ONE NUMERICAL OBSTRUCTION FOR RATIONAL MAPS BETWEEN HYPERSURFACES

ILYA KARZHEMANOV

ABSTRACT. Given a rational dominant map $\phi: Y \dasharrow X$ between two generic hypersurfaces $Y, X \subset \mathbb{P}^n$ of dimension $\geq 3$, we prove (under an addition assumption on $\phi$) a “Noether–Fano type” inequality $m_Y \geq m_X$ for certain (effectively computed) numerical invariants of $Y$ and $X$.

1. Introduction

1.1. Set-up. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface over $\mathbb{C}$ given by an equation $f = 0$ in some projective coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$. Identify $x_i$ with a basis of $H^0(X, \mathcal{O}(1))$. We will assume in what follows that $n \geq 4$. In particular, given two such hypersurfaces $X$ and $Y$, we have $\text{Pic} \, X = \text{Pic} \, Y = \mathbb{Z} \cdot L$ (Lefschetz), so that any rational map $\phi: Y \dasharrow X$ is induced by a self-map of $\mathbb{P}^n$.

Definition 1.2. Call $\phi$ symplectic if the corresponding map $\mathbb{P}^n \dasharrow \mathbb{P}^n$ preserves, up to a constant, the 2-form $\sum_{i=1}^{n} \frac{dx_i}{x_i} \wedge \frac{d\bar{x}_i}{\bar{x}_i}$ (cf. 2.1 below). Also, call $X$ symplectically unirational, if $Y := \mathbb{P}^{n-1} = (x_n = 0)$ and $\phi$ is symplectic.

Example 1.3. Take $X = Y = (x_n = 0) \simeq \mathbb{P}^{n-1}$ and $\phi$ to be the Frobenius morphism $x_i \mapsto x_i^d$, $0 \leq i \leq n-1$, for some integer $d \geq 2$. This $\phi$ is easily seen to be symplectic.

Fix a point $o \in X$ and consider the blowup $\sigma: \tilde{X} \dasharrow X$ of $o$. Let $\Sigma := \sigma^{-1}(o)$ be the exceptional divisor of $\sigma$. Define the quantity (cf. [1, Definition 1.6])

$$m_X(L; o) := \sup \{ \varepsilon \in \mathbb{Q} : \text{the linear system } |N(\sigma^* L - \varepsilon \Sigma)| \text{ is mobile for } N \gg 1 \}$$

It is called the mobility threshold (of $X$) and was first introduced by A. Corti in [17] (we will sometimes write simply $m_X$ when the point $o$ is irrelevant).

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1) Recall that “mobile” means no divisorial component in the base locus.
Example 1.4. Let $f := x_n$ and hence $X \simeq \mathbb{P}^{n-1}$ is a projective subspace. Then we get $m_X(L:o) = 1$ because the lines passing through $o$ sweep out a divisor (in particular, the above $\varepsilon$ is always $\leq 1$, while the opposite inequality is clear — consider the projection $X \rightarrow \mathbb{P}^{n-2}$ from $o$ to obtain $\varepsilon = 1$).

In this paper, we will be mostly using the following equivalent definition of $m_X$ (although the first one is more suitable for computations):

$$m_X(L:o) := \sup \frac{\text{mult}_o M}{N},$$

where sup is taken over all $N \geq 1$ and mobile linear systems $M \subseteq |L^{\otimes N}|$.

Let us assume from now on that $X = (f = 0)$ and $Y$ are generic. Here is our main result:

**Theorem 1.5.** If $\phi$ is symplectic, then there exist points $o \in X$, $o' \in Y$ such that $m_Y(L:o') \geq m_X(L:o)$.

Theorem 1.5 implies in particular that $1 = m_Y \geq m_X$ for $Y := \mathbb{P}^{n-1}$ and symplectically unirational $X$ (cf. Corollary 4.2 below). Note however that similar estimate does not hold for an arbitrary unirational $X$ (see 4.4).

1.6. Discussion. The main idea behind the proof of Theorem 1.5 is that birational invariants of $X$ appear from a (hidden) hyperbolic structure on hypersurfaces. Namely, we employ the so-called pairs-of-pants decomposition II from [15], which we recall briefly in Section 2. This brings further an analogy with hyperbolic manifolds (especially surfaces and 3-folds) and their geometric invariants — most common ones being various types of *volumes*.

This includes, as a basic example, the Euler characteristic of Riemann surface $M$. A finer invariant is the so-called *conformal volume* $V_c(M)$. One can show that $2V_c(M) \geq \lambda_1 \text{Vol}(M)$ for the first Laplacian eigenvalue $\lambda_1$ of $M$ (see [14, Theorem 1] and corollaries thereof). This was used, for instance, to obtain obstructions for existence of maps between Riemann surfaces (see e.g., the proof of the *Surface Coverings Theorem* in [6, §4] or that of [8, Theorem 2.A1]).

Further, if $M$ is a $d$-dimensional closed oriented hyperbolic manifold, then there is a *Gromov’s invariant* $\| [M] \|$ (see [20, Chapter 6]). It is defined in terms of certain (probability) measures on $M$ and coincides with $C_d \text{Vol}(M)$ for some absolute constant $C_d$. The most fundamental property of this invariant (used in the proof of Mostow’s rigidity theorem for example) is that for a map $M_1 \rightarrow M_2$ between
two hyperbolic $M_i$ one has $\| [M_1] \| \geq \| [M_2] \|$. The latter inequality and a close similarity between the definitions of $V_c(\ )$, $\| \ |$, etc. and that of $m_X$ (cf. 1.1 and 2.3 below) have motivated our approach towards the proof of Theorem 1.5.

Namely, on replacing $X$ by the complex $\Pi$ mentioned above, we recast $m_X$ in “probabilistic” terms (see Section 3). The argument here is an instance of the (Bernoulli) law of large numbers and allows one to give a conceptual explanation for the estimate $m_Y \geq m_X$ (cf. Remark 3.4). On the other hand, results in 2.3 and 2.7 together with initial definition of $m_X$ in 1.1 yield algebro-geometric applications (see Section 4).

Such line of thought — associating $X \rightarrow \Pi$ and extracting geometric properties of $X$ from combinatorics of $\Pi$ (and vice versa) — is not new. Classical case includes the Brunn - Minkowski inequality (see e. g. [3]). In a modern context (including the mirror symmetry) this viewpoint appears in [2] and [19] for instance. Finally, we mention the “motivic” part of the story, when one assigns to $X$ its stable birational volume $|X|_{sb}$ (see e. g. [16]) or its class $[X]_K$ in the connective $K$-theory (see [21]). In the latter case, given $\phi : Y \rightarrow X$ as above, the degree formula of [21] relating the classes $[Y]_K$, $[X]_K$ may be considered as a vast generalization of Noether – Fano inequality (see e. g. [11, Proposition 2]) and was another motivation for our Theorem 1.5.

2. Preliminaries

2.1. Pairs - of - pants complex. Consider the intersection

$$X^0 := X \cap \bigcap_{i=0}^{n} (x_i \neq 0)$$

of $X$ with the torus $(\mathbb{C}^*)^n \subset \mathbb{P}^n$ equipped with (affine) coordinates $x_1, \ldots, x_n$. We may identify $f$ with the Laurent polynomial defining $X^0$:

$$f = \sum_{j \in \Delta \cap \mathbb{Z}^n} a_j t^{-v(j)} x_j,$$

where $\Delta \subset \mathbb{R}^n$ is the Newton polyhedron of $f$, $v : \Delta \cap \mathbb{Z}^n \rightarrow \mathbb{R}$ is a piecewise affine function, $a_j \in \mathbb{C}$ and $t > 0$ is a real parameter (cf. [15, 6.4]).

Let us recall the balanced maximal dual polyhedral $\Delta$ - complex $\Pi$ associated with $X$. It corresponds to certain (dual) simplicial lattice subdivision of $\Delta$ associated with the corner locus of the Legendre transform of $v$; $\Pi$ may also be identified with...
its moment image (see \[15, 2.3 \text{ and Proposition 2.4}\]). Here are some properties of \( \Pi \) we will use below:

- there exists a smooth \( T^{n-1} \)-equivariant map (stratified \( T^{n-1} \)-fibration) \( \pi : X \to \Pi \), where \( T^{n-1} := (S^1)^{n-1} \) consists of the arguments of \( x_i \), so that for any open lattice simplex \( \Pi_0 \subset \Pi \) the preimage \( \pi^{-1}(\Pi_0) \) is an open pair-of-pants, symplectomorphic to \( H^0 := \left( \sum_{i=1}^{n} x_i = 1 \right) \subset (\mathbb{C}^*)^n \) equipped with the 2-form \( \Omega := \frac{1}{2\sqrt{-1}} \sum_{i=1}^{n} \frac{dx_i}{x_i} \wedge \frac{d\bar{x}_i}{\bar{x}_i} \);

- in fact, \( X \) is glued out of the tailored (or localized) pants \( Q^{n-1} \) (see \[15, 6.6 \text{ and Proposition 4.6}\]), isotopic to \( H^0 \), so that \( \pi \) is a deformation retraction under the Liouville flow associated with \( \Omega \), and \( \pi_*\Omega^{n-1} \) induces the Euclidean measure on \( \Pi \);

- \( \Pi \) is also obtained via the tropical degeneration (compatibly with \( \pi \)): recall that \( f \) depends on \( t \) (see (2.2)) and let \( t \to 0 \) (for each \( Q^{n-1} \)) in the preceding constructions — \( \Pi \) is then a Gromov–Hausdorff limit of amoebas \( A_t(X_0) \) (see \[15, 6.4\]).

2.3. Atomic (probability) measure. Choose a global section \( s \in H^0(X, L) \setminus \{0\} \) and a point \( o \in X \). We are going to construct a measure \( d\mu_{o,s} \) on \( \Pi \) so that \( \text{Vol}(\Pi) = \text{mult}_o \{ s = 0 \} \) with respect to it and \( d\mu_{o,s} \) is supported at the point \( \pi(o) \).

Identify \( s \) with a holomorphic function on a complex neighborhood \( U \subset X \) of \( o \) and consider the \((1,1)\)-current \( \tau := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |s| \). Then \( \tau \) acts on the \( L^1 \)-forms on \( U \) of degree \( 2n-2 \) via (Poincaré–Lelong)

\[
\omega \mapsto \tau(\omega) = \int_{s = 0} \omega.
\]

In particular, if \( \omega \) is the \((n-1)\)-st power of the Fubini–Studi form on \( \mathbb{P}^n \) restricted to \( U \), one may regard \( \tau(\omega) \) as the volume of the locus \( U \cap \{ s = 0 \} \).

Let us assume from now that \( U := B_o(r) \) is the Euclidean ball of radius \( r \) centered at \( o \). For all \( \xi \in \mathbb{C}, |\xi| < 1 \), we consider the dilations \( z \mapsto \xi z \) of \( U \) and the family of pushed-forward measures \( \xi_* (dm) \).

The following lemma is standard (cf. Remark 2.6 and Proposition 3.3 below):

\[\text{All considerations below apply literally to any } L^{\otimes N}, N \geq 1, \text{ in place of } L.\]
Lemma 2.4. The limit measure

\[ \frac{1}{(r\xi)^{2n-2}} \lim_{\xi \to 0} \xi_*(dm) := dm_{o,s} \]

exists and \( \int_U dm_{o,s} = \text{mult}_o \{ s = 0 \} \).

Proof. Indeed, Vol(\{ s = 0 \} \cap U) with respect to \( \frac{1}{(r\xi)^{2n-2}} \xi_*(dm) \) tends to \( \text{mult}_o \{ s = 0 \} \), as \( \xi \to 0 \). This implies that the limit of measures exists. \( \square \)

We may assume that for \( U = B_o(r) \) the radius \( r \to 0 \) as \( t \to 0 \) (cf. (2.2)). Then it follows from 2.1 and Lemma 2.4 that

\[ \pi(U) = \text{simplicial complex } \Pi_0 = \text{Gromov–Hausdorff limit of } A_t(U), \]

(2.5)

\[ s = \pi^* \ell \text{ for a piecewise linear function } \ell \text{ on } \Pi_0, \]

\[ d\mu_{o,s,n_0} := \pi_* dm_{o,s} = \text{measure on } \Pi_0 \text{ supported at } \pi(o) \text{ and such that } \]

\[ \text{Vol(} \Pi_0 \text{)} = \int_{\Pi_0} d\mu_{o,s,n_0} = \text{mult}_o \{ s = 0 \}; \]

note also that \( \pi(o) \) belongs to the corner locus of \( \ell \).

Finally, \( d\mu_{o,s,n_0} \) induces a measure \( d\mu_{o,s} \) on \( \Pi \supseteq \Pi_0 \) in the obvious way, which concludes the construction.

Remark 2.6. The atomic measure \( d\mu_{o,s} \) is an instance of the convexly derived measure from [5] (one may also treat the above “density function” \( \ell \) as a discrete version of the Hessian of \( \frac{\sqrt{-1}}{2\pi} \log |s| \)). The “mass concentration” concept of [5] will be used further to obtain intrinsic (bi) rational invariants of \( X \).

2.7. Rational maps: tropicalization. Let \( Y \subset \mathbb{P}^n \) be another hypersurface, similar to \( X \), with the maximal dual complex \( \Pi^Y \), projection \( \pi^Y : Y \to \Pi^Y \), etc. defined verbatim for \( Y \). Assume also that there exists a rational symplectic map \( \phi : Y \to X \). Then the constructions in 2.1 and 2.3 yield a map \( \Phi : \Pi^Y \to \Pi^X \), given by some PL functions with \( \mathbb{Z} \)-coefficients \( ^3 \) so that the following diagram commutes:

(2.8)

\[ Y \xrightarrow{\phi} X \]

\[ \pi^Y \downarrow \quad \downarrow \pi^X \]

\[ \Pi^Y \quad \Phi \quad \Pi^X. \]

Note that \( \Phi \) need not necessarily be a map of simplicial complexes.

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\(^3\) We use the notation \( \Pi^X =: \Pi \) and \( \pi^X := \pi \) in what follows.
The following lemma describes $\Phi$ as a map of measure spaces:

**Lemma 2.9.** There exists a positive number $\delta_\phi \in \mathbb{Z}$, depending only on $\phi$, such that $\Phi^* \pi_*^X \Omega^{n-1} = \delta_\phi \pi_*^Y \Omega^{n-1}$.  

**Proof.** We have

$$\pi_*^Y \phi^* \Omega^{n-1} = \delta_\phi \pi_*^Y \Omega^{n-1}$$

for some real $\delta_\phi > 0$ (cf. Definition 1.2). Now, since $\phi$ is the restriction of a rational self-map of $\mathbb{P}^n$ (see 1.1), it follows from 2.1 and (2.8) that $\pi_*^Y \Omega^{n-1}$ (resp. $\Phi^* \pi_*^X \Omega^{n-1}$) coincides with the measure induced by the standard one $dy_1 \wedge \ldots \wedge dy_n$ on $\mathbb{R}^n$ (resp. by $dl_1 \wedge \ldots \wedge dl_n$ for some piecewise linear functions $l_i = l_i(y)$ with $\mathbb{Z}$-coefficients). It remains to observe that $dl_1 \wedge \ldots \wedge dl_n = \delta_\phi dy_1 \wedge \ldots \wedge dy_n$ by construction. $\Box$

3. **Proof of Theorem 1.5**

3.1. **The entropy.** Let $\Pi \subset \mathbb{R}^n$ be a simplicial complex with the standard Borel measure $d\mu$. Fix some real number $M > 0$ and consider various measures $d\mu_\ell$ on $\Pi$, supported at the corner locus of PL functions $\ell$, such that $\int_\Pi d\mu_\ell \leq M$. Let $S := S(\Pi, M)$ be the set of all such measures (aka functions).

Further, given an integer $N > 0$ the measure space $(\Pi, N d\mu) := \Pi_N$ may be regarded as $\Pi \subset \mathbb{R}^{Nn}$, embedded diagonally, with the measure being $d\mu_N := \sum_{i=1}^N \pi_i^* d\mu$ for the $i$th factor projections $\pi_i : \mathbb{R}^{Nn} \to \mathbb{R}^n$. The affine structure on $\Pi_N$ is defined by the functions $\sum_{i=1}^N \pi_i^* \ell_i$ for various PL $\ell_i$. Note that

$$\frac{1}{N} \int_{\Pi_N} \sum_{i=1}^N \pi_i^* (d\mu_\ell_i) \leq M,$$

i.e. $\frac{1}{N} \sum_{i=1}^N \pi_i^* (d\mu_\ell_i) \in S$, provided $d\mu_\ell_i \in S$ for all $1 \leq i \leq n$.

Define the measures $d\mu_\ell^N$ on $\Pi_N \subset \mathbb{R}^{Nn}$ and the set $S(\Pi_N, M) \ni \frac{1}{N} d\mu_\ell^N$ similarly as above. Let also

$$C := \sup_{\ell \in S(\Pi_N; M), N} \frac{1}{N} \int_{\Pi_N} d\mu_\ell^N. \quad (3.2)$$

4) Here $\Phi^*$ is defined with respect to the (limiting) affine structure on $\Pi$ induced from the complex one on $X$ (cf. 2.1).
Proposition 3.3. There exists a number \( \text{ent}(d\mu, S) < \infty \), depending only on \( d\mu \) and \( S \), such that \( C = \text{ent}(d\mu, S)M \).

Proof. After normalizing we may assume that \( M = 1 \). Let us also assume for transparency that \( \Pi \) is a simplex.

All measures \( \frac{1}{N} d\mu_s^n \) can be identified with points (mass centers) in the dual simplex \( \Pi^* \subset \mathbb{R}^n \) (compare with the proof of [5, 4.4.A]). Let \( \mathcal{H}_{\mu, S} \subseteq \Pi^* \) be the convex hull of this set. Then

\[
\int_{\Pi} \cdot : \mathcal{H}_{\mu, S} \rightarrow \mathbb{R}_{\geq 0}
\]

is a bounded \((\leq 1)\) linear functional. By definition we obtain \( C = \max_{\mathcal{H}_{\mu, S}} \int_{\Pi} \cdot =: \text{ent}(d\mu, S) \) and the result follows. \( \square \)

Remark 3.4. The constant \( C = C^X \) resembles the value of logarithmic rate decay function at \( d\mu \) (see e.g. [7, Lecture 4]). This suggests \( C \) to be equal the “Boltzmann entropy” and the estimate \( C^Y \geq C^X \) in the setting of 2.7 (compare with [4, p. 7]).

In fact, taking \( d\mu_s := d\mu_{o,s} \) for various \( s \) as in 2.3, we will apply this probabilistic reasoning to (birational) geometry of \( X \) (see below).

3.5. The estimate. Let \( \Phi : \Pi^Y \rightarrow \Pi^X \) be as in 2.7. Although \( \Phi \) need not preserve the simplicial structures, we still can find a pair of \( k \)-simplices \( \Pi^X_0 \subseteq \Pi^X \) and \( \Pi^Y_0 \subseteq \Pi^Y \), \( 1 \leq k \leq n \), such that \( \Phi(\Pi^Y_0) = \Pi^X_0 \).

Identify both \( \Pi^X_0 \) and \( \Pi^Y_0 \) with a simplex \( \Pi \), carrying two (Borel) measures \( d\mu \) and \( \delta_0 d\mu \), induced by \( \pi^Y_0 \Omega^{n-1} \) and \( \Phi^* \pi^X_0 \Omega^{n-1} \), respectively (see Lemma 2.9).

Let us assume from now on that \( S := S_X \) consists of PL functions \( \ell \), obtained from various sections \( s = \pi^X \ell \in \mathcal{M} \) and mobile linear systems \( \mathcal{M} \subseteq |L \otimes N| \), so that \( d\mu_\ell = \frac{1}{N} d\mu_{o,s} \) for some \( o \in X \) satisfying \( \pi(o) \in \Pi \) (see 3.1 and (2.5)). It follows from 2.7 that \( M \) in 3.1 can be assumed to coincide with the mobility threshold \( M^X := m_X(L; o) \) (cf. 1.1). Same considerations apply to \( Y \), with \( S_Y, M^Y := m_Y(L; o) \), etc.

Lemma 3.6. In the previous setting, we have \( \text{ent}(d\mu, S_X) = 1 \), and similarly for \( S_Y \).

Proof. This follows from 3.2 (cf. Proposition 3.3), definition of \( m_X \) (cf. 2.5), and the fact that \( \pi^X_0 \Omega^{n-1} = d\mu \) on \( \Pi = \Pi^X_0 \) (see 2.1). \( \square \)

Proposition 3.7. For every \( \ell \in S(\Pi, M^X) \), we have \( d\mu_{\Phi_\ell} = d\mu_{\tilde{\ell}} \), where \( \tilde{\ell} \in S(\Pi, M^Y) \).
Proof. It follows from (2.5) and Lemma 2.4 that
\[ \int_{\Pi} \Pi d\mu = \frac{1}{N} \int_{\Pi} \Pi d\mu_{o,s} = \frac{1}{N} \int_{U} dm_{o,s} = \frac{1}{N} \int_{U\setminus Z} dm_{o,s} \]
for any closed subset $Z \subseteq U$. Recall that the rational transform $\phi^{-1}$ is naturally defined as a member of the mobile linear system $\phi^{-1}M$. In particular, if $\phi$ is a morphism over $U \setminus Z$, then
\[ \int_{\phi^{-1}(U\setminus Z)} dm_{o,\phi^{-1}s} = \text{mult}_{o}\{\phi^{-1}s = 0\} \]
This $\phi^{-1}s$ defines a PL function $\ell$ as earlier and we have
\[ \int_{\Pi} d\mu_{\phi \cdot \ell} = \int_{\Pi} d\mu_{\ell} \]
(cf. (2.8)). The identity $d\mu_{\phi \cdot \ell} = d\mu_{\ell}$ follows and $\ell \in S(\Pi, M^{Y})$ by construction.

□

Let $C := M^{Y}$ be as in Proposition 3.3 (ent$(d\mu, S)$ = 1 by Lemma 3.6) and $\delta_{\phi}$ as in Lemma 2.9. Then it follows from Proposition 3.7 (cf. Remark 3.4) that
\[ C \delta_{\phi} = \sup_{\ell \in S(\Pi_{N}, M^{Y}), N} \frac{1}{N} \int_{\Pi_{N}} \delta_{\phi} d\mu_{\ell}^{N} \geq \sup_{\ell \in S(\Pi_{N}, M^{X}), N} \frac{1}{N} \int_{\Pi_{N}} \delta_{\phi} d\mu_{\phi \cdot \ell}^{N} = \]
\[ = \sup_{\ell \in S(\Pi_{N}, M^{X}), N} \frac{1}{N} \int_{\Pi_{N}} \Phi^{\ast} d\mu_{\ell}^{N} = \text{ent}(\delta_{\phi} d\mu, S)M^{X} \]
(the last equality is due to the projection formula $\Phi^{\ast} d\mu = \delta_{\phi} d\mu$ and the change of variables in $\int$). Finally, since ent$(\delta_{\phi} d\mu, S) = \delta_{\phi} \text{ent}(d\mu, S)$, we conclude that $M^{Y} \geq M^{X}$.

4. SOME EXAMPLES AND APPLICATIONS

4.1. Soft. Setting $Y := \mathbb{P}^{n-1} = (x_{n} = 0)$ we arrive at the following immediate

Corollary 4.2. Suppose $X$ in Theorem 1.5 is symplectically unirational. Then there exists a point $o \in X$ such that $m_{X}(L; o) = 1$.

Proof. It suffices to prove that $m_{X} \geq 1$. This is done by considering the projection $X \rightarrow \mathbb{P}^{n-1}$ from $o$ and observing that the linear system $|\sigma^{\ast}L - a\Sigma|$ is mobile for some $a \geq 1$ (cf. 1.1).

5) There is a slight abuse of notation here — $o$ denotes a point in both $X$ and $Y$. 

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Suppose $X$ is a quadric. Although we do not know whether $X$ is symplectically unirational (cf. Example 1.3), it is obviously rational, and Corollary 4.2 confirms that $m_X = 1$ in this case (the latter equality can actually be proved directly by considering families of lines on $X$ as in Example 1.4).

Remark 4.3. It would be interesting to find out whether any birationally isomorphic hypersurfaces $X$ and $Y$ as in Theorem 1.5 always have $m_X(L; o) = m_Y(L; o')$ for some points $o$ and $o'$. It should also be possible to generalize all our considerations to the case of any smooth $X$ and $Y$.

Let us proceed with non-trivial examples distinguishing ordinary unirationality from the symplectic one.

4.4. Hard. Suppose $\deg f = 3$ (i.e. $X$ is a cubic). It is a classical fact that $X$ is unirational (see e.g. [10, Chapter 3, Corollary 1.18]). Fix a point $o \in X$. We may assume that $o = [1 : 0 : \ldots : 0]$, and hence $f = q_1 + q_2 + q_3$ in the affine chart ($x_0 \neq 0$), where $q_i = q_i(x_1, \ldots, x_n)$ are forms of degree $i$. Arguing as in [18, Section 1] we obtain that $q_1$ and $q_2$ are coprime. Thus the linear system $\mathcal{M} \subset |2L|$ spanned by $q_1^2$ and $q_2$ is mobile. We conclude that $m_X \geq 3/2$, since $\text{mult}_o \mathcal{M} = 3$ (cf. 1.1), and so $X$ is not symplectically unirational by Corollary 4.2.

Now assume only that $X$ is smooth (cf. Remark 1.3). Then it is possible to find a (Eckardt) point $o \in X$ for which $m_X(L; o) = 1$ (see [10, Chapter 5]). It would be interesting to study whether such cubics are symplectically unirational.

Further, consider the case $\deg f = 4 = n$, assuming again that the quartic $X$ is just smooth. Note that it is still unknown whether any such $X$ is unirational.\footnote{Recall that initially in 1.3 these hypersurfaces were assumed generic.}\footnote{Although a smooth quartic hypersurface $X$ is unirational when $n > 4$ (see [9, Corollary 3.8]).}

Here is a classical unirational example after Segre (cf. [12, 9.2]):

$$X = (x_0^4 + x_0x_4^3 + x_1^4 - 6x_1^3x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0).$$

We claim that $m_X(L; o) = 1$ for some $o \in X$. Indeed, take the hyperplane $\Pi := (x_1 - \alpha x_2 = 0)$, where $\alpha := \sqrt{3 + 2\sqrt{2}}$. Then $X \cap \Pi$ is a cone in $\mathbb{P}^3$ given by the equation $x_0^4 + x_0x_4^3 + x_1^4 + x_3^3x_4 = 0$. We take $o$ to be the vertex of this cone.
At the same time, if $X$ is generic, then one can show that $m_X \geq \frac{3}{2}$ by exactly the same argument as in the cubic case. Thus again symplectic version of the unirationality problem for $X$ is settled here.

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Laboratory of AGHA, Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Moscow Region, 141701, Russia

E-mail address: karzhemanov.iv@mipt.ru