Structured Latent Factor Analysis for Large-scale Data: Identifiability, Estimability, and Their Implications

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Abstract

Latent factor models are widely used to measure unobserved latent traits in social and behavioral sciences, including psychology, education, and marketing. When used in a confirmatory manner, design information is incorporated, yielding structured (confirmatory) latent factor models. Motivated by the applications of latent factor models to large-scale measurements which consist of many manifest variables (e.g. test items) and a large sample size, we study the properties of structured latent factor models under an asymptotic setting where both the number of manifest variables and the sample size grow to infinity. Specifically, under such an asymptotic regime, we provide a definition of the structural identifiability of the latent factors and establish necessary and sufficient conditions on the measurement design that ensure the structural identifiability under a general family of structured latent factor models. In addition, we propose an estimator that can consistently recover the latent factors under mild conditions. This estimator can be efficiently computed through parallel computing. Our results shed lights on the design of large-scale measurement and have important
implications on measurement validity. The properties of the proposed estimator are verified through simulation studies.

KEY WORDS: High-dimensional latent factor model, confirmatory factor analysis, identifiability of latent factors, structured low-rank matrix, large-scale psychological measurement

1 Introduction

Multivariate data analysis (Anderson, 2003) has been one of the most important topics in modern statistics that is widely encountered in different disciplines including psychology, education, marketing, economics, medicine, engineering, geography, and biology. Latent factor models, originated from the seminal work of Spearman (1904) on human intelligence, play a key role in multivariate data analysis (Anderson, 2003; Skrondal and Rabe-Hesketh, 2004; Bartholomew et al., 2008, 2011). Latent factor models capture and interpret the common dependence among multiple manifest variables through the use of low-dimensional latent factors.

A key application of latent factor models is in psychological measurement, where the latent factors are interpreted as the psychological traits (e.g., cognitive abilities, personality traits, and psychopathic traits) that are not directly observable. By modeling the relationship between the latent factors and the manifest variables, the ultimate goal of latent factor analysis in psychological measurement is to make statistical inference on the individual specific latent traits. Specifically, after fitting a latent factor model, each individual will be assigned latent factor scores, as an estimate of his/her levels on the corresponding psychological traits. Decision will be made based on the factor scores, such as deciding whether a student has mastered a certain skill, ranking the students according to their proficiency levels on a skill, describing the personality profile of an individual, and diagnosing whether a patient suffers from a certain mental health disorder.

Due to the confirmatory nature of psychological measurement, item design information,
i.e. how each item is associated with latent traits, is often available a priori and used in fitting a latent factor model. Such information is incorporated into the model through constraints on the model parameters, resulting in a *structured latent factor model*. A good item design is key to measurement validity, guaranteeing that the model-based measurement reflects what it is supposed to measure ([Wainer and Braun, 2013](#)).

This paper develops statistical theory and methods for the design and the scoring in psychological measurement, which are the central problems to the psychological measurement theory ([AERA et al., 2014](#)). Motivated by large-scale assessments, we adopt an asymptotic regime in which the sample size $N$ and the number of items $J$ grow to infinity. Under this regime, we provide insights into the design of measurement based on our theoretical development on the *structural identifiability* of latent factors, a concept proposed in this paper that is central to large-scale measurement. This notion of identifiability is different from the classic definition of identifiability and may shed lights on the identifiability of infinite-dimensional models for non-i.i.d. data that are widely encountered in modern statistics, including in network analysis and spatial-temporo statistics. Moreover, necessary and sufficient conditions are established for the structural identifiability of a latent factor, which formalizes the intuition *a latent factor can be identified when it is measured by sufficiently many items that distinguish it from the other factors*. This result explains the reason why the “simple structure” design is popular in psychological measurement ([Cattell, 2012](#)). Our asymptotic results also provide theoretical guarantee to the use of estimated factor scores in making decisions (e.g. classification and ranking) when the corresponding latent factors are structurally identifiable.

The rest of the paper is organized as follows. In Section 2 we introduce a generalized latent factor modeling framework, within which our research questions are formulated. In Section 3 we discuss the structural identifiability for latent factors, the relationship between structural identifiability and estimability, and provide an estimator that consistently estimates the structurally identifiable latent factors. Further implications of our theoretical
results on large-scale measurement are provided in Section 4 and extensions of our results to more complex settings are discussed in Section 5. A new perturbation bound on linear subspaces is presented in Section 6 that is useful to statistical analysis of low-rank matrix estimation. Our theoretical results are verified by simulation studies in Section 7. Finally, concluding remarks are provided in Section 8. The proofs of all the technical results are provided in the supplement.

2 Structured Latent Factor Analysis

2.1 Generalized Latent Factor Model

Consider that there are \( N \) individuals and \( J \) manifest variables (e.g. \( J \) test items). Let \( Y_{ij} \) be a random variable denoting the \( i \)th individual’s value on the \( j \)th manifest variable and let \( y_{ij} \) be its realization. For example, in educational tests, \( Y_{ij}s \) could be binary responses from the examinees, indicating whether the answers are correct or not. We further assumes that each individual \( i \) is associated with a \( K \)-dimensional latent vector, denoted as \( \theta_i = (\theta_{i1}, ..., \theta_{iK})^\top \) and each manifest variable \( j \) is associated with \( K \) parameters \( a_j = (a_{j1}, ..., a_{jK})^\top \). We give two concrete contexts. Consider an educational test of mathematics, with \( K = 3 \) dimensions of “algebra”, “geometry”, and “calculus”. Then \( \theta_{i1}, \theta_{i2}, \) and \( \theta_{i3} \) represent individual \( i \)'s proficiency levels on algebra, geometry, and calculus, respectively. In the measurement of Big Five personality factors (Goldberg, 1993), \( K = 5 \) personality factors are considered, including “openness to experience”, “conscientiousness”, “extraversion”, “agreeableness”, and “neuroticism”. Then \( \theta_{i1}, ..., \theta_{i5} \) represent individual \( i \)'s levels on the continuums of the five personality traits. The manifest parameter \( a_j \)s can be understood as the regression coefficients when regressing \( Y_{ij}s \) on \( \theta_i \)s, \( i = 1, ..., N \). The manifest parameter \( a_j \)s are also known as the factor loadings in the factor analysis literature (e.g. Bartholomew et al., 2008) and the discrimination parameters in the item response theory literature (e.g. Embretson and Reise, 2000). In many applications of latent factor models, especially in psychology and education,
the estimations of $\theta_i$s and $a_j$s are both of interest (e.g. Bartholomew et al., 2008).

Our development is under a generalized latent factor model framework (Skrondal and Rabe-Hesketh, 2004), which extends the generalized linear model framework (McCullagh and Nelder, 1989) to latent factor analysis. This general modeling framework allows for different types of manifest variables. Specifically, we assume that the distribution of $Y_{ij}$ given $\theta_i$ and $a_j$ is a member of the exponential family with natural parameter

$$m_{ij} = a_j^T \theta_i = a_{j1} \theta_{i1} + \cdots + a_{jK} \theta_{iK},$$

and possibly a scale (i.e. dispersion) parameter $\phi$. More precisely, the density/probability mass function takes the form:

$$f(y|a_j, \theta_i, \phi) = \exp \left( \frac{ym_{ij} - b(m_{ij})}{\phi} + c(y, \phi) \right),$$

where $b(\cdot)$ and $c(\cdot)$ are pre-specified functions that depend on the member of the exponential family. Given $\theta_i$ and $a_j$, $i = 1, ..., N$ and $j = 1, ..., J$, we assume all $Y_{ij}$s are independent. Consequently, the likelihood function, in which $\theta_i$s and $a_j$s are treated as fixed effects, can be written as

$$L(\theta_1, ..., \theta_N, a_1, ..., a_J, \phi) = \prod_{i=1}^N \prod_{j=1}^J \exp \left( \frac{y_{ij}m_{ij} - b(m_{ij})}{\phi} + c(y_{ij}, \phi) \right).$$

This likelihood function is known as the joint likelihood function in the literature of latent variable models (Skrondal and Rabe-Hesketh, 2004). Since the likelihood function depends on $a_j$ and $\theta_i$ only through $m_{ij}$s, it has rotational and scaling indeterminacy. That is, the likelihood remains unchanged when we replace $\theta_i$ and $a_j$ by $D\theta_i$ and $(D^{-1})^T a_j$, for all $i$ and $j$, where $D$ can be any invertible $K \times K$ matrix.

We remark that in the existing literature of latent factor models, there is often an intercept term indexed by $j$ in the specification of $\Pi$, which can be easily realized under our
formulation by constraining $\theta_{i1} = 1$, for all $i = 1, 2, ..., N$. In that case, $a_{j1}$ serves as the intercept term.

This family of models contains special cases, such as the linear factor analysis model (e.g. Anderson, 2003; Bartholomew et al., 2008), multidimensional item response theory model for binary responses that plays a key role in educational assessment (e.g. Reckase, 2009), and the Poisson model that is widely used to analyze multivariate count (e.g. Moustaki and Knott, 2000). We list their forms below.

1. Linear Factor Analysis: $Y_{ij} | \theta_i, a_j \sim N(a_j^\top \theta_i, \sigma^2)$, where the scale parameter $\phi = \sigma^2$.

2. Multidimensional Item Response Theory (MIRT): $Y_{ij} | \theta_i, a_j \sim \text{Bernoulli} \left( \frac{\exp(a_j^\top \theta_i)}{1 + \exp(a_j^\top \theta_i)} \right)$, where the scale parameter $\phi = 1$.

3. Poisson Factor Model: $Y_{ij} | \theta_i, a_j \sim \text{Poisson} \left( \exp(a_j^\top \theta_i) \right)$, where the scale parameter $\phi = 1$.

Usually, the likelihood function (2) is not used for maximum likelihood analysis. This is possibly because, under the conventional asymptotic setting that $J$ is fixed and $N$ grows to infinity, the number of parameters in (2) diverges due to the growing number of person parameters, resulting in inconsistent maximum likelihood estimation. This phenomenon is first pointed out in Neyman and Scott (1948) and is further investigated in subsequent developments, including Andersen (1970), Haberman (1977), Fischer (1981), and Ghosh (1995). Consequently, in generalized latent factor analysis, the person parameters $\theta_i$s are typically assumed to be random effects (i.e., independent and identically distributed samples from a distribution) and are integrated out from the joint likelihood function (2), while the manifest parameter $a_j$s are still regarded as fixed effects. The resulting likelihood function is known as the marginal likelihood.

The analysis of this paper focuses on the joint likelihood function (2) where both $\theta_i$s and $a_j$s are treated as fixed effects, under an asymptotic setting that both $N$ and $J$ grow to infinity. For ease of exposition, we assume the scale parameter is known in the rest of
the paper, while pointing out that it is straightforward to extend all the results to the case where it is unknown.

This asymptotic regime is motivated by large-scale assessments in psychology and education, where both the sample size and the number of manifest variables can be very large. Moreover, quantifying the estimation accuracy of $\theta_s$, which is a main focus of latent factor analysis in psychological and educational measurement, is more straightforward and thus easier to interpret under the fixed effect point of view. We point out that a similar asymptotic setting has been adopted in Haberman (1977) for the analysis of the Rasch model (Rasch, 1960), a simple unidimensional latent factor model that is widely used in educational measurement.

### 2.2 Confirmatory Structure

In this paper, we consider a confirmatory setting where domain knowledge is available for the manifest variables. For example, in a personality assessment in psychology, what personality factor each item measures is pre-specified. For instance, the item “I am the life of the party” measures trait “extraversion” and the item “I get stressed out easily” measures trait “neuroticism” in a Big Five personality test. In an educational assessment, the item design is also pre-specified in the test blueprint. For example, one item may measure both algebra and geometry and another may measure calculus solely. Such information is typically reflected by constraints on the manifest parameters $a_{jks}$. Specifically, for each item $j$, there is a pre-specified vector $q_j = (q_{j1}, ..., q_{jK})$, where $q_{jk} = 1$ means that latent factor $k$ is measured by manifest variable $j$ and thus no constraint is imposed on $a_{jyk}$, and $q_{jk} = 0$ implies that latent factor $k$ is independent with manifest variable $j$ and thus $a_{jyk}$ is set to 0.

Intuitively, a good design leads to superior measurement results. This intuition is formalized in this paper through asymptotic analysis which establishes a relationship between the design information given by $q_j$s and the structural identifiability of the latent factors that is defined in Section 3.
2.3 Research Questions

This paper focuses on the identifiability of the latent factors. To define the identifiability, consider a population of people where \( N = \infty \) and a population of manifest variables where \( J = \infty \). A latent factor \( k \) is a hypothetical construct, defined by the person population. More precisely, it is determined by the individual latent factor scores of the entire person population, denoted by \((\theta_{1k}^*, \theta_{2k}^* \ldots) \in \mathbb{R}^{Z^+}\), where \( \theta_{ik}^* \) denotes the true latent factor score of person \( i \) on latent factor \( k \) and \( \mathbb{R}^{Z^+} \) denotes the set of vectors with countably infinite real number components. The identifiability of the \( k \)th latent factor then is equivalent to the identifiability of a vector in \( \mathbb{R}^{Z^+} \) under the distribution of an infinite dimensional random matrix, \( \{Y_{ij} : i = 1, 2, \ldots, j = 1, 2, \ldots\} \).

The above setting is natural in the context of large-scale measurement, but is a nonstandard asymptotic setting in statistics. Under this setting, this paper addresses three research questions that are central to modern measurement theory. First, how should the identifiability of latent factors be suitably formalized? Second, under what design are the latent factors identifiable? Third, what is the relationship between the identifiability and estimability? In other words, we want to know whether and to what extend the scores of an identifiable latent factor can be recovered from data.

The identifiability of latent factor models is an important problem in statistics. Research on this topic dates back to Anderson and Rubin (1956) and has received much attention by statisticians under both low- and high-dimensional settings (e.g. Anderson, 1984; Bai et al., 2012). To the best of our knowledge, this is the first work characterizing the relationship between measurement design information (reflected by constraints) and the identifiability of latent factor models under a high-dimensional setting. In addition, our developments apply to a general model class. Both the incorporation of design information and the general model form make our problem technically challenging, involving the asymptotic analysis of a non-convex optimization problem. As will be shown in the rest of the paper, we tackle these challenges by proving useful probabilistic error bounds and by developing perturbation
bounds on the intersection of linear subspaces.

2.4 Preliminaries

In this section, we fix some notations used throughout this paper.

Notations.

a. $\mathbb{Z}_+$: the set of all positive integers.

b. $\mathbb{R}^{\mathbb{Z}_+}$: the set of vectors with countably infinite real number components.

c. $\mathbb{R}^{\mathbb{Z}_+ \times \{1, \ldots, K\}}$: the set of all the real matrices with countably infinite rows and $K$ columns.

d. $\{0, 1\}^{\mathbb{Z}_+ \times \{1, \ldots, K\}}$: the set of all the binary matrices with countably infinite rows and $K$ columns.

e. $\Theta$: the parameter matrix for the person population, $\Theta \in \mathbb{R}^{\mathbb{Z}_+ \times \{1, \ldots, K\}}$.

f. $A$: the parameter matrix for the manifest variable population, $A \in \mathbb{R}^{\mathbb{Z}_+ \times \{1, \ldots, K\}}$.

g. $Q$: the design matrix for the manifest variable population, $Q \in \{0, 1\}^{\mathbb{Z}_+ \times \{1, \ldots, K\}}$.

h. $\mathbf{0}$: the vector or matrix with all components being 0.

i. $P_{\Theta, A}$: the probability distribution of $(Y_{ij}, i, j \in \mathbb{Z}_+)$, given person and item parameters $\Theta$ and $A$.

j. $v_{[1:m]}$: the first $m$ components of a vector $v$.

k. $W_{[S_1, S_2]}$: the submatrix of a matrix $W$ formed by rows $S_1$ and columns $S_2$, where $S_1, S_2 \subset \mathbb{Z}_+$.

l. $W_{[1:m,k]}$: the first $m$ components of the $k$-th column of a matrix $W$.

m. $W_{[k]}$: the $k$-th column of a matrix $W$. 


n. \( \|v\| \): the Euclidian norm of a vector \( v \).

o. \( \sin \angle(u, v) \): the sine of the angle between two vectors,

\[
\sin \angle(u, v) = \text{sgn}(uv^\top) \sqrt{1 - \left( \frac{(uv^\top)^2}{\|u\|^2\|v\|^2} \right)},
\]

where \( u, v \in \mathbb{R}^m \), \( u, v \neq 0 \), and the function \( \text{sgn}(x) \) takes value 1, 0, and \(-1\) when \( x \) is positive, zero, and negative, respectively.

p. \( \|W\|_F \): the Frobenius norm of a matrix \( W = (w_{ij})_{m \times n} \), \( \|W\|_F \triangleq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij}^2} \).

q. \( \|W\|_2 \): the spectral norm of matrix \( W \), i.e., the largest singular value of matrix.

r. \( \sigma_1(W) \geq \sigma_2(W) \geq \ldots \geq \sigma_n(W) \): the singular values of a matrix \( W \in \mathbb{R}^{m \times n} \), in a descending order.

s. \( \gamma(W) \): a function mapping from \( \mathbb{R}^{Z_+ \times n} \) to \( \mathbb{R} \), defined as

\[
\gamma(W) \triangleq \lim_{m \to \infty} \inf m \left( \frac{\sigma_n(W_{[1:m,1:n]})}{\sqrt{m}} \right).
\]

3 Structural Identifiability and Theoretical Results

3.1 Structural Identifiability

We first formalize the definition of structural identifiability. For two vectors with countably infinite components \( w = (w_1, w_2, \ldots)^\top, z = (z_1, z_2, \ldots)^\top \in \mathbb{R}^{Z_+} \), we define

\[
\sin_+ \angle(w, z) = \lim_{n \to \infty} \sup |\sin \angle(w_{[1:n]}, z_{[1:n]})|,
\]
which quantifies the angle between two vectors $w$ and $z$ in $\mathbb{R}^{Z+}$. In particular, we say the angle between $w$ and $z$ is zero when $\sin_+ \angle (w, z)$ is zero.

**Definition 1 (Structural identifiability of a latent factor).** Consider the $k$th latent factor, where $k \in \{1, \ldots, K\}$, and a nonempty parameter space $S \subset \mathbb{R}^{Z+} \times \{1, \ldots, K\} \times \mathbb{R}^{Z+} \times \{1, \ldots, K\}$ for $(\Theta, A)$. We say the $k$-th latent factor is structurally identifiable in the parameter space $S$ if for any $(\Theta, A), (\Theta', A') \in S$, $P_{\Theta, A} = P_{\Theta', A'}$ implies $\sin_+ \angle (\Theta_{[k]}, \Theta'_{[k]}) = 0$.

We point out that the parameter space $S$ is essentially determined by the design information $q_{jks}$. As will be shown shortly in this section, a good design imposes suitable constraints on the parameter space, which further ensures the structure identifiability of the latent factors. This definition of identifiability avoids the consideration of the scale of the latent factor, which is not uniquely determined as the distribution of data only depends on $\{\theta_i^T a_j : i, j \in Z_+\}$. Moreover, the sine measure is a canonical way to quantify the distance between two linear spaces that has been used in, for example, the well-known sine theorems for matrix perturbation (Davis, 1963; Wedin, 1972). As will be shown in the sequel, this definition of structural identifiability naturally leads to a relationship between identifiability and estimability and has important implications on psychological measurement.

We now characterize the structural identifiability under suitable regularity conditions. We consider a design matrix $Q$ for the manifest variable population, where $Q \in \{0, 1\}^{Z+ \times \{1, \ldots, K\}}$. Throughout the paper, we consider design matrices satisfying the following stability assumption.

A1 The limit

$$p_Q(S) = \lim_{j \to \infty} \frac{|\{j : q_{jk} = 1, \text{ if } k \in S \text{ and } q_{jk} = 0, \text{ if } k \notin S, 1 \leq j \leq J\}|}{J}$$

exists for any subset $S \subset \{1, \ldots, K\}$. In addition, $p_Q(\emptyset) = 0$.

Note that $p_Q(S)$ is the proportion of manifest variables that are associated with and only with latent factors in $S$. In addition, $p_Q(\emptyset) = 0$ implies that there are few irrelevant manifest
variables. We also make the following assumption on the generalized latent factor model, that is satisfied under most of the widely used models, including the linear factor model, MIRT model, and the Poisson factor model listed above.

A2 The natural parameter space \( \{ \nu : |b(\nu)| < \infty \} = \mathbb{R} \).

Under the above assumptions, Theorem 1 provides a necessary and sufficient condition on the design matrix \( Q \) for the structural identifiability of the \( k \)-th latent factor. This result is established within the parameter space \( S_Q \subseteq \mathbb{R}^{Z^+ \times \{1, \ldots, K\}} \times \mathbb{R}^{Z^+ \times \{1, \ldots, K\}} \),

\[
S_Q = \left\{ (\Theta, A) \in \mathbb{R}^{Z^+ \times \{1, \ldots, K\}} \times \mathbb{R}^{Z^+ \times \{1, \ldots, K\}} : \| \theta_i \| \leq C, \| a_j \| \leq C, \gamma(\Theta) > 0, \right. \\
A_{[R_Q(S), S^c]} = 0 \text{ for all } S \subseteq \{1, \ldots, K\}, \gamma(A_{[R_Q(S), S^c]}) > 0 \text{ for all } S \text{ s.t. } p_Q(S) > 0 \left\}, \right.
\]

where \( C \) is a positive constant, the \( \gamma \) function is defined in (3) and

\[
R_Q(S) = \{ j : q_{jk} = 1, \text{ if } k \in S \text{ and } q_{jk} = 0, \text{ if } k \notin S \}
\]

denotes the set of manifest variables that are associated with and only with latent factors in \( S \). Discussions on the parameter space are provided after the statement of Theorem 1.

**Theorem 1.** Under Assumptions A1 and A2, the \( k \)-th latent factor is structurally identifiable in \( S_Q \) if and only if

\[
\{k\} = \bigcap_{k \in S, p_Q(S) > 0} S,
\]

where we define \( \bigcap_{k \in S, p_Q(S) > 0} S = \emptyset \) if \( p_Q(S) = 0 \) for all \( S \) that contains \( k \).

The following proposition guarantees that the parameter space is nontrivial.

**Proposition 1.** For all \( Q \) satisfying A1, \( S_Q \neq \emptyset \).

We further remark on the parameter space \( S_Q \). First, \( S_Q \) requires some regularities on each \( \theta_i \) and \( a_j \) (\( \| \theta_i \| \leq C, \| a_j \| \leq C \)) and the \( A \)-matrix satisfying the constraints imposed
by \( Q \) \((A_{[RQ(S),S]} = 0)\) for all \( S \). It further requires that there is enough variation among people, quantified by \( \gamma(\Theta) > 0 \), where the \( \gamma \) function is defined in (3). Note that this requirement is mild, in the sense that if \( \theta \)'s are i.i.d. with a strictly positive definite covariance matrix, then \( \gamma(\Theta) > 0 \) a.s., according to the strong law of large numbers. Furthermore, \( \gamma(A_{[RQ(S),S]}) > 0 \) for \( S \) satisfying \( p_Q(S) > 0 \) requires that each group of items (categorized by \( S \)) contains sufficient information if appearing frequently \( (p_Q(S) > 0) \). Similar to the justification for \( \Theta \), \( \gamma(A_{[RQ(S),S]}) > 0 \) can also be justified by considering that \( a_j \)'s are i.i.d. following a certain distribution for \( j \in R_Q(S) \).

We provide two examples to assist with understanding Theorem 1. If \( K = 2 \) and \( p_Q(\{1\}) = p_Q(\{1, 2\}) = 1/2 \), then the second latent factor is not structurally identifiable, even if it is associated with infinitely many manifest variables. In addition, having many manifest variables with a simple structure ensures the structural identifiability of a latent factor. That is, if \( p_Q(\{k\}) > 0 \), then the \( k \)th factor is structurally identifiable.

### 3.2 Identifiability and estimability

It is well known that for a fixed dimensional parametric model with i.i.d. samples, the identifiability of the model parameter is necessary for the existence of a consistent estimator. We extend this result to the infinite-dimensional parameter space under the current setting. We start with a generalized definition for the consistency of estimating a latent factor. An estimator given \( N \) individuals and \( J \) items is denoted by \((\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)})\), which only depends on \( Y_{[1:N,1:J]} \) for all \( N, J \in \mathbb{Z}_+ \).

**Definition 2** (Consistency for estimating latent factor \( k \)). The sequence of estimators \{\((\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)}), N, J \in \mathbb{Z}_+ \)\} is said to consistently estimate the latent factor \( k \) if

\[
\sin \angle (\hat{\Theta}^{(N,J)}_{[k]}, \Theta_{[1:N,k]}) \stackrel{P_{\Theta,A}}{\rightarrow} 0,
\]

for all \((\Theta, A) \in S_Q \).
The next proposition establishes the necessity of the structural identifiability of a latent factor on its estimability.

**Proposition 2.** If latent factor $k$ is not structurally identifiable in $S_Q$, then there does not exist a consistent estimator for latent factor $k$.

### 3.3 Estimation and Its Consistency

We further show that the structural identifiability and estimability are equivalent under our setting. For ease of exposition, let $Q$ be the true design matrix in $\{0, 1\}^{Z_{+} \times \{1, \ldots, K\}}$ satisfying assumption A1. In addition, let $(\Theta^*, A^*) \in S_Q$ be the true parameters for the person and the manifest variable populations. We provide an estimator $(\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)})$ such that

$$\sin \angle(\hat{\Theta}_{[k]}^{(N,J)}, \Theta^*_{[1:N,k]}) \xrightarrow{P_{(\Theta^*,A^*)}} 0, \quad N, J, \to \infty,$$

when $Q$ satisfies (4), which leads to the structural identifiability of latent factor $k$ under Theorem 1. Specifically, we consider the following estimator

$$(\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)}) = \arg\min l(\theta_1, \ldots, \theta_N, a_1, \ldots, a_J),$$

$$s.t. \|\theta_i\| \leq C', \|a_j\| \leq C',$$

$$a_j \in D_j, i = 1, \ldots, N; j = 1, \ldots, J,$$

where $l(\theta_1, \ldots, \theta_N, a_1, \ldots, a_J) = \sum_{i=1}^{N} \sum_{j=1}^{J} y_{ij}(\theta_i^\top a_j) - b(\theta_i^\top a_j)$, $C'$ is any constant greater than $C$ in the definition of $S_Q$, and $D_j = \{a \in \mathbb{R}^K : a_{jk} = 0 \text{ if } q_{jk} = 0\}$ imposes the constraint on $a_j$. Note that maximizing $l(\theta_1, \ldots, \theta_N, a_1, \ldots, a_J)$ is equivalent to maximizing the joint likelihood (2), due to the natural exponential family form. The next theorem provides an error bound on $(\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)})$.

**Theorem 2.** Under assumptions A1-A2 and $(\Theta^*, A^*) \in S_Q$, there exists $N_0$, $J_0$, and $\kappa_0 > 0$
such that for all $N \geq N_0$ and $J \geq J_0$, with probability $1 - 2/(N + J)$,

$$\frac{1}{N}\|\hat{\Theta}^{(N,J)}(\hat{A}^{(N,J)})^\top - \Theta_{[1:N,1:K]}^* A_{[1:J,1:K]}^*\|_F^2 \leq \kappa_0 \sqrt{\frac{N + J}{N}\log(NJ)}.
$$

Moreover, if $Q$ satisfies (4) and thus latent factor $k$ is structurally identifiable, then

$$|\sin \angle(\Theta_{[1:N,k]}^*, \hat{\Theta}^{(N,J)}_{[k]})| \leq \kappa_1 \left(\frac{N + J}{N}\log(NJ)\right)^{\frac{4}{4}}.
$$

with probability $1 - 2/(N + J)$, where $\kappa_1$ is a constant independent with $N$ and $J$.

Proposition 2 and Theorems 1 and 2 together imply that the structural identifiability and estimability over $S_Q$ are equivalent, which is summarized in the following corollary.

**Corollary 1.** Under Assumptions A1 and A2, there exists an estimator $(\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)})$ such that $\lim_{N,J \to \infty} \sin \angle(\hat{\Theta}^{(N,J)}_{[k]}, \Theta_{[1:N,k]}^* A_{[1:J,1:K]}^*) = 0$ in $P_{\Theta,A}$ for all $(\Theta, A) \in S_Q$ if and only if the design matrix $Q$ satisfies (4).

**Remark 1.** The error bound (6) holds even when one or more latent factors are not structurally identifiable. In particular, (6) holds when removing the constraint $a_j \in D_j$ from (5), which corresponds to the exploratory factor analysis setting where no design matrix $Q$ is pre-specified (or in other words, $q_{jk} = 1$ for all $j$ and $k$; see e.g. Chen et al., 2017).

**Remark 2.** The proposed estimator (5) and its error bound are related to low-rank matrix completion (e.g. Candès and Plan, 2010; Davenport et al., 2014), where a bound similar to (6) can typically be derived. The key differences are (a) the research on matrix completion is only interested in the estimation of $\Theta^* A^*\top$, while the current paper focuses on the estimation of $\Theta^*$ that is a fundamental problem of psychological measurement and (b) our results are derived under a generalized latent factor model that covers many models.

We end this section by providing an alternating minimization algorithm (Algorithm 1) for solving the optimization program (5), which is computationally efficient through our parallel computing implementation using Open Multi-Processing (OpenMP; Dagum and Menon,
Specifically, we adopt a projected gradient descent update (e.g., Parikh and Boyd, 2014) to handle the constraints, where the projections have closed-form solutions. Similar algorithms have been considered in other works, such as Udell et al. (2016) and Zhu et al. (2016), for solving optimization problems with respect to low-rank matrices. Convergence properties of this type of algorithms have also been studied (e.g., Zhao et al., 2015).

Algorithm 1: Alternating minimization algorithm

1. **Input:** Data \((y_{ij} : 1 \leq i \leq N, 1 \leq j \leq J)\), dimension \(K\), constraint parameter \(C'\), initial iteration number \(l = 1\), and initial value \(\theta_i^{(0)}\) and \(a_j^{(0)} \in D_j\), \(i = 1, ..., N\), \(j = 1, ..., J\).

2. **Alternating minimization:**
   
   for \(l = 1, 2, ...\) do
   
   for \(i = 1, 2, ..., N\) do
   
   \[ \theta_i^{(l)} = \text{Prox}_{x \in \mathbb{R}^K: |x| \leq C'}(\theta_i^{(l-1)} - \eta \hat{g}_i^{(l)}) \], where \(\hat{g}_i^{(l)} = -\frac{\partial l_i(\theta)}{\partial \theta} \bigg|_{\theta = \theta^{(l-1)}}\) and
   
   \[ l_i(\theta) = \sum_{j=1}^{J} y_{ij} (\theta^T a_j^{(l-1)}) - b((\theta^T a_j^{(l-1)}))^T \]. \(\eta > 0\) is a step size chosen by line search.
   
   end
   
   for \(j = 1, 2, ..., J\) do
   
   \[ a_j^{(l)} = \text{Prox}_{a \in \mathbb{R}^J: |a| \leq C'}(\text{Prox}_{D_j}(a_j^{(l-1)} - \eta \hat{g}_j^{(l)})) \], where \(\hat{g}_j^{(l)} = -\frac{\partial l_j(a)}{\partial a} \bigg|_{a = a_j^{(l-1)}}\) and
   
   \[ l_j(a) = \sum_{i=1}^{N} y_{ij} ((\theta_i^{(l)})^T a) - b((\theta_i^{(l)})^T a) \]. \(\eta > 0\) is a step size chosen by line search.
   
   end

3. **Output:** Iteratively perform Step 2 until convergence. Output \(\hat{\theta}_i = \theta_i^{(L)}, \hat{a}_j = a_j^{(L)}, i = 1, ..., N, j = 1, ..., J\), where \(L\) is the last iteration number.

4 Further Implications

In this section, we discuss the implications of the above results on large-scale measurement.
Table 1: An example of the design matrix $Q$, which has infinite rows and $K = 3$ columns. The rows of $Q$ are given by repeating the first $3 \times 3$ submatrix infinite times.

|   | 1 | 2 | 3 | 4 | 5 | 6 | ⋱ |
|---|---|---|---|---|---|---|---|
| $Q^\top$ | 1 | 1 | 0 | 1 | 1 | 0 | ⋱ |
|     | 1 | 0 | 1 | 1 | 0 | 1 | ⋱ |
|     | 0 | 1 | 1 | 0 | 1 | 1 | ⋱ |

4.1 On the design of tests.

According to Theorems 1 and 2, the key to the structural identifiability and consistent estimation of factor $k$ is

$$\{k\} = \bigcap_{k \in S, p_Q(S) > 0} S,$$

which provides insights on the measurement design. First, it implies that the “simple structure” design that is advocated in psychological measurement is a safe design. Under the simple structure design, each manifest variable is associated with one and only one factor. If each latent factor $k$ is associated with many manifest variables that only measure factor $k$, or more precisely $p_Q(\{k\}) > 0$, (8) is satisfied.

Second, our result implies that a simple structure is not necessary for a good measurement design. A latent factor can still be identified even when it is always measured together with some other factors. For example, consider the $Q$-matrix in Table 1. Under this design, all three factors satisfy (8) even when there is no item measuring a single latent factor.

Third, (8) is not satisfied when there exists a $k' \neq k$ and $k' \in \bigcap_{k \in S, p_Q(S) > 0} S$. That is, almost all manifest variables that are associated with factor $k$ are also associated with factor $k'$, in the asymptotic sense. Consequently, one cannot distinguish factor $k$ from factor $k'$, making factor $k$ structurally unidentifiable. We point out that in this case, factor $k'$ may still be structurally identifiable; for example, when $p_Q(\{k'\}) > 0$.

Finally, (8) is also not satisfied when $\bigcap_{k \in S, p_Q(S) > 0} S = \emptyset$. It implies that the factor $k$ is not structurally identifiable when the factor is not measured by a sufficient number of manifest variables.
4.2 Properties of Estimated Factor Scores

A useful result. Let \((\Theta^*, A^*) \in \mathcal{S}_Q\) be the true parameters for the person and the manifest variable populations. The following corollary is derived from Theorem 2 that establishes a relationship between the true person parameters and their estimates. This result is the key to the rest of the results in this section.

**Corollary 2.** Under Assumption A1-A2 and (4) is satisfied for some \(k\), then there exists a sequence of random variables \(c_{N,J} \in \{-1,1\}\), such that

\[
\left\| \Theta^*_{[1:N,k]} - c_{N,J} \hat{\Theta}^{(N,J)}_{[k]} \right\| \overset{P_{(\Theta^*, A^*)}}{\rightarrow} 0, \quad N, J, \rightarrow \infty.
\]

**Remark 3.** Corollary 2 follows directly from (7). It provides an alternative view on how \(\hat{\Theta}^{(N,J)}_{[k]}\) approximates \(\Theta^*_{[1:N,k]}\). Since the likelihood function depends on \(\Theta_{[1:N,1:K]}\) and \(A_{[1,J,1:K]}\) only through \(\Theta_{[1:N,1:K]}^t A_{[1,J,1:K]}^\top\), the scale of \(\Theta_{[1:N,k]}\) is not identifiable even when it is structurally identifiable. This phenomenon is intrinsic to latent variable models (e.g. Skrondal and Rabe-Hesketh, 2004). Corollary 2 states that \(\Theta^*_{[1:N,k]}\) and \(\hat{\Theta}^{(N,J)}_{[k]}\) are close in Euclidian distance after properly normalized. The normalized vectors \(\Theta^*_{[1:N,k]}/\|\Theta^*_{[1:N,k]}\|\) and \(c_{N,J} \hat{\Theta}^{(N,J)}_{[k]}/\|\hat{\Theta}^{(N,J)}_{[k]}\|\) are both of unit length. The value of \(c_{N,J}\) depends on the angle between \(\Theta^*_{[1:N,k]}\) and \(\hat{\Theta}^{(N,J)}_{[k]}\). Specifically, \(c_{N,J} = 1\) if \(\cos \angle(\Theta^*_{[1:N,k]}, \hat{\Theta}^{(N,J)}_{[k]}) > 0\) and \(c_{N,J} = -1\) otherwise. In practice, especially in psychological measurement, \(c_{N,J}\) can typically be determined by additional domain knowledge.

**On the distribution of person population.** In psychological measurement, the distribution of true factor scores is typically of interest, which may provide an overview of the population on the constructs being measured. Corollary 2 implies the following proposition on the empirical distribution of the factor scores.

**Proposition 3.** Suppose assumptions A1-A2 are satisfied and furthermore (4) is satisfied.
for factor $k$. We normalize $\theta_{ik}^*$ and $\hat{\theta}_{ik}^{(N,J)}$ by

$$v_i = \frac{\sqrt{N} \theta_{ik}^*}{\| \Theta^*_{[1:N,k]} \|} \quad \text{and} \quad \hat{v}_i = \frac{c_{N,J} \sqrt{N} \hat{\theta}_{ik}^{(N,J)}}{\| \hat{\Theta}^{(N,J)}_{[k]} \|} \quad \text{for} \quad i = 1, \ldots, N,$$

(9)

where $c_{N,J}$ is defined and discussed in Corollary 3. Let $F_N$ and $\hat{F}_{N,J}$ be the empirical measures of $v_1, \ldots, v_N$ and $\hat{v}_1, \ldots, \hat{v}_N$, respectively. Then,

$$\text{Wass}(F_N, \hat{F}_{N,J}) \xrightarrow{P(\theta^*, A^*)} 0, \quad N, J, \to \infty,$$

where $\text{Wass}(\cdot, \cdot)$ denotes the Wasserstein distance between two probability measures

$$\text{Wass}(\mu, \nu) = \sup_{h \text{ is 1-Lipschitz}} \left| \int h \mu - \int h \nu \right|.$$

We point out that the normalization in (9) is reasonable. Consider a random design setting where $\theta_{ik}^*$s are i.i.d. samples from some distribution with a finite second moment. Then $F_N$ converges weakly to the distribution of $\eta/\sqrt{E\eta^2}$, where $\eta$ is a random variable following the same distribution. Proposition 3 then implies that when factor $k$ is structurally identifiable and both $N$ and $J$ are large, the empirical distribution of $\hat{\theta}_{1k}^{(N,J)}, \hat{\theta}_{2k}^{(N,J)}, \ldots, \hat{\theta}_{Nk}^{(N,J)}$ approximates the empirical distribution of $\theta_{1k}^*, \theta_{2k}^*, \ldots, \theta_{Nk}^*$ accurately, up to a scaling. Specifically, for any 1-Lipschitz function $h$, $\int h(x) \hat{F}_{N,J}(dx)$ is a consistent estimator for $\int h(x) F_N(dx)$ according to the definition of Wasserstein distance. Furthermore, Corollary 2 states that under the regularity conditions, $\lim_{N \to \infty} \sum_{i=1}^N (v_i - \hat{v}_i)^2/N = 0$, implying that $\sum_{i=1}^N 1_{\{(v_i - \hat{v}_i)^2 \geq \epsilon\}}/N = 0$, for all $\epsilon > 0$. That is, most of the $\hat{v}_i$s will fall into a small neighborhood of the corresponding $v_i$s.

On ranking consistency. The estimated factor scores may also be used to rank individuals along a certain construct. In particular, in educational testing, the ranking provides an ordering of the students’ proficiency in a certain ability (e.g., calculus, algebra, etc.).
Our results also imply the validity of the ranking along a latent factor when it is structurally identifiable and $N$ and $J$ are sufficiently large. More precisely, we have the following proposition.

**Proposition 4.** Suppose assumptions A1-A2 are satisfied and furthermore (4) is satisfied for factor $k$. Consider $v_i$ and $\hat{v}_i$, the normalized versions of $\theta_{ik}^*$ and $\hat{\theta}_{ik}^{(N,J)}$ as defined in (9). In addition, assume that there exists a constant $\kappa_R$ such that for any sufficiently small $\epsilon > 0$ and sufficiently large $N$,

$$\frac{\sum_{i\neq i'} I\{|v_i - v_{i'}| \leq \epsilon\}}{N(N - 1)/2} \leq \kappa_R \epsilon.$$  (10)

Then,

$$\frac{\tau(v, \hat{v})}{N(N - 1)/2} \overset{p_{(\epsilon\theta, A^*)}}{\rightarrow} 0, \quad N, J, \rightarrow \infty,$$  (11)

where $\tau(v, \hat{v}) = \sum_{i \neq i'} I(v_i > v_{i'}, \hat{v}_i < \hat{v}_{i'}) + I(v_i < v_{i'}, \hat{v}_i > \hat{v}_{i'})$ is the number of inconsistent pairs according to the ranks of $v = (v_1, ..., v_N)$ and $\hat{v} = (\hat{v}_1, ..., \hat{v}_N)$.

We point out that (10) is a mild regularity condition on the empirical distribution $F_N$. It requires that the probability mass under $F_N$ does not concentrate in any small $\epsilon$-neighborhood, which further implies that the pairs of individuals who are difficult to distinguish along factor $k$, i.e., $(i, i')s$ that $v_i$ and $v_{i'}$ are close, take only a small proportion among all the $(N - 1)N/2$ pairs. In fact, it can be shown that (10) is true with probability tending to 1 as $N$ grows to infinity, when $\theta_{ik}^*$s are i.i.d. samples from a distribution with a bounded density function. Proposition 4 then implies that if we rank the individuals using $\hat{v}_i$ (assuming $c_{N,J}$ can be consistently estimated based on other information), the proportion of incorrectly ranked pairs converges to 0. Note that $\tau(v, \hat{v})$ is known as the Kendall’s tau distance ([Kendall and Gibbons, 1990](#)), a widely used measure for ranking consistency.

**On classification consistency.** Another common practice of utilizing estimated factor scores is to classify individuals into two or more groups along a certain construct. For example, in an educational mastery test, it is of interest to classify examinees into “mas-
tery" and “nonmastery” groups according to their proficiency in a certain ability (Lord, 1980; Bartroff et al., 2008). In measuring psychopathology, it is common to classify respondents into “diseased” and “non-diseased” groups based on a mental health disorder. We justify the validity of making classification based on the estimated factor score.

**Proposition 5.** Suppose assumptions A1-A2 are satisfied and furthermore (4) is satisfied for factor \( k \). Consider \( v_i \) and \( \hat{v}_i \), the normalized versions of \( \theta_{ik}^* \) and \( \hat{\theta}_{ik}^{(N,J)} \) as defined in (9). Let \( \tau_- < \tau_+ \) be the classification thresholds, then

\[
\sum_{i=1}^{N} I\{\hat{v}_i \geq \tau_+, v_i \leq \tau_-\} + I\{\hat{v}_i \leq \tau_-, v_i \geq \tau_+\} \rightarrow 0, \quad N, J \to \infty,.
\]

Considering two pre-specified thresholds \( \tau_- \) and \( \tau_+ \) is the well-known indifference zone formulation of educational mastery test (e.g. Bartroff et al., 2008). In that context, examinees with \( v_i \geq \tau_+ \) are classified into the “mastery” group and those with \( v_i \leq \tau_- \) are classified into the “nonmastery” group. The interval \( (\tau_-, \tau_+) \) is known as the indifference zone, within which no decision is made. Proposition [5] then implies that when factor \( k \) is structurally identifiable, the classification error tends to 0 as both \( N \) and \( J \) grow to infinity.

## 5 Extensions

### 5.1 Generalized latent factor models with intercepts

As mentioned in Section 2.1, intercepts can be easily incorporated in the generalized latent factor model by restricting \( \theta_{i1} = 1 \). Then, \( a_{j1} \)s are the intercept parameters and \( q_{j1} = 1 \) for all \( j \). Consequently, for any \( S \) satisfying \( p_Q(S) > 0, 1 \in S \) and thus the latent factors 2-K are not structurally identifiable according to Theorem 1. Interestingly, these factors are still structurally identifiable if we restrict to the following parameter space

\[
S_{Q,-} = \left\{ (\Theta, A) \in S_Q : \lim_{N \to \infty} \frac{1}{N} 1_N^T \Theta_{[1:N,m]} = 0 \text{ for } m \geq 2, \text{ and } \theta_{i1} = 1 \text{ for } i \in \mathbb{Z}_+ \right\}.
\]
which requires that $\Theta_{[k]}$ and $\Theta_{[1]}$ are asymptotically orthogonal, for all $k \geq 2$.

**Proposition 6.** Under Assumptions A1-A2, and assuming that $q_{j1} = 1$ for all $j \in \mathbb{Z}_+$ and $K \geq 2$, then the $k$th latent factor is structurally identifiable in $S_{Q,-}$ if and only if

$$\{1, k\} = \bigcap_{k \in S, p(S) > 0} S,$$

for $k \geq 2$.

The next proposition guarantees that $S_{Q,-}$ is also non-empty.

**Proposition 7.** For all $Q$ satisfying A1 and $q_{j1} = 1$ for all $j \in \mathbb{Z}_+$, and in addition $C > 1$, then $S_{Q,-} \neq \emptyset$.

**Remark 4.** When having intercepts in the model, similar consistency results can be established for the estimator

$$(\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)}) \in \arg \min -l(\theta_1, ..., \theta_N, a_1, ..., a_J),$$

s.t. $\|\theta_i\| \leq C, \|a_j\| \leq C, a_j \in D_j$, $\theta_{i1} = 1, \sum_{i' = 1}^N \theta_{i'k} = 0,$ $i = 1, ..., N, j = 1, ..., J, k = 2, ..., K.$

### 5.2 Extension to Missing Values

Our estimator can also handle missing data which are often encountered in practice. Let $\Omega = (\omega_{ij})_{N \times J}$ be the indicator matrix of nonmissing values, where $\omega_{ij} = 1$ if $Y_{ij}$ is observed and $\omega_{ij} = 0$ if $Y_{ij}$ is missing. When data are completely missing at random, the joint likelihood function becomes

$$L^\Omega(\theta_1, ..., \theta_N, a_1, ..., a_J, \phi) = \prod_{i,j: \omega_{ij} = 1} \exp\left(\frac{y_{ij}m_{ij} - b(m_{ij})}{\phi} + c(y_{ij}, \phi)\right)$$
and our estimator becomes

\[
(\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)}) \in \arg \min - \ell^\Omega(\theta_1, ..., \theta_N, a_1, ..., a_J),
\]

\[
s.t. \|\theta_i\| \leq C', \|a_j\| \leq C',
\]

\[
a_j \in \mathcal{D}_j, i = 1, ..., N, j = 1, ..., J,
\]

where \( \ell^\Omega(\Theta A^\top) = \sum_{i,j: \omega_{ij}=1} y_{ij} m_{ij} - b(m_{ij}) \). Moreover, results similar to Theorem 2 can be established even when having missing data. Specifically, we assume

A3 \( \omega_{ij} \)s in \( \Omega \) are independent and identically distributed Bernoulli random variables with

\[
P(\omega_{ij} = 1) = \frac{n}{N_J}.
\]

This assumption implies that data are completely missing at random and only about \( n \) entries of \( (Y_{ij})_{N \times J} \) are observed. We have the following result.

**Proposition 8.** Under assumptions A1-A3 and \((\Theta^*, A^*) \in S_Q\), there exists \( N_0, J_0, \) and \( \kappa_0 > 0 \) such that for \( N \geq N_0 \) and \( J \geq J_0, \) and \( n \geq (N + J) \log(JN), \) then there exists a constant \( \kappa_0 > 0 \) (independent with \( N \) and \( J \)) such that with probability \( 1 - 2/(N + J), \)

\[
\frac{1}{NJ} \|\hat{\Theta}^{(N,J)} (\hat{A}^{(N,J)})^\top - \Theta^*_{[1:N,1:K]} (A^*_{[1:N,1:K]})^\top\|_F^2 \leq \kappa_0 \sqrt{\frac{(N + J) \log(JN)}{n}}.
\]  

(14)

Moreover, if \( Q \) satisfies (4) and thus latent factor \( k \) is structurally identifiable, then there exists a constant \( \kappa_1 \) (independent with \( N \) and \( J \)) such that with probability \( 1 - 2/(N + J), \)

\[
|\sin \angle (\Theta^*_{[1:N,k]}, \hat{\Theta}^{(N,J)}_{[k]})| \leq \kappa_1 \left( \frac{(N + J) \log(JN)}{n} \right)^{\frac{1}{4}}.
\]  

(15)

**Remark 5.** Results similar to (14) have also been derived in the literature of matrix completion (e.g. Candès and Plan, 2010; Davenport et al., 2014) under specific statistical models with an underlying low rank structure. Proposition 8 extends existing results on matrix com-
6 A Useful Perturbation Bound on Linear Subspace

The standard approach (see, e.g., Davenport et al. (2014)) for bounding the error of the maximum likelihood estimator is by making use of the strong/weak convexity of the log-likelihood function. However, in the generalized latent factor model, the log-likelihood function is not convex in \((\Theta_{[1:N,1:K]}, A_{[1:J,1:K]})\). Thus, the standard approach is not applicable for proving (7) in Theorem 2.

For this reason, we develop new technical tools to handle this problem. In particular, we establish a new perturbation bound for the intersection of linear spaces, which may be of independent theoretical interest. Let \(R_pW_q\) denote the column space of a matrix \(W\). Under the conditions of Theorem 2, the result of (6) combined with the Davis-Kahan-Wedin sine theorem (see e.g. Stewart and Sun, 1990) allows us to bound \(|\sin(\Theta^{*}_{[1:N,S]}), R(\hat{\Theta}^{(N,J)}_{[S]}))|\), for any \(S\) satisfying \(pQ(S) > 0\), where \(\angle(L, M)\) denotes the largest principal angle between two linear spaces \(L\) and \(M\), i.e., \(\sin(\angle(L, M)) = \max_{u \in M, u \not= 0} \min_{v \in L, v \not= 0} |\angle(u, v)|\). Our strategy is to bound

\[ |\sin(\Theta^{*}_{[1:N,K]}, \hat{\Theta}^{(N,J)}_{[k]})| = \sin(\angle(R(\Theta^{*}_{[1:N,K]}), \hat{\Theta}^{(N,J)}_{[k]})) \]

by \(\sin(\angle(R(\Theta^{*}_{[1:N,K]}), \hat{\Theta}^{(N,J)}_{[S]}))\) under the assumptions of Theorem 2. Note that \(R(\Theta^{*}_{[1:N,K]} = \bigcap_{k \in S, pQ(S) > 0} R(\Theta^{*}_{[1:N,S]}))\) and similarly \(R(\hat{\Theta}^{(N,J)}_{[S]} = \bigcap_{k \in S, pQ(S) > 0} R(\hat{\Theta}^{(N,J)}_{[S]}))\).

Consequently, it remains to show that if the linear spaces are perturbed slightly, then their intersection does not change much. To this end, we establish a new perturbation bound on the intersection of general linear spaces in the next proposition.

**Proposition 9** (Perturbation bound for intersection of linear spaces). Let \(L, M, L', M'\) be
linear subspaces of a finite dimensional vector space. Then,

$$\|P_{L \cap M'} - P_{L \cap M}\| \leq 8 \max\{\alpha(\theta_{\text{min},+}(L, M)), \alpha(\theta_{\text{min},+}(L', M'))\}(\|P_L - P_{L'}\| + \|P_M - P_{M'}\|),$$

(16)

where we define $\theta_{\text{min},+}(L, M)$ as the smallest positive principal angle between $L$ and $M$ (defined as 0 if all the principal angles are 0), $P_M$ denotes the orthogonal projection onto a linear space $M$, and $\alpha(\theta) = \frac{2(1+\cos \theta)}{(1-\cos \theta)^2}$. Here, the norm $\| \cdot \|$ could be any unitary invariant, uniformly generated and normalized matrix norm. In particular, if we take $\| \cdot \|$ to be the spectral norm $\| \cdot \|_2$, then we have

$$\sin \angle(L' \cap M', L \cap M)$$

$$\leq 8 \max\{\alpha(\theta_{\text{min},+}(L, M)), \alpha(\theta_{\text{min},+}(L', M'))\}(\sin \angle(L, L') + \sin \angle(M, M')).$$

We refer the readers to Stewart and Sun (1990) for more details on principal angles between two linear spaces and on matrix norms. In particular, the spectral norm is a unitary invariant, uniformly generated and normalized matrix norm. The result in Proposition 9 holds for all linear subspaces. The right-hand side in (16) is finite if and only if $\theta_{\text{max},+}(L, M) \neq 0$ and $\theta_{\text{max},+}(L', M') \neq 0$. In our problem, $L = \mathcal{R}(\Theta_{[1:N,S_1]})$ and $M = \mathcal{R}(\Theta_{[1:N,S_2]})$ for $S_1, S_2 \subset \{1, ..., K\}$ and $S_1 \neq S_2$. The next lemma further bounds $\alpha(\theta_{\text{min},+}(L, M))$ when $L$ and $M$ are column spaces of a matrix, which is a key step in proving (7).

**Lemma 1.** Let $W \in \mathbb{R}^{N \times K}$ for some positive integer $N$ and $K$ and $W$ is not a zero matrix, and $S_1, S_2 \subset \{1, ..., K\}$ be such that $S_1 \setminus S_2 \neq \emptyset$ and $S_2 \setminus S_1 \neq \emptyset$, then

$$\cos(\theta_{\text{min},+}(\mathcal{R}(W_{[S_1]}), \mathcal{R}(W_{[S_2]}))) \leq 1 - \frac{\sigma_{S_1 \cup S_2}^2(W_{[S_1 \cup S_2]})}{\|W\|_2^2}. $$

(17)
Figure 1: The value of \( \frac{\| \hat{\Theta}^{(N,J)} (A^{(N,J)})^\top - \Theta^*_{[1:N,1:K]} A^{*\top}_{[1:N,1:K]} \|^2_F}{(NJ)} \) versus the number of manifest variables \( J \) under different simulation settings (solid line: simple structure; dashed line: mixed structure). The median, 25% quantile and 75% quantile based on the 50 independent replications are shown by the dot, lower bar, and upper bar, respectively.

7 Simulation Study

**Study I.** We first verify Theorem 2 and its implications when all latent factors are structurally identifiable. Specifically, we consider \( K = 5 \) under the three models discussed in Section 2.1, including the linear, the MIRT and the Poisson models. Two design structures are considered, including (1) a simple structure, where \( p_Q(\{k\}) = 1/5, k = 1, ..., 5 \) and (2) a mixed structure, where \( p_Q(S) = 1/5, S = \{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,1\}, \text{ and } \{5,1,2\} \). The true person parameters \( \theta^*_i \)'s and the true manifest parameters \( a^*_j \)'s are generated i.i.d. from distributions over the ball \( \{ x \in \mathbb{R}^K : \| x \| \leq 2.5 \} \), respectively, (i.e., \( C = 2.5 \) in \( S_Q \)). Under these settings, all the latent factors are structurally identifiable.

For each model and each design structure, a range of \( J \) values are considered and we let \( N = 25J \). Specifically, we consider \( J = 100, 200, ..., 1000 \) for the linear and the Poisson models and \( J = 200, 400, ..., 2000 \) for the MIRT model. For each combination of a model, a design structure, and a \( J \) value, 50 independent datasets are generated. For each dataset, we apply Algorithm 1 to solve (5), where \( C' = 1.2C \).

Results are shown in Figures 1-5. Figure 1 shows the trend of \( \frac{1}{NJ} \| \hat{\Theta}^{(N,J)} (A^{(N,J)})^\top - \Theta^*_{[1:N,1:K]} A^{*\top}_{[1:N,1:K]} \|^2_F \) -
Figure 2: The value of $|\sin \angle(\Theta^{*}_{[1:N,1]}; \hat{\Theta}^{(N,J)}_{[1]})|$ under different simulation settings (solid line: simple structure; dashed line: mixed structure). The median, 25% quantile and 75% quantile based on the 50 independent replications are shown by the dot, lower bar, and upper bar, respectively.

(a) Linear Factor Model  
(b) MIRT Model  
(c) Poisson Factor Model

Figure 2 is used to verify (7) in Theorem 2, showing the pattern that $|\sin \angle(\Theta^{*}_{[1:N,k]}; \hat{\Theta}^{(N,J)}_{[k]})|$ decreases as $J$ and $N$ increase. Moreover, Figure 3 provides evidence on the result of Proposition 3. Displayed in Figure 3 are the histograms of $v_i$s and $\hat{v}_i$s, respectively, based on a randomly selected dataset when $J = 1000$ under the Poisson model and the simple structure. According to this figure, little difference is observed between the empirical distribution of $v_i$s and that of $\hat{v}_i$s. Similar results are observed for other datasets under all these three models when $J$ and $N$ are large.

Finally, Figures 4 and 5 show results on the ranking and the classification, whose theoretical results are given in Propositions 4 and 5. The $y$-axes of the two figures show the normalized Kendall’s tau distance in (11) and the classification error in (12), respectively.
Figure 3: Comparison between the histogram of $v_i$s and that of $\hat{v}_i$s for the first latent factor under the Poisson Model and the simple structure.

Figure 4: The Kendall's tau ranking error calculated from $\Theta^{*}_{[1:N,1]}$ and $\hat{\Theta}^{(N,J)}_{[1]}$, under different simulation settings (solid line: simple structure; dashed line: mixed structure). The median, 25% quantile and 75% quantile based on the 50 independent replications are shown by the dot, lower bar, and upper bar, respectively.
Figure 5: The classification error calculated from $\Theta_{[1:N,1]}^*$ and $\hat{\Theta}^{(N,J)}_{[1]}$ with indifference zone $(0.13, 0.43)$ under different simulation settings (solid line: simple structure; dashed line: mixed structure). The median, 25% quantile and 75% quantile based on the 50 independent replications are shown by the dot, lower bar, and upper bar, respectively.

Specifically, $\tau_-$ and $\tau_+$ are chosen as 0.14 and 0.43 which are the 55% and 65% quantiles of the $\bar{v}_i$s. From these plots, both the ranking and the classification errors tend to zero, as $J$ and $N$ grow large.

**Study II.** We then provide an example, in which a latent factor is not identifiable. Specifically, we consider $K = 2$ and the same latent factor models as in Study I. The design structure is given by $p_Q({1}) = 1/2$ and $p_Q({1, 2}) = 1/2$. The true person parameters $\theta_i$s and the true manifest parameters $a_{j*}$s are generated i.i.d. from distributions over the ball $\{x \in \mathbb{R}^K : \|x\| \leq 3\}$, respectively, (i.e., $C = 3$ in $S_Q$). Under these settings, the first latent factor is structurally identifiable and the second factor is not. For each model, we consider $J = 100, 200, ..., 1000$. The rest of the simulation setting is the same as Study I. Results are shown in Figures 6 and 7. First, Figure 6 presents the patten that $\frac{1}{N_J} \| \hat{\Theta}^{(N,J)}(\hat{A}^{(N,J)})^T - \Theta_{[1:N,1:K]}^* A_{[1:J,1:K]}^T \|_F$ decays to 0 when $J$ increases, even when a latent factor is not structurally identifiable. This is consistent with the first part of Theorem 2. Second, Figure 7 shows the trend of $|\sin \angle(\Theta_{[1:N,k]}^*; \hat{\Theta}^{(N,J)}_{[k]})|$ as $J$ increases. In particular, the value of $|\sin \angle(\Theta_{[1:N,k]}^*; \hat{\Theta}^{(N,J)}_{[k]})|$ stays above 0.2 for most of the data sets for the factor
Figure 6: The value of \( \| \Theta^{(N,J)}(\hat{A}^{(N,J)})^\top - \Theta_{1:1,N,1:K}^* A_{1:1,J,1:K}^* \|_F^2/(N,J) \) versus the number of manifest variables \( J \) under different simulation settings (solid line: Linear Model; dashed line: MIRT Model; dotted line: Poisson Model). The median, 25% quantile and 75% quantile based on the 50 independent replications are shown by the dot, lower bar, and upper bar, respectively.

Figure 7: The value of \( | \sin \angle (\Theta_{1:1,N,1:1}^*, \hat{\Theta}^{(N,J)}_{1:1}) | \) under different simulation settings (solid line: the first latent trait; dashed line: the second latent trait). The median, 25% quantile and 75% quantile based on the 50 independent replications are shown by the dot, lower bar, and upper bar, respectively.
which is structurally unidentifiable, while it still decays towards 0 for the identifiable one.

8 Concluding Remarks

In this paper, we study a central problem in psychometrics that is the identifiability of latent factors in structured latent factor models. Motivated by large-scale psychological measurement that is becoming more and more popular these days, we adopt an asymptotic setting in which both the numbers of individuals and manifest variables grow to infinity. Under this asymptotic regime, the notion of identifiability of latent factors is formalized through the definition of structural identifiability. Under a generalized latent factor model that covers most of the popular latent factor models, necessary and sufficient conditions are established for the structural identifiability of a latent factor. Moreover, an estimator is proposed that can consistently recover all the structurally identifiable latent factors. This estimator can be efficiently computed through an alternating minimization algorithm which is substantially boosted by parallel computing. Our results have significant implications, including the design of test, inference on the distribution of latent factors, and the validity of making ranking and classification decisions based on the estimated factor scores, which provide theoretical guidance to large-scale psychological measurement.

There are many future directions along the current work. First, it is of interest to develop methods, such as information criteria, for model comparison under the current asymptotic regime. These methods can be used to select the design matrix $Q$ that best describes the data structure when there are multiple $Q$-matrices available, or to determine the underlying latent dimensions. Second, the current results may be further generalized by considering more general latent factor models beyond the exponential family. For example, we may establish similar identifiability and estimability results when the distribution of $Y_{ij}$ is a more complicated function or even an unknown function of $\theta_i$ and $a_j$. 

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A Technical Proofs

In this supplement, we provide proofs of theoretical results in the main manuscript.

A.1 Some facts about matrices

In the proof, we will use the following facts about matrices often.

- For any matrix $V \in \mathbb{R}^{m \times n}$, and a subset $S \subset \{1,..,n\}$,

$$\sigma_{|S|}(V[S]) \geq \sigma_n(V).$$

This is a direct result of Fact 3 in (Hogben, 2006, Chapter 17-7).

- For any two matrices $V \in \mathbb{R}^{m \times n}$, $W \in \mathbb{R}^{n \times l}$,

$$\sigma_n(V)\|W\|_2 \geq \sigma_n(VW) \geq \sigma_n(V)\sigma_n(W).$$

This is a direct result of Fact 7(b) in (Hogben, 2006, Chapter 17-8).

A.2 Proof of results in Section 3.1

In this section, we first prove Proposition 1, and then prove the necessary part of Theorem 1. The sufficiency part of Theorem 1 is implied by Proposition 2 together with Theorem 2, whose proof is provided in Section A.4.

Proof of Proposition 1. In this proof, we construct $(\Theta, A) \in \mathcal{S}_Q$. We first construct $A$. For $S$ with $p_Q(S) = 0$, we construct $A_{[R_Q(S), 1:K]} = 0$. For each $S$ such that $p_Q(S) > 0$, we construct
\[ A_{[R_Q(s),s]} \] as

\[
A_{[R_Q(s),s]} = C \begin{bmatrix}
I_{|S|} \\
I_{|S|} \\
\vdots
\end{bmatrix},
\]

where \( I_m \) denotes the \( m \times m \) identity matrix. We also let \( A_{[R_Q(s),s^c]} = 0 \) so that the Q-matrix requirement is satisfied. It is not hard to calculate that \( \gamma(A_{[R_Q(s),s]}) = C|S|^{-1/2} > 0 \). Now we construct \( \Theta \) as follows,

\[
\Theta = C \begin{bmatrix}
I_K \\
I_K \\
\vdots
\end{bmatrix}.
\]

We have \( \gamma(\Theta) = C/K^{1/2} > 0 \). Furthermore, from the construction, \( \|a_j\| \leq C \) and \( \|\theta_i\| = C \) for all \( i \in \mathbb{Z}_+ \). Thus, \( (\Theta, A) \in \mathcal{S}_Q \).

**Proof of Theorem 1, necessary part.** We prove by contradiction. If (4) is not satisfied, then there are two cases: 1) \( \{k, k'\} \subseteq \bigcap_{k \in S, pQ(s) > 0} S \) for some \( k' \neq k \), and 2) \( \emptyset = \bigcap_{k \in S, pQ(s) > 0} S \).

For these two cases, we will construct \( \tilde{\Theta}, \tilde{A} \) so that the Q-matrix requirement is satisfied. It is not hard to calculate that \( \gamma(A_{[R_Q(s),s]}) = C|S|^{-1/2} > 0 \). Now we construct \( \Theta \) as follows,

\[
\Theta = C \begin{bmatrix}
I_K \\
I_K \\
\vdots
\end{bmatrix}.
\]

We have \( \gamma(\Theta) = C/K^{1/2} > 0 \). Furthermore, from the construction, \( \|a_j\| \leq C \) and \( \|\theta_i\| = C \) for all \( i \in \mathbb{Z}_+ \). Thus, \( (\Theta, A) \in \mathcal{S}_Q \).

**Case 1:** \( \{k, k'\} \subseteq \bigcap_{k \in S, pQ(s) > 0} S \). Without loss of generality, we assume \( k = 1 \) and \( k' = 2 \). Let \( (\Theta, A) \) be constructed in the same way as we did in the proof of Proposition 11 on page 32.

Let \( \tilde{\Theta} = \frac{1}{2} \Theta \) and \( \tilde{A} = \frac{1}{2} A \), then \( (\tilde{\Theta}, \tilde{A}) \in \mathcal{S}_Q \). Now we construct \( (\Theta', A') \) as follows.

\[
\theta_{im}' = \begin{cases} 
\theta_{im}/2 \text{ if } m \neq k \\
(\theta_{ik} - \theta_{ik'})/2 \text{ if } m = k
\end{cases} \quad \text{and} \quad a_{jm}' = \begin{cases} 
a_{jm}/2 \text{ if } m \neq k' \\
(a_{jk} + a_{jk'})/2 \text{ if } m = k'
\end{cases}
\]

for all \( i, j \in \mathbb{Z}_+ \). That is, we subtract the \( k' \)th column from the \( k \)th column in the matrix \( \tilde{\Theta} \) and obtain \( \Theta' \). We add the \( k \)th column to the \( k' \)th column in the matrix \( \tilde{A} \) to construct \( A' \).

By construction, \( \sum_{m=1}^{K} \theta_{im}' a_{jm}' = \sum_{m=1}^{K} \tilde{\theta}_{im} \tilde{a}_{jm} \) for all \( i, j \in \mathbb{Z}_+ \). Thus, \( P(\tilde{\Theta}, \tilde{A}) = P(\Theta', A') \).
Now we verify that \((\Theta', A') \in \mathcal{S}_Q\). First, note that \(\Theta'\) is obtained by an invertible column transformation on \(\Theta\) and \(A\), so \(\gamma(\Theta') > 0\) given \(\gamma(\Theta) > 0\). Second, we verify that \(A'\) satisfies the Q-matrix requirement. That is, \(a'_{jm} = 0\) if \(q_{jm} = 0\) for all \(j \in \mathbb{Z}_+\) and all \(m \in \{1, ..., K\}\). According to the construction, it is automatically true if \(m \neq k'\) or \(q_{jk'} = 1\). For \(j\) such that \(q_{jk'} = 0\), there are two cases: \(p_Q(S_j) = 0\) or \(p_Q(S_j) > 0\), where we define

\[
S_j = \{m : q_{jm} = 1\}
\]

for all \(j \in \mathbb{Z}_+\). That is, \(S_j\) indicates the set of dimensions measured by item \(j\). For the former case, according to the construction of \(A\) in the proof of Proposition \(\square\) \(a_{jk} = a_{jk'} = 0\). So \(a'_{jk'} = 0\) as well. For the latter case, \(q_{jk'} = 0\), \(p_Q(S_j) > 0\), and \(k' \in \cap_{S: k \in S, p_Q(S) > 0} S\) implies \(k \notin S_j\). As a result, \(a_{jk} = a_{jk'} = 0\), and \(a'_{jk'} = 0\). Summarizing these two cases, we can see that \(a'_{jm} = 0\) for all \(j \in \mathbb{Z}_+\), \(m \in \{1, ..., K\}\) such that \(q_{jm} = 0\). That is, \(A'\) satisfies the Q-matrix restriction. Third, we verify that \(\gamma(A'_{[R_Q(S), S]}) > 0\) for all \(S\) such that \(p_Q(S) > 0\). This is true because \(A'_{[R_Q(S), S]}\) is either an invertible column transformation or equal to \(A_{[R_Q(S), S]}\), and \(\gamma(A_{[R_Q(S), S]}) > 0\). Lastly, it is easy to verify that \(\|\tilde{\Theta}_i\|, \|\tilde{\Theta}'_i\|, \|\tilde{a}_j\|, \|\tilde{a}'_j\| \leq C\) for all \(i, j \in \mathbb{Z}_+\). Therefore, we have verified that \((\Theta', A') \in \mathcal{S}_Q\).

Now we consider the angle between \(\Theta'_{[k]}, \tilde{\Theta}_{[k]}\). To see it clearer, we write down the \(k\) and
We can see that \( \lim_{N \to \infty} N^{-1} \langle \tilde{\Theta}_{[1:N,k]}, \Theta'_{[1:N,k]} \rangle = (C^2/4K) \), \( \lim_{N \to \infty} N^{-1/2} \| \tilde{\Theta}_{[1:N,k]} \| = C/(2\sqrt{K}) \), and \( \lim_{N \to \infty} N^{-1/2} \| \Theta'_{[1:N,k]} \| = C/\sqrt{2K} \). Thus,

\[
\lim_{N \to \infty} \cos \angle (\tilde{\Theta}_{[1:N,k]}, \Theta'_{[1:N,k]}) = \lim_{N \to \infty} \frac{\langle \tilde{\Theta}_{[1:N,k]}, \Theta'_{[1:N,k]} \rangle}{\| \tilde{\Theta}_{[1:N,k]} \| \| \Theta'_{[1:N,k]} \|} = \frac{1}{\sqrt{2}}.
\]

Consequently, \( \sin \angle (\tilde{\Theta}_{[k]}, \Theta'_{[k]}) = 1/\sqrt{2} > 0 \). This contradicts the definition of structural identifiability. We complete the proof for Case 1.

**Case 2:** We first note that if \( K = 1 \), then \( p_Q(\varnothing) > 0 \). This contradicts Assumption A1. In the rest of the proof, we assume \( K \geq 2 \). Without loss of generality, we assume \( k = 1 \). Let \( (\Theta, A) \) be constructed the same as the one in the proof of Proposition [1]. We let \( \tilde{\Theta} = \frac{1}{2} \Theta \) and \( \tilde{A} = \frac{1}{2} A \). Obviously, \( (\tilde{\Theta}, \tilde{A}) \in \mathcal{S}_Q \). We further let \( A' = \frac{1}{2} A = \tilde{A} \). We construct \( \Theta' \) as follows. \( \Theta'_m = \frac{1}{2} \Theta'_m \) for \( m \geq 2 \), and \( \theta'_i = C/2 \) for all \( i \in \mathbb{Z}_+ \). Since \( A_{[RQ(S),1,K]} = 0 \) for \( S \) with \( p_Q(S) = 0 \) and \( p_Q(S) = 0 \) for all \( k \in S \) in this case, we have \( a_{j1} = 0 \) for all \( j \in \mathbb{Z}_+ \). Combining this finding with the fact that \( \tilde{\Theta} \) and \( \Theta' \) are different only at the first column, we have \( \mathbf{a}_j^T \Theta'_i = \tilde{\mathbf{a}}_j^T \tilde{\Theta}_i \) for all \( i, j \in \mathbb{Z}_+ \). Thus, \( P_{\tilde{\Theta},\tilde{A}} = P_{\Theta',A'} \).

Note that \( A' = A/2 \), and \( \Theta' \) is an invertible column transformation of \( \Theta \). Thus, from \( (\Theta, A) \in \mathcal{S}_Q \), we have \( \gamma(\Theta') > 0 \), and \( (\Theta', A') \in \mathcal{S}_Q \).
We proceed to verify that \( \sin \angle (\Theta'_1, \tilde{\Theta}_1) > 0 \). Note that
\[
\lim_{N \to \infty} N^{-1} |\langle \Theta'_1, \tilde{\Theta}_1 \rangle| = C^2/(4K), \quad \lim_{N \to \infty} N^{-1/2} \|\Theta'_1\| = C/2, \quad \text{and}
\lim_{N \to \infty} N^{-1/2} \|\tilde{\Theta}_1\| = C/(2\sqrt{K}). \]
Therefore, we have \( \lim_{N \to \infty} \cos \angle (\tilde{\Theta}_1, \Theta'_1) = K^{-1/2} \). Thus, \( \sin \angle (\tilde{\Theta}_1, \Theta'_1) = \sqrt{K^{-1}} > 0 \). By contradiction, we complete the proof.

\[\square\]

### A.3 Proof of results in Section 3.2

In this section, we provide the proof of Proposition 2.

**Proof of Proposition 2.** We prove the proposition by contradiction. If the \( k \)-th dimension is not structurally identifiable, then there exists \( (\Theta, A), (\Theta', A') \in \mathcal{S}_Q \), with \( P_{\Theta, A} = P_{\Theta', A'} \) and \( \sin_+ \angle (\Theta[k], \Theta'[k]) > 0 \). On the other hand, by contradiction there is a consistent estimator \( \{ (\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)}), N, J \in \mathbb{Z}_+ \} \) for the \( k \)-th dimension, which means
\[
\lim_{N, J \to \infty} \sin \angle (\hat{\Theta}^{(N,J)}[k], \Theta^{[1:N,k]}) = 0 \text{ in } P_{\Theta, A} \text{ and } \lim_{N, J \to \infty} \sin \angle (\hat{\Theta}^{(N,J)}[k], \Theta'_1^{[1:N,k]}) = 0 \text{ in } P_{\Theta', A'}.
\]

Note that \( P_{\Theta', A'} = P_{\Theta, A} \), so the above display further implies
\[
\lim_{N, J \to \infty} \sin \angle (\hat{\Theta}^{(N,J)}[k], \Theta^{[1:N,k]}) = \lim_{N, J \to \infty} \sin \angle (\hat{\Theta}^{(N,J)}[k], \Theta'_1^{[1:N,k]}) = 0 \text{ in } P_{\Theta, A}.
\]

This implies \( \lim_{N, J \to \infty} \sin \angle (\Theta'_1^{[1:N,k]}, \Theta^{[1:N,k]}) = \lim_{N, J \to \infty} \sin \angle (\hat{\Theta}^{(N,J)}[k], \Theta'_1^{[1:N,k]}) = 0 \).

By definition, this gives \( \sin_+ \angle (\Theta[k], \Theta'[k]) = 0 \), which contradicts our assumption \( \sin_+ \angle (\Theta[k], \Theta'[k]) > 0 \).

\[\square\]

### A.4 Proofs of Results in Section 3.3

In this section, we provide the proof of Theorem 2, which consists of two parts: the proof of (6) and the proof of (7). Throughout this section, all the matrices considered are finite dimensional. For the ease of presentation, we drop the subscripts \( N \) and \( J \). We abuse
the notation a little bit and write \( \Theta^* := \Theta_{[1:N,1:K]}^*, A^* := A_{[1:J,1:K]}^*, \Theta := \Theta_{[1:N,1:K]}, A := A_{[1:J,1:K]}, \) and \((\hat{\Theta}, \hat{A}) = (\hat{\Theta}^{(N,J)}, \hat{A}^{(N,J)})\) within this section (Section A.4). Furthermore, all the probability considered in the current section are taken under the true value \( \Theta^* \) and \( A^* \). We will first present the main proof of the theorem and then present the proof of its supporting lemmas in Section A.4.1.

We start with the proof of (6).

**Proof of (6) in Theorem 2.** Let \( \hat{M} = \hat{\Theta} \hat{A}^\top \) and \( M^* = \Theta^* A^* \). Then

\[
\begin{align*}
l(\hat{M}) - l(M^*) & \geq 0. \quad (18)
\end{align*}
\]

On the other hand,

\[
\begin{align*}
l(\hat{M}) - l(M^*) & = \sum_{i=1}^{N} \sum_{j=1}^{J} Y_{ij}(\hat{m}_{ij} - m_{ij}^*) - (b(\hat{m}_{ij}) - b(m_{ij}^*)) \\
& = \sum_{i=1}^{N} \sum_{j=1}^{J} (Y_{ij} - b'(m_{ij}^*))(\hat{m}_{ij} - m_{ij}^*) - \sum_{i=1}^{N} \sum_{j=1}^{J} b(\hat{m}_{ij}) - b(m_{ij}^*) - b'(m_{ij}^*)(\hat{m}_{ij} - m_{ij}^*) \\
& = \sum_{i=1}^{N} \sum_{j=1}^{J} (Y_{ij} - b'(m_{ij}^*))(\hat{m}_{ij} - m_{ij}^*) - \sum_{i=1}^{N} \sum_{j=1}^{J} \frac{1}{2} b''(\hat{m}_{ij})(\hat{m}_{ij} - m_{ij}^*)^2 \\
& \leq \sum_{i=1}^{N} \sum_{j=1}^{J} (Y_{ij} - b'(m_{ij}^*))(\hat{m}_{ij} - m_{ij}^*) - \frac{1}{2} \min_{|\nu| \leq C^2} b''(\nu) \| \hat{M} - M^* \|_F^2, \quad (19)
\end{align*}
\]

for some \( \tilde{m}_{ij} = \eta m_{ij}^* + (1 - \eta)\hat{m}_{ij} \) and \( \eta \in (0, 1) \) in the third equation. Combining (18) and (19), we have

\[
\| \hat{M} - M^* \|_F^2 \leq \frac{2}{\min_{|\nu| \leq C^2} b''(\nu)} \sum_{i=1}^{N} \sum_{j=1}^{J} (Y_{ij} - b'(m_{ij}^*))(\hat{m}_{ij} - m_{ij}^*). \quad (20)
\]

Let

\[
C = \{ M : M = \Theta A^\top, \| a_j \| \leq C', \| \theta_i \| \leq C', a_j \in \mathcal{D}_j, \text{ for all } 1 \leq i \leq N \text{ and } 1 \leq j \leq J \}.
\]

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We further bound (20) by

\[ \| \tilde{M} - M^* \|^2_F \leq \frac{2}{\min_{|\nu| \leq C^2} b^a(\nu)} \sup_{\tilde{M} \in C} \left| \sum_{i=1}^N \sum_{j=1}^J (Y_{ij} - b^a(m_{ij}^*)) \tilde{m}_{ij} \right| = \frac{4}{\min_{|\nu| \leq C^2} b^a(\nu)} \sup_{\tilde{M} \in C} | \langle X, \tilde{M} \rangle |, \]

(21)

where we let \( X_{ij} = Y_{ij} - b^a(m_{ij}^*) \), and define \( \langle X, \tilde{M} \rangle = \sum_{i=1}^N \sum_{j=1}^J X_{ij} \tilde{m}_{ij} \). In what follows, we proceed to an upper bound of \( P(\sup_{\tilde{M} \in C} | \langle X, \tilde{M} \rangle | > t) \) for some positive \( t \) to be chosen later.

Let \( S = \{ (\Theta, A) : \| \Theta \| \leq C', \| a_j \| \leq C', a_j \in D_j \} \). For each \( \Theta \in R^K \), we define the ball \( B(\Theta, \delta) = \{ \Theta' : \| \Theta' - \Theta \| \leq \delta \} \). Now, for an \( N \times K \) matrix \( \Theta \), we define \( B(\Theta, \delta) = \{ \Theta' : \Theta'_i \in B(\Theta_i, \delta) \text{ for all } 1 \leq i \leq N \} \). Similarly, for a \( J \times K \) matrix \( A \), we define \( B(\Theta, \delta) = \{ A' : a_j' \in B(a_j, \delta) \text{ for all } 1 \leq j \leq J \} \). Furthermore, for a pair \( (\Theta, A) \), we define \( B((\Theta, A), \delta) = B(\Theta, \delta) \times B(A, \delta) \). For each \( \delta \), we consider a covering \( \{ B((\Theta^{(i)}, A^{(i)}), \delta), i = 1, ..., N(\delta) \} \) of the set \( S \), where \( (\Theta^{(i)}, A^{(i)}) \in S \). That is, \( S \subset \bigcup_{i=1}^{N(\delta)} B((\Theta^{(i)}, A^{(i)}), \delta) \), where \( N(\delta) \) denotes number of balls covering \( S \). In particular, we let \( N(\delta) \) be the smallest covering number and \( (\Theta^{(i)}, A^{(i)}) \) be the centers corresponding to the covering with the smallest covering number.

Now, the probability \( P(\sup_{\tilde{M} \in C} | \langle X, \tilde{M} \rangle | > t) \) can be controlled using the covering. We have for any \( \gamma > 0 \),

\[
P(\sup_{\tilde{M} \in C} | \langle X, \tilde{M} \rangle | > t)
\leq P(\sup_{\tilde{M} \in C} | \langle X, \tilde{M} \rangle | > t, \| X \|_F \leq \gamma) + P(\| X \|_F > \gamma)
\leq \sum_{i=1}^{N(\delta)} P(\sup_{(\Theta', A') \in S \cap B((\Theta^{(i)}, A^{(i)}), \delta)} | \langle X, \Theta' A'^T \rangle | > t, \| X \|_F \leq \gamma) + P(\| X \|_F > \gamma).
\]

(22)
Note that for all $\Theta', A' \in \mathcal{S} \cap B((\Theta^{(l)}, A^{(l)}), \delta)$,

$$\langle X, \Theta' A'^T \rangle \leq \langle X, \Theta^{(l)} A^{(l)T} \rangle + \|X\|_F \|\Theta' A'^T - \Theta^{(l)} A^{(l)T}\|_F$$

$$\leq \langle X, \Theta^{(l)} A^{(l)T} \rangle + \|X\|_F (\|A' - A^{(l)}\|_F \|\Theta^{(l)}\|_F + \|\Theta' - \Theta^{(l)}\|_F \|A'\|_F).$$

$$\leq \langle X, \Theta^{(l)} A^{(l)T} \rangle + \|X\|_F (\|A' - A^{(l)}\|_F \sqrt{NC'} + \|\Theta' - \Theta^{(l)}\|_F \sqrt{J'C'}).$$

(23)

The last inequality is due to the constraint on the set $\mathcal{S}$. That is, $\|\Theta^{(l)}\| \leq \sqrt{NC'}$. Similarly, $\|a'_j\| \leq C'$ for $1 \leq j \leq J$ implies $\|A'\| \leq \sqrt{JC'}$. Note that for $A' \in B(A^{(l)}, \delta)$,

$$\|A' - A^{(l)}\|_F^2 = \sum_{j=1}^{J} \|a'_j - a^{(l)}_j\|^2 \leq J\delta^2.$$ 

Thus, $\|A' - A^{(l)}\|_F \leq \sqrt{J}\delta$. Similarly, $\|\Theta' - \Theta^{(l)}\|_F \leq \sqrt{NJ}\delta$. Combine this with (23), we have

$$\langle X, \Theta' A'^T \rangle \leq \langle X, \Theta^{(l)} A^{(l)T} \rangle + 2\|X\|_F \sqrt{NJ}\delta C'.$$

Combine the above inequality with (22), we have for each $\gamma > 0$,

$$P(\sup_{M \in \mathcal{C}} |\langle X, M \rangle| > t)$$

$$\leq \sum_{l=1}^{N(\delta)} P\left(|\langle X, \Theta^{(l)} A^{(l)T} \rangle| + 2\|X\|_F \sqrt{NJ}\delta C' \geq t, \|X\|_F \leq \gamma\right) + P(\|X\|_F > \gamma)$$

$$\leq \sum_{l=1}^{N(\delta)} P\left(|\langle X, \Theta^{(l)} A^{(l)T} \rangle| \geq t - 2\gamma \sqrt{NJ}\delta C', \|X\|_F \leq \gamma\right) + P(\|X\|_F > \gamma)$$

$$\leq N(\delta) \sup_{(\Theta', A') \in \mathcal{S}} P\left(|\langle X, \Theta' A'^T \rangle| \geq t - 2\gamma \sqrt{NJ}\delta C'\right) + P(\|X\|_F \geq \gamma).$$

(24)

We proceed to bound $N(\delta), \sup_{(\Theta', A') \in \mathcal{S}} P\left(|\langle X, \Theta' A'^T \rangle| \geq t - 2\gamma \sqrt{NJ}\delta C'\right)$, and $P(\|X\|_F \geq \gamma)$.
separately. We start with $N(\delta)$. It is not hard to see that an upper bound of $N(\delta)$ is

$$N(\delta) \leq \left( \frac{C'K}{\delta} \right)^{K(N+J)}. \quad (25)$$

Next, we find an upper bound of $\sup_{(\Theta', A') \in S} P \left( |\langle X, \Theta' A'^T \rangle| \geq t - 2\gamma \sqrt{NJ} \delta C' \right)$. We use the following lemma on the marginal probability tail bound.

**Lemma 2.** There exist constants $\delta_0, \epsilon_0$ (depending only on the function $b$ and the constant $C'$, $\phi$), such that for $0 < t < \delta_0 N J$, and $\tilde{M} \in \mathcal{C}$,

$$P(|\langle X, \tilde{M} \rangle| > t) \leq 2 \exp \left( -\frac{\epsilon_0 t^2}{NJ} \right).$$

According to Lemma 2 for $0 \leq t - 2\gamma \sqrt{NJ} \delta C' \leq \delta_0 NJ$,

$$P \left( |\langle X, \Theta' A'^T \rangle| \geq t - 2\gamma \sqrt{NJ} \delta C' \right) \leq \exp \left\{ -\epsilon_0 \frac{\left( t - 2\gamma \sqrt{NJ} \delta C' \right)^2}{NJ} \right\}. \quad (26)$$

We proceed to an upper bound for $P(\|X\|_F > \gamma)$. By Chebyshev’s inequality, we have

$$P(\|X\|_F > \gamma) = P(\|X\|_F^2 > \gamma^2) \leq \gamma^{-2} E(\|X\|_F^2) = \gamma^{-2} \phi \sum_{i=1}^N \sum_{j=1}^J b''(m_{ij}^*) \leq \gamma^{-2} NJ \phi \sup_{|\nu| \leq C'^2} b''(\nu). \quad (27)$$

Combining (21), (25), (26), and (27), we arrive at that for $0 \leq t - 2\gamma \sqrt{NJ} \delta C' \leq \delta_0 NJ$,

$$P \left( \sup_{\tilde{M} \in \mathcal{C}} |\langle X, \tilde{M} \rangle| > t \right) \leq \left( \frac{C'K}{\delta} \right)^{K(N+J)} \exp \left\{ -\epsilon_0 \frac{\left( t - 2\gamma \sqrt{NJ} \delta C' \right)^2}{NJ} \right\} + \gamma^{-2} NJ \sup_{|\nu| \leq C'^2} b''(\nu)$$

$$= \exp \{ K(N + J)(\log C' + \log K - \log \delta) - \epsilon_0 \frac{\left( t - 2\gamma \sqrt{NJ} \delta C' \right)^2}{NJ} \} + \gamma^{-2} \phi NJ \sup_{|\nu| \leq C'^2} b''(\nu).$$

The above inequality holds for all $\delta, \gamma > 0$. Now we choose $t$, $\delta$, and $\gamma$. We choose $\gamma = \sqrt{NJ(N + J)\phi \sup_{|\nu| \leq C'^2} b''(\nu)}$, $\delta = (NJ)^{-1/2}$, and $t = 2\gamma \sqrt{NJ} \delta C' +$
\[
\frac{1}{\sqrt{\log_0}} \sqrt{2NJN+J)(\log C' + \log K - \log \delta)},
\]
then we have
\[
P(\sup_{\tilde{M} \in \mathcal{C}} |\langle X, \tilde{M} \rangle| > t) \leq \exp\{-K(N + J)(\log C' + \log K + \frac{1}{2} \log(NJ))\} + \frac{1}{N + J}.
\]

From the above display, we can see that for a fixed \(K\), there exists a positive constant \(\kappa_0\) (depending on \(C', b,\) and \(K\)) such that \(t \leq \kappa_0 \sqrt{Nj(N + J)\log(NJ)}\), thus,
\[
P(\sup_{\tilde{M} \in \mathcal{C}} |\langle X, \tilde{M} \rangle| \geq \kappa_0 \sqrt{Nj(N + J)\log(NJ)}) \leq \exp\{-(N + J)\log(NJ)\} + \frac{1}{N + J} \leq \frac{2}{N + J}
\]
for sufficiently large \(N\) and \(J\). Combining this with (21), we complete the proof.

\begin{proof} for (7) in Theorem \(\square\) Note that by proving (6), we have already proved that with probability \(1 - \frac{2}{N + J}\)
\[
\frac{1}{NJ} \|\Theta^* A^\top - \hat{\Theta} \hat{A}^\top\|_F^2 \leq \kappa_0 \sqrt{\frac{N + J}{Nj}} \log(NJ).
\] (28)

Note that the right-hand side of the above display tend to zero as \(N, J \to \infty\). On the event (28) happens and recall the definition of \(S_Q\), we can see that in order to prove (7), it is sufficient to prove the following statement. Let \(N_0, J_0\) and \(\sigma\) be such that
\[
s_K(\Theta^*) \geq \sigma \sqrt{N}, \quad \sigma_{|S|}(A^*_{\{R_Q(S) \cap \{1, \ldots, J\}, S\}}) \geq \sigma \sqrt{J}, \quad \text{and} \quad \frac{\{|j : S_j = S, 1 \leq j \leq J\}}{J} \geq \frac{p_Q(S)}{2},
\] (29)
and
\[
\sqrt{NJ} \sigma^2 > 2 \|\Theta^* A^\top - \hat{\Theta} \hat{A}^\top\|_2,
\] (30)
for all \(N \geq N_0, J \geq J_0,\) and \(S\) such that \(p_Q(S) > 0\). As (28) happens with probability at least \(1 - \frac{2}{N + J}\), the above display also happens with at least \(1 - \frac{2}{N + J}\) probability. In addition, \(\hat{a}_j \in D_j,\) and (4) holds. Then, then there exists a constant \(C_1\), independent of \(N, J, \Theta,\) and
A, and possibly depend on \( K, \sigma \) and \( C' \), such that

\[
|\sin \angle(\Theta^*_t, \hat{\Theta}_t)| \leq C_1 \frac{\|\Theta^* A^\top - \hat{\Theta}^\top \hat{A}^\top\|_2}{\sqrt{NJ}}.
\]

(31)

In what follows, we prove this statement. For each \( S \subset \{1, \ldots, K\} \), denote

\[
\rho(S) = \sin \angle(\mathcal{R}({\Theta^*_S}), \mathcal{R}(\hat{\Theta}_S)).
\]

The rest of the proof is based on Proposition 9 and Lemma 1 (proved in Section A.7.1). Consider two sets \( S_1, S_2 \subset \{1, \ldots, K\} \) that are not subset of each other. Let \( L = \mathcal{R}(\Theta^*_1) \), \( M = \mathcal{R}(\Theta^*_2) \), \( \hat{L} = \mathcal{R}(\hat{\Theta}_1) \), and \( \hat{M} = \mathcal{R}(\hat{\Theta}_2) \) in Proposition 9. Then, we have

\[
\rho(S_1 \cap S_2) \leq 8 \max\{\alpha(\theta_{\min,+}(\mathcal{R}(\Theta^*_1), \mathcal{R}(\Theta^*_2)), \alpha(\theta_{\min,+}(\mathcal{R}(\hat{\Theta}_1), \mathcal{R}(\hat{\Theta}_2)))\}(\rho(S_1) + \rho(S_2)).
\]

(32)

By Lemma 1, we have

\[
\cos \left( \theta_{\min,+}(\mathcal{R}(\Theta^*_1), \mathcal{R}(\Theta^*_2)) \right) \leq 1 - \frac{\sigma^2_{S_1 \cup S_2}(\Theta^*_1|_{S_1 \cup S_2})}{\|\Theta^*\|_2^2} \leq 1 - \frac{\sigma^2_K(\Theta^*)}{\|\Theta^*\|_2^2}.
\]

Recall

\[
\alpha(\theta) = \frac{2(1 + \cos \theta)}{(1 - \cos \theta)^3}.
\]

Thus, according to (29), and \( \|\Theta^*\|_2 \leq \|\Theta^*\|_F \leq \sqrt{NC'} \),

\[
\alpha(\theta_{\min,+}(\mathcal{R}(\Theta^*_1), \mathcal{R}(\Theta^*_2))) \leq 4\sigma^{-6}_K(\Theta^*)\|\Theta^*\|_2^6 \leq 4C'^6\sigma^{-6}.
\]

(33)

In order to get a similar result for \( \hat{\Theta} \), we need the following lemma.

**Lemma 3.** Given (29) and (30), we have

\[
\sigma_{|_{S \supset \mathcal{R}(S) > 0S}}(\hat{\Theta}[\cup S \supset \mathcal{R}(S) > 0S]) \geq \frac{\sigma^2\sqrt{N}}{2C'}.
\]
Using the above lemma and Lemma 1, we have
\[
\cos \theta_{\min,+}(\mathcal{R}(\hat{\Theta}[S_1]), \mathcal{R}(\hat{\Theta}[S_2])) \leq 1 - \frac{\sigma^2_{[\cup_{S \in\mathbb{S}}p_Q(S) > 0]}}{\|\hat{\Theta}\|^2} \leq 1 - \frac{\sigma^4}{4C^3}.
\]
Thus,
\[
\alpha(\theta_{\min,+}(\mathcal{R}(\hat{\Theta}[S_1]), \mathcal{R}(\hat{\Theta}[S_2]))) \leq 4\left(\frac{\sigma^4}{4C^3}\right)^{-3} \leq 256C^{12}\sigma^{-12}.
\]
Combining (32), (33), and (34), we arrive at an iteration inequality,
\[
\rho(S_1 \cap S_2) \leq 1024C^{12}\sigma^{-12}(\rho(S_1) + \rho(S_2))
\]
(35)
Now we can work on all the sets $S$ such that $k \in S$ and $p_Q(S) > 0$. Without loss of generality, we can arrange them as $S_1, \ldots, S_m$, where $m \leq 2^{K-1}$. We use (35) repeatedly, then we arrive at
\[
\rho(\{k\}) = \rho(\cap_{k \in S, S \subseteq \{1, \ldots, K\}} S) \leq 1024^m C^{12m} \sigma^{-12m} \sum_{i=1}^m \rho(S_i).
\]
(36)
We bound each $\rho(S_i)$ on the right-hand side of the above display by the following lemma.

**Lemma 4.** Given (29) and (30), we have
\[
\sin \angle(\mathcal{R}(\Theta^*_S), \mathcal{R}(\hat{\Theta}[S])) \leq \frac{2\|\Theta^*_S A^*\mathbb{T} - \hat{\Theta} \hat{A}^\mathbb{T}\|_2}{\sigma_{[S]}(\Theta^*_S A^*\mathbb{T}_{[R_Q(S), S]}),}
\]
(37)
for $S$ such that $p_Q(S) > 0$.

Combining (36) and (37), and notice that $\sigma_{[S]}(\Theta^*_S A^*\mathbb{T}_{[R_Q(S), S]}) > \sigma^2 \sqrt{N \mathcal{J}}$ we complete the proof of (31).

\[
43
\]
A.4.1 Proof of Lemmas 2 - 4

Proof of Lemma 2 We note that under assumption A2, \(E(e^{\lambda X_{ij}\tilde{m}_{ij}}) = \exp\{(b(m_{ij}^* + \lambda \phi \tilde{m}_{ij}) - b(m_{ij}^*))/\phi\}\). Thus, there exist constants \(\lambda_0\) and \(\kappa'\) (depending on \(b\)) uniformly for all \(i, j\) such that for all \(|\lambda| \leq \lambda_0\) and all \(|\tilde{m}_{ij}| \leq C'\), \(E(e^{\lambda X_{ij}\tilde{m}_{ij}}) \leq \kappa'\). That is, \(\tilde{m}_{ij}X_{ij}\) is sub-exponential. Therefore, based on the Bernstein inequality, there exist constants \(\delta_0\) and \(\epsilon_0\) (depending on \(\lambda_0, C'\), and \(\sup_{|\lambda| \leq \lambda_0, |m_{ij}| \leq C'} E(e^{\lambda X_{ij}\tilde{m}_{ij}})\)),

\[P(\langle X, \tilde{M} \rangle > t) \leq 2e^{-\epsilon_0 t^2/(NJ)},\]

for \(0 < t/(NJ) \leq \delta_0\).

Proof of Lemma 3 Let \(T = \cup_{S|Q(S) > 0} S\) and \(W = \cup_{S|Q(S) > 0} R_Q(S) \cap [1, J]\). Note that for \(j \in W\) \(a_{jm} = 0\) for \(m \notin T\), so we have

\[\Theta^* A_{[T]}^* = \Theta_{[T]} A_{[T]}^*\].

Thus, \((\Theta^* A^*)_{[1:N,W]} = \Theta_{[T]} A_{[W,T]}^*\). Similarly, \((\hat{\Theta} A^*)_{[1:N,W]} = \hat{\Theta}_{[T]} A_{[W,T]}^*\). Thus, \((\Theta^* A^* - \hat{\Theta} A^*)_{[1:N,W]} = \Theta_{[T]} A_{[W,T]}^* - \hat{\Theta}_{[T]} A_{[W,T]}^*\). That is, \(\Theta_{[T]} A_{[W,T]}^* - \hat{\Theta}_{[T]} A_{[W,T]}^*\) is a submatrix of \(\Theta^* A^* - \hat{\Theta} A^*\). Thus, we have the following bound

\[\|\Theta_{[T]} A_{[W,T]}^* - \hat{\Theta}_{[T]} A_{[W,T]}^*\|_2 \leq \|\Theta^* A^* - \hat{\Theta} A^*\|_2 \leq \frac{1}{2} \sigma^2 \sqrt{NJ} . \tag{38}\]
On the other hand, we have

\[
\sigma_{[T]}(A_{[W,T]}^*) = \min_{x \in \mathbb{R}^K, \|x_{[T]}\| = 1} x_{[T]}^T A_{[W,T]}^* A_{[W,T]} x_{[T]}
\]

\[
= \min_{x \in \mathbb{R}^K, \|x_{[T]}\| = 1} \sum_{j \in W} x_{[T]}^T a_{j,[T]}^* a_{j,[T]}^* x_{[T]}
\]

\[
= \min_{x \in \mathbb{R}^K, \|x_{[T]}\| = 1} \sum_{S: p_Q(S) > 0} \sum_{j \in R_Q(S)} x_{[S]}^T a_{j,[S]}^* a_{j,[S]}^* x_{[S]}
\]

\[
= \min_{x \in \mathbb{R}^K, \|x_{[T]}\| = 1} \sum_{S: p_Q(S) > 0} x_{[S]}^T A_{[R_Q(S),S]}^* A_{[R_Q(S),S]}^* x_{[S]}
\]

\[
\geq \min_{x \in \mathbb{R}^K, \|x_{[T]}\| = 1} \sum_{S: p_Q(S) > 0} \sigma_S(A_{[R_Q(S),S]}^*) \|x_{[S]}\|^2
\]

Here, the third equation holds because \( W = \cup_{S: p_Q(S) > 0} R_Q(S) \cap [1, J] \) and \( R_Q(S) \) are disjoint for different \( S \), and \( a_{jm} = 0 \) if \( j \in R_Q(S) \) and \( m \notin S \). Given (29) and the above display, we further have

\[
\sigma_{[T]}(A_{[W,T]}^*) \geq \min_{x \in \mathbb{R}^K, \|x_{[T]}\| = 1} \sum_{S: p_Q(S) > 0} \sqrt{J} \sigma \|x_{[S]}\|^2 \geq \sqrt{J} \sigma.
\]

Also, according to (29),

\[
\sigma_{[T]}(\Theta_{[T]}^*) \geq \sigma_K(\Theta^*) \geq \sqrt{N} \sigma.
\]

Combining the above two inequalities, we have

\[
\sigma_{[T]}(\Theta_{[T]}^* A_{[W,T]}^*) \geq \sigma_{[T]}(\Theta_{[T]}^*) \sigma_{[T]}(A_{[W,T]}^*) \geq \sqrt{NJ} \sigma^2.
\]

Combining the above inequality with (38), and Weyl’s perturbation theorem (see, e.g. Stewart and Sun (1990)), we arrive at

\[
\sigma_{[T]}(\hat{\Theta}_{[T]}^* A_{[W,T]}^*) \geq \frac{\sqrt{NJ} \sigma^2}{2}.
\]
Thus, we arrive at
\[
\sigma_{[T]}(\hat{\Theta}_{[T]}) \geq \frac{\sigma_{[T]}(\hat{\Theta}_{[T]} A_{[W,T]}^T)}{\|A\|_2} \geq \frac{\sqrt{NJ}\sigma^2}{2\sqrt{|W|C'}} \geq \frac{\sqrt{N}J\sigma^2}{2C'}.
\]

\[\square\]

**Proof of Lemma 4.** Similar to (38), we have

\[
\Theta^*_{[S]} A^*_{[R_Q(S),S]} - \hat{\Theta}_{[S]} A_{[R_Q(S),S]}^T
\]

is a submatrix of \(\Theta^* A^* - \hat{\Theta} \hat{A}^T\), and thus,

\[
\|\Theta^*_{[S]} A^*_{[R_Q(S),S]} - \hat{\Theta}_{[S]} A_{[R_Q(S),S]}^T\|_2 \leq \|\Theta^* A^* - \hat{\Theta} \hat{A}^T\|_2 \leq \frac{\sigma^2}{2} \sqrt{NJ}.
\]  

(39)

The rest of the proof is similar to that of Theorem 2 in Chen et al. (2017). We only state the main steps. We write the reduced singular value decomposition (SVD) of \(\Theta^*_{[S]} A^*_{[R_Q(S),S]}\) as

\[
\Theta^*_{[S]} A^*_{[R_Q(S),S]} = U_S \Sigma_S V_S^T,
\]

where \(U_S\) is an \(N \times M\) orthonormal matrix, \(\Sigma_S = \text{diag}(\sigma_1, \ldots, \sigma_M)\) is an \(M \times M\) matrix, \(V_S\) is a \(p_Q(S)J \times M\) orthonormal matrix, and \(M = |S|\), the cardinality of the set \(S\). By (29), we have

\[
\sigma_{|S|}(\Theta^*_{[S]} A^*_{[R_Q(S),S]}) \geq \sigma_{|S|}(\Theta^*_{[S]}) \sigma_{|S|}(A^*_{[R_Q(S),S]}) \geq \sqrt{NJ} \sigma^2,
\]  

(40)

which implies that both \(\Theta^*_{[S]}\) and \(A^*_{[R_Q(S),S]}\) have full rank, and \(\mathcal{R}(\Theta^*_{[S]}) = \mathcal{R}(U_S)\). By Weyl’s perturbation theorem (see, e.g., Stewart and Sun (1990)), we have

\[
|\sigma_{|S|}(\hat{\Theta}_{[S]} A_{[R_Q(S),S]}^T) - \sigma_{|S|}(\Theta^*_{[S]} A^*_{[R_Q(S),S]})| \leq \|\hat{\Theta}_{[S]} A_{[R_Q(S),S]}^T - \Theta^*_{[S]} A^*_{[R_Q(S),S]}\|_2.
\]
Combining the above display with (39) and (40), we have

$$\sigma |S| (\hat{\Theta}_S \hat{A}^T_{[R_Q(S), S]} \geq \sqrt{NJ} \sigma^2 - \| \hat{\Theta}_S \hat{A}^T_{[R_Q(S), S]} - \Theta^*_S \hat{A}^T_{[R_Q(S), S]} \|_2 > 0.$$ 

Thus, we also have $\hat{\Theta}_S$ and $\hat{A}_{[s, r]}$ have full rank. We write $\hat{\Theta}_S \hat{A}^T_{[s, r]} = \hat{U} \hat{S} \hat{V}^T$, the reduced singular value decomposition. Then, $\mathcal{R}(\hat{\Theta}_S) = \mathcal{R}(\hat{U}_S)$. By the modified Davis-Kahan-Wedin sine theorem (O’Rourke et al., 2013, Theorem 19) and (40), we have

$$\sin \angle(\mathcal{R}(\hat{U}_S), \mathcal{R}(\hat{U}_S)) \leq 2 \frac{\| \hat{\Theta}_S \hat{A}^T_{[R_Q(S), S]} - \Theta^*_S \hat{A}^T_{[R_Q(S), S]} \|_2}{\sigma |S| (\Theta^*_S \hat{A}^T_{[R_Q(S), S]})} \leq 2 \frac{\| \hat{\Theta}_S \hat{A}^T - \Theta^* \hat{A}^* \|_2}{\sigma |S| (\Theta^*_S \hat{A}^T_{[R_Q(S), S]})}.$$ 

This further implies

$$\sin \angle(\mathcal{R}(\hat{\Theta}_S), \mathcal{R}(\hat{\Theta}_S)) \leq 2 \frac{\| \hat{\Theta}_S \hat{A}^T - \Theta^* \hat{A}^* \|_2}{\sigma |S| (\Theta^*_S \hat{A}^T_{[R_Q(S), S]})}.$$ 

\[ \Box \]

A.5 Proof of Results in Section 4

In this section, we provide proofs of Corollary 2 and Proposition 3-5.

Proof of Corollary 2. Take

$$c_{N,J} = \begin{cases} 1 & \text{if } \Theta^T_{[1:N,k]} \hat{\Theta}_{[1:N,k]} > 0, \\ -1 & \text{otherwise}. \end{cases}$$ 

Note that

$$\frac{\Theta^{[1:N,k]}}{\| \Theta^{[1:N,k]} \|} - c_{N,J} \frac{\hat{\Theta}^{[1:N,k]}}{\| \hat{\Theta}^{[1:N,k]} \|} \| = 2 - 2c_{N,J} \cos \angle (\Theta^{[1:N,k]}, \hat{\Theta}^{[1:N,k]}) = 2 - 2 \cos \angle (\Theta^{[1:N,k]}, \hat{\Theta}^{[1:N,k]}).$$
The above display converges to 0 in probability because \( \sin \angle(\Theta_{[1:N,k]}, \hat{\Theta}_{[1:N,k]}) \to 0 \) in probability. This completes the proof.

**Proof of Proposition 3.** From Corollary 2 we know that \( \frac{1}{N} \sum_{i=1}^{N} (v_i - \hat{v}_i)^2 \to 0 \). Let \( h \) be a 1-Lipschitz function, we have

\[
| \int h(x) F_{N,J}(x) - \int h(x) \hat{F}_{N,J}(x) | = \left| \frac{1}{N} \sum_{i=1}^{N} h(v_i) - h(\hat{v}_i) \right| \leq \frac{1}{N} \sum_{i=1}^{N} |v_i - \hat{v}_i|.
\]

By Cauchy inequality, we further have

\[
\frac{1}{N} \sum_{i=1}^{N} |v_i - \hat{v}_i| \leq \sqrt{\frac{\sum_{i=1}^{N} (v_i - \hat{v}_i)^2}{N}}.
\]

Combining the above two displays, we have

\[
Wass(F_{N,J}, \hat{F}_{N,J}) = \sup_{h \text{ is 1- Lipschitz}} | \int h(x) F_{N,J}(x) - \int h(x) \hat{F}_{N,J}(x) | \leq \sqrt{\frac{\sum_{i=1}^{N} (v_i - \hat{v}_i)^2}{N}}.
\]

By Corollary 2 the right-hand side of the above display tend to 0 as \( N, J \to \infty \) in probability.
Proof of Proposition 4. For each $\epsilon > 0$, we have

$$
\sum_{i \neq j} I(v_i < v_j, \hat{v}_i > \hat{v}_j)
\leq \sum_{i \neq j} I(v_i < v_j, \hat{v}_i > \hat{v}_j, |v_i - v_j| > \epsilon) + \sum_{i \neq j} I(0 < v_i - v_j < \epsilon)
\leq \sum_{i \neq j} I(v_j - v_i > \epsilon, \hat{v}_i > \hat{v}_j) + \sum_{i \neq j} \kappa_R \epsilon
\leq \sum_{i \neq j} \frac{|v_j - v_i - (\hat{v}_j - \hat{v}_i)|}{\epsilon} + N(N - 1)\kappa_R \epsilon
\leq \sum_{i \neq j} \frac{|v_j - \hat{v}_j| + |v_i - \hat{v}_i|}{\epsilon} + N(N - 1)\kappa_R \epsilon
= 2(N - 1)/\epsilon \sum_{i=1}^{N} |v_i - \hat{v}_i| + N(N - 1)\kappa_R \epsilon
\leq \epsilon^{-1} 2(N - 1) \sqrt{N \sum_{i=1}^{N} (v_i - \hat{v}_i)^2 + N(N - 1)\kappa_R \epsilon}.
$$

Thus,

$$
\frac{\sum_{i \neq j} I(v_i < v_j, \hat{v}_i > \hat{v}_j)}{N(N - 1)} \leq 2\epsilon^{-1} \sqrt{\frac{\sum_{i=1}^{N} (v_i - \hat{v}_i)^2}{N}} + \kappa_R \epsilon.
$$

Take $\epsilon = (\sqrt{\sum_{i=1}^{N} (v_i - \hat{v}_i)^2}/\kappa)^{1/2}$ in the above inequality, then we have

$$
\frac{\sum_{i \neq j} I(v_i < v_j, \hat{v}_i > \hat{v}_j)}{N(N - 1)} \leq 3(\kappa_R \sum_{i=1}^{N} (v_i - \hat{v}_i)^2/N)^{1/2}.
$$

The right-hand side converge to 0 as $N, J \to \infty$ in probability, so as the left-hand side. This completes the proof. \qed
Proof of Proposition 5. We have

\[ N^{-1} \sum_{i=1}^{N} I\{ \hat{v}_i \geq \tau_+, v_i \leq \tau_- \} + I\{ \hat{v}_i \leq \tau_- , v_i \geq \tau_+ \} \]

\[ \leq N^{-1} \sum_{i=1}^{N} \frac{ |\hat{v}_i - v_i| }{ \tau_+ - \tau_- } \]

\[ \leq (\tau_+ - \tau_-)^{-1} \sqrt{ N^{-1} \sum_{i=1}^{N} (\hat{v}_i - v_i)^2 } . \]

We complete the proof by noting the right-hand side of the above display converge to 0 in probability. \( \square \)

A.6 Proofs of Results in Section 5

Proof of Proposition 7. We will construct \((\Theta, A) \in S_{Q,-}\). First, we note that \(S_{Q,-}\) does not impose additional requirement on \(A\) beyond those in \(S_Q\). So, we construct \(A\) in the same way as we did in the proof of Proposition 1. We proceed to construct \(\Theta\). We already have \(\theta_{i1} = 1\) for all \(i \in \mathbb{Z}_+\), so we only need to consider \(\Theta_{[2:K]}\). Let \(w_i = c_i (1, 1, ..., 1, -i, 0_{K-i}^\top)^\top \in \mathbb{R}^{K+1}\), where \(c_i = (i + i^2)^{-1/2}\) is the normalizing constants making the \(w_i\) to be a unit vector \((i = 1, ..., K)\). We can see \(w_i\) and \(w_j\) are orthogonal for \(i \neq j\). We consider a matrix

\[ E_m = [w_1, ..., w_m] \in \mathbb{R}^{(K+1) \times m} . \]

Now we construct \(\Theta_{[2:K]}\) as

\[ \Theta_{[2:K]} = (\sqrt{C^2 - 1/2}) \begin{bmatrix} E_{K-1} \\ E_{K-1} \\ \vdots \end{bmatrix} . \]
Then,

$$\Theta = \begin{bmatrix} 1_2 & (\sqrt{C^2 - 1}/2)E_{K-1} \\ 1_2 & (\sqrt{C^2 - 1}/2)E_{K-1} \\ \vdots & \vdots \end{bmatrix}.$$ 

We proceed to consider properties of Θ. We first consider \( \lim_{N \to \infty} \Theta_{[1:N,1,K]}^T \Theta_{[1:N,1,K]} \). We have

$$\lim_{N \to \infty} \frac{\Theta_{[1:N,1,K]}^T \Theta_{[1:N,1,K]}}{N} = \frac{1}{(K+1)} \begin{bmatrix} 1 & 0_{K-1} \\ 0_{K-1} & (C^2 - 1)/4I_{K-1} \end{bmatrix}.$$ 

Thus, \( \gamma(\Theta) = (K+1)^{-1/2} \min(\sqrt{C^2 - 1}/2, 1) > 0 \). Furthermore, it is easy to verify that \( \| \theta_i \| \leq C \) for all \( i \in \mathbb{Z}_+ \). Thus, \((\Theta, A) \in S_{Q,-}\). \qed

**Proof of Proposition 6.** **Necessary part:** We prove by contradiction. If (13) does not hold, then there are two cases: 1) \( \{1, k, k'\} \in S_{k',p_Q(s)>0} \) for some \( k' \neq k \), and 2) for all \( k \in S \), \( p_Q(S) = 0 \). For these two cases, we will construct \((\tilde{\Theta}, \tilde{A})\) and \((\Theta', A')\) such that \( P_{\tilde{\Theta},\tilde{A}} = P_{\Theta', A'} \) but \( \sin_+ \angle(\tilde{\Theta}_{[k]}, \Theta'_{[k]}) > 0 \).

**Case 1:** \( \{1, k, k'\} \in S_{k',p_Q(s)>0} \) for some \( k' \neq k \). First, we consider \( \tilde{\Theta} = \Theta/2 \) and \( \tilde{A} = A/2 \) where \((\Theta, A) \in S_{Q,-}\) is constructed in the same way as we did in the proof of Proposition 7. Now we construct \((\Theta', A') \in S_{Q,-}\) such that \( P_{\tilde{\Theta}, \tilde{A}} = P_{\Theta', A'} \). We use the same strategy as we did in the proof of Theorem 1.

\[
\theta'_{m} = \begin{cases} 
\theta_{m}/2 & \text{if } m \neq k \\
(\theta_{ik} - \theta_{ik'})/2 & \text{if } m = k
\end{cases}
\]

and

\[
a'_{m} = \begin{cases} 
a_{m}/2 & \text{if } m \neq k' \\
(a_{jk} + a_{jk'})/2 & \text{if } m = k'.
\end{cases}
\]
Then, similar as the proof of Theorem 11 we know $(\Theta', A') \in S_{Q,-}$. Now we consider the angle between $\tilde{\Theta}_{[k]}$ and $\Theta_{[k]}$. We have

$$\tilde{\Theta}_{[k]} = (C^2 - 1)^{1/2}/4 \begin{bmatrix} w_{k-1} \\ w_{k-1} \\ \vdots \end{bmatrix}, \text{ and } \Theta'_{[k]} = (C^2 - 1)^{1/2}/4 \begin{bmatrix} w_{k-1} - w_{k'-1} \\ w_{k-1} - w_{k'-1} \\ \vdots \end{bmatrix}. $$

Thus, we have

$$\lim_{N \to \infty} N^{-1} \langle \tilde{\Theta}_{[1:N,k]}, \Theta'_{[1:N,k]} \rangle = \frac{C^2 - 1}{4(K + 1)} \langle w_{k-1}, w_{k-1} - w_{k'-1} \rangle = (C^2 - 1)/(4(K + 1)), $$

and

$$\lim_{N \to \infty} N^{-1} \|\tilde{\Theta}_{[1:N,k]}\|^2 = (C^2 - 1)/(4(K + 1)), $$

and

$$\lim_{N \to \infty} N^{-1} \|\Theta'_{[1:N,k]}\|^2 = (C^2 - 1)/(2(K + 1)). $$

Thus, $\lim_{N \to \infty} \cos \angle(\tilde{\Theta}_{[k]}, \Theta'_{[k]}) = 1/\sqrt{2}$, and $\sin(\tilde{\Theta}_{[k]}, \Theta'_{[k]}) = 1/\sqrt{2} > 0$. This contradicts the identifiability assumption.

**Case 2: for all $k \in S$, $p_Q(S) = 0$.** Similar to Case 1, we let $\tilde{\Theta} = \Theta/2$ and $\tilde{A} = A/2$.

We proceed to construct $(\Theta', A')$. We let $A' = \tilde{A}$ and $\Theta'_{[m]} = \tilde{\Theta}_{[m]}$ for $m \neq k$. For $\Theta'_{[k]}$, we let

$$\Theta'_{[k]} = \frac{\sqrt{C^2 - 1}}{4} \begin{bmatrix} w_K \\ w_K \\ \vdots \end{bmatrix}. $$

Similar to the proof of Proposition 7 it is not hard to show that $(\Theta', A') \in S_{Q,-}$. On the other hand, $\langle \Theta'_{[k]}, \tilde{\Theta}_{[k]} \rangle = 0$. Thus, $\sin(\theta'_{[k], \tilde{\Theta}_{[k]}}) = 1 > 0$. Again, this contradicts our assumption.
**Sufficient part:** Let $(\Theta', A'), (\Theta, A) \in \mathcal{S}_{Q,-}$ such that $P_{\Theta', A'} = P_{\Theta, A}$. According to the definition of $\mathcal{S}_{Q,-}$, there exists $N_0, J_0$, $0 < \sigma < C$, and $0 < \varepsilon < \sigma/8$, such that for all $N \geq N_0$ and $J \geq J_0$, we have

$$\sigma_{|S|}(A_{[RQ(S) \cap [1,J],[S]]}) > \sqrt{J}\sigma, \quad \sigma_{|S|}(A'_{[RQ(S) \cap [1,J],[S]]}) > \sqrt{J}\sigma \quad \text{for} \quad S \quad \text{such that} \quad p_Q(S) > 0, \quad (41)$$

$$\sigma_K(\Theta_{[1:N,1:K]}) \geq \sigma_N, \quad \sigma_K(\Theta'_{[1:N,1:K]}) \geq \sigma_N, \quad (42)$$

$$\left|\frac{1}{N} 1_N \Theta_{[1:N,m]} \right| \leq \varepsilon, \quad \text{and} \quad \left|\frac{1}{N} 1_N \Theta'_{[1:N,m]} \right| \leq \varepsilon \quad \text{for all} \quad m \in \{2, ..., K\}. \quad (43)$$

Since $P_{\Theta', A'} = P_{\Theta, A}$ and the generalized latent factor model has a strictly convex loglikelihood function in $(a_j^\top \theta, i, j \in \mathbb{Z}_+)$, we have $a_j^\top \theta = a_j^\top \theta'$ for all $i, j \in \mathbb{Z}_+$. Consequently, for all $N, J$, we have

$$\Theta_{[1:N,1:K]}A_{[1:J,1:K]}^\top = \Theta'_{[1:N,1:K]}A'_{[1:J,1:K]}^\top$$

for all $N, J \in \mathbb{Z}_+$. Now we focus on $N \geq N_0$ and $J \geq J_0$. For all $S$ such that $p_Q(S) > 0$, we have

$$\left(\Theta_{[1:N,1:K]}A_{[1:J,1:K]}^\top\right)_{[RQ(S) \cap [1,J],1:K]} = \Theta_{[1:N,1:K]}A_{[RQ(S) \cap [1,J],1:K]}^\top = \Theta_{[1:N,S]}A_{[RQ(S) \cap [1,J],S]}^\top$$

Similarly,

$$\left(\Theta'_{[1:N,1:K]}A'_{[1:J,1:K]}^\top\right)_{[RQ(S) \cap [1,J],1:K]} = \Theta'_{[1:N,1:K]}A'_{[RQ(S) \cap [1,J],1:K]}^\top = \Theta'_{[1:N,S]}A'_{[RQ(S) \cap [1,J],S]}^\top.$$  

Therefore,

$$\Theta_{[1:N,S]}A_{[RQ(S) \cap [1,J],S]}^\top = \Theta'_{[1:N,S]}A'_{[RQ(S) \cap [1,J],S]}^\top.$$  

According to (41) and (42), we know $\Theta_{[1:N,S]}, \Theta'_{[1:N,S]}, A_{[RQ(S) \cap [1,J],S]}, A'_{[RQ(S) \cap [1,J],S]}$ all have full rank (rank $|S|$). Thus,

$$\mathcal{R}(\Theta_{[1:N,S]}) = \mathcal{R}(\Theta'_{[1:N,S]}).$$
Intersecting all $S$ such that $k \in S$ and $p_Q(S) > 0$, we have

$$
\mathcal{R}(\Theta_{[1:N, \cap S: k \in S, p_Q(S) > 0]}^S) \\
= \cap_{S: k \in S, p_Q(S) > 0} \mathcal{R}(\Theta_{[1:N,S]}^S) \\
= \cap_{S: k \in S, p_Q(S) > 0} \mathcal{R}(\Theta'_{[1:N,S]}) \\
= \mathcal{R}(\Theta'_{[1:N, \cap S: k \in S, p_Q(S) > 0]}^S).
$$

Given (13), we have $\cap_{S: k \in S, p_Q(S) > 0} S = \{1, k\}$. Thus, from the above display, we have

$$
\mathcal{R}(\Theta_{[1:N,\{1,k\}]}) = \mathcal{R}(\Theta'_{[1:N,\{1,k\}]})
$$

Recall $\Theta_{[1:N,1]} = \Theta'_{[1:N,1]} = 1_N$. Thus, the above display implies

$$
span(1_N, \Theta_{[1:N, k]} - \bar{\theta}_N 1_N) = span(1_N, \Theta'_{[1:N, k]} - \bar{\theta}'_N 1_N),
$$

where we write $\bar{\theta}_N = \frac{1}{N} 1_N^T \Theta_{[1:N, k]}$ and $\bar{\theta}'_N = \frac{1}{N} 1_N^T \Theta'_{[1:N, k]}$. Since $1_N$ is orthogonal to $\Theta_{[1:N, k]} - \bar{\theta}_N 1_N$ and $\Theta'_{[1:N, k]} - \bar{\theta}'_N 1_N$, from the above equation, we have

$$
span(\Theta_{[1:N, k]} - \bar{\theta}_N 1_N) = span(\Theta'_{[1:N, k]} - \bar{\theta}'_N 1_N).
$$

That is, there exists $\lambda_N \in \mathbb{R}$, such that

$$
\Theta_{[1:N, k]} - \bar{\theta}_N 1_N = \lambda_N (\Theta'_{[1:N, k]} - \bar{\theta}'_N 1_N). 
$$

Because $\|\theta_i\|, \|\theta'_i\| \leq C$ for all $i \in \mathbb{Z}_+$, we have

$$
|\lambda_N| = \frac{\|\Theta_{[1:N, k]} - \bar{\theta}_N 1_N\|}{\|\Theta'_{[1:N, k]} - \bar{\theta}'_N 1_N\|} \leq \frac{\sqrt{NC} + \sqrt{N} |\bar{\theta}_N|}{\sqrt{NC} - \sqrt{N} |\bar{\theta}_N|}.
$$

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According to (43), we further have
\[ |\lambda_N| \leq \frac{C + \varepsilon}{C - \varepsilon}. \] (45)

Now we find an upper bound on \(|\sin \angle(\Theta_{[1:N,k]}, \Theta'_{[1:N,k]})|\). We have
\[
|\sin \angle(\Theta_{[1:N,k]}, \Theta'_{[1:N,k]})| = \inf_{\lambda \in \mathbb{R}} \frac{\|\Theta_{[1:N,k]} - \lambda \Theta'_{[1:N,k]}\|}{\|\Theta_{[1:N,k]}\|} \\
\leq \frac{\|\Theta_{[1:N,k]} - \lambda_N \Theta'_{[1:N,k]}\|}{\|\Theta_{[1:N,k]}\|} \\
= \frac{\lambda_N (\theta_N - \theta'_{N})^T 1_N}{\|\Theta_{[1:N,k]}\|}.
\] (46)

The last equation in the above display is due to (44). According to (42), \(\|\Theta_{[1:N,k]}\| = \sigma_1(\Theta_{[1:N,1:K]}) \geq \sigma_K(\Theta_{[1:N,1:K]}) \geq \sigma \sqrt{N}\). Combining this with (46) and (45), we have
\[
|\sin \angle(\Theta_{[1:N,k]}, \Theta'_{[1:N,k]})| \leq \frac{|(\theta_N - \theta'_{N})^T (C + \varepsilon)|}{(C - \varepsilon) \sigma}.
\]

Taking limit in the above display, we arrive at
\[
\sin_+ \angle(\Theta_{[k]}, \Theta'_{[k]}) = \limsup_{N \to \infty} \frac{|(\theta_N - \theta'_{N})^T (C + \varepsilon)|}{(C - \varepsilon) \sigma} = 0.
\]

This completes our proof. \(\square\)

**Proof of Proposition 8.** The proof of Proposition 8 is similar to that of Theorem 2. We only state the difference. Throughout this proof, we will use \(\Theta^*, A^*, \hat{\Theta}, \text{ and } \hat{A}\) to represent \(\Theta^*_{[1:N,1:K]}, A^*_{[1:1,1:K]}, \hat{\Theta}^{(N,J)}, \text{ and } \hat{A}^{(N,J)}\), respectively, to simplify the notation.

We define \(M^* = \Theta^* A^{*\top}\) and \(\hat{M} = \hat{\Theta} \hat{A}^{\top}\). Then,
\[
l^{\Omega}(\hat{M}) - l^{\Omega}(M^*) \geq 0.
\]
Based on the above inequality, we obtain the following inequality which is similar to (20).

\[ \sum_{i=1}^{N} \sum_{j=1}^{J} (\hat{m}_{ij} - m_{ij}^*)^2 \omega_{ij} \leq \frac{2}{\min_{|\nu| \leq C^2} b'(\nu)} \sum_{i=1}^{N} \sum_{j=1}^{J} (Y_{ij} - b'(m_{ij}^*)) \omega_{ij} (\hat{m}_{ij} - m_{ij}^*). \]

Comparing the left-hand side of the above display with its conditional expectation given \( Y_{ij} \)'s, we further obtain the following inequality.

\[
\frac{n}{N} \sum_{i=1}^{N} \sum_{j=1}^{J} (\hat{m}_{ij} - m_{ij}^*)^2 
\leq \frac{2}{\min_{|\nu| \leq C^2} b'(\nu)} \sum_{i=1}^{N} \sum_{j=1}^{J} (Y_{ij} - b'(m_{ij}^*)) \omega_{ij} (\hat{m}_{ij} - m_{ij}^*) - \sum_{i=1}^{N} \sum_{j=1}^{J} (\hat{m}_{ij} - m_{ij}^*)^2 (\omega_{ij} - \frac{n}{N} J) \quad (47)
\]

\[
\leq \frac{4}{\min_{|\nu| \leq C^2} b'(\nu)} \sup_{\tilde{M} \in C} |\langle Z, \tilde{M} \rangle| + \left| \sum_{i=1}^{N} \sum_{j=1}^{J} (\hat{m}_{ij} - m_{ij}^*)^2 (\omega_{ij} - \frac{n}{N} J) \right|,
\]

where \( Z = (Z_{ij}, 1 \leq i \leq N, 1 \leq j \leq J) \) with \( Z_{ij} = X_{ij} \omega_{ij} = (Y_{ij} - b'(m_{ij}^*)) \omega_{ij} \). We establish probability tail bounds for the two terms on the right-hand side of the above display. We start with the second term. Note that

\[
E \left[ \sum_{i=1}^{N} \sum_{j=1}^{J} (\hat{m}_{ij} - m_{ij}^*)^2 (\omega_{ij} - \frac{n}{N} J) \right] = E \left\{ E \left[ \sum_{i=1}^{N} \sum_{j=1}^{J} (\hat{m}_{ij} - m_{ij}^*)^2 (\omega_{ij} - \frac{n}{N} J) | Y \right] \right\} = 0,
\]

and

\[
Var \left[ \sum_{i=1}^{N} \sum_{j=1}^{J} (\hat{m}_{ij} - m_{ij}^*)^2 (\omega_{ij} - \frac{n}{N} J) \right]
= \sum_{i=1}^{N} \sum_{j=1}^{J} E (\hat{m}_{ij} - m_{ij}^*)^4 (\omega_{ij} - \frac{n}{N} J)^2 \leq \sum_{i=1}^{N} \sum_{j=1}^{J} (2C^2)^4 \frac{n}{N} J \leq n 16 C^8.
\]

Thus, by Chebyshev’s inequality, for all \( s > 0 \),

\[
P \left( \sum_{i=1}^{N} \sum_{j=1}^{J} (\hat{m}_{ij} - m_{ij}^*)^2 (\omega_{ij} - \frac{n}{N} J) > s \right) \leq \frac{n}{t^2} 16 C^8. \quad (48)
\]

We proceed to the first term on the right-hand side of (47). Using the same arguments as
we arrive at
\[
P\left( \sup_{\tilde{M} \in \mathcal{C}} |\langle Z, \tilde{M} \rangle| > t \right) \leq N(\delta) \sup_{(\Theta', A') \in \mathcal{S}} P\left( |\langle Z, \Theta' A'^T \rangle| \geq t - 2\gamma \sqrt{NJ} \delta C' \right) + P(\|Z\|_F \geq \gamma) \tag{49}\]

for any positive \(\delta\) and \(\gamma\). \(N(\delta)\) is already bounded by (25). We proceed to bound
\[
P\left( |\langle Z, \Theta' A'^T \rangle| \geq t - 2\gamma \sqrt{NJ} \delta C' \right). \]
We use the following lemma which is similar to Lemma 2.

**Lemma 5.** There exist constant \(\delta_0, \varepsilon_0\) (depending on \(b, C'\)) such that for all \(\tilde{M} \in \mathcal{C} 0 < t < \delta_0 n,\)
\[
P(\|Z, \tilde{M}\| > t) \leq 2 \exp(-\frac{\varepsilon_0 t^2}{n}).
\]

Using the above lemma, we have for \(0 \leq t - 2\gamma \sqrt{NJ} \delta C' \leq \delta_0 n\)
\[
P\left( |\langle Z, \Theta' A'^T \rangle| \geq t - 2\gamma \sqrt{NJ} \delta C' \right) \leq 2 \exp\left\{ -\frac{(t - 2\gamma \sqrt{NJ} \delta C')^2}{n} \right\}. \tag{50}
\]

Now we consider \(P(\|Z\|_F \geq \gamma)\). By Chebyshev’s inequality, we have
\[
P(\|Z\|_F > \gamma) \leq \gamma^{-2} E(\|Z\|_F^2) = \gamma^{-2} \phi \frac{\gamma}{\sqrt{N}} \sum_{i=1}^{N} \sum_{j=1}^{J} b''(m_{ij}^*) \leq \gamma^{-2} \phi \sup_{|\nu| \leq C'} b''(\nu). \tag{51}
\]

Combining (49), (50) and (51), we have
\[
P(\sup_{\tilde{M} \in \mathcal{C}} |\langle X, \tilde{M} \rangle| > t)
\leq \left(\frac{CK}{\delta}\right)^{K(N+J)} \exp\{-\varepsilon_0 (t - 2\gamma \sqrt{NJ} \delta C')^2 / n\} + \gamma^{-2} \phi \sup_{|\nu| \leq C'} b''(\nu)
\leq \exp\{K(N + J)(\log C' + \log K - \log \delta) - \varepsilon_0 ((t - 2\gamma \sqrt{NJ} \delta C')^2 / n)\} + \gamma^{-2} \phi \sup_{|\nu| \leq C'^2} b''(\nu).
\]

In the above display, we choose \(\gamma = \sqrt{2n \phi (N + J) \sup_{|\nu| \leq C'} b(\nu)}, \delta = (NJ)^{-1/2}\) and \(t = \)
$$2\gamma \sqrt{NJ} \delta C' + \frac{1}{\sqrt{n}} \sqrt{2nK(N + J)(\log C' + \log K - \log \delta)},$$

then

$$P(\sup_{\tilde{M} \in \mathcal{C}} |\langle X, \tilde{M} \rangle| > t) \leq \exp\{-K(N + J)(\log C' + \log K + \frac{1}{2} \log(NJ))\} + \frac{1}{2(N + J)}.$$

Now we let $s = \sqrt{2n(N + J)16C^8}$ in (48) and obtain

$$P\left(\sum_{i=1}^{N} \sum_{j=1}^{J} (\tilde{m}_{ij} - m_{ij})^2 (\omega_{ij} - \frac{n}{NJ}) > s\right) \leq \frac{1}{2(N + J)}$$

Combining the above two inequalities with (47), we arrive at

$$P\left(\frac{n}{NJ} \sum_{i=1}^{N} \sum_{j=1}^{J} (\tilde{m}_{ij} - m_{ij})^2 \geq s + t\right) \leq \exp\{-K(N + J)(\log C' + \log K + \frac{1}{2} \log(NJ))\} + \frac{1}{N + J}.$$

Note that there exists constant $\kappa_1$ such that $s + t \leq \kappa_1 \sqrt{n(N + J) \log(NJ)}$. Thus,

$$P\left(\frac{\|\tilde{M} - M^*\|_2^2}{NJ} \geq \kappa_1 \sqrt{\frac{(N + J) \log(NJ)}{n}}\right)$$

$$\leq \exp\{-K(N + J)(\log C' + \log K + \frac{1}{2} \log(NJ))\} + \frac{1}{N + J} \leq \frac{2}{N + J},$$

for sufficiently large $N$ and $J$. This completes the proof of (14). To prove (15), we note that (29) and (31) still hold. Thus we could apply the same proof as the proof of (7). We omit the repetitive details.

\[\square\]
A.6.1 Proof of Lemma 5

Proof of Lemma 5 Consider the moment generating function for $|\lambda| < 1$,

$$E(e^{\lambda\langle Z, M \rangle}) = E \left( \exp \left[ \sum_{i=1}^{N} \sum_{j=1}^{J} \phi^{-1}\{b(m_{ij}^* + \phi \lambda \tilde{m}_{ij} \omega_{ij}) - b(m_{ij}^*)\} - \lambda b'(m_{ij}^*) \omega_{ij} \right] \right)$$

$$= E \left( \exp \left[ \sum_{i=1}^{N} \sum_{j=1}^{J} \frac{1}{2} b''(m_{ij}^\top) \phi \lambda^2 \tilde{m}_{ij}^2 \omega_{ij} \right] \right)$$

$$\leq E \left( \exp \left[ \kappa_2 \lambda^2 \sum_{i=1}^{N} \sum_{j=1}^{J} \omega_{ij} \right] \right).$$

where $\kappa_2 = \sup_{|\nu| \leq \phi C \nu^2} b(\nu) \phi$. Note that $\sum_{i=1}^{N} \sum_{j=1}^{J} \omega_{ij} \sim \text{Binomial}(NJ, n/(NJ))$. Thus,

$$E(\exp[\kappa_2 \lambda^2 \sum_{i=1}^{N} \sum_{j=1}^{J} \omega_{ij}]) = (1 - \frac{n}{NJ} + \frac{n}{NJ} e^{\kappa_2 \lambda^2})^{NJ} \leq e^{n(e^{\kappa_2 \lambda^2} - 1)}.$$ 

When $\lambda$ is sufficiently close to 0, the above display can be further bounded from above by $e^{2\kappa_2 n \lambda^2}$. Thus, $\langle Z, M \rangle$ is sub-exponential and we could use Bernstein’s inequality and arrive at

$$P(\langle Z, M \rangle > t) \leq 2 \exp\left( -\varepsilon_0 \frac{t^2}{n} \right)$$

for some $\varepsilon_0$, $\delta_0$ and $t \leq \delta_0 n$. \hfill \Box

A.7 Proofs of Results in Section 6

Proof of Proposition 9. Our proof is based on the following three results. The first one is proved in Anderson and Duffin (1969). It provides a characterization of the intersection of two spaces through the pseudo inverse of the sum of two projection matrices.

Lemma 6 (Theorem 8 of Anderson and Duffin (1969)).

$$P_{L \cap M} = 2P_L(P_L + P_M)^\top P_M,$$
where $P$ is the projection operator and $A^\dagger$ is the pseudo-inverse (or Moore-Penrose generalized inverse) of a matrix $A$.

The second result is from Stewart (1977). It provides perturbation bounds for the pseudo-inverse of a matrix.

**Lemma 7** (Theorem 3.3 of Stewart (1977)). For any two matrices $A$ and $B$,

$$
\| B^\dagger - A^\dagger \| \leq 3 \max(\| A^\dagger \|_2^2, \| B^\dagger \|_2^2) \| B - A \|.
$$

Here, the norm $\| \cdot \|$ could be any uniformly generated, normalized, and unitary invariant norm. See Stewart (1977) for the detailed definition.

In order to apply the above lemma to our problem, we need an upper bound of $\| (P_L + P_M)^\dagger \|_2$ and $\| (P_{L'} + P_{M'})^\dagger \|_2$. The next two Lemmas provide an upper bound through the smallest positive principal angle between $L$ and $M$.

**Lemma 8.** Let $L$ and $M$ be any two linear subspaces of a finite dimensional vector space. For all $x \in L + M$,

$$
\| P_L x + P_M x \| \geq \| x \| / \beta(\theta_{\min,+}(L, M)),
$$

where we define $\beta(\theta) = \sqrt{\alpha(\theta)}$.

**Lemma 9** (Spectral norm of the pseudo-inverse). For any matrix $A$,

$$
\| A^\dagger \|_2 = \frac{1}{\inf_{x \in \mathbb{R}(A^\dagger), \| x \| = 1} \| Ax \|}.
$$

Now we apply Lemma 8 and Lemma 9 to the operator $P_L + P_M$, and arrive at

$$
\| (P_L + P_M)^\dagger \|_2 \leq \beta(\theta_{\min,+}(L, M)).
$$
Similarly, we have
\[
\| (P_{L'} + P_{M'})^\dagger \|_2 \leq \beta(\theta_{\min,+}(L', M')).
\]
For the norm \( \| . \| \) discussed here, it has a nice property that \( \| AB \| \leq \| A \|_2 \| B \| \) and \( \| AB \| \leq \| A \|_2 \| B \| \) for any matrices \( A, B \). Now we can combine all the results to bound \( \| P_{L' \cap M'} - P_{L \cap M} \| \). We have
\[
\| P_{L' \cap M'} - P_{L \cap M} \|
\leq 2 \| P_L - P_{L'} \| \| (P_L + P_M)^\dagger \|_2 \| P_M \|_2 + 2 \| (P_L + P_M)^\dagger - (P_{L'} + P_{M'})^\dagger \|_2 \| P_{L'} \|_2 \| P_{M'} \|_2
\]
\[
+ 2 \| P_{M'} - P_M \| \| P_{L'} \|_2 \| (P_{L'} + P_{M'})^\dagger \|_2
\]
\[
\leq 2 \| P_L - P_{L'} \| \beta(\theta_{\min,+}(L, M)) + 2 \| (P_L + P_M)^\dagger - (P_{L'} + P_{M'})^\dagger \|
\]
\[
+ 2 \| P_{M'} - P_M \| \beta(\theta_{\min,+}(L', M'))
\]
\[
\leq 2 \| P_L - P_{L'} \| \beta(\theta_{\min,+}(L, M)) + 2 \| P_{M'} - P_M \| \beta(\theta_{\min,+}(L', M'))
\]
\[
+ 6 \max\{ \beta(\theta_{\min,+}(L, M))^2, \beta(\theta_{\min,+}(L', M'))^2 \} \| P_L - P_{L'} + P_M - P_{M'} \|
\]
\[
\leq \left[ 6 \max\{ \beta(\theta_{\min,+}(L, M))^2, \beta(\theta_{\min,+}(L', M'))^2 \} + 2 \max\{ \beta(\theta_{\min,+}(L, M)), \beta(\theta_{\min,+}(L', M')) \} \right]
\]
\[
\times \left( \| P_L - P_{L'} \| + \| P_{M} - P_{M'} \| \right)
\]
\[
\leq 8 \max\{ \beta(\theta_{\min,+}(L, M))^2, \beta(\theta_{\min,+}(L', M'))^2 \} (\| P_L - P_{L'} \| + \| P_{M} - P_{M'} \|)
\]
\[
\leq 8 \max\{ \alpha(\theta_{\min,+}(L, M)), \alpha(\theta_{\min,+}(L', M')) \} (\| P_L - P_{L'} \| + \| P_{M} - P_{M'} \|)
\]

where the last inequality holds, as \( \beta(\theta_{\min,+}(L, M)) \geq 1 \) and \( \beta(\theta_{\min,+}(L', M')) \geq 1 \).

\[\square\]

**Proof of Lemma** If \( \sigma_{|S_1 \cup S_2|}(W_{[S_1 \cup S_2]}) = 0 \), then the right-hand side of (17) is 1 and the lemma holds. In the rest of the proof, we assume \( \sigma_{|S_1 \cup S_2|}(W_{[S_1 \cup S_2]}) > 0 \).

Let \( X = W_{[S_1 \setminus S_2]} - P_{\mathcal{R}(W_{[S_1 \cap S_2 ]})} W_{[S_1 \setminus S_2] } \), \( Y = W_{[S_2 \setminus S_1]} - P_{\mathcal{R}(W_{[S_1 \cap S_2 ]})} W_{[S_2 \setminus S_1] } \) and \( Z = W_{[S_1 \cup S_2]} \). Then, \( \mathcal{R}(Z) = \mathcal{R}(W_{[S_1]}) \cap \mathcal{R}(W_{[S_2]}). \) Let \( \theta_1 \leq \theta_2 \leq \ldots \leq \theta_{\min(|S_1|, |S_2|)} \) be the principal angles between \( \mathcal{R}(W_{[S_1]}) \) and \( \mathcal{R}(W_{[S_2]}). \) Since \( \dim(\mathcal{R}(Z)) = |S_1 \cap S_2| \), we have \( \theta_1 = \theta_2 = \ldots = \theta_{|S_1 \cap S_2|} = 0 \), and \( \theta_{|S_1 \cap S_2|+1} = \theta_{\min,+}(\mathcal{R}(W_{[S_1]}), \mathcal{R}(W_{[S_2]})). \) We proceed to
find an upper bound of \( \cos(\theta_{|S_1 \cap S_2|+1}) \). In what follows, we find the upper bound according to the definition of \( \theta_{|S_1 \cap S_2|+1} \). Note that the space \( \mathcal{R}(X) \) and \( \mathcal{R}(Z) \) are orthogonal and \( \mathcal{R}(X) + \mathcal{R}(Z) = \mathcal{R}(W_{[S_1]}) \). Similarly, \( \mathcal{R}(Y) \) and \( \mathcal{R}(Z) \) are orthogonal and \( \mathcal{R}(Y) + \mathcal{R}(Z) = \mathcal{R}(W_{[S_2]}) \). We have

\[
2 - 2 \cos(\theta_{|S_1 \cap S_2|+1}) = 2 - \max_{u \in \mathcal{R}(X), v \in \mathcal{R}(Y), \|u\| = 1, \|v\| = 1} \langle u, v \rangle
\]

\[
= \min_{u \in \mathcal{R}(X), v \in \mathcal{R}(Y), \|u\| = 1, \|v\| = 1} \left\{ 2 - 2 \langle u, v \rangle \right\}
\]

\[
= \min_{u \in \mathcal{R}(X), v \in \mathcal{R}(Y), \|u\| = 1, \|v\| = 1} \| u - v \|^2.
\]

We use the following lemma to proceed.

**Lemma 10.** \( \sigma_{|S_1 \setminus S_2| \cup (S_2 \setminus S_1)}([X, Y]) \geq \sigma_{|S_1 \cup S_2|}(W_{[S_1 \cup S_2]}), \) and \( \|[X, Y]\|_2 \leq \|W_{[S_1 \cup S_2]}\|_2 \).

From the above lemma, we have \( \sigma_{|S_1 \setminus S_2|}(X) > \sigma_{|S_1 \setminus S_2|}(W_{[S_1 \cup S_2]}), \) and \( \sigma_{|S_2 \setminus S_1|}(Y) > 0. \) Similarly, \( \sigma_{|S_2 \setminus S_1|}(Y) > 0. \) Thus, \( X^\top X \) and \( Y^\top Y \) are invertible.

For \( u \in \mathcal{R}(X), v \in \mathcal{R}(Y), \) let \( w \in R^m, x \in R^m \) such that \( u = X(X^\top X)^{-1/2}w, \) and \( v = Y(Y^\top Y)^{-1/2}x. \) Then, it is easy to verify that \( \|u\| = \|w\| \) and \( \|v\| = \|x\| \).

Thus, (52) becomes

\[
2 - 2 \cos(\theta_{|S_1 \cap S_2|+1}) \geq \min_{\|w\| = 1, \|x\| = 1} \|X(X^\top X)^{-1/2}w - Y(Y^\top Y)^{-1/2}x\|^2
\]

\[
\geq \min_{\|w\|^2 + \|x\|^2 = 2} \|X(X^\top X)^{-1/2}w - Y(Y^\top Y)^{-1/2}x\|^2
\]

\[
= 2\lambda_{\min}(W_1),
\]

where

\[
W_1 = \begin{bmatrix}
I_{|S_1 \setminus S_2|} & (X^\top X)^{-1/2}X^\top Y(Y^\top Y)^{-1/2} \\
(Y^\top Y)^{-1/2}Y^\top X(X^\top X)^{-1/2} & I_{|S_2 \setminus S_1|}
\end{bmatrix}
\]

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and $\lambda_{\min}(W)$ denotes the smallest eigenvalue of $W$. Let

$$W_2 = \begin{bmatrix} X^TX & X^TY \\ Y^TX & Y^TY \end{bmatrix}$$

and

$$D = \begin{bmatrix} X^TX & 0_{|S_1\setminus S_2|,|S_2\setminus S_1|} \\ 0_{|S_2\setminus S_1|,|S_1\setminus S_2|} & Y^TY \end{bmatrix}.$$ 

Then, we have

$$W_1 = D^{-1/2}W_2D^{-1/2}.$$ 

Or, equivalently,

$$W_2 = D^{1/2}W_1D^{1/2}.$$ 

Note that $W_1, W_2, D$ are all symmetric positively definite matrices. Combine this fact with Lemma 10, we have

$$\lambda_{\min}(W_1) \geq \frac{\lambda_{\min}(W_2)}{\|D^{1/2}\|_2^2} \geq \frac{\lambda_{\min}([X,Y]^T[X,Y])}{\|X,Y\|_2^2} \geq \frac{\sigma_{|S_1\cup S_2|}(W_{|S_1\cup S_2|})^2}{\|W\|_2^2}. $$

Combining this with (53), we arrive at

$$\cos(\theta_{|S_1\cap S_2|+1}) \leq 1 - \frac{\sigma_{|S_1\cup S_2|}(W_{|S_1\cup S_2|})^2}{\|W\|_2^2}. $$

This completes our proof.

\[\square\]

### A.7.1 Proofs of Lemmas 8, 9, and 10

**Proof of Lemma 8** Our proof for this lemma has two steps:

1. We first prove the statement for $L \cap M = \{0\}$.

2. We generalize it to the case where $\dim(L \cap M) \geq 1$. 

Step 1, $L \cap M = \{0\}$ Let $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_m$ be the principal angles between $L$ and $M$ with $m = \min(\dim(L), \dim(M))$. Of particular interest is the smallest principal angle $\theta_1$, which is defined as

$$\theta_1 = \max_{u \in L, v \in M, |u| = 1, |v| = 1} \arccos \langle u, v \rangle,$$

or, equivalently,

$$\cos(\theta_1) = \max_{|u| = 1, u \in L, |v| = 1, v \in M} |\langle u, v \rangle|.$$

We start with a lower bound for $\|P_Lx\|^2 + \|P_Mx\|^2$ for $x \in L + M$. Since we assume $x \in L + M$ and $L \cap M = \{0\}$, there exists a unique pair $x \in L$ and $y \in M$ such that $x = x + y$.

$$\|P_Lx\|^2 + \|P_Mx\|^2 = \|x + P_Ly\|^2 + \|y + P_Mx\|^2 \tag{54}$$

$$\geq (\|x\| - \|P_Ly\|)^2 + (\|y\| - \|P_Mx\|)^2.$$

On the other hand,

$$\|P_Ly\| = \max_{u \in L, |u| = 1} |\langle u, y \rangle| \leq \max_{u \in L, |u| = 1, v \in M, |v| = 1} |\langle u, v \rangle| \|y\| = \cos(\theta_1)\|y\|.$$

Similarly,

$$\|P_Mx\| \leq \cos(\theta_1)\|x\|.$$

Combining this with (54), we have

$$\|P_Lx\|^2 + \|P_Mx\|^2 \geq \left(\|x\| - \|y\| \cos(\theta_1)\right)_+^2 + \left(\|y\| - \|x\| \cos(\theta_1)\right)_+^2$$

$$\geq \frac{1}{2} \left(\|x\| - \|y\| \cos(\theta_1)\right)_+^2 + \left(\|y\| - \|x\| \cos(\theta_1)\right)_+^2$$

$$\geq \frac{1}{2}(\|x\| + \|y\|)^2(1 - \cos(\theta_1))^2.$$
On the other hand,

\[ \|x\|^2 = \|y\|^2 + \|x\|^2 + 2\langle x, y \rangle \leq \|y\|^2 + \|x\|^2 + 2 \cos(\theta_1) \|x\| \|y\| \leq (1 + \cos(\theta_1))(\|x\|^2 + \|y\|^2). \tag{56} \]

Combining (55) and (56), we arrive at

\[ \|P_L x\|^2 + \|P_M x\|^2 \geq \frac{(1 - \cos(\theta_1))^2 \|x\|^2}{2(1 + \cos(\theta_1))}. \tag{57} \]

On the other hand, since \( P_L x \in L \) and \( P_M x \in M \), we have

\[
\|P_L x + P_M x\|^2 \\
= \|P_L x\|^2 + \|P_M x\|^2 + 2\langle P_L x, P_M x \rangle \\
\geq \|P_L x\|^2 + \|P_M x\|^2 - 2 \cos(\theta_1) \|P_L x\| \|P_M x\| \\
= (1 - \cos(\theta_1))(\|P_L x\|^2 + \|P_M x\|^2) + \cos(\theta_1)(\|P_L x\| - \|P_M x\|)^2 \\
\geq (1 - \cos(\theta_1))(\|P_L x\|^2 + \|P_M x\|^2). \tag{58}
\]

Now we combine (57) with (58). Then, we arrive at

\[ \|P_L x + P_M x\|^2 \geq (1 - \cos(\theta))(\|P_L x\|^2 + \|P_M x\|^2) \geq \frac{(1 - \cos(\theta_1))^3 \|x\|^2}{2(1 + \cos(\theta_1))} = \frac{\|x\|^2}{(\beta(\theta_1))^2}. \]

Thus, for all \( x \in L + M \),

\[ \|P_L x + P_M x\| \geq \|x\|/\beta(\theta_1), \]

**Step 2.** \( L \cap M = S \) and \( \dim(S) \geq 1 \). Let \( L^- \) and \( M^- \) be linear subspaces such that \( L^- \perp S, L^- + S = L, M^- \perp S \) and \( M^- + S = M \). It is not hard to show that \( L^- \cap M^- = \{0\} \).

Now, for \( x \in L + M = S + L^- + M^- \), there is a unique pair \((x_S, x_-)\) such that \( x = x_S + x_- \) and \( x_S \in S \) and \( x_- \in L^- + M^- \). We note that

\[ P_L x + P_M x = x_S + P_L x^- + P_M x^- . \]
Also,
\[ P_L x_- = P_{L^{-}} P_L x_- + P_S P_L x_- = P_{L^{-}} x_- + P_S x_- = P_{L^{-}} x_- . \]

Similarly,
\[ P_M x_- = P_{M^{-}} x_- . \]

Combining the above three equations, we have
\[ P_L x + P_M x = x_S + P_{L^{-}} x^- + P_{M^{-}} x^- . \]

Note that \( x_S \perp P_{L^{-}} x^- + P_{M^{-}} x^- \). Thus,
\[ \| P_L x + P_M x \|^2 = \| x_S \|^2 + \| P_{L^{-}} x^- + P_{M^{-}} x^- \|^2 . \]

According to Step 1 of the proof, we further have
\[ \| P_L x + P_M x \|^2 \geq \| x_S \|^2 + \frac{\| x_- \|^2}{\beta^2(\theta_1(L^{-}, M^-))} . \]

Recall that \( x = x_S + x_- \) and \( x_S \perp x_- \), and \( \beta(\theta) \geq 1 \) for all \( \theta \), we further bound the above display by
\[ \| P_L x + P_M x \|^2 \geq \frac{\| x \|^2}{\beta^2(\theta_1(L^{-}, M^-))} . \]

To complete the proof, the only thing we need to prove is \( \theta_1(L^{-}, M^-) = \theta_{\min,+}(L, M) \), which is obvious according to the iterative definition of principal angles between linear spaces.

Proof of Lemma 9. We write the SVD of \( A \),
\[ A = U \Sigma V^\top = U_1 \Sigma_1 V_1^\top , \]
where \( \Sigma = \text{diag}(\sigma_1, ..., \sigma_m, 0, ..., 0) \), and \( U_1 = U_{[1:m]} \), \( \Sigma_1 = \text{diag}(\sigma_1, ..., \sigma_m) \) and \( V_1 = V_{[1:m]} \).
Then, the pseudo inverse has the following form (Hogben, 2006, p. 5-13, fact 2),

\[ A^\dagger = U \Sigma^\dagger V^\top, \]

with \( \Sigma^\dagger = \text{diag}(1/\sigma_1, ..., 1/\sigma_m, 0, ..., 0) \). The spectral norm is unitary invariant, so

\[
\|A^\dagger\|_2 = \|\Sigma^\dagger\|_2 = \max_{i=1,\ldots,m} 1/\sigma_i = \frac{1}{\min_{1 \leq i \leq m} \sigma_i}.
\]  \hspace{1cm} (59)

On the other hand, consider \( x \in \mathcal{R}(A^\top) = \mathcal{R}(V_1) \) and \( \|x\| = 1 \). We write \( x = V_1 y \) for \( y \in \mathbb{R}^m \).

Then, \( \|y\| = \|x\| = 1 \). We have

\[
\|Ax\|^2 = x^\top V_1 \Sigma_1 U_1^\top U_1 \Sigma_1 V_1^\top x = x^\top V_1 \Sigma_1^2 V_1^\top x = y^\top \Sigma_1^2 y.
\]

Thus,

\[
\inf_{x \in \mathcal{R}(A^\top), \|x\|=1} \|Ax\|^2 = \inf_{y \in \mathbb{R}^m, \|y\|=1} y^\top \Sigma_1^2 y = \min_{i=1,\ldots,m} \sigma_i^2.
\]  \hspace{1cm} (60)

Combining (59) and (60), we completes the proof. \( \square \)

**Proof of Lemma 11** From definition, we have

\[
[X, Y] = (I_n - P_{\mathcal{R}(Z)})[W_{[S_1 \setminus S_2]}, W_{[S_2 \setminus S_1]}] = (I_n - P_{\mathcal{R}(Z)})H,
\]

where we write \( H = [W_{[S_1 \setminus S_2]}, W_{[S_2 \setminus S_1]}] = W_{([S_1 \setminus S_2] \cup (S_2 \setminus S_1))} \). Since \( \sigma_{|S_1 \cap S_2|}(Z) = \sigma_{|S_1 \setminus S_2|}(W_{[S_1 \setminus S_2]}) \geq \sigma_{|S_1 \cup S_2|}(W_{[S_1 \cup S_2]}) > 0 \), we know \( Z^\top Z \) is invertible and \( P_{\mathcal{R}(Z)} = Z(Z^\top Z)^{-1} Z^\top \). Thus we have

\[
[X, Y]^\top[X, Y] = H^\top (I_n - Z(Z^\top Z)^{-1} Z^\top) H = H^\top H - H^\top Z(Z^\top Z)^{-1} Z^\top H.
\]
Recall the formula of inverse of a block matrix, we have

\[
(H^\top H - H^\top Z(Z^\top Z)^{-1}Z^\top H)^{-1} = \left( \begin{bmatrix} H^\top H & H^\top Z \\ Z^\top H & Z^\top Z \end{bmatrix} \right)^{-1},
\]

where we write \( l = |(S_1 \setminus S_2) \cup (S_2 \setminus S_1)| \). Because \((H^\top H - H^\top Z(Z^\top Z)^{-1}Z^\top H)^{-1}\) is a submatrix of \( \begin{bmatrix} H^\top H & H^\top Z \\ Z^\top H & Z^\top Z \end{bmatrix} \), we have

\[
\| (H^\top H - H^\top Z(Z^\top Z)^{-1}Z^\top H)^{-1} \|_2 \leq \left\| \begin{bmatrix} H^\top H & H^\top Z \\ Z^\top H & Z^\top Z \end{bmatrix} \right\|_2 = \sigma_{|S_1 \cup S_2|}([H, Z])^{-1}.
\]

Note that \([H, Z] = W_{[S_1 \cup S_2]}\). Thus, from the above display we have

\[
\| (H^\top H - H^\top Z(Z^\top Z)^{-1}Z^\top H)^{-1} \|_2 \leq \sigma_{|S_1 \cup S_2|}(W_{[S_1 \cup S_2]})^{-1}.
\]

Note that \( \| (H^\top H - H^\top Z(Z^\top Z)^{-1}Z^\top H)^{-1} \|_2 = \sigma_l(H^\top H - H^\top Z(Z^\top Z)^{-1}Z^\top H)^{-1} = \sigma_l([X, Y])^{-1} \). Thus, we arrive at

\[
\sigma_l([X, Y])^{-1} \leq \sigma_{|S_1 \cup S_2|}(W_{[S_1 \cup S_2]})^{-1}.
\]

Consequently, \( \sigma_l([X, Y]) \geq \sigma_{|S_1 \cup S_2|}(W_{[S_1 \cup S_2]}) \). This proves the first statement of the lemma.

For the second statement of the lemma, we have

\[
\| [X, Y] \|_2 = \| (I_n - P_{\mathcal{R}(Z)})W_{[S_1 \setminus S_2 \cup S_2 \setminus S_1]} \|_2 \leq \| I_n - P_{\mathcal{R}(Z)} \|_2 \| W_{[S_1 \setminus S_2 \cup S_2 \setminus S_1]} \|_2 \leq \| W_{[S_1 \cup S_2]} \|_2.
\]

Here we used the fact that \( \| I_n - P_{\mathcal{R}(Z)} \|_2 \leq 1 \). \( \square \)
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