An algorithm for variational inequalities with equilibrium and fixed point constraints

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Abstract: In this paper, we propose a new hybrid extragradient-viscosity algorithm for solving variational inequality problems, where the constraint set is the common elements of the set of solutions of a pseudomonotone equilibrium problem and the set of fixed points of a demicontractive mapping. Using the hybrid extragradient-viscosity method and combining with hybrid plane cutting techniques, we obtain the algorithm for this problem. Under certain conditions on parameters, the convergence of the iteration sequences generated by the algorithms is obtained.

1. Introduction

Let \( \mathbb{R}^n \) be a \( n \)-dimensional Euclidean space with the inner product \( \langle \cdot, \cdot \rangle \) and associated norm \( \| \cdot \| \). Let \( C \) be a nonempty closed convex subset in \( \mathbb{R}^n \) and \( G: C \to \mathbb{R}^n \), \( T: C \to C \) be operators, and \( f: C \times C \to \mathbb{R} \) be a bifunction satisfying \( f(x, x) = 0 \) for every \( x \in C \). We consider the following variational inequality problem over the set is the common elements of the set of a pseudomonotone equilibrium problem and the set of fixed points of a demicontractive mapping (shortly \( \text{VIEFP}(C, f, T, G) \)):

Find \( x^* \in S \) such that \( \langle G(x^*), y - x^* \rangle \geq 0, \quad \forall y \in S, \quad (1) \)

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PUBLIC INTEREST STATEMENT

Variational inequality and equilibrium problems as well as fixed point problems are very useful and efficient tools in mathematics. They provided a unified framework for studying many problems arising in engineering, economics, and other fields. In this paper, we propose a new algorithm for solving variational inequality problems where the constraint set is the common elements of the set of solutions of a pseudomonotone equilibrium problem and the set of fixed points of a demicontractive mapping. One difficulty of this problem is that the constraint set is not given explicitly. The proposed method allows us to solve this problem by solving a sequence of convex programs in which they are much easier to solve.
where $S = S_f \cap \text{Fix}(T)$, $S_f = \{ u \in C : f(u,y) \geq 0, \forall y \in C \}$, i.e. $S_f$ is the solution set of the following equilibrium problem (EP$(C,f)$ for short)

\[
\text{Find } u \in C \text{ such that } f(u,y) \geq 0, \forall y \in C,
\]

and \(\text{Fix}(T)\) is the fixed points of the mapping $T$, i.e. $\text{Fix}(T) = \{ v \in C \text{ such that } T(v) = v \}$.

We call problem (1) the upper problem and (2) the lower one. Problem (1) is a special case of mathematical programs with equilibrium constraints. Sources for such problems can be found in Luo, Pang, and Ralph (1996), Migdalas, Pardalos, and Varbrand (1988), Muu and Oettli (1992). Bilevel variational inequalities were considered in Anh, Kim, and Muu (2012), Moudafi (2010) and Yao, Liou, and Kang (2010) suggested the use of the proximal point method for monotone bilevel equilibrium problems, which contain monotone variational inequalities as a special case. Recently, Ding (2010) used the auxiliary problem principle to monotone bilevel equilibrium problems. In those papers, the lower problem is required to be monotone. In this case, the subproblems to be solved are monotone.

It should be noticed that the solution set $S_f$ of the lower problem (2) is convex whenever $f$ is pseudomonotone on $C$. However, the main difficulty is that, even the constrained set $S_f$ is convex, it is not given explicitly as in a standard mathematical programming problem, and therefore the available methods of convex optimization and variational inequality cannot be applied directly to problem (1).

In this paper, we extend the hybrid extragradient-viscosity methods introduced by Maingé (2008b) for solving bilevel problem (1) when the lower problem is pseudomonotone with respect to its solution set equilibrium problems rather than monotone variational inequalities as in Maingé (2008b), the later pseudomonotonicity is somewhat more general than pseudomonotone. We show that the sequence of iterates generated by the proposed algorithm converges to the unique solution of the bilevel problem (1).

The paper is organized as follows. Section 2 contains some preliminaries on the Euclidean projection and equilibrium problems. Section 3 is devoted to presentation of the algorithm and its convergence. In Section 4, we describe a special case of variational inequalities with variational inequalities and fixed points constraints, where the lower variational inequality is pseudomonotone with respect to its solution set.

2. Preliminaries
In the rest of the paper, by $P_C$ we denote as the projection operator on $C$, that is

\[
P_C(x) \in C : \| x - P_C(x) \| \leq \| y - x \|, \quad \forall y \in C.
\]

The following well-known results on the projection operator onto a closed convex set will be used in the sequel.

**Lemma 2.1** Suppose that $C$ is a nonempty closed convex set in $\mathbb{R}^n$. Then,

1. $P_C(x)$ is singleton and well defined for every $x$;
2. $x = P_C(x)$ if and only if $(x - \pi, y - \pi) \leq 0, \forall y \in C$; and
3. $\| P_C(x) - P_C(y) \|^2 \leq \| x - y \|^2 - \| P_C(x) - x + y - P_C(y) \|^2, \quad \forall x, y \in C$.

We recall some well-known definitions which will be useful in the sequel (see e.g. Blum & Oettli, 1994; Facchinei & Pang, 2003; Konnov, 2001; Muu & Oettli, 1992; Solodov & Svaiter, 1999).
Definition 2.1 A mapping $T: C \rightarrow C$ is called

(1) nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$;

(2) quasi-nonexpansive if $\text{Fix}(T) \neq \emptyset$ and $\|T(x) - x'\| \leq \|x - x'\|$ for all $x \in C$ and $x' \in \text{Fix}(T)$;

(3) $\theta$-strict pseudocontractive if there exists $\theta \in (0, 1)$, such that

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \theta \|T(x) - T(y)\| \|x - y\|$$

for all $x, y \in C$; and

(4) $\mu$-demicontactive if $\text{Fix}(T)$ is not empty, and there exists $\mu \in [0, 1)$, such that

$$\|T(x) - T(x')\|^2 \leq \|x - x'\|^2 + \mu \|x - T(x)\|^2$$

for all $x \in C$ and $x' \in \text{Fix}(T)$.

Definition 2.2 A bifunction $\varphi: C \times C \rightarrow \mathbb{R}$ is said to be

(1) strongly monotone on $C$ with modulus $\beta > 0$, if

$$\varphi(x, y) + \varphi(y, x) \leq -\beta \|x - y\|^2, \quad \forall x, y \in C;$$

(2) monotone on $C$ if

$$\varphi(x, y) + \varphi(y, x) \leq 0, \quad \forall x, y \in C;$$

(3) pseudomonotone on $C$ if

$$\varphi(x, y) \geq 0 \implies \varphi(y, x) \leq 0, \quad \forall x, y \in C;$$

and

(4) pseudomonotone on $C$ with respect to $x^*$ if

$$\varphi(x^*, y) \geq 0 \implies \varphi(y, x^*) \leq 0, \quad \forall y \in C.$$

We say that $\varphi$ is pseudomonotone on $C$ with respect to a set $S$ if it is pseudomonotone on $C$ with respect to every point $x^* \in S$.

From Definition 2.2, it follows that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\forall x^* \in C$.

When $\varphi(x, y) = \langle \psi(x), y - x \rangle$, where $\psi: C \rightarrow \mathbb{R}^n$ is an operator, then the definition (1) becomes:

$$\langle \varphi(x) - \varphi(y), x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in C$$

that is $\psi$ is $\beta$-strongly monotone on $C$. Similarly, if $\psi$ satisfies (2) ((3), (4) resp.) on $C$, then $\varphi$ becomes monotone, (pseudomonotone, pseudomonotone with respect to $x^*$ resp.) on $C$.

In the sequel, we need the following blanket assumptions

(A1) $f(\cdot, y)$ is continuous on $\Omega$ for every $y \in C$;

(A2) $f(x, \cdot)$ is convex on $\Omega$ for every $x \in C$;

(A3) $f$ is pseudomonotone on $C$ with respect to the solution set $S_f$ of $\text{EP}(C, f)$;

(A4) $T$ is $\mu$-demicontactive and closed mapping;

(A5) $G$ is $L$-Lipschitz and $\beta$-strongly monotone on $C$;

(B1) $h(\cdot)$ is $\delta$-strongly convex, continuously differentiable on $\Omega$;

(B2) $\{ \lambda_k \}$ is a positive sequence, such that $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\sum_{k=0}^{\infty} \lambda_k^2 < \infty$; and

(B3) $\{ \mu_k \}$ is a positive sequences, such that $0 < \mu \leq \mu_k \leq \frac{1 + \mu}{2}$.

Lemma 2.2 Suppose Problem $\text{EP}(C, f)$ has a solution. Then, under assumptions (A1), (A2), and (A3), the solution set $S_f$ is closed, convex, and...
\[ f(x', y) \geq 0 \quad \forall y \in C \text{ if and only if } f(y, x') \leq 0 \quad \forall y \in C. \]

The following lemmas are well known from the auxiliary problem principle for equilibrium problems.

**Lemma 2.3 (Mastroeni, 2003)** Suppose that \( h \) is a continuously differentiable and strongly convex function on \( C \) with modulus \( \delta > 0 \). Then, under assumptions (A1) and (A2), a point \( x^* \in C \) is a solution of EP(C, f) if and only if it is a solution to the equilibrium problem:

\[
\text{Find } x^* \in C: f(x', y) + h(y) - h(x^*) - \langle \nabla h(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \quad (\text{AEP})
\]

The function

\[
d(x, y) := h(y) - h(x) - \langle \nabla h(x), y - x \rangle
\]

is called Bregman function. Such a function was used to define a generalized projection, called \( d \)-projection, which was used to develop algorithms for particular problems (see e.g. Censor & Lent, 1981).

An important case is \( h(x) := \frac{1}{2} \| x \|^2 \). In this case, \( d \)-projection becomes the Euclidean one.

**Lemma 2.4 (Mastroeni, 2003)** Under assumptions (A1) and (A2), a point \( x^* \in C \) is a solution of problem (AEP) if and only if

\[
x^* = \text{argmin} \{ f(x', y) + h(y) - h(x^*) - \langle \nabla h(x^*), y - x^* \rangle: y \in C \}. \quad (\text{CP})
\]

Note that since \( f(x, \cdot) \) is convex and \( h \) is strongly convex, Problem (CP) is a strongly convex program.

For each \( z \in C \), by \( \partial_z f(z, z) \), we denote the subgradient of the convex function \( f(z, \cdot) \) at \( z \), i.e.

\[
\partial_z f(z, z) := \{ w \in \mathbb{R}^n: f(z, y) \geq f(z, z) + \langle w, y - z \rangle, \quad \forall y \in C \}
\]

\[
= \{ w \in \mathbb{R}^n: f(z, y) \geq \langle w, y - z \rangle, \quad \forall y \in C \},
\]

and each \( w \in \partial_z f(z, z) \) we define the halfspace \( H_z \) as

\[
H_z := \{ x \in \mathbb{R}^n: \langle w, x - z \rangle \leq 0 \}. \quad (3)
\]

Note that when \( f(x, y) = \langle f(x), y - x \rangle \), where \( f: C \to \mathbb{R}^n \), this halfspace becomes the one introduced in Solodov and Svaiter (1999). The following lemma says that the hyperplane does not cut off any solution of problem EP(C, f).

**Lemma 2.5 (Dinh & Muu, 2015)** Under assumptions (A2) and (A3), one has \( S_z \subseteq H_z \) for every \( z \in C \).

**Lemma 2.6 (Dinh & Muu, 2015)** Under assumptions (A1) and (A2), if \( \{ z^k \} \subseteq C \) is a sequence, such that \( \{ z^k \} \) converges to \( z \) and the sequence \( \{ w^k \} \), with \( w^k \in \partial_z f(z^k, z^k) \) converges to \( w \), then \( w \in \partial_z f(z, z) \).

**Lemma 2.7** Suppose the bifunction \( f \) satisfies the assumptions (A1) and (A2), the function \( h \) satisfies the assumption (B1). If \( \{ x^k \} \subseteq C \) is bounded and \( \{ y^k \} \) is a sequence, such that

\[
y^k = \text{arg min} \{ f(x^k, y) + \frac{1}{\rho} [h(y) - h(x^k) - \langle \nabla h(x^k), y - x^k \rangle]: y \in C \},
\]

then \( \{ y^k \} \) is bounded.

**Proof** Firstly, we show that if \( \{ x^k \} \) converges to \( x^* \), then \( \{ y^k \} \) is bounded. Indeed,

\[
y^k = \text{arg min} \{ f(x^k, y) + \frac{1}{\rho} [h(y) - h(x^k) - \langle \nabla h(x^k), y - x^k \rangle]: y \in C \},
\]
and
\[ f(x^k, x^k) + \frac{1}{\rho} \left[ h(x^k) - h(x^k) - \langle \nabla h(x^k), x^k - x^k \rangle \right] = 0, \]
therefore
\[ f(x^k, y^k) + \frac{1}{\rho} \left[ h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle \right] \leq 0, \quad \forall k. \]

In addition, \( f(x^k, .) + \frac{1}{\rho} \left[ h(.) - h(x^k) - \langle \nabla h(x^k), . - x^k \rangle \right] \) is strongly convex on \( C \) with modulus \( \frac{\rho}{2} \); hence, for all \( y^k \in \partial f(x^k, x^k) \), we get
\[ f(x^k, y^k) + \frac{1}{\rho} \left[ h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle \right] \geq \langle w^k, y^k - x^k \rangle + \frac{\rho}{2} \| y^k - x^k \|^2. \]

This implies \( 0 \geq -\|w^k\|\|x^k - x^k\| + \frac{\rho}{2} \| y^k - x^k \|^2 \), so that
\[ \|y^k - x^k\| \leq \frac{\rho}{\rho} \|w^k\| \]
Because \( \{x^k\} \) converges to \( x^* \) and \( w^k \in \partial f(x^k, x^k) \), by Theorem 24.5 in Rockafellar (1970), the sequence \( \{w^k\} \) is bounded; combining with the boundedness of \( \{x^k\} \), we have \( \{y^k\} \) also bounded.

Now, we prove the Lemma 2.7. Suppose contradict that \( \{y^k\} \) is unbounded, i.e. there exists an subsequence \( \{y^k\} \subseteq \{y^k\} \), such that \( \lim_{k \to \infty} \|y^k\| = +\infty \). By the boundedness of \( \{x^k\} \), it implies \( \{x^k\} \) is also bounded; without loss of gerarality, we may assume that \( \lim_{k \to \infty} x^k = x^* \). By the same argument as above, we have \( \{y^k\} \) is bounded, which we contradict. Therefore, \( \{y^k\} \) is bounded. \( \square \)

The following lemma is in Solodov and Svaiter (1999) (see also Dinh & Muu, 2015).

**Lemma 2.8** (Dinh & Muu, 2015; Solodov & Svaiter, 1999) Suppose that \( x \in C \) and \( u = P_{C_{\alpha_0}}(x) \). Then,
\[ u = P_{\alpha_0, C}(x), \text{ where } \hat{x} = P_{\alpha_0}(x). \]

**Lemma 2.9** (Maingé, 2008b) Let \( T : C \to C \) be an \( \alpha \) demicontractive mapping, such that \( \text{Fix}(T) \) is non-empty. Then, \( T = (1 - \alpha I + \mu T \) is a quasi-nonexpansive mapping over \( C \) for every \( \mu \in [0;1 - \alpha] \). Furthermore,
\[ \| T(x) - x^* \|^2 \leq \|x - x^*\|^2 - \mu(1 - \alpha - \mu)\|T(x) - x\|^2 \] for all \( x \in C \) and \( x^* \in \text{Fix}(T). \)

**Lemma 2.10** (Lemma 3.1 Maingé, 2008a) Let \( \{\alpha_n\} \) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence \( \alpha_{n_k} \) of \( \{\alpha_n\} \), such that
\[ \alpha_{n_k} < \alpha_{n_{k+1}} \text{ for all } k \geq 0. \]

Also, consider the sequence of integers \( \{\sigma(k)\}_{k \geq k_0} \) defined by
\[ \sigma(k) = \max \left\{ j \leq k \mid \alpha_j < \alpha_{j+1} \right\}. \]

Then, \( \{\sigma(k)\}_{k \geq k_0} \) is a nondecreasing sequence verifying
\[ \lim_{k \to \infty} \alpha(k) = \infty \]
and, for all \( k \geq k_0 \), the following two inequalities hold:
3. A hybrid extragradient-viscosity algorithm for VIEFP \((C, f, T, G)\)

**Algorithm 1.**

**Initialization.** Pick \(x^0 \in C\), choose parameters \(\eta \in (0, 1)\), \(\rho > 0\) and \(\{\lambda_k\} \subset [0, 1)\); \(\{\mu_k\} \subset [0, 1)\).

**Iteration** \(k\). (\(k = 0, 1, 2, \ldots\)) Having \(x^k\) do the following steps:

**Step 1.** Solve the strongly convex program

\[
\min \left\{ f(x^k, y) + \frac{1}{\rho} \left[ h(y) - h(x^k) - \langle \nabla h(x^k), y - x^k \rangle \right] : y \in C \right\}
\]

\(CP(x^k)\)

to obtain its unique solution \(y^k\).

If \(y^k = x^k\), take \(u^k = x^k\) and go to **Step 4**. Otherwise, do **Step 2**.

**Step 2.** (Armijo linesearch rule) Find \(m_k\) as the smallest positive integer number \(m\) satisfying

\[
\begin{align*}
x^{k,m} &= (1 - \eta^m)x^k + \eta^my^k, \\
w^{k,m} &\in \partial f(x^{k,m}, z^{k,m}), \\
\langle w^{k,m}, x^k - y^k \rangle &\geq \frac{1}{\rho} \left[ h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle \right].
\end{align*}
\]

**Step 3.** Set \(\eta_k := \eta^{m_k}, z^k := z^{k,m_k}, w^k := w^{k,m_k}\). Take

\(C_k := \{ x \in C : \langle w^k, x - z^k \rangle \leq 0 \}\), \(u^k := P_{C_k}(x^k)\).

**Step 4.** Compute \(v^k = P_C(u^k - \lambda_k G(u^k))\).

**Step 5.** Set \(x^{k+1} = (1 - \mu_k)v^k + \mu_k T(v^k)\), and go to **Step 1** with \(k\) replaced by \(k + 1\).

**Remark 3.1**

1. If \(y^k = x^k\), then \(x^k\) is a solution to \(EP(C, f)\).
2. \(w^k \neq 0 \forall k\), indeed, at the beginning of **Step 2**, \(x^k \neq y^k\). By the Armijo linesearch rule and \(\delta\)-strong convexity of \(h\), we have

\[
\langle w^k, x^k - y^k \rangle \geq \frac{1}{\rho} \left[ h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle \right] \geq \frac{\delta}{\rho} \|x^k - y^k\|^2 > 0.
\]

Now, we are going to analyze the validity and convergence of the algorithm. Some parts in our proofs are based on the proof scheme in Maingé (2008b).

**Lemma 3.1** Under Assumptions \((A1)\) and \((A2)\), the linesearch rule (6) is well defined in the sense that at each iteration \(k\), there exists an integer number \(m > 0\), satisfying the inequality in (6) for every \(w^{k,m} \in \partial f(z^{k,m}, x^{k,m})\), and if, in addition assumptions \((A3)\) and \((A4)\) are satisfied, then for every \(x^k \in S\), one has

\[
\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \tilde{x}^k\|^2 - \left( \frac{\eta_k \delta}{\rho \|w^k\|} \right)^2 \|x^k - y^k\|^4 \\
- \|x^{k+1} - v^k\|^2 - 2 \lambda_k (u^k - x^*, G(u^k)) + \lambda_k^2 \|G(u^k)\|^2 \forall k,
\]

where \(\tilde{x}^k = P_{H_J}(x^k)\).
Proof First, we prove that there exists a positive integer \( m_0 \) such that

\[
\langle w^{k m_0}, x^k - y^k \rangle \geq \frac{1}{\rho} \left[ h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle \right]
\]

\( \forall w^{k m_0} \in \partial f(z^{k m_0}, z^{k m_0}). \)

Indeed, suppose by contradiction that for every positive integer \( m \) and \( z^{k m} = (1 - \eta^m)x^k + \eta^m y^k \), there exists \( w^{k m} \in \partial f(z^{k m}, z^{k m}) \), such that

\[
\langle w^{k m}, x^k - y^k \rangle < \frac{1}{\rho} \left[ h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle \right].
\]

Since \( z^{k m} \to x^* \) as \( m \to \infty \), by Theorem 24.5 in Rockafellar (1970), the sequence \( \{w^{k m}\}_{m=0}^{\infty} \) is bounded. Thus, we may assume that \( w^{k m} \to \tilde{w} \) for some \( \tilde{w} \). Taking the limit as \( m \to \infty \), from \( z^{k m} \to x^* \) and \( w^{k m} \to \tilde{w} \), by Lemma 2.6, it follows that \( \tilde{w} \in \partial f(x^*, x^*) \) and

\[
\langle \tilde{w}, x^k - y^k \rangle \leq \frac{1}{\rho} \left[ h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle \right].
\]

(9)

Since \( \tilde{w} \in \partial f(x^*, x^*) \), we have

\[
f(x^*, y^k) \geq f(x^*, x^*) + \langle \tilde{w}, y^k - x^* \rangle = \langle \tilde{w}, y^k - x^* \rangle.
\]

Combining with (9) yields

\[
f(x^*, y^k) + \frac{1}{\rho} \left[ h(y^k) - h(x^*) - \langle \nabla h(x^*), y^k - x^* \rangle \right] \geq 0,
\]

which contradicts to the fact that

\[
f(x^*, y^k) + \frac{1}{\rho} \left[ h(y^k) - h(x^*) - \langle \nabla h(x^*), y^k - x^* \rangle \right] < 0.
\]

Thus, the linesearch is well defined.

Now, we prove (8). For simplicity of notation, let \( d^k := x^k - y^k \), \( H_j := H_j \).

Since \( u^k = P_{C \cap H_j}(\tilde{x}^k) \) and \( x^* \in S \), by Lemma 2.5, \( x^* \in C \cap H_j \), we have

\[
\|u^k - \tilde{x}^k\|^2 \leq \langle x^* - \tilde{x}^k, u^k - \tilde{x}^k \rangle
\]

which together with

\[
\|u^k - x^*\|^2 = \|\tilde{x}^k - x^*\|^2 + \|u^k - \tilde{x}^k\|^2 + 2\langle u^k - \tilde{x}^k, \tilde{x}^k - x^* \rangle
\]

implies

\[
\|u^k - x^*\|^2 \leq \|\tilde{x}^k - x^*\|^2 - \|u^k - \tilde{x}^k\|^2.
\]

(10)

Replacing

\[
\tilde{x}^k = P_{H_j}(x^k) = x^k - \frac{\langle w^k, x^k - \tilde{x}^k \rangle}{\|w^k\|^2}w^k
\]

into (Equation 10), we obtain

\[
\|u^k - x^*\|^2 \leq \|\tilde{x}^k - x^*\|^2 - \|u^k - \tilde{x}^k\|^2 - 2\langle w^k, x^k - \tilde{x}^k \rangle \frac{(w^k, x^k - \tilde{x}^k)}{\|w^k\|^2} + \frac{(w^k, x^k - \tilde{x}^k)^2}{\|w^k\|^2}.
\]
Substituting $x^k = z^k + \eta_k d^k$ into the last inequality, we get

$$
\|u^k - x^k\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \tilde{x}^k\|^2 + \left(\frac{\eta_k \langle w^k, d^k \rangle}{\|w^k\|}\right)^2 - \frac{2\eta_k \langle w^k, d^k \rangle}{\|w^k\|^2} \langle w^k, x^k - x^*\rangle
$$

$$
= \|x^k - x^*\|^2 - \|u^k - \tilde{x}^k\|^2 - \left(\frac{\eta_k \langle w^k, d^k \rangle}{\|w^k\|}\right)^2 - \frac{2\eta_k \langle w^k, d^k \rangle}{\|w^k\|^2} \langle w^k, z^k - x^*\rangle.
$$

In addition, by the Armijo linesearch rule, using the $\delta$-strong convexity of $h$, we have

$$
\langle w^k, x^k - y^k \rangle \geq \frac{1}{\delta} \left[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle\right] \geq \frac{\delta}{\rho} \|x^k - y^k\|^2.
$$

Note that $x^k \in H_j$ can be written as:

$$
\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \tilde{x}^k\|^2 - \left(\frac{\eta_k \delta}{\rho \|w^k\|}\right)^2 \|x^k - y^k\|^2.
$$

(11)

From Lemma 2.9, we have

$$
\|x^{k+1} - x^*\|^2 \leq \|v^k - x^*\|^2 - \mu_k (1 - \mu - \mu_k) T(v^k - v^*)^2
$$

(12)

Replacing $T(v^k - v^*) = \frac{1}{\mu_k} (x^{k+1} - v^*)$ into (Equation 12), we get

$$
\|x^{k+1} - x^*\|^2 \leq \|v^k - x^*\|^2 - \frac{1 - \mu - \mu_k}{\mu_k} \|x^{k+1} - v^*\|^2
$$

$$
\leq \|v^k - x^*\|^2 - \|x^{k+1} - v^*\|^2,
$$

(13)

where the last inequality follows from $0 < \mu \leq \mu_k \leq \frac{1 - \mu}{\delta}$. We have

$$
\|x^{k+1} - x^*\|^2 \leq \|v^k - x^*\|^2 - \|x^{k+1} - v^*\|^2
$$

$$
= \|P_c(u^k - \mu_k G(u^k)) - P_c(x^*)\|^2 - \|x^{k+1} - v^*\|^2
$$

$$
\leq \|u^k - x^* - \mu_k G(u^k)\|^2 - \|x^{k+1} - v^*\|^2
$$

$$
= \|u^k - x^*\|^2 - 2\mu_k \|u^k - x^*, G(u^k)\|^2 + \mu_k^2 \|G(u^k)\|^2 - \|x^{k+1} - v^*\|^2,
$$

which together with (11) implies

$$
\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \tilde{x}^k\|^2 - \left(\frac{\eta_k \delta}{\rho \|w^k\|}\right)^2 \|x^k - y^k\|^2
$$

$$
- \|x^{k+1} - v^*\|^2 - 2\mu_k \|u^k - x^*, G(u^k)\|^2 + \mu_k^2 \|G(u^k)\|^2 \forall k
$$

(14)

as desired.

□

**Lemma 3.2** The sequences $\{x^k\}, \{u^k\},$ and $\{v^k\}$ generated by the Algorithm 1 are bounded under assumptions (A1), (A2), (A3), (A4), and (A5).

**Proof** From (13), we get

$$
\|x^{k+1} - x^*\| \leq \|v^k - x^*\|
$$

(15)

In addition,
\[ \| v^k - x^* \| = \| P_c(u^k - \hat{\lambda}_k G(u^k)) - P_c(x^*) \| \leq \| u^k - \hat{\lambda}_k G(u^k) - x^* \| \]
\[ \leq \| u^k - \hat{\lambda}_k G(u^k) - (x^* - \hat{\lambda}_k G(x^*)) \| + \hat{\lambda}_k \| G(x^*) \| \]
\[ = \left( 1 - L^2 \frac{\hat{\lambda}_k}{\beta} \right) \| u^k - x^* \| - L^2 \frac{\hat{\lambda}_k}{\beta} \left( \frac{\beta}{L^2} G - I \right) u^k - \left( \frac{\beta}{L^2} G - I \right) x^* \| \]
\[ + \hat{\lambda}_k \| G(x^*) \| \]
\[ \leq \left( 1 - L^2 \frac{\hat{\lambda}_k}{\beta} \right) \| u^k - x^* \| + L^2 \frac{\hat{\lambda}_k}{\beta} M_k + \hat{\lambda}_k \| G(x^*) \|. \] (16)

where \( M_k = \| \left( \frac{\beta}{L^2} G - I \right) u^k - \left( \frac{\beta}{L^2} G - I \right) x^* \| \).

Since \( G \) is \( L \)-Lipschitz and \( \beta \)-strongly monotone, we have
\[ M_k^2 = \left\| \frac{\beta}{L^2} G(u^k) - G(x^*) \right\| - (u^k - x^*)^2 \]
\[ = \frac{\beta^2}{L^4} \| G(u^k) - G(x^*) \|^2 - 2 \frac{\beta}{L^2} \langle G(u^k) - G(x^*), u^k - x^* \rangle + \| u^k - x^* \|^2 \]
\[ \leq \frac{\beta^2}{L^4} \| u^k - x^* \|^2 - 2 \frac{\beta^2}{L^2} \| u^k - x^* \|^2 + \| u^k - x^* \|^2 \]
\[ = (1 - \frac{\beta^2}{L^2}) \| u^k - x^* \|^2. \]

Hence, \( M_k \leq \sqrt{1 - \frac{\beta^2}{L^2}} \| u^k - x^* \| \). Then, combining with (16), we get
\[ \| v^k - x^* \| \leq \left( 1 - \hat{\lambda}_k \frac{L^2}{\beta} \left( 1 - \sqrt{1 - \frac{\beta^2}{L^2}} \right) \right) \| u^k - x^* \| + \hat{\lambda}_k \| G(x^*) \| \]
\[ = \left( 1 - \hat{\lambda}_k \frac{L^2}{\beta} \right) \| u^k - x^* \| + \hat{\lambda}_k \| G(x^*) \| \]
\[ = (1 - \chi_k) \| u^k - x^* \| + \gamma_k \left( \frac{\beta}{L^2} \| G(x^*) \| \right) \]
\[ \leq (1 - \chi_k) \| u^k - x^* \| + \gamma_k \left( \frac{\beta}{L^2} \| G(x^*) \| \right) \]

where, \( \gamma = 1 - \sqrt{1 - \frac{\beta^2}{L^2}}, \chi_k = \hat{\lambda}_k \frac{L^2}{\beta} \gamma \in (0;1) \), and the last inequality deduced from (11). Combining with (15), we obtain
\[ \| x^k - x^* \| \leq (1 - \chi_k) \| x^k - x^* \| + \gamma_k \left( \frac{\beta}{L^2} \| G(x^*) \| \right). \] (17)

By induction, it implies \( \| x^{k+1} - x^* \| \leq \max \{ \| x^1 - x^* \|, \frac{\beta}{L^2} \| G(x^*) \| \} \leq \ldots \leq \max \{ \| x^0 - x^* \|, \frac{\beta}{L^2} \| G(x^*) \| \} \).

Hence, \( \{ x^k \} \) is bounded, which, from (11), implies that \( \{ u^k \} \) is also bounded. Since \( \| v^k - u^k \| = \| P_c(u^k - \hat{\lambda}_k G(u^k)) - P_c(u^k) \| \leq \hat{\lambda}_k \| G(u^k) \| \) and by the boundedness of \( \{ u^k \} \), we can conclude that \( \{ v^k \} \) is bounded.

**Lemma 3.3** If the subsequence \( \{ v^k \} \subset \{ v^k \} \) converges to some \( \tilde{v} \) and

\[ \left( \frac{\eta_k}{\| w^k \|} \right)^2 \| y^k - x^k \| \rightarrow 0 \quad \text{and} \quad \| x^{k+1} - v^k \| \rightarrow 0 \quad \text{as} \; i \rightarrow \infty, \] (18)

then \( \tilde{x} \in S. \)

**Proof** By definition of \( \{ x^k \} \) in Algorithm 1, we have \( x^{k+1} = (1 - \mu_k) v^k + \mu_k T(v^k) \). Therefore, \( \| T(v^k) - v^k \| = \frac{1}{\mu_k} \| x^{k+1} - v^k \|. \) Taking the limit as \( i \rightarrow \infty \) and by the closedness of mapping \( T \), we get
$T(\bar{\nu}) = \bar{\nu}$, i.e. $\bar{\nu} \in \text{Fix}(T)$.

Now, we prove $\bar{\nu} \in S_f$. Indeed, from Lemma 3.2, $(x^k)$ is bounded. Without loss of generality, we may assume that $\lim_{i \to \infty} x^k = \bar{x}$. We will consider two distinct cases:

\textbf{Case 1.} $\inf \frac{\eta}{\|w^i\|} > 0$. Then, from (18), one has $\lim_{i \to \infty} \|y^h - x^h\| = 0$, thus $y^h \to \bar{x}$ and $z^h \to \bar{x}$.

According to the definition of $y^h$, we have

$$f(x^h, y) + \frac{1}{\rho}[h(y) - h(x^h) - (\nabla h(x^h), y - x^h)] \geq f(x^h, y^h) + \frac{1}{\rho}[h(y^h) - h(x^h) - (\nabla h(x^h), y^h - x^h)], \forall y \in C$$

by the continuity of $h$, $\forall h$, we get in the limit as $i \to \infty$ that

$$f(x, y) + \frac{1}{\rho}[h(y) - h(x^h) - (\nabla h(x^h), y - x^h)] \geq 0, \forall y \in C$$

this fact shows that $\bar{x} \in S_f$.

\textbf{Case 2.} $\lim_{i \to \infty} \frac{\eta}{\|w^i\|} = 0$. By the linesearch rule and $\rho$-strong convexity of $h$, we have

$$\langle w^i, x^h - y^h \rangle \geq \frac{1}{\rho}[h(y^h) - h(x^h) - (\nabla h(x^h), y^h - x^h)]$$

$$\geq \frac{\tau}{\rho}\|x^h - y^h\|^2.$$

Thus, $\|y^h - x^h\| \leq \sqrt{\frac{\tau}{\rho}}\|w^i\|$.

From the boundedness of $\{w^i\}$ and (18), it follows $n_i \to 0$, so that $z^h = (1 - n_i)x^h + n_iy^i \to \bar{x}$ as $i \to \infty$. Without loss of generality, we suppose that $w^i \to \bar{w} \in \partial f(\bar{x}, \bar{x})$ and $y^h \to \bar{y}$ as $i \to \infty$.

We have

$$f(x^h, y) + \frac{1}{\rho}[h(y) - h(x^h) - (\nabla h(x^h), y - x^h)]$$

$$\geq f(x^h, y^h) + \frac{1}{\rho}[h(y^h) - h(x^h) - (\nabla h(x^h), y^h - x^h)], \forall y \in C$$

letting $i \to \infty$, we obtain in the limit that

$$f(x, y) + \frac{1}{\rho}[h(y) - h(x) - (\nabla h(x), y - x)]$$

$$\geq f(x, y) + \frac{1}{\rho}[h(y) - h(x) - (\nabla h(x), y - x)], \forall y \in C.$$
\[
\langle \tilde{w}, \tilde{x} - \tilde{y} \rangle \leq \frac{1}{\rho} \left[ h(\tilde{y}) - h(\tilde{x}) - \langle \nabla h(\tilde{x}), \tilde{y} - \tilde{x} \rangle \right].
\]

Note that \( \tilde{w} \in \partial f(\tilde{x}, \tilde{x}) \); it follows from the last inequality that

\[
f(\tilde{x}, \tilde{y}) + \frac{1}{\rho} \left[ h(\tilde{y}) - h(\tilde{x}) - \langle \nabla h(\tilde{x}), \tilde{y} - \tilde{x} \rangle \right] \geq 0.
\]

Hence,

\[
f(\tilde{x}, y) + \frac{1}{\rho} \left[ h(y) - h(\tilde{x}) - \langle \nabla h(\tilde{x}), y - \tilde{x} \rangle \right] \geq 0, \quad \forall y \in C,
\]

which shows that \( \tilde{x} \in S_f \).

In addition, \( \|v^k - \tilde{x}\| \leq \|u^k - \tilde{x}\| + \lambda_k \|G(u^k)\| \) combining with (11), it implies

\[
\|v^k - \tilde{x}\| \leq \|x^k - \tilde{x}\| + \lambda_k \|G(u^k)\| \tag{20}
\]

taking the limit both sides of (20), we get \( \lim_{k \to \infty} v^k = \tilde{x} \). Hence, \( \tilde{v} = \tilde{x} \). Therefore, \( \tilde{v} \in S \). \( \square \)

Now, we are in a position to prove the convergence of the proposed algorithm.

**Theorem 3.4** Suppose that the set \( S = S_f \cap \text{Fix}(T) \) is nonempty and that the function \( h(\cdot, \cdot) \), the sequence \( \{\lambda_k\}, \{\mu_k\} \) satisfies the conditions (B1), (B2), and (B3), respectively. Then, under Assumptions (A1), (A2), (A3), (A4), and (A5), the sequence \( \{x^k\} \) generated by Algorithm 1 converges to the unique solution \( x^* \) of \( \text{VIEFP}(C, f, T, G) \).

**Proof** By Lemma 3.1, we have

\[
\|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2 + \|x^{k+1} - v^k\|^2 + \left( \frac{n_k \delta}{\rho \|w^k\|} \right)^2 \|x^k - y^k\|^4 \geq -2\lambda_k \langle u^k - x^*, G(u^k) \rangle + \lambda_k^2 \|G(u^k)\|^2 \forall k.
\]

(21)

From the boundedness of \( \{u^k\} \) and \( \{G(u^k)\} \), it implies that there exist positive numbers \( A, B \), such that

\[
\|u^k - x^*, G(u^k)\| \leq A, \quad \|G(u^k)\| \leq B \forall k.
\]

By setting \( a_k = \|x^k - x^*\|^2 \), and combining with the last inequalities, (21) becomes

\[
a_{k+1} - a_k + \|x^{k+1} - v^k\|^2 + \left( \frac{n_k \delta}{\rho \|w^k\|} \right)^2 \|x^k - y^k\|^4 \leq 2\lambda_k A + \lambda_k^2 B.
\]

(22)

We will consider two distinct cases:

**Case 1.** There exists \( k_0 \) such that \( \{a_k\} \) is decreasing when \( k \geq k_0 \).

Then, there exists \( \lim_{k \to \infty} a_k = a \), taking the limit on both sides of (22), we get

\[
\lim_{k \to \infty} \|x^{k+1} - v^k\|^2 = 0, \quad \text{and} \quad \lim_{k \to \infty} \left( \frac{n_k \delta}{\rho \|w^k\|} \right)^2 \|x^k - y^k\|^4 = 0.
\]

(23)

This implies \( \lim_{k \to \infty} \|v^k - x^*\| = a \). In addition,
\[
\|v^h - u^f\| = \|P_c(u^f - \lambda G(u^f)) - P_c(u^f)\| \\
\leq \|u^f - \lambda G(u^f) - u^f\| \\
= \lambda \|G(u^f)\| \to 0 \text{ as } \lambda \to \infty.
\]

(24)

Thus, \(\lim_{k \to \infty} \|u^k - x^\ast\| = a\). From the boundedness of \(\{u^k\}\), it implies that there exists \(\{u^k\} \subset \{u^f\}\) and \(u^k \to \tilde{u} \in C\), such that \(\lim \inf (u^k - x^\ast, G(x^\ast)) = \lim_{k \to \infty} (u^k - x^\ast, G(x^\ast))\).

Combining this fact with (23) and (24), we obtain

\[v^h \to \tilde{u}; x^{h+1} \to \tilde{u}, \text{ and } \left(\frac{n_{k+1} \delta}{\rho \|w^{h+1}\|}\right)^2 \|x^{h+1} - x^h\|^4 \to 0 \text{ as } i \to \infty.\]

By Lemma 3.3, we get \(\tilde{u} \in S\). Thus,

\[\lim \inf_{k \to \infty} (u^k - x^\ast, G(x^\ast)) = \lim_{k \to \infty} (u^k - x^\ast, G(x^\ast)) = \langle a - x^\ast, G(x^\ast)\rangle \geq 0.\]

Since \(G\) is \(\rho\)-strongly monotone, one has

\[\langle u^k - x^\ast, G(u^k)\rangle = \langle u^k - x^\ast, G(u^k) - G(x^\ast)\rangle + \langle u^k - x^\ast, G(u^k)\rangle \geq \rho \|u^k - x^\ast\|^2 + \langle u^k - x^\ast, G(u^k)\rangle.\]

Taking the limit as \(k \to \infty\) and remembering that \(a = \lim \|u^k - x^\ast\|^2\), we get

\[\lim \inf_{k \to \infty} (u^k - x^\ast, G(u^k)) \geq \beta a.\]

(25)

If \(a > 0\), then by choosing \(\epsilon = \frac{1}{2} \beta a\), from (25), it implies that there exists \(k_1 > 0\), such that

\[\langle u^k - x^\ast, G(u^k)\rangle \geq \frac{1}{2} \beta a, \quad \forall k \geq k_1.\]

From (21), we get

\[a_{k+1} - a_k \leq -\lambda \beta a + \lambda_j^2 B, \quad \forall k \geq k_1\]

and thus summing up from \(k_1\) to \(k\), we have

\[a_{k+1} - a_k \leq -\sum_{j=k_1}^{k} \lambda \beta a + B \sum_{j=k_1}^{k} \lambda_j^2\]

combining this fact with \(\sum_{k=1}^{\infty} \lambda_k = \infty\) and \(\sum_{k=1}^{\infty} \lambda_k^2 < \infty\), we obtain \(\lim \inf a_k = -\infty\), which is a contradiction.

Thus, we must have \(a = 0\), i.e. \(\lim_{k \to \infty} \|x^k - x^\ast\| = 0\).

Case 2. There exists a subsequence \(\{a_k\}_{k=0}^\infty \subset \{a_k\}_{k \geq 0}\) such that \(a_k \leq a_{k+1}\) for all \(i \geq 0\). In this situation, we consider the sequence of indices \(\{\sigma(k)\}\) defined as in Lemma 2.10. It follows that \(a_{\sigma(k+1)} - a_{\sigma(k)} \geq 0\), which by (22) amounts to

\[\|x^{\sigma(k+1)} - v^{\sigma(k)}\|^2 + \left(\frac{n_{\sigma(k)} \delta}{\rho \|w^{\sigma(k)}\|}\right)^2 \|x^{\sigma(k)} - y^{\sigma(k)}\|^4 \leq 2 \lambda \|A + \lambda_j^2 B\|^2.\]

Therefore,

\[\lim_{k \to \infty} \|x^{\sigma(k+1)} - v^{\sigma(k)}\|^2 = 0; \lim_{k \to \infty} \left(\frac{n_{\sigma(k)} \delta}{\rho \|w^{\sigma(k)}\|}\right)^2 \|x^{\sigma(k)} - y^{\sigma(k)}\|^4 = 0.\]
From the boundedness of \( \{v^{(k)}\} \), without loss of generality, we may assume that \( v^{(k)} \to \tilde{v} \). By Lemma 3.3, we get \( \tilde{v} \in S \).

In addition, 
\[
\|v^{(k)} - u^{(k)}\| = \|P_C(u^{(k)} - \lambda_{n(k)}G(u^{(k)})) - P_C(u^{(k)})\| \\
\leq \lambda_{n(k)}\|G(u^{(k)})\| \to 0 \text{ as } k \to \infty.
\]
Therefore, \( \lim_{k \to \infty} u^{(k)} = \tilde{v} \).

By (21), we get 
\[
2\lambda_{n(k)}\langle u^{(k)} - x^*, G(u^{(k)}) \rangle \leq a_{n(k)} - a_{n(k + 1)} - \left( \frac{n_{n(k)} \delta}{\rho \|w^{(k)}\|} \right)^2 \|x^{(k)} - y^{(k)}\|^2 \\
+ \lambda_{n(k)}^2 \|G(u^{(k)})\|^2 \leq \lambda_{n(k)}^2 B
\]
which implies 
\[
\langle u^{(k)} - x^*, G(u^{(k)}) \rangle \leq \frac{\lambda_{n(k)}}{2} B.
\] (26)

Since \( G \) is \( \rho \)-strongly monotone, we have 
\[
\rho \|u^{(k)} - x^*\|^2 \leq \langle u^{(k)} - x^*, G(u^{(k)}) - G(x^*) \rangle \\
= \langle u^{(k)} - x^*, G(u^{(k)}) - u^{(k)} - x^*, G(x^*) \rangle
\]
which combining with (26), we get 
\[
\|u^{(k)} - x^*\|^2 \leq \frac{1}{\rho} \left[ \frac{\lambda_{n(k)}^2}{2} B - \langle u^{(k)} - x^*, G(x^*) \rangle \right],
\]
so that 
\[
\lim_{k \to \infty} \|u^{(k)} - x^*\|^2 \leq -\langle u^{(k)} - x^*, G(x^*) \rangle \leq 0
\]
which amounts to 
\[
\lim_{k \to \infty} \|u^{(k)} - x^*\| = 0.
\] (27)

In addition, 
\[
\|x^{(k+1)} - u^{(k)}\| = \|P_C(u^{(k)} - \lambda_{n(k)}G(u^{(k)})) - P_C(u^{(k)})\| \\
\leq \lambda_{n(k)}\|G(u^{(k)})\| \to 0 \text{ as } k \to \infty
\]
which together with (27), one has \( \lim_{k \to \infty} x^{(k+1)} = x^* \), which means that \( \lim_{k \to \infty} a_{n(k+1)} = 0 \).

By (5) in Lemma 2.10, we have 
\[
0 \leq a_k \leq a_{n(k+1)} \to 0 \text{ as } k \to \infty.
\]
Thus, \( \{x^k\} \) converges to \( x^* \) \( \square \)
4. Application to variational inequalities with variational inequality and fixed point constraints

In this section, we consider the following variational inequality problem over the set that is the common elements of the solution set of a pseudomonotone variational inequality problem and the set of fixed points of a demicontractive mapping (shortly VIFP$(C, F, T, G)$):

Find $x^* \in S$ such that $(G(x^*), y - x^*) \geq 0, \forall y \in S,$  \hspace{1cm} (28)

where $S = S_F \cap \text{Fix}(T), S_F = \{ u \in C : \langle F(u), y - u \rangle \geq 0, \forall y \in C \},$

i.e. $S_F$ is the solution set of the following variational inequality problems VIP$(C, F)$ for short

Find $u \in C$ such that $(F(u), y - u) \geq 0, \forall y \in C,$  \hspace{1cm} (29)

and as before, $\text{Fix}(T)$ is the fixed point of the mapping $T$. This problem was considered by Maingé (2008b).

In the sequel, we always suppose that Assumptions (A1), (A2), (A3), (A4), and (A5) are satisfied.

The algorithm for this case takes the form:

Algorithm 2.

Initialization. Pick $x^0 \in C$, choose parameters $\eta \in (0, 1)$, $\rho > 0$ and $\{ \lambda_k \} \subset [0; 1); \{ \mu_k \} \subset [0; 1).$

Iteration $k. (k = 0, 1, 2, \ldots)$ Having $x^k$ do the following steps:

Step 1. $y^k = P_C(x^k - \frac{\rho}{2}F(x^k))$

If $y^k = x^k$, take $u^k = x^k$ and go to Step 4. Otherwise, do Step 2.

Step 2. (Armijo linesearch rule) Find $m_k$ as the smallest positive integer number $m$ satisfying

$\left\{ \begin{array}{l}
z^{k,m} = (1 - \eta^m)x^k + \eta^my^k, \\
\langle F(z^{k,m}), x^k - y^k \rangle \geq \frac{1}{\rho}||y^k - x^k||^2. \\
\end{array} \right.$ \hspace{1cm} (30)

Step 3. Set $\eta_k := \eta^{m_k}, z^k := z^{k,m_k}$. Take $C_k := \{ x \in C : \langle F(x^k), x - z^k \rangle \leq 0 \}, u^k := P_{C_k}(x^k).$ \hspace{1cm} (31)

Step 4. Compute $v^k = P_C(u^k - \lambda_kG(u^k)).$

Step 5. Set $x^{k+1} = (1 - \mu_k)v^k + \mu_kT(v^k)$, and go to Step 1 with $k$ is replaced by $k + 1$.

Similar to Theorem 3.1, we have the following theorem

**Theorem 4.1** Under assumptions (A1), (A2), (A3), (A4), and (A5) and (B1), (B2), and (B3), the sequence \{ $x^k$ \} generated by Algorithm 2 converges to the unique solution $x^*$ of VIFP$(C, F, T, G)$.

5. Conclusion

We have proposed a hybrid extragradient-viscosity algorithm for solving strongly monotone variational inequality problems over the set that is common points of the solution sets of a pseudomonotone equilibrium problem and the set of fixed points of a demicontractive mapping. The convergence of the proposed algorithm is obtained, and a special case of this problem is also considered.

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