Identifying and modelling delay feedback systems

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(August 18, 2019)

Systems with delayed feedback can possess chaotic attractors with extremely high dimension, even if only a few physical degrees of freedom are involved. We propose a state space reconstruction from time series data of a scalar observable, coming along with a novel method to identify and model such systems, if a single variable is fed back. Making use of special properties of the feedback structure, we can understand the structure of the system by constructing equivalent equations of motion in spaces with dimensions which can be much smaller than the dimension of the chaotic attractor. We verify our method using both numerical and experimental data.

Since it is known that simple nonlinear dynamical systems can produce extremely complicated, aperiodic motion, the search for low-dimensional deterministic structures in observed time series has become a common approach to the understanding of complex time dependent phenomena. The methods, summarised under the term non-linear time series analysis, have meanwhile acquired a high standard [1,2]. It is possible, in spite of problems with noise or non-stationarity, to extract estimates of the fractal dimensions, the entropy and the Lyapunov exponents from observed data. Often, the ultimate goal is the construction of model-equations from the data.

As a first step of this kind of data analysis, a state space has to be reconstructed from the data, since usually only a single observable is measured (scalar data) [3]. The mathematical basis for this reconstruction is given by the delay embedding theorems of Takens [4] and Sauer et al. [5]. The theorem, in the formulation of the latter authors, states that, loosely speaking, forming m-dimensional delay vectors \( \vec{v}_i = (x_i, x_{i-1}, \ldots, x_{i-m+1}) \) of the sequence of measurements \( \{x_j\} \) yields a one to one description of the attractor of the system, if \( m > 2D_f \), where \( D_f \) is the fractal dimension of the attractor. If \( D_f \) is larger than about 5, this method usually fails in practice, since the number \( N \) of observed measurements covers the high dimensional attractor insufficiently. Additionally, the time delay embedding method introduces distortions for data with large Kolmogorov-Sinai entropy, such that the typical minimal inter-point-distance which can be reached by reasonably large data sets (say, \( N = 10^6 \)) is still too large for a detection of the deterministic structure [6].

A special class of dynamical systems is that of the delayed feedback systems [7]. Despite a small number of physical variables, their phase space is infinite dimensional, namely the space of all differentiable functions on the time interval \([0, \tau_0]\), where \( \tau_0 \) is the delay time of the feedback. They can thus possess chaotic attractors of arbitrarily high dimension. The direct reconstruction of the attractor by the time delay embedding method is therefore usually impossible. Recently [8], a method was proposed to identify such systems from data when they involve only a single variable, i.e. when they can be described by a scalar delay-differential equation

\[
\dot{x}(t) = f(x(t), x(t-\tau)) .
\]  

The structure of the evolution equation (1) is obviously independent of the dimension of the resulting attractor. Indeed, in the space spanned by the vectors \((\dot{x}(t), x(t), x(t-\tau))\), Eq. (1) defines a constraint if \( \tau = \tau_0 \) and the data collapse onto a two dimensional manifold, defined by \( \dot{x}(t) - f(x(t), x(t-\tau_0)) = 0 \). This property was used in [8] to identify these systems and to recover the delay time \( \tau_0 \) and the function \( f \). Similar results were achieved for \( d \)-dimensional systems if all \( d \) components are measured [9].

In this paper we introduce a nontrivial generalisation for delay systems with more than one variable in the realistic situation that only a single observable of the system was measured. This method allows one to reconstruct state vectors with which we can determine the time delay \( \tau_0 \) and an equation of motion. We will thus be able to perform predictions and to model the system. The idea

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of our method is to combine a Takens–like embedding technique with the structure of time delayed feedback systems, and to use the ability of making forecasts as a criterion for a successful reconstruction of a state space. Thus the state space reconstruction implies the derivation of evolution equations in this space.

We start with the one-dimensional case \([1]\). In the following, \(\tau_0\) will be the time delay of the feedback measured in multiples of the sampling interval \(\Delta t\) and for simplicity will be assumed to be integer, whereas \(\tau\) will be a trial value used in the reconstruction. A successful reconstruction will lead to \(\tau = \tau_0\). We make the ansatz

\[
\dot{x}_i = g_i(x_i, x_{i-\tau}) ,
\]

where \(\dot{x}_i\) is an estimate for the time derivative computed from differences of the time series data \(\{x_j\}\). For \(g_i\) we choose a local linear model \(g_i(\vec{v}_i) = b_i + \vec{a}_i \cdot \vec{v}_i\), where the parameters \(b_i\) and \(\vec{a}_i\) depend on the spatial coordinates \(\vec{v}_i = (x_i, x_{i-\tau})\). This model represents the first terms of a Taylor series expansion of the unknown global nonlinear model around \(\vec{v}_i\). The parameters can be obtained for each time index \(j\) by the local least squares fit

\[
\sigma^2(\tau) = \frac{1}{N_{U_j}} \sum_{\vec{v}_i \in U_j} \left( \dot{x}_i - g_j(x_i, x_{i-\tau}) \right)^2 ,
\]

where \(U_j\) is a sufficiently small neighbourhood of \(\vec{v}_i = (x_j, x_{j-\tau})\) and \(N_{U_j}\) its cardinal number. The correct unknown value \(\tau_0\) of Eq.(1) is recovered if \(\sigma(\tau) = (\sigma_j(\tau))_j\) is at its absolute minimum, since only for this value of \(\tau\) all triples \((\dot{x}_i, x_i, x_{i-\tau})\) fulfil simultaneously a constraint, if the motion is chaotic.

Figure 3 shows the relative forecast errors for the Mackey–Glass [11] system

\[
\dot{x} = \frac{ax(t - \tau_0)}{1 + x^{10}(t - \tau_0)} - bx(t) ,
\]

with \(a = 0.1\), \(b = 0.2\) and \(\tau_0 = 100\). The figure demonstrates that the correct delay time is detected easily. Once we know \(\tau_0\), we can determine \(g_j\) to iterate the system by a simple Euler integration \(x_{j+1} = x_j + \Delta t g_j(\vec{v}_j)\), provided the numerical trajectory remains on the attractor formed by the observed data. It is plausible that one could as well replace the local linear models \(g_i\) by a single global nonlinear function \(g(\vec{v}) = \sum a_j f_j(\vec{v})\), e.g. a bivariate polynomial or radial basis functions, by minimising

\[
\sum_{i=2}^N (x_{i+1} - \sum a_j f_j(x_i, x_{i-\tau}))^2
\]

with respect to the coefficients \(a_j\), which is standard in nonlinear function fitting. We used the radial basis functions successfully for the Mackey–Glass data discussed above.

In general one has to expect a delayed feedback system to involve several variables. For the relevant situation that only one of these variables enters the feedback loop the equation of motion reads

\[
\dot{y}(t) = \tilde{f}(\tilde{y}(t), y(t - \tau_0)) ,
\]

where \(\tilde{y} \in \mathbb{R}^d\) and \(y(t - \tau_0)\) is the single component which is fed back into the system. In many laboratory experiments with time delayed feedback exactly this is the case. If we again possess only a scalar time series, we have to recover the unobserved variables of the system in addition to determining the delay time \(\tau_0\). We can do this by exploiting the ideas of Casdagli for input–output systems [12].

Consider a non-autonomous deterministic system with an arbitrary input \(\epsilon_t\) (which could even be noise) of the form

\[
\tilde{y}_{t+1} = \tilde{f}(\tilde{y}_t, \epsilon_t) ,
\]

where \(\tilde{y} \in \mathbb{R}^d\). If we possess a scalar time series of one component of \(\tilde{y}\), a standard time delay reconstruction cannot work, since each additional measurement contains an additional uncertainty, namely \(\epsilon_t\). If one simultaneously measures the input, one can form delay vectors

\[
\vec{v}_t = (y_t, y_{t-1}, \ldots, y_{t-m+1}, \epsilon_t, \ldots, \epsilon_{t-m+1}) .
\]

Casdagli argues that in this space the deterministic dynamics can be reconstructed, if \(m = 2d\), generically [12].

In our case the delayed feedback variable can be interpreted as non–autonomous input. Following Casdagli’s reasoning, we have to include these inputs in the delay vector,

\[
\vec{v}_t(\tau) = (x_i, x_{i-1}, \ldots, x_{i-m+1}, x_{i-\tau}, \ldots, x_{i-\tau-m+1}) .
\]

Here it is assumed that \(x\) is identical to or a unique function of only the delayed variable from Eq. (3), and \(m_1\) is the dimension we need to fully determine the actual state of the system, while \(m_2\) is the number of inputs required. In general we expect \(m_2 = m_1\). Due to strong correlations between successive measurements \(m_2 < m_1\) could be sufficient. Following Casdagli’s results we expect that \(m_1 = m_2 = 2d\) is sufficient for a unique representation of \((\tilde{y}(t), y(t - \tau)), t = i\Delta t\), of Eq. (3). Note that the fractal dimension of the attractor can be much larger than \(4d\) and that, in particular, the embedding we propose is independent of the attractor dimension.

In analogy to the 1-dimensional case we make an ansatz in this embedding space to recover the delay time and the dynamics of the \(d\)-dimensional system:

\[
\hat{x}_i = g_i(\vec{v}_i(\tau)) ,
\]

where \(g\) here is again a local linear model, namely \(g_i(\vec{v}_i(\tau)) = b_i + \vec{a}_i \vec{v}_i(\tau)\), and a successful reconstruction will lead to \(\tau = \tau_0\).

For a demonstration on numerically generated data we use a generalisation of the Mackey–Glass equation introduced in [13]:

\[
2
\[
\dot{x}(t) = \frac{ax(t - \tau_0)}{1 + x^{10}(t - \tau_0)} + y(t)
\]
\[
\dot{y}(t) = -\omega^2 x(t) - \rho y(t)
\]
(10)

where we choose the parameters to be \(a = 3\), \(\rho = 1.5\), \(\omega^2 = 1\) and \(\tau_0 = 10\). The Kaplan–Yorke dimension of this system is \(D_{KY} \approx 13.5\). The left panel of Fig. \(S\) shows its attractor in a two–dimensional representation. Figure \(S\) shows the one step prediction errors in an embedding space of the variable \(x\) as a function of \(\tau\). Already for an insufficient reconstruction \(m_1 = m_2 = 1\) (upper curve) we see indications of the correct \(\tau_0\). If we increase \(m_1\) and \(m_2\) to 2, the forecast errors decrease dramatically. At the correct value of \(\tau\) it is reduced by a factor of 20. Further increases of \(m_1\) and \(m_2\) do not yield better forecast errors.

This result can be verified analytically. An expression for the dynamics in delay space follows from rewriting Eq. (10) as second order in time

\[
\dot{x}(t) = -\omega^2 x(t) - \rho \dot{x}(t) + \omega^2 f(x(t - \tau_0)) + \frac{df(x(t - \tau_0))}{dx(t - \tau_0)} \dot{x}(t - \tau_0),
\]
(11)

where \(f\) is the first term of the rhs of Eq. (10), and substituting derivatives by differences. Using a simple Euler–scheme for this substitution, two ‘on–time’ measurements and two delayed measurements are involved in the forecast, i.e. \(m_1 = m_2 = 2\). The right panel of Fig. \(S\) shows data obtained by iterating the fitted dynamics \(g\) in Eq. (10). The properties of the attractor are reproduced well, although the dimension of the delay space is less than a third of the estimated Kaplan–Yorke dimension.

Experimental data from systems whose model equations are not fully known are a challenge. We apply our state space reconstruction scheme to data from a CO\(_2\)-laser experiment with delayed feedback performed at the Istituto Nazionale di Ottica in Firence/Italy. The output power of the laser is measured, which is the variable which is electronically fed back into the system. Experimental data are shown in the left panel of Fig. \(S\) in a representation which was proposed in [14]. The scalar time series \(\{x_i\}\) is represented in a space–time plane by decomposing the time index \(i\) into an artificial discrete time \(n\) and a space variable \(s\)

\[
i = s + n \tau_0 \quad \text{with} \quad s \in [0, \tau_0],
\]
(12)

This presentation allows one to observe that the system behaves periodically with defects travelling through it. Applying our analysis in \(m_1 = m_2 = 2\) dimensions we obtain a delay of \(\tau_0 = 800\), in perfect agreement with the value given by the experimentalists, and the forecast error drops by a factor of \(\approx 10\). Enlarging \(m_1\) and \(m_2\) does not improve the results, whereas \(m_1 = m_2 = 1\) turns out to be insufficient. Measurement noise may here distort the estimates of the derivatives. The right panel of Fig. \(S\) shows the presentation of the time series obtained from the integrated fitted equation of motion. Comparing the structures visually we see a good agreement, differing details are even to be expected for a perfect model because of the sensitive dependence on initial conditions. We thus conclude that the experimental data from the CO\(_2\)-laser are governed by a two-dimensional time-delay differential equation with a single delayed feedback variable. Therefore, we detect a considerable reduction of dimension in comparison to the six-dimensional model proposed in [15], which has been derived from first principles. Our finding is in agreement with the conjecture of [14] concerning the reduction of the dimension of a time-delay system close to a bifurcation. Without making use of the feedback structure, in standard embedding spaces with dimensions up to 6, we were unable to obtain models which reproduce the defect dynamics, and the optimal prediction error was much larger.

When only a single quantity is measured in a system with time delayed feedback, the full state space is not accessible and also the “true” equations of motion. We proposed a novel method for the reconstruction of a space which is equivalent to the physical one. In this space equations of motion attain a very natural form, mirroring the delay structure of the unknown dynamical system. We thus work in a space whose dimension is usually much smaller than the attractor’s dimension, although larger than the number of variables in the unknown space. We do not reconstruct the attractor by Takens’ embedding, but only the unobserved variables and then exploit the special structure of the evolution equations. This is of particular interest when \(\tau_0\) in the experiment is varied: The attractor dimension will vary, too, but the structure of our space remains invariant, and so do the equations of motion in this space. A successful reconstruction involves to detect the correct time delay \(\tau_0\), which is done here very efficiently by a minimisation of a kind of prediction error. A maximum of the auto-correlation function usually appears close to, but not exactly at \(\tau_0\) [17].

Although we only presented results for systems with a single delay time, our method can of course be easily generalised to systems with several delay times. Furthermore, once we have modelled the system we can estimate the Lyapunov spectrum from this model.

The problem of noise in experimental data will be subject to future works. However, the successful treatment of the laser data shows that well controlled laboratory experiments provide data whose noise contamination does not harm the applicability of the concept. On the other hand it is clear that the forecast error cannot be smaller than the noise level, so that too strong noise could completely destroy the detectability of the correct \(\tau_0\).

We thank the experimentalists at the INO for their excellent data. Moreover, we are indebted to R. Genesio, P. Grassberger, E. Olbrich, A. Politi and T. Schreiber for very stimulating discussions. A. G. acknowledges
support of the EC under the contract number ERBFMRXCT960010 and support from the MPIPKS.

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FIG. 1. One step prediction error for the Mackey–Glass system with \( \tau_0 = 100 \).

FIG. 2. One step prediction error for the two-dimensional generalized Mackey–Glass system with \( \tau_0 = 10 \).

FIG. 3. The attractor of the two dimensional Mackey–Glass system. The left panel shows the original data, the right panel the data obtained by iterating the fitted dynamics using \( m_1 = m_2 = 2 \).

FIG. 4. Output power of a \( \text{CO}_2 \) laser experiment. The left panel shows the original data, the right panel shows the data obtained from an iterated forecast in \( m_1 = m_2 = 2 \) dimensions. The representation of the data is described in the text.
The graph shows the function $\sigma(\tau)$ as a function of $\tau$. The graph indicates a peak at $\tau = 100$, with values ranging from $10^{-4}$ to 1 on a logarithmic scale. The x-axis represents $\tau$ from 90 to 110, and the y-axis represents $\sigma(\tau)$ from $10^{-4}$ to 1.
\[\sigma(\tau)\]

- \(m_1 = m_2 = 2\)
- \(m_1 = m_2 = 1\)
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