Abstract

This paper investigates cylindrically symmetric distribution of anisotropic fluid under the expansion-free condition, which requires the existence of vacuum cavity within the fluid distribution. We have discussed two family of solutions which further provide two exact models in each family. Some of these solutions satisfy Darmois junction condition while some show the presence of thin shell on both boundary surfaces. We also formulate a relation between the Weyl tensor and energy density.

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1 Introduction

In general relativity, the expansion scalar measures the change of small volumes of the fluid with respect to time. An explosion in the center leads to an overall expansion of the fluid thus making a cavity surrounding the center. Skripkin [1] was the pioneer who described the fascinating phenomenon of cavity formation by assuming non-dissipative isotropic fluid. This fluid was initially at rest but on explosion at the center, fluid ejects outwards thus...
forming the Minkowskian cavity inside. The expansion scalar vanishes for all such solutions.

Herrera et al. [2] discussed the physical meaning of expansion-free motion with the help of two different definitions for the radial velocity of a fluid element. It was found [3] that the Skripkin model does not satisfy Darboux junction conditions. In most cases, expansion-free models require pressure anisotropy and energy density inhomogeneity [4-5]. Some people [6]-[8] suggested that anisotropy plays a vital role for understanding the stability of highly compact bodies. Herrera et al. [9] described the instability of the cavity with the expansion-free fluid distribution. Sharif and his collaborators [10] have also discussed the dynamical instability of expansion-free gravitational collapse. The expansion-free condition also helps to explain voids.

The cavity evolution problem requires junction conditions which join two distinct solutions into one. Here we have two hypersurfaces delimiting the fluid, external and internal. The former delimits fluid distribution from cylindrically metric and the later delimits the Minkowskian cavity. The first set of junction conditions in general relativity was introduced by Darmois [11]. In a recent paper, Di Prisco et al. [12] found some exact analytic models of spherically symmetric spacetime under expansion-free condition some satisfying the junction conditions.

In this paper, we take the expansion-free cylindrically symmetric distribution of anisotropic fluid with a vacuum cavity. This paper is organized as follows. In the next section, we formulate the field equations and review some basic properties of anisotropic fluid. Section 3 provides the relationship between the Weyl tensor and energy density. In section 4, some analytic solutions are obtained under the expansion-free condition. We summarize the results in the last section.

2 Fluid Distribution and the Field Equations

Consider a cylindrically symmetric distribution of collapsing fluid bounded by a cylindrical surface $\Sigma^{(e)}$. The interior region is given by [13]

$$ds^2 = -A^2(t, r)dt^2 + B^2(t, r)dr^2 + C^2(t, r)d\phi^2 + dz^2,$$

where $-\infty \leq t \leq \infty$, $0 \leq r \leq \infty$, $0 \leq \phi \leq 2\pi$, $-\infty < z < \infty$ and the comoving coordinates are taken inside the hypersurface $\Sigma^{(e)}$. The energy-momentum
tensor $T^-_{\alpha\beta}$ for a locally anisotropic fluid is [14]

$$T^-_{\alpha\beta} = (\mu + P_r) V_\alpha V_\beta + P_r g_{\alpha\beta} + (P_z - P_r) S_\alpha S_\beta + (P_\phi - P_r) K_\alpha K_\beta, \quad (2)$$

where $\mu$, $P_r$, $P_\phi$, $P_z$, $V_\alpha$, $K_\alpha$ and $S_\alpha$ are the energy density, the principal stresses, four velocity and four-vectors, respectively, satisfying

$$V^\alpha V_\alpha = -1, \quad S^\alpha S_\alpha = K^\alpha K_\alpha = 1, \quad S^\alpha K_\alpha = V^\alpha K_\alpha = V^\alpha S_\alpha = 0,$$

with

$$V_\alpha = -A \delta^0_\alpha, \quad K_\alpha = C \delta^2_\alpha, \quad S_\alpha = \delta^3_\alpha.$$

The four acceleration and its non-vanishing components are

$$a_\alpha = V_{\alpha;\beta} V^\beta, \quad a_1 = \frac{A'}{A}, \quad a^2 = a^\alpha a_\alpha = \left( \frac{A'}{AB} \right)^2.$$

The shear tensor is defined as

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha} V_{\beta)) - \frac{1}{3} \Theta (g_{\alpha\beta} + V_\alpha V_\beta),$$

with non-zero components

$$\sigma_{11} = \frac{B^2}{3A} \left( \frac{2\dot{B}}{B} - \frac{\dot{C}}{C} \right), \quad \sigma_{22} = \frac{-C^2}{3A} \left( \frac{\dot{B}}{B} - \frac{2\dot{C}}{C} \right), \quad \sigma_{33} = \frac{-1}{3A} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right),$$

where dot and prime stand for $t$ and $r$ differentiation, respectively. The expansion scalar is

$$\Theta = V^\alpha_{;\alpha} = \frac{1}{A} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right). \quad (3)$$

The field equations for the interior spacetime [11] are

$$\kappa \mu A^2 = \left( \frac{A}{B} \right)^2 \left( \frac{B'C'' - C''}{BC} - \frac{C''}{C} \right) + \frac{\dot{B}\dot{C}}{BC}, \quad (4)$$

$$0 = \frac{\dot{C}}{C} - \frac{\dot{C}A'}{CA} - \frac{\dot{B}C'}{BC}, \quad (5)$$

$$\kappa P_r B^2 = \left( \frac{B}{A} \right)^2 \left( \frac{\dot{A}\dot{C}}{AC} - \frac{\ddot{C}}{C} \right) + \frac{A'C''}{AC}. \quad (6)$$
\[ \kappa P_\phi = \left( \frac{1}{AB} \right) \left( \frac{A''}{B} - \frac{\ddot{B}}{A} + \frac{\dot{A} \dot{B}}{A^2} - \frac{A'B'}{B^2} \right), \quad (7) \]

\[ \kappa P_z = \frac{A''}{AB^2} - \frac{\ddot{B}}{A^2 B} + \frac{\dot{A} \dot{B}}{A^3 B} - \frac{A'B'}{AB^3} + \frac{\dot{A} \dot{C}}{A^2 C} - \frac{\ddot{C}}{A^2 C} \]

\[ - \frac{B'C'}{B^2 C} + \frac{C''}{B^2 C} + \frac{A'C'}{AB^2 C} - \frac{\dot{B} \dot{C}}{A^2 BC}. \quad (8) \]

Thorne defined C-energy as [15]

\[ E(t, r) = \frac{1}{8} \left( 1 - l^{-2} \nabla^a \bar{r} \nabla_a \bar{r} \right), \quad (9) \]

where \( \rho, l \) and \( \bar{r} \) are the circumference radius, specific length and areal radius with the following relations

\[ \rho^2 = \xi_{(1)a} \xi^a_{(1)}, \quad l^2 = \xi_{(2)a} \xi^a_{(2)}, \quad \bar{r} = \rho l. \]

Here \( \xi_{(1)} = \frac{\partial}{\partial \phi}, \quad \xi_{(2)} = \frac{\partial}{\partial z} \) are the Killing vectors and \( E(t, r) \) represents the gravitational energy per unit specific length of the cylinder. The specific energy of the cylinder is given by \( \bar{E} = E l \). However, in view of Eq.(1), the specific length, \( l^2 \), turns out to be 1. Therefore, the specific energy of the cylinder in the interior region is written as

\[ \bar{E}(t, r) = E(t, r) = \frac{1}{8} \left[ 1 + \left( \frac{\dot{C}}{A} \right)^2 - \left( \frac{C'}{B} \right)^2 \right]. \quad (10) \]

Differentiating Eq.(10) with respect to \( t \) and \( r \), we get

\[ \dot{E} = -2\pi P_r C' \dot{C}, \quad E' = 2\pi \mu C'C, \quad (11) \]

which yields

\[ \dot{\mu} C' + P_r \dot{C} + (P_r + \mu)(C' + C' \frac{\dot{C}}{C}) = 0. \quad (12) \]

Integrating the second of Eq.(11), it follows that

\[ E = 2\pi \int_0^r \mu C'C dr. \quad (13) \]

Further, integration yields

\[ \frac{E}{C^2} = \pi \mu - \frac{\pi}{C^2} \int_0^r \mu C'^2 dr, \quad (14) \]
where we have used \(E(t, 0) = 0 = C(t, 0)\). The velocity of the collapsing fluid is \(U = \frac{\dot{C}}{A}\) for which Eq. (10) leads to
\[
\dot{E} \equiv \frac{C'}{B} = [1 + U^2 - 8E]^{1/2}. \tag{15}
\]

Now the electric and magnetic parts of the Weyl tensor are
\[
\hat{E}_{\alpha\beta} = C_{\alpha\mu\beta\nu}V^\mu V^\nu, \quad H_{\alpha\beta} = \tilde{C}_{\alpha\gamma\beta\delta}V^\gamma V^\delta = \frac{1}{2} \epsilon_{\alpha\gamma\epsilon\delta}C_{\beta\rho}^\epsilon V^\rho, \tag{16}
\]
where \(\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g} \eta_{\alpha\beta\gamma\delta}\). The non-vanishing components of the Weyl tensor are
\[
\hat{E}_{11} = \frac{B^2}{6A^2} \left( \frac{\ddot{C} - 2\dot{B}}{C} - \frac{\dot{A}C - 2\dot{A}\dot{B}}{AB} + \frac{\dot{B}C}{BC} \right)
+ \frac{B^2}{6} \left( \frac{2A'' - C''}{A} - \frac{2A'B'}{AB} + \frac{B'C'}{BC} - \frac{A'C''}{AC} \right),
\]
\[
\hat{E}_{22} = -\frac{2C^2}{6A^2} \left( \frac{\ddot{C}}{C} - \frac{\ddot{B}}{2B} - \frac{\dot{A}C - 2\dot{A}\dot{B}}{2AB} - \frac{\dot{B}C}{2BC} \right)
+ \frac{C^2}{6B^2} \left( \frac{A''}{A} + \frac{C''}{C} - \frac{2A'C'}{AC} - \frac{B'C'}{BC} - \frac{A'B'}{AB} \right),
\]
\[
\hat{E}_{33} = \frac{1}{6A^2} \left( \frac{\ddot{A}B}{AB} - \frac{\ddot{C}}{C} - \frac{\ddot{B}}{B} + \frac{\dot{A}C}{AC} + \frac{2\dot{B}C}{BC} \right)
+ \frac{1}{6B^2} \left( \frac{A''}{A} - \frac{2C''}{C} + \frac{A'C''}{AC} + \frac{2B'C'}{BC} - \frac{A'B'}{AB} \right),
\]
\[
H_{23} = H_{32} = \frac{1}{2A^2B^2C^2} \left( \frac{C\ddot{C}}{A} - \frac{C\dot{C}A'}{C} - \frac{C\dot{B}A'}{B} \right).
\]

The conservation of energy-momentum tensor gives
\[
\dot{\mu} + A\mu \Theta + \frac{|\dot{B}|}{B}P_r + \frac{\dot{C}}{C}P_\phi = 0, \tag{17}
\]
\[
P_r' + (\mu + P_r)\frac{A'}{A} + (P_r - P_\phi)\frac{C'}{C} = 0. \tag{18}
\]
For the junction conditions, we take the exterior spacetime as the cylindrically symmetric metric [16]
\[ ds^2_+ = -\left(\frac{-2M}{R}\right) dv^2 - 2dRd\nu + R^2(d\theta^2 + \alpha^2dz^2), \]
(19)
where \( M \) and \( \nu \) are the total mass and retarded time, respectively. Here \( \alpha^2 = -\frac{\Lambda}{3} \), where \( \Lambda \) is a cosmological constant. For smooth matching of the interior and exterior regions, Darmois conditions [11] lead to
\[ E - M \Sigma^{(e)} = \frac{1}{8}, \quad P_r \Sigma^{(e)} = 0. \]
(20)
Here \( \Sigma^{(e)} \) indicates that the quantities are evaluated at external hypersurface. This equation shows that the difference between two masses is equal to \( \frac{1}{8} \) as shown in the adiabatic case [17]. This is due to the least unsatisfactory definition of the C-energy [15].

Notice that the expansion-free models require two hypersurfaces, one is the boundary between the fluid distribution and the external cylindrically symmetric solution and other separating the central Minkowskian cavity from the fluid [12]. Taking \( \Sigma^{(i)} \) as the boundary surface of that internal vacuum cavity and the fluid distribution, then matching of the Minkowski spacetime within the cavity to the fluid gives
\[ E(t, r) \Sigma^{(i)} = 0, \quad P_r \Sigma^{(i)} = 0. \]
(21)

3 The Weyl Tensor and Matter Variables

In this section, we obtain a relation between the Weyl tensor and matter variables. The Weyl scalar \( \mathcal{C} \) in terms of the Kretchman scalar, \( \mathcal{R} \), is
\[ \mathcal{C}^2 = \mathcal{R} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2, \]
(22)
where the Kretchman scalar \( \mathcal{R} = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \) yields
\[ \mathcal{R} = 4 \left\{ \frac{1}{(AB)^4}(R^{0101})^2 + \frac{1}{(BC)^4}(R^{1212})^2 + \frac{1}{(AC)^4}(R^{0202})^2 \right. \\
- \left. \frac{1}{2A^2B^2C^4}(R^{1202})^2 \right\}. \]
(23)
The non-zero components of the Riemann tensor in terms of the Einstein tensor can be written as

\[
R^{0101} = \frac{1}{(ABC)^2} G_{22}, \quad R^{0202} = \frac{1}{(ABC)^2} G_{11}, \\
R^{0212} = \frac{1}{(ABC)^2} G_{01}, \quad R^{1212} = \frac{1}{(ABC)^2} G_{00}.
\]

Substituting these values in Eq. (23), we obtain

\[
\mathcal{R} = 4 \left\{ \frac{G_{00}^2}{A^4} + \frac{G_{11}^2}{B^4} + \frac{G_{22}^2}{C^4} - \frac{2G_{01}^2}{(AB)^2} \right\}.
\] (24)

The remaining terms of the Weyl scalar are

\[
R_{00} = A^2 \left( \frac{G_{11}}{B^2} + \frac{G_{22}}{C^2} \right), \quad R_{01} = G_{01}, \\
R_{22} = C^2 \left( \frac{G_{00}}{A^2} - \frac{G_{11}}{B^2} \right), \quad R_{11} = B^2 \left( \frac{G_{00}}{A^2} - \frac{G_{22}}{C^2} \right), \\
R = 2 \left( \frac{G_{00}}{A^2} - \frac{G_{11}}{B^2} + \frac{G_{22}}{C^2} \right), \\
R^{ab} R_{ab} = 2 \left\{ \frac{G_{00}^2}{A^4} + \frac{G_{11}^2}{B^4} + \frac{G_{22}^2}{C^4} - \frac{G_{01}^2}{(AB)^2} - \frac{G_{00} G_{11}}{(AB)^2} \\
+ \frac{G_{11} G_{22}}{(BC)^2} - \frac{G_{00} G_{22}}{(AC)^2} \right\}.
\]

Using the preceding equations, the Weyl scalar (22) takes the form

\[
C^2 = \frac{4}{3} \left[ \frac{G_{00}^2}{A^4} + \frac{G_{11}^2}{B^4} + \frac{G_{22}^2}{C^4} + \frac{G_{00} G_{11}}{(AB)^2} - \frac{G_{11} G_{22}}{(BC)^2} + \frac{G_{22} G_{00}}{(AC)^2} \right].
\]

Equations (4), (6) and (7) yield

\[
\frac{C \sqrt{3}}{2} = \left[ (\kappa(\mu + P_r - P_\phi))^2 - \kappa^2 \mu (P_r - 3P_\phi) + \kappa^2 P_r P_\phi \right]^{1/2}.
\] (25)

When we make use of Eq.(14), it follows that

\[
\frac{C \sqrt{3}}{2} = \left\{ \frac{\kappa E}{\pi C^2} + \frac{\kappa}{C^2} \int_0^r \mu' C^2 dr + \kappa(P_r - P_\phi) \right\}^2 + \kappa^2 P_r P_\phi \\
- \kappa^2 (P_r - 3P_\phi) \left( \frac{E}{\pi C^2} + \frac{1}{C^2} \int_0^r \mu' C^2 dr \right) \right]^{1/2},
\] (26)
which gives the relationship between the Weyl tensor and the fluid properties like energy density inhomogeneity and anisotropy of the pressure.

For pure dust $P_r = P_o = P_z = 0$, it follows that
\[
\frac{C\sqrt{3}C^2}{2} = \frac{\kappa E}{\pi} + \kappa \int_0^r \mu'C^2 dr.
\] (27)

Using Eq.(13) and differentiating Eq.(25) with respect to $r$, we have
\[
\left[ \frac{C\sqrt{3}}{2} \right]' = \kappa \mu',
\]
which yields $\mu' = 0$ if and only if $C = \text{constant}$. Thus it is also concluded that if the energy density is homogeneous, the metric is conformally flat and vice versa as in [18].

4 Exact Analytical Models

Here we use the expansion-free condition to investigate some exact analytical models. We also check the validity of junction conditions for the resulting models. The expansion-free condition, $\Theta = 0$, leads to
\[
B = \frac{\gamma}{C},
\] (28)
where $\gamma$ is an arbitrary function of $r$. Without loss of generality, we assume $\gamma = 1$. Using this value in Eq.(5), it follows that
\[
A = \frac{C\dot{C}}{\xi},
\] (29)
where $\xi$ is an arbitrary function of $t$. The physical variables $\mu$, $P_r$ and $\Pi$ can be written in terms of $C$ and $E$ as
\[
2\pi\mu = \frac{E'}{CC'}, \quad 2\pi P_r = -\frac{\dot{E}}{CC'}, \quad \Pi = (P_r - P_o) = \frac{C\dot{\mu}}{C}.
\] (30)

Also, using Eqs.(10), (28) and (29), we can write
\[
E = \frac{1}{8} \left( \frac{\xi^2}{C'^2} - C^2C'^2 + 1 \right).
\] (31)

We see that the metric coefficients $A$ and $B$ are now expressed in terms of $C$. In the following, we obtain some exact analytical models.
4.1 Solution I

For the sake of analytical models, we assume $E(t,r)$ of the form

$$2E(t,r) = \frac{1}{3}kC^3 + \frac{1}{6}lC^6 + \frac{1}{4},$$

(32)

where $j$, $k$ and $l$ are arbitrary functions of $t$. Using this value in Eq. (31), the energy density becomes

$$4\pi\mu = kC + lC^4.$$  (33)

Using Eqs. (30) and (33), we obtain a differential equation in terms of $C$

$$kC^2 + lC^5 + \frac{\xi^2}{2C^3} + \frac{CC'^2}{2} + \frac{C^2C''}{2} = 0,$$

(34)

which along with Eqs. (31) and (32) leads to

$$\frac{10}{3}kC^2 + \frac{8}{3}lC^5 + 3CC'^2 + C^2C'' = 0.$$  

For simplicity, we substitute $C^2 \equiv Z$ so that

$$aZ + bZ^{5/2} + \frac{Z'^2}{Z^{3/2}} + Z''Z^{1/2} = 0,$$

where $a(t) \equiv \frac{20}{3}k$, $b(t) \equiv \frac{16}{3}l$. Integrating this equation with respect to $Z$, it follows that

$$Z'^2 = -\frac{4}{7}aZ^{7/2} - \frac{2b}{5}Z^3.$$  (35)

The solution of this equation does not exist explicitly in terms of $Z$. However, we find some solutions by imposing the following constraints.

Case (i)

Here, we take $a \neq 0$ and $b = 0$. Consequently, Eq. (35) gives

$$Z = \left(\frac{a}{28}\right)^2 (r + \zeta)^4,$$

which can be written as

$$C = \left(\frac{5k}{21}\right) (r + \zeta)^2,$$  (36)
where $\zeta(t)$ is an arbitrary function. The corresponding physical variables turn out to be

$$4\pi\mu = \left(\frac{5k^2}{21}\right)(r + \zeta)^2,$$

(37)

$$4\pi P_r = -5k^2 (r + \zeta)^2 \left[\frac{k(r + \zeta)}{63 \left\{(r + \zeta) \dot{k} + 2k\dot{\zeta}\right\}} + \frac{1}{21}\right],$$

(38)

$$4\pi P_\phi = -10k^2 (r + \zeta)^2 \left[\frac{2k(r + \zeta)}{63 \left\{(r + \zeta) \dot{k} + 2k\dot{\zeta}\right\}} + \frac{1}{21}\right],$$

(39)

$$8\pi P_z = \frac{C}{C} \left(\dot{C}''C + 4\dot{C}'C' + 2C''C + \frac{C'^2}{C}\right)$$

$$+ \frac{1}{C^3} \left[\frac{\dot{\xi}}{\dot{C}} + \frac{\dot{C}^2}{C} + \ddot{C} - \dot{C} - \frac{\ddot{C}\xi^2}{C^2}\right].$$

(40)

When we use Darmois junction conditions, we obtain three independent equations with two functions $k(t)$ and $\zeta(t)$ which can be satisfied by any convenient choice of one of these functions. However, this does not lead to interesting solutions.

**Case (ii)**

In this case, we take $a = 0$ and $b \neq 0$. The assumption $a = 0$ gives $k = 0$ and hence Eq.(35) yields

$$Z = \left(\frac{10}{b}\right)(r + \zeta)^{-2},$$

or

$$C = (r + \zeta)^{-1} \left(\frac{15}{8f}\right)^{\frac{1}{2}}.$$
Consequently, the quantities $\mu$, $P_r$, $P_\phi$ and $P_z$ become

\begin{align*}
4\pi \mu &= \left( \frac{225}{64l^4} \right) (r + \zeta)^{-4}, \\
4\pi P_r &= \frac{-75}{64l^2 (r + \zeta)^4} \left[ 3l - \frac{\dot{l}(r + \zeta)}{(r + \zeta) \dot{l} + 2l\dot{\zeta}} \right], \\
4\pi P_\phi &= \frac{-75}{64l^2 (r + \zeta)^4} \left[ 15l - \frac{7\dot{l}(r + \zeta)}{(r + \zeta) \dot{l} + 2l\dot{\zeta}} \right], \\
8\pi P_z &= \frac{C}{C'} \left( \dot{C}''C + 4\dot{C}'C' + 2C''C + \frac{C'^2}{C} \dot{C} \right) \\
&+ \frac{1}{C^3} \left( \frac{\xi \ddot{\xi}}{C'} + \frac{\dot{C}^2}{C} + \ddot{C} - \frac{\dot{C} \xi^2}{C^2} \right). \quad (41)
\end{align*}

On $r = r_i$, this subfamily of solution does not satisfy Darmois conditions.

### 4.2 Solution II

The second family of solution is obtained by assuming $P_r = P_z = 0$. It follows from Eq.(12) that $\mu = \frac{d_1(r)}{CC'}$ which leads to

\begin{equation}
C'^2 = 2 \int \frac{d_1(r)}{\mu} dr + d_2(t), \quad (42)
\end{equation}

where $d_1$ and $d_2$ are integration functions. Consequently, the first of Eq.(30) yields $d_1 = \frac{E'}{2\pi}$. Under the expansion-free condition, Eq.(17) yields

\begin{equation}
P_\phi = -\frac{\ddot{\mu}C}{C}. \quad (43)
\end{equation}

We take the following equation of state to obtain different physical models.

**Case (i)**

In this case, we consider $P_\phi = \alpha \mu$, where $\alpha$ is a constant. Using this result in Eq.(18), we have

\begin{align*}
\dot{C} &= f(t)C^{\alpha-1}, \quad C' = g(r)C^{\alpha-1}, \quad (44)
\end{align*}
or
\[ C^{2-\alpha} = \psi(t) + \chi(r), \tag{45} \]
with \( \psi(t) = (2 - \alpha) \int f(t) dt \) and \( \chi(r) = (2 - \alpha) \int g(r) dr \). Here \( g(r) \) and \( f(t) \) are arbitrary functions. Without loss of generality we can choose \( \xi(t) = f(t) \), then from Eqs. (29) and (44), we have \( A = C^\alpha \). Using the constraints \( P_r = 0 \) and \( P_\theta = \alpha \mu \) in Eq. (11), it implies that \( E = E(r) \). Thus using Eq. (28) along with \( U = \frac{C}{\lambda} \), Eq. (10) becomes
\[ E(r) = \frac{1}{8} \left( \frac{\dot{C}^2}{C^{2\alpha}} - g^2(r) C^{2\alpha} + 1 \right). \tag{46} \]

Next, we evaluate this equation on the hypersurfaces, i.e., \( r = r_e \) and \( r = r_i \)
\[ \dot{C}^2 \Sigma^{(i)} = C^{2\alpha} \left( g^2 C^{2\alpha} - 1 \right), \quad \dot{C}^2 \Sigma^{(e)} = C^{2\alpha} \left( 8M + g^2 C^{2\alpha} \right). \tag{47} \]
From here we can easily evaluate \( C \) for an arbitrary value of \( \alpha \). For \( \alpha = 1/2 \), it follows
\[ C = \Sigma^{(i)} \left[ g^2 \cos^2(t + t_0) \right]^{-1}. \tag{48} \]
Equation (45) yields
\[ \psi(t) \Sigma^{(i)} = \left[ g^2 \cos^2(t + t_0) \right]^{-3/2} - \chi. \tag{49} \]
Thus the time dependence of all variables is fully determined. Now the radial dependence (\( d_i \) or \( \chi \)) can be obtained from the initial data.
\[ C_i^{2\alpha}(0) - C_e^{2\alpha}(0) = \frac{1}{2g^2} \left[ 1 + \left( 1 + 4g^2 C_i^{2\alpha}(0) \right)^{1/2} \right] \]
\[ + \frac{8M}{2g^2} - \frac{\sqrt{4M^2 + 4\dot{C}_e^2(0)}}{g}, \tag{50} \]
where \( C_i^{2\alpha}(0) \) and \( C_e^{2\alpha}(0) \) are calculated from Eq. (17) at \( t = 0 \). The differences \( C_i^{2\alpha}(0) - C_e^{2\alpha}(0) \) and \( C_i^{2\alpha}(0) - C_e^{2\alpha}(0) \) will be zero for \( \alpha = 0 \) and negative for any value of \( t \) (if \( \alpha > 0 \)) respectively. Also, from Eqs. (29) and \( U = \frac{C}{\lambda} \), we obtain
\[ U_i = \frac{f}{C_i} = \frac{\dot{C}_i}{C_i^\alpha}, \quad U_e = \frac{f}{C_e} = \frac{\dot{C}_e}{C_e^\alpha}. \tag{51} \]
As \( \alpha < 1 \) and \( C_e > C_i \), therefore we get
\[
\dot{C}_i > \dot{C}_e. \tag{52}
\]
Thus from Eqs.\((50)\) and \((52)\), we obtain \( C_i^{2\alpha} > C_e^{2\alpha} \), which has no physical significance as such models require the presence of thin shells at both boundaries.

**Case (ii)**

Here we assume that energy density is separable so that
\[
\mu = \mu_0(t) d_1 / r^2. \tag{53}
\]
Consequently, Eqs.\((42)\) and \((43)\) give
\[
C = \left( \frac{2r^3}{3\mu_0} + d_2(t) \right)^{\frac{1}{2}}, \quad P_\phi = -2\dot{\mu} \frac{2r^3}{3\mu_0} + d_2. \tag{54}
\]
Also, Eq.\((31)\) can be written as
\[
E = \frac{1}{8} \left[ 1 + \xi^2 \frac{C^2}{\mu_0^2} - \frac{r^4}{\mu_0^2} \right]. \tag{55}
\]
Evaluating the above equation by using Eq.\((20)\), we have
\[
C_i^2 = \xi^2 \left( \frac{r_i^4}{\mu_0^2} - 1 \right)^{-1}, \tag{56}
\]
implying that \( r_i^4 > \mu_0^2 \), for all \( t \). Using \( U = \dot{C} / A \) and Eq.\((29)\) in \((56)\), it follows that
\[
U_i^2 = \frac{r_i^4}{\mu_0^2} - 1. \tag{57}
\]
Thus, from the absence of superluminal velocities \( U < 1 \) and from Eq.\((57)\), we have to impose
\[
r_i^2 < \sqrt{2}\mu_0. \tag{58}
\]
The relation between \( U_i \) and \( U_e \) can be found from Eq.\((51)\) as
\[
U_e = U_i \left( \frac{C_i}{C_e} \right). \tag{59}
\]
This shows that the inner surface moves faster than the outer one as $C_e > C_i$. Further, we take $\xi/C = \dot{C}$, then

$$A_i = 1, \quad \dot{C}_i = U_i, \quad \xi = U_i C_i = U_e C_e.$$  \hspace{1cm} (60)

Using Eqs. (20), (55) and (60), we obtain

$$C_e^2 = \xi^2 \left( 8M + \frac{r_e^4}{\mu_0^2} \right)^{-1},$$  \hspace{1cm} (61)

which yields

$$8M + \frac{r_e^4}{\mu_0^2} = U_e^2.$$  \hspace{1cm} (62)

Thus, from the absence of superluminal velocities ($U < 1$) and from Eq. (62), we require $r_e^2 < \mu_0$, which is not the same as imposed on $r_i$ due to the least unsatisfactory definition of C-Energy. We can find the time dependence of $C_e$ if $\mu_0 = \mu_0(t)$ which implies time dependence of all variables.

5 Conclusions

We have found some exact analytical models with expansion-free condition, total of four solutions, out of which the last two models satisfy Darmois junction conditions on boundary surfaces. Such solutions may describe the possible importance of expansion-free condition to model situation where cavities are expecting to appear. Also, some of these models may contain essential features of a realistic situations. It is interesting to note that expansion-free condition might be helpful for the description of voids. During expansion, matter of the voids streams out which decreases the density of voids from inside. They are usually described as a vacuum cavity around the center. Relaxing Darmois conditions on both boundary surfaces enlarges the families of possible solutions that may show their importance in the study of voids with thin wall approximation \[20\]. We see from Eq. (17) that energy density changes with time for $\Theta = 0$. The results of this paper demonstrate that expansion-free condition helps a lot in the modeling of the cavity evolution.
References

[1] Skripkin, V.A.: Soviet Physics-Doklady 135(1960)1183.

[2] Herrera, L., Santos, N.O. and Wang, A.: Phys. Rev. D 78(2008)084026.

[3] Herrera, L., Le Denmat, G. and Santos, N.O.: Phys. Rev. D 79(2009)087505.

[4] Herrera, L., Santos, N.O.: Phys. Rep. 53(1997)286.

[5] Herrera, L., Di Prisco, A., Martin, J., Ospino, J., Santos, N.O. and Troconis, O.: Phys. Rev. D 69(2004)084026.

[6] Dev, K. and Gleiser, M.: Gen. Relativ. Gravit. 35(2003)1435.

[7] Ivanov, B.V.: Phys. Rev. D 65(2002)104011.

[8] Mak, M.K. and Harko, T.: Proc. R. Soc. London A 459(2003)393.

[9] Herrera, L., Santos, N.O. and Le Denmat, G.: Gen. Relativ. Gravit. 44(2012)1143.

[10] Sharif, M. and Kausar, H.R.: J. Cosmol. Astropart. Phys. 07(2011)022; Sharif, M. and Azam, M.: Gen. Relativ. Gravit. 44(2012)1181.

[11] Darmois, G.: Memorial des Sciences Mathematiques (Gautheir-Villars, 1927) Fasc. 25.

[12] Di Prisco, A., Herrera, L., Ospino, J., Santos, N.O. and Viña-Cervantes. V.M.: Int. J. Mod. Phys. D 20(2011)2351.

[13] Sharif, M. and Azam, M.: J. Cosmol. Astropart. Phys. 02(2012)043.

[14] Di Prisco, A., Herrera, L., MacCallum, H.A.M. and Santos, N.O.: Phys. Rev. D 80(2009)064031.

[15] Thorne, K.S.: Phys. Rev. B138(1965)251.

[16] Chao-Guang, H.: Acta Phys. Sin. 4(1995)617.

[17] Sharif, M. and Fatima, S.: Gen. Relativ. Gravit. 43(2011)127.

[18] Sharif, M. and Siddiqua, A.: Gen. Relativ. Gravit. 43(2011)73.
[19] Chaisi, M. and Maharaj, S.D.: Gen. Relativ. Gravit. 37(2005)1177.

[20] Pim, R. and Lake, K.: Astrophys. J. 330(1988)625.