EVOLUTIONARY SYSTEM, GLOBAL ATTRACTOR, TRAJECTORY ATTRACTOR AND APPLICATIONS TO THE NONAUTONOMOUS REACTION-DIFFUSION SYSTEMS

SONGSONG LU

ABSTRACT. This paper may be regarded as a continuation of a previous paper of Cheskidov and Lu [Adv. Math., 267(2014), 277-306], where develops a new framework of the evolutionary system that deals direct with the notion of a uniform global attractor due to Haraux, and by which a trajectory attractor is able to be defined for the original system under consideration. The notion of a trajectory attractor was previously constructed for a system without uniqueness by considering a family of auxiliary systems including the original one. In this paper, we define and prove the existence of a notion of a strongly compact strong trajectory attractor if the system is asymptotically compact. As a consequence, we obtain the strong equicontinuity of all complete trajectories on global attractor and the finite strong uniform tracking property. We remark that global attractor and trajectory attractor are two representational aspects of some single object, emphasizing the properties of attracting and uniform tracking trajectories respectively. Then we apply the theory to a general nonautonomous reaction-diffusion systems. In particular, we obtain the structure of uniform global attractors without any additional condition on nonlinearity other than those guarantee the existence of a uniform absorbing set, which answers part of open problems in [Asymptot. Anal., 54(2007), 197-210, Discrete Contin. Dyn. Syst. Ser. S, 2(2009), 55-66]. Finally, we construct some interesting examples of such nonlinearities. It is not known whether they can be handled by previous frameworks.

Keywords: evolutionary system, uniform global attractor, trajectory attractor, reaction-diffusion system, normal external force

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1. INTRODUCTION

A mathematical object describing the long-time behavior of an autonomous infinite dimensional dissipative dynamical system with uniqueness is the global attractor, which is a minimal closed set that attracts all the trajectories starting from a bounded set in the phase space. The global attractor consists of points on complete bounded trajectories, that is, its structural representation is given as a section at a
fixed time of the kernel, a collection of all complete bounded trajectories, of the system.

The natural generalization of the notion of a global attractor to the nonautonomous dynamical system is that of a uniform global attractor, initiated by Haraux [Ha91, defined with the attracting uniformly with respect to (w.r.t.) the initial time. Chepyzhov and Vishik [CV94, CV02] put forward an auxiliary notion of a uniform w.r.t. the symbol space attractor to study the structure of the uniform global attractor, based on the use of the so-called time symbol and constructing a symbol space as a suitable closure of the translation family of the symbol of the original system under consideration. However, the uniform w.r.t. the symbol space attractor does not always have to be identical to the original uniform global attractor (see [CL14]). On the other hand, the open problems in [Lu07, CL09] indicate that there may not exist such a required symbol space.

For the dynamical system without uniqueness, instead, the concept of a trajectory attractor, a global attractor in the trajectory space, was first introduced in [Se96] and further studied in [CV97, CV02, SY02]. Their constructions consider also a family of auxiliary systems containing the original one. The sections of such trajectory attractors are defined as, for the autonomous case, the global attractors, or for nonautonomous case, the uniform w.r.t the symbol space attractors. Similarly, the Open Problem 6.7 in [CL14] shows that it is not clear whether these trajectory attractors satisfy the minimality property for the original nonautonomous system due to the nonuniqueness feature of the system. In consequence, there is no information on the uniform global attractors.

Most recently, Cheskidov and Lu [CL14] developed, based on the previous studies [CF06, C09, CL09], a unified framework of evolutionary system to investigate the global attractors and the trajectory attractors for dissipative dynamical systems, no matter they are autonomous or nonautonomous, unique or nonunique. Especially, for the nonautonomous case, the new approach deals directly with the notion of a uniform global attractor and avoids the necessity of constructing a symbol space. Together with the advantage of a simultaneous use of weak and strong metric, this method is applicable to arbitrary dissipative PDE, without regard to uniqueness, autonomy and existence of a symbol space.

According to the theory in [CL14], an evolutionary system always processes a weak global attractor. Its structure is obtained via that of the weak global attractor for the closure of the evolutionary system under an assumption (see A1) satisfied by any dissipative PDE, since these two weak global attractors coincide. Moreover, a weak uniform tracking property is proved, which means that we can approximate in weak metric arbitrarily closely every trajectory of the evolutionary system for arbitrarily large time lengths by the trajectories on the weak global attractor after sufficiently long time. In particular, for the nonautonomous case, a trajectory attractor is naturally constructed for the originally considered system, rather that for
a family of systems. The sections of the trajectory attractor are the weak global attractor. In fact, in applications of previous frameworks (see e.g. [CV02, SY02]), the constructions of trajectory attractors are related to the phase spaces endowed with weaker topologies than the usual strong ones.

The weak global attractor becomes a strongly compact strong global attractor, if the evolutionary system is asymptotically compact. At the same time, the property of tracking is valid in strong metric and trajectories converge strongly toward the trajectory attractor. In this paper, we continue studying the properties of the evolutionary system, concerning on the trajectory attractor. More precisely, we define and prove the existence of a notion of a strongly compact strong trajectory attractor, which sheds light on some point of view.

We prove that the trajectory attractor is compact in the space of continuous functions of time with values in the phase space endowed with the usual strong topology under consideration. As a consequence, we obtain a finite strong uniform tracking property that for fixed accuracy and time length, a finite number of trajectories on the global attractor are able to capture in strong metric all trajectories after sufficiently large time. Moreover, all complete trajectories on global attractor is strongly equicontinuous. We remark that the weak and strong uniform tracking properties are essentially equivalent to the weak and strong trajectory attracting, respectively. With these properties in hand, together the strong convergence of the trajectories to the trajectory attractor, the concept of a strongly compact strong trajectory attractor crystalizes a new point of view that the global attractor, a section of the kernel of the closure of evolutionary system, and the trajectory attractor, the restriction of the same kernel to the semiaxis \([0, \infty)\), are two representational aspects of some single object, emphasizing the properties of attracting and uniform tracking trajectories respectively. It is convenient to call it (weak or strong) attractor. Notice again that such a weak/strong attractor is constructed for the original system we consider. Our theory shows that, even though the trajectories of an evolutionary system might not be strongly continuous, they are asymptotically strongly equicontinuous if the asymptotical compactness is provided.

Summarily, we obtain a strategy to study the dissipative dynamical systems. For instance, applying to PDEs, we first find out an absorbing set, endow a weak metric that makes it compact, and verify the condition A1. These steps are relatively easy. Then we focus on getting the asymptotical strong compactness. Once it succeed, the main results on the attractor hold in corresponding strong metrics. In this paper, we apply this procedure to a dissipative reaction-diffusion system (RDS) that is a fundamental model in the theory of infinite dimensional dynamical systems. It is quite general that covers many examples arising in physics, chemistry and biology etc. We just list a few: the RDS with polynomial nonlinearity, Ginzburg-Landau equation, Chafee-Infante equation, Fitz-Hugh-Nagumo equations and Lotka-Volterra competition system. See e.g. [M87, T88, CV96, Ro01, CV02, SY02] for more.
In the current paper, the RDS also serves as a canonical example that is nonautonomous, nonunique and lack of an appropriate symbol space.

The paper is organized as follows. In Section 2, we briefly recall the basic definitions of the theory of evolutionary system originally designed in \[\text{CF06, C09}\] for autonomous systems and developed in \[\text{CL09, CL14}\] especially for nonautonomous systems. Then we clarify the equivalence of the uniform tracking properties and those of trajectory attracting in Section 3. In particular, we define a strongly compact strong trajectory attractor as well as the finite strong uniform tracking property. Its existence and properties mentioned above are proved in Section 4. We present the results incorporating with the theory of \[\text{CL14}\].

In Section 5, we investigate the RDS with a fixed time-dependent nonlinearity and a driving force. The nonlinearity only satisfies the continuity, dissipativeness and growth conditions that do not guarantee the uniqueness of the solution and the assumption on the force is a translation boundedness condition, which is the weakest condition that ensures the existence of a bounded uniform absorbing ball. We take this ball as a phase space and the weak and strong metrics are those induced by the usual weak and strong topologies respectively. We verify that the weak solutions of RDS form an evolutionary system satisfying A1. Therefore, we obtain the structure of the weak attractor (both the uniform global attractor and the trajectory attractor). In addition, if the force is normal then the weak attractor is a strong attractor. The normality condition on the force was introduced in \[\text{LWZ05}\] and \[\text{Lu06}\] by different methods. Now, it follows naturally from the energy inequality of a criterion of asymptotical compactness, known for the 3D Navier-Stokes equations \[\text{CF06}\] and formulated in \[\text{C09}\] and generalized to general cases in \[\text{CL09, CL14}\]. It is worth to mention that the time-dependence of the nonlinearity is not imposed on any additional assumption, for instance, which promises the existence of a symbol space. Hence, we give an answer to part of open problems in \[\text{Lu07}\] and \[\text{CL09}\].

In Section 6, we first review concisely some results on RDS obtained by previous works \[\text{CV02, Lu07, CL09}\], supposing additional conditions on the nonlinearity. Then, we recove them using our framework and discuss their relation to those obtained in this paper. Finally, we construct some interesting examples of nonlinearities that only satisfy the continuity, dissipativeness and growth conditions. The extra assumptions imposed by previous works \[\text{CV02, Lu07, CL09}\] do not hold any more. The pointwise limit function of one example as \[t \to +\infty\] is discontinuous and the others have even no pointwise limit functions. These facts mean that the weak closure of the translation family of every example in the space of continuous functions of time with values in corresponding topological space is not complete. In other words, there does not exist a suitable symbol space required in previous studies.
2. Evolutionary system

Now we recall briefly the basic definitions on evolutionary systems (see [C09, CL14] for details). Assume that a set $X$ is endowed with two metrics $d_s(\cdot, \cdot)$ and $d_w(\cdot, \cdot)$ respectively, satisfying the following conditions:

1. $X$ is $d_s$-compact.
2. If $d_s(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$, then $d_w(u_n, v_n) \to 0$ as $n \to \infty$.

Hence, we will refer to $d_s$ as a strong metric and $d_w$ as a weak metric. Let $\overline{A^*}$ be the closure of a set $A \subset X$ in the topology generated by $d_s$. Here (the same below) $\bullet = s$ or $w$. Note that any strongly compact ($d_s$-compact) set is weakly compact ($d_w$-compact), and any weakly closed set is strongly closed.

Let

$$\mathcal{T} := \{I : I = [T, \infty) \subset \mathbb{R}, \text{ or } I = (-\infty, \infty)\},$$

and for each $I \subset \mathcal{T}$, let $\mathcal{F}(I)$ denote the set of all $X$-valued functions on $I$. Now we define an evolutionary system $\mathcal{E}$ as follows.

**Definition 2.1.** A map $\mathcal{E}$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathcal{F}(I)$ will be called an evolutionary system if the following conditions are satisfied:

1. $\mathcal{E}([0, \infty)) \neq \emptyset$.
2. $\mathcal{E}(I + s) = \{u(\cdot) : u(\cdot + s) \in \mathcal{E}(I)\}$ for all $s \in \mathbb{R}$.
3. $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)$ for all pairs $I_1, I_2 \in \mathcal{T}$, such that $I_2 \subset I_1$.
4. $\mathcal{E}((-\infty, \infty)) = \{u(\cdot) : u(\cdot)\vert_{[T, \infty)} \in \mathcal{E}([T, \infty)) \forall T \in \mathbb{R}\}$.

We will refer to $\mathcal{E}(I)$ as the set of all trajectories on the time interval $I$. $\mathcal{E}((-\infty, \infty))$ is called the kernel of $\mathcal{E}$ and its trajectories are called complete. Let $P(X)$ be the set of all subsets of $X$. For every $t \geq 0$, define a set-valued map

$$R(t) : P(X) \to P(X),$$

$$R(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}([0, \infty))\}, \quad A \subset X.$$ 

Note that the assumptions on $\mathcal{E}$ imply that $R(t)$ enjoys the following property:

$$R(t + s)A \subset R(t)R(s)A, \quad A \subset X, \quad t, s \geq 0.$$

**Definition 2.2.** A set $A_\bullet \subset X$ is a $d_\bullet$-global attractor if $A_\bullet$ is a minimal set that is

1. $d_\bullet$-closed.
2. $d_\bullet$-attracting: for any $B \subset X$ and $\epsilon > 0$, there exists $t_0$, such that

$$R(t)B \subset B_\bullet(A_\bullet, \epsilon) := \{u : \inf_{x \in A_\bullet} d_\bullet(u, x) < \epsilon\}, \quad \forall t \geq t_0.$$

**Definition 2.3.** The $\omega_\bullet$-limit of a set $A \subset X$ is

$$\omega_\bullet(A) := \bigcap_{T \geq 0} \bigcup_{t \geq T} R(t)A.$$
In order to extend the notion of invariance from a semiflow to an evolutionary system, we will need the following mapping:
\[ \tilde{R}(t)A := \{ u(t) : u(0) \in A, u \in \mathcal{E}((\mathbb{R}, \mathbb{R})) \}, \quad A \subset X, \quad t \in \mathbb{R}. \]

**Definition 2.4.** A set \( A \subset X \) is positively invariant if
\[ \tilde{R}(t)A \subset A, \quad \forall t \geq 0. \]

A is invariant if
\[ \tilde{R}(t)A = A, \quad \forall t \geq 0. \]

A is quasi-invariant if for every \( a \in A \) there exists a complete trajectory \( u \in \mathcal{E}((\mathbb{R}, \mathbb{R})) \) with \( u(0) = a \) and \( u(t) \in A \) for all \( t \in \mathbb{R} \).

Let \( \Sigma \) be a parameter set and \( \{ T(s) | s \geq 0 \} \) be a family of operators acting on \( \Sigma \) satisfying \( T(s)\Sigma = \Sigma \), \( \forall s \geq 0 \). Any element \( \sigma \in \Sigma \) will be called (time) symbol and \( \Sigma \) will be called (time) symbol space. For instance, in many applications \( \{ T(s) \} \) is the translation semigroup and \( \Sigma \) is the translation family of the time-dependent items of the considered system or its closure in some appropriate topological space (for more examples see [CV02, CL14], the appendix in [CLR13]).

**Definition 2.5.** A family of maps \( E_\sigma, \sigma \in \Sigma \) that for every \( \sigma \in \Sigma \) associates to each \( I \in \mathcal{T} \) a subset \( E_\sigma(I) \subset \mathcal{F}(I) \) will be called a nonautonomous evolutionary system if the following conditions are satisfied:

1. \( E_\sigma([\tau, \infty)) \neq \emptyset, \forall \tau \in \mathbb{R} \).
2. \( E_\sigma(I + s) = \{ u(\cdot) : u(\cdot + s) \in E_{T(s)\sigma}(I) \}, \forall s \geq 0 \).
3. \( \{ u(\cdot)|_{I_2} : u(\cdot) \in E_\sigma(I_1) \} \subset E_\sigma(I_2), \forall I_1, I_2 \in \mathcal{T}, I_2 \subset I_1 \).
4. \( E_\sigma((\mathbb{R}, \mathbb{R})) = \{ u(\cdot) : u(\cdot)|_{[\tau, \infty)} \in E_\sigma([\tau, \infty)), \forall \tau \in \mathbb{R} \}. \)

It is shown in [CL09, CL14] that a nonautonomous evolutionary system can be reduced to an (autonomous) evolutionary system. Consequently, the above notions of invariance, quasi-invariance, and a global attractor are applicable. The global attractor in the nonautonomous case will be conventionally called a uniform global attractor (or simply a global attractor). However, for some evolutionary systems constructed from nonautonomous dynamical systems the associated symbol spaces are not known. See [CL14] and the following sections below for more details. Thus, we will not distinguish between autonomous and nonautonomous evolutionary systems. If it is necessary, we denote an evolutionary system with a symbol space \( \Sigma \) by \( E_\Sigma \) and its attractor by \( A_\Sigma \).

**Definition 2.6.** Let \( \mathcal{E} \) be an evolutionary system. If a map \( \mathcal{E}^1 \) that associates to each \( I \in \mathcal{T} \) a subset \( \mathcal{E}^1(I) \subset \mathcal{F}(I) \) is also an evolutionary system, we will call it an evolutionary subsystem of \( \mathcal{E} \), and denote by \( \mathcal{E}^1 \subset \mathcal{E} \).

**Definition 2.7.** An evolutionary system \( \mathcal{E}_\Sigma \) is a system with uniqueness if for every \( u_0 \in X \) and \( \sigma \in \Sigma \), there is a unique trajectory \( u \in \mathcal{E}_\sigma([0, \infty)) \) such that \( u(0) = u_0 \).
3. Uniform Tracking Property and Trajectory Attractor

An important property of a global attractor is that it captures a long-time behavior of every trajectory of an evolutionary system.

Denote by \( C([a, b]; X) \) the space of \( d \)-continuous \( X \)-valued functions on \([a, b]\) endowed with the metric

\[
d_C([a, b]; X)(u, v) := \sup_{t \in [a, b]} d_X(u(t), v(t)).
\]

Let also \( C([a, \infty); X) \) be the space of \( d \)-continuous \( X \)-valued functions on \([a, \infty)\) endowed with the metric

\[
d_C([a, \infty); X)(u, v) := \sum_{l \in \mathbb{N}} \frac{1}{2^l} \left( \frac{1}{1 + d_C([a, a+l]; X)(u, v)} \right).
\]

Note that the convergence in \( C([a, \infty)) \) is equivalent to uniform convergence on compact sets.

Now we suppose that evolutionary systems \( E \) satisfy

\[
E([0, \infty)) \subset C([0, \infty); X_w).
\]

**Definition 3.1.** A set \( P \subset C([0, \infty); X_w) \) satisfies the weak uniform tracking property if for any \( \epsilon > 0 \), there exists \( t_0 \), such that for any \( t^* > t_0 \), every trajectory \( u \in E([0, \infty)) \) satisfies

(1) \[
d_C([t^*, \infty); X_w)(u, v) < \epsilon,
\]

for some trajectory \( v \in P \). Furthermore, if there exists a finite subset \( P_f \subset P \), such that (1) holds for some \( v \in P_f \), then \( P \) satisfies the finite weak uniform tracking property.

We may use the concept of trajectory attracting to restate the weak uniform tracking property. Define the translation operators \( T(s) \), \( s \geq 0 \),

\[
(T(s)u)(t) := u(t+s)|_{[0, \infty)}, \quad u \in C([0, \infty); X_w).
\]

Due to the property 3 of the evolutionary system (see Definitions 2.1), we have that,

\[
T(s)E([0, \infty)) \subset E([0, \infty)), \quad \forall s \geq 0.
\]

Note that \( E([0, \infty)) \) may not be closed in \( C([0, \infty); X_w) \). We consider the dynamics of the translation semigroup \( \{T(s)\}_{s \geq 0} \) acting on the phase space \( C([0, \infty); X_w) \). A set \( P \subset C([0, \infty); X_w) \) uniformly (weakly) attracts a set \( Q \subset E([0, \infty)) \) if for any \( \epsilon > 0 \) there exists \( t_0 \), such that

\[
T(t)Q \subset \{u \in C([0, \infty); X_w) : d_C([0, \infty); X_w)(P, u) < \epsilon\}, \quad \forall t \geq t_0.
\]

**Definition 3.2.** A set \( P \subset C([0, \infty); X_w) \) is a (weak) trajectory attracting set for an evolutionary system \( E \) if it uniformly (weakly) attracts \( E([0, \infty)) \).
Obviously, a trajectory attracting set is a set that satisfies the weak uniform tracking property, vice versa.

**Definition 3.3.** A set \( \mathcal{A}_w \subset C([0, \infty); X_w) \) is a (weak) trajectory attractor for an evolutionary system \( E \) if \( \mathcal{A}_w \) is a minimal (weak) trajectory attracting set that is

1. closed in \( C([0, \infty); X_w) \).
2. invariant: \( T(t)\mathcal{A}_w = \mathcal{A}_w, \forall t \geq 0 \).

It is said that \( \mathcal{A}_w \) is (weakly) compact if it is compact in \( C([0, \infty); X_w) \).

It is easy to see that if a trajectory attractor exists, it is unique, and if it is compact, it is the minimal weakly closed and invariant set satisfying the finite weak uniform tracking property.

The above definitions have strong versions.

**Definition 3.4.** A set \( P \subset C([0, \infty); X_w) \) satisfies the strong uniform tracking property if for any \( \epsilon > 0 \) and \( T > 0 \), there exists \( t_0 \), such that for any \( t^* > t_0 \), every trajectory \( u \in E([0, \infty)) \) satisfies

2. \( d_s(u(t), v(t)) < \epsilon, \forall t \in [t^*, t^* + T] \),

for some trajectory \( v \in P \). Furthermore, if there exists a finite subset \( P^f \subset P \), such that (2) holds for some \( v \in P^f \), then \( P \) satisfies the finite strong uniform tracking property.

**Definition 3.5.** A set \( P \subset C([0, \infty); X_w) \) is a strong trajectory attracting set for an evolutionary system \( E \) if it uniformly attracts \( E([0, \infty)) \) in \( L^\infty_{\text{loc}}((0, \infty); X_s) \).

Similarly, the strong uniform tracking property is in fact a uniform strong convergence of trajectories toward a strong trajectory attracting set.

**Definition 3.6.** A set \( \mathcal{A}_s \subset C([0, \infty); X_w) \) is a strong trajectory attractor for an evolutionary system \( E \) if \( \mathcal{A}_s \) is a minimal strong trajectory attracting set that is

1. closed in \( C([0, \infty); X_w) \).
2. invariant: \( T(t)\mathcal{A}_s = \mathcal{A}_s, \forall t \geq 0 \).

It is said that \( \mathcal{A}_s \) is strongly compact if it is compact in \( C([0, \infty); X_s) \).

Analogously, if a strong trajectory attractor exists, it is unique, and a strongly compact strong trajectory attractor is the minimal strongly closed and invariant set satisfying the finite strong uniform tracking property.

### 4. Attractor for Evolutionary System

We will investigate evolutionary systems \( E \) satisfying the following property:

A1 \( E([0, \infty)) \) is a precompact set in \( C([0, \infty); X_w) \).
Such kinds of evolutionary systems are closely related to the concept of the uniform with respect to (w.r.t.) the initial time global attractor for a nonautonomous dynamical system, initiated by Haraux [Ha91]. For instance, the uniform global attractor for the evolutionary system defined by a nonautonomous partial differential equation of mathematical physics, with $\Sigma$ in Definition 2.5 taken as the translation family of the time-dependent items of the original equation, is the uniform w.r.t. the initial time global attractors for the same equation due to Haraux, and the evolutionary system satisfies $A1$ in general. For more details see [CL09, CL14].

Let $\bar{E}(\tau, \infty) := E(\tau, \infty) \subset C(\tau, \infty); X_w), \forall \tau \in \mathbb{R}$.

It can be checked that $\bar{E}$ is also an evolutionary system. We call $\bar{E}$ the closure of the evolutionary system $E$, and add the top-script $\bar{}$ to the corresponding notations in previous sections for $E$. For example, we denote by $\bar{A}_w$ the uniform $d_*$-global attractor for $\bar{E}$. Obviously, $\bar{E}$ satisfies the following stronger version of $A1$:

$\bar{A}1 \ E([0, \infty))$ is a compact set in $C([0, \infty); X_w)$.

Note that for some evolutionary systems $E$, say, those generated by autonomous dynamical systems, $\bar{E} = E$. However, instead of condition $\bar{A}1$, the nonautonomous evolutionary systems $E$ usually only satisfy $A1$. Moreover, for some nonautonomous evolutionary systems, as we will see below, there may not exist symbol spaces associated to their closures $\bar{E}$.

With comments in previous section in mind, Theorems 3.5 and 4.3 in [CL14] are retold in following form.

**Theorem 4.1.** Let $E$ be an evolutionary system. Then

1. The weak global attractor $A_w$ exists, and $A_w = \omega_w(X)$.

Furthermore, assume that $E$ satisfies $A1$. Let $\bar{E}$ be the closure of $E$. Then

2. $A_w = \omega_w(X) = \bar{\omega}_w(X) = \bar{\omega}_d(X) = \bar{A}_w$.

3. $A_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. $\bar{E}$:

   $A_w = \{u_0 \in X : u_0 = u(0) \text{ for some } u \in \bar{E}((-\infty, \infty))\}$.

4. The weak trajectory attractor $\mathcal{A}_w$ exists, it is weakly compact, and

   $\mathcal{A}_w = \Pi_+ \bar{E}((-\infty, \infty)) := \{u(\cdot)|_{[0, \infty)} : u \in \bar{E}((-\infty, \infty))\}$.

   Hence, the finite weak uniform tracking property holds.

5. $A_w$ is a section of $\mathcal{A}_w$:

   $A_w = \mathcal{A}_w(t) := \{u(t) : u \in \mathcal{A}_w\}, \forall t \geq 0$.

**Definition 4.2.** The evolutionary system $E$ is asymptotically compact if for any $t_k \to \infty$ and any $x_k \in R(t_k)X$, the sequence $\{x_k\}$ is relatively strongly compact.
We have a stronger version of Theorems 3.6 and 4.4 in [CL14].

**Theorem 4.3.** Let \( E \) be an asymptotically compact evolutionary system. Then

1. The strong global attractor \( A_s \) exists, it is strongly compact, and \( A_s = A_w \).

Furthermore, assume that \( E \) satisfies A1. Let \( \bar{E} \) be the closure of \( E \). Then

2. The strong trajectory attractor \( A_s \) exists and \( A_s = A_w \), it is strongly compact. Hence, the finite strong uniform tracking property holds, i.e., for any \( \epsilon > 0 \) and \( T > 0 \), there exist \( t_0 \) and a finite subset \( P^f \subset A_s = \Pi_A \bar{E}((-\infty, \infty)) \), such that for any \( t^* > t_0 \), every trajectory \( u \in E([0, \infty)) \) satisfies \( d_s(u(t), v(t)) < \epsilon, \forall t \in [t^*, t^* + T] \), for some trajectory \( v \in P^f \).

**Proof.** Conclusion 1 is that of Theorem 3.6 in [CL14]. Due to Theorem 4.4 in [CL14] and Definition 3.5, the weak trajectory attractor \( A_w \) is a strong trajectory attracting set compact in \( C([0, \infty); X_w) \). If \( P \subset C([0, \infty); X_w) \) is any other strong trajectory attracting set compact in \( C([0, \infty); X_w) \), \( P \) is also a compact weak trajectory attracting set. Hence \( A_w \subset P \). This concludes that, according to Definition 3.6, \( A_w \) is indeed a strong trajectory attractor \( A_s \).

The remains is to demonstrate the strong compactness of \( A_s \) in \( C([0, \infty); X_s) \). First, we have \( A_s \subset C([0, \infty); X_s) \). In fact, thanks to Theorem 4.1 for every \( u \in A_s \),

\[
\{u(t) : t \in [0, \infty)\} \subset A_s,
\]

is compact in \( X_s \). Hence, any weakly convergent sequence \( u(n) \) with limit \( u(t_0) \) as \( n \to \infty \) does strongly converge to \( u(t_0) \), which implies \( u \in C([0, \infty); X_s) \).

Note that \( A_s \) is compact in \( C([0, \infty); X_w) \). Now take a sequence \( \{u_n(t)\} \subset A_s \) that converges to \( u(t) \) in \( C([0, \infty); X_w) \). We claim that the convergence is indeed in \( C([0, \infty); X_s) \). Otherwise, there exist \( \epsilon > 0, T > 0 \), and sequences \( \{n_j\}, n_j \to \infty \) as \( j \to \infty \) and \( \{n_j\} \subset [0, T] \), such that

\[
d_s(u(n_j(t)), u(t)) > \epsilon, \forall n_j.
\]

The sequences

\[
\{u_{n_j}(t)\}, \{u(t)\} \subset A_s
\]

are relatively strongly compact due to the strong compactness of \( A_s \). Passing to a subsequence and dropping a subindex, we obtain that \( \{u_{n_j}(t)\} \) and \( \{u(t)\} \) are strongly convergent with limits \( x \) and \( y \), respectively. We have

\[
d_w(x, y) \leq d_w(u_{n_j}(t), x) + d_w(u_{n_j}(t), u(t)) + d_w(u(t), y), \forall n_j.
\]

By the assumption,

\[
\lim_{j \to \infty} \sup_{t \in [0, T]} d_w(u_{n_j}(t), u(t)) = 0.
\]

It follows that \( x = y \), which is a contradiction to (3). \( \square \)
Corollary 4.4. Let $\mathcal{E}$ be an asymptotically compact evolutionary system satisfying A1, and $\overline{\mathcal{E}}$ be its closure. Then the kernel $\overline{\mathcal{E}}\left(\left(-\infty, \infty\right)\right)$ of $\mathcal{E}$ is equicontinuous on $\mathbb{R}$, i.e.,
\[
d_s \left(v(t_1), v(t_2)\right) \leq \theta \left(|t_1 - t_2|\right), \quad \forall t_1, t_2 \in \mathbb{R}, \forall v \in \mathcal{E}\left(\left(-\infty, \infty\right)\right),
\]
where $\theta(l)$ is a positive function tending to 0 as $l \to 0^+$.

Proof. First, by Theorem 4.3, $\Pi_{+}\overline{\mathcal{E}}\left(\left(-\infty, \infty\right)\right)$ is compact in $C\left([0, \infty); X_s\right)$.

Now, without loss of generality, we assume that $|t_1 - t_2| \leq 1$. Hence, $t_1, t_2$ belong to some interval $[T, T + 2]$. Denote by
\[
\Pi_{[a,b]}\overline{\mathcal{E}}\left(\left(-\infty, \infty\right)\right) := \{u(\cdot)|_{[a,b]} : u \in \overline{\mathcal{E}}\left(\left(-\infty, \infty\right)\right)\}.
\]
Notice that
\[
\{v(\cdot) + T : v(\cdot) \in \Pi_{[T, T + 2]}\overline{\mathcal{E}}\left(\left(-\infty, \infty\right)\right)\} = \Pi_{[0, 2]}\overline{\mathcal{E}}\left(\left(-\infty, \infty\right)\right).
\]
Thus, we need only to verify that $\Pi_{[0, 2]}\overline{\mathcal{E}}\left(\left(-\infty, \infty\right)\right)$ is equicontinuous on $[0, 2]$. Thanks to the Arzelá-Ascoli compactness criterion, this follows from the fact that $\Pi_{[0, 2]}\overline{\mathcal{E}}\left(\left(-\infty, \infty\right)\right)$ is compact in $C\left([0, 2]; X_s\right)$. \qed

Remark 4.5. We give some comments.

1. Theorem 4.3 and Corollary 4.4 indicate that the notion of a strongly compact strong trajectory attractor is an apt description of the deep results of finite strong uniform tracking property and equicontinuity.

2. Comparing with Theorem 4.1, Theorem 4.3 implies a strategy to study the long-time behavior of an evolutionary system. We first work on the phase space endowed with a weak topology. Then we need only to obtain the asymptotical compactness in a stronger topology to improve the results. In particular, the strong compactness of strong global attractor and strong trajectory attractor follow simultaneously.

3. Theorems 4.1 and 4.3 show that the notions of a global attractor and a trajectory attractor may be viewed as two different representational aspects of a single object, emphasizing the properties of attracting and uniform tracking trajectories respectively. Hence, it is convenient to call it weak/strong attractor for the evolutionary system.

Applying to, for instance, in [Lu06], the 2D Navier-Stokes equations with non-slip boundary condition, if external force is normal (see Section 5) in $L_{x}^{2}\left(\mathbb{R}; V^\prime\right)$, where $V^\prime$ is the dual of the space of divergence-free vector fields with square-integrable derivatives and vanishing on the boundary, Theorem 4.3 provides the existence of strongly compact strong trajectory attractor. As a consequence, the finite strong uniform tracking property holds. In 3D case, once the strong continuity of the trajectories on the weak trajectory attractor has been proved (see [CL14]), the similar result follows.
Now, we present some results for evolutionary systems satisfying these additional properties:

A2 (Energy inequality) Assume that $X$ is a set in some Banach space $H$ satisfying the Radon-Riesz property (see below) with the norm denoted by $|\cdot|$, such that $d(s, y) = |x - y|$ for $x, y \in X$ and $d_w$ induces the weak topology on $X$. Assume also that for any $\epsilon > 0$, there exists $\delta > 0$, such that for every $u \in E([0, \infty))$ and $t > 0$,

$$|u(t)| \leq |u(t_0)| + \epsilon,$$

for $t_0$ a.e. in $(t - \delta, t)$.

A3 (Strong convergence a.e.) Let $u_k \in E([0, \infty))$ be such that, $u_k$ is $d_C([0, T]; X_{w})$-Cauchy sequence in $C([0, T]; X_{w})$ for some $T > 0$. Then $u_k(t)$ is $d_s$-Cauchy sequence a.e. in $[0, T]$.

A Banach space $B$ satisfies the Radon-Riesz property if

$$x_n \rightarrow x \text{ strongly in } B \Leftrightarrow \begin{cases} x_n \rightharpoonup x \text{ weakly in } B \\ \|x_n\|_B \rightarrow \|x\|_B \end{cases}.$$ 

In many applications, $H$ in A2 is a uniformly convex separable Banach space and $X$ is a bounded closed set in $H$. Then the weak topology of $H$ is metrizable on $X$, and $X$ is compact w.r.t. such a metric $d_w$. Moreover, the Radon-Riesz property is automatically satisfied in this case.

**Theorem 4.6.** [CL14] Let $E$ be an evolutionary system satisfying A1, A2, and A3, and assume that its closure $\tilde{E}$ satisfies $\tilde{E}(((-\infty, \infty)) \subset C(((-\infty, \infty); X_{\delta})$. Then $E$ is asymptotically compact.

5. **ATTRACTORS FOR REACTION-DIFFUSION SYSTEMS**

We study the long time behavior of the the solutions of the following nonautonomous reaction-diffusion system (RDS):

$$\begin{align*}
\partial_t u - a \Delta u + f(u, t) &= g(x, t), & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega, \\
u|_{t=\tau} &= u_{\tau}, & \tau \in \mathbb{R}.
\end{align*}$$

(RDS)

Here $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a boundary $\partial \Omega$ of sufficient smoothness, $a = \{a_{ij}\}_{ij=1,\ldots,N}$ is an $N \times N$ real matrix with positive symmetric part $\frac{1}{2}(a + a^*) \geq \beta I$, $\beta > 0$; $u = u(x, t) = (u^1, \ldots, u^N)$ is the unknown function, $g = (g^1, \ldots, g^N)$ is the driving force and $f = (f^1, \ldots, f^N)$ is the interaction function. Denote the spaces $H = (L^2(\Omega))^N$ and $V = (H^1_0(\Omega))^N$ with norms $|\cdot|$ and $\|\cdot\|$, respectively. Let $V'$ be the dual of $V$. Assume that $g(s) = g(\cdot, s)$ is translation bounded (tr.b.) in
that for the sake of simplicity, we may assume $p_k := (p_k - 1)$, $r_k := \max\{1, n(1/2 - 1/p_k)\}$, $k = 1, \ldots, N$, and denote $p := (p_1, \ldots, p_N)$, $q := (q_1, \ldots, q_N)$, $r := (r_1, \ldots, r_N)$ and

- $L^p(\Omega) := L^{p_1}(\Omega) \times L^{p_2}(\Omega) \times \cdots \times L^{p_N}(\Omega)$,
- $H^{-r}(\Omega) := H^{-r_1}(\Omega) \times H^{-r_2}(\Omega) \times \cdots \times H^{-r_N}(\Omega)$,
- $L^p(\tau, T; L^p(\Omega)) := L^{p_1}(\tau, T; L^{p_1}(\Omega)) \times \cdots \times L^{p_N}(\tau, T; L^{p_N}(\Omega))$,
- $L^q(\tau, T; H^{-r}(\Omega)) := L^{q_1}(\tau, T; H^{-r_1}(\Omega)) \times \cdots \times L^{q_N}(\tau, T; H^{-r_N}(\Omega))$.

We recall the results on the existence of weak solutions to (RDS) (see e.g. [CV02]). Note that conditions (4)-(5) do not assure the uniqueness of the solutions.

**Theorem 5.1.** For every $u_r \in H$ and $g \in L^2_{\text{loc}}(\mathbb{R}; V')$, there exists a weak solution $u(t)$ of (RDS) satisfying

\begin{equation}
\|u\|_{H}^2 = \|g\|_{L^2_{\text{loc}}(\mathbb{R}; V')}^2 = \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|g(s)\|_{V'}^2 \, ds < \infty,
\end{equation}

and $f \in C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ satisfies the following conditions:

\begin{align}
(4) \quad & \sum_{i=1}^{N} \gamma_i |v^i|^{p_i} - C \leq \sum_{i=1}^{N} f^i(v, s)v^i = f(v, s) \cdot v, \; \gamma_k > 0, \; \forall v \in \mathbb{R}^N,
\end{align}

\begin{align}
(5) \quad & \sum_{i=1}^{N} |f^i(v, s)|^{p_i} \leq C \left( \sum_{i=1}^{N} |v^i|^{p_i} + 1 \right), \; \forall v \in \mathbb{R}^N,
\end{align}

where the letter $C$ denotes a constant which may be different in each occasion. \footnote{Note that conditions (4)-(5) do not assure the uniqueness of the solutions.}

Let $q_k := p_k/(p_k - 1)$, $r_k := \max\{1, n(1/2 - 1/p_k)\}$, $k = 1, \ldots, N$, and denote $p := (p_1, \ldots, p_N)$, $q := (q_1, \ldots, q_N)$, $r := (r_1, \ldots, r_N)$ and $L^p(\Omega) := L^{p_1}(\Omega) \times L^{p_2}(\Omega) \times \cdots \times L^{p_N}(\Omega)$, $H^{-r}(\Omega) := H^{-r_1}(\Omega) \times H^{-r_2}(\Omega) \times \cdots \times H^{-r_N}(\Omega)$, $L^p(\tau, T; L^p(\Omega)) := L^{p_1}(\tau, T; L^{p_1}(\Omega)) \times \cdots \times L^{p_N}(\tau, T; L^{p_N}(\Omega))$, $L^q(\tau, T; H^{-r}(\Omega)) := L^{q_1}(\tau, T; H^{-r_1}(\Omega)) \times \cdots \times L^{q_N}(\tau, T; H^{-r_N}(\Omega))$.

Moreover, the function $|u(t)|^2$ is absolutely continuous on $[\tau, \infty)$ and

\begin{equation}
\frac{d}{dt} |u(t)|^2 + (a \nabla u(t), \nabla u(t)) + (f(u(t), t), u(t)) = \langle g(t), u(t) \rangle,
\end{equation}

for a.e. $t \in [\tau, \infty)$.

Now, we consider a fixed interaction function $f_0$ and a driving force $g_0$, such that $f_0(v, t)$ satisfies (4)-(5), and $g_0 \in L^2_{\text{loc}}(\mathbb{R}; V')$. Let $\sigma_0 = (f_0, g_0)$ and $\Sigma := \Sigma_{(f_0, g_0)}$.
\[ \{ \sigma_0(\cdot + h) : h \in \mathbb{R} \} \]. Thus, for every \( \sigma = (f, g) \in \Sigma \), \( f \) satisfies (4)-(5) with the same constants, and \( \|g\|_{L^2_b}^2 \leq \|g_0\|_{L^2_b}^2 \).

Let \( u(t) \) be a weak solution of (RDS) with \( \sigma = (f, g) \in \Sigma \) on \( [\tau, \infty) \) guaranteed by Theorem 5.1. By (4) and (7), we have

\[
\frac{d}{dt}|u(t)|^2 + \lambda_1 \beta |u(t)|^2 \leq C + \beta^{-1} \|g_0\|_{V'}^2,\]

for a.e. \( t \in [\tau, \infty) \). Here \( \lambda_1 \) is the first eigenvalue of the Laplacian with Dirichlet boundary conditions. Thanks to the absolute continuity of \( |u(t)| \) and Gronwall’s inequality, (8) implies that

\[
|u(t)|^2 \leq |u(\tau)|^2 e^{-\lambda_1 \beta (t-\tau)} + C, \quad \forall t \in [\tau, \infty).
\]

Therefore there exists a closed (uniform w.r.t. \( \tau \in \mathbb{R} \)) absorbing ball \( B_s(0, R) \), where the radius \( R \) depends on \( \lambda_1, \beta \), the constant in (4) and \( \|g_0\|_{L^2_b}^2 \). We denote by \( X \) the absorbing ball

\[
X = \{ u \in H : |u| \leq R \}.
\]

That is, for any bounded set \( A \subset H \), there exists a time \( t_0 \geq \tau \), such that

\[ u(t) \in X, \quad \forall t \geq t_0, \]

for every weak solution \( u(t) \) with \( \sigma \in \Sigma \) and the initial data \( u(\tau) \in A \). It is known that \( X \) is weakly compact and metrizable with a metric \( d_w \) deducing the weak topology on \( X \).

Consider an evolutionary system for which a family of trajectories consists of all weak solutions of (RDS) with the fixed \( \sigma_0 \) in \( X \). More precisely, define

\[
\mathcal{E}([\tau, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a weak solution on } [\tau, \infty) \\
\text{with } \sigma \in \Sigma \text{ and } u(t) \in X \forall t \in [\tau, \infty) \}, \quad \tau \in \mathbb{R},
\]

\[
\mathcal{E}((-\infty, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a weak solution on } (-\infty, \infty) \\
\text{with } \sigma \in \Sigma \text{ and } u(t) \in X \forall t \in (-\infty, \infty) \}.
\]

Clearly, the properties 1–4 in Definition 2.1 hold for \( \mathcal{E} \) if we utilize the translation identity: the solutions of (RDS) with \( \sigma \) initiating at \( \tau + h \) are also the solutions of (RDS) with \( \sigma(\cdot + h) \) initiating at \( \tau \).

Thanks to Theorem 4.1, the weak global attractor \( A_w \) for this evolutionary system exists.

**Lemma 5.2.** Let \( u_n(t) \) be a sequence of weak solutions of (RDS) with \( \sigma_n \in \Sigma \), such that \( u_n(t) \in X \) for all \( t \geq t_0 \). Then

\[ u_n \text{ is bounded in } L^2(t_0, T; V), \]

\[ \partial_t u_n \text{ is bounded in } L^q(t_0, T; H^{-r}(\Omega)), \]
for all $T > t_0$. Moreover, there exists a subsequence $u_{n_j}$ converges in $C([t_0, T]; H_w)$ to some $\phi(t) \in C([t_0, T]; H)$, i.e.,

$$(u_{n_j}, v) \to (\phi, v) \text{ uniformly on } [t_0, T],$$

as $n_j \to \infty$, for all $v \in H$.

**Proof.** Take a sequence $u_n(s) \in E$. Standard estimates (see e.g. [CV02]) show that

(10) \( \{u_n\} \text{ is bounded in } L^2(t_0, T; V) \cap L^{\infty}(t_0, T; H) \cap L^p(t_0, T; L^p(\Omega)), \)

and

(11) \( \{\partial_s u_n\} \text{ is bounded in } L^q(t_0, T; H^{-r}(\Omega)), \)

for all $T > t_0$. By the embedding theorem (cf. Theorem II.1.4 in [CV02], Theorem 8.1 in [Ro01]), we obtain that

(12) \( \{u_n\} \text{ is precompact in } L^2(t_0, T; H). \)

Passing to a subsequence and dropping a subindex, we know from (10)-(12) that,

\[
\begin{align*}
  u_n(s) & \to \phi(s) \quad \text{weak-star in } L^\infty(t_0, T; H), \\
  u_n(s) & \to \phi(s) \quad \text{weakly in } L^2(t_0, T; V) \cap L^p(t_0, T; L^p(\Omega)), \\
  \Delta u_n(s) & \to \Delta \phi(s) \quad \text{weakly in } L^2(t_0, T; V'), \\
  \partial_s u_n(s) & \to \partial_s \phi(s) \quad \text{weakly in } L^q(t_0, T; H^{-r}(\Omega)), \\
  f_n(u_n(x, s), s) & \to \psi(s) \quad \text{weakly in } L^q(t_0, T; L^q(\Omega)),
\end{align*}
\]

for some

\[
\phi(s) \in L^\infty(t_0, T; H) \cap L^2(t_0, T; V) \cap L^p(t_0, T; L^p(\Omega)),
\]

and some

\[
\psi(s) \in L^q(t_0, T; L^q(\Omega)).
\]

Note that $g_0$ is translation compact in $L^2_{\text{loc}}(\mathbb{R}; V')$, i.e. the translation family $\{g_0(\cdot + h) : h \in \mathbb{R}\}$ is precompact in $L^2_{\text{loc}}(\mathbb{R}; V')$ (see [CV02]). Thus, passing to a subsequence and dropping a subindex again, we also have,

(15) \( g_n(s) \to g(s) \quad \text{weakly in } L^2(t_0, T; V') \)

with some $g(s) \in L^2(t_0, T; V')$. Passing the limits yields the following equality

(16) \( \partial_t \phi = a \Delta \phi - \psi + g \)

in the distribution sense of the space $\mathcal{D}'(t_0, T; H^{-r}(\Omega))$. Thanks to a vector version of Theorem II.1.8 in [CV02], (13)-(16) indicate that $\phi(s) \in C([t_0, T]; H)$. On the other hand, by the convergence of (13), we know that $u_n(s) \to \phi(s)$ in $C([t_0, T]; H_w)$. We complete the proof. \( \square \)
In particular, for some $C$ and dropping a subindex once more, we have that

$$
\text{Lemma 5.2, there exists a subsequence, still denoted by } \{u_n\} \text{, which converges in } C([0,1];H_w) \text{ to some } \phi^1 \in C([0,1];H) \text{ as } n \to \infty. \text{ Passing to a subsequence and dropping a subindex once more, we have that } u_n \to \phi^2 \text{ in } C([0,2];H_w) \text{ as } n \to \infty \text{ for some } \phi^2 \in C([0,2];H). \text{ Note that } \phi^1(s) = \phi^2(s) \text{ on } [0,1]. \text{ Continuing this diagonalization process, we obtain a subsequence } u_{n_j} \text{ of } u_{n} \text{ that converges in } C([0,\infty);H_w) \text{ to some } \phi \in C([0,\infty);H) \text{ as } n_j \to \infty. \text{ Therefore, A1 holds.}
$$

Now we give the definition of a normal function which was introduced in [LWZ05] and Lu [Lu06].

**Definition 5.3.** Let $\mathcal{B}$ be a Banach space. A function $\varphi(s) \in L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$ is said to be normal in $L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that

$$
\sup_{t \in \mathbb{R}} \int_t^{t+\delta} \| \varphi(s) \|^2_{\mathcal{B}} \, ds \leq \epsilon.
$$

Then, we have the following.

**Lemma 5.4.** The evolutionary system $\mathcal{E}$ of (RDS) satisfies A1 and A3. Moreover, if $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R};V')$ then A2 holds.

**Proof.** The proof is analogous to that of Lemma 3.4 in [CL09]. We write it out for completeness and for reader’s convenience. First, by Theorem 5.1, $\mathcal{E}([0,\infty)) \subset C([0,\infty);H)$. Now take any sequence $u_n \in \mathcal{E}([0,\infty))$, $n = 1, 2, \cdots$. Owing to Lemma 5.2 there exists a subsequence, still denoted by $u_n$, which converges in $C([0,1];H_w)$ to some $\phi^1 \in C([0,1];H)$ as $n \to \infty$. Passing to a subsequence and dropping a subindex once more, we have that $u_n \to \phi^2$ in $C([0,2];H_w)$ as $n \to \infty$ for some $\phi^2 \in C([0,2];H)$. Note that $\phi^1(s) = \phi^2(s)$ on $[0,1]$. Continuing this diagonalization process, we obtain a subsequence $u_{n_j}$ of $u_n$ that converges in $C([0,\infty);H_w)$ to some $\phi \in C([0,\infty);H)$ as $n_j \to \infty$. Therefore, A1 holds.

Let $u_k \in \mathcal{E}([0,\infty))$ be such that $u_k$ is a $d_{C([0,T];X_w)}$-Cauchy sequence in $C([0,T];X_w)$ for some $T > 0$. Thanks to Lemma 5.2 the sequence $\{u_k\}$ is bounded in $L^2(0,T;V)$. Hence, there exists some $\phi(s) \in C([0,T];H_w)$, such that

$$
\int_0^T |u_k(s) - \phi(s)|^2 \, ds \to 0, \quad \text{as} \quad k \to \infty.
$$

In particular, $|u_k(s)| \to |\phi(s)|$ as $k \to \infty$ a.e. on $[0,T]$, which means that $u_k(\cdot)$ is $d_{a}$-Cauchy sequence a.e. on $[0,T]$. Thus A3 is valid.

For any $u \in \mathcal{E}([0,\infty))$ and $t > 0$, it follows from (8) and the absolute continuity of $|u(t)|^2$ that

$$
|u(t)|^2 \leq |u(t_0)|^2 + C(t-t_0) + \frac{1}{\beta} \int_{t_0}^t \|g_0\|^2_{V'}, \, ds,
$$

for all $0 \leq t_0 < t$. Here $C$ is independent of $u$. Suppose now that $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R};V')$. Then given $\epsilon > 0$, there exists $0 < \delta < \frac{2\epsilon}{C}$, such that

$$
\sup_{t \in \mathbb{R}} \int_{t-\delta}^t \|g_0(s)\|^2_{V'} \, ds \leq \frac{\beta \epsilon}{2}.
$$

Hence, we obtain from (17) that

$$
|u(t)|^2 \leq |u(t_0)|^2 + \epsilon, \quad \forall t \in (t-\delta,t),
$$

which concludes that A2 holds. □
Let $\bar{E}$ be the closed evolutionary system of $E$, i.e.,

$$\bar{E}([\tau, \infty)) := E([\tau, \infty)), \forall \tau \in \mathbb{R}.$$ 

Then, it follows from Theorem 5.1, Lemma 5.4, Theorems 4.1, 4.6 and 4.3 that

**Theorem 5.5.** The uniform weak global attractor $A_w$ and the weak trajectory attractor $\mathfrak{A}_w$ for (RDS) exist, and

$$A_w = \omega_w(X) = \omega_w((-\infty, \infty)),$$

$$A_w = \Pi_+ \bar{E}((-\infty, \infty)) = \{u(\cdot)|_{[0, \infty)} : u \in \bar{E}((-\infty, \infty))\},$$

$$A_w = \mathfrak{A}_w(t) = \{u(t) : u \in A_w\}, \forall t \geq 0.$$

Hence, the finite weak uniform tracking property holds.

**Theorem 5.6.** If $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R}; V')$, then the uniform weak global attractor $A_w$ is a strongly compact strong global attractor $A_s$, and the weak trajectory attractor $\mathfrak{A}_w$ is a strongly compact strong trajectory attractor $A_s$. Hence, the finite strong uniform tracking property holds.

### 6. ON NONLINEARITY

In this section, we begin with a brief review of some additional assumptions, other than (4) and (5), on the nonlinearity to obtain the existence and the structure of uniform global attractors or trajectory attractors for (RDS) in some previous literature (see [CV94] [CV96] [CV97] [CV02] [Lu07] [CL09]). Then, we examine the results on attractors using our framework and discuss their relation to those obtained as in previous section. Finally, we construct some examples that do not satisfy these extra restrictions.

Let $\mathcal{M}_1$ be $C(\mathbb{R}^N; \mathbb{R}^N)$ endowed with finite weighted norm

$$\|\phi\|_{\mathcal{M}_1} = \sup_{v \in \mathbb{R}^N} \left( \sum_{i=1}^{N} \frac{|\phi^{i}(v)|}{(1 + \sum_{j=1}^{N}|v^j|^{p_j})^{\frac{p_j-1}{p_j}}} \right).$$

Chepyzhov and Vishik studied the uniform global attractor of (RDS) in [CV94] (see also [CV02]) assuming that $f_0(v, s)$ is translation compact in $C(\mathbb{R}; \mathcal{M}_1)$, that is, the closure of the translation family $\{f_0(\cdot, + h) : h \in \mathbb{R}\}$ in $C(\mathbb{R}; \mathcal{M}_1)$ is compact in $C(\mathbb{R}; \mathcal{M}_1)$. Later on, they investigated (RDS) in [CV96] [CV97] (see also [CV02]) by the method of the so-called trajectory attractor. The condition on $f_0(v, s)$ is translation compactness in $C(\mathbb{R}; \mathcal{M})$, where $\mathcal{M} = C(\mathbb{R}^N; \mathbb{R}^N)$ is endowed with the topology of local uniform convergence. This restriction on $f_0$ is equivalent to that $f_0(v, s)$ is bounded and uniformly continuous in every cylinder $Q(R) = \{(v, s) : \|v\|_{\mathbb{R}^N} \leq R, s \in \mathbb{R}\}, R > 0$. The section of the (weak) trajectory attractor at time $t = 0$ is the uniform w.r.t. the symbol space $\bar{\Sigma}$ global attractor. Here one component of $\bar{\Sigma}$ is the closure of $\{f_0(\cdot, + h) : h \in \mathbb{R}\}$ in $C(\mathbb{R}; \mathcal{M})$, ...
which is compact in $C(\mathbb{R}; \mathcal{M})$. However, as we will see below, its relation to the uniform global attractor for the original considered (RDS) with a fixed interaction function $f_0$ and a driving force $g_0$ is not yet clear.

In [Lu07], the author obtained the existence and the structure of the uniform global attractor of (RDS) with a weaker condition (see Theorems 6.1 and 6.2 below) on the nonlinearity.

Let $C^{p,u}(\mathbb{R}; \mathcal{M})$ denote the space $C(\mathbb{R}; \mathcal{M})$ endowed with the topology of following convergence: $\varphi_n(s) \to \varphi(s)$ as $n \to \infty$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$, if $\varphi_n(v, s)$ is uniformly bounded on any ball in $\mathbb{R}^N \times \mathbb{R}$ and for every $s \in \mathbb{R}$, $R > 0$,

$$\max_{\|v\|_{\mathbb{R}^N} \leq R} \|\varphi_n(v, s) - \varphi(v, s)\|_{\mathbb{R}^N} \to 0, \quad \text{as} \quad n \to \infty.$$ 

Denote by $C^{p,p}(\mathbb{R}; \mathcal{M})$ the space $C(\mathbb{R}; \mathcal{M})$ endowed with another topology of following convergence: $\varphi_n(s) \to \varphi(s)$ as $n \to \infty$ in $C^{p,p}(\mathbb{R}; \mathcal{M})$, if $\varphi_n(v, s)$ is uniformly bounded on any ball in $\mathbb{R}^N \times \mathbb{R}$ and for every $(v, s) \in \mathbb{R}^N \times \mathbb{R}$,

$$\|\varphi_n(v, s) - \varphi(v, s)\|_{\mathbb{R}^N} \to 0, \quad \text{as} \quad n \to \infty.$$ 

Note that $C^{p,p}(\mathbb{R}; \mathcal{M})$ is in fact $C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ with the usual weak topology.

Let $C^{p,u}_{tr.c.}(\mathbb{R}; \mathcal{M})$, $C^{p,p}_{tr.c.}(\mathbb{R}; \mathcal{M})$ and $C_{tr.c.}(\mathbb{R}; \mathcal{M})$ be the classes of the translation compact functions in $C^{p,u}(\mathbb{R}; \mathcal{M})$, $C^{p,p}(\mathbb{R}; \mathcal{M})$ and $C(\mathbb{R}; \mathcal{M})$, respectively.

The functions in $C^{p,u}_{tr.c.}(\mathbb{R}; \mathcal{M})$ are characterized by the following.

**Theorem 6.1.** [Lu07] $\varphi(s) \in C^{p,u}_{tr.c.}(\mathbb{R}; \mathcal{M})$ if and only if $\varphi(s) \in C^{p,p}_{tr.c.}(\mathbb{R}; \mathcal{M})$ and one of the following holds.

(i) $\{\varphi(s) | s \in \mathbb{R}\}$ is precompact in $\mathcal{M}$.

(ii) For any $R > 0$, $\varphi(v, s)$ is bounded in $Q(R) = \{(v, s) : \|v\|_{\mathbb{R}^N} \leq R, s \in \mathbb{R}\}$, and

$$(18) \quad \|\varphi(v_1, s) - \varphi(v_2, s)\|_{\mathbb{R}^N} \leq \theta(\|v_1 - v_2\|_{\mathbb{R}^N}, R), \quad \forall (v_1, s), (v_2, s) \in Q(R),$$

where $\theta(l, R)$ is a positive function tending to 0 as $l \to 0^+$.

By Arzelá-Ascoli compactness criterion, the conditions of (i) and (ii) imply that the family $\{\varphi(\cdot), t : t \in \mathbb{R}\}$ is equicontinuous on any ball $\{v : \|v\|_{\mathbb{R}^N} \leq R\}$.

Let $C^b(\mathbb{R}; \mathcal{M})$ be the space of bounded continuous functions with values in $\mathcal{M}$ and endowed with the uniform convergence topology on every cylinder $Q(R)$. We have the following relationships.

**Theorem 6.2.** [Lu07] $C_{tr.c.}(\mathbb{R}; \mathcal{M}) \subset C^{p,u}_{tr.c.}(\mathbb{R}; \mathcal{M}) \subset C^{p,p}_{tr.c.}(\mathbb{R}; \mathcal{M}) \subset C^b(\mathbb{R}; \mathcal{M})$ with all inclusions being proper and the former three sets being closed in $C^b(\mathbb{R}; \mathcal{M})$.

In [CL09], Cheskidov and Lu generalized the results in [Lu07] to (RDS) without uniqueness and considered in addition the uniform weak global attractors. More precisely, besides the conditions on $f_0$ and $g_0$ in Section 5, we suppose more that
Denote again by $\sigma_0 = (f_0, g_0)$. The family of (RDS) with $\sigma = (f, g)$ belonging to the following symbol space

$$\Sigma = \{(\sigma_0(\cdot + h) : h \in \mathbb{R})\} \subset C^0(\mathbb{R}; \mathcal{M}) \times L^2_{\text{loc}}(\mathbb{R}; V')$$

defines an evolutionary system $\mathcal{E}_{\Sigma}$ satisfying $\mathcal{A}$. The existence and the structure of its uniform global attractor $\mathcal{A}^0_{\Sigma}$ is presented in [CL09]. Now we apply Theorems 4.1 and 4.3 to $\mathcal{E}_{\Sigma}$, we can get further the trajectory attractor $\tilde{\mathcal{A}}^0_{\Sigma}$. Similarly defined as in Section 5, we have the evolutionary system $\mathcal{E}$ and its closure $\tilde{\mathcal{E}}$ for the originally considered (RDS) with fixed $f_0$ and $g_0$. Evidently, $\mathcal{E} \subset \tilde{\mathcal{E}} \subset \mathcal{E}_{\Sigma}$. Hence, $\mathcal{A} = \tilde{\mathcal{A}} = \tilde{\mathcal{A}}^0_{\Sigma}$ or $\mathcal{A}_* = \tilde{\mathcal{A}}_* = \tilde{\mathcal{A}}_w$. However, it is not known whether the following identities hold

$$\mathcal{A}_* = \tilde{\mathcal{A}}_* = \mathcal{A}^\Sigma_\bullet.$$

Or in the version of trajectory attractors,

$$\mathcal{A}_* = \tilde{\mathcal{A}}_* = \mathcal{A}^\Sigma_{\text{loc}}.$$

In [CL09] (see also [Lu07]), it is shown that (19) is valid, if we suppose further the following condition on nonlinearity $f_0$,

$$\left( f_0(v_1, s) - f_0(v_2, s), v_1 - v_2 \right) \geq -C\|v_1 - v_2\|^2_{\mathbb{R}^N}, \forall v_1, v_2 \in \mathbb{R}^N, \forall s \in \mathbb{R},$$

which guarantees the uniqueness of the solutions. In fact, the general result for evolutionary systems with uniqueness is proved in [CL14]. Contrarily, $\mathcal{A}^\Sigma_*$ or $\mathcal{A}^\Sigma_{\text{loc}}$ previously constructed for the system without uniqueness, might not satisfy the minimality property. That means that they might be bigger than the uniform global attractor or the trajectory attractor followed by our framework.

Now we construct several examples in $C(\mathbb{R} \times \mathbb{R} ; \mathbb{R})$ that satisfy conditions (4)-(5) but not those in Theorem 6.1 and (20). It is not clear how to obtain the results in Theorems 5.5 and 5.6 for the (RDS) with such kind of nonlinearities by previous frameworks. Let $T = \max\{0, t\}, t \in \mathbb{R}$.

**Example I.**

$$f(v, t) = \begin{cases} |v|^p, & \text{if } v \leq 0, \\
(1 + T)v, & \text{if } 0 \leq v \leq \frac{1}{1+T}, \\
|v - \frac{1}{1+T}|^p + 1, & \text{if } v > \frac{1}{1+T}. \end{cases}$$

Note that, the family $\{f(\cdot, t) : t \in \mathbb{R}\}$ is not equicontinuous in $[0, 1]$, which means that $f(v, t)$ does not satisfy (13). Moreover, the pointwise limit function of $f(\cdot, t)$, as $t \to +\infty$, is a discontinuous function,

$$f_\infty(v) = \begin{cases} |v|^p, & \text{if } v \leq 0, \\
|v|^p + 1, & \text{if } v > 0. \end{cases}$$
Hence, \( f(v, t) \) does not even belong to \( C^{p,p}_{tr.c.}(\mathbb{R}; \mathcal{M}) \), where \( \mathcal{M} = C(\mathbb{R}, \mathbb{R}) \). In fact, for any sequence \( \{ f(\cdot, \cdot + t_n), t_n \to +\infty \} \), the pointwise limit is \( f_\infty \).

Example II.

\[
\begin{align*}
   f(v, t) = \begin{cases} 
   |v + 2\pi|^p, & \text{if } v \leq -2\pi, \\
   \rho(v) \sin(1 + T)v, & \text{if } -2\pi < v < 2\pi, \\
   |v - 2\pi|^p, & \text{if } v \geq 2\pi,
   \end{cases}
\end{align*}
\]

where \( \rho(\cdot) \) is a continuous function supported in \([−2\pi, 2\pi]\). For instance, \( \rho(\cdot) \) is an infinitely differentiable function supported in \((-2\pi, 2\pi)\) and equals to 1 on \([-\pi, \pi]\). Note again that, the family \( \{ f(\cdot, t) | t \in \mathbb{R} \} \) is not equicontinuous in \([-2\pi, 2\pi]\). Moreover, there is no a pointwise limit function of any sequence \( \{ f(\cdot, \cdot + t_n) \} \), as \( t_n \to +\infty \). Hence, \( f(v, t) \notin C^{p,p}_{tr.c.}(\mathbb{R}; \mathcal{M}) \).

Example III.

\[
\begin{align*}
   f(v, t) = \begin{cases} 
   |v + 2|^p, & \text{if } v \leq -2, \\
   \rho(v) \sin T^2, & \text{if } -2 < v < 2, \\
   |v - 2|^p, & \text{if } v \geq 2,
   \end{cases}
\end{align*}
\]

where \( \rho(\cdot) \) is a continuous function supported in \([-2, 2]\). For example, \( \rho(\cdot) \) is an infinitely differentiable function supported in \((-2, 2)\) and equals to 1 on \([-1, 1]\). For any \( R > 0 \), the family \( \{ f(\cdot, t) | t \in \mathbb{R} \} \) is equicontinuous in \([-R, R]\). However, there is also no a pointwise limit function of any sequence \( \{ f(\cdot, \cdot + t_n) \} \), as \( t_n \to +\infty \). Hence, \( f(v, t) \notin C^{p,p}_{tr.c.}(\mathbb{R}; \mathcal{M}) \).

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DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: songsong.lu@yahoo.com; luss@mail.sysu.edu.cn