CLASSICAL PROOFS OF KATO TYPE SMOOTHING ESTIMATES FOR THE SCHRÖDINGER EQUATION WITH QUADRATIC POTENTIAL IN $\mathbb{R}^{n+1}$ WITH APPLICATION

XUWEN CHEN

Abstract. In this paper, we consider the Schrödinger equation with quadratic potential

$$i \frac{\partial}{\partial t} u = -\Delta u + |x|^2 u \text{ in } \mathbb{R}^{n+1}, \ u(x, 0) = f(x) \in L^2(\mathbb{R}^n).$$

Using Hermite functions and some other classical tools, we give an elementary proof of the Kato type smoothing estimate: for $i \neq j \neq k$, $\delta \in [0, 1]$, and $n \geq 3$

$$\int_0^{2\pi} \int\frac{|u(x, t)|^2}{\left(x_i^2 + x_j^2 + x_k^2\right)^\delta} \ dx dt \leq C \|f\|_2^2.$$

This is equivalent to proving a uniform $L^2(\mathbb{R}^n)$ boundedness result for a family of singularized Hermite projection kernels.

As an application of the above estimate, we also prove the $\mathbb{R}^9$ collapsing variable type Strichartz estimate

$$\int_0^{2\pi} \int\frac{|u(x, x, x, t)|^2}{\left(x_i^2 + x_j^2 + x_k^2\right)^\delta} \ dx dt \leq C \left\|(-\Delta + |x|^2)f\right\|_2^2$$

where $x \in \mathbb{R}^3$.

1. Introduction

In Bose-Einstein condensation (BEC), particles of integer spins ("Bosons") occupy a macroscopic quantum state often called the "condensation". In early lab experiments of BEC [1] [10], the particles were kept together by use of trapping potentials created by the effect of a magnetic field on the particle spins. In principle, the magnetic field has a complicated spatial structure. The interaction of the magnetic field with the spin is conveniently modeled by a quadratic potential. This captures salient features of the actual trap, especially the property that the external potential rises at large distances. In later experiments, e.g., [23], the trapping potential is produced by complicated laser fields; but mathematically, one can still use a quadratic potential as a simplified yet generic model. So the spin of the particle is removed in modeling and the effect of a trap is included in the form of a quadratic external potential. This physical background suggests that we study the Schrödinger equation with quadratic potential.

Date: 07/01/2010.
2000 Mathematics Subject Classification. Primary 35B45, 35Q41, 35A23; Secondary 42C10, 33C45.

Key words and phrases. Harmonic Oscillator, Kato Estimate, Hermite Functions.
\[ i \frac{\partial}{\partial t} u = -\Delta u + |x|^2 u \text{ in } \mathbb{R}^{n+1}, \quad (1.1) \]

with initial data

\[ u(x,0) = f(x) \in L^2(\mathbb{R}^n). \]

Many aspects of equation (1.1) which came from the study of the free Schrödinger equation

\[ i \frac{\partial}{\partial t} \phi = -\Delta \phi \text{ in } \mathbb{R}^{n+1} \quad (1.2) \]

have been studied by several authors. Its Strichartz estimates were proved by Koch and Tataru [4], Carles [6], Nandakumarana and Ratnakuma [19]. The well-posedness of its nonlinear energy critical version with radial initial data was studied by Killip, Visan and Zhang [15]. Bongioanni and Torrea, and Bongioanni and Rogers, proved results on the pointwise convergence to the initial data in [3] and [4]. Concerning the Kato \( \frac{1}{2} \)-smoothing effect, Doi (and later Bongioanni and Rogers) proved

\[ \int_0^T \int_{\mathbb{R}^n} \left| (I - (-\Delta + |x|^2))^\frac{1}{4} u(x,t) \right|^2 \frac{dxdt}{(1 + |x|^2)^{\frac{1}{4} + \varepsilon}} \leq C \| f \|_2^2 \quad (1.3) \]

in [9] (11), Robbiano and Zuily proved

\[ \int_0^T \int_{\Omega} \left| \chi(x)(I - (-\Delta + |x|^2))^\frac{1}{4} u(x,t) \right|^2 dxdt \leq C \| f \|_2^2 \]

for an external domain \( \Omega \) and \( \chi \in C_0^\infty(\Omega) \) in [21]. However, both [9] and [21] made extensive use of pseudo-differential techniques which did not suffice to prove the equation (1.1) counterpart to the Kato estimate

\[ \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{|\phi(x,t)|^2 dxdt}{|x|^{\frac{n}{2} - 2\alpha}} \leq C \| \phi(\cdot,0) \|_2^2 \quad (1.4) \]

in Kato and Yajima [14], or its generalization

\[ \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{||\nabla^\alpha \phi(x,t)||^2 dxdt}{|x|^{2 - 2\alpha}} \leq C \| \phi(\cdot,0) \|_2^2, \text{ for } \alpha \in [0, \frac{1}{2}) \quad (1.5) \]

in Kato and Yajima [14], and Ben-Artzi and Klainerman [2], where \( \phi \) is the solution to the free Schrödinger equation (1.2) in the case \( n \geq 3 \).

**Remark 1.** Doi proved [7, 3] type estimates in the case involving variable coefficients. Bongioanni and Rogers proved an estimate similar to (1.3) for equation (1.1) using Hermite functions. Their paper contains a series of results parallel to those in Vega [26].

**Remark 2.** The expository note [7] gives extensions of estimate (1.3) to a class of dispersive equations, and simultaneously arrives at the optimal constant for each \( \alpha \) and \( n \). The fact that \( \frac{n}{n-2} \) is the best constant achievable for the \( \alpha = 0 \) case is due to Simon [22].
This paper aims to prove Kato type smoothing estimates similar to [1.4] when \( n \geq 3 \) for equation 1.1 without using any pseudo-differential techniques.

In fact, we have

**Theorem 1.** Let \( u \) be the solution to equation 1.1 in the case \( n \geq 3 \), then for \( \delta \in [0, 1] \) and \( i \neq j \neq k \), one has the estimate

\[
2\pi \int_0^{2\pi} \int_{\mathbb{R}^n} \frac{|u(x, t)|^2}{(x_i^2 + x_j^2 + x_k^2)^\delta} dx dt \leq C \| f \|^2_2.
\]

(1.6)

In particular, when \( \delta = 1 \), the above estimate implies the Kato estimate 1.4 for equation 1.1 in the case \( n \geq 3 \) because of the trivial inequality \( (x_i^2 + x_j^2 + x_k^2) \leq |x|^2 \).

**Remark 3.** \( u \) naturally has period \( 2\pi \) in the time variable \( t \). We will show this in section 2. This was also shown in Nandakumarana and Ratnakumara [19].

**Remark 4.** Without lose of generality, from here on out we assume \( i = 1, j = 2, k = 3 \) for simplicity since the general case has an identical proof.

**Remark 5.** We shall also point out that the Kato type estimate

\[
2\pi \int_0^{2\pi} \int_{\mathbb{R}^n} \frac{|\nabla u(x, t)|^2}{|x|^{2-2\alpha}} dx dt \leq C \| f \|^2_2, \text{ for } \alpha \in [0, \frac{1}{2})
\]

which is the equation 1.1 counterpart to estimate 1.5 is still unproven.

**As an application of theorem 1**, we have the following collapsing variable type Strichartz estimate.

**Theorem 2.** Let \( u(x_1, x_2, x_3, t) \) solves equation 1.1 in the case \( n = 9 \) with \( x_i \in \mathbb{R}^3 \), then one has the estimate

\[
2\pi \int_0^{2\pi} \int_{\mathbb{R}^3} |u(x, x, x, t)|^2 dx dt \leq C \| (-\Delta + |x|^2)^s f \|^2_2.
\]

(1.7)

**Remark 6.** The ordinary Strichartz estimate gives

\[
\| \phi \|_{L^6_t L^3_x} \leq C \| \phi(\cdot, 0) \|_{H^s}
\]

for \( \phi \) satisfying equation 1.1 in the case \( n = 3 \). This leads us to consider estimate 1.7. Estimates similar to 1.7 were also considered in Grillakis and Margetis [13], and Klainerman and Machedon [17] in the setting of interacting Boson systems.

**Remark 7.** In Bongioanni and Torrea [3], Bongioanni and Rogers [4], and Thangavelu [25], \( \| (-\Delta + |x|^2)^{s} f \|^2_2 \) is called the Hermite-Soblev \( \mathcal{H}^s \) norm of \( f \). In sections 4 and 5 we will need the following lemma proved by Thangavelu concerning Hermite-Soblev spaces in [25].

**Lemma 1.** [25] The operator \( (I - \Delta)^{s/2} (-\Delta + |x|^2)^{s} \) is bounded on \( L^2(\mathbb{R}^n) \) for \( s \geq 0 \), or in other words

\[
\| (I - \Delta)^{s/2} f \|_2 \leq C \| (-\Delta + |x|^2)^{s} f \|_2, \text{ } s \geq 0.
\]
Theorem 1 will be deduced from the theorems below.

**Theorem 3.** Let $u$ be the solution to equation (1.1) in the case $n \geq 2$, then $\forall \delta \in [0, 1)$, one has the estimate
\[
\int_{\mathbb{R}^n} \int_{0}^{2\pi} \frac{|u(x,t)|^2}{(x_1^2 + x_2^2)^\delta} \, dx \, dt \leq C \|f\|_2^2,
\]
which implies estimate (1.6) in the case $n \geq 3$ and $\delta \in [0, 1)$.

**Theorem 4.** Say $g(-x) = -g(x) \in L^2(\mathbb{R})$, then we have the equality
\[
\int_{\mathbb{R}} \int_{0}^{2\pi} \frac{e^{-i\xi(-\Delta+|x|^2)\delta}}{|x|^2} \psi^2 \, dx \, dt = 4\pi \|g\|_2^2.
\]
In particular, estimate (1.8) is equivalent to estimate (1.6) in the 3d radial case i.e.
\[
\int_{\mathbb{R}^3} \int_{0}^{2\pi} \frac{e^{-i\xi(-\Delta+|x|^2)\delta}}{|x|^2} \psi^2 \, dx \, dt = 4\pi \|\psi\|_{L^2(\mathbb{R}^3)}^2
\]
if $\psi$ is a $L^2(\mathbb{R}^3)$ radial function.

**Remark 8.** There is an identity similar to equality (1.8) for the free Schrödinger equation (1.2). See the expository note [7].

**Theorem 5.** Say $d(\pm x_1, \pm x_2, \pm x_3) = d(x_1, x_2, x_3) \in L^2(\mathbb{R}^3)$, then one has the estimate
\[
\int_{\mathbb{R}^3} \int_{0}^{2\pi} \frac{e^{-i\xi(-\Delta+|x|^2)\delta}}{|x|^2} \psi^2 \, dx \, dt \leq C \|d\|_2^2.
\]
Because we can write $f$ as a sum of its $x_1$-odd part and $x_1$-even part by defining
\[
f_{odd}(x) = \frac{f(x_1, x_2, x_3) - f(-x_1, x_2, x_3)}{2}
\]
and
\[
f_{even}(x) = \frac{f(-x_1, x_2, x_3) + f(x_1, x_2, x_3)}{2}.
\]
So
\[
f(x) = f_{odd}(x) + f_{even,odd}(x) + f_{even,even,odd}(x) + f_{even,even,even}(x)
\]
if we iterate the procedure three times. The linearity of equation (1.1) and the fact that the terms in (1.8) are all linear combinations of $f$ shows that estimate (1.6) in the case when $n = 3$ and $\delta = 1$ indeed follows from theorems 4 and 5.

Moreover, theorems 1 and 3 are equivalent to the following uniform $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ boundedness result for a family of singularized Hermite projection kernels.

**Theorem 6.** For $n \geq 3$ and $\delta \in [0, 1)$, $(\delta \in [0, 1)$ when $n = 2)$, the singularized Hermite projection kernels $\left\{ \Phi_k(x,y) \right\}_{k} \int_{\mathbb{R}^n} \Phi_k(x,y) \, dy$ map $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ uniformly where $\Phi_k$ is the usual Hermite projection kernel with respect to the $k$-eigenspace defined in lemma 4, in other words, there exists a $C > 0$ depending only on $\delta$ and $n$ such that
\[
\left\| \int_{\mathbb{R}^n} \Phi_k(x,y) \, dy \right\|_2 \leq C \|f\|_2.
\]
Moreover the more singular family \( \{ \Phi_k(x,y) \}_{k \in \mathbb{N}} \) also maps \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) uniformly via the standard \( TT^\ast \) method.

Regularized Hermite projection kernels were studied in [3], [19], [20], and [24]. But, to the best of the author’s knowledge, theorem 6 might be the first result on the singularized Hermite projection kernels.

2. Some basics of Hermite functions and the proof of theorem 3

To prove theorems 1, 3, 4, 5 and 6, we will need the Hermite functions and some of their properties. For more details outside of lemma 5 whose proof is provided in the appendix I, we refer the reader to Thangavelu’s monograph [24].

**Definition 1.** [24] We define an \( n \)-dimensional Hermite function \( \Phi_\alpha(x) \) where \( \alpha \) is an \( n \)-multiindex by

\[
\Phi_\alpha(x) = \prod_{i=1}^{n} h_{\alpha_i}(x_i),
\]

where \( h_k \) are the one dimensional normalized Hermite functions defined by

\[
h_k(t) = \frac{(-1)^k}{(2^k k! \sqrt{\pi})^{1/2}} e^{\frac{t^2}{2}} dt, \quad t \in \mathbb{R}.
\]

Then we have the following well-known properties.

**Lemma 2.** [24] \( \Phi_\alpha \) are the eigenfunctions of the Fourier transform with eigenvalues \((-i)^{|\alpha|}\) i.e.

\[
\hat{\Phi}_\alpha(\xi) = (-i)^{|\alpha|} \Phi_\alpha(\xi)
\]

**Lemma 3.** [24] \( \Phi_\alpha \) are also the eigenfunctions of the Hermite operator \(-\Delta + |x|^2\) with eigenvalues \(2|\alpha| + n\). Moreover they form an orthonormal basis of \( L^2(\mathbb{R}^n) \).

As this lemma states, we can write

\[
u(x,t) = \sum_\alpha e^{-i\lambda_\alpha t} a_\alpha \Phi_\alpha(x), \quad (2.1)
\]

where \( a_\alpha \) are the Fourier-Hermite coefficients

\[
a_\alpha = \int_{\mathbb{R}^n} f(x) \Phi_\alpha(x) dx,
\]

with convergence in \( L^2(\mathbb{R}^n) \), i.e. \( u \) is naturally periodic \( 2\pi \) in the time variable \( t \) and we have

\[
\int_0^{2\pi} |\nu(x,t)|^2 dt = \int_0^{2\pi} \sum_{\alpha, \beta} e^{-i(\lambda_\alpha - \lambda_\beta)t} a_\alpha \overline{a_\beta} \Phi_\alpha(x) \Phi_\beta(x) dt
\]

\[
= 2\pi \sum_{\alpha, \beta} a_\alpha \overline{a_\beta} \Phi_\alpha(x) \Phi_\beta(x).
\]

But

\[
\int \sum_{\alpha, \beta} a_\alpha \overline{a_\beta} \Phi_\alpha(x) \Phi_\beta(x) dx_j = \sum_{\alpha, \beta} \delta_{a_j, \beta} a_\alpha \overline{a_\beta} \prod_{i=1}^{n} h_{\alpha_i}(x_i) \prod_{i=1}^{n} h_{\beta_i}(x_i).
\]
that is:

\[
\int_0^{2\pi} \int_{\mathbb{R}^{n+1}} \frac{|u(x,t)|^2}{|x|^{2\delta}} dx dt \leq 2\pi \int_{\mathbb{R}^n} \frac{1}{|x|^{2\delta}} \sum_{\alpha,\beta} \delta_{a_\alpha a_\beta} a_\alpha a_\beta \prod_{i=1}^n h_{a_\alpha}(x_i) \prod_{i=1}^n h_{\beta}(x_i) dx
\]

hence the estimate

\[
\int_0^{2\pi} \int_{\mathbb{R}^2} \frac{|u(x,t)|^2}{|x|^{2\delta}} dx dt \leq C \|f\|^2_2 \quad (2.2)
\]

for \(\delta \in [0,1)\) implies theorem 3 and we only need to prove theorem 1 in the case \(n = 3\) and \(\delta = 1\). We can now prove estimate 2.2 by the following lemma.

**Lemma 4.** [24] Let \(P_k\) be the Hermite projector corresponding to the \(k\)-eigenspace with kernel

\[
\Phi_k(x,y) = \sum_{|\alpha| = k} \Phi_\alpha(x) \Phi_\alpha(y)
\]

then there is a constant \(C \geq 0\) independent of \(k\) and \(x\) such that

\[
|\Phi_k(x,x)| \leq C k^{\frac{n}{2} - 1}. \quad (2.3)
\]

Therefore

\[
\int_0^{2\pi} \int_{\mathbb{R}^2} \frac{|u(x,t)|^2}{|x|^{2\delta}} dx dt \leq 2\pi \sum_k \int_{\mathbb{R}^2} \frac{|P_k f|^2}{|x|^{2\delta}} dx
\]

\[
\leq 2\pi \sum_k \int_{\mathbb{R}^2 - D} \frac{|P_k f|^2}{|x|^{2\delta}} dx + 2\pi \sum_k \int_{D} \frac{|P_k f|^2}{|x|^{2\delta}} dx
\]

\[
\leq 2\pi \|f\|^2_2 + 2\pi \sum_k \int_D \frac{1}{|x|^{2\delta}} \left( \sum_{|\alpha| = |\beta| = k} |a_\alpha|^2 |a_\beta|^2 \right)^{\frac{1}{2}} \left( \sum_{|\alpha| = |\beta| = k} |\Phi_\alpha(x)\Phi_\beta(x)|^2 \right)^{\frac{1}{2}} dx
\]

\[
\leq 2\pi \|f\|^2_2 + C \sum_k \left( \int_{\mathbb{R}^2} \frac{1}{r^{2\delta - 1}} dr d\theta \right)^{\frac{1}{2}} \int_0^{2\pi} \frac{1}{r^{2\delta - 1}} dr d\theta
\]

\[
\leq C \|f\|^2_2.
\]

**Remark 9.** In the above computation, we have also proved the Morawetz inequality

\[
\sup_{x \in \mathbb{R}^2} \int_0^{2\pi} |u(x,t)|^2 dt \leq C \|f\|^2_2
\]

which is identical to the well-known version for the free Schrödinger equation 1.2 in [8]. This is another Kato type smoothing estimate.

**Remark 10.** Estimate (2.3) is also the key ingredient to prove the regularized Hermite projection kernel estimates in [20]. But it does not yield theorem 1 in the case when \(n = 3\) and \(\delta = 1\). Lemma 4 will introduce a new tool for that purpose.
As
\[ \int_0^{2\pi} \int_{\mathbb{R}^3} \frac{|u(x,t)|^2}{|x|^{2\delta}} dx \, dt = 2\pi \sum_k \int_{\mathbb{R}} \frac{|P_k f|^2}{|x|^{2\delta}} \, dx, \]

theorem 6 implies theorems 1 and 3. However, the fact that \( e^{-i(2k+n)t} P_k f \) satisfies equation 1.1 with \( u(x,0) = P_k f(x) \) shows that theorems 1 and 3 also imply theorem 6.

We are left with the proofs of theorems 4 and 5 which will need the following tool.

**Lemma 5.** We define the "antiderivatives" of the 1-d Hermite functions to be
\[ X_{2k+1}(x) = \int_{-\infty}^{x} h_{2k+1}(t) \, dt \]
and
\[ X_{2k}(x) = \int_{-\infty}^{x} \operatorname{sign}(t) h_{2k}(t) \, dt \]
which are by definition absolutely continuous. Moreover
\[ \int_{\mathbb{R}} (X_{2k+1}(x))^2 \, dx = 2 \quad (2.5) \]
and
\[ \int_{\mathbb{R}} (X_{2k}(x))^2 \, dx = 2 \left( -1 + \sqrt{2} \sum_{i=0}^{k} \left( \frac{1}{2^i} \right) \right) \quad (2.6) \]
\[ \leq 3 \]
where
\[ \sum_{i=0}^{\infty} \left( \frac{1}{2^i} \right) = \sqrt{2} \]

i.e. \( \lim_{k \to \infty} \|X_{2k}\|_2^2 = 2 \) and \( X_k \in H^1(\mathbb{R}) \).

To the best of our knowledge, lemma 5 is new. The proof, which is a direct computation, is provided in the appendix I for completion. Now we can give the proofs of theorems 4 and 5.

**3. Proof of theorem 4**

We only need to prove
\[ \int_{\mathbb{R}} \frac{|P_{2k+1} g|^2}{|x|^2} \, dx = 2 \left| a_{2k+1} \right|^2 \]
because \( g(-x) = - g(x) \) implies \( a_{2k} = 0 \), \( \forall k \). In fact since \( h_{2k}(x) \) is even, we have
\[ a_{2k} = \int_{\mathbb{R}} g(x) h_{2k}(x) \, dx = 0. \]

One notices that
\[ h_{2k+1}(\xi) = \frac{d}{dx} X_{2k+1}(\xi) \]

where
\[ X_{2k+1}(x) = \int_{-\infty}^{x} h_{2k+1}(t) \, dt \]
(-i)^{-(2k+1)}h_{2k+1}(x) = x\hat{X}_{2k+1}(x)

hence

\int_{\mathbb{R}} \frac{|P_{2k+1}g|^2}{|x|^2} dx = |a_{2k+1}|^2 \int_{\mathbb{R}} |\hat{X}_{2k+1}(x)|^2 dx

= |a_{2k+1}|^2 \int_{\mathbb{R}} |X_{2k+1}(x)|^2 dx

= 2|a_{2k+1}|^2

via equality 2.5 Whence we have deduced theorem 4.

4. PROOF OF THEOREM 5

It suffices to prove that there exists a $C > 0$ independent of $k$ s.t.

\[ \int_{\mathbb{R}^3} \frac{|P_kd|^2}{|x|^2} dx \leq C \|P_kd\|_2^2. \]

Throughout this section, we will assume $k \neq 0$. In the case when $k = 0$, $P_kd$ has only one term

\[ P_0d = a_0h_0(x_1)h_0(x_2)h_0(x_3) = a_0 (\sqrt{\pi})^{-\frac{3}{2}} e^{-|x|^2} \]

and hence is a 3d radial function, and we dealt with this situation in theorem 4. In fact, it is easy to compute that

\[ \int_{\mathbb{R}^3} \frac{|P_0d|^2}{|x|^2} dx = 2|a_0|^2 \]

which matches theorem 4.

We write

\[ P_kd(x) = \sum_{\substack{\alpha, \alpha_1, \alpha_2, \alpha_3 \text{ are even} \\alpha \in I}} a_\alpha h_{\alpha_1}(x_1)h_{\alpha_2}(x_2)h_{\alpha_3}(x_3) \]

\[ = \sum_{\alpha \in I} + \sum_{\alpha \in I I} + \sum_{\alpha \in I I I} \]

where

\[ I = \{ \alpha : |\alpha| = k, \alpha_1, \alpha_2, \alpha_3 \text{ are even, and } \alpha_1 \geq \frac{\alpha_2 + \alpha_3}{2} \} \]

\[ II = \{ \alpha : |\alpha| = k, \alpha_1, \alpha_2, \alpha_3 \text{ are even, and } \alpha_2 \geq \frac{\alpha_1 + \alpha_3}{2} \} \]

\[ III = \{ \alpha : |\alpha| = k, \alpha_1, \alpha_2, \alpha_3 \text{ are even, and } \alpha_3 \geq \frac{\alpha_1 + \alpha_2}{2} \} \].
Remark 11. Suppose we have \( \alpha \) s.t.
\[
\alpha_1 < \frac{\alpha_2 + \alpha_3}{2}, \alpha_2 < \frac{\alpha_1 + \alpha_3}{2}, \text{and} \alpha_3 < \frac{\alpha_2 + \alpha_1}{2},
\]
then \( \alpha_1 + \alpha_2 + \alpha_3 < \alpha_1 + \alpha_2 + \alpha_3 \) which is a contradiction. So I, II, and III covers all cases. In some cases, I, II, III do not intersect trivially. In these cases, we just count the crossing terms once in one proper set. Moreover \( a_\alpha = 0, \forall \alpha \) with one odd index due to \( d(\pm x_1, \pm x_2, \pm x_3) = d(x_1, x_2, x_3) \), in fact since \( h_{\alpha_1}(x_1) \) is odd if \( \alpha_1 \) is odd, we have
\[
a_\alpha = \int_{\mathbb{R}^2} dx_2 dx_3 \int_{\mathbb{R}} dx_1 d(x_1) h_{\alpha_1}(x_1) h_{\alpha_2}(x_2) h_{\alpha_3}(x_3)
= \int_{\mathbb{R}^2} dx_2 dx_3 h_{\alpha_2}(x_2) h_{\alpha_3}(x_3) \cdot 0
= 0
\]
So it is enough to prove
\[
\int_{\mathbb{R}^3} \left| \frac{\sum_{\alpha \in I} a_\alpha}{|x|^2} \right|^2 \, dx \leq C \sum_{\alpha \in I} |a_\alpha|^2, \quad (4.1)
\]
\[
\int_{\mathbb{R}^3} \left| \frac{\sum_{\alpha \in II} a_\alpha}{|x|^2} \right|^2 \, dx \leq C \sum_{\alpha \in II} |a_\alpha|^2, \quad (4.2)
\]
and
\[
\int_{\mathbb{R}^3} \left| \frac{\sum_{\alpha \in III} a_\alpha}{|x|^2} \right|^2 \, dx \leq C \sum_{\alpha \in III} |a_\alpha|^2. \quad (4.3)
\]
In the following, we will only prove estimate 4.1 and the proofs of estimates 4.2 and 4.3 will be similar. To be more specific, we will use \( \alpha_1 \) and \( x_1 \) for estimate 4.1, \( \alpha_2 \) and \( x_2 \) for estimate 4.2, \( \alpha_3 \) and \( x_3 \) for estimate 4.3.
Define
\[
u_{k,I}(\xi) = \sum_{\alpha \in I} a_\alpha X_{\alpha_1}(\xi_1) h_{\alpha_2}(\xi_2) h_{\alpha_3}(\xi_3)
\]
then
\[
\|u_{k,I}\|_2^2
= \sum_{\alpha, \beta \in I} \int_{\mathbb{R}} d\xi_1 \int_{\mathbb{R}^2} d\xi_2 d\xi_3 a_{\alpha} a_{\beta} X_{\alpha_1}(\xi_1) h_{\alpha_2}(\xi_2) h_{\alpha_3}(\xi_3) X_{\beta_1}(\xi_1) h_{\beta_2}(\xi_2) h_{\beta_3}(\xi_3)
= \sum_{\alpha \in I} |a_{\alpha}|^2 \int_{\mathbb{R}} (X_{\alpha_1}(\xi_1))^2 d\xi_1
\leq 3 \sum_{\alpha \in I} |a_{\alpha}|^2,
\]
via formula 2.7.
Moreover,
\[
\sum_{\alpha \in I} a_\alpha h_{\alpha_1}(\xi_1) h_{\alpha_2}(\xi_2) h_{\alpha_3}(\xi_3) = \text{sign}(\xi_1) \frac{\partial}{\partial \xi_1} u_{k,I}(\xi), \xi_1 \neq 0
\]
where \( H \) is the Hilbert transform only with respect to the the first variable. Hence we have

\[
\int \left| \sum_{\alpha \in I} a_\alpha h_{\alpha_1}(x_1)h_{\alpha_2}(x_2)h_{\alpha_3}(x_3) \right|^2 dx \leq \int \left| x_1 H(\hat{u}_{k,1}(x)) - \int_{-\infty}^\infty \hat{u}_{k,1}(t,x_2,x_3)dt \right|^2 dx
\]

\[
\leq 2 \int \frac{|x_1 H(\hat{u}_{k,1}(x))|^2}{|x|^2} dx + 2 \int \frac{\int_{-\infty}^\infty |\hat{u}_{k,1}(t,x_2,x_3)|^2 dt}{|x|^2} dx_1dx_2dx_3
\]

\[
\leq 6 \sum_{\alpha \in I} |a_\alpha|^2 + 2\pi \int \left| \int_{-\infty}^\infty |\hat{u}_{k,1}(t,x_2,x_3)|^2 dt \right|^2 \frac{dx_2dx_3}{\sqrt{x_2^2 + x_3^2}}
\]

\[
= 6 \sum_{\alpha \in I} |a_\alpha|^2 + 2\pi (\text{MainTerm}_I)
\]

where

\[\text{MainTerm}_I = \int \int_{\mathbb{R}^2} |\hat{u}_{k,1}(t,x_2,x_3)|^2 \frac{dx_2dx_3}{\sqrt{x_2^2 + x_3^2}}\]

\[
= \int \sum_{\alpha \in I} \int_{-\infty}^\infty dt \int e^{\imath \xi_1(e^{\imath x_2} \xi_2 e^{\imath x_3} \xi_3) a_\alpha X_{\alpha_1}(\xi_1)}h_{\alpha_2}(\xi_2)h_{\alpha_3}(\xi_3) d\xi_1 dt \frac{dx_2dx_3}{\sqrt{x_2^2 + x_3^2}}
\]

\[
= \int \sum_{\alpha \in I} a_\alpha X_{\alpha_1}(0)(-\imath)^{\alpha_2 + \alpha_3}h_{\alpha_2}(x_2)h_{\alpha_3}(x_3) \frac{dx_2dx_3}{\sqrt{x_2^2 + x_3^2}}
\]

\[
\leq C \int \left( \nabla^2 \left( \sum_{\alpha \in I} a_\alpha X_{\alpha_1}(0)(-\imath)^{\alpha_2 + \alpha_3}h_{\alpha_2}(x_2)h_{\alpha_3}(x_3) \right) \right)^2 dx_2dx_3 \quad \text{(Hardy's inequality)}
\]

\[
\leq C \int \left( -\Delta + |(x_2,x_3)|^2 \right)^\frac{3}{2} \left( \sum_{\alpha \in I} a_\alpha X_{\alpha_1}(0)(-\imath)^{\alpha_2 + \alpha_3}h_{\alpha_2}(x_2)h_{\alpha_3}(x_3) \right)^2 dx_2dx_3 \quad \text{( Lemma 1)}
\]

\[
= C \sum_{\alpha \in I} |a_\alpha|^2 (X_{\alpha_1}(0))^2 (2\alpha_2 + 2\alpha_3 + 2) \frac{3}{2}.
\]

However, from Feldheim [11] and Busbridge [5], we know that given \( \alpha_1 \) even
by Stirling’s formula. The above inequality shows
\[(X_{\alpha_1}(0))^2 = \frac{1}{4}(\int h_{\alpha_1}(t)dt)^2 = \frac{\sqrt{2}2^{2\alpha_1}(\Gamma(\frac{1}{2}\alpha_1 + \frac{1}{2}))^2}{4\alpha_1!\sqrt{\pi}} \leq C\frac{1}{(\alpha_1)\frac{1}{2}}\]
for \(\alpha_1 \geq \frac{\alpha_2 + \alpha_3}{2}\) and \(\alpha_1 \neq 0\), or in other words, for \(\alpha \in I\) and \(k \neq 0\).

So
\[\text{MainTerm}_I \leq C \sum_{\alpha \in I} |a_{\alpha}|^2\]
i.e.
\[\int_{\mathbb{R}^3} \frac{|\sum_{\alpha \in I} j|^2}{|x|^2} dx \leq C \sum_{\alpha \in I} |a_{\alpha}|^2\]

**Remark 12.** If we apply this procedure to the case when \(n = 2\) and \(\delta = 1\), \(\text{MainTerm}_I\) will have \(|x_2|^{-1}\) as a singularity which forces \(\text{MainTerm}_I\) to be \(\infty\) whenever there is some \(a_{\alpha} \neq 0\). But this procedure does also prove estimate \(2.2\) when \(\delta < 1\) and hence theorem 3.

### 5. An Application of Theorem 1 / Proof of Theorem 2

To obtain theorem 2 aside from theorem 1 and lemma 1, an interaction Morawetz inequality is needed.

#### 5.1. Morawetz inequality
As in \[8\], define
\[T_{00} = |u|^2, \quad T_{0j} = T_{j0} = 2 \text{Im} \frac{\partial u}{\partial x_j}, \quad T_{jk} = T_{kj} = 4 \text{Re}(u_k \overline{u_j}) - \delta_{jk} \Delta (|u|^2)\]
where \(j, k\) mean summation from 1 to \(n\). Then a direct computation shows that
\[\partial_t T_{00} + \partial_j T_{0j} = 0, \quad \partial_t T_{0\alpha} + \partial_j T_{j\alpha} = -2V_k |u|^2, \quad \partial_t T_{\alpha\alpha} + \partial_j T_{j\alpha} = -2V_k |u|^2, \quad \partial_t T_{\alpha\beta} + \partial_j T_{j\beta} = -2V_k |u|^2, \]
for the equation
\[iu_t = -\Delta u + Vu.\]
Hence we have

\[
\partial_t M_0^a(t) = 4 \int_{\mathbb{R}^n} a_{kj} \text{Re}(u_k \overline{u_j}) dx - \int_{\mathbb{R}^n} \Delta a \Delta(|u|^2) dx - 2 \int_{\mathbb{R}^n} a_k V_k |u|^2 dx.
\]

\[
= 4 \int_{\mathbb{R}^n} a_{kj} \text{Re}(u_k \overline{u_j}) dx - \int_{\mathbb{R}^n} \Delta a \Delta(|u|^2) dx + 2 \int_{\mathbb{R}^n} a(x) V_{kk} |u|^2 dx
\]

if we define

\[
M_0^a(t) = \int_{\mathbb{R}^n} a_k(x) T_{0k}(t, x) dx
\]

to be the Morawetz action corresponding to a suitable \(a(x)\) which will be chosen momentarily.

Therefore, for equation (1.1) in the case \(n = 9\), we in fact have

\[
\begin{align*}
\int_0^{2\pi} \int_{\mathbb{R}^9} (\Delta^2 a) |u|^2 dxdt &= 4 \int_0^{2\pi} \int_{\mathbb{R}^9} a_{kj}(x) \text{Re}(u_k \overline{u_j}) dxdt \\
&\quad + 36 \int_0^{2\pi} \int_{\mathbb{R}^9} a(x) |u|^2 dxdt \\
&\quad + 4 \int_0^{2\pi} \int_{\mathbb{R}^9} a(x) V_{k} \text{Re}(u_k \overline{u_k}) dx
\end{align*}
\]

due to the facts that

\[
\int_0^{2\pi} \partial_t M_0^a(t) dt = 0
\]

and \(V_{kk} = 18\).

The \(a(x)\) we are going to pick is not non-strictly convex as in the usual cases in [8], but the following computation will help to simplify the technical problems arisen
from that:

\[
2 \int_{\mathbb{R}^3} a_{kj} \Re(u_k \overline{u_j}) dx
= \int_{\mathbb{R}^3} a_{kj} (u_k \overline{u_j} + u_j \overline{u_k}) dx
= - \int_{\mathbb{R}^3} a_k (u_{kj} \overline{u} + u_j \overline{u_k}) dx - \int_{\mathbb{R}^3} a_k (u_k \overline{\triangle u} + \overline{u_k} \triangle u) dx
= - \int_{\mathbb{R}^3} a_k (|u_j|^2) dx + \int_{\mathbb{R}^3} a (u_{kk} \overline{\triangle u} + u_k \overline{\triangle u_k} + \overline{u_k} \triangle u_k + \overline{u_k} \triangle u_k) dx
= \int_{\mathbb{R}^3} \triangle a |\nabla u|^2 dx + 2 \int_{\mathbb{R}^3} a |\triangle u|^2 dx + \int_{\mathbb{R}^3} a (\triangle |u_k|^2 - 2 |\nabla u_k|^2) dx
= 2 \int_{\mathbb{R}^3} \triangle a |\nabla u|^2 dx + 2 \int_{\mathbb{R}^3} a |\triangle u|^2 dx - 2 \int_{\mathbb{R}^3} a |\nabla^2 u|^2 dx.
\]

So

\[
2 \pi \int_0^{2\pi} \left( \int_{\mathbb{R}^3} (\triangle^2 a) |u|^2 dx dt \right) = 4 \int_0^{2\pi} \int_{\mathbb{R}^3} \triangle a |\nabla u|^2 dx dt \quad (5.1)
\]

\[
+ 4 \int_0^{2\pi} \int_{\mathbb{R}^3} a(x) |\triangle u|^2 dx dt
- 4 \int_0^{2\pi} \int_{\mathbb{R}^3} a(x) |\nabla^2 u|^2 dx dt
+ 36 \int_0^{2\pi} \int_{\mathbb{R}^3} a(x) |u|^2 dx dt
+ 4 \int_0^{2\pi} \int_{\mathbb{R}^3} a(x) V_k \Re(u \overline{u_k}) dx dt
\]

If we select

\[
a(x_1, x_2, x_3) = C \frac{1}{|x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2} \quad (5.2)
\]

where \( C \) is a suitable positive constant, then

\[
\triangle^2 a = \delta(x_1 - x_2) \delta(x_2 - x_3),
\]

\[
\triangle a(x_1, x_2, x_3) = -C \frac{1}{(|x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2)^2} < 0,
\]
and relation [5.1] reads
\[ \int_0^{2\pi} \int_{\mathbb{R}^3} |u(x, x, x, t)|^2 \ dx \ dt \leq 4 \int_0^{2\pi} \int_{\mathbb{R}^9} a |\triangle u|^2 \ dx \ dt + 36 \int_0^{2\pi} \int_{\mathbb{R}^9} a(x) |u|^2 \ dx \ dt \]
\[ + 4 \int_{\mathbb{R}^9} a(x) V_k \text{Re}(u \overline{u}) \ dx \ dt \]
\[ = 4A + 36B + 4D \]

**Remark 13.** Formula 5.2 is from Klainerman and Machedon’s private communication [16]. Thanks to Machedon for sharing this computation.

To prove estimate 1.7, it will suffice to show that \( A, B, \) and \( D \) are majorized by
\[ \| (\triangle + |x|^2) f \|_2^2. \]

### 5.2. Estimates for \( A, B, \) and \( D. \)

\[ A = \int_0^{2\pi} \int_{\mathbb{R}^9} a |\triangle u|^2 \ dx \ dt \]
\[ = C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|\triangle u|^2}{|x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2} \ dx \ dt \]
\[ \leq C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|\triangle u|^2}{|x_1 - x_2|^2} \ dx \ dt. \]

due to the well-known change of variables \( x_1 \to \frac{x_1 - x_2}{\sqrt{2}}, x_2 \to \frac{x_1 + x_2}{\sqrt{2}} \) which is compatible with \( (\triangle) \) and \( (\triangle + |x|^2) \), we only need to estimate

\[ \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|\triangle u|^2}{|x_1|^2} \ dx \ dt \] (5.3)
\[ \leq C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|(\triangle + |x|^2) u|^2}{|x_1|^2} \ dx \ dt + C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|x|^2 u|^2}{|x_1|^2} \ dx \ dt \]
\[ \leq C \left\| (\triangle + |x|^2) f \right\|_2^2 + C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|x_1|^4 |u|^2}{|x_1|^2} \ dx \ dt + C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|(x_2, x_3)|^2 u|^2}{|x_1|^2} \ dx \ dt \]
\[ \leq C \left\| (\triangle + |x|^2) f \right\|_2^2 + C \left\| \nabla f \right\|_2^2 + E \]
\[ \leq C \left\| (\triangle + |x|^2) f \right\|_2^2 + E \]
where
\[
\begin{align*}
E & \leq C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|(-\triangle_{x_2,x_3} + [(x_2,x_3)]^2)u|^2}{|x_1|^2} \, dx \, dt + C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|\triangle_{x_2,x_3}u|^2}{|x_1|^2} \, dx \, dt \\
& \leq C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|(-\triangle_{x_2,x_3} + [(x_2,x_3)]^2)u|^2}{|x_1|^2} \, dx \, dt \\
& \leq C \left\| (-\Delta + |x|^2)^{\frac{1}{2}} f \right\|^2_2
\end{align*}
\]
due to lemma 1 and theorem 1.

Then it is easy to see that
\[
B = 36 \int_0^{2\pi} \int_{\mathbb{R}^9} a(x) |u|^2 \, dx \, dt
\]
\[
\leq C \int_0^{2\pi} \int_{\mathbb{R}^9} \frac{|u|^2}{|x_1 - x_2|^2} \, dx \, dt
\]
\[
\leq C \left\| f \right\|^2_2
\]
because of theorem 1 and change of variables.
The only term left over is
\[
D = \int_0^{2\pi} \int_{\mathbb{R}^9} a(x) V_k \, \text{Re}(\overline{u_k} u_k) \, dx \, dt.
\]
A typical term in the sum reads
\[
\begin{align*}
\left| \int_0^{2\pi} \int_{\mathbb{R}^9} a(x) V_1 \, \text{Re}(\overline{u_1} u_1) \, dx \right| \\
\leq \left( 4 \int_0^{2\pi} \int_{\mathbb{R}^9} a(x) x_1^2 |u|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^{2\pi} \int_{\mathbb{R}^9} a(x) \left| \frac{\partial}{\partial x_1} u \right|^2 \, dx \, dt \right)^{\frac{1}{2}} \\
\leq C \left\| (-\Delta + |x|^2)^{\frac{1}{2}} f \right\|^2_2
\end{align*}
\]
using the same method as in estimate 5.3.

Hence we conclude
\[
\int_0^{2\pi} \int_{\mathbb{R}^9} |u(x,x,x,t)|^2 \, dx \, dt \leq C \left\| (-\Delta + |x|^2)^{\frac{1}{2}} f \right\|^2_2
\]
which is theorem 2.
Remark 14. If we choose not to ignore $\int_{\mathbb{R}^9} |\nabla u|^2 \, dx dt$ and $\int_{\mathbb{R}^9} a(x) |\nabla^2 u|^2 \, dx dt$ in relation 5.1, then in fact we have proven two additional Kato type smoothing estimates:

\[
\int_{\mathbb{R}^9} \frac{|\nabla u|^2}{(|x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2)^2} \, dx dt \leq C \left\| (\Delta + |x|^2)f \right\|_2^2
\]

and

\[
\int_{\mathbb{R}^9} \frac{|\nabla^2 u|^2}{(|x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2)^2} \, dx dt \leq C \left\| (\Delta + |x|^2)f \right\|_2^2.
\]

6. Appendix: Proof of Lemma 5 / Computation of the $L^2$ norms of the ”antiderivatives” of Hermite functions

In this section, we prove lemma 5 which yields the precise controlling constants. But we shall first prove that there exits a $C > 0$ s.t.

\[\|X_k\|_2^2 \leq C, \forall k\]

before we delve into the proof of lemma 5 which consists of many special function techniques.

6.1. Proof of the $L^2$ boundedness.

Lemma 6. [24] We have the following creation and annihilation relations

\[
\left( \frac{d}{dx} + x \right) \tilde{h}_k(x) = \tilde{h}_{k+1}(x)
\]

\[
\left( \frac{d}{dx} + x \right) \tilde{h}_k(x) = 2k \tilde{h}_{k-1}(x)
\]

where $\tilde{h}_k(x) = \frac{1}{c_k} h_k(x)$, and $c_k = \left( \frac{1}{2^k k! \sqrt{\pi}} \right)^{\frac{1}{2}}$ is the normalization constant, i.e. $\tilde{h}_k(x)$ is the unnormalized Hermite function of degree $k$. In this spirit, one has:

\[
\tilde{h}_{k+1}(x) = -2 \frac{d}{dx} \tilde{h}_k(x) + 2k \tilde{h}_{k-1}(x) \tag{6.1}
\]

or with the normalization factors

\[
h_{k+1}(x) = - \sqrt{\frac{2}{k+1}} \frac{d}{dx} \tilde{h}_k(x) + \sqrt{\frac{k}{k+1}} h_{k-1}(x). \tag{6.2}
\]

We will only consider the even case

\[
V_{2k} = \frac{\|X_{2k}\|^2}{2} = \int_0^\infty \left( \int_x^\infty h_{2k}(t) \, dt \right)^2 \, dx,
\]

since the odd case is similar. Iterating relation 6.2 yields

\[
h_{2k}(t) = \sum_{i=0}^{k-1} b_i \frac{d}{dt} h_{2k-1-2i}(t) + dh_0(t) \tag{6.3}
\]
because

\[
\begin{align*}
  h_{2k}(x) &= -\sqrt{2} \frac{d}{2k \, dx} h_{2k-1}(x) + \sqrt{\frac{2k-1}{2k}} h_{2k-2}(x) \\
  h_{2k-2}(x) &= -\sqrt{\frac{2}{2k-2}} \frac{d}{dx} h_{2k-3}(x) + \sqrt{\frac{2k-3}{2k-2}} h_{2k-4}(x) \\
  &\vdots \\
  h_4(x) &= -\sqrt{\frac{2}{4}} \frac{d}{dx} h_3(x) + \sqrt{\frac{3}{4}} h_2(x) \\
  h_2(x) &= -\sqrt{\frac{2}{2}} \frac{d}{dx} h_1(x) + \sqrt{\frac{1}{2}} h_0(x).
\end{align*}
\]

Therefore

\[
\begin{align*}
  \int_0^\infty \left( \int_x^\infty h_{2k}(t) \, dt \right)^2 \, dx &\leq 2 \int_0^\infty \left( \sum_{i=0}^{k-1} b_i h_{2k-1-2i}(x) \right)^2 \, dx + 2d^2 \int_0^\infty \left( \int_x^\infty h_0(t) \, dt \right)^2 \, dx \\
  &\leq 2 \int_0^\infty \left( \sum_{i=0}^{k-1} b_i h_{2k-1-2i}(x) \right)^2 \, dx + 2d^2 \int_0^\infty \left( \int_x^\infty h_0(t) \, dt \right)^2 \, dx \\
  &= 2 \left( \sum_{i=0}^{k-1} |b_i|^2 \right) + d^2 \int_0^\infty \left( \int_x^\infty h_0(t) \, dt \right)^2 \, dx,
\end{align*}
\]

where

\[
\begin{align*}
  \sum_{i=0}^{k-1} |b_i|^2 &= \frac{2}{2k} + \frac{2k-1}{2k} \frac{2}{2k-2} + \ldots + \frac{2k-1}{2k} \frac{2}{2k} \ldots \frac{2}{2} \\
  &= \frac{1}{k} + \frac{1}{k} \frac{2k-1}{2k} \frac{1}{2k-2} + \ldots + \frac{1}{k} \frac{2k-1}{2k} \frac{1}{2k-2} \frac{3}{2} \\
  &= \frac{1}{k} \sum_{i=0}^{k-1} \left( \prod_{l=0}^{i} \left(1 + \frac{1}{2k-2l}\right) \right).
\end{align*}
\]

Notice that

\[
\begin{align*}
  \ln \prod_{l=0}^{i} \left(1 + \frac{1}{2k-2l}\right) &\sim \sum_{l=0}^{i} \frac{1}{2k-2l} \\
  &\sim \frac{1}{2} \ln \frac{k-1}{k-i}
\end{align*}
\]
which implies
\[
\sum_{i=0}^{k-1} |b_i|^2 
\sim \frac{1}{k} \sum_{i=0}^{k-1} \left( \frac{k-1}{k-i} \right)^{\frac{1}{2}} 
\leq \frac{1}{k} \sum_{i=0}^{k-1} \left( \frac{k}{k-i} \right)^{\frac{1}{2}} 
\leq C \frac{1}{k^{\frac{1}{2}}} \int_0^k \left( \frac{1}{k-x} \right)^{\frac{1}{2}} dx 
\leq C
\]
i.e.
\[
\|X_{2k}\|_2^2 \leq C.
\]

Remark 15. For the odd case, formula 6.3 will read
\[
h_{2k+1}(t) = \sum_{i=0}^{k-1} b_i \frac{d}{dt} h_{2k-2i}(t) + dh_1(t).
\]

6.2. Proof of equalities 2.5 and 2.6. Below we will refer to the following lemmas as well as lemma 6.

Lemma 7. Write the degree \(k\) Hermite polynomial \(e^{\frac{x^2}{2}} \hat{h}_k(x)\) as \(H_k\),
\[
H_k(x) = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{k!}{i!(k-2i)!} (-1)^i (2x)^{k-2i}
\]
then every polynomial \(p(x)\) of degree \(\leq i\) is a finite linear combination of \(H_k, k \leq i\),
\[
p(x) = \sum_{k=0}^{i} \left( \int \hat{h}_k(x)p(x)e^{-\frac{1}{2}x^2} dx \right) H_k(x)
\]
In particular, given any polynomial \(p(x)\) of degree \(< k\), we have:
\[
\int \hat{h}_k(x)p(x)e^{-\frac{1}{2}x^2} dx = 0
\]
Proof. The first part of the statement is a well-known fact. To prove the second part, one only needs to notice that \(p(x)e^{-\frac{1}{2}x^2}\) is a finite linear combination of \(\hat{h}_i(x) = H_i e^{-\frac{1}{2}x^2}, i < k\), and then apply orthogonality.

Lemma 8. [24] If we define the degree \(k\) Laguerre polynomial of type \(\alpha\) by
\[
e^{-x}x^{\alpha} L_k^\alpha(x) = \frac{1}{k!} \frac{d^k}{dx^k} (e^{-x}x^\alpha),
\]
then
\[
H_{2k+1} = (-1)^k 2^{2k+1} k! L_k^\frac{1}{2}(x^2).x.
\]

Remark 16. Formula [6.4] is (1.1.53) in Thangavelu [24]. He missed a factor 2 on the right hand side. One can refer to page 1001 of [24].
At this point we can give the proof of formula 2.5.

6.2.1. **Proof of the odd formula 2.5.** By relation 6.1 we have
\[ \tilde{h}_{2k+1}(x) = -2 \frac{d}{dx} \tilde{h}_{2k}(x) + 4k \tilde{h}_{2k-1}(x) \]
and hence
\[ (\int_{-\infty}^{x} \tilde{h}_{2k+1}(t)dt)^2 = 4(\tilde{h}_{2k}(x))^2 - 16k \tilde{h}_{2k}(x) \int_{-\infty}^{x} \tilde{h}_{2k-1}(t)dt + 16k^2(\int_{-\infty}^{x} \tilde{h}_{2k-1}(t)dt)^2 \]
or with the normalization factors
\[ (X_{2k+1}(x))^2 = \frac{2}{2k+1}(h_{2k}(x))^2 - Junk(x) + \frac{2k}{2k+1}(X_{2k-1}(x))^2 \]
where
\[ Junk(x) = \frac{16k}{(c_{2k+1})^2} \tilde{h}_{2k}(x) \int_{-\infty}^{x} \tilde{h}_{2k-1}(t)dt. \]
So
\[ I_{2k+1} = \frac{2}{2k+1} + \int_{\mathbb{R}} Junk(x) dx + \frac{2k}{2k+1} I_{2k-1} \]
where
\[ I_{2k+1} = \|X_{2k+1}\|_2^2 \]
which is our target.

But \( \tilde{h}_{2k-1}(t) = e^{-\frac{t^2}{2}}(\sum_{i=0}^{k} b_{2i-1}x^{2i-1}) \), so \( \int_{-\infty}^{x} \tilde{h}_{2k-1}(t)dt = e^{-\frac{x^2}{2}}(\sum_{i=0}^{k-1} l_{2i}x^{2i}) \),
which implies \( \int_{\mathbb{R}} Junk(x) dx = 0 \) by lemma 7. Hence
\[ I_{2k+1} = \frac{2}{2k+1} + \frac{2k}{2k+1} I_{2k-1}. \]
(6.5)
The equalities that
\[ I_1 = \int_{-\infty}^{\infty} (\int_{-\infty}^{x} h_1(t)dt)^2 dx = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} (\int_{-\infty}^{x} 2xe^{-t^2} dt)^2 dx = 2 \]
and relation (6.5) tell us
\[ I_{2k+1} = 2. \]

6.2.2. **Proof of the even formula 2.6.** Applying relation 6.2 again, we have
\[ V_{2k+2} \]
\[ = \frac{1}{2} \frac{1}{2k+1} + \frac{\sqrt{2}}{2} \frac{2k+1}{2k+2} \int_{0}^{\infty} h_{2k+1}(x)(\int_{x}^{\infty} h_{2k}(t)dt)dx + \frac{2k+1}{2k+2} V_{2k} \]
\[ = \frac{1}{2} \frac{1}{2k+1} - \frac{\sqrt{2}}{2} \frac{2k+1}{2k+1} \int_{0}^{\infty} \frac{d}{dx} \int_{x}^{\infty} h_{2k+1}(t)dt)(\int_{x}^{\infty} h_{2k}(t)dt)dx + \frac{2k+1}{2k+2} V_{2k} \]
Just as the odd case, we are concerned with the middle term and would like to have
an explicit formula for it. Integrating by parts once, we have

\[
\int_0^\infty \left( \frac{d}{dx} \int_x^\infty h_{2k+1}(t) dt \right) (\int_x^\infty h_{2k}(t) dt) dx
\]

\[
= -(\int_0^\infty h_{2k+1}(t) dt) (\int_x^\infty h_{2k}(t) dt) + \int_0^\infty (\int_x^\infty h_{2k+1}(t) dt) h_{2k}(x) dx.
\]

Recall that we already know

\[
\int_0^\infty h_{2k}(t) dt = \frac{1}{2} \frac{2^{2k} \Gamma(\frac{1}{2}(2k) + \frac{1}{2})}{\sqrt{2^{2k}(2k)! \pi}}
\]

\[
= \frac{2^k \Gamma(k + \frac{1}{2})}{\sqrt{(2k)! \pi}}
\]

\[
= \frac{2^{-k+\frac{1}{2}}(\sqrt{\pi})^{\frac{3}{2}} \Gamma(2k)}{\Gamma(k) \sqrt{(2k)!}}
\]

from Feldheim [11], Busbridge [5] and the well-known formula for the gamma function

\[
\Gamma(z + \frac{1}{2}) = \frac{2^{1-2z} \pi^{\frac{1}{2}} \Gamma(2z)}{\Gamma(z)}
\]

so we would like to compute \(\int_0^\infty h_{2k+1}(t) dt\) and \(\int_0^\infty (\int_x^\infty h_{2k+1}(t) dt) h_{2k}(x) dx\). Using lemma [8] we have

\[
\int_0^\infty h_{2k+1}(t) dt = \frac{1}{c_{2k+1}} (-1)^k 2^{2k+1} k! \int_0^\infty L_k^\frac{1}{2}(x^2) e^{-x^2} dx
\]

\[
= \frac{(-1)^k 2^{2k+1} k!}{2 c_{2k+1}} \int_0^\infty L_k^\frac{1}{2}(u) e^{-\frac{u}{2}} du
\]

\[
= \frac{(-1)^k 2^{2k+1} k!}{c_{2k+1}} \sum_{i=0}^k \left( \frac{1}{2} + i - 1 \right) (-1)^{k-i}
\]

\[
= \frac{2^{k+1} k!}{\sqrt{2(2k+1)! \pi}} \sum_{i=0}^k \left( \frac{1}{2} + i - 1 \right) (-1)^i
\]

where the integral part, which has been worked out in page 809 of [12], is that

\[
\int_0^\infty L_k^\frac{1}{2}(u) e^{-\beta u} du = \sum_{i=0}^k \left( \alpha + i - 1 \right) \frac{(\beta - 1)^{k-i}}{\beta^{k-i+1}}.
\]
Hence

\[
\left( \int_0^\infty h_{2k+1}(t)dt \right) \left( \int_0^\infty h_{2k}(t)dt \right)
= \frac{2^{-k+\frac{3}{2}} \Gamma(2k)}{\Gamma(k) \sqrt{2(2k+1)}} \int_0^\infty \sum_{i=0}^{2k+1} \binom{\frac{1}{2} + i - 1}{i} (-1)^i \frac{2^{k+1} k!}{(2^{2k+1})! \sqrt{\pi}} (\frac{1}{2})^i
\]

\[
= \frac{1}{\sqrt{2k+1}} \sum_{i=0}^k \binom{\frac{1}{2} + i - 1}{i} (-1)^i
\]

due to the identity

\[
\binom{\alpha}{k} = \binom{k - \alpha - 1}{k} (-1)^k.
\]

For the last term, we have

\[
\int_0^\infty \int_x^\infty h_{2k+1}(t)h_{2k}(x)dx = \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty h_{2k+1}(t)dt h_{2k}(x)dx
\]

\[
= \frac{1}{2} \frac{\text{(the coefficient of } x^{2k} e^{-\frac{x^2}{2}} \text{ in } \int_x^\infty h_{2k+1}(t)dt)}{\text{(the coefficient of } x^{2k} e^{-\frac{x^2}{2}} \text{ in } h_{2k}(x))}
\]

\[
= \frac{1}{2} \frac{\Gamma(2k+1)}{2^{2k}}
\]

\[
= \frac{1}{\sqrt{2} \sqrt{2k+1}}
\]

via lemma 7.

At long last we have

\[
V_{2k+2} = -\frac{1}{2k+2} + \frac{\sqrt{2}}{k+1} \sum_{i=0}^k \binom{-\frac{1}{2}}{i} + \frac{2k+1}{2k+2} V_{2k}.
\]

Since

\[
\sum_{i=0}^k \binom{-\frac{1}{2}}{i} + \frac{2k+1}{2k+2} \sum_{i=0}^k \binom{\frac{1}{2}}{i} = \sum_{i=0}^{k+1} \binom{\frac{1}{2}}{i},
\]

a straight forward induction gives us formula \(2.6\). This concludes the proof of lemma 5.

REFERENCES

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, E. A. Cornell, Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor, Science, Vol. 269 (1995), 198–201.
[2] M. Ben-Artzi and S. Klainerman, Decay and Regularity for the Schrödinger Equation, J. Anal Math. Vol. 58 (1992), 25-37.
[3] B. Bongioanni and J. L. Torrea, Sobolev Spaces Associated to the Harmonic Oscillator, Proc. Indian Acad. Sci. (Math. Sci.), Vol. 116 (2006), 337-360.
[4] B. Bongioanni and K. M. Rogers, Regularity of the Schrödinger Equation for the Harmonic Oscillator, to appear in Ark. Mat.
[5] I. Busbridge, Some Integrals Involving Hermite Polynomials, J. London Math. Soc. 1948, 135-141.
[6] R. Carles, Global Existence Results for Nonlinear Schrödinger Equations with Quadratic Potentials, Discrete Contin. Dyn. Syst. Vol. 13 (2005), 385-398.
[7] X. Chen, Elementary Proofs for Kato Smoothing Estimates of Schrödinger-like Dispersive Equations, [arXiv:1007.1491v1].
[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Virial, Morawetz, and Interaction Morawetz Inequalities, preprint.
[9] S. Doi, Smoothness of Solutions for Schrödinger Equations with Unbounded Potentials, Publ. RIMS, Kyoto Univ. Vol. 41 (2005), 175-221.
[10] F. J. Dyson, Ground-state Energy of a Hard Sphere Gas. Phys. Rev. Vol. 106 (1957), 20-26.
[11] E. Feldheim, Quelques Nouvelles Relations Pour les Polynomes D’Hermite, J. London Math. Soc. 1938, 22-29.
[12] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products 7th Edition, Academic Press, Orlando, FL, 2007.
[13] M. G. Grillakis and D. Margetis, A Priori Estimates for Many-Body Hamiltonian Evolution of Interacting Boson System, J. Hyperb. Diff. Eqns., Vol. 5 (2008), 857-883.
[14] T. Kato and K. Yajima, Some Examples of Smooth Operators and the Associated Smoothing Effect, Rev. Math. Phys. Vol. 1 (1989), 481-496.
[15] R. Killip, M. Visan, and X. Zhang, Energy-critical NLS with Quadratic Potentials, Comm. PDE. Vol. 34 (2009), 1531-1565.
[16] S. Klainerman and M. Machedon, private communication.
[17] S. Klainerman and M. Machedon, On the Uniqueness of Solutions to the Gross-Pitaevskii Hierarchy, Commun. Math. Phys., Vol. 279 (2008), 169-185.
[18] H. Koch and D. Tataru, $L^p$ Eigenfunction Bounds for the Hermite Operator, Duke Math. J., Vol. 128 (2005), 369-392.
[19] A. K. Nandakumarana and P. K. Ratnakumar, Schrödinger Equation and the Oscillatory Semigroup for the Hermite Operator, J. Funct. Anal., Vol. 224 (2005), 371-385.
[20] P. Petrushev and Y. Xu, Decomposition of Spaces of Distributions Induced by Hermite Expansions, Journal of Fourier Analysis and Applications, Vol. 14 (2008), 372-414.
[21] L. Robbiano and C. Zuily, The Kato Smoothing Effect for Schrödinger Equations with Unbounded Potentials in Exterior Domains, IMRN, Vol. 2009, 1636-1698.
[22] B. Simon, Best Constants in Some Operator Smoothness Estimates, J. Funct. Anal., Vol. 107 (1992), 66-71.
[23] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, and W. Ketterle, Optical Confinement of a Bose-Einstein Condensate, Phys. Rev. Lett. Vol. 80 (1998), 2027-2030.
[24] S. Thangavelu, Lectures on Hermite and Laguerre Expansions, Princeton Univ. Press, Princeton, NJ, 1993.
[25] S. Thangavelu, Regularity of Twisted Spherical Means and Special Hermite Expansions, Proc. Indian Acad. Sci. (Math. Sci.), Vol. 103 (1993), 303-320.
[26] L. Vega, Schrödinger Equations: Pointwise Convergence to the Initial Data, Proc. Amer. Math. Soc. 102 (1988), 874-878.

Department of Mathematics, University of Maryland, College Park, MD 20742
E-mail address: chenxuwen@math.umd.edu