Optimal generation of single-qubit operation from an always-on interaction by algebraic decoupling

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We present a direct algebraic decoupling approach to generate arbitrary single-qubit operations in the presence of a constant interaction by applying local control signals. To overcome the difficulty of undesirable entanglement generated by the untunable interaction, we derive local control fields that are designed to both drive the qubit systems back to unentangled states at the end of the time interval over which the desired single-qubit operation is completed. This approach is seen to be particularly relevant for the physical implementation of solid-state quantum computation and for the design of low-power pulses in NMR.

Most schemes for implementation of quantum computation require achieving both single- and two-qubit operations in order to realize the speed-up associated with quantum algorithms \cite{1}. Typically, single-qubit operations are implemented by external local control fields applied to individual qubits, whereas two-qubit operations are generated by interplay of the couplings between qubits and local fields \cite{2}. To simplify engineering design and reduce decoherence channels, many physical proposals of quantum computation have “always-on” and fixed couplings between qubits. This is often the case in arrays of solid-state qubits, which constitute a very attractive research direction because of the inherent scalability of the required microfabrication techniques. For example, in superconducting systems the interaction between qubits is often coupled by a capacitor or inductor, whose value is fixed by the fabrication and generally cannot be tuned during the computations \cite{3,4}. Several variable coupling schemes have been suggested \cite{5,6,7,8,9}, but it is generally agreed that none of these is completely satisfactory. In contrast, proposals for using electron spins in quantum dots do in principle allow for electrical gating of the exchange spin-spin interaction \cite{10,11}, but engineering such control in practice still remains extremely challenging (See, e.g., Ref. \cite{12} and references therein.) Other spin-coupled implementations of current interest include endohedral fullerenes for which non-tunable magnetic dipolar coupling between electron spins of neighboring endohedral species provide the required qubit coupling \cite{13,14}.

While coupling between qubits is essential for the implementation of two-qubit operations, an always-on and untunable coupling leads to additional complications in implementation of single-qubit operations, because the qubits may become entangled. This general issue is also encountered in nuclear magnetic resonance. For the short and high power “hard” pulses, sophisticated refocusing schemes have been developed to decouple the subsystems \cite{15}. However, for the low-power pulses often used in homonuclear spin systems, the radio frequency signals can be of the same order of magnitude as the coupling strengths \cite{16,17}. In this “soft” pulse situation, the single-qubit rotations cannot be assumed to be implemented instantaneously as is the case with hard pulses, and consequently the interactions may strongly affect the intended operations. To overcome this general problem in the context of quantum information processing, Ref. \cite{18} employed an encoding of logical qubits, while logic operations are performed using an analogue of the NMR selective recoupling method. Ref. \cite{19} presented another encoding scheme to realize universal quantum computation on carefully designed interaction free subspaces. Recently, Ref. \cite{20} showed that one can also avoid this undesirable entanglement by tuning the transition energies of individual qubits. More complex schemes involving auxiliary degrees of freedom have also been developed \cite{13,14}. Here we present a direct approach for the design of local control signals in the presence of an always-on interaction. To eliminate the accompanying entanglement generated by this untunable interaction, we derive local control fields that drive the qubit systems back to unentangled states at the end of the time interval over which the desired single-qubit operation is completed.

We consider a two-qubit system with the following Hamiltonian:

$$
H = \frac{\omega_1}{2}(\cos \phi_1 \sigma_x^1 + \sin \phi_1 \sigma_y^1)
+ \frac{\omega_2}{2}(\cos \phi_2 \sigma_x^2 + \sin \phi_2 \sigma_y^2) + \frac{J}{2} \sigma_z^1 \sigma_z^2,
$$

(1)

where \(\sigma_x, \sigma_y, \) and \(\sigma_z\) are the Pauli matrices, \(J\) the always-on and untunable coupling strength, \(\omega_j\) and \(\phi_j\) the amplitude and phase of the external control fields, respectively. This Hamiltonian describes a wide range of two-qubit systems, e.g., solid-state quantum computation with superconducting circuits as proposed in \cite{13}, as well as NMR double-resonance \(J\) cross polarization with two RF fields in the XY plane of the doubly rotating frame \cite{21}. Note that the Hamiltonian in Eq. (1) can be transformed by local unitary operations into a Hamiltonian with YY coupling and \(\sigma_x, \sigma_z\) local terms, or into one with XX coupling and \(\sigma_y, \sigma_z\) local terms. Therefore, the results in this paper are also applicable to these two types of Hamiltonians. The spin coupling \(J\) here can be
neglected only for short and high-power pulses, since for low-power pulses \( \omega_1 \) and \( \omega_2 \) can be of the same order as \( J \). We now show how to design amplitude-modulated pulses that can generate an arbitrary single-qubit operation for such a system. The control pulses are derived by algebraically decoupling the two-qubit Hamiltonian into two unentangled single-qubit systems. This approach is easy to implement in the physical systems of interest and avoids the encoding overheads associated with the encoding schemes of Refs. 18, 19.

An arbitrary local unitary operation \( k_1 \) on two qubits, the target operation here, can be written using Euler’s XY decomposition as:

\[
k_1 = (e^{-i \omega_1 \sigma_x/2} \otimes e^{-i \omega_2 \sigma_x/2})(e^{-i \beta_1 \sigma_y/2} \otimes e^{-i \beta_2 \sigma_y/2})
\times (e^{-i \gamma_1 \sigma_z/2} \otimes e^{-i \gamma_2 \sigma_z/2}),
\]

where \( \alpha_j, \beta_j, \) and \( \gamma_j \) are the Euler angles. In order to generate \( k_1 \), we need to generate arbitrary \( \sigma_x \) and \( \sigma_y \) rotations on each qubit. However, we can simplify this problem to the generation of only \( \sigma_x \) rotations \( k_2 = e^{-i \gamma_1 \sigma_z/2} \otimes e^{-i \gamma_2 \sigma_z/2} \) from the following Hamiltonian:

\[
H_1 = \frac{\omega_1}{2} \sigma_x^1 + \frac{\omega_2}{2} \sigma_x^2 + \frac{J}{2} \sigma_z^1 \sigma_z^2,
\]

where \( H_1 \) is obtained from Eq. (11) by setting the phases \( \phi_1 = \phi_2 = 0 \) for external control pulses. This leads immediately to the implementations of the first and third operations in Eq. (2). For the second operation in Eq. (2), we can set \( \phi_1 = \phi_2 = \pi/2 \) in Eq. (11) to obtain a Hamiltonian:

\[
H_2 = \frac{\omega_1}{2} \sigma_y^1 + \frac{\omega_2}{2} \sigma_y^2 + \frac{J}{2} \sigma_z^1 \sigma_z^2 = V^\dagger H_1 V,
\]

where \( V = e^{i \pi/2 \sigma_z/2} \otimes e^{i \pi/2 \sigma_z/2}. \) Now if \( H_1 \) generates a quantum operation \( e^{-i \beta_1 \sigma_y/2} \otimes e^{-i \beta_2 \sigma_y/2}, \) \( H_2 \) will generate the second operation in Eq. (2), since \( e^{-i \beta_1 \sigma_y/2} \otimes e^{-i \beta_2 \sigma_y/2} = V^\dagger (e^{-i \beta_1 \sigma_y/2} \otimes e^{-i \beta_2 \sigma_y/2}) V. \)

Our task has therefore been simplified to the generation of an arbitrary local unitary operation \( e^{-i \gamma_1 \sigma_z/2} \otimes e^{-i \gamma_2 \sigma_z/2} \) from the Hamiltonian \( H_1 \) in Eq. (3). We now observe that the terms \( i \sigma_x^1 \sigma_x^2, i \sigma_x^2 \sigma_x^1, i \sigma_y^1 \sigma_y^2, i \sigma_y^2 \sigma_y^1, \) and \( i \sigma_z^1 \sigma_z^2 / 2 \) appearing in \( H_1 \) generate the following Lie algebra:

\[
\mathfrak{t}_1 = \frac{i}{2} \{ \sigma_x^1, \sigma_x^2, \sigma_y^1 \sigma_x^2, \sigma_y^2 \sigma_x^1, \sigma_y^1 \sigma_y^2, \sigma_y^2 \sigma_y^1, \sigma_x^1 \sigma_z^2, \sigma_z^1 \sigma_x^2, \sigma_z^2 \sigma_x^1 \sigma_x^2 \}.
\]

It is straightforward to show that \( \mathfrak{t}_1 \) satisfies the same commutation relations as \( \mathfrak{so}(4) \), where \( \mathfrak{so}(4) \) denotes the Lie algebra formed by all the \( 4 \times 4 \) real skew symmetric matrices. Therefore, \( \mathfrak{t}_1 \) is isomorphic to \( \mathfrak{so}(4) \). We also know that \( \mathfrak{so}(4) \) is isomorphic to \( \mathfrak{su}(2) \otimes \mathfrak{su}(2) \) \( \mathfrak{su}(2) \). To realize this fact, let

\[
\epsilon_x^1 = \frac{\sigma_1^1 - \sigma_2^2}{4}, \quad \epsilon_y^1 = \frac{\sigma_1^1 \sigma_2^2 + \sigma_2^1 \sigma_1^2}{4}, \quad \epsilon_z^1 = \frac{\sigma_1^1 \sigma_2^2 - \sigma_2^1 \sigma_1^2}{4},
\]

\[
\epsilon_x^2 = \frac{\sigma_1^2 + \sigma_2^1}{4}, \quad \epsilon_y^2 = \frac{\sigma_1^2 \sigma_1^2 - \sigma_2^1 \sigma_2^1}{4}, \quad \epsilon_z^2 = \frac{\sigma_1^2 \sigma_2^1 + \sigma_2^1 \sigma_1^2}{4}.
\]

and use \( \mathfrak{t}_2 \) to denote the Lie algebra generated by \( \{ i \epsilon_x^1, i \epsilon_y^1, i \epsilon_z^1, i \epsilon_y^2, i \epsilon_z^2 \} \). We have the following commutation relations for \( \mathfrak{t}_2 \):

\[
[i, j] =
\begin{array}{cccccc}
\epsilon_x^1 & \epsilon_y^1 & \epsilon_z^1 & \epsilon_y^2 & \epsilon_z^2 \\
\epsilon_x^1 & 0 & -i \epsilon_z^2 & i \epsilon_y^2 & 0 & 0 \\
i \epsilon_y^1 & i \epsilon_z^2 & 0 & 0 & 0 & 0 \\
i \epsilon_z^1 & -i \epsilon_y^2 & 0 & 0 & 0 & 0 \\
i \epsilon_y^2 & 0 & 0 & 0 & -i \epsilon_z^2 & i \epsilon_y^2 \\
i \epsilon_z^2 & 0 & 0 & -i \epsilon_y^2 & i \epsilon_z^2 & 0
\end{array}
\]

It is clear that the Lie algebra \( \mathfrak{t}_2 \) satisfies the same commutation relations as \( \mathfrak{su}(2) \otimes \mathfrak{su}(2) \), and therefore it is isomorphic to \( \mathfrak{su}(2) \otimes \mathfrak{su}(2) \). This isomorphism allows great simplification for the generation of single-qubit operations from Hamiltonian \( \mathfrak{t}_2 \), because it provides an algebraic way to decouple the entangling Hamiltonian into two unentangled single-qubit Hamiltonians. To our knowledge, this fact has not been recognized, although a transformation similar to \( \mathfrak{t}_2 \) was presented in [21].

We can now rewrite \( H_1 \) as

\[
H_1 = (\omega_1 - \omega_2)\epsilon_x^1 + J \epsilon_y^1 + (\omega_1 + \omega_2)\epsilon_z^2 - J \epsilon_y^2,
\]

and the desired generator \( k_2 \) as

\[
k_2 = e^{-i \gamma_1 \sigma_z/2} \otimes e^{-i \gamma_2 \sigma_z/2} = e^{-i((\gamma_1 - \gamma_2)\epsilon_x^1 + (\gamma_1 + \gamma_2)\epsilon_z^2)}.
\]

We now transform the parameters in \( k_2 \) and \( H_1 \) to reformulate the problem as generation of the local unitary

\[
e^{-i((\gamma_1 - \gamma_2)\sigma_x^1/2 + (\gamma_1 + \gamma_2)\sigma_z^2/2)}
\]

from the Hamiltonian

\[
\frac{\omega_1 - \omega_2}{2} \sigma_x^1 + \frac{J}{2} \sigma_y^1 + \frac{\omega_1 + \omega_2}{2} \sigma_x^2 - \frac{J}{2} \sigma_y^2.
\]

To obtain Eqs. (10) and (11), we simply replaced \( \epsilon_x^1 \) in Eqs. (7) and (8) by \( \sigma_x^1/2 \), which is warranted by the fact that \( \{ i \epsilon_x^1, i \epsilon_y^1, i \epsilon_z^1, i \epsilon_y^2, i \epsilon_z^2 \} \) is isomorphic to the Lie algebra \( \mathfrak{su}(2) \otimes \mathfrak{su}(2) \). Taking advantage of the commutation of \( \sigma_x^1 \) and \( \sigma_y^2 \), the problem then naturally decomposes into two well-defined problems of generating single-qubit rotations:

1. Generate \( e^{-i((\gamma_1 - \gamma_2)\sigma_x^1/2 + \gamma_1 \sigma_z^2/2)} \) from the Hamiltonian \( (\omega_1 + \omega_2)\sigma_x^2/2 + J \sigma_z^2/2 \); and
2. Generate \( e^{-i((\gamma_1 + \gamma_2)\sigma_z^2/2)} \) from the Hamiltonian \( (\omega_1 + \omega_2)\sigma_z^2/2 - J \sigma_y^2/2 \).

By making the transformation from \( \mathfrak{t}_1 \) to \( \mathfrak{su}(2) \otimes \mathfrak{su}(2) \), we have therefore arrived at two decoupled single-qubit quantum systems, solutions of which both reduce to a general steering problem on the Lie group \( \mathfrak{su}(2) \) with dynamics determined by the Schrödinger equation:

\[
i U = (\omega(t)/2 \sigma_x + J/2 \sigma_y)U, \quad U(0) = I.
\]
In general, \( \omega(t) \) is a time dependent external control field. The minimum energy control on the Lie group SU(2) has been studied in Ref. 22. Here we will use Lie-Poisson reduction to derive both the minimum energy and time optimal control. We will also give a good approximate solution to this problem.

Pontryagin’s Maximal Principle provides an important mathematical tool for solving the optimal steering problem on Lie groups [24]. We solve for a control field \( \omega \) that drives the system from the initial operation \( U(0) = I \) to a prescribed target operation \( U(T) = e^{-i\gamma/2\sigma_z} \), which also minimizes the cost function:

\[
J = \int_0^T L(\omega(t)) dt,
\]

where \( L(\omega(t)) \) is a general functional of the control field \( \omega(t) \) and is often referred as the running cost. The first step is to construct the control Hamiltonian:

\[
H(t) = L(\omega(t)) + \langle M, -iU(t)^\dagger (\frac{\omega(t)}{2} \sigma_x + \frac{J}{2} \sigma_y) U(t) \rangle.
\]

where \( M \) is a constant matrix in \( \mathfrak{su}(2) \), and \( \langle \cdot, \cdot \rangle \) is an inner product defined on \( \mathfrak{su}(2) \). For the ease of notation, we will suppress the time parameter unless otherwise specified. Proceeding further, we can write Eq. (13) as

\[
\mathcal{H} = L(\omega) = \frac{1}{2} \langle M, iU^\dagger \sigma_x U \rangle \omega - \frac{1}{2} \langle M, iU^\dagger \sigma_y U \rangle J.
\]  

We then let \( p_1 = \langle M, iU^\dagger \sigma_z U \rangle /2, \ p_2 = \langle M, iU^\dagger \sigma_y U \rangle /2, \) and \( p_3 = \langle M, iU^\dagger \sigma_z U \rangle /2 \). Taking the derivative of \( p_1 \), we obtain

\[
\dot{p}_1 = \frac{1}{2} \langle M, iU^\dagger \sigma_x U \rangle + \frac{1}{2} \langle M, iU^\dagger \sigma_x U \rangle = Jp_3.
\]

By similar means, we can get the dynamics for \( p_2 \) and \( p_3 \). Combining these together, we have the dynamics for \( p_3 \) as

\[
\dot{p}_1 = Jp_3, \\
\dot{p}_2 = -\omega p_3, \\
\dot{p}_3 = \omega p_2 - Jp_1.
\]  

Eq. (16) is indeed the Lie-Poisson equation on SU(2) [25]. It is easy to verify that the functions \( C_1 = p_1^2 + p_2^2 + p_3^2 \) and \( C_2 = p_1^2 + 2Jp_2 \) are both invariant along the system trajectory. For a general optimal control problem on Lie group SU(2), one usually needs to solve a set of six differential equations. However, here we can obtain the reduced dynamics of three differential equations in Eq. (16) by using the Lie-Poisson reduction theorem [24]. The two conditions for this theorem are the vector field is right invariant and the cost function being independent of the quantum states, and they are both satisfied in this case. This reduces the number of the differential equations to be solved by half. This approach is a general technique applicable also to analysis of optimal control on multi-qubit systems.

Now we derive the time optimal control, that is, the form of control field \( \omega \) that achieves the target operation in the minimum time possible. In this case, the cost function can be written as

\[
\min J = \int_0^T 1 \ dt,
\]

and the running cost \( L(\omega) = 1 \) is independent of control field \( \omega \). The corresponding control Hamiltonian is

\[
\mathcal{H} = 1 - \omega p_1 - Jp_2.
\]  

Let us assume that the control field \( \omega \) is restricted to an interval \([\omega_m, \omega_M]\) due to physical bounds on the available frequencies. From Pontryagin’s Maximum Principle, the optimal control field \( \bar{\omega}(t) \) will minimize the control Hamiltonian [17] pointwise. Therefore, \( \bar{\omega}(t) \) can take only extremal values:

\[
\bar{\omega}(t) = \begin{cases} 
\omega_m, & \text{if } p_1(t) \geq 0, \\
\omega_M, & \text{if } p_1(t) < 0.
\end{cases}
\]

This means that the time optimal control will switch back and forth between two extremal control values. This procedure is usually called Bang-Bang control in the control theory literature. (Note that the term “Bang-Bang control” has recently been adopted with a somewhat different meaning in the study of dynamical coupling of open quantum systems [27, 22, 24, 24], where it is referred to performing a set of instantaneously or as fast as physically possible hard pulses.) In Ref. 22, we have derived a constructive approach to achieve a desired single-qubit target quantum operation by switching between two such constant control fields. From the arguments above, we conclude now that this is indeed a time optimal control strategy.

Next we consider the minimum energy control which is encoded by the following cost function:

\[
\min J = \frac{1}{2} \int_0^T \omega^2(t) dt.
\]

The control Hamiltonian is now

\[
\mathcal{H} = \frac{1}{2} \omega^2 - \omega p_1 - Jp_2,
\]

and thus the minimum energy control is \( \bar{\omega}(t) = p_1(t) \). Taking the derivative of \( p_1 \) in Eq. (16), we have

\[
\ddot{\omega} = \left( \frac{C_2}{2} - J \right) \omega - \frac{1}{2} \dot{\omega}^2.
\]  

where \( C_2 = p_1^2 + 2Jp_2 \) is constant along the optimal trajectory. The solution of this differential equation is given by the Jacobi elliptic function \( \text{Cn} \) [24]:

\[
\bar{\omega}(t) = 2bk \text{Cn}(bt + f, k),
\]  

(19)
where \( b, f, \) and \( k \) are real numbers. We can numerically determine the values of these parameters such that the control function (14) steer the Hamiltonian (13) from the initial operator \( U(0) = I \) to the target operation \( U(T) = e^{-i\gamma/2\sigma_z} \).

Finally we give an approximate solution to the control problem in terms of a sinusoidal control function \( \omega = A \cos(vt) \), and show that this can be further fidelity optimized. Letting \( U_1 = e^{i\nu/2\sigma_y}U \), we transform the system (14) into the following form:

\[
i\dot{U}_1 = \left\{ \frac{A}{2}(1 + \cos 2vt)\sigma_y/2 + (J - \nu)\sigma_y/2 + A/2\sin 2vt\sigma_y/2 \right\} U_1
\]

(20)

The usual technique to solve this differential equation in the context of NMR is to drop the oscillating terms to get a time-independent Hamiltonian (rotating wave approximation). In order to get a more accurate solution, we use the Wei-Norman formula (20) to transform Eq. (20) to a dynamical system on \( \mathbb{R}^3 \). From Euler’s ZXZ decomposition, a general solution to Eq. (20) can be written as

\[
U_1(t) = e^{-i\alpha_1(t)\sigma_z/2}e^{-i\alpha_2(t)\sigma_z/2}e^{-i\alpha_3(t)\sigma_z/2}.
\]

(21)

Taking the time derivative of \( U_1 \), we have

\[
i\dot{U}_1 = \left\{ (\dot{\alpha}_2 \cos \alpha_1 + \dot{\alpha}_3 \sin \alpha_1 \sin \alpha_2)\sigma_y/2 + (\dot{\alpha}_2 \sin \alpha_1 - \dot{\alpha}_3 \cos \alpha_1 \sin \alpha_2)\sigma_y/2 \right\} U_1.
\]

(22)

Comparing Eqs. (20) and (22), we get

\[
\begin{bmatrix}
A/2(1 + \cos 2vt) \\
J - \nu \\
A/2\sin 2vt
\end{bmatrix}
= 
\begin{bmatrix}
0 & \cos \alpha_1 & \sin \alpha_1 \sin \alpha_2 \\
0 & \sin \alpha_1 & -\cos \alpha_1 \sin \alpha_2 \\
1 & 0 & \cos \alpha_2
\end{bmatrix}
\begin{bmatrix}
\dot{\alpha}_1 \\
\dot{\alpha}_2 \\
\dot{\alpha}_3
\end{bmatrix}.
\]

Therefore, we obtain a set of differential equations of the parameters \( \alpha_j \) in \( \mathbb{R}^3 \):

\[
\begin{align*}
\dot{\alpha}_1 &= -\frac{\sin \alpha_1}{\tan \alpha_2} \frac{\cos \alpha_1}{\sin \alpha_2} \frac{\cos \alpha_1}{\sin \alpha_2} \\
\dot{\alpha}_2 &= \frac{\cos \alpha_1}{\sin \alpha_2} \sin \alpha_1 \\
\dot{\alpha}_3 &= \frac{\sin \alpha_1}{\sin \alpha_2} - \frac{\cos \alpha_1}{\sin \alpha_2}
\end{align*}
\]

(23)

Rigorous solution of Eq. (23) cannot be given in terms of elementary functions. However, when \( \nu \) is close to \( J \), a very good approximate solution can be found as:

\[
\begin{align*}
\alpha_1(t) &= \frac{A}{2\nu}\sin^2 vt, \\
\alpha_2(t) &= \frac{A}{2}t + \frac{A}{4J}\sin 2vt, \\
\alpha_3(t) &= 0.
\end{align*}
\]

Hence a sinusoidal control function \( \omega = A \cos(vt) \) can achieve the following unitary operation at a final time \( T \):

\[
U(T) = e^{i\sigma_z/2}e^{-i\alpha_1(T)\sigma_z/2}e^{-i\alpha_2(T)\sigma_z/2}
\]

(24)

To generate an operation \( e^{-i\gamma/2\sigma_z} \) with \( \gamma \in [0, 2\pi] \), we only need that \( \nu = J, AT/2 = \gamma, \) and \( JT = 2n\pi \), where \( n \) is an integer. This leads to the conditions:

\[
\nu = J, \quad A = \frac{\gamma J}{n\pi}, \quad T = \frac{2n\pi}{J}.
\]

(25)

From Eq. (23), to implement a two-qubit local unitary operation \( e^{-i\gamma_1\sigma_z/2} \otimes e^{-i\gamma_2\sigma_x/2} \), we require that

\[
\omega_1 = \frac{\gamma_1 J}{n\pi} \cos Jt, \quad \omega_2 = \frac{\gamma_2 J}{n\pi} \cos Jt,
\]

with a time duration \( 2n\pi/J \). Note that this is an approximate solution. Numerical simulations reveal that the greater the value of \( n \), the better the approximation. To improve the accuracy, we can numerically search the variable parameters that maximize the fidelity of the actual achieved operation and the desired target operation, using parameter values given by Eq. (25) as a starting guess. This gives rise to a fidelity optimized control.

We illustrate the control strategies aforementioned with an example relevant to NMR experiments on homonuclear spin systems (16, 17). Let \( J = 200 \) Hz in Eq. (14), and our target the generation of a 90° rotation about \( x \)-axis on the first qubit, that is, the operation \( e^{-i\pi/4}x^1 \). Choosing \( \phi_1 = \phi_2 = 0, n = 1 \), the analytic approximation Eq. (24) yields

\[
\nu = 200 \text{ Hz}, \quad A = 100 \text{ Hz}, \quad T = 10\pi \text{ ms},
\]

(27)

which leads to an approximate solution:

\[
\omega_1 = 100 \cos 200t, \quad \omega_2 = 0.
\]

(28)

Numerical optimization via the maximization of the fidelity of actual achieved operation and the desired target operation, using the parameters in Eq. (27) as an initial guess, leads to the following fidelity optimized control function

\[
\omega_1 = 98.062 \cos 196.900t, \quad \omega_2 = 0,
\]

(29)

with a final time \( T = 31.911 \) ms. These two control functions, the analytic approximation and the fidelity optimized function, are shown in Fig. 1(A). The alternative minimum energy control function is determined numerically, resulting in parameters \( b = 210.744, f = -0.00549, \) and \( k = 0.00236 \), in Eq. (19), and the minimum energy control:

\[
\omega_1 = 99.678 \cos(210.744t - 0.00549, 0.00236).
\]

(30)

This minimum energy control solution is seen to be very close to the fidelity optimized control function of Eq. (29). The difference between these two control functions is plotted in Fig. 1(B).

In summary, we have presented a design of direct control pulses that implement any arbitrary single spin operation in a quantum system with an always-on interaction. This is crucial for the physical implementation of
solid-state quantum computations and for the low-power pulses design in NMR. We used an algebraic approach
to decouple the two-qubit system to overcome the difficulty of undesirable entanglement generated by the un-
tunable interaction. To generate the desired target quantum operations, we derived three control strategies, i.e.,
minimum time, minimum energy, and an approximate solution. The first two optimal strategies implement the
desired target quantum operation exactly, whereas the approximate control can be further optimized for perfect
fidelity. The advantage of the approximate control is that we can easily determine the control parameters, and it is
close to the minimum energy control.

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