CONTINUITY OF SPECTRAL RADIUS OVER HYPERBOLIC SYSTEMS

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Abstract. The continuity of joint and generalized spectral radius is proved for Hölder continuous cocycles over hyperbolic systems. We also prove the periodic approximation of Lyapunov exponents for non-invertible non-uniformly hyperbolic systems, and establish the Berger-Wang formula for general dynamical systems.

1. Introduction. Let $A$ be a compact subset of the space $M_d(\mathbb{R})$ of $d \times d$ real matrices. The joint spectral radius of $A$ was defined by Rota and Strang [23] as

$$\hat{\rho}(A) = \lim_{n \to \infty} \sup \{ \|A_n \cdots A_1\|^{1/n} : A_i \in A \}. \quad (1.1)$$

The joint spectral radius has applications in many areas including coding theory [18] and the theory of control and stability [2, 8]. Among research on the joint spectral radius, many important properties are revealed: it was proved by Wirth [27] that the function $A \mapsto \hat{\rho}(A)$ is continuous on the space of compact sets of $M_d(\mathbb{R})$, and is locally Lipschitz continuous on the space of irreducible compact sets of $M_d(\mathbb{R})$, where the explicit Lipschitz constant was given by Kozyakin [16]; the joint spectral radius can be also related to the generalized spectral radius defined by

$$\bar{\rho}(A) = \lim_{n \to \infty} \sup \sup \{ \rho(A_n \cdots A_1)^{1/n} : A_i \in A \},$$

where $\rho(A) = \lim_{n \to \infty} \|A^n\|^{1/n}$ denotes the ordinary spectral radius of a matrix $A$.

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In this paper, we study the joint spectral radius in more general content, from the point of view of dynamical systems.

1.1. Continuity of joint spectral radius. Consider \( f : X \to X \) is a continuous map of a compact metric space \( X \), \( A : X \to M_d(\mathbb{R}) \) is continuous.

**Definition 1.1.** Let \( f : X \to X \), \( A : X \to M_d(\mathbb{R}) \). A cocycle given by \( A \) is understood as a function \( \mathcal{A} : X \times \mathbb{Z}^+ \to M_d(\mathbb{R}) \) satisfying \( \mathcal{A}(x,0) = Id \), and \( \mathcal{A}(x,n) = A(f^{n-1}x) \cdots A(x) \).

Recall that if \( M \in M_d(\mathbb{R}) \), the singular values \( \alpha_1(M) \geq \cdots \geq \alpha_d(M) \) are the square roots of the eigenvalues of \( M^T M \).

**Definition 1.2.** For \( A : X \to M_d(\mathbb{R}) \), we define the joint spectral radius of \( A \) as

\[
\hat{\rho}_s(A) = \lim_{n \to \infty} \sup_{x \in X} \left( \varphi^n(A(x,n)) \right)^{\frac{1}{n}},
\]

where \( \varphi^n(M) \) is the singular value function of \( M \) defined by

\[
\varphi^n(M) = \begin{cases} 
\alpha_1(M) \cdots \alpha_{\lfloor s \rfloor}(M) \alpha_{\lfloor s \rfloor}+1(M)^{s-\lfloor s \rfloor}, & \text{if } 0 < s < d; \\
\left| \det(M) \right|^{s/d}, & \text{if } s \geq d.
\end{cases}
\]

**Remark.** The original definition of joint spectral radius is a particular case of Definition 1.2. In fact, if \( A \) is a compact subset of \( M_d(\mathbb{R}) \), denote \( X = A^\mathbb{Z} \), equipped with the product topology, let \( f : X \to X \) be the left-shift map defined by \( f((A_i)_{i \in \mathbb{Z}}) = (A_{i+1})_{i \in \mathbb{Z}} \), and define \( \mathcal{A} : X \to M_d(\mathbb{R}) \) by \( (A_i)_{i \in \mathbb{Z}} \mapsto A_0 \). Then \( \hat{\rho}(A) \) defined in (1.1) equals \( \hat{\rho}_1(A) \).

Our first result concerns the continuity of the joint spectral radius for dynamical systems with the closing property.

**Definition 1.3.** Let \( f : X \to X \) be a map of a compact metric space \( X \), we call \( f \) satisfying the closing property on a subset \( \Lambda \subseteq X \), if there is \( \lambda > 0 \), and for any \( \delta > 0 \), there exists \( \beta > 0 \), such that for any \( x, f^n x \in \Lambda \), \( n \geq 1 \) with \( d(x, f^n x) < \beta \), there can be found a periodic point \( p = f^n p \in X \), such that

\[
d(f^i p, f^i x) \leq \delta \cdot e^{-\lambda \min(i, n-i)}, \quad \text{for every } i = 0, \cdots, n.
\]

We note that if \( f \) is a symbolic dynamical system, a uniformly(non-uniformly) hyperbolic diffeomorphism, or even a uniform (non-uniformly) hyperbolic map, then \( f \) satisfies the above closing property [5, 15].

**Theorem A.** Let \( f \) be a continuous map of a compact metric space \( X \), and satisfy the closing property on \( X \). Then the function \( \hat{\rho}_s : C^\infty(X, M_d(\mathbb{R})) \to \mathbb{R} \) is continuous for every \( s > 0 \).

**Remark.** The condition that \( f \) satisfies the closing property on \( X \) is necessary for the continuity of the joint spectral radius. In fact, for any \( 0 \leq l \leq \infty \), Wang and You [25] constructed a cocycle over an irrational rotation for which the joint spectral radius is not continuous. Besides, for cocycle over full-shift, Dai, Huang and Huang [10] constructed an example whose Lyapunov exponent with respect to the Bernoulli measure is discontinuous.
1.2. Continuity of generalized spectral radius. Our next result states that the
generalized spectral radius also has the continuity property.

Definition 1.4. For \( A : X \to M_d(\mathbb{R}) \) and \( s > 0 \), define the generalized spectral
radius of \( A \) by

\[
\rho_s(A) = \limsup_{n \to \infty} \left( \sup_{x \in X} \rho_s(A(x,n)) \right)^{\frac{1}{n}},
\]

where \( \rho_s(A(x,n)) = \lim_{n \to \infty} \left( \varphi^s(A(x,n)) \right)^{\frac{1}{n}}. \)

Theorem B. Let \( f \) be a continuous map of a compact metric space \( X \), and satisfy
the closing property on \( X \). Then the function \( \rho_s : C^\infty(X, M_d(\mathbb{R})) \to \mathbb{R} \) is continuous
for every \( s > 0 \).

To deduce Theorem B, we generalize the Berger-Wang formula [4] to dynamical
setting, i.e. the generalized spectral radius equals the joint spectral radius. We note
that the classical Berger-Wang formula concerning the cocycle driven by full-shift
system, which has also been established for finite-type sub-shift system by Dai [9].
Here we consider general dynamical systems.

Theorem C. Let \( f \) be a continuous map of a compact metric space \( X \), \( A : X \to M_d(\mathbb{R}) \) be continuous. Then \( \hat{\rho}_s(A) = \rho_s(A) \) for every \( s > 0 \).

2. Lyapunov exponents and Lyapunov norm.

2.1. Lyapunov exponents. We start by studying the property of Lyapunov exponents which reflects the limit behavior of product of matrices in the dynamical
processes. Considering \( f \) is a homeomorphism of a compact metric space \( X \) preserving an
ergodic measure \( \mu \), and \( A : X \to M_d(\mathbb{R}) \) is continuous, the multiplicative ergodic
theorem [21, 12, 11, 1] states that there exists an \( f \)-invariant set \( \mathcal{R} \) with \( \mu(\mathcal{R}) = 1 \),
such that for each \( x \in \mathcal{R} \):

(i) there exist numbers \( \chi_1 > \cdots > \chi_k \geq -\infty \), and a measurable direct sum
decomposition \( \mathbb{R}^d = E_1(x) \oplus \cdots \oplus E_k(x) \), with \( \dim(E_i(x)) \) being constant
almost everywhere.

(ii) \( A(x)E_i(x) \subseteq E_i(fx) \) with equality when \( \chi_i > -\infty \).

(iii) \( \lim_{n \to \pm \infty} \frac{1}{n} \log \| A(x,n)u \| = \chi_i \), \( \forall \ u \in E_i(x) \setminus \{0\} \), if \( \chi_i > -\infty \); \n
\[
\lim_{n \to +\infty} \frac{1}{n} \log \| A(x,n)u \| = \chi_i, \quad \forall \ u \in E_i(x) \setminus \{0\}, \quad \text{if} \ \chi_i = -\infty.
\]

(iv) the convergence in (iii) is uniform with respect to \( u \in E_i(x) \cap S^{d-1} \) for each
fixed \( x \), where \( S^{d-1} \) is the \((d-1)\)-sphere.

(v) \( \lim_{n \to \pm \infty} \frac{1}{n} \log \sin \angle \left( E_1(f^n x), \bigoplus_{j=2}^k E_j(f^n x) \right) = 0 \),

where

\[
\mathcal{A}_i(x,n) := \begin{cases} A(f^{n-1}x)\|_{E_1(f^{n-1}x)}, \cdots A(x)\|_{E_i(x)}, & n > 0; \\ A(f^n x)\|_{E_1(f^n x)}^{-1}, \cdots A(f^{-1} x)\|_{E_i(f^{-1} x)}^{-1}, & n < 0. \end{cases}
\]

The numbers \( \chi_1 > \cdots > \chi_k \) are called the Lyapunov exponents of \( \mathcal{A} \), \( \mathbb{R}^d = E_1(x) \oplus \cdots \oplus E_k(x) \) is called the Oseledets decomposition, and \( R \) is called the regular
set of \( \mu \).

We remark that the part \( n \to -\infty \) in (iii) is proved by [1, Lemma 3.1], (iv) is
implied in the proof of [12, Theorem 4.1] and (v) can be taken as a corollary of
[11, Theorem 2].
2.2. Lyapunov norm. We assume in this subsection that $k > 1$, which implies $\chi_1 > -\infty$.

Letting $\mathbb{R}^d = E(x) \oplus F(x)$, where $E(x) = E_{\chi_1}(x)$, $F(x) = E_{\chi_2}(x) \oplus \cdots \oplus E_{\chi_k}(x)$, we denote $A_1(x) := A(x)|_{E(x)}$. Then for a fixed $\varepsilon > 0$, and a point $x \in \mathcal{R}$, we define the Lyapunov norm $\| \cdot \|_x$ as

$$
\|u_E\|_x = \sum_{n=-\infty}^{+\infty} \| A_1(x,n)u_E \| e^{-\chi_1 n - \varepsilon |n|}, \quad \text{if } u_E \in E(x);
$$

$$
\|u_F\|_x = \sum_{n=0}^{+\infty} \| A(x,n)u_F \| e^{-(b+\varepsilon)n}, \quad \text{if } u_F \in F(x),
$$

where $b = \chi_2 > -\infty$; $b = -2|\chi_1|$ if $\chi_2 = -\infty$. Now we define

$$
\|u\|_x = \|u_E\|_x + \|u_F\|_x, \quad \forall u = u_E + u_F \in \mathbb{R}^d, \quad (2.2)
$$

where $u_E \in E(x)$, $u_F \in F(x)$.

The next proposition gives some useful properties of the Lyapunov norm.

**Proposition 2.1.** Let $f$, $A$ and $\mu$ be as above. Then for any fixed $\varepsilon > 0$, the Lyapunov norm $\| \cdot \|_x = \| \cdot \|_x, \varepsilon$ defined above satisfies the following properties.

(i) For each point $x \in \mathcal{R}$,

$$
e^{\chi_1 - \varepsilon}\|u_E\|_x \leq \|A(x)u_E\|_{fx} \leq e^{\chi_1 + \varepsilon}\|u_E\|_x, \quad \forall u_E \in E(x),
$$

$$
\|A(x)u_F\|_{fx} \leq e^{b+\varepsilon}\|u_F\|_x, \quad \forall u_F \in F(x). \quad (2.3)
$$

(ii) There exists an $f$-invariant set $\mathcal{R}_\varepsilon \subset \mathcal{R}$ with $\mu(\mathcal{R}_\varepsilon) = 1$ and a measurable function $K_\varepsilon(x)$ such that for any $x \in \mathcal{R}_\varepsilon$,

$$
\|u\| \leq \|u\|_x \leq K_\varepsilon(x)\|u\|, \quad \forall u \in \mathbb{R}^d \text{ and } K_\varepsilon(fx) \leq K_\varepsilon(x)\varepsilon. \quad (2.5)
$$

**Proof.** (i) We will prove the inequality $\|A(x)u_E\|_{fx} \leq e^{\chi_1 + \varepsilon}\|u_E\|_x, \forall u_E \in E(x)$, the others can be proved analogously.

For any $x \in \mathcal{R}$ and $u_E \in E(x)$, by the definition we obtain that

$$
\|A(x)u_E\|_{fx} = \sum_{n=-\infty}^{+\infty} \|A_1(fx,n)A(x)u_E\| e^{-\chi_1 n - \varepsilon |n|}
$$

$$
= \sum_{n=-\infty}^{+\infty} \|A_1(x,n+1)u_E\| e^{-\chi_1 (n+1) - \varepsilon |n+1|} e^{\chi_1 + \varepsilon |n+1| - |n|}
$$

$$
\leq e^{\chi_1 + \varepsilon}\|u_E\|_x,
$$

and the inequality follows.

(ii) By the definition, $\|u\| = \|u_E + u_F\| \leq \|u_E\| + \|u_F\| \leq \|u_E\|_x + \|u_F\|_x = \|u\|_x$, this gives the lower bound, so it remains to estimate the upper bound.

For any $\varepsilon > 0$ and $x \in \mathcal{R}$, we define

$$
M_\varepsilon(x) = \sup \left\{ \frac{\|A(x,n)u_E\|}{e^{\chi_1 n + \varepsilon |n|} \|u_E\|} : n \in \mathbb{Z} \text{ and } u_E \in E(x) \right\},
$$

$$
M'_\varepsilon(x) = \sup \left\{ \frac{\|A(x,n)u_F\|}{e^{b n + \varepsilon |n|} \|u_F\|} : n \geq 0 \text{ and } u_F \in F(x) \right\}.
$$
Then for any \( u = u_E + u_F \in \mathbb{R}^d \) with \( u_E \in E(x) \) and \( u_F \in F(x) \), we have
\[
\|u_E\|_x \leq \sum_{n=-\infty}^{+\infty} M(x) e^{-\frac{\gamma}{2}|n|} \|u_E\| =: C(x) M(x) \|u_E\|,  
\]  
(2.7)
\[
\|u_F\|_x \leq \sum_{n=0}^{+\infty} M'(x) e^{-\frac{\gamma}{2}|n|} \|u_F\| =: C'(x) M'(x) \|u_F\|.  
\]  
(2.8)
Let \( P(x) : E(x) \oplus F(x) \rightarrow E(x) \) be the projection along \( F(x) \), and \( \gamma(x) := \mathcal{L}(E(x), F(x)) \). Then by [22, Lemma 3.1], we have \( \|u_E\| = \|P(x)u\| \leq \|P(x)||u| = \frac{1}{\sin \gamma(x)} \|u\| \). Similarly, we can also obtain \( \|u_F\| \leq \frac{1}{\sin \gamma(x)} \|u\| \). Hence we conclude by using (2.7) and (2.8) that
\[
\|u\|_x = \|u_E\|_x + \|u_F\|_x \leq \frac{1}{\sin \gamma(x)} \tilde{M}(x) \|u\|,
\]
where \( \tilde{M}(x) := C(x) M(x) + C'(x) M'(x) \).

**Claim.** \( \tilde{M}(x) \) is tempered on an \( f \)-invariant set \( \mathcal{R}_x \subset \mathcal{R} \) with \( \mu(\mathcal{R}_x) = 1 \), that is, for any \( x \in \mathcal{R}_x \)
\[
\lim_{n \to \pm\infty} \frac{1}{n} \log \tilde{M}(f^n x) = 0.
\]

**Proof.** It’s enough to prove that \( M(x) \) and \( M'(x) \) are tempered for \( \mu \)-a.e. \( x \in \mathcal{R} \). We will prove \( M(x) \) is tempered for \( \mu \)-a.e. \( x \in \mathcal{R} \), the other one can be proved analogously. Recall that \( A_1(x) := A(x)|_{E(x)} \). Since
\[
M(x) = \sup \left\{ \frac{\|A_1(x, n)\|}{e^{\chi_1 n + \frac{1}{2} \gamma |n|}} : n \in \mathbb{Z} \right\} 
\leq \sup \left\{ \frac{|A_1(f^n x, n - 1)| \cdot |A_1(x)|}{e^{\chi_1 (n-1) + \frac{1}{2} \gamma (|n-1|) - \frac{1}{2} \gamma |n-1|}} : n \in \mathbb{Z} \right\} 
\leq M(f x) \frac{|A_1(x)|}{e^{\chi_1 - \frac{1}{2} \gamma}},
\]
so,
\[
(\log M(x) - \log M(f x))^+ \leq \log^+ \|A(x)\| + |\chi_1 - \frac{1}{2} \gamma| \in L^1(X, \mu).
\]
Then the claim is proved by using [17, Lemma III.8].

Since \( \sin \gamma(x) \) is also tempered on \( \mathcal{R} \) by (v), we conclude by using [3, Lemma 3.5.7] that
\[
K_e(x) := \sum_{n \in \mathbb{Z}} \frac{1}{\sin \gamma(f^n x)} \tilde{M}(f^n x) e^{-\frac{1}{2} \gamma |n|}
\]
is a measurable function defined on \( \mathcal{R}_x \) and satisfies (2.5) and (2.6). This completes the proof of Proposition 2.1.

We define
\[
\mathcal{R}_{e,l}^A = \{ x \in \mathcal{R}_x : K_e(x) \leq l \},
\]
(2.9)
then \( \mu(\mathcal{R}_{e,l}^A) \to 1 \) as \( l \to \infty \). Moreover, without loss of generality, we may also assume by using Lusin’s theorem that the Lyapunov norm and Oseledet’s decomposition are continuous on \( \mathcal{R}_{e,l}^A \) by restricting to a compact subset of it.
3. Approximation of Lyapunov exponents. To prove Theorem A, we need to approximate the Lyapunov exponents of ergodic measures by those at periodic points.

**Theorem 3.1.** Let $f$ be a continuous map of a compact metric space $X$, satisfying the closing property on a Borel subset $\Lambda \subset X$, $\mu$ be an ergodic invariant measure for $f$ with $\mu(\Lambda) > 0$, and $A : X \to M_d(\mathbb{R})$ be H"older continuous. Then the Lyapunov exponents of $A$ with respect to $\mu$ can be approximated by the Lyapunov exponents of $A$ at periodic points.

We note that the approximation of Lyapunov exponents of hyperbolic measures for a diffeomorphism was proved by Wang and Sun [26]. Kalinin [13] proved the approximation when $f$ is a homeomorphism satisfying the closing property on $X$ and $A$ takes values in $GL_d(\mathbb{R})$ and furthermore, with Sadovskaya [14] considered invertible operators on a Banach space. Dai proved the approximation in [6] and [7] when $f$ is a continuous map and $A$ takes values in $GL_d(\mathbb{R})$. Backes [1] proved the approximation for semi-invertible cocycles: $f$ is a homeomorphism and $A$ takes values in $M_d(\mathbb{R})$. Theorem 3.1 deals with completely non-invertible cocycles since both $f$ and $A$ may be non-invertible.

To prove Theorem 3.1, we will need the following two lemmas.

**Lemma 3.2** ([7] Lemma 3.8). Suppose that $f$ is a continuous map of a compact metric space $X$, preserves an ergodic measure $\mu$ and satisfies the closing property on a Borel subset $\Lambda \subset X$ with $\mu(\Lambda) > 0$. Then there exist periodic points $\{p_j\}_{j \geq 1}$ with $p_j = f^{n_j}p_j$ such that $\mu_{p_j} \to \mu$ in the weak* topology, where $\mu_{p_j} = \frac{1}{n_j} \sum_{i=0}^{n_j-1} \delta_{f^i p_j}$.

**Lemma 3.3** ([20] Lemma A.4). Let $X$ be a compact metric space, $\{f_n\}_{n \geq 1}$ be a subadditive sequence of upper semi-continuous functions $f_n : X \to \mathbb{R} \cup \{-\infty\}$, then the function $\mathcal{M} \to \mathbb{R} \cup \{-\infty\}$ given by $\mu \mapsto \inf_{n \geq 1} \int f_n d\mu$ is also upper semi-continuous.

**Proof of Theorem 3.1.** Let $\chi_1 > \cdots > \chi_k \geq -\infty$ be the Lyapunov exponents of $A$ with respect to $\mu$ [21]. We will divide the proof into two cases: $\chi_1 = -\infty$ and $\chi_1 > -\infty$.

3.1. **Case I:** $\chi_1 = -\infty$. It implies $k = 1$.

Let $\mu_{p_j}$ be as in Lemma 3.2, and $\lambda_1(A, \mu_{p_j}) \geq \cdots \geq \lambda_d(A, \mu_{p_j})$ be the Lyapunov exponents of $A$ with respect to $\mu_{p_j}$ counted with multiplicities. Since the function $f_n(x) := \frac{1}{n} \log \|A(x, n)\|$ is continuous for every $n \geq 1$, and $\chi_1 = \inf_{n \geq 1} \int f_n d\mu$, by Lemma 3.2 and Lemma 3.3, we obtain

$$\limsup_{j \to \infty} \lambda_1(A, \mu_{p_j}) \leq \chi_1 = -\infty.$$ 

Thus we conclude that $\lim_{j \to \infty} \lambda_i(A, \mu_{p_j}) = -\infty$ for every $1 \leq i \leq d$, which completes the proof of case I.

3.2. **Case II:** $\chi_1 > -\infty$. Since $f$ may be not invertible, we consider the natural extension of $f$. Let

$$\hat{X} = \{(x_n)_{n \leq 0} : f(x_n) = x_{n+1}, \forall n < 0\}$$

be the space of pre-orbits, endowed with the metric

$$d(\hat{x}, \hat{y}) = \sum_{n=-\infty}^{0} 2^n d(x_n, y_n), \text{ where } \hat{x} = (x_n)_{n \leq 0} \text{ and } \hat{y} = (y_n)_{n \leq 0}.$$
The natural extension \( \hat{f} : \hat{X} \to \hat{X} \) of \( f \) is defined by 
\[
\hat{f}(\cdots, x_n, \cdots, x_0) \mapsto (\cdots, x_n, \cdots, x_0, f(x_0)).
\]
Then \( \hat{f} \) is a homeomorphism. Let \( \pi : \hat{X} \to X \) be the \( \sigma \)-th projection map, and \( \hat{\mu} \) be the unique ergodic \( \hat{f} \)-invariant measure on \( \hat{X} \) with \( \pi_* \hat{\mu} = \mu \) [24, Proposition 2.4.4]. Let \( \hat{A} = A \circ \pi : \hat{X} \to M_d(\mathbb{R}) \). Noting that \( \int \log \|A\| d\hat{\mu} = \int \log \|A\| d\mu \), it is clear that \( A \) and \( \hat{A} \) have the same Lyapunov exponents which we denote by \( \chi_1 \geq \cdots \geq \chi_k \geq -\infty \).

Since \( \hat{f} \) is invertible, by the multiplicative ergodic theorem stated in the subsection 2.1, there is a measurable direct sum decomposition \( \mathbb{R}^d = E_1(\hat{x}) \oplus \cdots \oplus E_k(\hat{x}) \), and a regular set \( R \) with \( \hat{\mu}(R) = 1 \). Let \( \mathbb{R}^d = E(\hat{x}) \oplus F(\hat{x}) \), where \( E(\hat{x}) = E_1(\hat{x}) \), \( F(\hat{x}) = E_2(\hat{x}) \oplus \cdots \oplus E_k(\hat{x}) \). If \( k = 1 \), let \( E(\hat{x}) = \mathbb{R}^d, F(\hat{x}) = \{0\} \), then the conclusion in Lemma 3.4 below also holds, and the proof is simpler. Thus we may assume \( k > 1 \). For any \( \varepsilon > 0 \), the Lyapunov norm \( \| \cdot \|_e = \| \cdot \|_{\mathbb{R}^d, \varepsilon} \) is defined as in (2.2). Suppose \( \hat{A} : X \to M_d(\mathbb{R}) \) is \( \alpha \)-Hölder, let \( \varepsilon_0 = \min\{\lambda\alpha, (\chi_1 - b)/4\} \), where \( \lambda \) is given by Definition 1.3 and \( b \) is given by (2.1).

Denote \( \hat{x}_i = f^i\hat{x}, \hat{p}_i = f^i\hat{p} \), and let \( x = \pi(\hat{x}), p = \pi(\hat{p}) \). Let \( \mathcal{R}^\theta_{\hat{x}_i} \subset \hat{X} \) be defined as in (2.9). For any \( u = u_E + u_F \in \mathbb{R}^d \), where \( u_E \in E(\hat{x}_i), u_F \in F(\hat{x}_i) \), we consider the cone 
\[
K^\theta_i := \{ u \in \mathbb{R}^d : \|u_F\|_{\hat{x}_i} \leq \theta \|u_E\|_{\hat{x}_i} \}.
\]

**Lemma 3.4.** For any \( 0 < \varepsilon < \varepsilon_0, l \geq 1 \) and \( \theta > 0 \), there exist \( \delta, \eta > 0 \), such that for any \( \hat{x}, f^n\hat{x} \in \mathcal{R}^\theta_{\hat{x}_i}, p = f^n p \in X \) with \( d(f^n x, f^n p) \leq \delta \cdot e^{-\lambda \min(i, n - i)} \) for all \( 0 \leq i \leq n \), we have \( \hat{A}(\hat{p}_i)(K^\theta_i) \subset K^\theta_{\hat{x}_{i+1}} \), for every \( 0 \leq i \leq n - 1 \). Moreover, if \( \theta \leq e^\varepsilon - 1 \), then for any \( u \in K^\theta_i \), \( 0 \leq i \leq n - 1 \), we have 
\[
e^{\chi_1 - 3\varepsilon}\|u\|_{\hat{x}_i} \leq \|\hat{A}(\hat{p}_i)u\|_{\hat{x}_{i+1}} \leq e^{\chi_1 + 3\varepsilon}\|u\|_{\hat{x}_i}.
\]

**Proof.** For any \( 0 \leq i \leq n - 1 \), let \( u = u_E + u_F \in K^\theta_i \). Since 
\[
\hat{A}(\hat{p}_i)u = (\hat{A}(\hat{p}_i) - \hat{A}(\hat{x}_i))u + \hat{A}(\hat{x}_i)u = v + \hat{A}(\hat{x}_i)u,
\]
we denote \( v = v_E + v_F \), where \( v_E \in E(\hat{x}_{i+1}), v_F \in F(\hat{x}_{i+1}) \), then by (2.5), (2.6) and \( u \in K^\theta_i \), we have 
\[
\|v_E\|_{\hat{x}_{i+1}} \leq \|v\|_{\hat{x}_{i+1}} \leq K \|\hat{A}(\hat{p}_i) - \hat{A}(\hat{x}_i)\|\|u\|_{\hat{x}_i} \leq K \|\hat{A}(\hat{p}_i) - \hat{A}(\hat{x}_i)\|\|u\|_{\hat{x}_i} \leq e^{\varepsilon}c_0\delta^\alpha e^{(\varepsilon - \alpha\lambda)\min(i, n - i)}\|u\|_{\hat{x}_i} \leq (1 + \theta)e^{\varepsilon}c_0\delta^\alpha\|u_E\|_{\hat{x}_i},
\]
since \( \varepsilon - \alpha\lambda < 0 \). Similarly, we can also obtain 
\[
\|v_F\|_{\hat{x}_{i+1}} \leq (1 + \theta)e^{\varepsilon}c_0\delta^\alpha\|u_E\|_{\hat{x}_i}.
\]
Let \( \hat{A}(\hat{p}_i)u = (\hat{A}(\hat{p}_i)u)_E + (\hat{A}(\hat{p}_i)u)_F \), where \( (\hat{A}(\hat{p}_i)u)_E \in E(\hat{x}_{i+1}), (\hat{A}(\hat{p}_i)u)_F \in F(\hat{x}_{i+1}) \), then by (2.3) and (3.1), we get 
\[
\|((\hat{A}(\hat{p}_i)u)_E)_{\hat{x}_{i+1}} \geq ((\hat{A}(\hat{x}_i)u)_E)_{\hat{x}_{i+1}} - \|v_E\|_{\hat{x}_{i+1}} \geq e^{\chi_1 - \varepsilon}\|u_E\|_{\hat{x}_i} - (1 + \theta)e^{\varepsilon}c_0\delta^\alpha\|u_E\|_{\hat{x}_i} \geq e^{\chi_1 - 2\varepsilon}\|u_E\|_{\hat{x}_i},
\]
if $\delta$ is small enough. Similarly, by (2.4), (3.2) and $u \in K_0^\theta$, we can also obtain
\[
\|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}} \leq e^{b+\varepsilon}\|u_F\|_{\hat{x}_i} + (1 + \theta)e^{\varepsilon\delta_0}\|u_E\|_{\hat{x}_i} \\
\leq \theta e^{b+2\varepsilon}\|u_E\|_{\hat{x}_i} + (1 + \theta)e^{2\varepsilon}\|u_E\|_{\hat{x}_i}
\]
(3.4) for $\delta$ small enough. Let $\eta = e^{b-\chi_1+4\varepsilon} < 1$, then we have
\[
\|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}} \leq \theta \eta \|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}},
\]
which implies $\hat{A}(\hat{\mu}_u)(K_0^\theta) \subset K_{i+1}^{\theta \eta}$. Moreover, if $\theta \leq e^\varepsilon - 1$, then for any $u \in K_0^\theta$, by (3.3), we have
\[
\|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}} \geq e^{\chi_1 - 2\varepsilon}\|u_E\|_{\hat{x}_i} \geq \frac{1}{1 + \theta} e^{\chi_1 - 2\varepsilon}\|u\|_{\hat{x}_i} \geq e^{\chi_1 - 3\varepsilon}\|u\|_{\hat{x}_i}.
\]
Similar to (3.3), by (2.3) and (3.1), we can also get
\[
\|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}} \leq \|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}} + \|u_E\|_{\hat{x}_{i+1}} \leq e^{\chi_1 + 2\varepsilon}\|u_E\|_{\hat{x}_i}
\]
for $\delta$ small enough. Thus we conclude by using (3.4) that
\[
\|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}} \leq \|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}} + \|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{i+1}} + \|u_E\|_{\hat{x}_{i+1}} \leq e^{\chi_1 + 2\varepsilon}\|u_E\|_{\hat{x}_i} + \theta e^{2\varepsilon}\|u\|_{\hat{x}_i} \\
\leq e^{\chi_1 + 3\varepsilon}\|u\|_{\hat{x}_i}.
\]

First we will give the approximation of the largest Lyapunov exponent of $A$ with respect to $\mu$. Recall that $\mu_p := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i\mu}$ represents the periodic measure, and $\chi_1(\hat{A}, \mu_p) = \chi_1(\hat{A}, \hat{\mu})$ is the largest Lyapunov exponent of $A$ with respect to $\mu_p$.

**Proposition 3.5.** Under the assumption of “Case II”, for any $0 < \varepsilon < \varepsilon_0$, there exists a periodic point $p$ such that $|\chi_1(\hat{A}, \mu_p) - \chi_1| \leq 4\varepsilon$. Moreover, $\mu_p \to \mu$ in the weak* topology as $\varepsilon \to 0$.

**Proof.** Let $\hat{\Lambda} = \pi^{-1}\Lambda$, then $\hat{\mu}(\hat{\Lambda}) = \mu(\Lambda) > 0$. Let
\[
B(\mu) = \{\hat{x} \in \hat{X} : \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i\hat{x}} \to \hat{\mu} \text{ in the weak}^* \text{ topology}\},
\]
then $\hat{\mu}(B(\mu)) = 1$. Take $l$ large enough such that $\hat{\mu}(\hat{\Lambda} \cap R_{\varepsilon,l}^{\hat{A}} \cap B(\hat{\mu})) > 0$. Then the Poincaré’s recurrence theorem implies that for $\mu$-a.e. $\hat{x} \in \hat{\Lambda} \cap R_{\varepsilon,l}^{\hat{A}} \cap B(\hat{\mu})$, there exist infinitely many $\hat{f}^n\hat{x} \in \hat{\Lambda} \cap R_{\varepsilon,l}^{\hat{A}} \cap B(\hat{\mu})$ with $\hat{f}^n\hat{x} \to \hat{x}$. Since $f$ satisfies the closing property on $\Lambda$, take $\hat{x}, \hat{f}^n\hat{x} \in \hat{\Lambda} \cap R_{\varepsilon,l}^{\hat{A}} \cap B(\hat{\mu})$ with $d(x, f^n(x)) \leq \delta \leq e^{-\lambda \min\{i, n-i\}}$ small enough, then there exists $p = f^n p \in X$ with $d(f^i x, f^i p) \leq \delta \cdot e^{-\lambda \min\{i, n-i\}}$ for all $0 \leq i \leq n$, where $\delta$ is given by Lemma 3.4. Moreover, similar to the proof of Lemma 3.2, one has $\mu_p \to \mu$ as $\varepsilon \to 0$. Take $0 < \theta \leq e^\varepsilon - 1$, then Lemma 3.4 gives $\hat{A}(\hat{\mu}, n)(K_0^\theta) \subset K_n^{\theta \delta},$ and
\[
e^{\chi_1 - 3\varepsilon)n}\|u\|_{\hat{x}_0} \leq \|\hat{A}(\hat{\mu}_u)u\|_{\hat{x}_{n}} \leq e^{\chi_1 + 3\varepsilon)n}\|u\|_{\hat{x}_{n}}, \quad \forall u \in K_0^\theta.
\]
(3.5) Since the Lyapunov norm and the Oseledets decomposition are continuous on $R_{\varepsilon,l}^{\hat{A}}$, we have
\[
\hat{A}(\hat{\mu}, n)(K_0^\theta) \subset K_n^{\theta \delta} \subset K_0^\theta,
\]
Therefore, taking any $u \in K_0^\theta$, by (3.6) it follows that
\[
\chi_1(A, \mu_p) = \chi_1(\hat{A}, \hat{\mu}_p) \geq \lim_{n \to \infty} \frac{1}{n_j} \log \|\hat{A}(\hat{\rho}, nj)u\|_{\hat{z}_n}
\geq \chi_1 - 3\varepsilon - \frac{1}{n} \log l
\geq \chi_1 - 4\varepsilon,
\]
if $n$ is large enough with respect to $l$. This gives the lower bound.

To estimate the upper bound, notice that (3.5) also implies
\[
\|\hat{A}(\hat{\rho}, n)u\|_{\hat{z}_0} \leq l\|\hat{A}(\hat{\rho}, n)u\|_{\hat{z}_n} \leq l e^{(\chi_1 + 3\varepsilon)n} \|u\|_{\hat{z}_0}, \forall u \in K_0^\theta.
\]
Let $\mathbb{R}^d = E_1(\hat{\rho}) \oplus \cdots \oplus E_{k'}(\hat{\rho})$ be the Oseledets decomposition of $\hat{A}$ at the periodic point $\hat{\rho}$, with $E_1(\hat{\rho})$ corresponding to the largest Lyapunov exponent $\chi_1(\hat{A}, \hat{\mu}_p)$. Then for any $u \in K_0^\theta \setminus (E_2(\hat{\rho}) \oplus \cdots \oplus E_{k'}(\hat{\rho}))$, we have
\[
\chi_1(A, \mu_p) = \chi_1(\hat{A}, \hat{\mu}_p) = \lim_{n \to \infty} \frac{1}{n_j} \log \hat{A}(\hat{\rho}, nj)u \|_{\hat{z}_0}
\leq \chi_1 + 3\varepsilon + \frac{1}{n} \log l
\leq \chi_1 + 4\varepsilon,
\]
for $n$ large enough with respect to $l$. This completes the proof. \qed

We now estimate all Lyapunov exponents of $A$ with respect to $\mu$. Without loss of generality, we may assume $\chi_k = -\infty$. Let $\lambda_1 \geq \cdots \geq \lambda_d$ be the Lyapunov exponents of $A$ with respect to $\mu$ counted with multiplicities. We consider the $i$-fold exterior power $\Lambda^i A$ of $A$. One sees that the largest Lyapunov exponent of $\Lambda^i A$ with respect to $\hat{\mu}$ is $\lambda_1 + \lambda_2 + \cdots + \lambda_i$. Let
\[
\hat{R}_{\varepsilon, l} = \hat{A} \cap \bigcap_{i=1}^{i_0} \mathcal{R}_{\varepsilon, l}^{\Lambda^i A},
\]
where $i_0 := d - \dim(E_{\chi_k}(\hat{x}))$. Then $\hat{\mu}(\hat{R}_{\varepsilon, l}) > 0$ for $l$ large enough. By similar arguments as in Proposition 3.5, taking $\varepsilon_j < \frac{1}{2}$ small enough, there exists a sequence of periodic points $\{p_j\}_{j \geq 1}$ with $\mu_{p_j} \to \mu$ as $j \to \infty$, such that
\[
|\lambda_1(A, \mu_{p_j}) + \cdots + \lambda_i(A, \mu_{p_j}) - (\lambda_1 + \cdots + \lambda_i)| \leq 4\varepsilon_j, \quad \forall 1 \leq i \leq i_0,
\]
where $\lambda_1(A, \mu_{p_j}) \geq \cdots \geq \lambda_d(A, \mu_{p_j})$ are the Lyapunov exponents of $A$ with respect to $\mu_{p_j}$ counted with multiplicities. Moreover, by Lemma 3.3,
\[
\limsup_{j \to \infty} (\lambda_1(A, \mu_{p_j}) + \cdots + \lambda_{i_0+1}(A, \mu_{p_j})) \leq \lambda_1 + \cdots + \lambda_{i_0+1} = -\infty.
\]

Then (3.7) and (3.8) imply that $\lim_{j \to \infty} \lambda_i(A, \mu_{p_j}) = \lambda_i$, $\forall 1 \leq i \leq d$. This completes the proof of case II. \qed
4. **Proof of Theorem A.** To begin the proof of Theorem A, we first state the following lemma.

**Lemma 4.1.** Let $f : X \to X$ be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$, $A : X \to M_d(\mathbb{R})$ satisfy $\log^+ \|A(x)\| \in L^1(X, \mu)$ and have Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_d$ with respect to $\mu$ counted with multiplicities. Then for any $0 < s < d$, we have

$$\inf_{n \geq 1} \frac{1}{n} \int \log \varphi^s(A(x, n)) d\mu = (s - m) \cdot \lambda_1(A^{m+1}A, \mu) + (1 - s + m) \cdot \lambda_1(A^m A, \mu),$$

where $m = \lfloor s \rfloor$, and $\lambda_1(A^t A, \mu)$ is the largest Lyapunov exponent of $A^t A$ with respect to $\mu$.

**Proof.** For a matrix $M \in M_d(\mathbb{R})$, by [3, Section 3.1], we have

$$\varphi^s(M) = \alpha_1(M) \cdot \alpha_2(M) \cdots \alpha_m(M) \cdot \alpha_{m+1}(M)^{s-m}$$

$$= \left( \alpha_1(M) \cdot \alpha_2(M) \cdots \alpha_m(M) \right)^{s-m} \cdot \left( \alpha_1(M) \cdot \alpha_2(M) \cdots \alpha_m(M) \right)^{1-s+m} \quad (4.1)$$

Thus

$$\inf_{n \geq 1} \frac{1}{n} \int \log \varphi^s(A(x, n)) d\mu = (s - m) \cdot \lambda_1(A^{m+1}A, \mu) + (1 - s + m) \cdot \lambda_1(A^m A, \mu).$$

We now prove Theorem A. We assume $s < d$, the case $s \geq d$ can be considered analogously.

Since $X$ is compact, $f : X \to X$ and $A : X \to M_d(\mathbb{R})$ are continuous, by [20, A.3], one has

$$\log \hat{\rho}_s(A) = \max_{\mu \in \mathcal{M}_{inv}(X, f)} \left\{ \inf_{n \geq 1} \frac{1}{n} \int \log \varphi^s(A(x, n)) d\mu \right\}, \quad (4.2)$$

where $\mathcal{M}_{inv}(X, f)$ represents the set of $f$-invariant Borel probability measures on $X$. Since $(A, \mu) \mapsto \int \frac{1}{n} \log \varphi^s(A(x, n)) d\mu$ is continuous for every $n \geq 1$, $(A, \mu) \mapsto \inf_{n \geq 1} \frac{1}{n} \int \log \varphi^s(A(x, n)) d\mu$ is upper semi-continuous. Together with the compactness of $\mathcal{M}_{inv}(X, f)$, it follows that the function $A \mapsto \hat{\rho}_s(A)$ is upper semi-continuous. So it remains to show the function $A \mapsto \hat{\rho}_s(A)$ is lower semi-continuous.

By (4.2) and Lemma 4.1,

$$\log \hat{\rho}_s(A) \geq \sup_{p=f^{k}, k \geq 1} \left\{ \inf_{n \geq 1} \frac{1}{n} \int \log \varphi^s(A(x, n)) d\mu_p \right\}$$

$$= \sup_{p=f^{k}, k \geq 1} \left\{ (s - m) \cdot \lambda_1(A^{m+1}A, \mu_p) + (1 - s + m) \cdot \lambda_1(A^m A, \mu_p) \right\}. \quad (4.3)$$

Denote the set of measures admitting the maximum $\log \hat{\rho}_s(A)$ by

$$\mathcal{M}_{max}(A) := \left\{ \mu \in \mathcal{M}_{inv}(X, f) : \inf_{n \geq 1} \frac{1}{n} \int \log \varphi^s(A(x, n)) d\mu = \log \hat{\rho}_s(A) \right\}.$$ 

Take $\mu \in \mathcal{M}_{max}(A)$. We may suppose that $\mu$ is ergodic, otherwise we replace $\mu$ by its ergodic component in $\mathcal{M}_{max}(A)$. By Lemma 4.1, we have

$$\log \hat{\rho}_s(A) = (s - m) \cdot \lambda_1(A^{m+1}A, \mu) + (1 - s + m) \cdot \lambda_1(A^m A, \mu).$$
Let 
\[
B(x) = \begin{bmatrix} A^{m+1}(x) & 0 \\
0 & A^m(x) \end{bmatrix},
\]
then \(B(x)\) is Hölder continuous. By Theorem 3.1, we deduce that the Lyapunov exponents of \(\Lambda^{m+1}(x)\) and \(\Lambda^m(x)\) with respect to \(\mu\) can be approximated by those at the same periodic points. Thus,
\[
\log \hat{\rho}_s(A) = (s - m) \cdot \lambda_1(\Lambda^{m+1}(x), \mu) + (1 - s + m) \cdot \lambda_1(\Lambda^m(x), \mu)
\leq \sup_{p = f^k p, k \geq 1} \left\{ (s - m) \cdot \lambda_1(\Lambda^{m+1}, \mu_p) + (1 - s + m) \cdot \lambda_1(\Lambda^m, \mu_p) \right\}.
\]
Using (4.3),
\[
\log \hat{\rho}_s(A) = \sup_{p = f^k p, k \geq 1} \left\{ (s - m) \cdot \lambda_1(\Lambda^{m+1}, \mu_p) + (1 - s + m) \cdot \lambda_1(\Lambda^m, \mu_p) \right\}
= \sup_{p = f^k p, k \geq 1} \left\{ (s - m) \cdot \frac{1}{k} \log \rho(\Lambda^{m+1}(p, k)) + (1 - s + m) \cdot \frac{1}{k} \log \rho(\Lambda^m(p, k)) \right\}.
\]
Since \(A \mapsto (s - m) \cdot \frac{1}{k} \log \rho(\Lambda^{m+1}(p, k)) + (1 - s + m) \cdot \frac{1}{k} \log \rho(\Lambda^m(p, k))\) is continuous, the function \(A \mapsto \hat{\rho}_s(A)\) is lower semi-continuous. This completes the proof of Theorem A.

5. **Proof of Theorem C.** In this section, we assume \(s < d\). In fact, if \(s \geq d\), then by the definition of \(\rho_s\) and \(\varphi^s\), for any matrix \(B \in M_d(\mathbb{R})\), we have \(\rho_s(B) = \varphi^s(B)\), thus the conclusion of Theorem C is trivial.

The proof of Theorem C relies on our next result which generalizes Theorem 1.6 in [19].

**Theorem 5.1.** Let \(f : X \to X\) be a measure-preserving transformation of a probability space \((X, \mathcal{B}, \mu), A : X \to M_d(\mathbb{R})\) satisfy \(\log^+ \|A(x)\| \in L^1(X, \mu)\). Then for \(\mu\)-a.e. \(x \in X\),
\[
\lim_{n \to \infty} (\varphi^s(A(x, n)))^\frac{1}{s} = \limsup_{n \to \infty} (\rho_s(A(x, n)))^\frac{1}{s}.
\]

Assuming Theorem 5.1 for the time being, we may give the proof of theorem C first by using Theorem 5.1.

**Proof of Theorem C.** For a matrix \(M \in M_d(\mathbb{R})\), by (4.1), one has
\[
\rho_s(M) = \lim_{n \to \infty} \left( \|A^{m+1} M^n\|^{s-m} \cdot \|A^m M^n\|^{1-s+m} \right)^{1/n}
= \rho(\Lambda^{m+1} M)^{s-m} \cdot \rho(\Lambda^m M)^{1-s+m}
\leq \|\Lambda^{m+1} M\|^{s-m} \cdot \|\Lambda^m M\|^{1-s+m}
= \varphi^s(M),
\]
which implies
\[
\varphi_s(A) = \limsup_{n \to \infty} \sup_{x \in X} (\rho_s(A(x, n)))^\frac{1}{s} \leq \limsup_{n \to \infty} \sup_{x \in X} (\varphi^s(A(x, n)))^\frac{1}{s} = \hat{\rho}_s(A).
\]
On the other hand, since \(\mathcal{M}_{inv}(X, f)\) is compact, there exists \(\mu \in \mathcal{M}_{inv}(X, f)\) such that \(\log \hat{\rho}_s(A) = \lim_{n \to \infty} \frac{1}{n} \int \log \varphi^s(A(x, n)) d\mu\), moreover, \(\mu\) can be taken as an ergodic measure by using the ergodic decomposition theorem. The sub-additive
ergodic theorem gives \( \log \hat{\rho}_s(A) = \lim_{n \to \infty} \frac{1}{n} \log \varphi^s(A(x, n)) \) for \( \mu \)-a.e. \( x \in X \). By Theorem 5.1, we conclude that for \( \mu \)-a.e. \( x \in X \),

\[
\bar{\rho}_s(A) = \limsup_{n \to \infty} \sup_{y \in X} \left( \rho_s(A(y, n)) \right)^{\frac{1}{n}} = \limsup_{n \to \infty} \left( \varphi^s(A(x, n)) \right)^{\frac{1}{n}}
\]

This completes the proof of Theorem C.

Next we prove Theorem 5.1, which is a particular case of the following proposition.

**Proposition 5.2.** Let \( f : X \to X \) be an ergodic measure-preserving transformation of a probability space \( (X, \mathcal{B}, \mu) \), \( C : X \to M_d(\mathbb{R}) \) and \( D : X \to M_d(\mathbb{R}) \) satisfy \( \log^+ ||C(x)||, \log^+ ||D(x)|| \in L^1(X, \mu) \). Then for any \( \beta, \gamma > 0 \), and \( \mu \)-a.e. \( x \in X \), we have

\[
\frac{1}{n} \log \rho(C(x, n))^{\beta} \cdot \rho(D(x, n))^{\gamma} = \beta \cdot \chi_1(C, \mu) + \gamma \cdot \chi_1(D, \mu),
\]

(5.2)

where \( \chi_1(C, \mu), \chi_1(D, \mu) \) are the largest Lyapunov exponents of \( C \) and \( D \), respectively.

**Proof of Proposition 5.2.** We assume that \( f \) is invertible first, the general case will be deduced later.

For \( \tau = C \) or \( D \), let \( \chi_1(\tau, \mu) > \cdots > \chi_k(\tau, \mu) \) be the Lyapunov exponents of \( \tau \) with respect to \( \mu \), \( \mathcal{R}^\tau \) be the regular set of \( \tau \), \( \mathbb{R}^d = E^1_\tau(x) \oplus \cdots \oplus E^k_\tau(x) \) be the corresponding Oseledets decomposition. Let \( \mathbb{R}^d = E^\tau(x) \oplus F^\tau(x) \), where \( E^\tau(x) = E^1_\tau(x) \oplus \cdots \oplus E^k_\tau(x) \) (if \( k_\tau = 1 \), then let \( E^\tau(x) = \mathbb{R}^d \), \( F^\tau(x) = \{0\} \)).

By Lusin’s theorem, there exists a compact subset \( \Lambda_1 \subset \mathcal{R}^C \cap \mathcal{R}^D \) with \( \mu(\bigcup_{j \geq 1} \Lambda_j) = 1 \), such that the decomposition \( \mathbb{R}^d = E^\tau(x) \oplus F^\tau(x) \) is continuous on \( \Lambda_1 \). Let

\[
\tilde{\Lambda}_1 := \{ x \in \Lambda_1 : \text{there exist infinitely many } f^{n_j}(x) \in \Lambda_1, \text{ and } f^{n_j}(x) \to x \}.
\]

Let \( \tilde{\Lambda} = \bigcup_{j \geq 1} \tilde{\Lambda}_j \), then \( \mu(\tilde{\Lambda}) = 1 \).

For any \( u \in \mathbb{R}^d \), let \( u = u_{E^\tau} + u_{F^\tau} \) with \( u_{E^\tau} \in E^\tau(x) \), \( u_{F^\tau} \in F^\tau(x) \). We consider the cone

\[
K^\tau(x, \delta) = \{ u \in \mathbb{R}^d : \| u_{F^\tau} \| \leq \delta \cdot \| u_{E^\tau} \| \}.
\]

**Claim.** For any \( x \in \tilde{\Lambda} \), we have

(i) for \( \varepsilon > 0 \) small enough, there exists \( N > 0 \), such that for any \( n \geq N \) and any \( \tau = C \) or \( D \), \( u \in K^\tau(x, 1) \), then

\[
\| \tau(x, n)u \| \geq e^{n(\chi_1(\tau, \mu) - 3\varepsilon)} \| u \|.
\]

(ii) there exists a sequence \( \{ n_j \}_{j \geq 1} \), such that for any \( \tau = C \) or \( D \), \( j \geq 1 \), then

\[
\tau(x, n_j)(K^\tau(x, 1)) \subset K^\tau(x, 1).
\]

**Proof of the Claim.** By the multiplicative ergodic theorem, for any \( x \in \tilde{\Lambda} \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \| \tau(x, n)v \| = \chi_1(\tau, \mu), \quad \forall \; v \in E^\tau(x) \cap S^{d-1},
\]

and the convergence is uniform for \( v \). Thus for any \( \varepsilon > 0 \), there exists \( N_0 > 0 \), such that for any \( n \geq N_0 \) and any \( \tau = C \) or \( D \), \( v \in E^\tau(x) \), we have

\[
\| \tau(x, n)v \| \geq e^{n(\chi_1(\tau, \mu) - \varepsilon)} \| v \|.
\]

(5.3)
Now for any $u = u_{E^*} + u_{F^*} \in K^*(x, 1)$, by (5.3), we have
\[ \|\tau(x,n)u_{E^*}\| \geq e^{n\chi_1(\tau,\mu) - \varepsilon}\|u_{E^*}\|, \quad \forall n \geq N_0. \]
Similarly, there exists $N_1 > 0$, such that for any $n \geq N_1$ and any $\tau = \mathcal{C}$ or $\mathcal{D}$,
\[ \|\tau(x,n)u_{F^*}\| \leq e^{n\chi_2(\tau,\mu) + \varepsilon}\|u_{F^*}\| \leq e^{n\chi_2(\tau,\mu) + \varepsilon}\|u_{E^*}\|. \]
Take $\varepsilon < \min_{\tau = \mathcal{C}, \mathcal{D}} \left\{ \frac{1}{3}(\chi_1(\tau, \mu) - \chi_2(\tau, \mu)) \right\}$, then there exists $N > \max\{N_0, N_1\}$ large enough such that for any $n \geq N$, $u \in K^*(x, 1)$ and $\tau = \mathcal{C}, \mathcal{D}$, we have
\[ \|\tau(x,n)u\| \geq \|\tau(x,n)u_{E^*}\| - \|\tau(x,n)u_{F^*}\| \geq e^{n\chi_1(\tau,\mu) - 3\varepsilon}\|u\|, \]
which implies (i). Moreover, since
\[ \|\tau(x,n)u_{E^*}\| \leq e^{n\chi_2(\tau,\mu) + \varepsilon}\|u_{E^*}\| \leq e^{n\chi_2(\tau,\mu) - \chi_1(\tau,\mu) + 2\varepsilon}\|\tau(x,n)u_{E^*}\|, \]
we have $\tau(x,n)(K^*(x, 1)) \subset K^*(f^n, e^{n\chi_2(\tau,\mu) - \chi_1(\tau,\mu) + 2\varepsilon})$ for $n$ large enough. Observe that $x \in \hat{A}_1$ for some $l \geq 1$. Hence there exists $\{f^n: x\}_{j \geq 1} \subset \Lambda_l$ with $f^n: x \to x$ as $j \to \infty$. Since the Oseledets’s decomposition is continuous on $\Lambda_l$,
\[ K^*(f^n, e^{n\chi_2(\tau,\mu) - \chi_1(\tau,\mu) + 2\varepsilon}) \subset K^*(x, 1) \]
for $n_j$ large enough, which gives rise to $\tau(x,n_j)(K^*(x, 1)) \subset K^*(x, 1)$. \qed

We continue the proof of Proposition 5.2. For any $\varepsilon > 0$, take $u_{\tau} \in K^*(x, 1) \setminus \{0\}$, $n_j \to N$, then by the claim, for any $\tau = \mathcal{C}, \mathcal{D}$, we have
\[ \frac{1}{n_j} \log \rho(\tau(x, n_j)) \geq \limsup_{i \to \infty} \frac{1}{i \cdot n_j} \log \|\tau(x, n_j)^iu_{\tau}\| \geq \chi_1(\tau, \mu) - 3\varepsilon. \]
Hence
\[ \limsup_{n \to \infty} \frac{1}{n} \log \rho(\mathcal{C}(x, n))^\beta \cdot \rho(\mathcal{D}(x, n))^\gamma \geq \limsup_{j \to \infty} \frac{1}{n_j} \log \rho(\mathcal{C}(x, n_j))^\beta \cdot \rho(\mathcal{D}(x, n_j))^\gamma \geq \beta(\chi_1(\mathcal{C}, \mu) - 3\varepsilon) + \gamma(\chi_1(\mathcal{D}, \mu) - 3\varepsilon). \]
This estimates the lower bound. The upper bound holds naturally.

If $f$ is not invertible, we consider the natural extension of $f$. Let $\hat{X}, \hat{f}, \pi, \hat{\mu}$ and $\hat{\tau} = \tau \circ \pi$ be as in subsection 3.2. Then $\hat{f}$ is invertible and $\hat{\mu}$ is ergodic with $\pi_*\hat{\mu} = \mu$. Since
\[ \int_X \log^+ \hat{\tau}(\hat{x})d\hat{\mu} = \int_X \log^+ \tau(x)d\pi_*\hat{\mu} = \int_X \log^+ \tau(x)d\mu < \infty, \]
for $\hat{\mu}$-a.e. $\hat{x} \in \hat{X}$,
\[ \limsup_{n \to \infty} \frac{1}{n} \log \rho(\hat{\mathcal{C}}(\hat{x}, n))^\beta \cdot \rho(\hat{\mathcal{D}}(\hat{x}, n))^\gamma = \beta \cdot \chi_1(\hat{\mathcal{C}}, \hat{\mu}) + \gamma \cdot \chi_1(\hat{\mathcal{D}}, \hat{\mu}), \]
which implies the (5.2). This completes the proof of Proposition 5.2. \qed

Proof of Theorem 5.1. We may assume that $\mu$ is ergodic by using the ergodic decomposition theorem. For a matrix $M \in M_d(\mathbb{R})$, by (5.1),
\[ \rho_s(M) = \rho(\Lambda^{m+1}M)^{s-m} \cdot \rho(\Lambda^mM)^{1-s+m}. \]
Thus we conclude by using Proposition 5.2 that for $\mu$-a.e. $x \in X$,
\[
\limsup_{n \to \infty} \frac{1}{n} \log \rho_s(A(x, n)) = \limsup_{n \to \infty} \frac{1}{n} \log (\Lambda_{m+1} A(x, n))^{s-m} \cdot \rho(\Lambda_m A(x, n))^{1-s+m} = (s-m) \cdot \chi_1 (\Lambda_{m+1} A, \mu) + (1-s+m) \cdot \chi_1 (\Lambda_m A, \mu).
\]
Therefore, the conclusion of Theorem 5.1 is obtained by Lemma 4.1.

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