BPS States of Strings in 3-Form Flux

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Abstract

We count the BPS states of strings in uniform 3-form fluxes, using supersymmetric quantum mechanics derived from the \( \kappa \)-symmetric action for D-branes. This problem is relevant to the stringy physics of warped compactifications. We work on a type IIB \( T^6/\mathbb{Z}_2 \) orientifold with imaginary self-dual, quantized, 3-form flux. Ignoring the orientifold projection, the number of short multiplets living on a single string is the square of the units of 3-form flux present on the torus; the orientifold removes roughly half of the multiplets. We review the well-known case of a superparticle on \( T^2 \) as a pedagogical example.

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I. INTRODUCTION

In recent years, following the realization of [1, 2] that “warped product” compactifications can lead to a solution of the hierarchy problem, string and M theory compactifications with nonvanishing fluxes on the internal manifold [3] have garnered much interest [4, 5, 6, 7] (see therein for related references). Of particular interest is the type IIB version of [3] (discussed from the point of view of supersymmetry in [8, 9]), which has been applied to Calabi-Yau compactifications with orientifolds [10, 11]. In these compactifications, the 3-form $G = F - \tau H$ (with $\tau = C + i/g_s$) must be imaginary self-dual on the internal manifold, $\ast_6 G = iG$.

The simplest examples of the warped Calabi-Yau compactifications are the $T^6/Z_2$ form studied in [12, 13, 14], where, with different 3-form backgrounds, it is possible to have any degree of supersymmetry up to $\mathcal{N} = 4$ ($\mathcal{N} = 4$ corresponds to no 3-form flux). For smaller degrees of SUSY, the low energy physics is described by the superHiggs effect in supergravity [14, 15, 16, 17, 18, 19], so the SUSY breaking is necessarily accompanied by gauge symmetry breaking.

Frey and Polchinski [14] studied the U-dualities of the $\mathcal{N} = 3$ string theory compactifications and found a dilemma related to the superHiggs effect, as follows. D5- and NS5-branes wrapped on the torus carry magnetic charges for the broken vectors and are confined; the minimal unbroken magnetic charges are therefore bound states of the two types of 5-branes. On the other hand, the minimal electric charges are individual D- and F-strings, so there appear to be twice as many electric BPS multiplets as magnetic for each unbroken symmetry. In such a case, it would appear difficult to have the usual electric-magnetic duality transformation for an effective 4D gauge theory. One suggested resolution is that the combined number of BPS states of the two strings is equal to the number of BPS states of the 5-brane bound state.

In this paper, we carry out the first steps needed to compare the number of electric and magnetic BPS states by carrying out the ground state quantization of a D-string in 3-form flux. To that end, we find the action for the D-string in background fluxes from the known supersymmetric and $\kappa$-symmetric Born-Infeld and Wess Zumino actions [20, 21], expanded to second order in the bosonic and fermionic world-volume coordinates. For simplicity, we will focus on the $\mathcal{N} = 3$ background discussed in [14], but the techniques we use are readily
generalizable to the other models described in [10, 13]. Also, because of the RR flux and the finite fixed value of the string coupling, little is known about such compactifications beyond the low energy supergravity. This calculation sheds a little light on the stringy additions to the spectrum.

The paper is organized as follows. In section II, we review the simple example of a superparticle on a $T^2/Z_2$ orbifold, and in section III we describe our supergravity background and find the action and Hamiltonian of a D-string in that background. We then find the supercharges of the quantum mechanics on the string in section IV, followed by counting the BPS ground states in section V. Finally, we comment on states isolated at singularities, bound states, and the number of BPS states of an F-string in section VI, and section VII gives a concluding discussion. Some conventions are given in appendix A, and the size of the corrections to our model are shown in appendix B.

II. EXAMPLE OF PARTICLE ON $T^2$

The counting of the spectrum of BPS states for the D-string on the $T^6/Z_2$ orientifold with constant fluxes will turn out to be very similar to the case of spin 1/2 particle on a torus subject to a constant perpendicular magnetic field, so in this section we review the latter, simpler, case (cf. [22] and references therein). We work in the context of supersymmetric quantum mechanics and calculate a Witten index [23].

The supersymmetric Lagrangian for a spin 1/2 particle with mass $m$ and charge $e$ is

$$\mathcal{L} = \frac{m}{2} \dot{x}^2 + eA \cdot \dot{x} + \frac{i}{2} \psi_1 \dot{\psi}_1 + \frac{i}{2} \psi_2 \dot{\psi}_2 - \frac{ie}{m} \psi_1 \psi_2 B$$

(1)

where $\psi_1$ and $\psi_2$ are two real Grassman variables, obeying

$$\{\psi_i, \psi_j\} = \delta_{ij}.$$  

(2)

We should note here that we are using conventions such that $(\psi \phi) = \phi^\dagger \psi^\dagger$, leading to the real anticommutator [2]. The Lagrangian [II] is invariant under the supersymmetry transformation

$$\delta x^i = i \psi_i \epsilon \quad \delta \psi_i = -m \dot{x}^i \epsilon$$

(3)

1 The anticommutator follows from the theory of systems with constraints, as detailed in [24]; this is important for the D-string, since it gives constants in the anticommutator. Note especially that the Dirac quantization does not give a factor of 1/2 in the anticommutator.
up to a total derivative, as can be verified by a simple calculation.

We find it convenient to work in the operator formalism, with Hamiltonian

\[ H = \frac{1}{2m} \left( \vec{p} - e\vec{A} \right)^2 + i \frac{e}{m} \psi_1 \psi_2 B \]  

(4)

Then the supersymmetry (3) is generated by the supercharge operator

\[ Q = i \left( \vec{p} - e\vec{A} \right) \cdot \psi \]  

(5)

satisfying \( Q^2 = mH \).

In terms of the complex combinations

\[ z = \frac{1}{2}(x^1 + ix^2), \quad \psi_\pm = \frac{1}{2}(\psi_1 \pm i\psi_2), \]  

(6)

the supercharge (3) can be rewritten as

\[ Q = i(p - eA)_z \psi_+ + i(p - eA)_\bar{z} \psi_- . \]  

(7)

Supersymmetric ground states should be annihilated by the supercharge. If we start with a state \( |-\rangle \) that is annihilated by \( \psi_- \), then, from (7), supersymmetric wave functions obey

\[ (p - eA)_z \phi_- = 0 . \]  

(8)

Let’s use a gauge for the potential in which

\[ A_1 = -\frac{1}{2}Bx^2, \quad A_2 = \frac{1}{2}Bx^1 \Rightarrow A_z = -iB\bar{z} . \]  

(9)

This is periodic up to a gauge transformation,

\[ A_i(x^1 + 2\pi R, x^2) = A_i(x^1, x^2) + \partial_i \lambda_1, \quad A_i(x^1, x^2 + 2\pi R) = A_i(x^1, x^2) + \partial_i \lambda_2 \]  

(10)

with \( \lambda_1 = B\pi R x^2 \) and \( \lambda_2 = -B\pi R x^1 \). When going around the torus, the wave function picks up a phase determined by these gauge transformations,

\[ \phi(x^1 + 2\pi R, x^2) = e^{i\lambda_1} \phi(x^1, x^2), \quad \phi(x^1, x^2 + 2\pi R) = e^{i\lambda_2} \phi(x^1, x^2) . \]  

(11)

Single-valuedness of the wavefunction implies that the magnetic field is quantized,

\[ B = \frac{n}{2\pi eR^2} \]  

(12)
where \( n \in \mathbb{Z} \), and \( R \) is the radius of the square torus. If there is a \( \mathbb{Z}_2 \) orbifold, any flux of the form (12) is still allowed, but some of the fixed points will carry flux, also. These special fixed points will not affect our discussion below, since the translations and reflections automatically give the wavefunction the correct boundary conditions at those fixed points. This is discussed in detail in [14].

From (8), the wave function should satisfy

\[
\frac{\partial \phi_-}{\partial z} = eB\bar{z}\phi_-
\]

whose solution is

\[
\phi_- = e^{eB|z|^2}F(\bar{z}) .
\]

The periodicity conditions (11), written in complex coordinates, are

\[
\phi(z + \pi R, \bar{z} + \pi R) = e^{eB\pi(z-\bar{z})}\phi(z, \bar{z}), \quad \phi(\bar{z} + i\pi R, \bar{z} - i\pi R) = e^{-ieB\pi(z+\bar{z})}\phi(z, \bar{z}) .
\]

Inserting (14), we get from the first condition in (15)

\[
F(\bar{z} + \pi R) = e^{-eB2\pi R\bar{z}-eB\pi^2 R^2}F(\bar{z}) = e^{-\frac{n}{R^2}-\frac{m}{R^2}}F(\bar{z}) ,
\]

where in the last equality we used the quantization condition (12). Defining

\[
F(\bar{z}) = e^{-\frac{n}{\pi R^2}z^2}G(\bar{z})
\]

the condition (16) implies that \( G(\bar{z}) \) is periodic, with period \( \pi R \). Writing \( G(\bar{z}) \) as a sum of Fourier modes, we get for the wave function

\[
\phi_- = e^{\frac{n}{2\pi R^2}|z|^2-\frac{n}{2\pi R^2}z^2} \sum_{-\infty}^{\infty} C_m e^{\frac{2\pi i}{R}m\bar{z}} .
\]

The second periodicity condition in (15) implies the recursion relation

\[
C_{m+n} = C_m e^{\pi(n+2m)} .
\]

So only \( n \) of the coefficients \( C_m \) are free. With this recursion relation, the series in (18) converges if \( n < 0 \). So, for a magnetic field in the negative 3 direction, there are \( n \) ground states.

If instead we had started with a ground state annihilated by \( \psi_+ \), then the wave function would have been

\[
\phi_+ = e^{-\frac{n}{2\pi R^2}|z|^2+\frac{n}{2\pi R^2}z^2} \sum_{-\infty}^{\infty} C_m e^{\frac{2\pi i}{R}mz} .
\]
and the constraint on the components $C_m$ turns out to be

$$C_{m+n} = C_m e^{-\pi(n+2m)},$$

so in this case the sum in (20) converges for positive $n$.

Let’s take a minute to note the relation of the wavefunctions to the well-known theta functions on the torus (see chapter 7 of [25] for a review). For example, with $n > 0$, the recursion relation (21) has solution $C_m = De^{-\pi m^2/n}$ with constant $D$, so the wavefunction (20) is a Gaussian times the theta function

$$\phi_+ = \sum_{k=0}^{n-1} D_k \exp \left[ -\frac{n}{2\pi R^2} |z|^2 + \frac{n}{2\pi R^2} z^2 \right] \vartheta \left[ \begin{array}{c} k/n \\ 0 \end{array} \right] \left( \frac{n}{\pi R}, in \right).$$

(We use the notation of [25].) The sum now has $n$ independent coefficients $D_k$.

A spin $1/2$ particle in a constant perpendicular magnetic field on a torus has then a finite number of ground states, given by the number of units of magnetic flux. On a $\mathbb{Z}_2$ orbifold, we must have $z \simeq -z$, or $C_m = C_{-m}$, which force relationships between the $D_k$ coefficients. Thus the number of supersymmetric ground states becomes $(n+1)/2$ for $n$ odd and $n/2 + 1$ for $n$ even. If we think of the states $|\pm\rangle$ as respectively bosonic and fermionic, the number of ground states is therefore the Witten index $\text{tr}(-1)^F$.

### III. D-STRING QUANTUM MECHANICS

In this section, we find the Lagrangian and Hamiltonian for a D-string extended along one direction of a $T^6/\mathbb{Z}_2$ orientifold in imaginary self-dual 3-form flux, including the fermionic degrees of freedom.

#### A. Supergravity background

We start by describing the supergravity background, which was given in [13, 14]. In this section, we use Greek indices for spacetime coordinates and Latin for torus coordinates. For

2 It seems naively that the $T^2/\mathbb{Z}_2$ orbifold can be smoothly deformed to $S^2$, but in that case an even number of flux units would give $n/2$ states. Apparently the orbifold limit of the sphere is singular; we leave the resolution of this problem as an exercise for the reader.
a general amount of SUSY up to $N = 4$ ($N = 4$ corresponding to $G_3 = 0$), the fields have the form

\begin{align}
&ds^2 = Z^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + Z^{1/2} ds_6^2, \\
&\tau = C + i g_s = \text{constant} \\
&\tilde{F}_5 = (1 + *) d\chi_4, \quad \chi_4 = \frac{1}{Z g_s} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&*_6 G_3 = i G_3 \\
&-\hat{\nabla}^2 Z = (2\pi)^4 \alpha'^2 g_s \rho_3 + \frac{g_s}{2} |\hat{G}|^2.
\end{align}

Here, $ds^2 = g_{mn} dy^m dy^n$ ($m, n = 4, \ldots 9$) is the unwarped metric on the torus, which we describe with the complex pairs $y^m = (x^m + t^{mn} x^n)/2$, with here $m = 4, 5, 6$, $n = 7, 8, 9$. The Kähler moduli $g_{mn}$ are free, so we choose them to give the unit metric for simplicity. The complex structure is given by the period matrix $i [13]$, which will be fixed by the 3-forms; we will consider a case with $t^{mn} = i \delta^{m+3,n}$. We will find it convenient to have the proper radii implicit in the coordinates: $x^{4,7} \simeq x^{4,7} + 2\pi R_1$, $x^{5,8} \simeq x^{5,8} + 2\pi R_2$, $x^{6,9} \simeq x^{6,9} + 2\pi R_3$. Additionally, the dilaton-axion is fixed by the fluxes, and we will study cases with vanishing RR scalar. We should note also that (27) implies that $Z = 1 + \mathcal{O}(\alpha'^2/R^4)$ in the large radius limit.

As in [14], these values for the fixed moduli are consistent with constant 3-forms

\begin{align}
H_{mnpp} &= \frac{\alpha'}{2\pi R_1 R_2 R_3} h_{mnpp}, \quad F_{mnpp} = \frac{\alpha'}{2\pi R_1 R_2 R_3} f_{mnpp}, \quad h_{mnpp}, f_{mnpp} \in \mathbb{Z}
\end{align}

given by

\begin{align}
&h_1 = h_{456} = -h_{489} = -h_{759} = -h_{786}, \\
&h_2 = h_{789} = -h_{756} = -h_{486} = -h_{459}, \\
&f_1 = f_{456} = -f_{489} = -f_{759} = -f_{786}, \\
&f_2 = f_{789} = -f_{756} = -f_{486} = -f_{459}.
\end{align}

As in the case of the superparticle, these integers can be odd, as long as some of the orientifold fixed planes carry flux. These will not affect the quantization of the string [14]. With vanishing RR scalar, imaginary self-duality gives

\begin{align}
f_2 &= \frac{h_1}{g_s}, \quad f_1 = -\frac{h_2}{g_s}.
\end{align}
This background has 4D $\mathcal{N} = 3$ supersymmetry. In this background, a D-string and orthogonal F-string carry the same central charge (for example, a D-string wrapped on $x^4$ and an F-string on $x^7$) and fall in short multiplets of the $\mathcal{N} = 3$ theory. These preserve 4 of the 12 supercharges preserved by the background and have 16 states; we will be counting the number of multiplets associated with each string. We should caution the reader that it is impossible to be certain that these BPS states are truly D- or F-strings, since we are working at $g_s \sim 1$, but we see in this paper that the physics of D-strings appears to make sense even at this strong coupling.

### B. Action and Hamiltonian

Here we find the supersymmetric action for a D-string wrapped (without loss of generality) on $x^4$ in the self-dual 3-form flux described in section III A. Because the number of BPS states is stable under small perturbations (in the 4D $\mathcal{N} = 3$ case we consider, for example, a long multiplet is 4 times the size of the BPS multiplet), we will work in the large radius limit of the compactification and ignore the warp factor. Also, because motion along the string is quantized by the compactification, we will set those derivatives to zero. Finally, our D-string will not intersect any D3-branes, so we ignore states associated with 1-3 strings. We will estimate the corrections to our simple model in appendix B. Because we are studying only one type of electric charge in the 4D theory, the D-string will not be wound in any other direction or carry any dissolved F-strings.

The supersymmetric action for a D1-brane in background fluxes was worked out in [20, 21]. It looks like the bosonic Dirac-Born-Infeld and Wess-Zumino actions, but the spacetime fields live in superspace.

$$S = -\frac{1}{2\pi\alpha'} \int d^2\zeta e^{-\Phi} \sqrt{-\det (g_{\mu\nu} + \mathcal{F}_{\mu\nu})} + \frac{1}{2\pi\alpha'} \int e^{\mathcal{F}} \wedge C .$$  \hspace{1cm} (31)

The fields in boldface are superfields; as usual,

$$\mathcal{F} = 2\pi\alpha' F - B, \text{ and } C = \oplus_n C_{(n)}$$  \hspace{1cm} (32)

is the collection of all RR potentials pulled back to the world-volume. The expansions of the superfields in terms of components fields was developed in [26], [27], and [28] for 11-dimensional, IIA, and IIB supergravities respectively, using a method known as gauge
completion. The expansions of the fields that we will need, as well as our conventions, are listed in appendix A.

Without getting into algebraic details, the action (31) for a D-string in our background is
\[
S = -\frac{1}{2\pi\alpha'g_s} \int d^2\zeta \left[ (1 - F_{04}^2)^{1/2} + \frac{1}{2} (1 - F_{04}^2)^{-1/2} (\dot{x}^m)^2 - g_s C_{m4}\dot{x}^m + i\frac{g_{1/2}^s}{2} \Theta \dot{\Theta} + i\frac{g_{3/2}^s}{16} \overline{\Theta} \Gamma_0 \Theta F_{4mn} + i\frac{g_{1/2}^s}{48} \overline{\Theta} \Gamma^{mnp} \Theta H_{mnp} - i\frac{g_{1/2}^s}{16} \overline{\Theta} \Gamma^{4mn} \Theta H_{4mn} \right]
\] (33)

where we work to second order in the world-volume coordinates and fermions.\(^3\) As noted above, there are corrections to this action, arising from the expansion of the D-brane action; we consider these in appendix B.

As discussed in appendix A, the fermion Θ is a 10D (Majorana-Weyl) superspace coordinate; let us now expand it in terms of 2D spinors. We do so by noting that the 10-dimensional gamma matrices can be decomposed into \(SO(1,1) \otimes SO(8)\) pieces as
\[
\Gamma^\parallel = \gamma^\parallel \otimes 1, \quad \Gamma^\perp = \gamma^{(2)} \otimes \gamma^\perp,
\] (34)

where “\(\parallel\)” and “\(\perp\)” mean along the D-string and perpendicular to it; \(\Gamma\) is a 32x32 Dirac matrix, \(\gamma^\parallel\) and \(\gamma^\perp\) are its 2x2 and 16x16 blocks, and \(\gamma^{(2)}\) is the chirality matrix in \(SO(1,1)\). Therefore, a Majorana-Weyl spinor \(\Theta\) can be decomposed into
\[
\Theta = |−\rangle \otimes \psi^\alpha u_\alpha \oplus |+\rangle \otimes \phi^\dot{\alpha} v^{\dot{\alpha}},
\] (35)

where \(|+\rangle\) and \(|−\rangle\) are the eigenfunctions of \(\gamma^{(2)}\), and \(\alpha\) and \(\dot{\alpha}\) are indices in the 8 and \(8'\) representations of \(SO(8)\). The \(\psi\)s and \(\phi\)s are 2D Majorana-Weyl fermions. The spin raising (lowering) gamma matrices are the (anti)holomorphic gamma matrices defined with respect to the coordinates
\[
\begin{align*}
z^{0\pm} &= \frac{1}{2}(\pm x^0 + x^4) \ , \quad z^1 = \frac{1}{2}(x^1 + ix^2) \ , \quad z^2 = \frac{1}{2}(x^3 + ix^7) \ , \\
z^3 &= \frac{1}{2}(x^5 + ix^8) \ , \quad z^4 = \frac{1}{2}(x^6 + ix^9) .
\end{align*}
\] (36)

We will label the complex coordinates 1,2,3,4 with indices \(i,j,\ldots\).

\(^3\) \(F_{4mn}\) in the second fermionic term would be \(F'_{4mn} = F_{4mn} - CH_{4mn}\), and also the Chern-Simons term would couple the velocity \(\dot{x}^m\) to \(C_{m4} + CF_{m4}\), if the RR scalar is nonvanishing.
Using the basis (A2) for the Majorana-Weyl spinors and integrating along the string, the fermionic Lagrangian from eq (33) can be written

\[ L_f = -i \frac{R_1}{2 \alpha' g_s^{1/2}} \left[ \psi^\alpha \dot{\psi}^\alpha + \phi^{\dot{\alpha}} \dot{\phi}^{\dot{\alpha}} \right] \]

\[ + \frac{\alpha'}{2 \pi R_1 R_2 R_3} h_1 \left( \psi^1 \psi^4 + \psi^2 \psi^3 + \phi^{\dot{1}} \phi^{\dot{1}} + \phi^{\dot{2}} \phi^{\dot{2}} + \psi^1 \phi^{\dot{3}} + \phi^{\dot{1}} \psi^3 + \psi^4 \phi^{\dot{2}} \right) \]

\[ + \frac{\alpha'}{2 \pi R_1 R_2 R_3} h_2 \left( \psi^3 \psi^4 + \psi^2 \psi^3 + \phi^{\dot{2}} \phi^{\dot{2}} + \psi^1 \phi^{\dot{3}} + \psi^2 \phi^{\dot{3}} + \phi^{\dot{2}} \psi^3 + \phi^{\dot{1}} \psi^4 \right) \]

(37)

As well as some algebraic simplification, arriving at this result requires 3-form self-duality to relate the NSNS and RR fluxes as in (30). We should note that the fermions \( \psi^\alpha \) and \( \phi^{\dot{\alpha}} \) with \( \alpha, \dot{\alpha} = 5, 6, 7, 8 \) enter only through their kinetic terms.

Now to convert to the Hamiltonian formalism, we start by finding the canonical momentum for the world-volume gauge field \( F = dA \). Following the discussion in [23], the Wilson lines are periodic variables, so the momentum

\[ p_A = \frac{2 \pi R_1}{g_s} \frac{F_{04}}{[1 - F_{04}^2]^{1/2}} \]

(38)

is quantized in units of \( 2 \pi R_1 \). Further, it is this canonical momentum (up to constants) which couples the D-string to \( B \), so \( p_A \) and therefore \( \mathcal{F} \), not the gauge field strength \( F \), vanish for a D-string with no F-string charge. This issue is somewhat more complicated in the presence of the RR scalar \( C \), but we have now removed the NSNS flux from the problem.

The canonical momenta for the collective coordinates now simplify to

\[ p_m = \frac{R_1}{\alpha' g_s} \dot{x}_m + \frac{R_1}{\alpha'} C_{m4} , \]

(39)

as in the usual quantum mechanics with a gauge field proportional to \( C_{m4} \). Thus, the total Hamiltonian is a constant mass \( m = R_1/\alpha' g_s \) and a dynamical Hamiltonian

\[ H = \frac{1}{2m} (\vec{p} - \vec{A})^2 + i \frac{C}{2} h_1 \left( \psi^1 \psi^4 + \psi^2 \psi^3 + \phi^{\dot{1}} \phi^{\dot{1}} + \phi^{\dot{2}} \phi^{\dot{2}} + \psi^1 \phi^{\dot{3}} + \phi^{\dot{1}} \psi^3 + \psi^4 \phi^{\dot{2}} \right) \]

\[ + i \frac{C}{2} h_2 \left( \psi^3 \psi^4 + \psi^2 \psi^3 + \phi^{\dot{2}} \phi^{\dot{2}} + \psi^1 \phi^{\dot{3}} + \psi^2 \phi^{\dot{3}} + \phi^{\dot{2}} \psi^3 + \phi^{\dot{1}} \psi^4 \right) \]

(40)

with \( C = (2 \pi R_2 R_3 g_s^{1/2})^{-1} \). The gauge field is defined as \( A_m = (R_1/\alpha') C_{m4} \).

### IV. SUPERCHARGES

In this section, we demonstrate that the Hamiltonian (40) is, in fact, supersymmetric, and we identify the 4 supercharges that belong to the unbroken supersymmetries of the
$\mathcal{N} = 3$ 4D effective theory. We will proceed by first finding the spacetime supersymmetries that leave both the background (see section III A) and the BPS states of the string invariant by starting with the 10D theory. We will then relate those to world-volume supercharges, which should be of the form given in eq. (3), $Q \sim i(p - A)\psi$. Finally, we will check that these do actually commute with the Hamiltonian.

In the IIB string theory, the supercharges are Majorana-Weyl spinors with positive chirality, one coming from each side of the string, and the superalgebra contains NSNS and RR charges as tensorial central charges (see [29]). In this notation, the supercharges preserved by a BPS state are given by $\bar{\epsilon}_1 Q + \bar{\epsilon}_2 \tilde{Q}$ ($\epsilon_{1,2}$ have negative chirality), where

$$
\begin{pmatrix}
\bar{Q} \\
\tilde{Q}
\end{pmatrix} = 0.
$$

(41)

In contrast, the supercharges of supergravity backgrounds (as in [8, 4]) are denoted following the conventions of [30, 31]. In this form, the supersymmetry parameters are given by a single negative chirality Weyl spinor, so that the preserved supercharges are given by $\bar{\epsilon} Q + \bar{\epsilon}^* Q^*$. To relate these two formalisms, we consider the supersymmetries preserved by a D3-brane in flat spacetime. In the form of (11), it is easy to see that $\epsilon_2 = i\gamma(4)\epsilon_1$, whereas the supergravity formalism gives $\epsilon = \gamma(4)\epsilon$ (see, for example, [8, 4]), where $\gamma(4)$ is the chirality along the D3-brane. The conventions agree if we take $\epsilon = \epsilon_1 - i\epsilon_2$ and $Q = (Q + i\tilde{Q})/2$.

The supersymmetries preserved by our background (section III A) are expressed conveniently by $SO(3,1) \otimes SO(6)$ decomposition $\epsilon^A = \eta \otimes \chi^A$, where $\chi^A$ are 3 of the 4 negative chirality spinors in 6D (those that don’t have the three spins parallel) [14]. We can re-write these in the $SO(1,1) \otimes SO(8)$ basis (12) as

$$
\begin{align*}
\epsilon^1 &= \epsilon_1^1 \left[ \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle - \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (u_1 - iu_2) - \left\langle \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle + \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (v_1 - iv_2) + \epsilon_3^1 \left[ \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle - \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (u_3 - iu_4) + \left\langle \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle + \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (v_3 - iv_4), \\
\epsilon^2 &= \epsilon_5^2 \left[ \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle - \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (u_5 - iu_6) + \left\langle \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle + \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (v_5 - iv_6) + \epsilon_7^2 \left[ \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle - \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (u_7 + iu_8) + \left\langle \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle + \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (v_7 + iv_8), \\
\epsilon^3 &= \epsilon_9^3 \left[ \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle - \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (u_5 + iu_6) + \left\langle \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle + \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (v_5 + iv_6) + \epsilon_7^2 \left[ \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle - \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (u_7 - iu_8) + \left\langle \begin{array}{c} \text{\scriptsize}\begin{array}{c} \left\langle \left\langle + \right\rangle \right\rangle \end{array} \end{array} \right] \otimes (v_7 - iv_8)
\end{align*}
$$

(42)

with complex Grassman numbers $\epsilon^A_\alpha$.

Now, let’s intersect these with the supersymmetries preserved by the D-string. Using the superalgebra with central charges, it is easy to find that a D-string wrapped on $x^4$ has supersymmetries $\epsilon_1 = \gamma(2)\epsilon_2$ [29]. Then, using $\epsilon = \epsilon_1 - i\epsilon_2$, we have $\gamma(2)\epsilon = -i\epsilon^*$, so the coefficients of the $u$ spinors must satisfy $\epsilon = ie^*$ and the coefficients of $v$ spinors satisfy
\[ \epsilon = -i\epsilon^*. \] We can obtain spinors that satisfy these constraints by taking linear combinations of the latter two spinors (12) such that \( \epsilon_5^2 = \pm \epsilon_2^3 \) and \( \epsilon_2^3 = \pm \epsilon_7^3 \). In the end, we find 4 different one component spinors \( \epsilon^A \) with

\[
\begin{align*}
\epsilon^1 &= \epsilon^1 (|\rangle \otimes u_5 + i|\rangle \otimes v_6) \ , \\
\epsilon^2 &= \epsilon^2 (|\rangle \otimes u_6 - i|\rangle \otimes v_6) \ , \\
\epsilon^3 &= \epsilon^1 (|\rangle \otimes u_7 + i|\rangle \otimes v_8) \ , \\
\epsilon^4 &= \epsilon^4 (|\rangle \otimes u_8 - i|\rangle \otimes v_7). 
\end{align*}
\]

The coefficients are \( \epsilon^A = \epsilon^A e^{i\pi/4} \) with \( \epsilon^A \) real.

Now we can actually find the worldvolume supercharges. As discussed in [32, 33], the worldvolume supersymmetries are not given simply by their action on the spacetime fields (including \( \Theta \) as a superspace coordinate) because that transformation would in general change the \( \kappa \)-symmetry gauge. The supersymmetry transformations of the worldvolume fields, including a \( \kappa \) transformation to keep the same gauge, were found in [33]. However, we will not follow this approach. Instead, we will take the ansatz

\[
-\frac{1}{\sqrt{2}} Q^A \epsilon^A = i(p - A)_i \bar{\Theta} \Gamma^i \text{Re} (\epsilon^A) + i(p - A)_i \bar{\Theta} \Gamma^i \text{Re} (\epsilon^A)
\]

for the supercharges. We know that the real part of the spinor should be used because the supercharges should be real (as the spacetime ones are), and we follow [32] in using the spacetime supersymmetry parameters. The prefactors are included for convenience.

Using this ansatz, we can write the supercharges in terms of the 2D fermions \( \psi, \phi \) and complex coordinates \( z \) as

\[
\begin{align*}
Q^1 &= i p_1 (-i \psi_7 - \psi_8 + \phi_7 - i \phi_8) + i p_1 (i \psi_7 - \psi_8 + \phi_7 + i \phi_8) \\
&\quad + i p_2 (-i \psi_5 + \psi_6 - \phi_5 + i \phi_6) + i p_2 (i \psi_5 + \psi_6 - \phi_5 - i \phi_6) \\
&\quad + i(p - A)_3 (-i \psi_3 - \psi_4 - \phi_3 + i \phi_4) + i(p - A)_3 (i \psi_3 - \psi_4 - \phi_3 - i \phi_4) \\
&\quad + i(p - A)_4 (i \psi_1 + \psi_2 + \phi_1 - i \phi_2) + i(p - A)_4 (-i \psi_1 + \psi_2 + \phi_1 + i \phi_2), \\
Q^2 &= i p_1 (-\psi_7 + i \psi_8 - i \phi_7 - \phi_8) + i p_1 (-\psi_7 - i \psi_8 + i \phi_7 - \phi_8) \\
&\quad + i p_2 (-\psi_5 - i \psi_6 - i \phi_5 + \phi_6) + i p_2 (\psi_5 + i \psi_6 + i \phi_5 - \phi_6) \\
&\quad + i(p - A)_3 (-\psi_3 + i \psi_4 + i \phi_3 + \phi_4) + i(p - A)_3 (\psi_3 + i \psi_4 - i \phi_3 + \phi_4) \\
&\quad + i(p - A)_4 (-\psi_1 + i \psi_2 + i \phi_1 + \phi_2) + i(p - A)_4 (\psi_1 - i \psi_2 - i \phi_1 + \phi_2), \\
Q^3 &= i p_1 (i \psi_5 + \psi_6 - \phi_5 + i \phi_6) + i p_1 (-i \psi_5 + \psi_6 - \phi_5 - i \phi_6) \\
&\quad + i p_2 (-i \psi_7 + \psi_8 - \phi_7 + i \phi_8) + i p_2 (i \psi_5 + \psi_8 - \phi_7 - i \phi_8) \\
&\quad + i p_3 (-i \psi_5 + \psi_6 - \phi_5 + i \phi_6) + i p_3 (i \psi_5 + \psi_6 - \phi_5 - i \phi_6) \\
&\quad + i p_4 (-i \psi_7 + \psi_8 - \phi_7 + i \phi_8) + i p_4 (i \psi_5 + \psi_8 - \phi_7 - i \phi_8).
\end{align*}
\]
\[ +i(p - A)_3(-i\psi_1 - \psi_2 - \phi_1 + i\phi_2) + i(p - A)_3(i\psi_1 - \psi_2 - \phi_1 - i\phi_2) \\
+ i(p - A)_4(-i\psi_3 - \psi_4 - \phi_3 + i\phi_4) + i(p - A)_4(i\psi_3 - \psi_4 - \phi_3 - i\phi_4), \]

\[ Q^4 = ip_1(\psi_5 - i\psi_6 + i\phi_5 + \phi_6) + ip_1(\psi_5 + i\psi_6 - i\phi_5 + \phi_6) \\
+ ip_2(-\psi_7 - i\psi_8 - i\phi_4 - \phi_8) + ip_2(-\psi_7 + i\psi_8 + i\phi_4 - \phi_8) \\
+ i(p - A)_3(\psi_1 - i\psi_2 - i\phi_1 - \phi_2) + i(p - A)_3(\psi_1 + i\psi_2 + i\phi_1 - \phi_2) \\
+ i(p - A)_4(-\psi_3 + i\psi_4 + i\phi_3 + \phi_4) + i(p - A)_4(-\psi_3 - i\psi_4 - i\phi_3 + \phi_4). \] (45)

It is a straightforward but tedious calculation to show that each of these commute with the Hamiltonian (40). We need to note that canonical quantization gives the anticommutators \{Q, H\} and \{Q, H\} below Eq.(40) in the basis \( \{\psi_\alpha, \psi_\beta\} = \{\phi_{\alpha i}, \phi_{\beta j}\} = \delta_{\alpha\beta}/(mg_s^{1/2}) \) [24]. The magnetic field \( F = dA \) where \( A \) was defined below Eq.(40) in the \( \bar{z} \) coordinates is also necessary:

\[ F_{34} = -\frac{1}{\pi R_2 R_3 g_s} (h_2 - ih_1), \quad F_{3\bar{4}} = -\frac{1}{\pi R_2 R_3 g_s} (h_2 + ih_1). \] (46)

This arises in \([Q, H]\) from commutators \([p, A]\); there are no commutators mixing holomorphic and antiholomorphic indices.

V. SUPERSYMMETRIC GROUND STATES

To find the states annihilated by the supercharges (45), it’s easier to work in the complex basis

\[ w^1 = \frac{1}{2}(\bar{x}^5 + i\bar{x}^6), \quad w^2 = \frac{1}{2}(\bar{x}^8 + i\bar{x}^9) \] (47)

where

\[ \bar{x}^5 = \frac{1}{\sqrt{2h(h-h_2)}}(h_1 \bar{x}^5 + (h - h_2) \bar{x}^8), \quad \bar{x}^6 = \frac{1}{\sqrt{2h(h-h_2)}}(h_1 \bar{x}^6 + (h - h_2) \bar{x}^9) \]
\[ \bar{x}^8 = \frac{1}{\sqrt{2h(h-h_2)}}((h - h_2) \bar{x}^5 - h_1 \bar{x}^8), \quad \bar{x}^9 = \frac{1}{\sqrt{2h(h-h_2)}}((h - h_2) \bar{x}^6 - h_1 \bar{x}^9) \] (48)

and \( h = \sqrt{h_1^2 + h_2^2} \). In the basis (47), the nonzero components of the magnetic field are

\[ F_{w^1\bar{w}^1} = -\frac{1}{\pi R_2 R_3 g_s} \frac{ih}{h}, \quad F_{w^2\bar{w}^2} = \frac{1}{\pi R_2 R_3 g_s} \frac{ih}{h} \] (49)

We can use a gauge where the potential is

\[ A_{w^1} = \frac{1}{2\pi R_2 R_3 g_s} \frac{ih}{w^1}, \quad A_{\bar{w}^1} = -\frac{1}{2\pi R_2 R_3 g_s} \frac{ih}{w^1} \]
\[ A_{w^2} = -\frac{1}{2\pi R_2 R_3 g_s} \frac{ih}{w^2}, \quad A_{\bar{w}^2} = \frac{1}{2\pi R_2 R_3 g_s} \frac{ih}{w^2}. \] (50)
The supercharges (51) can be rewritten (up to sign)

\[
\begin{align*}
Q^1 &= (p - A)_{\omega^1}(\lambda_1 - \lambda_4) + (p - A)_{\bar{\omega}^1}(-\bar{\lambda}_1 + \bar{\lambda}_4) \\
&\quad + (p - A)_{\omega^2}(\lambda_2 + \lambda_3) + (p - A)_{\bar{\omega}^2}(-\bar{\lambda}_2 + \bar{\lambda}_3) + \\
Q^2 &= (p - A)_{\omega^1}(\lambda_2 + \lambda_3) + (p - A)_{\bar{\omega}^1}(-\lambda_2 - \lambda_3) \\
&\quad + (p - A)_{\omega^2}(-\bar{\lambda}_1 + \bar{\lambda}_4) + (p - A)_{\bar{\omega}^2}(\lambda_1 - \lambda_4) + \\
Q^3 &= i(p - A)_{\omega^1}(\lambda_1 + \lambda_4) + i(p - A)_{\bar{\omega}^1}(-\bar{\lambda}_1 + \bar{\lambda}_4) \\
&\quad + i(p - A)_{\omega^2}(-\lambda_2 + \lambda_3) + i(p - A)_{\bar{\omega}^2}(-\bar{\lambda}_2 - \bar{\lambda}_3) + \\
Q^4 &= i(p - A)_{\omega^1}(-\bar{\lambda}_2 - \bar{\lambda}_3) + i(p - A)_{\bar{\omega}^1}(-\lambda_2 - \lambda_3) \\
&\quad + i(p - A)_{\omega^2}(\bar{\lambda}_1 - \bar{\lambda}_4) + i(p - A)_{\bar{\omega}^2}(\lambda_1 - \lambda_4) + \cdots
\end{align*}
\]

(51)

where \( + \cdots \) are terms involving momenta in the noncompact and \( x^7 \) directions, which will give zero when acting on the ground state wave-functions. The fermions \( \lambda_\alpha \) in (52) are defined in terms of the fermions in (55) as

\[
\begin{align*}
\lambda_1 &= \frac{1}{\sqrt{2h(h-h_2)}} (h_1(\psi_2 + i\psi_4) + (h - h_2)(\psi_1 + i\psi_3)) \\
\lambda_2 &= \frac{1}{\sqrt{2h(h-h_2)}} ((h - h_2)(\psi_2 + i\psi_4) - h_1(\psi_1 + i\psi_3)) \\
\lambda_3 &= \frac{1}{\sqrt{2h(h-h_2)}} (h_1(\phi_2 + i\phi_4) + (h - h_2)(\phi_1 + i\phi_3)) \\
\lambda_4 &= \frac{1}{\sqrt{2h(h-h_2)}} ((h - h_2)(\phi_2 + i\phi_4) - h_1(\phi_1 + i\phi_3)).
\end{align*}
\]

(53)

These spinors satisfy the oscillator algebra

\[
\{\lambda_\alpha, \lambda_\beta\} = \{\bar{\lambda}_\alpha, \bar{\lambda}_\beta\} = 0, \quad \{\lambda_\alpha, \bar{\lambda}_\beta\} = k\delta_{\alpha\beta}
\]

(54)

where \( k \) is a real constant that can be absorbed in the definition of the spinors. So \( \lambda, \bar{\lambda} \) are raising and lowering operators.

As in the \( T^2 \) example, to build the wave functions, we start with a state that is annihilated by half of the fermionic operators appearing in the supercharges (52). There are only two possible states, \( |+\rangle \) and \( |-\rangle \), satisfying

\[
\begin{align*}
\bar{\lambda}_1|+\rangle = \bar{\lambda}_4|+\rangle = \lambda_2|+\rangle = \lambda_3|+\rangle &= 0, \\
\lambda_1|-\rangle = \lambda_4|-\rangle = \bar{\lambda}_2|-\rangle = \bar{\lambda}_3|-\rangle &= 0 .
\end{align*}
\]

(55)
It turns out that it is impossible for any other spin choices to preserve all four supersymmetries; the wave function would have to satisfy incompatible differential equations. From the quantum mechanics superalgebra, partial supersymmetry breaking is not allowed [23].

From (52), the wave function corresponding to the first state must satisfy

\[(p - A)_{w^1} \phi_+ = (p - A)_{w^2} \phi_+ = 0.\]  (56)

Using (50) for the potential, the solution is

\[\phi_+ = e^{-\frac{1}{2\pi R_2 R_3 g_3}} \sqrt{\frac{h}{2h(h-h_2)}} F(\bar{w}^1, w^2).\]  (57)

As in the $T^2$ example, when going around the torus, the wave function picks up a phase given by the gauge transformations (cf. Eqs (11) and (11)). When $x^5 \to x^5 + 2\pi R_2$, we get the following condition on the function $F(\bar{w}^1, w^2)$

\[F(\bar{w}^1 + \frac{h_1}{\sqrt{2h(h-h_2)}} \pi R_2, w^2 + \frac{h-h_2}{\sqrt{2h(h-h_2)}} \pi R_2) = e^{-\frac{1}{2\pi R_2 R_3 g_3}} e^{\frac{h}{2\pi R_2 R_3 g_3}} F(\bar{w}^1, w^2)\]  (58)

where

\[v = \frac{1}{\sqrt{2h(h-h_2)}} (h_1 \bar{w}^1 + (h-h_2)w_2), \quad u = \frac{1}{\sqrt{2h(h-h_2)}} ((h-h_2)\bar{w}^1 - h_1 w_2)\]  (59)

(we defined $u$ for future use).

In a similar fashion to the $T^2$ example, we define the function $G$ as

\[F(\bar{w}^1, w^2) = e^{\frac{1}{2\pi R_2 R_3 g_3} \frac{h}{2} ((\bar{w}^1)^2 + (w^2)^2)} G(\bar{w}^1, w^2)\]  (60)

which should be periodic when $\bar{w}^1 \to \bar{w}^1 + \frac{h_1}{\sqrt{2h(h-h_2)}} \pi R_2$ and $w^2 \to \frac{h-h_2}{\sqrt{2h(h-h_2)}} \pi R_2$, or, using the variables $u$ and $v$ in (59), $v \to v + \pi R_2$ and $u \to u$. So we can decompose $G$ in terms of Fourier modes. Putting everything together, our wave function is

\[\phi_+ = e^{-\frac{1}{2\pi R_2 R_3 g_3} \frac{h}{2} ((\bar{w}^1)^2 + (w^2)^2)} e^{\frac{1}{2\pi R_2 R_3 g_3} \frac{h}{2} ((\bar{w}^1)^2 + (w^2)^2)} \sum_m e^{2im\frac{\bar{w}^1}{R_2} g_m(u)}\]  (61)

where $g_m(u)$ is to be determined from the other periodicity conditions.

When going around the torus in the $x^8$ direction, i.e. when $x^8 \to x^8 + 2\pi R_2$, the condition on the wave function forces $g_m(u)$ to have periodicity $\pi R_2$. So our Fourier mode decomposition in (11) is a double sum, i.e.

\[\phi_+ = e^{-\frac{1}{2\pi R_2 R_3 g_3} \frac{h}{2} ((\bar{w}^1)^2 + (w^2)^2)} e^{\frac{1}{2\pi R_2 R_3 g_3} \frac{h}{2} ((\bar{w}^1)^2 + (w^2)^2)} \sum_{m,n} C_{m,n} e^{2i\frac{(m\bar{w}^1 + n w^2)}{R_2}}\]  (62)
Finally, similar to the $T^2$ case, going around the torus in the $x^6$ and $x^9$ gives us two recursive relations for the coefficients $C_{m,n}$, of the form

$$C_{m+n\frac{h_2}{g_s}, n+\frac{h_1}{g_s}} = C_{m,n} e^{-\frac{2\pi R_2}{R_3} h_s (h_2 m + h_1 n)}$$

$$C_{m+n\frac{h_1}{g_s} - n\frac{h_2}{g_s}} = C_{m,n} e^{-\frac{2\pi R_2}{R_3} h_s (h_1 m - h_2 n)}.$$  

The series defined by these recursive relations converges for any sign of $h_1$ and $h_2$ and factorizes into theta functions as

$$\phi_+ = \sum_{k,l} D_{k,l} \exp \left[ -\frac{1}{2\pi R_2 R_3 g_s} \frac{h}{|w|^2 + |\bar{w}|^2 - (\bar{w}^1)^2 - (w^2)^2} \right]$$

$$\times \vartheta \left[ \frac{g_s(h_2 k + h_1 l)}{h^2} \right] \left( \frac{h_2 v + h_1 u}{\pi R_2 g_s}, \frac{ih R_3}{2R_2 g_s} \right) \vartheta \left[ \frac{g_s(h_1 k - h_2 l)}{h^2} \right] \left( \frac{h_1 v - h_2 u}{\pi R_2 g_s}, \frac{ih R_3}{2R_2 g_s} \right) \right]$$

after solving the recursion, where the sum on $k, l$ is over points in the unit cell for $m, n$ as in (62) (see figure 1).

If instead of working with the first choice of ground state in (55) we start with the second possibility, we get a similar solution to (62), but the recursive relations are such that the series doesn’t converge for any sign of $h_1$ and $h_2$. At first glance, the fact that the $|+\rangle$ state is normalizable for any magnetic field, while the $|-\rangle$ is not, appears contradictory with the results for a particle on $T^2$. Physically, we expect that, as in the particle case, a change of sign of the magnetic field would be compensated by a change of spin. This is indeed still the case in the current scenario; our change of basis (53) reverses the physical spin of the string (given by the $\psi$ and $\phi$ variables) as the field is reversed.

The number of ground states is given by the number of independent coefficients. This number is equal to $\frac{h^2}{g_s} = \frac{h_1^2 + h_2^2}{g_s^2} = f_2^2 + f_1^2$, as can be seen from figure 1.

The orientifold projection forces $C_{m,n} = C_{-m,-n}$. Then, some of the elements in the fundamental lattice are related to one another. For the case $\frac{h_1}{g_s} = 2$, $\frac{h_2}{g_s} = 3$, we indicate in figure 3 all those coefficients that are related after the orientifold projection and a translation along the lattice basis vectors. In this case, the number of independent coefficients is 7 (6 interior points, plus the origin).

There is no general formula for the number of ground states that survive after the projection, but nevertheless there are only a finite number of cases to consider in the string compactification. The 3-form flux carries D3-brane charge, as can be seen from the equation

16
For $h_1/g_s = 2$, $h_2/g_s = 3$, the independent coefficients are those that lie inside the fundamental cell. The number of BPS states in this case is $13 = \left(\frac{h_1}{g_s}\right)^2 + \left(\frac{h_2}{g_s}\right)^2$.

Tadpole cancellation then imposes conditions on this flux of the form $g_s(f_1^2 + f_2^2) \leq 8$. Besides, the self duality condition on the 3-form flux requires $g_s \geq \frac{1}{\min f_{1,2}}$, leaving only the possibilities considered in table I (the number of ground states...
### Table I: Possible combinations of 3-form fluxes, and the number of ground states obtained before and after the orientifold projection.

| \( \left( \frac{h_1}{g_s}, \frac{h_2}{g_s} \right) \) | \# indep. \( C_{m,n} \) before orientifold | \# indep. \( C_{m,n} \) after orientifold |
|------------------------------------------------|------------------------------------------|------------------------------------------|
| \((0, 1)\)                                      | 1                                       | 1                                       |
| \((0, 2)\)                                      | 4                                       | 4                                       |
| \((0, 3)\)                                      | 9                                       | 5                                       |
| \((0, 4)\)                                      | 16                                      | 10                                      |
| \((1, 1)\)                                      | 2                                       | 2                                       |
| \((1, 2)\)                                      | 5                                       | 3                                       |
| \((2, 3)\)                                      | 13                                      | 7                                       |
| \((3, 3)\)                                      | 18                                      | 10                                      |
| \((4, 4)\)                                      | 32                                      | 17                                      |

For the pair \( \left( \frac{h_1}{g_s}, \frac{h_2}{g_s} \right) = (1, 0) \), for example, is equal to that for \( \left( \frac{h_1}{g_s}, \frac{h_2}{g_s} \right) = (0, 1) \).

There is one final issue to consider; in some cases, a point \( (m, n) \) maps to itself under the \( \mathbb{Z}_2 \) projection followed by several recursion relations among the Fourier coefficients. In order to verify that those points truly survive the orientifold, we should check that the necessary recursion relations truly give \( C_{-m,-n} = C_{m,n} \). First, we note that, in this case, \( \pm (m, n) \) are separated by integer numbers of the recursion shift vectors:

\[
(m, n) - q(h_2/g_s, h_1/g_s) - p(h_1/g_s, -h_2/g_s) = (-m, -n), \quad p, q \in \mathbb{Z}.
\]  

(65)

Since the recursion relations (63) are independent of position along the orthogonal shift vector, we have then

\[
C_{-m,-n} = \exp \left\{ -\frac{\pi R_3}{R_2} \left[ (p + q) \frac{h}{g_s} - \frac{2h_2}{h} \sum_{i=0}^{q-1} \left( m - i \frac{h_2}{g_s} \right) - \frac{2h_1}{h} \sum_{i=0}^{q-1} \left( n - i \frac{h_1}{g_s} \right) \right. \\
\left. - \frac{2h_1}{h} \sum_{j=0}^{p-1} \left( m - j \frac{h_1}{g_s} \right) + \frac{2h_2}{h} \sum_{j=0}^{p-1} \left( n + j \frac{h_2}{g_s} \right) \right] \right\} C_{m,n}.
\]  

(66)

Carrying out the arithmetic sums and using (65) shows easily that the exponent indeed vanishes.
VI. OTHER BPS STATES

There are other BPS states that we can consider. In this section, we will find BPS states localized at singularities of the D-string moduli space, show how to count the states of F-strings, and discuss briefly the problem of threshold bound states.\(^4\)

A. Singularities in Moduli Space

As the D-string moves on the torus, it encounters two types of singularities. First, at \(x^5 = \cdots = x^9 = 0\), it passes through two orientifold planes and doubles back on itself. (Note that the D-string does not end on an O3-plane; this is because, with vanishing flux, it is T-dual to a type I D5-brane, which always has 2 CP indices.) Also, if tadpole cancellation so requires, the string could intersect a D3-brane (which we specifically ignored in our calculation). These are direct analogs of the singularities in the moduli space of type I and heterotic 5-branes mentioned in \([34]\).

To count states localized at the singularities, we should be able to ignore the 3-form background. The reason is the localized states do not have zero modes that can move through the fluxes. Therefore, the counting should be the same as without 3-forms, which is dual to the \(SO(32)\) heterotic 5-brane.\(^5\) Translating the results of \([36, 37]\) to IIB language, we find that there are no localized states at the O3-planes and that there is one BPS multiplet attached to each D3-brane.

B. Counting for an F-string

We can obtain the number of BPS states for an F-string by starting with the Green-Schwarz action for superstrings and accounting for the background fields; because of supersymmetry, we expect it to suffice for the calculation of our index. However, it is easier to proceed by applying S-duality and rotating the background fields, and then using the results from the previous section. We want to find the number of BPS states for an F-string wrapped in the 7-direction, so we will S-dualize the background, and then rotate in the 4-7

\(^4\) We are grateful to our referee for comments that led to the analysis in sections VI A and VI C.

\(^5\) For an \(E_8\) application, see \([37]\).
plane.

Under an $SL(2, \mathbb{Z})$ transformation,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} H_{(3)} \\ F_{(3)} \end{pmatrix} \rightarrow \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H_{(3)} \\ F_{(3)} \end{pmatrix}.$$  

(67)

An S-duality is $\tau \rightarrow -\frac{1}{\tau}$. Then, $H_{(3)} \rightarrow F_{(3)}$ and $F_{(3)} \rightarrow -H_{(3)}$. So $h_1$ and $h_2$ from the previous sections get mapped to $f_1$ and $f_2$ respectively. Using the duality condition on the 3-form flux $\{3\}$, the action of S-duality is $h_1 \rightarrow -\frac{h_2}{g_s}$ and $h_2 \rightarrow \frac{h_1}{g_s}$.

A rotation in the 4-7 plane interchanges $h_1$ with $h_2$ again. From $\{23\}$, we see that a rotation of $\pi/2$ gives $h_1 \rightarrow -h_2$ and $h_2 \rightarrow h_1$. A rotation of $-\pi/2$ does the same thing with opposite signs.

Then, the combined action of an S-duality and a rotation, gives $h_{1,2} \rightarrow -\frac{h_{1,2}}{g_s}$ in one case and $h_{1,2} \rightarrow \frac{h_{1,2}}{g_s}$ for the opposite rotation. Then, the number of BPS states of the rotated F-string is just $h_1^2 + h_2^2$ plus one state at each D3-brane. (In the D-string, the number of states is $f_1^2 + f_2^2$ plus one for each D3-brane.) So the total number of BPS multiplets for a given BPS charge should be $f_1^2 + f_2^2 + h_1^2 + h_2^2 + 2N_{D3}$.

C. Bound States

So far we have considered only states of one D- or F-string at a time, which are the states of minimal BPS electric charge. However, it would be very interesting to consider the BPS spectrum of multiple strings, since there could be threshold bound states. We will largely leave this question for the future, but we can make some comments.

Consider, for example, the case of two BPS charges. If they are widely separated in the noncompact dimensions, then the spectrum should just be the direct product of the spectra of the individual charges (appropriately symmetrized). As the two charges become coincident, we expect that there would, in addition, be BPS multiplets associated with threshold bound states. The counting of multiplets breaks down according to the nature of the strings. A bound state of two D-strings, for example, would have twice the charge but would be otherwise identical to our previous analysis, so there would be four times as many nonlocalized states. A bound state of a D-string and F-string seems more complicated in that the two strings feel different fluxes.

Another question that we leave to future work is the nature of the bound states. For two
D-strings, we really should take into account the non-Abelian nature of the worldvolume theory; we can do this using the D-brane action of [38], and supersymmetrizing. It is possible that the bound state is a “polarized” configuration, or it is possible that there are polarized and unpolarized bound states. We should also mention that we could start with, instead of the action of [38], super-Yang-Mills with a superpotential due to the 3-forms along with the appropriate velocity coupling to the vector potential. The problem is to guess how the superpotential depends on the 3-form flux.

VII. CONCLUSIONS

We were able to count the number of short multiplets for both a D- and an F-string wrapped on a compact direction of a $T^6/Z_2$ orientifold, on a background containing constant 3-form fluxes. We showed that the number of states is proportional to the square of the units of 3-form flux enclosed on $T^3$ cycles inside the $T^6$. Modulo algebraic complications, the counting of ground states follows two copies of that of a superparticle on a torus with constant perpendicular magnetic field.

This is a nice result obtained from the $\kappa$-symmetric action for a D1-brane in background fluxes, which was our starting point. We showed that the Hamiltonian obtained from this action preserved the amount of supersymmetries expected, and from the supercharges we obtained the supersymmetric ground states. By S-duality, we were able to find the corresponding number of states for the F-string case.

One of the motivations of this work was the mismatch found in [14] between the number of states carrying minimal electric (D- and F-strings) versus magnetic (bound state of D5- and NS5-brane) charges in the $\mathcal{N} = 3$ theory. It was then conjectured that the sum of the BPS states of the D- and F-string should be equal to those of the bound state of 5-branes. In this paper we computed the former, while the latter is left for future work.

It is our hope that we have introduced a useful piece of technology for the study of string and D-brane physics in flux backgrounds.
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APPENDIX A: CONVENTIONS

In the following, Greek indices refer to 0, 1, 2, 3 directions, while Roman indices \( m, n, \ldots \) refer to 4 to 9 directions.

There are two spacetime Majorana-Weyl spinors \( \Theta^1 \) and \( \Theta^2 \) in the \( \kappa \)-symmetric action. The \( \kappa \)-symmetry can be used to fix a gauge, keeping only 16 of those 32 degrees of freedom. We used the gauge \( \Theta^2 = 0 \).

The spinor \( \Theta \) in Eqs (33) and (A3) below, is a Majorana-Weyl spinor in a Dirac basis:

\[
\Theta = \begin{pmatrix} 0 \\ \theta \end{pmatrix}.
\]  

(A1)

It can be decomposed into a Majorana-Weyl spinor in \( SO(1, 1) \) and a Majorana-Weyl spinor in \( SO(8) \) as in Eq.(35). A basis of Majorana-Weyl spinors in \( SO(8) \), given by spins in the complex planes defined in eq. (36) is

\[
\begin{align*}
\begin{array}{ll}
u_1 & = |+++-\rangle + |----\rangle, \quad v_1 = |+-+-\rangle - |---+\rangle, \\
u_2 & = i (|+++-\rangle - |----\rangle), \quad v_2 = i (|+-+-\rangle + |---+\rangle), \\
u_3 & = |++--\rangle - |---+\rangle, \quad v_3 = |+-+-\rangle + |---+\rangle, \\
u_4 & = i (|++--\rangle + |---+\rangle), \quad v_4 = i (|+-+-\rangle - |---+\rangle), \\
u_5 & = |+-++\rangle + |---+\rangle, \quad v_5 = |+-+-\rangle - |---+\rangle, \\
u_6 & = i (|+-++\rangle - |---+\rangle), \quad v_7 = i (|+-+-\rangle + |---+\rangle), \\
u_7 & = |+-++\rangle - |---+\rangle, \quad v_7 = |+-+-\rangle - |---+\rangle, \\
u_8 & = i (|+-++\rangle - |---+\rangle), \quad v_8 = i (|+-+-\rangle - |---+\rangle).
\end{array}
\end{align*}
\]  

(A2)

The \( u_\alpha \) (\( v_\dot{\alpha} \)) form an \( 8 \) (\( 8' \)).

The expansions of the spacetime fields in terms of the world-volume spinor \( \Theta \) in the gauge \( \Theta^2 = 0 \) are, for constant dilaton-axion [28]

\[
\begin{align*}
e_m^a = e_m^a + \frac{i}{8} \Theta^{abc} \Theta w_{mbc} - \frac{i}{16} \Theta^{anp} \Theta H_{mnp}.
\end{align*}
\]
We have introduced a factor of $i$ in each fermion bilinear, so that the action (33) matches usual quantum field theory conventions and gives a real anticommutator.

APPENDIX B: CORRECTIONS TO THE QUANTUM MECHANICS

Here we will estimate a number of possible corrections to the Hamiltonian given in section III B and argue that they will not change the counting of BPS states. Because a single long multiplet could only be formed from four short multiplets (of differing spins), we anticipate that it would be difficult to lift the supersymmetric states given in section V; this is essentially the argument of [23]. We will mainly here be concerned if any of the long multiplets could be made BPS by corrections to our Hamiltonian, eqn (40). We work in the framework of perturbation theory and concentrate on the bosonic terms. We take all the coordinate radii to be similar $\sim R$.

First we note the structure of the Hilbert space once we include excited states. The center of mass modes considered in the text have excited states with energy of the order of the magnetic field, $\sim \alpha'/R^3$. Then there are modes that move around the D-string, which form a tower of supersymmetric harmonic oscillators with frequency $\sim 1/R$. Each oscillator has a unique supersymmetric ground state, and we don’t expect perturbations to the Hamiltonian to break the supersymmetry.\footnote{We should note that the oscillator amplitudes are not periodic, unlike center of mass modes. This is clear}
Just like the center of mass modes, the oscillators couple to the 3-form; we treat this as a perturbation. Because the perturbation couples different oscillators, it appears only in second order perturbation theory. Since the magnetic field is of order $\alpha' / R^3$, the energy shift is of order $\alpha'^2 / R^5$. The 3-form flux $F_{mn7}$ also couples to the oscillators; this perturbs the excited state energies by $\sim \alpha'^3 / R^7$. There are also higher derivative terms in the D-brane action, coming from the Born-Infeld determinant. One typical term is $(\dot{X} \partial_4 X)^2$. This can contribute at first order in perturbation theory, giving an energy shift $\alpha' / R^2$ times the energy of the state.

We also briefly consider bulk supergravity effects. At large but finite radius, there is a metric warp factor; it’s lowest order contribution is $\alpha'^2 / R^4$ times the energy of the state. The interaction with bulk modes is a little trickier to understand. The massless moduli of the compactification should not affect the BPS spectrum, since the same supercharges are preserved in the spacetime. The BPS particle mass and excited state energies will change, however. The massive scalars of the 4D effective theory, such as the dilaton, should also not change the BPS spectrum. One argument for this is that the dilaton enters into the Hamiltonian only through the mass of the BPS particle (when one takes into account the normalization of the fermions), which affects only the excited states of the center of mass modes. Another argument is that there are some cancellations among different terms in the perturbation theory for the dilaton.

Finally, we should note that the 4D gauge fields can change the BPS spectrum rather violently. For example, a BPS particle in the field of an oppositely charged particle should have no supersymmetric ground states.

---

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