A Rigorous Theory of Prethermalization without Temperature

Wen Wei Ho\(^1\) and Wojciech De Roeck\(^2\)

\(^1\)Department of Physics, Stanford University, Stanford, CA 94305, USA
\(^2\)Instituut Theoretische Fysica, KU Leuven, 3001 Leuven, Belgium

(Dated: December 9, 2020)

Prethermalization refers to the physical phenomenon where a system evolves toward some long-lived non-equilibrium steady state before eventual thermalization sets in. One general scenario where this occurs is in driven systems with dynamics governed by an effective Hamiltonian (in some rotating frame), such that ergodicity of the latter is responsible for the approach to the prethermal state. This begs the question whether it is possible to have a prethermal state not associated to any effective Hamiltonian. Here, we answer this question in the affirmative. We exhibit a natural class of systems in which the prethermal state is defined by emergent, global symmetries, but where the dynamics that takes the system to this state has no additional conservation laws, in particular energy. We explain how novel prethermal phases of matter can nevertheless emerge under such settings, distinct from those previously discussed.

Introduction.—Recent years have seen dramatic progress towards understanding universal features of quantum many-body systems out of equilibrium. This has led to discoveries of exotic nonequilibrium physical phenomena, like ergodicity-breaking scenarios of many-body localization [1–4] and quantum many-body scars [5, 6], as well as novel phases of matter realized only in driven settings, like the discrete time crystal (DTC) [7–9], anomalous Floquet insulator [10, 11] and chiral Floquet phases [12, 13].

A key development in the theory of non-equilibrium many-body systems has been the establishment of the paradigm of prethermalization [13–30]. This is the phenomenon where the system evolves towards a long-lived, quasistationary state, which might exhibit interesting features not found in the true equilibrium state realized at extremely late times. Perhaps the cleanest incarnation of the phenomenon is exemplified by a system periodically driven at a high frequency: it has been shown that despite time-translation symmetry being explicitly broken, there nevertheless exists a static, quasilocal energy operator \( H_{\text{eff}} \), an ‘effective Hamiltonian’, that is approximately conserved for times exponentially long in the driving frequency, and is moreover an effective generator of stroboscopic dynamics [20–23, 31–33]. The system therefore resists heating towards the featureless, infinite-temperature state expected on grounds of entropy maximization in the absence of any global conservation laws, and instead equilibrates to a thermal state with well-defined temperature set by its (approximately conserved) energy, at least for such ‘prethermal times’.

The prethermal regime of a time-periodic (Floquet) system can in fact exhibit much richer structure than just conservation of energy. Ref. [34] identified a class of strong, high-frequency driving that leads to an additional \( Z_n \) symmetry of \( H_{\text{eff}} \), whose presence underpins the existence of Floquet prethermal phases. It is important to note that this symmetry is an emergent and robust one, and not tied to any exact, microscopic conservation laws nor a particular fine-tuned driving protocol.

Our results represent a rigorous realization of an exotic scenario of “prethermalization without temperature” coined by Ref. [38], who envisaged a scenario where an emergent charge conservation leads to nontrivial dynamics for long times, despite dynamics occurring either (i) due to an effective Hamiltonian but at energies corre-
sponding to high or infinite temperatures, or (ii) in the absence of an effective Hamiltonian description such that temperature is not well defined in the prethermal state. Ref. [38] analyzed the former situation. Here, we provide general conditions showing that the latter scenario can in fact occur. Indeed, one of the nontrivial physical consequences of our work is that just the conservation of an emergent charge in the prethermal state is already sufficient to sharply define distinct prethermal phases of matter, for example those distinguished by the absence or presence of a non-zero plateau of a local ‘order parameter’ in dynamics. This represents a novel class of prethermal phases different than has been previously discussed, for instance by [34, 36], which are based on the existence of an effective Hamiltonian.

**Key ideas.—** The general setting behind our theorems is encapsulated by the following class of driven many-body Hamiltonians

\[ G(t) = \nu N + H(t), \]

where \( N \) is a term that has uniform spectral gaps which remain open in the thermodynamic limit (for example, a Zeeman field on quantum spins or a non-vanishing bandgap of lattice fermions, see [13, 39]). \( H(t) \) represents interactions or couplings that depend periodically on time with frequency \( \omega \). Here, \( \nu \) is the amplitude of \( N \), taken to be much larger than all other local energy scales, which include \( \omega \) and the local bandwidth of \( V(t) \) assumed to be bounded at any time by \( J \). We do not need any relation of \( \omega \) to \( J \). What we will show is that under an additional assumption of sufficient smoothness of the drive, there is a dressed version of \( N \) that is conserved to exponentially long times in \( \nu \).

To intuitively understand why this might be true, consider temporarily that \( H(t) \) is time-independent. Then we are back to a previously considered situation [22] where we know there is an emergent charge which is approximately conserved. Schematically, the large spectral gaps of \( \nu N \) entail that we can ‘integrate out’ processes coupling different subspaces of \( N \), leading to only effective couplings within them (this is akin to a Schrieffer-Wolff transformation [40]). As transitions are local in real space by assumption, such a procedure is always well-defined initially and can be carried out to high-orders \( \sim \nu/J \) until a many-body resonance is encountered, which gives a bound on the rate of loss of conservation [22].

Eq. (1) represents a generalization in which there can be additional processes that couple states separated in energy by multiples of the drive frequency \( \omega \) (Fig. 1). In particular, there can be direct, resonant transitions between states belonging to different \( N \) sectors with energy difference \( \nu \), with accompanying absorption or emission of \( n \sim \nu/\omega \) quanta of drive energy \( \omega \) effected by the Fourier modes \( H_n \) of the Hamiltonian. This can in principle result in a rapid loss of conservation of \( N \). Our key observation is that we can suppress such deleterious processes if we impose that \( H(t) \) is sufficiently smooth in time, such that there is a fast decay of \( H_n \) with Fourier number \( n \sim \nu/\omega \). Such processes can also be suppressed if the drive is smooth (local in Fourier space) so that the Fourier modes decay fast with \( n \).

**Statement of theorems.—** We state here our formal theorems in a manner as self-contained as possible, relegating the complete mathematical details to the Supplemental Material (SM) [41]. We consider a quantum many-body Hamiltonian \( G \) on a lattice with locally bounded Hilbert space, for example of quantum spins or fermions. We also take it to be parameterized by angles \( \theta \in [0, 2\pi) \) (Floquet case) or \( \theta = (\theta_1, \ldots, \theta_m) \in [0, 2\pi)^m \) (quasiperiodic case, \( m \geq 2 \)). Indeed, we can define, in a slight abuse of notation, a time-periodic Hamiltonian via \( G(t) := G(\theta t) \) where \( \theta t = \omega t + \theta_0 \mod 2\pi \) (\( \omega \): frequency, \( \theta_0 \): arbitrary initial phase), and a time-quasiperiodic one \( G(t) := G(\theta t) \) where \( \theta t = \omega t + \theta_0 \mod 2\pi \) for each argument. Here \( \omega = (\omega_1, \ldots, \omega_m) \) is a vector of frequencies and \( \theta_0 \) an arbitrary vector of initial phases. In both cases, our object of interest is the unitary propagator \( U(t) \) satisfying the Schrödinger equation \( i\partial_t U(t) = G(t)U(t) \) with initial condition \( U(0) = I \).

We furthermore assume the Hamiltonians \( G(\theta), G(\theta) \)

![FIG. 1. Processes out of a given sector of \( N \). The operator \( \nu N \) has large spectral gaps \( \nu \), depicted as a tower of energies. There are processes effected by the Fourier mode \( H_n \) (also present in the absence of a drive). The local nature of interactions entail they enter only with amplitude \( J \), hence are off-resonant and heavily suppressed. In the presence of the drive there can be processes involving involving multiples of the frequency \( \omega \), in principle leading to a resonant coupling effected by \( H_n \) with Fourier number \( n \sim \nu/\omega \). Such processes can also be suppressed if the drive is smooth (local in Fourier space) so that the Fourier modes decay fast with \( n \).]
are sufficiently local in real space and smooth in $\theta, \theta$ (this translates to the driven Hamiltonian $G(t)$ being smooth in time). We measure this via a local norm $\|G(\cdot)\|_\kappa$, parameterized by a decay constant $\kappa > 0$, which takes into account the decay of local terms making up $G(\cdot)$ in both spatial extent and Fourier space (see SM for details [41]; similar norms were used in [22, 36]). We can now state our first theorem.

Theorem 1. Approximate $U(1)$-conservation in Floquet systems. Let $N$ (i) be a sum of local terms that mutually commute $\mathcal{E}$ (ii) has integer spectrum, and $H(\theta)$ be a many-body Hamiltonian where $\theta \in [0, 2\pi)$, with local norm $\|H(\theta)\|_{\kappa_0} < \infty$ for some $\kappa_0 > 0$. Let $\omega > 0$ and define the local energy scale $\nu_0 := \max\{2\|H(\theta)\|_{\kappa_0}, \omega\}$. We consider the Hamiltonian

$$G(\theta) = \nu N + H(\theta)$$

and (correspondingly, dynamics under the time-periodic (Floquet) Hamiltonian $G(t) := G(\theta_t)$ with fundamental frequency $\omega$), where the amplitude $\nu$ is assumed large, specifically $\nu > C\nu_0$ for some constant $C$ depending only on $\kappa_0$ but not the volume of the system. Then, there is a small unitary $e^{A(\theta)}$ effected by a quasilocally, anisotermic operator $A(\theta)$, such that the unitary propagator corresponding to $G(t)$ can be written

$$U(t) = e^{A(\theta_t)} T \exp \left(-i \int_0^t ds \nu N + D(\theta_s) + V(\theta_s) \right) e^{-A(\theta_0)},$$

where $T$ represents time-ordering, and $D(\theta), V(\theta)$ are quasilocally, many-body Hamiltonians satisfying

$$\|D(\theta) - \langle H(\theta) \rangle\|_\kappa \leq C'(\nu_0/\nu),$$

$$\|V(\theta)\|_\kappa \leq \nu_0 2^{-n_*},$$

$$[D(\theta), N] = 0.$$ (5)

Here $\kappa = \kappa_0/4$, $\langle \cdot \rangle$ represents the symmetrization operation $\langle O(\theta) \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \phi^{i\phi N} O(\theta) e^{-i\phi N}$, and $n_* = \lfloor \nu_0/\nu \rfloor$. $C', c$ are numerical constants independent of volume.

Unpacking the theorem, it says that there is a small (close to identity) time-periodic change of frame such that dynamics is generated by a time-periodic Hamiltonian $\nu N + D(\theta)$ which conserves $N$, a $U(1)$ charge. Corrections to this (explicit symmetry-breaking terms $V(\theta_t)$) are very weak, being exponentially suppressed in $\nu$, and can be ignored. This statement can be made precise for the case of local observables invoking Lieb-Robinson bounds [42], see [41]. One obvious consequence is that in the laboratory frame, the dressed charge $\tilde{N} = e^{A(\theta_0)} N e^{-A(\theta_0)}$, which is a sum of quasilocals, is approximately conserved at stroboscopic times $t = ZT$ ($T$: period) up to a prethermal time $\tau$ that is exponentially long in $\nu/\nu_0$ [41]. Relatedly, the original charge $N$ is approximately conserved for similar times, albeit up to a bounded error of $O(\nu_0/\nu)$.

Crucially, the theorem does not require any relation of the drive frequency $\omega$ to local energy scales of the system $\|H(\theta)\|_{\kappa_0}$. In particular, $\omega$ could be smaller or even comparable to $\|H(\theta)\|_{\kappa_0}$, such that a further ‘high-frequency’ (e.g. Magnus) expansion on the time-dependent $D(\theta_t)$ to obtain an effective Hamiltonian might not make sense. It is generally expected that there is then no notion of energy which is (approximately) conserved. Thus, our result pertains to one in which there is a long-lived emergent charge conservation in a driven system, without necessarily accompanying emergent energy conservation, as claimed.

Remark 1: while Theorem 1 specifies a set-up where $\nu$ is constant in time, we can actually apply it to a larger class of cases where $\nu = \nu(t)$ is time-periodic, see [41]. Remark 2: We can extend our result to conservation of multiple $U(1)$ charges, by upgrading $N$ to $r$ mutually commuting $U(1)$ charges $N_1, \ldots, N_r$, and promoting $\nu \mapsto \nu = (\nu_1, \ldots, \nu_r)$. Similarly, we can promote $\theta \mapsto \theta$ and achieve results of $U(1)$ charge conservation in quasiperiodically-driven systems [41]. Remark 3: Technically speaking, our theorem applies to systems with interactions decaying at least exponentially with distance in real space, implicit in the definition of local norm. This restriction may be lifted to encompass long-range interactions combining techniques of [43].

Let us provide here a sketch of the proof; details are in [41]. The proof technique relies on a rigorous implementation of Schrieffer-Wolff transformations, i.e. many-body versions of Kolmogorov-Arnold-Moser (KAM) or Nekoroshev techniques, and it goes back at least to [44]; we have been especially influenced by [22, 36, 45–49]. The logic is to iteratively renormalize the Hamiltonian such that terms off-diagonal in $N$ have reduced amplitude, possible because the large energy scale $\nu$ allows to ‘integrate out’ such processes. More precisely, we introduce a sequence of small unitaries $e^{-A_0(\theta)} e^{-A_1(\theta)} \ldots$ with antithermionic $A_0(\theta), A_1(\theta), \ldots$, so that at step $n + 1$ we have a rotated Hamiltonian

$$\nu N + H_{n+1}(\theta) \equiv e^{-A_n(\theta)} (\nu N + H_n(\theta) - \lambda \partial_\theta) e^{A_n(\theta)}.$$ (6)

(The original Hamiltonian $H(\theta)$ is labeled $H_0(\theta)$). If the Hamiltonian were time-independent, the last term $i\lambda \partial_\theta e^{A_0(\theta)}$, a ‘gauge potential’, would not exist and we would reduce to the analysis of [22]. There it was shown how a choice of $A_n$ satisfying $[\nu N, A_n] = -V_n$ performed the renormalization, where $V_n := H_n - \lambda H_n$ is the off-diagonal part of $H_n$ (we term the diagonal part $D_n := \langle H_n \rangle$). Indeed, $A_n$ is $1/\nu$ small, so expanding $\nu N + H_{n+1}$ (still pretending it is time-independent) yields $\nu N + D_n + V_n = [A_n, \nu N] + O(\nu^{-1})$ and we see the strength of $V_n$ in $H_{n+1}$ is reduced by a factor $1/\nu$ relative to $V_n$. Of course, all objects are many-body operators and so we should measure amplitudes via the local norm $\|\cdot\|_\kappa$. The price to pay of the renormalization is
a slight decrease of the locality of the Hamiltonian $H_{n+1}$ resulting in a smaller reduction factor than just $1/\nu$.

In our present case, we must account for the effect of the gauge potential. Now, suppose we continue to choose the previous solution of $A_n$ for each $\theta$, simply promoting $A_n \mapsto A_n(\theta)$. Then the key point is: this should $V_n(\theta)$ be a smooth function of $\theta$, so will $A_n(\theta)$. Its derivative will then be bounded, and we can estimate $\|w e^{-A_n(\theta)} \partial_\theta e^{A_n(\theta)}\|_\text{local} \sim \|\partial_\theta A_n(\theta)\|_\text{local} \omega \leq C \times \|\nu_n(\theta)\|_\text{local} \omega / \nu$ which will be small should $\omega \ll \nu$. That is to say, the size of the off-diagonal terms $V_{n+1}(\theta)$ will still be $\sim 1/\nu$ smaller than $V_n(\theta)$ even in the time-dependent scenario. Iterating the procedure up to the optimal order $n$, then yields Theorem 1.

A slight modification leads us to:

**Theorem 2.** Approximate $\mathbb{Z}_n$-charge conservation in quasiperiodically-driven systems away from the high-frequency limit. Consider $N$ (i) a sum of local terms that mutually commute and (ii) has integer eigenvalue spacings. Fix a non-zero integer $n$ and let $H(\theta)$ be a many-body Hamiltonian on $\theta \in [0, 2\pi]^2$, assuming that $\|H(\theta)\|_{\text{local}}$ for some $\kappa_0 > 0$. We introduce a frequency vector $\omega = (\nu, \omega)$, define $v_0 := \max\{2\|H(\theta)\|_{\kappa_0}, \omega\}$, and consider the Hamiltonian

$$G(\theta) = \frac{\nu}{n} N + H(\theta)$$

(and corresponding, the time-quasiperiodic Hamiltonian $G(t) := G(\theta(t))$). We take $\nu > C v_0$ for some constant $C$ depending on $\kappa_0$ but not on the system’s volume. Then, there is a small time-quasiperiodic unitary $e^{A(\theta_1)}$ affected by a quaslocal, antithermion operator $A(\theta)$ such that the unitary propagator can be written

$$U(t) = e^{A(\theta_1)} T \exp \left(-i \int_0^t d\theta \frac{\nu}{n} N + D(\theta_1) + V(\theta_1)\right) e^{-A(\theta_0)},$$

where $D(\theta), V(\theta)$ are quasilocally Hamiltonians satisfying

$$\|D(\theta) - \langle H(\theta) \rangle\|_{\kappa} \leq C' (v_0/\nu),$$

$$\|V(\theta)\|_{\kappa} \leq v_0 2^{-n},$$

$$|D(\theta), g| = 0.$$  (9)

Here $D(\theta) = D'(\theta_2)$ has dependence only on $\theta_2 \in [0, 2\pi)$, $\kappa = \kappa_0/4$, $g = e^{-n N}$ is a generator of the $\mathbb{Z}_n$ group satisfying $g^\nu = 1$, $\langle \cdot \rangle$ is the symmetrization operation $\langle O(\theta) \rangle = \frac{1}{2\pi n} \int_0^{2\pi n} d\theta e^{-i\pi N} O(\theta) e^{i\pi N}$, $n_* = \lceil \nu/v_0 \rceil$, and $C', c$ are numerical constants.

Theorem 2 (proof given in the SM [41]) specifies that in this strongly-driven set-up, there is a small time-quasiperiodic change of frame where dynamics is generated by a $\mathbb{Z}_n$-symmetric, but now time-periodic (in $T_2 = 2\pi/\omega$) Hamiltonian $D'(\omega t + (\theta_0)_2)$, with small corrections. Therefore, similar to Theorem 1, the Heisenberg time evolution of a local operator is essentially governed just by this symmetric time-dependent Hamiltonian, for exponentially long times [41]. Note that in this scenario, we have utilized that one of the drive frequencies, $\nu$, is the large energy scale, while the other drive frequency $\omega$ need not be: it can again be comparable to or smaller than local energy scales. This is thus not captured by the high-frequency regime of [36] and hence a different kind of $\mathbb{Z}_n$ symmetry conservation from the one identified there. We remark that the amplitude of $N$ in Eq. (6) can also be made time-periodic in $T_1 = 2\pi/\nu$ as long as its time-average equals $\nu/n$. Moreover, Theorem 2 can be upgraded to encompass multiple long-lived emergent $\mathbb{Z}_n$ charges [41].

**Discussion.**—We now spell out the physical consequences, focusing on the case of an emergent $U(1)$-charge conservation in a Floquet system without energy conservation. The prethermal state reached (after a relaxation time $t_* \sim \nu_{\text{avg}}^{-1}$, which is much shorter than the prethermal time $\tau \sim \nu_{\text{avg}}^{-1} e^{(\nu/v_0)}$) is simply one determined by a fixed density of the conserved charge $N$. Despite this apparent simplicity, this setting does allow for distinct dynamical phases with a dynamical phase transition separating them. Indeed, consider as a paradigmatic case

$$N = -\frac{1}{2} \sum_{(i,j)} \sigma_i^x \sigma_j^x,$$  (10)

where $\sigma^z$ are spin-1/2 Pauli-matrices and the summation is over nearest-neighbors on a lattice, i.e. the classical Ising Hamiltonian, which can be understood as the total number of domain walls in the system. In an equilibrium ensemble determined by domain wall density $n \approx N/V$ with $V$ the volume, we have ferromagnetic ordering if $n < n_c(d)$ a critical density, provided spatial dimension $d > 1$. This ordering stems from the $\mathbb{Z}_2$ (spin-flip) symmetry that $N$ additionally possesses.

Now, the key point is that such ordering will still be preserved dynamically, even if the time-dependent Hamiltonian $D(\theta)$ driving dynamics is not $\mathbb{Z}_2$-symmetric, as long as it does preserve $N$. In particular, we could have $D(\theta) = h(\theta) \sum_i \sigma_i^z$. This stability is due to the kinetic barrier that separates the positive and negative magnetization (as measured by $N_m = \sum_i \sigma_i^z$) sectors within the ensemble of constant $N$. Indeed, one can argue that if the domain wall density is low enough, it is difficult in $d > 1$ to change the total magnetization of the system significantly without also changing the number of domain walls at the same time. As our Theorem guarantees $N$ (appropriately) conserved in dynamics, it means that an initial state with low enough domain wall density and large net magnetization will have its magnetization survive up to at least the parametrically-long prethermal timescale, while a state with high domain wall density and large net magnetization will have its magnetization decay rapidly. Furthermore, such behavior is robust to changes in the drive protocol, justifying their terminology as realizing prethermal ‘phases of matter’.
Our Theorem can also be directly applied to constrain dynamics in experimentally-relevant systems. Consider an ensemble of Rydberg atoms interacting via strong, repulsive Van der Waals forces between Rydberg states [50–53]. The effective Hamiltonian is one acting on a collection of two-level systems spanned by states $|g\rangle, |r\rangle$:

$$H = \frac{\Omega(t)}{2} \sum_i \sigma_i^x - \Delta(t) \sum_i n_i + \sum_{i<j} V_{ij} n_i n_j,$$

(11)

where $\sigma_i^x = |g\rangle_i \langle r| + \text{h.c.}$, $n_i = |r\rangle_i \langle r|$, and $V_{ij} \propto \frac{1}{|i-j|^3}$. The atoms can be arranged in such a way that the interaction between nearest-neighbor pairs $\langle ij \rangle$ is dominant, so we can identify $\nu N = \sum_{\langle ij \rangle} V_{ij} n_i n_j$. Now it is natural to argue that due to the large separation of energy scales we can effectively work with states with definite $\nu$ so we can identify $\nu N = \sum_{\langle ij \rangle} V_{ij} n_i n_j$. This is covered by Theorems in [22, 43]. Our present Theorems guarantee that this intuition in fact continues to hold for a large class of time-dependent scenarios, thereby allowing for an analysis of dynamics still within the Rydberg-blockaded space.

Lastly, let us briefly comment on extensions beyond our work. Besides having a large energy scale, a key idea was that the drive should be smooth. Such a treatment hence excludes step-drives, for example one where $N$’s amplitude varies as $+2/3\nu$ for half a period and $-1/3\nu$ for the other half, but is otherwise the dominant energy scale instantaneously. The work of [54] covers such a scenario and presents similar results as us (albeit with different assumptions), indicating a prethermalization mechanism different from the one identified here.

**Conclusion.**—We have exhibited the occurrence of a novel prethermalization scenario. Its distinguishing feature is that the prethermal state is defined by a conserved, emergent charge, a $U(1)$ or $Z_n$-charge, without any reference to an effective, static Hamiltonian. Despite its simplicity, this setting already allows for nontrivial examples of dynamical phases.

**Acknowledgments.**—W.W.H. is supported in part by the Stanford Institute for Theoretical Physics at Stanford University. W.D.R. acknowledges the support of the Flemish Research Fund FWO undergrants G098919N and G076216N, and the support of KULeuven University under internal grant C14/16/062.

[1] D. Basko, I. Aleiner, and B. Altshuler, *Annals of Physics* **321**, 1126 (2006).
[2] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, *Phys. Rev. Lett.* **95**, 206603 (2005).
[3] R. Nandkishore and D. A. Huse, *Annual Review of Condensed Matter Physics* **6**, 15 (2015).
[4] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, *Rev. Mod. Phys.* **91**, 012001 (2019).
[5] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, *Nature Physics* **14**, 745 (2018).
[6] W. W. Ho, S. Choi, H. Pichler, and M. D. Lukin, *Phys. Rev. Lett.* **122**, 040603 (2019).
[7] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, *Phys. Rev. Lett.* **116**, 250401 (2016).
[8] D. V. Else, B. Bauer, and C. Nayak, *Phys. Rev. Lett.* **117**, 090402 (2016).
[9] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, *Phys. Rev. Lett.* **118**, 030401 (2017).
[10] G. Titum, E. Berg, M. S. Rudner, G. Refael, and N. H. Lindner, *Phys. Rev. X* **6**, 021013 (2016).
[11] F. Nathan, D. Abanin, E. Berg, N. H. Lindner, and M. S. Rudner, *Phys. Rev. B* **99**, 195133 (2019).
[12] H. C. Po, L. Fidkowski, T. Morimoto, A. C. Potter, and A. Vishwanath, *Phys. Rev. X* **6**, 041070 (2016).
[13] N. H. Lindner, E. Berg, and M. S. Rudner, *Phys. Rev. X* **7**, 011018 (2017).
[14] J. Berges, S. Borsáiny, and C. Wetterich, *Phys. Rev. Lett.* **93**, 142002 (2004).
[15] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature* **440**, 900 (2006).
[16] M. Moosiek and S. Kehrein, *Phys. Rev. Lett.* **100**, 175702 (2008).
[17] M. Eckstein, M. Kollar, and P. Werner, *Phys. Rev. Lett.* **103**, 056403 (2009).
[18] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, *Science* **337**, 1318–1322 (2012).
[19] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, *Phys. Rev. Lett.* **115**, 180601 (2015).
[20] T. Mori, T. Kuwahara, and K. Saito, *Phys. Rev. Lett.* **116**, 120401 (2016).
[21] T. Kuwahara, T. Mori, and K. Saito, *Annals of Physics* **367**, 96 (2016).
[22] D. Abanin, W. De Roeck, W. W. Ho, and F. Huse, *Communications in Mathematical Physics* **354**, 809 (2017).
[23] D. A. Abanin, W. De Roeck, W. W. Ho, and F. Huse, *Phys. Rev. B* **95**, 014112 (2017).
[24] O. Howell, P. Weinberg, D. Sels, A. Polkovnikov, and M. Bukov, *Phys. Rev. Lett.* **122**, 016002 (2019).
[25] T. Mori, *Phys. Rev. B* **98**, 104303 (2018).
[26] A. Rajak, R. Citro, and E. G. Dalla Torre, *Journal of Physics A: Mathematical and Theoretical* **51**, 465001 (2018).
[27] K. Mallayya, M. Rigol, and W. De Roeck, *Phys. Rev. X* **9**, 021027 (2019).
[28] P. Reimann and L. Dabelow, *Phys. Rev. Lett.* **122**, 080603 (2019).
[29] F. Huse, *Annals of Physics* **51**, 465001 (2018).
[30] A. Rubio-Abadal, M. Ippoliti, S. Hollerith, D. Wei, J. Rui, S. L. Sondhi, V. Khemani, C. Gross, and I. Bloch, *Phys. Rev. X* **10**, 021044 (2020).
[31] A. Lazarides, A. Das, and R. Moessner, *Phys. Rev. E* **90**, 012110 (2014).
[32] L. D’Alessio and M. Rigol, *Phys. Rev. X* **4**, 041048 (2014).
[33] M. Bukov, L. D’Alessio, and A. Polkovnikov, *Advances
in Physics 64, 139–226 (2015).

[34] D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. X 7, 011026 (2017).

[35] P. T. Dumitrescu, R. Vasseur, and A. C. Potter, Phys. Rev. Lett. 120, 070602 (2018).

[36] D. V. Else, W. W. Ho, and P. T. Dumitrescu, Phys. Rev. X 10, 021032 (2020).

[37] S. Blanes, F. Casas, J. Oteo, and J. Ros, Physics Reports 470, 151–238 (2009).

[38] D. J. Luitz, R. Moessner, S. L. Sondhi, and V. Khemani, Phys. Rev. X 10, 021046 (2020).

[39] T. Gulden, E. Berg, M. S. Rudner, and N. Lindner, SciPost Physics 9 (2020), 10.21468/scipostphys.9.1.015.

[40] S. Bravyi, D. P. DiVincenzo, and D. Loss, Annals of Physics 326, 2793–2826 (2011).

[41] See Supplemental Material online for proofs of theorems, extensions of results, and details on physical consequences.

[42] E. H. Lieb and D. W. Robinson, Communications in Mathematical Physics 28, 251 (1972).

[43] F. Machado, D. V.Else, G. D. Kahanamoku-Meyer, C. Nayak, and N. Y. Yao, Phys. Rev. X 10, 011043 (2020).

[44] N. Datta, R. Fernández, and J. Fröhlich, Journal of Statistical Physics 84, 455 (1996).

[45] G. Benettin, J. Fröhlich, and A. Giorgilli, Communications in Mathematical Physics 119, 95 (1988).

[46] N. Cuneo, J.-P. Eckmann, and C. E. Wayne, Nonlinearity 30, R81–R117 (2017).

[47] A. Giorgilli, S. Paleari, and T. Penati, Annales Henri Poincaré 16, 897–959 (2014).

[48] J. Fröhlich, T. Spencer, and C. E. Wayne, Journal of Statistical Physics 42, 247 (1986).

[49] J. Z. Imbrie, Journal of Statistical Physics 163, 998–1048 (2016).

[50] H. Labuhn, D. Barredo, S. Ravets, S. de Léséleuc, T. Macri, T. Lahaye, and A. Browaeys, Nature 534, 667–670 (2016).

[51] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletic, and M. D. Lukin, Nature 551, 579 EP (2017), article.

[52] H. Kim, Y. Park, K. Kim, H.-S. Sim, and J. Ahn, Phys. Rev. Lett. 120, 180502 (2018).

[53] A. Browaeys and T. Lahaye, Nature Physics 16, 132–142 (2020).

[54] W. D. Roeck and V. Verreet, “Very slow heating for weakly driven quantum many-body systems,” (2019), arXiv:1911.01998 [cond-mat.stat-mech].
In this Supplemental Material, we provide (i) the proofs of Theorems 1 and 2 of the main text, corollaries on physical consequences such as long-lived charge conservation, and (ii) extensions of the theorems.

I. PROOFS OF THEOREMS 1 AND 2

A. Mathematical setting

We consider a large but finite graph Λ, equipped with the graph distance. The vertices $i$ of this graph are our ‘sites’. We assume there is a finite Hilbert space $C^d$ attached to each site $i \in \Lambda$ and we take $d$ to be fixed; the total Hilbert space $H$ is hence $(C^d)^{\otimes \Lambda}$. We say that an operator $O = O_S$ in $B \equiv B(H)$ (the space of bounded operators on $H$) is supported in a set $S$ if it is of the form $O_S \otimes I_S^c$ ($I$: identity operator, $S^c$: complement of $S$), with a slight abuse of notation. This is the setting for quantum spin systems. One can also consider lattice fermions, if one makes some modifications in the definition to deal with the fact that the fermionic space is not naturally given as a tensor product (due to anticommutation relations). For operators $O \in B$, we use the standard operator norm $\|O\| = \sup_{\psi \neq 0, \langle \psi | \psi \rangle \neq 0} \langle \psi | O^\dagger O | \psi \rangle / \langle \psi | \psi \rangle$. Also, given an operator $A$, we will freely use the notation $\text{ad}_A$ to denote the superoperator acting on $B$ as $\text{ad}_A(B) = [A, B]$.

1. The ‘number’ operator $N$

The operator $N$ plays a central role in our analysis. We assume it is given as a sum of local terms $N = \sum_{S \subseteq \Lambda} N_S$ satisfying the following conditions:

1. All local terms mutually commute: $[N_S, N_{S'}]$.
2. All of the $N_S$ have integer spectrum.
3. There is a fixed range $R$ such that $N_S = 0$ whenever $\text{diam}(S) > R$ (‘diam’ stands for diameter as defined in any metric space).

With these definitions in hand, we need to refine the notion of support of operators, following [1]. We say that $O \in B$ is ‘strongly supported’ in $S$ if $O$ is supported in $S$ and, for any $S' \not\subseteq S$ we have $[O, N_{S'}] = 0$. Here are the important consequences:

1. For any function $f$, if $O$ is strongly supported in $S$, then $f(\text{ad}_N)O$ is strongly supported in $S$.
2. If $A, B$ are strongly supported in $S_A, S_B$, then $[A, B]$ is strongly supported in $S_A \cup S_B$.

We write $B_S \subset B$ for the algebra of operators strongly supported in $S$.

2. Colored potential and norm

We will manipulate operators that are not only sums of local terms (on the graph $\Lambda$), but also parameterized by angles $\theta \in [0, 2\pi]^m$ ($m = 1$: Floquet; $m \geq 2$: quasiperiodic). To that end we introduce the notion of a ‘colored potential’ $\Phi$, as was done in [2]. This is a function

$$2^\Lambda \times \mathbb{Z}^m \to B : (Z, n) \mapsto \Phi_{Z, n}$$

(1)
such that $\Phi_{Z,n} \in B_{Z}$ (i.e. it has strong support in the set $Z$) and that $\Phi_{Z,n} = 0$ unless $Z$ is a connected set. We define a weighted norm as

$$\|\Phi\|_\kappa = \sup_x \sum_{Z \ni x} e^{|Z|} \|\Phi_Z\|$$

(2)

for any $\kappa > 0$ where $x \in Z$ iff $x \in Z$; also $|Z| = |Z| + |n|$.

We will also need the $\cup$ operation acting on colored sets as $(Z_1, n_1) \cup (Z_2, n_2) = (Z_1 \cup Z_2, n_1 + n_2)$. We declare two colored sets $Z_1, Z_2$ to be disjoint iff $Z_1, Z_2$ are disjoint. This means in particular, that, in the definition of the norm above, the condition $\sum_{Z \ni x}$ can be recast as $\sum_{(Z, n), (Z, n) \sim (\{x\}, n')}$, for any $n'$, with the binary relation $\sim$ indicating that the colored sets are not disjoint. Such a formulation is necessary when we apply abstract cluster expansion results in Lemma 1.

A potential $\Phi$ defines an many-body operator $H_\Phi$, that depends periodically on a variable $\theta \in [0, 2\pi]^m$, by

$$H_\Phi(\theta) = \sum_{Z, n} \Phi_{Z,n} e^{in\theta}$$

Since one can make the relation between potentials and many-body operators one-to-one (allowing for the addition of a constant to the many-body operator), we will in practice simply conflate $H$ and $H_\Phi$, and so we view the above norms $\|\cdot\|_\kappa$ as a local norm on many-body operators.

**B. Proof of Theorem 1.**

We now give the proof of Theorem 1 in the main text. We take $m$, the number of components of $\theta$, to be $m = 1$ and so we write simply $\theta$ instead of $\theta_\theta$.

**Theorem 1.** Approximate $U(1)$-conservation in Floquet systems.

Let $N$ (i) be a sum of local terms that mutually commute & (ii) has integer spectrum, and $H(\theta)$ be a many-body Hamiltonian where $\theta \in [0, 2\pi]$, with local norm $\|H(\theta)\|_{\kappa_0} < \infty$ for some $\kappa_0 > 0$. Let $\nu > 0$ and define the local energy scale $\nu_0 := \max\{2\|H(\theta)\|_{\kappa_0, \omega}\}$. We consider the Hamiltonian

$$G(\theta) = \nu N + H(\theta)$$

(3)

(and correspondingly, the time-periodic (Floquet) Hamiltonian $G(t) := G(\theta \omega)$ with fundamental frequency $\omega$), where the amplitude $\nu$ is assumed large, specifically $\nu > C\nu_0$ for some constant $C$ depending only on $\kappa_0$ but not the volume of the system. Explicitly it is given as

$$C^{-1} = \min\left\{1, \frac{\kappa_0}{12\pi}, \frac{1}{2A}, \frac{64\sqrt{2}}{\nu_0} \right\}$$

(4)

where

$$A = \left(\frac{216\pi}{\kappa_0^2} + \left(1 + \frac{72\pi}{\kappa_0^2}\right) \frac{4\pi}{e\kappa_0}\right),$$

$$x = \min\left\{\frac{1}{6\pi\kappa_0}, \frac{-(108\pi + 4\pi\kappa_0)}{e\kappa_0} + \sqrt{(108\pi + 4\pi\kappa_0)^2 + 288\pi\kappa_0} \right\}$$

(5)

Then, there is a small unitary $e^{A(\theta)}$ effected by a quasilocal, antihermitian operator $A(\theta)$, such that the unitary propagator corresponding to $G(t)$ can be written

$$U(t) = e^{A(\theta)} T \exp \left(-i \int_0^t ds \nu N + D(\theta_s) + V(\theta_s)\right) e^{-A(\theta_0)},$$

where $T$ represents time-ordering, and $D(\theta), V(\theta)$ are quasilocal, many-body Hamiltonians satisfying

$$\|D(\theta) - \langle H(\theta) \rangle\|_\kappa \leq C'\nu_0/\nu,$$

(6)

$$\|V(\theta)\|_\kappa \leq \nu_0 2^{-n^*},$$

(7)

$$[D(\theta), N] = 0.$$

(8)
Here $\kappa = \kappa_0/4$, $\langle \cdot \rangle$ represents the symmetrization operation $\langle O(\theta) \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i\phi N} O(\theta) e^{-i\phi N}$, and

$$n_\ast = \left[ \frac{32\kappa^2}{32\sqrt{2}} \left( \frac{\nu}{\nu_0} \right)^2 \right].$$  \hspace{1cm} (9)

$C'$ is a numerical constant independent of volume.

Proof. We relabel the initial $G(\theta), H(\theta) \mapsto G_0(\theta), H_0(\theta)$ and we will define renormalized operators $G_n(\theta), H_n(\theta)$. At each step we will also split the operator $H_n(\theta)$ into a term diagonal in $N$ and a term completely off-diagonal in $N$:

$$H_n(\theta) = D_n(\theta) + V_n(\theta)$$  \hspace{1cm} (10)

where $D_n(\theta) := \langle H_n(\theta) \rangle, V_n(\theta) := H_n(\theta) - \langle H_n(\theta) \rangle$. Clearly, $[D_n(\theta), N] = 0$.

To define the renormalized Hamiltonians, at the $n$-th order, we introduce an antihermitian operator $A_n(\theta)$ defined via

$$A_n(\theta) = \frac{i\nu}{2\pi} \int_0^{2\pi} dt \int_0^t ds e^{i\nu N} V_n(\theta)e^{-i\nu N}$$  \hspace{1cm} (11)

which satisfies

$$[\nu N, A_n(\theta)] = -V_n(\theta).$$  \hspace{1cm} (12)

We use this to define the next $G_{n+1}(\theta)$:

$$G_{n+1}(\theta) := e^{-A_n(\theta)} G_n(\theta) e^{A_n(\theta)} - i\omega e^{-A_n}(\partial_\theta e^{A_n(\theta)})$$  \hspace{1cm} (13)

$$= \nu N + H_{n+1}(\theta)$$  \hspace{1cm} (14)

$$= \nu N + D_{n+1}(\theta) + V_{n+1}(\theta).$$  \hspace{1cm} (15)

By introducing notation

$$\gamma_n(O) := e^{-A_n} O e^{A_n}$$  \hspace{1cm} (16)

$$\alpha_n(O) := \int_0^1 ds e^{sA_n} O e^{sA_n},$$  \hspace{1cm} (17)

we can write $H_{n+1}(\theta)$ as

$$H_{n+1}(\theta) = (\gamma_n(H_n(\theta)) + \gamma(\nu N) - \nu N) - i\omega \alpha_n(\partial_\theta A_n(\theta))$$

$$= \gamma_n(H_n(\theta)) - \alpha_n([A_n(\theta), \nu N]) + i\omega \alpha_n(\partial_\theta A_n(\theta))$$

$$= \gamma_n(D_n(\theta)) + (\gamma_n(V_n(\theta)) - V_n(\theta)) + (V_n(\theta) - [A_n(\theta), \nu N])$$

$$- (\alpha_n([A_n(\theta), \nu N]) - [A_n(\theta), \nu N]) - i\omega \alpha_n(\partial_\theta A_n(\theta))$$

$$= \gamma_n(D_n(\theta)) + (\gamma_n(V_n(\theta)) - V_n(\theta)) + (\alpha_n(V_n(\theta) - V_n(\theta)) - i\omega \alpha_n(\partial_\theta A_n(\theta))$$  \hspace{1cm} (18)

It is useful to introduce

$$W_n(\theta) := (\gamma_n(D_n(\theta)) - D_n(\theta)) + (\gamma_n(V_n(\theta)) - V_n(\theta)) + (\alpha_n(V_n(\theta) - V_n(\theta)) - i\omega \alpha_n(\partial_\theta A_n(\theta))$$  \hspace{1cm} (19)

so that

$$D_{n+1}(\theta) = D_n(\theta) + \langle W_n(\theta) \rangle,$$  \hspace{1cm} (20)

$$V_{n+1}(\theta) = W_n(\theta) - \langle W_n(\theta) \rangle.$$  \hspace{1cm} (21)

This concludes the recursion formulae defining the renormalization procedure of the Hamiltonians. The aim next is to provide bounds on the renormalized Hamiltonians. Note that formally, all manipulations have been similar to [3], however, the main difference is that we have an additional term, the gauge potential $-i\omega e^{-A_n(\theta)} \partial_\theta e^{A_n(\theta)}$, which we have to account for. We assume $A_n(\theta)$ is given by the choice Eq. (11).

We shall have to make use of two lemmas:

**Lemma 1.** Let $Z(\theta), Q(\theta)$ be colored potentials on $S^1$ and assume that $3\|Q(\theta)\|_\kappa \leq \kappa - \kappa'$, with $0 < \kappa' < \kappa$. Then

$$\|e^{Q(\theta)} Z(\theta) e^{-Q(\theta)} - Z(\theta)\|_{\kappa'} \leq \frac{18}{(\kappa - \kappa')\kappa'} \|Q(\theta)\|_\kappa \|Z(\theta)\|_\kappa,$$  \hspace{1cm} (22)

$$\|e^{Q(\theta)} Z(\theta) e^{-Q(\theta)}\|_{\kappa'} \leq \left(1 + \frac{18}{(\kappa - \kappa')\kappa'} \|Q(\theta)\|_\kappa \right) \|Z(\theta)\|_\kappa.$$  \hspace{1cm} (23)
Proof. Equivalent to Sec. 5.1 of [3], replacing sets by colored sets. It is important to use the new notions of union \( \cup \) and disjointness for colored sets. In particular, the notion of disjointness is crucial to set up the cluster expansion. \( \square \)

Lemma 2. For \( 0 < \kappa' < \kappa \),
\[
\| \partial_\theta O(\theta) \|_{\kappa'} \leq \frac{1}{e(\kappa - \kappa')} \| O(\theta) \|_\kappa. \tag{24}
\]

Proof. Using \( e^y > ye \) for any \( y > 0 \), we have
\[
\| \partial_\theta O(\theta) \|_{\kappa'} = \sup_x \sum_{Z \ni x,n} e^{\kappa'(|Z|+|n|)} \| i mO_{Z,n} \|
\leq \sup_x \sum_{Z \ni x,n} e^{\kappa'(|Z|+|n|)} \frac{1}{e(\kappa - \kappa')} e^{(\kappa - \kappa')|n|} \| O_{Z,n} \|
\leq \frac{1}{e(\kappa - \kappa')} \sup_x \sum_{Z \ni x,n} e^{\kappa(|Z|+|n|)} \| O_{Z,n} \|
\equiv \frac{1}{e(\kappa - \kappa')} \| O(\theta) \|_\kappa. \tag{25}
\]

Armed with these lemmas, we now bound the renormalized Hamiltonians. Suppose there is a sequence of strictly decreasing decay constants \( \kappa_0 > \kappa_1 > \kappa_2 > \cdots > 0 \), we then have
\[
\| D_{n+1}(\theta) \|_{\kappa_{n+1}} \leq \| D_n(\theta) \|_{\kappa_n} + w_n/2,
\| V_{n+1}(\theta) \|_{\kappa_{n+1}} \leq w_n,
\| D_{n+1}(\theta) - D_n(\theta) \|_{\kappa_{n+1}} \leq w_n/2, \tag{26}
\]
where \( w_n = 2\| W_n(\theta) \|_{\kappa_{n+1}} \). Furthermore,
\[
\| A_n(\theta) \|_\kappa \leq \frac{\pi}{\nu} \| V_n(\theta) \|_\kappa \tag{27}
\]
for any \( \kappa > 0 \), which follows from Eq. (11) viewed as the pointwise (in \( \mathbf{Z} \)) definition of a colored potential.

Now from Lemma 1, provided we have
\[
3\| A_n(\theta) \|_{\kappa'_{n}} < \kappa'_{n} - \kappa_{n+1} \tag{28}
\]
for some intermediate \( \kappa'_{n} \) (to be determined) such that \( \kappa_{n+1} < \kappa'_{n} < \kappa_{n} \), which we note is satisfied if \( \frac{3\pi}{\nu} \| V_n(\theta) \|_{\kappa'_n} < \kappa'_n - \kappa_{n+1} \), then
\[
\| W_n(\theta) \|_{\kappa_{n+1}} \leq \frac{18}{(\kappa'_{n} - \kappa_{n+1})\kappa_{n+1}} \| A_n(\theta) \|_{\kappa'_{n}} (\| D_n(\theta) \|_{\kappa_{n}} + 2\| V_n(\theta) \|_{\kappa_{n}}) + \omega \alpha_n(\partial_\theta A_n(\theta)) \|_{\kappa_{n+1}}
\leq \frac{18}{\nu(\kappa'_{n} - \kappa_{n+1})\kappa_{n+1}} \| V_n(\theta) \|_{\kappa_{n}} (\| D_n(\theta) \|_{\kappa_{n}} + 2\| V_n(\theta) \|_{\kappa_{n}}) + \omega \alpha_n(\partial_\theta A_n(\theta)) \|_{\kappa_{n+1}}
\leq \frac{18}{\nu(\kappa'_{n} - \kappa_{n+1})\kappa_{n+1}} \| V_n(\theta) \|_{\kappa_{n}} (\| D_n(\theta) \|_{\kappa_{n}} + 2\| V_n(\theta) \|_{\kappa_{n}})
+ \omega \left( 1 + \frac{18}{(\kappa'_{n} - \kappa_{n+1})\kappa_{n+1}} \| A_n(\theta) \|_{\kappa'_{n}} \right) \| \partial_\theta A_n(\theta) \|_{\kappa'_{n}}
\leq \frac{18}{\nu(\kappa'_{n} - \kappa_{n+1})\kappa_{n+1}} \| V_n(\theta) \|_{\kappa_{n}} (\| D_n(\theta) \|_{\kappa_{n}} + 2\| V_n(\theta) \|_{\kappa_{n}})
+ \omega \left( 1 + \frac{18}{\nu(\kappa'_{n} - \kappa_{n+1})\kappa_{n+1}} \| V_n(\theta) \|_{\kappa_{n}} \right) \| \partial_\theta A_n(\theta) \|_{\kappa'_{n}}. \tag{29}
\]

We now need to estimate \( \| \partial_\theta A_n(\theta) \|_{\kappa'_{n}} \). We first work out \( \partial_\theta A_n(\theta) \):
\[
\partial_\theta A_n(\theta) = \frac{\nu}{2\pi} \int_0^{2\pi} dt \int_0^t ds e^{is\nu N} \partial_\theta V_n(\theta) e^{-is\nu N}. \tag{30}
\]
Therefore we have
\[
\|\partial_\theta A_n(\theta)\|_{\kappa'_n} \leq \frac{\pi}{\nu} \|\partial_\theta V_n(\theta)\|_{\kappa'_n} \leq \frac{1}{\nu} e(\kappa_n - \kappa'_n) \|V_n(\theta)\|_{\kappa_n}
\] (31)
from Lemma 2. Plugging this in we have
\[
\|W_n(\theta)\|_{\kappa_{n+1}} \leq \frac{18\pi}{\nu(\kappa'_n - \kappa_{n+1})\kappa_{n+1}} \|V_n(\theta)\|_{\kappa_n} (\|D_n(\theta)\|_{\kappa_n} + 2\|V_n(\theta)\|_{\kappa_n}) \\
+ \frac{\omega}{\nu} \left(1 + \frac{18\pi}{\nu(\kappa'_n - \kappa_{n+1})\kappa_{n+1}}\right) \|V_n(\theta)\|_{\kappa_n} \frac{\pi}{e(\kappa_n - \kappa'_n)} \|V_n(\theta)\|_{\kappa_n}.
\] (32)

Now we make the choice that \( \kappa'_n = (\kappa_n + \kappa_{n+1})/2 \) so that
\[
\|W_n(\theta)\|_{\kappa_{n+1}} \leq \frac{36\pi}{\nu(\kappa_n - \kappa_{n+1})\kappa_{n+1}} \|V_n(\theta)\|_{\kappa_n} (\|D_n(\theta)\|_{\kappa_n} + 2\|V_n(\theta)\|_{\kappa_n}) \\
+ \frac{\omega}{\nu} \left(1 + \frac{36\pi}{\nu(\kappa_n - \kappa_{n+1})\kappa_{n+1}}\right) \|V_n(\theta)\|_{\kappa_n} \frac{2\pi}{e(\kappa_n - \kappa_{n+1})} \|V_n(\theta)\|_{\kappa_n}
\] (33)
and the requirement for Lemma 1 to hold can be satisfied if
\[
\frac{6\pi}{\nu} \|V_n(\theta)\|_{\kappa_n} < \kappa_n - \kappa_{n+1}.
\] (34)

These are the ultimate expressions and now our aim is to start an inductive process and choose the decay constants \( \kappa_n \) appropriately. Let us first define \( \kappa_1 = \kappa_0/2 \). Then
\[
\|W_0(\theta)\|_{\kappa_1} \leq \left(\frac{216\pi}{\kappa_0^2} + \left(1 + \frac{72\pi \nu_0}{\kappa_0^2} \frac{4\pi}{\epsilon\kappa_0}\right) \frac{\nu_0^2}{\nu} \right) E_0 \leq A_0 \frac{\nu_0}{\nu}
\] (35)
(here we made use of the assumption \( \nu > C\nu_0 \)). The requirement (for Lemma 1 to hold) reads
\[
\frac{6\pi\nu_0}{\nu} < \frac{\kappa_0}{2} \text{ or } \frac{\nu_0}{\nu} < \frac{\kappa_0}{12\pi}
\] (36)
and is satisfied by similar assumption of the set-up of the problem. Why did we take \( \kappa_1 - \kappa_0 \) to be independent of system parameters \( \nu, \omega \)? Well, in doing so, we have ensured \( W_0(\theta) \) is small in \( 1/\nu \), so vanishes as \( \nu \to \infty \) holding everything else fixed.

Now we impose the inductive hypothesis that for some \( n \),
\[
\|D_n(\theta)\|_{\kappa_n} \leq 2\nu_0, \\
\|V_n(\theta)\|_{\kappa_n} \leq \nu_0 \left(\frac{1}{2}\right)^n.
\] (37)
(38)
Clearly this is true for \( n = 0 \) by definition, and also true for \( n = 1 \), since the assumption \( \nu_0/\nu < 1/(2A) \) guarantees
\[
\|V_1(\theta)\|_{\kappa_1} \leq \nu_0 \leq A_0 (\nu_0/\nu) \leq \nu_0 (1/2).
\]
Plugging in the induction hypothesis into Eq. (33) we have
\[
\|W_n(\theta)\|_{\kappa_{n+1}} \leq \frac{\nu_0}{\nu} \left(\frac{108\pi}{(\kappa_n - \kappa_{n+1})\kappa_{n+1}} + \left(1 + \frac{36\pi}{(\kappa_n - \kappa_{n+1})\kappa_{n+1}} \frac{\nu_0}{\nu} \right) \frac{2\pi}{e(\kappa_n - \kappa_{n+1})}\right) \|V_n(\theta)\|_{\kappa_n}.
\] (39)
Let us now impose the condition that
\[
\frac{1}{(\kappa_n - \kappa_{n+1})\kappa_{n+1}} \frac{\nu_0}{\nu} \leq x
\] (40)
for an \( x > 0 \) to be determined. Then
\[
\|W_n(\theta)\|_{\kappa_{n+1}} \leq (108\pi x + (1 + 36\pi x) (2\pi \kappa_1 x/e)) \|V_n(\theta)\|_{\kappa_n} \leq \frac{1}{2} \|V_n(\theta)\|_{\kappa_n} \leq \nu_0 \left(\frac{1}{2}\right)^{n+1},
\] (41)
if we choose \( x > 0 \) to be at most the positive root of the quadratic equation
\[
P(x', \nu_0/\nu) := 108\pi x' + (1 + 36\pi x') (2\pi\kappa_1 x'/e) - \frac{1}{2}
\]
\[
= (72\pi\kappa_1/e) x'^2 + [108\pi + 2\pi\kappa_1/e] x' - \frac{1}{2}
\]
which yields
\[
x = \frac{-(108\pi + 4\pi\kappa_0/e) + \sqrt{(108\pi + 4\pi\kappa_0/e)^2 + 288\pi\kappa_0/e}}{288\pi\kappa_0/e}.
\]
We also have to satisfy the requirement for Lemma 1, which reads 6\( \nu_0/(1/2)^n/\nu < \kappa_n - \kappa_{n+1} \). The choice Eq. (40) works, provided we take \( x \leq \frac{1}{6\pi\kappa_0} \), since
\[
\frac{6\pi}{\nu} \| V_n(\theta) \|_{\kappa_n} \leq \frac{6\pi \nu_0}{\nu} \left( \frac{1}{2} \right)^n < \frac{1}{6\pi x} < \frac{1}{\kappa_1 x} \nu < \frac{1}{\kappa_{n+1} x} \nu \leq (\kappa_n - \kappa_{n+1}).
\]
Therefore, our ultimate choice is
\[
x = \min \left\{ \frac{-(108\pi + 4\pi\kappa_0/e) + \sqrt{(108\pi + 4\pi\kappa_0/e)^2 + 288\pi\kappa_0/e}}{288\pi\kappa_0/e}, \frac{1}{6\pi\kappa_0} \right\}.
\]
We now define the decay rates on \( n = 1, 2, 3, \ldots \) as
\[
\kappa_n := \kappa(n) \text{ for } n = 1, 2, 3, \ldots \text{ where }
\]
\[
\kappa(y)^2 := \kappa_1^2 - 2B \left( \frac{\nu_0}{\nu} \right) (y - 1), \quad y \in \mathbb{R}.
\]
Now \( \kappa(y) \) is a concave down function, so we have that
\[
\kappa_n - \kappa_{n+1} \geq -\kappa'(n) = \frac{B(\nu_0/\nu)}{\kappa_n}.
\]
Moreover, assuming we only look at \( n^* \) such that \( \kappa_n \geq \kappa_1/2 = \kappa_0/4 \) then
\[
\left( \frac{\kappa_n}{\kappa_1} \right)^2 = 1 - \frac{2B(\nu_0/\nu)}{\kappa_n^2} \geq 1 - \frac{8B(\nu_0/\nu)}{\kappa_1^2}.
\]
So if we impose
\[
\frac{8B(\nu_0/\nu)}{\kappa_1^2} \leq \frac{1}{2}
\]
we would then have
\[
\left( \frac{\kappa_n}{\kappa_1} \right)^2 \geq \frac{1}{2}.
\]
Combining Eq. (48) and (51) we would have
\[
\frac{1}{\kappa_n + 1 (\kappa_n - \kappa_{n+1})} \leq \frac{\sqrt{2}}{B(\nu_0/\nu)}
\]
so we should pick
\[
B = \frac{\sqrt{2}}{x}.
\]
Imposition Eq. (50) therefore reads
\[
\frac{\nu_0}{\nu} \leq \frac{x}{64\sqrt{2}} \kappa_1^2.
\]
Therefore, the maximal \( n^* \) to which the iteration procedure can be carried out to, is
\[
n^* := \left\lfloor \frac{3x\kappa_0^2 \left( \frac{\nu}{\nu_0} \right)}{32\sqrt{2}} \right\rfloor.
\]
This gives the claimed bound on \( V(\theta) \) in the main text defined as \( V(\theta) := V_{n^*}(\theta) \) (abusing a little notation). Similarly, \( D(\theta) \) defined as \( D(\theta) := D_{n^*}(\theta) \) satisfies \( [D(\theta), N] = 0 \). Lastly the estimate Eq. (6) follows from summing the bound on \( \| D_{n+1}(\theta) - D_n(\theta) \|_{\kappa_{n+1}} \).
C. Proof of Theorem 2.

We state Theorem 2 again.

Theorem 2. Approximate $Z_n$-charge conservation in quasiperiodically-driven systems away from the high-frequency limit.

Consider $N$ (i) a sum of local terms that mutually commute and (ii) has integer eigenvalue spacings. Fix a non-zero integer $n$ and let $H(\theta)$ be a many-body Hamiltonian on $\theta \in [0, 2\pi]^2$, assuming that $\| H(\theta) \|_{c_0}$ for some $c_0 > 0$. We introduce a frequency vector $\omega = (\nu, \omega)$, define $\nu_0 := \max \{ 2\| H(\theta) \|_{c_0}, \omega \}$, and consider the Hamiltonian

$$G(\theta) = \frac{\nu}{n} N + H(\theta)$$  \hspace{1cm} (56)

(and corresponding, the time-quasiperiodic Hamiltonian $G(t) := G(\theta(t))$). We take $\nu > C\nu_0$ for some constant $C$ depending on $c_0$ but not on the system’s volume. Then, there is a small time-quasiperiodic unitary $e^{A(\theta_1)}$ affected by a quasilocal, antidermition operator $A(\theta)$ such that the unitary propagator can be written

$$U(t) = e^{A(\theta_1)T} \exp \left( -i \int_0^t \frac{\nu_1}{n} N + D(\theta_1) + V(\theta_1) \right) e^{-A(\theta_0)},$$

where $D(\theta), V(\theta)$ are quasilocal Hamiltonians satisfying

$$\| D(\theta) \|_\epsilon \leq C(\nu_0/\nu),$$  \hspace{1cm} (57)

$$\| V(\theta) \|_\epsilon \leq \nu_0 2^{-n},$$  \hspace{1cm} (58)

$$[D(\theta), g] = 0.$$  \hspace{1cm} (59)

Here $D(\theta) = D'(\theta_2)$ has dependence only on $\theta_2 \in [0, 2\pi)$, $\kappa = \kappa_0/4$, $g = e^{-i \frac{2\pi}{n} N}$ is a generator of the $Z_n$ group satisfying $g^n = \mathbb{I}$, $\langle \cdot \rangle$ is the symmetrization operation $\langle O(\theta) \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i \frac{2\pi}{n} N} O(\theta) e^{i \frac{2\pi}{n} N}$, $n_* = \lfloor c(\nu/\nu_0) \rfloor$, and $C', c$ are numerical constants.

Proof. To begin, we move into the rotating frame of $\nu N/n$ and base our analysis on the related Hamiltonian

$$H_0(\theta) = e^{-i \frac{2\pi}{n} N} H(\theta) e^{i \frac{2\pi}{n} N}. \hspace{1cm} (60)$$

In this formulation, the large amplitude appears only as a driving frequency (of frequency $\nu/n$), but crucially $H_0(\theta)$ now obeys the so-called twisted time-translation property [2]:

$$H_0(\theta_1, \theta_2) = g H_0(\theta_1 + 2\pi, \theta_2) g^\dagger$$  \hspace{1cm} (61)

where $g = e^{i \frac{2\pi}{n} N}$ is a $Z_n$ generator, which satisfies $g^n = \mathbb{I}$.

For ease of convenience we rescale $\theta_1 \mapsto \theta_1/n$ so that $H_0(\theta)$ is defined on the “standard torus” $T^2 = [0, 2\pi)^2$, then the twisted time-translation property reads

$$H_0(\theta_1, \theta_2) = g H_0(\theta_1 + 2\pi/n, \theta_2) g^\dagger.$$  \hspace{1cm} (62)

We define the operation

$$\langle O(\theta) \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta O(\theta) \equiv O'(\theta_2)$$  \hspace{1cm} (63)

which we can interpret as performing a ‘Born-Oppenheimer’ approximation (treating $\theta_1$ as the fast mode and integrating it out, while treating $\theta_2$ as slow and frozen). We will introduce a sequence of small, frame transformations $e^{D_k(\theta)}$, $k = 1, 2, 3, \cdots$ up to an optimal order $k_*$ to renormalize the Hamiltonians, getting $H_k(\theta)$ at each stage, which we split according to

$$D_k(\theta) = \langle H_k(\theta) \rangle \equiv D_k(\theta_2)$$  \hspace{1cm} (64)

$$V_k(\theta) = H_k(\theta) - \langle H_k(\theta) \rangle.$$  \hspace{1cm} (65)

Note $D_k$ is only a function of $\theta_2$. Also, $[D_k(\theta_2), g] = 0$ because, if $O(\theta_1, \theta_2) = g O(\theta_1 + 2\pi/n, \theta_2) g^\dagger$, then

$$g \langle O(\theta) \rangle g^\dagger = \frac{1}{2\pi} \int_0^{2\pi} d\theta_1 g O(\theta_1, \theta_2) g^\dagger = \frac{1}{2\pi} \int_0^{2\pi} d\theta_1 O(\theta_1 - 2\pi/n, \theta_2) = \langle O(\theta) \rangle.$$  \hspace{1cm} (66)
The renormalization procedure at level $k + 1$ involves defining a Hamiltonian $H_{k+1}(\theta)$ from the previous one at level $k$ via

$$H_{k+1}(\theta) = e^{-B_k(\theta)} \left( H_k(\theta) - i \frac{\nu}{n} \partial_{\theta_1} - i \omega \partial_{\theta_2} \right) e^{B_k(\theta)}$$

$$= e^{-B_k(\theta)} \left( D_k(\theta) + V_k(\theta) - i \frac{\nu}{n} \partial_{\theta_1} - i \omega \partial_{\theta_2} \right) e^{B_k(\theta)}.$$  
(67)

We choose $B_k(\theta)$ to satisfy

$$V_k(\theta) - i \frac{\nu}{n} \partial_{\theta_1} B_k(\theta) = 0,$$  
(68)

which we take as solution

$$B_k(\theta) = -i \frac{n}{2 \pi \nu} \int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_1 + 2\pi/n} d\theta_1' V_k(\theta_1', \theta_2).$$  
(69)

The reason for the outer integral is to ensure $A_k(\theta)$ has also the twisted time-translation symmetry (should $V_k(\theta)$ have such a property too):

$$B_k(\theta_1 + 2\pi/n, \theta_2) = -i \frac{n}{2 \pi \nu} \int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_1 + 2\pi/n} d\theta_1' V_k(\theta_1', \theta_2)$$

$$= -i \frac{n}{2 \pi \nu} \int_0^{2\pi} d\phi \int_{\theta_1 - 2\pi/n}^{\theta_1} d\theta_1' V_k(\theta_1' + 2\pi/n, \theta_2)$$

$$= -i \frac{n}{2 \pi \nu} \int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_1} d\theta_1' g V_k(\theta_1', \theta_2) g$$

$$= g^1 B_k(\theta_1, \theta_2) g.$$  
(70)

In such a case, $H_{k+1}(\theta)$ is then guaranteed to also have a twisted time-translation symmetry.

Now, introducing similar notation as used before

$$\gamma_k(O) := e^{-B_k} O e^{B_k},$$  
(71)

$$\alpha_k(O) := \int_0^1 ds e^{-s B_k} O e^{s B_k},$$  
(72)

we can write $H_{k+1}(\theta)$ as

$$H_{k+1}(\theta) = \gamma_k(H_k(\theta)) - i \frac{\nu}{n} \alpha_k(\partial_{\theta_1} B_k(\theta)) - i \omega \alpha_k(\partial_{\theta_2} B_k(\theta))$$

$$= \gamma_k(D_k(\theta)) + (\gamma_k(V_k(\theta)) - V_k(\theta)) + \left( V_k(\theta) - i \frac{\nu}{n} \partial_{\theta_1} B_k(\theta) \right)$$

$$- \left( i \frac{\nu}{n} \alpha_k(\partial_{\theta_1} B_k(\theta)) - i \frac{\nu}{n} \partial_{\theta_1} B_k(\theta) \right) - i \omega \alpha_k(\partial_{\theta_2} B_k(\theta))$$

$$= \gamma_k(D_k(\theta)) + (\gamma_k(V_k(\theta)) - V_k(\theta)) + (\alpha_k(V_k(\theta)) - V_k(\theta)) - i \omega \alpha_k(\partial_{\theta_2} B_k(\theta)).$$  
(73)

It is useful to also introduce similarly

$$W_k(\theta) := (\gamma_k(D_k(\theta)) - D_k(\theta)) + (\gamma_k(V_k(\theta)) - V_k(\theta)) + (\alpha_k(V_k(\theta)) - V_k(\theta)) - i \omega \alpha_k(\partial_{\theta_2} B_k(\theta))$$  
(74)

so that

$$D_{k+1}(\theta) = D_k(\theta) + \langle W_k(\theta) \rangle,$$  
(75)

$$V_{k+1}(\theta) = W_k(\theta) - \langle W_k(\theta) \rangle.$$  
(76)

Then,

$$\|B_k(\theta)\|_\kappa \leq \frac{n \pi}{\nu} \|V_k(\theta)\|_\kappa,$$  
(77)

$$\|\partial_{\theta_2} B_k(\theta)\|_{\kappa'} \leq \frac{n \pi}{\nu} \|\partial_{\theta_2} V_k(\theta)\|_{\kappa'} \leq \frac{n \pi}{\nu} \frac{1}{e^{(\kappa - \kappa')}} \|V_k(\theta)\|_\kappa.$$  
(78)
for any $0 < \kappa' < \kappa$. The mathematical setup is the same as that of Theorem 1, and therefore all bounds from the previous $U(1)$ case can be copied, verbatim, replacing $\nu \mapsto \nu/n$. In particular there is an optimal order $k_*$ going as $\nu/\nu_0$ to which the renormalization procedure can be carried out to, which minimizes $V_{k_*}(\theta)$’s local norm.

We now state the form of the unitary. There exists a frame transformation $e^{B(\theta)} = e^{B_0(\theta)}e^{B_1(\theta)}\ldots e^{B_{k_*}(\theta)}$ where $B_k(\theta)$ are defined on $T^2 = [0, 2\pi]^2$ such that

$$U(t) = e^{-i\frac{\nu}{n}N}e^{B(\theta)}\mathcal{T}\exp\left(-i\int_0^t ds D_{k_*}(\theta_s) + V_{k_*}(\theta_s)\right)e^{-B(\theta_0)}. \quad (79)$$

Here $D_{k_*}(\theta) = D_{k_*}'(\theta_2)$ is only a function of $\theta_2$ and has a $\mathbb{Z}_n$ symmetry $[D_{k_*}(\theta), g] = 0$. Importantly, the flow here is of the form $\theta_1 = (\nu/n, \omega)t + \theta_0$.

If we want to work in the original coordinates we simply scale back $\theta_1 \mapsto n\theta_1$. In the original coordinates $e^{B(\theta)}$ is not invariant under translations by $2\pi$ in the $\theta_1$ direction, but a related object is:

$$e^{A(\theta)} := e^{-i\frac{\nu}{n}N}e^{B(\theta)}e^{i\frac{\nu}{n}N}. \quad (80)$$

To wit,

$$e^{A(\theta_1 + 2\pi, \theta_2)} = e^{-i\frac{\nu}{n}N}e^{-i\frac{\nu}{n}N}e^{B(\theta_1 + 2\pi, \theta_2)}e^{i\frac{\nu}{n}N}e^{i\frac{\nu}{n}N} = e^{-i\frac{\nu}{n}N}g\{e^{B(\theta)}g\}^{\frac{\nu}{n}}N = e^{A(\theta)}.$$

Therefore we have the final result

$$U(t) = e^{A(\theta)}\mathcal{T}\exp\left(-i\int_0^t ds \frac{\nu}{n}N + D_{k_*}(\theta_s) + V_{k_*}(\theta_s)\right)e^{-A(\theta_0)} \quad (82)$$

where $\theta_1 = \omega t + \theta_0 = (\nu, \omega)t + \theta_0$, the original flow (or the ‘original’ driving frequencies). Note we have ‘reinserted’ $e^{-i\frac{\nu}{n}N}$ into the time-ordered exponential at the expense of redefining $V_{k_*}(\theta)$. However, in doing so, all objects are now defined on the standard torus $T^2$ of the original problem. □

In pedestrian terms, this says: (i) there is a small time-quasiperiodic change of frame $e^{A(\theta_1)}$, such that dynamics is decomposed into two parts: (ii) there is a ‘large’ overall oscillating envelope $e^{-i\frac{\nu}{n}N}$, which realizes the generator $g$ of a $\mathbb{Z}_n$ group raised to the power of $m$, at time which is an $m$-multiple of the period $T_1 = 2\pi/\nu$, and (iii) there is dynamics of a time-periodic Hamiltonian $\frac{\nu}{n}N + D_{k_*}(\theta_t) = \frac{\nu}{n}N + D_{k_*}'(\omega_2 t + (\theta_0)_2)$ with period $T_2 = 2\pi/\omega$. Corrections to this arise due to exponentially small terms.

### D. Long-lived charge conservation

Here we prove the statement that from Theorems 1 and 2 there is accompanying long-lived charge conservation in dynamics. To do so, it is convenient to make our setup a bit more explicit. We did not yet exclude that the local terms $N_S$ of $N$ grow unboundedly as the support set moves away from the origin of the lattice $\Lambda$. We do so now by requiring that $\sup_S \|N_S\| \leq n_0$ for some $n_0 \in \mathbb{N}$. Additionally, we will use a simple lemma.

**Lemma 3.** For any observable $O$ supported in $S$, and a Hamiltonian $H$, we have, for any $\kappa$  

$$\| [O, G] \| \leq 2 |S| \| O \| \| H \|_{\kappa} \quad (83)$$

**Proof.** By using the definition Eq. (2) directly, we find

$$\|[O, G]\| \leq 2\| O \| \sum_{x \in S} \sum_{Z \geq x} \| H_{Z, n} \| \leq 2\| H \|_{\kappa} \quad \square$$

Now for Theorem 1 we consider a time-dependent dressed charge

$$\tilde{N}(t) := e^{A(\theta_t)}N e^{-A(\theta_t)} \quad (84)$$
and its Heisenberg time evolution $U(t)\dagger \tilde{N}(t)U(t)$. This can be written as $e^{A(\theta_0)}U(t)\dagger NU(t)e^{-A(\theta_0)}$ where $U(t)$ is generated by the Hamiltonian
\begin{equation}
H'(t) = D_{n,*}(\theta_t) + e^{-i\nu Nt}V_{n,*}(\theta_t)e^{i\nu Nt}.
\end{equation}
The only symmetry-breaking processes are contained within the last term, so we can ask the question how similar Heisenberg time evolution of $N$ is under $H'(t)$, to Heisenberg time evolution just under $D_{n,*}(\theta_t)$. Formally, let the time evolution operator $U_1(t)$ be generated by $D_{n,*}(\theta_t)$, and $U_2(t)$ by $D_{n,*}(\theta_t) + e^{-i\nu Nt}V_{n,*}(\theta_t)e^{i\nu Nt}$. That is,
\begin{equation}
i\partial_t U_1(t) = D_{n,*}(\theta_t)U_1(t),
\end{equation}
\begin{equation}
i\partial_t U_2(t) = (D_{n,*}(\theta_t) + e^{-i\nu Nt}V_{n,*}(\theta_t)e^{i\nu Nt})U_2(t),
\end{equation}
with initial conditions $U_1(0) = U_2(0) = I$. We can write the difference in Heisenberg time evolution as
\begin{equation}
U_1(t)\dagger NU_1(t) - U_2(t)\dagger NU_2(t) = N - U_2(t)\dagger NU_2(t)
= - \int_0^t dsU_2(s)\dagger U_1(t)\dagger (t-s)NU_1(t-s)e^{-i\nu Ns}V_{n,*}(\theta_s)e^{i\nu Ns}U_2(s)
= - \int_0^t dsU_2(s)\dagger |N, e^{-i\nu Ns}V_{n,*}(\theta_s)e^{i\nu Ns}|U_2(s).
\end{equation}
Bounding the expression using the standard operator norm, utilizing Lemma 3, gives
\begin{equation}
\|N - U_2(t)\dagger NU_2(t)\| \leq 2d\|\nu\|_0 d^R 2^{-n_*},
\end{equation}
where $R$ is the maximal range of $N$ and $d$ is the spatial dimension of $A$. Therefore for $\tilde{N}$ we have
\begin{equation}
\frac{1}{|A|}\|\tilde{N}(0) - U(t)\dagger \tilde{N}(t)U(t)\| \leq 2d\|\nu\|_0 d^R 2^{-n_*}.
\end{equation}
This bound means that $\tilde{N}(t)$ is conserved in Heisenberg time-evolution up to the prethermal timescale $\tau \sim e^{c\nu/\nu_0}$. Focusing only on stroboscopic times $t = ZT$ ($T$: period) gives the claimed result in the main text of long-lived conservation of $N := \tilde{N}(0)$ (in again a slight abuse of notation). If we instead work with the original charge $N$, then we have an additional bounded error of $O(\nu_0/\nu)$ in Eq. (90) arising from the small frame change in Eq. (84).

For Theorem 2, it is less straightforward to express the charge conservation, because there is no infinitesimal generator associated to the symmetry group. The most meaningful way to express that the charge is approximately conserved is by exhibiting that the dynamics of local observables is generated by a charge-conserving generator, up to an error that is very small in norm. Such a statement can easily be deduced from the smallness of $V$: Let $\bar{U}(t)$ be the unitary propagator $U(t)$ but with the error term $V$ set to zero. Then, by the Duhamel formula,
\begin{equation}
U(t)\dagger OU(t) - \bar{U}(t)\dagger O\bar{U}(t) = i\int_0^t dsU(t)\dagger U(s)[V(\theta_s), U(s)\dagger O\bar{U}(s)]U(s)\dagger U(t).
\end{equation}
To bound the right-hand side, we invoke Lemma 3 and the Lieb-Robinson bound [4] to argue that $U(s)\dagger O\bar{U}(s)$ is a local observable if $O$ is. We refer to [3] for the details and we simply state the result, namely
\begin{equation}
\|U(t)\dagger OU(t) - \bar{U}(t)\dagger O\bar{U}(t)\| \leq \nu_0 2^{-n_*}(Ct + C')^d\|O\||S|
\end{equation}
for some constants $C, C'$, and with $t$ the spatial dimension of the lattice $A$ and $S$ the support of $O$.

In particular, if the observable $O$ is symmetric, $[O, g] = 0$, then $\bar{U}(t)\dagger O\bar{U}(t)$ is $g$-symmetric and hence
\begin{equation}
\|\|U(t)\dagger OU(t), g\| \leq \nu_0 2^{-n_*}(Ct + C')^d\|O\||S|,
\end{equation}
so that indeed, $O$ approximately remains $g$-symmetric in the rotated frame.

II. EXTENSIONS

A. Non-constant amplitude $\nu$

Our theorems specify having the amplitude in front of the $N$ operator be constant: $\nu$ for Theorem 1 and $\nu/n$ for some fixed positive integer $n$ for Theorem 2. We can straightforwardly extend the domain of Theorems to encompass a large class of time-dependent amplitudes $\nu \mapsto \nu(t)$. There are actually two ways to encompass such situations.
We focus on the setting given in Theorem 1. Case (i): Suppose
\[ \nu(t) = \bar{\nu}(1 + f(t)) \quad (94) \]
where \( \bar{\nu} \) is a constant, and \( f(t) \) is a sufficiently smooth, time-periodic function with zero time-average satisfying \( |f(t)| < 1 \). We can always reparameterize time by defining
\[ t' = t + \int_0^t ds f(s). \quad (95) \]
Note that \( t(t' + T) = t(t') + T \). Then, the solution of the Schrödinger equation
\[ i\partial_t U(t) = (\nu(t)N + H(t))U(t), \quad U(0) = \mathbb{I} \quad (96) \]
can be obtained by solving the related equation
\[ i\partial_{t'} U'(t') = \left( \bar{\nu}N + \left[ \frac{H(t(t'))}{(1 + f(t(t')))} \right] \right) U'(t'), \quad U'(0) = \mathbb{I} \quad (97) \]
via \( U(t) := U'(t'(t)) \). Note that the Hamiltonian in the square parenthesis is time-periodic in \( t' \)-variables, so that we can apply Theorem 1 as quoted in the main text.

Case (ii): Suppose \( \nu(t) \) is similarly as in Eq. (94) for some smooth \( f(t) \). We can simply eliminate \( f(t) \) in Eq. (94) by going into the rotating frame associated with it,
\[ U_0(t) = e^{-i \int_0^t ds f(s)N}, \quad (98) \]
so that in the rotating frame, the Hamiltonian is
\[ G(t) = \bar{\nu}N + U_0(t)H(t)U_0(t). \quad (99) \]
Once again, Theorem 1 can be applied in such a formulation.

In similar fashion, we can apply Theorem 2 in situations where the amplitude \( \nu(t) \) is time-dependent, specifically when it is time-periodic with period \( T_1 = 2\pi/\nu \), as long as its time average is \( \nu/n \).

### B. Extensions to multiple charge conservation, drives with more fundamental frequencies

We are not only limited to a single \( U(1) \)-charge conservation in Floquet systems nor a single \( \mathbb{Z}_n \)-charge conservation in a two-tone quasiperiodic system. Let us focus on extensions to the former scenario. We can in fact allow for multiple \( U(1) \) charge conservation, by introducing \( r > 1 \) mutually commuting \( N \) operators: \( N_1, \cdots , N_r \), each appearing with amplitudes \( \nu_1, \cdots , \nu_r \), so that the set-up is
\[ G(\theta) = \sum_{k=1}^r \nu_kN_k + H(\theta). \quad (100) \]
(From this we consider a Floquet system with Hamiltonian \( G(t) := G(\theta_t) = \omega t + \theta_0 \)). Explicitly, we need the condition that \( \|H(\theta)\|_{\nu_k}, \omega \ll \nu_k \) for all \( k \), and we additionally require a Diophantine condition on the frequency vector \( \nu = (\nu_1, \cdots , \nu_r) \):
\[ \frac{\nu \cdot n}{|\nu|} \geq \frac{c}{|n|^\gamma} \quad (101) \]
where \( n = (n_1, \cdots , n_r) \in \mathbb{Z}^r \) is any non-zero integer vector, for some \( \gamma > 0 \). This is so that one can control potentially dangerous resonances due to the small energy differences that can arise from a linear combination of multiple fundamental frequencies (see [2] for more discussion). It turns that for almost all choices of \( \nu \), the exponent satisfies \( \gamma > r - 1 \), with the constant \( c \) depending on the choice of ratios \( \nu_i/\nu_j \) but important not the overall frequency scale \( |\nu| \), as explained in [2]. We expect that the proof of Theorem 1 can be carried over with appropriate changes accounting for the multiple frequencies, adopting the techniques of [2], resulting in a prethermal lifetime of the multiple charges that is weaker than a pure exponential but still superpolynomially long in the large driving amplitude \( |\nu| \).
Furthermore, we can also extend the situation to allow for a drive with multiple fundamental frequencies (time-quasiperiodic driving), by promoting \( \theta \mapsto \theta = (\theta_1, \cdots, \theta_p) \) for some integer \( p \geq 2 \). That is, we can consider

\[
G(\theta) = \sum_{k=1}^{r} \nu_k N_k + H(\theta)
\]  

(102)

with a flow \( \omega_t + \theta_0 \) defining a time-quasiperiodic Hamiltonian \( G(t) := G(\theta_t) \), where \( \omega = (\omega_1, \cdots, \omega_p) \) is a vector of drive frequencies. In this case, we require \( \|H(\theta)\|_{\kappa_0}, \omega_l \ll \nu_k \) for all \( l, k \), and a Diophantine condition on both frequency vectors \( \nu \) and \( \omega \), separately, to achieve multiple long-lived \( U(1) \)-charge conservation. Similar considerations also apply to the setting of Theorem 2 of the main text. We can consider a frequency vector \( \omega = (\nu_1, \cdots, \nu_r, \omega_1, \cdots, \omega_p) \) and a Hamiltonian

\[
G(\theta) = \sum_{k=1}^{r} \frac{\nu_k}{n_k} N_k + H(\theta),
\]  

(103)

with some fixed set of non-zero integers \( (n_1, \cdots, n_r) \), as well as a Diophantine condition on the set of frequencies \( (\nu_1, \cdots, \nu_r) \) and \( (\omega_1, \cdots, \omega_p) \), separately, to achieve a result of multiple long-lived \( Z_n \) charge conservation in time-quasiperiodic drives with more than two fundamental frequencies.

[1] D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. X 7, 011026 (2017).
[2] D. V. Else, W. W. Ho, and P. T. Dumitrescu, Phys. Rev. X 10, 021032 (2020).
[3] D. Abanin, W. De Roeck, W. W. Ho, and F. Huveneers, Communications in Mathematical Physics 354, 809 (2017).
[4] E. H. Lieb and D. W. Robinson, Communications in Mathematical Physics 28, 251 (1972).