On the power of one bit:
How to explore a graph when you cannot backtrack?

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Abstract

We consider the problem of exploration of an anonymous, port-labelled, undirected graph with \( n \) nodes, \( m \) edges, by a single mobile agent. Initially the agent does not know the topology of the graph nor any of the global parameters. Moreover, the agent does not know the incoming port when entering to a vertex thus it cannot backtrack its moves. We study a \((M_a, M_w)\)-agent model with two types of memory: \( M_a \) bits of internal memory at the agent and \( M_w \) bits of local memory at each node that is modifiable by the agent upon its visit to that node. In such a model, condition \( M_a + M_w \geq \log_2 d \) has to be satisfied at each node of degree \( d \) for the agent to be able to traverse each edge of the graph. As our main result we show an algorithm for \((1, O(\log d))\)-agent exploring any graph with return in optimal time \( O(m) \). We also show that exploration using \((0, \infty)\)-agent sometimes requires \( \Omega(n^3) \) steps. On the other hand, any algorithm for \((\infty, 0)\)-agent is a subject to known lower bounds for Universal Traversal Sequence thus requires \( \Omega(n^4) \) steps in some graphs. We also observe that neither \((0, \infty)\)-agent nor \((\infty, 0)\)-agent can stop after completing the task without the knowledge of some global parameter of the graph. This shows separation between the model with two types of memory and the models with only one in terms of exploration time and the stop property.

1 Introduction

Imagine you want to visit every vertex (or every edge) of an initially unknown graph, being physically located in this graph. This problem, called graph exploration is among the basic problems investigated under a context of a mobile agent in graphs. A common question arises: “What are the minimal capabilities of the agent for it to be able to complete the graph exploration?”, where we are interested in necessary memory and model assumptions to be either able to complete this task at

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all, or to be able to complete this task efficiently. One approach to solve the graph exploration by a mobile agent would be to implement a standard Depth-First-Search (DFS) algorithm. However, to perform the DFS, the agent needs to mark the visited nodes (or remember their position in its internal memory) and needs to be able to backtrack its moves. However, assumption of knowledge of the incoming port can be a strong one, and one has to ask: “What is the price of having no direct backtracking capability?”

When exploring an unknown port-labelled graph with a mobile agent, three prevalent approaches appear in literature: usage of randomness, existence of global memory and existence of local memory. Global memory can be modelled as internal state memory of the agent that remains intact when the agent traverses an edge. Local memory is a state of a node that can be modified by the agent upon its visit to the node. Randomness can be used either for amplification of any of global/local memory approaches, or as a sole source of oblivious agent movement.

It turns out that each of these approaches is a subject to certain limitations. A agent allowed to use only randomness can only do the random walk and thus achieves the worst-case cover time of $\Omega(n^3)$ [10]. A deterministic agent with only global memory cannot explore faster than $\Omega(n^4)$ in the worst case (due to lower bounds for Universal Traversal Sequences [5]). Finally we show in this paper that any deterministic agent using only local memory explores some graphs in time $\Omega(n^3)$.

It turns out that speeding up the exploration can be achieved by using combined approaches. It is possible to explore certain graphs faster using a random walks that prefers unvisited edges [4] which can be implemented using the local memory. In this work we consider a combined approach that uses both global memory and local memory but the approach is purely deterministic.

1.1 Preliminaries

The graph $G = (V,E)$ is assumed to be anonymous, there is no global labelling of the nodes nor the edges. In order to navigate in the graph, the agent needs to locally distinguish between the edges outgoing from its position thus we assume that all edges outgoing from a fixed vertex with degree $d$ are distinctly labelled using numbers $\{1, 2, \ldots, d\}$.

The agent is modelled as a finite state machine with $M_a$ bits of state memory. Each node contains a label of $M_w$ bits which can be read and modified by the agent upon its visit to that node. Such a model will be referred to as $(M_a, M_w)$-agent. Size of the memory on each node can depend on the degree of the node.

Let us denote by $S_w = \{0, 1, \ldots, 2^{M_w} - 1\}$ the set of states of a node and by $S_a = \{0, 1, \ldots, 2^{M_a} - 1\}$ the set of states of the agent. A $(M_a, M_w)$-agent is then defined as a function $f : S_a \times S_w \times \mathbb{N} \to S_a \times S_w \times \mathbb{N}$ whose input is a triple $(s_a, s_w, d) \in S_a \times S_w \times \mathbb{N}$, where $s_a$ is the current state of the agent, $s_w$ is the state of the node currently occupied by the agent and $d$ is the degree of the node. The output of function $f$ is a triple $(s'_a, s'_w, p) \in S_a \times S_w \times \{1, 2, \ldots, d\}$, where $s'_a$ is a new state of the agent, $s'_w$ is the new state of the currently occupied node and $p$ is the port number through which the agent exits the current node in the current step. We say that the agent is located at a node $v$ at the beginning of some step $t$, then traverses the chosen arc during step $t$ and appears at the other end of the arc at the beginning of step $t+1$. Initially, each node is in a starting state $s_w^0$ and the agent is in a starting state $s_a^0$.

Observe that the label of the port through which the agent entered to the current node is not part of the input. Thus the agent cannot easily backtrack its moves. We call this feature of the
model unknown inport. Most of the existing research in graph exploration assumes known inport thus in this paper we would like also to discuss the importance of this assumption.

**Graph exploration problem.** The goal of the agent is to visit all vertices of graph $G$. We assume that the initial position of the agent in the graph as well as the port-labelling of the edges can be chosen by an adversary. Initially the agent has no knowledge about the topology of $G$ or even its size.

Concerning the termination of the algorithm, three variants are considered in literature. In *perpetual exploration*, the agent is required to repeatedly visit all vertices of the graph. In *exploration with stop*, the agent has to stop after visiting all vertices. Finally in *exploration with return*, it is required that the agent terminates the algorithm in its starting position.

We start with a simple lower bound on the total memory on the agent and on each node of any algorithm that explores all unknown graphs.

**Observation 1.1.** If a $(M_a, M_w)$-agent explores all graphs in the model with unknown inport, then $M_a + M_w \geq \log_2 d$ holds for any node with degree $d$.

*Proof.* Since the graph is unknown in advance, the agent has to traverse all outgoing edges from each node thus it has to traverse each edge in both directions. Take any node $v$ and let $d$ be its degree. The number of different outputs of the algorithm at $v$ has to be at least $d$ for the agent to traverse all edges outgoing from $v$. As we consider only deterministic algorithms thus the total number of different inputs also has to be at least $d$. This implies that there exists no algorithm where the total number of bits at the agent and at $v$ is less than $\log_2 d$. \hfill \Box

### 1.2 Our results

As the main result of this paper we present an algorithm for $(1, O(\log d))$-agent that explores any graph with return in time $O(m)$, in model with unknown inport. Time complexity of the algorithm is asymptotically optimal, as any agent needs $\Omega(m)$ time, since $n$ is unknown to the agent.

We also prove that any algorithm for an agent with 0 bits of memory requires time $\Omega(n^3)$ for some graphs regardless of the sizes of node memory. Our results show that the only way to achieve deterministic exploration in time $O(m)$ of all graphs is to have memory both at the agent and on the nodes. The results also show a twofold separation between the model with memory on both parties (agent and nodes) and the model with memory only on one. Exploration in time $O(m)$ is not possible by any $(\infty, 0)$-agent nor by any $(0, \infty)$-agent, but is possible by a $(1, O(\log d))$-agent. Moreover, exploration with stop is not possible by any $(\infty, 0)$-agent \[9\] nor by any $(0, \infty)$-agent (by Observation 3.4) but in the model with memory on both parties it is even possible to perform exploration with return.

When considering the memory efficiency of our algorithm one can observe that removing the bit from the agent leads to $\Omega(n^3)$ lower bound and total memory of $\Omega(\log d)$ is needed by Observation 1.1.

Our algorithm can also be used for perpetual exploration of the graph, because after the agent has completed the exploration, all nodes of the graph are in the initial state.
Graph | $(1, O(\log d))$-agent | $(\infty, 0)$-agent | $(0, \infty)$-agent |
|---|---|---|---|
| 2-regular graph: | $O(n)$ | $\Omega(n^{1.51})$ | $\Omega(n^2)$ |
| General graph: | $O(m)$ | $\Omega(n^4)$ | $\Omega(n^3)$ |

Table 1: Summary of the optimal exploration time depending on the location of the memory by a $(M_a, M_w)$–agent, where $M_a$ is number of bits of internal memory at the agent and $M_w$ is number of bits of local memory at each node. The lower bounds for general graphs hold for $\Delta = \Omega(n)$.

Observe that with known inport and $\lceil \log_2 d \rceil$-bits of memory at each node with degree $d$, an oblivious agent can execute DFS and thus explore in $O(m)$. Thus one can interpret our results as showing that lack of the knowledge of inport can be overcame by adding one bit of memory to the agent.

Summary of our results is presented in Table 1.

### 1.3 Related work

**Memory both on the agent and on the nodes.** Labelling schemes for exploration of undirected graphs in the model with known inport were considered by Cohen et al. [6]. They showed a labelling scheme which assigns one of three states to each node such that an agent with $O(1)$ bits using the labels can explore the graph in time $O(m)$. The labels can be assigned online by the agent in time $O(mD)$. The second labeling scheme uses only 2 states on each node and allows an agent with $O(\log \Delta)$-bits to explore in time $O(\Delta O(1)m)$, where $\Delta$ is the maximum degree of $G$. In the model with known inport, Observation 1.1 does not hold because the number of different inputs to the algorithm at some node is always at least the degree of the node.

In exploration of directed graphs, considered by Fraigniaud and Ilcinkas [11], the inport cannot be known as any edge may be only in one direction. The authors showed an algorithm for exploration with return for an agent with $O(1)$ bits of memory and with $O(\log d)$ bits of memory per node of degree $d$.

**Memory only on the nodes.** Behaviour of deterministic, oblivious agent, who does not know the incoming port, is fully controlled by the state of the current node in which the agent is located. One of the most prevalent strategies for exploration assisted by the environment is the Rotor-Router, introduced by Priezzhev et al. [16]. The edges outgoing from each node $v$ are arranged in a fixed cyclic order known as a port ordering, which does not change during the exploration. Each node $v$ maintains a pointer which indicates the edge to be traversed by the agent during its next visit to $v$. The next time when an agent enters node $v$, it is directed along the edge indicated by the pointer, which is then advanced to the next edge in the port ordering. When exploring a graph using a Rotor-Router mechanism with arbitrary initialization, time $\Theta(mD)$ is always sufficient and sometimes required for any graph [19] [3]. Since the Rotor-Router requires no special initialization, it can be implemented in a graph with $\lceil \log_2 d \rceil$-bits of memory at each node.
with degree $d$. An oblivious agent can simply exit the node $v$ via port $w(v) + 1$, where $w(v)$ is the value on the whiteboard, and increment the value $w(v)$ modulo $\deg(v)$. Thus exploration in time $O(mD)$ is possible by oblivious agents with $\lceil \log_2 d \rceil$-bit node memory, which is minimum possible by Observation 1.1.

**Memory only on the agent.** When no node memory is available but the agent has internal memory, the only information the agent has are: its own state and the degree of the current node. Thus when exploring a regular graph, the agent does not learn anything about the graph and the problem reduces to a well-known problem of finding a Universal Traversal Sequence (UTS). A sequence $Q = \{q_1, q_2, \ldots, q_l\}$ of port labels $q_i \in \{0, 1, \ldots, d - 1\}$ is $(n, d)$-universal if an agent following it will explore any $d$-regular graph of a size $n$. Aleliunas et al. [1] showed, using the probabilistic method, that there exist $(n, d)$-UTSs of length $O(d^2 n^3 \log n)$. Note that a mobile agent, given $n$, can always compute a distinguished $(n, n)$-UTS (e.g. a lexicographically minimal) and thus, by remembering the current position in the sequence in its internal memory and recomputing the sequence in each node, perform exploration with unknown import in polynomial time using $O(\log n)$ bits of internal memory. If termination of the algorithm is not required then the assumption about known $n$ can be dropped as the agent can approximate $n$ by doubling.

Borodin et al. [5] showed a lower bound of $\Omega(d^2 n^2 + dn^2 \log n)$ for any $(n, d)$-UTS for $3 \leq d \leq n/3 - 2$. Lower bound of $\Omega(d^{2.49} n^{2.51})$ was shown by Dai and Flannery [7]. For 2-regular graphs UTSs of length $O(n^2)$ exist [1], the best known explicit construction gives length $O(n^{1.03})$ [13] and the known lower bound is $\Omega(n^{1.51})$ [7].

Knowledge of the import simplifies the problem of universal sequences. A Universal Exploration Sequence (UXS) introduced by Koucky [14] is a sequence $Q = \{q_1, q_2, \ldots, q_l\}$ of port increments. An agent entering to a node $v$ in step $i$ via port $p$ leaves it via port $(p + q_i) \mod \deg(v)$. Universal Exploration Sequences can be constructed by a log-space machine [17] whereas currently the most space-efficient algorithm for construction of Universal Traversal Sequences requires space $O(\log^2 n)$ [15].

Exploration with known import of all graphs cannot be achieved by an agent with a constant number of bits of memory even if the agent is allowed to use a finite number of pebbles that can be dropped and removed from nodes [18]. With no pebbles $\Theta(D \log \Delta)$ bits are always sufficient and sometimes necessary [12].

With known import, exploration with stop of all trees requires $\Omega(\log \log \log n)$ bits of internal memory [8]. For exploration with return of trees, $\Theta(\log n)$ bits are necessary [8] and sufficient [2].

## 2 Efficient exploration

The main idea of our exploration algorithm is to speed up the running time of simple Rotor-Router exploration algorithm by making use of states of the agent and additionally states of the nodes.

It was observed in previous work on Rotor-Router [19] [3], that exploration using Rotor-Router has the following regular structure. Rotor-Router algorithm in subsequent phases (where $i$-th phase starts with the $i$-th traversal through outgoing port 1 of starting vertex) traverses Eulerian cycles of subgraphs of $\vec{G}$ (directed symmetric version of $G$). In each phase Rotor-Router
traverses all arcs that were visited in previous phases. Moreover each border vertex \( v \) (i.e. a vertex whose set of outgoing arcs contains both traversed and not traversed arcs) \( v \) is, informally speaking a root of a new exploration subphase, where some neighborhood of \( v \) is explored (including all unvisited neighbors of \( v \)), potentially creating new border vertices.

The main reason why \textsc{Rotor-Router} can be sometimes inefficient is that it uses one pass through all already traversed arcs (which can take up to \( \Omega(m) \) steps) to visit all border vertices and explore their neighborhood. There always exists initialization of the system, such that \( \Omega(D) \) phases are required to explore the whole graph leading to total worst-case exploration time of \( \Omega(mD) \) \cite{3}.

We present algorithm \textsc{One-bit}, which works as follow. The agent can be in one of two states, let us call them EXPLORATION MODE and REVISIT MODE. Thus, EXPLORATION MODE is used to perform exploration subphase in a fashion resembling an exploration subphase from \textsc{Rotor-Router}, starting each time from a border vertex \( v \), and ending in the same vertex \( v \), traversing arcs for the first time. The algorithm then detects, during one of returns to \( v \), that every neighbour of \( v \) was visited, and switches to REVISIT MODE. If, during EXPLORATION MODE a sequence of vertices \( v_0 = v, v_1, \ldots, v_k = v \) was visited, then during this mode, agent will follow the same sequence of vertices, traversing each arc for the second time. However, whenever in a REVISIT MODE the agent detects that its current location is a new border vertex, lets call it \( v_i \), then current REVISIT MODE will be interrupted by a subsequent phase of EXPLORATION MODE starting in \( v_i \) and ending in \( v_i \), followed by REVISIT MODE starting in \( v_i \) and ending in \( v_i \). Only after, the agent will fall back to original REVISIT MODE traversal and continue to vertex \( v_{i+1} \) (see Figure 1).

Thus, by the recursive running of exploration modes inside modes, the traversal of agent following \textsc{One-bit} can be interpreted as a DFS style of traversal, in the same sense as traversal of agent following \textsc{Rotor-Router} can be interpreted as in a BFS style.

Figure 1: Exploration by algorithm \textsc{One-bit}. (1) agent is in \( u_2 \) in EXPLORATION MODE, (2) after returning to border vertex \( u_2 \) in REVISIT MODE, the agent interrupts REVISIT MODE and starts exploring new edges in EXPLORATION MODE from \( u_2 \), (3) the agent is in REVISIT MODE in \( u_3 \), it will switch to EXPLORATION MODE and explore new edges, (4) the agent continues interrupted REVISIT MODE until discovering a new border vertex \( u_4 \) in which it will start EXPLORATION MODE.
function ONE-BIT($v$)
  if pointer($v$) = 2deg($v$) then
    pointer($v$) ← 0
    STOP()
  end if
  if (agentmode = 0) ∧ (pointer($v$) ≤ deg($v$)) then
    shift($v$) ← pointer($v$)  \(\triangleright\) marking of the node $v$
  end if
  pointer($v$) ← pointer($v$) + 1
  if 1 ≤ pointer($v$) ≤ deg($v$) then
    agentmode ← 1
    tmp ← pointer($v$)  \(\triangleright\) a temporary variable
  else
    agentmode ← 0
    tmp ← pointer($v$) + shift($v$)
    if (pointer($v$) = 2deg($v$)) ∧ (shift($v$) > 0) then
      shift($v$) ← 0
      pointer($v$) ← 0
    end if
  end if
  PORT(tmp)$^1$
end function

Implementation details.

The memory of algorithm consists of 1 bit at the agent and $\lceil \log_2(2d + 1) \rceil + \lceil \log_2(d + 1) \rceil$ bits at each a node with degree $d$. The memory is organized as follows:

- agent stores variable agentmode $\in \{0, 1\}$,
- on each vertex $v$, two variables are stored: pointer($v$) $\in \{0, 1, \ldots, 2\text{deg}(v)\}$ and shift($v$) $\in \{0, 1, \ldots, \text{deg}(v)\}$.

Variable agentmode stores information whether the agent is in EXPLORATION MODE (if agentmode = 1) or REVISIT MODE (if agentmode = 0). Local variable pointer($v$) stores the total number of visits in vertex $v$. Later we will prove that this number is always bounded by $2 \cdot \text{deg}(v)$. The memory both at the agent and on each node is initially set to 0 thus the agent is initialized to be in REVISIT MODE.

In each step $t$, the mode of exploration is determined by the value of pointer($v$), where $v$ is the location of the agent at the beginning of step $t$. The first deg($v$) traversals of outgoing arcs from $v$ will be performed in EXPLORATION MODE, and following second deg($v$) traversals will be performed in REVISIT MODE. Agent detects that it entered border vertex by the fact that it enters to a vertex in REVISIT MODE, and so far only EXPLORATION MODE exploration from this vertex took place (pointer($v$) ≤ deg($v$)). Thus, the current value of the pointer $p =$ pointer($v$)

$^1$Agent will leave through port $p \in \{1..\text{deg}(v)\}$, where $p =$ tmp mod deg($v$).
is stored in auxiliary variable $\text{shift}(v)$, so that future explorations in REVISIT MODE from vertex $v$ will start from port $p+1$ instead of starting from 1. Thus, the sequence of outgoing ports chosen by algorithm will be:

\[
\begin{array}{c|c|c|c}
\text{A1} & \text{A2} & \text{B1} & \text{B2} \\
1,2,\ldots,p,p+1,\ldots,\text{deg}(v) & p+1,\ldots,\text{deg}(v),1,2,\ldots,p \\
\text{EXPLORATION MODE} & \text{REVISIT MODE} & & \\
\end{array}
\]

Thus, the initial traversals, denoted as A1, happen in the same phase of EXPLORATION MODE exploration in which vertex is explored.

Following traversals denoted by A2 and B1 will be launched after arrival in $v$ while in phase of REVISIT MODE exploration. Subsequently, traversals from A2 will be part of a recursive subphase of EXPLORATION MODE exploration and traversals from B1 will be part of a recursive subphase of REVISIT MODE exploration. Finally, traversals from B2 will be part of the continued exploration from REVISIT MODE, the same one in which we arrived in $v$.

**Proof of correctness.**

We now proceed to prove that the presented algorithm explores any graph in time linear in the number of edges.

**Definition 2.1.** Let us call a time step in which a value is assigned to variable $\text{shift}(v)$ as marking of node $v$, and denote it by $t_m(v)$. We will also call a time step in which $\text{pointer}(v) + \text{shift}(v) = 2 \cdot \text{deg}(v) + 1$ as unmarking of $v$, and denote it by $t_u(v)$. (To complete the definition, we need to define unmarking step for starting vertex $v_0$, being the step in which we call $\text{STOP}()$.)

We will later show that the timesteps of marking and unmarking are uniquely defined for each node.

Location of the agent at the beginning of $i$-th timestep will be denoted by $v(i)$. Let $e(i)$ denote the arc traversed during $i$-th timestep. Thus $e(i) = (v(i), v(i + 1))$ is the arc through which the agent leaves $v(i)$ in step $i$ and $e(i-1) = (v(i-1), v(i))$ is the arc through which the agent enters to $v(i)$ in step $i - 1$. Let $\tilde{E}$ be the set of arcs of the directed symmetric version of $G$. Let $\tilde{E}_\geq 1(t)$ be the set of arcs traversed by the agent before the beginning of timestep $t$. Let $\tilde{E}_0(t)$ be the set of arcs not traversed by the agent before the beginning of timestep $t$.

**Theorem 2.2.** An $(1, O(\log d))$-agent following algorithm ONE-BIT explores with return any graph in time $4m$, by traversing every edge 4 times, twice in each direction. Moreover state of the memory after the exploration is completed is equal to the initial state.

**Proof.** The following lemma is true for any algorithm in which the agent traverses all arcs outgoing from a vertex $v$ once during first $\text{deg}(v)$ visits to $v$.

**Lemma 2.3.** The graph $(V, \tilde{E}_\geq 1(t))$ is semi-Eulerian for any $t \geq 0$. In other words, at most one vertex can have more outgoing arcs traversed than incoming, and at most one can have more incoming arcs traversed than outgoing, and the difference if exists is equal to one. Additionally, if any vertex has larger indegree than outdegree in $(V, \tilde{E}_\geq 1(t))$, then it is the location of the agent at the beginning of timestep $t$. 

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Proof. We will call the vertex with more outgoing arcs as the source and the one with more incoming as the sink.

We proceed by induction over \( t \).

For \( t = 0 \) we have \( \vec{E}_{\geq 1}(0) = \emptyset \), thus the lemma holds.

Let us assume that the claim holds for some \( t \geq 0 \). We will consider two cases. First assume that there is neither sink nor source in \((V, \vec{E}_{\geq 1}(t))\). During timestep \( t \), the agent can either traverse an arc from \( \vec{E}_0(t) \) or from \( \vec{E}_{\geq 1}(t) \). If it traverses an arc from \( \vec{E}_0(t) \), then in \( t + 1 \) there is neither sink nor source because \( \vec{E}_{\geq 1}(t + 1) = \vec{E}_{\geq 1}(t) \). If it traverses an arc from \( \vec{E}_{\geq 1}(t) \) then \( \vec{E}_{\geq 1}(t + 1) = \vec{E}_{\geq 1}(t) \cup \{e(t)\} \) and in step \( t + 1 \), node \( v(t+1) \) is the sink and \( v(t) \) is the source in graph \((V, \vec{E}_{\geq 1}(t+1))\).

On the other hand assume that in \((V, \vec{E}_{\geq 1}(t))\) there exists a source and a sink, respectively in vertices \( u \) and \( v(t) \) (agent location at timestep \( t \)). Since \( v(t) \) is a sink then at least one its outgoing arcs has never been traversed. By the definition of the algorithm, arc \( e(t) \) is traversed for the first time in step \( t \). If \( v(t + 1) \neq u \), then \( \vec{E}_{\geq 1}(t + 1) = \vec{E}_{\geq 1}(t) \cup \{e(t)\} \) has one source in \( u \) and one sink in \( v(t + 1) \). If \( v(t + 1) = u \), then there are no sources and no sinks in step \( t + 1 \). \( \square \)

We say that the agent is in EXPLORATION MODE (REVISIT MODE) in step \( t \), if sets its value of agentmode to \( 1 \) \((0)\) in step \( t \).

**Lemma 2.4.** Take any vertex \( v \) and a smallest \( k \geq 0 \) such in step \( t_m(v) + k \) the agent is in REVISIT MODE, then \( v = v(t_m(v) + k) \).

Proof. Assume that \( k = 0 \) since for \( k = 0 \) the claim is trivial. Denote by \( v_i = v(t_m(v) + i) \) and \( v = v_0 \). By the definition of the algorithm, in each of steps \( t_m(v), t_m(v) + 1, \ldots, t_m(v) + k - 1 \) the agent is in EXPLORATION MODE. Thus each arc \( e(t_m(v) + i) = (v_i, v_{i+1}) \), for \( i = 0, 1, \ldots, k - 1 \) is traversed for the first time in timestep \( t_m(v) + i \). Observe that since in timestep \( t_m(v) - 1 \), the agent is in REVISIT MODE then traversal of arc \( e(t_m(v) - 1) \) during timestep \( t_m(v) - 1 \) is not the first traversal of this arc. Thus, by Lemma 2.3 each vertex in \((V, \vec{E}_{\geq 1}(t_m(v)))\) has the same outdegree as indegree, thus \((V, \vec{E}_r(t_m(v)))\) is Eulerian. Also by Lemma 2.3 in timestep \( t_m(v) + i \) if \( v_i \neq v \), then \( v \) is the single source, and \( v_i \) is the single sink in \((V, \vec{E}_{\geq 1}(t_m(v) + i))\). Thus if \( v_i \neq v \), then in timestep \( t_m(v) + i \) the vertex \( v_i \) has at least one untraversed outgoing arc, so the agent will traverse an arc that has not been traversed before. Thus EXPLORATION MODE cannot end in a vertex different than \( v \). \( \square \)

We denote by \( \vec{E}_r(t) \subseteq \vec{E}_0(t) \) the arcs reachable from \( v(t) \) by traversing alongside arcs from \( \vec{E}_0(t) \). Now we proceed to following lemma:

**Lemma 2.5.** Let \( v \) be arbitrary vertex in \( G \). Between timesteps \( t_m(v) \) and \( t_u(v) \) agent traverses all of arcs from \( \vec{E}_r(t_m(v)) \), each of those arcs are traversed exactly twice, and no additional arcs are traversed.

Proof. We proceed by induction (induction I) on the size of \( \vec{E}_r(t_m(v)) \), that is we assume that the Lemma is true for any vertex \( v' \) such that \( |\vec{E}_r(t_m(v'))| \leq k \) (for some integer \( k \)) and we proceed to prove it for \( v \) such that \( |\vec{E}_r(t_m(v))| = k + 1 \).

(Base of induction) If \( \vec{E}_r(t_m(v)) = \emptyset \), then at the timestep \( t_m(v) \) every outgoing arc from \( v \) has been traversed at least once. Thus, since we triggered marking of the vertex, this vertex
has been visited already $\text{deg}(v)$ times, and we set $\text{shift}(v) = \text{deg}(v)$. Then immediately we set $\text{pointer}(v) = \text{deg}(v) + 1$, unmarking the vertex, thus $t_m(v) = t_u(v)$.

(Inductive step) Take such $k$ that in steps $t_m(v) + i$, the agent is in EXPLORATION MODE for $i \in \{0, 1, \ldots, k - 1\}$ and in step $t_m(v) + k$, the agent is in REVISIT MODE. Let us additionally denote $v_i = v(t_m(v) + i)$ for $i = 0, 1, \ldots, k$.

By Lemma 2.4 we know that $v_k = v_0 = v$, and that at timestep $t_m(v) + k$ agent is visiting $v$ for $(\text{deg}(v) + 1)$-th time. Let us analyze the agent trajectory in time steps $t \geq t_m(v) + k$, during REVISIT MODE exploration phase. The value of $\text{shift}(v)$ was stored at the $t_m(v)$-th timestep, and this vertex will not be marked for the second time, since at timestep $t_m(v) + k$ already $\text{pointer}(v) > \text{deg}(v)$. We claim that in the timesteps $(t_m(v) + k), (t_m(v) + k + 1), \ldots, (t_u(v) - 1)$ the agent will traverse in REVISIT MODE for the second time the arcs from $\mathcal{V} = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\}$, in the same chronological order, but possibly traversing some more arcs not in $\mathcal{V}$ in between. We prove this by induction on the position of the agent (induction II).

First observe that at the beginning of timestep $t_m(v) + k$, the agent is in $v$. Knowing that it is $(\text{deg}(v) + 1)$-st visit to $v$ and by the definition of the algorithm we have $\text{pointer}(v) = \text{deg}(v) + 1$, thus the agent will choose port number $\text{shift}(v) + 1$. Observe that it is the same port as the port chosen during step $t_m(v)$ and thus it corresponds to arc $(v_0, v_1)$.

Let us fix the timestep $t_m(v) + k < t' \leq t_u(v)$ such that $v(t') = v_i$ and $e(t' - 1) = (v_{i-1}, v_i)$, and let us assume agent already traversed in REVISIT MODE in chronological order exactly $\{(v_0, v_1), \ldots, (v_{i-1}, v_i)\}$. There are possible cases:

- $v_i = v, i \neq k$. Let us assume this is $p$-th occurrence of $v$ in sequence $v_0, \ldots, v_k$. When leaving $v$ at timestep $t'$, the agent will choose port value $(\text{pointer}(v) + \text{shift}(v)) \mod \text{deg}(v)$, where $\text{pointer}(v) = \text{deg}(v) + p$, and the stored value of $\text{shift}(v)$ will be the same as at the timestep $t_m(v) + i$ when the chosen port was $\text{shift}(v) + p$. Thus during step $t'$, the agent will traverse edge $(v_i, v_{i+1})$.

- $v_i \neq v$, and $\text{pointer}(v_i) \leq \text{deg}(v_i)$. We want to show that at the beginning of step $t'$, the value of $\text{pointer}(v_i)$ is equal to the number of occurrences of $v_i$ in sequence $v_0, \ldots, v_k$ (let us denote it by $p$). We know that it is at least $p$, as this is the number of visits to $v_i$ between timesteps $t_m(v)$ and $t_m(v) + k$.

Take the following set of vertices $\mathcal{M} = \{v_0, v_1, \ldots, v_{i-1}\} \setminus \{v_i\}$. Observe that for each $v' \in \mathcal{M}$, by inductive assumption (II) there exists a time step $t'' < t' - 1$ in which the agent enters $v'$ in REVISIT MODE, thus $t_m(v') < t'$. We know by the inductive assumption (I), that between steps $t_m(v')$ and $t_u(v')$ only arcs from set $\overline{E}_v(t_m(v'))$ are traversed. But observe that $(v_{i-1}, v_i) \notin \overline{E}_v(t_m(v'))$, because $(v_{i-1}, v_i)$ was traversed for the first time before step $t_m(v) + k$, and $t_m(v') > t_m(v) + k$. This shows that at time $t'$, node $v'$ is already unmarked (i.e. $t' > t_u(v')$). Thus $v_i \notin \mathcal{M}$ as otherwise all arcs outgoing from $v_i$ would be traversed twice by step $t'$, contradicting with the fact that $\text{pointer}(v_i) \leq \text{deg}(v_i)$. Moreover this shows that for every node $v' \in \mathcal{M}$, node $v_i$ is not reachable from $v'$ via arcs from $\overline{E}_v(t_m(v'))$. Otherwise by inductive assumption (I) all arcs outgoing from $v_i$ would be traversed twice by step $t'$. Thus $v_i$ was not visited during recursive EXPLORATION MODE explorations launched from any node from $\mathcal{M}$. Thus, during timestep $t' - 1$ agent enters $v_i$ for the first time after round $t_m(v) + k$, through the arc $(v_{i-1}, v_i) \in \mathcal{V}$.

By the definition of the algorithm, in step $t'$, node $v_i$ is marked, storing $\text{shift}(v_i) = p$. By
inductive assumption (I) between marking of $v_i$ at $t' = t_m(v_i)$ and unmarking of $v_i$ at $t'' = t_u(v_i)$ all outgoing and all incoming arcs from $E_r(t')$ will be traversed exactly twice, ending in $v_i$ and unmarking it $(\text{pointer}(v_i) + \text{shift}(v_i) = 2\deg(v_i) + 1)$. Thus in round $t''$, the agent chooses port number 1. The corresponding arc leads to $v_{i+1}$ because in step $t_m(v) + i$, the agent also took port 1 and traversed arc $(v_i, v_{i+1})$.

- $v_i \neq v$, and $\text{pointer}(v_i) > \deg(v_i)$. Along the similar lines as in the previous case since arc $(v_{i-1}, v_i)$ was traversed before step $t_m(v) + k$ then it implies that $t_u(v_i) < t'$. At timestep $t_u(v_i)$ we have that $\text{pointer}(v_i) + \text{shift}(v_i) = 2\deg(v_i) + 1$. Upon $r$-th visit to $v_i$ in a time step that is greater than $t_u(v_i)$, the chosen outport is $r$, because $\text{pointer}(v_i) + \text{shift}(v_i) = 2\deg(v_i) + 1 + r$. Observe that $r$ is also the outport chosen upon $r$-th visit to $v_i$, during EXPLORATION MODE. Thus in step $t'$, the agent traverses arc $(v_i, v_{i+1})$

That completes the inductive proof (II).

Thus, at the end of the REVISIT MODE exploration phase, $\text{pointer}(v) + \text{shift}(v) = 2\deg(v) + 1$, thus the vertex $v$ will be unmarked. Thus we proved that agent traversed twice arcs from $V = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\}$, and additionally some other arcs from $E_r(t_m(v))$. What is left is to prove that arcs from $E_r(t_m(v))$ are traversed between marking and unmaking of $v$. However, we know that for every vertex $v_i \in \{v_1, \ldots, v_{k-1}\}$, such that $v_i \neq v$, $v_i$ was marked after $v$ ($t_m(v_i) > t_m(v)$) and unmarked before $v$ ($t_u(v_i) < t_u(v)$). So by inductive assumption (I), every untraversed arc reachable from $v_i$ by untraversed arcs be traversed exactly twice between timestep $t_m(v)$ and $t_u(v)$. This implies that between marking and unmarking of node $v$, all arcs from $E_r(t_m(v))$ are traversed, which completes the inductive proof (I).

Let us denote by $v_0$, the location of the agent at the beginning of the first step of the algorithm. Observe the agent will mark $v_0$, and store value $\text{shift}(v_0) = 0$. Moreover, it will be the only vertex marked with stored value $\text{shift}$ equal to 0. By Lemma 2.5, the whole graph $G$ is be explored, traversing each edge exactly twice in each direction thus the total exploration time is $4m$. Since for each $v \neq v_0$, $\text{pointer}(v)$ will reach $2\deg(v)$, each value of $\text{pointer}(v)$ and $\text{shift}(v)$ will be reset to 0. Additionally, the agent detects after $4m$ steps of the algorithm that it made $2\deg(v_0)$ visits to $v_0$ because it is the only vertex with $\text{shift}$ already set to 0 while making the last visit to it. Thus the agent will set $\text{pointer}(v_0)$ to 0, and call $\text{STOP}()$ which proves the correct termination of the algorithm.

3 Lower bounds

In this section we provide lower bound on the number of steps of graph exploration for oblivious agent. First we need the following observation, which helps to reason about behaviour of oblivious agents in port-labelled graphs.

Lemma 3.1. Behaviour of any $(0, N_w)$-agent (i.e., oblivious) $A$ in graph $G$ with arbitrary size of node memory is fully characterized by the collection of functions $\text{port}_d(i)$ for $d = 1, 2, \ldots$. For a fixed $d$, the function denotes the outport chosen by the agent upon its $i$-th visit to any node with degree $d$. 

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Theorem 3.3. For any value of $n$ and any any $(0, \infty)$-agent $A$ there exists graph $G$ with $n$ vertices, such that $A$ needs at least $\Omega(n^3)$ time steps to visit all vertices of $G$.

Proof. Fix agent $A$ and size of the graph $n$. Assume that $n$ divisible by 3. Let $d = n/3$. Consider the sequence $a_i = \text{port}_d(i) \in \{1, 2, \ldots d\}$ for $i = 1, 2, \ldots$, where $\text{port}_d(i)$ is defined in Lemma 3.1. In
the prefix of length \( d \cdot (d - 1) \) of sequence \( \{a_i\} \) there exists a value \( p \) that appears at most \( d - 1 \) times.

Consider a graph \( G_1 \) constructed from the clique \( K_d \) by attaching one node \( v' \) to each node \( v \) of the clique (see Figure 2 for an example). Observe that each node coming from clique \( K_d \) has an additional neighbor thus its degree in \( G_1 \) is \( d \). For each \( v \) from the clique, the port leading to \( v' \) is \( p \). All other ports are set arbitrarily. Consider a walk of agent \( \mathcal{A} \) on graph \( G_1 \) starting from an arbitrary vertex \( v_s \) for \( d^2 \cdot (d - 1) \) steps. There exists a vertex \( v^* \) from the original clique \( K_d \) that was visited at most \( d \cdot (d - 1) \) times.

Construct a graph \( G \) by modifying \( G_1 \). Replace the one additional node attached to \( v^* \) with a path \( P \) of \( d + 1 \) nodes. Set the worst-case port-labelling of path \( P \), as in the Theorem 3.2 depending on function \( \text{port}_2(i) \) of \( \mathcal{A} \). Denote by \( v_f \), the first node of \( P \) that is connected to \( v^* \) (see Figure 3 for an example). The total number of nodes in \( G \) is: \( n/3 \) clique nodes from \( G_1 \), \( n/3 - 1 \) nodes \( v' \) from \( G_1 \) and \( n/3 + 1 \) path nodes, thus we have \( n \) nodes in total.

Consider agent \( \mathcal{A} \) exploring the graph \( G \) starting from vertex \( v_s \). Since the agent is oblivious, its moves between vertices in \( G \) that come from original graph \( G_1 \) are the same as in the graph \( G_1 \). Thus within \( d^2 \cdot (d - 1) \) steps in \( G \), node \( v^* \) is visited at most \( d \cdot (d - 1) \) times. Since \( p \) is the port leading from \( v^* \) to \( v_f \) then after \( d \cdot (d - 1) \) visits to \( v^* \), agent visited \( v_f \) at most \( d - 1 \). But by Theorem 3.2, the agent needs to visit \( v_f \) at least \( d \) times to explore path \( P \). Thus the agent needs time at least \( d^2 \cdot (d - 1) = \Omega(n^3) \) to explore graph \( G \).

If \( n \) is not divisible by 3 we can add the remaining vertices to the path and the exploration time will be at least \( \left\lfloor \frac{n}{3} \right\rfloor^2 \cdot (\left\lfloor \frac{n}{3} \right\rfloor - 1) = \Omega(n^3) \)

The theorem shows that, even with unbounded node memory, the oblivious agents need \( \Omega(n^3) \)
steps to explore some graphs. Since the Rotor-Router explores any graph in time $O(mD) = O(n^3)$ \cite{BT01} there is no strategy for oblivious agents that would be faster in the worst-case. Observe also that the Rotor-Router can be implemented using node memory of minimum possible size $\lceil \log_2 d \rceil$ at nodes of degree $d$. By Observation \cite{AKL99} agent with less memory cannot traverse all outgoing edges. Thus the Rotor-Router is both time and space optimal strategy for oblivious agents. We conclude with the following observation that exploration with stop is impossible with an oblivious agent.

**Observation 3.4.** Any $(0, \infty)$-agent without the knowledge of an upper bound on any global parameter cannot perform exploration with stop.

**Proof.** Assume on the contrary that a $(0, \infty)$-agent explores all graphs with stop. Thus the agent has to stop after $k_1$ visits to a node of degree 1 and after $k_2$ visits on a node of degree 2. Since the agent does not known any global parameter of the graph, then $k_1$ and $k_2$ are independent of $n$. Thus we can construct a path of $n = \max\{k_1, k_2\} + 1$ vertices $v_1, \ldots, v_n$, place the agent initially at node $v_n$, choose the worst-case port-labelling and observe that by Theorem \cite{AKL99} the agent will terminate before reaching to $v_1$. 

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