Moment and exponential tail estimations for norms of random variables and random operators in mixed (anisotropic) Lebesgue - Riesz spaces.

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Abstract.

We study the random variables (r.v.) with values in the so-called mixed (anisotropic) Lebesgue - Riesz spaces: formulate the sufficient conditions for belonging of the r.v. to these spaces, estimate the tail of norms distribution, especially deduce the exponential decreasing tails of them, etc.

We obtain as a consequence the estimations of the norms of random integral operators acting between these spaces.

Key words and phrases.

Random variables and vectors (r.v.), distribution and tails of distributions, Tchebychev - Markov inequality, moment, kernel, Hölder's inequality, Lebesgue - Riesz and Grand Lebesgue Spaces (GLS), norms, Young - Fenchel transform, Young inequality, exponential decreasing function, random and ordinary linear integral operators, measure, mixed (anisotropic) spaces, permutation inequality, factorable functions.
1 Statement of problem. Notations. Previous works.

We recall here the definition of the so-called anisotropic Lebesgue (Lebesgue-Riesz) spaces, which appeared in the famous article of Benedek A. and Panzone R. [6]. More detail information about this spaces with described applications see in the classical books of Besov O.V., Ilin V.P., Nikolskii S.M. [7], chapter 1,2; Leoni G. [23], chapter 11.

Let \( (\Omega = \{ \omega \}, \mathcal{B}, \mathcal{P}) \) be certain probability space with expectation \( E \) and variation \( \text{Var} \), and let

\[
(X_k = \{ x_k \}, A_k, \mu_k), \quad k = 1, 2, \ldots, l, \quad l < \infty
\]

be measurable spaces with sigma - finite separable non - trivial measures \( \mu_k \). The separability denotes that the metric space \( A_k \) relative the distance

\[
\rho_k(D_1, D_2) = \mu_k(D_1 \setminus D_2) + \mu_k(D_2 \setminus D_1) = \mu_k(D_1 \Delta D_2), \quad D_{1,2} \subset A_k
\]

is separable.

Denote \( X := \otimes_{k=1}^l X_k \), so that \( x \in X \iff x = \bar{x} = \{ x_1, x_2, \ldots, x_l \} \).

Let also \( \bar{p} = p = (p_1, p_2, \ldots, p_l) \) be \( l \) dimensional fixed numerical vector such that \( 1 \leq p_j < \infty \). Recall that the anisotropic (mixed) Lebesgue - Riesz space

\[
L(p) = L_p = L(\bar{p}, \{ X_k \}, \{ \mu_k \})
\]

consists by definition on all the total measurable real valued function \( f = f(x_1, x_2, \ldots, x_l) = f(\bar{x}) \):

\[
f : \otimes_{k=1}^l X_k \to R,
\]

having a finite norm \( ||f||_{\bar{p}} \overset{def}{=} \)

\[
\left( \int_{X_1} \mu_1(dx_1) \left( \int_{X_{l-1}} \mu_{l-1}(dx_{l-1}) \ldots \left( \int_{X_1} \mu_1(dx_1) |f(\bar{x})|^{p_1} \right)^{p_2/p_1} \right)^{p_3/p_2} \ldots \right)^{1/p_l}.
\]

In particular, for the one - dimensional numerical valued r.v. \( \xi = \xi(\omega) \) as well as for the number \( p, p \in [1, \infty) \) we obtain the classical Lebesgue - Riesz \( L_p(\Omega) \) norm

\[
||\xi||_p = (E|\xi|^p)^{1/p},
\]

as well as ones
$$||f||_{p,X_k} = \left[ \int_{X_k} |f(x_k)|^p \mu_k(dx_k) \right]^{1/p}.$$  

Note that in general case \( ||f||_{p_1,p_2} \neq ||f||_{p_2,p_1} \), but \( ||f||_{p,p} = ||f||_p \).

Observe also that if \( f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2) \), (condition of factorization), then

$$||f||_{p_1,p_2} = ||g_1||_{p_1} \cdot ||g_2||_{p_2},$$  \( (3) \)

(formula of factorization).

Note that under conditions separability of measures \( \mu_k \) these spaces are also separable and Banach spaces.

These spaces arises in the Theory of Approximation, Functional Analysis, theory of Partial Differential Equations, theory of Random Processes etc.

Let for example \( l = 2; \) we agree to rewrite for clarity the expression for \( ||f||_{p_1,p_2} \) as follows:

$$||f||_{p_1,p_2} := ||f||_{p_1,X_1; p_2,X_2}.$$  

Analogously,

$$||f||_{p_1,p_2,p_3} = ||f||_{p_1,X_1; p_2,X_2; p_3,X_3}.$$  

Notice that the last expression may be rewritten as follows:

$$||f||_{p_1,p_2,p_3} = || || || ||f||_{p_1,X_1} ||f||_{p_2,X_2} ||f||_{p_3,X_3}.$$  \( (4) \)

Let us recall also the following important fact: the so - called permutation inequality, in the terminology of an original article [6]; see also the monograph [7], chapter 1, pp. 24 - 26. Indeed, let \((Z, B, \nu)\) be another measurable space and \( \phi : (X, Z) = X \otimes Z \rightarrow R \) be common measurable function. In what follows as before \( X = \otimes_{k=1}^l X_k \).

Let also \( r = \text{const} \geq \overline{p} \), where

$$\overline{p} := \max_{k=1}^d p_k.$$  

It is true the following permutation inequality (in our notations):

$$||\phi||_{p,X; r,Z} \leq ||\phi||_{r,Z; p,X}.$$  \( (5) \)

In what follows \( Z = \Omega, \nu = P \).
2 Estimation of the distribution for the norm of random field.

Let $\eta = \eta(x, \omega)$, $x \in X$ be separable total measurable numerical valued random field (r.f.). Define the following random variable

$$\zeta = \zeta(\omega) \overset{def}{=} \|\eta\|_{p, X} = \|\eta\|_{\bar{p}, X}. \quad (6)$$

We intent to estimate in this section the Lebesgue - Riesz probabilistic norms of the r.v. $\zeta : |\zeta|_{r, \Omega}$, $r \geq 1$.

**Theorem 2.1.** Denote

$$\psi[p](r) \overset{def}{=} \| \|\eta\|_{r, \Omega} \|_{p, X}. \quad (7)$$

Suppose $r \geq \bar{p}$; then

$$[E|\zeta|^r]^{1/r} = \|\zeta\|_{r, \Omega} \leq \psi_p(r). \quad (7)$$

**Proof** is simple. We have

$$\|\zeta\|_{r, \Omega} = \| \|\eta\|_{p, X} \|_{r, \Omega}. \quad (7)$$

One can apply the permutation inequality (5):

$$\| \|\eta\|_{p, X} \|_{r, \Omega} \leq \| \|\eta\|_{r, \Omega} \|_{\bar{p}, \Omega} = \psi[p](r), \quad (7)$$

Q.E.D.

Suppose in addition that the introduced above function $\psi[p](r)$ is finite for some non-trivial segment $r \in [a, b)$, where $a = \bar{p}$, $a < b = \text{const} \leq \infty$. The proposition of theorem (2.1) may be reformulated as follows. Introduce as ordinary the so-called Grand Lebesgue Space $G_p$ builded on our probability space consisting on all the r.v.'s $\xi$ having a finite norm

$$\|\xi\|_{G_p}[\cdot] \overset{def}{=} \sup_{r \in (a, b)} \left\{ \frac{\|\xi\|_r}{\psi[p](r)} \right\}. \quad (8)$$

The general theory of these spaces is represented in many works, see e.g. [1], [3], [4], [8], [10], [12], [13], [14], [15], [16], [17], [18],[21], [25],[26], [27], [28] etc. In particular, these spaces are complete, Banach functional and rearrangement invariant.

We have from the proposition (7) of theorem 2.1

$$\|\zeta\|_{G_p}[\cdot] \leq 1. \quad (9)$$
Introduce the following function
\[ g_p(u) := \sup_{r \in (a,b)} \left( ur - r \ln \psi[p](r) \right) - \]
the (regional) Young-Fenchel transformation for the function \( r \to r \ln \psi[p](r) \), relative the variable \( r \), for the values \( r \in (a,b) \).

It follows from (9) the following exponential decreasing (in general case) tail estimation for the distribution of the random variable \( \zeta \):
\[ P(|\zeta| > u) \leq \exp(-g_p(u)), \ u \geq 1. \]
The inverse conclusion also holds true under appropriate natural conditions, see [12], [21], [28].

**Example 2.1.** Let \( \exists C(p) < \infty \ \forall r \geq 1 \ \Rightarrow \psi[p](r) \leq C(p) \ r^{1/m}, \ m = \text{const} > 0; \ \text{then} \ \exists C_2(m,p) > 0 \ \Rightarrow \]
\[ P(|\zeta| > u) \leq \exp(-C_2(m,p) \ u^m), \ u \geq 1, \]
and inverse conclusion is also true.

**Example 2.2.** Let us clarify slightly the applicability of our theorem yet in the case \( d = l = 1 \). Namely, let again \( \eta = \eta(x) = \eta(x,\omega) \) be separable measurable numerical valued random field; put
\[ \zeta = \zeta(\omega) = ||\eta||_{p,X}, \ p \in [1,\infty). \]
How one can estimate the tail of distribution of the r.v. \( \zeta \)?

Answer. Let \( r \) be arbitrary number greatest than \( p : \ r \geq p \). We propose by virtue of Theorem 2.1
\[ \left[ E|\zeta|^r \right]^{1/r} \leq \left\{ \int_X \mu(dx) \left[ E|\eta(x)|^r \right]^{p/r} \right\}^{1/p}. \]

3 Estimation of the norm of random operators.

Let also
\[ L(q) = L_q = L(\vec{q}) = L(\vec{q}, \{Y_j\}, \{\nu_j\}), \ j = 1, 2, \ldots, d; \ d < \infty \]
be another "\( d \)" - dimensional mixed (anisotropic) space and put as before \( Y := \otimes_{j=1}^d Y_j \).
Let \( K = K(x, y, \omega), \ y \in Y, \ x \in X, \ \omega \in \Omega \) be certain kernel, i.e. a numerical valued total measurable function, on the other words, random kernel. The function \( K(\cdot, \cdot, \cdot) \) is named also a random field (r.f.).

Introduce the following important linear random integral operator \( U \) having the kernel \( K \):

\[
U[g](x) \overset{def}{=} \int_{Y} K(x, y, \omega) \ g(y) \ \prod_{j=1}^{d} \nu_j(dy_j), \ y \in Y, \ x \in X, \quad (11)
\]

or briefly

\[
U[g](x) \overset{def}{=} \int_{Y} K(x, y, \omega) \ g(y) \ \nu(dy), \ y \in Y, \ x \in X, \ \omega \in \Omega. \quad (12)
\]

Our target in this section is investigation of conditions for the finiteness a.e. of this operator acting between two mixes Lebesgue spaces and estimate the distribution of its norm.

There are several publications devoted the theme of the random operators acting between different Banach spaces: [5], [11], [19], [30], [31], a classical monograph [29] and so one. The case of stochastic integral operators is considered in particular in [5], [29].

Let us recall the following simple estimation for the norm of linear deterministic integral kernel operator of the form

\[
f(x) = V[g](x) \overset{def}{=} \int_{Y} V(x, y) \ g(y) \ \nu(dy), \ x \in X, \ y \in Y. \quad (13)
\]

It follows immediately by virtue of Hölder’s inequality that

\[
\|f\|_{r,X} \leq \| \ V(\cdot, \cdot) \|_{q,Y} \ \|g\|_{p,Y} \times \|g\|_{p,Y}, \quad (14)
\]

where as before \( p = \vec{p} = \{p_j\}, \ q = \vec{q} = \{q_j\}, \ i = 1, 2, \ldots, d; \)

\[
\frac{1}{p_j} + \frac{1}{q_j} = 1, \ p_j, q_j \in (1, \infty), \quad (15)
\]

and \( r = \vec{r} = \{ r_k \}, \ k = 1, 2, \ldots, l; \ 1 < r_k < \infty; \) see e.g. the classical monograph [20], chapter XI, section 3.

Remark 3.1. There are many works devoted to the operators norm estimates for concrete form of these linear operators: Fourier, Laplace, Pseudo-differential, singular, convolution, Bochner, Riesz, fractional, Hardy-Littlewood, Hausdorff etc. The fundamental investigation of this theory may be found in particular in the famous book of G.O.Okikiolu [24]; see also the brief review in an article [27].
Let us return to the random integral operator \( U \), described above in (11), (12). We have by virtue of (14) under the same notations

\[
||U[g]||_{r,X} \leq ||K||_{q,Y} ||_{r,X} \times ||g||_{p,Y}.
\]

(16)

Further, let \( s \) be some number such that

\[
s \geq \max \{ \max r_k, \max q_j \}.
\]

(17)

We have taking the norm \( || \cdot ||_{s,\Omega} \) from both the sides of the inequality (16) taking into account the permutation inequality

\[
|| ||U[g]||_{r,X} ||_{s,\Omega} \leq || ||K||_{s,\Omega} ||_{q,Y} ||_{r,X} \times ||g||_{p,Y}.
\]

(18)

To summarize. Denote for these values of the parameters

\[
\theta(s) = \theta_{p,r}(s) := || ||K||_{s,\Omega} ||_{q,Y} ||_{r,X},
\]

(19)

\[
A := \max \{ \max r_k, \max q_j \}.
\]

(20)

**Theorem 3.1.** Suppose that for some values \( a \geq A, b \in (a, \infty] \implies \theta(s) < \infty \). Then

\[
|| ||U[g]||_{r,X} ||_{G_{a,b}} \leq 1,
\]

(21)

with correspondent tail estimation.

**Example 3.1.** Assume that the random kernel \( K(x, y, \omega) \) allows a factorization:

\[
|K(x, y, \omega)| \leq K_0(x, y) \tau(\omega),
\]

where the non-negative (measurable) functions \( K_0(\cdot, \cdot), \tau \) are such that

\[
h(q, r) := || ||K_0||_{q,Y} ||_{r,X} < \infty
\]

and

\[
\exists(a, b) = \text{const}, 1 \leq a < b \leq \infty, \forall s \in (a, b) \implies \rho(s) := ||\tau||_{s,\Omega} < \infty.
\]

Then

\[
\theta_{p,r}(s) \leq h(q, r) \cdot \rho(s), \ s \in (a, b).
\]

(22)
4 Concluding remarks.

A. The essential unimprovability of obtained estimations may be illustrated, for instance, in the one-dimensional case \( l = d = 1 \), see e.g. [24], chapters 3,4; [27]. Namely, define the factorable r.f. \( \eta(x, \omega) = \tau(\omega) \cdot h(x) \), \( \omega \in \Omega \), \( x \in X \), where for definiteness

\[
|| \tau ||_{r,\Omega} = 1 = || h ||_{p,X}.
\]

Then both the sides of inequality (7) are equal to 1, by virtue of (3):

\[
|| h \cdot \tau ||_{p,X} ||_{r,\omega} = || h(\cdot) ||_{p,X} \times || \tau ||_{r,\Omega} = 1 =
\]

\[
|| h \cdot \tau ||_{r,\Omega} ||_{p,X},
\]

still without the restriction \( r \geq p \).

B. Let us ground the unimprovability of the integral operators norm estimations (13), (14). Indeed, assume as above \( d = l = 1 \); \( \nu(Y) = 1 \), and set \( g(y) = 1 \); and let the kernel \( V(\cdot) \) be degenerate: \( V(x, y) = v(x) \). Then both the hand sides of (13) are equal:

\[
f(x) = v(x),
\]

\[
|| f ||_{r,X} = || v ||_{r,X} = || V(\cdot, \cdot) ||_{q,Y} ||_{r,X} = || V(\cdot, \cdot) ||_{p,Y} \times || g ||_{p,Y}.
\]

C. The unimprovability of the assertion of theorem 3.1. about the random linear operators follows formally from ones for deterministic operator. One can use also the slightly modified previous example, choosing for instance

\[
K(x, y, \omega) := V(x) \cdot \xi(\omega),
\]

where the non-zero r.v. \( \xi \) belongs to the Lebesgue - Riesz space \( L_r(\Omega), r > 1 \), and as before \( \nu(Y) = 1, g(y) = 1 \).

D. The unimprovability of both the propositions of theorem 2.1 and 3.1 in the multidimensional case may be shown by means of consideration of the factorable functions, i.e. when

\[
g(y) = \prod_{j=1}^{d} g_j(y_j),
\]

\[
f(x) = \prod_{k=1}^{d} f_k(x_k),
\]

\[
V(x, y) = \prod_{k=1}^{d} V_k(x_k)
\]

etc.

E. It is interest in our opinion to investigate analogously the case of non-linear random operators, as well as consider a possibility when \( l = \infty \) or \( d = \infty \).

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