Differential calculi on finite groups

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Abstract

A brief review of bicovariant differential calculi on finite groups is given, with some new developments on diffeomorphisms and integration. We illustrate the general theory with the example of the nonabelian finite group $S_3$. 

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1 Introduction

Differential calculi can be constructed on spaces that are more general than differentiable manifolds. Indeed the algebraic construction of differential calculus in terms of Hopf structures allows to extend the usual differential geometric quantities (connection, curvature, metric, vielbein etc.) to a variety of interesting spaces that include quantum groups, noncommutative spacetimes (i.e. quantum cosets), and discrete spaces.

In this contribution we concentrate on the differential geometry of finite group “manifolds”. As we will discuss, these spaces can be visualized as collections of points, corresponding to the finite group elements, and connected by oriented links according to the particular differential calculus we build on them. Although functions $f \in \text{Fun}(G)$ on finite groups $G$ commute, the calculi that are constructed on $\text{Fun}(G)$ by algebraic means are in general noncommutative, in the sense that differentials do not commute with functions, and the exterior product does not coincide with the usual antisymmetrization of the tensor product.

Among the physical motivations for finding differential calculi on finite groups we mention the possibility of using finite group spaces as internal spaces for Kaluza-Klein compactifications of Yang-Mills, (super)gravity or superstring theories (for example Connes’ reconstruction of the standard model in terms of noncommutative geometry [1] can be recovered as Kaluza-Klein compactification of Yang-Mills theory on an appropriate discrete internal space). Differential calculi on discrete spaces can be of use in the study of integrable models, see for ex. ref. [2]. Finally gauge and gravity theories on finite group spaces may be used as lattice approximations. For example the action for pure Yang-Mills $f F \wedge * F$ considered on the finite group space $\mathbb{Z}^N \times \mathbb{Z}^N \times \mathbb{Z}^N \times \mathbb{Z}^N$, yields the usual Wilson action of lattice gauge theories, and $N \to \infty$ gives the continuum limit [3]. New lattice theories can be found by choosing different finite groups.

A brief review of the differential calculus on finite groups is presented. Most of this material is not new, and draws on the treatment of ref.s [4, 5, 6, 7], where the Hopf algebraic approach of Woronowicz [8] for the construction of differential calculi is adapted to the setting of finite groups. Some developments on Lie derivative, diffeomorphisms and integration are new. The general theory is illustrated in the case of $S_3$.

2 Differential calculus on finite groups

Let $G$ be a finite group of order $n$ with generic element $g$ and unit $e$. Consider $\text{Fun}(G)$, the set of complex functions on $G$. An element $f$ of $\text{Fun}(G)$ is specified by its values $f_g \equiv f(g)$ on the group elements $g$, and can be written as

$$f = \sum_{g \in G} f_g x^g, \quad f_g \in \mathbb{C}$$  \hspace{1cm} (2.1)
where the functions $x^g$ are defined by

$$x^g(g') = \delta_{g,g'}^g$$  \hspace{1cm} (2.2)

Thus $\text{Fun}(G)$ is a n-dimensional vector space, and the n functions $x^g$ provide a basis. $\text{Fun}(G)$ is also a commutative algebra, with the usual pointwise sum and product $[(f + h)(g) = f(g) + h(g), (f \cdot h)(g) = f(g)h(g), (\lambda f)(g) = \lambda f(g), f, h \in \text{Fun}(G), \lambda \in \mathbb{C}]$ and unit $I$ defined by $I(g) = 1, \forall g \in G$. In particular:

$$x^gx^{g'} = \delta_{g,g'}x^g, \quad \sum_{g \in G} x^g = I$$  \hspace{1cm} (2.3)

Consider now the left multiplication by $g_1$:

$$L_{g_1}g_2 = g_1g_2, \quad \forall g_1, g_2 \in G$$  \hspace{1cm} (2.4)

This induces the left action (pullback) $L_{g_1}$ on $\text{Fun}(G)$:

$$L_{g_1}f(g_2) \equiv f(g_1g_2)|_{g_2}, \quad L_{g_1} : \text{Fun}(G) \to \text{Fun}(G)$$  \hspace{1cm} (2.5)

where $f(g_1g_2)|_{g_2}$ means $f(g_1g_2)$ seen as a function of $g_2$. Similarly we can define the right action on $\text{Fun}(G)$ as:

$$(R_{g_1}f)(g_2) = f(g_2g_1)|_{g_2}$$  \hspace{1cm} (2.6)

For the basis functions we find easily:

$$L_{g_1}x^g = x^{g^{-1}g_1}, \quad R_{g_1}x^g = x^{gg_1^{-1}}$$  \hspace{1cm} (2.7)

Moreover:

$$L_{g_1}L_{g_2} = L_{g_1g_2}, \quad R_{g_1}R_{g_2} = R_{g_2g_1}, \quad L_{g_1}R_{g_2} = R_{g_2}L_{g_1}$$  \hspace{1cm} (2.8-2.9)

**Bicovariant differential calculus**

Differential calculi can be constructed on Hopf algebras $A$ by algebraic means, using the costructures of $A$ [3]. In the case of finite groups $G$, differential calculi on $A = \text{Fun}(G)$ have been discussed in ref.s [5, 6, 7]. Here we give the main results derived in [7], to which we refer for a more detailed treatment.

A first-order differential calculus on $A$ is defined by

i) a linear map $d: A \to \Gamma$, satisfying the Leibniz rule

$$d(ab) = (da)b + a(db), \quad \forall a, b \in A;$$  \hspace{1cm} (2.10)
The “space of 1-forms” $\Gamma$ is an appropriate bimodule on $A$, which essentially means that its elements can be multiplied on the left and on the right by elements of $A$ [more precisely $A$ is a left module if $\forall a, b \in A, \forall \rho, \rho' \in \Gamma$ we have: $a(\rho + \rho') = a\rho + a\rho'$, $(a + b)\rho = a\rho + b\rho$, $a(b\rho) = (ab)\rho$, $I\rho = \rho$. Similarly one defines a right module. A left and right module is a bimodule if $a(\rho b) = (a\rho)b$. From the Leibniz rule $da = d(Ia) = (dI)a + Ida$ we deduce $dI = 0$.

ii) the possibility of expressing any $\rho \in \Gamma$ as
\[
\rho = \sum_k a_k db_k
\]
for some $a_k, b_k$ belonging to $A$.

To build a first order differential calculus on $Fun(G)$ we need to extend the algebra $A = Fun(G)$ to a differential algebra of elements $x^g, dx^g$ (it is sufficient to consider the basis elements and their differentials). Note however that the $dx^g$ are not linearly independent. In fact from $0 = dI = d(\sum_{g \in G} x^g) = \sum_{g \in G} dx^g$ we see that only $n - 1$ differentials are independent. Every element $\rho = adb$ of $\Gamma$ can be expressed as a linear combination (with complex coefficients) of terms of the type $x^g dx^g$. Moreover $\rho b \in \Gamma$ (i.e. $\Gamma$ is also a right module) since the Leibniz rule and the multiplication rule (2.3) yield the commutations:
\[
dx^g x^g' = -x^g dx^g' + \delta^g_{g'} dx^g
\]
allowing to reorder functions to the left of differentials.

**Partial derivatives**

Consider the differential of a function $f \in Fun(g)$:
\[
df = \sum_{g \in G} f_g dx^g = \sum_{g \neq e} f_g dx^g + f_e dx^e = \sum_{g \neq e} (f_g - f_e) dx^g \equiv \sum_{g \neq e} \partial_g f dx^g
\]
We have used $dx^e = -\sum_{g \neq e} dx^g$ (from $\sum_{g \in G} dx^g = 0$). The partial derivatives of $f$ have been defined in analogy with the usual differential calculus, and are given by
\[
\partial_g f = f_g - f_e = f(g) - f(e)
\]
Not unexpectedly, they take here the form of finite differences (discrete partial derivatives at the origin $e$).

**Left and right covariance**

A differential calculus is left or right covariant if the left or right action of $G$ ($L_g$, or $R_g$) commutes with the exterior derivative $d$. Requiring left and right covariance in fact defines the action of $L_g$ and $R_g$ on differentials: $L_g db \equiv d(L_g b), \forall b \in Fun(G)$
and similarly for $R_g db$. More generally, on elements of $\Gamma$ (one-forms) we define $L_g$ as:

$$L_g(adb) \equiv (L_g a)L_g db = (L_g a)d(L_g b)$$

(2.15)

and similar for $R_g$. Computing for example the left and right action on the differentials $dx^g$ yields:

$$L_g(dx^{g_1}) \equiv d(L_g x^{g_1}) = dx^{g_1}g^{-1}, \quad R_g(dx^{g_1}) \equiv d(R_g x^{g_1}) = dx^{g_1}g^{-1}$$

(2.16)

A differential calculus is called \textit{bicovariant} if it is both left and right covariant.

\textbf{Left invariant one forms}

As in usual Lie group manifolds, we can introduce a basis in $\Gamma$ of left-invariant one-forms $\theta^g$:

$$\theta^g \equiv \sum_{h \in G} x^{hg}dx^h \quad (= \sum_{h \in G} x^h dx^{hg^{-1}}),$$

(2.17)

It is immediate to check that $L_h \theta^g = \theta^g$. The relations (2.17) can be inverted:

$$dx^h = \sum_{g \in G} (x^{hg} - x^h)\theta^g$$

(2.18)

From $0 = dI = d\sum_{g \in G} x^g = \sum_{g \in G} dx^g = 0$ one finds:

$$\sum_{g \in G} \theta^g = \sum_{g \in G} \sum_{h \in G} x^h dx^{hg^{-1}} = \sum_{h \in G} x^h \sum_{g \in G} dx^{hg^{-1}} = 0$$

(2.19)

Therefore we can take as basis of the cotangent space $\Gamma$ the $n-1$ linearly independent left-invariant one-forms $\theta^g$ with $g \neq e$ (but smaller sets of $\theta^g$ can be consistently chosen as basis, see later).

The commutations between the basic 1-forms $\theta^g$ and functions $f \in Fun(G)$ are given by:

$$f \theta^g = \theta^g R_g f$$

(2.20)

Thus functions do commute between themselves (i.e. $Fun(G)$ is a commutative algebra) but do not commute with the basis of one-forms $\theta^g$. In this sense the differential geometry of $Fun(G)$ is noncommutative, the noncommutativity being milder than in the case of quantum groups $Fun_q(G)$ (which are noncommutative algebras).

The right action of $G$ on the elements $\theta^g$ is given by:

$$R_h \theta^g = \theta^{ad(h)g}, \quad \forall h \in G$$

(2.21)

where $ad$ is the adjoint action of $G$ on $G$, i.e. $ad(h)g \equiv hgh^{-1}$. Then \textit{bicovariant calculi are in 1-1 correspondence with unions of conjugacy classes (different from $\{e\}$)} [5]: if $\theta^g$ is set to zero, one must set to zero all the $\theta^{ad(h)g}$, $\forall h \in G$ corresponding to the whole conjugation class of $g$. 

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We denote by \( G' \) the subset corresponding to the union of conjugacy classes that characterizes the bicovariant calculus on \( G \) (\( G' = \{ g \in G | \theta^g \neq 0 \} \)). Unless otherwise indicated, repeated indices are summed on \( G' \) in the following.

A bi-invariant (i.e. left and right invariant) one-form \( \Theta \) is obtained by summing on all \( \theta^g \) with \( g \neq e \):

\[
\Theta = \sum_{g \neq e} \theta^g \quad (2.22)
\]

**Exterior product**

For a bicovariant differential calculus on a Hopf algebra \( A \) an exterior product, compatible with the left and right actions of \( G \), can be defined by

\[
\theta^g_1 \wedge \theta^g_2 = \theta^g_1 \otimes \theta^g_2 - \theta^{g_1^{-1}g_2g_1} \otimes \theta^{g_1} \quad (2.23)
\]

where the tensor product between elements \( \rho, \rho' \in \Gamma \) is defined to have the properties

\[
\rho a \otimes \rho' = \rho \otimes \rho' a, \quad a(\rho \otimes \rho') = (a \rho) \otimes \rho' \quad \text{and} \quad (\rho \otimes \rho') a = \rho \otimes (\rho' a).
\]

Note that:

\[
\theta^g \wedge \theta^g = 0 \quad \text{(no sum on} \ g) \quad (2.24)
\]

Left and right actions on \( \Gamma \otimes \Gamma \) are simply defined by:

\[
\mathcal{L}_h(\rho \otimes \rho') = \mathcal{L}_h\rho \otimes \mathcal{L}_h\rho', \quad \mathcal{R}_h(\rho \otimes \rho') = \mathcal{R}_h\rho \otimes \mathcal{R}_h\rho' \quad (2.25)
\]

(with the obvious generalization to \( \Gamma \otimes \ldots \otimes \Gamma \)) so that for example:

\[
\mathcal{L}_h(\theta^i \otimes \theta^j) = \theta^i \otimes \theta^j, \quad \mathcal{R}_h(\theta^i \otimes \theta^j) = \theta^{ad(h)i} \otimes \theta^{ad(h)j} \quad (2.26)
\]

We can generalize the definition (2.28) to exterior products of \( n \) one-forms:

\[
\theta^{i_1} \wedge \ldots \wedge \theta^{i_n} \equiv W^{i_1 i_2}_{j_1 k_1} W^{k_1 i_3}_{j_2 k_2} \ldots W^{k_{n-2} i_n}_{j_{n-1} k_n} \theta^{j_1} \otimes \ldots \otimes \theta^{j_n} \quad (2.27)
\]

where the matrix \( W \) is defined by:

\[
\theta^i \wedge \theta^j = W^{ij}_{kl} \theta^k \otimes \theta^l = \theta^i \otimes \theta^j - \Lambda^{ij}_{kl} \theta^k \otimes \theta^l \quad (2.28)
\]

and \( \Lambda^{ij}_{kl} \) is the braiding matrix defined by (2.23). The space of \( n \)-forms \( \Gamma^\wedge n \) is therefore defined as in the usual case but with the new permutation operator \( \Lambda \), and can be shown to be a bicovariant bimodule, with left and right action defined as for \( \Gamma \otimes \ldots \otimes \Gamma \) with the tensor product replaced by the wedge product.

**Exterior derivative**

Having the exterior product we can define the exterior derivative

\[
d : \Gamma \to \Gamma \wedge \Gamma \quad (2.29)
\]

\[
d(a_k da_k) = da_k \wedge db_k, \quad (2.30)
\]
which can easily be extended to $\Gamma^\wedge n$ ($d : \Gamma^\wedge n \to \Gamma^\wedge(n+1)$), and has the following properties:

$$d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^k \rho \wedge d\rho'$$

(2.31)

$$d(d\rho) = 0$$

(2.32)

$$L_g(d\rho) = dL_g\rho$$

(2.33)

$$R_g(d\rho) = dR_g\rho$$

(2.34)

where $\rho \in \Gamma^\wedge k$, $\rho' \in \Gamma^\wedge n$. The last two properties express the fact that $d$ commutes with the left and right action of $G$.

**Tangent vectors**

Using (2.18) to expand $df$ on the basis of the left-invariant one-forms $\theta^g$ defines the (left-invariant) tangent vectors $t_g$:

$$df = \sum_{g \in G} f_g dx^g = \sum_{h \in G'} (R_{h^{-1}} f - f)\theta^h \equiv \sum_{h \in G'} (t_h f)\theta^h$$

(2.35)

so that the “flat” partial derivatives $t_h f$ are given by

$$t_h f = R_{h^{-1}} f - f$$

(2.36)

The Leibniz rule for the flat partial derivatives $t_g$ reads:

$$t_g(ff') = (t_g f)R_{g^{-1}}f' + ftt'_g$$

(2.37)

In analogy with ordinary differential calculus, the operators $t_g$ appearing in (2.33) are called (left-invariant) tangent vectors, and in our case are given by

$$t_g = R_{g^{-1}} - id$$

(2.38)

They satisfy the composition rule:

$$t_g t_g' = \sum_h C^h_{g,g'} t_h$$

(2.39)

where the structure constants are:

$$C^h_{g,g'} = \delta^h_{g'g} - \delta^h_g - \delta^h_{g'}$$

(2.40)

and have the property:

$$C^{ad(h)g_1}_{ad(h)g_2,ad(h)g_3} = C^{g_1}_{g_2,g_3}$$

(2.41)

**Note 2.1** : The exterior derivative on any $f \in Fun(G)$ can be expressed as a commutator of $f$ with the bi-invariant one-form $\Theta$:

$$df = [\Theta, f]$$

(2.42)

as one proves by using (2.20) and (2.35).
Note 2.2: From the fusion rules (2.39) we deduce the “deformed Lie algebra” (cf. ref.s [8, 9, 11]):

\[ t_{g_1} t_{g_2} - \Lambda_{g_1 g_2}^{g_3 g_4} t_{g_3} t_{g_4} = C_{g_1 g_2}^{g_3} t_{g_3} \]  

(2.43)

where the \( C \) structure constants are given by:

\[ C_{g_1 g_2}^{g_3} = C_{g_1 g_2}^{g_3} - \Lambda_{g_1 g_2}^{g_3 g_4} - \delta_{g_1}^{g_3} - \delta_{g_2}^{g_4} \]  

(2.44)

and besides property (2.41) they also satisfy:

\[ C_{g_1 g_2}^{g_3} = C_{g_4 g_2}^{g_1} \]  

(2.45)

Moreover the following identities hold:

i) deformed Jacobi identities:

\[ C_{h_1 g_1}^{h_2} C_{h_2}^{g_2} - \Lambda_{g_1 g_2}^{g_3 g_4} C_{h_1 g_3}^{h_2} C_{h_2 g_4}^{g_2} = C_{h_1 g_2}^{h_3} C_{h_2 h_3}^{g_2} \]  

(2.46)

ii) fusion identities:

\[ C_{h_1 g}^{h_2} = C_{g_1 g}^{h_2} \]  

(2.47)

Thus the \( C \) structure constants are a representation (the adjoint representation) of the tangent vectors \( t \).

\[ \text{Cartan-Maurer equations, connection and curvature} \]

From the definition (2.17) and eq. (2.20) we deduce the Cartan-Maurer equations:

\[ d\theta^g + \sum_{g_1 g_2} C_{g_1 g_2}^{g_3} \theta^{g_1} \wedge \theta^{g_2} = 0 \]  

(2.48)

where the structure constants \( C_{g_1 g_2}^{g_3} \) are those given in (2.40).

Parallel transport of the vielbein \( \theta^g \) can be defined as in ordinary Lie group manifolds:

\[ \nabla \theta^g = -\omega^g_{g'} \otimes \theta^{g'} \]  

(2.49)

where \( \omega^g_{g_2} \) is the connection one-form:

\[ \omega^g_{g_2} = \Gamma^g_{g_1 g_2} \theta^{g_3} \]  

(2.50)

Thus parallel transport is a map from \( \Gamma \) to \( \Gamma \otimes \Gamma \); by definition it must satisfy:

\[ \nabla(a \rho) = (da) \otimes \rho + a \nabla \rho, \quad \forall a \in A, \quad \rho \in \Gamma \]  

(2.51)

and it is a simple matter to verify that this relation is satisfied with the usual parallel transport of Riemannian manifolds. As for the exterior differential, \( \nabla \) can be extended to a map \( \nabla : \Gamma^{\wedge n} \otimes \Gamma \rightarrow \Gamma^{\wedge(n+1)} \otimes \Gamma \) by defining:

\[ \nabla(\varphi \otimes \rho) = d\varphi \otimes \rho + (-1)^n \varphi \nabla \rho \]  

(2.52)
Requiring parallel transport to commute with the left and right action of \( G \) means:

\[
\mathcal{L}_h (\nabla \theta^g) = \nabla (\mathcal{L}_h \theta^g) = \nabla \theta^g \\
\mathcal{R}_h (\nabla \theta^g) = \nabla (\mathcal{R}_h \theta^g) = \nabla \theta^{ad(h)g}
\]

(2.53) \hspace{1cm} (2.54)

Recalling that \( \mathcal{L}_h (a \rho) = (\mathcal{L}_h a)(\mathcal{L}_h \rho) \) and \( \mathcal{L}_h (\rho \otimes \rho') = (\mathcal{L}_h \rho) \otimes (\mathcal{L}_h \rho'), \ \forall a \in A, \ \rho, \ \rho' \in \Gamma \) (and similar for \( \mathcal{R}_h \)), and substituting (2.49) yields respectively:

\[
\Gamma_{g_1 g_3, g_2} \in C
\]

(2.55) and

\[
\Gamma_{ad(h)g_1 \ ad(h)g_2} = \Gamma_{g_1 g_3, g_2}
\]

(2.56)

Therefore the same situation arises as in the case of Lie groups, for which parallel transport on the group manifold commutes with left and right action iff the connection components are \( ad(G) \) - conserved constant tensors. As for Lie groups, condition (2.56) is satisfied if one takes \( \Gamma \) proportional to the structure constants. In our case, we can take any combination of the \( C \) or \( C \) structure constants, since both are \( ad(G) \) conserved constant tensors. As we see below, the \( C \) constants can be used to define a torsionless connection, while the \( C \) constants define a parallelizing connection.

As usual, the curvature arises from \( \nabla^2 \):

\[
\nabla^2 \theta^g = - R^g \ g' \otimes \theta^g
\]

(2.57)

\[
R^{g_1}_{g_2} \equiv d \omega^{g_1}_{g_2} + \omega^{g_1}_{g_3} \wedge \omega^{g_3}_{g_2}
\]

(2.58)

The torsion \( R^g \) is defined by:

\[
R^g \equiv d \theta^{g_1} + \omega^{g_1}_{g_2} \wedge \theta^{g_2}
\]

(2.59)

Using the expression of \( \omega \) in terms of \( \Gamma \) and the Cartan-Maurer equations yields

\[
R^{g_1}_{g_2} = ( - \Gamma^{g_1}_{h,g_2} C^{h}_{g_3,g_4} + \Gamma^{g_1}_{g_3,h} \Gamma^{h}_{g_4,g_2} ) \theta^{g_3} \wedge \theta^{g_4} =
\]

\[
( - \Gamma^{g_1}_{h,g_2} C^{h}_{g_3,g_4} + \Gamma^{g_1}_{g_3,h} \Gamma^{h}_{g_4,g_2} - \Gamma^{g_1}_{g_4,h} \Gamma^{h}_{g_4 g_3 g_4 -1 , g_2} ) \theta^{g_3} \otimes \theta^{g_4}
\]

(2.60)

\[
R^{g_1} = ( - C^{g_1}_{g_2,g_3} + \Gamma^{g_1}_{g_2,g_3} ) \theta^{g_2} \wedge \theta^{g_3} =
\]

\[
( - C^{g_1}_{g_2,g_3} + \Gamma^{g_1}_{g_2,g_3} - \Gamma^{g_1}_{g_3,g_2,g_3 -1 } ) \theta^{g_2} \otimes \theta^{g_3}
\]

(2.61)

Thus a connection satisfying:

\[
\Gamma^{g_1}_{g_2,g_3} - \Gamma^{g_1}_{g_3,g_2,g_3 -1 } = C^{g_1}_{g_2,g_3}
\]

(2.62)

corresponds to a vanishing torsion \( R^g = 0 \) and could be referred to as a “Riemannian” connection.
On the other hand, the choice:

\[ \Gamma_{g_1 g_2 g_3}^{g_1} = C_{g_5 g_2}^{-1} \]  

(2.63)
corresponds to a vanishing curvature \( R_{g'_p}^{g' q} = 0 \), as can be checked by using the fusion equations (2.47) and property (2.45). Then (2.63) can be called the parallelizing connection: finite groups are parallelizable.

**Tensor transformations**

Under the familiar transformation of the connection 1-form:

\[ (\omega^i_j)' = a^i_k \omega^k_l (a^{-1})^l_j + a^i_k d(a^{-1})^k_j \]  

(2.64)
the curvature 2-form transforms homogeneously:

\[ (R^i_j)' = a^i_k R^k_l (a^{-1})^l_j \]  

(2.65)
The transformation rule (2.64) can be seen as induced by the change of basis \( \theta^i = a^i_j \theta^j \), with \( a^i_j \) invertible \( x \)-dependent matrix (use eq. (2.51) with \( a^\rho = a^i_j \theta^j \)).

**Metric**

The metric tensor \( \gamma \) can be defined as an element of \( \Gamma \otimes \Gamma \):

\[ \gamma = \gamma_{i,j} \theta^i \otimes \theta^j \]  

(2.66)
Requiring it to be invariant under left and right action of \( G \) means:

\[ \mathcal{L}_h(\gamma) = \gamma = \mathcal{R}_h(\gamma) \]  

(2.67)
or equivalently, by recalling \( \mathcal{L}_h(\theta^i \otimes \theta^j) = \theta^i \otimes \theta^j \), \( \mathcal{R}_h(\theta^i \otimes \theta^j) = \theta^{ad(h)i} \otimes \theta^{ad(h)j} \):

\[ \gamma_{i,j} \in C, \quad \gamma_{ad(h)i, ad(h)j} = \gamma_{i,j} \]  

(2.68)
These properties are analogous to the ones satisfied by the Killing metric of Lie groups, which is indeed constant and invariant under the adjoint action of the Lie group.

On finite \( G \) there are various choices of biinvariant metrics. One can simply take \( \gamma_{i,j} = \delta_{i,j} \), or \( \gamma_{i,j} = C^k_{i,l} C^l_{k,j} \).

For any biinvariant metric \( \gamma_{ij} \) there are tensor transformations \( a^i_j \) under which \( \gamma_{ij} \) is invariant, i.e.:

\[ a^h_{h'} \gamma_{h,k} a^k_{k'} = \gamma_{h',k'} \Leftrightarrow a^h_{h'} \gamma_{h,k} \gamma_{h',k'} (a^{-1})^k_k \]  

(2.69)
These transformations are simply given by the matrices that rotate the indices according to the adjoint action of \( G \):

\[ a^h_{h'} (g) = \delta^h_{h'} (\alpha(g)) h \]  

(2.70)
where $\alpha(g) : G \mapsto G$ is an arbitrary mapping. Then these matrices are functions of $G$ via this mapping, and their action leaves $\gamma$ invariant because of its biinvariance \((2.68)\). Indeed substituting these matrices in \((2.69)\) yields:

$$a^h_{\;h'}(g) \gamma_{h,k} a^k_{\;k'}(g) = \gamma_{ad([\alpha(g)]^{-1})h',ad([\alpha(g)]^{-1})k'} = \gamma_{h',k'}$$

\((2.71)\)

proving the invariance of $\gamma$.

Consider now a contravariant vector $\varphi^i$ transforming as $(\varphi^i)' = a^i_j(\varphi^j)$. Then using \((2.69)\) one can easily see that

$$(\varphi^k \gamma_{k,i})' = \varphi^k \gamma_{k',i'}(a^{-1})^{i'}_i$$

\((2.72)\)

i.e. the vector $\varphi_i \equiv \varphi^k \gamma_{k,i}$ indeed transforms as a covariant vector.

### Lie derivative and diffeomorphisms

The notion of diffeomorphisms, or general coordinate transformations, is fundamental in gravity theories. Is there such a notion in the setting of differential calculi on Hopf algebras? The answer is affirmative, and has been discussed in detail in ref.s \([9, 10, 11]\). As for differentiable manifolds, it relies on the existence of the Lie derivative.

Let us review the situation for Lie group manifolds. The Lie derivative $l_{t_i}$ along a left-invariant tangent vector $t_i$ is related to the infinitesimal right translations generated by $t_i$:

$$l_{t_i} \rho = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mathcal{R}_{\exp[\varepsilon t_i]} \rho - \rho]$$

\((2.73)\)

$\rho$ being an arbitrary tensor field. Introducing the coordinate dependence

$$l_{t_i} \rho(y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\rho(y + \varepsilon t_i) - \rho(y)]$$

\((2.74)\)

identifies the Lie derivative $l_{t_i}$ as a directional derivative along $t_i$. Note the difference in meaning of the symbol $t_i$ in the r.h.s. of these two equations: a group generator in the first, and the corresponding tangent vector in the second.

For finite groups the Lie derivative takes the form:

$$l_{t_g} \rho = [\mathcal{R}_{g^{-1}} \rho - \rho]$$

\((2.75)\)

so that the Lie derivative is simply given by

$$l_{t_g} = \mathcal{R}_{g^{-1}} - id = t_g$$

\((2.76)\)

cf. the definition of $t_g$ in \((2.38)\). For example

$$l_{t_g}(\vartheta^{g_1} \otimes \vartheta^{g_2}) = \vartheta^{ad(g^{-1})g_1} \otimes \vartheta^{ad(g^{-1})g_2} - \vartheta^{g_1} \otimes \vartheta^{g_2}$$

\((2.77)\)
As in the case of differentiable manifolds, the Cartan formula for the Lie derivative acting on p-forms holds:

\[ l_g = i_g d + di_g \]  

(2.78)

see ref.s [9, 11, 7].

Exploiting this formula, diffeomorphisms (Lie derivatives) along generic tangent vectors \( V \) can also be consistently defined via the operator:

\[ l_V = i_V d + di_V \]  

(2.79)

This requires a suitable definition of the contraction operator \( i_V \) along generic tangent vectors \( V \), discussed in ref. [11, 7].

We have then a way of defining “diffeomorphisms” along arbitrary (and \( x \)-dependent) tangent vectors for any tensor \( \rho \):

\[ \delta \rho = l_V \rho \]  

(2.80)

and of testing the invariance of candidate lagrangians under the generalized Lie derivative.

**Haar measure and integration**

Since we want to be able to define actions (integrals on \( p \)-forms) we must now define integration of \( p \)-forms on finite groups.

Let us start with integration of functions \( f \). We define the integral map \( h \) as a linear functional \( h : Fun(G) \rightarrow \mathbb{C} \) satisfying the left and right invariance conditions:

\[ h(\mathcal{L}_{g} f) = 0 = h(\mathcal{R}_{g} f) \]  

(2.81)

Then this map is uniquely determined (up to a normalization constant), and is simply given by the “sum over \( G \)” rule:

\[ h(f) = \sum_{g \in G} f(g) \]  

(2.82)

Next we turn to define the integral of a \( p \)-form. Within the differential calculus we have a basis of left-invariant 1-forms, which may allow the definition of a biinvariant volume element. In general for a differential calculus with \( n \) independent tangent vectors, there is an integer \( p \geq n \) such that the linear space of \( p \)-forms is 1-dimensional, and \((p + 1)\)-forms vanish identically. We will see explicit examples in the next Section. This means that every product of \( p \) basis one-forms \( \theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} \) is proportional to one of these products, that can be chosen to define the volume form \( vol \):

\[ \theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} = \epsilon^{g_1, g_2, ..., g_p} vol \]  

(2.83)

where \( \epsilon^{g_1, g_2, ..., g_p} \) is the proportionality constant. Note that the volume \( p \)-form is obviously left invariant. We can prove that it is also right invariant with the following
argument. Suppose that \( \text{vol} \) be given by \( \theta^{h_1} \wedge \theta^{h_2} \wedge \ldots \wedge \theta^{h_p} \) where \( h_1, h_2, \ldots h_p \) are given group element labels. Then the right action on \( \text{vol} \) yields:

\[
\mathcal{R}_g[\theta^{h_1} \wedge \ldots \wedge \theta^{h_p}] = \theta^{ad(g)h_1} \wedge \ldots \wedge \theta^{ad(g)h_p} = \epsilon^{ad(g)h_1 \ldots ad(g)h_p \text{vol}} \quad (2.84)
\]

Recall now that the “epsilon tensor” \( \epsilon \) is necessarily made out of the \( W \) tensors of eq. \((2.28)\), defining the wedge product. These tensors are invariant under the adjoint action \( ad(g) \), and so is the \( \epsilon \) tensor. Therefore

\[
\epsilon^{ad(g)h_1 \ldots ad(g)h_p} = \epsilon^{h_1 \ldots h_p} = 1
\]

and \( \mathcal{R}_g \text{vol} = \text{vol} \). This will be verified in the examples of next Section.

Having identified the volume \( p \)-form it is natural to set

\[
\int f \text{vol} \equiv h(f) = \sum_{g \in G} f(g) \quad (2.85)
\]

and define the integral on a \( p \)-form \( \rho \) as:

\[
\int \rho = \int \rho_{g_1, \ldots, g_p} \theta^{g_1} \wedge \ldots \wedge \theta^{g_p} = \int \rho_{g_1, \ldots, g_p} \epsilon^{g_1 \ldots g_p} \text{vol} \equiv \sum_{g \in G} \rho_{g_1, \ldots, g_p}(g) \epsilon^{g_1 \ldots g_p} \quad (2.86)
\]

Due to the biinvariance of the volume form, the integral map \( \int : \Gamma^p \rightarrow \mathcal{C} \) satisfies the biinvariance conditions:

\[
\int \mathcal{L}_g f = \int f = \int \mathcal{R}_g f \quad (2.87)
\]

Moreover, under the assumption that the volume form belongs to a nontrivial cohomology class, that is \( d(\text{vol}) = 0 \) but \( \text{vol} \neq d\rho \), the important property holds:

\[
\int df = 0 \quad (2.88)
\]

with \( f \) any \((p-1)\)-form: \( f = f_{g_2, \ldots, g_p} \theta^{g_2} \wedge \ldots \wedge \theta^{g_p} \). This property, which allows integration by parts, has a simple proof. Rewrite \( \int df \) as:

\[
\int df = \int (df_{g_2, \ldots, g_p}) \theta^{g_2} \wedge \ldots \wedge \theta^{g_p} + \int f_{g_2, \ldots, g_p} d(\theta^{g_2} \wedge \ldots \wedge \theta^{g_p}) \quad (2.89)
\]

Under the cohomology assumption the second term in the r.h.s. vanishes, since \( d(\theta^{g_2} \wedge \ldots \wedge \theta^{g_p}) = 0 \) (otherwise, being a \( p \)-form, it should be proportional to \( \text{vol} \), and this would contradict the assumption \( \text{vol} \neq d\rho \)). Using now (2.35) and (2.85):

\[
\int df = \int (t_{g_1, f_{g_2, \ldots, g_p}}) \theta^{g_1} \wedge \theta^{g_2} \wedge \ldots \wedge \theta^{g_p} = \int [\mathcal{R}_{g_1}^{-1} f_{g_2, \ldots, g_p} - f_{g_2, \ldots, g_p}] \epsilon^{g_1 \ldots g_p} \text{vol} = 0 \quad (2.90)
\]

Q.E.D.
3 Bicovariant calculus on $S_3$

In this Section we illustrate the general theory on the particular example of the permutation group $S_3$.

Elements: $a = (12)$, $b = (23)$, $c = (13)$, $ab = (132)$, $ba = (123)$, $e$.

Nontrivial conjugation classes: $I = [a,b,c]$, $II = [ab,ba]$.

There are 3 bicovariant calculi $BC_I$, $BC_{II}$, $BC_{I+II}$ corresponding to the possible unions of the conjugation classes \[3\]. They have respectively dimension 3, 2 and 5. We examine here the $BC_I$ and $BC_{II}$ calculi.

$BC_I$ differential calculus

Basis of the 3-dimensional vector space of one-forms:

$$\theta^a, \theta^b, \theta^c$$

(3.1)

Basis of the 4-dimensional vector space of two-forms:

$$\theta^a \wedge \theta^b, \theta^b \wedge \theta^c, \theta^a \wedge \theta^c, \theta^c \wedge \theta^b$$

(3.2)

Every wedge product of two $\theta$ can be expressed as linear combination of the basis elements:

$$\theta^b \wedge \theta^a = -\theta^a \wedge \theta^c - \theta^c \wedge \theta^b, \quad \theta^c \wedge \theta^a = -\theta^a \wedge \theta^b - \theta^b \wedge \theta^c$$

(3.3)

Basis of the 3-dimensional vector space of three-forms:

$$\theta^a \wedge \theta^b \wedge \theta^c, \theta^a \wedge \theta^c \wedge \theta^b, \theta^b \wedge \theta^a \wedge \theta^c$$

(3.4)

and we have:

$$\theta^c \wedge \theta^b \wedge \theta^a = -\theta^c \wedge \theta^a \wedge \theta^b = -\theta^a \wedge \theta^b \wedge \theta^c$$

$$\theta^b \wedge \theta^c \wedge \theta^a = -\theta^b \wedge \theta^a \wedge \theta^c = -\theta^a \wedge \theta^c \wedge \theta^b$$

$$\theta^c \wedge \theta^a \wedge \theta^b = -\theta^c \wedge \theta^b \wedge \theta^a = -\theta^b \wedge \theta^a \wedge \theta^c$$

(3.5)

Basis of the 1-dimensional vector space of four-forms:

$$\text{vol} = \theta^a \wedge \theta^b \wedge \theta^c$$

(3.6)

and we have:

$$\theta^{g_1} \wedge \theta^{g_2} \wedge \theta^{g_3} \wedge \theta^{g_4} = \epsilon^{g_1,g_2,g_3,g_4} \text{vol}$$

(3.7)

where the nonvanishing components of the $\epsilon$ tensor are:

$$\epsilon_{abac} = \epsilon_{acab} = \epsilon_{cbca} = \epsilon_{cabc} = \epsilon_{abc} = \epsilon_{bca} = 1$$

$$\epsilon_{baca} = \epsilon_{caba} = \epsilon_{abcb} = \epsilon_{cbab} = \epsilon_{abc} = \epsilon_{baca} = -1$$

(3.8)

(3.9)
Cartan-Maurer equations:

\[
\begin{align*}
    d\theta^a + \theta^b \wedge \theta^c + \theta^c \wedge \theta^b &= 0 \\
    d\theta^b + \theta^a \wedge \theta^c + \theta^c \wedge \theta^a &= 0 \\
    d\theta^c + \theta^a \wedge \theta^b + \theta^b \wedge \theta^a &= 0
\end{align*}
\]

The exterior derivative on any three-form of the type \( \theta \wedge \theta \wedge \theta \) vanishes, as one can easily check by using the Cartan-Maurer equations and the equalities between exterior products given above. Then, as shown in the previous section, integration of a total differential vanishes on the “group manifold” of \( S_3 \) corresponding to the \( BC_I \) bicovariant calculus. This “group manifold” has three independent directions, associated to the cotangent basis \( \theta^a, \theta^b, \theta^c \). Note however that the volume element is of order four in the left-invariant one-forms \( \theta \).

**BC\( _{II} \) differential calculus**

Basis of the 2-dimensional vector space of one-forms:

\[
\theta^{ab}, \quad \theta^{ba} \quad (3.11)
\]

Basis of the 1-dimensional vector space of two-forms:

\[
vol = \theta^{ab} \wedge \theta^{ba} = -\theta^{ba} \wedge \theta^{ab} \quad (3.12)
\]

so that:

\[
\theta^{g_1} \wedge \theta^{g_2} = \epsilon^{g_1,g_2} vol \quad (3.13)
\]

where the \( \epsilon \) tensor is the usual 2-dimensional Levi-Civita tensor.

Cartan-Maurer equations:

\[
d\theta^{ab} = 0, \quad d\theta^{ba} = 0 \quad (3.14)
\]

Thus the exterior derivative on any one-form \( \theta^g \) vanishes and integration of a total differential vanishes on the group manifold of \( S_3 \) corresponding to the \( BC_{II} \) bicovariant calculus. This group manifold has two independent directions, associated to the cotangent basis \( \theta^{ab}, \theta^{ba} \).

**Visualization of the \( S_3 \) group “manifold”**

We can draw a picture of the group manifold of \( S_3 \). It is made out of 6 points, corresponding to the group elements and identified with the functions \( x^e, x^a, x^b, x^c, x^{ab}, x^{ba} \).

\( BC_I \) - calculus:

From each of the six points \( x^g \) one can move in three directions, associated to the tangent vectors \( t_a, t_b, t_c \), reaching three other points whose “coordinates” are

\[
\mathcal{R}_a x^g = x^g, \quad \mathcal{R}_b x^g = x^{gb}, \quad \mathcal{R}_c x^g = x^{gc} \quad (3.15)
\]
The 6 points and the “moves” along the 3 directions are illustrated in Fig. 1. The links are not oriented since the three group elements $a, b, c$ coincide with their inverses.

$BC_{II}$ - calculus:

From each of the six points $x^g$ one can move in two directions, associated to the tangent vectors $t_{ab}, t_{ba}$, reaching two other points whose “coordinates” are

$$R_{ab}x^g = x^{gab}, \quad R_{ba}x^g = x^{gab}$$

(3.16)

The 6 points and the “moves” along the 3 directions are illustrated in Fig. 1. The arrow convention on a link labeled (in italic) by a group element $h$ is as follows: one moves in the direction of the arrow via the action of $R_h$ on $x^g$. (In this case $h = ab$). To move in the opposite direction just take the inverse of $h$.

The pictures in Fig. 1 characterize the bicovariant calculi $BC_{I}$ and $BC_{II}$ on $S_3$, and were drawn in ref. [5] as examples of digraphs, used to characterize different calculi on sets. Here we emphasize their geometrical meaning as finite group “manifolds”.

![Fig. 1: $S_3$ group manifold, and moves of the points under the group action](image)

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