Quantum Probability Estimation for Randomness with Quantum Side Information

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We develop a quantum version of the probability estimation framework [arXiv:1709.06159] for randomness generation with quantum side information. We show that most of the properties of probability estimation hold for quantum probability estimation (QPE). This includes asymptotic optimality at constant error and randomness expansion with logarithmic input entropy. QPE is implemented by constructing model-dependent quantum estimation factors (QEFs), which yield statistical confidence upper bounds on data-conditional normalized Rényi powers. This leads to conditional min-entropy estimates for randomness generation. The bounds are valid for relevant models of sequences of experimental trials without requiring independent and identical or stationary behavior. QEFs may be adapted to changing conditions during the sequence and trials can be stopped any time, such as when the results so far are satisfactory. QEFs can be constructed from entropy estimators to improve the bounds for conditional min-entropy of classical-quantum states from the entropy accumulation framework [Dupuis, Fawzi and Renner, arXiv:1607.01796]. QEFs are applicable to a larger class of models, including models permitting measurement devices with super-quantum but non-signaling behaviors and semi-device dependent models. The improved bounds are relevant for finite data or error bounds of the form $e^{-\kappa s}$, where $s$ is the number of random bits produced. We give a general construction of entropy estimators based on maximum probability estimators, which exist for many configurations. For the class of $(k,2,2)$ Bell-test configurations we provide schemas for directly optimizing QEFs to overcome the limitations of entropy-estimator-based constructions. We obtain and apply QEFs for examples involving the $(2,2,2)$ Bell-test configuration to demonstrate substantial improvements in finite-data efficiency.
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1. OVERVIEW

1.1. Introduction

For a relevant overview of the problem of device-independent randomness generation and expansion and how probability estimation (PE) solves this problem for classical side-information, see Ref. [1]. Here we establish the mathematical foundations for quantum probability estimation (QPE), which implements most of the features of PE from Ref. [1] for quantum side-information. The features implemented include: (i) Sound conditional min-entropy estimation for general models covering device-independent and device-dependent configurations without assuming stationarity or independence of trials. (ii) Forward adaptability to changing experimental conditions and the ability to stop acquiring trials early when satisfied. (iii) Asymptotically optimal rates at constant error bounds. (iv) Uncomplicated and clean exponential expansion with highly biased inputs. (v) Accessible constructions for available experimental configurations. We did not implement a generalization to “soft” estimators that would allow use of information not intended to be part of the extractor input. In addition, while we have general effective methods for PE optimization, effective methods for unrestricted QPE optimization presently exist only for special configurations, which include standard Bell-test configurations.

The first insight of the PE framework is that it is possible to directly estimate the data-dependent side information and input conditional probabilities for a sequence of trials. The estimate is a traditional statistical one, giving confidence upper bounds on the conditional probability of the data. The second insight is that these estimates can be used to estimate conditional min-entropy for use with classical-proof strong randomness extractors to produce near-uniform random bits, or directly to prove soundness of bits extracted with arbitrary strong randomness extractors. The third insight is that probability estimates can be obtained by martingale methods from probability estimation factors (PEFs) that are computed for each trial.

In the presence of quantum side information, instead of estimating conditional probabilities, we estimate conditional Rényi powers for the observed data given the inputs and the side information. The conditional Rényi powers are non-commutative generalizations of the conditional probabilities estimated in PE. Rényi entropies have played major roles in previous works showing that it is possible to generate randomness in a device-independent way
with \( E \) holding quantum side information \([2, 5]\). Most of the properties of Rényi entropies rest on properties established for Rényi powers, so estimating the latter may be viewed as more fundamental. The conditional Rényi powers are estimated via quantum estimation factors (QEFs), replacing PEFs in PE. We prove that chaining QEFs by multiplying them for a sequence of trials yields QEFs for the sequence as a whole. As a result, QEFs (more precisely, their inverses) may be seen as accumulating conditional Rényi power estimates, so the framework could alternatively be called “Rényi power accumulation”. The trials, their models and the QEFs in a chain can depend arbitrarily on data from previous trials, as a result of which it is also possible to stop trials whenever sufficient Rényi power has been accumulated. Other approaches to randomness generation have not explicitly developed these capabilities to the same extent. Because the Rényi power estimates depend on the specific data observed, they imply but are separate from any entropy estimates for the state as a whole. A main result is that like PEFs for PE, QEFs yield a conditional min-entropy estimate that can be used directly with quantum-proof strong extractors.

The conceptual principles of QPE rest on statistical estimates of probabilities rather than entropic analyses, and the proofs of the mathematical results characterizing QEFs and establishing their chainability reflect these principles. However, given the common goals of the entropy accumulation framework \([4]\) and QPE, it is not surprising that there are connections between the two. Every QEF yields an entropy estimator, which is equivalent to an instance of affine min-tradeoff functions as defined in the entropy accumulation framework. Conversely, QEFs can be constructed from entropy estimators. However, the construction is not reversible in the sense that QEFs obtained from entropy estimators belong to a restricted class of QEFs with strictly worse performance than the original QEFs from which the entropy estimator was derived. In the examples of Sect. 8.4 the performance is substantially worse.

Our construction of QEFs from entropy estimators and its consequences for conditional min-entropy estimation parallel the corresponding results in Ref. \([4]\). A corollary of our construction is an improved version of the entropy accumulation theorem (EAT, Thm. 4.4 of Ref. \([4]\)) for the case of conditional min-entropy of classical-quantum states. The EAT is formulated for quantum-quantum states, but for randomness generation there is no need to estimate conditional min-entropy for such states, so we do not pursue this generalization here. Neither do we consider extensions to estimating smooth max-entropy, which is another capability of the EAT. Unlike the original EAT, our construction leads to exponential randomness expansion without protocol complications, where the input entropy is a simple logarithm of the output entropy. We remark that there is now a refinement of the EAT which yields “second-order” improvements similar to ours and also achieves exponential randomness expansion \([6]\).

The QPE framework has more flexibility for models of the quantum side information. In particular, we can obtain randomness secure against any non-signaling devices, quantum or otherwise, provided the side information is still quantum. At the time of writing, there are few min-tradeoff functions suitable for use with the EAT. We provide a large family of entropy estimators from which QEFs can be constructed and optimized. In general, we prefer to optimize QEFs directly whenever possible, and we show that the optimization problem can be solved numerically for the important class of \((k, 2, 2)\)-Bell-test configurations.

Like entropy accumulation, QPE is asymptotically optimal at constant error bounds. This does not imply optimality for finite data, for randomness expansion, or when error bounds decrease exponentially with the randomness produced. For this regime, we do not
know what the optimal rates are, but like PE for classical side information, QPE performs substantially better than other methods developed so far for quantum side information. For this, we consider two closely related problems. Suppose we are given a model for the side information after any sequence of trials, and we anticipate a particular distribution for the results from each trial. The first problem is to determine the minimum number of trials \( n \) required to obtain \( k \) random bits at a given error bound \( \epsilon \). The second is to determine the asymptotic rate of random bits that can be produced given that the error bound is of the form \( e^{-\kappa n} \). For the EAT and QPE, the solutions of the two problems are essentially equivalent, but the second problem has the advantage of a clear asymptotic formulation not affected by finite \( n \). For \( \kappa = 0 \), the maximum rate is determined by the asymptotic equipartition property \([7]\).

The problems of the previous paragraph are motivated by relevant applications such as randomness beacons \([8]\) or low-latency randomness generation. In these cases, a fixed-size block of random bits, uniform within a given error bound, needs to be produced within a short time. This is typically far from an asymptotic regime, where the amount of randomness generated is much larger than the log-error bound and there is a long delay from protocol initiation to randomness availability. A relevant finite problem for benchmarking purposes is to produce 512 random bits certified to be within \( 2^{-64} \) of uniform. The performance of a particular protocol is determined by the resources required. We usually fix the observed trial distribution, assume that it is independent and identical, then ask for trade-off curves for the number of trials and the number of initial random bits required. The initial random bits are needed for input choices and for the extractor seed. Under many circumstances, the initial random bits may come from a public source. Here, the assumptions on the trial distribution are a completeness property, where in an ideal setup we expect to be able to configure the experiment so that overall frequencies approach the assumed ones. Soundness of the protocols does not depend on the specific distributions, only on the model.

Since the completion of this preprint, parts of this work have been published. Ref. \([9]\) covers the basic theory of QEFs for randomness generation and Ref. \([10]\) describes an experimental implementation for repeated and low-latency production of blocks of 512 random bits.

### 1.2. Summary of Main Results

The purpose of this manuscript is to provide the mathematical foundations for quantum probability estimation. The technical results in the manuscript may be difficult to interpret without having worked through the parts leading up to them. For accessibility, in this section we summarize the main results without precise definitions.

We consider systems consisting of classical variables \( C \) and \( Z \) and a quantum system containing the side information \( E \). For the present purposes, these symbols may be treated as system labels. In quantum terms, a joint state of the systems may be written as \( \rho_{CZE} = \sum_{cz} |cz\rangle\langle cz| \otimes \rho_E(cz) \) with respect to the classical basis of \( C \) and \( Z \), where \( \sum_{cz} \text{tr}(\rho_E(cz)) = 1 \). We treat \( Z \) as the input and \( C \) as the output system. In a typical Bell test, \( Z \) is the sequence of measurement settings choices (or inputs) and \( C \) is the sequence of measurement outcomes (or outputs), where the inputs and outputs may contain choices and results from multiple devices. The joint state given is the final state after the experiment, which consists of a sequence of trials generating results \( C_i Z_i \) so that \( C = (C_i)_i^1 \) and \( Z = (Z_i)_i^1 \). A model for the experiment is the set of final states that can occur and is normally constructed...
by chaining models for each trial. The models must be chained while satisfying a Markov
condition on the inputs similar to the Markov condition required for EAT channel chains [5].
To avoid the Markov condition one can drop the use of explicit inputs by including them in
$C$. For example, see Protocol [3], which requires that the conditional min-entropy witnessed
exceeds the number of bits required for the inputs. For Bell tests, the trial models are
constrained by non-signaling conditions and, for quantum devices, by the requirement that
the results can be achieved with measurements of separate quantum systems according
to the configuration. We develop a general framework for models and their construction
in Sect. 3. We explain how models capture standard configurations for device-dependent
and device-independent randomness generation in Sect. 3.5. Configurations modeled with
explicit quantum systems and quantum processes producing the data are readily accounted
for, as are scenarios where the devices may exhibit unspecified super-quantum behaviors, as
long as the side information is still quantum.

Let $\alpha > 1$ and $\beta = \alpha - 1$. Given $\rho_{CZE}$ as above, define $\rho(z) = \sum_c \rho(cz)$, where we omit
the $E$ system label when this is the only quantum system in play. For a given state $\rho_{CZE}$,
the normalized, sandwiched, conditional $\alpha$-Rényi power for value $cz$ of $CZ$ is given by

$$\hat{R}_\alpha(\rho(cz)|\rho(z)) = \frac{1}{\text{tr}(\rho(cz))} \text{tr} \left( (\rho(z)^{-\beta/(2\alpha)}\rho(cz)\rho(z)^{-\beta/(2\alpha)})^\alpha \right). \quad (1.1)$$

If $E$ is one-dimensional, then $\mu(cz) = \rho(cz)$ is a probability distribution and the conditional
Rényi power becomes $(\mu(cz)/\mu(z))^\beta$, a power of the probability of $c$ conditional on $z$. In
the probability estimation framework [1], the main goal is to estimate such conditional
probabilities. Here, the non-commutative generalization is to estimate the conditional Rényi
powers.

The success of probability estimation framework rests on the construction of probabil-
ty estimation factors (PEFs) which yield probability estimates via a martingale analysis. Quantum estimation factors (QEFs) with power $\beta$ are functions $F : cz \mapsto F(cz) \geq 0$ such
that for all states $\rho_{CZE}$ in the model, $F$ satisfies the QEF inequality

$$\sum_{cz} \text{tr}(\rho(cz)) F(cz) \hat{R}_\alpha(\rho(cz)|\rho(z)) \leq 1. \quad (1.2)$$

We do not use an explicit martingale analysis for QEFs. Instead we show directly that QEFs
for the trial models can be multiplied to yield QEFs for the sequence of trials. QEFs for later
trials may depend on data from earlier trials, so we refer to this procedure as QEF chaining.
QEFs and their variations are defined in Sect. 4.1. That they can be chained is Thm. 4.12. It
appears that the sandwiched Rényi powers are particularly well suited for chaining. We have
not succeeded in chaining other quantities that yield conditional min-entropy estimates.

The main result for QEFs is that they yield confidence upper bounds on the conditional Rényi
powers:

**Theorem.** (Thm. 4.14) If $F$ is a QEF with power $\beta$ for a model, and $\rho_{CZE}$ is a state in the
model, then $[0, 1/(\epsilon F(cz))]$ is a significance-level $\epsilon$ confidence interval for $\hat{R}_\alpha(\rho(cz)|\rho(z))$
with respect to the probability distribution $\text{tr}(\rho(cz))$ induced on $CZ$ by $\rho_{CZE}$.

For randomness generation, QEFs are used to estimate conditional min-entropy with an
error bound. If the estimate is larger than a protocol threshold, a quantum-proof strong
extractor can be applied to the outputs to obtain a string of nearly uniform random bits.
The number of bits is somewhat less than the estimate in order to take into account extractor constraints. Let $H_{\infty}(C|ZE, \Phi')$ denote the smooth quantum conditional min-entropy for the state of CZE conditional on the event $\Phi'$ defined as a set of values $cz$ of CZ. The smoothness parameter $\epsilon$ is an error bound that chains directly with error bounds of extractors. It is defined with respect to purified distance, but may be interpreted as total variation distance for chaining with protocols whose error bounds use the latter distance. The conditional Rényi power estimate provided by a QEF implies a conditional min-entropy estimate suitable for randomness generation protocols:

**Theorem.** (Thm. 4.18) Suppose that $F$ is a QEF with power $\beta$ for a model, and $pc_{CZE}$ is a state in the model. Fix $1 \geq p > 0$ and $\epsilon > 0$ and write $\Phi = \{cz : F(cz) \geq 1/(p^3(\epsilon^2/2))\}$. Let $\Phi' \subseteq \Phi$ and let $\kappa = \sum_{cz \in \Phi'} \text{tr}(\rho(cz))$ be the probability of the event $\Phi'$ according to the state. Then $H_{\infty}(C|ZE, \Phi') \geq -\log(p) + \frac{\alpha}{\beta} \log(\kappa)$.

Here we used the convention $\log(0) = -\infty$. We formulated the theorem to parallel the statements of the EAT and the propositions that lead to the EAT in Ref. [4]. If CZ is generated by a sequence of trials chained with identical models and $F$ is obtained by multiplying identical trial-wise QEFs $F_0$, then we can define a rate $h$ by $h = -\log(p)/n$. The event $\Phi$ can alternatively be expressed as $\Phi = \{cz : \frac{\log(F_0(cz))}{\beta} \geq nh - 2\log(\epsilon/\sqrt{2})/\beta\}$. This identifies $h$ as the targeted conditional min-entropy rate, and we can interpret $\frac{\log(F_0(cz))}{\beta}$ as the trial-wise contributions to the final conditional min-entropy. When configuring an experiment, the goal is therefore to maximize the expected values of $\log(F_0(cz))/\beta$.

Comparing the bounds to the corresponding ones for PEFs in Ref. [1], the main difference is the change in the threshold requirement replacing the term $\epsilon$ by $\epsilon^2/2$. An interpretation is that for the same witnessed rate and for a positive conditional min-entropy bound, twice as many trials are required to satisfy the error bound with quantum side information than with classical side information. A similar phenomenon occurs when comparing parameters of quantum-proof to classical-proof strong extractors, for example, see Ref. [11].

The QPE framework was motivated and developed as a generalization of the PE framework [1] to quantum side-information, which in turn arose from a program [12, 13] for randomness generation based on test supermartingales [14] constructed from trial-wise test factors [15]. This led to the development of conditional Rényi power estimates. To obtain conditional min-entropy estimates suitable for randomness generation we take advantage of the connection between Rényi relative entropy and conditional min-entropy [7], which is also used to prove the EAT from its prequel.

Explicit protocols for randomness generation that compose the conditional min-entropy estimate with quantum-proof randomness extractors are given in Sect. [5]. For the soundness of the protocols, the power $\beta$, the smoothness $\epsilon$ and the target entropy $-\log(p)$ must be chosen before the protocol, in particular before or at least independently of the data being generated by the experiment. For QEFs, it is possible to optimize and update trial-wise QEFs (with $\beta$ fixed in advance) before each trial, but after the data is obtained no further optimization is possible. These considerations apply to all randomness generation protocols. For example, to apply the EAT, the number of trials, the target conditional min-entropy rate $h$ and the affine min-tradeoff function are fixed before the protocol and temptation to optimize them after the protocol in view of the trial results must be resisted.

In Ref. [1], effective algorithms for optimizing PEFs are described and implemented. We do not have such algorithms for QEFs but offer two general theoretical constructions and a schema for optimizing QEFs for Bell-test configurations with two input choices and two
possible outputs for each station. The first construction is based on a relationship between QEFs and entropy estimators. The function $K : cz \mapsto K(cz) \in \mathbb{R}$ is an entropy estimator for a model if for all states $\rho_{CZE}$ of the model,

$$\sum_{cz} K(cz) \text{tr}(\rho(cz)) \leq H_1(C|ZE),$$

(1.3)

where $H_1(C|ZE)$ is the quantum conditional entropy of the state. Every QEF yields an entropy estimator.

**Theorem.** (Thm. 6.2) Suppose that $F$ is a QEF with power $\beta$ for a model. Then $K : cz \mapsto \log(F(cz))/\beta$ is an entropy estimator for the model.

In the examples of Sect. 8.4, the entropy estimators so obtained can have comparable performance to existing min-tradeoff functions when used with EAT, but only at small powers. We infer that the QEF and entropy-estimator or min-tradeoff-function optimization problems are not well matched.

It is possible to obtain QEFs from entropy estimators:

**Theorem.** (Thm. 6.3) Let $K$ be an entropy estimator for a model. Then there exists $\tilde{c} : \beta \in (0, 1/2] \mapsto \tilde{c}(\beta) \in (0, u]$ such that $F : cz \mapsto e^{\beta K(cz)}/(1 + \tilde{c}(\beta)\beta^2/2)$ is a QEF with power $\beta$ for the model. The upper bound $u$ depends on the model and the image of $K$.

The QEFs so obtained belong to the special class of Petz QEFs (QEFPs). Because the construction is essentially model-agnostic, it does not yield optimal QEFs. In particular, the strategy of optimizing entropy estimators and then determining QEFs accordingly does not yield good QEFs for finite data. A function $\tilde{c}$ is explicitly obtained in Thm. 6.3. This theorem can substitute for the EAT prequel, Prop. 4.5 of Ref. [4] to obtain improvements on the EAT bounds for conditional min-entropy. (Similar improvements are also obtained in Ref. [6].) For this we optimize $\beta$ given the number of trials and a targeted conditional min-entropy rate, see the handicapped comparison in Sect. 6.3. We also include examples that demonstrates the broad applicability of QEFs and the significant improvements achievable by direct QEF construction, see Sect. 8.4.

The connection between entropy estimators and QEFPs relies on a Rényi relative entropy bounding technique from Ref. [7] that is also used for the connection between Rényi relative entropy and min-tradeoff functions that is needed for the proof of the EAT in Ref. [4]. This suggests the view that the EAT fundamentally rests on QPE via QEFPs. Our work makes this connection explicit, thereby enabling extensions, improvements and broader applicability of the results.

An application of entropy estimators and their QEFPs is a proof that asymptotically optimal conditional min-entropy rates are achieved with QEFPs. As suggested in Ref. [16], this follows from the quantum asymptotic equipartition property [7]. We provide the necessary convexity arguments to determine entropy estimators that witness achievability of optimal rates.

To remedy the lack of availability of general entropy estimators, we show how entropy estimators can be obtained from max-prob estimators. The function $B : cz \mapsto B(cz) \in \mathbb{R}$ is a max-prob estimator for a model if for all states $\rho_{CZE}$ of the model, $\sum_{cz} \text{tr}(\rho(cz))B(cz) \geq \max_{cz}(\text{tr}(\rho(cz))/\text{tr}(\rho(z)))$. Note that the definition depends only on the classical probability distributions of $CZ$ that are allowed by the model and can therefore be designed for general
non-signaling distributions. In particular, it is of foundational interest that they can be used for sound and complete randomness generation assuming only non-signaling constraints on the experimental devices, which may have super-quantum capabilities. However, if super-quantum devices are reused in subsequent protocols, composability may be compromised in ways that are not accounted for by a quantum analysis.

Max-prob estimators are used in probability estimation to directly construct PEFs for exponential randomness expansion. Non-trivial max-prob estimators exist for Bell-test configurations. For QEFs, the direct construction from max-prob estimators fails, but it is possible to obtain entropy estimators by a similar method. The QEFPs then derived from these entropy estimators can be used for exponential randomness expansion.

**Theorem.** (Thm. 7.8 and its proof) Suppose that $B$ is a max-prob estimator for a trial model with $Z$ uniformly distributed such that there exists $\rho_{CZE}$ in the model satisfying
\[ \sum_{cz} \text{tr}(\rho(cz))B(cz) < 1. \]
Then there is a configuration with highly biased probability distributions of independent and identical trial inputs and QEFPs for this configuration such that for $n$ trials, the conditional min-entropy witnessed is at least $ng$ and the input entropy is $\log(n)g'$ for some constants $g, g' > 0$. The bias of the input distribution depends on $n$.

In Sect. 8 we consider the standard $(k, 2, 2)$-Bell-test configurations involving $k$ stations, two input choices at each station and two possible outputs for each input. It is well-known that the quantum devices in such configurations can be reduced to devices measuring one qubit in each station. For $k = 2$ the reduction is well explained in [17], Sect. 2.4.1, where the main mathematical results needed are from Ref. [18] and Ref. [19]. We establish a general form of this observation for arbitrary $k$ and suitable for use with QEF optimization. As a result, the QEF optimization problem for $(k, 2, 2)$-Bell-test configurations can be effectively solved by numerical methods, after exploiting concavity and convexity properties of the relevant quantities.

Finally, in Sect. 8.4 we construct QEFs from PEFs for examples involving $(2, 2, 2)$-Bell-test configurations. We apply QEFs to the data from the first demonstration of certified conditional min-entropy with respect to classical side information [20]. Our analysis shows that QEFs would have yielded more bits while being secure against quantum side information. To illustrate the excellent finite-data performance of QEFs, we consider the minimum number of trials required for three families of standard quantum states of the devices to show orders of magnitude improvement over EAT. We highlight the improvement by determining the number of trials required for the reference example of 512 bits with error bound $2^{-64}$ with the distributions observed in the loophole-free Bell test used previously for randomness generation with classical side-information in Ref. [13].

## 2. PRELIMINARIES

### 2.1. Basics

Let $\mathcal{H}$ be a finite dimensional Hilbert space. $B(\mathcal{H})$ is the set of operators on $\mathcal{H}$, $A(\mathcal{H})$ the subset of self-adjoint (equivalently, Hermitian) operators, $S(\mathcal{H})$ the subset of Hermitian, positive semidefinite operators, $S_1(\mathcal{H}) = \{ A \in S(\mathcal{H}) : \text{tr}(A) = 1 \}$ the set of density operators, and $S_{\leq 1}(\mathcal{H}) = \{ A \in S(\mathcal{H}) : \text{tr}(A) \leq 1 \}$. For vectors $|\psi\rangle \in \mathcal{H}$, we abbreviate $\hat{\psi} = |\psi\rangle\langle\psi|$. If $|\psi\rangle$ is normalized, then $\hat{\psi}$ is the projector onto the one-dimensional subspace
spanned by $|\psi\rangle$. For $\sigma \in B(H)$ we write $\sigma \geq 0$ if $\sigma \in S(H)$. The comparison $\sigma \geq \tau$ is equivalent to $\sigma - \tau \geq 0$. For $\sigma \in A(H)$, the support of $\sigma$ is the span of the eigenvectors of $\sigma$ with non-zero eigenvalues. The support of $\sigma$ is denoted by Supp($\sigma$), and the projector onto the support of $\sigma$ is denoted by $[\sigma \neq 0]$. For $\sigma, \tau \in S(H)$, we write $\sigma \ll \tau$ if Supp($\sigma$) $\subseteq$ Supp($\tau$). Equivalently, $\sigma \ll \tau$ iff there exists $\lambda > 0$ such that $\sigma < \lambda \tau$. For Hermitian $\sigma$, the spectrum Spec($\sigma$) is the family of eigenvalues of $\sigma$ accounting for multiplicity. To be specific, we treat the spectrum as a vector of real numbers in descending order. We use the fact that Spec($A^\dagger A$) $=$ Spec($AA^\dagger$). For $\sigma \in A(H)$ without full support, we define $\sigma^{-1}$ as the relative inverse. That is, given a spectral decomposition of $\sigma$ in the form $\sigma = \sum_j \lambda_j \hat{j}$ with $\text{tr}(\hat{j} \hat{i}) = \delta_{i,j}$ and $\lambda_j \neq 0$, we have $\sigma^{-1} = \sum_j \lambda_j^{-1} \hat{j}$. If $\Pi$ is the projector onto the support of $\sigma$, then $\sigma \sigma^{-1} = \Pi \sigma^{-1} \Pi = \Pi$ and Supp$(\sigma^{-1})$ $=$ Supp($\sigma$). We define $\log(\sigma)$ in the same relative way. For $\sigma \in A(H)$ with spectral decomposition $\sigma = \sum_j \lambda_j \hat{j}$, the positive part of $\sigma$ is defined as $[\sigma]_+ = \sum_{j, \lambda_j > 0} \lambda_j \hat{j}$. The absolute value is $|\sigma| = \sum_j |\lambda_j| \hat{j} = [\sigma]_+ + [-\sigma]_+$. For arbitrary $\sigma \in B(H)$, define $|\sigma| = \sqrt{\sigma^\dagger \sigma}$. The projector onto the support of $[\sigma]_+$ is denoted by $[\sigma > 0]$. We need two properties of positive parts:

**Lemma 2.1.** $\text{tr}([\sigma]_+)$ is monotone in $\sigma$, and for $\sigma \geq 0$, $\tau \geq 0$ we have $\text{tr}(\sigma [\sigma - \tau > 0]) \geq \text{tr}([\sigma - \tau]_+)$. 

*Proof.* Since $[\sigma]_+ = f(\sigma)$ with $f(x) = (|x| + x)/2$ and $f$ is continuous and monotone increasing, $\text{tr}([\sigma]_+)$ is monotone in $\sigma$ according to Ref. [21], Thm. 2.10. Let $|i\rangle$ be an orthonormal basis of eigenvectors of $\sigma - \tau$ with $(\sigma - \tau)|i\rangle = \lambda_i |i\rangle$. Write $\sigma_{ii} = \text{tr}(\sigma^i \hat{i})$ and $\tau_{ii} = \text{tr}(\tau^i \hat{i})$. Then

$$\begin{align*}
\text{tr}([\sigma - \tau]_+) &= \sum_{i: \lambda_i > 0} \sigma_{ii} - \tau_{ii} \\
&\leq \sum_{i: \lambda_i > 0} \sigma_{ii} \\
&= \sum_{i: \lambda_i > 0} \text{tr}(\sigma^i \hat{i}) \\
&= \text{tr} \left( \sigma \sum_{i: \lambda_i > 0} \hat{i} \right) \\
&= \text{tr}(\sigma [\sigma - \tau > 0]).
\end{align*}$$

(2.1)

\[\square\]

A linear map $\mathcal{E} : B(H) \rightarrow B(H')$ is positive if $\mathcal{E}(S(H)) \subseteq S(H')$. The map $\mathcal{E}$ is a pure completely positive map (pCP map) if it is of the form $\mathcal{E}(\rho) = A\rho A^\dagger$ for some $A \in B(H)$. A completely positive map (CP map) is a positive linear combination of pCP maps. A CP map $\mathcal{E} : B(H) \rightarrow B(H')$ can be expressed non-uniquely in the form $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$. $\mathcal{E}$ is trace-preserving if $\text{tr}(\mathcal{E}(\rho)) = \text{tr}(\rho)$ or equivalently, $\sum_i A_i^\dagger A_i = 1$. A quantum operation is a CP map that is trace-preserving. Quantum operations are also referred to as CPTP maps.

For $n \in \mathbb{N}$, $[n] = \{k \in \mathbb{N} : 1 \leq k \leq n\}$. For maps $f : X \rightarrow Y$, we extend $f$ to subsets $\mathcal{X}$ of $X$ according to $f(\mathcal{X}) = \{f(x) : x \in \mathcal{X}\}$. For a formula $\phi$ with free variables, the expression
The values of the variables where \( \phi \) is true and \([\phi] = 1\) for values of the variables where \( \phi \) is false and \([\phi] = 0\) otherwise. There should be no confusion with the case where \([\ldots]\) is applied to a comparison of a given Hermitian operator and a real number to define a projector.

A subset \( \mathcal{C} \) of a vector space is convex if \( \sum_{i=1}^{k} \lambda_i c_i \in \mathcal{C} \) whenever \( c_i \in \mathcal{C} \), \( \lambda_i \geq 0 \) for all \( i \in [k] \) and \( \sum_{i=1}^{k} \lambda_i = 1 \). Vectors \( \sum_{i=1}^{k} \lambda_i c_i \) with \( \lambda_i \geq 0 \) and \( \sum_{i=1}^{k} \lambda_i = 1 \) are referred to as convex combinations of the \( c_i \). For any \( \mathcal{C} \), the convex closure \( \text{Cvx}(\mathcal{C}) \) of \( \mathcal{C} \) is the set of all convex combinations of members of \( \mathcal{C} \). We write \( \text{Cone}(\mathcal{C}) = [0, \infty)\text{Cvx}(\mathcal{C}) \) for the convex cone generated by \( \mathcal{C} \). The set of extreme points of \( \mathcal{C} \) is denoted by \( \text{Extr}(\mathcal{C}) \).

### 2.2. Systems

We distinguish between systems and their state spaces. We denote and label quantum systems with \( A, B, \ldots, E, \ldots, U, V, W, X, Y, Z \). In this work, \( E \) plays a distinguished role as a universal quantum system for defining models or as the system carrying the quantum state information. We often use \( U, V, W \) to denote generic quantum systems. For a quantum system \( U \), its Hilbert space is \( \mathcal{H}(U) \) with dimension \( \dim(U) \). \( \mathcal{S}(U) \) is the set of positive semidefinite operators on \( \mathcal{H}(U) \), and \( \mathcal{S}_1(U) = \{ \rho \in \mathcal{S}(U) : \text{tr}(\rho) = 1 \} \) is the set of density operators of \( U \). Members of \( \mathcal{S}_1(U) \) are referred to as the states of \( U \). States \( \rho \) are considered to be normalized by the condition \( \text{tr}(\rho) = 1 \), and general members of \( \mathcal{S}(U) \) are referred to as unnormalized states. We abbreviate \( \mathcal{B}(\mathcal{H}(U)) = \mathcal{B}(U) \) and \( \mathcal{A}(\mathcal{H}(U)) = \mathcal{A}(U) \). If \( \dim(U) = 1 \), we call \( U \) trivial and \( \mathcal{S}(U) = [0, \infty) \). The set of systems in play has a joint state. We use juxtaposition to combine systems, so \( UV \) combines systems \( U \) and \( V \). Its Hilbert space is \( \mathcal{H}(UV) = \mathcal{H}(U) \otimes \mathcal{H}(V) \).

We need to refer to subsystem factorizations of quantum state spaces. For a Hilbert space \( \mathcal{H} \), a factorization of \( \mathcal{H} \) is a representation of \( \mathcal{H} \) in the form \( \mathcal{H} = \bigoplus_k \mathcal{H}_k \otimes \mathcal{C}_k \oplus \mathcal{R} \). Technically, such factorizations are realized by an isomorphism, but we freely identify the two sides without making this isomorphism explicit. Given this factorization, states of \( \mathcal{H}_k \otimes \mathcal{C}_k \) are also states of \( \mathcal{H} \), and we construct unnormalized states of the form \( \sigma_k \otimes \tau_k \) accordingly with \( \sigma_k \in \mathcal{S}(\mathcal{H}_k) \) and \( \tau_k \in \mathcal{S}(\mathcal{C}_k) \). The state space membership may be left implicit when the factorization is clear and the index sets match, here by using the same index-symbol \( k \) with implicit index set \( K \).

We identify classical systems with classical variables (CVs). Notationally and operationally we treat CVs as random variables (RVs) without specified probability distributions. CVs are denoted by capital letters \( A, B, C, \ldots, U, V, W, X, Y, Z, \Omega \). In this work, \( A, B, C, X, Y, Z \) play a distinguished role, and \( U, V, W \) are often used as generic CVs. Like RVs, as mathematical objects CVs are functions from an underlying set \( \Omega \), which we assume is finite. Accordingly, a CV \( U \) has an associated space of values denoted by \( \text{Rng}(U) \). Values of CVs are denoted by the corresponding lower case letter. Thus the symbol \( u \) denotes values of \( U \). This implies that in a CV context, the symbol \( u \) is typed and always refers to a member of \( \text{Rng}(U) \). This simplifies notation. For example, \( \sum u \ldots = \sum_{u \in \text{Rng}(U)} u \ldots \) and \( \{ u : \ldots \} = \{ u \in \text{Rng}(U) : \ldots \} \). If we need distinct symbols of this type we use primed symbols such as \( u' \) or explicitly specify the symbols' membership. In a context where a CV \( U \) has an associated state, possibly joint with other CVs and quantum systems, we refer to the process of obtaining a value \( u \) of \( U \) as instantiating \( U \), with the connotation that the value was not available for inspection before it was instantiated.
The CV $U$ is trivial if $|\text{Rng}(U)| = 1$. We freely construct CVs by concatenation denoted by juxtaposition. For example, if $U$ and $V$ are CVs, then $UV$ is a CV with values $uv$. If $u$ and $v$ are strings or sequences, then $uv$ is the concatenation of the two strings or sequences. Otherwise, $uv$ may be interpreted as the pair or two-element sequence with first element $u$ and second element $v$. Any of the typical mathematical realizations of these concepts may be used.

The CV $F$ is determined by the CV $U$ if for some function $F$ on $\text{Rng}(U)$, for all $\omega \in \Omega$, $F(\omega) = F(U(\omega))$. We introduce such determined CVs as $F(U)$, which specifies that $F$ is a CV determined by $U$ as well as a function $u \mapsto F(u)$. This overloads the symbol $F$. Its meaning is determined by the type of the argument. The special expression $F(U)$ may be considered to refer to both meanings while emphasizing the type of the argument of $F$ as a value of $U$. Thus, given an expression $F(u)$, we may define $F(U)$ by specifying a function $F : u \mapsto F(u)$ and call $F(U)$ a function of $U$, or we may specify $F(U)$ by an identity of the form $F(U) = F(U)$, which we also consider equivalent to the statement $\forall u : F(u) = F(u)$. We remark that in expressions such as $F(U)$ or $F(U) = F(U)$, the symbol $U$ plays the role of a free variable with arbitrary values in $\text{Rng}(U)$. We may introduce objects such as $\rho(U)$ that are primarily functions of CV values and not intended to be interpreted as determined by CVs themselves.

When considering sequences of trials for randomness generation, the final state involves a CV consisting of a sequence of individual trial CVs. We use boldface to distinguish such CVs. A sequence CV $U$ is defined in terms of the trial CVs $U_i$ by $U = U_1U_2\ldots U_N$ and has values $u = u_1u_2\ldots u_N$. Here, $N$ is an absolute upper bound on the number of trials that might be considered before a protocol stops. We always assume that such an upper bound exists. The actual number of trials considered is denoted by $n$. To refer to initial and final segments of $U$ we use the notation $U_{\leq k} = U_1\ldots U_k$ and similarly for $U_{< k}$, $U_{\geq k}$ and $U_{> k}$. The length of $U$ is denoted by $|U|$. Similarly, if $U$ is a string, the number of letters in $U$ is denoted by $|U| = \log_l(|\text{Rng}(U)|)$, where $l$ is the size of the alphabet of the string. We may treat string CVs as sequence CVs without using the explicit boldface.

A CV’s state is a probability distribution on its values. $S(U)$ is the set of unnormalized, non-negative distributions on $U$, and $S_1(U)$ is the set of probability distributions on $U$. If $U$ is a CV, then $U$ is its quantization. The Hilbert space of $U$ has a classical basis whose members are $|u\rangle$. Probability distributions $\mu(U)$ of $U$ are associated with the corresponding states $\sum_u \mu(u)\hat{u}$ diagonal in the classical basis. Probabilities and expectations with respect to the probability distribution $\mu(U)$ are expressed as $P_{\mu(U)}(\phi) = \sum_u \mu(u)\langle \phi | \hat{u} \rangle$ and $E_{\mu(U)}(G(U)) = \sum_u \mu(u)G(u)$.

### 2.3. Classical-Quantum States

We study joint states of classical-quantum systems. For a CV $U$ and a quantum system $V$, $UV$ is the joint system. We define the set of $S(V)$-valued distributions of $U$ as

$$S(UV) = \{ \rho : u \mapsto \rho(u) \in S(V) \}.$$  \hspace{1cm} (2.2)

The members of $S(UV)$ may be considered as CVs with values in $S(V)$, so we denote these members by $\rho(U)$. If $V$ is clear from context or generic, we refer to $\rho(U)$ as a state-valued distribution, or just a distribution of $U$ or a state of $UV$, although the values are unnormalized states of $V$. 
For the purpose of universality, we may consider $V$ with infinite-dimensional $H(V)$. However, by default we assume that the values $\rho(u)$ of distributions are finite rank. A $S(V)$-valued distribution $\rho(U)$ is normalized if $\text{tr}(\sum_u \rho(u)) = 1$. The set of normalized distributions of $U$ is denoted by $S_1(\text{UV})$. The set of sub-normalized distributions is $S_{\leq 1}(\text{UV}) = \{\rho(U) \in S(\text{UV}) : \text{tr}(\sum_u \rho(u)) \leq 1\}$. The set $S_1(\text{UV})$ is the set of states of UV. For finite-dimensional $H(V)$, it is consistent with the conventional, quantized definition of the set of classical-quantum states of UV as the set of density operators of the form $\sum_u \hat{u} \otimes \rho(u)$. If $V$ is trivial and $\rho(U)$ is normalized, then $\rho(U)$ is a probability distribution. Our notational choices are designed to be compatible with those in Ref. [1] when specialized to trivial $V$ for handling classical side information. We use symbols such as $\rho, \sigma, \tau, \chi, \zeta, \xi$ for general states and $\mu, \nu$ for probability distributions.

In this work we normally consider finite CVs and density operators with finite support. The soundness of randomness generation protocols is relative to a model, which is a set of state-valued distributions, see Sect. 3. Some models are most conveniently formulated with states in an infinite-dimensional Hilbert space, but we define them so that the relevant state-valued distributions have finite support in the Hilbert space. The support of a distribution $\rho(U)$ is the linear span of the supports of the $\rho(u)$. The projector onto the support is the smallest projector $\Pi$ such that for all $u$, $\Pi \rho(u) = \rho(u)$. While the technical arguments are restricted to effectively finite dimensional situations, in most cases the consequences for randomness generation extend to countable-dimension side information. To verify this requires approximating a model’s infinite-support trace-class states by model states with finite-dimensional support.

A positive map $\mathcal{E} : B(H(V)) \to B(H(W))$ induces a map $S(\text{UV}) \to S(\text{VW})$ defined by $\rho(U) \mapsto (\mathcal{E} \circ \rho)(U) = \mathcal{E}(\rho(U))$. If $\mathcal{E}$ is trace-preserving, then the map restricts to $S_1(\text{UV}) \to S_1(\text{UW})$.

We adapt RV and probability distribution conventions to denote and manipulate state-valued distributions. If $\rho(\text{UV})$ is a $S(\text{W})$-valued distribution, then $\rho(\text{uw})$ refers to the value of the distribution at $\text{uw}$. According to marginalization conventions, $\rho(U)$ is the marginal state-valued distribution of $U$ and defined as $\rho(U) = \sum_v \rho(Uv)$. With this, $\rho() = \sum_{uv} \rho(\text{uw})$ is the marginal state of $\text{W}$. We abbreviate $\rho = \rho()$ whenever the meaning is clear from context. Conventions for events apply: If $\mathcal{X}, \mathcal{Y} \subseteq \text{Rng}(\text{UV})$, then $\rho(\mathcal{X}) = \sum_{uv \in \mathcal{X}} \rho(uv)$ and $\rho(\mathcal{X} \cap \mathcal{Y}) = \rho(\mathcal{X} \cap \mathcal{Y})$. We can specify subsets using logical expressions in the CVs. If $\phi(U,V)$ is such a logical formula with free variables $U$ and $V$, we define $\{\phi\} = \{\phi(U,V)\} = \{uv : \phi(u,v)\}$. In arguments of a distribution, the curly brackets are normally omitted. With this, we have the identities $\rho(u) = \rho(U = u) = \rho(\{U = u\})$. Thus, our conventions imply that the expression $\rho(V,U = u)$ defines a distribution $\sigma(V)$ depending on $V$ only, but since this can be confusing we circumvent such expressions whenever possible.

We also adapt the usual conventions for conditioning. We define conditioning on a CV event according to the states obtained conditionally on observing the event. If $\rho(\text{UV}) \in S(\text{UVW})$ and $\phi(U,V)$ is a formula with free variables $U$ and $V$, then $\rho(\text{UV}|\phi) = [\phi(U,V)] \rho(\text{UV})/\text{tr}(\rho(\phi))$. We define $\rho(\text{uw}|\phi) = 0$ if $\text{tr}(\rho(\phi)) = 0$. Note that if $\text{tr}(\rho(\phi)) \neq 0$, then $\text{tr}(\rho(\text{UV}|\phi)) = 1$ and therefore $\rho(UV|\phi) \in S_1(\text{UV})$. In view of conventions for point events, the expression $\rho(U|v)$ is interpreted as $\rho(U|v) = (u \mapsto \rho(u|V = v) = \rho(uv)/\text{tr}(\rho(\text{uw})))$.

For chaining purposes, we distinguish distributions $\rho(\text{UV})$ for which $\rho(\text{UV}) = \mu(V)\rho(U)$ for a probability distribution $\mu(V)$. In this case $\rho(U|v) = \rho(U)$ is independent of $v$. We define $S(\{U|V\}E) = \{\tau(\text{UV}) : \tau(U|v) \text{ is independent of } v\}$. Members of this set of distributions
may be written as \( \sigma(U|V) \in \mathcal{S}(U|V) \), the idea being that up to normalization, \( \sigma(U|V) \) could have been obtained by conditioning on \( V \), where \( V \) is independent of \( E \). In this situation \( \sigma(UV) \) is unspecified until we provide the probability distribution \( \mu(V) \), at which point we can define \( \sigma(UV) = \mu(V) \sigma(U|V) \).

If \( \rho(X) \in \mathcal{S}(XU) \) and \( \rho(Y) \in \mathcal{S}(YV) \), then \( \rho(X) \otimes \rho(Y) \in \mathcal{S}(XYUV) \). If \( F(U) \) is a function of \( U \), then \( F \) pushes distributions forward according to \( (F \ast \rho)(f) = \rho(F(U) = f) \).

For clarity, the marginalization conventions do not apply when distributions are expressed in terms of compound constructions such as \( E(\rho(UV)) \), \( P(\rho(X)) \) or \( \mathcal{E}(\rho(U);\ldots) \) without an explicit final CV argument of the form \( \ldots(UV\ldots) \). The CV arguments of the proper construction are bound variables and not intended to be substituted by values. The construction’s expression refers to a distribution with CVs determined by the specific expression.

We occasionally define state-valued distributions using anonymous mapping notation, which includes the equivalence \( \rho(UV) = (uv \mapsto \rho(uv)) \). For example, the expression \( u \mapsto \rho/|\text{Rng}(U)| \) defines the uniform distribution on \( U \) independent of \( E \) with the reduced density matrix of \( E \) the state \( \rho \). In quantized terms this is the joint state \( 1_U/|\text{Rng}(U)| \otimes \rho \), a notation with similar complexity. The uniform probability distribution of \( V \) is defined as \( \text{Unif}(V) : v \mapsto 1/|\text{Rng}(V)| \) or equivalently \( \text{Unif}(V) = (v \mapsto 1/|\text{Rng}(V)|) \). Here, the quantum system is trivial.

We define POVMs of \( V \) with outcomes \( U \) as linear maps \( P : \mathcal{S}(V) \to \mathcal{S}(U) \) of the form \( P(\rho)(U) = \text{tr}(P_u \rho) \) with \( P_u \in \mathcal{S}(V) \) for all \( u \) and \( \sum_u P_u = 1_V \). Without confusion and following tradition, we refer to families of operators \( P_U = (P_{u})_u \) satisfying these conditions as POVMs. The term “POVM” is an abbreviation for “positive, operator-valued measure”.

We can naturally apply \( P \) to members of \( \mathcal{S}(XVW) \) by defining \( P(\rho(X))(XU) \in \mathcal{S}(XUW) \) according to

\[
P(\rho(X))(xu) = \text{tr}_V((P_u \otimes 1_W)\rho(x)). \tag{2.3}
\]

POVMs defined in this way remove the quantum system being measured. POVMs do not specify what happens to the measured system, so if we want to retain the measured system, we need to consider quantum operations with classical outputs.

For the purpose of explicit conditioning on inputs, we make use of the concept of short quantum Markov chains \([22]\). We define these chains for the class of states used here. For the general definition, see the references.

**Definition 2.2.** The distribution \( \rho(UVW) \in \mathcal{S}(UVWE) \) is a short quantum Markov chain over \( WE \), written as \( \rho(UVW) \in U \leftrightarrow WE \leftrightarrow V \), if for all \( w \), there is a factorization \( \mathcal{H}(E) = \bigoplus_k \mathcal{U}_{w,k} \otimes \mathcal{V}_{w,k} \oplus \mathcal{R} \) such that \( \rho(UW) = \bigoplus_k \sigma_{w,k}(U) \otimes \tau_{w,k}(V) \).

The definition is symmetric in \( U \) and \( V \). That is, \( \rho(UVW) \in U \leftrightarrow WE \leftrightarrow V \) iff \( \rho(UVW) \in V \leftrightarrow WE \leftrightarrow U \).

### 2.4. Distances

We use the half trace distance as the extension of total variation (TV) distance from probability distributions to states for compatibility with classical protocols and conventions. Purified distance is more natural when dealing with quantum side information, partly because it is well-behaved with respect to extension to previously traced-out quantum systems, see Ref. \([23]\), Cor. 3.6, Pg. 52. Since purified distance is an upper bound on half trace distance, this usually does not complicate comparisons.
Lemma 2.4. Let \( \rho(U), \sigma(U) \in S_1(UW) \). Then

\[
\text{TV}(\rho(U), \sigma(U)) = \sum_u \text{tr}([\rho(u) - \sigma(u)]_+).
\] (2.5)

Proof. In general \(|\xi - \chi| = [\xi - \chi]_+ + [\chi - \xi]_+\) and

\[
\text{tr}(\xi) - \text{tr}(\chi) = \text{tr}(\xi - \chi) = \text{tr}([\xi - \chi]_+) - \text{tr}([\chi - \xi]_+).
\]

Since \(\sum_u \text{tr}(\rho(u)) = \sum_u \text{tr}(\sigma(u))\), we find that

\[
\sum_u \text{tr}([\rho(u) - \sigma(u)]_+) = \sum_u \text{tr}([\sigma(u) - \rho(u)]_+)
\]

and

\[
\text{TV}(\rho(U), \sigma(U)) = \sum_u \frac{1}{2} \text{tr}([\rho(u) - \sigma(u)]_+) = \sum_u \frac{1}{2} \text{tr}([\rho(u) - \sigma(u)]_+) + \sum_u \frac{1}{2} \text{tr}([\sigma(u) - \rho(u)]_+)
\]

\[
= \sum_u \text{tr}([\rho(u) - \sigma(u)]_+).
\]

\[
\text{TV}(\rho(U), \sigma(U)) = \sum_u \frac{1}{2} \text{tr}([\rho(u) - \sigma(u)]_+) + \frac{1}{2} \text{tr}([\sigma(u) - \rho(u)]_+).
\] (2.6)

Lemma 2.5. Let \( \rho(U), \sigma(U) \in S_1(UW) \). If there exists \( \tau(U) \in S(UW) \) with \( \text{tr}(\tau) \geq 1 - \epsilon \), \( \tau(U) \leq \rho(U) \) and \( \tau(U) \leq \sigma(U) \), then \( \text{TV}(\rho(U), \sigma(U)) \leq \epsilon \).

Proof. Suppose that \( \tau(U) \) has the given properties. Then the TV distance is

\[
\text{TV}(\rho(U), \sigma(U)) = \sum_u \text{tr}([\rho(u) - \sigma(u)]_+)
\]

\[
= \sum_u \text{tr}([\rho(u) - \tau(u)]_+) + \sum_u \text{tr}([\sigma(u) - \tau(u)]_+).
\]
Proof. The first statement follows from property (v) of Tbl. 3.1, Pg. 48, and the second from Prop. 3.3, Pg. 50 of Ref. [23], in view of the two remarks after Defs. 2.3 and 2.7. □

Lemma 2.6. Let $\tau(UV) \in S_{\leq 1}(UVW)$ and $\sigma(V) \in S_{1}(VW)$ with $\tau(UV) \leq p\sigma(V)$ and $p|\text{Rng}(U)| \geq 1$. Then there exists $\rho(UV) \in S_{1}(UVW)$ such that $\tau(UV) \leq \rho(UV) \leq p\sigma(V)$.

Proof. Let $\epsilon = 1 - \text{tr}(\tau)$ and $\delta = \sum_{uv} \text{tr}(p\sigma(u) - \tau(uv)) = |\text{Rng}(U)|p - (1 - \epsilon) \geq \epsilon$. Let $\xi(UV) = (\epsilon/\delta)(p\sigma(V) - \tau(UV))$. Then $\text{tr}(\xi) = \epsilon$ and $0 \leq \xi(UV) \leq p\sigma(V) - \tau(UV)$. Define $\rho(UV) = \tau(UV) + \xi(UV)$. Then $\rho(UV)$ satisfies the desired conditions. □

Definition 2.7. For $\sigma \in S_{1}(\mathcal{H})$ and $\tau \in S_{\leq 1}(\mathcal{H})$, the purified distance between $\sigma$ and $\tau$ is given by

$$PD(\sigma, \tau) = \sqrt{1 - \left(\text{tr}(|\sqrt{\sigma} \sqrt{\tau}|)\right)^2}.$$  \hspace{1cm} (2.8)

For $\sigma(U) \in S_{1}(UW)$ and $\tau(U) \in S_{\leq 1}(VW)$,

$$PD(\sigma(U), \tau(U)) = \sqrt{1 - \left(\sum_u \text{tr}(|\sqrt{\sigma(u)} \sqrt{\tau(u)}|)\right)^2}.$$  \hspace{1cm} (2.9)

The fidelity between $\sigma(U)$ and $\tau(U)$ is $F(\sigma(U), \tau(U)) = \sum_u \text{tr}(|\sqrt{\sigma(u)} \sqrt{\tau(u)}|)$.

The definition of purified distance can be extended to $S_{\leq 1}(\mathcal{H})$ in the first argument, but the expression becomes more involved. We do not need the extension. The relevant properties of purified distance can be determined from Tbl. 3.1, Pg. 48 in Ref. [23] and the subsequent sections, given the definition of purified distance in terms of fidelity (Def. 3.3, Pg. 49). We remark that the extension of purified distance to distributions is consistent with the definition of purified distance for the quantization of the distributions, see property (vi) in the referenced table. That is, the purified distance between $\rho(U) \in S_{1}(UW)$ and $\sigma(U) \in S_{1}(UW)$ is the same as that between the quantized states $\sum_u \hat{u} \otimes \rho(u)$ and $\sum_u \hat{u} \otimes \sigma(u)$.

The purified distance satisfies the triangle inequality (as it should) and the data-processing inequality, see Ref. [23], Prop. 3.2, Pg. 50 and Thm. 3.4, Pg. 51. We also need the following relationships:

Lemma 2.8. If $\rho(U), \sigma(U) \in S_{1}(UW)$ and $\tau(U) \in S_{\leq 1}(UW)$ such that $\tau(U) \leq \sigma(U)$, then $PD(\rho(U), \sigma(U)) \leq PD(\rho(U), \tau(U))$ and $TV(\rho(U), \sigma(U)) \leq PD(\rho(U), \sigma(U)) \leq \sqrt{2TV(\rho(U), \sigma(U))}$.

Proof. The first statement follows from property (v) of Tbl. 3.1, Pg. 48, and the second from Prop. 3.3, Pg. 50 of Ref. [23], in view of the two remarks after Defs. 2.3 and 2.7. □
2.5. Rényi Powers

We adopt the convention that the trace has higher priority than power so that $\text{tr}(A)^\alpha = (\text{tr}(A))^\alpha$. Since many works have the opposite convention, we often use the additional parentheses to disambiguate.

**Definition 2.9.** Let $0 \leq \rho \ll \sigma$, $\alpha > 1$ and $\beta = \alpha - 1$. The sandwiched Rényi power of order $\alpha$ of $\rho$ conditional on $\sigma$ is defined as

$$ \mathcal{R}_\alpha(\rho|\sigma) = \text{tr}\left(\left(\sigma^{-\beta/(2\alpha)} \rho \sigma^{-\beta/(2\alpha)}\right)^\alpha\right). $$

The Petz Rényi power of order $\alpha$ of $\rho$ conditional on $\sigma$ is defined as

$$ \mathcal{P}_\alpha(\rho|\sigma) = \text{tr}(\rho^\alpha \sigma^{-\beta}). $$

Both Rényi powers are defined to be identically 0 if both $\rho = 0$ and $\sigma = 0$.

The normalized Rényi powers are defined by

$$ \hat{\mathcal{R}}_\alpha(\rho|\sigma) = \frac{1}{\text{tr}(\rho)} \mathcal{R}_\alpha(\rho|\sigma), $$

$$ \hat{\mathcal{P}}_\alpha(\rho|\sigma) = \frac{1}{\text{tr}(\rho)} \mathcal{P}_\alpha(\rho|\sigma), $$

for $\text{tr}(\rho) > 0$. For $\text{tr}(\rho) = 0$ they are defined to be identically 1.

Throughout this work, we use the convention that the symbols $\alpha$ and $\beta$ satisfy $\alpha > 1$ and $\beta = \alpha - 1 > 0$. We normally do not reiterate these constraints on $\alpha$ and $\beta$. For Petz Rényi powers we generally also assume $\alpha \leq 2$. By default, Rényi powers are sandwiched. We only consider Rényi powers of order $\alpha > 1$, but they are well-defined and useful for $0 < \alpha < 1$. A pedagogical introduction to Rényi powers and their properties is in Ref. [24]. See Sect. 4.3 for the sandwiched Rényi powers and Sect. 4.4 for the Petz Rényi powers. The focus in Ref. [24] and most other references is on Rényi divergences, which are entropic quantities obtained from the Rényi powers, although many of the fundamental properties are derived by an analysis of the latter. The divergences share a set of properties given in Sect. 4.1.1 and 4.1.2 of Ref. [24] and labeled (I)-(X). The next lemmas give properties of Rényi powers that we need. The Roman numerals in the headings refer to the labels used in Ref. [24] for related properties of Rényi divergences.

**Lemma 2.10.** We have $\mathcal{P}_\alpha(\rho|\sigma) \geq \mathcal{R}_\alpha(\rho|\sigma)$.

**Proof.** This follows from the Araki-Lieb-Thirring inequality $\text{tr}(B^\gamma A^\gamma B^\gamma) \geq \text{tr}((BAB)^\gamma)$ for all $\gamma \geq 1$, $A \geq 0$ and $B \geq 0$, where we set $\gamma = \alpha$, $A = \rho$ and $B = \sigma^{-\beta/(2\alpha)}$. See Ref. [26], Pg. 258. 

**Lemma 2.11.** (I) Continuity of Rényi powers. Suppose that $0 < \rho \ll \sigma$. The Rényi powers $\mathcal{R}_\alpha(\rho'|\sigma')$ and $\mathcal{P}_\alpha(\rho'|\sigma')$ are continuous at $\rho' = \rho$, $\sigma' = \sigma$ in each of $\rho'$ and $\sigma'$.

Given appropriate conditions on the support of $\sigma$, joint continuity also holds.
Proof. For the sandwiched Rényi entropy this is shown in Ref. [27], Sect. IV.B. For the Petz Rényi entropy, this is stated as an exercise at the end of Sect. 4.4.1 in Ref. [24]. □

Lemma 2.12. (X) Dominance property of Rényi powers. For \( 0 \leq \rho \ll \sigma \leq \sigma' \), \( R_\alpha (\rho|\sigma') \leq R_\alpha (\rho|\sigma) \). If \( \alpha \leq 2 \), then \( P_\alpha (\rho|\sigma') \leq P_\alpha (\rho|\sigma) \).

Proof. The relevant arguments can be found in Sects. 4.3 and 4.4 of Ref. [24]. □

Lemma 2.13. Let \( 0 \leq \rho_i \) and \( \rho = \sum_i \rho_i \). Then
\[
\sum_i R_\alpha (\rho_i|\rho) \leq \text{tr}(\rho).
\] (2.13)

If \( \alpha \leq 2 \), then
\[
\sum_i P_\alpha (\rho_i|\rho) \leq \text{tr}(\rho).
\] (2.14)

Proof. By Lem. 2.12 we have
\[
\sum_i R_\alpha (\rho_i|\rho) \leq \sum_i R_\alpha (\rho_i|\rho_i) = \sum_i \text{tr}(\rho_i) = \text{tr}(\rho),
\] (2.15)

and similarly for the Petz Rényi power when \( \alpha \leq 2 \). □

Lemma 2.14. Log-convexity of Rényi powers: For \( 0 \leq \rho \ll \sigma \) the function \( \alpha \mapsto \log(R_\alpha (\rho|\sigma)) \) is convex, and so is \( \alpha \mapsto \log(P_\alpha (\rho|\sigma)) \).

Proof. These are the first halves of Cor. 4.2, Pg. 56 (sandwiched Rényi power) and of Cor. 4.3, Pg. 62 (Petz Rényi power) of Ref. [24]. □

Lemma 2.15. Monotonicity of Rényi powers: For \( 0 \leq \rho \ll \sigma \) the function \( \alpha \mapsto \tilde{R}_\alpha (\rho|\sigma)^{1/\beta} \) is non-decreasing, and so is \( \alpha \mapsto \tilde{P}_\alpha (\rho|\sigma)^{1/\beta} \).

Proof. These are the second halves of Cor. 4.2, Pg. 56 (sandwiched Rényi power) and Cor. 4.3, Pg. 62 (Petz Rényi power) of Ref. [24]. □

Lemma 2.16. Joint convexity of Rényi powers: The function \( \rho, \sigma \mapsto R_\alpha (\rho|\sigma) \) is jointly convex in \( \rho \) and \( \sigma \) on its domain, and similarly for the Petz Rényi powers when \( \alpha \leq 2 \).

Proof. For the sandwiched Rényi powers, see Prop. 3 of Ref. [28]. For the Petz Rényi powers, this is Prop. 4.8, Pg. 61 in Ref. [24]. □

Lemma 2.17. (VIII) Data-processing inequality for Rényi powers: Let \( \mathcal{E} \) be a quantum operation and \( 0 \leq \rho \ll \sigma \). Then \( R_\alpha (\mathcal{E}(\rho)|\mathcal{E}(\sigma)) \leq R_\alpha (\rho|\sigma) \) and similarly for the Petz Rényi powers when \( \alpha \leq 2 \).

Proof. For the sandwiched Rényi powers, see Ref. [28, 29]. For the Petz Rényi powers, see Sect. 4.4.1 of Ref. [24]. □
2.6. Quantum Relative Entropy

Most of this work concerns estimation of Rényi powers so Rényi entropies and divergences play a secondary role. However, according to the quantum asymptotic equipartition property [7], the asymptotic rate for randomness generation is determined by quantum relative entropies. The quantum relative entropy arises naturally as a limit of Rényi divergences.

Throughout this work, logarithms are base $e$ and entropies are expressed in nats (the natural units of information) unless explicitly specified otherwise. This simplifies calculus; conversion is only needed when composing with extractors to specify the relationships between certified conditional min-entropy and lengths of bit strings. For results mentioning entropies, the conversion between nits and bits usually just requires replacing $\log$ base $e$ with $\log$ base $2$. Exceptions are the theorems of Sect. 6.3 stating EAT and QEFP bounds, which are not intended to be used in applications.

\begin{definition}
Let $0 \leq \rho \ll \sigma$ and $\alpha > 1$. The sandwiched Rényi divergence of order $\alpha$ for $\rho$ given $\sigma$ is
\[
\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\beta} \log \left( \hat{R}_\alpha(\rho|\sigma) \right).
\]
(This is Def. 4.3, Pg. 53 in Ref. [24].) \hfill (2.16)
\end{definition}

\begin{lemma}
Let $0 < \rho \ll \sigma$. The limit of $\tilde{D}_\alpha(\rho|\sigma)$ as $\alpha \searrow 1$ exists and satisfies
\[
\tilde{D}_1(\rho|\sigma) \doteq \lim_{\alpha \searrow 1} \tilde{D}_\alpha(\rho|\sigma) = \frac{\text{tr}(\rho(\log(\rho) - \log(\sigma)))}{\text{tr}(\rho)},
\]
which is the quantum relative entropy.
\end{lemma}

\begin{proof}
This is Prop. 4.5, Pg. 57 of Ref. [24]. \hfill \square
\end{proof}

2.7. Min-Entropy

Quantum min-entropy characterizes the randomness that is available in a given system. We define the relevant quantities for the family of classical-quantum states treated in this work, where $C$ is the output CV, $Z$ is the input CV and $E$ is the system containing the quantum side information. We can instantiate these variables in each context as we wish. For example, we can consider the situation where we let $Z$ be a trivial CV, which is equivalent to just leaving it out.

\begin{definition}
Let $\rho(CZ) \in S_{\leq 1}(CZE)$. Then $\rho(CZ)$ has max-prob $p$ given $ZE$ if there exists $\sigma(Z) \in S_1(ZE)$ such that $\rho(CZ) \leq p\sigma(Z)$. The exact max-prob of $\rho(CZ)$ given $ZE$ is
\[
P_{\text{max}}(\rho(CZ)|ZE) = \inf \{ p : \text{there exists } \sigma(Z) \in S_1(ZE) \text{ such that } \rho(CZ) \leq p\sigma(Z) \}.
\]
The quantity $H_\infty(\rho(CZ)|ZE) = -\log(P_{\text{max}}(\rho(CZ)|ZE))$ is called the conditional min-entropy of $\rho(CZ)$ given $ZE$.
\end{definition}

When writing conditional quantities like $P_{\text{max}}$, we put the state with its CV arguments first. The conditioned systems are always classical and consist of every CV that does not occur in the conditioner.

We need a lemma to switch between conditioning on a CV and conditioning on its quantization.
Lemma 2.21. Let $\rho(CZ) \in S_1(CZE)$ and define $\tau(C) = \sum_z \hat{z} \otimes \rho(cz) \in S_1(CZE)$. Then $P_{\max}(\rho(CZ)|ZE) = P_{\max}(\tau(C)|ZE)$.

Proof. For $\sigma(Z) \in S_1(ZE)$ such that $\rho(CZ) \leq p\sigma(Z)$, we have that $\tau(C) \leq p\sum_z \hat{z} \otimes \sigma(z)$. This implies that $P_{\max}(\rho(CZ)|ZE) \geq P_{\max}(\tau(C)|ZE)$. For the reverse inequality, consider $\sigma' \in S_1(ZE)$ such that $\tau(C) \leq p\sigma'$. Since the map $\xi \mapsto \hat{z}\xi$ is positive, it preserves operator ordering and $\hat{z} \otimes \rho(cz) = \hat{z}\tau(C)\hat{z} \leq p\hat{z}\sigma'\hat{z}$. With $\sigma(Z)$ defined by $\sigma(z) = \text{tr}_Z \hat{z}\sigma'\hat{z}$, it follows that $\rho(CZ) \leq p\sigma(Z)$. \hfill \Box

Definition 2.22. Let $\rho(CZ) \in S_1(CZE)$. System ZE’s guessing probability for $\rho(CZ)$ is

$$G_{\max}(\rho(CZ)|ZE) = \sup \left\{ \sum_{cz} \text{tr}(P_{cz}\rho(cz)) : \text{for all } z \ (P_{cz})_c \text{ is a POVM} \right\}. \quad (2.19)$$

Lemma 2.23. Let $\rho(CZ) \in S_1(CZE)$. Then

$$G_{\max}(\rho(CZ)|ZE) = P_{\max}(\rho(CZ)|ZE). \quad (2.20)$$

Proof. Let $\tau(C) = \sum_z \hat{z} \otimes \rho(cz)$. According to Ref. [30], Thm. 1, $P_{\max}(\tau(C)|ZE) = G_{\max}(\tau(C)|ZE)$. According to Lem. 2.21 it suffices to show that $G_{\max}(\tau(C)|ZE) = G_{\max}(\rho(CZ)|ZE)$. Let $(P_{cz})_c$ be $z$-indexed POVMs. Then $P_c = (\sum_z \hat{z} \otimes P_{cz})_c$ is a POVM, and

$$\sum_c \text{tr}(P_{cz}\rho(cz)) = \sum_c \text{tr}(\text{tr}_Z(P_c\tau(c))) = \sum_c \text{tr}(P_c\tau(c)), \quad (2.21)$$

from which it follows that $G_{\max}(\tau(C)|ZE) \geq G_{\max}(\rho(CZ)|ZE)$. For the reverse inequality, let $(P_c)_c$ be a POVM on ZE. We have

$$\sum_c \text{tr}(P_c\tau(c)) = \sum_{cz} \text{tr}(P_c(\hat{z} \otimes \rho(cz))) = \sum_{cz} \text{tr}(\text{tr}_Z(P_c(\hat{z} \otimes 1_E))\rho(cz)). \quad (2.22)$$

Let $P_{cz} = \text{tr}_Z(P_c(\hat{z} \otimes 1_E)) = \text{tr}_Z((\hat{z} \otimes 1_E)P_c(\hat{z} \otimes 1_E))$. Then $(P_{cz})_c$ is a POVM for each $z$, and Eq. 2.22 and arbitrariness of $(P_c)_c$ implies that $G_{\max}(\tau(C)|ZE) \leq G_{\max}(\rho(CZ)|ZE)$.

Definition 2.24. Let $\rho(CZ) \in S_1(CZE)$. The conditional entropy of $\rho(CZ)$ given ZE is

$$H_1(\rho(CZ)|ZE) = -\sum_{cz} \text{tr}(\rho(cz)(\log(\rho(cz)) - \log(\rho(z)))) = -\sum_{cz} \text{tr}(\rho(cz))\tilde{D}_1(\rho(cz)\|\rho(z)). \quad (2.23)$$

Lemma 2.25. $H_1(\rho(CZ)|ZE) \geq H_\infty(\rho(CZ)|ZE)$.

Proof. Define $\tilde{H}_\alpha(\rho(CZ)|ZE) = -\sum_{cz} \text{tr}(\rho(cz))\tilde{D}_\alpha(\rho(cz)\|\rho(z))$. The lemma follows from $H_1 = \lim_{\alpha \searrow 1} \tilde{H}_\alpha$, $H_\infty = \lim_{\alpha \nearrow \infty} \tilde{H}_\alpha$ and monotonicity of $\tilde{H}_\alpha$ in $\alpha$. These facts can be found in Ref. [24]. The first limit is an application of Lem. 2.19. For the second limit, see Ref. [24], Def. 4.2, Pg. 52 and the comment at the beginning of Sect. 4.3.2. That $\tilde{H}_\alpha$ is non-increasing in $\alpha$ follows from Lem. 2.15 \hfill \Box
2.8. Smooth Min-Entropy

**Definition 2.26.** Let \( \rho(CZ) \in S_1(CZE) \). Then \( \rho(CZ) \) has \( \epsilon \)-smooth max-prob \( p \) given \( ZE \) if there exists a \( \rho'(CZ) \in S_{\leq 1}(CZE) \) with \( PD(\rho(CZ), \rho'(CZ)) \leq \epsilon \) and \( P_{\max}(\rho'(CZ)|ZE) \leq p \). The exact \( \epsilon \)-smooth max-prob of \( C \) given \( ZE \) at \( \rho(CZ) \) is

\[
P_{\max}^\epsilon(\rho(CZ)|ZE) = \inf \{ P_{\max}(\rho'(CZ)|ZE) : \rho'(CZ) \in S_{\leq 1}(CZE), PD(\rho'(CZ), \rho(CZ)) \leq \epsilon \}.
\]

The quantity \( H_\infty^\epsilon(\rho(CZ)|ZE) = -\log(P_{\max}^\epsilon(\rho(CZ)|ZE)) \) is called the smooth conditional min-entropy of \( \rho(CZ) \) given \( ZE \). Here, the smoothing is with respect to the purified distance, as in Refs. \[23, 30\].

For relevant cases, the witness \( \rho'(CZ) \) in the definition of \( \epsilon \)-smooth max-prob can be assumed to be normalized states. This observation is formalized by the next lemma.

**Lemma 2.27.** Suppose that \( \rho(CZ) \in S_1(CZE) \) has \( \epsilon \)-smooth max-prob \( p \) given \( ZE \) with \( p|Rng(C)| \geq 1 \). Then there exists \( \rho''(CZ) \in S_1(CZE) \) such that \( PD(\rho(CZ), \rho''(CZ)) \leq \epsilon \) and \( P_{\max}(\rho''(CZ)|ZE) \leq p \).

**Proof.** Let \( \rho'(CZ) \in S_{\leq 1}(CZE) \) and \( \sigma(Z) \in S_1(ZE) \) such that \( PD(\rho(CZ), \rho'(CZ)) \leq \epsilon \) and \( \rho'(CZ) \leq p\sigma(Z) \). By Lem. \[2.6\], there exists \( \rho''(CZ) \in S_1(CZE) \) such that \( \rho'(CZ) \leq \rho''(CZ) \) and \( \rho''(CZ) \leq p\sigma(Z) \). So \( P_{\max}(\rho''(CZ)|ZE) \leq p \), and by Lem. \[2.8\]

\[
PD(\rho(CZ), \rho''(CZ)) \leq PD(\rho(CZ), \rho'(CZ)) \leq \epsilon.
\]

\[ \square \]

**Definition 2.28.** Let \( \rho(CZ) \in S_1(CZE) \). Then \( \rho(CZ) \) has TV:\( \epsilon \)-smooth max-prob \( p \) given \( ZE \) if there exists a \( \rho'(CZ) \in S_1(CZE) \) with \( TV(\rho(CZ), \rho'(CZ)) \leq \epsilon \) and \( P_{\max}(\rho'(CZ)|ZE) \leq p \). The TV:exact \( \epsilon \)-smooth max-prob of \( C \) given \( ZE \) at \( \rho(CZ) \) is

\[
P_{\max}^{TV,\epsilon}(\rho(CZ)|ZE) = \inf \{ P_{\max}(\rho'(CZ)|ZE) : \rho'(CZ) \in S_1(CZE), TV(\rho'(CZ), \rho(CZ)) \leq \epsilon \}.
\]

The quantity \( H_{\infty}^{TV,\epsilon}(\rho(CZ)|ZE) = -\log(P_{\max}^{TV,\epsilon}(\rho(CZ)|ZE)) \) is called the TV:smooth conditional min-entropy of \( \rho(CZ) \) given \( ZE \). Here, the smoothing is with respect to the TV distance, as first proposed in Ref. \[31\].

We remark that the definitions are monotonic in the smoothness parameter \( \epsilon \). For example, if \( P_{\max}^\epsilon(\rho(CZ)|ZE) \leq p \) and \( \epsilon' > \epsilon \), then \( P_{\max}^{\epsilon'}(\rho(CZ)|ZE) \leq p \). Besides using TV distance instead of purified distance, the second definition requires that the state being compared is normalized. This is unproblematic for max-prob bounds greater than \( 1/|Rng(C)| \), and smaller bounds are generally not helpful, see the next lemma. As explained in Ref. \[23\], when dealing with quantum information, purified distance is preferred and the fact that it exceeds TV distance means that there are few complications when chaining with classical protocols or extractors, or for interpreting results in familiar probabilistic terms.

We can readily switch from smoothing with purified distance to smoothing with TV distance by applying the next lemma. Switching in the other direction involves a square-root increase of smoothing parameter; we do not consider this switch here.
Lemma 2.29. Let $\rho(CZ) \in S_1(CZE)$ have $\epsilon$-smooth max-prob $p$ given $ZE$ with $p|\text{Rng}(C)| \geq 1$. Then $P^\text{TV}\epsilon(\rho(CZ)|ZE) \leq p$. It follows that

$$P^\text{TV}\epsilon(\rho(CZ)|ZE) \leq \max(P^\epsilon_{\text{max}}(\rho(CZ)|ZE), 1/|\text{Rng}(C)|). \quad (2.27)$$

Proof. By Lem. 2.27 there exists $\rho''(CZ) \in S_1(CZE)$ such that $\text{PD}(\rho(CZ), \rho''(CZ)) \leq \epsilon$ and $P^\epsilon_{\text{max}}(\rho''(CZ)|ZE) \leq p$. By Lem. 2.8

$$\text{TV}(\rho(CZ), \rho''(CZ)) \leq \text{PD}(\rho(CZ), \rho''(CZ)). \quad (2.28)$$

Hence $P^\text{TV}\epsilon(\rho(CZ)|ZE) \leq p$. For Eq. 2.27 we set $p = \max(1/|\text{Rng}(C)|, P^\epsilon_{\text{max}}(\rho(CZ)|ZE))$ and apply the result just proven. \qed

Lemma 2.30. Let $\rho(CZ) \in S_1(CZE)$. Then

$$P^\epsilon_{\text{max}}(\rho(CZ)|ZE) \leq |\text{Rng}(Z)| P^\epsilon_{\text{max}}(\rho(CZ)|E). \quad (2.29)$$

Proof. This is Lem. 6.8, Pg. 95 of Ref. [24]. Consider an arbitrary $p > P^\epsilon_{\text{max}}(\rho(CZ)|E)$. Then there exist $\rho'(CZ) \in S_{\leq 1}(CZE)$ and $\tau \in S_1(E)$ such that $\text{PD}(\rho(CZ), \rho'(CZ)) \leq \epsilon$ and $\rho'(CZ) \leq p\tau$, which we can rewrite as $\rho'(CZ) \leq p|\text{Rng}(Z)| \tau/|\text{Rng}(Z)|$. Define $\sigma(Z) = \tau/|\text{Rng}(Z)|$, which is in $S_1(ZE)$. Therefore $\rho'(CZ)$ and $\sigma(Z)$ witness that $P^\epsilon_{\text{max}}(\rho(CZ)|ZE) \leq |\text{Rng}(Z)|p$. Letting $p \backslash \alpha P^\epsilon_{\text{max}}(\rho(CZ)|ZE)$ proves the lemma. \qed

Lemma 2.31. Let $\rho(CZH) \in S_1(CZHE)$, and suppose that $Z = Z(H)$ is determined by $H$, then $P^\epsilon_{\text{max}}(\rho(CH)|E) \leq P^\epsilon_{\text{max}}(\rho(CZ)|E)$ and $P^\epsilon_{\text{max}}(\rho(CZH)|HE) \leq P^\epsilon_{\text{max}}(\rho(CH)|HE)$. Proof. These are instances of data-processing inequalities for smooth conditional min-entropy. Since $Z$ is determined by $H$, the first statement is a consequence of Prop. 6.4, Pg. 96 of Ref. [24], according to which applying a function to a classical system does not increase the $\epsilon$-smooth conditional min-entropy of the system conditional on other systems. For the second, the transformation $h \mapsto Z(h)$ can be considered as a CPTP map of system $H$ to the system $Z$, where these are the systems in the conditioners of the smooth max-probs being compared. The inequality is therefore obtained from Thm. 6.2, Pg. 95 of Ref. [24], according to which a CPTP process applied to the conditioning system does not decrease the $\epsilon$-smooth conditional min-entropy. \qed

2.9. Extractors

For randomness generation protocols, we assume that a quantum-proof strong extractor $\mathcal{E}$ is available.

Definition 2.32. Let $C$, $S$ and $R$ be CVs. Define $n = \log_2(|\text{Rng}(C)|)$, $k_s = \log_2(|\text{Rng}(S)|)$ and $k_o = \log_2(|\text{Rng}(R)|)$. Here $S$ is a seed CV with probability distribution $\mu(S) = \text{Unif}(S)$ and independent of all other systems. Consider a function $\mathcal{E} : (C, S; n, k_s, k_o, k_i, \epsilon_x) \mapsto \text{Rng}(R)$. Define $\tilde{\mathcal{E}} : c, s \mapsto \mathcal{E}(c, s)s$, where the parameters are implicit. The function $\mathcal{E}$ is a quantum-proof strong extractor with parameters $(n, k_s, k_o, k_i, \epsilon_x)$ if for every $\rho(CS) \in S_1(CSE)$ of the form $\rho(CS) = \rho(C)\text{Unif}(S)$ that satisfies $P^\epsilon_{\text{max}}(\rho(C)|E) \leq 2^{-k_i}$, the extractor and seed output $\tilde{\mathcal{E}}$ is close to uniform and independent of $E$ with distance

$$\text{PD}(\rho(\tilde{\mathcal{E}}), \text{Unif}(RS)|\rho) \leq \epsilon_x. \quad (2.30)$$
This definition of quantum-proof extractors differs from others such as Ref. [11] by requiring small purified distance instead of small TV distance. With this change we can take advantage of extensions to previously traced-out quantum systems.

In this work, we use the term extractor to refer to a function $E$ that is a quantum-proof strong extractor provided that the parameters $(n, k_s, k_o, k_i, \epsilon_x)$ satisfy constraints that we refer to as the extractor constraints. (The convention in this manuscript for parameters and their ordering differs from that in Ref. [1].) We assume that the extractor constraints include the conditions $1 \leq k_i \leq n$, $k_s \geq 0$, $k_o \leq k_i$, and $0 < \epsilon_x \leq 1$. We generally deal with bit strings $C$, $S$ and $R$, so we also assume that $n$, $k_s$ and $k_o$ are integers.

A specific quantum-proof strong extractor with reasonably low seed requirements is the TMPS extractor based on Ref. [11], which we applied in Ref. [12] using the implementation available at https://github.com/usnistgov/libtrevisan. Simplified constraints for this extractor include $2 \leq k_o \leq k_i \leq n$ and

$$k_o + 4 \log_2(k_o) \leq k_i - 4 \log_2(1/\delta_x) - 6,$$

$$k_s \geq 36 \log_2(k_o)(\log_2(4nk_o^2/\delta_x^2))^2. \quad (2.31)$$

Here, $\delta_x$ is the desired error in terms of TV distance. To ensure that the purified distance is at most $\epsilon_x$, we set $\delta_x = \epsilon_x^2/2$, see Lem. 2.8. See Ref. [12] for the smaller expression for $k_s$ in terms of $\delta_x$ used by the implementation. Better extractors exist in theory, but full implementations are still rare.

3. MODELS

3.1. Definitions

Definition 3.1. A model $\mathcal{C}(U)$ for $UE$ is a subset of $\mathcal{S}(UE)$ closed under multiplication by non-negative real numbers. The set of normalized distributions in $\mathcal{C}(U)$ is $\mathcal{N}(\mathcal{C}(U)) = \{\rho(U) \in \mathcal{C} : \text{tr}(\rho) = 1\}$. The model $\mathcal{C}(U)$ is null if its only member is the zero distribution given by $u \mapsto 0$.

If $\mathcal{C}(U)$ is not null, we can reconstruct $\mathcal{C}(U)$ from $\mathcal{N}(\mathcal{C}(U))$ by $\mathcal{C}(U) = [0, \infty)\mathcal{N}(\mathcal{C}(U))$. We normally omit “for $UE$” when introducing a model. In this case, the default quantum system is $E$.

Expressions of the form $\mathcal{C}(U)$ with $U$ a CV are reserved for models. We may subscript $\mathcal{C}$ to distinguish models in context. The notation $\mathcal{C}(U)$ indicates the CV or CVs that the members of the model depend on and does not indicate function application or a CV construction. We adapt the marginalization conventions for state-valued distributions for models. Thus if $\mathcal{C}(UV)$ is a model, then $\mathcal{C}(U) = \{\rho(U) : \rho(UV) \in \mathcal{C}(UV)\}$ and $\mathcal{C} = \{\rho : \rho(UV) \in \mathcal{C}(UV)\}$. When a model is expressed in terms a compound construction such as $\mathcal{E}(\mathcal{C}(UV)), \mathcal{M}(\mathcal{C}(UV)) \ldots$ or $\mathcal{C}(U) \circ \mathcal{C}_U(V)$ without a final CV argument, the marginalization conventions do not apply.

A classical model $\mathcal{C}(U)$ is a model for $U$, which means that the quantum system is trivial and the model consists of a set of unnormalized distributions on $U$. In this case, $\mathcal{N}(\mathcal{C}(U))$ consists of probability distributions and is a standard statistical model. For any model $\mathcal{C}(U)$, $\text{tr}(\mathcal{C}(U))$ is a classical model.
We consider several closure properties and operations on models. First we define $V$-conditional quantum operations on $B(E)$ as a family $\mathcal{E}_V$ of $v$-dependent quantum operations $\mathcal{E}_v$ on $B(E)$. As an operation, $\mathcal{E}_V$ transforms members of $\mathcal{S}(UVE)$ according to $\mathcal{E}_V : \rho(uv) \mapsto \mathcal{E}_v(\rho(uv))$. Among the many closure properties that can be satisfied by models, we distinguish the following:

**Definition 3.2.** The model $\mathcal{C}(U)$ is closed under the linear map $\mathcal{E} : B(E) \to B(E)$ if $\mathcal{E}(\mathcal{C}(U)) \subseteq \mathcal{C}(U)$. $\mathcal{C}(U)$ is pCP-closed if it is closed under pCP maps, CP-closed if it is closed under CP maps and CPTP-closed if it is closed under trace-preserving CP maps. The model $\mathcal{C}(UV)$ is closed under $V$-conditional quantum operations if $\mathcal{E}_V(\mathcal{C}(UV)) \subseteq \mathcal{C}(UV)$ for every $V$-conditional quantum operation $\mathcal{E}_V$.

We may also consider closedness under special families of CP maps, for instance the family of CP maps that preserve the projectors of a partition of unity. For each closedness property in the definitions above, there is a corresponding closure operation. We use suggestive notation for closure operations. For example $\text{Cvx}(\mathcal{C}(U))$ is convex closure, $\text{pCP}(\mathcal{C}(U))$ is closure under pCP maps, and $\text{CPTP}_V(\mathcal{C}(UV))$ is closure under $V$-conditional CPTP maps.

### 3.2. General Constructions

Models $\mathcal{C}(U)$ arise from constraints on the physical processes that result in the distributions $\rho(U)$ in $\mathcal{C}(U)$. It is possible to associate quantum models to classical models.

**Definition 3.3.** Let $\mathcal{C}(U)$ be a classical model. Then the maximal extension of $\mathcal{C}(U)$ to $E$ is defined as

$$
\mathcal{M}(\mathcal{C}(U); E) = \{ \rho(U) : \text{tr}(\sigma \rho(U)) \in \mathcal{C}(U) \text{ for all } \sigma \in \mathcal{S}(E) \}. \tag{3.1}
$$

In this definition, if $\mathcal{C}(U)$ is convex closed, one can restrict $\sigma$ to pure states when verifying membership in $\mathcal{M}(\mathcal{C}(U); E)$ according to Eq. 3.1.

**Lemma 3.4.** If $\mathcal{C}(U)$ is a classical model, then $\mathcal{M}(\mathcal{C}(U); E)$ is CP-closed.

**Proof.** Let $\rho(U) \in \mathcal{M}(\mathcal{C}(U); E)$ and let $\mathcal{E} : \tau \mapsto \sum_i A_i \tau A_i^\dagger$ be a CP map. Given $\sigma \in \mathcal{S}(E)$, let $\chi = \sum_i A_i^\dagger \sigma A_i \in \mathcal{S}(E)$ and evaluate

$$
\text{tr}(\sigma \mathcal{E}(\rho(U))) = \text{tr} \left( \sigma \sum_i A_i \rho(U) A_i^\dagger \right) = \sum_i \text{tr}(\sigma A_i \rho(U) A_i^\dagger) = \sum_i \text{tr}(A_i^\dagger \sigma A_i \rho(U)) = \text{tr} \left( \sum_i A_i^\dagger \sigma A_i \rho(U) \right) = \text{tr}(\chi \rho(U)) \in \mathcal{C}(U). \tag{3.2}
$$

Since $\sigma \in \mathcal{S}(E)$ is arbitrary, it follows that $\mathcal{E}(\rho(U)) \in \mathcal{M}(\mathcal{C}(U); E)$. \qed
If \( C(U) \) is the classical model arising from a Bell-test configuration with only non-signaling assumptions and no additional quantum constraints, then the maximal extension of \( C(U) \) to \( E \) makes no physical assumptions on the protocol devices other than non-signaling and therefore allows the devices to exhibit super-quantum correlations. The models obtained when the devices and \( E \) are jointly quantum are more constrained. They arise from families of POVMs as follows.

**Definition 3.5.** Let \( \mathfrak{P}(U) \) be a family of POVMs of \( D \) with outcomes \( U \). The model for \( UE \) induced by \( \mathfrak{P}(U) \) is defined by

\[
\mathcal{M}(\mathfrak{P}(U); E) = \{ \mathcal{P}(\sigma)(U) : \sigma \in \mathcal{S}(DE), \mathcal{P} \in \mathfrak{P}(U) \}.
\]

Expressions of the form \( \mathfrak{P}(U) \) with \( U \) a CV are reserved for families of POVMs. The notation \( \mathfrak{P}(U) \) indicates the outcome CV of the members and does not indicate function application or a CV construction. We may subscript \( \mathfrak{P} \) to distinguish families in context. For Bell-test configurations, adding all quantum constraints to a classical non-signaling model \( C(V) \) and constructing the maximal extension of \( C(V) \) need not be equivalent to inducing a model from a suitably constrained set of POVMs. Further research is required to explore the relationships between maximal extensions and induced models.

**Lemma 3.6.** For any family \( \mathfrak{P}(U) \) of POVMs of \( D \) with outcomes \( U \), the induced model \( \mathcal{M}(\mathfrak{P}) \) is CP-closed.

**Proof.** It suffices to observe that by definition, CP maps on \( \mathcal{S}(E) \) preserve \( \mathcal{S}(E) \), and POVMs of \( D \) with outcomes \( U \) commute with CP maps on \( \mathcal{S}(E) \). \( \square \)

For induced models, \( D \) consists of the devices used by a protocol and the POVMs can be constrained by partial trust in device behavior. For example, in many situations, the trust involves assumptions that \( Z \) is an input with known probability distribution, and that there exists a system decomposition of the devices according to protocol parties, with the POVMs acting independently on the subsystems. In partially device-dependent applications, one may also trust the form of the specific measurements or the dimensions of the subsystems. It is possible to generalize the definition of induced models by restricting the measured states of \( D(E) \) to a model of \( DE \) or of \( WDE \) for some CV \( W \).

Both maximal extensions and induced models are defined uniformly, independent of the dimension of \( E \). We can take the state space of \( E \) to be an infinite dimensional Hilbert space, but according to our finiteness assumptions, we restrict to states with finite support.

### 3.3. Chaining Models

**Definition 3.7.** Let \( C(U) \) be a model for \( UE \) and for each \( u \), let \( C_u(V) \) be model for \( VE \). We write \( C_u(V) \) for the \( u \)-indexed family of models consisting of the \( C_u(V) \). The result of chaining \( C(U) \) and \( C_u(V) \) is the model for \( UV \) defined by

\[
C(U) \circ C_u(V) = \{ \rho(UV) : \rho(U) \in C(U) \text{ and for all } u, \rho(uV) \in C_u(V) \}.
\]

Chained models can be null unless \( C(U) \) and \( C_u(V) \) are sufficiently rich. The next lemma shows that quantum operations distribute over chaining.
Lemma 3.8. Let $\mathcal{C}(U)$ be a model for $UE$, $\mathcal{C}_U(V)$ a family of models for $VE$ and $\mathcal{E} : \mathcal{B}(E) \to \mathcal{B}(E)$ a positive linear map. Then $\mathcal{E}(\mathcal{C}(U) \circ \mathcal{C}_U(V)) \subseteq \mathcal{E}(\mathcal{C}(U)) \circ \mathcal{E}(\mathcal{C}_U(V))$. In particular, if $\mathcal{C}(U)$ and the $\mathcal{C}_u(V)$ are closed under $\mathcal{E}$, then so is $\mathcal{C}(U) \circ \mathcal{C}_U(V)$.

Proof. Let $\rho(UV) \in \mathcal{C}(U) \circ \mathcal{C}_U(V)$ and consider $\rho'(UV) = \mathcal{E}(\rho(UV))$. Since $\rho'(U) = \mathcal{E}(\rho(U))$, we have $\rho'(U) \in \mathcal{E}(\mathcal{C}(U))$. Similarly, for each $u$, $\rho'(uV) = \mathcal{E}(\rho(uV)) \in \mathcal{E}(\mathcal{C}_u(V))$.

When the CV over which a model is defined consists of inputs and outputs where we later condition on the inputs, we need to restrict the composed models so that future inputs are effectively independent of the past outputs given $E$ and the past inputs. Because $E$ is quantum, this is formulated by means of a short quantum Markov chain. In the next definition, $\mathcal{C}Z$ and $CZ$ are separate CVs with no relationship assumed. In an experiment consisting of a sequence of trials, $\mathcal{C}Z$ are the outputs and inputs of the trials so far, and $CZ$ is the output and input of the next trial.

Definition 3.9. Let $\mathcal{C}(CZ)$ be a model for $CZE$ and $\mathcal{C}_{CZ}(CZ)$ a family of models for $CZE$. The set of models obtained by chaining $\mathcal{C}(CZ)$ and $\mathcal{C}_{CZ}(CZ)$ with conditionally independent inputs is written as $\mathcal{C}(CZ) \circ_{Z|CZ} \mathcal{C}_{CZ}(CZ)$ and consists of the members $\rho(CZCZ)$ of $\mathcal{C}(CZ) \circ \mathcal{C}_{CZ}(CZ)$ such that $\rho(CZCZ) \in Z \leftrightarrow ZE \leftrightarrow C$.

3.4. Input-Output Models

When considering models for $CZE$, $Z$ is normally an input CV that can be freely chosen in some sense. We may expect conditional distributions of $C$ given $Z = z$ are independent of $z$. For classical side information, this idea was captured with some generality by models that are free for $Z$ in Ref. [1]. For quantum side information, the conditional constraints are captured by models for $(C|Z)E$ according to the next definition.

Definition 3.10. $\mathcal{C}(C|Z)$ is a model for $(C|Z)E$ if $\mathcal{C}(C|Z) \subseteq \mathcal{S}((C|Z)E)$ and $\mathcal{C}(C|Z)$ is closed under multiplication by non-negative real numbers.

By default, the quantum system for $\mathcal{C}(C|Z)$ is $E$ and we normally omit the phrase “for $(C|Z)E$”.

If $\mathcal{C}(Z)$ is a classical model for $Z$ and $\mathcal{C}(C|Z)$ is a model for $(C|Z)E$, then we can formalize the idea that we freely choose inputs according to $\mathcal{C}(Z)$ with the conditional distributions constrained by $\mathcal{C}(C|Z)$ as follows:

Definition 3.11. Let $\mathcal{C}(Z)$ and $\mathcal{C}(C|Z)$ be models where $\mathcal{C}(Z)$ is classical. The free-for-$Z$ chaining of $\mathcal{C}(Z)$ with $\mathcal{C}(C|Z)$ is defined as

$$\mathcal{C}(Z) \ltimes \mathcal{C}(C|Z) = \{ \nu(Z) \rho(C|Z) : \nu(Z) \in \mathcal{C}(Z), \rho(C|Z) \in \mathcal{C}(C|Z) \}. \tag{3.5}$$

If $\mathcal{C}(Z) = [0, \infty)\mu(Z)$, we abbreviate $\mathcal{C}(Z) \ltimes \mathcal{C}(C|Z) = \mu(Z) \ltimes \mathcal{C}(C|Z)$.

Here is a more general form of free-for-$Z$ chaining that allows for quantum side information on $Z$.

Definition 3.12. Let $\mathcal{C}(Z)$ be a model for $ZV$ and $\mathcal{C}(C|Z)$ a model for $(C|Z)W$. The free-for-$Z$ chaining of $\mathcal{C}(Z)$ with $\mathcal{C}(C|Z)$ is the model of $CZ\ltimes W$ given by

$$\mathcal{C}(Z) \ltimes \mathcal{C}(C|Z) = \{ \sigma(Z) \otimes \rho(C|Z) : \sigma(Z) \in \mathcal{C}(Z), \rho(C|Z) \in \mathcal{C}(C|Z) \}. \tag{3.6}$$
3.5. Constructing Models for Experimental Configurations

The models introduced above can represent all experimental configurations involving quantum side information. In particular, they can represent configurations involving a sequence of trials with devices that perform measurements based on random input choices. The simplest case is where the side information is in a quantum system E that has no interaction with the experimental devices after the experiment starts. If E has independent dynamics during the experiment and protocol, we can time-shift the dynamics to the initial state and then treat E as being static. From the point of view of the experimenter, the initial state of E is a density operator \( \rho \). If the devices are quantum, then \( \rho \) is the marginal state of E for the initial joint quantum state of the devices and E. The joint state can depend on initial, classical information that the experiment may depend on. We condition on all such information and omit it from further consideration. By the end of the experiment classical information and omit it from further consideration. By the end of the experiment classical data CZ is obtained, which includes the inputs Z and outputs C of the devices. The inputs come from a random source, which must be modeled along with everything else, but is often constrained to produce random bits independently of E and the devices. The relevant part of the final state is the joint state of CZ and E, which can be described by \( \rho(CZ) \) and satisfies that \( \sum_{cz} \rho(cz) = \rho \). The model must be formulated so that any such final state that may be encountered is in the model.

We construct models by chaining individual trials. Given that E does not interact with the results cz of the experiment so far, the (unnormalized) state of E is \( \sigma = \rho(cz) \), where \( \rho(CZ) \) is in the model \( \mathcal{C}(CZ) \) for the past. The model \( \mathcal{C}_{cz}(CZ) \) for the next trial may depend on the past and constrains on the results CZ of the next trial. The state of E given the next trial results cz and the past is \( \sigma(cz) \), and we require that \( \sigma(CZ) \) is in \( \mathcal{C}_{cz}(CZ) \). Thus chaining \( \mathcal{C}(CZ) \) with \( \mathcal{C}_{cz}(CZ) \) according to Def. 3.7 yields the model for the results including CZ.

When chaining, the trial models are motivated by physical constraints on the devices used. For quantum experiments, the current state \( \rho_D = \rho(cz) \) of E must be related to a joint state \( \rho_{ED} \) of E and the devices D by performing a measurement on the devices D and then tracing out D. We make no assumptions on the joint state and its dependence on \( cz \) other than the requirement that \( \rho(CZ) \) is in the model for the past results. The experiment is constructed to constrain the way in which the devices can use fresh random input \( Z = z \) to perform a measurement during the next trial. The constraints are typically described by constraints on the z-dependent POVMs that are applied. These may be modeled by a single family \( \Psi \) of POVMs, where the z-dependence is transferred to structural constraints on the POVMs. For example, consider the experimental configuration of a two-station, l-input, m-output Bell test (the \((2, l, m)\)-Bell-test configuration) with inputs X, Y and outputs A, B where the input distribution is uniform. In this case, we have a factorization \( V \otimes W \) of the devices’ Hilbert space for this trial and write the POVM in the form \( P_{XA} \otimes Q_{YB} \) where \( \sum_a P_{xa} = 1/l \), \( \sum_b Q_{yb} = 1/l \). With \( \Psi \) the set of all such POVMs, the trial model becomes the model induced by \( \Psi \) according to Def. 3.5 and this model chains as desired with the past. See Sect. 8 for a detailed analysis of \((k, 2, 2)\)-Bell-test configurations.

In the trial model considered in the previous paragraph, the observable probability distributions of the inputs and outputs form the set of quantum-realizable distributions for this configuration, which is a subset of non-signaling distributions. The distribution \( \mu(ABXY) \) is non-signaling if \( \mu(A|XY) = \mu(A|X) \) and \( \mu(B|XY) = \mu(B|Y) \), so a station’s observed output distribution does not depend on the inputs of the other station. We can drop the
assumption that the devices are quantum and consider the trial model where the only restriction is that conditional on $E$, the observed probability distributions are non-signaling. This idea is captured by the maximal extension of the non-signaling distributions according to Def. 3.3. While it is not realistic at this time to think that super-quantum devices exist and can be exploited by an otherwise quantum entity $E$, that randomness can be generated for this model is of fundamental interest. Caution is required when reusing super-quantum devices in multiple protocols as composability may be compromised in ways that are not yet accounted for.

We remark that there is no restriction on the dynamics of the devices between trials, nor is there any reason to explicitly represent this dynamics. The model keeps track only of the state of $E$, and with the formulation of the trial models as maximal extensions or induced models, any quantum systems or quantum operations that the devices use over the course of the experiment are subsumed by the trial models and the chaining constructions.

If the inputs are published or may become known to $E$, final probabilities and entropies are conditioned on the inputs. For randomness generation, one option is to estimate the joint min-entropy of inputs and outputs conditional on the side information and eliminate the input entropy by subtracting the number of bits that generated the inputs before applying an extractor, see Protocol 3. For input distributions with low entropy per trial, this is inefficient, so we need a direct method of conditioning on inputs. Direct methods developed so far require that model chaining is restricted to chaining with conditionally independent inputs according to Def. 3.9, which imposes an additional restriction on the relationship between the next input and the past. The conditional independence restriction is satisfied if the input distribution is fixed and the inputs are assumed to be independent of the devices and $E$. More generally, it is satisfied if the source for the inputs has only classical initial correlations with the devices and $E$, so that given a classical part of $E$ the input distribution is independent of the devices and the quantum part of $E$. It is desirable to have models that can capture restricted interactions between $E$ and the devices. Consider the case where $E$ controls the source of the states used by the devices for producing the outputs. We assume that the interaction is representable by a strictly one-way communication, which means that for a given trial, $E$ includes a subsystem $S$ that is prepared and then transferred permanently to the devices. All such transfers can be time-shifted to before the protocol to return to the situation of the strictly non-interacting $E$ already discussed. A more challenging and interesting situation is where $E$ learns the inputs of the past trials before preparing a state and transferring it to the devices for the next trial. For this situation we can start with the model for the past trials, close under $Z$-conditional quantum operations, then use chaining, with conditionally independent inputs if necessary. The $Z$-conditional quantum operations model the change of state of $E$ when $E$ prepares a state in a source subsystem after having learned the previous inputs and transfers the subsystem to the devices.

We finish this section with EAT models, which are the models that are determined by EAT channel chains as required to apply the EAT for randomness generation. The term “EAT channel” is from Refs. 5, 16, but for an authoritative definition and statement of the EAT, see Ref. 4. An EAT channel chain is a sequence of CPTP maps $N_i$ composed in a specific way. As defined in Ref. 16 (Def. 5), $N_i$ is a CPTP map transforming system $R_{i-1}$ into $C_iZ_iR_i$, where $C_i$ here is $A_iB_i$ there and $Z_i$ here is $I_i$ there. The systems $R_i$ represent the devices used for trial $i$. The definition of EAT channels also includes a CV $X_i$ that is determined by $C_i$ and $Z_i$. Because it is determined, $X_i$ plays no role in our treatment.
For the EAT the CVs $X_i$ indirectly enable the possibility that the affine (or convex) min-tradeoff function used in the EAT can quantify the final conditional min-entropy in a way that depends on $i$. This in turn allows use of different types of trials in a single sequence, provided that the type of the $i$th trial is determined by information that was or could have been public before the start of the experiment. For QPE this is readily accounted for by the built-in option for dependence on the past of both the models and the QEFs.

The initial state of an EAT channel chain is a joint state of $R_0E$. An experiment consists of applying the $N_i$ sequentially to the system $R_{i-1}$ without touching $E$ or the previously generated CVs. That is, for the $i$th trial, $N_i \otimes 1_E$ is applied to the quantum systems. The Markov chain condition applies at each step, namely for the state after applying $N_i$ it is required that

$$C_{<i} \leftrightarrow Z_{<i}E \leftrightarrow Z_i.$$  

(3.7)

Since after time-shifting one-way communications there is no interaction between $E$ and the devices (or the CVs) after the initial state is determined, this fits the non-interacting scenario introduced above. Each $N_i$ can be expressed as a POVM $P^{(i)}_{C_iZ_i}$ of $R_i$ with outcome $C_iZ_i$ followed by an outcome-conditional CPTP map to transform $R_i$ into $R_i$. As far as the EAT is concerned, the relevant properties are captured by associating with each trial the model induced by $P_i = \{P^{(i)}_{C_iZ_i}\}$ on $C_iZ_iE$ in the sense that the EAT applies to chains of these models. After the experiment is formulated in terms of models in our framework, the Markov chain condition for the EAT channel chain is equivalent to the requirement that the model is chained with conditionally independent inputs.

4. QUANTUM ESTIMATION FACTORS

4.1. Definition and Equivalent Conditions

Definition 4.1. The real-valued function $F(CZ)$ is a quantum estimation factor (QEF) with power $\beta > 0$ for $C|Z$ and the model $C(CZ)$ if $F(CZ) \geq 0$ and for all $\rho(CZ) \in C(CZ)$ with $\rho \neq 0$, $F(CZ)$ satisfies the QEF inequality with power $\beta$ at $\rho(CZ)$ for $C|Z$ given by

$$\sum_{cz} F(cz) R_\alpha (\rho(cz)|\rho(z)) \leq R_\alpha (\rho|\rho) = \text{tr}(\rho).$$  

(4.1)

The real-valued function $F(CZ)$ is a Petz quantum estimation factor (QEFP) with power $\beta > 0$ for $C|Z$ and the model $C(CZ)$ if $F(CZ) \geq 0$ and for all $\rho(CZ) \in C(CZ)$ with $\rho \neq 0$, $F(CZ)$ satisfies the QEFP inequality with power $\beta$ at $\rho(CZ)$ for $C|Z$ given by

$$\sum_{cz} F(cz) P_\alpha (\rho(cz)|\rho(z)) \leq P_\alpha (\rho|\rho) = \text{tr}(\rho).$$  

(4.2)

Both sides of the QEF and QEFP inequalities are positive homogeneous of degree 1 in $\rho(CZ)$. It follows that for $F(CZ)$ to be a QEF (or QEFP), it is necessary and sufficient that the QEF (or QEFP) inequality holds for normalized distributions in $N(C(CZ))$. For normalized $\rho(CZ)$, the right-hand side of the QEF and QEFP inequalities evaluate to 1.

We use QEFPs primarily as a tool for constructing QEFs.

Lemma 4.2. If $F(CZ)$ is a QEFP with power $\beta \leq 1$, then $F(CZ)$ is a QEF with power $\beta$. 

Proof. It suffices to apply the inequality $P_\alpha (\sigma | \tau ) \geq R_\alpha (\sigma | \tau )$ (Lem. 2.10) to each summand of the QEF inequality.

The next lemmas give conditions for QEFs that can be used when $C$ is closed under appropriate pCP maps. The first is an alternative form that may be useful for finding QEFs, particularly for the special cases in Sect. 4.2. The second is needed when constructing QEFs by QEF chaining. We remind that according to our marginalization convention, if $\rho(CZ)$ is a state of $CZE$, we write the marginal state of $E$ as $\rho = \sum_{cz} \rho(cz)$.

**Lemma 4.3.** Let $C(CZ)$ be a model such that for all $\tau(CZ) \in C(CZ)$ we have the closure condition $\tau^{\beta/2} \tau(CZ) \tau^{\gamma/2} \in C(CZ)$ for $\gamma = \beta$ and $\gamma = -\beta/\alpha$. Then $F(CZ)$ is a QEF with power $\beta$ for $C|Z$ and $C(CZ)$ iff $F \geq 0$ and for all $\tau(CZ) \in C$,

$$
\sum_{cz} F(cz) R_\alpha \left( \tau^{\beta/2} \tau(cz) \tau^{\beta/2} | \tau^{\beta/2} \tau(z) \tau^{\beta/2} \right) \leq R_\alpha (\tau | 1) = \text{tr}(\tau^\alpha).
$$

The closure condition in the lemma is satisfied if $C(CZ)$ is pCP-closed.

Proof. Suppose that $F(CZ)$ is a QEF with power $\beta$ for $C|Z$ and $C(CZ)$. Then $F(CZ) \geq 0$. For any $\tau(CZ) \in C$, define $\rho(CZ) = \tau^{\beta/2} \tau(CZ) \tau^{\beta/2} \in C(CZ)$. Since $\rho = \tau^\alpha$, the right-hand side of Eq. 4.3 is $\text{tr}(\rho)$, matching the right-hand side of the QEF inequality at $\rho(CZ)$. Since $\rho(Z) = \tau^{\beta/2} \tau(Z) \tau^{\beta/2}$, the left-hand side of Eq. 4.3 matches that of the QEF inequality. Since the QEF inequality at $\rho(CZ)$ is satisfied by assumption, so is Eq. 4.3.

Suppose that $F(CZ)$ satisfies the condition in the lemma. Then $F(CZ) \geq 0$. To show that $F(CZ)$ is a QEF, consider any $\rho(CZ) \in C(CZ)$. To verify the QEF inequality at $\rho(CZ)$, we reverse the transformation of the previous paragraph by defining $\tau(CZ) = \rho^{-\beta/(2\alpha)} \rho(CZ) \rho^{-\beta/(2\alpha)} \in C(CZ)$. We have $\tau = \rho^{1/\alpha}$, so $\tau^{\beta/2} \tau(CZ) \tau^{\beta/2} = \rho(CZ)$ and $\tau^{\beta/2} \tau(z) \tau^{\beta/2} = \rho(Z)$. The expressions in Eq. 4.3 are therefore identical to the corresponding ones in the QEF inequality at $\rho(CZ)$, so the former implies the latter, as desired.

**Lemma 4.4.** Let $F(CZ)$ be a QEF with power $\beta$ for $C|Z$ and $C(CZ)$. Consider $\sigma(CZ) \in C(CZ)$ and $\zeta(Z) \in S(ZE)$ such that $\sigma(Z) \ll \zeta(Z)$ and define

$$
\xi(UZ) = \sigma(Z) [U = 0] + \zeta(Z) [U = 1],
$$

$$
\chi = \zeta^{\beta/(2\alpha)} \sigma^{\beta/(2\alpha)},
$$

$$
\rho(CZ) = \chi^{\beta/2} \zeta^{\beta/(2\alpha)} \sigma(CZ) \zeta^{\beta/(2\alpha)} \chi^{\beta/2},
$$

where $\text{Rng}(U) = \{0, 1\}$. If $\xi(UZ) \in U \leftrightarrow E \leftrightarrow Z$ and $\rho(CZ) \in C(CZ)$, then

$$
\sum_{cz} F(cz) R_\alpha (\sigma(cz) | \zeta(z)) \leq R_\alpha (\sigma | \zeta).
$$

The condition $\rho(CZ) \in C(CZ)$ is satisfied if $C(CZ)$ is pCP-closed. The main purpose of the lemma is to enable a change in the conditioner in the QEF inequality from the marginal state to another one. This requires conditions on the relationship between the two conditioners. The conditions are expressed by introducing the auxiliary CV $U$ and state $\xi(UZ)$ and include the short Markov chain condition in the lemma. The lemma simplifies
in the absence of inputs or when the input distribution is fixed and known, see the next section.

Proof. By the definition of short quantum Markov chains, there is a factorization \( \mathcal{H}(E) = \bigoplus_i U_i \otimes \mathcal{Z}_i \oplus \mathcal{R} \) such that \( \sigma(Z) = \bigoplus_i \sigma_i \otimes \xi_i(Z) \) and \( \zeta(Z) = \bigoplus_i \zeta_i \otimes \xi_i(Z) \), where \( \sigma(Z) \ll \zeta(Z) \) implies \( \sigma_i \ll \zeta_i \) for each \( i \). In order to derive the inequality in Eq. 4.5 from the QEF inequality, we can assure a match of the right-hand sides with

\[
\rho = \left( \zeta^{-\beta/(2\alpha)} \sigma \zeta^{-\beta/(2\alpha)} \right)^\alpha = \left( \zeta^{-\beta/(2\alpha)} \sigma \zeta^{-\beta/(2\alpha)} \right)^{\beta/2} = \left( \zeta^{-\beta/(2\alpha)} \sigma \zeta^{-\beta/(2\alpha)} \right) \left( \zeta^{-\beta/(2\alpha)} \sigma \zeta^{-\beta/(2\alpha)} \right)^{\beta/2}. \tag{4.6}
\]

This motivates the definitions of \( \chi \) and \( \rho(CZ) \). The support assumptions ensure that the supports of \( \sigma \) and \( \sigma(CZ) \) are contained in that of \( \zeta \).

For a match of the left-hand sides of the target inequalities, we need to verify that \( \mathcal{R}_\alpha (\rho(CZ)|\rho(Z)) = \mathcal{R}_\alpha (\sigma(CZ)|\zeta(Z)) \). For this it suffices that

\[
\rho(Z)^{-\beta/(2\alpha)} \rho(CZ) \rho(Z)^{-\beta/(2\alpha)} \sim_U \zeta(Z)^{-\beta/(2\alpha)} \sigma(CZ) \zeta(Z)^{-\beta/(2\alpha)}, \tag{4.7}
\]

where \( \sim_U \) denotes equality up to conjugation by a unitary operator, or equivalently, that the two sides have the same spectrum with multiplicities. The support assumptions ensure that the support of \( \sigma(CZ) \) is contained in that of \( \zeta(Z) \) for the right-hand side of the spectral equivalence. Starting from the left-hand side, we get

\[
\rho(Z)^{-\beta/(2\alpha)} \rho(CZ) \rho(Z)^{-\beta/(2\alpha)} = \rho(Z)^{-\beta/(2\alpha)} \chi^{\beta/2} \zeta^{-\beta/(2\alpha)} \sigma(CZ) \zeta^{-\beta/(2\alpha)} \chi^{\beta/2} \rho(Z)^{-\beta/(2\alpha)} \sim_U \sigma(CZ)^{1/2} \zeta^{-\beta/(2\alpha)} \chi^{\beta/2} \rho(Z)^{-\beta/\alpha} \chi^{\beta/2} \zeta^{-\beta/(2\alpha)} \sigma(CZ)^{1/2} \rho(Z)^{-\beta/(2\alpha)} \rho(CZ) \rho(Z)^{-\beta/(2\alpha)} \sim_U \sigma(CZ)^{1/2} \zeta^{-\beta/(2\alpha)} \chi^{\beta/2} \left( \chi^{\beta/2} \zeta^{-\beta/(2\alpha)} \sigma(Z) \zeta^{-\beta/(2\alpha)} \chi^{\beta/2} \right)^{-\beta/\alpha} \chi^{\beta/2} \zeta^{-\beta/(2\alpha)} \sigma(CZ)^{1/2}, \tag{4.8}
\]

where the equivalence in the third line follows from \( A^\dagger A \sim_U AA^\dagger \) for all operators \( A \). The expression between the two terms \( \sigma(CZ)^{1/2} \) factors with respect to the representation of \( \mathcal{H}(E) \), so we can compute each factor separately. First determine

\[
\chi = \bigoplus_i \zeta_i^{-\beta/(2\alpha)} \sigma_i \zeta_i^{-\beta/(2\alpha)} \otimes \xi_i^{1/\alpha} \tag{4.9}
\]

and define \( \chi_i = \zeta_i^{-\beta/(2\alpha)} \sigma_i \zeta_i^{-\beta/(2\alpha)} \) so that \( \chi = \bigoplus_i \chi_i \otimes \xi_i^{1/\alpha} \). From this,

\[
\chi^{\beta/2} \zeta^{-\beta/(2\alpha)} = \bigoplus_i \chi_i^{\beta/2} \zeta_i^{-\beta/(2\alpha)} \otimes \mathbf{1}_i, \tag{4.10}
\]

where \( \mathbf{1}_i \) is the projector onto the support of \( \xi_i \) in \( \mathcal{Z}_i \). Since \( \sigma(Z) = \bigoplus_i \sigma_i \otimes \xi_i(Z) \), we have
for the inner expression on the right-hand side of Eq. \(4.8\)

\[
\left( \chi^{\beta/2} \zeta^{-\beta/(2\alpha)} \sigma(Z)^{-\beta/(2\alpha)} \chi^{\beta/2} \right)^{-\beta/\alpha} = \bigoplus_i \left( \chi_i^{\beta/2} \zeta_i^{-\beta/(2\alpha)} \sigma_i^{\beta/(2\alpha)} \chi_i^{\beta/2} \right)^{-\beta/\alpha} \otimes \xi_i(Z)^{-\beta/\alpha}
\]

\[
= \bigoplus_i \left( \chi_i^{\beta/2} \chi_i^{-\beta/\alpha} \right) \otimes \xi_i(Z)^{-\beta/\alpha}
\]

\[
= \bigoplus_i \chi_i^{-\beta} \otimes \xi_i(Z)^{-\beta/\alpha}. \quad (4.11)
\]

Define the support projectors \(\Pi_i = [\chi_i > 0]\) and \(\Pi = [\chi > 0] = \bigoplus_i \Pi_i \otimes 1_i\). Substituting the identities obtained and continuing from the end of Eq. \(4.8\) we get

\[
\rho(Z)^{-\beta/(2\alpha)} \rho(CZ) \rho(Z)^{-\beta/(2\alpha)}
\]

\[
\sim_U \sigma(CZ)^{1/2} \left( \bigoplus_i \zeta_i^{-\beta/(2\alpha)} \chi_i^{-\beta/\alpha} \chi_i^{-\beta/\alpha} \zeta_i^{-\beta/(2\alpha)} \otimes \xi_i(Z)^{-\beta/\alpha} \right) \sigma(CZ)^{1/2}
\]

\[
= \sigma(CZ)^{1/2} \left( \bigoplus_i \zeta_i^{-\beta/(2\alpha)} \Pi_i \zeta_i^{-\beta/(2\alpha)} \otimes \xi_i(Z)^{-\beta/\alpha} \right) \sigma(CZ)^{1/2}
\]

\[
= \sigma(CZ)^{1/2} \left( \bigoplus_i \zeta_i^{-\beta/(2\alpha)} \Pi_i \zeta_i^{-\beta/(2\alpha)} \otimes \xi_i(Z)^{-\beta/(2\alpha)} 1_i \xi_i(Z)^{-\beta/(2\alpha)} \right) \sigma(CZ)^{1/2}
\]

\[
= \sigma(CZ)^{1/2} \zeta(Z)^{-\beta/(2\alpha)} \Pi \zeta(Z)^{-\beta/(2\alpha)} \sigma(CZ)^{1/2}
\]

\[
\sim_U \Pi \zeta(Z)^{-\beta/(2\alpha)} \sigma(CZ) \zeta(Z)^{-\beta/(2\alpha)} \Pi. \quad (4.12)
\]

The support of \(\zeta(Z)^{-\beta/(2\alpha)} \sigma(CZ) \zeta(Z)^{-\beta/(2\alpha)}\) is contained in that of \(\zeta(Z)^{-\beta/(2\alpha)} \sigma(Z) \zeta(Z)^{-\beta/(2\alpha)}\), which is the direct sum of the supports of \(\zeta_i^{-\beta/(2\alpha)} \sigma_i \zeta_i^{-\beta/(2\alpha)} \otimes \xi_i(Z)^{1/\alpha}\) and therefore contained in the support of \(\chi\). The support projector \(\Pi\) can therefore be eliminated from the final expression in Eq. \(4.12\) to finish the proof of the lemma.

\[
4.2. \text{ QEF Conditions for Special Cases}
\]

The conditions in Eqs. \(4.1\), \(4.3\) and \(4.5\) simplify when the probability distribution of \(Z\) is given and independent of \(E\).

**Lemma 4.5.** Let \(\mu(Z)\) be a probability distribution and \(C(CZ) = \mu(Z) \times C(C|Z)\). Consider \(F(CZ) \geq 0\). Then \(F(CZ)\) is a QEF with power \(\beta\) for \(C|Z\) and \(C(CZ)\) iff for all \(\rho(CZ) \in C(CZ)\),

\[
\sum_{cz} F(cz) \mu(z) R_\alpha (\rho(c|z)|\rho) \leq \text{tr}(\rho). \quad (4.13)
\]

If \(C(C|Z)\) is pCP-closed, then \(F(CZ)\) is a QEF with power \(\beta\) for \(C|Z\) and \(C(CZ)\) iff for all \(\tau(CZ) \in C(CZ)\),

\[
\sum_{cz} F(cz) \mu(z) \text{tr}(\tau(c|z)^\alpha) \leq \text{tr}(\tau^\alpha). \quad (4.14)
\]

If \(C(C|Z)\) is pCP-closed and \(F(CZ)\) is a QEF with power \(\beta\) for \(C|Z\) and \(C(CZ)\), then for
all $\sigma(CZ) \in C(CZ)$ and $\zeta \gg \sigma$,

$$\sum_{cz} F(cz) \mu(z) R_\alpha (\sigma(cz)|\zeta) \leq R_\alpha (\sigma|\zeta).$$

(4.15)

**Proof.** The first equivalence follows by substitution in the QEF definition and the second by substitution in Lem. 4.3. For the last claim, define $\zeta(Z) = \mu(Z)\zeta$. The distribution $\xi(UZ)$ defined in Lem. 4.4 can be written as $\xi(UZ) = (\sigma[U = 0] + \zeta[U = 1]) \mu(Z)$, which satisfies $\xi(UZ) \in U \leftrightarrow E \leftrightarrow Z$ with respect to the trivial factorization $H(E) = H(E) \otimes \mathbb{C}$. The claim then follows by substitution in Eq. 4.5.

The QEF conditions further simplify in the absence of inputs, namely when $Z$ is trivial and can be omitted.

**Lemma 4.6.** Let $C(\cdot)$ be a model and $F(C) \geq 0$. Then $F(C)$ is a QEF with power $\beta$ for $C(\cdot)$ iff for all $\rho(C) \in C(\cdot)$,

$$\sum_c F(c) R_\alpha (\rho(c)|\rho) \leq \text{tr}(\rho).$$

(4.16)

If $C$ is pCP-closed, then $F(C)$ is a QEF with power $\beta$ for $C(\cdot)$ iff for all $\tau(C) \in C(\cdot)$,

$$\sum_c F(c) \text{tr}(\tau(c)^\alpha) \leq \text{tr}(\tau^\alpha).$$

(4.17)

If $C$ is pCP-closed, and $F(C)$ is a QEF with power $\beta$ for $C(\cdot)$, then for all $\sigma(C) \in C(\cdot)$ and $\zeta \gg \sigma$,

$$\sum_c F(c) R_\alpha (\sigma(c)|\zeta) \leq R_\alpha (\sigma|\zeta).$$

(4.18)

**Proof.** Apply Lem. 4.5 and simplify.

4.3. QEF Properties

**Lemma 4.7.** For $C|Z$ and all models, the function $F(CZ) = 1$ is a QEF with power $\beta$ for each $\beta > 0$, and a QEFP with power $\beta$ for each $\beta \in (0, 1]$.

**Proof.** It suffices to verify Eq. 4.11

$$\sum_{cz} F(cz) R_\alpha (\rho(cz)|\rho(z)) = \sum_{cz} R_\alpha (\rho(cz)|\rho(z))$$

$$= \sum_z \sum_c R_\alpha (\rho(cz)|\rho(z))$$

$$\leq \sum_z R_\alpha (\rho(z)|\rho(z))$$

$$= \sum_z \text{tr}(\rho(z))$$

$$= \text{tr}(\rho),$$

(4.19)
where we applied Lem. 2.13 for the inequality in the third line. In this argument, we can replace the sandwiched by the Petz Rényi power provided \( \beta \leq 1 \).

**Lemma 4.8.** Let \( F(CZ) \) be a QEF with power \( \beta \) for \( C|Z \) and \( C(CZ) \). Then for all \( \beta' \geq \beta \), \( F(CZ) \) is a QEF with power \( \beta' \) for \( C|Z \) and \( C(CZ) \).

**Proof.** Consider any \( \rho(CZ) \in C(CZ) \). All expressions in the calculation below are homogeneous of the same degree, so we may assume that \( \tr(\rho) = 1 \). If not, it suffices to rescale \( \rho(CZ) \) to ensure this condition. In view of the QEF inequality, it suffices to show that the function \( g_{cz} : \beta' \mapsto R_{1+\beta'}(\rho(cz)|\rho(z)) \) is non-increasing for all \( cz \). According to Lem. 4.7, \( \sum_{cz} R_{1+\beta'}(\rho(cz)|\rho(z)) \leq 1 \), and since the summands are non-negative, for each \( cz \) we have \( R_{1+\beta'}(\rho(cz)|\rho(z)) \leq 1 \). For the \( cz \) with \( \rho(cz) = 0 \), \( R_{1+\beta'}(\rho(cz)|\rho(z)) = 0 \) for all \( \beta' \) and \( g_{cz} \) is non-increasing. For the \( cz \) with \( \rho(cz) > 0 \) the function \( \log(g_{cz}) \) is non-positive. Log-convexity of Rényi powers (Lem. 2.14) implies that the slope of \( \log(g_{cz}) \) is non-decreasing. In view of \( -\infty < \log(g_{cz}) \leq 0 \), the slope of \( \log(g_{cz}) \) at any \( \beta' \) cannot become positive, otherwise when \( \beta' \nearrow \infty \) the value of \( \log(g_{cz}) \) would become positive. Thus \( \log(g_{cz}) \) is non-increasing and since \( x \mapsto \log(x) \) is order-preserving, \( g_{cz} \) is also non-increasing.

**Lemma 4.9.** Let \( F(CZ) \) be a QEF with power \( \beta \) for \( C|Z \) and \( C(CZ) \). Then for \( 0 < \gamma \leq 1 \), \( F(CZ)^{\gamma} \) is a QEF with power \( \gamma \beta \) for \( C|Z \) and \( C(CZ) \). This also holds with “QEF” replaced by “QEFP”.

The transformation \( F \mapsto F^{\gamma} \) in the lemma is referred to as power reduction by \( \gamma \).

**Proof.** Consider any \( \rho(CZ) \in C(CZ) \). All expressions in the calculation below are homogeneous of the same degree, so we may assume that \( \tr(\rho) = 1 \). Define the probability distribution \( \mu(CZ) \) by \( \mu(cz) = \tr(\rho(cz)) \). We check the QEF inequality at \( \rho(CZ) \):

\[
\sum_{cz} F(cz)^{\gamma} R_{1+\gamma\beta}(\rho(cz)|\rho(z)) = \sum_{cz} F(cz)^{\gamma} \mu(cz) \hat{R}_{1+\gamma\beta}(\rho(cz)|\rho(z))
= \sum_{cz} \mu(cz) \left( F(cz) \hat{R}_{1+\gamma\beta}(\rho(cz)|\rho(z)) \right)^{1/\gamma} 
\leq \left( \sum_{cz} \mu(cz) F(cz) \hat{R}_{1+\gamma\beta}(\rho(cz)|\rho(z)) ^{1/\gamma} \right)^{\gamma}, \tag{4.20}
\]

since for \( \gamma \in (0, 1] \) the function \( x \mapsto x^{\gamma} \) is concave and the sums are expectations with respect to \( \mu(CZ) \). By monotonicity of Rényi powers (Lem. 2.15), we have \( \hat{R}_{1+\gamma\beta}(\rho(cz)|\rho(z)) ^{1/(\gamma\beta)} \leq \hat{R}_{1+\beta}(\rho(cz)|\rho(z)) ^{1/\beta} \), so we can continue where we left off to get

\[
\sum_{cz} F(cz)^{\gamma} R_{1+\gamma\beta}(\rho(cz)|\rho(z)) \leq \left( \sum_{cz} F(cz) \mu(cz) \hat{R}_{1+\beta}(\rho(cz)|\rho(z)) \right)^{\gamma}
= \left( \sum_{cz} F(cz) R_{1+\beta}(\rho(cz)|\rho(z)) \right)^{\gamma}
\leq 1, \tag{4.21}
\]

since \( F \) is assumed to be a QEF with power \( \beta \). The lemma follows by arbitrariness of \( \rho(CZ) \in C(CZ) \). In this argument, we can replace the sandwiched by the Petz Rényi power. \( \square \)
Since the inequality in Eq. 4.1 is linear in $F(CZ)$, the set of QEFs is convex. By positive homogeneity of the QEF inequality in $\rho(CZ)$, it suffices to check the trace-normalized $\rho(CZ) \in \mathcal{N}(\mathcal{C}(CZ))$. Further, as a consequence of the next lemma, it suffices to check the QEF inequalities on any subset of $\mathcal{N}(\mathcal{C}(CZ))$ whose convex closure contains $\mathcal{N}(\mathcal{C}(CZ))$.

**Lemma 4.10.** $F(CZ)$ is a QEF with power $\beta$ for $C|Z$ and $\mathcal{C}(CZ)$ iff $F(CZ)$ is a QEF with power $\beta$ for $C|Z$ and $\text{Cone}(\mathcal{C}(CZ))$. This also holds with “QEF” replaced by “QEFP” provided $\beta \leq 1$.

**Proof.** It suffices to check that if the QEF inequality holds at $\rho_i(CZ) \in \mathcal{C}(CZ)$ for $i \in I$, then it holds at every convex combination $\rho(CZ) = \sum_i \lambda_i \rho_i(CZ)$. By joint convexity of conditional Rényi powers (Lem. 2.16),

$$
\mathcal{R}_\alpha (\rho(CZ)|\rho(Z)) \leq \sum_i \lambda_i \mathcal{R}_\alpha (\rho_i(CZ)|\rho_i(Z)) .
$$

(4.22)

Therefore

$$
\sum_{cz} F(cz) \mathcal{R}_\alpha (\rho(CZ)|\rho(Z)) \leq \sum_{cz} F(cz) \sum_i \lambda_i \mathcal{R}_\alpha (\rho_i(cz)|\rho_i(z))
$$

$$
= \sum_i \lambda_i \sum_{cz} F(cz) \mathcal{R}_\alpha (\rho_i(cz)|\rho_i(z))
$$

$$
\leq \sum_i \lambda_i \text{tr}(\rho_i)
$$

$$
= \text{tr} \left( \sum_i \lambda_i \rho_i \right)
$$

$$
= \text{tr}(\rho).
$$

(4.23)

In this argument, we can replace the sandwiched by the Petz Rényi power provided $\beta \leq 1$.

It may be difficult to determine manageable subsets of $\mathcal{N}(\mathcal{C}(CZ))$ whose convex closure contains $\mathcal{N}(\mathcal{C}(CZ))$. If $\text{Cone}(\mathcal{C}'(CZ)) \supset \mathcal{C}(CZ)$, then any QEF for $\mathcal{C}'(CZ)$ is a QEF for $\mathcal{C}(CZ)$, so a strategy for constructing QEFs is to find better behaved models $\mathcal{C}'(CZ)$ whose convex closure contains $\mathcal{C}(CZ)$.

According to the next lemma, QEFs of a model are QEFs of the closure of the model under $Z$-conditional quantum operations.

**Lemma 4.11.** Let $\mathcal{C}(CZ)$ be a model for $CZE$ and let $\text{CPTP}_Z(\mathcal{C}(CZ))$ be the set of distributions that can be obtained by applying a $Z$-conditional quantum operation to members of $\mathcal{C}(CZ)$. Then $F(CZ)$ is a QEF with power $\beta$ for $C|Z$ and $\mathcal{C}(CZ)$ iff $F(CZ)$ is a QEF with power $\beta$ for $C|Z$ and $\text{CPTP}_Z(\mathcal{C}(CZ))$. This also holds with “QEF” replaced by “QEFP” provided $\beta \leq 1$.

**Proof.** The lemma follows from the data-processing inequality for Rényi powers (Lem. 2.17). It suffices to check that if the QEF inequality holds at $\rho(CZ) \in \mathcal{C}(CZ)$ and $\mathcal{E}_Z$ is a $Z$-conditional quantum operation, then it holds at $\sigma(CZ) = \mathcal{E}_Z(\rho(CZ))$:

$$
\sum_{cz} F(cz) \mathcal{R}_\alpha (\sigma(cz)|\sigma(z)) = \sum_{cz} F(cz) \mathcal{R}_\alpha (\mathcal{E}_Z(cz)|\mathcal{E}_Z(z))
$$
\[ \leq \sum_{cz} F(cz) R_\alpha (\rho(cz)|\rho(z)) \]
\[ \leq \text{tr}(\rho) \]
\[ = \sum_z \text{tr}(\rho(z)) \]
\[ = \sum_z \text{tr}(E_z(\rho(z))) \]
\[ = \text{tr}(\sigma), \quad (4.24) \]

since each \( E_z \) is trace-preserving. Again, in this argument, we can replace the sandwiched by the Petz Rényi power provided \( \beta \leq 1 \).

\[ \square \]

4.4. Chaining QEFs

The next theorem shows that QEFs can be chained with conditionally independent inputs. We do not know whether this is true for QEFPs.

**Theorem 4.12.** Let \( C(CZ) \) be a model for \( CZE \) and for each \( cz \), let \( C_{cz}(CZ) \) be a pCP-closed model for \( CZ \). If \( G \) is a QEF with power \( \beta \) for \( C|Z \) and \( C(CZ) \), and for each \( cz, F_{cz} \) is a QEF with power \( \beta \) for \( C|Z \) and \( C_{cz}(CZ) \), then \( G(CZ) F_{cz}(CZ) \) is a QEF with power \( \beta \) for \( C|Z \) and \( C(CZ) \circ_{Z} C_{cz}(CZ) \).

For the models constructed for experiments consisting of sequences of trials discussed in Sect. 3.5, the trial models are maximal extensions or induced and therefore pCP-closed since pCP maps are special cases of CP maps (Lems. 3.4 and 3.6). The pCP-closure condition can be weakened by taking advantage of the specific membership condition in Lem. 4.4 as indicated in the proof.

**Proof.** Consider any \( \sigma(CZ|Z) \in C(CZ) \circ_{Z} C_{cz}(CZ) \). We show below that for each \( cz, \)

\[ \sum_{cz} F_{cz}(cz) R_\alpha (\sigma(czcz)|\sigma(zz)) \leq R_\alpha (\sigma(cz)|\sigma(z)). \quad (4.25) \]

Once this is shown, the theorem follows from

\[ \sum_{cz \in c_{cz}} G(cz) F_{cz}(cz) R_\alpha (\sigma(czcz)|\sigma(zz)) \]
\[ = \sum_{cz} G(cz) \sum_{cz} F_{cz}(cz) R_\alpha (\sigma(czcz)|\sigma(zz)) \]
\[ \leq \sum_{cz} G(cz) R_\alpha (\sigma(cz)|\sigma(z)) \]
\[ \leq R_\alpha (\sigma|\sigma), \quad (4.26) \]

where we applied Eq. 4.25, model chaining Def. 3.7, and the assumption that \( G \) is a QEF for \( C|Z \) and \( C(CZ) \). Thus \( G(CZ) F_{cz}(CZ) \) is a QEF as claimed.

To show Eq. 4.25 we apply Lem. 4.4 with \( \sigma(CZ) \) there replaced by \( \sigma(czCZ) \) here, \( \zeta(Z) \) there by \( \sigma(zZ) \) here, and \( F(CZ) \) there by \( F_{cz}(CZ) \) here. By definition of chaining, \( \sigma(czCZ) \in C_{cz}(CZ) \). We verify that the Markov chain condition there follows from
σ(CZz) ∈ C ↔ ZE ↔ Z according to the definition of chaining with conditionally independent inputs. For each z, there is a factorization Η(E) = ⊔i D i ⊗ Z i ⊕ R for which σ(CZz) = ⊔i σi(C) ⊗ ζi(Z) for some σi(C) and ζi(Z) that depend implicitly on z. This implies σ(zZ) = ⊔i σi ⊗ ζi(Z). To verify the Markov chain condition of Lem. 4.4, we define ξ(U) = σ czZ [U = 0] + σ(zZ) [U = 1]. Then

\[ \xi(U) = \bigoplus_i (\sigma_i [U = 0] + \sigma_i [U = 1]) \otimes \zeta_i(Z), \] (4.27)

which implies ξ(U) ∈ U ↔ E ↔ Z. The membership condition of Lem. 4.4 is satisfied since the Ccz(CZ) are assumed to be pCP-closed. For the purpose of weakening this condition the explicit distributions that need to be in Ccz(CZ) are

\[ \rho(czCZ) = \chi(cz)^{\beta/2} \sigma(z)^{-\beta/(2\alpha)} \sigma(czZ) \sigma(z)^{-\beta/(2\alpha)} \chi(cz)^{\beta/2}, \] (4.28)

where

\[ \chi(cz) = \sigma(z)^{-\beta/(2\alpha)} \sigma(cz) \sigma(z)^{-\beta/(2\alpha)}. \] (4.29)

Although it is an immediate consequence of the results so far, we give the next corollary for emphasis, and so that we can use it explicitly when discussing models relevant to experimental configurations.

**Corollary 4.13.** In Thm. 4.12, we may close C(CZ) under Z-conditional quantum operations and positive combinations before chaining.

**Proof.** This follows from Thm. 4.12 after applying Lems. 4.10 and 4.11.

The Z-conditional quantum operations on C(CZ) may affect the quantum Markov chain condition, but in chaining with conditionally independent inputs, only cases where the condition survives are passed on to the chained model. Since chaining is monotone in the models being chained, no states are lost by closing C(CZ) before chaining.

In Sect. 3.5 we mentioned some situations where the quantum Markov chain condition applies, such as when the distribution of the inputs is fixed and independent of E. When such situations do not apply, we rely on physical constraints satisfied by the experiments to make sure that the actual states after the trials satisfy the quantum Markov chain condition. Alternatively, we use the strategy where input entropy is eliminated when the extractor is applied and QEFs are designed without conditioning on inputs.

### 4.5. QEFs as Estimators

QEFs and QEFPs can be interpreted as estimators of normalized Rényi powers. We formalize this interpretation for QEFs. Let F(CZ) be a QEF with power β for C|Z and C(CZ). Consider ρ(CZ) ∈ N(C(CZ)). We can interpret 1/(εF(CZ)) as a level-ε confidence upper bound on \( R_\alpha (\rho(CZ)|\rho(Z)) \) in the following sense:
Theorem 4.14. Let $F(CZ)$ be a QEF with power $\beta$ for $C|Z$ and $C(CZ)$. Then for all $\rho(CZ) \in \mathcal{N}(C(CZ))$,

$$
\mathbb{P}_{\mu(CZ)} \left( 1/(\epsilon F(CZ)) < \hat{R}_\alpha (\rho(CZ)|\rho(Z)) \right) \leq \epsilon,
$$

where $\mu(CZ) = \text{tr}(\rho(CZ))$.

According to the theorem, the interval $[0, 1/(\epsilon F(CZ))]$ has coverage probability at least $1 - \epsilon$ for $\hat{R}_\alpha (\rho(CZ)|\rho(Z))$ which is what is required of a confidence interval at level $\epsilon$ (or confidence level $1 - \epsilon$).

**Proof.** According to the QEF inequality at $\rho(CZ)$,

$$
\mathbb{E}_{\mu(CZ)} \left( F(CZ) \hat{R}_\alpha (\rho(CZ)|\rho(Z)) \right) = \sum_{cz} \mu(cz) F(cz) \hat{R}_\alpha (\rho(cz)|\rho(z))
= \sum_{cz} F(cz) \text{tr}(\rho(cz)) \hat{R}_\alpha (\rho(cz)|\rho(z))
= \sum_{cz} F(cz) R_\alpha (\rho(cz)|\rho(z))
\leq 1.
$$

(4.31)

Since $F(CZ) \hat{R}_\alpha (\rho(CZ)|\rho(Z)) \geq 0$ and by the Markov inequality,

$$
\mathbb{P}_{\mu(CZ)} (F(CZ) \hat{R}_\alpha (\rho(CZ)|\rho(Z)) > 1/\epsilon) \leq \epsilon.
$$

(4.32)

The theorem follows by rearranging the inequality defining the event in the probability on the left-hand side.

We remark that the normalized $\alpha$-Rényi powers generalize the $\beta$-power of conditional probabilities when $\mathbb{E}$ is trivial. This motivates our terminology and the description of the framework as “quantum probability estimation”.

Lemma 4.15. Let $0 \leq \rho \ll \sigma$ and $p \geq 0$. Then

$$
p^\beta \text{tr}([\rho - p\sigma]_+) \leq p^\beta \text{tr}(\rho \| \rho - p\sigma > 0\|) \leq R_\alpha (\rho|\sigma).
$$

(4.33)

This lemma is one step in the proof of Prop. 6.2, Pg. 95 of Ref. [23], where it is applied with Petz Rényi entropy in mind. That it works for sandwiched Rényi entropy is established in the proof of Lem. B.4., Ref. [4].

**Proof.** The first inequality of the lemma follows from Lem. 2.1. For the second inequality, let $(|i\rangle)_{i=1}^k$ be an eigenbasis of $[\rho - p\sigma]_+$ ordered so that $|i\rangle$ has positive eigenvalue iff $i \in [l]$, where $l$ is the number of positive eigenvalues of $[\rho - p\sigma]_+$ counting multiplicity. Write $\rho_{ii} = \langle i| \rho |i\rangle$ and $\sigma_{ii} = \langle i| \sigma |i\rangle$. Because $\rho - p\sigma = [\rho - p\sigma]_+ - [p\sigma - \rho]_+$ and since $[\rho - p\sigma]_+$ and $[p\sigma - \rho]_+$ have orthogonal supports, we have $\text{tr}([\rho - p\sigma]_+) = \sum_{i=1}^l (\rho_{ii} - p\sigma_{ii})$ and for
each \( i \in [l] \), \( \rho_{ii} \geq p\sigma_{ii} \). Since \( \rho \ll \sigma \), \( \rho_{ii} > 0 \) implies \( \sigma_{ii} > 0 \). From the data-processing inequality for Rényi powers (Lem. 2.17) with respect to decoherence in the \((|i\rangle)_i \) basis,

\[
\mathcal{R}_\alpha (\rho|\sigma) \geq \mathcal{R}_\alpha \left( \sum_i \rho_{ii}^{\frac{1}{\beta}} \left| \sum_i \sigma_{ii}^{\frac{1}{\beta}} \right| \right)
\]

\[
= \sum_{i=1}^{k} \mathcal{R}_\alpha \left( \rho_{ii}^{\frac{1}{\beta}} | \sigma_{ii}^{\frac{1}{\beta}} \right)
\]

\[
= \sum_{i=1}^{k} \rho_{ii}^{\frac{\beta}{\sigma_{ii}^{\beta}}}
\]

\[
\geq \sum_{i=1}^{l} \rho_{ii}^{\frac{\beta}{\sigma_{ii}^{\beta}}}
\]

\[
\geq \sum_{i=1}^{l} \rho_{ii} p^{\beta},
\]

where the last inequality follows from \( \rho_{ii} \geq p\sigma_{ii} \) for all \( i \in [l] \). Continuing

\[
\mathcal{R}_\alpha (\rho|\sigma) \geq p^{\beta} \sum_{i=1}^{l} \text{tr} \left( \rho_i^\frac{1}{\beta} \right)
\]

\[
= p^{\beta} \text{tr} \left( \rho \sum_{i=1}^{l} \frac{1}{\beta} \right)
\]

\[
\geq p^{\beta} \text{tr} (\rho [\rho - p\sigma > 0]).
\]

The next theorem suggests another way in which QEFs can be interpreted as estimators. The statement is not far from a conditional min-entropy estimate.

**Theorem 4.16.** Let \( \rho(CZ) \in \mathcal{S}_1(CZE) \) and suppose that \( F(CZ) \geq 0 \) satisfies the QEF inequality with power \( \beta \) at \( \rho(CZ) \) for \( C|Z \). Then for all \( \epsilon > 0 \),

\[
\sum_{cz} \text{tr} \left( \rho(cz) - \frac{1}{(\epsilon F(cz))^{\frac{1}{\beta}}} \rho(z) \right) \leq \sum_{cz} \text{tr} \left( \rho(cz) - \frac{1}{(\epsilon F(cz))^{\frac{1}{\beta}}} \rho(z) > 0 \right) \leq \epsilon.
\]

The theorem does not require \( F(CZ) \) to be a QEF for a specific model.

**Proof.** The first inequality is an application of Lem. 2.1. For the second, we apply Lem. 4.15 as follows:

\[
\sum_{cz} \text{tr} \left( \rho(cz) - \frac{1}{(\epsilon F(cz))^{\frac{1}{\beta}}} \rho(z) > 0 \right) \leq \epsilon F(cz) \text{tr} \left( \rho(cz) - \frac{1}{(\epsilon F(cz))^{\frac{1}{\beta}}} \rho(z) > 0 \right)
\]

\[
\sum_{cz} (\epsilon F(cz) - 1) \text{tr} \left( \rho(cz) - \frac{1}{(\epsilon F(cz))^{\frac{1}{\beta}}} \rho(z) > 0 \right)
\]
\[ \leq \sum_{cz} \epsilon F(cz) R_\alpha (\rho(cz)|\rho(z)) \]
\[ \leq \epsilon, \quad (4.37) \]

according to the QEF inequality and since \( \text{tr}(\rho) = 1 \).

### 4.6. Entropy Estimates From QEFs

**Theorem 4.17.** Let \( \rho(CZ) \in S_1(CZE) \) and suppose that \( F(CZ) \geq 0 \) satisfies the QEF inequality with power \( \beta \) at \( \rho(CZ) \) for \( C|Z \). Fix \( 1 \geq p > 0 \) and \( \epsilon > 0 \) and write \( \phi(CZ) = (F(CZ) \geq 1/(p^\beta \epsilon)) \). Let \( \phi'(CZ) \) satisfy \( \{\phi'(CZ)\} \subseteq \{\phi(CZ)\} \), and define \( \kappa = \text{tr}(\rho(\phi')) \).

Then

\[ \kappa \sum_{cz: \phi'(cz)} \text{tr}\left( \left[ \rho(cz|\phi') - \frac{p}{\kappa} \rho(z) \right]_+ \right) \leq \epsilon. \quad (4.38) \]

The quantity \( \kappa \) is the probability that \( \phi' \) holds at \( \rho(CZ) \). Again, the theorem does not require \( F(CZ) \) to be a QEF for a specific model.

**Proof.** Without loss of generality, let \( \kappa > 0 \). Define \( p(cz) = 1/(\epsilon F(cz))^{1/\beta} \). For \( cz \) satisfying \( \phi'(cz) \), we have \( p \geq p(cz) \). By Thm. 4.16

\[ \epsilon \geq \sum_{cz} \text{tr}\left( \left[ \rho(cz) - p(cz)\rho(z) \right]_+ \right) \]
\[ \geq \sum_{cz: \phi'(cz)} \text{tr}\left( \left[ \rho(cz) - p(cz)\rho(z) \right]_+ \right) \]
\[ = \sum_{cz: \phi'(cz)} \text{tr}\left( \left[ \rho(cz) \left[ \phi'(cz) \right] - p(cz)\rho(z) \right]_+ \right) \]
\[ = \kappa \sum_{cz: \phi'(cz)} \text{tr}\left( \left[ \frac{1}{\kappa} \rho(cz) \left[ \phi'(cz) \right] - p(cz)\rho(z) \right]_+ \right) \]
\[ = \kappa \sum_{cz: \phi'(cz)} \text{tr}\left( \left[ \frac{1}{\kappa} \rho(cz) \left[ \phi'(cz) \right] - \frac{p(cz)}{\kappa} \rho(z) \right]_+ \right) \]
\[ \geq \kappa \sum_{cz: \phi'(cz)} \text{tr}\left( \left[ \rho(cz|\phi') - \frac{p(cz)}{\kappa} \rho(z) \right]_+ \right), \quad (4.39) \]

since \( \text{tr}(\left[ \chi \right]_+) \) is monotone in \( \chi \). \qed

We can obtain a conditional min-entropy bound from Thm. 4.17 after applying Lem. 6.1, Pg. 94 of Ref. [23] and Lem. 4.15, in the spirit of Prop. 6.2, Pg. 95 of the same reference. This proposition was extended to sandwiched Rényi entropies by Lem. B.4 of Ref. [4]. The statement of Lem. B.4 contains an unnecessary restriction \( \alpha \leq 2 \): The data processing inequality for sandwiched Rényi entropy applies for all \( \alpha > 1 \). The same result for all
α > 1 is a consequence of Prop. 6.5, Pg. 99 of Ref. [24]. Instead of deriving a conditional min-entropy bound from Thm. 4.17, we apply this Prop. 6.5 to the conditional Rényi power bound in the first part of the next theorem, in order to obtain the conditional max-prob bound in the second part.

**Theorem 4.18.** Let ρ(CZ) ∈ S₁(CZE) and suppose that F(CZ) ≥ 0 satisfies the QEF inequality with power β at ρ(CZ) for C|Z. Fix δ,q ∈ (0,1], and set p = q/δ^{1/β}. Write φ(CZ) = (F(CZ) ≥ 1/(qδ)). Let φ'(CZ) satisfy {φ'(CZ)} ⊆ {φ(CZ)}, and define κ = tr(ρ(φ')). Then

\[ \sum_{cz} R_\alpha (\rho(cz|\phi')|\rho(z)) \leq \frac{q^\beta}{\kappa^\alpha} \]  \hspace{1cm} (4.40)

and

\[ P_{\text{max}}^{\sqrt{2\delta}} (\rho(cz|\phi')|ZE) \leq P_{\text{max}}^{\sqrt{2\delta}-\delta^2} (\rho(cz|\phi')|ZE) \leq \frac{p}{\kappa^{\alpha/\beta}}. \]  \hspace{1cm} (4.41)

Again, the theorem does not require F(CZ) to be a QEF for a specific model.

**Proof.** For the first part, it suffices to rewrite the QEF inequality and drop terms:

\[ 1 \geq \sum_{cz} F(cz) R_\alpha (\rho(cz)|\rho(z)) \]
\[ \geq \sum_{cz} F(cz) [\phi'(cz)] R_\alpha (\rho(cz)|\rho(z)) \]
\[ \geq \sum_{cz} \frac{1}{q^\beta} [\phi'(cz)] R_\alpha (\rho(cz)|\rho(z)) \]
\[ = \sum_{cz} \frac{1}{q^\beta} R_\alpha ([\phi'(cz)] \rho(cz)/\kappa|\rho(z)) \]
\[ = \sum_{cz} \frac{\kappa^\alpha}{q^\beta} R_\alpha (\rho(cz|\phi')|\rho(z)). \]  \hspace{1cm} (4.42)

The claimed inequality is obtained by multiplying both sides by q^\beta/\kappa^\alpha.

For the second part, we interpret Eq. 4.40 as a sandwiched α-Rényi relative entropy bound. According to Def. 2.9 we have

\[ R_\alpha \left( \sum_{cz} \hat{c} \otimes \hat{z} \otimes \rho(cz|\phi') \left| \sum_{z} 1 \otimes \hat{z} \otimes \rho(z) \right. \right) \]
\[ = \text{tr} \left( \left( \sum_{z} 1 \otimes \hat{z} \otimes \rho(z)^{-\beta/(2\alpha)} \right) \left( \sum_{cz} \hat{c} \otimes \hat{z} \otimes \rho(cz|\phi') \right) \left( \sum_{z} 1 \otimes \hat{z} \otimes \rho(z)^{-\beta/(2\alpha)} \right)^\alpha \right) \]
\[ = \text{tr} \left( \sum_{cz} \hat{c} \otimes \hat{z} \otimes \left( \rho(z)^{-\beta/(2\alpha)} \rho(cz|\phi') \rho(z)^{-\beta/(2\alpha)} \right)^\alpha \right) \]
\[ = \sum_{cz} \text{tr} \left( \left( \rho(z)^{-\beta/(2\alpha)} \rho(cz|\phi') \rho(z)^{-\beta/(2\alpha)} \right)^\alpha \right) \]
\[= \sum_{cz} \mathcal{R}_\alpha (\rho(cz|\phi')|\rho(z)) \, . \] (4.43)

We can now apply Prop. 6.5, Pg. 99 of Ref. [24]. We convert to our notation, and substitute for \(\epsilon\) in the reference according to \(\delta = 1 - \sqrt{1 - \epsilon^2}\) (equivalently, \(\epsilon = \sqrt{2\delta - \delta^2}\)), the operator \(\rho\) there by \(\sum_{cz} \hat{c} \otimes \hat{z} \otimes \rho(cz|\phi')\) here, and \(\sigma\) there by \(\sum_z 1 \otimes \hat{z} \otimes \rho(z)\) here. This gives

\[
\inf_{\rho'} \inf \{p' : \rho'(CZ) \leq p' \rho(Z), \rho'(CZ) \in S_{\leq 1}(CZE), \text{PD}(\rho'(CZ), \rho(CZ|\phi')) \leq \sqrt{2\delta - \delta^2}\}
\]

\[
\leq \left( \frac{1}{\delta} \sum_{cz} \hat{c} \otimes \hat{z} \otimes \rho(cz|\phi') \right)^{1/\beta} \left( \sum_z 1 \otimes \hat{z} \otimes \rho(z) \right)^{1/\beta} \, . \quad (4.44)
\]

Taking note of the definition of \(P^e_{\text{max}}\) in Def. [2.26], we get

\[
P^\sqrt{2\delta - \delta^2}_{\text{max}}(\rho(CZ|\phi')|Z\text{E})
\]

\[
\leq \inf_{\rho'} \inf \{p' : \rho'(CZ) \leq p' \rho(Z), \rho'(CZ) \in S_{\leq 1}(CZE), \text{PD}(\rho'(CZ), \rho(CZ|\phi')) \leq \sqrt{2\delta - \delta^2}\}.
\]

Combining Eqs. 4.43, 4.44 and 4.45 we get

\[
P^\sqrt{2\delta - \delta^2}_{\text{max}}(\rho(CZ|\phi')|Z\text{E}) \leq \left( \frac{1}{\delta} \sum_{cz} \mathcal{R}_\alpha (\rho(cz|\phi')|\rho(z)) \right)^{1/\beta} \, . \quad (4.46)
\]

Continuing from the right-hand side and applying Eq. [4.40], we get

\[
P^\sqrt{2\delta - \delta^2}_{\text{max}}(\rho(CZ|\phi')|Z\text{E}) \leq \left( \frac{q^\beta}{\delta^\kappa} \right)^{1/\beta}
\]

\[
= \frac{p}{\kappa^{\alpha/\beta}} \, .
\]

(4.47)

Since \(P^e_{\text{max}}\) is monotonic in the smoothness parameter \(\epsilon\), the proof of the second part of the theorem is complete.

We also use a simplified version of Thm. [4.18] where \(F(CZ)\) has a uniform lower bound:

**Corollary 4.19.** Fix \(\delta, p \in (0, 1]\). Let \(\rho(CZ) \in S_1(CZE)\) and suppose that \(F(CZ) \geq 1/(p^\beta \delta)\) satisfies the QEF inequality with power \(\beta\) at \(\rho(CZ)\) for \(C|Z\). Then \(P_{\text{max}}^e(\rho(CZ)|Z\text{E}) \leq p\).

**Proof.** It suffices to apply Eq. [4.41] with \(\kappa = 1\).

---

5. **QEF-BASED RANDOMNESS GENERATION PROTOCOLS**

5.1. **Protocol Soundness and Completeness**

A generic randomness generation protocol \(G\) produces three outputs: a bit string of length \(k_o\), a length \(k_u\) bit string consisting of potentially reusable random bits and a “flag”
indicating failure or success. We write \( G = (G_X, G_S, G_P) \) accordingly where \( G_X \) is the bit string of length \( k_0 \), \( G_S \) the bit string of length \( k_u \) and \( \text{Rng}(G_P) = \{0, 1\} \). The values 0 and 1 of \( G_P \) indicate failure and success, respectively. The outputs \( G_X, G_S \) and \( G_P \) are determined by CVs associated with a sequence of trials involving the devices of the protocols and a seed bit-string CV. Parameters of \( G \) include \( k_0, k_u, \) the length \( k_s \) of the seed CV, and a target error bound \( \epsilon \). Other parameters may be relevant before the protocol is invoked, such as the maximum number of trials \( N \) and, after it has executed, the number \( n \) of trials actually performed and the number of bits \( k_z \) of input randomness used.

Informally, a protocol is \( \epsilon \)-sound if its output is within \( \epsilon \) of an ideal protocol. The distance measure used determines the protocol’s composability properties. There is some variation in the soundness definitions for randomness generation protocols in the literature. We prove soundness with respect to TV distance including the devices, which is better behaved for composability analyses. There is some variation in the soundness definitions for randomness generation protocols in the literature. We prove soundness with respect to purified distance, which is stronger than other definitions. It implies soundness with respect to TV distance including the devices, which is better behaved for composability analyses.

**Definition 5.1.** Let \( CZ \) and \( S \) be CVs, where \( S \) is a length \( k_s \) bit string, and let \( \sigma(S) \) be the uniform distribution, that is \( \sigma(S) = \text{Unif}(S) \). A randomness generation protocol \( G = (G_X, G_S, G_P) \) determined by \( CZS \) is \( \epsilon \)-sound for \( C|Z \) at \( \rho(CZ) \in S_1(CZE) \) if there exists \( \tau(Z) \in S_1(ZE) \) such that

\[
PD\left( (\rho \otimes \sigma)|G_XG_SZG_P = 1) , \text{Unif}(G_XG_S) \otimes \tau(Z) \right) \text{tr}( (\rho \otimes \sigma)(G_P = 1)) \leq \epsilon. \tag{5.1}
\]

\( G \) is \( \epsilon \)-sound for \( C|Z \) and model \( C(CZ) \) if it is \( \epsilon \)-sound for \( C|Z \) at all \( \rho(CZ) \in C(CZ) \). \( G \) is \( \kappa \)-complete for \( C|Z \) and model \( C(CZ) \) if there exists \( \rho(CZ) \in C(CZ) \) such that \( \text{tr}( (\rho \otimes \sigma)(G_P = 1)) \geq \kappa. \)

If required for clarity, we may refer to the soundness in this definition as **PD soundness**. Completeness is important to ensure that protocols can be usefully realized. For our protocols and models with extractable randomness, completeness is readily achieved with an exponentially good completeness parameter. In practice, completeness parameters cannot be relied on to be exponentially good. Further, the idea of device-independent protocols is that the devices are minimally trusted, so regardless of completeness or other expectations of the experimental configuration, provisions for failure must be made to mitigate denial-of-service and mundane device faults. Soundness makes sure that any randomness produced has guaranteed performance even in the context of probabilities of success that are temporarily or permanently far from 1.

It is possible to consider soundness statements involving seed CVs whose distributions are not uniform, but this requires extractors satisfying stronger conditions than the quantum-proof strong extractors considered here. See [32] and the references therein for recent work with less-than-perfect seeds.

It may be desirable to have the purified distance conditional on success be bounded by \( \delta \) given that the success probability is larger than some small threshold \( \kappa \). For this it suffices to choose the soundness error \( \epsilon \) as \( \epsilon \leq \delta \kappa \). If one wishes to be equally conservative for both \( \delta \) and \( \kappa \), it makes sense to set \( \epsilon = \delta^2 \).

The purified distance allows for extension to the devices to enable analysis of protocol composition involving the same devices, where the devices may have memory. This kind of composition can introduce the possibility of memory attacks, whereby the devices leak information about past results through leakage channels enabled by later protocols [33]. For our randomness generation protocols, such a leakage channel is introduced by the success
variable $G_P$: The devices can modify their future behavior so that the variables $G_P$ in later protocols depend on the past results. This favors protocols with no possibility of failure such as Protocol 2 below. A detailed discussion of memory attacks for randomness generation is in the supplemental material of Ref. [33]. We note that our protocols have fixed length outputs, which avoids leakage channels based on the length of the output but does not eliminate implementation-dependent leakage channels such as variations in timing or side-effects of using randomness.

We do not formally analyze composition of randomness-generation protocols with the same devices, and unrestricted composability is not assured. But to support such composition, we require that the devices are permanently isolated from $E$ and that they never gain knowledge of seeds used for randomness extraction. The latter supports the following strategy to mitigate $G_P$-based leakage channels: Anticipate the number of future instances of the protocol and reduce the number of bits extracted from the current protocol accordingly, similarly to how settings entropy is eliminated in Protocol 3 below. The requirements may be difficult to guarantee in a practical setting but can be weakened once the randomness generated is used, see the discussion in Ref. [33].

PD soundness implies a strong TV distance-based soundness. Let $D$ be the system of the devices and let $\rho'(G_XG_SZG_P) \in S_1(G_XG_SZG_PDE)$ be the final state of the protocol. Thus $\rho(G_XG_SZG_P) = \text{tr}_D \rho'(G_XG_SZG_P)$. Here, information about $C$ may be contained in the quantum part of the state carried by $D$. If the protocol is PD $\epsilon$-sound, then the extension property of purified distance (Ref. [23], Cor. 3.6, Pg. 52) and the relationship between TV and purified distances (Lem. 2.8) imply that there exists a state $\tau'(G_XG_SZ) \in S_1(G_XG_SZDE)$ such that

$$\text{TV}(\rho'(G_XG_SZ|G_P = 1), \tau'(G_XG_SZ)) \text{tr}((\rho \otimes \sigma)(G_P = 1)) \leq \epsilon \quad (5.2)$$

and

$$\text{tr}_D \tau'(G_XG_SZ) = \text{Unif}(G_XG_S) \otimes \tau(Z), \quad (5.3)$$

with $\tau(Z)$ witnessing PD $\epsilon$-soundness. This construction can be used to justify the informal idea that $\epsilon$-soundness relates to how close the protocol endstate is from that of an ideal protocol. From $\tau'(G_XG_SZ)$, we can construct an ideal protocol endstate $\xi(G_XG_SZG_P) \in S_1(G_XG_SZG_PDE)$ that includes the devices and is $\epsilon$-close in TV distance to the actual state:

$$\xi(G_XG_SZG_P) = \text{tr}((\rho \otimes \sigma)(G_P = 1))\tau'(G_XG_SZ) + \rho'(G_XG_SZ, G_P = 0). \quad (5.4)$$

This state satisfies $\epsilon$-soundness with $\epsilon = 0$ and agrees with the protocol endstate conditionally on failure. The existence of this state motivates the definition of soundness and our use of purified distance. But for composability analysis, we use TV $\epsilon$-soundness including the devices, which we define as existence of the state $\tau'$ satisfying Eqs. (5.2) and (5.3).

The protocols below are proved to be PD $\epsilon$-sound regardless of the incoming state and dependence on initial classical variables that may be public and on variables determined from initial information. TV soundness extends to such initial variables without changing the error bound. From the previous paragraph, the TV soundness error is uniformly bounded by $\epsilon$ given these initial variables, as is the distance from an ideal protocol conditional on the initial variables. We can define an unconditional ideal protocol by having it act as the conditional ideal protocol given the initial variables. The probability distribution for the initial variables is the same for the actual and the ideal protocol. The TV distance between two states classical on $R$ with identical marginal distribution on $R$ is the expected $R$-conditional TV distance. It follows that the distance between the two is the expected
distance conditional on the initial variables, which is less than \( \epsilon \).

5.2. Protocols with QEFs

We define three sound randomness generation protocols given a QEF. Whether they are complete depends on the model and the QEF. The results established later show that if a trial model permits proper randomness generation in principle, then completeness with exponentially good completeness parameter is readily achieved for sequences of independent and identical (i.i.d.) trials, each constrained by the trial model. This generally follows from large deviation results applied to sums of i.i.d. RVs. For our protocols, these RVs are the logarithms of the QEFs. We do not explore the relevant arguments further here.

In this section we consider monolithic QEFs and \( CZ \), meaning that we do not explicitly subdivide the results into a sequence of trials. Thus, \( CZ \) stands for all results, whether or not they were obtained in a sequence of trials, and the QEFs are the final QEFs, obtained by chaining if necessary. Protocol-related issues when the QEFs are determined by chaining are discussed in Sect. 5.3. Anticipating the amount of conditional min-entropy that can be certified is the topic of Sect. 5.4.

The first protocol directly composes Thm. 4.18 on the relationship between QEFs and smooth max-prob with a quantum-proof strong extractor. The protocol is displayed in...
Protocol 1: Input-conditional randomness generation.

**Input**: Number of bits of randomness \( k_o \) to be generated. Error bound \( \epsilon \in (0, 1] \).

**Given**: Access to CVs \( CZ \) and \( S \), where \( S \) is uniformly distributed and independent of all other systems. All CVs are bit strings. A QEF \( F(CZ) \) with power \( \beta \) for \( CZ \) and model \( C(CZ) \). A quantum-proof strong extractor \( E \).

**Output**: Length \( k_o \) bit string \( G_X, G_S = S, G_P \in \{0, 1\} \).

Define \( n = |C|, k_s = |S| \);
Define \( X = \{(k_i, \epsilon_x) : (n, k_s, k_o, k_i, \epsilon_x < \epsilon) \} \); // See the paragraph after Def. 2.32.
Get an instance \( s \) of \( S \);

- **if** \( X \) is empty **then**
  - Return \( G_P = 0, G_X = 0^{-k_o}, G_S = s \); // Protocol failed.
- **else**
  - Choose \( (k_i, \epsilon_x) \in X \);
  - Set \( \epsilon_h = (\epsilon - \epsilon_x) \);
  - If \( \alpha > 2 \), then set \( p = 2^{-k_i}\epsilon(\alpha-2)/\beta \), otherwise set \( p = 2^{-k_i} \);
  - Set \( f_{\min} = 1/(p^\beta(\epsilon_h^2/2)) \); // Choose \((k_i, \epsilon_x)\) to minimize \( f_{\min} \).
  - Get an instance \( cz \) of \( CZ \);
  - Compute \( f = F(cz) \).
  - **if** \( f < f_{\min} \) **then**
    - Return \( G_P = 0, G_X = 0^{-k_o}, G_S = s \); // Protocol failed.
  - **else**
    - Return \( G_P = 1, G_X = E(\epsilon, s; n, k_s, k_o, k_i, \epsilon_x) \), \( G_S = s \); // Protocol succeeded.

**Theorem 5.2.** Protocol\[1\] is an \( \epsilon \)-sound randomness generation protocol for \( C|Z \) and model \( C(CZ) \).

**Proof.** According to our modeling assumptions, the model applies conditionally on the past, which includes the protocol inputs \( k_o \) and \( \epsilon \) and the specific choice for \( (k_i, k_s, \epsilon_x) \) made in the protocol, as these parameters are determined before \( CZ \) is instatiated. Let \( \rho(CZ) \in C(CZ) \) be the specific state from which \( CZ \) is instantiated to \( cz \) in the protocol. Let \( \phi(CZ) = (F(CZ) \geq f_{\min}) = (G_P = 1) \). Define \( \kappa = tr(\rho(\phi)) \). First consider the case \( \kappa \in [\epsilon, 1] \). In Thm. 4.18 set \( \delta = (\epsilon_h/\kappa)^2/2 \) and \( p \) there to \( pk^{2/\beta} \) here. With these substitutions, \( q \) there satisfies \( q = (pk^{2/\beta})(\epsilon_h/\kappa)^2/2 \). Define \( \alpha > 2 \), then set \( p = 2^{-k_i} \), and if \( \alpha > 2 \), then \( pk^{2/\beta} \leq p \). Hence, when \( \epsilon \leq \kappa \leq 1 \) we have \( P_{\max}^k(\rho(CZ) | \phi)(Z) \leq 2^{-k_i} \). That is, there exists \( \rho'(CZ) \in S_1(CZE) \) such that \( P_{\max}(\rho'(CZ) | Z) \leq 2^{-k_i} \) and PD(\( \rho'(CZ), \rho(CZ) \phi) \leq \epsilon_h/\kappa \) (see Lem. 2.27)
the extractor constraints ensure that $2^{-k_i} \leq 2^n = |Rng(C)|$. As in the definition of soundness, let $\sigma(S) = \text{Unif}(S)$. Because the parameters $n, k_s, k_o, k_i, \epsilon_x$ satisfy the extractor constraints, we get

$$\text{PD}((\rho' \otimes \sigma)(G_XG_SZ), \text{Unif}(G_XG_S) \otimes \rho'(Z)) \leq \epsilon_x.$$  \hspace{1cm} (5.6)

Since $\text{PD}((CZ), \rho(CZ|\phi)) \leq \epsilon_h/\kappa$ and the purified distance satisfies the data-processing inequality,

$$\text{PD}((\rho \otimes \sigma)(G_XG_SZ|G_P = 1), (\rho' \otimes \sigma)(G_XG_SZ)) \leq \epsilon_h/\kappa.$$  \hspace{1cm} (5.7)

The triangle inequality for the purified distance together with Eqs. 5.6 and 5.7 yield

$$\text{PD}((\rho \otimes \sigma)(G_XG_SZ|G_P = 1), \text{Unif}(G_XG_S) \otimes \rho'(Z)) \leq \epsilon_x + \epsilon_h/\kappa.$$  \hspace{1cm} (5.8)

We multiply both sides by $\kappa$ for

$$\text{PD}((\rho \otimes \sigma)(G_XG_SZ|G_P = 1), \text{Unif}(G_XG_S) \otimes \rho'(Z)) \kappa \leq \epsilon_x \kappa + \epsilon_h \leq \epsilon_x + \epsilon_h = \epsilon.$$  \hspace{1cm} (5.9)

For $\kappa < \epsilon$, since the purified distance cannot be larger than one,

$$\text{PD}((\rho \otimes \sigma)(G_XG_SZ|G_P = 1), \text{Unif}(G_XG_S) \otimes \rho(Z|G_P = 1)) \kappa \leq \kappa < \epsilon,$$  \hspace{1cm} (5.10)

so the condition for $\epsilon$-soundness is satisfied for the full range of values of $\kappa$. \hfill \Box

Next we define a protocol that avoids failure by taking advantage of banked randomness. It has the advantage of simplicity at the cost of occasionally producing randomness that is not entirely fresh, which adds effective latency. Of course, in situations where we can experimentally ensure completeness, it is possible to make the probability of requiring banked
randomness extremely small. The protocol is displayed in Protocol 2.

---

**Protocol 2:** Input-conditional randomness generation with banked randomness.

**Input:** Number of bits of randomness \( k_0 \) to be generated. Error bound \( \epsilon \in (0, 1] \).

**Given:** Access to CVs \( CZ, \) \( S \) and \( B \), where \( |B| = k_0 \) and \( SB \) is uniformly distributed and independent of all other systems. All CVs are bit strings. A QEF \( F(CZ) \) with power \( \beta \) for \( C|Z \) and model \( C(CZ) \). A quantum-proof strong extractor \( \mathcal{E} \).

**Output:** Length \( k_0 \) bit string \( \mathcal{G}_X \), \( \mathcal{G}_S = S \), \( \mathcal{G}_P \in \{0, 1\} \).

1. Define \( n = |C|, k_s = |S| \);
2. Define \( \mathcal{X} = \{(k_i, \epsilon_x) : (n + k_o, k_s, k_o, k_i, \epsilon_x < \epsilon) \text{ satisfies the extractor constraints for } \mathcal{E}\} \);
3. Get an instance \( s \) of \( S \);
4. **if** \( \mathcal{X} \) is empty **then**
   - Get an instance \( b_{\leq k_0} \) of \( B_{\leq k_0} \);
   - Return \( \mathcal{G}_P = 1, \mathcal{G}_X = b_{\leq k_0}, \mathcal{G}_S = s \); // Return only banked randomness.
5. **else**
   - Choose \( (k_i, \epsilon_x) \in \mathcal{X} \);
   - Set \( \epsilon_h = (\epsilon - \epsilon_x) \);
   - Set \( p = 2^{-k_i} \);
   - Set \( f_{\min} = 1/(p^{\beta}(\epsilon_n^2/2)) \); // Choose \( (k_i, \epsilon_x) \) to minimize \( f_{\min} \)
   - Get an instance \( cz \) of \( CZ \);
   - Compute \( f = F(cz) \).
6. **if** \( f \geq f_{\min} \) **then**
   - Return \( \mathcal{G}_P = 1, \mathcal{G}_X = \mathcal{E}(c0^{k_0}, s; n + k_o, k_s, k_o, k_i, \epsilon_x), \mathcal{G}_S = s \); // No banked randomness needed.
7. **else**
   - Set \( k_b = \lceil \log_2(f_{\min}/f) / \beta \rceil \);
   - Get an instance \( b_{\leq k_b} \) of \( B_{\leq k_b} \);
   - Return \( \mathcal{G}_P = 1, \mathcal{G}_X = \mathcal{E}(c0^{k_0}, s; n + k_o, k_s, k_o, k_i, \epsilon_x), \mathcal{G}_S = s \); // Needed \( k_b \) bits of banked randomness.
8. **end**

---

**Theorem 5.3.** Protocol 3 is a complete and \( \epsilon \)-sound randomness generation protocol for \( C|Z \) and model \( C(CZ) \).

**Proof.** If \( f \geq f_{\min} \) in the protocol, set \( k_b = 0 \). The protocol can be thought of as one that adds a final trial conditionally on \( F(cz) < f_{\min} \), where the final trial has output \( B' = B_{\leq k_b}0^{k_0-k_b} \), which is a bit string of length \( k_0 \) and model \( \{\text{Unif}(B_{\leq k_b}) : \rho \in S(E)\} \). We can define \( G_{cz}(b') = 2^{\beta k_b} \), which is a QEF with power \( \beta \) for the last trial, and chain \( F \) with \( G_{cz} \) to get a QEF \( F'(CZB') = F(CZ)G_{CZ}(B') \) with power \( \beta \) for \( CB'|Z \) and the chained model. By construction, \( F'(CZB') \geq f_{\min} \), so we can apply Cor. 4.19 to show that for any \( \rho(CZB') \) in the chained model, \( P_{\max}^{\rho}(\rho(CZB')|Z\mathcal{E}) \leq p \). The theorem follows because \( \mathcal{E} \) is a quantum-proof strong extractor, its parameters satisfy the extractor constraints, the incoming smooth max-prob is less than \( 2^{-k_i} \), and the data-processing and triangle inequalities for the purified distance. \( \square \)
The third protocol conditions on inputs indirectly by exploiting the privacy amplification capabilities of extractors. We give a version not relying on banked randomness. The only difference to the first protocol is that the conditional min-entropy certified internally needs to also account for the maximum number of bits that contribute to the inputs. An advantage is that the models for which this protocol works need not involve chaining with explicitly conditional inputs. The protocol is displayed in Protocol 3.

**Protocol 3:** Randomness generation with implicit input conditioning.

**Input:** Number of bits of randomness \( k_o \) to be generated. Error bound \( \epsilon \in (0, 1] \).

**Given:** Access to CVs \( CZ \) and \( S \), where \( Z = Z(H) \) for a CV \( H \) and \( S \) is uniformly distributed and independent of all other systems. All CVs are bit strings. A QEF \( F(CZ) \) with power \( \beta \) for \( CZ \) and model \( \mathcal{C}(CZ) \). A quantum-proof strong extractor \( E \).

**Output:** Length \( k_o \) bit string \( G_X \), \( G_S = S \), \( G_P \in \{0, 1\} \).

Define \( n = |C| \), \( k_s = |S| \), \( k_z = |H| \);

Define \( \mathcal{X} = \{(k_i, \epsilon_x) : (n, k_s, k_o, k_i, \epsilon_x < \epsilon) \) satisfies the extractor constraints for \( E \}\); Get an instance \( s \) of \( S \);

if \( \mathcal{X} \) is empty then

\[ \text{Return } G_P = 0, G_X = 0^{-k_o}, G_S = s ; \quad \text{// Protocol failed.} \]

else

Choose \( (k_i, \epsilon_x) \in \mathcal{X} \);

Set \( \epsilon_h = (\epsilon - \epsilon_x) \);

If \( \alpha > 2 \), set \( p = 2^{-k_i-k_z} \epsilon(\alpha - 2)/\beta \), otherwise set \( p = 2^{-k_i-k_z} \);

Set \( f_{\text{min}} = 1/(p^\beta(\epsilon_h^2/2)) \); // Choose \( (k_i, \epsilon_x) \) to minimize \( f_{\text{min}} \.

Get an instance \( cz \) of \( CZ \);

Compute \( f = F(cz) \).

if \( f < f_{\text{min}} \) then

\[ \text{Return } G_P = 0, G_X = 0^{-k_o}, G_S = s ; \quad \text{// Protocol failed.} \]

else

\[ \text{Return } G_P = 1, G_X = \mathcal{E}(c, s; n, k_s, k_o, k_i, \epsilon_x), G_S = s ; \quad \text{// Protocol succeeded.} \]

end

end

**Theorem 5.4.** Protocol 3 is an \( \epsilon \)-sound randomness generation protocol for \( C|Z \) and model \( \mathcal{C}(CZ) \).

**Proof.** The proof follows that of Protocol 1. For the initial part, \( CZ \) are both considered output and there is no explicit input. For the case \( \kappa \in [\epsilon, 1] \), the max-prob established for this protocol is

\[ P_{\text{max}}^{\kappa}(\rho(CZ|\phi)|E) \leq P_{\text{max}}^{\kappa}(\rho(CH|\phi)|HE) \leq P_{\text{max}}^{\kappa}(\rho(CH|\phi)|E) \leq 2^{-k_z} P_{\text{max}}^{\kappa}(\rho(CZ|\phi)|E) \leq 2^{-k_i} . \tag{5.11} \]

Since \( Z \) is determined by \( H \) and invoking Lem. 2.30 and Lem. 2.31 we get

\[ P_{\text{max}}^{\kappa}(\rho(CZ|\phi)|ZE) \leq P_{\text{max}}^{\kappa}(\rho(CH|\phi)|HE) \leq 2^{k_z} P_{\text{max}}^{\kappa}(\rho(CH|\phi)|E) \leq 2^{k_z} P_{\text{max}}^{\kappa}(\rho(CZ|\phi)|E) \leq 2^{-k_i} . \tag{5.12} \]
The rest of the proof of Protocol 1 now applies without change.

5.3. Trial-Wise QEF Computation for Protocols

For the applications we have in mind, the QEFs $F(CZ)$ used by the protocols arise by chaining trial-wise QEFs $F_i(C_iZ_i)$ for a sequence of trials, where the final model is an appropriate chaining of the trial models. An advantage of QEFs is that they can be adapted while the trials are acquired. A consequence is that one can stop acquiring trials as soon as the chained QEF witnesses sufficiently small Rényi power. For definiteness, we let $k$ be the number of trials performed (or analyzed) so far. According to QEF chaining, the next trial’s QEF $F_{k+1}(C_{k+1}Z_{k+1})$ can depend arbitrarily on $(cz) \leq k$, the results from trials so far. In particular, one can check the statistics of recent trials to see whether the observed probability distribution of $CZ$ changed and if so, adapt the next trial’s QEFs accordingly. Further, if the chained QEF so far, $\prod_{i=1}^{k} F_i(c_i z_i)$, already exceeds the threshold for the protocol, then one can set all future QEFs $F_i(C_iZ_i)$ with $i > k$ to 1. Since this eliminates any contribution from future trials to the final chained QEF value, it is not necessary to perform the future trials at this point. Since the trial models can also depend on the past, one can change the configuration between trials. If there is a change in trial model, it must also be determined by $(cz) \leq k$ and the next trial’s QEF needs to take the change into account. Changes that do not affect the model are not so restricted. For example, there are no restrictions on device recalibration between trials.

Unlike QEFs, soft PEFs as defined in Ref. 1 can directly use available information not determined by $(cz) \leq k$ to choose the next trial’s model and PEF. We have not implemented softening for QEFs. However, this is not a fundamental obstacle. A feature of the CV $CZ$ as used in the randomness generation protocols above is that $C$ must be provided to the extractor, while $Z$ must be conditioned on. A simple method to enable use of information obtained during an experiment besides $(cz) \leq k$ is the following: Periodically, at predictable intervals, insert special trials with output consisting of the information that one wishes to use in future trials, but no input. These trials’ outputs are ultimately included in the extractor input or conditioned on via the method in Protocol 3 which can add a moderate amount of complexity to the extractor calculation. The QEFs for the special trials are set to 1, so these trials contribute no conditional min-entropy. Future trial’s models and QEFs can then depend on the special trials’ outputs in addition to the normal trial results.

We remark that when computing chained QEF given by $\prod_{i=1}^{n} F_i(C_iZ_i)$ with floating point numbers, to avoid overflow of the mantissa, it is good practice to work with the logarithm of the QEF and add the logarithms of the trial-wise QEFs $F_i(C_iZ_i)$.

5.4. QEF Rates and Optimization

Consider a trial model $C(CZ)$, a non-negative function $F(CZ)$, a probability distribution $\nu(CZ)$ and a QEF power $\beta$. We treat $\nu(CZ)$ as the design or the predicted probability distribution for $CZ$.

**Definition 5.5.** The log-prob rate of $F(CZ)$ at $\nu(CZ)$ is $\sum_{cz} \nu(cz) \log(F(cz))/\beta = E_{\nu(CZ)}(\log(F(CZ)))/\beta$.

If $F(CZ)$ is a QEF with power $\beta$ for $C|Z$ and $C(CZ)$, then from Thm. 4.18 we can see that the log-prob rate can be interpreted as the expected conditional min-entropy of $C|ZE$.
witnessed by $F(CZ)$ without adjusting for the error bound or for probability of success. It is a useful predictor of the smooth conditional min-entropy witnessed in a sequence of trials with trial models $\mathcal{C}(C_iZ_i)$ identical to $\mathcal{C}(CZ)$ except for the change of CVs, where the experiment is configured so that the marginal trial distributions are $\nu(C_iZ_i)$, or at least close to $\nu(C_iZ_i)$ conditionally on the past. In such a sequence of trials, $\log(F(C_iZ_i))/\beta$ are approximately i.i.d. RVs and their mean is typically close to the log-prob rate at $\nu(CZ)$. If the error bound and lower bound on probability of success are constant, then the asymptotic smooth conditional min-entropy rate according to Thm. 4.18 for the chained QEF $\prod_{i=1}^n F(C_iZ_i)$ is the log-prob rate of the trial-wise QEF $F(CZ)$. We emphasize that the assumption on the trial distributions is a completeness assumption and not required for sound conditional min-entropy estimation with QEFs. If the experiment does not perform according to expectation, the worst that can happen is that we do not witness the expected amount of conditional min-entropy.

The log-prob rate neglects the reduction of conditional min-entropy due to the error bound, which is a problem for finite data or when the error bound grows with number of trials.

**Definition 5.6.** Given an error bound $\epsilon$ and $n$ trials, the error bound rate of $\epsilon$ is $r = |\log(\epsilon)/n|$. Let $\bar{\kappa} \geq \epsilon$ be the smallest probability of success that we need to protect against. The expected quantum net log-prob of $F(CZ)$ at $\nu(CZ)$ is

$$n\mathbb{E}_{\nu(CZ)} \left( \log(F(CZ)) \right)/\beta + \log \left( e^{2\bar{\kappa}\beta(\beta > 1)/2} \right)/\beta. \quad (5.13)$$

The quantum net log-prob rate of $F(CZ)$ at $\nu(CZ)$ is

$$\mathbb{E}_{\nu(CZ)} \left( \log(F(CZ)) \right)/\beta - 2r/\beta. \quad (5.14)$$

The expected quantum net log-prob reflects the smooth conditional entropy one can aim for if the experiment is designed for trials with i.i.d. observable distributions $\nu(CZ)$ for each trial. The dependence on $\bar{\kappa}$ is motivated by the reference protocol Protocol. But accounts for $\bar{\kappa}$ and neglects the extractor constraints: Let $O_F$ be the log-prob rate of $F(CZ)$ at $\nu(CZ)$. Let $\kappa$ be the probability of success of the protocol, assume $\kappa \geq \bar{\kappa}$ and consider the proof of Thm. 5.2. To motivate the definition of expected quantum net log-prob, we neglect the extractor constraints and the error $\epsilon_x$, set $\epsilon_h = \epsilon$, and choose $f_{\min} = e^{nO_F}$, which is the maximum $f_{\min}$ at which we can hope to have a reasonable probability of success for completeness. For soundness, we set $\delta = (\epsilon/\kappa)^2/2$. When applying Thm. 4.18 we determine $p$ and $q$ by $f_{\min} = q^{-\beta}$ and $p = q^{-1/\beta}$, so $p = (f_{\min}^0)^{-1/\beta}$. On success, the $(\epsilon/\kappa)$-smooth conditional min-entropy is given by the negative logarithm of the right-hand side of Eq. 4.41 which evaluates to

$$nO_F + \log(\delta\kappa^\alpha)/\beta = nO_F + \log(e^{2\kappa^{\alpha-2}/2})/\beta \geq nO_F + \log(e^{2\bar{\kappa}\beta(\beta > 1)/2})/\beta, \quad (5.15)$$

which is the expected quantum net log-prob. If we set $\bar{\kappa} = \epsilon$, the right-hand side is the amount of randomness that would be obtained in Protocol if the extractor constraints are neglected, $\epsilon_x = 0$, $f_{\min}$ is chosen as above and $k_i = k_o$. The proof of Thm. 5.2 makes it clear that there is nothing to be gained by considering $\bar{\kappa} < \epsilon$: For success probabilities smaller than $\epsilon$, $\epsilon$-soundness is automatically satisfied.

The quantum net log-prob rate does not take into account the bound on the probability
of success, effectively assuming that this bound is constant. The quantum net log-prob rate accounts for the asymptotic contribution of the error bound to the conditional min-entropy witnessed by $F(CZ)$ according to Thm. 4.18, where the error bound $\epsilon$ for $n$ trials is determined by the error bound rate $r$ according to $\epsilon = e^{-rn}$. It is distinguished from the net log-prob rate as defined for PEFs in Ref. [1] by the factor of 2 multiplying $r$, which originates in Thm. 4.18. It reflects a doubling of the number of trials required to satisfy error bounds for quantum side information compared to what is required for classical side information in the PE and QPE frameworks.

Given an experimental configuration with target $\nu(CZ)$, a first goal is to maximize the log-prob rate subject to $F(CZ)$ being a QEF with power $\beta$ for $C|Z$ and $C(CZ)$. The power $\beta$ can then be varied to maximize the expected quantum net log-prob. Define

$$Q_\alpha(F(CZ), \rho(CZ)) = \sum_{cz} F(cz) R_\alpha(\rho(cz)|\rho(z)).$$

(5.16)

The power-$\beta$ QEF condition for $C|Z$ and $C(CZ)$ is $Q_\alpha(F(CZ), \rho(CZ)) \leq 1$ for all $\rho(CZ) \in \mathcal{N}(\mathcal{C})$. If the probability distribution of $Z$ is fixed, given by $\mu(Z)$, then for $\rho(CZ) \in \mathcal{C}(CZ)$, $\rho(z) = \mu(z)\rho$ and according to Eq. 4.13 the expression for $Q_\alpha$ simplifies to

$$Q_\alpha(F(CZ), \rho(CZ)) = \sum_{cz} \mu(z) F(cz) \tr\left( (\rho^{-\beta/(2\alpha)}\rho(c|z)\rho^{-\beta/(2\alpha)})^\alpha \right).$$

(5.17)

QEFs are optimized by maximizing the log-prob rate. Instead of requiring $F(CZ)$ to be a QEF with power $\beta$ for $C|Z$ and $C(CZ)$, we formulate the QEF optimization problem as follows:

Maximize: $\sum_{cz} \nu(cz) \log(F(cz)) - \log(f_{\text{max}})$

Variables: $F(CZ), f_{\text{max}}$

Subject to: $F(CZ) \geq 0, \sum_{cz} F(cz) = 1,$

$$f_{\text{max}} = \max\{Q_\alpha(F(CZ), \rho(CZ)) : \rho(CZ) \in \mathcal{N}(\mathcal{C})\}. \tag{5.18}$$

Every feasible solution $(F(CZ), f_{\text{max}})$ determines the QEF $F(CZ)/f_{\text{max}}$ with power $\beta$ for $C|Z$ and $C(CZ)$ whose log-prob rate is the objective function divided by $\beta$.

6. QEFs and Entropy Estimators

6.1. Entropy Estimators from QEFs

Definition 6.1. The function $K(CZ)$ is an entropy estimator for $C|Z$ and $C(CZ)$ if for all $\rho(CZ) \in \mathcal{C}(CZ)$,

$$\sum_{cz} K(cz) \tr(\rho(cz)) \leq -\sum_{cz} \tr(\rho(cz) (\log(\rho(cz)) - \log(\rho(z)))) \tag{6.1}$$

The entropy estimate of $K(CZ)$ at $\rho(CZ)$ is $\sum_{cz} K(cz) \tr(\rho(cz))$. 

Both sides of Eq. 6.1 are positive homogeneous of degree 1 in \( \rho(CZ) \), which implies that  
\( K(CZ) \) is an entropy estimator for \( C(CZ) \) iff it is an entropy estimator for \( \mathcal{N}(C(CZ)) \). For normalized states, the right-hand side of Eq. 6.1 is the conditional entropy \( H_1(\rho(CZ)|ZE) \) of \( C|ZE \) with respect to \( \rho(CZ) \).

**Theorem 6.2.** Let \( F(CZ)^\beta \) be a QEF with power \( \beta \) for \( C|Z \) and \( C(CZ) \). Then \( K(CZ) = \log(F(CZ)) \) is an entropy estimator for \( C|Z \) and \( C(CZ) \).

**Proof.** Without loss of generality, consider \( \rho(CZ) \in \mathcal{N}(C(CZ)) \). By power reduction, \( F(CZ)^\gamma \) is a QEF with power \( \gamma \) for \( C|Z \) and \( C(CZ) \) for all \( 0 < \gamma \leq \beta \). Hence

\[
1 \geq \sum_{cz} F(cz)^\gamma \mathcal{R}_{1+\gamma} (\rho(cz)|\rho(z)) = \sum_{cz} \text{tr}(\rho(cz)) F(cz)^\gamma \mathcal{R}_{1+\gamma} (\rho(cz)|\rho(z)) = \sum_{cz} \text{tr}(\rho(cz)) \exp \left( \gamma \log(F(cz)) + \gamma \tilde{D}_{1+\gamma} (\rho(cz)||\rho(z)) \right) \geq \sum_{cz} \text{tr}(\rho(cz)) \left( 1 + \gamma \left( \log(F(cz)) + \tilde{D}_{1+\gamma} (\rho(cz)||\rho(z)) \right) \right) = 1 + \gamma \left( \sum_{cz} \text{tr}(\rho(cz)) \log(F(cz)) + \text{tr}(\rho(cz)) \tilde{D}_{1+\gamma} (\rho(cz)||\rho(z)) \right). \tag{6.2}
\]

Subtracting 1 on both sides and dropping the positive quantity \( \gamma \) gives

\[
\sum_{cz} \log(F(cz)) \text{tr}(\rho(cz)) \leq \sum_{cz} - \text{tr}(\rho(cz)) \tilde{D}_{1+\gamma} (\rho(cz)||\rho(z)) \tag{6.3}
\]

where the right-hand side converges to \(-\sum_{cz} \text{tr}(\rho(cz)(\log(\rho(cz)) - \log(\rho(z))))\) as \( \gamma \searrow 0 \) (Eq. 2.17), so \( F(CZ) \) satisfies the entropy-estimator inequality Eq. 6.1 \( \square \)

### 6.2. QEFs from Entropy Estimators

**Theorem 6.3.** Let \( K(CZ) \) be an entropy estimator for \( C|Z \) and \( C(CZ) \). Define \( F(CZ)^\beta = e^{\beta K(CZ)} \) and

\[
c_p(\beta) = c_p(\beta; K(CZ)) = \sup \left\{ \sum_{cz} F(cz) P_{1+\beta} (\tau(cz)|\tau(z)) : \tau(CZ) \in \mathcal{N}(C(CZ)) \right\} - 1. \tag{6.4}
\]

Then for \( \beta \leq 1 \), \( F(CZ)^\beta/(1 + c_p(\beta)) \) is a QEFP with power \( \beta \) for \( C|Z \) and \( C(CZ) \). The function \( c_p(\beta) \) can be extended to \( \beta = 0 \) by taking the limit \( \beta \searrow 0 \) and satisfies \( c_p(0) = 0 \) and \( c_p \) is convex. Let \( \iota_0 \approx 2.065339 \) be the positive solution to \( 2\coth(x) = x \). Define \( \|x\| = \max(\iota_0, x), N = |\text{Rng}(C)|, k_{\text{max}}(z) = \max_c K(cz), \) and \( \bar{w}_\gamma(z) = (1 - \gamma) \max_c \left( \max \left( \log(N) - K(cz), K(cz) \right) \right) + \log(2) \). For \( \beta < 1/2 \), an upper bound on \( c_p \)
is given by \( c_P(\beta) \leq \frac{\beta^2}{2} \sup \{ e(\beta, \nu(Z)) : \nu(CZ) \in \text{tr}(\mathcal{N}(CZ)) \} \), where for \( \beta \in [0, 1) \),

\[
c(\beta, \nu(Z)) = \frac{1}{3} \sum_z \nu(z) \left( 2\|\bar{w}_0(z)\| (\|\bar{w}_0(z)\| + 2 \coth(\|\bar{w}_0(z)\|)) \\
+ \frac{\text{e}^{\kappa_{\max}(z)\beta}}{(1 - \beta)^2} (\|\bar{w}_\beta(z)\| (\|\bar{w}_\beta(z)\| + 2 \coth(\|\bar{w}_\beta(z)\|))) \right). \tag{6.5}
\]

We demonstrate by example in Sect. 8.4 that direct constructions of QEFs have much better performance than constructions from entropy estimators. Direct constructions for \((k, 2, 2)\) Bell-test configurations are given in Sect. 8.1. If it is necessary to construct QEFs from entropy estimators by applying Thm. 6.3, the bound can be improved according to expressions obtained in the proof, where we develop bounds suitable for numerical implementation. Beyond taking advantage of input probability constraints, the bounds are agnostic with regard to specific properties of \( C(Z) \) and are therefore necessarily suboptimal.

The proof of Thm. 6.3 is an elaboration on the techniques for bounding Rényi entropies in Ref. [7], see the proof of Lem. 8 in this reference. The same techniques also contribute to the proof of the entropy accumulation theorem in [4], with similar results for estimating conditional min-entropy. See the comparison in the next section. Much of the complexity of the proof below arises from squeezing out the best bounds possible given the constraints of written text. The proof is presented to enable numerical improvements and to provide information on limitations of the technique. Improved bounds are readily obtained but matter primarily when \( \beta \) is not small. See relevant remarks in the proof.

**Proof.** By definition of \( c_P(\beta) \), \( F(CZ)^\beta / (1 + c_P(\beta)) \) satisfies the QEFP inequality with power \( \beta \) at all \( \tau(CZ) \in \mathcal{N}(C(Z)) \), so the first claim is immediate.

To determine an upper bound on \( c_P(\beta) \), consider any \( \tau(CZ) \in \mathcal{N}(C(Z)) \). The right-hand side of the QEFP inequality with power \( \beta \) at \( \tau(CZ) \) for \( C|Z \) and \( F(CZ)^\beta \) is equivalent to

\[
h(\beta) = \sum_{cz} F(cz)^\beta \text{tr}(\tau(cz)^{1+\beta} \tau(z)^{-\beta}). \tag{6.6}
\]

The goal is to determine a bound on \( h(\beta) \) that depends on \( \beta \) and the values of \( K(C(Z)) \). In general, we may also take advantage of constraints on the probability distribution \( \text{tr}(\tau(CZ)) \).

The \( cz \)-term in the sum for \( h(\beta) \) is of the form

\[
g(\beta) = g(\beta; a, \rho|\sigma) = \text{tr}((\rho^a \sigma)^\beta \sigma^{-\beta}), \tag{6.7}
\]

where \( a = K(cz) \), \( \rho = \tau(z) \) and \( \rho \ll \sigma = \tau(z) \). We bound \( g(\beta) \) by Taylor expansion with a second-order remainder. For this, it is convenient to express

\[
\text{tr}((\rho^a \sigma)^\beta \sigma^{-\beta}) = \text{tr}(\xi ((\rho^a \sigma)^\beta \otimes (\sigma^{-T})^\beta)) = \text{tr}(\xi (\rho \otimes 1)((\rho^a \sigma \otimes \sigma^{-T})^\beta)), \tag{6.8}
\]

where \( \xi = |\phi\rangle\langle\phi| \) with \( \phi = \sum_i |i\rangle \otimes |i\rangle \) for some orthonormal basis \( (|i\rangle_i) \). Write \( g^{(k)}(\gamma) \) for the \( k \)’th derivative of \( g(\beta) \). For the Taylor expansion, we compute for \( 0 \leq \gamma \leq \beta \)

\[
g(0) = \text{tr}(\rho), \]
\[
g^{(k)}(\gamma) = \text{tr}(\xi (\rho \otimes 1) \log((\rho^a \sigma \otimes \sigma^{-T})^k)((\rho^a \sigma \otimes \sigma^{-T})^\gamma)). \tag{6.9}
\]
The factors after $\xi$ in the trace commute and multiply to a positive semidefinite operator for every $k$, so $g^{(2)}(\gamma) \geq 0$ for all $l \in \mathbb{N}$. In particular, the second derivative is non-negative and convex. We have

$$g(\beta) = g(0) + \beta g^{(1)}(0) + \int_0^\beta d\gamma (\beta - \gamma) g^{(2)}(\gamma), \quad (6.10)$$

where by convexity we can replace $g^{(2)}(\gamma)$ by $(1 - \gamma/\beta) g^{(2)}(0) + (\gamma/\beta) g^{(2)}(\beta) = g^{(2)}(0) + (\gamma/\beta)(g^{(2)}(\beta) - g^{(2)}(0))$ for an upper bound. Since $\int_0^\beta d\gamma (\beta - \gamma)^2 / \beta = \beta^2 / 6$, we have the bound

$$g(\beta) \leq g(0) + \beta g^{(1)}(0) + \frac{\beta^2}{2} \left( \frac{2}{3} g^{(2)}(0) + \frac{1}{3} g^{(2)}(\beta) \right). \quad (6.11)$$

To expand Eq. 6.6 in orders of $\beta$, we substitute $\rho$ by $\tau(cz)$, $\sigma$ by $\tau(z)$, $a$ by $K(cz)$ and replace the corresponding terms of Eq. 6.6 to obtain the bound

$$h(\beta) = h(0) + \beta h^{(1)}(0) + \int_0^\beta d\gamma (\beta - \gamma) h^{(2)}(\gamma)$$

$$= h(0) + \beta \sum_{cz} \left( K(cz) \text{tr}(\tau(cz)) + \text{tr}(\tau(cz)(\log(\tau(cz)) - \log(\tau(z)))) \right)$$

$$+ \sum_{cz} \int_0^\beta d\gamma (\beta - \gamma) g^{(2)}(\gamma; K(cz), \tau(cz)|\tau(z))$$

$$\leq 1 + \sum_{cz} \int_0^\beta d\gamma (\beta - \gamma) g^{(2)}(\gamma; K(cz), \tau(cz)|\tau(z))$$

$$\leq 1 + \sum_{cz} \frac{\beta^2}{2} \left( \frac{2}{3} g^{(2)}(0; K(cz), \tau(cz)|\tau(z)) + \frac{1}{3} g^{(2)}(\beta; \tau(cz)|\tau(z)) \right), \quad (6.12)$$

since $K(CZ)$ is an entropy estimator, which implies that $h^{(1)}(0) \leq 0$. The results so far also establish that $h^{(2)}(\gamma) \geq 0$, so $h(\gamma)$ is convex. Since the supremum of convex functions is convex, so is $c_P$. Since $h(0) = 1$, the claim $c_P(\beta) \searrow 0$ follows once we established that $c_P(\beta)$ is finite.

Write $h_2(\beta; K(cz), \tau(cz)|\tau(z))$ for the coefficient of $\beta^2 / 2$ of the $cz$-summand in the last line of Eq. 6.12. To prove the theorem, we determine a bound $b(\beta) \geq \sum_{cz} h_2(\beta; K(cz), \tau(cz)|\tau(z))$ expressed as an expectation over the probability distribution $\text{tr}(\tau(Z))$ with no other dependence on $\tau(CZ)$. Then

$$h(\beta) \leq 1 + b(\beta) \beta^2 / 2. \quad (6.13)$$

To determine $b(\beta)$, we apply the following lemma.

**Lemma 6.4.** For each $a \in \mathbb{R}$, let $\mu_a$ be a positive measure on $[-1,1]$ such that for $y \in (0, \infty)$

$$\int_{[-1,1]} y^x d\mu_a(\chi) \geq \log(e^a y)^2 \left( 2/3 + 1/3(e^a y)^3 \right). \quad (6.14)$$

Let $k_{\max}(z) = \max_c K(cz)$, $k_{\min}(z) = \min_c K(cz)$, $\bar{k}(z) = \sum_c K(cz)/N$. Given $z$, let $p(a) = (a - k_{\min}(z))/(k_{\max}(z) - k_{\min}(z))$ so that $a = (1 - p(a))k_{\min}(z) + p(a)k_{\max}(z)$. Write $\bar{\mu}_{z,a} =
\[ p(a)\mu_{k_{\text{max}}(z)} + (1 - p(a))\mu_{k_{\text{min}}(z)}. \]

Then

\[
\sum_{c} h_2(\beta; K(cz), \tau(cz)\mid \tau(z)) \leq \text{tr}(\tau(z)) \left( N\bar{\mu}_{z, k(z)}(\{1\}) + \int_{(-1,0)} N^{-x}d(\mu_{k_{\text{min}}(z)} \lor \mu_{k_{\text{max}}(z)})(\chi) + \max \left( \int_{[0,1]} d\mu_{\alpha}(\chi) : \alpha \in \{k_{\text{min}}(z), k_{\text{max}}(z)\} \right) \right).
\]

(6.15)

When we apply this lemma, the measures are sums of point measures at values of \( \chi \) that depend on \( \beta \) but not on \( K(cz) \). If \([\tau(cz), \tau(z)] = 0\) for all \( c \), then the lemma can be improved by restricting \( y \) to \( y \in (0, 1] \) in the first inequality in the lemma. See the remark in the proof for the explanation.

**Proof.** For this proof, \( z \) can be held fixed, so we omit it, writing \( K(c) \) for \( K(cz) \), \( \tau(c) \) for \( \tau(cz), \tau \) for \( \tau(z) \), and similarly for the measures to be found.

For the moment, we fix \( c \) and write \( a = K(c) \). We express

\[
h_2(\beta; a, \rho\mid \sigma) = \text{tr}\left( \xi(\rho \otimes 1) \log \left( e^{a\rho \otimes \sigma^{-T}} \right)^2 \left( (2/3)1 \otimes 1 + (1/3)(e^{a\rho \otimes \sigma^{-T}})\beta \right) \right).
\]

(6.16)

Since \( \xi \) is positive semidefinite, an upper bound on \( h_2(\beta; a, \rho\mid \sigma) \) can be obtained by determining an operator upper bound on

\[
X_a = (\rho \otimes 1) \log \left( e^{a\rho \otimes \sigma^{-T}} \right)^2 \left( (2/3)1 \otimes 1 + (1/3)(e^{a\rho \otimes \sigma^{-T}})\beta \right).
\]

(6.17)

For this, we can work in a joint eigenbasis of the form \((|i\rangle_1 \otimes |j\rangle_2)_{ij}\) of \( \rho \otimes 1 \) and \( 1 \otimes T \). We identify the corresponding eigenvalues as \( \rho_i \) and \( \sigma_j \), which are also the diagonal elements in this basis. The operator \( X_a \) is also diagonal, with diagonal elements

\[
x_{ij} = \rho_i \log(e^{a\rho_i}/\sigma_j)^2((2/3) + (1/3)(e^{a\rho_i}/\sigma_j)^\beta).
\]

(6.18)

Write \( y_{ij} = \rho_i/\sigma_j \). The terms where \( y_{ij} = 0 \) do not contribute to the relevant sums because of the additional factor of \( \rho_i \). Remark: If \( [\rho, \sigma] = 0 \), then we can choose a common eigenbasis for \( \rho \) and \( \sigma \) in the expression for \( \xi \) to see that \( y_{ij} = 0 \) for \( i \neq j \), and if \( \sigma \geq \rho \), \( y_{ij} \in [0, 1] \).

The constraint on \( \mu_a \) can be reexpressed with the change of variables \( y = e^t \) in terms of \( t \in \mathbb{R} \) as

\[
\int_{[-1,1]} e^{tx}d\mu_a(\chi) \geq (t + a)^2 \left( 2/3 + (1/3)e^{(t+a)\beta} \right).
\]

(6.19)

We prove that the right-hand side is convex in \( a \). With the change of variables \( x = (t + a)\beta \), this is equivalent to \( v(x) = x^2(2/3 + (1/3)e^x) \) being convex in \( x \). Compute \( v^{(2)}(x) = 4/3 + (1/3)(2 + 4x + x^2)e^x = 4/3 + (1/3)((x + 2)^2 - 2)e^x \). For \( x \geq 0 \), \( (x + 2)^2 - 2 > 0 \), so to show that \( v^{(2)} > 0 \), it suffices to consider \( x < 0 \). Then \( e^x \in (0, 1] \) and \( (x + 2)^2 - 2 \geq -2 \) so \( v^{(2)}(x) \geq 4/3 + (1/3)(-2) = 2/3 > 0 \) as claimed.
Applying convexity, for $a \in [k_{\min}, k_{\max}]$,

$$
\int_{[-1,1]} e^{tx}d(p(a)\mu_{k_{\max}} + (1-p(a))\mu_{k_{\min}})(\chi)
\geq p(a)(t + k_{\max})^2 \left( \frac{2}{3} + \frac{1}{3}e^{(t+k_{\max})\beta} \right) 
+ (1-p(a))(t + k_{\min})^2 \left( \frac{2}{3} + \frac{1}{3}e^{(t+k_{\min})\beta} \right) 
\geq (t + a)^2 \left( \frac{2}{3} + \frac{1}{3}e^{(t+a)\beta} \right). 
\tag{6.20}
$$

With $\bar{\mu}_a = p(a)\mu_{k_{\max}} + (1-p(a))\mu_{k_{\min}}$ as defined in the statement of the lemma, we get

$$
x_{ij} = \rho_i \log(e^a y_{ij})^2 \left( \frac{2}{3} + \frac{1}{3}(e^a y_{ij})^\beta \right) 
\leq \rho_i \int_{[-1,1]} y_{ij}^\chi d\bar{\mu}_a(\chi) 
= \int_{[-1,1]} \rho_i^{1+\chi} \sigma_j^{-\chi'} d\bar{\mu}_a(\chi). 
\tag{6.21}
$$

It follows that $X_a \leq \int_{[-1,1]} \rho^{1+\chi} \otimes (\sigma^{-T})^\chi' d\bar{\mu}_a(\chi)$ and

$$
h_2(\beta; a, \rho|\sigma) = \text{tr}(X_a) 
\leq \int_{[-1,1]} \text{tr}(\xi (\rho^{1+\chi} \otimes (\sigma^{-T})^\chi)) d\bar{\mu}_a(\chi) 
= \int_{[-1,1]} \text{tr}(\rho^{1+\chi} \sigma^{-\chi}) d\bar{\mu}_a(\chi). 
\tag{6.22}
$$

Substituting accordingly we get

$$
\sum_c h_2(\beta; K(c), \tau(c)|\tau) 
\leq \sum_c \int_{[-1,1]} \text{tr}(\tau(c)^{1+\chi}_{\tau^{-\chi}}) d\bar{\mu}_{K(c)}(\chi) 
= \sum_c \text{tr}(\tau)\bar{\mu}_{K(c)}(\{-1\}) + \int_{(-1,0)} \sum_c \text{tr}(\tau(c)^{1+\chi}_{\tau^{-\chi}}) d\bar{\mu}_{K(c)}(\chi) 
+ \int_{[0,1]} \sum_c \text{tr}(\tau(c)^{1+\chi}_{\tau^{-\chi}}) d\bar{\mu}_{K(c)}(\chi). 
\tag{6.23}
$$

For $0 \leq \chi \leq 1$, we have $\text{tr}(\tau(c)^{1+\chi}_{\tau^{-\chi}}) \leq \text{tr}(\tau(c))$, as can be seen by applying Lem. 2.12 with $\rho$ there replaced by $\tau(c)$ here, $\sigma$ there with $\tau(c)$ here, $\sigma'$ there with $\tau$ here, and $\beta$ there with $\chi$ here. For $-1 < \chi < 0$, the dimension bounds on Rényi powers imply that $\sum_c \text{tr}(\tau(c)^{1+\chi}_{\tau^{-\chi}}) \leq \text{tr}(\tau)|N^{-\chi}$ (Ref. [24], Sect. 5.3.5). We can now bound each summand at the end of Eq. 6.23 By linearity of $\bar{\mu}_a$ in $a$,

$$
\sum_c \text{tr}(\tau)\bar{\mu}_{K(c)}(\{-1\}) = \text{tr}(\tau)|N\bar{\mu}_k. 
\tag{6.24}
$$
For \( a \in [k_{\text{min}}, k_{\text{max}}] \), \( \bar{\mu}_a \leq \mu_{k_{\text{min}}} \lor \mu_{k_{\text{max}}} \), where \( \mu_{k_{\text{min}}} \lor \mu_{k_{\text{max}}} \) is independent of \( c \). Therefore

\[
\int_{(-1,0)} \sum_c \text{tr} \left( \tau(c)^{1+\chi \tau^{-\gamma}} \right) d\bar{\mu}_K(c)(\chi) \leq \int_{(-1,0)} \sum_c \text{tr} \left( \tau(c)^{1+\chi \tau^{-\gamma}} \right) d(\mu_{k_{\text{min}}} \lor \mu_{k_{\text{max}}})(\chi)
\]

\[
\leq \text{tr}(\tau) \int_{(-1,0)} N^{-\chi} d(\mu_{k_{\text{min}}} \lor \mu_{k_{\text{max}}})(\chi). \tag{6.25}
\]

Since \( \text{tr}(\tau(C)) / \text{tr}(\tau) \) is a probability distribution and for \( a \in [k_{\text{min}}, k_{\text{max}}] \), the integral \( \int_{[0,1]} d\bar{\mu}_a(\chi) \) is between \( \int_{[0,1]} d\bar{\mu}_{k_{\text{min}}}(\chi) \) and \( \int_{[0,1]} d\bar{\mu}_{k_{\text{max}}}(\chi) \),

\[
\int_{[0,1]} \sum_c \text{tr} \left( \tau(c)^{1+\chi \tau^{-\gamma}} \right) d\bar{\mu}_K(c)(\chi) \leq \int_{[0,1]} \sum_c \text{tr} \left( \tau(c) \right) d\bar{\mu}_K(c)(\chi)
\]

\[
= \text{tr}(\tau) \sum_c \frac{\text{tr} \left( \tau(c) \right)}{\text{tr}(\tau)} \int_{[0,1]} d\bar{\mu}_K(c)(\chi)
\]

\[
\leq \text{tr}(\tau) \max_c \int_{[0,1]} d\bar{\mu}_K(c)(\chi)
\]

\[
\leq \text{tr}(\tau) \max \left( \int_{[0,1]} d\bar{\mu}_a(\chi) : a \in \{k_{\text{min}}, k_{\text{max}}\} \right) . \tag{6.26}
\]

Inserting these summands back into the right-hand side of Eq. 6.23 gives the lemma. ~

Motivated by the above lemma we consider the reparametrized constraint in Eq. 6.19. Reparametrizing a second time by replacing \( t + a \) by \( t \) gives

\[
\int_{[-1,1]} e^{(t-a)\chi} d\mu_a(\chi) \geq t^2 (2/3 + (1/3)e^{t\beta}). \tag{6.27}
\]

To simplify the problem, we express \( \mu_a \) in terms of a sum of measures \( \nu \) satisfying

\[
\int_{[-1,1]} e^{(t-a)\chi} d\nu(\chi) \geq t^2 e^{t^\gamma}. \tag{6.28}
\]

for \( \gamma = 0 \) or \( \gamma = \beta \). Replacing \( d\nu(\chi) \) by \( e^{\alpha x} d\mu(\chi) \) and dividing both sides by \( e^{t^\gamma} \), we can equivalently determine \( \mu \) such that for all \( t \in \mathbb{R} \),

\[
\int_{[-1,1]} e^{t(\chi-\gamma)} d\mu(\chi) \geq t^2 . \tag{6.29}
\]

In view of the form of Eq. 6.15 and in view of the reparametrization of measures, we wish to minimize

\[
\int_{[-1,1]} N^{-\chi[\chi \leq 0]} e^{\alpha x} d\mu(\chi). \tag{6.30}
\]

Let \( \delta_x \) denote the delta-function probability distribution defined by \( \int f(y) d\delta_x(y) = f(x) \). We converge on the choice

\[
\mu = \lambda_1 (\delta_{-1+2\gamma} + \delta_1) + \lambda_0 \delta_\gamma , \tag{6.31}
\]
for which the constraints are

$$2 \cosh((1 - \gamma)t)\lambda_1 + \lambda_0 \geq t^2$$  \hspace{1cm} (6.32)

for all $t \in \mathbb{R}$, where we determine $\lambda_1 \geq 0$ and $\lambda_0 \geq 0$ so that this inequality is tight. We naturally arrived at this choice after considering more general forms that satisfy the constraints. See the comment after the proof for a discussion. Subject to the constraints, we minimize

$$\lambda_1 \left( N^{(1-2\gamma)\gamma \leq 1/2} e^{-(1-2\gamma) + e^a} \right) + \lambda_0 e^{\gamma a} = e^{\gamma a} \left( \lambda_1 \left( N^{(1-2\gamma)\gamma \leq 1/2} e^{-(1-\gamma) + e^{(1-\gamma)a}} \right) + \lambda_0 \right)$$

$$= e^{\gamma a} \left( 2 \cosh(w(a))\lambda_1 + \lambda_0 \right),$$  \hspace{1cm} (6.33)

where $e^{w(a)}$ is the larger of the two solutions $x$ to the identity $x + 1/x = e^{l(a)}$ with $l(a) = \log(N^{(1-2\gamma)\gamma \leq 1/2} e^{(1-\gamma)a} + e^{(1-\gamma)a}) \geq \log(2)$. Thus $e^{w(a)} = \left( e^{l(a)} + \sqrt{e^{2l(a)} - 4} \right)/2 \leq e^{l(a)}$, where the upper bound is a good approximation for large $l(a)$. The function $a \mapsto e^{l(a)}$ is convex and symmetric around its minimum at $a = \log(N^{(1-2\gamma)\gamma \leq 1/2} (2(1 - \gamma)))$, and as a result $w(a)$ is also minimized at and symmetric about this value, and monotone on each side. From $\log(e^x + e^{-x}) \leq |x| + \log(2)$, we obtain that for $1/2 \leq \gamma \leq 1$, $l(a) \leq (1 - \gamma)|a| + \log(2)$, and for $0 \leq \gamma \leq 1/2$,

$$w(a) \leq l(a) = \log(N^{(1-2\gamma) e^{-(1-\gamma)a} + e^{(1-\gamma)a}})$$

$$= \log\left( N^{(1-2\gamma)/2} e^{-(1-\gamma)a + N^{-1-2\gamma}/2 e^{-(1-\gamma)a}} \right)$$

$$\leq (1 - 2\gamma) \log(N)/2 + |(1 - 2\gamma) \log(N)/2 - (1 - \gamma)a| + \log(2)$$

$$= \max((1 - 2\gamma) \log(N) - (1 - \gamma)a, (1 - \gamma)a) + \log(2)$$

$$\leq \max((1 - \gamma) \log(N) - (1 - \gamma)a, (1 - \gamma)a) + \log(2)$$

$$= (1 - \gamma) \max(\log(N) - a, a) + \log(2).$$  \hspace{1cm} (6.34)

For the last inequality we opted for a simpler expression at the cost of worse bounds when $\gamma$ is not small. The better bound is readily taken into account by changing the next definitions and the corresponding ones in the theorem statement. Let $\tilde{w}(a) = (1 - \gamma) \max(\log(N) - a, a) + \log(2)$ so that $w(a) \leq \tilde{w}(a)$. We also define $\tilde{w}(z) = \max_c \tilde{w}(K(cz)) = \max(\tilde{w}(k_{\max}(z)), \tilde{w}(k_{\min}(z)))$, consistent with the theorem statement, but suppressing the subscript $\gamma$ for the moment.

The minimization problem defined by the constraints in Eq. 6.32 and the objective function in Eq. 6.33 can be transformed to an instance of

Minimize: $2 \cosh(v) a_1 + a_0$

Variables: $a_1, a_0$

Subject to: $2 \cosh(s) a_1 + a_0 \geq s^2$ or all $s \in \mathbb{R}$,

$$a_1 \geq 0, a_0 \geq 0,$$  \hspace{1cm} (6.35)

where $v \geq 0$; the transformation is below. To satisfy the constraint, we determine the minimum value $f(s_0)$ of $f(s) = f(s; a_1) = 2 \cosh(s) a_1 - s^2$ and set $a_0 = -f(s_0)$ if $f(s_0) < 0$ and $a_0 = 0$ otherwise. In the process of determining the minimum of $f(s)$, we parametrize
\( a_1 \) and \( a_0 \) in terms of \( s_0 \) to enable minimizing the objective function.

The minimum of \( f(s) \) is achieved at a critical point \( s_0 \) satisfying \( f^{(1)}(s_0) = 2\sinh(s_0)a_1 - 2s_0 = 0 \). One such critical point is \( s_0 = 0 \). By symmetry it suffices to consider \( s_0 \geq 0 \). Consider \( f^{(2)}(s) = 2\cosh(s)a_1 - 2 \). This is positive for \( a_1 \geq 1 \) and \( s > 0 \), in which case there are no positive critical points and the minimum of \( f(s) \) is \( f(0) = 2a_1 \). If \( a_1 < 1 \), then the slope \( f^{(2)} \) of \( f^{(1)} \) is increasing for \( s \geq 0 \), negative at \( s = 0 \), and positive for \( s \) large enough, from which it follows that there is exactly one critical point \( s_0 > 0 \) for \( f \), which determines the minimum of \( f \). By making use of the critical-point equation to express

\[
a_1 = a_1(s_0) = s_0/\sinh(s_0), \tag{6.36}
\]

we have \( f(s_0; a_1(s_0)) = 2s_0 \coth(s_0) - s_0^2 \). The function \( x \in (0, \infty) \mapsto 2x \coth(x) - x^2 \) approaches 2 as \( x \searrow 0 \) and has derivative \( 2\coth(x) - 2x/\sinh(x)^2 - 2x = 2\coth(x)(1 - x \coth(x)) \) which is negative for \( x > 0 \). Negativity follows from \( \sinh(x) = \int_0^x \cosh(t)dt \leq x \cosh(x) \). Therefore \( f(s_0; a_1(s_0)) \) is decreasing in \( s_0 \), thus negative for \( s_0 > t_0 \) where \( t_0 > 0 \) uniquely satisfies \( 2\coth(t_0) = t_0 \). By numerical calculation, \( t_0 \in (2.065338, 2.065339) \). We now constrain \( s_0 \geq t_0 \) and parametrize \( a_1 \) and \( a_0 \) in terms of \( s_0 \), with \( a_1 \) given in Eq. 6.36 and \( a_0 \geq 0 \) given by

\[
a_0 = a_0(s_0) = -f(s_0; a_1(s_0)) = s_0(s_0 - 2\coth(s_0)). \tag{6.37}
\]

Here \( a_0 \) is increasing and \( a_1 \) is decreasing in \( s_0 \) for \( s_0 > 0 \). For the latter, the function \( x \mapsto x/\sinh(x) \) has derivative \( (\sinh(x) - x \cosh(x))/\sinh(x)^2 \leq 0 \).

It remains to minimize \( 2\cosh(v)a_1 + a_0 \) over \( s_0 \geq t_0 \). Rewrite

\[
2\cosh(v)a_1 + a_0 = 2\cosh(v)a_1 + s_0(s_0 - 2\coth(s_0))
\]

\[
= 2\cosh(v)a_1 + a_1 \sinh(s_0)(s_0 - 2\coth(s_0))
\]

\[
= a_1(2\cosh(v) + s_0 \sinh(s_0) - 2\cosh(s_0)), \tag{6.38}
\]

and differentiate by \( s_0 \)

\[
\frac{d}{ds_0}(2\cosh(v)a_1 + a_0) = \left( \frac{d}{ds_0}a_1 \right)(2\cosh(v) + s_0 \sinh(s_0) - 2\cosh(s_0))
\]

\[
+ a_1(\sinh(s_0) + s_0 \cosh(s_0) - 2 \sinh(s_0))
\]

\[
= \left( \frac{d}{ds_0}a_1 \right)(2\cosh(v) + s_0 \sinh(s_0) - 2\cosh(s_0))
\]

\[
+ a_1(s_0 \cosh(s_0) - \sinh(s_0)). \tag{6.39}
\]

Since \( \frac{d}{ds_0}a_1 = (\sinh(s_0) - s_0 \cosh(s_0))/\sinh(s_0)^2 \), we can replace the second factor of the second summand by \( -a_1 \sinh(s_0)^2 \frac{d}{ds_0}a_1 \) to get

\[
\frac{d}{ds_0}(2\cosh(v)a_1 + a_0) = \left( \frac{d}{ds_0}a_1 \right)(2\cosh(v) + s_0 \sinh(s_0) - 2\cosh(s_0) - a_1 \sinh(s_0)^2)
\]

\[
= \left( \frac{d}{ds_0}a_1 \right)(2\cosh(v) + s_0 \sinh(s_0) - 2\cosh(s_0) - s_0 \sinh(s_0))
\]
\[ \frac{d}{ds_0} a_1 \left( 2 \cosh(v) - 2 \cosh(s_0) \right). \] (6.40)

There is one critical point \( s_0 = v \) which gives the desired minimizing \( s_0 \) provided \( v \geq t_0 \). Accordingly, we set \( s_0 = \max(t_0, v) \). Define \( \|x\| = \max(t_0, x) \). Substituting for \( a_0 \) and \( a_1 \) gives

\[
a_0 = \|v\| (\|v\| - 2 \coth(\|v\|)) \\
= \max(0, v(v - 2 \coth(v))), \\
a_1 = \|v\| \csch(\|v\|) \\
= \min(t_0 \csch(t_0), v \csch(v)), \\
2 \cosh(v) a_1 + a_0 \leq 2 \cosh(\|v\|) a_1 + a_0 \\
= \|v\|^2. \] (6.41)

To return to Eqs. 6.32 and 6.33, we identify \( s = (1 - \gamma)t \) to match constraints. In Eq. 6.32 this requires multiplying both sides by \((1 - \gamma)^2\) to match the constraint of Eq. 6.35, after which we must identify \( \lambda_1(1 - \gamma)^2 = a_1 \) and \( \lambda_0(1 - \gamma)^2 = a_0 \). For the objective function, we consider Eq. 6.33 to identify \( v = w(a) \), as the positive prefactor \( e^{\gamma a}/(1 - \gamma)^2 \) does not affect the optimizing variables. Since \( s_0 = \|w(a)\| \), this yields

\[
\lambda_{0, \gamma}(a) = \frac{1}{(1 - \gamma)^2} \|w_{\gamma}(a)\| (\|w_{\gamma}(a)\| - 2 \coth(\|w_{\gamma}(a)\|)), \\
\lambda_{1, \gamma}(a) = \frac{1}{(1 - \gamma)^2} \|w_{\gamma}(a)\| \csch(\|w_{\gamma}(a)\|), \] (6.42)

where we now make the parameter \( \gamma \) explicit with subscripts and make \( a \) visible as an argument of the \( \lambda_i \). To apply Lem. 6.4, we expand

\[
\mu_{a, \gamma} = \lambda_{1, \gamma}(a) \delta_{-1+2\gamma} + \lambda_{0, \gamma}(a) \delta_{\gamma} + \lambda_{1, \gamma}(a) \delta_1 \] (6.43)

according to Eq. 6.31, where we now depend the dependence on \( a \) visible as a subscript. We then apply the replacement \( d\nu_{a, \gamma}(\chi) \) by \( e^{\alpha x} d\mu_{a, \gamma}(\chi) \) used to arrive at the constraint of Eq. 6.29 and finally express the \( d\mu_{a}(\chi) \) required for applying Lem. 6.4 as the weighted combination of \( d\nu_{a, \gamma}(\chi) = e^{\alpha x} d\mu_{a, \gamma}(\chi) \) with \( \gamma = 0 \) and \( \gamma = \beta \) suggested by the form of Eq. 6.27. This gives

\[
d\mu_{a}(\chi) = \frac{1}{3} e^{\alpha x} \left( 2 d\mu_{a, 0}(\chi) + d\mu_{a, \beta}(\chi) \right). \] (6.44)

The construction above ensures that \( \mu_a \) satisfies the condition in Lem. 6.4. Expanding in terms of the parameters found we get

\[
\mu_a = \frac{2 \lambda_{1, 0}(a)}{3} e^{-a} \delta_{-1} \\
+ \frac{\lambda_{1, \beta}(a)}{3} e^{-a(1-2\beta)} \delta_{-1+2\beta} \\
+ \frac{2 \lambda_{0, 0}(a)}{3} \delta_0 + \frac{\lambda_{0, \beta}(a)}{3} e^{a\beta} \delta_\beta + \frac{2 \lambda_{1, 0}(a) + \lambda_{1, \beta}(a)}{3} e^{a} \delta_{1}. \] (6.45)
For $\beta < 1/2$, the terms of Lem. 6.4 behind $\text{tr}(\tau(z))$ are

$$N \tilde{\mu}_{z,k(z)}\{1\} = \frac{2N}{3} \left( \lambda_{1,0}(k_{\min}(z)) e^{-k_{\min}(z)} + \frac{k(z) - k_{\min}(z)}{k_{\max}(z) - k_{\min}(z)} \left( \lambda_{1,0}(k_{\max}(z)) e^{-k_{\max}(z)} - \lambda_{1,0}(k_{\min}(z)) e^{-k_{\min}(z)} \right) \right),$$

$$\int_{(-1,0)} N^{-\chi} d(\mu_{k_{\min}(z)} \vee \mu_{k_{\max}(z)})(\chi) = \frac{1}{3} \max \left( \int_{[0,1]} e^{a_\chi} d\mu_{a,0}(\chi) + \int_{[0,1]} 2e^{a_\chi} d\mu_{a,\beta}(\chi) : a \in \{k_{\min}(z), k_{\max}(z)\} \right),$$

$$\max \left( \int_{[0,1]} d\mu_a(\chi) : a \in \{k_{\min}(z), k_{\max}(z)\} \right) = \frac{1}{3} \max \left( \int_{[0,1]} e^{a_\chi} d\mu_{a,0}(\chi) + \int_{[0,1]} 2e^{a_\chi} d\mu_{a,\beta}(\chi) : a \in \{k_{\min}(z), k_{\max}(z)\} \right),$$

$$= \frac{1}{3} \max \left( \int_{[0,1]} e^{a_\chi} d\mu_{a,0}(\chi) + \int_{[0,1]} 2e^{a_\chi} d\mu_{a,\beta}(\chi) : a \in \{k_{\min}(z), k_{\max}(z)\} \right).$$

These expressions are ready to implement for specific applications. It remains to obtain the bound in the statement of the theorem. For this, we use the bound $w_\gamma(a) \leq \tilde{w}_\gamma(a)$ obtained earlier.

We first simplify the third expression in Eq. 6.46 by means of the inequality

$$\int_{[0,1]} e^{a_\chi} d\mu_{a,\gamma}(\chi) \leq \int_{[-1,1]} N^{-\chi |x| \leq 0} e^{a_\chi} d\mu_{a,\gamma}(\chi).$$

The right-hand side is the quantity in Eq. 6.30 that was evaluated in Eq. 6.33 and then minimized. It is related to the third quantity given and bounded in Eq. 6.41 by the conversion from the $a_i$ to the $\lambda_i$ and a scale, namely by a factor of $e^{\gamma a}/(1 - \gamma)^2$. This gives

$$\int_{[0,1]} e^{a_\chi} d\mu_{a,\gamma}(\chi) \leq e^{\gamma a} (2 \cosh(w_\gamma(a)) \lambda_{1,\gamma}(a) + \lambda_{0,\gamma}(a))$$

$$\leq \frac{e^{\gamma a}}{(1 - \gamma)^2} \|w_\gamma(a)\|^2$$

$$\leq \frac{e^{\gamma k_{\max}(z)}}{(1 - \gamma)^2} \|w_\gamma(a)\|^2$$

$$\leq \frac{e^{\gamma k_{\max}(z)}}{(1 - \gamma)^2} \|\tilde{w}_\gamma(a)\|^2.$$  

The third expression is therefore bounded by

$$\frac{1}{3} \max \left( 2\|\tilde{w}_0(a)\|^2 + \frac{e^{\beta k_{\max}(z)}}{(1 - \beta)^2} \|\tilde{w}_\beta(a)\|^2, a \in \{k_{\min}(z), k_{\max}(z)\} \right).$$
where $\bar{w}_\beta(z)$ is as defined in the theorem statement.

Next, the first expression of Eq. 6.46 is bounded by

$$N \bar{\mu}_{z,k(z)}(\{-1\}) \leq \frac{2}{3} \max \left(N e^{-a} \lambda_{1,0}(a) : a \in \{k_{\min}(z), k_{\max}(z)\}\right),$$

which differs from the second expression only in the initial factor and a replacement of $\beta$ by 0. In view of the definition of $w_\gamma(a)$ after Eq. 6.33 and the expression for $\lambda_{1,\gamma}(a)$ in Eq. 6.42

$$N^{1-2\gamma} e^{-a(1-2\gamma)} \lambda_{1,\gamma}(a) \leq e^{a\gamma} \left(N^{1-2\gamma} e^{-a(1-\gamma)} + e^{a(1-\gamma)}\right) \lambda_{1,\gamma}(a)$$

$$= e^{a\gamma} 2 \cosh(w_\gamma(a)) \lambda_{1,\gamma}(a)$$

$$\leq e^{k_{\max}(z)\gamma} 2 \cosh(w_\gamma(a)) \lambda_{1,\gamma}(a)$$

$$= \frac{e^{k_{\max}(z)\gamma}}{(1-\gamma)^2} 2 \cosh(\|w_\gamma(a)\|) \csc(\|w_\gamma(a)\|)$$

$$\leq \frac{e^{k_{\max}(z)\gamma}}{(1-\gamma)^2} 2 \|w_\gamma(a)\| \coth(\|w_\gamma(a)\|)$$

$$\leq \frac{e^{k_{\max}(z)\gamma}}{(1-\gamma)^2} 2 \|\bar{w}_\gamma(a)\| \coth(\|\bar{w}_\gamma(a)\|),$$

where the last inequality follows from monotonicity of $x \coth(x)$. With this we can combine the bounds for the first and second expressions to

$$\max \left(\frac{4}{3} \|\bar{w}_0(a)\| \coth(\|\bar{w}_0(a)\|) : a \in \{k_{\min}, k_{\max}\}\right)$$

$$+ \max \left(\frac{2e^{k_{\max}(z)\beta}}{3(1-\beta)^2} \|\bar{w}_\beta(a)\| \coth(\|\bar{w}_\beta(a)\|) : a \in \{k_{\min}, k_{\max}\}\right)$$

$$= \frac{1}{3} \left(4 \|\bar{w}_0(z)\| \coth(\|\bar{w}_0(z)\|) + 2 \frac{e^{k_{\max}(z)\beta}}{(1-\beta)^2} \|\bar{w}_\beta(z)\| \coth(\|\bar{w}_\beta(z)\|)\right).$$

By combining the bounds on all three expressions we get

$$\sum c h_2(\beta; K(cz), \tau(cz)|\tau(z)) \leq \frac{\text{tr}(\tau(z))}{3} \left(2 \|\bar{w}_0(z)\| (\|\bar{w}_0(z)\| + 2 \coth(\|\bar{w}_0(z)\|))\right)$$
\[ + \frac{e^{k_{\max}(z)\beta}}{(1-\beta)^2} \left[ \|\tilde{w}_\beta(z)\| \left( \|\tilde{w}_\beta(z)\| + 2 \coth(\|\tilde{w}_\beta(z)\|) \right) \right]. \] 

(6.53)

The bound in the theorem statement follows.

The form of the measure \( \mu \) in Eq. (6.31) for the proof of Thm. 6.3 is guided by its potential for closed-form determination of optimal parameters. It is not optimal for minimizing Eq. (6.30) subject to Eq. (6.29), which is the intent at that point in the proof. An optimal measure \( \mu \) is of the form \( b_{-\delta_1} - b_0 \delta_0 + b_+ \delta_1 \). To see this, reparametrize \( \mu \) by 

\[
\frac{d\mu'}{d\chi} = N\chi^{J_1}e^{-\gamma \chi}d\mu'(\chi).
\]

The problem in terms of \( \mu' \) is to

\[
\text{Minimize: } \int_{[-1,1]} d\mu'(\chi)
\]

Variable: The positive measure \( \mu' \)

Subject to: 

\[
\int_{[-1,1]} N\chi^{J_1}e^{(\chi-\gamma)-\alpha \chi}d\mu'(\chi) \geq t^2 \text{ for all } t \in \mathbb{R}. \tag{6.54}
\]

Let \( \mu' \) be a feasible solution. For any positive measure \( \nu \), reals \( c \leq d \), measurable \( I \subseteq [c,d] \), and real parameter \( s \), convexity of \( x \mapsto e^{sx} \) implies

\[
e^{sc} \int_I \frac{d-x}{d-c}d\nu(\chi) + e^{sd} \int_I \frac{\chi-c}{d-c}d\nu(\chi) = \int_I \left( \frac{d-x}{d-c}e^{sc} + \frac{\chi-c}{d-c}e^{sd} \right) d\nu(\chi)
\]

\[
\geq \int_I e^{s(c(d-\chi)/(d-c)+d(\chi-c)/(d-c))}d\nu(\chi)
\]

\[
= \int_I e^{sx}d\nu(\chi). \tag{6.55}
\]

By applying this inequality with \( \nu = \mu' \), first with \( c = -1, \ d = 0, \ I = [-1,0] \) and \( s = \log(N) + t - a \), then with \( c = 0, \ d = 1, \ I = (0,1] \) and \( s = t - a \), we find that the measure

\[
\mu'' = \delta_{-1} \int_{[-1,0]} (-\chi)d\mu'(\chi) + \delta_1 \int_{[0,1]} \chi d\mu'(\chi)
\]

\[
+ \delta_0 \left( \mu'(\{0\}) + \int_{[-1,0]} (\chi + 1) d\mu'(\chi) + \int_{[0,1]} (1 - \chi) d\mu'(\chi) \right) \tag{6.56}
\]

is a feasible solution with the same value for the objective function. The measure \( \mu'' \) can be interpreted as a redistribution of \( \mu' \) to point measures at \(-1, 0 \) and \( 1 \). It is possible to apply this technique to improve the bound in Thm. 6.3 by redistributing the contribution of \( \lambda_{1,\gamma}\delta_{-1+2\beta} \) to \( \chi = -1 \) and \( \chi = 0 \) and of \( \lambda_{0,\gamma}\delta_0 \) to \( \chi = 0 \) and \( \chi = 1 \). This mostly helps when \( \tilde{w}(a) \) is not large.
6.3. Comparison to the EAT

The entropy accumulation theorem (EAT) is the main result of Ref. [4] (Thm. 4.4). It uses a different framework for describing models, where models are obtained from an explicit quantum representation of the devices. The EAT estimates conditional min-entropy from min-tradeoff functions applied to the observed frequencies of a CV. The estimate can be used with quantum-entangled states. Here we consider the case of classical-quantum states matching our scenarios, a restriction also used in Ref. [4] for the same reasons. Models \( \mathcal{C}(CZ) \) in our framework that fit the conditions of the EAT are EAT models as introduced and related to EAT channel chains in Sect. 3.5. EAT models are chained with conditionally independent inputs from models induced by POVMs associated with a given class of quantum processes. With our notation, the following is an instance of the EAT:

**Theorem 6.5.** Entropy Accumulation Theorem for Conditional Min-Entropy [4]: Let \( K(CZ) \) be an entropy estimator for \( C|Z \) and \( \mathcal{C}(CZ) \), where \( \mathcal{C}(CZ) \) is the trial model for EAT model \( \mathcal{C}(CZ) \) with \( n \) trials. Fix \( \epsilon \in (0, 1) \) and an entropy goal \( h \) per trial. Let \( \phi(CZ) = (\sum_{i=1}^{n} K(C_iZ_i) \geq nh) \). Suppose \( \{ \phi'(CZ) \} \subseteq \{ \phi(CZ) \} \) and \( \rho(CZ) \in \mathcal{C}(CZ) \). Define \( \kappa = \text{tr}(\rho(CZ)) \), \( k_\infty = \max_{cz} |K(cz)| \) and \( N = |\text{Rng}(C)| \). Then

\[
H_\infty^e(C|ZE; \rho(CZ|\phi')) \geq nh - 2\sqrt{\log_2(\epsilon)}(\log(1 + 2N) + \lceil k_\infty \rceil)\sqrt{\log(2^\kappa^2/2)}\sqrt{n}. \quad (6.57)
\]

The EAT in Ref. [4] is expressed in terms of bits. We convert terms on both sides of the inequality to nats and change to logarithms base \( e \), which requires a factor of \( \sqrt{\log_2(e)} \) for the error term. The version of the EAT given here omits the possibility that the trials may vary according to a predetermined schedule, which can be taken into account in the min-tradeoff functions that substitute for entropy estimators in Ref. [4]. For the purpose of this comparison, we consider only the case where each trial is constrained by the same model.

The EAT is formulated for affine min-tradeoff functions, not entropy estimators. For the models under consideration, affine min-tradeoff functions correspond to entropy estimators. With our notation, an affine min-tradeoff function for \( C|Z \) and \( \mathcal{C}(CZ) \) can be written as a linear function \( f : \mu(CZ) \mapsto f(\mu(CZ)) \in \mathbb{R} \) such that for all \( \rho(CZ) \in \mathcal{N}(\mathcal{C}(CZ)) \),

\[
f(\text{tr}(\rho(CZ))) \leq H_1(\rho(CZ)|ZE). \quad \text{Since } f \text{ is linear, for probability distributions } \mu(CZ) \text{ we can write}
\]

\[
f(\mu(CZ)) = \sum_{cz} a_{cz}\mu(cz) + a_0 = \sum_{cz}(a_{cz} + a_0)\mu(cz), \quad \text{so } f(\mu(CZ)) = \mathbb{E}_{\mu(CZ)}K(CZ)
\]

where \( K(CZ) : c \mapsto a_{cz} + a_0 \) is an entropy estimator.

A version of the EAT with a better coefficient of the \( \sqrt{n} \) term can be obtained by combining Thms. 4.18 and 6.3.

**Theorem 6.6.** Let \( 0 < \beta_{max} < 1/2 \). Suppose that \( c(\beta) \) is a continuous, non-decreasing function of \( \beta \in [0, \beta_{max}] \) satisfying \( \bar{c}(\beta) \geq c(\beta) \equiv \sup \{ c(\beta, \nu(Z)) : \nu(CZ) \in \text{tr}(\mathcal{N}(\mathcal{C}(CZ))) \} \) with \( c(\beta, \nu(Z)) \) as defined in Thm. 6.3. Define

\[
\beta = \frac{\sqrt{2|\log(2^\kappa^2/2)|}}{\sqrt{n}c(0)}. \quad (6.58)
\]

For \( \bar{\beta} \leq \beta_{max} \) and with the notation and assumptions of Thm. 6.5

\[
H_\infty^e(C|ZE; \rho(CZ|\phi')) \geq nh - \sqrt{2\bar{c}(\bar{\beta})}\sqrt{\log(2^\kappa^2/2)}\sqrt{n}. \quad (6.59)
\]
Proof. Let $G(CZ)$ be the QEF with power $\beta$ for $C|Z$ and $\mathcal{C}(CZ)$ obtained from chaining the QEFP given by $e^{\beta K(CZ)/(1 + cP(\beta))}$ in Thm. 6.3. Then $G(CZ) = \prod_{i=1}^{n} e^{\beta K(C_iZ_i)/(1 + cP(\beta))}$ and

$$\log(G(CZ))/\beta = \sum_{i=1}^{n} K(C_iZ_i) - n \log(1 + cP(\beta))/\beta \geq \sum_{i=1}^{n} K(C_iZ_i) - ncP(\beta)/\beta. \quad (6.60)$$

The targeted threshold is $\sum_{i=1}^{n} K(C_iZ_i) \geq nh$. The threshold in Thm. 4.18 with $F(CZ)$ there replaced by $G(CZ)$ here is equivalent to

$$\sum_{i=1}^{n} \log(G(C_iZ_i))/\beta \geq -\log(p) - \log(\delta)/\beta. \quad (6.61)$$

We set $\delta = \epsilon^2/2$ to achieve the error bound and determine $p$ by $-\log(p) = nh - ncP(\beta)/\beta + \log(\delta)/\beta$. The event $\{\phi\}$ here is defined as $\{\sum_{i=1}^{n} K(C_iZ_i) \geq nh\}$, and from Eq. 6.60 $\phi$ implies

$$\log(G(CZ))/\beta \geq nh - ncP(\beta)/\beta = -\log(p) - \log(\delta)/\beta = \log\left(\frac{1}{p^\beta}\right), \quad (6.62)$$

which matches the expression for $\phi$ in Thm. 4.18. The event $\phi'$ thus satisfies the conditions of Thm. 4.18. The conditional min-entropy bound is

$$-\log\left(p/\kappa^\alpha/\beta\right) = nh - ncP(\beta)/\beta + \log(\delta)/\beta + \log(\kappa^\alpha)/\beta$$

$$= nh - ncP(\beta)/\beta - |\log(\epsilon^2/2)|/\beta$$

$$\geq nh - n\beta c(\beta)/2 - |\log(\epsilon^2/2)|/\beta$$

$$\geq nh - n\beta \tilde{c}(\beta)/2 - |\log(\epsilon^2/2)|/\beta$$

$$\geq nh - n\beta \tilde{c}(\beta)/2 - |\log(\epsilon^2/2)|/\beta, \quad (6.63)$$

provided that $\beta \leq \beta_{\text{max}}$.

For the next step we need to extend the validity of the inequality $\tilde{c}(\beta) \geq c(\beta)$ to all $\beta < 1$. For $\beta > \beta_{\text{max}}$, we define $\tilde{c}(\beta) = \max(\tilde{c}(\beta_{\text{max}}), \max c([0, \beta]))$, which is still continuous and non-decreasing. The quantity $-\log\left(p/\kappa^\alpha/\beta\right)$ is a lower bound on the left-hand side of Eq. 6.59 so we could choose $\beta \leq \beta_{\text{max}}$ to maximize the last expression in Eq. 6.63. Instead, we choose it for convenience, using the case where $\tilde{c}(\beta)$ is independent of $\beta$ as a template. Since $c(\beta)$ diverges as $\beta \nearrow 1$, $c(0) > 0$, and $\tilde{c}(\beta) \geq c(\beta)$, the function $f(\beta) = \sqrt{2|\log(\epsilon^2/2)|/\sqrt{n\tilde{c}(\beta)}}$ is positive at $\beta = 0$ and goes to 0 as $\beta \nearrow 1$. Moreover, since $\tilde{c}$ is continuous, so is $f$. Accordingly there is a solution $\beta_0 < 1$ to the fixed-point equation

$$\beta_0 = f(\beta_0) = \frac{\sqrt{2|\log(\epsilon^2/2)|}}{\sqrt{n\tilde{c}(\beta_0)}}. \quad (6.64)$$

Since $\tilde{c}(\beta)$ is non-decreasing in $\beta$, we have $\beta_0 \leq \bar{\beta}$. Thus from Eq. 6.63 we obtain

$$-\log\left(p/\kappa^\alpha/\beta\right) \geq nh - n\beta_0 \tilde{c}(\beta_0)/2 - |\log(\epsilon^2/2)|/\beta_0$$
\[ = nh - \sqrt{2} \sqrt{c(\beta_0)} \sqrt{|\log(e^2\kappa^2/2)|} \sqrt{n} \]
\[ \geq nh - \sqrt{2} \sqrt{c(\beta)} \sqrt{|\log(e^2\kappa^2/2)|} \sqrt{n}. \] (6.65)

The condition on \( \bar{\beta} \) in the statement of the theorem is required to stay within the domain of the unextended function \( \bar{c} \).

For comparison to the EAT, we determine a bound \( \bar{c}(\beta) \geq c(\beta) \) satisfying the conditions in Thm. 6.6. For a handicapped but direct comparison, we make conservative estimates in terms of parameters that occur in the EAT to obtain moderate improvements over the EAT. The main advantage of Thm. 6.6 is that one can choose \( \bar{c}(\beta) \) less conservatively, taking advantage of the average over inputs in the expression for \( c(\beta, \nu(Z)) \) in Thm. 6.3, which enables effective use of estimators that are heavily weighted toward rare inputs. This enables the clean exponential-expansion results of Sect. 7.3.

Let \( k_{\text{max}} = \max_{cz} K(cz) \) and \( \bar{w}_\gamma = \max_z \bar{w}_\gamma(z) = (1-\gamma) \max_{cz} (\max(\log(N) - K(cz), K(cz))) + \log(2) = (1-\gamma) \max_{cz} (\log(N)/2 + |\log(N)/2 - K(cz)|) + \log(2). \) We may assume that \( k_{\text{max}} \geq 0 \) as the entropy estimator is otherwise useless. We have

\[
c(\beta) = \sum_z \tau(z) \frac{1}{3} \left( 2\|\bar{w}_0(z)\| (\|\bar{w}_0(z)\| + 2 \coth(\|\bar{w}_0(z)\|)) 
+ \frac{e^{k_{\text{max}}(cz)\beta}}{(1-\beta)^2} \|\bar{w}_\beta(z)\| (\|\bar{w}_\beta(z)\| + 2 \coth(\|\bar{w}_\beta(z)\|)) \right)
\leq \frac{1}{3} \left( 2\|\bar{w}_0\| (\|\bar{w}_0\| + 2 \coth(\|\bar{w}_0\|)) + \frac{e^{k_{\text{max}}\beta}}{(1-\beta)^2} \|\bar{w}_\beta\| (\|\bar{w}_\beta\| + 2 \coth(\|\bar{w}_\beta\|)) \right).
\] (6.66)

For a more specific comparison based on the parameters of Thm. 6.5, namely \( k_{\infty}, N, \epsilon, \kappa \) and \( n \), we use \( \bar{w}_\gamma' = (1-\gamma)(\log(N) + k_{\infty}) + \log(2) \geq \bar{w}_\gamma. \) Define \( \bar{c}(\beta) \) as the last expression of Eq. 6.66, with \( k_{\text{max}}, \bar{w}_0 \) and \( \bar{w}_\beta \) replaced by \( k_{\infty}, \bar{w}_0' \) and \( \bar{w}_\beta' \), respectively. Then \( \bar{c}(\beta) \) is non-decreasing in \( \beta \) for \( \beta < 1 \) and we can apply Thm. 6.6 with any \( \beta_{\text{max}} < 1/2 \).

We first consider the asymptotic behavior as \( n \to \infty \). For simplicity, assume that \( N \geq 4 \), so that \( \log(2N) \geq \epsilon_0 \). We compare the coefficients \( u_{\text{EAT}} \) and \( u_{\text{QEF}} \) of the \( \sqrt{|\log(e^2\kappa^2/2)|} \sqrt{n} \) terms in the conditional min-entropy bounds. In Thm. 6.6, \( \bar{\beta} = O(1/\sqrt{n}) \), so for large \( n \) and with \( \bar{c}(\beta) \) as defined in the previous paragraph, we can set \( \bar{\beta} = 0 \). This gives

\[
u_{\text{QEF}} = -\sqrt{2} \sqrt{(\log(2N) + k_{\infty})(\log(2N) + k_{\infty} + 2 \coth(\log(2N) + k_{\infty}))},
\]

where \( 2 \leq 2 \coth(\log(2N) + k_{\infty}) \leq 2 \coth(\log(8)) \approx 2.0635 \). This may be compared to

\[
u_{\text{EAT}} = -2 \sqrt{\log_2(e)} (\log(1 + 2N) + \lceil k_{\infty} \rceil).
\]

(6.67)

(6.68)

The terms involving \( N \) and \( k_{\infty} \) are similar and approach each other for large \( N \) or \( k_{\infty} \). The constant initial factors in Eq. 6.67 and Eq. 6.68 are \( \sqrt{2} \) and \( 2 \sqrt{\log_2(e)} \) respectively, which implies that \( u_{\text{EAT}}/u_{\text{QEF}} \) approaches \( \sqrt{2 \log_2(e)} \approx 1.699 \). Of course, for large \( n \), the relative difference in conditional min-entropy witnessed disappears.

For applications such as low-latency generation of a block of random bits, optimal ran-
domness expansion, or randomness with exponentially small error, the above asymptotic regime is not relevant. For the next comparison, we parameterize the error term with \( l = |\log(e^2 \kappa^2/2)| \). We consider the problem of determining the smallest \( n \) for which there is positive conditional min-entropy given \( l_e \) and the threshold rate \( h \) for the entropy estimators in Thms. \[ \text{6.5 and 6.6} \] This problem is closely related to the problem where given an error bound rate \( r \), we wish to determine the infimum of the threshold rates \( h \) such that if \( l_e = rn \), the asymptotic conditional min-entropy is positive. For the EAT, given \( l_e \) and \( h \), the smallest value of \( n \) for which the conditional min-entropy lower-bound is positive is at least

\[
N_{\min, \text{EAT}}(h, l_e) = 4 \log_2(e)(\log(1 + 2N) + k_\infty)^2 l_e/h^2. \tag{6.69}
\]

If we set \( l_e = rn \), then the smallest \( h \) for which the entropy lower-bound is non-negative is at least

\[
h_{\min, \text{EAT}}(r) = (4 \log_2(e)(\log(1 + 2N) + k_\infty)^2 r)^{1/2}. \tag{6.70}
\]

The two expressions are related by \( N_{\min, \text{EAT}}(h, l)h^2/l = h_{\min, \text{EAT}}(r)^2/r \). In general, suppose we are given a function \( h_{\min} : r \mapsto h_{\min}(r) \) such that for all \( h > h_{\min}(r) \), the asymptotic conditional min-entropy with error bound \( l_e = rn \) is positive. Then we can estimate the minimum \( n \) required for positive entropy given \( l_e \) and \( h \) from \( r_{\max}(h) = \sup \{ r : h_{\min}(r) \leq h \} \) by computing \( n \) according to \( n = l_e/r_{\max}(h) \). The estimate may be off because an asymptotic computation of \( h_{\min}(r) \) neglects lower-order terms, but in the case of the EAT, it gives a valid answer. In view of these considerations, we compare the EAT and QEF constructions by determining which has larger \( r_{\max}(h) \). For this, we determine \( h_{\min, \text{QEF}}(r) \) according to Thm. \[ \text{6.6} \]

\[
h_{\min, \text{QEF}}(r) = (2\bar{c}(\bar{\beta}) r)^{1/2}, \tag{6.71}
\]

where we now use the function \( \bar{c} \) introduced after Eq. \[ \text{6.66} \] and \( \bar{\beta} \) is given in terms of \( r \) by

\[
\bar{\beta} = \frac{\sqrt{2r}}{\sqrt{c(0)}}. \tag{6.72}
\]

Eq. \[ \text{6.71} \] requires \( \bar{\beta} \leq \beta_{\max} \), where \( \beta_{\max} < 1/2 \), so we restrict \( r \) accordingly. An analytic comparison of the two expressions for \( r_{\max} \) derived from \( h_{\min} \) is not simple, but we can plot specific examples for a visual comparison. For this we consider relevant values of \( N = 2, 4, 8 \) and \( k_\infty = 1, \log(N) \) and plot \( r_{\max} \) as a function of \( h \in (0, \log(N)) \), see Fig. \[ 1 \] The values of \( r_{\max} \) for QEFs are up to a factor of 2 larger than those for the EAT. Such improvements in rates can be significant in resource-limited applications.

The values of \( h \) occurring in the comparison have not been constrained. But since they play the role of a threshold rate for an entropy estimator, the probability that the entropy estimate exceeds \( nh \) must be sufficiently large. Values of \( h \) for which this is not the case in a given situation are not relevant. For a given trial distribution, this normally requires that \( h \) is below the expected value of the entropy estimator.

To finish this section, we remove the handicap to demonstrate the broad applicability and finite-data efficiency of QEFs. Let \( p \in (0, 1) \) and consider the trial model \( C(C) \) with \( \text{Rng}(C) = \{0, 1\} \), no inputs, and no quantum correlations, defined by \( C(C) = \text{Cvx}(\{\mu(C)p : \mu(1) = p\mu(0)\}) \). The extremal states of \( N(C(C)) \) are of the form \( [C = 0] \hat{\psi} \) and \( ((1 - p)[C = 0] + p[C = 1])\hat{\psi} \). This model is equivalent to a classical-side-information model and may be relevant for semi-device-dependent randomness generation. For the extremal states, if the number of times that \( C = 1 \) is observed in \( n \) trials is \( k \), then the
probability of the experiment’s output is at most $p^k$ from $E$’s point of view. Converting this information to a conditional min-entropy estimate without using QEFs or the EAT requires taking into account the probability that $k$ exceeds some threshold. We do not attempt this conversion, but it suggests that it is natural to analyze this model directly rather than to use QEFs or invoke the EAT. However, QEFs and the EAT are applicable and, according to the optimality theorem Thm. 6.7, achieve the asymptotically optimal rate for randomness generation.

QEFs for $C(C)$ can be written in the form $F(C) : c \mapsto ([c = 0] + f [c = 1])/m$ where $f$ and $m$ are constrained so that the QEF inequality with power $\beta$ is satisfied. The QEF inequalities for the two extremal states are

$$\frac{1}{m} \leq 1 \quad \frac{(1-p)^\alpha + fp^\alpha}{m} \leq 1. \quad (6.73)$$

Thus $m \geq 1$, and given $m$, we choose $f$ as large as possible, which gives $f = (m-(1-p)^\alpha)/p^\alpha$. The log-prob rate of $F(C)$ at $\mu(C) : c \mapsto (1-q)c + q[c = 1]$ with $q \in [0, p]$ is

$$L_{q,\beta} (m) = (q \log((m-(1-p)^\alpha)/p^\alpha) - \log(m)) / \beta. \quad (6.74)$$

To maximize the log-prob rate with respect to $m$, compute

$$\beta \frac{d}{dm} L_{q,\beta} (m) = \frac{q}{m - (1-p)^\alpha} - \frac{1}{m} = \frac{-(1-q)m + (1-p)^\alpha}{(m-(1-p)^\alpha)m}. \quad (6.75)$$

Since $1-q \geq 1-p$ and $\alpha > 1$, $\frac{d}{dm} L_q (m) \leq 0$ for $m \geq 1$, so the maximum is achieved at $m = 1$.

To illustrate the asymptotic optimality of QEFs established in Sect. 6.5, we compute the limit $\beta \searrow 0$ of the log-prob rate. Rearranging terms and the estimates $(1-p)^\beta = 1 + \beta \log(1-p) + O(\beta^2)$ and $\log(1+\beta d + O(\beta^2)) = \beta d + O(\beta^2)$ give

$$L_{q,\beta} (1) = \frac{q}{\beta} (\log(1-(1-p)^{1+\beta}) - (1+\beta) \log(p))$$

$$= \frac{q}{\beta} (\log(p + (1-p)(1-(1-p)^\beta)) - (1+\beta) \log(p))$$

$$= \frac{q}{\beta} (\log(p + (1-p)(-\beta \log(1-p) + O(\beta^2))) - (1+\beta) \log(p))$$

$$= \frac{q}{\beta} (\log(p) - (1+\beta) \log(p))$$

$$= \frac{q}{\beta} (-\beta(1-p) \log(1-p)/p + O(\beta^2) - \beta \log(p))$$

$$= -\frac{q}{p} ((1-p) \log(1-p) + p \log(p)) + O(\beta), \quad (6.76)$$

so $L_{q,0+} (1) = (q/p) H(p)$, where $H(p)$ is the Shannon entropy of the distribution $(1-p)[C = 0] + p[C = 1]$ in nits. The log-prob rate $L_{q,0+} (1)$ can be recognized as the minimum
conditional entropy for states whose output distribution is \((1 - q) [C = 0] + q [C = 1]\) given the model \(C(C)\), see Sect. 6.5.

For comparing to the EAT, we fix \(q \in (0, p]\) and consider the simplified QEF \(F_\beta(C) : c \mapsto [c = 0] + p^{-\beta} [c = 1]\). Because for \(m = 1\), \(f = (1 - (1 - p)^{\alpha})/p^\alpha \geq (1 - (1 - p))/p^\alpha = p^{-\beta}\), this QEF satisfies the QEF inequalities. Given \(q\), from the previous paragraph, the optimal log-prob rate is \(h = (q/p)H(p)\). The log-prob rate of \(F_\beta(C)\) is \(h_F = q \log(p)| \leq h\). For small \(p\), the ratio of the two rates approaches 1. We determine the minimum \(n\) such that positive conditional min-entropy can be certified. Let \(\epsilon\) be the error bound and \(\kappa\) the minimum probability of success that we need to protect against. For the EAT with rate \(h\),

\[
n_{\text{min,EAT}} \geq 4 \log_2(e)(\log(1 + 2N) + k_\infty)^2|\log(e^2\kappa^2/2)| \frac{1}{h^2} \\
> 4 \log_2(e) \log(5)^2|\log(e^2\kappa^2/2)| \frac{1}{h^2},
\]

where \(4 \log_2(e) \log(5)^2 \approx 14.95\) and we set \(k_\infty = 0\) for a lower bound. For the QEF \(F_\beta(C)\) with power \(\beta\), we apply Thm. 4.18 with \(\delta = \epsilon^2/2\) and threshold \(q\) determined by \(- \log(q) = nh\) so that \(- \log(p) = nh + \log(\delta)/\beta\). The smooth conditional min-entropy is bounded below by \(- \log(p) + \log(\kappa^{\alpha/\beta})\). From this

\[
n_{\text{min,QEF}} = \left(\left|\log(e^2\kappa/2)\right|/\beta + \left|\log(\kappa)\right|\right) \frac{1}{h},
\]

where \(\beta\) can be chosen arbitrarily large. Choosing \(\beta\) large increases the per-trial variance of \(\log(F_\beta(C))\) and may reduce the probability of success for completeness, so it is not appropriate to take the limit as \(\beta \to \infty\) without taking this issue into consideration. For the explicit QEF \(F_\beta(C)\), one can choose any \(h \leq h_F\), but to satisfy completeness with reasonable probabilities of success given \(q\) and \(\beta\), the number of trials needs to be at least some multiple of \(1/q\). For both the EAT and QEFs, useful values of \(h\) are bounded by the optimal log-prob rate \((q/p)H(p)\). It is therefore clear that \(n_{\text{min,EAT}}\) has quadratically worse dependence on \(q\) for small \(q\), and always depends on \(\epsilon\) with significantly worse prefactors. In contrast, \(n_{\text{min,QEF}}\)'s dependence on \(\epsilon\) can be suppressed by choosing large \(\beta\). The effect of the term \(|\log(\kappa)|\) depends on what is considered the minimum safe probability of success and the protocol.

We find similar, practical advantages of QEF for the (2,2,2) Bell-test configuration in Sect. 8.4, with clear advantages for all useful states. As in the example above, the advantages can be particularly large at states with low conditional entropy.

### 6.4. Entropy Estimator Optimization Problem

According to the above results, we can construct QEFs from entropy estimators, but the construction does not lead to a simple objective function for entropy estimators. However, one can seek optimal entropy estimates. Consider candidates for entropy estimators \(K(CZ)\). The entropy estimate at \(\nu\) is \(\hat{\mathcal{E}}(K; \nu) = \sum_{cz} \nu(cz)K(cz)\). The entropy estimator condition is

\[
\sum_{cz} \text{tr}(\rho(cz))K(cz) \leq -\sum_{cz} \text{tr}(\rho(cz)(\log(\rho(cz)) - \log(\rho(z))))
\]

\[(6.79)\]
for all $\rho(CZ) \in \mathcal{N}(\mathcal{C}(CZ))$. When the probability distribution of $Z$ is fixed at $\mu$, the constraint becomes

$$\sum_{cz} \mu(z) \text{tr}(\rho(c|z)K(cz)) \leq -\sum_{cz} \mu(z) \text{tr}(\rho(c|z)(\log(\rho(c|z)) - \log(\rho))). \quad (6.80)$$

The problem is then to maximize:

**Maximize:** $\sum_{cz} \nu(cz)K(cz)$

**Variables:** $K(CZ)$

**Subject to:**

$$\sum_{cz} \mu(z) \text{tr}(\rho(c|z)K(cz)) \leq -\sum_{cz} \mu(z) \text{tr}(\rho(c|z)(\log(\rho(c|z)) - \log(\rho))) \quad \text{for all } \rho(CZ) \in \mathcal{N}(\mathcal{C}). \quad (6.81)$$

### 6.5. Optimality of QEFs

In this section, we show that given the model $\mathcal{C}(CZ)$ and a probability distribution $\mu(CZ) \in \text{tr}(\mathcal{N}(\mathcal{C}(CZ)))$ consistent with the model, entropy estimators witness the maximum possible entropy rates at $\mu(CZ)$. By Thm. 6.3 with $\beta \searrow 0$, these entropy rates are asymptotically achieved by log-prob rates of QEFs. Thus QEFs are asymptotically optimal. Optimality for min-tradeoff functions is mentioned in Ref. [16]. For classical side information a proof is in Ref. [1]. Here we generalize this proof for quantum side information.

The optimality statement concerns the experimentally desirable situation where the observed statistics are i.i.d. for each trial. QEFs and entropy estimators are designed for $\mu(CZ)$ but must be valid regardless of how the observed statistics arise in the model. The proof of optimality requires relating information theoretic upper bounds on achievable rates to lower bounds achieved by entropy estimators. Both are for i.i.d. states of the form $\rho(CZ) = \bigotimes_{i=1}^{n} \rho(C_iZ_i)$ in the chained model determined by the fixed trial model $\mathcal{C}(CZ)$, where the $\rho(C_iZ_i)$ are obtained from a fixed $\rho(CZ) \in \mathcal{C}(CZ)$ by substitution of CVs. We abbreviate the expression for such states $\rho(CZ)$ as $\rho(CZ)^{\otimes n}$. Because QEFs remain valid under CPTP maps and convex closure, we may assume that $\mathcal{C}(CZ)$ is closed in both respects. This ensures that the states $\rho(CZ)$ and their tensor products are rich enough to witness the upper bounds without appealing to “mixed strategies” for $E$.

**Theorem 6.7.** Let $\mathcal{C}(CZ)$ be a CPTP-closed and convex closed model and $\mu(CZ)$ a distribution in the relative interior of $\text{tr}(\mathcal{N}(\mathcal{C}(CZ)))$. Define

$$g_{\text{QEF}}(\mu(CZ)) = \sup \left\{ E_{\mu(CZ)}(K(CZ)) : K(CZ) \text{ is an entropy estimator for } C|Z \text{ and } \mathcal{C}(CZ) \right\}, \quad (6.82)$$

and

$$s_\infty(\mu(CZ)) = \inf \left\{ \lim_{n \to \infty} \frac{1}{n} H^\infty_\infty(C|ZE; \rho(CZ)^{\otimes n}) : \right\}.$$
\[ \rho(CZ) \in \mathcal{N}(\mathcal{C}(CZ)), \epsilon > 0, \text{tr}(\rho(CZ)) = \mu(CZ) \]  

(6.83)

Then \( g_{\text{QEF}}(\mu(CZ)) = s_{\infty}(\mu(CZ)) \).

The definition of \( s_{\infty}(\mu(CZ)) \) assumes constant error bound, but error bounds decreasing sub-exponentially in \( n \) can be used. We use the condition that \( \mu(CZ) \) is in the relative interior to avoid issues that can arise at the boundary in the absence of compactness of \( \mathcal{C}(CZ) \). Since every \( \mu(CZ) \in \text{tr}(\mathcal{N}(\mathcal{C}(CZ))) \) is arbitrarily close to distributions in the relative interior, the restriction does not have practical significance.

**Proof.** According to the quantum asymptotic equipartition property, Thm. 1 of Ref. [7],

\[ s_{\infty}(\mu(CZ)) = \inf \{ H_1(\rho(CZ)|ZE) : \rho(CZ) \in \mathcal{N}(\mathcal{C}(CZ)), \text{tr}(\rho(CZ)) = \mu(CZ) \}. \]

(6.84)

Therefore, by the definition of entropy estimators, \( g_{\text{QEF}} \leq s_{\infty} \).

We claim that \( s_{\infty}(\nu(CZ)) \) is a convex function of \( \nu(CZ) \in \text{tr}(\mathcal{N}(\mathcal{C}(CZ))) \). Suppose that \( \nu(CZ) = \lambda \nu_1(CZ) + (1 - \lambda) \nu_2(CZ) \) with \( \nu_i(CZ) \in \mathcal{N}(\mathcal{C}(CZ)) \) and \( \lambda \in [0, 1] \). Let \( \rho_i(CZ) \in \mathcal{N}(\mathcal{C}(CZ)) \) satisfy \( \text{tr}(\rho_i(CZ)) = \nu_i(CZ) \) and \( H_1(\rho_i(CZ)|ZE) \leq s_{\infty}(\nu_i(CZ)) + \delta \) with \( \delta > 0 \) arbitrarily small. Define \( \rho(CZ) = \lambda \rho_1(CZ) \oplus (1 - \lambda) \rho_2(CZ) \). Because of the closure properties of \( \mathcal{C}(CZ) \), we have \( \rho(CZ) \in \mathcal{N}(\mathcal{C}(CZ)) \). By additivity of \( H_1 \) over direct sums, \( H_1(\rho(CZ)|ZE) = \lambda H_1(\rho_1(CZ)|ZE) + (1 - \lambda) H_1(\rho_2(CZ)|ZE) \). It follows that \( s_{\infty}(\nu(CZ)) \leq \lambda s_{\infty}(\nu_1(CZ)) + (1 - \lambda) s_{\infty}(\nu_2(CZ)) + \delta \). Letting \( \delta \rightarrow 0 \) proves the claim.

For concepts and properties used next, see Ref. [34], particularly Ch. 3 on the convex conjugate of convex functions. For \( \nu (CZ) \) in the relative interior of \( \text{tr}(\mathcal{N}(\mathcal{C}(CZ))) \), for every \( \delta > 0 \) there exists an “affine underestimator” \( \nu'(CZ) \mapsto \sum_{cZ} K(cZ) \nu'(cZ) \) of \( s_{\infty} \) satisfying \( \sum_{cZ} K(cZ) \nu'(cZ) \leq s_{\infty}(\nu'(CZ)) \) for all \( \nu'(CZ) \in \text{tr}(\mathcal{N}(\mathcal{C}(CZ))) \) and \( \sum_{cZ} K(cZ) \nu(cZ) \geq s_{\infty}(\nu(cZ)) - \delta \). This observation follows from Exercise 3.28 of Ref. [34]. Since \( \sum_{cZ} K(cZ) \mu(cZ) = E_{\nu(CZ)}(K(CZ)) \leq g_{\text{QEF}}(\mu(CZ)) \) and we can let \( \delta \rightarrow 0 \), this completes the proof of the theorem.

\[ \Box \]

### 7. QEFS AND MAX-PROB ESTIMATORS

#### 7.1. Max-Prob Estimators

So far we have shown how to determine QEFS from entropy estimators. There are presently few explicitly computable entropy estimators. Examples can be obtained from the affine min-tradeoff functions given in Refs. [5] [32]. In this section, we assume that the inputs \( Z \) are coming from a separate and well-characterized source.

**Definition 7.1.** \( B(CZ) \) is a max-prob estimator for \( C|Z \) and \( \mathcal{C}(CZ) \) if for all \( \nu(CZ) \in \text{tr}(\mathcal{C}(CZ)) \),

\[ E_{\nu(CZ)}(B(CZ)) \geq \max_c \nu(c|Z) \text{ for all } \nu(CZ) \in \text{tr}(\mathcal{C}(CZ)) \].

(7.1)

Like entropy estimators, max-prob estimators for a model are max-prob estimators for any submodel. Because the definition of max-prob estimators depends only on \( \text{tr}(\mathcal{C}(CZ)) \), they are also max-prob estimators of maximal extensions obtained from \( \text{tr}(\mathcal{C}(CZ)) \) provided that the input distribution is fixed.
Lemma 7.2. Let $\mu(Z)$ be a probability distribution of $Z$ and $C(C|Z)$ a model for $(C|Z)E$. If $B(CZ)$ is a max-prob estimator for $C|Z$ and $\mu(Z) \propto C(C|Z)$, then $B(CZ)$ is a max-prob estimator for $C|Z$ and $\mathcal{M}(\text{Cvx}(\mu(Z) \propto \text{tr}(C(C|Z))); E)$, the maximal extension of the model $\text{Cvx}(\mu(Z) \propto \text{tr}(C(C|Z)))$ for $CZE$.

Proof. Let $\nu(CZ)$ be a probability distribution in $\text{tr}(\mathcal{M}(\text{Cvx}(\mu(Z) \propto \text{tr}(C(C|Z))); E))$. By definition of maximal extensions, $\nu(CZ) \in \text{Cvx}(\mu(Z) \propto \text{tr}(C(C|Z)))$. We have $\text{Cvx}(\mu(Z) \propto \text{tr}(C(C|Z))) = \mu(Z) \propto \text{Cvx}(\text{tr}(C(C|Z))) = \mu(Z) \propto \text{tr}(\text{Cvx}(C(C|Z)))$. We can therefore express $\nu(CZ)$ as a convex combination $\nu(CZ) = \sum_i \lambda_i \mu(Z) \text{tr}(\tau_i(C|Z))$ with $\tau_i(C|Z) \in C(C|Z)$. Write $\nu_i(CZ) = \mu(Z) \text{tr}(\tau_i(C|Z)) \in \text{tr}(\mu(Z) \propto C(C|Z))$. Then $\nu(C|Z) = \sum_i \lambda_i \nu_i(C|Z)$. Since $B(CZ)$ is a max-prob estimator and maxima are subadditive,

$$E_{\nu(CZ)}(B(CZ)) = \sum_i \lambda_i (E_{\nu_i(CZ)}B(CZ))$$

$$\geq \sum_i \lambda_i \max_{cz} \nu_i(c|z)$$

$$= \sum_i \max_{cz} (\lambda_i \nu_i(c|z))$$

$$\geq \max_{cz} \left( \sum_i \lambda_i \nu_i(c|z) \right)$$

$$= \max_{cz} (\nu(c|z)),$$

(7.2)

as required for the lemma.

\[ \square \]

7.2. Entropy Estimators From One-Trial Max-Prob

Theorem 7.3. Let $B(CZ)$ be a max-prob estimator for $CZ$ and $C(CZ)$. For $0 < \bar{b} \leq 1$, define

$$K(CZ) = - \log(\bar{b}) + 1 - B(CZ)/\bar{b}.$$  

(7.3)

Then $K$ is an entropy estimator for $CZ$ and $C'(CZ) = \mathcal{M}(\text{Cvx}(\text{tr}(C(C|Z))); E)$.

For this theorem, $Z$ is considered as part of the output, not input. If it is necessary to condition on $Z$ later, this can be done according to Protocol \[3\]. Thm. \[7.4\] below considers the case where $Z$ is input to be conditioned on explicitly with a known probability distribution. An advantage of Thm. \[7.3\] is that it is applicable even when the distribution of $Z$ is not predetermined at each trial. As discussed in Ref. \[1\], it is a good idea to choose $\bar{b} = E_{\nu(CZ)}(B(CZ))$ where $\nu(CZ)$ is the anticipated trial probability distribution of $CZ$.

Proof. According to Lem. \[3.4\], $C'(CZ)$ is pCP-closed. Lem. \[7.2\] with $C$ there replaced by $CZ$ here and $Z$ there set to the trivial CV here implies that $B(CZ)$ is a maximum probability estimator for $CZ$ and $C'(CZ)$. Let $\tau(CZ) \in \mathcal{N}(C'(CZ))$ and $\nu(CZ) = \text{tr}(\tau(CZ))$. Below, we show that $E_{\nu(CZ)}(B(CZ)) \geq P_{\text{max}}(\tau(CZ)|E)$. Given this inequality and since $- \log P_{\text{max}}(\tau(CZ)|E) \leq H_{\text{1}}(\tau(CZ)|E)$ (Lem. \[2.25\] with the above replacements), for the theorem it suffices to show that $E_{\nu(CZ)}(K(CZ)) \leq - \log(E_{\nu(CZ)}(B(CZ)))$. From

$$- \log(x) = - \log(\bar{b}) - \log(x/\bar{b}) \geq - \log(\bar{b}) - (x/\bar{b} - 1) = - \log(\bar{b}) + 1 - x/\bar{b},$$  

(7.4)
we get

\[-\log(\mathbb{E}_\nu(B(CZ))) \geq -\log(\bar{b}) + 1 - \mathbb{E}_\nu(B(CZ))/\bar{b}
\]
\[= \mathbb{E}_\nu(-\log(\bar{b}) + 1 - B(CZ)/\bar{b})
\]
\[= \mathbb{E}_\nu(K(CZ)). \tag{7.5}
\]

To prove that \(\mathbb{E}_\nu(B(CZ)) \geq P_{\text{max}}(\tau(CZ)|E)\), we apply the relationship between maximum guessing probability and \(P_{\text{max}}\), Lem. 2.23. Let \((\Pi_{cz})_c\) be a POVM with guessing probability \(p = \sum_c \text{tr}(\Pi_{cz}\tau(cz))\). Let \(\nu_{c'z'}(CZ) = \text{tr}(\Pi_{c'z'}\tau(CZ))/\text{tr}(\Pi_{c'z'}\tau)\). By pCP-closure, \(\nu_{c'z'}(CZ) \in \text{tr}(\mathcal{C}'(CZ))\). For each \(c'z'\) we have \(\mathbb{E}_{\nu_{c'z'}}(B(CZ)) \geq \max_c (\nu_{c'z'}(cz)) \geq \nu_{c'z'}(c'z')\). Consequently

\[p = \sum_{c'z'} \nu_{c'z'}(c'z') \text{tr}(\Pi_{c'z'}\tau)
\]
\[\leq \sum_{c'z'} \mathbb{E}_{\nu_{c'z'}}(B(CZ)) \text{tr}(\Pi_{c'z'}\tau)
\]
\[= \sum_{c'z'} \sum_c B(cz) \nu_{c'z'}(cz) \text{tr}(\Pi_{c'z'}\tau(cz))
\]
\[= \sum_{c'z'} \sum_{cz} B(cz) \text{tr}(\Pi_{c'z'}\tau(cz))
\]
\[= \sum_{cz} B(cz) \text{tr}\left(\sum_{c'z'} \Pi_{c'z'}\tau(cz)\right)
\]
\[= \sum_{cz} B(cz) \text{tr}(\tau(cz))
\]
\[= \mathbb{E}_\nu(B(CZ)). \tag{7.6}
\]

The claim follows because the POVM can be chosen so that \(p\) is arbitrarily close to \(P_{\text{max}}(\tau(CZ)|E)\).

**Theorem 7.4.** Let \(\mathcal{C}(C|Z)\) be a model for \((C|Z)|E\), \(\mu\) a probability distribution of \(Z\) and \(B(CZ)\) a max-prob estimator for \(C|Z\) and \(\mu(Z) \propto \mathcal{C}(C|Z)\). Let \(\mathcal{C}'(C|Z)\) be the closure under CP maps of \(\mathcal{C}(C|Z)\). For \(0 < b \leq 1\), define

\[K(CZ) = -\log(\bar{b}) + 1 - B(CZ)/\bar{b}. \tag{7.7}
\]

Then \(K(CZ)\) is an entropy estimator for \(C|Z\) and \(\mu(Z) \propto \mathcal{C}'(C|Z)\).

**Proof.** CP maps preserve the form of members of models for \((C|Z)|E\), so the CP closure \(\mathcal{C}'(C|Z)\) is a model for \((C|Z)|E\) as required for the lemma statement. Since \(\mu(Z) \propto \mathcal{C}'(C|Z) \subseteq \mathcal{M}(\text{Cvx}(\mu(Z) \propto \mathcal{C}(C|Z))); E\) and by Lem. 7.2, \(B(CZ)\) is a max-prob estimator for \(C|Z\) and \(\mu(Z) \propto \mathcal{C}'(C|Z)\).

Let \(\tau(CZ) \in \mathcal{C}'(C|Z)\) with \(\text{tr}(\tau(|z)) = 1\). Define \(\nu(CZ) = \text{tr}(\mu(Z)\tau(CZ))\) and \(\tau(CZ) = \mu(Z)\tau(CZ)\). We have \(\tau = \tau(|z)\), independent of \(z\). Below, we show that \(\mathbb{E}_\nu(B(CZ)) \geq P_{\text{max}}(\tau(CZ)|ZE)\). Given this inequality and since \(-\log P_{\text{max}}(\tau(CZ)|ZE) \leq H_1(\tau(CZ)|ZE)\) (Lem. 2.25), for the theorem it suffices to show that \(\mathbb{E}_\nu(K(CZ)) \leq -\log(\mathbb{E}_\nu(B(CZ)))\), which follows from the same calculation as that given in the proof of Thm. 7.3.
To prove that $\mathbb{E}_{\nu(CZ)}(B(CZ)) \geq P_{\max}(\mu(Z)\tau(CZ)|ZE)$, we again apply the relationship between maximum guessing probability and $P_{\max}$, Lem. 2.23. For each $z$, let $(\Pi_{c|z})_c$ be a POVM with guessing probability $p_z = \sum_c \text{tr}(\Pi_{c|z}\tau(c|z))$ and overall guessing probability $p = \sum_z \mu(z)p_z$. Let $\nu_{c'|z'}(C|Z) = \text{tr}(\Pi_{c'|z'}\tau(C|Z))/\text{tr}(\Pi_{c'|z'}\tau)$. By pCP-closure, $\nu_{c'|z'} \in \text{tr}(\mathcal{C}'(C|Z))$. For each $c'|z'$,

$$
\mathbb{E}_{\nu(Z)}(B(CZ)) \geq \max_c(\nu_{c'|z'}(c|z)) \geq \nu_{c'|z'}(c'|z').
$$

Consequently

$$
p = \sum_{z'} \mu(z')p_{z'} = \sum_{c',z'} \mu(z')\nu_{c'|z'}(c'|z') \text{tr}(\Pi_{c'|z'}\tau)
\leq \sum_{c',z'} \mu(z')\mathbb{E}_{\nu(Z)}(B(CZ)) \text{tr}(\Pi_{c'|z'}\tau)
= \sum_{c',z'} \mu(z') \sum_{cz} B(cz) \mu(z) \nu_{c'|z'}(c|z) \text{tr}(\Pi_{c'|z'}\tau(c|z))
= \sum_{c',z'} \mu(z') \sum_{cz} B(cz) \mu(z) \text{tr}(\Pi_{c'|z'}\tau(c|z))
= \sum_{cz} B(cz) \mu(z) \text{tr}(\sum_{z'} \mu(z')\tau(c|z))
= \sum_{cz} B(cz) \mu(z) \text{tr}(\tau(c|z))
= \mathbb{E}_{\nu(Z)}(B(CZ)).
$$

The claim follows because the POVMs can be chosen so that $p$ is arbitrarily close to $P_{\max}(\tau(CZ)|ZE)$.

### 7.3. Exponential Expansion by Spot-Checking

Let $\mathcal{C}(C|Z)$ be a model for $(C|Z)E$ and $B(CZ)$ a max-prob estimator for $C|ZE$ and $\operatorname{Unif}(Z) \propto \mathcal{C}(C|Z)$. For this section, we fix $\rho(C|Z) \in \mathcal{C}(C|Z)$ with $\text{tr}(\rho(z)) = 1$ and $\nu(CZ) = \operatorname{Unif}(Z)\text{tr}(\rho(C|Z))$. Define $\tilde{b} = \mathbb{E}_{\nu(CZ)}(B(CZ))$. We assume that $\tilde{b} < 1$ as otherwise $\operatorname{Unif}_Z(Z)p(C|Z)$ has only trivial max-prob witnessed by $B(CZ)$. By definition, $\tilde{b} > 0$.

The following repeats the treatment of spot-checking input distributions in Ref. [1]. To simplify the analysis, we take advantage of the fact that for configurations such as those of Bell tests, we can hide the choice of whether or not to apply a test trial from the devices. This corresponds to appending a test bit $T$ to $Z$, where $T = 1$ indicates a test trial and $T = 0$ indicates a fixed one, with $Z = z_0$. The model $\mathcal{C}(C|ZT)$ is obtained from $\mathcal{C}(C|Z)$ by constraining $\sigma(C|Z0) = \sigma(C|Z1)$ and $\sigma(C|Z0) \in \mathcal{C}(C|Z)$. For any $\sigma(C|Z) \in \mathcal{C}(C|Z)$ there is a corresponding $\tilde{\sigma}(C|ZT) \in \mathcal{C}(C|ZT)$ defined by $\tilde{\sigma}(C|Zt) = \sigma(C|Z)$ for $t \in \{0, 1\}$. The map $\sigma(C|Z) \mapsto \tilde{\sigma}(C|ZT)$ is a bijection between $\mathcal{C}(C|Z)$ and $\mathcal{C}(C|ZT)$.
Let \( q = 1/|\text{Rng}(Z)| \). Let \( \mu_r \) be the probability distribution of \( ZT \) defined by \( \mu_r(z1) = rq \) and \( \mu_r(z0) = (1 - r) [z = z_0] \) for some value \( z_0 \) of \( Z \). Since we are interested in the case where \( r \) is small, we assume \( 0 < r < 1/2 \). The entropy of the distribution \( \mu_r \) is given by \( S(\mu_r) = H(r) + r \log(1/q), \) where \( H(r) = -r \log(r) - (1 - r) \log(1 - r) \). Let \( \nu_r(CZT) = \mu_r(ZT)\text{tr}(\rho(C|Z)) \). Define \( B_r(CZT) \) by

\[
B_r(CZ0) = 1, \\
B_r(CZ1) = 1 + (B(CZ) - 1)\frac{1}{r}.
\]

Let

\[
K_r(CZT) = -\log(\bar{b}) + 1 - B_r(CZT)/\bar{b}.
\]

Setting \( B_r(CZT) \) to 1 when \( T = 0 \) is convenient, we did not explore optimality of this choice.

**Lemma 7.5.** \( B_r \) is a max-prob estimator for \( C|ZT \) and \( \mu_r(ZT) \propto C(C|ZT) \).

**Proof.** Let \( \sigma(C|Z) \in \mathcal{C}(C|Z) \) with \( \text{tr}(\sigma(|z)) = 1 \). For the duration of this proof, define \( \nu(CZ) = \text{tr}(\text{Unif}(Z)\sigma(C|Z)) = q\text{tr}(\sigma(C|Z)) \) and \( \nu_r(CZT) = \text{tr}(\mu_r(ZT)\sigma(C|Z)) \). All normalized members of \( \mu_r(ZT) \propto \mathcal{C}(C|ZT) \) are of the form \( \mu_r(ZT)\sigma(C|Z) \), so it suffices to confirm that \( \mathbb{E}_{\nu_r(CZT)}(B_r(CZT)) \geq \max_{czt} \nu_r(c|zt) \). We have

\[
\mathbb{E}_{\nu_r(CZT)}(B_r(CZT)) = \sum_{czt} \nu_r(czt) B_r(czt)
\]

\[
= \sum_{czt} \nu_r(czt0) B_r(czt0) + \sum_{czt} \nu_r(czt1) B_r(czt1)
\]

\[
= \sum_{czt} \mu_r(z0) \text{tr}(\sigma(c|z)) + \sum_{czt} \mu_r(z1) \text{tr}(\sigma(c|z))(1 + (B(cz) - 1)/r)
\]

\[
= \sum_{czt} \mu_r(z0) + \sum_{czt} rq \text{tr}(\sigma(c|z))(1 + (B(cz) - 1)/r)
\]

\[
= (1 - r) + \sum_{czt} r\text{Unif}(z) \text{tr}(\sigma(c|z))(1 + (B(cz) - 1)/r)
\]

\[
= (1 - r) + r \sum_{czt} \nu(czt)(1 + (B(cz) - 1)/r)
\]

\[
= \mathbb{E}_{\nu(CZ)}(B(CZ))
\]

\[
\geq \max_{czt} \nu_r(c|zt).
\]

since \( \nu_r(c|z1) = \nu(c|z) \) and \( \nu_r(c|z0) = [z = z_0] \nu(c|z) \), according to our convention that zero-probability conditionals are 0.

**Lemma 7.6.** For \( \nu_r(CZT) = \text{tr}(\mu_r(ZT)\rho(C|Z)) \), \( \mathbb{E}_{\nu_r(CZT)}(K_r(CZT)) = -\log(\bar{b}) \).

**Proof.**

\[
\mathbb{E}_{\nu_r(CZT)}(K_r(CZT)) = -\log(\bar{b}) + 1 - \mathbb{E}_{\nu_r(CZT)}(B_r(CZT))/\bar{b}
\]

\[
= -\log(\bar{b}),
\]

(7.13)
where the last equality follows from the more general Eq. \[7.12\] and the definition of \( \bar{b} \).

**Theorem 7.7.** With the notation of this section, there exist constants \( d \) and \( d' \) independent of \( r \) such that for \( 0 < \beta \leq dr \),

\[
F_{r,\beta}(CZT) = \frac{e^{\beta K_r(CZT)}}{1 + d' \beta^2/r} \tag{7.14}
\]

is a QEFP with power \( \beta \) for \( C|ZT \) and \( \mu_r(ZT) \propto C|ZT \). The log-prob rate \( g_{r,\beta} \) of \( F_{r,\beta} \) at \( \mu_r(ZT) \rho(C|Z) \) satisfies

\[
g_{r,\beta} \geq -\log(\bar{b}) - d' \beta/r. \tag{7.15}
\]

This theorem extends Thm. 51 from Ref. 1 to QEFPs constructed from max-prob estimators. We do not intend the constants obtained in the proof to be used in practice. If necessary in an application, the QEFPs and the values of \( \beta \) obtained according to the strategy here can be optimized with numerical methods with the expressions obtained in Thm. 6.3 and its proof.

**Proof.** The bound on the log-prob rate follows from Eq. \[7.14\] by direct computation: With \( \nu_r(CZT) = \text{tr}(\mu_r(ZT) \rho(C|Z)) \),

\[
g_{r,\beta} = \mathbb{E}_{\nu_r}(\log(F_{r,\beta}(CZT))/\beta) = \mathbb{E}_{\nu_r}(K_r(CZT)) - \log(1 + d' \beta^2/r)/\beta
\]

\[
= -\log(\bar{b}) - \log(1 + d' \beta^2/r)/\beta
\]

\[
\geq -\log(\bar{b}) - d' \beta/r, \tag{7.16}
\]

where we applied Lem. 7.6 in the second-last step.

For the main statement of the theorem, we apply Thm. 6.3, where \( Z \) there becomes \( ZT \) here. Consider the bound \( c(\beta) \) on \( c_{\rho}(\beta, K_r(CZT)) \) from this theorem. Let \( k_{\infty}(zt) = \max_e [K_r(zt)] \). By definition of \( K_r(zt) \), \( k_{\infty}(z0) \leq -\log(\bar{b}) + 1/\bar{b} \) and since \( B_r(\bar{b}z1) \leq (|B(\bar{b}z1)| + 1)/r \), we have \( k_{\infty}(z1) \leq -\log(\bar{b}) + 1 + (|B(\bar{b}z1)| + 1)/(\bar{b}r) \leq (\log(\bar{b}) + 1 + (|B(\bar{b}z1)| + 1)/(\bar{b}r)). \) Define \( d = (2 \max_e (-\log(\bar{b}) + 1 + (|B(\bar{b}z1)| + 1)/(\bar{b}r))^{-1} \) so that \( k_{\infty}(z1) \leq 1/(2dr) \) and \( k_{\infty}(z0) \leq 1/(2d) \). We increase the bound \( c(\beta) \) by replacing \( \bar{w}_\gamma(zt) \) and \( e^{k_{\max}(zt)\beta} \) in the expression for \( c(\beta) \) by \( c(zt) = k_{\infty}(zt) + \log(2N) + \mu_0 \geq \bar{w}_\gamma(zt) \) and \( e^{k_{\infty}(zt)\beta} \), respectively. With this replacement, the \( ||...|| \) operation can be omitted and the terms combined for

\[
c_{\rho}(\beta, K_r(CZT)) \leq \frac{\beta^2}{6} \sum_{zt} \mu_r(zt) \left( \left( 2 + \frac{e^{k_{\infty}(zt)\beta}}{(1 - \beta)^2} \right) v(zt)(v(zt) + \mu_0) \right), \tag{7.17}
\]

noting that \( \tau(Z) \) in Thm. 6.3 becomes \( \mu_r(ZT) \) here and \( 2 \coth(v(zt)) \leq \mu_0 \) by definition of \( \mu_0 \). For \( \beta \leq dr \), we have \( e^{k_{\infty}(zt)\beta} \leq e^{k_{\infty}(zt)dr} \leq e^{1/2} \leq 2 \), and \( (1 - \beta) \geq 1/2 \), so we can weaken the bound to

\[
c_{\rho}(\beta, K_r(CZT)) \leq \frac{\beta^2}{6} \sum_{zt} \mu_r(zt) 10 v(zt)(v(zt) + \mu_0). \tag{7.18}
\]

We have that \( v(z0) + \mu_0 \leq 1/(2d) + \log(2N) + 2\mu_0 \) and \( v(z1) + \mu_0 \leq (1/(2dr)) + \log(2N) + 2\mu_0 \leq (1/(2d) + \log(2N) + 2\mu_0)/r \). After separating the sum over \( zt \) for \( t = 0 \) and \( t = 1 \), the bound...
It now suffices to set \( d' = 10(1/(2d) + \log(2N) + 2\epsilon_0)^2/3 \).

Thm. 7.7 implies exponential expansion via the argument used to prove exponential expansion in Ref. [1], Thm 53. We formulate the theorem with power-law error-bound rates to match the conclusion of Cor. 1.3 in Ref. [2]. Standard exponential expansion is obtained by setting the parameter \( \gamma \) to \( \gamma = 1 \).

**Theorem 7.8.** Let \( F_{r,\beta}(CZT) \) be the family of QEFPs of Thm. 7.7 for model \( \mu_r(ZT) \times \mathcal{C}(C|ZT) \). Suppose that \( \mathcal{C}(CZT) \) is obtained by chaining \( \mu_r(ZT) \times \mathcal{C}(C|ZT) \) \( n \) times, and \( \rho(CZT) \in \mathcal{C}(CZT) \) satisfies that \( \text{tr}(\rho(CZT)) \) is i.i.d. with trial distribution \( \text{tr}(\rho(CZT)) \), where \( \bar{b} < 1 \). Then given \( \gamma \in (0, 1] \), \( l_e > 0 \), and error bound \( \epsilon(n) \) defined by \( \log(2/\epsilon(n))^2 = l_e n^{1-\gamma} \) for \( \gamma \in [0, 1] \), the expected quantum net log-prob \( g_{\text{net}}(n) \) at \( \rho(CZT) \) satisfies

\[
g_{\text{net}}(n) = e^{\Omega(n^{\gamma} S_{\text{net}}(n))},
\]

where \( S_{\text{net}}(n) \) is the net input entropy for \( n \) trials.

Proof. We repeat the proof Thm. 53 in [1] with minor modifications to obtain the more general statement of Thm. 7.8. We determine constants \( 0 < c < 1 \) and \( 0 < c' \) for which the testing rate \( r_n = c'/n^\gamma \) and the power \( \beta_n = cr_n = cc'/n^\gamma \) achieve the goal of the theorem. It suffices to prove the theorem assuming that \( n \) is sufficiently large. Let \( d \) and \( d' \) be the constants in Thm. 7.7. We require that \( c \leq d \). From this theorem, Def. 5.6 and since \( \beta_n < 1 \) for sufficiently large \( n \), the expected quantum net log-prob for power \( \beta_n \) is bounded by

\[
g_{\text{net}} = ng_{\text{r},\beta_n} = \frac{\log(2/\epsilon(n)^2)}{\beta_n} \geq n \left(-\log(b) - \frac{d' \beta_n}{r_n}\right) - \frac{l_e n^{1-\gamma}}{\beta_n}. \tag{7.21}
\]

The input entropy per trial \( S(\nu_r) = H(r) + r \log(1/q) \) is bounded above by \(-2r \log(r)\), provided we take \( r \leq q/e \). For this, note that \( r \log(1/q) \leq r \log(1/(re)) = -r \log(r) - r \) and \(- (1-r) \log(1-r) \leq r \) since the function \( x \mapsto -(1-x) \log(1-x) \) is concave for \( 0 \leq x < 1 \),
is equal to 0 at $x = 1$, and has slope 1 at $x = 0$. For $r_n = c'/n^\gamma$, the expected number of test trials is $c'n^{1-\gamma}$ and the total input entropy satisfies $nS(\nu_n) \leq 2c'n^{1-\gamma}(\gamma \log(n) - \log(c'))$, or equivalently $n \geq c'^{1/\gamma}e^{\gamma S(\nu_n)/(2\gamma c')}$. For sufficiently large $n$, $r \leq q/e$ is satisfied. Write $g_0 = -\log(\bar{\theta})$. Substituting the expressions for $\beta_n$ and $r_n$ in Eq. 7.21

$$g_{net} \geq n g_0 \left( 1 - \frac{d'c}{g_0} - \frac{l_c}{cc'g_0} \right).$$

We first set $c = \min(d, g_0/(3d'))$, which ensures that $d'c/g_0 \leq 1/3$. We then set $c' = 3l_c/(cg_0)$. This gives the inequality

$$g_{net} \geq n g_0 \frac{g_0}{3} \geq c'^{1/\gamma}e^{\gamma S(\nu_n)/(2\gamma c')} \frac{g_0}{3},$$

which implies the theorem.

8. QEFS FOR $(k,2,2)$-BELL-TEST CONFIGURATIONS

8.1. $(k,2,2)$-Bell-Test Configurations

We consider models induced by POVMs that are physically achievable on the device side of $(k,2,2)$-Bell-test configurations. A $(k,2,2)$-Bell-test configuration involves $k$ stations (or parties or devices), where each applies one of two binary-outcome measurements in each trial. The trial CVs $C$ and $Z$ are both $k$-bit strings. For this section, we do not need to consider sequences of trials directly, so $C_i$ and $Z_i$ refer to the $i$'th bits of these strings. For optimizing the log-prob, we assume a fixed input distribution given by $\mu(Z)$. To define the induced model, the total device Hilbert space is $\mathcal{H}(D) = \bigotimes_{i=1}^k \mathcal{V}^{(i)}$. The set of POVMs $\Psi$ consists of the families of positive semidefinite operators $\{\mu(z)P_{c|z}\}_{c\in\mathcal{Z}}$ with $P_{c|z}$ of the form $P_{c|z} = \bigotimes_{i=1}^k P_{c_{i|z_i}}^{(i)}$ where $P_{0|z_i}^{(i)} + P_{1|z_i}^{(i)} = 1_{\mathcal{V}^{(i)}}$. Let $\mathcal{C}_{k22}(CZ)$ be the union of the $\mathcal{M}(\Psi;E)$ over choices for $\mathcal{V}^{(i)}$.

For qubits (Hilbert space of dimension 2), let $\phi \in (-\pi, \pi]$ and

$$Q_{c|0,\phi} = \frac{1}{2}(1 + (-1)^c \sigma_z),$$

$$Q_{c|1,\phi} = \frac{1}{2}(1 + (-1)^c (\cos(\phi) \sigma_z + \sin(\phi) \sigma_x)).$$

For $\theta$ a vector of length $k$, we define $Z$-indexed POVM operators $P_{C|Z;\theta}$ by $P_{c|z;\theta} = \bigotimes_{i=1}^k Q_{c_{i|z_i};\theta_i}$ for each $c$ and $z$, where $\theta_i \in (-\pi, \pi]$.

**Theorem 8.1.** Let $\mu(Z)$ be a fixed input distribution. The model $\mathcal{C}_{k22}(CZ)$ consists of positive combinations of members $\rho(CZ)$ expressible in the form

$$\rho(CZ) = \mu(Z)U^{1/2}P_{C|Z;\theta}^{1/2}U^{\dagger}$$

for an operator $\tau \geq 0$ and an isometry $U$ from $(\mathbb{C}^2)^{\otimes k}$ into $\mathcal{H}(E)$.

The theorem follows from a well-known analysis of this situation for $k = 2$ going back to Ref. [18] and Ref. [19]; a nice version of this analysis is in Ref. [17], Sect. 2.4.1.
Proof. According to the definition of induced models, we consider an initial state $\chi$ of DE and a POVM $\mu(Z)P_{C|Z} \in \Psi$. This gives the generic state $\rho(CZ) = \mu(Z) \text{tr}_D((P_{C|Z} \otimes 1)\chi)$ in $C_{k22}(CZ)$. The usual dilation argument shows that we can extend $V^{(i)}$ and $P_{i,1}^{(i)}$ so that $P_{0,1}^{(i)}$ and $P_{1,1}^{(i)}$ are pairs of orthogonal and complete projectors. For $\mu(Z)P_{C|Z}$ replaced by the extended POVM, and $V^{(i)}$ replaced by its dilation, we still have $\rho(CZ) = \mu(Z) \text{tr}_D((P_{C|Z} \otimes 1)\chi)$. With this, $A^{(i)}_{z_i} = (P_{0,1}^{(i)} - P_{1,1}^{(i)})$ are observables with eigenvalues in $\{-1, 1\}$. Since there are two such observables for each $V^{(i)}$, Lem. 2 of Ref. [17] now applies so that $V^{(i)} = \oplus_j \psi_j^{(i)}$ with $\psi_j^{(i)}$ of dimension one or two and $A^{(i)}_{z_i} = \oplus_j A^{(i)}_{z_i,j}$. On the one-dimensional summands, $A^{(i)}_{b_j} = \pm 1$. We can add a second dimension on which the state has no support and extend $A^{(i)}_{b_j}$ to the added dimension so that $A^{(i)}_{b_j} = \pm \sigma_z$. We also extend the POVM operators so that their relationship to the $\pm 1$ eigenspaces of the $A^{(i)}_{b_j}$ is unchanged. According to the proof of the referenced lemma, we may assume that on the two-dimensional summands, $A^{(i)}_{z_i,j}$ act as conjugated Pauli matrices. Thus, after extending the one-dimensional summands as described, for all $j$ we may choose logical bases such that $A^{(i)}_{0,j} = \sigma_z$ and $A^{(i)}_{1,j} = \cos(\theta_j)\sigma_z + \sin(\theta_j)\sigma_x$, for some $\theta_j$.

The reasoning so far shows that $P_{C|Z} = \oplus_i P_{C|Z_1}$ with $P_{C|Z_1}$ acting on tensor products of two-dimensional subspaces of the subsystems $V^{(i)}$. The direct sum is over $l$ defined as sequences of $k$ indices, with each index labeling a direct summand of the corresponding subsystem. The transition elements of $\chi$ between the direct summands tensored with $\mathcal{H}(E)$ do not contribute to $\rho(CZ)$, so by zeroing these transition elements with the appropriate decoherence superoperator we may assume $\chi = \oplus_i \chi_l$. Now $\rho(CZ)$ is a positive combination of the $\rho_l(CZ) = \mu(Z) \text{tr}_D((P_{C|Z_1} \otimes 1)\chi_l)$, which is in $C_{k22}(CZ)$. With this we have reduced the problem to one where the $V^{(i)}$ are two-dimensional and $P_{C|Z} = P_{C|Z,\theta}$ for some $\theta$ as defined after Eq. 8.11.

Next, we may assume that $\chi$ is pure. If not we purify $\chi$ with the addition of another system $E'$. Then $\chi = \sum_m \chi_m$ where the $\chi_m$ are the unnormalized pure states obtained from the purification of $\chi$ by projecting onto the $m$’th basis state of $\mathcal{H}(E')$ for some choice of orthonormal basis and tracing out the system $E'$, and $\rho(CZ)$ is the sum of the $\rho_m(CZ) = \mu(Z) \text{tr}_D((P_{C|Z,\theta} \otimes 1)\chi_m)$ with $\chi_m$ pure, which again are in $C_{k22}(CZ)$.

Consider $U \otimes W$ with $\dim(W) \geq \dim(U) = d$ and a given orthonormal basis $\{ |y\rangle_U \}_{y \in I}$ of $U$. Every pure state $|\psi\rangle$ of $U \otimes W$ can be written in the form $(1 \otimes \tau^{1/2}) \sum_y |y\rangle_U \otimes |y\rangle_W / \sqrt{d}$, where the $|y\rangle_W$ are orthonormal and we can choose $\tau^{1/2}$ to be positive semidefinite and preserve the subspace spanned by the $|y\rangle_W$: One way to determine the $|y\rangle_W$ and $\tau$ is to let $\{ |y'\rangle_W \}_{y' \in I'}$ be a Schmidt basis for the pure state $|\psi\rangle$ and $\lambda_{y'} \geq 0$ the corresponding Schmidt amplitudes, where the label set $I'$ is disjoint from $I$ but $|I'| = |I'| = d$. With $|y'\rangle_W$, the corresponding partial Schmidt basis of $W$, define $\tau^{1/2}$ by $\tau^{1/2} |y'\rangle_W = \sqrt{\lambda_{y'}} |y'\rangle_W$ and $\tau^{1/2} |\varphi\rangle = 0$ for $|\varphi\rangle$ orthogonal to the $|y'\rangle_W$. With this, $|\psi\rangle = (1 \otimes \tau^{1/2}) \sum_{y' \in I'} |y'\rangle_U \otimes |y'\rangle_W / \sqrt{d}$. By the properties of maximally entangled states, there exists a partial orthonormal basis $\{ |y\rangle_W \}_{y \in I}$ of $W$ such that $\sum_y |y\rangle_U \otimes |y\rangle_W / \sqrt{d} = \sum_{y'} |y'\rangle_U \otimes |y'\rangle_W / \sqrt{d}$.

For $x$ a $k$-bit string, let $|x\rangle_D$ be the corresponding logical basis element of $\otimes_{i=1}^k V_i$ considered as $k$ qubits. Applying the observation of the previous paragraph and the reduction to $k$-qubits and pure states from before, define $|\psi\rangle_{DE} = \sum_x |x\rangle_D \otimes |x\rangle_E / 2^{k/2}$ so that
\( \chi = (1 \otimes \tau^{1/2}) |\psi\rangle \langle \psi| (1 \otimes \tau^{1/2}) \) for some positive semidefinite \( \tau^{1/2} \). Now

\[
\rho(cz) = \mu(z) \text{tr}_D \left( (P_{c|z;\theta} \otimes 1)(1 \otimes \tau^{1/2}) |\psi\rangle \langle \psi| (1 \otimes \tau^{1/2}) \right) \\
= \mu(z) \tau^{1/2} \text{tr}_D \left( (P_{c|z;\theta} \otimes 1) |\psi\rangle \langle \psi| \right) \tau^{1/2} \\
= \mu(z) U^T \tau^{1/2} P_{c|z;\theta}^T \tilde{\tau}^{1/2} U^T, \tag{8.3}
\]

where \( U \) is the isometry that maps \( |x\rangle_D \) to \( |x\rangle_E \), \( \tilde{\tau}^{1/2} = U^T \tau^{1/2} U \), and the transpose is taken with respect to the basis \( |x\rangle_D \). To complete the proof, since \( P_{c|z;\theta} \) is real and symmetric in this basis, \( P_{c|z;\theta}^T = P_{c|z;\theta} \).

With Thm. 8.1 and Lem. 4.10, the QEF optimization problem Prob. 5.18 for \( C_{k22}(CZ) \) and anticipated probability distribution \( \nu(CZ) \in \text{tr}(\mathcal{N}(C_{k22}(CZ))) \) simplifies to a finite-dimensional problem. Let \( \rho(CZ) \in C_{k22}(CZ) \) have the form given in Thm. 8.1 with \( \text{tr}(\rho) = \text{tr}(\tau) = 1 \). In view of the simplification of Eq. 4.13 for the fixed input distribution \( \mu(Z) \), the QEF inequality with power \( \beta \) for \( F(CZ) \) and \( C_{k22}(CZ) \) at \( \rho(CZ) \) is

\[
1 \geq \sum_{cz} F(cz) \mu(z) \mathcal{R}_\alpha \left( \rho(c|z)|\rho \right) \\
= \sum_{cz} F(cz) \mu(z) \text{tr} \left( \left( \tau^{-\beta/(2\alpha)} \tau^{1/2} P_{c|z;\theta}^T \tau^{1/2} \tau^{-\beta/(2\alpha)} \right)^\alpha \right) \\
= \sum_{cz} F(cz) \mu(z) \text{tr} \left( \left( \tau^{1/(2\alpha)} P_{c|z;\theta} \tau^{1/(2\alpha)} \right)^\alpha \right) \\
= \sum_{cz} F(cz) \mu(z) \left( \text{tr} \left( \tau^{1/(2\alpha)} P_{c|z;\theta} \tau^{1/(2\alpha)} \right) \right)^\alpha \\
= \sum_{cz} F(cz) \mu(z) \left( \text{tr} \left( P_{c|z;\theta} \tau^{1/\alpha} P_{c|z;\theta} \right) \right)^\alpha, \tag{8.4}
\]

where we used the fact that for a rank 1 projector \( \Pi \), \( \text{tr}(\chi^{1/2} \Pi \chi^{1/2})^\alpha = (\text{tr}(\chi^{1/2} \Pi \chi^{1/2}))^\alpha \) and cyclicity of the trace. In the last expression, one of the projectors in the argument of the trace can be omitted. The QEF optimization problem of Eq. 5.18 now reduces to the following:

Maximize: \( \sum_{cz} \nu(cz) \log(F(cz)) - \log(f_{\text{max}}) \)

Variables: \( F(CZ), f_{\text{max}} \)

Subject to: \( F(CZ) \geq 0 \), \( \sum_{cz} F(cz) = 1 \),

\[
f_{\text{max}} \geq \sum_{cz} \mu(z) F(cz) \left( \text{tr} \left( P_{c|z;\theta} \tau^{1/\alpha} P_{c|z;\theta} \right) \right)^\alpha \quad \text{for all } \theta \text{ and } \tau \geq 0 \text{ with } \text{tr}(\tau) = 1. \tag{8.5}
\]

As in Eq. 5.18, the variable \( F(CZ) \) in this optimization problem is not a QEF, but every feasible solution \( (F(CZ), f_{\text{max}}) \) determines the QEF \( F(CZ)/f_{\text{max}} \) with power \( \beta \).

Define \( Q_\alpha(F(CZ), \theta, \tau) = \sum_{cz} \mu(z) F(cz) \left( \text{tr} \left( P_{c|z;\theta} \tau^{1/\alpha} P_{c|z;\theta} \right) \right)^\alpha \).
Lemma 8.2. In Prob. 8.5, $Q_\alpha(F(CZ), \theta, \tau)$ is concave in the density operator $\tau$, the operator $\tau$ may be restricted to be real, and it suffices to consider $\theta$ with $\theta_i \in [0, \pi]$.

Proof. For the first claim, we apply the general fact that $A \mapsto \text{tr}\left( (K^\dagger A^{1/\alpha} K)^\alpha \right)$ is a concave function in $A \geq 0$ given $\alpha \geq 1$, see Ref. [21], Thm. 7.2. The concavity of $(\text{tr}(P_{cz;\theta}^{\tau} 1/\alpha P_{cz;\theta}))^\alpha$ is obtained with $K = P_{cz;\theta}$ and $A = \tau$, and since $K$ is now rank 1, $\text{tr}\left( (K^\dagger A^{1/\alpha} K)^\alpha \right) = (\text{tr}(K^\dagger A^{1/\alpha} K))^\alpha$. It follows that $Q_\alpha(F(CZ), \theta, \tau)$ is a positive linear combination of concave functions and is therefore itself concave. Concavity implies that the set of $\tau$ over which $Q_\alpha(F(CZ), \theta, \tau)$ needs to be maximized can be restricted to real matrices. This follows from $Q_\alpha(F(CZ), \theta, \tau) = Q_\alpha(F(CZ), \theta, \bar{\tau})$, which is a consequence of $P_{cz;\theta}$ being real, so by concavity $Q_\alpha(F(CZ), \theta, (\tau + \bar{\tau})/2) \geq Q_\alpha(F(CZ), \theta, \tau)$. (Here we used mathematics conventions to denote conjugates of complex quantities by an overline). For the last claim, for each $i$, let $\sigma_z^{(i)}$ be $\sigma_z$ acting on the $i$'th subsystem. By periodicity, we may assume $\theta_i \in [-\pi, \pi]$. Fix $i$ and define $\theta'$ by $\theta'_i = -\theta_i$ and $\theta'_l = \theta_l$ for $l \neq i$. Then

$$\text{tr}\left( P_{cz;\theta}^{\tau} 1/\alpha P_{cz;\theta} \right) = \text{tr}\left( \sigma_z^{(i)} P_{cz;\theta}^{\tau} 1/\alpha P_{cz;\theta} \sigma_z^{(i)} \right) = \text{tr}\left( P_{cz;\theta}^{\tau} \sigma_z^{(i)} 1/\alpha \sigma_z^{(i)} \right) = \text{tr}\left( P_{cz;\theta'}(\sigma_z^{(i)} \tau \sigma_z^{(i)}) 1/\alpha P_{cz;\theta'} \right).$$

(8.6)

Since $\tau \mapsto \sigma_z^{(i)} \tau \sigma_z^{(i)}$ is a bijection of density matrices, the maximum over $\tau$ of the above expression does not change when $\theta$ is changed to $\theta'$. Therefore, if any $\theta_i \in [-\pi, 0)$, we can replace it with $\theta'_i = -\theta_i$.

\[ \square \]

8.2. Schemas for QEF Optimization

With the help of Lem. 8.2, Prob. 8.5 can be attacked by numerical methods. An algorithm for solving Prob. 8.5 needs to certify that $f_{\text{max}}$ exceeds $Q(F(CZ), \theta, \tau)$ for all $\theta$ and density operators $\tau \geq 0$. By concavity, given $\theta$, the maximum in $\tau$ is unique, but the dependence of this maximum on $\theta$ is less well behaved. We give a strategy for ensuring that $f_{\text{max}}$ satisfies its constraint for all $\tau$ and $\theta$ with arbitrarily small slack.

Let $H_1$ denote the displaced half unit circle in $\mathbb{R}^3$ consisting of the points of the form $(\cos(\theta), \sin(\theta), 1)$ with $\theta \in [0, \pi]$, and let $R_1$ be the set of semidefinite operators operators $\chi$ on $k$ qubits that are real with respect to the logical basis and satisfy $\text{tr}(\chi^\alpha) = 1$. For the purpose of distinguishing factors in tensor products, for each $i$ let $H_1^{(i)}$ be an identified copy of $H_1$. Define $\mathcal{R}_1 = R_1 \otimes (\bigotimes_{i=1}^k H_1^{(i)})$. Let $\mathcal{R}$ be the tensor product of the vector spaces containing $R_1$ and the $H_1^{(i)}$. Write $r_i = (u_i, v_i, w_i)$ for a point in linear span of $H_1^{(i)}$. For each $cz$ the map

$$L_{cz} : \chi, r_1, \ldots, r_k \mapsto 2^{-k} \text{tr}\left( \chi \times \bigotimes_i \left( (1 + (-1)^\alpha \sigma_z) \left[ z_i = 0 \right] + (w_i + (-1)^\alpha (u_i \sigma_z + v_i \sigma_x) [z_i = 1] ) \right) \right)$$

(8.7)

is multilinear with respect to $\chi$ and each of the $r_i$. It therefore lifts to a linear map $\tilde{L}_{cz}$ on
\[ \mathcal{R} \] so that \( \tilde{L}_{cz}(\chi \otimes r_1 \otimes \ldots \otimes r_k) = L_{cz}(\chi, r_1, \ldots, r_k) \). This map satisfies
\[
\tilde{L}_{cz}(\chi \otimes (\cos(\theta_1), \sin(\theta_1), 1) \otimes \ldots \otimes (\cos(\theta_k), \sin(\theta_k), 1)) = \text{tr}(\chi P_{\epsilon z; \theta}). \tag{8.8}
\]
Since \( x \mapsto |x|^\alpha \) is convex and the compositions of linear and convex maps are convex, the map \( |\tilde{L}_{cz}|^\alpha \) is convex. Since positive linear combinations of convex maps are convex, the map
\[
\tilde{Q}_\alpha : F(CZ), u \in \mathcal{R} \mapsto \sum_{cz} \mu(z)F(cz)|\tilde{L}_{cz}(u)|^\alpha \tag{8.9}
\]
is convex.

In Prob. 8.5 with \( F(CZ) \) fixed, we can set \( f_{\text{max}} \) to \( f_{\text{max}}(F(CZ)) = \max_{u \in \mathcal{R}} \tilde{Q}(F(CZ), u) \). Given an algorithm to determine \( f_{\text{max}}(F(CZ)) \), any generic local search algorithm can be used to optimize \( F(CZ) \), so we focus on algorithms for \( f_{\text{max}} \). A certified upper bound on \( f_{\text{max}} \) suffices, and such a bound can be obtained by maximizing \( \tilde{Q}_\alpha \) over any convex set \( \mathcal{R}' \supseteq \text{Cvx}(\mathcal{R}_1) \). For example, if \( \mathcal{P}^{(i)} \) are convex polygons satisfying \( \text{Cvx}(H_{1}^{(i)}) \subseteq \mathcal{P}^{(i)} \), then we can let \( \mathcal{R}' = \text{Cvx}(\mathcal{R}_1) \otimes \bigotimes_{i=1}^{k} \mathcal{P}^{(i)} \). Because \( \tilde{Q}_\alpha(F(CZ), u) \) is convex in \( u \), the maximum is achieved on an extreme point of \( \mathcal{R}' \) and the upper bound becomes tight in the limit where the \( \mathcal{P}^{(i)} \) converge to \( \text{Cvx}(H_{1}^{(i)}) \). The extreme points of \( \mathcal{R}' \) are tensor products of some \( \chi \in \text{Cvx}(\mathcal{R}_1) \) with members of the finite sets \( \text{Extr}(\mathcal{P}^{(i)}) \). Provided we can effectively maximize over \( \chi \in \text{Cvx}(\mathcal{R}_1) \), there are finitely many tensor products of extreme points of \( \text{Extr}(\mathcal{P}) \) to check. Let \( r = \bigotimes_{i=1}^{k} r_i \) be in \( \bigotimes_{i=1}^{k} \text{Extr}(\mathcal{P}^{(i)}) \), where \( r_i = (u_i, v_i, 1) = ((1 + \epsilon_i)\cos(\theta_i), (1 + \epsilon_i)\sin(\theta_i), 1) \). Then
\[
\tilde{L}_{cz}(\chi \otimes r) = \text{tr} \left( \chi \otimes \bigotimes_{i=1}^{k} P_i \right), \tag{8.10}
\]
where for \( z_i = 0 \), \( P_i = (1 + (-1)^{\epsilon_i} \sigma_z)/2 \) and for \( z_i = 1 \), \( P_i = (1 + (1 + \epsilon_i)(-1)^{\alpha_i} \sigma_z)/2 \) with \( \sigma_{z_i} = \cos(\theta_i)\sigma_z + \sin(\theta_i)\sigma_x \). An issue is that \( P_i \) is not positive semidefinite, so the concavity property with respect to \( \tau \) with \( \tau^{1/\alpha} = \chi \) does not apply and maximizing over \( \chi \in \text{Cvx}(\mathcal{R}_1) \) is more difficult. To avoid this difficulty we give an algorithm that uses inner approximations of \( \text{Cvx}(H_{1}^{(i)}) \) instead.

For the simplest algorithm, let \( \mathcal{X} = (j\pi/m)_{j=0}^{m} \) evenly divide \([0, \pi]\) with \( m \geq 2 \). Write \( r(\theta) = \bigotimes_{i=1}^{k} (\cos(\theta_i), \sin(\theta_i), 1) \). Let \( \mathcal{X}^l \) denote the \( l \)-fold cartesian product of \( \mathcal{X} \) with itself. For each \( r (\mathcal{X}^k) \), compute \( f_{\text{max}}(r) = \max \{ \tilde{Q}_\alpha(F(CZ), \tau^{1/\alpha} \otimes r) : \tau^{1/\alpha} \in \mathcal{R}_1 \} \}, \) where the maximization is concave over real \( 2^k \times 2^k \) density matrices \( \tau \). How to perform this maximization will be explained later. Given that \( f_{\text{max}}(r) \) has been determined for all \( r \in \mathcal{R}^k \), a lower bound on \( f_{\text{max}} \) is given by \( f_{\text{max}} \geq \max \{ f_{\text{max}}(r) : r \in \mathcal{R}^k \} \). An upper bound can be obtained by recursively applying the next lemma.

**Lemma 8.3.** Consider \( \theta, \theta' \) so that \( \theta' - \theta = \phi e_i \) where \( \phi \in (0, \pi/2] \) and \( e_i = ([j = i])_{j=1}^{k} \). Let \( f = f_{\text{max}}(r(\theta)) \) and \( f' = f_{\text{max}}(r(\theta')) \). For \( \varphi \in [0, \phi] \) and \( \theta'' = \theta + \varphi e_i \),
\[
f_{\text{max}}(r(\theta'')) \leq u(\varphi) = \frac{(\sin(\phi - \varphi) + \sin(\varphi))^{\beta}(\sin(\phi - \varphi)f + \sin(\varphi)f')}{\sin(\phi)^\alpha}. \tag{8.11}
\]
The bound \( u(\varphi) \) is log-concave in \( \varphi \) and satisfies
\[
u(\varphi) \leq \left( \frac{\phi}{\sin(\phi)} \right)^{\alpha} \max(f, f'). \tag{8.12}\]

If only upper bounds \( u \) and \( u' \) respectively on \( f \) and \( f' \) are known, then upper bounds
on \( f_{\text{max}}(r(\theta'')) \) can be obtained from Eqs. \( 8.11 \) and \( 8.12 \) with the replacement of \( f \) and \( f' \) by their upper bounds \( u \) and \( u' \).

**Proof.** Write \( f'' = f_{\text{max}}(r(\theta'')) \). Let \( \chi \) witness \( f'' \) in the sense that \( f'' = \tilde{Q}_\alpha(F(CZ), \chi \otimes r(\theta'')) \). For each \( cz \), consider the contribution \( f''(cz) = \mu(z)F(cz)\tilde{L}_{cz}(\chi \otimes r(\theta''))^{\alpha} \) to \( f'' \). If \( z_i = 0 \), then
\[
f''(cz) = \mu(z)F(cz)\tilde{L}_{cz}(\chi \otimes r(\theta))^{\alpha} = \mu(z)F(cz)\tilde{L}_{cz}(\chi \otimes r(\theta'))^{\alpha}, \tag{8.13}\]

since for \( z_i = 0 \), the \( i \)'th factor \( P_{c_i|z_i}^{(i)} \) of \( P_{c|z}^{(i)} \) does not depend on \( \psi_i \). For \( z_i = 1 \), the \( i \)'th factor of \( P_{c|z}^{(i)} \) is \((1 + \cos(\theta_i + \varphi)\sigma_z + \sin(\theta_i + \varphi)\sigma_x)/2\). Let \( a = (\cos(\theta_i), \sin(\theta_i)) \), \( a' = (\cos(\theta_i + \phi), \sin(\theta_i + \phi)) \) and \( a'' = (\cos(\theta_i + \phi), \sin(\theta_i + \varphi)) \). Then there exist \( \lambda \in [0, 1] \) and \( b \in (0, 1] \) such that \( \lambda a + (1 - \lambda)a' = ba'' \). The values of \( \lambda \) and \( b \) will be determined later. Given such \( \lambda \) and \( b \), we have
\[
P_{c_i|z_i; \theta_i + \varphi}^{(i)} \leq P_{c_i|z_i; \theta_i + \varphi}^{(i)} + (1/b - 1)1 = (\lambda P_{c_i|z_i; \theta_i}^{(i)} + (1 - \lambda)P_{c_i|z_i; \theta_i + \varphi}^{(i)})/b. \tag{8.14}\]

The operator inequality extends to
\[
\chi \otimes P_{c|z; \theta''} \leq (\lambda(\chi \otimes P_{c|z; \theta}) + (1 - \lambda)(\chi \otimes P_{c|z; \theta'}))/b. \tag{8.15}\]

By operator monotonicity, homogeneity and convexity it follows that
\[
f''(cz) \leq \mu(z)F(cz) \left( \lambda\tilde{L}_{cz}(\chi \otimes r(\theta))^{\alpha} + (1 - \lambda)\tilde{L}_{cz}(\chi \otimes r(\theta'))^{\alpha} \right)/b^\alpha. \tag{8.16}\]

Since \( b < 1 \), this inequality is also satisfied for \( z_i = 0 \). Since \( f \geq \tilde{Q}_\alpha(F(CZ), \chi \otimes r(\theta)) \) and similarly for \( f' \), after summing over \( cz \) to add the contributions to \( f'' \), we conclude that
\[
f'' \leq (\lambda f + (1 - \lambda)f')/b^\alpha. \tag{8.17}\]

To determine \( \lambda \) and \( b \) in terms of \( \phi \) and \( \varphi \), we solve a geometrical problem involving chords. For this paragraph we use notational conventions from plane geometry. Let \( O \) be the center of a unit circle and \( A, B \) and \( C \) points on the circumference with \( C \) between \( A \) and \( B \). Write \( \angle AOB = \phi \) and \( \angle AOC = \varphi \). Let \( M \) be the intersection of the lines \( OC \) and \( AB \). Let \( x = AM \), \( y = MB \) and \( b = OM \) be the lengths of the respective line segments. Then \( b\sin(\varphi) + b\sin(\phi - \varphi) = \sin(\phi) \) since the \( b\sin(\phi)/2 \) is the area of \( \triangle OAB \), \( b\sin(\varphi)/2 \) the area of \( \triangle OAM \) and \( b\sin(\phi - \varphi)/2 \) the area of \( \triangle OMB \). Thus \( b = \sin(\phi)/\sin(\varphi) + \sin(\phi - \varphi) \). Since \( \angle OAB = (\pi/2 - \phi)/2 \), \( x\sin(\pi/2 - \phi)/2 = b\sin(\varphi) \) and \( y\sin(\pi/2 - \phi)/2 = b\sin(\phi - \varphi). \) From this we determine \( \lambda = y/(x + y) = \sin(\phi - \varphi)/(\sin(\varphi) + \sin(\phi - \varphi)) \). Summarizing, we have
\[
b = \frac{\sin(\phi)}{\sin(\varphi) + \sin(\phi - \varphi)} \in (0, 1].
\[
\lambda = \frac{\sin(\phi - \varphi)}{\sin(\varphi) + \sin(\phi - \varphi)} \in [0, 1].
\] (8.18)

By rotational symmetry, the desired identity \(\lambda a + (1 - \lambda)a' = ba''\) is satisfied with \(a, a'\) and \(a''\) as defined before Eq. 8.14. It is possible to maximize the upper bound \((\lambda f + (1 - \lambda)f')/b^\alpha\) on \(f''\) over \(\varphi \in [0, \varphi]\). In terms of \(\varphi\), the bound is
\[
u(\varphi) = \frac{\lambda f + (1 - \lambda)f'}{b^\alpha} = \frac{(\sin(\phi - \varphi) + \sin(\varphi))\beta(\sin(\phi - \varphi)f + \sin(\varphi)f')}{\sin(\phi)^\alpha}.
\] (8.19)

To show that the function \(u(\varphi)\) has a unique maximum we prove log-concavity in \(\varphi\). Consider
\[v(\varphi) = \log(\sin(\phi)^\alpha u(\varphi)) = \beta \log(\sin(\phi - \varphi) + \sin(\varphi)) + \log(\sin(\phi - \varphi)f + \sin(\varphi)f').\] (8.20)

As a functions of \(\varphi\), both \(\sin(\phi - \varphi)\) and \(\sin(\varphi)\) are concave for the values of \(\phi\) and \(\varphi\) under consideration. Therefore, any linear combination \(g(\varphi) = c \sin(\phi - \varphi) + c' \sin(\varphi)\) with \(c, c' \geq 0\) is concave. Since \(\log\) is monotone increasing and concave, \(\log(g(\varphi))\) is concave for any concave \(g(\varphi)\). Consequently, \(v(\varphi)\) is the sum of two concave functions and therefore also concave.

We use the small angle approximation to upper bound \(u(\varphi)\). Applying the inequalities \(\sin(\phi - \varphi) \leq (\phi - \varphi)\) and \(\sin(\varphi) \leq \varphi\) gives
\[u(\varphi) \leq \left(\frac{\phi}{\sin(\phi)}\right)^\alpha \max(f, f').\] (8.21)

The maximum of the bound \(u(\varphi)\) defined in Eq. 8.11 can be found as follows: With \(v(\varphi)\) as defined in Eq. 8.20 and considering the concavity of \(v(\varphi)\), if the derivative \(v^{(1)}(0) \leq 0\) the maximum of \(u(\varphi)\) is \(f\), if \(v^{(1)}(\phi) \geq 0\), the maximum is \(f'\), and otherwise there is a unique critical point \(\varphi_0\) between \(0\) and \(\phi\) for \(v(\varphi)\), and the maximum of \(u(\varphi)\) is \(u(\varphi_0)\). The critical point is found by solving \(v^{(1)}(\varphi_0) = 0\).

We can now determine an upper bound on \(f_{\max}\) from the values of \(f_{\max}(r)\) for \(r \in \{\mathcal{X}^k\}\). Write \(\theta > l = (\theta_{l+1})_{k-l}^1\) and \(\theta \leq l = (\theta_i)_{i=1}^l\) so that \(\theta = \theta_{\leq l}\theta > l\) with our concatenation conventions. For any \(l\) define
\[f_{\max}(\theta_{\geq l}) = \max_{\chi, \theta \leq l} \tilde{Q}_\alpha(F(CZ), \chi \otimes r(\theta_{\leq l}\theta > l)),\] (8.22)
where we are overloading the symbol \(f_{\max}\) by making it depend on the type and length of the argument. The upper bound on \(f_{\max}\) can be obtained recursively, where at the \(l\)th step we obtain upper bound \(v(\theta_{\geq l})\) on \(f_{\max}(\theta_{\geq l})\), so that the \(k\)th step yields an upper bound on \(f_{\max}\). To initialize the procedure (the 0th step), we determine \(f_{\max}(\theta)\) for all \(\theta = \theta_{\geq 0} \in \mathcal{X}^k\). This requires a method for maximizing \(r \in \mathcal{S}_1(\mathcal{H}) \mapsto \tilde{Q}_\alpha(F(CZ), \tau^{1/\alpha} \otimes r)\) for given \(r\), and such a method is given later in this section. Let \(v(\theta) = f_{\max}(\theta)\). For the \(l\)th step, fix \(\theta_{> l} \in \mathcal{X}_{k-l}\). From the previous steps, for all \(\theta_l \in \mathcal{X}\), we have determined upper bounds \(v(\theta_{\geq l}) \geq f_{\max}(\theta_{\geq l})\). For any pair of successive \(\psi, \psi' \in \mathcal{X}\), we can apply Lem. 8.3 to obtain a bound \(u(\psi, \psi') \geq f_{\max}(\psi''\theta_{> l})\) for all \(\psi''\in [\psi, \psi']\). The maximum of
these bounds is an upper bound on \( f_{\text{max}}(\theta, \gamma) \). After having determined \( u(\psi, \psi') \), we can set \( u(\theta, \gamma) = \max\{ u(\psi, \psi') : \psi, \psi' \text{ are successive pairs in } \mathcal{X} \} \).

The upper and lower bounds on \( f_{\text{max}} \) obtained converge with the resolution \( m \) used for \( \mathcal{X} \). It is possible to start at low resolution, and refine the subdivision \( \mathcal{X} \) if the gap between lower and upper bounds is too large. However, not all intervals need refinement and we can significantly reduce the work required by selectively refining a cubical grid in \( \otimes_{i=1}^{k} H_{(i)}^{(i)} \). The grid-refinement algorithm’s state contains two data structures. Let \( [0, \pi]^{l} \) denote the \( l \)-fold cartesian product of \([0, \pi] \) with itself. The first data structure is \( \mathcal{T} \) and contains the pairs of \( \theta \in [0, \pi]^{k} \) and the corresponding values \( f_{\text{max}}(\mathcal{T}) \) for which \( f_{\text{max}}(\theta) \) has been determined. The second is \( \mathcal{K} \) and consists of cuboidal regions in \([0, \pi]^{k} \), where each region \( K \) is specified by its \( 2^{k} \) vertices. The region \( K \) comes with an upper bound \( f_{\text{max}}(K) \geq \max_{\theta \in \mathcal{K}} f_{\text{max}}(\theta) \).

The structure \( \mathcal{K} \) may be organized as a priority heap, where the priority of the region \( K \) is given by the maximum of \( f(\mathcal{T}) \) over the \( \theta \) in \( \mathcal{T} \). The algorithm is initialized with a grid \( \mathcal{X} \) for some resolution \( m \geq 2 \). For this, it computes \( f_{\text{max}}(\mathcal{T}) \) for each \( \theta \in \mathcal{X} \) and adds \( (\theta, f_{\text{max}}(\theta)) \) to \( \mathcal{T} \). It then iterates over the cubical regions \( K \) defined by \( \mathcal{X} \), computes \( f_{\text{max}}(K) \) and adds \( (K, f_{\text{max}}(K)) \) to \( \mathcal{K} \). We can compute \( f_{\text{max}}(K) \) for \( K \) consisting of the convex closure of \( \{ \theta + \sum_{i \in I} \varphi_{i} e_{i} : I \subset [k] \} \) according to the strategy for computing the global \( f_{\text{max}} \) given \( \mathcal{X} \). For this, we replace \( \mathcal{X} \) by \( \prod I_{i} \{ \theta_{i}, \theta_{i} + \varphi_{i} \} \), which is the cartesian product of the sets \( \{ \theta_{i}, \theta_{i} + \varphi_{i} \} \). The strategy gives the value of \( f_{\text{max}}(K) \) for the region \( K \) covered by the convex closure of \( \prod I_{i} \{ \theta_{i}, \theta_{i} + \varphi_{i} \} \). After initialization, the algorithm updates the structures in each step by refining the top region \( K \) in \( \mathcal{K} \). If \( K \) is the convex closure of \( \{ \theta + \sum_{i \in J} \varphi_{i} e_{i} : I \subset [k] \} \), a possible refinement strategy is to divide each of \( K \)’s edges in two for \( 2^{k} \) subregions defined as the convex closures \( K_{J} \) of \( \{ \theta + \sum_{i \in J} \varphi_{i} e_{i} / 2 + \sum_{i \in I} \varphi_{i} e_{i} / 2 : I \subset [k] \} \) for \( J \subset [k] \). For each new vertex \( \theta' \), if the vertex is not in \( \mathcal{T} \), the algorithm computes \( f_{\text{max}}(\theta') \) and adds \( (\theta', f_{\text{max}}(\theta')) \) to \( \mathcal{T} \). For each \( K_{J} \) the algorithm computes \( f_{\text{max}}(K_{J}) \) and adds \( (K_{J}, f_{\text{max}}(K_{J})) \) to \( \mathcal{K} \). The original region \( K \) is removed from \( \mathcal{K} \) at the beginning of the refinement cycle.

To complete the schema for determining \( f_{\text{max}} \), we return to the problem of maximizing the concave, homogeneous-of-degree-1 function \( g : \tau \in S_{1}(H) \mapsto \hat{Q}_{\alpha}(F(CZ), \tau^{1/\alpha} \otimes r) \) for fixed \( r \in \otimes_{i=1}^{k} H_{1}^{(i)} \). It can in principle be maximized by any method for concave maximization over a domain defined by semi-definite constraints. Here we have a special domain and we can take advantage of this. Further, \( g \) is differentiable at full rank \( \tau \). Write \( g \) in the form
\[
g(\tau) = \sum_{cz} \left( \text{tr} \left( \tau^{1/\alpha} Q_{cz} \right) \right)^{\alpha} \tag{8.23}
\]
for a family of positive semidefinite operators \( Q_{cz} \). Each \( Q_{cz} \) is a product of \( (\mu(z)F(cz))^{1/\alpha} \) and a rank-1 projector \( P_{cz,\theta} \). We begin by reducing the problem to the case where it suffices to consider operators \( \tau \) with full support on one of the irreducible subspaces generated by the \( Q_{cz} \). Let \( \Pi_{0} \) be the null-space projector for \( \tau \). Suppose that \( \Pi_{0} \neq 0 \), and consider changing \( \tau \) to \( \tau' = (1 - \epsilon)\tau + \epsilon\Pi_{0} / \text{tr}(\Pi_{0}) \). Then
\[
\tau^{1/\alpha} = (1 - \epsilon)^{1/\alpha} \tau^{1/\alpha} + (\epsilon / \text{tr}(\Pi_{0}))^{1/\alpha} \Pi_{0} = \tau^{1/\alpha} + \gamma \epsilon^{1/\alpha} \Pi_{0} + O(\epsilon), \tag{8.24}
\]

for a family of positive semidefinite operators \( Q_{cz} \). Each \( Q_{cz} \) is a product of \( (\mu(z)F(cz))^{1/\alpha} \) and a rank-1 projector \( P_{cz,\theta} \). We begin by reducing the problem to the case where it suffices to consider operators \( \tau \) with full support on one of the irreducible subspaces generated by the \( Q_{cz} \). Let \( \Pi_{0} \) be the null-space projector for \( \tau \). Suppose that \( \Pi_{0} \neq 0 \), and consider changing \( \tau \) to \( \tau' = (1 - \epsilon)\tau + \epsilon\Pi_{0} / \text{tr}(\Pi_{0}) \). Then
with \( \gamma = (\text{tr}(\Pi_0))^{-1/\alpha} \). Consider the set \( I \) of \( cz \) such that \( \text{tr}(\tau Q_{cz}) > 0 \) and \( \text{tr}(Q_{cz} \Pi_0) > 0 \). For \( cz \in I \), \( \text{tr}(\tau^{1/\alpha} Q_{cz}) > 0 \) and

\[
\left( \text{tr}(\tau^{1/\alpha} Q_{cz}) \right)^\alpha = \left( \text{tr}(\tau^{1/\alpha} Q_{cz}) + \gamma \epsilon^{1/\alpha} \text{tr}(\Pi_0 Q_{cz}) + O(\epsilon) \right)^\alpha = \left( \text{tr}(\tau^{1/\alpha} Q_{cz}) \right)^\alpha + \alpha \left( \text{tr}(\tau^{1/\alpha} Q_{cz}) \right)^{\beta \gamma^{1/\alpha}} \text{tr}(\Pi_0 Q_{cz}) + o(\epsilon^{1/\alpha}).
\]

(8.25)

If \( \text{tr}(Q_{cz} \tau) = 0 \) or \( \text{tr}(Q_{cz} \Pi_0) = 0 \), then \( \left( \text{tr}(\tau^{1/\alpha} Q_{cz}) \right)^\alpha = \left( \text{tr}(\tau^{1/\alpha} Q_{cz}) \right)^\alpha + O(\epsilon) \). It follows that if \( I \) is not empty, for small enough \( \epsilon > 0 \), \( g(\tau') - g(\tau) \) is dominated by positive terms of order \( \epsilon^{1/\alpha} \) and, unless \( I \) is empty, \( \tau \) does not maximize \( g \). The set \( I \) is empty iff for all \( cz \) either \( \text{tr}(Q_{cz} \tau) = 0 \) or \( \text{tr}(Q_{cz} \Pi_0) = 0 \), which implies that every \( Q_{cz} \) is supported in \( 1 - \Pi_0 \) or in \( \Pi_0 \). In other words, the \( Q_{cz} \) can be block-diagonized with respect to \( \Pi_0 \). Let \( \{\Pi_i\} \) be a maximal complete set of projectors for which the \( Q_{cz} \) are block-diagonal. Equivalently, the \( \Pi_i \) project onto the irreducible subspaces of the algebra generated by the \( Q_{cz} \) and generate the center of this algebra. For an orthogonal \( U \) that commutes with all \( Q_{cz} \), \( \left( \text{tr}(U \tau U^T)^{1/\alpha} Q_{cz} \right)^\alpha = \left( \text{tr}(\tau)^{1/\alpha} Q_{cz} \right)^\alpha \) for all \( cz \). Since averaging over such \( U \) is decoherence of \( \tau \) with respect to the center of the algebra generated by the \( Q_{cz} \) and by concavity, the maximum of \( g \) is achieved for \( \tau \) block-diagonal with respect to the \( \Pi_i \). We can then write \( \tau \) as a mixture \( \tau = \bigoplus_i \mu(i) \tau_i \) where the \( \tau_i \) are density matrices supported in the \( i \)th irreducible subspace and \( \mu \) is a probability distribution. With this, \( g(\tau) = \sum_i \mu(i) g(\tau_i) \), so \( g(\tau) \leq \max_i g(\tau_i) \), and the problem reduces to the case where \( \tau \) has full support in one of the irreducible subspaces. We remark that for determining \( f_{\text{max}} \) it may be necessary to check for reducibility of the \( Q_{cz} \). In particular, for the cases where \( F(cz) \) has zeros or if any of the angles defining the \( Q_{cz} \) are 0 or \( \pi \), the algebra generated by the \( Q_{cz} \) may not be complete, in which case the \( Q_{cz} \) can be jointly block diagonalized.

The previous paragraph implies that it suffices to consider the general problem of maximizing a concave, homogeneous- of-degree-1 and differentiable function \( g : \tau \in S_1(H) \mapsto g(\tau) \) over real positive density operators. Let \( \nabla g \) be the derivative expressed as a Hermitian operator so that for positive semidefinite \( \tau + \epsilon \Delta \), \( g(\tau + \epsilon \Delta) = g(\tau) + \epsilon \text{tr}(\Delta \nabla g) + o(\epsilon) \). An iterative maximization algorithm updates \( \tau \) to \( \tau' \) to approach the maximum. For this problem, given a density operator \( \Delta \), we can update \( \tau' = (1 - \epsilon) \tau + \epsilon \Delta \) to satisfy the constraints. By degree-1 homogeneity, \( \text{tr}(\tau \nabla g(\tau)) = g(\tau) \). Thus \( g(\tau') = (1 - \epsilon) g(\tau) + \epsilon \text{tr}(\Delta \nabla g(\tau)) + o(\epsilon) \).

Write \( \nabla g(\tau) = \sum_{i=1}^d \lambda_i \Pi_i \) with \( \Pi_i \) a complete family of orthogonal projectors onto the distinct eigenvalue eigenspaces of \( \nabla g(\tau) \). We order the eigenvalues so that \( \lambda_1 \) is the maximum eigenvalue. Then we have \( \text{tr}(\Delta \nabla g(\tau)) \leq \lambda_1 \), so it is natural to choose directions \( \Delta \) supported in \( \Pi_1 \). The maximum is achieved if \( \lambda_1 = g(\tau) \), in which case necessarily \( \tau \) is supported in \( \Pi_1 \), and \( \Pi_1 = 1 \) since \( \tau \) has full support. That is, \( \Pi_1 = 1 \) is a necessary and sufficient condition for maximum \( g(\tau) \). If this condition is not satisfied, an update option is to set \( \Delta = \Pi_1 / \text{tr}(\Pi_1) \). An alternative is to set \( \Delta = \|\nabla g(\tau) - g(\tau)\| / \text{tr}(\|\nabla g(\tau) - g(\tau)\|) \). One can choose \( \epsilon \) according to a schedule such as one of those used in the Frank-Wolfe algorithm [36], or one can choose \( \epsilon \) by performing a one-dimensional maximization in the direction \( \Delta \). Concave maximization over density matrices is also a task for maximum-likelihood state tomography, where a common strategy is the \( R \rho R \) algorithm [37]. A diluted version of this algorithm [38] could be used here also. However, the methods discussed so far do not have good convergence properties, so some exploration may be required to determine the best update strategy. Convergence issues can be mitigated by taking advantage of the fact that \( \lambda_1 \) is also an upper bound on the maximum value of \( g \), so \( \lambda_1 - g(\tau) \) is the gap and
can be used as a stopping criterion, noting that we often do not require extremely small
gaps between upper and lower bounds in our applications.

For computing $\nabla g$, it suffices to consider the coefficients of the form $g_P(\tau) = \text{tr}(\tau^{1/\alpha} P)^\alpha$ of $\mu(z) F(c z)$ in the sum for $\bar{Q}_\alpha$. Here $P$ is a projector. We can write the gradient in the form

$$\nabla \tau g_P(\tau) = \alpha \text{tr}(\tau^{1/\alpha} P)^\beta X,$$

(8.26)

where $X \doteq \nabla \tau \text{tr}(\tau^{1/\alpha} P)$. To compute $X$ requires perturbation techniques. Write $\tau' = \tau + \epsilon \Delta$ and express $\tau = \sum_i \lambda_i \Pi_i$ in terms of its eigenspace projectors, where the $\lambda_i$ are positive. This enables a unique decomposition of $\Delta$ in the form $\Delta = \sum \tau + S$, where the support of $\Delta$ is in $\Pi_i$ and $S$ is skew-symmetric with $\Pi_i S \Pi_i = 0$ for each $i$. To compute $\Delta_i$ and $S$ in terms of $\Delta$, define $\Delta_{ij} = \sum_{i,j} \lambda_i \Delta_{ij}$. Then $\Delta_i = \Delta_{ii}$ and $S = \sum_{i \neq j} \Delta_{ij}$ with $S_{ij} = \Delta_{ij}/(\lambda_j - \lambda_i)$. For orthogonal $U$, $(U \tau U^T)^1/\alpha = U \tau^{1/\alpha} U^T$. With $U = e^{S}, \gamma > 0$ and $Y$ commuting with $\tau$, we have $U(\tau + \epsilon Y)^\gamma U^T = \tau^\gamma + \epsilon \gamma \tau^\gamma Y + \epsilon[S, \tau^\gamma] + O(\epsilon^2)$, where we used the assumption that $\tau$ is positive. For sufficiently small $\epsilon$, we can expand

$$(\tau + \epsilon \Delta)^{1/\alpha} = \left(\tau + \sum_i \epsilon \Delta_i + \epsilon[S, \tau]\right)^{1/\alpha}$$

$$= \left(U(\tau + \sum_i \epsilon \Delta_i)U^T + O(\epsilon^2)\right)^{1/\alpha}$$

$$= \left(U(\tau + \sum_i \epsilon \Delta_i + O(\epsilon^2))U^T\right)^{1/\alpha}$$

$$= U \left(\tau + \sum_i \epsilon \Delta_i + O(\epsilon^2)\right)^{1/\alpha} U^T$$

$$= U \left(\tau + \sum_i \epsilon \Delta_i\right)^{1/\alpha} U^T$$

$$= \tau^{1/\alpha} + \epsilon \frac{1}{\alpha} \tau^{-\beta/\alpha} \sum_i \Delta_i + \epsilon[S, \tau^{1/\alpha}] + O(\epsilon^2)$$

$$= \tau^{1/\alpha} + \epsilon \left(\frac{1}{\alpha} \sum_i \lambda_i^{-\beta/\alpha} \Delta_i + [S, \tau^{1/\alpha}]\right) + O(\epsilon^2).$$

(8.27)

Expressed with the $\Delta_{ij}$ this is

$$(\tau + \epsilon \Delta)^{1/\alpha} = \tau^{1/\alpha} + \epsilon \left(\sum_i \frac{1}{\alpha} \lambda_i^{-\beta/\alpha} \Delta_{ii} + \sum_{i \neq j} \frac{1}{\lambda_j - \lambda_i} (\lambda_j^{1/\alpha} - \lambda_i^{1/\alpha}) \Delta_{ij}\right) + O(\epsilon^2)$$

$$= \tau^{1/\alpha} + \epsilon \left(\sum_i \frac{1}{\alpha} \lambda_i^{-\beta/\alpha} \Pi_i \Delta_{ii} + \sum_{i \neq j} \frac{1}{\lambda_j - \lambda_i} (\lambda_j^{1/\alpha} - \lambda_i^{1/\alpha}) \Pi_i \Delta_{ij}\right) + O(\epsilon^2).$$

(8.28)
With this, 

$$g_P(\tau + \epsilon \Delta)^{1/\alpha} = \text{tr}((\tau + \epsilon \Delta)^{1/\alpha} P)$$

$$= \text{tr}(\tau^{1/\alpha} P) + \epsilon \left( \text{tr} \left( \sum_i \frac{\lambda_i^{1/\alpha} - \lambda_i^{1/\alpha}}{\lambda_j - \lambda_i} \Pi_i \Pi \Delta \right) + \text{tr} \left( \sum_{i \neq j} \frac{\lambda_j^{1/\alpha} - \lambda_i^{1/\alpha}}{\lambda_j - \lambda_i} \Pi_j \Pi \Delta \right) \right)$$

$$+ o(\epsilon)$$

$$= \text{tr}(\tau^{1/\alpha} P) + \epsilon \text{tr} \left( \left( \sum_i \frac{\lambda_i^{1/\alpha} - \lambda_i^{1/\alpha}}{\lambda_j - \lambda_i} P_{ii} + \sum_{i \neq j} \frac{\lambda_j^{1/\alpha} - \lambda_i^{1/\alpha}}{\lambda_j - \lambda_i} P_{ji} \right) \Delta \right) + o(\epsilon),$$

(8.29)

where $P_{ij} \doteq \Pi_i \Pi_j$. With this equation and the definition of the gradient, we can determine that $X$ in Eq. 8.26 is given by

$$X = \sum_i \frac{\lambda_i^{1/\alpha} - \lambda_i^{1/\alpha}}{\lambda_j - \lambda_i} P_{ii} + \sum_{i \neq j} \frac{\lambda_j^{1/\alpha} - \lambda_i^{1/\alpha}}{\lambda_j - \lambda_i} P_{ji}.$$  \hspace{1cm} (8.30)

Note that the limit of $(\lambda_j^{1/\alpha} - \lambda_i^{1/\alpha})/(\lambda_j - \lambda_i)$ as $\lambda_j \rightarrow \lambda_i$ is $\lambda_i^{-\beta/\alpha}/\alpha$, so the potentially problematic term for near-degenerate eigenvalues can be stably computed. The simplest way to avoid precision problems with this expression is to always collapse nearby eigenvalues of $\tau$, where $\lambda_i$ and $\lambda_j$ should be considered nearby if $|\lambda_i^{1/\alpha} - \lambda_j^{1/\alpha}| \leq \sqrt{\delta}$ with $\delta$ the machine precision. This limits numerical errors in the computation of $X$ to approximately $\sqrt{\delta}$. However, the numerical error has less effect on the validity of the upper bound on $g$ if we replace $\tau$ by $\tilde{\tau}$ where $\tilde{\tau}$ is $\tau$ with nearby eigenvalues collapsed and rescaled to satisfy the constraint $\text{tr}(\tilde{\tau}) = 1$ before determining the upper bound from the maximum eigenvalue of the gradient.
A protocol-style outline of QEF optimization is given in Protocol 4.

**Protocol 4**: Schema for QEF optimization for the \((k,2,2)\)-Bell-test configuration with known input distribution \(\mu(Z)\).

**Input**: The targeted trial probability distribution \(\nu(CZ)\) and an initial candidate \(F_0(CZ) \geq 0, \sum cz F_0(cz) = 1\) with its \(f_{0,\text{max}}\).

// The input distribution is \(\mu(Z) = \nu(Z)\).

// Recommendation: \(F_0(CZ)\) can be obtained by rescaling a good PEF with power \(\beta\) at \(\nu(CZ)\).

**Output**: Best \(F(CZ), f_{\text{max}}(F(CZ))\) found and its log-prob rate \(r_{F(CZ)}\).

Initialize an empty list \(L\) of triples of candidates \(F(CZ), f_{\text{max}}(F(CZ))\) and their log-prob rates \(r_{F(CZ)}\);

while stopping criteria are not satisfied do

  // Stopping criteria may be satisfied if resource limits are reached or log-prob rates are not improving sufficiently anymore.

  if \(L\) is empty then

    Set \(F(CZ) = F_0(CZ)\);

  else

    Determine the next candidate \(F(CZ) \geq 0, \sum cz F(cz) = 1\) by using the triples in \(L\) as a discrete sample of the QEF landscape;

  end

  Compute \(f_{\text{max}}(F(CZ))\); // Strategies are given in the text.

  Compute \(r_{F(CZ)}\) and add \((F(CZ), f_{\text{max}}(F(CZ)), r_{F(CZ)})\) to \(L\);

end

---

**8.3. Optimal PEFs for Comparison**

In Protocol 4, we suggested starting QEF optimization with a good PEF previously determined for the \((k,2,2)\)-Bell-test configuration at trial probability distribution \(\nu(CZ)\). In Ref. [1], we gave algorithms for determining such PEFs with respect to polytope envelopes of the classical-side-information models. The simplest such polytope is the non-signaling polytope, which can be restricted with Tsirelson’s bounds or other linear inequalities obtained from the hierarchy of semidefinite programs in Ref. [39]. The schema for QEF optimization suggests optimizing PEFs directly using the reduction enabled by Thm. 8.1. The PEF optimization problem then reduces to an analog of the QEF optimization problem Prob. 8.5 as follows:

Maximize: \(\sum cz \nu(cz) \log(F'(cz)) - \log(f'_{\text{max}})\)

Variables: \(F'(CZ), f'_{\text{max}}\)

Subject to: \(F'(CZ) \geq 0, \sum cz F'(cz) = 1,\)

\[ f'_{\text{max}} \geq \sum cz \mu(z) F'(cz) \text{tr}(\tau P_{cz;\theta})^\alpha \text{ for all } \tau \geq 0 \text{ with } \text{tr}(\tau) = 1 \text{ and } \theta. \] (8.31)
The PEF constraint is obtained since \( \nu'(cz) = \mu(z) \text{tr}(\tau P_{cz;\theta}) \) defines the trial probability distribution for the model state under consideration. The coefficient of \( F'(CZ) \) is \( \nu'(cz)\nu'(c|z) \). The PEF constraint on \( f'_{\text{max}} \) is convex in \( \tau \), so we cannot use the same argument to restrict \( \tau \) to real density operators. However, convexity implies that \( \tau \) can be restricted to pure states. In solving Prob. 8.31 we can set \( f'_{\text{max}} \) to the maximum value of \( Q'_\alpha(F'(CZ), \theta, \tau) = \sum_{cz} \mu(z) F'(cz) \text{tr}(\tau P_{cz;\theta}) \) over \( \tau \) and \( \theta \).

**Lemma 8.4.** In Prob. 8.31, the operator \( \tau \) may be restricted to pure states \( \hat{\psi} \) with \(|\psi\rangle \) real, and it suffices to consider \( \theta \) with \( \theta_i \in [0, \pi] \).

**Proof.** We noted before the lemma that \( \tau \) may be assumed to be pure. That we only need to consider \( \theta_i \in [0, \pi] \) follows by the same argument as that used to prove the corresponding statement of Lem. 8.2. Suppose \( \tau \) is not real. Then the conditional probabilities \( \nu'(c|z) = \text{tr}(\tau P_{cz;\theta}) \) contributing to \( Q'_\alpha \) satisfy

\[
\text{tr}(\tau P_{cz;\theta}) = \text{tr}(\tau P_{cz;\theta}) = \text{tr}\left(\frac{1}{2}(\tau + \tau) P_{cz;\theta}\right),
\]

so the set of constraints on \( f'_{\text{max}} \) is unchanged if we restrict \( \tau \) to real density matrices. Since real density matrices can be diagonalized over the reals, they are mixtures of real pure states and by convexity we can further restrict to real pure states. \( \square \)

While we cannot take advantage of concavity to simplify maximizing \( Q'_\alpha(F'(CZ), \theta, \tau) \) with respect to \( \tau \), we can take advantage of convexity as before, but need to extend the strategy used to optimize over \( \theta \) to also include \( \tau \). With the notation of Sect. 8.2 \( Q'_\alpha(F'(CZ), \theta, \hat{\psi}) = \hat{Q}_\alpha(F'(CZ), \hat{\psi} \otimes r(\theta)) \) (see Eq. 8.9), and \( \hat{Q}_\alpha(F'(CZ), u) \) is convex in \( u \). If we can maximize over real \(|\psi\rangle\) for given \( \theta \), then the schemas for maximizing over \( \theta \) in Sect. 8.2 can also be used here. To perform the maximization over \(|\psi\rangle\), we describe an inner approximation generalizing the one used to maximize over the \( \theta_i \in [0, \pi] \). The real pure states \(|\psi\rangle\) can be identified with points in the sphere \( S_{2k-1} \). We reduce the inner-most maximization problem to one of maximizing over \(|\psi\rangle\) contained in convex cones spanned by small sets of points on the sphere with large overlaps as vectors. Refinement involves subdividing the cones. In the case of \( k = 2 \), we suggest sets of points defining the eight corners of a cuboid. For describing the technique, we fix \( \theta \) and \( F'(CZ) \), and omit them from expressions. In particular, we abbreviate \( \hat{Q}_\alpha(F'(CZ), \hat{\psi} \otimes r(\theta)) \) as \( \hat{Q}_\alpha(\hat{\psi}) \). The general goal is to upper bound a non-negative, convex function \( \hat{Q}_\alpha(\hat{\psi}) \) homogeneous of degree \( \alpha \) in \( \hat{\psi} \) over \(|\psi\rangle \in S_{2k-1} \), where the function \( \hat{Q}_\alpha(\tau) \) is operator monotone in \( \tau \). We switch to mathematical notation for real vectors, omitting kets and bras.

**Lemma 8.5.** Fix \( \epsilon \in (0, 1) \). Let \( I \) be a finite index set and for \( i \in I \), let \( x_i \) be real unit vectors with \( x_i^T x_j \geq 1 - \epsilon \) for all \( j \in I \). If \( y \) is a unit vector that is a positive combination of the \( x_i \), then there is a convex combination \( \rho \) of the \( x_i x_i^T \) such that \( yy^T \leq \rho/(1 - \epsilon) \).

**Proof.** Write \( y \) as an explicit positive combination \( y = \sum_i \lambda_i x_i \). Define

\[
\rho' = \sum_i \lambda_i x_i x_i^T y.
\]
Then for any real vector \( z \), \( z^T \rho' z \geq 0 \), that is \( \rho' \geq 0 \). Moreover, \( \rho' y = \sum_i \lambda_i x_i = y \) so that \( y \) is a unit eigenvector with eigenvalue 1 of \( \rho' \). Therefore \( \rho' \geq yy^T \). Let \( \lambda = \sum_i \lambda_i \). Compute
\[
x_i^T y = \sum_j \lambda_j x_i^T x_j \geq \sum_j \lambda_j (1 - \epsilon) = \lambda (1 - \epsilon),
\]
which gives
\[
\text{tr}(\rho') = \sum_i \frac{\lambda_i}{x_i^T y} \leq \sum_i \frac{\lambda_i}{\lambda (1 - \epsilon)} = \frac{1}{1 - \epsilon}.
\]
To complete the proof of the lemma, we set \( \rho = \rho' / \text{tr}(\rho') \).

\[\text{Lemma 8.6.} \] Fix \( \epsilon \in (0, 1) \). Let \( I \) be a finite index set and for \( i \in I \), let \( x_i \) be real unit vectors with \( x_i^T x_j \geq 1 - \epsilon \) for all \( j \in I \). Let \( q_i = \hat{Q}_\alpha(x_i x_i^T) \). Then for all unit vectors \( y \) in the positive convex cone generated by \( x_i, \hat{Q}_\alpha(yy^T) \leq \max_i q_i / (1 - \epsilon)^\alpha \).

\[\text{Proof.} \] Let \( \rho = \sum_i \lambda_i x_i x_i^T \) be a convex combination of \( x_i x_i^T \) with \( yy^T \leq \rho' / (1 - \epsilon) \) according to Lem. 8.5. Then by monotonicity, homogeneity of degree \( \alpha \) and convexity of \( \hat{Q}_\alpha \), we have
\[
\hat{Q}_\alpha(yy^T) \leq \frac{1}{(1 - \epsilon)^\alpha} \hat{Q}_\alpha(\rho)
\]
\[
\leq \frac{1}{(1 - \epsilon)^\alpha} \sum_i \lambda_i \hat{Q}_\alpha(x_i x_i^T)
\]
\[
\leq \frac{1}{(1 - \epsilon)^\alpha} \max_i q_i.
\]
We describe the \( \hat{\psi} \)-maximization strategy for the case \( k = 2 \), so that \( |\psi\rangle \in S_3 \subset \mathbb{R}^4 \). We parametrize \( x \in S_3 \) with angles \( \phi_1 \in [0, \pi/2], \phi_2 \in [0, 2\pi] \) and \( \phi_3 \in [0, 2\pi] \) according to
\[
x(\phi_1, \phi_2, \phi_3) = \sin(\phi_1)(\sin(\phi_2), \cos(\phi_2), 0, 0)^T + \cos(\phi_1)(0, 0, \sin(\phi_3), \cos(\phi_3))^T.
\]
Because \( x \) and \(-x\) correspond to the same density matrix, we can restrict \( \phi_2 \) to \([0, \pi]\). To start the maximization, we can choose points according to a cubical grid on \([0, \pi/2] \times [0, \pi] \times [0, 2\pi]\). For this, fix \( m \geq 2 \) and let \( x_{i,j,k} = (i\pi/(2m), j\pi/(2m), k\pi/(2m)) \) for \( i \in \{0, \ldots, m\}, j \in \{0, \ldots, 2m\} \) and \( k \in \{0, \ldots, 4m\} \). We identify a set of facets, where each facet is defined by the eight corners of the cubes in the cubical grid. The facets may be identified with the sets of points defined by \( f_{i,j,k} = \{ x_{i+b_1,j+b_2,k+b_3} : b_1, b_2, b_3 \in \{0, 1\} \} \) for \( i \in \{0, \ldots, m-1\}, j \in \{0, \ldots, 2m-1\} \) and \( k \in \{0, \ldots, 4m-1\} \). The positive convex cones generated by the \( f_{i,j,k} \) cover the half space of \( \mathbb{R}^4 \) with non-negative first coordinate. Thus we can first compute \( \hat{Q}_\alpha \) for all \( x_{i,j,k} x_{i,j,k}^T \) to get a lower bound and then compute an upper bound for each facet according to Lem. 8.6. Facets whose upper bounds are below one of the values of \( \hat{Q}_\alpha \) obtained can be abandoned. Facets for which the upper bound exceeds the maximum value of \( \hat{Q}_\alpha \) over all vertices by more than the tolerance can be refined by dividing the angle intervals determining the facet’s cube in half. This determines 19 new points and 8 subfacets.

The strategy of the previous paragraph can be combined with that for maximizing over the \( \theta \) by covering \( S_3 \times [0, \pi]^2 \) with an initial cubical grid and refining cuboids as described in Sect. 8.2. In this case the cuboids are five-dimensional.
8.4. Examples

In Ref. [1] we analyzed PEF performance on photonic and atomic experimental data from published experiments, and in Ref. [40] we determined PEF finite-data performance in comparison to other methods, in particular trial-wise guessing probability [20, 11-16] and entropy accumulation [4, 5]. Here we repeat some of these analyses and perform comparisons with QEFs instead. For this, we do not optimize QEFs. Instead, we compute optimal PEFs $F'(CZ)$ for $C|Z$ with appropriate parameters, determine an upper bound on $f_{\text{max}}$ for each $F'(CZ)$ according to the methods in Sect. 8.2, and obtain a QEF $F(CZ)$ by dividing the PEF by $f_{\text{max}}$, that is $F(CZ) = F'(CZ)/f_{\text{max}}$. Throughout, we assume that the PEFs are for the classical trial model $T$ where the input distribution is uniform and the input-conditional output distributions satisfy non-signaling and Tsirelson’s bounds, see Ref. [1], Sect. VIII for details. This classical trial model includes tr($C_{222}(CZ)$) with the uniform input distribution. In each case, we optimize the expected net log$_2$-prob for $T$ at a trial distribution $\nu(CZ)$, where the expected net log$_2$-prob is computed according to Eq. (5.13) with $\bar{\kappa} = 1$. When obtaining a bound on $f_{\text{max}}$, we stopped refining the evaluation grid when the difference between lower and upper bounds on $f_{\text{max}}$ was smaller than a stopping criterion determined by the application. We set the stopping criterion so that the difference between the upper and lower bounds on $f_{\text{max}}$ has negligible impact on the QEF’s performance. For all PEFs checked, we found that $f_{\text{max}}$ was indistinguishable from 1 at numerical precision. We conjecture that these PEFs are QEFs with the same power $\beta$ for $C|Z$ and $C_{222}(CZ)$ with the uniform input distribution.

We first reconsider the results from the first experiment to demonstrate certified conditional min-entropy with a Bell test [20]. The experiment established entangled states of two ions in two separate ion-traps by entanglement swapping with photons as intermediaries. From the results of the experiment, the authors claimed 42 bits of conditional min-entropy at a smoothness error bounded by 0.01. That the claim did not take into account probability of success or quantum side information was clarified in subsequent papers [41, 42]. A question is whether the experiment could have certified positive conditional min-entropy with respect to quantum side information. To answer this question we repeated the analysis of Ref. [1], Sect. VIII.E with modifications for quantum side information. The experiment consisted of 3016 trials, of which we used the first 1000 for training. We optimized a PEF on the training set by maximizing the expected net log$_2$-prob in the remaining 2016 trials, where the expected net log$_2$-prob is computed according to Eq. (5.13) with $\bar{\kappa} = 1$. When obtaining a bound on $f_{\text{max}}$, we stopped refining the evaluation grid when the difference between lower and upper bounds on $f_{\text{max}}$ was smaller than a stopping criterion determined by the application. We set the stopping criterion so that the difference between the upper and lower bounds on $f_{\text{max}}$ has negligible impact on the QEF’s performance. For all PEFs checked, we found that $f_{\text{max}}$ was indistinguishable from 1 at numerical precision. We conjecture that these PEFs are QEFs with the same power $\beta$ for $C|Z$ and $C_{222}(CZ)$ with the uniform input distribution.

We then divided the PEF used by the upper bound on $f_{\text{max}}$ to construct a valid QEF. After applying this QEF to the remaining 2016 trials, we found that it witnesses 127.86 bits of quantum net log$_2$-prob at smoothness error $\epsilon = 0.01$ and presumed lower bound $\kappa = 1$ of the success probability. For the observed frequencies in this experiment, entropy accumulation requires 54688 trials to certify any random bits at $\epsilon = 0.01$ and $\kappa = 1$ with the min-tradeoff functions given in Ref. [5]. Here, the assignment of $\kappa = 1$ is purely formal for comparison with respect to the soundness criteria implicit in Ref. [20]. These soundness criteria are now considered inadequate. With modern soundness criteria and at $\epsilon = 0.03$ and $\kappa = 0.03$, the number of bits witnessed by the QEF is 72.70. This number is derived from the experimental QEF value. In a protocol, the number of bits to
be produced needs to be decided before the experiment and would have been less to ensure sufficiently high probability of success.

Next we compare the finite-data efficiency of QEFs to that of entropy accumulation with the min-tradeoff functions given in the EAT references for computed trial results distributions with uniform inputs. We consider the families of distributions, $\mathcal{P}_E = \{\nu_{E,\theta}\}_{0 \leq \theta \leq \pi/4}$, $\mathcal{P}_W = \{\nu_{W,p}\}_{1/\sqrt{2} < p \leq 1}$ and $\mathcal{P}_P = \{\nu_{P,p}\}_{2/3 < \eta \leq 1}$ studied in Ref. [40]. They are defined as follows: For the first and third, the two-party device to be measured is initially in the unbalanced Bell state defined by $|\Psi_{\theta}\rangle = \cos(\theta) |00\rangle + \sin(\theta) |11\rangle$. For the second, the initial state is the Werner state $p|\Psi_{\pi/4}\rangle<\Psi_{\pi/4}+(1-p)1/4$. To compute $\nu_{E,\theta}$ and $\nu_{W,p}$, the input-dependent measurements are chosen so as to maximize the expected CHSH value $\hat{I}$ defined by $\hat{I} = \mathbb{E}(4(1-2XY)(-1)^{A+B})$ with $A, B, X, Y \in \{0, 1\}$, where $X$ and $Y$ are the inputs and $A$ and $B$ are the outputs of Alice and Bob, respectively. For local realistic distributions, $\hat{I} \leq 2$ and for quantum distributions, $\hat{I} \leq 2\sqrt{2}$. To compute $\nu_{P,p}$, we use detectors of efficiency $\eta \in (2/3, 1]$ and choose both the state $|\Psi_{\theta}\rangle$ and the input-dependent measurements such that the statistical strength for rejecting local realism [48, 49] is maximized. The value of $\hat{I}$ for each family is monotonic in the parameters. That is, for $\nu_{E,\theta}$, $\hat{I}$ increases with $\theta$ for $\theta \in [0, \pi/4]$, for $\nu_{W,p}$ it increases with $p$ for $p \in (1/\sqrt{2}, 1]$, and for $\nu_{P,p}$ it increases with $\eta \in (2/3, 1]$. The family $\mathcal{P}_E$ and $\mathcal{P}_W$ represent the best and worst cases for conditional min-entropy as a function of $\hat{I}$, while $\mathcal{P}_P$ is experimentally relevant, particularly for photonic experiments.

Entropy accumulation is formulated to yield smooth min-entropy estimates and we compare performances accordingly. Specifically, we consider protocols for certifying $\epsilon$-smooth min-entropy conditional on success that satisfy the following: For specified values of $\sigma$, $\epsilon$ and $\kappa$, for all states in the model, if the probability of success is at least $\kappa$, then the $\epsilon$-smooth min-entropy of the output conditional on success is at least $\sigma$. A QEF protocol is determined by the application of Thm. [4.18] to all states in the model for which the probability of success is at least $\kappa$, and where $p$ and $\delta$ satisfy $-\log_2(p/\kappa^{\alpha/\beta}) \geq \sigma$ and $\delta = \epsilon^2/2$. Here, we refer to the quantity $\log_2(F(CZ))/\beta + \log_2(\epsilon^2/2)/\beta + \alpha \log_2(\kappa)/\beta$ in such a protocol as its min-entropy estimate. We remark that for randomness generation, the quantum net log-prob has better dependence on the probability of success parameter. Both entropy accumulation and QEFs give valid estimates regardless of the experimental distributions provided that the model is satisfied. But the performances are determined by the actual trial distributions. EAT protocols also have an associated min-entropy estimate determined from an affine min-tradeoff function.

We assume that for the “honest” devices, namely the devices as designed, the trials are i.i.d. with distribution $\nu$ in one of the families $\mathcal{P}_E$, $\mathcal{P}_W$ and $\mathcal{P}_P$. We are interested in the minimum number of trials required for a protocol with parameters $\sigma$, $\epsilon$ and $\kappa$ as described in the previous paragraph. To be useful, such a protocol should have a large probability of success greater than $\kappa$ for honest devices. For QEFs, the probability of success is determined by the distribution of the min-entropy estimate, which is obtained from a sum of i.i.d. random variables for honest devices. In the absence of specific information of the QEF defining these random variable, the probability of success cannot be estimated. Instead, we set $\sigma$ to the expectation of the min-entropy estimate. Generically, this implies an honest probability of success near $1/2$, at least for large enough $n$. For the EAT, we use the same strategy, setting $\sigma$ to the expectation of the EAT min-entropy estimate. For both QEFs and the EAT, the probability of success can be made close to 1 by reducing $\sigma$, provided the number of trials is large enough. For a representative comparison, we formally set
If $\epsilon = 10^{-6}$ and $\kappa = 1$ to determine the minimum number of trials required for positive $\sigma$. The assignment $\kappa = 1$ is singular but chosen as a convenient reference point for values of $\kappa$ that are not small. The improvements obtained by QEFs are as significant for all meaningful assignments with the same value for the product $\epsilon \kappa$.

First consider QEFs. Suppose that $F(CZ)$ is a trial-wise QEF with power $\beta$ and $\log_2$-prob rate $g$. According to Thm. 4.18, the expected $\epsilon$-smooth conditional min-entropy estimate in bits for $n$ trials is

$$ng + \frac{\log_2(\epsilon^2/2)}{\beta} + \frac{\alpha \log_2(\kappa)}{\beta}, \quad (8.38)$$

so the minimum number of trials required for positive $\epsilon$-smooth conditional min-entropy is

$$n_{\text{min,QEF}}(F(CZ); \beta, \epsilon, \kappa) = \frac{1}{g\beta} |\log_2(\epsilon^2\kappa^\alpha/2)|. \quad (8.39)$$

For simplicity we do not require that the number of trials is an integer. Except for the replacement of the error bound $\epsilon$ by $\epsilon^2/2$, this agrees with the expressions in Ref. [40].

For entropy accumulation, we can apply Thm. 6.5 with an entropy estimator, where the entropy estimator can be derived either from the QEF $F(CZ)$, or from the min-tradeoff function given in Ref. [5]. With the QEF, from Thm. 6.5 in terms of bits, with $h$ replaced by the $\log_2$-prob rate $g$ and $k_\infty = \lceil \max \log_2(F(CZ))/\beta \rceil$, the expected $\epsilon$-smooth conditional min-entropy estimate is

$$ng - 2 (\log_2(9) + [k_\infty]) \sqrt{1 - 2 \log_2(\epsilon \kappa)} \sqrt{n}, \quad (8.40)$$

which implies that the minimum number of trials is

$$n_{\text{min,EAT}}(F(CZ); \beta, \epsilon, \kappa) = \frac{4}{g^2} (\log_2(9) + [k_\infty])^2 (1 - 2 \log_2(\epsilon \kappa)). \quad (8.41)$$

We write $n_{\text{min,EAT}}(T; \epsilon, \kappa)$ for the same quantity but computed for the min-tradeoff function $T$ given in Ref. [5]. An explicit but involved expression for $n_{\text{min,EAT}}(T; \epsilon, \kappa)$ is given in Ref. [40], which we do not repeat here. Its evaluation involves optimizing over additional parameters.

For the comparison at a given distribution $\nu$, we first minimize the expression for $n_{\text{min,QEF}}(F'(CZ); \beta, \epsilon, \kappa)$ over $\beta$ and PEFs $F'(CZ)$ for $T$. The minimum found is witnessed by PEF $F'(CZ)$ and $\beta$. We then compute $f_{\text{max}}$ for $F'(CZ)$, which determines a valid QEF $F(CZ) = F'(CZ)/f_{\text{max}}$ with the same power $\beta$. This determines $n_{\nu,QEF} = n_{\text{min,QEF}}(F(CZ); \beta, \epsilon, \kappa)$. We then obtain $n_{\nu,F,EAT} = n_{\text{min,EAT}}(F(CZ); \beta, \epsilon, \kappa)$ according to the above formula and $n_{\nu,T,EAT} = n_{\text{min,EAT}}(T; \epsilon, \kappa)$ according to the instructions in Ref. [40]. The QEF advantages are determined by the ratios $f_{\nu,F} = n_{\nu,F,EAT}/n_{\nu,QEF}$ and $f_{\nu,T} = n_{\nu,T,EAT}/n_{\nu,QEF}$. For the distributions $\nu_{W,p}$, the advantage $f_{\nu,T}$ depends weakly on $\hat{I}$: $f_{\nu,T}$ increases from 36.9 at $\hat{I} = 2.008$ to 38.2 at $\hat{I} = 2\sqrt{2}$. For the other distributions, $f_{\nu,T}$ can be much larger, particularly at $\hat{I}$ near 2, as shown in Fig. 2. We also find that $f_{\nu,F}$ is systematically larger than $f_{\nu,T}$ by factors of at least two near maximum $\hat{I}$ and growing substantially toward minimum $\hat{I}$. Thus, determining the entropy estimator from the QEFs found and applying the EAT performs worse than applying the EAT with the min-tradeoff function from Ref. [5]. This suggests that the problem of optimizing QEFs and that of optimizing entropy estimators or min-tradeoff functions are not well matched.
With entropy estimators determined from QEFs optimized for powers near zero, the EAT performance improves substantially. In some cases, the performance is better than the EAT with the min-tradeoff function given in Ref. [5]. We remark that this comparison does not take advantage of the improvements to the EAT implied by Thm. 6.3.

For the last example, we consider the problem of producing 512 bits at smoothness error $\epsilon = 2^{-64}$ and probability of success parameter $\kappa = 2^{-64}$ with trials whose results distribution matches that observed in the photonic loophole-free randomness generation experiment reported in Ref. [13]. For this, we do not consider the overhead of extracting the random bits and ask for the minimum number of trials for which 512 bits of smooth conditional min-entropy can be certified at the given $\epsilon, \kappa$. We optimized the minimum number of trials required according to Eq. [8.39] over PEFs and powers, assuming that the PEFs are QEFs. We confirmed that the best PEF found has $f_{\text{max}} \leq 1 + 9.88 \times 10^{-9}$, which we verified with Mathematica at a precision of 10$^{-32}$. The QEF thus found requires $6.97 \times 10^7$ trials on average. For entropy accumulation, $2.89 \times 10^{11}$ trials are required, as reported in Ref. [40]. Given the trial rate in the experiment of Ref. [13], this would require 11.62 minutes of experimental time with QEFs, and 802.1 hours with entropy accumulation.

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arXiv revision notes:

V1. Original submission.

V2. First revision.

1. Clarified soundness definitions in Sect. 5.1 and corrected the discussion of dependence and extension to initial classical variables.

2. Added remarks on and references to Dupuis and Fawzi’s second-order improvement of the EAT [6].

3. Clarified definition of expected quantum net log-prob.

4. Miscellaneous clarifications and minor corrections.

V3. Second revision.

1. Clarified the comparison to EAT in Sect. 8.4 and improved the treatment and discussion of the probability of success parameter \( \kappa \).

2. Added references to our published papers based on this work. Ref. [9] covers the basic theory of QEFs for randomness generation and Ref. [10] describes an experimental implementation for repeated and low-latency production of blocks of 512 random bits.
FIG. 1. Maximum error bound rates versus entropy threshold rates. The left plot has $k_\infty = 1$, the right has $k_\infty = \log(N)$, where $N = |\text{Rng}(C)|$. Three pairs of curves are shown in each plot, for $N = 2, 4, 8$ as labeled. The dashed lines show the EAT curves, and the solid lines show the QEFP curves according to the handicapped calculations in the text. From the maximum error bound rate $r_{\text{max}}$ one can estimate the minimum number $n_{\text{min}}$ of trials required for positive smooth conditional min-entropy with an error bound of $\epsilon$ at probability of success $\kappa = 1$. The estimate is given by $n_{\text{min}} = |\log(\epsilon^2/2)|/r_{\text{max}}$. The higher QEFP curves imply about half the number of trials are required. Further improvements are possible by taking full advantage of Thm. 6.3 and its proof. Achievable entropy threshold rates are determined by the entropy estimator and the trial probability distribution.
FIG. 2. QEF advantage factors for $P_E$ and $P_P$ as a function of $\hat{I}$. Shown are values for $f_{\nu_{E,\theta},T}$ and $f_{\nu_{P,\eta},T}$. We verified that the quantity $f_{\text{max}}$ is indistinguishable from 1 at high precision for each of the points indicated by open circles.