Efficiently expressing feasibility problems in Linear Systems, as feasibility problems in Asymptotic-Linear-Programs

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Abstract: We present a polynomial-time algorithm that obtains a set of Asymptotic Linear Programs (ALPs) from a given linear system S, such that one of these ALPs admits a feasible solution if and only if S admits a feasible solution. We also show how to use the same algorithm to determine whether or not S admits a non-trivial solution for any desired subset of its variables. S is allowed to consist of linear constraints over real variables with integer coefficients, where each constraint has either a lesser-than-or-equal-to (≤), or a not-equal-to (≠) relational operator. Each constraint of the obtained ALPs has a lesser-than-or-equal-to (≤) relational operator, and the coefficients of its variables vary linearly with respect to the time parameter that tends to positive infinity.

1. Introduction

In our previous paper [1], we showed how to efficiently convert any given linear system with simultaneous constraints having lesser-than-or-equal-to (≤) and lesser-than (<) relational operators, into an Asymptotic Linear Program (ALP), such that (The ALP has a feasible solution) ↔ (The given linear system has a non-trivial feasible solution). In that paper [1], we showed that one way of modeling n Inequations (i.e. constraints with not-equal-to relational operators), was to iteratively consider the rest of the constraints with two cases (for example, for x≠0, consider x<0 and x>0), which would lead to an overall exponential complexity of \(O(2^n)\). We posed an open question, on efficiently (i.e. within polynomial-time) modeling Inequations as an ALP. In this paper, we show this is possible in \(O(n^2)\) time.

2. The foundation for efficiently modeling Inequations

Definition: Let \(\langle y_1, y_2, \ldots, y_N\rangle\), \(\langle x_1, x_2, \ldots, x_N\rangle\) and \(\langle z_1, z_2, \ldots, z_N\rangle\) be three vectors of real variables, and let \(K\) be a real variable. Let the variables of these three vectors be related as follows:

\[
y_i = (\text{Summation}(x_j, \text{over all integers } j \in [1,N], \text{and } j \neq i)), \text{ for all integers } i \in [1,N].
\]

For example, for \(N=5\):

\[
\begin{align*}
y_1 &= x_2 + x_3 + x_4 + x_5 \\
y_2 &= x_1 + x_3 + x_4 + x_5 \\
y_3 &= x_1 + x_2 + x_4 + x_5 \\
y_4 &= x_1 + x_2 + x_3 + x_5 \\
y_5 &= x_1 + x_2 + x_3 + x_4 \\
x_i &= z_j/(K+i) \\
x_2 &= z_2/(K+2) \\
x_3 &= z_3/(K+3) \\
x_4 &= z_4/(K+4) \\
x_5 &= z_5/(K+5)
\end{align*}
\]

We will conveniently assume that \(N>1\), because if there is only one not-equal-to constraint (say \(t\neq0\)) in a linear system, one can easily solve the system by considering the remaining problem with two cases (\(t<0\)) and then with (\(t>0\)).

We now state and prove two Theorems, which will form the foundation for efficiently modeling inequations as an ALP.

Theorem-1: For all real values of the elements of vector \(\langle y_1, y_2, \ldots, y_N\rangle\), the following statement is true: - (There exists a positive real \(r\), such that for all \(K > r\), there exists a real solution to the vectors \(\langle x_1, x_2, \ldots, x_N\rangle\) and \(\langle z_1, z_2, \ldots, z_N\rangle\))

Proof: For \(K>0\), it is trivial to see that \(\langle x_1, x_2, \ldots, x_N\rangle\) has a real solution) ↔ \(\langle z_1, z_2, \ldots, z_N\rangle\) has a real solution). Next, to prove that \(\langle x_1, x_2, \ldots, x_N\rangle\) has a real solution) for all real values of \(\langle y_1, y_2, \ldots, y_N\rangle\), we need to show that the determinant of square matrix \(A\) (see Figure-1) defined by \(a_{ij} = 1\) if \(i\neq j\), \(0\) if \(i=j\), for all integers \(i\) and \(j\) in \([1,N]\), is non-zero.
Figure-1: The Determinant of square-Matrix A of dimension N

To show that the determinant of Matrix A is indeed non-zero, iteratively apply the following rule to Matrix A:
Row $i = \sum_{j=i+1}^{N} \text{(Row}_j\text{)} - (N-i-1)\text{Row}_i$, for all integers $i \in [1,N]$. We then get the Matrix B defined by (for all integers $j \in [1,N]$):
$$
\begin{align*}
\{b_{i,j} \text{ (for all integers } i \text{ in } [1,N-2])\} &= \begin{cases} 
(N-i) & \text{if } i=j, \\
1 & \text{if } i>j, \\
0 & \text{if } i<j
\end{cases}, \text{ and,}
\{b_{i,j} \text{ (for all integers } i \text{ in } [N-1,N])\} &= \begin{cases} 
1 & \text{if } i\neq j, \\
0 & \text{if } i=j
\end{cases}.
\end{align*}
$$

This Matrix B is a left diagonal matrix, whose determinant is equal to $-(N-1)!$, which is obviously non-zero for $N>1$.

Hence Proved

Theorem-2: There exists a positive real $\gamma$, such that for all $K>\gamma$, the following statement is true:
((Atleast two elements of vector $<z_1, z_2, \ldots, z_N>$ are non-zero) $\iff$ (All elements of vector $<y_1, y_2, \ldots, y_N>$ are non-zero))

Proof: Consider the two boolean statements within the 'if-and-only-if' in the Theorem to be P and Q. It is well known that to prove (P $\iff$ Q), it suffices to prove two statements: (Q $\implies$ P) and ((not Q) $\implies$ (not P)). It is easy to see that (All the elements of $<z_1, z_2, \ldots, z_N>$ are zero) $\implies$ (All the elements of $<y_1, y_2, \ldots, y_N>$ are zero). Also, if exactly one of the elements of $<z_1, z_2, \ldots, z_N>$ is non-zero, this would mean that exactly (N-1) elements of $<y_1, y_2, \ldots, y_N>$ are non-zero, because each variable $z_i$ appears in the defining equations of $\{y_j \text{ for all } j \in [1,N] \text{ where } j \neq i\}$. To put it explicitly, we have:
$$
y_j = \text{(Summation}(z_j/(K+j), \text{over all integers } j \text{ between } 1 \text{ and } N \text{, and } j \neq i), \text{for all integers } i \in [1,N].$$
Thus ((not Q) $\implies$ (not P)). Next, from Theorem-1 of paper [1], and the above defining relationship, it follows that (P $\implies$ Q) for all $K>\gamma$, where $\gamma$ is a positive real that is a function of the elements of $<y_1, y_2, \ldots, y_N>$.

Hence Proved

3. Converting Linear Feasibility Problems into an ALP

The Linear System we consider is a set of simultaneous linear constraints over a vector of real variables $<x_1, x_2, \ldots, x_N>$ (i.e. each variable is initially allowed to take the values of zero, positive Reals, or negative Reals). We shall refer to our Linear System as $S_{\text{linear}}$, having $P$ linear constraints with lesser-than-or-equal-to relational operators, $Q$ linear constraints with lesser-than relational operators, and $R$ linear constraints with not-equal-to relational operators:
$$
\begin{align*}
a_{1,1}x_1 + a_{1,2}x_2 + & \ldots + a_{1,N}x_N \leq p_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + & \ldots + a_{2,N}x_N \leq p_2 \\
& \ldots \\
a_{P,1}x_1 + a_{P,2}x_2 + & \ldots + a_{P,N}x_N \leq p_P \\
b_{1,1}x_1 + b_{1,2}x_2 + & \ldots + b_{1,N}x_N < q_1 \\
b_{2,1}x_1 + b_{2,2}x_2 + & \ldots + b_{2,N}x_N < q_2 \\
& \ldots \\
b_{Q,1}x_1 + b_{Q,2}x_2 + & \ldots + b_{Q,N}x_N < q_Q \\
c_{1,1}x_1 + c_{1,2}x_2 + & \ldots + c_{1,N}x_N \neq r_1 \\
c_{2,1}x_1 + c_{2,2}x_2 + & \ldots + c_{2,N}x_N \neq r_2 \\
& \ldots \\
c_{R,1}x_1 + c_{R,2}x_2 + & \ldots + c_{R,N}x_N \neq r_R
\end{align*}
$$
In $S_{\text{linear}}$, for all integers $i$ in $\{1, N\}$, for all integers $j$ in $\{1, P\}$, for all integers $k$ in $\{1, Q\}$, for all integers $l$ in $\{1, R\}$: $x_i$ is a real variable, and the elements of $f \{a_{i,j}, p_j, b_{i,j}, q_j, c_{i,j}, r_i\}$ belong to the set of integers. $S_{\text{linear}}$ is able to express most linear systems, except linear discrete systems (for example, if $x$ is constrained to integer values).

Let $P_{\text{linear}}$ be the problem of deciding whether or not $S_{\text{linear}}$ admits a feasible solution. Our Algorithm for $P_{\text{linear}}$ is as follows:

**Step-1:** Replace each constraint having a lesser-than relation operator with 2 simultaneous constraints. For example, $(x < a)$ can be replaced with a set of constraints of the form $((a-x) > 0) \text{ AND } ((Ke > 1))$, where $e$ is a real variable introduced, and where $K$ is the time parameter of our ALP (i.e. a real number that is assumed to tend to positive infinity).

**Step-2:** $S_{\text{linear}}$ now consists of constraints with only lesser-than-or-equal-to and not-equal-to relational operators. Divide $S_{\text{linear}}$ into two sets of constraints: - $S_{\text{linear}}\_{\text{subset without inequations}}$ (that has the constraints with only the lesser-than-or-equal-to operators) and $S_{\text{linear}}\_{\text{subset inequations}}$ (that has the $R$ constraints with only the not-equal-to operators).

**Step-3:** Write out the $R$ inequations as follows:

$$c_{1,1}x_1 + c_{1,2}x_2 + \ldots + c_{1,N}x_N = f_1$$
$$c_{2,1}x_1 + c_{2,2}x_2 + \ldots + c_{2,N}x_N = f_2$$
$$\ldots$$
$$c_{R,1}x_1 + c_{R,2}x_2 + \ldots + c_{R,N}x_N = f_R$$

$f_1 \neq 0$
$f_2 \neq 0$
$$\ldots$$
$f_k \neq 0$

**Step-4:** Write $f_i =$ (Summation($y_{ij}$, over all integers $j$ between $I$ and $R$, and $j \neq i$)), for all integers $i$ in $\{1, R\}$. Also, write $(K+i)y_i = z_i$ for all integers $i$ in $\{1, R\}$. Here $<y_1, y_2, \ldots, y_R>$ and $<z_1, z_2, \ldots, z_R>$ are the vectors of real variables introduced.

**Step-5:** Write each of the constraints with equal-to relational operators obtained in Step-3 and Step-4 (and any other such constraints initially present in $S_{\text{linear}}$), as two simultaneous constraints with lesser-than-or-equal-to operators. For example, $(x = a)$ can be expressed as a set of two constraints $((x-a) \leq 0) \text{ AND } ((a-x) \leq 0))$. Add these constraints to $S_{\text{linear}}\_{\text{subset without inequations}}$.

**Step-6:** Consider $R^2C_2$ cases ($= R(R-1)/2$ cases) by taking all possible combinations of $2$ elements from the vector $<z_1, z_2, \ldots, z_R>$ to be not-equal-to-zero. For each of these $R(R-1)/2$ cases, there will be $4$ separate cases, involving each of these $2$ elements being $> 0$ and $< 0$. For example, if $z_2$ and $z_3$ are selected, we have $4$ separate cases: - $(z_2<0, z_3<0)$, $(z_2<0, z_3>0)$, $(z_2>0, z_3<0)$ and $(z_2>0, z_3>0)$. We thus have a total of $2R(R-1)$ separate cases.

**Step-7:** For each of these $2R(R-1)$ separate cases, convert the $2$ constraints with lesser-than operators into constraints with lesser-than-or-equal-to operators using the procedure described in Step-1.

**Step-8:** Write ALP as the union of $S_{\text{linear}}\_{\text{subset without inequations}}$ with the constraints with lesser-than-or-equal-to operators of Case described above in Step-7, for all integers $i$ in $\{1, 2R(R-1))\}$.

**Step-9:** (For at least one of the integers $i$ in $\{1, 2R(R-1))\}$, ALP is feasible) $\leftrightarrow$ ($S_{\text{linear}}$ is feasible).

**A Note on the Asymptotic Linear Program (ALP)**

An ALP [2][3][4] is a linear program, where the coefficients of the variables in the constraints are rational Polynomials involving a single real variable called the time parameter. The author of [4] proved that as this time parameter grows beyond a certain positive value, the Linear Program gets constant (i.e. steady-state) properties of feasibility or infeasibility. In other words, as this time parameter tends to positive infinity, the Asymptotic Linear Program becomes either feasible or infeasible.

**Start of Example demonstrating Algorithm for $P_{\text{linear}}$:**

Consider $S_{\text{linear}}$ to be defined by the following set of $7$ linear constraints over the real variable vector $<x_1, x_2, x_3>$:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \leq p_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \leq p_2$$
$$b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 \leq q_1$$
$$b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 \leq q_2$$
$$c_{1,1}x_1 + c_{1,2}x_2 + c_{1,3}x_3 \neq r_1$$
$$c_{2,1}x_1 + c_{2,2}x_2 + c_{2,3}x_3 \neq r_2$$
$$c_{3,1}x_1 + c_{3,2}x_2 + c_{3,3}x_3 \neq r_3$$

After Step-1, $S_{\text{linear}}$ becomes the following:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \leq p_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \leq p_2$$
$$b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 - q_1 \leq -e$$
$$b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 - q_2 \leq -e$$
$$c_{1,1}x_1 + c_{1,2}x_2 + c_{1,3}x_3 \neq r_1$$
$$c_{2,1}x_1 + c_{2,2}x_2 + c_{2,3}x_3 \neq r_2$$
$$c_{3,1}x_1 + c_{3,2}x_2 + c_{3,3}x_3 \neq r_3$$
\[ c_{1,1}x_1 + c_{1,2}x_2 + c_{1,3}x_3 \neq r_3 \]

\(Ke \geq 1\), where \(e\) is the real variable introduced, and \(K\) is the large positive real.

As per Step-2, \(S_{\text{linear subset without ineqations}}\) consists of the following:

\[
\begin{align*}
a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 & \leq p_1 \\
 a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 & \leq p_2 \\
b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 - q_1 & \leq -e \\
b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 - q_2 & \leq -e \\
Ke & \geq 1,
\end{align*}
\]

and \(S_{\text{linear subset ineqations}}\) consists of the following:

\[
\begin{align*}
c_{1,1}x_1 + c_{1,2}x_2 + c_{1,3}x_3 - r_1 & \neq 0 \\
c_{2,1}x_1 + c_{2,2}x_2 + c_{2,3}x_3 - r_2 & \neq 0 \\
c_{3,1}x_1 + c_{3,2}x_2 + c_{3,3}x_3 - r_3 & \neq 0.
\end{align*}
\]

As per Step-3, we have:

\[
\begin{align*}
c_{1,1}x_1 + c_{1,2}x_2 + c_{1,3}x_3 - r_1 & = f_1 \\
c_{2,1}x_1 + c_{2,2}x_2 + c_{2,3}x_3 - r_2 & = f_2 \\
c_{3,1}x_1 + c_{3,2}x_2 + c_{3,3}x_3 - r_3 & = f_3 \\
f_1 & \neq 0 \\
f_2 & \neq 0 \\
f_3 & \neq 0.
\end{align*}
\]

As per Step-4, we have:

\[
\begin{align*}
f_1 & = y_1 + y_2 \\
f_2 & = y_1 + y_3 \\
f_3 & = y_1 + y_2 \\
(K+1)y_1 & = z_1 \\
(K+2)y_2 & = z_2 \\
(K+3)y_3 & = z_3, \text{ where } <y_1, y_2, y_3> \text{ and } <z_1, z_2, z_3> \text{ are the vectors of real variables introduced.}
\end{align*}
\]

As per Step-5, \(S_{\text{linear subset without ineqations}}\) now becomes:

\[
\begin{align*}
a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 & \leq p_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 & \leq p_2 \\
b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 - q_1 & \leq -e \\
b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 - q_2 & \leq -e \\
1 - Ke & \leq 0 \\
c_{1,1}x_1 + c_{1,2}x_2 + c_{1,3}x_3 - r_1 - f_1 & \leq 0 \\
c_{2,1}x_1 + c_{2,2}x_2 + c_{2,3}x_3 - r_2 - f_2 & \leq 0 \\
c_{3,1}x_1 + c_{3,2}x_2 + c_{3,3}x_3 - r_3 - f_3 & \leq 0 \\
-c_{1,1}x_1 - c_{1,2}x_2 - c_{1,3}x_3 + r_1 + f_1 & \leq 0 \\
-c_{2,1}x_1 - c_{2,2}x_2 - c_{2,3}x_3 + r_2 + f_2 & \leq 0 \\
-c_{3,1}x_1 - c_{3,2}x_2 - c_{3,3}x_3 + r_3 + f_3 & \leq 0 \\
y_2 + y_3 - f_1 & \leq 0 \\
y_1 + y_3 - f_2 & \leq 0 \\
y_1 + y_2 - f_3 & \leq 0 \\
y_2 - y_3 + f_1 & \leq 0 \\
y_1 - y_3 + f_2 & \leq 0 \\
y_1 - y_2 + f_3 & \leq 0 \\
(K+1)y_1 - z_1 & \leq 0 \\
(K+2)y_2 - z_2 & \leq 0 \\
(K+3)y_3 - z_3 & \leq 0 \\
-(K+1)y_1 + z_1 & \leq 0 \\
-(K+2)y_2 + z_2 & \leq 0 \\
-(K+3)y_3 + z_3 & \leq 0.
\end{align*}
\]

As per Step-6, we have a total of 12 cases:

Case 1: \(z_1 < 0, z_2 < 0\)
Case 2: \(z_1 < 0, z_3 < 0\)
Case 3: \(z_2 < 0, z_1 < 0\)
Case 4: \(z_3 < 0, z_2 > 0\)
Case 5: \(z_1 < 0, z_3 > 0\)
Case 6: \(z_2 < 0, z_3 > 0\)
Case 7: \(z_1 > 0, z_2 < 0\)
Case 8: \(z_3 > 0, z_1 < 0\)
Case 9: \(z_2 > 0, z_3 < 0\)
Case 10: \(z_1 > 0, z_2 > 0\)
Case 1: \( z_1 > 0, z_3 > 0 \)  
Case 12: \( z_2 > 0, z_3 > 0 \)

As per Step-7, the constraints with lesser-than operators can be converted to constraints with lesser-than-or-equal-to operators. So we have:

- Case 1: \( z_1 \leq e, z_2 \leq e, l \leq Ke \)
- Case 2: \( z_1 \leq e, z_2 \leq e, l \leq Ke \)
- Case 3: \( z_1 \leq e, z_2 \geq e, l \leq Ke \)
- Case 4: \( z_1 \geq e, z_2 \leq e, l \leq Ke \)
- Case 5: \( z_1 \geq e, z_2 \geq e, l \leq Ke \)

As per Step-8, we generate a set of 12 separate ALPs. \( \{ \text{ALP}_1, \text{ALP}_2, \ldots, \text{ALP}_{11}, \text{ALP}_{12} \} \). Here \( \text{ALP}_i \) is the union of \( S_{\text{linear subset without inequations}} \) with the constraints of Case, described above in Step-7, for all integers \( i \) in \( \{1, 12\} \). For example, \( \text{ALP}_{11} \) is shown below:

\[
\begin{align*}
 a_{1,1} x_1 + a_{1,2} x_2 + a_{1,3} x_3 &\leq p_1 \\
 a_{2,1} x_1 + a_{2,2} x_2 + a_{2,3} x_3 &\leq p_2 \\
 b_{1,1} x_1 + b_{1,2} x_2 + b_{1,3} x_3 - q_1 &\leq -e \\
 b_{2,1} x_1 + b_{2,2} x_2 + b_{2,3} x_3 - q_2 &\leq -e \\
 c_{1,1} x_1 + c_{1,2} x_2 + c_{1,3} x_3 - r_1 - f_1 &\leq 0 \\
 c_{2,1} x_1 + c_{2,2} x_2 + c_{2,3} x_3 - r_2 - f_2 &\leq 0 \\
 c_{3,1} x_1 + c_{3,2} x_2 + c_{3,3} x_3 - rind - f_3 &\leq 0 \\
 -c_{1,1} x_1 - c_{1,2} x_2 - c_{1,3} x_3 + r_1 + f_1 &\leq 0 \\
 -c_{2,1} x_1 - c_{2,2} x_2 - c_{2,3} x_3 + r_2 + f_2 &\leq 0 \\
 -c_{3,1} x_1 - c_{3,2} x_2 - c_{3,3} x_3 + r_3 + f_3 &\leq 0 \\
 y_2 + y_3 - f_1 &\leq 0 \\
 y_1 + y_3 - f_2 &\leq 0 \\
 y_1 + y_2 - f_3 &\leq 0 \\
 -y_2 - y_3 + f_1 &\leq 0 \\
 -y_1 - y_3 + f_2 &\leq 0 \\
 -y_1 - y_2 + f_3 &\leq 0 \\
 (K+1)y_j - z_j &\leq 0 \\
 (K+2)y_j - z_2 &\leq 0 \\
 (K+3)y_j - z_3 &\leq 0 \\
 -(K+1)y_j + z_1 &\leq 0 \\
 -(K+2)y_j + z_2 &\leq 0 \\
 -(K+3)y_j + z_3 &\leq 0 \\
 e - z_1 &\leq 0 \\
 e - z_2 &\leq 0 \\
 l - Ke &\leq 0
\end{align*}
\]

Finally, in Step-9, we determine feasibility of the 12 ALPs. (Atleast one of \( \{ \text{ALP}_1, \text{ALP}_2, \ldots, \text{ALP}_{11}, \text{ALP}_{12} \} \) is feasible) \( \leftrightarrow \) \( S_{\text{linear is feasible}} \).

End of Example demonstrating Algorithm for \( P_{\text{linear}} \).

4. Deciding non-triviality of a subset of variables of \( S_{\text{linear}} \)

(A vector of reals \( \langle \mu_1, \mu_2, \ldots, \mu_N \rangle \) is said to be non-trivial) \( \leftrightarrow \) (For at least one integer \( i \) in \( \{1, N\} \), \( \mu_i \neq 0 \)). If it is desired to determine whether or not \( S_{\text{linear}} \) permits a non-trivial solution for a subset of the variables of the vector \( \langle x_1, x_2, \ldots, x_N \rangle \), we can introduce an additional constraint with a not-equal-to operator using Theorem-1 of the paper \( [1] \). For example, if it is desired to determine whether or not \( S_{\text{linear}} \) permits a non-trivial solution for \( \langle x_3, x_5, x_{13}, x_N \rangle \), we have:

\[
(x_2 / (K+1)) + (x_3 / (K+2)) + (x_{13} / (K+3)) + (x_N / (K+4)) \neq 0,
\]

which may be expressed using the following set of 5 simultaneous constraints, where \( \langle w_2, w_3, w_{13}, w_N \rangle \) is the vector of variables introduced:

\[
\begin{align*}
 w_2 + w_3 + w_{13} + w_N &\neq 0 \\
 x_2 &\neq (K+1) w_2 \\
 x_3 &\neq (K+2) w_3 \\
 x_{13} &\neq (K+3) w_{13} \\
 x_N &\neq (K+N) w_N
\end{align*}
\]

We call the union of \( S_{\text{linear}} \) and the above set of 5 constraints, as \( \text{linear-non-trivial}_{2,5,13,N_z} \), which is shown next:
\[
\begin{align*}
& a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,N}x_N \leq p_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,N}x_N \leq p_2 \\
& \ldots \\
& b_{P,1}x_1 + b_{P,2}x_2 + \ldots + b_{P,N}x_N < q_P \\
& c_{1,1}x_1 + c_{1,2}x_2 + \ldots + c_{1,N}x_N \neq r_1 \\
& c_{2,1}x_1 + c_{2,2}x_2 + \ldots + c_{2,N}x_N \neq r_2 \\
& \ldots \\
& c_{R,1}x_1 + c_{R,2}x_2 + \ldots + c_{R,N}x_N \neq r_R \\
& w_2 + w_5 + w_{13} + w_N \neq 0 \\
& x_2 = (K+1)w_2 \\
& x_5 = (K+2)w_5 \\
& x_{13} = (K+3)w_{13} \\
& x_N = (K+N)w_N
\end{align*}
\]

We now apply the algorithm for \( P_{\text{linear}} \) to \( S_{\text{linear non-trivial 2_5_13_N}} \). (\( S_{\text{linear non-trivial 2_5_13_N}} \) has a feasible solution) \( \leftrightarrow \) (\( S_{\text{linear}} \) permits a non-trivial feasible solution for \( <x_2, x_5, x_{13}, x_N> \)).

5. Conclusion

In this paper, we developed the foundations for expressing the feasibility of a set of Inequations within Linear Systems, using ALPs, within polynomial time, thus answering an open question we posed in our previous paper. We then developed a polynomial-time algorithm to express as ALP problems, the feasibility of a set of linear constraints over real variables with integer coefficients, each constraint having one of 4 types of relational operators (\( =, \leq, <, \text{and} \neq \)). The resulting ALP problems have linear constraints (with the \( \leq \) operator) over real variables with coefficients that vary linearly with the time parameter \( K \) that tends to positive infinity. We also showed how to efficiently (within polynomial-time) convert the question of whether or not the linear system allows a subset of its variables to be non-trivial, into the question of whether or not another linear system (with \( =, \leq, <, \text{and} \neq \) relational operators) has a feasible solution, thus allowing our polynomial-time algorithm to be used for determining feasibility of the non-trivial solution of the desired subset of variables of the original linear system.

6. Future Work

If it possible to express (within polynomial-time) the question of whether or not linear constraints over binary-variables (i.e. the variables are allowed to take the values of either 0 or 1), as ALPs, this would prove that Asymptotic-Linear-Programming is NP-hard. So this is an important open problem. Another open problem is whether or not a weakly-polynomial-time algorithm exists for Asymptotic-Linear-Programming (just as weakly-polynomial-time algorithms already exist for Ordinary-Linear-Programming).

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