BOUNDARY STABILIZATION OF PARABOLIC NONLINEAR EQUATIONS

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Abstract. We design here a feedback stabilizing Dirichlet boundary controller for the equilibrium solutions to parabolic equations. These results extend those ones in [1], providing a feedback controller expressed in terms of the eigenfunctions $\phi_j$ corresponding to the unstable eigenvalues $\{\lambda_j\}_{j=1}^N$ of the operator corresponding to the linearized equation. In [1], the stabilizability result is conditioned by the require of linear independence of $\{\frac{\partial}{\partial \nu} \phi_j\}_{j=1}^N$, on the part of the boundary where control acts. In this work, we design a similar control as in [1], and show that it assures the stability of the system without any a priori assumption. At the end of the paper, some examples are provided. More exactly, boundary stabilization of the heat equation and the Fitzhugh-Nagumo equation.

Keywords: parabolic-like equations, feedback controller, eigenvalue.

1. Introduction

We consider the following parabolic boundary stabilization problem in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3, \ldots$, with a smooth boundary $\partial \Omega$, split in two, i.e., $\partial \Omega = \Gamma_1 \cup \Gamma_2$, such that the Lebesgue measure $\sigma(\Gamma_1) \neq 0$:

$$
\begin{align*}
&y(t, x) - \Delta y(t, x) + f(x, y) = 0, \quad t > 0, \quad x \in \Omega, \\
y(t, x) = u(t, x), \quad t > 0, \quad x \in \Gamma_1, \\
&\frac{\partial}{\partial \nu} y(t, x) = 0, \quad t > 0, \quad x \in \Gamma_2, \\
y(0, x) = y_o(x), \quad x \in \Omega.
\end{align*}
$$

(1.1)

Here, $\nu$ stands for the outward unit normal on the boundary $\partial \Omega$; $y_o \in L^2(\Omega)$ is given (the initial data); and $f = f(x, y)$ is a nonlinear function such that $f, f_y \in C(\overline{\Omega} \times \mathbb{R})$ (here, $f_y$ stands for the partial derivative of $f$ with respect to $y$, i.e., $\frac{\partial}{\partial y} f$). On the part of the boundary $\Gamma_1$ it is applied the control $u$, while $\Gamma_2$ is insulated.

Let $y_e \in C^1(\overline{\Omega})$ be an equilibrium solution to (1.1), that is, $y_e$ satisfies

$$
-\Delta y_e + f(x, y_e) = 0 \text{ in } \Omega; \quad y_e = 0 \text{ on } \Gamma_1, \quad \frac{\partial}{\partial \nu} y_e = 0 \text{ on } \Gamma_2.
$$

Then, defining the fluctuation variable $y := y - y_e$, equation (1.1) can be rewritten as

$$
\begin{align*}
&y(t, x) - \Delta y(t, x) + f(x, y + y_e) - f(x, y_e) = 0, \quad t > 0, \quad x \in \Omega, \\
y(t, x) = u(t, x) - y_e(x), \quad t > 0, \quad x \in \Gamma_1, \\
&\frac{\partial}{\partial \nu} y(t, x) = 0, \quad t > 0, \quad x \in \Gamma_2, \\
y(0, x) = y_o(x) := y_o(x) - y_e(x), \quad x \in \Omega.
\end{align*}
$$

(1.2)
Our main concern here is the design of a boundary feedback controller $u$ that stabilizes exponentially the equilibrium state $y_e$ in (1.1), or, equivalently, the zero solution in (1.2).

The problem of boundary feedback stabilization of this kind of parabolic-like equations was first solved in the pioneering work [11]. Then, several others methods were proposed for deriving new types of controls, like backstepping approach, see [3, 4, 7, 12], or the proportional type controllers designed in [1, 2, 10]. The last one provides a simple feedback stabilizer expressed in terms of the unstable eigenfunctions for the linear system, which is easily implementable. However, the stabilizing procedure is conditioned by the requirement that the system

$$\{ \frac{\partial}{\partial \nu} \phi_i \}_{i=1}^N$$

is linearly independent in $L^2(\Gamma_1)$ (here, $\phi_i$, $i = 1, ..., N$, are the first $N$ eigenfunctions of the linearized operator $A$ defined below). In the present work, we shall extend these results. More exactly, we propose a similar type control as that one in [1], and show that it assures the stability of the system without any a priori assumption. Briefly, the idea is the following: in the case when the system

$$\{ \frac{\partial}{\partial \nu} \phi_i \}_{i=1}^N$$

is not linearly independent, the corresponding Gram matrix is singular, and the approach in [1] fails because the controller is not well-defined. However, this issue may be overcome as follows: for the beginning, in order to clarify the method, we shall work under the assumption that the $N \in \mathbb{N}$ unstable eigenvalues are simple. We involve $N$ matrices obtained from the gramian, multiplied by some diagonal matrices, and show that their sum is invertible, even if each one is not. Therefore, we may well-define $u$ as the sum of those $N$ different controllers obtained from those matrices, and arrive to similar stabilization results as in [1]. Then, doing some additional tricks we show that, in fact, we may drop the hypothesis of simple eigenvalues as-well, and prove that the same type of controller assures the stability. Finally, via a fixed point argument, locally stabilization results may be deduced for the fully nonlinear system.

On this subject, it should be mentioned also the work [5], where controllers, of the above type, are designed. Instead of the eigenfunctions system, they show that there exists a (possible different) family of functions for which, the corresponding proportional controller assures the stability. That result is based on some unique continuation property for the operators involved. However, in comparison to that result, here, the form of the feedback is given explicitly and no a priori condition on the linear operator is imposed.

2. Notations, the feedback law and the main results

2.1. The case of simple eigenvalues. The main ingredient toward the proposed goal is the stabilization of the linearized system corresponding to (1.2), given by,

$$\begin{cases}
y_{t}(t, x) - \Delta y(t, x) + a(x)y(t, x) = 0, & t > 0, \ x \in \Omega, \\
y(t, x) = v(t, x), & t > 0, \ x \in \Gamma_1, \\
\frac{\partial}{\partial \nu} y(t, x) = 0, & t > 0, \ x \in \Gamma_2, \\
y(0, x) = y_0, & x \in \Omega,
\end{cases}$$

(2.1)

where we have denoted by $a(x) = f_y(x, y_0(x))$, $x \in \Omega$. Let us define the operator $A : D(A) \to L^2(\Omega)$ as

$$Ay := -\Delta y + a(x)y, \ \forall y \in D(A) = \left\{ y \in H^2(\Omega) : y|_{\Gamma_1} = 0, \ \frac{\partial}{\partial \nu} y|_{\Gamma_2} = 0 \right\}.$$
Notice that \( A \) is self-adjoint, moreover, it can be shown that \( A \) has compact resolvent, therefore, it has a countable set of eigenvalues, denoted by \( \{ \lambda_i \}_{i=1}^{\infty} \) (repeated according to their multiplicity) such that, given \( \rho > 0 \), there exists only a finite number of eigenvalues \( \{ \lambda_i \}_{i=1}^{N} \) with \( \lambda_i < \rho \), \( i = 1, \ldots, N \) (we call them the unstable eigenvalues). So, \( \lambda_i \geq \rho \), for all \( i \geq N + 1 \) (for more details, see \cite{1}).

Let us denote by \( \{ \phi_i \}_{i=1}^{\infty} \) the corresponding eigenfunctions, that can be chosen in such a way to form an orthonormal basis of the space \( D(A) \). In this subsection we shall assume that

\[
(H) \quad \text{All the eigenvalues } \lambda_i, \ i = 1, \ldots, N, \text{ are simple,} \\
\lambda_1 < \lambda_2 < \ldots < \lambda_N. \quad (2.2)
\]

hence, we may (eventually) redefine the first \( N \) eigenvalues such that

\[
\lambda_1 < \lambda_2 < \ldots < \lambda_N. \quad (2.3)
\]

In the following subsection we shall drop this assumption, showing that the below proposed feedback law assures the stability, without any constraints.

One of the main results of this work, Theorem 2.1 below, amounts to saying that there is a boundary feedback controller of the form

\[
u = \left( T \right. \left. A \left( \begin{array}{c}
\langle y(t) - y_e, \phi_1 \rangle \\
\langle y(t) - y_e, \phi_2 \rangle \\
\vdots \\
\langle y(t) - y_e, \phi_N \rangle
\end{array} \right), \left( \begin{array}{c}
1 \\
1 \\
\vdots \\
1
\end{array} \right) \right) + y_e, \quad (2.4)
\]

which globally exponentially stabilizes the linear system (2.1) (and locally the nonlinear system (1.2), that is Theorem 2.2 below). Here \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( L^2(\Omega) \); while \( \langle \cdot, \cdot \rangle_N \) denotes the scalar product in \( \mathbb{R}^N \). Moreover, \( T \) is defined as

\[
T := \left( \begin{array}{cccc}
\frac{1}{\gamma_1 - \lambda_1} \frac{\partial}{\partial \nu} \phi_1(x) & \frac{1}{\gamma_2 - \lambda_2} \frac{\partial}{\partial \nu} \phi_2(x) & \ldots & \frac{1}{\gamma_N - \lambda_N} \frac{\partial}{\partial \nu} \phi_N(x) \\
\frac{1}{\gamma_1 - \lambda_1} \frac{\partial}{\partial \nu} \phi_1(x) & \frac{1}{\gamma_2 - \lambda_2} \frac{\partial}{\partial \nu} \phi_2(x) & \ldots & \frac{1}{\gamma_N - \lambda_N} \frac{\partial}{\partial \nu} \phi_N(x) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\gamma_1 - \lambda_1} \frac{\partial}{\partial \nu} \phi_1(x) & \frac{1}{\gamma_2 - \lambda_2} \frac{\partial}{\partial \nu} \phi_2(x) & \ldots & \frac{1}{\gamma_N - \lambda_N} \frac{\partial}{\partial \nu} \phi_N(x)
\end{array} \right) \quad (2.5)
\]

for

\[
B := \left( \begin{array}{cccc}
\langle \frac{\partial}{\partial \nu} \phi_1, \frac{\partial}{\partial \nu} \phi_1 \rangle_0 & \langle \frac{\partial}{\partial \nu} \phi_1, \frac{\partial}{\partial \nu} \phi_2 \rangle_0 & \ldots & \langle \frac{\partial}{\partial \nu} \phi_1, \frac{\partial}{\partial \nu} \phi_N \rangle_0 \\
\langle \frac{\partial}{\partial \nu} \phi_2, \frac{\partial}{\partial \nu} \phi_1 \rangle_0 & \langle \frac{\partial}{\partial \nu} \phi_2, \frac{\partial}{\partial \nu} \phi_2 \rangle_0 & \ldots & \langle \frac{\partial}{\partial \nu} \phi_2, \frac{\partial}{\partial \nu} \phi_N \rangle_0 \\
\vdots & \vdots & \ddots & \vdots \\
\langle \frac{\partial}{\partial \nu} \phi_N, \frac{\partial}{\partial \nu} \phi_1 \rangle_0 & \langle \frac{\partial}{\partial \nu} \phi_N, \frac{\partial}{\partial \nu} \phi_2 \rangle_0 & \ldots & \langle \frac{\partial}{\partial \nu} \phi_N, \frac{\partial}{\partial \nu} \phi_N \rangle_0
\end{array} \right) \quad (2.8)
\]

and

\[
\Lambda_{\gamma_k} := \left( \begin{array}{ccccc}
\frac{1}{\gamma_k - \lambda_1} & 0 & \ldots & 0 \\
0 & \frac{1}{\gamma_k - \lambda_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{\gamma_k - \lambda_N}
\end{array} \right), \ k = 1, \ldots, N \quad (2.9)
\]

(by \( \langle \cdot, \cdot \rangle_0 \) we mean the scalar product in \( L^2(\Gamma_1) \)). Note that \( B \) is nothing but the Gram matrix of the system \( \{ \frac{\partial}{\partial \nu} \phi_1, \ldots, \frac{\partial}{\partial \nu} \phi_N \} \) in the Hilbert space \( L^2(\Gamma_1) \). Since it
Lemma 2.1. For any $\rho < \gamma_1 < \gamma_2 < \ldots < \gamma_N$, we have

$$\begin{vmatrix}
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N}
\end{vmatrix} \neq 0.$$  \hspace{1cm} (2.10)

Proof. Let us prove this by mathematical induction over $N$. Step 1, for $N = 2$, we have

$$\begin{vmatrix}
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2}
\end{vmatrix} = -\frac{(\lambda_1 - \lambda_2)(\gamma_1 - \gamma_2)}{(\gamma_1 - \lambda_1)(\gamma_2 - \lambda_2)(\gamma_1 - \lambda_2)} \neq 0,$$

since $\lambda_1 < \lambda_2$ and $\gamma_1 < \gamma_2$.

Step 2, we assume that for $N = 1$ the claim is true and prove it for $N$. To this end we have

$$\begin{vmatrix}
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N}
\end{vmatrix} = (-1)^{N-1} \frac{\lambda_N - \lambda_1}{\gamma_N - \lambda_N} \prod_{k=1}^{N-1} \frac{\lambda_N - \lambda_k}{\gamma_N - \lambda_N}.$$  

(Subtracting from the first column the $N^{th}$ one, ..., from the $(N-1)^{th}$ column the $N^{th}$ one)

$$= (-1)^{N-1} \frac{\lambda_N - \lambda_1}{\gamma_N - \lambda_N} \prod_{k=1}^{N-1} \frac{\lambda_N - \lambda_k}{\gamma_N - \lambda_N} \begin{vmatrix}
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\cdots & \cdots & \cdots & \cdots
\end{vmatrix}.$$  

(Subtracting the $N^{th}$ line from the first one, ..., the $(N-1)^{th}$ line from the $(N-1)^{th}$ one)

$$= (-1)^{N-1} \frac{\lambda_N - \lambda_1}{\gamma_N - \lambda_N} \prod_{k=1}^{N-1} \frac{\lambda_N - \lambda_k}{\gamma_N - \lambda_N} \begin{vmatrix}
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{\gamma_1 - \lambda_1} & \frac{1}{\gamma_2 - \lambda_2} & \cdots & \frac{1}{\gamma_N - \lambda_N} \\
\cdots & \cdots & \cdots & \cdots
\end{vmatrix} \neq 0,$$

by the inductive hypothesis and the fact that $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ and $\gamma_1 < \gamma_2 < \ldots < \gamma_N$.

Lemma 2.2. The sum $B_1 + B_2 + \ldots + B_N$ is an invertible matrix, where $B_k$, $k = 1, \ldots, N$ are introduced in relation (2.7).
Proof. Arguing by contradiction, let us assume that there is \( z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} \in \mathbb{R}^N \), nonzero, such that \((B_1 + \ldots + B_N)z = 0\). It follows that

\[
\sum_{k=1}^{N} \langle B_k z, z \rangle_N = 0,
\]

or, equivalently,

\[
\sum_{k=1}^{N} \int_{\Gamma_1} \left( \sum_{i=1}^{N} z_i \frac{1}{\gamma_k - \lambda_i} \frac{\partial}{\partial \nu} \phi_i(x) \right)^2 \, dx = 0.
\]

Because the Lebesgue measure of \( \Gamma_1 \) is nonzero, we deduce from the above that

\[
\sum_{i=1}^{N} z_i \frac{1}{\gamma_k - \lambda_i} \frac{\partial}{\partial \nu} \phi_i(x) = 0, \quad \text{a.e. on } \Gamma_1,
\]

for all \( k = 1, \ldots, N \). This gives a \( N \times N \) linear homogeneous system with the unknowns \( z_i, i = 1, \ldots, N \), for almost all \( x \in \Gamma_1 \). By the unique continuation property of the eigenfunctions \( \{ \phi_i \}_{i=1}^{N} \), we know that, for all \( i = 1, 2, \ldots, N \), \( \frac{\partial}{\partial \nu} \phi_i \neq 0 \) on \( \Gamma_1 \).

So, there is some \( \bar{x} \in \Gamma_1 \) such that \( \frac{\partial}{\partial \nu} \phi_i(\bar{x}) \neq 0, \forall i = 1, \ldots, N \). The determinant of the matrix of the corresponding system is

\[
\begin{vmatrix}
\frac{1}{\gamma_1 - \lambda_1} \frac{\partial}{\partial \nu} \phi_1(\bar{x}) & \frac{1}{\gamma_1 - \lambda_2} \frac{\partial}{\partial \nu} \phi_2(\bar{x}) & \ldots & \frac{1}{\gamma_1 - \lambda_N} \frac{\partial}{\partial \nu} \phi_N(\bar{x}) \\
\frac{1}{\gamma_2 - \lambda_1} \frac{\partial}{\partial \nu} \phi_1(\bar{x}) & \frac{1}{\gamma_2 - \lambda_2} \frac{\partial}{\partial \nu} \phi_2(\bar{x}) & \ldots & \frac{1}{\gamma_2 - \lambda_N} \frac{\partial}{\partial \nu} \phi_N(\bar{x}) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\gamma_N - \lambda_1} \frac{\partial}{\partial \nu} \phi_1(\bar{x}) & \frac{1}{\gamma_N - \lambda_2} \frac{\partial}{\partial \nu} \phi_2(\bar{x}) & \ldots & \frac{1}{\gamma_N - \lambda_N} \frac{\partial}{\partial \nu} \phi_N(\bar{x}) \\
\end{vmatrix}
\]

= \prod_{i=1}^{N} \frac{\partial}{\partial \nu} \phi_i(\bar{x}) \neq 0,

by Lemma \ref{T1} Hence, necessarily \( z = 0 \). This is in contradiction with our assumption. We conclude that the sum \( B_1 + \ldots + B_N \) is indeed an invertible matrix. \( \Box \)

We are now able to state and prove the first stabilization result for the linearized system \( \ref{T1} \), under the hypothesis \( (H) \).

**Theorem 2.1.** Assume that hypothesis \( (H) \) holds. The feedback \( u \), given by \( \ref{T1} \), exponentially stabilizes the linearized system \( \ref{T1} \). More precisely, the solution \( y \) to the system

\[
\begin{align*}
\bigg\{ y_t(t, x) - \Delta y(t, x) + f(y(x, y_0(x)))y(t, x) &= 0, \quad t > 0, \quad x \in \Omega, \\
y(t_0, x) &= \begin{pmatrix} y(t_0, \phi_1) \\ y(t_0, \phi_2) \\ \vdots \\ y(t_0, \phi_N) \end{pmatrix}, \\
\frac{\partial}{\partial \nu} y(t, x) &= \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \forall x \in \Gamma_1, \\
y(0, x) &= y_0, \quad x \in \Omega,
\end{align*}
\]

\( \tag{2.11} \)
satisfies the exponential decay
\[ \|y(t)\|^2 \leq Ce^{-\mu t}\|y_0\|^2, \quad t \geq 0, \]  \hspace{1cm} (2.12) \[ \text{for a prescribed } \mu > 0, \text{ and a constant } C > 0. \text{ Here } T \text{ is introduced in relation (2.5), while } A \text{ is introduced in relation (2.4). } \| \cdot \| \text{ denotes the norm in } L^2(\Omega). \]

**Proof.** The proof is based on similar ideas with that ones in the proof of [1, Theorem 3.1]. We shall prove only the fact that the first \( N \) modes of the solution \( y \) are exponentially decaying. The rest of it will be omitted because it is almost identically the counterpart in the proof of [1, Theorem 3.1].

Firstly, in order to "lift" the boundary conditions, we introduce the Dirichlet map, as follows: given \( \alpha \in L^2(\Gamma_1) \) and \( \gamma > 0 \), we denote by \( D_\gamma \alpha := y \), the solution to the equation
\[
\begin{cases}
-\Delta y(x) + a(x)y(x) - 2 \sum_{k=1}^{N} \lambda_k \langle y, \phi_k \rangle \phi_k(x) + \gamma y(x) = 0, & \text{for } x \in \Omega, \\
y = \alpha, & \text{on } \Gamma_1, \\
\frac{\partial y}{\partial \nu} = 0, & \text{on } \Gamma_2.
\end{cases} \hspace{1cm} (2.13)
\]
For \( \gamma > 0 \) large enough equation (2.13) has a unique solution, defining so the map \( D_\gamma \in L(L^2(\Gamma_1), H^2(\Omega)) \) (see, e.g., [9]). Besides this, we have
\[
\|D_\gamma \alpha\|_{H^2} \leq \frac{C}{\gamma} \|\alpha\|_{L^2(\Gamma_1)}, \quad \forall \alpha \in L^2(\Gamma_1).
\]

In the following, we need to compute the scalar product \( \langle D_\gamma \alpha, \phi_i \rangle \), for all \( i \in \{1, ..., N\} \). To this end, we scalarly multiply equation (2.13) by \( \phi_i \), to get, via Green’s formula, that
\[
0 = -\int_\Omega \Delta y(x) \phi_i(x) \, dx + \int_\Omega a(x)y(x) \phi_i(x) \, dx + (\gamma - 2\lambda_i) \int_\Omega y(x) \phi_i(x) \, dx \\
= \int_{\Gamma_1} \alpha(x) \frac{\partial}{\partial \nu} \phi_i(x) \, d\sigma + \int_\Omega y(x)(-\Delta \phi_i(x) + a(x)\phi_i(x)) \, dx + (\gamma - 2\lambda_i) \int_\Omega y(x) \phi_i(x) \, dx.
\hspace{1cm} (2.14)
\]
It yields that
\[
\langle D_\gamma \alpha, \phi_i \rangle = -\frac{1}{\gamma - \lambda_i} \left\langle \alpha, \frac{\partial}{\partial \nu} \phi_i \right\rangle \, \bigg|_0, \quad i = 1, ..., N. \hspace{1cm} (2.15)
\]

Next, we choose \( N \) constants \( \rho < \gamma_1 < \gamma_2 < \ldots < \gamma_N \), large enough, such that
\[
\text{Eq. (2.13), corresponding to each } \gamma_k, \ k = 1, ..., N, \text{ has a unique solution},
\hspace{1cm} (2.16)
\]
and introduce the feedbacks
\[
v_k(t, x) = \begin{bmatrix}
\langle y(t), \phi_1 \rangle \\
\langle y(t), \phi_2 \rangle \\
\vdots \\
\langle y(t), \phi_N \rangle
\end{bmatrix}, \quad t \geq 0, \ x \in \Gamma_1, \hspace{1cm} (2.17)
\]
for \( k = 1, 2, ..., N; \) and \( v = v_1 + v_2 + ... + v_N \). It is easy to see that we have
\[
v = \begin{bmatrix}
T A 
\begin{bmatrix}
\langle y(t), \phi_1 \rangle \\
\langle y(t), \phi_2 \rangle \\
\vdots \\
\langle y(t), \phi_N \rangle
\end{bmatrix}, \\
1 \\
1 \\
\end{bmatrix}. \hspace{1cm} (2.11)
\]
that is exactly the boundary feedback plugged in (2.11).

For latter purpose, we show that
\[
\begin{pmatrix}
\langle D_{\gamma_k}v_k, \phi_1 \rangle \\
\langle D_{\gamma_k}v_k, \phi_2 \rangle \\
\vdots \\
\langle D_{\gamma_k}v_k, \phi_N \rangle
\end{pmatrix}
= -B_k A
\begin{pmatrix}
\langle y(t), \phi_1 \rangle \\
\langle y(t), \phi_2 \rangle \\
\vdots \\
\langle y(t), \phi_N \rangle
\end{pmatrix},
\]
(2.18)  
where \(B_k\) are introduced in (2.24) above, for \(k = 1, \ldots, N\). This is indeed so. We have
\[
\langle D_{\gamma_k}v_k, \phi_i \rangle = A
\begin{pmatrix}
\langle y(t), \phi_1 \rangle \\
\langle y(t), \phi_2 \rangle \\
\vdots \\
\langle y(t), \phi_N \rangle
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\gamma_k - \lambda_1} \langle D_{\gamma_k} \partial_{\nu} \phi_1, \phi_1 \rangle \\
\frac{1}{\gamma_k - \lambda_2} \langle D_{\gamma_k} \partial_{\nu} \phi_2, \phi_1 \rangle \\
\vdots \\
\frac{1}{\gamma_k - \lambda_N} \langle D_{\gamma_k} \partial_{\nu} \phi_N, \phi_1 \rangle
\end{pmatrix}, \ i = 1, \ldots, N,
\]
(2.19)  
It then follows by relation (2.15), that
\[
\langle D_{\gamma_k}v_k, \phi_i \rangle = A
\begin{pmatrix}
\langle y(t), \phi_1 \rangle \\
\langle y(t), \phi_2 \rangle \\
\vdots \\
\langle y(t), \phi_N \rangle
\end{pmatrix}
\begin{pmatrix}
\frac{1}{(\gamma_k - \lambda_1)(\gamma_k - \lambda_2)} \langle \partial_{\nu} \phi_1, \partial_{\nu} \phi_1 \rangle_0 \\
\frac{1}{(\gamma_k - \lambda_2)(\gamma_k - \lambda_3)} \langle \partial_{\nu} \phi_2, \partial_{\nu} \phi_1 \rangle_0 \\
\vdots \\
\frac{1}{(\gamma_k - \lambda_N)(\gamma_k - \lambda_1)} \langle \partial_{\nu} \phi_N, \partial_{\nu} \phi_1 \rangle_0
\end{pmatrix}, \ i = 1, \ldots, N,
\]
(2.19)  
from where we immediately obtain (2.18).

Now, let us return to the linear system
\[
\begin{align*}
y(t, x) - \Delta y(t, x) + a(x) y(t, x) &= 0, \ t > 0, \ x \in \Omega, \\
y(t, x) &= v(t, x) = v_1(t, x) + \ldots + v_N(t, x), \ t > 0, \ x \in \Gamma_1, \\
\frac{\partial}{\partial \nu} y(t, x) &= 0, \ t > 0, \ x \in \Gamma_2, \\
y(0, x) &= y_0(x), \ x \in \Omega.
\end{align*}
\]
(2.20)  
Let us denote by \(z(t, x) := y(t, x) - D_{\gamma_1}v_1(t, x) - \ldots - D_{\gamma_N}v_N(t, x), \ t \geq 0, \ x \in \Omega\).

We claim that the feedbacks \(v_k, k = 1, \ldots, N,\) can be expressed only in terms of \(z,\) as
\[
v_k(t, x) = \frac{1}{2} A
\begin{pmatrix}
\langle z(t), \phi_1 \rangle \\
\langle z(t), \phi_2 \rangle \\
\vdots \\
\langle z(t), \phi_N \rangle
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\gamma_k - \lambda_1} \partial_{\nu} \phi_1 \\
\frac{1}{\gamma_k - \lambda_2} \partial_{\nu} \phi_2 \\
\vdots \\
\frac{1}{\gamma_k - \lambda_N} \partial_{\nu} \phi_N
\end{pmatrix}, \ n = 1, \ldots, N.
\]
(2.21)
To see this we do the following straightforward computations

\[
\frac{1}{2} \left< A \begin{pmatrix} \langle z(t), \phi_1 \rangle \\ \langle z(t), \phi_2 \rangle \\ \vdots \\ \langle z(t), \phi_N \rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_k - \lambda_1} \frac{\partial}{\partial v} \phi_1 \\ \frac{1}{\gamma_k - \lambda_2} \frac{\partial}{\partial v} \phi_2 \\ \vdots \\ \frac{1}{\gamma_k - \lambda_N} \frac{\partial}{\partial v} \phi_N \end{pmatrix} \right> = \frac{1}{2} \left< A \begin{pmatrix} \langle y(t), \phi_1 \rangle \\ \langle y(t), \phi_2 \rangle \\ \vdots \\ \langle y(t), \phi_N \rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_k - \lambda_1} \frac{\partial}{\partial v} \phi_1 \\ \frac{1}{\gamma_k - \lambda_2} \frac{\partial}{\partial v} \phi_2 \\ \vdots \\ \frac{1}{\gamma_k - \lambda_N} \frac{\partial}{\partial v} \phi_N \end{pmatrix} \right>^N
\]

(taking into account relation (2.18))

\[
= \frac{1}{2} \left< [J + A(B_1 + \ldots + B_N)] A \begin{pmatrix} \langle y(t), \phi_1 \rangle \\ \langle y(t), \phi_2 \rangle \\ \vdots \\ \langle y(t), \phi_N \rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_k - \lambda_1} \frac{\partial}{\partial v} \phi_1 \\ \frac{1}{\gamma_k - \lambda_2} \frac{\partial}{\partial v} \phi_2 \\ \vdots \\ \frac{1}{\gamma_k - \lambda_N} \frac{\partial}{\partial v} \phi_N \end{pmatrix} \right>^N
\]

\[
= v_k,
\]

since \( A = (B_1 + \ldots + B_N)^{-1} \). Moreover, likewise in (2.18), we have now

\[
\begin{pmatrix} \langle D_{\gamma_k} v_k, \phi_1 \rangle \\ \langle D_{\gamma_k} v_k, \phi_2 \rangle \\ \vdots \\ \langle D_{\gamma_k} v_k, \phi_N \rangle \end{pmatrix} = -\frac{1}{2} B_k A \begin{pmatrix} \langle z(t), \phi_1 \rangle \\ \langle z(t), \phi_2 \rangle \\ \vdots \\ \langle z(t), \phi_N \rangle \end{pmatrix}, \quad k = 1, \ldots, N. \tag{2.22}
\]

Finally, equation (2.20) may be rewritten in terms of \( z \) as follows

\[
\begin{cases}
    z_t(t, x) + A z(t, x) = R(\langle z, \phi_1 \rangle, \ldots, \langle z, \phi_N \rangle), \quad t > 0, \quad x \in \Omega, \\
    z(0, x) = z_0(x), \quad x \in \Omega,
\end{cases} \tag{2.23}
\]

where

\[
R(\langle z, \phi_1 \rangle, \ldots, \langle z, \phi_N \rangle) := - \left( \sum_{i=1}^N D_{\gamma_i} v_i \right)_t - 2 \sum_{i,j=1}^N \lambda_j \langle D_{\gamma_i} v_i, \phi_j \rangle \phi_j + \sum_{i=1}^N \gamma_i D_{\gamma_i} v_i. \tag{2.24}
\]

By (2.22), and using the fact that \( A \) is the inverse of the sum of \( B_k, \ k = 1, \ldots, N, \) we see immediately that

\[
\begin{pmatrix} \langle R, \phi_1 \rangle \\ \langle R, \phi_2 \rangle \\ \vdots \\ \langle R, \phi_N \rangle \end{pmatrix} = \frac{1}{2} \sum_{k=1}^N B_k A Z_t + \Lambda \sum_{k=1}^N B_k A Z - \frac{1}{2} \sum_{k=1}^N \gamma_k B_k A Z \tag{2.25}
\]

\[
= \frac{1}{2} Z_t + \Lambda Z - \frac{1}{2} \sum_{k=2}^N (\gamma_1 - \gamma_k) B_k A Z,
\]

where we have denoted by \( Z(t) := \begin{pmatrix} \langle z(t), \phi_1 \rangle \\ \langle z(t), \phi_2 \rangle \\ \vdots \\ \langle z(t), \phi_N \rangle \end{pmatrix}, \ t \geq 0; \) and by \( \Lambda := \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_N \end{pmatrix} \).
We have now, by (2.23), scalarly multiplied by $\phi_i$, $i = 1, \ldots, N$, together with (2.25), that
\[ Z_t + \Lambda Z = \frac{1}{2} Z_t + \Lambda Z - \frac{1}{2} \gamma_1 Z + \frac{1}{2} \sum_{k=2}^{N} (\gamma_1 - \gamma_k) B_k A Z, \quad t > 0; \quad Z(0) = Z_0. \] (2.26)

Or, equivalently,
\[ Z_t = -\gamma_1 Z + \sum_{k=2}^{N} (\gamma_1 - \gamma_k) B_k A Z, \quad t > 0; \quad Z(0) = Z_0. \] (2.27)

Recall that $B_j$, $j = 1, \ldots, N$, are positive semidefinite symmetric matrices, therefore, \[ \langle B_j q, q \rangle_N \geq 0, \quad \forall q \in \mathbb{R}^N, \quad j = 1, \ldots, N. \] Consequently, $A = (B_1 + \ldots + B_N)^{-1}$ is a positive definite symmetric matrix, thus one can define another positive definite symmetric matrix, denoted by $A^\frac{1}{2}$, such that $A^\frac{1}{2} A^\frac{1}{2} = A$ (the square root of $A$; for details see [6]). Let us scalarly multiply equation (2.27) by $A Z$, to get
\[ \frac{1}{2} \frac{d}{dt} \| A^\frac{1}{2} Z(t) \|_N^2 = -\gamma_1 \| A^\frac{1}{2} Z(t) \|_N^2 + \sum_{k=2}^{N} (\gamma_1 - \gamma_k) \langle B_k A Z(t), A Z(t) \rangle_N, \] (2.28)
that leads to
\[ \frac{1}{2} \frac{d}{dt} \| A^\frac{1}{2} Z(t) \|_N^2 \leq -\gamma_1 \| A^\frac{1}{2} Z(t) \|_N^2, \quad t \geq 0, \]
since $\gamma_1 - \gamma_k < 0$, $k = 2, \ldots, N$. Here $\| \cdot \|_N$ stands for the euclidian norm in $\mathbb{R}^N$. The above relation implies the exponential decay of $Z$ in the $\| A^\frac{1}{2} \cdot \|_N$-norm, i.e.,
\[ \| A^\frac{1}{2} Z(t) \|_N^2 \leq e^{-2\gamma_1 t} \| A^\frac{1}{2} Z_0 \|_N^2, \quad t \geq 0, \]
where using the fact that $A^\frac{1}{2}$ is a positive definite symmetric matrix, we finally arrive to
\[ \| Z(t) \|_N^2 \leq C e^{-2\gamma_1 t} \| Z_0 \|_N^2, \quad t \geq 0, \] (2.29)
for some positive constant $C$.

The rest of the proof follows by identical arguments as in the proof of [1, Theorem 3.1], therefore it is omitted. However, we note that since $\gamma_1$ and $\rho$ may be taken arbitrarily large, the decaying constant $\mu$ in (2.12) may be taken arbitrarily large. □

Now, we focus on the stabilization of the nonlinear system. To this end, assuming, in addition, about the nonlinear function $f$ that
\[ |f_y(x, y)| \leq C(|y|^m + 1), \quad \forall x \in \overline{\Omega}, \quad y \in \mathbb{R}, \] (2.30)
where $0 < m < \infty$ for $d = 1, 2$, and $m = 3$ for $d = 3$, the feedback (2.4) locally stabilizes the solution $y_e$ into the nonlinear system (1.1). More exactly, we have
Theorem 2.2. Assume hypothesis (H) holds true. Let \( 1 \leq d \leq 3 \) and \( f \) obeying assumption (2.30). The solution to the closed-loop nonlinear system

\[
\begin{align*}
    y(t, x) &= \begin{cases}
        y(t) - \Delta y(t, x) + f(x, y) = 0, & t > 0, \ x \in \Omega, \\
        y(t, x) = T A \left( \begin{array}{c}
            \langle y(t) - y_e, \phi_1 \rangle \\
            \langle y(t) - y_e, \phi_2 \rangle \\
            \vdots \\
            \langle y(t) - y_e, \phi_N \rangle
        \end{array} \right) + y_e(x), & t > 0, \ x \in \Gamma_1, \\
        \frac{\partial}{\partial t} y(t, x) = 0, & t > 0, \ x \in \Gamma_2, \\
        y(0, x) = y_0, & x \in \Omega,
    \end{cases}
\end{align*}
\]

(2.31)

satisfies the exponential decay

\[
\| y(t) - y_e \|^2 \leq C e^{-\mu t} \| y_0 - y_e \|^2, \quad t \geq 0,
\]

for a prescribed \( \mu > 0 \), and a constant \( C > 0 \), provided that \( \| y_0 - y_e \| \) is small enough. Here \( T \) is introduced in relation (2.23), while \( A \) is introduced in relation (2.6).

Proof. The proof is almost identical with that one in [1, Theorem 4.1], therefore it is omitted. □

2.2. The case of general eigenvalues. As announced above, we shall see in the following that, doing some minor changes in the controller’s form (2.4), we will be able to show that the present approach works equally well for the case of general unstable eigenvalues (i.e., not necessarily simple). Indeed, let us assume, for example, that the first unstable eigenvalue has its multiplicity equal to 2, i.e., \( \lambda_1 = \lambda_2 \) (and the others unstable eigenvalues are simple), and denote the common value by \( \lambda \). Moreover, let us denote by \( \phi \) the sum of the first two corresponding eigenfunctions, that is \( \phi = \phi_1 + \phi_2 \). We claim that the feedback

\[
u = \begin{cases}
    T A \left( \begin{array}{c}
        \langle y(t) - y_e, \phi \rangle \\
        \langle y(t) - y_e, \phi_3 \rangle \\
        \vdots \\
        \langle y(t) - y_e, \phi_N \rangle
    \end{array} \right) + y_e, & (2.32)
\end{cases}
\]

globally exponentially stabilizes the linear system (2.1) (and locally the nonlinear system (1.2)). Here, \( T \) is defined as

\[
T := \begin{pmatrix}
    1/\gamma_1 \partial_x \phi(x) & 1/\gamma_2 \partial_x \phi_2(x) & \ldots & 1/\gamma_N \partial_x \phi_N(x) \\
    1/\gamma_1 \partial_x \phi(x) & 1/\gamma_2 \partial_x \phi_2(x) & \ldots & 1/\gamma_N \partial_x \phi_N(x) \\
    \cdots & \cdots & \cdots & \cdots \\
    1/\gamma_1 \partial_x \phi(x) & 1/\gamma_2 \partial_x \phi_2(x) & \ldots & 1/\gamma_N \partial_x \phi_N(x)
\end{pmatrix}
\]

(2.33)

with \( \rho < \gamma < \ldots < \gamma_N \) chosen large enough, and

\[
A = (B + B_3 + \ldots + B_N)^{-1},
\]

(2.34)

where we have denoted by

\[
B = \Lambda \mathbf{B} \Lambda, \ B_k := \Lambda_{\gamma_k} \mathbf{B} \Lambda_{\gamma_k}, \ k = 3, \ldots, N,
\]

(2.35)
Theorem 2.3. Let $\gamma > 0$ and $\lambda, \lambda_3, ..., \lambda_N$ be mutually disjoint. Then for the system

$$\begin{equation}
B := \begin{pmatrix}
\frac{1}{\gamma_1} & 0 & \ldots & 0 \\
0 & \frac{1}{\gamma_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{\gamma_N}
\end{pmatrix},
\Lambda_\gamma := \begin{pmatrix}
\frac{1}{\gamma_1} & 0 & \ldots & 0 \\
0 & \frac{1}{\gamma_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{\gamma_N}
\end{pmatrix},
\Lambda_{\gamma k} := \begin{pmatrix}
\frac{1}{\gamma_k} & 0 & \ldots & 0 \\
0 & \frac{1}{\gamma_k-\lambda_3} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{\gamma_k-\lambda_N}
\end{pmatrix}, \quad k = 3, ..., N.
\end{equation}
$$

Thus, this time $B$ is the Gram matrix of the system $\{\frac{\partial \phi_i}{\partial y}, \frac{\partial \phi_j}{\partial y}, \phi \in \mathcal{V}, i = 3, ..., N\}$. By the above definitions, taking into account that $\lambda, \lambda_3, ..., \lambda_N$ are mutually disjoint and $\gamma, \gamma_3, ..., \gamma_N$ are also mutually disjoint, one may show that the matrix $A$ is well-defined, that is, the sum $B + B_3 + ... + B_N$ is indeed invertible (this follows similarly as in Lemma 2.1 and Lemma 2.2 above).

As in the above subsection, we begin with a stabilization result for the linearized system. This is

**Theorem 2.3.** The feedback $u$, given by (2.22), exponentially stabilizes the linearized system (2.7). More precisely, the solution $y$ to the system

$$\begin{cases}
y(t, x) - \Delta y(t, x) + a(y, x) + \frac{\partial y}{\partial y} \phi(t, x) y(t, x) = 0, & t > 0, \ x \in \Omega, \\
y(t, x) = T A \begin{pmatrix} \gamma_1(t, \phi) \\ \gamma_2(t, \phi_3) \\ \vdots \\ \gamma_N(t, \phi_N) \end{pmatrix}, & t > 0, \ x \in \Gamma_1, \\
\frac{\partial y}{\partial y}(t, x) = 0, & t > 0, \ x \in \Gamma_2, \\
y(0, x) = y_0, & x \in \Omega,
\end{cases}
$$

satisfies the exponential decay

$$\|y(t)\|^2 \leq Ce^{-\mu t} \|y_0\|^2, \quad t \geq 0,$$

for a prescribed $\mu > 0$, and a constant $C > 0$. Here $T$ is introduced in relation (2.28), while $A$ is introduced in relation (2.34). $\|\cdot\|$ denotes the norm in $L^2(\Omega)$.

**Proof.** The proof is almost identical with that one of Theorem 2.1 that is why we shall skip most of computational details.

This time, the Dirichlet operator looks like $D_\gamma \alpha := y$, where $y$ is solution to

$$\begin{cases}
-\Delta y(x) + a(x) y(x) - \lambda \langle y, \phi \rangle \phi - 2 \sum_{k=3}^N \lambda_k \langle y, \phi_k \rangle \phi_k(x) + \gamma y(x) = 0, & x \in \Omega, \\
y = \alpha \text{ on } \Gamma_1, \\
\frac{\partial y}{\partial y} = 0 \text{ on } \Gamma_2.
\end{cases}
$$

Likewise in (2.14) - (2.15) we get

$$\langle D_\gamma \alpha, \phi \rangle = -\frac{1}{\gamma_1} \langle \alpha, \frac{\partial \phi}{\partial y} \rangle_0 \quad \text{and} \quad \langle D_\gamma \alpha, \phi_i \rangle = -\frac{1}{\gamma_{i-1}} \langle \alpha, \frac{\partial \phi_i}{\partial y} \rangle_0, \quad i = 3, ..., N.
$$

(2.41)
Next, we choose $N - 1$ constants $\rho < \gamma < \gamma_3 < \ldots < \gamma_N$, large enough, and introduce the feedbacks

$$v(t, x) = \left\langle A \begin{pmatrix} \langle y(t), \phi \rangle \\ \langle y(t), \phi_3 \rangle \\ \vdots \\ \langle y(t), \phi_N \rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma - \lambda_3} \frac{\partial}{\partial r} \phi(x) \\ \frac{1}{\gamma - \lambda_3} \frac{\partial}{\partial r} \phi_3(x) \\ \vdots \\ \frac{1}{\gamma - \lambda_N} \frac{\partial}{\partial r} \phi_N(x) \end{pmatrix} \right\rangle, \quad t \geq 0, \ x \in \Gamma_1, \quad (2.42)$$

and

$$v_k(t, x) = \left\langle A \begin{pmatrix} \langle y(t), \phi \rangle \\ \langle y(t), \phi_3 \rangle \\ \vdots \\ \langle y(t), \phi_N \rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_1 - \lambda_3} \frac{\partial}{\partial r} \phi(x) \\ \frac{1}{\gamma_1 - \lambda_3} \frac{\partial}{\partial r} \phi_3(x) \\ \vdots \\ \frac{1}{\gamma_1 - \lambda_N} \frac{\partial}{\partial r} \phi_N(x) \end{pmatrix} \right\rangle, \quad t \geq 0, \ x \in \Gamma_1, \quad (2.43)$$

for $k = 3, \ldots, N$; and $v = v_3 + \ldots + v_N$. It is clear that

$$v = \left\langle T A \begin{pmatrix} \langle y(t), \phi \rangle \\ \langle y(t), \phi_3 \rangle \\ \vdots \\ \langle y(t), \phi_N \rangle \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle, \quad N - 1$$

which is the boundary feedback plugged in $L^2(S)$. Now, let us denote by $z(t, x) := y(t, x) - D_{\gamma_1} v(t, x) - D_{\gamma_3} v_3(t, x) - \ldots - D_{\gamma_N} v_N(t, x)$. Similarly as in $[2.21]$ we have

$$v(t, x) = \frac{1}{2} \left\langle A \begin{pmatrix} \langle z(t), \phi \rangle \\ \langle z(t), \phi_3 \rangle \\ \vdots \\ \langle z(t), \phi_N \rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma - \lambda_3} \frac{\partial}{\partial r} \phi(x) \\ \frac{1}{\gamma - \lambda_3} \frac{\partial}{\partial r} \phi_3(x) \\ \vdots \\ \frac{1}{\gamma - \lambda_N} \frac{\partial}{\partial r} \phi_N(x) \end{pmatrix} \right\rangle; \quad (2.44)$$

and

$$v_k(t, x) = \frac{1}{2} \left\langle A \begin{pmatrix} \langle z(t), \phi \rangle \\ \langle z(t), \phi_3 \rangle \\ \vdots \\ \langle z(t), \phi_N \rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_1 - \lambda_3} \frac{\partial}{\partial r} \phi(x) \\ \frac{1}{\gamma_1 - \lambda_3} \frac{\partial}{\partial r} \phi_3(x) \\ \vdots \\ \frac{1}{\gamma_1 - \lambda_N} \frac{\partial}{\partial r} \phi_N(x) \end{pmatrix} \right\rangle, \quad k = 3, \ldots, N. \quad (2.45)$$

Besides this, as in $[2.22]$, we also have that

$$\begin{pmatrix} \langle D_{\gamma} v, \phi \rangle \\ \langle D_{\gamma} v, \phi_3 \rangle \\ \vdots \\ \langle D_{\gamma} v, \phi_N \rangle \end{pmatrix} = -\frac{1}{2} BA \begin{pmatrix} \langle z(t), \phi \rangle \\ \langle z(t), \phi_3 \rangle \\ \vdots \\ \langle z(t), \phi_N \rangle \end{pmatrix}; \quad (2.46)$$

and

$$\begin{pmatrix} \langle D_{\gamma_k} v_k, \phi \rangle \\ \langle D_{\gamma_k} v_k, \phi_3 \rangle \\ \vdots \\ \langle D_{\gamma_k} v_k, \phi_N \rangle \end{pmatrix} = -\frac{1}{2} B_k A \begin{pmatrix} \langle z(t), \phi \rangle \\ \langle z(t), \phi_3 \rangle \\ \vdots \\ \langle z(t), \phi_N \rangle \end{pmatrix}, \quad k = 3, \ldots, N. \quad (2.47)$$
In terms of the new variable \( z \), the linearization (2.38) may be rewritten as

\[
\begin{aligned}
\begin{cases}
  z_t(t, x) + A z(t, x) = R(\langle z, \phi \rangle, \langle z, \phi_3 \rangle, \ldots, \langle z, \phi_N \rangle), \quad t > 0, \ x \in \Omega, \\
  z(0, x) = z_0(x) := y_0 - D_{\gamma} v(0, x) - D_{\gamma_3} v_3(0, x) - \ldots - D_{\gamma_N} v_N(0, x), \ x \in \Omega,
\end{cases}
\end{aligned}
\]

(2.48)

where

\[
R(\langle z, \phi \rangle, \langle z, \phi_3 \rangle, \ldots, \langle z, \phi_N \rangle) := -(D_{\gamma} v)_t - \left( \sum_{i=3}^{N} D_{\gamma_i} v_i \right)_t
\]

(2.49)

\[
\begin{aligned}
- \lambda \left( D_{\gamma} v + \sum_{i=3}^{N} D_{\gamma_i} v_i, \phi \right) \\
- 2 \sum_{j=3}^{N} \lambda_j \left( D_{\gamma} v + \sum_{i=3}^{N} D_{\gamma_i} v_i, \phi_j \right) + \gamma D_{\gamma} v + \sum_{i=3}^{N} \gamma_i D_{\gamma_i} v_i.
\end{aligned}
\]

Then, simple computations as in (2.49) give

\[
\begin{pmatrix}
  \langle R, \phi \rangle \\
  \langle R, \phi_3 \rangle \\
  \vdots \\
  \langle R, \phi_N \rangle
\end{pmatrix} = \frac{1}{2} \mathcal{Z}_t + \Lambda \mathcal{Z} - \frac{1}{2} \gamma \mathcal{Z} + \frac{1}{2} \sum_{k=3}^{N} (\gamma - \gamma_k) B_k A \mathcal{Z},
\]

(2.50)

where we have denoted by \( \mathcal{Z}(t) := \left( \langle z(t), \phi \rangle, \langle z(t), \phi_3 \rangle, \ldots, \langle z(t), \phi_N \rangle \right) \), \( t \geq 0 \); and by \( \Lambda := \left( \begin{array}{ccc}
  \lambda & 0 & \ldots & 0 \\
  0 & \lambda_3 & \ldots & 0 \\
  \ldots & \ldots & \ldots & \ldots \\
  0 & 0 & \ldots & \lambda_N
\end{array} \right) \).

Instead of considering system (2.48) with the initial data \( z_0 \), we shall consider the following two systems for which the initial data \( z_0 \) is split into its positive and negative part, respectively. More precisely,

\[
\begin{aligned}
\begin{cases}
  P_t(t, x) + A P(t, x) = R(\langle P, \phi \rangle, \langle P, \phi_3 \rangle, \ldots, \langle P, \phi_N \rangle), \quad t > 0, \ x \in \Omega, \\
  P(0, x) = \sum_{k=1}^{\infty} \langle z_0, \phi_k \rangle^+ \phi_k, \ x \in \Omega,
\end{cases}
\end{aligned}
\]

(2.51)

and

\[
\begin{aligned}
\begin{cases}
  M_t(t, x) + A M(t, x) = R(\langle M, \phi \rangle, \langle M, \phi_3 \rangle, \ldots, \langle M, \phi_N \rangle), \quad t > 0, \ x \in \Omega, \\
  M(0, x) = \sum_{k=1}^{\infty} \langle z_0, \phi_k \rangle^- \phi_k, \ x \in \Omega.
\end{cases}
\end{aligned}
\]

(2.52)

(Here, given a real number \( r \), we denote by \( r^+ := \max \{ r, 0 \} \) and \( r^- := -\min \{ r, 0 \} \). Notice that both \( r^+ \) and \( r^- \) are nonnegative, and \( r = r^+ - r^- \).)

Scalarly multiplying (2.51) by \( \phi, \phi_3, \ldots, \phi_N \), respectively, and taking into account of (2.49), we get as in (2.49), (2.50), that

\[
\mathcal{P}_t = -\gamma \mathcal{P} + \sum_{k=3}^{N} (\gamma - \gamma_k) B_k A \mathcal{P}, \ t > 0; \ \mathcal{P}(0) = \mathcal{Z}_o^+,
\]

(2.53)
where $\mathcal{P}(t) := \begin{pmatrix} \langle P(t), \phi \rangle \\ \langle P(t), \phi_3 \rangle \\ \vdots \\ \langle P(t), \phi_N \rangle \end{pmatrix}$, $t \geq 0$. But this implies as in (2.29) the following exponential decay

$$\| \mathcal{P}(t) \|^2_{N-1} \leq C e^{-2\eta t} \| \mathcal{Z}_0 \|^2_{N-1}, \ t \geq 0.$$
satisfies the exponential decay
\[ \|y(t) - y_c\|^2 \leq Ce^{-\mu t}\|y_0 - y_c\|^2, \quad t \geq 0, \]
for a prescribed \( \mu > 0 \), and a constant \( C > 0 \), provided that \( \|y_0 - y_c\| \) is small enough. Here \( T \) is introduced in relation (2.33), while \( A \) is introduced in relation (2.34).

3. EXAMPLES

Example 1. We consider first a problem discussed in [3], and also [1]. More precisely, the stabilization of the heat equation on the rod \((0,1)\)
\[
\begin{cases}
y_t - y_{xx} - \lambda y, & x \in (0,1), \ t > 0, \\
y_x(t,0) = 0, \ y(t,1) = u(t), & t > 0,
\end{cases}
\]
with the Dirichlet actuation \( u \) in \( x = 1 \) and with \( \lambda \) a positive constant parameter. First of all, we notice that the stabilizing results using the backstepping method in [3] work for arbitrarily level of instability, while in [1] they are applicable under the condition that the parameter \( \lambda \) do not exceed the bound \( \frac{(2\pi - 1)^2}{4} \). In the present paper, Theorem 2.1 above holds true without any a priori condition on \( \lambda \). This is indeed so, because in our case the operator \( A_y = -y'' - \lambda y, \ \forall y \in D(A) = \{ y \in H^2(0,1) : y'(0) = y(1) = 0 \} \) has the eigenvalues \( \lambda_j = \frac{(2j-1)^2\pi^2}{4} - \lambda \) with the corresponding eigenfunctions \( \phi_j = \cos \frac{(2j-1)\pi}{2} x, \ j = 1,2,\ldots \). It is clear that hypothesis (H) holds true for any \( N \in \mathbb{N} \). Let us pick some \( N \) such that \( \lambda_j > \rho > 0, \ \forall j = N+1, N+2,\ldots \). Since system (3.1) is linear and Theorem 2.1 is applicable we get that the feedback
\[
u = \left<TA \left( \begin{array}{c}
\int_{0}^{1} y(x) \cos \frac{\pi}{N} x \, dx \\
\int_{0}^{1} y(x) \cos \frac{3\pi}{N} x \, dx \\
\vdots \\
\int_{0}^{1} y(x) \cos \frac{(2N-1)\pi}{2N} x \, dx
\end{array} \right), \begin{array}{c}
1 \\
1 \\
\vdots \\
1
\end{array} \right>,
\]
where
\[
T = \begin{pmatrix}
-\frac{2\pi}{N} & \frac{6\pi}{N} & \cdots & (-1)^N \frac{2(2N-1)\pi}{N} \\
\frac{2\pi}{N} & \frac{6\pi}{N} & \cdots & (-1)^N \frac{2(2N-1)\pi}{N} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{2\pi}{N} & \frac{6\pi}{N} & \cdots & (-1)^N \frac{2(2N-1)\pi}{N}
\end{pmatrix},
\]
and
\[
A = \left\{ \begin{array}{c}
\frac{1}{l_k - \pi^2} \\
\frac{-3}{l_k - \pi^2} \\
\vdots \\
\frac{(-1)^N (2N-1)}{l_k - (2N-1)^2 \pi^2} \\
\frac{(2N-1)^2 \pi^2}{l_k - (2N-1)^2 \pi^2}
\end{array} \right\}^{-1},
\]
(here we have denoted by \( l_k := 4(\gamma_k + \lambda), \ k = 1,\ldots, N \), assures the exponential stability of the null solution of (3.1). Here, \( \gamma_1 < \gamma_2 < \ldots < \gamma_N \) are some positive constants, large enough.
Example 2. We consider now the classical Fitzhugh-Nagumo equation that describes the dynamics of electrical potential across cell membrane (for details, see [8])

\[
\begin{aligned}
\begin{cases}
y_t - y_{xx} + y(y - 1)(y - a) = 0, & 0 < x < l, \ t > 0, \\
y_x(t, l) = 0, & y(t, 0) = a(t), \ t > 0,
\end{cases}
\end{aligned}
\]  

where \(0 < a < \frac{1}{2}\). It is known that the equilibrium \(y_e = a\) is unstable. The linearized operator \(\mathcal{A}\) is given by

\[
\mathcal{A}y = -y'' - a(1 - a)y, \ \forall y \in \mathcal{D}(\mathcal{A}) = \left\{ y \in H^2(0, l) : y(0) = y'(l) = 0 \right\}.
\]

It has the simple eigenvalues \(\lambda_j = \frac{(2j - 1)^2 \pi^2}{4l^2} - a(1 - a), \ j = 1, 2, \ldots\), with the corresponding eigenfunctions \(\phi_j = \sin \frac{(2j - 1)\pi}{2l}x, \ j = 1, 2, \ldots\). Since hypothesis (H) is fulfilled for any \(N \in \mathbb{N}\), we chose one such that \(\lambda_j > \rho > 0, \ j = N + 1, N + 2, \ldots\). It is easy to see that \(f(y) = y(y - 1)(y - a)\) obeys the hypothesis of Theorem 2.2 and so we get that the feedback

\[
u = \left\langle T A \begin{pmatrix} f_0'(y(x) - a) \sin \frac{2\pi}{l} x dx \\ f_0'(y(x) - a) \sin \frac{3\pi}{2l} x dx \\ \vdots \\ f_0'(y(x) - a) \sin \frac{(2N - 1)\pi}{2l} x dx \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle + a,
\]

where

\[
T = \begin{pmatrix}
\frac{2l}{l_1 - \pi^2} & \frac{6l}{l_1 - (3\pi)^2} & \cdots & \frac{2l(2N - 1)}{l_1 - (2N - 1)^2\pi^2} \\
\frac{2l}{l_2 - \pi^2} & \frac{6l}{l_2 - (3\pi)^2} & \cdots & \frac{2l(2N - 1)}{l_2 - (2N - 1)^2\pi^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{2l}{l_N - \pi^2} & \frac{6l}{l_N - (3\pi)^2} & \cdots & \frac{2l(2N - 1)}{l_N - (2N - 1)^2\pi^2}
\end{pmatrix},
\]

with \(l_k := 4l^2(\gamma_k + a(1 - a)), \ k = 1, 2, \ldots, N\); and

\[
A = \left\{ (2l)^2 \pi^2 \sum_{k=1}^{N} \begin{pmatrix}
\frac{(l_k - \pi^2)^2}{l_k - \pi^2} \\
\frac{3}{l_k - \pi^2} \\
\vdots \\
\frac{2N - 1}{l_k - (2N - 1)^2\pi^2}
\end{pmatrix} \right\}^{-1},
\]

locally stabilizes the solution \(y_e = a\) in [5, 2].

Example 3. Finally we consider the periodic heat equation in \((0, \pi)^2\)

\[
\begin{aligned}
\begin{cases}
y_t - \Delta y - \lambda y = 0, & x \in (0, \pi)^2, \ t > 0, \\
y(t, x_1, 0) = u(t), \ y_x(t, x_1, \pi) = 0, & x_1 \in (0, \pi), \\
y_x(t, 0, x_2) = y_x(t, \pi, x_2) = 0, & x_2 \in (0, \pi).
\end{cases}
\end{aligned}
\]

In this case the operator

\[
\mathcal{A}y = -\Delta y - \lambda y,
\]

for all

\[
y \in \mathcal{D}(\mathcal{A}) = \left\{ y \in H^2((0, \pi)^2) : y(x_1, 0) = y_x(x_1, \pi) = 0, \ y_{x_1}(0, x_2) = y_{x_1}(\pi, x_2) = 0 \right\},
\]

has the eigenvalues

\[
\lambda_k = k_1^2 + \left( \frac{2k_2 + 1}{2} \right)^2 - \lambda, \ \forall k = (k_1, k_2) \in \mathbb{N}^2,
\]
with the corresponding eigenfunctions

\[ \phi_k = \cos k_1 x_1 \sin \frac{2k_2 + 1}{2} x_2, \quad \forall k \in \mathbb{N}^2. \]

Ordering the eigenvalues set as an increasing sequence and redefine them, we have

\[ \lambda_1 = 1.25 - \lambda, \quad \phi_1 = \cos x_1 \sin \frac{3}{2} x_2; \quad \lambda_2 = 3.25 - \lambda, \quad \phi_2 = \cos x_1 \sin \frac{5}{2} x_2; \quad \lambda_3 = 4.25 - \lambda, \quad \phi_3 = \cos 2x_1 \sin \frac{5}{2} x_2; \quad \lambda_4 = 6.25 - \lambda, \quad \phi_4 = \cos 2x_1 \sin \frac{7}{2} x_2; \quad \lambda_5 = 7.25 - \lambda, \quad \phi_5 = \cos x_1 \sin \frac{7}{2} x_2; \quad \lambda_6 = 9.25 - \lambda, \quad \phi_6 = \cos 3x_1 \sin \frac{5}{2} x_2; \quad \lambda_7 = 10.25 - \lambda, \quad \phi_7 = \cos 2x_1 \sin \frac{7}{2} x_2; \quad \lambda_8 = 11.25 - \lambda, \quad \phi_8 = \cos 3x_1 \sin \frac{7}{2} x_2; \quad \lambda_9 = 13.25 - \lambda, \quad \phi_9 = \cos x_1 \sin \frac{9}{2} x_2; \quad \lambda_{10} = 15.25 - \lambda, \quad \phi_{10} = \cos 3x_1 \sin \frac{9}{2} x_2; \quad \lambda_{11} = 16.25 - \lambda, \quad \phi_{11} = \cos 4x_1 \sin \frac{9}{2} x_2; \quad \lambda_{12} = 16.25 - \lambda, \quad \phi_{12} = \cos 2x_1 \sin \frac{9}{2} x_2. \]

It is clear that since \( \lambda_{11} = \lambda_{12} \), hypothesis \((H)\) fails to hold. So, we apply this time the result in Theorem 2.3. That is, we denote by \( \lambda := 16.25 - \lambda \) and by \( \phi = \phi_{11} + \phi_{12} \), then the corresponding control \( u \) of the

\[ form \quad (2.52) \quad assures \quad the \quad stability \quad of \quad the \quad null \quad solution \quad in \quad the \quad system \quad (3.3). \]

We will not write it down here explicitly since it involves three 11 order square matrices.

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