TOWARDS SELECTING A FINITE-RANGE REGULARIZATION SCALE

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Extensive studies have demonstrated that finite-range regularization (FRR) offers significantly improved chiral extrapolations for lattice QCD. These studies have typically relied on selecting the finite-regularization scale based upon phenomenological input. Here we report on a preliminary investigation of a procedure to determine a preferred range of FRR scale based on nonperturbative lattice results — without any phenomenological prejudice.

1. Background

Modern lattice QCD results are beyond the power-counting regime (PCR) of the chiral expansion. Any results from implementation of the chiral effective field theory (EFT) in this domain are therefore model dependent. From the perspective of lattice practitioners, the goal is focussed on ab initio studies of nonperturbative QCD — without phenomenological input. Thereby, even when working outside the PCR, it can be attractive to choose dimensional regularisation (or a similar minimal subtraction-style scheme) to avoid phenomenological bias. As results are dependent upon this choice of how the EFT is applied, then this application is implicitly a model. Nevertheless, the ab initio demand is maintained by not introducing additional information from prior knowledge.

Finite-range regularisation (FRR) has been demonstrated to offer stable and robust chiral extrapolation of lattice simulation results performed at moderate quark masses. The FRR procedure necessarily involves introducing a preference for a regularisation scale that is “low” in favour of one that is infinite — whereby this assumption of prior knowledge means one can no longer assert ab initio.

Here we highlight an initial investigative study in search of a technique which will enable the results of lattice simulations to select a preferred regularisation scale. By removing the need to artificially select the regularisation scale by hand, one can
utilise the benefits of FRR and maintain \textit{ab initio} status.

2. Convergence and the power-counting regime

The success of effective field theories, such as chiral perturbation theory, lies in the presence of a small expansion parameter and the reliance of a series expansion of natural-sized coefficients. A simple example function is the geometric series

\[ \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \]  

The radius of convergence of this series expansion about \( x = 0 \) is \( |x| < 1 \). Given a truncated expansion, the deviation from the exact result, at a given value of \( x \), can be estimated by considering the size of the first neglected term in the series — without necessarily having knowledge of the full result.

This toy example can provide insight into the natural convergence radius of the quark-mass expansion of the nucleon mass. This expansion can be written as

\[ M_N = M_N^0 + c_2 m_\pi^2 + \chi_\pi m_\pi^3 + \ldots, \]  

where \( m_\pi \propto \sqrt{m_q} \). The \( m_\pi^3 \) term is nonanalytic in the quark mass (i.e. \( \sim m_q^{3/2} \)), and it’s coefficient is known model-independently,

\[ \chi_\pi = -\frac{3 g_A^2}{32\pi f_\pi^2}. \]  

Strictly, \( g_A \) and \( f_\pi \) are to be evaluated in the chiral limit. Assuming the physical values do not differ too significantly, this suggests a numerical value of \( \chi_\pi \simeq -5.6 \text{ GeV}^{-2} \). The term in \( m_\pi^2 \) is also known phenomenologically through the pion-nucleon sigma term, whereby

\[ \sigma_N = m_\pi^2 \frac{dM_N}{dm_\pi^2} \simeq 45 \text{ MeV}. \]  

Using this and the physical nucleon mass, together with the leading nonanalytic term gives, \( c_2 \simeq 3.5 \text{ GeV}^{-1} \) and \( M_N^0 \simeq 0.89 \text{ GeV} \).

Using these numerical estimates in Eq. (2) one finds that the second and third term reach 100\% of the leading term at pion masses of 0.51 GeV and 0.54 GeV, respectively. This suggests that at a pion mass of the order \( m_\pi \sim 0.5 \text{ GeV} \) one is at the radius of convergence of this series — where an infinite number of terms are required to reproduce the exact result. Of practical importance is how precisely can the curve be reproduced with a finite number of terms.

The first few terms of the expansion Eq. (2) looks very much like the geometric series, setting \( m_R = 0.54 \text{ GeV} \),

\[ M_N = \left(1 + 1.1 \left(\frac{m_\pi}{m_R}\right)^2 - 1.0 \left(\frac{m_\pi}{m_R}\right)^3 + \ldots\right) [0.89 \text{ GeV}]. \]  

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The suggests that using Eq. (1) with
\[ x = \frac{m_\pi}{m_R} \]
(ignoring the term linear in \( x \)) is a good way to estimate the precision of a given truncation.Demanding that the series expansion in Eq. (1) up to \( x^3 \) is accurate to the 1% level, this limits one to \( x < 0.34 \). Thereby 1% precision is enabled up to \( m_\pi/m_R < 0.34 \) or \( m_\pi < 0.18 \) GeV. Continuing the natural-size argument means that at order \( m_\pi^4 \) demanding this level of precision restricts one to \( m_\pi < 0.23 \) GeV. Expansions to higher order in the nucleon mass have also recently been revisited \(^8\) , \(^9\), but even going to \( m_\pi^5 \) or \( m_\pi^6 \) still only provides a valid expansion up to \( m_\pi < 0.27 \) and 0.30 GeV, respectively. Further, going to increasing order also comes at the cost of fitting more parameters and thereby requiring even more simulation results below the quoted thresholds.

3. Numerical approach to FRR scale determination

With the aim of performing precision \textit{ab initio} studies, it is evident that lattice results from beyond the (1%) PCR must be used. The first option would be to just to use the truncated forms of a DR expansion. In using lattice results up to \( m_\pi \sim 0.5 \) GeV to constrain an extrapolation based upon a fourth-order expansion would lead to a systematic uncertainty of 35% (based on the geometric series discussed above). A second option is to introduce a finite-range regularisation scale with the aim of both incorporating more data, to improve statistical precision, and minimising the systematic uncertainty.

The physical interpretation for the success of FRR lies in the suppression of rapidly-varying chiral logs at moderate quark masses. Mathematically, the success is based on using a separation of scales to introduce an effective resummation of higher-order terms in the chiral series, while maintaining the model-independence of the expansion to the order one is working.

To continue with the nucleon mass example, the chiral expansion, Eq. (2), can be rewritten in an unrenormalised form
\[
M_N = a_0 + a_2 m_\pi^2 + a_4 m_\pi^4 + \Sigma(m_\pi, \Lambda),
\]
where \( \Sigma \) denotes the meson-loop corrections, which contains the chiral nonanalyticities, and \( a_i \) are regularisation-scale dependent expansion coefficients. In adjusting the regularisation scale \( \Lambda \), one can shift strength between the loop contributions and the residual series composed of the \( a_i \)’s. FRR works well because with an appropriately chosen \( \Lambda \), the residual series typically shows much better convergence than the renormalised form. We wish to exploit this feature in using lattice results to choose an “optimal” range for \( \Lambda \).

Further, any criteria to determine \( \Lambda \) should produce certain desired limits. Firstly, if one really is using data in the PCR, then there should be no preference for a finite \( \Lambda \), and \( \Lambda \to \infty \) should display equally good convergence properties. In fact, the data should always reject a \( \Lambda \) which is too low. In the PCR, then letting \( \Lambda \to 0 \) would give a curve inconsistent with the nonanalyticities of the data. Further, if the expansion is extended to quark masses beyond the range accessible
within FRR, then a solution consistent with $\Lambda \to 0$ is a clear indication that the data is inconsistent with the chiral behaviour. In this domain, it may also be evident that no amount of tuning $\Lambda$ can stabilize the convergence.

The technique to constrain $\Lambda$ we investigate here, is to numerically adjust $\Lambda$ such as to ensure the residual series is satisfactorily converging over the range upon which the lattice results are fit. In the case above, Eq. (6), a preferred value of $\Lambda$ is one which gives numerical evidence of convergence of the series

$$\left\{ a_0, a_2 m_\pi^2, a_4 m_\pi^4, \ldots \right\},$$

where $m_{\pi,\text{max}}$ denotes the largest pion mass at which the expansion in used in a fit. A potential convergence criteria is to ensure that

$$R \equiv \left| \frac{a_4 m_{\pi,\text{max}}^4}{a_2 m_{\pi,\text{max}}^2} \right|,$$

remains small. Here, Eq. (6) is assumed to be working to just leading nonanalytic order, i.e. fully renormalised to $O(m_\pi^3)$. The $m_\pi^4$ term is introduced in the fit to reduce the sensitivity to the ultra-violet cutoff. It is desired that this term only contribute of the order 1% of the total result — corresponding to roughly 10% of the $m_\pi^2$ term, suggesting an initial test scale of $R < 0.1$.

4. Rho-meson in quenched-QCD

Quenched QCD offers a useful framework to test methods and techniques where the cost of dynamical simulations remain too computationally expensive. Further, in the case of the rho-meson, neglecting the dynamical sea quarks removes the decay channel of the rho. Thereby, one does not have to deal with complications arising through extrapolating through the 2-$\pi$ decay threshold.

On the assumption that disconnected contributions in the $\omega$ propagator are negligible, the only leading one-loop diagrams contributing to the $\rho$ mass are those associated with the flavor-singlet $\eta'$. The $\eta'$ only appears in the low-energy EFT as a result of quenching, whereby it remains degenerate with the Goldstone pion.

Our expansion for the rho-meson mass in quenched QCD is summarized as

$$m_\rho = \sqrt{\hat{m}_\rho^2 + \Sigma},$$

where the residual, unrenormalised expansion is defined by

$$\hat{m}_\rho = a_0 + a_2 m_\pi^2 + a_4 m_\pi^4,$$

and the chiral loop corrections are denoted by $\Sigma$. Further details will appear in a forthcoming manuscript.

5. Preliminary investigation

Here we wish to test the hypothesis to determine $\Lambda$ by constraining the quantity $R$, Eq. (8). To do so, we construct some test, or pseudo, data in order to perform
hypothesised fits. Firstly, we fit Eq. (9) to new, unpublished, quenched, overlap results of the Kentucky Group\cite{11} using a dipole regulator of scale $\Lambda = 0.8\,\text{GeV}$. This input simply enables a scale to be set for the test data. A more detailed study would investigate alternative constraints at this initial step.

With this fit we construct a dense set of pseudo-data between the physical pion mass and some upper value $m_{\pi,\text{max}}$. We then refit this pseudo-data with alternative regularization scales and assess the convergence criteria. The convergence test is plotted for varying $\Lambda$ for a test curve which using an upper $m_{\pi,\text{max}} = 0.3\,\text{GeV}$.

![Convergence criteria](image)

**Fig. 1.** Convergence criteria, $R$, plotted against regularization scale.

The first indication of support for the procedure is the failure to produce a converging series if the regularisation scale is chosen as too small. This reflects the fact that the analytic expansion cannot describe the analyticities of the data. The existence of a lower bound as determined by this criteria is then a signature that the chiral nonanalyticities are present in the data.

There is a window of $\Lambda$ values where the series is satisfactorily converging to meet the demands of 1% precision. This happens to be in the vicinity of the originally input scale of $\Lambda = 0.8\,\text{GeV}$ and success of the fit should come as no surprise. From our initial tests using pseudo-data constrained using $\Lambda = 2.0\,\text{GeV}$, this convergence criteria reflects this and chooses an optimal regulator in this vicinity. Thereby, with good enough data, the intrinsic preferred scale can reveal itself through this criteria.

The curve indicates that should a large regularisation scale be attempted to fit our pseudo-data, the residual series is not so well converging. Even here, with an upper pion mass of 300 MeV, truncation of the residual series suggests a $(0.25)^2$ or 6% uncertainty.

Another desirable feature of this procedure is that as $m_{\pi,\text{max}}$ is reduced, the
sensitivity to $\Lambda$ must disappear through the additional factor of $m_{r,\text{max}}^2$ in $\mathcal{R}$. Thereby, within the PCR the lower bound will remain, yet no significant upper bound will persist.

On face value, this looks like a reasonable procedure to allow lattice data itself to make an *ab initio* determination of an optimal FRR scale. A weakness is apparent in considering the renormalisation of the residual series. For this argument, considering just the renormalisation of the single-hairpin graph. The Taylor expansion of this diagram, up to normalisation, can be expressed as

$$b_0 + b_2 m_{r}^2 + m_{r}^3 + b_4 m_{r}^4 + \ldots$$  \hspace{1cm} (11)

The leading term $b_0$ behaves as $\Lambda^3$, while $b_2 \sim \Lambda$ and $b_4 \sim \Lambda^{-1}$. Since for any fit the renormalised parameters are essentially stabilised by the data, this indicates that for very large $\Lambda$ the residual series coefficients will behave as $a_{i(0,2,4)} \sim \Lambda^{3,1,-1}$. Thereby, $a_2$ will diverge and $a_4$ saturate to a constant, and consequently our described convergence criteria $\mathcal{R}$ will approach zero, regardless of the data.

While a promising approach, it appears that the criteria described by Eq. (8) does not lead to a comprehensive test of convergence. A potential modification could be to normalise the $a_4 m_{r,\text{max}}^4$ term to a renormalised quantity, such as the rho mass. We anticipate further studies in this direction will lead to a reliable scale determination procedure that will facilitate *ab initio* studies with FRR for all $\Lambda$.

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