Unorthodox properties of critical clusters

A. Robledo *
Instituto de Física,
Universidad Nacional Autónoma de México,
Apartado Postal 20-364, México 01000 D.F., Mexico.

January 9, 2022

Abstract

We look at the properties of clusters of order parameter \( \phi \) at critical points in thermal systems and consider their significance to statistical-mechanical ground rules. These properties have been previously obtained through the saddle-point approximation in a coarse-grained partition function. We examine both static and dynamical aspects of a single large cluster and indicate that these properties fall outside the canonical Boltzmann-Gibbs (BG) scheme. Specifically: 1) The faster than exponential growth with cluster size of the space-integrated \( \phi \) suggests nonextensivity of the BG entropy but extensivity of a \( q \)-entropy expression. 2) The finding that the time evolution of \( \phi \) is described by the dynamics of an intermittent nonlinear map implies an atypical sensitivity to initial conditions compatible with \( q \)-statistics and displays an 'aging' scaling property. 3) Both, the approach to criticality and the infinite-size cluster limit at criticality manifest through a crossover from canonical to \( q \)-statistics and we discuss the nonuniform convergence associated to these features.

Key words: critical clusters, saddle-point approximation, \( q \)-entropy, intermittency, aging

PACS: 64.60.Ht, 75.10.Hk, 05.45.Ac, 05.10.Cc

*E-mail address: robledo@fisica.unam.mx
1 Introduction

An unanticipated – and remarkable – relationship between intermittency and critical phenomena has been recently suggested \[1\] \[2\]. This development brings together fields of research in nonlinear dynamics and condensed matter physics, specifically, the dynamics in the proximity of an incipiently chaotic attractor \[3\] appears associated to the dynamics of fluctuations of an equilibrium state with well-known scaling properties \[4\]. Here we examine this connection in some detail with special attention to several unorthodox properties, such as, the extensivity of entropy of fractal clusters, the anomalous - faster than exponential - sensitivity to initial conditions, and the aging scaling features in time evolution.

Our interest is on the local fluctuations of a system undergoing a second order phase transition, for example, in the Ising model as the magnetization fluctuates and generates magnetic domains on all size scales at its critical point. In particular, the object of study is a single cluster of order parameter $\phi$ at criticality. This is described by a coarse-grained free energy or effective action, like in the Landau-Ginzburg-Wilson (LGW) continuous spin model portrayal of the equilibrium configurations of Ising spins at the critical temperature and zero external field. As we shall see, a cluster of radius $R$ is an unstable configuration whose amplitude grows in time and eventually collapses when an instability is reached. This process has been shown \[1\] \[2\] to be described by a nonlinear map with tangency and feedback features \[3\], such that the time evolution of the cluster is given in the nonlinear system as a laminar episode of intermittent dynamics.

The method employed to determine the cluster’s order parameter profile $\phi(r)$ assumes the saddle-point approximation of the coarse-grained partition function $Z$, so that $\phi(r)$ is its dominant configuration and is determined by solving the corresponding Euler-Lagrange equation. The procedure is equivalent to the density functional approach for stationary states in equilibrium nonuniform fluids. The solution found for $\phi(r)$ is similar to an instanton in field theories \[5\], and from its thermal average (evaluated by integrating over its amplitude $\phi_0$, the remaining degree of freedom after its size $R$ has been fixed) interesting properties have been derived. These are the fractal dimension of the cluster \[6\] \[7\] and the intermittent behavior in its time evolution \[1\] \[2\]. Both types of properties are given in terms of the critical isotherm exponent $\delta$.

As we describe below, the dominance of $\phi(r)$ in $Z$ depends on a condition
that can be expressed as an inequality between two lengths in space. This is \( r_0 \gg R \), where \( r_0 \) is the location of a divergence in the expression for \( \phi(r) \) that decreases as an inverse power of the cluster amplitude \( \phi_0 \). When \( r_0 \gg R \) the profile is almost horizontal but for \( r_0 \lesssim R \) the profile increases from its center faster than an exponential. It is this feature that gives the cluster some atypical properties that we present and discuss here. These properties relate to the dependence of i) the number of cluster configurations on size \( R \), and ii) the sensitivity to initial conditions \( \xi_t \) of order-parameter evolution on time \( t \).

The above-mentioned properties appear to be at odds with the usual Boltzmann-Gibbs (BG) statistics but suggest compatibility \([8]\) with one of its generalizations known as \( q \)-statistics \([9] \ [10] \). A condition for these properties to arise is criticality but also is the circumstance that phase space has only been partially represented by selecting only dominant configurations. Hence, the motivation to examine this problem rests on explaining the physical and methodological basis under which proposed generalizations of the BG statistics may apply. For this reason, we begin in Section 2 by recalling the basic framework for the Landau approach, and, in Section 3, we review details of the derivation of the dominant configuration. Then, in Section 4 we consider an estimate for the numbers of configurations that give rise to a cluster and check on the extensivity of entropy expressions. In section 5 we review the connection between the cluster time evolution and the nonlinear dynamics of intermittency, so that in Section 6 we can discuss again the manifestation of \( q \)-statistics. In Section 7 we summarize and discuss our results.

2 A Landau approach for single clusters

We start our account by recalling that the Landau approach is backed by a two-stage calculation strategy for the partition function \( Z \) of the system. In the first, a sum is made over the microscopic configurations that lead to a specific form for the order parameter \( \phi(r) \) (where \( r \) is the spatial position) to obtain a partial, \( \phi(r) \)-dependent, result \( Z_\phi \). The second stage consists of summing up \( Z_\phi \) over all possible forms \( \phi(r) \), so that

\[
Z = \int D[\phi] Z_\phi, \tag{1}
\]
where a path integration over the configurations for $\phi$ is indicated. The partial partition function $Z_\phi$ is then written as

$$Z_\phi = \Omega[\phi] \exp(-E[\phi]/kT),$$

where $E[\phi]$ is the energy of the system when the order parameter takes the fixed form $\phi(r)$, and $\Omega[\phi]$ is the number of microscopic configurations for that $\phi(r)$. Taking logarithms to both sides of Eq. (2) gives

$$\Psi[\phi] = E[\phi] - S[\phi],$$

where $\Psi[\phi]$ is the so-called LGW free energy and $S[\phi] = \ln \Omega[\phi]$ is an ‘entropy’ term associated to $\Omega[\phi]$ (Boltzmann’s constant has been taken as unity). In practice the evaluation of

$$Z = \int D[\phi] \exp(-\Psi[\phi])$$

is considered with $\Psi[\phi]$ written as a spatial integral over some function of $\phi(r)$ that involves a square gradient (perhaps higher-order derivatives) and powers of $\phi(r)$.

At criticality the LGW free energy takes the form

$$\Psi_c[\phi] = a \int dr^d \left[ \frac{1}{2} \nabla^2 \phi + b |\phi|^\delta \right],$$

$\delta$ is the critical isotherm exponent and $d$ is the spatial dimension ($\delta = 5$ for the $d = 3$ Ising model with short range interactions). The coefficients $a$ and $b$ are dimensionless couplings and $\phi = \lambda^{-d} \phi$ and $r = \lambda \tilde{r}$ are dimensionless quantities, $\lambda$ is the ultraviolet cutoff that fixes the coarse graining scale, $\phi$ is the order parameter (e.g. magnetization per unit volume) and $\tilde{r}$ represents a spatial position coordinate with dimension of length. Out of criticality $\Psi[\phi]$ takes the form $\Psi[\phi] = \int dr^d \left[ \frac{1}{2} \nabla^2 \phi + 1/2 r_0 \phi^2 + u_0 \phi^4 \right]$, where $r_0 = a_0 t$, $a_0$ and $u_0 > 0$ are constants and $t$ is the reduced temperature distance to the critical point. The ordering field has been set to zero [11].

To study the coarse-grained configurations of a single cluster, a subsystem of finite size $R$ is considered in the neighborhood of $r = 0$ and there are no restrictions imposed to the value of $\phi$ at its boundaries with the rest of the (infinite-sized) physical system. In this case the integration in Eq. (4) is performed over the subsystem’s volume $V$. 

4
3 Cluster properties from dominant configurations

We consider a one dimensional system with unspecified range of interactions, as in this case the procedure is more transparent and the expressions for the relevant quantities more easily derived. The analysis can be carried out for higher dimensions with no further significant assumptions and with comparable results [1]-[7]. The LGW free energy reads now

\[ \Psi_c[\phi] = a \int_0^R dx \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + b |\phi|^{\delta+1} \right], \]  

and we adopt the saddle-point approximation - valid for \( a \gg 1 \) - to circumvent the nontrivial task of carrying out the path integration in Eq. (1). The saddle-point configurations are obtained from the Euler-Lagrange equation

\[ \frac{d^2\phi}{dx^2} = -\frac{dV}{d\phi}, \]  

where \( V = -b |\phi|^{\delta+1} \). It is helpful to recall the classical motion analog of this problem, a particle in time \( x \) under a potential \( V \). Integration of Eq. (6) yields

\[ U = \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 - b |\phi|^{\delta+1}, \]  

where the constant \( U \) is the total particle’s energy. Subsequent integration of Eq. (11) with \( U = 0 \) leads to profiles for critical clusters of the form [1]-[7]

\[ \phi(x) = A |x - x_0|^{-2/(\delta-1)}, \]  

where

\[ A = \left[ \sqrt{b/2} (\delta - 1) \right]^{-2/(\delta-1)} \]  

and

\[ x_0 = \left[ \sqrt{b/2} (\delta - 1) \right]^{-1} \phi_0^{-\delta-1/2}, \]  

where \( x_0 \) is a system-dependent reference position and \( \phi_0 = \phi(0) \). The value of \( \phi \) at the edge of the cluster is \( \phi_R = \phi(R) \) and the cluster free energy is

\[ \Psi_c[\phi] = 2ab \int_0^R dx \phi(x)^{\delta+1}. \]
This family of solutions give the largest contributions to $Z$ and are the analogs of instantons in semiclassical quantum theories, i.e. configurations that minimize the Yang-Mills action $\tilde{5}$.

Similar solutions are obtained for small $U \approx 0$ $\tilde{1}$ $\tilde{2}$ where now the position of the singularity $x_0$ depends also on $U$. These solutions enter $Z$ with a weight $\exp(-\alpha R |U|)$ and therefore their relevance diminishes as $|U|$ increases.

4 Extensivity of critical cluster entropy

We comment now on the nature of the profile $\phi(x)$ in Eq. $\tilde{8}$. This function, as well as those solutions for $U \approx 0$, can be rewritten in the form

$$\phi(x) = \phi_0 \exp_q(kx), \quad (11)$$

where $\exp_q(x)$ is the $q$-exponential function $\exp_q(x) \equiv [1 - (q - 1)x]^{-1/(q-1)}$ with $q = (1 + \delta)/2$ and $k = \sqrt{2b}\phi_0^{(6-1)/2}$. Because $\delta > 1$ one has $q > 1$ and $\phi(x)$ grows faster than an exponential as $x \to x_0$ and diverges at $x_0$. It is important to notice that only configurations with $R \ll x_0$ have a nonvanishing contribution to the path integration in Eq. $\tilde{1}$ $\tilde{1}$ $\tilde{2}$ and that these configurations vanish for the infinite cluster size system. There are some characteristics of nonuniform convergence in relation to the limits $R \to \infty$ and $x_0 \to \infty$, a feature that is significant for our connection with $q$-statistics. By taking $\delta = 1$ the system is set out of criticality, then $q = 1$ and the profile $\phi(x)$ becomes the exponential $\phi(x) = \phi_0 \exp(k_0 x)$, $k_0 = \sqrt{a_0I}$. (Note that in this case $x_0 \to 0$).

The quantity

$$\Phi(R) = \int_0^R dx \phi(x), \quad (12)$$

or total 'magnetization' of the cluster, is given by

$$\Phi(R) = \Phi_0 \left\{ [\exp_q(kR)]^{2-q} - 1 \right\}, \quad R < x_0, \quad (13)$$

where $\Phi_0 = \text{sgn}(3 - \delta)[2\phi_0/(\delta - 3)k]$ with $q = (1 + \delta)/2$. (We have not shown the special case $\delta = 3$). For $\delta = 1$, one has $\Phi(R) = \phi_0 k_0^{-1}[\exp(k_0 R) - 1]$. The rate at which $\Phi(R)$ grows with $R$, $d\Phi(R)/dR$, is necessarily equal to $\phi_R$, the
value of $\phi(x)$ at the edge of the cluster, therefore
\[
\frac{d\Phi(R)}{dR} = \phi_0 \exp_q(kR), \quad R < x_0,
\]
while for $\delta = 1$ it is $d\Phi(R)/dR = \phi_0 \exp(k_0R)$.

The expressions above may be used to estimate the dependence on cluster size $R$ of the number of microscopic configurations $\Omega[\phi]$ that make up the partial partition function $Z_\phi$ in Eq. (2) for the dominant coarse-grained $\phi(x)$. This dependence may be obtained in a way analogous to that of how is obtained the dependence with time of the number of configurations $\Omega$ for an ensemble of trajectories in a one-dimensional dynamical system. Here ‘trajectory positions’ are given by the values of $\phi$ in microscopic configurations and ‘time’ is given by the cluster size $R$. Initially adjacent positions stay adjacent and $\Omega$ is almost constant but at later times they spread and $\Omega$ increases rapidly. For chaotic orbits the increment is exponential [3] but for marginally chaotic orbits at the tangent bifurcation $\Omega$ increases as a $q$-exponential with $q > 1$ [12] [13]. The ensemble of trajectories is initially contained in the interval $[0, \phi_0]$ and at time $R$ they occupy the interval $[0, \phi_R]$, therefore we assume
\[
\Omega(R) \sim \phi_0^{-1} d\Phi(R)/dR = \phi_0^{-1} \phi_R.
\]
The results in the following Sections provide a justification for this choice.

Then, it is significant to note that the Tsallis entropy [9],
\[
S_q = \ln_q \Omega = \frac{\Omega^{1-q} - 1}{1 - q},
\]
(where $\ln_q y \equiv (y^{1-q} - 1)/(1 - q)$ is the inverse of $\exp_q(y)$), when evaluated for $\Omega \sim \exp_q(kR)$ complies with the extensivity property $S_q \sim R$ [14], while the BG entropy
\[
S_{BG}(t) = \ln \Omega,
\]
obtained from $S_q$ when $q = 1$, when evaluated for $\Omega \sim \exp(k_0R)$ complies also with the extensivity property $S_1 \sim R$.

5 Cluster instability and intermittency

The profile $\phi(x)$ given by Eqs. (8) or (11) describes a fluctuation of the critical equilibrium state of the infinite system with average $\langle \phi \rangle = 0$. In a coarse-grained time scale the cluster is expected to evolve by increasing its amplitude
φ₀ and size R because the subsystem studied represents an environment with unevenness in the states of the microscopic degrees of freedom (e.g. more spins up than down). Increments in φ₀ for fixed R takes the position x₀ for the singularity closer to R and the almost constant shape φ(x) ≃ φ₀ for x₀ ≫ R is eventually replaced by a faster than exponential shape φ(x), while, as mentioned, the dominance of this configuration in Z decreases accordingly and rapidly so. When the divergence is reached at x₀ = R the profile φ(x) no longer describes the spatial region where the subsystem is located. But a subsequent fluctuation would again be represented by a cluster φ(x) of the same type. From this renewal process we obtain a picture of intermittency. A similar situation would occur if R is increased for fixed φ₀.

Indeed, a link was revealed [1] [2] between the fluctuation properties of a critical cluster described by Eqs. (8) and (11) and the dynamics of marginally chaotic intermittent maps. This connection was demonstrated in different ways in Refs. [1] [2] by considering the properties of the thermal average

\[ \langle \Phi(R) \rangle = Z^{-1} \int D[\phi] \Phi(R) \exp(-\Psi_c[\phi]), \] (18)

for fixed R. When x₀ ≫ R the profile is basically flat φ(x) ≃ φ₀, the LGW free energy is \( \Psi_c \approx 2abR\phi_0^{\delta+1} \), and the path integral in Eq. (18) becomes an ordinary integral over 0 ≤ φ ≤ φ₀. One obtains

\[ \langle \Phi(R) \rangle \approx \frac{\phi_0 R}{2} \exp \left(-uR\phi_0^{\delta+1}\right), \] (19)

where \( u = 2ab(\delta + 1)/(\delta + 2)(\delta + 3) \).

The procedure that resembles the picture given in the beginning of this Section is to consider the value of \( \langle \Phi \rangle \) at successive times \( t = 0, 1, \ldots \), and assume that this quantity changes by a fixed amount \( \mu \) per unit time, that is

\[ \langle \Phi_{t+1} \rangle = \langle \Phi_t \rangle + \mu. \] (20)

Making use of Eq. (19) one obtains [6] [7] for small values of φₜ the map

\[ \phi_{t+1} = \epsilon + \phi_t + \nu\phi_t^{\delta+1}, \] (21)

where the shift parameter is \( \epsilon \sim R^{-1} \) and the amplitude of the nonlinear term is \( \nu = u\mu \).

Eq. (21) can be recognized as that describing the intermittency route to chaos in the vicinity of a tangent bifurcation [3]. The complete form of
the map \[1\] \[2\] displays a ‘superexponentially’ decreasing region that takes back the iterate close to the origin in approximately one step. Thus the parameters of the thermal system determine the dynamics of the map. The mean number of iterations in the laminar region was seen to be related to the mean magnetization within a critical cluster of radius \(R\). There is a corresponding power law dependence of the duration of the laminar region on the shift parameter \(\epsilon\) of the map \[1\]. For \(\epsilon > 0\) the (small) Lyapunov exponent is simply related to the critical exponent \(\delta\) \[2\].

6 Intermittency and \(q\)-statistics

At the tangent bifurcation, the intermittency route to chaos, the ordinary Lyapunov exponent \(\lambda_1\) vanishes and the sensitivity to initial conditions \(\xi_t \equiv |dx_t/dx_{in}|\) (where \(x_t\) is the orbit position at time \(t\) given the initial position \(x_{in}\) at time \(t = 0\) ) is no longer given by the BG law \(\xi_t = \exp(\lambda_1 t)\) but acquires either a power or a super-exponential law \[12\] \[13\]. In this case \(\xi_t\) is given by the \(q\)-exponential expression,

\[
\xi_t = \exp_Q(\lambda_Q t) \equiv [1 - (Q - 1)\lambda_Q t]^{1/(Q-1)},
\]

containing the entropic index \(Q\) and the \(q\)-generalized Lyapunov coefficient \(\lambda_Q\). In the limit \(Q \to 1\) Eq. \[22\] reduces to the ordinary BG law. See \[12\] and \[13\] for a more rigorous description of the theory and related issues.

Assisted by the known renormalization group (RG) treatment for the tangent bifurcation \[3\], the formula for \(\xi_t\) has been rigorously derived \[12\] \[13\] and found to comply with Eq. \[22\]. The tangent bifurcation is usually studied by means of the map

\[
f(x) = \epsilon + x + \nu |x|^z + o(|x|^z), \quad \nu > 0, \tag{23}
\]

with nonlinearity \(z > 1\) in the limit \(\epsilon \to 0\). The associated RG fixed-point map \(x' = f^*(x)\) was found to be

\[
x' = x \exp_\nu(\nu x^{z-1}) = x[1 - (z - 1)\nu x^{z-1}]^{1/(z-1)}, \quad \epsilon = 0, \tag{24}
\]

as it satisfies \(f^*(f^*(x)) = \alpha^{-1}f^*(\alpha x)\) with \(\alpha = 2^{1/(z-1)}\) and has a power-series expansion in \(x\) that coincides with Eq. \[23\] in the two lowest-order terms. (Above \(x^{z-1} \equiv |x|^{z-1}\text{sgn}(x)\)). The long time dynamics is readily derived from the static solution Eq. \[24\], one obtains

\[
\xi_t(x_{in}) = [1 - (z - 1)\nu x_{in}^{z-1}]^{-z/(z-1)}, \tag{25}
\]

9
and so, \( Q = 2 - z^{-1} \) and \( \lambda_Q(x) = z\nu x_{in}^{z-1} \) \([12][13]\). When \( Q > 1 \) the left-hand side \((x < 0)\) of the tangent bifurcation map, Eq. \( (23) \), exhibits a weak insensitivity to initial conditions, i.e. power-law convergence of orbits. However at the right-hand side \((x > 0)\) of the bifurcation the argument of the \( q \)-exponential becomes positive and this results in a ‘super-strong’ sensitivity to initial conditions, i.e. a sensitivity that grows faster than exponential \([13]\). Comparison of Eq. \( (23) \) with Eq. \( (21) \) indicates the simple relation \( z = \delta + 1 \).

There is an interesting scaling property displayed by \( \xi_t \) in Eq. \( (25) \) similar to the scaling property known as aging in systems close to glass formation. This property is observed in two-time functions (e.g. time correlations) for which there is no time translation invariance but scaling is observed in terms of a time ratio variable \( t/t_w \) where \( t_w \) is a ‘waiting time’ assigned to the time interval for preparation or hold of the system before time evolution is observed through time \( t \). This property can be seen immediately in \( \xi_t \) if one assigns a waiting time \( t_w \) to the initial position \( x_{in} \) as \( t_w = x_{in}^{1-z} \). Eq. \( (25) \) reads now

\[
\xi_t(t_w) = [1 - (z - 1)\nu t/t_w]^{-z/(z-1)} . \tag{26}
\]

The sensitivity \( \xi_t \) for this critical attractor is dependent on the initial position \( x_{in} \) or, equivalently, on its waiting time \( t_w \), the closer \( x_{in} \) is to the point of tangency the longer \( t_w \) but the sensitivity of all trajectories fall on the same \( q \)-exponential curve when plotted against \( t/t_w \). Aging has also been observed for the properties of the same map but in a different context \([15]\).

### 7 Summary and discussion

We have examined the study of clusters at criticality in thermal systems by means of the saddle-point approximation in the LGW free energy model \([1][2][6][7]\). The retention of only one coarse-grained configuration leads to cluster properties that are physically reasonable but also appear to fall outside the limits of validity of the canonical theory. The fractal geometry and the intermittent behavior of critical clusters obtained from this method \([1][2][6][7]\) are both consistent with equivalent properties found for clusters at the critical points of the \( d = 2 \) Ising and Potts models \([16][17]\). On the other hand, we found that the entropy expression that provides the property of extensivity for our estimate of the number of cluster configurations is not the usual BG expression but that of \( q \)-statistics. Likewise, the nonlinear map and its corresponding sensitivity to initial conditions linked to the intermittency
of clusters do not follow the fully-chaotic trajectories of BG statistics but display the features of $q$-statistics.

With regards to the extensivity of entropy, what our assumptions and results mean basically is that extensivity of entropy (BG or $q$-generalized) of a cluster and the Kolmogorov-Sinai [3] linear growth with time of entropy (BG or $q$-generalized) of trajectories for the attractor of a nonlinear map are equivalent. The crossover from the $S_q$ to the $S_{BG}$ expressions is obtained when the system is taken out of criticality, because $\delta \to 1$ makes $q \to 1$. When at criticality the crossover between $q$-statistics and BG statistics in the dynamical behavior is related to the subsystem’s size in the following way. We keep in mind that the map shift parameter depends on the domain size as $\epsilon \sim R^{-1}$. So, the time evolution of $\phi$ displays laminar episodes of duration $< t > \sim \epsilon^{-\delta/(\delta+1)}$ and the Lyapunov coefficient in this regime is $\lambda_1 \sim \epsilon^2$. Within the first laminar episode the dynamical evolution of $\phi(x)$ obeys $q$-statistics, but for very large times the occurrence of many different laminar episodes leads to an increasingly chaotic orbit consistent with the small $\lambda_1 > 0$ and BG statistics is recovered. As $R$ increases ($R \ll x_0$ always) the time duration of the $q$-statistical regime increases and in the limit $R \to \infty$ there is only one infinitely long laminar $q$-statistical episode with $\lambda_1 = 0$ and with no crossover to BG statistics. On the other hand when $R > x_0$ the clusters $\phi(x)$ are no longer dominant, for the infinite subsystem $R \to \infty$ their contribution to $Z$ vanishes and no departure from BG statistics is expected to occur [8].

In view of the results presented here the departure from BG statistics and the applicability of $q$-statistics is due in part to the presence of the long-ranged correlations in space and in time that take place at criticality. These correlations give the integrand in the LGW $\Psi_c$ a power-law dependence of the form $|\phi|^{\delta+1}$ with $\delta > 1$ (commonly $\delta \geq 3$ as $\delta = 3$ gives the Gaussian critical point) and this in turn determines the $q$-exponential expression for $\phi(x)$ and the properties derived from it. On the other hand, the neglect of all coarse-grained configurations other than the most dominant implies that phase space has not been properly sampled, and that the ergodic and mixing properties characteristic of equilibrium BG statistics are not guaranteed. In this respect it is perhaps significant to notice that the most important instance for which $q$-statistics has been to date rigorously proved to hold is the so-called onset of chaos in unimodal nonlinear maps [18], [19]. This marginally chaotic attractor is nonergodic and nonmixing and its dynamical properties exhibit infinitely-ranged time correlations. The crossover to BG statistics in this model can be
induced via perturbation with noise. Recently [20], the dynamical properties of the noise-perturbed onset of chaos have been proved to be analogous to those of supercooled liquids close to vitrification.

Acknowledgments. It is with much pleasure and appreciation that I dedicate this work to Benjamin Widom. Partial support by DGAPA-UNAM and CONACyT (Mexican Agencies) is acknowledged.

References

[1] Y.F. Contoyiannis and F.K. Diakonos, Phys. Lett. A268, 286 (2000).
[2] Y.F. Contoyiannis, F.K. Diakonos, and A. Malakis, Phys. Rev. Lett. 89, 035701 (2002).
[3] See, for example, H.G. Schuster, Deterministic Chaos. An Introduction, 2nd Revised Edition (VCH Publishers, Weinheim, 1988).
[4] See, for example, H.E. Stanley, Introduction to Phase Transitions and Critical Phenomena (Oxford University Press, New York, 1987).
[5] See, for example, S. Coleman, Aspects of Symmetry: Selected Erice Lectures (Cambridge University Press, 1985).
[6] N.G. Antoniou, Y.F. Contoyiannis, F.K. Diakonos, and C.G. Papadopoulos, Phys. Rev. Lett. 81, 4289 (1998).
[7] N.G. Antoniou, Y.F. Contoyiannis, and F.K. Diakonos, Phys. Rev. E 62, 3125 (2000).
[8] A. Robledo, Physica A 344, 631 (2004).
[9] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
[10] For a recent review see, Nonextensive Entropy – Interdisciplinary Applications, M. Gell-Mann and C. Tsallis, eds., (Oxford University Press, New York, 2004), in press. See http://tsallis.cat.cbpf.br/biblio.htm for full bibliography.
[11] See, for example, P.M. Chaikin and T.C. Lubensky, Principles of Condensed Matter Physics (Cambridge University Press, Cambridge, 1995).
[12] A. Robledo, Physica A 314, 437 (2002); A. Robledo, Physica D 193, 153 (2004).

[13] F. Baldovin and A. Robledo, Europhys. Lett. 60, 518 (2002).

[14] C. Tsallis, M. Gell-Mann and Y. Sato, cond-mat/0502274.

[15] E. Barkai, Phys. Rev. Lett. 90, 104101 (2003).

[16] A. L. Stella and C. Vanderzande, Phys. Rev. Lett. 62, 1067 (1989); C Vanderzande and A. L. Stella, J. Phys. A: Math. Gen. 22, L445 (1989); A. Coniglio, Phys. Rev. Lett. 62, 3054-3057 (1989).

[17] S. Gupta, P. Lacock and H. Satz, Nucl. Phys. B 362, 583 (1991); Z. Burda, J. Wosiek and K. Zalewski, Phys. Lett. B 266, 439 (1991).

[18] F. Baldovin and A. Robledo, Phys. Rev. E 66, 045104(R) (2002); F. Baldovin and A. Robledo, Phys. Rev. E 69, 045202(R).

[19] E. Mayoral and A. Robledo, cond-mat/0501366.

[20] A. Robledo Phys. Lett. A 328, 467 (2004); F. Baldovin and A. Robledo, Fluctns. and Noise Lett. 5, xxx (2005).