Quantizations of the Witt algebra and of simple Lie algebras in characteristic $p$

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Abstract

We quantize explicitly the Witt algebra in characteristic 0 equipped with its Lie bialgebra structures discovered by Taft. Then, we study the reduction modulo $p$ of our formulas. This gives $p - 1$ families of polynomial noncocommutative deformations of a restricted enveloping algebra of a simple Lie algebra in characteristic $p$ (of Cartan type). In particular, this yields new families of noncommutative noncocommutative Hopf algebras of dimension $p^p$ in char $p$.

Keywords: Witt algebra, Lie bialgebra, finite dimensional Hopf algebra in non-zero characteristic.

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“Il est faux de croire que rien n’arrive jamais. En vérité, tout arrive toujours mais seulement quand on ne le désire plus.”

1 Introduction

This article comes within the scope of the general program of quantization of simple Lie bialgebras in non-zero characteristic.

In the following, we use the general method of quantization by Drinfeld’s twist to quantize explicitly the Lie bialgebra structures discovered by Taft on the Witt algebra in characteristic zero [4]. Then, we study the case of the Witt algebra in characteristic $p$ where $p$ is a prime number and we show that our formulas lead to quantizations of its restricted enveloping algebra.

We start by recalling the definitions of the Witt algebra in characteristic 0 and of its natural structure of triangular Lie bialgebra of Taft. In Section 2, by using the twist discovered by Giaquinto and Zhang [2], we quantize explicitly this structure. In Section 3, we remark that although the twist used for
the quantization involves terms with negative \( p \)-adic valuation, conjugation by this twist preserves the nonnegative \( p \)-adic valuation part of the tensor square of the enveloping algebra of the Witt algebra. Moreover, the reduction modulo \( p \) of these formulas is compatible with the structure of \( p \)-Lie algebra on the Witt algebra. We use this fact to equip the restricted enveloping algebra of the Witt algebra with non-commutative and non-cocommutative Hopf algebra structures. We do not expect these Hopf algebras to be triangular since the twist which helps to define it is not defined on \( \mathbb{F}_p \). We thus get new examples of Hopf algebra of dimension \( p^2 \) in characteristic \( p \). These Hopf algebras contain the Radford algebra as a Hopf-subalgebra.

1.1 Definition of the Witt algebra

Definition 1 One denotes by \( W \) the \( \mathbb{Q} \)-Lie algebra given by generators: \( L_r, r \in \mathbb{Z} \) and relations:

\[ [L_r, L_s] = (s - r)L_{r+s}. \]

This Lie algebra is isomorphic to the Lie algebra of vector fields on the circle and generators \( L_r, r \in \mathbb{Z} \) correspond to operators \( x^{r+1} \frac{d}{dx} \) in \( \text{Der}(\mathbb{Q}[x, x^{-1}]) \). The following proposition is due to Taft [4].

Proposition 1 There is a triangular Lie bialgebra structure on \( W \) given by the \( r \)-matrix \( r - r^{21} \) with \( r := L_0 \otimes L_i \) and \( i \in \mathbb{Z} \).

In fact, it has been proved by Ng and Taft that it was the only Lie bialgebra structure on the “positive” part of the Witt algebra [1]. The purpose of this article is to quantize this structure and to investigate the case of the non-zero characteristic.

Notations.

In Section 2, one fixes \( i \in \mathbb{Z} \). In Section 3, \( p \) is an odd prime number. We set:

\[
\begin{align*}
    h &:= \frac{1}{i}L_0 \\
    e &= iL_i
\end{align*}
\]

For all element \( x \) of an unitary \( R \)-algebra (\( R \) a ring) and \( a \in R \), we also set:

\[
x^{(n)} := x(x + 1) \ldots (x + n - 1)
\] (1)

2 Quantification of Taft’s structure

Let us denote by \( (U(W), m, \Delta_0, S_0, \varepsilon) \) the natural Hopf algebra structure on \( U(W) \) i.e.,

\[
\begin{align*}
    \Delta_0(L_k) &= L_k \otimes 1 + 1 \otimes L_k \\
    S_0(L_k) &= -L_k \\
    \varepsilon(L_k) &= 0.
\end{align*}
\]

In order to quantize the Taft’s structure, we recall the twist \( F \) of [2] defined by:

\[
F := \sum_{r=0}^{\infty} \frac{1}{r!} h^{(r)} \otimes e^r t^r.
\]
where \( t \) denotes a formal variable.

The following proposition has been proved in [2].

**Proposition 2** We have:

\[
(\Delta_0 \otimes \text{Id})(F) \cdot (1 \otimes F) = (F \otimes 1) \cdot (\text{Id} \otimes \Delta_0)(F)
\]

\[
(\varepsilon \otimes \text{Id})(F) = (\text{Id} \otimes \varepsilon)(F) = 1
\]

\[
F = 1 \quad (t)
\]

In other words, \( F \) is a twist for \((U(W), m, \Delta_0, S_0, \varepsilon)\) i.e., \( F \) is invertible, \( u := m \circ (S \otimes \text{Id})(F) \) is invertible and \((U(W)[[t]], m, \Delta, S, \varepsilon)\) is a Hopf algebra with \( \Delta := F^{-1} \Delta_0 F \) and \( S := u^{-1} S_0 u \).

We note that \( F = 1 + r \cdot t \quad (t^2) \). Therefore, the Hopf algebra constructed with \( F \) quantizes the Taft’s structure. The following theorem gives explicitly the quantization.

**Theorem 1** There exists a structure of non-commutative and non-cocommutative Hopf algebra on \( U(W)[[t]] \) denoted by \((U(W)[[t]], m, \Delta, S, \varepsilon)\) which leaves the product of \( U(W)[[t]] \) undeformed but with a deformed comultiplication defined by:

\[
\Delta(L_k) := L_k \otimes (1 - et)^{\frac{k}{2}} + \sum_{l=0}^{\infty} (-1)^l \prod_{j=-1}^{l-2} \frac{(k + ji)}{l!} h(l) \otimes (1 - et)^{-l} L_{k+li} t^l
\]

(2)

\[
S(L_k) := -(1 - et)^{-\frac{k}{2}} \sum_{l=0}^{\infty} d_l \prod_{j=-1}^{l-2} \frac{(k + ji)}{l!} L_{k+li}(h + 1)^{(l)} t^l
\]

(3)

\[
\varepsilon(L_k) := 0
\]

(4)

with \( k \in \mathbb{Z} \).

One can simplify formulas of Theorem 4 by introducing the operator \( d^{(l)} \) \((l \in \mathbb{N})\) on \( U(W) \) defined by \( d^{(l)} := \frac{1}{l} \text{ad}(e)^l \). Indeed, it is easy to see that

\[
d^{(l)}(L_k) = \sum_{l=0}^{\infty} d_l \prod_{j=-1}^{l-2} \frac{(k + ji)}{l!} L_{k+li}
\]

(5)

We have then:

\[
\Delta(L_k) := L_k \otimes (1 - et)^{\frac{k}{2}} + \sum_{l=0}^{\infty} (-1)^l h(l) \otimes (1 - et)^{-l} d^{(l)}(L_k) t^l
\]

(6)

\[
S(L_k) := -(1 - et)^{-\frac{k}{2}} \sum_{l=0}^{\infty} d^{(l)}(L_k) \cdot (h + 1)^{(l)} t^l.
\]

(7)

We can also give a general formula for the antipode.

**Proposition 3** For any homogeneous element \( x \in U(W) \) with respect to the graduation given by \(|L_k| = k\), one has:

\[
S(x) = (1 - et)^{-\frac{|x|}{2}} \sum_{n=0}^{\infty} d^{(n)}(S_0(x)) \cdot (h + 1)^{(n)} t^n.
\]

(8)
In fact, by applying Lemma 1 below with \( a = i \) and \( k \) replaced by \( k - i \), we remark that the structure coefficients of \( \Delta \) and \( S \) in \([2]\) and \([3]\) belong to \( \mathbb{Z} \).

**Lemma 1** For all integers \( a, k \) and \( l \), \( \alpha \left( \prod_{i=0}^{l} \frac{(k+j_\alpha)}{i} \right) \) is an integer.

This fact allows us to consider the reduction modulo \( p \) of our formulas.

## 3 Quantizations of restricted enveloping algebras

We show that formulas above can be used to define an Hopf algebra structure on the restricted enveloping algebra of the Witt algebra in characteristic \( p \) \([3]\).

**Definition 2** One denotes by \( \mathfrak{D} \) the \( \mathbb{F}_p \)-Lie algebra given by generators: \( D_k, k \in \mathbb{Z} \) and relations:

\[
[D_k, D_l] = (l - k)D_{k+l} \quad D_{k+p} = D_k
\]

for \( k, l \in \mathbb{Z} \). The algebra \( \mathfrak{D} \) is called the Witt algebra in characteristic \( p \).

The algebra \( \mathfrak{D} \) is isomorphic to \( \text{Der}(\mathbb{F}_p[X]/(X^p - 1)) \) and \( D_k \) corresponds to the operator \( X^{k+1} \frac{d}{dX} \). It is well known that \( \mathfrak{D} \) is a simple restricted Lie algebra \([3]\). Its structure of \( p \)-Lie algebra is given by \( D_0^{(0)} = D_0 \) and \( D_k^{(0)} = 0 \) for \( k \) non divisible by \( p \). Its restricted enveloping algebra \( U_c(\mathfrak{D}) \) is isomorphic to \( U(\mathfrak{D})/I \) where \( I \) is the ideal of \( U(\mathfrak{D}) \) generated by \( D_p^r - D_r \) with \( p \) divides \( r \) and \( D_p^r \) with \( k \) non divisible by \( p \). A basis of this algebra is given by monomials \( \prod_{k=0}^{r-1} D_k^{\alpha_k} \) with \( \alpha_k \in \{0, \ldots, p-1\} \) \([3]\). So, \( \dim_{\mathbb{F}_p} U_c(\mathfrak{D}) = p^p \).

**Note 1** The Witt algebra in characteristic \( p \) is also sometimes defined as the Lie algebra \( \text{witt} \) given by generators: \( e_k, k \in \{-1, \ldots, p-2\} \) and relations:

\[
[e_k, e_l] = \begin{cases} 
(l - k)e_{k+l} & \text{if } k + l < p - 2; \\
0 & \text{otherwise.} 
\end{cases} \tag{9}
\]

The Lie algebra \( \text{witt} \) is isomorphic to \( \text{Der}(\mathbb{F}_p[Y]/(Y^p)) \) and \( e_k \) stands for the operator \( Y^{k+1} \frac{d}{dY} \). This Lie algebra is also isomorphic to \( \mathfrak{D} \) as it can be seen by setting \( X = Y + 1 \). The corresponding Lie algebra morphism maps \( e_k \) to \( \sum_{l=-1}^{k} (-1)^l \left( \begin{array}{c} k + 1 \\ l + 1 \end{array} \right) D_l \).

On \( \mathbb{F}_p \), the twist \( F \) does not make sense because of the coefficient \( \frac{1}{m} \) in the definition of \( F \). However, as we have seen at the end of Section 2, formulas of Theorem 1 are defined on \( \mathbb{Z} \). Moreover, Lemma 2 below shows that these coefficients are compatible with the reduction modulo \( p \).

**Lemma 2** Let \( \alpha, \kappa, l \) be integers. Then, the residue class of \( \alpha \left( \prod_{i=0}^{l} \frac{(\kappa+j_\alpha)}{i} \right) \) modulo \( p \) depends only on \( l \) and on the residue class of \( \alpha \) and \( \kappa \).

By definition, for \( l \in \mathbb{N} \) and \( a, k \in \mathbb{F}_p \), we will denote by \( N(a, k, l) \) the common residue modulo \( p \) of all integers of the form \( \alpha \left( \prod_{i=0}^{l} \frac{(\kappa+j_\alpha)}{i} \right) \) with \( \text{cl}(\alpha) = a, \text{cl}(\kappa) = k \).

Therefore, Theorem 2 below make sense in \( \mathbb{F}_p \).
Theorem 2 Let \( i \) be an element of \( \mathbb{F}_p - \{0\} \). Then, the Hopf algebra \( (U_c(\mathfrak{G})[t], m, \Delta, S, \varepsilon) \) is a polynomial deformation of the restricted enveloping algebra of \( \mathfrak{G} \). The algebra structure is undeformed and the coalgebra structure is given by:

\[
\Delta(D_k) := D_k \otimes (1 - et)^k + \sum_{l=0}^{p-1} (-1)^l N(i, k - i, l) h(l) \otimes (1 - et)^{-l} D_{k+l} t^l
\]  

(10)

\[
S(D_k) := -(1 - et)^{-k} \sum_{l=0}^{p-1} N(i, k - i, l) D_{k+l}(h + 1)^{(l)} t^l
\]  

(11)

\[
\varepsilon(D_k) := 0
\]  

(12)

with, \( k \in \mathbb{F}_p, h := \frac{1}{i} D_0 \) and \( e := i D_1 \).

The above sums are finite, which explains why they define a polynomial deformation. Thus, we can specialize \( t \) to any element of \( \mathbb{F}_p \). Since \( i \) is an arbitrary element of \( \mathbb{F}_p - \{0\} \), this gives \( p - 1 \) new families of non-commutative and non-cocommutative Hopf algebras of dimension \( p^\alpha \) in characteristic \( p \).

Note 2 If we set \( \alpha := (1 - et)^{-1} \), then we have: \( [h, \alpha] = \alpha^2 - \alpha; h^p = h; \alpha^p = 1; \Delta(h) = h \otimes \alpha + 1 \otimes h, \alpha \) is “group-like”, \( S(h) = h\alpha^{-1} \) and \( \varepsilon(h) = 0 \).

So, the sub-algebra generated by \( h \) and \( e \) is a sub-Hopf algebra of \( U_c(\mathfrak{G}) \) isomorphic to the Radford algebra.

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References

[1] S-H Ng E. Taft. Classification of the lie bialgebra structures on the witt and virasoro algebras. \textit{J. Pure Appl. Algebra}, 2000.

[2] A. Giaquinto and J. Zhang. Bialgebra action, twists and universal deformation formulas. \textit{J. Pure Appl. Algebra}, 1998.

[3] N. Jacobson. Lie algebras. \textit{Dover}, 1962.

[4] E. Taft. Witt and virasoro algebras as bialgebras. \textit{J. Pure Appl. Algebra}, 1993.