On algebras of gauge transformations in a general setting

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Abstract We consider a Lagrangian system on a fiber bundle and its gauge transformations depending on derivatives of dynamic variables and gauge parameters of arbitrary order. We say that gauge transformations form an algebra if they generate a nilpotent BRST operator.

1 Introduction

In a general setting, one can say that a family of gauge transformations depending on parameters is an algebra if the Lie bracket of arbitrary two gauge transformations depending on different parameter functions is again a gauge transformation depending on some parameter function. The goal is to formulate this condition in strict mathematical terms. For instance, gauge transformations in gauge theory on a principal bundle form a finite-dimensional real (or complex) Lie algebra. Gauge transformations of a certain class of field theories constitute sh-Lie algebras [4]. It may happen that gauge transformations are not assembled into an algebra or form an algebra on-shell [6].

We consider a Lagrangian system on a smooth fiber bundle $Y \to X$ subject to gauge transformations depending both on derivatives of dynamic variables of arbitrary order and a finite family of gauge parameters and their derivatives of arbitrary order.

Let $J^r Y$, $r = 1, \ldots$, be finite order jet manifolds of sections of $Y \to X$. In the sequel, the index $r = 0$ stands for $Y$. Given bundle coordinates $(x^\lambda, y^i)$ on $Y$, jet manifolds $J^r Y$ are endowed with the adapted coordinates $(x^\lambda, y^i, y^i_{\Lambda})$, where $\Lambda = (\lambda_k \ldots \lambda_1)$, $k = 1, \ldots, r$, is a symmetric multi-index. We use the notation $\lambda + \Lambda = (\lambda \lambda_k \ldots \lambda_1)$ and

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda|} y^i_{\lambda+\Lambda} \partial^\Lambda_i, \quad d_\Lambda = d_\lambda \circ \cdots \circ d_\lambda.$$

(1)

In order to describe gauge transformations depending on parameters, let us consider Lagrangian formalism on the bundle product

$$E = Y \times_X V,$$

(2)

where $V \to X$ is a vector bundle whose sections are gauge parameter functions [2]. Let $V \to X$ be coordinated by $(x^\lambda, \xi^r)$. Then gauge transformations are represented by a differential operator

$$v = \sum_{0 \leq |\Lambda| \leq m} v^i_{r,\Lambda}(x^\lambda, y^i_{\Xi}) \xi^r \partial_i.$$

(3)
on $E$ (2) which is linear on $V$ and takes its values into the vertical tangent bundle $VY$ of $Y \to X$. Given a section $\xi(x)$ of $V \to X$, the pull-back

$$\xi^* v = \sum_{0 \leq |\Lambda| \leq m} v^{i,\Lambda}_r(x^\lambda, y^i_{\Sigma}) d_{\Lambda}(x) \partial_i$$

(4)

of $v$ (3) onto $Y$ is a gauge transformation depending on a parameter function $\xi(x)$.

By means of a replacement of even gauge parameters $\xi^r$ and their jets $\xi^r_{\Lambda}$ with the odd ghosts $c^r$ and their jets $c^r_{\Lambda}$, the operator (3) defines a graded derivation

$$v = \sum_{0 \leq |\Lambda| \leq m} v^{i,\Lambda}_r(x^\lambda, y^i_{\Sigma}) c^r_{\Lambda} \partial_i$$

(5)

of the algebra of the original even fields and odd ghosts. Its extension

$$v = \sum_{0 \leq |\Lambda| \leq m} v^{i,\Lambda}_r c^r_{\Lambda} \partial_i + u^r \partial_r$$

(6)

to ghosts is called the BRST transformation if it is nilpotent.

We say that gauge transformations (4) make up an algebra if they generate a BRST transformation (6). One can think of the nilpotency conditions (36) – (37) as being the generalized commutation relations and Jacobi identity, respectively. This definition is especially convenient for BRST theory and BV quantization [1].

2 Gauge systems on fiber bundles

In Lagrangian formalism on a fiber bundle $Y \to X$, Lagrangians and their Euler–Lagrange operators are represented by elements of the following graded differential algebra (henceforth GDA).

With the inverse system of jet manifolds

$$X \leftarrow \cdots \leftarrow J^{r-1}Y \leftarrow J^rY \leftarrow \cdots \leftarrow J^1Y \leftarrow Y$$

(7)

one has the direct system

$$\mathcal{O}^*X \xrightarrow{\pi^*} \mathcal{O}^*Y \xrightarrow{\pi^*_{r-1}} \cdots \xrightarrow{\pi^*_{r-1}} \mathcal{O}^*_rY \xrightarrow{\pi^*_{r-1}} \cdots$$

(8)

of GDAs $\mathcal{O}^*_rY$ of exterior forms on jet manifolds $J^rY$ with respect to the pull-back monomorphisms $\pi^*_{r-1}$. Its direct limit $\mathcal{O}^*_\infty Y$ is a GDA consisting of all exterior forms on finite order jet manifolds modulo the pull-back identification.

The projective limit $(J^\infty Y, \pi^\infty_r : J^\infty Y \to J^rY)$ of the inverse system (7) is a Fréchet manifold. A bundle atlas $\{(U_Y; x^\lambda, y^i)\}$ of $Y \to X$ yields the coordinate atlas

$$\{((\pi^\infty_0)^{-1}(U_Y); x^\lambda, y^i_{\Lambda})\}, \quad y^i_{\lambda+\Lambda} = \frac{\partial_{\nu} d_{\mu} y^i_{\lambda}}{\partial x^{x^\nu} x^{\lambda}}, \quad 0 \leq |\Lambda|,$$

(9)
of $J^\infty Y$, where $d_\mu$ are the total derivatives (1). Then $\mathcal{O}_\infty^* Y$ can be written in a coordinate form where the horizontal one-forms $\{dx^\lambda\}$ and the contact one-forms $\{\theta^i_\lambda = y^i_{\lambda+\Lambda} dx^\lambda\}$ are generating elements of the $\mathcal{O}_\infty^0 U_Y$-algebra $\mathcal{O}_\infty^* U_Y$.

There is the canonical decomposition $\mathcal{O}_\infty^* Y = \oplus \mathcal{O}^{k,m}_\infty Y$ of $\mathcal{O}_\infty^* Y$ into $\mathcal{O}_\infty^0 Y$-modules $\mathcal{O}_\infty^{k,m} Y$ of $k$-contact and $m$-horizontal forms together with the corresponding projectors $h_k : \mathcal{O}_\infty^* Y \to \mathcal{O}_\infty^{k,*} Y$ and $h^m : \mathcal{O}_\infty^* Y \to \mathcal{O}_\infty^{* m} Y$. Accordingly, the exterior differential on $\mathcal{O}_\infty^* Y$ is split into the sum $d = d_H + d_V$ of the nilpotent total and vertical differentials

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \quad d_V(\phi) = \theta^i_\lambda \wedge \partial^i_\lambda \phi, \quad \phi \in \mathcal{O}_\infty^* Y.$$  

Any finite order Lagrangian $L = L_\omega : J^r Y \to \Lambda^n T^* X$, $\omega = dx^1 \wedge \cdots \wedge dx^n$, $n = \dim X$, (10) is an element of $\mathcal{O}_\infty^{0,n} Y$, while

$$\delta L = \epsilon_i \theta^i \wedge \omega = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda(\partial^i_\Lambda L) \theta^i \wedge \omega \in \mathcal{O}_\infty^{1,n} Y$$  

is its Euler–Lagrange operator taking values into the vector bundle

$$T^* Y \wedge^n T^* X = V^* Y \otimes^n T^* X.$$  

A Lagrangian system on a fiber bundle $Y \to X$ is said to be a gauge theory if its Lagrangian $L$ admits a family of variational symmetries parameterized by elements of a vector bundle $V \to X$ and its jet manifolds as follows.

Let $\mathfrak{d} \mathcal{O}_\infty^0 Y$ be the $\mathcal{O}_\infty^0 Y$-module of derivations of the $\mathbb{R}$-ring $\mathcal{O}_\infty^0 Y$. Any $\vartheta \in \mathfrak{d} \mathcal{O}_\infty^0 Y$ yields the graded derivation (the interior product) $\vartheta \lrcorner \phi$ of the GDA $\mathcal{O}_\infty^* Y$ given by the relations

$$\vartheta \lrcorner df = \vartheta(f), \quad f \in \mathcal{O}_\infty^0 Y,$$
$$\vartheta \lrcorner (\phi \wedge \sigma) = (\vartheta \lrcorner \phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (\vartheta \lrcorner \sigma), \quad \phi, \sigma \in \mathcal{O}_\infty^* Y,$$

and its derivation (the Lie derivative)

$$L_\vartheta \phi = \vartheta \lrcorner d\phi + d(\vartheta \lrcorner \phi), \quad \phi \in \mathcal{O}_\infty^* Y,$$
$$L_\vartheta (\phi \wedge \phi') = L_\vartheta (\phi) \wedge \phi' + \phi \wedge L_\vartheta (\phi').$$  

Relative to an atlas (9), a derivation $\vartheta \in \mathfrak{d} \mathcal{O}_\infty^0$ reads

$$\vartheta = \vartheta^\lambda \partial_\lambda + \vartheta_i \partial_i + \sum_{|\Lambda| > 0} \vartheta^i_\Lambda \partial^i_\Lambda,$$  

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where \( \{ \partial_\lambda, \partial_\Lambda^i \} \) is the dual to the basis \( \{ dx^\lambda, dy^i_\Lambda \} \) with respect to the interior product \( \lceil [5] \rceil \).

A derivation \( \vartheta \) is called contact if the Lie derivative \( L_\vartheta \) \((13)\) preserves the contact ideal of the GDA \( \mathcal{O}_\infty^* \mathcal{Y} \) generated by contact forms. A derivation \( \nu \) \((14)\) is contact iff

\[
\vartheta^i_\Lambda = d_\Lambda (\vartheta^i - y^i_\mu \vartheta^\mu), \quad 0 < |\Lambda|.
\]

Any contact derivation admits the horizontal splitting

\[
\vartheta = \vartheta^H + \vartheta^V = \vartheta^\lambda d_\lambda + \left( \sum_{0 < |\Lambda|} d_\Lambda \nu^i_\Lambda \partial^i_\Lambda \right), \quad \nu^i = \vartheta^i - y^i_\mu \vartheta^\mu.
\]

Its vertical part \( \vartheta^V \) is completely determined by the first summand

\[
\nu = \nu^i (x^\lambda, y^i_\Lambda) \partial_i, \quad 0 \leq |\Lambda| \leq k.
\]

This is a section of the pull-back \( VY \times J^k Y \to J^k Y \), i.e., a \( k \)-order \( VY \)-valued differential operator on \( Y \). One calls \( \nu \) \((17)\) a generalized vector field on \( Y \).

**Proposition 1.** The Lie derivative of a Lagrangian \( L \) \((10)\) along a contact derivation \( \vartheta \) \((16)\) fulfills the first variational formula

\[
L_\vartheta L = v \lceil [\delta L] + d_H (h_0(\vartheta | \Xi_L)) + \mathcal{L}_\nu (\vartheta^H | \omega),
\]

where \( \Xi_L \) is a Lepagean equivalent of \( L \) \([5]\).

A contact derivation \( \vartheta \) \((16)\) is called variational if the Lie derivative \((18)\) is \( d_H \)-exact, i.e., \( L_\vartheta L = d_H \sigma, \sigma \in \mathcal{O}_0^{0,n-1} \). A glance at the expression \((18)\) shows that: (i) \( \vartheta \) \((16)\) is variational only if it is projected onto \( X \); (ii) \( \vartheta \) is variational iff its vertical part \( \vartheta^V \) is well; (iii) it is variational iff \( v \lceil [\delta L] \) is \( d_H \)-exact.

By virtue of item (ii), we can restrict our consideration to vertical contact derivations \( \vartheta = \vartheta^V \). A generalized vector field \( \nu \) \((17)\) is called a variational symmetry of a Lagrangian \( L \) if it generates a variational contact derivation.

Turn now to the notion of a gauge symmetry \([2]\). Let us consider the bundle product \( E \) \((2)\) coordinated by \((x^\lambda, y^i, \xi^r)\). Given a Lagrangian \( L \) on \( Y \), let us consider its pull-back, say again \( L \), onto \( E \). Let \( \vartheta_E \) be a contact derivation of the \( \mathbb{R} \)-ring \( \mathcal{O}_\infty^0 E \), whose restriction

\[
\vartheta = \vartheta_E | \mathcal{O}_\infty^0 Y = \sum_{0 \leq |\Lambda|} d_\Lambda \nu^i_\Lambda \partial^i_\Lambda
\]

to \( \mathcal{O}_\infty^0 Y \subset \mathcal{O}_\infty^0 E \) is linear in coordinates \( \xi^r_\Xi \). It is determined by a generalized vector field \( \nu_E \) on \( E \) whose projection

\[
\nu : J^k E \xrightarrow{\nu_E} VE \to E \times VY
\]
is a linear \( VY \)-valued differential operator \( \upsilon \) on \( E \). Let \( \vartheta_E \) be a variational symmetry of a Lagrangian \( L \) on \( E \), i.e.,

\[
\upsilon_E \delta L = \upsilon \delta L = d_H \sigma.
\]

Then one says that \( \upsilon \) is a gauge symmetry of a Lagrangian \( L \).

Note that any differential operator \( \upsilon \) defines a generalized vector field \( \upsilon_E = \upsilon \) on \( E \) which lives in \( VY \) and, consequently, generates a contact derivation \( \vartheta_E = \vartheta \).

### 3 Graded Lagrangian systems

In order to introduce a BRST operator, let us consider Lagrangian systems of even and odd variables. We describe odd variables and their jets on a smooth manifold \( X \) as generating elements of the structure ring of a graded manifold whose body is \( X \). This definition reproduces the heuristic notion of jets of ghosts in the field-antifield BRST theory.

Recall that any graded manifold \( (\mathfrak{A}, X) \) with a body \( X \) is isomorphic to the one whose structure sheaf \( \mathfrak{A}_Q \) is formed by germs of sections of the exterior product

\[
\wedge Q^* = \mathbb{R} \oplus Q^* \oplus \mathbb{R} \frac{1}{2} Q^* \oplus \cdots,
\]

where \( Q^* \) is the dual of some real vector bundle \( Q \to X \) of fiber dimension \( m \). In field models, a vector bundle \( Q \) is usually given from the beginning. Therefore, we consider graded manifolds \( (X, \mathfrak{A}_Q) \) where the above mentioned isomorphism holds, and call \( (X, \mathfrak{A}_Q) \) the simple graded manifold constructed from \( Q \). The structure ring \( \mathfrak{A}_Q \) of sections of \( \mathfrak{A}_Q \) consists of sections of the exterior bundle (21) called graded functions. Given bundle coordinates \((x^\lambda, q^a)\) on \( Q \) with transition functions \( q^b_a = \rho^a_b q^b \), let \( \{c^a\} \) be the corresponding fiber bases for \( Q^* \to X \), together with transition functions \( c^a_b = \rho^a_b c^b \). Then \((x^\lambda, c^a)\) is called the local basis for the graded manifold \((X, \mathfrak{A}_Q)\). With respect to this basis, graded functions read

\[
f = \sum_{k=0}^m \frac{1}{k!} f_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k},
\]

where \( f_{a_1 \cdots a_k} \) are local smooth real functions on \( X \).

Given a graded manifold \((X, \mathfrak{A}_Q)\), let \( \mathfrak{dA}_Q \) be the \( \mathfrak{A}_Q \)-module of \( \mathbb{Z}_2 \)-graded derivations of the \( \mathbb{Z}_2 \)-graded ring of \( \mathfrak{A}_Q \), i.e.,

\[
u(f f') = u(f)f' + (-1)^{|u||f|} f u(f'), \quad u \in \mathfrak{dA}_Q, \quad f, f' \in \mathfrak{A}_Q,
\]

where \([,]\) denotes the Grassmann parity. Its elements are called \( \mathbb{Z}_2 \)-graded (or, simply, graded) vector fields on \((X, \mathfrak{A}_Q)\). Due to the canonical splitting \( VQ = Q \times Q \), the vertical tangent bundle \( VQ \to Q \) of \( Q \to X \) can be provided with the fiber bases \( \{\partial_a\} \) which is the
are local graded functions. It acts on $A$ by the rule

$$u(f_{a\ldots b}c^a\cdots c^b) = u^\lambda \partial_\lambda (f_{a\ldots b}) c^a\cdots c^b + u^d f_{a\ldots b} \partial_d (c^a\cdots c^b). \quad (22)$$

This rule implies the corresponding transformation law

$$u^\lambda = u^\lambda, \quad u^a = \rho^a \partial_\lambda (\rho^a) c^\lambda.$$  

Then one can show [7, 8] that graded vector fields on a simple graded manifold can be represented by sections of the vector bundle $\mathcal{V}_Q \to X$ which is locally isomorphic to the vector bundle $\wedge Q^* \otimes_X (Q \otimes_X T_X)$.

Using this fact, we can introduce graded exterior forms on the graded manifold $(X, \mathfrak{A}_Q)$ as sections of the exterior bundle $\wedge \mathcal{V}_Q^*$, where $\mathcal{V}_Q^* \to X$ is the $\wedge Q^*$-dual of $\mathcal{V}_Q$. Relative to the dual local bases $\{dx^a\}$ for $T^*X$ and $\{dc^b\}$ for $Q^*$, graded one-forms read

$$\phi = \phi_\lambda dx^\lambda + \phi_a dc^a, \quad \phi'_a = \rho^{-1b} \phi_b, \quad \phi'_\lambda = \phi_\lambda + \rho^{-1b} \partial_\lambda (\rho^b) \phi_b c^\lambda.$$  

The duality morphism is given by the interior product

$$u \cdot \phi = u^\lambda \phi_\lambda + (-1)^{[\phi]} |u^a \phi_a.$$  

Graded exterior forms constitute the bigraded differential algebra (henceforth BGDA) $C^*_Q$ with respect to the bigraded exterior product $\wedge$ and the exterior differential $d$. The standard formulae of a BGDA hold.

Since the jet bundle $J^r Q \to X$ of a vector bundle $Q \to X$ is a vector bundle, let us consider the simple graded manifold $(X, \mathfrak{A}_{J^r Q})$ constructed from $J^r Q \to X$. Its local basis is $\{x^\lambda, c^a_{\lambda}\}$, $0 \leq |\lambda| \leq r$, together with the transition functions

$$c^a_{\lambda+\Lambda} = d_\lambda (\rho^a_{\lambda} c^\Lambda_{\lambda}), \quad d_\lambda = \partial_\lambda + \sum_{|\lambda| < r} c^a_{\lambda+\Lambda} \partial_a^\Lambda, \quad (23)$$

where $\partial_a^\Lambda$ are the duals of $c^a_{\lambda}$. Let $C^*_{J^r Q}$ be the BGDA of graded exterior forms on the graded manifold $(X, \mathfrak{A}_{J^r Q})$. A linear bundle morphism $\pi_{r-1}^{-1} : J^r Q \to J^{r-1} Q$ yields the corresponding monomorphism of BGDAs $C^*_{J^{r-1} Q} \to C^*_{J^r Q}$. Hence, there is the direct system of BGDAs

$$C^*_Q \xrightarrow{\pi^*_1} C^*_Q \xrightarrow{\pi^*_2} \cdots \xrightarrow{\pi^*_r} C^*_Q \to \cdots \quad (24)$$

Its direct limit $C^\infty_Q$ consists of graded exterior forms on graded manifolds $(X, \mathfrak{A}_{J^r Q})$, $r \in \mathbb{N}$, modulo the pull-back identification, and it inherits the BGDA operations intertwined by the monomorphisms $\pi^{-1}_{r-1}$. It is a $C^\infty(X)$-algebra locally generated by the elements $(1, c^a_{\lambda}, dx^\lambda, \theta^a_\Lambda = dc^a_{\lambda} - c^a_{\lambda+\Lambda} dx^\lambda)$, $0 \leq |\lambda|$.
In order to regard even and odd dynamic variables on the same footing, let $Y \to X$ be hereafter an affine bundle, and let $\mathcal{P}_\infty^* Y \subset \mathcal{O}_\infty^* Y$ be the $C^\infty(X)$-subalgebra of exterior forms whose coefficients are polynomial in the fiber coordinates $y^i_\Lambda$ on jet bundles $J^rY \to X$. Let us consider the product

$$S_\infty^* = C_\infty^* Q \wedge \mathcal{P}_\infty^* Y \quad (25)$$

of graded algebras $C_\infty^* Q$ and $\mathcal{P}_\infty^* Y$ over their common graded subalgebra $\mathcal{O}_X^*$ of exterior forms on $X$ [5]. It consists of the elements

$$\sum_i \psi_i \otimes \phi_i, \quad \sum_i \phi_i \otimes \psi_i, \quad \psi \in C_\infty^* Q, \quad \phi \in \mathcal{P}_\infty^* Y,$$

modulo the commutation relations

$$\psi \otimes \phi = (-1)^{\psi|\phi} \phi \otimes \psi, \quad \psi \in C_\infty^* Q, \quad \phi \in \mathcal{P}_\infty^* Y;$$

$$\begin{aligned}
(\psi \wedge \sigma) \otimes \phi &= \psi \otimes (\sigma \wedge \phi), \\
\sigma &\in \mathcal{O}_X^*.
\end{aligned} \quad (26)$$

They are endowed with the total form degree $|\psi| + |\phi|$ and the total Grassmann parity $[\psi]$. Their multiplication

$$(\psi \otimes \phi) \wedge (\psi' \otimes \phi') := (-1)^{\psi'|\phi} (\psi \wedge \psi') \otimes (\phi \wedge \phi').$$

obeys the relation

$$\begin{aligned}
\varphi \wedge \varphi' &= (-1)^{|\varphi|+|\varphi'|} \varphi' \wedge \varphi, \\
\varphi, \varphi' &\in S_\infty^*,
\end{aligned}$$

and makes $S_\infty^*$ (25) into a bigraded $C^\infty(X)$-algebra. For instance, elements of the ring $S_\infty^0$ are polynomials of $c^a_\Lambda$ and $y^i_\Lambda$ with coefficients in $C_\infty^*(X)$.

The algebra $S_\infty^*$ is provided with the exterior differential

$$d(\psi \otimes \phi) := (d_C \psi) \otimes \phi + (-1)^{\psi} \psi \otimes (d_P \phi), \quad \psi \in C_\infty^*, \quad \phi \in \mathcal{P}_\infty^*, \quad (28)$$

where $d_C$ and $d_P$ are exterior differentials on the differential algebras $C_\infty^* Q$ and $\mathcal{P}_\infty^* Y$, respectively. It obeys the relations

$$d(\varphi \wedge \varphi') = d\varphi \wedge \varphi' + (-1)^{|\varphi|} \varphi \wedge d\varphi', \quad \varphi, \varphi' \in S_\infty^*,$$

and makes $S_\infty^*$ into a BGDA, which is locally generated by the elements

$$(1, c^a_\Lambda, \dot{y}^i_\Lambda, d x^\lambda, \theta^a_\Lambda = d c^a_\Lambda - c^a_{\lambda+\Lambda} d x^\lambda, \theta^i_\Lambda = d y^i_\Lambda - y^i_{\lambda+\Lambda} d x^\lambda), \quad 0 \leq |\Lambda|. \quad (29)$$

Hereafter, let the collective symbols $s^a_\Lambda$ and $\theta^A_\Lambda$ stand both for even and odd generating elements $c^a_\Lambda$, $y^i_\Lambda$, $\theta^a_\Lambda$, $\theta^i_\Lambda$ of the $C^\infty(X)$-algebra $S_\infty^*$ which, thus, is locally generated by $(1, s^A_\Lambda, d x^\lambda, \theta^A_\Lambda)$, $|\Lambda| \geq 0$. We agree to call elements of $S_\infty^*$ the graded exterior forms on $X$. 




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Similarly to $O_\infty^*Y$, the BGDA $S^*_\infty$ is decomposed into $S_{\infty}^0$-modules $S_{\infty}^{k,r}$ of $k$-contact and $r$-horizontal graded forms together with the corresponding projections $h_k$ and $h^r$. Accordingly, the exterior differential $d$ on $S^*_\infty$ is split into the sum $d = d_H + d_V$ of the total and vertical differentials

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \quad d_V(\phi) = \theta_A^\lambda \wedge \partial_\lambda^A \phi, \quad \phi \in S^*_\infty.$$

One can think of the elements

$$L = L\omega \in S_{\infty}^{0,n}, \quad \delta(L) = \sum_{|A|\geq 0} (-1)^{|A|} \theta_A^\lambda \wedge d_\lambda(\partial_\lambda^A L) \in S_{\infty}^{0,n}$$

as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

### 4 BRST symmetry

A graded derivation $\vartheta \in \mathfrak{d}S^0_\infty$ of the $\mathbb{R}$-ring $S^0_\infty$ is said to be contact if the Lie derivative $L_\vartheta$ preserves the ideal of contact graded forms of the BGDA $S^*_\infty$. With respect to the local basis $(x^\lambda, s^A_\lambda, dx^\lambda, \theta_A^\lambda)$ for the BGDA $S^*_\infty$, any contact graded derivation takes the form

$$\vartheta = \vartheta_H + \vartheta_V = \vartheta^\lambda d_\lambda + (\vartheta^A \partial_A + \sum_{|A|>0} d_\lambda \vartheta^A \partial_\lambda^A), \quad \text{(29)}$$

where $\vartheta^\lambda$, $\vartheta^A$ are local graded functions [5]. The interior product $\vartheta | \phi$ and the Lie derivative $L_\vartheta \phi$, $\phi \in S^*_\infty$, are defined by the same formulae

$$\vartheta | \phi = \vartheta^\lambda \phi_\lambda + (-1)^{|\vartheta|\phi} \vartheta^A \phi_A, \quad \phi \in S^1_\infty,$n
$$\vartheta | (\phi \wedge \sigma) = (\vartheta | \phi) \wedge \sigma + (-1)^{|\vartheta|\phi} (\vartheta | \sigma) \wedge (\vartheta | \phi), \quad \phi, \sigma \in S^*_\infty.$n
$$L_\vartheta \phi = \vartheta | d_\phi + d(\vartheta | \phi), \quad L_\vartheta (\phi \wedge \sigma) = L_\vartheta (\phi) \wedge \sigma + (-1)^{|\vartheta|\phi} \phi \wedge L_\vartheta (\sigma).$$

as those on a graded manifold. One can justify that any vertical contact graded derivation $\vartheta$ (29) satisfies the relations

$$\vartheta | d_H \phi = -d_H (\vartheta | \phi), \quad L_\vartheta (d_H \phi) = d_H (L_\vartheta \phi), \quad \phi \in S^*_\infty. \quad \text{(30)}$$

**Proposition 2.** The Lie derivative $L_\vartheta L$ of a Lagrangian $L$ along a contact graded derivation $\vartheta$ (29) fulfills the first variational formula

$$L_\vartheta L = \vartheta_V | \delta L + d_H (h_0 (\vartheta | \Xi_L)) + d_V (\vartheta_H | \omega) L, \quad \text{(31)}$$

where $\Xi_L = \Xi + L$ is a Lepagean equivalent of a graded Lagrangian $L$ [5].
A contact graded derivation \( \vartheta \) is said to be variational if the Lie derivative (31) is \( d_H \)-exact. A glance at the expression (31) shows that: (i) A contact graded derivation \( \vartheta \) is variational only if it is projected onto \( X \), and (ii) \( \vartheta \) is variational iff its vertical part \( \vartheta_V \) is well. Therefore, we restrict our consideration to vertical contact graded derivations

\[
\vartheta = \sum_{0 \leq |\Lambda|} d_{\Lambda} v^{A} \partial^A_{\Lambda}.
\] (32)

Such a derivation is completely defined by its first summand

\[
v = v^{A}(x^\lambda, s^{A}_A) \partial_A, \quad 0 \leq |\Lambda| \leq k,
\] (33)

which is also a graded derivation of \( \mathcal{S}_{\infty}^0 \). It is called the generalized graded vector field. A glance at the first variational formula (31) shows that \( \vartheta (32) \) is variational iff \( v \right| \delta L \) is \( d_H \)-exact.

A vertical contact graded derivation \( \vartheta (32) \) is said to be nilpotent if

\[
\mathbf{L}_v (\mathbf{L}_v \phi) = \sum_{|\Sigma| \geq 0, |\Lambda| \geq 0} (v^{A}_B \partial^\Sigma_B (v^{A}_A) \partial^A_{\Lambda} + (-1)^{|\Lambda|} [v^A_B]_s v^{A}_A \partial^\Sigma_B \partial^A_{\Lambda}) \phi = 0
\] (34)

for any horizontal graded form \( \phi \in \mathcal{S}_{\infty}^{0,*} \) or, equivalently, \( \vartheta \circ \vartheta)(f) = 0 \) for any graded function \( f \in \mathcal{S}_{\infty}^0 \). One can show that \( \vartheta \) is nilpotent only if it is odd and iff the equality

\[
\vartheta(v^{A}) = \sum_{|\Sigma| \geq 0} v^{A}_B \partial^\Sigma_B (v^{A}) = 0
\] (35)

holds for all \( v^{A} \) [5].

Return now to the original gauge system on a fiber bundle \( Y \) with a Lagrangian \( L \) (10) and a gauge symmetry \( v \) (3). For the sake of simplicity, \( Y \to X \) is assumed to be affine. Let us consider the BGDA \( \mathcal{S}_{\infty}^* = C_{\infty}^* V \land \mathcal{P}_{\infty}^0 Y \) locally generated by \( (1, e^r, dx^\lambda, y^i, \theta^r, \theta^i) \). Let \( L \in \mathcal{O}_{\infty}^{0,n} Y \) be a polynomial in \( y^i_{\Lambda}, 0 \leq |L| \). Then it is a graded Lagrangian \( L \in \mathcal{P}_{\infty}^{0,n} Y \subset \mathcal{S}_{\infty}^{0,n} \) in \( \mathcal{S}_{\infty}^* \). Its gauge symmetry \( v \) (3) gives rise to the generalized vector field \( v_E = v \) on \( E \), and the latter defines the generalized graded vector field \( v \) (33) by the formula (5). It is easily justified that the contact graded derivation \( \vartheta \) (32) generated by \( v \) (5) is variational for \( L \). It is odd, but need not be nilpotent. However, one can try to find a nilpotent contact graded derivation (32) generated by some generalized graded vector field (6) which coincides with \( \vartheta \) on \( \mathcal{P}_{\infty}^{0,n} Y \). We agree to call it the BRST operator.

In this case, the nilpotency conditions (35) read

\[
\sum_{\Sigma} d_{\Sigma} \left( \sum_{\Xi} v^{i,\Xi}_{r} e^r_{\Xi} \right) \sum_{\Lambda} \partial_i^\Sigma (v^{j,\Lambda}_{s} e^s_{\Lambda}) c^r_{\Lambda} + \sum_{\Lambda} d_{\Lambda}(u^r) v^{j,\Lambda}_{r} = 0,
\] (36)

\[
\sum_{\Lambda} \sum_{\Xi} d_{\Lambda}(u^r) e^r_{\Xi} \partial_i^\Lambda + d_{\Lambda}(u^r) \partial_r^\Lambda) u^q = 0
\] (37)
for all indices \( j \) and \( q \). They are equations for graded functions \( u^r \in \mathcal{S}_0^\infty \). Since these functions are polynomials

\[
\begin{align*}
  u^r &= u^r_{(0)} + \sum_{\Gamma} u^r_{(1)p} c^p_T + \sum_{\Gamma_1, \Gamma_2} u^r_{(2)p_1p_2} c^p_1 c^p_2 + \cdots 
\end{align*}
\]

in \( c^s_\Lambda \), the equations (36) – (37) take the form

\[
\begin{align*}
  \sum_{\Sigma} d_{\Sigma}(\sum_{\Xi} v^r_{i_r} e^r_{\Xi}) \sum_{\Lambda} \partial_i^{\Sigma} (v^j_{s_r} c^s_\Lambda) c^s_\Lambda + \sum_{\Lambda} d_\Lambda(u^r_{(2)}) v^j_{s_r} &= 0, \\
  \sum_{\Lambda} d_\Lambda(u^r_{(k\neq 2)}) v^j_{s_r} &= 0, \\
  \sum_{\Sigma} d_\Lambda(v^r_{i_r} e^r_{\Xi}) \partial_i^{\Lambda} u^q_{(k-1)} + \sum_{m+n-1=k} d_\Lambda(u^r_{(m)}) \partial_i^{\Lambda} u^q_{(n)} &= 0.
\end{align*}
\]

If the equations (39) – (41) have a solution, i.e., the (nilpotent) BRST operator exists, one can think of the equalities (39) and (41) (and, consequently, the nilpotency conditions (36) – (37)) as being the generalized commutation relations and generalized Jacobi identities of gauge transformations, respectively.

Indeed, the relation (39) for components \( v^i_r \) takes the form of the familiar Lie bracket

\[
\sum_{\Sigma} [d_{\Sigma}(v^i_r) \partial_i^{\Sigma} v^j_q - d_{\Sigma}(v^j_q) \partial_i^{\Sigma} v^i_p] = -2u^r_{(2)pq} v^j_r,
\]

where \(-2u^r_{(2)pq}\) are generalized structure constants depending on dynamic variables \( y^i \) and their jets \( y^i_\Lambda \) in general. For instance, let us assume that all \( v^i_r \) are linear in \( y^i_\Lambda \). Then \( u^r_{(2)pq} \) are independent of these variables. Let \( u^r_{(m\neq 2)} = 0 \). In this case, the relation (41) reduces to the familiar Jacobi identity

\[
u^r_{(2)pq} u^j_{(2)rs} + u^r_{(2)qs} u^j_{(2)rp} + u^r_{(2)sp} u^j_{(2)rq} = 0.
\]

Let us note that any Lagrangian \( L \) have gauge symmetries. In particular, there always exist trivial gauge symmetries

\[
u = \sum_{\Lambda} T^j_{r,i} \mathcal{E}^r_j \mathcal{E}^r_i \partial_i, \quad T^j_{r,i} = -T^j_{r,i}.
\]

vanishing on-shell. In a general setting, one therefore can require that the nilpotency conditions (36) – (37) hold on-shell, i.e., gauge transformations form an algebra on-shell.

5 Example

Let us consider the gauge theory of principal connections on a principal bundle \( P \rightarrow X \) with a structure Lie group \( G \). These connections are represented by sections of the quotient

\[
C = J^1 P/G \rightarrow X.
\]
This is an affine bundle coordinated by \((x^\lambda, a^r_\lambda)\) such that, given a section \(A \rightarrow X\), its components \(A^r_\lambda = a^r_\lambda \circ A\) are coefficients of the familiar local connection form (i.e., gauge potentials). Let \(J^\infty C\) be the infinite order jet manifold of \(C \rightarrow X\) coordinated by \((x^\lambda, a^r_\lambda\Lambda\), \(0 \leq |\Lambda|\). We consider the GDA \(O^*_\infty C\).

Infinitesimal generators of one-parameter groups of automorphisms of a principal bundle \(P\) are \(G\)-invariant projectable vector fields on \(P \rightarrow X\). They are associated to sections of the vector bundle \(T_G P = TP/G \rightarrow X\). This bundle is provided with the coordinates\((x^\lambda, \hat{x}^\lambda, \xi^r)\) with respect to the fibre bases \(\{\partial_\lambda, e_r\}\) for \(T_G P\), where \(\{e_r\}\) is the basis for the right Lie algebra \(g\) of \(G\) such that \([e_p, e_q] = c_r^{pq} e_r\).

If \(u = u^\lambda \partial_\lambda + u^r e_r, \quad v = v^\lambda \partial_\lambda + v^r e_r, \quad (43)\)
are sections of \(T_G P \rightarrow X\), their bracket reads
\[ [u, v] = (u^\mu \partial_\mu v^\lambda - v^\mu \partial_\mu u^\lambda)\partial_\lambda + (u^\lambda \partial_\lambda v^r - v^\lambda \partial_\lambda u^r + c_r^{pq} u^p v^q) e_r. \quad (44)\]

Any section \(u\) of the vector bundle \(T_G P \rightarrow X\) yields the vector field
\[ u_C = u^\lambda \partial_\lambda + (c_r^{pq} a^p_\lambda u^q + \partial_\lambda u^r - a^r_\mu \partial_\lambda u^\mu)\partial_r \quad (45)\]
on the bundle of principal connections \(C\) \((42)\). It is an infinitesimal generator of a one-parameter group of automorphisms of \(C\) \([7]\). Let us consider the bundle product
\[ E = C \times T_G P, \quad (46)\]
coordinated by \((x^\lambda, \tau^\lambda = \hat{x}^\lambda, \xi^r, a^r_\lambda)\). It can be provided with the generalized vector field
\[ v_E = v = (c_r^{pq} a^p_\lambda \xi^q + \xi^r - a^r_\mu \tau^\mu - \tau^\mu a^r_\mu)\partial_r. \quad (47)\]

Following the procedure in Sections 3 – 4, we replace parameters \(\xi^r\) and \(\tau^\lambda\) with the odd ghosts \(c^r\) and \(c^\lambda\), respectively, and obtain the generalized graded vector field
\[ v = (c_r^{pq} a^p_\lambda c^q + c^r_\lambda - a^r_\mu c^\mu - c^\mu a^r_\mu)\partial_r + \left(-\frac{1}{2} e_r^{pq} c^p c^q - c^r e^r_\mu\right) \partial_r + c^r_\lambda e^\mu \partial_\lambda \quad (48)\]
such that the vertical contact graded derivations \((32)\) generated by \(v\) \((48)\) is nilpotent, i.e., it is a BRST operator.

**References**

[1] G.Barnich, F.Brandt and M.Henneaux, Local BRST cohomology in gauge theories, *Phys. Rep.* 338, 439 (2000).
[2] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, Noether’s second theorem in a general setting. Reducible gauge theories, *E-print arXiv*: math.DG/0411070.

[3] F.Brandt, Jet coordinates for local BRST cohomology, *Lett. Math. Phys.* 55, 149 (2001).

[4] R.Fulp, T.Lada and J.Stasheff, Sh-Lie algebras induced by gauge transformations, *Comm. Math. Phys.* 231, 25 (2002); *E-print arXiv*: math.QA/0012106.

[5] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology, *Commun. Math. Phys.* (accepted); *E-print arXiv*: hep-th/0407185.

[6] J.Gomis, J.París, J. and S.Samuel, Antibracket, antifields and gauge theory quantization, *Phys. Rep* 295, 1 (1995).

[7] L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).

[8] G.Sardanashvily, SUSY-extended field theory, *Int. J. Mod. Phys. A* 15, 3095 (2000).