Dynamical relaxation of correlators in periodically driven integrable quantum systems

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We show that the correlation functions of a class of periodically driven integrable closed quantum systems approach their steady state value as \( n^{-(\alpha+1)/\beta} \), where \( n \) is the number of drive cycles and \( \alpha \) and \( \beta \) denote positive integers. We find that generically \( \beta = 2 \) within a dynamical phase characterized by a fixed \( \alpha \); however, its value can change to \( \beta = 3 \) or \( \beta = 4 \) either at critical drive frequencies separating two dynamical phases or at special points within a phase. We show that such decays are realized in both driven Su-Schrieffer-Heeger (SSH) and one-dimensional (1D) transverse field Ising models, discuss the role of symmetries of the Floquet spectrum in determining \( \beta \), and chart out the values of \( \alpha \) and \( \beta \) realized in these models. We analyze the SSH model for a continuous drive protocol using a Floquet perturbation theory which provides analytical insight into the behavior of the correlation functions in terms of its Floquet Hamiltonian. This is supplemented by an exact numerical study of a similar behavior for the 1D Ising model driven by a square pulse protocol. For both models, we find a crossover timescale \( n_c \) which diverges at the transition. We also unravel a long-time oscillatory behavior of the correlators when the critical drive frequency, \( \omega_c \), is approached from below (\( \omega < \omega_c \)). We tie such behavior to the presence of multiple stationary points in the Floquet spectrum of these models and provide an analytic expression for the time period of these oscillations.

I. INTRODUCTION

Non-equilibrium dynamics of closed quantum systems has been the subject of intense research activity in the recent past\(^1\)–\(^7\). Theoretical studies on the subject focussed initially on quench\(^8\)–\(^10\) and ramp\(^11\)–\(^16\) protocols. However, recently the focus in the field has shifted to periodically driven systems\(^8\)–\(^7\). More recently quasi-periodic and aperiodically driven systems have also been studied\(^17\)–\(^21\). The experimental signatures of such dynamics have been investigated in the context of ultracold atoms in optical lattices\(^22\)–\(^26\).

Quantum systems driven out of equilibrium via a periodic protocol host several phenomena which are not seen in those driven by a quench or a ramp. These include the generation of drive-induced topological states of matter\(^28\)–\(^30\), realization of Floquet time crystals\(^31\)–\(^33\), and phenomena such as dynamical localization\(^34\)–\(^37\), dynamical freezing\(^38\)–\(^40\), and drive-induced tuning of ergodicity\(^41\),\(^42\). These properties of periodically driven systems, having a time period \( T \), are most easily understood from their Floquet Hamiltonian \( H_F \) which is related to their unitary evolution operator \( U \) via the relation \( U(T,0) = \exp[-iH_FT/\hbar] \).

The presence of dynamical transitions constitutes yet another interesting phenomenon in periodically driven closed quantum system\(^43\)–\(^48\). Such transitions can be categorized into two distinct classes. The first involves non-analyticities of the return probability of its wave function; these non-analyticities show up as cusps in Loschmidt echoes\(^43\). Such transitions can be related to Fischer zeroes of the complex partition function of the driven system\(^43\),\(^44\). In contrast, the second class of transitions constitutes a change in the long-time behavior of the correlation functions of a periodically driven integrable quantum system as a function of the drive frequency\(^46\),\(^47\). Such a transition results from a change in the extrema of the Floquet Hamiltonian \( H_F \) as a function of the drive parameters; the signature of such transitions can be deciphered from the study of local correlation functions of such models\(^46\),\(^47\). The study of such transitions has also been extended to integrable models with long-ranged interactions\(^47\) and those coupled to an external bath\(^48\). The characteristics of the correlation function in the two dynamical phases across the transition have been studied in detail. It was shown that for a \( d \)-dimensional integrable system after \( n \) drive cycles and for large \( n \), these correlators decay as \( n^{-(d+2)/2} \) in the high-frequency regime and as \( n^{-d/2} \) in the low-frequency regime. However, the behavior of the system at a dynamical critical point and its vicinity has not been studied previously.

In this work, we study the properties of correlation functions for general driven 1D integrable quantum systems which have a simple representation in terms of free fermions. Our analysis holds for several 1D spin systems such as the Ising model in a transverse field, the \( XY \) model, and the 1D Kitaev chain. All of these models allow for a simple fermionic representation via a Jordan-Wigner transformation leading to a quadratic, exactly solvable Hamiltonian\(^49\). In addition, it is also applicable to charge- or spin-density wave systems described by the SSH model\(^50\).

The central points that emerge from such a study are as follows. First, we show that all local fermionic correlation functions of such driven models decay to their steady state value according to the relation

\[
C_x(nT) \sim n^{-(\alpha+1)/\beta},
\]

where \( \alpha \) and \( \beta \) are positive integers and \( x \) indicates the spatial coordinate. We note that only the case of \( \beta = 2 \)
and \( \alpha = 0, 2 \) has been discussed in earlier studies; these are reproduced as special cases of the general result given by Eq. (1). We show that such a result is tied to the stationary point structure and symmetry properties of the Floquet spectrum of the system. We identify the condition for the existence of anomalous powers \((\beta \neq 2)\) for the driven system and estimate a crossover scale, \( n_c \), after which the system is expected to deviate from the anomalous \((\beta \neq 2)\) scaling towards the generic \((\beta = 2)\) one. This crossover scale diverges at specific points in the parameter space of the driven system; we chart out the condition for the realization of such points in terms of its Floquet spectrum. Second, we provide specific example of such decay with \( \beta \neq 2 \) in the context of simple models. To this end, we study the driven SSH model using a continuous drive protocol. We find the realization of decay exponents \(-1/3\) corresponding to \( \beta = 3 \) and \( \alpha = 0 \). We analytically calculate the corresponding Floquet Hamiltonian within a Floquet perturbation theory (FPT) which provides insight into the structure of the Floquet spectrum and the correlation functions of the model. Such analytical results are shown to match closely with exact numerical studies. Third, we identify a long-time coherent oscillation of the correlation function of such models when the drive frequency is near to but less than a critical drive frequency. We show that the oscillation is a consequence of the presence of multiple stationary points in the Floquet spectrum of the SSH model; consequently, it is absent at drive frequencies higher than the critical frequencies. We provide an analytic expression for the time period of the oscillation which matches our numerical results. Fourth, we analyze the Ising model driven by a square pulse protocol and show the existence of anomalous decay exponents corresponding to \( \beta = 4 \) at the first dynamical transition. We provide a detailed analysis of the crossover scale around this transition. Furthermore, we note that at the reentrant transitions present in this model, the correlation functions show a decay characterized by an exponent of \(-1/3\) which is similar to that in the SSH model. In addition, near the first transition, we unravel the long-time oscillatory nature of the correlation functions when the critical drive frequency is approached from below (lower frequency); this feature is absent when the transition is approached from above. We provide an explanation of such a behavior using the properties of the Floquet spectrum of the driven models. Finally, our analysis identifies a crossover scale \( n_c \) which diverges at the dynamical transition characterized by the critical drive frequency \( \omega_c \): \( n_c \sim |\omega - \omega_c|^{-\beta_0/(\beta_0-a_0)} \), where \( a_0 = 1 \) or \( 2 \) depending on the symmetry of the model, and \( \beta_0 > a_0 \) corresponds to the order of the second term in expansion of the Floquet energy around the transition point. For \( n > n_c \), the decay of the correlation function follows a generic exponent corresponding to \( \beta = 2 \); below \( n_c \), the decay is characterized by \( \beta > 2 \). We validate such a power-law divergence of \( n_c \) from exact numerics for both the Ising and the SSH model.

The plan of the rest of the paper is as follows. In Sec. II, we analyze the correlation functions of a driven fermion model and provide a detailed derivation of Eq. (1). This is followed, in Sec. III, by a study of the driven SSH model which provides concrete examples of the scaling laws discussed. Next, in Sec. IV, we study the scaling behavior of the correlation functions of the periodically driven 1D Ising model in a transverse field. Finally, we summarize our results and conclude in Sec. V.

II. GENERAL RESULTS

In this section, we shall discuss the general behavior of correlation functions of periodically driven 1D integrable models. In what follows, we shall consider a 1D integrable model whose Hamiltonian is given by

\[
H = \sum_k \psi_k^\dagger H_k \psi_k, \quad H_k = \vec{\sigma} \cdot \vec{h}(k,t),
\]

where \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) denotes the standard Pauli matrices, and \( \vec{h}(k,t) = (h_x(k,t), h_y(k,t)) \) is the Hamiltonian density in momentum space. The time-dependence of \( \vec{h}(k,t) \) is fixed by the drive; in this work, we shall consider the case where the drive is characterized by a time period \( T = 2\pi/\omega \), where \( \omega \) is the drive frequency. In what follows, we shall consider \( \psi_k = (a_k, b_k)^T \) to be a two-component fermionic field characterized by annihilation operators \( a_k \) and \( b_k \). The exact nature of these operators depend on the model and shall be discussed in detail in subsequent sections for the SSH and the Ising models.

The unitary evolution operator for such systems can be expressed in term of their Floquet Hamiltonian

\[
U(T,0) = \prod_k U_k(T,0) = Te^{-\int_0^T dtH(t)/\hbar} = e^{-iH_FT/\hbar},
\]

where \( H_F \) is the Floquet Hamiltonian of the system. Thus \( U_k \) for such models can be resolved in terms of the Floquet eigenvalues and eigenvectors. Since \( U_k(T,0) \) is a \( 2 \times 2 \) matrix, we find

\[
U_k(T,0) = \sum_{j=1,2} e^{-i\epsilon_F^{(j)}(k)T/\hbar}|n_j(k)\rangle\langle n_j(k)|,
\]

where \( \epsilon_F^{(j)}(k) \), for \( j = 1, 2 \), are the Floquet eigenvalues and \( |n_j(k)\rangle \) are the corresponding eigenvectors.

To compute the correlation functions for such a driven system, we start from an initial state \( |\psi_{in}^k\rangle \) and compute the expectation value

\[
C_k(nT) = \langle \psi_{in}^k | (U_k^n)^* O_k(U_k^n) | \psi_{in}^k \rangle,
\]

where \( O_k \) is a generic quadratic operator constructed out of \( \psi_k \) and \( \psi_k^\dagger \). The specific forms of these operators shall be discussed in subsequent sections in the context of the
SSH and Ising models. We note that for the integrable models treated here, the correlations of \( C_k \) constitute the most general independent correlation functions; all quartic or higher order correlation of fermionic operators can be expressed in terms of \( C_k \).

Using \( \text{Eq. (4)} \), we can express these correlations as

\[
C_k(nT) = C_{0k} + \delta C_k(nT),
\]

\[
C_{0k} = \sum_j |\alpha_j(k)|^2 O_{j1j2}(k),
\]

\[
\delta C_k(nT) = e^{-in\Delta(k)T/\hbar} f(k) + \text{H.c.},
\]

\[
f(k) = \alpha_2^2(k)\alpha_4(k)O_{12}(k),
\]

\[
C_x(nT) = \int_{\text{BZ}} \frac{dk}{2\pi} e^{ikx} C_k(nT),
\]

where the integral is taken over the Brillouin zone. Here the Floquet energy gap \( \Delta(k) \), the overlap \( \alpha_j(k) \) of the initial state with the Floquet eigenstates, and the matrix elements \( O_{j1j2}(k) \) are given by

\[
\Delta(k) = \epsilon_F^{(1)}(k) - \epsilon_F^{(2)}(k), \quad \alpha_j(k) = \langle \psi_{kn}^{|}\rangle |n_j(k)\rangle,
\]

\[
O_{j1j2}(k) = \langle n_{j2}(k)|O_{k2}|n_{j2}(k)\rangle.
\]

(7)

We note that the Fourier transform of \( C_{0k} \) (Eq. (6)) denotes the steady state value of \( C_x \) in real space which is independent of \( n \). Thus

\[
\delta C_x(nT) = \int_{\text{BZ}} \frac{dk}{2\pi} e^{ikx} (f(k)e^{-in\Delta(k)T/\hbar} + \text{H.c.})
\]

represents the deviation of \( C_x(nT) \) from its steady state value in real space. Since such a steady state is reached for large \( n \) in any driven system, we expect \( \delta C_x(nT) \) to be a decaying function of \( n \) for large \( n \).

To understand the nature of this decay, we note that for large \( n \), the integral for \( \delta C_x(nT) \) can be evaluated within a stationary point approximation. To this end, let us assume that the leading contribution to the integral comes from a stationary point at \( k = k_0 \). Around this point, let us assume that

\[
\Delta(k) \simeq \Delta(k_0) + \Delta^{(\beta)}(k_0)\delta k^{\beta} + \cdots
\]

\[
f(k) \simeq f(k_0) + f^{(\alpha)}(k_\delta k^{\alpha}) + \cdots
\]

\[
\Delta^{(\beta)}(k_0) = \frac{\partial^\beta \Delta(k_0)}{\partial k^\beta} \bigg|_{k=k_0}, \quad f^{(\alpha)}(k_0) = \frac{\partial^\alpha f(k)}{\partial k^\alpha} \bigg|_{k=k_0},
\]

(9)

where \( \alpha \) and \( \beta \) denote the leading powers for expansion \( \Delta(k) \) and \( f(k) \) respectively around \( k = k_0 \). We note that since \( k_0 \) is a stationary point, \( \beta \geq 2 \). Substituting Eq. (9) in Eq. (8), we find the leading behavior of the correlation to be given by

\[
\delta C_x(nT) \sim A(k_0; n, T) + \int_{-\infty}^{\infty} \frac{d\delta k}{2\pi} e^{-i\delta kx}
\]

\[
\times \left(f^{(\alpha)}(k_0)\delta k^{\alpha}e^{i\Delta^{(\beta)}(k_0)\delta k^\beta T/\hbar} + \text{H.c.}\right),
\]

(10)

where we have included \( f(k_0) \equiv f(0)(k_0) \) by allowing the exponent \( \alpha \) to have zero value in the second term. Here \( A(k_0; n, T) \) is the value of the integral obtained from the first term of the stationary point expansion. This term is non-zero if \( f(k_0) \) is finite, but it does not contribute to the decay of the correlator since its an oscillatory function of \( n \). A scaling \( \delta k \rightarrow \delta k' = n^{1/\beta} \delta k \) and \( x \rightarrow x' = x/n^{1/\beta} \) in the integral in Eq. (10) leads to

\[
\delta C_x(nT) = A(k_0; n, T) + n^{-\alpha(\alpha+1)/\beta} g(k_0; x'),
\]

(11)

\[
g(k_0; z) = \int_{-\infty}^{\infty} \frac{dy}{2\pi} f^{(\alpha)}(k_0)g^\alpha e^{i(yz-\Delta^{(\beta)}(k_0)y^\beta T/\hbar)},
\]

(12)

Since \( g(k_0; z) \) is an oscillatory function of \( z \), it does not contribute to the decay of the correlator. Thus we find the general result that the leading decay of the correlator is given by

\[
C_x(nT) \sim n^{-\alpha(\alpha+1)/\beta}
\]

(13)

which is the main result of this section. For multiple stationary points, it is easy to see that the leading behavior is given by the one which allows for the slowest decay.

We note that for any stationary point expansion, generically, we expect \( \beta = 2 \) since the second derivative of the energy gap need not vanish at the stationary point. In this case, we find that the correlators would decay as \( \delta C_x(nT) \sim n^{-3/2} \) if \( f(k_0) \) vanishes at the point and \( f(k_0 + \delta k) \sim (\delta k)^2 \) as \( \delta k \rightarrow 0 \) and as \( \delta C_x(nT) \sim n^{-1/2} \) if \( f(k_0) \) is finite (this corresponds to \( \alpha = 0 \)). These two behaviors correspond to two dynamical phases; the former behavior is seen for high drive frequencies where the stationary point typically occurs at the edge of the Brillouin zone, while the latter occurs at lower frequencies where additional stationary points which correspond to \( \alpha = 0 \) appear inside the Brillouin zone. As noted in Ref. 46, these two phases are separated by a dynamical phase transition characterized by a critical drive frequency \( \omega_c \).

The decay of the correlators exactly at the transition allows for richer behavior which we explore next. We note that exactly at the transition point, the Floquet energy gap \( \Delta(k) \) must have a point of inflection which necessitates its second derivative to also vanish. Thus for this case \( \beta > 2 \). Depending on the symmetry of model, we find that either the third or the fourth derivative of the Floquet gap contributes to the lowest non-vanishing term in the expansion of \( \Delta(k) \) about \( k = k_0 \). The former behavior corresponds to \( \beta = 3 \) and occurs if the Floquet energy is odd under the transformation \( k \rightarrow -k \). This leads to

\[
C_x(nT) \sim n^{-\alpha(\alpha+1)/3}.
\]

(14)

In contrast, if the Floquet energy is even under \( k \rightarrow -k \), its fourth derivative contributes to the lowest non-vanishing term. This yields \( \beta = 4 \) and leads to

\[
C_x(nT) \sim n^{-\alpha(\alpha+1)/4}.
\]

Thus the decay of the correlators may follow a different power law at the critical point between two dynamical
phases. We note that the presence of two distinct dynamical phases across a transition is a sufficient condition for such behavior; however it is not a necessary condition and we shall discuss an example of such anomalous decay without the presence of distinct dynamical phases in the next section.

Finally, we discuss the crossover scale \( n_c \) which denotes the number of drive cycles after which the system crosses over to a decay characterized by \( \beta = 2 \). We note that \( n_c \) diverges at a dynamical transition and tends to zero far away from it. To estimate \( n_c \), we note that near a transition we can always write

\[
\delta C_2(nT) \sim A(k_0; n, T) + \int_{-\infty}^{\infty} \frac{d\delta k}{2\pi} e^{-i\delta k x} \times (f\alpha(k_0)\delta k^\alpha e^{-i(n(c_1\delta k^{a_0} + c_2\delta k^{\beta_0}) + \ldots)T/\hbar} + \text{H.c.}),
\]

where \( c_1 \) and \( c_2 \) are the coefficients of expansions of the Floquet spectrum around \( k = k_0 \), \( \beta_0 \) denotes the lowest integer larger than \( a_0 \) for which \( c_2 \neq 0, a_0 = 1 \) or 2 depending on the symmetry of the Floquet spectrum, and the ellipsis indicates higher order terms in the expansion of \( \Delta(k) \) around \( k = k_0 \) which we shall ignore. A simple scaling \( k \to \delta k' = n^{1/\beta_0} \delta k \) yields

\[
\delta C_x(nT) \sim A(k_0; n, T) + \int_{-\infty}^{\infty} \frac{d\delta k'}{2\pi} e^{-i\delta k'n^{1/\beta_0}x} [f\alpha(k_0)n^{-(\alpha+1)/\beta_0}(\delta k')^\alpha e^{-i(c_1n^{1-a_0/\beta_0}\delta k')a_0 + c_2(\delta k')^{\beta_0})T/\hbar} + \text{H.c.}].
\]

Thus the behavior of the integral is governed by the coefficient of \( (\delta k')^{a_0} \) in the exponent after

\[
n_c \simeq (c_2/c_1)^{\beta_0/(\beta_0-a_0)}
\]

drive cycles. Hence the crossover scale is also controlled by the symmetry of the model which renders \( a_0 = 1(2) \) and \( \beta_0 = 3(4) \) for models whose Floquet spectrum is odd (even) under \( k \to -k \) near the transition point. Furthermore, for a generic transition point for these integrable models \( c_1 \sim |\omega - \omega_c| \) and \( c_2 \) is a constant. Thus we find

\[
n_c \sim |\omega - \omega_c|^{-\beta_0/(\beta_0-a_0)}
\]

which shows that \( n_c \) diverges at the transition point where \( c_1 = 0 \) and it is small away from the transition where generically \( c_1 \ll c_2 \). We explore this crossover physics in detail in Secs. III and IV in the context of specific models.

### III. SSH MODEL

In this section, we will study the effect of periodic driving in the Su-Schrieffer-Heeger (SSH) model. We will show that the long-time behavior of the correlation function can show transitions between different power laws for some special choices of the driving parameters. We shall analyze the driven SSH model within first-order FPT; this is done so as to obtain simple analytical insights. The results obtained from FPT shall be compared with exact numerics towards the end of the section.

The SSH model is a tight-binding model of non-interacting electrons in 1D in which the nearest-neighbor hopping has different strengths on alternate bonds.\(^{50}\). We will ignore the spin of the electron since it will not play any role in this paper. In second-quantized notation, the Hamiltonian for a system with \( N \) sites (where \( N \) is even) and periodic boundary conditions is given by

\[
H = \sum_{n=1}^{N/2} [\gamma_1 a_n^\dagger b_n + \gamma_2 b_n^\dagger a_{n+1} + \text{H.c.}],
\]

where \( a_{N/2+1} \equiv a_1 \). (We will set both Planck’s constant \( \hbar \) and the spacing \( a \) between nearest-neighbor sites to 1). Transforming to momentum space, we find that

\[
H = \sum_k [\gamma_1 a_k^\dagger b_k + \gamma_2 b_k^\dagger a_k e^{i2k} + \text{H.c.}],
\]

where \( k \) takes \( N/2 \) equally spaced values lying in the range \([-\pi/2, \pi/2]\). This can be written in terms of a \( 2 \times 2 \) matrix \( H_k \) as

\[
H = \sum_k \begin{pmatrix} a_k^\dagger & b_k^\dagger \end{pmatrix} H_k \begin{pmatrix} a_k \\ b_k \end{pmatrix},
\]

\[
H_k = \begin{pmatrix} 0 & \gamma_1 + \gamma_2 e^{-i2k} \\ \gamma_1 + \gamma_2 e^{i2k} & 0 \end{pmatrix}.
\]

The energy-momentum dispersion is given by \( E_{k\pm} = \pm E_k \), where

\[
E_k = \sqrt{\gamma_1^2 + \gamma_2^2 + 2\gamma_1\gamma_2 \cos(2k)}.
\]
We see that the spectrum has a minimum gap equal to $E_{k^+} - E_{k^−} = 2|\gamma_1 \pm \gamma_2|$ at $k = 0$ and $\pm \pi / 2$ respectively, depending on whether $\gamma_1$ and $\gamma_2$ have opposite signs or the same sign.

We will now consider driving this system periodically in time by adding a term to the hopping which is of the form $a \sin(\omega t)$, where $a$ and $\omega$ are the driving amplitude and frequency respectively. The Hamiltonian in momentum space is therefore given by

$$H = \sum_k \left[ (\gamma_1 + a \sin(\omega t)) a_k^\dagger b_k + (\gamma_2 + a \sin(\omega t)) b_k^\dagger a_k e^{2ik} + \text{H.c.} \right].$$

This system can be analytically studied by several methods such as the Floquet-Magnus expansion which works in the limit $\omega$ is much larger than all the other parameters, $a$, $\gamma_1$ and $\gamma_2$, and FPT which is valid in the limit that both $a$ and $\omega$ are much larger than $\gamma_1$ and $\gamma_2$. We will use FPT which proceeds as follows.

For each value of $k$, we consider the Floquet operator

$$U_k = T \exp[-i \int_0^T dt H_k(t)].$$

where $T$ denotes time-ordering. Note that $U_k$ is an SU(2) matrix since $H_k(t)$ is a Hermitian and traceless matrix for all times $t$. We can write the Floquet operator as

$$U_k = e^{-iH_{Fk}T},$$

where $H_{Fk}$ is time-independent and is called the Floquet Hamiltonian. Assuming $a \gg \gamma_1$, $\gamma_2$, we write

$$H_k(t) = H_0(t) + V,$$

$$H_0(t) = \begin{pmatrix} 0 & a \sin(\omega t)(1 + e^{2ik}) \\ a \sin(\omega t)(1 + e^{2ik}) & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & \gamma_1 + \gamma_2 e^{-2ik} \\ \gamma_1 + \gamma_2 e^{2ik} & 0 \end{pmatrix}. $$

We will find the form of $H_{Fk}$ only to first order in the perturbation $V$.

The instantaneous eigenvalues of $H_0(t)$ are given by $E_{k^+} = 2a \sin(\omega t) \cos k$ and $E_{k^−} = -2a \sin(\omega t) \cos k$. These satisfy the condition

$$e^{i\int_0^T dt (E_{k^+} - E_{k^−})} = 1.$$  

We will therefore have to carry out degenerate FPT. The eigenfunctions corresponding to $E_{k^\pm}$ are given by

$$|+\rangle_k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{ik} \end{pmatrix},$$

$$|-\rangle_k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{ik} \end{pmatrix}. $$

We begin with the Schrödinger equation

$$i \frac{d|\psi\rangle}{dt} = (H_0 + V)|\psi\rangle.$$  

We assume that $|\psi(t)\rangle$ has the expansion

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-i \int_0^t dt' E_n(t') |n\rangle},$$

where $|n\rangle = |+\rangle$ and $|-\rangle$. Eq. (29) then implies that

$$\frac{dc_m}{dt} = -i \sum_n \langle m|V|n\rangle e^{i \int_0^t dt' (E_m(t') - E_n(t'))} c_n.$$  

Integrating this equation, and keeping terms only to first order in $V$, we find that

$$c_m(T) = c_m(0) - i \sum_n \int_0^T dt' \langle m|V|n\rangle e^{i \int_0^t dt' (E_m(t') - E_n(t'))} c_n(0).$$

This can be written as

$$c_m(T) = \sum_n \left( I - iH^{(1)}_{Fk}T \right)_{mn} c_n(0),$$

where $I$ denotes the identity matrix and $H^{(1)}_{Fk}$ is the Floquet Hamiltonian to first order in $V$. We then find that

$$\langle +|H^{(1)}_{Fk}|+\rangle = (\gamma_1 + \gamma_2) \cos k,$$

$$\langle -|H^{(1)}_{Fk}|-\rangle = - (\gamma_1 + \gamma_2) \cos k,$$

$$\langle +|H^{(1)}_{Fk}|-\rangle = -i (\gamma_1 - \gamma_2) \sin k J_0 \left( \frac{4a \cos k}{\omega} \right),$$

$$\langle -|H^{(1)}_{Fk}|+\rangle = i (\gamma_1 - \gamma_2) \sin k J_0 \left( \frac{4a \cos k}{\omega} \right).$$

$H^{(1)}_{Fk}$ then takes the following form in the $|+\rangle_k$, $|-\rangle_k$ basis

$$H^{(1)}_{Fk} = (\gamma_1 + \gamma_2) \cos k \sigma_z + (\gamma_1 - \gamma_2) \sin k J_0 \left( \frac{4a \cos k}{\omega} \right) \sigma_y.$$  

We now change basis to

$$|\uparrow\rangle_k = a_k^\dagger |0\rangle,$$

$$|\downarrow\rangle_k = b_k^\dagger |0\rangle,$$

so that

$$|+\rangle_k = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_k + e^{ik} |\downarrow\rangle_k \right),$$

$$|-\rangle_k = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_k - e^{ik} |\downarrow\rangle_k \right).$$
In the $|\uparrow\rangle_k, |\uparrow\rangle_k$ basis, we get

$$H_{Fk}^{(1)} = \left[ (\gamma_1 + \gamma_2) \cos^2 k + (\gamma_1 - \gamma_2) J_0 \left( \frac{4a}{\omega} \cos k \right) \sin^2 k \right] \sigma_x$$

$$+ \sin k \cos k \left[ (\gamma_1 + \gamma_2) - (\gamma_1 - \gamma_2) J_0 \left( \frac{4a}{\omega} \cos k \right) \right] \sigma_y. \quad (38)$$

Before proceeding further, we make two comments about the exact form of $H_{Fk}$ to all orders based on certain symmetries. First, $H_{Fk}$ must be an odd function of $\gamma_1, \gamma_2$. To see this, we note that Eq. (24) can be written as a product of $N_t$ factors in which $t$ increases from 0 to $T$ as we go from right to left in steps of $T/N_t$ (eventually we take the limit $N_t \to \infty$). We then use the fact that the driving term satisfies $\sin(\omega(T-t)) = -\sin(\omega t)$ to see that

$$[U_k(\gamma_1, \gamma_2)]^{-1} = U_k(-\gamma_1, -\gamma_2), \quad (39)$$

if we hold $a, \omega$ fixed. Eq. (25) then implies that

$$H_F(-\gamma_1, -\gamma_2) = -H_F(\gamma_1, \gamma_2). \quad (40)$$

Hence $H_F$ can only have odd powers of $\gamma_1, \gamma_2$. This implies that if $\gamma_1, \gamma_2 \ll a, \omega$, the first-order Floquet Hamiltonian will be a very good approximation to the exact Floquet Hamiltonian since the next correction is of third order in $\gamma_1, \gamma_2$. Second, let us consider the special case $\gamma_2 = -\gamma_1$ which will be considered in more detail below. We then find that after doing a unitary transformation,

$$H_k(t) \rightarrow V_k H_k(t) V_k^\dagger,$$

where

$$V_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{-ik} \end{pmatrix}. \quad (41)$$

we obtain

$$H_k(t) = 2a \sin(\omega t) \cos k \sigma_x - 2\gamma_1 \sin k \sigma_y. \quad (42)$$

We now use Eq. (42) to calculate the Floquet operator in and Floquet Hamiltonian in Eqs. (24) and (25). Then an argument similar to the one above shows that

$$[U_k]^{-1} = U_{-k}, \quad (43)$$

where we have held $\gamma_1, \gamma_2$ fixed and only changed $k \rightarrow -k$. Eq. (43) implies that

$$H_{F,-k} = -H_{F,k}. \quad (44)$$

This means that the eigenvalues of $H_F$ (quasienergies) must be odd functions of $k$ if $\gamma_2 = -\gamma_1$.

The eigenvalues of the first-order Floquet Hamiltonian in Eq. (38) are given by $\pm E_k$ (with the Floquet energy gap being $\Delta(k) = 2E(k)$) where

$$E_k = \sqrt{(\gamma_1 + \gamma_2)^2 \cos^2 k + (\gamma_1 - \gamma_2)^2 \sin^2 k \left[ J_0 \left( \frac{4a}{\omega} \cos k \right) \right]^2}. \quad (45)$$

For general values of $\gamma_1, \gamma_2$, we see that $E_k$ is non-zero for all values of $k$. However, for $\gamma_2 = \gamma_1$ it vanishes if $k = \pi/2$ (in fact, $E_k$ does not depend on the driving if $\gamma_2 = \gamma_1$), while for $\gamma_2 = -\gamma_1$ it vanishes when either $k = 0$ or $J_0((4a/\omega) \cos k) = 0$. Thus driving can lead to non-trivial zeros of the Floquet energy for special values of $(a/\omega) \cos k$. In the rest of this section we will therefore consider the case $\gamma_2 = \gamma_1$. Setting $\gamma_1 = 1$, we have

$$E_k = 2 \sin k J_0 \left( \frac{4a}{\omega} \cos k \right). \quad (46)$$

Fig. 1 shows plots of $E_k$ and $dE_k/dk$ for a system with $\gamma_1 = 1, \gamma_2 = -1, a = 6$ and $\omega = 4a/\mu_1$, where $\mu_1 \simeq 2.4048$ is the first zero of $J_0(z)$.

We now consider an operator of the form $a_j^\dagger b_j$ where $j$ denotes a particular unit cell. Starting from an initial state $\Psi(0)$, we will look at the correlation function at stroboscopic instances of time $t = nT$,

$$C_n = \langle \Psi(nT) | a_j^\dagger b_j | \Psi(nT) \rangle, \quad (47)$$

and we will study how this behaves for large values of $n$. We take the initial state to be a half-filled state given by a product in momentum space

$$\Psi(0) = \prod_k \langle a_k^\dagger + e^{i\phi} b_k^\dagger \rangle / \sqrt{2} | \text{vac} \rangle. \quad (48)$$
For simplicity we have taken the phase \( \phi \) to be independent of \( k \).

\[
C_n = \frac{2}{N} \sum_k \langle \Psi(nT)|a_k^\dagger b_k|\Psi(nT)\rangle \\
= \frac{2}{N} \sum_k \langle \Psi(0)|(U_k^\dagger)^n a_k^\dagger b_k(U_k)^n|\Psi(0)\rangle.
\]

(49)

Using Eqs. (48) and (54), we obtain

\[
C_n = A + \frac{2}{N} \sum_k f(k) \cos(2nTE_k),
\]

\[
A = \frac{1}{2N} \sum_k \left( e^{i\phi} - e^{i(2k-\phi)} \right),
\]

\[
f(k) = \frac{1}{4} \left( e^{i\phi} + e^{i(2k-\phi)} \right).
\]

(55)

For \( N \to \infty \), these quantities have the integral forms

\[
C_n = A + \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} dk \left( e^{i\phi} + e^{-i\phi} \cos(2k) \right) \times \cos(2nTE_k),
\]

\[
A = \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} dk \left( e^{i\phi} - e^{i(2k-\phi)} \right) = \frac{e^{i\phi}}{4},
\]

(56)

where we have used the relation \( \cos(2nTE_k) = \cos(2nTE_{-k}) \) (since \( E_{-k} = -E_k \)) to write the first equation in Eq. (56).

We will now study the form of the \( n \)-dependent part of \( C_n \), called \( \delta C_n \), for large \( n \). The dominant contributions will come from regions around the values of \( k \) where \( E_k \) has an extremum, namely, \( dE_k/dk = 0 \). One such point is \( k = \pi/2 \). Expanding around it to second order, we find that \( E_k = 2 - (1 + 8a^2/\omega^2)(k - \pi/2)^2 \), where we have used the expansion \( J_0(z) = 1 - z^2/4 \) for small \( z \).

We first assume that \( f(k = \pi/2) \neq 0 \); this will be true if \( \phi \) is not an integer multiple of \( \pi \). Near \( k' = k - \pi/2 \), the \( n \)-dependent term in Eq. (56) then takes the form

\[
\delta C_n \simeq \frac{i}{2\pi} \int dk' \sin \phi \times \text{Re} \exp[i4nT - i2nT(1 + 8a^2/\omega^2)k'^2],
\]

(57)
where \( \text{Re} \) denotes real part. We thus see that \( C_n \) will oscillate as \( \cos(4nT) \) (which implies that its absolute value will vary periodically with \( n \) with a period \( \Delta n = \pi/(4T) = \omega/8 \) multiplied by an integral of the form \( \int dk' \exp[\imath \omega k'^2] \) which, by a scaling argument, will decay as \( 1/n^{1/2} \) for large \( n \). However, in the special case that \( \phi \) is an integer multiple of \( \pi \), both \( e^{i\phi} + e^{-i\phi} \cos(2k) \) and its first derivative vanish at \( k = \pi/2 \). We then get a factor of \( k^2 \) appearing in the integrand of Eq. (57). The integral will therefore be of the form \( \int dk' k^2 \exp[\imath \omega k'^2] \) which will decay as \( 1/n^{3/2} \) for large \( n \). Below we will see plots showing a \( 1/n^{1/2} \) decay (for \( \phi = \pi/4 \)) and a \( 1/n^{3/2} \) decay (for \( \phi = 0 \)).

Next, we consider if there are any other values of \( k \) where \( dE_k/dk = 0 \). We find that such points exist if \( \omega < \omega_1 \), where \( \omega_1 = 4a/\mu_1 \) with \( \mu_1 \approx 2.4048 \) being the first zero of \( J_0(z) \). This is because, as \( k \) goes from 0 to \( \pi/2 \), \( \sin k \) goes from 0 to 1, taking only positive values in between, while \( J_0((4a/\omega) \cos k) \) goes between \( J_0(4a/\omega) \) and 1, crossing zero \( p \) times in between if \( 4a/\omega \) is larger than the first \( p \) zeros of \( J_0(z) \). This implies that \( E_k \) in Eq. (46) will go between 0 and 2, crossing zero \( p \) times in between; hence \( E_k \) will have \( p \) extrema where \( dE_k/dk = 0 \). Next, if \( k = k_0 \) is one of the points where \( dE_k/dk = 0 \), and \( k_0 \) is not equal to either 0 or \( \pi/2 \), the factor \( f(k) \) in Eq. (47) is not zero, but the argument of \( \cos(2E_k nT + \alpha n(k - k_0)^2) \) goes from 0 to 1, taking only positive \( \alpha \) values in between a period \( \Delta n = \omega/(4E_k) \) multiplied by an integral of the form \( \int dk' \exp[\imath \omega k'^2] \). By a scaling argument, this will again decay as \( 1/n^{1/2} \) for large \( n \).

We thus conclude that \( \delta C_n \) will generally oscillate and decay as \( 1/n^{1/2} \). This is what we see in Figs. 2 (a) and (b) for a system with \( \gamma_1 = 1, \gamma_2 = -1, a = 6, \omega = 4a/|\mu_1 + \epsilon| \), where \( \epsilon = 0.1 \), and \( \phi = \pi/4 \) for the initial state (see Eq. (48)). If \( 4a/\omega \) is larger than \( p \) zeros of \( J_0(z) \) (where \( p \) can be 1, 2, 3, 4, \ldots), there will be \( p \) terms in \( \delta C_n \), all of which decay as \( 1/n^{1/2} \) but which oscillate with \( p \) different periods \( \Delta n \).

Interestingly, a different scaling of \( \delta C_n \) versus \( n \) arises if \( \omega \) is exactly equal to \( \omega_p = 4a/\mu_p \) with \( \mu_p \) being the \( p \)-th zero of \( J_0(z) \). (We will call the \( \omega_p \)'s critical frequencies, \( \omega_1 \) being the largest such frequency). Then both \( E_k \) and its first two derivatives vanish at \( k = 0 \) as we will now show. For definiteness, we consider the neighborhood of \( \omega_1 \), namely, we take \( \omega = 4a/|\mu_1 + \epsilon| \), where \( |\epsilon| \ll 1 \). We now expand Eq. (46) around \( k = 0 \) up to order \( k^3 \). Using the property \( dJ_0(z)/dz = -J_1(z) \) and \( J_1(\mu_1) \equiv \nu_1 \simeq 0.519 \), we find that

\[
E_k \simeq \nu_1 (-2\epsilon k + \mu_1 k^3). \tag{58}
\]

Eq. (47) then gives, in the region around \( k = 0 \),

\[
\delta C_n \simeq \frac{\cos \phi}{2\pi} \int dk \text{ Re} \exp[i2nT \nu_1 (-2\epsilon k + \mu_1 k^3)]. \tag{59}
\]

Defining a scaled variable \( k' = k n^{1/3} \), we get

\[
\delta C_n \simeq \frac{\cos \phi}{2\pi n^{1/3}} \int dk' \times \text{Re} \exp[i2T \nu_1 (-2\epsilon n^{2/3} k' + \mu_1 k'^3)]. \tag{60}
\]

We see from Eq. (60) that if \( \epsilon = 0 \), i.e., \( \omega = \omega_1 \) exactly, \( \delta C_n \) will go as \( 1/n^{1/3} \) times a factor which does not oscillate with \( n \) at large \( n \). This can be seen in Fig. 2 (c). If \( n \) is not very large we see some oscillations which arise due to the stationary point at \( k = \pi/2 \). Further, if \( \epsilon \) is non-zero but small, then we will still get the \( 1/n^{1/3} \) scaling if \( |\epsilon| n^{2/3} \ll 1 \) since the term of order \( k' \) will dominate over the term of order \( k^3 \). But if \( |\epsilon| n^{2/3} \gg 1 \), the \( k' \) term will dominate over the \( k^3 \) term, and we do not expect to get the \( 1/n^{1/3} \) scaling anymore. We will then...
get the other scaling, namely, an oscillating function of $n$ times $1/n^{1/2}$. Hence a crossover will occur between a non-oscillating function of $n$ times $1/n^{1/3}$ and an oscillating function of $n$ times $1/n^{1/2}$ at a crossover $n_c$ which scales with $\epsilon$ as $1/|\epsilon|^{3/2}$. This is shown in Fig. 3. Since $|\epsilon| \approx |\omega - \omega_1|$, we see that $n_c \sim 1/|\omega - \omega_1|^{3/2}$. This corresponds to $\beta_0 = 3$ and $\alpha_0 = 1$ (Eq. (18)).

We now consider the oscillations which appear in $\delta C_n$ when $n$ is larger than the crossover scale and $\omega$ not equal to a critical frequency. These are shown in Figs. 2 (a) and (b) for $\omega$ close to the value $4a/\mu_1$. For $\epsilon < 0$, $|\delta C_n|$ goes as an oscillating function of $n$ times $1/n^{1/2}$ due to the integral over the region around $k = \pi/2$; the oscillation period is $\Delta n = \omega/8$ which is independent of $\epsilon$ and too small to be visible in Fig. 2 (a). But for $\epsilon > 0$, we see in Fig. 2 (b) that the oscillations in $\delta C_n$ have quite a large period, about $\Delta n = 215$. We will now derive this. For $\epsilon > 0$, we see from Eq. (58) that $dE_k/dk = 0$ at $k = \pm k_0$, where

$$ k_0 = \sqrt{\frac{2\epsilon}{3\mu_1}}. \quad (61) $$

Expanding around the stationary point at $k_0$, we find that the argument of the exponential in Eq. (59) is given by

$$ -i2nT \nu_1 \frac{4}{3} \sqrt{\frac{2}{3\mu_1}} \epsilon^{3/2} + \text{a term of order } (k - k_0)^2. \quad (62) $$

The Gaussian integral involving the term of order $(k - k_0)^2$ will give a scaling like $1/n^{1/2}$ while the first term in Eq. (62) implies that $|\delta C_n|$ will oscillate with $n$ with period

$$ \Delta n = \frac{\pi}{(4/3)2T \nu_1 \sqrt{\frac{2}{3\mu_1}} \epsilon^{3/2}} = \frac{a}{(4/3)\mu_1 \nu_1 \sqrt{\frac{2}{3\mu_1}} \epsilon^{3/2}}, \quad (63) $$

where we have used $T = 2\pi/\omega = \pi\mu_1/(2a)$ to derive the second line. In Fig. 4 we show a plot of $\Delta n$ versus $\epsilon$, for $\gamma_1 = 1$, $\gamma_2 = -1$, $a = 6$, $\omega = 4a/\mu_1 + \epsilon$ for $\epsilon > 0$ (so that $\omega < 4a/\mu_1$), and $\phi = \pi/4$ for the initial state. The best fit is given by $\Delta n = 6.81/\epsilon^{3/2}$ which agrees well with the value of $6.85/\epsilon^{3/2}$ that we find from Eq. (63).

Finally, we compare our results for first-order FPT with that found from exact numerics. The latter is shown in Fig. 5 for $\gamma_1 = -\gamma_2 = 1$ and $a = 6$. We note that these values of $a$, $\gamma_1$, and $\gamma_2$, the first-order FPT yields a critical drive frequency to be $\omega_c \simeq 9.9799$ whereas the exact numerics leads to $\omega_c \simeq 9.9794$; this reflects the accuracy of FPT for these parameters. (This occurs since the expansion parameter for FPT is $\gamma_1/a = 1/6$ and only odd powers of this parameter appear. So the third-order term is about 36 times smaller than the first-order
term). The top left panel of Fig. 5 displays the Floquet energy and its derivative as a function of $k$ showing that the exact Floquet energies closely resemble the first-order theory. The top right (bottom left) panel indicates that the crossover from $n^{-1/3}$ to $n^{-1/2}$ behavior at $\omega = 4a/(\mu_1 + 0.1)$ is present in the exact theory and is almost identical to that obtained within first-order FPT. Finally, the bottom right panel shows that the $n^{-1/3}$ decay of the correlation function at $\omega = \omega_c = 9.7994$ is reproduced within exact numerics. The reason for this near exact match can be traced to large value of $a$ which shifts the transition to high frequency where first-order FPT naturally produces accurate results.

We end this section by noting that it is not necessary for a dynamical phase transition to have different power laws for $\omega < \omega_c$ and $\omega > \omega_c$. We have seen above that the power law $(1/n^{1/2})$ is the same on the two sides of $\omega_c$ for a general initial state, but there is a different power law $(1/n^{1/3})$ exactly at $\omega_c$. However, for a special choice of initial state ($\phi = 0$), the power law is different on the two sides, being $1/n^{1/2}$ for $\omega < \omega_c$ and $1/n^{3/2}$ for $\omega > \omega_c$.

**IV. ISING MODEL**

For the one-dimensional $S = 1/2$ Ising model with $L$ spins and periodic boundary conditions, the Hamiltonian reads as

$$H = -\frac{1}{2} \sum_{j=1}^{L} (g\tau_j^x + \tau_j^z\tau_{j+1}^z), \quad (64)$$

where $\tau_j^x, \tau_j^z$ denote the Pauli matrices for the physical spins on site $j$, we have set the Ising nearest-neighbor interaction to $J = 1/2$, and $g = h/J$ is the dimensionless magnetic field. Carrying out a Jordan-Wigner transformation from spins to spinless fermions with

$$\tau_i^x = 1 - 2c_i^\dagger c_i,$$

$$\tau_i^z = -\prod_{j<i} (1 - 2c_j^\dagger c_j) (c_i^\dagger + c_i), \quad (65)$$

FIG. 5: Top left panel: Plot of the exact numerical Floquet energy (black solid line) and its derivative (blue dot-dashed line) as a function of $k$, for $\gamma_1 = 1$, $\gamma_2 = -1$ and $a = 6$. Top right panel: Log-log plot of $\delta C_n$ computed from exact numerics (for $\phi = 0$ for the initial state) showing $n^{-1/3}$ to $n^{-1/2}$ crossover at $\omega = 4a/\mu_1 + 0.1$. Bottom left panel: Same as top right panel but for $\omega = 4a/(\mu_1 - 0.1)$ showing crossover from $n^{-1/3}$ to $n^{-3/2}$ behavior. Bottom right panel: Same as top right panel but for $\omega_c = 4a/\mu_1$ showing $n^{-1/3}$ decay.
where \( c_i^\dagger (c_i) \) creates (destroys) a spinless fermion on site \( i \) allows one to rewrite \( H \) in Eq. (64) as

\[
H = g \sum_{j=1}^{L} c_j^\dagger c_j - \sum_{j=1}^{L-1} (c_j^\dagger c_{j+1} + c_j^\dagger c_{j+1} + \text{H.c.})/2 \\
+ (-1)^{N_F} (c_L^\dagger c_1 + c_1^\dagger c_L + \text{H.c.})/2,
\]

where \( N_F \) denotes the number of fermions. For the rest, we restrict to even \( N_F \) which implies that \( c_{L+1} = -c_1 \). Further using

\[
c_k = \frac{\exp(i\pi/4)}{\sqrt{L}} \sum_j \exp(-ikj)c_j,
\]

where \( k = 2\pi m/L \) with \( m = -(L - 1)/2, \ldots, -1/2, 1/2, \ldots, (L - 1)/2 \), Eq. (66) can be written as \( H = \sum_{k>0} H_k \) where

\[
H_k = (g - \cos k)[c_k c_k^\dagger - c_{-k} c_{-k}^\dagger] \\
+ \sin k[c_{-k} c_k + c_k^\dagger c_{-k}^\dagger].
\]

This can be recast in the form of Eq. (2) by noting that since the fermions can be created or destroyed only in pairs, one can introduce “pseudospins” \( | \uparrow \rangle_k = c_k^\dagger c_{-k}^\dagger |0 \rangle \) and \( | \downarrow \rangle_k = |0 \rangle \) where \( |0 \rangle \) represents the fermion vacuum which gives

\[
h_z(k, t) = g(t) - \cos k, \\
h_x(k, t) = \sin k, \\
h_y(k, t) = 0.
\]

We concentrate on a square pulse protocol with \( g(t) = g_i \) for \( 0 \leq t < T/2 \) and \( g(t) = g_f \) for \( T/2 \leq t < T \). Further, without any loss of generality, we choose the initial state to be \( |0 \rangle \) for all \( k \) which represents the fermion vacuum or \( \tau_F = +1 \) in terms of the physical spins to study relaxation of local quantities to their final steady state values as a function of \( n \), the number of drive cycles. For the choice of initial state and for \( L \to \infty \), \( \delta C_{ij}(n) = \langle c_i^\dagger c_j \rangle_n - \langle c_i^\dagger c_j \rangle_{\infty} \) and \( \delta F_{ij}(n) = \langle c_i^\dagger c_j \rangle_n - \langle c_j^\dagger c_i \rangle_{\infty} \) equal \cite{46}

\[
\delta C_{ij}(n) = \int_0^\pi \frac{dk}{2\pi} f_1(k) \cos(2n\phi(k))
\]

\[
\delta F_{ij}(n) = \int_0^\pi \frac{dk}{2\pi} [f_2(k) \cos(2n\phi(k)) + f_3(k) \sin(2n\phi(k))],
\]

with

\[
f_1(k) = -(1 - \tilde{n}_f^2(k)) \cos(k(i-j)), \\
f_2(k) = -i\tilde{n}_f(k)f_3(k), \\
f_3(k) = i(n_x(k) + in_y(k)) \sin(k(i-j)).
\]

In Eq. (72), we used the fact that the Floquet unitary at each \( k \) mode can be written as a \( 2 \times 2 \) matrix of the form \( U_k = \exp[-i\phi(k)\vec{\sigma} \cdot \vec{n}(k)] \) where \( \vec{n}(k) = (n_x(k), n_y(k), n_z(k)) \) represents a unit vector and \( \phi(k) \in [0, \pi] \) in the reduced zone scheme. The Floquet Hamiltonian can be expressed as

\[
H_F(k) = \vec{\sigma} \cdot \epsilon(k) = \Delta(k)\vec{\sigma} \cdot \vec{n}(k)/2
\]

where \( \epsilon(k) = (\epsilon_x(k), \epsilon_y(k), \epsilon_z(k)), \Delta(k) = 2|\epsilon(k)| \) is the Floquet energy gap, and \( \vec{n}(k) = \epsilon(k)/|\epsilon(k)| \). This fixes \( \phi(k) = T\Delta(k)/2 \) where each component of \( \epsilon(k) \) is restricted to \([-\pi/T, \pi/T]\) in the reduced zone scheme. The expression of \( \epsilon(k) \) has been computed in Ref. 46. For the square pulse protocol which we focus on in this work, we find

\[
\Delta(k) = 2 \arccos(M_k/T), \\
M_k = \cos \Phi_i(k) \cos \Phi_f(k) \\
- \tilde{N}_i(k) \cdot \tilde{N}_f(k) \sin \Phi_i(k) \sin \Phi_f(k), \\
\Phi_i(f)(k) = (T/2)\sqrt{(g_i(f) - \cos k)^2 + \sin^2 k}, \\
N_i(f)(k) = (\sin k, 0, (g_i(f) - \cos k)) T/(2\Phi_i(f)(k)).
\]

The square pulse protocol allows for analytic expressions for \( U_k \). From Eq. (71), the stationary points \( \partial \Delta(k)/dk = 0 \) in \([0, \pi]\) determine the behavior of the relaxation of local quantities. As shown in Ref. 46, the number of stationary points in \( k \in (0, \pi) \) is 0 for \( \omega = 2\pi/T \to \infty \) while it scales as \( 1/\omega \) as \( \omega \to 0 \). Importantly, \( f_{1,2,3}(k) \) in Eq. (72) vanish at \( k = 0 \) and \( k = \pi \) for any \((g_i, g_f, T)\) while these are generally non-zero when \( k \neq 0, \pi \). Lastly, keeping \( g_i, g_f, T \) fixed, a series expansion of \( \Delta(k) \) around \( k = 0 \) and \( k = \pi \) respectively yields only even powers.

For the rest, we focus on \( \delta C_{ii}(n) \) which also equals \((1 - \langle \tau_F^2 \rangle)/2 \) from Eq. (65) (since the initial state and the drive protocol are both translationally invariant, the dependence on the site index \( i \) can be dropped) with the other local fermionic correlators also showing similar decays in time. Let us quickly recapitulate the relaxation behavior in the two dynamical phases that are distinguished by whether the stationary points occur only at \( k = 0, \pi \) versus the appearance of extra stationary points in \( k \in (0, \pi) \). We denote the number of stationary points in \( k \in (0, \pi) \) by \( N_b \). First, \( \alpha = 2 \) (\( \alpha = 0 \)) for stationary points with \( k = 0 \) or \( \pi \) \((k \neq 0, \pi)\) from the behavior of \( f_1(k) \). Second, \( \beta = 2 \) in both cases. This immediately gives a relaxation of \( n^{-3/2} \) \((n^{-1/2}) \) when \( N_b = 0 \) \((N_b \neq 0)\) from Eq. (1).

We now focus on the relaxation behavior exactly at the dynamical critical points. As discussed in Ref. 46, these come in two varieties – critical points where \( N_b \) changes by 1 (e.g., from \( N_b = 0 \) to \( N_b = 1 \)) and critical points where \( N_b \) changes by two (e.g., from \( N_b = 2 \) to \( N_b = 0 \)). The former class arises due to an extra stationary point entering from either \( k = 0 \) or \( \pi \) and the latter class arises due to two stationary points in \( k \in (0, \pi) \) coalescing to one at the critical point (Fig. 6 (top left)) as the drive frequency is tuned keeping \( g_i, g_f \) fixed. The first dynamical transition as \( \omega \) is reduced from very large values always belongs to the first category, while some
other dynamical transitions may belong to the second category as \( \omega \) as lowered further. For the first category, while \( \alpha = 2 \) since the extra stationary point emerges from either \( k = 0 \) or \( \pi \), \( \beta = 4 \) since although the critical point requires that \( d^2 \Delta(k)/dk^2 \) at \( k = 0 \) or \( \pi \), \( d^3 \Delta(k)/dk^3 = 0 \) at these two momenta. This implies a critical relaxation of \( n^{-3/4} \) from Eq. (1). For the second category, the single stationary point in \( k \in (0, \pi) \) also becomes a minimum of \( d\Delta(k)/dk \) (Fig. 6 (top left)). Thus, \( \alpha = 0 \) and \( \beta = 3 \) for this point since \( k \neq 0, \pi \) and \( f_1(k) \neq 0 \) generically for \( k \in (0, \pi) \). This gives a critical relaxation of \( n^{-1/3} \) from Eq. (1) for these critical points.

We now show results for \( g_i = 2 \) and \( g_f = 0 \) where both types of dynamical critical points can be accessed by tuning the drive frequency \( \omega \) by using a system size of \( L = 8 \times 10^5 \) to minimize finite-size effects. For \( \omega \approx 3.63821 \), we encounter the first dynamical critical transition where \( N_b \) changes from 0 to 1 across the transition while for \( \omega \approx 1.49853 \), we encounter a dynamical transition where \( N_b \) changes from 2 to 0. Fig. 6 (top right) shows the relaxation to be \( n^{-3/4} \) for the former case and Fig. 6 (bottom left) shows the relaxation to be \( n^{-1/3} \) for the latter case, completely in accord with our theoretical expectation. Furthermore, we expect a diverging dynamical crossover timescale \( n_c \) in the vicinity of the critical points in both the dynamical phases where the relaxation of local quantities scale as \( n^{-3/4} (n^{-1/3}) \) for \( n \ll n_c \) before crossing over to \( n^{-3/2} \) or \( n^{-1/2} \) for
n \gg n_c$. We extract $n_c$ from our numerical data and show its behavior in the vicinity of the first dynamical phase transition in Fig. 6 (bottom right). Expectedly, $n_c$ shows a divergence as the critical point is approached from both sides. The crossover scale $n_c$ is determined by fitting the early (late) time data for $\delta C_{ii}(n)$ to $n^{-3/4}$ ($n^{-3/2}$ or $n^{-1/2}$) and extracting the crossing point of the fitted lines in a log-log plot (see Fig. 7 (left and right panels)).

We now discuss how $n_c$ diverges near the first dynamical phase transition as $\omega$ approaches $\omega_c$ from above. Referring to Eq. (17), we see that here $\beta_0 = 4$ and $a_0 = 2$ since the extra stationary point enters from $k = \pi$ for the square pulse protocol\[46\] where only even powers contribute. Thus, $n_c \sim (c_2/c_1)^2$ and the divergence occurs since $c_1 = 0$ exactly at the critical point. Furthermore, $c_1$ changes sign as $\omega$ is changed from above to below the critical frequency which implies that $c_1 \sim (\omega - \omega_c)^{-2}$ as one approaches the dynamical critical point from above.

In contrast, for approaching the point from below, we need to take into account the fact that there are two stationary points (at $k = \pi$ and $\pi - k_0$ where $k_0 \sim \sqrt{\omega_c - \omega}$) which approach each other as one nears the critical point. Numerically, $c_1$ is small and $|c_1|$ is of the same order for both the stationary points. The characteristic around this stationary point controls $n_c$ and numerically we find that the same scaling (as the one when the critical point is approached from above) holds in this case. This is shown in the right panel of Fig. 7. A plot of the correlation function for two representative values of $\omega < \omega_c$ is shown in the left panel of Fig. 8. The plot reveals a long-time oscillation of the correlation function, similar to that identified for the SSH model in the previous section, with $\Delta n = 1400 (260)$ for $\omega = 3.6361 (3.63505)$. The time period $\Delta n$ of these oscillations diverges as $\omega$ approaches $\omega_c$ in accordance with that found for the SSH model in Sec. III. An analysis along the same line as in the SSH model predicts $1/k_0^4$ divergence, where $k_0 \sim \sqrt{|\omega - \omega_c|}$ is the distance between the extrema (at $k = \pi$ and $k = \pi - k_0$) in the Floquet Brillouin zone. This fits the data for large $k_0$; however it breaks down when $k_0$ is small where a much faster divergence is encountered; this is probably due to the proximity of the two symmetry-unrelated stationary points in the Brillouin zone as well as the small value of $d\Delta(k)/dk$ near them. These features probably invalidate an analysis based on the premise that the contribution to the correlation function come only from the two stationary points.

V. DISCUSSION

In this work, we have studied the dynamical relaxation of correlation functions to their steady state values in driven 1D integrable quantum models as a function of the number of drive cycles $n$. We summarize the generic behavior of such relaxation by identifying a general power
FIG. 8: Left panel: The behavior of $\delta C_{ii}(n)$ for two representative values of $\omega < \omega_c \approx 3.638$ showing long-time coherent oscillations corresponding to $\Delta n \approx 1400$ ($\omega = 3.6361$) and $\Delta n \approx 260$ ($\omega = 3.63505$). For both the plots $g_i = 2$ and $g_f = 0$ and $\omega_c$ corresponds to the first transition frequency. Right panel: A plot of the oscillation period $\Delta n$ to the distance $k_0$ between the two extrema of $H_F$ (at $k = \pi$ and $k = \pi - k_0$) in the Floquet Brillouin zone showing $1/k_0^4$ (the dashed blue line corresponds to $0.0135/k_0^4$) behavior at larger $k_0$.

law in terms of two positive integers $\alpha$ and $\beta$. The exponents corresponding to $\beta = 2$ and different $\alpha$ characterize different dynamical phases; this was identified in Ref. 46. Here, we find the presence of other possible exponents characterized by $\beta = 3$ and $\beta = 4$. These anomalous exponents typically occur at the dynamical transition between two dynamical phases; however, they may also occur at special points within a dynamical phase.

We provide a general analysis of the behavior of such correlation functions in terms of the Floquet spectrum of the driven model and show that their occurrence is tied to points of inflections in the Floquet spectrum. At these points, for a Floquet spectrum which is odd under $k \to -k$, the correlation functions decay with $\beta = 3$; for an even spectrum, we find a decay with $\beta = 4$.

This analysis also points to the absence of such anomalous powers ($\beta \neq 2$) for dynamical transitions in higher dimensional integrable models. The presence of the anomalous exponent requires the existence of a point of inflection in the Floquet spectrum; for $d > 1$, this requires vanishing of multiple derivatives $\partial^2 \Delta / \partial k_i \partial k_j$ at such a point. Since the transition can be reached by tuning a single parameter, namely the drive frequency, multiple derivatives cannot generically vanish at the transition. Thus we expect such anomalous exponents to be realized only for 1D models.

We have studied two concrete models to show the existence of such anomalous decay. The first one involves the SSH model driven by a continuous protocol; this model realizes decay of correlations with $\beta = 3$ leading to a $n^{-1/3}$ behavior. We analyze the driven SSH model within first-order FPT to gain analytical insight into the problem; the results of the first-order FPT agrees almost identically with the exact numerical study. We also study the correlation functions of the 1D transverse field Ising model. The model shows a reentrant transition between two dynamical phases at several drive frequencies. We show that the correlation function decays with $\beta = 4$ at the first (highest frequency) transition leading to a $n^{-3/4}$ behavior. In contrast, the subsequent transitions at lower drive frequency exhibit $n^{-1/3}$ decay and correspond to $\beta = 3$.

Near these transitions which host relaxation with anomalous power laws, we find a crossover scale, $n_c$, after which the correlators decay to their steady state values with exponents corresponding to $\beta = 2$. Such crossover scales can be identified at both sides of the transition. It was found that $n_c \sim (\omega - \omega_c)^{-\beta_0/(\beta_0 - \alpha_0)}$; thus it exhibits a power law divergence at the transition. This behavior has been confirmed from exact numerics for both the Ising and the SSH model. The former model exhibits $\beta_0 = 4$ and $\alpha_0 = 2$ leading to $n_c \sim (\omega - \omega_c)^{-2}$ at the first dynamical transition, while the second model corresponds to $\beta_0 = 3$ and $\alpha_0 = 1$ leading to $n_c \sim |\omega - \omega_c|^{-3/2}$.

Finally, our analysis shows a long-time oscillatory behavior of the correlation functions near the transition at $\omega_c$. Such a behavior is seen when the transition is approached from below $\omega_c$ and is seen in both models. Our FPT analysis for the SSH model shows that such an oscillation results from the presence of two stationary points (at $k = \pm k_0$) and provides an analytical estimate of the time period of such oscillations. This estimate shows a near-exact match with results from exact numerics. However, for the Ising model, a similar analysis fails to capture the time period when the two stationary points are close to each other (small $k_0$); this failure could be due to proximity of symmetry unrelated stationary points and the flat nature of $\Delta(k)$ around $k = \pi$ near the transition. This leads to near-zero values of $d\Delta(k)/dk$ for several values of $k$ between the two stationary points (at $k = \pi$ and
π – $k_0$); as a result, the correlators receive contribution from all these momenta. This may invalidate an analysis based on contributions from only the two stationary points; we leave a further study of this issue for future work.

In conclusion, we have studied the dynamical relaxation of correlation function of driven 1D quantum integrable models. We have identified anomalous power laws characterizing the decay of these correlators to their steady state value as a function of the number of drive cycles and a diverging crossover timescale as the dynamical transition is approached from both sides. Our analysis also reveals a long-time oscillatory behavior of these correlation functions near a dynamical transition when the transition is approached from the low-frequency side.

Note added: While this manuscript was in preparation we came to know about a similar work unraveling anomalous power laws by Makki, Bandopadhyay, Maity and Dutta (unpublished). Our results agree wherever a comparison is possible.

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