Sharp Global Existence for Semilinear Wave Equation with Small Data

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The global existence in time for nonlinear wave equation with small data usually require high Sobolev regularity, when one dealt with them by classical energy method (see [1], [4] for example). The purpose of this note is to give the sharp regularity global existence for semilinear equation with the power nonlinearity of the derivative, the counterpart of quasilinear equation or the quadratic nonlinearity seems still unreachable.

Consider the following Cauchy problem (denote \( \square := \partial_t^2 - \Delta \) and \( \partial = (\partial_t, \partial_x) \))

\[
\begin{align*}
\square u &= \sum_{|\alpha|=k} c_\alpha (\partial u)^\alpha := N(u) \\
u(0, x) &= u_0 \in H^s, \quad \partial_t u(0, x) = u_1 \in H^{s-1}
\end{align*}
\] (0.1)

Let \( s_c = \frac{n+2}{2} - \frac{1}{k-1} \) be the scaling index, we have

**Theorem 1.** Let \( \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \leq \epsilon \) with \( \epsilon \) small enough, and

\[
\begin{align*}
&\text{if } k - 1 = \frac{n}{n-1} \lor 2 \text{ and } n \neq 3 \\
&\text{or } k - 1 > \frac{n}{n-1} \lor 2,
\end{align*}
\] (0.2)

then the equation (0.1) has a unique global solution in \( C^1_t H^s \) such that \( \partial u \in L^\infty_t H^{s-1} \cap L^{k-1}_t L^\infty \).

Moreover, if \( k = n = 3 \), then the lifespan \( T^\ast \) of the solution with \( s > 2 \) is at least of order \( \exp(\epsilon^{-2}) \) with \( \epsilon \ll 1 \).

We will prove a similar result for the initial data which are spherical symmetric in addition. For such purpose, we introduce a concept here. We say that the equation (0.1) is **radial**, if \( u(t, x) \) is any solution of the equation, then for any rotation \( S \) in \( \mathbb{R}^n \), \( u(t, Sx) \) is still a solution of the same equation. For example, when \( k = 2 \), the radial equation must take the form of

\[
\square u = c_1 (\partial_t u)^2 + c_2 |\nabla u|^2 .
\]

**Theorem 2.** Let \( n \geq 2 \) and \( k > \frac{n+1}{n-1} \lor 2 \), and consider the radial equation, then there exists a global solution in time for \( s \geq s_c \) with small radial data.

**Remark 1.** The requirement for regularity in Theorem[1] and[2] are essentially sharp. Since for the equation

\[
\square u = |\partial_t u|^{k-1} \partial_t u ,
\]

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it’s well-known that the problem is ill posed in $H^s$ for $s < s_c$ (see Theorem 2 in [3] for example),
in the sense that, there is a sequence of data $f_j, g_j \in C_0^\infty(B_{R_j})$, for which the lifespan of the
solutions $u_j$ tends to zero as the data’s norm and $R_j$ goes to 0, under the condition that the
solutions obey finite speed of propagation. Note that the initial data $f_j, g_j$ can be radial functions.
Thus for such $s$, we can not hope any existence results as in these Theorems.

Remark 2. For the case $n = k = 3$, we have almost global existence in general and global
existence for the radial data. Thus a natural question is: To what extent does the result of global
existence depend on the radial symmetry? The answer is that it is very little. In fact, in [5], the
existence for the radial data. Thus a natural question is: To what extent does the result of global

Proposition 3. For the case $s > s_0$, we can not hope any existence results as in these Theorems.

Remark 3. It’s regret that such argument can not apply to the more interesting case $k = 2$, since
it’s well known that the corresponding $L^1L^\infty$ Strichartz estimate is not hold true in general. For
the local result for semilinear and quasilinear equation, one can refer to [6], [7] and references therein.

We will use the Strichartz estimates to prove the result. For the details of the Strichartz
estimates, one may consult [2] and references therein.

**Proposition 3 (Strichartz Estimate).** Let $u$ be the solution of the linear wave equation and
$q < \infty$, then for $(q, n) \neq (2, 3)$

$$
\|\partial u\|_{L^q L^\infty \cap L^\infty H^{s-1}} \leq C_q \|\partial u(0)\|_{H^{s-1}}
$$

(0.3)

with $s > \frac{n+2}{2} - \frac{1}{q}$ and $q > \frac{4}{n-1} \lor 2$ or $s > \frac{n+2}{2} - \frac{1}{q}$ and $q = \frac{4}{n-1} \lor 2$. For the case $(q, n) = (2, 3)$
and $s > 2$, we have

$$
\|\partial u\|_{L^2([0, T], L^\infty)} + (\ln(1 + T))^{1/2} \|\partial u\|_{L^\infty([0, T], H^{s-1})} \leq C(\ln(1 + T))^{1/2} \|\partial u(0)\|_{H^{s-1}}.
$$

(0.4)

Moreover, if $u$ is spatial radial function, then we have (0.3) with $s > \frac{n+2}{2} - \frac{1}{q}$ for all $q > \frac{2}{n-1}$
and $q \geq 2$.

We’ll use Picard’s iteration argument to give the proof. First, we give the proof for the case
$k - 1 \geq \frac{4}{n-1} \lor 2$ and $(n, k) \neq (3, 3)$.

Let $u(0) = 0$ and then define $u^{(m+1)}$ ($m \in \mathbb{N}$) to be the solution of the problem

$$
\Box u^{(m+1)} = N(u^{(m)})
$$

with the given data $(u_0, u_1)$. We’ll see below that $(\partial u^{(m)}, \partial_x u^{(m)})$ is a Cauchy sequence in
$C_t H^{s-1} \cap L^k L^\infty$ with the norm $L^\infty H^{s-1} \cap L^k L^\infty$ if $\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} = \epsilon$ is small enough.

We claim that for any $m \in \mathbb{N}$, $u^{(m)} \in CH^s \cap C^1 H^{s-1}$ and

$$
\|\partial u^{(m)}\|_{L^\infty H^{s-1} \cap L^k L^\infty} \leq M \epsilon
$$

(0.5)

with $M$ large enough. In fact, it’s true for $m = 0$, and we assume it’s true for some $m$, then by
Proposition 3 with $q = k - 1$ and $s$ as in (0.2),

$$
\|\partial u^{(m+1)}\|_{L^\infty H^{s-1} \cap L^k L^\infty} \leq C(\epsilon + \|N(u^{(m)})\|_{L^1 H^{s-1}}) \\
\leq C(\epsilon + \|\partial u^{(m)}\|_{L^k L^\infty} \|\partial u^{(m)}\|_{L^\infty H^{s-1}}) \\
\leq C(\epsilon + (M \epsilon)^k) \leq M \epsilon.
$$
Thus we get (0.5) by induction.

Now we show that \((\partial_t u^{(m)}, \partial_x u^{(m)})\) is a Cauchy sequence in \(C_t H^{s-1} \cap L_t^{k-1} L^\infty\) with norm \(L_t^s H^{s-1} \cap L_t^{k-1} L^\infty\). Note that for any \(m \in \mathbb{N}_+\), \(u^{(m+1)} - u^{(m)}\) is the solution of equation
\[
\Box(u^{(m+1)} - u^{(m)}) = N(u^{(m)}) - N(u^{(m-1)})
\]
with the null data. Then
\[
\|\Box(u^{(m+1)} - u^{(m)})\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty} \leq C \|N(u^{(m)}) - N(u^{(m-1)})\|_{L_t^1 H^{s-1}} \\
\leq C e^{-k} \|\Box(u^{(m)} - u^{(m-1)})\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty} \\
\leq \frac{1}{2} \|\Box(u^{(m)} - u^{(m-1)})\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty}.
\]

Thus we have
\[
\|\Box(u^{(m+1)} - u^{(m)})\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty} \leq 2^{-m} \|\Box(u^{(1)} - u^{(0)})\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty} \leq 2^{-m} M \epsilon.
\]

by induction and (0.5). So
\[
\|\Box(u^{(m)} - u^{(l)})\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty} \leq 2^{1 - \max(m, l)} M \epsilon. \tag{0.6}
\]

Therefore, there exist \(u^i, i \in \{0, 1, \cdots, n\}\), such that
\[
\partial_i u^{(m)} \to u^i \text{ in } C H^{s-1} \cap L^{k-1} L^\infty.
\]

Now we define
\[
u(t) = u_0 + \int_0^t u^0 \in C H^{s-1}.
\]

Since
\[
u^{(m)}(t) = u_0 + \int_0^t \partial_t u^{(m)},
\]

thus for any \(0 < T < \infty, t \in [0, T]\),
\[
\partial_t u^{(m)}(t) = \partial_t u_0 + \int_0^t \partial_t \partial_t u^{(m)} \to \partial_t u_0 + \int_0^t \partial_t u^0 \to \partial_t u(t) \text{ in } C([0, T], H^{s-2})
\]

and so \(\partial_t u = u^i\),
\[
\partial_t u^{(m)} \to \partial_t u \text{ in } C H^{s-1} \cap L^{k-1} L^\infty.
\]

Then we can get the solution \(u \in C H^s \cap C^1 H^{s-1}\) of equation (0.1).

For the uniqueness and continuous dependence of the initial data, it’s essentially as the above proof. Let \(\|(u_0, u_1)\|_{H^s \times H^{s-1}} \leq \epsilon\) and \(\|(v_0, v_1)\|_{H^s \times H^{s-1}} \leq \epsilon\). Assume \(u\) and \(v\) are two solutions of (0.1) with data \((u_0, u_1)\) and \((v_0, v_1)\) respectively, then \(u - v\) is the solution of equation
\[
\Box(u - v) = N(u) - N(v)
\]

with the data \((u_0 - v_0, u_1 - v_1)\).
\[
\|\Box(u - v)\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty} \leq C \|\Box(u_0 - v_0, u_1 - v_1)\|_{H^s \times H^{s-1}} + \|N(u) - N(v)\|_{L_t^1 H^{s-1}} \\
\leq C \|\Box(u_0 - v_0, u_1 - v_1)\|_{H^s \times H^{s-1}} + C e^{-k} \|\Box(u - v)\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty} \\
\leq C \|\Box(u_0 - v_0, u_1 - v_1)\|_{H^s \times H^{s-1}} + \frac{1}{2} \|\Box(u - v)\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty}.
\]

Thus we have
\[
\|\Box(u - v)\|_{L_t^\infty H^{s-1} \cap L_t^{k-1} L^\infty} \leq C \|u_0 - v_0, u_1 - v_1\|_{H^s \times H^{s-1}}. \tag{0.7}
\]
This complete the proof for the case $k - 1 \geq \frac{4}{n - 1} \lor 2$ and $(n, k) \neq (3, 3)$.

For the case $n = k = 3$, it remains to claim alternatively that

$$\|\partial u^{(m)}\|_{L^\infty_{[0,T]}H^{s-1}} \leq M\epsilon, \quad \|\partial u^{(m)}\|_{L^2_{[0,T]}L^\infty} \leq c \ll 1 \quad (0.8)$$

if $\ln(1 + T) \ll \epsilon^{-2}$. In fact, let

$$A_m := \|\partial u^{(m)}\|_{L^2([0,T],L^\infty)} + (\ln(1 + T))^{1/2}\|\partial u^{(m)}\|_{L^\infty([0,T],H^{s-1})},$$

then by inductive assumption,

$$A_{m+1} \leq C\ln(1 + T)^{1/2}(\epsilon + \|N(u^{(m)})\|_{L^1_{[0,T]}H^{s-1}})$$

$$\leq C\ln(1 + T)^{1/2}(\epsilon + \|\partial u^{(m)}\|^2_{L^2_{[0,T]}L^\infty}\|\partial u^{(m)}\|_{L^\infty_{[0,T]}H^{s-1}})$$

$$\leq C\ln(1 + T)^{1/2}(\epsilon + c^2M\epsilon)$$

$$\leq M\epsilon \ln(1 + T)^{1/2} \ll 1.$$

Thus we have (0.8) for any $m$.

For the radial cases, it only needs to replace the usual Strichartz estimate by the required radial $L^{k-1}L^\infty$ estimate in Proposition 3.

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