Ring potential generated from the central hyperbolic Manning–Rosen potential using the transformation method

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An exactly solved ring potential is generated from the central hyperbolic Manning–Rosen potential using the transformation method in the framework of non-relativistic quantum mechanics. The basis of the method is coordinate redesignation and extended transformation, comprising a coordinate transformation supplemented by a functional transformation. Application of the coordinate transformation entailing redesignation of the radial coordinate with polar angle to the Schrödinger radial equation for the exactly solved central hyperbolic Manning–Rosen potential in 3D Euclidean space yields a second-order homogeneous angular differential equation. The functional transformation is indispensable in molding the angular differential equation to the Schrödinger angular equation form and, in the process of retrieving the standard Schrödinger angular equation by invoking a plausible ansatz, a new exactly solved ring potential is generated. The transformed angular wave functions for the generated ring potential are normalizable.

Subject Index A02, A13, A64

1. Introduction

The study of quantum systems (QSs) with exactly solved potentials contributes new physical ideas and/or mathematical techniques to quantum mechanics. The study of exactly solved potentials plays a deciding role in evaluating the effectiveness of the approximate methods implemented in the study of practical QSs with potentials having some sort of perturbations. With the advent of quantum mechanics, many researchers have studied QSs with exactly solved central potentials in different contexts. In the last two decades, the study of QSs with non-central potentials has also drawn much attention in chemistry for the theoretical investigation of molecules like benzene with some axial symmetry and in nuclear physics for the study of the interaction between deformed nuclei. Each non-central potential in a spherical polar coordinate system comprises two components for the separability of the Schrödinger stationary-state wave equation, one being a radial-coordinate-dependent central potential and the other a polar-angle-dependent ring potential. Various methods, e.g. the standard approach [1], the factorization method [2], the partial wave method [3], the Uvarov–Nikiforov (NU) method [4–6], the exact path integral approach [7], etc., have been employed to study QSs analytically with non-central potentials.

Here, we report a scheme based on the transformation method to generate an exactly solved ring potential from an already known physical QS with the exactly solved central hyperbolic Manning–Rosen potential. The basis of the method is redesignation of a radial coordinate with polar angle
and an extended transformation [8] consisting of a coordinate transformation supplemented by a
functional transformation. Coordinate transformation changes the characteristics of a differential
equation, while subsequent functional transformation along with a suitable ansatz is essential to
retrieve a differential equation of a particular form, but of the same order. The extended trans-
formation is performed on the second-order angular differential equation, obtained through the
redesignation of the radial coordinate with polar angle in the Schrödinger radial equation for the
hyperbolic Manning–Rosen potential, to retrieve the Schrödinger angular equation form and, by
invoking a plausible ansatz, a new exactly solved ring potential is generated. The angular wave func-
tions corresponding to the generated ring potential are normalizable, since the generated potential is
mapped from the exactly solved hyperbolic Manning–Rosen potential for a physical QS.

The organization of the paper is as follows: The method is discussed in Sect. 2, implementation
of the method in a practical QS with the central hyperbolic Manning–Rosen potential is shown in
Sect. 3, and conclusions are given in Sect. 4.

2. Formalism

We start with the Schrödinger radial equation in 3D Euclidean space:

$$\frac{d^2}{dr^2} R(r) + \frac{2}{r} \frac{d}{dr} R(r) + \left[ E_n - V_{ES}(r) - \frac{\lambda}{r^2} \right] R(r) = 0,$$

where the normalized radial wave functions $R(r)$ and the energy eigenvalues $E_n$ are known for a
physical QS with an exactly solved central potential $V_{ES}(r)$, henceforth called the parent potential. $\lambda$
is a real and dimensionless constant. In the case of a central potential, $\lambda = l(l + 1)$ and the admissible
values for the orbital quantum number $l$ are $0, 1, 2, 3, \ldots, (n - 1)$, if $R(r)$ is to be the wave function
for a physical QS, but $l$ needs to be redefined [3] in the presence of the polar-angle ($\theta$)-dependent
ring potential.

The radial coordinate ($r$) is first redesignated exclusively in a mathematical sense by the polar
angle ($\theta$) in the Schrödinger radial equation (1) for a parent potential $V_{ES}(r)$ in 3D Euclidean space
for an s-wave ($l = 0$) to obtain a second-order $\theta$ differential equation as

$$\frac{d^2}{d\theta^2} R(\theta) + \frac{2}{\theta} \frac{d}{d\theta} R(\theta) + \left[ E_n - V_{ES}(r = \theta) \right] R(\theta) = 0.$$

The standard extended transformation comprising a coordinate transformation followed by a func-
tional transformation as

$$\theta \rightarrow g(\theta)$$

and

$$\chi(\theta) = f(\theta)^{-1} R[g(\theta)]$$

is performed on the above $\theta$ differential equation (2). Here, the coordinate transformation function
$g(\theta)$, which will be specified by an ansatz, is a differentiable function of at least class $C^2$, required for
generating a new potential by modifying the spatial character of the parent potential, while $f(\theta)^{-1}$ is
the modulating function that plays a role in molding the $\theta$ differential equation (2) to the Schrödinger
θ equation form. Application of the coordinate transformation as in Eq. (3) to Eq. (2) yields

$$\frac{d^2}{dg^2}R(g) + \frac{2}{g} \frac{d}{dg} R(g) + [E_n - V_{ES}(r = g(\theta))]R(g) = 0, \quad (5)$$

then using the relations \( \frac{d}{dg} = \frac{1}{g} \frac{d}{d\theta} \) and \( \frac{d^2}{dg^2} = \frac{1}{g^2} \frac{d^2}{d\theta^2} - \frac{g''}{g^3} \frac{d}{d\theta} \) and subjecting Eq. (5) to functional transformation as in Eq. (4), we have

$$\frac{d^2}{d\theta^2}\chi(\theta) + \left( \frac{d}{d\theta} \ln \frac{f^2 g^2}{g'} \right) \frac{d}{d\theta} \chi(\theta)$$

$$+ \left\{ \left( \frac{d}{d\theta} \ln f \right) \left( \frac{d}{d\theta} \ln \frac{f' g^2}{g} \right) + g^2 [E_n - V_{ES}(r = g(\theta))] \right\} \chi(\theta) = 0. \quad (6)$$

To retrieve the Schrödinger θ equation form, the coefficient of the first-order derivative in Eq. (6) above is made equal to \( \cot \theta \), fixing the functional form of \( f(\theta) \) as

$$f(\theta) = n_c g^{1/2}(\theta) g^{-1}(\theta) \sin^2 \theta, \quad (7)$$

which changes Eq. (6) to

$$\frac{d^2}{d\theta^2}\chi(\theta) + \cot \theta \frac{d}{d\theta} \chi(\theta)$$

$$+ \left\{ \frac{1}{2} \{g, \theta\} - \frac{1}{4} (1 + \csc^2 \theta) + g^2 [E_n - V_{ES}(r = g(\theta))] \right\} \chi(\theta) = 0, \quad (8)$$

where the Schwarzian derivative symbol \( \{g, \theta\} = \frac{g'''}{g} - \frac{3}{2} \left( \frac{g''}{g} \right)^2 \) and the constant \( n_c \) in Eq. (7) will contribute to the normalization constant \( N_c \).

To insert a term \( \frac{m^2}{\sin^2 \theta} \) as it appears in the Schrödinger θ equation (here \( m \) is the magnetic quantum number), we invoke the following ansatz:

$$g^2 E_n = -\frac{m^2}{\sin^2 \theta}, \quad (9)$$

which specifies the functional form of \( g(\theta) \) as

$$g(\theta) = \eta \ln | C \tan \frac{\theta}{2} |, \quad (10)$$

where

$$\eta = \pm \sqrt{\frac{m^2}{-E_n}}. \quad (11)$$

The integration constant \( C = 1 \) attributes the local property \( g \left( \frac{\pi}{2} \right) = 0 \). In the transformation of the Schrödinger radial equation in 1D or 3D Euclidean space to Schrödinger θ equation form using the transformation function given by Eq. (10), the Schwarzian derivative \( \{g, \theta\} \) emerges as \( \frac{1}{2} (1 + \csc^2 \theta) \),
reducing Eq. (8) to
\[
\frac{d}{d\theta^2} \chi(\theta) + \cot \theta \frac{d}{d\theta} \chi(\theta) + \left[ -\frac{m^2}{\sin^2 \theta} - \frac{\eta^2}{\sin^2 \theta} V_{ES} (r = g(\theta)) \right] \chi(\theta) = 0. \tag{12}
\]

To mold Eq. (12) above to the standard Schrödinger equation with a ring potential \( V(\theta) \), i.e.
\[
\frac{d^2}{d\theta^2} \chi(\theta) + \cot \theta \frac{d}{d\theta} \chi(\theta) + \left[ \lambda - \frac{m^2}{2} \sin^2 \theta - V(\theta) \right] \chi(\theta) = 0, \tag{13}
\]
the following identity must be prescribed:
\[
\lambda - V(\theta) = -\frac{\eta^2}{\sin^2 \theta} V_{ES}[r = g(\theta)]. \tag{14}
\]
The constant terms on the right-hand side of Eq. (14) above are combined to construct \( \lambda = l(l + 1) \), while the \( \theta \)-dependent terms are assembled to generate the ring potential \( V(\theta) \). The angular wave functions for the generated ring potential \( V(\theta) \) are obtained by using Eq. (7) in Eq. (4) as
\[
\chi(\theta) = N_c g^{-\frac{1}{2}}(\theta) g(\theta) \sin^{-\frac{1}{2}} \theta R[g(\theta)], \tag{15}
\]
and are known since both the transformation function \( g(\theta) \) given by Eq. (10) and the radial wave function \( R(r) \) for the parent potential \( V_{ES}(r) \) are known.

### 3. Application with the hyperbolic Manning–Rosen potential

The central hyperbolic Manning–Rosen potential \([9]\) is given by
\[
V_{ES}(r) = \alpha_r \frac{\cosh^2 \frac{r}{2b}}{2b} + \beta_r \frac{\cosh \frac{r}{2b}}{\sinh \frac{r}{2b}} + \gamma_r, \tag{16}
\]
where
\[
\alpha_r = \frac{\alpha(\alpha - 1)}{4b^2}, \quad \beta_r = \frac{1}{2b^2} [\alpha(\alpha - 1) - A], \quad \text{and} \quad \gamma_r = \alpha_r + \frac{A}{2b^2}.
\]
The radial wave functions are
\[
R_n(r) = r^{-1} (1 - \omega)^a \omega^b \omega^c 2F_1(a, b; c; \omega), \tag{17}
\]
where
\[
\omega = \exp \left( -\frac{r}{b} \right),
\]
\[
a = -n = \alpha + \delta - \sqrt{A + \alpha(\alpha - 1) + \delta^2}, \tag{19}
\]
\[
b = \alpha + \delta + \sqrt{A + \alpha(\alpha - 1) + \delta^2}, \tag{20}
\]
\[
c = 2\delta + 1, \tag{21}
\]
and the discrete energy eigenvalues are
\[
E_n = -\frac{\delta^2}{b^2} = -\frac{1}{b^2} \left[ \frac{A - \alpha}{2(\alpha + n)} - \frac{n(n + 2\alpha)}{2(\alpha + n)} \right]^2 \tag{22}
\]
with \( n = 0, 1, 2, 3, \ldots, n_{\text{max}} \) and \( n_{\text{max}} = \sqrt{A + \alpha(\alpha - 1) - \alpha} \).
Taking the transformation function \( g(\theta) \) from Eq. (10) and the expression for the Manning–Rosen potential from Eq. (16), and using \( \frac{n}{2\theta} = 1 \) in Eq. (14) only for mathematical simplifications, the ring potential generated is

\[
V(\theta) = \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\beta_\theta}{\sin^2 \theta} \cos \theta. \tag{23}
\]

We select \( l = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \alpha_\theta} \) to obtain \( \lambda = l(l + 1) \) in the Schrödinger \( \theta \) equation through the identity given by Eq. (14). The angular wave functions corresponding to the generated ring potential are obtained by using Eqs. (10) and (17) in Eq. (15) as

\[
\chi^m_l(n) = N_c (1 - n)^\sigma v^\kappa _2 F_1(a, b; c; n), \tag{24}
\]

choosing

\[
\sigma = \frac{1}{2} \pm \sqrt{\frac{1}{4} \alpha_\theta + \frac{1}{4}}\kappa = \pm \frac{1}{2} \sqrt{m^2 - 2\sqrt{\alpha_\theta + A}}, \quad \text{and} \quad A = \alpha_\theta + \frac{1}{8} \beta_\theta, \tag{25}
\]

where

\[
n = \cot^2 \theta \frac{2}{2}, \tag{26}
\]

\[
a = \sigma + \kappa - \sqrt{A + \sigma(\sigma - 1) + \kappa^2}, \tag{27}
\]

\[
b = \sigma + \kappa + \sqrt{A + \sigma(\sigma - 1) + \kappa^2}, \tag{28}
\]

The normalization condition for the angular wave function \( \chi(\theta) \) for a physical QS is

\[
I(0, \pi) = \int_0^\pi |\chi(\theta)|^2 \sin \theta d\theta = \text{finite}. \tag{29}
\]

The above normalization integral for the angular wave functions in Eq. (14) corresponding to the generated ring potential in Eq. (23) yields

\[
I(0, \pi) = \int_{g(0)}^{g(\pi)} \left( \frac{1}{\cosh^2 \frac{g}{2h}} \right) |R(g)|^2 g^2 dg, \tag{30}
\]

which is equivalent to the expectation value of a term in the hyperbolic Manning–Rosen potential and is hence finite, indicating that the angular wave functions for the generated ring potential are normalizable.

4. Conclusions

We report a transformation method for the generation of an exactly solved ring potential from a QS with the exactly solved central hyperbolic Manning–Rosen potential in a non-relativistic regime. The generation of the exactly solved ring potential using the transformation method is only realizable through the implication of the extended transformation, which is a combination of a coordinate transformation and a functional transformation. The coordinate redesignation mechanism, along with a suitable ansatz, is mandatory for generating the ring potential. The angular wave functions for the generated ring potential are analytically verified and are also normalizable. Since the transformation function is a natural logarithm of a tangent, the method has the capability of generating ring potentials from hyperbolic central potentials. Work is currently ongoing to generate/construct new ring potentials from exactly solved power-law, non-power-law, and trigonometric central potentials by using the transformation method with a plausible ansatz.
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