On the Correlation between Angle and Distance Distributions in Finite Wireless Networks

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Abstract—Directional beamforming will play a paramount role in 5G and beyond networks in order to combat the higher path losses incurred at millimeter wave bands. Appropriate modeling and analysis of the angles and distances between transmitters and receivers in these networks is thus essential to understand performance and limiting factors. Most existing literature considers either infinite and uniform networks, where nodes are drawn according to a Poisson point process, or finite networks with the reference receiver placed at the origin of a disk. Under either of these assumptions, the distance and azimuth angle between transmitter and receiver are independent, and the angle follows a uniform distribution between 0 and $2\pi$. Here, we consider a more realistic case of finite networks where the reference node is placed at any arbitrary location. We obtain the joint distribution between the distance and azimuth angle and demonstrate that these random variables do exhibit certain correlation, which depends on the shape of the region and the location of the reference node. To conduct the analysis, we present a general mathematical framework which is specialized to exemplify the case of a rectangular region. We then also derive the statistics for the 3D case where, considering antenna heights, the joint distribution of distance, azimuth and zenith angles is obtained. Finally, we describe some immediate applications of the present work, including the analysis of directional beamforming, the design of analog codebooks and wireless routing algorithms.

Index Terms—stochastic geometry, finite networks, angle distribution, beam management, millimeter-wave.

I. INTRODUCTION

A. Motivation and Scope

The need of greater bandwidths to accommodate the ever increasing demand of data rates have led to the use of higher frequency bands, e.g., millimeter wave (mmW) bands. Key to compensate the higher path loss experienced at these bands is directional beamforming, which uses a massive number of antenna elements that can be conveniently packed due to the smaller wave lengths. The enabling role of directional beamforming in the realization of 5G and beyond networks is indeed unquestionable, and appropriate modeling and analysis is needed to identify performance trends, trade-offs and limiting factors [1–3]. This has motivated a number of works that analyze directional beamforming on a wide set of scenarios such as 5G cellular networks [4–9], vehicular networks [10–12], device-to-device (D2D) communications [13–16], unmanned aerial vehicles (UAVs) based networks [17–21], and wireless communications empowered with reconfigurable intelligent surfaces (RISs) [22–25], among others.

In these works, assuming a random location of the transmitters/receivers, the analyses typically require the joint statistical distribution of the polar coordinates (distance and azimuth angle) in 2D scenarios; and that of the spherical coordinates, which also include the zenith (elevation) angle in 3D scenarios. The latter case is considered when the height of the nodes is relevant (e.g., UAV scenarios). In 2D scenarios it is assumed that the distance, $R$, and azimuth angle, $\Theta$, between the transmitter and receiver are independent random variables (RVs), where $\Theta$ follows a uniform distribution between 0 and $2\pi$. Those works investigating 3D scenarios make the same assumptions for the distance and azimuth angle, but they consider the zenith angle, $\Psi$, correlated with the distance. Such correlation comes from the fact that $\Psi$ can be expressed in simple terms as a trigonometric function of the distance and the height of the antennas.

While these assumptions on the distance and azimuth angle are valid for infinite networks with uniform node distributions, e.g., governed by a uniform Poisson Point Process (PPP), they do not hold for finite networks. Indeed, as it will be shown in this work, for finite networks the azimuth angle and distance between a randomly placed node and the reference node are correlated. Modeling this correlation is very challenging as it depends on both the shape of the region (network) where the nodes are located, and on the position of the arbitrarily placed reference node. The only exception is the simplified case where the reference node is at the center of a disk; in such case, the azimuth angles are independent of the distance and they follow a uniform distribution within $[0, 2\pi]$.

B. Related Work

Considering finite networks is crucial to investigate the system-level performance of directional beamforming, which typically aims at boosting the throughput in limited regions with an increased demand for high data rates (hot spots). Nevertheless, the analysis of finite networks, where node locations are modeled by a binomial point process (BPP), is substantially more complex than that of infinite networks, modeled by...
a PPP. In particular: i) in finite networks, the distribution of the signal to interference plus noise ratio (SINR) depends on the position of the reference node and, therefore, the performance is location dependent; ii) the distances of all nodes in the network towards a reference node are correlated; iii) unlike for PPP-based infinite networks, the selection of a randomly placed node in finite networks changes the distribution of the underlying point process [20].

Despite its relevance, the analysis of directional beamforming in finite networks is scarce. Remarkably, previous results have been obtained under modeling assumptions that ignore the correlation between the distance and azimuth angle. To analyze the case of 2D indoor mmW systems, the model in [27] assumes that the transmitting nodes are randomly placed within a disk. The probe receiver is placed at a given distance from the center of the disk and perfect beam alignment is assumed between the probe transmitter and receiver. However, the steering directions of the interfering beams are assumed uniform and independent, thus ignoring the said distance-angle correlation. In [28], 2D finite mmW systems with sectored antenna patterns are investigated; once again, a uniform distribution is assumed for the steering angles of the interfering beams. The uplink of an indoor wireless system operating at the terahertz band is analyzed in [29], considering a 3D rectangular region. The probe receiver is arbitrarily placed whereas the interfering transmitters are randomly placed within the 3D region. However, all transmit and receive beams are assumed to be perfectly aligned. Again, this avoids the need to compute the joint distribution, and neglects the said correlation.

Moreover, existing results on the distance distributions in finite regions do not account for the angle distribution. The distribution of distances from random points, uniformly placed within a regular L-polygon and a disk, to the origin (center of the region) is derived in [30]. Leveraging the rotational symmetry of regular polygons, the analysis is extended in [31] to compute the distance distribution between an arbitrary location and a random point uniformly distributed within the regular polygon. The distance distributions of two random points uniformly placed within a rectangular and an hexagonal region is derived in [32]. To make the problem tractable, a space decomposition method is proposed to divide the entire plane in regions after extending the edges of the polygons to infinity. In [33] the distance distribution between two random points is also derived for the case of convex polygons. This work is further extended in [34], considering arbitrarily shaped regions including convex polygons, as well as concave and disjoint regions. The analysis uses a mathematical tool called Kinematic Measure in integral geometry as well as decomposition and recursion methods. The distance distribution between two random points in the 3D case is obtained in [35], where the region forms a cube. The analysis is based on the convolution of 3 distributions for the 3 Cartesian coordinates, characterized by the Kumaraswamy distribution [36].

C. Main Contributions

To our knowledge, the azimuth angle distribution has not been derived yet for finite networks, despite its relevance to accurately model and analyze directional beamforming. This motivated us to investigate the correlation between the distance and azimuth angle. The main contributions of this work are:

1) We present a mathematical framework to obtain the joint distribution of the distance and azimuth angle in 2D arbitrarily shaped networks. We consider the angle and distance between an arbitrary (fixed) point, and a uniformly distributed random point.
2) We particularize the proposed framework to the case of a rectangular region, since this kind of region matches many practical constructions such as buildings, shopping centers, stadiums or main squares where finite networks are normally deployed. We derive the joint cumulative density function (CDF) and joint probability density function (PDF) of the distance and angle, and compute the marginals for the azimuth angle.
3) We extend the obtained results to the case of 3D networks, where both the height of the reference node and that of the random node are considered.
4) Finally, we discuss some of the applications of the present work, including the analysis of directional beamforming, the design of the beam-patterns of analog codebooks and the design of wireless routing algorithms.

The derivation of the joint distance and azimuth angle distribution is significantly more challenging than that of the marginal distance distribution, considered so far in the existing literature. For the derivation of the marginal distance distribution, say, in a regular polygon, one only needs to compute the intersection between the polygon (region of interest) and a disk centered at the reference node. To derive the joint distribution, however, we need to compute the intersection of 3 regions: the region of interest, a disk of radius $r$ and a given sector. Besides, extending the results to the 3D case requires the inclusion of an additional angle, the zenith (elevation) angle, which complicates the analysis. Moreover, it should be noted that the analysis of a rectangular region is substantially more complex than the case of a regular L-polygon or a disk, since rotational symmetry properties no longer hold.

D. Notation and Paper Organization

The following notation is used through the text. $\mathbb{Z}$ and $\mathbb{R}$ stands for the set of integers and real numbers respectively, $\mathbb{R}^+$ represents the non-negative real numbers and $\mathbb{R}^d$ stands for the d-dimensional Euclidean space. Other sets are represented with fraktur font, e.g., $\mathcal{R} \subseteq \mathbb{R}^d$, with $d > 0$, whereas boolean expressions are written with calligraphic font, e.g., $C = \{x \leq a\}$. The Lebesgue measure of the compact set $\mathcal{R}$ is denoted by $|\mathcal{R}|$, which represents the area or the volume for $d = 2$ or $d = 3$, respectively. The intersection and union of sets are represented with symbols $\cap$ and $\cup$, respectively, whereas $\land$ and $\lor$ stand for the and and or logical operations. The empty set is denoted by $\emptyset$, whereas $\mathcal{R}$ represents the complement of the set $\mathcal{R}$. In addition, the upper bar represents negation of a logical expression, i.e., $\bar{C}$ is true if $C$ is false. If $X$ is a RV, then $F_X(x) = \Pr(X \leq x)$ represents its CDF where $\Pr(\cdot)$ denotes the probability measure.
Fig. 1: Coordinate system including: 1) spherical coordinates for distance, $d$, azimuth angle, $\theta$, and zenith angle, $\varphi$, related to a 3D point placed at $u = (u_x, u_y, u_z)$; and 2) polar coordinates for distance, $r$, and azimuth angle, $\theta$, related to the projection of the 3D point in the $xy$ plane.

The rest of the paper is structured as follows: the proposed mathematical framework and related results are presented in Section II. Section III derives the main theoretical results, particularizing the proposed framework to the analysis of a rectangular finite network in the 2D and 3D cases, whereas in Section IV some applications of the results are discussed. The theoretical expressions are validated in Section V and finally, the main conclusions are drawn in Section VI.

II. MATHEMATICAL FRAMEWORK

A. Mathematical Preliminaries and Definitions

We consider points that belong to either the 2D or 3D euclidean space. They are represented using Cartesian coordinates, as well as either polar (2D) or spherical (3D) coordinates. Fig. 1 illustrates the conventional spherical and polar coordinate systems, formalized in the next two definitions. Note that 2D points can be seen as projections of 3D points in the $xy$ plane.

**Definition 1** (Spherical coordinate system). An arbitrary point $u \in \mathbb{R}^3$ expressed as $(u_x, u_y, u_z)$ in Cartesian coordinates can be written as $(d, \theta, \varphi)$, being $d$ the distance, $\theta$ the azimuth angle, and $\varphi$ the zenith angle. The following relations hold:

$$u_x = d \cos(\theta) \sin(\varphi); \quad u_y = d \sin(\theta) \sin(\varphi); \quad u_z = d \cos(\varphi).$$

**Definition 2** (Polar coordinate system). An arbitrary point $u \in \mathbb{R}^2$ expressed as $(u_x, u_y)$ in Cartesian coordinates can be written as $(r, \theta)$, being $r$ the distance and $\theta$ the azimuth angle. The following relations hold:

$$x = r \cos(\theta); \quad y = r \sin(\theta);$$

We now present some mathematical functions used in the analysis of joint angle and distance distributions.

**Definition 3** (Indicator function). The indicator function evaluated over a set, $\mathbb{C} \subset \mathbb{R}^d$, with $d > 0$, is defined as

$$\mathbb{I}_\mathbb{C}(x) = \int_\mathbb{C} \delta(x - u) \, du = \begin{cases} 1 & \text{if } x \in \mathbb{C} \\ 0 & \text{otherwise} \end{cases}$$

where $\delta(x)$ is the Dirac delta function.

The expression $x \in \mathbb{C}$ in the above definition can also be viewed as a boolean expression representing an event \( \mathbb{C} \), that is true if $x$ belongs to the set $\mathbb{C}$ and false otherwise. Thus, the indicator function can alternatively be written as

$$\mathbb{I}_\mathbb{C}(x) = \mathbb{I}(C(x)),$$

where $C = \{x \in \mathbb{C}\}$.

**Definition 4** (Properties of the indicator function). The logical and ($\land$) and or ($\lor$) operations on two boolean expressions, $C_1$ and $C_2$, satisfy the following relations:

$$\mathbb{I}(C_1 \land C_2) = \mathbb{I}(C_1) \mathbb{I}(C_2),$$
$$\mathbb{I}(C_1 \lor C_2) = \mathbb{I}(C_1) + \mathbb{I}(C_2) - \mathbb{I}(C_1 \land C_2).$$

Equivalently, the union and intersection of two sets, $\mathbb{C}_1$ and $\mathbb{C}_2$, lead to the following equalities:

$$\mathbb{I}_{\mathbb{C}_1 \cap \mathbb{C}_2}(x) = \mathbb{I}_{\mathbb{C}_1}(x) \mathbb{I}_{\mathbb{C}_2}(x),$$
$$\mathbb{I}_{\mathbb{C}_1 \cup \mathbb{C}_2}(x) = \mathbb{I}_{\mathbb{C}_1}(x) + \mathbb{I}_{\mathbb{C}_2}(x) - \mathbb{I}_{\mathbb{C}_1 \cap \mathbb{C}_2}(x).$$

The expression $x \in \mathbb{C}$ in the above definition can also be viewed as a boolean expression representing an event $\mathbb{C}$, that is true if $x$ belongs to the set $\mathbb{C}$ and false otherwise. Thus, the indicator function can alternatively be written as $\mathbb{I}_\mathbb{C}(x) = \mathbb{I}(C(x))$, where $C = \{x \in \mathbb{C}\}$.

**Definition 5** (Positive part operator). For any real number (or real-valued function) $x$, the positive part of $x$ is defined as

$$(x)^+ = \max\{0, x\} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

**Definition 6** ($\mathbb{F}$ operator). For a function $f$ on the real domain, i.e., $f : \mathbb{R} \mapsto \mathbb{R}$, the operator $\mathbb{F}(f; a; b)$ is defined as

$$\mathbb{F}(f; a; b) = \begin{cases} f(b) - f(a) & \text{if } b \geq a, \\ 0 & \text{otherwise}. \end{cases}$$

This definition allows us to write the result of definite integrals in compact form, as illustrated next.

**Proposition 1.** The definite integral

$$\int_{b_1(u)}^{b_2(u)} \mathbb{I}_\mathbb{F}(t) g(t) \, dt,$$ with $g(t) = \frac{df(t)}{dt}$ and $\mathbb{F} = \{a_1(v), a_2(v)\}$ admits

$$\int_{b_1(u)}^{b_2(u)} \mathbb{I}_\mathbb{F}(t) g(t) \, dt = \mathbb{F}\left(f; \max(a_1(v), b_1(u)), \min(a_2(v), b_2(u))\right).$$

**Proof.** The proof comes after (i) applying the indicator function to the integration limits; (ii) considering that if $a_1(v) > a_2(v)$, then $\mathbb{F}$ reduces to an empty set and the result of the integral is 0; and (iii) substituting the $\mathbb{F}$ operator in the resulting expression.

**Corollary 1.** If $f(t)$ is a non-decreasing function, then the definite integral in Proposition 7 can be further expressed as

$$\int_{b_1(u)}^{b_2(u)} \mathbb{I}_\mathbb{F}(t) g(t) \, dt = \left(f(\min(a_2(v), b_2(u))) - f(\max(a_1(v), b_1(u)))\right)^+.\quad (9)$$

\(^1\)The indicator function of a subset (or event) maps elements of the subset (i.e., the event that $x$ falls in the subset) to one and zero otherwise.
B. Joint distribution in arbitrarily shaped networks

In this section, we present the procedure to derive the joint distribution of the distance, $R$, and azimuth angle, $\Theta$, between a random point in the arbitrary region of interest $\mathcal{R}(o) \subset \mathbb{R}^2$, and the reference point $u = (u_x, u_y) \in \mathbb{R}^2$. The region is modeled as a compact set whose center of mass is located at the origin $o = (0, 0)$, and random points are uniformly distributed within the region.

The joint CDF of the distance and azimuth angle can be written as

$$ F_{R, \Theta}(r, \theta) = \frac{\mathbb{I}(u, r, \theta)}{\mathbb{I}(o, r) \cap \mathcal{R}(o) \cap \mathbb{I}(u, \theta)} $$

where $\mathbb{I}(u, r)$ is a disk of radius $r$ centered at $u$ and $\mathbb{I}(o, \theta)$ is the sector spanning $[0, \theta]$ with origin at $o$. The compact set $\mathbb{I}(u, r, \theta)$ represents the intersection of both regions with $\mathbb{I}(o, \theta)$ the region of interest, and (a) follows from the fact that the Lebesgue measure is invariant to translations. Fig. 2 illustrates an example of such regions. The disk $\mathbb{I}(o, r)$, sector $\mathbb{I}(o, \theta)$, and the translated region of interest (centered at $-u$) $\mathcal{R}(-u)$ are drawn in red, blue, and gray color, respectively.

The main challenge to derive the joint CDF is the computation of the area of $\mathbb{I}(u, r, \theta)$, which can be expressed as

$$ [\mathbb{I}(u, r, \theta)] = \int_{\phi=0}^{\theta} \int_{\rho=0}^{r} \mathbb{I}(\mathbb{I}(o, r) \cap \mathcal{R}(o) \cap \mathbb{I}(u, \theta)) \rho \, d\rho \, d\phi $$

where (a) comes after applying Definition 4 to the indicator function, and translating the resulting indicator functions for the disk and the sector into corresponding changes on the integration limits over $\rho$ and $\phi$, respectively; while (b) comes after the fact that $\mathcal{R}(u) = \{ \rho \in \mathbb{R}^+, \phi \in [0, 2\pi] \mid \rho \in \mathbb{B}(\phi) \}$.

The set $\mathbb{B}(\phi) \subset \mathbb{R}^+$ represents, for a given azimuth angle $\phi$, the range of values of $\rho$ that belong to the compact set $\mathcal{R}(u)$. Note that, if the arbitrary region is composed of concave sets and/or the union of disjoint sets, $\mathbb{B}(\phi)$ will be the union of disjoint intervals, where the number of intervals and their limits depend on the azimuth angle, $\phi$. This is illustrated in Fig. 2. In the shown example, for $\phi_1$, we have $\mathbb{B}(\phi_1) = (0, \beta_1(\phi_1)] \cap [\beta_2(\phi_1), \beta_3(\phi_1)]$; whereas, for $\phi_2$, $\mathbb{B}(\phi_2) = (0, \beta_1(\phi_2))$.

If $\mathcal{R}(u)$ is a convex region, we can always write $\mathbb{B}(\phi) = [0, \beta(\phi)]$, since the line segment between any two points in a convex set will always lie within the set $\mathbb{B}(\phi)$. In such case,

$$ [\mathbb{I}(u, r, \theta)] = \int_{\phi=0}^{\theta} \int_{\rho=0}^{\min(r, \beta(\phi))} \rho \, d\rho \, d\phi $$

where $\mathbb{I}(r, \phi)$ stands for the boolean expression (or event) $r \leq \beta(\phi)$ and $\mathbb{I}(\bar{r}, \phi)$ represents its complement, i.e., $r > \beta(\phi)$. Note that (12) is significantly more tractable than the initial problem to compute the overlap area $[\mathbb{I}(u, r, \theta)]$. Based on (12), the procedure to compute this area is summarized as:

- **Step 1:** Derive $\beta(\phi)$ to write the convex set $\mathcal{R}(u) = \{ \rho \in \mathbb{R}^+, \phi \in [0, 2\pi] \mid \mathbb{I}(r, \phi) \}$ with $\mathbb{I}(r, \phi) = \{ \rho \leq \beta(\phi) \}$

- **Step 2:** Solve the inequalities defining the events $\mathbb{I}(r, \phi)$ and $\mathbb{I}(\bar{r}, \phi)$, i.e., $r \leq \beta(\phi)$ and $r > \beta(\phi)$ to derive the disjoint sets of azimuth angles which satisfy either of the two events. These sets are written as $\mathcal{I}_\mathbb{B}(r) = \{ \phi \in [0, 2\pi] \mid \mathbb{I}(r, \phi) = 1 \}$ and $\mathcal{I}_\mathbb{I}(r) = \{ \phi \in [0, 2\pi] \mid \mathbb{I}(\bar{r}, \phi) = 1 \}$.

- **Step 3:** Use $\mathbb{I}(\mathbb{I}(r, \phi)) = \mathbb{I}(\mathbb{I}(r, \phi)) \mathbb{I}(\mathbb{I}(\bar{r}, \phi)) = \mathbb{I}(\mathbb{I}(\bar{r}, \phi)) \mathbb{I}(\mathbb{I}(r, \phi))$ in (12) and modify the integration limits according to $\mathcal{I}_\mathbb{B}(r)$ and $\mathcal{I}_\mathbb{I}(r)$ to solve the integrals.

In the next section we follow the above procedure to derive the joint CDF of distance and angle in the practically relevant case of a rectangle. As mentioned earlier, the rectangular region may be used to represent many finite area networks of interest, typically deployed in, e.g., buildings, shopping centers, stadiums or main squares. Recall that rotational symmetry does not hold for rectangles, making the computation of the overlap area more challenging than for regular L-polygons.

III. RECTANGULAR REGIONS

A. Angular and distance distributions in 2D networks

Let us assume a rectangular region, $\mathcal{R}(o)$, centered at the origin $o$, whose side lengths are $\ell_x$ and $\ell_y$ on the $x$ and $y$ axis, respectively. We aim at deriving the joint distribution of the distance and azimuth angle for the link between a random point (uniformly distributed in the rectangle) and a reference
point, \(u = (u_x, u_y)\). As pointed out in Section II-B, this is equivalent to computing the overlap area of a disk and a sector after translating the regions by \(\pm u\). To that end, we follow the three steps provided in Section II-B, starting with the derivation of \(\beta(\phi)\), given in the following lemma.

**Lemma 1 (Step 1).** A rectangular region of sides \(\ell_x\) and \(\ell_y\) centered at \(\pm u\) can be expressed in polar coordinates as

\[
\mathcal{R}(\pm u) = \{ \rho > 0, \phi \in [0, 2\pi) \mid B(\rho, \phi) \},
\]

where \(B(\rho, \phi) = \{ \rho \leq \beta(\phi) \}\) and

\[
\beta(\phi) = \min \left( \frac{h_x(\phi)}{\cos(\phi)} \frac{h_y(\phi)}{\sin(\phi)} \right),
\]

where

\[
h_x(\phi) = h_x 1 - (A_e(\phi)) + h_x 1 - (A_e(\phi)), \quad h_y(\phi) = h_y 1 - (A_e(\phi)) + h_y 1 - (A_e(\phi)),
\]

with \(h_x = \frac{\ell_x}{2} - u_x, h_y = \frac{\ell_y}{2} - u_y, h_y = \frac{\ell_y}{2} - u_y, h_y = \frac{\ell_y}{2} - u_y,\) and

\[
A_e(\phi) = (\phi \in \Omega_1) \lor (\phi \in \Omega_4); \quad A_e(\phi) = (\phi \in \Omega_2) \land (\phi \in \Omega_3);
\]

where \(\Omega_i, i = \{1, 2, 3, 4\},\) represent the 4 angular quadrants as \(\Omega_i = \left[\frac{(i-1)\pi}{2}, \frac{i\pi}{2}\right]\).

**Proof.** See Appendix A. \(\square\)

The next two lemmas give the sets \(\mathcal{X}(r)\) and \(\mathcal{W}(r)\), following Step 2 of the proposed procedure.

**Lemma 2 (Step 2).** The sets \(\mathcal{X}(r)\) and \(\mathcal{W}(r)\) are expressed as the union of 8 disjoint sets as

\[
\mathcal{X}(r) = \bigcup_{i=1}^{8} \mathcal{X}_i(r); \quad \mathcal{W}(r) = \bigcup_{i=1}^{8} \mathcal{W}_i(r).
\]

with

\[
\mathcal{X}_i(r) = \begin{cases} X_i^{(\\geq)}(r), X_i^{(\\leq)}(r), & \text{if } r \geq h_i, \\ X_i^{(\\geq)}(r), X_i^{(\\leq)}(r), & \text{if } r < h_i, \end{cases}
\]

and

\[
\mathcal{W}_i(r) = \begin{cases} W_i^{(\\geq)}(r), W_i^{(\\leq)}(r), & \text{if } r \geq h_i, \\ 0, & \text{if } r < h_i, \end{cases}
\]

where \(X_i^{(\\geq)}(r), X_i^{(\\leq)}(r), \text{ and } W_i^{(\\geq)}(r), W_i^{(\\leq)}(r), i = \{1, \ldots, 8\}, j = \{1, 2\},\) are given in (20) and (21), and \(W_i^{(\\geq)}(r), W_i^{(\\leq)}(r), i = \{1, \ldots, 8\}, j = \{1, 2\}\) are \(h_i, h_2, h_3, h_4, h_5, h_6, h_7, h_8 = [h_x, h_x, -h_x, -h_x, h_y, h_y, -h_y, -h_y] .\)

**Proof.** See Appendix B. \(\square\)

**Corollary 1.** The subsets \(\mathcal{X}(r)\) and \(\mathcal{W}(r)\) in (18) and (19) are mutually disjoint, i.e.,

\[
\bigcup_{i=1}^{8} \mathcal{X}_i(r) = \bigcup_{i=1}^{8} \mathcal{W}_i(r) = 0.
\]

Each subset is restricted to a given quadrant as follows:

\[
\begin{align*}
\mathcal{X}_1(r) &\subset \mathcal{Q}_1, \quad \mathcal{X}_2(r) \subset \mathcal{Q}_1, \quad \mathcal{X}_3(r) \subset \mathcal{Q}_1, \quad \mathcal{X}_4(r) \subset \mathcal{Q}_2, \\
\mathcal{X}_5(r) \subset \mathcal{Q}_1, \quad \mathcal{X}_6(r) \subset \mathcal{Q}_2, \quad \mathcal{X}_7(r) \subset \mathcal{Q}_3, \quad \mathcal{X}_8(r) \subset \mathcal{Q}_4,
\end{align*}
\]

**Proof.** The proof is given with Appendix C. \(\square\)

It only remains to complete step 3 of the framework to derive the joint CDF of the distance and angle, given next.

**Theorem 1.** The joint CDF of the distance and angle of random points, uniformly distributed in a rectangle \(\mathcal{R}(o)\), towards a reference point \(u = (u_x, u_y)\), can be written as

\[
F_{R,\Theta}(r, \theta) = \frac{r^2}{2\ell_x \ell_y} \left[ \sum_{i=1}^{8} \mathbb{I}(r < h_i) \left( \min(\theta, \chi_{i,1}^{(\geq)}(r)) - \chi_{i,1}^{(\leq)}(r) \right) + \mathbb{I}(r \geq h_i) \left( \min(\theta, \chi_{i,2}^{(\geq)}(r)) - \chi_{i,2}^{(\leq)}(r) \right) \right] + \sum_{i=1}^{4} \frac{h_i^2}{2\ell_x \ell_y} \mathbb{I}(r \geq h_i) \mathbb{E} \left( \tan(\mu_{i,1}(r)); \min(\theta, \mu_{i,2}(r)) \right) - \sum_{i=1}^{8} \frac{h_i^2}{2\ell_x \ell_y} \mathbb{I}(r \geq h_i) \mathbb{E} \left( \tan^{-1}(\mu_{i,1}(r)); \min(\theta, \mu_{i,2}(r)) \right),
\]

with \(h_i, \chi_{i,1}^{(\geq)}(r), \chi_{i,1}^{(\leq)}(r)\) and \(\mu_{i,1}(r)\) given in Lemma 2 with (20) and (21) for \(i = \{1, \ldots, 8\}, j = \{1, 2\}\).

**Proof.** See Appendix D. \(\square\)

The joint CDF of Theorem 1 is given in closed-form as the sum of 16 simple terms involving compositions of trigonometric and \(\min(x, y)\) functions. With this, we next derive the joint PDF and the marginal distribution for the azimuth angle.

**Corollary 3.** In the settings of Theorem 1, the joint PDF of the distance and angle is given by

\[
f_{R,\Theta}(r, \theta, \phi) = \frac{r^2}{2\ell_x \ell_y} \sum_{i=1}^{8} \mathbb{I}(r < h_i) \mathbb{I} \left( \chi_{i,1}^{(\geq)}(r) \chi_{i,1}^{(\leq)}(r) \right) + \mathbb{I}(r \geq h_i) \mathbb{I} \left( \chi_{i,2}^{(\geq)}(r) \chi_{i,2}^{(\leq)}(r) \right) \end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial \theta} \left( \min(\theta, \chi_{i,1}(r)) - \chi_{i,1}(r) \right) &\mathbb{I}(\theta < \chi_{i,1}(r)) \mathbb{I}(\min(\theta, \chi_{i,2}(r)) > \chi_{i,1}(r)),
\end{align*}
\]

**Proof.** The joint PDF of distance and azimuth angle is computed as \(f_{R,\Theta}(r, \theta) = \frac{\partial^2 F_{R,\Theta}(r, \theta)}{\partial \theta \partial \phi} \). It can be noticed that the derivative of the two summations multiplied by \(\frac{h_i^2}{2\ell_x \ell_y}\) in (24) are 0 since they do not have terms that depend simultaneously on the \(r\) and \(\theta\) variables. The partial derivative with respect to \(\theta\) of the terms \(\min(\theta, \chi_{i,1}(r)) - \chi_{i,1}(r)\) can be written as follows

\[
\frac{\partial}{\partial \theta} \left( \min(\theta, \chi_{i,1}(r)) - \chi_{i,1}(r) \right) = \mathbb{I}(\theta < \chi_{i,1}(r)) \mathbb{I}(\min(\theta, \chi_{i,2}(r)) > \chi_{i,1}(r)).
\]
where it has been expressed the term \(\min(\theta, \chi_{1,2}(r))\) as \(\min(\{x < \chi_{1,2}(r)\} \cup \{x \geq \chi_{1,2}(r)\})\), it has been computed the derivative with respect to \(\theta\) and it has been multiplied by \(\min(\theta, \chi_{1,2}(r))\) as per Definition \[5\]. The superscript \(<\) or \(>\) in \(\chi_{1,1}(r)\) and \(\chi_{1,2}(r)\) has been omitted to refer to both cases. Finally, multiplying by \(\frac{\pi^2}{r^2}\), deriving with respect to \(r\) and manipulating the resulting expression completes the proof. 

\[
\begin{align*}
\chi_{1,1}(r) &= 0, \\
\chi_{1,2}(r) &= \min\left(\frac{h_y^+}{h_x^+}, \frac{h_y^-}{h_x^-}\right), \\
\chi_{2,1}(r) &= \arctan\left(\frac{h_y^-}{h_x^+}\right) + 2\pi, \\
\chi_{2,2}(r) &= 2\pi, \\
\chi_{3,1}(r) &= 2\pi - \arcsin\left(\frac{h_y^-}{h_x^-}\right), \\
\chi_{3,2}(r) &= \arctan\left(\frac{h_y^-}{h_x^-}\right) + \pi, \\
\chi_{4,1}(r) &= \arctan\left(\frac{h_y^+}{h_x^-}\right) + \pi, \\
\chi_{4,2}(r) &= \arcsin\left(\frac{h_y^+}{h_x^-}\right), \\
\chi_{5,1}(r) &= \pi, \\
\chi_{5,2}(r) &= \frac{\pi}{2}, \\
\chi_{6,1}(r) &= \pi - \arcsin\left(\frac{h_y^-}{h_x^-}\right), \\
\chi_{6,2}(r) &= \arctan\left(\frac{h_y^-}{h_x^-}\right) + \pi, \\
\chi_{7,1}(r) &= 2\pi + \arcsin\left(\frac{h_y^-}{h_x^-}\right), \\
\chi_{7,2}(r) &= \arctan\left(\frac{h_y^-}{h_x^-}\right) + 2\pi. \\
\end{align*}
\]

\[
\begin{align*}
\mu_{1,1}(r) &= 0, \\
\mu_{2,1}(r) &= \max\left(2\pi - \arcsin\left(\frac{h_y^-}{h_x^-}\right), \arctan\left(\frac{h_y^-}{h_x^+}\right) + 2\pi\right), \\
\mu_{3,1}(r) &= \pi, \\
\mu_{4,1}(r) &= \max\left(\arctan\left(\frac{h_y^+}{h_x^-}\right) + \pi, \arcsin\left(\frac{h_y^+}{h_x^-}\right)\right), \\
\mu_{5,1}(r) &= \max\left(\arctan\left(\frac{h_y^+}{h_x^-}\right), \arcsin\left(\frac{h_y^+}{h_x^-}\right)\right), \\
\mu_{6,1}(r) &= \pi, \\
\mu_{7,1}(r) &= \max\left(\arctan\left(\frac{h_y^-}{h_x^-}\right) + \pi, \pi - \arcsin\left(\frac{h_y^-}{h_x^-}\right)\right), \\
\mu_{8,1}(r) &= \frac{3\pi}{2}, \\
\mu_{1,2}(r) &= \min\left(\arcsin\left(\frac{h_y^-}{h_x^-}\right), \arctan\left(\frac{h_y^+}{h_x^-}\right)\right), \\
\mu_{2,2}(r) &= 2\pi, \\
\mu_{3,2}(r) &= \min\left(2\pi - \arcsin\left(\frac{h_y^-}{h_x^-}\right), \arctan\left(\frac{h_y^+}{h_x^-}\right) + \pi\right), \\
\mu_{4,2}(r) &= \pi, \\
\mu_{5,2}(r) &= \frac{\pi}{2}, \\
\mu_{6,2}(r) &= \min\left(\pi - \arcsin\left(\frac{h_y^-}{h_x^-}\right), \arctan\left(\frac{h_y^+}{h_x^-}\right) + \pi\right), \\
\mu_{7,2}(r) &= \frac{3\pi}{2}, \\
\mu_{8,2}(r) &= \min\left(2\pi + \arcsin\left(\frac{h_y^-}{h_x^-}\right), \arctan\left(\frac{h_y^+}{h_x^-}\right) + 2\pi\right). \\
\end{align*}
\]

**Corollary 4.** The marginal CDF of the angle between a reference point at \(u = (u_x, u_y)\) and uniformly distributed random points placed within the rectangle \(\mathcal{R}(\alpha)\) is given by

\[
\begin{align*}
F_0(\theta) &= \frac{1}{2\ell_x\ell_y} \left\{ \sum_{i=1}^{4} h_i^2 \tan^{-1}(\epsilon_{1,i}(\theta); \epsilon_{1,2}(\theta)) \\
&\quad - \sum_{i=5}^{8} h_i^2 \tan^{-1}(\epsilon_{1,i}(\theta); \epsilon_{1,2}(\theta)) \right\}, \\
\end{align*}
\]

where \(h_i\) for \(i = 1, \ldots, 8\) is given in Lemma \[2\] and

\[
\begin{align*}
\epsilon_{1,1}(\theta) &= 0, & \epsilon_{1,2}(\theta) &= \min\left(\theta, \arctan\left(\frac{h_y^+}{h_x^-}\right)\right), \\
\epsilon_{2,1}(\theta) &= \arctan\left(\frac{h_y^-}{h_x^+}\right) + 2\pi, & \epsilon_{2,2}(\theta) &= \theta.
\end{align*}
\]
\[
\begin{align*}
\tau_{1,1} &= 0, \\
\tau_{1,2} &= \tan\left(\frac{h_y}{h_x}\right), \\
\tau_{4,1} &= \tan\left(\frac{h_y}{h_x}\right) + \pi, \\
\tau_{4,2} &= \pi, \\
\tau_{5,1} &= \tan\left(\frac{h_y}{h_x}\right), \\
\tau_{5,2} &= \frac{\pi}{2}, \\
\tau_{7,1} &= \tan\left(\frac{h_y}{h_x}\right) + \pi, \\
\tau_{7,2} &= \frac{3\pi}{2}, \\
\tau_{8,1} &= \frac{3\pi}{2}, \\
\tau_{8,2} &= \tan\left(\frac{h_y}{h_x}\right) + 2\pi.
\end{align*}
\]

\( \epsilon_{3,1}(\theta) = \pi, \quad \epsilon_{3,2}(\theta) = \min\left(\theta, \tan\left(\frac{h_y}{h_x}\right) + \pi\right), \)

\( \epsilon_{4,1}(\theta) = \tan\left(\frac{h_y}{h_x}\right) + \pi, \quad \epsilon_{4,2}(\theta) = \min(\theta, \pi), \)

\( \epsilon_{5,1}(\theta) = \tan\left(\frac{h_y}{h_x}\right), \quad \epsilon_{5,2}(\theta) = \min\left(\theta, \frac{\pi}{2}\right), \)

\( \epsilon_{6,1}(\theta) = \frac{\pi}{2}, \quad \epsilon_{6,2}(\theta) = \tan\left(\frac{h_y}{h_x}\right) + \pi, \)

\( \epsilon_{7,1}(\theta) = \tan\left(\frac{h_y}{h_x}\right) + \pi, \quad \epsilon_{7,2}(\theta) = \min\left(\theta, \frac{3\pi}{2}\right), \)

\( \epsilon_{8,1}(\theta) = \frac{3\pi}{2}, \quad \epsilon_{8,2}(\theta) = \tan\left(\frac{h_y}{h_x}\right) + 2\pi. \)

Proof. The marginal CDF is computed as \( F_{\Theta}(\theta) = \lim_{r \to \infty} F_{R, \Theta}(r, \theta). \) It can be shown that the summation multiplied by \( \frac{2}{\pi h_i} \) in \( \frac{2}{\pi h_i} \) is 0 since the terms multiplied by \( \mathbb{1}(r < h_i) \) are 0 due to the indicator function and the terms multiplied by \( \mathbb{1}(r \geq h_i) \) are 0 since the argument of the \( \Theta^* \) operator are negative when \( r \to \infty. \) This can be checked from \( (20) \) since \( \lim_{r \to \infty} \frac{h_i}{r} = \frac{2}{\pi} \) and \( \lim_{r \to \infty} \sinh(\frac{h_i}{r}) = 0 \) and \( \tan(\pi) = 0 \) when \( x < 0. \) Finally, deriving the limit when \( r \to \infty \) of the term \( \mu_{i,1}(r) \) and \( \min(\theta, \mu_{i,2}(r)) \) on the two summations multiplied by \( \frac{h_i^2}{\pi h_i} \) completes the proof.

Corollary 5. The marginal PDF of the azimuth angle, \( \Theta, \) can be expressed as

\[
f_\Theta(\theta) = \frac{1}{2\ell_x \ell_y} \sum_{i=1}^{4} \frac{h_i^2}{\cos^2(\theta)} \left(\tau_{i,1}, \tau_{i,2}\right)(\theta) - \sum_{i=5}^{8} \frac{h_i^2}{\sin^2(\theta)} \left(\tau_{i,1}, \tau_{i,2}\right)(\theta), \tag{28}
\]

where \( h_i \) for \( i = \{1, \ldots, 8\} \) are given in Lemma 2 and the terms \( \tau_{i,j} \) with \( j \in \{1, 2\} \) are given in (29).

Proof. The result is readily obtained from the derivative of \( F_\Theta(\theta), \) tanking into account Definition 4.

B. Extension to 3D Networks: impact of height

For the 3D case, we consider an arbitrary reference point placed at \( u = (u_x, u_y, u_z) \in \mathbb{R}^3 \) and random points \( V = \{V_x, V_y, V_z\} \) within a 2D convex region \( \mathcal{R}(o), \) with center of mass located at the origin. Similarly, the projection (on the xy plane) of random points \( V \) lies in \( \mathcal{R}(o), \) i.e., \( (V_x, V_y) \in \mathcal{R}(o). \) The setting models a scenario where the position of a reference node (e.g., access point) and the height of its antennas are known, whereas the positions of the users are random with uniform distribution, but the height of their antennas is fixed and known. This is a realistic scenario in many applications. For instance in terrestrial networks, the access points (APs) or base stations (BSs) are placed at specific locations; the users are typically modeled with random location but their antennas are assumed to have a deterministic (fixed) height (since it is very similar for all users). In aerial networks such as UAV-based networks, the height of the UVAs and the BSs, which provide the backhaul links, are considered deterministic [39], and this actually is a parameter to optimize in the network design.

We are interested in the joint distribution of the distance, azimuth and zenith angles of the link between \( u \) and \( V, \) given in the next theorem.

Theorem 2. The joint distribution of distance (\( D \)), azimuth (\( \Theta \)) and zenith (\( \Psi \)) angles for the link between a reference point \( u = (u_x, u_y, u_z) \in \mathbb{R}^3 \) and random points \( V = \{V_x, V_y, V_z\} \) uniformly distributed in a convex region \( \mathcal{R}(o) \subset \mathbb{R}^2, \) and deterministic \( v_z, \) is expressed as

\[
F_{D, \Theta, \Psi}(d, \theta, \psi) = \mathbb{1}(d > |u_z - v_z|) \left[ \frac{\pi}{2}, \pi \right](\psi)
\]

\[
F_{R, \Theta}(\sqrt{d^2 - (u_z - v_z)^2}, \theta) - F_{R, \Theta}(\sqrt{(u_z - v_z)^2}, \theta)
\]

\[
+ \mathbb{1}_{[0, \frac{\pi}{2}])(\psi) F_{R, \Theta}(\sqrt{d^2 - (u_z - v_z)^2}, \theta)
\]

\[
+ \mathbb{1}(d > |u_z - v_z|) \left( \mathbb{1}_{[0, \frac{\pi}{2}])(\psi) F_{R, \Theta}(\min(\sqrt{d^2 - (u_z - v_z)^2}, (v_z - u_z) \tan(\psi)), \theta) \right), \tag{30}
\]

where \( F_{R, \Theta}(r, \theta) \) is the joint distribution of distance and azimuth angle in the xy plane, given in Theorem 1.

Proof. The proof is given in Appendix E.
statistics relating to the angular domain are also instrumental. In particular, the joint distribution of azimuth and zenith angles can be exploited for the design of analog codebooks. We characterize this distribution in the following two corollaries, providing the joint CDF and PDF.

**Corollary 6.** The joint CDF of azimuth ($\Theta$) and zenith ($\Psi$) angles is given by

\[
F_{\Theta, \Psi}(\theta, \psi) = \mathbb{I}(u_z \geq v_z) \mathbb{I}\left(\psi \in \left[0, \frac{\pi}{2}\right]\right) \left(F_{\Theta}(\theta) - F_{\Theta}((u_z - v_z) \tan(\pi - \psi), \theta)\right) + \mathbb{I}(v_z < u_z) \mathbb{I}\left(\psi \in \left[\frac{\pi}{2}, \pi\right]\right) F_{\Theta, \Psi}(\theta, \psi).
\]

**Proof.** The result is obtained by computing the limit $F_{\Theta, \Psi}(\theta, \psi) = \lim_{\delta \to 0} F_{D, \Theta, \Psi}(\theta, \psi, \delta)$, with $F_{D, \Theta, \Psi}(\theta, \psi, \delta)$ given in Theorem 2. \hfill \Box

**Corollary 7.** The joint PDF of azimuth ($\Theta$) and zenith ($\Psi$) angles is given by

\[
\begin{align*}
&f_{\Theta, \Psi}(\theta, \psi) = \mathbb{I}(u_z \geq v_z) \left(\frac{v_z - u_z}{\cos^2(\psi)}\right) f_{\Theta}(\theta, \psi) \\
&f_{\Theta, \Psi}(v_z - u_z) \tan(\pi - \psi), \theta) - \mathbb{I}(u_z \geq v_z) \left(\frac{v_z - u_z}{\cos^2(\psi)}\right) f_{\Theta, \Psi}(\theta, \psi) \\
&f_{\Theta, \Psi}(u_z - v_z) \tan(\pi - \psi), \theta) \left(\frac{v_z - u_z}{\cos^2(\psi)}\right) f_{\Theta}(\theta, \psi).
\end{align*}
\]

**Proof.** The result is obtained from the partial derivatives of the joint CDF (given in Corollary 6) with respect to the azimuth and zenith angles, i.e., $f_{\Theta, \Psi}(\theta, \psi) = \frac{\partial^2 F_{\Theta, \Psi}(\theta, \psi)}{\partial \Theta \partial \Psi}$.

**IV. APPLICATIONS**

The proposed mathematical framework allows for modeling a wide number of scenarios of practical relevance in wireless communications. As illustrated in Fig. 5, a rectangle with $L_x \gg L_y$ can model the road to consider the case of vehicular communications. In this case, a mounted AP can be considered as the reference node (placed at $u$) that communicates with vehicles randomly located along the road. The case of indoor WiFi-based communications or outdoor cellular communications can also be considered. For terrestrial communications, e.g., a BS with given antenna height at a reference location a hot-spot of users with smaller antenna height; in indoor scenarios, the reference node could be an AP mounted on the ceiling. Aerial-to-terrestrial communications, e.g., based on UAVs can also be considered. For instance, a BS located at a reference position $u$, giving backhaul access to a network of UAVs flying at a given altitude, much higher than the BS.

A rectangular region is an appropriate modeling choice for these aforementioned scenarios, where our results might be directly applied. More specifically, the joint distance and angle distribution given in Theorem 1 and 2 are needed to compute the distribution of the SNR when users are placed within a finite area and directional radiation patterns are used. In the general 3D case, the SNR for the link between a transmit node placed at $u$ and a randomly located node can be expressed as

\[
\text{SNR} = \frac{g_t(\Theta_r, \Psi_r) g_r(\Theta_r, \Psi_r) (\tau D)^\alpha |\beta|^2 \rho_r}{N_0},
\]

where $\Theta_r, \Psi_r$, and $\Theta_r, \Psi_r$ are the transmit and receive azimuth and zenith angles, respectively; $g_t(\cdot)$ and $g_r(\cdot)$ represent the transmit and receive antenna gains (radiation patterns); $|\beta|$ is the fast-fading amplitude; $\tau, \alpha$ are the path loss slope and exponent; and $\rho_r, N_0$ are the transmit power spectral density (PSD) and noise PSD, respectively. If we consider for instance the downlink with a BS placed at $u$ and random nodes, the joint distribution of Theorem 2 would model the transmit angles $\Theta_r, \Psi_r$ and the distance $D$, whereas the receive angles, $\Theta_r, \Psi_r$, would be obtained from the transmit angles after simple trigonometric transformations. Importantly, our results do not make any assumption on the radiation patterns, as opposed to previous works restricted to a sector model, e.g., [28], or assuming perfect beam alignment, e.g., [27, 29]. Our results hold for any radiation patterns $g_t(\cdot)$ and $g_r(\cdot)$, which can be related either to single-element antennas e.g., horn antennas, [40], or antenna arrays, e.g., uniform planar arrays (UPAs). For the latter, to compute the SNR in (33), the product of radiation patterns $g_t(\cdot)g_r(\cdot)$ should be replaced by the product of array response vectors, $|a_t(\Theta_r, \Psi_r)|^2 |a_r(\Theta_r, \Psi_r)|^2$. Therefore, our results can be applied to the analysis of emerging techniques related to directional beamforming, such as beam management procedures in 5G and beyond, and the analysis of RIS-empowered networks.

Moreover, our results can be applied to the design and optimization of new emerging techniques. In the 5G 3GPP New Radio (NR) standard, the optimal transmit beam is determined by a process that includes beam-sweeping and beam-refinement, using a pre-defined set of $m$ analog beams that form the analog codebook [45, 46]. The angular distribution of the users, as per Corollary 7, can be exploited to design the optimal set of $m$ beams. Finally, as another example, our results can be applied to the design of wireless routing, where the marginal distribution of azimuth angles of Corollary 5 can be used. In essence, wireless routing aims to transmit a message between two nodes $A$ and $B$ in a wireless multi-hop network [30]. The origin and destination nodes cannot communicate directly due to the limited transmit power that establishes a maximum communicating range, $r_{\text{max}}$. The problem is to find the optimal path that minimizes the number of hops. In this scenario, for each node and its given location $u$, the marginal PDF of azimuth angles can be used to find the optimal transmit direction towards the next node.
TABLE I: Main Parameters

| Scenario | Shape $(\ell_x, \ell_y)$ | Reference location $u$ | Height $v_z$ |
|----------|--------------------------|------------------------|-------------|
| $O$      | $\ell_x = 200, \ell_y = 100$ m | $u = (30, 25, 10)$ m | $v_z = 1.5$ m |
| $A$      | $\ell_x = 200, \ell_y = 9.75$ m | $u = (0, \frac{\sqrt{2}}{2})$ | - |
| $B$      | $\ell_x = 3, \ell_y = 5$ m | $u = (0.5, 1.25, 3)$ m | $v_z = 1.5$ m |
| $C$      | $\ell_x = 200, \ell_y = 100$ m | $u = (30, 25, 10)$ m | $v_z = 120$ m |

V. NUMERICAL RESULTS

A. Validation

We now evaluate the theoretical expressions previously derived for the cases of four exemplary regions that model different scenarios as summarized in Table I. The shape of the region of scenario $O$ in Table I which is drawn in Fig. 4 matches with a typical public square in many cities, (e.g., the Dam Square in Amsterdam). A BS placed at $(30, 25)$ with an antenna height of $10$ m is considered. The scenario $A$ in Table I models a road segment of $200$ m with $3$ lanes of $3.25$ m wide each, and a road side unit (RSU) located at the edge of the road on the center of the segment. An indoor office scenario is considered in $B$, with a rectangular room of $3 \times 5$ meters and an AP placed on the ceiling at a height of $3$ m. Lastly, scenario $C$ models a UAV-based network with drones flying at a height of $120$ m, that communicate with a BS with an antenna height of $10$ m.

Throughout this section, theoretical results are validated and double-checked with Monte Carlo (MC) simulation results. The empirical distributions used in this section have been estimated using $10^5$ realizations of random points. We first consider a 2D scenario (A in Table I), for which we represent the joint CDF of the distance on the $xy$ plane, $R$, and the azimuth angle, $\Theta$, which is given by Theorem 1. We particularize the expression for a set of three angles per quadrant for the variable $\theta = \theta_0$ (see Fig. 5) for the first and second quadrants, and Fig. 6 for the third and fourth quadrants, which constitute an appropriate validation set. We observe a perfect match between simulation and theoretical results.

For the 3D case, the joint CDF of distance, $D$, azimuth and zenith angles, $\Theta, \Psi$ as per Theorem 2 is validated with Fig. 7 and Fig. 8. In this case, six azimuth angle values $\theta = \theta_0$ within the first and fourth quadrants are considered. It is important to remark that, while the azimuth angle ranges from $0$ to $2\pi$, the zenith angles range from $\psi_{\text{min}}$ to $\pi$ if the height of the reference node is greater than that of the random nodes, whereas such angles range from $0$ to $\psi_{\text{max}}$ otherwise. Both maximum and minimum angles are given by (57) as $\psi_{\text{min}} = \pi - \tan(R_{\text{max}}/(u_z - v_z))$, and $\psi_{\text{max}} = \tan(R_{\text{max}}/(v_z - u_z))$, where $R_{\text{max}}$ is the distance between the reference point, $u$, and the farthest vertex of the region. Thus, in the case of scenario $O$ we have $\psi_{\text{min}} = 93.2^\circ$. We have chosen two zenith values equally spaced within the range of $\Psi \in (\pi, \psi_{\text{min}})$ to validate the expression of Theorem 2 with Fig. 7 and Fig. 8. As expected, it is observed a perfect match between theoretical and simulation results.

B. Reference Scenarios

After validating the 2D and 3D distributions in the previous subsection, we now pay special attention to the scenarios defined in Table I. Fig. 9 illustrates scenarios $A$, $B$ and $C$ as per Table I. Scenario $A$ represents a road segment of $200$ m with $3$ lanes of $3.25$ m each, with a RSU placed at the edge of the road, as shown in Fig. 9(a). The joint PDF of distance and azimuth angle, which is given with Corollary 3 is evaluated numerically for such a case and illustrated.
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Fig. 7: Joint CDF of distance, $D$, azimuth and zenith angles, $\Theta$, $\Psi$, particularized for azimuth angles within the first quadrant. Theoretical results are represented with solid lines; markers denote MC simulations.

Fig. 8: Joint CDF of distance, $D$, azimuth and zenith angles, $\Theta$, $\Psi$, particularized for azimuth angles within the fourth quadrant. Theoretical results are represented with solid lines; markers denote MC simulations.

Fig. 9: Sketch of sample scenarios A, B, C as per Table I: (a) 2D scenario which models a road with 3 lanes of 3.25 m wide each, and a road side unit located at the edge of the road; (b) indoor office scenario, with a room of $3 \times 5$ meters, and an AP placed on the ceiling at a height of 3 m; (c) UAV-based network, with drones flying at a height of 120 m, that communicate with a BS with an antenna height of 10 m. The reference node is drawn in black, whereas the random nodes are drawn in blue.

Fig. 10: Joint PDF of distance on the $xy$ plane, $R$, and azimuth angle, $\Theta$, for scenario A.

with Fig. [10] It is observed how the shape of the region and the location of the arbitrary node determine the joint distribution. In addition, it is observed the high correlation between the distance and azimuth angle. Longer distances, $R$, are associated with azimuth angles close to either $\pi$ or $2\pi$ radians, since those directions point to the two extremes of the road segment. The minimum distance values of $R$ are related to azimuth angles around $3\pi/2$ radians, since in this direction the reference node points to the perpendicular direction of the road segment, which has a width of only 9.75 m. It is also observed that the joint PDF is null for azimuth angles between 0 and $\pi$ radians since those directions point outside the road, and thus they are forbidden locations for the vehicles (i.e., random nodes). Therefore, this joint PDF identifies the direction (or azimuth angles) where a higher user density is expected, and this information can be used to the design of the beam patterns in such scenario.

The joint CDF of azimuth, $\Theta$, and zenith, $\Psi$, angles, which is given with Corollary [7] is illustrated with Fig. [11] for scenario B. This scenario, which is shown in Fig. [9](b), models an office room of $3 \times 5$ m, with an AP placed on the ceiling at a height of 3 m. The geometry of this scenario sets a range of the zenith angles as $\Psi \in (0.63\pi, \pi)$ radians, which is observed in Fig. [11]. It is observed that the highest node density is obtained for
\( \theta = 1.34\pi, \) and \( \psi = 0.66\pi \) radians. Hence, this is the direction where it is most probable to find users, and such information can be exploited for beam design and wireless routing as identified in section IV.

Lastly, scenario C (cfr. Fig. 9(c)) represents a UAV-based network, with drones flying at a height of 120 m, that communicate with a BS with an antenna height of 10 m. Here, the random nodes are located at a higher distance than the reference node, which is the BS, and thus the range of the zenith angle is \( \Psi \in (0, \psi_{\max}) \), with \( \psi_{\max} = 0.26\pi \) radians. This is observed in Fig. 12, which represents the joint PDF of azimuth and zenith angles for this scenario. In this scenario the direction of maximum node density is given by the pair on angles, \( \theta = 1.16\pi, \psi = 0.26\pi \) radians. On the other hand, it is observed that the azimuth direction where the user density is minimal is around \( \theta = \pi/2 \), which is due to the fact that the reference node is placed close to the edge of the region where the drones are placed on the positive y axis direction.

All these results (Figs. 11 through 12) highlight that there exists a correlation between the distance, azimuth and zenith angles that depend on the shape of the region and the location of the reference node. The next subsection firstly investigates the effect of the shape of the region on the marginal PDF of the azimuth angle, and then the location of the reference node.

### C. Impact of shape and location on the distribution

Fig. 13 shows the marginal PDF of the azimuth angle, which is given with Corollary 4 for a reference node that is being placed at the center of mass of its region. This figure investigates the effect of a modification in the shape of the region. For this purpose, the marginal PDF is evaluated for different lengths of its side \( \ell_y \) on the y axis, while the other side, \( \ell_x \), remains constant. The PDF of a uniform distribution, which is the classical assumption for the user distribution in the literature, is also included for comparison. It is observed that when the shape is regular, i.e., a square, and the reference node is located at the center of mass, the marginal PDF approximates a uniform distribution. Nevertheless, as the shape becomes more irregular, i.e., one side becomes greater than the other one, the marginal PDF largely differs from the uniform distribution.

As the \( \ell_y \) side is reduced, the node density increases in the positive and negative directions of the x axis, which lead to a greater node density for azimuth angles close to \( \theta = 0, \theta = 2\pi, \) and \( \theta = \pi \) radians.

Finally, the effect of the node location is investigated in Fig. 14 for an square region with sides of 200 m. Initially, a
reference node placed at the center of mass is considered, and then its position is modified to approach the edge of the region in the positive y axis direction. Again, the uniform distribution is included for comparison purposes. Again, it is observed that the distribution becomes more different to the uniform distribution as the node moves away from the center of mass. Besides of this, as the reference node approaches the edge in the positive y axis direction, the node density increases in the direction of negative y axis, i.e., $\theta = 3\pi/2$, while it reduces in the direction of positive y axis, i.e., $\theta = \pi/2$ radians. In the extreme case where the reference node is placed at the edge of the region, i.e., $(0, 100)$, the marginal PDF is null for angles in the range $(0, \pi)$, since those directions point outside the region.

VI. Conclusion

In this paper, we have proven that in 2D networks there exists a non-negligible correlation between the distance and azimuth angle that depends on the shape of the region where the random nodes are distributed, and also on the location of the reference node. We have first proposed a mathematical framework for the analysis of such joint distribution in arbitrarily-shaped regions. Then, we have particularized this framework for the relevant case of a rectangular region, which can model many practical scenarios of interest where finite networks play an important role. We extended their results to consider the 3D case where the zenith angle must be considered jointly to the distance and azimuth angle. To illustrate the importance of the proposed framework, a number of relevant applications are identified, such as the analysis of directional beamforming, the design of analog codebooks or wireless routing algorithms. Finally, we have presented some numerical results to validate the theoretical expressions, and to shed light on the dependencies between the RVs under consideration and the effect of the shape of the region and location of the reference node.

APPENDIX A

Proof of Lemma 1

The rectangular region can be expressed in Cartesian coordinates as follows

$$\mathcal{R}(-u) = \{(x, y) \in \mathbb{R}^2 \mid -\frac{\ell_x}{2} - u_x \leq x \leq \frac{\ell_x}{2} - u_x \}
\land \quad -\frac{\ell_y}{2} - u_y \leq y \leq \frac{\ell_y}{2} - u_y \}. \quad (34)$$

This set can be written in polar coordinates using (4) in the form of (13), where the condition $\mathcal{B}(\rho, \phi)$ is expressed as

$$\mathcal{B}(\rho, \phi) = -\frac{\ell_x}{2} - u_x \leq \rho \cos(\phi) \leq \frac{\ell_x}{2} - u_x
\land \quad -\frac{\ell_y}{2} - u_y \leq \rho \sin(\phi) \leq \frac{\ell_y}{2} - u_y. \quad (35)$$

Then, we isolate the variable $\rho$ from $\mathcal{B}(\rho, \phi)$ as follows

$$\begin{align*}
\left(\cos(\phi) \geq 0\right) \land \left(\rho \leq \frac{h_x^+}{\cos(\phi)} \right) \lor \left(\mathcal{A}_{c}(\phi) \land \left(\rho \leq \frac{h_x^-}{\cos(\phi)} \right) \right) \\
\land \\
\left(\sin(\phi) \geq 0\right) \land \left(\rho \leq \frac{h_y^+}{\sin(\phi)} \right) \lor \left(\mathcal{A}_{s}(\phi) \land \left(\rho \leq \frac{h_y^-}{\sin(\phi)} \right) \right)
\end{align*}$$

(36)

after some manipulations over the inequalities in (35). It should be noticed that each of the former inequalities yields two inequalities but one of them is always true, and thus it is discarded. Then, applying the distributive and associative properties of the logical operators leads to

$$\mathcal{B}(\rho, \phi) = \mathcal{A}_{c}(\phi) \land \mathcal{A}_{s}(\phi) \land \mathcal{A}_{c+}(\rho, \phi) \land \mathcal{A}_{s+}(\rho, \phi)
= \phi \in \mathcal{D}_{i}
\lor \mathcal{A}_{c}(\phi) \land \mathcal{A}_{s}(\phi) \land \mathcal{A}_{c+}(\rho, \phi) \land \mathcal{A}_{s+}(\rho, \phi)
= \phi \in \mathcal{D}_{i}
\lor \mathcal{A}_{c}(\phi) \land \mathcal{A}_{s}(\phi) \land \mathcal{A}_{c+}(\rho, \phi) \land \mathcal{A}_{s+}(\rho, \phi)
= \phi \in \mathcal{D}_{i}
\lor \mathcal{A}_{c}(\phi) \land \mathcal{A}_{s}(\phi) \land \mathcal{A}_{c+}(\rho, \phi) \land \mathcal{A}_{s+}(\rho, \phi). \quad (37)$$

It is identified that the and operation of the logical expressions $\mathcal{A}_{c}(\phi)$ and $\mathcal{A}_{s}(\phi)$ and its negations is equivalent to checking whether the angle $\phi$ belongs to each of the angular quadrants. For instance, the first term can be expressed as follows: $\mathcal{A}_{c}(\phi) \land \mathcal{A}_{s}(\phi) = \phi \in \mathcal{D}_{i}$. The remaining terms are manipulated as follows

$$\mathcal{A}_{c+}(\rho, \phi) \land \mathcal{A}_{s+}(\rho, \phi) = \rho \leq \frac{h_x^+}{\cos(\phi)} \land \rho \leq \frac{h_y^+}{\sin(\phi)}
= \rho \leq \min\left\{\frac{h_x^+}{\cos(\phi)}, \frac{h_y^+}{\sin(\phi)}\right\}. \quad (38)$$

The terms $\mathcal{A}_{c+}(\rho, \phi) \land \mathcal{A}_{s+}(\rho, \phi)$ lead to similar expressions to (38) but using different combinations of the parameters $h_x^+, h_y^+, h_x^-, h_y^-$. Details are omitted for the sake of compactness. Leveraging this fact and substituting the functions $h_x(\phi)$ and $h_y(\phi)$ in the above expression completes the proof.

APPENDIX B

Proof of Lemma 2

The boolean expression $\mathcal{B}(r, \phi) = r \leq \beta(\phi)$ can be expressed as

$$\begin{align*}
\mathcal{B}(r, \phi) &\overset{(a)}{=} r \leq \left(\frac{h_x(\phi)}{\cos(\phi)} \mathbb{1}(C(\phi)) + \frac{h_y(\phi)}{\sin(\phi)} \mathbb{1}(C(\phi)) \right) \\
&\overset{(b)}{=} \left(\frac{h_x(\phi)}{\cos(\phi)} \geq r \right) \lor \left(\frac{h_y(\phi)}{\sin(\phi)} \leq r \right) \land \mathbb{1}(C(\phi)),
\end{align*}$$

(39)
where (a) comes after expressing the \( \min(x, y) \) function has the sum of two indicator functions with \( C(\phi) = \frac{h_x(\phi)}{\cos(\phi)} \leq \frac{h_y(\phi)}{\sin(\phi)} \), and (b) after reordering and applying the inequality in both terms. Below, we isolate \( \phi \) in the boolean expressions \( C(\phi), D(\phi), E(\phi) \). The term \( C(\phi) \) can be written as

\[
C(\phi) = \left( \tan(\phi) \leq \frac{h_y(\phi)}{h_x(\phi)} \right) \land \mathcal{A}_1(\phi) \land \mathcal{C}_1(\phi) \lor \left( \tan(\phi) \geq \frac{h_x(\phi)}{h_y(\phi)} \right) \land \mathcal{A}_2(\phi) \land \mathcal{C}_2(\phi),
\]

where the expression of \( C(\phi) \) has been manipulated to group the \( \sin(x) \) and \( \cos(x) \) as a tangent function, and this latter term is isolated by splitting the inequality into the cases where the tangent is positive (\( \mathcal{A}_1(\phi) = 1 \)) and negative (\( \mathcal{A}_1(\phi) = 0 \)). Thus, \( \mathcal{A}_1(\phi) = \psi(\phi) \in \mathcal{S}_1 \cup \mathcal{S}_3 \) and \( \mathcal{A}_1(\phi) = \phi \in \mathcal{S}_2 \cup \mathcal{S}_4 \). The solution of the inequality \( \mathcal{F}_1(\phi) \) is \( \phi \in \left\{-\frac{\pi}{2} + \pi n, \tan\left(\frac{h_y(\phi)}{h_x(\phi)}\right) + n\pi, n \in \mathbb{Z} \right\} \), whereas the solution of \( \mathcal{F}_2(\phi) \) is \( \phi \in \left\{\tan\left(\frac{h_x(\phi)}{h_y(\phi)}\right) + n\pi, \frac{\pi}{2} + n\pi, n \in \mathbb{Z} \right\} \). Thus, the intersection of these solutions with the intervals that define \( \mathcal{A}_1(\phi) \) and \( \mathcal{A}_2(\phi) \) leads to

\[
C_1(\phi) = \phi \in \left[0, \tan\left(\frac{h_y(\phi)}{h_x(\phi)}\right)\right] \lor \left(\pi, \tan\left(\frac{h_x(\phi)}{h_y(\phi)}\right)\right),
\]

\[
C_2(\phi) = \phi \in \left[\tan\left(\frac{h_x(\phi)}{h_y(\phi)}\right), \frac{\pi}{2}\right] \lor \left(\tan\left(\frac{h_x(\phi)}{h_y(\phi)}\right), 2\pi\right) \lor \left(\pi, \tan\left(\frac{h_y(\phi)}{h_x(\phi)}\right)\right) + 2\pi, 2\pi \right),
\]

Similarly, \( \tilde{C}(\phi) \) can be expressed as follows

\[
\tilde{C}(\phi) = \frac{h_x(\phi)}{\cos(\phi)} > \frac{h_y(\phi)}{\sin(\phi)} = C_3(\phi) \lor C_4(\phi),
\]

with

\[
C_3(\phi) = \phi \in \left[\pi, \tan\left(\frac{h_x(\phi)}{h_y(\phi)}\right)\right],
\]

\[
C_4(\phi) = \phi \in \left[\pi, \tan\left(\frac{h_y(\phi)}{h_x(\phi)}\right)\right] \lor \left(\pi, \tan\left(\frac{h_x(\phi)}{h_y(\phi)}\right)\right) + 2\pi, 2\pi \right) \lor \left(\pi, \tan\left(\frac{h_y(\phi)}{h_x(\phi)}\right)\right) + 2\pi, 2\pi \right).
\]

We now isolate \( \phi \) on \( D(\phi) \) as follows

\[
D(\phi) = \left(\cos(\phi) \leq \frac{h_x(\phi)}{\cos(\phi)} \right) \land \mathcal{A}_1(\phi) \lor \left(\cos(\phi) \geq \frac{h_x(\phi)}{\sin(\phi)} \right) \land \mathcal{A}_2(\phi).
\]

Solving the two inequalities leads to

\[
D_1(\phi) = (r < h_x) \land \mathcal{A}_1(\phi) \lor (r \geq h_x)
\]

\[
\land \phi \in \left[\cos\left(\frac{h_x(\phi)}{r}\right) + \frac{\pi}{2}, \pi \right \lor \left(\frac{3\pi}{2}, 2\pi - \cos\left(\frac{h_x(\phi)}{r}\right)\right),
\]

\[
D_2(\phi) = (r < -h_x) \land \mathcal{A}_2(\phi) \lor (r \geq -h_x)
\]

\[
\land \phi \in \left[\frac{\pi}{2}, \cos\left(\frac{h_x(\phi)}{r}\right)\right] \lor \left(2\pi - \cos\left(\frac{h_x(\phi)}{r}\right), \frac{3\pi}{2}\right).
\]

Analogously, \( E(\phi) \) can be written as

\[
E(\phi) = \left(\sin(\phi) \leq \frac{h_x(\phi)}{r} \right) \land \mathcal{A}_3(\phi) \lor \left(\sin(\phi) > \frac{h_x(\phi)}{r} \right) \land \mathcal{A}_4(\phi).
\]

Then, we solve the above inequalities which yields to

\[
E_1(\phi) = (r < h_y) \land \mathcal{A}_1(\phi) \lor (r > h_y)
\]

\[
\land \phi \in \left[0, \cos\left(\frac{h_x(\phi)}{r}\right)\right \lor \left(\frac{3\pi}{2}, 2\pi - \cos\left(\frac{h_x(\phi)}{r}\right)\right),
\]

\[
E_2(\phi) = (r < -h_y) \land \mathcal{A}_3(\phi) \lor (r \geq -h_y)
\]

\[
\land \phi \in \left[2\pi + \cos\left(\frac{h_x(\phi)}{r}\right), 2\pi \right \lor \left(\frac{3\pi}{2}, \pi - \sin\left(\frac{h_x(\phi)}{r}\right)\right).
\]

Now that we have written the boolean expressions \( C(\phi), \tilde{C}(\phi), D(\phi), E(\phi), \) as the \( \lor \) of two boolean expressions, we can write \( B(\phi) \) as follows

\[
B(\phi) = \tilde{C}(\phi) \land D(\phi) \lor C(\phi) \land D(\phi) \lor C(\phi) \land \tilde{C}(\phi) \land D(\phi).
\]

where expressions of \( C(\phi), \tilde{C}(\phi), D(\phi), E(\phi), \) and \( \tilde{E}(\phi) \) in \( (59) \) are used, and the distributive and associative properties of the logical operators are applied. Now, we identify in \( (48) \) the terms ranging from \( X_1(\phi) \) up to \( X_8(\phi) \). This terms, \( X_i(\phi) \in [1, 8] \subset \mathbb{Z} \), are related to the regions \( X_i(\phi) \in [1, 8] \subset \mathbb{Z} \) given in \( (18) \) and \( (20) \) as follows:

\[
X_i(\phi) = \{ \phi \in [0, 2\pi] \mid X_i(\phi) = 1 \}.
\]

Finally, substituting \( (41), (43), (45) \), and \( (47) \) in \( (48) \) leads to the and operation of 8 terms with the following form:

\[
X_i(\phi) = (r < h_i) \land \mathcal{A}_i(\phi) \lor (r \geq h_i) \land \mathcal{A}_i(\phi)
\]

\[
\lor (r \geq h_i) \land \mathcal{A}_i(\phi) \lor X_i(\phi).
\]

Identifying the resulting terms \( h_i, X_i(\phi) \), \( X_i(\phi) \), \( X_i(\phi) \), and \( X_i(\phi) \) on each of the 8 expressions completes the proof for \( M_{\phi}(r) \). Following a similar approach for \( M_{\phi}(r) \), we can write \( B(\phi) \) as

\[
B(\phi) = \tilde{C}(\phi) \land \mathcal{C}(\phi) \lor E(\phi) \land \tilde{C}(\phi),
\]

with \( D(\phi) = D_3(\phi) \lor D_4(\phi) \) and \( E(\phi) = E_3(\phi) \lor E_4(\phi) \). Then, it can be shown that \( B(\phi) \) can be expressed as

\[
B(\phi) = D_3(\phi) \land \mathcal{C}(\phi) \lor \ldots \lor E_4(\phi) \land \mathcal{C}_4(\phi),
\]

where \( M_{\phi}(r) = \{ \phi \in [0, 2\pi] \mid M_i(\phi) = 1 \} \), where the details are omitted due to space limitations.
APPENDIX C
PROOF OF COROLLARY 2

The proof comes after realizing that the boolean expression $C_1(\phi)$ and $C_2(\phi)$, which are given in (41), are restricted to the events $A_1(\phi)$ and $A_2(\phi)$, respectively; the expressions $D_1(\phi)$ and $D_2(\phi)$ from (45) to the events $A_3(\phi)$ and $A_4(\phi)$; $E_1(\phi)$ and $E_2(\phi)$ from (47) to the events $A_3(\phi)$ and $A_4(\phi)$; and finally $C_1(\phi)$ and $C_2(\phi)$ from (43) to $A_4(\phi)$ and $A_2(\phi)$. Expressing these events as intervals of the angle $\phi$ and substituting on each of the boolean expressions $X_i(r, \phi)$ in (48) completes the proof. The same process is followed for the sets $M_i(r)$ with $i \in \{1, 2, 3, 4\}$.

APPENDIX D
PROOF OF THEOREM 1

Using Lemma 1 and 2, the overlap area given by (12) can be written as follows:

$$|3(u, r, \theta)| = \frac{r^2}{2} \int_{\phi=0}^{\theta} \mathbb{1}_{\Xi(\phi)}(\phi) \, d\phi$$

(53)

$$+ \int_{\phi=0}^{\theta} h_2^2(\phi) \cos^2(\phi) \mathbb{1}(C(\phi) \land \tilde{B}(r, \phi)) \, d\phi$$

(54)

$$+ \int_{\phi=0}^{\theta} h_2^2(\phi) \sin^2(\phi) \mathbb{1}(C(\phi) \land \tilde{B}(r, \phi)) \, d\phi,$$

where the equality $\mathbb{1}(B(\phi) = \mathbb{1}_{\Xi(\phi)}(\phi)$ is used, the term $\beta(\phi)$ in (12) is expressed as the sum of two indicator functions, and after some manipulations. The term $\mathbb{1}(C(\phi) \land \tilde{B}(r, \phi))$ can be manipulated as follows:

$$\mathbb{1}(C(\phi) \land \tilde{B}(r, \phi))$$

(55)

$$= \mathbb{1}(C(\phi) \land (\tilde{D}(r, \phi) \land C(\phi) \lor \tilde{E}(r, \phi) \lor \tilde{C}(\phi)))$$

(56)

$$\mathbb{1}(D_3(r, \phi) \lor D_4(r, \phi)) \land (C_1(\phi) \lor C_2(\phi)))$$

where (a) comes after expressing $\tilde{B}(r, \phi)$ as $\tilde{D}(r, \phi) \land C(\phi) \lor \tilde{E}(r, \phi) \lor \tilde{C}(\phi)$; (b) after applying distributive, associative and absorption properties of the logical operators; and (c) after some manipulations, applying Corollary 2 and identifying the terms $D_k(r, \phi) \land C_j(\phi) = M_{ij}$ with $k \in \{3, 4\}$, $j \in \{1, 2\}$ and $i \in \{1, 2, 3, 4\}$.

Analogously, the term $\mathbb{1}(\tilde{C}(\phi) \land \tilde{B}(r, \phi))$ can be written as:

$$\mathbb{1}(\tilde{C}(\phi) \land \tilde{B}(r, \phi))$$

(57)

$$= \sum_{i=1}^{8} \mathbb{1}(M_{ij}(r, \phi)).$$

Finally, substituting (56) and (57) in (55) and (10) and applying Lemma 2, Proposition 1, and Corollary 2 completes the proof after some additional manipulations.

APPENDIX E
PROOF OF THEOREM 2

In this proof, we also apply the transition invariant property of the regions considered, and thus we assume that all the points are translated by $-u \in \mathbb{R}^2$, which involves that the reference point after translation is placed at the origin.

Then, the first step of the proof is to identify the relation between the random variables in polar and spherical coordinates. It can be proven that: (i) the distance, $D$, and the zenith angle, $\Psi$, in spherical coordinates are expressed as two functions that depend on the distance, $R$ in the $xy$ plane and the height of the nodes $u_z$ and $v_z$; and ii) the azimuth angle is equivalent in spherical and polar coordinates.

Thus, the determination of the distribution of distance, azimuth and zenith angles can be posed as a standard RV transformation problem from the polar coordinates, $R$, and $\Theta$ of the 2D case. With this approach, the RV transformation is written as follows:

$$D = f(R, u_z, v_z) = \sqrt{R^2 + (u_z - v_z)^2},$$

(58)

$$\Theta = \Theta,$$

(59)

$$\Psi = g(R, u_z, v_z) = \begin{cases} \pi - \arctan\left(\frac{R}{u_z - v_z}\right) & \text{if } u_z \geq v_z, \\ \arctan\left(\frac{R}{u_z - v_z}\right) & \text{if } u_z < v_z, \end{cases}$$

Hence, the CDF of the distance and angle distribution can be expressed as follows

$$F_{D, \bar{\Theta}, \bar{\Psi}}(d, \bar{\Theta}, \bar{\Psi}) = \int_{d' \geq d} \int_{\Theta' \geq \Theta} \int_{\Psi' \geq \Psi} f_{D, \bar{\Theta}, \bar{\Psi}}(d', \Theta', \Psi') \, dr \, d\Theta \, d\Psi.$$  

The next step is to isolate the variable $R$ on the Boolean expressions $F(d')$ and $G(\psi)$, which yield to the following inequalities

$$F(d') = d' > |u_z - v_z| \wedge R \leq \sqrt{d^2 - (u_z - v_z)^2},$$

(59)

$$G(\psi) = (u_z \geq v_z) \wedge \left( R \leq (u_z - v_z) \tan(\pi - \psi) \wedge \psi \in \left[\frac{\pi}{2}, \pi\right] \right.$$

(60)

$$\lor (u_z < v_z) \wedge \left( R \leq (v_z - u_z) \tan(\psi) \wedge \psi \in \left[0, \frac{\pi}{2}\right] \right).$$

Finally, substituting (59) and (60) into (58), applying the indicator functions over the limits of the integrals, and identifying the resulting expression in terms of the joint CDF of distance and azimuth angle completes the proof.

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