The fast signal diffusion limit in a Keller–Segel system

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Abstract. This paper deals with convergence of a solution for the parabolic-parabolic
Keller–Segel system
\[
\begin{cases}
(u_\lambda)_t = \Delta u - \chi \nabla \cdot (u_\lambda \nabla v_\lambda) & \text{in } \Omega \times (0, \infty), \\
(\lambda v_\lambda)_t = \Delta v - v + u_\lambda & \text{in } \Omega \times (0, \infty)
\end{cases}
\]
to that for the parabolic-elliptic Keller–Segel system
\[
\begin{cases}
u_t = \Delta v - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0, \infty), \\
0 = \Delta v - v + u & \text{in } \Omega \times (0, \infty)
\end{cases}
\]
as \(\lambda \searrow 0\), where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) \((n \geq 2)\) with smooth boundary, \(\chi, \lambda > 0\) are constants. In chemotaxis systems parabolic-elliptic systems often provided some guide to
methods and results for parabolic-parabolic systems. However, there have not been rich results on the relation between parabolic-elliptic systems and parabolic-parabolic
systems. Namely, it still remains to analyze on the following question except some cases: Does a solution of the parabolic-parabolic system converge to that of the parabolic-elliptic
system as \(\lambda \searrow 0\)? In the case that \(\Omega\) is the whole space \(\mathbb{R}^n\), or \(\Omega\) is a bounded domain and \(\chi\) is a strong signal sensitivity, some positive answer
was shown in the author’s previous paper (Math. Nachr., to appear). Therefore one can expect a positive answer to this question also in the Keller–Segel system in a bounded
domain \(\Omega\) in some cases. This paper gives some positive answer in the 2-dimensional and the higher-dimensional Keller–Segel system.

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1. Introduction

The subject of this work is to construct a new approach to a parabolic-elliptic Keller–Segel system from its parabolic-parabolic case, and to use the parabolic-parabolic case as a step to establish new results in the parabolic-elliptic case. In this paper our aim is, by considering that the parabolic-elliptic system is as a limit of its parabolic-parabolic case, to establish a result such that only dealing with the parabolic-parabolic Keller–Segel system is enough to obtain new properties for solutions of its parabolic-elliptic case. As a related work, in the study of a chemotaxis system with signal-dependent sensitivity, some result on this subject has already been obtained ([18]); however, in this study we could not attain a result on a minimal Keller–Segel system from a technical reason. Thus the subject of this paper is a challenging problem for a progress of the chemotaxis system.

Before an introduction of a problem in this paper, we will recall some related works on the chemotaxis system. Here chemotaxis is the property such that species move towards higher concentration of a chemical substance when they plunge into hunger. Keller–Segel [12, 13] studied the migration of the species which have chemotaxis, and proposed the following problem:

$$u_t = \Delta u - \nabla \cdot (u \chi(v) \nabla v), \quad \lambda v_t = \Delta v - v + u \quad \text{in } \Omega \times (0, \infty),$$

where $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a bounded domain, $\lambda = 0$ (the parabolic-elliptic system) or $\lambda > 0$ (the parabolic-parabolic system) is a constant and $\chi$ is a function. This problem is called a chemotaxis system, and especially, is called a (minimal) Keller–Segel system in the case that $\chi$ is a constant function. About the Keller–Segel system, Nanjundiah [25] first asserted that we could expect existence of a blow-up solution to the Keller–Segel system. Moreover, Childress–Percus [4] claimed the following conjecture:

- In the 1-dimensional setting, global existence holds.
- In the 2-dimensional setting, there is a critical number $c$ such that if an initial data $u_{\text{init}}$ satisfies $\|u_{\text{init}}\|_{L^1(\Omega)} < c$ then global existence holds, and for any $m > c$ there are initial data $u_{\text{init}}, v_{\text{init}}$ such that $\|u_{\text{init}}\|_{L^1(\Omega)} = m$ and the corresponding solution blows up in finite time.
- In the higher-dimensional setting, there are many blow-up solutions.

Here we first focus on the 2-dimensional setting. The study of the 2-dimensional Keller–Segel system is supported by the interaction between the parabolic-elliptic case and the parabolic-parabolic case. In order to verify the Childress–Percus conjecture Nagai [20] tried to deal with the parabolic-elliptic case which is a simplified problem of the parabolic-parabolic Keller–Segel system, and shown that, in the radial setting, $8\pi$ is the critical value in the Childress–Percus conjecture. Subsequently, Nagai–Senba–Yoshida [24] established global existence and boundedness of radial solutions in the parabolic-parabolic Keller–Segel system under the condition that $\|u_{\text{init}}\|_{L^1(\Omega)} < 8\pi$, and also obtained existence of global bounded nonradial solutions to the parabolic-parabolic system under the condition that $\|u_{\text{init}}\|_{L^1(\Omega)} < 4\pi$. Here Senba–Suzuki [30] asserted that arguments in proofs of these results could also be applied to the parabolic-elliptic case, which meant that global existence and boundedness of solutions to the parabolic-elliptic system were shown under
the condition that \( \| u_{\text{init}} \|_{L^1(\Omega)} < 4\pi \). Therefore in the both cases of the parabolic-elliptic system and the parabolic-parabolic system, \( 8\pi \) is the critical value in the Childress–Percus conjecture in the radial case, and \( 4\pi \) is the critical value in the nonradial case. Indeed, existence of blow-up solutions such that \( \| u_{\text{init}} \|_{L^1(\Omega)} \) is larger than the critical value was shown \((\cite{10, 11, 17, 20, 21})\): The radial parabolic-elliptic case was treated by a combination of the results in \cite{10, 20}; the nonradial parabolic-elliptic case is in \cite{21}; the radial parabolic-parabolic case can be found in \cite{17}; the nonradial parabolic-parabolic case is in \cite{11}. Moreover, related works which deal with blow-up asymptotics of solutions to the parabolic-elliptic case can be found in \cite{10, 29, 31} and to the parabolic-parabolic case are in \cite{16, 23}. In summary, in the 2-dimensional setting, the study of the Keller–Segel system was developed by the interaction between the parabolic-elliptic system and the parabolic-parabolic system, and it is shown that the Childress–Percus conjecture is true. On the other hand, the other dimensional cases have also been studied only in the parabolic-parabolic system, and it is shown that the Childress–Percus conjecture is valid also in the other dimensional cases; in the 1-dimensional setting Osaki–Yagi \cite{26} showed global existence and boundedness of classical solutions; in the higher-dimensional case Winkler \cite{36} obtained that for all \( m > 0 \) there are initial data \( u_{\text{init}}, v_{\text{init}} \) such that \( \| u_{\text{init}} \|_{L^1(\Omega)} = m \) and the corresponding solution blows up in finite time. Here global existence of bounded solutions to the higher-dimensional parabolic-parabolic Keller–Segel system also holds under some smallness condition for initial data \( u_{\text{init}}, v_{\text{init}} \) with respect to some Lebesgue norm; Winkler first established global existence and boundedness in the higher-dimensional parabolic-parabolic Keller–Segel system under the condition that \( \| u_{\text{init}} \|_{L^p(\Omega)} \) and \( \| \nabla v_{\text{init}} \|_{L^q(\Omega)} \) are sufficiently small with some \( p > \frac{n}{2} \) and \( q > n \); Cao \cite{3} obtained global existence of bounded solutions to the parabolic-parabolic system under the smallness conditions for initial data in optimal spaces: \( \| u_{\text{init}} \|_{L^{2n}(\Omega)} \) and \( \| \nabla v_{\text{init}} \|_{L^n(\Omega)} \) are small enough.

As we mentioned before, the interaction between the parabolic-elliptic system and the parabolic-parabolic system made progress on researches of the Keller–Segel system. The similar things occurred in the study of the chemotaxis system with signal-dependent sensitivity which is the case that \( \chi \) is a function. In the parabolic-elliptic system with \( \chi(v) = \frac{\chi_0}{v} \) \((\chi_0 > 0)\) Nagai–Senba \cite{22} first showed that if \( n = 2 \), or \( n \geq 3 \) and \( \chi_0 < \frac{2}{n-2} \) then a radial solution is global and bounded, and if \( n \geq 3 \) and \( \chi_0 > \frac{2n}{n-2} \) then there exists some initial data such that a radial solution blows up in finite time. In the nonradial case Biler \cite{2} obtained global existence of solutions to the parabolic-elliptic system with \( \chi(v) = \frac{\chi_0}{v} \) \((\chi_0 > 0)\) under the conditions that \( n = 2 \) and \( \chi_0 \leq 1 \), or \( n \geq 3 \) and \( \chi_0 < \frac{2}{n} \). Thanks to these results, we can expect that conditions for global existence in the above system were determined by a dimension of a domain and a smallness of \( \chi \) in some sense. Indeed, global existence and boundedness of solutions to the parabolic-elliptic system with \( \chi(v) = \frac{\chi_0}{v} \) \((\chi_0 > 0, k \geq 1)\) were derived under some smallness conditions for \( \chi_0 \) \((\cite{9})\). On the other hand, also in the parabolic-parabolic case, it was shown that some smallness condition for \( \chi \) leads to global existence and boundedness; in the case that \( \chi(v) = \frac{\chi_0}{v} \) \((\chi_0 > 0)\) Winkler \cite{35} obtained global existence of classical solutions under the condition that \( \chi_0 < \sqrt{\frac{2}{n}} \) and Fujie \cite{5} established boundedness of these solutions; moreover, Lankeit \cite{14} improved these results in the 2-dimensional setting; in the case that \( \chi(v) \leq \frac{\chi_0}{(a+v)^2} \)
some smallness condition for $\chi_0$ yields global existence and boundedness ([19]). In the case that $\chi$ is a more general sensitivity, Fujie–Senba [6] first established global existence and boundedness in the two-dimensional parabolic-elliptic system, and then they also showed existence of radially symmetric bounded solutions to the parabolic-parabolic system in a two-dimensional ball under the condition that $\lambda$ is sufficiently small ([7]). Recently, in the nonradial setting, a sufficient condition of sensitivity functions for global existence and boundedness in the parabolic-parabolic system was studied by Fujie–Senba [8].

In summary parabolic-elliptic chemotaxis systems often gave us some guide to how we could deal with parabolic-parabolic chemotaxis systems; however, there have not been rich results on the relation between the both systems. Namely, it still remains to analyze on the following question except some cases:

**Does a solution of the parabolic-parabolic system converge to that of the parabolic-elliptic problem as $\lambda \downarrow 0$?**

If we can obtain some positive answer to this question, then we can see that solutions of both systems have some similar properties; thus an answer will enable us to establish approaches to obtain properties for solutions of the chemotaxis systems. Here, in the case that $\Omega$ is the whole space $\mathbb{R}^n$, there are some positive answers to this question in 2-dimensional case ([28]) and n-dimensional case ([15]). Moreover, in the case that $\Omega$ is a bounded domain and $\chi(v) \leq \frac{\chi_0}{(a+v)^k}$ ($\chi_0 > 0, a \geq 0, k > 1$), a positive answer to this question is also shown under the condition that $\chi_0$ is small [18]. Therefore we can expect a positive answer to this question also in the Keller–Segel system in a bounded domain $\Omega$ in some case. The purpose of this paper is to give some positive answer to this question.

In order to attain this purpose, this paper investigates the fast signal diffusion limit, which namely is convergence of a solution for the parabolic-parabolic Keller–Segel system to that of the parabolic-elliptic Keller–Segel system

$$
\begin{cases}
(u_\lambda)_t = \Delta u_\lambda - \chi \nabla \cdot (u_\lambda \nabla v_\lambda), & x \in \Omega, \ t > 0, \\
\lambda(v_\lambda)_t = \Delta v_\lambda - v_\lambda + u_\lambda, & x \in \Omega, \ t > 0, \\
\nabla u_\lambda \cdot \nu = \nabla v_\lambda \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
u_\lambda(x,0) = u_{\text{init}}(x), \ v_\lambda(x,0) = v_{\text{init}}(x), & x \in \Omega
\end{cases}
$$

(1.1)

to that of the parabolic-elliptic Keller–Segel system

$$
\begin{cases}
u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\
0 = \Delta v - v + u, & x \in \Omega, \ t > 0, \\
\nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
u(x,0) = u_{\text{init}}(x), & x \in \Omega
\end{cases}
$$

(1.2)

as $\lambda \downarrow 0$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial \Omega$ and $\nu$ is the outward normal vector to $\partial \Omega$; $\chi, \lambda > 0$ is a constant; the initial functions $u_{\text{init}}, v_{\text{init}}$ are assumed to be nonnegative functions. The unknown functions $u_\lambda$ and $u$ represent the
population density of the species and $v_\lambda$ and $v$ show the concentration of the chemical substance at place $x$ and time $t$.

Now the main results read as follows. The first theorem is concerned with global existence and the fast signal diffusion limit of solutions for the higher-dimensional Keller–Segel system under smallness conditions for the initial data.

**Theorem 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n (n \geq 3)$ with smooth boundary and let $\chi > 0$ be a constant. Assume that $u_{\text{init}}$ and $v_{\text{init}}$ satisfy

$$0 \leq u_{\text{init}} \in C(\overline{\Omega}), \quad 0 \leq v_{\text{init}} \in W^{1,q}(\Omega)$$

with some $q > n$. Then for all $p > \frac{n}{2}$ there exists $\varepsilon_0 = \varepsilon_0(p, q, \chi, |\Omega|) > 0$ such that, if $u_{\text{init}}$ and $v_{\text{init}}$ satisfy

$$\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_0 \quad \text{and} \quad \|\nabla v_{\text{init}}\|_{L^q(\Omega)} < \varepsilon_0,$$

then for all $\lambda > 0$ the problem (1.1) possesses a unique global bounded solution $(u_\lambda, v_\lambda)$ which is a pair of nonnegative functions

$$u_\lambda, v_\lambda \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)).$$

Moreover, if $u_{\text{init}}$ and $v_{\text{init}}$ satisfy (1.4), then there are unique functions

$$u \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \text{ and } v \in C^{2,0}(\overline{\Omega} \times (0, \infty)) \cap L^\infty(0, \infty; W^{1,q}(\Omega))$$

such that the solution $(u_\lambda, v_\lambda)$ of (1.1) satisfies

$$u_\lambda \to u \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \quad v_\lambda \to v \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))$$

as $\lambda \searrow 0$, and the pair of the functions $(u, v)$ solves (1.2) classically.

As an application of this result, we can establish a new result which provides global existence and boundedness in the higher-dimensional parabolic-elliptic Keller–Segel system (1.2) under some smallness condition for initial data $u_{\text{init}}$.

**Corollary 1.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n (n \geq 3)$ with smooth boundary and let $\chi > 0$ be a constant. Then for all $p > \frac{n}{2}$ there exists $\varepsilon_1 = \varepsilon(p, \chi, |\Omega|) > 0$ such that, if $u_{\text{init}} \in C(\overline{\Omega})$ satisfies that

$$\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_1$$

holds, then the problem (1.2) possesses a unique global bounded classical solution.

**Remark 1.1.** In these results we assume the smallness conditions for $\|u_{\text{init}}\|_{L^p(\Omega)}$ and $\|\nabla v_{\text{init}}\|_{L^q(\Omega)}$ with some $p > \frac{n}{2}$ and $q > n$, instead of $p = \frac{n}{2}$ and $q = n$ which are the conditions assumed in [3]; we could not attain fast signal diffusion limit under the smallness conditions in optimal spaces.
In the 2-dimensional setting, it is known that global existence and boundedness in (1.1) hold under the condition that \( \|u_{\text{init}}\|_{L^1(\Omega)} \leq \frac{4\pi}{\chi} \) ([24]). Thanks to this previous work, we attain the fast signal diffusion limit in the 2-dimensional Keller–Segel system under the smallness conditions for the initial data in the optimal space.

**Theorem 1.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary and let \( \chi > 0 \) be a constant. Assume that \( u_{\text{init}} \in C(\overline{\Omega}) \) satisfies \( \|u_{\text{init}}\|_{L^1(\Omega)} < \frac{4\pi}{\chi} \). Then there exist unique functions

\[
\begin{align*}
    u & \in C(\overline{\Omega} \times [0, \infty)) \cap C^2,1(\overline{\Omega} \times (0, \infty)) \\
    v & \in C^2(\overline{\Omega} \times (0, \infty)) \cap L^\infty(0, \infty; W^{1,q}(\Omega))
\end{align*}
\]

such that for all \( v_{\text{init}} \in W^{1,q}(\Omega) \) (\( q > 2 \)) the global bounded classical solution \((u_\lambda, v_\lambda)\) of (1.1) satisfies

\[
\begin{align*}
    u_\lambda & \to u \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \\
    v_\lambda & \to v \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\end{align*}
\]

as \( \lambda \searrow 0 \), and the pair of the functions \((u, v)\) solves (1.2) classically.

This result tells us a new method to obtain global existence and boundedness in the 2-dimensional parabolic-elliptic Keller–Segel system (1.2).

**Corollary 1.4.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary and let \( \chi > 0 \) be a constant. If \( u_{\text{init}} \in C(\overline{\Omega}) \) satisfies \( \|u_{\text{init}}\|_{L^p(\Omega)} < \frac{4\pi}{\chi} \), then the problem (1.2) possesses a unique global bounded classical solution.

In the proof of these main results difficulties are caused by the facts that \( v_\lambda \) satisfies a parabolic equation and \( v \) satisfies an elliptic equation. Thus we cannot use methods only for parabolic equations and only for elliptic equations when we would like to obtain some error estimate for solutions of (1.1) and those of (1.2), and it seems to be difficult to combine these methods. Therefore we rely on a compactness method to obtain convergence of a solution \((u_\lambda, v_\lambda)\) as \( \lambda \searrow 0 \), which is the same strategy as that of the proof of [18, Theorem 1.3]. In order to use a compactness method some estimate for the solution uniformly in time and \( \lambda \) is required. In the chemotaxis system with signal dependent sensitivity the boundedness of \( \int_{\Omega} u_\lambda^p(\cdot, t) \exp\{ -r \int_0^{v_\lambda(\cdot, t)} \chi(s) \, ds \} \) with some \( r > 0 \) and the fact \( \int_0^\infty \chi(s) \, ds < \infty \) lead to the desired estimate ([18]). Nevertheless, in the Keller–Segel setting, it is difficult to obtain the boundedness of \( \int_0^{v_\lambda(\cdot, t)} \chi \, ds = \chi v_\lambda(\cdot, t) \). Thus we should give the other method to obtain the desired estimate in the Keller–Segel setting. However, in the higher-dimensional case, a construction of some estimate for the solution uniformly in time and \( \lambda \) is a challenging problem: Indeed, in the previous works [3, 34] the following inequality was obtained:

\[
\|u_\lambda(\cdot, t) - e^{t\Delta}u_{\text{init}}\|_{L^\infty(\Omega)} \leq C(1 + t^{-\alpha})e^{-\beta t} \quad \text{for all } t > 0
\]

with some \( C, \alpha, \beta > 0 \), which could not lead to the uniform-in-time estimate for the solution. This is one of the reason why we could not attain fast signal diffusion limit under the smallness conditions in optimal spaces in the higher-dimensional setting. To
establish the $L^\infty$-estimate for $u_\lambda$ uniformly in time and $\lambda$ we modified the method in [34]. Let $\varepsilon > 0$ be a constant fixed later and put

$$T_\lambda := \sup \left\{ \hat{T} > 0 \mid \| u_\lambda(\cdot, t) - e^{t\Delta} u_{\text{init}} \|_{L^q(\Omega)} < \varepsilon \quad \text{for all } t \in (0, \hat{T}) \text{ and all } \lambda > 0 \right\} \leq \infty$$

with some $\theta > \frac{n}{2}$, which is different from a setting in [34]. Then, under the conditions that $\| u_{\text{init}} \|_{L^p(\Omega)} \leq \varepsilon$ and $\| \nabla v_{\text{init}} \|_{L^q(\Omega)} \leq \varepsilon$, we can see that

$$\| u_\lambda(\cdot, t) - e^{t\Delta} u_{\text{init}} \|_{L^q(\Omega)} \leq C(\varepsilon) \varepsilon \quad \text{for all } t \in (0, T_\lambda) \text{ and all } \lambda > 0,$$

where $C(\varepsilon) > 0$ is a constant such that $C(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$. Thus by choosing $\varepsilon > 0$ satisfying $C(\varepsilon) < 1$, we can obtain the $L^q$-estimate for $u_\lambda$ uniformly in time and $\lambda$, which with the standard $L^p$-$L^q$ estimate for the Neumann heat semigroup on bounded domains implies the desired estimate for $u_\lambda$. This strategy enables us to pass to the fast signal diffusion limit; however, it also lets us assume that $\| u_{\text{init}} \|_{L^p(\Omega)}$ and $\| \nabla v_{\text{init}} \|_{L^q(\Omega)}$ are small with some $p > \frac{n}{2}$ and $q > n$ in Theorem 1.1. On the other hand, in the 2-dimensional setting, by using a combination of an argument in the proof of [24, Theorem 1.1] and a compactness method we can show fast signal diffusion limit under the smallness conditions for the initial data in optimal spaces.

This paper is organized as follows. In Section 2 we collect basic facts which will be used later. In Section 3 we prove global existence and uniform-in-$\lambda$ boundedness in (1.1); we divide the section into Sections 3.1 and 3.2 according to the higher-dimensional setting and the 2-dimensional setting, respectively. Section 4 is devoted to the proofs of the main results according to arguments in [33]; we show convergence of the solution $(u_\lambda, v_\lambda)$ for (1.1) as $\lambda \searrow 0$ by using the uniform-in-$\lambda$ estimate established in Section 3.

2. Preliminaries

In this section we collect results which will be used later. We first recall the well-known result concerned with local existence of solutions to (1.1) (see e.g., [1, Lemma 3.1]).

**Lemma 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary, and let $\chi > 0$ be a constant. Then for all $\lambda > 0$ and any $u_{\text{init}}, v_{\text{init}}$ satisfying (1.3) there exists $T_{\text{max}, \lambda} \in (0, \infty]$ such that the problem (1.1) possesses a unique solution $(u_\lambda, v_\lambda)$ fulfilling

$$u_\lambda \in C(\overline{\Omega} \times [0, T_{\text{max}, \lambda})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}, \lambda})),$$

$$v_\lambda \in C(\overline{\Omega} \times [0, T_{\text{max}, \lambda})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}, \lambda})) \cap L^\infty([0, T_{\text{max}, \lambda}); W^{1,q}(\Omega)),$$

$$u_\lambda(x, t) \geq 0 \quad \text{and} \quad \varepsilon(x, t) \geq 0 \quad \text{for all } x \in \Omega, \ t > 0 \text{ and all } \lambda > 0,$$

$$\int_{\Omega} u_\lambda(\cdot, t) = \int_{\Omega} u_{\text{init}} \quad \text{for all } t \in (0, T_{\text{max}, \lambda}) \text{ and all } \lambda > 0.$$

Moreover, either $T_{\text{max}, \lambda} = \infty$ or

$$\limsup_{t \to T_{\text{max}, \lambda}} (\| u_\lambda(\cdot, t) \|_{L^\infty(\Omega)} + \| v_\lambda(\cdot, t) \|_{W^{1,q}(\Omega)}) = \infty.$$
We next introduced the $L^p$-$L^q$ estimates for the Neumann heat semigroup on bounded domains which are often utilized to estimate terms coming from the variation-of-constants representation for the solutions. The following lemma and its proof can be found in [3, Lemma 2.1] (or see [34, Lemma 1.3]).

**Lemma 2.2.** Let $(e^{t\Delta})_{t \geq 0}$ be the Neumann heat semigroup in $\Omega$, and let $\alpha > 0$ denote the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under Neumann boundary conditions. There are $k_1, k_2, k_3, k_4 > 0$ only depending on $|\Omega|$ satisfying the following properties.

(i) If $1 \leq q \leq p \leq \infty$, then
\[
\|e^{t\Delta}\varphi\|_{L^p(\Omega)} \leq k_1 (1 + t^{\frac{n}{2} - \frac{1}{p}}) e^{-\alpha t} \|\varphi\|_{L^q(\Omega)} \quad \text{for all } t > 0
\]
holds for all $\varphi \in L^q(\Omega)$ satisfying $\int_{\Omega} \varphi = 0$.

(ii) If $1 \leq q \leq p \leq \infty$, then
\[
\|\nabla e^{t\Delta}\varphi\|_{L^p(\Omega)} \leq k_2 (1 + t^{-\frac{1}{2} - \frac{1}{q} (\frac{1}{q} - \frac{1}{p})}) e^{-\alpha t} \|\varphi\|_{L^q(\Omega)} \quad \text{for all } t > 0
\]
is true for all $\varphi \in L^q(\Omega)$.

(iii) If $2 \leq q \leq p \leq \infty$, then
\[
\|\nabla e^{t\Delta}\varphi\|_{L^p(\Omega)} \leq k_3 (1 + t^{-\frac{1}{2} - \frac{1}{q} (\frac{1}{q} - \frac{1}{p})}) e^{-\alpha t} \|\nabla \varphi\|_{L^q(\Omega)} \quad \text{for all } t > 0
\]
is valid for all $\varphi \in W^{1,p}(\Omega)$.

(iv) If $1 < q \leq p \leq \infty$, then
\[
\|e^{t\Delta}\nabla \cdot \varphi\|_{L^p(\Omega)} \leq k_4 (1 + t^{-\frac{1}{2} - \frac{1}{q} (\frac{1}{q} - \frac{1}{p})}) e^{-\alpha t} \|\varphi\|_{L^q(\Omega)} \quad \text{for all } t > 0
\]
holds for all $\varphi \in (W^{1,p}(\Omega))^n$.

We finally give the following result which plays an important role in obtaining uniform-in-$\lambda$ boundedness of solutions to (1.1).

**Lemma 2.3.** Let $\lambda > 0$. If there exist $p > \frac{n}{2}$ and $M > 0$ such that
\[
\|u_\lambda(\cdot, t)\|_{L^p(\Omega)} \leq M \quad \text{for all } t \in (0, T_{\text{max}, \lambda}),
\]
then there exists $C = C(p, M) > 0$ such that
\[
\|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\lambda(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}, \lambda}).
\]
Moreover, if $p$ and $M$ are independent of $\lambda \in (0, \lambda_0)$ with some $\lambda_0 > 0$, then $C$ is also independent of $\lambda \in (0, \lambda_0)$.

**Proof.** The proof is a combination of [18, Lemmas 2.3 and 2.4] (the proof is based on an application of the $L^p$-$L^q$ estimates for the Neumann heat semigroup in the proof of [1, Lemma 3.2]).
3. Uniform-in-$\lambda$ boundedness

In this section we establish global existence of solutions to (1.1) and their uniform-in-$\lambda$ boundedness.

3.1. The higher-dimensional setting

In this subsection we will deal with the higher-dimensional Keller–Segel system (1.1). Aided by Lemma 2.3, we shall only verify the $L^{p_0}$-estimate for $u_\lambda$ with some $p_0 > \frac{n}{2}$.

We first prove the following lemma which enables us to pick appropriate constants in the proof of Lemma 3.2.

**Lemma 3.1.** Let $p > \frac{n}{2}$ and $q > n$. Then there are constants $\theta, q_0, \mu > 0$ such that

\[
\theta \in I_1 := \left( p, \min \left\{ \frac{npq}{(np + nq - pq)_+}, \frac{np}{2(n - p)_+} \right\} \right), \\
q_0 \in I_2 := \left( \max \left\{ 1, \frac{np\theta}{p\theta + np - n\theta} \right\}, \min \left\{ q, \frac{np}{(n - p)_+} \right\} \right), \\
\mu \in I_3 := \left( \max \left\{ 1, \frac{n\theta}{n + \theta}, \frac{np\theta}{p\theta + np - n\theta} \right\}, \min \left\{ q, \frac{q_0\theta}{q_0 + \theta} \right\} \right).
\]

**Proof.** Since we have from the conditions $q > n$ and $p > \frac{n}{2}$ that

\[(np + nq - pq)p < npq \quad \text{and} \quad 2(n - p) < n,
\]

we can verify that

\[I_1 = \left( p, \min \left\{ \frac{npq}{(np + nq - pq)_+}, \frac{np}{2(n - p)_+} \right\} \right) \neq \emptyset.
\]

Thus we can take $\theta \in I_1$. We next see that

\[I_2 = \left( \max \left\{ 1, \frac{np\theta}{p\theta + np - n\theta} \right\}, \min \left\{ q, \frac{np}{(n - p)_+} \right\} \right) \neq \emptyset.
\]

Noticing from the fact $\theta \in I_1 \subset (p, \frac{np}{(n - p)_+})$ that

\[p\theta + np - n\theta > 0 \quad \text{and} \quad \frac{np\theta}{p\theta + np - n\theta} \geq 1,
\]

we will only confirm that

\[
\frac{np\theta}{p\theta + np - n\theta} < \min \left\{ q, \frac{np}{(n - p)_+} \right\} \quad \text{(3.1)}
\]

holds. Here since the fact $\theta < \min\left\{ \frac{npq}{(np + nq - pq)_+}, \frac{np}{2(n - p)_+} \right\}$ implies

\[np\theta < q(p\theta + np - n\theta) \quad \text{and} \quad \theta(n - p) < p\theta + np - n\theta,
\]
we can verify that (3.1) is true, which tells us that \( I_2 \neq \emptyset \). Therefore we can choose \( q_0 \in I_2 \). We finally confirm that

\[
I_3 = \left( \max \left\{ 1, \frac{n\theta}{n + \theta}, \frac{np\theta}{p\theta + 2np - n\theta} \right\}, \min \left\{ q_0, \frac{q_0\theta}{q_0 + \theta} \right\} \right) \neq \emptyset.
\]

Here we note that \( p\theta + 2np - n\theta > p\theta + np - n\theta > 0 \) and \( \frac{q_0\theta}{q_0 + \theta} < q_0 \). Since the facts \( \theta > p > \frac{n}{2} \geq \frac{n}{n+1} \) \((n \geq 3)\) and \( p\theta + 2np - n\theta < p(n + \theta) \) derive that

\[
1 \leq \frac{n\theta}{n + \theta} < \frac{np\theta}{p\theta + 2np - n\theta}.
\]

we shall only see that

\[
\frac{np\theta}{p\theta + 2np - n\theta} < \frac{q_0\theta}{q_0 + \theta},
\]

(3.2)

Now aided by the relation \( \frac{np\theta}{p\theta + np - n\theta} < q_0 \), we establish that

\[
np(\theta + q_0) < q_0(p\theta + 2np - n\theta)
\]

holds. Therefore we have (3.2), which enables us to find a constant \( \mu \in I_3 \). This completes the proof. \( \Box \)

Then we can show the following lemma which entails the desired estimate for \( u_\lambda \).

**Lemma 3.2.** Let \( p > \frac{n}{2} \). Then there exists a positive constant \( \varepsilon_0 = \varepsilon_0(p, q, \chi, |\Omega|) \) such that, if \( u_{\text{init}} \) and \( v_{\text{init}} \) satisfy

\[
\| u_{\text{init}} \|_{L^p(\Omega)} < \varepsilon_0 \quad \text{and} \quad \| \nabla v_{\text{init}} \|_{L^q(\Omega)} < \varepsilon_0,
\]

then there exist \( p_0 > \frac{n}{2} \) and \( C > 0 \) which are independent of \( \lambda > 0 \) such that

\[
\| u_\lambda(\cdot, t) \|_{L^{p_0}(\Omega)} \leq C
\]

for all \( t \in (0, T_{\text{max}, \lambda}) \) and all \( \lambda > 0 \).

**Proof.** Let \( \varepsilon > 0 \) be a constant fixed later, and assume that \( u_{\text{init}} \) and \( v_{\text{init}} \) satisfy

\[
\| u_{\text{init}} \|_{L^p(\Omega)} \leq \varepsilon \quad \text{and} \quad \| \nabla v_{\text{init}} \|_{L^q(\Omega)} \leq \varepsilon \quad (3.3)
\]

with some \( p > \frac{n}{2} \). Then invoking to Lemma 3.1, we can take \( \theta, q_0, \mu \geq 1 \) such that

\[
\theta \in I_1, \quad q_0 \in I_2 \quad \text{and} \quad \mu \in I_3,
\]

where \( I_1, I_2, I_3 \) are intervals defined in Lemma 3.1. Now we put

\[
T_\lambda := \sup \left\{ \hat{T} \in (0, T_{\text{max}, \lambda}) \mid \| u_\lambda(\cdot, t) - e^{t\Delta}u_{\text{init}} \|_{L^p(\Omega)} < \varepsilon \quad \text{for all} \ t \in (0, \hat{T}) \right\}.
\]
Then because $u_{\lambda}(\cdot, 0) - e^{0,\Delta} u_{\text{init}} = 0$ and the function $t \mapsto u_{\lambda}(\cdot, t) - e^{t,\Delta} u_{\text{init}}$ is continuous on $[0, T_{\text{max}, \lambda})$, $T_{\lambda}$ is well-defined and positive with $T_{\lambda} \leq T_{\text{max}, \lambda}$. We first note from the standard $L^p$-$L^q$ estimate for the Neumann heat semigroup that there is $C_1 = C_1(|\Omega|) > 0$ such that for all $r \in [1, \theta)$,

$$
\|u_{\lambda}(\cdot, t)\|_{L^r(\Omega)} \leq \|u_{\lambda}(\cdot, t) - e^{t,\Delta} u_{\text{init}}\|_{L^r(\Omega)} + \|e^{t,\Delta} u_{\text{init}}\|_{L^r(\Omega)} \\
\leq |\Omega|^\frac{1}{p} \|u_{\lambda}(\cdot, t) - e^{t,\Delta} u_{\text{init}}\|_{L^p(\Omega)} + C_1 \|u_{\text{init}}\|_{L^r(\Omega)} \\
\leq |\Omega|^\frac{1}{p} \frac{1}{\theta} \varepsilon + C_1 \|u_{\text{init}}\|_{L^r(\Omega)} \tag{3.4}
$$

for all $t \in (0, T_{\lambda})$, which with the relation $p < \theta$ and (3.3) tells us that

$$
\|u_{\lambda}(\cdot, t)\|_{L^p(\Omega)} \leq (|\Omega|^\frac{1}{p} \frac{1}{\theta} + C_1) \varepsilon 
$$

for all $t \in (0, T_{\lambda})$. We then obtain from the variation-of-constants representation for $v_{\lambda}$, the fact $q_0 < q$ and Lemma 2.2 (ii), (iii) that

$$
\|\nabla v_{\lambda}(\cdot, t)\|_{L^q(\Omega)} \leq \|\nabla e^{\frac{\Delta}{\lambda}(\cdot - 1)} v_{\text{init}}\|_{L^q(\Omega)} + \frac{1}{\lambda} \int_0^t \|\nabla e^{\frac{\Delta}{\lambda}(\cdot - 1)} u_{\lambda}(\cdot, s)\|_{L^q(\Omega)} ds \\
\leq C_2 \|\nabla v_{\text{init}}\|_{L^q(\Omega)} + \frac{C_3 \varepsilon}{\lambda} \int_0^t \left(1 + \left(\frac{t-s}{\lambda}\right)^{-\frac{1}{2} - \frac{p}{2}(\frac{1}{p} - \frac{1}{q_0})}\right) e^{-\alpha s(\frac{s}{\lambda})} ds \\
\leq C_2 |\Omega|^\frac{1}{q_0} \frac{1}{\theta} \|\nabla v_{\text{init}}\|_{L^q(\Omega)} + C_3 \varepsilon \int_0^t \left(1 + \sigma^{-\frac{1}{2} - \frac{p}{2}(\frac{1}{p} - \frac{1}{q_0})}\right) e^{-\alpha \sigma} d\sigma 
$$

for all $t \in (0, T_{\lambda})$ with some $C_2 = C_2(|\Omega|) > 0$ and $C_3 = C_3(p, q, |\Omega|) > 0$. Since the fact $q_0 < \frac{np}{(n-p)_+}$ implies $\frac{1}{2} + \frac{n}{2}(\frac{1}{p} - \frac{1}{q_0}) < 1$, from (3.3) we infer that

$$
C_2 |\Omega|^\frac{1}{q_0} \frac{1}{\theta} \|\nabla v_{\text{init}}\|_{L^q(\Omega)} + C_3 \varepsilon \int_0^t \left(1 + \sigma^{-\frac{1}{2} - \frac{p}{2}(\frac{1}{p} - \frac{1}{q_0})}\right) e^{-\alpha \sigma} d\sigma \leq C_4 \varepsilon 
$$

holds with $C_4 := C_2 |\Omega|^\frac{1}{q_0} \frac{1}{\theta} + C_3 \int_0^\infty (1 + \sigma^{-\frac{1}{2} - \frac{p}{2}(\frac{1}{p} - \frac{1}{q_0})}) e^{-\alpha \sigma} d\sigma < \infty$, which means that

$$
\|\nabla v_{\lambda}(\cdot, t)\|_{L^q(\Omega)} \leq C_4 \varepsilon \tag{3.5}
$$

for all $t \in (0, T_{\lambda})$. Finally, in order to show $T_{\lambda} = T_{\text{max, \lambda}}$, we will show that

$$
\|u_{\lambda}(\cdot, t) - e^{t,\Delta} u_{\text{init}}\|_{L^p(\Omega)} \leq C \varepsilon 
$$

for all $t \in (0, T_{\lambda})$ with some $C < 1$. Employing the variation-of-constant formula for $u_{\lambda}$ and Lemma 2.2 (iv), we see that

$$
\|u_{\lambda}(\cdot, t) - e^{t,\Delta} u_{\text{init}}\|_{L^p(\Omega)} \leq \chi \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{p}{2}(\frac{1}{p} - \frac{1}{q_0})}) e^{-\alpha (t-s)} \|u_{\lambda}(\cdot, s) \nabla v_{\lambda}(\cdot, s)\|_{L^p(\Omega)} ds 
$$

\[\tag{3.6}\]
Now thanks to the facts \( \mu < q_0, 1 < \frac{q_0 \mu}{q_0 - \mu} < \theta \) and (3.4)–(3.5), we derive from the Hölder inequality and the interpolation inequality that
\[
\| u_\lambda(\cdot, s) \nabla v_\lambda(\cdot, s) \|_{L^p(\Omega)} \\
\quad \leq \| u_\lambda(\cdot, s) \|_{L^{q_0}(\Omega)}^{\frac{q_0 \mu}{q_0 - \mu}} \| \nabla v_\lambda(\cdot, s) \|_{L^{q_0}(\Omega)} \\
\quad \leq \| u_\lambda(\cdot, s) \|_{L^1(\Omega)} \| u_\lambda(\cdot, s) \|_{L^{p}(\Omega)}^{1-a} \| \nabla v_\lambda(\cdot, s) \|_{L^{0}(\Omega)} \\
\quad \leq C_5\varepsilon^{1+a} \left( \| u_\lambda(\cdot, s) - e^{s\Delta} u_{\text{init}} \|_{L^{p}(\Omega)}^{1-a} + \| e^{s\Delta} (u_{\text{init}} - \overline{u}_{\text{init}}) \|_{L^{p}(\Omega)}^{1-a} + \| e^{s\Delta} \overline{u}_{\text{init}} \|_{L^{p}(\Omega)}^{1-a} \right) \tag{3.7}
\]
with some \( C_5 = C_5(p, q, |\Omega|) > 0 \), where \( a = \frac{(q_0 - \mu)\theta - q_0 \mu}{q_0 \mu (\theta - 1)} \in (0, 1) \) and \( \overline{u}_{\text{init}} := \frac{1}{|\Omega|} \int_{\Omega} u_{\text{init}} \).
Here from the Hölder inequality, the Young inequality and Lemma 2.2 (i) we can find \( C_6 = C_6(p, q, |\Omega|) > 0 \) and \( C_7 = C_7(p, q, |\Omega|) > 0 \) such that
\[
\| e^{s\Delta} \overline{u}_{\text{init}} \|_{L^{p}(\Omega)}^{1-a} = \| \overline{u}_{\text{init}} \|_{L^{p}(\Omega)}^{1-a} \leq C_6 \| u_{\text{init}} \|_{L^{p}(\Omega)}^{1-a} \leq C_6\varepsilon^{1-a} \tag{3.8}
\]
and
\[
\| e^{s\Delta} (u_{\text{init}} - \overline{u}_{\text{init}}) \|_{L^{p}(\Omega)}^{1-a} \leq (1 - a) \| e^{s\Delta} (u_{\text{init}} - \overline{u}_{\text{init}}) \|_{L^{p}(\Omega)} + a \\
\quad \leq C_7(1 + s^{-\frac{n}{p} - \frac{1}{\theta}}) e^{-as} \| u_{\text{init}} - \overline{u}_{\text{init}} \|_{L^{p}(\Omega)} + a \\
\quad \leq C_7(1 + \|\Omega\|^{-1+\frac{1}{\theta}}) e(1 + s^{-\frac{n}{p} - \frac{1}{\theta}}) e^{-as} + a. \tag{3.9}
\]
Since the fact \( \frac{1}{2} + \frac{n}{2} (\frac{1}{\mu} - \frac{1}{\theta}) < 1 \) leads to \( \int_0^\infty (1 + \sigma^{-\frac{1}{2} - \frac{n}{p} (\frac{1}{\mu} - \frac{1}{\theta})) e^{-\sigma} \) \( d\sigma < \infty \) and the relation
\[
1 - \frac{1}{2} - \frac{n}{2} (\frac{1}{\mu} - \frac{1}{\theta}) - \frac{n}{2} (\frac{1}{p} - \frac{1}{\theta}) > 0
\]
leads to \( (1 + \frac{1}{\theta} - \frac{n}{2} (\frac{1}{\mu} - \frac{1}{\theta})) e^{-\sigma} \) \( d\sigma \), which means that
\[
\int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{n}{2} (\frac{1}{\mu} - \frac{1}{\theta})}) e^{-a(t-s)} (1 + s^{-\frac{n}{2} (\frac{1}{\mu} - \frac{1}{\theta})}) e^{-as} \, ds \\
\quad \leq C_8(1 + t^{\min\{0,1+\frac{1}{2} - \frac{n}{2} (\frac{1}{\mu} - \frac{1}{\theta}),\frac{1}{2} - \frac{n}{2} (\frac{1}{\mu} - \frac{1}{\theta})\}}) e^{-at} \leq 2C_8
\]
for all \( t > 0 \) with some \( C_8 = C_8(p, q) > 0 \). Thus if we take \( \varepsilon > 0 \) satisfying
\[
C_9\varepsilon a(2\varepsilon^{-1-a} + 1) < 1,
\]
then the continuity of the function \( t \mapsto \| u_\lambda(\cdot, t) - e^{t\Delta} u_{\text{init}} \|_{L^p(\Omega)} \) concludes that
\[
T_\lambda = T_{\max, \lambda},
\]
which namely means that
\[
\| u_\lambda(\cdot, t) - e^{t\Delta} u_{\text{init}} \|_{L^p(\Omega)} \leq \varepsilon \tag{3.10}
\]
for all $t \in (0, T_{\text{max}, \lambda})$. Here, since $\varepsilon > 0$ is independent of $\lambda > 0$, we note that (3.10) holds for all $t \in (0, T_{\text{max}, \lambda})$ and all $\lambda > 0$, which together with the maximum principle
\[
\|\epsilon^t u_{\text{init}}\|_{L^\infty(\Omega)} \leq \|u_{\text{init}}\|_{L^\infty(\Omega)} \quad \text{for all } t > 0
\]
enables us to see that
\[
\|u_\lambda(\cdot, t)\|_{L^\theta(\Omega)} \leq \varepsilon + |\Omega|^{\frac{1}{\theta}} \|u_{\text{init}}\|_{L^\infty(\Omega)} \quad (3.11)
\]
for all $t \in (0, T_{\text{max}, \lambda})$ and all $\lambda > 0$. Noticing that $\theta > p > \frac{n}{2}$ holds and $\theta, \varepsilon$ are independent of $\lambda$, from (3.11) we can attain the goal of the proof. \hfill \Box

Here we are in the position to prove global existence and uniform-in-$\lambda$ boundedness in the higher-dimensional Keller–Segel system (1.1).

**Lemma 3.3.** Let $p > \frac{n}{2}$. Assume that $u_{\text{init}}$ and $v_{\text{init}}$ satisfy
\[
\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_0 \quad \text{and} \quad \|\nabla v_{\text{init}}\|_{L^q(\Omega)} < \varepsilon_0,
\]
where $\varepsilon_0$ is the constant defined in Lemma 3.2. Then $T_{\text{max}, \lambda} = \infty$ holds, and there exists $C > 0$ independent of $\lambda > 0$ such that
\[
\|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} \leq C
\]
for all $t \in (0, \infty)$ and all $\lambda > 0$.

**Proof.** A combination of Lemmas 2.3 and 3.2, along with the extensibility criterion directly leads to this lemma. \hfill \Box

### 3.2. The 2-dimensional setting

In this subsection we will show uniform-in-$\lambda$ boundedness in the 2-dimensional Keller–Segel system. The proof is mainly based on arguments in the proof of [24, Theorem 1.1]. Thus we will only give short proofs.

**Lemma 3.4.** Assume that $u_{\text{init}}$ satisfies
\[
\|u_{\text{init}}\|_{L^1(\Omega)} < \frac{4\pi}{\lambda}.
\]
Then for all $\lambda_0 > 0$ there exists $C > 0$ such that
\[
\|u_\lambda(\cdot, t)\|_{L^2(\Omega)} \leq C
\]
for all $t \in (0, T_{\text{max}, \lambda})$ and all $\lambda \in (0, \lambda_0)$.

**Proof.** Let $\lambda_0 > 0$ be an arbitrary constant. From straightforward calculations we can verify that the function
\[
W_\lambda := \int_\Omega \left( u_\lambda \log u_\lambda - \lambda u_\lambda v_\lambda + \frac{\lambda}{2} (|\nabla v_\lambda|^2 + v_\lambda^2) \right)
\]
satisfies
\[
\frac{dW_\lambda}{dt} + \chi \lambda \int_\Omega |(v_\lambda)_t|^2 + \int_\Omega u_\lambda \nabla \cdot (\log u_\lambda - \chi v_\lambda) = 0
\]  
(3.12)
for all \( \lambda \in (0, \lambda_0) \). Then by virtue of the Jensen inequality and the Trudinger–Moser inequality, the same argument as in the proof of [24, Lemma 3.4] derives that there is \( C_1 > 0 \) such that
\[
\int_\Omega u_\lambda(\cdot, t)v_\lambda(\cdot, t) \leq C_1 \quad \text{and} \quad |W_\lambda(t)| \leq C_1
\]
(3.13)
for all \( t \in (0, T_{\max,\lambda}) \) and \( \lambda \in (0, \lambda_0) \) under the condition that \( \|u_{\text{init}}\|_{L^1(\Omega)} < \frac{4\pi}{\chi} \). Thanks to (3.13), the relation (3.12) implies that
\[
\int \left| u_\lambda(\cdot, t) \log u_\lambda(\cdot, t) \right| \leq \max \left\{ W_\lambda(0) + C_1, \frac{1}{e} \right\}
\]  
(3.14)
and
\[
\lambda \int_0^t \|(v_\lambda)_t(\cdot, s)\|_{L^2(\Omega)}^2 \, ds \leq \frac{1}{\chi} (|W_\lambda(0)| + C_1)
\]  
(3.15)
for all \( t \in (0, T_{\max,\lambda}) \) and \( \lambda \in (0, \lambda_0) \). Now we shall show the \( L^2 \)-boundedness of \( u_\lambda \). Multiplying the first equation in (1.1) by \( \frac{1}{2} u_\lambda \) and integrating it over \( \Omega \), we infer from integration by parts that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u_\lambda^2 = - \int_\Omega |\nabla u_\lambda|^2 + \chi \int_\Omega u_\lambda \nabla u_\lambda \cdot \nabla v_\lambda
\]
\[
= - \int_\Omega |\nabla u_\lambda|^2 - \frac{\chi \lambda}{2} \int_\Omega u_\lambda^2(v_\lambda)_t + \frac{\chi}{2} \int_\Omega u_\lambda^3 - \frac{\chi}{2} \int_\Omega u_\lambda^2 v_\lambda.
\]  
(3.16)
Let \( \varepsilon > 0 \) be a constant fixed later. Since the Gagliardo–Nirenberg inequality and its application (see [24, Lemma 3.5]) derive
\[
\frac{\chi}{2} \int_\Omega u_\lambda^3 \leq \varepsilon \|u_\lambda\|_{L^2(\Omega)}^2 \|u_\lambda \log u_\lambda\|_{L^1(\Omega)} + C_2 (\|u_\lambda \log u_\lambda\|_{L^1(\Omega)}^3 + \|u_\lambda\|_{L^1(\Omega)}^2)
\]
and
\[
-\frac{\chi \lambda}{2} \int_\Omega u_\lambda^2(v_\lambda)_t \leq C_3 \lambda \|(v_\lambda)_t\|_{L^2(\Omega)}^2 \left( \|\nabla u_\lambda\|_{L^2(\Omega)} \|u_\lambda\|_{L^2(\Omega)} + \|u_\lambda\|_{L^2(\Omega)}^2 \right)
\]
\[
\leq \varepsilon \|\nabla u_\lambda\|_{L^2(\Omega)}^2 + \left( C_4 \lambda^2 \|(v_\lambda)_t\|_{L^2(\Omega)}^2 + \frac{1}{4} \right) \|u_\lambda\|_{L^2(\Omega)}^2
\]
with some \( C_2, C_3, C_4 > 0 \), the relation (3.16) with the nonnegativity of \( v_\lambda \) tells us that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u_\lambda^2 + (1 - \varepsilon - \varepsilon \|u_\lambda \log u_\lambda\|_{L^1(\Omega)}) \int_\Omega |\nabla u_\lambda|^2
\]
\[
\leq \left( C_4 \lambda^2 \|(v_\lambda)_t\|_{L^2(\Omega)}^2 + \frac{1}{4} \right) \int_\Omega u_\lambda^2 + C_2 (\|u_\lambda \log u_\lambda\|_{L^1(\Omega)}^3 + \|u_\lambda\|_{L^1(\Omega)}^2).
\]
Noticing from the boundedness of \( \| u_\lambda \log u_\lambda \|_{L^1(\Omega)} \) (from (3.14)) that there is \( \varepsilon > 0 \) such that
\[
1 - \varepsilon - \varepsilon \| u_\lambda (\cdot, t) \log u_\lambda (\cdot, t) \|_{L^1(\Omega)} \geq \frac{1}{2}
\]
for all \( t \in (0, T_{\max, \lambda}) \) and \( \lambda \in (0, \lambda_0) \), we infer from the application of the Gagliardo–Nirenberg inequality
\[
\| u_\lambda \|_{L^2(\Omega)} \leq \| \nabla u_\lambda \|_{L^2(\Omega)}^2 + C_5 \| u_\lambda \|_{L^1(\Omega)}^2
\]
with some \( C_5 > 0 \) that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u_\lambda^2 + \frac{1}{2} \left( \frac{1}{2} - C_4 \lambda^2 \| (v_\lambda)_t \|_{L^2(\Omega)}^2 \right) \int_{\Omega} u_\lambda^2 \leq C_6 (\| u_\lambda \log u_\lambda \|_{L^3(\Omega)}^3 + \| u_\lambda \|_{L^1(\Omega)}^2)
\]
\[
\leq \frac{1}{2} L
\]
with some \( C_6 > 0 \) and \( L > 0 \). Now we put
\[
y(t) := \int_{\Omega} u_\lambda^2 (\cdot, t) \quad \text{and} \quad \phi(t) := \frac{1}{2} t - \frac{C_4 \lambda^2}{2} \int_0^t \| (v_\lambda)_t (\cdot, s) \|_{L^2(\Omega)}^2 \, ds.
\]
Then from the differential inequality (3.17), we establish that
\[
y(t) \leq y(0) e^{-\phi(t)} + L e^{-\phi(t)} \int_0^t e^{\phi(s)} \, ds \quad \text{for all } t \in (0, T_{\max, \lambda}) \text{ and } \lambda \in (0, \lambda_0).
\]
Thus the boundedness of \( \phi(t) \)
\[
\frac{1}{2} t - \frac{C_4 \lambda_0}{2 \chi} (|W_\lambda(0)| + C_1) \leq \phi(t) \leq \frac{1}{2} t \quad (t \in (0, T_{\max, \lambda}), \lambda \in (0, \lambda_0))
\]
(from (3.15)) entails that there is \( C_7 = C_7(\lambda_0) > 0 \) such that
\[
y(t) \leq y(0) e^{-\phi(t)} + L e^{-\phi(t)} \int_0^t e^{\phi(s)} \, ds \leq C_7
\]
for all \( t > 0 \) and \( \lambda \in (0, \lambda_0) \), which means the end of the proof.

Thanks to Lemma 3.4, we attain global existence and uniform-in-\( \lambda \in (0, \lambda_0) \) boundedness of the solution \((u_\lambda, v_\lambda)\) to the 2-dimensional Keller–Segel system.

**Lemma 3.5.** Assume that \( u_{\text{init}} \) satisfies
\[
\| u_{\text{init}} \|_{L^1(\Omega)} < \frac{4\pi}{\chi}.
\]
Then \( T_{\max, \lambda} = \infty \) holds, and for all \( \lambda_0 > 0 \) there exists \( C > 0 \) such that
\[
\| u_\lambda (\cdot, t) \|_{L^\infty(\Omega)} \leq C
\]
for all \( t \in (0, \infty) \) and all \( \lambda \in (0, \lambda_0) \).

**Proof.** A combination of Lemmas 2.3 and 3.4, along with the extensibility criterion leads to this lemma.
4. Convergence

In this section we will show that solutions of (1.1) converge to those of (1.2). Here we assume that there exists a unique global classical solution \((u_\lambda, v_\lambda)\) of (1.1) such that for all \(\lambda_0 > 0\) there is \(C > 0\) independent of \(\lambda \in (0, \lambda_0)\) such that,

\[
\|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\lambda(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C
\]

for all \(t > 0\) and all \(\lambda \in (0, \lambda_0)\), which is established by Lemmas 3.3 and 3.5. Arguments in this section are based on those in the proof of [33, Theorem 1.1]; thus I shall only show brief proofs. We first confirm the following lemma which is a cornerstone of this work.

**Lemma 4.1.** For all sequences of numbers \(\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \lambda_0)\) satisfying \(\lambda_n \searrow 0\) as \(n \to \infty\) there exist a subsequence \(\lambda_{n_j} \searrow 0\) and functions

\[
\begin{align*}
u_{\lambda_{n_j}} &\to u \quad \text{in } C_{\text{loc}}^\alpha(\overline{\Omega} \times [0, \infty)), \\
v_{\lambda_{n_j}} &\rightharpoonup v \quad \text{in } L^\infty(0, \infty; W^{1,2}_{\text{loc}}(\Omega))
\end{align*}
\]

such that as \(j \to \infty\). Moreover, \((u, v)\) solves (1.2) classically.

**Remark 4.1.** This lemma also gives that global existence and boundedness in (1.2) hold under the condition that there is a unique global bounded solution in (1.1) which is bounded uniformly in \(\lambda \in (0, \lambda_0)\).

**Proof.** From the assumption in this section and the standard parabolic regularity argument [27, Theorem 1.3] we see that \(\{u_\lambda\}_{\lambda \in (0, \lambda_0)}\) is bounded in \(C^{\alpha,\frac{\alpha}{2}}_{\text{loc}}(\overline{\Omega} \times [0, \infty))\) with some \(\alpha \in (0, 1)\). Thus the Arzelà–Ascoli theorem and the boundedness of \(\|\nabla v_\lambda\|_{L^\infty(0, \infty; W^{1,q}(\Omega))}\) yields that we can find a subsequence \(\lambda_{n_j} \searrow 0\) and functions

\[
\begin{align*}
u_{\lambda_{n_j}} &\to u \quad \text{in } C_{\text{loc}}^\alpha(\overline{\Omega} \times [0, \infty)), \\
v_{\lambda_{n_j}} &\rightharpoonup v \quad \text{in } L^\infty(0, \infty; W^{1,2}(\Omega))
\end{align*}
\]

satisfying

\[
\begin{align*}
u_{\lambda_{n_j}} &\to u \quad \text{in } C_{\text{loc}}^\alpha(\overline{\Omega} \times [0, \infty)) \quad \text{and} \quad v_{\lambda_{n_j}} \rightharpoonup v \quad \text{in } L^\infty(0, \infty; W^{1,q}(\Omega))
\end{align*}
\]

as \(j \to \infty\). Then arguments similar to those in the proof of [33, Theorem 1.1] enable us to attain this lemma.

We next verify the following lemma which implies that the pair of functions \((u, v)\) provided by Lemma 4.1 is independent of a choice of a sequence \(\lambda_n \searrow 0\).

**Lemma 4.2.** A solution \((\overline{u}, \overline{v})\) of (1.2) satisfying

\[
\overline{u} \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{and} \quad \overline{v} \in C^{2,0}(\overline{\Omega} \times (0, \infty)) \cap L^\infty(0, \infty; W^{1,q}(\Omega))
\]

is unique.
Proof. Let \((\overline{u}_1, \overline{v}_1)\) and \((\overline{u}_2, \overline{v}_2)\) be solutions to (1.2) and put \(y(x, t) := \overline{u}_1(x, t) - \overline{u}_2(x, t)\) for \((x, t) \in \Omega \times (0, \infty)\). Then aided by the Gronwall-type argument similar to that in the proof of [32, Lemma 2.1], we infer that \(y(x, t) = 0\), which concludes the proof. □

Finally we shall establish convergence of the solution \((u_\lambda, v_\lambda)\) for (1.1) as \(\lambda \searrow 0\).

**Lemma 4.3.** The solution \((u_\lambda, v_\lambda)\) of (1.1) with \(\lambda \in (0, \lambda_0)\) satisfies that
\[
\begin{align*}
  u_\lambda &\to u \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \\
  v_\lambda &\to v \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\end{align*}
\]
as \(\lambda \searrow 0\), where \((u, v)\) is the solution of (1.2) provided by Lemma 4.1.

Proof. Lemmas 4.1 and 4.2 yield that there exists the pair of the functions \((u, v)\) such that for any sequences \(\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \lambda_0)\) satisfying \(\lambda_n \searrow 0\) as \(n \to \infty\) there is a subsequence \(\lambda_{n_j} \searrow 0\) such that
\[
\begin{align*}
  u_{\lambda_{n_j}} &\to u \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \\
  v_{\lambda_{n_j}} &\to v \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\end{align*}
\]
as \(j \to \infty\), which enables us to see this lemma. □

**Proof of Theorem 1.1.** Lemmas 3.3 and 4.3 directly show Theorem 1.1. □

**Proof of Corollary 1.2.** Put \(\varepsilon_1 = \varepsilon_1(p, \chi, |\Omega|) := \sup_{q \in (n, \infty)} \varepsilon_0(p, q, \chi, |\Omega|)\) and let \(u_{\text{init}}\) satisfy \(\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_1\). Then we can pick \(q > n\) such that \(\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_0(p, q, \chi, |\Omega|)\). Now we choose \(v_{\text{init}} \in W^{1,q}(\Omega)\) satisfying \(\|\nabla v_{\text{init}}\|_{L^q(\Omega)} < \varepsilon_0\). By virtue of Theorem 1.1, we can prove Corollary 1.2. □

**Proof of Theorem 1.3.** From Lemmas 3.5 and 4.3 we can see Theorem 1.3. □

**Proof of Corollary 1.4.** Theorem 1.3 directly leads to Corollary 1.4. □

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