Statistics of leading digits leads to unification of quantum correlations

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received 15 December 2015; accepted in final form 9 May 2016
published online 31 May 2016

PACS 03.67.Mn – Entanglement measures, witnesses, and other characterizations
PACS 75.10.Jm – Quantized spin models, including quantum spin frustration
PACS 03.65.Aa – Quantum systems with finite Hilbert space

Abstract – We show that the frequency distribution of the first significant digits of the numbers in the data sets generated from a large class of measures of quantum correlations, which are either entanglement measures or belong to the information-theoretic paradigm, exhibit a universal behavior. In particular, for Haar uniformly simulated arbitrary two-qubit states, we find that the first-digit distributions corresponding to a collection of chosen computable quantum correlation quantifiers tend to follow the first-digit law, known as Benford’s law, when the rank of the states increases. Considering a two-qubit state which is obtained from a system governed by paradigmatic spin Hamiltonians, namely, the $XY$ model in a transverse field, and the $XXZ$ model, we show that entanglement as well as information-theoretic measures violate Benford’s law. We quantitatively discuss the violation of Benford’s law by using a violation parameter, and demonstrate that the violation parameter can signal quantum phase transitions occurring in these models. We also comment on the universality of the statistics of the first significant digits corresponding to appropriate measures of quantum correlations in the case of multipartite systems as well as systems in higher dimensions.

Introduction. – During the advancement of quantum information science, the search for correlations having truly quantum nature has been in focus. Quantum entanglement [1], a manifestation of quantum correlation, has emerged as the key ingredient in several quantum protocols [2–4] and computation [5]. Over time, exciting results such as non-zero non-classical efficiency of quantum states having vanishing entanglement, and locally indistinguishable orthogonal product states [6–8] have led to the development of quantum correlations beyond the “entanglement-separability” paradigm, broadly known as the quantum information-theoretic measures [9]. This has led to a large set of bona fide measures of quantum correlations (see [1,9], and references therein), each of which, belonging usually to either of the two paradigms, has its own degree of importance, applicability, and computability.

The measures of quantum correlations, belonging to the entanglement-separability domain, are significantly different from those of the quantum information-theoretic origin, as indicated by their several properties (see [1,9], and references therein). Interestingly, despite such differences, measures from the two domains show some similar behavior in a variety of physical scenarios, such as decoherence [10,11], monogamy [12–14], and signaling cooperative phenomena such as quantum phase transitions (QPTs) [15,16]. In the case of pure states, all the bipartite measures belonging to the two classes behave quite similarly [1,9]. It is thus natural to ask whether an interlink exists between the values obtained from the different measures of quantum correlations having drastically different origins. There has been efforts to relate the measures of quantum correlations belonging to two different paradigms [17], although no conclusive result, as yet, exists.

In this paper, we show that the statistical properties of the different measures of quantum correlation, belonging to either of the paradigms, exhibit a universal behavior in the sense that the frequency distribution of the first significant digits occurring in a dataset corresponding to a quantum correlation measure exhibits a decaying profile irrespective of which measure is used. In the case of arbitrary two-qubit states with increasing rank, we quantitatively demonstrate, by using standard distance metrics, that the observed frequency distributions obtained
from all the quantum correlations tend to follow Benford’s law [18–20] – an empirical law which states that the first significant digit in any data is more often 1, and which has been studied in various fields like biology, geology, finance models, etc. [18–23]. The universality is found to be retained even when specific subsets of the set of two-qubit states are considered, but it is hindered when the quantum states correspond to physical systems governed by specific Hamiltonians [23], as demonstrated by considering the well-known XY chain in a transverse external magnetic field [16,24], and the XZX chain [25,26]. Indeed, it is possible to find a suitable quantum correlation measure whose Benford’s violation parameter faithfully detects quantum-critical points in these models. We also investigate the digit distributions of distance-based entanglement and monogamy-based quantum correlation measures for three-qubit pure states, and computable quantum correlation measures for bipartite systems in higher dimensions. We show that the first-digit distribution for the geometric measure of entanglement can distinguish between the two inequivalent classes of three-qubit pure states, namely, the Greenberger-Horne-Zeilinger (GHZ) class [27] and the W class [28].

Statistics of leading digits. – The study of the statistics of leading digits started in 1881 [18], when the astronomer Simon Newcomb observed that the occurrence of the digits from 1 to 9 as the first significant digit of the numbers in a given set of data is not randomly distributed. Specifically, the frequency distribution, \( p_0(d) \), of the first significant digits, \( d (d \in \{1, 2, \ldots, 9\}) \), is governed by an empirical law given by \( p_0(d) = \log_{10} \left( \frac{d}{9} \right) \). It is widely known as Benford’s law due to its rediscovery by Frank Benford in 1938, who verified the law for a wide range of natural datasets [19]. Since then, the frequency distribution of the first significant digits of the numbers occurring in datasets of various origins has attracted a lot of attention of the scientific community, and Benford’s law has been tested in diverse areas of science [20]. Mathematical insight regarding the scale invariance of Benford’s law has also been obtained in recent studies [21].

Interestingly, not all naturally occurring datasets obey Benford’s law. The deviation of the observed frequency distribution, \( p_0(d) \), can be quantified by computing the distance of \( p_0(d) \) from \( p_0(d) \). This quantity, known as the “Benford violation parameter” (BVP), \( \nu \) [22,23], depends on the two distributions as well as on the distance metric used to quantify the separation between them. For the present study, we consider three specific metrics, namely, the mean deviation (MD), the standard deviation (SD), and the Bhattacharyya metric (BM), in terms of which the corresponding BVP are given by [23]

\[
\nu_{md} = \sum_{d=1}^{9} \left[ |p_0(d) - p_0(d)|/p_0(d) \right],
\nu_{sd} = \sum_{d=1}^{9} \left[ \frac{p_0(d) - p_0(d)}{p_0(d)} \right]^2/3, \quad \nu_{bm} = -\ln \sum_{d=1}^{9} \left[ p_0(d)p_0(d) \right]^{\frac{1}{2}},
\]

where \( \nu_{md}, \nu_{sd}, \) and \( \nu_{bm} \) correspond to MD, SD, and BM, respectively. The concept of violation of Benford’s law has been used in several scenarios as diverse as economics, election processes, digital image manipulation, seismology, and quantum phase transition [22,23].

In the case of a data collected from a specific physical phenomenon, the measured quantity, \( q \), usually has its own range of values, which might lead to a trivial violation of Benford’s law. For example, in a data corresponding to \( q \) having values \( 2 \leq q < 3 \), the first significant digit will always be \( d = 2 \). Such triviality can be avoided by suitable shifting and scaling of \( q \) achieved by “Benford’s quantity” (BQ), \( q_b \), defined as \( q_b = (q - q_{min})/(q_{max} - q_{min}) \), where \( q_{min} \) and \( q_{max} \) are, respectively, the minimum and the maximum values of \( q \) in the dataset [23]. The definition of \( q_b \) implies a mapping of the actual range of \( q \) onto the range [0, 1] of \( q_b \). Unless otherwise stated, we use BQ to compute the frequency distributions, \( p_b(d) \), and the subsequent values of \( \nu \).

Universality in quantum correlations. – In the present study, we use two broad genres of measures of quantum correlations, namely, the entanglement measures, and the quantum information theoretic measures. Our choice of bipartite quantum correlations belonging to the first category includes entanglement of formation (EoF) [29], concurrence, \( C \) [12,29], negativity, \( N \) [30], logarithmic negativity, LN [31], and relative entropy of entanglement, \( S_R \) [32], while in the second category, we choose quantum discord (QD), \( D \) [33], and one-way quantum work deficit (QWD), WD [34]. In the multiparty cases, we restrict ourselves to pure states due to the lack of computable multipartite measures of quantum correlations in the case of mixed multiparty states, and also due to computational difficulties of numerical generation of multipartite mixed state. In this domain, we consider the generalized geometric measure (GGM), \( G \) [35,36], and monogamy scores, \( \delta \) [12–14], corresponding to squares of different bipartite quantum correlations, such as \( C, N, D, \) and WD.

We first consider the bipartite measures, and study the behavior of \( p_b(d) \) in the case of quantum states in \( C^2 \otimes C^2 \). To compute \( p_b(d) \), one can either generate arbitrary pure as well as mixed two-qubit states Haar uniformly in the state space, or consider a parametrization of an arbitrary two-qubit state in terms of the correlation matrix and the local Bloch vectors. Let us first concentrate on the case in which to determine \( p_b(d) \) corresponding to different measures of quantum correlations in \( C^2 \otimes C^2 \) systems, we Haar uniformly generate a sample of \( 10^6 \) two-qubit states, \( \rho_{12} \), for each of the ranks \( r = 2, 3, \) and 4. The rank-1 states form the set of pure states, on which we shall comment later. We find that the values of \( \nu \), for different measures of quantum correlations, decrease with the increase of ranks of the states in the case of arbitrary two-qubit states, irrespective of the distance metrics (see fig. 1). We find that \( \nu \) is minimum and considerably low for \( r = 4 \), the maximum rank possible for a two-qubit state, implying a \( p_b(d) \) very close to \( p_0(d) \), irrespective of the distance measures. Hence, it is reasonable to conclude that the frequency
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distribution of the first significant digits in a data corresponding to a given measure of quantum correlation closely mimics Benford’s law when the two-qubit quantum states have full rank. Interestingly, we note that BVP for entanglement measures shows monotonic behavior with respect to the rank, while that for quantum information-theoretic measures shows non-monotonicity with the increase of \( r \). Figures 2(a) and (b) depict the histogram representations of the profiles of \( p_0(d) \) originated from different measures with \( d \) in the cases of two-qubit mixed states of rank \( r = 2 \) and \( r = 4 \), respectively.

Table 1: The values of \( \nu_{\text{nd}}, \nu_{\text{sd}}, \) and \( \nu_{\text{bm}} \) for LN, LN, C, EoF, D, and WD in the case of arbitrary two-qubit states of rank 4, with \( x = 10^2 \) and \( y = 10^3 \).

| BVP  | NN | LN | C  | EoF | D   | WD |
|------|----|----|----|-----|-----|----|
| \( \nu_{\text{nd}} \) | 1.770 | 2.340 | 2.431 | 0.768 | 3.497 | 4.960 |
| \( x\nu_{\text{sd}} \) | 2.086 | 2.751 | 2.806 | 1.091 | 5.250 | 6.231 |
| \( y\nu_{\text{bm}} \) | 4.838 | 8.026 | 8.770 | 0.982 | 23.487 | 43.074 |

Fig. 1: (Color online) Variations of BVP (ordinate), namely, \( \nu_{\text{nd}}, \nu_{\text{sd}}, \) and \( \nu_{\text{bm}} \), corresponding to LN and EoF with increasing rank, \( r \) (abscissa), in the case of arbitrary two-qubit states. Insert: trends of \( \nu_{\text{nd}}, \nu_{\text{sd}}, \) and \( \nu_{\text{bm}} \) (ordinate), corresponding to D, against \( r \) (abscissa), in the case of arbitrary two-qubit states. To compare the trends of the graphs, the values of \( \nu_{\text{nd}} \) and \( \nu_{\text{bm}} \) are multiplied by a factor \( x = 50 \) in the case of all the measures of quantum correlation. All quantities are dimensionless.

Fig. 2: (Color online) Histograms representation of \( p_0(d) \) (ordinate), of the first significant digit, \( d \) (abscissa), corresponding to NN, LN, C, EoF, D, and WD in the case of arbitrary two-qubit states of rank (a) \( r = 2 \), and (b) \( r = 4 \). The distributions closely obey Benford’s law for \( r = 4 \), confirming the observation in fig. 1. All the quantities plotted are dimensionless.

two-qubit states, we therefore find a quantifier using Benford’s law, where patterns unify all the quantum correlation measures, and erase their origin. On the other hand, violation parameters of Benford’s law are capable to distinguish them.

It is well known that an arbitrary two-qubit state, up to local unitary transformation, can be expanded in terms of nine real parameters and the Pauli matrices, \( \sigma^x, \sigma^y, \sigma^z (\alpha, \beta, x, y, z) \), as [37] \( \rho_{12} = \frac{1}{4} (I_1 \otimes I_2 + \sum_{\alpha} \sigma^\alpha \sigma^\alpha \otimes \sigma^\alpha + \sum_{\beta} (\sigma^\beta \otimes I_2 + \sum_{\gamma} (\sigma^\gamma \otimes \sigma^\gamma)) \), where \( c^{\alpha\beta} = \langle \sigma^\alpha \sigma^\beta \rangle \), called the “classical correlators”, are the diagonal elements of the correlation matrix with \( |c^{\alpha\alpha}| \leq 1 \), \( c_1^\alpha = \langle \sigma_1^\alpha \otimes I_2 \rangle \) and \( c_2^\alpha = \langle I_1 \otimes \sigma^\alpha_2 \rangle \), called the “magnetizations”, are the elements of the two local Bloch vectors where \( |c_1^\alpha, c_2^\alpha| \leq 1 \), and \( I_{1(2)} \) is the identity operator in the Hilbert space of qubit 1 (2). A two-qubit state of the form \( \rho_{12} \) can have a maximum rank, \( r = 4 \). To compute \( p_0(d) \), we randomly generate \( 10^6 \) two-qubit states in the parameter space by choosing the relevant parameters from uniform distributions in their allowed ranges to check the effect of different simulations of states on Benford’s law. We simulate sets of \( 10^6 \) random states of the form \( \rho_{12} \) for each of the measures, by choosing the parameters from uniform distributions in appropriate ranges. We observe that \( p_0(d) \), corresponding to all the quantum correlation measures, closely follow Benford’s law.

It is now natural to ask whether \( p_0(d) \) corresponding to two-qubit states generated within specific subsets of the complete set of arbitrary two-qubit states possesses such universal feature. To address this question, we consider the following three special instances, two of which can be obtained as special cases of \( \rho_{12} \): 1) Bell-diagonal (BD) states, obtained from \( \rho_{12} \) with \( c_1^\alpha = c_2^\alpha = 0 \). 2) Two-qubit states with single magnetization, \( e.g., \), the \( z \) magnetization, obtained by setting \( c_1^{z,y} = c_2^{z,y} = 0, c_1^{x,y} \neq c_2^{x,y} \) are \( c_1^{z,y} \neq 0 \). Here, \( \rho_{12} \), written in the computational basis \( \{|00\}, \{|01\}, \{|10\}, \{|11\} \) assumes the form of an X-state [38] with real matrix elements. 3) Generic two-qubit \( X \) states with complex off-diagonal matrix elements. In the first case, \( p_0(d) \) is computed by generating a set of \( 10^6 \) random states by choosing the three diagonal correlators from a uniform distribution in their allowed ranges, while in the cases 2) and 3), \( p_0(d) \) is examined by generating a set of \( 10^6 \) states by choosing the real matrix elements (for case 2)), or the real and imaginary parts of the matrix.
elements (in case 3)) from both uniform and normal distributions. In all the three cases listed above, we observe that \( p_\theta(d) \) for both entanglement and information-theoretic measures, closely follows Benford’s law. For example, we find that the BVP, \( v_{bm} \), corresponding to transverse correlation measures (see [15], and references herein). It also follows Benford’s law. For example, we observe that the BVP, \( v_{bm} \), corresponding to different correlation measures, obeys such universality. Hence the universality of the variation of \( p_\theta(d) \) against \( d \) is retained in the case of entanglement measures as well as quantum information-theoretic measures, when arbitrary two-qubit states in the parameter space are considered. Such analysis indicates that the quantum correlation measures, irrespective of the choice of the measures, tend to follow Benford’s law, especially when the states are of full rank, independent of the simulation process.

For a given physical system, the allowed quantum states are governed by the Hamiltonian describing the system. We now ask what happens to such unifying features of \( p_\theta(d) \) corresponding to different quantum correlations, if it is constrained by the system Hamiltonian. To address this question, we consider a quantum spin model in one dimension, given by the Hamiltonian \( H = \sum_{i=1}^{N} \hat{H}_i \) with \( \hat{H}_i = J/2 \left[ (1+\gamma)\sigma^x_i \sigma^x_{i+1} + (1-\gamma)\sigma^y_i \sigma^y_{i+1} + \Delta \sigma^z_i \sigma^z_{i+1} \right] + h \sigma^z_i \). Here, \( J \) and \( h \) are, respectively, the strengths of the exchange interaction and the transverse magnetic field. The anisotropies in the strengths of the exchange interaction are given by in the \( z \)-direction, and in the \( x \)-, \( y \)-direction. The total number of quantum spin-(1/2) particles in the system is \( N \). Note here that the present case is an example where the observable, \( q_i \), may have an allowed range of values in the corresponding phase, and, thus, may result in a trivial violation of Benford’s law. To avoid this, we employ the procedure described in [23], where we determine \( p_\theta(d) \) corresponding to a fixed value of say \( \lambda = \lambda_0 \), by considering a dataset of \( n \) values of \( q_i \), which is computed for a sample of \( n \) values of \( \lambda \), and is normalized in such a way that \( 0 \leq q \leq 1 \), in an interval of width \( \epsilon \) around \( \lambda_0 \).

We consider two specific cases of the Hamiltonian \( H \): a) the anisotropic XY model in a transverse field, represented by \( \Delta = 0 \) [24], and b) the XXZ model, given by \( \Delta = 0 \) and \( h = 0 \). In the case a), with a choice of the system parameter as \( \lambda \equiv J/h \), the model, with increasing \( \lambda \), undergoes a QPT at \( \lambda = 1 \) from an antiferromagnetic phase to a quantum paramagnetic phase [16]. In the case b), two QPTs take place at \( \Delta = \pm 1 \). For \( \Delta = -1 \), the ground state of the model undergoes a Kosterlitz-Thouless (KT) QPT, while at \( \Delta = 1 \), a QPT from the metallic phase (0 < \( \Delta \) < 1) to the insulating phase (\( \Delta > 1 \)) occurs [25,26]. The QPTs in both the models have been detected by standard condensed-matter physics techniques [16,24,25] as well as several quantum correlation measures (see [15], and references herein). It has been shown that BVPs corresponding to transverse magnetization and two-site LN can also signal the occurrence of the QPT in the transverse-field XY model [23]. In the present study, we find that for the XXZ model, all the QPTs on the \( \Delta \) axis described above are signalled by the BVP corresponding to different entanglement measures even when the system size is small, but not by the same corresponding to information theoretic measures. Figure 3 depicts the variation of \( v_{bm} \) corresponding to nearest-neighbour concurrence and quantum discord for the ground state of the XXZ model with \( \Delta \).

The QPT at \( \Delta = 1 \) is signaled by a sharp kink in the case of concurrence, while no change in the case of quantum discord takes place (see fig. 3).

We choose specific values of the system parameters \( \lambda \), in different phases of the models, and determine \( p_\theta(d) \) corresponding to different bipartite quantum correlation measures. Following [23], in each case, \( p_\theta(d) \) is computed from a sample of \( n = 5 \times 10^3 \) data points in the vicinity of the fixed value of \( \lambda \), with a small fixed interval \( \epsilon = 0.1 \). We observe that the universality in the behavior of \( p_\theta(d) \), as discussed in the case of arbitrary bipartite states, is washed out as the states are governed by specific spin Hamiltonians such as the XY model in a transverse field, or the XXZ model. Curiously, in specific phases of the XY model, the profiles of \( p_\theta(d) \) corresponding to the entanglement measures are found to be similar for both finite system size as well as in the thermodynamic limit. On the other hand, the profiles of \( p_\theta(d) \) corresponding to QD are found to be different across QPT (see fig. 4 for the XY model). However, note that even in the absence of universality of \( p_\theta(d) \) in the case of quantum states governed by a specific Hamiltonian, the values of the BVP corresponding to different values of the chosen system parameters in both the models remain small as demonstrated in table 2 for the antiferromagnetic and the paramagnetic phases of the XY model.

**Universality in higher dimensions and higher number of parties.**—We now address the question as to whether the universal behavior of \( p_\theta(d) \) is generic when systems with a higher number of parties, or in higher dimensions are considered. However, generating multipartite quantum states as well as states in higher dimensions having all possible ranks are themselves non-trivial problems. In our study, we comment on the nature of \( p_\theta(d) \) in
the cases of certain paradigmatic classes of states. More specifically, we consider pure states in three-qubit systems, the complete set of which is constructed by the union of two independent classes of states, namely, the GHZ class and the W class [27,28]. In the case of higher-dimensional systems, we simulate rank-2 states in $C^2 \otimes C^3$ and $C^2 \otimes C^4$.

To check for unifying features as in the case of two-qubit systems, we Haar uniformly generate sets of $10^6$ states in each of the cases including tripartite pure states of the GHZ and the W classes, and higher-dimensional states in $C^2 \otimes C^3$ and $C^2 \otimes C^4$ systems, and determine the relevant quantum correlations. It is important to note that the measures that we consider for two-qubit systems do not have a direct generalization for three-qubit systems, and hence we choose a distance-based entanglement measure, namely, GGM [36], and monogamy-based measures [12–14] originated from bipartite quantum correlations.

The profiles of $p_0(d)$ and the corresponding values of $\nu$ for different quantum correlations, in all the mentioned cases, are found to be inconclusive in determining whether such universality exists. Note that the multiparty states considered here are pure (rank-1) states, while all the higher-dimensional bipartite states have rank 2, and in the case of qubit systems, the universal feature becomes prominent in the case of states with higher rank. Hence it is reasonable to infer that conclusive results on the

universality of the behavior of $p_0(d)$ in the case of multipartite systems as well as systems in higher dimensions can only be obtained once the corresponding states up to their full rank can be considered. For the three-qubit pure states, $p_0(d)$ corresponding to the W class exhibits considerable unifying behavior compared to the GHZ class (for comparison, see figs. 5(a) and (b)). For example, $\nu_{bnm}$, obtained from the simulation of the W-class states for $\delta_{xy}$, $\delta_{zz}$, $\delta_{W2}$, and $\delta_{W2}$, are, respectively, given by $2.006 \times 10^{-3}$, $3.832 \times 10^{-3}$, $9.846 \times 10^{-3}$, and $31.593 \times 10^{-3}$, showing a very low Benford’s violation similar to the two-qubit states. Note here that the monogamy scores corresponding to $D^2$ and $W^2$ are computed by performing measurement over the nodal observer. Interestingly, we observe that although the value of GGM cannot distinguish these two inequivalent classes, its first-digit distribution can.

Concluding remarks. – In this paper, we showed that statistics of the first significant digits occurring in the data corresponding to different measures of quantum correlations, belonging to both entanglement-separability and the information-theoretic paradigms, exhibit a universal feature. More specifically, according to our data analysis, the frequency distributions of the first significant digits in the numbers corresponding to all measures of quantum correlations, considered here, tend to obey Benford’s law, when states with higher ranks are considered, irrespective of their origin and nature, in general. This feature becomes prominent when quantum states with full rank are considered. We also discuss the effect when states corresponding to specific Hamiltonians, namely, the $XY$ model in a transverse field, and the $XXZ$ model, are considered. We find that although the universality is washed out in such cases, BVP can detect QPTs occurring in these models. We finally analyse the first significant digit distribution of different computable quantum correlation measures for multipartite as well as higher-dimensional systems, and show that first-digit distribution of geometric measure of multipartite entanglement for the W-class states closely follows Benford’s law, while it is not the case for the GHZ class, indicating yet another quantity of completely different root from the ones known in the literature [14,28] that can identify these two classes.
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