Exact result for nonreciprocity in one-dimensional wave transmission

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For sound waves impinging on a one-dimensional medium, we show that nonlinearity can lead to nonreciprocal transmission, without dissipation or broken time reversal invariance. Placing quasi-monochromatic filters at the ends of the nonlinear medium, nonreciprocity can be obtained without the generation of higher harmonics outside the medium. Remarkably, in this configuration the nonreciprocity is found to be proportional to the net energy flow when monochromatic sources of equal strength (at the filter frequency) are simultaneously turned on at both ends. This result is conjectured to be general for one dimensional scattering. It is also shown that although simultaneous monochromatic sources lead to net energy flow, with sources of small but non-zero bandwidth there is no net energy transport, in accordance with the second law of thermodynamics.

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The reciprocity theorem has a long history in acoustics and optics. For the case of a linear medium with time reversal invariance, the theorem can be proved \cite{1,2}: for one dimensional systems, it amounts to the transmission coefficient being the same when waves are incident from the left or from the right. In the absence of time reversal invariance, reciprocity is no longer necessary. In optics, this can be achieved by intrinsic magnetization in the scatterer or with an external magnetic field \cite{3}. In acoustics, nonreciprocity can be caused by — and be used to detect — the motion of objects, as in acoustic tomography \cite{4}.

For nonlinear media, however, even if time reversal invariance is not broken, nonreciprocity is possible \cite{5}. Photonic structures with diode like behavior have been proposed, where the effect of the nonlinearity is strengthened by the existence of a bandgap \cite{6,7}. Such passive diode like behavior can be useful in the field of optical communications.

In this paper, we examine nonreciprocity in one-dimensional nonlinear media for longitudinal waves such as sound. As with light, we find that nonlinearity is sufficient to result in nonreciprocity, even for dissipationless systems without broken time reversal invariance. We go on to consider a different configuration: when monochromatic filters are placed at the two ends of the nonlinear medium. This confines the higher harmonics to the nonlinear medium. This setup, apart from possible advantages from a communications perspective, allows us to examine constraints from the second law of thermodynamics.

With filters, we obtain the unexpected result that the nonreciprocity is now proportional to the net energy transport from one side of the system to the other when two monochromatic sources of equal strength are simultaneously connected to the two ends of the device. We conjecture that this proportionality is general for any scattering process with two input and two output channels (at the same frequency) that is invariant under time-translation and time-reversal and is perturbatively accessible (explained later in this paper).

We consider a system that can be modelled as two adjacent layers, in each of which longitudinal waves propagate in accordance with a nonlinear wave equation. Outside the system, both to the left and the right, the linear wave equation is satisfied. Thus we have

$$\dot{y}^2 = B_i \partial_x^2 y + \mu_i \partial_x (\partial_x y)^2$$  \hspace{1cm} (1)

where $y$ is the displacement of the wave, and $n_i, B_i, \mu_i$ vary from region to region. Inside the scatterer, the two layers have parameters $(n_1, B_1, \mu_1)$ and $(n_2, B_2, \mu_2)$. Outside the scatterer, $n = B = 1$ and $\mu = 0$. The scatterer is taken to cover the region $-1 < x < 1$, with the boundary between the two layers at $x = 0$. The form of Eq. (1) retains the leading order nonlinearity in the elasticity of the medium; the energy density of the wave is $\frac{1}{2}n_y^2 + \mu (\partial_x y)^2 + \frac{1}{2} (\partial_x y)^2$. At the three boundaries between the four regions, $y$ and $B \partial_x y + \mu (\partial_x y)^2$ (the force exerted on the boundary from the two regions it separates) are continuous. We also consider an alternative to Eq. (1)

$$\dot{y}^3 = B_i \partial_x^2 y + \mu_i \partial_x (\partial_x y)^3$$  \hspace{1cm} (2)

which is slightly easier to work with, but which has an accidental $y \to -y$ symmetry. Eqs. (1) and (2) are in the class of Fermi Pasta Ulam (FPU) wave equations \cite{8}.

For both Eqs. (1) and (2), we first use perturbation theory to obtain an analytical solution. The incoming wave amplitudes from the left and right are $a_1$ and $a_2$ respectively. Eqs. (1) and (2) can then be solved to linear order in $a_1,2$, and then iteratively to successive higher orders in perturbation theory. For the case without filters, one imposes the requirement that all frequency components of the solution are purely outgoing outside the scattering medium, except for the component at frequency $\omega$.
whose incoming part is specified. For the case with filters, except for the component at frequency $\omega$ which is unaffected by the filters, all other components are confined to the scattering medium and have zero amplitude at $x = \pm 1$. These conditions are sufficient to solve Eqs. (1) and (2), order by order.

The equations were solved to third order using Mathematica$^{TM}$, with $\mu = 1$ and various specific values chosen for $n_{1,2}, B_{1,2}$ and $\omega$. This third order solution yields the leading $O(|a|^4)$ correction to the outgoing power to the left (or to the right) for Eq. (1). On the other hand, a similar third order solution to Eq. (2) yields the outgoing power to $O(|a|^6)$, with two nonlinear contributions to the component at frequency $\omega$.

For the case with filters, the outgoing wave is entirely at frequency $\omega$. Expressing the outgoing amplitudes $b_{1,2}$ as an expansion in powers of $a_{\pm 1,2}$, time translational invariance requires that each term in the expansion should have one extra power of the unconjugated variables as compared to the conjugated ones. Thus $b_{1}(a_1, a_2) = M_{ij}a_j + N_{ijkl}a_ja_ka_l + \ldots$. The outgoing power to the left is $|b_1|^2$. It is possible to verify for both Eq. (1) and Eq. (2) that $|b_1(0, a)|^2 - |b_2(a, 0)|^2$ is not equal to zero, demonstrating nonreciprocity in the transmission coefficient. (Since the system is nondissipative, this is equivalent to nonreciprocity in the reflection coefficient.) With $\mu_{1,2} = 1$ and various values of $n_{1,2}, B_{1,2}$, and $\omega$ all $\sim O(1)$, the coefficient of $|a|^4$ in $|b_1(0, a)|^2 - |b_2(a, 0)|^2$ is $O(1)$. In units where $\mu = 1$, the amplitude of the incoming wave will be small, so this is a weak effect. However, it should be possible to enhance the effect by constructing more complicated structures, for instance acoustic analogs of Ref. [6]. For the case without filters, one has to consider the outgoing power at different harmonics separately, but nonreciprocity is still found.

With filters, one might consider a monochromatic source as a blackbody (white noise) source from which only waves of one frequency are allowed to escape [11]. If one connects a blackbody at each end of the nonlinear scatterer, with both blackbodies at the same temperature, there should be no net flow of energy from one side to another. With the filters, this would seem to be equivalent to choosing $a_{\pm 1,2}$ to be equal in magnitude, but with a random relative phase. Surprisingly, it is possible to verify through the perturbation expansion of the previous paragraph that if $a_{\pm 1,2}$ are indeed of equal strength, the phase averaged outgoing power is not the same on both sides of the scatterer. In fact, for all choices of the parameters we have tried, we have verified that

$$
|b_1(0, a\sqrt{2})|^2 - |b_2(a\sqrt{2}, 0)|^2 = 2\left(|b_1(a, ae^{i\phi})|^2\right)_{\phi} - \left(|b_2(a, ae^{i\phi})|^2\right)_{\phi}.
$$

This has been verified to third order for both Eqs. (1) and (2), i.e. the $O(|a|^4)$ terms for Eq. (1), and the $O(|a|^6)$ terms for Eq. (2). We have been unable to find any result resembling Eq. (3) for the case without filters, either including or excluding higher harmonics in the outgoing wave.

Since our analytical treatment is only perturbative, we turn to numerical simulations. The nonlinear medium is modelled by a chain of $N$ particles with anharmonic springs connecting them. Thus if $y_i$ are the displacements of the particles from their equilibrium positions,

$$
m_i\ddot{y}_i = -\partial_{y_i}[V(y_i - y_{i-1}) + V(y_{i+1} - y_i)]
$$

for all the particles except the first and the last one, with

$$
V(y) = \frac{1}{2}y^2 + \frac{\epsilon}{4}y^4.
$$

The first and the last particle must be coupled to the external environment. This coupling is through incoming and outgoing waves, with — as in any scattering problem — the incoming waves specified and the outgoing waves determined by the scattering medium. Beyond the left boundary of the medium, the external waves can be expressed as $f_i(x - vt) + f_o(x + vt)$. The force exerted by these waves on the boundary of the medium is proportional to $-\partial_x[f_i + f_o]$. By continuity, the velocity of the boundary is equal to the velocity just outside the scattering medium, which is $\partial_t[f_i + f_o]$. From the form of $f_i$ and $f_o$, it is easy to see that $-v\partial_x[f_i + f_o] = -\partial_t[f_i + f_o] + 2\partial_t f_i$. Thus the external force acting on the boundary is a sum of a term proportional to the velocity of the boundary, and a term specified by the incoming waves. For monochromatic waves, we have

$$
m_1\ddot{y}_1 = -m_1\omega_0^2y_1 - V'(y_1 - y_2) - \kappa\dot{y} + A_1\cos(\omega t)
m_N\ddot{y}_N = -m_N\omega_0^2y_N - V'(y_N - y_{N-1}) - \kappa\dot{y}_N + A_2\cos(\omega t + \phi)
$$

where $\phi$ is the relative phase between the incoming waves from the left and the right. Thus the coupling to the external environment is seen as an effective damping and forcing term in the equation of motion for the first and last particle.

The first term on the right hand side of Eqs. (4) makes these particles act as filters if $\omega = \omega_0$ and $m_{1, N}$ are very large. Due to the nonlinearity of the medium the incoming waves at frequency $\omega_0$ produce a response at all multiples of $\omega_0$. If the excitations are resolved into frequency components, for the component at $\omega_0$ the left hand side of Eqs. (4) cancels the first term on the right hand side. The forcing and effective damping term from the external environment must balance the $\partial_t V(y)$ term from the interior, as they would have if the terminal particles had been missing. On the other hand, at any higher harmonic, $m_{1, N}(\omega^2 - \omega_0^2)$ diverges in the $m_{1, N} \rightarrow \infty$ limit, so that $y_{1, N}(n\omega_0) \rightarrow 0$ for $n \neq \pm 1$. Thus for the component at $\omega_0$ the terminal particles are transparent, whereas for higher harmonics the terminal particles act as fixed boundaries for $m_{1, N} \rightarrow \infty$, confining the higher harmonics to the nonlinear medium.

Eqs. (4) and (5) together with Eq. (6) were numerically simulated for a chain of $4 + 2$ particles, in units where
FIG. 1: Numerical results for a 4 + 2 particle chain with asymmetric mass-distribution. The particle masses are 100, 1.7, 1.4, 1.9, 1.3 and 100. The left-to-right current \( J_{10} = |b_2(1,0)|^2 \) is plotted as a function of the nonlinearity \( \epsilon \). The nonreciprocity, \( J_{10} - J_{01} \), is shown for 7 different values of \( \epsilon \), and compared with the phase averaged current \( J_{11} \) with both sources on (right vertical scale). The two quantities are equal in the perturbative state, but not in the non-perturbative state.

The non-zero right hand side of Eq. (3) might seem to contradict the second law of thermodynamics. If two blackbody sound sources at the same temperature were connected at the ends, no net energy flow would be possible. The filters would only allow waves at frequency \( \omega_0 \) to enter or exit the system, seemingly equivalent to monochromatic sources. However, as can be seen from our numerical implementation, any filter has a non-zero (albeit arbitrarily small) bandwidth. For a nonlinear medium, the different frequency channels interact with each other. Thus even with filters, a blackbody and monochromatic source are not strictly equivalent [14]. This may seem like a quibble, but from the discussion before Eqs. (9) it is clear that blackbody sources at the ends would correspond to white noise being applied to the terminal particles (which is then filtered by them). Eqs. (4) and (6) are then generalized Langevin equations (with damping and noise only in Eq. (6)), which can be rigorously proved to reach thermal equilibrium [15]. In view of our explicit results for monochromatic radiation, and the Langevin description for blackbody sources, we must conclude that narrow and zero bandwidth filters are not equivalent beyond linear order [16]. The situation here is different from the one considered in Ref. [17], where non-equilibrium energy sources were used; since the sources had to be maintained out of equilibrium, second law arguments were inapplicable there.

We note in passing that Eq. (4) is the standard FPU system [6], which is difficult to equilibrate [15], but the open boundaries in Eq. (6) seem to be sufficient to equilibrate the system with thermal (blackbody) sources.

We return to the possible basis of Eq. (3). With filters, the scatterer can be viewed as generating a mapping from the two complex input amplitudes to the two complex output amplitudes. This mapping has to satisfy the properties that i) since the system is nondissipative, \(|b_1|^2 + |b_2|^2 = |a_1|^2 + |a_2|^2\) ii) from time translation invariance, if \(a_{1,2} \rightarrow a_{1,2}e^{i\alpha}\) then \(b_{1,2} \rightarrow b_{1,2}e^{i\alpha}\) iii) from time reversal invariance, if \(a_{1,2} \rightarrow b_{1,2}^\ast\) then \(b_{1,2} \rightarrow a_{1,2}^\ast\) iv) the mapping is perturbatively accessible from the zero amplitude limit. We conjecture that these constraints may be sufficient to yield Eq. (3). This would imply that the equation is valid for any one dimensional two-channel scattering problem that satisfies the conditions above. In view of our numerical results, the fourth condition is essential; mappings that violate Eq. (3) can in fact be constructed without it [18].

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[10] This is not possible without filters, since the incoming wave is at frequency $\omega$, while the outgoing wave is at all multiples thereof; in order to achieve this with a blackbody, one would need a (passive) one-way filter that would be completely transparent to the higher harmonics in one direction, irrespective of their intensity, and opaque in the other direction. Even with the nonreciprocal devices being considered here, this is not possible.

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[12] Increasing $\epsilon$ slowly is equivalent to increasing the amplitude of the incident wave slowly. Alternatively, one can consider the case when the system starts from rest and the incident wave is switched on. The transition between the two phases occurs somewhere in the middle of the hysteresis loop of Figure 1.

[13] As $m_{1,N} \to \infty$, the noise in the transmitted power vanishes outside the nonlinear medium. At finite $m_{1,N}$ it is a suppressed version of the power spectrum inside the medium.

[14] We thank Peter Young for pointing this out to us.

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[16] For a linear system the different frequencies are independent, and there is no non-reciprocity even without time reversal invariance. A dissipative system may be viewed as one where there are (an infinite number of) extra channels that the system is coupled to in addition to those for the waves being scattered, and would not be covered by this result.

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