Solving Time-Space-Fractional Cauchy Problem with Constant Coefficients by Finite-Difference Method

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Abstract In this chapter, we present the time-space-fractional Cauchy equation with constant coefficients, the space and time-fractional derivative are described in the Riemann-Liouville sense and Caputo sense, respectively. The implicit scheme is introduced to solve time-space-fractional Cauchy problem in a matrix form by utilising fractionally Grünwald formulas for discretization of Riemann-Liouville fractional integral, and L1-algorithm for the discretization of time-Caputo fractional derivative, additionally, we provided a proof of the von Neuman type stability analysis for the fractional Cauchy equation of fractional order. Several numerical examples are introduced to illustrate the behaviour of approximate solution for various values of fractional order.

1 Introduction

Many phenomena in non-Brownian motion, fluid flows, chemical science, management theory, signal process, fibre optics, systems identification, elastic materials, polymers and others, are well described by a fractional differential equation. In specific, the partial differential equations (PDE) of fractional order are progressively
used to model issues in finance, viscoelasticity, mathematical biology and chemistry [1–8]. Different partial differential equations of fractional order are studied and resolved by several powerful methods [9–17]. Consequently, considerable attention has been given to the answer of partial differential equations of fractional order. Several powerful strategies are established and developed to induce numerical and analytical solutions of fractional differential equations, like finite-difference technique [7], finite volume technique [9], finite element technique [11], homotopy perturbation technique [13] and the fractional sub-equation technique [2].

Recently, many scholars introduced methods for solving fractional differential equations. Momani developed a domain decomposition technique to approximate solution for the fractional convection-diffusion equation with a nonlinear source term [12]. Dehghan et al introduced a numerical solution for a class of fractional convection-diffusion equations using the flatlet oblique multi-wavelets [8]. Saadatmandi et al studied the sinc-Legendre collocation technique for a category of fractional convection-diffusion equations with variable coefficients [14]. Liu et al introduced the finite volume technique for solving the fractional diffusion equations [18], and Yang et al proposed the finite volume technique to the fractional diffusion equations [19], all of that are without theoretical analysis.

Meerschaert and Tadjeran proposed the finite-difference technique for the resolution of the fractional advection-dispersion flow equations [16]. Baeumer and Meerschaert obtained the solution for fractional Cauchy equations by subordinating the solution of the original Cauchy equation [20]. Pskhu introduced a fundamental solution of a higher order Cauchy equation with time-fractional derivative [21]. Recently, Hejazi et al utilised the finite volume technique and finite-difference technique for solving the space-fractional advection-dispersion equation [17]. They used fractionally shifted Grünwald formula for the fractional derivative and verified the stability and convergence of the scheme, whose order is $O(\tau + h)$.

During this Chapter, we propose a finite-difference technique to get a new approximate solution for the time-space-fractional Cauchy equation with constant coefficients, space-fractional derivative and time-fractional derivative are described within the Riemann-Liouville sense and Caputo sense, respectively.

Consider the time-space-fractional Cauchy equation of the shape

$$\frac{\partial^\gamma u(x, t)}{\partial t^\gamma} + \epsilon \frac{\partial^\beta u(x, t)}{\partial x^\beta} = g(x, t),$$

subject to the initial condition

$$u(x, 0) = f(x),$$

where $t > 0, x \in [a, b], 0 < \beta \leq 1, 0 < \gamma \leq 1, g(x, t)$ is a given function provided that $u(x, t), g(x, t)$, and $f(x)$ are smooth enough, $\epsilon$ is a positive parameter,
Solving Time-Space-Fractional Cauchy Problem … 27

\( \beta \) is a parameter describing the order of the space fractional, and \( \gamma \) is a parameter describing the order of the time-fractional, the space-fractional derivative and time-fractional derivative are described in the Riemann-Liouville sense and Caputo sense, respectively.

The starting point for a finite-difference discretization is a partition of the computational domain \( [a, b] \) into a finite number of sub-domains \( V_i, i = 0, 1, 2, \ldots, N \), known as control volumes CVs, the union of all CVs should cover the whole domain. We introduce the implicit scheme by discretization of the Riemann-Liouville fractional integral, and time-Caputo fractional derivative. For another numerical scheme, see [22–31].

This chapter introduces a finite-difference technique for solving the time-space-fractional Cauchy equation with constant coefficients and contains the following sections: Sect. 2 is devoted to mathematical preliminaries. The description of a modified finite-difference technique is presented in Sect. 3. The von Neuman type stability analysis and consistency are proved in Sect. 4. Whilst the numerical experiments are given in Sect. 5. Finally, a brief conclusion is outlined in the last section.

2 Preliminaries

Throughout the past decade, fractional calculus has been applied to virtually every field of engineering, economics, science and another field. People like Liouville, Riemann and Weyl created major contributions to the idea of fractional calculus [32–39]. The story of the fractional calculus continued with contributions from Fourier, Abel, Leibniz, Grünwald and Letnikov. Over the years, several definitions found that are acceptable for the concept of fractional derivatives and integrals [40–48].

**Definition 1** The Riemann-Liouville integral of fractional order \( \alpha > 0 \), \( J_a^\alpha u(x) \) is defined by

\[
J_a^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} u(t) \, dt, \quad t > a,
\]

provided that \( u \in L_1[a, b] \). For \( \alpha = 0 \), we have \( J_a^0 u(x) = u(x) \) is the identity operator.

**Definition 2** Let \( n \in \mathbb{N} \) be the smallest integer that exceeds \( \alpha \), then the Riemann-Liouville fractional derivative of order \( \alpha > 0 \) is defined by

\[
D_a^\alpha u(x) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d^n}{dx^n} \right) \left[ \int_a^x \frac{u(t)}{(x - t)^{\alpha+1-n}} \, dt \right].
\]

(2)

provided that \( D_a^\alpha u(x) = D_a^n J_a^{(n-\alpha)} u(x) \). For \( \alpha = 0 \), we have \( D_a^0 u(x) = u(x) \) is the identity operator. For \( \alpha \in \mathbb{N} \), \( D_a^\alpha u(x) = \frac{d^\alpha u(x)}{dx^\alpha} \).
Definition 3 Let $\alpha > 0$, $u \in C^\alpha[a, b]$. Then,

$$\tilde{D}_a^\alpha u(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} (-1)^k \binom{\alpha}{k} u(x - kh), \quad (3)$$

where $a < x \leq b$, with $h = \frac{x-a}{N}$ is called the Grünwald-Letnikov fractional derivative of order $\alpha$ of the function $u$.

Definition 4 Let $n$ be the smallest integer that exceeds $\alpha$, then the Caputo fractional derivative of order $\alpha > 0$ is defined by

$$D_a^\alpha u(x) = \begin{cases} 
\frac{1}{\Gamma(n-\alpha)} \int_0^x (u^{(n)}(t) \frac{1}{(x-t)^{\alpha+1-n}} dt, & n-1 < \alpha < n, \\
\frac{d^n}{dx^n} u(x), & \alpha = n,
\end{cases} \quad (4)$$

provided that $D_a^\alpha u(x) = D_a^{-(n-\alpha)} D^n u(x)$ whenever $D^n u \in L_1[a, b]$.

The following theorem shows the relation between this definition and the Riemann-Liouville fractional derivatives:

Theorem 1 Let $\alpha > 0$, $n = \alpha$ and $u \in C^n[a, b]$. Then,

$$\tilde{D}_a^\alpha u(x) = D_a^\alpha u(x), \quad a < x \leq b.$$ 

Theorem 2 Let $\alpha > 0$, and $u \in C[a, b]$. Then,

$$J_a^\alpha u(x) = \lim_{h \to 0} h^\alpha \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} (-1)^k \binom{-\alpha}{k} u(x - kh), \quad (5)$$

where $a < x \leq b$, $(-1)^k \binom{-\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k+1)}$, and the function $\Gamma(x)$ is defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

If we define weights $w_0^\alpha = 1$, $w_1^\alpha = \alpha$ and $w_k^\alpha = \left(1 - \frac{(1-\alpha)}{k}\right)w_{k-1}^\alpha$, $k = 2, 3, \ldots$, then we may rewrite (5) as

$$J_a^\alpha u(x) = \lim_{h \to 0} h^\alpha \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} w_k^\alpha u(x - kh), \quad (6)$$

This formula is used to approximations the fractional integrals $J_a^\alpha u(x)$.

Lemma 1 Let $0 < \alpha < 1$. Then, we have

(1) $w_0^\alpha = 1$, and $w_j^\alpha > 0$ for $j = 1, 2, \ldots$;
(2) $w_j^\alpha - w_{j+1}^\alpha > 0$ for $j = 0, 1, \ldots$;
(3) $\lim_{j \to \infty} w_j^\alpha = 0$. 

Proof} For the first part, let $w_0^\alpha = 1$ and $w_1^\alpha = \alpha > 0$, thus from the recursive definition

$$w_j^\alpha = \left(1 - \frac{(1 - \alpha)}{j}\right)w_{j-1}^\alpha, \quad k = 2, 3, \ldots, \quad (7)$$

and since $0 < \alpha < 1$, we have $0 < \frac{1 - \alpha}{j} < \frac{1}{j} < 1$ for $j \geq 2$. So the coefficient \( \left(1 - \frac{(1-\alpha)}{k}\right) \) in (7) is strictly between zero and one.

Now, the second part can be done for $j \geq 2$ such that

$$w_j^\alpha - w_{j+1}^\alpha = \left(1 - \frac{(1 - \alpha)}{j}\right)w_{j-1}^\alpha - \left(1 - \frac{(1 - \alpha)}{j+1}\right)w_j^\alpha$$

$$= \left(1 - \frac{(1 - \alpha)}{j}\right)w_{j-1}^\alpha - \left(1 - \frac{(1 - \alpha)}{j+1}\right)\left(1 - \frac{(1 - \alpha)}{j}\right)w_{j-1}^\alpha$$

$$= \left[1 - \left(1 - \frac{(1 - \alpha)}{j+1}\right)\right]\left(1 - \frac{(1 - \alpha)}{j}\right)w_{j-1}^\alpha$$

$$= \frac{(1 - \alpha)}{j+1}\left(1 - \frac{(1 - \alpha)}{j}\right)w_{j-1}^\alpha > 0.$$ \(\text{Finally, from 1 and 2 we have for } j \geq 2\)

$$0 < w_{j+1}^\alpha < w_j^\alpha < w_1^\alpha = \alpha < 1 = w_0^\alpha.$$ \(\text{So, } \lim_{j \to \infty} w_j^\alpha = 0.\)

Whenever we use a numerical technique to solve a differential equation, we would like to make sure that the numerical solution obtained is a sufficiently good approximation to the actuality solution, some necessary definition and remarks are introduced to discuss the stability analysis [27, 49–54].

To analyse the stability of difference scheme for IVP, suppose that we are given a vector in $\ell_2$, $v = (...) , v_{-1}, v_0, v_1,...)^T$, and define the discrete fourier transform of $v$ as follows:

**Definition 5** The discrete Fourier transform of $v \in \ell_2$ is the operation $\hat{v} \in L_2[-\pi, \pi]$ defined by

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-i\xi m} v_m, \quad \xi \in [-\pi, \pi].$$

**Definition 6** The symbol of difference scheme $v^{n+1} = Qv^n$ is the coefficient of $\hat{v}^n$ in the equation $\hat{v}^{n+1} = \rho(\xi)\hat{v}^n$, where $\hat{v}^{n+1} = \rho(\xi)\hat{v}^n$ is the discrete Fourier transform of the discrete scheme.
Remark 1 For simplification, we can get the discrete Fourier transform of the difference scheme by replacing \( v^n_j \) in the difference scheme by

\[
\hat{v}^n_j = \hat{v}^n \exp(ij\xi), i = \sqrt{-1}.
\]

Remark 2 The difference scheme \( v^{n+1} = Qv^n \) is stable with respect to \( \ell_2, h \) norm if and only if there exist positive constants \( \tau_0, h_0 \) and \( C \) so that \( |\rho(\xi)| \leq 1 + C\tau, \) for \( 0 < \tau \leq \tau_0, 0 < h \leq h_0 \) and all \( \xi \in [-\pi, \pi]. \)

Remark 3 If \( \rho \) satisfies the inequality in Remark 2, then \( \rho \) is said to be satisfied the von Neumann condition.

Remark 4 The difference scheme that is stable under a set of conditions is called conditionally stable, otherwise is called unconditionally stable scheme.

3 Modified Finite-Difference Method

In this section, we propose a new finite-difference method for solving the time-space-fractional Cauchy equation of the shape:

\[
\frac{\partial^\gamma u(x, t)}{\partial t^\gamma} + \epsilon \frac{\partial^\beta u(x, t)}{\partial x^\beta} = g(x, t),
\]

subject to the initial condition \( u(x, 0) = f(x) \) for \( t > 0, x \in [a, b], 0 < \beta \leq 1, 0 < \gamma \leq 1, g(x, t) \) is a given function provided that \( u(x, t), g(x, t), \) and \( f(x) \) are smooth enough, \( \epsilon \) is a positive parameter, \( \beta \) is a parameter describing the order of the space fractional and \( \gamma \) is a parameter describing the order of the time-fractional, the space-fractional derivative and time-fractional derivative are described in the Riemann-Liouville sense and Caputo sense, respectively.

Using the definition of Riemann-Liouville fractional derivative where \( 0 < \beta \leq 1, \)

we have

\[
\frac{\partial^\gamma u(x, t)}{\partial t^\gamma} + \epsilon \frac{\partial}{\partial x} J_a^{1-\beta} u(x, t) = g(x, t),
\]

where \( J_a^{1-\beta} \) is the Riemann-Liouville integral with respect to \( x, \) take \( \alpha = 1 - \beta, \) we have \( 0 \leq \alpha < 1. \) Let \( \Omega = [a, b] \) be a finite domain that is discretised with \( N + 1 \) uniformly spaced nodes \( x_i = a + ih, i = 0, 1, \ldots, N, \) where the spatial step \( h = \frac{b-a}{N}, \) we approximate the \( \alpha \) order fractional Riemann-Liouville integral with standard Grünwald formula and approximate the first derivative with central difference formula:
\[ J_d^\alpha u(x, t) = h^\alpha \sum_{j=0}^{N} w_j^\alpha u(x - jh, t) + o(h), \]  
(10)

\[ \frac{\partial u(x, t)}{\partial x} \bigg|_{x=x_i} = u(x_{i+1}, t) - u(x_{i-1}, t) \frac{2h}{2} + O(h^2). \]  
(11)

A finite-difference discretization is applied by evaluating Eq. (9) at \( x = x_i \), and using the above equations.

\[ \frac{\partial^\gamma u(x_i, t)}{\partial t^\gamma} = -\frac{\epsilon}{2h} \left[ h^\alpha \sum_{j=0}^{i+1} w_j^\alpha u(x_i-j+1, t) - h^\alpha \sum_{j=0}^{i-1} w_j^\alpha u(x_i-j-1, t) \right] + g(x_i, t). \]  
(12)

Letting \( t_n = n\tau, n = 0, 1, 2, \ldots \), where \( \tau \) is the time step, and discretise the Caputo time-fractional derivative using L1-algorithm,

\[ \frac{\partial^\gamma u(x_i, t_{n+1})}{\partial t^\gamma} = \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{s=0}^{n} b_s^\gamma \left[ u(x_i, t_{n+1-s}) - u(x_i, t_{n-s}) \right] + O(\tau^{2-\gamma}). \]  
(13)

where \( b_s^\gamma = (s + 1)^{1-\gamma} - s^{1-\gamma}, s = 0, 1, \ldots, n. \)

Letting \( u^n_i \approx u(x_i, t_n) \) denote the numerical solution, we have

\[ \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{s=0}^{n} b_s^\gamma \left[ u_{n+1-s}^i - u_{n-s}^i \right] \]

\[ = -\frac{\epsilon}{2h} \left[ h^\alpha \sum_{j=0}^{i+1} w_j^\alpha u^1_{n+1-j+1} - h^\alpha \sum_{j=0}^{i-1} w_j^\alpha u^1_{n+1-j-1} \right] + g(x_i, t_{n+1}). \]  
(13)

Collecting like terms, we can rewrite Eq. (13) as:

\[ \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{s=0}^{n} b_s^\gamma \left[ u_{n+1-s}^i - u_{n-s}^i \right] = \frac{1}{h} \sum_{j=0}^{N} b_{ij} u_{n+1}^j + g_{n+1}^i, \]  
(14)

where \( i = 0, 1, \ldots, N, \) and \( b_{ij} = \begin{cases} \frac{\epsilon h^\alpha}{2} [w_{j+1}^u - w_{j-1}^u], & j < i - 1, \\ \frac{\epsilon h^\alpha}{2} [w_j^u - w_{j}^u], & j = i - 1, \\ \frac{\epsilon h^\alpha}{2} w_j^u, & j = i, \\ \frac{\epsilon h^\alpha}{2} w_j^u, & j = i + 1, \\ 0, & j > i + 1. \end{cases} \)

Denoting the numerical solution vector \( U^n = [u_0^n, u_1^n, \ldots, u_N^n] \) and source vector \( g^{n+1} = [g(x_0, t_{n+1}), g(x_1, t_{n+1}), \ldots, g(x_N, t_{n+1})] \), we have the following vector equation:
\[
\left( I + \frac{\Gamma(2 - \gamma)\tau^\gamma}{h} A \right) U^{n+1} = b^n U^0 + \sum_{s=0}^{n-1} \left( b^s_s - b^{s+1}_s \right) U^{n-s} + \tau^\gamma \Gamma(2 - \gamma) g^{n+1},
\]
where the matrix \( A \) has elements \( a_{ij} = b_{ij} \).

In particular, for \( \gamma = 1 \) we can use the standard backward difference to approximate the time derivative in Eq. (12)

\[
\frac{du(x_i, t)}{dt} \bigg|_{t=t_{n+1}} = \frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\tau} + O(\tau), \tag{15}
\]
yields the numerical solution

\[
\frac{u_i^{n+1} - u_i^n}{\tau} = -\frac{\epsilon}{2h} \left[ h^\alpha \sum_{j=0}^{i+1} w^\alpha_{ij} u_j^{n+1} + h^\alpha \sum_{j=0}^{i-1} w^\alpha_{ij} u_j^{n+1} \right] + g_i^{n+1}, \tag{16}
\]

Anyhow, we can rewrite Eq. (16) as a vector equation in the form

\[
\left( I + \frac{\tau}{h} A \right) U^{n+1} = U^n + \tau g^{n+1}, \tag{17}
\]
where the matrix \( A \) has elements \( a_{ij} = b_{ij} \).

In the next section, we prove that this scheme is conditionally stable, and it is first-order accurate in time and second-order accurate in space.

### 4 Stability Analysis

In this section, the stability analysis for the proposed numerical scheme is presented as in the following theorems:

**Theorem 3** The numerical scheme (16) is conditionally stable.

**Proof** To debate stability, consider the homogeneous scheme

\[
\frac{u_m^{n+1} - u_m^n}{\tau} = \frac{1}{h} \sum_{j=0}^{N} b_{mj} u_j^{n+1}, \quad m = 0, 1, 2, \ldots, N.
\]

Substitution of \( u_m^n = \hat{u}^n \exp(i m \xi), i = \sqrt{-1} \) into numerical scheme

\[
\hat{u}^{n+1} \exp(i m \xi) - \hat{u}^n \exp(i m \xi) = r \sum_{j=0}^{N} b_{mj} \hat{u}^{n+1} \exp(i j \xi), \quad r = \frac{\tau}{h}.
\]
\[
\dot{u}_{i}^{n+1} = \frac{1}{1 - r \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi)} \dot{u}_{i}^{n}.
\]

The symbol of numerical scheme is
\[
\rho(\xi) = \frac{1}{1 - r \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi)},
\]
and satisfies the von Neumann condition if
\[
\left| 1 - r \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi) \right| \geq 1.
\]

By using Reverse Triangle Inequality, we have got
\[
\left| 1 - r \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi) \right| \geq \left| 1 - \left| r \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi) \right| \right|.
\]

So, the von Neumann condition satisfies if \( \left| 1 - \left| r \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi) \right| \right| \geq 1 \). That is equivalent to \( 1 - \left| r \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi) \right| \geq 1 \), impossible hold, or \( 1 - \left| r \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi) \right| \leq -1 \). Which is equivalent to \( \left| \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi) \right| \geq \frac{2}{r} \). Hence, the symbol of numerical scheme satisfies the von Neumann condition if \( \left| \sum_{j=0}^{N} b_{mj} \exp(i(j - m)\xi) \right| \geq \frac{2}{r} \), \( \forall m = 0, 1, 2, \ldots, N \). So, the numerical scheme is conditionally stable.

**Theorem 4** The numerical scheme (16) is consistent with second-order accuracy in direction of space and first order in direction of time.

**Proof** By using Eqs. (10), (11) and (15), we can write Eq. (8) at \((x_i, t_{n+1})\) as follows:

\[
\frac{u_{i}^{n+1} - u_{i}^{n}}{\tau} + \mathcal{O}(\tau) = -\frac{\epsilon}{2h} \left[ h^{\alpha} \sum_{j=0}^{i+1} w_{j}^{\alpha} u_{i-j+1}^{n+1} + o(1) - h^{\alpha} \sum_{j=0}^{i-1} w_{j}^{\alpha} u_{i-j-1}^{n} + o(1) \right] + \mathcal{O}(h^2) + g_{i}^{n+1}.
\]

Thus, we get that
\[
\frac{\partial u(x_i, t)}{\partial t} \bigg|_{t=t_{n+1}} = -\epsilon \frac{\partial}{\partial x} \left[ J_a^{1-\beta} u(x, t_{n+1}) \right]_{x=x_i} + g_i^{n+1},
\]

which is Eq. (8) at \((x_i, t_{n+1})\), \(\gamma = 1\).

**Theorem 5** The numerical scheme (14) is consistent with second-order accuracy in space and \(2-\gamma\) order in time.

**Proof** we can write Eq. (8) at \((x_i, t_{n+1})\) as follows:

\[
\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{s=0}^{n} b_s^\gamma \left[ u(x_i, t_{n+1-s}) - u(x_i, t_{n-s}) \right] + O(\tau^{2-\gamma})
\]

\[
= -\frac{\epsilon}{2h} \left[ h^\alpha \sum_{j=0}^{i+1} w_j^{n+1} u_{i-j+1} + o(1) - h^\alpha \sum_{j=0}^{i-1} w_j^{n+1} u_{i-j-1} + o(1) \right] + O(h^2) + g_i^{n+1}
\]

Thus, we have

\[
\frac{\partial^\gamma u(x_i, t)}{\partial t^\gamma} \bigg|_{t=t_{n+1}} = -\epsilon \frac{\partial}{\partial x} \left[ J_a^{1-\beta} u(x, t_{n+1}) \right]_{x=x_i} + g_i^{n+1}.
\]

which is Eq. (8) at \((x_i, t_{n+1})\), \(0 < \gamma < 1\).

### 5 Numerical Experiments

In this section, in order to solve the fractional Cauchy equation using the finite-difference discretization scheme (FDDS), the equation is presented in a discrete specific form. Anyhow, we consider four illustrated examples to demonstrate the performance and efficiency of the proposed algorithm. The computations are performed by Wolfram-Mathematica software 11.

**Example 1** Consider the following homogeneous fractional Cauchy equation:

\[
\frac{\partial^\gamma u(x, t)}{\partial t^\gamma} + \epsilon \frac{\partial^\beta u(x, t)}{\partial x^\beta} = 0,
\]

subject to the initial condition

\[
u(x, 0) = \sin(\pi x),
\]

where \(\epsilon = 1 \times 10^{-3}, t \geq 0, x \in [1, 4], \gamma = 1,\) and \(0 < \beta \leq 1\).

In particular, the exact solution of IVPs (18) and (19) at \(\beta = 1, \gamma = 1\) is given by \(u(x, t) = \sin(\pi (x - \epsilon t))\). Following the FDD algorithm, using \(h = 0.05\) and
\( \tau = 0.01 \), the numerical results of FDDS with varying fractional order \( \beta \) such that \( \beta \in \{0.75, 0.85, 0.95, 1 \} \), \( \gamma = 1 \), compared with exact solution are given in Table 1 at the time \( t = 0.5 \) and \( x \in [1, 1.5] \). In the light of showing the agreement between the FDDS and exact solutions, the absolute error of IVPs (18) and (19) are listed in Table 2 for \( \beta = 1, \gamma = 1 \) when \( t = 0.5 \) and \( x \in [1, 1.5] \) with \( h = 0.1 \). Table 3 is

Table 1: Numerical results for Example 1 at \( t = 0.5 \), \( \gamma = 1 \) with varying \( \beta \)

| \( x \) | Exact | \( \beta = 1 \) | \( \beta = 0.95 \) | \( \beta = 0.85 \) | \( \beta = 0.75 \) |
|------|-------|----------------|----------------|----------------|----------------|
| 1.05 | -0.154883 | -0.156431 | -0.156431 | -0.156432 | -0.156432 |
| 1.10 | -0.307523 | -0.309013 | -0.309014 | -0.309014 | -0.309015 |
| 1.15 | -0.452590 | -0.453987 | -0.454008 | -0.454031 | -0.454036 |
| 1.20 | -0.586514 | -0.587782 | -0.587832 | -0.587886 | -0.587900 |
| 1.25 | -0.705995 | -0.707104 | -0.707186 | -0.707276 | -0.707301 |
| 1.30 | -0.808093 | -0.809015 | -0.809130 | -0.809257 | -0.809294 |
| 1.35 | -0.890292 | -0.891005 | -0.891152 | -0.891317 | -0.891366 |
| 1.40 | -0.950570 | -0.951055 | -0.951232 | -0.951433 | -0.951495 |
| 1.45 | -0.987441 | -0.987688 | -0.987892 | -0.988124 | -0.988199 |
| 1.50 | -0.999999 | -1.000000 | -1.000230 | -1.000490 | -1.000570 |

Table 2: Absolute errors for Example 1 at \( \beta = 1, \gamma = 1 \)

| \( x \) | Exact | FDDS | Absolute error |
|------|-------|------|----------------|
| \( 1.1 \) | -0.307523 | -0.309013 | 1.49065 \times 10^{-3} |
| \( 1.2 \) | -0.586514 | -0.587782 | 1.26842 \times 10^{-3} |
| \( 1.3 \) | -0.808093 | -0.809015 | 0.92203 \times 10^{-3} |
| \( 1.4 \) | -0.950570 | -0.951055 | 0.48539 \times 10^{-3} |
| \( 1.5 \) | -0.999999 | -1.000000 | 1.23369 \times 10^{-6} |

Table 3: FDDS of Example 1 at \( \beta = 0.95, \gamma = 1 \) with varying time \( T \)

| \( x \) | \( T = 0.5 \) | \( T = 1.0 \) |
|------|-------------|-------------|
| 1.05 | -0.156431 | -0.156428 |
| 1.10 | -0.309014 | -0.309010 |
| 1.15 | -0.454008 | -0.454027 |
| 1.20 | -0.587832 | -0.587880 |
| 1.25 | -0.707186 | -0.707267 |
| 1.30 | -0.809130 | -0.809245 |
| 1.35 | -0.891152 | -0.891300 |
| 1.40 | -0.951232 | -0.951412 |
| 1.45 | -0.987892 | -0.988099 |
| 1.50 | -1.000230 | -1.000460 |
devoted to the FDDS approximate solutions at $\beta = 0.95$, $\gamma = 1$ with varying times $t$ such that $t = 0.5$ and $t = 1.0$ over the interval $[1, 1.5]$ with $h = 0.05$.

From these tables, it can be noted that the FDDS approximate solutions are in good agreement with the exact solutions over the domain of interest. Anyhow, more iteration leads to more accurate solutions. For further analysis, the 2D-plot of the FDDS and exact solution for Example 1 are drawn in Fig. 1 at $t = 0.5$ and $x \in [1, 3.5]$. Whilst, the surface plot of the approximate solution at $\beta = 0.95$, $\gamma = 1$ is shown in Fig. 2.

**Example 2** Consider the following non-homogeneous fractional Cauchy equation:

\[
\frac{\partial^\gamma u(x, t)}{\partial t^\gamma} + \epsilon \frac{\partial^\beta u(x, t)}{\partial x^\beta} = g(x, t),
\]

subject to the initial condition

\[
u(x, 0) = x^2 \sin(x),
\]

where $g(x, t) = \sin(2x)$, $\epsilon = 2$, $t \geq 0$, $x \in [0, 0.9]$, $\beta = 0.85$, and $0 < \gamma \leq 1$.
Following the FDD algorithm, using \( h = 0.05 \) and \( \tau = 0.025 \), the numerical results of FDDS with varying fractional order \( \gamma \) such that \( \gamma \in \{0.75, 0.85, 0.95, 1\} \), \( \beta = 0.85 \) are given in Table 4 at the time \( t = 0.5 \) and \( x \in [0, 0.5] \). Table 5 is devoted to the FDDS approximate solutions at \( \beta = 0.85 \) and \( \gamma = 0.95 \) with varying times \( t \) such that \( t = 0.25 \) and \( t = 0.5 \) over the interval \([0, 0.4]\) with \( h = 0.05 \), the 2D-plot of the FDDS for Example 5.2 is drawn in Fig. 3 at \( t = 0.5 \) and \( x \in [0, 0.9] \). Figure 4 shown the FDDS approximate solutions at \( \beta = 0.85 \) and \( \gamma = 0.95 \) with varying times \( t \) such that \( t = 0.25 \) and \( t = 0.5 \) over the interval \([0, 0.4]\) . Whilst, the surface plot of the approximate solution at \( \beta = 0.85, \gamma = 1 \) is shown in Fig. 5 at \( t = 1.0 \).

**Example 3** Consider the following homogeneous fractional Cauchy equation:

\[
\frac{\partial^\gamma u(x, t)}{\partial t^\gamma} + \epsilon \frac{\partial^\beta u(x, t)}{\partial x^\beta} = 0, \tag{22}
\]

| Table 4 | Numerical results for Example 2 at \( t = 0.5 \), \( \beta = 0.85 \), with varying \( \gamma \) |
|---------|-----------------------------------------------|
| \( x \) | \( \gamma = 0.75 \) | \( \gamma = 0.85 \) | \( \gamma = 0.95 \) | \( \gamma = 1.0 \) |
| 0.00 | 0.059580 | 0.054150 | 0.049026 | 0.046615 |
| 0.05 | 0.119865 | 0.108810 | 0.098405 | 0.093520 |
| 0.10 | 0.206253 | 0.183249 | 0.162686 | 0.153359 |
| 0.15 | 0.304010 | 0.266338 | 0.233562 | 0.218978 |
| 0.20 | 0.417262 | 0.360035 | 0.311871 | 0.290902 |
| 0.25 | 0.542894 | 0.462352 | 0.396428 | 0.368253 |
| 0.30 | 0.681071 | 0.573192 | 0.487204 | 0.451077 |
| 0.35 | 0.831202 | 0.692349 | 0.584309 | 0.539609 |
| 0.40 | 0.993351 | 0.820042 | 0.688155 | 0.634342 |
| 0.45 | 1.167720 | 0.956699 | 0.799340 | 0.735934 |
| 0.50 | 1.354800 | 1.102950 | 0.918616 | 0.845175 |

| Table 5 | FDDS of Example 2 at \( \beta = 0.85, \gamma = 0.95 \) with varying time \( T \) |
|---------|-----------------------------------------------|
| \( x \) | \( T = 0.25 \) | \( T = 0.5 \) |
| 0.0 | 0.024218 | 0.049026 |
| 0.05 | 0.048509 | 0.098405 |
| 0.1 | 0.076892 | 0.162686 |
| 0.15 | 0.107694 | 0.233562 |
| 0.2 | 0.141272 | 0.311871 |
| 0.25 | 0.177948 | 0.396428 |
| 0.3 | 0.218268 | 0.487204 |
| 0.35 | 0.262869 | 0.584309 |
| 0.4 | 0.312441 | 0.688155 |
subject to the initial condition

\[ u(x, 0) = \sin(x^2), \quad (23) \]

where \( \epsilon = \pi, t \geq 0, x \in [0, 0.95], \beta = 0.75, \) and \( 0 < \gamma \leq 1. \)

Following the FDD algorithm, using \( h = 0.05 \) and \( \tau = 0.025, \) numerical results of FDDS with varying fractional order \( \gamma \) such that \( \gamma \in \{0.75, 0.85, 0.95, 1\}, \beta = 0.75 \) are given in Table 6 at the time \( t = 0.5 \) and \( x \in [0, 0.5]. \)
Table 6  Numerical results for Example 3 at $t = 0.5$, $\beta = 0.75$, with varying $\gamma$

| $x$ | $\gamma = 0.75$ | $\gamma = 0.85$ | $\gamma = 0.95$ | $\gamma = 1.0$ |
|-----|-----------------|-----------------|-----------------|----------------|
| 0.05| 0.002266        | 0.002287        | 0.002307        | 0.002316       |
| 0.10| 0.009407        | 0.009473        | 0.009533        | 0.009560       |
| 0.15| 0.024289        | 0.024174        | 0.024053        | 0.023992       |
| 0.20| 0.051221        | 0.050313        | 0.049423        | 0.048994       |
| 0.25| 0.094644        | 0.091597        | 0.088738        | 0.087400       |
| 0.30| 0.160305        | 0.152586        | 0.145614        | 0.142435       |
| 0.35| 0.254323        | 0.237849        | 0.223484        | 0.217086       |
| 0.40| 0.383737        | 0.352317        | 0.325831        | 0.314297       |
| 0.45| 0.556090        | 0.500969        | 0.455986        | 0.436812       |
| 0.50| 0.779532        | 0.688842        | 0.617112        | 0.587159       |

Table 7  FDDS of Example 3 at $\beta = 0.75$, $\gamma = 0.95$ with varying time $T$

| $x$ | $T = 0.25$ | $T = 0.5$ |
|-----|------------|-----------|
| 0.05| 0.0024045  | 0.0023072 |
| 0.10| 0.0097809  | 0.0095331 |
| 0.15| 0.0233115  | 0.0240534 |
| 0.20| 0.0447471  | 0.0494235 |
| 0.25| 0.0752926  | 0.0887387 |
| 0.30| 0.1161080  | 0.1456140 |
| 0.35| 0.1681450  | 0.2234840 |
| 0.40| 0.2321930  | 0.3258310 |

Table 7 is devoted to the FDDS approximate solutions at $\beta = 0.75$ and $\gamma = 0.95$ with varying times $t$ such that $t = 0.25$ and $t = 0.5$ over the interval $[0, 0.4]$ with $h = 0.05$, the 2D-plot of the FDDS for Example 5.3 is drawn in Fig. 6 at $t = 0.5$ and $x \in [0, 0.95]$. Figure 7 shown the FDDS approximate solutions at $\beta = 0.75$ and $\gamma = 0.95$ with varying times $t$ such that $t = 0.25$ and $t = 0.5$ over the interval

Fig. 6  FDDS for Example 3 at $\beta = 0.75$, $t = 0.5$ with varying $\gamma$
Fig. 7  FDDS for Example 3 at $\beta = 0.75$ and $\gamma = 0.95$ with varying times.

Fig. 8  Surface plot of FDDS solution for Example 3, $\beta = 0.75$, $\gamma = 1$ at $t = 1$.

[0, 0.4]. Whilst, the surface plot of the approximate solution at $\beta = 0.75$, $\gamma = 1$ is shown in Fig. 8 at $t = 1.0$.

**Example 4** Consider the following homogeneous fractional Cauchy equation:

$$\frac{\partial^\gamma u(x, t)}{\partial t^\gamma} + \epsilon \frac{\partial^\beta u(x, t)}{\partial x^\beta} = 0, \tag{24}$$

subject to the initial condition

$$u(x, 0) = e^{\xi x}. \tag{25}$$

where $\epsilon = 0.1$, $\xi = 1.1771243444677$, $t \geq 0$, $x \in [-2, 1]$, $\gamma = 1$ and $0 < \beta \leq 1$.

In particular, the exact solution of IVPs (24) and (25) at $\beta = 1$ is given by $u(x, t) = e^{\xi(x-\epsilon t)}$. Following the FDDS algorithm, using $h = 0.0625$ and $\tau = 0.01$, the numerical results of the exact and FDDS for different values of fractional order $\beta$ such that $\beta \in [0.75, 0.85, 0.95, 1]$, $\gamma = 1$ are given in Table 8 at the time $t = 0.5$ and $x \in [-2, -1.25]$. Table 9 is devoted to the FDDS approximate solutions at $\beta = 0.95$ with varying times $t$ such that $t = 0.5$ and $t = 1$ over the interval $[-2, -1.5]$ with $h = 0.0625$. 
Table 8  Numerical results for Example 4 at $t = 0.5, \gamma = 1$ with varying $\beta$

| $x$    | Exact           | $\beta = 1$ | $\beta = 0.95$ | $\beta = 0.85$ | $\beta = 0.75$ |
|--------|-----------------|-------------|----------------|----------------|----------------|
| $-2.0000$ | 0.089537        | 0.094808    | 0.094822       | 0.094847       | 0.094868       |
| $-1.9375$ | 0.096373        | 0.102192    | 0.102184       | 0.102175       | 0.102171       |
| $-1.8750$ | 0.103730        | 0.109994    | 0.111023       | 0.112139       | 0.112442       |
| $-1.8125$ | 0.111649        | 0.118391    | 0.119907       | 0.121624       | 0.122163       |
| $-1.7500$ | 0.120173        | 0.127429    | 0.129290       | 0.131461       | 0.132197       |
| $-1.6875$ | 0.129347        | 0.137158    | 0.139306       | 0.141862       | 0.142774       |
| $-1.6250$ | 0.139222        | 0.147629    | 0.150043       | 0.152954       | 0.154028       |
| $-1.5625$ | 0.149851        | 0.158900    | 0.161572       | 0.164827       | 0.166060       |
| $-1.5000$ | 0.161291        | 0.171031    | 0.173965       | 0.177565       | 0.178954       |
| $-1.4375$ | 0.173605        | 0.184088    | 0.187291       | 0.191244       | 0.192792       |
| $-1.3750$ | 0.186859        | 0.198142    | 0.201627       | 0.205946       | 0.207658       |
| $-1.3125$ | 0.201124        | 0.213269    | 0.217050       | 0.221754       | 0.223637       |
| $-1.2500$ | 0.216479        | 0.229551    | 0.233647       | 0.238756       | 0.240817       |

Table 9  FDDS of Example 4 at $\beta = 0.95$ with varying time $t$

| $x$    | $t = 0.5$ | $t = 1$ |
|--------|-----------|---------|
| $-2.0000$ | 0.094822  | 0.094677 |
| $-1.9375$ | 0.102184  | 0.102151 |
| $-1.8750$ | 0.111023  | 0.112044 |
| $-1.8125$ | 0.119907  | 0.121425 |
| $-1.7500$ | 0.129290  | 0.131169 |
| $-1.6875$ | 0.139306  | 0.141487 |
| $-1.6250$ | 0.150043  | 0.152500 |
| $-1.5625$ | 0.161572  | 0.164299 |
| $-1.5000$ | 0.173965  | 0.176962 |

Fig. 9  Solution behaviour Example 4 for different values of $\beta$
Figure 9 displays the approximate solutions of IVPs (24) and (25) for different values of fractional order $\beta$ such that $\beta \in \{0.75, 0.85, 0.95, 1\}$, $\gamma = 1$ at time $t = 0.5$ and $x \in [-2, 1]$. The 2D-plot of the FDDS and exact solution for Example 5.4 are drawn in Fig. 10 at $t = 0.5$ and $x \in [-2, 1]$. Figure 11 shown the FDDS approximate solutions at $\beta = 0.95$ and $\gamma = 1.0$ with varying times $t$ such that $t = 0.5$ and $t = 1$ over the interval $[-2, 1]$. Whilst, the surface plot of the FDDS approximate solution at $\beta = 0.95$, $\gamma = 1$ is shown in Fig. 12. From these graphs, it can be concluded that
the behaviour of the FDDS approximate solutions are in good agreement with each other at different values of $\beta$.

6 Conclusion

In this chapter, a new finite-difference technique has been developed for solving linear Cauchy equation of fractional order. We introduce the implicit scheme by discretization of the space-Riemann-Liouville fractional integral, and time-Caputo fractional derivative, the solution obtained using this technique shows that this approach can solve the problem effectively. The basic idea of this approach can be further utilised to resolve the linear Cauchy equation of fractional order with a variable coefficient or apply the finite volume method by using the same discretization.

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