GREEN'S FUNCTION FOR SECOND ORDER PARABOLIC EQUATIONS
WITH SINGULAR LOWER ORDER COEFFICIENTS

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ABSTRACT. We construct Green’s functions for second order parabolic operators of the form $Pu = \partial_t u - \operatorname{div}(Au) + b \cdot \nabla u + cu$ on $(\mathbb{R}^n, \infty) \times \Omega$, where $\Omega$ is an open connected set in $\mathbb{R}^n$. It is not necessary that $\Omega$ be bounded and $\Omega = \mathbb{R}^n$ is not excluded. We assume that the leading coefficients $A$ are bounded and measurable and the lower order coefficients $b$ and $c$ belong to critical mixed norm Lebesgue spaces and satisfy the conditions $d - \operatorname{div} b \geq 0$ and $\operatorname{div} (b - c) \geq 0$. We show that the Green’s function has the Gaussian bound in the entire $(\mathbb{R}^n, \infty) \times \Omega$.

1. INTRODUCTION

In this paper, we are concerned with Green’s functions of the second order parabolic equations of divergence form

$$Pu = \partial_t u - \sum_{i=1}^{n} D_i(a^i(t, x)D_j u) + b^i(t, x)u + \sum_{i=1}^{n} c^i(t, x)D_i u + d(t, x)u$$

in a cylindrical domain $\mathcal{D} = (a, b) \times \Omega \subset \mathbb{R}^{n+1}$, where $-\infty < a < b \leq +\infty$ and $\Omega$ is an open connected set in $\mathbb{R}^n$ with $n \geq 1$. It is not necessary that $\Omega$ be bounded and $\Omega = \mathbb{R}^n$ is not excluded. In the case when $\Omega = \mathbb{R}^n$, the Green’s function is usually called the fundamental solution.

By Green’s function for the operator $P$, we mean a function $G(t, x, s, y)$ which satisfies the following:

$$PG(t, s, s, y) = 0 \quad \text{in} \quad (s, \infty) \times \Omega,$$

$$G(t, x, s, y) = \delta_y(x) \quad \text{on} \quad \{t = s\} \times \Omega,$$

where $\delta_y(\cdot)$ is a Dirac delta function. See Theorem 5.1 for the precise definition.

Before describing the remaining assumption on $P$, we introduce the function space $L_{p,q}(\mathcal{D})$, the usual Lebesgue space with mixed norm. Let $t$ denote points on the real line $\mathbb{R}$ and $x = (x_1, \ldots, x_n)$ denote points in the $n$-dimensional Euclidean space $\mathbb{R}^n$. For $f \in L_{p,q}(\mathcal{D})$ with $1 \leq p, q < \infty$, we define

$$\|f\|_{L_{p,q}(\mathcal{D})} = \|f\|_{L_{p,q}(\mathcal{D})} := \left( \int_{a}^{b} \left( \int_{\Omega} |f(t, x)|^p \, dx \right)^{q/p} \, dt \right)^{1/q}.$$

In case either $p$ or $q$ is infinite, $\|f\|_{L_{p,q}(\mathcal{D})}$ is defined in a similar fashion using essential supremum rather than integrals. We denote $L_{p,q}(\mathcal{D})$ by $L_{p}(\mathcal{D})$ and the norm $\|\cdot\|_{L_{p,q}(\mathcal{D})}$

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by \( \| \cdot \|_{L^p(\mathcal{D})} \). Throughout the rest of the paper, we shall adopt the usual summation convention over repeated indices.

We assume that the coefficients of \( P \) are defined in \( \mathcal{D} = (-\infty, \infty) \times \Omega \) and satisfy the following conditions which will be referred to collectively as (H).

(H1) There exists a constant \( \nu \in (0, 1) \) such that for all \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) and for all \( (t, x) \in \mathcal{D} \), we have

\[
\nu|\xi|^2 \leq A\xi \cdot \xi = a^i(t, x)\xi_i\xi_j \quad \text{and} \quad \sum_{i,j=1}^n |a^{ij}(t, x)|^2 \leq \nu^{-2}.
\]

(H2) \( b = (b^1, \ldots, b^n) \), \( c = (c^1, \ldots, c^n) \) are contained in some \( L_{p, q}(\mathcal{D}) \) and \( d \) is contained in some \( L_{p/2, q/2}(\mathcal{D}) \) where \( p \) and \( q \) are such that

\[
2 \leq p, q \leq \infty \quad \text{and} \quad \frac{n}{p} + \frac{2}{q} = 1.
\]

There exists a constant \( \Theta \geq 0 \) such that

\[
\|b - c\|_{L_{p, q}(\mathcal{D})} \leq \Theta.
\]

(H3) The following inequalities hold in the sense of distributions:

\[
d - \text{div} b \geq 0 \quad \text{and} \quad \text{div}(b - c) \geq 0.
\]

It should be noted that we deal with the “critical” mixed norm spaces in (H2) and in the case when \( p = n \) and \( q = \infty \), it can be weakened to \( (b - c)1_{\mathcal{D}} \in L_{\infty}(BMO^{-1}) \); see [4,12]. The condition (H3) allows us to obtain the “global” energy inequality and also “scale invariant” local boundedness estimate for weak solutions.

The goal of this paper is to show that if \( P \) satisfies the condition (H), then there exists the Green’s function \( G(t, x, s, y) \) and it has the following Gaussian bound: there exist constants \( C = C(n, \nu, p, \Theta) \) and \( \kappa = \kappa(n, \nu, \Theta) > 0 \) such that for all \( t, s \) satisfying \(-\infty < s < t < +\infty \) and \( x, y \in \Omega \), we have

\[
|G(t, x, s, y)| \leq \frac{C}{(t-s)^{n/2}} \exp \left\{ -\frac{\kappa|x - y|^2}{t-s} \right\}. \tag{1.2}
\]

We will give some brief history regarding the Gaussian bounds for fundamental solutions of parabolic equations with measurable coefficients, starting with the case when there are no lower order terms present. Since the groundbreaking work of Nash [20], where he established certain estimates of the fundamental solutions in proving Hölder continuity of weak solutions, there have been many important works in this field. By employing the parabolic Harnack inequality of Moser [19], Aronson [1] established two-sided Gaussian bounds for the fundamental solutions. Fabes and Strook [10] showed that the Nash’s method could be used to prove Aronson’s Gaussian bounds and as a consequence, they gave a new proof of Moser’s parabolic Harnack inequality.

In the elliptic setting, Littman, Stampacchia, and Weinberger [18] and Grüter and Widman [12] studied Green’s functions of elliptic equations in divergence form with measurable coefficients and showed that the Green’s function \( G(x, y) \) has a pointwise bound

\[
|G(x, y)| \leq C|x - y|^{2-n} \quad (n \geq 3). \tag{1.3}
\]

Later, Hofmann and Kim [14] gave an approach that also works for elliptic systems, where it is shown that the Green’s function has pointwise bound (1.3) if weak
solutions of the elliptic system satisfy certain scale invariant Hölder continuity estimates. Recently, Kim and Sakellaris [15] studied Green’s function of elliptic operators of the form

\[ Lu = -\text{div}(A \nabla u + bu) + c \cdot \nabla u + du, \]  

(1.4)

where the principal coefficients \( A \) satisfy (H1), the lower order coefficients \( b, c, \) and \( d \) satisfy (in the critical setting) the conditions that \( b, c \in L_n, \) \( d \in L_{n/2}, \) \( d - \text{div} b \geq 0, \) and \( d - \text{div} c \geq 0. \) Assuming that \( \Omega \) has a finite measure, they established pointwise bounds (1.3) for the Green’s function.

We would like to mention that our investigation is largely motivated by [15]. As it is well known, the Gaussian bound (1.2) for the “heat kernel” of the elliptic operator \( L \) yields the pointwise bound (1.3) for the Green’s function of \( L. \) Therefore, in the elliptic context, our result says that if (H1) and (H3) hold and if \( b, c \in L_n(\Omega) \) with \( \|b - c\|_L \leq \Theta \) and \( d \in L_{n/2}(\Omega), \) then the Green’s function for the elliptic operator \( L \) has the pointwise bound (1.3) with constant \( C = C(n, \nu, \Theta). \) This gives a new proof for a result in [15] dispensing with the assumption that \( |\Omega| < \infty. \)

There are also many previous results in the literature regarding Gaussian bounds for fundamental solutions for parabolic equations with lower order terms. To name a few, we mention [2, 22, 23, 24, 26]. However, there are very few in the literature dealing with global Gaussian bound (1.2). For example, Aronson [2] considered parabolic equations of the form (1.1) with coefficients \( b, c \in L_p, d \in L_{p/2}, q < 1, \) which obviously satisfy (H3). As a matter of fact, in the proof of our main theorem, the only place where we strongly use the scalar property of weak solutions is in the proof of Lemma 2.15, and as long as we have the local boundedness estimate in Lemma 2.15 essentially the same proof carries over to the systems. Therefore, we recover the result in [9] as a corollary. See Remark 3.5.

The organization of the paper is as follows. In section 2, we introduce some notation and function spaces. Then we prove the energy inequality which in turn implies the existence and uniqueness of weak solutions of Cauchy problems. We present the main result in section 3 and provide the proof in section 4. In section 5, we prove the local boundedness estimate for weak solutions, which plays a key role in establishing the Gaussian bound.

2. Preliminaries

In this section, we first recall some frequently used notation and function spaces in [17]. Then for reader’s convenience, we give the definition of weak solutions to second order parabolic equations, and present some auxiliary estimates which will be used later.
2.1. **Notation and function spaces.** The adjoint operator $P^*$ of $P$ is defined by

$$P^* u = -\partial_t u - \text{div}(A^T \nabla u) + b \cdot \nabla u + du,$$

(2.1)

where $A^T$ is the transpose of $A$. Note that the coefficients $A^T = (a^{ij})$ satisfy the same ellipticity condition (H1).

We denote points in $\mathbb{R}^{n+1}$ by $X = (t, x)$, $Y = (s, y)$, $X_0 = (t_0, x_0)$, etc. We define the “parabolic distance” between the points $X = (t, x)$ and $Y = (s, y)$ in $\mathbb{R}^{n+1}$ as

$$|X - Y| = \max(\sqrt{|t - s|}, |x - y|).$$

For $U \subset \mathbb{R}^{n+1}$, we write $U(t_0)$ for the set of all points $(t_0, x)$ in $U$ and $I(U)$ for the set of all $t$ such that $U(t)$ is nonempty. We define

$$|\bar{u}|_U := \|Du\|_{L^p(U)} + \text{ess sup}_{t \in I(U)} \|u(t, \cdot)\|_{L^{2, \infty}(U(t))}.$$

In the rest of this section, we restrict ourselves to the case when $U$ is a cylindrical domain $\mathcal{D} = (a, b) \times \Omega$ with $-\infty < a < b < +\infty$. We define the lateral boundary $\partial_x \mathcal{D}$ and the parabolic boundary $\partial_p \mathcal{D}$ of $\mathcal{D}$ by

$$\partial_x \mathcal{D} = [a, b] \times \partial \Omega \quad \text{and} \quad \partial_p \mathcal{D} = \partial_x \mathcal{D} \cup \{t = a\} \times \Omega,$$

respectively. We denote

$$W^{1, 0}_2(\mathcal{D}) := \{u : u, Du \in L^2(\mathcal{D})\}, \quad W^{1, 1}_2(\mathcal{D}) := \{u : u, \partial_t u, Du \in L^2(\mathcal{D})\}.$$

We define $V_2(\mathcal{D})$ as the Banach space consisting of all elements of $W^{1, 0}_2(\mathcal{D})$ having a finite norm

$$||u||_{V_2(\mathcal{D})} := |\bar{u}|_{\mathcal{D}} = \|Du\|_{L^2(\mathcal{D})} + \|u\|_{L^{2, \infty}(\mathcal{D})}.$$

We define $V^{1, 0}_2(\mathcal{D})$ as the Banach space consisting of all elements of $V_2(\mathcal{D})$ which are continuous in $t$ in the norm of $L^2(\Omega)$, with the norm

$$||u||_{V^{1, 0}_2(\mathcal{D})} := |\bar{u}|_{\mathcal{D}} = \|Du\|_{L^2(\mathcal{D})} + \max_{a \leq t \leq b} \|u(t, \cdot)\|_{L^2(\Omega)}.$$

The space $V^{1, 0}_2(\mathcal{D})$ is obtained by completing the set $W^{1, 1}_2(\mathcal{D})$ in the norm of $V_2(\mathcal{D})$.

We say that a function $u$ in $W^{1, 0}_2(\mathcal{D})$ vanishes on $S \subset \partial_x \mathcal{D}$ if $u$ is the limit in $W^{1, 0}_2(\mathcal{D})$ of functions from $C^{1, 1}_c(\mathcal{D} \setminus S)$, the set of all continuously differentiable functions with compact supports in $\mathcal{D} \setminus S$. We denote by $\hat{V}^{1, 0}_2(\mathcal{D})$ the set of functions in $W^{1, 0}_2(\mathcal{D})$ that vanish on the lateral boundary $\partial_p \mathcal{D}$. We define

$$\hat{V}_2(\mathcal{D}) := V_2(\mathcal{D}) \cap \hat{V}^{1, 0}_2(\mathcal{D}) \quad \text{and} \quad \hat{V}^{1, 0}_2(\mathcal{D}) := V^{1, 0}_2(\mathcal{D}) \cap \hat{V}^{1, 0}_2(\mathcal{D}).$$

2.2. **Embedding inequalities.** Let the exponents $\tilde{\rho}$ and $\tilde{\theta}$ satisfy

$$\frac{n}{\tilde{\rho}} + \frac{2}{\tilde{\theta}} = \frac{n}{2}$$

with

$$\begin{cases}
\tilde{\rho} \in [2, \frac{2n}{n-2}], & \tilde{\theta} \in [2, \infty] \quad \text{if} \quad n \geq 3, \\
\tilde{\rho} \in [2, \infty), & \tilde{\theta} \in (2, \infty) \quad \text{if} \quad n = 2, \\
\tilde{\rho} \in [2, \infty), & \tilde{\theta} \in [4, \infty) \quad \text{if} \quad n = 1.
\end{cases}$$

(2.2)

By a well-known embedding theorem, there exists a constant $\beta = \beta(n, \tilde{\rho})$ such that

$$||u||_{L^{\tilde{\rho}, \tilde{\theta}}(\mathcal{D})} \leq \beta |\bar{u}|_{\mathcal{D}}, \quad \forall u \in \hat{V}_2(\mathcal{D}).$$

(2.3)
We emphasize that the constant \( \beta \) in (2.3) is independent of \( \mathcal{D} \). In particular, if we take \( \bar{p} = \bar{q} = 2(n + 2)/n \) in (2.2), then \( \beta = \beta(n) \). See [17, pp. 74-75].

2.3. Weak solutions. Let \( \mathcal{D} = (a, b) \times \Omega \), where \(-\infty < a < b < +\infty \). Let \( f \in L^{p', q'}(\mathcal{D}) \), where \( p' \) and \( q' \) are Hölder conjugates of \( p \) and \( q \), respectively, and \( p = q = \frac{2n}{n - 1} \) with ranges specified in (2.2). We say that \( u \in V_2(\mathcal{D}) \) is a weak solution of \( Pu = f \) in \( \mathcal{D} \) if for almost all \( t_1 \in (a, b) \) the identity

\[
I(t_1; u, \phi) := \int_{\Omega} u(t_1, x) \phi(t_1, x) \, dx - \int_{a}^{t_1} \int_{\Omega} u \phi_t \, dx \, dt + \int_{a}^{t_1} \int_{\Omega} (a^{ij} D_i u + b^j u) D_j \phi \, dx \, dt
\]

\[
+ \int_{a}^{t_1} \int_{\Omega} (c D_i u \phi + d u \phi) \, dx \, dt - \int_{a}^{t_1} \int_{\Omega} f \phi \, dx \, dt = 0
\]

holds for all \( \phi \in C^{1,1}_c(\overline{\mathcal{D}} \setminus \partial_p \mathcal{D}) \).

For a given function \( \psi_0 \in L_2(\Omega) \), we say that \( u \in \tilde{V}_2(\mathcal{D}) \) is a weak solution of the problem

\[
Pu = f \quad \text{in} \quad \mathcal{D}, \quad u(a, \cdot) = \psi_0 \quad \text{on} \quad \Omega,
\]

if for all \( t_1 \in [a, b] \) the identity

\[
I(t_1; u, \phi) = \int_{\Omega} \psi_0(x) \phi(a, x) \, dx
\]

holds for all \( \phi \in C^{1,1}_c(\overline{\mathcal{D}} \setminus \partial_p \mathcal{D}) \).

For the adjoint operator \( P^* \) given by (2.1), we similarly define weak solutions of \( P^* u = f \) in \( \mathcal{D} \) and weak solutions of the corresponding backward problem

\[
P^* u = f \quad \text{in} \quad \mathcal{D}, \quad u(b, \cdot) = \psi_0 \quad \text{on} \quad \Omega.
\]

2.4. Energy inequality. Under the condition (H), we can derive the following “global” energy inequality.

**Lemma 2.7.** Suppose the coefficients of the operator \( P \) satisfy the condition (H). Let \( \mathcal{D} = (a, b) \times \Omega \), where \(-\infty < a < b < +\infty \). Let \( f \in L^{2(n+2)/(n+4)}(\mathcal{D}) \) and \( \psi_0 \in L_2(\Omega) \) be given. If \( u \in \tilde{V}_2(\mathcal{D}) \) is a weak solution of the problem (2.3), then we have

\[
\|u\|_{\mathcal{D}} \leq C \left( \|\psi_0\|_{L_2(\Omega)} + \|f\|_{L^{2(n+2)/(n+4)}(\mathcal{D})} \right),
\]

where \( C \) is a constant depending only on \( n \) and \( \nu \). The same estimate is true if \( u \in \tilde{V}_2(\mathcal{D}) \) is a weak solution of the corresponding backward problem (2.6).

**Proof.** By taking \( \phi = u_h \) in (2.5), where \( u_h \) is the Steklov average of \( u \) (see [17, §III.2]), integrating by parts, and taking \( h \to 0 \), we have for all \( t_1 \in [a, b] \) that

\[
\frac{1}{2} \int_{\Omega} u^2(t_1, x) \, dx + \int_{a}^{t_1} \int_{\Omega} (a^{ij} D_i u + b^j u) D_j u \, dx \, dt
\]

\[
= \frac{1}{2} \int_{\Omega} \psi_0(x) u(a, x) \, dx + \int_{a}^{t_1} \int_{\Omega} f u \, dx \, dt.
\]

Since the condition (H3) implies that

\[
\int_{a}^{t_1} \int_{\Omega} d^2 u + 2b' D_i u \geq 0 \quad \text{and} \quad -\int_{a}^{t_1} \int_{\Omega} (b' - c') D_i u \geq 0,
\]

we have by the Hölder inequality,

\[
\|u\|_{L_2(\mathcal{D})} \leq C \left( \|\psi_0\|_{L_2(\Omega)} + \|f\|_{L^{2(n+2)/(n+4)}(\mathcal{D})} \right).
\]

This completes the proof.
it follows from (2.9) and the condition (H1) that
\[ \frac{1}{2} \int_{\Omega} u'^2(t_1, x) \, dx + \nu \int_{t_1}^{t} \int_{\Omega} |Du|^2 \, dx dt \leq \int_{t_1}^{t} \int_{\Omega} fu \, dx dt + \frac{1}{2} \int_{\Omega} \psi_0(x)u(a, x) \, dx. \]  
(2.10)

By Hölder’s inequality and the embedding (2.3), we have
\[ \int_{t_1}^{t} \int_{\Omega} fu \, dx dt \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq \beta \|f\|_{L^1(\Omega)}\|u\|_{L^1(\Omega)} \]  
(2.11)

where \( \beta = \beta(n) \). Now, the estimate (2.8) follows from the standard argument involving Young’s inequality.

For the corresponding backward problem (2.6), similar to (2.10), we have
\[ \frac{1}{2} \int_{\Omega} u'^2(t_1, x) \, dx + \nu \int_{t}^{t_1} \int_{\Omega} |Du|^2 \, dx dt \leq \int_{t}^{t_1} \int_{\Omega} fu \, dx dt + \frac{1}{2} \int_{\Omega} \psi_0(x)u(b, x) \, dx \]
for all \( t_1 \in [a, b] \). Hence, the energy inequality (2.8) is also valid for the backward problem (2.6).

2.5. Existence and uniqueness of weak solutions. With the energy inequality (2.8) available, we can construct a weak solution of the problem (2.4) by Galerkin’s method. Uniqueness is also a consequence of the energy inequality. See [17, §III.4] for the details. We state these observations in the following lemma for the reference.

Lemma 2.12. Suppose the coefficients of the operator \( P \) satisfy the condition (H). Let \( \mathcal{D} = (a, b) \times \Omega \), where \( -\infty < a < b < +\infty \). Let \( f \in L^2((a+b)/2, \Omega) \) and \( \psi_0 \in L^2(\Omega) \) be given. Then there exists a unique weak solution \( u \in \tilde{V}^{1,0}_2(\mathcal{D}) \) of the problem (2.4). The same is true for the backward problem (2.6).

We note that the condition that \( f \in L^2((a+b)/2, \Omega) \) in Lemmas 2.7 and 2.12 can be replaced by \( f \in L^{p', q'}(\mathcal{D}) \), where \( p' \) and \( q' \) are Hölder conjugates of \( p \) and \( q \), respectively, and \( p \) and \( q \) satisfy \( \frac{1}{p'} + \frac{1}{q'} = \frac{1}{2} \) with ranges specified in (2.2). This is because the inequality (2.11) remains valid with \( L^{p', q'} \) norm of \( f \). However, we do not use this fact in the paper.

If \( \mathcal{D} = (a, \infty) \times \Omega \) and \( f \in L^2((a+b)/2, \Omega) \), then by letting \( b \to \infty \) in Lemmas 2.7 and 2.12, we can say that \( u \) is the weak solutions in \( \tilde{V}^{1,0}_2(\mathcal{D}) \) of the problem (2.4).

2.6. Local boundedness property. The following lemma says that we have “scale-invariant” local boundedness property for weak solutions of \( Pu = f \) in \( Q^*_r(X_0) \) that vanish on \( S^*_r(X_0) \), where
\[ Q^*_r(X_0) = (t_0 - r^2, t_0) \times (B_r(x_0) \cap \Omega), \]
\[ S^*_r(X_0) = (t_0 - r^2, t_0) \times (B_r(x_0) \cap \partial \Omega), \]
and \(-\infty < t_0 < +\infty \) and \( X_0 \in \Omega \). When dealing with the adjoint operator \( P^* \), we replace \( Q^*_r(X_0) \) and \( S^*_r(X_0) \) with \( Q'^*_r(X_0) \) and \( S'^*_r(X_0) \), where
\[ Q'^*_r(X_0) = (t_0, t_0 + r^2) \times (B_r(x_0) \cap \Omega), \]
\[ S'^*_r(X_0) = (t_0, t_0 + r^2) \times (B_r(x_0) \cap \partial \Omega). \]

Lemma 2.15. Suppose the coefficients of the operator \( P \) satisfy the condition (H). Let \( Q^*_r = Q^*_r(X_0) \) and \( S^*_r = S^*_r(X_0) \). If \( u \in \tilde{V}^{1,0}_2(Q^*_r) \) is a weak solution of \( Pu = f \) in \( Q^*_r \) vanishing on \( S^*_r \), where \( f \in L^2(Q^*_r) \), then we have
\[ \|u\|_{L^2(Q^*_r)} \leq N_0 \left( r^{-1} \|u\|_{L^2(Q^*_r)} + r^2 \|f\|_{L^2(Q^*_r)} \right), \]
(2.16)
where $N_0$ is a constant that depends only on $n, \nu, p,$ and $\Theta$. The corresponding statement
is valid for the weak solution of $P^* u = f$ in $Q^+_t(X_0)$ vanishing on $S^+_t(X_0)$.

We emphasize that the constant $N_0$ in the lemma is independent of $r$. The proof
will be given in section 5.

3. Main results

Theorem 3.1. Suppose the the coefficients of operator $P$ satisfy the condition (H) and let
$\mathcal{D} = (-\infty, \infty) \times \Omega$. Then, there exists a unique Green’s function $G(x, y) = G(t, x, s, y)$ on
$\mathcal{D} \times \mathcal{D}$ which satisfies $G(t, x, s, y) \equiv 0$ for $t < s$, and has the following property: For any
$\psi_0 \in L^2(\Omega)$, the function $u$ given by

$$u(t, x) := \int_{\Omega} G(t, x, s, y) \psi_0(y) \, dy \quad (t > s, \ x \in \Omega) \tag{3.2}$$

is the unique weak solution in $W^{1,2}_1((s, \infty) \times \Omega)$ of the problem

$$Pu = 0 \quad \text{in} \quad (s, \infty) \times \Omega, \quad u(s, \cdot) = \psi_0 \quad \text{on} \quad \Omega.$$

Moreover, the Green’s function satisfies the following Gaussian bound: For all $t > s$ and
$x, y \in \Omega$, we have

$$|G(t, x, s, y)| \leq \frac{C}{(t - s)^{n/2}} \exp \left\{ \frac{\kappa|x - y|^2}{t - s} \right\}, \tag{3.3}$$

where $C = C(n, \nu, p, \Theta)$ and $\kappa = \kappa(n, \nu, \Theta)$ are positive constants.

Corollary 3.4. Let $\Omega$ be an open connected set in $\mathbb{R}^n$ with $n \geq 3$. Suppose the coefficients of elliptic operator $L$ in $\{1, 2\}$ satisfy the condition (H1) and (H3). In place of (H2), assume
that $b, c \in L^\infty(\Omega)$, $d \in L^2(\Omega)$, and that $(b - c)1_{\Omega} \in \text{BMO}(\mathbb{R}^n)$, that is, there are functions
$\Phi^i$ in $\mathbb{R}^n$ and a positive constant $\Theta$ such that

$$(b^i - c^i)1_{\Omega} = D_i \Phi^i, \quad \sum_{i,j=1}^n \|\Phi^i\|_{\text{BMO}(\mathbb{R}^n)}^2 \leq \Theta^2.$$

Then there exists the Green’s function $G(x, y)$ on $\Omega \times \Omega$ and it has the bound

$$|G(x, y)| \leq C|x - y|^{2-n},$$

where $C = C(n, \nu, \Theta)$.

Proof. Let $K_i(x, y) = \tilde{G}(t, x, 0, y)$, where $\tilde{G}(t, x, s, y)$ is the Green’s function for the
operator $P$ with time independent coefficients; as mentioned in the introduction,
when $p = n$, the condition (H2) can be relaxed to the weaker condition $4, 12$. Let

$$G(x, y) = \int_0^\infty K_i(x, y) \, dt.$$

Then, it is known that $G(x, y)$ becomes the Green’s function for the operator $L$; see,
e.g., [8]. From the Gaussian bound (3.3), it follows

$$\int_0^\infty |K_i(x, y)| \, dt \leq C \int_0^\infty t^{-n/2} e^{-|y|^2/t} \, dt \leq C|x - y|^{2-n}. \quad \blacksquare$$

Remark 3.5. Consider the second-order parabolic systems of divergence form

$$P_i u^i = \partial_i u^i - D_{i\alpha} (a_{ij}^\alpha (t, x) D_j u^i) + b_{ij}^\alpha (t, x) u^i + c_{ij}^\alpha (t, x) D_n u^i + d_{ij} (t, x) u^i, \quad i = 1, \ldots, m,$

where the coefficients satisfy the following conditions analogous to (H).
(H1') There exists a constant $v \in (0, 1)$ such that for all $(t, x) \in \mathcal{D}$, we have
\[ v \sum_{i=1}^{m} \sum_{a=1}^{n} |\xi_a^i|^2 \leq a_{ij}^a(t, x)\xi_a^i \xi_j^i \quad \text{and} \quad \sum_{i,j=1}^{m} \sum_{a,b=1}^{n} |a_{ij}^a(t, x)|^2 \leq v^{-2}. \]

(H2') $b^a = (b_i^a)$, $c^a = (c_i^a)$ are symmetric and belong to some $L_p(\mathcal{D})$, and $d = (d_{ij})$ is contained in some $L_{p/2,q/2}(\mathcal{D})$, where $p$ and $q$ are such that
\[ 2 \leq p, q \leq \infty \quad \text{and} \quad \frac{n}{p} + \frac{2}{q} = 1. \]

There exists a constant $\Theta > 0$ such that
\[ \sum_{a=1}^{n} \|b^a - c^a\|_{L_{p,q}(\mathcal{D})}^2 \leq \Theta^2. \]

(H3') The following inequalities hold in the sense of distributions:
\[ d - D_a b^a \geq 0 \quad \text{and} \quad D_a (b^a - c^a) \geq 0, \]
where $M \geq 0$ for a matrix $M$ means that $M\xi \cdot \xi \geq 0$ for all $\xi \in \mathbb{R}^n$.

Also, we assume that local boundedness property holds for the operator $P_{ij}$ and its adjoint operator. Then the conclusion of Theorem 3.1 is true. See [2].

4. Proof of Theorem 3.1

4.1. Construction of the Green’s function. The proof for construction of Green’s function is a modification of that given in [9, 3]. For reader’s convenience we present main steps here. Let $Y = (s, y) \in \mathcal{D}$. For $\varepsilon > 0$, fix $a \in (-\infty, s - \varepsilon^2)$ and $b \in (s, \infty)$. We consider the problem
\[ P_Y \varphi = \frac{1}{|Q_+^\varepsilon(Y)|} 1_{Q_+^\varepsilon(Y)} \quad \text{in} \quad (a, b) \times \Omega, \quad \varphi(a, \cdot) = 0 \quad \text{on} \quad \Omega, \quad \varphi(b, \cdot) \in \mathcal{D}, \quad \varphi(b, \cdot) \equiv 0 \quad \text{in} \quad (a, b) \times \Omega \quad \text{and} \quad \lim_{b \to \infty} \varphi(b, \cdot) = 0 \quad \text{then the energy estimate (2.8), we have}
\[
\int_{Q_+^\varepsilon(Y)}^\varepsilon v \leq (Q_+^\varepsilon(Y))^{- \frac{1}{2(p-2q)}}. \quad (4.2)
\]

We define the “approximate” Green’s function $G^\varepsilon(\cdot, Y)$ for $P$ in $\mathcal{D}$ by
\[ G^\varepsilon(\cdot, Y) = \varphi. \]

Next, for $f \in C_0^\infty(\mathcal{D})$, choose a number $b$ such that $f \equiv 0$ in $[b, \infty) \times \Omega$. For any $a < b$, consider the backward problem
\[ P^* u = f \quad \text{in} \quad (a, b) \times \Omega, \quad u(b, \cdot) = 0 \quad \text{on} \quad \Omega. \quad (4.3)\]

By Lemma 2.12 again, we obtain a unique weak solution $u \in V^{1,0}_2((a, b) \times \Omega)$ of the problem (4.3). Again, we may extend $u$ to entire $\mathcal{D}$ by setting $u \equiv 0$ in $(b, \infty) \times \Omega$ and letting $a \to -\infty$. The energy inequality (2.8) then tells us that
\[
\|u\|_{\mathcal{D}} \leq C \|f\|_{L_{p,q}((a+b))}(\mathcal{D}). \quad (4.4)
\]
Notice from (4.1) and (4.3) that we have
\[ \int_G G'(\cdot, Y) f = \int_{Q_1(Y)} u. \] (4.5)

Now, we assume that \( f \) is supported in \( Q^+_p(X_0) \), where it is defined in (2.14). By Lemma 2.15 combined with (4.4) and (2.3), we have
\[ \|u\|_{L^p(Q^+_1(X_0))} \leq CR^2 \|f\|_{L^p(Q^+_1(X_0))}. \] (4.6)

If \( Q^+_1(Y) \subset Q^+_p(X_0) \), then (4.5) together with (4.6) yields
\[ \left| \int_{Q^+_1(Y)} G'(\cdot, Y) f \right| \leq \int_{Q^+_1(Y)} \|u\| \leq CR^2 \|f\|_{L^p(Q^+_1(X_0))}. \]

By duality, it follows that if \( Q^+_1(Y) \subset Q^+_p(X_0) \), then
\[ \|G'(\cdot, Y)\|_{L^p(Q^+_1(X_0))} \leq CR^2. \]

Therefore, the same proof of [9, Lemma 3.6] yields the following lemma.

**Lemma 4.7.** Let \( X = (t, x), Y = (s, y) \in \mathscr{D} \) with \( X \neq Y \). Then we have
\[ |G'(X, Y)| \leq C|X - Y|^{-n}, \quad \forall \varepsilon \leq \frac{1}{4}|X - Y|, \]
where \( C = C(n, v, p, \Theta) \).

For \( \rho \) and \( R \) satisfying \( \varepsilon < \rho < R \), let \( \zeta : \mathbb{R}^{n+1} \rightarrow [0, 1] \) be a smooth function satisfying \( \zeta(X) = 0 \) for \( |X - Y| < \rho \), \( \zeta(X) = 1 \) for \( |X - Y| \geq R \), and
\[ \max \left( \|D\zeta\|_{L^{\infty}}, \|D^2\zeta\|_{L^1}, |\partial_t\zeta|_{L^1} \right) \leq \frac{4}{(R - \rho)^2}. \]

Recall that \( v_{\varepsilon} \in \mathscr{V}^1_2(\mathscr{D}) \) and it satisfies (4.1). Testing the equation with \( \zeta^2 v_{\varepsilon} \), letting \( a \to -\infty, b \to \infty \), and using that \( d - \text{div} \, b \geq 0 \) in (H3), we obtain
\[
\frac{1}{2} \int_\Omega \zeta^2 v_{\varepsilon}^2(t, x) \, dx + \int_{-\infty}^t \int_\Omega \zeta^2 \partial_t^j D_{ij}^p D_{ij} v_{\varepsilon} v_{\varepsilon} \, dx dt + \int_{-\infty}^t \int_\Omega 2 \zeta \partial_t^j D_{ij}^p D_{ij} v_{\varepsilon} \, dx dt \\
\leq \int_{-\infty}^t \int_\Omega \zeta^2 \partial_t \zeta v_{\varepsilon}^2 \, dx dt + \int_{-\infty}^t \int_\Omega \zeta^2 v_{\varepsilon} (b' - c') D_{ij}^p D_{ij} v_{\varepsilon} \, dx dt
\]
for all \( t \). By the assumption that \( \text{div}(b - c) \geq 0 \) in (H3), we have
\[ \int_{-\infty}^t \int_\Omega \zeta^2 v_{\varepsilon} (b' - c') D_{ij}^p D_{ij} v_{\varepsilon} \, dx dt \leq - \int_{-\infty}^t \int_\Omega \zeta (b' - c') v_{\varepsilon}^2 D_{ij}^p D_{ij} \zeta \, dx dt. \]

Then by the condition (H1) and Young’s inequality, we obtain
\[ \frac{1}{2} \|\zeta v_{\varepsilon}\|_{L^2(\mathscr{D})}^2 + \frac{3}{4} \int_\mathscr{D} \zeta^2 |Dv_{\varepsilon}|^2 \, dx dt \leq 2 \int_\mathscr{D} \zeta |Dv_{\varepsilon}|^2 \, dx dt + \frac{8}{3} \int_\mathscr{D} |D\zeta|^2 v_{\varepsilon}^2 \, dx dt - \int_\mathscr{D} \zeta (b' - c') v_{\varepsilon}^2 D_{ij}^p D_{ij} \zeta \, dx dt. \] (4.8)

In the case when \( p > n \) so that \( q < \infty \), we use Hölder’s inequality, the embedding (2.3), and Young’s inequality, to find that
\[ -2 \int_\mathscr{D} \zeta (b' - c') v_{\varepsilon}^2 D_{ij} \zeta \, dx dt \leq 2 \|b - c\|_{L^\infty(\mathscr{D})} \|\zeta v_{\varepsilon}\|_{L^2(\mathscr{D})} \|Dv_{\varepsilon}\|_{L^2(\mathscr{D})} \]
\[ \leq 2\beta \Theta \|\zeta v_{\varepsilon}\|_{\mathscr{D}} \|D\zeta v_{\varepsilon}\|_{L^2(\mathscr{D})}. \]
We note that (4.12) is a consequence of the embedding $\mathcal{E}_2$. For $0 < \nu < 1$, we get from (4.8), (4.9), and (4.10) that
\[
\|\zeta \nu_i \|_{L^2(\Omega)}^2 + \int_\Omega \zeta^2 |D\zeta \nu_i|^2 \, dx \leq C \int_\Omega (|\partial_i \zeta| + |D\zeta|^2) \nu_i^2 \, dx dt.
\] (4.10)

Then by using (4.10) again, taking $\rho = \frac{1}{2} r$ and $R = r$ for $\zeta$, and using Lemma 4.7, we get
\[
\|\zeta \nu_i \|_{L^2(\Omega)}^2 + \int_\Omega \zeta^2 |D\zeta \nu_i|^2 \, dx dt \leq C \int_\Omega (|\partial_i \zeta| + |D\zeta|^2) \nu_i^2 \, dx dt.
\]

provided $6 \epsilon \leq r$. If $r > 6 \epsilon$, then thanks to (4.2), the same inequality is obviously true. Therefore, we have
\[
\|\zeta \nu_i \|_{L^2(\Omega)}^2 + \int_\Omega \zeta^2 |D\zeta \nu_i|^2 \, dx dt \leq C \int_\Omega (|\partial_i \zeta| + |D\zeta|^2) \nu_i^2 \, dx dt.
\] (4.11)

which corresponds to (3.20). With the uniform estimate (4.11) at hand, we may invoke the same compactness argument as presented in Section 3.3 and obtain a Green’s function $G(\cdot, Y)$ from the family $\{v_i\}$ in the case when $p > n$.

In the case when $p = n$ and $q = \infty$, we use the following facts.

1. There exist functions $\Phi^{ij}$ on $\mathbb{R}^{n+1}$ satisfying
\[
(b^i - c^i)1_{\|Y\|} = D_i \Phi^{ij} \sup_{t \in \Omega} \sum_{i,j=1}^n \|\Phi^{ij}(t, \cdot)\|_{\text{BMO}(\mathbb{R}^n)}^2 \leq C_n \|b - c\|_{L^2(\Omega)}^2 \leq C_n \Theta^2,
\] (4.12)

where $C_n$ is a constant which depends only on $n$.

2. For $f, g \in W_2^1(\mathbb{R}^n)$ and $j = 1, \ldots, n$, we have
\[
\|D_j(fg)\|_{\text{BMO}(\mathbb{R}^n)} \leq C_n \left( \|D_j f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)} \|D_j g\|_{L^2(\mathbb{R}^n)} \right),
\] (4.13)

where $\|\cdot\|_{\text{BMO}(\mathbb{R}^n)}$ denotes the Hardy norm.

We note that (4.12) is a consequence of the embedding $L_n(\mathbb{R}^n) \hookrightarrow \text{BMO}^{-1}(\mathbb{R}^n)$. See, e.g., [16]. Estimates of type (4.13) are originally due to Coifman et al. [4] and usually referred to as “compensated compactness”. See [22, Proposition 3.2] for the proof of (4.13).

By setting $v_i(t, \cdot) = 0$ outside $\Omega$ and applying (4.12), we get
\[
\begin{align*}
-2 \int_{\Omega} \zeta (b^i - c^i) \nu_i^2 D_i \zeta &= 2 \int_{\Omega} \Phi^{ij} D_j (\zeta \nu_i^2 D_i \zeta) \leq 2 \|\Phi^{ij}\|_{\text{BMO}(\mathbb{R}^n)} \|D_j (\zeta \nu_i^2 D_i \zeta)\|_{\text{BMO}(\mathbb{R}^n)} \\
&\leq C_n \Theta \left( \sum_{i,j=1}^n \|D_j (\zeta \nu_i^2 D_i \zeta)\|_{\text{BMO}(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}} =: \text{RHS}. \quad (4.14)
\end{align*}
\]
Fix a smooth function \( \tilde{\zeta} : \mathbb{R}^{n+1} \to [0, 1] \) such that \( \tilde{\zeta}(X) = 1 \) for \( |X - Y| < R \), \( \tilde{\zeta}(X) = 0 \) for \( |X - Y| \geq 2R \), and \( \|D\tilde{\zeta}\|_{L_2} \leq 2/R \). Then, since
\[
\tilde{\zeta}v^2 D_i \tilde{\zeta} = \tilde{\zeta}v^2 D_i \zeta,
\]
by taking \( f = \tilde{\zeta}v \) and \( g = v D_i \zeta \) in (4.13), and using Young's inequality, the right hand side of (4.14) is bounded by
\[
\text{RHS} \leq C \left( \|D(\tilde{\zeta}v)\|_{L_2} + \|\tilde{\zeta}v D_i \zeta\|_{L_2} + |||D\tilde{\zeta}v_i\|_{L_2} + \|\tilde{\zeta}v D_i \zeta\|_{L_2} \right)
\]
\[
\leq C \|D(\tilde{\zeta}v)\|_{L_2}^2 + \|\tilde{\zeta}v D_i \zeta\|_{L_2}^2 + \frac{v}{4} \|\tilde{\zeta} D_i v\|_{L_2}^2
\]
\[
+ \frac{C}{(R - \rho)^2} \|\tilde{\zeta}v\|_{L_2}^2 + \frac{v(R - \rho)^2}{16} \|D_i \zeta\|_{L_2}^2 + (R - \rho)^2 \|D^2 \zeta\varepsilon\|_{L_2}^2,
\]
where \( C = C(n, v, \Theta) \). Then, by integrating (4.14) with respect to \( t \) over \( (-\infty, \infty) \), and using the properties of \( \zeta \) and \( \tilde{\zeta} \), we have
\[
-2 \int_{\mathbb{R}} \zeta(b - c)v^2 D_i \zeta \leq \frac{v}{4} \int_{\mathbb{R}} \zeta v^2 |Dv_i|^2
\]
\[
+ \frac{C}{(R - \rho)^2} \int_{|X| < |X - Y| \leq 2R} |Dv_i|^2 + \frac{v}{4} \int_{|X| < |X - Y| \leq 2R} |Dv_i|^2.
\]
Putting (4.15) back to (4.8) and using the properties of \( \tilde{\zeta} \), we obtain
\[
\frac{1}{2} \|\zeta v\|_{L_2, \mathbb{R}}^2 + \frac{v}{2} \int_{\mathbb{R}} \zeta v^2 |Dv_i|^2
\]
\[
\leq \frac{C}{(R - \rho)^2} \int_{|X| < |X - Y| \leq 2R} |Dv_i|^2 + \frac{v}{4} \int_{|X| < |X - Y| \leq 2R} |Dv_i|^2.
\]
In particular, (4.16) implies that
\[
\int_{|X| < |X - Y| \leq 2R} |Dv_i|^2 \leq \frac{C}{(R - \rho)^2} \int_{|X| < |X - Y| \leq 2R} v^2 + \frac{1}{2} \int_{|X| < |X - Y| \leq 2R} |Dv_i|^2.
\]
Since the above inequality is true for all \( \rho \) and \( R \) satisfying \( \varepsilon < \rho < R \), a well-known iteration argument yields (see [11] Lemma 5.1) that for any \( r \) satisfying \( \varepsilon < r < \infty \), we have
\[
\int_{|X| \leq |X - Y| < 2r} |Dv_i|^2 \leq \frac{C}{r^2} \int_{|X| \leq |X - Y| < 4r} v^2.
\]
Then, by taking \( \rho = 2r \) and \( R = 4r \), we get from (4.16) and Lemma 4.7 that
\[
|v_i|_{\mathbb{R} \cap |X - Y| > 4r} \leq \|\zeta v_i\|_{\mathbb{R} \cap |X - Y| > 4r} \leq \frac{C}{r^2} \int_{|X| \leq |X - Y| < 2r} v^2 + \int_{|X| \leq |X - Y| < 2r} |Dv_i|^2
\]
\[
\leq \frac{C}{r^2} \int_{|X| \leq |X - Y| < 8r} v^2
\]
\[
\leq \frac{C}{r^2} \int_{|X| \leq |X - Y| < 8r} |X - Y|^{-2n} dX \leq \frac{C}{r^2} \leq C Q_r(Y)^{-n}
\]
provided that \( 3\varepsilon \leq r \). Again, thanks to (4.2), we get the uniform estimate (4.11), which allows us to construct a Green’s function \( G'(. Y) \) out of the family \( \{G'(. Y)\} \) in the case when \( p = n \).

Also, by parallel reasonings, we can construct a Green’s function \( G'_{X, Y} \) for the adjoint operator \( P' \). We refer to [3 Section 3.5] for the proof of the representation
4.2. Gaussian estimates. We now prove the Gaussian estimate (3.3).

4.2.1. Case when $p = n$. In the case when $p = n$ and $q = \infty$, we follow the argument in [9], which is an adaptation of the techniques in [5, 3, 13], to obtain Gaussian bound (3.3). Here, we shall make strong use of (4.12) and (4.13).

Now, let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a bounded $C^2$ function satisfying

$$|D\psi| \leq \gamma_1, \quad |D^2\psi| \leq \gamma_2,$$

for some positive constants $\gamma_1$ and $\gamma_2$ to be fixed later. For $t > s$, we define an operator $P_{s \to t}$ on $L_2(\Omega)$ as follows. For a given $f \in L_2(\Omega)$, let $u \in \hat{V}^2_2((s, \infty) \times \Omega)$ be the weak solution of the problem

$$P_{s \to t}u = 0, \quad u(s,.) = e^{-\psi} f.$$

Then we define

$$P_{s \to t}^\psi f(x) = e^{\psi(x)}u(t, x).$$

It follows from (3.2) that

$$P_{s \to t}^\psi f(x) = e^{\psi(x)} \int_\Omega G(t, x, s, y)e^{-\psi(y)} f(y) dy. \quad (4.18)$$

Denote

$$I(t) := \|P_{s \to t}^\psi f\|^2_{L_2(\Omega)} = \int_\Omega e^{2\psi(x)}|u(t, x)|^2 dx, \quad t \geq s. \quad (4.19)$$

By using the equation (4.17) and the condition (H), we have

$$I'(t) = -2 \int_\Omega (a^{ij}D_i u + b^i u)D_j(e^{2\psi} u) + c^i D_i u e^{2\psi} u + d u e^{2\psi} u$$

$$= -2 \int_\Omega a^{ij}D_i u (e^{2\psi} D_j u + 2e^{2\psi} u D_j \psi) + (c^i - b^i)D_i u e^{2\psi} u + b^i D_i (e^{2\psi} u^2) + d e^{2\psi} u^2$$

$$\leq -2 \int_\Omega a^{ij}D_i u (e^{2\psi} D_j u + 2e^{2\psi} u D_j \psi) + (c^i - b^i)D_i u e^{2\psi} u$$

$$= -2 \int_\Omega a^{ij}D_i u (e^{2\psi} D_j u + 2e^{2\psi} u D_j \psi) + \int_\Omega (b^j - c^j)(D_j (e^{2\psi} u^2) - 2u^2 e^{2\psi} D_j \psi)$$

$$\leq -2\nu \int_\Omega e^{2\psi} |Du|^2 + \frac{4 \gamma_1}{\nu} \int_\Omega e^{2\psi} |u| |Du| - 2 \int_\Omega (b^j - c^j) e^{2\psi} u^2 D_j \psi. \quad (4.20)$$

By setting $u = 0$ outside $\Omega$ and applying (4.12), we have

$$-2 \int_\Omega (b^j - c^j) e^{2\psi} u^2 D_j \psi = 2 \int_\Omega \Phi^{ij} D_i (e^{2\psi} u^2 D_j \psi)$$

$$\leq C_n \|\Phi^{ij}\|_{BMO(\mathbb{R}^n)} \|D_j (e^{2\psi} u^2 D_j \psi)\|_{\mathcal{H}^1(\mathbb{R}^n)}. \quad (4.21)$$

By taking $f = e^\psi u$ and $g = e^\psi u D_j \psi$ in (4.13), we have

$$\|D_j (e^{2\psi} u^2 D_j \psi)\|_{\mathcal{H}^1} \leq C_n \left\{ \left( \|e^\psi u D^2 \psi\|_{L_2} + \|e^\psi Du\|_{L_2} \right) \|e^\psi u D^2 \psi\|_{L_2}$$

$$+ \|e^\psi u\|_{L_2} \left( \|e^\psi u D^2 \psi\|_{L_2} + \|e^\psi Du\|_{L_2} + \|e^\psi Du D_\psi\|_{L_2} \right) \right\} \leq C_n \left( 2 \gamma_1^2 + \gamma_2 \right) \|e^\psi u\|_{L_2}^2 + 2 \gamma_1 \|e^\psi Du\|_{L_2} \|e^\psi u\|_{L_2}. \quad (4.22)$$
Substituting the above into (4.21) and using (4.12), we obtain
\[-2 \int_\Omega (b^i - c^i)v^2u^2D_i\psi \leq C_n \Theta \left(2\gamma_1^2 + \gamma_2 \right) \int_\Omega e^{2\psi}u^2 + 2\gamma_1 \left(\int_\Omega e^{2\psi}|Du|^2\right)^{1/2} \left(\int_\Omega e^{2\psi}u^2\right)^{1/2}.
\]

Coming back to (4.20) and using Young’s inequality and Hölder’s inequality, we obtain the differential inequality
\[I'(t) \leq \left((4/v^3 + C_n^2 \Theta^2/v + 2C_n \Theta)\gamma_1^2 + C_n \Theta \gamma_2 \right) I(t). \quad (4.22)
\]
Recalling the initial condition \(I(s) = \|f\|_L_2(\Omega)\) and the definition (4.19), we obtain the \(L_2 \to L_2\) estimate
\[\|P_{s-t}^{\psi}f\|_{L_2(\Omega)} \leq e^{(\lambda_1^2 + \mu_1^2)(t-s)}\|f\|_{L_2(\Omega)}, \quad \forall t \geq s, \quad (4.23)
\]
where we set
\[2\lambda := 4/v^3 + C_n^2 \Theta^2/v + 2C_n \Theta \quad \text{and} \quad 2\mu := C_n \Theta.
\]

With (4.23) and Lemma 2.15 at hand, we replicate the same arguments in [9, p. 3028] to obtain the \(L_2 \to L_\infty\) estimate
\[\|P_{s-t}^{\psi}f\|_{L_\infty(\Omega)} \leq C(t-s)^{-\frac{3}{4}v/\sqrt{2(\lambda_1^2 + \mu_1^2)+\mu_1^2}(t-s)}\|f\|_{L_2(\Omega)}, \quad \forall t \geq s. \quad (4.24)
\]

Let the operator \(Q_{s-t}^{\psi}\) on \(L_2(\Omega)\) for \(s < t\) be given by
\[Q_{s-t}^{\psi}g(y) = e^{-\psi(y)}v(s, y)
\]
and denote
\[J(s) := \|Q_{s-t}^{\psi}f\|_{L_2(\Omega)}^2 = \int_\Omega e^{-2\psi}|v(s, y)|^2 \, dy, \quad s \leq t,
\]
where \(v \in \mathcal{V}_{2}^1((-\infty, t) \times \Omega)\) is the weak solution of the backward problem
\[\begin{cases}
    P^e v = 0, \\
    v(t, \cdot) = e^{\psi}g.
\end{cases} \quad (4.25)
\]

Then similar to (4.20), we have
\[J'(s) = 2 \int_\Omega \left( a^{ij}D_i v + c^i v D_i (e^{-2\psi} v) + b^i D_i (e^{-2\psi} v) + e^{-2\psi} D_i v \right) \]}
\[\geq 2 \int_\Omega a^{ij} D_i v (e^{-2\psi} v - 2e^{-2\psi} D_i v) + (c^i - b^i) v D_i (e^{-2\psi} v) \]}
\[= 2 \int_\Omega a^{ij} D_i v (e^{-2\psi} D_i v - 2e^{-2\psi} v D_i v) - 2 \int_\Omega (b^i - c^i)(D_i (e^{-2\psi} v) - e^{-2\psi} v D_i v) \]}
\[\geq 2\nu \int_\Omega e^{-2\psi} |Dv|^2 - \frac{4\gamma_1^1}{v} \int_\Omega e^{-2\psi} |Dv| |Dv| + 2 \int_\Omega (b^i - c^i)e^{-2\psi} v D_i v. \quad (4.26)
\]
Therefore, similar to (4.22), we have
\[J'(s) \geq - \left((4/v^3 + C_n^2 \Theta^2/v + 2C_n \Theta)\gamma_1^2 + C_n \Theta \gamma_2 \right) J(s), \]
and thus, similar to (4.24), we obtain
\[ \|Q_{t-s}\|_{L_2(\Omega)} \leq C(t-s)^{-\frac{\delta}{2}} e^{\gamma_1 \sqrt{\frac{t-s}{n} + (\gamma_1^2 + \mu_2^2)(t-s)}} \|g\|_{L_2(\Omega)}, \quad \forall s \leq t. \] (4.27)

From (4.17), (4.25), and the definitions of \( P_{t-s}^\psi f \) and \( Q_{t-s}^\psi g \), we obtain the duality relation
\[ \int_\Omega (P_{t-s}^\psi f) g = \int_\Omega f (Q_{t-s}^\psi g). \] (4.28)

This combined with (4.27) yields the \( L_1 \rightarrow L_2 \) estimate
\[ \|P_{t-s}^\psi f\|_{L_2(\Omega)} \leq C(t-s)^{-\frac{\delta}{2}} e^{\gamma_1 \sqrt{\frac{t-s}{n} + (\gamma_1^2 + \mu_2^2)(t-s)}} \|f\|_{L_1(\Omega)}, \quad \forall f \in C_c^\infty(\Omega). \] (4.29)

Then by noting \( P_{t-s}^\psi f = P_{t-(s/2)}^\psi (P_{t-(s/2)}^\psi f) \), we find from (4.24) and (4.29) that
\[ \|P_{t-s}^\psi f\|_{L_2(\Omega)} \leq C(t-s)^{-\frac{\delta}{2}} e^{\gamma_1 \sqrt{2(t-s) + (\gamma_1^2 + \mu_2^2)(t-s)}} \|f\|_{L_1(\Omega)}, \quad \forall f \in C_c^\infty(\Omega). \]

For fixed \( x, y \in \Omega \) with \( x \neq y \), we obtain from the above estimate and (4.13) that
\[ e^{\gamma_1 t-\gamma_1 y} |G(t-s, x, y)| \leq C(t-s)^{-\frac{\delta}{2}} e^{\gamma_1 \sqrt{2(t-s) + (\gamma_1^2 + \mu_2^2)(t-s)}}. \] (4.30)

This corresponds to [9, (3.19)] and by choosing an appropriate \( \psi \), we obtain the Gaussian bound (3.3). See [9, p. 3028] for details.

4.2.2. Case when \( p > n \). In the case when \( p > n \), we combine the argument of Aronson [2] with techniques in [3,13]. Let \( L(t) \) be as in (4.19). It follows from (4.20) that
\[ I'(t) \leq -2v \int_\Omega e^{\psi t} |D u|^2 + \frac{4\gamma_1}{v} \int_\Omega e^{\psi t} |D u| |D u| + 2\gamma_1 \int_\Omega |b - c e^{\psi t} u|^2. \]

Let \( \delta > 0 \) be a number to be fixed later. By integrating the above inequality in \( t \) over \([t_1, t_2]\), where \( s \leq t_1 \leq t_2 \leq t_1 + \delta \), and denoting
\[ S = [t_1, t_2] \times \Omega, \]
we have
\[ I(t_2) + 2\nu \int_\Omega e^{\psi t} |D u|^2 \leq I(t_1) + 2\gamma_1 \int_\Omega |b - c e^{\psi t} u|^2. \] (4.31)

By Young’s inequality, we have
\[ \frac{4\gamma_1}{v} \int_\Omega e^{\psi t} |D u| |D u| \leq \nu \int_\Omega e^{\psi t} |D u|^2 + \frac{4\gamma_1^2 \delta}{v^2} \|e^{\psi t} u\|_{L_2(S),S}^2. \] (4.32)

Also, by Hölder’s inequality, the condition (H), the embedding (2.3), and Young’s inequality, we estimate
\[ 2\gamma_1 \int_\Omega |b - c e^{\psi t} u|^2 \leq 2\gamma_1 \|b - c\|_{L_6(S)} \|e^{\psi t} u\|_{L_6(S)} \|e^{\psi t} u\|_{L_2(S)} \|e^{\psi t} u\|_{L_2(S),S} \]
\[ \leq 2\gamma_1 \Theta \sqrt{\delta} \|e^{\psi t} u\|_{L_2(S),S} \|e^{\psi t} u\|_{L_2(S)} \leq \frac{\nu}{4} \|e^{\psi t} u\|_{L_2(S)}^2 + \frac{4\gamma_1^2 \Theta^2 \delta}{v} \|e^{\psi t} u\|_{L_2(S)}^2, \]
where we use the fact that \((\hat{\rho}, \hat{\eta}) = (\frac{2p}{p+2}, \frac{2p}{q})\) satisfy (2.2). Note that
\[ \|e^{\psi t} u\|_{L_2(S)}^2 \leq 2\|e^{\psi t} u\|_{L_2(S)}^2 + 2\|D(e^{\psi t} u)\|_{L_2(S)}^2 \]
\[ \leq 2\|e^{\psi t} u\|_{L_2(S)}^2 + 4\gamma_1^2 \delta \|e^{\psi t} u\|_{L_2(S)}^2 + 4\|e^{\psi t} D u\|_{L_2(S)}^2. \]
Combining the above inequalities, we have

\[
2\gamma_1 \int_S |b - c| e^{\psi} |u|^2 \leq \nu \int_S e^{\psi} |Du|^2 + \left( \frac{\nu}{2} + \nu \gamma_1^2 \delta + \frac{4\gamma_1^2 \Theta^2 \delta}{\nu} \right) \|e^{\psi} u\|_{L^2(S)}^2. \tag{4.33}
\]

By substituting (4.32) and (4.33) back to (4.31), we obtain

\[
I(t_2) \leq I(t_1) + \left( \frac{\nu}{2} + \nu \gamma_1^2 \delta + \frac{4\gamma_1^2 \Theta^2 \delta}{\nu} \right) \|e^{\psi} u\|_{L^2(S)}^2. \tag{4.34}
\]

Recall that \(\nu \in (0, 1)\). We choose

\[
\delta = \frac{(3 - 2\nu)\nu^3}{4(4^4 + 4\Theta^2\nu^2 + 4\gamma_1^2)} \quad \text{so that} \quad \frac{\nu}{2} + \nu \gamma_1^2 \delta + \frac{4\gamma_1^2 \Theta^2 \delta}{\nu} + \frac{4\gamma_1^2 \delta}{\nu^3} = \frac{3}{4}. \tag{4.35}
\]

Then, we take the supremum over \(t_2 \in [t_1, t_1 + \delta]\) in (4.34) to get

\[
\max_{t_1 \leq t \leq t_1 + \delta} I(t) \leq 4I(t_1).
\]

In particular, by take \(t_1 = s\) and iterating, we have

\[
I(t) \leq 4^j I(s) = 4^j \|f\|_{L^2(\Omega)}^2 \quad \text{if} \quad s + (j - 1)\delta \leq t \leq s + j\delta,
\]

which combined with (4.35) yields

\[
I(t) \leq 4e^{\nu\gamma_1^2(t-s)} \|f\|_{L^2(\Omega)}^2 \quad \forall t \geq s, \quad \text{where} \quad \mu = \frac{2(4^4 + 4\Theta^2\nu^2 + 4) \ln 4}{(3 - 2\nu)\nu^3}.
\tag{4.36}
\]

which is equivalent to

\[
\|P_{\nu}\|_{L^2(\Omega)} \leq 2e^{\nu(\gamma_1^2(t-s))} \|f\|_{L^2(\Omega)} \quad \forall t \geq s. \tag{4.37}
\]

With the \(L_2 \to L_2\) estimate (4.37) and Lemma 2.15 at hand, we replicate the same argument in [9] p. 3028 to obtain the \(L_2 \to L_\infty\) estimate (c.f. (4.24))

\[
\|P_{\nu}\|_{L_\infty(\Omega)} \leq C(t-s)^{-\frac{\mu}{2} \sqrt{e^{\nu(\gamma_1^2)\gamma_1^2}(t-s)} \|f\|_{L^2(\Omega)}}, \quad \forall t \geq s.
\]

Similarly, we obtain from (4.26) that

\[
I(s) \leq 4e^{\nu\gamma_1^2(t-s)} \|g\|_{L^2(\Omega)}^2 \quad \forall s \leq t,
\]

which, combined with Lemma 2.15 and duality relation (4.28), yields the \(L_1 \to L_2\) estimate (c.f. (4.29))

\[
\|P_{\nu}\|_{L_1(\Omega)} \leq C(t-s)^{-\frac{\mu}{2} \sqrt{e^{\nu(\gamma_1^2)\gamma_1^2}}(t-s)} \|f\|_{L^1(\Omega)} \quad \forall f \in C_\infty(\Omega), t \geq s.
\]

Then, similar to (4.30), for \(x \neq y\), we have

\[
e^{\nu(x-s)}G(t, x, s, y) \leq C(t-s)^{-\frac{\mu}{2} \sqrt{e^{\nu(\gamma_1^2)\gamma_1^2}}}(t-s), \quad t > s.
\]

This corresponds to [3] (5.8). Note that \(\mu\), which is specified in (4.36), depends only on \(\nu\) and \(\Theta\). By choosing the function \(\psi\) appropriately, we obtain the Gaussian bound (5.3). See [3] p. 1670 for details.

5. Proof of Lemma 2.15

The proof is based on an original idea of De Giorgi [6] in the parabolic context as appears in [17]. See Seregin et al. [25] and Nazarov and Ural’tseva [21] for related results. We restrict ourselves to the case when \(u\) is a weak solution of \(Pu = f\) in \(\Omega^-\). The proof for the other case requires just a routine adjustment and we leave the details to the readers.
5.1. Case when $p > n$. We shall first treat the case when $p > n$ so that $q < \infty$. Let us denote

$$v = (u - k)_+ = \max(u - k, 0),$$

where $k > 0$ is to be chosen, and let $\zeta : \mathbb{R}^{n+1} \to [0, 1]$ be a smooth cut-off function such that

$$\text{supp}(\zeta) \cap \{t \leq t_0\} \subset (t_0 - r^2, t_0] \times B_r(x_0).$$

In what follows we shall write $\Omega_+ = \Omega \cap B_r(x_0)$ and $Q^+ = Q^+(X_0)$ for brevity. By testing $Pu = f$ with $\zeta^2 v$, using the assumption that $d - \text{div} b \geq 0$ together with $\zeta^2 uv \geq 0$, and noting that $Du = Dv$ on the set $\{u > k\} = \{v > 0\}$, we obtain

$$\frac{1}{2} \int_{Q^+} \zeta^2 v^2(t_1, x) \, dx + \int_{t_0 - t_1}^{t_1} \int_{\Omega_+} \zeta^2 \partial_i D_i v D_j v \, dx \, dt + \int_{t_0 - t_1}^{t_1} \int_{\Omega_+} 2 \partial_i \zeta D_{ij} v D_i v \, dx \, dt$$

$$\leq \int_{t_0 - t_1}^{t_1} \int_{\Omega_+} \zeta \partial_i \zeta \partial_i \zeta \, dx \, dt + \int_{t_0 - t_1}^{t_1} \int_{\Omega_+} \zeta \partial_i \zeta \partial_i \zeta \, dx \, dt + \int_{t_0 - t_1}^{t_1} \int_{\Omega_+} f \zeta^2 v \, dx \, dt$$

for all $t_1$ satisfying $t_0 - r^2 \leq t_1 \leq t_0$. By the assumption that $\text{div}(b - c) \geq 0$, we have

$$\int_{t_0 - r^2}^{t_1} \int_{\Omega_+} \zeta \partial_i \zeta \partial_i \zeta \, dx \, dt \leq \int_{t_0 - r^2}^{t_1} \int_{\Omega_+} \zeta (b - c) \partial_i \zeta \, dx \, dt.$$

Then by the condition (H1) and Young’s inequality, we obtain

$$\frac{1}{2} \|\zeta v\|_{L^2(Q^+)}^2 + \frac{v}{2} \int_{Q^+} \zeta^2 |Dv|^2$$

$$\leq 2 \int_{Q^+} |\partial_i \zeta|^2 + \frac{4}{v} \int_{Q^+} |D\zeta|^2 v^2 - 2 \int_{Q^+} \zeta (b - c) \partial_i \zeta + 2 \int_{Q^+} f \zeta^2 v. \quad (5.1)$$

We estimate the last two terms as follows. Note that $\zeta v \in \tilde{L}^2(Q^+)$). By using Hölder’s inequality, Young’s inequality, and the embedding (2.3), we have

$$\int_{Q^+} f \zeta^2 v \leq \|f\|_{L_p(Q^+)} \|\zeta v\|_{L^2(\{x \in Q^+ \cap \{u > k\}\})}$$

$$\leq \beta \|f\|_{L_p(Q^+)} \|\zeta v\|_{L^2(\{x \in Q^+ \cap \{u > k\}\})}$$

$$\leq \frac{v}{64} \|\zeta v\|_{L^2(Q^+)}^2 + \frac{16 \beta^2}{v} \|f\|_{L^2(Q^+)}^2 \|\zeta v\|_{L^2(\{x \in Q^+ \cap \{u > k\}\})}^2. \quad (5.2)$$

By Hölder’s inequality, the embedding (2.3), and Young’s inequality, we obtain

$$\int_{Q^+} \zeta (b - c) \partial_i \zeta \leq \|b - c\|_{L^p(\{x \in Q^+ \cap \{u > k\}\})} \|\zeta v\|_{L^2(\{x \in Q^+ \cap \{u > k\}\})} \|D\zeta v\|_{L^2(Q^+)}$$

$$\leq \beta \Theta \|\zeta v\|_{L^2(Q^+)} \|D\zeta v\|_{L^2(Q^+)}$$

$$\leq \frac{v}{64} \|\zeta v\|_{L^2(Q^+)}^2 + \frac{16 \beta^2 \Theta^2}{v} \|D\zeta v\|_{L^2(Q^+)}^2 \|\zeta v\|_{L^2(Q^+)}^2. \quad (5.3)$$

where we used the fact that the pair $(\frac{2p}{p+2}, \frac{2p}{q+2})$ satisfy the condition (2.2).

It follows from (5.1), (5.2), and (5.3) that

$$\frac{1}{2} \|\zeta v\|_{L^2(Q^+)}^2 + \frac{v}{2} \|\zeta Dv\|_{L^2(Q^+)}^2 \leq C(v, \beta, \Theta) \left(\|\partial_i \zeta\|_{L^\infty} + \|D\zeta\|_{L^2(Q^+)}^2 \right) \|\zeta v\|_{L^2(Q^+)}^2$$

$$+ \frac{32 \beta^2}{v} \|f\|_{L^2(Q^+)}^2 \|\zeta v\|_{L^2(\{x \in Q^+ \cap \{u > k\}\})}^2 + \frac{v}{16} \|\zeta v\|_{L^2(Q^+)}^2.$$
Then, it follows from (5.8) that
\[ |\{\zeta \}^2 \leq 2 |\{\zeta \}^2 + 2 |D(\zeta \})^2 | + 4 |\{\zeta \}^2 | + 4 |D\zeta |^2 | + 4 |D\zeta| |^2 \]
and \(0 < \nu < 1\), it follows that
\[ |\zeta(u-k) + I_{Q_r}^2 | \leq C \left( |\partial_t \zeta^2 | + |D\zeta |^2 \right) |(u-k) + I_{Q_r}^2 \]
where \(C = C(v, \beta, \Theta) = C(v, \beta, \Theta)\). On the other hand, by Hölder’s inequality we have
\[ |\zeta(u-k) + I_{Q_r}^2 | \leq |Q_r^{-1} \cup |u > k)| |\zeta(u-k) + I_{Q_r}^2 \]
It follows from (5.5), (5.4), and the embedding (2.3) that
\[ |\zeta(u-k) + I_{Q_r}^2 | \leq C \left( |\partial_t \zeta^2 | + |D\zeta |^2 \right) |(u-k) + I_{Q_r}^2 \]
Now, for \(m = 1, 2, \ldots\), we set
\[ r_m = r \left( \frac{1}{2} + \frac{1}{2m} \right), \quad k_m = k \left( 1 - \frac{1}{2m} \right), \quad Q_m := Q_{r_m} = (t_0 - r_m, t_0 + 1 \times (B_{r_m}(x_0) \cap \Omega), \quad \text{and let } \zeta_m : \mathbb{R}^{n+1} \rightarrow [0, 1] \text{ be smooth cut-off functions such that } \zeta_m = 1 \text{ on } Q_m, \quad \text{supp}(\zeta_m) \cap \{ t < t_0 \} \subset Q_m, \text{ and } \]
\[ |\zeta_m^2 | + |D\zeta_m |^2 + |D^2\zeta_m | \leq \frac{100 \cdot 2^{2m}}{r^2}. \]
By taking \(\zeta = \zeta_m\), \(r = r_m\), and \(k = k_{m+1}\) in (5.6), and then using obvious inequalities
\[ (k_{m+1} - k_m)^2 |Q_m \cap |u > k_{m+1}| | \int_{Q_m \cap |u > k_{m+1}|} (u - k_m)^2 \leq \int_{Q_m} (u - k_m)^2 | \]
and
\[ ||(u - k_{m+1})_+ ||_{L^2(Q_m)} \leq ||(u - k_m)_+ ||_{L^2(Q_m)}, \]
we have
\[ ||(u - k_{m+1})_+ ||_{L^2(Q_m)} \leq C \left( \frac{2^{2m}}{r^2 \left( \frac{2^{2m}}{k_m} \right) } \right) \left( \frac{2^{2m}}{k_m} \right) + \frac{100 \cdot 2^{2m}}{r^2}. \]
Let us denote
\[ Y_m := \frac{1}{k^{2m+2}} \| u - k_m \|_{L^2(Q_m)} \]
and assume \(k \geq r^2 \| f \|_{L^2(Q_m)} \).
Then, it follows from (5.8) that
\[ Y_{m+1} \leq C \left( \frac{2^{2m+2}}{r^{2m+2}} Y_m \right) + C \left( \frac{2^{2m+2}}{r^{2m+2}} k^{-2} r^2 \| f \|_{L^2(Q_m)} \right) \frac{2^{2m+2}}{r^{2m+2}} \]
\[ \leq K^{2m+2} \left( \frac{2^{2m+2}}{r^{2m+2}} Y_m \right). \]
where $K = K(n, v, p, \Theta) > 0$. By a well-known lemma on fast geometric convergence (see, e.g., [7, Lemma 15.1]), it follows that $Y_m \to 0$ provided

$$Y_1 = \frac{1}{K^{2n+2}} \| u_+ \|^2_{L^2(Q_r)} \leq \delta^2$$

for some $\delta = \delta(n, K) = \delta(n, v, p, \Theta) > 0$. Therefore, by taking

$$k = \max \left( r^2 \| f \|_{L^\infty(Q_r)}, \delta^{-1} r^{-\frac{2n}{p+2}} \| u_+ \|^2_{L^2(Q_r)} \right),$$

we see that

$$u \leq k \text{ in } Q_r.$$

By applying the same estimate to $-u$, we obtain (2.16).

5.2. Case when $p = n$. We now treat the case when $p = n$ and $q = \infty$. We proceed the same as in the case when $p > n$ until we reach (5.3), where by using (4.12), we instead obtain

$$- \int_{Q_r} \zeta b^i - c^i v^2 D_i \zeta = \int_{Q_r} (\Phi^{ij} - \Theta^{ij}(t)) D_j (\zeta v^2 D_i \zeta) = \int_{Q_r} (\Phi^{ij} - \Theta^{ij}(t)) (v^2 D_j (\zeta D_i \zeta) + 2vD_j \zeta D_i \zeta),$$

(5.9)

where we set

$$\Theta^{ij}(t) = \frac{1}{|B_r|} \int_{B_r} \Phi^{ij}(t, x) \, dx.$$

Fix a number $s \in (2, \frac{2(n+1)}{n})$. By using Hölder’s inequality and the John-Nirenberg inequality, we estimate

$$\int_{Q_r} (\Phi^{ij} - \Theta^{ij}(t)) v^2 D_j (\zeta D_i \zeta) \, dx \, dt$$

$$\leq \| D_j (\zeta D_i \zeta) \|_{L^\infty} \int_{Q_r} |\Phi^{ij} - \Theta^{ij}(t)| v^2 \, dx \, dt$$

$$\leq \| D_j (\zeta D_i \zeta) \|_{L^\infty} \left( \int_{B_r} \int_{B_r} |\Phi^{ij}(t, x) - \Theta^{ij}(t)| \frac{2n}{p+2} \, dx \, dt \right)^\frac{p+2}{2n} \left( \int_{Q_r} |v|^s \, dx \, dt \right)^{\frac{2n}{s}}$$

$$\leq \| D_j (\zeta D_i \zeta) \|_{L^\infty} (r^2 |B_r|)^{\frac{2n}{p+2}} \left( \int_{B_r} \int_{B_r} |\Phi^{ij}(t, x) - \Theta^{ij}(t)| \frac{2n}{p+2} \, dx \, dt \right)^\frac{p+2}{2n} \left( \| \zeta \|_{L^2(Q_r)} \right)^2$$

$$\leq C \left( \| D_i \zeta \|_{L^2}^2 + \| D^2 \zeta \|_{L^\infty} \right) r^{\frac{2n(n+1)}{p+2}} \Theta^{ij} \| v \|_{L^2(Q_r)}^s$$

(5.10)

where $C = C(n, s)$. Similarly, we estimate

$$\int_{Q_r} (\Phi^{ij} - \Theta^{ij}(t)) vD_j \zeta D_i \zeta \, dx \, dt$$

$$\leq \left( \int_{B_r} \int_{B_r} |\Phi^{ij}(t, x) - \Theta^{ij}(t)| \frac{2n}{p+2} \, dx \, dt \right)^\frac{p+2}{2n} \left( \int_{Q_r} |\zeta D \zeta|^2 \right)^{\frac{1}{2}} \left( \int_{Q_r} |\zeta D \zeta|^2 \right)^{\frac{1}{2}}$$

$$\leq C r^{\frac{2n(n+1)}{p+2}} \Theta \| \zeta D \zeta \|_{L^2(Q_r)} \| D \zeta \|_{L^2(Q_r)} \| \zeta \|_{L^\infty(Q_r)}$$

$$\leq \frac{v}{2} \| \zeta D \zeta \|_{L^2(Q_r)}^2 + \frac{2C^2 \Theta^2}{v} r^{\frac{2n(n+1)}{p+2}} \| D \zeta \|_{L^2(Q_r)} \| \zeta \|_{L^\infty(Q_r)}$$

(5.11)

where we used Young’s inequality at the last step.
Coming back to (5.9) and using (5.10) and (5.11), we obtain

\[- \int_{Q_{r/2}} \zeta(b′ - c′)v^2 \mathrm{D} \zeta \leq \frac{\nu}{8} \int_{Q_{r/2}} \zeta^2 |\mathcal{D}v|^2 + C(n, s, \Theta, \nu) \left( \|\mathcal{D} \zeta\|_{L^\infty}^2 + \|\mathcal{D}^2 \zeta\|_{L^\infty} \right) r \frac{(s+2m-2)}{\nu} \|v\|_{L^2(Q_{r/2})}^2. \tag{5.12} \]

Using (5.12) instead of (5.4), we obtain similar to (5.4) that

\[ \|\zeta(u - k)\|_{L^2(Q_{r/2})} \leq C \left( \|\partial_1 \zeta\|_{L^\infty} + \|\mathcal{D} \zeta\|_{L^\infty} + \|\mathcal{D}^2 \zeta\|_{L^\infty} \right) r \frac{(s+2m-2)}{\nu} \|u - k\|_{L^2(Q_{r/2})} \]

\[ + Cr^{s+4-2(n+2)/\nu} \|f\|_{L^2(Q_{r/2})} \|Q_r \cap \{u > k\}\|^2. \]

Also, similar to (5.5), we have

\[ \|\zeta(u - k)\|_{L^2(Q_{r/2})} \leq C \|\zeta\|_{L^\infty} \|u - k\|_{L^2(Q_{r/2})} \]

\[ \leq C \left( \|\partial_1 \zeta\|_{L^\infty} + \|\mathcal{D} \zeta\|_{L^\infty} + \|\mathcal{D}^2 \zeta\|_{L^\infty} \right) r \frac{(s+2m-2)}{\nu} \|u - k\|_{L^2(Q_{r/2})} \]

\[ + Cr^{s+4-2(n+2)/\nu} \|f\|_{L^2(Q_{r/2})} \|Q_r \cap \{u > k\}\|^2. \]

Take \( r_m, k_m, Q_m \), and \( \zeta_m \) as before. By setting \( \zeta = \zeta_m, r = r_m, \) and \( k = k_{m+1} \) in the preceding two inequalities, applying the embedding (2.3), and using

\[ (k_{m+1} - k_m)^4 \|Q_m \cap \{u > k_{m+1}\}\| \leq \int_{Q_m} (u - k_m)^4, \]

instead of (5.7), we obtain

\[ \|u - k_{m+1}\|_{L^2(Q_{r/2})}^2 \leq \frac{C2^{2m}}{r^2} \frac{(2m)^{2m}}{k^2} \|u - k_m\|_{L^2(Q_m)} \]

\[ + Cr^{s+4-2(n+2)/\nu} \|f\|_{L^2(Q_{r/2})} \frac{(2m)^{2m}}{k^2} \|u - k_m\|_{L^2(Q_m)} \]

\[ \leq C2^{4m+2s/n} \|f\|_{L^2(Q_{r/2})} Y_m^2 k^{-2} \]

\[ \leq C2^{4m+2s/n} \|f\|_{L^2(Q_{r/2})} Y_m^2 \]

which corresponds to (5.8). Now, if we set

\[ Y_m := \frac{1}{k^2} \|u - k_m\|_{L^2(Q_m)}, \]

then it follows from (5.13) that

\[ Y_{m+1} \leq C2^{4m+2s/n} Y_m^2 \]

\[ \leq C2^{4m+2s/n} \]

provided that \( k \geq \sqrt{r^2 \|f\|_{L^2(Q_{r/2})}} \). By the same argument involving fast geometric convergence as above, we see that

\[ u \leq C \left( r^2 \|f\|_{L^2(Q_{r/2})} + r^{-\frac{2m}{s}} \|u^+\|_{L^2(Q_{r/2})} \right) \]

in \( Q_{r/2} \).

By applying the same estimate to \(-u\), and applying a well-known covering argument (see, e.g., [11] pp. 80–82), we can replace the number \( s \) by 2 and get the estimate (2.16).

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