SOME \(q\)-EXPONENTIAL FORMULAS FOR
FINITE-DIMENSIONAL \(\square_q\)-MODULES

YANG YANG

Abstract. We consider the algebra \(\square_q\) which is a mild generalization of the quantum algebra \(U_q(\mathfrak{sl}_2)\). The algebra \(\square_q\) is defined by generators and relations. The generators are \(\{x_i\}_{i \in \mathbb{Z}_4}\), where \(\mathbb{Z}_4\) is the cyclic group of order 4. For \(i \in \mathbb{Z}_4\) the generators \(x_i, x_{i+1}\) satisfy a \(q\)-Weyl relation, and \(x_i, x_{i+2}\) satisfy a cubic \(q\)-Serre relation. For \(i \in \mathbb{Z}_4\) we show that the action of \(x_i\) is invertible on every nonzero finite-dimensional \(\square_q\)-module. We view \(x_{i-1}\) as an operator that acts on nonzero finite-dimensional \(\square_q\)-modules.

For \(i \in \mathbb{Z}_4\), define \(n_{i,i+1} = q(1 - x_i x_{i+1})/(q - q^{-1})\). We show that the action of \(n_{i,i+1}\) is nilpotent on every nonzero finite-dimensional \(\square_q\)-module. We view the \(q\)-exponential \(\exp_q(n_{i,i+1})\) as an operator that acts on nonzero finite-dimensional \(\square_q\)-modules. In our main results, for \(i, j \in \mathbb{Z}_4\) we express each of \(\exp_q(n_{i,i+1}) x_j \exp_q(n_{i,i+1})^{-1}\) and \(\exp_q(n_{i,i+1})^{-1} x_j \exp_q(n_{i,i+1})\) as a polynomial in \(\{x_k\}_{k \in \mathbb{Z}_4}\).

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1. Introduction

This paper is about a certain algebra \(\square_q\); we will recall the definition shortly. Broadly speaking it can be viewed as a generalization of the quantum algebra \(U_q(\mathfrak{sl}_2)\). In order to motivate our results we make some comments about \(U_q(\mathfrak{sl}_2)\).

We will work with the equitable presentation, which was introduced in [10] and investigated further in [11, 12, 13, 14, 15, 16, 17, 18, 19]. Let \(\mathbb{F}\) denote an algebraically closed field. Fix \(0 \neq q \in \mathbb{F}\) that is not a root of unity. In the equitable presentation, the \(\mathbb{F}\)-algebra \(U_q(\mathfrak{sl}_2)\) has generators \(x, y, z\) and relations \(yy^{-1} = y^{-1} y = 1, qxy - qyx = 1, qyz - qzy = 1, qzx - qxz = 1\).

Define \(n_x = q(1 - yz)/(q - q^{-1}), n_y = q(1 - zx)/(q - q^{-1}), n_z = q(1 - xy)/(q - q^{-1})\).

On every nonzero finite-dimensional \(U_q(\mathfrak{sl}_2)\)-module, \(x, y, z\) are invertible (see [18 Lemma 5.15]) and \(n_x, n_y, n_z\) are nilpotent (see [18 Lemma 5.14]). Recall from [13] p. 204 the \(q\)-exponential function

\[\exp_q(T) = \sum_{n \in \mathbb{N}} q^{\binom{n}{2}} [n]_q^n T^n.\]
In [10] Sections 5, 6 it was shown that the following equations hold on every nonzero finite-dimensional \(U_q(\mathfrak{sl}_2)\)-module:

\[
\begin{align*}
\exp_q(n_x) x \exp_q(n_x)^{-1} &= x + z - z^{-1}, \\
\exp_q(n_x) y \exp_q(n_x)^{-1} &= z^{-1}, \\
\exp_q(n_x) z \exp_q(n_x)^{-1} &= yz, \\
\exp_q(n_x)^{-1} x \exp_q(n_x) &= x + y - y^{-1}, \\
\exp_q(n_x)^{-1} y \exp_q(n_x) &= yz, \\
\exp_q(n_x)^{-1} z \exp_q(n_x) &= y^{-1}.
\end{align*}
\]

Cyclically permuting \(x, y, z\) in the above equations, we get 12 more equations. Our goal in this paper is to find analogous equations that apply to \(\Box_q\).

We now discuss the algebra \(\Box_q\). This algebra was introduced in [19] Definition 5.1. We mention some algebras that are related to \(\Box_q\). For the positive part \(\mathcal{A}_q\) of \(U_q(\mathfrak{sl}_2)\) (see [8] Definition 1.1), there exists an injective algebra homomorphism from \(\mathcal{A}_q\) to \(\Box_q\) (see [19] Proposition 5.5]). For the \(O\)-Onsager algebra \(\mathcal{O}_q\) (see [2] Section 2], there exists an injective algebra homomorphism from \(\mathcal{O}_q\) to \(\Box_q\) (see [19] Proposition 11.9]). For the quantum loop algebra \(U_q(L(\mathfrak{sl}_2))\), there exists an injective algebra homomorphism from \(\mathcal{O}_q\) to \(U_q(L(\mathfrak{sl}_2))\) (see [19] Proposition 5.5] and [5] Propositions 4.1, 4.3]). For the \(q\)-tetrahedron algebra \(\mathfrak{L}_q\) (see [9] Definition 6.1]), there exists an injective algebra homomorphism from \(\mathcal{O}_q\) to \(\mathfrak{L}_q\) (see [19] Proposition 5.5] and [9] Propositions 4.1])

The \(\mathbb{F}\)-algebra \(\Box_q\) is defined as follows (formal definitions start in Section 2). The generators are \(\{x_i\}_{i \in \mathbb{Z}_4}\), where \(\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}\) is the cyclic group of order 4. The relations are

\[
\begin{align*}
q x_i x_{i+1} - q^{-1} x_{i+1} x_i &= 1, \\
x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} - x_i x_{i+2} x_i &= 0,
\end{align*}
\]

for \(i \in \mathbb{Z}_4\). We will state our main results after some preliminary remarks. We show that for \(i \in \mathbb{Z}_4\) the action of \(x_i\) is invertible on every nonzero finite-dimensional \(\Box_q\)-module. We view \(x_i^{-1}\) as an operator that acts on nonzero finite-dimensional \(\Box_q\)-modules. For \(i \in \mathbb{Z}_4\) define

\[
n_{i,i+1} = \frac{q(1 - x_i x_{i+1})}{q - q^{-1}}.
\]

We show that the action of \(n_{i,i+1}\) is nilpotent on every nonzero finite-dimensional \(\Box_q\)-module. We view the \(q\)-exponential \(\exp_q(n_{i,i+1})\) as an operator that acts on nonzero finite-dimensional \(\Box_q\)-modules. For \(i, j \in \mathbb{Z}_4\) consider the two expressions

\[
\begin{align*}
(1.1) \quad \exp_q(n_{i,i+1}) x_j \exp_q(n_{i,i+1})^{-1}, & \quad \exp_q(n_{i,i+1})^{-1} x_j \exp_q(n_{i,i+1}).
\end{align*}
\]

For each expression in (1.1), expand both \(q\)-exponential terms. This yields a double sum with infinitely many terms. A natural question is, to what extent can this double sum be simplified? In our main results we will show that in fact, each double sum is a polynomial in \(\{x_{k}^{\pm 1}\}_{k \in \mathbb{Z}_4}\). These results are Theorems 8.1, 8.2 and Theorems 9.3, 9.6.

We mention another motivation for studying (1.1). Near the top of page 2 we gave 18 equations for \(U_q(\mathfrak{sl}_2)\). These equations were used to construct a rotator for
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$U_q(\mathfrak{sl}_2)$ (see [18] Definition 9.5). These equations were also used to describe the Lusztig operators (see [11][12] for $U_q(\mathfrak{sl}_2)$ in the equitable presentation (see [18] Theorem 9.9)). We hope to obtain similar results for $\square_q$.

The paper is organized as follows. Section 2 contains the preliminaries. Section 3 contains some basic facts about $\square_q$. In Section 4 we describe some isomorphisms and antiisomorphisms for $\square_q$. In Section 5 we show that the action of each $x_i$ is invertible on every nonzero finite-dimensional $\square_q$-module. In Section 6 we show that the action of each $n_{i,i+1}$ is nilpotent on every finite-dimensional $\square_q$-module. In Section 7 we review the $q$-exponential function, and apply it to $n_{i,i+1}$. In Sections 8 and 9 we prove our main results.

2. Preliminaries

Throughout the paper, we fix the following notation. Let $\mathbb{F}$ denote an algebraically closed field. Recall the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots \}$ and the ring of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}$. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. We will be discussing algebras. An algebra is meant to be associative and have a 1.

Let $V$ denote a nonzero finite-dimensional vector space over $\mathbb{F}$. Let $\text{End}(V)$ denote the $\mathbb{F}$-algebra consisting of the $\mathbb{F}$-linear maps from $V$ to $V$. An element $A \in \text{End}(V)$ is called nilpotent whenever there exists a positive integer $n$ such that $A^n = 0$. By an eigenvalue of $A$, we mean a root of the characteristic polynomial of $A$.

Fix $0 \neq q \in \mathbb{F}$ such that $q$ is not a root of unity. For $n \in \mathbb{Z}$ define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$  

For $n \in \mathbb{N}$ define

$$[n]_q! = \prod_{i=1}^{n} [i]_q.$$  

We interpret $[0]_q! = 1$.

3. The algebra $\square_q$

In this section, we recall the algebra $\square_q$.

Definition 3.1. [19] Definition 5.1] Let $\square_q$ denote the $\mathbb{F}$-algebra with generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

$$(3.1) \quad \frac{qx_ix_{i+1} - q^{-1}x_{i+1}x_i}{q - q^{-1}} = 1,$$

$$(3.2) \quad x_i^3x_{i+2} - [3]_q x_i^2x_{i+2}x_i + [3]_q x_ix_{i+2}^2 - x_{i+2}x_i^2 = 0.$$  

The structure of the algebra $\square_q$ is analyzed in [19]. We don’t need the full strength of the results in [19], but we will use the following fact.

Lemma 3.2. The elements $\{x_i\}_{i \in \mathbb{Z}_4}$ are linearly independent in $\square_q$.

Proof. By [19] Proposition 5.5.]  

We now give some formulas for later use.
Lemma 3.3. For $i \in \mathbb{Z}_4$ and $n \in \mathbb{N}$ the following relations hold in $\square_q$:

\[\begin{align*}
q^n x_i^n x_{i+1} - q^{-n} x_{i+1} x_i^n &= (q^n - q^{-n}) x_i^{n-1}, \\
q^n x_i x_{i+1} - q^{-n} x_{i+1} x_i &= (q^n - q^{-n}) x_{i+1}^{n-1}.
\end{align*}\]  

Proof. Use (3.1) and induction on $n$. \hfill \Box

Lemma 3.4. For $i \in \mathbb{Z}_4$ and $n \in \mathbb{N}$, the following relation holds in $\square_q$:

\[x_i^n x_{i+2} = \frac{[n-1]_q[n-2]_q}{[2]_q} x_{i+2} x_i^n - [n]_q [n-2]_q x_i x_{i+2} x_i^{n-1} + \frac{[n]_q [n-1]_q}{[2]_q} x_i^2 x_{i+2} x_i^{n-2}.\]  

Proof. Use (3.2) and induction on $n$. \hfill \Box

For $i \in \mathbb{Z}_4$ we define an element $n_{i,i+1} \in \square_q$. Later we will show that $n_{i,i+1}$ is nilpotent on every finite-dimensional $\square_q$-module.

Lemma 3.5. For $i \in \mathbb{Z}_4$,

\[q(1 - x_i x_{i+1}) = q^{-1}(1 - x_{i+1} x_i).\]  

Proof. This equation is a reformulation of (3.1). \hfill \Box

Definition 3.6. For $i \in \mathbb{Z}_4$ define

\[n_{i,i+1} = \frac{q(1 - x_i x_{i+1})}{q - q^{-1}} = \frac{q^{-1}(1 - x_{i+1} x_i)}{q - q^{-1}}.\]  

We now describe some basic properties of $n_{i,i+1}$ for later use.

Lemma 3.7. For $i \in \mathbb{Z}_4$, the following relations hold in $\square_q$:

\[x_i n_{i,i+1} = q^{-2} n_{i,i+1} x_i, \quad x_{i+1} n_{i,i+1} = q^2 n_{i,i+1} x_{i+1}.\]  

Proof. To verify (3.7), eliminate $n_{i,i+1}$ using the first equality in (3.6) and simplify the result using (3.1). \hfill \Box

Corollary 3.8. For $i \in \mathbb{Z}_4$ and $n \in \mathbb{N}$, the following relations hold in $\square_q$:

\[x_i^n n_{i,i+1} = q^{-2n} n_{i,i+1} x_i^n, \quad x_{i+1}^n n_{i,i+1} = q^{2n} n_{i,i+1} x_{i+1}^n.\]  

Proof. By (3.7) and induction on $n$. \hfill \Box

Lemma 3.9. For $i \in \mathbb{Z}_4$ and $n \in \mathbb{N}$, the following relations hold in $\square_q$:

\[x_i^n x_{i+1}^n \left(1 - (q-2n - q^{-2n-2}) n_{i,i+1}\right) = x_i^{n+1} x_{i+1}^{n+1}; \]

\[x_{i+1}^n x_i^n \left(1 - (q^{2n+2} - q^{2n}) n_{i,i+1}\right) = x_{i+1}^{n+1} x_i^{n+1}.\]  

Proof. In order to verify these equations, eliminate $n_{i,i+1}$ using the first equality in (3.6) and simplify the result using (3.4). \hfill \Box
4. Some isomorphisms and antiisomorphisms for $\Box_q$

In this section, we introduce some isomorphisms and antiisomorphisms for $\Box_q$. By an automorphism of $\Box_q$ we mean an $F$-algebra isomorphism from $\Box_q$ to $\Box_q$.

**Lemma 4.1.** There exists an automorphism $\rho$ of $\Box_q$ that sends $x_i \mapsto x_{i+1}$ for $i \in \mathbb{Z}_4$.

*Proof.* By Definition 3.1.

**Lemma 4.2.** The map $\rho$ from Lemma 4.1 sends $n_{i,i+1} \mapsto n_{i+1,i+2}$ for $i \in \mathbb{Z}_4$.

*Proof.* By the definition of $\rho$ and Definition 3.6.

We recall the notion of antiisomorphism. Given $F$-algebras $A, B$, a map $\gamma : A \to B$ is called an antiisomorphism whenever $\gamma$ is an isomorphism of $F$-vector spaces and $(ab)\gamma = b\gamma a\gamma$ for all $a, b \in A$. An antiisomorphism $\gamma : A \to A$ is called an antiautomorphism of $A$.

**Lemma 4.3.** There exists an antiautomorphism $\phi$ of $\Box_q$ that sends $x_0 \leftrightarrow x_1$, $x_2 \leftrightarrow x_3$.

Moreover there exists an antiautomorphism $\varphi$ of $\Box_q$ that sends $x_1 \leftrightarrow x_2$, $x_3 \leftrightarrow x_0$.

*Proof.* By Definition 3.1.

**Lemma 4.4.** The map $\phi$ from Lemma 4.3 sends $n_{0,1} \mapsto n_{0,1}$, $n_{2,3} \mapsto n_{2,3}$, $n_{1,2} \leftrightarrow n_{3,0}$.

Moreover the map $\varphi$ from Lemma 4.3 sends $n_{1,2} \mapsto n_{1,2}$, $n_{3,0} \mapsto n_{3,0}$, $n_{0,1} \leftrightarrow n_{2,3}$.

*Proof.* By the definitions of $\phi, \varphi$ and Definition 3.6.

**Lemma 4.5.** The maps $\rho$ from Lemma 4.1 and $\phi, \varphi$ from Lemma 4.3 satisfy the following relations:

\[\rho^4 = \phi^2 = \varphi^2 = (\rho \phi)^2 = (\rho \varphi)^2 = 1,\]

\[\rho \phi = \phi \rho, \quad \rho \varphi = \varphi \rho.\]

*Proof.* By the definitions of $\rho, \phi, \varphi$.

Recall that the dihedral group $D_4$ has the following group presentation:

\[D_4 = \{ x, y \mid x^4 = y^2 = (xy)^2 = 1 \}.\]

The group $D_4$ has 8 elements and is the group of symmetries of a square. Consider the group $\text{AAut}(\Box_q)$ consisting of the automorphisms and antiautomorphisms of $\Box_q$. The group operation is composition.

**Lemma 4.6.** Let $G$ denote the subgroup of $\text{AAut}(\Box_q)$ generated by the maps $\rho$ from Lemma 4.1 and $\phi, \varphi$ from Lemma 4.3. Then $G$ is isomorphic to $D_4$.

*Proof.* By (4.1) there exists a group homomorphism $f : D_4 \to G$ that sends $x \mapsto \rho$ and $y \mapsto \phi$. By (4.2) the element $\varphi$ is in the image of $f$. Therefore $f$ is surjective. By Lemma 3.2 the map $f$ is injective. By these comments $f$ is an isomorphism. The result follows.
We now relate $\square_q$ and $\square_{q^{-1}}$.

**Lemma 4.7.** There exists an antiisomorphism $\dagger : \square_q \rightarrow \square_{q^{-1}}$ that sends $x_i \mapsto x_i$ for $i \in \mathbb{Z}_4$. Moreover $\dagger^2 = 1$.

*Proof.* By Definition 3.1. □

In Definition 3.6 we discussed an element $n_{i,i+1} \in \square_q$. We retain the notation $n_{i,i+1}$ for the corresponding element in $\square_{q^{-1}}$.

**Lemma 4.8.** The map $\dagger$ from Lemma 4.7 sends $n_{i,i+1} \mapsto -n_{i,i+1}$ for $i \in \mathbb{Z}_4$.

*Proof.* By the definition of $\dagger$ and Definition 3.6. □

5. The Element $x_i$ is Invertible on Finite-Dimensional $\square_q$-Modules

Let $V$ denote a nonzero finite-dimensional $\square_q$-module. In this section, we show that for $i \in \mathbb{Z}_4$ the action of $x_i$ on $V$ is invertible.

We first show that the action of $x_i$ on $V$ is not nilpotent.

**Lemma 5.1.** Let $V$ denote a nonzero finite-dimensional $\square_q$-module. For $i \in \mathbb{Z}_4$, the action of $x_i$ on $V$ is not nilpotent.

*Proof.* Assume that $x_i$ is nilpotent on $V$. Then there exists a minimal positive integer $n$ such that $x_i^n = 0$ on $V$. By (3.1), we have $n \neq 1$. By (3.3) and since $q$ is not a root of unity, we have $x_i^{n-1} = 0$ on $V$. This contradicts the minimality of $n$. The result follows. □

We will use the following notation. Let $V$ denote a finite-dimensional vector space over $\mathbb{F}$ and let $A \in \text{End}(V)$. For $\theta \in \mathbb{F}$ define

$$V_A(\theta) = \{v \in V \mid \exists n \in \mathbb{N}, (A - \theta I)^n v = 0\}.$$ 

Observe that $\theta$ is an eigenvalue of $A$ if and only if $V_A(\theta) \neq 0$, and in this case $V_A(\theta)$ is the corresponding generalized eigenspace. The sum $V = \sum_{\theta \in \mathbb{F}} V_A(\theta)$ is direct.

**Proposition 5.2.** Let $V$ denote a nonzero finite-dimensional $\square_q$-module. For $i \in \mathbb{Z}_4$ the action of $x_i$ on $V$ is invertible.

*Proof.* To show $x_i$ is invertible on $V$, it suffices to show that 0 is not an eigenvalue of $x_i$. Consider the subspace $W = V_{x_i}(0)$. We first show that $W$ is $\square_q$-invariant. By construction, $W$ is $x_i$-invariant. Pick $v \in W$. By the definition of $W$, there exists $m \in \mathbb{N}$ such that $x_i^m v = 0$. By (3.3) with $n - 1 = m$, we have $x_i^{m+1} x_i v = 0$. Therefore $x_i + 1 v \in W$. By (3.4) with $n - 1 = m$, we have $x_i^{m+1} x_i v = 0$. Therefore $x_i + 1 v \in W$. By (3.5) with $n - 2 = m$, we have $x_i^{m+2} x_i + 1 v = 0$. Therefore $x_i + 2 v \in W$. We have shown that $W$ is invariant under $x_i$ for $j \in \mathbb{Z}_4$. Therefore $W$ is $\square_q$-invariant. By construction $x_i$ is nilpotent on $W$. Therefore $W = 0$ in view of Lemma 5.1. The result follows. □

Motivated by Proposition 5.2, we make the following definition.

**Definition 5.3.** For $i \in \mathbb{Z}_4$, let $x_i^{-1}$ denote the operator that acts on every nonzero finite-dimensional $\square_q$-module as the inverse of $x_i$.

We now give some formulas involving the operators $x_i^{-1}$. 
Lemma 5.4. For \( i \in \mathbb{Z}_4 \) the following relations hold on every nonzero finite-dimension \( \Box_q \)-module:

\[
\begin{align*}
(5.1) \quad q x_{i+1} x_i^{-1} - q^{-1} x_i^{-1} x_{i+1} & = (q - q^{-1}) x_i^{-2}, \\
(5.2) \quad q x_{i+1}^{-1} x_i - q^{-1} x_i x_{i+1} & = (q - q^{-1}) x_{i+1}^{-2}.
\end{align*}
\]

Proof. By (3.1) and Definition 5.3. \( \square \)

Lemma 5.5. For \( i \in \mathbb{Z}_4 \) the following relations hold on every nonzero finite-dimension \( \Box_q \)-module:

\[
\begin{align*}
(5.3) \quad \frac{q x_i^{-2} x_{i+1}^{-1} + q^{-1} x_{i+1}^{-1} x_i x_i^{-1}}{q + q^{-1}} & = x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} x_i^{-1}, \\
(5.4) \quad \frac{q x_i^{-1} x_{i+1}^{-2} + q^{-1} x_{i+1}^{-2} x_i^{-1}}{q + q^{-1}} & = x_i^{-1} x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1}.
\end{align*}
\]

Proof. We first show (5.3). In (5.1) multiply each term on the left by \( x_{i+1}^{-1} \) and on the right by \( x_i^{-1} x_{i+1}^{-1} \) to get

\[
(5.5) \quad q x_i^{-1} x_{i+1}^{-1} - q^{-1} x_{i+1}^{-1} x_i x_i^{-1} x_{i+1}^{-1} = (q - q^{-1}) x_{i+1}^{-1} x_i x_i^{-1} x_{i+1}^{-1}.
\]

Similarly in (5.1), multiply each term on the left by \( x_{i+1}^{-1} x_i^{-1} \) and on the right by \( x_{i+1}^{-1} \) to get

\[
(5.6) \quad q x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} x_i^{-1} - q^{-1} x_i^{-1} x_{i+1}^{-1} x_i^{-1} = (q - q^{-1}) x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} x_i^{-1}.
\]

Subtract (5.5) from (5.6) and solve for \( x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} x_i^{-1} \) to get (5.3). To get (5.4), apply the map \( \phi \) from Lemma 4.3 to each side of (5.3). \( \square \)

6. The Element \( n_{i,i+1} \) is Nilpotent on Finite-Dimensional \( \Box_q \)-Modules

Let \( V \) denote a finite-dimensional \( \Box_q \)-module. In this section, we show that for \( i \in \mathbb{Z}_4 \) the action of \( n_{i,i+1} \) on \( V \) is nilpotent.

Lemma 6.1. Let \( V \) denote a finite-dimensional \( \Box_q \)-module and let \( \theta \in \mathbb{F} \). Then for \( i \in \mathbb{Z}_4 \), we have \( n_{i,i+1} V_x(\theta) \subseteq V_x(q^{-2}\theta) \).

Proof. Pick \( v \in V_x(\theta) \). We show \( n_{i,i+1} v \in V_x(q^{-2}\theta) \). By the definition of \( V_x(\theta) \), there exists \( n \in \mathbb{N} \) such that \( (x_i - \theta I)^n v = 0 \). By this and the left equation in (3.8), we have \( (x_i - q^{-2}\theta I)^n n_{i,i+1} v = 0 \). Therefore \( n_{i,i+1} v \in V_x(q^{-2}\theta) \). The result follows. \( \square \)

Proposition 6.2. Let \( V \) denote a finite-dimensional \( \Box_q \)-module. For \( i \in \mathbb{Z}_4 \) the action of \( n_{i,i+1} \) on \( V \) is nilpotent.

Proof. Assume that \( V \) is nonzero; otherwise the result is trivial. It suffices to show that for each eigenvalue \( \theta \) of \( x_i \), there exists a positive integer \( m \) such that \( n_{i,i+1}^m V_x(\theta) = 0 \). By Proposition 5.2 the scalar 0 is not an eigenvalue of \( x_i \). Therefore \( \theta \neq 0 \). Since \( V \) has finite positive dimension and \( q \) is not a root of unity, there exists a positive integer \( j \) such that \( \theta q^{-2j} \) is an eigenvalue of \( x_i \), for \( 0 \leq j \leq m - 1 \), but \( \theta q^{-2m} \) is not an eigenvalue of \( x_i \). By this and Lemma 6.1, we have \( n_{i,i+1}^m V_x(\theta) \subseteq V_x(\theta q^{-2m}) = 0 \). Therefore \( n_{i,i+1}^m V_x(\theta) = 0 \). The result follows. \( \square \)
7. The $q$-exponential function

In this section we obtain some results involving the $q$-exponential function.

**Definition 7.1.** [13, p. 204] Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\psi \in \text{End}(V)$ be nilpotent. Define

$$\exp_q(\psi) = \sum_{n \in \mathbb{N}} \frac{q^{\binom{n}{2}}}{[n]_q!} \psi^n.$$ (7.1)

The following result is well known and readily verified.

**Lemma 7.2.** [13, p. 204] Referring to Definition 7.1, the map $\exp_q(\psi)$ is invertible and its inverse is

$$\exp_q^{-1}(-\psi) = \sum_{n \in \mathbb{N}} (-1)^n \frac{q^{-\binom{n}{2}}}{[n]_q!} \psi^n.$$ (7.2)

We mention an identity for later use.

**Lemma 7.3.** Referring to Definition 7.1,

$$\exp_q(q^2 \psi)(1 - (q^2 - 1)\psi) = \exp_q(\psi).$$ (7.3)

**Proof.** To verify (7.3), express each side as a power series in $\psi$ using (7.1). □

Pick $i \in \mathbb{Z}_4$. By Proposition 6.2, the action of $n_{i,i+1}$ on every nonzero finite-dimensional $\boxtimes_q$-module is nilpotent. We view $\exp_q(n_{i,i+1})$ as an operator that acts on every nonzero finite-dimensional $\boxtimes_q$-module. For $i, j \in \mathbb{Z}_4$, consider the following two expressions:

$$\exp_q(n_{i,i+1}) x_j \exp_q(n_{i,i+1})^{-1}, \quad \exp_q(n_{i,i+1})^{-1} x_j \exp_q(n_{i,i+1}).$$ (7.4)

For each expression in (7.4), expand both $q$-exponential terms using Definition 7.1 and Lemma 7.2. This yields a double sum with infinitely many terms. We will show that in fact, each double sum is a polynomial in $\{x_k^{\pm 1}\}_{k \in \mathbb{Z}_4}$. We now give some formulas for later use.

**Lemma 7.4.** For $i \in \mathbb{Z}_4$ and $r \in \mathbb{Z}$, the following relations hold on every nonzero finite-dimensional $\boxtimes_q$-module:

$$\exp_q(q^2 n_{i,i+1}) = \exp_q(n_{i,i+1}) x_i^{-r} x_{i+1}^{-r},$$ (7.5)

$$\exp_q(q^2 n_{i,i+1})^{-1} x_{i+1}^{-r} x_i^{-r} \exp_q(n_{i,i+1}).$$ (7.6)

**Proof.** To show (7.5) for $r \geq 0$, use induction on $r$. The calculation is routine using (7.2) with $\psi = q^2 n_{i,i+1}$ along with (3.10). We similarly show (7.5) for $r < 0$ by induction on $r = -1, -2, \ldots$ using (3.9) and (7.2). To get (7.6), apply the map $\phi$ from Lemma 4.3 to each side of (7.5). □

8. Some $q$-exponential formulas, part I

In this section, we analyze (7.3) for the case $j = i$ or $j = i + 1$. The following Theorem 8.1 is a variation of [10, Lemma 5.8, 5.9].

**Theorem 8.1.** For $i \in \mathbb{Z}_4$, the following relations hold on every nonzero finite-dimensional $\boxtimes_q$-module:

$$\exp_q(n_{i,i+1}) x_i \exp_q(n_{i,i+1})^{-1} = x_{i+1}^{-1},$$ (8.1)

$$\exp_q(n_{i,i+1})^{-1} x_{i+1} \exp_q(n_{i,i+1}) = x_i^{-1}.$$ (8.2)
Proof. We first verify (8.2). By the equation on the right in (3.8) and Definition 7.1, we have
\[ x_{i+1} \exp_q(n_{i+1}) x_{i+1}^{-1} = \exp_q(q^n x_{i+1}). \]
Using this and (7.4) with \( r = 1 \) we routinely obtain (8.2). To get (8.3), apply the map \( \phi \) from Lemma 4.3 to each side of (8.2).

The following Theorem 8.2 is a variation of [10, Lemma 6.1, 6.2].

**Theorem 8.2.** For \( i \in \mathbb{Z}_4 \), the following relations hold on every nonzero finite-dimension \( \square_q \)-module:

\[
\begin{align*}
(8.3) & \quad \exp_q(n_{i+1})^{-1} x_i \exp_q(n_{i,i+1}) = x_i x_{i+1} x_i, \\
(8.4) & \quad \exp_q(n_{i+1}) x_{i+1} \exp_q(n_{i,i+1})^{-1} = x_{i+1} x_i x_{i+1}.
\end{align*}
\]

**Proof.** We first verify (8.3). By (3.8) the element \( x_i x_{i+1} \) commutes with \( n_{i+1} \).
Therefore \( \exp_q(n_{i+1})^{-1} x_i x_{i+1} \exp_q(n_{i,i+1}) = x_i x_{i+1} \) in view of Definition 7.1. Combine this equation with (8.2) to get (8.3). To get (8.4), apply the map \( \phi \) from Lemma 4.3 to each side of (8.3). □

9. Some \( q \)-exponential formulas, part II

In this section, we analyze (7.3) for the case \( j = i + 2 \) or \( j = i + 3 \).

**Lemma 9.1.** For \( i \in \mathbb{Z}_4 \), the following relations hold in \( \square_q \):

\[
\begin{align*}
(9.1) & \quad \sum_{m=0}^{3} (-1)^m q^{3-2m} \frac{n_{i+1}^{3-m}}{[3-m]_q} x_{i+2}^{3-m} x_{i+1}^{m} = -(q-q^{-1})^2 n_{i,i+1} x_i n_{i,i+1}, \\
(9.2) & \quad \sum_{m=0}^{3} (-1)^m q^{3-2m} \frac{n_{i,i+1}^{m}}{[m]_q} x_{i+3}^{m} x_{i+1}^{3-m} = -(q-q^{-1})^2 n_{i,i+1} x_{i+1} n_{i,i+1}.
\end{align*}
\]

**Proof.** To verify (9.1) let \( \Theta \) denote the left-hand side of (9.1) minus the right-hand side of (9.2). We will show that \( \Theta = 0 \). To do this we first eliminate each occurrence of \( n_{i,i+1} \) in \( \Theta \) using the second equality in (3.6). In the resulting equation, we simplify things using the following principle: for each occurrence of \( x_{i+1} \), move it to the far left using (3.1). The above simplification yields the following results.

The expression \( q^3(q-q^{-1})^2 n_{i,i+1} x_{i+2} \) is a weighted sum involving the following terms and coefficients:

\[
\begin{array}{c|cccc}
\text{term} & x_{i+2} & x_{i+1} x_i x_{i+2} & x_i^2 x_{i+2} & x_{i+1}^2 x_{i+2} \\
\text{coeff.} & 1 & -q^2 x_3 [3]_q & -q^4 x_3 [3]_q & -q^6
\end{array}
\]

The expression \( q^3(q-q^{-1})^2 n_{i,i+1} x_{i+2} n_{i,i+1} \) is a weighted sum involving the following terms and coefficients:

\[
\begin{array}{c|cccc}
\text{term} & x_{i+2} & x_i x_{i+1} x_i x_{i+2} & x_{i+1} x_{i+2} x_i & x_{i+1} x_i x_{i+2} \\
\text{coeff.} & 1 & q^2 - 1 & -q^2 & q^2
\end{array}
\]

\[
\begin{array}{c|cccc}
\text{term} & x_i^2 x_{i+2} x_{i+1} x_i^2 x_{i+2} & x_i^2 x_{i+1} x_i x_{i+2} x_i & x_i^2 x_{i+1} x_i x_{i+2} x_i & x_i^2 x_{i+1} x_i x_{i+2} x_i \\
\text{coeff.} & q^2 & q^2 + q^4 & 1 - q^2 & q^4
\end{array}
\]
The expression $q^3(q - q^{-1})^3 n_{i,i+2} x_{i+1}^2 n_{i,i+1}^2$ is a weighted sum involving the following terms and coefficients:

| term | $x_{i+2}$ | $x_i$ | $x_{i+1} x_{i+2}$ | $x_{i+1} x_{i+2} x_i$ | $x_{i+1} x_i^2$ |
|------|-----------|------|---------------------|----------------------|-----------------|
| coeff. | $1$ | $q^2 - q^{-2}$ | $-1$ | $-1 - q^{-2}$ | $q^{-1}(q^{-1} + q)(q^{-2} - q^2)$ |

| term | $x_{i+1} x_{i+2} x_i^2$ | $x_{i+1} x_{i+2} x_i$ | $x_{i+1} x_i^2$ | $x_{i+1} x_i^2 x_i$ | $x_{i+1} x_i^2 x_i^2$ |
|------|---------------------|----------------------|-----------------|------------------|------------------|
| coeff. | $q^{-2}$ | $q^{-2} + 1$ | $1 - q^{-4}$ | $-q^{-2}$ | |

The expression $q^3(q - q^{-1})^3 x_{i+2} n_{i,i+1}^3$ is a weighted sum involving the following terms and coefficients:

| term | $x_{i+1} x_{i+2} x_i^2$ | $x_{i+1} x_i^2$ | $x_{i+1} x_i^2 x_i$ | $x_{i+1} x_i^2 x_i^2$ |
|------|---------------------|-----------------|------------------|------------------|
| coeff. | $[3]_q$ | $1 - q^{-6}$ | $-1$ | |

The expression $q^4(q - q^{-1})^2 n_{i,i+1} x_{i+1} x_i^2 n_{i,i+1}^2$ is a weighted sum involving the following terms and coefficients:

| term | $x_i$ | $x_{i+1} x_i^2$ | $x_{i+1} x_i^2 x_i$ | $x_{i+1} x_i^2 x_i^2$ |
|------|------|-----------------|------------------|------------------|
| coeff. | $1$ | $-1 - q^{-2}$ | $q^{-2}$ | |

By the above comments $\Theta$ is equal to

$$-x_{i+1}^3(x_i x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^2),$$

$$q^6(q - q^{-1})^3 [3]_q.$$ (9.3)

The expression (9.3) is 0 by (3.2). Therefore $\Theta = 0$. We have shown (9.1). To get (9.2) apply the map $\phi$ from Lemma 4.3 to each side of (9.1). $\square$

**Lemma 9.2.** For $i \in \mathbb{Z}_4$ and $m \in \mathbb{N}$, the following relations hold in $\square_q$:

$$x_{i+2} n_{i,i+1}^m = a_m(q) n_{i,i+1}^{m-1} x_{i+1} x_{i+2} + b_m(q) n_{i,i+1}^{m-2} x_{i+1} x_{i+2} n_{i,i+1},$$

(9.4)

$$n_{i,i+1} x_{i+3} = a_m(q) x_{i+3} n_{i,i+1} + b_m(q) x_{i+1} x_{i+3} n_{i,i+1},$$

(9.5)

$$n_{i,i+1} x_{i+2} = a_m(q^-1) x_{i+2} n_{i,i+1} + b_m(q^-1) x_{i+1} x_{i+2} n_{i,i+1},$$

(9.6)

$$n_{i,i+1} x_{i+3} = a_m(q^-1) x_{i+3} n_{i,i+1} + b_m(q^-1) x_{i+1} x_{i+3} n_{i,i+1},$$

(9.7)
where
\[ a_m(q) = q^{2m} \frac{[m-1]q[m-2]}{[2]q}, \]
\[ b_m(q) = -q^{2m-2} [m]q[m-2], \]
\[ c_m(q) = q^{2m-4} [m]q[m-1], \]
\[ d_m(q) = q^{m-5} [3m]q - q^{-3} [3]q [2m]q + q^{-m-1} [3]q [m]q. \]

**Proof.** To get (9.3), use (3.2), (9.1) and induction on \( m \). To get (9.5), apply the map \( \phi \) from Lemma 4.7 to each side of (9.4). Concerning (9.6), apply the map \( \dagger \) from Lemma 4.3 to each side of (9.4). Concerning (9.6), apply the map \( \phi \) from Lemma 4.7 to each side of (9.4). This yields an equation that holds in \( \square_q \). In this equation replace \( q \) by \( q^{-1} \). This gives (9.6). To get (9.7), apply \( \phi \) to each side of (9.6).

We now analyze (7.3) for the case \( j = i+3 \).

**Theorem 9.3.** For \( i \in \mathbb{Z}_4 \), the following relation holds on every nonzero finite-dimensional \( \square_q \)-module:
\[
\exp_q(n_{i,i+1})^{-1} x_{i+3} \exp_q(n_{i,i+1}) = x_{i+3} - x_i^{-1} + \frac{q x_i x_{i+1} x_{i+3}}{q - q^{-1}} - \frac{x_i x_{i+3} x_{i+1}}{q(q - q^{-1})} + \frac{q^2 x_i^2 x_{i+1}^2 x_{i+3}}{(q - q^{-1})(q^2 - q^{-2})} + \frac{q x_i^2 x_{i+1} x_{i+3}^2}{(q - q^{-1})(q^2 - q^{-2})} - \frac{q^2 x_i^2 x_{i+1} x_{i+3}^2}{q(q - q^{-1})^2}.
\]

**Proof.** For \( m \in \mathbb{N} \) multiply each side of (9.7) by \( q^{2n}/[m]^q \). Sum the resulting equations over \( m \in \mathbb{N} \) and evaluate the result using (7.1) to get
\[
x_{i+3} \exp_q(n_{i,i+1}) = q^3 \exp_q(q^{-4} n_{i,i+1}) x_{i+3} + \exp_q(q^{-2} n_{i,i+1}) x_{i+3} \frac{q x_i x_{i+1} x_{i+3}}{q - q^{-1}^2} - \exp_q(n_{i,i+1}) x_{i+3} \frac{q x_i x_{i+1} x_{i+3} x_{i+1}}{q(q-1)} + \exp_q(q^{-2} n_{i,i+1}) x_{i+3} \frac{q x_i x_{i+1} x_{i+3} x_{i+1} x_{i+1}}{q(q-1)^2} - \exp_q(q^{-2} n_{i,i+1}) x_{i+1} + (1 + q^2) \exp_q(n_{i,i+1}) x_{i+1} - q^2 \exp_q(q^{-2} n_{i,i+1}) x_{i+1}.
\]

In the above equation multiply each term on the left by \( \exp_q(n_{i,i+1})^{-1} \) and use (7.3) to get that \( \exp_q(n_{i,i+1})^{-1} x_{i+3} \exp_q(n_{i,i+1}) \) is equal to
\[
\frac{q^3 x_i x_{i+1} x_{i+3}}{(q - q^{-1})(q^2 - q^{-2})} - \frac{x_i x_{i+1} x_{i+3}}{(q - q^{-1})^2} + \frac{x_{i+3}}{q(q - q^{-1})} - \frac{x_{i+3} n_{i,i+1}}{q(q - q^{-1})} + \frac{q x_i x_{i+1} x_{i+3} n_{i,i+1}}{q(q - q^{-1})} + \frac{q^2 x_{i+3}^2 n_{i,i+1}^2}{q(q - q^{-1})} - x_i^{-1} + (1 + q^2) x_{i+1} - q^2 x_i x_{i+1}.
\]

For notational convenience let \( \Psi \) denote the above expression. In \( \Psi \) we first eliminate every occurrence of \( n_{i,i+1} \) using the second equality in (3.9). In the resulting expression, we simplify things using the following principle: for each occurrence of
x_i$, move it to the far left using (3.1). The above simplification yields the following results.

The expression $-q^{-1}(q - q^{-1})^{-1}x_{i+3}n_{i,i+1}$ is a weighted sum involving the following terms and coefficients:

| term | coeff. |
|------|--------|
| $x_{i+3}$ | $-(q - q^{-1})^{-2}$ |
| $x_i x_{i+3} x_{i+1}$ | $q^{-2} (q - q^{-1})^{-2}$ |
| $x_{i+1}$ | $q^{-1} (q - q^{-1})^{-1}$ |

The expression $q(q - q^{-1})^{-1}x_i x_{i+1} x_{i+3} n_{i,i+1}$ is a weighted sum involving the following terms and coefficients:

| term | coeff. |
|------|--------|
| $x_i x_{i+1} x_{i+3}$ | $q^{-2} (q - q^{-1})^{-2}$ |
| $x_i^2 x_{i+1} x_{i+3} x_{i+1}$ | $-q^{-2} (q - q^{-1})^{-2}$ |
| $x_i x_{i+3} x_{i+1}$ | $q(q - q^{-1})^{-1}$ |
| $x_{i+1}$ | $-q(q - q^{-1})^{-1}$ |

The expression $q(q + q^{-1})^{-1}x_{i+3} n_{i,i+1}^2$ is a weighted sum involving the following terms and coefficients:

| term | coeff. |
|------|--------|
| $x_{i+3}$ | $q^{-1} (q - q^{-1})^{-1} (q^2 - q^{-2})^{-1}$ |
| $x_i x_{i+3} x_{i+1}$ | $-q^{-1} (q - q^{-1})^{-2}$ |
| $x_{i+1}$ | $-q^2 (q - q^{-1})^{-1}$ |

Evaluating $\Psi$ using the above comments, we get the result. \[ \square \]

**Theorem 9.4.** For $i \in \mathbb{Z}_4$, the following relation holds on every nonzero finite-dimensional $\square_q$-module:

$$
\exp_q(n_{i,i+1})x_{i+3}\exp_q(n_{i,i+1})^{-1} = x_{i+1} - \frac{x_{i+3}}{(q - q^{-1})^2} + \frac{qx_{i+1} x_{i+3} x_{i+1}}{(q - q^{-1})(q^2 - q^{-2})} + \frac{q^{-1} x_i x_{i+3} x_{i+1}}{(q - q^{-1})(q^2 - q^{-2})}.
$$

**Proof.** For $m \in \mathbb{N}$ multiply each side of (9.3) by $q^{-2m}q(n)/[m]_q!$. Sum the resulting equations over $m \in \mathbb{N}$ and evaluate the result using (7.1) to get

$$
\exp_q(q^{-2}n_{i,i+1})x_{i+3} = \frac{x_{i+3}\exp_q(q^{2}n_{i,i+1})}{q^3(q - q^{-1})(q^2 - q^{-2})} + \frac{x_{i+3}\exp_q(n_{i,i+1})}{(q - q^{-1})^2} - \frac{x_{i+1}\exp_q(q^{-2}n_{i,i+1})}{q^3(q - q^{-1})} + \frac{x_{i+1}\exp_q(n_{i,i+1})}{q^2} - \frac{n_{i,i+1} x_{i+3}\exp_q(n_{i,i+1})}{q(q - q^{-1})} + \frac{n_{i,i+1} x_{i+3}\exp_q(q^{2}n_{i,i+1})}{q(q + q^{-1})} + \frac{1}{q^3(q + q^{-1})}.
$$
In the above equation multiply each term on the left by \( x_{i+1}^{-1} x_i^{-1} \) and on the right by \( \exp_q(n_{i+1})^{-1} \), and then use (7.5) to get that \( \exp_q(n_{i+1}) x_{i+1} \exp_q(n_{i+1})^{-1} \) is equal to

\[
\frac{x_{i+1}^{-1} x_i^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1}}{q^3 (q - q^{-1}) (q^2 - q^{-2})} \quad \frac{x_{i+1}^{-1} x_i^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1}}{(q - q^{-1})^2} \quad \frac{x_{i+1}^{-1} x_i^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1}}{(q - q^{-1}) (q^2 - q^{-2})} \quad \frac{x_{i+1}^{-1} x_i^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1}}{q(q - q^{-1})} \quad \frac{x_{i+1}^{-1} x_i^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1}}{q^3(q + q^{-1})} \quad \frac{x_{i+1}^{-1} x_i^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1}}{q^3(q + q^{-1})}.
\]

For notational convenience let \( \Phi \) denote the above expression. In \( \Phi \) we first eliminate every occurrence of \( n_{i+1} \) using the first equality in (3.3). In the resulting expression, we simplify things using (5.1), (5.1), and (5.2). Our guiding principle is to bring \( x_i, x_i^{-1} \) together for cancellation, and also to bring \( x_{i+1}, x_{i+1}^{-1} \) together for cancellation. The above simplification yields the following results.

The expression \( q^3(q - q^{-1})^{-1}(q^2 - q^{-2})^{-1} x_{i+1}^{-1} x_i^{-1} x_{i+3} x_{i+1} x_{i+1}^{-1} \) is a weighted sum involving the following terms and coefficients:

| Term | Coefficient |
|------|-------------|
| \( x_{i+1}^{-1} x_{i+3} x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} \) | \( q(q - q^{-1})^{-1}(q^2 - q^{-2})^{-1} q^3(q^2 - q^{-2})^{-1} - q^3(q + q^{-1})^{-1} \) |

The expression \( q^{-1}(q - q^{-1})^{-1} x_{i+1}^{-1} x_i^{-1} n_{i+1} x_{i+3} \) is a weighted sum involving the following terms and coefficients:

| Term | Coefficient |
|------|-------------|
| \( x_{i+1}^{-1} x_{i+3} x_{i+1}^{-1} \) | \( (q - q^{-1})^{-1} \) |

The expression \( -q^{-1}(q - q^{-1})^{-1} x_{i+1}^{-1} x_i^{-1} n_{i+1} x_{i+3} x_i^{-1} x_{i+1}^{-1} \) is a weighted sum involving the following terms and coefficients:

| Term | Coefficient |
|------|-------------|
| \( x_{i+1}^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1} \) | \( -q^{-2}(q - q^{-1})^{-2} \) |

The expression \( q^{-3}(q + q^{-1})^{-1} x_{i+1}^{-1} x_i^{-1} n_{i+1}^2 x_{i+3} x_i^{-1} x_{i+1}^{-1} \) is a weighted sum involving the following terms and coefficients:

| Term | Coefficient |
|------|-------------|
| \( x_{i+1}^{-1} x_{i+3} x_i^{-1} x_{i+1}^{-1} x_{i+1}^{-1} \) | \( q^{-1}(q - q^{-1})^{-1}(q^2 - q^{-2})^{-1} - q^{-2}(q - q^{-1})^{-2} \) |

| Term | Coefficient |
|------|-------------|
| \( -q^{-2}(q^2 - q^{-2})^{-1} x_{i+1} x_{i+3} x_i^{-1} x_{i+1}^{-1} \) | \( -q^{-3}(q + q^{-1})^{-1} q^{-1}(q - q^{-1})^{-1}(q^2 - q^{-2})^{-1} \) |

The expression \( q^{-2} x_{i+1}^{-1} x_i^{-1} x_{i+1} x_i x_{i+1} \) is a weighted sum involving the following terms and coefficients:
\begin{align*}
\text{term} & \quad x_{i+1} & x_{i}^{-1} & x_{i+1}x_{i}^{-2} \\
\text{coeff.} & \quad 1 & 1 - q^2 & (q - q^{-1})^2
\end{align*}

The expression $-q^{-3}(q + q^{-1})x_{i+1}^{-1}x_{i}^{-1}x_{i+1}$ is a weighted sum involving the following terms and coefficients:

\begin{align*}
\text{term} & \quad x_{i}^{-1} & x_{i+1}^{-1}x_{i}^{-2} \\
\text{coeff.} & \quad -q^{-1}(q + q^{-1}) & q^{-2}(q^2 - q^{-2})
\end{align*}

Using (5.3) the expression $q^{-4}x_{i+1}^{-1}x_{i}^{-1}x_{i+1}^{-1}x_{i}^{-1}x_{i+1}$ is a weighted sum involving the following terms and coefficients:

\begin{align*}
\text{term} & \quad x_{i}^{-2} & x_{i+1}^{-1} & x_{i}^{-1}x_{i+1}^{-1}x_{i}^{-1}x_{i+1} \\
\text{coeff.} & \quad q^{-3}(q + q^{-1})^{-1} & q^{-4}(q + q^{-1})^{-1}
\end{align*}

Evaluating $\Phi$ using the above comments we get the result. \hfill \Box

We now analyze (7.3) for the case $j = i + 2$.

**Theorem 9.5.** For $i \in \mathbb{Z}_4$, the following relation holds on every nonzero finite-dimensional $\mathbb{F}_q$-module:

\[
\exp q(n_{i,i+1})^{-1}x_{i+2}\exp q(n_{i,i+1}) = x_i - \frac{x_{i+2}}{(q - q^{-1})^2} + \frac{qx_ix_{i+1}x_{i+1}^{-1}}{(q - q^{-1})(q^2 - q^{-2})} + \frac{q^{-1}x_{i+1}^{-1}x_{i+2}x_{i}}{(q - q^{-1})(q^2 - q^{-2})}.
\]

*Proof.* Apply the map $\phi$ from Lemma 4.3 to each side of the equation in Theorem 9.4. \hfill \Box

**Theorem 9.6.** For $i \in \mathbb{Z}_4$, the following relation holds on every nonzero finite-dimensional $\mathbb{F}_q$-module:

\[
\exp q(n_{i,i+1})x_{i+2}\exp q(n_{i,i+1})^{-1} = x_{i+2} - x_{i+1}^{-1} + \frac{qx_{i+1}x_{i}x_{i+1}}{q - q^{-1}} - \frac{x_{i}x_{i+1}x_{i+1}}{q(q - q^{-1})} + \frac{q^2x_{i+1}x_{i+2}x_{i+1}^{2}}{(q - q^{-1})(q^2 - q^{-2})} + \frac{qx_{i+1}x_{i}x_{i+1}^{2}}{(q - q^{-1})(q^2 - q^{-2})} - \frac{q^2x_{i}x_{i+2}x_{i+1}^{2}}{(q - q^{-1})^2}.
\]

*Proof.* Apply the map $\phi$ from Lemma 4.3 to each side of the equation in Theorem 9.3. \hfill \Box

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Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

E-mail address: yyang@math.wisc.edu