Singularity theorems assuming trapped submanifolds of arbitrary dimension

Gregory J Galloway\textsuperscript{1} and José M M Senovilla\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, University of Miami, Coral Gables, FL, U.S.A.
\textsuperscript{2}Física Teórica, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
E-mail: josemm.senovilla@ehu.es

Abstract. New singularity theorems are proven in Lorentzian manifolds of arbitrary dimension $n$ if they contain closed trapped submanifolds of arbitrary co-dimension. The timelike or null convergence conditions must be generalized to a condition on sectional curvatures, or tidal forces, which reduces to the former in the cases of co-dimension 1, 2 or $n$. Applications to higher dimensional theories and to the case of trapped circles are briefly mentioned.

1. Introduction

The classical Hawking-Penrose singularity theorem \cite{6} is

\textbf{Theorem 1 (Hawking and Penrose)} Spacetime \textit{(in 4 dimensions)} is causal geodesically incomplete if the strong-energy, causality and generic conditions hold and if there is one of the following:

\begin{itemize}
\item a closed achronal set without edge, (co-dimension 1)
\item a closed trapped surface, (co-dimension 2)
\item a point with re-converging light cone. (co-dimension 4)
\end{itemize}

What about co-dimension 3 —a closed spacelike curve? In order to include this missing case we need a unification of the concept of trapping for submanifolds of arbitrary co-dimension. This is given by the mean curvature vector $\vec{H}$ of the submanifold $\zeta$ \cite{7, 8, 9}. Recall that, for any vector field $\vec{n}$ orthogonal to $\zeta$, $\theta(\vec{n}) \equiv n_\mu H^\mu$ is called the expansion of the submanifold along $\vec{n}$. A spacelike $\zeta$ is said to be future-trapped if $\vec{H}$ is timelike and future-pointing on all $\zeta$. This is equivalent to having $\theta(\vec{n}) < 0$ for all possible future normal vectors $\vec{n}$.

2. Existence of focal points

The only intermediate result that is not immediately available in order to prove singularity theorems —such as theorem 1— based on trapped submanifolds $\zeta$ of arbitrary dimension is the existence of points focal to $\zeta$. To that end, we use the following notation:

\begin{itemize}
\item $n_\mu$: \textit{future-pointing} normal to the spacelike submanifold $\zeta$
\item $e_A$: vector fields tangent to $\zeta$. They obviously satisfy $n_\mu e_A^\mu = 0$
\item $\gamma$: geodesic curve tangent to $n^\mu$ at $\zeta$
\end{itemize}

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• $u$: affine parameter along $\gamma$, setting $u = 0$ at $\zeta$
• $N^\mu$: geodesic vector field tangent to $\gamma$ such that $N_\mu|_{u=0} = n_\mu$
• $\vec{E}_A$: vector fields defined by parallelly propagating $\vec{e}_A$ along $\gamma$, and such that $\vec{E}_A|_{u=0} = \vec{e}_A$.
  
  Observe that $N_\mu E_\mu = 0$ for all $u$
• $P^{\mu\sigma} \equiv \gamma^{AB} E_\mu^A E_\sigma^B$ (at $u = 0$ this is the projector to $\zeta$), where $\gamma^{AB}$ is the inverse of the first fundamental form $\gamma_{AB} \equiv g_{\mu\nu} e_\mu^A e_\nu^B$ of $\zeta$ in the spacetime. Here $g_{\mu\nu}$ denotes the Lorentzian metric in the manifold.

**Proposition 1** Let $\zeta$ be a spacelike submanifold of co-dimension $m$ in a Lorentzian manifold of dimension $n$, and let $n_\mu$ be a future-pointing normal to $\zeta$. If $\theta(\vec{n}) \equiv (m - n)c < 0$ and the curvature tensor satisfies the inequality

$$R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} \geq 0$$

along $\gamma$, then there is a point focal to $\zeta$ along $\gamma$ at or before $\gamma|_{u=1/c}$, provided $\gamma$ arrives that far.

The proof can be found in [4].

**Remarks:**

(i) Spacelike hypersurfaces: $m = 1$. There is a unique timelike orthogonal direction $n_\mu$. Then $P_{\mu\nu} = g_{\mu\nu} - (N_\rho N^\rho)^{-1} N_\mu N_\nu$ and (1) reduces to

$$R_{\mu\nu} N^\mu N^\nu \geq 0.$$

This is the *timelike convergence condition* along $\gamma$ [5].

(ii) Spacelike ‘surfaces’: $m = 2$. Now, there are two linearly independent null normals on $\zeta$, say $n_\mu$ and $\ell_\mu$. Define $L_\mu$ parallelly propagating $\ell_\mu$ on $\gamma$. Then, $P_{\mu\nu} = g_{\mu\nu} - (N_\rho L^\rho)^{-1} (N_\mu L_\nu + N_\nu L_\mu)$ and again (1) reduces to

$$R_{\mu\nu} N^\mu N^\nu \geq 0.$$

This is the *null convergence condition* along $\gamma$ [5].

**2.1. The curvature condition (1)**

For co-dimension $m > 2$, the interpretation of condition (1) can be given physically in terms of *tidal forces*, or geometrically in terms of *sectional curvatures* [4].

The sectional curvature $k(n,e)$ relative to the plane spanned by $\vec{n}$ and $\vec{e}$ ($n_\mu e^\mu = 0$) is defined by [1, 2, 7, 9]

$$R_{\mu\nu\rho\sigma} n^\mu e^\nu n^\rho e^\sigma = k(n,e)(n_\rho n^\rho)(e_\rho e^\rho)$$

if $n_\mu$ is not null, and by [1]

$$R_{\mu\nu\rho\sigma} n^\mu e^\nu n^\rho e^\sigma = -k(n,e)(e_\rho e^\rho)$$

if it is null. Observe that the second of these coincide with the first for the case of a *unit* timelike normal. (There is a different sign with respect to the literature [1, 4]).

Hence (1) states that: the *sum of the $n - m$ sectional curvatures* relative to a set of mutually orthogonal planes aligned with $n_\mu$ is non-positive, and remains so along $\gamma$. These planes are timelike (respectively null) for a timelike (resp. null) normal $n_\mu$.

In physical terms, this is a statement about the attractiveness of the gravitational field on average. The tidal force in directions initially tangent to $\zeta$ is attractive on average.
3. New singularity theorems
We concentrate on the generalization of the more elaborated Hawking-Penrose theorem, but many other theorems can be similarly generalized, see [4].

Recall that for any set $\zeta$, $E^+(\zeta) \equiv J^+(\zeta) \cap I^+(\zeta)$, using the standard notation for the causal $J^+(\zeta)$ and chronological $I^+(\zeta)$ futures of $\zeta$, see e.g. [1, 6, 5, 7, 10, 12]. For the proofs of the following statements we refer the reader to [4].

Proposition 2 If $(\mathcal{V}, g)$ is strongly causal and there is a closed $f$-trapped submanifold $\zeta$ of arbitrary co-dimension $m > 1$ such that condition (1) holds along every null geodesic emanating orthogonally from $\zeta$, then either $E^+(E^+(\zeta) \cap \zeta)$ is compact, or the spacetime is null geodesically incomplete, or both.

Thus, $\eta \equiv E^+(\zeta) \cap \zeta$ provides the compact achronal set in the standard Hawking-Penrose Lemma [6, 5, 12], the achronality of $\eta$ being a direct consequence of the achronality of $E^+(\zeta)$. (The case with $m = 1$ is not included as it is trivial. If $\zeta$ is an achronal hypersurface, then $E^+(\zeta) = \zeta$ and so this set is compact).

The generalization of the Hawking-Penrose theorem follows from the above.

Theorem 2 If the chronology, generic and strong energy conditions hold and there is a closed $f$-trapped submanifold $\zeta$ of arbitrary co-dimension such that condition (1) holds along every null geodesic emanating orthogonally from $\zeta$ then the spacetime is causal geodesically incomplete.

Remark: For co-dimension $m = 1$ there are no null geodesics orthogonal to $\zeta$ so that there is no need to assume (1). For co-dimension $m = 2$ the condition (1) is actually included in the strong energy condition as explained after Proposition 1. Something analogous occurs for co-dimension $m = n$. These three cases cover the original Hawking-Penrose theorem, which was valid in arbitrary dimension (but only for co-dimension $m = 1, 2, n$).

4. Selected applications
The main obvious application of these theorems is to higher dimensional spacetimes, and thereby they can be relevant in Kaluza-Klein, string, supergravity, or M-type theories. For instance, in dimension $n = 11$, say, there are now 10 inequivalent possibilities for the boundary condition in the singularity theorems. This can be of special relevance in connection with the compactified extra-dimensions.

As an example, consider the arguments by Penrose [11] on the classical instability of compactified spatial extra-dimensions, which seem to imply that some sort of singularities may develop within a tiny fraction of a second. These arguments can be extended and probably clarified or reinforced by using our generalized Theorems.

Even in the traditional case of $n = 4$, the new theorems have applications to the cases with closed trapped curves. These are curves whose proper acceleration vector is timelike. An obvious relevant example is the case of spacetimes with whole cylindrical symmetry [13]

$$ds^2 = -A^2 dt^2 + B^2 d\rho^2 + F^2 d\varphi^2 + E^2 dz^2,$$

where $\partial_\varphi, \partial_z$ are spacelike commuting Killing vectors, whence all the functions depend only on $t$ and $\rho$, and the axis is located at $\rho \to 0$. The coordinate $\varphi$ is closed with standard periodicity $2\pi$. The cylinders given by constant values of $t$ and $\rho$ are geometrically preferred; however, they are not compact in general.

On the other hand, the spacelike curves defined by constant values of $t, \rho$ and $z$ are certainly closed. Their mean curvature vector is proportional to $dF$. Thus, the causal character of the gradient of the norm of the circular Killing vector $\partial_\varphi$ characterizes the trapping of these closed circles. Many results on incompleteness of geodesics can be derived thereof. There also appears
a new type of horizon, defined as the set of points where \( dF \) is null, acting as the boundary which separates the trapped from the untrapped circles.

Finally, an important remark is that the curvature condition (1) in Proposition 1 can be weakened. As happens in the case of proper Riemannian spaces [3], it is sufficient that it holds on the average in the sense that only a milder version, using the integral along the geodesic, is needed. This is discussed in [4].

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