On the dimension of ideals in group algebras, and group codes

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February 18, 2020

Abstract

Several relations and bounds for the dimension of principal ideals in group algebras are determined by analyzing minimal polynomials of regular representations. These results are used in the two last sections. First, in the context of semisimple group algebras, to compute, for any abelian code, an element with Hamming weight equal to its dimension. Finally, to get bounds on the minimum distance of certain MDS group codes. A relation between a class of group codes and MDS codes is presented. Examples illustrating the main results are provided.

Keywords. group algebras, principal ideals, primitive idempotents, MDS group codes, abelian codes.

1 Introduction

The group algebra $FG$ of a finite group $G$ over the field $F$ is the set of formal linear combinations of elements in $G$ with coefficients in $F$, i.e., $FG := \left\{ \sum_{g \in G} a_g g : a_g \in F \right\}$. This set is a ring with the usual sum of vectors and the multiplication given by extending the operation of $G$. If $F = \mathbb{F}_q$, a group code is an ideal of $FG$, and an abelian group code is a group code over a commutative group algebra. The Hamming weight $wt_G(x)$ of an element $x \in FG$ is the number of non-zero coefficients in its coordinate vector with respect to the basis $G$. The minimum weight of a group code is the minimum Hamming weight of its non-zero elements.
Finding ways to compute the dimension of ideals of finite-dimensional $F$-algebras is itself of interest. In the context of group coding theory, this is crucial because the dimension is a parameter needed, apart from the minimum distance, to determine how good or bad is a group code for error correction. However, unlike the minimum distance, this aspect is mostly algebraic and thus can be approached using algebraic methods.

In the literature, several cases appear in which the dimension of group codes is being explored. For instance, in [13] R. A. Ferraz, M. Guerreiro, y C. Polcino, determine relations to compute the dimension and minimum weight of minimal abelian codes (only containing themselves and the ideal 0) in $\mathbb{F}_2(C_{p^n} \times C_p)$ where $p$ is an odd prime number and $n \geq 3$. Later, in [10], F. S. Dutra, R. A. Ferraz, and C. Polcino determine the dimension and minimal distance of ideals in the semisimple group algebra of a dihedral group. Recently, in [11], M. Elia and E. Gorla addressed the problem of determining the dimension of a principal group code by studying the characteristic polynomial of the right/left regular representations of a generator.

In this work, we focus on the determination of relations (such that bounds, identities, and congruences) for the dimension of principal ideals in group algebras by studying the minimal polynomial of the right regular representation determined by a generator of the ideal and use these relations to study the dimension of semisimple abelian codes. The manuscript is organized as follows. In Section 2 preliminary results that will be needed throughout the manuscript are presented. In Section 3 by using The Primary Decomposition Theorem, bounds, formulas, and congruences for the dimension of some principal ideals in group algebras are presented. These results are later used in the last two sections, first, in Section 4 to study the dimension of abelian codes in semisimple group algebras. In this case, a formula and a bound for the dimension of certain abelian codes are given, and a linear transformation is determined with the property that its evaluation in the generator idempotent of an ideal has Hamming weight equal to the dimension of the ideal. Finally, in Section 5 to compute bounds on the minimum distance of some MDS group codes that are principal ideals. Examples are included illustrating the main results.

2 Preliminaries

Through this work, $G$ will denote a finite group, $F$ a field, $R = FG$ the group algebra of $G$ over $F$, and for $b \in R$, $r_b$ ($l_b$) will denote the right (left) regular representation of $b$, i.e., the $F$-endomorphism of $R$ given by
\( r_b(w) = wb \ (l_b(w) = bw) \). Also, \( m_b(x) \) and \( p_b(x) \) will denote the minimal and characteristic polynomial of \( r_b \). Further, every module is considered a left module, unless stated otherwise.

Observe that \( G \) is an isomorphic group to \( \rho(G) := \{ r_g : g \in G \} \) with the composition. Thus \( \varphi: FG \rightarrow F\rho(G) \) given by \( \varphi(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g r_g \) is an isomorphism of \( F \)-algebras.

**Lemma 2.1.** Let \( b \in R \), and \( \kappa(x) \) be a polynomial that annihilates \( r_b \). Let \( \kappa(x) = f_0(x) f_1(x) \) be a decomposition into coprime factors. If \( u_0(x), u_1(x) \in F[x] \) are such that \( u_0(x) f_0(x) + u_1(x) f_1(x) = 1 \), then \( \varphi^{-1}(u_i(r_b) f_i(r_b)) \) is an idempotent generator of \( Rf_i(b) \) for \( i = 0, 1 \).

**Proof.** Since \( f_0(x) \) and \( f_1(x) \) are coprime, there exist \( u_0(x), u_1(x) \in F[x] \) such that \( 1 = u_0(x) f_0(x) + u_1(x) f_1(x) \). Let \( E_i := u_i(r_b) f_i(r_b) \) for \( i = 0, 1 \). Then \( id = E_0 + E_1 \), \( E_0 = E_0^2 + E_0 E_1 \) and \( f_0(r_b) = f_0(r_b) E_0 + f_0(r_b) E_1 \), but \( E_0 E_1 = (u_0(r_b) u_1(r_b)) \kappa(r_b) = 0 \) and \( f_0(r_b) E_1 = u_1(r_b) \kappa(r_b) = 0 \), so \( E_0 \) is an idempotent of \( F \rho(G) \) and \( f_0(r_b) \in F \rho(G) E_0 \). By a similar argument, \( E_1 \) is an idempotent of \( F \rho(G) \) and \( f_1(r_b) \in F \rho(G) E_1 \). Now the result follows from the fact that \( \varphi^{-1}(E_i) = u_i(b) f_i(b) \in Rf_i(b) \) for all \( i \). \( \square \)

In [11] Corollary 5] is proposed to compute idempotent generators of a projective ideal by solving a system of multivariate quadratic and linear equations over the field. Lemma 2.1 is an alternative to solve that problem using the Euclidean Algorithm when a special type of generator element is known.

Recall that one of the equivalences of being a projective module is the following [6, pg 29]. \( P \) is a projective \( A \)-module if there exists an \( A \)-module \( P' \) such that \( A^n \cong P \oplus P' \) for some \( n \in \mathbb{Z}^+ \).

**Lemma 2.2.** Let \( J \) be a non-trivial principal ideal of \( R \). The following statements are equivalent:

1. \( J \) has a generator \( b \) such that \( r_b \) is annihilated by a polynomial \( \kappa(x) = xh(x) \) where \( x \mid h(x) \).

2. \( J \) is a projective \( R \)-submodule of \( R \).

3. \( J \) has an idempotent generator.

**Proof.** By [7, Lemma 2.1, part (a)] 2) implies 3). It is clear that 3) implies 1). Suppose \( J \) has a generator \( b \) such that \( xh(x) \) annihilated by a polynomial \( \kappa(x) = xh(x) \) where \( x \mid h(x) \). Then, by Lemma 2.1, \( J \) is generated by an idempotent, so \( J \) is projective (by [7, Lemma 2.1, part (b)]). \( \square \)
Now we will see that the dimensions of the left and the right ideal generated by an element in \( R \) are the same. Recall that the mapping \( * : R \to R \) given by \( u^* = \sum_{g \in G} \lambda_g g^{-1} \), for \( u = \sum_{g \in G} \lambda_g g \), is an antiautomorphism of \( F \)-algebras (see [19, Proposition 3.2.11], [18, pg 5]).

**Lemma 2.3.** Let \( a \in R \), \([r_a]_G\) and \([l_a]_G\) be the matrices of \( r_a \) and \( l_a \) in the basis \( G \), respectively. Then, \( \dim(Ra) = \text{rank}([r_a]_G) = \text{rank}([l_a]_G) = \dim(aR) \).

**Proof.** Since \( Ra = \text{span}_F(\{ga : g \in G\}) \), \( \dim(Ra) = \dim(\text{span}_F(\{[ga]_G : g \in G\}) \) where \([ga]_G\) is the coordinate vector of \( ga \) with respect to the basis \( G \). But these vectors are precisely the columns of \([r_a]_G\), so that \( \dim(Ra) = \text{rank}([r_a]_G) \). By doing an analogous reasoning with \( aR \) and \( l_a \), we get that \( \dim(aR) = \text{rank}([l_a]_G) \). On the other hand, \((Ra)^* = a^*R \) because \((\ )^* \) is an antiautomorphism of \( F \)-algebras. This implies that \( \text{rank}([r_a]_G) = \dim(Ra) = \text{dim}(a^*R) = \dim(aR) = \text{rank}([l_a]_G) \).

If \( F = \mathbb{F}_q \), the ideals \( Ra \) and \( a^*R \) define equivalent group codes because \((\ )^* \) restricted to \( G \) is a permutation.

**Lemma 2.3** is a different version of [11, Proposition 1]. However, **Lemma 2.3** mentions the equality of the dimensions between the left and right ideals generated by the same element, while this Proposition does not.

### 3 Dimension of ideals in group algebras

For convention, when an integer is matched with a modular class, what we mean is that the reduction of this number to the respective module is equal to the modular class. This notation is the same used in [18, Lemma 1.2].

**Lemma 3.1.** Let \( A \in \mathcal{M}_{n \times n}(F) \) be a matrix with minimal polynomial \( m_A(x) = x(x-a)^s \) for some integer \( s \geq 1 \) and \( a \in F - \{0\} \). Then the following statements hold:

1. \( \text{trace}(A)a^{-1} \) lies in the prime field of \( F \).
2. \( \text{rank}(A) = \text{trace}(A)a^{-1} \).

**Proof.** By The Primary Decomposition Theorem [15, Theorem 12, Ch. 6], \( n = \dim(\ker(A)) + \dim(\ker(A-aI)^s) \), then \( \text{rank}(A) = n - \dim(\ker(A)) = \dim(\ker(A-aI)^s) \). If \( N \) is the Jordan canonical form of \( A \), then \( \text{trace}(A) = \text{trace}(N) \) is equal to the sum of \( a \) as many times as it appears in the diagonal of \( N \), this implies that \( \text{trace}(A)a^{-1} \) lies in the prime subfield of \( F \). On the
other hand, the number of non-zero entries in the diagonal of $N$ is equal to \( \dim(\ker(A - aI)^s) \). Thus, if \( \text{char}(F) = 0 \), \( \dim(A) = \text{trace}(A)a^{-1} \). If \( \text{char}(F) = p > 0 \) and \( u = \dim(A) < p \), then \( \dim(A) = \text{trace}(A)a^{-1} \). If \( u = pc + r \) where \( 0 \leq r < p \), then \( \text{trace}(A)a^{-1} = \frac{r}{p} \) which is equal to the reduction of \( u \) modulo \( p \), finishing the proof.

Lemma 3.1(part 1) is related to [19, Theorem 7.2.1, 7.2.2]. In fact, if we restrict these last to finite-dimensional group algebras, then Lemma 3.1(part 1) implies both of them.

For any \( x \in R \), the coefficient of \( x \) at 1 will be denoted by \( \lambda_1(x) \), this is called the trace of \( x \) (see, [19, pg 221], [18, pg 31]).

**Theorem 3.2.** Let \( b \in R \) be such that \( m_b(x) = x^np_1(x)^{r_1} \cdots p_t(x)^{r_t} \) where the \( p_i \) are monic irreducible distinct factors and \( p_i \neq x \) for all \( i \). Let \( \zeta_n = \dim(\ker(r^n_b)/\ker(r_b)) \). Then

1. \( \dim(Rb) = \zeta_n + \sum_{i=1}^{t} \dim(ker(p_i(r_b)^{r_i})) \). Moreover, If \( p_b(x) = x^uh(x) \) with \( x \nmid h(x) \), then \( \dim(Rb) = \zeta_n + |G| - u \).

2. If \( \text{char}(F) = p > 0 \) and \( |G|_p \) the \( p \)-part of \( |G| \), then

\[
\dim(Rb) \geq \begin{cases} 
(t + 1)|G|_p & \text{if } |G|_p \mid \dim(\ker(r_b)) \text{ and } n > 1 \\
t|G|_p & \text{otherwise}
\end{cases}
\]

Moreover, if \( n = 1 \), then \( |G|_p \mid \dim(Rb) \) and \( \dim(Rb) \in [t|G|_p, |G| - |G|_p]\).

3. If \( m_b(x) = x(x-a)^s \) for some \( s \geq 1 \) and \( a \in F - \{0\} \), then \( \dim(Rb) = |G|\lambda_1(b)a^{-1} \).

**Proof.** Let \( U_0 = \ker(r_b) \) and \( U_1 = \ker(r^n_b) \).

1. Let \( W \subset U_1 \) be such that \( U_1 = U_0 \oplus W \). Then, by [15, Theorem 12, Ch. 6] (part i), \( R = (U_0 \oplus W) \oplus \ker(p_1(r_b)^{r_1}) \oplus \cdots \oplus \ker(p_t(r_b)^{r_t}) \). Thus, by The rank-nullity Theorem,

\[
\dim(Rb) = \dim(Im(r_b))
\]
\[
= \dim(W) + \sum_{i=1}^{t} \dim(\ker(p_i(r_b)^{r_i}))
\]
\[
= \zeta_n + \sum_{i=1}^{t} \dim(\ker(p_i(r_b)^{r_i})).
\]
Let \( p_b(x) = x^n h(x) \) with \( x \nmid h(x) \). Let \( U = \ker(r_b^G) \), \( m_1(x) \) and \( m_2(x) \) be the minimal polynomials of \( r_b|_{U_1} \) and \( r_b|_U \), respectively. By [15, Theorem 12, Ch. 6] (part \( iii \)), \( m_1(x) = x^n \). Since \( U_1 \subseteq U \subseteq R \) is a chain of \( r_b \)-invariant spaces, then \( x^n \mid m_2(x) \mid m_b(x) \). So as \( m_2(x) \mid x^G \), \( m_2(x) \mid x^n \), implying that \( m_2(x) = x^n \), and thus \( U_1 = U \). Hence \( \dim(U_1) = \dim(U) \) which is equal to the algebraic multiplicity \( u \) (see [3, pp 171-172]). Therefore \( \sum_{i=1}^{t} \dim(\ker(p_i(r_b)^{\ast})) = |G| - u \).

2. Let \( \text{char}(F) = p > 0 \) and \( |G|_p \) be the \( p \)-part of \( |G| \). Since \( r_b \) is a morphism of \( R \)-modules, by [15, Theorem 12, Ch. 6] (part \( i \)), \( R = U_1 \oplus \ker(p_1(r_b)^{\ast}) \oplus \cdots \oplus \ker(p_t(r_b)^{\ast}) \) is a decomposition of \( R \) as sum of ideals. So \( U_1 \) and \( \ker(p_i(r_b)^{\ast}) \) are projective \( R \)-modules for all \( i \), and by [14, Chap. VII, Corollary 7.16], \( |G|_p \) divides \( \dim(U_1) \) and \( \dim(\ker(p_i(r_b)^{\ast})) \) for all \( i \). Thus, \( t|G|_p \leq \dim(Rb) \). In addition, if \( |G|_p \mid \dim(U_0) \), \( |G|_p \) is a common divisor of \( \dim(U_0) \) and \( \dim(U_1) \). Thus, if \( n > 1 \), \( |G|_p \mid \zeta_n \neq 0 \), implying that \( (t + 1)|G|_p \leq \dim(Rb) \).

If \( n = 1 \), \( Rb \) is projective (by Lemma [2.2]), and thus \( |G|_p \) divides \( \dim(Rb) \). So, as \( Rb \neq R \), \( \dim(Rb) \in \{ t|G|_p, |G| - |G|_p \} \).

3. Let \( m_b(x) = x(x - a)^s \) for some \( s \geq 1 \) and \( a \in F - \{0\} \). Then, by Lemmas [2.3] and [3.1] \( \dim(Rb) = \rank([r_b]_G) = \trace([r_b]_G) a^{-1} \). Finally, by [19, Lemma 7.1.1], \( \trace([r_b]_G) = |G|\lambda_1(b) \), hence \( \dim(Rb) = |G|\lambda_1(b) a^{-1} \).

\( \square \)

Theorem 3.2 (Part 3) is a more general version of [18, Lemma 1.2, part \( ii \)], which is only valid for idempotents. So, the benefit of this result when compared with [18, Lemma 1.2, part \( ii \)] is that it can be applied to a larger amount of elements of \( R \), apart from the idempotents. However, By Lemma [2.2] any element that satisfies the hypothesis of Theorem 3.2 (part 3) generates a projective ideal, implying that this theorem can be applied only to ideals generated by idempotents.

By our convention, if \( \text{char}(F) = 0 \) in Theorem 3.2 (part 3), we get an explicit formula for the dimension of \( Rb \). However, if \( \text{char}(F) = p > 0 \), we only get the class of the dimension module \( p \) (which is \( |G|\lambda_1(b) a^{-1} \)). Thus we have the two following Corollaries.

**Corollary 3.3.** Let \( b \in R \) and \( m_b(x) = x(x - a)^s \) for some \( s \geq 1 \) and \( a \in F - \{0\} \). If \( \text{char}(F) = 0 \), then \( \dim(Rb) = |G|\lambda_1(b) a^{-1} \).
Corollary 3.4. Let \( b \in R \), \( J = Rb \), and \( m_b(x) = x(x - a)^s \) for some \( s \geq 1 \) and \( a \in F - \{0\} \). Let \( \text{char}(F) = p > 0 \), and \( r \) be the minimum positive integer in the class \( |G|\lambda_1(b)a^{-1} \). Then the following holds:

1. \( r \leq \dim(J) \). Moreover, if \( \dim(J) \leq p \), then \( \dim(J) = r \). In particular, if \( |G| - 1 \leq p \) and \( |G| \neq p \), then the dimension of any non-trivial ideal can be computed in this way.

2. If \( \lambda_1(b) = a = 1 \), \( |G| \geq p \) and \( c \) is the quotient of dividing \( |G| \) by \( p \), then \( \dim(J) = |G| - pt \) for some \( 0 \leq t \leq c \).

3. \( \dim(J) \) is multiple of \( p \) iff \( \lambda_1(b) = 0 \) or \( p \mid |G| \).

4. If \( |G| - 1 > p \), \( c \) is the quotient of dividing \( |G| - 1 \) by \( p \), and \( \lambda_1(b) = 0 \), then \( \dim(J) = pt \) for some \( 1 \leq t \leq c \).

5. If \( \dim(J) = 1 \), then \( \lambda_1(b) = |G|^{-1}a \).

Proof. 1. As \( \dim(J) \) is a positive integer number in the class \( |G|\lambda_1(b)a^{-1} \), then \( r \leq \dim(J) \). Suppose that \( \dim(J) \leq p \), then \( \dim(J) \) is the minimum positive integer in the class \( |G|\lambda_1(b)a^{-1} \), which implies \( \dim(J) = r \). If \( |G| - 1 \leq p \) and \( |G| \neq p \), \( p \nmid |G| \), so that \( R \) is semisimple (by \cite[Theorem 3.4.7]{19}). Hence any non-trivial ideal is principal generated by a non-trivial idempotent and has dimension less than or equal to \( p \).

2. Suppose that \( \lambda_1(b) = a = 1 \). As \( J \) is a proper ideal, then \( \dim(J) = |G| - pt \) for an integer \( 1 \leq t \), but the minimum possible value for \( \dim(J) \) is \( r \) (by part 1) in such case \( t \) would attain the value of \( c \).

3. \( \dim(J) = |G|\lambda_1(b)a^{-1} = 0 \) iff \( |G| \) is multiple of \( p \) or \( \lambda_1(b) = 0 \).

4. Since \( \lambda_1(b) = 0 \), \( \dim(J) \) is multiple of \( p \), but the greatest multiple of \( p \) less than or equal to \( |G| - 1 \) is \( pc \), and thus \( \dim(J) = pt \) for some \( 1 \leq t \leq c \).

5. If \( 1 = \dim(J) = |G|\lambda_1(b)a^{-1} \), then \( \lambda_1(b) = |G|^{-1}a \).

In \cite[Theorem 6]{11} M. Elia and E. Gorla give a lower bound on the dimension of a principal ideal in any group algebra when the multiplicity of 0 as a root of the characteristic polynomial of the regular left/right representation is known. They also point out that this bound turns out to be the exact dimension when is applied to the characteristic polynomial of an
idempotent. We noticed that the only elements for which this equality holds are those with right regular representation having minimal polynomial with 0 as a simple root. So we are going to restate their result.

**Theorem 3.5.**  Let \( b \in R \) be such that \( p_b(x) = x^nh(x) \) where \( x \not\equiv h(x) \), then

\[
|G| - u \leq \dim(Rb) \leq |G| - 1.
\]

Moreover, \( \dim(Rb) = |G| - u \) iff 0 is a simple root of \( m_b(x) \).

**Proof.** By Theorem 3.2 (part 1), \( |G| - u \leq \dim(Rb) \). Besides, as 0 is an eigenvalue of \( Rb \), \( Rb \subseteq R \), and so \( \dim(Rb) \leq |G| - 1 \).

Let \( m_b(x) = x^uw(x) \) with \( x \not\equiv w(x) \). If \( n = 1 \), by Theorem 3.2 (part 1), \( \dim(Rb) = |G| - u \). Conversely, if \( \dim(Rb) = |G| - u \), \( \dim(\ker(Rb)) = u \). As \( \ker(r_b) \subseteq \ker(r_b^u) \subseteq \ker(r_b^{G}) \), then \( \ker(r_b) = \ker(r_b^u) = \ker(r_b^{G}) \). Thus the minimal polynomial of \( r_b|_{\ker(Rb)} = r_b|_{\ker(r_b^u)} \) is \( x = x^n \), and hence \( n = 1 \).

**Corollary 3.6.** Let \( b, b' \in R \) such that \( p_b(x) = x^uh(x) \) and \( p_{b'}(x) = x^{u'}h'(x) \). Then the following hold:

1. If \( Rb = Rb' \) and \( m_b(x) \) has 0 as a simple root, then \( u \leq u' \).

2. If \( r_b \) is diagonalizable \( \dim(Rb) = |G| - u \). In particular, if \( b \) is idempotent this holds.

**Proof.** 1. By Theorem 3.5 \( \dim(Rb) = |G| - u \). Thus \( |G| - u' \leq \dim(Rb') \) = \( \dim(Rb) = |G| - u \), and thus \( u \leq u' \).

2. It follows from Theorem 3.5 and the fact that \( r_b \) is diagonalizable iff all the roots of \( m_b(x) \) are simple roots. In particular, if \( b \) is idempotent, \( m_b(x) = x^2 - x \equiv x(x - 1) \), and hence \( r_b \) is diagonalizable.

**Example 3.7.** Let \( G = \langle x, y \mid x^3 = y^2 = (xy)^3 = 1 \rangle = \{ 1, x, x^2y, y, x^2yx, x^2, yx, xy, xyx, yxy, yx^2, xyx^2 \} \) be the alternating group of degree 4 and \( R = \mathbb{F}_2G \). If \( b = x + x^2yx \) then \( m_b(z) = z(z^2 + z + 1)^2 \). Thus, by Theorem 3.2 (part 2), \( 4 \mid \dim(Rb) \) and \( 4 \leq \dim(Rb) \leq 8 \), so that \( \dim(Rb) \) is equal to 4 or 8. Alternatively, we can use Theorem 3.5 to compute \( \dim(Rb) \). Since \( p_b(z) = z^4(z^2 + z + 1)^4 \) and 0 is a simple root of \( m_b(z) \), then \( \dim(Rb) = 12 - 4 = 8 \).

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1Every example in this work was done using SageMath [21]
On the other hand, if $b' = 1 + x + y + x^2yx$, then $m_{b'}(z) = z^2(z^2 + z + 1)^2$ and $p_{b'}(z) = z^4(z^2 + z + 1)^4$. So, by Theorem 3.2, $8 = 12 - 4 \leq \dim(Rb)$. In fact, by using Lemma 2.3, we get that $\dim(Rb) = \text{rank}([Rb]_G) = 9$. Thus, by Theorem 3.5, we get that $\dim(Rb) = \text{rank}([Rb]_G) = 9$. This happened because the multiplicity of $z$ as a root of $m_{b'}(z)$ is not $1$ but $2$.

Example 3.8. Let $G = \langle x, y \mid x^4 = 1, \ x^2 = y^2 = (xy)^2, \ yxy^{-1} = x^{-1} \rangle = \{1, x, y, x^2, x^3y, xy, x^3, x^2y\}$ be the quaternion group and $R = F_3G$. Let $b_0 = x + 2y + 2x^2 + 2x^3y + xy + x^2y$ and $b_1 = 1 + x + y + x^3y$, then $m_{b_i}(z) = z(z - 1)^2$ for $i = 0, 1$. Thus, by Corollary 3.4(part 4), $\dim(Rb_0) = 3t$ where $1 \leq t \leq 2$, i.e., $\dim(Rb_0)$ is equal to $3$ or $6$. Besides, by Corollary 3.4(part 2), $\dim(Rb_1) = 8 - 3t$ where $1 \leq t \leq 2$, i.e., $\dim(Rb_1)$ is equal to $5$ or $2$. Let $b_2 = 2 + 2x + y + x^3y$, then $m_{b_2}(z) = z(z - 2)^2$. Thus, by Theorem 3.4(part 3), $\dim(Rb_2) = |G|\lambda_1(b_2)2^{-1} = 2$ so that $\dim(Rb_2)$ is equal to $2$ or $5$. In fact, by using Lemma 2.3, we get that $\dim(Rb_i)$ is equal to $6, 5, 5$ for $i = 0, 1, 2$, respectively.

4 Dimension of abelian codes

Recall that $F$ is a splitting field for the group $G$ (the algebra $FG$) if $End_{FG}(V) = F$ for every irreducible $FG$-module $V$ [9, pg 22]. This section, $F = F_q$, $p = \text{char}(F_q)$, $G$ is abelian of order relatively prime to $q$, $e$ is a non-trivial idempotent of $R$, and $I = Re$. The following result is a consequence of Theorem 3.5 above.

Corollary 4.1. Let $b \in R$ such that $p_b = x^u h(x)$ with $x \nmid h(x)$. Then $\dim(Rb) = |G| - u$.

Proof. Let $F$ be an extension of $F$ that is a splitting field for $G$ (this exists by [9, Proposition 7.13]). Then the minimal polynomial of $r_b$ (seen as a $F$-automorphism of $FG$) splits into distinct linear factors (by [13, Proposition 2.7]). So $0$ is a simple root of $m_b(x) \in F[x]$. Thus, by Theorem 3.5, $\dim(Rb) = |G| - u$.

The mapping $\alpha : G \to G$ given by $\alpha(x) = x^q$ is an automorphism of $G$, so the group $H := \langle \alpha \rangle$ acts on $G$ by evaluation. The orbits under this action are called $q$-orbits (also known as $q$-subsets). Let $\{U_j\}_{j=1}^w$ be the collection of the $q$-orbits of $G$. It is well-known (see, e.g., [12, Theorem 1.3]) that if $R = \bigoplus_{j=1}^w I_j$ is a decomposition of $R$ into minimal ideals, then there is a bijection between these ideals $I_j$ and the $q$-orbits under which, the size of a $q$-orbit equals the dimension of the corresponding ideal. We summarize this in the following theorem.
Theorem 4.2. Assuming the previous notation, the following holds: \( r = w \) and for a proper indexation, \( \dim(I_j) = |U_j| \) for \( j = 1, \ldots, r \).

Theorem 4.3. \((q\text{-orbits bound})\) If \( Y = \{ |U_j| : |U_j| = |G|\lambda_1(e), j = 1, \ldots, r \} \) and \( I \) is a minimal ideal, then

\[
\min(Y) \leq \dim(I) \leq \max(Y)
\]

Proof. It follows from Theorems 4.2 and 3.2 (part 3).

By Theorem 4.2 \( 1 \leq |Y| \). If \( |Y| = 1 \), the bound in Theorem 4.3 gives us the exact dimension. The following two results will offer a solution to the problem of computing the dimension of any abelian code, but first we will introduce a set-up.

Let \( m \) be the exponent of \( G \). Let \( \theta \) be a \( m \)-th primitive root of unity in some extension field of \( F \), then \( F := F(\theta) \) is a splitting field for \( G \) (see [9, Corollary 24.11]). Let \( R = FG \) and \( R = \oplus_{j=1}^t R\epsilon_j \) be the decomposition of \( R \) into minimal ideals where \( \epsilon_j \) is idempotent for all \( j \). Then \( \dim_F(R\epsilon_j) = 1 \) for all \( j \) (see [9 Corollary 4.4]). This implies that \( \eta := \{ \epsilon_j \}_{j=1}^t \) is a basis for \( R \) as a \( F \)-vector space. On the other hand, \( \alpha \) can be extended linearly to an \( F \)-algebra automorphism of \( R \), and thus \( H \) acts on \( \eta \) by evaluation (because \( \alpha \) sends primitive idempotents into primitive idempotents). Let \( U \) be the \( F \)-automorphism of \( R \) that sends \( G \) into \( \eta \) and \( A = [U]_G \), then \( A[\alpha|_G = [\alpha]_\eta A \) (see [15, Theorem 14, Ch. 3]). Hence \( U \) defines is an isomorphism of \( H \)-sets, and so

\[
G \cong \eta
\]

as \( H \)-sets.

Theorem 4.4. The following statements hold:

1. If \( f \in R \) is a primitive idempotent, and \( D \) is the inverse of \( U \), then \( \dim_F(Rf) = \mathrm{wt}_G(D(f)) \).

2. \( U \) induces a bijection between the \( q \)-orbits and the minimal ideals of \( R \), under which, the size of a \( q \)-orbit equals the dimension of the corresponding ideal.

Proof. Note that \( \alpha \) acting in an element of \( \eta \) is the same as the inverse of the Frobenius automorphism acting by evaluation on the coefficients of this element. Let \( * \) and \( \odot \) denote the actions of \( H \) and \( Gal(F/F) \) in \( \eta \), respectively. If \( e = \sum_{g \in G} a_g g \in \eta \) and \( \phi \in Gal(F/F) \) denotes the Frobenius automorphism, then \( \phi^{-1} \odot e = \phi^{-1} \odot e^q = \phi^{-1} \odot (\sum_{g \in G} \phi(a_g)g^q) = \alpha * e \). Thus the actions of \( H \) and \( Gal(F/F) \) generate the same orbits.
1. Let $D$ be the $\mathbf{F}$-automorphism of $\mathbf{R}$ given by $D(x) := U^{-1}(x)$, where $U$ is the change of basis transformation given above. Then $D|_\eta : \eta \to G$ is an isomorphism of $H$-sets. Since $H$ and $\text{Gal}(F/F)$ generate the same orbits in $\eta$, by the Galois descending argument (see [3] Proposition 7.18), there exists $O(e) \in \eta/H$ such that $f = \sum_{z \in O(e)} z$. Therefore $D(f) = \sum_{z \in O(e)} D(z)$, which is the sum of the elements of $G$ belonging to the $q$-orbit $D(O(e))$, and so $\text{wt}_G(D(f)) = |O(e)|$. On the other hand, by [3] Theorem III.8, $\dim_F(Rf) = \dim_F(Rf)$. Thus, since $Rf = \sum_{z \in O(e)} Rz$, $\dim_F(Rf) = |O(e)| = \text{dim}_G(D(f))$ (because $\dim_F(Rz) = 1$ for all $z \in O(e)$).

2. Since $U|_G : G \to \eta$ is isomorphism of $H$-sets, $U$ induces a size-preserving bijective correspondence $\tilde{U} : G/H \to \eta/H$ given by $\tilde{U}(S) = U(S)$. Let $\eta'$ the collection of the primitive idempotents of $R$. Since $H$ and $\text{Gal}(F/F)$ generate the same orbits in $\eta$, by the Galois descending argument, $v : \eta/H \to \eta'$ given by $v(O) = \sum_{o \in O} o$ is a bijection. Thus $v \circ \tilde{U}$ is a bijection between the $q$-orbits and the primitive idempotents of $R$. Now, by a similar argument to the one presented at the end of the proof of part 1, if $f = v(O)$ where $O \in \eta/H$, then $\dim_F(Rf) = |O| = |\tilde{U}^{-1}(O)| = |\tilde{U}^{-1} \circ v^{-1}(f)|$.

Observe that Theorem 4.3 (part 2) implies Theorem 4.2. The $\mathbf{F}$-automorphism $D$ presented in Theorem 4.4 (part 1) will be called the dimension indicator of $R$ associated with $\mathbf{F}$, or simply the indicator of $R$. Note that as any abelian code is the direct sum of minimal ideals, the indicator of $R$ can be also applied to compute the dimension of any abelian code. We will see that the indicator of $R$ is related to the discrete Fourier transform. The group of characters $G^*$ of $G$ is the set of the group homomorphisms from $G$ to $\mathbf{F} - \{0\}$ with the multiplication of functions. It is well-known that $G^* \cong G$ (see, e.g., [5] Section 1.1). The discrete Fourier transformation $\epsilon$ (see [5] Section II.1), [8] Section 2) is the isomorphism of $\mathbf{F}$-algebras that goes from $FG^*$ to its Artin-Wedderburn decomposition $\mathbf{F}[G]$ given by $\epsilon(f) = (f(g))_{g \in G}$.

Our final step is to provide a way to explicitly compute the indicator of $R$ (i.e., to compute $A^{-1}$) using tensor product algebras and Corollary 3.14.

Let $G = C_{n_1} \times \cdots \times C_{n_s}$ be a decomposition of $G$ as a product of cyclic groups with $C_{n_i} = \langle x_i \rangle = \{1, x_i, \ldots, x_i^{n_i-1} \}$ for all $i$. Let $R_i := F C_{n_i}$, $l_i :
$R_i \to R_i$ be the $\mathbf{F}$-linear transformation given by $l_i(y) = x_iy$, $\{\gamma_{ij_i}\}_{j_i=1}^{n_i}$, and
$\{e_{ij_i}\}_{j_i=1}^{n_i}$ be the spectrum of $l_i$ and the collection of primitive idempotents of $R_i$ for all $i$, respectively. Let $c_i \equiv |C_{n_i}|^{-1} \, \text{mod} \, p$ for $i = 1, \ldots, s$.

**Theorem 4.5.** Assuming the previous notation, the following holds: $[e_{ij_i}]_{C_{n_i}} = c_i(1, \gamma_{ij_i}^{n_i-1}, \gamma_{ij_i}^{n_i-2}, \ldots, \gamma_{ij_i})$ for $i = 1, \ldots, s$ and $j_i = 1, \ldots, n_i$. Besides, the coordinate vectors of the primitive idempotents of $R$ with respect to $G$ are given by $\{[e_{1j_1}]_{C_{n_1}} \otimes \cdots \otimes [e_{sj_s}]_{C_{n_s}} : j_i = 1, \ldots, n_i \text{ for } i = 1, \ldots, s\}$, where $\otimes$ denotes the Kronecker product of vectors.

**Proof.** Let $\beta = C_{n_1} \otimes \cdots \otimes C_{n_s}$ be the typical basis for the tensor product $R$. As $\mathbf{F}$ is a splitting field for every $C_{n_i}$, then every minimal ideal has dimension 1 as a $\mathbf{F}$-vector space. Thus, since the ideals of $R_i$ are $l_i$-invariant vector subspaces, every primitive idempotent in $R_i$ is an eigenvector of $l_i$. Suppose $l_i(e_{ij_i}) = \gamma_{ij_i}e_{ij_i}$ where $j_i = 1, \ldots, n_i$ for $i = 1, \ldots, s$. The minimal polynomial of $l_i$ is $x_i^{n_i} - 1$, and thus $l_i$ has as many distinct eigenvalues as $n_i$, implying that every eigenspace of $l_i$ in $R_i$ has dimension 1 for all $i$. Thus, as $1 + \gamma_{ij_i}^{n_i-1}x_i + \gamma_{ij_i}^{n_i-2}x_i^2 + \ldots + \gamma_{ij_i}x_i^{n_i-1}$ is an eigenvector of $l_i$ associated to the eigenvalue $\gamma_{ij_i}$ for all $i$, then $e_{ij_i}$ must be a multiple of this element, and thus $e_{ij_i} = c_i(1 + \gamma_{ij_i}^{n_i-1}x_i + \gamma_{ij_i}^{n_i-2}x_i^2 + \ldots + \gamma_{ij_i}x_i^{n_i-1})$ for all $i$ (by Corollary 3.3 (part 5)). On the other hand, if $T = R_1 \otimes \cdots \otimes R_s$, tensor products of the form $e_{1j_1} \otimes \cdots \otimes e_{sj_s}$ are primitive idempotents of $T$ because the set $\{e_{1j_1} \otimes \cdots \otimes e_{sj_s} : j_i = 1, \ldots, n_i \text{ for } i = 1, \ldots, s\}$ is a set of orthogonal idempotents with a suitable size, so it must be the set of the primitive idempotents of $T$. In addition, by the definition of $\beta$, we have that $[e_{1j_1} \otimes \cdots \otimes e_{sj_s}]_{\beta} = [e_{1j_1}]_{C_{n_1}} \otimes \cdots \otimes [e_{sj_s}]_{C_{n_s}}$ for $j_i = 1, \ldots, n_i$ and $i = 1, \ldots, s$. Thus, since $\chi : \mathbf{R} \to T$ given by $\chi(x_1^{s_1} \cdots x_s^{s_s}) = x_1^{s_1} \otimes \cdots \otimes x_s^{s_s}$ is an isomorphism of $\mathbf{F}$-algebras, the coordinate vectors of the primitive idempotents of $\mathbf{R}$ with respect to $G$ are the same coordinate vectors of the primitive idempotents of $\chi(\mathbf{R}) = T$ with respect to $\chi(G) = \beta$, finishing the proof.

$\square$

Thanks to Theorem 4.5, we are able to compute all the indicator of $R$, as the inverse of the $\mathbf{F}$-linear transformation of $\mathbf{R}$ whose matrix with respect to $G$ has as its columns the coordinate vectors of the primitive idempotents of $\mathbf{R}$ with respect to $G$. Since $A = G_\ast [\lambda]_\mu$, this could have also been achieved using characters theory (see [8 Corollary II.2]), but we were mainly motivated from the fact that the indicator can be obtained as an application of Theorem 3.2 (part 3), with an approach that is independent of the classic one.
Example 4.6. Let $F = \mathbb{F}_3$ and $G = C_2 \times C_4$ where $C_2 = \{1, x_1\}$ and $C_4 = \{1, x_2, x_2^2, x_2^3\}$ are the cyclic groups of order 2 and 4. Let $\alpha$ be the 4-th primitive root of the unity whose minimal polynomial over $F$ is $z^2 - z - 1$. As we mentioned before, $F = F(\alpha)$ is a splitting field for $G$. Let $l_i$ be as in Theorem 4.5, and $\sigma(l_i)$ denote the spectrum of $l_i$ for $i = 1, 2$. Then $\sigma(l_1) = \{1, 2\}$ and $\sigma(l_2) = \{1, 2, \alpha^2, \alpha^6\}$. Thus, by Theorem 4.5, the coordinate vectors of the primitive idempotents of $R_1 = FC_2$ and $R_2 = FC_4$ are $\{(2, 2), (2, 1)\}$ and $\{(1, 1, 1), (1, 2, 1, 2), (1, \alpha^6, 2, \alpha^2), (1, \alpha^2, 2, \alpha^6)\}$, respectively.

Let $\beta$ be as in Theorem 4.5, i.e., $\beta = \{1 \otimes 1, 1 \otimes x_2, 1 \otimes x_2^2, 1 \otimes x_2^3, x_1 \otimes 1, x_1 \otimes x_2, x_1 \otimes x_2^2, x_1 \otimes x_2^3\}$, then

$$(2, 2) \otimes (1, 1, 1) = (2, 2, 2, 2, 2, 2, 2)$$
$$(2, 2) \otimes (1, 2, 1, 2) = (2, 1, 2, 1, 2, 1, 2, 1)$$
$$(2, 2) \otimes (1, \alpha^6, 2, \alpha^2) = (2, \alpha^2, 1, \alpha^6, 2, \alpha^2, 1, \alpha^6)$$
$$(2, 2) \otimes (1, \alpha^2, 2, \alpha^6) = (2, \alpha^6, 1, \alpha^2, 2, \alpha^6, 1, \alpha^2)$$
$$(2, 1) \otimes (1, 1, 1, 1) = (2, 2, 2, 1, 1, 1, 1)$$
$$(2, 1) \otimes (1, 2, 1, 2) = (2, 1, 2, 1, 1, 2, 1, 2)$$
$$(2, 1) \otimes (1, \alpha^6, 2, \alpha^2) = (2, \alpha^2, 1, \alpha^6, 1, \alpha^6, 2, \alpha^2)$$
$$(2, 1) \otimes (1, \alpha^2, 2, \alpha^6) = (2, \alpha^6, 1, \alpha^2, 1, \alpha^2, 2, \alpha^6)$$

are the coordinate vectors of the primitive idempotents of $R_1 \otimes R_2$ with respect to $\beta$ (because $2\alpha^6 = \alpha^2$). If we suppose that $G$ has the ordering determined by $\beta$, i.e., $G = \{1, x_2, x_2^3, x_2^3, x_1, x_1 x_2, x_1 x_2^2, x_1 x_2^3\}$, then $\beta[x]_G = Id$. Hence, these are also the coordinate vectors of the primitive idempotents of $R$, and

$$A^{-1} = \begin{pmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2
2 & 1 & \alpha^2 & \alpha^6 & 2 & 1 & \alpha^2 & \alpha^6
2 & 2 & 1 & 1 & 2 & 2 & 1 & 1
2 & 1 & \alpha^6 & \alpha^2 & 2 & 1 & \alpha^6 & \alpha^2
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1
2 & 1 & \alpha^2 & \alpha^6 & 1 & 2 & \alpha^6 & \alpha^2
2 & 2 & 2 & 1 & 1 & 1 & 2 & 2
2 & 1 & \alpha^6 & \alpha^2 & 1 & 2 & \alpha^2 & \alpha^6
\end{pmatrix}^{-1}$$

13
Let $v_1 = 20002111$, $v_2 = 01111101$, $v_3 = 11212101$, $v_4 = 21111111$, and $v_5 = 00200202$. By straight computation, one obtains that the element $e_i \in R$ such that $[e_i]_G = v_i$ is idempotent for all $i$. We computed $\dim F(Re_i)$ using Lemma 2.3 and this coincided with $\text{wt}_G(D(e_i))$ for all $i$. For instance, $e_1$ generates an $[8, 4, 4]$-abelian code and $[D(e_1)]_G = 10000111$. $e_2$ generates an $[8, 3, 4]$-abelian code and $[D(e_2)]_G = 01000011$. $e_3$ generates an $[8, 5, 2]$-abelian code and $[D(e_3)]_G = 01111100$. $e_4$ generates an $[8, 7, 2]$-abelian code and $[D(e_4)]_G = 01111111$. Finally, $e_5$ generates an $[8, 6, 2]$-abelian code and $[D(e_5)]_G = 01111011$.

5 MDS group codes

The Singleton Bound states that if an $[n, k, d]$ linear code over $\mathbb{F}_q$ exists, then $k \leq n - d + 1$. A code for which equality is attained in the Singleton Bound is called maximum distance separable, abbreviated MDS. These codes are optimal in the sense that they achieve the maximum possible minimum distance for a given length and dimension, thus they are of great interest for error correction. $C$ is said to be a trivial MDS code over $\mathbb{F}_q$ if $C = \mathbb{F}_q^n$ or $C$ is monomially equivalent to the repetition code or its dual (see [16, pp 71-72]). $C$ is an MDS group code if $C$ is an ideal of $\mathbb{F}_qG$ such that its parameters satisfy the equality in the Singleton Bound.

In this section, $F = \mathbb{F}_q$ and $p = \text{char}(F)$ unless it would be stated otherwise. Observe that Corollary 3.4 (part 1) gives a way to easily compute the dimension of certain group codes, this leads us to our next definition. Let $J$ be an ideal of $R$, if $J$ principal generated by an idempotent, and $\dim_{\mathbb{F}_q}(J) \leq p$, then it will be said that $J$ is a easily computable dimension group code, abbreviated ECD. If any non-trivial ideal of $R$ is an ECD group code, then it will be said that $R$ is an easily computable dimension group algebra abbreviated ECD. A consequence of Maschke's
Theorem (see Theorem 3.4.7) is that $R$ is ECD iff $|G| \leq p + 1$ and $|G| \neq p$. The following is a direct consequence of Theorem 3.2 and Corollary 3.4 (part 1).

**Corollary 5.1.** Let $b \in R$ be such that $m_b(x) = x^n p_1(x)^{r_1} \cdots p_t(x)^{r_t}$ where the $p_i$ are monic irreducible distinct factors with $p_i \neq x$ for all $i$. Let $p_b(x) = x^n h(x)$ where $x \nmid h(x)$. Let $d$ be the minimum distance of $Rb$. Then the following statements hold:

1. If $\zeta_n = \text{dim}(\ker(r^n_b)/\ker(r_b))$, then $d \leq u - \zeta_n + 1$. Besides, $Rb$ is and MDS group code iff is an $[|G|, |G| - u + \zeta_n + 1]$-code. In particular, if $n = 1$, $d \leq u + 1$; and $Rb$ is and MDS group code iff is an $[|G|, |G| - u, u + 1]$-code.

2. 

$$d \leq \begin{cases} |G| - (t + 1)|G|_p + 1 & \text{if } |G|_p \mid \text{dim}(\ker(r_b)) \text{ and } n > 1 \\ |G| - t|G|_p + 1 & \text{otherwise} \end{cases}$$

In addition, if $n = 1$ and $Rb$ is an MDS group code, then $d \equiv 1 \pmod{p}$ and $|G|_p + 1 \leq d \leq |G| - t|G|_p + 1$.

3. If $m_b(x) = x(x - a)^s$ for some integer $s \geq 1$ and $a \in F - \{0\}$, $r$ is the minimum positive integer in the class $|G|\alpha^{-1}(b)\alpha^{-1}$; and $Rb$ is an ECD group code, then $d \leq |G| - r + 1$. Besides, $Rb$ is an MDS and ECD group code iff is an $[|G|, |G| - r + 1]$-code.

**Proof.** It follows from the Singleton Bound, Theorem 3.2 and Corollary 3.4 (part 1). \qed

**Example 5.2.** Let $G$ be a group of order $mp^l$ with $l \geq 1$, $m \neq 1$, and $p \nmid m$. Let $b \in R$, $m_b(x) = x^n p_1(x)^{r_1} \cdots p_t(x)^{r_t}$ where the $p_i$ are monic irreducible distinct factors with $p_i \neq x$ for all $i$. Then, by Chap. VII, Corollary 7.16, $p^l$ divides $\text{dim}(\ker(r^n_b))$ and $\text{dim}(\ker(p_i(r^n_b)))$ for $i = 1, \ldots, t$, thus $1 \leq t < m$. So if $m = 2$, then $m_b(x)$ has only two irreducible divisors. Thus if $n = 1$ and $Rb$ is an MDS group code, then $Rb$ is projective (by Lemma 2.2) and its minimum distance $d$ must be $p^l + 1$ (by Corollary 5.4, part 2).

Let $G = \langle a, b \mid a^3 = b^2 = 1, bab^{-1} = a^2 \rangle = \{1, b, a, a^2, ba, ba^2, ba^2 \}$ be the symmetric group of degree 3, and $R = \mathbb{F}_9G$. If $\alpha$ is an element of $\mathbb{F}_9$ with minimal polynomial $z^2 + z + 1$, then $b = (2\alpha + 2) + (\alpha + 1)b + \alpha a + (2\alpha + 1)a^2 + (\alpha + 1)ba^2 + ba$ is such that $m_b(x) = x(x + 2\alpha)^2$. In this case, $Rb$
is an MDS $[6, 3, 4]$-code, and so $d = 3 + 1$ as stated in Corollary 5.1 (part 2). On the other hand, $b' = (\alpha + 1) + ab + 2a + 2\alpha^2 + 2ba$ is such that $m_{b'}(x) = x^2(x + \alpha + 2)^2$. In this case, $Rb'$ is an MDS $[6, 4, 3]$-code, and so $d = 3 \leq 3 + 1$, this happened because the multiplicity of 0 as a root of $m_{b'}(x)$ is not 1 but 2.

Now we will study the relation of MDS and ECD group codes. For that purpose we recall the MDS-Conjecture.

**MDS-Conjecture** [16, pg 265]: If there is a non-trivial $[n, k]$ MDS code over $\mathbb{F}_q$, then

$$n \leq \begin{cases} q + 2 & \text{if } q \text{ even, and } k = 3 \text{ or } k = q - 1 \\ q + 1 & \text{otherwise} \end{cases}$$

**Lemma 5.3.** Let $p$ be a prime number. If the MDS-Conjecture is true, then the only non-trivial MDS group codes in the non-semisimple group algebra $\mathbb{F}_p G$ exist when $G = C_p$ and $p$ is odd; and are equivalent to extended Reed-Solomon codes.

**Proof.** If $\mathbb{F}_p G$ is non-semisimple, by [19, Theorem 3.4.7], $p \mid |G|$. Suppose that the MDS-Conjecture is true and that there exists an MDS $[|G|, k]$ group code $C$ in $\mathbb{F}_p G$. Then if $p = 2$, then every MDS group code in $\mathbb{F}_2 G$ is trivial (by [17, Theorem 2.4.4]). If $p$ is an odd prime, then $p \mid |G|$ and $|G| \leq p + 1$. Since the equality is not possible, $p \mid |G|$ and $|G| < p + 1$, thus $G = C_p$. Now, the assertion follows from [20, Theorem 1].

**Theorem 5.4.** The following statements hold:

1. If $C$ is an MDS and ECD group code in $R$, then $|G| \leq q + 1$

2. Let $p$ be an odd prime. Suppose that the MDS-Conjecture is true and $G \neq C_p$. If there exists a non-trivial MDS group code in $\mathbb{F}_p G$, then $\mathbb{F}_p G$ is an ECD group algebra.

**Proof.**

1. It follows from [2] Corollary 9.1.

2. If there exists a non-trivial MDS group code in $\mathbb{F}_p G$, by Lemma 5.3, $\mathbb{F}_p G$ is semisimple or $G = C_p$ with $p$ odd. So $\mathbb{F}_p G$ is semisimple, and by the MDS-Conjecture, $|G| - 1 \leq p$, implying that $R$ is an ECD group algebra.
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