COMBINED EFFECTS OF THE SPATIAL HETEROGENEITY
AND THE FUNCTIONAL RESPONSE

YU-XIA WANG
School of Mathematical Sciences
University of Electronic Science and Technology of China
Chengdu, Sichuan 611731, China

WAN-TONG LI
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China

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Abstract. This paper deals with a predator-prey model with Beddington-DeAngelis functional response, in which a protection zone is created for the prey species. Whether the combination of the protection zone and the Beddington-DeAngelis functional response can yield new results or not is of interest. The result reveals that they jointly produce a new critical value, which is smaller than that determined by either the protection zone or the functional response singly. As a result, rather different stationary patterns can be found and the combined effects are very prominent. Then the effect of the parameter \( k \) in the Beddington-DeAngelis functional response is studied. The result deduces that as \( k \) is large enough, there exists a unique positive stationary solution and it is linearly stable except a special case. Actually, we can obtain that the positive stationary solution is globally asymptotically stable.

1. Introduction. In many scientific experiments, the spatial heterogeneity has been observed to influence the population dynamics. For example, Huffaker [22] found that a predator-prey system consisting of two species of mites could collapse to extinct quickly in small homogeneous environment, but would persist longer in suitable heterogeneous environment. In recent years, the effect of the spatial heterogeneity has attracted the attention of both mathematicians and ecologists. Whereas, the spatial heterogeneity brings new mathematical questions and it seems difficult to address this question. The reason is that if the coefficients are not constant, then the known mathematical tools are either unapplicable or not enough to observe the effect of the spatial heterogeneity.

To address this problem, Du et al. [7, 11, 12, 13, 14, 15, 16, 17, 18] have done a series of works. On the one hand, since the behavior of ecosystems is usually very sensitive to certain coefficient function becoming small in part of the underlying spatial region, they assume that the intraspecific coefficient possesses certain spatial degeneracy in the model. From the viewpoint of ecology, it means that there exists a favorable subregion for the weak species, in which the growth of the species is...
unbounded. By taking this kind of spatial degeneracy into account, Du et al. have successfully revealed the effect of the spatial heterogeneity on the predator-prey model and the competition model\[7, 11, 12, 13, 17, 18\]. On the other hand, in most predator-prey interactions, the prey population would extinguish if the growth rate of the predator is too large or the predation rate is too high. Then human interference is often needed to save the endangered prey species and a natural idea is to set protection zones for the prey, where the prey species can enter and leave freely while the predator is blocked out. By introducing protection zone to the model, Du et al. have also found the effect of protection zone on the predator-prey model and competition model \[14, 15, 16\]. In addition, one can also see \[5, 21, 20, 33, 35\] and references therein.

Besides the spatial heterogeneity, it is well known that the predator-prey model is usually sensitive to its functional response. Since the Lotka-Volterra functional response, more and more functional responses have been proposed, such as the Holling-II functional response and so on. In particular, taking the effects of feeding saturation and intraspecific consumer interference into account, Beddington [1] and DeAngelis et al. \[8\] introduced the Beddington-DeAngelis functional response

\[
\frac{u}{1 + mu + kv},
\]

where the parameter \(k\) measures the mutual interference between predators. For more biological background, One can refer to \[9, 29\] and references therein. For the investigation of the Beddington-DeAngelis functional response, one can refer to \[2, 3, 19\].

In \[33\], Wang and Li consider a predator-prey model with Beddington-DeAngelis functional response in heterogeneous environment, where the spatial heterogeneity is caused by the spatial degeneracy of the intraspecific coefficient of the prey. Comparing to the Lotka-Volterra or Holling-II functional response, they show that the Beddington-DeAngelis functional response can generate rather different stationary pattern. More precisely, Du et al. \[7, 17\] show that for the predator-prey model with either Lotka-Volterra or Holling-II functional response, there exists only one critical value which is produced by the spatial degeneracy. Whereas, Wang and Li show that except the critical value produced by the spatial degeneracy, the Beddington-DeAngelis functional response also generates a critical value. Thus, the Beddington-DeAngelis functional response can behave rather different behavior. Then we are very interested in that if the spatial heterogeneity is created by a protection zone for the prey species, can the Beddington-DeAngelis functional response jointly with the spatial heterogeneity generate new results?

For this purpose, in this paper, we consider the following predator-prey system with Beddington-DeAngelis functional response and a protection zone for the prey species:

\[
\begin{aligned}
&u_t - \Delta u = u \left( \lambda - u - \frac{a(x)v}{1 + mu + kv} \right), \quad x \in \Omega, \quad t > 0, \\
v_t - \Delta v = v \left( \mu - v + \frac{cu}{1 + mu + kv} \right), \quad x \in \Omega \setminus \overline{\Omega_0}, \quad t > 0, \\
\partial_{\nu}u = 0, \quad x \in \partial \Omega, \quad t > 0, \quad \partial_{\nu}v = 0, \quad x \in \partial \Omega \cup \partial \Omega_0, \quad t > 0, \\
&u(x, 0) = u_0(x) \geq 0, \quad x \in \Omega, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega \setminus \overline{\Omega_0},
\end{aligned}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial \Omega$, $\Omega_0$ is an open and connected subdomain of $\Omega$ with smooth boundary $\partial \Omega_0$. $u(x, t)$ and $v(x, t)$ represent the population density of the prey and predator species, respectively; $\lambda, c, m, k$ are positive constants and $\mu$ is a real constant, which may take negative values; the function $a(x)$ satisfies $a(x) \equiv 0$ in $\Omega_0$ and $a(x) = a > 0$ in $\Omega \setminus \Omega_0$. We see that in this model, the prey species $u$ lives in the larger habitat $\Omega$, and $\Omega_0$ is its protection zone, where $u$ can leave and enter freely, while $v$ can only live outside. This is the reason why homogeneous Neumann boundary condition is assumed for $v$ on the boundary $\partial \Omega_0$. In addition, it should be pointed out that to focus our attention on the effects on the spatial heterogeneity and the functional response, we have rescaled the coefficients of system (1).

To investigate the combined effects of the protection zone and the Beddington-DeAngelis functional response, we try to give some results about the stationary problem of (1) and its dynamical behavior. It is clear that the stationary problem of (1) is given by the following system:

$$
\begin{align*}
-\Delta u &= u \left( \lambda - u - \frac{a(x)v}{1 + mu + kv} \right), & x \in \Omega, \\
-\Delta v &= v \left( \mu - v + \frac{cu}{1 + mu + kv} \right), & x \in \Omega_1,
\end{align*}
$$

where we write $\Omega_1 = \Omega \setminus \Omega_0$ for convenience of notations. It should be remarked that He and Zheng [20] also study the predator-prey model with Beddington-DeAngelis functional response. But different from them, we use $\mu$ as the bifurcation parameter to obtain the positive stationary solution. More importantly, this paper would like to focus on the positive stationary solution and its dynamical behavior for large $k$, which are not discussed in [20].

By virtue of the bifurcation theory and some estimates, the positive solution set of (2) bifurcating from its semitrivial solution set can be obtained. To be precise, the result shows that together with the Beddington-DeAngelis functional response, the protection zone produces one critical value

$$
\lambda^*_N \left( \frac{a(x)}{k}, \Omega \right),
$$

and we denote it by $\lambda^*$. As $\lambda$ crosses the critical value, the global continuum of positive stationary solution undergoes a drastic change.

To see the combined effects of the protection zone and the Beddington-DeAngelis functional response, on the one hand, we compare our result with that of the homogeneous case. Due to Remark 2 in [33], one can see that the Beddington-DeAngelis functional response in homogeneous environment creates one critical value $\frac{a}{k}$, which is larger than the one obtained in the heterogeneous environment. As a result, no matter how small the protection zone is, the effect of the protection zone can be very profound. More precisely, regardless of the size of the protection zone, there exists $\lambda$ such that

$$
\lambda^* < \lambda < \frac{a}{k}.
$$

Then as $\lambda$ falls in this range, the global bifurcation continuum is bounded in the homogeneous environment, but unbounded in the heterogeneous environment. So when the predator has a large enough birth rate $\mu$, no positive stationary solution exists in the homogeneous environment. However, so long as a protection zone
is set for the prey, the existence of positive stationary solution for large \( \mu \) can be guaranteed. Thus, no matter how small the protection zone is, it can save the prey species. On the other hand, we compare our result with that of the model with different functional response. Du and Shi [16] show that for the Holling-II functional response, the protection zone itself produces one critical value \( \lambda_{D}^{0}(\Omega_{0}) \), which is also larger than the one obtained in this paper. So the effect of the Beddington-DeAngelis functional response can also be rather important, even though \( k \) is small.

More precisely, regardless of the value of \( k \), there exists \( \lambda \) such that

\[
\lambda^{*} < \lambda < \lambda_{D}^{0}(\Omega_{0}).
\]

For fixed \( \lambda \) in this range, the predator-prey model with Holling-II functional response has no positive stationary solution if the birth rate \( \mu \) of the predator is large enough; whereas, the predator-prey model with Beddington-DeAngelis functional response always has at least one positive stationary solution for any \( \mu > -\frac{c}{1+m\lambda} \).

Thus, we see that no matter how weak of the mutual interference between predators, when a protection zone cannot save the prey species with Holling-II functional response, it can save the prey species with Beddington-DeAngelis functional response. In summary, we see that the combination of the Beddington-DeAngelis functional response and the protection zone is much more favorable for the prey species.

Due to the result in [33], we can also see that the effects of the spatial heterogeneity caused by the protection zone and spatial degeneracy are different. To be precise, the Beddington-DeAngelis functional response and the spatial degeneracy together produce two critical values \( a/k \) and \( \lambda_{D}^{0}(\Omega_{0}) \); whereas the Beddington-DeAngelis functional response and the protection zone together produce one critical value \( \lambda^{*} \), which is the smallest of the three critical values. Thus, as \( \lambda \) falls in the range of

\[
\lambda^{*} < \lambda < \min\{a/k, \lambda_{D}^{0}(\Omega_{0})\},
\]

positive stationary solution does not exist for large \( \mu \) in case of the spatial degeneracy and exists for large \( \mu \) in case of the protection zone. Therefore, setting up protection zones for the prey may be more effective.

Since the Beddington-DeAngelis functional response can induce rather different stationary patterns, the effect of large \( k \) is further studied. By various estimates, we show that as \( k \) is large enough, (1) has a positive stationary solution if and only if \( \mu > -\frac{c}{1+m\lambda} \). Moreover, the positive stationary solution is unique and globally asymptotically stable. It should be pointed out that for technical reason, the uniqueness and the global stability are only given in the case of \( \mu \neq 0 \). For the case of \( \mu = 0 \), we have done it in the other paper [34], which shows that the uniqueness and global stability still hold true.

In addition, we remark that for the study of the spatial heterogeneity, one can refer to the series work by Hutson et al. and Lou et al. [10, 23, 26]. For the cross-diffusive system in heterogeneous environment, one can see [24, 27, 32, 31] and references therein.

Finally, we give the notations used in this paper. Let \( O \) be a bounded domain with smooth boundary. The usual norms of the spaces \( L^{p}(O) \) for \( p \in [1, \infty) \) and \( C(\overline{O}) \) are defined by

\[
\|u\|_{p,O} = \left( \int_{O} |u(x)|^{p}dx \right)^{1/p} \quad \text{and} \quad \|u\|_{\infty,O} = \max_{\overline{O}} |u(x)|.
\]

For a continuous function \( \phi \), let \( \lambda_{D}^{0}(\phi,O) \) and \( \lambda_{N}^{0}(\phi,O) \) be the principal eigenvalue of \(-\Delta + \phi \) subject to Dirichlet and Neumann boundary conditions over a domain \( O \).
Then simple computations deduce that condition. Moreover, one can see that only when \( \mu \) respectively. If the potential function \( \phi \) is omitted, then we understand that \( \phi = 0 \). It is well known that the following properties hold:

(i): \( \lambda^1_\mu(\phi, O) > \lambda^N_\mu(\phi, O) \);
(ii): \( \lambda^1_\mu(\phi_1, O) > \lambda^B_\mu(\phi_2, O) \) if \( \phi_1 \geq \phi_2 \) and \( \phi_1 \neq \phi_2 \), where \( B = D \) or \( B = N \);
(iii): \( \lambda^0_\mu(\phi, O_1) \geq \lambda^0_\mu(\phi, O_2) \) if \( O_1 \subset O_2 \).

This paper is organized as follows. In Section 2, we show the global bifurcation of positive solutions of (2) bifurcating from its semitrivial solution set. In Section 3, the effects of large \( k \) on the positive stationary solution set is given. In Section 4, the dynamical behavior is deduced.

2. Global bifurcation of positive stationary solution. In this section, we regard \( \mu \) as the bifurcation parameter and apply the bifurcation theory to show a basic understanding of the positive solution set of (2). For the framework of the bifurcation analysis, we set up the following Banach spaces:

\[
X = W^{2,p}_\nu(\Omega) \times W^{2,p}_\nu(\Omega_1), \quad Y = L^p(\Omega) \times L^p(\Omega_1),
\]

where

\[
W^{2,p}_\nu(\Omega) = \{ u \in W^{2,p}(\Omega) : \partial_{\nu}u = 0 \text{ on } \partial\Omega \}
\]

and \( p > 1 \).

First, we consider the local bifurcation solution along the semitrivial solution set. It is clear that (2) has two semitrivial solution sets

\[
\Gamma_u = \{ (\mu, u, v) = (\mu, \lambda, 0) : \mu \in \mathbb{R} \}
\]

and

\[
\Gamma_v = \{ (\mu, u, v) = (\mu, 0, \mu) : \mu > 0 \}.
\]

To show the local bifurcation along \( \Gamma_u \), let \( \lambda = \mu - u \) and define \( F : \mathbb{R} \times X \to Y \) by

\[
F(\mu, U, v) = \begin{pmatrix}
\Delta U - (\lambda - U) \left( U - \frac{a(x)v}{1 + m(\lambda - U) + kv} \right) \\
\Delta v + v \left( \mu - v + \frac{c(\lambda - U)}{1 + m(\lambda - U) + kv} \right)
\end{pmatrix},
\]

Then simple computations deduce that

\[
F_{(U,v)}(\mu, 0, 0) = \begin{pmatrix}
\Delta \phi - \lambda \phi + \frac{\lambda a(x)}{1 + m \lambda} \psi \\
\Delta \psi + (\mu + \frac{c \lambda}{1 + m \lambda}) \psi
\end{pmatrix}.
\]

One can see that only when \( \mu = \mu_1 = -\frac{c \lambda}{1 + m \lambda} \) that \( F_{(U,v)}(\mu, 0, 0) [\phi, \psi] = (0, 0)^T \) has a solution \( (\phi, \psi) \) with \( \psi > 0 \), and we can obtain that

\[
\text{Ker} \ (F_{(U,v)}(\mu_1, 0, 0)) = \text{span} \{ (\phi_1, \psi_1) \},
\]

where \( \phi_1 = (-\Delta + \lambda I)^{-1}_\Omega \frac{\lambda a(x)}{1 + m \lambda}, \psi_1 = 1, I \) is the identity operator, and \( (-\Delta + \lambda I)^{-1}_\Omega \) is the inverse operator of \( -\Delta + \lambda I \) in \( \Omega \) subject to homogeneous Neumann boundary condition. Moreover,

\[
\text{Range} \ (F_{(U,v)}(\mu_1, 0, 0)) = \{ (f, g) \in Y : \int_{\Omega_1} g(x)dx = 0 \},
\]

which asserts that the range of \( F_{(U,v)}(\mu_1, 0, 0) \) is of co-dimension one. In addition,

\[
F_{\mu(U,v)}(\mu_1, 0, 0)(\phi_1, \psi_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin \text{Range} \ (F_{(U,v)}(\mu_1, 0, 0)).
\]
Thus, the local bifurcation result of Crandall-Rabinowitz [4] concludes that for some \( \delta > 0 \), the positive solution set of (2) near \((\mu_1, \lambda, 0)\) is a smooth curve

\[
\Gamma_1 = \{(\mu_1(s), u_1(s), v_1(s)) : s \in [0, \delta)\},
\]

where \( \mu_1(0) = \mu_1 = -\frac{\lambda}{1 + \mu} \), \( u_1(s) = \lambda - s\phi_1(x) - o(|s|) \), \( v_1(s) = s + o(|s|) \).

Next we show the local bifurcation along \( \Gamma_v \). Let \( V = v - \mu \) and define

\[
F(\mu, u, V) = \begin{pmatrix}
\Delta u + u \left( \lambda - u - \frac{a(x)(V + \mu)}{1 + mu + k(V + \mu)} \right) \\
\Delta V + (V + \mu) \left( -V + \frac{cu}{1 + mu + k(V + \mu)} \right)
\end{pmatrix}.
\]

A simple computation yields that

\[
F(u, V)(\mu, 0, 0) = \begin{pmatrix}
\Delta \phi + \left( \lambda - \frac{\mu a(x)}{1 + k\mu} \right) \phi \\
\Delta \psi - \mu \psi + \frac{c\psi}{1 + k\mu} \phi
\end{pmatrix}.
\]

Then \( \mu \) is one bifurcation point if and only if \( \mu \) satisfies

\[
\lambda = \lambda_1^N \left( \frac{\mu a(x)}{1 + k\mu}, \Omega \right).
\]

If such \( \mu \) exists, we denote it by \( \mu_2 \). By a similar discussion as above, we can know that there exist some \( \sigma > 0 \) and positive functions \( \phi_2(x) \) and \( \psi_2(x) \), such that the positive solution set of (2) near \((\mu_2, 0, \mu_2)\) is a smooth curve

\[
\Gamma_2 = \{(\mu_2(s), u_2(s), v_2(s)) : s \in [0, \sigma)\},
\]

where \( \mu_2(0) = \mu_2, u_2(s) = s\phi_2(x) + o(|s|), v_2(s) = \mu_2(s) + s\psi_2(x) + o(|s|) \).

For the result of \( \mu_2 \), we have the following lemma:

**Lemma 2.1.** (I) If \( 0 < \lambda < \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) \), then there exists a unique positive number \( \mu_2 \) such that

\[
\lambda = \lambda_1^N \left( \frac{\mu_2 a(x)}{1 + k\mu_2}, \Omega \right);
\]

(II) if \( \lambda \geq \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) \), then

\[
\lambda > \lambda_1^N \left( \frac{\mu a(x)}{1 + k\mu}, \Omega \right) \quad \text{for any } \mu > 0.
\]

**Proof.** Setting

\[
h(\mu) = \lambda_1^N \left( \frac{\mu a(x)}{1 + k\mu}, \Omega \right),
\]

we know that \( h(\mu) \) is a continuous increasing function with respect to \( \mu \). Moreover,

\[
h(0) = 0, \quad \lim_{\mu \to +\infty} h(\mu) = \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right).
\]

Then we can easily obtain the conclusion of the lemma. \( \square \)

**Remark 1.** By virtue of the property of the eigenvalues, we can see that on the one hand, for any \( k > 0 \),

\[
\lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) < \lambda_1^D \left( \frac{a(x)}{k}, \Omega \right) \leq \lambda_1^D \left( \frac{a(x)}{k}, \Omega_0 \right) = \lambda_1^D (\Omega_0).
\]
On the other hand, for any subdomain $\Omega_0$,

$$
\lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) < \frac{a}{k}.
$$

For the global bifurcation, we have the following theorem:

**Theorem 2.2.** (I) If $0 < \lambda < \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right)$, then

(i) a bounded global continuum $\Gamma$ of positive solution of (2) bifurcates from $\Gamma_u$ at $\mu = \mu_1$ and meets $\Gamma_v$ at $\mu = \mu_2$;

(ii) there exists a positive number $\overline{\mu} \geq \mu_2$ such that (2) has no positive solution as $\mu \leq \mu_1$ or $\mu \geq \overline{\mu}$. Moreover, (2) has at least one positive solution for $\mu \in (\mu_1, \mu_2)$.

(II) If $\lambda \geq \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right)$, then

(i) an unbounded global continuum $\Gamma$ of positive solution of (2) bifurcates from $\Gamma_u$ at $\mu = \mu_1$ and $\text{Proj}_\mu \Gamma = (\mu_1, \infty)$;

(ii) (2) has at least one positive solution if and only if $\mu > \mu_1$.

**Proof.** First, we do some estimates of the positive solution of (2). Suppose that $(u, v)$ is a positive solution of (2). Then a standard comparison argument shows that

$$
0 < u < \lambda.
$$

Since

$$
v(\mu - v) < -\Delta v < v \left( \mu + \frac{c}{m} - v \right),
$$

we obtain that

$$
\max\{\mu, 0\} < v < \mu + \frac{c}{m}.
$$

Next, we do some estimates of the bifurcation parameter $\mu$. From the equation of $v$, we can see that

$$
\mu = \lambda_1^N \left( v - \frac{cu}{1 + mu + kv}, \Omega_1 \right)
= \lambda_1^N \left( -\frac{cu}{1 + mu}, \Omega_1 \right)
= \lambda_1^N \left( -\frac{c\lambda}{1 + m\lambda}, \Omega_1 \right)
= \frac{c\lambda}{1 + m\lambda} = \mu_1.
$$

Thus, no positive solution of (2) exists when $\mu \leq \mu_1$.

In particular, if $0 < \lambda < \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right)$, we assert that there exists $\overline{\mu}$ such that (2) has no positive solution for $\mu \geq \overline{\mu}$. Otherwise, we may assume that there exists a sequence $\{\mu_n\}$ with $\mu_n > 0$ and $\mu_n \to +\infty$ as $n \to \infty$ such that (2) at $\mu = \mu_n$ has a positive solution $(u_n, v_n)$. Then one sees that

$$
\lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) > \lambda = \lambda_1^N \left( u_n + \frac{a(x)v_n}{1 + mu_n + kv_n}, \Omega \right).
$$
\[ \lambda \rightarrow \lambda_1^N \left( \frac{a(x)\mu_n}{1 + m\lambda + k\mu_n}, \Omega \right) \]

\[ \rightarrow \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) \quad \text{as } n \to \infty, \]

which is a contradiction.

Summarizing the above results and applying a standard global bifurcation analysis, we see that the conclusion of the theorem holds true. Thus, the proof of the theorem is complete. \(\square\)

By virtue of Theorem 2.2, we see that if \( \lambda \geq \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) \), then (2) has at least one positive solution if and only if \( \mu > -\frac{e\lambda_1}{1+mx} \), which implies that the exact coexistence region of the two species is obtained. Moreover, at this time, it is an interesting question to investigate the asymptotic behavior of the positive solution as \( \mu \to +\infty \). By a similar analysis of [33, Theorem 4.1], we can deduce the following theorem:

**Theorem 2.3.** Assume that \( \lambda > \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) \), then as \( \mu \) is large enough, (2) has a unique positive solution, and it is linearly stable.

We point out that the authors in [20] also give a slightly different proof of the result.

### 3. Effects of large \( k \) on positive stationary solution set.

In this section, we try to show the effects of large \( k \) on the positive stationary solution set of (1).

First, for the asymptotic behavior of the positive stationary solution, we have the following result:

**Lemma 3.1.** For any positive solution \((u_k, v_k)\) of (2), as \( k \to +\infty \), it holds that

\[
\begin{cases}
(u_k, v_k) \to (\lambda, 0) & \text{for } \mu \leq 0, \\
(u_k, v_k) \to (\lambda, \mu) & \text{for } \mu > 0.
\end{cases}
\]

**Proof.** As \( 0 < u_k < \lambda, \max\{\mu, 0\} < v_k < \mu + c/m \), we know that \((u_k, v_k)\) is uniformly bounded. Moreover,

\[
\left\| \lambda - u_k - \frac{a(x)v_k}{1 + m\mu_k + k\mu_k} \right\|_{\infty, \Omega} \leq 2\lambda + \frac{a}{k} \to 2\lambda \quad \text{as } k \to +\infty,
\]

\[
\left\| \mu - v_k + \frac{cu_k}{1 + m\mu_k + k\mu_k} \right\|_{\infty, \Omega_1} \leq 2\mu + 2\frac{c}{m}.
\]

Then the standard elliptic regularity theory yields that, subject to a subsequence if necessary, \((u_k, v_k) \to (\tilde{u}, \tilde{v})\) in \( C^1(\Omega) \times C^1(\Omega_1) \) as \( k \to +\infty \). As

\[
\left\| \frac{a(x)u_kv_k}{1 + m\mu_k + k\mu_k} \right\|_{\infty, \Omega} \leq \frac{a\lambda}{k} \to 0 \quad \text{as } k \to +\infty,
\]

\[
\left\| \frac{u_kv_k}{1 + m\mu_k + k\mu_k} \right\|_{\infty, \Omega_1} \leq \frac{\lambda}{k} \to 0 \quad \text{as } k \to +\infty,
\]

one sees that \((\tilde{u}, \tilde{v})\) is a nonnegative solution of the following problem:

\[
\begin{align*}
-\Delta u &= u(\lambda - u), & x &\in \Omega, \\
-\Delta v &= v(\mu - v), & x &\in \Omega_1, \\
\partial_\nu u |_{\partial \Omega} &= \partial_\nu v |_{\partial \Omega_1} &= 0.
\end{align*}
\]
Thus, $\hat{u} = 0$ or $\hat{u} = \lambda$. If $\hat{u} = 0$, by setting $\hat{u}_n = \frac{u_n}{\|u_n\|_{\infty, \Omega}}$, we can deduce that subject to a subsequence if necessary, $\hat{u}_n \rightarrow \hat{u}$ in $C^1(\Omega)$, and $\hat{u}$ satisfies

$$-\Delta \hat{u} = \lambda \hat{u}, \quad x \in \Omega, \quad \partial_\nu \hat{u} = 0, \quad x \in \partial \Omega.$$ 

Then the Harnack inequality deduces that $\hat{u} > 0$ since $\hat{u} \geq 0$. Thus, $\lambda = 0$, which is a contradiction. So, we have that $\hat{u} = \lambda$.

On the other hand, if $\mu \leq 0$, then $v = 0$ is the only nonnegative solution of the second equation of (3). Thus, in case of $\mu \leq 0$, we have that $(\hat{u}, \hat{v}) = (\lambda, 0)$. While if $\mu > 0$, $v_k > \mu$, we obtain that $\bar{v} = \mu$.

For any sequence $\{k_n\}$ with $k_n \rightarrow +\infty$ as $n \rightarrow \infty$, we can deduce that subject to a subsequence, the corresponding positive solution $(u_n, v_n)$ satisfies $(u_n, v_n) \rightarrow (\hat{u}, \hat{v})$ uniformly in $\Omega \times \bar{\Omega}$. So, one can see that $(u_k, v_k) \rightarrow (\hat{u}, \hat{v})$ uniformly in $\Omega \times \Omega$ as $k \rightarrow +\infty$. Thus, the proof of the lemma is complete.

**Lemma 3.2.** Assume that $\mu > -\frac{c\lambda}{1 + m\lambda}$. Let $(u_k, v_k)$ be a positive solution of (2) and set $w_k = kv_k$. Then as $k \rightarrow +\infty$,

(I) we have that for $\mu < 0$,

$$w_k \rightarrow w = \frac{\mu(1 + m\lambda) + c\lambda}{-\mu} \quad \text{uniformly in } \Omega_1.$$

(II) for $\mu = 0$,

$$\left\{ \begin{array}{l}
\|v_k\|_{\infty, \Omega_1} \rightarrow +\infty, \\
\|v_k\|_{2, \Omega_1} \rightarrow c\lambda.
\end{array} \right.$$ 

**Proof.** (I) Assume that $\mu < 0$. First, we claim that for any positive constant $k_0$, if $k \geq k_0$, there exist two positive constants $C_1$ and $C_2$ independent of $k$ such that $C_1 \leq w_k \leq C_2$. If $C_1$ does not exist, then there exists a sequence $\{(k_n, w_n)\}$ with $k_n \rightarrow +\infty$ as $n \rightarrow \infty$ such that $\min_{\Omega_1} w_n \rightarrow 0$ as $n \rightarrow \infty$, then the Harnack inequality deduces that $w_n \rightarrow 0$ uniformly in $\Omega_1$ as $n \rightarrow \infty$. Setting

$$\bar{w}_n = \frac{v_n}{\|v_n\|_{\infty, \Omega_1}},$$ 

we obtain that $\bar{w}_n$ satisfies

$$-\Delta \bar{w}_n = \bar{w}_n \left( \mu - \frac{w_n}{k_n} + \frac{c\mu}{1 + m\mu + w_n} \right), \quad x \in \Omega_1.$$ 

(5)

Then together with Lemma 3.1, we see that subject to a subsequence if necessary, $\bar{w}_n \rightarrow \bar{v}$ for some nonnegative function $\bar{v}$ in $C^1(\Omega_1)$, which satisfies

$$-\Delta \bar{v} = \bar{v} \left( \mu + \frac{c\lambda}{1 + m\lambda} \right), \quad x \in \Omega_1, \quad \partial_\nu \bar{v}|_{\Omega_1} = 0.$$ 

As $\bar{v} \geq 0$, the Harnack inequality deduces that $\bar{v} > 0$, $x \in \Omega_1$. Thus, we have that

$$\mu = -\frac{c\lambda}{1 + m\lambda},$$ 

which is a contradiction. So, the desired $C_1$ is established. If the positive constant $C_2$ does not exist, then there exists a sequence $\{(k_n, w_n)\}$ with $k_n \rightarrow +\infty$ such that $\|w_n\|_{\infty, \Omega_1} \rightarrow +\infty$ as $n \rightarrow \infty$, then the Harnack inequality deduces that $w_n \rightarrow \infty$ uniformly in $\Omega_1$ as $n \rightarrow \infty$. So, as $n \rightarrow \infty$,

$$\frac{c\mu}{1 + m\mu + w_n} \rightarrow 0 \quad \text{uniformly in } \Omega_1.$$
Then for \( \hat{v}_n \) defined by (4), we also have that subject to a subsequence if necessary, \( \hat{v}_n \to \hat{v} \) for some nonnegative function \( \hat{v} \) in \( C^1(\Omega_1) \). By virtue of Lemma 3.1, we can obtain that \( \hat{v} \) satisfies

\[
-\Delta \hat{v} = \mu \hat{v}, \quad x \in \Omega_1, \quad \partial_n \hat{v}|_{\partial \Omega_1} = 0,
\]

Since \( \hat{v} \geq 0 \), the Harnack inequality deduces that \( \hat{v} > 0, x \in \Omega_1 \). So \( \mu = 0 \), which is a contradiction. Thus, the claim is proved.

By standard elliptic regularity theory, we can see that subject to a subsequence if necessary, \( w_k \to w \) in \( C^1(\Omega_1) \) as \( k \to +\infty \). Moreover, \( w \) is a positive solution of the following problem

\[
- \Delta w = w \left( \mu + \frac{c\lambda}{1 + m\lambda + w} \right), \quad x \in \Omega_1, \quad \partial_n w|_{\partial \Omega_1} = 0. \tag{6}
\]

It is clear that (6) has a unique positive solution for \( 0 < \mu > -\frac{c\lambda}{1 + m\lambda} \), which is just the positive constant

\[
\frac{\mu(1 + m\lambda) + c\lambda}{-\mu}.
\]

So, the conclusion of (I) is proved.

(II) Assume that \( \mu = 0 \). If the asymptotic behavior of \( k\|v_k\|_{\infty, \Omega_1} \) does not hold, then there exists a sequence \( \{k_n, v_n\} \) with \( k_n \to +\infty \) as \( n \to \infty \) such that

\[
\lim_{n \to \infty} k_n\|v_n\|_{\infty, \Omega_1} = l
\]

for some nonnegative number \( l \). Then for \( \hat{v}_n \) defined by (4), we can obtain that subject to a subsequence if necessary, \( \hat{v}_n \to \hat{v} \) for some nonnegative function \( \hat{v} \) in \( C^1(\Omega_1) \). Moreover, \( \hat{v} \) satisfies the following equation weakly:

\[
-\Delta \hat{v} = \frac{c\lambda}{1 + m\lambda + \hat{v}} \hat{v}, \quad x \in \Omega_1, \quad \partial_n \hat{v}|_{\partial \Omega_1} = 0.
\]

Then the Harnack inequality deduces that \( \hat{v} > 0, x \in \Omega_1 \). While

\[
\int_{\Omega_1} \frac{c\lambda}{1 + m\lambda + \hat{v}} \hat{v} dx = 0,
\]

which is a contradiction to the positivity of \( \hat{v} \). Thus, it follows that \( k\|v_k\|_{\infty, \Omega_1} \to +\infty \) as \( k \to +\infty \).

For the asymptotic behavior of \( k\|v_k\|_{\infty, \Omega_1}^2 \) as \( k \to +\infty \), let \( (k_n, v_n) \) be any sequence with \( k_n \to +\infty \) as \( n \to \infty \). Since \( v_n \) satisfies

\[
-\Delta v_n = v_n \left( -v_n + \frac{c_k n}{1 + m_k n + k_n v_n} \right), \quad x \in \Omega_1,
\]

integrating the equation over \( \Omega_1 \) yields that

\[
\int_{\Omega_1} v_n^2 dx = c \int_{\Omega_1} \frac{u_n v_n}{1 + m_u n + k_n v_n} dx.
\]

Thus,

\[
k_n\|v_n\|_{\infty, \Omega_1}^2 \int_{\Omega_1} \hat{v}_n^2 dx = c \int_{\Omega_1} \frac{k_n u_n v_n}{1 + m u_n + k_n v_n} dx. \tag{7}
\]

As \( k_n\|v_n\|_{\infty, \Omega_1} \to +\infty \) as \( n \to \infty \), we can know that \( \hat{v}_n \to \hat{v} \) in \( C^1(\Omega_1) \), which satisfies

\[
-\Delta \hat{v} = 0, \quad x \in \Omega_1, \quad \partial_n \hat{v}|_{\partial \Omega_1} = 0.
\]
Clearly, \( \hat{v} \equiv 1 \). Moreover, since
\[
k_n v_n = k_n \|v_n\|_{\infty, \Omega_1} \hat{v}_n \to +\infty \quad \text{as} \quad n \to \infty,
\]
by letting \( n \to \infty \) in (7), we deduce that
\[
\lim_{n \to \infty} k_n \|v_n\|_{\infty, \Omega_1}^2 = c \lambda.
\]

Thus, the proof of the lemma is complete. \( \Box \)

**Remark 2.** It should be pointed out that if \( \mu > 0 \), for any positive solution \( (u_k, v_k) \) of (2), as \( k \to +\infty \), subject to a subsequence, \( v_k \to \mu \) and \( k v_k \to +\infty \) uniformly in \( \Omega_1 \).

For the stability of positive solutions of (2), we have the following theorem:

**Theorem 3.3.** Assume that \( \mu > -\frac{c \lambda}{1 + m \lambda} \). Then for sufficiently large \( k \), any positive solution \( (u_k, v_k) \) of (2) is nondegenerate and linearly stable.

The proof of Theorem 3.3 lies on the following two lemmas:

**Lemma 3.4.** Assume that \( 0 \neq \mu > -\frac{c \lambda}{1 + m \lambda} \). Then there exists a positive number \( K \) such that for \( k > K \), any positive solution \( (u_k, v_k) \) of (2) is nondegenerate and linearly stable.

**Proof.** If the assertion fails, then there exist a sequence \( \{k_n\} \) with \( k_n \to +\infty \) as \( n \to \infty \), a corresponding positive solution \( (u_n, v_n) \) such that the linearized problem of (2) at \( (u, v) = (u_n, v_n) \)

\[
\begin{aligned}
-\Delta \phi_n &= \left( \lambda - 2 u_n - \frac{a(x) v_n (1 + k_n v_n)}{(1 + m u_n + k_n v_n)^2} \right) \phi_n - \frac{a(x) u_n (1 + m u_n)}{(1 + m u_n + k_n v_n)^2} \psi_n + \eta_n \phi_n, \quad x \in \Omega, \\
-\Delta \psi_n &= \left( \mu - 2 v_n + \frac{c u_n (1 + m u_n)}{(1 + m u_n + k_n v_n)^2} \right) \psi_n + \frac{c v_n (1 + k_n v_n)}{(1 + m u_n + k_n v_n)^2} \phi_n + \eta_n \psi_n, \quad x \in \Omega, \\
\partial_{\nu} \phi_n |_{\partial \Omega_1} &= \partial_{\nu} \psi_n |_{\partial \Omega_1} = 0,
\end{aligned}
\]

has an eigenvalue \( \eta_n \) with \( \text{Re} \eta_n \leq 0 \) for any \( n \geq 1 \), and the corresponding eigenfunction pair \( (\phi_n, \psi_n) \) with \( \|\phi_n\|_{2, \Omega} + \|\psi_n\|_{2, \Omega_1} = 1 \).

(i) First, we show that \( \eta_n \) is uniformly bounded. It suffices to show that \( \text{Re} \eta_n \) is bounded below and \( |\text{Im} \eta_n| \) is bounded.

Suppose that \( \text{Re} \eta_n \to -\infty \) as \( n \to \infty \). Then by virtue of the Kato’s inequality, we can obtain
\[
-\Delta |\phi_n| \leq - \text{Re} \left( \frac{\phi_n}{|\phi_n|} \Delta \phi_n \right) \leq \left( \lambda - 2 u_n - \frac{a(x) v_n (1 + k_n v_n)}{(1 + m u_n + k_n v_n)^2} \right) |\phi_n| + \frac{a(x) u_n (1 + m u_n)}{(1 + m u_n + k_n v_n)^2} |\psi_n| + (\text{Re} \eta_n) |\phi_n|,
\]

for any positive solution \( (u_k, v_k) \) of (2).
and

\[-\Delta |\psi_n| \leq - \text{Re} \left( \frac{\overline{\psi_n}}{|\psi_n|} \Delta \psi_n \right) \]
\[\leq \left( \mu - 2v_n + \frac{cu_n(1 + m\mu_n)}{(1 + m\mu_n + k_n v_n)^2} \right) |\psi_n| \]
\[+ \frac{cv_n(1 + k_n v_n)}{(1 + m\mu_n + k_n v_n)^2} |\phi_n| + (\text{Re} \eta_n) |\psi_n|.
\]

By multiplying both sides of (9) by $|\phi_n|$ and integrating over $\Omega$, one sees that

\[0 \leq \int_{\Omega} |\nabla \phi_n|^2 \leq (\lambda + \text{Re} \eta_n) \int_{\Omega} |\phi_n|^2 + \frac{a}{m} \|\phi_n\|_{2,\Omega} \|\psi_n\|_{2,\Omega_1}.
\]

Thus,

\[-\lambda - \text{Re} \eta_n \|\phi_n\|_{2,\Omega}^2 \leq \frac{a}{m} \|\phi_n\|_{2,\Omega} \|\psi_n\|_{2,\Omega_1}.
\]

Since $-\lambda - \text{Re} \eta_n \to +\infty$ as $n \to \infty$, we deduce that

\[\|\phi_n\|_{2,\Omega} \to 0 \quad \text{and} \quad \|\psi_n\|_{2,\Omega_1} \to 1
\]

as $n \to \infty$. Similarly, we can show that

\[-\mu - \frac{c}{m} - \text{Re} \eta_n \|\psi_n\|_{2,\Omega_1}^2 \leq \frac{c}{k_n} \|\phi_n\|_{2,\Omega} \|\psi_n\|_{2,\Omega_1},
\]

which yields that $\|\psi_n\|_{2,\Omega_1} \to 0$ as $n \to \infty$. So, we obtain a contradiction. Therefore, $\text{Re} \eta_n$ is bounded below.

Next, we prove that $|\text{Im} \eta_n|$ is uniformly bounded. Multiplying both sides of the first equation and the second equation of (8) by $\overline{\phi_n}$ and $\overline{\psi_n}$ and integrating over $\Omega$ and $\Omega_1$ respectively, we can obtain that

\[|\text{Im} \eta_n| \int_{\Omega} |\phi_n|^2 dx = \text{Im} \int_{\Omega} \frac{au_n(1 + m\mu_n)}{(1 + m\mu_n + k_n v_n)^2} \overline{\phi_n} \psi_n dx \leq \frac{a}{m} \|\phi_n\|_{2,\Omega} \|\psi_n\|_{2,\Omega_1}
\]

and

\[|\text{Im} \eta_n| \int_{\Omega_1} |\psi_n|^2 dx = \text{Im} \int_{\Omega_1} \frac{-cv_n(1 + k_n v_n)}{(1 + m\mu_n + k_n v_n)^2} \overline{\phi_n} \psi_n dx \leq \frac{c}{k_n} \|\phi_n\|_{2,\Omega} \|\psi_n\|_{2,\Omega_1}.
\]

Then if $|\text{Im} \eta_n| \to \infty$, it holds that

\[\|\phi_n\|_{2,\Omega} \to 0 \quad \text{and} \quad \|\psi_n\|_{2,\Omega_1} \to 0,
\]

which is a contradiction. Thus, $|\text{Im} \eta_n|$ is uniformly bounded.

(ii) If we can prove that subject to a subsequence if necessary, it holds that $\text{Re} \eta_n \to 0$ for sufficiently large $n$, then a contradiction is derived. Since $\eta_n$ is uniformly bounded, we may assume that subject to a subsequence if necessary, $\eta_n \to \eta$ as $n \to \infty$ with $\text{Re} \eta \leq 0$. Due to (8), one can see that $(\phi_n, \psi_n)$ is bounded in $H^1(\Omega) \times H^1(\Omega_1)$. Then subject to a subsequence of the subsequence if necessary, $(\phi_n, \psi_n) \to (\phi, \psi)$ weakly in $H^1(\Omega) \times H^1(\Omega_1)$ and strongly in $L^2(\Omega) \times L^2(\Omega_1)$.

If $\mu > 0$, Lemma 3.1 deduces that along a subsequence of the above subsequence if necessary, $(u_n, v_n) \to (\lambda, \mu)$ as $n \to \infty$. Then by letting $n \to \infty$ in (8), we get that $(\phi, \psi)$ satisfies

\[
\begin{align*}
- \Delta \phi &= -\lambda \phi + \eta \phi, & x \in \Omega, \\
- \Delta \psi &= -\mu \psi + \eta \psi, & x \in \Omega_1, \\
\partial_n \phi |_{\partial \Omega} &= \partial_n \psi |_{\partial \Omega_1} = 0.
\end{align*}
\]
Obviously, $\eta \in \mathbb{R}$. If $\phi \neq 0$, then
\[ \eta \geq \lambda > 0. \]
If $\psi \neq 0$, then
\[ \eta \geq \mu > 0. \]
Thus, when $\mu > 0$, we derive a contradiction to that $\text{Re} \eta \leq 0$.

If $\mu < 0$, by virtue of Lemma 3.2, we may assume that $(\phi, \psi)$ satisfies
\[
\begin{cases}
- \Delta \phi = -\lambda \phi - \frac{a(x)\lambda(1 + m\lambda)}{(1 + m\lambda + w)^2} \psi + \eta \phi, & x \in \Omega, \\
- \Delta \psi = \left( \mu + \frac{c\lambda(1 + m\lambda)}{(1 + m\lambda + w)^2} \right) \psi + \eta \psi, & x \in \Omega_1, \\
\partial_\nu \phi |_{\partial \Omega} = \partial_\nu \psi |_{\partial \Omega_1} = 0,
\end{cases}
\]
where $w$ is the positive constant given by Lemma 3.2. If $\psi \equiv 0$, then
\[ \text{Re} \eta \geq \lambda > 0. \]
If $\psi \neq 0$, then
\[ \text{Re} \eta \geq \lambda_1^N \left( -\mu - \frac{c\lambda(1 + m\lambda)}{(1 + m\lambda + w)^2}, \Omega_1 \right). \]
On the other hand, noting that $w$ is constant, Lemma 3.2 asserts that
\[ \mu = -\frac{c\lambda}{1 + m\lambda + w} < -\frac{c\lambda(1 + m\lambda)}{(1 + m\lambda + w)^2}, \]
which deduces that
\[ \text{Re} \eta \geq \lambda_1^N \left( -\mu - \frac{c\lambda(1 + m\lambda)}{(1 + m\lambda + w)^2}, \Omega_1 \right) = -\mu - \frac{c\lambda(1 + m\lambda)}{(1 + m\lambda + w)^2} > 0. \]
So we obtain a contradiction to that $\text{Re} \eta \leq 0$. Therefore, the proof of the lemma is complete.

**Lemma 3.5.** Assume that $\mu = 0$. Then there exists a positive number $\hat{K}$ such that for $k > \hat{K}$, any positive solution $(u_k, v_k)$ of (2) is nondegenerate and linearly stable.

**Proof.** If not, then there exist a sequence $\{k_n\}$ with $k_n \to +\infty$ as $n \to \infty$, a corresponding positive solution $(u_n, v_n)$ such that the linearized operator of (2) at $(u, v) = (u_n, v_n)$ with $\mu = 0$ has an eigenvalue $\eta_n$ with $\text{Re} \eta_n \leq 0$ for any $n \geq 1$, and a corresponding eigenfunction pair $(\phi_n, \psi_n)$ with $\|\phi_n\|_{2, \Omega} + \|\psi_n\|_{2, \Omega_1} = 1$.

First, by Lemma 3.2 and the same discussion in the proof of Lemma 3.4, we can deduce that $\eta_n$ is uniformly bounded. Moreover, subject to a subsequence if necessary, $(\phi_n, \psi_n) \to (\phi, \psi)$ weakly in $H^1(\Omega) \times H^1(\Omega_1)$ and strongly in $L^2(\Omega) \times L^2(\Omega_1)$, $(u_n, v_n) \to (\lambda, 0)$ uniformly and $\eta_n \to \eta$ as $n \to \infty$. So, letting $n \to \infty$ in (8) with $\mu = 0$, we get that $(\phi, \psi, \eta)$ satisfies
\[
\begin{cases}
- \Delta \phi + \lambda \phi = \eta \phi, & x \in \Omega, \\
- \Delta \psi = \eta \psi, & x \in \Omega_1, \\
\partial_\nu \phi |_{\partial \Omega} = \partial_\nu \psi |_{\partial \Omega_1} = 0.
\end{cases}
\]
As $\text{Re} \eta_n \leq 0$ for any $n \geq 1$, we see that $\text{Re} \eta \leq 0$. So, it follows that $\eta = 0$ and $(\phi, \psi) = \left( 0, |\Omega_1|^{-\frac{1}{2}} \right)$. 


On the other hand, since \((\phi_n, \psi_n, \eta_n)\) satisfies (8) with \(\mu = 0\), multiplying the equation of \(\psi_n\) in (8) by \(v_n\) and the equation of \(v_n\) by \(\psi_n\) and integrating over \(\Omega_1\), we can obtain that
\[
\eta_n \int_{\Omega_1} \psi_n v_n dx = \int_{\Omega_1} \psi_n v_n \left( v_n + \frac{c k n u v_n}{(1 + m u_n + k_n v_n)^2} \right) dx - \int_{\Omega_1} c v_n^2 (1 + k_n v_n) (1 + m u_n + k_n v_n)^2 \phi_n dx.
\]
Then
\[
\eta_n k_n \|v_n\|_{\infty, \Omega_1} \int_{\Omega_1} \psi_n v_n dx = \int_{\Omega_1} \psi_n \left( k_n \|v_n\|_{\infty, \Omega_1}^2 \psi_n^2 + \frac{c u_n (k_n v_n)^2}{(1 + m u_n + k_n v_n)^2} \right) dx - \int_{\Omega_1} c k_n v_n (1 + k_n v_n) v_n \phi_n dx.
\]
Letting \(n \to \infty\), we obtain that
\[
\lim_{n \to \infty} \eta_n k_n \|v_n\|_{\infty, \Omega_1} = 2c \lambda.
\]
Then, it follows that Re \(\eta_n\) is positive for sufficiently large \(n\), which is a contradiction to the assumption. So, the proof of the lemma is complete.

Now we give the uniqueness of the positive solution of (2). More precisely,

**Theorem 3.6.** Assume that \(-\frac{c \lambda}{1 + \mu m} < \mu \neq 0\). Then for sufficiently large \(k\), (2) has a unique positive solution \((u, v)\), and it is linearly stable.

**Proof.** Since \(\lambda \frac{a(x)}{k}, \Omega \to 0\) as \(k \to +\infty\), we see that for any fixed \(\lambda > 0\), if \(k\) is large enough, it holds that
\[
\lambda > \lambda \left( \frac{a(x)}{k}, \Omega \right).
\]
Thus, Theorem 2.2 deduces that for any fixed \(\lambda > 0\) and \(\mu > -\frac{c \lambda}{1 + \mu m}\), if \(k\) is large enough such that (11) holds, then (2) has at least one positive solution. Thus, the existence of the positive solution is shown. Moreover, the stability of the positive solution is given by Theorem 3.3. So, we only need to prove the uniqueness of the positive solution.

To show the uniqueness, we apply the theory of the fixed point index\([6, 25, 28]\). For any \(t \in [0, 1]\), define an operator \(A_t\) by
\[
A_t(k, u, v) = \left( \begin{array}{c} (-\Delta + I)_{\Omega_1}^{-1} \left( u \left( \lambda - u - \frac{t a(x) v}{1 + m u + k v} \right) + M u \right) \\ (-\Delta + I)_{\Omega_1}^{-1} \left( v \left( \mu - v + \frac{t c u}{1 + m u + k v} \right) + M v \right) \end{array} \right),
\]
where \(M\) is a large positive constant. Then we can see that \((u, v)\) is a fixed point of \(A_t(k, u, v)\) if and only if it satisfies
\[
\begin{align*}
\begin{cases}
-\Delta u &= u \left( \lambda - u - \frac{t a(x) v}{1 + m u + k v} \right), & x \in \Omega, \\
-\Delta v &= v \left( \mu - v + \frac{t c u}{1 + m u + k v} \right), & x \in \Omega_1, \\
\partial_x u &= 0, & x \in \partial \Omega, \\
\partial_x v &= 0, & x \in \partial \Omega_1.
\end{cases}
\end{align*}
\]
It is easy to see that any nonnegative fixed point of \(A_t\) satisfies
\[
0 \leq u \leq \lambda, \quad \max \{\mu, 0\} \leq v \leq \mu + c/m.
\]
Thus, for
\[ W = K_1 \times K_2, \]
where
\[ K_1 = \{ w \in C(\overline{\Omega}) : w \geq 0, x \in \overline{\Omega}, \partial_x w|_{\partial \Omega} = 0 \}, \]
\[ K_2 = \{ w \in C(\overline{\Omega}_1) : w \geq 0, x \in \overline{\Omega}_1, \partial_x w|_{\partial \Omega_1} = 0 \}, \]
and
\[ D = \left\{ (u, v) \in W : u < \lambda + 1, v < \mu + \frac{c}{m} + 1 \right\}, \]
we see that \( A_t \) has no nonnegative fixed point on \( \partial D \). In addition, \( A_t \) is compact, and if \( M \) is large enough, the operator \( A_t \) is also positive. Thus, the degree \( \deg_W(I - A_t, D) \) is well defined. Moreover, noting that any nonnegative fixed point of \( A_t \) is in \( D^o \), the degree \( \deg_W(I - A_t, D) \) is independent of \( t \). Thus,
\[ \deg_W(I - A_1, D) = \deg_W(I - A_0, D). \]

When \( t = 0 \), by some routine computations, we can obtain that for any \( \mu > 0 \),
\[ \deg_W(I - A_0, D) = \text{index}(A_0, (0, 0)) + \text{index}(A_0, (\lambda, 0)) + \text{index}(A_0, (0, \mu)) + \text{index}(A_0, (\lambda, \mu)) = 0 + 0 + 0 + 1 = 1; \]
while for \( \mu < 0 \),
\[ \deg_W(I - A_0, D) = \text{index}(A_0, (0, 0)) + \text{index}(A_0, (\lambda, 0)) = 0 + 1 = 1. \]
Thus, we have that for \( \mu \neq 0 \),
\[ \deg_W(I - A_0, D) = 1. \]

When \( t = 1 \), any nonnegative fixed point of \( A_1 \) is a nonnegative solution of (2). By the compactness of \( A_1 \), we know that there exist only finitely many isolated positive fixed points of \( A_1 \), which we denote by \( (u_i, v_i) \) for \( i = 1, 2, \cdots, \sigma \). Moreover, Theorem 3.6 yields that for any \( i \),
\[ \text{index}(A_1, (u_i, v_i)) = 1. \]
Thus, when \( -\frac{c}{1 + \mu} < \mu < 0 \), by some computations, we have that
\[ 1 = \deg_W(I - A_1, D) = \text{index}(A_1, (0, 0)) + \text{index}(A_1, (\lambda, 0)) + \text{index}(A_1, (0, \mu)) + \sum_{i=1}^{\sigma} \text{index}(A_1, (u_i, v_i)) = \sigma, \]
which asserts that \( \sigma = 1 \). When \( \mu > 0 \), noting that
\[ \lambda > \lambda_1^N \left( \frac{a(x)}{k}, \Omega \right) > \lambda_1^N \left( \frac{a(x)\mu}{1 + k\mu}, \Omega \right), \]
we can obtain that
\[ 1 = \deg_W(I - A_1, D) = \text{index}(A_1, (0, 0)) + \text{index}(A_1, (\lambda, 0)) + \text{index}(A_1, (0, \mu)) + \sum_{i=1}^{\sigma} \text{index}(A_1, (u_i, v_i)) = \sigma, \]
which also asserts that $\sigma = 1$. Thus, the uniqueness is proved. The proof of the theorem is complete. 

4. Dynamical behavior of large $k$. In this section, we show the dynamics of (1) and assume that the initial data are nonnegative and nonzero.

**Theorem 4.1.** Let $(u(x,t), v(x,t))$ be a solution of (1),

(I) If $\mu \leq -\frac{c\lambda}{1+m\lambda}$, then

\[
\lim_{t \to \infty} (u(x,t), v(x,t)) = (\lambda, 0) \quad \text{uniformly in } \Omega \times \Omega_1.
\]

(II) If $\mu > -\frac{c\lambda}{1+m\lambda}$, then

\[
0 \leq \lim_{t \to \infty} u(x,t) \leq \lim_{t \to \infty} u(x,t) \leq \lambda \quad \text{uniformly in } \Omega,
\]

\[
\max\{0, \mu\} \leq \lim_{t \to \infty} v(x,t) \leq \lim_{t \to \infty} v(x,t) \leq \mu + \frac{c\lambda}{1+m\lambda} \quad \text{uniformly in } \Omega_1.
\]

**Proof.** By the strong maximum principle, we see that any solution $(u(x,t), v(x,t))$ of (1) is positive. Then $u$ satisfies

\[
u_t - \Delta u < u(\lambda - u).
\]

Thus, the comparison principle deduces that

\[
0 \leq \lim_{t \to \infty} u(x,t) \leq \lim_{t \to \infty} u(x,t) \leq \lambda.
\] (13)

Then for any $\varepsilon > 0$, there exists $T_1 > 0$ such that for $t > T_1$,

\[
u(x,t) < \lambda + \varepsilon.
\]

Then

\[
(\mu - v)v < v_t - \Delta v < \left(\mu + \frac{c(\lambda + \varepsilon)}{1+m(\lambda + \varepsilon)} - v\right) v, \quad t > T_1.
\]

Thus, the comparison principle yields that

\[
\max\{0, \mu\} \leq \lim_{t \to \infty} v(x,t) \leq \lim_{t \to \infty} v(x,t) \leq \mu + \frac{c(\lambda + \varepsilon)}{1+m(\lambda + \varepsilon)}.
\]

Due to the arbitrariness of $\varepsilon$, we see that

\[
\max\{0, \mu\} \leq \lim_{t \to \infty} v(x,t) \leq \lim_{t \to \infty} v(x,t) \leq \mu + \frac{c\lambda}{1+m\lambda}.
\] (14)

If $\mu \leq -\frac{c\lambda}{1+m\lambda}$, then (14) deduces that $v(x,t) \to 0$ uniformly in $\Omega_1$ as $t \to \infty$. Then for any $\varepsilon > 0$, there exists $T_2 > 0$ such that for $t > T_2$, it holds that $v(x,t) < \varepsilon/a$. Then $u(x,t)$ satisfies

\[
u_t - \Delta u > u(\lambda - \varepsilon - u), \quad t > T_2.
\]

Thus, by virtue of the comparison principle, we obtain that

\[
\lim_{t \to \infty} u(x,t) \geq \lambda - \varepsilon.
\]

Together with the arbitrariness of $\varepsilon$ and (13), we see that $u(x,t) \to \lambda$ uniformly in $\Omega$ as $t \to \infty$. The proof of the theorem is complete. 

By virtue of Theorem 3.6, we see that for any fixed $\lambda > 0$ and $0 \neq \mu > -\frac{c\lambda}{1+m\lambda}$, (1) has a unique positive stationary solution as $k$ is large enough, and we denote it by $(u_k(x), v_k(x))$. In the following, we can further show that $(u_k(x), v_k(x))$ is actually globally asymptotically stable.
Theorem 4.2. Assume that $\mu > -\frac{c\lambda}{1 + m\lambda}$ and $\mu \neq 0$. Then as $k$ is large enough, any solution $(u(x,t), v(x,t))$ of (1) satisfies
\[
\lim_{t \to \infty} (u(x,t), v(x,t)) = (u_k(x), v_k(x)) \quad \text{uniformly in } \Omega \times \Omega_1.
\] (15)

The proof of Theorem 4.2 is given by the following two lemmas:

Lemma 4.3. If $\mu > 0$, then there exists $K$ such that for $k > K$, any solution $(u(x,t), v(x,t))$ of (1) satisfies (15).

Proof. First, by the comparison principle, any solution $(u(x,t), v(x,t))$ of (1) is globally bounded and positive. Since
\[
\left| \frac{a(x)v}{1 + mu + kv} \right| < \frac{a}{k} \to 0 \text{ as } k \to +\infty,
\]
we know that for any $\varepsilon_1 > 0$, there exists $K_1 = K_1(\varepsilon_1) > 0$ such that when $k > K_1$,
\[
u \left( \lambda - \frac{\varepsilon_1}{4} - u \right) < u_t - \Delta u < u(\lambda - u).
\]
Hence, there exists $T_1$ such that for $t > T_1$,
\[
\lambda - \frac{\varepsilon_1}{2} < u(x,t) < \lambda + \frac{\varepsilon_1}{2}, \quad \forall x \in \Omega.
\]
On the other hand, since $\lim_{t \to \infty} v(x,t) \geq \mu$, there exists $T_2$ such that for $t > T_2$,
\[
v(x,t) \geq \mu > 0.
\]
Then
\[
\left| \frac{u}{1 + mu + kv} \right| \to 0 \text{ as } k \to +\infty.
\]

Then similar discussion as above, we can know that there exists $K_2 = K_2(\varepsilon_1) > 0$ such that when $k > K_2$, there exists $T_3 > T_2$ such that for $t > T_3$,
\[
\mu - \frac{\varepsilon_1}{2} < v(x,t) < \mu + \frac{\varepsilon_1}{2}, \quad \forall x \in \Omega_1.
\]
While $u_k(x) \to \lambda$ and $v_k(x) \to \mu$ as $k \to +\infty$, we see that for any $\varepsilon_1 > 0$, there exists $K^* \geq \max\{K_1, K_2\}$ such that for $k > K^*$, there exists $T = \max\{T_1, T_2, T_3\}$ so that for $t > T$,
\[
|u(x,t) - u_k(x)| < \varepsilon_1, \quad \forall x \in \Omega, \quad |v(x,t) - v_k(x)| < \varepsilon_1, \quad \forall x \in \Omega_1. \quad (16)
\]

Second, set $\Phi(x,t) = u(x,t) - u_k(x), \Psi(x,t) = v(x,t) - v_k(x)$. Then we can know that
\[
\begin{aligned}
\Phi_t - \Delta \Phi &= \Phi (\lambda - 2u_k - \Phi) - a(x) [p(u_k + \Phi, v_k + \Psi) - p(u_k, v_k)], \\
\Psi_t - \Delta \Psi &= \Psi (\mu - 2v_k - \Psi) + c [p(u_k + \Phi, v_k + \Psi) - p(u_k, v_k)],
\end{aligned} \quad (17)
\]

Here, $p(u, v) = \frac{uv}{1 + mu + kv}$. By virtue of Taylor’s expansion formula, there exists some $0 \leq \theta \leq 1$ so that
\[
p(u_k + \Phi, v_k + \Psi) = p(u_k, v_k) + \frac{v_k(1 + kv_k)}{(1 + mu_k + kv_k)^2} \Phi + \frac{u_k(1 + mu_k)}{(1 + mu_k + kv_k)^2} \Psi \\
+ \frac{1}{2} \left[ p_{uu}(u_k + \theta \Phi, v_k + \theta \Psi) \Phi^2 + 2p_{uv}(u_k + \theta \Phi, v_k + \theta \Psi) \Phi \Psi \\
+ p_{vv}(u_k + \theta \Phi, v_k + \theta \Psi) \Psi^2 \right].
\]
Moreover, $|p_{u0}|, |p_{uv}|$ and $|p_{vv}|$ are uniformly bounded with respect to $k$. Multiplying both sides of the first equation of (17) by $\Phi$ and integrating over $\Omega$, we can obtain that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \Phi^2 dx \leq \int_{\Omega} \left( \lambda - 2u_k - \Phi - \frac{a(x)v_k(1 + kv_k)}{(1 + mu_k + kv_k)^2} \right) \Phi^2 dx$$

$$- \int_{\Omega} \frac{a v_k(1 + mu_k)}{(1 + mu_k + kv_k)^2} \Phi \Psi dx + M \int_{\Omega} \left( |\Phi|^3 + |\Phi|^2 |\Psi| + |\Phi||\Psi|^2 \right) dx,$$

where $M$ is a large positive constant. By virtue of (16), and noting $u_k \to \lambda$ and $v_k \to \mu$ as $k \to \infty$, one sees that for sufficiently small $0 < \varepsilon < \min \left\{ \frac{\lambda}{12}, \frac{\mu}{14} \right\}$, by setting $\varepsilon_1 = \varepsilon/M$, there exists $K^*$ such that when $k > K^*$,

$$\frac{d}{dt} \left( \int_{\Omega} \Phi^2 dx \right) \leq -\lambda \int_{\Omega} \Phi^2 dx + 2\varepsilon \int_{\Omega} |\Phi||\Psi| dx + \varepsilon \int_{\Omega} \Psi^2 dx.$$

Similarly,

$$\frac{d}{dt} \left( \int_{\Omega} \Psi^2 dx \right) \leq -\mu \int_{\Omega} \Psi^2 dx + 2\varepsilon \int_{\Omega} |\Phi||\Psi| dx + \varepsilon \int_{\Omega} \Phi^2 dx.$$

Now we define

$$f(t) = \int_{\Omega} \Phi^2 dx, \quad g(t) = \int_{\Omega} \Psi^2 dx.$$

One sees that

$$\frac{d}{dt} \left( f(t) + g(t) \right) \leq -\lambda f(t) + (-\mu + 7\varepsilon) g(t)$$

$$< -c^* \left( f(t) + g(t) \right),$$

where

$$c^* = \frac{1}{2} \min\{\lambda, \mu\}.$$

Then it is clear that

$$\lim_{t \to \infty} (f(t) + g(t)) = 0.$$

Thus,

$$\lim_{t \to \infty} f(t) = 0, \quad \lim_{t \to \infty} g(t) = 0.$$

Therefore, the proof of the lemma is complete.

**Lemma 4.4.** If $0 > \mu > -\frac{c\lambda}{1 + m\lambda}$, then there exists $\tilde{K}$ such that for $k > \tilde{K}$, any solution $(u(x,t), v(x,t))$ of (1) satisfies (15).

**Proof.** First, due to the proof of Lemma 4.3, we can know that for any $\varepsilon_1 > 0$, there exists $K_1 = K_1(\varepsilon_1) > 0$ such that when $k > K_1$, there exists $T_1$ such that for $t > T_1$,

$$\lambda - \frac{\varepsilon_1}{2} < u(x,t) < \lambda + \frac{\varepsilon_1}{2}, \quad \forall x \in \Omega.$$
Then we have that
\[ v(\mu - v) < v_t - \Delta v < v \left( \mu + \frac{c \left( \lambda + \frac{\varepsilon}{2} \right)}{1 + m \left( \lambda + \frac{\varepsilon}{2} \right) + kv} - v \right), \quad t > T_1. \]
By some computations, we can know that there exist two constants \( v_1(k) < 0 < v_2(k) \) such that
\[ v \left( \mu + \frac{c \left( \lambda + \frac{\varepsilon}{2} \right)}{1 + m \left( \lambda + \frac{\varepsilon}{2} \right) + kv} - v \right) = \frac{kv}{1 + m \left( \lambda + \frac{\varepsilon}{2} \right) + kv} (v_1(k)) (v_2(k) - v). \]
Then by virtue of [30, Lemma 1], we obtain that
\[ \limsup_{t \to \infty} \max_{\Omega_t} v(\cdot, t) \leq v_2(k). \]
On the other hand, \( v_2(k) \to 0 \) as \( k \to +\infty \). Thus, there exists \( K_2 = K_2(\varepsilon_1) > 0 \) such that when \( k > K_2 \), there exists \( T_2 \) such that for \( t > T_2 \),
\[ 0 < v(x, t) < \frac{\varepsilon_1}{2}, \quad \forall x \in \Omega_t. \]
Since \( u_k(x) \to \lambda \) and \( v_k(x) \to 0 \) as \( k \to +\infty \), we obtain that for any \( \varepsilon_1 > 0 \), there exists \( K_2^* \) such that when \( k > K_2^* \), there exists \( T_2^* \) such that for \( t > T_2^* \), (16) holds.
Second, set \( \Phi(x, t) = u(x, t) - u_k(x), \Psi(x, t) = v(x, t) - v_k(x) \). Then \( (\Phi(x, t), \Psi(x, t)) \) satisfies (17). Multiplying the first equation of (17) by \( \Phi \) and integrating over \( \Omega \), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \Phi^2 dx = - \int_{\Omega} |\nabla \Phi|^2 dx + \int_{\Omega} (\lambda - 2u_k + \Phi) \Phi^2 dx \]
\[ - \int_{\Omega} a (p(u, v) - p(u_k, v_k)) \Phi dx. \]
Since \( p(u, v) \to 0 \), \( p(u_k, v_k) \to 0 \), \( u_k \to \lambda \) and \( v_k \to 0 \) as \( k \to +\infty \), we can see that for any \( \varepsilon > 0 \), there exists \( K_2^{*} \) such that when \( k > K_2^{*} \), there exists \( T_2^{*} \) such that for \( t > T_2^{*} \),
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \Phi^2 dx \leq (-\lambda + 2\varepsilon) \int_{\Omega} \Phi^2 dx + \varepsilon \int_{\Omega} |\Phi| dx \]
\[ \leq (-\lambda + 2\varepsilon) \int_{\Omega} \Phi^2 dx + \varepsilon |\Omega|^{\frac{1}{2}} \left( \int_{\Omega} \Phi^2 dx \right)^{\frac{1}{2}}. \quad (18) \]
Then we deduce that
\[ \lim_{t \to \infty} f(t) = \lim_{t \to \infty} \int_{\Omega} \Phi^2 dx = 0. \]
Otherwise, we may assume that \( \lim_{t \to \infty} f(t) = \ell > 0 \). Then there exists a sequence of \( t_n \to \infty \) such that
\[ \lim_{n \to \infty} f(t_n) = \ell, \quad \lim_{n \to \infty} f'(t_n) = 0. \]
Letting \( n \to \infty \) in (18), we obtain that
\[ 0 \leq a \left( -\lambda + 2\varepsilon + \frac{|\Omega|^{\frac{1}{2}}}{a^2} \varepsilon \right), \]
which is impossible for sufficiently small $\varepsilon > 0$. Thus, it follows that $\lim_{t \to \infty} f(t) = 0$. Similarly, we can show that
\[
\lim_{t \to \infty} g(t) = \lim_{t \to \infty} \int_{\Omega} \Psi^2 dx = 0.
\]
Therefore, the proof of the lemma is complete.

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E-mail address: wangyux10@163.com (Y.-X. Wang, Corresponding author)
E-mail address: wtli@lzu.edu.cn (W.-T. Li)