Rank One symmetric Spaces and Rigidity

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Abstract

In this paper we show that if we have two nonelementary, irreducible representations from some group $G$ (not necessarily finitely generated) into $\text{Iso}^+(H^n_F)$ and $\text{Iso}^+(H^n_K)$ such that $\dim_R(F) \times n = \dim_R(K) \times m$ where $F, K = R, C, H, O$ and have the same marked length spectrum, then $F = k, n = m$ and two representations are conjugate. In the last section we show that if we have a representation from a finitely generated group into $\text{SL}_2(C)$, sitting smoothly in the space of representations then we can find smooth coordinate chart by translation lengths of finite number of hyperbolic elements around this representation.

1 Introduction

Let $F$ to be $R, C, H$. The inner product on $F^{n+1}$ is given as:

$$< (z_1, \cdots, z_{n+1}), (w_1, \cdots, w_{n+1}) > = \sum_{i=1}^{n} z_i \overline{w_i} - z_{n+1} \overline{w_{n+1}}$$

Let $GL(n + 1, F)$ act on $F^{n+1}$ from the right and let $P^n_F$ be the space of left $F$-lines. Then $G = O_F(n, 1)$ is a subgroup of $GL(n + 1, F)$ preserving this inner product. The induced action of $O_F(n, 1)$ on $P^n_F$ gives three invariant subsets $D_+, D_0, D_-$ and the action is positive definite on $D_+$. A component of $D_+$ is called hyperbolic space. An isotropy group of $O_F(n, 1)$ is a maximal compact subgroup $K = O_F(1) \times O_F(n)$. If we numerate them;

$$O_R(n, 1) = O(n, 1), K = O(1) \times O(n)$$
\[ O_C(n, 1) = U(n, 1), K = U(1) \times U(n) \]

\[ O_H(n, 1) = Sp(n, 1), K = Sp(1) \times Sp(n) \]

The hyperbolic Cayley plane is similarly realized and the isometry group is the real form of \( F_4 \) of real rank one. The maximal compact subgroup is \( Spin(9) \).

We can compactify the hyperbolic space by adding \( D_0 \). If \( \dim_R F = k \) then \( D_0 \) is a \( (kn - 1) \) dimensional sphere denoted by \( \partial H^n_F \).

Take the Iwasawa decomposition \( G = KAN \). Then \( A \) and \( N \) fix \( \xi \) in the sphere at infinity and \( N \) is a nilpotent group obtained as an extension

\[ 0 \to ImF \to N \to F^{n-1} \to 0. \]

Horospheres based at \( \xi \) are the orbits of \( N \). The stabilizer in \( O_F(n, 1) \) of \( \xi \) is a parabolic subgroup with Langlands decomposition \( MAN \). For \( F = R, C, H \) \( M \) is a subgroup of \( K \) of the form \( O_F(1) \times O_F(n - 1) \). For Cayley plane \( M \) is \( Spin(7) \). \( K \) actually acts transitively on \( \partial H^n_F \) giving \( K/M = \partial H^n_F \). The centralizer of \( M \) in \( K \) acts freely on \( K/M \) with quotient \( P^n_{F-1} \). This gives Hopf fibration \( \partial H^n_F \to P^n_{F-1} \).

Since \( K \) acts transitively on \( \partial H^n_F \), killing form on its Lie algebra determines a Riemannian metric on \( \partial H^n_F \). Define the orthogonal complements to the vertical tangent spaces of the Hopf fibration \( \partial H^n_F \to P^n_{F-1} \). Then there is \( K \) invariant Riemannian metric on the horizontal bundle and if we define the distance of two points as the minimum of the lengths of the paths joining these two points and staying horizontal bundle it is Carnot-Caratheodary metric. Mitchell [Mi] calculated that the Hausdorff dimension of \( \partial H^n_F \) in this metric is \( k(n + 1) - 2 \).
2 Boundary of Rank one symmetric Spaces

The boundary of rank one symmetric space $H^m$ is a one point compactification of nilpotent group $N$ in Iwasawa decomposition, denoted by $N \cup \infty$. We will introduce nice coordinates and left invariant distance to introduce cross ratio of four points. We will define everything in Cayley numbers because that is the most general case among four.

Cayley number is a pair of quaternions $(q_1, q_2)$. The multiplication is defined as:

$$(q_1, q_2)(p_1, p_2) = (q_1p_1 - \bar{p_2}q_2, q_2p_1 + p_2\bar{q_1})$$

Also we define $(\bar{q_1}, q_2) = (\bar{q_1}, -q_2)$.

Then it has the following properties.

1) $q\bar{q} = |q|^2$
2) $|qp| = |q| |p|
3) $q^{-1} = \bar{q} / |q|^2$
4) $\bar{pq} = \bar{q}p$

Even though Cayley numbers are not commutative nor associative, using Artin’s theorem saying that the subalgebra generated by two elements is associative, we have following extra properties.

1) $(xy)y^{-1} = x$
2) $(xy)^{-1} = y^{-1}x^{-1}$

Here is some useful fact. Aut$(O) = G_2$ and it is compact group. It fixes real number and acts transitively on unit pure imaginaries. The stabilizer of $i$ is a copy of SU$(3)$ and it acts transitively on the unit pure imaginaries orthogonal to $i$. The stabilizer of $i, j$ which fixes $k$ since $k = ij$, acts transitively on the unit pure imaginaries orthogonal to $i, j, k$ and it is a copy of SU$(2)$. For more extensive informations, see [Fr].
Now an element of $N$ is denoted as $[(t, q), k]$ where

1) $F = R(\text{Real}), (t, q) = 0, k \in R^{m-1}$.
2) $F = C(\text{Complex}), t$ is real, $q = 0, k \in C^{m-1}$.
3) $F = H(\text{Quaternion}), t \in R^3, q = 0, k \in H^{m-1}$.
4) $F = O(\text{Cayley}), t \in R^3, q$ is a quaternion, $k$ is a Cayley number.

The multiplication is defined as:

$$[(t, q), k][(t', q'), k'] = [(t + t', q + q') + 2Im < k, k'>, k + k']$$

We define the gauge of $[(t, q), k]$ as;

$$A([(t, q), k]) = (|k|^2 + t, q)$$

Then define $||[(t, q), k]|| = (|k|^4 + |t|^2 + |q|^2)^{1/4}$ and $d(g, g') = |g'^{-1}g|$. This is a left invariant distance and the houseorff dimension of the boundary with respect to this metric is $\dim_R(F) \times (m + 1) - 2$ which agrees with Mitchell’s calculation. Even though this metric is not riemannian if we take the inner metric of this it becomes Carnot-Caratheodory metric. For references see [Gr][Pa][Mi].

The action of $Iso(H^m_F)$ extends continuously to $\partial(H^m_F)$. Let $Sim(N)$ denote the subgroup of $Iso(H^m_F)$ which fixes $\infty$. Then it is isomorphic to $N \times (O_F(m - 1) \cdot O_F(1) \times R)$ and $N \times O_F(m - 1) \cdot O_F(1)$ acts as isometries with respect to the metric given above. Note that $N$ part comes from left action of the group itself and $O_F(m - 1) \cdot O_F(1) \times R$ part comes from hyperbolic isometries of the form in $O_F(m, 1)$

$$\begin{bmatrix}
M & 0 & 0 \\
0 & \nu \cosh(s) & \nu \sinh(s) \\
0 & \nu \sinh(s) & \nu \cosh(s)
\end{bmatrix}$$
where \( O_F(m-1) \cdot O_F(1) = O(m-1), U(m-1), Sp(m-1) \cdot Sp(1), O_O(1) \cdot O_O(1) \)
depending on \( F = R, C, H, O \) respectively. These actions will be described in
next section.

We define the cross ratio of four points as;

\[
[g_1, g_2, g_3, g_4] = \frac{|A(g_3^{-1}g_1)| |A(g_4^{-1}g_2)|}{|A(g_4^{-1}g_1)| |A(g_3^{-1}g_2)|}
\]

Let’s diverge to the unit ball model for a while so that we can get some
connection between the metric of hyperbolic space and the metric defined
above. In Real, complex, Quaternion case for \( x, y \in H^m_F \)

\[
cosh(d(x, y)) = \frac{|1-<x, y>|}{(1-<x, x>)^{1/2}(1-<y, y>)^{1/2}}
\]

For Cayley hyperbolic case if we define

\[
R < v, w >= Re(v_1 \bar{v}_2)(\bar{w}_1w_2) - Re(\bar{v}_2 w_2)(\bar{w}_1v_1)
\]

Then for \( x, y \in H^2_O \) the distance is

\[
cosh(d(x, y)) = \frac{(|1-<x, y>|^2 + 2R <x, y>)^{1/2}}{(1-<x, x>)^{1/2}(1-<y, y>)^{1/2}}
\]

For details see[Mo]. Let \( << x, y >> = 1-<x, y> \) for \( F = R, C, H \) and \( << x, y >> = (|1-<x, y>|^2 + 2R <x, y>)^{1/2} \) for Cayley case. Then the
cross ratio of the four points in the boundary of hyperbolic space is defined
as

\[
[x, y, z, w] = \frac{<< z, x >><< w, y >>}{<< w, x >><< z, y >>}
\]

Note that this is a limit of

\[
\frac{\cosh(d(z_i, x_i)) \cosh(d(w_i, y_i))}{\cosh(d(w_i, x_i)) \cosh(d(z_i, y_i))}
\]
where $x_i, y_i, z_i, w_i$ tend to $x, y, z, w$. So it is invariant under isometries of hyperbolic space.

To make connection between these two definitions of cross ratio we introduce generalize projection from $N \cup \infty$ to the boundary of unit ball.

$$w_1 = 2(1 + |k|^2 - t_i - q_i)^{-1} k$$

$$w_2 = (1 + |k|^2 - t_i - q_i)^{-1}(1 - |k|^2 + t_i, q_i)$$

Note that $(0, 0)$ corresponds to $(0, 1), \infty$ to $(0, -1)$.

Let $g_i = [(t_i, q_i), (c_i, d_i)]$ for $i = 1, 2$. By the general projection these two points correspond to

$$x_i = 1/((1 + |k_i|^2)^2 + |t_i|^2 + |q_i|^2)[2(c_i + |k_i|^2)c_i + t_i c_i - \bar{d}_i q_i, q_i c_i + d_i +$$

$$d_i |k_i|^2 - d_i t_i), (1 - |k_i|^4 + 2t_i - |t_i|^2 - |q_i|^2, 2q_i)]$$

Since $(0, 0)$ and $\infty$ correspond to $(0, 1)$ and $(0, -1)$

$$<< x_i, (0, -1) >> = 2/((1 + |k_i|^2)^2 + |t_i|^2 + |q_i|^2)$$

$$<< x_i, (0, 1) >> = 2\sqrt{((|k_i|^4 + |k_i|^2 + |t_i|^2 + |q_i|^2)^2 + |t_i|^2 + |q_i|^2)}$$

$$((1 + |k_i|^4 + |t_i|^2 + |q_i|^2)^2 + |t_i|^2 + |q_i|^2)$$

Since $((|k_i|^4 + |k_i|^2 + |t_i|^2 + |q_i|^2)^2 + |t_i|^2 + |q_i|^2) = (|k_i|^4 + |t_i|^2 + |q_i|^2)((1 + |k_i|^4 + |k_i|^2 + |t_i|^2 + |q_i|^2)$ we get

$$[\infty, 0, x_1, x_2] = \frac{(k_2^4 + |t_2|^2 + |q_2|^2)^{1/2}}{(k_1^4 + |t_1|^2 + |q_1|^2)^{1/2}}$$

$$= |A(g_2^{-1})|/|A(g_1^{-1})|$$

so we are done.

For the case $[0, g_1, \infty, g_2]$, just note that $[0, g_1, \infty, g_2] = [g_2^{-1}, g_2^{-1} g_1, \infty, 0]$ By using above fact we are done again.
In fact for $F = R, C, H$ it is easy to show that two definitions of cross ratio agree by brutal calculation but in Cayley Hyperbolic case calculation is quite hard. But to prove our main theorem we will just need what we proved.

3 Action of Isometries on $\partial H^m_F$

In this section we want to incode the action of hyperbolic isometry on the boundary. A hyperbolic isometry in $O_F(m, 1)$ has the form of:

$$\begin{bmatrix} M & 0 & 0 \\ 0 & \nu \cosh(s) & \nu \sinh(s) \\ 0 & \nu \sinh(s) & \nu \cosh(s) \end{bmatrix}$$

where $\nu = 1, M \in O_F(m-1)$ for $F = R, C$ and $|\nu| = 1, M \in Sp(m-1)$ for $F = H$ and $|M| = |\nu| = 1$ for Cayley hyperbolic case.

Then this element send $[(t, q), k]$ to:

$$(t', q') = e^{-2s} \nu^{-1}(t, q)$$

$$k' = e^{-s}[[\nu^{-1}(e^{2s} + |k|^2 - t, -q)^{-1}]]$$

$$\{[\nu^{-1}(e^{2s} + |k|^2 - t, -q)^{-1}(1 + |k|^2 - t, -q)]((1 + |k|^2 - t, -q)^{-1} k)A]\}$$

First note that the matrix acts on the unit ball as:

$$w'_1 = (w_2 \nu \sinh(s) + \nu \cosh(s))^{-1}(w_1 A)$$

$$w'_2 = (w_2 \nu \sinh(s) + \nu \cosh(s))^{-1}(w_2 \nu \cosh(s) + \nu \sinh(s))$$

using coordinate change between unit ball model and $F^{m+1}$ sending $w$ to $(w, 1)$. To check above claim we use Artin’s theorem crucially. By using generalized projection and above equations we can show

$$w'_2 = \{(1 + |k|^2 - t, -q)^{-1}(1 - |k|^2 + t, q)\nu \sinh(s) + \nu \cosh(s)\}^{-1}$$

$$\{[(1 + |k|^2 - t, -q)^{-1}(1 - |k|^2 + t, q)\nu \cosh(s) + \nu \sinh(s)]$$

$$7$$
\[ \nu^{-1}((e^{2s} + |k|^2 - t, -q)^{-1}(1 + |k|^2 - t, -q))([(1 + |k|^2 - t, -q)^{-1}(e^{2s} - |k|^2 + t, q)] \]
equal to

\[ [(\nu^{-1}(e^{2s} + |k|^2 - (t, q))^{-1})\nu][\nu^{-1}(e^{2s} - |k|^2 + (t, q)\nu] \]

To prove this, set \( r_1 = e^{2s} + |k|^2, r_2 = e^{2s} - |k|^2, r_3 = 1 + |k|^2, (t, q) = Q \) and using identity \(-QQ = |Q|^2\) we can show

\[ [\nu^{-1}((r_1 + Q)^{-1}(r_3 - Q))][(r_3 + Q)(r_2 + Q)]\nu = [(\nu^{-1}(r_1 - Q)^{-1})\nu][\nu^{-1}(r_2 + Q)\nu]. \]

For \( w'_1 \), it is easy to check.

### 4 Marked length spectrum determines representation

First we will make very simple but important observation to prove the theorem.

**Lemma 1** Let \( a, b \) be two hyperbolic isometries such that \( x_1 \) is a repelling fixed point of \( a \), \( x_3 \) is an attracting fixed point of \( a \) and \( x_4 \) is an attracting fixed point of \( b \), \( x_2 \) is a repelling fixed point of \( b \). Then

\[ \lim_{n \to \infty} e^{l(a^n) + l(b^n) - l(a^nb^n)} = |[x_1, x_2, x_3, x_4]|. \]

**Proof:** Choose \( x_1^n \) on the axis of \( a \), \( z_1^n \) on the axis of \( b^n \), \( x_2^n \) on the axis of \( b^na^n \), \( x_2^n \) on the axis of \( a^nb^n \) so that \( d(x_1^n, z_1^n) \) and \( d(x_2^n, w_2^n) \) go to zero. If we put \( x_3^n = a^n(x_1^n), z_3^n = a^n(z_1^n), x_4^n = b^n(x_2^n), w_3^n = b^n(w_2^n) \) and \( d_{13}^n = d(x_1^n, x_3^n), d_{24}^n = d(x_2^n, x_4^n), d_{23}^n = d(w_2^n, z_3^n), d_{14}^n = d(z_1^n, w_4^n) \) then

\[ \lim \sqrt{<x_1^n, x_3^n> <x_2^n, x_4^n> <x_1^n, x_2^n> <x_2^n, x_4^n> <x_4^n, x_1^n> <x_2^n, x_3^n> <x_3^n, x_4^n> <x_1^n, x_4^n>} \]

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Theorem 1

Let $\rho : G \to Iso^{\pm}(H^n_K), \phi : G \to Iso^{\pm}(H^n_F)$ be two nonelementary, irreducible representations having the same marked length spectrum such that $n \times \dim_R(K) = m \times \dim_R(F)$ where $K, F$ is real, complex, quaternion or Cayley fields. Then $n = m, K = F$ and they are conjugate.

Proof: After we conjugate representations we may assume that 0 and $\infty$ are in the limit sets and they are the two fixed points of hyperbolic isometries $a$ and $a'$ in $\rho$ and $\phi$ respectively. Furthermore we have

$$\frac{|x_i|}{|x_j|} = ||[x_i, x_j, [0, 0], \infty]||^{1/2} = ||[y_i, y_j, [0, 0], \infty]||^{1/2} = \frac{|y_i|}{|y_j|}$$
So after scaling we can assume that $|x_i| = |y_i|$. Similarly

$$\frac{d(x_i, x_j)}{|x_j|} = \sqrt{[[0, 0], x_i, \infty, x_j]} = \sqrt{[[0, 0], y_i, \infty, y_j]} \implies \frac{d(y_i, y_j)}{|y_j|}$$

and hence $d(x_i, x_j) = d(y_i, y_j)$.

Let $[0, 1], [0, z]$ are the two points in the limit set of $\rho$ and $[c, d], [(t, q), k]$ be the corresponding points in the limit set of $\phi$.

Then $\alpha$ sends $[(t, q), k]$ to

$$(t', q') = e^{-2s}\nu^{-1}(t, q)\nu$$

$$k' = e^{-s}\{[\nu^{-1}(e^{2s} + |k|^2 - t, -q)^{-1}]^{-1}$$

$$\{[\nu^{-1}((e^{2s} + |k|^2 - t, -q)^{-1}(1 + |k|^2 - t, -q))][((1 + |k|^2 - t, -q)^{-1}k)M]\}$$

where $\nu = 1, M \in O(m - 1), U(m - 1)$ in real and complex case, $|\nu| = 1, M \in Sp(m - 1), |M| = 1$ in quaternion and cayley cases.

Similarly $\alpha'$ sends $[(t, q), k]$ to

$$(t', q') = e^{-2s}\nu^{-1}(t, q)\nu$$

$$k' = e^{-s}\{[\mu^{-1}(e^{2s} + |k|^2 - t, -q)^{-1}]^{-1}$$

$$\{[\mu^{-1}((e^{2s} + |k|^2 - t, -q)^{-1}(1 + |k|^2 - t, -q))][((1 + |k|^2 - t, -q)^{-1}k)N]\}$$

Now

$$d([0, z], [0, 1])^4 = d([(t, q), k], [c, d])^4$$

gives us that $|(t, q)|$ is a function of $z, k, c, d$. But

$$d([0, z], a^n([0, 1]))^4 = d([(t, q), k], a^n([c, d]))^4$$
gives that \(|(t,q)|\) is a function of \(e^{-ns}, z, k, c, d\) for all \(n\). This is because right side of equation has the higher power of \(e^{-s}\). This is not possible unless \(|(t,q)| = 0\).

By this way we can conclude that all the limit sets of \(\phi\) corresponding to \([0, z]\) of \(\rho\) are the form of \([0, k]\). Then by further conjugation we can assume that \([0,1]\) is also the corresponding point of \(\phi\) to \([0,1]\) of \(\rho\).

Next we want to show that either \(M = \bar{N}, \nu = \mu\) or \(M = \bar{N}, \nu = \bar{\mu}\). This can be seen from the equation
\[
d(a^n([0,1]), [0,1])^4 = d(a^n([0,1]), [0,1])^4
\]
for all \(n\). Now by conjugating the representation by the map \((k \to \bar{k})\) we can assume that \(M = N, \nu = \mu\).

If \([0, w]\) is a point in the limit set of \(\rho\) and \([0, w']\) is the corresponding point of \(\phi\) then
\[
d(a^n([0, w]), [0,1])^4 = d(a^n([0, w']), [0,1])^4
\]
gives that \(w = w'\).

These all show that \(K = F, m = n\). So far we conjugated two representations to get that two limit sets are equal on the set \(K^{m-1} = \{[0,k] : k \in R^{m-1}, C^{m-1}, Q^{m-1}, O\}\) and two spaces are actually the same one of the four hyperbolic spaces.

For any point \(x\) of the limit set of \(\rho\) not in the set \(K^{m-1}\) there are only two possible corresponding points \(y\) (either \(x\) itself or reflection of \(x\) along \(K^{m-1}\)) such that the distance between \(x\) and every point in the limit set of \(\rho\) lying on the set \(K^{m-1}\) is equal to the distance between \(y\) and corresponding point of \(\phi\). By taking reflection along this set if necessary, we can assume that \(x = y\). Now for any point in the limit set of \(\rho\) off the set \(K^{m-1} \cup x\)
there is a unique corresponding point as above. So two limit sets are actually equal. This shows that we found an isometry such that after we conjugate one representation by this isometry the limit sets are equal. But this actually shows that two representations are equal. The reason is the following. Note that the end points of hyperbolic isometry $aba^{-1}$ are the images of end points of $b$ under $a$. But since every isometry is determined by images of finitely many points on $\partial H^2_F$ and since two representations have the same limit sets after conjugation each element should be the same.

**Corollary** Let $M, N$ be rank one locally symmetric manifolds of the same dimension and homotopically equivalent. If none of them have totally geodesically embedded submanifold of dimension greater than one and if they have the same marked length spectrum then they are isometric.

Since three manifolds get special attention we will write down the corollary for hyperbolic three manifolds. Here we do not need the assumption about irreducibility.

**Corollary** If we have two nonelementary orientable hyperbolic three manifolds having the same marked length spectrum then they are isometric.

**Proof:** Since every orientation preserving Mobius map is determined by the images of three distinct points whether they are on the line or not we don’t have to worry about irreducibility.

5 Space of Representations into $SL_2(C)$

5.1 Backgroud

Let $G$ be a finitely generated group. Space of representations from $G$ to $SL_2(C)$ is a set of homomorphisms from $G$ into $SL_2(C)$. We say that two
representations are equivalent if they are conjugated by some element of $SL_2(C)$. The character of a representation $\rho : G \to C$ is the function $\chi_\rho : G \to C$ such that $\chi_\rho(g) = tr(\rho(g))$. If $G = \{g_i, i = 1, \cdots, n; r_1 = \cdots = r_k = 1\}$ then space of representations $R(G)$ is a subset of $SL_2(C) \subset C^{4n}$, which are set of all points $(\rho(g_1), \cdots, \rho(g_n))$ satisfying the relator relations. It is easy to see that $R(G)$ is an affine algebraic set in $C^{4n}$. See [CS] for details. If we have a finite volume complete hyperbolic 3-manifold then its holonomy representation is discrete, faithful and irreducible.

For each $g \in G$ we define a regular function $\tau_g$ on $R(G)$ by $\tau_g(\rho) = tr\rho(g)$. If $T$ is a ring generated by all functions $\tau_g, g \in G$ then it is finitely generated. See [CS] for a proof.

Also it is shown in [CS] that character space of an irreducible component of $R(G)$ is an affine variety.

Thurston ([Th] chapter 5) showed that geometric structure of compact manifold $M$ are determined by holonomy representations of $\pi_1(M)$ near some geometric structure and its complex dimension is at least $3 \times (\#(generators) - \#(relators) - 1)$.

**Theorem 2** ([Th],[CS]) Let $N$ be a compact orientable 3-manifold. Let $\rho : \pi_1(N) \to SL_2(C)$ be an irreducible representation such that for each torus component $T$ of $\partial N, \rho(im(\pi_1(T) \to \pi_1(N)))$ not trivial. Let $R$ be an irreducible component of $R(\pi_1(N))$ containing $\rho$. Then $X_0 = character(R)$ has dimension $\geq s - 3\chi(N)$, where $s$ is the number of tori component of $\partial N$.

5.2 Local smooth coordinate chart of Representation space by lengths of finite number of loops

Let $G$ be a group and $\mathcal{R}$ be the representation space of $G$ into $SL_2(C)$, where $\mathcal{R}$ is the set of equivalence classes of homomorphisms from $G$ into $SL_2(C)$.
and two representations are equivalent if they are conjugate.

Suppose $\rho$ is a representation sitting smoothly in $\mathbb{R}$. We want to find nice coordinate chart around $\rho$ such that each coordinate is a translation length of some element in $G$.

**Lemma 2** Let $\alpha$ be an isometry in $SL_2(C)$. Then

$$ |\text{tr}(\alpha) - 2| + |\text{tr}(\alpha) + 2| = 2(e^{\frac{i\pi}{2}} + e^{-\frac{i\pi}{2}}) $$

**Proof:** If $\alpha$ is

$$ \alpha = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} $$

Put $x = |\lambda + \lambda^{-1} - 2|, y = |\lambda + \lambda^{-1} + 2|$. Then $x^2 + y^2 = 2|\lambda|^2 + 2\lambda\lambda^{-1} + 2\lambda^{-1}\lambda + 2|\lambda^{-1}|^2 + 8$ and

$$ x^2 + y^2 + xy = 4(|\lambda|^2 + 2 + |\lambda^{-1}|^2). $$

By using $|\lambda| = e^{\frac{i\pi}{2}}$ claim follows. ■

Now let $\hat{G}$ be a subset of $G$ consisting of all elements whose image under $\rho$ is either hyperbolic or elliptic. Notice that if $a, b \in \hat{G}$ then $a^n, b^n, a^n b^n$ are all in $\hat{G}$ for large $n$. Also if $a \in G$ and $b \in \hat{G}$ then $aba^{-1}$ is in $\hat{G}$. This implies that marked length spectrum on $G'$ determines representation up to conjugacy. Choose finite set $S$ of $G$ so that traces of those elements determines representation up to conjugation and trace map from the neighborhood to $C^*S$ where $C^* = C - \{2, -2\}$ is an immersion. This is possible if the representation is not elementary. See the argument after the example below for justification.
Lemma 3 (Vogt) Let $G$ be a free group with three generators. Let $trX_i = x_i, trX_iX_j = y_{ij}, trX_iX_jX_k = z_{ijk}$. Define

$$P = x_1y_{23} + x_2y_{13} + x_3y_{12} - x_1x_2x_3$$

$$Q = x_1^2 + x_2^2 + x_3^2 + y_{12}^2 + y_{13}^2 + y_{23}^2 + y_{12}y_{13}y_{23} - x_1x_2y_{12} - x_1x_3y_{13} - x_2x_3y_{23} - 4$$

$$\Delta(X_1, X_2, X_3) = P^2 - 4Q$$

Then $z_{123}$ and $z_{213}$ are roots of the quadratic equation for $z$:

$$z^2 - Pz + Q = 0$$

Given prescribed values $x_i, y_{ij}$ there exists only one conjugacy class if and only if $\Delta(X_1, X_2, X_3) = 0$.

See [Ma 1] for the proof.

Example. Let $G$ be a free group with three generators. I want to show that trace map is not an immersion sometimes. Let

$$X = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

and

$$Y = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$$

Let $Z$ be a third element which does not commute with any of two elements and $\Delta(X, Y, Z) \neq 0$. Since $G$ is free neighborhood of this representation is a regular neighborhood of $SL_2(C) \times SL_2(C) \times SL_2(C)$ and obviously sitting smoothly in $\mathbb{R}$. Let $tr$ be a map from this neighborhood to $C^7$ such that

$$tr(\rho) = (trX, trY, trZ, trXY, trXZ, trYZ, trXYZ)$$
Then elementary calculation shows that
\[ \text{dtr}_{(X,Y,Z)}(\xi_1, \xi_2, \xi_3) = (\text{tr}(X\xi_1), \text{tr}(Y\xi_2), \text{tr}(Z\xi_3), \text{tr}(X\xi_2) \]
\[ + \text{tr}(Y\xi_1), \text{tr}(X\xi_3) + \text{tr}(Z\xi_1), \text{tr}(Y\xi_3) + \text{tr}(Z\xi_2), \ldots) \]
where \( \xi_i \in \mathfrak{sl}_2(C) \). Then dimension of kernel at \((X,Y,Z)\) is 4. To see this note that there are two degrees of freedom from \( \text{tr}(X\xi_1) \) and so on, so there are six degrees of freedom from \( \xi_1, \xi_2, \xi_3 \).

\( \text{tr}(X\xi_1), \text{tr}(Y\xi_2) \) equal to zero implies \( \text{tr}(X\xi_2) + \text{tr}(Y\xi_1) \) is zero since \( X \) and \( Y \) commute. But it should satisfy \( \text{tr}(X\xi_3) + \text{tr}(Z\xi_1) = \text{tr}(X\xi_3) + \text{tr}(Z\xi_1) = \text{tr}(Y\xi_3) + \text{tr}(Z\xi_2) = 0 \) then the last term vanishes automatically. Here is the reseason. If \( X_i(t) \) are curves passing through \( X, Y, Z \) respectively and with tangent vectors \( \xi_1, \xi_2, \xi_3 \) set

\[ \text{tr}(X_i(t)) = x_i(t), \text{tr}X_i(t)X_j(t) = y_{ij}(t), \text{tr}X_1(t)X_2(t)X_3(t) = z(t). \]

By lemma 3 \( z(t)^2 - P(t)z(t) + Q(t) = 0 \). If we differentiate this equation at 0 we get \( z'(0)(2z(0) - P(0) = 0 \) by using \( P'(0) = Q'(0) = 0 \). But \( 2z(0) - P(0) \neq 0 \) since \( \Delta \) is not equal to zero. So \( \text{tr} \) is not an embedding at this point.

This example has an obvious geometric meaning. For \( \text{tr}(X\xi_1) \) to be zero, diagonal elements of \( \xi_1 \) should be equal to zero. This implies that we deform \( X \) by conjugation. The same is true for \( Y \). Since \( X, Y \) have the same axis, conjugating \( X \) by those elements conjugating \( Y \) does not change trace of \( X \) so infinitesimal change of \( \text{tr}(X) \) is equal to zero. Same argument holds for \( Y \).

More explicitly let
\[ g_t = \begin{pmatrix} x_1(t) & x_2(t) \\ x_3(t) & x_4(t) \end{pmatrix} \]
and
\[ h_t = \begin{pmatrix} y_1(t) & y_2(t) \\ y_3(t) & y_4(t) \end{pmatrix} \]
such that \( g_0 = h_0 = I \) and \( x'_2(0) = -y'_2(0) = x'_3(0) = -y'_3(0) \). Let
\[ Z = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \]
for simplicity. Set \( X_t = g_t X g_t^{-1}, Y_t = h_t Y h_t^{-1}, Z_t = g_t Z g_t^{-1} \). Then \( \xi(t) = (X_t, Y_t, Z_t) \) is a 1-parameter family of deformation which is not a conjugation deformation. For
\[ X'(0) = \begin{pmatrix} 0 & -x'_2 \alpha + x'_2 \alpha^{-1} \\ x'_3 \alpha - x'_3 \alpha^{-1} & 0 \end{pmatrix} \]
and similarly for \( Y'(0) \) there is no conjugation deformation giving tangent vectors \( X'(0) \) and \( Y'(0) \). It is easy to see that \( dtr_X Y Z(\xi'(0)) = 0 \). For if we calculate \( \frac{dtr(Y_t Z_t)}{dt} \) at \( t = 0 \), it is equal to
\[ \beta(x'_2(0)b - x'_3(0)b) + b(-y'_2(0)\beta + x'_2(0)\beta^{-1}) + b(y'_3(0)\beta - y'_3(0)\beta^{-1}) + \beta^{-1}(-x'_2(0)b + x'_3(0)b). \]
If we look at above example carefully, it shows if no two elements commute then trace map is an immersion, which implies that every tangent vector in the kernel of \( dtr \) comes from some conjugation deformation.

Observe the following calculation.
\[ g_t = \begin{pmatrix} x_1(t) & x_2(t) \\ x_3(t) & x_4(t) \end{pmatrix} \]
\[ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
Then conjugation of \( X \) by \( g_t \) gives tangent vector at \( X \)
\[ \begin{pmatrix} x'_2 c - x'_3 b \\ x'_3 a + 2x'_4 c - x'_3 d \\ -x'_2 a + 2x'_4 b + x'_4 d \\ -x'_2 c + x'_4 b \end{pmatrix} \]
Notice that two entries are linearly independent. So if \( g_t, h_t \) give the same infinitesimal deformation on two different element which do not commute then they give the same infinitesimal deformation on every element.

By using this fact we can show that if the representation is not elementary then we can show that trace map from a neighborhood of a representation \( \rho \) sitting smoothly in representation space to \( C^N \) for large enough \( N \) is a smooth immersion.

To see this if \( S = \{X_1, \ldots, X_n\} \) is the finite set containing all generators of the group such that its traces determine representation uniquely up to conjugacy (which is possible by [CS]) and their images under \( \rho \) are all hyperbolic isometries. By abusing the notation we will identify \( X_i \) with \( \rho(X_i) \). This can be seen that if \( S \) is such a set possibly containing parabolic or elliptic elements then choose a hyperbolic element and multiply those elements which are not hyperbolic in \( S \) by this element and its inverse to make them hyperbolic and then throw in this element. Then using the formula 
\[
tr(XY) + tr(XY^{-1}) = tr(X)tr(Y)
\]
we see that all the traces of old elements can be obtained by new elements. This is a new set consisting entirely of hyperbolic elements.

Since representation is not elementary there is \( X_k \) which does not commute with \( X_1 \). Choose any \( X_i, i \neq 1, k \). If it commutes with \( X_1 \) replace \( X_1, X_i, X_k \) by \( X_1 X_i, X_1 X_k, X_k \). It is easy to check that no two of three does commute. By renaming them we get \( (X_1, \ldots, X_n) \) such that no two of \( X_1, X_2, X_3 \) commute and \( X_i, i \neq 1, 2, 3 \) commutes with at most one of three \( X_1, X_2, X_3 \).

If \( (\xi_1, \ldots, \xi_n) \) is in the kernel of the differential of the trace map, there is \( g_t \) conjugating \( X_1, X_2, X_3 \), giving tangent vectors \( \xi_1, \xi_2, \xi_3 \) because no two of three commute. Similary \( h_t \) conjugating \( X_1, X_2, X_4 \) giving \( \xi_1, \xi_2, \xi_4 \) since no
two of three commute. But $h_i$ can be replaced by $g_i$ since they give the same infinitesimal deformation on $X_1, X_2$ which do not commute. In this way we can see that this tangent vector comes from conjugation deformation.

By lemma 2 the map $f$ from $C^*S$ to $R\hat{G}$ sending each component $\alpha$ of an element in $C^*S$ to $2(e^{i\alpha} + e^{-i\alpha})$ is $C^\infty$ since there is no parabolic element in $\hat{G}$ and norm function is smooth at $C^*$. Furthermore by adding more elements in $S$ we can make this function an immersion. To see this consider the map $g$ from $R^2$ to $R$ such that $g(x, y) = |(x, y) - 2| + |(x, y) + 2|$ as in the above lemma. It is easy to see that if $(x, y) \neq 2, -2$ then dimension of the kernel of $dg$ is one. Suppose $dg(\xi) = 0$ at $\alpha$. Then find an element $\beta$ so that $\text{trace}(\beta)$ is a polynomial in $\alpha$ and other elements in $S$. Choosing $\beta$ carefully it is possible to ensure that $dg(\xi) \neq 0$ at $\beta$. So adding $\beta$ to $S$ for each such an $\alpha$ we can show the map is an immersion.

**Proposition 5.1** Let $\rho$ be an nonelementary representation sitting smoothly inside the representation space. Then neighborhood of $\rho$, denoted by $N(\rho)$, is parametrized by translation lengths of finitely many hyperbolic elements in $G$.

**Proof:** So far we got

$$f \circ tr : N(\rho) \to C^*S \to R\hat{G}$$

is an immersion. But since marked length spectrum on $R\hat{G}$ determines representation it is one to one also.

Since dimension of representation space is finite around $\rho$ we can find finite subspace $R^N$ such that projection of the image of the neighborhood to this subspace is injective. ■

Specially if the representation is an holonomy of geometrically finite 3-manifold then it is automatically sitting smoothly in representation space
because Teichmuller space is open in representation space and it satisfies all the necessary condition for above considerations, so small neighborhood of this representation is parametrized smoothly by lengths of finitely many geodesics.

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