Mechanical buckling analysis of functionally graded plates using an accurate shear deformation theory

Berrabah Hamza Madjida and Bouderba Bachirb

aDepartment of Civil Engineering, Mechanical Engineering Materials and Structures Laboratory, University of Relizane, Relizane, Algeria; bDepartment of Science and Technology, Mechanical Engineering Materials and Structures Laboratory, University of Tissemsilt, Tissemsilt, Algeria

ABSTRACT
In this present study, we are interested in the use of a precise theory of shear deformation for the buckling analysis of plates with functional gradation simply supported such as the refined theory of plates with four variables. Several parameters of comparison have been used, dimensional and nondimensional. The displacement field is compatible with this study, the nonuse of shear correction factors is satisfied, and the choice of material is very precise in such a way are variable according to the thickness of the plate and on the other hand to make comparison with other researcher and confirms that this study gives precise results and converges. The transverse shear stresses vary through the thickness, the results found is also studied and discussed.

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1. Introduction
Functionally graded materials (FGMs) and their excellent properties have led to a wide and rapid technological revolution for researchers and industry in various fields and for their utilization particularly at very high temperatures. The basic concept of FGMs has been presented by Koizumi and other researchers through studies referencing in the literature [1–4].

For years, several studies were based on numerous forms of structure such as plates that are widely used in many fields of engineering, tunnels, dams, and buildings. Most of the plated structures can withstand tensile loads and are poor resistant to compressive forces. Usually, the buckling phenomena observed in compressed plates occur quite suddenly and can lead to catastrophic structural rupture. It is therefore important to know the buckling capacities of the plates to avoid premature failure. The subject of this research is the buckling behavior of structural elements in isotropic materials subjected to mechanical loads. Subsequently, many scientists developed equilibrium and stability equations for plates and shells in laminated and functionally graduated composite materials and used them to determine the buckling and the vibrational behavior of the structures [5].

To determine static and dynamic analysis of plate structures, a number of plate theories are available based on considering the transverse shear deformation of plate. The classical plate theory (CPT) in which the transverse shear deformation effects are neglected and the normal to the mid-plane remains straight and normal to the middle surface during the deformation. As a result, the CPT usually underestimates deflection and overestimates the natural frequencies and buckling loads for thick plates [6]. Singh and Harsha investigated the responses of FGM plate subjected to uniform, linear, and nonlinear in-plane loads. New nonlinear in-plane load models are proposed based on trigonometric and exponential function. Nondimensional critical buckling loads are evaluated using nonpolynomial based higher-order shear deformation theory. Navier’s method, which assures minimum numerical error, is employed to get an accurate explicit solution [7].

Nowadays, these types of materials are still treated as modern materials that, through varying different properties throughout their thickness, can carry loads in hard conditions, especially in high-temperature environment. The gradual changes in volume fraction of the components and nonhomogenous structure allow continuous graded macroscopic properties to be obtained [8]. On the basis of a new theory of shear deformation and of the theory of modified torque stresses, the hygro-thermal buckling of microplates and porous microbeams in sandwich FGM is studied. Contrary to the classical theory of elasticity, implies a parameter of the material length scale and can thus capture the effect of small size. The theory of four variable shear deformation with a new shape function is used to derive the stability equations governing microplates and microbeams from the principle of virtual work [9].

Two new higher-order transverse shear deformation theories (NHSDTs) with five variables have been proposed for the analysis of FGM plate. A governing differential equation
(GDE) of the FGM plate is developed using energy principle [10].

A study of the pre- and post-buckling state of square plates built FGMs and pure ceramics is presented. In contrast to the theoretical approach, the structure under consideration contains a finite number of layers with a step-variable change in mechanical properties across the thickness. An influence of ceramics content on a wall and a number of finite layers of the step-variable FGM on the buckling and post-critical state was scrutinized by Czechowski and Kołakowski [10].

The FGMs are microscopically inhomogeneous, in which the mechanical properties vary smoothly and continuously from one surface to the other. They are usually made from a mixture of ceramics and metals to attain the significant requirement of material properties. The determination of accurate behavior of the FGMs largely depends on the theory used to model the structure because in the FGMs, material properties vary continuously as a function of position in the preferred direction. Various concepts have been developed to inculcate the appropriate analysis of the FGM plates. The classical Kirchhoff plate theory neglects transverse shear deformation and gives acceptable results for relatively thin plates. To circumvent this problem an earlier attempts were made by Reissner [11] and Mindlin [12]. However, a shear correction factor must be incorporated to overcome the problem of a constant transverse shear stress distribution and its value depends on various parameters, such as applied loads, boundary conditions, and geometric parameters, etc. The inaccuracy occurs due to neglecting the effects of transverse shear and normal strains in the plate [12]. Due to continuous variation in material properties, the first-order shear deformation theory and higher-order shear deformation theory may be conveniently used in the analysis. It is noted that the first-order shear deformation theory proposed by Mindlin [12] does not satisfy the parabolic variation of transverse shear strain in the thickness direction. Subsequently, many higher-order theories were proposed, notable among them are the studies by Matasunaga [13], Nelson and Larch [14], Reissner [15], Lo et al. [16], and Reddy [17]. The higher order theories assume the in-plane displacements as a cubic expression of the thickness coordinate and the out-of-plane displacement to be constant. Thus, the development of higher-order shear deformation theory to assimilate the behavior of FGM structures has been of high importance to the researchers.

In the literature, studies of the buckling of FGM structures are as follows, several researchers have studied different phenomena such as buckling analysis of FGM plates under uniform, linear, and nonlinear in-plane loading [7]. The thermal effect for FGM plates is interested by Kiani et al. [18] under a work to determine by thermal buckling of clamped thin rectangular FGM plates resting on Pasternak elastic foundation.

Yaghoobi and Torabi examined the exact solution for thermal buckling of functionally graded plates resting on elastic foundations with various boundary conditions [19]. Czechowski and Kołakowski studied the buckling and post-buckling of a step-variable FGM box [8]. A new quasi-3D higher-order shear deformation theory (HSDT) for buckling and vibration of FG plate was considered by Sekkal et al. [20].

In addition, much study has been done to the semi-analytical solution for the circular plates as the research of Alipour and Shariyat [21] in a form of demonstration described semi-analytical solution for buckling analysis of variable thickness for two-directional FGM circular plates with nonuniform elastic foundations.

Li et al. have study correspondence relations between deflection, buckling load, and frequencies of thin FGM plates and those of corresponding homogeneous plates [22]. Farahmand et al. developed Navier’s solution for buckling analysis of size-dependent FGM micro plates [23]. A theory of transverse shear deformation for homogeneous monoclinic plates has been studied by Soldatos [24].

Karama et al. were interested in the mechanical behavior of laminated composite beams by the new model of multi-layer laminated composite structures with continuity of transverse shear stresses [25]. Reddy et al. used the theory of high-order shear deformation to analyze the buckling of FGM plates [26]. On the other hand, the simple refined theory is used for the analysis of buckling of plates with functional gradation by Thai and Choi [27].

The aspect of the new theory of shear deformation for several structures such as laminated composite plates is studied by Aydogdu [28]. Mantari et al. studied the static and mechanical buckling analyses of thick functionally graded (FG) plates. For this purpose, Carrera’s unified formulation (CUF) and the principle of virtual displacement are employed in the numerical finite strip method (FSM). Since this formulation is capable of considering the effects of shear deformations in a realistic manner, it is suitable for analyzing the structures in which these deformations play a major role and cannot be ignored [29]. The numerical performance of a set of physically nonlinear models applied together with CUF for the analysis of beams is investigated by Arruda et al. The main objective of their work is to assess the numerical efficiency of CUF when nonlinear material analysis is applied to 1D elements using the equivalent single layer (ESL) formulation [30]. Carrera et al. investigated the large deflection and post-buckling of composite plates by employing the CUF. As a consequence, the geometrically nonlinear governing equations and the relevant incremental equations are derived in terms of fundamental nuclei, which are invariant of the theory approximation order [31]. The unified formulation of a full geometrically nonlinear refined plate theory in a total Lagrangian approach is developed to study the post-buckling and large-deflection analysis of sandwich FG plate with FG porous (FGP) core. The plate has three layers so that the upper and lower layers are FG and the middle layer (core) is the FGP, which is considered with four cases in terms of the porosity core distribution [32]. New higher-order models of orthotropic micropolar plates and shells have been developed using CUF. Here, a complete linear expansion case (CLEC) has been considered in detail. The stress and strain tensors,
as well as the vectors of displacements and rotation, have been presented as linear expansion in terms of the shell thickness coordinates [33]. Metal–ceramic FG materials are suitable for constructing structures subjected to a very high-temperature gradient. The composition of the FG materials constituents impacts the thermal buckling of the FG plates. Different forms of the composition for the FG plates have been proposed in the literature [34].

In this study, accurate shear deformation theory is presented for mechanical buckling analysis of FGM plates. Numerical examples covering the effects of the gradient index, plate aspect ratio, side-to-thickness ratio on the critical buckling load of FG plates are investigated and discussed.

2. Formulation

Consider a FG plate of thickness $h$, side length $a$ in the $x$-direction, and $b$ in the $y$-direction as shown in Figure 1.

The assumptions of the present theory are as follows; see Shimpi [35] and Bouterba et al. [36]:

- The transverse displacements are partitioned into bending and shear components;
- The in-plane displacement is partitioned into extension, bending, and shears components;
- The bending parts of the in-plane displacements are similar to those given by CPT;
- The shear parts of the in-plane displacements give rise to the nonlinear variations of shear strains and hence to shear stresses through the thickness of the plate in such a way that the shear stresses vanish on the top and bottom surfaces of the plate.

Based on these assumptions, the following displacement field relations can be obtained

\[
\begin{align*}
U(x,y,z) & = u(x,y) - z \frac{\partial w_b}{\partial x} - \psi(z) \frac{\partial w_b}{\partial x} \\
V(x,y,z) & = v(x,y) - z \frac{\partial w_b}{\partial y} - \psi(z) \frac{\partial w_b}{\partial y} \\
W(x,y,z) & = w_b(x,y) + w_s(x,y)
\end{align*}
\]

where $U$, $V$, $W$ are displacements in the $x$, $y$, $z$ directions, $u$, $v$ and $w_b$, $w_s$ are mid-plane displacements and $\psi(z)$ is a shape function that represents the distribution of the transverse shear strain and stress through the thickness, as presented in Table 1.

Consider a FG plate made of ceramic and metal, the material properties of FGM such as Young’s modulus $E$ are assumed to vary through the plate thickness with a power law distribution of the volume fraction of the two materials [44]

\[
E(z) = E_m + (E_m - E_c) \frac{V_C(z)}{E_m} = E_c - E_m
\]

where $E_c$ and $E_m$ are the corresponding properties of the ceramic and metal, respectively, and $k$ is the volume fraction exponent that takes values greater than or equal to zero. The volume fraction $V_C(z)$ follows a simple power law as [45, 46]:

\[
V_C(z) = (z/h + 0.5)^k V_C + V_m = 1;
\]

The kinematic relations can be obtained as follows:

\[
\begin{align*}
\begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} & = \begin{bmatrix} e^0_x \\ e^0_y \\ \gamma^0_{xy} \end{bmatrix} + z \begin{bmatrix} k^b_x \\ k^b_y \\ k^b_{xy} \end{bmatrix} + \psi(z) \begin{bmatrix} k^s_x \\ k^s_y \\ k^s_{xy} \end{bmatrix}, \\
\nu & = g(z) \begin{bmatrix} 
\end{bmatrix},
\end{align*}
\]

where:

\[
\begin{align*}
\begin{bmatrix} e^0_x \\ e^0_y \\ \gamma^0_{xy} \end{bmatrix} & = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}, \\
\begin{bmatrix} k^b_x \\ k^b_y \\ k^b_{xy} \end{bmatrix} & = \begin{bmatrix} \frac{\partial^2 w_b}{\partial x^2} \\ \frac{\partial^2 w_b}{\partial y^2} \\ -2 \frac{\partial^2 w_b}{\partial x \partial y} \end{bmatrix}, \\
\begin{bmatrix} k^s_x \\ k^s_y \\ k^s_{xy} \end{bmatrix} & = \begin{bmatrix} \frac{\partial^2 w_s}{\partial x^2} \\ \frac{\partial^2 w_s}{\partial y^2} \\ -2 \frac{\partial^2 w_s}{\partial x \partial y} \end{bmatrix},
\end{align*}
\]

And $\psi(z)$ is given by [9]:

\[
\psi(z) = z - h \tan^{-1} \left( \frac{z}{h} \right) + \left( \frac{16z^3}{15h^2} \right), \quad g(z) = 1 - \frac{\partial \psi(z)}{\partial z}.
\]
The linear constitutive relations are:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} \quad \text{and}
\begin{bmatrix}
\tau_{yz} \\
\tau_{zx}
\end{bmatrix} =
\begin{bmatrix}
C_{44} & 0 \\
0 & C_{55}
\end{bmatrix}
\begin{bmatrix}
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\]  

(5)

where \((\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{zx})\) and \((\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})\) are the stress and strain components, respectively.

Stiffness coefficients, \(C_{ij}\), can be expressed as

\[
C_{11} = C_{22} = \frac{E(z)}{1 - \nu^2}, \quad C_{12} = \nu \frac{E(z)}{1 - \nu^2}, \quad C_{44} = C_{55} = C_{66}
\]  

(6)

The strain energy of the plate can be written

\[
U = \frac{1}{2} \int \left[ (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx}) dV. \right.
\]

(7)

Substituting Eqs. (3) and (5) into Eq. (7) and integrating through the thickness of the plate, the strain energy of the plate can be rewritten as

\[
U = \frac{1}{2} \int_A \left[ N_x \phi_x + N_y \phi_y + N_{xy} \phi_{xy} + M_{xz} \phi_z + M_{yz} \phi_y + M_{yz} \phi_y + S_{yz} \phi_{zy} + S_{xz} \phi_{xz} \right] \frac{dz}{h}.
\]

(8)

Where the resultants forces, moments, and shear forces, which are all defined by

\[
\begin{align*}
N_x &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy})dz, \\
N_y &= \frac{1}{2} \int h/2 \int \sigma_y dz, \\
N_{xy} &= \frac{1}{2} \int h/2 \int \tau_{xy} dz, \\
M_{xz} &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy})dz, \\
M_{yz} &= \frac{1}{2} \int h/2 \int \sigma_y dz, \\
M_{yz} &= \frac{1}{2} \int h/2 \int \tau_{xy} dz, \\
S_{yz} &= \frac{1}{2} \int h/2 \int \sigma_y dz, \\
S_{xz} &= \frac{1}{2} \int h/2 \int \tau_{xy} dz.
\end{align*}
\]

(9)

Substituting Eq. (5) into Eq. (9) and integrating through the thickness of the plate, the stress resultants are given as

\[
\begin{bmatrix}
N \\
M^b \\
M^p \\
S_{yz} \\
S_{xz}
\end{bmatrix} =
\begin{bmatrix}
A & B & B' \\
B & D & D' \\
B' & D' & H' \\
A_4 & 0 & 0 \\
0 & A_{55} & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon \\
k^b \\
k^p \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix},
\]

(10)

where

\[
N = \{N_x, N_y, N_{xy}\}^t, M^b = \{M_{xz}^b, M_{yz}^b, M_{yz}^b\}^t, M^p = \{M_{xz}^p, M_{yz}^p, M_{yz}^p\}^t,
\]

\[
\varepsilon = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}, k^b = \{k_x^b, k_y^b, k_y^b\}, k^p = \{k_x^p, k_y^p, k_y^p\}
\]

\[
A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \end{bmatrix},
\]

\[
D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \end{bmatrix}, B' = \begin{bmatrix} B_{11}' & B_{12}' & 0 \\ B_{12}' & B_{22}' & 0 \end{bmatrix}
\]

\[
D' = \begin{bmatrix} D_{11}' & D_{12}' & 0 \\ D_{12}' & D_{22}' & 0 \end{bmatrix},
\]

(10a)

where \(A_{ij}, B_{ij}\), etc., are the plate stiffness, defined by

\[
\{A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}\} = \sum_{n=1}^{n_{hi}} \left\{ 1, z, z^2, z^3, z^4, z^5 \right\} C_{ij} dz,
\]

(11)

\[
(i, j = 1, 2, 6)
\]

\[
B_{ij} = -\frac{4}{3} B_{ij} + \frac{5}{3h^2} E_{ij}, \quad (i, j = 1, 2, 6)
\]

\[
D'_{ij} = -\frac{4}{3} D'_{ij} + \frac{5}{3h^2} F_{ij}, \quad (i, j = 1, 2, 6)
\]

\[
H_{ij} = \frac{1}{12} D_{ij} - \frac{5}{6h^2} E_{ij} + \frac{25}{9h^4} F_{ij}, \quad (i, j = 1, 2, 6)
\]

(11a)

\[
\{A_{ij}, D_{ij}, F_{ij}\} = \sum_{n=1}^{n_{hi}} \left\{ 1, z^2, z^4 \right\} C_{ij} dz, \quad (i, j = 4, 5)
\]

\[
A_{ij} = \frac{25}{16} A_{ij} - \frac{25}{6h^2} D_{ij} + \frac{25}{9h^4} F_{ij}, \quad (i, j = 4, 5)
\]

(11b)

The work done by applied forces can be written as

\[
V = \frac{1}{2} \int_A \left[ N_x^0 \frac{\partial(w_x + w_z)}{\partial x} + N_y^0 \frac{\partial(w_x + w_z)}{\partial y} \\
+ 2N_{xy}^0 \frac{\partial(w_x + w_z)}{\partial x} \frac{\partial(w_x + w_z)}{\partial y} \right] dxdy
\]

(12)

where \(N_x^0, N_y^0, N_{xy}^0\) are in-plane pre-buckling forces.

The governing equations of equilibrium can be obtained as follows

\[
\delta u : \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \quad \delta v : \frac{\partial N_y}{\partial x} + \frac{\partial N_x}{\partial y} = 0
\]

\[
\delta w_x : \frac{\partial^2 M_{xz}}{\partial x^2} + \frac{\partial^2 M_{yz}}{\partial x \partial y} + \frac{\partial^2 M_{yz}}{\partial y^2} + P(w) = 0
\]

\[
\delta w_y : \frac{\partial^2 M_{xz}}{\partial y^2} + \frac{\partial^2 M_{yz}}{\partial x \partial y} + \frac{\partial^2 M_{yz}}{\partial x^2} + \frac{\partial S_{yz}}{\partial x} + \frac{\partial S_{yz}}{\partial y} + P(w) = 0
\]

(13)

where
Substituting from Eq. (6) into Eq. (13), we obtain the following equation,

\[ P(w) = N_x^0 \frac{\partial^2 (w_b + w_s)}{\partial x^2} + N_y^0 \frac{\partial^2 (w_b + w_s)}{\partial y^2} + 2N_{xy}^0 \frac{\partial^2 (w_b + w_s)}{\partial x \partial y}. \]  

(14)

Substituting from Eq. (6) into Eq. (13), we obtain the following equation,

\[
\begin{align*}
A_{11}d_{11}u + A_{66}d_{22}u + (A_{12} + A_{66})d_{12}v - B_{11}d_{11}w_b - (B_{12} + 2B_{66})d_{12}w_b &= 0, \\
A_{22}d_{22}v + A_{66}d_{11}v + (A_{12} + A_{66})d_{12}u - B_{22}d_{22}w_b - (B_{12} + 2B_{66})d_{11}w_b &= 0, \\
B_{11}d_{1111}u + (B_{12} + 2B_{66})d_{1222}u + (B_{12} + 2B_{66})d_{1222}v + B_{22}d_{2222}v - D_{11}d_{1111}w_b - 2D_{12}d_{1111}w_b - 2D_{66}d_{1222}w_b &= 0, \\
- D_{22}d_{2222}w_b - D_{11}d_{1111}w_b + 2D_{12}d_{1111}w_b &= 0, \\
B_{11}d_{1111}u + (B_{12} + 2B_{66})d_{1222}u + (B_{12} + 2B_{66})d_{1222}v + B_{22}d_{2222}v - D_{11}d_{1111}w_b - 2D_{12}d_{1111}w_b - 2D_{66}d_{1222}w_b &= 0, \\
- D_{22}d_{2222}w_b - D_{11}d_{1111}w_b &= 0, \\
B_{11}d_{1111}u + (B_{12} + 2B_{66})d_{1222}u + (B_{12} + 2B_{66})d_{1222}v + B_{22}d_{2222}v - D_{11}d_{1111}w_b - 2D_{12}d_{1111}w_b - 2D_{66}d_{1222}w_b &= 0, \\
- D_{22}d_{2222}w_b - D_{11}d_{1111}w_b &= 0, \\
B_{11}d_{1111}u + (B_{12} + 2B_{66})d_{1222}u + (B_{12} + 2B_{66})d_{1222}v + B_{22}d_{2222}v - D_{11}d_{1111}w_b - 2D_{12}d_{1111}w_b - 2D_{66}d_{1222}w_b &= 0, \\
- D_{22}d_{2222}w_b - D_{11}d_{1111}w_b &= 0, \\
B_{11}d_{1111}u + (B_{12} + 2B_{66})d_{1222}u + (B_{12} + 2B_{66})d_{1222}v + B_{22}d_{2222}v - D_{11}d_{1111}w_b - 2D_{12}d_{1111}w_b - 2D_{66}d_{1222}w_b &= 0, \\
- D_{22}d_{2222}w_b - D_{11}d_{1111}w_b &= 0.
\end{align*}
\]  

(15)

where \(d_{ij}, \ d_{ijl}, \) and \(d_{ijlm}\) are the following differential operators:

\[
\begin{align*}
d_{ij} &= \frac{\partial^2}{\partial x_i \partial x_j}, \ d_{ijl} &= \frac{\partial^3}{\partial x_i \partial x_j \partial x_l}, \ d_{ijlm} &= \frac{\partial^4}{\partial x_i \partial x_j \partial x_l \partial x_m}, \ d_i &= \frac{\partial}{\partial x_i}, (i, j, l, m = 1, 2).
\end{align*}
\]  

(16)

Consider a simply supported rectangular plate with length \(a\) and width \(b\) which is subjected to in-plane loading in two directions\((N_x^0 = \gamma_1 N_{cr}, N_y^0 = \gamma_2 N_{cr}, N_{xy}^0 = 0)\). Based on the Navier method, the following expansions of displacements \((u, v, w_b, w_s)\) are obtained

\[
\begin{bmatrix}
    u \\
    v \\
    w_b \\
    w_s
\end{bmatrix} = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix}
    U_{mn} \cos(\lambda x) \sin(\mu y) \\
    V_{mn} \sin(\lambda x) \cos(\mu y) \\
    W_{bmn} \sin(\lambda x) \sin(\mu y) \\
    W_{snn} \cos(\lambda x) \sin(\mu y)
\end{bmatrix}
\]  

(17)

where \(\lambda = m\pi/a, \mu = n\pi/b\), and \((U_{mn}, V_{mn}, W_{bmn}, W_{snn})\) are unknown functions to be determined. Substituting Eq. (17) into Eq. (15), the closed-form solution of buckling load \(N_{cr}\) can be obtained from

\[
\begin{bmatrix}
    K_{11} & K_{12} & K_{13} & K_{14} \\
    K_{12} & K_{22} & K_{23} & K_{24} \\
    K_{13} & K_{23} & K_{33} + \bar{K} & K_{34} + \bar{K} \\
    K_{14} & K_{24} & K_{34} + \bar{K} & K_{44} + \bar{K}
\end{bmatrix} \begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{bmn} \\
    W_{snn}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]  

(18)

where

\[
\begin{align*}
K_{11} &= A_{11}\lambda^2 + A_{66}\mu^2, \quad K_{12} = \lambda \mu (A_{12} + A_{66}), \quad K_{13} = -\lambda \left[ B_{11}\lambda^2 + (B_{12} + 2B_{66}) \mu^2 \right], \quad K_{14} = -\lambda \left[ B_{11}\lambda^2 + (B_{12} + 2B_{66}) \mu^2 \right], \\
K_{22} &= A_{66}\lambda^2 + A_{22}\mu^2, \\
K_{23} &= -\mu \left[ (B_{12} + 2B_{66}) \lambda^2 + B_{22}\mu^2 \right], \quad K_{24} = -\mu \left[ (B_{12} + 2B_{66}) \lambda^2 + B_{22}\mu^2 \right], \quad K_{33} = D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2 \mu^2 + D_{22}\mu^4, \\
K_{34} &= D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2 \mu^2 + D_{22}\mu^4, \quad K_{44} = H_{11}\lambda^4 + 2(H_{12} + 2H_{66})\lambda^2 \mu^2 + H_{22}\mu^4 + A_{12}\lambda^2 + A_{44}\mu^2, \\
\bar{K} &= N_{cr} (\gamma_1 \lambda^2 + \gamma_2 \mu^2).
\end{align*}
\]  

(19)

3. Numerical results and discussions

For numerical study and investigation, it is assumed that FGM plate is made of Alumina \((\text{Al}_2\text{O}_3)\) as the ceramic part \((E_c = 380 \text{ GPa})\) and aluminum \((\text{Al})\) as metal part \((E_m = 70 \text{ GPa})\). Also, the Poisson ratio is constant and equal to \(\nu = 0.3\).
In Table 2, the effect of the uniaxial compressive load for a FG plate simply supported \((\lambda_1 = -1, \lambda_2 = 0)\) is clear, the results by present theory refined shear deformation theory (RSDT) converge; in first-order shear deformation theory (FSDT) and HSDT, we take three digits after the decimal point for more precision, the values of \((b/h)\) increases, the results for CPT remains constant for the three values of \(k\), the transverse shear deformation effects of plate are not considered in the CPT, the values of nondimensional critical buckling load predicted by CPT are independent of
thickness ratio, which accounts for the transverse shear deformation effects, are dependent of thickness ratio. While the CPT overestimates the nondimensional critical buckling load of FG plate. The amplification of the critical load compared to the thickness is very clear, in this table the values of FSDT and HSDT are closer compared to the other. For the ratio $h/b=20$ and a value of $k=0.1$ there is also a change in critical buckling mode, on the other hand for the other theories the values increases with the increase of $h/b$ and $k$.

$$N = N_{cr} \frac{a^2}{Em^h}.$$  \hspace{1cm} (20)

Table 3 shows the plate FG subjected to uniaxial compression along the axis $x$ ($\lambda_1 = -1$, $\lambda_2 = 0$). The fixing mode for this plate is always simply supported, the ratio

| a/h | Theory | 0 | 0.5 | 1 | 2 | 5 | 10 | 20 | 100 |
|-----|--------|---|-----|---|---|---|----|----|-----|
| 0.5 | Reddy et al.* | 5.371 | 3.527 | 2.715 | 2.109 | 1.700 | 1.527 | 1.364 | 1.097 |
| | Thai and Choi* | 5.376 | 3.539 | 2.733 | 2.116 | 1.719 | 1.537 | 1.369 | 1.099 |
| | Present | 5.376 | 3.538 | 2.731 | 2.116 | 1.718 | 1.537 | 1.369 | 1.099 |
| 1 | Reddy et al.* | 5.918 | 3.850 | 2.961 | 2.302 | 1.925 | 1.747 | 1.548 | 1.218 |
| | Thai and Choi* | 5.926 | 3.857 | 2.969 | 2.312 | 1.933 | 1.752 | 1.551 | 1.220 |
| | Present | 5.924 | 3.865 | 2.968 | 2.311 | 1.933 | 1.751 | 1.559 | 1.220 |
| 10 | Reddy et al.* | 6.072 | 3.940 | 3.029 | 2.362 | 1.991 | 1.812 | 1.602 | 1.253 |
| | Thai and Choi* | 6.079 | 3.857 | 3.034 | 2.367 | 1.996 | 1.815 | 1.604 | 1.255 |
| | Present | 6.079 | 3.945 | 3.044 | 2.366 | 1.995 | 1.815 | 1.604 | 1.254 |
| 50 | Reddy et al.* | 6.117 | 3.966 | 3.049 | 2.379 | 2.011 | 1.831 | 1.618 | 1.263 |
| | Thai and Choi* | 6.124 | 3.971 | 3.053 | 2.382 | 2.014 | 1.834 | 1.620 | 1.265 |
| | Present | 6.124 | 3.970 | 3.053 | 2.382 | 2.013 | 1.833 | 1.619 | 1.264 |
| 100 | Reddy et al.* | 6.123 | 3.970 | 3.052 | 2.382 | 2.014 | 1.834 | 1.620 | 1.265 |
| | Thai and Choi* | 6.131 | 3.974 | 3.056 | 2.385 | 2.016 | 1.837 | 1.622 | 1.266 |
| | Present | 6.130 | 3.974 | 3.056 | 2.386 | 2.016 | 1.836 | 1.622 | 1.266 |

*Taken from Reddy et al. [26] and Thai and Choi [27].
Table 5. Comparison of nondimensionalized critical buckling load \( (N) \) of simply supported \( \text{Al}_2\text{O}_3 \) plate subjected to biaxial compression and tension \( (\lambda_1 = -1, \lambda_2 = 1) \).

| \( a/h \) | Theory | 0 | 0.5 | 1 | 2 | 5 | 10 | 20 | 50 | 100 |
|----------|--------|---|-----|---|---|----|-----|-----|-----|-----|
| 0.5      | 5       | Reddy et al. \(^a\) | 8.953 | 5.879 | 4.525 | 3.487 | 2.833 | 2.545 | 2.274 | 1.829 |
|          |        | Thai and Choi \(^a\) | 8.960 | 5.898 | 4.555 | 3.527 | 2.865 | 2.562 | 2.282 | 1.832 |
|          |        | Present \(^a\) | 8.9604 | 5.8980 | 4.5551 | 3.5268 | 2.8646 | 2.5617 | 2.2820 | 1.8316 |
| 10       | Reddy et al. \(^a\) | 9.863 | 6.416 | 4.934 | 3.837 | 3.208 | 2.911 | 2.580 | 2.031 |
|          | Thai and Choi \(^a\) | 9.874 | 6.426 | 4.948 | 3.853 | 3.222 | 2.900 | 2.585 | 2.033 |
|          | Present \(^a\) | 9.8738 | 6.4275 | 4.9481 | 3.8526 | 3.2199 | 2.9195 | 2.5849 | 2.0344 |
| 20       | Reddy et al. \(^a\) | 10.120 | 6.566 | 5.049 | 3.936 | 3.319 | 3.020 | 2.670 | 2.089 |
|          | Thai and Choi \(^a\) | 10.132 | 6.575 | 5.057 | 3.944 | 3.326 | 3.025 | 2.674 | 2.091 |
|          | Present \(^a\) | 10.1324 | 6.5753 | 5.0574 | 3.9442 | 3.3259 | 3.0253 | 2.6739 | 2.0911 |
| 50       | Reddy et al. \(^a\) | 10.195 | 6.610 | 5.082 | 3.965 | 3.352 | 3.052 | 2.697 | 2.105 |
|          | Thai and Choi \(^a\) | 10.207 | 6.618 | 5.089 | 3.971 | 3.356 | 3.056 | 2.700 | 2.108 |
|          | Present \(^a\) | 10.2073 | 6.6179 | 5.0889 | 3.9706 | 3.3562 | 3.0564 | 2.6999 | 2.1079 |
| 100      | Reddy et al. \(^a\) | 10.206 | 6.606 | 5.067 | 3.969 | 3.356 | 3.057 | 2.700 | 2.108 |
|          | Thai and Choi \(^a\) | 10.218 | 6.624 | 5.093 | 3.974 | 3.361 | 3.061 | 2.704 | 2.110 |
|          | Present \(^a\) | 10.2181 | 6.6240 | 5.0934 | 3.9744 | 3.3606 | 3.0609 | 2.7037 | 2.1103 |

\(^a\)Mode for plate is \((m,n) = (2,1)\).
\(^b\)Mode for plate is \((m,n) = (1,2)\).
\(^c\)From Reddy et al. [26] and Thai and Choi [27].

\((a/b)\) varies with a regular step of 0.5 and for each step of \((a/b)\) the ratio \((a/h)\) varies from 5 to 100, for fixed values of \((a/b)\) and \(k\), nondimensionalized critical buckling load increases with the ratio \((a/h)\), for fixed values of \((a/b)\) and \((a/h)\), nondimensionalized critical buckling load decreases with the increase in \(k\) value, from a value of 1 for \((a/b)\) the results in the table are almost doubled. For the ratio \((a/b) = 2\), \((a/h) = 5\), and all the values of \(k\) there is also a change in critical buckling mode, for a fixed value of \(k\) and increases in the values of the ratios \((a/b)\) and \((a/h)\), we have an overestimation of nondimensionalized critical buckling load; on the other hand for fixed values of the ratios \((a/b)\)
and \((a/h)\) and ascending values of \(k\), we have an underestimation of nondimensionalized critical buckling load.

Table 4 presents the results obtained by our theory and that of Reddy et al. [26] and Thai and Choi [27]. For nondimensionalized critical buckling load \((\tilde{N})\) of simply supported Al/Al₂O₃ plate subjected to biaxial compression \((k_1 = -1, k_2 = -1)\), for fixed values of \((a/b)\) and \(k\), nondimensionalized critical buckling load increases with the ratio \((a/h)\), for fixed values of \((a/b)\) and \((a/h)\), nondimensionalized critical buckling load decreases with the increase in \(k\) value. For a fixed value of \(k\) and increases in the values of the ratios \((a/b)\) and \((a/h)\), we have an overestimation of nondimensionalized critical buckling load; on the other hand for fixed values of the ratios \((a/b)\) and \((a/h)\) and ascending values of \(k\) we have an underestimation of nondimensionalized critical buckling load. For the ratio \((a/b = 1.5), (a/h = 5)\), and all the values of \(k\) there is also a change in critical buckling mode.

For Figures 2 and 3, we studied the effect of side-to-thickness ratios \((a/h)\) on nondimensionalized critical buckling load \((\tilde{N})\) under uniaxial compression for a simply supported FG plate for various material variation parameters \((k)\), \((a = 10 \times h)\).

Table 5 shows a comparison between the theory cited in the latter, for nondimensionalized critical buckling load \((\tilde{N})\) of simply supported Al/Al₂O₃ plate subjected to biaxial compression and tension \((\lambda_1 = -1, \lambda_2 = 1)\). For fixed values of \((a/b)\) and \(k\), nondimensionalized critical buckling load increases with the ratio \((a/h)\), for fixed values of \((a/b)\) and \((a/h)\), nondimensionalized critical buckling load decreases with the increase in \(k\) value. For a fixed value of \(k\) and increases in the values of the ratios \((a/b)\) and \((a/h)\), we have an overestimation of nondimensionalized critical buckling load; on the other hand for fixed values of the ratios \((a/b)\) and \((a/h)\) and ascending values of \(k\) we have an underestimation of nondimensionalized critical buckling load. For the ratio \((a/b = 1.5), (a/h = 5)\), and all the values of \(k\) there is also a change in critical buckling mode.

For Figures 2 and 3, we studied the effect of side-to-thickness ratios \((a/h)\) on nondimensionalized critical buckling load \((\tilde{N})\) under uniaxial compression for a simply supported FG plate for various material variation parameters \((k)\). The variation of this load is quite clear as a function of these ratios, these figures show that the critical dimensionless buckling loads can go up to a value of 20 for the plates rich in ceramic, on the other hand for the plates rich in metal does not exceed the value of 4 that is to say it is lower. The critical buckling loads of FG plates are intermediate to those of ceramic and metal. For values of \(k\) between 0.5 and 10, Figure 2 shows that the effect of shear
deformation is significant for a ratio \((a/h)\) less than 20 and decreases with values greater than 20. For Figure 3, the increase in the ratio \((b/a)\) decreases the critical buckling load in such a way that it is almost convergent, which explains the decrease in the rigidity of the plate.

The effect of side-to-thickness ratio \((a/h)\) for values varying from 0 to 100, aspect ratio \((b/a)\) for values varying from 0 to 30, and power law index values \((k)\) for values varying from 0.5 to 10 on nondimensionalized critical buckling load for a simply supported FG plate under biaxial compression is shown in Figures 4 and 5. The biaxial nondimensionalized critical buckling load is better converged for values greater than 30.

4. Conclusions

In this study, the plates with functional gradation were developed by the refined theory of four variables or one to study the behavior of mechanical buckling. For this theory, the number of primary variables is still lower than that of the theories of shear deformation plates of first order and of higher level that implies the nonuse of the shear correction factor. We can conclude that the present theory is precise and effective for predicting the mechanical buckling load of functionally graded plates simply supported. For the certain ratio values \((a/b)\), \((a/h)\), and all the values of \(k\), there is also a change in critical buckling mode, in certain cases the increase in the ratio \((b/a)\) implies the reduction in the rigidity of the plate, in addition, the results found present a convergence with other similarities and especially that of Reddy et al. This study remains useful to evaluate other future plate theories. Hence, it can be said that the proposed refined plate theory is accurate and simple in solving the buckling behavior of FG plates.

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