1 Introduction

In a series of papers, Mweene has developed a generalized description of angular momentum which contains the standard results in a certain limit. He has thereby obtained new generalized expressions for the operators and eigenvectors for spin 1/2[1-4], spin 1[5], spin 3/2[6], spin 2[7] and spin 5/2[8]. Applying this approach to angular momentum addition, he has shown how the standard results for various states that arise from combining two values of angular momentum come about from a consideration of probability amplitudes for measurements on these systems. This had led to generalized results for angular momentum addition which also reduce to the standard results in an appropriate limit[9-10].

In a further development of the approach, Mweene has shown that the usual spherical harmonics are just special forms of more generalized quantities and he has obtained the generalized spherical harmonics for the case $l = 1[11]$. In this paper, we give the generalized spherical harmonics for $l = 2$.

This paper is organized as follows. This Introduction is followed in Section 2 by a brief review of the theory underlying the work. Section 3 contains the derivation of the generalized spherical harmonics and their probability amplitudes. Section 4 is a discussion of some of the properties of these quantities. The Discussion and Conclusion in Section 5 closes the paper.

2 Theoretical Background

This work is inspired by the interpretation of quantum mechanics due to Landé[12-15]. According to Landé, if a quantum system possesses three sets of observables $A$, $B$ and $C$ with respective eigenvalue spectra $A_1$, $A_2$, ..., $A_N$, $B_1$, $B_2$, ..., $B_N$ and $C_1$, $C_2$, ..., $C_N$ - where $N$ is the multiplicity of each spectrum, which is necessarily the same for each observable[13] - then three sets of probability amplitudes can be defined for the system. One set, which is denoted by $\psi(A_i; C_j)$, relates to measurement of the observable $C$ if the system is in a state corresponding to eigenvalues of the observable $A$: thus $|\psi(A_i; C_j)|^2$ gives the probability for obtaining the value $C_j$ if the initial state corresponds to the eigenvalue $A_i$ of the observable $A$. The second set $\chi(A_i; B_j)$ relates to measurement of $B$ when the system is in a state corresponding to the eigenvalue $A$. Finally, the set $\phi(B_i; C_j)$ describes measure-
ments of $C$ when the initial state belongs to the observable $B$. Since the three sets of probability amplitudes belong to one system, they are interdependent, and the law of interdependence is [12,13]

$$\psi(A_i; C_j) = \sum_j \chi(A_i; B_j)\phi(B_j; C_n)$$ (1)

Another aspect of Landé’s interpretation of quantum mechanics is that every eigenfunction or wave function of a quantum system is first and foremost a probability amplitude and that every such probability amplitude connects two well-defined states - one corresponding to the state in which the system is before a measurement, and the other to the state that comes about as a result of the measurement [12,13]. It is therefore always possible to identify an initial and a final state for any wave function or eigenfunction.

Mweene has argued that for an eigenfunction resulting from solution of a differential eigenvalue equation, the initial state corresponds to the eigenvalue while the final state corresponds to the eigenvalue defined by the continuous variable in terms of which the differential operator is defined [16]. For the Schrödinger equation, the eigenfunctions $\psi_{E_i}(x)$ should really be written as $\psi(E_i; x)$ to emphasize that the initial state in the probability amplitude corresponds to the eigenvalue $E_i$ while the final state corresponds to the eigenvalue $x$. Another example comes from the solution of Legendre’s equation. The spherical harmonics $Y_{lm}(\theta, \varphi)$ should really be written as $Y(l, m; \theta, \varphi)$ to emphasize that in this case the initial state is defined by the eigenvalues $m\hbar$ and $l(l + 1)\hbar^2$ while the final state corresponds to the angular position $(\theta, \varphi)$. Since $m\hbar$ is an angular momentum projection, it must be defined with respect to some axis. Owing to the absence of another set of angles in the expressions of the spherical harmonics which could define this direction, it must be the $z$ axis [11]. But since an axis of quantization can be chosen arbitrarily, it is possible to define spherical harmonics with respect to any other direction as the axis of initial quantization. The functions resulting from this are the generalized spherical harmonics and have already been worked out for the case $l = 1$. In this work, we obtain them for $l = 2$. 
3 Generalized Spherical Harmonics

3.1 Probability Amplitudes

The generalized spherical harmonics connect states of angular momentum projection in the arbitrary direction \( \hat{a} \) defined by the polar angles \((\theta', \varphi')\) to states of the angular position \((\theta, \varphi)\). We denote them by \( Y(l, m(\hat{a}); \theta, \varphi) \). To derive them, we use the probability addition law Eq. (1). We start off by writing

\[
Y(l, m(\hat{a}); \theta, \varphi) = \sum_j \chi(l, m(\hat{a}); B_j) \phi(B_j; \theta, \varphi) \tag{2}
\]

If we choose the observable \( B \) carefully, we should find that both the probability amplitudes \( \chi(l, m(\hat{a}); B_j) \) and \( \phi(B_j; \theta, \varphi) \) are known. If \( B \) is chosen to be the spin projection with respect to the \( z \) direction, it is found that

\[
\phi(B_j; \theta, \varphi) = Y(l, m(\hat{k}); \theta, \varphi) \tag{3}
\]

are the standard spherical harmonics, while \( \chi(l, m_i(\hat{a}); l, m_f(\hat{k})) \) are just spin probability amplitudes connecting states such that the initial one corresponds to the spin projection being \( m_i \hbar \) in the direction \( \hat{a} \) while the final state corresponds to the spin projection being \( m_f \hbar \) along the \( z \) axis. These have already been worked out\[7\] and are as given below. If the initial spin projection in the direction \( \hat{a} \) is \( 2 \hbar \), these probability amplitudes are

\[
\chi(2, 2(\hat{a}); 2, 2(\hat{k})) = \cos^4 \frac{\theta'}{2} e^{-2i\varphi'} \tag{4}
\]

\[
\chi(2, 2(\hat{a}); 2, 1(\hat{k})) = 2 \sin \frac{\theta'}{2} \cos^3 \frac{\theta'}{2} e^{-i\varphi'} \tag{5}
\]

\[
\chi(2, 2(\hat{a}); 2, 0(\hat{k})) = \sqrt{6} \sin^2 \frac{\theta'}{2} \cos^2 \frac{\theta'}{2} \tag{6}
\]

\[
\chi(2, 2(\hat{a}); 2, (-1)(\hat{k})) = 2 \sin^3 \frac{\theta'}{2} \cos \frac{\theta'}{2} e^{i\varphi'} \tag{7}
\]

and

\[
\chi(2, 2(\hat{a}); 2, (-2)(\hat{k})) = \sin^4 \frac{\theta'}{2} e^{2i\varphi'} \tag{8}
\]

If the initial spin projection in the direction \( \hat{a} \) is \( \hbar \), the probability amplitudes are

\[
\chi(2, 1(\hat{a}); 2, 2(\hat{k})) = 2 \sin \frac{\theta'}{2} \cos^3 \frac{\theta'}{2} e^{-2i\varphi'} \tag{9}
\]
\[ \chi(2, 1^\hat{a}; 2, 1^\hat{k}) = -(3 \sin^2 \frac{\theta'}{2} - \cos^2 \frac{\theta'}{2}) \cos^2 \frac{\theta'}{2} e^{-i\phi'} \] (10)

\[ \chi(2, 1^\hat{a}; 2, 0^\hat{k}) = -\sqrt{6} \cos \frac{\theta'}{2} \sin \frac{\theta'}{2} \cos \theta' \] (11)

\[ \chi(2, 1^\hat{a}; 2, (-1)^\hat{k}) = (3 \cos^2 \frac{\theta'}{2} - \sin^2 \frac{\theta'}{2}) \sin^2 \frac{\theta'}{2} e^{i\phi'} \] (12)

and

\[ \chi(2, 1^\hat{a}; 2, (-2)^\hat{k}) = -2 \sin^3 \frac{\theta'}{2} \cos \frac{\theta'}{2} e^{2i\phi'} \] (13)

If the initial spin projection in the direction \( \hat{a} \) is 0, the probability amplitudes are

\[ \chi(2, 0^\hat{a}; 2, 2^\hat{k}) = \sqrt{6} \sin^3 \frac{\theta'}{2} \cos^2 \frac{\theta'}{2} e^{-2i\phi'} \] (14)

\[ \chi(2, 0^\hat{a}; 2, 1^\hat{k}) = -\sqrt{6} \sin \frac{\theta'}{2} \cos \frac{\theta'}{2} \cos \theta' e^{-i\phi'} \] (15)

\[ \chi(2, 0^\hat{a}; 2, 0^\hat{k}) = \frac{1}{2} (2 \cos^2 \theta' - \sin^2 \theta') \] (16)

\[ \chi(2, 0^\hat{a}; 2, (-1)^\hat{k}) = \sqrt{6} \sin \frac{\theta'}{2} \cos \frac{\theta'}{2} \cos \theta' e^{i\phi'} \] (17)

and

\[ \chi(2, 0^\hat{a}; 2, (-2)^\hat{k}) = \sqrt{6} \sin^3 \frac{\theta'}{2} \cos^2 \frac{\theta'}{2} e^{2i\phi'} \] (18)

If the initial spin projection in the direction \( \hat{a} \) is \(-\hat{h}\), the probability amplitudes are

\[ \chi(2, (-1)^\hat{a}; 2, 2^\hat{k}) = 2 \cos \frac{\theta'}{2} \sin^3 \frac{\theta'}{2} e^{-2i\phi'} \] (19)

\[ \chi(2, (-1)^\hat{a}; 2, 1^\hat{k}) = -(3 \cos^2 \frac{\theta'}{2} - \sin^2 \frac{\theta'}{2}) \sin^2 \frac{\theta'}{2} e^{-i\phi'} \] (20)

\[ \chi(2, (-1)^\hat{a}; 2, 0^\hat{k}) = \sqrt{6} \sin \frac{\theta'}{2} \cos \frac{\theta'}{2} \cos \theta' \] (21)

\[ \chi(2, (-1)^\hat{a}; 2, (-1)^\hat{k}) = (3 \sin^2 \frac{\theta'}{2} - \cos^2 \frac{\theta'}{2}) \cos^2 \frac{\theta'}{2} e^{i\phi'} \] (22)

and

\[ \chi(2, (-1)^\hat{a}; 2, (-2)^\hat{k}) = -2 \sin \frac{\theta'}{2} \cos^3 \frac{\theta'}{2} e^{2i\phi'} \] (23)
Finally, if the initial spin projection in the direction $\hat{a}$ is $-2\hbar$, the probability amplitudes are

\begin{align}
\chi(2, (-2)\hat{a}; 2, 2\hat{k}) &= \sin^4 \frac{\theta'}{2} e^{-2i\varphi'} \quad (24) \\
\chi(2, (-2)\hat{a}; 2, 1\hat{k}) &= -2 \cos \frac{\theta'}{2} \sin^3 \frac{\theta'}{2} e^{-i\varphi'} \quad (25) \\
\chi(2, (-2)\hat{a}; 2, 0\hat{k}) &= \sqrt{6} \sin^2 \frac{\theta'}{2} \cos^2 \frac{\theta'}{2} \quad (26) \\
\chi(2, (-2)\hat{a}; 2, (-1)\hat{k}) &= -2 \sin \frac{\theta'}{2} \cos^3 \frac{\theta'}{2} e^{i\varphi'} \quad (27)
\end{align}

and

\begin{align}
\chi(2, (-2)\hat{a}; 2, (-2)\hat{k}) &= \cos^4 \frac{\theta'}{2} e^{2i\varphi'} \quad (28)
\end{align}

The ordinary spherical harmonics $Y_{2m}(\theta, \varphi) = Y(2, m\hat{k}; \theta, \varphi)$ for $l = 2$ are

\begin{align}
Y(2, 2\hat{k}; \theta, \varphi) &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\varphi} \quad (29) \\
Y(2, 1\hat{k}; \theta, \varphi) &= -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\varphi} \quad (30) \\
Y(2, 0\hat{k}; \theta, \varphi) &= \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) \quad (31) \\
Y(2, (-1)\hat{k}; \theta, \varphi) &= \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{-i\varphi} \quad (32) \\
Y(2, (-2)\hat{k}; \theta, \varphi) &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{-2i\varphi} \quad (33)
\end{align}

Using Eq. (2), the generalized spherical harmonics for $l = 2$ are found to be
\[ Y(2, 2^{(a)}; \theta, \varphi) = \sqrt{\frac{15}{32\pi}} \{ \sin^2 \theta (\cos^2 \frac{\theta'}{2} e^{2i(\varphi - \varphi')} + \sin^2 \frac{\theta'}{2} e^{-2i(\varphi - \varphi')}) + \sin 2\theta \sin \theta' (-\cos^2 \frac{\theta'}{2} e^{i(\varphi - \varphi')} + \sin^2 \frac{\theta'}{2} e^{-i(\varphi - \varphi')} ) + \frac{1}{2} \sin^2 \theta' (3 \cos^2 \theta - 1) \} \] (34)

\[ Y(2, 1^{(a)}; \theta, \varphi) = \sqrt{\frac{15}{32\pi}} \{ \sin \theta' \sin^2 \theta (\cos^2 \frac{\theta'}{2} e^{2i(\varphi - \varphi')} - \sin^2 \frac{\theta'}{2} e^{-2i(\varphi - \varphi')}) - \sin 2\theta [(3 \sin^2 \frac{\theta'}{2} - \cos^2 \frac{\theta'}{2}) \cos^2 \frac{\theta'}{2} e^{i(\varphi - \varphi')} + (3 \cos^2 \frac{\theta'}{2} - \sin^2 \frac{\theta'}{2}) \sin^2 \frac{\theta'}{2} e^{-i(\varphi - \varphi')} ] - \frac{1}{2} \sin 2\theta' (3 \cos^2 \theta - 1) \} \] (35)

\[ Y(2, 0^{(a)}; \theta, \varphi) = \sqrt{\frac{45}{256\pi}} \{ \sin^2 \theta' \sin^2 \theta (e^{2i(\varphi - \varphi')} + e^{-2i(\varphi - \varphi')}) + \sin 2\theta \sin 2\theta' (e^{i(\varphi - \varphi')} + e^{-i(\varphi - \varphi')} ) + \frac{2}{3} (3 \cos^2 \theta - 1)(2 \cos^2 \theta' - \sin^2 \theta') \} \] (36)

\[ Y(2, (-1)^{(a)}; \theta, \varphi) = \sqrt{\frac{15}{32\pi}} \{ \sin \theta' \sin^2 \theta (\sin^2 \frac{\theta'}{2} e^{2i(\varphi - \varphi')} - \cos^2 \frac{\theta'}{2} e^{-2i(\varphi - \varphi')}) + \sin 2\theta [(3 \cos^2 \frac{\theta'}{2} - \sin^2 \frac{\theta'}{2}) \sin^2 \frac{\theta'}{2} e^{i(\varphi - \varphi')} + (3 \sin^2 \frac{\theta'}{2} - \cos^2 \frac{\theta'}{2}) \cos^2 \frac{\theta'}{2} e^{-i(\varphi - \varphi')} ] + \frac{1}{2} \sin 2\theta' (3 \cos^2 \theta - 1) \} \] (37)

\[ Y(2, (-2)^{(a)}; \theta, \varphi) = \sqrt{\frac{15}{32\pi}} \{ \sin^2 \theta (\sin^4 \frac{\theta'}{2} e^{2i(\varphi - \varphi')} + \cos^4 \frac{\theta'}{2} e^{-2i(\varphi - \varphi')} ) \]
\[ + \sin 2\theta \sin \theta' (\sin^2 \frac{\theta'}{2} e^{i(\varphi - \varphi')} - \cos^2 \frac{\theta'}{2} e^{-i(\varphi - \varphi')}) + \frac{1}{2} \sin^2 \theta' (3 \cos^2 \theta - 1) \] 

(38)

### 3.2 Probability Amplitudes for the \( x' \) Direction

The results we have presented refer to the direction \( \hat{a} \) as the direction of initial quantization. We may think of the vector \( \hat{a} \) as defining a new \( z' \) axis, which we denote by \( z' \), since in the limit \( \theta' = \varphi' = 0 \), the results corresponding to it reduce to those for the \( z \) axis. This \( z' \) axis corresponds to a new coordinate system in which the unit vector in the \( x' \) direction is \( \hat{u} \) and that in the \( y' \) direction is \( \hat{v} \) [11]. From the results for the \( \hat{a} \) or \( z' \) axis, we can obtain the probability amplitudes and probabilities densities for the \( x' \) direction by applying the transformation \( \theta' \rightarrow \theta' - \pi/2 \) to them [3,5]. We are justified in associating the results so obtained with the \( x' \) axis since in the limit \( \theta' = \varphi' = 0 \), they reduce to those for the \( x \) direction. When we make these argument changes, \( \hat{a} \) becomes \( \hat{u} \). Applying this prescription to the generalized spherical harmonics, we obtain the results

\[
Y(2, 2^{(\hat{u})}; \theta, \varphi) = \sqrt{\frac{15}{128\pi}} \left\{ \sin^2 \theta [(1 + \sin \theta')^2 e^{2i(\varphi - \varphi')} + (1 - \sin \theta')^2 e^{-2i(\varphi - \varphi')}] + \sin 2\theta \cos \theta' [(1 + \sin \theta') e^{i(\varphi - \varphi')} - (1 - \sin \theta') e^{-i(\varphi - \varphi')}] + \cos^2 \theta' (3 \cos^2 \theta - 1) \right\}
\]

(39)

\[
Y(2, 1^{(\hat{u})}; \theta, \varphi) = \sqrt{\frac{15}{128\pi}} \left\{ -\sin^2 \theta \cos \theta' [(1 + \sin \theta') e^{2i(\varphi - \varphi')} - (1 - \sin \theta') e^{-2i(\varphi - \varphi')}] - \sin 2\theta [(1 - 2 \sin \theta' \cos \theta') e^{i(\varphi - \varphi')} + (1 + 2 \sin \theta' \cos \theta') e^{-i(\varphi - \varphi')}] + \sin 2\theta' (3 \cos^2 \theta - 1) \right\}
\]

(40)
\[- \sin 2\theta \sin 2\theta'[e^{i(\varphi - \varphi')} + e^{-i(\varphi - \varphi')}]
\] + \frac{2}{3}(2 \sin^2 \theta' - \cos^2 \theta')(3 \cos^2 \theta - 1) \right \}

Y(2, (-1)\hat{\mathbf{a}}; \theta, \varphi) = \sqrt{\frac{15}{128\pi}} \{ - \sin^2 \theta \cos \theta'[1 - \sin \theta']e^{2i(\varphi - \varphi')}
- (1 + \sin \theta')e^{-2i(\varphi - \varphi')}
+ \sin 2\theta[(1 + 2 \sin \theta')(1 - \sin \theta')e^{i(\varphi - \varphi')}
+ (1 - 2 \sin \theta')(1 + \sin \theta')e^{-i(\varphi - \varphi')}
- \sin 2\theta'(3 \cos^2 \theta - 1) \right \}

Y(2, (-2)\hat{\mathbf{a}}; \theta, \varphi) = \sqrt{\frac{15}{128\pi}} \left\{ \frac{1}{2} \sin^2 \theta[(1 - \sin \theta')^2 e^{2i(\varphi - \varphi')}
+ (1 + \sin \theta')^2 e^{-2i(\varphi - \varphi')}
- \sin 2\theta \cos \theta'(1 - \sin \theta')e^{i(\varphi - \varphi')}
- (1 + \sin \theta')e^{-i(\varphi - \varphi')} + \cos^2 \theta'(3 \cos^2 \theta - 1) \right \}

3.3 Probability Amplitudes for the \(y'\) Direction

The prescription for obtaining the probability amplitudes and probability densities corresponding to \(y'\) is to set \(\theta' = \pi/2\), \(\varphi' \rightarrow \varphi' - \pi/2\) in the expressions corresponding to the \(z'\) direction. As well as transforming the unit vector \(\hat{\mathbf{a}}\) to the unit vector \(\hat{\mathbf{v}}\), this yields the probability amplitudes:

Y(2, 2\hat{\mathbf{v}}; \theta, \varphi) = - \sqrt{\frac{15}{128\pi}} \left\{ \frac{1}{2} \sin^2 \theta( e^{2i(\varphi - \varphi')} + e^{-2i(\varphi - \varphi')})
+ i \sin 2\theta [e^{i(\varphi - \varphi')} + e^{-i(\varphi - \varphi')} - (3 \cos^2 \theta - 1) \right \}

Y(2, 1\hat{\mathbf{v}}; \theta, \varphi) = \sqrt{\frac{15}{128\pi}} \{ \sin^2 \theta(- e^{2i(\varphi - \varphi')} + e^{-2i(\varphi - \varphi')})
+ i \sin 2\theta [e^{i(\varphi - \varphi')} - e^{-i(\varphi - \varphi')}] \right \}
\[ Y(2,0^2; \theta, \varphi) = -\sqrt{\frac{45}{256\pi}} \{ \sin^2 \theta (e^{2i(\varphi-\varphi')} + e^{-2i(\varphi-\varphi')}) + \frac{2}{3} (3 \cos^2 \theta - 1) \} \] (46)

\[ Y(2,-1^2; \theta, \varphi) = \sqrt{\frac{15}{128\pi}} \{ \sin^2 \theta (-e^{-2i(\varphi-\varphi')} + e^{2i(\varphi-\varphi')}) \\
+ i \sin 2\theta [e^{i(\varphi-\varphi')} - e^{-i(\varphi-\varphi')} \} \} \] (47)

\[ Y(2,-2^2; \theta, \varphi) = \sqrt{\frac{15}{128\pi}} \{ -\frac{1}{2} \sin^2 \theta (e^{2i(\varphi-\varphi')} + e^{-2i(\varphi-\varphi')}) \\
+ i \sin 2\theta [e^{i(\varphi-\varphi')} + e^{-i(\varphi-\varphi')} + 3 \cos^2 \theta - 1] \} \] (48)

We emphasize that the unit vectors \( \hat{u}, \hat{v} \) and \( \hat{a} \) define a system of mutually orthogonal coordinate axes.

## 4 General Properties of the Generalized Spherical harmonics

The generalized quantities presented here reduce to the standard quantities in the limit \( \theta' = \varphi' = 0 \), which corresponds to the arbitrary vector \( \hat{a} \) pointing in the direction of the \( z \) axis. Thus, in this limit, we get

\[ Y(2, m^{(a)}; \theta, \varphi) \rightarrow Y_{2m}(\theta, \varphi) \] (49)

A property of special interest with regard to the ordinary spherical harmonics is their behaviour under the parity operation \( r \rightarrow -r \), a reflection in the origin. Under this operation, the spherical polar coordinates \( (r, \theta, \varphi) \) transform thus: \( r \rightarrow r, \theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi \). Thus if \( \rho \) is the parity operator defined by

\[ \rho \Psi(r) = \Psi(-r). \] (50)
then

\[ \rho Y(l, m\hat{k}; \theta, \varphi) = Y(l, m\hat{k}; \pi - \theta, \varphi + \pi) \quad (51) \]

As is well-known however,

\[ Y(l, m\hat{k}; \pi - \theta, \varphi + \pi) = (-1)^l Y(l, m\hat{k}; \theta, \varphi) \quad (52) \]

so that \( Y(l, m\hat{k}; \theta, \varphi) \) has even parity if \( l \) is even and odd parity if \( l \) is odd.

The generalized spherical harmonics have the form

\[ Y(l, m\hat{a}; \theta, \varphi) = \sum_j c_j Y(l, m_j\hat{k}; \theta, \varphi) \quad (53) \]

where \( c_j = \chi(l, m_i\hat{a}; l, m_f\hat{k}) \) is a constant with respect to the angles \((\theta, \varphi)\). Hence,

\[ \rho Y(l, m\hat{a}; \theta, \varphi) = \sum_j c_j \rho Y(l, m_j\hat{k}; \theta, \varphi) = (-1)^l Y(l, m\hat{a}; \theta, \varphi) \quad (54) \]

Thus, the generalized spherical harmonics have the same parity as the corresponding standard spherical harmonics.

The generalized spherical harmonics for value of \( l \) can be shown to be orthonormal:

\[ \int \int Y^*(l, m\hat{a}; \theta, \varphi) Y(l, m\hat{a}; \theta, \varphi) d\Omega = \delta_{m'm} \quad (55) \]

Thus, since

\[ Y(l, m\hat{a}; \theta, \varphi) = \sum_j \chi(l, m; l, m_j\hat{k}) Y(m_j\hat{k}; \theta, \varphi) \quad (56) \]

and

\[ Y^*(l, m\hat{a}; \theta, \varphi) = \sum_j \chi^*(l, m\hat{a}; l, m_j\hat{k}) Y^*(m_j\hat{k}; \theta, \varphi) \quad (57) \]

the overlap integral is

\[ I = \int \int \sum_j \chi^*(l, m\hat{a}; l, m_j\hat{k}) Y^*(l, m_j\hat{k}; \theta, \varphi) \]

\[ \times \sum_j \chi(l, m\hat{a}; l, m_j\hat{k}) Y(l, m_j\hat{k}; \theta, \varphi) d\Omega \quad (58) \]
\[= \sum_{j^\prime} \sum_j \chi^*(l, m^{(\hat{a})}; l, m^{(\hat{k})}_j) \chi(l, m^{(\hat{a})}; l, m^{(\hat{k})}_j) \times \int\int Y^*(l, m^{(\hat{k})}_j; \theta, \varphi) Y(l, m^{(\hat{k})}_j; \theta, \varphi) d\Omega \]  
\[= \sum_{j^\prime} \sum_j \chi^*(l, m^{(\hat{a})}; l, m^{(\hat{k})}_j) \chi(l, m^{(\hat{a})}; l, m^{(\hat{k})}_j) \delta_{m_{j^\prime} m_{j'}} \]  
\[= \delta_{m^\prime m} \] (59)

In the proof, we have used the result

\[\sum_j \chi^*(l, m^{(\hat{a})}; l, m^{(\hat{k})}_j) \chi(l, m; l, m^{(\hat{k})}_j) = \delta_{m^\prime m} \] (60)

which is just the orthonormality relation for the spin probability amplitudes.

5 Discussion and Conclusion

This work has extended the derivation of the new generalized spherical harmonics to the case \(l = 2\). The expressions for the functions have been derived, as well as the corresponding probability densities for the \(z'\) direction. By means of simple transformations the corresponding expressions for the \(x'\) and \(y'\) directions have been obtained.

Now, for the case \(l = 1\), it has been shown that the generalized spherical harmonics satisfy the eigenvalue equation[11]

\[L_{\hat{a}} Y(1, m^{(\hat{a})}; \theta, \varphi) = m \hbar Y(1, m^{(\hat{a})}; \theta, \varphi) \] (62)

where

\[L_{\hat{a}} = i \hbar \{ \sin \theta' \sin(\varphi - \varphi') \frac{\partial}{\partial \theta} + [\sin \theta' \cot \theta \cos(\varphi - \varphi') - \cos \theta' \frac{\partial}{\partial \varphi} \} \] (63)

We note that \(L_{\hat{a}}\) can also be written as \(L_{z'}\) since as argued in the section on probability amplitudes and probability densities, it is convenient to think of the vector \(\hat{a}\) as defining a new \(z\) direction, denoted by \(z'\).
It is expected that all generalized spherical harmonics satisfy the eigen-value equation, Eq. (62). This is tedious to prove in practice, and has not been done for the present case \( l = 2 \). This will be tackled in the near future, since it is an important part of the proof of the correctness of the philosophy underlying this work.

6 References

1. Mweene H. V., "Derivation of Spin Vectors and Operators From First Principles", quant-ph/9905012
2. Mweene H. V., "Generalized Spin-1/2 Operators and Their Eigenvectors", quant-ph/9906002
3. Mweene H. V., "Alternative Forms of Generalized Vectors and Operators for Spin 1/2", quant-ph/9907031
4. Mweene H. V., "Spin Description and Calculations in the Landé Interpretation of Quantum Mechanics", quant-ph/9907033
5. Mweene H. V., "Vectors and Operators for Spin 1 Derived From First Principles", quant-ph/9906043
6. Mweene H. V., Unposted results on spin 3/2 systems.
7. Mweene H. V., "Generalized Probability Amplitudes for Spin Projection Measurements on Spin 2 Systems", quant-ph/0502005
8. Mweene H. V., Unposted results on spin 5/2 systems.
9. Mweene H. V., "New Treatment of Systems of Compounded Angular Momentum", quant-ph/9907082
10. Mweene H. V., "Derivation of Standard Treatment of Spin Addition From Probability Amplitudes", quant-ph/0003056
11. Mweene H. V., "Generalized Spherical Harmonics", quant-ph/0211135
12. Landé A., "From Dualism To Unity in Quantum Physics", Cambridge University Press, 1960.
13. Landé A., "New Foundations of Quantum Mechanics", Cambridge University Press, 1965.
14. Landé A., "Foundations of Quantum Theory," Yale University Press, 1955.
15. Landé A., "Quantum Mechanics in a New Key," Exposition Press, 1973.
16. Mweene H. V., "Proposed Differential Equation for Spin 1/2", Proc. Third Int. Workshop on Contemporary Problems in Mathematical Physics,
Cotonou 2003, ed. J. Govaerts, M. N. Hounkounnou and A. Z. Msezane (World Scientific, 2004), quant-ph/0411060.