Moduli spaces of $G_2$ and $Spin(7)$—instantons on product manifolds

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Abstract

Let $X$ be a closed 6−dimensional manifold with a half-closed SU(3)−structure. With respect to the product $G_2$−structure and on certain vector-bundles, we describe the moduli of $G_2$−instantons on $X \times S^1$ in terms of the moduli of Hermitian Yang-Mills connections on $X$. In dimension 8, similar result holds for moduli of $Spin(7)$−instantons. A generalization and an example are given.

1 Introduction

1.1 Motivation and Background

Following the programs of Donaldson-Thomas [11] and Donaldson-Segal [10], it is tempting to generalize the classical gauge theory in dimensions 2, 3, 4 to dimensions 6, 7, 8. In the classification of holonomy groups by Berger and Simon ([1], [19]), these higher dimensions correspond to the special holonomy groups $SU(3)$, $G_2$, and $Spin(7)$. Based on the programs in [11] and [10], Walpuski [22] and Joyce [13] discussed the possible enumerative invariant on 7−dimensional manifolds "counting" $G_2$−instantons. Related to the above, we study the full moduli of $G_2$−instantons in some special cases.

1.2 Half-closed $SU(3)$-structure

Throughout this article, we understand $S^1$ as the smooth Riemannian manifold $\mathbb{R}/2\pi \mathbb{Z}$, so its length is $2\pi$. Let $t$ be the coordinate variable of $\mathbb{R}$ such that $dt$ descends to the smooth closed (but not exact) 1-form on $S^1$.

All manifolds, bundles, gauges, connections, sections etc are assumed to be smooth unless otherwise specified.

We seek for a dimension reduction for moduli of $G_2$−instantons on a product manifold $X \times S^1$, where $X$ is a 6−manifold satisfying the following conditions.

Definition 1.1. Given a 6-dimensional manifold $X$, we say that $(J,g_X,\omega,\Omega)$ is a $SU(3)$−structure (c.f. [14] (3.1) and the enclosing section) if

1. $J$ is an almost complex structure, $\Omega$ is a no-where vanishing $(3,0)$−form.
2. $g_X$ is a Hermitian metric on $X$ i.e. $g_X(J\cdot, J\cdot) = g_X(\cdot, \cdot)$. $\omega = g(J\cdot, \cdot)$ is the associated real positive $(1,1)$−form.
3. $|\Omega|^2_{g_X} = 8$ i.e. $\frac{1}{8} |\Omega|^2_{g_X} = \frac{1}{4} Re\Omega \wedge Im\Omega$.

A $SU(3)$−structure is called half-closed if $dRe\Omega = 0$.

Remark 1.2. The half-closed condition is not restricted to $Re\Omega$. Given a $SU(3)$−structure such that $Im\Omega$ is closed, then $(\cdot, \cdot, \sqrt{-1}\Omega)$ is half-closed. Given $J$, $\Omega$ as in Definition 1.11 such that $dRe\Omega = 0$, by Lemma 5.3 there exist abundant Hermitian metrics $g_X$ such that $(J,g_X,\omega,\Omega)$ is half-closed.
Remark 1.3. A half-closed $SU(3)$–structure is said to be Calabi-Yau if $J$ is integrable, $\omega$ is closed (Kähler), and $\Omega$ is holomorphic. Then $(g_X, \omega)$ must be Ricci flat by Definition 1.1.

Another class of half-closed $SU(3)$–structures consists of nearly-Kähler 6–manifolds, including $S^6$, $S^3 \times S^3$ etc (see [13] 3.2 and [10]).

Definition 1.4. Let $\mathbb{R}^7$ be the 7–dimensional Euclidean vector space with the co-frame $e^i$, $1 \leq i \leq 7$, we define the Euclidean associative 3–form as
\[
\phi_{Euc} = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \text{ where } e^{ijk} \triangleq e^i \wedge e^j \wedge e^k. \tag{1}
\]

Given a 7–manifold $M$, a $G_2$–structure $\phi$ is a smooth 3–form such that at every point $p$, there exists a co-frame $e^i$, $1 \leq i \leq 7$ such that $\phi(p) = \phi_{Euc}$. $\phi$ determines a Riemannian metric $g_\phi$. We let $\phi \triangleq *_{g_\phi} \phi$.

Given a 6–manifold $X$ with a $SU(3)$–structure $(J, g_X, \omega, \Omega)$, on the 7–manifold $M \times S^1$, the 3–form
\[
\phi = dt \wedge \omega + Re\Omega. \tag{2}
\]
is a $G_2$–structure whose induced metric is the product $g_X + dt \otimes dt$.

Definition 1.5. Let $\mathbb{R}^8$ be the 8–dimensional Euclidean vector space with the co-frame $e^i$, $0 \leq i \leq 7$, we define the Euclidean Cayley 4–form as
\[
\Psi_{Euc} \triangleq e^0 \wedge \phi_{Euc} + \Psi_{Euc}. \tag{3}
\]

Given an 8–dimensional manifold $M^8$, a $Spin(7)$–structure $\Psi$ is a 4–form such that at every point $p$, there exists a co-frame $e^i$, $0 \leq i \leq 7$ such that $\Psi(p) = \Psi_{Euc}$.

Let $M$ be a 7–manifold with a $G_2$–structure $\phi$. On the 8–dimensional manifold $M \times S^1$, the 4–form
\[
\Psi = dt \wedge \phi + \psi \tag{4}
\]
is a $Spin(7)$–structure. The induced metric is the product $g_\phi + dt \otimes dt$. The orientation is defined by $dt \wedge \phi \wedge \psi$, so $\Psi$ is self-dual.

1.3 Instantons and iso-trivial connections

The following definition is the foundation of our discussion in this note.

Definition 1.6. (Iso-trivial connections) Let $Y$ be a closed $n$–dimensional smooth manifold, and $E \rightarrow Y$ be a smooth $U(m)$ complex vector-bundle (of rank $m$).

Let $\mathfrak{G}$ denote the space of all smooth $U(m)$–gauges. Given a connection $A$ and $\phi \in \mathfrak{G}$, we adopt the convention $\phi(A) = A + \phi^{-1} d_A \phi$ i.e. $d_{\phi(A)} \triangleq \phi^{-1} \cdot d_A \cdot \phi$. Then
\[
\phi[u(A)] = (u \phi)(A) \text{ i.e. the gauge-action is a right multiplication.} \tag{5}
\]

Convention: unless otherwise specified, all gauges and connections are assumed to be unitary.

Let $\pi$ denote the projection from $Y \times S^1$ (or $Y \times I$ for any interval $I \subset \mathbb{R}$) to $Y$. A smooth section (or connection) $v$ to $\pi^* E \rightarrow Y \times (0,2\pi)$ is called smoothly periodic, if $v$ extends to a smooth section (connection) to $\pi^* E \rightarrow Y \times S^1$. The “$E$” here is a dummy notation, in practice the definition could apply to $End E$ where $E$ is a specific bundle.

Given a smooth connection $B$ on $E \rightarrow Y$, we define the stabilizer group as
\[
\Gamma_B \triangleq \{ u \in \mathfrak{G} | d_B u = 0 \}. \tag{6}
\]

$B$ is said to be irreducible if $\Gamma_B = Center[U(m)]$ [which is homeomorphic to $U(1)$ and $S^1$].

A smooth gauge $u$ on $\pi^* E \rightarrow Y \times [0,2\pi]$ (see Definition 1.5) is said to be $B$–admissible if $u(0) = 1d$, $u(2\pi) \in \Gamma_B$, and $\chi_u \triangleq u^{-1} du \frac{d}{dt}$ is smoothly periodic.

A smooth connection $A$ on $\pi^* E \rightarrow Y \times S^1$ is said to be iso-trivial with respect to $B$ (or iso-trivial for short), if there exists a smooth connection $B$ on $E \rightarrow Y$ and a $B$–admissible gauge $u$ such that $A = u(B)$.
Remark 1.7. Conversely, by Claim 2.3, we routinely verify that for any connection $B$ on $E \to Y$ and a $B$--admissible gauge $u$, $u(B)$ is a smooth connection on $\pi^*E \to Y \times S^1$.

Remark 1.8. Iso-triviality is preserved by gauge-transformations on $Y \times S^1$.

Definition 1.9. Given an almost complex 6--manifold $X$ with a positive real (1,1)--form $\omega$, and a bundle $E \to X$ as in Definition 1.5, we say that $A$ is Hermitian Yang-Mills if $F_A$ is (1,1) and $\overline{\partial}_E F_A, \omega = \mu I_{E}$ for a real number $\mu$. The $\mu$ is called the slope of $A$.

When $\omega$ is co-closed i.e. $d(\omega \wedge \omega) = 0$, let $\deg E \triangleq \frac{1}{2} \int_X c_1(E) \wedge \omega \wedge \omega$, the slope of any Hermitian Yang-Mills connection on $E$ must be $\frac{\deg E}{\text{rank} E}$.

Remark 1.10. The contraction of two forms, in any context, is with respect to the underlying Riemannian metric. For example, in Definition 1.9, $\lrcorner$ denotes the contraction with respect to (the Riemannian metric of) $\omega$; in (7) and (8), $\lrcorner = \lrcorner_{g_{\phi}}$ means the contraction with respect to the metric of $\phi$.

Definition 1.11. Let $(M, \phi)$ be a 7--manifold with a $G_2$--structure. A connection “$A$” on $E \to M$ is called a $G_2$--instanton if

\[ \star (F_A \wedge \psi) = 0 \] (which is equivalent to $F_A, \omega \phi = 0$).

"$A$" is called a projective $G_2$--instanton if there is a harmonic $\mathbb{R}$--valued 1--form $\theta$ such that

\[ \sqrt{\frac{1}{2\pi}} \star (F_A \wedge \psi) = \theta I_{E} \] [which is equivalent to \( \sqrt{\frac{1}{2\pi}} (F_A, \phi) = \theta I_{E} \)].

Let $(M^8, \Psi)$ be an 8--manifold with a $\text{Spin}(7)$--structure. A connection “$A$” on a bundle $E \to M^8$ is called a $\text{Spin}(7)$--instanton if

\[ \ast_{g_{\Phi}} (F_A \wedge \Psi) + F_A = 0. \] (9)

1.4 Moduli spaces

Definition 1.12. In view of Definition 1.9 and 1.11, let

\[ \mathcal{M}_{X,E,\omega\rightarrow \text{HY}}M, \mathcal{M}_{X,E,\omega\rightarrow \text{HY}}M^0, \mathcal{M}_{M,E,\phi}, \mathcal{M}_{M,E,\phi}^{\text{proj}}, \mathcal{M}_{M^8,E,\Psi}, \] (10)

denote the set of all gauge equivalence-classes of smooth Hermitian Yang-Mills connections on $E \to X$, Hermitian Yang-Mills connections with 0--slope on $E \to X$, $G_2$--instantons on $E \to M$, projective $G_2$--instantons on $E \to M$, and $\text{Spin}(7)$--instantons on $E \to M^8$ respectively.

Let

\[ \mathcal{M}_{X,E,\omega\rightarrow \text{HY}}M^{\text{irred}}, \mathcal{M}_{X,E,\omega\rightarrow \text{HY}}M^0, \mathcal{M}_{M,E,\phi}^{\text{irred}}, \mathcal{M}_{M,E,\phi}^{\text{proj,irred}}, \mathcal{M}_{M^8,E,\Psi}^{\text{irred}}, \] (11)

denote respectively the subsets of all irreducible (gauge equivalence-classes of) connections.

Definition 1.13. Given a holomorphic vector bundle $E$ on a Kähler 3--fold $X_{kah}$, let the condition of “slope-stability” be as [21] the definition in page S261, section 1. We say that a holomorphic bundle $E \to X_{kah}$ is poly-stable if $E = \bigoplus_{i=1}^{k_0} E_i$, $k_0 \leq \text{rank} E$, each $E_i$ is a stable bundle (locally free sheaf) such that $\mu(E_i) = \mu(E)$.

Donaldson-Uhlenbeck-Yau Theorem ([7], [21], [8]) implies that the holomorphic bundle (structure) $E$ admits a Hermitian Yang-Mills connection if and only if $E$ is poly-stable (also see the presentation in [15] Theorem 8.3]). Moreover, let $\mathcal{M}_{AG}^{\text{irred}} X_{kah}, E, \omega\rightarrow \text{stable}$ denote the set of all isomorphism classes of $\omega$--slope-stable holomorphic structures $E$ on $E \to X_{kah}$, which is an algebro-geometric moduli. Their theorem implies that

\[ \mathcal{M}_{AG}^{\text{irred}} X_{kah}, E, \omega\rightarrow \text{HY}M \cong \text{Space of stable holomorphic structures on } E \text{ isomorphisms} \quad (\sim \text{" is bijective to"}) \]
\[ \cong \mathcal{M}_{X_{kah}, E, \omega\rightarrow \text{stable}}^{\text{AG}}. \] (12)
Definition 1.14. (Topology of the moduli spaces) Let $|\cdot|$ denote the standard norm (metric) for complex matrices, and $E \to Y$ be a bundle as in Definition 1.6. Using a Riemannian metric on $Y$ (which should be clear from the context in practice), $|\cdot|$ extends to a norm on $\Omega^k(adE)_p$ for any $p \in Y$. We still denote this norm by $|\cdot|$.

- Let $\Lambda_{E,Y}$ ($\Lambda_{E,Y}^{irred}$) denote the space of all (irreducible) gauge equivalence classes of smooth $U(m)$-connections respectively. Similarly to [2] (4.2.3), we define a metric (distance) on $\Lambda_{E,Y}$ as the following.

$$d_{\Lambda_{E,Y}}([A_1], [A_2]) \triangleq \inf_{g \in G} \| A_1 - g(A_2) \|, \text{ where } \| \cdot \| \triangleq \sup_{p \in Y} |\cdot| \text{ is bi-invariant.} \quad (13)$$

In general, this metric induces a metric topology on any subset of $\Lambda_{E,Y}$, including those in Definition 1.12 and Theorem 1.15.

- Let $G$ be a compact subgroup of $\mathfrak{g}$ and $CON(G)$ denote the space of all conjugacy classes of $G$. We consider the following metric and the associated topology on $CON(G)$.

$$d_{CON(G)}(x, y) = \inf_{g \in G} \| x - gyg^{-1} \|. \quad (14)$$

1.5 Main Statement

Theorem 1.15. Given a 6-dimensional manifold $X$ with a half-closed $SU(3)$-structure $(J, g_X, \omega, \Omega)$ and a $U(m)$ complex vector-bundle $E \to X$, on the pullback $\pi^*(E) \to X \times S^1$ and with respect to the product $G_2$-structure (2) on $X \times S^1$, the following is true.

1. $\pi^*E \to X \times S^1$ admits a $G_2$-instanton if and only if $E \to X$ admits a Hermitian Yang-Mills connection with 0-slope.

Consequently, when $(X, J, g_X, \omega, \Omega)$ is Calabi-Yau, $\pi^*E \to X \times S^1$ admits a $G_2$-instanton if and only if $E \to X$ admits a poly-stable holomorphic structure and deg $E = 0$.

2. A connection on $\pi^*E \to X \times S^1$ is a $G_2$-instanton if and only if it is iso-trivial with respect to a Hermitian Yang-Mills connection with 0-slope on $E \to X$.

3. $\mathfrak{M}_{X \times S^1, \pi^*E, \phi}$, if non-empty, admits a continuous surjective map $\rho$ to $\mathfrak{M}_{X, E, \omega, HYM - 0}$. For any $[B] \in \mathfrak{M}_{X, E, \omega, HYM - 0}$, $\rho^{-1}([B])$ is homeomorphic to $CON(\Gamma_B)$.

4. $\mathfrak{M}^{irred}_{X \times S^1, \pi^*E, \phi} = \rho^{-1}(\mathfrak{M}^{irred}_{X, E, \omega, HYM - 0})$. $\mathfrak{M}^{irred}_{X \times S^1, \pi^*E, \phi}$ is homeomorphic to $S^1 \times \mathfrak{M}^{irred}_{X, E, \omega, HYM - 0}$.

Consequently, when $(X, J, g_X, \omega, \Omega)$ is Calabi-Yau and deg $E = 0$, $\mathfrak{M}^{irred}_{X \times S^1, \pi^*E, \phi}$ is bijective to $S^1 \times \mathfrak{M}^{AG}_{X, E, [\omega] - \text{stable}}$.

II: Given a 7-dimensional manifold $M$ with a co-closed $G_2$-structure $\phi$, and a $U(m)$ complex vector-bundle $E \to M$, on the pullback $\pi^*E \to M \times S^1$ and with respect to the product $\text{Spin}(7)$-structure on $M \times S^1$ in (3), the following is true.

1. $\pi^*E \to M \times S^1$ admits a $\text{Spin}(7)$-instanton if and only if $E \to M$ admits a $G_2$-instanton.

2. A connection on $\pi^*E \to M \times S^1$ is a $\text{Spin}(7)$-instanton if and only if it is iso-trivial with respect to a $G_2$-instanton on $E \to M$.

3. $\mathfrak{M}_{M \times S^1, \pi^*E, \phi}$, if non-empty, admits a continuous surjective map $\rho$ to $\mathfrak{M}_{M, E, \phi}$. For any $[B] \in \mathfrak{M}_{M, E, \phi}$, $\rho^{-1}([B])$ is homeomorphic to $CON(\Gamma_B)$.

4. $\mathfrak{M}^{irred}_{M \times S^1, \pi^*E, \phi} = \rho^{-1}(\mathfrak{M}^{irred}_{M, E, \phi})$, and they are homeomorphic to $S^1 \times \mathfrak{M}^{irred}_{M, E, \phi}$.

III (projective version of II): Under the same conditions and setting in 1, we assume additionally that $H^1(X, \mathbb{R}) = 0$. Then the following is true.
1. \( \pi^* E \to X \times S^1 \) admits a projective \( G_2 \)-instanton if and only if \( E \to X \) admits a Hermitian Yang-Mills connection.

Consequently, when \( (X, J, g_X, \omega, \Omega) \) is Calabi-Yau, \( \pi^* E \to X \times S^1 \) admits a projective \( G_2 \)-instanton if and only if \( E \to X \) admits a poly-stable holomorphic structure.

2. A connection on \( \pi^* E \to X \times S^1 \) is a projective \( G_2 \)-instanton if and only if it is iso-trivial with respect to a Hermitian Yang-Mills connection on \( E \to X \).

3. \( \mathcal{M}_{X \times S^1, \pi^* E, \phi}^{\text{proj}} \), if non-empty, admits a continuous surjective map \( \rho \) to \( \mathcal{M}_{X, E, \omega, \text{HYM}} \).

For any \( [B] \in \mathcal{M}_{X, E, \omega, \text{HYM}} \), \( \rho^{-1}([B]) \) is homeomorphic to \( \text{CON}(\Gamma_B) \).

4. \( \mathcal{M}_{X \times S^1, \pi^* E, \phi}^{\text{proj, irred}} = \rho^{-1}(\mathcal{M}_{X, E, \omega, \text{HYM}}^{\text{irred}}) \), and they are homeomorphic to \( S^1 \times \mathcal{M}_{X, E, \omega, \text{HYM}}^{\text{irred}} \).

Consequently, when \( (X, J, g_X, \omega) \) is Calabi-Yau, \( \mathcal{M}_{X \times S^1, \pi^* E, \phi}^{\text{proj, irred}} \) is bijective to \( S^1 \times \mathcal{M}_{X, E, [\omega] - \text{stable}}^{\text{AG}} \).

Remark 1.16. The pullback of any Hermitian Yang-Mills connection \( B \) with \( 0 \)-slope on \( E \to X \) to \( \pi^* E \to X \times S^1 \) is a \( G_2 \)-instanton (see [22], Example 1.93) for example). Proposition 2.6 and Lemma 2.7 imply that there exist \( G_2 \)-instantons on the product manifold which is not gauge equivalent to any such pullback.

Remark 1.17. Investigations by Walpuski, Sá Earp, Nordström, Menet etc show that the moduli of \( G_2 \)-instantons on certain closed 7–manifolds are non-empty (see [22], [16] and the references therein). The point of our work here is the full moduli.

Remark 1.18. When the \( G_2 \)-structure \( \phi \) on \( X \times S^1 \) is not co-closed, it seems natural to work with \( G_2 \)-monopoles rather than instantons (see [10], (25) and the enclosing page). However, the proof of Theorem 1.15 indicates that it is reasonable to work with instantons.

Remark 1.19. The structure group (of the bundle) being \( U(m) \) is crucial in Lemma 2.7. Even if it is \( SU(m) \) or \( SO(m) \), we do not know whether there is any a \( B \)-admissible gauge connecting \( I_{E} \) to an arbitrary element in \( \Gamma_B \).

Similarly to the 3–dimensional case, modulo gauge, a \( G_2 \)-instanton on \( X \times S^1 \) can be understood as a “periodic” orbit of the gradient flow of the Chern-Simons functional on \( X \). The difficulty is that the Chern-Simons functional is not necessarily gauge invariant. Our point is that this issue can be overcome.

The proof of Theorem 1.15 (and II, III) can be sketched by the following diagram.

For Calabi-Yau links, related results of Theorem 1.15 are obtained by Calvo-Andrade-Rodríguez Díaz-Sá Earp [2].

5
1.6 Simple examples

Except for trivial bundles on Calabi-Yau manifolds $X \times S^1$ (where all instantons with respect to the product $G_2$--structure are flat), it is hard to find an explicit moduli of $G_2$--instantons. Nevertheless, using the projective instantons, we do get an explicit moduli on a non-trivial bundle.

**Corollary 1.20.** There exist a smooth anti-canonical hyper-surface $X_{CY}$ in $CP^1 \times CP^1 \times CP^2$, a Kähler-metric $\omega$ on $X_{CY}$, a nowhere-vanishing holomorphic $(3,0)$--form $\Omega$ on $X_{CY}$, and a $U(2)$--bundle $E \to X_{CY}$ with the following property. Let $\phi$ be as (2), then $\mathfrak{m}^{\text{proj}}_{X_{CY} \times S^1, \pi^* E, \phi}$ and $\mathfrak{m}^{\text{proj,irred}}_{X_{CY} \times S^1, \pi^* E, \phi}$ are both homeomorphic to $S^1$.

The above example might only be a drop in those which could be produced by Theorem 1.15. For instance, by understanding the full moduli of stable structures on Jardim’s instanton bundles [12], we can hope to produce new explicit moduli spaces of instanton bundles [12], we can hope to produce new explicit moduli spaces of instanton bundles.

**Remark 2.1.** We need the following classical existence and uniqueness result for ordinary differential equations. The author is supported by Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics.

2 Preliminary on iso-trivial connections

For any $GL(E)$--valued gauge $\phi$ on $Y$ (i.e. any automorphism of $E$), and any section $\chi$ to $\text{End}(E)$, routine calculation shows that

$$\frac{\partial \phi(A_0)}{\partial t} = d_{\phi(A_0)} \chi + \phi^{-1}(\frac{\partial A_0}{\partial t}) \phi \text{ if } \frac{\partial \phi}{\partial t} = \phi \chi,$$

where we used the identity

$$\phi^{-1}(d_{\phi(A_0)} \chi) \phi = d_{\phi(A_0)} (\phi^{-1} \chi \phi).$$

**Remark 2.1.** In the context of (15) and (16), $Y$ is a dummy notation for an arbitrary manifold. However, at many places in the below, $Y$ and $Y \times S^1$ mean two different manifolds. Thus when necessary, we add the subscript $Y$ to emphasize that it is not on $Y \times S^1$, please see Remark 2.10.

**Lemma 2.2.** Let $E \to Y$ be a bundle as in Definition 1.6, and $I \subset \mathbb{R}$ be an open interval. Suppose $s(t)$ is a smooth $t$--family of gauge transformations on $E$ such that $\frac{\partial A(t)}{\partial t} = \chi s$, then

$$\frac{ds(t)|A(t)}{dt} = ds(A) \chi + s^{-1} \frac{\partial A}{\partial t} s.$$

We need the following classical existence and uniqueness result for ordinary differential equations.

**Lemma 2.3.** Let $E \to Y$ be a bundle as in Definition 1.6. Let $\chi_i$ ($i = 1, 2$) be smooth sections to $\pi^* \text{End} E \to Y \times (a_0, b)$, $-\infty < a < b < +\infty, \infty > \epsilon > 0$. Then for any smooth section $s_0$ to $\text{End} E \to Y$, the initial value problem

$$\frac{ds}{dt} = \chi_1 s + s \chi_2, \quad s(a) = s_0$$

admits an unique smooth solution $s$ on $Y \times (a_0, b)$. Moreover, when $\chi_i$ are all $u(m)$--valued and $s_0$ is a $U(m)$--valued gauge on $Y$, $s$ is a $U(m)$--valued gauge on $Y \times (a_0, b)$.
For the reader’s interest, Lemma 2.3 can be proved by the existence, uniqueness (see [3, Theorem 3.1]), and Gronwall-inequality (see [4, Page 12]).

Claim 2.4. Under the setting of Definition 1.6 and 1.7, suppose φ is smooth section to \( \pi^* \text{End} E \to Y \times [0, 2\pi] \). Then φ extends to a smooth (periodic) section on \( \pi^* \text{End} E \to Y \times S^1 \) if and only if \( \frac{\partial \phi}{\partial t}(0) = \frac{\partial \phi}{\partial t}(2\pi) \) for any \( k \geq 0 \).

The proof of Claim 2.4 in view of Definition 1.6 is a routine (but interesting) exercise on multi-variable calculus. We note that the "only if" in Claim 2.4 is obvious. The point is to show the "if" by the patching condition.

Remark 2.5. When the underlying manifold is \( Y \times S^1 \), we add \( Y \) as a subscript if the operation (gauge transformation, derivative etc) is on \( Y \). For example, see (19), (30), and (40). Hence in the setting of Definition 1.6 (iso-trivial connections), we have

\[
u(B) = u_Y(B) + \chi_u dt. \tag{18}\]

In other cases, we omit the domain in the notation for the operation.

Proposition 2.6. In the setting of Definition 1.6, two iso-trivial connections \( u(B) \) and \( v(\tilde{B}) \) are gauge equivalent if and only if there is a gauge \( g \) on \( Y \) with the following properties.

1. \( B \) is gauge equivalent to \( \tilde{B} \) as connections on \( Y \) i.e. \( g(B) = \tilde{B}, \)

2. \( u(2\pi)g = gv(2\pi) \).

Proof. We first show the “only if”. On \( Y \times (0, 2\pi) \), \((us)(B) = v(\tilde{B})\) means \( g(B) = \tilde{B} \) where \( g \triangleq usv^{-1} \). Then \( (18) \) yields that

\[
v(B) = \tilde{B}, \quad \frac{\partial g}{\partial t} = 0 \quad i.e. \quad g \text{ is independent of } t \in (0, 2\pi). \tag{19}\]

Let \( t \to 0 \), we find that \( g = s(0) \). Let

\[
\phi = gus^{-1}u^{-1}, \quad \text{we find that} \quad B = \phi(B). \tag{20}\]

The same argument as for (19) yields that \( \phi \) is independent of \( t \in (0, 2\pi) \). Because \( s(2\pi) = s(0) = g \), let \( t \to 0 \) in (20) we find that \( \phi = Id \) i.e. \( u = gus^{-1} \) for all \( t \in (0, 2\pi) \).

Let \( t \to 2\pi \), we find that \( u(2\pi)g = gv(2\pi) \).

The proof of the “if” is simply by taking \( s = u^{-1}gv \). Because both \( \chi_u \) and \( \chi_v \) are smoothly periodic, by \( s(2\pi) = (u^{-1}gv)(2\pi) = g = s(0) \) and Claim 2.3, \( s \) is smoothly periodic.

Lemma 2.7. Still in the setting of Definition 1.6, for any connection \( B \) on \( E \to Y \), and any \( a \in \Gamma_B \), there is a \( B \)-admissible gauge \( u \) on the pullback \( \pi^*E \to Y \times [0, 2\pi] \) such that \( u(2\pi) = a \). Consequently, \( u(B) \) is an iso-trivial connection on \( \pi^*E \to Y \times S^1 \).

Proof. Step 1: we first show that there exists an automorphism \( \tau \) satisfying all requirements for being \( B \)-admissible except being \( U(m) \)-valued.

Claim 2.8. There exists a smooth curve \( \gamma(t) : [0, 2\pi] \to C \) such that the following holds

- \( \gamma(t) = 1 \) when \( t \in [0, \frac{1}{10}] \).
- \( \gamma(t) = 0 \) when \( t \in [-\frac{1}{10}, 2\pi, 2\pi] \).
- \( \tau \triangleq a + \gamma(t)(Id - a) \) is a section to \( \text{Aut}(E) \) i.e. it is invertible for every \( t \in [0, 2\pi] \).

To prove Claim 2.8, we note that at any \( p \in Y \), \( det[a + x(Id - a)] \) is a degree \( m \) polynomial in \( x \). As a section to \( \text{End}(E) \to Y \), we find that

\[
\nabla_B[a + x(Id - a)] = 0. \tag{21}\]

Claim 2.9. \( H \in C^\infty[Y, \text{End} E], \) \( d_BH = 0 \implies \text{det}H \) is a constant on \( Y \).
It suffices to show \( \det H \) is a constant on any smooth curve \( l(t), \ t \in [0, t_0] \) connecting two arbitrary distinct points \( p, q \in Y \). Parallel transport yields a \( B \)-parallel frame \( S(t) = [s_1(t), \ldots, s_n(t), s_m(t)] \) along \( l(t) \). Let \( h \) be the matrix of \( H \) under \( S(t) \) i.e. \( HS = Sh \) on \( l(t) \), then \( d_B H = 0 \) implies that

\[
0 = \nabla_{B,i(t)} HS = \nabla_{B,i(t)} Sh = S \frac{\partial h}{\partial t},
\]

which means that \( h \) is independent of \( t \). Using that \( \det H = \det h \) on \( l(t) \), and that at any point, \( \det H \) is independent of frame, the proof of Claim 2.9 is complete.

Applying Claim 2.9 to \( H = \tau \equiv a + \gamma(t)(Id - a) \) with (21), the roots \( x_i, i = 1\ldots m \) of \( \det[a + x(Id - a)] \equiv 0 \) (counted with multiplicities) must be constants on \( Y \). We note that \( \mathbb{C} \setminus \cup_{i=1}^m x_i \) is path connected. Because \( \det[a + x(Id - a)] \neq 0 \) when \( x = 1 \) and when \( x = 0 \), there is a \( \gamma(t) \) which not only satisfies the first desired condition (item) in Claim 2.8 but also avoids the roots \( \cup_{i=1}^m x_i \). Then the second desired condition in Claim 2.8 holds.

Step 2: we then improve \( \tau \) to be \( U(m) \)-valued. The following key ingredient holds by elementary proof. Let \( \text{Herm}_{m \times m}^+ (\text{Herm}^+_{m \times m}) \) denote the set of all \( m \times m \) (positive definite) Hermitian matrices.

**Claim 2.10.** For any \( H \in \text{Herm}^+_{m \times m} \), there exists a unique \( h \in \text{Herm}^+_{m \times m} \) such that \( H = h^2 \). We denote \( h \) by \( \sqrt{H} \).

Let \( N \in \text{GL}(m, \mathbb{C}) \) be an invertible complex matrix, then we define

\[
P(N) = (\sqrt{NN^*}) \cdot N^* \circ^{-1}.
\]

It is routine to verify that

\[
P(N) \in U(m) \text{ for any } N \in \text{GL}(m, \mathbb{C}). \quad P(N) = N \text{ if } N \in U(m).
\]

(23)

\[
P(g^{-1}Ng) = g^{-1}P(N)g \text{ if } g \in U(m).
\]

(24)

Let \( u \equiv P(\tau) \), the following is true.

- In any coordinate chart of \( E \) (under the defining trivialization), \( u \) is \( U(m) \)-valued.
- By (24), \( u \) is a globally defined section to \( \text{Aut}(E) \to Y \).
- By Lemma 5.4 \( P \) is analytic in \( N \in GL(m, \mathbb{C}) \). Then \( u \) is smooth since \( \tau \) is.
- Because \( \tau = Id_E \) when \( t \) is close to 0 and \( \tau = a \) when \( t \) is close to 2\( \pi \), by (23) and that \( a \) is \( U(m) \)-valued, so does \( u \). Because \( \chi_t \equiv \tau^{-1} \frac{\partial \tau}{\partial t} = 0 \) when \( t \) is close to 0 or 2\( \pi \), by (23), so does \( \chi_u \). Then Claim 2.8 says that \( u \) is smoothly periodic.

The above precisely means that \( u \) is \( B \)-admissible [see Definition 1.4]. The proof of Lemma 2.7 is complete.

**Lemma 2.11.** Given an isotrivial connection \( u(B) \) on \( E \to Y \times S^1 \), for any gauge \( v \) on \( Y \times S^1 \), the following two conditions are equivalent.

1. \( du(B)v = 0 \).
2. There is a \( t \)-independent element \( b \in \Gamma_B \) such that \( bu(2\pi) = u(2\pi)b \) and \( v = u^{-1}bu \).

Consequently, \( u(B) \) is reducible on \( Y \times S^1 \iff B \) is reducible on \( Y \).

**Proof.** Routine computation shows that

\[
du(B)v = dy_{u(B)}v + \frac{\partial v}{\partial t} + [\chi_u, v] dt.
\]

(25)

Then

\[
du(B)v = 0 \iff \begin{cases} 
\frac{\partial v}{\partial t} + [\chi_u, v] = 0 \\
dy_{u(B)}v = 0 \end{cases} \iff \partial(\text{wv}u^{-1}) = 0.
\]

(26)
Let $b \triangleq v(0)$, then $b \in \Gamma_B$ and $v = u^{-1}bu$ for all $(p, t) \in X \times S^1$.  
(27)

"1 $\implies$ 2": Because $v(0) = v(2\pi) = b$, it follows from evaluating (27) at $t = 2\pi$.

"2 $\implies$ 1": The conditions in "2" implies that $v(0) = v(2\pi) = b$. Because $\chi_\nu$ is smoothly periodic, $\frac{\partial}{\partial t}v$ is periodic for any $k \geq 0$ and satisfies the conditions in Claim 2.4. By Claim 2.4, $v$ is smoothly periodic. Applying (16) to the easy equation $u^{-1}(d_YB)b_u = 0$, we directly verify the 2 equivalent conditions in (26) of that $d_{u(B)}v = 0$. This means "1" is true.

For the last conclusion in Lemma 2.11, we first prove "$\implies$". Suppose $u(B)$ is reducible, then there exists a point $(p, t) \in X \times S^1$ and a $v$ such that $d_{u(B)}v = 0$ but $v(p, t) \notin \text{Center}[U(m)]$. By "2", $b \notin \text{Center}[U(m)]$. This is because if not, $v = b \in \text{Center}[U(m)]$ at $(p, t)$, which is a contradiction. This means that $B$ is reducible.

We then prove "$\impliedby$". Suppose $B$ is reducible.

If $u(2\pi) \in \text{Center}[U(m)]$, let $b$ be an arbitrary element in the non-empty set $\Gamma_B \setminus \text{Center}[U(m)]$. Then $bu(2\pi) = u(2\pi)b$, the implication "2 $\implies$ 1" says that $v \triangleq u^{-1}bu$ satisfies $d_{u(B)}v = 0$, and $v(0) \notin \text{Center}[U(m)]$, hence $u(B)$ is reducible on $Y \times S^1$.

If $u(2\pi) \notin \text{Center}[U(m)]$, let $b = u(2\pi) \in \Gamma_B \setminus \text{Center}[U(m)]$, then $bu(2\pi) = u(2\pi)b$ holds. Let $v \triangleq u^{-1}bu$, the interesting transmigration is that we still get $v(0) \notin \text{Center}[U(m)]$ and $d_{u(B)}v = 0$. Hence $u(B)$ is reducible on $Y \times S^1$.

\[\square\]

3 Chern-Simons functionals and proof of Theorem 1.15

I1, I2, II1, II2

In general, let $E \to Y$ be a bundle as in Definition 1.6 given a closed $(n-3)$–form $H$ on $Y$, and a smooth (reference) connection $A_0$ on $E$, we define the Chern-Simons functional as follows.

$$CS_{Y,H} = \int_Y \text{Tr}(a \wedge d_Aa + \frac{2}{3}a \wedge a \wedge a + 2a \wedge F_A) \wedge H \text{dvol}_Y.$$  
(28)

Any smooth connection $A$ on the pullback $\pi^*E \to Y \times S^1$ can be written as

$$A = A_Y + \chi dt,$$  
(29)

where $A_Y = A_Y(t)$ is a smooth connection on $\pi^*E \to Y \times S^1$ without $dt$–component, and $\chi$ is a smooth section to $\pi^*(adE) \to Y \times S^1$. In this case, the curvature of $A$ on $Y \times S^1$ splits as

$$F_A = F_{Y,A_Y} + (d_{Y,A_Y}\chi - \frac{\partial A_Y}{\partial t}) \wedge dt.$$  
(30)

Lemma 3.1. Let $E \to Y$ be a bundle as in Definition 1.6. Suppose $A_Y$ is a smooth connection on $\pi^*E \to Y \times [0, 2\pi]$ without $dt$–component.

I : Suppose $\frac{\partial A_Y}{\partial t} = b + d_A\chi$ for two arbitrary smooth sections $b$ and $\chi$ to $\pi^*\text{End}E \to Y \times S^1$. Let $s$ be the solution to the following equation produced by Lemma 2.6

$$\frac{\partial s}{\partial t} = -\chi s, \quad s(0) = I_{d}, \quad t \in [0, 2\pi).$$  
(31)

Then

$$\frac{\partial s_Y(A_Y)}{\partial t} = s^{-1}bs.$$  
(32)

II : Suppose further that $A_Y$ is smoothly periodic. The following conditions are equivalent.

1. $\frac{\partial A_Y}{\partial t} = d_A\chi$ for a $u(m)$–valued section $\chi$ to $\pi^*\text{End}E \to Y \times S^1$.

2. There exists a smooth gauge $u$ on $\pi^*E \to Y \times [0, 2\pi]$ such that $A_Y = u_Y[A_Y(0)]$, $u(0) = I_{d}, u(2\pi) \in \Gamma_{A_Y(0)}$, and $u^{-1}\frac{\partial A_Y}{\partial t}$ is smoothly periodic.
Moreover, the correspondence is given by \( \chi = u^{-1} \frac{\partial u}{\partial t} \).

**Proof.** Via routine calculation, \( \parallel \) is a direct corollary of \( \parallel \).

For \( \parallel \), we first show that \( 1 \implies 2 \). Let \( b = 0 \) in \( \parallel \), we find

\[
\frac{\partial s_Y(A_Y)}{\partial t} = 0 \text{ i.e. } s_Y(A_Y) \text{ is independent of } t.
\]

Then \( A_Y = s_Y^{-1}[A_Y(0)] \). Let \( u \triangleq s^{-1} \), by \( \parallel \), \( u(0) = Id \). Because \( A_Y \) is smoothly periodic, we have that \( u(2\pi) \in \Gamma_{A_Y(0)} \). Moreover, we compute

\[
u^{-1} \frac{\partial u}{\partial t} = - \frac{\partial s}{\partial t} s^{-1} = \chi \text{ by } \parallel .
\]

The implication \( 2 \implies 1 \) directly follows from \( \parallel \). \( \square \)

**Lemma 3.2.** In the setting of Definition \( \parallel \) \( \parallel \), and the paragraph above \( \parallel \), suppose \( H \) is closed. Let \( A_0 \) be an arbitrary smooth connection on \( E \rightarrow Y \). In an arbitrary direction \( v \), the variation of the Chern-Simons functional \( \parallel \) is given by the following.

\[
\delta CS_{Y,H}(a) \triangleq \frac{dCS_{Y,H}(a + v)}{dx}|_{x = 0} = 2 \int_Y Tr(v \wedge F_{b_0 + a} \wedge H)
\]

\[
= \left\{ \begin{array}{ll}
-2 \int_Y <v, \ast(F_{b_0 + a} \wedge H) > dvol_Y & \text{when } \dim Y \text{ is odd}, \\
2 \int_Y <v, \ast(F_{b_0 + a} \wedge H) > dvol_Y & \text{when } \dim Y \text{ is even}.
\end{array} \right.
\]

**Proof.** It is absolutely standard. For the reader’s convenience, we still give the full detail. We calculate

\[
\delta Tr(a \wedge d_{A_0} a + \frac{2}{3} a \wedge a \wedge a + 2a \wedge F_{A_0}) = 2 \int_Y Tr(v \wedge F_{b_0 + a} \wedge H)
\]

\[
= Tr(v \wedge d_{A_0} a + (d_{A_0} a) \wedge v - d_{A_0} (a \wedge v) + \frac{2}{3}[v \wedge a \wedge a + a \wedge v \wedge a + a \wedge a \wedge v]
\]

\[
+ 2v \wedge F_{A_0}) = Tr(2v \wedge d_{A_0} a + 2v \wedge a \wedge a + 2v \wedge F_{A_0}) - dTr(a \wedge v).
\]

Because \( H \) is closed, the proof is complete by plugging \( \parallel \) in the following

\[
\delta CS_{Y,H} = \int_Y \delta\{Tr(a \wedge d_{A_0} a + \frac{2}{3} a \wedge a \wedge a + 2a \wedge F_{A_0})\} \wedge H.
\]

\( \square \)

**Lemma 3.3.** (see \( \parallel \)) In the setting of Definition \( \parallel \) \( \parallel \), and the paragraph above it \( \parallel \), suppose \( H \) is closed. For any connection \( A \) on \( E \rightarrow Y \), and any smooth gauge \( s \) on \( \pi^*E \rightarrow Y \times I \), where \( I \) is a bounded open interval in \( t \), we have \( \frac{d}{dt} CS_{Y,H}[s_Y(A)] = 0 \) in \( I \). Consequently, \( CS_{Y,H} \) is constant along any smooth gauge orbit.

**Proof.** Let \( \chi \triangleq \frac{ds}{dt} s^{-1} \), then

\[
\frac{\partial}{\partial t}[s_Y(A)] = \frac{d}{dt}(s^{-1} dy_{\cdot} A s) = -s^{-1} \frac{ds}{dt} s^{-1} dy_{\cdot} A s + s^{-1} dy_{\cdot} A (\chi s) = s^{-1}(d_{Y,A} \chi) s.
\]

Because conjugation by a unitary gauge preserves the inner-product, by Lemma \( \parallel \), we calculate

\[
\frac{d}{dt} CS_{Y,H}[s_Y(A)] = (-1)^n 2 \int_Y <s^{-1}(d_{Y,A} \chi) s, \ast(F_{s_Y(A)} \wedge H) > dvol
\]

\[
= (-1)^n 2 \int_Y <d_{Y,A} \chi, \ast(F_A \wedge H) > = 2 \int_Y <\chi, d_A(F_A \wedge H) > dvol
\]

\[
= 0.
\]

\( \square \)
Next we use the routine results established so far to prove Theorem 1.15. We first need to calculate the $G_2$ and $\text{Spin}(7)$–instanton equations with respect to the splitting (40).

$G_2$–case: In the setting of Theorem 1.15, let $Y = X$ [the manifold with a $SU(3)$–structure], (29) reads $A = A_X + \chi dt$. The instanton equation (7) implies via (30) that

$$F_{X,A_X} \wedge \text{Re} \Omega + J(\frac{\partial A_X}{\partial t} - d_{X,A_X} \chi) + (F_{X,A_X} \wedge \omega) dt = 0,$$

(41)

where $J(\eta) \triangleq \eta \wedge \omega$ for an arbitrary $1$–form $\eta$. Then

$$F_{X,A_X} \wedge \text{Re} \Omega + J(\frac{\partial A_X}{\partial t} - d_{X,A_X} \chi), \quad F_{X,A_X} \wedge \omega = 0,$$

(42)

Applying $J$ to both hand sides of the first equation in (42), using that $J(F_{\omega} \text{Re} \Omega) = F_{\omega} \text{Im} \Omega$, we find

$$F_{\omega} \text{Im} \Omega - (\frac{\partial A_X}{\partial t} - d_{X,A_X} \chi) = 0 \text{ for any } F \in \Lambda^2 X.$$

(43)

Using $\ast \text{Re} \Omega = \text{Im} \Omega$, we find $F_{\omega} \text{Im} \Omega = \ast_X(F_{X,A_X} \wedge \text{Re} \Omega)$. Hence (42) [therefore (7)] is equivalent to

$$\frac{\partial A_X}{\partial t} = \ast_X(F_{X,A_X} \wedge \text{Re} \Omega) + d_{X,A_X} \chi$$

(44)

$$F_{X,A_X} \wedge \omega = 0.$$

(45)

$\text{Spin}(7)$–case. In the setting of Theorem 1.15, on the $8$–dimensional manifold $M \times S^1$, we still write the connection as $A = A_M + \chi dt$ [in view of (29)]. Then we still have

$$F_A = F_{M,A_M} + (d_{M,A_M} - \frac{\partial A_M}{\partial t}) \wedge dt.$$

(46)

the orientation is $dt \wedge \phi_{Euc} \wedge \psi_{Euc}$.

Given a $2$–form $F$ on $\mathbb{R}^8 = \mathbb{R} \times \mathbb{R}^7$, we write $F = F_{R^7} + F_0 \wedge e^0$, where $e^0$ stands for the coordinate vector of the $\mathbb{R}$ in the Cartesian product. The algebraic equation $\ast s(F \wedge \Psi_{Euc}) + F = 0$ is equivalent to the following equations on $\mathbb{R}^7$.

$$\ast_t(F_{R^7} \wedge \psi_{Euc}) = F_0,$$

(47)

$$\ast_t(F_{R^7} \wedge \phi_{Euc}) + F_{R^7} = \ast_t(\psi_{Euc} \wedge F_0).$$

(48)

Using the algebraic identity $\ast_t(\theta \wedge \phi_{Euc}) = \ast_t(\theta \wedge \phi_{Euc}) + \theta$ for any $\theta \in \Lambda^2 \mathbb{R}^7$, and contracting both hand sides of (47) with $\phi$, we find that (47) implies (48). This means that (48) is a redundant condition, and that (47) is equivalent to the following equation on $M$.

$$\ast_\phi(F_{M,A_M} \wedge \psi) = d_{M,A_M} \chi - \frac{\partial A_M}{\partial t}.$$  

(49)

Proof of Theorem 1.15 I, II, III, II, III, II: We only prove the first 2 statements in I, the proof for (the first 2 statements in each of) II, III are the same.

By Lemma 3.1 and (44), we find that

$$\frac{\partial s_X(A_X)}{\partial t} = \ast_X(F_{s_X(A_X) \wedge \text{Re} \Omega}), \quad \text{where } s \text{ is given by Lemma 4.11} \text{ with } b \triangleq \ast_X(F_{X,A_X} \wedge \text{Re} \Omega).$$  

(50)

Hence Lemma 3.2 yields

$$\frac{dCS_{X,\text{Re} \Omega}[s_X(A_X)]}{dt} = 2 \int_X |F_{s_X(A_X) \wedge \text{Re} \Omega}|^2 dv_{X}.$$

(51)
We recall that $A_X$ is smoothly periodic and $s$ is a smooth gauge on $Y \times (-1, 2\pi +1)$ [because $\chi$ is smoothly periodic in $t$, the gauge $s$ in Lemma 3.1] produced by Lemma 2.3 actually exists smoothly for all $t \in (-\infty, +\infty)$, via Lemma 3.3 we obtain

$$CS_{X,Re\Omega}[s_X(A_X)(0)] = CS_{X,Re\Omega}[A_X(0)] = CS_{X,Re\Omega}[A_X(2\pi)]$$

$$= CS_{X,Re\Omega}[s_X(A_X)(2\pi)] [\text{because } s(0) = Id],$$

(52)

where Lemma 3.3 is only used for the last equality among the 3 equalities above. Integrating (51) over $t \in [0, 2\pi]$, using (52), we find

$$2\int_0^{2\pi} \int_X |F_{s_X(A_X)} \wedge Re\Omega|^2 dvol_X dt = CS_{X,Re\Omega}[s_X(A_X)(2\pi)] - CS_{X,Re\Omega}[s_X(A_X)(0)] = 0$$

Therefore $F_{s_X(A_X)} \wedge Re\Omega = 0$ everywhere, which in turn implies that

$$F_{X,AX} \wedge Re\Omega = 0 \text{ over } X \times \{t\} \text{ for any } t \in S^1.$$  

(53)

(55) with (53) imply that $A_X(t)$ is Hermitian Yang-Mills with $0$–slope for all $t \in S^1$. This and Remark 1.16 complete the proof of Theorem 1.15. Moreover, plugging (53) back into (44), we find

$$\frac{\partial A_X}{\partial t} = d_{X,AX} \chi.$$

(54)

Then the proof of the “only if” in Theorem 1.15 II is complete by Lemma 3.1 II. By (44), (54), and Remark 1.7, we directly verify the “if” in I.2.

The proof of Theorem 1.15 I (1 and 2) is by repeating exactly the above argument, changing the manifold $X$ into the 7–dimensional $M$, changing the closed form $Re\Omega$ into the co-associative form $\psi$ on $M$, and using (49) instead of (44), (45).

To prove Theorem 1.15 III (1 and 2), we observe that by Kunneth-Theorem for the Hodge-DeRham cohomology and the condition that $H^1(X, \mathbb{R}) = 0$, $H^1(X \times S^1, \mathbb{R})$ is spanned by $dt$. Then on $X \times S^1$, $A$ is a projective $G_2$–instanton if and only if

$$\frac{\sqrt{-1}}{2\pi} (F_A \wedge \psi) = \mu dt \otimes Id_E, \text{ for some real number } \mu.$$

(55)

By the calculation from (11)–(15), $A$ is a projective $G_2$–instanton if and only if (44) and $\frac{\sqrt{-1}}{2\pi} F_{X,AX} \wedge Re\Omega = \mu Id_E$ [instead of (15)] hold true. The rest of the proof is identical to that of Theorem 1.15 II (1 and 2) above.

4 Topology of the moduli: proof of Theorem 1.15 I, I, II, II, III, III, IV

Let the bundle and manifold $E \rightarrow Y$ be as in Definition 1.6 and $\mathcal{M}_{Y,\pi^*E}^{isotrivial}$, $\mathcal{M}_{Y,\pi^*E}^{isotrivial, irreducible}$ denote the set of (irreducible) gauge equivalence classes of iso-trivial connections on $\pi^*E \rightarrow Y \times S^1$, respectively. The proof for these topologic statements, by our formulation, does not essentially involve the instanton or Hermitian Yang-Mills condition.

**Definition 4.1.** We define the map $\rho : \mathcal{M}_{Y,\pi^*E}^{isotrivial} \rightarrow \Lambda_{E,Y}$ as $\rho(A) = A(0)$. For any $[B] \in \Lambda_{E,Y}$ and $B$ representing $[B]$, we define the map $\tau_B : \rho^{-1}([B]) \rightarrow CON(\Gamma_B)$ as

$$\tau_B([u(B)]) = [u(2\pi)].$$

(56)

We define the map $\tau : \mathcal{M}_{Y,\pi^*E}^{isotrivial, irreducible} \rightarrow Center[\mathcal{U}(m)] \times \Lambda_{E,Y}^{irreducible}$ as

$$\tau([u(B)]) = [u(2\pi), [B]].$$

(57)
Remark 4.2. The maps in (59) and (61) are well defined i.e. they do not depend on the representative chosen in $[B]$.

Remark 4.3. From here to the end of the proof of Proposition 4.8 in view of Remark 2.1 and 2.5 $|| \cdot ||$ means the norm on $Y \times [0, 2\pi]$ or $Y \times S^1$ (the whole manifold), and $|| \cdot ||_Y$ means the norm on $Y$ (the cross-section).

Lemma 4.4. In view of Definition 4.1 and the above paragraph, suppose $B_1, B$ are smooth connections on $E \to Y$, and $[u_i(B_1)], [u(B)] \in \mathcal{M}_{\text{isotrivial}}^{\text{irred}}$. Then

$$\lim_{i \to \infty} d_{\Lambda_{E,Y}}([u_i(B_1)], [u(B)]) = 0$$

if and only if there exists smooth gauges $g_i$ on $\pi^*E \to Y \times S^1$ such that

$$\lim_{i \to \infty} \|B_i - \eta_i Y(B)\| = 0 \quad \text{and} \quad \lim_{i \to \infty} \|\eta_i^{-1} \frac{\partial \eta_i}{\partial t}\| = 0, \quad \text{where} \quad \eta_i \triangleq ug_iu_i^{-1}. \quad (58)$$

Proof. It suffices to observe that $\|u_i(B_1) - g_i(u(B))\| = \|B_i - ug_iu_i^{-1}(B)\|$, then use $B_i - \eta_i(B) = (B_i - B - \eta_i^{-1}d_{Y,B}\eta_i) - \eta_i^{-1}\frac{\partial \eta_i}{\partial t}dt.$

Lemma 4.4 directly implies

Corollary 4.5. In the same setting as Definition 4.1 and Lemma 4.4, the map $\rho: \mathcal{M}_{\text{isotrivial}}^{\text{irred}} \to \Lambda_{E,Y}$ is continuous. Consequently, for any subset $\mathcal{M} \subset \mathcal{M}_{\text{isotrivial}}^{\text{irred}}$, under the induced topology, the map $\rho: \mathcal{M} \to \rho(\mathcal{M})$ is continuous.

Lemma 4.6. In the same setting as Definition 4.1 and Lemma 4.4

1. both $\tau_B$ and $\tau$ are bijective.

2. $\tau_B^{-1}: \text{CON}(\Gamma_B) \to \rho^{-1}(\{B\})$ is continuous for any $[B] \in \Lambda_{E,Y}$.

3. $\tau^{-1}: \text{Center}([U(m)] \times \Lambda_{E,Y}^{\text{irred}}) \to \mathcal{M}_{\text{isotrivial}, \text{irred}}^{\text{irred}}$ is continuous.

Remark 4.7. For any gauge $\phi$, because $\Gamma_{\phi(B)} = \phi^{-1}(\Gamma_B)\phi$ is homeomorphic to $\Gamma_B$ as compact sub-groups of $\Phi$, statement 2 in Lemma 4.6 is independent of the representative chosen in $[B]$.

Proof. The fact that $\tau_B$ is surjective follows directly from Lemma 2.7 that $\tau_B$ is injective follows directly from Proposition 2.6. Similarly, Proposition 2.6, Lemma 2.7, 2.11 imply that $\tau$ is a bijection.

Next, we prove statement 2. Statement 3 follows by similar argument.

Suppose $[a_i] \to [a]$ in $\text{CON}(\Gamma_B)$. It means that there exists gauges $b_i \in \Gamma_B$ such that

$$\lim_{i \to \infty} \|b_i^{-1}a_ib_i - a\|_Y = 0. \quad (59)$$

Because $d_B(b_i^{-1}a_ib_i - a) = 0$ (and $B$ is smooth), we find

$$\lim_{i \to \infty} \|b_i^{-1}a_i - a\|_{C^0(Y \times [0, 2\pi], \pi^*E \otimes \mathcal{E})} = 0 \quad \text{for all} \quad k \geq 0. \quad (60)$$

As Claim 2.8 and below (21), let

$$u_i \triangleq P(b_i^{-1}a_ib_i) + \gamma(t)[Id - (b_i^{-1}a_i)b_i]), \quad u = P[a + \gamma(t)(Id - a)], \quad (61)$$

where $\gamma(t)$ avoids a small enough open neighborhood of all the roots of $det(a + x[Id - a])$ (in terms of $x$, see the material from Claim 2.8 to Claim 2.9). Then $u$ is a unitary gauge, and when $i$ is large enough, so is $u_i$. Moreover, (60) implies that

$$\lim_{i \to \infty} \|u_i - u\|_{C^0(Y \times [0, 2\pi], \pi^*E \otimes \mathcal{E})} = 0 \quad \text{(see Definition 5.1), which in turn implies}\n
\lim_{i \to \infty} \|u_i(B) - u(B)\| = 0. \quad \text{Hence} \quad \lim_{i \to \infty} d_{\Lambda_{E,Y}^{\text{irred}}}([u_i(B)], [u(B)]) = 0. \quad \square$$

The continuity in the other direction is by another approach.
Proposition 4.8. In view of Lemma 4.6, let \( \eta \in \Lambda_{E,Y} \) be as in (65), condition (62) yields
\[
\lim_{i \to \infty} \| \eta_i^{-1} d_{B,Y} \eta_i \| = 0, \quad \lim_{i \to \infty} \| \eta_i^{-1} \partial \eta_i / \partial t \| = 0.
\]
By Lemma 5.2 and the first condition in (64), there exists \( a \in \Gamma_B \) such that
\[
\lim_{i \to \infty} \| \eta_i(0) - a \|_Y = 0.
\]
Integrating the second condition in (64) with respect to \( t \), we find
\[
\lim_{i \to \infty} \| \eta_i(2\pi) - \eta_i(0) \|_Y = 0.
\]
Then triangle-inequality yields
\[
\lim_{i \to \infty} \| \eta_i(2\pi) - a \|_Y = 0.
\]
Using (65), (66), \( \eta_i(0) = g_i(0) = g_i(2\pi) \), and that
\[
\| a - a^{-1} u - u \|_Y(2\pi) = \| u a u^{-1} - a - u_i \|_Y(2\pi) = \| u a u^{-1} - u g_i u_i^{-1} + \eta_i - a \|_Y(2\pi),
\]
we find \( \lim_{i \to \infty} \| a - a^{-1} u - u \|_Y(2\pi) = 0 \), therefore (63) is true.

Next we prove "2" similarly to the fiber-wise case above. Suppose
\[
\lim_{i \to \infty} \| u_i(B_i) - g_i[u(B)]\| = 0, \quad [\text{note the slight difference from (62)}],
\]
we need to show
\[
\lim_{i \to \infty} d_{\Lambda_{E,Y}}(B_i, [B]) = 0 \quad \text{and} \quad \lim_{i \to \infty} \| u_i(2\pi) - u(2\pi) \|_Y = 0.
\]
Still let \( \eta_i \) be as in (65), the condition (67) and Lemma 4.4 yield that
\[
\lim_{i \to \infty} \| B_i - \eta_i(B) \| = 0 \quad [\text{note } B_i - \eta_i(B) = B_i - B - \eta_i^{-1} d_{B,Y} \eta_i],
\]
\[
\lim_{i \to \infty} \| \eta_i^{-1} \partial \eta_i / \partial t \| = 0.
\]
The first condition in (66) is proved. It remains to prove the second using irreducibility.
Again, integrating (67) with respect to \( t \), we find \( \lim_{i \to \infty} \| \eta_i(0) - \eta_i(2\pi) \|_Y = 0 \). Hence
\[
\lim_{i \to \infty} \| g_i(0) - u(2\pi) g_i(2\pi) u_i^{-1}(2\pi) \|_Y = 0
\]
Because \( a(2\pi), u_i(2\pi) \in \text{Center}(\Gamma_B) \), using (71) and that
\[
\| u_i(2\pi) - u(2\pi) \|_Y = \| Id - u(2\pi) u_i^{-1}(2\pi) \|_Y = \| g_i(0) - g_i(0) u(2\pi) u_i^{-1}(2\pi) \|_Y
\]
\[
= \| g_i(0) - u(2\pi) g_i(2\pi) u_i^{-1}(2\pi) \|_Y \quad [\text{using } g_i(0) = g_i(2\pi) \text{ and } u(2\pi) g_i(0) = g_i(0) u(2\pi)]
\]
we find \( \lim_{i \to \infty} \| u_i(2\pi) - u(2\pi) \|_Y = 0 \). Hence the second condition in (65) holds.  \[\square\]
Lemma 5.2. ω Yau [24] showed that there exists a unique property. Suppose Ω

Definition 5.1. The proof is then complete by (72) and Theorem 1.15

Proof of Theorem 5.1. By [24] Theorem 4.8 and page 418 Example 1, there is a bundle E → X_CY as in Corollary 1.20 and a Kähler-class [ω] such that the following holds.

• M^{AG}{X_CY,E,π,E,Ω} stable consists of one point.

• Any poly-stable holomorphic structure on E is stable, therefore simple. By [15] VII Proposition 4.14 and the Donaldson-Uhlenbeck-Yau Theorem (stated in the second paragraph of Definition 1.13), we obtain

M^{irred}_{X_CY,E,ω-HYM} = M^{X_CY,E,ω-HYM}, and both of them consist of one point.

Let Ω₀ be a trivialization of K_{X_CY}, there exists c₀ ∈ ℂ (unique up a unitary factor) such that Ω = c₀Ω₀ satisfies

\[ \int_{X_{CY}} \bar{\omega}^3 = \frac{\sqrt{-1}}{8} \int_{X_{CY}} Ω \wedge \bar{Ω} = \frac{1}{4} \int_{X_{CY}} ReΩ \wedge ImΩ. \]

Yau [24] showed that there exists a unique ω ∈ [ω] satisfying the volume-form equation in Definition 1.13. The proof is then complete by (72) and Theorem 1.15.

5 Appendix

Definition 5.1. Let \|\cdot\|_{C^k[Y,E]} denote the C^k-norms of a section to the bundle EndE → Y (defined as the weighted sum of the C^k-norms of the matrix-valued functions in coordinate charts with respect to the partition of unity). We define the C^∞[Y,E]-topology by the following.

\[ \lim_{j \to \infty} φ_j = φ_∞ \iff \lim_{j \to \infty} φ_j = φ_∞ \text{ in } C^k[Y,E] \text{ for every } k. \]

This is a metric topology by [17] Section 1.46.

Suppose s is a continuous section to π^*E → Y × [0, 2π] which is smooth on Y × (0, 2π). s is said to be smooth on π^*E → Y × [0, 2π] if under the C^∞[Y,E]-topology, for any k ≥ 0, both \lim_{t \to 0} \frac{∂s}{∂t} and \lim_{t \to 2π} \frac{∂s}{∂t} exist. Then for any k ≥ 0, \frac{∂s}{∂t} extends continuously to Y × [0, 2π]. The values at the end points are still denoted by \frac{∂s}{∂t}(0) and \frac{∂s}{∂t}(2π) respectively. The C^k-norm on Y × [0, 2π] is defined naturally as

\[ \|s\|_{C^k(Y×[0,2π],π*E)} ≜ \sup_{0 ≤ i+j ≤ k, t_0 \in [0,2π]} \|\frac{∂^i s}{∂t^j}(t_0)\|_{C^j[Y,E]}. \]

A smooth connection on π^*E → Y × [0, 2π] is defined similarly.

Lemma 5.2. In the setting of Proposition 4.8, for any ε > 0, there is a δ with the following property: Suppose η is a gauge on E → Y and \|η^{-1}d_Bη\| < δ, then there is an \alpha ∈ Γ_B such that d_Bα = 0 and \|η - α\| < ε.
Proof. If not, there is an $\epsilon > 0$ and a sequence $\eta_j$ such that
\[ ||\eta_j^{-1}d_B\eta_j|| \to 0, \]
but for any $j$, $||s - \eta_j|| < \epsilon \implies d_A s \neq 0$.
\(\Box\) implies that $||\eta_j|| + ||\nabla_B \eta_j|| \leq C$, where $C$ is independent of $j$. Then Arzela-Ascoli Theorem implies that $\eta_j$ sub-converges to $a$ in $C^0[Y, \text{End}(E)]$ for any $a \in [0, 1]$. [18 Theorem 7.17] implies that $a$ admits all partial derivatives at any point $p \in Y$ under any coordinate chart, and $d_B a = 0$. Hence $a$ is smooth and $a \in \Gamma_B$. This is a contradiction to the line below (70).

\[ \Box \]

Lemma 5.3. Let $X$ be a closed 6-dimensional manifold with an almost complex structure $J$ and a no-where vanishing $(3,0)$-form $\Omega$. For any conformal class $[g]$ of Hermitian metrics, there is a unique Hermitian metric $g$ such that $|\Omega|^2_g = 8$ i.e. $\frac{\omega^2}{2} = \frac{1}{4} \Re \Omega \wedge \Im \Omega$ where $\omega \triangleq g(J\cdot, \cdot)$ is the associated $(1,1)$-form of $g$. Consequently, at an arbitrary point $p$, there exists a unitary frame $v^1, v^2, v^3 \in T^1,0_p(X)$ with respect to $g$ such that
\[ \omega|_p = \frac{\sqrt{-1}}{2} \Sigma_{i=1}^3 v^i \wedge \overline{v}^i, \quad \Omega|_p = v^1 \wedge v^2 \wedge v^3. \]

Proof. For any $p$, let $u^1, u^2, u^3 \in T^1,0_p(X)$ be a unitary frame such that $\omega = \sqrt{-1} \Sigma_{i=1}^3 u^i \wedge \overline{u}^i$, $\Omega = c_0(u^1 \wedge u^2 \wedge u^3)$. Then $|c_0|^2 = |\Omega|^2_8$ is smooth. Let $h^i \triangleq |c_0|^2 u^i$, we find
\[ \Omega = c_1(h^1 \wedge h^2 \wedge h^3), \quad c_1 = \frac{c_0}{|c_0|} \text{ thus } |c_1| = 1. \]
We define $\omega \triangleq |c_0|^2 \omega = \sqrt{-1} \Sigma_{i=1}^3 h^i \wedge \overline{h}^i$. (78)

This means that $|\Omega|^2_g = 8$, where $g = \omega(\cdot, J \cdot)$ is the corresponding Hermitian metric. Finally, let $c_2$ be an arbitrary cubic root of $c_1$ and $v_i \triangleq c_2 h_i$, the existence in Lemma 5.3 is proved.

Suppose $\tilde{g} = e^{2f} g$ is another Hermitian metric satisfying (77) everywhere, then
\[ 8 = |\Omega|^2_{\tilde{g}} = e^{-6f} |\Omega|^2_g = 8e^{-6f} \implies f = 0. \]
Hence the uniqueness of $g$ in the conformal class follows. \(\Box\)

Lemma 5.4. In view of Claim 2.10, the map $\sqrt{\cdot} : \text{Herm}^+_m \rightarrow \text{Herm}^+_m$ is real-analytic.

Remark 5.5. For lack of reference, we still give the full proof.

Proof. The idea is to interpret $\sqrt{\cdot}$ as an implicit function, then use the implicit function theorem. We consider $F(H, h) \triangleq H - h^2 : \text{Herm}^+_m \oplus \text{Herm}^+_m \rightarrow \text{Herm}^+_m$. For any $H_0$, $h_0$ such that $F(H_0, h_0) = 0$, it suffices to show that the linearization $L_{h_h} : \text{Herm}^+_m \rightarrow \text{Herm}^+_m$ with respect to $h$ is invertible. We calculate
\[ -L_{h_h} g = h_0 g + gh_0, \text{ where } g \text{ is the variation of } h. \]
(80)

Suppose
\[ h_0 g + gh_0 = 0. \]
(81)

For any eigenvalue $\mu$ of $h$, let $v$ be a corresponding eigenvector. Because $h_0$ and $g$ are both Hermitian, $\mu$ must be real, and we compute
\[ 0 = (h_0 g v, v) + (gh_0 v, v) = 2\mu(h_0 v, v) \text{ where } "(\cdot, \cdot)" \text{ is the Euclidean Hermitian product}. \]
Because $h_0$ is positive definite Hermitian, $(h_0 v, v) > 0$. Then $\mu = 0$. Because $\mu$ is an arbitrary eigenvalue of $g$, we have $g = 0$. Therefore $\text{Ker} L_{h_h} = \{0\}$, and $L_{h_h}$ is an linear isomorphism from $\text{Herm}^+_m$ to itself.

By the analytic implicit function theorem (see [23] Page 1081) and the uniqueness of square root in Claim 2.10 $h(H) = \sqrt{H}$ is real-analytic near $H_0$. Because $H_0$ is arbitrary, the proof is complete. \(\Box\)
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