Correlators of Special States in $c=1$ Liouville Theory

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We generalize the ground ring structure to all other special BRST invariant operators in the right branch in the $c=1$ Liouville theory. We also discuss correlation functions of special states on the sphere.
1. Introduction

Two dimensional string theory is believed to be a good toy model for investigating issues such as a complete string field formulation, gauge symmetries as well as exact blackhole solutions in higher dimensional strings. Most of all, it is exactly solvable perturbatively [1]. It is remarkable that in addition to the tachyon field, which should have been the only physical mode in two dimensions, there are infinitely many discrete physical modes as remnants of transverse excited fields. These physical states were first seen in the calculation of the tachyon two point function in the matrix model [2] as poles at special external momenta, and later were confirmed by Liouville calculation [3] and by free field BRST analysis [4]. Much has been done for correlation functions of tachyons in the matrix model [2] [5] [6] and also in the Liouville theory on the sphere [7]. All results are in remarkable agreement. It remains to be done for correlation functions of special states and mixed correlation functions of special states and tachyon states. It is the purpose of this paper to calculate the former on the sphere in the Liouville theory.

The special states of standard ghost number can be easily seen and constructed in light of the fact that it has long been known that in c=1 conformal field theory there are additional primary states at special momenta [8]. Consider a compactified scalar of radius $\sqrt{2}$. The enlarged current algebra is an $SU(2)$ current algebra of level 1. The $SU(2)$ currents are $J^{\pm}(z) = \exp(\pm i\sqrt{2}X(z))$ and $J^3(z) = i/\sqrt{2}\partial X(z)$. Denote the usual primary field at momentum $s\sqrt{2}$ by $V_{s,s} = \exp(is\sqrt{2}X(z))$, where $s$ is a positive integer or half integer. Now the additional primary fields are $V_{s,n} = \exp(i\sqrt{2}(1-s)\phi)$, where $\phi$ is the Liouville field. We shall consider an uncompactified scalar $X$ in which case we have to have the same momenta in the left and right sectors. Denote, following Witten [9], these states by $W_{s,n} = V_{s,n}\exp(\sqrt{2}(1-s)\phi)$. The corresponding BRST invariant states are $Y_{s,n}^\pm = cW_{s,n}^\pm$. The other special states of non-standard ghost numbers are to be considered as companions of $Y_{s,n}^\pm$, as far as the so-called relative cohomology is concerned [4]. The companion of $Y_{s,n}^+$ has ghost number zero and we denote it by $O_{s-1,n}$. The companion of $Y_{s,n}^-$ has ghost number two and will not concern us in this paper. The $Y^-$’s do not satisfy Seiberg’s condition of a microscopic state [10], and are important in the 2d stringy blackhole [11].
All other BRST invariant states are irrelevant as far as correlation functions are concerned

Let $Y_{s,n}^\pm$ stand for the symmetric left and right combination $Y_{s,n}^\pm Y_{s,n}^\pm$, and $W_{s,n}^\pm$ for $W_{s,n}^\pm W_{s,n}^\pm$. First of all, we are interested in calculating correlation functions on the sphere of the following form

$$\langle Y_{s_1,n_1}^+, Y_{s_2,n_2}^+, Y_{s_3,n_3}^+ \int W_{s_4,n_4}^+ \cdots \int W_{s_N,n_N}^+ \rangle_\mu.$$  \hspace{1cm} (1.1)

Positions of three operators are fixed, so we use $Y^+$ instead of $W^+$. The above correlator is symmetric in indices $(s_i, n_i)$ by construction. The subscript $\mu$ indicates the expectation value is defined with a cosmological term in the action, as in (3.1). One may further insert a number of operators $O$. Since these operators are of conformal dimension $(0,0)$ and have ghost number zero, the conservation of ghost number will not be violated and the correlator is independent of the positions of insertions of $O$. One of our main results is that the correlation in (1.1) is always zero when all $n_i = 0$. We conjecture that all the Liouville bulk correlators in (1.1) are zero. Throughout this paper all correlators are defined on the sphere.

Witten recently discovered a ground ring structure among operators $O$, and one naturally expects this ring structure be helpful in determining correlator (1.1). This turns out to be partly true. We shall generalize the ground ring structure to all BRST invariant operators in the right branch in section 2. What we will show is that the generators of the ground ring together with a couple of other operators generate all the special operators in the right branch. We then use this fact to devise a reshuffling argument in section 4. We will show by this reshuffling argument that correlator (1.1) when $n_i = 0$ can be reduced to a simple one which depends only on the number of inserted operators and the sum of $s_i$. And this simple correlator turns out to be zero if we rescale all operators by a infinitely small factor. We have not been able to generalize the reshuffling argument to a general case. Section 3 is devoted to some explicit calculations of three point correlators of the type in (1.1) and mixed type, namely with some $Y^-$ operators. We use a certain regularization in calculating correlators. The fact that some correlators are regularized to be zero is confirmed in section 4 by use of the reshuffling argument.

Some un-rescaled three point correlators of one $Y^-$ operator and two $Y^+$ operators calculated in section 3 are relevant to the perturbed OPE considered in [12].

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1 I am grateful to K. Li for an enlightening discussion on this issue.
We should stress that all correlators (amplitudes) considered in this paper are Liouville bulk correlators. The discrepancy between our result about the correlators of the $Y^+$'s and the matrix model calculations in [13] indicates that the nonvanishing results in the matrix model are to be explained by the tachyon wall effects.

2. Generalization of The Ground Ring

Lian and Zuckerman calculated the free field BRST cohomology in the $c=1$ Liouville theory [4]. In addition to the usual tachyon operators, they find discrete operators with various ghost numbers. We shall use superscript $+$ to indicate a discrete operator with a Liouville momentum $p < \sqrt{2}$, an operator in the right branch; and use superscript $-$ to indicate an operator with a Liouville momentum $p \geq \sqrt{2}$, an operator in the wrong branch. As we showed in the introduction, one can construct operators $Y^+_s,n$ and $Y^-_s,n$ explicitly. These operators have ghost number one, the standard ghost number for a BRST invariant operator.

We now briefly recall the main results about special operators in [4]. The minus operators are simpler. They do not involve oscillators of the Liouville field. In the relative cohomology which consists of states annihilated by the zero mode of the anti-ghost $b$, there are operators of ghost number one and two. The ghost number one operators are just $Y^-_s,n$. We will not be interested in the ghost number two operators. According to [4], associated with each operator in the relative cohomology, there is an operator in the absolute cohomology with a ghost number increased by one. To be more precise, this operator is not annihilated by $b_0$. We see that the only ghost number one operators are $Y^-_s,n$. As for the plus operators, there are operators of ghost number zero and one in the relative cohomology. The ghost number one operators are $Y^+_s,n$. There are no explicit formulas for operators of ghost number zero. However, Witten discovered [9] a ring structure for these operators which we shall describe presently. Therefore one needs not know the detailed formulas. Again, associated with these operators are operators of ghost number one which are not annihilated by $b_0$. The ghost number two counterpart of $Y^+_s,n$ is just $\partial c Y^+_s,n$. The above is the complete list of the special operators in the right branch.

Consider operators with ghost number zero. Following Witten [9], we denote them by $O_{s,n}$. The momenta of $O_{s,n}$ are $\sqrt{2}(n,-s)$. $s$ is an positive half integer or integer, $n = -s, -s+1, \cdots, s$. Note that instead of using a pure imaginary Liouville momentum, we use the real Liouville momentum convention. Now the ring structure of these $O$ operators
is obvious. The product $O(z)O'(0)$ is again BRST invariant. The singular terms in the product must be BRST exact, since there are no BRST nontrivial operators with negative dimensions. The limit $z \to 0$ is well defined and the resulting operator must be a third $O$ operator, since there are no other operators with ghost number zero. The ring so defined is a commutative ring and called the ground ring in [9]. Witten further showed that $O_{s,n}$ can be written in a form $x^{s+n}y^{s-n}$. $x$ and $y$ are defined as

\begin{align*}
x &= O_{\frac{1}{2},\frac{1}{2}} = \left( cb + \frac{i}{\sqrt{2}}(\partial X - i\partial \phi) \right) e^{(iX - \phi)/\sqrt{2}} \\
y &= O_{\frac{1}{2},-\frac{1}{2}} = \left( cb - \frac{i}{\sqrt{2}}(\partial X + i\partial \phi) \right) e^{(-iX - \phi)/\sqrt{2}}.
\end{align*}

A natural question is how to use this ring structure in calculating correlation functions. To answer this question, one must first generalize this ring structure to the whole ring of physical operators in the right branch. To fix notation, we use the following normalization for $W_{s,n}^+$

\begin{equation}
W_{s,n}^+ = (-1)^{2n+1} \frac{1}{2} (s + n)![(s - n)!]^{1/2} \left[ \frac{1}{2\pi} \oint J^- \right]^{s-n} e^{i\sqrt{2}sX + \sqrt{2}(1-s)\phi},
\end{equation}

where the meaning of the contour integrals is the following. The first contour integral surrounds the position of $W_{s,n}^+$. The next contour integral surrounds the first one, and so forth. The normalization in (2.2) is motivated by an explicit calculation of Klebanov and Polyakov [12]. Define operators

$$Q_{s,n}^+ = \frac{1}{2\pi i} \oint W_{s,n}^+.$$ 

Klebanov and Polyakov showed that these operators satisfy $w_\infty$ algebra

\begin{equation}
[Q_{s,n}^+, Q_{s',n'}^+] = (ns' - n's)Q_{s+s'-1,n+n'}^+,
\end{equation}

or equivalently, the following OPE

\begin{equation}
W_{s,n}^+(z)W_{s',n'}^+(0) = \cdots + \frac{1}{z}(ns' - n's)W_{s+s'-1,n+n'}^+ + \cdots.
\end{equation}

Now consider a product $Y_{1,0}^+ x^{s+n-1} y^{s-n-1}, |n| < s$. This operator has the same ghost number and momenta as $Y_{s,n}^+$. One may try to identify these two operators up to a proportionality constant. Unfortunately, as we have learned, there is another operator with the same ghost number and momenta in the absolute cohomology. We call this operator
where $Q$ is the diffeomorphism BRST operator. $X^+_{0,0}$ is considered as a nontrivial BRST invariant operator in [4], since there is no state in the Fock space corresponding to operator $\phi$.

It is easy to see that the term proportional to $\partial c$ in the product $X^+_{0,0}x^{s+n}y^{s-n}$ is $(s + 1)\partial cx^ny^m$. The latter is defined as a normal ordered product. By a theorem in [4], the counterpart of operator $Q$ is $\frac{1}{\sqrt{2}}c\partial \phi$, corresponding to the identity operator. Indeed, the differential operators on the left hand side of the above equation form the $w_\infty$ algebra, then $Q^+_{s,n}$ are proportional to them with factors $e^{an}$. Since $Q^+_{1/2,1/2}$ is just $1/2\partial y$, so $a = 0$.

Calculate $Y^+_{1/2,1/2}x^{s+n-1}y^{s-n+1}$, using (2.7). We find the product $a_{s,n}Y^+_{s,n}y$ contains a term $(s + n)/2a_{s,n}X^+_{s-1,n+1/2}$ by use of (2.8). The product $(s - n)/2X^+_{s-1,n}y$ is simply $(s - n)(2s + 1)/(4s)X^+_{s-1,n+1/2}$, by the definition (2.5).

It remains for us to determine the coefficient $a_{s,n}$. To this end, we first show that

$$Q^+_{s,n} = \frac{1}{2}((s + n)x^{s+n-1}y^{s-n}\partial y - (s - n)x^{s+n}y^{s-n-1}\partial x),$$

when $Q^+_{s,n}$ acts on the ground ring. Indeed, the differential operators on the left hand side of the above equation form the $w_\infty$ algebra, then $Q^+_{s,n}$ are proportional to them with factors $e^{an}$. Since $Q^+_{1/2,1/2}$ is just $1/2\partial y$, so $a = 0$.

Calculate $Y^+_{1/2,1/2}x^{s+n-1}y^{s-n+1}$, using (2.7). We find the product $a_{s,n}X^+_{s-1,n}y$ contains a term $(s + n)/2a_{s,n}X^+_{s-1,n+1/2}$ by use of (2.8). The product $(s - n)/2X^+_{s-1,n}y$ is simply $(s - n)(2s + 1)/(4s)X^+_{s-1,n+1/2}$, by the definition (2.5). Since we expect
that $Y_{1/2,1/2}^{s+n-1}y^{s-n+1}$ must contain $(s-n+1)/2X_{s-1/2,n+1/2}^+$ from (2.4), we find $a_{s,n} = 1/(2s)$. Finally, we obtain

$$Y_{1/2,1/2}^{s+n-1}y^{s-n} = \frac{1}{2s}Y_{s,n}^+ + \frac{s-n}{2}X_{s-1,n}^+.$$ (2.9)

(2.6) is a easy consequence of (2.9). Rewrite

$$Y_{1/2,1/2}^{s+n-1}y^{s-n} = Y_{1/2,1/2}^+yx^{s+n-1}y^{s-n-1} = \frac{1}{2}(Y_{1,0}^+ + X_{0,0}^+)x^{s+n-1}y^{s-n-1}.$$ Using (2.9) again and the definition (2.5) we find (2.6).

(2.7) and (2.6) tell us how those operators of ghost number one are generated by $x, y$ together with $Y_{1,0}^+$ and $X_{0,0}^+$. The only operators of ghost number two in the right branch are $\partial cY_{c,n}^+$. One might think this operator can be obtained by simply multiplying $\partial c$ to the right hand side of (2.6). This is wrong, since $\partial c$ is not a BRST closed operator hence the product is not well defined. Nevertheless, we expect from (2.4) that $Y_{s,n}^+Y_{s',n'}^+ = (ns' - n's)\partial cY_{s+s'-1,n+n'}^+$. Using (2.6) and the basic products $Y_{1,0}^+Y_{1,0}^+ = X_{0,0}^+X_{0,0}^+ = 0$ and $X_{0,0}^+Y_{1,0}^+ = -Y_{1,0}^+X_{0,0}^+ = \partial cY_{1,0}^+$, we find

$$Y_{s,n}^+Y_{s',n'}^+ = (ns' - n's)\partial cY_{1,0}^+x^{s+s'+n+n'-2}y^{s'-n-n'-2}. \tag{2.10}$$

The above equation tells us $\partial cY_{s,n}^+ = (\partial cY_{1,0}^+)x^{s+n-1}y^{s-n-1}$. Products similar to (2.10) are listed below:

$$Y_{s,n}^+O_{s',n'} = \frac{s}{s+s'}Y_{s+s',n+n'}^+ + (ns' - n's)X_{s+s'-1,n+n'}^+$$

$$X_{s,n}^+O_{s',n'} = \frac{s+s'+1}{s+1}X_{s+s',n+n},$$

$$Y_{s,n}^+X_{s',n'}^+ = -\frac{s}{s'+1}\partial cY_{s+s',n+n'}^+$$

$$X_{s,n}^+X_{s',n'}^+ = 0. \tag{2.11}$$

We now turn to the ghost number one operators in the wrong branch, namely $Y_{s,n}^-$. Unfortunately in this case it is not possible to generate all operators by a single $Y^-$ operator and $x, y$. The reason is simple. Multiplying $Y_{s,n}^-$ by $x$ or $y$ lowers $s$ to $s - 1/2$. While one may use $Y_{s,0}^-x^ny^m$ to generate another $Y^-$. This product, if not zero, will be proportional to $Y_{s-(n+m)/2, -(n-m)/2}^-$, since there is no other ghost number one operator with the same momenta in the wrong branch. We shall show this is indeed the case. First of all, we shall argue that the product is zero whenever the condition $|(n-m)/2| \leq s - (n + m)/2$ is not
satisfied. Consider first the case when the inequality is saturated and \( s - (n + m)/2 > 0 \). The product is proportional to \( Y_{|n-m|/2,\pm|n-m|/2} \). It is obvious that any further multiplication of \( x \) or \( y \) which makes the inequality violated will annihilate the operator. Next consider the case when the inequality is saturated and \( s - (n + m)/2 = 0 \). We get an operator proportional to the cosmological operator \( \exp(\sqrt{2} \phi) \). It is easy to see from (2.1) that this operator destroys \( x \) and \( y \). In other words, one can not get an \( Y^+ \) operator from a product \( Y_{s,0}^{-} x^m y^n \).

To fix the normalization of \( Y_{s,n}^{-} \), again we follow [12]. Define

\[
W_{s,n}^{-} = (-1)^{2n+1} \frac{2}{(2s)!} [(s - n)!]^{-1/2} \left( \frac{1}{2\pi i} \oint J^{-} \right)^{s-n} e^{i\sqrt{2}sX + \sqrt{2}(1+s)\phi}. \tag{2.12}
\]

It was shown in [12] that

\[
W_{s,n}^{+}(z)W_{s',n'}^{-} = \cdots - \frac{1}{2} (ns' + n's + n) W_{s'-s+1, n+n'}^{-} + \cdots. \tag{2.13}
\]

We shall use (2.13) to prove that \( Y_{s,0}^{-} x^n y^m \) is not zero if \(|(n - m)/2| \leq s - (n + m)/2 \). We divide our proof into three steps.

First consider \( Y_{s,0}^{-} x^m y^n \) and \( Y_{s,-s}^{-} x^m y^n \). These operators must be proportional to \( Y_{s-1/2, s-1/2}^{-} \) and \( Y_{s-1/2, -(s-1/2)}^{-} \) respectively, assuming they are not zero. Using formula (2.12) and the definition in (2.1), it is fairly easy to show that \( Y_{s,0}^{-} x^m y^n = Y_{s-1/2, 1/2}^{-} \) and \( Y_{s,-s}^{-} x^m y^n = -Y_{s-1/2, -(s-1/2)}^{-} \). This in particular implies that \( Y_{s,0}^{-} y^{2s-1} = (1)^{s-1} Y_{1/2, 1/2}^{-} \), which of course is nonvanishing.

Next we prove \( Y_{s,0}^{-} y^n \) is not zero when \( m \leq s \). It suffices to prove \( Y_{s,0}^{-} y^n \) not zero. Consider the commutator \([Q_{(s+1)/2,(s+1)/2}, Y_{s,0}^{-} y^n] \). Using the OPE in (2.13) and the differential operator realization of \( Q \) operators when acting on the ground ring, we obtain

\[
[Q_{(s+1)/2,(s+1)/2}, Y_{s,0}^{-} y^n] = -\frac{1}{2} (s + 1)^2 Y_{(s+1)/2,(s+1)/2}^{-} y^n + \frac{s(s + 1)}{2} Y_{s,0}^{-} x^s y^{s-1}. \tag{2.14}
\]

The first term on the r.h.s. is not zero, as we learned before. Suppose \( Y_{s,0}^{-} y^n \) vanishes. Then the second term on the r.h.s. must be nonzero, in order to cancel the first term. Therefore, \( Y_{s,0}^{-} x^s y^{s-1} \neq 0 \). This implies \( Y_{s,0}^{-} x^s \neq 0 \). However, we know the symmetry between \( x \) and \( y \) under reflection of the matter field \( X \rightarrow -X \). \( Y_{s,0}^{-} y^n \) must be non-vanishing too. So the only consistent solution is for both of them to be non-zero.

It remains to show that \( Y_{s,0}^{-} x^n y^m \) is not zero. Without loss of generality, we can assume \( m > n \) (The case \( n > m \) can be similarly considered. The \( n = m \) case is a
consequence of $n \neq m$ cases). Now the only condition is $m - n \leq s$. Consider the commutator $[Q_{n+1,0}, Y_{s,0}^{-}y^{m-n}]$. It must be nonzero, as indicated by (2.13). Note that $Q_{n+1,0}$ commutes with $Y_{s,0}^{-}$. We then find

$$[Q_{n+1,0}, Y_{s,0}^{-}y^{m-n}] = \frac{m-n}{2}(n+1)Y_{s,0}^{-}x^ny^m.$$  \hfill (2.15)

The immediate consequence of the above equation is that $Y_{s,0}^{-}x^ny^m$ must be non-vanishing. After a few calculations, we find

$$Y_{s,n}^{-} = (-1)^{2n}Y_{s,0}^{-}x^{S+n-s}y^{S-n-s},$$  \hfill (2.16)

where $S$ is an arbitrary integer such that $S \pm n - s \geq 0$.

3. Three Point Correlators of Special Operators

We shall calculate various three point functions of special states in this section. The action under consideration is

$$S = \frac{1}{2\pi} \int \left( \partial X \overline{\partial} X + \partial \phi \overline{\partial} \phi - \frac{1}{\sqrt{2}} R\phi + 2\mu \Gamma(\epsilon)e^{\sqrt{2}\phi} \right),$$  \hfill (3.1)

where we have suppressed the action of the ghosts. We have put a short distance cut-off $\Gamma(\epsilon)$, $\epsilon \rightarrow 0^+$ explicitly into the cosmological term. The origin of this cut-off is of course the short distance divergence. In calculating an amplitude of tachyons, one needs to perform a multi-complex integral. The integrand usually displays poles when two complex points approach each other. There are two equivalent regularization prescriptions. One prescription is to simply introduce a cut-off in the integral such that any two of complex positions are separated by a minimal distance. Another is to shift the exponent in each factor $|z_i - z_j|^{\alpha_{ij}}$ by a small amount proportional to $\epsilon$. It turns out that $\Gamma(\epsilon)$ will appear in the place of $|\log \lambda|$, $\lambda$ is the short distance cut-off. The reason for us to put a factor $\Gamma(\epsilon)$ in the cosmological term is that the cosmological term decouples from the tachyon amplitude thereby needs an infinity rescaling factor. This is true for those special tachyon vertices in the wrong branch [7], namely when the Liouville momentum $p \geq \sqrt{2}$. For those special tachyon vertices in the right branch, an infinitely small factor $[\Gamma(\epsilon)]^{-1}$ is needed in rescaling vertices, however. One expects this phenomenon persist for special operators we are considering.
We are going to use the same trick as in [14] to calculate three point correlators. We first perform integration over the Liouville zero mode, effectively bringing down a power of the cosmological term. The power is always a positive integer for three $\mathcal{Y}^+$ operators. So this ad hoc trick is equivalent to expanding $e^{-S}$ in terms of the cosmological term. We expect a scaling behavior $(s!)^{-1}(-\mu)^s|\log\mu|$ in this case. The term $|\log\mu|$ is just the Liouville volume, resulting from the integration over the Liouville zero mode. The power of the cosmological term we bring down may become zero or a negative integer in other cases. Then we need an analytic continuation technique used in the second paper in [14].

3.1 Three Point Correlators of $\mathcal{Y}^+$

We are going to calculate
$$\langle \prod_{i=1}^{3} \mathcal{Y}_{s_i,n_i}^+ \rangle_{\mu},$$
where the subscript $\mu$ indicates the correlator is defined with action (3.1). We integrate over the Liouville zero mode to obtain
$$\frac{1}{s!}(-\mu/\pi)^s|\log\mu|\Gamma^s(e)\langle \prod_{i=1}^{3} \mathcal{Y}_{s_i,n_i}^+ \left( \int e^{\sqrt{2} \phi} \right)^s \rangle$$
$$s = -1 + \sum_i s_i.$$  (3.2)

The expectation value in (3.2) is to be calculated without the cosmological term in the action. The integration of the Liouville zero mode is represented by $|\log\mu|$. We have used $\mathcal{Y}_{s_i,n_i}^+$ instead of $\mathcal{W}_{s_i,n_i}^+$ to indicate that positions of these operators are fixed. To calculate the expectation value in (3.2), we may first calculate the correlation function without integrations of the cosmological term, also let positions of $\mathcal{W}_{s_i,n_i}^+$ not be fixed. Let $z_i$ be positions of $\mathcal{W}^+$ and $w_i$ positions of $e^{\sqrt{2} \phi}$. Now the part from the matter sector $X$ is just
$$C_{s_1,n_1,s_2,n_2}^{s_3,n_3} \prod_{i<j}^{3} |z_i - z_j|^{-2\Delta_{ij}},$$
$$\Delta_{12} = s_1^2 + s_2^2 - s_3^2,$$
other $\Delta_{ij}$ are obtained from $\Delta_{12}$ by permutations. The structure constant $C$ is proportional to the Clebsch-Gordon coefficient. It is not zero only when $\sum_i n_i = 0$ and $s_3 = |s_1 - s_2|, \ldots, s_1 + s_2$. Next we calculate the contribution from the Liouville sector. The result is
$$\prod_{i<j}^{3} |z_i - z_j|^{-4(1-s_i)(1-s_j)} \prod_{i,j} |w_i - z_j|^{-4(1-s_j)} \prod_{i<j}^{s} |w_i - w_j|^{-4}.$$
Our strategy is to fix $z_i$ and integrate over $w_i$. We immediately find that the multi-complex integral is divergent. To regularize the integral, we should shift the exponents in $|w_i - w_j|^{-4}$ to $-4 + 4\epsilon$. To have a $SL(2,\mathbb{C})$ invariant integrant, other exponents must be shifted accordingly. For example, the exponent in $|w_i - z_j|^{-4(1-s_j)}$ is shifted to $-4(1-s_j) - 4\epsilon(s_j - 2/3)$. After doing shifting and fixing $z_i$, the integral we need to do is

$$
\int \prod_i^s d^2w_i |w_i|^{-4(1-s_1)}|1-w_i|^{-4(1-s_2)}\prod_{i<j}^s |w_i - w_j|^{-4+4\epsilon}. \quad (3.3)
$$

This integral is not symmetric in all $s_i$ superficially. As we soon see, the final result is symmetric. The integral is performed in [15]. We simply write down the final answer

$$
(-1)^{1+s} \frac{\pi^s}{(s!)^2} \left( \frac{\Gamma(1/3)}{\Gamma(2/3)} \right)^3 \prod_{j=1}^s \Gamma^{-8}(j) \prod_{i=1}^3 \left( \frac{\Gamma(2s_i - 1/3)}{\Gamma(s + 1 - 2s_i + 1/3)} \right) \prod_{j=1}^{2s_i-1} \Gamma^2(s + 1 - j) \Gamma^2(j)^{2-(s)}(\epsilon). \quad (3.4)
$$

Note that there is a factor $\Gamma^{-s}(\epsilon)$ appearing in the above formula and is to be cancelled by the factor $\Gamma^s(\epsilon)$ in (3.2). Note also that we have assumed that each $s_i \geq 1$, so $s \geq 2$. The above result tells us that the multi-complex integral goes to zero in general.

Collecting (3.2) and (3.4) together, we find

$$
\langle \prod_{i=1}^3 \mathcal{W}_{s_i,n_i}^+ \rangle_\mu = -\mu^s |\log \mu| F(s_i, n_i) \Gamma^2(\epsilon)
$$

$$
F(s_i, n_i) = C_{s_1,n_1,s_2,n_2}^{s_3,n_3} \left( \frac{\Gamma(1/3)}{\Gamma(2/3)s!} \right)^3 \prod_{j=1}^s \Gamma^{-8}(j) \prod_{i=1}^3 \left( \frac{\Gamma(2s_i - 1/3)}{\Gamma(s + 1 - 2s_i + 1/3)} \right) \prod_{j=1}^{2s_i-1} \Gamma^2(s + 1 - j) \Gamma^2(j) \quad (3.5)
$$

The final result is divergent, as we expected. To have a finite result, one should rescale operators $\mathcal{W}^+$. If one rescale each operator by a factor $\Gamma^{-1}(\epsilon)$, then one has a vanishing result from (3.3). We can not attribute the factor $\Gamma^{-1}(\epsilon)$ in the rescaled correlator to the partition function. The reason is the following. The correlators of tachyon vertices of generic momenta are finite, no matter how one rescale the partition function. To agree with the matrix model calculation, one therefore requires a finite partition function.
3.2 Three Point Correlators of Mixed Type

Next we calculate three point correlators of mixed type, i.e. with presence of both $Y^+$ and $Y^-$ operators. First, consider correlators of two $Y^+$ with one $Y^-$. We shall calculate

$$\langle Y^+_{s_1,n_1} Y^+_{s_2,n_2} Y^-_{s_3,n_3} \rangle_{\mu}.$$ 

Again we perform the integration over the Liouville zero mode first, we will get a formula similar to (3.2). Now $s = s_1 + s_2 - s_3 - 1$, which can be positive, zero and negative. We would have exactly same formula as in (3.2) if $s$ is positive. The structure constant $C$ must be replaced by another $\tilde{C}$, since we use different normalization for the matter part of $Y^-$ (see section 2). We repeat steps in the previous calculations and finally reach (3.3). The integral is not given by (3.4) since this time $s_3$ appears in $s$ with a different sign. It is not hard to use a formula in [15] to obtain the regularized integral

$$(-\pi)^s \prod_{i=1}^s \Gamma^2(2s_1 - i)\Gamma^2(2s_2 - i)\Gamma^{-2}(2s_3 + i + 1)\Gamma^{-2}(i + 1)$$

$$\frac{\Gamma(2s_1 - 1/3)\Gamma(2s_2 - 1/3)\Gamma(2s_3 + 4/3)}{\Gamma(2s_1 - s - 1/3)\Gamma(2s_2 - s - 1/3)\Gamma(2s_3 + s + 4/3)} \Gamma^{-s}(\epsilon).$$

The arguments in gamma functions in the product are all positive, since for example $2s_1 - s = s_1 + s_3 - s_2 + 1$ is always positive by the fusion rules in the matter sector. It is notable that the integral goes to zero for positive $s$ for which (3.6) is valid. Note that the above formula is also good for $s = 0$, provided we forget about the product $\prod_{i=1}^s$. It is just 1 as expected. We write down the correlator

$$\langle Y^+_{s_1,n_1} Y^+_{s_2,n_2} Y^-_{s_3,n_3} \rangle_{\mu} = \frac{\mu^s}{s!} |\log \mu| G(s_i, n_i)$$

$$G(s_i, n_i) = \tilde{C}_{s_1,n_1,s_2,n_2} \frac{\Gamma(2s_1 - 1/3)\Gamma(2s_2 - 1/3)\Gamma(2s_3 + 4/3)}{\Gamma(2s_1 - s - 1/3)\Gamma(2s_2 - s - 1/3)\Gamma(2s_3 + s + 4/3)}$$

$$\prod_{i=1}^s \Gamma^2(2s_1 - i)\Gamma^2(2s_2 - i)\Gamma^{-2}(2s_3 + i + 1)\Gamma^{-2}(i + 1).$$

The regulator $\Gamma(\epsilon)$ simply disappears in the correlator.

There are two implications of formula (3.7). First, as one should rescale $Y^+$ operators by $\Gamma^{-1}(\epsilon)$, one should also rescale $Y^-$ operators by $\Gamma(\epsilon)$. We already know that this is necessary for the cosmological term. If we do so, the rescaled correlator will be zero again, as (3.7) tells us. However, if one does not rescale any operator at all, we will get perturbed
OPE of two $W^+$ operators from (3.7). The $s = 0$ case corresponds to the leading term in the OPE and is given in [12]. It is just the unperturbed OPE. Thus it would be interesting to use (3.7) to check the conjecture made in [12] about the full OPE and to construct a space-time action for special states with a nonzero cosmological constant $\mu$.

The negative $s$ is unique, $s = -1$. This is because the maximal value of $s_3$ is $s_1 + s_2$. To define the “integral”, we need use a kind of analytic continuation. Indeed every term in (3.6) is perfectly defined except for those in the product. The product is of form $\prod_{i=1}^{-1} f(i)$. If we are bold enough to use Dotsenko’s suggestion [16] to define a product $\prod_{i=0}^{-n} f^{-1}(-i)$, we can rewrite the undefined product in (3.6). There is a subtlety to notice, however. If we start with the formula in [15] for the integral and use Dotsenko’s prescription, the result is different from the one we can read from (3.6). The correct answer is

$$\frac{(-\pi)^{-1}}{\Gamma^2(2s_3 + 1)(2s_3 + 1/3)} \frac{\Gamma^2(2s + 1)(2s + 1/3)}{\Gamma^2(2s_1)(2s_2)(2s_1 - 1/3)(2s_2 - 1/3)} \Gamma(\epsilon).$$  (3.8)

The result is divergent as $\Gamma(\epsilon)$. Had we started with (3.6), we would have obtained a divergent factor $\Gamma(\epsilon)$ and the same numeric coefficient as in (3.8). Using (3.8) we obtain

$$\langle Y_{s_1,n_1} Y_{s_2,n_2} Y_{s_3,n_3} \rangle_{\mu} = \tilde{c}_{s_1+n_1,s_2,n_2} \mu^{-1} \frac{\Gamma^2(2s_3 + 1)(2s_3 + 1/3)}{\Gamma^2(2s_1)(2s_2)(2s_1 - 1/3)(2s_2 - 1/3)} \Gamma(\epsilon).$$  (3.9)

The integration of the Liouville zero mode gives $\Gamma(1) = 1$ instead of the Liouville volume $|\log \mu|$. Unlike (3.7), the correlator in (3.9) is divergent and the rescaled correlator is finite.

4. A Reshuffling Argument

We have learned from (3.4) that the “bare” expectation value in (3.2) is zero if $s > 2$. This is the generic case since we consider $s_i \geq 1$ only. It would be nice if we can confirm this result by using the product representation of $Y^+$ operators in section 2. We shall use a reshuffling argument which unfortunately applies to three point correlators only thus far. Let us consider a four point correlator of the form

$$\langle O_{s_0,n_0} O_{s_0,n_0} \prod_{i=1}^{3} Y_{s_i,n_i}^{+} \rangle_{\mu},$$
where we have inserted another operator $O\overline{O}$. All positions of operators in the correlator are fixed, since all operators are of conformal dimension $(0, 0)$ and the total ghost number in each sector is three.

Again we integrate the Liouville zero mode first and obtain

$$\frac{1}{s!}(-\mu/\pi)^s|\log\mu|\Gamma^s(\epsilon)\langle O_{s_0,n_0}\overline{O}_{s_0,n_0}\prod_{i=1}^3 Y^+_{s_i,n_i} \left(\int e^{\sqrt{2}\phi}\right)^s \rangle. \quad (4.1)$$

We have assumed $s = -1\sum_{i=0}^3 s_i$ be positive. The expectation value in (4.1) is again the "bare" expectation value, without the cosmological term in the action. Now our reshuffling argument goes as follows. Since $O\overline{O}$ is BRST invariant, the expectation value in (4.1) should not depend on the position of its insertion. We move the position and let it hit any of three $Y^+$ operators. The resulting operator of the product of this $Y^+$ operator with $O\overline{O}$ is not another $Y^+$ in general, as formula (2.6) in section 2 tells us. The resulting operator is rather a sum of one $Y^+$ operator and one operator $X^+ = X^+X^+$. In any case, this operation tells us one can move $x$ and $y$ (together with $\bar{x}$ and $\bar{y}$ in the anti-holomorphic sector) from one $Y^+$ to another. This is our reshuffling argument. One may worry about contact terms caused by integrations of the cosmological term in (4.1). As we learned in section 2, the cosmological term kills other operators when one considers a product with its presence. Therefore the contact terms are zero.

The main difficulties in generalizing the resuffling argument to a general correlator are obvious. Some positions of $Y^+$ operators must be integrated. One can not reshuffle $x$ and $y$ factors among all $Y^+$ operators, since some reshuffling will result in a $X^+$ operator and integration of this operator over the surface is not well defined \[17\]. Another issue is that one should worry about contact terms. These difficulties disappear if we consider correlators of operators of zero momenta in the $X$ direction exclusively. We shall discuss these correlators in the end of this section.

Come back to the three point correlator, we would like to show that it is zero by using the reshuffling argument. From (3.5) we learn that the dependence of the bare correlator on $n_i$ is through the structure constant $C$. One can always choose $n_1 = 0$ and a certain $n_2 = -n_3 = n$ such that the structure constant is not zero (there is always an integer $s_i$, let $s_1$ be an integer). To prove the vanishing of the bare correlator, we assume $n_1 = 0$. By use of the reshuffling argument and (2.6) in section 2, we get

$$\langle Y^+_{s_1,0} Y^+_{s_2,n} Y^+_{s_3,-n} (\int e^{\sqrt{2}\phi} )^s \rangle =$$

$$s_1((Y^+_{s_1,0} x^{s_2-2} y^{s_3-2}) (s_2 Y^+_{1,0} + n X^+_{1,0}) (s_3 Y^+_{1,0} - n X^+_{1,0}) (\int e^{\sqrt{2}\phi} )^s \rangle, \quad (4.2)$$
where we omitted the anti-holomorphic sector. Now $Y_{1,0}^+ x^{s-2} y^{s-2}$ is proportional to $Y_{s-1,0}^+$. The matter part in $Y_{s-1,0}^+$ is proportional to

$$\left( \oint J^- \right)^{s-1} \exp(i\sqrt{2}(s-1)X).$$

It is easy to see that for the matter part on the right hand side of (4.2), one needs only to evaluate correlators

$$\langle \left( \oint J^- \right)^{s-1} e^{i\sqrt{2}(s-1)X} \partial X \partial X \rangle$$

$$\langle \left( \oint J^- \right)^{s-1} e^{i\sqrt{2}(s-1)X} \partial X \rangle.$$ (4.3)

In both correlators, there are a number of insertions of contour integral. We can now deform these contour integrals to act on $\partial X$. Note that $\oint J^- \partial X \propto \exp(-i\sqrt{2}X)$ and $\oint J^- \partial X = 0$, we conclude that these correlators in (4.3) are zero if $s > 3$. When $s = 3$, the first correlator is not zero, one needs to go back to calculations in the previous section. When $s = 2$, both correlators are not zero. Indeed in (3.4) the multi-complex integral is not zero for $s = 2$.

A remark is in order. One may wonder why we do not just calculate the OPE in (4.2) to show directly that the correlator is zero, using the fact that $Y_{1,0}^+ Y_{1,0}^+ = X_{0,0}^+ X_{0,0}^+ = 0$. The answer is the following. These products are zero up to BRST commutators, for example $Y_{1,0}^+ Y_{1,0}^+ = [Q, -1/2\partial c]$. Since $\partial c$ is not annihilated by $b_0$, the usual argument for decoupling of a BRST commutator does not go through.

We use the same argument to show that the bare correlator of two $Y^+$ operators and one $Y^-$ operator is zero when $s > 0$, in agreement with (3.4). Choose the representation $Y_{s_3,n_3}^- = (-1)^{2n_3} Y_{s_0}^- x^{S+n_3-s_3} y^{S-n_3-s_3}$ as in section 2. Applying the reshuffling argument, we find

$$\langle Y_{s_3,n_3}^- Y_{s_1,n_2}^+ Y_{s_2,n_2}^+ \left( \int e^{\sqrt{2} \phi} \right)^s \rangle$$

$$\langle (-1)^{2n_3} \langle Y_{S_0}^- x^{S+s-1} y^{S+s-1} \rangle (s_1 Y_{1,0}^+ + n_1 X_{0,0}^+) (s_2 Y_{1,0}^+ + n_2 X_{0,0}^+) \left( e^{\sqrt{2} \phi} \right)^s \rangle,$$ (4.4)

where we suppressed the anti-holomorphic sector again. As we showed in section 2, the product $Y_{S_0}^- x^{S+s-1} y^{S+s-1}$ is simply zero when $s > 1$. When $s = 1$, the product is just the
cosmological term. One should calculate the multi-complex integral again. It is interesting to note that the argument does not force the bare correlator be zero when $s = 0, -1$, as we know it is not zero for these values. Applying the reshuffling argument, we will find that bare correlators of two $\mathcal{Y}^-$ operators and one $\mathcal{Y}^+$ operator are zero.

Finally we consider correlators of $\mathcal{Y}^+$ operators with zero $X$ momentum. We do not have to worry about $\mathcal{X}^+$ operators since they do not appear here. The only things worrying us are contact terms. It is easy to see that the contact term of two such $\mathcal{Y}^+$ operators is zero. With the resuffling argument, the following correlator

$$
\langle \prod_{i=1}^{3} \mathcal{Y}^+_{s_i,0} \prod_{i=4}^{N} \mathcal{W}^+_{s_i,0} \left( \int e^{\sqrt{2}\phi} \right)^s \rangle
$$

(4.5)

is reduced to

$$
\frac{1}{s - 1} \prod_{i=1}^{N} s_i \left( \frac{\mathcal{Y}^+_{s-1,0}(\mathcal{Y}^+_{1,0})^2}{\mathcal{W}^+_{1,0}} \right)^N \left( \int e^{\sqrt{2}\phi} \right)^s.
$$

(4.6)

Now the matter part of $\mathcal{Y}^+_{s-1,0}$ is proportional to $(\oint J^-)^{s-1} \exp(i\sqrt{2}(s-1)X)$. We can apply the contour deformation to (4.6) again. An immediate consequence is that the correlator is zero when $s - 1 > N - 1$. When $s - 1 = N - 1$, the correlator can be explicitly calculated. We have to evaluate a regularized multi-complex integral again. We shall not do it here. Suffices it to say that the regularized integral is proportional to $\Gamma^{-1}(\epsilon)$, which is just zero. $N = 3$ is a special case, and from (3.4) we see that the regularized integral indeed goes to zero. It can be proven that even when $s - 1 < N - 1$, the regularized multi-complex integral is zero.

To summarize, we have shown that for all three point correlators and correlators of $\mathcal{Y}^+$ with zero $X$ momentum the bare correlators are zero whenever the scaling exponent $s > 0$. We would like to conjecture that this is always the case and furthermore the rescaled correlators are zero whenever $s > 0$.

5. Conclusion

We have shown that the generators $x$ and $y$ of the ground ring can be used to generate other BRST invariant operators. This fact makes possible a reshuffling argument by which some calculations of three point correlators in section 3 are confirmed by an independent
method. We also show that bare correlators of operators with zero $X$ momentum vanish. This leads us to conjecture that indeed all Liouville bulk correlators of operators in the right branch are zero. If this is true, then the nontrivial results obtained in [13] must be explained in other way, maybe by tachyon wall effects. It would be interesting to make further use of the ground ring structure.

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References

[1] D.J. Gross and N. Miljkovic, Phys. Lett. 238B (1990) 217; E.Brezin, V. Kazakov and Al. B. Zamolodchikov, Nucl. Phys. B 338 (1990) 673; P. Ginsparg and J. Zinn-Justin, Phys. Lett. 240B (1990) 333; G. Parisi, Phys. Lett. 238B (1990) 209.
[2] D.J. Gross, I.R. Klebanov and M.J. Newman, Nucl. Phys. B350 (1991) 621
[3] A.M. Polyakov, Mod. Phys. Lett. A6 (1991) 635.
[4] B. Lian and G. Zuckerman, Yale preprint YCTP-P18-91; P. Bouwknegt, J. McCarthy and K. Pilch, CERN preprint CERN-TH.6162/91.
[5] G. Moore, Yale and Rutgers preprint YCTP-P8-91, RU-91-12; K. Demeterfi, A. Jevicki and J.P. Rodrigues, Brown preprints BROWN-HET-795 and BROWN-HET-803 (1991); D.J. Gross and I.R. Klebanov, Nucl. Phys. B359 (1991) 3.
[6] G. Moore and N. Seiberg, Rutgers and Yale preprint RU-91-29, YCTP-P19-91.
[7] P. Di Francesco and D. Kutasov, Phys. Lett. 261B (1991) 385.
[8] V.G. Kac, in Group Theoretical Methods in Physics, ed. W. Beiglbock et al. (Springer-Verlag, 1979); G. Segal, Comm. Math. Phys. 80 (1981) 301; R. Dijkgraaf, E. Verlinde and H. Verlinde, Comm. Math. Phys. 115 (1988) 649.
[9] E. Witten, IAS preprint IASSNS-HEP-91/51.
[10] N. Seiberg, Notes on Quantum Liouville Theory and Quantum Gravity, in Common Trends in Mathematics and Quantum Field Theory, Proceedings of the 1990 Yukawa International Seminar, ed. T Eguchi et al.
[11] E. Witten, IAS preprint IASSNS-91/12.
[12] I.R. Klebanov and A.M. Polyakov, Princeton preprint, Sept. 1991.
[13] D.J. Gross and I.R. Klebanov, Nucl. Phys. B352 (1991) 671; U.H. Danielsson and D.J. Gross, Princeton preprint PUPT-1258 (1991).
[14] A. Gupta, S.P. Trivedi and M.B. Wise, Nucl. Phys. B340 (1990) 475; M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051.
[15] Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B251 (1985) 691.
[16] V.S. Dotsenko, preprint PAR-LPTHE 91-18.
[17] J. Polchinski, Nucl. Phys. B307 (1988) 61; P. Nelson, Phys. Rev. Lett. 62 (1989) 993.