On the Synchronizing Probability Function and the Triple Rendezvous Time for Synchronizing Automata

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Abstract. We push further a recently proposed approach for studying synchronizing automata and Černý’s conjecture, namely, the synchronizing probability function. In this approach, the synchronizing phenomenon is reinterpreted as a Two-Player game, in which the optimal strategies of the players can be obtained through a Linear Program. Our analysis mainly focuses on the concept of triple rendezvous time, the length of the shortest word mapping three states onto a single one. It represents an intermediate step in the synchronizing process, and is a good proxy of its overall length. Our contribution is twofold. First, using the synchronizing probability function and properties of linear programming, we provide a new upper bound on the triple rendezvous time. Second, we disprove a conjecture on the synchronizing probability function by exhibiting a family of counterexamples. We discuss the game theoretic approach and possible further work in the light of our results.

Keywords: automata and logic, synchronization, Černý’s conjecture, game theory, synchronizing probability function, triple rendezvous time.

1 Synchronizing Automata and Černý’s Conjecture

Synchronizing automata have been the source of intense research in the past 50 years. An automaton is called synchronizing if there exists a sequence of letters which maps all the states onto a single one (see the next subsection for rigorous definitions). Figure 1 shows an example of such an automaton. The interest for the subject appeared in computers and relay control systems in the 60s. The aim was to restore control over these devices without knowing their current state. In the 80s and 90s, synchronizing automata found applications in robotics and industry. From a theoretical perspective, the synchronizing property is also linked with active research topics in engineering, like the consensus theory and the primitivity of matrix sets (see [24]).

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Fig. 1. A synchronizing automaton. The word \textit{abbbabba} maps any state onto state 0.

In this paper, we will represent automata as sets of matrices. A set of states of an automaton with \(n\) possible states will be represented by its \(1 \times n\) characteristic vector and the letters of the automaton will be represented as \(n \times n\) matrices, acting multiplicatively on the characteristic vector of states:

\begin{definition}
A (deterministic, finite state, complete) automaton (DFA) is a set of \(m\) column-stochastic matrices \(\Sigma \subset \{0, 1\}^{n \times n}\) (where \(m, n\) are respectively the number of letters in the alphabet, and the number of states of the automaton). Each letter corresponds to a matrix \(L \in \Sigma\) with binary entries, which satisfies \(L^T e = e^T\), where \(e\) is the \(1 \times n\) all-ones vector. We write \(\Sigma^t\) for the set of matrices which are products of length \(t\) of matrices taken in \(\Sigma\). We refer to these matrices as words of length \(t\).
\end{definition}

\begin{definition}
An automaton \(\Sigma \subset \{0, 1\}^{n \times n}\) is synchronizing if there is an index \(1 \leq i \leq n\) and a finite product \(W = L_{c_1} \cdots L_{c_s}\) : \(L_{c_j} \in \Sigma\) which satisfy
\(\quad W = e^Te_i,\)
where \(e_i\) is the \(i\)th standard basis vector (1 \(\times n\)).
In this case, the sequence of letters \(L_{c_1} \cdots L_{c_s}\) is said to be a synchronizing word.
\end{definition}

\begin{example}
The two letters of the automaton in Fig. 1 are the following matrices:
\[
\begin{align*}
a &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \\
b &= \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]
\end{example}

Jan Černý stated his conjecture on DFA in 1964. Although it is very simple in its formulation, it has not been proven since then.

\begin{conjecture}[Černý’s conjecture, 1964]
Let \(\Sigma \subset \{0, 1\}^{n \times n}\) be a synchronizing automaton. Then, there is a synchronizing word of length at most \((n - 1)^2\).
\end{conjecture}

\footnote{A vector \(x \subset \{0, 1\}^n\) for which \(x_i = 1\) if state \(i\) is in the set, and 0 otherwise.}
In [7], Černý proposes an infinite family of automata attaining this bound, for any number of states. We refer to this family as the Černý family of automata. The automaton in Fig. 1 is the automaton of the family with four states. Its shortest synchronizing word is \( aabbabbbba \). The length of this word indeed corresponds to the bound in Conjecture 3. Synchronizing automata attaining the bound of Conjecture 3 or getting close to it are very infrequent (see [1], [14], [18] for examples).

Since its formulation, Conjecture 3 has been the subject of intense research. On the one hand theoretical research is aiming to prove the conjecture, on the other hand numerical research is aiming to design efficient algorithms to find synchronizing words in automata (see [19], [21]). Černý’s Conjecture has been proven to hold for several families of automata [2, 3, 6, 7, 9, 10, 15, 22]. However the best known general upper bound is \( \left( \frac{n^3 - n}{6} \right) \), obtained by Pin and Frankl [11] [16]. However the best general upper bound on the length of a shortest synchronizing word for an automaton with \( n \) states is equal to \( \left( \frac{n^3 - n}{6} \right) \), obtained by Pin and Frankl [11] [16]. This bound has been holding for more than 30 years. A state of the art overview is given by Volkov [25].

Recently, several research efforts have tried to shed light on the problem by making use of probabilistic approaches (see [13], [20]). The main tool we will focus on, the synchronizing probability function (SPF), was introduced by the second author in 2012 [13]. This tool allows the reformulation of the synchronizing property as a game theoretical problem whose solution can be obtained through convex optimization. This new link between convex optimization (a mature discipline with strong theoretical basis, see [5], [17]) and synchronizing automata is arguably promising towards a better understanding of the synchronizing phenomenon.

The paper is organised as follows. In Section 2 we recall the main properties of the synchronizing probability function. In Section 3, we introduce the concept of triple rendezvous time, and, making use of the synchronizing probability function we obtain a new upper bound on this value. In Section 4, we refute a recent conjecture on the synchronizing probability function (Conjecture 2 in [13]) by presenting a particular family of automata which doesn’t satisfy it.

2 A Game Theoretical Framework and the Synchronizing Probability Function

In this section, we recall the definition and the properties of the synchronizing probability function needed to develop our results. A more complete introduction to the SPF and the details of the proofs can be found in [13]. This concept is based on a reformulation of the synchronization over an automaton as a Two-Player game with the following rules:

1. The length \( t \) is chosen.

\(^2\) A bound of \( n(7n^2 + 6n - 16)/48 \) was proposed by Trahtman [23], but its proof was incomplete.
2. Player Two secretly chooses a state for the automaton.
3. Player One chooses a word of length at most $t$. It is applied to the automaton, and changes the state of the automaton accordingly.
4. Player One guesses what the final state of the automaton is. If it is the right final state, he wins. Otherwise, Player Two wins.

The policy of Player Two is defined as a probability distribution over the states, that is, any vector $p \in \mathbb{R}^+ \times \mathbb{R}^+$, $e_p^T = 1$. Player Two chooses the state $i$ with probability $p_i$, in which case the automaton will end up at the state corresponding to $e_i A$, where $A$ is the matrix representation of the word chosen by Player One. Since Player One wants to maximize the probability of choosing the right final state, he will pick up the state where the probability for the automaton to end is maximal, that is,

$$\arg\max_i (pA)e_i^T.$$  

Therefore the probability of winning for Player One is

$$\max_{i,A} (pA)e_i^T. \tag{1}$$

The aim of Player Two is to minimize that probability.

In the following, $\Sigma \subseteq \{0,1\}^n$ is the set of products of length at most $t$ of matrices taken in $\Sigma$. By convention, and for the ease of notation, it contains the product of length zero, which is the identity matrix.

**Definition 5 (SPF, Definition 2 in [13]).** Let $n \in \mathbb{N}$ and $\Sigma \subseteq \{0,1\}^n \times \mathbb{R}$ be an automaton. The synchronizing probability function (SPF) of $\Sigma$ is the function $k_\Sigma : \mathbb{N} \rightarrow \mathbb{R}^+$:

$$k_\Sigma(t) = \min_{p \in \mathbb{R}^+\setminus\{0\}} \left\{ \max_{A \in \Sigma \subseteq \{0,1\}^n} \left\{ \max_i (pA)e_i^T \right\} \right\}. \tag{2}$$

Conjecture (4) can now be reformulated in terms of the SPF:

**Proposition 6 (Proposition 1 in [13]).** The following conjecture is equivalent to Conjecture (4):

If $\Sigma \subseteq \{0,1\}^n$ is a synchronizing automaton, then,

$$\forall t \geq (n - 1)^2, \quad k_\Sigma(t) = 1.$$  

If there is no ambiguity on the automaton, we use $k(t)$ for $k_\Sigma(t)$. This function is non-decreasing, cannot take the same value for more than $n$ consecutive values of $t$ (Theorem 3 in [13]) and returns value one iff there is a synchronizing word of length smaller or equal to $t$. Figure 2 represents the SPF of the automaton presented in Fig. 1.

In order to use the SPF, we need an explicit algorithmic construction of the optimal strategies for both players, which allows us to compute the SPF value. Each basic strategy of Player One, i.e. the choice of a word and a final state, is equivalent to choosing a column in this word. Therefore, we consider the set of all the different columns reached in words of length at most $t$. 


Fig. 2. The synchronizing probability function of the automaton on four states presented in Fig. There is a synchronizing word of length nine, therefore $k(9) = 1$.

**Definition 7.** We call reachable columns the set $A(t)$ of all the different columns in the matrices (words) in $\Sigma^\leq t$. We represent $A(t)$ as a $n \times M(t)$ matrix, where $M(t)$ is the number of different columns.

When there is no ambiguity on $t$, we use $A$ for $A(t)$. We notice that if $t = 0$, the reachable columns are the columns of the identity matrix, by definition of $\Sigma^0$.

**Example 8.** Let us consider the automaton from Fig. At $t = 3$, $A(3)$ is the following matrix:

$$A(3) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}.$$

The first four columns are obtained with products of length 0, the fifth column is in the word $a$, the sixth in the word $ba$ and the last in the word $bba$.

It turns out that the SPF can be computed efficiently thanks to the following linear programs.

**Theorem 9 (Theorem 1 in [13]).** The synchronizing probability function $k_{\Sigma}(t)$ of $\Sigma$ is given by

$$\begin{align*}
\min_{p, k} & \quad k \\
\text{s.t.} & \quad pA \leq k e^T \\
& \quad e p^T = 1 \\
& \quad p \geq 0.
\end{align*}$$

The following inequalities are entrywise.

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$^3$ The following inequalities are entrywise.
It is also given by:

$$\begin{align*}
\max_{q,k} & \quad k \\
\text{s.t.} & \quad Aq \geq ke^T \\
& \quad eq = 1 \\
& \quad q \geq 0.
\end{align*}$$

(4)

In the equations above, $A$ denotes the set of reachable columns at time $t$ (see Def.7), $q$ is a $M(t) \times 1$ vector, $e$ represents all-ones vectors of the appropriate dimension, $1$ is a scalar, and $0$ represents zero vectors of the appropriate dimension.

The linear Program (4) is the dual of Program (3). For any primal feasible $p$ and any dual feasible $q$, the objective value $k(p)$ of Program (3) and the objective value $k(q)$ of Program (4) satisfy $k(q) \leq k(p)$. Therefore, if the objective value $k$ is the same for both programs with feasible solutions $p$ and $q$, this value is the optimum (see [5], [17] for more details on convex optimization and linear programming).

The linear Program (4) is the dual of Program (3). In the primal (3), the optimal objective value $k(p)$ is obtained with the strategy of Player Two, $p$, which is a probability distribution on the states. In the dual (4), the optimal objective value $k(q)$ is obtained with the strategy of Player One, $q$, which is a probability distribution on the set of all the possible columns obtained in words of length at most $t$ (i.e. columns contained in $A(t)$). For any primal feasible solution $p$ and any dual feasible solution $q$, the objective value $k(p)$ of Program (3) and the objective value $k(q)$ of Program (4) satisfy $k(p) \leq k(q)$. Therefore, if the objective value $k$ is the same for both programs with feasible solutions $p$ and $q$, this value is the optimum (see [5], [17] for more details on convex optimization and linear programming).

Example 10. Let us consider the automaton of Fig. 1 and $t = 3$. The set of reachable columns is given in Example 8. On the one hand $p = (1/4, 1/4, 1/4, 1/4)$ is an admissible solution for (3) (i.e. a probability distribution on the nodes for player two), which gives as objective value $k(p) = 1/2$. On the other hand $q = (0, 0, 0, 0, 1/2, 0, 1/2)^T$ is an admissible solution for the dual (i.e. a probability distribution on the possible columns), which also gives the objective value $k(q) = 1/2$. Therefore, the SPF at $t = 3$ is equal to $k(3) = 1/2$, as shown in Fig. 2. In other words, this means that if both players play optimally, with words of length at most 3, Player One has a probability $1/2$ to win the game. △

In the following, our main arguments will be based on the dimension of the set of optimal strategies of (4):

Definition 11 (Definition 3 in [13]). Let $\Sigma$ represent an automaton and $t$ be a positive integer. The polytopes $P_t$ and $Q_t$ are the sets of optimal solutions of respectively (3) and (4).
Example 12. In Example 8, $P_3$ and $Q_3$ are the following sets:

$P_3 = \{(1/4 + x, 1/4 - x, 1/4 + y, 1/4 - y) | -1/4 \leq x \leq 1/4,$ 
$-1/4 \leq y \leq 1/4,$ 
$x - y \leq 0\}$

$Q_3 = \{(0, 0, 0, 0, 1/2, 0, 1/2)^T\}$.

There are two parameters in the set of solutions $P_3$, therefore it is a polytope
of dimension two. On the other hand, $Q_3$ is a singleton, which means that there
is only one optimal strategy for Player One.

If the SPF doesn’t increase, we have the following result on $P(t)$:

Lemma 13 (Lemma 1 in [13]). If $k(t) = k(t + 1)$ then $P_{t+1} \subset P_t$.

With these tools in hand, we can get to our contributions.

3 A New Bound on the Triple Rendezvous Time

The triple rendezvous time is equal to the length of the shortest word
mapping three states of the automaton onto a single one. Although it is a very natural
concept, we are not aware of any attempts to bound its value for synchronizing
automata. In what follows, the weight of a vector is the number of its non-zero
elements.

Definition 14. For a synchronizing automaton $\Sigma$, the triple rendezvous time
$T_3, \Sigma$ is defined as the smallest integer $t$ such that $A(t)$ contains a column of
weight superior or equal to 3.

In other words, it is the length of the shortest word $W$ such that for three
of the possible initial states, applying this word to the automaton leaves it in
the same final state, i.e. such that there exists states $q_i, q_j$, and $q_k$ with $q_iW = q_jW = q_kW$. In the following, we will use $T_3$ for $T_3, \Sigma$ when there is no ambiguity
on the automaton.

Our motivations for studying $T_3$ are multiple. There are empirical evidences
that $T_3$ is correlated with the length of the shortest synchronizing word. Indeed,
for the known automata achieving the bound of Conjecture 4, $T_3$ is close to
$n$. Moreover, numerical experiments showed that automata with small $T_3$ have
short synchronizing word. In addition, the triple rendezvous time is directly
linked with the synchronizing probability function evolution (Proposition 6 and
Conjecture 4 in [13]). It is also related to the k-extension property developed
in [13].

Figure 3 shows the automaton of the Cerný family with 6 states and Kari’s
automaton [14], two automata achieving the bound of Conjecture 4 with syn-
chronizing words of length 25. Figure 4 shows their SPF. For these automata,
$T_3$ is equal to 7 and 5 respectively.

$T_3$ is the smallest number such that there is a pair of states in the set of pairs of
states which are synchronized by some single letter, which is $(T_3 - 1)$-extendable.
Fig. 3. On the left the automaton of the Cerný family with 6 states, on the right Kari’s automaton.

Fig. 4. Synchronizing probability function for the automaton of the Cerný family with 6 states in solid, and for Kari’s automaton in dashed.

For the known automata achieving the bound of Conjecture [4] the synchronizing probability function is growing close to linearly. This consideration led to the following conjecture:

**Conjecture 15 (Conjecture 2 in [13])** In a synchronizing automaton $\Sigma$ with $n$ states, for any $1 \leq j \leq n - 1$,

$$k_\Sigma(1 + (j - 1)(n + 1)) \geq j/(n - 1).$$

This conjecture is stronger than Černý’s conjecture (Theorem 4 in [13]). Conjecture [15] would also imply that the following conjecture about the triple rendezvous time is true:

**Conjecture 16 (Conjecture 4 in [13])** In a synchronizing automaton $\Sigma$ with $n$ states,
In Section 4, we provide a family of automata which are counterexamples for both Conjecture 15 and Conjecture 16.

We now focus on bounding $T_3$. A first upper bound can be easily obtained without using the SPF:

**Proposition 17.** In a synchronizing automaton $\Sigma$ with $n$ states,

$$T_{3, \Sigma} \leq \frac{n(n - 1)}{2} + 1.$$

**Proof.** For any positive integer $t$ smaller than the length of the shortest synchronizing word, the matrix $A(t + 1)$ must contain columns that are not in $A(t)$ (Lemma 1 in [13]). However, there are only $n(n - 1)/2$ possible different columns of weight two. As $A(0)$ includes the $n$ columns of weight one, $A(n(n - 1)/2 + 1)$ includes at least $n + n(n - 1)/2 + 1$ columns, in which one must be of weight superior or equal to 3. \qed

In order to obtain a better upper bound on $T_3$, we study the evolution of the SPF and $A(t)$ for $t < T_3$. In that setting, $A(t)$ only contains columns of weight one or two. We will associate the graph $G(t)$ with $A(t)$, $A(t)$ being the incidence matrix of $G(t)$ (ignoring columns of weight one in $A(t)$).

**Example 18.** For the automaton from Example 8 at $t = 3$, the graph $G(3)$ associated with $A(3)$ is shown in Fig. 5.

![Fig. 5. Graph $G(3)$ associated with $A(3)$ for the automaton from Example 8.](image)

In the graph $G(t)$ associated with $A(t)$, we call a singleton a vertex which is disconnected from the rest of the graph, a pair two vertices which are connected to each other and disconnected from the rest of the graph, and a cycle a set of
vertices connected between them as a cycle\(^5\) and disconnected from the rest of the graph. We call a cycle \textit{odd} (resp. \textit{even}) if it contains an odd (resp. even) number of vertices.

We also use the reverse correspondence. With a singleton or a pair is associated the column in \(A(t)\) corresponding to its characteristic vector, and with a cycle of \(c\) vertices is associated the set of \(c\) columns in \(A(t)\) corresponding to the characteristic vectors of the \(c\) edges of the cycle.

In the following, based on this matrix-graph approach, we study the values that \(k(t)\) can take with \(t < T_3\), and the maximal dimension that \(P_t\) could take for each value \(k(t)\). To do so, we start from the matrix \(A(t)\). We prove that it is possible to extract a matrix \(A'\) from \(A(t)\) by keeping only some of its columns, satisfying the following properties. The matrix \(A'\) is such that its associated graph \(G'\) is composed of disjoint singletons, pairs and odd cycles, and such that the optimal objective value for Program (3) and Program (4) is the same if \(A(t)\) is replaced by \(A'\). This structure allows us to compute easily the value \(k(t)\) and the dimension of the solution set \(P_t\) associated with (3) (with \(A'\) instead of \(A(t)\)). Replacing \(A(t)\) with \(A'\) can only increase the dimension of \(P_t\) as it reduces the amount of constraints in (3), while still achieving the same objective value by definition. We call \textit{support} of the strategy \(q\) the set of columns in \(A(t)\) corresponding to non zero entries in \(q\).

In the proof of Theorem 19 hereunder, we split the program into two subprograms and introduce some notation to get to our arguments. The reader may refer to Example 20 to have some illustrations of the concepts while reading the proof.

\textbf{Lemma 19.} If \(t < T_3\), there exists an optimal solution \(q\) for Program (4) such that its support is associated with a graph composed of disjoint singletons, pairs and odd cycles.

\textit{Proof (sketch).}

The sketch of the proof is the following. We proceed by induction on the number of variables of Program (4). If there is only one or two states, it is trivially true. Otherwise, we use the fact that the graph \(G(t)\) associated with \(A(t)\) can either be connected or disconnected.

If \(G(t)\) is disconnected, we can define two subprograms with the structure of (4) with less variables, for which we know by induction that there exist optimal solutions satisfying the lemma. From these solutions, we can build an optimal solution of the original program whose support will also be of the right shape.

If \(G(t)\) is connected, we show that we can either find an optimal solution with a support associated with an odd cycle including all the vertices, or find a solution with a support associated with a disconnected graph. In this latter case, we will again be able to split the program as in the disconnected case.

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\(^5\) \(c\) vertices are forming a \textit{cycle} if we can number them from 1 to \(c\) in such a way that node 1 is only connected to nodes 2 and \(c\), each node \(1 < i < c\) is only connected to nodes \(i - 1\) and \(i + 1\), and the node \(c\) is only connected to nodes \(c - 1\) and 1.
Full proof. We suppose by induction that the Lemma holds if \( A(t) \) is associated with a graph with \( n-1 \) vertices or less (so if \( A(t) \) has \( n-1 \) rows or less). Let us take a graph \( G(t) \) associated with a matrix \( A(t) \) on \( n \) vertices. Let \( A \) be a minimal subset of columns of \( A(t) \), in other words such that Program (4) based on \( A(t) \) and Program (4)’ based on \( A \) admit the same optimal objective value, and such that removing any column of \( A \) would change the optimal objective value. Let \( G \) be the graph associated with \( A \). We will prove that either \( G \) is already composed of disjoint singletons, pairs and odd cycles, or that we can build an optimal solution \( q' \) to (4)’ such that the graph associated with its support is composed of disjoint singletons, pairs and odd cycles. As the program built on \( A \) reaches the same optimal objective value as the original program, there is a vector \( q \) solution of (4)’ with optimal objective value \( k \) such that \( Aq \geq ke \).

Two cases can occur: either \( G \) is connected, or it is disconnected.

Disconnected case. If \( G \) is disconnected, then the graph can be split into two separate graphs \( G_1, G_2 \) with respectively \( n_1 \) and \( n_2 \) vertices. We will now separate (4)’ into two subprograms, Subprogram 1 associated with \( G_1 \), and Subprogram 2 associated with \( G_2 \), solve them separately, and rebuild a solution for the original program.

For these two graphs, let us construct \( A_1 \) the adjacency matrix of \( G_1 \) and \( A_2 \) the adjacency matrix of \( G_2 \). They have respectively \( n_1 \) and \( n_2 \) rows.

Now define \( q_1 \) and \( q_2 \) the subvectors of \( q \) of dimension \( 1 \times n_1 \) and \( 1 \times n_2 \) corresponding to the graphs \( G_1 \) and \( G_2 \) respectively. That is, the entries of \( q_1 \) are the same as the entries of \( q \) corresponding to columns in Subprogram 1, and the entries of \( q_2 \) are the same as the entries of \( q \) corresponding to columns in Subprogram 2.

Then define \( w_1 = 1/(eq_1) \), \( w_2 = 1/(eq_2) \), the inverse of the weight associated with each subprogram in the strategy \( q \).

We have that \( q_1w_1 \) and \( q_2w_2 \) are feasible solutions for Subprogram 1 and for Subprogram 2 respectively. Let \( k(G_1) \) and \( k(G_2) \) be the objective value attained by these two subprograms. We have that \( k(G_1) \geq w_1k \) and \( k(G_2) \geq w_2k \).

By induction, for each of the subprogram, there are vectors \( \Pi_1 \) and \( \Pi_2 \) which are optimal solutions and have a support associated with a graph composed of singletons, pairs and odd cycles.

Let now expand \( \Pi_1 \) and \( \Pi_2 \), which are respectively \( 1 \times n_1 \) and \( 1 \times n_2 \) vectors to \( q'_1 \) and \( q'_2 \), both \( 1 \times n \) vectors, with the entries corresponding to the related subprogram equal to \( \Pi_1 \) and \( \Pi_2 \), and the other entries equal to zero. For these strategies, we have that \( Aq'_1 \geq w_1ke \) and \( Aq'_2 \geq w_2ke \) (with the original \( A \) from Program (4)’). Now defining \( q' = q'_1/w_1 + q'_2/w_2 \), we have \( e^Tq' = 1 \) and \( Aq' \geq k \), which implies that \( q' \) is an optimal solution for (4).

Since the union of two graphs composed of disjoint singletons, pairs and odd cycles is also composed of disjoint singletons, pairs and odd cycles, the graph associated with the support of the strategy \( q' \) is composed of disjoint singletons, pairs and odd cycles, as wanted.

Connected case. For the connected case, if \( G \) is connected with \( n \geq 3 \), we claim that it has no vertex of degree one.
Indeed, consider the edge \((v_1, v_2)\) with the vertex \(v_2\) that would be of degree one. If \(v_1\) is also of degree one, then \((v_1, v_2)\) is a disconnected pair and \(G\) is not connected. If \(v_1\) has another adjacent edge, one can change its corresponding value in the strategy \(q\) to zero without changing \(k(t)\). Indeed, we can do the following: set the entry of the strategy \(q\) corresponding to the edge \((v_1, v_2)\) equal to \(k\) (because \(v_2\) is of degree one), set the of entries of \(q\) corresponding to the edges \((v_1, v_i)\) to zero, and set the values that were originally associated with these edges on the entries corresponding to the singletons \(v_i\). That way, we kept the objective \(k(t)\) while still satisfying the constraints of \((4)\).

This would imply that the value associated with edges \((v_1, v_i)\) is 0, and that this edge can be removed from \(A\) without changing \(k\), which is a contradiction with the definition of \(A\). Therefore if \(G\) is connected with \(n \geq 3\), it has no vertex of degree one.

We now notice that a connected graph with less than \(n\) edges and no vertex of degree one cannot have vertices of degree larger than two (because the sum of the degrees is equal to twice the number of edges). So all the vertices are in the same cycle. If the cycle is odd, we have our result. If it is even, let us number its edges. Set the weight \(2/k\) in the strategy \(q\) for edges with even numbers, and zero for edges with odd numbers. This gives a solution for \(q\) achieving the same objective \(k\). It implies that the odd edges can be removed from \(A\), which is again a contradiction. Therefore there is no even cycle in \(G\), which is what we wanted to prove.

\[\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.
\]

A possible minimal subset of the columns of \(A(3)\) is the following:

\[\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

Which corresponds to the solution of the original program \(q = (0, 0, 0, 0, 1/2, 0, 1/2)^T\).

The associated graph \(G\) is represented in Fig. 6. It is composed of two pairs. It already has the right structure, however we will continue the analysis to illustrate the subprograms.

As the graph is disconnected, we split it in two subprograms. Subprogram 1 includes vertices 0 and 3, Subprogram 2 includes vertices 1 and 2.

Therefore we have \(A_1 = (1,1)^T\) and \(A_2 = (1,1)^T\), the adjacency matrix of both subgraphs. Optimal solutions of these subprograms are \(\Pi_1 = (1)\) and

\[\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.
\]
Fig. 6. Graph $G'$ associated with $A'$.

$H_2 = (1)$, which each have a support forming a single pair (in the proof, by induction it is possible to find a solution with support associated to a graph composed of singletons, pairs and odd cycles).

The inverse of the weight associated with each of these subprograms in the strategy $q$ is $w_1 = 2$ for Subprogram 1 and $w_2 = 2$ for Subprogram 2. Expanding these solutions gives us $q'_1 = (1, 0)^T$, $q'_2 = (0, 1)^T$. This leads to the solution of the original program $q' = q'_1/w_1 + q'_2/w_2 = (1/2, 1/2)^T$, which has a support composed of two pairs.

For programs with this particular structure, we can compute $k(t)$ and the dimension of $P_t$:

**Lemma 21.** If the graph $G(t)$ associated with $A(t)$ is composed of disjoint odd cycles, pairs and singletons, then the optimum of Program (4) is given by $2/(n + n_1)$, where $n_1$ is the number of singletons. Moreover, the dimension of $P_t$ is the number of pairs.

**Proof.** To make notations concise, we define $K = n + n_1$. Our claim is that $k(t) = 2/K$. We provide an admissible solution for the primal (3), as well as for the dual (4), with the same objective value. Therefore this value is optimal.

A solution of (3) can be built as follows:

\[ p_i = \begin{cases} 
2/K & \text{if state } i \text{ corresponds to a singleton vertex,} \\
1/K & \text{otherwise.} 
\end{cases} \tag{5} \]

The sum of the coefficients is $\sum_{i=1}^n (p_i) = (n - n_1)/K + 2n_1/K = 1$, and $p$ is a feasible solution for (3) with objective value of $2/K$.

For (4), a solution can be built as follows:

\[ q_i = \begin{cases} 
2/K & \text{if column } i \text{ corresponds to a pair,} \\
1/K & \text{if column } i \text{ corresponds to an edge of an odd cycle,} \\
2/K & \text{if column } i \text{ corresponds to a singleton.} 
\end{cases} \tag{6} \]

The sum of the coefficients is 1, and $q$ is a feasible solution for (4) with objective value of $2/K$. Summarizing, equation (5) describes a solution for (3), and equation (6) describes a solution for (4), achieving the same objective value.
As the programs are dual, this implies that both strategies are optimal, and $k(t) = 2/K$.

We can now give an explicit expression for $P_t$. Reordering the vertices such that the first $f$ indexes correspond to singletons, the next $g$ indexes correspond to vertices in pairs, grouped by pair (two vertices in the same pair have indices $f + 2j - 1$ and $f + 2j, 1 \leq j \leq g/2$), and the last $h$ indexes correspond to vertices in odd cycles, we have the following set of optimal solutions for (3):

$$P_t = \{(p_1, ..., p_{f+g+h}) | p_i = 2/K, \quad 1 \leq i \leq f,$$

$$p_{f+2j-1} = 1/K + x_j, \quad 1 \leq j \leq g/2,$$

$$p_{f+2j} = 1/K - x_j, \quad -1/K \leq x_j, \leq 1/K,$$

$$p_k = 1/K, \quad f + g + 1 \leq k \leq f + g + h\}.$$ (7)

Indeed, the total value assigned to each pair in the strategy $p$ must be $k(t)$, and it can be split in any way between both vertices as long as none of the values is negative. The value assigned to singletons must be $k(t)$. For the odd cycles, the total value assigned to each pair in the cycle must be $k(t)$ (as it cannot be more than that for any pair, and in the optimal solution this value is effectively reached). However the only way to achieve that is to assign $k(t)/2$ to each vertex of the cycle.

The dimension of $P_t$ is $g/2$, the number of pairs (see [12] for details). □

Each point of the polytope $P_t$ presented in (7) can be mapped uniquely on a vector $(x_1, ..., x_{g/2})$. The parameters $x_j$ are independant and can vary between $-1/K$ and $1/K$, this describes a polyhedron of dimension equal to $g/2$, the number of pairs. The dimension of $P_t$ is therefore the number of pairs. □

We now present the main result of this section, which provides a universal upper bound on the triple rendezvous time for synchronizing automata. The main steps of the reasoning are as follows: starting from any original program obtained from an automaton and a value $t$, we can from Lemma 19 replace $A(t)$ with an other matrix to obtain a new optimization program with the same objective value, higher dimension for $P_t$, and the same structure as in Lemma 21.

For this program, from Lemma 21, we can easily compute the value of the SPF and an upper bound on the dimension of $P_t$. Then, making use of the lemma 13 on the evolution of $P_t$, we then obtain a lower bound on the SPF growing rate before $T_3$:

**Corollary 1** If $t < T_3$, then $k(t)$ can only take the values $2/(n+s), 0 \leq s \leq n-1$, and this value cannot be optimal at more than $\lfloor (n-s)/2 \rfloor + 1$ consecutive values of $t$.

**Proof.** By Lemma 19, for any $t < T_3$, we can replace $A(t)$ with a matrix $A$ which is a subset of the columns of $A(t)$, such that the graph $G$ associated is composed of singletons, pairs and odd cycles, and such that Program (4)', based on this matrix, achieves the same optimal objective. The set of optimal solutions $P_t'$ of this program has a dimension superior or equal to the dimension of the set of
optimal solution of the original program $P_t$. Let $s$ be the number of singletons in $\mathcal{G}$. There are $n - s$ vertices which are either in pairs or in odd cycles in $\mathcal{G}$. As the dimension of $P'_t$ is the number of pairs, it is at most $(n - s)/2$. Lemma 13 states that when $k(t)$ does not increase, the dimension of $P_t$ has to decrease. As the dimension of $P_t$ is bounded by the dimension of $P'_t$, $k(t)$ cannot stay at the same value for more than $\lfloor (n - s)/2 \rfloor + 1$ consecutive values of $t$. \hfill \Box

With Corollary 41, we can now obtain the main result of this section:

**Theorem 22.** In a synchronizing automaton $\Sigma$ with $n$ states,

$$T_{3,\Sigma} \leq \frac{n(n + 4)}{4} - \frac{n \mod 2}{4}.$$

**Proof.** By Corollary 41 the different values that the function $k(t)$ can take before $T_3$ are of the shape $2/(n + s)$, $0 \leq s \leq n - 1$, and this value can be the same for $\lfloor (n - s)/2 \rfloor + 1$ steps at most. Summing over all possible values for $k(t)$, one gets

$$\sum_{s=0}^{n-1} (\lfloor (n-s)/2 \rfloor + 1) = \sum_{s=1}^{n} (\lfloor s/2 \rfloor + 1) = \frac{n(n + 4)}{4} - \frac{n \mod 2}{4}.$$

\hfill \Box

This result, based on the count of the number of possible steps, provides us with a better upper bound on the triple rendezvous time. We will now consider another approach of the problem. Coupling the two approaches, we will again improve the bound.

This second approach is based on the observation that, if the synchronizing probability function at $t = n$ is small, then $T_3$ is also small.

**Lemma 23.** For $1 \leq s \leq n/2$, if $k(n) < \frac{1}{n-s}$, then for any word $W \in \Sigma^{\leq n}$, there are less than $s$ zeros entries in $eW$, with $e$ the all ones $1 \times n$ vector.

**Proof.** Suppose that there is a word $L \in \Sigma^{\leq n}$ such that $eL$ has more than $s$ zero entries. This also implies there are less than $n - s$ non zero entries in $eL$. Let $P$ be an optimal probability distribution on the states for Player Two for problem [3]. Applying the word $L$ to this probability distribution, we obtain $PL$, a probability distribution on the states. The zero entries of $eL$ are also zero entries of $PL$. Therefore, $PL$ has less than $n - s$ non-zero entries. As the sum of the entries is equal to one, the average of the non-zero entries is higher than $1/(n-s)$. This implies that one of the entries has a value higher or equal to $1/(n-s)$. Picking the corresponding state, Player One has a probability higher or equal to $1/(n-s)$ to win the game. As $P$ is an optimal strategy for Player Two, it implies that $k(n) \geq \frac{1}{n-s}$. Therefore we have the contrapositive, if $k(n) < \frac{1}{n-s}$, for any word $W \in \Sigma^{\leq n}$ there are less than $s$ zeros entries in $eW$. \hfill \Box
This can now be linked with the triple rendezvous time:

**Lemma 24.** For any strongly connected synchronizing automaton, and for any integer \(1 \leq s \leq n/2\), either

\[
k(n) > \frac{1}{n-s}
\]

or \(T_3 \leq n(s+2)/2\)

**Proof.** Let \(v\) be the first vertex to be of degree more than \(s\) in the graph \(G(t)\) associated with \(A(t)\) when \(t\) increases. This happens at the latest at \(t = sn/2\) as there is at least one new edge in the graph at every increase of \(t\) and there are only \(n\) vertices. This means that there exists a state \(q\), corresponding to \(v\), \(s\) states \(q_i\) corresponding to the vertices connected to \(v\), and words \(W_i \in \Sigma^{\leq sn/2}\), with \(1 \leq i \leq s\), such that \(qW_i = q_iW_i\).

Let us now construct the following word. We start with a non-permutation matrix, which gives weight two to one of the states. Since the graph is strongly connected, we can then send this state to state \(v\) with at most \(n-1\) letters. Let us call this word \(W_1\).

If \(k(n) \leq \frac{1}{n-s}\), then by Lemma 24 there are less than \(s\) zeros in any product of length \(n\). So, one of the vertices connected to \(v\) in \(G(sn/2)\) is of weight at least one after applying the word \(W_1\). Let us call \(W_2\) the word of length at most \(sn/2\) merging that state and \(v\).

The word \(W_1W_2\) is such that three states are mapped to a single state. Therefore either \(T_3 \leq sn/2 + n\) or \(k(n) > \frac{1}{n-s}\). \(\square\)

Coupling Theorem 1 and Lemma 24, we obtain the following upper bound:

**Proposition 25.** For any \(1 \leq s \leq n/2\),

\[
T_3 \leq \max \{n(s+2)/2, (n(n+4) - (2s-1)(2s+3) + 1)/4\} \tag{8}
\]

**Proof.** If

\[
k(n) \leq \frac{1}{n-s},
\]

we apply Theorem 1.

Otherwise, the values that can be taken by the synchronizing probability function between \(n\) and \(T_3\), are

\[
k(t) = \frac{2}{n+r},
\]

with \(0 \leq r \leq n-2s\).
We will now sum over all values of \( r \) the number of steps where they can occur. This number is smaller than
\[
\sum_{r=0}^{n-2s} \left\lfloor \frac{n-r}{2} \right\rfloor + 1 = \sum_{r=2s}^{n} \left\lfloor \frac{r}{2} \right\rfloor + 1
= \frac{(n)(n+4) - n \text{mod} 2}{4} - \frac{(2s-1)(2s+3) + 1}{4}
\leq (n(n+4) - (2s-1)(2s+3) + 1)/4. \tag{9}
\]

As this inequality on \( T_3 \) holds for all \( s \), we can find the expression of \( s \) as a function of \( n \) which minimize the maximum of the two functions. This leads to the following result.

**Corollary 2** There is a row of weight three at time \( (\sqrt{5n^2 + 4n - 12} - n + 6)n/8 \).

*Proof.* As \( n(s+2)/2 \) is an increasing function of \( s \), and \( (n(n+4) - (2s-1)(2s+3) + 1) \) is decreasing, the minimum of the maximum of the two is at the intersection of the two functions, or at one of the boundary of the domain, if the intersection is not on the domain. The value of \( s \) that minimizes the expression in (8) is the solution of
\[
n(s + 2)/2 = (n(n + 4) - (2s - 1)(2s + 3) + 1)/4
\]
under the constraint \( 1 \leq s \leq n/2 \). This expression can be re-written as a second degree equation:
\[
s^2 + s(n + 2)/2 + (1 - n^2/4) = 0,
\]
which has the solution
\[
s = (\sqrt{5n^2 + 4n - 12} - (n + 2))/4.
\]
Plugging this value into (8) we obtain
\[
T_3 \leq n(\sqrt{5n^2 + 4n - 12} - n + 6)/8.
\]
\[\square\]
This expression tends toward $\sqrt{5} - \frac{1}{8}n^2 = n^2/(6.4...)$ when $n$ grows, and therefore is about 50% better than the two other methods taken separately, which led to $n^2/4$.

4 A Counterexample to a Conjecture on the Synchronizing Probability Function

In this section, we present an infinite family of automata which are counterexamples to Conjecture [15] and Conjecture [19]. This family provides us with a lower bound on the maximum value of the triple rendezvous time for automata with $n$ states for every odd integer $n \geq 9$. The automaton with nine states and two letters in Fig. 7 is the first of the family.

![Automaton with 9 states and $k(11) = 2/9$.]

Conjecture [15] would imply that, for $j = 2$ and $n = 9$, $k(11) \geq 2/8$. However, the synchronizing probability function of the automaton in Fig. 7 at $t = 11$ is $k(11) = 2/9$, disproving the conjecture. Figure 8 represents the SPF of this automaton, compared with the SPF of the automaton of the Černý family on 9 states. At $t = 11$, the SPF of Černý’s automaton is larger than the SPF of the new automaton (from Fig. 7). This automaton also has the particularity that its triple rendezvous time equals 12, which is the number of states plus 3. Indeed, on the one hand it can be verified that the matrix $A(11)$ contains only columns of weight two. On the other hand, for the three initial states 0, 4 and 6 of the automaton, the automaton ends at state 4 after application of the word $abbabababba$, which is twelve letters long. Therefore this automaton is also a counterexample to Conjecture [19].

We can now extend this automaton with 9 states to an infinite family of automata with an odd number of states. Figure 9 shows the automata of this family with 11 and 13 states.

The recursive process to build the automaton of the family with $n$ states from the one with $n − 2$ states is the following:
1. We start from the automaton of the family with \( n - 2 \) states (with states numbered from 0 to \( n - 3 \), with \( n - 4 \) and \( n - 3 \) added last).

2. We remove the self loops from states \( n - 4 \) and \( n - 3 \).

3. We add state \( n - 2 \) with a self loop labelled as the self loop removed from state \( n - 3 \), and state \( n - 1 \) with a self loop with the other label.

4. We add the connections between states \( n - 2 \) and \( n - 4 \) in both directions, labelled as the self loop removed from state \( n - 4 \), and we add the connections in both directions between \( n - 1 \) and \( n - 3 \), with the other label.

All the automata of this family are such that \( T_3 = n + 3 \), and \( k(n+2) = 2/n \).

The graph in Fig. 10 presents the synchronizing probability function of these automata with 9, 11 and 13 states. The overall synchronizing time is not reaching the limit of Conjecture 4 but \( k(t) \) grows slowly for the first values of \( t \).
Fig. 10. The synchronizing probability function for the described automata for 9, 11 and 13 states (from the left to the right).

5 Conclusion

In this paper, we pushed further the study of the synchronizing probability function as a tool to represent the synchronization of an automaton. Our results are twofold and somewhat antagonistic: on the one hand, we managed to prove a non-trivial upper bound on the triple rendezvous time thanks to the synchronizing probability function. This result shows that this tool can effectively help in understanding synchronizing automata. On the other hand, we refuted Conjecture 15 formulated in [13], by providing an infinite family of automata for which $T_3 = n + 3$ (with $n$ being the number of states of the automaton). Conjecture 15 was stated as a tentative roadmap toward a proof of Cerny’s conjecture with the help of the synchronizing probability function, and in that sense our counterexample is a negative result towards that direction.

A natural continuation to this research would be to find non-trivial bounds for $T_s$, with $3 < s \leq n$ (i.e. the smallest number such that the reachable column set includes a column of weight at least $s$). Another research question is how to narrow the gap between $n + 3$ and $n^2/(6.4...)$ for the triple rendezvous time.

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