$ is called thin whenever $\dim E_i^*W \leq 1$ for $0 \leq i \leq D$.

Lemma 10.2. Let $W$ denote a thin irreducible $T$-module with endpoint $r$ and diameter $d$. Then

$$R^i E_i^*W = E_{r+i}^*W \quad (0 \leq i \leq d).$$  \hspace{1cm} (24)

Proof. Similar to the proof of [4, Lemma 2.7].  \hfill $\Box$

Lemma 10.3. Let $W$ denote a thin irreducible $T$-module with endpoint $r$ and diameter $d$. Pick a nonzero $u \in E_r^*W$. Then:

(i) for $0 \leq i \leq d$, $R^i u$ is a basis for $E_{r+i}^*W$;

(ii) the vectors $\{R^i u\}_{i=0}^d$ form a basis for $W$.

Proof. (i) By Lemma 10.2 and since $\dim E_{r+i}^*W = 1$ for $0 \leq i \leq d$.

(ii) Use (i) and (9).  \hfill $\Box$

Definition 10.4. The basis in Lemma 10.3(ii) is called standard.

Lemma 10.5. Let $W$ denote a thin irreducible $T$-module with diameter $d$. Let $\{v_i\}_{i=0}^d$ denote a standard basis for $W$. Then $Rv_i = v_{i+1}$ $(0 \leq i \leq d - 1)$ and $Rv_d = 0$.

Proof. By construction.  \hfill $\Box$

Lemma 10.6. Let $W$ denote a thin irreducible $T$-module with diameter $d$. Then there exist scalars $\{a_i(W)\}_{i=0}^d$, $\{x_i(W)\}_{i=1}^d$ in $\mathbb{R}$ that satisfy the following. For any standard basis $\{v_i\}_{i=0}^d$ of $W$,

$$Fv_i = a_i(W)v_i \quad (0 \leq i \leq d),$$  \hspace{1cm} (25)

$$Lv_i = x_i(W)v_{i-1} \quad (1 \leq i \leq d).$$  \hspace{1cm} (26)

Proof. By Lemma 1.4 and since $F, L \in T$.  \hfill $\Box$

Proposition 10.7. Let $U, W$ denote thin irreducible $T$-modules. Then the following are equivalent:

(i) $U$ and $W$ are quasi-isomorphic;

(ii) $U$ and $W$ have the same diameter $d$ and

$$a_i(U) = a_i(W) \quad (0 \leq i \leq d),$$

$$x_i(U) = x_i(W) \quad (1 \leq i \leq d).$$

Proof. Use Definition 8.1.  \hfill $\Box$
11 Examples

In this section we discuss two examples. The first example concerns the Hamming graphs \[1, p. 261\]. The second example concerns the bipartite dual polar graphs \[1, p. 274\]. We show that for the first example \(Q = T\), and for the second example \(Q \neq T\).

We now recall the Hamming graphs. Fix integers \(D \geq 1\) and \(N \geq 2\). For the Hamming graph \(H(D, N)\) the vertex set consists of the \(D\)-tuples of elements from \(\{1, \ldots, N\}\). Two vertices of \(H(D, N)\) are adjacent whenever they differ in precisely one coordinate. The graph \(H(D, N)\) is distance-regular and \(Q\)-polynomial \[1, Section 9.2\].

**Proposition 11.1.** Fix a vertex \(x\) of \(H(D, N)\). Let \(Q = Q(x)\) and \(T = T(x)\). Then \(Q = T\).

**Proof.** Define \(A^* = \sum_{i=0}^{D} \theta_i^* E_i^*\), where \(\theta_i^* = (N - 1)(D - i) - i\) for \(0 \leq i \leq D\). In \[20\], \(A^*\) is called the dual adjacency matrix. Observe that \(\{\theta_i^*\}_{i=0}^{D}\) are mutually distinct. Therefore \(A^*\) generates \(M^*\). Consequently \(A, A^*\) generate \(T\). By construction \(A \in Q\). By combinatorial counting we obtain

\[A^* = F + LR - RL.\]

So \(A^* \in Q\). By these comments \(Q = T\). \(\Box\)

We now turn our attention to the bipartite dual polar graphs. Fix a prime power \(q\) and an integer \(D \geq 2\). The bipartite dual polar graph \(D_D(q)\) is described in \[1, Section 9.4\]; see also \[14, 22, 25\]. This graph is distance-regular and \(Q\)-polynomial \[1, Section 9.4\].

**Proposition 11.2.** Fix a vertex \(x\) of \(D_D(q)\). Let \(Q = Q(x)\) and \(T = T(x)\). Then \(Q \neq T\).

**Proof.** The irreducible \(T\)-modules are described in \[2\]. By \[2\, Lemma 9.2\] these modules are thin. By \[2\, Theorem 15.6\], up to isomorphism there exists a unique irreducible \(T\)-module \(U\) with endpoint 1 and diameter \(D - 2\). By \[2\, Theorem 15.6\], up to isomorphism there exists a unique irreducible \(T\)-module \(W\) with endpoint 2 and diameter \(D - 2\). By \[22\, p. 200\], \(a_i(U) = a_i(W) = 0\) for \(0 \leq i \leq D - 2\). Also by \[22\, p. 200\],

\[x_i(U) = x_i(W) = \frac{q^{i+1}(q^i - 1)(q^{D-i-1} - 1)}{(q - 1)^2} \quad (1 \leq i \leq D - 2).\]

By these comments \(U\) and \(W\) satisfy the conditions of Proposition \[10.7(ii)\]. By Proposition \[10.7\] the \(T\)-modules \(U\) and \(W\) are quasi-isomorphic. By this and Theorem \[9.1\] \(Q \neq T\). \(\Box\)

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