Characterization of substitution invariant 3iet words

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Abstract

We study infinite words coding an orbit under an exchange of three intervals which have full complexity $C(n) = 2^n + 1$ for all $n \in \mathbb{N}$ (non-degenerate 3iet words). In terms of parameters of the interval exchange and the starting point of the orbit we characterize those 3iet words which are invariant under a primitive substitution. Thus, we generalize the result recently obtained for sturmian words.

1 Introduction

We study invariance under substitution of infinite words coding exchange of three intervals with permutation (321). These words, which are here called 3iet words, are one of the possible generalizations of sturmian words to a three-letter alphabet. Our main result provides necessary and sufficient conditions on the parameters of a 3iet word to be invariant under substitution.

A sturmian word $(u_n)_{n \in \mathbb{N}}$ over the alphabet $\{0, 1\}$ is defined as

$$u_n = \left\lfloor (n+1)\alpha + x_0 \right\rfloor - \left\lfloor n\alpha + x_0 \right\rfloor \quad \text{for all } n \in \mathbb{N},$$

or

$$u_n = \left\lceil (n+1)\alpha + x_0 \right\rceil - \left\lceil n\alpha + x_0 \right\rceil \quad \text{for all } n \in \mathbb{N},$$

where $\alpha \in (0, 1)$ is an irrational number called the slope, and $x_0 \in [0, 1)$ is called the intercept.

There are many various equivalent definitions of sturmian words, among others also as an infinite word coding an exchange of 2 intervals of length $\alpha$ and $1 - \alpha$. A direct generalization of this definition are infinite words coding exchange of $k$ intervals, as introduced by Stepin and Katok [12].

Definition 1.1. Let $\alpha_1, \ldots, \alpha_k$ be positive real numbers and let $\pi$ be a permutation over the set $\{1, 2, \ldots, k\}$. Denote $I = I_1 \cup I_2 \cup \cdots \cup I_k$, where $I_j := \left[\sum_{i<j} \alpha_i, \sum_{i \leq j} \alpha_i\right)$. Put $t_j := \sum_{\pi(i)<\pi(j)} \alpha_i - \sum_{i<j} \alpha_i$. The mapping $T : I \mapsto I$ given by the prescription

$$T(x) = x + t_j \quad \text{for } x \in I_j$$

will be called $k$-interval exchange transformation (k-iet) with permutation $\pi$ and parameters $\alpha_1, \ldots, \alpha_k$.

Keane [13] has studied in which case a k-iet satisfy the so-called minimality condition, i.e., when the orbit $\{T^n(x_0) \mid n \in \mathbb{Z}\}$ of every point $x_0 \in I$ is dense in $I$. It is easy to see that minimality condition can be satisfied only if the permutation $\pi$ is irreducible, i.e.,

$$\pi\{1, 2, \ldots, j\} \neq \{1, 2, \ldots, j\} \quad \text{for all } j < k.$$
Keane has also derived a sufficient condition for minimality: Denote \( \beta_j \) the left end-point of the interval \( I_j \), i.e., \( \beta_j = \sum_{i<j} \alpha_i \). If the orbits of points \( \beta_1, \ldots, \beta_k \) under the transformation \( T \) are infinite and disjoint, then \( T \) satisfies the minimality property. In the literature, this sufficient condition is known under the notation i.d.o.c. However, in general, i.d.o.c. is not a necessary condition for minimality property.

To the orbit of every point \( x_0 \in I \), one can naturally associate an infinite word \( u = (u_n)_{n \in \mathbb{Z}} \) in a \( k \)-letter alphabet \( \mathcal{A} = \{1,2,\ldots,k\} \). For \( n \in \mathbb{Z} \) put

\[
\begin{align*}
    u_n &= 1 & \text{if } T^n(x_0) \in I_1, \\
    &\vdots
\end{align*}
\]

Infinite words coding \( k \)-iet with i.d.o.c. are called here non-degenerate \( k \)-iet words. Non-degenerate \( k \)-iet words are studied in [10]. The authors give a combinatorial characterization of the language of infinite words which correspond to \( k \)-iet with permutation

\[
\pi(1) = k, \quad \pi(2) = k - 1, \quad \ldots, \quad \pi(k) = 1
\]  

or to permutations in some sense equivalent with it.

For \( k = 2 \), the only irreducible permutation is of the form \( (1) \). The minimality property for parameters \( \alpha_1, \alpha_2 \) means that they are linearly independent over \( \mathbb{Q} \). Infinite words coding 2iet with the minimality property are precisely the sturmian words.

In this paper we concentrate on infinite words coding exchange of 3 intervals under the permutation \( (1) \). The transformation which we study is thus given by a triple of positive parameters \( \alpha_1, \alpha_2, \alpha_3 \) and the prescription

\[
\tilde{T}(x) := \begin{cases} 
    x + \alpha_2 + \alpha_3 & \text{for } x \in [0, \alpha_1), \\
    x - \alpha_1 + \alpha_3 & \text{for } x \in [\alpha_1, \alpha_1 + \alpha_2), \\
    x - \alpha_1 - \alpha_2 & \text{for } x \in [\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3). 
\end{cases}
\]  

For such a transformation, the minimality property is equivalent to the following condition (as proved in [3]): numbers \( \alpha_1 + \alpha_2 \) and \( \alpha_2 + \alpha_3 \) are linearly independent over \( \mathbb{Q} \). It is known [11] that infinite words coding (2) are non-degenerate if and only if (2) satisfies the minimality property and

\[
\alpha_1 + \alpha_2 + \alpha_3 \notin (\alpha_1 + \alpha_2)\mathbb{Z} + (\alpha_2 + \alpha_3)\mathbb{Z}.
\]  

The central problem of this paper is the substitution invariance of given infinite words. For sturmian words this question was extensively studied; Chapter X. of [14] gives references to authors who gave some contributions to its solution. Complete answer to this question was first provided by Yasutomi [17], other proofs of the same result are given in [5, 6]. Crucial for stating this result is the notion of a Sturm number. The original definition of a Sturm number used continued fractions. In 1998, Allauzen [2] has provided a simple characterization of Sturm numbers:

\[
A \text{ quadratic irrational number } \alpha \text{ with conjugate } \alpha' \text{ is called a Sturm number if } \\
\alpha \in (0,1) \quad \text{and} \quad \alpha' \notin (0,1).
\]

\[\text{Let us mention that the question of expressing the minimality property in terms of parameters } \alpha_1, \ldots, \alpha_k \text{ has not been solved for general } k.\]
Theorem 1.2 (5). Let \( \alpha \) be an irrational number, \( \alpha \in (0, 1) \), \( x_0 \in [0, 1) \). The Sturmian word with slope \( \alpha \) and intercept \( x_0 \) is invariant under a substitution if and only if the following three conditions are satisfied:

(i) \( \alpha \) is a Sturm number,

(ii) \( x_0 \in \mathbb{Q}(\alpha) \),

(iii) \( \min(\alpha', 1 - \alpha') \leq x'_0 \leq \max(\alpha', 1 - \alpha') \), where \( x'_0 \) denotes the image of \( x_0 \) under the Galois automorphism of the quadratic field \( \mathbb{Q}(\alpha) \).

Let us mention that one can study also weaker property than substitution invariance; namely the substitutivity. For an infinite word \( u \) coding an exchange of \( k \) intervals, Boshernitzan and Carol [8] have shown that belonging of lengths of all intervals \( I_1, \ldots, I_k \) to the same quadratic field is a sufficient condition for substitutivity of \( u \). For \( k = 2 \) in [7], and for \( k = 3 \) in [1] the respective authors show that such condition is also necessary.

However, quadraticity of parameters is not sufficient for the property of substitution invariance. Already in [4] it is shown that substitution invariance of 3iet words implies that a certain parameter of the 3iet is a Sturm number, namely

\[ \varepsilon = \frac{\alpha_1 + \alpha_2}{\alpha_1 + 2\alpha_2 + \alpha_3} \]

The main result of this paper is given as Theorem 6.3, where a necessary and sufficient condition for substitution invariance is expressed using simple inequalities for other parameters of the 3iet word.

2 Basic notions of combinatorics on words

We will deal with infinite words over a finite alphabet, say \( \mathcal{A} = \{1, 2, \ldots, k\} \). We consider either right sided infinite words

\[ u = (u_n)_{n \in \mathbb{N}} = u_0 u_1 u_2 u_3 \cdots, \quad u_i \in \mathcal{A}, \]

or pointed bidirectional infinite words,

\[ u = (u_n)_{n \in \mathbb{Z}} = \cdots u_{-2} u_{-1} | u_0 u_1 u_2 u_3 \cdots, \quad u_i \in \mathcal{A}, \]

A finite word \( w = w_0 w_1 \cdots w_{n-1} \) of length \( |w| = n \) is a factor of an infinite word \( u = (u_n) \) if \( w = u_i u_{i+1} \cdots u_{i+n-1} \) for some \( i \).

The (factor) complexity of \( u = (u_n)_{n \in \mathbb{N}} \) is the function \( C : \mathbb{N} \mapsto \mathbb{N} \),

\[ C(n) := \# \{ u_i \cdots u_{i+n-1} | i \in \mathbb{N} \}, \]

analogously for \( u = (u_n)_{n \in \mathbb{Z}} \). Obviously, every infinite word satisfies \( 1 \leq C(n) \leq k^n \) for all \( n \in \mathbb{N} \). It is not difficult to show [15] that an infinite word \( u = (u_n)_{n \in \mathbb{N}} \) is eventually periodic if and only if there exists \( n_0 \) such that \( C(n_0) \leq n_0 \). Obviously, the aperiodic words of minimal complexity satisfy \( C(n) = n + 1 \) for all \( n \in \mathbb{N} \). Such infinite words are called Sturmian words. The definition of Sturmian words can be extended also to bidirectional infinite words \( (u_n)_{n \in \mathbb{Z}} \), requiring except of \( C(n) = n + 1 \) for all \( n \in \mathbb{N} \) also the irrationality of the densities of letters.
In our paper we study invariance of infinite words under substitution. A substitution is a mapping \( \varphi : \mathcal{A}^* \to \mathcal{A}^* \), where \( \mathcal{A}^* \) is the monoid of all finite words including the empty word, satisfying \( \varphi(vw) = \varphi(v)\varphi(w) \) for all \( v, w \in \mathcal{A}^* \). In fact, a substitution is a special case of a morphism \( \mathcal{A}^* \to \mathcal{B}^* \), where \( \mathcal{A} = \mathcal{B} \). Obviously, \( \varphi \) is uniquely determined, if defined on all the letters of the alphabet. A substitution \( \varphi \) is called primitive, if there exists \( n \in \mathbb{N} \) such that \( \varphi^n(a) \) contains \( b \) for all letters \( a, b \in \mathcal{A} \).

The action of \( \varphi \) can be naturally extended to infinite words. For a pointed bidirectional infinite word \( u = (u_n)_{n \in \mathbb{Z}} \) we in particular have

\[
\varphi(\cdots u_{-2}u_{-1}|u_0u_1u_2\cdots) = \cdots \varphi(u_{-2})\varphi(u_{-1})|\varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots
\]

An infinite word \( u \) is said to be a fixed point of \( \varphi \) (or invariant under \( \varphi \)), if \( \varphi(u) = u \).

### 3 Exchange of three intervals and cut-and-project sets

Our aim is to study substitution invariance of words coding an exchange of three intervals \( (2) \). The main tool is the fact that the orbit of an arbitrary point under this transformation can be geometrically represented by a so-called cut-and-project sequence.

**Definition 3.1.** Let \( \varepsilon, \eta \in \mathbb{R}, \varepsilon \neq -\eta, \varepsilon, \eta \text{ irrational}, \) and let \( \Omega = [c, c + l], c \in \mathbb{R}, l > 0 \). The set

\[
\Sigma_{\varepsilon, \eta}(\Omega) := \{ a + b\varepsilon \mid a, b \in \mathbb{Z}, \ a - b\varepsilon \in \Omega \}
\]

is called a cut-and-project set with parameters \( \varepsilon, \eta \) and acceptance window \( \Omega \).

The above definition is a very special case of a general cut-and-project set, introduced in \( [16] \). In the definition we have used an interval \( \Omega \), closed from the left and open from the right. One can also consider an interval \( \tilde{\Omega} = (\hat{c}, \hat{c} + \hat{l}] \). However, by doing this, we do not obtain anything new, since \( \Sigma_{\varepsilon, \eta}(\Omega) = -\Sigma_{\varepsilon, \eta}(\tilde{-\Omega}) \).

For simplicity of notation, we denote the additive group \( \{ a + b\varepsilon \mid a, b \in \mathbb{Z} \} = \mathbb{Z} + \varepsilon\mathbb{Z} =: \mathbb{Z}[\varepsilon] \) and analogously for \( \mathbb{Z}[\eta] \). The morphism of these groups \( x = a + b\eta \mapsto x^* = a - b\varepsilon \) will be called the star map. In this formalism, the cut-and-project set \( \Sigma_{\varepsilon, \eta}(\Omega) \) can be rewritten as

\[
\Sigma_{\varepsilon, \eta}(\Omega) = \{ x \in \mathbb{Z}[\eta] \mid x^* \in \Omega \}.
\]

The relation between the set \( \Sigma_{\varepsilon, \eta}(\Omega) \) and the exchange of 3 intervals is explained by the following theorem proved in \( [11] \).

**Theorem 3.2** \(([11])\). Let \( \Sigma_{\varepsilon, \eta}(\Omega) \) be defined by (4). Then there exist positive numbers \( \Delta_1, \Delta_2 \in \mathbb{Z}[\eta] \) and a strictly increasing sequence \( (s_n)_{n \in \mathbb{Z}} \) such that

1. \( \Sigma_{\varepsilon, \eta}(\Omega) = \{ s_n \mid n \in \mathbb{Z} \} \subset \mathbb{Z}[\eta] \).
2. \( \Delta_1 > 0, \Delta_2 < 0, \Delta_1^* - \Delta_2^* \geq l > \max(\Delta_1^*, -\Delta_2^*) \).
3. \( s_{n+1} - s_n \in \{ \Delta_1, \Delta_2, \Delta_1 + \Delta_2 \} \), for all \( n \in \mathbb{Z} \), and, moreover,

\[
s_{n+1} = \begin{cases} 
  s_n + \Delta_1 & \text{if } s_n^* \in [c, c + l - \Delta_1^*], \\
  s_n + \Delta_1 + \Delta_2 & \text{if } s_n^* \in [c + l - \Delta_1^*, c - \Delta_2^*], \\
  s_n + \Delta_2 & \text{if } s_n^* \in [c - \Delta_2^*, c + l]. 
\end{cases}
\]
4. Numbers $\Delta_1$ and $\Delta_2$ depend only on parameters $\varepsilon, \eta$ and the length $l$ of the interval $\Omega$. In particular, they do not depend on the position $c$ of $\Omega$ on the real line.

We see that the set $\{s_n^* | n \in \mathbb{Z}\}$ is an orbit under the 3iet with permutation $\pi = (321)$ and parameters $l - \Delta_1^* \Delta_1^* - \Delta_2^* - l$ and $l + \Delta_2^*$ (if $l < \Delta_1^* - \Delta_2^*$) and it is an orbit under the 2iet with permutation $\pi = (21)$ and parameters $l - \Delta_1^*$ and $l + \Delta_2^*$ (if $l = \Delta_1^* - \Delta_2^*$). Thus every cut-and-project sequence can be viewed as a geometric representation of an orbit of a point under exchange of two or three intervals.

The determination of $\Delta_1$, $\Delta_2$ is in general laborious; the values $\Delta_1$, $\Delta_2$ depend on the continued fraction expansions of parameters $\varepsilon$ of $\eta$, according to the length $l$ of the acceptance window $\Omega = [c, c + l]$.

In case that

$$\varepsilon \in (0, 1), \quad \eta > 0 \quad \text{and} \quad 1 \geq l > \max(1 - \varepsilon, \varepsilon),$$

one has

$$\Delta_1 = 1 + \eta \quad \text{and} \quad \Delta_2 = -\varepsilon,$$

i.e., the corresponding triple of shifts in the prescription of the exchange of intervals is $\Delta_1^* = 1 - \varepsilon$, $\Delta_1^* + \Delta_2^* - 1 = 2\varepsilon$, $\Delta_2^* = -\varepsilon$. In fact, without loss of generality, we can limit our consideration to cut-and-project sequences with parameters satisfying (5), since in [11] it is shown that every cut-and-project sequence is equal to $\mu \Sigma_{\varepsilon, \eta}(\Omega)$, where $\varepsilon$, $\eta$ and length $l$ of the interval $\Omega$ satisfy (5) and $\mu \in \mathbb{R}$. By that, we have shown how to interpret a cut-and-project set as an orbit under an exchange of 3 (or 2) intervals with the permutation (321) (or (21)).

On the other hand, let us show that every exchange of three intervals with permutation (321) can be represented geometrically using a cut-and-project scheme. First realize that studying the orbit of a point $x_0 \in I$ under the 3iet $\tilde{T}$ of (2), we can, without loss of generality, substitute $\tilde{T}$ by the transformation $T(x) = \frac{1}{\mu}(\mu(x - c)) + c$ for arbitrary $\mu, c \in \mathbb{R}, \mu \neq 0$, and instead of the orbit of $x_0$ under $\tilde{T}$ consider the orbit of the point $y_0 = c + \frac{a_3}{\mu}$ under the transformation $T$. In particular, putting $\mu = \alpha_1 + 2\alpha_2 + \alpha_3$ and $c = -x_0\mu^{-1}$, we have the orbit of $y_0 = 0$ under the mapping $T : [c, c + l] \mapsto [c, c + l]$

$$T(x) = \begin{cases} x + 1 - \varepsilon & \text{for } x \in [c, c + l - 1 + \varepsilon), \\
 x + 1 - 2\varepsilon & \text{for } x \in [c + l - 1 + \varepsilon, c + \varepsilon) \\
 x - \varepsilon & \text{for } x \in [c + \varepsilon, c + l), \end{cases}$$

(7)

where we have denoted by $\varepsilon$ and $l$ the new parameters

$$\varepsilon := \frac{\alpha_1 + \alpha_2}{\alpha_1 + 2\alpha_2 + \alpha_3} \quad \text{and} \quad l := \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_1 + 2\alpha_2 + \alpha_3}.$$  

(8)

Let us mention that under such parameters, the minimality property of the transformation $T$ in (7) is equivalent to the requirement that $\varepsilon$ be irrational.

For the above defined values of $\varepsilon, l, c$ and arbitrary irrational $\eta > 0$ put $\Omega = [c, c + l]$ and consider the cut-and-project set $\Sigma_{\varepsilon, \eta}(\Omega)$. Since $0 \in \Omega$, we have also $0 \in \Sigma_{\varepsilon, \eta}(\Omega)$. The strictly increasing sequence $(s_n)_{n \in \mathbb{N}}$ from Theorem 3.2 can be indexed in such a way that $s_0 = 0$. Since our parameters $\varepsilon, l, \eta$ satisfy (5) (and $l < 1$), the right neighbor $s_{n+1}$ of the point $s_n$ is given by the position of $s_n^*$ in the interval $[c, c + l)$, namely by the transformation $T(x)$. In particular, we have $s_{n+1} = T(s_n^*)$. Therefore the set

$$\{s_n^* | n \in \mathbb{N}\} = (\Sigma_{\varepsilon, \eta}(c, c + l))^* = \mathbb{Z}[\varepsilon] \cap \Omega$$

(9)
is the orbit of the point 0 under the transformation \( T \).

Note that we have decided to consider instead of an orbit of an arbitrary point under a 3iet \( \tilde{T} \) with the domain being an interval starting at 0, the orbit of 0 under the 3iet \( T \) given by (7), with parameters \( \varepsilon, l, c \) satisfying

\[
\varepsilon \in (0, 1), \quad 1 > l > \max(1 - \varepsilon, \varepsilon), \quad 0 \in [c, c + l).
\]

Let us summarize the advantages of such new notation:

- Points of the sequence \( (T^n(0))_{n \in \mathbb{Z}} \subset \mathbb{Z}[\varepsilon] \) which has a chaotic behavior in the interval \( [c, c + l) \) can be, using the star map \( * : \mathbb{Z}[\eta] \mapsto \mathbb{Z}[\varepsilon] \), represented by a strictly increasing sequence \( (s_n)_{n \in \mathbb{Z}} \) such that \( s_n^* = T^n(0) \) for all \( n \in \mathbb{Z} \).

- The orbit of 0 can be simply expressed as

\[
\{T^n(0) \mid n \in \mathbb{Z}\} = \mathbb{Z}[\varepsilon] \cap [c, c + l).
\]

For the orbit of an arbitrary point \( x_0 \in [c, c + l) \) under \( T \), one can write

\[
\{T^n(x_0) \mid n \in \mathbb{Z}\} = x_0 + (\mathbb{Z}[\varepsilon] \cap [c - x_0, c + l - x_0)) = (x_0 + \mathbb{Z}[\varepsilon]) \cap [c, c + l).
\]

Further advantages of the presented point of view on 3iets by cut-and-project sequences will be clear from the following section.

**Remark 3.3.** To conclude the section, let us stress that for the 3iet \( T \) the parameter \( \eta \) was chosen arbitrarily, except the requirement of irrationality and positiveness. Then adjacency of points \( x, y, x < y \), in the set \( \Sigma_{\varepsilon, \eta}(\Omega) \) indicates that their star map images \( x^*, y^* \) are consecutive iterations of \( T \), i.e., \( T(x^*) = y^* \). Choosing the parameter \( \eta < 0 \), we obtain again a cut-and-project set \( \Sigma_{\varepsilon, \eta}(\Omega) \) but with different \( \Delta_1, \Delta_2 \). Therefore the corresponding 3iet is different from \( T \). From the definition of a cut-and-project set, it can be easily shown that

\[
\Sigma_{\varepsilon, \eta}(\Omega) = \Sigma_{1 - \varepsilon, 1 - \eta}(\Omega).
\]

Therefore in case that \( \eta < -1 \), the corresponding cut-and-project set represents a 3iet, in which we interchange the lengths of the first and last intervals, i.e., the mapping \( T^{-1} \). In fact, the ‘dangerous’ choice for the irrational parameter \( \eta \) is \( \eta \in (-1, 0) \).

### 4 First return map

Let \( T : I \mapsto I \) be a \( k \)-interval exchange transformation with minimality property and let \( J \) be an interval \( J \subset I \), \( J \) closed from the left and open from the right, say \([c, c + l)\).

The minimality property of \( T \) ensures that for every \( z \in J \) there exists a positive integer \( i \in \mathbb{N} \) such that \( T^i(z) \in J \). The minimal such \( i \) is called the return time of \( z \) and denoted by \( r(z) \).

To every \( z \in J \) we associate a ‘return name’, i.e., a finite word \( w = v_0v_1 \cdots v_{r(z) - 1} \) in the alphabet \( \{1, \ldots, k\} \), whose length is equal to the return time of \( z \) and for all \( i, 0 \leq i < r(z) \) we have

\[
v_i = X \quad \text{if} \quad T^i(z) \in I_X.
\]
To the given subinterval $J$ of $I$, we define the map $T_J : J \mapsto J$ by the prescription

$$T_J(z) = T^r(z),$$

which is called the first return map.

Since for a fixed interval $J$ the return time $r(z)$ is bounded, there exist only finitely many return names. It is obvious, that points $z \in J$ with the same return name form an interval, and $J$ is thus a finite disjoint union of such subintervals, say $J_1, \ldots, J_p$. The boundary points of these intervals can be easily described by the notion of ancestor in $J$.

The minimality property of $T$ ensures that for every $y \in I$ there exists $z \in J$ such that $y \in \{z, T(z), \ldots, T^{r(z)-1}(z)\}$. Such $z$ is uniquely determined and we call it the ancestor of $y$ in the interval $J$. We denote $z = \text{anc}_J(y)$.

The boundary points of the intervals $J_1, \ldots, J_p$ are then exactly the following points:

- $\hat{c}$, $\hat{c} + \hat{l}$ (i.e., the boundary points of $J$ itself);
- $\text{anc}_J(\hat{c} + \hat{l})$;
- $\text{anc}_J(\alpha_1 + \alpha_2 + \cdots + \alpha_i)$ for $i = 1, 2, \ldots, k - 1$; 
- and the point $z \in J$ such that $T^r(z) = \hat{c}$.

This implies that for a $k$-iet the number of different return names is at most $k + 2$. It is obvious, that the first return map $T_J$ is again a $m$-iet for some $m \leq k + 2$. In fact, it is known that $m \leq k + 1$ (see [9], Chap. 5). For a 3iet which we study in this paper, we can say even more. The following theorem is a direct consequence of Theorem 3.2.

**Theorem 4.1.** Let $T : I \mapsto I$ be a 3iet with permutation (321) and satisfying minimality property, and let $J \subset I$ be an interval. Then the first return map $T_J$ is either a 3iet with permutation (321) or a 2iet with permutation (21).

**5 First return map and substitution invariance**

Let us now see how the notions of first return map, return time and return name are related to substitution invariance of words coding 3iet. We will focus on non-degenerate 3iet words. Let us mention that non-degeneracy in terms of parameters $\varepsilon, l$ of (11) means that $l \notin \mathbb{Z}[\varepsilon]$, cf. (3).

Consider a 3iet $T : [c, c+l) \mapsto [e, c+l)$ of (7) with parameters (9) and an interval $J \subset [c, c+l)$ such that $0 \in J$. Let $w_1, \ldots, w_p$ be all possible return names of points $z \in J$. Then the infinite word $u = (u_n)_{n \in \mathbb{Z}}$ coding 0 under the transformation $T$ can be written as a concatenation

$$u = \cdots w_{j_{-2}}w_{j_{-1}}w_{j_0}w_{j_1}w_{j_2} \cdots , \quad \text{with } j_i \in \{1, \ldots, p\}. \quad \text{(12)}$$

The starting letters of the blocks $w_{jm}$ correspond to positions $n$ in the infinite word $u$ if and only if $T^n(0) \in J$. More formally, we have

$$w_{jm}w_{jm+1}w_{jm+2} \cdots = u_nu_{n+1}u_{n+2}u_{n+3} \cdots \iff T^n(0) \in J.$$ 

Suppose we have an interval $J \subset I$, $0 \in J$ such that the first return map $T_J$ satisfies

1. $T_J$ is homothetic with $T$, i.e.,

$$T_J(x) = \nu T(\frac{x}{\nu}), \quad \text{for } x \in J \text{ and some } \nu \in (-1, 1),$$

which means that $T_J$ is an exchange of intervals $J_1 = \nu I_1$, $J_2 = \nu I_2$, and $J_3 = \nu I_3$.
P2. the set of return names defined by $J$ has three elements.

Then the sequence of indices $(j_m)_{m \in \mathbb{Z}}$ defining the ordering of finite words $w_1, w_2, w_3$ in the concatenation (12) equals to the infinite word $u$. In particular, it means that $u$ is invariant under the substitution

$$
1 \mapsto \varphi(1) = w_1,
2 \mapsto \varphi(2) = w_2,
3 \mapsto \varphi(3) = w_3.
$$

We stand therefore in front of the following questions: How to decide, for which $3$iet $J \subset I$ with properties P1. and P2. exists? What can be said in case that such $J$ does not exist?

In case that $u = (u_n)_{n \in \mathbb{Z}}$ is a non-degenerate $3$iet word coding the orbit of 0 under the transformation $T$ defined by (11), the second question is solved by the paper [4], as follows.

The existence of a substitution $\varphi$ over the alphabet $\{1, 2, 3\}$, under which the word $u$ is invariant, means that $u$ can be written as a concatenation of blocks $\varphi(1), \varphi(2), \varphi(3)$, i.e.,

$$
\begin{align*}
  u &= \cdots u_{-2}u_{-1}|u_0u_1u_2| & \cdots | \varphi(u_{-2})\varphi(u_{-1}) | \varphi(u_0)\varphi(u_1)\varphi(u_2)| \cdots.
\end{align*}
$$

(13)

In [4] one considers a non-degenerate $3$iet word $u$ invariant under a primitive substitution $\varphi$ and studies for $i = 1, 2, 3$ the set $E_{\varphi(i)}$ of points $T^n(0)$ such that the block $\varphi(i)$ starts at position $n$ in the concatenation (13). Formally,

$$
E_{\varphi(i)} = \{T^n(0) \mid \exists m \in \mathbb{Z}, \varphi(i)\varphi(u_m)\varphi(u_{m+1})| \cdots = u_nu_{n+1}u_{n+2}\cdots\}.
$$

As a result, several properties of a matrix of substitution $\varphi$ are described. Recall that for a substitution $\varphi$ over the alphabet $A = \{1, 2, \ldots, k\}$ one defines the substitution matrix $M_\varphi$ by

$$(M_\varphi)_{ij} = \text{number of letters } i \text{ in the word } \varphi(j), \quad 1 \leq i, j \leq k.$$  

Such matrix has obviously non-negative integer entries and if the substitution $\varphi$ is primitive, the matrix $M_\varphi$ is primitive as well, and therefore one can apply the Perron-Frobenius theorem.

We summarize several statements of [4] in the following theorem.

**Theorem 5.1** ([4]). Let $u = (u_n)_{n \in \mathbb{Z}}$ be a non-degenerate $3$iet word with parameters $\varepsilon, l, c$ satisfying (9). Let $\varphi$ be a primitive substitution such that $\varphi(u) = u$. Then

(i) $\varepsilon$ is a Sturm number, i.e., $\varepsilon$ is a quadratic irrational in $(0, 1)$ such that its algebraic conjugate $\varepsilon'$ satisfies $\varepsilon' \notin (0, 1)$;

(ii) the dominant eigenvalue $\Lambda$ of the matrix $M_\varphi$ of the substitution $\varphi$ is a quadratic unit in $\mathbb{Q}(\varepsilon)$;

(iii) the column vector $(1 - \varepsilon, 1 - 2\varepsilon, -\varepsilon)^T$ is a right eigenvector of $M_\varphi$ corresponding to the eigenvalue $\Lambda'$, i.e., to the algebraic conjugate of $\Lambda$;

(iv) parameters $c, l \in \mathbb{Q}(\varepsilon)$;

(v) $E_{\varphi(i)} = \Lambda'(I_i \cap \mathbb{Z}[\varepsilon])$ for $i = 1, 2, 3$. 

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The statement (v) in particular says that the existence of a substitution \( \varphi \) under which a non-degenerate 3iet word \( u \) is invariant forces existence of an interval \( J \subset I \) with properties P1. and P2. We have already explained that existence of an interval \( J \) with properties P1. and P2. forces substitution invariance. We have thus the following statement.

**Proposition 5.2.** Let \( u = (u_n)_{n \in \mathbb{Z}} \) be a non-degenerate 3iet word with parameters \( \varepsilon, l, c \) satisfying (9). Then there exists a primitive substitution \( \varphi \) under which \( u \) is invariant, if and only if there exists an interval \( J \subset I \) with properties P1. and P2.

Let us first derive two simple observations which complement results of [4].

**Lemma 5.3.** For \( \Lambda, \Lambda' \) and \( \varepsilon \) from Theorem 5.1 we have

\[
\Lambda \mathbb{Z}[\varepsilon] = \Lambda' \mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon].
\]

*Proof.* Statement (iii) of Theorem 5.1 implies

\[
M_{\varphi} \begin{pmatrix} 1 - \varepsilon \\ 1 - 2\varepsilon \\ -\varepsilon \end{pmatrix} = \Lambda' \begin{pmatrix} 1 - \varepsilon \\ 1 - 2\varepsilon \\ -\varepsilon \end{pmatrix}.
\]

Since \( M_{\varphi} \) is an integer matrix, we obtain from the third row of the above equality that \( \Lambda' \varepsilon \in \mathbb{Z}[\varepsilon] \). Subtracting third row from the first one we get \( \Lambda' \varepsilon \in \mathbb{Z}[\varepsilon] \). Since \( \mathbb{Z}[\varepsilon] \) is closed under addition, we have \( \Lambda' \mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\varepsilon] \).

Since \( \Lambda \) is a quadratic integer, we have \( \Lambda + \Lambda' \in \mathbb{Z} \). This implies that \( \Lambda \in \mathbb{Z} - \Lambda' \in \mathbb{Z}[\varepsilon] \), whence \( \Lambda \varepsilon \in \varepsilon \mathbb{Z} - \Lambda' \varepsilon \in \mathbb{Z}[\varepsilon] \), and thus \( \Lambda \mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\varepsilon] \).

Now since \( \Lambda \) is a unit, we have \( \Lambda \Lambda' = \pm 1 \), and therefore multiplying \( \Lambda \mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\varepsilon] \) by \( \Lambda' \) we obtain \( \mathbb{Z}[\varepsilon] \subseteq \Lambda' \mathbb{Z}[\varepsilon] \). \( \square \)

It is obvious that in our considerations, \( \varepsilon \) must be a quadratic irrational. When putting a 3iet with such a parameter into context of cut-and-project sets, we need to specify the slope of the second projection, i.e., the parameter \( \eta \). Choosing \( \eta = -\varepsilon' \), where \( \varepsilon' \) is the algebraic conjugate of \( \varepsilon \), the star map \( x = a + b\eta \mapsto x^* = a - b\varepsilon \) becomes the Galois automorphism in \( \mathbb{Q}(\varepsilon) \). We will use the notation \( x = a + b\varepsilon, a, b \in \mathbb{Q} \mapsto x' = a + b\varepsilon' \), as is usual. Recall that for \( x, y \in \mathbb{Q}(\varepsilon) \) we have

\[
(x + y)' = x' + y' \quad \text{and} \quad (xy)' = x'y'.
\]

With such notation, \( \Sigma_{\varepsilon,-\varepsilon'}(\Omega) \) can be rewritten in the form

\[
\Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \{ x \in \mathbb{Z}[\varepsilon'] | x' \in \Omega \}.
\]

**Lemma 5.4.** Let \( \varepsilon \) be a quadratic irrational and let \( \Lambda \) be a quadratic unit in \( \mathbb{Q}(\varepsilon) \) such that

\[
\Lambda \mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon].
\]

\[ \Lambda \Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \Sigma_{\varepsilon,-\varepsilon'}(\Lambda' \Omega) \].
• If moreover \( \varepsilon' < 0, \Lambda > 1, \Lambda' \in (0, 1) \) and \( T : [c, c + \ell) \mapsto [c, c + \ell) \) is a \( \mathcal{S} \)-iet with parameters satisfying (\textit{iii}), then the first return map \( T_J \) for the interval \( J = \Lambda'[c, c + \ell) \) is a \( \mathcal{S} \)-iet homothetic with \( T \).

Proof. Since \( \Lambda \Lambda' = \pm 1 \), multiplying of (\textit{v}) by \( \Lambda' \) leads to \( \Lambda'Z[\varepsilon] = Z[\varepsilon] = Z[\varepsilon'] \). By algebraic conjugation we obtain \( \Lambda Z[\varepsilon'] = Z[\varepsilon'] = Z[\varepsilon'] \). Note that in general \( Z[\varepsilon] \neq Z[\varepsilon'] \). From (\textit{v}) we obtain

\[
\Lambda \Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \Lambda\{x \in Z[\varepsilon'] \mid x' \in \Omega\} = \{\Lambda x \in Z[\varepsilon'] \mid \Lambda'x' \in \Lambda'\Omega\} = y \in Z[\varepsilon'] \mid y' \in \Lambda'\Omega\} = \Sigma_{\varepsilon,-\varepsilon'}(\Lambda'\Omega).
\]

This however means that the distances between adjacent elements of the cut-and-project set \( \Sigma_{\varepsilon,-\varepsilon'}(\Lambda'\Omega) \) are \( \Lambda \) multiples of the distances between adjacent elements of the cut-and-project set \( \Sigma_{\varepsilon,-\varepsilon'}(\Omega) \). Since the star map images (in our case the images under the Galois automorphism) of the distances between neighbors in a cut-and-project set correspond to translations in the corresponding \( \mathcal{S} \)-iet (see Theorem 3.2), the factor of homothety between the two \( \mathcal{S} \)-iets is \( \Lambda \).

If the parameter \( \eta = -\varepsilon' > 0 \), the \( \mathcal{S} \)-iet mappings corresponding to \( \Sigma_{\varepsilon,-\varepsilon'}(\Omega) \) and \( \Sigma_{\varepsilon,-\varepsilon'}(\Lambda'\Omega) \) are precisely \( T \) and \( T_J \) respectively, see Remark 3.3.\( \square \)

Using Lemma 5.3 and statement (v) of Theorem 5.1 we obtain

\[
E_{\varphi(i)} = (\Lambda'\Omega) \cap Z[\varepsilon] = (\Lambda'\Omega) \cap \{T^n(0) \mid n \in \mathbb{Z}\}.
\]

We are now in position to prove the main theorem of this section, which provides a necessary and sufficient condition for substitution invariance of a non-degenerate \( \mathcal{S} \)-iet word.

**Proposition 5.5.** Let \( u \) be a non-degenerate \( \mathcal{S} \)-iet word with parameters \( \varepsilon, l, c \), such that \( \varepsilon \) is a Sturm number having \( \varepsilon' < 0 \) and \( l, c \in \mathbb{Q}(\varepsilon), l \notin \mathbb{Z}[\varepsilon] \). Then \( u \) is invariant under a primitive substitution if and only if there exists a quadratic unit \( \Lambda \in \mathbb{Q}(\varepsilon), \Lambda > 1, \) with conjugate \( \Lambda' \in (0, 1), \) such that

\begin{itemize}
  \item \textit{C1.} \( \Lambda Z[\varepsilon] = Z[\varepsilon] \), and
  \item \textit{C2.} for the interval \( J = \Lambda'[c, c + \ell) \), one has
    \[
    \text{anc}_J(c + \varepsilon), \text{anc}_J(c + \ell - (1 - \varepsilon)) \in \{\Lambda'c, \Lambda'(c + \varepsilon), \Lambda'(c + l - (1 - \varepsilon))\}.
    \]
\end{itemize}

Proof. Let \( u \) be invariant under a primitive substitution \( \varphi \). We search for \( \Lambda \) with properties C1. and C2. of the proposition. According to Theorem 5.1, the dominant eigenvalue of the matrix \( M_\varphi \) is a quadratic unit in \( \mathbb{Q}(\varepsilon) \), i.e., its conjugate belongs to the interval \( (-1, 1) \). If the conjugate is positive, we use for \( \Lambda \) the dominant eigenvalue of \( M_\varphi \). Otherwise, since \( u \) is invariant also under the substitution \( \varphi^2 \), we take for \( \Lambda \) the dominant eigenvalue of the matrix \( M_{\varphi^2} = M_\varphi^2 \).

The validity of property C1. follows from Lemma 5.3. Equation (\textit{ii}) states that the interval \( J = \Lambda'\Omega \) defines only three return names and that the subintervals corresponding to these return names are \( \Lambda'\Omega_1, \Lambda'\Omega_2 \) and \( \Lambda'\Omega_3 \). Since \( I = [c, c + \ell) \), these are \( \Lambda'\Omega_1 = [\Lambda'c, \Lambda'(c + l - 1 + \varepsilon)] \), \( \Lambda'\Omega_2 = [\Lambda'(c + l - 1 + \varepsilon), \Lambda'(c + \varepsilon)] \), and \( \Lambda'\Omega_1 = [\Lambda'(c + \varepsilon), \Lambda'(c + l)] \). The list (\textit{iv}) defines the boundary points of subintervals determining the return names. Property C2. follows.

For the opposite implication, realize that by Lemma 5.3 property C1. ensures that \( T_J \) is a \( \mathcal{S} \)-iet with subintervals \( \Lambda'[c, c + l - 1 + \varepsilon) \Lambda'(c + l - 1 + \varepsilon, c + \varepsilon) \), and \( \Lambda'[c + \varepsilon, c + l) \). This, together with property C2., forces that points of the list (\textit{iv}) belong to the set \( \{\Lambda'c, \Lambda'(c + \varepsilon), \Lambda'(c + l - 1 + \varepsilon)\} \), and thus the interval \( J = \Lambda'\Omega \) defines three return names. Hence according to Proposition 5.2 the infinite word \( u \) is invariant under a primitive substitution.\( \square \)
Remark 5.6. The proof of the above proposition directly implies that in case that \( u \) is invariant under a substitution \( \varphi \), the scaling factor \( \Lambda \) from Proposition 5.5 can be taken to be the dominant eigenvalue of the substitution matrix \( M_\varphi \) or \( M_\varphi^2 = M_\varphi^2 \).

6 Characterization of substitution invariant 3iet words

We now have to solve the question, when for a given Sturm number \( \varepsilon \) and parameters \( c, l \in \mathbb{Q}(\varepsilon) \) satisfying (9) there exists \( \Lambda \) with properties C1. and C2. of Proposition 5.5. Finding \( \Lambda \) having the first of the properties is simple.

Lemma 6.1. Let \( \varepsilon \) be irrational, solution of the equation \( Ax^2 + Bx + C = 0 \). Then there exists a quadratic unit \( \Lambda \in \mathbb{Q}(\varepsilon) \) such that
\[
\Lambda > 1, \quad \Lambda' \in (0, 1), \quad \text{and} \quad \Lambda \mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]. \tag{17}
\]

Proof. Let the pair of integers \( X, Y \) be a non-trivial solution of the Pell equation
\[
X^2 - (B^2 - 4AC)Y^2 = 1.
\]
Put \( \gamma := X + BY + 2AY \varepsilon \). Using \( A \varepsilon^2 = -B \varepsilon - C \), we easily verify that \( \gamma \varepsilon \in \mathbb{Z}[\varepsilon] \). Using \( A(\varepsilon + \varepsilon') = -B \) and \( A \varepsilon \varepsilon' = C \), we derive that \( \gamma \gamma' = 1 \). This implies
\[
\gamma \mathbb{Z}[\varepsilon] = \gamma' \mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon].
\]
Finally, we put \( \Lambda = \max\{|\gamma|, |\gamma'|\} \).

In Lemma 6.1 we have found \( \Lambda \) with property C1. It is more difficult to decide when \( \Lambda \) satisfies also property C2. of Proposition 5.5. By definition of the map \( T \), it follows that \( x \) and \( T(x) \) differ by an element of \( \mathbb{Z}[\varepsilon] \). Therefore for arbitrary \( z_0 \) and its ancestor \( \text{anc}_J(z_0) \) we have \( z_0 - \text{anc}_J(z_0) \in \mathbb{Z}[\varepsilon] \). It is useful to introduce an equivalence on \( \mathbb{Q}(\varepsilon) \) as follows. We say that elements \( x, y \in \mathbb{Q}(\varepsilon) \) are equivalent if their difference belongs to \( \mathbb{Z}[\varepsilon] \). Formally,
\[
x - y \in \mathbb{Z}[\varepsilon] \iff x \sim y.
\]
For the parameters \( c, l \in \mathbb{Q}(\varepsilon) \), one can find \( q \in \mathbb{N} \) such that \( c, l \in \frac{1}{q} \mathbb{Z}[\varepsilon] \). Clearly, \( \text{anc}_J(c + \varepsilon) \) and \( \text{anc}_J(c + l - 1 + \varepsilon) \) also belong to the set \( \frac{1}{q} \mathbb{Z}[\varepsilon] \). The set to which belong ancestors of \( c + \varepsilon \) and \( c + l - 1 + \varepsilon \) can be restricted even more. For, the equivalence \( \sim \) divides the set \( \frac{1}{q} \mathbb{Z}[\varepsilon] \) into \( q^2 \) classes of equivalence of the form
\[
T_{ij} := \frac{i + j \varepsilon}{q} + \mathbb{Z}[\varepsilon], \quad \text{where} \quad 0 \leq i, j \leq q - 1.
\]
Relation \( \Lambda' \mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon] \) implies
\[
z \in \mathbb{Z}[\varepsilon] \iff \Lambda' z \in \mathbb{Z}[\varepsilon].
\]
Therefore the mapping \( \psi(T_{ij}) = \Lambda' T_{ij} \) is a bijection on the set of \( q^2 \) classes of equivalence. For every bijection \( \psi \) on a finite set, there exists an iteration \( s \in \mathbb{N}, s \geq 1 \), such that \( \psi^s = \text{id} \). Denoting \( L := \Lambda^s \), the number \( L \) has obviously all properties of \( \Lambda \), namely
a) $L$ is a quadratic unit in $\mathbb{Q}(\varepsilon)$;

b) $L > 1$, $L' \in (0, 1)$;

c) $LZ[\varepsilon] = Z[\varepsilon]$;

and moreover

d) $L'\left(\frac{i+j}{q} + Z[\varepsilon]\right) = \frac{i+j}{q} + Z[\varepsilon]$, for all $i, j, 1 \leq i, j \leq q - 1$.

Having a quadratic unit $\Lambda$ with properties of the number $L$ in items a) – d), it is less difficult to decide about validity of the condition

$$\text{anc}_{\Lambda}(c + \varepsilon), \text{anc}_{\Lambda}(c + l - 1 + \varepsilon) \in \{\Lambda'c, \Lambda'(c + \varepsilon), \Lambda'(c + l - 1 + \varepsilon)\}.$$ (18)

Non-degeneracy of the infinite word $u$ implies that $l \notin Z[\varepsilon]$, and therefore $c + \varepsilon \not\sim c + l - 1 + \varepsilon$. Since for every $z_0 \in \frac{1}{q}Z[\varepsilon]$ we have now

$$z_0 \sim \text{anc}_{\Lambda}(z_0) \sim \Lambda'z_0,$$

the condition (18) in fact means

$$\text{anc}_{\Lambda}(c + l - 1 + \varepsilon) = \Lambda'(c + l - 1 + \varepsilon)$$ (19)

and

$$\text{anc}_{\Lambda}(c + \varepsilon) \in \{\Lambda'c, \Lambda'(c + \varepsilon)\}.$$ (20)

**Lemma 6.2.** Let $\varepsilon$ be a Sturm number with $\varepsilon' < 0$. Let $l, c \in \frac{1}{q}Z[\varepsilon]$. Let $\Lambda$ satisfy properties of $L$ in a) – d) and let $J = \Lambda'[c, c + l]$. Then for arbitrary $z_0 \in \frac{1}{q}Z[\varepsilon] \cap [c, c + l]$, one has

$$\text{anc}_{\Lambda}(z_0) = \Lambda'z_0 \iff z_0' \leq 0 \leq (T(z_0))'.$$

**Proof.** The transformation $T$ preserves the classes of equivalence and thus for the orbit of a point $z_0$ it holds that

$$\{T^n(z_0) \mid n \in \mathbb{Z}\} \subset z_0 + Z[\varepsilon].$$

As $(T^{n+1}(z_0) - T^n(z_0))' \in \{1 - \varepsilon', 1 - 2\varepsilon, -\varepsilon'\}$, the assumption $\varepsilon' < 0$ implies that the sequence $(s_n)_{n \in \mathbb{Z}}$,

$$s_n := (T^n(z_0))'$$

is strictly increasing. By (18) we have moreover

$$\{T^n(z_0) \mid n \in \mathbb{Z}\} = \{s_n' \mid n \in \mathbb{Z}\} = (z_0 + Z[\varepsilon]) \cap [c, c + l].$$

Since $0 \in [c, c + l]$ and $\Lambda' \in (0, 1)$, it is $\Lambda'[c, c + l] \subset [c, c + l]$. This inclusion together with property d) implies

$$\{s_n' \mid n \in \mathbb{Z}\} \supset \Lambda'((z_0 + Z[\varepsilon]) \cap [c, c + l]) = \{\Lambda's_n' \mid n \in \mathbb{Z}\}.$$

The strictly increasing sequence $(\Lambda s_n)_{n \in \mathbb{Z}}$ is therefore a subsequence of the strictly increasing sequence $(s_n)_{n \in \mathbb{Z}}$. Thus there exists a unique index $m$ such that

$$\Lambda s_m \leq s_0 < s_1 \leq \Lambda s_{m+1}.$$ (21)
For determination of the ancestor of the point \( z_0 = s'_0 \) by definition, we search for the maximal non-positive index \( k \in \mathbb{Z} \) such that \( T^k(z_0) \in \Lambda'(c,c+l) \), i.e., such that \( T^k(z_0) \) is an element of the sequence \( (\Lambda's'_n)_{n\in\mathbb{Z}} \). Since both \( (s_n)_{n\in\mathbb{Z}} \) and \( (\Lambda s_n)_{n\in\mathbb{Z}} \) are strictly increasing, we have \( (T^k(z_0))' = \Lambda s_m \) and thus \( \text{anc}_J(s'_0) = \Lambda' s'_m \). Denoting \( s'_m = y_0 \), equation (21) can be rewritten

\[
\Lambda y_0 \leq z'_0 < (T(z_0))' \leq \Lambda(T(y_0))'.
\]

On the other hand, recall that \( \{s'_n \mid n \in \mathbb{Z}\} = (z_0 + \mathbb{Z}[\varepsilon]) \cap [c,c+l] \) and the index \( m \) for which (21) holds, is determined uniquely. Therefore we can claim that \( \text{anc}_J(z_0) = \Lambda'y_0 \) if and only if \( y_0 \) verifies inequalities (22). Thus \( \text{anc}_J(z_0) = \Lambda'z_0 \) if and only if

\[
\Lambda z'_0 \leq z'_0 < (T(z_0))' \leq \Lambda(T(z_0))'.
\]

Note that strict inequality in the middle is trivial and it is satisfied by arbitrary \( z_0 \). Since \( \Lambda > 1 \) we have \( \Lambda z'_0 \leq z'_0 \Leftrightarrow z'_0 \leq 0 \) and \( (T(z_0))' \leq \Lambda(T(z_0))' \Leftrightarrow 0 \leq (T(z_0))' \), which completes the proof. \( \square \)

**Theorem 6.3.** Let \( u \) be a non-degenerate 3iet word coding the orbit of the point \( x_0 \) under a 3iet with permutation (321) and parameters \( \alpha_1, \alpha_2, \alpha_3 \). Put

\[
\varepsilon := \frac{\alpha_1 + \alpha_2}{\alpha_1 + 2\alpha_2 + \alpha_3}, \quad l := \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_1 + 2\alpha_2 + \alpha_3}, \quad \text{and} \quad c := \frac{-x_0}{\alpha_1 + 2\alpha_2 + \alpha_3}.
\]

Then \( u \) is invariant under a primitive substitution if and only if

1. \( \varepsilon \) is a Sturm number;
2. \( c, l \in \mathbb{Q}(\varepsilon) \);
3. \( \min(\varepsilon', 1 - \varepsilon') \leq c', \quad c' + l' \leq \max(\varepsilon', 1 - \varepsilon') \).

**Proof.** Theorem 5.1 claims that items 1. and 2. are necessary conditions for existence of a primitive substitution under which \( u \) be invariant. Therefore we shall prove the following statement:

If \( \varepsilon \) is a Sturm number and \( c, l \in \mathbb{Q}(\varepsilon) \), then \( u \) is invariant under a primitive substitution if and only if condition 3. holds.

Note that the infinite word

\[
\cdots u_{-3}u_{-2}u_{-1}|u_0u_1u_2\cdots
\]

is substitution invariant if and only if

\[
\cdots u_2 u_1 u_0|u_{-1}u_{-2}u_{-3}\cdots
\]

is substitution invariant. At the same time, \( \cdots u_2 u_1 u_0|u_{-1}u_{-2}u_{-3}\cdots \) is a 3iet word coding the transformation \( T^{-1} \), i.e. the 3iet with parameters \( 1 - \varepsilon, l, c \). The fact that \( \varepsilon \) is a Sturm number means either \( \varepsilon' < 0 \) or \( \varepsilon' > 1 \). Instead of parameters \( \varepsilon, l, c \) we can thus have \( 1 - \varepsilon, l, c \), and therefore limit our study (without loss of generality) to Sturm number \( \varepsilon \) satisfying \( \varepsilon' < 0 \). In that case, inequalities in item 3. of the theorem are of the form

\[
\varepsilon' \leq c' + l' \leq 1 - \varepsilon' \quad \text{and} \quad \varepsilon' \leq -c' \leq 1 - \varepsilon'.
\]

(24)
For the implication \( \Rightarrow \), suppose that \( u \) is invariant under a primitive substitution \( \varphi \). Denote \( q \in \mathbb{N} \), such that \( c, l \in \frac{1}{q} \mathbb{Z}[\varepsilon] \). Since \( u \) is invariant under an arbitrary power of the substitution \( \varphi \), we can use Proposition 5.5 and Remark 5.6 to find a number \( \Lambda \) with properties of the number \( L \) described in a) – d), and such that for the interval \( J = \Lambda'[c, c + l] \) equalities (19) and (20) hold.

When applying Lemma 6.2 on \( z_0 = c + l - 1 + \varepsilon \), equality (19) is equivalent to

\[
c' + l' - 1 + \varepsilon' \leq 0 \leq (T(c + l - 1 + \varepsilon))' = c' + l' - \varepsilon',
\]

which is one of the inequalities in (24).

Now, let us study validity of (20). Since \( T(c + \varepsilon) = c \), we have \( \text{anc}_J(c) = \text{anc}_J(c + \varepsilon) \). Relation (20) states that \( \text{anc}_J(z_0) = \Lambda'z_0 \) holds either for \( z_0 = c \) or for \( z_0 = c + \varepsilon \). Therefore we have

\[
c' + \varepsilon' \leq 0 \leq (T(c + \varepsilon))' \quad \text{or} \quad c' \leq 0 \leq (T(c))'.
\]

Since \( (T(c + \varepsilon))' = c' \) and \( (T(c))' = c' + 1 - \varepsilon' \), verifying at least one of the inequalities in (25) means

\[
c' + \varepsilon' \leq 0 \leq c' + 1 - \varepsilon',
\]

which is the other inequality in (24).

In the opposite implication, we take \( \varepsilon \) Sturm and \( c, l \in \frac{1}{q} \mathbb{Z}[\varepsilon] \) and with the use of Lemma 6.1 we find \( \Lambda \) with properties of \( L \) given in a) – d). By Lemma 6.2, validity of inequalities (24) is equivalent to validity of (19) and (20). Therefore using Proposition 5.5, \( u \) is invariant under a primitive substitution.

**Remark 6.4.** Note that in the proof of the theorem we have applied Lemma 6.2 only to points \( z_0 = c + l - 1 + \varepsilon \) and \( z_0 = c + \varepsilon \). Realize that in fact, we do not need that \( \Lambda \) satisfies property d) of \( L \), but only that

\[
d' \quad \Lambda'(c + \mathbb{Z}[\varepsilon]) = c + \mathbb{Z}[\varepsilon] \quad \text{and} \quad \Lambda'(c + l + \mathbb{Z}[\varepsilon]) = c + l + \mathbb{Z}[\varepsilon].
\]

This can be important when we search for minimal \( \Lambda > 1 \) with desired properties.

### 7 Characterization of substitution invariant 3iet words using Sturmian words

Comparing Theorems 6.3 and 1.2 we immediately see a striking narrow connection between 3iet words and Sturmian words, namely that the 3iet word \( u = (u_n)_{n \in \mathbb{N}} \) is invariant under a primitive substitution if and only if the Sturmian word with slope \( \varepsilon \) and intercept \( -c \) and the Sturmian word with slope \( \varepsilon \) and intercept \( \ell + c \) are both invariant under a substitution.

In fact, as shown in [3], these two Sturmian words appear naturally as images of the given 3iet word by the following morphisms.

Let us denote by \( \sigma_{01} : \{A, B, C\}^* \rightarrow \{0, 1\}^* \) the morphism given by

\[
A \mapsto 0, \quad B \mapsto 01, \quad C \mapsto 1,
\]

and by \( \sigma_{10} : \{A, B, C\}^* \rightarrow \{0, 1\}^* \) the morphism given by

\[
A \mapsto 0, \quad B \mapsto 10, \quad C \mapsto 1.
\]
One can verify that if $u = (u_n)_{n \in \mathbb{N}}$ is a non-degenerate 3iet word with parameters $\varepsilon, \ell, c$ satisfying (9), then the infinite word

$$\sigma_0(u) = \sigma_0(u_0)\sigma_0(u_1)\sigma_0(u_2)\ldots$$

is the Sturmian word with slope $\varepsilon$ and intercept $-c$ and the infinite word $\sigma_{10}(u)$ is the Sturmian word with slope $1 - \varepsilon$ and intercept $\ell + c$.

With the definition of morphisms $\sigma_{01}$ and $\sigma_{10}$, we can give the characterization of substitution invariant non-degenerate 3iet words without use of any parameters.

**Corollary 7.1.** Let $u = (u_n)_{n \in \mathbb{N}}$ be a non-degenerate 3iet word. Then $u$ is invariant under a primitive substitution if and only if both Sturmian words $\sigma_{10}(u)$ and $\sigma_{01}(u)$ are invariant under substitution.

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**References**

[1] B. Adamczewski, *Codages de rotations et phénomènes d’autosimilarité*, J. Théor. Nombres Bordeaux 14 (2002), 351–386.

[2] C. Allauzen, *Une caractérisation simple des nombres de Sturm*, J. Théor. Nombres Bordeaux 10 (1998), 237–241.

[3] P. Ambrož, Z. Masáková, E. Pelantová, *Matrices of 3iet preserving morphisms*, submitted to Theor. Comp. Sc. (2007), 26pp.

[4] P. Arnoux, V. Berthé, Z. Masáková, E. Pelantová, *Sturm numbers and substitution invariance of 3iet words*, preprint (2007)

[5] P. Baláži, Z. Masáková, E. Pelantová, *Complete characterization of substitution invariant Sturmian sequences*, Integers 5 (2005), A14, 23 pp. (electronic)

[6] V. Berthé, H. Ei, S. Ito, H. Rao, *Invertible substitutions and Sturmian words: an application of Rauzy fractals*, to appear in RAIRO Theoret. Informatics Appl., (2006).

[7] V. Berthé, C. Holton, L.Q. Zamboni, *Initial powers of Sturmian words*, Acta Arithmetica 122 (2006), 315–347.

[8] M.D. Boshernitzan, C.R. Carroll, *An extension of Lagrange’s theorem to interval exchange transformations over quadratic fields*, J. Anal. Math. 72 (1997), 21–44.

[9] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, *Ergodic Theory*, Berlin, Heidelberg, New York: Springer 1982.

[10] S. Ferenczi, L.Q. Zamboni, *Combinatorial structure of symmetric k-interval exchange transformations*, preprint (2006).
[11] L.S. Guimond, Z. Masáková, E. Pelantová, *Combinatorial properties of infinite words associated with cut-and-project sequences*, J. Théor. Nombres Bordeaux 15 (2003), 697–725.

[12] A.B. Katok, A.M. Stepin, *Approximations in ergodic theory*, Uspehi Mat. Nauk 22 (1967), 81–106.

[13] M. Kean, *Interval exchange transformations*, Math. Z. 141 (1975), 25–31.

[14] M. Lothaire, *Algebraic Combinatorics on Words*, volume 90 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, (2002).

[15] M. Morse, G. A. Hedlund, *Symbolic dynamics II. Sturmian trajectories*, Amer. J. Math. 62 (1940), 1–42.

[16] R.V. Moody, *Meyer Sets and Their Duals*, in *The Mathematics of Aperiodic Order*, Proceedings of the NATO-Advanced Study Institute on Long-range Aperiodic Order, ed. R.V. Moody, NATO ASI Series C489, 403–441, Kluwer Acad. Press, 1997.

[17] S. Yasutomi, *On Sturmian sequences which are invariant under some substitutions*, Number theory and its applications (Kyoto, 1997), Dev. Math. 2, 347–373, Kluwer Acad. Publ., Dordrecht, 1999.