Upper bounds for the probability of unusually small components in critical random graphs

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Abstract

We describe a methodology, mostly based on an estimate for the probability that a (mean zero) $Z$-valued random walk remains below a constant barrier over a finite time interval and Kolmogorov’s maximal inequality, to derive upper bounds for the probability of observing unusually small maximal components in two classical random graphs when considered near criticality. Specifically, we consider the random graph $G(n, d, p)$ obtained by performing $p$-bond percolation on a (simple) random $d$-regular graph, as well as the Erdős-Rényi random graph $G(n, p)$, and show that, near criticality, the probability of observing a largest component containing less than $n^{2/3}/A$ vertices decays as $A^{-\epsilon}$ for some $\epsilon > 0$ in both models. Even though this result is not new, our approach is quite robust since it yields very similar proofs for both models considered here. We also provide a short, random-walk-based proof of the fact that, in the random graph obtained through critical percolation on any $d$-regular graph with $d \geq 3$, the largest component contains less than $A n^{2/3}$ with probability at least $1 - 3 c_d/A^{1/2}$, for some explicit constant $c_d$ that depends on $d$ which, in turn, is allowed to depend on $n$.

Keywords: Random graph, random walk, ballot theorem

1 Introduction

The scope of the present paper is to describe a robust probabilistic argument which yields polynomial upper bounds for the probability of observing unusually small maximal components in two classical models of random graphs when considered near criticality.

Even though the main result is not new, we believe our methodology to be interesting because it highlights a common proof strategy which works for both models under consideration and, to the best of our knowledge, together with the martingale approach of Nachmias and Peres [14] (which we discuss later) it seems to be the only other probabilistic methodology to derive bounds for the probability of small components. Moreover, we believe that the approach we are going to describe has the potential to be applied to study other models of random graphs at criticality.

Before stating the main result and discussing the proof ideas, we recall the definition of the two models considered here.

Let $G(n, p)$ be the Erdős-Rényi random graph, which is obtained by performing $p$-bond percolation on $K_n$, the complete graph on $n$ vertices; that is, each edge of $K_n$ is independently retained with probability $p$ and deleted with probability $1 - p$.

Let $d \geq 3$ be a fixed integer, and let $n \in \mathbb{N}$ be such that $dn$ is even. We let $G(n, d)$ be a (random, simple) $d$-regular graph sampled uniformly at random from the set of all simple $d$-regular graphs on $[n]$, and then denote by $G(n, d, p)$ the random graph obtained by performing $p$-bond percolation on a realisation of $G(n, d)$.

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It is well known (see e.g. the monographs [4], [11] and [12] for more details) that the $G(n, \gamma/n)$ random graph undergoes a fascinating phase transition as $\gamma$ passes 1. Specifically, if $\gamma \leq 1 - \varepsilon$ for some constant $\varepsilon \in (0, 1)$, then $|C_{\text{max}}|$ is of order $\log n$; if $\gamma = 1$ (critical case), then $|C_{\text{max}}|$ is of order $n^{2/3}$; and if $\gamma \geq 1 + \varepsilon$ for some constant $\varepsilon > 0$, then $|C_{\text{max}}|$ is of order $n$.

A similar phenomenon occurs in the $G(n, d, p)$ model. Indeed, Goerdt [10] showed that also the $G(n, d, \gamma \lambda/(d - 1))$ random graph undergoes a phase transition as $\gamma$ passes 1: specifically, $|C_{\text{max}}|$ is of order $\log(n)$ when $\gamma \leq 1 - \varepsilon$, and of order $n$ when $\gamma \geq 1 + \varepsilon$ (another proof was provided by Alon, Benjamini and Stacey [2]).

Nachmias and Peres [14, 15] provided a probabilistic analysis of the two models $G(n, p)$ and $G(n, d, p)$ near criticality. Amongst other results they proved that, in $G(n, 1/n)$, for any $0 < \delta < 1/10$ and $n > 200/\delta^{3/5}$ we have

$$P\left(|C_{\text{max}}| < \lfloor \delta n^{2/3} \rfloor \right) \leq 15\delta^{3/5},$$

while in $G(n, d, (1 + \lambda n^{-1/3})/(d - 1))$, for any $\lambda \in \mathbb{R}$ and $d \geq 3$ there exists a positive constant $D(\lambda, d)$ such that, for $\delta > 0$ small enough and all sufficiently large $n$, then

$$P\left(|C_{\text{max}}| < \lfloor \delta n^{2/3} \rfloor \right) \leq D(\lambda, d)\delta^{1/2}.$$  

Even though the problem of bounding the probability of observing unusually large maximal components by means of probabilistic arguments has received increasing interest during the last decade (see e.g. [7, 8, 9] and references therein), the complementary problem of determining bounds for the probability of observing small maximal clusters by means of probabilistic techniques have received far less attention.

The methodology used in [14] and [15] to prove the bounds [1] and [2] was based on a martingale analysis of the random processes arising from two suitably defined algorithmic procedures to reveal the connected components of the random graphs $G(n, p)$ and $G(n, d, p)$.

Here we illustrate an alternative probabilistic argument, mostly based on an estimate for the probability that a mean zero random walk stays below a constant barrier over a finite (discrete) time interval and Kolmogorov’s maximal inequality, of the fact that $|C_{\text{max}}|$ is unlikely to be much smaller than $n^{2/3}$ in both models $G(n, d, p)$ and $G(n, p)$ when considered near criticality.

More precisely, we start by analysing the $G(n, d, p)$ random graph, for which we describe an exploration process (simpler than the one used in [15]) to reveal the components of $G(n, d, p)$ that allows us to transform the problem of bounding the probability of observing a maximal component containing less than $T \approx n^{2/3}/A$ vertices to the simpler task of bounding the probability that the positive excursions of a random walk never last for more than $T$ steps. Subsequently we control the latter quantity by means of an estimate for the probability that a random walk stays below a constant barrier over a finite (discrete) time interval (see Proposition 3 below) and Kolmogorov’s maximal inequality.

We remark that our approach leads to a simplified analysis compared to [15], where the authors relied on a very detailed exploration process to reveal the components of $G(n, d, p)$ and needed to establish several preliminary estimates, related to the random process arising from such algorithmic procedure, before having at their disposal all the necessary tools to actually prove [2].

Subsequently we use the same methodology developed for the $G(n, d, p)$ model to study the near-critical $G(n, p)$ random graph, showing in particular that the proofs are very similar and hence that our approach is quite robust since it can be easily adapted to study both models.

Even though our main result is not new, we believe the arguments presented here to be interesting because, as we said at the beginning of this section, the martingale analysis of Nachmias and Peres [14, 15] was (to the best of our knowledge) the only probabilistic approach available in the literature to derive upper bounds for the probability of observing
unusually small maximal components. In this regard, our methodology could be an interesting alternative because it yields very similar proofs for the two models considered here, whereas the martingale argument somehow requires computations that are more model-dependent.

**Theorem 1.1.** There exists a constant $n_0 \in \mathbb{N}$ such that, for any $1 \vee (\lambda \wedge 0)^{16} < A = A(n) < n^{2/3}$ and all $n \geq n_0$, the following statements hold. In $\mathbb{G}(n,d,p)$ with $p = (1 + \lambda n^{-1/3})/(d-1)$, $d \geq 3$ fixed and $\lambda \in \mathbb{R}$, we have

$$\mathbb{P}\left( |C_{\text{max}}| < n^{2/3}/A \right) \leq \frac{C_1}{A^{4/9}}$$

for some finite constant $C_1 = C_1(d) > 0$ which depends solely on $d$; similarly, in $\mathbb{G}(n,p)$ where $p = (1 + \lambda n^{-1/3})/n$ and $\lambda \in \mathbb{R}$, we have

$$\mathbb{P}\left( |C_{\text{max}}| < n^{2/3}/A \right) \leq \frac{C_2}{A^{4/9}}$$

for some finite constant $C_2 > 0$.

**Remark 1.2.** We remark that our assumption on $A$, which in particular it is allowed to depend on $n$, is not restrictive. Indeed, if $A \geq n^{2/3}$, then $n^{2/3}/A \leq 1$ and hence we obtain that $\mathbb{P}\left( |C_{\text{max}}| < n^{2/3}/A \right) \leq \mathbb{P}\left( |C_{\text{max}}| = 0 \right) = 0$, because the component of any given vertex $u \in [n]$ contains at least $u$, and hence $|C_{\text{max}}| \geq 1$ by definition. We also note that we allow $A$ in (3) and (4) to depend on $n$, and therefore our result provides information about the whole lower tail of $|C_{\text{max}}|$ in both models. Finally, we remark that in our proofs also $\lambda$ could be allowed to depend on $n$; however, in order to simplify the computations, we refrained to do so.

Next we state a result which says that the largest component in the (random) graph obtained by critical $p$-bond percolation on any $d$-regular graph contains less than $An^{2/3}$ vertices with probability at least $1 - 3c_4 A^{4/9}$, where $c_4$ is an explicit constant which depends on $d \geq 3$. A similar bound was established in [15] for the case $p \leq (d-1)^{-1}$, and a simpler proof was subsequently provided in [8] for the same values of $p$ but for the case $d \geq 4$. We remark that, in our next result, $d$ is allowed to depend on $n$.

**Proposition 1.3.** Let $\mathbb{G}_d$ be any $d$-regular graph, where $3 \leq d = d(n) < n - 1$. Denote by $\mathbb{G}_d(p)$ the (random) graph obtained by $p$-bond percolation on $\mathbb{G}_d$. If $p \leq (d-1)^{-1}$ then, for any $A \geq 1$ and for all large enough $n$, we have

$$\mathbb{P}\left( |C_{\text{max}}(\mathbb{G}_d(p))| > An^{2/3} \right) \leq \frac{3}{A^{4/9}} \sqrt{\frac{d-2}{d-1}} \left( 1 - \frac{1}{d-1} \right)^{-2(d-1)} .$$

**Proof Ideas.** Both proofs of (3) and (4) start by describing an algorithmic procedure to reveal the connected components of the specific (random) graph under investigation and which reduces the study of components sizes to the analysis of the trajectory of a random process arising from such procedure.

However, contrary to Nachmias and Peres [14][15], we do not directly analyse the random processes arising from the algorithm used to reveal the components in the two random graphs, but rather we start both proofs by introducing smaller processes that allow us to perform a simpler random walk analysis.

More precisely, let $T = T(n) \in \mathbb{N}$, and let $\mathbb{G}$ be one of the two random graphs considered here. In order to bound from above the probability

$$\mathbb{P}(\|C_{\text{max}}(\mathbb{G})\| < T) ,$$

we start by describing a procedure which allows us to rewrite (5) as the probability that the positive excursions of a (discrete time) $\mathbb{N}_0$-valued random process never last for more than
for at least $T$ steps, we turn the process into a mean zero random walk (with iid increments) and subsequently, in order to prove that $S_t$ remains positive for at least $T$ steps, we use a simple (random walk) estimate, namely Proposition 2.1 below, which states that the probability that $S_t$ remains below $h$ for $T$ steps is $O(h/\sqrt{T})$ (provided that $h \geq \log(T)$). Furthermore, in order to show that $S_t$ remains positive for $T$ steps once it has reached level $h$, we provide a very short argument based on Kolmogorov’s maximal inequality.

Notation. We write $\mathbb{N} = \{1, 2, \ldots \}$ and define $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$. Given two sequences of real numbers $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ we write: (1) $x_n = O(y_n)$ if there exist $N \in \mathbb{N}$ and $C \in [0, \infty)$ such that $x_n \leq C y_n$ for all $n \geq N$; (2) either $x_n = o(y_n)$ or $x_n \ll y_n$ if $x_n/y_n \to 0$ as $n \to \infty$; and (3) either $x_n = \Theta(y_n)$ or $x_n \asymp y_n$ if $x_n = O(y_n)$ and $y_n = O(x_n)$. Sometimes we write $O_d(\cdot), \Theta_d(\cdot)$ to indicate that the constants involved depend on the specific parameter $d$. The abbreviation iid means independent and identically distributed. We write Ber$(\cdot), \text{Bin}(\cdot, \cdot), \text{Poi}(\cdot)$ and $\text{Nor}(\cdot, \cdot)$ to denote the Bernoulli, binomial, Poisson and normal distributions, respectively. Let $G = (V,E)$ be any (undirected, possibly random) multigraph. Given two vertices $u, v \in V$, we write $u \sim v$ if $\{u, v\} \in E$ and say that vertices $u$ and $v$ are neighbours. We often write $uv$ as shorthand for the edge $\{u, v\}$. We write $u \leftrightarrow v$ if there exists a path of occupied edges connecting vertices $u$ and $v$, where we adopt the convention that $v \leftrightarrow v$ for every $v \in V$. We denote by $\mathcal{C}(v) := \{u \in V : u \leftrightarrow v\}$ the component containing vertex $v \in V$. We define the largest component $\mathcal{C}_{\text{max}}$ to be some cluster $\mathcal{C}(v)$ for which $|\mathcal{C}(v)|$ is maximal, so that $|\mathcal{C}_{\text{max}}| = \max_{v \in V} |\mathcal{C}(v)|$. Sometimes we write $\mathcal{C}_{\text{max}}(G)$ to denote a largest component in $G$.

2 Preliminaries

In this section we state some results that are needed throughout our proofs. More specifically, we start by introducing a few lemmas that are needed to prove the bounds stated in Theorem 1.1, and subsequently we proceed by stating a few facts which we use to prove Proposition 1.3.

2.1 Preliminary results for Theorem 1.1

The most important tool to establish our bounds stated in Theorem 1.1 is the following simple result, which provides an estimate for the probability that a (mean zero) random walk remains below a constant barrier $h > 0$ over a discrete time interval.

Proposition 2.1. Let $(X_i)_{i \geq 1}$ be a sequence of iid, $\mathbb{Z}$-valued random variables such that $\mathbb{E}(X_1) = 0$, $\mathbb{V}(X_1) = \sigma^2$, and $\mathbb{E}(e^{b|X_1|}) < \infty$ for some $b > 0$. Let $N \in \mathbb{N}$ and $h > 0$. Define $S_0 := 0$ and $S_t := \sum_{i=1}^t X_i$ for $t \geq 1$. There exist finite constants $c_1, c_2 > 0$ such that

$$
\Pr(S_t < h \ \forall t \in [N]) \leq c_1 \frac{h}{N^{1/2}} + c_2 \frac{\log(N)}{N^{1/2}}.
$$

To prove Proposition 2.1 we use Theorem 2.2 and Lemma 2.3 below. The first result is a powerful coupling of Brownian motion and random walk for which the paths are very close to each other, while Lemma 2.3 states that it is very unlikely for Brownian motion to be below some level $a > 0$ at discrete times $t - 1, t \in [L]$ and simultaneously to go above $a + x$ at some time $s \in (t - 1, t)$, when $x$ is large.
Theorem 2.2 (Theorem 7.1.1 in [13]). Suppose that \((X_i)_{i \geq 1}\) is a sequence of iid random variables satisfying the hypothesis of Proposition 2.7 and let \(N \in \mathbb{N}\). Then one can define, on the same probability space \((\Omega, \mathcal{H}, P)\), a Brownian motion \((B_s)_{s \geq 0}\) with variance parameter \(\sigma^2\) and a random walk \(S_t\) with increment distribution \(X_1\) such that the following holds. For each \(\gamma < \infty\), there is a finite constant \(c_{i,\gamma} > 0\) such that

\[
P \left( \max_{1 \leq t \leq N} |S_t - B_t| \geq c_{i,\gamma} \log(N) \right) \leq \frac{c_{i,\gamma}}{N^{1/\gamma}}.
\]

We remark that there exist stronger versions of Theorem 2.2 (see e.g. [5] and references therein), but the version stated here suffices for our purposes.

Lemma 2.3. Let \(a, x > 0\) and \(N \in \mathbb{N}\). Let \((B_s)_{s \geq 0}\) be a (standard) Brownian Motion. Then

\[
P \left( \exists t \in [N] : B_{t-1} < a, \max_{s \in (t-1, t]} B_s \geq a + x, B_t < a \right) \leq \frac{N}{\sqrt{8\pi x}} e^{-2x^2}.
\]

Proof. A union bound yields

\[
P \left( \exists t \in [N] : B_{t-1} < a, \max_{s \in (t-1, t]} B_s \geq a + x, B_t < a \right)
\leq \sum_{t \in [N]} \int_{-\infty}^{a} P \left( B_{t-1} \in dy, \max_{s \in (t-1, t]} B_s \geq a + x, B_t < a \right)
\leq \sup_{y \leq a} P_y \left( \max_{s \in (0, 1]} B_s \geq a + x, B_1 < a \right) \sum_{t \in [N]} \int_{-\infty}^{a} P(B_{t-1} \in dy)
\leq N \sup_{y \leq a} P_y \left( \max_{s \in (0, 1]} B_s \geq a + x, B_1 < a \right),
\]

where \(P_y(\cdot)\) denotes the law of a Brownian motion started at \(y\). Now using the fact that

\[
P \left( \max_{0 \leq s \leq t} B_s > \omega, B_t \leq \ell \right) = P(B_t > 2\omega - \ell)
\]

for \(t, \omega > 0\) and \(\ell \leq \omega\), we obtain (taking \(t = 1, \omega = a + x - y\) and \(\ell = a - y\))

\[
P_y \left( \max_{s \in (0, 1]} B_s \geq a + x, B_1 < a \right) = P \left( \max_{s \in (0, 1]} B_s \geq a + x - y, B_1 < a - y \right)
= P(B_1 > 2(a + x - y) - (a - y)) = P(B_1 > a + 2x - y).
\]

Hence the expression on the right-hand side of (6) equals \(N \sup_{y \leq a} P(B_1 > a + 2x - y) = N \mathbb{P}(B_1 > 2x)\). It is well known that, if \(X\) is a random variable with the \(\text{Nor}(0, 1)\) distribution, then \(P(X \geq w) \leq e^{-w^2/2}/(\sqrt{2\pi})\) for every \(w > 0\). Therefore we obtain \(P(B_1 > 2x) \leq e^{-2x^2}/(\sqrt{8\pi})\), and the proof is complete.

Proof of Proposition 2.7. By Theorem 2.2 we know that there is a Brownian motion \((B_s)_{s \geq 0}\) such that, keeping the notation \(S_t\) for the coupled random walk,

\[
P \left( S_t < h \ \forall t \in [N] \right) \leq P \left( B_t < h + C \log(N) \ \forall t \in [N] \right)
+ P \left( S_t < h \ \forall t \in [N], \exists t \in [N] : B_t \geq h + C \log(N) \right)
\leq P \left( B_t < h + C \log(N) \ \forall t \in [N] \right)
+ P \left( \exists t \in [N] : B_t - S_t \geq C \log(N) \right)
\leq P \left( B_t < h + C \log(N) \ \forall t \in [N] \right)
+ P \left( \max_{1 \leq t \leq N} |B_t - S_t| \geq C \log(N) \right)
\leq P \left( B_t < h + C \log(N) \ \forall t \in [N] \right) + C/N,
\]

(7)
for some finite constant $C > 0$. In order to apply standard results concerning first passage times of Brownian motion, we would like the event within the probability in (7) to be true for every $s \in [0, N] \subset \mathbb{R}$, and not only at discrete times $t \in [N]$. To switch from a discrete to a continuous interval, we use Lemma 2.3 in the following way. Defining $\Phi = \Phi_C(h, N) := h + C \log(N)$ we see that, given any $z > 0$, the probability in (7) equals

$$
\Pr \left( B_t < \Phi \quad \forall t \in [N], \max_{s \in (t-1, t)} B_s < \Phi + z \quad \forall t \in [N] \right) + \Pr \left( B_t < \Phi \quad \forall t \in [N], \exists t \in [N] : \max_{s \in (t-1, t)} B_s \geq \Phi + z \right). \quad (8)
$$

By Lemma 2.3 the second probability in (8) can be bounded from above by

$$
\Pr \left( \exists t \in [N] : B_{t-1} < \Phi, \max_{s \in (t-1, t)} B_s \geq \Phi + z, B_t < \Phi \right) \leq \frac{N}{\sqrt{8\pi}} e^{-2z^2}. \quad (9)
$$

On the other hand, setting $\tau_{\Phi+z}^B := \inf \{ s \geq 0 : B_s = \Phi + z \}$, we see that the first probability in (8) is at most

$$
\Pr \left( B_s < \Phi + z \quad \forall s \in [0, N] \right) = \Pr \left( \tau_{\Phi+z}^B > N \right). \quad (9)
$$

Since the law of $\tau_{\Phi+z}^B$ has density (with respect to the Lebesgue measure on $\mathbb{R}$) given by

$$
f_{\tau_{\Phi+z}^B}(y) = \frac{\Phi + z}{\sqrt{2\pi}y^{3/2}} e^{-\frac{(\Phi+z)^2}{2y}} \, dy,
$$

we obtain that

$$
\Pr \left( \tau_{\Phi+z}^B > N \right) = \int_N^{\infty} \frac{\Phi + z}{\sqrt{2\pi}y^{3/2}} e^{-\frac{(\Phi+z)^2}{2y}} \, dy \leq \frac{\Phi + z}{\sqrt{2\pi}} \int_N^{\infty} y^{3/2} \, dy = \sqrt{\frac{2}{\pi}} \Phi + z.
$$

Summarizing, and recalling the definition of $\Phi$, we arrive at

$$
\Pr \left( S_t < h \quad \forall t \in [N] \right) \leq \sqrt{\frac{2}{\pi}} \frac{h + C \log(N) + z}{N^{1/2}} + \frac{1}{\sqrt{8\pi}} \frac{Ne^{-z^2}}{z} + C. \quad (10)
$$

Taking $z = \log^{1/2}(N) > 0$ we see that $\frac{Ne^{-z^2}}{z} = O(1/N \log^{1/2}(N))$, whence we obtain

$$
\Pr \left( S_t < h \quad \forall t \in [N] \right) \leq c_1 \frac{h}{N^{1/2}} + c_2 \frac{\log(N)}{N^{1/2}}
$$

for some finite constants $c_1, c_2 > 0$, which is the desired result. \hfill \square

Next we state a lemma which shows how to change the mean of a sequence of iid Poisson random variables.

**Lemma 2.4.** Let $(Y_i)_{i \in [N]}$, $N \in \mathbb{N}$, be a sequence of iid random variables such that each $Y_i$ has the Pois($a$) distribution, for some $a > 0$. Let $c > 0$ and set $b := \log(c/a) \in \mathbb{R}$. For $B \in \mathcal{F}_N := \sigma(\{Y_i : i \in [N]\})$ define

$$
\widehat{P}(B) = \widehat{P}_b(B) := \mathbb{E} \left[ e^{b \sum_{i=1}^N Y_i} 1_B \right] \mathbb{E} \left[ e^{by_1} \right]^{-N}.
$$

Then, under $\widehat{P}$, the random variables $Y_1, \ldots, Y_N$ are iid and each $Y_i$ has the Pois($c$) distribution.
Proof. Let us start by showing that, under \( \hat{P} \), we have \( Y_j \sim \text{Poi}(c) \). Note that, for \( k \in \mathbb{N}_0 \),
\[
\hat{P}(Y_j = k) = \mathbb{E}\left[ e^{b \sum_{i=1}^N Y_i} \mathbb{I}(Y_j = k) \right] \mathbb{E}\left[ e^{b Y_j} \right]^{-N} = e^{bk} \mathbb{E}\left[ e^{b \sum_{i \neq j} Y_i} \mathbb{I}(Y_j = k) \right] \mathbb{E}\left[ e^{b Y_j} \right]^{-N},
\]
and since the \( Y_i \) are independent we can write
\[
e^{bk} \mathbb{E}\left[ e^{b \sum_{i \neq j} Y_i} \mathbb{I}(Y_j = k) \right] \mathbb{E}\left[ e^{b Y_j} \right]^{-N} = e^{bk} \mathbb{P}(Y_j = k) \mathbb{E}\left[ e^{b \sum_{i \neq j} Y_i} \mathbb{I}(Y_j = k) \right] \mathbb{E}\left[ e^{b Y_j} \right]^{-N} = e^{bk} \mathbb{P}(Y_j = k) \mathbb{E}\left[ e^{b Y_j} \right]^{-N(N-1)} = e^{bk} e^{-a^\frac{a^k}{k!}} \mathbb{E}\left[ e^{b Y_j} \right]^{-1}.
\]
Moreover,
\[
\mathbb{E}\left[ e^{b Y_j} \right] = e^{a(e^b - 1)} = e^{-a^\frac{a^k}{k!}} e^{-a^c},
\]
and so we arrive at \( \hat{P}(Y_j = k) = e^{-a^c}(e^k/k!) \), as desired. To prove \( \hat{P} \)-independence of the \( Y_i \), we compute that, for \( k_1, \ldots, k_N \geq 0 \),
\[
\hat{P}(Y_i = k_i \forall i \in [N]) = e^{b \sum_{i=1}^N k_i} \mathbb{P}(Y_i = k_i \forall i \in [N]) \mathbb{E}\left[ e^{b Y_i} \right]^{-N}
= e^{b \sum_{i=1}^N k_i} \prod_{i=1}^N \mathbb{P}(Y_i = k_i) \mathbb{E}\left[ e^{b Y_i} \right]^{-N}
= \prod_{i=1}^N \mathbb{E}\left[ e^{b Y_i} \mathbb{I}(Y_i = k_i) \right] \mathbb{E}\left[ e^{b Y_i} \right]^{-N}.
\]
Using again the independence of the \( Y_i \) we can write
\[
\mathbb{E}\left[ e^{b \sum_{i \neq j} Y_j} \right] \mathbb{E}\left[ e^{b Y_j} \mathbb{I}(Y_j = k_i) \right] = \mathbb{E}\left[ e^{b \sum_{i \neq j} Y_j} \mathbb{I}(Y_j = k_i) \mathbb{I}(Y_i = k_i) \right],
\]
whence multiplying both the numerator and the denominator of the ratio in (10) by \( \mathbb{E}\left[ e^{b \sum_{i \neq j} Y_j} \right] \)
we obtain that
\[
\prod_{i=1}^N \mathbb{E}\left[ e^{b Y_i} \mathbb{I}(Y_i = k_i) \right] \mathbb{E}\left[ e^{b Y_i} \right]^{-N} \prod_{i=1}^N \mathbb{E}\left[ e^{b \sum_{i \neq j} Y_j} \mathbb{I}(Y_j = k_i) \right] \mathbb{E}\left[ e^{b Y_i} \right]^{-N} = \prod_{i=1}^N \hat{P}(Y_i = k_i) \cdot \hat{P}(Y_j = k_j) \quad \square.
\]

We conclude this section by recalling Kolmogorov’s maximal inequality.

**Theorem 2.5.** Let \( (X_i)_{i \geq 1} \) be a sequence of independent random variables with \( \mathbb{E}(X_i) = 0 \) for each \( i \). Then, given any finite \( \alpha > 0 \) and \( N \in \mathbb{N} \), we have
\[
\mathbb{P}\left( \max_{1 \leq k \leq N} \left| \sum_{i=1}^k X_i \right| > \alpha \right) \leq \frac{\mathbb{V}(\sum_{i=1}^N X_i)}{\alpha^2}.
\]

### 2.2 Preliminary results for Proposition 1.3

We start by stating a ballot-type result, introduced in [8] with the purpose of establishing a sharp upper bound for the probability of observing unusually large maximal components in the near-critical \( G(n, p) \) random graph.

In its original form, the ballot theorem concerns the probability that a (simple) random walk \( S_t = \sum_{i=1}^t X_i \) stays positive for all times \( t \in [n] \), given that \( S_n = k \in \mathbb{N} \), and says that the answer is \( k/n \); see e.g. [1] and references therein. However, here we will be interested in computing probabilities of the type \( \mathbb{P}(S_t > 0 \forall t \in [n], S_n = j) \), where \( j \geq 1 \) and \( X_1, \ldots, X_n \) are iid random variables taking values in \( \{-1, 0, 1, 2, \ldots\} \). To this end, we will make use of the following
Lemma 2.6. (Corollary 2.4 in [3]) Fix $n \in \mathbb{N}$ and let $(X_i)_{i \geq 1}$ be i.i.d. random variables taking values in $\mathbb{Z}$, whose distribution may depend on $n$. Let $h \in \mathbb{N}$, and suppose that $\mathbb{P}(X_1 = h) > 0$. Define $S_t = \sum_{i=1}^{t} X_i$ for $t \in \mathbb{N}_0$. Then for any $j \geq 1$ we have

$$\mathbb{P}(h + S_t > 0 \forall t \in [n], h + S_n = j) \leq \mathbb{P}(X_1 = h)^{-1} \frac{j}{n+1} \mathbb{P}(S_{n+1} = j).$$

Next we state a result which provides precise upper bounds for the (point) probabilities that a binomial random variables with parameters $N$ and $p$ takes a given value $j \geq Np + x$.

Lemma 2.7 (Theorem 1.2 of [4]). Let $\text{Bin}(N, p)$ be a binomial random variable of parameters $N$ and $p$. Suppose that $pN \geq 1$ and $1 \leq x(1 - p)N/3$. Then if $j \geq pN + x$, we have

$$\mathbb{P}(\text{Bin}(N, p) = j) < \frac{1}{\sqrt{2\pi p(1-p)N}} \exp \left( -\frac{x^2}{2p(1-p)N} + \frac{x}{(1-p)N} + \frac{x^3}{pN^2} \right).$$

We conclude by recalling a standard concentration bound for the binomial distribution.

Lemma 2.8. (Theorem 2.1 in [7]) Let $\text{Bin}(N, p)$ be a binomial random variables of parameters $N$ and $p$. Then, for any $h \geq 0$, we have that

$$\mathbb{P}(\text{Bin}(N, p) \geq Np + h) \leq \exp \left\{ -\frac{h^2}{2Np} + \frac{3h}{2} \right\}.$$  

3 Proof of Theorem 1.1 — $G(n, d, p)$

In order to derive our result for the $G(n, d, p)$ model, i.e., the random graph obtained through $p$-bond percolation on a (simple) $d$-regular graph selected uniformly at random from the set of all simple $d$-regular graphs on $[n]$, the idea is to analyse the percolated version of a random $d$-regular multigraph by means of the configuration model, which is an algorithmic procedure introduced by Bollobás [3] that gives us a way of choosing a graph uniformly at random from the set of all simple $d$-regular multigraphs on $[n]$, provided that $dn$ is even (see [11] for a detailed introduction to this model). In particular we will show that, for such a random multigraph, $\mathbb{P}(|C_{\text{max}}| < \nu^{2/3}/A)$ decays as $A^{-1/8}$. Before explaining how to relate this result to the $G(n, d, p)$ random graph (which is our object of interest), let us first describe the configuration model.

Start with $dn$ stubs (or half-edges), labelled $(v, i)$ for $v \in [n]$ and $i \in [d]$. Choose a stub $(v_0, i_0)$ in some way (the manner of choosing may be deterministic or random) and pair it with another half-edge $(w_0, j_0)$ selected uniformly at random. Say that these two stubs are matched (or paired) and put $v_0w_0 \in E$. Then, at each subsequent step $k \in \{1, \ldots, nd/2 - 1\}$ (recall that $dn$ is even), choose an half-edge $(v_k, i_k)$ in some way from the set of unmatched stubs and pair it with another half-edge $(w_k, j_k)$ selected uniformly at random from the set of all unpaired stubs. Say that these two stubs are matched and put $v_kw_k \in E$. At the end of this procedure, the resulting multipart graph $G'(n, d)$ is uniformly chosen amongst all $d$-regular multigraphs on $[n]$. However, with probability converging to $\exp\{1 - d^2/4\}$ it is a simple graph and, conditioning on this event, it is uniformly chosen amongst all $d$-regular (simple) graphs on the vertex set $[n]$.

To obtain our result for the $G(n, d, p)$ model we argue as follows. Writing $S_n$ for the event that the multipart graph $G'(n, d)$ generated by means of the configuration model is simple, like we said earlier we have that $\mathbb{P}(S_n) \to c_d := \exp\{1 - d^2/4\}$ as $n \to \infty$. Moreover, since the (conditional) law of $G'(n, d)$ given $S_n$ coincides with that of $G(n, d)$ (that is, the graph selected uniformly at random from the set of all simple $d$-regular graphs with $n$ vertices on which we perform $p$-bond percolation to obtain $G(n, d, p)$), denoting by $G'(p)$ the
p-percolated version of \( G'(n, d) \) we obtain (for \( T \in \mathbb{N} \))
\[
\mathbb{P}(|C_{\text{max}}(G(n, d, p))| < T) = \mathbb{P}(|C_{\text{max}}(G'(p))| < T | S_n)
\leq \frac{\mathbb{P}(|C_{\text{max}}(G'(p))| < T)}{\mathbb{P}(S_n)} \leq 2c_d \mathbb{P}(|C_{\text{max}}(G'(p))| < T)
\]
for all large enough \( n \). Therefore, we can deduce our result for the \( G(n, d, p) \) model by studying the random graph \( G'(p) \).

3.1 An exploration process

Our algorithmic procedure, which is taken from \cite{9}, uses the configuration model to generate components of \( G'(p) \), the p-percolated version of a uniformly random \( d \)-regular multigraph \( G'(n, d) \).

During our exploration process, each stub of the \( d \)-regular multigraph \( G'(n, d) \) is either active, unseen or explored, and its status changes during the course of the procedure. We denote by \( A_t, U_t \) and \( E_t \) the sets of active, unseen and explored half-edges at the end of the \( t \)-th step of the exploration process, respectively.

Given a stub \( h \) of \( G'(n, d) \), we denote by \( v(h) \) the vertex incident to \( h \) (in other words, if \( h = (u, i) \) for some \( i \in [d] \) then \( v(h) = u \)) and we write \( S(h) \) for the set of all half-edges incident to \( v(h) \) in \( G'(n, d) \) (that is, \( S(h) = \{(v(h), i) : i \in [d]\}; note in particular that \( h \in S(h) \)).

Let \( V_n \) be a node selected uniformly at random from the vertex set \([n]\). The exploration process works as follows. At step \( t = 0 \) we declare active all half-edges incident to \( V_n \), while all the other \( d(n - 1) \) stubs are declared unseen. Since there are \( d \) half-edges incident to \( V_n \) we have that \(|A_0| = d, |U_0| = d(n - 1) \) and \(|E_0| = 0 \). We let \((J_t)_{t \in \mathbb{N}}\) be a sequence of iid \( \text{Ber}(p) \) random variables, independent of the random pairing which we discuss next; we will use the \( J_t \) to decide whether keeping the edges at those steps where a matching actually occurs.

For every \( t \geq 1 \) and as long as \(|U_{t-1}| \geq 1 \), the algorithm proceeds as follows.

(a) If \(|A_{t-1}| \geq 1 \), we choose (in an arbitrary way) one of the active stubs, say \( e_t \), and we pair it with a half-edge \( h_t \) selected uniformly at random from \([dn]\) \( \setminus \{E_{t-1} \cup \{e_t\}\} \), the set of all unexplored stubs after having removed \( e_t \).

(a.1) If \( h_t \in U_{t-1} \) and \( J_t = 1 \), then all the unseen stubs in the set \( S(h_t) \setminus \{h_t\} \) are declared active, while \( e_t \) and \( h_t \) are declared explored. In other terms, we update \( A_t := (A_{t-1} \setminus \{e_t\}) \cup (U_{t-1} \cap S(h_t) \setminus \{h_t\}), U_t := U_{t-1} \setminus S(h_t) \) and \( E_t := E_{t-1} \cup \{e_t, h_t\} \).

(a.2) If \( h_t \in U_{t-1} \) but \( J_t = 0 \), then we simply declare \( e_t \) and \( h_t \) explored while the status of all other stubs remain unchanged. Thus we update \( A_t := A_{t-1} \setminus \{e_t\}, U_t := U_{t-1} \setminus \{h_t\} \) and \( E_t := E_{t-1} \cup \{e_t, h_t\} \).

(a.3) If \( h_t \in A_{t-1} \), then we simply declare \( e_t \) and \( h_t \) explored while the status of all other half-edges remain unchanged. In other terms, we update \( A_t := A_{t-1} \setminus \{e_t, h_t\}, U_t := U_{t-1} \) and \( E_t := E_{t-1} \cup \{e_t, h_t\} \).

(b) If \(|A_{t-1}| = 0 \) then (independently of the value of \( J_t \)) we pick in an arbitrary way a stub \( e_t \in U_{t-1} \neq \emptyset \) and we declare active all the unseen stubs in \( S(e_t) \) (thus \( e_t \) at least is declared active).

In the next subsection we discuss how to use such procedure to actually study component sizes in the random graph generated by the algorithm.
3.1.1 Relating the exploration process to component sizes

Recall that \( G'(p) \) is the \( p \)-percolated version of \( G(n, d) \), the random \( d \)-regular multigraph on \( n \) vertices constructed via the configuration model.

The goal here is to obtain an upper bound for the probability that \( \text{C}_{\text{max}}(G'(p)) \) is smaller than \( T \in \mathbb{N} \) in terms of the probability that all the positive excursions of \( \{ |A_t| \} \) never last for more than \( T \) steps.

Our argument is closely related to the one used in [13] to prove Lemma 10 in that paper. However, since we have used a different exploration process and because we only need part of the result established in [13], we decided to report the full argument here.

Let \( 0 = t_0 < t_1 < \cdots \) be the times at which the set of active stubs becomes empty, so that \( |A_t| = 0 \) for all \( j \geq 1 \). For \( j \geq 1 \), we denote by \( \mathcal{C}_j \) the \( j \)-th explored component (in \( G'(p) \)). Define

\[
\Upsilon_{j}^{(\text{UR})} := \left\{ \{ t \in (t_{j-1}, t_j) : h_t \in \mathcal{U}_{t-1}, J_t = 1 \} \right\},
\]

the number of steps during the exploration of \( \mathcal{C}_j \) in which we pick an unseen stub and retain the corresponding edge.

We claim that

\[
\mathbb{P}(|\text{C}_{\text{max}}(G'(p))| < T) \leq \mathbb{P}(t_j - t_{j-1} \leq (d-1)T \ \forall j \geq 1).
\] (12)

To see this, let us start by observing that at time \( t > t_{j-1} \) we add a vertex to \( \mathcal{C}_j \) if, and only if, \( h_t \in \mathcal{U}_{t-1} \) and \( J_t = 1 \); that is if, and only if, \( h_t \) is unseen and the edge \( e_t h_t \) is retained in the percolation. Thus \( |\mathcal{C}_j| = \Upsilon_{j}^{(\text{UR})} + 1 \), where the +1 comes from counting \( V_n \) during the exploration of \( C(V_n) \), whereas it comes from counting \( v(e_{t_{j+1}}) \) during the exploration of \( \mathcal{C}_{j+1} \) (for \( j \geq 1 \)).

We would like to express the component sizes \( |\mathcal{C}_j| \) in terms of the random distances \( t_j - t_{j-1} \). To this end, we need to introduce a few quantities. Specifically, we define

\[
\Upsilon_{j}^{(\text{UR})} := \left\{ \{ t \in (t_{j-1}, t_j) : h_t \in \mathcal{U}_{t-1}, J_t = 0 \} \right\} \text{ and } \Upsilon_{j}^{(\text{A})} := \left\{ \{ t \in (t_{j-1}, t_j) : h_t \in \mathcal{A}_{t-1} \} \right\}.
\]

Since at each step \( t \in (t_{j-1}, t_j) \) either we pick an unseen or an active stub, we have that \( t_j - t_{j-1} = \Upsilon_{j}^{(\text{UR})} + \Upsilon_{j}^{(\text{A})} \). Moreover, for \( j \geq 1 \) we have that

\[
0 = |A_t| = \left| S(e_{t_{j-1}+1}) \cap \mathcal{U}_{t_{j-1}} \right| - 2\Upsilon_{j}^{(\text{A})} - \Upsilon_{j}^{(\text{UR})} + \sum_{m=1}^{d} (m-2)N_m^{(i)},
\] (13)

where (for \( m \in [d] \)) we denote by \( N_m^{(i)} \) the number of steps \( t \) during the exploration of \( \mathcal{C}_j \) in which \( h_t \) is incident to a vertex in \( Y_{t-1}^{(m)} \), the set of vertices having \( m \) unseen stubs at the end of step \( t-1 \), and the edge \( e_t h_t \) is retained in the percolation; formally,

\[
N_m^{(i)} := \left\{ \{ t \in (t_{j-2}, t_j) : h_t \in \mathcal{Y}_{t-1}^{(m)}, J_t = 1 \} \right\}.
\]

Note that the sum in (13) is at most \( (d-2)\Upsilon_{j}^{(\text{UR})} \) and \( |S(e_{t_{j-1}+1}) \cap \mathcal{U}_{t_{j-1}}| \leq d \), so that we obtain \( 2\Upsilon_{j}^{(\text{A})} + \Upsilon_{j}^{(\text{UR})} \leq d + (d-2)\Upsilon_{j}^{(\text{UR})} \). Thus, since \( t_j - t_{j-1} = \Upsilon_{j}^{(\text{UR})} + \Upsilon_{j}^{(\text{A})} \), we arrive at

\[
\Upsilon_{j}^{(\text{UR})} + t_j - t_{j-1} = 2\Upsilon_{j}^{(\text{A})} + \Upsilon_{j}^{(\text{UR})} \leq d + (d-2)\Upsilon_{j}^{(\text{UR})} = d + (d-1)\Upsilon_{j}^{(\text{UR})} - \Upsilon_{j}^{(\text{UR})};
\]

whence we obtain, since \( \Upsilon_{j}^{(\text{UR})} = |\mathcal{C}(j)| - 1 \),

\[
t_j - t_{j-1} = \Upsilon_{j}^{(\text{A})} + t_j - t_{j-1} - \Upsilon_{j}^{(\text{UR})} + (\Upsilon_{j}^{(\text{UR})} - \Upsilon_{j}^{(\text{A})}) \leq d + (d-1)\Upsilon_{j}^{(\text{UR})} - \Upsilon_{j}^{(\text{A})} + (\Upsilon_{j}^{(\text{UR})} - \Upsilon_{j}^{(\text{A})}) \leq d + (d-1)|\mathcal{C}_j| - 1 = 1 + (d-1)|\mathcal{C}_j|.
\] (14)

Therefore it follows from (14) that, if \( |\mathcal{C}_j| < T \), then \( t_j - t_{j-1} < 1 + (d-1)T \) (i.e. \( t_j - t_{j-1} \leq (d-1)T \)), establishing (12).
3.2 Upper bound for $\mathbb{P}(|C_{\max}| < n^{2/3}/A)$ in $\mathbb{G}'(p)$

Denoting by $\eta_t$ the number of half-edges that we add to the set of active stubs at time $t$ we see that, if $|A_{t-1}| \geq 1$, then

$$\eta_t = \mathbb{I}_{(h_t \in U_{t-1})} |S(h_t) \cap U_{t-1} \setminus \{h_t\}| - \mathbb{I}_{(h_t \in A_{t-1})} - 1.$$  \hfill (15)

In words, assuming $|A_{t-1}| \geq 1$, the number of active stubs at the end of step $t$ increases by $m-2 \in \{0, 1, \ldots, d-2\}$ if $v(h_t)$ has $m \in \{2, \ldots, d\}$ unseen stubs at the end of step $t-1$ and $J_t = 1$; it decreases by one if $h_t$ is unseen and $J_t = 0$, or if $h_t$ is the unique unseen stub incident to $v(h_t)$ and $J_t = 1$; and it decreases by two if $h_t$ is an active stub.

On the other hand, suppose that $|A_{t-1}| = 0$ (but $|U_{t-1}| \geq 1$). Then, recalling that at time $t$ we pick a stub $e_t$ from $U_{t-1}$ and declare active all the unseen stubs in $S(e_t)$, we obtain

$$\eta_t = |S(e_t) \cap U_{t-1}|.$$  \hfill (16)

Recalling that $V^{(d)}_{i-1}$ is the set of vertices which, at the end of step $i-1$, have $d$ unseen stubs, we claim that

$$\eta_t \geq \begin{cases} J_i(d-1)\mathbb{I}_{(v(h_t) \in V^{(a)}_{i-1})} - 1 - \mathbb{I}_{(h_t \in A_{i-1})}, & \text{if } |A_{i-1}| \geq 1 \\ J_i(d-1)\mathbb{I}_{(v(e_t) \in V^{(a)}_{i-1})} - 1, & \text{if } |A_{i-1}| = 0. \end{cases}$$

To see this note that, when $|A_{i-1}| \geq 1$, if $h_t \in A_{i-1}$ then

$$\eta_t = -2 = J_i(d-1)\mathbb{I}_{(v(h_t) \in V^{(a)}_{i-1})} - 1 - \mathbb{I}_{(h_t \in A_{i-1})};$$

also, if $h_t$ is in $U_{t-1}$, node $v(h_t)$ has $d$ unseen stubs at the end of step $i-1$ and $J_t = 1$, then

$$\eta_t = d - 2 = J_i(d-1)\mathbb{I}_{(v(h_t) \in V^{(a)}_{i-1})} - 1 - \mathbb{I}_{(h_t \in A_{i-1})};$$

finally, in all the other cases we have that

$$\eta_t \geq 0 = J_i(d-1)\mathbb{I}_{e_t} - 1 - \mathbb{I}_{(h_t \in A_{i-1})}.$$

Next, suppose that $|A_{i-1}| = 0$. If $v(e_t)$ has $d$ unseen stubs at the end of step $i-1$, then

$$\eta_t = d > J_i(d-1) - 1 = J_i(d-1)\mathbb{I}_{(v(e_t) \in V^{(a)}_{i-1})} - 1;$$

in all the other cases (i.e. if $v(e_t) \in V^{(m)}_{i-1}$ with $1 \leq m \leq d-1$) then

$$\eta_t \geq 1 > -1 = J_i(d-1)\mathbb{I}_{(v(e_t) \in V^{(a)}_{i-1})} - 1,$$

thus proving the claim. Define now $\xi_i := J_i(d-1)$ and

$$X_i^{(1)} := \xi_i \mathbb{I}_{(v(h_t) \in V^{(a)}_{i-1})} + \mathbb{I}_{(h_t \in A_{i-1})}, \quad X_i^{(2)} := \xi_i \mathbb{I}_{(v(e_t) \in V^{(a)}_{i-1})}.$$

Then, setting $X_i := X_i^{(1)}\mathbb{I}_{(|A_{i-1}| \geq 1)} + X_i^{(2)}\mathbb{I}_{(|A_{i-1}| = 0)}$, we see that (for any $i$) we have

$$\eta_i = \eta_i \mathbb{I}_{(|A_{i-1}| \geq 1)} + \eta_i \mathbb{I}_{(|A_{i-1}| = 0)} \geq (\xi_i \mathbb{I}_{(v(h_t) \in V^{(a)}_{i-1})} - 1 - \mathbb{I}_{(h_t \in A_{i-1})}) \mathbb{I}_{(|A_{i-1}| \geq 1)} + \left(\xi_i \mathbb{I}_{(v(e_t) \in V^{(a)}_{i-1})} - 1\right) \mathbb{I}_{(|A_{i-1}| = 0)}$$

$$= (\xi - 1) \mathbb{I}_{(|A_{i-1}| \geq 1)} - X_i^{(1)} \mathbb{I}_{(|A_{i-1}| \geq 1)} + (\xi - 1) \mathbb{I}_{(|A_{i-1}| = 0)} - X_i^{(2)} \mathbb{I}_{(|A_{i-1}| = 0)}$$

$$= \xi - 1 - X_i.$$
Therefore, defining for $0 \leq t \leq (d - 1)T$ (with $T = T(n) \in \mathbb{N}$)
\begin{equation}
S_t := d + \sum_{i=1}^{t} (\xi_i - 1 - X_i),
\end{equation}
we see that $d + \sum_{i=1}^{t} \eta_i \geq S_t$ for all $t \leq (d - 1)T$. This implies that each time $t \leq (d - 1)T$ such that $d + \sum_{i=1}^{t} \eta_i = 0$ also satisfies $S_t \leq 0$. Consequently, setting $\tau_0 := 0$ and defining recursively $\tau_j := \inf \{ t > \tau_{j-1} : S_t \leq 0 \}$ for $j \geq 1$, we obtain that
\begin{equation}
t_j - t_{j-1} \leq (d - 1)T \ \forall j \geq 1 \text{ implies } \tau_j - \tau_{j-1} \leq (d - 1)T \ \forall j \geq 1.
\end{equation}

To formally establish (15), we proceed as follows. First of all note that, by definition of $t_1$, we have $0 = d + \sum_{i=1}^{t_1} \eta_i \geq S_{t_1}$ and hence the first visit to 0 by $S_t$ must occur before time $t_1$, i.e. $\tau_1 \leq t_1$. Next, suppose toward a contradiction that there is $j_0 > 1$ such that $\tau_{j_0} - \tau_{j_0-1} > (d - 1)T$. Let $t_1$ be the last time before $\tau_{j_0-1}$ at which the process $d + \sum_{i=1}^{t} \eta_i$ reaches 0. Since $\tau_{j_0} > \tau_{j_0-1} + (d - 1)T$, it follows that $S_t > 0$ for all $t \in \{ \tau_{j_0-1} + 1, \ldots, \tau_{j_0-1} + (d - 1)T \} =: I$ and consequently $d + \sum_{i=1}^{t} \eta_i > 0$ for all $t \in I$. But then this implies that $t_{i+1} > \tau_{j_0-1} + (d - 1)T$ resulting in the (strict) inequality $t_{i+1} - t_i \geq t_{i+1} - \tau_{j_0-1} > \tau_{j_0-1} + (d - 1)T - \tau_{j_0-1} = (d - 1)T$, which contradicts our initial assumption that $t_j - t_{j-1} \leq (d - 1)T$ for all $j$. This proves (18) which in turn implies that
\begin{equation}
\mathbb{P}(t_j - t_{j-1} \leq (d - 1)T \ \forall j \geq 1) \leq \mathbb{P}(\tau_j - \tau_{j-1} \leq (d - 1)T \ \forall j \geq 1).
\end{equation}

Following Nachmias and Peres [14] [15], now the idea is to show that (with high probability) the process $S_t$ reaches some level $h > 0$ before time $n \gg T' = T'(n) \ll n$ and then stays positive for at least $(d - 1)T$ steps. Hence we bound
\begin{equation}
\mathbb{P}(\tau_j - \tau_{j-1} \leq (d - 1)T \ \forall j \geq 1) \\
\leq \mathbb{P}(S_t < h \ \forall t \in [T']) + \mathbb{P}(\tau_j - \tau_{j-1} \leq (d - 1)T \ \forall j \geq 1, \exists t \in [T'] : S_t \geq h),
\end{equation}
and study these two terms separately.

**Proposition 3.1.** Let $T' = [n^{2/3}A^{-1/4}]$ and $h = [4(d-1)n^{1/3}A^{-1/4}]$. Then, for all large enough $n$, we have that
\begin{equation}
\mathbb{P}(S_t < h \ \forall t \in [T']) \leq \frac{C}{A^{1/8}},
\end{equation}
where $C = C_d > 0$ is a finite constant that depends solely on $d$.

**Proposition 3.2.** Let $T' = [n^{2/3}A^{-1/4}]$, $T = [n^{2/3}A^{-1}]$ and $h = [4(d-1)n^{1/3}A^{-1/4}]$. Then, for all large enough $n$, we have that
\begin{equation}
\mathbb{P}(\tau_j - \tau_{j-1} \leq (d - 1)T \ \forall j \geq 1, \exists t \in [T'] : S_t \geq h) \leq \frac{C'}{A^{1/4}},
\end{equation}
where $C' = C'_d > 0$ is a finite constant that depends solely on $d$.

Before proving Propositions 3.1 and 3.2 which together with (11), (12), (19) and (20) establish the statement in Theorem 1.1 concerning the $G(n,d,p)$ random graph, we first would like to derive a rough estimate for $\sum_{K=1}^{K} X_i$ ($K \in \mathbb{N}$) by means of Markov’s inequality, where we recall that $X_i$ was defined prior to (17). With the purpose of finding an upper bound for the expected value of this (random) sum, we first need to establish upper and lower bounds for the number of active stubs available at time $i$ and the number of vertices having $d$ unseen half-edges at time $i$, respectively.

Note that, at time $i \geq 0$, the number of active stubs is at most $d(i + 1)$. To see this, first of all observe that $|A_0| = d$ (since at time $t = 0$ all stubs incident to $V_0$ are declared active, whereas all other half-edges are unseen) and so the inequality is trivially true. Next,
suppose that the inequality holds for some \( i \geq 1 \) and note that at each step \( j \) we can turn to activate at most \( d \) unseen stubs (with equality if, at time \( j - 1 \), we have \(|A_{j-1}| = 0\) and \( e_j \) is incident to a vertex in \( V_{j-1}^{d(j)} \)). Therefore, using the inductive hypothesis we obtain

\(|A_{i+1}| \leq |A_i| + d \leq (i + 1)\) \( d = d(i + 1) + 1 \), thus establishing the claim.

Next we bound from below the number of vertices that, at time \( i \), have \( d \) unseen stubs attached to them. Recalling that \(|Y_0^{(d)}| = n - 1\), we obtain (for \( i \geq 1 \))

\[
|Y_i^{(d)}| = |Y_0^{(d)}| - \sum_{j=1}^{i} \left( \mathbb{I}_{\{|A_{j-1}| = 0, e_j \in V_{j-1}^{d(j)}\}} + \mathbb{I}_{\{|A_{j-1}| \geq 1, e_j \in V_{j-1}^{d(j)}\}} \right) 
\]

\[
= n - 1 - 2i + \sum_{j=1}^{i} \left( \mathbb{I}_{\{|A_{j-1}| \geq 1\}} + \mathbb{I}_{\{|A_{j-1}| = 0\}} \right) 
\]

\[
\geq n - 1 - 2i \geq n - 3i. 
\]

Now write

\[
\mathbb{E}(X_i) = \mathbb{E}\left(X_i^{(1)}\mathbb{I}_{\{|A_{i-1}| \geq 1\}} + X_i^{(2)}\mathbb{I}_{\{|A_{i-1}| = 0\}}\right) 
\]

\[
= \mathbb{E}\left(X_i^{(1)}\mathbb{I}_{\{|A_{i-1}| \geq 1\}}\right) + \mathbb{E}\left(X_i^{(2)}\mathbb{I}_{\{|A_{i-1}| = 0\}}\right); \quad (21) 
\]

we control these two expectations separately, using our bounds on \(|A_i|\) and \(|Y_i^{(d)}|\). Note that, since at step \( i \) there are at least \( dn - 2(i - 1) - 1 \) unpaired stubs that still has to be matched, writing \( G_i \) for the \( \sigma \)-algebra generated by the exploration process up to time \( j \) and recalling that \( p(d - 1) = 1 + \lambda n^{-1/3} = 1 + o(1) \), we have (for large enough \( n \) and since \( d \geq 3 \))

\[
\mathbb{E}\left(X_i^{(1)}\mathbb{I}_{\{|A_{i-1}| \geq 1\}}\right) = \mathbb{E}\left[\mathbb{I}_{\{|A_{i-1}| \geq 1\}} \mathbb{E}(X_i^{(1)}|G_{i-1})\right] 
\]

\[
\leq (1 + o(1))\mathbb{E} \left[ \frac{n - |A_{i-1}|}{dn - 2(i - 1) - 1} + \frac{|A_{i-1}|}{dn - 2(i - 1) - 1} \right] 
\]

\[
\leq (1 + o(1))\frac{3i}{dn - 2(i - 1) - 1} + \frac{di}{dn - 2(i - 1) - 1} \leq \frac{3di}{dn - 2(i - 1) - 1}. 
\]

Concerning the second expectation on the right-hand side of (21), a similar computation yields (for large enough \( n \))

\[
\mathbb{E}(X_i^{(2)}\mathbb{I}_{\{|A_{i-1}| = 0\}}) \leq \frac{3di}{dn - 2(i - 1) - 1}. 
\]

Therefore, for all large enough \( n \) we can bound \( \mathbb{E}(X_i) \leq 6di(dn - 2(i - 1) - 1) \) so that, if \( i \leq K \ll n \), then \( \mathbb{E}(X_i) \leq 12di/dn = 12i/n \) for all \( i \in [K] \). Thus \( \mathbb{E}\left(\sum_{i=1}^{K} X_i\right) = O(K^2/n) \) for all large enough \( n \) and hence, given any \( H \in \mathbb{N} \), by Markov’s inequality we finally obtain

\[
\mathbb{P}\left(\sum_{i=1}^{K} X_i \geq H\right) = O\left(\frac{K^2}{Hn}\right). \quad (22) 
\]

### 3.2.1 Bound on \( \mathbb{P}(S_t < h \ \forall t \in [T']) \) — Proof of Proposition 3.1

Using the definition of \( S_t \) given in (17) together with the fact that \( \sum_{i=1}^{t} X_i \leq \sum_{i=1}^{T'} X_i \) for all \( t \leq T' \) (where the last inequality follows from the fact that each \( X_i \) is non-negative),
we obtain

\[
P(S_t < h \ \forall t \in [T']) \leq \mathbb{P} \left( d + \sum_{i=1}^{t} (\xi_i - 1) < h + \sum_{i=1}^{T'} X_i \ \forall t \in [T'] \right)
\]

\[
\leq \mathbb{P} \left( \sum_{i=1}^{t} (\xi_i - 1) < 2h \ \forall t \in [T'] \right) + \mathbb{P} \left( \sum_{i=1}^{T'} X_i \geq h \right). \tag{23}
\]

Recall that \( \xi_i = (d - 1)J_i \), where the \( J_i \) are iid random variables having the Ber\( (p) \) distribution with \( p = (1 + \lambda n^{-1/3})/(d - 1) \), so that \( \mathbb{E}(\xi_i - 1) = \lambda n^{-1/3} \). We would like to apply Proposition 2.1 to bound the first probability on the right-hand side of (23) and, to this end, we need a sequence of (iid) mean zero random variables (whose distributions do not depend on \( n \)). The random variables \( \xi_i - 1 \) are iid but, unless \( \lambda = 0 \) (in which case \( \mathbb{E}(\xi_i - 1) = 0 \) and \( \mathbb{P}(\xi_i - 1 = 0) \) does not depend on \( n \) since \( d \) is fixed), they do not have mean zero and moreover the law of \( \xi_i - 1 \) depends on \( n \). In order to bound from above the first probability on the right-hand side of (23), let us start by assuming that \( \lambda \leq 0 \). With the purpose of removing the (vanishing) drift we use a simple change of measure. As in [9] we set

\[
\gamma := \frac{1}{d - 1} \log \left( \frac{1 - p}{p(d - 2)} \right)
\]

and define a new probability measure \( \hat{\mathbb{P}} = \mathbb{P}_\gamma \) through

\[
\hat{\mathbb{P}}(B) := \mathbb{E} \left( e^{\gamma \sum'_{i=1} (\xi_i - 1) 1_B} \right) \mathbb{E} \left( e^{\gamma (\xi_i - 1)} \right)^{-T'}, \ B \in \mathcal{F}_{T'} \coloneqq \sigma(\{\xi_1, \ldots, \xi_{T'}\}). \tag{24}
\]

Then, under \( \hat{\mathbb{P}} \), the sequence \( (\xi_i - 1)_{i\in[T']} \) is iid with \( \hat{\mathbb{P}}(\xi_i - 1 = d - 2) = (d - 1)^{-1} = 1 - \hat{\mathbb{P}}(\xi_i - 1 = -1) \). In particular, denoting by \( \hat{\mathbb{E}} \) the expectation operator with respect to \( \hat{\mathbb{P}} \), we have \( \hat{\mathbb{E}}(\xi_i - 1) = 0 \) and \( \hat{\mathbb{E}} \left( (\xi_i - 1)^2 \right) = d - 2 \).

As we will see shortly, we need a lower bound for the (random) sum \( \sum'_{i=1} (\xi_i - 1) \) within the event \( \{\sum'_{i=1} (\xi_i - 1) < 2h \ \forall t \in [T']\} \) (appearing in (23)) whose probability we are trying to bound. To this end, we let \( m = m(n) \leq n^{1/3} \) be a positive quantity to be specified later and bound from above the first probability in (23) by

\[
P \left( \sum_{i=1}^{t} (\xi_i - 1) < 2h \ \forall t \in [T'], \sum_{i=1}^{T'} \xi_i \geq T'(d - 1)p - m \right)
\]

\[
+ \mathbb{P} \left( \sum_{i=1}^{T'} \xi_i < T'(d - 1)p - m \right). \tag{25}
\]

By Chebyshev’s inequality, since \( \forall \sum'_{i=1} \xi_i = p(1 - p)(d - 1)^2T' = O_d(T') \) for all sufficiently large \( n \), it follows that the second probability in (25) is at most \( O_d(T'/n^3) \). On the other hand, the first probability in (25) equals

\[
\hat{\mathbb{E}} \left( e^{-\gamma \sum'_{i=1} (\xi_i - 1)} 1_{\{\sum'_{i=1} (\xi_i - 1) < 2h \ \forall t \in [T'], \sum'_{i=1} \xi_i \geq T'(d - 1)p - m \}} \right) \mathbb{E} \left( e^{\gamma (\xi_i - 1)} \right)^{-T'}. \tag{26}
\]

A simple calculation shows that \( [0, \infty) \ni \gamma = -\lambda n^{-1/3}/(d - 2) + O_d(12^2/n^{2/3}) \) as \( n \to \infty \). In order to provide an upper bound for \( \exp\{-\gamma \sum'_{i=1} (\xi_i - 1)\} \) within the last expectation, let us start by observing that, on the event appearing as argument of the indicator function in (26), we have \( T'\lambda n^{-1/3} - m \leq \sum'_{i=1} (\xi_i - 1) < 2h \). Recall that \( \lambda \leq 0, m \leq n^{1/3} \). Using the
asymptotic for \( \gamma \) together with the fact that
\[
O_d(\lambda^{2}n^{-2/3}) \left| \sum_{i=1}^{T'} (\xi_i - 1) \right| \leq O_d(\lambda^{2}n^{-2/3}) \left( 2h + T'|\lambda|n^{-1/3} + m \right) = O_d(n^{-1/3})
\]
we can write
\[
- \gamma \sum_{i=1}^{T'} (\xi_i - 1) \leq \frac{\lambda}{n^{1/3}(d-2)} \left( T'\lambda n^{-1/3} - m \right) + O_d(n^{1/3})
\]
\[
= \frac{\lambda^2 T'}{n^{2/3}(d-2)} + \frac{m(\lambda)}{n^{1/3}(d-2)} + O_d(n^{1/3})
\]
\[
= \frac{(\lambda \wedge 0)^2 T'}{n^{2/3}(d-2)} + \frac{m(\lambda \wedge 0)}{n^{1/3}(d-2)} + O_d(n^{1/3}).
\]
Therefore we arrive at
\[
e^{-\gamma \sum_{i=1}^{T'} (\xi_i - 1)} \leq \exp \left\{ \frac{(\lambda \wedge 0)^2 T'}{n^{2/3}(d-2)} + \frac{m(\lambda \wedge 0)}{n^{1/3}(d-2)} + O_d(n^{1/3}) \right\} = e^\Psi.
\]
We next bound the second expectation \[26\]. Since \( e^x = 1 + x + x^2/2 + O(x^3) \) for all \( x \in (-1,1) \) and since \( |\gamma|(d-1) < 1 \) for all sufficiently large \( n \), a simple computation yields
\[
E \left( e^{\gamma(\xi_1 - 1)} \right)^{T'} = e^{-\gamma T'} \left[ 1 + p \left( e^{\gamma(d-1) - 1} \right) \right]^{T'} \leq e^{-\gamma T'} e^{T'p(e^{(d-1) - 1})}
\]
\[
\leq e^{T'(\lambda \wedge 0)^2/2(n-1)(d-2) - \lambda \gamma} \leq 0 \text{ and hence } E \left( e^{\gamma(\xi_1 - 1)} \right)^{T'} \leq e^{o(1)}.
\]
Now note that, since \( d \geq 3 \), the expression within round brackets in the last exponential term satisfies \( \lambda \gamma - 2^{-1}(d-1)(d-2)^{-1} \lambda \gamma \leq 0 \) and hence \( E \left( e^{\gamma(\xi_1 - 1)} \right)^{T'} \leq e^{o(1)} \). Therefore the expression in \[26\] is at most
\[
e^{\Psi + o(1)} \hat{P} \left( \sum_{i=1}^{t} (\xi_i - 1) < 2h \ \forall t \in [T'] \right).
\]
As we have already pointed out, under \( \hat{P} \) the random variables \( \xi_i - 1 \) are iid with mean zero and \( \hat{P}(\xi_1 - 1 = 0) \) does not depend on \( n \). An application of Proposition \[2.1\] then yields
\[
\hat{P} \left( \sum_{i=1}^{t} (\xi_i - 1) < 2h \ \forall t \in [T'] \right) \leq c_1 \frac{h}{(T')^{1/2}} + c_2 \frac{\log(T')}{(T')^{1/2}}
\]
for some finite constants \( c_1, c_2 > 0 \) which depend on \( d \). Now suppose that \( \lambda > 0 \). In this case, we can couple the \( J_i \sim \Ber((d-1)^{-1}(1 + \lambda n^{-1/3})) \) with iid random variables \( J'_i \sim \Ber((d-1)^{-1}) \) such that \( J'_i \leq J_i \) so that, setting \( \xi'_i := (d-1)J'_i \), we can bound
\[
\hat{P} \left( \sum_{i=1}^{t} (\xi'_i - 1) < 2h \ \forall t \in [T'] \right) \leq \hat{P} \left( \sum_{i=1}^{t} (\xi'_i - 1) < 2h \ \forall t \in [T'] \right).
\]
Since \( E(\xi'_i - 1) = 0 \) and the distribution of \( \xi'_i \) does not depend on \( n \), we can directly apply Proposition \[2.1\] to obtain the same bound as in \[27\].

Now using \[22\], we know that the second probability on the right-hand side of \[23\] is at most \( O(T' h^2/n^2 \log(n)) \). Summarizing, we have shown that
\[
\hat{P}(S_t < h \ \forall t \in [T']) \leq \exp \{ \Psi + o(1) \} \left( c_1 \frac{h}{(T')^{1/2}} + c_2 \frac{\log(T')}{(T')^{1/2}} \right) + O_d \left( \frac{T'}{h^2 n^2} \right).
\]
Recall that \( T' = [n^{2/3}/A^{1/4}] \) and \( h = [4(d-1)n^{1/3}A^{-1/4}(1 \lor |\lambda \lor 0|)^{-1}] \). Since \( 1 < A < n^{2/3} \) (so that \( A^{1/4} < n^{1/6} \)) we obtain
\[
\log(T') \bigg/ (T')^{1/2} \leq 2 \log(n) \leq O \left( A^{1/8} \frac{\log(n)}{n^{1/4}} \right) \leq O \left( A^{-1/8} \frac{\log(n)}{n^{1/3}} \right) \leq \frac{c}{A^{1/8}} \frac{\log(n)}{n^{1/6}} \ll A^{-1/8},
\] (30)
whence
\[
c_2 \frac{h}{(T')^{1/2}} + c_2 \frac{\log(T')}{(T')^{1/2}} \leq 2c_1 \frac{h}{(T')^{1/2}}
\]
when \( n \) is large enough. Therefore, taking \( m = n^{1/3}/A^{1/16} \) it is not difficult to see that the expression on the right-hand side of (29) is at most
\[
e^{\frac{(\lambda \land 0)^2}{A^{1/4}(d-2)} + \frac{|\lambda \land 0|}{A^{1/16}(d-2)}} + e^{\frac{(\lambda \land 0)^2}{A^{1/16}(d-2)} + \frac{|\lambda \land 0|}{A^{1/16}(d-2)}} + \frac{c}{A^{1/8}} \frac{\log(n)}{n^{1/6}} + O_d(A^{-1/8}) + O_d(A^{-1/8}).
\]
Recall that \( A > 1 \lor (\lambda \land 0)^1 \). If \( |\lambda \land 0| \leq 1 \), then
\[
\frac{(\lambda \land 0)^2}{A^{1/4}(d-2)} + \frac{|\lambda \land 0|}{A^{1/16}(d-2)} \leq \frac{(\lambda \land 0)^2}{(1 \lor (\lambda \land 0)^{1/4}(d-2)} + \frac{|\lambda \land 0|}{(1 \lor (\lambda \land 0)^{1/4}(d-2)} \leq \frac{(\lambda \land 0)^2}{d-2} + \frac{|\lambda \land 0|}{d-2} \leq 2(d-2)^{-2}.
\]
Similarly, if \( |\lambda \land 0| > 1 \) then
\[
\frac{(\lambda \land 0)^2}{A^{1/4}(d-2)} + \frac{|\lambda \land 0|}{A^{1/16}(d-2)} \leq \frac{(\lambda \land 0)^2}{(1 \lor (\lambda \land 0)^{1/4}(d-2)} + \frac{|\lambda \land 0|}{(1 \lor (\lambda \land 0)^{1/4}(d-2)} \leq \frac{1}{(\lambda \land 0)^2(d-2)} + \frac{|\lambda \land 0|}{(d-2)(\lambda \land 0)} = 2(d-2)^{-2}.
\]
Therefore we conclude that the the right-hand side of (29) is at most \( C/A^{1/8} \) for some constant \( C = C_d \) which depends solely on \( d \), as required.

3.2.2 Bound on \( P(\tau_j - \tau_{j-1} < (d-1)T \forall j \geq 1, \exists t \in [T'] : S_t \geq h) \) – Proof of Proposition 3.2

Recall from (17) that \( S_t = d + \sum_{i=1}^t (\xi_i - 1 - X_i) \). Setting \( \tau_h = \inf \{ t \geq 1 : S_t \geq h \} \) and \( T'' := (d-1)T \in N \) we obtain that
\[
P(\tau_j - \tau_{j-1} \leq (d-1)T \forall j \geq 1, \exists t \in [T'] : S_t \geq h)
\]
\[
\leq P(\exists t \in [T''] : S_{t+\tau_h} = 0, \tau_h \leq T', S_{\tau_h} \geq h)
\]
\[
= \sum_{m \geq h} P(\exists t \in [T''] : S_{t+\tau_h} = 0, \tau_h \leq T', S_{\tau_h} = m).
\] (31)
Observe that, on the event \{ \( S_{\tau_h} = m \) \}, we have \( S_{t+\tau_h} = m + \sum_{i=\tau_h+1}^{\tau_h+t} (\xi_i - 1 - X_i) \) for \( t \in [T''] \); thus, if \( S_{t+\tau_h} = 0 \) (and \( S_{\tau_h} = m \)) then \( \sum_{i=\tau_h+1}^{\tau_h+t} (\xi_i - 1 - X_i) = -m \). Now since
$m \geq h$, if $\sum_{i=1}^{\tau_h + t} (\xi_i - 1 - X_i) = -m$ then $\sum_{i=1}^{\tau_h + t} (\xi_i - 1 - X_i) \leq -h$ and so the sum on the right-hand side of (31) is at most

$$P \left( \tau_h \leq T', \exists t \in [T''] : \sum_{j=1}^{\tau_h + t} (\xi_j - 1 - X_j) \leq -h \right),$$

(32)

Moreover, since the $X_i$ are non-negative we have $\sum_{j=1}^{T''} X_{j + \tau_h} \leq \sum_{j=1}^{T'' + \tau_h} X_j$ for $t \leq T''$, and making the change of variables $j = i - \tau_h$ we see that the probability in (32) equals

$$P \left( \tau_h \leq T', \exists t \in [T''] : \sum_{j=1}^{t} (\xi_{j + \tau_h} - 1 - X_{j + \tau_h}) \leq -h \right)$$

$$\leq P \left( \tau_h \leq T', \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - 1) \leq -h + \sum_{j=1}^{T''} X_{j + \tau_h} \right).$$

(33)

Now observe that, on the event $\{ \tau_h \leq T' \}$, we have $\sum_{j=1}^{T''} X_{j + \tau_h} \leq \sum_{j=1}^{T'' + \tau_h} X_j \leq \sum_{j=1}^{T'' + T'} X_j$. Moreover, thanks to (22) and noticing that $T'' \leq T'$, we see that with probability at least $1 - O((T'' + T')^2 / hn)$ we have $\sum_{j=1}^{T'' + T'} X_j < h/2$. Therefore, the probability on the right-hand side of (33) is at most

$$P \left( \tau_h \leq T', \exists t \in [T''] : \sum_{j=1}^{t} (\xi_{j + \tau_h} - 1) \leq -h + \sum_{j=1}^{T'' + T'} X_j \right)$$

$$\leq P \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - 1) \leq -h/2 + O \left( \frac{(T'' + T')^2}{hn} \right) \right)$$

$$= P \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - 1) \leq -h/2 + O \left( \frac{(T'' + T')^2}{hn} \right) \right),$$

(34)

where the last equality follows from the fact that the $\xi_j$ are iid random variables. We would like to bound the last probability using Kolmogorov’s maximal inequality and, to this end, we need to turn the increments $\xi_j - 1$ into mean zero random variables. Note that, since $E[\xi_j] = np = 1 + \lambda n^{-1/3}$ for all $j$, we have

$$P \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - 1) \leq -h/2 \right)$$

$$= P \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - (d - 1)p) \leq -h/2 - \lambda tn^{-1/3} \right).$$

(35)

Suppose first that $\lambda \leq 0$. Then the expression on the right-hand side of (35) equals

$$P \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - (d - 1)p) \leq -h/2 + |\lambda| tn^{-1/3} \right)$$

$$\leq P \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - (d - 1)p) \leq -h/2 + |\lambda| T'' n^{-1/3} \right)$$

$$= P \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - (d - 1)p) \leq -h/2 \left[ 1 - \frac{2|\lambda| T''}{hn^{1/3}} \right] \right).$$

(36)
Recalling the values of \( T'' \) and \( h \) we can bound
\[
\frac{2|\lambda|T''}{hn^{1/3}} \leq \frac{T''|\lambda|}{hn^{1/3}} \leq \frac{|\lambda \land 0|}{2A^{3/4}} \leq \frac{|\lambda \land 0|}{2(1 \lor (\lambda \land 0)^{10})^{3/4}}.
\]
If \( |\lambda \land 0| \leq 1 \), then the last expression is at most 1/2. Similarly, if \( |\lambda \land 0| > 1 \) we see that the last expression is at most 1/2|\( \lambda \land 0 |^{11} \leq 1/2 \). Thus the probability on the right-hand side of \((36)\) is at most
\[
\mathbb{P} \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - (d - 1)p) \leq -h/4 \right) \leq \mathbb{P} \left( \max_{t \in [T'']} \left| \sum_{j=1}^{t} (\xi_j - (d - 1)p) \right| \leq -h/4 \right)
\leq \frac{16}{h^2} \mathcal{V} \left( \sum_{j=1}^{T''} (\xi_j - (d - 1)p) \right) = O_d \left( \frac{T''}{h^2} \right).
\]
If \( \lambda > 0 \), then the probability on the right-hand side of \((35)\) is at most
\[
\mathbb{P} \left( \exists t \in [T''] : \sum_{j=1}^{t} (\xi_j - (d - 1)p) \leq -h/2 \right) = O_d \left( \frac{T''}{h^2} \right)
\]
too. Thus, summarizing, we have shown that when \( n \) is sufficiently large
\[
\mathbb{P} \left( \tau_j - \tau_{j-1} \leq (d - 1)T \ \forall j \geq 1, \exists t \in [T''] : S_t \geq h \right)
\leq O_d \left( \frac{T''}{h^2} \right) + O_d \left( \frac{T''}{m^2} \right) + O \left( \frac{(T'' + T')^2}{hn} \right). \tag{37}
\]
Taking \( T, T' \) and \( h \) as in the statement of the proposition, setting e.g. \( m = n^{1/3}A^{-1/4} \) and recalling that \( T'' = (d - 1)T \) we see that the expression on the right-hand side of \((37)\) is at most \( C'/A^{1/4} \) for some finite constant \( C' = C'_d > 0 \) that depends on \( d \), which is the desired result.

4 Proof of Theorem 1.1 — \( \mathbb{G}(n, p) \)

Our goal here is to adapt the methodology of Section 3 to study the near-critical \( \mathbb{G}(n, p) \) random graph where \( p = (1 + \lambda n^{-1/3})/n \) and \( \lambda \in \mathbb{R} \). In particular, here we show that the argument we have described for the \( \mathbb{G}(n, d, p) \) model easily adapt to the \( \mathbb{G}(n, p) \) random graph and, in this sense, that our method is sufficiently robust. We start by describing an exploration process, different in many aspects from the one used in Section 3 and that applies to any (undirected, simple) graph on \( n \) vertices, which we use to sequentially discovers the connected components of \( \mathbb{G}(n, p) \) and that reduces the study of component sizes to the analysis of the trajectory of a stochastic process.

4.1 An exploration process

A main ingredient to prove our result is an exploration process, which is an algorithmic procedure to sequentially discover the components of an undirected graph, and which reduces the study of component sizes to the analysis of a stochastic process. Our description closely follows the one appearing in [8] and [14].

Let \( \mathbb{G} \) be any (undirected) graph with vertex set \([n]\), and let \( v \in [n] \) be any given node in \([n]\). Fix an ordering of the \( n \) vertices with \( v \) first. At each time \( t \in \{0\} \cup [n] \) of the procedure, each vertex will be active, explored or unseen; the (possibly random) number of active vertices will be denoted by \( Y_t \). At time \( t = 0 \), vertex \( v \) is declared to be active.
whereas all other vertices are in status unseen, so that $Y_0 = 1$. At each step $t \in [n]$ of the procedure, if $Y_{t-1} > 0$ (i.e. if the previous step finished with at least one active node) then we let $u_t$ be the first active vertex (here the term first refers to the ordering that we have fixed at the very beginning of the procedure); if $Y_{t-1} = 0$ (i.e. if the set of active vertices becomes empty at the end of time $t - 1$), we let $u_t$ be the first unseen vertex. Note that at time $t = 1$ we have $u_1 = v$, and at each step of the exploration we explore one node, so that at time $t$ we have exactly $t$ explored vertices. Denote by $\eta_t$ the (possibly random) number of unseen neighbours of $u_t$ in $G$ and change the status of these vertices to active. Then, set $u_t$ itself explored. From this description it is clear that the number of active vertices satisfies the following important recursion:

- if $Y_{t-1} > 0$, then $Y_t = Y_{t-1} + \eta_t - 1$;
- if $Y_{t-1} = 0$, then $Y_t = \eta_t$.

If we know let $G$ be distributed as $G(n, p)$, $p \in (0, 1)$, and start the exploration process from a vertex $v_0$ selected uniformly at random from $[n]$, then we see that the $\eta_t$ are random variables with $\eta_t \sim \text{Bin}(n - 1, p)$ and

$$\eta_t \mid \{\eta_j : 1 \leq j \leq i - 1\} \sim \text{Bin}(n - (i - 1) - Y_{i-1} - \mathbb{1}_{\{Y_{i-1} = 0\}}, p)$$

for every $2 \leq i \leq n$. Moreover, setting $\tau_1 := \min\{t \geq 1 : Y_1 = 0\}$, we have that $|C_{V_0}(G)| = \tau_1$. Indeed, denoting by $\mathcal{E}_i$ the set of explored vertices at the end of step $i$, we have $\mathcal{E}_{t+1} = \mathcal{E}_t + \{\mathcal{E}_t\}$, whence $|C_{V_t}(G)| = |\mathcal{E}_{\tau_1}| = \tau_1$, where the last identity follows from the fact that at each step $i$ during the exploration of $C_{V_1}(G)$ we explore exactly one vertex.

With the purpose of constructing a smaller process, we observe that $Y_t \geq 1 + \sum_{i=1}^{t} (\eta_i - 1)$ for all $t \in \{0\} \cup [n]$. Indeed, this is clearly true for $t = 0$. Assume now that the inequality holds for $t \geq 1$ we see that and note that if $Y_t \geq 1$, then $Y_{t+1} = Y_t + (\eta_{t+1} - 1) \geq 1 + \sum_{i=1}^{t+1} (\eta_i - 1)$, while if $Y_t = 0$ then we obtain $Y_{t+1} = \eta_{t+1} > \eta_{t+1} - 1 = Y_t + (\eta_{t+1} - 1) \geq 1 + \sum_{i=1}^{t+1} (\eta_i - 1)$ too.

### 4.2 Upper bound for $\mathbb{P}(\|C_{\max}\| < n^{2/3}/A)$ in $G(n, p)$

We now specialize the exploration process described in Subsection 4.1 to the near-critical Erdős-Rényi random graph; that is, we now take $G = G(n, p)$ with $p = (1 + \lambda n^{-1/3})/n$, and start the procedure from a vertex $V_0$ selected uniformly at random from $[n]$. Let us define $U_t := n - t - Y_{t-1} - \mathbb{1}_{\{Y_{t-1} = 0\}}$ (the number of unseen vertices at time $t$), $\mathcal{F}_t := \{\Omega, \emptyset\}$ and, for $t \in [n]$, $\mathcal{F}_t := \sigma(\{\eta_j : 1 \leq j \leq t\})$, the $\sigma$-algebra generated by the random variables $\eta_1, \ldots, \eta_t$ which provide the number of unseen vertices that become active during the first $t$ steps of the procedure. Note that $\eta_t$ has the $\text{Bin}(n - 1, p)$ distribution whereas, for $2 \leq t \leq n$ and given $\mathcal{F}_{t-1}$, we see that $\eta_t$ has the $\text{Bin}(U_{t-1}, p)$ distribution.

Define $Y'_t := 1 + \sum_{i=1}^{t} (\eta_i - 1)$, $\tau_0 := 0$, $\tau_0' := 0$ and recursively $\tau_j := \inf\{t > \tau_{j-1} : Y_t = 0\}$ and $\tau_j' := \inf\{t > \tau_{j-1}' : Y'_t = 0\}$, for $j \geq 1$. Denote by $|C_t|$ the size of the $j$-th explored component.

Since the excursions of $(Y_t)_t$ encodes the component structure of $G(n, p)$ and because $Y_t \geq 1 + \sum_{i=1}^{t} (\eta_i - 1) = Y'_t$, we see that $Y_t = 0$ implies $Y'_t \leq 0$. Hence, given any $T \in \mathbb{N}$, we can bound (as in Subsection 3.2)

$$\mathbb{P}(\|C_{\max}\| < T) = \mathbb{P}(\tau_j - \tau_{j-1} < T \text{ } \forall j \geq 1) \leq \mathbb{P}(\tau_j' - \tau_{j-1}' < T \text{ } \forall j \geq 1).$$

(38)

Proceeding as in the previous section, the idea is to show that (with high probability) $Y'_t$ reaches some level $h > 0$ before time $N \ni T^* = T^*(n) \ll n$ and then stays positive for at least $T$ steps. Consequently, we bound from above the probability on the right-hand side of (38) by

$$\mathbb{P}(Y'_t < h \text{ } \forall t \in [T^*]) + \mathbb{P}(\tau_j' - \tau_{j-1}' < T \text{ } \forall j \geq 1, \exists i \in [T^*] : Y_{i}' \geq h)$$

(39)

and estimate these two terms separately.
Proposition 4.1. Let \( T' = \lfloor n^{2/3} A^{-1/4} \rfloor \) and \( h = 12n^{1/3} A^{-1/4} \). Then, for all large enough \( n \), we have that
\[
P(Y'_t < h \ \forall t \in [T']) \leq \frac{C}{A^{1/8}},
\]
where \( C > 0 \) is some finite constant.

Proposition 4.2. Let \( T' = \lfloor n^{2/3} A^{-1/4} \rfloor \), \( T = \lfloor n^{2/3} A^{-1} \rfloor \) and \( h = 12n^{1/3} A^{1/4} \). Then, for all large enough \( n \), we have that
\[
P(\tau'_j - \tau'_{j-1} < T \ \forall j \geq 1, \exists t \in [T'] : Y'_t \geq h) \leq \frac{C'}{A^{1/4}},
\]
where \( C' > 0 \) is some finite constant.

Before proving Propositions 4.1 and 4.2, which together with (38) and (39) establish the statement in Theorem 1.1 concerning the \( \xi(n,p) \) model, we first need to establish a rough estimate for the number of active vertices at all times \( \ell \in [K] \) (for any given \( K \in \mathbb{N} \)), along the lines of what we did for the random sum \( \sum_{i=1}^{K} X_i \) in Section 3.2. To this end, let \( H \in \mathbb{N} \) and note that, for each \( i \in [K] \), can construct a sequence \((\xi_j)_{j \in [K]}\) of iid random variables with the Bin\((n,p)\) distribution such that \( Y_t = \sum_{j=1}^{\ell} \xi_j \) for every \( \ell \). Therefore, by Markov’s inequality we obtain (for all sufficiently large \( n \))
\[
P(\exists \ell \in [K] : Y_t \geq H) \leq \mathbb{P}\left(\exists \ell \in [K] : \sum_{j=1}^{\ell} \xi_j \geq H\right) \leq \mathbb{P}\left(\sum_{j=1}^{K} \xi_j \geq H\right) \leq \frac{2K}{H}. \tag{40}
\]

4.2.1 Bound on \( P(Y'_t < h \ \forall t \in [T']) \) – Proof of Proposition 4.1

With the purpose of coupling the \( \eta_i \) with smaller, independent random variables, we bound
\[
P(Y'_t < h \ \forall t \in [T']) \leq P(Y'_t < h \ \forall t \in [T'], Y_t \leq H \ \forall t \in [T']) + P(\exists t \in [T'] : Y_t \geq H),
\]
where \( N \ni H = H(n) \ll n \) is some positive integer to be specified later. We already know from (40) that the second probability on the right-hand side of the last inequality is at most \( 2T'/H \). On the other hand, recalling that \( \eta_i \) is a random variable with the Bin\((n-1,p)\) distribution while, given \( \mathcal{F}_{i-1} \) and for \( 2 \leq i \leq T' \), the random variable \( \eta_i \) has the Bin\((n-\delta_i-\sum_{i=1}^{\ell} \xi_j)\) distribution, on the event \( \{Y'_t < h \ \forall t \in [T'], Y_t \leq H \ \forall t \in [T']\} \) we can construct iid random variables \( \delta_i \) such that each \( \delta_i \) has the Bin\((n-\delta_i - T' - H, p)\) distribution and satisfies \( \delta_i \leq \eta_i \) for \( i \in [T'] \). Consequently we can bound
\[
P(Y'_t < h \ \forall t \in [T'], Y_t \leq H \ \forall t \in [T']) \leq P\left(\sum_{i=1}^{t} (\delta_i - 1) < h \ \forall t \in [T']\right). \tag{41}
\]

Now, with the purpose of constructing a random walk over the (discrete) interval \([T']\), we introduce a sequence of iid random variables \((L_i)_{i \in [T']}\), independent of \((\delta_i)_{i \in [T']}\), such that each \( L_i \) has the Bin\((T' + H, p)\) distribution, and set \( X_i := \delta_i + L_i \). Then each \( X_i \) has the Bin\((n, p)\) distribution and we rewrite the probability on the right-hand side of (41) as
\[
P\left(\sum_{i=1}^{t} (X_i - 1) < h + \sum_{i=1}^{t} L_i \ \forall t \in [T']\right) \leq P\left(\sum_{i=1}^{t} (X_i - 1) < h + \sum_{i=1}^{T'} L_i \ \forall t \in [T']\right), \tag{42}
\]
where the last inequality follows from the fact that \( L_i \geq 0 \) for each \( i \). Since \( E\left(\sum_{i=1}^{T'} L_i\right) = T'(T' + H)p \) and the \( L_i \) are independent, we can use Chebyshev’s inequality to bound
\[
P\left(\sum_{i=1}^{T'} L_i \geq T'(T' + H)p + x\right) \leq \frac{T'(T' + H)p}{x^2}.
\]
Therefore, taking \( x \leq h \) and \( T' \leq H \), so that \( h + T'(T' + H)p + x \leq 2(h + T'Hp) =: M \), we see that (42) is at most
\[
P \left( \sum_{i=1}^{t} (X_i - 1) < h + T'(T' + H)p + x \quad \forall t \in [T'] \right) + O \left( \frac{T'Hp}{x^2} \right)
\leq P \left( \sum_{i=1}^{t} (X_i - 1) < M \quad \forall t \in [T'] \right) + O \left( \frac{T'Hp}{x^2} \right). \tag{43}
\]

Recall that each \( X_i \) has the Bin\((n, p)\) distribution, and these random variables are also independent. Hence, by a standard coupling between binomial and Poisson random variables, there exist sequences \((\hat{X}_i)_{i \in [T']}, (\Delta_i)_{i \in [T']}\) of iid random variables (defined on a common probability space) such that each \( \Delta_i \) has the Pois\((np)\) distribution, \((\hat{X}_i)_{i \in [T']} \overset{d}{=} (X_i)_{i \in [T']}\) and
\[
P \left( \exists t \in [T'] : \hat{X}_i \neq \Delta_i \right) \leq \sum_{i=1}^{T'} P(\hat{X}_i \neq \Delta_i) \leq T' \left( \frac{1 + \lambda n^{-1/3}}{n} \right)^2 \leq 2T' n^{-1/3}
\]
for all sufficiently large \( n \). Therefore, we can bound from above the probability in (43) by
\[
P \left( \sum_{i=1}^{t} (\Delta_i - 1) < M \quad \forall t \in [T'] \right) + O \left( \frac{T'}{n} \right). \tag{44}
\]

We would like to apply Proposition 2.1 to estimate the probability in (44) and, to this end, we need a sequence of iid mean zero random variables (whose distribution does not depend on \( n \)). The random variables \( \Delta_i - 1 \) are iid but, unless \( \lambda \neq 0 \) (in which case \( \mathbb{E}(\Delta_i - 1) = 0 \) and \( P(\Delta_i - 1 = k) \) does not depend on \( n \) since each \( \Delta_i \) has the Pois\((1)\) distribution), they do not have mean zero and moreover the law of \( \Delta_i - 1 \) depends on \( n \). In order to remove the drift, we use Lemma 2.4. Recall that, in our case, we have \( \mathbb{E}(\Delta_i) = np = 1 + \lambda n^{-1/3} \). Let us start by assuming that \( \lambda \leq 0 \). In order to turn the \( \Delta_i \) into (iid) random variables with the Pois\((1)\) distribution, we take \( a = 1 + \lambda n^{-1/3} \) and \( b = \log \left( \frac{1}{1 + \lambda n^{-1/3}} \right) = - \log \left( 1 + \lambda n^{-1/3} \right) \) in Lemma 2.4.

As it occurred in the proof of Proposition 3.1, we will need a lower bound for \( \sum_{i=1}^{T'} (\Delta_i - 1) \) within the event which appears in (44). To this end, let \( m = m(n) \) be a positive quantity to be specified later and bound from above the probability in (44) by
\[
P \left( \sum_{i=1}^{t} (\Delta_i - 1) < M \quad \forall t \in [T'], \sum_{i=1}^{T'} \Delta_i \geq T'np - m \right) + P \left( \sum_{i=1}^{T'} \Delta_i < T'np - m \right) \tag{45}
\]
By Chebyshev’s inequality, the second probability in (45) is at most \( npT'/m^2 = O(T'/m^2) \), whereas the first probability in (45) equals
\[
\hat{E} \left( e^{-b \sum_{i=1}^{T'} \Delta_i} \mathbb{1}_{\{\sum_{i=1}^{T'} (\Delta_i - 1) < M \quad \forall t \in [T'], \sum_{i=1}^{T'} \Delta_i \geq T'np - m \}} \right) \hat{E} \left( e^{b \Delta_1} \right)^{T'}. \tag{46}
\]
Recalling that \( b = - \log \left( 1 + \lambda n^{-1/3} \right) \), a little algebra shows that
\[
\mathbb{E} \left[ e^{b \Delta_1} \right]^{T'} \leq e^{\log(1+\lambda n^{-1/3})T'-\lambda T'n^{-1/3}} \]
and so the expression in (46) is at most
\[
e^{\log(1+\lambda n^{-1/3})T'-\lambda T'n^{-1/3}}, \hat{E} \left( e^{\log(1+\lambda n^{-1/3}) \sum_{i=1}^{T'} (\Delta_i - 1) \mathbb{1}_{\{\sum_{i=1}^{T'} (\Delta_i - 1) < M \quad \forall t \in [T'], \sum_{i=1}^{T'} \Delta_i \geq T'np - m \}}} \right). \tag{47}
\]
Since \((-\infty, 0] \ni \log \left(1 + \lambda n^{-1/3}\right) = \lambda n^{-1/3} - O(\lambda^2 n^{-2/3})\) as \(n \to \infty\), we see that the exponential term multiplying the expectation in (47) is at most 1. Moreover, on the event which appears as argument of the indicator function in (47), we have that \(T^* \lambda n^{-1/3} - m < \sum_{i=1}^{T'} (\Delta_i - 1) < M\) and therefore

\[
\left| \sum_{i=1}^{T'} (\Delta_i - 1) \right| \leq T' |\lambda| n^{-1/3} + m + M.
\]

It follows that

\[
\log \left(1 + \lambda n^{-1/3}\right) \sum_{i=1}^{T'} (\Delta_i - 1) \leq \lambda n^{-1/3} \sum_{i=1}^{T'} (\Delta_i - 1) + O(n^{-1/3})
\]

\[
\leq \frac{\lambda^2 T'}{n^{2/3}} + \frac{m(-\lambda)}{n^{1/3}} + O(n^{-1/3})
\]

\[
= \frac{(\lambda \wedge 0)^2 T'}{n^{2/3}} + \frac{m|\lambda \wedge 0|}{n^{1/3}} + O(n^{-1/3}),
\]

whence

\[
e^{\log(1 + \lambda n^{-1/3}) \sum_{i=1}^{T'} (\Delta_i - 1)} \leq \exp \left\{ \frac{(\lambda \wedge 0)^2 T'}{n^{2/3}} + \frac{m|\lambda \wedge 0|}{n^{1/3}} + O(n^{-1/3}) \right\}.
\]

(48)

Denoting by \(\Phi\) the argument of the exponential function in (48) and plugging the last two estimates into (47), we conclude that the first probability in (45) is at most

\[
\exp \{\Phi\} \hat{P} \left( \sum_{i=1}^{t} (\Delta_i - 1) < M \ \forall t \in [T'] \right).
\]

(49)

Since, under \(\hat{P}\), the increments \(\Delta_i - 1\) are i.i.d. with mean zero and \(\hat{P}(\Delta_1 - 1 = k)\) does not depend on \(n\) (being \(\Delta_1\) a random variable with the Poi(1) distribution), we can apply Proposition 2.1 to bound

\[
\hat{P} \left( \sum_{i=1}^{t} (\Delta_i - 1) < M \ \forall t \in [T'] \right) \leq c_1 \frac{M}{(T')^{1/2}} + c_2 \frac{\log(T')}{(T')^{1/2}}
\]

(50)

for some finite constants \(c_1, c_2 > 0\). Now suppose that \(\lambda > 0\). Then we can couple the \(\Delta_i \sim \text{Poi}(1 + \lambda n^{-1/3})\) with i.i.d. random variables \(\Delta'_i \sim \text{Poi}(1)\) such that \(\Delta_i \geq \Delta'_i\) for all \(i\), so that

\[
\mathbb{P} \left( \sum_{i=1}^{t} (\Delta_i - 1) < M \ \forall t \in [T'] \right) \leq \mathbb{P} \left( \sum_{i=1}^{t} (\Delta'_i - 1) < M \ \forall t \in [T'] \right),
\]

and the probability on the right-hand side of the last inequality can be directly bounded by means of Proposition 2.1 (since \(\Delta'_i \sim \text{Poi}(1)\) does not depend on \(n\)) to retrieve the same bound as in (50).

Summarizing, we have shown that

\[
\mathbb{P} \left( Y_t < h \ \forall t \in [T'] \right) \leq \exp \{\Phi\} \left( \frac{c_1 M}{(T')^{1/2}} + \frac{c_2 \log(T')}{(T')^{1/2}} \right)
\]

\[+ O \left( \frac{T'}{n} \right) + O \left( \frac{T'Hp}{x^2} \right) + O \left( \frac{T'}{H} \right) + O \left( \frac{T'}{m^2} \right).
\]

(51)

Therefore, taking \(T', h\) as in the statement of the proposition and setting e.g. \(x = n^{1/6}, H = [n^{2/3}]\) and \(m = n^{1/3}/A^{1/16}\) (as it occurred while analysing the \(\hat{\Theta}(n,d,p)\) model) we see that
the expression on the right-hand side of (51) is at most $C/A^{1/8}$ for some finite constant $C > 0$, where we have used that (by definition of $M$)

$$\frac{M}{(T')^{1/2}} = \frac{2(h + T'Hp)}{(T')^{1/2}} < A^{-1/8} \gg \frac{\log(T')}{(T')^{1/2}},$$

as shown in (50). This completes the proof of the proposition.

### 4.2.2 Bound on $P(\tau'_j - \tau'_{j-1} < T \forall j \geq 1, \exists t \in [T'] : Y'_t \geq h)$ - Proof of Proposition 3.2

Define $\tau_h := \inf \{ t \geq 1 : 1 + \sum_{i=1}^{t}(\eta_i - 1) \geq h \}$. Proceeding in the same way as we did to prove Proposition 3.2 we arrive at

$$P(\tau'_j - \tau'_{j-1} < T \forall j \geq 1, \exists t \in [T'] : Y'_t \geq h) \leq P(\tau_h \leq T', \exists t \in [T] : \sum_{j=1}^{t}(\eta_{j+\tau_h} - 1) \leq -h). \quad (52)$$

On the event $\{\tau_h \leq T'\}$ we have $j + \tau_h \leq T + T'$ for all $j \in [T]$. Also, by (40) we know that $Y_t < H$ for all $\ell \in [T + T']$ with probability at least $1 - O((T + T')/H)$, whence the probability on the right-hand side of (52) is at most

$$P(\tau_h \leq T', \exists t \in [T] : \sum_{j=1}^{t}(\eta_{j+\tau_h} - 1) \leq -h, Y_t < H \forall \ell \in [T + T']) + O\left(\frac{T + T'}{H}\right). \quad (53)$$

On the event $\{\tau_h \leq T', Y_t < H \forall \ell \in [T + T']\}$ we can construct a sequence $(\delta_j)_{j \in [T]}$ of iid random variables such that each $\delta_j$ has the Bin$(n - T - T' - H, p)$ distribution and satisfies $\delta_j \leq \eta_{j+\tau_h}$ for all $j \in [T]$. Hence the probability in (53) is at most

$$P\left(\exists t \in [T] : \sum_{j=1}^{t}(\delta_j - 1) \leq -h\right). \quad (54)$$

In order to obtain a random walk on $[T]$, we proceed as we did in the proof of Proposition 4.1. Let $(W_j)_{j \in [T]}$ be a sequence of iid random variables with the Bin$(T + T' + H, p)$ distribution, also independent of $(\delta_j)_{j \in [T]}$. Then each $X_j := \delta_j + W_j$ has the Bin$(n, p)$ distribution, the $X_j$ are independent and the probability in (54) equals

$$P\left(\exists t \in [T] : \sum_{j=1}^{t}(X_j - 1) \leq -h + \sum_{j=1}^{T}W_j\right) \leq P\left(\exists t \in [T] : \sum_{j=1}^{t}(X_j - 1) \leq -h + \sum_{j=1}^{T}W_j\right). \quad (55)$$

By Chebyshev’s inequality, we obtain

$$P\left(\sum_{j=1}^{T}W_j \geq \mathbb{E}\left[\sum_{j=1}^{T}W_j\right] + q\right) \leq \frac{T(T + T' + H)p}{q^2}. \quad (56)$$

Taking $T, T' \leq H$ we see that $\mathbb{E}\left[\sum_{j=1}^{T}W_j\right] = T(T + T' + H)p \leq 3THp$ and hence we can bound from above the probability on the right-hand side of (55) by

$$P\left(\exists t \in [T] : \sum_{j=1}^{t}(X_j - 1) \leq -h + 3THp + q\right) + O\left(\frac{THp}{q^2}\right). \quad (56)$$
Let $H = \lceil n^{2/3}/C_0 \rceil$ for some finite numerical constant $C_0 \geq 10$, and recall that $T^* = \lceil n^{2/3}/A^{1/4} \rceil$, $h = \lceil n^{1/3}/A^{1/4} \rceil$. Taking $q = q(n) \leq h/3$ we see that

$$-h + 3 \frac{T H}{n} + q \leq -h + 3 \frac{n^{1/3}}{C_0 A} + \frac{h}{3} \leq -\frac{2}{3} h + 3 \frac{n^{1/3}}{C_0 A} \leq -\frac{h}{3},$$

whence the probability in (56) is at most

$$P\left( \exists t \in [T] : \sum_{j=1}^{t} (X_j - 1) \leq -h/3 \right). \quad (57)$$

We would like to bound the last probability using Kolmogorov’s inequality and, to this end, we first need to turn the increments $X_j - 1$ into mean zero random variables. Proceeding in the same way as we did while analysing the $O(n, d, p)$ random graph, we write

$$P\left( \exists t \in [T] : \sum_{j=1}^{t} (X_j - 1) \leq -h/3 \right) = P\left( \exists t \in [T] : \sum_{j=1}^{t} (X_j - np) \leq -h/3 - t\lambda n^{-1/3} \right). \quad (58)$$

Suppose first that $\lambda \leq 0$. Then the probability on the right-hand side of (58) is at most

$$P\left( \exists t \in [T] : \sum_{j=1}^{t} (X_j - np) \leq -h/3 + T|\lambda| n^{-1/3} \right) = P\left( \exists t \in [T] : \sum_{j=1}^{t} \left| X_j - np \right| \leq -h/3 \left( 1 - \frac{3T|\lambda|}{n^{1/3}h} \right) \right). \quad (59)$$

Recalling the definitions of $T$ and $h$, a quick computation shows that $3T|\lambda|/n^{1/3}h \leq 1/2$, whence the probability on the right-hand side of (59) is at most

$$P\left( \exists t \in [T] : \sum_{j=1}^{t} \left| X_j - np \right| \leq -h/6 \right) \leq P\left( \max_{t \in [T]} \left| \sum_{j=1}^{t} (X_j - np) \right| > h/6 \right) = O\left( \frac{T}{h^2} \right).$$

If $\lambda > 0$, then the probability on the right-hand side of (58) is at most

$$P\left( \exists t \in [T] : \sum_{j=1}^{t} (X_j - np) \leq -h/3 \right) = O\left( \frac{T}{h^2} \right)$$

too. Summarizing, we have shown that

$$P\left( \tau'_j - \tau'_{j-1} < T \forall j \geq 1, \exists t \in [T'] : Y^t_i \geq h \right) \leq O\left( \frac{T}{h^2} \right) + O\left( \frac{T}{m^2} \right) + O\left( \frac{4T}{n} \right) + O\left( \frac{THp}{q^2} \right) + O\left( \frac{T'}{H} \right).$$

Taking $T, T', h$ as in the statement of the proposition and recalling that $H = \lceil n^{2/3}/C_0 \rceil$, if we take $q = h/3$ and $m = n^{1/3}/A^{1/4}$ we see that there exists a finite constant $C' > 0$ such that the expression on the right-hand side of the last inequality is at most $C'/A^{1/4}$, completing the proof of the proposition.
5 Proof of Proposition 1.3

Here we let $G_d$ be any $d$-regular graph on $[n]$ and denote by $G_d(p)$ the $p$-percolated version of $G_d$, where $p \leq 1/(d-1)$, and $3 \leq d = d(n) < n - 1$ is allowed to depend on $n$. By a monotonicity argument, we can focus on the (critical) case $p = (d - 1)^{-1}$.

We are interested in providing an upper bound for $P(|C_{\max}(G_d(p))| > An^{2/3})$. To this end observe that, denoting by $V_n$ a node selected uniformly at random from the vertex set $[n]$ (in $G_d(p)$) and independently of the percolation process, if

$$P(|C(V_n)| > |An^{2/3}|) \leq \frac{c}{\sqrt{An^{1/3}}} \quad (60)$$

for some finite constant $c > 0$ then, denoting by

$$N := \sum_{u \in [n]} 1_{|C(u)| > |An^{2/3}|}$$

the number of nodes lying in components containing more than $|An^{2/3}|$ vertices, we obtain (using Markov’s inequality)

$$P(|C_{\max}(G_d(p))| > An^{2/3}) \leq P(|C_{\max}(G_d(p))| > |An^{2/3}|)$$

$$\leq P(N > |An^{2/3}|) \leq \frac{\mathbb{E}(N)}{|An^{2/3}|} = \frac{n}{|An^{2/3}|} \sum_{u \in [n]} P(|C(V_n)| > |An^{2/3}|, V_n = u)$$

$$= \frac{n}{|An^{2/3}|} \sum_{u \in [n]} P(|C(V_n)| > |An^{2/3}|) \leq \frac{c}{|An^{2/3}|} \sqrt{An^{1/3}}$$

$$\leq \frac{c}{An^{1/3}} \left(1 + O\left(1/(An^{2/3})\right)\right).$$

Therefore, in order to obtain the bound stated in Proposition 1.3 we need to prove (60) with $c < 3\sqrt{(d-2)(d-1)^{-1}(1 - 1/(d-1))^{-2(d-1)}}$. To achieve this we use once more an exploration process, in which vertices are sequentially explored one by one and where, at each step, the unseen neighbours of the node under examination are declared active whereas the vertex itself becomes explored. However, rather than proceeding as in [6], where the exploration process was started from two active vertices and that caused the stronger requirement $d \geq 4$ (recall that here we are only assuming $d \geq 3$), we use the same exploration process of Section 4.1 started from a node $V_n$ selected uniformly at random from $[n]$ in $G_d(p)$. That is, at time $t = 0$ we declare $V_n$ active, whereas all the remaining $n - 1$ nodes are in status unseen, and proceed as we did for the $G(n, p)$ model.

Setting $\tau := \min\{t \geq 1 : Y_t = 0\}$ (with $Y_t$ denoting the number of active vertices at time $t$), we have the equality $|C(V_n)| = \tau$. Indeed, denoting by $E_i$ the set of explored vertices at the end of step $i$, we have $E_\tau = C(V_n)$, whence $|C(V_n)| = |E_\tau| = \tau$, where the last identity follows from the fact that at each step $i$ during the exploration of $C(V_n)$ we explore exactly one vertex. Therefore we obtain the useful identity

$$P(|C(V_n)| > |An^{2/3}|) = P(\tau > |An^{2/3}|). \quad (61)$$

Since $G_d$ is $d$-regular and edges are retained independently with probability $p$, it is not difficult to see that the random variables $Y_t$ can be coupled with independent random variables $X_i$ satisfying $Y_t \leq X_t$, and such that $X_1$ has the $\text{Bin}(d,p)$ distribution, whereas the $X_i$ (for $i \geq 2$) have the $\text{Bin}(d - 1, p)$ distribution. We claim that, for all $0 \leq t \leq \tau$, we have $Y_t \leq 1 + \sum_{i=1}^{t}(X_i - 1)$. To see this, note that the inequality is trivially true for $t = 0$. Next, suppose that it holds for some $1 \leq t \leq \tau - 1$, and observe that (since $Y_t \geq 1$)

$$Y_{t+1} = Y_t + \eta_{t+1} - 1 \leq Y_t + X_{t+1} - 1 \leq 1 + \sum_{i=1}^{t+1}(X_i - 1),$$

where the last inequality follows from the inductive hypothesis.
Now suppose that $\tau > \lfloor An^{2/3} \rfloor$; then $Y_t > 0$ for all $t \leq \lfloor An^{2/3} \rfloor$, whence (as $\lfloor An^{2/3} \rfloor < \tau$) we obtain $1 + \sum_{i=1}^{t}(X_i - 1) > 0$ for all $t \leq \lfloor An^{2/3} \rfloor$. It follows that, setting $S_t := 1 + \sum_{i=1}^{t}(X_i - 1)$,

$$\mathbb{P}(\tau > \lfloor An^{2/3} \rfloor) \leq \mathbb{P}\left( S_t > 0 \forall 1 \leq t \leq \lfloor An^{2/3} \rfloor \right). \tag{62}$$

In order to apply Lemma 2.6 (which requires a random walk with iid increments), we first need to substitute $X_1$ (which has the Bin($d-1, p$) distribution) with a random variable having the Bin$(d-1, p)$ distribution. To this end, first of all observe that we can assume each $X_1$ to be of the form $X_1 := \sum_{j=1}^{d} I_j$, with $(I_j)_{j \in [d]}$ a sequence of iid random variables (also independent of the $X_i, i \geq 2$) having the Ber($p$) distribution. Then, setting $k = k(A, n) := \lfloor An^{2/3} \rfloor$, we can rewrite the probability on the right-hand side of (62) as

$$\mathbb{P}(S_t > 0 \forall t \in [k]) = \mathbb{P}\left( I_d + \sum_{j=1}^{d-1} I_j + \sum_{i=2}^{t}(X_i - 1) > 0 \forall t \in [k] \right) = \mathbb{P}\left( 1 + I_d + \left( \sum_{j=1}^{d-1} I_j - 1 \right) + \sum_{i=2}^{t}(X_i - 1) > 0 \forall t \in [k] \right) = \mathbb{P}\left( 1 + I_d + \sum_{i=1}^{t}(X_i' - 1) > 0 \forall t \in [k] \right),$$

where we set $X_i' := \sum_{j=1}^{d-1} I_j$ and $X_i' := X_i$ for $2 \leq i \leq k$. Now let $(J_j)_{j \in [d-2]}$ be a sequence of iid random variables with the Ber($p$) distribution, independent of all other random quantities involved. Define $Y_0 := \sum_{j=1}^{d-2} J_j + I_d$. Then, since

$$\mathbb{P}\left( \sum_{j=1}^{d-2} J_j = 1 \right) = \mathbb{P}(\text{Bin}(d-2, p) = 1) = (d-2)p(1-p)^{d-3} = \frac{d-2}{d-2} \left( 1 - \frac{1}{d-1} \right)^{d-1}$$

and $Y_0 - I_d = \sum_{j=1}^{d-2} J_j$, using independence between the $I_j$ and the $J_j$ we can write

$$\mathbb{P}\left( 1 + I_d + \sum_{i=1}^{t}(X_i' - 1) > 0 \forall t \in [k] \right) = \frac{d-2}{(d-1)(1 - 1/(d-1))^{d-2}} \mathbb{P}\left( 1 + I_d + \sum_{i=1}^{t}(X_i' - 1) > 0 \forall t \in [k], Y_0 - I_d = 1 \right) \geq \frac{d-2}{(d-1)(1 - 1/(d-1))^{d-2}} \mathbb{P}\left( Y_0 + \sum_{i=1}^{t}(X_i' - 1) > 0 \forall t \in [k], Y_0 = I_d + 1 \right) \geq \frac{d-2}{(d-1)(1 - 1/(d-1))^{d-2}} \mathbb{P}\left( Y_0 + \sum_{i=1}^{t}(X_i' - 1) > 0 \forall t \in \{0\} \cup [k] \right) = \frac{d-2}{(d-1)(1 - 1/(d-1))^{d-2}} \mathbb{P}\left( 1 + (Y_0 - 1) + \sum_{i=1}^{t}(X_i' - 1) > 0 \forall t \in \{0\} \cup [k] \right).$$

Setting $X''_0 := Y_0, X''_i := X'_i$ for $i \in [k]$ and $S'_t := \sum_{i=0}^{t}(X''_i - 1)$ for $t \in \{0\} \cup [k]$, we arrive at

$$\frac{d-2}{(d-1)(1 - 1/(d-1))^{d-2}} \mathbb{P}\left( 1 + (Y_0 - 1) + \sum_{i=1}^{t}(X'_i - 1) > 0 \forall t \in \{0\} \cup [k] \right) = \frac{d-2}{(d-1)(1 - 1/(d-1))^{d-2}} \mathbb{P}(1 + S'_t > 0 \forall t \in [k]). \tag{63}$$
At this stage, we are ready to apply Lemma 2.6 since the $X''_i$ are iid having the Bin$(d-1, p)$ distribution and $P(X''_i = 1) = P(\text{Bin}(d-1, p) = 2) = 2^{-1}(d-1)(d-2)p^2(1-p)^{d-3} > 0$ for $d \geq 3$. Before doing this, however, we let $\omega = \omega(n) \leq \sqrt{k}$ be some non-negative quantity that we specify later and write

$$P \left( 1 + S'_t > 0 \forall t \in [k] \right)$$

$$\leq P \left( 1 + S'_t > 0 \forall t \in [k], 1 + S'_k \leq \omega((1-p)k)^{1/2} \right) + P \left( 1 + S'_k > \omega((1-p)k)^{1/2} \right);$$

by means of (2.8) we bound

$$P \left( 1 + S'_k > \omega((1-p)k)^{1/2} \right) \leq P \left( S'_k \geq \omega((1-p)k)^{1/2} \right) \leq \exp \left\{ -\frac{\omega^2(1-p)k}{2k + \frac{3}{2}\omega((1-p)k)^{1/2}} \right\} \leq \exp \left\{ -\frac{\omega^2(1-p)k}{3k} \right\} = e^{-\frac{\omega^2(d-2)}{3(d-1)}}.$$

Taking $\omega = n^{1/9}$ we arrive at

$$P \left( 1 + S'_t > 0 \forall t \in [k] \right) \leq P \left( 1 + S'_t > 0 \forall t \in [k], 1 + S'_k \leq \omega((1-p)k)^{1/2} \right) + e^{-\frac{n^{2/9}(d-2)}{3(d-1)}}.$$

Next, we apply Lemma 2.6 and write (using the fact that $2^{-1}(d-1)(d-2)p^2(1-p)^{d-3} = (d-2)(d-1)^{-1}(1-1/(d-1))^{d-1}$)

$$P \left( 1 + S'_t > 0 \forall t \in [k], 1 + S'_k \leq \omega((1-p)k)^{1/2} \right) = \sum_{j=1}^{\omega((1-p)k)^{1/2}} P \left( 1 + S'_t > 0 \forall t \in [k], 1 + S'_k = j \right) \leq \frac{d-1}{(d-2)(d-1)^{d-1}} \sum_{j=1}^{\omega((1-p)k)^{1/2}} \frac{j}{k+1} P(S'_{k+1} = j).$$

We split the last sum into two parts: we set $\mathcal{I}_1 := \{1, \ldots, \lfloor Ck^{1/2} \rfloor \}$ and $\mathcal{I}_2 := \{ \lfloor Ck^{1/2} \rfloor + 1, \ldots, \lfloor \omega((1-p)k)^{1/2} \rfloor \}$, where $C > 0$ is some finite constant to be specified later (recall that $\omega = n^{1/9}$). Then, since

$$\sum_{j \in \mathcal{I}_1} \frac{j}{k+1} P(S'_{k+1} = j) \leq \frac{Ck^{1/2}}{k+1} \sum_{j \in \mathcal{I}_1} P(S'_{k+1} = j) = \frac{Ck^{1/2}}{k+1} P(S'_{k+1} \in \mathcal{I}_1) \leq \frac{Ck^{1/2}}{k+1} \leq \frac{C}{\sqrt{k}},$$

we can bound

$$\sum_{j=1}^{\lfloor \omega((1-p)k)^{1/2} \rfloor} \frac{j}{k+1} P(S'_{k+1} = j) \leq \frac{C}{\sqrt{k}} + \sum_{j \in \mathcal{I}_2} \frac{j}{k+1} P(S'_{k+1} = j).$$

(64)

There remains to bound the sum on the right-hand side of (64). To this end, note that for $j \in \mathcal{I}_2$ we can use Lemma 2.7 to bound

$$P(S'_{k+1} = j) = P(\text{Bin}((d-1)(k+1), p) = (d-1)(k+1)p + j)$$

$$\leq \frac{(2\pi)^{-1/2}}{[p(1-p)(d-1)(k+1)]^{1/2}} \exp \left\{ -\frac{j^2}{2p(1-p)(d-1)(k+1) + \Psi} \right\}.$$

(65)
where we set
\[
\Psi := \frac{j}{(1 - p)(d - 1)(k + 1)} + \frac{j^3}{p^2(d - 1)^2(k + 1)^2}.
\] (66)

We claim that \(\Psi\) is bounded from above by \(2/\sqrt{A}\). To see this, note that the denominator in the first ratio on the right-hand side of (66) satisfies \((1 - p)(d - 1)(k + 1) = (d - 2)(k + 1)\), whereas the denominator in the second ratio satisfies \(p^2(d - 1)^2(k + 1)^2 = (k + 1)^2\). Since \(j \leq [\omega((1 - p)k)^{1/2}]\) for all \(j \in \mathbb{Z}_2\) and \(k = [An^{2/3}]\), we see that the first ratio in (66) is at most
\[
\frac{\omega((1 - p)k)^{1/2}}{k(d - 2)} \leq \frac{\omega}{\sqrt{k}} = O\left(\frac{1}{A^{1/2}n^{2/3}}\right) \leq \frac{1}{n^{2/3}} \leq 1.
\]

On the other hand, the second ratio on the right-hand side of (66) is bounded from above by \(\omega^3k^{-1/2} \leq A^{-1/2}(1 + o(1))\). Thus we conclude that \(\Psi \leq (1 + o(1))/\sqrt{A} \leq 2/\sqrt{A}\) for all large enough \(n\), establishing the claim.

Consequently, since the denominator of the ratio in (65) is such that
\[
p(1 - p)(d - 1)(k + 1) = \frac{d - 2}{d - 1}(k + 1),
\]
when \(n\) is large we can bound
\[
\frac{(2\pi)^{-1/2}}{[p(1 - p)(d - 1)(k + 1)]^{1/2}} \exp\left\{\frac{j^2}{2(p(1 - p)(d - 1)(k + 1)) + \Psi}\right\} \leq \frac{(2\pi)^{-1/2}e^{2\sqrt{A}}}{\left[\frac{d - 2}{d - 1}(k + 1)\right]^{1/2}} e^{-\frac{j^2}{2}\frac{1}{d - 1}} e^{-\frac{\Psi}{\sqrt{A}}} e^{\frac{\Psi}{\sqrt{A}}}. \]

Therefore we arrive at
\[
\sum_{j \in \mathbb{Z}_2} \frac{j}{k + 1} \mathbb{P}(S'_{k+1} = j) \leq \sum_{j \in \mathbb{Z}_2} \frac{j}{k} \left[\frac{d - 2}{d - 1}\right]^{1/2} e^{-\frac{j^2}{2}\frac{1}{d - 1}} e^{\frac{\Psi}{\sqrt{A}}}. \] (67)

With the purpose of bounding the sum in (67) note that, since \(j \geq [C\sqrt{k}]\), we obtain
\[
j^2(d - 1) \geq C(d - 1)j \frac{j}{2(d - 2)k} = j(d - 1) \frac{j}{2(d - 2)k}
\]
and hence (since we also have \(j \leq [\omega((1 - p)k)^{1/2}]\)) the exponential term in (67) satisfies
\[
e^{-\frac{j^2}{2(d - 2)k} e^{\frac{\Psi}{\sqrt{A}}}} \leq e^{-\frac{C(d - 1)j}{2(d - 2)\sqrt{k}}} e^{-\frac{\Psi}{\sqrt{A}}} e^{\frac{\Psi}{\sqrt{A}}} = e^{-\frac{C(d - 1)j}{2(d - 2)\sqrt{k}}} \left(1 + O\left(\frac{\omega}{2\sqrt{k}} \left(\frac{d - 1}{d - 2}\right)^{1/2}\right)\right)
\]
where we have used that \(\omega \ll \sqrt{k}\). Thus the expression on the right-hand side of (67) is at most
\[
\left(1 + O\left(\frac{\omega}{2\sqrt{k}} \left(\frac{d - 1}{d - 2}\right)^{1/2}\right)\right) \sum_{j \in \mathbb{Z}_2} \frac{j}{k} \left[\frac{d - 2}{d - 1}\right]^{1/2} e^{-\frac{C(d - 1)j}{2(d - 2)\sqrt{k}}} e^{\frac{\Psi}{\sqrt{A}}}. \]

A simple computation then shows that the integral above is at most
\[
\frac{2(d - 2)}{d - 1} \left(1 + \frac{2(d - 2)}{C^2(d - 1)}\right) ke^{-\frac{C^2(d - 1)}{2(d - 2)}} \left(1 + O\left(\frac{d - 1}{(d - 2)\sqrt{k}}\right)\right).
\]

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Therefore, taking $C = 2[(d - 2)/(d - 1)]^{1/2}$ we arrive at

$$\sum_{j \in I_2} \frac{j}{k+1} \mathbb{P}(S_{k+1}' = j) \leq \left(1 + O\left(\frac{\omega}{2\sqrt{k}} \left(\frac{d-1}{d-2}\right)^{1/2}\right)\right) \sqrt{\frac{d-2}{(d-1)^2}} \frac{3}{\sqrt{k}} e^{2/k}$$

and hence

$$\sum_{j=1}^{\lfloor (1-p)k \rfloor^{1/2}} \frac{j}{k+1} \mathbb{P}(S_{k+1}' = j) \leq \frac{2 \sqrt{d-2}}{d-1} + \left(1 + O\left(\frac{\omega}{2\sqrt{k}} \left(\frac{d-1}{d-2}\right)^{1/2}\right)\right) \sqrt{\frac{d-2}{(d-1)^2}} \frac{3}{\sqrt{k}} e^{2/k} \leq 2.5 \sqrt{\frac{d-2}{d-1}} \frac{1}{\sqrt{k}}$$

for all large enough $n$. Summarizing and putting all pieces together we have shown that (when $n$ is large enough)

$$\mathbb{P}(|\mathcal{C}(V_n)| > \lfloor An^{2/3} \rfloor) \leq \frac{2.6}{\sqrt{k}} \frac{\sqrt{d-2}}{d-1} \left(1 - \frac{1}{d-1}\right)^{-2(d-1)},$$

completing the proof of the proposition.

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