Stability and duality in $\mathcal{N} = 2$ supergravity

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April 29th, 2010
Introduction

dualities

BPS black holes

stability
Outline

- BPS-states and wall-crossing
- Review supergravity and dualities
- (Multi)-center black hole solutions
- Partition functions
- Convergence and modularity
- Conclusions

Based on arXiv:0906.1767 and 1003.1570
$\mathcal{N} = 2$ algebra

\[ \{ Q^I_\alpha, Q^J_\beta \} = 2\varepsilon_{\alpha\beta} Z^{IJ}, \]

central charge: $Z : (L, C_X) \rightarrow \mathbb{C}$, where:
- $L$: lattice of electro-magnetic charges
- $C_X$: moduli space

BPS states:
- invariant under half of the susy generators,
- their mass satisfies $M = |Z(\Gamma, t)|$ with $\Gamma \in L$ and $t \in C_X$

Supersymmetric index:

\[ \Omega(\Gamma; t) = \frac{1}{2} \text{Tr}_{\mathcal{H}(\Gamma, t)} (2J_3)^2 (-1)^{2J_3} \]

Is generically a protected quantity.
Ω(Γ₁ + Γ₂; t) is only locally constant as function of t; it might jump across walls where \( Z(Γ₁, t) || Z(Γ₂, t) \).
Wall-crossing formulas:

Primitive:
\[ \Delta \Omega(\Gamma_1 + \Gamma_2; t_s \rightarrow t_u) = (-1)^{\langle \Gamma_1, \Gamma_2 \rangle} |\langle \Gamma_1, \Gamma_2 \rangle| \Omega(\Gamma_1; t_{ms}) \Omega(\Gamma_2; t_{ms}), \]

Denef, Moore (2007)

Kontsevich-Soibelman formula:
\[
\prod_{\Gamma \in L, Z(\Gamma,t) \in V} T_{\Gamma}^{\Omega(\Gamma; t)}
\]
Partition function:

Mixed ensemble:

\[ Z(\tau, C, t) = \sum_{Q} \Omega(P, Q; t) e^{-2\pi \tau_2 M(\Gamma, t) + 2\pi i C^A Q_A} \]

\( \tau_2 \in \mathbb{R}_+, \ C^A \in \mathbb{R}^{b_2+1} \)
dualities

BPS black holes

stability
$\mathcal{N} = 2$ supergravity I

Relevant field content:

vector multiplets:
- $U(1)$ field strengths $F^A = dC^A, A = 0, \ldots, b_2$ sourced by electro-magnetic charges $\Gamma = (P, Q) \in L$,
- complex scalars $X^A$
- fermions

Compactification

compactify 10d space-time on a Calabi-Yau 3-fold $X$ (6 real dimensions) $\implies \mathcal{N} = 2$ supergravity in $\mathbb{R}^{1,3}$
Properties of $X$:

- Betti numbers $b_n = \dim H_n(X)$: $b_0 = b_6 = 1$, $b_2 = b_4$, $b_3$, $b_1 = b_5 = 0$
- triple intersection product of 4-cycles: $d_{abc}$
- Kähler moduli: $t^a = X^a/X^0 = B^a + iJ^a$, $a = 1, \ldots, b_2$
- Kähler cone:
  $C_X = \{ J : Q \cdot J > 0, P \cdot J^2, J^3 > 0$ for $Q, P$ effective $\}$

Fees

Charges

$\Gamma = (P^0, P^a, Q_a, Q_0) = \text{D6-D4-D2-D0 branes} \in H_6 \oplus H_4 \oplus H_2 \oplus H_0$

Symplectic inner product:

$\langle \Gamma_1, \Gamma_2 \rangle = I_{12} = -P_0^0 Q_{0,2} + P_1 \cdot Q_2 - P_2 \cdot Q_1 + P_2^0 Q_{0,1}$
Electric-magnetic duality:

- Electric-magnetic duality is a symplectic group $Sp(2b_2 + 2, \mathbb{Z})$: $K^T IK = I$. Acts on the vector multiplets, e.g. $K \Gamma$

- Large volume limit $J \to \infty$: subgroup of translations $K(k) \sim k^a \in \mathbb{Z}^{b_2}$
  
  \[ Q_a \to Q_a + d_{abc} k^b P^c \]
  \[ Q_0 \to Q_0 + k \cdot Q + \frac{1}{2} d_{abc} k^a k^b P^c \]
  \[ t^a \to t^a + k^a \]

- Large gauge transformations $C^a \to C^a + m^a$, also $\mathbb{Z}^{b_2}$
**Dualities II**

- $SL(2, \mathbb{Z})$ duality group: $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, $ad - bc = 1$
  
  More manifest in IIB supergravity $\rightarrow$ T-duality on $S^1_t$

- How does it act?
  
  $\tau = \tau_1 + i\tau_2 = C_0 + i\beta/g_s$
  
  $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$, $C \rightarrow aC + bB$, $B \rightarrow cC + dB$, $J \rightarrow |c\tau + d|J$

- $S$-duality + EM-duality + large gauge transformations $\rightarrow$

  Jacobi group $SL(2, \mathbb{Z}) \ltimes (\mathbb{Z}^b)^2$
Dualities $\Rightarrow$ modular properties of partition function:

$$ Z(\gamma(\tau, C, t)) \sim Z(\tau, C, t), \quad \gamma \in SL(2, \mathbb{Z}) $$

$$ Z(\tau, C + k, t + \ell) \sim Z(\tau, C, t) $$

$\Rightarrow$ Partition function is useful to test the compatibility of stability with duality.
dualities

BPS black holes

stability
Moduli “flow” by attractor mechanism. Ferrara, Kallosh and Strominger (1995)

Mass: $M = |Z(\Gamma, t)|$

Entropy: $S_{\text{BH}}(\Gamma) = \pi |Z(\Gamma, t(\Gamma))|^2$
Goal: construction of partition function for such BPS-states. 
⇒ test dualities.
Large volume limit:

\[
\lim J \rightarrow \infty
\]

D-branes $\rightarrow$ coherent sheaves.
No D6-branes: \( P^0 = 0 \)

D4-brane wraps divisor in \( X \).

\[
Z(\Gamma, t) = -\frac{1}{2} P \cdot t^2 + Q \cdot t - Q_0
\]
Lattice \( \Lambda \) for \( Q_a \in H_2(X, \mathbb{Z}) \)

- quadratic form \( D_{ab} = D_{abc} P^c : H_2(X, \mathbb{Z}) \otimes H_2(X, \mathbb{Z}) \to \mathbb{Z} \),
- signature \((1, b_2 - 1)\)
- projection to \( \Lambda_+ \): \( Q_+ = Q \cdot J/|J| \); such that \( Q^2 = Q_-^2 + Q_+^2 \)

Entropy from MSW CFT Maldacena, Strominger, Witten (1997):

Entropy: \( S_{BH} = \pi \sqrt{\frac{2}{3} P^3 \hat{Q}_0} \)

Lower bound: \( \hat{Q}_0 = -Q_0 + \frac{1}{2} Q^2 \geq -c_R/24 \approx -P^3/24 \)

To every black hole center a lattice \( \Lambda_i \), index \( \Omega(\Gamma_i) = \Omega(\Gamma_i, t(\Gamma_i)) \)
and central charge \( c_{R_i} \) is associated.
Mass for single center

\[
\lim_{J \to \infty} M(\Gamma, t) = \frac{1}{2} P \cdot J^2 + Q^2_+ - Q_0
\]

\[
= \frac{1}{2} P \cdot J^2 + Q^2_+ - \frac{1}{2} Q^2 + \hat{Q}_0
\]

\[
\begin{vmatrix}
Q^2 & Q \cdot J \\
Q \cdot J & J^2
\end{vmatrix} < 0 \quad \text{implies} \quad Q^2_+ - \frac{1}{2} Q^2 > 0,
\]

\[
\Rightarrow \quad M(\Gamma, t) \text{ bounded from below.}
\]
Mass for 2 centers

\[
\lim_{J \to \infty} M(\Gamma, t) = \frac{1}{2} P \cdot J^2 + Q_+^2 - Q_0
\]

\[
= \frac{1}{2} P \cdot J^2 + Q_+^2 - \frac{1}{2} (Q_1)^2_1 - \frac{1}{2} (Q_2)^2_2 + \hat{Q}_{0,1} + \hat{Q}_{0,2}
\]

signature \((2b_2 - 1, 1)\)

\Rightarrow \text{not bounded from below for generic } Q_1 \in \Lambda_1, \ Q_2 \in \Lambda_2.

**Stability**

\[
(P_1 \cdot Q_2 - P_2 \cdot Q_1) \text{Im}(Z(\Gamma_1, t)\bar{Z}(\Gamma_2, t)) < 0
\]

\Rightarrow \ Q_+^2 - \frac{1}{2} (Q_1)^2_1 - \frac{1}{2} (Q_2)^2_2 > 0

\Rightarrow \text{bounded from below.}
Partition function for single center

For single center:

Mass bounded from below
⇒ \( \mathcal{Z}_P(\tau, C, t) = \sum Q_i \Omega(\Gamma) e^{-\pi \tau_2 M(\Gamma, t) + 2\pi i C' Q_i} \) is convergent.

**S-duality/modularity**

\[
S : \quad \mathcal{Z}_P(-1/\tau, -B, C + i|\tau|J) = \tau^{1/2} \bar{\tau}^{-3/2} \varepsilon(S) \mathcal{Z}_P(\tau, C, t), \\
T : \quad \mathcal{Z}_P(\tau + 1, C + B, t) = \varepsilon(T) \mathcal{Z}_P(\tau, C, t),
\]
Electric-magnetic duality

\[ \mathcal{Z}_P(\tau, C, t + k) = (-1)^{P \cdot k} e(C \cdot k/2) \mathcal{Z}_P(\tau, C, t), \]
\[ \mathcal{Z}_P(\tau, C + k, t) = (-1)^{P \cdot k} e(-B \cdot k/2) \mathcal{Z}_P(\tau, C, t). \]

Theta function decomposition:

\[ \mathcal{Z}_P(\tau, C, t) = \sum_{\mu \in \Lambda^*/\Lambda} h_{P, \mu}(\tau) \Theta_{\mu}(\tau, C, B), \]

vector-valued modular form:

\[ h_{P, \mu}(\tau) = \sum_{Q_0} \Omega(\Gamma) q^{-Q_0 + \frac{1}{2} Q^2}, \quad Q \in \mu + \Lambda \]

Siegel-Narain theta function:

\[ \Theta_{\mu}(\tau, 0, 0) = \sum_{Q \in \Lambda_{+} + \mu} (-1)^{P \cdot Q} \exp \left( \pi i (\tau Q^2_+ + \bar{\tau} Q^2_-) \right) \]
Partition function for 2 centers

How to implement:

\[ \Omega_{P_1 \leftrightarrow P_2}(\Gamma; t) = \frac{1}{2} \left( \text{sgn}(\text{Im} \ Z(\Gamma_1, t)\bar{Z}(\Gamma_2, t)) + \text{sgn}(\langle \Gamma_1, \Gamma_2 \rangle) \right) \times \langle \Gamma_1, \Gamma_2 \rangle (-1)^{\langle \Gamma_1, \Gamma_2 \rangle} \Omega(\Gamma_1) \Omega(\Gamma_2). \]

Partition function:

\[ \mathcal{Z}_{P_1 \leftrightarrow P_2}(\tau, C, t) = \sum_{(\mu_1, \mu_2) \in \Lambda_1^*/\Lambda_1 \oplus \Lambda_2^*/\Lambda_2} h_{P_1, \mu_1}(\tau) h_{P_2, \mu_2}(\tau) \psi_{(\mu_1, \mu_2)}(\tau, C, B) \]

\( \psi_{(\mu_1, \mu_2)}(\tau, C, B) \) is a combination of a Siegel-Narain theta function and indefinite theta function.
Indefinite theta function

Sums only over negative definite lattice points (Göttsche, Zagier (1996); Zwegers (2002)):

\[ \Theta_\mu(\tau) = \sum_{Q \in \mu + \Lambda} \frac{1}{2} (\text{sgn}(Q \cdot J) - \text{sgn}(Q \cdot P)) \exp(2\pi i \bar{\tau} Q^2 / 2) \]
Modular invariant? No, but **mock** modular invariant.

\[ \Theta_\mu(\tau) \rightarrow \Theta^*_\mu(\tau) \]

by replacing

\[ \text{sgn}(x) \]

with

\[ 2 \int_0^{\sqrt{2\tau_2}x} e^{-\pi u^2} du \]

approaches \( \text{sgn}(x) \) for \( \tau_2 \rightarrow \infty \).
Similarly $\Psi_{(\mu_1, \mu_2)}(\tau, C, B) \rightarrow \Psi^*_{(\mu_1, \mu_2)}(\tau, C, B)$

$\Rightarrow$ then $\mathcal{Z}_{P_1 \leftrightarrow P_2}(\tau, C, t)$:

- has same modular properties as $\mathcal{Z}_P(\tau, C, t)$ → evidence for compatibility of stability and duality
- is continuous across walls, reminiscent of results by Gaiotto, Moore, Neitzke (2008); Joyce (2006)
Flow trees are schematic representations of supergravity solutions.

Analysis of more complicated BPS objects is possible.
- Also the contribution of flow trees with 3 endpoints is convergent.
- Partition functions with manifest S-duality, are generating functions of
  \[ \tilde{\Omega}(\Gamma; t) = \sum_{m|\Gamma} \frac{\Omega(\Gamma; t)}{m^2} \]

  instead of \( \Omega(\Gamma; t) \)
- \( \tilde{\Omega}(\Gamma; t) \) seem most natural to determine the contribution of flow trees.
Evidence is given for:
- the convergence of the BPS partition function in the mixed ensemble,
- the compatibility of stability and duality
- this also makes the partition function continuous of \( t \)

Open problems:
- modularity for \( N \geq 3 \)
- relax \( P^0 = 0 \) and \( J \to \infty \)