COMPRESSION OF UNIFORM EMBEDDINGS INTO HILBERT SPACE

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Abstract. If one tries to embed a metric space uniformly in Hilbert space, how close to quasi-isometric could the embedding be? We answer this question for finite dimensional CAT(0) cube complexes and for hyperbolic groups. In particular, we show that the Hilbert space compression of any hyperbolic group is 1.

1. Introduction

The study of uniform embeddings of metric spaces into a Hilbert space was originated by Gromov [10].

Definition 1.1. Let \((X, d)\) be a metric space and \(\mathcal{H}\) be a Hilbert space. A map \(f: X \to \mathcal{H}\) is said to be a uniform embedding if there exist non-decreasing functions \(\rho, \delta: \mathbb{R}_+ \to \mathbb{R}_+\) such that

1. \(\rho(d_X(x, y)) \leq d_\mathcal{H}(f(x), f(y)) \leq \delta(d_X(x, y))\) for all \(x, y \in X\);
2. \(\lim_{t \to +\infty} \rho(t) = +\infty\).

Gromov asked whether a finitely generated group (viewed as a metric space with a word length metric) that is uniformly embeddable into a Hilbert space satisfies the Novikov Conjecture. Yu [17] answered this question affirmatively, showing that for a finitely generated group equipped with a word length metric, uniform embeddability into a Hilbert space implies the Coarse Baum-Connes Conjecture and the Novikov Conjecture. Uniform embeddability of finitely generated groups into a Hilbert space has been studied extensively since then (see [7, 11, 13, 15, 16, 17] and references therein).

The definition above suggests that there exist two useful real-valued functions associated to the map \(f\): its dilatation
\[\delta_f(t) = \sup\{d_\mathcal{H}(f(x), f(y)) \mid d_X(x, y) \leq t\}\]
and its compression
\[\rho_f(t) = \inf\{d_\mathcal{H}(f(x), f(y)) \mid d_X(x, y) \geq t\}.

If \(X\) is a quasi-geodesic metric space (in particular, a finitely generated group with a word metric), then the dilatation of its uniform embedding in a Hilbert space is dominated by a linear function [10, 12]. We consider the following question: how close to a linear function can the compression of a uniform embedding of a group be?
into a Hilbert space be? In other words, how close to a quasi-isometric embedding can a uniform embedding be?

We are interested in asymptotic behavior of the compression function. Therefore it is convenient to introduce a (partial) relation on functions $g, h: \mathbb{R}_+ \to \mathbb{R}_+$ as follows: we write $f \preceq g$ if there exist numbers $C$ and $M$ such that $f(t) \leq C g(t)$ for all $t \geq M$. We write $f \sim g$ if $f \preceq g \preceq f$.

Guentner and Kaminker [12] introduced a geometric invariant $R(G)$ of a finitely generated group $G$ called Hilbert space compression. $R(G)$ is the supremum of all numbers $\alpha \geq 0$ for which there exists a uniform embedding $f: G \to \mathcal{K}$ with the compression $\rho_f(t) \geq t^\alpha$. This invariant is a number between 0 and 1 and it parameterizes the difference between $G$ being uniformly embeddable in a Hilbert space and being exact [12].

It was shown in [12] is $R(F_2) = 1$ where $F_2$ is a free group on two generators. The authors proved it by providing a sequence of uniform embeddings $f_n: F_2 \to \mathcal{K}$ with $\rho_{f_n}(t) \geq t^{1 - \frac{1}{n}}$. A question remained of whether there is one uniform embedding $f: F_2 \to \mathcal{K}$ with $\rho_f(t) \geq t^{1 - \frac{1}{n}}$ for all $n$. We construct such an embedding in Section 2 by modifying the construction from [12]. Moreover we estimate the compression function of our embedding as

$$\rho_f(t) \geq \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}$$

It follows from results of Bourgain [3] that for any uniform embedding $f: F_2 \to \mathcal{K}$ we have

$$\rho_f(t) \preceq \frac{t}{\sqrt{\ln t}}.$$

It is not known whether there exists a uniform embedding of $F_2$ in $\mathcal{K}$ with the compression function $\sim \frac{t}{\sqrt{\ln t}}$.

We show in Section 3 that any hyperbolic group embeds uniformly in a Hilbert space with the compression function $\rho_f(t) \geq \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}$. In particular, the Hilbert space compression of any hyperbolic group is equal to 1. Notice that for any uniform embedding of a non-elementary hyperbolic group in a Hilbert space we have $\rho_f(t) \preceq \frac{t}{\sqrt{\ln t}}$.

Finally, Section 4 is devoted to uniform embeddings of groups acting on CAT(0) cubical complexes. It was proven by Campbell andNiblo [5] that any discrete group $G$ acting properly, co-compactly on a finite dimensional CAT(0) cubical complex has Hilbert space compression $R(G) = 1$. They provided a sequence $f_n$ of uniform embeddings with compressions $\rho_{f_n}(t) \geq t^{1 - \frac{1}{n}}$. We modify their construction to embed such a group uniformly in a Hilbert space with compression $\rho_f(t) \geq \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}$. It is proved recently by Sageev and Wise [16] that groups acting properly and co-compactly on finite dimensional CAT(0) cubical complexes satisfy the Tits alternative provided the orders of finite subgroups are uniformly bounded. We note that for such a group $G$ either a quasi-isometric embedding in a Hilbert space exists or any uniform embedding of $G$ in Hilbert space has compression $\rho_f(t) \preceq \frac{t}{\sqrt{\ln t}}$.

The authors wish to thank Mark Sapir, Graham Niblo, Alexander Ol’shanskii, Denis Osin, Michah Sageev and Sergey Borodachov for interesting conversations during the course of this research.
2. Uniform embeddings of trees

All trees considered in this section are locally finite, with geodesic metric and with all edges of length 1.

We start with an observation (for the proof see, for example, [12, Proposition 2.9]).

**Lemma 2.1.** Let $X$ and $Y$ be metric spaces, and assume that $X$ is geodesic. If $f: X \to Y$ is a uniform embedding, then $\delta_f(t) \leq C \cdot t$ for some constant $C > 0$.

Results of Bourgain [3] (see [13] for the theorem we use and its short proof) imply the following

**Theorem 2.2.** For any uniform embedding $f$ of a complete infinite binary tree into a Hilbert space,

$$\rho_f(t) \leq \frac{\delta_f(t)}{\sqrt{\ln t}}.$$  

Since complete infinite binary tree embeds isometrically into the Cayley graph of the free group $F_2$ relative to the standard generating set, in view of Lemma 2.1 we have

**Corollary 2.3.** If $f: F_2 \to \mathcal{H}$ is a uniform embedding, then $\rho_f(t) \leq \frac{t}{\sqrt{\ln t}}$.

In particular,

**Corollary 2.4.** There is no quasi-isometric embedding of the free group $F_2$ into Hilbert space.

We need the following technical lemma.

**Lemma 2.5.** Let $\xi(t) = \frac{\sqrt{t}}{\sqrt{\ln t \cdot \ln \ln t}}$. Then

1. $\sum_{j=2}^{\infty} (\xi(j+1) - \xi(j))^2 < \infty$;
2. there exists a constant $C$ such that for any large enough integer $N$,

$$\sum_{i=1}^{N} \xi^2(i) \geq \frac{1}{2} N \cdot \xi^2(N) - C$$

**Proof.** (1) It is enough to estimate the sum starting with $j = 4$:

$$\sum_{j=4}^{\infty} (\xi(j+1) - \xi(j))^2 = \sum_{j=4}^{\infty} \int_{j}^{j+1} [\xi'(t)]^2 dt \leq \sum_{j=4}^{\infty} \int_{j}^{j+1} \frac{1}{t \cdot \ln t} \cdot (\ln \ln t)^2 dt = \int_{4}^{\infty} [\xi'(t)]^2 dt$$

In view of the inequality $\xi'(t) \leq \frac{\xi(t)}{t \cdot \ln t}$ we get

$$\int_{4}^{\infty} [\xi'(t)]^2 dt \leq \int_{\ln \ln 4}^{\infty} \frac{1}{u^2} \cdot \frac{1}{t \cdot \ln t \cdot (\ln \ln t)^2} dt = \int_{\ln \ln 4}^{\infty} \frac{du}{u^2} < \infty$$

(2) One can check that there exists a constant $C$ such that for any large enough integer $N$, $\xi(t)$ is increasing for $t \geq M$. Therefore, for any $N \geq M$,

$$\sum_{i=1}^{N} \xi^2(i) \geq \sum_{i=1}^{M} \xi^2(i) + \int_{M}^{N} \xi^2(t) dt$$
The inequality $\xi^2(t) \geq \frac{1}{2}(t \cdot \xi^2(t))'$ implies that

$$\int_M^N \xi^2(t) \, dt \geq \int_M^N \frac{1}{2}(t \cdot \xi^2(t))' \, dt = \frac{1}{2} N \cdot \xi^2(N) - \frac{1}{2} M \cdot \xi^2(M)$$

\[ \square \]

**Theorem 2.6.** Let $T$ be a locally finite tree with geodesic metric and all edges of length 1. There exists a uniform embedding $f$ of $T$ into Hilbert space $\mathcal{H}$ with

$$\rho_f(t) \geq \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}$$

**Proof.** We are going to define our embedding on vertices of $T$ and extend it linearly to edges. Let us fix a basis in the space $\mathcal{H}$ and enumerate the elements of this basis by all edges of the tree $T$. If $v$ is an edge of the tree, we denote by $\vec{e}_v$ the corresponding unit vector of the basis.

Fix a base vertex $O$ of $T$. For any vertex $V$ consider the geodesic path $[VO]$ joining $V$ and $O$. Let us enumerate the edges of $[VO]$ by $v_1, v_2, ..., v_{\|V\|}$ in the order from $V$ to $O$, where $\|V\|$ the length of $[VO]$. Fix a weight function

$$\xi(t) = \frac{\sqrt{t}}{\sqrt{\ln t \cdot \ln \ln t}}$$

and define the embedding $f$ on the vertex $V$ as

$$f(V) = \sum_{i=1}^{\|V\|} \xi(i) \cdot \vec{e}_{v_i}.$$ 

In order to show that $f$ is a uniform embedding we need to estimate its dilatation $\delta_f$ from above and its compression $\rho_f$ from below.

First we show that $\delta_f(t) \leq t$. To prove this we show that there is a constant $C$ such that the expansion of any edge of $T$ under the map $f$ does not exceed $C$. Let $U$ and $V$ be adjacent vertices of the tree $T$. We are going to find a constant $C$ such that $d(f(U), f(V)) \leq C \cdot d(U, V) = C$. Recall that we denote by $v_1, v_2, ..., v_{\|V\|}$ (respectively by $u_1, u_2, ..., u_{\|U\|}$) the edges of the path $[VO]$ (respectively $[UO]$) in the order from $V$ (resp. $U$) to $O$. We may assume that $\|U\| = \|V\| - 1$ and therefore $u_i = v_{i+1}$ for every $i = 1, ..., \|U\|$. Since

$$f(V) - f(U) = \sum_{i=1}^{\|V\|} \xi(i) \cdot \vec{e}_{v_i} - \sum_{j=1}^{\|U\|} \xi(j) \cdot \vec{e}_{u_j} = \xi(1) \cdot \vec{e}_{v_1} + \sum_{j=1}^{\|U\|} (\xi(j+1) - \xi(j)) \vec{e}_{u_j},$$

we have

$$d(f(U), f(V)) = \sqrt{\xi^2(1) + \sum_{j=1}^{\|U\|} (\xi(j+1) - \xi(j))^2}$$

and the existence of the constant $C$ follows from part (1) of Lemma 2.5.

Now we estimate the function $\rho_f$. Let $U$ and $V$ be any vertices of the tree $T$ and $S$ be the vertex of $T$ such that $[UO] \cap [VO] = [SO]$. Then the path from $U$ to $V$ is a union of paths from $U$ to $S$ and from $S$ to $V$. Without loss of generality we assume that $|SV| \geq \frac{1}{2}|UV|$. 


We estimate \(d(f(U), f(V))\) by looking at coordinates corresponding to edges of the path from \(V\) to \(S\) (these are edges \(v_1, v_2, ..., v_{|SV|}\)). These coordinates of the point \(f(U)\) are all 0. So,

\[
[d(f(U), f(V))]^2 \geq \sum_{i=1}^{|SV|} \xi^2(i).
\]

Part (2) of Lemma 2.5 implies

\[
\sum_{i=1}^{|SV|} \xi^2(i) \geq \frac{1}{2} |SV| \cdot \xi^2(|SV|) + \text{const}
\]

Since the function \(\xi^2(t)\) increases for large \(t\), we can put \(\frac{|UV|}{2}\) instead of \(|SV|\):

\[
d(f(U), f(V)) \geq \sqrt{\frac{1}{4} |UV| \cdot \xi^2\left(\frac{|UV|}{2}\right) + \text{const}}
\]

and finally

\[
\rho_f(t) \geq \sqrt{t} \cdot \xi(t) = \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}.
\]

3. Uniform embeddings of hyperbolic groups

Let us recall one of many equivalent definitions of a word hyperbolic group (see [11, 4] for more details). A geodesic triangle in a metric space is called \(\delta\)-thin if any of its sides is contained in \(\delta\)-neighborhood of the union of the other two sides. A geodesic metric space is called hyperbolic if there exists \(\delta > 0\) such that any geodesic triangle in this space is \(\delta\)-thin. A finitely generated group is called hyperbolic (in a sense of M. Gromov) if it is hyperbolic as a metric space with word metric.

Recall that a hyperbolic group is elementary if it is virtually cyclic. Any elementary hyperbolic group can be quasi-isometrically embedded into a Hilbert space. Any non-elementary hyperbolic group contains a quasi-isometric image of the free group \(F_2\) [11].

**Theorem 3.1.** Any hyperbolic group admits a uniform embedding into Hilbert space with the compression

\[
\rho(t) \geq \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}.
\]

If \(G\) is a non-elementary hyperbolic group, then for any uniform embedding \(f\) of \(G\) in Hilbert space

\[
\rho_f(t) \leq \frac{t}{\sqrt{\ln t}}
\]

**Proof.** We construct a uniform embedding as a composition of uniform embeddings as follows. Any word hyperbolic group embeds quasi-isometrically into a hyperbolic space \(\mathbb{H}^n\) for some \(n\) [2]. The space \(\mathbb{H}^n\) embeds quasi-isometrically into a finite product \(\prod_{i=1}^k T_i\) of locally finite trees equipped with the \(l_1\)-metric [9]. A finite product of metric spaces with \(l_1\)-metric is quasi-isometric to the same product equipped with \(l_2\)-metric. By Theorem 2.6 each tree \(T_i\) embeds into Hilbert space...
\[ \mathcal{K}_i \text{ with compression function } \geq \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}. \] Therefore, the product \( \prod_{i=1}^{k} T_i \) embeds into the Hilbert space \( \mathcal{K} = \prod_{i=1}^{k} \mathcal{K}_i \) with compression function \( \geq \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}. \)

Assuming that \( G \) is a non-elementary hyperbolic group, one can find an undistorted subgroup in \( G \) isomorphic to the free group on two generators (115). Now the required upper bound on the compression follows from corollary 2.3. \( \square \)

4. Uniform embeddings of \( \text{CAT}(0) \) cubical complexes

In this section we use the method of [5] to generalize the construction of uniform embedding from Theorem 2.6. Recall that a cubical complex is a polyhedral complex with geodesic metric in which each cell is isometric to a Euclidean unit cube \([0,1]^n\) for some \( n \) and the gluing maps are isometries. We refer the reader to the book [4] for definition and properties of \( \text{CAT}(0) \) spaces.

Let us recall some notions that we will use in this Section (see [5] and references therein for more details). A midplane of a cube \([0,1]^n\) is the intersection of the cube with a codimension 1 hyperplane parallel to a coordinate hyperplane and passing through the center of the cube. There are \( n \) midplanes in every \( n \)-dimensional cube. Given an edge in a \( \text{CAT}(0) \) cubical complex, there is a unique codimension 1 hyperplane in the complex which cuts the edge transversely in its midpoint. This is obtained by developing the midplanes in the cubes containing the edge. Any hyperplane in a \( \text{CAT}(0) \) cubical complex separates it into two components.

Let \( K \) be a \( \text{CAT}(0) \) cubical complex with metric \( d \). We denote by \( K^{(1)} \) its 1-skeleton (the union of all vertices and edges of \( K \)) equipped with geodesic metric \( d_1 \). Then the \( d_1 \)-distance between two vertices of \( K^{(1)} \) is equal to the number of hyperplanes in \( K \) separating them. In case if \( K \) is a tree, any hyperplane is a midpoint of some edge and the distance between two vertices \( U \) and \( V \) is simply the number of midpoints separating \( U \) and \( V \).

In case of a tree the hyperplanes which separate two vertices are linearly ordered. In a higher dimensional cubical complex they are not, but the notion of normal cube path introduced in [14] allows one to introduce a partial order and to modify the argument from Theorem 2.6. A cube path is a sequence of cubes \( \mathcal{C} = \{C_1, \ldots, C_n\} \), each of dimension at least 1, such that the intersection \( C_i \cap C_{i+1} \) is a single vertex \( V_i \) and \( C_i \) is the (unique) cube of minimal dimension containing \( V_i \) and \( V_{i+1} \). We denote by \( V_0 \) the vertex of \( C_1 \) which is diagonally opposite to \( V_1 \), and by \( V_n \) the vertex of \( C_n \) diagonally opposite to \( V_{n-1} \). The vertices \( V_0 \) and \( V_n \) are called the initial vertex and the terminal vertex respectively. A cube path is called normal if \( C_{i+1} \cap \text{st}(C_i) = V_i \) for each \( i \), where \( \text{st}(C_i) \) is the union of all cubes which contain \( C_i \) as a face (including \( C_i \) itself).

For any two vertices \( U \) and \( V \) of a \( \text{CAT}(0) \) cubical complex there is a unique normal cube path \( \mathcal{C} = \{C_1, \ldots, C_n\} \) from \( U \) to \( V \). A hyperplane separates \( U \) and \( V \) if and only if it intersects (exactly) one of the cubes in the path \( \mathcal{C} \).

Suppose that \( U \) and \( V \) are adjacent vertices of a \( \text{CAT}(0) \) cubical complex. Take any vertex \( O \) and look at the normal cube paths \( \mathcal{C}_U = \{u_1, \ldots, u_n\} \) from \( U \) to \( O \) and \( \mathcal{C}_V = \{v_1, \ldots, v_n\} \) from \( V \) to \( O \). The following key property of normal cube paths basically says that these two paths stay close to each other: if a hyperplane \( h \) separates both \( U \) and \( V \) from \( O \) and intersects a cube \( u_i \), then it also intersects one of the cubes \( v_{i-1}, v_i, v_{i+1} \).
Theorem 4.1. Let $K$ be a finite-dimensional, locally finite CAT(0) cubical complex. There exists a uniform embedding $f$ of $K$ in Hilbert space $\mathcal{H}$ with the compression

$$\rho_f(t) \geq \frac{t}{\sqrt{\ln t} \cdot \ln \ln t}$$

Proof. If a cubical complex $K$ is finite dimensional, then it is quasi-isometric to its 1-skeleton $K^{(1)}$. We will construct a uniform embedding $f$ of $K^{(1)}$ into Hilbert space with

$$\rho_f(t) \geq \frac{t}{\sqrt{\ln t} \cdot \ln \ln t}$$

Our embedding is going to be linear on all edges of $K^{(1)}$, therefore it is enough to define it on vertices. We fix a base vertex $O \in K^{(1)}$. Our uniform embedding $f: K^{(1)} \to \mathcal{H}$ sends $O$ to the origin of $\mathcal{H}$. Let us fix a countable basis in the space $\mathcal{H}$. We would like to enumerate the elements of the basis by all hyperplanes in $K^{(1)}$. If $h$ is a hyperplane in $K^{(1)}$, we denote by $\vec{e}_h$ the corresponding unit vector of the basis.

Let $V$ be any vertex of $K^{(1)}$. We denote by $\|V\|$ the number of cubes in the (unique) normal cube path $\mathcal{C}_V = \{v_1, \ldots, v_{\|V\|}\}$ from $V$ to $O$. We are going to define the map $f$ on the vertex $V$ in such a way that the point $f(V)$ has non-zero coordinates with respect to basic vectors corresponding to the hyperplanes separating $V$ and $O$. To define the map $f$ we fix a function $\xi: \mathbb{R} \to \mathbb{R}$ called weight function of the embedding $f$ and put

$$f(V) = \sum_{i=1}^{\|V\|} \sum_{h \cap v_i \neq \emptyset} \xi(i) \cdot \vec{e}_h$$

for any vertex $V$ of $K^{(1)}$.

Let us define a function $N_V$ on the set of all hyperplanes:

$$N_V(h) = \begin{cases} i & \text{if } h \cap v_i \neq \emptyset \\ 0 & \text{if } h \cap v_j = \emptyset \text{ for all } v_j \in \mathcal{C}_V \end{cases}$$

Assuming that $\xi(0) = 0$ we can write

$$f(V) = \sum_h \xi(N_V(h)) \cdot \vec{e}_h$$

We are going to use essentially the same weight function we used in Theorem 2.6. It is convenient here to assume that $\xi(t)$ is non-decreasing. So, we use the formula

$$\xi(t) = \begin{cases} \sqrt{\ln t} & \text{if } t \geq M \\ 0 & \text{if } t < M \end{cases}$$

where $M$ is some fixed positive integer such that the function $\sqrt{\ln t}$ increases for $t > M$.

In order to show that $f$ is a uniform embedding we need to estimate its dilatation $\delta_f$ from above and its compression $\rho_f$ from below.

First we show that $\delta_f(t) \leq t$. To prove this we show that there is a constant $C$ such that the expansion of any edge of $K^{(1)}$ under the map $f$ does not exceed $C$. 


Let \( U \) and \( V \) be adjacent vertices of \( K^{(1)} \). We are going to find a constant \( C \) such that \( d(f(U), f(V)) \leq C \cdot d(U, V) = C \). Clearly

\[
[d(f(U), f(V))]^2 = \sum_h (\xi(N_U(h)) - \xi(N_V(h)))^2
\]

By the key property of normal cube paths \( N_U(h) \) is equal to one of \( N_V(h) - 1 \), \( N_V(h) \), \( N_V(h) + 1 \). Denote by \( h_{UV} \) the hyperplane separating \( U \) and \( V \). Without loss of generality we can assume that \( N_U(h_{UV}) = 0 \) and \( N_V(h_{UV}) = 1 \). Therefore

\[
[d(f(U), f(V))]^2 = \xi^2(1) + \sum_{i=1}^{\|U\|} \sum_{N_U(h)=i} (\xi(i) - \xi(i \pm 1))^2
\]

Since the complex \( K \) has finite dimension \( n \), the number of hyperplanes with \( N_U(h) = i \) is at most \( n \). Thus

\[
[d(f(U), f(V))]^2 \leq \sum_{i=1}^{\|U\|} 2n (\xi(i) - \xi(i + 1))^2 \leq 2n \sum_{i=1}^{\infty} (\xi(i + 1) - \xi(i))^2
\]

part (1) of Lemma 2.5 implies that the last sum is finite and therefore the distance \( d(f(U), f(V)) \) is bounded by some constant \( C \).

Now we estimate the compression function \( \rho_f \). Let \( U \) and \( V \) be any vertices of the complex \( K \). There are \( d_1(U, V) \) hyperplanes in \( K \) separating \( U \) from \( V \). Without loss of generality we may assume that at least half of them separate \( U \) from \( O \); denote by \( H \) the set of these hyperplanes. Clearly

\[
[d(f(U), f(V))]^2 \geq \sum_{h \in H} \xi^2(N_U(h)) = \sum_{i} \sum_{N_U(h)=i, h \in H} \xi^2(i)
\]

There are at least \( \lfloor \frac{d_1(U, V)}{2} \rfloor \) hyperplanes in the set \( H \) (where \( \lfloor t \rfloor \) denotes the largest integer smaller than \( t \)). Since there are at most \( n = \dim K \) hyperplanes with \( N_U(h) = i \) for every \( i \) and the function \( \xi \) is non-decreasing, we have

\[
\sum_i \sum_{N_U(h)=i, h \in H} \xi^2(i) \geq \sum_{i=1}^{\lfloor d_1(U, V) \rfloor} n \cdot \xi^2(i)
\]

Using part (2) of Lemma 2.5 we estimate the last sum and finally

\[
[d(f(U), f(V))]^2 \geq n \cdot \frac{1}{2d_1(U, V)^2} \cdot \xi^2(\lfloor \frac{d_1(U, V)}{2n} \rfloor) + \text{const}
\]

Therefore

\[
\rho_f(t) \geq \sqrt{t} \cdot \xi(t) \sim \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}.
\]

\[\Box\]

**Theorem 4.2.** If a group \( G \) acts properly and cocompactly on a finite dimensional \( \text{CAT}(0) \) cubical complex, then \( G \) admits a uniform embedding into Hilbert space with compression

\[
\rho(t) \geq \frac{t}{\sqrt{\ln t \cdot \ln \ln t}}.
\]
Suppose that the orders of finite subgroups of G are uniformly bounded. Then, either G can be quasi-isometrically embedded into Hilbert space or for any uniform embedding f into Hilbert space

\[ \rho_f(t) \leq \frac{t}{\sqrt{\ln t}}. \]

Proof. Since the action is proper and cocompact, the group G embeds quasi-isometrically into a finite dimensional CAT(0) cubical complex and the first statement of the theorem follows from Theorem 4.1.

If G acts properly and cocompactly on a finite dimensional CAT(0) cubical complex and the orders of finite subgroups of G are uniformly bounded, then either G is virtually a finitely generated abelian group or G has a rank 2 free subgroup [10]. In the former case G embeds quasi-isometrically into Hilbert space. In the latter case one can choose a quasi-isometrically embedded rank 2 free subgroup in G (the same proof works as in [16]). Application of Corollary 2.3 completes the proof. □

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