THREE-DIMENSIONAL 2-CRITICAL BOOTSTRAP PERCOLATION:
THE STABLE SETS APPROACH

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Abstract. Consider a \( p \)-random subset \( A \) of initially infected vertices in the discrete cube \([L]^3\), and assume that the neighbourhood of each vertex consists of the \( a_i \) nearest neighbours in the \( \pm e_i \)-directions for each \( i \in \{1, 2, 3\} \), where \( a_1 \leq a_2 \leq a_3 \). Suppose we infect any healthy vertex \( v \in [L]^3 \) already having \( r \) infected neighbours, and that infected sites remain infected forever. In this paper we determine \( \log \) of the critical length for percolation up to a constant factor, for all \( r \in \{a_3 + 1, \ldots, a_3 + a_2\} \) with \( a_3 \geq a_1 + a_2 \). We moreover give upper bounds for all remaining cases \( a_3 < a_1 + a_2 \) and believe that they are tight up to a constant factor.

1. Introduction

The study of bootstrap processes on graphs was initiated in 1979 by Chalupa, Leath and Reich [11], and is motivated by problems arising from statistical physics, such as the Glauber dynamics of the zero-temperature Ising model, and kinetically constrained spin models of the liquid-glass transition (see, e.g., [6,16,18–20]). The \( r \)-neighbour bootstrap process on a locally finite graph \( G \) is a monotone cellular automata on the configuration space \( \{0, 1\}^V(G) \) (we call vertices in state 1 “infected”), evolving in discrete time in the following way: 0 becomes 1 when it has at least \( r \) neighbours in state 1, and infected vertices remain infected forever. Throughout this paper, \( A \) denotes the initially infected set, and we write \([A] = G\) if the state of each vertex is eventually 1.

We will focus on anisotropic bootstrap models, which are \( d \)-dimensional analogues of a family of (two-dimensional) processes studied by Duminil-Copin, van Enter and Hulshof [12,13,15]. In these models the graph \( G \) has vertex set \([L]^d\), and the neighbourhood of each vertex consists of the \( a_i \) nearest neighbours in the \( -e_i \) and \( e_i \)-directions for each \( i \in [d] \), where \( a_1 \leq \cdots \leq a_d \) and \( e_i \in \mathbb{Z}^d \) denotes the \( i \)-th canonical unit vector. In other words, \( u,v \in [L]^d \) are neighbours if (see Figure 1 for \( d = 3 \))

\[
u - v \in N_{a_1,\ldots,a_d} := \{\pm e_1, \ldots, \pm a_1 e_1\} \cup \cdots \cup \{\pm e_d, \ldots, \pm a_d e_d\}.
\]

We also call this process the \( N_{a_1,\ldots,a_d} \)-model. Our initially infected set \( A \) is chosen according to the Bernoulli product measure \( \mathbb{P}_p = \otimes_{v \in [L]^d} \text{Ber}(p) \), and we are interested in the so-called critical length for percolation, for small values of \( p \)

\[
L_c(N_{a_1,\ldots,a_d}, p) := \min\{L \in \mathbb{N} : \mathbb{P}_p([A] = [L]^d) \geq 1/2\}.
\]

The analysis of these bootstrap processes for \( a_1 = \cdots = a_d = 1 \) was initiated by Aizenman and Lebowitz [1] in 1988, who determined the magnitude of the critical length up to a constant factor in the exponent for the \( N_2 \)-model (in other words, they determined...
the ‘metastability threshold’ for percolation). In the case \( d = 2 \), Holroyd \([17]\) determined (asymptotically, as \( p \to 0 \)) the constant in the exponent (this is usually called a sharp metastability threshold).

For the general \( \mathcal{N}_r^{1,...,1} \)-model with \( 2 \leq r \leq d \), the threshold was determined by Cerf and Cirillo \([9]\) and Cerf and Manzo \([10]\), and the sharp threshold by Balogh, Bollobás and Morris \([3]\) and Balogh, Bollobás, Duminil-Copin and Morris \([2]\): for all \( d \geq r \geq 2 \) there exists a computable constant \( \lambda(d, r) \) such that, as \( p \to 0 \),

\[
L_c(\mathcal{N}_r^{1,...,1}, p) = \exp\left( -\frac{\lambda(d, r) + o(1)}{p^{1/(d-r+1)}} \right).
\]

In dimension \( d = 2 \), we write \( a_1 = a, a_2 = b \), and the \( \mathcal{N}_r^{a,b} \)-model is called isotropic when \( a = b \) and anisotropic when \( a < b \). Hulshof and van Enter \([15]\) determined the threshold for the first interesting anisotropic model given by the family \( \mathcal{N}_3^{a,b,c} \), and the corresponding sharp threshold was determined by Duminil-Copin and van Enter \([12]\).

The threshold was also determined in the general case \( r = a + b \) by van Enter and Fey \([14]\) and the proof can be extended to all \( b + 1 \leq r \leq a + b \): as \( p \to 0 \),

\[
\log L_c(\mathcal{N}_r^{a,b,c}, p) = \begin{cases} 
\Theta\left( p^{-(r-b)} \right) & \text{if } b = a, \\
\Theta\left( p^{-(r-b)}(\log p)^2 \right) & \text{if } b > a.
\end{cases}
\]  

(3)

1.1. Anisotropic bootstrap percolation on \([L]^3\). In this paper we consider the three-dimensional analogue of the anisotropic bootstrap process studied by Duminil-Copin, van Enter and Hulshof. In dimension \( d = 3 \), we write \( a_1 = a, a_2 = b \) and \( a_3 = c \).

These models were studied by van Enter and Fey \([14]\), and the present author \([4]\) for \( r \in \{1 + b + c, \ldots, a + b + c\} \); they determined the following bounds on the critical length, as \( p \to 0 \),

\[
\log \log L_c(\mathcal{N}_r^{a,b,c}, p) = \begin{cases} 
\Theta\left( p^{-(r-(b+c))} \right) & \text{if } b = a, \\
\Theta\left( p^{-(r-(b+c))}(\log \frac{1}{p})^2 \right) & \text{if } b > a.
\end{cases}
\]  

(4)

We moreover determined the magnitude of the critical length up to a constant factor in the exponent in the cases \( r \in \{c + 1, c + 2\}, r \leq a + c \), for all triples \( (a, b, c) \), except for \( c = a + b - 1 \) when \( r = c + 2 \) (see Section 6 in \([5]\)):

set \( s := r - c \in \{1, 2\} \), then, as \( p \to 0 \),
corresponds to two out of four possibilities for the stable set
Proposition 1.2. Consider the sequences
and believe that they tell us the right order of the threshold. This range of values of
$r$
critical length is given by the following.
of $N$
For every $r$
The following is our main result.
$r$
$determining log of the critical length up to a constant factor for these values of $r$
$c < a$
We also have upper bounds for the remaining values $c < r$
$c < a$
Finally, the range $c < r$
$c < a$
Here there are some numerical values of $t$
$c < r$
$\alpha$
\begin{align*}
\log L_c (N_r^{a,b,c}, p) &= \begin{cases}
\Theta \left(p^{-s/2}\right) & \text{if } c = b = a, \\
\Theta \left(p^{-s/2} (\log \frac{1}{p})^{1/2}\right) & \text{if } c = b > a, \\
\Theta \left(p^{-s/2} (\log \frac{1}{p})^{3/2}\right) & \text{if } c \in \{b + 1, \ldots, a + b - s\}, \\
\Theta \left(p^{-s}\right) & \text{if } c = a + b, \\
\Theta \left(p^{-s} (\log \frac{1}{p})^2\right) & \text{if } c > a + b.
\end{cases}
\end{align*}
(5)

In this paper we extend the last two cases in [5] to all values $c < r \leq b + c$, by
determining log of the critical length up to a constant factor for these values of $r$
and $c$.
The following is our main result.

**Theorem 1.1.** For every $r \in \{c + 1, \ldots, b + c\}$, as $p \to 0$,
\[
\log L_c (N_r^{a,b,c}, p) = \begin{cases}
\Theta \left(p^{-(r-c)}\right) & \text{if } c = a + b, \\
\Theta \left(p^{-(r-c)} (\log \frac{1}{p})^2\right) & \text{if } c > a + b.
\end{cases}
\]
(6)

We also have upper bounds for the remaining values $c < a + b$ when $c < r \leq a + c$, and believe that they tell us the right order of the threshold. This range of values of $r$
corresponds to two out of four possibilities for the stable set of $N_r^{a,b,c}$ (see Remark 1.4).

**Proposition 1.2.** Consider the sequences $\{\alpha_s\}_{s \geq 2}$ and $\{t_s\}_{s \geq 2}$ given by
\[
\alpha_s = \frac{t + 1}{t + 2}(s - t/2), \quad \text{and } t = t_s := \left\lfloor \frac{\sqrt{9 + 8s - 5}}{2} \right\rfloor.
\]
(7)
(i) For every $r \in \{c + 2, \ldots, a + c\}$, if $a + b - (r - c) < c < a + b$, as $p \to 0$,
\[
\log L_c (N_r^{a,b,c}, p) = \begin{cases}
O \left(p^{-\alpha_{r-c}} (\log \frac{1}{p})^2\right) & \text{if } r < a + b + \alpha_{r-c}, \\
O \left(p^{-(r-(a+b))} (\log \frac{1}{p})^2\right) & \text{if } r \geq a + b + \alpha_{r-c}.
\end{cases}
\]
(8)
(ii) For every $r \in \{c + 3, \ldots, a + c\}$, as $p \to 0$,
\[
\log L_c (N_r^{a,b,c}, p) = \begin{cases}
O \left(p^{-\alpha_{r-c}}\right) & \text{if } c = b = a, \\
O \left(p^{-\alpha_{r-c}} (\log \frac{1}{p})^{(t_r-c+1)/(t_r-c+2)}\right) & \text{if } c = b > a, \\
O \left(p^{-\alpha_{r-c}} (\log \frac{1}{p})^{(t_r-c+3)/(t_r-c+2)}\right) & \text{if } b < c \leq a + b - (r - c).
\end{cases}
\]
(9)

Here there are some numerical values of $t_s$ and $\alpha_s$, for $s = 2, 3, \ldots, 14$.

| $s$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $t_s$ | $0$ | $1$ | $1$ | $1$ | $2$ | $2$ | $2$ | $2$ | $3$ | $3$ | $3$ | $3$ | $3$ |
| $\alpha_s$ | $1$ | $5/3$ | $7/3$ | $3$ | $15/4$ | $18/4$ | $21/4$ | $6$ | $34/5$ | $38/5$ | $42/5$ | $46/5$ | $10$ |

**Table 1.** Some values of $t_s$ and $\alpha_s$.

Finally, the range $a + c < r \leq b + c$ corresponds to a third possibility for the stable set
of $N_r^{a,b,c}$; in these cases with $c < a + b$ (based on upper bounds), we conjecture that the
critical length is given by the following.
Conjecture 1.3. For \( r \in \{a + c + 1, \ldots, b + c\} \), as \( p \to 0 \),

\[
\log L_c(\mathcal{N}_r^{a,b,c}, p) = \begin{cases} 
\Theta \left( p^{-(r-c-a+\alpha_1)} \right) & \text{if } c = b > a, \\
\Theta \left( p^{-(r-a-b)} \right) & \text{if } b < c < a + b,
\end{cases}
\]

where \( \alpha_1 = 1/2 \).

1.2. The stable sets approach. The model we study here is a special case of the following extremely general class of \( d \)-dimensional monotone cellular automata, which were introduced by Bollobás, Smith and Uzzell [8].

Let \( \mathcal{U} = \{X_1, \ldots, X_m\} \) be an arbitrary finite family of finite subsets of \( \mathbb{Z}_d \setminus \{0\} \). We call \( \mathcal{U} \) the update family, each \( X \in \mathcal{U} \) an update rule, and the process itself \( \mathcal{U} \)-bootstrap percolation. Let \( \Lambda \) be either \( \mathbb{Z}_d \) or \( \mathbb{Z}_d^d \) (the \( d \)-dimensional torus of sidelength \( L \)). Given a set \( A \subset \Lambda \) of initially infected sites, set \( A_0 = A \), and define for each \( t \geq 0 \),

\[
A_{t+1} = A_t \cup \{x \in \Lambda : x + X \subset A_t \text{ for some } X \in \mathcal{U}\}.
\]

The set of eventually infected sites is the closure of \( A \), denoted by \( [A]_{\mathcal{U}} = \bigcup_{t \geq 0} A_t \), and we say that there is percolation when \( [A]_{\mathcal{U}} = \Lambda \).

For instance, our \( \mathcal{N}_r^{a_1, \ldots, a_d} \)-model is the same as \( \mathcal{N}_r^{a_1, \ldots, a_d} \)-bootstrap percolation, where \( \mathcal{N}_r^{a_1, \ldots, a_d} \) is the family consisting of all subsets of size \( r \) of the neighbourhood \( N_{a_1, \ldots, a_d} \) in \( [1] \).

Let \( S^{d-1} \) be the unit \((d-1)\)-sphere and denote the discrete half space orthogonal to \( u \in S^{d-1} \) as \( \mathbb{H}^d_u := \{x \in \mathbb{Z}_d : \langle x, u \rangle < 0\} \). The stable set \( \mathcal{S} = \mathcal{S}(\mathcal{U}) \) is the set of all \( u \in S^{d-1} \) such that no rule \( X \in \mathcal{U} \) is contained in \( \mathbb{H}^d_u \). Let \( \mu \) denote the Lebesgue measure on \( S^{d-1} \). The following classification of families was proposed in [8] for \( d = 2 \) and extended to all dimensions in [7]: A family \( \mathcal{U} \) is

- subcritical if for every hemisphere \( \mathcal{H} \subset S^{d-1} \) we have \( \mu(\mathcal{H} \cap \mathcal{S}) > 0 \).
- critical if there exists a hemisphere \( \mathcal{H} \subset S^{d-1} \) such that \( \mu(\mathcal{H} \cap \mathcal{S}) = 0 \), and every open hemisphere in \( S^{d-1} \) has non-empty intersection with \( \mathcal{S} \);  
- supercritical otherwise.

In general, we are mostly interested in critical families. It is easy to check that the family \( \mathcal{N}_r^{a,b,c} \) is critical if and only if \( r \in \{c + 1, \ldots, a + b + c\} \).

Remark 1.4 (The stable sets approach). For each \( i = 1, 2, 3 \), let us denote by \( S^2_i := \{(u_1, u_2, u_3) \in S^2 : u_i = 0\} \) the unit circle contained in \( S^2 \) that is orthogonal to the vector \( e_i \). Then, straightforward calculations lead us to

\[
\mathcal{S}(\mathcal{N}_r^{a,b,c}) = \begin{cases} 
\{\pm e_1, \pm e_2, \pm e_3\} & \text{for } c < r \leq a + b, \\
\{\pm e_3\} \cup S^2_1 & \text{for } a + b < r \leq a + c, \\
S^2_1 \cup S^2_3 & \text{for } a + c < r \leq b + c, \\
S^2_1 \cup S^2_3 & \text{for } b + c < r \leq a + b + c.
\end{cases}
\]

The critical length \( L_c(\mathcal{N}_r^{a,b,c}, p) \) is determined in the case \( \mathcal{S}(\mathcal{N}_r^{a,b,c}) = S^2_1 \cup S^2_3 \cup S^2_3 \) and by [4] it is doubly exponential in \( p \), as \( p \to 0 \). On the other hand, we have shown in [5] that \( L_c(\mathcal{N}_r^{a,b,c}, p) \) is singly exponential in the first 3 cases in (11).

Given [5] and Theorem 1.1, it only remains to determine \( L_c(\mathcal{N}_r^{a,b,c}, p) \) in the first 3 cases in (11) for \( c < a + b \), where we believe that the magnitude is given by Proposition
Note that the cases in (10) correspond to $S(N^a_{r,b,c}) = S^1_2 \cup S^1_3$, the cases in (6) and (8) correspond to $S(N^a_{r,b,c}) = \{\pm e_3\} \cup S^1_3$, while cases in (9) correspond to $S(N^a_{r,b,c}) = \{\pm e_1, \pm e_2, \pm e_3\}$.

2. Proof of Theorem 1.1: Upper bounds

It is known that (see Proposition A.1 in [5]) for every $r \in \{c+1, \ldots, c+b\}$, as $p \to 0$,

$$\log L_c(N^a_{r,b,c}, p) = O\left(p^{-(r-c)}(\log p)^2\right).$$

(12)

In particular, the the upper bound for the second case in (9) follows. Therefore, it only remains to cover the first case $c = a + b$ when $r - c \in \{3, \ldots, b\}$, since the sub-cases $r - c \in \{1, 2\}$ are covered by (5).

Definition 2.1. A rectangular block is a set of the form $R = [l] \times [h] \times [w] \subset \mathbb{Z}^3$. A rectangular block $R$ is internally filled if $R \subset [A \cap R]$, and denote this event by $I^*(R)$.

When $l, h, w \geq c$, for simplicity we denote the event

$I(l, h, w) := I^*([l] \times [h] \times [w])$.

Throughout this section we will assume that

$c = a + b$.

As usual in bootstrap percolation, we actually prove a stronger proposition.

Proposition 2.2. Consider $N^a_{r,b,c}$-bootstrap percolation with $r \in \{c+3, \ldots, b+c\}$. There exists a constant $\Gamma > 0$ such that, if

$$L = \exp \left(\Gamma p^{-(r-c)}\right),$$

then $\mathbb{P}_p(I^*([L]^3)) \to 1$, as $p \to 0$.

2.1. The upper bounds for $r \in \{a+c+1, \ldots, b+c\}$. We start with $S(N^a_{r,b,c}) = S^1_2 \cup S^1_3$. Let us consider the cases $a + c < r \leq b + c$, then the processes induced on the faces orthogonal to $e_3$ and $e_2$, namely $N^a_{r-c, b}$ and $N^a_{r-b, c}$ respectively, are supercritical, while the induced process (orthogonal to $e_1$) $N^a_{r-a, c}$ is critical. This means, that the most likely way to grow is to start with some small initially infected rectangular block and grow simultaneously along the $e_3$ and $e_2$-directions, until we reach a volume of the order of $L_c(N^a_{r,c}, p)$, only then we can grow along the $e_1$-direction.

Lemma 2.3 (Regime critical $e_1$-process). Consider $N^a_{r,b,c}$-bootstrap percolation with $a + c < r \leq b + c$, and fix integers $l, h, w \geq c$. If $p$ is small enough, then

(i) $\mathbb{P}_p(I(l, h + 1, w) I(l, h, w)) \geq \left(1 - e^{-\Omega(p^{r-b} w)}\right)^a \left(1 - e^{-\Omega(p^{r-(a+b) w})}\right)^l$,

(ii) $\mathbb{P}_p(I(l, h, w + 1) I(l, h, w)) \geq \left(1 - e^{-\Omega(p^{r-c} h)}\right)^a \left(1 - e^{-\Omega(p^{r-((a+c) h})}\right)^l$.

Proof. See Lemma 2.6 in [5].

Next, we show that if a rectangle $R \subset [L]^3$ of a well chosen size is internally filled, then it can grow and fill $[L]^3$ with high probability (in the literature $R$ is called a critical droplet), for $L$ larger than the critical length (up to a constant factor in the exponent).
Lemma 2.4. Set \( L = \exp(\Gamma p^{-(r-c)}) \) and fix \( \varepsilon > 0 \), \( h = p^{-(r-c+\varepsilon)} \), and
\[
R := [a] \times [h] \times [p^{-a}h].
\]
Conditionally on \( I^\bullet(R) \), the probability that \([L]^3\) is internally filled goes to 1 as \( p \to 0 \).

Proof. Start with \( R \) and apply Lemma 2.3 until we reach an internally filled rectangular block \( R' = [a] \times [w]^2 \), with \( w \approx L_c(\mathcal{N}_{r-a}^c, p) \), then it becomes easy to grow in all directions.

Now, we prove the upper bound for the critical length.

Proof of Proposition 2.2 (\( r > a + c \)). Set \( L = \exp(\Gamma p^{-(r-c)}) \), where \( \Gamma \) is a constant to be chosen. Fix a small \( \varepsilon > 0 \), and consider the rectangle
\[
R := [a] \times [p^{-(r-c+\varepsilon)}] \times [p^{-(r-c+b+\varepsilon)}] \subset [L]^3.
\]
As usual, by using Lemma 2.4 considering disjoint copies of \( R \) in \([L]^3\) and taking \( \Gamma > 0 \) large, it is enough to show that there exists a constant \( C > 0 \) such that
\[
\mathbb{P}_p(I^\bullet(R)) \geq \exp(-Cp^{-(r-c)}).
\]
(13)
To do so, set \( h = p^{-2\varepsilon} \), then for every \( k = 1, \ldots, n := p^{-(r-c+c)} \) set
\[
h_k = h_k, \quad w_k = p^{-a}h_k, \quad R_k = [a] \times [h_k] \times [w_k], \quad \text{and} \quad R_k' = [a] \times [h_k] \times [w_{k+1}].
\]
Note that \( R_n = R, h_{k+1} = h_k + h \) and \( w_{k+1} = w_k + p^{-a}h \), so by Lemma 2.3
\[
\mathbb{P}_p(I^\bullet(R_n)) \geq \mathbb{P}_p(R_1 \subset A) \prod_{k=1}^{n-1} \mathbb{P}_p(I^\bullet(R_k')|I^\bullet(R_k)) \mathbb{P}_p(I^\bullet(R_{k+1})|I^\bullet(R_k'))
\]
\[
\geq p^{|R_1|} \prod_{k=1}^{n} \left[ \left( 1 - e^{-\Omega(p^{r-c}h_k)} \right)^a \left( 1 - e^{-\Omega(p^{r-c-a}h_k)} \right) \right] p^{-a}h
\]
\[
\times \prod_{k=1}^{n} \left[ \left( 1 - e^{-\Omega(p^{r-c+b}w_{k+1})} \right)^a \left( 1 - e^{-\Omega(p^{r-c-b}w_{k+1})} \right) \right] h_k
\]
\[
\geq p^{|R_1|} p^{O(n)} \prod_{k=1}^{n} \left[ 1 - e^{-\Omega(p^{r-2a-b}h_k)} \right] 2p^{-a}h.
\]
Finally, note that \( |R_1| = ap^{-a-4\varepsilon} \ll O(n) \), since \( r > a + c \), thus
\[
\mathbb{P}_p(I^\bullet(R)) \geq p^{O(n)} \exp \left( -\Omega \left( p^{-a}p^{-(r-2a-b)} \int_0^\infty [-\log(1 - e^{-x})] \, dx \right) \right)
\]
\[
\geq \exp \left( -Cp^{-(r-(a+b))} \right),
\]
for some constant \( C > 0 \), as we claimed.

Before we prove Proposition 2.2 when \( r \leq a + c \), we will have a quick discussion about supercritical two-dimensional families.

2.2. The supercritical families \( \mathcal{N}_{s,s}^c \). In this section we consider two-dimensional supercritical \( \mathcal{N}_{s,s}^c \)-bootstrap percolation and assume that \( s \geq 3 \).
Definition 2.5. An \( s \)-pattern is a union of \( t + 1 \) sets of vertices:
\[
S_0 \cup S_1 \cup \cdots \cup S_t,
\]
where \( t = t_s < s \) is the biggest integer satisfying
\[
|S_0| + |S_1| + \cdots + |S_t| < s - t,
\] (14)
and for each \( i = 0, 1, \ldots, t \), \( S_i \subset [i+1] \times \mathbb{Z} \) is a copy of \( \{1\} \times [s-i] \) (so that \( |S_i| = s-i \)) in the following restricted sense (recall that \( c_2 = (0,1) \))
\[
S_i = m(s-i)e_2 + \{i+1\} \times [s-i], \text{ for some integer } m \geq 0.
\]

Remark 2.6. The restrictions above are made to guarantee that when two \( s \)-patterns intersect in some column \( i \), this necessarily implies that they coincide in column \( i \). This fact and independence imply that the probability of existing a set \( S_i \) inside \( \{i+1\} \times [k] \) is at least
\[
1 - \exp \left( -\Omega \left( p |S_i| k \right) \right).
\] (15)

The next step is to provide a lower bound for the probability of the event \( I^*([l] \times [k]) \). This is the main lemma for \( s \geq 3 \).

Lemma 2.7 (Supercritical induced process). Fix \( m \in [s] \cup \{0\} \). Under \( N_s^{s,s} \)-bootstrap percolation, there exists \( \delta > 0 \) such that, if \( k = \Omega(p^{-m}) \) then
\[
\mathbb{P}_p(I^*([l] \times [k])) \geq 1 - \left( 1 - \delta \prod_{i=m+1}^s \left[ 1 - \exp \left( -\Omega(kp^i) \right) \right] \right)^{l/s}. \] (16)

If moreover, \( k \leq (2/3)p^{-(m+1)} \), then
\[
\mathbb{P}_p(I^*([l] \times [k])) \geq 1 - \exp \left( -\Omega \left( k^{s-m}lp^{\sum_{i=m+1}^s i} \right) \right). \] (17)

Proof. Partition the rectangle \( R = [l] \times [k] \) into \( l/s \) copies of \( R' = [s] \times [k] \), and note that \( R \) is internally filled if we can find \( s \) restricted (in the sense of Definition 2.5) sets
\[
S_0 \cup S_1 \cup \cdots \cup S_{s-1}
\]
in the rectangle \( R' \) (or any of its disjoint copies), so by Remark 2.6 it follows that
\[
\mathbb{P}_p(I^*(R)) \geq 1 - \left( 1 - \prod_{i=1}^s \left[ 1 - \exp \left( -\Omega(kp^i) \right) \right] \right)^{l/s}.
\]

Now, if \( k = \Omega(p^{-m}) \) then \( \prod_{i=1}^m \left[ 1 - \exp \left( -\Omega(kp^i) \right) \right] \geq \delta \), this proves (16).

Finally, if \( k \leq (2/3)p^{-(m+1)} \), then for every \( i \geq m+1 \) we have \( kp^i \leq 2/3 \), hence
\[
\prod_{i=m+1}^s \left[ 1 - \exp \left( -\Omega(kp^i) \right) \right] \geq \Omega \left( \prod_{i=m+1}^s kp^i \right) = \Omega \left( k^{s-m}lp^{\sum_{i=m+1}^s i} \right),
\]
and (17) follows by applying \( 1 - q \geq e^{-2q} \) for \( q \) small. \( \square \)

2.3. The upper bounds for \( r \in \{c+3, \ldots, a+c\} \). Now, let us consider our \( N_r^{a,b,c} \)-bootstrap percolation with \( c + 2 < r \leq a + c \), set
\[
s := r - c \geq 3.
\]
This is a consequence of Lemma\textsuperscript{2.7}.

**Corollary 2.8.** Consider $\mathcal{N}_{a,b,c}^{s}$-bootstrap percolation and fix $m \in [s] \cup \{0\}$. If $p$ is small enough and $p^{-m} < h \leq p^{-(m+1)}$, then
\[
\mathbb{P}_{p}(I(l, h, w + 1)|I(l, h, w)) \geq 1 - \exp\left(-\Omega\left(l^{s-m}lp^\sum_{i=m+1}^{l}i\right)\right). \tag{18}
\]

**Proof.** The induced $\mathcal{N}_{r_{a,b}}^{s}$-process along the $e_{3}$-direction (on the $[l] \times [h]$ face) is dominated by the $\mathcal{N}_{s,s}^{s}$-process, since $s = r - c$ and $s \leq a \leq b$, so Lemma\textsuperscript{2.7} applies.

This corollary tells us the cost of growing along the (easiest) $e_{3}$-direction, and we are also interested in computing the cost of growing along the $e_{1}$ and $e_{2}$ (harder) directions. Note that $r > c \geq a+b$ in Theorem\textsuperscript{1.1} so we just need to cover the cases $a+b < r \leq a+c$, where, all induced processes $\mathcal{N}_{r_{a,c}}^{s}, \mathcal{N}_{r_{b,c}}^{s}$ and $\mathcal{N}_{r_{a,b}}^{s}$ are supercritical.

**Lemma 2.9** (Regime supercritical $e_{1}$-process). Consider $\mathcal{N}_{a,b,c}^{s}$-bootstrap percolation with $a+b < r \leq a+c$, and fix integers $l, h, w \geq c$. If $p$ is small enough, then
\[
\text{(i) } \mathbb{P}_{p}(I(l, h+1, w)|I(l, h, w)) \geq \left(1 - e^{-\Omega(p^{r-b}w)}\right)^{a} \left(1 - e^{-\Omega(p^{r-a}w)}\right)^{b},
\]
\[
\text{(ii) } \mathbb{P}_{p}(I(l+1, h, w)|I(l, h, w)) \geq \left(1 - e^{-\Omega(p^{r-b}w)}\right)^{b} \left(1 - e^{-\Omega(p^{r-a}w)}\right)^{h}.
\]

**Proof.** See Lemma\textsuperscript{2.6} in [5].

Next, we show the candidate to be our critical droplet.

**Lemma 2.10.** Set $L = \exp(\Gamma p^{-s})$ and fix $\varepsilon > 0$, $l = p^{-(1+\varepsilon)}$, and
\[
R := [l] \times [l^{s-1}] \times [l^{s}].
\]
Conditionally on $I^{*}(R)$, the probability that $[L]^{3}$ is internally filled goes to $1$ as $p \to 0$.

**Proof.** Follows from Lemma\textsuperscript{2.9} and Corollary\textsuperscript{2.8} with $m \geq s-1$.

Now, we prove the upper bound for the critical length.

**Proof of Proposition\textsuperscript{2.2}** ($r \leq a+c$). Set $L = \exp(\Gamma p^{-s})$, where $\Gamma$ is a constant to be chosen. Fix a small $\varepsilon > 0$, and consider the rectangle
\[
R := [p^{-(1+\varepsilon)}] \times [p^{-(1+\varepsilon)(s-1)}] \times [p^{-(1+\varepsilon)s}] \subset [L]^{3}.
\]
As usual, by using Lemma\textsuperscript{2.10} considering disjoint copies of $R$ in $[L]^{3}$ and taking $\Gamma > 0$ large, it is enough to show that there exists a constant $C > 0$ such that
\[
\mathbb{P}_{p}(I^{*}(R)) \geq \exp(-Cp^{-s}). \tag{19}
\]
To do so, set $l = p^{-\varepsilon}$, then for every $k = 1, \ldots, n := p^{-1}$ set
\[
l_{k} = kl, \ h_{k} = l_{k}^{-1}, \ w_{k} = l_{k}, \ R_{k} = [l_{k}] \times [h_{k}] \times [w_{k}],
\]
\[
R_{k}' = [l_{k}] \times [h_{k}] \times [w_{k+1}] \text{ and } R_{k}'' = [l_{k}] \times [h_{k+1}] \times [w_{k+1}].
\]
Note that $R_{n} = R, l_{k+1} = l_{k} + l, h_{k+1} = h_{k} + O(k^{s-2}l^{s-1})$ and $w_{k+1} = w_{k} + O(k^{s-1}l^{s})$, so by Lemma\textsuperscript{2.9} and Corollary\textsuperscript{2.8} with $m \leq s-1$, we have
\[ \mathbb{P}_p(I^*(R_0^n)) \geq \mathbb{P}_p(R_1 \subset A) \prod_{k=1}^{n-1} \mathbb{P}_p(I^*(R_k) | I^*(R_k^c)) \mathbb{P}_p(I^*(R_k^c) | I^*(R_k)) \mathbb{P}_p(I^*(R_{k+1}) | I^*(R_k)) \]

\[ \geq p^{l_{R_1}} \prod_{m=0}^{s-1} \prod_{k=1}^{l-1} \left( 1 - \exp \left[ -\Omega \left( h_s^{k-m+1} p \sum_{i=m+1}^{s} \right) \right] \right)^{O(k^{s-1})} \]

\[ \times \prod_{k=1}^{n} \left( 1 - e^{-\Omega(p^{r-a} w_{k+1})} \right) \left( 1 - e^{-\Omega(p^{r-a} w_{k+1})} \right) \]

\[ \geq p^{l_{R_1}} \prod_{m=0}^{s-1} \prod_{k=1}^{l-1} \left( 1 - \exp \left[ -\Omega \left( (kl)^{(s-1)(s-m)+1} p \sum_{i=m+1}^{s} \right) \right] \right)^{O(k^{s-1})} \]

\[ \times p^{O((l)^{s-1})} \times p^{O(l)} \prod_{k=1}^{n} \left( 1 - e^{-\Omega(p^{r-a})} \right)^{O(k^{s-1})} \]

\[ \geq \prod_{m=0}^{s-1} \exp \left( -\Omega \left( p^{-s \sum_{i=m+1}^{s} i} \int_0^\infty z^{s-1} \left[ -\log(1 - e^{-z}) \right] dz \right) \right) \]

\[ \times p^{O(p^{-1}(s-1))} \exp \left( -\Omega \left( p^{-s} \int_0^\infty z^{s-1} \left[ -\log(1 - e^{-z}) \right] dz \right) \right) \]

\[ \geq \exp(-Cp^{-a}), \]

for some constant \( C > 0 \), since \( m \leq s - 1 \) implies \( (s - m - 1)(s - m - 2) \geq 0 \), so that \( \sum_{i=m+1}^{s} i \leq (s - 1)(s - m) + 1 \), and we are finished. \( \square \)

3. Lower bounds

To prove the lower bounds, we will use a technique introduced in [5] called the beams process.

3.1. Subcritical two-dimensional families. In this section, we recall an exponential decay property that holds for subcritical families \( \mathcal{U} \) with \( \mathcal{S}(\mathcal{U}) = S^1 \). Consider \( \mathcal{U} \)-bootstrap percolation in \( \mathbb{Z}^2 \) with \( \mathcal{U} \) subcritical.

Definition 3.1. We define the component (or cluster) of 0 \( \in \mathbb{Z}^2 \) as the connected component containing 0 in the graph induced by \( \langle A \rangle_{\mathcal{U}} \), and we denote it by \( K = K(\mathcal{U}, A) \). If \( 0 \notin [A]_{\mathcal{U}} \), then we set \( K = \emptyset \).

The following result was proved in [3].

Theorem 3.2 (Exponential decay for the cluster size). Consider subcritical \( \mathcal{U} \)-bootstrap percolation on \( \mathbb{Z}^2 \) with \( \mathcal{S}(\mathcal{U}) = S^1 \). If \( \varepsilon > 0 \) is small and \( C = C(\varepsilon) := \log(1/\varepsilon) \), then

\[ \mathbb{P}_\varepsilon(\vert K \vert \geq n) \leq \varepsilon^{\Omega(n)} = e^{-\Omega(Cn)}, \]

for every \( n \in \mathbb{N} \).
Proof. See Thorem 4.11 in [5].

Observe that $S(N_{m}^{a,b}) = S^1$ if and only if $m \geq a + b + 1$, in particular, our exponential decay result (Theorem 3.2) holds for these families.

3.2. The beams process. From now on we set

$$m := a + b + 1.$$ (21)

Definition 3.3. A beam is a finite subset of $\mathbb{Z}^3$ of the form $H \times [w]$, where $H \subset \mathbb{Z}^2$ is connected and $\langle H \rangle_{N_{m}^{a,b}} = H$.

In order to introduce the beams process, we need more definitions.

Definition 3.4. Given a beam $H \times [w]$ and sets $S_1, S_2 \subset H \times [w]$, we say that $H \times [w]$ is generated by $S_1 \cup S_2$ if the sets $H_1, H_2 \subset H$ given by $H_i := \{x \in R : (\{x\} \times [w]) \cap S_i \neq \emptyset\}, \quad i = 1, 2,$

are connected and there exists a path $P \subset H$ with minimal size ($P$ could be $\emptyset$), connecting $H_1$ to $H_2$, such that $H = \langle H_1 \cup H_2 \cup P \rangle_{N_{m}^{a,b}}$. Moreover, we denote

$$B(S_1 \cup S_2) := H \times [w].$$

In this definition $\langle S_1 \cup S_2 \rangle \subset H \times [w]$ for each $r \geq m$, and generated beams could depend on the choice of the path $P$. However, such minimal paths are not relevant for our purposes.

Example 3.5. In Figure 2 we show (disconnected) sets $S_1, S_2$ on the picture to the left, and the beam $B(S_1 \cup S_2)$ with respect to the subcritical family $N_{4}^{1,2}$ to the right. $S_1$ consists of the left-most isolated vertex union the copy of $\{e_3, 2e_3, 3e_3\}$ (three consecutive vertices) on the top right-most side, while $S_2$ consists of all remaining vertices.

Following the notation in Definition 3.4 note that $H_1$ and $H_2$ are connected, and we can take $P = \emptyset$ since $\langle H_1 \cup H_2 \rangle_{N_{4}^{1,2}} = H$ is already connected.

![Figure 2. A generated beam w.r.t. the subcritical family $N_{4}^{1,2}$.](image)

Next, let us consider the following coarser process.

Definition 3.6 (Coarse bootstrap percolation). Partition $[L]^2$ as $L^2/(b + 1)^2$ copies of $\boxplus := [b + 1]^2$ in the obvious way, and think of $\boxplus$ as a single vertex in the new scaled grid $[L/(b + 1)]^2$. Given a two-dimensional family $\mathcal{U}$, suppose we have some fully infected copies of $\boxplus \in [L/(b + 1)]^2$ and denote this initially infected set by $A$, then we define coarse $\mathcal{U}$-bootstrap percolation to be the result of applying $\mathcal{U}$-bootstrap percolation to the new rescaled vertices. We denote the closure of this process by $[A]_{b}$. 

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To avoid trivialities, we assume that $b + 1$ divides $L$.

**Definition 3.7.** A coarse beam is a finite set of the form $H \times [w]$, where $H \subset \mathbb{Z}^2$ is connected and $[H]_b = H$ under coarse $N_m^{a,b}$-bootstrap percolation.

**Notation 3.8.** Given sets $S_1, S_2 \subset [L]^2$, we partition $[L]^2$ as in Definition 3.6 and denote by $B_b(S_1 \cup S_2)$ the coarse beam generated by $S_1 \cup S_2$ which is constructed in the (coarse) analogous way, as we did in Definition 3.4 using coarse paths when needed. Note that every coarse beam is a beam.

**Definition 3.9.** Let $G_3 = (V, E)$ be the graph with vertex set $[L]^3$ and edge set given by $E = \{ uv : \|u - v\|_\infty \leq 2c \}$. We say that a set $S \subset [L]^3$ is strongly connected if it is connected in the graph $G_3$.

We use the beams process to show an Aizenman-Lebowitz-type lemma which says that when $[L]^3$ is internally filled, then it contains covered beams of all relevant intermediate sizes (see Lemma 3.11 below).

**Definition 3.10 (The coarse beams process).** Let $A = \{ x_1, \ldots, x_{|A|} \} \subset [L]^3$ and fix $r \geq c + 1$. Set $B_b := \{ S_1, \ldots, S_{|A|} \}$, where $S_i = \{ x_i \}$ for each $i = 1, \ldots, |A|$, and repeat until STOP:

1. If there exist distinct beams $S_1, S_2 \in B$ such that $S_1 \cup S_2$ is strongly connected, and $\langle S_1 \cup S_2 \rangle \neq S_1 \cup S_2$, then remove them from $B$, and replace by a coarse beam $B_b(S_1 \cup S_2)$.

2. If there do not exist such a family of sets in $B$, then STOP.

We call any beam $S = B_b(S_1 \cup S_2)$ added to the collection $B$ a covered beam, and denote the event that $S$ is covered by $I_b^*(S)$.

Consider $N_m^{a,b,c}$-bootstrap percolation with $r \geq c + 1$, and let $\kappa, \lambda$ be large constants depending on $b, c$ and $r$. The following is a beams version of the Aizenman-Lebowitz lemma.

**Lemma 3.11.** If $[L]^3$ is internally filled then for every $h, k = \kappa, \ldots, L$, there exists a covered (coarse) beam $H \times [w] \subset [L]^3$ satisfying $w \leq \lambda k$, $|H| \leq \lambda h$, and either $w \geq k$ or $|H| \geq h$.

*Proof.* See Lemma 5.13 in [5]. \qed

Let $B_{h,k} = B_{h,k}(L)$ be the collection of all connected sets of the form $H \times [w] \subset [L]^3$ satisfying $|H| \leq h$ and $w \leq k$. The proof of Lemma 5.2 in [5] implies the following upper bound for $B_{h,k}$.

**Lemma 3.12 (Upper bound on the number of beams).** For all $h, k \leq L$,

$$|B_{h,k}| \leq L^4(3e)^h.$$ 

3.3. The lower bounds. The proofs in this section work for all values $r > a + b$. However, by [4], they are useless when $r > b + c$. 

3.3.1. The case \( c = a + b \). In this section we prove the following.

**Proposition 3.13.** Under \( N_{p}^{a,b,c} \)-bootstrap percolation with \( r \in \{c + 2, \ldots, b + c\} \) and \( c = a + b \), there is a constant \( \gamma = \gamma(a,b) > 0 \) such that, if

\[
L < \exp(\gamma p^{-(r-c)}),
\]

then \( \mathbb{P}_{p}[\mathcal{I}^{*}([L]^{3})] \to 0, \) as \( p \to 0 \).

**Proof.** Set \( s = r - c \) and take \( L < \exp(\gamma p^{-s}) \), where \( \gamma > 0 \) is some small constant. Let us show that \( \mathbb{P}_{p}[\mathcal{I}^{*}([L]^{3})] \) goes to 0, as \( p \to 0 \). Fix \( \varepsilon > 0 \).

If \( [L]^{3} \) is internally filled, by Lemma \ref{lem:3.11} with

\[
h = k = \varepsilon/\lambda p^{s},
\]

there exists a covered (coarse) beam \( S = H \times [w] \subset [L]^{3} \) satisfying \( w, |H| \leq \varepsilon/p^{s} \), and moreover, either \( w \geq k \) or \( |H| \geq h \), hence, by union bound, \( \mathbb{P}_{p}[\mathcal{I}^{*}([L]^{3})] \) is at most

\[
\sum_{S \in B_{h,k,k}} (\mathbb{P}_{p}[\mathcal{I}^{*}(S) \cap \{w \geq k\}] + \mathbb{P}_{p}[\mathcal{I}^{*}(S) \cap \{|H| \geq h\}]).
\]

To bound the first term, we use the fact that \( H \times [w] \) is covered; this implies that every copy (inside \( S \)) of the slab \( H \times [rs] \) must contain \( s \) vertices in \( A \) within constant distance. Therefore, by considering the \( w/rs \) disjoint slabs (and applying the FKG inequality to the \( O(|H|) \) distinct subsets \( Z \) of \( s \) vertices within constant distance, \( Z \subset H \times [rs] \), if \( \varepsilon \) is small, then there exists some \( c_{1} = c_{1}(\varepsilon, r) > 0 \) such that

\[
\mathbb{P}_{p}[\mathcal{I}^{*}(H \times [w]) \cap \{w \geq k\}] \leq (1 - e^{-\Omega(p^{s}|H|)})^{k/rs} = (1 - e^{-\Omega(\varepsilon)})^{k/rs} \leq e^{-c_{1}/\lambda p^{s}}.
\]

To bound the second term we use the fact that if \( [L]^{3} \) is internally filled, then every copy of \( \mathbb{I} \times [L] \) (as in Definition \ref{def:3.6}) should contain at least

\[
t := r - (a + b)
\]

vertices \( v_{0}, v_{1}, \ldots, v_{t-1} \) with \( ||v_{i} - v_{l}|| = O(1) \) for all \( i, l < t \) such that, some \( l < t \) satisfies \( v_{0}, v_{1}, \ldots, v_{l} \in A \) and \( \forall i > l, v_{i} \) got infected by using \( r - i \) infected neighbours in \( v_{i} + N_{a,b} \times \{0\} \), where \( N_{a,b} \) is given by \( \big[1\big] \) (so, \( v_{i} \in \{v_{0}, \ldots, v_{t-1}\} \cup ([L]^{3} \setminus (\mathbb{I} \times [L])) \), otherwise, there is no way to fully infect such a copy).

Moreover, by our choice of \( t \), \( N_{r-1}^{a,b} \) is subcritical for all \( i \). Therefore, by monotonicity we can couple the process on \([L]^{2} \times [w]\) (wlog we are assuming that \( S \subset [L]^{2} \times [w] \)) having initial infected set \( A \), with coarse \( N_{r-1}^{a,b} \)-bootstrap percolation \( (m = a + b + 1) \) on \([L/(b + 1)]^{2} \times \{0\} \subset \mathbb{Z}^{2} \) and initial infected set

\[
A' := \{\mathbb{I} \in [L]^{2} : |A \cap (\mathbb{I} \times [w])|_{O(1)} \geq t\},
\]

where the subindex \( O(1) \) in the cardinality symbol means that the vertices participating in the intersection are within constant distance.

Now, by applying Markov’s inequality and the fact that \( c = a + b \) implies \( t = s \),

\[
\mathbb{P}_{p}(|A \cap (\mathbb{I} \times [w])|_{O(1)} \geq t) = O(wp^{t}) \leq O(\varepsilon).
\]

In particular, under (coarse) \( N_{r-1}^{a,b} \)-bootstrap percolation with initial infected set \( \varepsilon \)-random, there should exist a connected component of size at least \( |H| \geq h \) inside \([L]^{2}\). On the
other hand, there are at most $L^2$ possible ways to locate the origin in $H$, so if $\mathcal{K}$ denotes the (coarse) cluster of $0$, Theorem \ref{thm:main} implies

$$
\mathbb{P}_p[I^*(S) \cap \{|H| \geq h\}] \leq \sum_{\Xi \subset [L]^2} \mathbb{P}_x(\{|\mathcal{K}| \geq h\} \cap \{\Xi = 0\}) \leq L^2 \mathbb{P}_x(\{|\mathcal{K}| \geq h\}) \\
\leq e^{2\gamma/p^*} e^{-\Omega(C\gamma/p^*)} = e^{-\Omega(p^{-r})},
$$

where $C = -\log \varepsilon$ and, we choose small $\varepsilon > 0$ such that $C\varepsilon > 0$ and $\gamma \ll C\varepsilon$ at first. By Lemma \ref{lem:main} we conclude that

$$
\mathbb{P}_p[I^*([L]^3)] \leq \sum_{S \in B_{p^*,p^*}} \left\{ e^{-c_1/p^*} + e^{-\Omega(p^{-r})} \right\} \leq L^4(3e)^{p^{-r}} e^{-\Omega(p^{-r})} \to 0,
$$

for $\gamma > 0$ small enough, since $L < e^{\gamma p^{-r}}$.

\[\square\]

3.3.2. The case $c > a + b$. In this section we prove the lower bound corresponding to our last case $c > a + b$. We will show the following.

**Proposition 3.14.** Under $\mathcal{N}_{a,b,c}^{r,c}$-bootstrap percolation with $r \in \{c + 2, \ldots, b + c\}$ and $c > a + b$, there is a constant $\gamma = \gamma(c) > 0$ such that, if

$$
L < \exp(\gamma p^{-r}(\log p)^2),
$$

then $\mathbb{P}_p[I^*([L]^3)] \to 0$, as $p \to 0$.

**Proof.** Set $s = r - c$ and take $L < \exp(\gamma p^{-s}(\log p)^2)$, where $\gamma > 0$ is some small constant. Let us show that $\mathbb{P}_p[I^*([L]^3)]$ goes to $0$, as $p \to 0$. Fix $\delta > 0$ and set

$$
h = (\delta p^{-s} \log \frac{1}{p})/\lambda, \quad k = p^{-s-1/2}/\lambda,
$$

If $[L]^3$ is internally filled, by Lemma \ref{lem:main} with there exists a covered (coarse) beam $S = H \times [w] \subset [L]^3$ satisfying $w \leq \lambda k$, $|H| \leq \lambda h$, and moreover, either $w \geq k$ or $|H| \geq h$, hence,

$$
\mathbb{P}_p[I^*([L]^3)] \leq \sum_{S \in B_{\lambda h, \lambda k}} \left\{ \mathbb{P}_p[I^*(S) \cap \{w \geq k\}] + \mathbb{P}_p[I^*(S) \cap \{|H| \geq h\}] \right\}.
$$

To bound the first term, we use the fact that $H \times [w]$ is covered; this implies that every copy (inside $S$) of the slab $H \times [rs]$ must contain $s$ vertices in $A$ within constant distance. Therefore, if $\delta$ is small enough, then

$$
\mathbb{P}_p[I^*(H \times [w]) \cap \{w \geq k\}] \leq (1 - e^{-\Omega(p^{|w|H})})^{k/rs} \leq (1 - p^{O(\delta)})^{k/rs} \leq e^{-\Omega(p^{-s}(\log p)^2)}.
$$

To bound the second term we use the fact that if $[L]^3$ is internally filled, then every copy of $\Xi \times [L]$ (as in Definition \ref{def:main}) should contain at least

$$
t := r - (a + b)
$$

vertices $v_0, v_1, \ldots, v_{t-1}$ with $\|v_i - v_l\| = O(1)$ for all $i, l < t$ such that, some $l < t$ satisfies $v_0, v_1, \ldots, v_l \in A$ and $\forall i > l,$ $v_i$ get infected by using $r - i$ infected neighbours in $v_i + a,b \times \{0\}$. Moreover, $\mathcal{N}_{m_i}$ is subcritical for all $i$. Therefore, by monotonicity we can couple the process on $[L]^2 \times [w]$ having initial infected set $A$, with coarse $\mathcal{N}_{m_i}^{a,b}$-bootstrap
percolation on $[L/(b+1)]^2 \times \{0\} \subset \mathbb{Z}^2$ and initial infected set

$$A' := \{ \bigoplus \in [L]^2 : |A \cap (\bigoplus \times [w])|_{O(1)} \geq t \},$$

where the subindex $O(1)$ in the cardinality symbol means that the vertices participating in the intersection are within constant distance.

Now, by applying Markov’s inequality and the fact that $c > a + b$ implies $t \geq s + 1$,

$$\mathbb{P}_p(|A \cap (\bigoplus \times [w])|_{O(1)} \geq t) = O(w^t) = O(p^{-s - 1/2 + t}) \leq O(p^{1/2}).$$

In particular, under (coarse) $\mathcal{N}_m^{a,b}$-bootstrap percolation with initial infected set $p^{1/2}$-random, there should exist a connected component of size at least $|H| \geq h$ inside $[L]^2$.

So, if $\mathcal{K}$ denotes the (coarse) cluster of 0, Theorem 3.2 implies for $\gamma \ll \delta$,

$$\mathbb{P}_p[I^*_b(S) \cap \{|H| \geq h\}] \leq \mathcal{L} \mathbb{P}_{p^{1/2}}(|\mathcal{K}| \geq h) \leq \exp(-\gamma p^{-2s}(\log p)^2) e^{-\Omega(Ch)} = \exp(-\Omega(p^{-s}(\log p)^2),$$

where $C = -\log(p^{1/2}) = \Theta(\log \frac{1}{p})$. By Lemma 3.12 we conclude that

$$\mathbb{P}_p[I^*(\mathcal{L}^3)] \leq \sum_{S \in B_{p^{-s}(\log \frac{1}{p})}^s} \exp(-\Omega(p^{-s}(\log \frac{1}{p})^2) \leq L^4(3e)(p^{s - 1/2}) \exp(-\Omega(p^{-s}(\log \frac{1}{p})^2) \rightarrow 0,$

for $\gamma > 0$ small enough, and we are done. \qed

4. PROOF OF PROPOSITION 1.2

Now, we proceed to prove all upper bounds of Proposition 1.2 in increasing order of difficulty. Most of the proofs are analogous versions of the cases $c < a + b$ in (5), thus, we will sketch some of them and only point out what are the new ideas.

Continuing with Definition 2.5, given $s \geq 3$ and $t = t_s$ given by (14), we also define

$$\alpha_s := \frac{s + (s - 1) + \cdots + (s - t)}{t + 2}. \quad (22)$$

Note that inequality (14) holds for $t = 1$, since $s + (s - 1) \leq 3(s - 1)$, thus, $t_s$ is well defined for all $s \geq 3$; moreover, it is easy to see that

$$t_s := \max\{t \in [s - 1] : \alpha_s < s - t\} = \left[\frac{\sqrt{9 + 8s} - 5}{2}\right].$$

Finally, we replace to obtain $\alpha_s = \frac{(t_s + 1)s - (t_s + 1)t_s/2}{t_s + 2}$, which is the same as (7).

Remark 4.1. By the maximality of $t = t_s$, it follows that $s - t - 1 \leq \alpha_s < s - t$. Thus, $\alpha_s$ is integer if and only if $\alpha_s = s - t - 1$, which occurs if and only if

$$s = \frac{(t + 1)(t + 4)}{2}. \quad (23)$$

4.1. Cases in Proposition 1.2 (i). Since the cases $c \geq a + b$ are covered by Theorem 1.1 from now on we assume $c < a + b$, and set

$$s := r - c \in \{3, \ldots, a\}. \quad (24)$$

Case $c = a + b - s + m$ and $m \in [s - 1]$. In this section we consider the families

$$\mathcal{N}_{a+b+m}^{a+b-s+m}.$$
corresponding to the case \( r = c + s = a + b + m \) with \( m \in [s - 1] \). We are only considering the cases \( r \leq a + c \), thus, we assume \( a \geq s \). Set
\[
M := \max\{\alpha_s, m\}.
\]
We will show the following.

**Proposition 4.2.** Consider \( \mathcal{N}^{a,b,a+b-s+m} \)-bootstrap percolation. There exists a constant \( \Gamma = \Gamma(b) > 0 \) such that, if
\[
L = \exp \left( \Gamma p^{-M} (\log \frac{1}{p})^2 \right),
\]
then \( \mathbb{P}_p (I^*([L]^3)) \to 1, \) as \( p \to 0 \).

By Lemma 2.9 we have
\[
\mathbb{P}_p (I^*([h + 1]^2 \times [w]) | I^*([h]^2 \times [w])) \geq \left( 1 - e^{-\Omega(p^{h+m}w)} \right) (1 - e^{-\Omega(p^m w)})^{2h}
\]
(26)

Now, we show our candidate for critical droplet.

**Lemma 4.3.** Set \( L = \exp \left( \Gamma p^{-M} (\log \frac{1}{p})^2 \right), \) \( h = C_s p^{-\alpha_s} (\log \frac{1}{p})^{1/(t+2)} \) for some large constant \( C_s \), and
\[
R_1 := [h]^2 \times \left[ C_s p^{-m} \log \frac{1}{p} \right].
\]
Conditionally on \( I^*(R_1) \), the probability that \([L]^3 \) is internally filled goes to 1 as \( p \to 0 \).

**Proof.** Consider the rectangles \( R_2 \subset R_3 \subset R_4 \subset R_5 := [L]^3 \) containing \( R_1 \), defined by \( R_2 := [h]^2 \times [p^{-(t+1)}] \), \( R_3 := [p^{-2s}]^2 \times [p^{-(t+1)}] \), with \( \tau := b + m \), and \( R_4 := [p^{-2s}]^2 \times [L] \).

The rest of the proof is as always; we apply Corollary 4.5 to deduce that
\[
\mathbb{P}_p (I^*(R_2) | I^*(R_1)) \geq \left( 1 - e^{-\Omega(p^{\alpha_s(t+2)h^{t+2}}} \right)^{p^{-\tau}} = (1 - p^{C_s})^{p^{-\tau}} \to 1,
\]
if \( C_s \) is large. The rest of the proof is straightforward by using (26). \(\square\)

Now, we prove the upper bound for the critical length. Note that \( M < 2\alpha_s \).

**Proof of Proposition 4.2.** Set \( L = \exp \left( \Gamma p^{-M} (\log \frac{1}{p})^2 \right), \) where \( \Gamma \) is a constant to be chosen. Set \( w := m(2\alpha_s - M)p^{-m} \log \frac{1}{p} \) and consider the rectangle
\[
R := \left[ C_s p^{-\alpha_s} (\log \frac{1}{p})^{1/(t+2)} \right]^2 \times [w] \subset [L]^3.
\]
We need to show that there exists \( C' > 0 \) satisfying
\[
\mathbb{P}_p (I^*(R)) \geq \exp \left( -C' p^{-M} (\log \frac{1}{p})^2 \right).
\]
(27)

In fact, start with \( R_c := [c]^2 \times [w] \subset A \), and then grow from \( R_k = [k]^2 \times [w] \) to \( R_{k+1} \), for
\[
k = c, \ldots, K := C_s p^{-\alpha_s} (\log \frac{1}{p})^{1/(t+2)}
\]
to obtain
\[ \mathbb{P}_p(I^*(R)) \geq \mathbb{P}_p(R \subset A) \prod_{k=c}^{K} \mathbb{P}_p(I^*(R_{k+1})|I^*(R_k)) \]
\[ \geq p^{|R_c|} \prod_{k=c}^{K} \left( 1 - e^{-\Omega(p^{k+m})} \right)^c (1 - e^{-\Omega(p^m)})^{2h} \geq p^{|R_c|} |C|^K (1 - p^{2\alpha_s-M})^{C_1K^2} \]
\[ \geq e^{-\Omega(w+K) \log \frac{1}{p}} \exp \left( -\Omega(p^{2\alpha_s-M} p^{-2\alpha_s} (\log \frac{1}{p})^{2/(t+2)} \right) \geq \exp \left( -C'' p^{-M} (\log \frac{1}{p})^2 \right), \]
with all \( C \)'s being positive constants depending on \( s \) and \( c \). \( \square \)

4.2. **Cases in Proposition 1.2 (ii).** These cases correspond to the regime
\[ r \leq a + b. \]
This time, for a given rectangle \( R \), growing along the \( e_1 \) and \( e_2 \) (harder) directions is easier, because it is enough to find a copy of some ‘pattern’ with constant size in all faces of \( R \), no matter its size.

**Lemma 4.4 (Regime \( r \leq a + b \)).** Fix \( h, w \geq c \). If \( p \) is small enough, then
\[ \mathbb{P}_p(I^*([h+1]^2 \times [w])|I^*([h]^2 \times [w])) \geq \left( 1 - e^{-\Omega(s(h))} \right)^2. \] (28)

**Proof.** See Lemma 2.4 in [5] (the induced 2-dimensional processes \( N^{b,c}_{r-a} \) and \( N^{s,c}_{r-b} \) are supercritical). \( \square \)

We also know the cost of growing along the (easiest) \( e_3 \)-direction.

**Corollary 4.5 (Supercritical process).** Consider \( N^{s,s}_{r-a} \)-bootstrap percolation.
\( \text{(a)} \) Suppose that \( p^{-(s-t-1)} \leq k < \varepsilon p^{-(s-t)} \) (in particular, if \( p^{-\alpha_s} \leq k < p^{-\alpha_s-\delta} \) and \( 0 < \delta < s-t-\alpha_s \)), then
\[ \mathbb{P}_p(I^*([k]^2)) \geq 1 - \exp \left( -\Omega \left( p^{\alpha_s(t+2)} k^{t+2} \right) \right). \] (29)
\( \text{(b)} \) If \( k \geq p^{-s} \), then
\[ \mathbb{P}_p(I^*([k]^2)) \geq 1 - \exp \left( -\Omega \left( k \right) \right). \] (30)

**Proof.** We use Lemma 2.7 with \( l = k \). For (a) take \( m = s-t-1 \) (recall that \( s-t-1 \leq \alpha_s < s-t \)). For (b) consider \( m = s \). \( \square \)

Now, we prove the three cases in (ii).

Case \( c \in \{ b+1, \ldots, a+b-s \} \). In this section we consider the families
\[ N^{\alpha,b,c}_{c+a}, \]
with \( c \in I_a := \{ b+1, \ldots, a+b-s \} \) (here \( a > s \), otherwise this case does not exist). We have to prove the following.

**Proposition 4.6.** Fix \( c \in I_a \) and consider \( N^{\alpha,b,c}_{c+a} \)-bootstrap percolation. There exists a constant \( \Gamma = \Gamma(c) > 0 \) such that, if
\[ L = \exp \left( \Gamma p^{-\alpha_s} (\log \frac{1}{p})^{(t+3)/(t+2)} \right), \] (31)
then \( \mathbb{P}_p(I^*([L]^2)) \to 1 \), as \( p \to 0. \)
The following result gives us the size of a critical droplet $R_1$.

**Lemma 4.7.** Set $L = \exp \left( \Gamma p^{-\alpha_s}(\log \frac{1}{p})^{(t+3)/(t+2)} \right)$, $h = C_s p^{-\alpha_s}(\log \frac{1}{p})^{1/(t+2)}$ for some large constant $C_s$, and

$$R_1 := [h]^2 \times [c].$$

Conditionally on $I^*(R_1)$, the probability that $[L]^3$ is internally filled goes to 1, as $p \to 0$.

**Proof.** Consider the rectangles $R_2 \subset R_3 \subset R_4 \subset R_5 := [L]^3$ containing $R_1$, defined by

$$R_2 := [h]^2 \times [p^{-\alpha_s}+\alpha_s, \delta], \quad R_3 := [p^{-\alpha_s} \times [p^{-\alpha_s}+\alpha_s, \delta], \quad R_4 := [p^{-\alpha_s} \times [L].$$

As usual,

$$\mathbb{P}_p(I^*(L^3)|I^*(R_1)) \geq \prod_{k=1}^4 \mathbb{P}_p(I^*(R_{k+1})|I^*(R_k)).$$

By applying (28) and Corollary 4.5 (a) we obtain

$$\mathbb{P}_p(I^*(R_2)|I^*(R_1)) \geq \left( 1 - e^{-\Omega(p^{-\alpha_s}(h+2))} \right)^{-\alpha_s} \geq \left( 1 - e^{-\Omega(C_s \log \frac{1}{p})} \right)^{-\alpha_s} \rightarrow 1,$$

if $C_s$ is large, and

$$\mathbb{P}_p(I^*(R_3)|I^*(R_2)) \geq \left( 1 - e^{-\Omega(p^{-\alpha_s}(p^{-\alpha_s}+\alpha_s, \delta))} \right)^{2p^{-\alpha_s}} \geq \left( 1 - e^{-\Omega(p^{-\alpha_s})} \right)^{2p^{-\alpha_s}} \rightarrow 1,$$

and now we can use item (b) to get

$$\mathbb{P}_p(I^*(R_4)|I^*(R_3)) \geq \left( 1 - e^{-\Omega(p^{-\alpha_s})} \right)^L \rightarrow 1,$$

since $\alpha_s < s$. Finally, by (28) it is clear that $\mathbb{P}_p(I^*(R_5)|I^*(R_4)) \to 1$. \hfill \Box

The proof of Proposition 4.6 is straightforward.

**Proof of Proposition 4.6.** Set $L = \exp \left( \Gamma p^{-\alpha_s}(\log \frac{1}{p})^{(t+3)/(t+2)} \right)$, where $\Gamma$ is a constant to be chosen. Consider the rectangle

$$R := \left[ C_s p^{-\alpha_s}(\log \frac{1}{p})^{1/(t+2)} \right]^2 \times [c] \subset [L]^3.$$

As usual, it is enough to show that there exists a constant $C' > 0$ such that

$$\mathbb{P}_p(I^*(R)) \geq \exp \left( -C' p^{-\alpha_s}(\log \frac{1}{p})^{(t+3)/(t+2)} \right),$$

We fill $R$ in the same way as before: start with $[c]^3 \subset A$, and then grow from $R_k = [k]^2 \times [c]$ to $R_{k+1}$, for $k = c, \ldots, m := C_s p^{-\alpha_s}(\log \frac{1}{p})^{1/(t+2)}$

$$\mathbb{P}_p(I^*(R)) \geq p^3 \prod_{k=c}^m \left( 1 - e^{-\Omega(p^{-\alpha_s}(\log \frac{1}{p})^k)} \right)^2 \geq p^{c^2+m} \geq \exp \left( -C' p^{-\alpha_s}(\log \frac{1}{p})^{(t+3)/(t+2)} \right),$$

for $C' > c^3$, and we are done. \hfill \Box
Finally, we deal with the cases that involve integration of the functions
\[
f_d(x) = -\log (1 - e^{-x^d}).
\]
We first consider the isotropic case.

**Case** \(c = b = a\) and \(r \in \{c + 3, \ldots, 2c\}\). In this section, we consider \(N_{c,c,c}\)-bootstrap percolation with \(3 \leq s \leq c\). We will prove the following.

**Proposition 4.8.** Consider \(N_{c,c,c}\)-bootstrap percolation. There exists a constant \(\Gamma > 0\) such that, if
\[
P \geq \exp (\Gamma p^{-\alpha_s}),
\]
then \(P_p(I^*([L]^{3})) \to 1\), as \(p \to 0\).

The induced process in all three directions is coupled by \(N_s,s\)-bootstrap percolation, and recall that (Corollary 4.5) for \(k \geq p^{-s}\)
\[
P_p(I^*([k]^2)) \geq 1 - \exp (-\Omega (k)).
\]
This time we need to use the full strength of Lemma 2.7, which corresponds to all sizes \(k \geq p^{-(s-t)}\).

**Corollary 4.9.** Consider \(N_s,s\)-bootstrap percolation and \(m \in \{s-t, \ldots, s-1\}\). Suppose that \(\varepsilon p^{-m} \leq k < \varepsilon p^{-(m+1)}\), then
\[
P_p(I^*([k]^2)) \geq 1 - \exp (-\Omega (p^{-\delta(s,m)/2})),
\]
where \(\delta(s,m) := -m^2 + (2s+3)m - s(s+1) > 0\).

**Proof.** By setting \(l = k\) and applying Lemma 2.7 we get
\[
P_p(I^*([k]^2)) \geq 1 - \exp (-\Omega (p^{-s(m+1)/2})).
\]
The inequality \(\delta(s,m) > 0\) is equivalent to
\[
2s + 3 - \sqrt{9 + 8s} < 2m,
\]
which holds whenever \(m \geq s-t\).

Now, we can set the size of a rectangle that will grow w.h.p.

**Lemma 4.10.** Set \(L = \exp(\Gamma p^{-\alpha_s})\) and \(R_1 := \varepsilon p^{-(s-t)}\). Conditionally on \(I^*(R_1)\), the probability that \([L]^3\) is internally filled goes to 1 as \(p \to 0\).

**Proof.** By (34) and Corollary 4.9 we have
\[
P_p(I^*([L]^3) | I^*(R_1)) \geq \left(1 - e^{-\Omega (p^{-s})}\right) \prod_{m=s-t}^{m=s-l} \prod_{h=\exp^{-m}}^{m=\exp^{-m}} P_p(I^*([h+1]^3) | I^*([R]^3)) \notag
\]
\[
\geq \exp \left(-2Le^{-\Omega (p^{-s})}\right) \prod_{m=s-t}^{m=s-l} \left(1 - e^{-\Omega (p^{-s})}\right) p^{-\exp^{-m}},
\]
where \(\delta = \delta(s,m)/2 > 0\), and every factor goes to 1, as \(p \to 0\).

Now, we are ready to show the upper bound.
Proof of Proposition 4.8. Set \( L = \exp(\Gamma p^{-\alpha_s}) \), where \( \Gamma \) is a constant to be chosen. Consider the rectangle
\[ R := [e^{p^{-s-t}}]^3 \subset [L]^3. \]
As before, we only need to show that there is a constant \( C' > 0 \), such that
\[ \mathbb{P}_p(I^*(R)) \geq \exp(-C' p^{-\alpha_s}). \]  
(36)
Recall that
\[ s - t - 1 \leq \alpha_s < s - t. \]
It is enough to consider the (hardest) case \( s - t - 1 = \alpha_s \), since we can use the same idea to deduce the case \( s - t - 1 < \alpha_s \) (indeed, fewer steps are needed).

Remarkably, when \( \alpha_s = s - t - 1 \) we can apply Lemma 2.7 one more time for \( m = s - t - 2 \) to get the lower bound needed to obtain right exponents. More precisely, since \( \alpha_s \) is integer, by \([23]\) we know that
\[ s = \frac{(t + 1)(t + 4)}{2}, \]
thus, under \( \mathcal{N}^{s,s}_\alpha \)-bootstrap percolation, if \( e^{p^{-(s-t-2)}} = K_1 \leq k < K_2 = e^{p^{-(s-t-1)}}, \)
\[ \mathbb{P}_p(I^*([k]^2)) \geq 1 - \exp \left( -\Omega \left( k^{s-m+1} p^{\sum_{i=s-t-1}^1} \right) \right) = 1 - \exp \left( -\Omega \left( k^{t+3} p^{\frac{1}{2}(t+1)(t+2)(t+3)} \right) \right) \]
\[ \geq 1 - \exp \left( -\Omega \left( k^{t+2} p^{\alpha_s(t+2)} \right) \right). \]

While, by Corollary 4.5 we already computed the following matching ratio in the exponents: if \( e^{p^{-(s-t-1)}} = K_2 \leq k < K_3 = e^{p^{-(s-t)}}, \)
\[ \mathbb{P}_p(I^*([k]^2)) \geq 1 - \exp \left( -\Omega \left( k^{t+2} p^{\alpha_s(t+2)} \right) \right). \]

On the other hand, by the discussion we had in the heuristics, we know that the is a lower bound which holds for all values of \( k \), namely,
\[ \mathbb{P}_p(I^*([k]^2)) \geq 1 - \exp \left( -\Omega(p^{R_1}|k|^2) \right) \geq p^{C_s}, \]
for some large constant \( C_s > 0 \). This implies for \( R_1 := [p^{-(s-t-2)}]^3 \) that
\[ \mathbb{P}_p(I^*(R_1)) \geq \mathbb{P}_p([c]^3 \subset A) \prod_{h=c}^{K_1} p^{3C_s} \geq p^{C_s+3C_s p^{-(s-t-2)}.} \]

Finally, by setting \( R_2 := [p^{-(s-t-1)}]^3 \) we obtain
\[ \mathbb{P}_p(I^*(R)) \geq \mathbb{P}_p(I^*(R_1)) \mathbb{P}_p(I^*(R_2)|I^*(R_1)) \mathbb{P}_p(I^*(R)|I^*(R_2)) \]
\[ \geq \mathbb{P}_p(I^*(R_1)) \prod_{k=K_1}^{K_2} \left( 1 - e^{-\Omega(k^{t+3} p^{\alpha_s(t+3)})} \right)^3 \prod_{k=K_2}^{K_3} \left( 1 - e^{-\Omega(k^{t+2} p^{\alpha_s(t+2)})} \right)^3 \]
\[ \geq \mathbb{P}_p(I^*(R_1)) \exp \left( 3 \int_0^\infty \log(1 - e^{-\Omega(z^{t+3} p^{\alpha_s(t+3)})}) \, dz \right) \]
\[ \times \exp \left( 3 \int_0^\infty \log(1 - e^{-\Omega(z^{t+2} p^{\alpha_s(t+2)})}) \, dz \right) \]
\[ e^{-p^{−αs+1/2}} \exp \left(\tilde{C} p^{−αs} \left(\int_0^\infty \log(1 - e^{-y}) \, dy + \int_0^\infty \log(1 - e^{-y+2}) \, dy\right)\right) \]
\[ \geq \exp(-C' p^{−αs}), \]
for some constants $\tilde{C}, C' > 0$. \hfill \Box

**Case $c = b > a$.** In this section we cover the last case $c = b > a$. Consider the families $\mathcal{N}_{c+s}^{a,c,c}$. The corresponding upper bound goes as follows.

**Proposition 4.11.** Consider $\mathcal{N}_{c+s}^{a,c,c}$-bootstrap percolation with $c > a$. There exists a constant $\Gamma > 0$ such that, if

\[ L = \exp \left(\Gamma p^{−αs}\left(\log \frac{1}{p}\right)^{(t+1)/(t+2)}\right), \tag{37} \]

then $\mathbb{P}_p(I^*(\lfloor L^3 \rfloor)) \to 1$, as $p \to 0$.

The proof is a combination of all ideas we have already seen, thus, we will only sketch it.

**Sketch of the proof.** By following the proof of Corollary 4.9, we can see that under $\mathcal{N}_{s,s}$-bootstrap percolation, if $m \geq s - t$, then there exists a constant $\gamma > 0$ such that, for all $\varepsilon p^{-m} \leq w < \varepsilon p^{-(m+1)}$ and $w^{1-\gamma} \leq l \leq w$,

\[ \mathbb{P}_p(I^*(\lfloor l \times [w] \rfloor)) \geq 1 - \exp \left(-\Omega\left(p^{-\delta/2}\right)\right), \tag{38} \]

where $\delta = \delta(s, m, \gamma) := -m^2 + (2s + 3 - \gamma)m - s(s + 1) > 0$ (apply continuity as $\gamma \to 0$). This implies that the rectangle

\[ R = \left[p^{−αs}\left(\log \frac{1}{p}\right)^{-1/(t+2)}\right] \times \left[\varepsilon p^{-\delta}\right]^2 \]

is internally filled with probability at least

\[ \mathbb{P}_p(I^*(R)) \geq \exp(-C p^{−αs}\left(\log \frac{1}{p}\right)^{(t+1)/(t+2)}), \tag{39} \]

and $R$ can grow with high probability. \hfill \Box

5. **Future work**

In dimension $d = 3$, a problem which remains open is the determination of the threshold for $c + 3 \leq r \leq b + c$ and $c < a + b$, and we think that the order of the critical length should be given by Proposition 1.2 and Conjecture 1.3. We believe that the techniques used in 5 and here can be adapted to cover these cases (though significant technical obstacles remain).

For dimensions $d \geq 4$, it is an open problem to determine $L_c(\mathcal{N}_{r}^{a_1,\ldots,a_d}, p)$ for all values $a_1 \leq \cdots \leq a_d$ and all $r \in \{a_d + 1, \ldots, a_d + a_d - 1\}$. 
Acknowledgements

The author would like to thank Janko Gravner and Rob Morris for their stimulating conversations on this project, and their many invaluable suggestions.

References

[1] M. Aizenman and J.L. Lebowitz. Metastability effects in bootstrap percolation. J. Phys. A., 21(19):3801–3813, 1988.
[2] J. Balogh, B. Bollobás, H. Duminil-Copin, and R. Morris. The sharp threshold for bootstrap percolation in all dimensions. Trans. Amer. Math. Soc., 364(5):2667–2701, 2012.
[3] J. Balogh, B. Bollobás, and R. Morris. Bootstrap percolation in three dimensions. Ann. Prob., 37(4):1329–1380, 2009.
[4] D. Blanquicett. The $d$-dimensional bootstrap percolation models with threshold at least double exponential. Submitted, arXiv:2201.09029.
[5] D. Blanquicett. Anisotropic bootstrap percolation in three dimensions. Ann. Prob., 48(5):2591–2614, 2020.
[6] D. Blanquicett. Fixation for Two-Dimensional $U$-Ising and $U$-Voter Dynamics. J. Stat. Phys., 182(21), 2021.
[7] B. Bollobás, H. Duminil-Copin, R. Morris, and P. Smith. Universality of two-dimensional critical cellular automata. Proc. Lond. Math. Soc., to appear, arXiv:1406.6680.
[8] B. Bollobás, P.J. Smith, and A.J. Uzzell. Monotone cellular automata in a random environment. Combin. Probab. Computing, 24(4):687–722, 2015.
[9] R. Cerf and E.N.M. Cirillo. Finite size scaling in three-dimensional bootstrap percolation. Ann. Prob., 27(4):1837–1850, 1999.
[10] R. Cerf and F. Manzo. The threshold regime of finite volume bootstrap percolation. Stochastic Proc. Appl., 101(1):69–82, 2002.
[11] J. Chalupa, P.L. Leath, and G.R. Reich. Bootstrap percolation on a Bethe lattice. J. Phys. C., 12(1):L31–L35, 1979.
[12] H. Duminil-Copin and A.C.D. van Enter. Sharp metastability threshold for an anisotropic bootstrap percolation model. Ann. Prob., 41(3A):1218–1242, 2013.
[13] H. Duminil-Copin, A.C.D. van Enter, and W.J.T. Hulshof. Higher order corrections for anisotropic bootstrap percolation. Prob. Theory Rel. Fields, 172:191–243, 2018.
[14] A.C.D. van Enter and A. Fey. Metastability thresholds for anisotropic bootstrap percolation in three dimensions. J. Stat. Phys., 147(1):97–112, 2012.
[15] A.C.D. van Enter and W.J.T. Hulshof. Finite-size effects for anisotropic bootstrap percolation: logarithmic corrections. J. Stat. Phys., 128(6):1383–1389, 2007.
[16] L.R. Fontes, R.H. Schonmann, and V. Sidoravicius. Stretched Exponential Fixation in Stochastic Ising Models at Zero Temperature. Commun. Math. Phys., 228(3):495–518, 2002.
[17] A. Holroyd. Sharp metastability threshold for two-dimensional bootstrap percolation. Prob. Theory Rel. Fields, 125(2):195–224, 2003.
[18] F. Martinelli, R. Morris, and C. Toninelli. Universality results for kinetically constrained spin models in two dimensions. Commun. Math. Phys., Oct 2018.
[19] R. Morris. Zero-temperature Glauber dynamics on $\mathbb{Z}^d$. Prob. Theory Rel. Fields, 149(3-4):417–434, 2011.
[20] R. Morris. Bootstrap percolation, and other automata. European J. Combin., 66:250–263, 2017.

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