ANALYTIC SOLUTIONS TO THE CONSTRAINT EQUATION FOR A FORCE-FREE MAGNETOSPHERE AROUND A KERR BLACK HOLE

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ABSTRACT

The Blandford-Znajek constraint equation for a stationary, axisymmetric black hole force-free magnetosphere is cast in a 3+1 absolute space and time formulation, following the work of Komissarov. We derive an analytic solution for fields and currents to the constraint equation in the far-field limit that satisfies the Znajek condition at the event horizon. This solution generalizes the Blandford-Znajek monopole solution for a slowly rotating black hole to black holes with arbitrary angular momentum. Energy and angular momentum extraction through this solution occurs mostly along the equatorial plane. We also present a nonphysical, reverse-jet–like solution.

Subject headings: black hole physics — gravitation

1. INTRODUCTION

Penrose (1969) recognized the possibility of extracting the spin energy of a black hole using particle decay in negative energy orbits within the ergosphere. On the basis of studies of force-free pulsar magnetospheres, Blandford & Znajek (1977) proposed that rotational energy could be extracted through currents flowing in the black hole’s magnetosphere. In this picture, strong electric and magnetic fields are induced by gravito-MHD (GMHD) processes. Blandford & Znajek (1977) derived the equations for a stationary, axisymmetric force-free magnetosphere in curved spacetime and reduced the set of equations to a central constraint equation relating toroidal magnetic field $H_x$ to the charge density $\rho$ and toroidal current density $J_x$. They also found a perturbative solution to the constraint equation valid in the limit $a/M \ll 1$, where $M$ is the black hole mass and $a$ is the angular momentum per unit mass.

Thorne & MacDonald (1982) and MacDonald & Thorne (1982) developed this theory in a more intuitive “3+1” formulation that led to the membrane paradigm (Thorne et al. 1986), where the equations of GMHD were written using the familiar electric and magnetic 3-vectors in absolute space whose time dependence is governed by Maxwell-type equations. Komissarov (2004) recently presented the essential equations of this formulation in a form useful for numerical studies and helped resolve questions (Punsly & Coroniti 1990; Punsly 2001) relating to energy extraction in the membrane paradigm.

The equations presented by Komissarov (2004) provide a useful starting point to search for analytic solutions. Here we use these equations to rederive the constraint equation of Blandford & Znajek (1977) in the 3+1 form. This brings forth a clear understanding of the nature of the poloidal functions defining the currents and fields. We have discovered an analytical solution valid for arbitrary angular momentum that reduces to the monopole solution of Blandford & Znajek (1977) in the limit of $a/M \ll 1$. This solution, which satisfies the Znajek (1977) regularity condition, permits energy extraction preferentially along the equatorial direction of the Kerr black hole.

The modified Maxwell’s equations in curved spacetime are given in § 2, and the equations for a force-free magnetosphere are presented in § 3. In § 4, we construct the form of fields and currents for a given poloidal function $\Omega$. The governing constraint equation for this function is given in § 5. Solutions to this equation are derived in § 6, and we summarize in § 7.

2. ELECTRODYNAMICS IN ABSOLUTE SPACE

While Maxwell’s equations preserve all of their elegance in a covariant formalism on a four-dimensional manifold, they distract from some of the simple (far-field) solutions that they might permit. With this in mind, we briefly state the essential equations of electrodynamics in an absolute three-dimensional space. The recent paper by Komissarov (2004) explains how these equations are derived.

The construction of absolute space is facilitated by noting that an arbitrary spacetime metric can be written in the form

$$ds^2 = (\beta^2 - \alpha^2) dt^2 + 2\beta \alpha dx^i dt + \gamma_{ij} dx^i dx^j .$$

The functions $x^i$ serve as coordinates for our spacelike hypersurfaces defined by constant values of $t$. Consider one such hypersurface $\Sigma$ defined by the region $t = 0$. We can think of electric and magnetic fields $(\mathbf{E} \text{ and } \mathbf{B})$ as objects existing in our absolute space $\Sigma$. The time evolution equations for $\mathbf{E}$ and $\mathbf{B}$ in the presence of a charge density $\rho$ and electric current density vector $\mathbf{J}$ in our absolute (curved) space endowed with a metric $\gamma_{ij}$ are given by the following set of Maxwell’s equations:

$$\nabla \cdot \mathbf{B} = 0 ,$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 ,$$

and their inhomogeneous counterparts,

$$\nabla \cdot \mathbf{D} = \rho ,$$

$$-\partial_t \mathbf{D} + \nabla \times \mathbf{H} = \mathbf{J} .$$

It is important to remember that $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{J}$ are vectors in our three-dimensional absolute space $\Sigma$ and in general are time-dependent. Also, $\nabla$ is the covariant derivative induced by the metric $\gamma_{ij}$ on $\Sigma$. As usual, the curl of a vector field is defined by the expression

$$(\nabla \times \mathbf{A})^i = \epsilon^{ijk} \nabla_j A_k ,$$

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where \( \epsilon^{ijk} \) is the completely antisymmetric pseudotensor such that \( \epsilon_{123} = (\gamma)^{-1/2} \) and \( \gamma = \det(\gamma_{ij}) \). It is easily seen that Maxwell’s equations imply the continuity equation
\[
\partial_t \rho + \nabla \cdot J = 0. \tag{7}
\]
Unlike its flat space counterparts, even in regions of negligible electric and magnetic susceptibilities, \( E \neq D \) and \( B \neq H \). Indeed, it can be shown that they instead satisfy the constitutive relations
\[
E = \alpha D + \beta \times B \tag{8}
\]
and
\[
H = \alpha B - \beta \times D. \tag{9}
\]
Of interest are spacetimes admitting Killing fields corresponding to axial symmetry \((m)\) and stationarity. Consequently, Noether’s theorem implies energy and angular momentum conservation laws. They can be stated in the form
\[
\partial_t e + \nabla \cdot S = -(E \cdot J) \tag{10}
\]
and
\[
\partial_t l + \nabla \cdot L = -(\rho E + J \times B) \cdot m. \tag{11}
\]
Here
\[
e = \frac{1}{2} (E \cdot D + B \cdot H) \tag{12}
\]
is the volume density of energy,
\[
l = (D \times B) \cdot m \tag{13}
\]
is the density of angular momentum,
\[
S = E \times H \tag{14}
\]
is the flux of energy, and
\[
L = -(E \cdot m) D - (H \cdot m) B + \frac{1}{2} (E \cdot D + B \cdot H) m \tag{15}
\]
is the flux of angular momentum.

3. STATIONARY, AXISYMMETRIC, FORCE-FREE MAGNETOSPHERES

The condition that the magnetosphere is force-free brings about enough structure into Maxwell’s equations to enable the introduction of a streaming function that will help us visualize the field structure in geometric terms. It is traditional to use spheroidal spatial coordinates given by \( x = (\rho, \theta, \varphi) \) such that \( m = \partial_\varphi \). Assumptions of stationarity and axisymmetry imply that \( \partial_\rho g_{\mu \nu} = 0 = \partial_\theta g_{\varphi \rho} \).

In our absolute space framework, the force-free condition reduces to
\[
E \cdot J = 0, \tag{16}
\]
and
\[
\rho E + J \times B = 0. \tag{17}
\]
These restrictions, along with Maxwell’s equations and equations (8) and (9), imply that
\[
E_T = 0, \tag{18}
\]
and
\[
E_P \cdot B_P = 0. \tag{19}
\]
The poloidal and toroidal components \((A_P \text{ and } A_T)\) of a vector field are defined such that \( A = A_P + A_T \), where \( A_P = A' \partial_\rho + A^\theta \partial_\theta \text{ and } A_T = A^\varphi \partial_\varphi \). Equations (18) and (19) imply that there exists a vector \( \omega = \Omega \partial_\varphi \) such that
\[
E = -\omega \times B, \tag{20}
\]
From the vanishing of the curl of \( E \) under the stationarity condition (eq. [3]), one finds that
\[
B \cdot \nabla \Omega = 0. \tag{21}
\]
It can also be shown that
\[
B \cdot \nabla H_\varphi = 0. \tag{22}
\]

4. EXPLICIT EXPRESSIONS FOR FIELDS AND CURRENTS

To simplify calculations, we assume that the spatial coordinates are orthogonal, and that the shift vector \( \beta \) is purely toroidal, i.e., \( \beta = (0,0,\beta_\varphi) \). The Kerr solution written in Boyer-Lindquist (though not Kerr-Schild) coordinates can be written in this form.

Surfaces of constant \( \Omega \) are referred to as poloidal surfaces (not to be confused with poloidal components of a vector). From equation (21) it is clear that \( B \) is tangent to poloidal surfaces. Since \( \Omega \) does not have any \( \varphi \)-dependence, and since equation (21) has nothing to say about the toroidal component of \( B \), it is clear that \( B_P \) will entertain solutions of the type
\[
B_P = \frac{\Lambda}{\sqrt{\gamma}} (-\Omega_\theta \partial_\rho + \Omega_\rho \partial_\theta), \tag{23}
\]
where, for the moment, \( \Lambda \) is an arbitrary function. This must be so because in the two-dimensional subspace given by \( \Omega = \) constant, there is a unique vector (modulo magnitude) that is perpendicular to \( \nabla \Omega \). The condition that \( B \) is divergence-free means that \( \Lambda \) satisfies
\[
\Lambda_\rho \Omega_\theta = \Lambda_\theta \Omega_\rho. \tag{24}
\]
Consequently, \( \Lambda \) is a poloidal function (a function that is constant on poloidal surfaces). In the notation of the original paper by Blandford & Znajek (1977), \( \Lambda \, d\Omega \equiv -dA_\varphi \). The electric field is immediately calculated from equation (20) and, as expected, is the gradient of a scalar function:
\[
E_P = \Lambda \, d(\Omega^2/2) = d \int \Lambda \Omega \, d\Omega. \tag{25}
\]
From equation (8), we see that
\[
D = D_P = \frac{\Lambda}{\alpha} (\Omega + \beta^\varphi) \, d\Omega. \tag{26}
\]
Similarly, the expression for $H_p$ can be calculated from equation (8), giving

$$H_p = \frac{1}{\alpha^2 - \beta^2 - \beta_s^2 \Omega} \frac{B_p}{\alpha}. \quad (27)$$

The electric charge is determined by the divergence of the $D_p$ (eq. [4]). Explicitly,

$$\sqrt{\gamma} \rho = \partial_r \left[ \frac{\Lambda}{\alpha \sqrt{\gamma}} \left( \gamma_{\varphi r} \Omega + \beta_r \right) \gamma_{\vartheta \vartheta} \right] r \theta$$

$$+ \partial_\theta \left[ \frac{\Lambda}{\alpha \sqrt{\gamma}} \left( \gamma_{\varphi \vartheta} \Omega + \beta_\vartheta \right) \gamma_{r \vartheta} \Omega \right]. \quad (28)$$

The toroidal component of the electric current density vector can be obtained from the derivatives of components of $H_p$:

$$\sqrt{\gamma} J^r = H_{\vartheta r} - H_{r \vartheta}$$

$$= \partial_r \left[ \frac{\Lambda}{\alpha \sqrt{\gamma}} \left( \alpha^2 - \beta^2 - \beta_s^2 \Omega \right) \gamma_{\vartheta \vartheta} \right] r \theta$$

$$+ \partial_\theta \left[ \frac{\Lambda}{\alpha \sqrt{\gamma}} \left( \alpha^2 - \beta^2 - \beta_s^2 \Omega \right) \gamma_{r \vartheta} \Omega \right]. \quad (29)$$

It is clear from the above discussion that the poloidal fields and, consequently, the toroidal current $J^t$ are uniquely described by the poloidal functions $\Omega$ and $\Lambda$. On the other hand, the toroidal fields and the poloidal currents can be determined from the poloidal function $H_p$. In particular, from equation (9), it is clear that $H_p = \alpha B_p$. Maxwell’s equation (eq. [5]) implies that

$$\sqrt{\gamma} J^r = H_{\vartheta r} - H_{r \vartheta}.$$ 

Thus, we see that fields and currents separate into two distinct categories: objects that are determined by $\Omega$ and $\Lambda$, and those that are determined by $H_p$. Outside of the fact that $H_p$ is a poloidal function (by definition, $\Omega$ is), it is not yet clear as to how these two functions are dynamically related. This issue is resolved in the following section.

5. THE CONSTRAINT EQUATION

The expressions for the fields and currents given in the previous section naturally satisfies equation (16). Since the toroidal component of the electric field vanishes, it is easily checked that, from equation (17), $(J \times B)_p = 0$ (as shown below in eq. [32]). Thus, the only remaining requirement for a force-free solution is

$$\rho E_p + (J \times B)_p = 0. \quad (31)$$

The implication of the above equation is most easily understood by projecting the equation onto $E_p, B_p$, which serve as basis vectors for poloidal vector fields. The above equation yields no constraint when projected onto $B_p$; i.e.,

$$\rho E \cdot B_p + (J \times B) \cdot B_p = (J \times B) \cdot (B - B_T)$$

$$= -B^2 (J \times B)_r$$

$$= -B^2 \Lambda \frac{1}{\sqrt{\gamma}} (H_{\varphi \vartheta} \Omega - H_{\varphi r} \Omega) = 0. \quad (32)$$

Projecting equation (17) onto $E_p$ gives

$$\rho E \cdot E_p + (J \times B) \cdot E_p = \rho E^2 + [(J_p + J_T) \times (B_p + B_T)] \cdot E_p$$

$$= \rho E^2 + [(J_p \times B_T) + (J_T \times B_p)] \cdot E_p = 0, \quad (33)$$

since $J_p$ is parallel to $B_p$. With the help of the relations

$$J_p = \frac{1}{\sqrt{\gamma}} \frac{dH_p}{d\Omega} \left( \Omega_r \partial_r - \Omega \partial_\vartheta \right) = -\frac{dH_p}{\Lambda d\Omega} B_p,$$

$$E \cdot B_T - E_\vartheta B_p = \frac{\Lambda^2}{\sqrt{\gamma}} \left[ \gamma_{\vartheta \vartheta} (\Omega \partial_r)^2 + \gamma_{r \vartheta} (\Omega \partial_\vartheta)^2 \right],$$

and

$$E^2 = \frac{\Omega^2}{\gamma_{r \vartheta} \gamma_{\vartheta \vartheta}} \left[ \gamma_{\vartheta \vartheta} (\Omega \partial_r)^2 + \gamma_{r \vartheta} (\Omega \partial_\vartheta)^2 \right], \quad (34)$$

equation (33) reduces to the manageable form

$$\frac{1}{2\Lambda} \frac{dH_p^2}{d\Omega} = \alpha (\rho \Omega \gamma_{\varphi \vartheta} - J_\varphi). \quad (35)$$

This is the final and only constraint equation. If $\Omega$ and $\Lambda$ are picked such that the right-hand side of the above equation is a poloidal function, then $H_p$ continues to be poloidal function. The poloidal functions $\Omega$, $\Lambda$, and $H_p$ then uniquely determine all currents and fields. It is important to realize that $\Omega$ is not to be thought of as a potential: physically relevant quantities like the electric field depend on $\Omega$ directly and are invariant under transformations of the type $\Omega \to \Omega + \text{const}$. The charge density $\rho$ and the toroidal current $J_\varphi$ are functions of $\Omega$ and $\Lambda$ (see eqs. [28] and [29]).

6. ASYMPTOTIC SOLUTIONS
AND ENERGY EXTRACTION

Inserting equations (28) and (29) into equation (35), our constraint equation gives

$$\frac{1}{2\Lambda} \frac{dH_p^2}{d\Omega} = \alpha \frac{\gamma_{\varphi \vartheta}}{\sqrt{\gamma}} \left\{ \partial_r \left[ \frac{\Lambda}{\alpha \sqrt{\gamma}} (\gamma_{\varphi \vartheta} \Omega + \beta_\vartheta) \gamma_{\vartheta \vartheta} \Omega_r \right] \right.$$ 

$$+ \partial_\vartheta \left[ \frac{\Lambda}{\alpha \sqrt{\gamma}} (\gamma_{\varphi \vartheta} + \beta_\vartheta \Omega) \gamma_{r \vartheta} \Omega \right]$$

$$+ \partial_r \left[ \frac{\Lambda}{\alpha \sqrt{\gamma}} (\beta^2 - \alpha^2 + \beta_s^2 \Omega) \gamma_{\vartheta \vartheta} \right]$$

$$+ \partial_\vartheta \left[ \frac{\Lambda}{\alpha \sqrt{\gamma}} (\beta^2 - \alpha^2 + \beta_s \Omega) \gamma_{r \vartheta} \Omega \right] \right\}. \quad (36)$$

Therefore, equation (35) is equivalent to equation (3.14) of Blandford & Znajek (1977) written in the 3+1 formalism. While searching for solutions for the fields and currents that might permit extraction of energy and angular momentum from a rotating black hole, it is advantageous to observe that

$$\frac{d^2 E}{dA dt} = S^s \sqrt{\gamma_{rr}} = -H_{\varphi r} \Omega B^s \sqrt{\gamma_{rr}}, \quad (37)$$

$$\frac{d^2 L}{dA dt} = L^r \sqrt{\gamma_{rr}} = -H_{\varphi \vartheta} B^s \sqrt{\gamma_{rr}}, \quad (38)$$
The choice of $\Omega$ is determined by the Znajek regularity condition applied at the event horizon \( r = M + (M^2 - a^2)^{1/2} \) so as to make $B_\varphi$ finite in the well-behaved (even near the event horizon) Kerr–Schild coordinate system (see Znajek 1977; Komissarov 2004). The Znajek condition can be written as

\[
H_\varphi = \frac{\sin^2 \theta}{\alpha_+} (2r_+M\Omega - a)B_\varphi = \frac{\sin \theta}{\rho_+^2} (2r_+M\Omega - a)f, \tag{45}
\]

where the subscript plus sign indicates that the relevant quantities are to be evaluated at the event horizon and $\rho_+^2 = r_+^2 + a^2 \cos^2 \theta$. From equations (44) and (45), we see that

\[
\pm H_0^2 = \frac{\sin^2 \theta}{\rho_+^2} \left[ (4r_+^2M^2 - \rho_+^4)\Omega^2 - 4r_+Ma\Omega + a^2 \right] f^2. \tag{46}
\]

We consider the solution for $H_\varphi$ such that $H_0^2 = 0$ (it is easily seen that when $H_0 \neq 0$, the resulting solution does not permit a finite rate of energy extraction, and this type of situation is dealt with in §6.2). This is possible if and only if the quantity in the square brackets in the above equation vanishes identically. Solving the resulting quadratic equation for $\Omega$, we find two solutions, namely,

\[
\Omega_+ = 2Mr_+ + \rho_+^2, \tag{47}
\]

and

\[
\Omega_- = \frac{a}{2Mr_+ - \rho_+^2} = \frac{1}{a \sin^2 \theta}. \tag{48}
\]

From equation (43) and the definition of $f$, we see that the only nonvanishing poloidal component of the magnetic field is given by

\[
B'_\pm = \frac{1}{\sqrt{\gamma}} f = \frac{1}{\sqrt{\gamma}} \frac{B_0 \sin \theta (2Mr_+ \pm \rho_+^2)}{\sqrt{(a \sin \theta)^a - (2Mr_+ \pm \rho_+^2)^a}}. \tag{49}
\]

where we have relabeled $C_1$ as $B_0$. It is clear that $\Omega_-$ is an unphysical solution since $B'_-\gamma$ as given above is undefined everywhere.

### 6.1. The $\Omega_+$ Solution

In this case, the nonvanishing components of the fields are

\[
B' = \frac{1}{\sqrt{\gamma}} B_0 \sin \theta \sqrt{a\Omega_H},
\]

\[
E_\varphi = -\sqrt{\gamma} \Omega_+ B'_\varphi
\]

\[
\alpha B_\varphi = H_\varphi = -\sqrt{\gamma} \Omega_+ B'_\varphi \sin \theta, \tag{50}
\]

where $\Omega_H = a/2Mr_+$ is the angular velocity of the event horizon. When $|a| \ll M$

\[
B'_+ \to \frac{1}{\sqrt{\gamma}} B_0 \sin \theta, \text{ and } \Omega_+ \to \frac{a}{8MT}. \tag{51}
\]

This is precisely the Blandford & Znajek (1977) monopole solution (Komissarov 2004). Therefore, the solutions for the fields
and currents corresponding to \( \Omega = \Omega_+ \) generalize the Blandford-Znajek monopole solution to accommodate the case of a black hole for all values of \( a^2 < M^2 \).

A parallel approach to the study of the force-free magnetosphere has been developed via the Grad-Shafranov equation (see, e.g., eq. [6.4] of MacDonald & Thorne 1982). In our notation, the Grad-Shafranov equation takes the form (Uzdensky 2005)

\[
\nabla \cdot \left( \alpha \nabla \psi \left[ 1 - \frac{(\Omega_+ + \beta^2) \gamma_{\psi \psi}}{\alpha^2} \right] \right) + \frac{I}{\alpha \gamma_{\psi \psi}} \frac{d\psi}{d\psi} = 0.
\]

(52)

Here \( I = H_0 \) and \( \psi = A_\phi \). By straightforward substitution and evaluation of the various terms in equation (52), it is not difficult to see that our solution satisfies the Grad-Shafranov equation to order \( 1/r^2 \).

From equations (50) and (37), the angular dependence of energy extraction can be calculated. In the limit \( r \gg M \), the result is

\[
\frac{d^2 \mathcal{E}}{dr^2} = \frac{a \Omega_H}{r^2} \left( \frac{B_0}{2} \right) \sin^2 \theta \rho_+^2.
\]

(53)

From the above equation, it is clear that most of the energy extraction happens along the equatorial plane. The total rate of energy extraction can be obtained by integrating the above result, giving

\[
\frac{d\mathcal{E}}{dt} = \frac{\pi B_0^2}{ar_+} \left( \arctan \frac{a}{r_+} - \frac{a}{2M} \right).
\]

(54)

In similar fashion, we see by integrating equation (38) that

\[
\frac{d\mathcal{L}}{dt} = \frac{2\pi}{3} B_0^2 \Omega_H + \frac{1}{\Omega_H} \frac{d\mathcal{E}}{dt}.
\]

(55)

As a result of energy and angular momentum extraction from the black hole, the mass and the total angular momentum \( (J = aM) \) of the black hole changes by the amount

\[
\frac{\delta M}{\delta t} = -\frac{d\mathcal{E}}{dt}, \quad \text{and} \quad \frac{\delta J}{\delta t} = -\frac{d\mathcal{L}}{dt},
\]

(56)

respectively. From equation (55) and the above definitions, it is clear that

\[
\frac{\delta J}{\delta t} + \frac{2\pi}{3} B_0^2 \Omega_H = \frac{1}{\Omega_H} \frac{\delta M}{\delta t}.
\]

(57)

Therefore, we get the familiar inequality (Christodoulou 1970)

\[
\frac{\delta J}{\delta t} \leq \frac{1}{\Omega_H} \frac{\delta M}{\delta t},
\]

(58)

which ensures that the irreducible mass of the black hole is nondecreasing if the black hole evolves along a Kerr sequence in a reversible way. This process therefore cannot lead to the formation of a naked singularity.

6.2. A Jet-Type Solution

It is easily seen that \( \Omega = \Omega_- \) removes all the \( r \)-dependence in the right-hand side of equation (39) to order \( 1/r^3 \). As shown by equations (48) and (49), \( \Omega = \Omega_- \) is not a physical solution for the condition \( H_0 = 0 \). We now let \( H_0 \neq 0 \) and impose the Znajek condition, equation (45), for this case. Because our solutions involve both \( \Omega_+ \) and \( \Omega_- \), the results continue to be valid only to order \( r^{-2} \).

From equation (46) we see that

\[
f^2 = \pm \frac{H_0^2 \rho_+^4}{a^2} \frac{\Omega_+ \Omega_-^2}{(\Omega - \Omega_+)(\Omega - \Omega_-)}.
\]

(59)

Here the \( \pm \) factor is to ensure that \( f^2 \geq 0 \). Similarly we find from equation (43) that

\[
f^2 = \frac{B_0^2}{a^2 \sin^2 \theta (\Omega - \Omega_-)(\Omega + \Omega_-)}.
\]

(60)

Equating the right-hand sides of the last two equations, we see that

\[
B_0^2 |\Omega - \Omega_-| = \frac{H_0^2 \rho_+^4}{\sin^2 \theta (2Mr_+ + \rho_+^2)} |\Omega + \Omega_-|.
\]

(61)

It is important to remember that any \( \Omega \) satisfying the above equation is consistent with equation (39) (to order \( 1/r^3 \)) and with equation (45). The above equation has the unique solution

\[
\Omega_p = \frac{\tilde{A} \Omega_+ + \tilde{B} \Omega_-}{\tilde{A} - \tilde{B}},
\]

(62)

where

\[
\tilde{A} = \begin{cases} +1 & \text{if } \Omega_p - \Omega_+ \geq 0, \\ -1 & \text{otherwise,} \end{cases}
\]

\[
\tilde{B} = \begin{cases} +1 & \text{if } \Omega_p + \Omega_- \geq 0, \\ -1 & \text{otherwise.} \end{cases}
\]

(63)

All other poloidal fields quantities are now uniquely determined by noting that \( f \) is given by equation (60). It is important to see whether we can indeed satisfy the above conditions. A quick calculation shows that

\[
\Omega_p - \Omega_+ = \frac{\tilde{B}(\Omega_+ + \Omega_-)}{\tilde{A} - \tilde{B}}, \quad \text{and}
\]

\[
\Omega_p + \Omega_- = \frac{\tilde{A}(\Omega_+ + \Omega_-)}{\tilde{A} - \tilde{B}}.
\]

(64)

Therefore, the choice \( \tilde{A} = -B_0^2 \) and \( \tilde{B} = +H_0^2 \rho_+^4 \Omega_+ \Omega_- \) is a valid one. We pick this choice for the remainder of the paper. Consequently, we have

\[
\Omega_p = -\Omega_+ \Omega_- \frac{(H_0^2 \rho_+^4 - B_0^2 a^2 \sin^4 \theta)}{(H_0^2 \rho_+^4 \Omega_+ + B_0^2 \rho_+^2 \sin^2 \theta)}.
\]

(65)

Note that as \( \theta \to 0 \) and \( \pi \), \( \Omega_p \to -\Omega_- \to -\infty \). The form of \( f \) is determined by equation (60), and upon substitution of the explicit form of \( \Omega (=\Omega_p) \), we find that in the limit as \( \theta \to 0 \) and \( \pi \), \( f \to \pm H_0(a/\Omega_H)^{1/2}/2 \).
The expression for the rate of total energy extraction is given by

\[
\frac{d^2 E}{dA dt} = -H_\theta \Omega_p \frac{1}{\sqrt{\gamma}} f \approx -\frac{1}{r^2} \frac{f^2 \Omega_p}{\rho^2} \left( \frac{2Mr_+ \Omega_p - a}{\rho^2} \right).
\]

(66)

As \( \theta \to 0 \) and \( \pi \),

\[
\frac{d^2 E}{dA dt} \to -\frac{1}{r^2} \frac{H^2_0 a}{4\Omega_H} \Omega_-(\Omega_+ + \Omega_H). \]

(67)

A solution of this type has the following features: Energy extraction is less than zero near the poles; i.e., energy is being fed into the system, indicating a reverse jet type situation. Also, the total rate of energy and angular momentum “insertion” is not calculable since the above integral is divergent along the poles. This solution is therefore unphysical.

7. DISCUSSION

On the basis of the 3+1 equations as written by Komissarov (2004), we have rederived the constraint equation relating the toroidal magnetic field to the charge and current densities in a force-free magnetosphere around a spinning black hole. Known solutions to the constraint equation for the force-free magnetosphere include the monopole and the parabolic solutions obtained in the original paper by Blandford & Znajek (1977), and the solution by Beskin et al. (1992) for a black hole surrounded by a magnetized, conducting accretion disk. We have discovered a solution to the constraint equation that generalizes the “monopole solution” originally derived by Blandford & Znajek (1977). This solution satisfies the Znajek (1977) regularity condition at the event horizon, even in the limit \( a/M \ll 1 \) (contrary to the statement of Blandford and Znajek).

Komissarov (2001) has used a time-dependent numerical simulation to calculate the electromagnetic extraction of energy for a monopole magnetic field at different values of \( a/M \). Our value of \( \Omega_+ / \Omega_H \) at \( \theta = 0.5 \) ranges from 0.5 to 0.58 when \( a/M \) varies from 0.1 to 0.9, in comparison with the numerical value of 0.52 for \( a/M = 0.9 \) at \( r = 10 \). The value of \( H_\theta \) for our \( \Omega_+ \) solution is \( \approx 25\% \) larger than the numerical value of Komissarov (2001) when \( a/M = 0.9 \). These discrepancies, although not large, may reflect the finite value of \( r = 10 \) used in Komissarov’s work, whereas our solution holds in the asymptotic limit of large \( r \).

For the \( \Omega_+ \) solution, energy and angular momentum is extracted preferentially along the equatorial directions of the spinning black hole. Thus, it does not account for galactic black holes and active galactic nuclei that display radio jets. Time-dependent numerical solutions employing accretion of magnetized plasma into the ergosphere seem to indicate the presence of such jetlike features (Semenov et al. 2004). In future work, analytic solutions that exhibit jetlike structures will be studied using the techniques developed in this paper.

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