Statistical immersions between statistical manifolds of constant curvature

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Abstract

The condition for the curvature of a statistical manifold to admit a kind of standard hypersurface is given. We study the statistical hypersurfaces of some types of the statistical manifolds \((M, \nabla, g)\), which enable \((M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbb{R}\) to admit the structure of a constant curvature.

Keywords: statistical manifold of a constant curvature, statistical submanifold, Hessian structure, statistical hypersurface

MSC 2010: 53A15, 53C42, 53B05

1 Introduction

Since Lauritzen introduced the notation of statistical manifolds in 1987 [5], the geometry of statistical manifolds has been developed in close relations with affine differential geometry and Hessian geometry as well as information geometry (see, for example, [2, 4, 8]). In this paper we study the hypersurface of statistical manifolds.

Let \(M\) be an \(n\)-dimensional manifold, \(\nabla\) a torsion-free affine connection on \(M\), \(g\) a Riemannian metric on \(M\), and \(R\) a curvature tensor field on \(M\). We denote by \(TM\) the set of vector fields on \(M\), and by \(TM^{(r,s)}\) the set of tensor fields of type \((r,s)\) on \(M\).

Definition 1.1. A pair \((\nabla, g)\) is called a statistical structure on \(M\) if \((M, \nabla, g)\) is a statistical manifold, that is, \(\nabla\) is a torsion-free affine connection and for all \(X,Y,Z \in T(M)\), \((\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z)\).

Let \(\nabla^o\) be a Levi-Civita connection of \(g\). Certainly, a pair \((\nabla^o, g)\) is a statistical structure, which is called a Riemannian statistical structure or a trivial statistical structure (see [3]).

On the other hand, the statistical structure is also introduced from affine differential geometry which was proposed by Blasche (see [6]). Recently the relation between statistical structures and Hessian geometry has been studied (see [3, 7]).

For all \(\alpha \in \mathbb{R}\), a connection \(\nabla^{(\alpha)}\) is defined by

\[
\nabla^{(\alpha)} = \frac{1 + \alpha}{2} \nabla + \frac{1 - \alpha}{2} \nabla^* \]

where \(\nabla\) and \(\nabla^*\) are dual connections on \(M\). We study a statistical hypersurface of a statistical manifold \((M, \nabla, g)\) which enables \((M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbb{R}\) to admit the structure of a constant curvature.
In section 3, a statistical manifold \((M, \nabla, g)\), which enables \((M, \nabla^{(\alpha)}, g)\) \(\forall \alpha \in \mathbb{R}\) to admit the structure of a constant curvature, is considered. In section 4, we study characteristics of statistical immersions between statistical manifolds \((M, \nabla, g)\) which enable \((M, \nabla^{(\alpha)}, g)\) \(\forall \alpha \in \mathbb{R}\) to admit the structure of a constant curvature.

2 Preliminaries

A statistical manifold \((M, \nabla, g)\) is said to be of constant curvature \(k \in \mathbb{R}\) if

\[
R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}, \forall X,Y,Z \in TM
\]

holds, where \(R\) is the curvature tensor field of \(\nabla\). A pair \((\nabla, g)\) is called a Hessian structure if a statistical manifold \((M, \nabla, g)\) is of constant curvature 0.

A Riemannian metric \(g\) on a flat manifold \((M, g)\) is called a Hessian metric if \(g\) can be locally expressed by

\[
g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j},
\]

where \(\{x^1, \ldots, x^n\}\) is an affine coordinate system with respect to \(\nabla\). Then \((M, \nabla, g)\) is called a Hessian manifold (see [7]).

Let \((M, \nabla, g)\) be a Hessian manifold and \(K(X,Y) := \nabla_X Y - \nabla_Y X\) be the difference tensor between the Levi-Civita connection \(\nabla\) of \(g\) and \(\nabla\). A covariant differential of differential tensor \(K\) is called a Hessian curvature tensor for \((\nabla, g)\). A Hessian manifold \((M, \nabla, g)\) is said to be of constant Hessian curvature \(c \in \mathbb{R}\) if

\[
(\nabla_X K)(Y,Z) = -\frac{c}{2} \{g(X,Y)Z + g(X,Z)Y\}, \forall X,Y,Z \in TM
\]

holds (see [7]).

Example 2.1. ([3]) Let \((H, \tilde{g})\) be the upper half space:

\[
H := \{y = (y^1, \ldots, y^{n+1})^T \in \mathbb{R}^{n+1} | y^{n+1} > 0\}, \tilde{g} := (y^{n+1})^{-2} \sum_{i=1}^{n+1} dy^i dy^i.
\]

We define an affine connection \(\tilde{\nabla}\) on \(H\) by the following relations:

\[
\tilde{\nabla} \frac{\partial}{\partial y^{n+1}} = (y^{n+1})^{-1} \frac{\partial}{\partial y^{n+1}}, \tilde{\nabla} \frac{\partial}{\partial y^i} = 2\delta_{ij}(y^{n+1})^{-1} \frac{\partial}{\partial y^{n+1}};
\]

\[
\tilde{\nabla} \frac{\partial}{\partial y^i} = 0,
\]

where \(i, j = 1, \ldots, n\). Then \((H, \tilde{\nabla}, \tilde{g})\) is a Hessian manifold of constant Hessian curvature 4.

Let \((\tilde{M}, \tilde{\nabla}, \tilde{g})\) be a statistical manifold and \(f : M \to \tilde{M}\) be an immersion. We define \(g\) and \(\nabla\) on \(M\) by

\[
g = f^* \tilde{g}, \quad g(\nabla_X Y, Z) = \tilde{g}((\nabla_X f)Y, fZ), \quad \forall X, Y, Z \in TM.
\]

Then the pair \((\nabla, g)\) is a statistical structure on \(M\), which is called the one by \(f\) from \((\tilde{\nabla}, \tilde{g})\) (see [3]).
Let $(M, \nabla, g)$ and $(\overline{M}, \overline{\nabla}, \overline{g})$ be two statistical manifolds. An immersion $f : M \to \overline{M}$ is called a statistical immersion if $(\nabla, g)$ coincides with the induced statistical structure (see [3]).

Let $f : (M, \nabla, g) \to (\overline{M}, \overline{\nabla}, \overline{g})$ be a statistical immersion of codimension one, and $\xi$ a unit normal vector field of $f$. Then we define $h, h^* \in TM^{(0,2)}$, $\tau, \tau^* \in TM^*$ and $A, A^* \in TM^{(1,1)}$ by the following Gauss and Weingarten formulae:

$$\tilde{\nabla}_X f = f_* \nabla_X Y + h(X,Y)\xi, \quad \tilde{\nabla}_X \xi = -f_* A^* X + \tau^*(X)\xi,$$

$$\tilde{\nabla}_X^* f = f_* \nabla_X^* Y + h^*(X,Y)\xi, \quad \tilde{\nabla}_X^* \xi = -f_* A X + \tau(X)\xi, \quad \forall X, Y \in TM,$$

where $\tilde{\nabla}^*$ is the dual connection of $\tilde{\nabla}$ with respect to $\overline{g}$.

In addition, we define $I \in TM^{(0,2)}$ and $S \in TM^{(1,1)}$ by using the Riemannian Gauss and Weingarten formulae:

$$\tilde{\nabla}_X f = f_* \nabla_X Y + I(X,Y)\xi, \quad \tilde{\nabla}_X \xi = -f_* SX.$$

For more details on the Gauss, Codazzi and Ricci formulae on statistical hypersurfaces, we refer to [3].

### 3 The condition that a statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbb{R}$

In this section we consider a condition that a statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbb{R}$.

**Theorem 3.1.** A statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbb{R}$ iff there exist $\alpha_1, \alpha_2 \in \mathbb{R}(|\alpha_1| \neq |\alpha_2|)$ such that statistical manifolds $(M, \nabla^{(\alpha_1)}, g)$ and $(M, \nabla^{(\alpha_2)}, g)$ are of constant curvature.

**Proof.** Necessity is obvious. We find sufficiency. Without loss of generality, we assume $\alpha_1 \neq 0$. Then since

$$\nabla^{(\alpha)} = \frac{\alpha_1 + \alpha}{2\alpha_1} \nabla^{(\alpha_1)} + \frac{\alpha_1 - \alpha}{2\alpha_1} \nabla^{(-\alpha_1)}$$

holds for all $\alpha \in \mathbb{R}$, the following relation

$$R^{(\alpha)}(X,Y)Z = \frac{\alpha_1 + \alpha}{2\alpha_1} R^{(\alpha_1)}(X,Y)Z + \frac{\alpha_1 - \alpha}{2\alpha_1} R^{(-\alpha_1)}(X,Y)Z + \frac{\alpha_1^2 - \alpha^2}{4\alpha_1^2} [K(Y, K(Z, X)) - K(X, K(Y, Z))]$$

holds, where $K(X,Y) := \nabla_X Y - \nabla_Y X$ is the difference tensor field of a statistical manifold.

From the relations

$$R^{(\alpha_1)}(X,Y)Z = k_1 \{g(Y,Z)X - g(X,Z)Y\},$$

$$R^{(\alpha_2)}(X,Y)Z = k_2 \{g(Y,Z)X - g(X,Z)Y\},$$

the relation

$$R^{(\alpha)}(X,Y)Z = \frac{k_2\alpha_1^2 - k_1\alpha_2^2 + (k_1 - k_2)\alpha^2}{\alpha_1^2 - \alpha_2^2} \{g(Y,Z)X - g(X,Z)Y\}$$

holds, that is, a statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature $\frac{k_2\alpha_1^2 - k_1\alpha_2^2 + (k_1 - k_2)\alpha^2}{\alpha_1^2 - \alpha_2^2}$.
Example 3.1. Let \((M, g)\) be a family of normal distributions:

\[
M := \left\{ p(x, \theta) \bigg| p(x, \theta) = \frac{1}{\sqrt{2\pi \theta^2}} \exp \left\{ -\frac{1}{2\theta^2} (x - \theta^1)^2 \right\} \right\}, \quad g := 2(\theta^2)^{-2} \sum d\theta^i d\theta^i,
\]

\(x \in \mathbb{R}, \quad \theta^1 \in \mathbb{R}, \quad \theta^2 > 0.\)

We define an \(\alpha\)-connection by the following relations:

\[
\nabla^{(\alpha)} \nabla^{(\alpha)} = (-1 + 2\alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^1}, \quad \nabla^{(\alpha)} \frac{\partial}{\partial \theta^2} = (1 + \alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^2},
\]

\[
\nabla^{(\alpha)} \frac{\partial}{\partial \theta^1} = \nabla^{(\alpha)} \nabla^{(\alpha)} = 0.
\]

Then the statistical manifold \((M, \nabla^{(1)}, g)\) is of constant curvature \((-\frac{1}{2})\), and the statistical manifold \((M, \nabla^{(1)}, g)\) is of constant curvature 0. Hence for all \(\alpha \in \mathbb{R}\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is of constant curvature \(\frac{\alpha^2 - 1}{2}\).

Example 3.2. Let \((M, g)\) be a family of random walk distributions (\([1]\)):

\[
M := \left\{ p(x; \theta^1, \theta^2) \bigg| p(x; \theta^1, \theta^2) = \frac{\theta^2}{\sqrt{2\pi \theta^2}} \exp \left\{ -\frac{\theta^2 x^2}{2} \frac{\theta^2}{\theta^2 - \theta^2} - \frac{\theta^2}{2(\theta^2)^2} \right\}, \quad x, \mu, \lambda > 0 \right\},
\]

\[g := \frac{\theta^2}{(\theta^1)^3} (d\theta^1)^2 + \frac{1}{2(\theta^2)^2} (d\theta^2)^2.\]

We define an \(\alpha\)-connection by the following relations:

\[
\nabla^{(\alpha)} \nabla^{(\alpha)} = \frac{-3(1 + \alpha)(\theta^1)^{-1} \frac{\partial}{\partial \theta^1} + (-1 + \alpha)(\theta^1)^{-3}(\theta^2)^2 \frac{\partial}{\partial \theta^2}}{2},
\]

\[
\nabla^{(\alpha)} \frac{\partial}{\partial \theta^2} = \nabla^{(\alpha)} \nabla^{(\alpha)} = \frac{(1 + \alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^1}}{2},
\]

\[
\nabla^{(\alpha)} \frac{\partial}{\partial \theta^1} = (-1 + \alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^2}.
\]

Then the statistical manifold \((M, \nabla^{(1)}, g)\) is of constant curvature \((-\frac{1}{2})\), and the statistical manifold \((M, \nabla^{(1)}, g)\) is of constant curvature 0. Hence for all \(\alpha \in \mathbb{R}\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is of constant curvature \(\frac{\alpha^2 - 1}{2}\).

Theorem 3.1 implies the following fact.

Corollary 3.1. If there exist \(\alpha_1, \alpha_2 \in \mathbb{R}(|\alpha_1| \neq |\alpha_2|)\) such that the statistical manifold \((M, \nabla^{(\alpha_1)}, g)\) is of constant curvature \(k_1\) and the statistical manifold \((M, \nabla^{(\alpha_2)}, g)\) is of constant curvature \(k_2\), and \(k_1 \neq k_2\), then for \(\alpha \in \mathbb{R}\) satisfying that \(\alpha^2 = (k_2 \alpha_2^2 - k_1 \alpha_2^2)/(k_2 - k_1)\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is flat.

Example 3.3. \(k_1 = -1/2, k_2 = 0, \alpha_1 = 0\) and \(\alpha_2 = 1\) hold in example 3.1 and example 3.2. Hence for \(\alpha \in \mathbb{R}\) satisfying that \(\alpha^2 = 1\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is flat.

Theorem 3.2. If the Hessian manifold \((M, \nabla, g)\) is of constant Hessian curvature, then for all \(\alpha \in \mathbb{R}\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is of constant curvature.
Proof. If the Hessian manifold \((M, \nabla, g)\) is of constant Hessian curvature, then for all \(X, Y, Z \in TM\),
\[
(\nabla K)(Y, Z; X) = -\frac{c}{2} \{g(X, Y)Z + g(X, Z)Y\}, c \in \mathbb{R}
\]
holds. On the other hand, the curvature tensor \(R^c\) of Levi-Civita connection \(\nabla^o\) is written by
\[
R^c(X, Y)Z = R(X, Y)Z - (\nabla K)(Y, Z; X) + (\nabla K)(Z, X; Y)
+ K(X, K(Y, Z)) - K(Y, K(Z, X)),
\]
where \(R\) is the curvature tensor of \(\nabla\) and \(K(X, Y) = \nabla_X Y - \nabla_Y X\) is difference tensor. Then
\[
(\nabla K)(Y, Z; X) - (\nabla K)(Z, X; Y)
= 2\{K(X, K(Y, Z)) - K(Y, K(Z, X))\} + \frac{1}{2}\{(R(X, Y)Z - R^*(X, Y)Z\}
\]
implies
\[
R^c(X, Y)Z = -\frac{c}{4} \{g(Y, Z)X - g(X, Z)Y\},
\]
where \(R^*\) is curvature tensor of dual connection \(\nabla^*\), that is, the statistical manifold \((M, \nabla^o, g)\) is of constant curvature. On the other hand, the statistical manifold \((M, \nabla, g)\) is flat, that is, constant curvature 0. Therefore we finish the proof of theorem by applying Theorem 3.1.

Hitherto we found some conditions that for any \(\alpha \in \mathbb{R}\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is of constant curvature.

4 The hypersurfaces of statistical manifolds of constant curvature

We consider statistical hypersurfaces of some type of statistical manifolds, which enable for any \(\alpha \in \mathbb{R}\) a statistical manifold \((M, \nabla^{(\alpha)}, g)\) to be of constant curvature.

Theorem 4.1. Let \((M, \nabla, g)\) be a trivial statistical manifold of constant curvature \(k\), \((\tilde{M}, \tilde{\nabla}, \tilde{g})\) a statistical manifold of constant curvature \(\tilde{k}\) with a Riemannian manifold of constant curvature \(\tilde{k} (\neq \tilde{k}) (\tilde{M}, \tilde{\nabla}, \tilde{g})\), and \(f : M \rightarrow \tilde{M}\) a statistical immersion of codimension one. Then \(f : M \rightarrow \tilde{M}\) is equiaffine, that is, \(\tau^*\) vanishes.

Proof. If \((\tilde{M}, \tilde{\nabla}, \tilde{g})\) is a statistical manifold of constant curvature \(\tilde{k}\) with a Riemannian manifold of constant curvature \(\tilde{k} (\neq \tilde{k}) (\tilde{M}, \tilde{\nabla}, \tilde{g})\), the following equation
\[
(\tilde{\nabla}_X \tilde{K})(f, Y, f, Z) - (\nabla_Y \tilde{K})(f, X, f, Z)
= 2\{\tilde{\nabla}(f, X, f, Y)f, Z - \tilde{\nabla}^o(f, X, f, Y)f, Z\}
= 2\{\tilde{k} - \tilde{k}\}(f, Y, f, Z)f, X - \tilde{g}(f, X, f, Z)f, Y\}
\]
holds by Eq.(2.2) and Eq.(2.3) in [3]. By above equation and equation Eq.(3.6) in [3], we have

\[ -2(\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} = (\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z) - b(Y, Z)A^*X + b(X, Z)A^*Y + h(X, Z)B^*Y - h(Y, Z)B^*X \]

\[ 0 = (\nabla_X b)(Y, Z) - (\nabla_Y b)(X, Z) + \tau^*(X)b(Y, Z) - \tau^*(Y)b(X, Z) \tag{4.2} \]

By above equation and equation Eq.(3.6) in [3], we have

\[ 0 = -\tau^*(Y)A^*X + \tau^*(X)A^*Y - (\nabla_X B^*)Y + (\nabla_Y B^*)X + \tau^*(X)B^*Y - \tau^*(Y)B^*X \]

By above equation and equation Eq.(4.2) in [3], we have

\[ 0 = -h(X, B^*Y) + h(Y, B^*X) + (\nabla_X \tau^*)(Y) - (\nabla_Y \tau^*)(X) + b(Y, A^*X) - b(X, A^*Y). \]

By \( K = 0, B^* = A^* - S \) and Gauss equation (3.3) in [3], from Eq.(4.2), we have

\[ -2(\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} = -b(Y, Z)A^*X + b(X, Z)A^*Y + h(X, Z)A^*X - h(Y, Z)A^*Y \]

\[ = -b(Y, Z)A^*X + b(X, Z)A^*Y - h(X, Z)SY - h(Y, Z)SX + b(Y, Z)SX + h(Y, Z)SY + \tilde{R}(X, Y)Z - R(X, Y)Z. \]

By \( b = h - II, B^* = A^* - S \) and Riemannian Gauss equation (3.5) in [3], we have

\[ -2(\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} = -h(Y, Z)A^*X + h(X, Z)A^*Y - h(X, Z)SY - h(Y, Z)SX + h(X, Z)SX - h(Y, Z)SY \]

\[ = -b(Y, Z)B^*Y + h(X, Z)B^*X + h(Y, Z)B^*X - h(X, Z)B^*Y - h(Y, Z)SY + h(X, Z)SX - h(Y, Z)SX + \tilde{R}(X, Y)Z - R(X, Y)Z. \]

Since \((M, \nabla, g)\) is Riemannian manifold, clearly \( R^e(X, Y)Z = R(X, Y)Z \). Hence we have

\[ 0 = (\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} - b(Y, Z)B^*X + b(X, Z)B^*Y. \]

And since \( b(Y, Z) = g(BY, Z), b(X, Z) = g(BX, Z) \), from above equation we have

\[ 0 = (\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} - g(BY, Z)B^*X + g(BX, Z)B^*Y. \tag{4.3} \]

From Eq.(4.2), \( B^* = A^* - S \) and Codazzi equation on \( A \) we get

\[ 0 = -\tau^*(Y)A^*X + \tau^*(X)A^*Y - (\nabla_X A^*)Y + (\nabla_Y S)X - (\nabla_Y S)X \]

\[ + \tau^*(X)B^*Y - \tau^*(Y)B^*X \]

\[ = (\nabla_X S)Y - (\nabla_Y S)X + \tau^*(X)B^*Y - \tau^*(Y)B^*X \]

and by \( \nabla = \nabla^o \) and Codazzi equation on \( S \), we also get

\[ 0 = \tau^*(X)B^*Y - \tau^*(Y)B^*X. \tag{4.4} \]

From Eq.(4.2), \( B^* = A^* - S \) and Ricci equation we have

\[ b(X, B^*Y) - b(Y, B^*X) = 0, \]
and since \( b(X, B^c Y) = g(BX, B^c Y) \) and \( b(Y, B^c X) = g(BY, B^c X) \), we have
\[
g(BX, B^c Y) - g(BY, B^c X) = 0.
\]
Since \( g(BX, B^c Y) = g(B^c Y, BX) = b^*(BX, Y) = g(B^c BX, Y) \), we have
\[
0 = -g([B, B^c]X, Y). \tag{4.5}
\]
From Eq.(4.5), \( B \) and \( B^c \) are simultaneously diagonalizable.

In the case that \( B^c \) is of the form \( \lambda^c I \), we see easily that \( \tau^c \) vanishes from Eq.(4.4) if \( \lambda^c \neq 0 \) and \( \bar{k} = \bar{k} \) from Eq.(4.3) otherwise. In the case that \( B^c \) is not of the form \( \lambda^c I \), there are \( \lambda_1^c, \lambda_2^c \) with \( \lambda_1^c \neq \lambda_2^c \) such that \( B^c X_j = \lambda_j^c X_j \), where \( g(X_i, X_j) = \delta_{ij} \), \( i, j = 1, 2 \). Besides there are \( \lambda_1, \lambda_2 \) such that \( BX_j = \lambda_j X_j \). Eq.(4.3) implies that
\[
(\bar{k} - \bar{k}) \{ g(X_j, Z) X_i - g(X_i, Z) X_j \} + \lambda_j \lambda_i^c g(X_j, Z) X_i - \lambda_i \lambda_j^c g(X_i, Z) X_j = (\bar{k} - \bar{k}) + \lambda_j \lambda_i^c g(X_j, Z) X_i - (\bar{k} - \bar{k} + \lambda_j \lambda_i^c) g(X_i, Z) X_j = 0
\]
and hence \( \bar{k} - \bar{k} + \lambda_j \lambda_i^c = \bar{k} - \bar{k} + \lambda_i \lambda_j^c = 0 \), which means that
\[
\lambda_j \lambda_i^c = \lambda_i \lambda_j^c = -(\bar{k} - \bar{k}) \neq 0.
\]
By Eq.(4.4) we have \( \lambda_2^c \tau^c(X_1) X_2 - \lambda_1^c \tau^c(X_2) X_1 = 0 \), which implies that \( \tau^c \) vanishes. \( \square \)

**Example 4.1.** Suppose \( \bar{M} \) be \( \mathbb{R}^3 \). We define Riemannian metric and an Affine connection by the following relations:
\[
\tilde{g} = a \sum d\theta^i d\theta^i,
\]
\[
\tilde{\nabla} \frac{\partial}{\partial \theta^1} = \tilde{\nabla} \frac{\partial}{\partial \theta^2} = \tilde{\nabla} \frac{\partial}{\partial \theta^3} = \frac{\partial}{\partial \theta^1}, \tilde{\nabla} \frac{\partial}{\partial \theta^2} = \frac{\partial}{\partial \theta^3} = \frac{\partial}{\partial \theta^2}.
\]
Then \( (\bar{M}, \tilde{\nabla}, \tilde{g}) \) is a statistical manifold of constant curvature \(-\frac{j^2}{4a}\) with a trivial statistical manifold of constant curvature \(0\) \( (\bar{M}, \nabla, \bar{g}) \). Suppose \( M \) be \( \mathbb{R}^2 \), and \( (\nabla, g) \) an induced statistical structure from \( (\tilde{\nabla}, \tilde{g}) \) by an immersion \( f : (x, y) \in \mathbb{R}^2 \mapsto (0, x, y) \). We remark that \( (M, \nabla, g) \) is a trivial statistical manifold of constant curvature \(0\).

Theorem 3.2 and Theorem 4.1 imply the following fact.

**Corollary 4.1.** Let \( (M, \nabla, g) \) be a trivial statistical manifold of constant curvature \( k \), \( (\bar{M}, \tilde{\nabla}, \tilde{g}) \) a Hessian manifold of constant Hessian curvature \( \bar{c} \), and \( f : M \rightarrow \bar{M} \) a statistical immersion of codimension one. Then \( f : M \rightarrow \bar{M} \) is equiaffine, that is, \( \tau^c \) vanishes.

We consider a shape operator of statistical immersion of a trivial statistical manifold of constant curvature into a Hessian manifold of constant Hessian curvature.
Lemma 4.1. Let \((M, \nabla, g)\) be a trivial statistical manifold of constant curvature \(k\), \((\tilde{M}, \tilde{\nabla}, \tilde{g})\) a Hessian manifold of constant Hessian curvature \(\tilde{c}\), and \(f : M \to \tilde{M}\) a statistical immersion of codimension one. Then the following holds:

\[
A^* = kv\tilde{c}^{-1}I, B^* = -\frac{1}{2}\nu I, h = \tilde{c}v^{-1}g, A = \tilde{c}v^{-1}I, B = [2\tilde{c}^2 - (2k + \tilde{c})\nu^2](2\nu^{-1})I.
\]

Proof. Combining Eq.(2.3) and Eq.(3.6) in [3] with Eq.(2.1), we have

\[
\frac{\tilde{c}}{2}\{g(Y,Z)X - g(X,Z)Y\} = 2(k - k_0)\{g(Y,Z)X - g(X,Z)Y\}
\]

\[-b(Y,Z)A^*X + b(X,Z)A^*Y + h(X,Z)B^*Y = h(Y,Z)B^*X
\]

\[0 = h(X,K(Y,Z)) - h(Y,K(X,Z)) + (\nabla_X b)(Y,Z) - (\nabla_Y b)(X,Z)
\]

\[+\tau^*(X)b(Y,Z) - \tau^*(Y)b(X,Z) - \tau^*(Y)b(X,Y) + \tau^*(X)h(Y,Z)
\]

Taking the trace of (4.6) \((X)\) with respect to \(X\), we have

\[-\tilde{c}g(Y,Z) = -\text{tr}A^*b(Y,Z) + h(B^*Z,Y) + h(B^*Y,Z)
\]

and taking the trace of (4.6) \((Y)\) with respect to \(Y\), we have

\[-\frac{\tilde{c}}{2}(n + 1)g(X,Z) = -b(A^*X, Z) + h(X, B^*Z) + \text{tr}B^*h(X, Z).
\]

Using the above equation and Eq.(4.6) \((Y)\), we have

\[-\frac{\tilde{c}}{2}(n + 2)g(X,Y) = -b(A^*X, Y) + h(X, B^*Y) + \text{tr}B^*h(Y, X).
\]

\[+h(X,Y)\nu - h(X,B^*Y) + (\nabla_X \tau^*)Y + b(Y,A^*X)
\]

\[= \text{tr}B^*h(X,Y) - h(X,Y)\nu + (\nabla_X \tau^*)Y
\]

and since from Corollary 4.1 \(\tau^* = 0\) holds, we have

\[h = \frac{\tilde{c}}{2}(n + 2)(\nu - \text{tr}B^*)^{-1}g.
\]

Hence we have

\[(\nu - \text{tr}B^*)h(X,Y) = \frac{\tilde{c}}{2}(n + 2)g(X,Y).
\]

If \(\tilde{c} \neq 0\) holds, \(h\) is non-degenerated.

Since \(\tilde{\nabla}\) is flat in Gaussian equation in [3], we obtain

\[k\{g(Y,Z)X - g(X,Z)Y\} = h(Y,Z)A^*X - h(X,Z)A^*Y
\]

and taking the trace of above equation with respect to \(X\), we have

\[k(n - 1)g(Y,Z) = \text{tr}A^*h(Y,Z) - h(A^*Y,Z) = h((\text{tr}A^*I - A^*)Y, Z).
\]

Since the above equation and Eq.(4.7) imply that

\[k(n - 1)I = \frac{\tilde{c}}{2}(n + 2)(\nu - \text{tr}B^*)^{-1}(\text{tr}A^*I - A^*),
\]
there is \( a \in \mathbb{R} \) such that \( A^* = aI \) and \( trA^* = an \). Therefore the above equation implies that

\[
k(n-1)I = \frac{\tilde{c}}{2}(n+2)\nu^{-1}(na-a)I
\]

and thus since

\[
2k(\nu - trB^*) = \tilde{c}(n+2)a,
\]

we have

\[
A^* = 2k(\nu - trB^*)[\tilde{c}(n+2)]^{-1}I.
\] (4.8)

If \( k \neq 0 \) holds, then since \( A^* \) is non-degenerated, by Eq.(4.8) we have

\[
B^* = -\nu I, \quad trB^* = -\frac{n\nu}{2}
\]

and

\[
A^* = \frac{2k(\nu + \frac{n\nu}{2})}{\tilde{c}(n+2)}I = \frac{k\nu}{\tilde{c}}I, \quad h = \frac{\tilde{c}}{2}(n+2)(\nu + \frac{n\nu}{2})^{-1}g = \frac{\tilde{c}}{\nu}g.
\]

Since \( h(X,Y) = g(AX,Y) \), we have \( A = \frac{\tilde{c}}{\nu}I \) and

\[
B = B^* + (A - A^*) = -\nu I + (\frac{\tilde{c}}{\nu} - \frac{k\nu}{\tilde{c}})I = \frac{-\nu^2\tilde{c} + 2\tilde{c}^2 - 2k\nu^2}{2\nu\tilde{c}}I = \frac{2\tilde{c}^2 - (2k + \tilde{c})\nu^2}{2\nu\tilde{c}}I.
\]

Exercise 4.2. Let \((M, \nabla, g)\) be a trivial statistical manifold of constant curvature \( k \), \((\tilde{M}, \tilde{\nabla}, \tilde{g})\) a Hessian manifold of constant Hessian curvature \( \tilde{c} \). If there is a statistical immersion of codimension one \( f : M \to \tilde{M} \), \( 2k + \tilde{c} \) is of non-negative. Moreover, when \( \tilde{c} \) is positive, the Riemannian shape operator of \( f : M \to \tilde{M} \) is given by \( S = \pm \frac{\tilde{c}}{\nu}I \).

Proof. By Lemma 4.1 and Eq.(4.2), we have

\[
\frac{\tilde{c}}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{2\tilde{c}^2 - (2k + \tilde{c})\nu^2}{2\nu\tilde{c}}(-\frac{\nu}{2})\{g(Y,Z)X - g(X,Z)Y\}
\]

and thus conclude that

\[
\frac{\tilde{c}}{4} - \frac{2\tilde{c}^2 - (2k + \tilde{c})\nu^2}{4\tilde{c}} = 0.
\]

Since \( \tilde{c}^2 = (2k + \tilde{c})\nu^2 \), we have \( 2k + \tilde{c} \geq 0 \) and

\[
\nu = \pm \frac{|\tilde{c}|}{\sqrt{2k} + \tilde{c}}.
\]

Thus the Riemannian shape operator \( S \) is given by

\[
S = A^* - B^* = (\frac{k\nu}{\tilde{c}} + \frac{\nu}{2})I = \frac{2k + \tilde{c}}{2\tilde{c}}(\pm \frac{|\tilde{c}|}{\sqrt{2k} + \tilde{c}})I = \pm \frac{|\tilde{c}|}{2\tilde{c}}\sqrt{2k + \tilde{c}}I.
\]

When \( \tilde{c} \) is positive, we have \( S = \pm \frac{1}{2}\sqrt{2k + \tilde{c}}I \).
Example 4.2. Let $(H, \hat{\nabla}, \hat{g})$ be the $(n+1)$-dimensional upper half Hessian space of constant Hessian curvature 4 as in Example 2.1. For a constant $y_0 > 0$, write the following immersion by $f$:
\[(y^1, \cdots, y^n)^T (\in \mathbb{R}^n) \mapsto (y^1, \cdots, y^n, y_0)^T \in H.\]
Let $(\nabla, g)$ be the statistical structure on $\mathbb{R}^n$ induced by $f$ from $(\hat{\nabla}, \hat{g})$. Then $(\mathbb{R}^n, \nabla, g)$ is a trivial statistical manifold of constant curvature 0 and $f$ is a statistical immersion of a trivial statistical manifold of constant curvature into Hessian manifold of constant Hessian curvature.

Acknowledgement The authors would like to thank the anonymous referees for their helpful comments and suggestions.

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