New Bounds for the Sine Function and Tangent Function

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Abstract: Using the power series expansion technique, this paper established two new inequalities for the sine function and tangent function bounded by the functions \(x^2(\sin(\lambda x)/(\lambda x))^\alpha\) and \(x^2(\tan(\mu x)/(\mu x))^\beta\). These results are better than the ones in the previous literature.

Keywords: Mitrinović–Adamović-type inequality; Becker–Stark-type inequality; circular functions

1. Introduction

Because of the fact the functions \(\cos x\) and \((\sin x)/x\) are less than 1 for \(x \in (0, \pi/2)\), in order to determine this relationship \((\sin x)/x\) and the weighted geometric mean of \(\cos x\) and 1, we examine the Taylor expansion of the following function:

\[
\frac{\sin x}{x} - (\cos x)^{\beta} = \left(\frac{1}{2} \beta - \frac{1}{6}\right) x^2 + \left(\frac{-1}{8} \beta^2 + \frac{1}{12} \beta + \frac{1}{120}\right) x^4 + \left(\frac{1}{48} \beta^3 - \frac{1}{24} \beta^2 + \frac{1}{45} \beta - \frac{1}{5040}\right) x^6 + O(x^8).
\]

When choosing \(\beta = 1/3\) in above formula we can obtain the following fact

\[
\frac{\sin x}{x} - (\cos x)^{1/3} = \frac{1}{45} x^4 + \frac{19}{5670} x^6 + O(x^8),
\]

which will motivate us to prove the following inequality

\[
\frac{\sin x}{x} > (\cos x)^{1/3}
\]

holds for \(0 < x < \pi/2\). The above inequality was confirmed by Mitrinović and Adamović in [1], so we call it Mitrinović–Adamović inequality. On the other hand, the relationship between \((\sin x)/x\) and the weighted arithmetic mean of \(\cos x\) and 1 has been discussed in Zhu [2] just published, described as the following inequality similarly:

\[
\frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3} \cos x
\]

or

\[
\frac{3 \sin x}{2 + \cos x} < x.
\]

In 1451, using a geometrical method Nicolaus De Cusa (1401–1464) discovered (3), and in 1664 when considering the estimation of \(\pi\) Christian Huygens (1629–1695) confirmed (2). In view of the above historical facts (see [3–10]), we call the inequality (2) Cusa-Huygens inequality.

In 2018, Zhu [11] shown two improvements to (3) as follows: the inequalities

\[
\frac{1}{180} x^5 < x - \frac{3 \sin x}{2 + \cos x}
\]
and
\[
\frac{1}{2100} x^7 < x - \frac{3 \sin x}{2 + \cos x} \left[ 1 + \frac{(1 - \cos x)^2}{9(3 + 2 \cos x)} \right]
\] (5)

hold for all \(x \in (0, \pi]\), where \(1/180\) and \(1/2100\) are the best constants in previous inequalities, respectively. Two results of previous proposition are corrections of Theorem 3.4.20 from monograph Mitrinović [9]. Malešević et al. [12] made a bilateral supplement to the above two inequalities. Chen and Cheung [13] obtained the bounds for \((\sin x)/x\) in term of \((2 + \cos x)/3\) as follows

\[
\left( \frac{2 + \cos x}{3} \right)^{\theta_0} < \frac{\sin x}{x} < \left( \frac{2 + \cos x}{3} \right)^{\theta_0}
\] (6)

holds for all \(0 < x < \pi/2\), where \(\theta_0 = 1\) and \(\theta_0 = (\ln \pi - \ln 2)/(\ln 3 - \ln 2)\) are the best possible constants in (6). The double inequality (6) was proved by Bagul [14] and Zhu [15] in different ways. In Zhu [15] some new improvements to inequality (2) can be found:

\[
\left( 1 - \frac{x^3}{3!} \right) \frac{2 + \cos x}{3} < \frac{\sin x}{x} < \left( 1 - \frac{x^4}{180} \right) \frac{\cos x + 2}{3},
\] (7)

\[
\left[ 1 + \frac{8(\pi - 3)}{\pi^3} x^2 \right] \frac{2 + \cos x}{3} - \frac{8(\pi - 3)}{\pi^3} x^2 \frac{\sin x}{x} < \left( 1 + \frac{1}{30} x^2 \right) \frac{2 + \cos x}{3} - \frac{1}{30} x^2,
\] (8)

and

\[
\left[ 1 + \frac{1}{30} x^2 + \frac{2(240 \pi - \pi^3 - 720)}{15 \pi^5} x^4 \right] \frac{2 + \cos x}{3} - \left[ \frac{1}{30} x^2 + \frac{2(240 \pi - \pi^3 - 720)}{15 \pi^5} x^4 \right] \frac{\sin x}{x} < \left( 1 + \frac{1}{30} x^2 + \frac{1}{840} x^4 \right) 2 + \cos x - \frac{1}{30} x^2
\] (9)

hold for \(0 < x < \pi/2\).

Bercu [16] used the truncations of Fourier cosine series to the inequality (2) and obtained an enhanced form of (2):

\[
\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\frac{1}{45} (1 - \cos x)^2, \quad 0 < x < \frac{\pi}{2}.
\] (10)

Bagul et al. [17] draw two conclusions about the improvement of inequality (2):

\[
- \left( \frac{2}{3} - \frac{2}{\pi} \right) \left( x - \sin x \right) < \frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left( \frac{2}{3} - \frac{2}{\pi} \right) \frac{1}{(\pi/2 - 1)} (x - \sin x)^2
\] (11)

and

\[
- \left( \frac{2}{3} - \frac{2}{\pi} \right) (x - x \cos x) < \frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left( \frac{2}{3} - \frac{2}{\pi} \right) (x - x \cos x)^2
\] (12)

hold for \(0 < x < \pi/2\).

Recently, Zhu [2] improved the famous inequality (2) using two different technology paths and draw two results as follows: for \(0 < x < \pi/2\), the two inequalities

\[
\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\frac{1}{180} x^4 \left( \frac{\sin x}{x} \right)^{2/7}
\] (13)
and

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left( \frac{2}{3} - \frac{2}{\pi} \right) \left( \sin x - x \cos x \right) \left( \frac{\pi^2 - 12}{(3\pi^2 - \pi^2)} \right)$$

(14)

hold with the best constant $-1/180$ and $(2/3 - 2/\pi)$ respectively.

Inequalities on two functions $(\sin x)/x$ and $(\tan x)/x$ arouse great enthusiasm of researchers. Interested readers can refer to these literatures [18–61] and monograph [62] which was edited by Rassias and Andrica.

This paper focuses on some new bounds for the functions $\sin(x)/x$ and $\tan(x)/x$ and wants to improve the following inequalities:

$$1 - \frac{\sin x}{x} > 0 \quad \text{and} \quad \frac{\tan x}{x} - 1 > 0.$$  

(15)

Recently, Wu and Bercu [63] thought of Fourier series technology to approximate these two functions. They considered the power series expansion of the following function

$$1 - \frac{\sin x}{x} = -(a + b \cos x + c \cos 2x + d \cos 3x) + x^2 \left( \frac{1}{2} b + 2c + \frac{9}{2} d + \frac{1}{6} \right) - x^4 \left( \frac{1}{24} b + \frac{2}{3} c + \frac{27}{8} d + \frac{1}{120} \right) + O(x^8).$$

To obtain a slightly higher precision approximation, they let

$$\begin{cases} 
    a + b + c + d & = 0 \\
    \frac{1}{2} b + 2c + \frac{9}{2} d + \frac{1}{6} & = 0 \\
    \frac{1}{24} b + \frac{2}{3} c + \frac{27}{8} d + \frac{1}{120} & = 0 \\
    \frac{1}{720} b + \frac{4}{45} c + 81d + \frac{1}{5040} & = 0
\end{cases}$$

(16)

to obtain these constants

$$a = \frac{359}{945}, \ b = -\frac{167}{420}, \ c = \frac{2}{105}, \ d = -\frac{1}{756}$$

and find

$$1 - \frac{\sin x}{x} = \left( \frac{359}{945} - \frac{167}{420} \cos x + \frac{2}{105} \cos 2x - \frac{1}{756} \cos 3x \right) = \frac{23}{226800} x^8 + O(x^{10}),$$

which leads them to prove

$$1 - \frac{\sin x}{x} > \frac{359}{945} - \frac{167}{420} \cos x + \frac{2}{105} \cos 2x - \frac{1}{756} \cos 3x$$

$$= \frac{(1 - \cos x)(-31 \cos x + 5 \cos^2 x + 341)}{945}.$$

This technique can be used to deal with the approximation problem of another function $\tan(x)/x - 1$, and then they obtain the following results.
Proposition 1 ([63]). The following inequalities

\[
1 - \frac{\sin x}{x} > \frac{1 - \cos x}{945} \left( -31 \cos x + 5 \cos^2 x + 341 \right),
\]

\[
\tan x - 1 > \frac{1 - \cos x}{945} \left( 604 \cos^2 x - 1817 \cos x + 1843 \right),
\]

\[
1 - \frac{\sin x}{x} > 1 - \left( \frac{2 + \cos x}{3} - \frac{1}{180} x^4 + \frac{1}{3780} x^6 \right)
\]

\[
> 1 - \left( 1 + \frac{1 - \cos x}{945} \left( -5 \cos^2 x + 31 \cos x - 341 \right) \right)
\]

(18)

hold for all \( x \in (0, \pi/2) \).

In this paper, we want to obtain an approximation with appropriate accuracy about these two functions. We examine the power series expansion of function in the following form

\[
1 - \frac{\sin x}{x} - ax^2 \left( \frac{\sin bx}{bx} \right)^c
\]

\[
= -x^2 \left( a - \frac{1}{6} \right) + x^4 \left( \frac{1}{6} ab^2 c - \frac{1}{120} \right) + x^6 \left[ a \left( \frac{1}{180} b^4 c - \frac{1}{72} b^4 c^2 \right) + \frac{1}{5040} \right]
\]

\[
+ x^8 \left[ \frac{1}{45360} ab^6 c \left( 35c^2 - 42c + 16 \right) - \frac{1}{362880} \right] + O(x^{10}),
\]

and let

\[
\begin{align*}
& a - \frac{1}{6} = 0 \\
& \frac{1}{6} ab^2 c - \frac{1}{120} = 0 \\
& a \left( \frac{1}{180} b^4 c - \frac{1}{72} b^4 c^2 \right) + \frac{1}{5040} = 0
\end{align*}
\]

(16)

(17)

(18)

(19)

(20)

(21)

to determine

\[
a = \frac{1}{6}, \quad b = \pm \frac{\sqrt{7}}{14}, \quad c = \frac{42}{5}.
\]

We can obtain that

\[
1 - \frac{\sin x}{x} - \frac{1}{6} x^2 \left( \frac{\sin \sqrt{7} x}{\sqrt{7} x} \right)^{\frac{42}{5}} = \frac{1}{4116000} x^{8} + O\left(x^{10}\right).
\]

In the same way, we obtain

\[
\tan x - 1 - \frac{1}{3} x^2 \left( \frac{\tan \sqrt{7} x}{\sqrt{7} x} \right)^{\frac{31}{12}} = \frac{12416}{18907875} x^{8} + O\left(x^{10}\right).
\]

With the above foreshadowing, we can now announce the main work of this paper which established two inequalities of exponential type for the functions \( 1 - \frac{\sin x}{x} \) and \( \frac{\tan x}{x} - 1 \) bounded by the function \( x^2 (\sin(\lambda x) / (\lambda x))^\alpha \) and \( x^2 (\tan(\mu x) / (\mu x))^\beta \) as follows.
Theorem 1. Let $0 < |x| < \pi/2$, $\phi = 1$ and

$$\varphi = \left[ \frac{24^5 \pi^{27} (\pi - 2)^5}{44^{27} 21 \left( \sin \frac{\sqrt{x}}{2 \sqrt{\pi}} \right)^{22}} \right]^{1/5} > 1.$$ 

Then the double inequality

$$\phi \left[ \frac{1}{6} x^2 \left( \frac{\sin \frac{x}{1 - 2x^2}}{2x / x} \right)^{\frac{4}{3}} \right] < 1 - \frac{\sin x}{x} < \varphi \left[ \frac{1}{6} x^2 \left( \frac{\sin \frac{x}{1 - 2x^2}}{2x / x} \right)^{\frac{4}{3}} \right]$$

holds with the best constants $\phi$ and $\varphi$.

Theorem 2. Let $0 < |x| < \pi/2$. Then

$$\frac{\tan x}{x} - 1 > \frac{1}{3} x^2 \left( \frac{\tan \sqrt{\frac{x}{2}}}{{\sqrt{\frac{x}{2}}} x} \right)^{\frac{294}{\pi}}$$

holds with the best constant $1/3$.

2. Lemmas

The proof of the main conclusions (Theorems 1 and 2) of this paper needs the following lemmas as the basis.

Lemma 1. Let $n \geq 3$, $n \in \mathbb{N}$,

$$\sigma_1 = \frac{\sqrt{7} + 14}{14} \approx 1.1890,$$

$$\sigma_2 = \frac{14 - \sqrt{7}}{14} \approx 0.81102,$$

and for $k \geq 4$,

$$a_k = -\frac{27}{2} \sqrt{7} \frac{x^2}{(2k + 2)!} + \frac{27}{2} \sqrt{7} \frac{x^2}{(2k + 2)!} - 21 \frac{x^2}{(2k)!} + 32 \sqrt{7} \frac{x^2}{(2k + 1)!} + \frac{5 \sqrt{7} + 21}{2} \frac{x^2}{(2k + 1)!} + \frac{21 - 5 \sqrt{7}}{2} \frac{x^2}{(2k + 1)!}.$$ 

Then $2a_{2n} - 5a_{2n+1} > 0$.

Proof. Since

$$a_{2n} = -\frac{27}{2} \sqrt{7} \frac{x^2}{(4n + 2)!} + \frac{27}{2} \sqrt{7} \frac{x^2}{(4n + 2)!} - 21 \frac{x^2}{(4n)!} + 32 \sqrt{7} \frac{x^2}{(4n + 1)!} + \frac{5 \sqrt{7} + 21}{2} \frac{x^2}{(4n + 1)!} + \frac{21 - 5 \sqrt{7}}{2} \frac{x^2}{(4n + 1)!}.$$
we compute to obtain

\[(4n + 4)! (2a_{2n} - 5a_{2n+1}) = \left[ \frac{5}{2} u(n) c_1^{4n+1} - 512 v(n) c_3^{4n} \right] + \frac{5}{2} w(n) c_2^{4n+1}, \]

where

\[ u(n) = \left( \frac{2688}{5} - 128\sqrt{7} \right) n^3 - \left( \frac{2304}{5} \sqrt{7} - \frac{5616}{5} \right) n^2 - \left( \frac{17 559}{35} \sqrt{7} - \frac{3277}{5} \right) n \]

\[ - \left( \frac{19 459}{140} \sqrt{7} - \frac{31 019}{280} \right), \]

\[ v(n) = (n + 1) \left( 21n^3 + \frac{55}{2} n^2 + \frac{1193}{128} n + \frac{1445}{3584} \right), \]

\[ w(n) = \left( \frac{128 \sqrt{7} + \frac{2688}{5} }{109 760} \right) n^3 + \left( \frac{2304}{5} \sqrt{7} + \frac{5616}{5} \right) n^2 + \left( \frac{17 559}{35} \sqrt{7} + \frac{3277}{5} \right) n \]

\[ + \left( \frac{14 436 973}{109 760} \sqrt{7} + \frac{75 065}{784} \right) \]

are positive for \( n \geq 3 \). In order to prove Lemma 1, it suffices to prove that for \( n \geq 3 \),

\[ \frac{5}{2} u(n) c_1^{4n+1} - 512 v(n) c_3^{4n} > 0 \iff \left( \frac{\sigma_1}{\sigma_3} \right)^{4n} > \frac{1024 v(n)}{5 \sigma_1 u(n)}. \]

To note the fact

\[ \frac{\sigma_1}{\sigma_3} = 1 + 2\sqrt{7} \approx 6.2915 > 6, \]

we only need to prove

\[ 6^{4n} > \frac{1024 v(n)}{5 \sigma_1 u(n)}. \quad (23) \]

By mathematical induction, we can prove the inequality (23). First, the inequality (23) is obviously true for \( n = 3 \). Assume that (23) holds for \( n = m \geq 3 \), that is,

\[ 6^{4m} > \frac{1024 v(m)}{5 \sigma_1 u(m)} \]

holds. In the following, we shall prove that (23) holds for \( n = m + 1 \). Since

\[ 6^{4(m+1)} = 6^4 \cdot 6^{4m} > 1296 \left( \frac{1024 v(m)}{5 \sigma_1 u(m)} \right) \]

we can complete the proof of (23) when showing that

\[ 1296 \left( \frac{1024 v(m)}{5 \sigma_1 u(m)} \right) > \frac{1024 v(m + 1)}{5 \sigma_1 u(m + 1)}, \]

or

\[ \frac{A}{B} := \frac{1296v(m)}{u(m)} > \frac{v(m + 1)}{u(m + 1)} := \frac{C}{D} \]
In fact,

\[
AD - BC = 1296v(m)u(m+1) - v(m+1)u(m)
\]

\[
= \left( \frac{48280 304 638 987 025}{200 704} - \frac{7757 241 843 454 045 \sqrt{7}}{100 035} \right)
\]

\[
+ \left( \frac{12 227 207 591 977 687}{28 672} - \frac{1898 338 908 715 519 \sqrt{7}}{14 336} \right)(m - 3)
\]

\[
+ \left( \frac{11 559 549 671 546 923}{35 840} - \frac{10 809 910 365 147 \sqrt{7}}{112} \right)(m - 3)^2
\]

\[
+ \left( \frac{17 268 465 300 163}{128} - \frac{24 831 245 768 359 \sqrt{7}}{640} \right)(m - 3)^3
\]

\[
+ \left( \frac{674 131 077 463}{20} - \frac{18 578 001 121 \sqrt{7}}{2} \right)(m - 3)^4
\]

\[
+ \left( \frac{5030 226 642}{5} - \frac{104 111 168 \sqrt{7}}{14} \right)(m - 3)^5
\]

\[
+ \left( \frac{12 227 207 591}{5} - \frac{1 041 111 168 \sqrt{7}}{14} \right)(m - 3)^6
\]

\[
+ \left( \frac{14 620 032}{5} - \frac{1 857 8001 121 \sqrt{7}}{14} \right)(m - 3)^7
\]

> 0

holds for all \(m \geq 3\). This completes the proof of Lemma 1. \(\Box\)

It is not difficult to prove the following conclusion in the similar way.

**Lemma 2.** Let \(n \geq 2, n \in \mathbb{N}\),

\[
\rho_1 = \frac{2}{\sqrt[4]{43}} + 2 \approx 3.873 6,
\]

\[
\rho_2 = \frac{2}{\sqrt[4]{43}} \approx 1.873 6,
\]

\[
\rho_3 = 2 - \frac{2}{\sqrt[4]{43}} \approx 0.126 45,
\]

and for \(k \geq 4\),

\[
b_k = \frac{17 \sqrt[4]{43}}{903 \cdot 2 (2k+1)!} - \frac{117 \sqrt[4]{43}}{4816 (2k+2)!} \rho_1^{2k+2}
\]

\[
+ \frac{1}{2 \sqrt[4]{43}} - \frac{1}{2 \sqrt[4]{43}} (2k)! - \frac{1}{2 \sqrt[4]{43}} (2k+1)!
\]

\[
+ \frac{283 \sqrt[4]{43}}{3612} \rho_2^{2k+1} - \frac{17 \sqrt[4]{43}}{903 \cdot 2 (2k+1)!} \rho_1^{2k+2}
\]

\[
+ \frac{117 \sqrt[4]{43}}{4816 (2k+2)!} \rho_3^{2k+2}.
\]

Then \(2b_{2n} - 5b_{2n+2} > 0\).

**3. Proofs of Theorems 1 and 2**

**Proof of Theorem 1.** Let

\[
F(x) = \ln \left( 1 - \frac{\sin x}{x} \right) - \ln \left[ \frac{1}{6} x^2 \left( \frac{\sin \frac{1}{2 \sqrt{7}} x}{\frac{1}{2 \sqrt{7}} x} \right)^{\frac{2}{5}} \right],
\]
where \(0 < |x| < \pi/2\). Since this function \(F(x)\) is even, let’s consider the problem on interval \((0, \pi/2)\). We compute to obtain

\[
F'(x) = \frac{2\sqrt{7}}{35} \frac{f(x)}{x(x \cos \sigma_1 x - \cos \sigma_2 x + 2x \sin \sigma_3 x)},
\]

where

\[
f(x) = \frac{27}{2} \sqrt{7} \cos \sigma_1 x - \frac{27}{2} \sqrt{7} \cos \sigma_2 x - 21x^2 \cos \sigma_3 x + 32\sqrt{7}x \sin \sigma_3 x \\
+ \left( \frac{5}{2} \sqrt{7} + \frac{21}{2} \right) x(x \sin \sigma_2 x) - \left( \frac{5}{2} \sqrt{7} - \frac{21}{2} \right) x(x \sin \sigma_1 x).
\]

(26)

and \(\sigma_i (i = 1, 2, 3)\) are defined as (21). Substituting

\[
\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}, \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}
\]

into (26), we can obtain that

\[
f(x) := \sum_{k=4}^{\infty} \frac{(-1)^k}{(2k + 1)!} a_k x^{2k+2} = \sum_{n=2}^{\infty} \left( a_{2n} x^{4n+2} - a_{2n+1} x^{4n+4} \right)
= \left( \frac{1}{274400} x^{10} - \frac{163}{1521273600} x^{12} \right) + \sum_{n=3}^{\infty} \left( a_{2n} x^{4n+2} - a_{2n+1} x^{4n+4} \right),
\]

where \(a_k\) is defined by (22). Since

\[
\frac{1}{274400} x^{10} - \frac{163}{1521273600} x^{12} > 0
\]

for all \(x \in (0, \pi/2)\), we can determine the positive definiteness of the function \(f(x)\) on \((0, \pi/2)\) when proving

\[
a_{2n} x^{4n+2} - a_{2n+1} x^{4n+4} > 0 \iff x^2 < \frac{a_{2n}}{a_{2n+1}}
\]

(28)

for \(n \geq 3\). Since

\[
x^2 < \left( \frac{\pi}{2} \right)^2 \approx 2.4674 < \frac{5}{2},
\]

we can prove (28) when proving that for \(n \geq 3\),

\[
\frac{5}{2} < \frac{a_{2n}}{a_{2n+1}}
\]

or

\[
2a_{2n} - 5a_{2n+1} > 0,
\]

which comes from Lemma 1.

So \(f(x) > 0\) and \(F(x)\) is increasing on \((0, \pi/2)\). In view of

\[
F(0^+) = 0, \ F(1^-) = \frac{1}{5} \ln \left( \frac{245 \pi^{27} (\pi - 2)^5}{442721 \left( \sin \frac{\sqrt{7}}{2} \pi \right)^{42}} \right) > 0,
\]

the proof of Theorem 1 is complete. \(\square\)
Proof of Theorem 2. Let
\[ G(x) = \ln \left( \frac{\tan x}{x} - 1 \right) - \ln \left[ \frac{1}{3} x^2 \left( \frac{\tan \sqrt{43} x}{\sqrt{43} x} \right)^{\frac{2n}{3x}} \right], \quad 0 < x < \frac{\pi}{2}. \]

Then
\[ G'(x) = \frac{1806\sqrt{43}}{9245} \frac{g(x)}{x\left( \tan \frac{43}{7} x \right) \left( \tan x - x \right) \cos^2 \frac{\sqrt{43}}{\sqrt{7}} x \cos^2 x}, \]

where
\[ g(x) = \frac{117\sqrt{43}}{4816} \cos \rho_1 x - \frac{117\sqrt{43}}{4816} \cos \rho_3 x - \frac{1}{2} x \sin 2x + \frac{1}{2} x^2 \cos 2x + \frac{1}{2} x^2 \]
\[ + \frac{283\sqrt{43}}{3612} x \sin \rho_2 x + \frac{17\sqrt{43}}{903 \cdot 2} x \sin \rho_1 x - \frac{17\sqrt{43}}{903 \cdot 2} x \sin \rho_3 x, \tag{29} \]

where \( \rho_i \) \( (i = 1, 2, 3) \) are defined as (24). Substituting (27) into (29), we can obtain that
\[ g(x) = \sum_{k=4}^{\infty} (-1)^k b_k x^{2k+2}, \]

where \( b_k \) is defined by (25). We can rewrite \( g(x) \) as
\[ g(x) = \sum_{n=2}^{\infty} \left( b_{2n} x^{4n+2} - b_{2n+1} x^{4n+4} \right). \]

Then, we determine the positive definiteness of the function \( g(x) \) on \((0, \pi/2)\) when proving
\[ b_{2n} x^{4n+2} - b_{2n+1} x^{4n+4} > 0 \iff x^2 < \frac{b_{2n}}{b_{2n+1}}, \tag{30} \]

for \( n \geq 3 \). Since
\[ x^2 < \left( \frac{\pi}{2} \right)^2 = \frac{\pi^2}{4} \approx 2.4674 < \frac{5}{2}, \]
we can prove (30) when proving that for \( n \geq 3 \),
\[ \frac{5}{2} < \frac{b_{2n}}{b_{2n+1}}, \]

which is the result of Lemma 2. So \( g(x) > 0 \) and \( G(x) \) is strictly increasing on \((0, \pi/2)\). Therefore \( G(x) > G(0^+) = 0 \). Considering the reason
\[ \lim_{x \to 0^+} \frac{\tan x}{x^2 \left( \frac{\tan \sqrt{43} x}{\sqrt{43} x} \right)^{\frac{2n}{3x}}} = \frac{1}{5}, \]
the proof of Theorem 2 is completed. \( \Box \)

4. Comparison of New and Old Results

When letting \( \phi = 1 \) in (19) we can obtain
\[ 1 - \frac{\sin x}{x} > \frac{1}{6} x^2 \left( \frac{1}{\sqrt{7}} x \right)^{\frac{42}{5}}. \tag{31} \]
By using the similar proof method of Theorem 1 it is not difficult to prove the following results:

\[
\frac{1}{6} x^2 \left( \frac{\sin \frac{1}{2\sqrt{7}} x}{\frac{1}{2\sqrt{7}} x} \right)^{4/5} > 1 - \left( \frac{2 + \cos x}{3} - \frac{1}{180} x^4 + \frac{1}{3780} x^6 \right),
\]

(32)

\[
\frac{1}{3} x^2 \left( \frac{\tan \frac{\sqrt{21}}{2 \sqrt{7}} x}{\frac{\sqrt{21}}{2 \sqrt{7}} x} \right)^{234/215} > \frac{(1 - \cos x) (604 \cos^2 x - 1817 \cos x + 1843)}{945}
\]

(33)

hold for all \( x \in (0, \pi/2) \). So new results (19) and (20) are better that the old ones (16) and (17), respectively. In addition, there are deeper conclusions:

\[
1 - \frac{\sin x}{x} > \frac{1}{6} x^2 \left( \frac{\sin \frac{1}{2\sqrt{7}} x}{\frac{1}{2\sqrt{7}} x} \right)^{4/5}
\]

\[
> 1 - \left( \frac{2 + \cos x}{3} - \frac{1}{180} x^4 + \frac{1}{3780} x^6 \right)
\]

(34)

\[
> \frac{(1 - \cos x) (5 \cos^2 x - 31 \cos x + 341)}{945} > 1 - \frac{2 + \cos x}{3}.
\]

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