ON THE DYNAMICS OF RATIONAL SOLUTIONS FOR 1-D GENERALIZED VOLterra SYSTEM

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April 1, 2022

Abstract

The Hirota bilinear formalism and soliton solutions for a generalized Volterra system is presented. Also, starting from the soliton solutions, we obtain a class of nonsingular rational solutions using the "long wave limit" procedure of Ablowitz and Satsuma, and appropriate "gauge" transformations. Their properties are also discussed and it is shown that these solutions interact elastically with no phase shift.

1 Introduction

It is known that with the development of soliton theory significant progress has been made in finding special solutions of integrable nonlinear evolution equations which include soliton solutions, rational solutions, similarity solutions and so on, using I. S. T., Hirota bilinear formalism, Backlund transformations and Wronskian formalism.

The class of rational solutions was firstly investigated for KdV equation. A very simple way to find them was developed by Ablowitz and Satsuma, and consist in taking the "long wave" limit in the multisoliton solution. This method was used successfully to find the rational solutions also for other completely integrable systems. Although, for continuous case, many results have been obtained with respect to finding them, a relatively less work for discrete systems has been done.

Unlike the (2+1)-dimensional continuous systems where the dynamics and phenomenology of rational solutions ("lump" solutions) are well studied,
[11], the dynamics of (1+1)-dimensional nonsingular rational solutions was very little investigated [12]. In this paper we’ll derive the bilinear formalism for a (1+1)-dimensional discrete generalized Volterra system, which represent a coupled system of generalized Volterra equations proposed by M. Wadati [13], [14]. From its soliton solutions we’ll obtain a class of real and complex rational nonsingular solutions using Ablowitz-Satsuma limiting procedure. Here the crucial step is the existence of many nontrivial arbitrary constants and ”gauge” transformations. It is shown that these solutions collide elastically with no phase shift using a ”head-on” collision of two 1-rational nonsingular solutions. Also their real and complex structure is preserved. The properties, generalizations and possible developments are also discussed.

The paper is organized as follows. In the section 2 is constructed the bilinear form of the system by a reduction to a variant of Ablowitz-Ladik system with nonzero boundary condition. In section 3 the construction of rational solutions is presented. In section 4 properties and dynamics are discussed and, finally, conclusions are given in section 5.

2 Bilinear formalism

The generalized Volterra system under consideration is:

\[
\begin{align*}
\dot{Q}_n &= (a + bQ_n + cQ^2_n)(R_{n+1} - R_{n-1}) \\
\dot{R}_n &= (a + bR_n + cR^2_n)(Q_{n+1} - Q_{n-1})
\end{align*}
\] (1)

with \(Q_n, R_n \to 0\) as \(n\) goes to infinity and \(a, b, c\) are real constants.

For waves propagating in one direction we may put \(Q_n = R_n\) and we’ll find a generalized Volterra equation proposed firstly by M. Wadati [13], [14].

By the following transformations:

\[
\begin{align*}
&u_n = \frac{2c}{\sqrt{4ac - b^2}}(Q_n + b/2c) \\
v_n = \frac{2c}{\sqrt{4ac - b^2}}(R_n + b/2c) \\
t \to t\left(\frac{4ac - b^2}{4c}\right)
\end{align*}
\] (2)

and putting \(\alpha_0 = \frac{b}{\sqrt{4ac - b^2}}\)

the system (1) becomes:

\[
\begin{align*}
\dot{u}_n &= (1 + u_n^2)(v_{n+1} - v_{n-1}) \\
\dot{v}_n &= (1 + v_n^2)(u_{n+1} - u_{n-1})
\end{align*}
\] (3)
with $u_n, v_n \to \alpha_0$ as $n$ goes to infinity. In order to have $\alpha_0$ real we consider that $c, b$ are nonzero and $4ac - b^2$ is positive. The system (3) represent a variant of Ablowitz-Ladik system \cite{15}, with nonvanishing boundary conditions.

We are constructing the bilinear form of this system by taking:

$$u_n = \alpha_0 - \frac{i}{2} \frac{d}{dt} \log \left( \frac{f_n}{g_n} \right)$$

$$v_n = \alpha_0 - \frac{i}{2} \frac{d}{dt} \log \left( \frac{f'_n}{g'_n} \right)$$

(4)

where we assume that $f_n/g_n$ and $f'_n/g'_n \to \text{const.}$ as $n$ goes to infinity.

Plugging (4) into (3), integrating with respect to $t$ and taking into account the nonzero boundary conditions we’ll get:

$$\tan^{-1} u_n = \tan^{-1} \alpha_0 - \frac{i}{2} \log \left( \frac{f_{n+1}g'_{n-1}}{g_{n+1}f'_{n-1}} \right)$$

$$\tan^{-1} v_n = \tan^{-1} \alpha_0 - \frac{i}{2} \log \left( \frac{f_{n+1}g'_{n-1}}{g_{n+1}f'_{n-1}} \right)$$

Using (4) again we’ll find after some algebra:

$$D_t f_n g_n = 2(1 + \alpha_0^2) \sinh D_n f'_n g'_n$$

$$D_t f'_n g'_n = 2(1 + \alpha_0^2) \sinh D_n f_n g_n$$

$$\cosh D_n + i\alpha_0 \sinh D_n f'_n g'_n = f_n g_n$$

$$\cosh D_n + i\alpha_0 \sinh D_n f_n g_n = f'_n g'_n$$

(5)

which represent the Hirota form of the system (3). We can see that for $\alpha_0 = 0$ we can recover the self-dual network form proposed by Hirota \cite{2}.

The 1-soliton solution are given by:

$$f_n = 1 + \exp(\eta_1 + \phi_1)$$

$$f'_n = 1 + \exp(\eta'_1 + \psi'_1)$$

$$g_n = 1 + \exp(\eta_1 + \psi_1)$$

$$g'_n = 1 + \exp(\eta'_1 + \psi'_1)$$

(6)

where

$$\eta_j = 2\epsilon_j(1 + \alpha_0^2) \sinh P_j t + P_j n + \eta_0^j$$

$$\exp \phi_j = \frac{\alpha_0}{2} K_j \left( \frac{\epsilon_j + \cosh P_j}{\sinh P_j} + \frac{i}{2} K_j \right)$$

$$\exp \psi_j = \frac{\alpha_0}{2} K_j \left( \frac{\epsilon_j + \cosh P_j}{\sinh P_j} - \frac{i}{2} K_j \right)$$

$$\exp \phi'_j = \frac{\alpha_0}{2} K_j \left( \frac{1 + \epsilon_j \cosh P_j}{\sinh P_j} + \frac{i\epsilon_j}{2} K_j \right)$$

$$\exp \psi'_j = \frac{\alpha_0}{2} K_j \left( \frac{1 + \epsilon_j \cosh P_j}{\sinh P_j} - \frac{i\epsilon_j}{2} K_j \right)$$

(7)
where the index \( j \) labels the number of the soliton i.e. for 1-soliton solution \( j = 1 \), for 2-soliton \( j = 1, 2 \) and so on and \( \epsilon_j = \pm 1 \) indicates the propagation direction. On the other hand we’ve choosen the phases such that any multisoliton solution to be real. \( K_j \) represents arbitrary real constants which does not depend on \( n \) or \( t \). The procedure of finding rational solutions relies mainly on this arbitrariness.

Now we can proceed to the 2-soliton solution which has the form:

\[
\begin{align*}
    f_n &= 1 + \exp(\eta_1 + \phi_1) + \exp(\eta_2 + \phi_2) + \exp(\eta_1 + \eta_2 + \phi_1 + \phi_2 + A_{12}) \\
    f'_n &= 1 + \exp(\eta_1 + \phi'_1) + \exp(\eta_2 + \phi'_2) + \exp(\eta_1 + \eta_2 + \phi'_1 + \phi'_2 + A_{12}) \\
    g_n &= 1 + \exp(\eta_1 + \psi_1) + \exp(\eta_2 + \psi_2) + \exp(\eta_1 + \eta_2 + \psi_1 + \psi_2 + A_{12}) \\
    g'_n &= 1 + \exp(\eta_1 + \psi'_1) + \exp(\eta_2 + \psi'_2) + \exp(\eta_1 + \eta_2 + \psi'_1 + \psi'_2 + A_{12})
\end{align*}
\]

(8)

where

\[
\exp A_{12} = \left[\frac{\sinh \left(\frac{P_1 - P_2}{2}\right)}{\sinh \left(\frac{P_1 + P_2}{2}\right)}\right]^2
\]

for solitons propagating in the same directions (\( \epsilon_1\epsilon_2 = 1 \)) and:

\[
\exp A_{12} = \left[\frac{\cosh \left(\frac{P_1 - P_2}{2}\right)}{\cosh \left(\frac{P_1 + P_2}{2}\right)}\right]^2
\]

for solitons propagating in opposite directions (\( \epsilon_1\epsilon_2 = -1 \)).

Using the classical procedure we can construct any \( N \)-soliton solution but for our purposes 1-soliton and 2-soliton solutions are sufficient.

3 Rational Solutions

The ”long wave limit” method of Ablowitz and Satsuma we’ll use henceforth consists in three main steps. In order to obtain a \( N \)-rational solution from \( N \)-soliton solution we have to:

- expand the Hirota form of the \( N \)-soliton solution in power series of \( P_j \) with \( j = 1, \ldots, N \)
- choose the phase constants such that all \( O(P_j^k) \) with \( k \) less than \( N \) to be zero.
- making \( P_j \to 0 \) we recover the \( N \)-rational solution from the \( O(P_j^l) \) terms with \( l \geq N \)

We are focusing on \( f_n \) and \( g_n \). The procedure for \( f'_n \) and \( g'_n \) works in the same manner. Thus, for 1-rational solutions we’ll expand \( f_n \) and \( g_n \) given by (6). We’ll use the following notations:

\[
\theta_j = 2\epsilon_j(1 + \alpha_0^2)t + n
\]
Jeff = \exp \eta^0_j

P_j = \delta \bar{p}_j

and the expansions will be in power of \( \delta \) and \( \bar{p}_j \) are \( O(1) \) terms.

So, up to \( O(\delta) \):

\[
f_n = 1 + \zeta_1 (1 + \delta \bar{p}_1 \theta_1 + O(\delta^2)) \left[ \frac{\alpha_0 (\epsilon_1 + 1)}{2 \bar{p}_1} K_1 \delta^{-1} + \frac{i}{2} K_1 + \frac{K_1 \bar{p}_1 \alpha_0}{12} \delta + O(\delta^2) \right]
\]

\[
g_n = (f_n)^*
\]

where (*) means changing the sign of the terms multiplicated with \( i \) (complex conjugate as if \( \zeta_j \) would be real for \( j = 1, 2 \)).

For \( \epsilon_1 = 1 \) we must take \( K_1 \) proportional with \( \delta \). To cancel \( O(1) \) terms we’re taking \( \zeta_1 = -1/\alpha_0 \). Thus when \( \delta \to 0 \) we obtain:

\[
u_n = \alpha_0 - \frac{i}{2} \frac{d}{dt} \log \left( \frac{\theta_1 + i/2\alpha_0}{\theta_1 - i/2\alpha_0} \right) = \alpha_0 - \frac{4\alpha_0 (1 + \alpha_0^2)}{4\alpha_0^2 \theta_1^2 + 1}
\]

In the same manner \( v_n = u_n \) so, up to a constant and time scaling factor,

\[
Q_n = R_n \sim -\frac{4\alpha_0 (1 + \alpha_0^2)}{4\alpha_0^2 \theta_1^2 + 1}
\]

which is a nonsingular real rational solution.

For \( \epsilon_1 = -1 \) the situation is dramatically changed. In this case the expansions becomes:

\[
f_n = 1 + \frac{i}{2} K_1 \zeta_1 + \frac{i}{2} K_1 \zeta_1 \bar{p}_1 (\theta_1 - i\alpha_0 / 2) \delta + O(\delta^2)
\]

\[
g_n = (f_n)^*
\]

We have no divergent terms in \( \delta \) so we can take \( K_1 \sim O(1) \). But in this case the \( O(1) \) terms appear to be complex conjugate in \( f_n \) and \( g_n \). Accordingly there is no \( \zeta_1 \) to cancel both of them. To cope with this situation, we can see that \( u_n \) and \( v_n \) are invariant under the following ”gauge” transformation:

\[
\frac{f_n}{g_n} \to \frac{f_n}{g_n} \Lambda
\]

where \( \Lambda \) is an arbitrary complex function which does not depend on \( t \).

Now, taking \( \zeta_1 = 2i/K_1 \) we’ll cancel the \( O(1) \) terms only in \( f_n \). So,

\[
f_n = -\bar{p}_1 (\theta_1 - i\alpha_0 / 2) \delta + O(\delta^2)
\]

\[
g_n = 2 + \bar{p}_1 (\theta_1 + i\alpha_0 / 2) \delta + O(\delta^2)
\]

Choosing \( \Lambda = 1/\delta \) and then \( \delta \to 0 \) we’ll find:

\[
\frac{f_n}{g_n} = -\frac{1}{2} \bar{p}_1 (\theta_1 - i\alpha_0 / 2)
\]
and:
\[ u_n = \alpha_0 - \frac{i}{2} \frac{d}{dt} \log (\theta_1 - i\alpha_0/2) \]
\[ v_n = (u_n)^* \] (11)

which represents complex solitary waves. The physical significance of this type of solutions is not clear. The form of \( Q_n \) and \( R_n \) will be given up to constants:

\[ Q_n = (R_n)^* \sim -\frac{\alpha_0}{2} \frac{1 + \alpha_0^2}{\theta_1^2 + \alpha_0^2/4} + i \frac{\theta_1(1 + \alpha_0^2)}{\theta_1^2 + \alpha_0^2/4} \]

In order to study ”head-on” collision of the solutions (9) and (11) we’ll construct the 2-rational solution from 2-soliton solution with solitons having opposite directions i. e. \( f_n \) and \( g_n \) are given by (8) and phase factors by (7) with \( \epsilon_1 \epsilon_2 = -1 \). Expanding \( f_n \) and \( g_n \) up to \( O(\delta^2) \) we’ll take \( K_2 \sim O(1) \) in order to cut divergent behaviour of \( \delta^{-1} \) type in \( \exp \phi_1 \)

\[ f_n = M_0 + M_1 \delta + M_2 \delta^2 + O(\delta^3) \]
\[ g_n = (f_n)^* \]

where:

\[ M_0 = 1 + \zeta_1 \alpha_0 + \frac{i}{2} K_2 \zeta_2 + \frac{i}{2} K_2 \zeta_1 \zeta_2 \]
\[ M_1 = \frac{i}{2} \bar{p}_1 \zeta_1 + \alpha_0 \bar{p}_1 \zeta_1 \theta_1 + \frac{1}{4} \alpha_0 \zeta_2 K_2 \bar{p}_2 + \frac{i}{2} \bar{p}_2 \zeta_2 \theta_2 K_2 + \]
\[ + \frac{K_2}{4} (\alpha_0^2 \bar{p}_2 - 1) \bar{p}_1 \zeta_1 \zeta_2 + \frac{i}{2} \alpha_0 K_2 \zeta_1 \zeta_2 (\bar{p}_1 \theta_1 + \bar{p}_2 \theta_2) \] (12)

The expression of \( M_2 \) is very complicated. Anyway after choosing the phase constants it will be strongly simplified.

Thus we can cancel \( O(1) \) terms both in \( f_n \) and \( g_n \) by taking \( \zeta_1 = -1/\alpha_0 \) but we cannot do this for \( O(\delta) \) terms. Taking \( K_2 \sim O(1) \) and \( \zeta_2 = 2i/K_2 \) we’ll do it for \( f_n \) only. Introducing a ”gauge” transformation in the form \( \Lambda = 1/\delta \) we will recover \( f_n \) from \( M_2 \) and \( g_n \) from \((M_1)^*\). So, the rational solution will be:

\[ f_n = \theta_1 \theta_2 - \frac{3}{4} + \frac{i}{2} \frac{\theta_2}{\alpha_0} - \frac{\alpha_0 \theta_1}{\alpha_0} \]
\[ g_n = \theta_1 - \frac{i}{2 \alpha_0} \] (13)

and

\[ f_n' = \theta_1 + \frac{i}{2 \alpha_0} \]
\[ g_n' = \theta_1 \theta_2 - \frac{3}{4} - \frac{i}{2} \frac{\theta_2}{\alpha_0} - \frac{\alpha_0 \theta_1}{\alpha_0} \] (14)
4 Dynamics and Properties

The solutions (13) and (14) represent a "head-on" collision of two rational solitary waves one being real and the other complex, having same but opposite velocities. To see this we’re keeping $\theta_1$ fixed in (13) and make $t \rightarrow \pm \infty$. We’ll find that (13) becomes:

$$u_n \rightarrow \alpha_0 - \frac{i}{2} \frac{d}{dt} \log \left( \frac{\theta_1 + i/2\alpha_0}{\theta_1 - i/2\alpha_0} \right)$$

which represent exactly the real solitary wave(9).

Also, keeping $\theta_2$ being fixed and making $t \rightarrow \pm \infty$ we’ll get:

$$u_n \rightarrow \alpha_0 - \frac{i}{2} \frac{d}{dt} \log \left( \frac{\theta_2 - i\alpha_0/2}{\theta_2 + i\alpha_0/2} \right)$$

which means that we have no phase shift at this "head-on" collision so, we have a (1+1)-dimensional discrete "lump-type" dynamics. Also, the real and complex character is preserved. We don’t know exactly if does exist real nonsingular rational solutions for both directions.

5 Conclusions

To summarize, we’ve showed that rational nonsingular solutions both real and complex exists even for discrete systems. So this system posses both solitons and nonsingular (1+1)-dimensional discrete rational solutions decaying to zero as $O(n^{-2})$ and their character being real and complex, depending on the propagation direction.

We’ve discussed the dynamics of this rational solutions and we’ve showed that we have no phase shift at the interaction, specific for the (2+1)-dimensional continuous "lumps" solutions. Unfortunately we don’t know any condition of existence for this type of solutions and why they occur in (1+1)-dimensional systems. Its continuum limit could be viewed as a coupled pair of mKdV equations with nonzero boundary conditions. We can see that the form of 1-rational solution (the real one) has the same structure as those of mKdV given by Ono [12] and Ablowitz-Satsuma [7], so the continuum does not affect it drastically. Probably a setting up of this type of solutions in a I. S. T. framework will clarify much more about their form, dynamics, stability.

AKNOWLEDGEMENTS: The author would like to express his sincere thanks
to Prof. J. Hietarinta and Prof. M. J. Ablowitz for valuable discussions during the Summer School "Painleve property, one century later", Cargese, Corsica 1996. I am also very indebted to Prof. R. Conte for his kindness and hospitality which allowed me to participate at this Summer School.

References

[1] M. J. Ablowitz, H. Segur *Solitons and the Inverse Scattering Transform* SIAM, Philadelphia, (1981)

[2] R. Hirota, J. Satsuma, Progr. Theor. Phys. Suppl. 59 (1976) 64

[3] C. Rogers, W. Shadwick *Backlund Transformations and their Applications* New York, Academic, Mathematics in Science and Engineering 161

[4] H. Airault, H. P. McKean, J. Moser, Comm. Pure. Appl. Math. 30 (1977) 1

[5] D. Grecu, V. Cionga, Rev. Roum. Phys. 33 (1988) 3.

[6] J. J. C. Nimmo, Phys. Lett. 99 A, (1983) 281

[7] M. J. Ablowitz, J. Satsuma, J. Math. Phys. 19, 10 (1978) 2180

J. Satsuma, M. J. Ablowitz, J. Math. Phys. 20, 7, (1979) 1496

[8] A. S. Cărstea, D. Grecu, to appear in Progr. Theor. Phys., July (1996)

[9] X. B. Hu, P. Clarkson, J. Phys. A: Math. Gen 28 (1995) 5009

[10] H. C. Freeman *Soliton Interactions in two dimensions*, Adv. Appl. Mech. 20 (1980) 1-37

[11] B. D. Konopelchenko *Introduction to multidimensional integrable systems*, Plenum Press, New York (1980)

[12] H. Ono, J. Phys. Soc. Japan 41 (1976) 1817

[13] M. Wadati, Progr. Theor. Phys. Suppl. 59 (1976) 36

[14] M. Wadati, in *Solitons* ed. R. Bullough and P. J. Caudrey, Springer-Verlag, Berlin (1980)

[15] M. J. Ablowitz, J. Ladik, J. Math. Phys. 16, 3, (1975)