On Passivity, Feedback Passivity, And Feedback Passivity Over Erasure Network: A Piecewise Affine Approximation Approach

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Abstract—In this paper, we deal with the problem of passivity and feedback passification of smooth discrete-time nonlinear systems by considering their piecewise affine approximations. Sufficient conditions are derived for passivity and feedback passivity. These results are then extended to systems that operate over Gilbert-Elliott type communication channels. As a special case, results for feedback passivity of piecewise affine systems over a lossy channel are also derived.

Index Terms—Piecewise affine approximation, passivity, nonlinear systems, networked control systems.

I. INTRODUCTION

Passivity is an important tool to assess stability of nonlinear systems. In the recent past, passivity of smooth discrete-time nonlinear systems has been investigated in great detail [1]–[4].

Various problems relating to smooth nonlinear systems can be studied using a piecewise affine (PWA) approximation. The intuition for using such an approach stems from the fact that any smooth map can be locally approximated by an affine map with an arbitrary accuracy [5].

Control of smooth nonlinear systems using PWA approximations was introduced in [6]. Therein, controllability and stabilization of PWA systems and nonlinear systems are studied. Dissipativity and passivity for a PWA system are studied in [5] and [7]–[9] respectively. Passivity of a smooth nonlinear system by linearizing around the origin is studied in [10]. However, the results presented in [10] are valid only inside a small neighbourhood of the origin.

Systems whose subsystems, viz: controllers, actuators, and sensors are connected through a communication network are termed as networked control systems (NCSs). Communication over a network results in data packet loss which can negatively impact the performance of a system [11]. Usually, an independent and identically distributed (i.i.d.) Bernoulli process or a Markov process is used to model packet losses. The Bernoulli process model is used more often due to its mathematical tractability. However, packet loss in a realistic communication network could be temporally correlated.

Thus, a Markov process based model represents a more mathematical tractability. However, packet loss in a realistic communication network could be temporally correlated. Thus, a Markov process based model represents a more tractable model. Feedback passivity of a nonlinear system with packet losses has been studied in [12]. Their analysis is based on the frequency of packet losses that occur in the channel.

The main contributions of this paper can be summarized as follows. We address three problems relating to smooth nonlinear systems using a PWA approximation. To start with, we derive conditions under which a smooth discrete-time nonlinear system becomes passive. Unlike [10], our results are valid for any neighbourhood (bigger or smaller) of the origin. Then, we derive conditions under which a piecewise linear state-feedback control law is sufficient to ensure feedback passivity. Finally, we consider the problem of feedback passivity of smooth nonlinear systems over lossy communication network. Results corresponding to the feedback passivity of a PWA system with packet losses are also presented. One of the main features of the approach presented in this paper is that it enables us to design the controllers by solving certain linear matrix inequalities (LMI) which are subjected to additional constraints. As LMI is easy to solve, our approach provides a more efficient way of controller design. Unlike [12], we consider feedback passivity over a Gilbert-Elliott type communication channel where packet losses are modeled as a two-state Markov chain. Further, their approach is different than the one presented here.

Although there is literature available on the passivity of PWA systems, to the best of the authors’ knowledge there is no literature that studies the passivity and feedback passivity of smooth nonlinear systems using a PWA approximation. Further, feedback passivity of a PWA system with packet loss has not been addressed as yet.

The paper is structured as follows. Section II describes the problem along with some important notations. Section III, contains the main results. In section IV, we demonstrate our results using a numerical example. Finally, section V presents the conclusion.

II. PROBLEM FORMULATION

Consider the smooth discrete-time nonlinear system:

\[
\begin{align*}
x_{k+1} &= f(x_k) + B_1 u_k + D_1 w_k \\
z_k &= h(x_k) + B_2 u_k + D_2 w_k,
\end{align*}
\]

(1)

wherein \(x_k \in \mathbb{R}^n\) is the state vector, \(u_k \in \mathbb{R}^m\) is the control input to the actuators, \(w_k \in \mathbb{R}^r\) is the external input (or disturbance), \(z_k \in \mathbb{R}^s\) is the output, \(f : \mathbb{R}^n \to \mathbb{R}^n\), \(h : \mathbb{R}^n \to \mathbb{R}^s\) are smooth maps, \(B_1 \in \mathbb{R}^{n \times m}\), \(D_1 \in \mathbb{R}^{n \times s}\), \(B_2 \in \mathbb{R}^{s \times m}\), \(D_2 \in \mathbb{R}^{s \times s}\) are constant matrices. Many systems like power networks [13], permanent magnet synchronous motors [14], and nonlinear RLC circuits [15], can be modeled by Equation (1).

As the maps \(f(\cdot)\) and \(h(\cdot)\) are smooth maps, it is possible to partition the state space into small regions and
approximate the nonlinear system (1) in each region by an affine system with arbitrary accuracy. In particular, one can consider a polyhedral partition \( \mathcal{X} = \{ X_i \}_{i \in \mathcal{N}} \) of the state space, which is indexed by the set \( \mathcal{N} \). \( \mathcal{N}_0 \subseteq \mathcal{N} \) denotes the index set for all cells that contains the origin while \( \mathcal{N}_1 \subseteq \mathcal{N} \) is the index set for the cells that do not contain the origin. Each polyhedral cell can be characterized by an equation of the form [5]:

\[
E(i)x_k + e(i) \geq 0 \quad x_k \in X_i, \quad \text{or} \quad E(i)\bar{x}_k \geq 0,
\]

where \( E(i) := [E(i), e(i)] \), \( \bar{x}(k) := [x(k)]^T \). Thus, each element of the partition is characterised by the corresponding matrix \( E(i) \). Note that \( e(i) = 0 \) if \( i \in \mathcal{N}_0 \).

Let \( \mathcal{X} = \{ X_i \}_{i \in \mathcal{N}} \) be a polyhedral partition of the state space. Consider matrices \( A(i), C(i), \) and vectors \( a(i), c(i) \) for all \( i \in \mathcal{N} \) such that:

\[
\begin{align*}
||f(x_k) - A(i)x_k - a(i)|| &\leq ||m(x_k,i)|| \leq e(i)||x_k|| \quad \text{for } x_k \in X_i \setminus \{ x = 0 \} \\
||h(x_k) - C(i)x_k - c(i)|| &\leq ||n(x_k,i)|| \leq \delta(i)||x_k|| \quad \text{for } x_k \in X_i \setminus \{ x = 0 \}
\end{align*}
\]

Let \( \bar{x}(i) = \bar{A}(i)\bar{x}_k + \bar{B}(i)w_k + \bar{D}(i)x_k + \bar{m}(x_k,i) \).

One can write (1) in terms of the variable \( \bar{x}_k \) as:

\[
\begin{align*}
\bar{x}_{k+1} &= \bar{A}(i)\bar{x}_k + \bar{B}(i)w_k + \bar{D}(i)x_k + \bar{m}(x_k,i) \\
\bar{z}_k &= \bar{C}(i)\bar{x}_k + \bar{B}(i)w_k + \bar{D}(i)x_k + \bar{m}(x_k,i)
\end{align*}
\]

where \( \bar{A}(i) := \begin{bmatrix} A(i) & a(i) \end{bmatrix} \), \( \bar{C}(i) := \begin{bmatrix} C(i) & c(i) \end{bmatrix} \),

\[
\begin{align*}
\bar{B}(i) &:= \begin{bmatrix} B(i) & 0_{1\times n} \end{bmatrix}, \\
\bar{D}(i) &:= \begin{bmatrix} D(i) & 0_{1\times n} \end{bmatrix}, \\
\bar{m}(x_k,i) &:= \begin{bmatrix} m(x_k,i) \\
0_{n\times 1} \end{bmatrix}
\end{align*}
\]

Define passivity for nonlinear system (1) with \( w_k \equiv 0 \) as:

**Definition 1:** [16] The system (1) with \( w_k \equiv 0 \) is said to be passive if there exists a nonnegative function \( V : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) with \( V(0) = 0 \), called the storage function, such that for all \( x_k \in \mathbb{R}^n \), \( w \in \mathbb{R}^n \), and for all \( k \in \mathbb{Z}^+ \):

\[
V(x_{k+1}) - V(x_k) \leq T_k w_k.
\]

### III. MAIN RESULTS

#### A. Passivity and passification of the nonlinear system (1)

Consider a piecewise quadratic storage function of the form

\[
V(x_k) = \frac{1}{2} x_k^T P(i)x_k \quad \text{if } x_k \in X_i
\]

If \( i \in \mathcal{N}_0 \), then \( \bar{P}(i) \) is given by \( \bar{P}(i) = \text{diag}(P(i), 0) \). For each \( i \in \mathcal{N}_0, \) \( P(i) \) is chosen such that \( x_k^T P(i)x_k > 0 \) for all \( x_k \neq 0 \).

Note that to find a nonnegative storage function \( V(x_k) \) of the form given in (7), it is not necessary to look for matrices \( \bar{P}(i), i \in \mathcal{N}_0, \) and \( \bar{P}(i), i \in \mathcal{N}_1 \), which are positive definite. Instead, matrices \( P(i), i \in \mathcal{N}_0 \), and \( \bar{P}(i), i \in \mathcal{N}_1 \) which result in \( x_k^T P(i)x_k > 0 \), when \( x_k \neq 0 \) in \( X_i, i \in \mathcal{N}_0 \), and \( \bar{x}_k^T \bar{P}(i)\bar{x}_k > 0 \), when \( x_k \neq 0 \) in \( X_i, i \in \mathcal{N}_1 \), respectively, will ensure nonnegativity of \( V(x_k) \). This will enhance the flexibility in the selection of matrices \( P(i) \).

Following theorem presents conditions for passivity of system (1) with \( w_k \equiv 0 \).

**Theorem 1:** The system (1), with \( w_k \equiv 0 \), is passive if there exist symmetric matrices with positive entries \( W(i), \bar{R}(i) \), and symmetric matrices \( P(i) \) such that the following inequalities are satisfied for all \( i, j \in \mathcal{N}_i \):

\[
P(i) - E(i)^T R(i) (E(i) > 0, \begin{bmatrix} A_{11}(i,j) & A_{12}(i,j) \\
A_{21}(i,j) & A_{22}(i,j) \end{bmatrix} \leq 0,
\]

where

\[
P(i) = \begin{bmatrix} P(i) & 0_{n \times 1} \\
0_{1 \times n} & 0 \end{bmatrix}, \quad \begin{bmatrix} A_{11}(i,j) & A_{12}(i,j) \\
A_{21}(i,j) & A_{22}(i,j) \end{bmatrix} = A_{11}(i) + A_{12}(i), \quad A_{21}(i,j) = A_{21}(i), \quad A_{22}(i,j) = 2\bar{A}(i)T A_{11}(i) + A_{12}(i)^T A_{11}(i),
\]

\[
\begin{align*}
\rho_1(i,j) &= 2\epsilon(i)||\bar{P}(i)||A_{11}(i)||A_{12}(i)||, \\
\rho_2(i,j) &= 2\epsilon(i)||\bar{P}(i)||A_{11}(i)||A_{22}(i)\|\
\end{align*}
\]

**Proof:** With a polyhedral partition \( \{ X_i \}_{i \in \mathcal{N}} \), one can find matrices \( E(i) \) and \( \bar{E}(i) \), for \( i \in \mathcal{N} \), such that \( E(i)x_k \geq 0 \) and \( \bar{E}(i)\bar{x}_k \geq 0 \). As a piecewise affine approximation is being employed, one can consider a piecewise quadratic storage function of the form \( V(x_k) = \frac{1}{2} x_k^T P(i)x_k \), for \( x_k \in X_i \), to study the passivity of the system (1).

The first inequality in (8) ensures that \( V(x_k) \) is positive definite. Further,

\[
V(x_{k+1}) - V(x_k) - T_k w_k
\]

\[
= \frac{1}{2} \bar{x}_{k+1}^T \bar{P}(i)\bar{x}_{k+1} - \frac{1}{2} \bar{x}_k^T \bar{P}(i)\bar{x}_k - T_k w_k
\]

\[
= \frac{1}{2} \left( \bar{A}(i)\bar{x}_k + \bar{D}(i)x_k + \bar{m}(x_k,i) \right)^T \bar{P}(i) \left( \bar{A}(i)\bar{x}_k + \bar{D}(i)x_k + \bar{m}(x_k,i) \right) - \bar{x}_k^T \bar{P}(i)\bar{x}_k - \bar{C}(i)\bar{x}_k + \bar{D}(i)x_k + \bar{m}(x_k,i)
\]

\[
= \frac{1}{2} \left( \bar{A}(i)\bar{x}_k + \bar{D}(i)x_k \right)^T \bar{P}(i) \left( \bar{A}(i)\bar{x}_k + \bar{D}(i)x_k \right) - \bar{x}_k^T \bar{P}(i)\bar{x}_k
\]

\[
= 2w_k^T \bar{D}(i)^T \bar{P}(i)\bar{x}_k + 2w_k^T \bar{D}(i)^T \bar{P}(i)\bar{x}_k
\]

\[
\leq 2||\bar{P}(i)||\left( ||\bar{x}_k||^2 + ||\bar{m}(x_k,i)||^2 \right) \quad \text{as } ||x_k|| \leq ||\bar{x}_k||
\]

**Using basic properties of 2-norm and induced 2-norm:**

\[
2w_k^T \bar{D}(i)^T \bar{P}(i)\bar{x}_k \leq ||2w_k^T \bar{D}(i)^T \bar{P}(i)\bar{x}_k||
\]

\[
\leq 2||\bar{P}(i)||\left( ||\bar{x}_k||^2 + ||\bar{m}(x_k,i)||^2 \right)
\]

\[
\leq 2\epsilon(i)||\bar{P}(i)||A_{11}(i)||A_{22}(i)||\bar{x}_k||
\]

\[
\leq 2\epsilon(i)||\bar{P}(i)||A_{11}(i)||A_{22}(i)||\bar{x}_k||
\]
As \((||\bar{x}_k||-||w_k||)^2 \geq 2||\bar{x}_k||||w_k|| \leq (||\bar{x}_k||^2+||w_k||^2)\).

Thus, (11) implies:
\[
2w_k^TD_k^TP(i)\tilde{m}(x_k,i) \leq \rho_2(i,j)(||\bar{x}_k||^2 + ||w_k||^2)
\]

(12)

Similarly, \(\tilde{m}^T(x_k,i)P(i)\tilde{m}(x_k,i) \leq \rho_3(||\bar{x}_k||^2)

- 2w_k^T\tilde{m}(x_k,i) \leq \rho_4(i)(||\bar{x}_k||^2 + ||w_k||^2)

(14)

From (9), (10), (12), (13), (14), and using the fact that \(W(i)\) has only positive elements:
\[
V(x_{k+1}) - V(x_k) - z_k^Tw_k
\leq \frac{1}{2} \begin{bmatrix} \bar{x}_k^T & \tilde{m}^T(x_k,i) \\
\end{bmatrix} \begin{bmatrix} \Lambda_{11}(i,j) & \Lambda_{12}(i,j) \\
\Lambda_{12}(i,j) & \Lambda_{22}(i,j) \\
\end{bmatrix} \begin{bmatrix} \bar{x}_k \\
\end{bmatrix} + \rho_4(i) z_k^Tw_k
\]

If the second inequality in (8) is satisfied for \(i, j \in \mathcal{N}\), then:
\[
V(x_{k+1}) - V(x_k) \leq z_k^Tw_k
\]

Hence, the nonlinear system (1), with \(u_k \equiv 0\), is passive. □

Remark 1: Note that, for a specific \(i \in \mathcal{N}\), the conditions given by (8) need to be satisfied for all \(j \in \mathcal{N}\). This is due to the fact that one may not have any knowledge about the external input \(w_k\). Consequently, \(x_{k+1}\) is unknown, and hence the cell where \(x_{k+1}\) lies is also unknown. □

For a piecewise state feedback law \(u_k = K(i)x_k\), where \(i\) denotes the index of the state space partition, let \(\mathcal{H}_K(i)\) and \(\mathcal{S}_K(i)\) denote the matrices \(A(i) + B_1(i)K(i), \mathcal{C}_K(i)\) and \(\mathcal{D}_K(i)\), respectively. Let \(\mathcal{G}_K(i)\) and \(\mathcal{G}_K(i)\) denote the matrices \(\mathcal{C}(i) + B_2(i)K(i)\) and \([\mathcal{C}_K(i) \mathcal{C}_K(i)]\), respectively.

Now, we derive results for feedback passivity as follows.

Theorem 2: The nonlinear system (1), with \(u_k = K(i)x_k \equiv W(i)U^{-1}(i)x_k\), is passive if there exist matrices \(T(i) > 0\), \(U(i), W(i), R(i) > 0\), \(G(i) > 0\) and scalars \(q, r, h > 0\) such that for all \(i \in \mathcal{N}\):
\[
T(i) = \begin{bmatrix} T(j) & 0 \nn \end{bmatrix} > 0,
\]

\[
\Omega_{11}(i) \quad \Omega_{12}(i) \\
0_{n \times n} \quad 0_{n \times n}
\]

(15a)

\[
\Omega_{22}(i) \quad \Omega_{23}(i) \\
0_{n \times n} \quad 0_{n \times n}
\]

(15b)

\[
\gamma_1(i,j) + \gamma_2(i,j) + ||\mathcal{G}_K(i)|| I_{n+1} \leq \mathcal{L}(i)
\]

(15c)

\[
||\tilde{m}(j)|| \leq \mathcal{L}(i)
\]

(15d)

where
\[
\Omega_{11}(i) = \begin{bmatrix} U(i) + U^T(i) - T(i) \end{bmatrix} - R(i)
\]

\[
\Omega_{12}(i) = U^T(i)C(i) - W^T(i)B_2^T
\]

\[
\Omega_{14}(i) = U^T(i)A(i) - W^T(i)B_1^T
\]

\[
\Omega_{22} = 2q - (h + r), \quad \Omega_{33} = \begin{bmatrix} D_2^2 + D_2 \end{bmatrix} - G(i), \quad \Omega_{34} = D_1^2.
\]

\[
\gamma_1(i,j) = 2\epsilon(\rho_2(i,j)||\bar{x}_k||||T^{-1}(j)||)
\]

\[
\gamma_2(i,j) = \epsilon(\rho_4(i)||T^{-1}(j)||||D_1||)
\]

\[
\mathcal{L}(i) = U^T(i)R(i)U^{-1}(i) - \frac{q}{2}
\]

(16)

Proof: Suppose, LMIs (15a) and (15b) are satisfied. As all principal submatrices of a positive semidefinite symmetric matrix are also positive semidefinite, from (15b), one gets:
\[
\Omega_{11}(i) \quad 0_{n \times 1} \\
0_{1 \times n} \quad \Omega_{22} \geq 0
\]

\[
\Rightarrow \hat{U}(i) + \hat{U}^T(i) \geq \hat{T}(i) + \begin{bmatrix} R(i) & 0_{1 \times n} \\
0_{1 \times n} & r \end{bmatrix}
\]

where, \(\hat{U}(i) := \text{diag}\{U(i), q\}, \forall i \in \mathcal{N}\). So, \(\forall i \in \mathcal{N}\), one gets that \(\hat{U}(i)\) is non-singular as \(\hat{T}(i) > 0\). The following inequality can be proved easily.
\[
\hat{U}^T(i)\hat{T}^{-1}(i)\hat{U}(i) \geq \hat{U}(i) + \hat{U}^T(i) - \hat{T}(i)
\]

which implies:
\[
U^T(i)T^{-1}(i)U(i) - R(i) \geq U(i) + U^T(i) - T(i) - R(i)
\]

\[
q^2 - r \geq 2q - (h + r)
\]

Therefore,
\[
\Omega(i,j) = \begin{bmatrix} \hat{S}(i) & 0_{n \times 1} & \hat{S}_3(i) \\
0_{1 \times n} & \frac{q}{2} - r & 0_{1 \times n} \\
\end{bmatrix} \geq 0,
\]

(17)

where \(\hat{S}(i) = \begin{bmatrix} U^T(i)T^{-1}(i)U(i) - R(i) \end{bmatrix}\). Define \(\mathcal{P}(i)\) as \(\mathcal{P}(i) = \text{diag}\{U^{-1}(i), \frac{1}{q}, I_n, 1\}\). Then from (17):
\[
\mathcal{P}(i)\Omega(i,j) \mathcal{P}(i) \geq 0
\]

\[
\Rightarrow \begin{bmatrix} \hat{T}^{-1}(i) - \mathcal{L}(i) \end{bmatrix} \begin{bmatrix} \hat{S}(i) & 0_{n \times 1} & \hat{S}_3(i) \\
0_{1 \times n} & \frac{q}{2} - r & 0_{1 \times n} \\
\end{bmatrix} \begin{bmatrix} \hat{S}(i) & 0_{n \times 1} & \hat{S}_3(i) \\
0_{1 \times n} & \frac{q}{2} - r & 0_{1 \times n} \\
\end{bmatrix} \geq 0,
\]

(18)

As \(\hat{T}(j) > 0\), from (18), we get the following inequality using Schur complement:
\[
\mathcal{S}(i,j) = \begin{bmatrix} \mathcal{S}_{11}(i,j) & \mathcal{S}_{12}(i,j) \\
\mathcal{S}_{12}(i,j) & \mathcal{S}_{22}(i,j) \end{bmatrix} \leq 0,
\]

(19)

where, \(\mathcal{S}_{11}(i,j) = \hat{S}_K(i)\hat{T}^{-1}(j)\hat{S}_K(i) - \hat{T}^{-1}(j) + \mathcal{L}(i), \mathcal{S}_{12}(i,j) = \mathcal{S}_K(i)\hat{T}^{-1}(j)D_1 - \mathcal{D}_K(i), \mathcal{S}_{22}(i,j) = \hat{D}_T^T\hat{T}^{-1}(j)D_1 - (\hat{D}_2^2 + D_2) + G(i)\).

Now, consider a piecewise quadratic storage function of the form \(V(x_k) = \frac{1}{2}x_k^T\hat{T}^{-1}(j)x_k\) for \(x_k \in \mathcal{X}_i\). Substituting
\[ u_k = K(i)x_k \text{ for } x_k \in \mathcal{X}_i, \] and using (1) we get:

\[ V(x_{k+1}) - V(x_k) - z_k^T w_k = \frac{1}{2} T_k - T_{k+1} (j)x_{k+1} - \frac{1}{2} x_k^T T_k - 1 (i)x_k - z_k^T w_k = \frac{1}{2} \left[ \left( \omega K(i) \hat{x}_k + D_1 w_k + m(x_k, i) \right)^T T_k - 1 (j)(\omega K(i) \hat{x}_k + D_1 w_k + m(x_k, i) - x_k^T T_k - 1 (i)x_k - \left( \omega K(i) \hat{x}_k + D_2 w_k + n(x_k, i) \right)^T T_k - 1 (i)x_k \right] - w_k^T \left( \omega K(i) \hat{x}_k + D_2 w_k + n(x_k, i) \right) w_k \]

\[ = \frac{1}{2} \left[ \left( \omega K(i) \hat{x}_k + D_1 w_k \right)^T T_k - 1 (j)(\omega K(i) \hat{x}_k + D_1 w_k) - x_k^T T_k - 1 (i)x_k - \left( \omega K(i) \hat{x}_k + D_2 w_k \right)^T T_k - 1 (i)x_k - w_k^T \left( \omega K(i) \hat{x}_k + D_2 w_k \right) + 2x_k^T \omega K(i) \hat{x}_k - 1 (j)m(x_k, i) + 2w_k^T D_2^T T_k - 1 (j)m(x_k, i) + m^T (x_k, i) T_k - 1 (j)m(x_k, i) - 2w_k^T n(x_k, i) \right]. \]

Similar to (10), (12), (13), (14), the following inequalities can be derived.

Also, it is easy to show that:

\[ (\omega K(i) \hat{x}_k + D_1 w_k)^T T_k - 1 (j)(\omega K(i) \hat{x}_k + D_1 w_k) - x_k^T T_k - 1 (i)x_k = (\omega(i) \hat{x}_k + D_1 w_k)^T T_k - 1 (j)(\omega(i) \hat{x}_k + D_1 w_k) - x_k^T T_k - 1 (i)x_k, \]

Then, one gets the following:

\[ V(x_{k+1}) - V(x_k) - z_k^T w_k = \frac{1}{2} x_k^T A_{12}(i, j) x_k + \frac{1}{2} x_k^T A_{12}(i, j) x_k + \frac{1}{2} x_k^T A_{12}(i, j) x_k + w_k^T A_{12}(i, j) x_k, \]

where \( A_{12}(i, j) = \omega K(i) \omega(i) \hat{x}_k + D_1 w_k - x_k^T \bar{T} - 1 (i) + \left( \gamma_1(i, j) + \gamma_2(i, j) + \gamma_3(i, j) + \gamma_4(i, j) \right) I_{n+1} \).

Similarly, the system (21) becomes equivalent to the following closed-loop PWA affine system with a piecewise linear state-feedback control law:

\[ x_{k+1} = (A(i) + v_B K(i)) x_k + a(i) + D_1 w_k + m(x_k, i) \]

\[ z_k = (C(i) + v_B K(i)) x_k + c(i) + D_2 w_k + n(x_k, i), \]

(20)

Note that if one puts \( m(x_k, i) \equiv 0 \), then the nonlinear system (20) becomes passive with respect to the control input \( v_k \).

B. Feedback passivity over erasure network

In this section, we deal with the problem of feedback passivity over a Gilbert-Elliott type communication channel.

Suppose \( u_k \) is the controller output and is sent to the actuators through a lossy network. Then, under the zero-input scheme [17], one can relate \( u_k \) (as defined in (1)) with \( u_k' \) by the expression \( u_k = v_k u_k' \), where \( v_k \) is binary random variable, and can either be 0 or 1. It represents the packet loss condition in the channel. At a time index \( k \), \( v_k = 0 \) (\( v_k = 1 \)) denotes a packet being lost (a successful packet delivery) from the controller side to the actuator side.

In this work we consider a TCP-like protocol wherein packet reception is acknowledged. Under such a protocol, with perfect state knowledge, one can define an information set given by: \( I_k = \{ x_0, x_1, ..., x_k, v_0, v_1, ..., v_k \} \).

The Gilbert-Elliott type channel model is basically a two-state Markov chain \( \{ v_k \} \), where \( v_k = 0 \) and \( v_k = 1 \) represent the two states of the Markov chain. At a time stage \( k \geq 1 \), the packet arrival probabilities are given as: \( Pr(v_k = 1 | v_{k-1} = 0) = \alpha \) and \( Pr(v_k = 1 | v_{k-1} = 1) = 1 - \beta \). At \( k = 0 \), packet arrival probabilities are given by: \( Pr(v_0 = 1) = \alpha / (\alpha + \beta) \) \[ Pr(v_0 = 0) = \beta / (\alpha + \beta) \] [18].

Consider a piecewise linear state-feedback control law \( u_k' = K(i) x_k \), where \( i \) is the index of the partition. Then, the nonlinear system (1), with a polyhedral partition \( \{ \mathcal{X}_i \}_{i \in \mathcal{N}} \), takes the form:

\[ x_{k+1} = (A(i) + v_B K(i)) x_k + a(i) + D_1 w_k + m(x_k, i) \]

\[ z_k = (C(i) + v_B K(i)) x_k + c(i) + D_2 w_k + n(x_k, i), \]

if \( x_k \in \mathcal{X}_i \).

Observe that due to the randomness of packet losses \( v_k \), the closed-loop systems (20) and (21) become stochastic in nature. Thus, to analyze the feedback passivity of the system (20) and (21), one needs a notion of stochastic passivity. Stochastic passivity, in the spirit of [19], is defined as follows:

\[ Definition 2: \] The system (20) (similarly the system (21)) is said to be passive in the stochastic sense if there exists a nonnegative function \( V: \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^+ \) with \( V(0,.) = 0 \), called the storage function, such that for all \( x_k \in \mathbb{R}^n \), \( w \in \mathbb{R}^a \) and for all \( k \in \mathbb{Z}^+ \):

\[ \mathbb{E} \left[ V(x_{k+1}, s_{k+1}) | I_k \right] - V(x_k, s_k) \leq \mathbb{E} \left[ z_k^T w_k | I_k \right], \]

where, \( s_k \in \mathcal{N} \) denotes the cell in which \( x_k \) lies in.

Following theorem presents results for feedback passivity with random packet losses.

\[ Theorem 3: \] Consider the nonlinear system (20) with given control packet arrival probabilities \( \alpha \) and \( 1 - \beta \). With a control input \( u_K = K(i) x_k = W(i)U^{-1}(i)x_k \), the nonlinear system becomes stochastically passive if, for all \( i, j \in \mathcal{N} \), there exist matrices \( T(i), U(i), W(i), R(i) > 0 \), \( G(i) > 0 \), and positive scalars \( h, r, q \) such that:

\[ T(j) = \begin{bmatrix} T(j)_{11} & 0 \\ 0 & T(j)_{22} \end{bmatrix} > 0, \]

(22a)
\[
\begin{bmatrix}
\Omega_{11}(i) & 0_{n \times 1} & \Omega_{12}(i) & \Omega_{14}(i) & 0_{n \times 1} & \Omega_{16}(i) & 0_{n \times 1} \\
0_{n \times 1} & \Omega_{22}(i) & \Omega_{24}(i) & 0_{n \times 1} & \Omega_{26}(i) & 0_{n \times 1} \\
\Omega_{22}(i) & \Omega_{24}(i) & \Omega_{33} & D_1^T & D_1 & 0_{n \times 1} \\
\Omega_{24}(i) & \Omega_{26}(i) & D_1 & \Omega_{44}(j) & 0_{n \times 1} & 0_{n \times 1} \\
0_{1 \times n} & q_{1 \times 1} & 0_{1 \times n} & \Omega_{45} & 0_{1 \times n} & 0 \\
0_{1 \times n} & q & 0_{1 \times n} & 0_{1 \times n} & \Omega_{66}(l) & 0_{1 \times n} \\
0_{1 \times n} & q & 0_{1 \times n} & 0_{1 \times n} & 0 & \Omega_{77} \\
\end{bmatrix} \geq 0
\]  

(22b)

\[
\begin{bmatrix}
\rho_1(i, j) + \rho_2(i, j) + \rho_3(i, j) + \rho_4(i, l) + \rho_5(i, l) + \rho_6(i, l) + \rho_7(i, l) \mid I_{n \times 1} \leq \mathcal{L}(i) \\
\end{bmatrix},
\]

(22c)

where

\[
\Omega_{11}(i) = \left[ U(i) + U^T(i) - T(i) \right] - R(i)
\]

\[
\Omega_{22} = 2h - (h + r)
\]

\[
\Omega_{13}(i) = U^T(i)C^T(i) + \tilde{p}_k W^T(i)B_2^T
\]

\[
\Omega_{14}(i) = U^T(i)A^T(i) + W^T(i)B_1^T, \quad \Omega_{16}(i) = U^T(i)A^T(i)
\]

\[
\Omega_{33} = \left(D_1^T + D_2^T - G(i), \Omega_{44}(j) = \frac{1}{\tilde{p}_k}T(j), \Omega_{55} = \frac{h}{\tilde{p}_k} \right]
\]

\[
\Omega_{66}(l) = \frac{1}{1 - \tilde{p}_k}T(l), \quad \Omega_{77} = \frac{h}{1 - \tilde{p}_k}
\]

\[
\mathcal{L}(i) \text{ is defined as in Theorem 2,}
\]

\[
\rho_1(i, j) = 2\tilde{p}_k\epsilon(i)\|\mathcal{A}_K(i)\|\|T^{-1}(j)\|
\]

\[
\rho_2(i, j) = \tilde{p}_k\epsilon(i)\|T^{-1}(j)\|\|D_1\|
\]

\[
\rho_3(i, j) = \tilde{p}_k\epsilon^2(i)\|T^{-1}(j)\|
\]

\[
\rho_4(i, l) = 2(1 - \tilde{p}_k)\epsilon(i)\|\tilde{A}(i)\|\|T^{-1}(l)\|
\]

\[
\rho_5(i, l) = (1 - \tilde{p}_k)\epsilon(i)\|T^{-1}(l)\|\|D_1\|
\]

\[
\rho_6(i, j) = (1 - \tilde{p}_k)\epsilon^2(i)\|T^{-1}(l)\|
\]

\[
\rho_7(i, j) = \delta(i), \quad \tilde{p}_k = \begin{cases} \alpha, & \text{if } v_{k-1} = 0 \\ 1 - \beta, & \text{if } v_{k-1} = 1 \end{cases}
\]

**Proof:** Assume that LMIs (22a) and (22b) are satisfied. Using the same line of argument as used in the proof for Theorem 2, one gets that \( \hat{U}(i) := \text{diag}(U(i), q) \) is non-singular. Consider the following:

\[
\Omega(i, j) = \begin{bmatrix}
\Omega_1(i, j) & 0_{n \times 1} & \Omega_2(i, j) & \Omega_3(i, j) & 0_{n \times 1} & \Omega_4(i, j) & 0_{n \times 1} \\
0_{n \times 1} & \Omega_2(i, j) & q(i, j) & \Omega_3(i, j) & 0_{n \times 1} & \Omega_4(i, j) & 0_{n \times 1} \\
\Omega_2(i, j) & \Omega_3(i, j) & \Omega_4(i, j) & 0_{n \times 1} & \Omega_5(i, j) & 0_{n \times 1} \\
0_{1 \times n} & q_{1 \times 1} & 0_{1 \times n} & \Omega_5(i, j) & 0_{1 \times n} & 0 \\
0_{1 \times n} & q & 0_{1 \times n} & 0_{1 \times n} & \Omega_6(i, j) & 0_{1 \times n} \\
0_{1 \times n} & q & 0_{1 \times n} & 0_{1 \times n} & 0 & \Omega_7(i, j) \\
\end{bmatrix}
\]

where \( \Omega_1(i, j) = [U^T(i)T^{-1}(i)U(i)] - R(i), \quad \Omega_2 = \frac{q^T}{h} - r. \)

From (22b), using same reasoning as used in the proof for Theorem 2, it can be proved that \( \Omega'(i, j) \geq 0 \).

Then, with \( \mathcal{P}(i) = \text{diag}\left\{ U^{-1}(i), \frac{1}{q}, I_1, I_n, 1, I_n, 1 \right\} \),

\[
\mathcal{P}^T \Omega'(i, j) \mathcal{P} \geq 0
\]

Or

\[
\begin{bmatrix}
\tilde{T}^{-1}(i) - \mathcal{L}(i) & \mathcal{A}_K(i)^T & \mathcal{A}_K(i) & \hat{A}(i) \\
\mathcal{A}_K(i) & D_1 & \Lambda_1(j) & \Lambda_3 \\
\hat{A}(i) & \hat{D}_1 & \Lambda_2(l) & \Lambda_3 \\
\end{bmatrix} \geq 0,
\]

(24)

where,

\[
\Lambda_1(j) = \tilde{p}_k^{-1}\tilde{T}(j), \quad \Lambda_2(l) = \left(1 - \tilde{p}_k^{-1}\right)T(l), \quad \Lambda_3 = 0_{(n+1) \times (n+1)}
\]

\[
\mathcal{C}_K(i) = C(i) + \tilde{p}_k B_2(i) K(i), \quad \mathcal{C}_K(i) = \left[ \mathcal{C}_K(i), c(i) \right]
\]

Using Schur complement, as \( \tilde{T}(j) > 0 \) for all \( j \in N \), implies:

\[
\begin{bmatrix}
\mathcal{D}_K(i) & \hat{A}(i) & \Lambda_1 & \Lambda_2 \\
\hat{D}_1 & \hat{D}_1 & \Lambda_3 \\
\end{bmatrix} \geq 0
\]

(25)

where,

\[
\begin{bmatrix}
\mathcal{D}_K(i) & \hat{A}(i) & \Lambda_1 & \Lambda_2 \end{bmatrix} \geq 0
\]

Using a piecewise quadratic storage function of the form \( V(x_k, i) = \frac{1}{2}x_k^T T^{-1}(i) x_k \) if \( x_k \in \mathcal{X}_i \). Assume that \( x_{k+1} \in \mathcal{X}_i \), \( x_{k+1} = 1 \), and \( x_{k+1} \in \mathcal{X}_j \) if \( v_k = 0 \). Note that \( s_{k+1} \) can either be \( j \) or \( l \) depending on \( v_k \). Thus, with a control law \( u_k = W(i)U^{-1}(i)x_k = K(i)x_k \) if \( x_k \in \mathcal{X}_i \), one gets:

\[
\begin{bmatrix}
\mathcal{F}_K(i) \tilde{x}_k + D_1 w_k + m(x_k, i) \\
\hat{A}(i) \tilde{x}_k + D_1 w_k + m(x_k, i) \\
\hat{A}(i) \tilde{x}_k + D_2 w_k + m(x_k, i) \\
\hat{A}(i) \tilde{x}_k + D_2 w_k + m(x_k, i) \\
\end{bmatrix} \geq 0
\]


Using the same line of argument as used in the proof for Theorem 2, we get:
\[
\mathbb{E}
\left[
V(x_{k+1}, x_{k+1}) - V(x_k, i) - \mathbb{E}
\left[
\begin{bmatrix}
\mathcal{J}_1(i, j) & \mathcal{J}_2(i, j)
\end{bmatrix}
\begin{bmatrix}
x_k^T w_k
\end{bmatrix}
\right]
\right] \leq \left[\begin{bmatrix}
\bar{x}_k^T & \bar{w}_k^T
\end{bmatrix}
\right]\left[
\begin{bmatrix}
\mathcal{J}_1(i, j) & \mathcal{J}_2(i, j)
\end{bmatrix}
\begin{bmatrix}
x_k^T w_k
\end{bmatrix}
\right].
\]
Now, from \cite{25}:
\[
\mathbb{E}
\left[
V(x_{k+1}, x_{k+1}) - V(x_k, i) - \mathbb{E}
\left[
\begin{bmatrix}
\bar{x}_k^T w_k
\end{bmatrix}
\right]
\right] \leq 0.
\]
Hence, closed-loop system \cite{20} is stochastically passive. \hfill \Box

It is straightforward to see that, if one puts \(\epsilon(i) = 0\) and \(\delta(i) = 0\) in the above theorem, for all \(i \in \mathcal{N}\), then the result corresponds to the result for feedback passivity of an PWA system over a Gilbert-Elliott type channel.

**Corollary 4:** Consider the PWA system given by \cite{21} with given control packet arrival probabilities \(\alpha\) and \(1 - \beta\). The PWA system, with a piecewise linear state-feedback control law of the form \(u_k = K(i) x_k = W(i) U(i) x_k\), becomes stochastically passive if, for all \(i, j \in \mathcal{N}\), there exists a matrices \(T(i) > 0, U(i), W(i)\), and positive scalars \(q, h\) such that:
\[
\dot{T}(j) = T(j) \begin{bmatrix}
0_{nxn} & 0_{nx1} \\
0_{1xn} & h
\end{bmatrix}
\geq 0,
\]
where
\[
\begin{bmatrix}
\Omega_{11}(i) & \Omega_{13}(i) & \Omega_{14}(i) & \Omega_{16}(i) & \Omega_{18}(i) \\
\Omega_{21}(i) & \Omega_{22}(i) & \Omega_{24}(i) & \Omega_{26}(i) & \Omega_{28}(i) \\
\Omega_{31}(i) & \Omega_{32}(i) & \Omega_{34}(i) & \Omega_{36}(i) & \Omega_{38}(i) \\
\Omega_{41}(i) & \Omega_{42}(i) & \Omega_{44}(i) & \Omega_{46}(i) & \Omega_{48}(i) \\
\Omega_{51}(i) & \Omega_{52}(i) & \Omega_{54}(i) & \Omega_{56}(i) & \Omega_{58}(i)
\end{bmatrix}
\]
\[
\geq 0
\]
\[
\Omega_{66}(i) = \frac{1}{1 - \frac{\rho_k}{p_k}} T(l), \quad \Omega_{77} = \frac{h}{1 - \frac{\rho_k}{p_k}}.
\]

### IV. Numerical Example

Consider the nonlinear system \cite{1} with the following system parameters:
\[
f(x_k) = [4 \sin(x_k) + x_k^2, x_k^4, x_k^3, x_k^5]^T, \quad B_1 = [2, 0, 1]^T, \quad D_1 = [1, 0.5, 0]^T, \quad C = [1 0 0], \quad B_2 = 0.1, \quad D_2 = 2,
\]
where \(x_k = [x_k^1, x_k^2, x_k^3]^T\). External input is assumed to be \(w_k = 0.02 \sin(0.2 \pi k) \exp(-k/25)\).

To demonstrate the results presented in Theorem 2, we calculate a piecewise linear state-feedback law that passes the system in the region \(-0.82 \leq x_k^1 \leq 0.82\) (note that nonlinearity exists only in \(x_k^1\)). The region \(-0.82 \leq x_k^1 \leq 0.82\) is partitioned into 14 cells, which are given by: \(-0.82 \leq x_k^1 < -0.78, -0.78 \leq x_k^1 < -0.74, -0.74 \leq x_k^1 < -0.7, -0.7 \leq x_k^1 < -0.65, -0.65 \leq x_k^1 < -0.6, -0.6 \leq x_k^1 < -0.55, -0.55 \leq x_k^1 < -0.5, -0.5 \leq x_k^1 < -0.45, -0.45 \leq x_k^1 < -0.4, -0.4 \leq x_k^1 < -0.34, -0.34 \leq x_k^1 < -0.28, -0.28 \leq x_k^1 < -0.13, -0.13 \leq x_k^1 < 0, 0 \leq x_k^1 < 0.13, 0.13 \leq x_k^1 < 0.28, 0.28 \leq x_k^1 < 0.34, 0.34 \leq x_k^1 < 0.4, 0.4 \leq x_k^1 < 0.45, 0.45 \leq x_k^1 < 0.5, 0.5 \leq x_k^1 < 0.55, 0.55 \leq x_k^1 < 0.6, 0.6 \leq x_k^1 < 0.65, 0.65 \leq x_k^1 < 0.7, 0.7 \leq x_k^1 < 0.74, 0.74 \leq x_k^1 < 0.78, 0.78 \leq x_k^1 < 0.82\).

Piecewise linear approximations are computed using the Taylor series expansion and the error term \(\epsilon(i)\) is calculated in each cell. We use \textit{lmisolver} function available in SCILAB to solve the LMIs in each of the cells. Solving \cite{15b} we then calculate the controller gain \(K(i)\) such that the closed system becomes feedback passive. The given cells are designed in such a way that the corresponding error term \(\epsilon(i)\), in each \(i \in \mathcal{N}\), is almost the maximum value that satisfies the conditions given by \cite{15c} and \cite{15d}. From figure 1 one can see that difference in storage function at each stage is less than or equal to the supply rate.

For feedback passivity with packet losses, we consider the region \(-0.5 \leq x_k^1 \leq 0.5\). The region \(-0.5 \leq x_k^1 \leq 0\) is partitioned into cells: \(-0.5 \leq x_k^1 < -0.47, -0.47 \leq x_k^1 < -0.44, -0.44 \leq x_k^1 < -0.41, -0.41 \leq x_k^1 < -0.38, -0.38 \leq x_k^1 < -0.35, -0.35 \leq x_k^1 < -0.32, -0.32 \leq x_k^1 < -0.29, -0.29 \leq x_k^1 < -0.26, -0.26 \leq x_k^1 < -0.23, -0.23 \leq x_k^1 < -0.2, -0.2 \leq x_k^1 < -0.17, -0.17 \leq x_k^1 < -0.14, -0.14 \leq x_k^1 < -0.11, -0.11 \leq x_k^1 < -0.07, -0.07 \leq x_k^1 < 0\). The region \(0 \leq x_k^1 \leq 0.5\) is partitioned in the similar fashion as the partition of the region \(-0.5 \leq x_k^1 \leq 0\), i.e., \(0 \leq x_k^1 < 0.07, 0.07 \leq x_k^1 < 0.11\) and so on. Then, solving LMI \cite{22} in Theorem 3, we calculate the controller gain \(K(i)\) in each cells with a control packet arrival probability \(\alpha = 0.95\) and \(1 - \beta = 0.96\). Figure 2 demonstrates that the closed-loop system is passive with the quadratic storage function \(V(x_k) = \frac{1}{2} x_k^T T^{-1}(i) x_k\).

It is to be noted that, for feedback passivity with packet losses, we have to consider smaller cells as compared to feedback passivity. This is due to the fact that condition \cite{22} contains the term \(\rho_h(i, l)\) which comes from open loop system dynamics.

![Figure 1](image-url)
In this paper, using a piecewise affine approximation approach, we have first derived sufficient conditions that ensure passivity of a smooth nonlinear system. Then, we have designed a piecewise linear state-feedback control law with which the closed-loop system becomes passive. Finally, with random control packet losses, a piecewise linear state-feedback control law is derived that makes the closed-loop system passive. Moreover, the problem of controller design for feedback passification of a PWA system over a lossy communication channel is also addressed as a special case of Theorem 3.

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