HIGHER DIMENSIONAL MÖBIUS BANDS AND THEIR BOUNDARIES

CHADY EL MIR AND JACQUES LAFONTAINE

Abstract. We give a characterisation of Bieberbach manifolds which are geodesic boundaries of a compact flat manifold, and discuss the low dimensional cases, up to dimension 4.

Keywords: flat manifolds, geodesic boundary, Bieberbach group.

2010 MSC: 53C20, 53C22, 53C23.

1. Introduction

1.1. Presentation of our results. A flat Riemannian manifold is a Riemannian manifold locally isometric to the Euclidean space $E^n$. If such a manifold is complete, then the exponential map is a Riemannian covering: complete flat manifolds are the quotients of the Euclidean space by a subgroup of affine isometries acting properly and freely.

In particular, flat compact manifolds are the quotients $E^n/\Gamma$, where $\Gamma$ is a discrete, co-compact and fixed points free group of affine isometries of $E^n$. Following [2], we call such groups Bieberbach groups.

Bieberbach Theorem shows, more generally, that a crystallographic group, i.e., a discrete and co-compact sub-group $\Gamma$ of affine isometries of $E^n$, is an extension of a finite group $G$ by a lattice $\Lambda$ of $E^n$. This lattice is the subgroup of elements of $\Gamma$ that are translations. Note that a crystallographic group is a Bieberbach group if and only if it is torsion free (see [11], p.99).

Theorem 1.1 (Bieberbach Theorem). i) Let $\Gamma$ be a discrete and co-compact sub-group of affine isometries of $E^n$, then the sub-group $\Lambda$ of the elements of $\Gamma$ that are translations is a normal sub-group of finite index in $\Gamma$ and a lattice of $E^n$.

ii) The number of crystallographic groups in dimension $n$ is finite up to an isomorphism. Two crystallographic groups on $E^n$ are isomorphic if and only if they are conjugate by an element of the affine group.

This shows that we have the following exact sequence of groups

$$0 \rightarrow \Lambda \rightarrow \Gamma \rightarrow G \rightarrow e$$

where $G$ is finite.
The manifold \( E^n/\Gamma \) admits a Riemannian covering by the flat torus \( E^n/\Lambda \). The group \( G \) is the *holonomy group* of the manifold. It admits a natural realization as a subgroup of \( O(n) \).

There are two types of Bieberbach groups in dimension 2, 10 in dimension 3 and 74 in dimension 4. Obtaining a precise classification in any dimension seems difficult and maybe hopeless. Some classes have been studied by Charlap, see [2], and more recently by Szczepański, cf. [9].

Bieberbach manifolds are known to be boundaries (see [7] and [8]). Here we address the more precise (and indeed very different) question of realizing a Bieberbach manifold as the *totally geodesic boundary* of a compact flat manifold. In dimension 1, the answer, although well-known, is not completely trivial, since it involves the Möbius band. The general situation, as the main result of this paper shows (see Theorem 2.3), is basically the same: an \( n \)-dimensional compact flat manifold with geodesic boundary is either a product \( I \times M \), where \( M \) is an \((n - 1)\)-dimensional Bieberbach manifold and \( I \) a compact interval (if the boundary is not connected) or a non-trivial Riemannian \( I \)-bundle over an \((n - 1)\)-dimensional Bieberbach \( M' \) (if the boundary is connected); this boundary is a non-trivial \( \mathbb{Z}_2 \)-bundle over \( M' \). In particular it admits a fixed-point free isometric involution. In the rest of the paper we give some applications of this result.

In section 3, we give first a classification of 3-dimensional flat manifolds with geodesic boundary; those that have a connected boundary are somewhat the 3-dimensional Möbius bands. There are 3 types of such manifolds. In the same way as the Klein bottle can be realized by gluing two Möbius bands, some Bieberbach manifolds can be obtained from them by a suitable gluing, see 3.3. Then, using Theorem 2.3 we specify, among the ten types (up to an affine diffeomorphism) of 3-dimensional Bieberbach manifolds those that are the geodesic boundaries of a compact flat manifold. This enables us to give a classification of flat compact 4-manifolds with connected geodesic boundary (cf. 3.4).

In section 4, we show that Bieberbach manifolds whose holonomy group is isomorphic to \( \mathbb{Z}_{2m+1} \) or \( \mathbb{Z}_2 \) can be realised as a geodesic boundary of a compact flat manifold.

In section 5, we study the 4-dimensional manifolds with a holonomy group isomorphic to \( \mathbb{Z}_3 \) (Generalized Hantzsche-Wendt manifolds) and give lowest-dimensional examples of non-orientable Bieberbach manifolds that are not a geodesic boundaries of a compact flat manifold.

1.2. *Notations.* All the manifolds we consider are supposed to be connected if the contrary is not explicitly stated. We shall identify \( E^n \) with \( \mathbb{R}^n \) equipped with its standard orthonormal basis.

The translation of vector \( \vec{v} \in \mathbb{R}^n \) is denoted \( t_v \). If \( V \) is an affine subspace of \( \mathbb{R}^n \), and \( a \) a vector parallel to \( V \), we denote by \( \sigma_{V,a} \) the glide reflection obtained by composing the orthogonal reflection with respect to \( V \) and the translation \( t_a \). If \( V \) is a line, we shall content ourselves with mentioning the length of \( a \).
The flat rectangular torus whose fundamental domain is a rectangle of sides $a$ and $b$ will be denoted $T_{a,b}$, and the flat Klein bottle whose fundamental group is generated by $(x, y) \mapsto (x + a/2, -y)$ and $(x, y) \mapsto (x, y + b)$ by $K_{a,b}$.

2. The structure of flat manifolds with geodesic boundary

Theorem 2.1. The universal (Riemannian) cover of a compact $n$-dimensional flat Riemannian manifold with geodesic boundary is isometric to $I \times \mathbb{R}^{n-1}$.

Proof. Let $M$ be such a manifold. In a neighborhood of any component $S$ of the boundary, the metric can be written $g_S + dt^2$ ($g_S$ is the induced metric and $t$ the distance to $S$). Therefore by gluing two copies of $M$ along the boundary, we get a compact flat boundaryless manifold $N$. Let $p : \mathbb{R}^n \to N$ its universal Riemannian covering. The inverse image $p^{-1}$ of the boundary is a (disjoint) union of complete totally geodesic hypersurfaces of $\mathbb{R}^n$ that is to say of parallel hyperplanes.

In fact, there are infinitely many such hyperplanes: indeed, the deck transforms of the covering $p : \mathbb{R}^n \to N$ act on this family of hyperplanes, and form a Bieberbach group in dimension $n$, which contains a lattice of rank $n$. Now, $p^{-1}(M)$ is a disjoint union of bands limited by two such hyperplane.

Let $B \simeq I \times \mathbb{R}^{n-1}$ be one of them. The restriction of $p$ to $B$ gives a universal cover of $M$. □

Now, we study the manifolds themselves. We can suppose for simplicity that $I = [-1, 1]$.

The fundamental group $\Gamma$ of $M$, viewed as a sub-group of $\text{Isom}(\mathbb{R}^n)$, leaves the boundary $\{1\} \times \mathbb{R}^{n-1} \cup \{-1\} \times \mathbb{R}^{n-1}$ globally invariant and admits a compact fundamental domain. The hyperplane $P = \{0\} \times \mathbb{R}^{n-1}$ is also globally invariant, and the restriction to this hyperplane of the action of $\Gamma$ still admits a fundamental domain. Two cases are possible.

(i) Each connected component of the boundary of $I \times \mathbb{R}^{n-1}$ is $\Gamma$-invariant. Then, any parallel hyperplane to these components is also $\Gamma$-invariant. In this situation, $\Gamma$ is isomorphic to its restriction to $\mathbb{R}^{n-1}$, which is a $(n-1)$-dimensional Bieberbach group that we denote $\Gamma'$. Then $M$ is isometric to $I \times (\mathbb{R}^{n-1}/\Gamma')$. Its boundary has two connected components and can be identified with $\mathbb{R}^{n-1}/\Gamma' \sqcup \mathbb{R}^{n-1}/\Gamma'$.

(ii) There are elements of $\Gamma$ which exchange the components. Then the subgroup $\Gamma_0$ of $\Gamma$ which preserves these components is a normal subgroup of index 2. We have the following property, which is trivial in case i).

Lemma 2.2. The restriction map $\gamma \mapsto \gamma|_P$ is injective.

Proof. Suppose that $\gamma|_P$ is the identity map. If $\gamma \in \Gamma_0$, clearly $\gamma = \text{Id}_P$. If $\gamma \in \Gamma \setminus \Gamma_0$ then $\gamma$ must be orthogonal reflection with respect to $P$. This is impossible, since $\Gamma$ acts freely. □
Therefore the restriction to $P$ maps $\Gamma$ and $\Gamma_0$ isomorphically onto Bieberbach groups in dimension $(n - 1)$, that we shall denote $\Gamma'$ and $\Gamma_0'$.

Denoting by $(x, y)$ the elements of $\mathbb{R} \times \mathbb{R}^{n-1}$, the actions of $\Gamma$ on $\mathbb{R}^n$ and of $\Gamma'$ on $\mathbb{R}^{n-1}$ are related as follows

$$\gamma \cdot (x, y) = \begin{cases} (x, \gamma' \cdot y) & \text{if } \gamma \in \Gamma_0 \\ (-x, \gamma' \cdot y) & \text{if } \gamma \notin \Gamma_0 \end{cases}$$

The boundary of $M$ is connected and isometric to $\mathbb{R}^{n-1}/\Gamma_0'$, and $M$ admits a Riemannian covering of order 2 isometric to $I \times \mathbb{R}^{n-1}/\Gamma_0$. This situation is exactly what happens in dimension 2 with the Möbius band.

**Remark.** There is of course a strong contrast with the hyperbolic case. In dimension 2, if a compact hyperbolic manifold has a (non-empty) geodesic boundary, this boundary has at least 3 components. For higher dimension, see [5].

A consequence of this discussion is a characterisation of connected geodesic boundaries of Bieberbach manifolds.

**Theorem 2.3.** For a Bieberbach manifold $\mathbb{R}^{n-1}/\Gamma_0$ the following properties are equivalent.

(i) It can be realized as the totally geodesic boundary of a compact flat $n$-dimensional manifold.

(ii) It admits a fixed point free isometric involution.

(iii) There is an exact sequence

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

where $\Gamma$ is a $(n - 1)$-dimensional Bieberbach group.

**Proof.** We have just seen that i) implies iii). If iii) is true, then $\Gamma$ normalises $\Gamma_0$ and defines an isometric involution, which is fixed-point free since $\Gamma$ is a Bieberbach group. Now, let $\sigma$ be such an isometry of $\mathbb{R}^{n-1}/\Gamma_0$. Take the manifold $[-1, 1] \times \mathbb{R}^{n-1}/\Gamma_0$ equipped with the product metric. The map $(x, y) \mapsto (-x, \sigma(y))$ is a fixed point free isometric involution, and the quotient manifold has the required property. \qed

**Remark 2.4.** This theorem allows to check whether a given Bieberbach $\mathbb{R}^{n-1}/\Gamma_0$ manifold is (or is not) the geodesic boundary of an $n$-dimensional compact flat manifold. One must prove (or disprove) the existence of a $\gamma \in \text{Isom}(\mathbb{R}^{n-1})$ with the following properties:

(i) $\gamma$ normalizes $\Gamma_0$ (it must go to the quotient as an isometry);

(ii) $\gamma^2 \in \Gamma_0$ (involutiveness);

(iii) The sub-group of $\text{Isom}(\mathbb{R}^{n-1})$ generated by $\gamma$ and $\Gamma_0$ acts freely.

An involutive fixed points free isometry which is not a translation doubles the order of the holonomy group. Therefore, the case where the holonomy group has maximal order in a given dimension is more simple, since it would be enough to only consider the translations $\gamma \in \Lambda/2 \setminus \Lambda$.

**Definition 1.** An isometry of a Bieberbach manifold is called *admissible* if it is involutive and does not admit fixed points.
To summarize this discussion, a compact flat n-dimensional manifold with connected geodesic boundary is a non-trivial I-bundle over a (n−1)-dimensional Bieberbach manifold (that one could call the soul) and the boundary is a two-fold non-trivial covering of the soul.

3. **Two and three dimensional boundaries**

3.1. **Two dimensional boundaries.** We will study flat manifolds with geodesic boundary in dimension 3. We follow the notations of the previous section.

In the case i), i.e., if each connected component of the boundary of $I \times \mathbb{R}^2$ is stable by $\Gamma$, then $M$ is the product of a flat torus or a flat Klein bottle by a segment.

In the case ii), i.e., if the sub-group of $\Gamma$ which keeps stable the connected components of the boundary of $I \times \mathbb{R}^2$ is a sub-group $\Gamma_0$ of index 2, then $\Gamma'$ is either a lattice of the Euclidean plane or the fundamental group of a Klein bottle.

- If $\Gamma'$ is a lattice, then its sub-groups of index 2 are of the form $\mathbb{Z}v+\mathbb{Z}v'$ for a suitable basis $(v, v')$ of $\Gamma$. The manifold is the quotient of $[-d, d] \times \mathbb{R}^2$ by the sub-group of $\text{Isom}(\mathbb{R}^3)$ generated by $t_v$ and the glide reflection $\sigma_{P,v'}$, where $P$ is the plane $x = 0$ with $v$ and $v'$ parallel to $P$. This manifold is not orientable; its boundary is the flat torus $\mathbb{R}^2/\mathbb{Z}v+\mathbb{Z}v'$, whereas the plane $P$ gives in the quotient the flat torus $\mathbb{R}^2/\mathbb{Z}v+\mathbb{Z}v'$. The boundary and the soul are both tori. The similarity with the Möbius band is such that we suggest to call this manifold the *solid Möbius band*. We denote it by $TT_{v,v',d}$.

- If $\Gamma'$ is the fundamental group of the Klein bottle $K_{a,b}$, generated by the glide reflection $(y, z) \mapsto (y + \frac{a}{2}, -z)$ and the translation $(y, z) \mapsto (y, z + b)$, there are two types of sub-groups of index 2.

  - First, $\Gamma_0$ can be the lattice associated to $\Gamma$. The manifold is then the quotient of $[-d, d] \times \mathbb{R}^2$ by the sub-group of $\text{Isom}(\mathbb{R}^3)$ generated by $t_{(0,0,b)} : (x, y, z) \mapsto (x, y, z + b)$ and $\sigma_{D,(0,\frac{a}{2},0)} : (x, y, z) \mapsto (-x, y + \frac{a}{2}, -z)$ where $D$ is the straight line $x = z = 0$. This manifold is orientable, and its boundary is the torus $T_{a,b}$. The boundary is a rectangular torus and the soul is a Klein bottle. We denote it by $TK_{a,b,d}$.

  - Finally, $\Gamma_0$ can be generated by the glide reflection of displacement $\frac{a}{2}$ and the translation $(x, y, z) \mapsto (x, y, z + 2b)$. The manifold is then the quotient of $[-d, d] \times \mathbb{R}^2$ by the sub-group of $\text{Isom}(\mathbb{R}^3)$ generated by $\sigma_{P,(0,0,b)} : (x, y, z) \mapsto (-x, y, z + b)$ and $\sigma_{P',(0,\frac{a}{2},0)}(x, y, z) \mapsto (x, y + \frac{a}{2}, -z)$ where $P$ and $P'$ are respectively the planes $x = 0$ et $z = 0$. This manifold is not orientable; the boundary and the soul are both Klein bottles. We denote it by $KK_{a,b,d}$. 
Therefore, we have three types of compact flat manifolds with connected geodesic boundary. They can be thought of as 3-dimensional Mobius bands. The following table summarizes this discussion.

| Generators of $\Gamma$ | Orientability | Boundary |
|-------------------------|---------------|----------|
| $t_v$ and $\sigma_{P,\frac{a}{2}}$ | non-orientable | $T^2 = \mathbb{R}^2/\mathbb{Z}v + \mathbb{Z}v'$ |
| $t_{(0,0,b)}$ and $\sigma_{D,(0,\frac{a}{2},0)}$ | orientable | $T_{a,b}$ |
| $\sigma_{P,(0,0,b)}$ and $\sigma_{P',(0,\frac{a}{2},0)}$ | non-orientable | $K_{a,2b}$ |

By gluing two such manifolds along isometric boundaries, some 3-dimensional Bieberbach manifolds can be obtained. We will give the details in 3.3 after giving a precise description of these manifolds.

### 3.2. Three dimensional boundaries.

Three dimensional Bieberbach manifolds have been classified by Hantzsche and Wendt in 1935 (See [11] Th. 3.5.5 (orientable case) and Th. 3.5.9 (non orientable case)). There exist ten compact and flat manifolds of dimension 3 up to an affine diffeomorphism, six are orientable and four are non-orientable.

In the orientable case, these types are characterized by the holonomy group $G$. Besides flat tori, they are as follows (we use the notations of Thurston, cf. [10], 4.3.

- The manifold $C_2$: the group $\Gamma$ is generated by two orthogonal translations $\{a_1, a_2\}$ and a glide reflection $\sigma_{L,a_3/2}$, where the line $L$ is orthogonal to $a_1$ and $a_2$. Its holonomy group is isomorphic to $\mathbb{Z}_2$.
- The manifold $C_3$: the group $\Gamma$ is generated by independent translations $\{a_1, a_2\}$ which generate a flat hexagonal lattice and a screw motion $(R, t_{a_3/3})$ of angle $2\pi/3$ around a vector $a_3$ orthogonal to $a_1$ and $a_2$. Its holonomy group is isomorphic to $\mathbb{Z}_3$.
- The manifold $C_4$: the group $\Gamma$ is generated by two translations $\{a_1, a_2\}$ which generate a flat square lattice and a screw motion $(R, t_{a_3/4})$ of angle $\pi/2$ around a vector $a_3$ orthogonal to $a_1$ and $a_2$. Its holonomy group is isomorphic to $\mathbb{Z}_4$.
- The manifold $C_6$: the group $\Gamma$ is generated by two translations $\{a_1, a_2\}$ which generate a flat hexagonal lattice and a screw motion $(R, t_{a_3/6})$ of angle $\pi/3$ around a vector $a_3$ orthogonal to $a_1$ and $a_2$. Its holonomy group is isomorphic to $\mathbb{Z}_6$.
- The manifold $C_2,2$: let $L$, $M$ and $N$ be three disjoint lines pairwise orthogonal. The group $\Gamma$ is generated by the glide reflections $\sigma_{L,2d(M,N)}$, $\sigma_{M,2d(N,L)}$ and $\sigma_{N,2d(L,M)}$ (see the very suggestive figure of [10], p. 236; in fact two reflections suffice). Its holonomy group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proposition 3.1.** Among orientable 3-dimensional Bieberbach manifolds, only the torus, the manifold $C_2$ and the manifold $C_3$ are the geodesic boundary of a compact flat 4-dimensional manifold.

**Proof.** For the manifold $C_2$, the translation $a_1/2$ (and also $a_2/2$) goes to the quotient and does not admit fixed points. It is therefore a geodesic
boundary. Note that $C_2$ covers $C_4$ if the corresponding $\{a_1, a_2\}$ lattice is square. The manifold $C_3$ is also a geodesic boundary since it always covers the manifold $C_6$. Also, the translation $a_3/2$ goes to the quotient and does not admit fixed points (this is a particular case of Theorem 4.1).

The manifolds $C_4$, $C_6$, and $C_{2,2}$ do not admit any admissible involution. Indeed, their holonomy group, viewed as a subgroup of $O(3)$, is maximal. Therefore, an isometric involution must be given by an element of $\Lambda/\Gamma$ together with a translation in $\Lambda$. For $C_4$ and $C_6$, such an involution is given by $t_{a_3/2}$ but it has fixed points. The situation is the same for $C_{2,2}$: the subgroup of $\text{Isom}(\mathbb{R}^3)$ generated by $\Gamma$ together with a translation in $\Lambda/\Gamma$ has fixed points.

\begin{remark}
\text{We have seen in the proof of the previous proposition (prop. 3.1) that there are sometimes two ways to obtain the manifolds $C_2$ and $C_3$ as totally geodesic boundaries. In fact, the manifold $C_2$ (if the corresponding lattice is orthogonal) and the manifold $C_3$ admit two types of admissible isometries. The first type belongs to the neutral component of the isometry group. The second type, i.e., fixed point free involutions which do not belong to the neutral component of the isometry group, give nice examples of compact 4-dimensional manifolds with connected geodesic boundary, namely one with soul $C_4$ and boundary $C_2$, and one with soul $C_6$ and boundary $C_3$.}
\end{remark}

Now, we study the four types of non-orientable Bieberbach manifolds in dimension three. Each one of them can be obtained with a suspension of the Klein bottle (cf. [3] for details). They are as follows (we use the notations of Wolf, cf. [11], ch.3) as we did in [3]:

- The manifold $B_1$: the group $\Gamma$ is generated by two orthogonal translations $\{a_1, a_2\}$ and a glide reflection $\sigma_{P,a_3/2}$ with respect to a plane $P$ generated by $\{a_2, a_3\}$ where $a_3$ is orthogonal to $a_1$. Its holonomy group is isomorphic to $\mathbb{Z}_2$.
  
  The lattice $\Lambda$ corresponding to $B_1$ is generated by the basis $\{a_1, a_2, a_3\}$.

- The manifold $B_2$: the group $\Gamma$ is generated by two glide reflections $\sigma_{P_1,a_1/2}$ and $\sigma_{P_2,a_2/2}$ with respect to two parallel planes $P_1$ and $P_2$, where $a_1$ and $a_2$ are linearly independent vectors parallel to these planes. Its holonomy group is isomorphic to $\mathbb{Z}_2$. The lattice $\Lambda$ corresponding to $B_2$ is generated by $\{a_1, a_2, a_3\}$ where $a_3 = \frac{a_1 + a_2}{2} + 2d$ and $d$ is the vector (orthogonal to $P_1$) that sends $P_1$ to $P_2$.

- The manifold $B_3$: $\Gamma$ is generated by a translation $a_3$, a glide reflection $\sigma_{P,a_1/2}$ with respect to a plane $P$ perpendicular to $a_3$ and generated by two orthogonal vectors $\{a_1, a_2\}$ and a screw motion $(R, t_{a_3/2})$ of angle $\pi$ around a straight line $L \subset P$ and parallel to $a_2$. Its holonomy group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- The manifold $B_4$: the group $\Gamma$ is generated by a glide reflection $\sigma_{P,a_1/2}$ with respect to a plane $P$ generated by two orthogonal vectors $\{a_1, a_2\}$, a screw motion $(R, t_{a_2/2})$ of angle $\pi$ around a straight line $L$ parallel to $a_2$ but not contained in $P$. Its holonomy group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. 

The lattice Λ corresponding to \( B_4 \) is generated by the orthogonal basis \{a_1, a_2, a_3\} where \(|a_3| = 4 \text{dist}(P, L)\).

It should be noted that the orientable cover of \( B_1 \) and \( B_2 \) is a torus, whereas the orientable cover of \( B_3 \) and \( B_4 \) is \( C_2 \) (with an orthogonal lattice).

**Proposition 3.3.** Every non-orientable 3-dimensional Bieberbach manifolds is the geodesic boundary of a compact flat 4-dimensional manifold.

**Proof.** The manifold \( B_1 \) admits a self-cover: the translation \( a_1/2 \) goes to the quotient and does not admit fixed points. It also covers the manifolds \( B_3 \) and \( B_4 \) if the lattice \( \Lambda \) is orthogonal. The manifold \( B_2 \) also admits an admissible isometry: the translation \( 2d \) (and also the translation \( a_1+a_2 \), see section 4.2 below) goes to the quotient and does not admit fixed points.

The manifolds \( B_1 \) admits a self-cover: the translation \( a_3/2 \) goes to the quotient and does not admit fixed points. The manifold \( B_4 \) also admits an admissible isometry: the translation \( a_3/2 \) goes to the quotient and does not admit fixed points. Note that the quotient of \( B_4 \) by the translation \( a_3/2 \) is a manifold of type \( B_3 \). □

As a consequence of that discussion, we have the following

**Corollary 3.4.** Up to diffeomorphisms, there are fourteen types of compact flat 4- with connected boundary. Namely

- Self-coverings: \((T^3, T^3), (C_2, C_2), (B_1, B_1), (B_3, B_3)\);
- Orientation coverings: \((T^3, B_1), (T^3, B_2), (C_2, B_3), (C_2, B_4)\);
- Miscellaneous: \((T^3, C_2), (C_2, C_4), (C_3, C_6), (B_1, B_3), (B_1, B_4), (B_2, B_1), (B_4, B_3)\)

(in that list have denoted by \((A, B)\) the total space of a \( \mathbb{Z}_2 \)-fibration of \( A \) over \( B \)). Moreover, if two such manifolds are diffeomorphic, they are affinely diffeomorphic.

**Proof.** Many cases have already been covered. A systematic use of remark 2.3 gives the admissible involutions we did not encounter yet. They concern \( T^3 \) and \( B_1 \), and are suitable glide reflections with respect to a line. □

### 3.3. Gluing along the boundary.

It is well known that the Klein bottle can be realized by gluing two isometric Möbius bands along their boundary. A similar phenomena occurs in higher dimension.

- \( B_1 \) can be obtained by gluing two copies of \( TT_{v,v',d} \);
- \( B_2 \) can be obtained by gluing \( TT_{v,v',d} \) and \( TT_{v',v,d} \);
- \( B_3 \) can be obtained by gluing two copies of \( KK_{a,b,d} \);
- \( B_4 \) can be obtained by gluing \( TT_{a,b,d} \) and \( TK_{a,b,d} \);
- \( C_2 \) can be obtained by gluing two copies of \( TK_{a,b,d} \);
- \( C_{2,2} \) can be obtained by gluing \( TK_{a,b,d} \) and \( TK_{b,a,d} \).
Te see that, we proceed backward: we check that $B_1$ and $B_2$ admit totally geodesic foliations by tori, with 2 exceptional leaves which are tori of “half-size”. Indeed, both fundamental groups contain glide reflections. Foliate $\mathbb{R}^3$ by parallel planes to the reflection plans and go to the quotient.

Concerning $B_3$, the same procedure gives a foliation by Klein bottles, with two exceptional leaves which are Klein bottles of half size.

The procedure is still the same for $B_4$, but the situation is more involved: we also obtain a foliation whose generic leaves are tori, but there are 2 exceptional leaves, namely a (half-size) torus and a Klein bottle.

Concerning $C_2$, we just foliate $\mathbb{R}^3$ with orthogonal planes to the axis of a screw motion of the fundamental group and go to the quotient. This time we get a foliation whose generic leaves are tori, with 2 exceptional leaves which are (generally non isometric) Klein bottles.

4. Bieberbach manifolds with particular Holonomy groups

In this section, we will give a general result for Bieberbach manifolds with a holonomy group isomorphic to $\mathbb{Z}_{2m+1}$ or $\mathbb{Z}_2$: they all can be realised as the geodesic boundary of a compact flat manifold.

4.1. Manifolds with cyclic Holonomy group of odd order.

**Theorem 4.1.** Every Bieberbach manifolds with a cyclic holonomy group of odd order is the totally geodesic boundary of a compact flat manifold.

**Proof.** We will show that such a manifold admits an admissible translation. Let $\gamma$ be a lift in $\Gamma$ of a generator of the holonomy group (supposed to be of order $N$). Its linear part must fix some non-trivial vector space $V$, and $\gamma^N$ is a non-trivial translation $t_a$, where $a \in V$. Then, using oddness, we see that the translation $t_{a/2}$ does not admit fixed points. 

4.2. Manifolds with $\mathbb{Z}_2$ holonomy group. Suppose that $\mathbb{R}^n/\Gamma$ has holonomy $\mathbb{Z}_2$. Its realization as a subgroup of $O(n)$ is just $\{I, \sigma_P\}$, where $\sigma_P$ is an orthogonal reflexion with respect to a $k$-dimensional subspace $P$. The lattice $\Lambda$ has index 2 in $\Gamma$, and its non trivial class $\mathcal{R}$ is a set of glide-reflections of type $\sigma_{Q,a}$, where the $k$-dimensional affine subspaces $Q$ are parallel to $P$.

It is clear that if $\sigma_{Q,a} \in \mathcal{R}$, then $2a \in \Lambda$.

Now, for $\sigma_{Q_1,a_1}$ and $\sigma_{Q_2,a_2}$ in $\mathcal{R}$, let $d$ be the translation orthogonal to the $Q_i$ which sends $Q_1$ onto $Q_2$.

**Lemma 4.2.** We have the following properties:

(i) the vector $a_1 + a_2 + 2d$ belongs to $\Lambda$.

(ii) the set of vectors $d$ is a lattice of dimension $(n-k)$.

**Proof.** The first claim follows from the fact that $\sigma_{Q_1,a_1} \circ \sigma_{Q_2,a_2} = a_1 + a_2 + 2d$. Now, we have a transitive action (by translations) of $\Lambda/2$ on the set of $k$-planes $Q$ such that $\sigma_{Q,a} \in \mathcal{R}$. Pick one of them $Q_0$. The third claim amounts to say that the image of $\Lambda + Q_0$ is a lattice in $\mathbb{R}/Q_0$. It is clearly cocompact. Now, if $a_1 + a_2 + 2d \in \Lambda$ and $2(a_1 + a_2) \in \Lambda$, then $4d \in \Lambda$, which proves discreteness. 

□
Theorem 4.3. If an $n$-dimensional Bieberbach manifold has holonomy $\mathbb{Z}_2$, then it is a geodesic boundary.

Proof. The case when $\sigma_P$ is a reflection with respect to a line parallel to a vector $a_0 \in \Lambda$ is completely analogous to that of the manifold $C_2$, which we described in 3.2. The lattice $\Lambda$ is an orthogonal sum $\Lambda_0 \oplus \mathbb{Z}^{a_0}$, and $\Gamma \setminus \Lambda$ is composed with glide reflections $\sigma_{D,a}$, where the vectors $a$ are odd integer multiples of $\frac{a_0}{2}$. The lines $D$ are translated of one of them by a vector of $\Lambda_0/2$. Any translation of $\Lambda_0/2 \setminus \Lambda$ goes to the quotient as a fixed point free isometric involution of $\mathbb{R}^n/\Gamma$.

Now, we consider the case where $\sigma_P$ is a reflection with respect to a $k$-dimensional plane, where $k \geq 2$. For an affine $k$-plane $Q$, set

$$T_Q = \{ v \in \mathbb{R}^n, \sigma_{Q,v} \in \Gamma \}$$

Lemma 4.2 shows that the affine $k$-planes $Q$ such that $T_Q$ is not empty are parallel. Following the notations of the previous lemma, the distance of two neighboring $k$-planes is $|d|$, and the vector $4d$ is in $\Lambda$. Therefore, by composing a glide-reflection with a translation $t_{4d}$, we see that for two $k$-planes whose distance is $2|d|$ the sets $T_Q$ are the same. There are two possibilities.

(i) $T_Q$ is always the same for all the $Q$’s in $R$, and can be written as $a/2 + \Lambda_0$, where $a \in \Lambda$ and $\Lambda_0$ is the orthogonal projection of $\Lambda$ onto $P$. In this case, $\Lambda$ itself is the orthogonal sum $\Lambda_0 \oplus 2\mathbb{Z}d$. The situation is the exact analogue of that of the manifold $B_1$ in dimension 3.

(ii) there are exactly two possibilities for $T_Q$, namely $a_1/2 + \Lambda_0$ and $a_2/2 + \Lambda_0$, where $a_1$ and $a_2$ are independent and belong to $\Lambda$. The lattice $\Lambda$ is the (not orthogonal!) sum $\Lambda_0 \oplus \mathbb{Z}(\frac{a_1 + a_2}{2} + 2d)$. This is the situation of $B_2$ in dimension 3.

The translation $t_d$ in the first case, $t_{2d}$ in the second case, gives by going to the quotient a fixed point free isometric involution. \square

5. Hantzsche-Wendt Manifolds of Dimension 4

In this section, we shall study Generalized Hantzsche-Wendt n-manifolds, i.e., Bieberbach n-manifolds with holonomy isomorphic to $\mathbb{Z}^{n-1}$. In dimension 3, there are three Generalized Hantzsche-Wendt manifolds. The manifolds $B_3$ and $B_4$ are a geodesic boundary whereas the manifold $C_{2,2}$ is not. This shows that the data of the holonomy group of a Bieberbach manifold is generally not sufficient to decide whether it satisfies that property.

In dimension 4, there are 12 Generalized Hantzsche-Wendt manifolds up to an affine isometry. We shall determine which among these manifolds are or are not geodesic boundaries of a compact flat manifold. To see that, a rough description of these manifolds is necessary. In what follows, we view $\mathbb{R}^4$ as the Euclidean spaces of dimension 4 equipped with its standard orthonormal basis. All these manifolds are quotients of flat tori $\mathbb{R}^n/\Lambda$, where $\Lambda$ is the orthogonal lattice generated by $ae_1, be_2, ce_3, de_4$. 
In our description, we use an affine isometries of the Euclidean space which clearly go to the quotient as isometries of the torus $\mathbb{R}^n/\Lambda$, and will abu-
sively use the same notation for an affine isometry and the corre-
spending quotient map. The 4-dimensional Hantzsche-Wendt mani-
folds will be then the quotient of $\mathbb{R}^n/\Lambda$ by the generators (affine isometries) given in the table below.

**Notations:** A symmetry with respect to the hyperplane $yzt$ will be denoted by $S_x$ (similarly for symmetries with respect to $xzt$, $xt$, $yt$ and $zt$).

A symmetry with respect to the plane $zt$ will be denoted by $S_{xy}$ (similarly for symmetries with respect to $xz$, $xt$, $yz$, $yt$ and $zt$).

A translation of vector $\frac{1}{2}e_1$ will be denoted by $T_x$ (similarly for translations $\frac{1}{2}e_2$, $\frac{1}{2}e_3$ and $\frac{1}{2}e_4$). The composition of a translation $T_x$ and a translation $T_y$ will be denoted $T_{xy}$.

| H-W Manifold | Generators | H-W Manifold | Generators |
|--------------|------------|--------------|------------|
| $H - W_1$    | $T_y S_z$, $T_z S_{xt}$, $T_x S_{yt}$ | $H - W_2$    | $T_y S_z$, $T_y S_{xt}$, $T_x S_{yt}$ |
| $H - W_3$    | $T_x S_z$, $T_y S_{xx}$, $T_x S_{yz}$ | $H - W_4$    | $T_y S_z$, $T_y S_{xz}$, $T_x S_{yz}$ |
| $H - W_5$    | $T_z S_z$, $T_x S_{xz}$, $T_x S_{yz}$ | $H - W_6$    | $T_y S_z$, $T_x S_{xx}$, $T_x S_{yz}$ |
| $H - W_7$    | $T_y S_z$, $T_y S_{xz}$, $T_x S_{yz}$ | $H - W_8$    | $T_y S_z$, $T_y S_{xx}$, $T_x S_{yz}$ |
| $H - W_9$    | $T_y S_z$, $T_y S_{xx}$, $T_x S_{yz}$ | $H - W_{10}$ | $T_y S_z$, $T_y S_{xz}$, $T_x S_{yz}$ |
| $H - W_{11}$ | $T_y S_z$, $T_y S_{xx}$, $T_x S_{yz}$ | $H - W_{12}$ | $T_y S_z$, $T_y S_{xz}$, $T_x S_{yz}$ |

Now, we come to the main result of this section.

**Theorem 5.1.** Among 4-dimensional generalized Hantzsche-Wendt mani-
folds, only four of them (all non-orientable) are not totally geodesic bound-
daries, namely the manifolds with vanishing Betti number $H - W_1$ and $H - W_2$ and the manifolds $H - W_3$ and $H - W_4$.

**Proof.** In dimension 4, there does not exist a Bieberbach manifold with a holonomy group of order 16. Therefore, it would be enough to check, for each $H - W_i$, if it admits an admissible translation.

For the manifolds $H - W_i$, where $i \in \{5, 6, 7, 8, 9, 11, 12\}$, the translation $T_z$ is admissible. For the manifolds $H - W_{10}$, the translation $T_y$ is admissible.

Now, for the manifolds $H - W_i$, $i = 1, 2, 3$ or 4, direct inspection shows that, in the four cases we have to consider, a translation in $\Lambda/2 \setminus \Lambda$ that goes to the quotient always admits fixed points.

As a conclusion, it seems that if a Bieberbach manifold has an holonomy group which is a product of $\mathbb{Z}_2$, the situation becomes very arbitrary even if the order of the holonomy becomes maximal (in the class of the manifolds whose holonomy is a product of $\mathbb{Z}_2$) in a given dimension.
Acknowledgements: The authors would like to thank Andrzej Szczepański for giving the description of 4-dimensional Bieberbach manifolds with zero first Betti number by the use of the useful package CARAT.

References

[1] Brown, H.; Bulow, R.; Neubuser, J.; Wondratschek, H.; Zassenhaus, H; Crystallographic Groups of four-Dimensional space, 1st edition, Wiley, New York (1978).
[2] Charlap, L.S.; Bieberbach Groups and Flat Manifolds, Springer Universitext, Berlin (1986).
[3] Elmir, C.; Lafontaine, J.; Sur la Géométrie Systolique des Variétés de Bieberbach, Geom. Dedicata. 136, 95–110 (2008).
[4] Elmir, C.; Lafontaine, J.; The Systolic Constant of Orientable Bieberbach 3-manifolds, Ann. Math. Toulouse, Sr. 6 Vol. 22 no. 3, 623–648 (2013).
[5] Frigerio, R.; Petronio, C.; Construction and Recognition of Hyperbolic 3-manifolds with Geodesic Boundary, TAMS. 13, 171–184 (2001).
[6] Gallot, S.; Hulin, D.; Lafontaine, J.; Riemannian Geometry, 3rd edition, Springer, Berlin Heidelberg (2004).
[7] Gordon, M.; The Unoriented Cobordism Classes of Compact Flat Riemannian Manifolds, J. Differential Geom. 15, no. 1, 8190 (1980)
[8] Hamrick, G.; Royster, D; Flat Riemannian Manifolds are Boundaries, Invent. Math. 66, 405–413 (1982).
[9] Szczepański, A.; Geometry of crystallographic groups, ADM, World Scientific 2012.
[10] Thurston, W.P.; Three-Dimensional Geometry and Topology, edited by S. Levy, Princeton University Press, Princeton (1997).
[11] Wolf, J.A.; Spaces of Constant Curvature, Publish or Perish, Boston (1974).

E-mail address: chady.mir@gmail.com

Current address:
Université Libanaise, Laboratoire de Mathématiques et Applications (LaMA-Liban), Tripoli, Liban

E-mail address: jacques.lafontaine@univ-montp2.fr

Current address:
Université de Montpellier, Institut Montpellierain Alexander Grothendieck (UMR 5149), CC 0051, Place Eugène Bataillon, F-34095, Montpellier Cedex 5, France