BRST Cohomologies of Mixed and Second Class Constraints

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Article

Abstract: The cohomological resolution of mixed constraints is constructed and shown to give Gupta–Bleuler space of physical states. The differential space and so-called anomalous BRST complex is constructed in detail. Special structures associated with anomaly are demonstrated and proved to be of important significance. Finally, the formalism is applied to a spinorial system with second class constraints. In this case, the Laplace operators are proved to define the (irreducibility) equations for fields carrying arbitrary (high) spin.

Keywords: constrained systems; anomalies; complex polarization; cohomologies; Kähler structures

1. Introduction

This paper is devoted to the presentation of ideas underlying the cohomological (BRST) approach to mixed class constraints (Dirac classification, see [1,2]). The method presented here is a straightforward generalization of the approach proposed a long time ago by Gupta and Bleuler in the context of quantum electrodynamics [3,4]. There were several attempts [5–8] to join the BRST formalism with ideas of [3,4]. Unfortunately, they were premature at that time and not very successful.

The main idea of the review presented here is to exploit the methods of complex geometry applied to (complex) polarized constraints of second class. The details of the construction and important results are presented within the abstract context of centrally extended simple Lie algebra of constraints realized by operators acting on some Hilbert space. Since the algebra is assumed to be simple and of finite dimension, the central extension is absolutely trivial. This simplifying assumption does not exclude that the formalism presented here can be applied in a more general case (e.g., string theory), see [9–11].

The steps of the construction are as follows: First, one introduces the natural, complex polarization of constraints. Next, the differential space corresponding to full algebra of constraints is introduced. It is not a complex, as the BRST differential operator \( D \) constructed according to standard rules is not nilpotent. Its square \( D^2 \neq 0 \) is called the anomaly. It is then almost an immediate idea to introduce a subspace of the original differential space as the kernel of the anomaly \( D^2 \). This subspace equipped with original differential \( D \) becomes a complex called the anomalous BRST complex. It is shown that an anomalous complex provides the cohomological resolution of polarized constraints of the [3,4] type. The anomaly defines in a natural way the pairing of the cochains of Kähler type and Hodge type conjugations.

It is moreover shown that the Laplace operator constructed in the natural way with the use of pairing corresponding to anomaly provides, as in the standard case, the contracting homotopy for the anomalous cochains. This unexpected result confirms the value of the presented approach. It should be stressed that we are not aware of the mathematical significance of our constructions. The physical value seems to be indisputable, especially in the context of examples presented at the end of paper.
2. A Toy Model

Assume that one considers the quantum model of a particle with fixed angular momentum. The Hilbert space \( \mathcal{H} \) carries the representation of angular momentum operators \( M_i \) \( (i = 1, 2, 3) \) with well-known commutation relations of \( \text{so}(3) \) Lie algebra:

\[
[ M_i, M_j ] = i \epsilon_{ijk} M_k .
\]

The operators are assumed to be self-adjoint. For one reason or another, one wants to fix the angular momentum i.e., to impose the constraints \( \vec{M} - \vec{\lambda} = 0 \), where \( \vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \) is a fixed vector in space. By these constraints, we mean conditions on the vectors in the space of states. We will denote \( \hat{M}_i = M_i - \lambda_i \). It is possible to choose the coordinate system such that \( \lambda_1 = \lambda_2 = 0, \lambda_3 = \lambda \). One is then left with three constraints \( \hat{M}_1 = 0, \hat{M}_2 = 0 \) and \( \hat{M}_3 = M_3 - \lambda = 0 \), represented by non-commuting operators. Instead of Equation (1), one obtains centrally extended algebra:

\[
[ \hat{M}_i, \hat{M}_j ] = i \epsilon_{ijk} (\hat{M}_k + \delta_{k3}\lambda) .
\]

It is clear that the above constraints are of mixed class and admit as a solution zero vector only.

In order to obtain the sensible result, one can apply the Gupta–Bleuler [3,4] approach which amounts to polarization of (2). Instead of \( M_1 \) and \( M_2 \), one introduces a new basis for angular momentum: the well-known jump operators \( \hat{M}_\pm = \hat{M}_1 \pm i\hat{M}_2 \) with important property \( \hat{M}_\pm^\dagger = \hat{M}_\mp \):

\[
[ \hat{M}_3, \hat{M}_\pm ] = \pm \hat{M}_\pm , \quad [ \hat{M}_+, \hat{M}_- ] = 2M_3 + 2\lambda .
\]

The constraints are now to be imposed on the averages in the space of states:

\[
(\Psi, \hat{M}_\pm \Psi) = 0 = (\Psi, \hat{M}_3 \Psi) \quad \text{or} \quad (\Psi, \hat{M}_- \Psi) = 0 = (\Psi, \hat{M}_3 \Psi) .
\]

The above conditions are solved by vectors from the kernel of appropriate Borel subalgebras of \( \text{so}(3) \), namely \( b_\pm \) spanned by \( \hat{M}_\pm, \hat{M}_3 \), respectively. This is the idea of Gupta–Bleuler complex polarization of constraints. The polarized constraints

\[
\hat{M}_\pm \Psi = 0 = \hat{M}_3 \Psi ,
\]

where plus or minus is chosen and fixed) give as a solution highest (lowest) weight state vectors of angular momentum \( \lambda \) in the third direction. It should be mentioned finally that \( \lambda \) must be positive (negative) integer for the solutions to exist.

In the next section, the same ideas are applied for the general compact Lie algebra of constraints. The natural BRST formalism is developed there. It allows one to define the pairing of Kähler type and to introduce the corresponding Laplace operators. They play a significant role in the last chapter where they serve as generators of kinematical (irreducibility) equations for spinning fields. In the next section, the above example is generalized and appropriate complex is constructed.

3. Centrally Extended Lie Algebra

One starts with the family of skew-symmetric operators \( T_x \) representing the Lie algebra \( g \) on some Hilbert space \( \mathcal{H} \) \( (x \in g) \). The original structural relations of \( g \) (represented by \( T_{(c)} \) operators) are modified by a central term (an anomaly):

\[
[ L_x, L_y ] = L_{\{x,y\}} + c(x,y) ; \quad x,y \in g ,
\]

where \( c \) is the antisymmetric bilinear function on \( g \times g \) satisfying the cocycle identity, i.e., the Jacobi identity for the relations (6). The centrally extended algebra (6) will be denoted by \( \hat{g} \).
In order to push forward the considerations, it will be assumed that $g$ is simple and consequently the cocycle $c$ is trivial i.e., of the form:

$$c(x, y) = \langle \lambda, [x, y] \rangle; \quad \lambda \in g^*$$

with $g^*$ being a dual of $g$. In fact, in this last case, the operators satisfying Equations (6) are obtained by the shift of the original ones of $g$:

$$\mathcal{L}_x = T_x - \langle \lambda, x \rangle; \quad x \in g.$$

Hence, the constraints defined by $\mathcal{L}_x \approx 0$ imposed on the (state) vectors of $\mathcal{H}$ define the common eigenvectors of the corresponding Lie algebra elements with the respective (non zero) eigenvalues $\langle \lambda, x \rangle$. It is clear that they are contradictory in the general case and they admit zero vector as the unique solution. The way to overcome this situation is so-called procedure of polarization [12,13]. On the other hand, this procedure is in perfect agreement with the original ideas of [3,4], where it was postulated that, instead of the strong conditions on the physical states being in the kernel of all constraints operators, one should replace them by weaker equations on the average values of constraints: they have to be zero for the physical state vectors. Hence, one defines:

$$\mathcal{H}_{GB} = \{ \varphi \in \mathcal{H}; \quad (\varphi, \mathcal{L}_x \varphi) = 0; \quad x \in g \},$$

where $(\cdot, \cdot)$ denotes the scalar product in the Hilbert space $\mathcal{H}$. The above definition suggests the idea of polarization of the Lie algebra of constraints. The rough definition can be provisionally formulated as follows. The polarization of the Lie algebra $g$ is a maximal anomaly free Lie subalgebra $g_+$ such that

$$g = g_+ + (g_+)^*,$$

where $(\cdot)^*$ denotes the conjugation of operators in $\mathcal{H}$. The sign $+$ instead of $\oplus$ means that it is not assumed that $g_+ \cap (g_+)^* = \{ 0 \}$, so there are non zero common elements in general. From the above definition, it follows that the subspace

$$\mathcal{H}_0 = \ker g_+ \subset \mathcal{H}$$

coincides with that of (9).

It is convenient to make technical assumption on the cocycle in (7), namely that $\lambda$ is regular in $g^*$, i.e., $\lambda$ belongs to the interior of some Weyl chamber [12,13]. Then, there exists Cartan subalgebra [14] $\mathfrak{h} \subset g$ such that $\lambda \in \mathfrak{h}^*$ in this case [12,13]. One may choose a basis $\{ h_1 \}$ of $\mathfrak{h}$ such that $\langle \lambda, h_1 \rangle \geq 0$, i.e., $\lambda$ is dominant. It is then natural to introduce the corresponding root decomposition of $g$ [14] with the split of the root system $R$ into positive and negative roots:

$$g = \mathfrak{h} \oplus \bigoplus_{\alpha > 0} (C_{\tau_\alpha} + \overline{C}_{\tau_{-\alpha}}).$$

The structural relations of $g$ in Chevalley basis [14]

$$[\tau_\alpha, \tau_\beta] = N_{\alpha \beta} \tau_{\alpha + \beta} + \delta_{\alpha, -\beta} h_\alpha, \quad N_{\alpha \beta} = 0 \quad \text{if} \quad \alpha + \beta \notin R$$

(13)

$$[h, \tau_\alpha] = \alpha(h) \tau_\alpha; \quad h \in \mathfrak{h}.$$
allow one to identify the subspaces of (10), namely \( g_\pm = b_\pm \) - Borel subalgebras spanned by Cartan subalgebra \( h \) and positive and negative root vectors, respectively. The structural relations (6) read:

\[
[ \mathcal{L}_a, \mathcal{L}_b ] = N_{a\beta} \mathcal{L}_{a+b} - \delta_{a,-\beta} ( \mathcal{L}_h a + <a, h_a > ) , \quad [ \mathcal{L}_h, \mathcal{L}_a ] = a (h) \mathcal{L}_a .
\]  

(15)

From the above, it follows that the space of Gupta–Bleuler states (11) can be identified with

\[
\delta_0 = \ker b_+ = \{ \varphi \in \delta \} ; \quad \mathcal{L}_a \varphi = \mathcal{L}_h \varphi = 0 , \quad a > 0 , \quad h \in h ,
\]

(16)

which is natural generalization of the original approach of [3,4] designed for quantum electrodynamics. In other words: the elements of \( \delta_0 \) are the highest weight vectors of weight \( \lambda \) with respect to the original Lie algebra action \( \tau \) on \( \delta \). The rules of conjugation of constraint operators:

\[
\mathcal{L}^*_a = - \mathcal{L}^-_a \quad \text{and} \quad \mathcal{L}^*_h = \mathcal{L}_h ; \quad h \in h ,
\]

(17)

are compatible with properties of real structure constants [14].

3.1. Ghosts

The first step in the cohomological (BRST) description of constrained systems consists of dressing the Hilbert space with ghosts: \( \delta \rightarrow \text{\Lambda} g^* \otimes \delta \). The resulting extended space is nothing else than the space of \( \delta \) valued (say left) invariant forms on the corresponding group. The factor \( \text{\Lambda} g^* \) is called the ghost sector in the physical literature. The ghost and anti-ghost modes are simply multiplication operators by basis forms \( c \) and dual substitution operators \( b \) by Lie algebra elements. It is clear that they satisfy the following anti-commutation rules:

\[
c^i b_j + b_j c^i = \delta^i_j , \quad i, j = 1 \ldots r ,
\]

(18)

for \( c_i, b_i \) associated with the basis vectors of \( h \) (we recall that \( r = \dim h \) is the rank of the Lie algebra \( g \)).

Similarly, for the root vectors, one writes the formulæ:

\[
c^\alpha b_\beta + b_\beta c^\alpha = \delta^\alpha_{\beta} , \quad \alpha, \beta \in R .
\]

(19)

This somewhat unusual convention originating from string theory means that \( c^\alpha \) are the multiplication operators by the forms dual to the root vectors of opposite sign \(-\alpha\). There is a representation of the original Lie algebra \( g \) on \( \text{\Lambda} g^* \)—the canonical extension of the coadjoint one. The corresponding operators are expressed in terms of elementary ghost modes by famous Koszul formulæ [15]:

\[
t_a = - \sum_{\beta \in R} N_{a\beta} c^{-\beta} t_{a+\beta} + \sum_{i=1}^r h_i^a c^\alpha b_i + \sum_{i=1}^r a(h_i) c^i b_a ; \quad a \in R
\]

\[
t_i = - \sum_{\beta \in R} \beta(h_i) c^\beta b_{-\beta} ; \quad 1 \leq i \leq r .
\]

(20)

The numbers \( h_i^a \) are the coordinates of \( h_a \) on the basis of Cartan subalgebra. The exterior differential of invariant forms is expressed in terms of elementary ghost modes as follows [15]:

\[
d_{\text{nil}} = \frac{1}{2} \sum_{a \in R} c^{-a} t_a + \frac{1}{2} \sum_{i=1}^r c^i t_i \quad \text{and, obviously,} \quad (d_{\text{nil}})^2 = 0 .
\]

(21)

The above differential is extended canonically onto \( \delta \)-valued differential forms from \( \text{\Lambda} g^* \otimes \delta \):

\[
D = \sum_{a \in R} c^{-a} \otimes t_a + \sum_{i=1}^r c^i \otimes t_i + d_{\text{nil}} \otimes 1 .
\]

(22)
The space $\mathcal{H}$ is identified with $\wedge^0 g^* \otimes \mathcal{H}$ and $g$-invariant elements of $\mathcal{H}$ are determined by single equation $D\varphi = 0$ imposed on 0-forms. This equation splits into a family of conditions $t.\varphi = 0$ corresponding to all basis elements of $g$. It is clear that, at this stage of construction, the operators $t.$ cannot be simply replaced by the shifted ones of (8) as they would imply unsatisfactory conditions on the states from $\mathcal{H}$. In order to enforce conditions corresponding to $g_+$ Lie subalgebra, one has to change the identification of $\mathcal{H}$ in $\wedge g^* \otimes \mathcal{H}$. For this reason, one introduces so-called ghost vacuum $\omega$—the distinguished non zero element of $\wedge g^*$ such that the Hilbert space is identified with $\omega \otimes \mathcal{H}$.

The ghost vacuum is described by the set of the following conditions:

$\alpha = 0$ ; $\beta > 0$ and $b_{\beta} = 0$ ; $1 \leq i \leq r$. (23)

The above equations fix the vacuum $\omega$ up to scalar factor and give $\omega \sim c_1 \ldots c_n$. (24)

The Hilbert space factor of $\wedge^0 g^* \otimes \mathcal{H}$ carries the scalar product by assumption. There is, however, a minor problem: the vacuum element is neither closed under (21) nor it is $h$-invariant under a coadjoint action of (20). Despite this unwanted property of (24), there is a way to by pass this difficulty. One has to introduce so-called normal ordering of operators expressed in ghost modes, which consists of moving all $b_{-\alpha}$ anti-ghosts to the left in all expressions with appropriate change of sign. The ghosts corresponding to $h$ elements should be antisymmetrized. The normally ordered operators are denoted by $: \cdot :$ as usual in the physical literature. The normally ordered counterparts of (20) are given by the following formulae:

$l_i = : t_i : = l_i = \sum_{\beta > 0} \beta (H_i) c^{-\beta} b_\beta - \sum_{\beta > 0} \beta (h_i) b_{-\beta} c^\beta ; 1 \leq i \leq r$. (27)

and the operators corresponding to the root vectors are unaffected. In fact, there is a simple relation similar to that of (8), namely:

$l_i = t_i - 2 < \varrho, H_i > , \ \varrho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. (28)

The structural relations of normally ordered operators corresponding to root vectors are changed by central terms corresponding to lowest dominant weight $\varrho$ analogously to those of (6) and (7). The normal ordering rule changes the differential (21) accordingly:

$d = : d_{nil} : = d_{nil} - 2 \varrho , \varrho = \sum_{i=1}^r < \varrho, H_i > c^i$. (29)

where this time $\varrho = \sum_{i=1}^r < \varrho, H_i > c^i$. The normally ordered differential is not nilpotent anymore and one has instead:

$d^2 = -2 \sum_{\alpha > 0} c^{-\alpha} c^\alpha < \varrho, H_{\alpha} >$. (30)
In fact, (29) looks like a covariant differential with connection form $\varrho$ and curvature given by (30). Hence, the space of invariant differential forms $\bigwedge \mathfrak{g}^*$ equipped with differential $d$ of (29) is not a complex, but the differential space with $d$ raising the degree of an element by 1 remains. Although it is not essential [16], it is worth introducing at this place the grading of this space—in correspondence with the choice of vacuum element (24). The ghost number grading is defined by a normally ordered degree operator:

$$: \deg := \text{gh} = \sum_{a>0} e^{-a} b_a - \sum_{a>0} b_{-a} c^a + \frac{1}{2} \sum_{i=1}^l (c^i b_i - b_i c^i) ,$$ (31)

The ghost number of the vacuum $\omega$ equals $-\frac{1}{2} r$ and consequently the space $\bigwedge \mathfrak{g}^*$ (denoted here and in the sequel by $\mathcal{D}$ in order to simplify the notation) splits into direct sum

$$\mathcal{D} = \bigoplus_{k=-m-\frac{1}{2}r}^{k=m+\frac{1}{2}r} \mathcal{D}^k , \text{ where } r = \text{rank } \mathfrak{g} \text{ and } 2m + r = \dim \mathfrak{g} ,$$ (32)

of eigenspaces of the ghost number operator (31).

3.2. $\mathfrak{g}$-Differential Space

We are now in a position to dress the original Hilbert space with ghosts and to equip it with appropriate BRST differential. As in the standard case, the BRST differential space is defined to be

$$\mathcal{D}(\mathfrak{g}) := \mathcal{D} \otimes \mathfrak{g} = \bigoplus_k \mathcal{D}^k(\mathfrak{g}) ,$$ (33)

with the decomposition inherited from that of (32). The differential acting on the space (33) is introduced in a standard way

$$D = \sum_{a \in \mathbb{R}} e^{-a} \otimes L_a + \sum_{i=1}^l c^i \otimes L_i + d \otimes 1 .$$ (34)

The operators $L_i$, satisfying Equations (15), carry the central extension—an anomaly that contributes to the one already present in the relations of normally ordered ones (27) of coadjoint action:

$$L_{\lambda(\cdot)}^{\text{tot}} := b_{\lambda(\cdot)} D + D b_{\lambda(\cdot)} = 1 \otimes L_{\lambda(\cdot)} + l_{\lambda(\cdot)} \otimes 1 ,$$ (35)

according to definitions (8) and (27) with structural relations

$$[ L_a^{\text{tot}}, L_{\beta}^{\text{tot}} ] = N_{a\beta} L_{a+\beta}^{\text{tot}} - \delta_{a,-\beta} ( L_{H_a}^{\text{tot}} + < \lambda + 2\varrho, H_a > ) .$$ (36)

The relations of all remaining generators are intact with respect to those of $\mathfrak{g}$. The differential (34) is not nilpotent accordingly

$$D^2 = - \sum_{a>0} e^{-a} c^a < \chi + 2\varrho, H_a > ,$$ (37)

and, moreover, in contrast to conventional case [15], it is not invariant $[ L_a^{\text{tot}}, D ] = -c^a < \chi + 2\varrho, H_a >$. It is worth stressing that the anomaly above is never zero as $\lambda$ was assumed to be regular. This situation is in contrast to the one in string theory where non-trivial anomalies of coadjoint action on semi-infinite forms contribute $-26$ (bosonic strings) or $-10$ (fermionic RNS strings) [17,18], which
can be interpreted as the source of critical dimensions. It is convenient to introduce the compact notation for (non-zero) coefficients of the curvature:

\[ \mu_\alpha := \text{sign}(\alpha) < \lambda + 2q, \quad \alpha \in R ; \quad D^2 = - \sum_{\alpha > 0} \mu_\alpha c^{-\alpha} c^\alpha . \] (38)

Since \( D \) is not nilpotent, the structure \( (D(\mathfrak{h}), D) \) is not a complex. There is however simple way to overcome this drawback. This can be done by introducing an anomalous complex.

3.3. Anomalous Complex

The differential space \( \mathfrak{A} \) of the anomalous complex is defined in as natural way as it is possible, simply

\[ \mathfrak{A} = \ker D^2 . \] (39)

It is endowed with the original differential \( D \mid_{\mathfrak{A}} \) restricted to (39)—the kernel of the curvature. One is now in a position to define the anomalous cohomology spaces:

\[ H^r = Z^r \setminus B^r, \quad Z^r = \ker D \mid_{\mathfrak{A}^r}, \quad B^r = \text{im} , D \mid_{\mathfrak{A}^{r-1}} \] (40)

as quotients of closed cochains divided by coboundaries. In order to gain more information on the content of \( \mathfrak{A} \), it is useful to introduce the following operators:

\[ J^+ := -D^2 = \sum_{\alpha > 0} \mu_\alpha c^{-\alpha} c^\alpha, \quad J^- := \sum_{\alpha > 0} \frac{1}{\mu_\alpha} b_{-\alpha} b_\alpha, \quad \text{gh}_0 := \sum_{\alpha > 0} (c^{-\alpha} b_\alpha - b_{-\alpha} c^\alpha) , \] (41)

which are quite well known in complex differential geometry [19,20]. Their structural relations are that of \( \text{sl}(2) \):

\[ [J^+, J^-] = \text{gh}_0 \quad [\text{gh}_0, J^\pm] = \pm 2 J^\pm . \] (42)

Consequently, the elements of \( \mathfrak{A} \) are the highest weight vectors of the above \( \text{sl}(2) \) Lie algebra. Their weights are determined by \( \text{gh}_0 \)—the so-called relative ghost number operator. The detailed information on the structure of \( \mathfrak{A} \) can be drawn from [14]. It is not, however, necessary for further considerations. It is then natural to introduce relative complex as differential space

\[ \mathfrak{A}_0 = \{ \Psi \in \mathfrak{A} ; \quad L^\text{tot}_h \Psi = 0, \quad h \in \mathfrak{h} , \quad b_i \Psi = 0, \quad i = 1 \ldots r \} , \] (44)

equipped with differential \( D_0 = D \mid_{\mathfrak{A}_0} \). Closer look at the structure of \( D \) clearly indicates that it acts inside \( \mathfrak{A}_0 \). The space of relative cochains is naturally graded by eigenvalues of \( \text{gh}_0 \) which are all
integers. In fact, the space \( \mathfrak{A}_0 \) admits richer grading by the eigenvalues of two operators being the elements of \( \mathfrak{gh} \), namely

\[
\mathfrak{gh}_0 = \overline{\mathfrak{gh}} - \mathfrak{gh}, \quad \text{where} \quad \overline{\mathfrak{gh}} = \sum_{a>0} c^{-a} b_a \quad \text{and} \quad \mathfrak{gh} = \sum_{a>0} b_a c^a.
\]  

(45)

From the commutation relations of generators of \( \mathfrak{sl}(2) \) (42) and the above definitions, it follows that

\[
[\overline{\mathfrak{gh}}, J^\pm] = \pm J^\pm, \quad [\mathfrak{gh}, J^\pm] = \mp J^\pm.
\]  

(46)

Any element \( \Psi^k \in \mathfrak{A}_0^k \) can decompose into bi-homogenous \( \Psi_{p, q}^p \) components such that \( \mathfrak{gh} \Psi_{p, q}^p = p \Psi_{p, q}^p \) and \( \mathfrak{gh} \Psi_{p, q}^p = q \Psi_{p, q}^p \). Consequently, any space of relative ghost number \( k \) can be further decomposed as

\[
\mathfrak{A}_0^k = \bigoplus_{p, q | p - q = k} \mathfrak{A}_{p, q}^p.
\]  

(47)

The relative differential splits accordingly [19,20]

\[
D_0 = \overline{\mathcal{D}} + \mathcal{D}, \quad \text{where} \quad \overline{\mathcal{D}} : \mathfrak{A}_q^p \to \mathfrak{A}_q^{p+1} \quad \text{and} \quad \mathcal{D} : \mathfrak{A}_q^p \to \mathfrak{A}_q^{p-1},
\]  

(48)

By counting the bidegrees of the above operators, one obtains immediately

\[
\overline{\mathcal{D}}^2 = 0 = \mathcal{D}^2, \quad \text{and} \quad \mathcal{D} \overline{\mathcal{D}} + \overline{\mathcal{D}} \mathcal{D} = -J^+ \quad (\equiv 0 \text{ on } \mathfrak{A}).
\]  

(49)

The components of (48) can be quite simply determined. For example, for the part of bidegree \((1, 0)\), we have

\[
\overline{\mathcal{D}} = \sum_{a>0} c^{-a} \otimes L_a + \overline{\mathcal{D}} + \sum_{a>0} c^{-a} t_a, \quad \text{where}
\]

\[
\overline{\mathcal{D}} = -\frac{1}{2} \sum_{a, \beta>0} N_a \beta c^{-a} e^{-\beta} b_{a+\beta} \quad \text{and} \quad t_a = -\sum_{\beta>0} N_{a-\beta} e^{\beta} b_{a+\beta}.
\]  

(50)

An analogous expression for the remaining component \( \mathcal{D} \) can be obtained by \( * \) conjugation according to (26) of the above formulæ. The structure of the bigraded complex (differential space) can be illustrated in the following way:

\[
\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\uparrow \mathcal{D} & \uparrow \mathcal{D} & \uparrow \mathcal{D} & \uparrow \mathcal{D} & \uparrow \mathcal{D} & \uparrow \mathcal{D} & \uparrow \mathcal{D} & \uparrow \mathcal{D} & \uparrow \mathcal{D} \\
\mathfrak{A}_q^p \rightarrow \mathfrak{A}_q^{p+1} \rightarrow \mathfrak{A}_q^{p+2} \rightarrow \cdots & \mathfrak{A}_{q-1}^p \rightarrow \mathfrak{A}_{q-1}^{p+1} \rightarrow \mathfrak{A}_{q-1}^{p+2} \rightarrow \cdots \\
\downarrow \overline{\mathcal{D}} & \downarrow \overline{\mathcal{D}} & \downarrow \overline{\mathcal{D}} & \downarrow \overline{\mathcal{D}} & \downarrow \overline{\mathcal{D}} & \downarrow \overline{\mathcal{D}} & \downarrow \overline{\mathcal{D}} & \downarrow \overline{\mathcal{D}} & \downarrow \overline{\mathcal{D}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]  

(51)

The space \( \mathcal{D}(\mathfrak{g}) \) admits identical bigrading too. Hence, the anomalous complex \( \mathfrak{A} \) in the above diagram can be replaced by total differential space \( \mathcal{D}(\mathfrak{g}) \). The bigraded structure will not be exploited in the sequel, although it is worth mentioning that bigraded cohomology spaces can be introduced. It was already mentioned above that the space \( \mathfrak{A} \) of anomalous cochains is asymmetric with respect to ghost number. For this reason, the definition of non-degenerate pairing requires an external
operation acting on $\mathcal{D}(\mathcal{H})_0$—the space of relative cochains. The pairing is associated with a Hodge-star type operation, which takes the elements of $\mathfrak{A}$ out of the space.

$$\star(b_{-n_1...b_{-n_k}c^{−\beta_1}...c^{−\beta_k}\omega \otimes \varphi}) = (-1)^{(r+1)(k+1)}\prod_{ij=1}^{k} \frac{H_{\beta_i}^{\alpha_i}}{H_{\beta_j}^{\alpha_j}} b_{-\beta_1...c^{−\alpha_1}...c^{−\alpha_k}\omega \otimes \varphi},$$  \hspace{1cm} (52)

where $r = \text{rank}(\mathfrak{g})$. The above mapping is an isomorphism of the subspaces of opposite bidegrees and it extends to relate the subspaces of $\mathcal{D}(\mathcal{H})$ of opposite ghost number. The $\star$ defines a nondegenerate and positive inner product on the differential space $\mathcal{D}(\mathcal{H})$ as well as on $\mathfrak{A}$:

$$\langle \Psi, \Psi' \rangle = (\star \Psi, v(\hbar) \Psi'),$$  \hspace{1cm} (53)

where $(\cdot, \cdot)$ denotes the natural pairing on $\mathcal{D}(\mathcal{H})$ ((25) and (26)). The inner product induces a new conjugation of the operators

$$\mathfrak{B} \rightarrow \mathfrak{B}^\dagger = (-1)^{\text{deg} (\mathfrak{B})} \star \mathfrak{B}^\star,$$  \hspace{1cm} (54)

where $\text{deg}(\mathfrak{B})$ is the total degree of the operator in $(b,c)$ ghosts and $\star$ denotes the conjugation with respect to the natural, original pairing (26). By straightforward calculation, it can be checked that, for (48), one has

$$\mathcal{D}^\dagger = \frac{1}{2} \sum_{\alpha, \beta > 0} N_{a_{\beta}} \frac{H_{a+\beta}}{H_{a} H_{\beta}} c^{a+\beta} b_{-a} b_{-\beta} + \sum_{a > 0, \beta > a} N_{a_{\beta}} \frac{H_{a-\beta}}{H_{a} H_{\beta}} c^{a-\beta} b_{-a} b_{-\beta} - \sum_{a > 0} \frac{1}{H_{a}} b_{-a} \otimes \mathcal{L}_{a},$$  \hspace{1cm} (55)

and respective formulae for $\mathcal{D}^\dagger$. With direct but rather tedious calculation with the help of cocycle property of $J^+$, the curvature (anomaly) operator gives important identities

$$[J^+, \mathcal{D}^\dagger] = -\mathcal{D}, \quad [J^+, \mathcal{D}^\dagger] = \mathcal{D}.$$  \hspace{1cm} (56)

As in classical textbooks [19,20], it is useful to introduce the following operator with opposite curvature:

$$D_0^\circ = \mathcal{D} - \overline{\mathcal{D}} \quad \text{then} \quad D_0 D_0^\circ + D_0^\circ D_0 = 0, \quad D_0^2 = J^+.$$  \hspace{1cm} (57)

From (56) and the above definition, it can be calculated that

$$[J^+, D_0^\circ] = D_0^\circ \quad \text{and} \quad [J^+, D_0^\circ] = -D_0.$$  \hspace{1cm} (58)

The positive inner product allows one to define the family of Laplace operators

$$\triangle = D_0 D_0^\dagger + D_0^\dagger D_0 \quad \text{and} \quad \triangle^\circ = D_0^\circ D_0^\dagger + D_0^\dagger D_0^\circ,$$  \hspace{1cm} (59)

$$\square = D \mathcal{D}^\dagger + \mathcal{D}^\dagger D \quad \text{and} \quad \square = \overline{\mathcal{D}} \overline{\mathcal{D}}^\dagger + \overline{\mathcal{D}}^\dagger \overline{\mathcal{D}}.$$  \hspace{1cm} (60)

The Laplace operators defined above satisfy part of the standard relations of complex geometry [19,20]:

$$\triangle = \square + \square = \triangle^\circ \quad \text{and} \quad [J^{\pm}, \triangle] = 0,$$  \hspace{1cm} (61)

but some of them are broken

$$[J^{\pm}, \square] = \mp J^{\pm}, \quad [J^{\pm}, \square] = \pm J^{\pm} \quad \text{and} \quad [gh_0, \square] = 0 = [gh_0, \square].$$  \hspace{1cm} (62)
The small Laplace operators are not equal \([19,20]\) but are related by quite simple identity:

\[
\Box = \Box - gh_0 .
\]  

The proof of (61) and (62) requires clever application of (56) and definitions ((59),(60)). Having the Laplace operator(s) at one’s disposal, one may define the space of anomalous harmonic cochains. Namely:

\[
\mathbb{H}(\Box) := \ker \Box|_{\mathfrak{a}_0} .
\]

Assuming positivity of the scalar product defined by Hodge-star operation (53), it is possible to prove that every relative cohomology class has harmonic representative, i.e.,

\[
H^m_0 \simeq \mathbb{H}^m(\Box) ,
\]

where \(\mathbb{H}^m(\Box)\) denotes the space of harmonic cochains of degree \(m\). From the fact that \(\mathbb{H}^m(\Box) = 0\) for \(m > 0\) follows the vanishing theorem for relative cohomology,

\[
H^m_0 = 0 ; \ m > 0 ,
\]

and important identification of the G-B subspace defined in (11)

\[
\mathbb{H}_0^0(\Box) \simeq \mathfrak{a}_0 .
\]

One can define a real Lagrange density, which, via variational principle, enforces the equations for G–B states:

\[
L(\Psi) \sim < \Psi, \Box \Psi >
\]

In the next section, it will be shown that, in the case of spinorial model, the Lagrange densities above induce the irreducibility (kinematical) equations for classical fields with arbitrary spin. Before passing to this topic, it is necessary to mention the results on absolute cohomology of anomalous complex. From (67), one may draw the conclusion on absolute cohomology spaces:

\[
H^{-\frac{1}{2}r+s} \simeq \wedge^s \mathfrak{h}^* \mathbb{H}(\Box) , \ 0 \leq s \leq r .
\]

The cohomology classes of all remaining degrees do vanish.

4. Spinorial System and Field Equations

The space of states of quantum spinning particle is given as direct sum

\[
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = \int_{\mathcal{O}_+} \mathfrak{g}_+(p) \oplus \int_{\mathcal{O}_-} \mathfrak{g}_-(p)
\]

of direct integrals of momentum depending Fock-like spaces carrying the representations of the operator algebras

\[
[L^A, L^B] = p^{AB} , \ K^A, K^B = -p^{AB} ,
\]

where the \(L\)'s and \(K\)'s are Weyl spinors and \(p^{AB} = p\sigma^{AB}\) is Hermitian momentum matrix. The momentum is assumed to be massive i.e., \(m^2(p) = p_0^2 - \vec{p}^2 > 0\) to provide the invertibility of the matrix. The domains of integration \(\mathcal{O}_\pm\) are the interiors of future pointed light cone \((p^0 > 0)\) and past pointed light cone \((p^0 < 0)\), respectively. The relations (71) are complete to define Fock-like space if they are supplemented by the conjugation rules \(L^A = L^\tilde{A}, \ K^A = K^\tilde{A}\). It should be added that the indices \(\tilde{A}\) denote the same as dotted indices in the original textbooks \([21–23]\). The operators of the same spinorial character commute \([L^A, L^B] = 0\) and the same concerns \(K\)'s. Finally, it has to be said that operators of \(L\) family commute with these of the set \(K\).
The spaces $\mathcal{F}_\pm(p)$ are generated from the local vacuum vectors $\omega_\pm(p)$ normalized by formal condition $(\omega_\pm(p), \omega_\pm(p')) = \delta(p-p')$ and satisfying

$$L^A \omega_+(p) = K^A \omega_+(p) = 0 \quad \text{and} \quad L^A \omega_-(p) = K^A \omega_-(p) = 0$$ \quad (72)

These conditions are enforced by signs in (71) and supports of vacuum states located on $O_\pm$, respectively. The spaces $\mathcal{F}_\pm(p)$ are most transparently realized as spaces of functions of momentum variables and Weyl spinors coordinates $(z^A, z^\bar{A})$. The operators (71) are then represented by

$$2L_i^A = \text{im}(p) \frac{1}{2} p^{AB} \frac{\partial}{\partial z_i^B} - m(p) \frac{1}{2} e^{AB} \frac{\partial}{\partial z_i^B} + (m(p) \frac{1}{2} p^A z_i^B + \text{im}(p) \frac{1}{2} z_i^A) \ , \ L_i^{\bar{A}} = L_i^{\bar{A}} \ ,$$ \quad (73)

while the complementary modes are given as

$$2K_i^A = \text{im}(p) \frac{1}{2} p^{AB} \frac{\partial}{\partial z_i^B} - m(p) \frac{1}{2} e^{AB} \frac{\partial}{\partial z_i^B} - (m(p) \frac{1}{2} p^A z_i^B + \text{im}(p) \frac{1}{2} z_i^A) \ , \ K_i^{\bar{A}} = K_i^{\bar{A}} \ .$$ \quad (74)

The symbols $e^{AB}$ and $e^{AB}$ denote the coordinates of antisymmetric $\text{Sl}(2; \mathbb{C})$ invariant tensors. They serve to raise and lower the spinorial indices. The conditions (72) are solved by Gauss exponents

$$\omega_\pm(p) = \exp(\pm \sum_{A,B} z^A p_{AB} z^B \frac{m(p)}{m(p)}) \ ,$$ \quad (75)

and finally the spaces of states are generated by dense sets of the following vectors

$$\Psi_\pm(p,z,z) = \mathcal{P}_\pm(z,z) \omega_\pm(p) \ ,$$ \quad (76)

where $\mathcal{P}_\pm(z,z)$ are the polynomials in Weyl coordinates with $p$-dependent coefficients. It has to be clarified at this point that the operators $(L_i^A, L_i^{\bar{A}})$ represent the complex polarized second class constraints of the reparametrization invariant model of spinning particle, which originally contains a Majorana spinor put along the particle trajectory. The complex polarization splits Majorana spinors into Weyl components. There is supplementary constraint following from reparametrization invariance—the canonical Hamiltonian,

$$H_D = \frac{1}{2} (p^2 + m_0^2) + \frac{1}{m(p)} \sum_{AB} (L^A p_{AB} L^{\bar{B}} - K^A p_{AB} K^{\bar{B}}) \ ,$$ \quad (77)

which correlates the mass of the particle with its spin. The commutation relations of $H_D$ with second class constraints reads

$$[H_D, L^A] = m(p) L^A \ , \ [H_D, L^{\bar{A}}] = -m(p) L^{\bar{A}} \ .$$ \quad (78)

As it was shown in the previous section, it is possible to construct the corresponding BRST differential space and the anomalous complex. The ghosts and antighosts operators corresponding to $L$ operators are denoted by $(c_A, b_A)$ and $(c_{\bar{A}}, b_{\bar{A}})$, respectively, while these for $H_D$ by $(c_0, b_0)$. The ghost vacuum states have to be chosen in agreement with (72)—separately for both disjoint momentum regions $O_\pm$ . Hence, the differential space (33) splits into two sectors subordinate to these regions:

$$\mathcal{D}(\mathcal{F}) = \int_{O_+} \mathcal{D}_+(p) \oplus \int_{O_-} \mathcal{D}_-(p) \ , \ \mathcal{D}_{\pm}(p) = \mathcal{G}_\pm \otimes \mathcal{F}_\pm(p)$$ \quad (79)
where $G_{\pm}$ are the ghost sectors generated out of respective ghost vacua. The BRST differential constructed according to [15] is given as follows:

$$D = \sum_A c_A L^A + \sum_A c_A \bar{L}^A + c_0 H_D - m(p) c_0 \left( \sum_A : c_A b^A : - \sum_A : c_A \bar{b}^A : \right)$$  \hspace{1cm} (80)

and one may compute that it is anomalous (not nilpotent):

$$D^2 = \sum_{A, \bar{A}} c_A c_{\bar{A}} p_\bar{A} \bar{A} .$$  \hspace{1cm} (81)

One may now introduce the machinery developed in the previous chapter: the anomalous complex and $\mathfrak{sl}(2, \mathbb{C})$ algebra generated by curvature. The most important notions are the pairing and the corresponding Laplace operator. In agreement with (52), one puts

$$\ast (b^A_1 \ldots b^A_{i-1} c_{B_1} \ldots c_{B_j} \omega^B \otimes \Psi(p)) =$$

$$= (-1)^{j+pq} \sum_{B_1 \ldots B_j} \left( \sum_{A_1 \ldots A_j} p_{B_1} b^A_{i-1} b^A_i \ldots b^A_{j-1} b^A_j c_{A_1} \ldots c_{A_j} \omega^B \otimes \Psi(p) \right).$$  \hspace{1cm} (82)

The Laplace operator constructed according to recipe (59) is quite simple in spinorial case

$$\Delta = 2 \mathcal{R} + \text{deg}, \quad \text{with} \quad \mathcal{R} = -\theta(p^0) \frac{1}{m^2} \sum_{A, B, i} L^A_i p_{A B} L^B_i + \theta(-p^0) \frac{1}{m^2} \sum_{A, \bar{B}, i} \bar{L}^A_i p_{A \bar{B}} \bar{L}^\bar{B}_i .$$

The mark deg denotes total ghost degree. All the statements on the connection of harmonic cochains with cohomology classes made in the previous section remain in effect in the presented example. The positive scalar product induced by star operation (82) allows one to define the Lagrange density which induces proper equations on the spinorial states via variational principle

$$\mathcal{L}[\Psi](p) = (\Psi(p), \Delta \Psi(p)) .$$  \hspace{1cm} (84)

Using explicit realization (73) of constraints in terms of Weyl coordinates, one can rewrite the Laplace operator as differential operation acting on ghost-free vectors (denoted as restriction $|_{\text{matter}}$):

$$\Delta |_{\text{matter}} = -\text{sign}(p^0) \frac{1}{m(p)} \sum_{A, \bar{A}} \left( p^{A \bar{B}} \frac{\partial^2}{\partial z^A \partial \bar{z}^\bar{B}} + i z^A p^{A \bar{B}} \frac{\partial}{\partial z^\bar{B}} + i \bar{z}^{\bar{B}} p^{A \bar{B}} \frac{\partial}{\partial z^A} + z^A p_{A \bar{B}} \bar{z}^{\bar{B}} \right) - 2 .$$  \hspace{1cm} (85)

This expression can be further simplified for the vectors of the form of (76). The basis (dense set) states are of the following form:

$$\Psi^j_\pm (p, z, \bar{z}) = \mathcal{P}^j_\pm (z, \bar{z}) \omega^j_\pm (p) = \sum_{p_{\bar{A}}, p_{\bar{A}} = 2j} \sum_{A_p, \bar{A}_q} \left( \Psi_{A_p \bar{B}_q}(p) \ z^{A_p} \ \bar{z}^{\bar{A}_q} \right) \omega_{\pm} (p) .$$  \hspace{1cm} (86)

The second sum runs over the set of multi-indices $A_p = (A_1 \ldots A_q)$ and $\bar{B}_q = (\bar{B}_1 \ldots \bar{B}_q)$ with $A_i = 1, 2$ and $\bar{B}_j = 1, 2$. Where the overbar notation is used in the standard way, see Ref. [19]. The number $j$ is nothing else than the spin of the state [21–23]. It can be checked by direct calculation that Laplace operator can be transformed to simpler form when acting on (86), namely

$$\Delta |_{\text{matter}} \Psi^j_\pm (p, z, \bar{z}) = \left( \Delta_{\pm} \mathcal{P}^j_\pm (z, \bar{z}) \right) \omega_{\pm} (p) ,$$
where a part acting on polynomials is given by

\[ \Delta_\pm = \left( 2 \mp \frac{1}{m(p)} \sum_{A,B} \left( p^{A\beta} \frac{\partial^2}{\partial z^A \partial \bar{z}^B} + i z^A p^\beta \bar{z}^B + i \bar{z}^B p_A \frac{\partial}{\partial z^A} \right) \right). \]  \hspace{1cm} (87)

In order to construct the kinematical, irreducibility equations for spinning fields explicitly, one needs the expression of the scalar product induced by Hodge star \((82)\). It is given by the following formulae for states \((86)\) with fixed spin:

\[ (\Psi^j_\pm, \Phi^j_\pm)(p) \propto \sum (\Psi_{AB} \Phi_{A'B'} \bar{P}^{A'A'} \bar{P}^{B'B'}) \]  \hspace{1cm} (88)

where \(P_{\bar{A}A'} = p_{\bar{A}1} a_{1A'} \ldots p_{\bar{A}p} a_{pA'}\) and \(P_{A\bar{A}} = \overline{P^{A\bar{A}'}}\), and the markings and the summation limits are the same as in \((86)\).

The next subsections are devoted to demonstrate the power of the method by applying the formalism to the states with fixed spin. It is shown there that the abstract Lagrange densities \((84)\) give the well known and correct Lagrange functions for spinning fields. This feature of the model is promising as a universal tool to construct the field theories with high spin.

4.1. Dirac Field

From \((86)\), it follows that, for spin \(\frac{1}{2}\) vectors, one has:

\[ \Psi^{\frac{1}{2}}(p) = \left( \Psi_A(p) z^A + \Phi_{\bar{A}}(p) \bar{z}^{\bar{A}} \right) \omega_\pm(p). \]  \hspace{1cm} (89)

It can be directly calculated that the Lagrange density \((84)\) for this state can be written in the form

\[ \mathcal{L}[\Psi^{\frac{1}{2}}](p) = \left( p^{A\bar{A}} \Psi_A(p) \Phi_{\bar{A}}(p) + p^{B\bar{B}} \bar{\Phi}_{B}(p) \bar{\Psi}_{\bar{B}}(p) \right) + im \left( e^{A\bar{A}} \Psi_A(p) \Phi_{\bar{A}}(p) + e^{B\bar{B}} \bar{\Phi}_{B}(p) \bar{\Psi}_{\bar{B}}(p) \right). \]  \hspace{1cm} (90)

It should be added that the formulae \((88)\) has to be used and the mass function \(m(p)\) is replaced by fixed value \(m\). In order to check that \((90)\) is indeed the Dirac density, it should be transformed into better known form by representing it in terms of bispinors. Introducing

\[ \Psi = \left( \begin{array}{c} \Psi_A \\ \Phi_{\bar{A}} \end{array} \right) \quad \text{and} \quad \Psi = \Psi^C C \quad \text{where} \quad C = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \]  \hspace{1cm} (91)

and momentum matrix

\[ \gamma^\mu = \left( \begin{array}{cc} 0 & p_{AB} \\ p_{AB} & 0 \end{array} \right), \]  \hspace{1cm} (92)

the density \((90)\) can be rewritten in well known form:

\[ (\Psi^{\frac{1}{2}}(p), \Delta \Psi^{\frac{1}{2}}(p)) = \Psi [\gamma^\mu + im] \Psi. \]  \hspace{1cm} (93)

Applying Fourier transform (with \(p_\mu \rightarrow -i \partial_\mu\)), one obtains finally

\[ \mathcal{L}[\Psi^{\frac{1}{2}}](x) = -i \Psi(x) [\gamma^\mu - m] \Psi(x). \]  \hspace{1cm} (94)

up to total divergence terms.
4.2. Proca Fields

As it was mentioned under (86), the most general state of spin 1 is expressed by the spin tensors of degree 2:

\[ \Psi^1(p) = \left( \Psi_{AB}(p) z^A z^B + \Phi(p) z^A z^B \right) \omega_\pm(p). \]  

(95)

The state (95) contains, in fact, a spin zero component given by trace (or divergence) \( p^{AB} Y_{AB}(p) \). The spin-tensors of the above expansion can be transformed into spacetime tensors

\[ F^{\mu\nu}_\pm(p) = \epsilon^{AB} \Psi_{AB}(p), \quad F^{\mu\nu}_-(p) = \epsilon^{AB} \Phi_{AB}(p), \quad A_\mu(p) = \epsilon^{AB} Y_{AB}(p). \]  

(96)

The fields \( F \) as well as \( A \) are complex. In addition, antisymmetric tensors \( F^{\pm} \) are self-dual and anti-self-dual, respectively, with duality eigenvalues \( \pm i \). The Lagrange density (68) is obtained in the form

\[ \mathcal{L}[\Psi^1](p) \propto p^{AB} p^{CD} \Psi_A \Psi_B + p^{AB} p^{CD} \Phi_A \Phi_B + \frac{1}{2} p^{AB} p^{CD} \Phi_A Y_{BD} + \right. \]

\[ \left. + \frac{im}{2} \left( e^{AB} p^{CD} \Psi_A \Phi_B + e^{AB} p^{CD} \Phi_A \Psi_B + e^{AB} p^{CD} \Phi_A \Phi_B + e^{AB} p^{CD} \Psi_A \Phi_B \right) \right]. \]  

(97)

Some terms in the expression above can be identified as the traces of the products of complex matrices from spacetime Clifford algebra \( F = \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \) and \( A = A_\mu \gamma^\mu \) and their complex conjugates. It should be added that \( F = F^{(+)} + F^{(-)} \). The manipulations within the Clifford algebra combined with Fourier transform and mass shell condition \( \Box = m^2 \) finally give the following Lagrange density:

\[ \mathcal{L}[\Psi^1](x) \propto -\frac{1}{2} m^2 F_{\mu\nu} \Phi^{\mu\nu} + m^2 A_\mu \Phi^\mu + \frac{1}{2} m \left( \Phi^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + (\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu) F^{\mu\nu} \right). \]  

(98)

It should be mentioned that (98) represents the Proca model in a reducible way. By Euler–Lagrange equations, one can reduce the model. The field \( F \) can be expressed in terms of \( A \): \( m F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and, vice versa, \( m A_\mu = \partial^\nu F_{\nu\mu} \).

5. Conclusions and Outlook

The presented formalism is shown to be an appropriate cohomological description of systems with mixed and second class constraints. There have been attempts of second class cohomological formulation and mixed class systems for quite a long time, see [5–7]. In the light of results presented, they seemed premature at that time. Here, it is shown that mixed and second class constrained systems can be consistently formulated within cohomological formalism of BRST type. Moreover, the presented approach opens new interesting questions on possible geometrical meaning of cohomologies restricted to the kernel of the square of non-nilpotent differential. It is also shown that the cohomological approach, when specified and applied to spinning systems, gives a unique and uniform formulation of kinematical (irreducibility) equations for massive, classical fields carrying arbitrary spin. The construction of corresponding Lagrange densities for fields is obtained within presented formalism too. There is an open question on analogous results for massless fields. Both this problem and that raised above are left to future investigation by authors.

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