TIED LINKS

FRANCESCA AICARDI AND JESÚS JUYUMAYA

Abstract. In this paper we introduce the tied links, i.e., ordinary links provided with some ‘ties’ between strands. The motivation to introduce such objects is due to a diagrammatical interpretation of the defining generators of the so-called algebra of braids and ties; indeed, one half of such generators can be interpreted as the usual generators of the braid algebra, and the remaining generators can be interpreted as ties between consecutive strands; this interpretation leads to define the tied braids. We define an invariant polynomial for the tied links via a Skein relation. Also, we introduce the monoid of tied braids and we prove the corresponding theorems of Alexander and Markov for tied links. Then, we prove that the invariant of tied links that we have defined can be obtained also by using the Jones recipe.

Introduction

Since the seminal works of Jones [8] on the famous invariants for classical links, called nowadays the Jones polynomial, several others classes of knotted objects have been proposed. E.g., singular links [4, 6, 15], framed links [12] and virtual knots [11]. This paper introduces and studies a new class of knotted objects, the tied links. The tied links are the closure of the tied braids; the tied braids come from a diagrammatical interpretation of the defining generators of the so-called algebra of braids and ties [1, 2, 5, 9, 14], in short bt–algebra.

Let $n$ be a positive integer, the bt–algebra $E_n$ is a one parameter unital associative algebra, defined originally by the generators $T_1, \ldots, T_{n-1}$ and $E_1, \ldots, E_{n-1}$ and certain relations, for details see Definition 4. The defining relations of $E_n$ make evident that the generators $T_i$’s can be interpreted diagrammatically as the usual braids generators. On the contrary, an immediate diagrammatical interpretation of the generators $E_i$’s is not evident. However, in [1] we proposed to interpret the generator $E_i$ as a tie that connects the $i$–strand with the $(i+1)$–strand, making thus the bt–algebra a diagram algebra. In [2] we have proved that the bt–algebra supports a Markov trace. Consequently, using this Markov trace, the classical theorems of Markov and Alexander for classical links together with certain representation of the classical braid groups in the bt–algebra, we have defined an invariant, at three parameters, for classical links; an analogous route yields an invariant for singular links; see [2] for details. Now, associated with the bt–algebra we have the tied braid monoid which is defined as the monoid having an analogous presentation of the bt–algebra, but considering only the monomials relations. We call tied braids the elements
of this monoid and tied links the closure of the tied braids. Notice that the classical links can be regarded as tied links since the braid group can be considered obviously as a submonoid of the tied braid monoid.

This paper is largely inspired from our work [2] on the bt–algebra. More precisely, we introduce and study the tied braid monoid and the tied links. In particular, we define an invariants for tied links which ‘contains’ the invariants for classical links defined in [2]. This invariants is defined in two ways, one by Skein relation and the other as the normalization and the rescaling of the composition of a representation of the tied monoid in the bt–algebra with the Markov trace defined on it.

This work is organized as follows. In Section 1 we define tied links, their diagrams and their isotopy classes. In Section 2 we prove the first main result (Theorem 1) of the paper, that is, the construction, via skein relation, of an invariant polynomial for tied links which we shall denote $F$. The proof of the existence of such invariant is an adaptation of the proof of the existence [13] of the Homflypt polynomial for classical knots. Further, we provide the value of the invariant on simple examples of tied links. Section 3 is devoted to the understanding of the tied links as algebraic objects; thus we introduce the monoid of tied braids (Definition 2) and we prove the analogous of the Alexander theorem (Theorem 2) and of the Markov theorem (Theorem 3) for tied links. Section 4 has as goal the construction of the invariant $F$ through the Jones recipe, i.e., using the trace on the bt–algebra: this is done in Theorem 5. Finally, Section 5 is devoted to the computer algorithm to obtain the value of $F$ for any tied link put in the form of a closed tied braid; the algorithm calculates the trace of the tied braid and then the polynomial for the tied link via the Jones recipe.

1. Tied links

In this section we introduce the concepts of tied links and of their diagrams.

A tied (oriented) link $L(P)$ with $n$ components is a set $L$ of $n$ disjoint smooth (oriented) closed curves embedded in $S^3$, and a set $P$ of ties, i.e., unordered pairs of points $(p_r, p_s)$ of such curves between which there exists a tie. We shall denote $T$ the set of oriented tied links.

**Remark 1.** If $P$ is the empty set the tied link $L(P)$ is nothing else the classical link $L$.

A tied link diagram is like the diagram of a link, provided with ties, depicted as springs connecting pairs of points lying on the curves. The ties have no ‘physical consistency’: they can freely move (provided only that the endpoints move continuously on the curves), overstepping other ties as well as the link curves.
Definition 1. Two oriented tied links $L(P)$ and $L'(P')$ are tie-isotopic if

1. the links $L$ and $L'$ are ambient isotopic.
2. if there exists in $P$ a tie $(p_r(i), p_s(j))$ between two points $p_r$, $p_s$ belonging respectively to the components $i$ and $j$, then in $P'$ there is at least one tie $(p'_r(i'), p'_s(j'))$ between the images $i'$ and $j'$ of the components $i$ and $j$, if $i$ and $j$ are distinct.
3. if there exists in $P$ a tie $(p_r(i), p_s(j))$ between the components $i$ and $j$ ($i \neq j$), and a tie $(p_t(j), p_v(k))$ between the components $j$ and $k$ ($j \neq k$), then in $P'$ there are at least two ties connecting two components among $i'$, $j'$, and $k'$, images respectively of the components $i$, $j$ and $k$.

Figure 1. The diagram of a tied link

Figure 2. Diagrams of equivalent tied links

In other words the tie-isotopy says that, remaining in the same equivalence class, it is allowed to move any tie between two components letting its extremes to move along the two whole components. Moreover, ties can be destroyed or created between two components, provided that either there exists already one tie between them, or both the components are tied to a third component, or the two components coincide.
Definition 1. We call essential a tie between two components, if it cannot be destroyed (i.e., removing it, the components remain untied). An essential tie is therefore between distinct components.

Remark 2. The ties define an equivalence relation between the components of the link. Indeed, each component can be tied with itself (reflection); a tie is an unordered pair (symmetry), two different components tied to a third component are tied (transitivity). A tied link with $n$ components is therefore nothing else a link where the components are colored with $m \leq n$ distinct colors [3]: two components have the same color if there is a tie between them. However, we deal with diagrams that are made by links with ties, and that may look arbitrarily complicated.

2. An invariant for tied links

Let us denote also by $\mathcal{T}$ the set of oriented tied links diagrams and let $A$ be a commutative ring. Notice that an invariant of tied links is a function from $\mathcal{T}$ to $A$ that takes one constant value on each class of tie-isotopic links.

Remark 3. In the sequel, by $TL$ we indicate an oriented tied link as well as its diagram, if there is no risk of confusion.

The theorem we will state here, is a counterpart of the theorem stated at page 112 in [13] for classical links. Let $u, z$ and $t$ be three non-zero indeterminates in the field of complex number $\mathbb{C}$. Let

$$W = \frac{z + t - ut}{uz} \quad \text{and} \quad w := \sqrt{W} \quad (1)$$

Let $A$ be the field of the rational functions on $\mathbb{Q}$ in the variables $u, z, t, w$. We have the following theorem.

Theorem 1. There exists a function $F: \mathcal{T} \rightarrow A$, invariant of oriented tied links, uniquely defined by the following three conditions on tied-links diagrams:

I The value of $F$ is equal to 1 on the unknotted circle (no matter if tied with itself).

II Let $TL$ be a tied link. By $TL^o$ we denote the tied link consisting of $TL$ and the unknotted circle (no matter if tied with itself), unlinked to $TL$. Then

$$F(TL^o) = \frac{1}{wz}F(TL).$$

III (Skein rule) Let $TL_+, TL_-, TL_{\sim}, TL_{+, \sim}$ be the diagrams of tied links, that are identical outside a small disk into which two strands enter and exit, whereas inside the disk the two strands look as shown in Fig. 3. Then the following identity holds:

$$\frac{1}{w}F(TL_+) - wF(TL_-) = (1 - u^{-1}) F(TL_{\sim}) + \frac{1}{w}(1 - u^{-1})F(TL_{+, \sim}). \quad (2)$$
Figure 3. The disks where $TL_+, TL_-, TL_\sim$ and $TL_{+\sim}$ are different.

**Remark 4.** The following three Skein rules, all equivalent to the Skein rule above, will be used in the sequel. The first one is obtained from III, simply adding a tie between the two strands inside the disk. The rules Va, b follows from III and IV.

- **IV**
  \[
  \frac{1}{uw} F(TL_{+\sim}) - wF(TL_{-\sim}) = (1 - u^{-1})F(TL_\sim).
  \]

- **Va**
  \[
  \frac{1}{w} F(TL_+) = w[F(TL_-) + (u - 1)F(TL_{-\sim})] + (u - 1)F(TL_\sim).
  \]

- **Vb**
  \[
  wF(TL_-) = \frac{1}{w}[F(TL_+) + (u^{-1} - 1)F(TL_{+\sim})] + (u^{-1} - 1)F(TL_\sim).
  \]

Theorem 1 is proved by the same procedure used in [13]. We will outline the parts where the presence of ties modifies the demonstration.

According to [13], the fact that the Skein rule, together with the value of the invariant on the unknotted circle, are sufficient to define the value of the invariant on any tied link is proved the following way.

Let $T^n$ be the set of diagrams of tied links with $n$ crossings, and $TL \in T^n$. Ordering the components and fixing a point in each component, for every diagram $TL$ one constructs an associated standard ascending diagram $TL'$. $TL$ and $TL'$ are identical except for a finite number of crossing, here called ‘deciding’, where the signs are opposite. Moreover, we define here $\widetilde{TL'}$, adding to $TL'$ a tie between the strands near to each deciding crossing. $TL'$ and $\widetilde{TL}$ are by construction collections of unknotted and unlinked components; $TL'$ has the same ties as $TL$, $\widetilde{TL'}$ may have more ties. The procedure defining $TL'$ allows to get an ordered sequence of deciding crossings, whose order depends on the ordering of the components, and on the choice of the base points.

The induction hypothesis states that to each tied link diagram $TL$ in $T^n$ there is associated a function $F(TL)$ which is independent of the ordering of the components, independent of the choices of the base points, and invariant under Reidemeister moves that do not increase the number of crossing beyond $n$. Moreover, by induction hypothesis

- a) The value of $F$ on any diagram with $n$ crossings of the tied link $TL_0^n$, composed of $c$ components unknotted and untied, is $\frac{1}{(wz)^n}$. 


b) The value of $F$ on the tied link with $n$ crossings $\tilde{T}L_{0}^{c,m}$ consisting of $c$ components unknotted and untied, connected by $m$ essential ties ($m \leq c - 1$), is $\frac{v^{m}}{(wz)^{c-1}}$. Observe that these values are independent of $n$.

One starts with zero crossings: the tied link is thus a collection of $c \geq 1$ curves unknotted and unlinked, with $m \leq c - 1$ essential ties between them. The value of $F$ on such tied link is given by $\frac{v^{m}}{(wz)^{c-1}}$.

Now, let $TL$ be in $T^{n+1}$. If $TL$ consists of $c$ components unknotted and untied, connected by $m$ essential ties ($m \leq c - 1$), let’s define

$$F(TL) = \frac{v^{m}}{(wz)^{c-1}}. \quad (3)$$

Otherwise, consider the first deciding crossing $P$. If in a neighborhood of $P$ the tied link looks like $TL_{+,-}$ (or $TL_{-,-}$), use Skein rule IV to write the value of $F$ in terms of $TL_{-,-}$ and $TL_{-}$ (respectively, $TL_{+,-}$ and $TL_{+}$). If in a neighborhood of $P$ the tied link looks like $TL_{+}$ (respectively, $TL_{-}$), use Skein rule V-a (resp., V-b) to write the value of $F$ in terms of the value of $F$ on the tied links $TL_{-}, TL_{-,-}$ and $TL_{+}$ (respectively, $TL_{+}, TL_{+,-}$ and $TL_{+}$).

Observe that if the tied links $TL_{\sigma}$ or $TL_{\sigma,-}$ ($\sigma = \pm$) are coinciding with the original tied link in a neighborhood of the crossing $P$, then $TL_{-\sigma}$ and $TL_{-\sigma,-}$ in the neighborhood of the same crossing coincide with the associated tied link $TL'$ or $\tilde{TL}'$. On the other hand, $TL_{\sim}$ represents a tied link diagram with $n$ crossings, for which the value of $F$ is known, and invariant according to the induction hypothesis. Then we apply the same procedure to the second deciding crossing, which is present in all the diagrams, obtained by the application of the Skein rules, and that result to have $n + 1$ crossings, and so on. The procedure ends with the last deciding crossing, so obtaining unlinked and unknotted tied links, with $n + 1$ crossings, where the value of $F$ is given by (3), depending only on the number of components and the number of essential ties.

Observe that the Skein rule IV could be avoided: this should make longer the procedure.

It remains to prove that

(i) The procedure is independent on the order of the deciding points.

(ii) The procedure is independent on the order of components, and on the choice of base-points.

(iii) The function is invariant under Reidemeister moves.

Following the proof done in \[12\] for classical links, we observe that the proofs of points (ii) and (iii) remain the same also in the presence of ties. Indeed, ties can be always moved far from the chosen fixed points and from the part of link involved in the Reidemeister moves.

The proof of point (i) consists in a verification that the value of the invariant does not change if we interchange any two deciding points in the procedure of calculation. So, let $TL$ be the diagram of a tied link and let $p$ and $q$ the first two deciding crossings that will be interchanged.

Denote by $\epsilon_{p}$ the sign at the crossing $p$, by $\sigma_{p}TL$ the tied link obtained by $TL$ changing the sign at the crossing $p$, by $\tilde{\sigma}_{p}TL$ the tied link obtained by $TL$ changing the sign at $p$ and adding a tie near $p$, and by $\rho_{p}TL$ the tied link obtained by $TL$ removing the crossing $p$ and adding a tie. Similarly for the point $q$.

Then, by Skein Va,b it follows that, if $q$ follows $p$:

$$F(TL) = w^{2\epsilon_{p}}[F(\sigma_{p}TL) + (w^{\epsilon_{p}} - 1)F(\tilde{\sigma}_{p}TL)] + w^{\epsilon_{p}}(w^{\epsilon_{p}} - 1)\rho_{p}TL.$$
then

\[ F(TL) = w^{2\xi_p} \{ w^{2\xi_q} [F(\sigma_q \sigma_p TL) + (u^{\xi_q} - 1) F(\tilde{\sigma}_q \sigma_p TL)] + w^{\xi_q} (u^{\xi_q} - 1) \rho_q \sigma_p TL \} + \\
+ (u^{\xi_p} - 1) w^{2\xi_p} \{ w^{2\xi_q} [F(\sigma_q \tilde{\sigma}_p TL) + (u^{\xi_q} - 1) F(\tilde{\sigma}_q \tilde{\sigma}_p TL)] + w^{\xi_q} (u^{\xi_q} - 1) \rho_q \tilde{\sigma}_p TL \} + \\
+ (u^{\xi_p} - 1) w^{\xi_p} \{ w^{2\xi_q} [F(\sigma_q \rho_p TL) + (u^{\xi_q} - 1) F(\tilde{\sigma}_q \rho_p TL)] + w^{\xi_q} (u^{\xi_q} - 1) \rho_q \rho_p TL \} \]

Observe that, if \( p \) follows \( q \), \( F(TL) \) is obtained from the above expression exchanging \( p \) with \( q \). But that expression is symmetrical under such interchange, since \( \sigma_p \sigma_q TL = \sigma_q \sigma_p TL \), \( \sigma_p \rho_q TL = \rho_q \sigma_p TL \), \( \rho_p \rho_q TL = \rho_q \rho_p TL \), and so on.

□

Remark 5. The necessity of II in the definition of \( F \) (Theorem 1) is due to the fact that by the sole Skein relation we cannot calculate the value of \( F \) on \( c \) unknotted and unlinked circles, without ties between them.

The value \( X_c \) of \( F \) on \( c \) unknotted and unlinked circles, all tied together, is instead calculable using Skein rule IV recursively \( c - 1 \) times, i.e.

\[ \frac{1}{uw} X_c - wX_c = (1 - u^{-1}) X_{c+1}, \quad \text{from which} \quad X_{c+1} = \frac{t}{wz} X_c. \]

The initial value \( X_1 = 1 \) of \( F \) is given by rule I, see figure 4.

Figure 4. The values of \( F \) on the unknotted and unlinked tied circles, involved in Skein rule IV

Remark 6. Because of Remark 1, we observe that the polynomial \( F \) in particular provides an invariant polynomial for classical links.

2.1. Properties of the Polynomial \( F \). Here we list some properties of the polynomial \( F \), which can be easily verified.

(i) \( F \) is multiplicative on the connected sum of tied links.
(ii) The value of \( F \) does not change if the orientations of all curves of the link are reversed.
(iii) Let \( TL^\pm \) be the link diagram obtained from \( TL \) by changing the signs of all crossings.

Then \( F(TL^\pm) \) is obtained from \( F(TL) \) by the following changes: \( w \to 1/w \) and \( u \to 1/u \).
Let \( \Gamma \) be a knot or a link whose components are all linked together. Then \( F(\Gamma) \) is defined by I and IV, and is a Homflypt–type polynomial (see [13]) with

\[
\ell = i \sqrt{\frac{z}{z + t - ut}} \quad \text{and} \quad m = i \left( \frac{1}{\sqrt{u}} - \sqrt{u} \right).
\]

In particular, for \( t = -z(u + 1) \) the invariant \( F \), restricted to classical links, becomes the Jones polynomial in the variable \( u \).

Item (i) follows from the defining relation I of \( F \), and by the same arguments proving the multiplicity of the invariants obtained by Skein relations (see [13]). Items (ii) is evident, observing that the value of \( F \) on the unlinked circles is independent of their orientation s, and that the Skein relations are invariant under the inversion of the orientations of the strands. Item (iii) is clear looking at the symmetrical Skein relations Va and Vb.

As for item (iv), observe that if the components of the links are all connected, or there is a unique component, then adding a tie anywhere does not alter the isotopy-class of the links: therefore, comparing the Skein relation IV, multiplied by \( u^{-1/2} \), with the general Skein relation of the Homflypt polynomial (see [13]), that here reads:

\[
\ell F(\Gamma_{+,\sim}) + \ell^{-1} F(\Gamma_{-,\sim}) + mF(\Gamma_{\sim}) = 0,
\]

we obtain the expressions of \( \ell \) and \( m \) in terms of \( z, t \) and \( u \). Moreover, being for the Jones polynomial \( V(\Gamma) \) in the variable \( u \)

\[
\ell = iu^{-1} \quad \text{and} \quad m = i(u^{-1/2} - u^{1/2}),
\]

we observe that the expression of \( m \) is the same for \( F(\Gamma) \) and \( V(\Gamma) \), whereas the expressions of \( \ell \) coincide under the equality \( t = -z(u + 1) \).

### 2.2. Examples

We give here some examples of tied links with their polynomial \( F \).

Let \( L^+, L^-, \tilde{L}^+, \tilde{L}^-, TF^+, TF^-, H \) be the tied links shown in figure

Using rules I–III we get

\[
\begin{align*}
F(L^+) &= \frac{u}{z}(1 + ut + uz - t - z) ; \\
F(L^-) &= \frac{u^2 + z + t - uz - ut}{uw(z + t - ut)} ; \\
F(\tilde{L}^+) &= \frac{u}{z}(ut + uz - z) ; \\
F(\tilde{L}^-) &= \frac{u^2 + z + t - uz - ut}{uw(z + t - ut)} ; \\
F(TF^+) &= \frac{-u^3 tz - u^2 t^2 - 2u^2 t^2 + 3u^2 t^2 + u^2 t^2 + uz - uz^2 - ut^2 + z^2}{uz^2} ; \\
F(TF^-) &= \frac{z(-u^3 t + u^2 t - ut + u^2 t + uz - uz^2 + t)}{u(z + t - ut)^2} ;
\end{align*}
\]
F(H) = \frac{u^3t^2+u^3tz-2u^2t^2-4u^2tz-u^2z^2+ut^2+3uz^2+4utz-z^2-tz}{uz(z^2-ut)}.

**Remark 7.** Observe that the last three links are in fact knots (links with one component). For these links, whatever tie should connect the component with itself, i.e., the tie should be non-essential.

### 3. The tied braid monoid

The study of classical links through the braid group is based on the classical theorems of Alexander and Markov. The Alexander theorem says that every link can be obtained by closing a braid. The Markov theorem says when two braids yield isotopic links. These theorems are repeated for singular knots [6, 7], framed knots [12], virtual knots [11] and $p$–adic framed links [10]. That is, in each of these classes of knotted objects a convenient analogous of the braid group is defined and analogous Alexander and Markov theorems are established. In this section we introduce the monoid of tied braids which plays the role of the braid group for tied links. Thus we will establish the Alexander theorem and Markov theorem for tied links.

#### 3.1. Now we define the monoid of tied braids and we discuss the diagrammatic interpretation for its defining generators.

**Definition 2.** The tied braid monoid $TB_n$ is the monoid generated by usual braids $\sigma_1, \ldots, \sigma_{n-1}$ and the generators $\eta_i, \ldots, \eta_{n-1}$, called ties, such the $\sigma_i$’s satisfy braid relations among them together with the following relations:

\[
\begin{align*}
\eta_i \eta_j &= \eta_j \eta_i \quad \text{for all } i, j & (4) \\
\eta_i \sigma_i &= \sigma_i \eta_i \quad \text{for all } i & (5) \\
\eta_i \sigma_j &= \sigma_j \eta_i \quad \text{for } |i - j| > 1 & (6) \\
\eta_i \sigma_j \sigma_i &= \sigma_j \sigma_i \eta_j \quad \text{for } |i - j| = 1 & (7) \\
\eta_i \sigma_j \sigma_i^{-1} &= \sigma_j \sigma_i^{-1} \eta_j \quad \text{for } |i - j| = 1 & (8) \\
\eta_i \sigma_i \eta_j &= \eta_j \sigma_i \eta_j = \sigma_i \eta_j \eta_i \quad \text{for } |i - j| = 1 & (9) \\
\eta_i \eta_i &= \eta_i \quad \text{for all } i & (10)
\end{align*}
\]

In terms of diagrams the defining generator $\eta_i$ correspond to a cord connecting the $i$ with $(i+1)$-strands and $\sigma_i$ is represented as the usual braid diagram:

![Figure 5. Diagrammatic representation of $\sigma_i$ and $\eta_i$](image)

Now we examine the defining relations of $TB_n$ in terms of these diagrammatic interpretations of the defining generators. Relations (4) and (6) are trivial. Relation (5) corresponds to the sliding of the cord through to the crossing. Consider now relations (7) and (8) in terms of
diagrams. The first one corresponds to the sliding of the cord from up to down behind or in front of a strand. The second one corresponds to the same sliding but bypassing the strand.

\[ \eta_2\sigma_1\sigma_2 = \eta_1\sigma_2\eta_1 = \sigma_2\eta_2\eta_1 \]  

(11)

Finally, we see relations (9) and (10) in diagrams. Without loss of generality, let’s assume \( i = 2, j = 1 \) in (9), so that

\[ \eta_{i,j+1} = \sigma_j^{-1}\eta_i\sigma_j, \quad j - i = 1 \]  

(12)

The transparency property plays an important role in the introduction of the elements

\[ \eta_{i,j+1} = \sigma_j^{-1}\eta_i\sigma_j, \quad j - i = 1 \]  

as we will see now.

Multiply both terms of relation (7) at left by \( \sigma_j^{-1} \) and at right by \( \sigma_i^{-1} \), obtaining

\[ \sigma_j^{-1}\eta_i\sigma_j = \sigma_i\eta_j\sigma_i^{-1} \quad |j - i| = 1. \]  

(13)

Then multiply both terms of relation (8) at left by \( \sigma_j^{-1} \) and at right by \( \sigma_i \), obtaining

\[ \sigma_j^{-1}\eta_i\sigma_j = \sigma_i^{-1}\eta_j\sigma_i \quad |j - i| = 1. \]  

(14)

Taking, for instance, \( i = 1, j = 2 \) and \( i = 2, j = 1 \) in equations (7) and (8), equations (13) and (14) imply that \( \eta_{1,3} \) has four equivalent expressions:

\[ \eta_{1,3} = \sigma_2^{-1}\eta_1\sigma_2 = \sigma_2\eta_1\sigma_2^{-1} = \sigma_1\eta_2\sigma_2^{-1} = \sigma_1^{-1}\eta_2\sigma_1, \]  

(15)
which are depicted in the following figure.

![Figure 8](image)

**Figure 8.** \(\eta_{1,3}\) and its equivalent diagrams

Observe now that, if the cords are provided with *elasticity*, each one of the elements representing \(\eta_{1,3}\) in figure 8 can be transformed, by a Reidemeister move of second type in which the cord is stretched, in the following compact diagram (i.e., a cord connecting strand 1 with strand 3). From now on, the cord, having elastic property, will be represented as a spring.

![Figure 9](image)

**Figure 9.** A compact diagram for \(\eta_{13}\)

Observe that it does not matter whether the tie in \(\eta_{1,3}\) is in front or behind the strand 2, because of the transparency of the cords.

Another advantage of considering the compact diagram (figure 9) is that the equation:

\[
\eta_i \sigma_j = \sigma_j \sigma_i^{-1} \eta_i \sigma_j = \sigma_j \eta_{i,j+1}
\]  

(16)

can be interpreted as the sliding of the tie up and down along the braid under stretching or contracting. In other words, while the elements \(\eta_i\) do not commute with the \(\sigma_j\), when \(|j - i| = 1\), equation (16) provides a sort of commutation between \(\sigma_i\) and ties (see example in figure).

![Figure 10](image)

**Figure 10.** \(\eta_1 \sigma_2 = \sigma_2 \eta_{1,3}\)

Using equations (16) and (14), we obtain from (11)

\[
\sigma_2 \eta_{1,3} \eta_1 = \eta_1 \sigma_2 \eta_1 = \eta_1 \eta_{1,3} \sigma_2.
\]  

(17)
Comparing now (11) with (17) we get

\[ \eta_1\eta_{1,3} = \eta_1\eta_2 = \eta_{1,3}\eta_1. \]

On the other hand, starting with \( i = 1 \) and \( j = 2 \) in equation (9) we obtain also:

\[ \eta_2\eta_{1,3} = \eta_1\eta_2 = \eta_{1,3}\eta_2. \]

In diagrams, we have therefore the following equivalences:

In diagrams, we have therefore the following equivalences:

I.e., if \( j - i = 1 \), \( \eta_{i,j+1} \) commute with \( \eta_i, \eta_j \) and

\[ \eta_{i,j+1}\eta_i = \eta_i\eta_{i,j+1}\eta_j = \eta_i\eta_j. \]  \hspace{1cm} (18)

We are now ready to generalize the elements \( \eta_{i,j+1} \) (\( j - i = 1 \)), to the elements \( \eta_{i,k} \), for every \( k \geq i \), i.e., \( \eta_{i,k} \) will be represented by a spring connecting the strand \( i \) with the strand \( k \). We shall say also that such a tie has length equal to \( k - i \).

Of course, each \( \eta_i \) is a tie of length 1.

\[ \eta_{i,i+1} = \eta_i. \]
We define also the tie of zero length, as the monoid unit:

\[ \eta_{i,i} = 1. \]  

(19)

In virtue of the transparency of the tie, there are different expressions of \( \eta_{i,k} \) in terms of the \( \sigma_j \) and \( \eta_j \). An example is given in figure 14 where the three diagrams are all equivalent to \( \eta_{2,7} \). The proof of the equivalence is based on the equations (13) and (14).

There are in fact \((k - i - 1)^2\) equivalent expressions of \( \eta_{i,k} \). Let \( s_i \) denote either the element \( \sigma_i \) or \( \sigma_i^{-1} \), and let \( \bar{s}_i = s_i^{-1} \).

Given a pair \( i, k \) such that \( k - i > 1 \), the following \( 2^{k-i-1} \) expressions of \( \eta_{i,k} \), obtained for all possible choices of \( s_l = \sigma_l \) or \( s_l = \sigma_l^{-1} \):

\[ \eta_{i,k} = s_is_{i+1}s_{i+2}\cdots s_{k-2}\eta_k\bar{s}_{k-2}\cdots \bar{s}_{i+1}\bar{s}_i \]

are all equivalent. Moreover, for every \( j \) such that \( i \leq j < k - 1 \) there are similarly \( 2^{k-i-1} \) equivalent expressions:

\[ \eta_{i,k} = s_is_{i+1}\cdots s_j-1\bar{s}_{k-1}\bar{s}_{k-2}\cdots \bar{s}_{j+1}\eta_j\bar{s}_{j+1}\cdots \bar{s}_{k-1}\bar{s}_{j-1}\cdots \bar{s}_{i+1}\bar{s}_i. \]

Figure 14. \( \sigma_6^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\eta_2\sigma_3\sigma_4\sigma_5\eta_6^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1} \sim \sigma_2^{-1}\sigma_3^{-1}\sigma_6\eta_4\sigma_5^{-1}\sigma_6^{-1}\sigma_3\sigma_2 \)

Similarly, the generalization of (18) to all \( \eta_{i,k} \) reads (for \( i \leq k \leq m \)) (see examples in figure 15):

\[ \eta_{i,k}\eta_{k,m} = \eta_{i,k}\eta_{k,m} = \eta_{k,m}\eta_{i,k} \]  

(20)

In particular, if \( k = m \), we get

\[ \eta_{i,k}\eta_{k,k} = \eta_{i,k}\eta_{i,k}, \]
i.e., by (19), all ties \( \eta_{i,k} \) are idempotent. This follows as well from any expression of \( \eta_{i,k} \), in virtue of (10).

Figure 15. Examples relation (20)
3.3. **The Markov and Alexander theorems for tied links.** By taking the obvious monomorphism of monoids $TB_n$ into $TB_{n+1}$, we can consider the inductive limit $TB_\infty$ associated to the inclusions chain $TB_2 \subset \cdots \subset TB_{n-1} \subset TB_n \subset \cdots$. As in the classical case, given a tied braid $\tau$ we denote by $\hat{\tau}$ its closure, which is a tied link. We have then a map from $TB_\infty$ to $T$. We are going to prove now that in fact this map is surjective (Theorem 2). Later, we define the Markov moves for tied braids and then we prove a Markov theorem for tied links (Theorem 3).

**Theorem 2** (Alexander theorem for tied links). *Every oriented tied link can be obtained by closing a tied braid.*

**Proof.** Given a tied link $L(P)$, one fixes a center $O$ in the plane of the diagram of $L(P)$ and proceed according to the Alexander procedure for classical links. The ties do not prevent the procedure because of their transparency, so that the tied link that we obtain is isotopy equivalent to $L(P)$. Such a tied link has, however, ties connecting pairs of points in any direction. Using the property that the ends of the ties can slide freely along the strands of the link, and that the ties are transparent, we arrange them so that the ends of each tie lie on one halfline originating at $O$, non coinciding with the halfline where we open $L(P)$ (see figure 16). The braid obtained will have horizontal ties connecting two points of different strands. This is by construction a tied braid whose closure is isotopy equivalent to $L(P)$. \hfill \square

![Figure 16. Arranging the ties](image)

**Definition 3.** *Two tied braids in the monoid $TB_\infty$ are $\sim_T$ equivalent if one can be obtained from the other by a finite sequence of moves belonging to the following set of operations (or moves):*

1. $\alpha \beta$ can be exchanged with $\beta \alpha$
2. $\alpha$ can be exchanged with $\alpha \sigma_n$ or $\alpha \sigma_n^{-1}$
3. $\alpha$ can be exchanged with $\alpha \sigma_n \eta_n$ or $\alpha \sigma_n^{-1} \eta_n$

*for all $\alpha, \beta \in TB_n$.*

**Theorem 3** (Markov Theorem for tied links). *Two tied braids have isotopic closure if and only if they are $\sim_T$-equivalent.*

**Proof.** If the tied braids have no ties, then the theorem coincides with the classical Markov theorem, using items (1) and (2). If there are ties, it is sufficient to remark that the properties of transparency, idem-potency, and property (20) reflect exactly the properties of tied links stated in definition 1. Item (2) of the theorem reflects the invariance of the isotopy class of a
link by adding a loop (the Reidemeister move of third type). Item (3) says the same, when there is an additional tie between a strand and a loop added to this strand. Indeed, such a tie is not essential and can be destroyed.

\[ \square \]

4. Construction of $F$ via the Jones recipe

The procedure to construct the Homflypt polynomial done in [8], leads to a generic way to construct an invariant of knotted objects, which is called Jones recipe. The main objective of this section is to use the Jones recipe to construct the invariant $F$, see Theorem 3. To do that, we firstly note that Theorems 3 and 2 allows to see the set of tied links as the set of equivalence classes, under $\sim_T$, of $TB_\infty$. Secondly, in Proposition 4 below we define a representation of the $TB_\infty$ in the so-called bt–algebra. This representation together with a Markov trace, supported by the bt–algebra, are the main ingredients in the Jones recipe for the construction of the invariant $F$.

In order to show the first ingredient we shall recall the definition of the bt–algebra. Let $n$ be a positive integer. The bt–algebra, denoted $E_n = E_n(u)$, appeared firstly in [9] and [1] and was studied in different contexts in [14], [5] and [2].

**Definition 4.** The algebra $E_n$ is the associative unital $\mathbb{C}(u)$–algebra generated by $T_1, \ldots, T_{n-1}, E_1, \ldots, E_{n-1}$ subject to the following relations:

\begin{align*}
T_i T_j & = T_j T_i \quad \text{for all } |i - j| > 1 \quad (21) \\
T_i T_j T_i & = T_j T_i T_j \quad \text{for all } |i - j| = 1 \quad (22) \\
T_i^2 & = 1 + (u - 1)E_i (1 + T_i) \quad \text{for all } i \quad (23) \\
E_i E_j & = E_j E_i \quad \text{for all } i, j \quad (24) \\
E_i^2 & = E_i \quad \text{for all } i \quad (25) \\
E_i T_i & = T_i E_i \quad \text{for all } i \quad (26) \\
E_i E_j T_i & = T_i E_i E_j = E_j T_i E_j \quad \text{for } |i - j| = 1 \quad (27) \\
E_i T_j T_i & = T_j T_i E_j \quad \text{for } |i - j| = 1. \quad (28)
\end{align*}

The relations (23) and (25) imply that $T_i$ is invertible. Moreover, we have

\[ T_i^{-1} = T_i + (u^{-1} - 1)E_i + (u^{-1} - 1)E_i T_i. \quad (29) \]

**Proposition 1.** The mapping $\sigma_i \mapsto T_i$ and $\eta_i \mapsto E_i$ defines a monoid representation, denoted $\varpi_n$, of $TB_n$ in $E_n$.

**Proof.** It is enough to prove that the defining relations of $TB_n$, suitably translated by replacing $\sigma_i$ by $T_i$ and $\eta_i$ by $E_i$, are satisfied in $E_n$. Thus, it remains only to check that the translation of relation (3) holds in $E_n$. From (29), we have

\[ E_i T_j T_i^{-1} = E_i T_j T_i + (u^{-1} - 1)E_i T_j E_i + (u^{-1} - 1)E_i T_j E_i T_i \]

Now, having in mind relations (23) and (27), it follows that

\[ E_i T_j T_i^{-1} = T_j T_i^{-1}E_i \quad \text{for } |i - j| = 1. \quad (30) \]

\[ \square \]
Remark 8. The proposition above says in particular that the bt–algebra can be defined as the quotient of the monoid algebra of $TB_n$ by the two–sided ideal generated by the elements $T_i^2 - 1 - (u - 1)T_i(1 + T_i)$, for all $i$.

We shall recall now the second ingredient, that is, a Markov trace on the bt–algebra. Let $A$ and $B$ be two non–zero indeterminates in $\mathbb{C}$. We have:

Theorem 4. [2, Theorem 3] There exist a family $\{\rho_n\}_{n \in \mathbb{N}}$ of Markov traces on the bt–algebra. I.e. for all $n \in \mathbb{N}$, $\rho_n : E_n \rightarrow \mathbb{C}(u, A, B)$ is the linear map uniquely defined by the following rules:

(i) $\rho_n(1) = 1$
(ii) $\rho_n(XY) = \rho_n(YX)$
(iii) $\rho_{n+1}(XT_n) = \rho_{n+1}(XE_nT_n) = A\rho_n(X)$
(iv) $\rho_{n+1}(XE_n) = B\rho_n(X)$

where $X, Y \in E_n$.

In [2, Section 5], by using the Jones recipe, we have defined an invariant polynomial $\bar{\Delta}$ for classical links. Extending the field of definition to $\mathbb{C}(u, A, B, \sqrt{L})$, the invariant $\bar{\Delta}$ is a Laurent polynomial in $\sqrt{L}, B$ and $u$, where

$$L := \frac{A + B - uB}{uA}$$

$\bar{\Delta}$ is essentially the composition of the natural representation of the braid group $B_n$ in $E_n$ with the Markov trace above, see [2, Theorem 4]. The invariant of tied links that we will define now is nothing more that an extension of $\bar{\Delta}$ to tied links. To be precise, we simply replace in the definition of $\bar{\Delta}$ the representation of the braid group in $E_n$ by the representation $\varpi_n$ of the tied braid monoid in $E_n$. Thus, we will denote also by $\bar{\Delta}$ this invariant of tied links. More precisely, set

(31) $\bar{\Delta} = \frac{1 - Lu}{\sqrt{L}(1 - u)B}$

thus,

(32) $\sqrt{L} \bar{\Delta} A = 1$ or equivalently $\bar{\Delta} = \frac{1}{A \sqrt{L}}$.

The invariant $\bar{\Delta}$ is defined as follows

(33) $\bar{\Delta}(\theta) := \bar{\Delta}^{n-1}(\sqrt{L})^{e(\theta)}(\rho_n \circ \varpi_n)(\theta)$ $(\theta \in TB_n)$,

where $e(\theta)$ denotes the exponent of the tied braid $\theta$. That is, if $\theta = g_1^{e_1}g_2^{e_2} \ldots g_m^{e_m} \in TB_n$, where the $g_i$’s are defining generators of $TB_n$, then

(34) $e(\theta) := \sum_{i=1}^{N} \epsilon_i$,

where $\epsilon_i = 1$ if $g_i = T_k$ and $\epsilon_i = 0$ if $g_i = E_k$.

Theorem 5. Let $TL$ be a tied–link diagram obtained by closing the tied braid $\theta \in TB_n$. Let $A = z, B = t$. Then

$$\Delta(\theta) = F(TL).$$
Proof. We use here the results of Section 2 under the equations $A = z$ and $B = t$. So, in particular, $W = L$. If $TL$ is a collection of $c$ unknotted, unlinked curves, the value of $F(TL)$ is $1/(\sqrt[3]{A})^{c-1}$. Such link is indeed the closure of the trivial braid $\theta$ with $c$ threads, where $\rho_c(\theta) = 1$ and $e(\theta) = 0$. Therefore $\Delta(\theta) = D^{c-1} = F(TL)$. Observe that the values of $\Delta$ is 1 on a single closed unknotted curve.

If $TL$ is a collection of $c$ unknotted, unlinked curves with $m$ essential ties, the value of $F(TL)$ is $B^m/(\sqrt[3]{A})^{c-1}$. Such link is indeed the closure of the braid $\theta$ with $c$ vertical threads, of which $m$ pairs are connected by a tie. Of course it is possible to arrange the ties so that they have all length one and connect the last $m + 1 \leq c$ threads. Then using item definition (iv) of the trace we obtain $\rho_c(\theta) = B^m$. Moreover, $e(\theta) = 0$. Therefore $\Delta(\theta) = D^{c-1}B^m = F(TL)$.

Suppose now that four tied braids in $E_n(u)$ are given, $\theta_+, \theta_-, \theta_\sim$, and $\theta_{+,\sim}$, that are all identical except for the neighborhood of a $T_i$ element, exactly as for the tied links, see Figure 3. Now, formula (29) of the inverse of $T_i$ and the linearity of the trace imply that:

$$\rho_n(\theta_+) - \rho_n(\theta_-) = (1 - u^{-1})\rho_n(\theta_\sim) + (1 - u^{-1})\rho_n(\theta_{+,\sim}).$$

(35)

Let now $TL_+, TL_-, TL_\sim$ and $TL_{+,\sim}$ the tied links obtained from the closure of the tied braids above. The polynomial $\Delta$ for these links is obtained multiplying the trace by the factor $D^{n-1}(\sqrt[3]{U})^{\rho_k} (k = +, -, \sim$ or $+, \sim)$. Now, let us denote $e(\theta_\sim) = d$. By the definition is evident that $e(\theta_+) = e(\theta_{+,\sim}) = d + 1$ and $e(\theta_-) = d - 1$. Therefore (35) can be written in terms of the polynomial $\Delta$:

$$\frac{1}{\sqrt[3]{U}}\Delta(\theta_+) - \sqrt[3]{U}\Delta(\theta_-) = (1 - u^{-1})\frac{1}{\sqrt[3]{U}}\Delta(\theta_\sim) + (1 - u^{-1})\frac{1}{\sqrt[3]{U}}\Delta(\theta_{+,\sim}),$$

(36)

which coincides with the Skein rule III for the polynomial $F$.

Since $\Delta$ is a topological invariants for links (see 2) and satisfies the same rules I,II, and III as the polynomial $F$, invariant for tied links, it coincides with $F$.

\[ \square \]

5. Computer computation of $F$

In this section we show how to calculate the polynomial $F$ for a tied link or a classical link by means of the Theorem 5. Indeed, if a (tied) link $L$ is put in the form of (tied) braid $X$, then we calculate the trace of $X$, and then we normalize it by (33) to obtain the invariant $F$.

An element of the algebra $E_n$ is a linear combination of words, i.e., finite expressions in the generators $T_1, \ldots, T_{n-1}$, and the $E_1, \ldots, E_{n-1}$. The coefficients are Laurent polynomials in the parameter $u$. An addend is a single word with a coefficient.

A word is simple if the consecutive generators in it are different and appear to the first power. The trace of an element of the algebra is obtained as a linear combination of traces of words. A word of $E_n$ containing a sole element in the set $\{E_{n-1}, T_{n-1}, E_{n-1}T_{n-1}\}$ is said $p$-reducible. Indeed, it is reduced by the trace properties, stated in Theorem 3, to a coefficient times the trace of a word of $E_{n-1}$. Therefore, to calculate the trace of a word, one needs to transform every word into a word or a linear combination of words $p$-reducible.

We list here a series of procedures used by the algorithm.

5.1. Simplification of an addend. Iterations of the following procedures reduce a word into a linear combination of simple words.
S1 $k$ consecutive copies of the same generator $E_i$ are replaced by a unique $E_i$ because of the relation

$$E_i^k = E_i \quad \text{for every } k > 0.$$ 

S2 Consecutive powers of the same generator $T_i$ are replaced by $T_i$ to the algebraic sum of the exponents of such powers.

S3 An addend whose word contains powers of the $T_i$'s with exponents different from one is transformed into a sum of addends containing $T_i$'s to the first power. We use the following relations that follows from relations (23), (25), (26) and (29).

\[
T_i^{2m} = 1 + \sum_{k=0}^{2m-1} (-1)^{k+1} u^k (E_i + E_i T_i) \quad \text{for every } m > 0,
\]

\[
T_i^{2m+1} = T_i + \sum_{k=1}^{2m} (-1)^k u^k (E_i + E_i T_i) \quad \text{for every } m > 0,
\]

\[
T_i^{-2m} = 1 + \sum_{k=0}^{2m} (-1)^k u^{-k} (E_i + E_i T_i) \quad \text{for every } m > 0,
\]

\[
T_i^{-(2m+1)} = T_i + \sum_{k=1}^{2m+1} (-1)^{k+1} u^{-k} (E_i + E_i T_i) \quad \text{for every } m > 0.
\]

5.2. Reduction of a word. The following procedures are used to make a word $\rho$-reducible

R1 Denote by $X_i, Y_i, Z_i$ elements in the set $G_i := \{T_i, E_i, E_i T_i = T_i E_i\}$. A word of type $X_i Y_{i-1} Z_i$ is said reducible. Using the defining relations of the $b t$–algebra, this procedure transforms a reducible word $X_i Y_{i-1} Z_i$ into a word or into a linear combination of words that contain one and only one element of $G_i$. There are in all 27 cases, listed here:
successive element to
element from 
combination of words with 
$T$-reducible. The iteration of the following procedures transform 
\textbf{STEP 1}

Let $m$ be the maximum index of the elements of the simple word $X$ and let $G_m := \{T_m, E_m, E_mT_m = T_mE_m\}$. We suppose, moreover, that $X$ contains $r > 1$ elements from $G_m$, so that is not $\rho$-reducible. The iteration of the following procedures transform $X$ into a word or a linear combination of words with $r - 1$ elements from $G_m$.

Let $X = x_1x_2 \ldots x_t$, where each $x_j$ is an element from a $g_k, k \leq m$, and let $g_m$ be the first element from $G_m$ encountered in the simple word $X$. Observe that $X$ is simple, so that the successive element to $g_m$ belongs to $G_k, k \leq m - 1$. Write

$$X = X_1g_mY_1$$

\textbf{STEP 1}
R1 Let $y$ be the first element of $Y_1$. If $y = g_i$, with $i < m - 1$, then let $X_2 = X_1y$ and $Y_1 = yY_2$, so that
\[ X = X_2g_mY_2. \]
Rename $X_1 \leftarrow X_2$ and $Y_1 = Y_2$ and repeat, until $y \in G_{m-1}$ or $y \in G_m$. If $y \in G_m$, then the simplified $X$ is a word or a linear combination of words with at most $r - 1$ elements from $G_m$. Otherwise

R2 If $y = g_{m-1} \in G_{m-1}$, then write $Y_1 = g_{m-1}Y_2$, so that
\[ X = X_1g_mg_{m-1}Y_2. \]
Rename $Y_1 \leftarrow Y_2$. Go to step 2.

STEP 2

R4 If the first element $y$ of $Y_1$ belongs to $G_m$, then reduce the word by R1, so that the simplified $X$ is a word or a linear combination of words with at most $r - 1$ elements from $G_m$. Otherwise

R5 If $y = g_j$, with $i < m - 2$, then let $X_2 = X_1y$ and $Y_1 = yY_2$, so that
\[ X = X_2g_mg_{m-1}Y_2. \]
Rename $X_1 \leftarrow X_2$ and $Y_1 \leftarrow Y_2$. Let $y$ be the first element of $Y_1$. If $y = g_i$, with $i < m - 2$, then repeat R5. If $z \in G_m$, then go to R4. If $y \in G_{m-1}$, then simplify. The simplified words are of type $X = X_1g_mY_1$ (in that case go to R2) or of type $X = X_1g_mg_{m-1}Y_1$ (in that case go to R4). Otherwise

R6 If $y = g_i$, with $i = m - 2$, then write $Y_1 = g_{m-2}Y_2$, so that
\[ X = X_1g_mg_{m-1}g_{m-2}Y_2. \]
Rename $Y_1 \leftarrow Y_2$. Go to next step.

STEP n At step n every simple word $X$ with $r > 1$ elements from $G_m$ is written as
\[ X = X_1Z_{m,n}Y_1 \quad \text{where} \quad Z_{m,n} = g_mg_{m-1}g_{m-2}\ldots g_{m-n+1}. \]

R7 Let $k = m - n + 1$. If the fist element $y$ of $Y_1$ is $g_i$ with $i < k - 1$, then it is put at the end of $X_1$, since it commutes wit every $g_j$, $k \leq j \leq m$, and proceed by analyzing the successive element.

R8 If $y = g_j$, with $j = k$, then simplify the word. The simplified words have to be processed by step $n$ or $n - 1$.

R9 If $y \in G_j$, with $k < j < m$, then $y$ commutes with all elements of $Z_{m,n}$ with index less than $j - 1$, so that, writing $y = g'_j$, we get
\[ X = X_1g_mg_{m-1}\ldots g_{i+1}g_jg_{j-1}g'_jg_{j-2}\ldots g_k. \]

The subword $g_jg_{j-1}g'_j$ is processed by R1, replacing it by subwords of type $g'_{j-1}g''_j$ or $g''_{j-1}g'_jg''_{j-1}$. Since $g'_{j-1}$ commutes with $g_{i+1}, g_{i+2}, \ldots, g_m$, it is put at the end of $X_1$. If the element $g''_{j-1}$ is absent, then the subword $g_{j-2}\ldots g_k$ commutes with $g_mg_{m-2}\ldots g_j$ and is put at the end of $X_1$. Go to step $m-j+1$. Otherwise, go to R8.

R10 If $y \in G_m$, then $y$ commutes with all elements of $Z_{m,n}$ with index less than $m - 1$, so that, writing $y = g'm$, we get
\[ X = X_1g_mg_{m-1}g'_mg_{m-2}\ldots g_k. \]
The word is reduced by R1 and the reduced words have at most \( r - 1 \) elements from \( G_m \).

R11 If \( y = g_{k-1} \), then write \( Y_1 = g_{k-1}Y_2 \), so that

\[
X = X_1 g_m g_{m-1} g_{m-2} \ldots g_{m-n} Y_2.
\]

Rename \( Y_1 \leftarrow Y_2 \). Go to step \( n+1 \).

Since the number of elements between \( g_m \) and the second occurrence in \( X \) of an element from \( G_m \) is finite, the case \( y \in G_m \) happens for some step \( n \geq 1 \), so that the number \( r \) of occurrences of elements from \( G_m \) is diminished. When \( r = 1 \) the word is \( \rho \)-reducible.

5.3. **Contraction.** Suppose the word \( X \) be \( \rho \)-reducible. Then we write \( X \) as \( W_{m-1}g_mV_{m-1} \), with \( g_m \in G_m \).

C1 The words \( W_{m-1} \) and \( V_{m-1} \) are two simple words of \( E_m \). Let \( \alpha \) be a coefficient. This procedure transforms the following simple addends:

\[
\begin{align*}
\alpha W_{m-1}E_m V_{m-1} & \rightarrow \alpha B W_{m-1} V_{m-1} \\
\alpha W_{m-1}T_m V_{m-1} & \rightarrow \alpha A W_{m-1} V_{m-1}, \\
\alpha W_{m-1}E_m T_m V_{m-1} & \rightarrow \alpha A W_{m-1} V_{m-1}.
\end{align*}
\]

This procedure applies the properties of the trace. Indeed, we have for instance (see Theorem 5(i),(ii) and (iii)),

\[
\rho_{m+1}(\alpha W_{m-1}T_m V_{m-1}) = (by \ (ii)) = \alpha \rho_{m+1}(V_{m-1}W_{m-1}T_m) = (by \ (iii)) \rho_{m}(V_{m-1}W_{m-1}) = (by \ (ii)) = \alpha A \rho_{m}(W_{m-1}V_{m-1}).
\]

When \( m = 1 \), \( W_{m-1} = V_{m-1} = 1 \), so that \( \rho_{m}(W_{m-1}V_{m-1}) = 1 \) (by Theorem 5(i)). Therefore the output of the procedure is a polynomial and increments the trace.

**References**

[1] F. Aicardi, J. Juyumaya, *An algebra involving braids and ties*. Preprint ICTP IC/2000/179, Trieste.
[2] F. Aicardi, J. Juyumaya, *Markov trace on the algebra of braids and ties*. Preprint 2014.
[3] F. Aicardi, *Colored Links with an Invariant Polynomial*, submitted to ArJM, december 2014.
[4] J.C. Baez, *Link invariants of finite type and perturbation theory*, Lett. Math. Phys. 26 (1992), no. 1, 43–51.
[5] E. O. Banjo, *The generic representation theory of the Juyumaya algebra of braids and ties*. Algebr. Represent. Theory 16 (2013), no. 5, 1385–1395.
[6] J.S. Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. (N.S.) 28 (1993), no. 2, 253–287.
[7] B. Gemein, *Singular braids and Markov’s theorem*, J. Knot Theory Ramifications 6 (1997), no. 4, 441–454.
[8] V.F.R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. Math. 126 (1987), 335–388.
[9] J. Juyumaya, *Another algebra from the Yokonuma-Hecke algebra*. Preprint ICTP IC/1999/160.
[10] J. Juyumaya, S. Lambropoulou, *p-adic framed braids II*, Adv Math, 234 (2013), 149–191.
[11] L. Kauffman, *Virtual Knot Theory*, European J. Combin 20 (1999), 663–690.
[12] K.H. Ko, L. Smolinsky, *The framed braid group and 3-manifolds*, Proceedings of the AMS, 115, No. 2, 541–551 (1992).
[13] W.B.R. Lickorish, K.C. Millet, *A Polynomial Invariant of Oriented Links*. Topology Vol. 26, No 1 (1987), 107–141.
[14] S. Ryom-Hansen, *On the representation theory of an algebra of braids and ties*. J. Algebr. Comb. 33 (2011), 57–79.
[15] L. Smolin, *Knot theory, loop space and the diffeomorphism group*, New perspectives in canonical gravity, 245–266, Monogr. Textbooks Phys. Sci. Lecture Notes, 5, Bibliopolis, Naples, 1988.
ICTP, Strada Costiera, 11, 34100 Trieste, Italy.
E-mail address: faicardi@ictp.it

Instituto de Matemáticas, Universidad de Valparaíso, Gran Bretaña 1091, Valparaíso, Chile.
E-mail address: juyumaya@uvach.cl