THE TELESCOPE CONJECTURE FOR HEREDITARY RINGS VIA
EXT-ORTHOGONAL PAIRS

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Dedicated to Helmut Lenzing on the occasion of his 70th birthday.

Abstract. For the module category of a hereditary ring, the Ext-orthogonal pairs of subcategories are studied. For each Ext-orthogonal pair that is generated by a single module, a 5-term exact sequence is constructed. The pairs of finite type are characterized and two consequences for the class of hereditary rings are established: homological epimorphisms and universal localizations coincide, and the telescope conjecture for the derived category holds true. However, we present examples showing that neither of these two statements is true in general for rings of global dimension 2.

1. Introduction

In this paper, we prove the telescope conjecture for the derived category of any hereditary ring. To achieve this, we study Ext-orthogonal pairs of subcategories for hereditary module categories.

The telescope conjecture for the derived category of a module category is also called smashing conjecture. It is the analogue of the telescope conjecture from stable homotopy theory which is due to Bousfield and Ravenel [6, 28]. In each case one deals with a compactly generated triangulated category. The conjecture then claims that a localizing subcategory is generated by compact objects provided it is smashing, that is, the localizing subcategory arises as the kernel of a localization functor that preserves arbitrary coproducts [24]. In this general form, the telescope conjecture seems to be wide open. For the stable homotopy category, we refer to the work of Mahowald, Ravenel, and Shick [22] for more details. In our case, the conjecture takes the following form and is proved in §7:

Theorem A. Let $A$ be a hereditary ring. For a localizing subcategory $C$ of $\mathbf{D}(\text{Mod } A)$ the following conditions are equivalent:

(1) There exists a localization functor $L: \mathbf{D}(\text{Mod } A) \to \mathbf{D}(\text{Mod } A)$ that preserves coproducts and such that $C = \text{Ker } L$.

(2) The localizing subcategory $C$ is generated by perfect complexes.

For the derived category of a module category, only two results seem to be known so far. Neeman proved the conjecture for the derived category of a commutative noetherian ring [25], essentially by classifying all localizing subcategories; see [16] for a treatment of this approach in the context of axiomatic stable homotopy theory. On the other hand, Keller gave an explicit example of a commutative ring where the conjecture does

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not hold [17]. In fact, an analysis of Keller’s argument [18] shows that there are such examples having global dimension 2; see Example 7.8.

The approach for hereditary rings presented here is completely different from Neeman’s. In particular, we are working in a non-commutative setting and without using any noetherianess assumption. The main idea here is to exploit the very close connection between the module category and the derived category in the hereditary case. Unfortunately, this approach cannot be extended directly even to global dimension 2, as mentioned above.

At a first glance, the telescope conjecture seems to be a rather abstract statement about unbounded derived categories. However in the context of a fixed hereditary ring, it turns out that smashing localizing subcategories are in bijective correspondence to various natural structures; see §8:

**Theorem B.** For a hereditary ring $A$ there are bijections between the following sets:

1. Extension closed abelian subcategories of $\text{Mod} A$ that are closed under products and coproducts.
2. Extension closed abelian subcategories of $\text{mod} A$.
3. Homological epimorphisms $A \to B$ (up to isomorphism).
4. Universal localizations $A \to B$ (up to isomorphism).
5. Localizing subcategories of $\mathbf{D}(\text{Mod} A)$ that are closed under products.
6. Localization functors $\mathbf{D}(\text{Mod} A) \to \mathbf{D}(\text{Mod} A)$ preserving coproducts (up to natural isomorphism).
7. Thick subcategories of $\mathbf{D}^b(\text{mod} A)$.

This reveals that the telescope conjecture and its proof are related to interesting recent work by some other authors. In [34], Schofield describes for any hereditary ring its universal localizations in terms of appropriate subcategories of finitely presented modules. This is a consequence of the present work since we show that homological epimorphisms and universal ring coincide for any hereditary ring; see §6. However, as we mention at the end of §6 the identification between homological epimorphisms and universal localizations also fails already for rings of global dimension 2.

In [27], Nicolás and Saorín establish for a differential graded algebra a correspondence between recollements for its derived category and differential graded homological epimorphisms. This correspondence specializes for a hereditary ring to the above mentioned bijection between smashing localizing subcategories and homological epimorphisms.

The link between the structures mentioned in Theorem B is provided by so-called Ext-orthogonal pairs. This concept seems to be new, but it is based on the notion of a perpendicular category which is one of the fundamental tools for studying hereditary categories arising in representation theory [32, 13].

Given any abelian category $\mathcal{A}$, we call a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories Ext-orthogonal if $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal to each other with respect to the bifunctor $\bigoplus_{n \geq 0} \text{Ext}^n_{\mathcal{A}}(\cdot, \cdot)$. This concept is the analogue of a torsion pair and a cotorsion pair where one considers instead the bifunctors $\text{Hom}_{\mathcal{A}}(\cdot, \cdot)$ and $\prod_{n > 0} \text{Ext}^n_{\mathcal{A}}(\cdot, \cdot)$, respectively [9, 30].

Torsion and cotorsion pairs are most interesting when they are complete. For a torsion pair this means that each object $M$ in $\mathcal{A}$ admits a short exact sequence $0 \to X_M \to M \to Y^M \to 0$ with $X_M \in \mathcal{X}$ and $Y^M \in \mathcal{Y}$. In the second case this means that each object $M$
admits short exact sequences $0 \to Y_M \to X_M \to M \to 0$ and $0 \to M \to Y^M \to X^M \to 0$
with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

It turns out that there is also a reasonable notion of completeness for Ext-orthogonal pairs. In that case each object $M$ in $\mathcal{A}$ admits a 5-term exact sequence

$$0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. This notion of a complete Ext-orthogonal pair is meaningful also for non-hereditary module categories, see Example 4.5.

In this work, however, we study Ext-orthogonal pairs mainly for the module category of a hereditary ring. As already mentioned, this assumption implies a close connection between the module category and its derived category, which we exploit in both directions. We use Bousfield localization functors which exist for the derived category to establish the completeness of certain Ext-orthogonal pairs for the module category; see §2. On the other hand, we are able to prove the telescope conjecture for the derived category by showing first a similar result for Ext-orthogonal pairs; see §5 and §7.

Specific examples of Ext-orthogonal pairs arise in the representation theory of finite dimensional algebras via perpendicular categories; see [31]. Note that a perpendicular category is always a part of an Ext-orthogonal pair. Schöfled introduced perpendicular categories for representations of quivers [32] and this fits into our set-up because the path algebra of any quiver is hereditary. In fact, the concept of a perpendicular category is fundamental for studying hereditary categories arising in representation theory [13]. It is therefore somewhat surprising that the 5-term exact sequence for a complete Ext-orthogonal pair seems to appear for the first time in this work.

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### 2. Ext-orthogonal pairs

Let $\mathcal{A}$ be an abelian category. Given a pair of objects $X, Y \in \mathcal{A}$, set

$$\text{Ext}^*_{\mathcal{A}}(X, Y) = \prod_{n \in \mathbb{Z}} \text{Ext}^n_{\mathcal{A}}(X, Y).$$

For a subcategory $\mathcal{C}$ of $\mathcal{A}$ we consider its full Ext-orthogonal subcategories

$$\mathcal{C} = \{X \in \mathcal{A} \mid \text{Ext}^*_{\mathcal{A}}(X, C) = 0 \text{ for all } C \in \mathcal{C}\},$$

$$\mathcal{C}^\perp = \{Y \in \mathcal{A} \mid \text{Ext}^*_\mathcal{A}(C, Y) = 0 \text{ for all } C \in \mathcal{C}\}.$$

If $\mathcal{C} = \{X\}$ is a singleton, we write $\perp X$ instead of $\perp\{X\}$, and similarly with $X^\perp$.

**Definition 2.1.** An Ext-orthogonal pair for $\mathcal{A}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories such that $\mathcal{X}^\perp = \mathcal{Y}$ and $\mathcal{X} = \mathcal{Y}^\perp$. An Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ is called complete if there exists for each object $M \in \mathcal{A}$ an exact sequence

$$0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. The pair $(\mathcal{X}, \mathcal{Y})$ is generated by a subcategory $\mathcal{C}$ of $\mathcal{A}$ if $\mathcal{Y} = \mathcal{C}^\perp$. 
The definition can be extended to the derived category $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ if we put for each pair of complexes $X, Y \in \mathbf{D}(\mathcal{A})$ and $n \in \mathbb{Z}$

$$\operatorname{Ext}^n_{\mathcal{A}}(X, Y) = \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, Y[n]).$$

Thus an $\operatorname{Ext}$-orthogonal pair for $\mathbf{D}(\mathcal{A})$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathbf{D}(\mathcal{A})$ such that $\mathcal{X}^\perp = \mathcal{Y}$ and $\mathcal{X} = \perp \mathcal{Y}$.

Recall that an abelian subcategory of $\mathcal{A}$ is a full subcategory $\mathcal{C}$ such that the category $\mathcal{C}$ is abelian and the inclusion functor $\mathcal{C} \to \mathcal{A}$ is exact. Moreover, we will always assume that an abelian subcategory $\mathcal{C}$ is closed under taking isomorphic objects in the original category $\mathcal{A}$. Suppose $\mathcal{A}$ is hereditary, that is, $\operatorname{Ext}^n_{\mathcal{A}}(-, -)$ vanishes for all $n > 1$. Then a simple calculation shows that for any subcategory $\mathcal{C}$ of $\mathcal{A}$, the subcategories $\mathcal{C}^\perp$ and $\perp \mathcal{C}$ are extension closed abelian subcategories; see [13, Proposition 1.1].

The following result establishes the completeness for certain $\operatorname{Ext}$-orthogonal pairs. Recall that an abelian category is a Grothendieck category if it has a set of generators and admits colimits that are exact when taken over filtered categories.

**Theorem 2.2.** Let $\mathcal{A}$ be a hereditary Grothendieck category and $X$ an object in $\mathcal{A}$. Set $\mathcal{Y} = X^\perp$ and let $\mathcal{X}$ denote the smallest extension closed abelian subcategory of $\mathcal{A}$ that is closed under taking coproducts and contains $X$. Then $(\mathcal{X}, \mathcal{Y})$ is a complete $\operatorname{Ext}$-orthogonal pair for $\mathcal{A}$. Thus there exists for each object $M \in \mathcal{A}$ an exact sequence

$$0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. This sequence is natural and induces bijections $\operatorname{Hom}_\mathcal{A}(X, X_M) \to \operatorname{Hom}_\mathcal{A}(X, M)$ and $\operatorname{Hom}_\mathcal{A}(Y^M, Y) \to \operatorname{Hom}_\mathcal{A}(M, Y)$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

The proof uses derived categories and Bousfield localization functors. Thus we need to collect some basic facts about hereditary abelian categories and their derived categories.

**The derived category of a hereditary abelian category.** Let $\mathcal{A}$ be a hereditary abelian category and let $\mathbf{D}(\mathcal{A})$ denote its derived category. We assume that $\mathcal{A}$ admits coproducts and that the coproduct of any set of exact sequences is again exact. Thus the category $\mathbf{D}(\mathcal{A})$ admits coproducts, and for each integer $n$ these coproducts are preserved by the functor $H^n: \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ which takes a complex to its cohomology in degree $n$.

It is well-known that each complex is quasi-isomorphic to its cohomology. That is:

**Lemma 2.3.** Given a complex $X$ in $\mathbf{D}(\mathcal{A})$, there are (non-canonical) isomorphisms

$$\prod_{n \in \mathbb{Z}} (H^nX)[-n] \cong X \cong \prod_{n \in \mathbb{Z}} (H^nX)[-n].$$

**Proof.** See for instance [19, §1.6].

A full subcategory $\mathcal{C}$ of $\mathbf{D}(\mathcal{A})$ is called thick if it is a triangulated subcategory which is, in addition, closed under taking direct summands. A thick subcategory is localizing if it is closed under taking coproducts. Note that for each full subcategory $\mathcal{C}$ the subcategories $\mathcal{C}^\perp$ and $\perp \mathcal{C}$ are thick.

To a full subcategory $\mathcal{C}$ of $\mathbf{D}(\mathcal{A})$ we assign the full subcategory

$$H^0\mathcal{C} = \{ M \in \mathcal{A} \mid M = H^0X \text{ for some } X \in \mathcal{C} \},$$
and given a full subcategory $\mathcal{X}$ of $\mathcal{A}$, we define the full subcategory
\[ D_\mathcal{X}(\mathcal{A}) = \{ X \in D(\mathcal{A}) \mid H^n X \in \mathcal{X} \text{ for all } n \in \mathbb{Z} \}. \]

Both assignments induce mutually inverse bijections between appropriate subcategories. This is a useful fact which we recall from [7, Theorem 6.1].

**Proposition 2.4.** The functor $H^0: D(\mathcal{A}) \to \mathcal{A}$ induces a bijection between the localizing subcategories of $D(\mathcal{A})$ and the extension closed abelian subcategories of $\mathcal{A}$ that are closed under coproducts. The inverse map sends a full subcategory $\mathcal{X}$ of $\mathcal{A}$ to $D_\mathcal{X}(\mathcal{A})$. □

**Remark 2.5.** The bijection in Proposition 2.4 has an analogue for thick subcategories. Given any hereditary abelian category $\mathcal{B}$, the functor $H^0: D^b(\mathcal{B}) \to \mathcal{B}$ induces a bijection between the thick subcategories of $D^b(\mathcal{B})$ and the extension closed abelian subcategories of $\mathcal{B}$; see [7, Theorem 5.1].

Next we extend these maps to bijections between Ext-orthogonal pairs.

**Proposition 2.6.** The functor $H^0: D(\mathcal{A}) \to \mathcal{A}$ induces a bijection between the Ext-orthogonal pairs for $D(\mathcal{A})$ and the Ext-orthogonal pairs for $\mathcal{A}$. The inverse map sends a pair $(\mathcal{X}, \mathcal{Y})$ for $\mathcal{A}$ to $(D_\mathcal{X}(\mathcal{A}), D_\mathcal{Y}(\mathcal{A}))$.

**Proof.** First observe that for each pair of complexes $X, Y \in D(\mathcal{A})$, we have $\text{Ext}^*_\mathcal{A}(X, Y) = 0$ if and only if $\text{Ext}^*_\mathcal{A}(H^p X, H^q Y) = 0$ for all $p, q \in \mathbb{Z}$. This is a consequence of Lemma 2.3. It follows that $H^0$ and its inverse send Ext-orthogonal pairs to Ext-orthogonal pairs. Each Ext-orthogonal pair is determined by its first half, and therefore an application of Proposition 2.4 shows that both maps are mutually inverse. □

**Localization functors.** Let $\mathcal{T}$ be a triangulated category. A localization functor $L: \mathcal{T} \to \mathcal{T}$ is an exact functor that admits a natural transformation $\eta: \text{Id}_\mathcal{T} \to L$ such that $L\eta_X$ is an isomorphism and $\eta_L Y = \eta_X Y$ for all objects $X \in \mathcal{T}$. Basic facts about localization functors one finds, for example, in [3, §3].

**Proposition 2.7.** Let $\mathcal{A}$ be a hereditary abelian category. For a full subcategory $\mathcal{X}$ of $\mathcal{A}$ the following are equivalent.

1. There exists a localization functor $L: D(\mathcal{A}) \to D(\mathcal{A})$ such that $\ker L = D_\mathcal{X}(\mathcal{A})$.
2. There exists a complete Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for $\mathcal{A}$.

**Proof.** (1) $\Rightarrow$ (2): The kernel $\ker L$ and the essential image $\text{Im} L$ of a localization functor $L$ form an Ext-orthogonal pair for $D(\mathcal{A})$; see for instance [3, Lemma 3.3]. Then it follows from Proposition 2.6 that the pair $(\mathcal{X}, \mathcal{Y}) = (H^0 \ker L, H^0 \text{Im} L)$ is Ext-orthogonal for $\mathcal{A}$.

The localization functor $L$ comes equipped with a natural transformation $\eta: \text{Id}_{D(\mathcal{A})} \to L$, and for each complex $M$ we complete the morphism $\eta_M: M \to LM$ to an exact triangle
\[ \Gamma M \to M \to LM \to \Gamma M[1]. \]

Note that $\Gamma M \in \ker L$ and $LM \in \text{Im} L$ since $L\eta_M$ is an isomorphism and $L$ is exact. Now suppose that $M$ is concentrated in degree zero. Applying $H^0$ to this triangle yields an exact sequence
\[ 0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0 \]
with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. 

Thus, $L \eta_M$ is an isomorphism. □
(2) ⇒ (1): Let \((X, Y)\) be an Ext-orthogonal pair for \(\mathcal{A}\). This pair induces an Ext-orthogonal pair \((\mathcal{D}X(\mathcal{A}), \mathcal{D}Y(\mathcal{A}))\) for \(\mathcal{D}(\mathcal{A})\) by Proposition 2.6. In order to construct a localization functor \(L: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})\) such that \(\text{Ker } L = \mathcal{D}X(\mathcal{A})\), it is sufficient to construct for each object \(M\) in \(\mathcal{D}(\mathcal{A})\) an exact triangle \(X \to M \to Y \to X[1]\) with \(X \in \mathcal{D}X(\mathcal{A})\) and \(Y \in \mathcal{D}Y(\mathcal{A})\). Then one defines \(LM = Y\) and the morphism \(M \to Y\) induces a natural transformation \(\eta: \text{Id}_{\mathcal{D}(\mathcal{A})} \to L\) having the required properties. In view of Lemma 2.3 it is sufficient to assume that \(M\) is a complex concentrated in degree zero.

Suppose that \(M\) admits an approximation sequence

\[\varepsilon_M: 0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0\]

with \(X_M, X^M \in \mathcal{X}\) and \(Y_M, Y^M \in \mathcal{Y}\). Let \(M'\) denote the image of \(X_M \to M\) and \(M''\) the image of \(M \to Y^M\). Then \(\varepsilon_M\) induces the following three exact sequences

\[\alpha_M: 0 \to M' \to M \to M'' \to 0,\]

\[\beta_M: 0 \to Y_M \to X_M \to M' \to 0,\]

\[\gamma_M: 0 \to M'' \to Y^M \to X^M \to 0.\]

In \(\mathcal{D}(\mathcal{A})\) these three exact sequence give rise to the following commuting square

\[
\begin{array}{ccc}
X^M[-2] & \xrightarrow{\gamma_M} & M''[-1] \\
\downarrow 0 & & \downarrow \alpha_M \\
X_M & \xrightarrow{\beta_M} & M'
\end{array}
\]

where \(\beta_M\) is the second morphism in \(\beta_M\). Commutativity of the diagram is clear since \(\text{Hom}_{\mathcal{D}(\mathcal{A})}(U[-2], V) = 0\) for any \(U, V \in \mathcal{A}\). An application of the octahedral axiom shows that this square can be extended as follows to a diagram where each row and each column is an exact triangle.

\[
\begin{array}{ccccccc}
X^M[-2] & \to & M''[-1] & \to & Y^M[-1] & \to & X^M[-1] \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
X_M & \to & M' & \to & Y_M[1] & \to & X_M[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_M \oplus X^M[-1] & \to & M & \to & Y_M[1] \oplus Y^M & \to & X_M[1] \oplus X^M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^M[-1] & \to & M'' & \to & Y^M & \to & X^M
\end{array}
\]

The first and third column are split exact triangles, and this explains the objects appearing in the third row. In particular, this yields the desired exact triangle \(X \to M \to Y \to X[1]\) with \(X \in \mathcal{D}X(\mathcal{A})\) and \(Y \in \mathcal{D}Y(\mathcal{A})\).

Remark 2.8. The proof of the implication (2) ⇒ (1) comes as a special case of a more general result on the existence of exact triangles with a specified long exact sequence of cohomology objects. We refer to work of Neeman [23] for more details.
Next we formulate the functorial properties of the 5-term exact sequence constructed in Proposition 2.7.

**Lemma 2.9.** Let $\mathcal{A}$ be an abelian category and $(\mathcal{X}, \mathcal{Y})$ an Ext-orthogonal pair for $\mathcal{A}$. Suppose there is an exact sequence

$$
\varepsilon_M: 0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0
$$

in $\mathcal{A}$ with $X_M, Y^M \in \mathcal{X}$ and $Y_M, X^M \in \mathcal{Y}$.

1. The sequence $\varepsilon_M$ induces for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ bijections $\text{Hom}_\mathcal{A}(X, X_M) \to \text{Hom}_\mathcal{A}(X, M)$ and $\text{Hom}_\mathcal{A}(Y^M, Y) \to \text{Hom}_\mathcal{A}(M, Y)$.

2. Let $\varepsilon_N: 0 \to Y_N \to X_N \to N \to Y^N \to X^N \to 0$ be an exact sequence in $\mathcal{A}$ with $X_N, X^N \in \mathcal{X}$ and $Y_N, Y^N \in \mathcal{Y}$. Then each morphism $M \to N$ extends uniquely to a morphism $\varepsilon_M \to \varepsilon_N$ of exact sequences.

3. Any exact sequence $0 \to Y' \to X' \to M \to Y'' \to X'' \to 0$ in $\mathcal{A}$ with $X', Y'' \in \mathcal{X}$ and $Y', Y'' \in \mathcal{Y}$ is uniquely isomorphic to $\varepsilon_M$.

**Proof.** We prove part (1). Then parts (2) and (3) are immediate consequences.

Fix an object $X \in \mathcal{X}$. The map $\mu: \text{Hom}_\mathcal{A}(X, X_M) \to \text{Hom}_\mathcal{A}(X, M)$ is injective because $\text{Hom}_\mathcal{A}(X, Y_M) = 0$. Any morphism $X \to M$ factors through the kernel $M'$ of $M \to Y^M$ since $\text{Hom}_\mathcal{A}(X, Y^M) = 0$. The induced morphism $X \to M'$ factors through $X_M \to M'$ since $\text{Ext}^1_\mathcal{A}(X, Y_M) = 0$. Thus $\mu$ is surjective. The argument for the other map $\text{Hom}_\mathcal{A}(Y^M, Y) \to \text{Hom}_\mathcal{A}(M, Y)$ is dual. \(\square\)

**Ext-orthogonal pairs for Grothendieck categories.** Now we give the proof of Theorem 2.2. The basic idea is to establish a localization functor for $\mathcal{D}(\mathcal{A})$ and to derive the exact approximation sequence in $\mathcal{A}$ by taking the cohomology of some appropriate exact triangle as in Proposition 2.7.

**Proof of Theorem 2.2.** Let $\mathcal{X}$ denote the smallest extension closed abelian subcategory of $\mathcal{A}$ that contains $X$ and is closed under coproducts. Then Proposition 2.4 implies that $\mathcal{D}_\mathcal{X}(\mathcal{A})$ is the smallest localization subcategory of $\mathcal{D}(\mathcal{A})$ containing $X$. Thus there exists a localization functor $L: \mathcal{D}(\mathcal{A}) \to \mathcal{D}_\mathcal{X}(\mathcal{A})$ with $\text{Ker} L = \mathcal{D}_\mathcal{X}(\mathcal{A})$. This is a result which goes back to Bousfield’s work in algebraic topology, [6]. In the context of derived categories we refer to [2] Theorem 5.7. Now apply Proposition 2.7 to get the 5-term exact sequence for each object $M$ in $\mathcal{A}$. The properties of this sequence follow from Lemma 2.9. \(\square\)

**Remark 2.10.** We do not know an example of an Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for a hereditary Grothendieck category such that the pair $(\mathcal{X}, \mathcal{Y})$ is not complete.

Ext-orthogonal pairs naturally arise also for non-hereditary abelian categories. Here we mention one such class of examples, but we do not know whether or when exactly they are complete:

**Example 2.11.** Let $\mathcal{A}$ be any Grothendieck category and $\mathcal{X}$ a localizing subcategory. That is, $\mathcal{X}$ is a full subcategory closed under taking coproducts and such that for any exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$ we have $M \in \mathcal{X}$ if and only if $M', M'' \in \mathcal{X}$. Set $\mathcal{Y} = \mathcal{X}^\perp$ and let $\mathcal{Y}_{\text{inj}}$ denote the full subcategory of injective objects of $\mathcal{A}$ contained in $\mathcal{Y}$. Then $\mathcal{X} = \mathcal{Y}_{\text{inj}}^\perp$ and therefore $(\mathcal{X}, \mathcal{Y})$ is an Ext-orthogonal pair for $\mathcal{A}$; see [11], III.4 for details.
**Torsion and cotorsion pairs.** We also sketch an interpretation of an Ext-orthogonal pair in terms of torsion and cotorsion pairs. Here, a pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of $\mathcal{A}$ is called a torsion pair if $\mathcal{U}$ and $\mathcal{V}$ are orthogonal to each other with respect to $\text{Hom}_\mathcal{A}(-, -)$. Analogously, a pair of full subcategories is a cotorsion pair if both categories are orthogonal to each other with respect to $\prod_{n>0} \text{Ext}^n_\mathcal{A}(-, -)$.

Let $\mathcal{A}$ be an abelian category and $(\mathcal{X}, \mathcal{Y})$ an Ext-orthogonal pair. The subcategory $\mathcal{X}$ generates a torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$ and a cotorsion pair $(\mathcal{X}_1, \mathcal{Y}_1)$ for $\mathcal{A}$, if one defines the corresponding full subcategories of $\mathcal{A}$ as follows:

- $\mathcal{Y}_0 = \{ Y \in \mathcal{A} \mid \text{Hom}_\mathcal{A}(X, Y) = 0 \text{ for all } X \in \mathcal{X} \}$,
- $\mathcal{X}_0 = \{ X \in \mathcal{A} \mid \text{Hom}_\mathcal{A}(X, Y) = 0 \text{ for all } Y \in \mathcal{Y}_0 \}$,
- $\mathcal{Y}_1 = \{ Y \in \mathcal{A} \mid \text{Ext}_\mathcal{A}^n(X, Y) = 0 \text{ for all } X \in \mathcal{X}, \ n > 0 \}$,
- $\mathcal{X}_1 = \{ X \in \mathcal{A} \mid \text{Ext}_\mathcal{A}^n(X, Y) = 0 \text{ for all } Y \in \mathcal{Y}_1, \ n > 0 \}$.

Note that $\mathcal{X} = \mathcal{X}_0 \cap \mathcal{X}_1$ and $\mathcal{Y} = \mathcal{Y}_0 \cap \mathcal{Y}_1$. In particular, one recovers the pair $(\mathcal{X}, \mathcal{Y})$ from $(\mathcal{X}_0, \mathcal{Y}_0)$ and $(\mathcal{X}_1, \mathcal{Y}_1)$.

Suppose an object $M \in \mathcal{A}$ admits an approximation sequence

$$\varepsilon_M : 0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. We give the following interpretation of this sequence. Let $M'$ denote the image of $X_M \to M$ and $M''$ the image of $M \to Y^M$. Then there are three short exact sequences:

- $\alpha_M : 0 \to M' \to M \to M'' \to 0$,
- $\beta_M : 0 \to Y_M \to X_M \to M' \to 0$,
- $\gamma_M : 0 \to M'' \to Y^M \to X^M \to 0$.

The sequence $\alpha_M$ is the approximation sequence of $M$ with respect to the torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$, that is, $M' \in \mathcal{X}_0$ and $M'' \in \mathcal{Y}_0$. On the other hand, $\beta_M$ and $\gamma_M$ are approximation sequences of $M'$ and $M''$ respectively, with respect to the cotorsion pair $(\mathcal{X}_1, \mathcal{Y}_1)$, that is, $X_M, X^M \in \mathcal{X}_1$ and $Y_M, Y^M \in \mathcal{Y}_1$. Thus the 5-term exact sequence $\varepsilon_M$ is obtained by splicing together three short exact approximation sequences.

Suppose finally that the Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ is complete. It is not hard to see that then the associated torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$ has an explicit description: we have $\mathcal{X}_0 = \text{Fac} \mathcal{X}$ and $\mathcal{Y}_0 = \text{Sub} \mathcal{Y}$, where

$$\text{Fac} \mathcal{X} = \{ X/U \mid U \subseteq X, \ X \in \mathcal{X} \} \quad \text{and} \quad \text{Sub} \mathcal{Y} = \{ U \mid U \subseteq Y, \ Y \in \mathcal{Y} \}.$$
or equivalently, if restriction induces isomorphisms
\[ \text{Ext}^*_B(X, Y) \cong \text{Ext}^*_A(X, Y) \]
for all \( B \)-modules \( X, Y \); see [13] for details. The first observation is that every homological epimorphism naturally induces two complete Ext-orthogonal pairs:

**Proposition 3.1.** Let \( A \) be a hereditary ring and \( f: A \to B \) a homological epimorphism. Denote by \( \mathcal{Y} \) the category of \( A \)-modules which are restrictions of modules over \( B \). Set \( \mathcal{X} = \perp \mathcal{Y} \) and \( \mathcal{X} \perp = \mathcal{Z} \). Then \( (\mathcal{X}, \mathcal{Y}) \) and \( (\mathcal{Y}, \mathcal{Z}) \) are complete Ext-orthogonal pairs for \( \text{Mod} A \) with \( \mathcal{Y} = (\text{Ker} f \oplus \text{Coker} f)^\perp \) and \( \mathcal{Z} = B^\perp \).

**Proof.** We wish to apply Theorem 2.2 which provides a construction for complete Ext-orthogonal pairs.

First observe that \( \mathcal{Y} \) is the smallest extension closed abelian subcategory of \( \text{Mod} A \) closed under coproducts and containing \( B \). This yields \( \mathcal{Z} = B^\perp \).

Next we show that \( \mathcal{Y} = (\text{Ker} f \oplus \text{Coker} f)^\perp \). In fact, an \( A \)-module \( Y \) is the restriction of a \( B \)-module if and only if \( f \) induces an isomorphism \( \text{Hom}_A(B, Y) \to \text{Hom}_A(A, Y) \). Using the assumptions on \( A \) and \( f \), a simple calculation shows that this implies \( \mathcal{Y} = (\text{Ker} f \oplus \text{Coker} f)^\perp \).

It remains to apply Theorem 2.2. Thus \( (\mathcal{X}, \mathcal{Y}) \) and \( (\mathcal{Y}, \mathcal{Z}) \) are complete Ext-orthogonal pairs. \( \square \)

Now we use a crucial theorem of Gabriel and de la Peña. It identifies, only by their closure properties, the full subcategories of a module category \( \text{Mod} A \) that arise as the images of the restriction functors \( \text{Mod} B \to \text{Mod} A \) for ring epimorphisms \( A \to B \). In our version, we identify in a similar way the essential images of the restriction functors of homological epimorphisms, provided \( A \) is hereditary.

**Proposition 3.2.** Let \( A \) be a hereditary ring and \( \mathcal{Y} \) an extension closed abelian subcategory of \( \text{Mod} A \) that is closed under taking products and coproducts. Then there exists a homological epimorphism \( f: A \to B \) such that the restriction functor \( \text{Mod} B \to \text{Mod} A \) induces an equivalence \( \text{Mod} B \xrightarrow{\sim} \mathcal{Y} \).

**Proof.** It follows from [12, Theorem 1.2] that there exists a ring epimorphism \( f: A \to B \) such that the restriction functor \( \text{Mod} B \to \text{Mod} A \) induces an equivalence \( \text{Mod} B \xrightarrow{\sim} \mathcal{Y} \). To be more specific, one constructs a left adjoint \( F: \text{Mod} A \to \mathcal{Y} \) for the inclusion \( \mathcal{Y} \to \text{Mod} A \). Then \( FA \) is a small projective generator for \( \mathcal{Y} \), because \( A \) has this property for \( \text{Mod} A \) and the inclusion of \( \mathcal{Y} \) is an exact functor that preserves coproducts. Thus one takes for \( f \) the induced map \( A \cong \text{End}_A(A) \to \text{End}_A(FA) \).

We claim that restriction via \( f \) induces an isomorphism
\[ \text{Ext}^n_B(X, Y) \cong \text{Ext}^n_A(X, Y) \]
for all \( B \)-modules \( X, Y \) and all \( n \geq 0 \). This is clear for \( n = 0,1 \) since \( \mathcal{Y} \) is extension closed. On the other hand, the isomorphism for \( n = 1 \) implies that \( \text{Ext}^1_B(X, -) \) is right exact since \( A \) is hereditary. It follows that \( B \) is hereditary and \( \text{Ext}^n_B(-, -) \) vanishes for all \( n > 1 \). \( \square \)

We get as an immediate consequence that any class \( \mathcal{Y} \) satisfying the assumptions of Proposition 3.2 belongs to two complete cotorsion pairs. In order to obtain more information about the corresponding 5-term approximation sequences, we prefer, however, to postpone this corollary after the following lemma:
Lemma 3.3. Let $A \to B$ be a homological epimorphism and denote by $\mathcal{Y}$ the category of $A$-modules which are restrictions of modules over $B$.

1. The functor $\mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ sending a complex $X$ to $X \otimes^L_A B$ is a localization functor with essential image equal to $\mathbf{D}_\mathcal{Y}(\operatorname{Mod} A)$.

2. The functor $\mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ sending a complex $X$ to the cone (which is in this case functorial) of the natural morphism $\mathbf{R}\text{Hom}_A(B, X) \to X$ is a localization functor with kernel equal to $\mathbf{D}_\mathcal{Y}(\operatorname{Mod} A)$.

Proof. Restriction along $f : A \to B$ identifies $\operatorname{Mod} B$ with $\mathcal{Y}$. The functor induces an isomorphism

$$\operatorname{Ext}_B^n(X, Y) \cong \operatorname{Ext}_A^n(X, Y)$$

for all $B$-modules $X, Y$ and all $n \geq 0$, because $f$ is a homological epimorphism. This isomorphism implies that the induced functor $f_* : \mathbf{D}(\operatorname{Mod} B) \to \mathbf{D}(\operatorname{Mod} A)$ is fully faithful with essential image $\mathbf{D}_\mathcal{Y}(\operatorname{Mod} A)$. Moreover, $f_*$ is naturally isomorphic to both $\mathbf{R}\text{Hom}_B(A, -)$ and $- \otimes^L_B A$. It follows that:

1. The functor $f_*$ admits a left adjoint $f^* = - \otimes^L_A B$ and we therefore have a localization functor $L : \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ sending a complex $X$ to $f_* f^*(X)$; see [4, Lemma 3.1]. It remains to note that the essential images of $L$ and $f_*$ coincide.

2. The functor $f_*$ admits a right adjoint $f^! = \mathbf{R}\text{Hom}_A(B, -)$ and we therefore have a colocalization functor $\Gamma : \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ sending a complex $X$ to $f_* f^!(X)$. Note that the adjunction morphism $\Gamma X \to X$ is an isomorphism if and only if $X$ belongs to $\mathbf{D}_\mathcal{Y}(\operatorname{Mod} A)$. Completing $\Gamma X \to X$ to a triangle yields a well defined localization functor $\mathbf{D}(\operatorname{Mod} B) \to \mathbf{D}(\operatorname{Mod} A)$ with kernel $\mathbf{D}_\mathcal{Y}(\operatorname{Mod} A)$; see [4, Lemma 3.3].

Now we state the above mentioned immediate consequence of Propositions 3.1 and 3.2 but with an alternative and more explicit proof.

Corollary 3.4. Let $A$ be a hereditary ring and $\mathcal{Y}$ an extension closed abelian subcategory of $\operatorname{Mod} A$ that is closed under taking products and coproducts. Set $\mathcal{X} = \perp \mathcal{Y}$ and $\mathcal{Z} = \mathcal{Y} \perp$. Then $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are both complete Ext-orthogonal pairs.

Proof. There exists a homological epimorphism $f : A \to B$ such that restriction identifies $\operatorname{Mod} B$ with $\mathcal{Y}$; see Proposition 3.2. Then Lemma 3.3 produces two localization functors $L_1, L_2 : \mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ with $\operatorname{Im} L_1 = \mathbf{D}_\mathcal{Y}(\operatorname{Mod} A) = \operatorname{Ker} L_2$. Thus

$$\operatorname{Ker} L_1 = \perp (\operatorname{Im} L_1) = \mathbf{D}_\mathcal{X}(\operatorname{Mod} A) \quad \text{and} \quad \operatorname{Im} L_2 = (\operatorname{Ker} L_2) \perp = \mathbf{D}_\mathcal{Z}(\operatorname{Mod} A),$$

where in both cases the first equality follows from [4, Lemma 3.3] and the second from Proposition 2.6. It remains to apply Proposition 2.6, which yields in both cases for each $A$-module the desired 5-term exact sequence.

Remark 3.5. The proof of Lemma 3.3 and Corollary 3.4 yields for any $A$-module $M$ an explicit description of some terms of the 5-term exact sequence $\varepsilon_M$, using the homological epimorphism $A \to B$. In the first case, we have

$$\varepsilon_M : \ 0 \to \operatorname{Tor}_1^A(M, B) \to X_M \to M \to M \otimes_A B \to X^M \to 0,$$

and in the second case, we have

$$\varepsilon_M : \ 0 \to Z_M \to \operatorname{Hom}_A(B, M) \to M \to Z^M \to \operatorname{Ext}_A^1(B, M) \to 0.$$
We also mention another consequence of the above discussion, which is immediately implied by Corollary 3.3. It reflects the fact that given a homological epimorphism $A \to B$ and the fully faithful functor $f_* : D(\text{Mod} B) \to D(\text{Mod} A)$ having both a left and a right adjoint, there exists a corresponding recollement of the derived category $D(\text{Mod} A)$; see [20, §4.13].

**Corollary 3.6.** Let $A$ be a hereditary ring and $(\mathcal{X}, \mathcal{Y})$ an Ext-orthogonal pair for the category of $A$-modules.

1. There is an Ext-orthogonal pair $(\mathcal{W}, \mathcal{X})$ if and only if $\mathcal{X}$ is closed under products.
2. There is an Ext-orthogonal pair $(\mathcal{Y}, \mathcal{Z})$ if and only if $\mathcal{Y}$ is closed under coproducts.

4. Examples

We present a number of examples of Ext-orthogonal pairs which illustrate the results of this work. The first example is classical and provides one of the motivations for studying perpendicular categories in representation theory of finite dimensional algebras. We refer to Schofield’s work [33, 32] which contains some explicit calculations; see also [13, 14].

**Example 4.1.** Let $A$ be a finite dimensional hereditary algebra over a field $k$ and $X$ a finite dimensional $A$-module. Then $X^\perp = \mathcal{Y}$ identifies via a homological epimorphism $A \to B$ with the category of modules over a $k$-algebra $B$ and this yields a complete Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$. If $X$ is exceptional, that is, $\text{Ext}^1_A(X, X) = 0$, then $B$ is finite dimensional (see the proposition below) and often can be constructed explicitly. We refer to [33] for particular examples. Note that in this case for each finite dimensional $A$-module $M$ the corresponding 5-term exact sequence $\varepsilon_M$ consists of finite dimensional modules. Moreover, the category $\mathcal{X}$ is equivalent to the module category of another finite dimensional algebra. We do not know of a criterion on $X$ that characterizes the fact that $B$ is finite dimensional; see however the following proposition.

**Proposition 4.2.** Let $A$ be a finite dimensional hereditary algebra over a field $k$ and $(\mathcal{X}, \mathcal{Y})$ a complete Ext-orthogonal pair such that $\mathcal{Y}$ is closed under coproducts. Fix a homological epimorphism $A \to B$ inducing an equivalence $\text{Mod} B \cong \mathcal{Y}$. Then the following are equivalent.

1. There exists an exceptional module $X \in \text{mod} A$ such that $\mathcal{Y} = X^\perp$.
2. The algebra $B$ is finite dimensional over $k$.
3. For each $M \in \text{mod} A$, the 5-term exact sequence $\varepsilon_M$ belongs to $\text{mod} A$.

**Proof.** (1) $\Rightarrow$ (2): This follows, for example, from [13, Proposition 3.2].

(2) $\Rightarrow$ (3): This follows from Remark 3.5.

(3) $\Rightarrow$ (1): Let $\mathcal{X}_{fp} = \mathcal{X} \cap \text{mod} A$ and $\mathcal{Y}_{fp} = \mathcal{Y} \cap \text{mod} A$. The assumption on $(\mathcal{X}, \mathcal{Y})$ implies that $(\mathcal{X}_{fp}, \mathcal{Y}_{fp})$ is a complete Ext-orthogonal pair for $\text{mod} A$. Moreover, every object in $\mathcal{X}$ is a filtered colimit of objects in $\mathcal{X}_{fp}$. To see this, we first express $X$ as a filtered colimit $\lim M_i$ of finitely presented modules. Then, using the forthcoming Lemma 5.3(2), we see that $\varepsilon_X = \lim \varepsilon_{M_i}$, from which it easily follows that $X \cong \lim X_{M_i}$.

Now choose an injective cogenerator $Q$ in $\text{mod} A$ and let $X = X_Q$ be the module from the 5-term exact sequence $\varepsilon_Q$. This module is the image of $Q$ under a right adjoint of the inclusion $\mathcal{X}_{fp} \to \text{mod} A$. Note that a right adjoint of an exact functor preserves injectivity. It follows that $X$ is an exceptional object and that $\mathcal{X}_{fp}$ is the smallest
extension closed abelian subcategory of mod A containing X. Thus $X^\perp = X_{lp}^\perp = X^\perp = Y$, using the fact that $X = \lim X_{lp}$. □

As a special case, any finitely generated projective module generates an Ext-orthogonal pair that can be described explicitly; see [13, §5]. For cyclic projective modules, this is discussed in more generality in the following example.

**Example 4.3.** Let $A$ be a hereditary ring and $e^2 = e \in A$ an idempotent. Let $\mathcal{X}$ denote the category of $A$-modules $M$ such that the natural map $Me \otimes_{e A} e A \to M$ is an isomorphism, and let $\mathcal{Y} = e A^\perp = \{ M \in \text{Mod} A \mid Me = 0 \}$. Thus $- \otimes_{e A} e A$ identifies $\text{Mod} e A e$ with $\mathcal{X}$ and restriction via $A \to A/AeA$ identifies $\text{Mod} A/AeA$ with $\mathcal{Y}$. Then $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair for $\text{Mod} A$, and for each $A$-module $M$ the 5-term exact sequence $\varepsilon_M$ is of the form

$$0 \to \text{Tor}_1^A(M, A/AeA) \to Me \otimes_{e A} e A \to M \to M \otimes_A A/AeA \to 0 \to 0.$$  

The next example arises from the work of Reiten and Ringel on infinite dimensional representations of canonical algebras; see [29] which is our reference for all concepts and results in the following discussion. Note that these algebras are not necessarily hereditary. The example shows the interplay between Ext-orthogonal pairs and (co)torsion pairs.

**Example 4.4.** Let $A$ be a finite dimensional canonical algebra over a field $k$. Take for example a tame hereditary algebra, or, more specifically, the Kronecker algebra $[k \ k^2 \ k]$. For such algebras, there is the concept of a *separating tubular family*. We fix such a family and denote by $\mathcal{T}$ the category of finite dimensional modules belonging to this family. There is also a particular *generic module over A* which depends in some cases on the choice of the tubular family; it is denoted by $G$. Then the full subcategory $\mathcal{X} = \lim \mathcal{T}$ consisting of all filtered colimits of modules in $\mathcal{T}$ and the full subcategory $\mathcal{Y} = \text{Add} G$ consisting of all coproducts of copies of $G$ form an Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for $\text{Mod} A$. Note that the endomorphism ring $D = \text{End}_A(G)$ of $G$ is a division ring and that the canonical map $A \to B$ with $B = \text{End}_D(G)$ is a homological epimorphism which induces an equivalence $\text{Mod} B \cong \mathcal{Y}$. In the particular case of the Kronecker algebra $A = [k \ k^2 \ k]$, a direct computation shows that $B = M_2(k(x))$.

The category of $A$-modules which are generated by $\mathcal{T}$ and the category of $A$-modules which are cogenerated by $G$ form a torsion pair $(\text{Fac} \mathcal{X}, \text{Sub} \mathcal{Y})$ for $\text{Mod} A$ which equals the torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$ generated by $\mathcal{X}$. On the other hand, let $\mathcal{C}$ denote the category of $A$-modules which are cogenerated by $\mathcal{X}$, and let $\mathcal{D}$ denote the category of $A$-modules $M$ satisfying $\text{Hom}_A(M, \mathcal{T}) = 0$. Then the pair $(\mathcal{C}, \mathcal{D})$ forms a cotorsion pair for $\text{Mod} A$ which identifies with the cotorsion pair $(\mathcal{X}_1, \mathcal{Y}_1)$ generated by $\mathcal{X}$.

If $A$ is hereditary, then the Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ is complete by Corollary [3, 4] see also Remark [5, 5] for an explicit description of the 5-term approximation sequence $\varepsilon_M$ for each $A$-module $M$. Alternatively, one obtains the sequence $\varepsilon_M$ by splicing together appropriate approximation sequences which arise from $(\mathcal{X}_0, \mathcal{Y}_0)$ and $(\mathcal{X}_1, \mathcal{Y}_1)$.

The following example of an Ext-orthogonal pair arises from a localizing subcategory; it is a specialization of Example [2, 11] and provides a simple (and not necessarily hereditary) model for the previous example.

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1The first author is grateful to Lidia Angeleri Hügel for suggesting this example.
Example 4.5. Let \( A \) be an integral domain with quotient field \( Q \). Let \( \mathcal{X} \) denote the category of torsion modules and \( \mathcal{Y} \) the category of torsion free divisible modules. Note that the modules in \( \mathcal{Y} \) are precisely the coproducts of copies of \( Q \). Then \( (\mathcal{X}, \mathcal{Y}) \) is a complete Ext-orthogonal pair for \( \text{Mod} A \), and for each \( A \)-module \( M \) the 5-term exact sequence \( \varepsilon_M \) is of the form
\[
0 \to 0 \to tM \to M \to M \otimes_A Q \to \bar{M} \to 0.
\]

We conclude the section by showing that there are examples of abelian categories that admit only trivial Ext-orthogonal pairs.

Example 4.6. Let \( A \) be a local artinian ring and set \( \mathcal{A} = \text{Mod} A \). Then \( \text{Hom}_A(X, Y) \neq 0 \) for any pair \( X, Y \) of non-zero \( A \)-modules. This is because the unique (up to isomorphism) simple module \( S \) is a submodule of \( Y \) and a factor of \( X \). Thus if \( (\mathcal{X}, \mathcal{Y}) \) is an Ext-orthogonal pair for \( A \), then \( \mathcal{X} = A \) or \( \mathcal{Y} = A \).

5. Ext-orthogonal pairs of finite type

At this point, we use the results from §3 to characterize for hereditary rings the Ext-orthogonal pairs of finite type. Those are, by definition, the Ext-orthogonal pairs generated by a set of finitely presented modules.

Theorem 5.1. Let \( A \) be a hereditary ring and \( (\mathcal{X}, \mathcal{Y}) \) an Ext-orthogonal pair for the module category of \( A \). Then the following are equivalent.

1. The subcategory \( \mathcal{Y} \) is closed under taking coproducts.
2. Every module in \( \mathcal{X} \) is a filtered colimit of finitely presented modules from \( \mathcal{X} \).
3. There exists a category \( \mathcal{C} \) of finitely presented modules such that \( \mathcal{C} = \mathcal{Y} \).

We need some preparations for the proof of this result. The first lemma is a slight modification of [3, Proposition 2.1].

Lemma 5.2. Let \( A \) be a ring and \( \mathcal{Y} \) a full subcategory of its module category. Denote by \( \mathcal{X} \) the category of \( A \)-modules \( X \) of projective dimension at most 1 satisfying \( \text{Ext}^1_A(X, Y) = 0 \) for all \( Y \in \mathcal{Y} \). Then any module in \( \mathcal{X} \) is a filtered colimit of finitely presented modules from \( \mathcal{X} \).

Proof. Let \( X \in \mathcal{X} \). Choose an exact sequence \( 0 \to P \xrightarrow{\phi} Q \to X \to 0 \) such that \( P \) is free and \( Q \) is projective. Note that \( \text{Ext}^1_A(X, Y) = 0 \) implies that every morphism \( P \to Y \) factors through \( \phi \). The commuting diagrams of \( A \)-module morphisms
\[
\begin{array}{ccc}
0 & \xrightarrow{\phi_i} & P_i & \xrightarrow{\phi_i} & Q_i & \xrightarrow{\phi_i} & X_i & \to 0 \\
0 & \xrightarrow{\phi} & P & \xrightarrow{\phi} & Q & \xrightarrow{\phi} & X & \to 0
\end{array}
\]
with \( P_i \) and \( Q_i \) finitely generated projective form a filtered system of exact sequences such that \( \varinjlim \phi_i = \phi \). Note that \( P \) is a filtered colimit of its finitely generated direct summands since \( P \) is free. Thus there is a cofinal subsystem such that each morphism \( P_i \to P \) is a split monomorphism. Therefore we may without loss of generality assume that each morphism \( P_i \to P \) is a split monomorphism.

Clearly \( \varinjlim X_i = X \), and it remains to prove that \( \text{Ext}^1_A(X_i, Y) = 0 \) for all \( i \). This is equivalent to showing that each morphism \( \mu: P_i \to Y \) with \( Y \in \mathcal{Y} \) factors through
φ. For this, we first factor each such µ through the split monomorphism P_i → P, then through φ, and finally compose the morphism Q → Y which we have obtained with the morphism Q_i → Q. The result is a morphism ν: Q_i → Y such that νφ_i = µ, as desired.

The second lemma establishes some necessary properties of the 5-term sequences.

Lemma 5.3. Let A be a hereditary ring and (X, Y) a complete Ext-orthogonal pair for Mod A. Let M be an A-module and ε_M the corresponding 5-term exact sequence.

(1) If Ext^1_A(M, Y) = 0, then Y_M = 0.
(2) Suppose that Y is closed under coproducts and let M = lim_{→} M_i be a filtered colimit of A-modules M_i. Then ε_M = lim_{→} ε_{M_i}.

Proof. We use the uniqueness of the 5-term exact sequences guaranteed by Lemma 2.9. If Ext^1_A(M, Y) = 0, then the image of the morphism X_M → M belongs to X. Thus X_M → M is a monomorphism since ε_M is unique, and this yields (1).

To prove (2), one uses that X and Y are closed under taking colimits and that taking filtered colimits is exact. Thus lim_{→} ε_{M_i} is an exact sequence with middle term M and all other terms in X or Y. Now the uniqueness of ε_M implies that ε_M = lim_{→} ε_{M_i}.

Finally, the following lemma is needed for hereditary rings which are not noetherian.

Lemma 5.4. Let M be a finitely presented module over a hereditary ring and N ⊆ M any submodule. Then N is a direct sum of finitely presented modules.

Proof. We combine two results. Over a hereditary ring, any submodule of a finitely presented module is a direct sum of a finitely presented module and a projective module; see [8, Theorem 5.1.6]. In addition, one uses that any projective module is a direct sum of finitely generated projective modules; see [1].

Proof of Theorem 5.1. (1) ⇒ (2): Suppose that Y is closed under taking coproducts. We apply Corollary 3.4 to obtain for each module M the natural exact sequence ε_M. Here note that we a priori did not assume completeness of (X, Y). Now suppose that M belongs to X. Then one can write M = lim_{→} M_i as a filtered colimit of finitely presented modules with Ext^1_A(M_i, Y) = 0 for all i; see Lemma 5.3. Next we apply Lemma 5.3. Thus

lim_{→} X_{M_i} → Y \rightarrow M_i → M

and each X_{M_i} is a submodule of the finitely presented module M_i. Finally, each X_{M_i} is a filtered colimit of finitely presented direct summands by Lemma 5.4. Thus M is a filtered colimit of finitely presented modules from X.

(2) ⇒ (3): Let X_{fp} denote the full subcategory that is formed by all finitely presented modules in X. Observe that Y is closed under taking coproducts and cokernels. Thus X_{fp}\perp = X\perp = Y provided that X = lim_{→} X_{fp}.

(3) ⇒ (1): Use that for each finitely presented A-module X, the functor Ext^1_A(X, −) preserves all coproducts.

Note that Theorem 5.1 gives rise to a bijection between extension closed abelian subcategories of finitely presented modules and Ext-orthogonal pairs of finite type. We will state this explicitly in §8, but we in fact prove it here by the following proposition.
Proposition 5.5. Let $A$ be a hereditary ring and $C$ a category of finitely presented $A$-modules. Then $\perp (C^\perp ) \cap \text{mod } A$ equals the smallest extension closed abelian subcategory of mod $A$ containing $C$.

Proof. Let $D$ denote the smallest extension closed abelian subcategory of mod $A$ containing $C$. We claim that the category $\lim D$ which is formed by all filtered colimits of modules in $D$ is an extension closed abelian subcategory of Mod $A$.

Assume for the moment that the claim holds. Then Theorem 6.1 implies that $\mathcal{X} = \perp (C^\perp )$ equals the smallest extension closed abelian subcategory of Mod $A$ closed under coproducts and containing $C$. Our claim then implies $\mathcal{X} = \lim D$, so $\mathcal{X} \cap \text{mod } A = D$ and we are finished.

Therefore, it only remains to prove the claim. First observe that every morphism in $\lim D$ can be written as a filtered colimit of morphisms in $D$. Using that taking filtered colimits is exact, it follows immediately that $\lim D$ is closed under kernels and cokernels in Mod $A$.

It remains to show that $\lim D$ is closed under extensions. To this end let $\eta : 0 \to L \to M \to N \to 0$ be an exact sequence with $L$ and $N$ in $\lim D$. We can without loss of generality assume that $N$ belongs to $D$, because otherwise the sequence $\eta$ is a filtered colimit of the pull-back exact sequences with the last terms in $D$. Next we choose a morphism $\phi : M' \to M$ with $M'$ finitely presented. All we need to do now is to show that $\phi$ factors through an object in $D$; see [21]. We may, moreover, assume that the composite of $\phi$ with $M \to N$ is an epimorphism. This is because otherwise we can take an epimorphism $P \to N$ with $P$ finitely generated projective, factor it through $M \to N$, and replace $\phi$ by $\phi' : M' \otimes P \to M$. Finally, denote by $L'$ the kernel of $\phi$, which is necessarily a finitely presented module. The induced map $L' \to L$ then factors through an object $L''$ in $D$ since $L$ belongs to $\lim D$. Forming the push-out exact sequence $0 \to L' \to M' \to N \to 0$ along the morphism $L' \to L''$ gives an exact sequence $0 \to L'' \to M'' \to N \to 0$. Now $\phi$ factors through $M''$ which belongs to $D$. \hfill \Box

6. Universal localizations

A ring homomorphism $A \to B$ is called a universal localization if there exists a set $\Sigma$ of morphisms between finitely generated projective $A$-modules such that

1. $\sigma \otimes_A B$ is an isomorphism of $B$-modules for all $\sigma \in \Sigma$, and
2. every ring homomorphism $A \to B'$ such that $\sigma \otimes_A B'$ is an isomorphism of $B$-modules for all $\sigma \in \Sigma$ factors uniquely through $A \to B$.

Let $A$ be a ring and $\Sigma$ a set of morphisms between finitely generated projective $A$-modules. Then there exists a universal localization inverting $\Sigma$ and this is unique up to a unique isomorphism; see [31] for details. The universal localization is denoted by $A \to A_\Sigma$ and restriction identifies Mod $A_\Sigma$ with the full subcategory consisting of all $A$-modules $M$ such that $\text{Hom}_A(\sigma, M)$ is an isomorphism for all $\sigma \in \Sigma$. Note that $\text{Hom}_A(\sigma, M)$ is an isomorphism if and only if $M$ belongs to $\{\text{Ker } \sigma, \text{Coker } \sigma\}^\perp$, provided that $A$ is hereditary. The main result of this section is then the following theorem.

Theorem 6.1. Let $A$ be a hereditary ring. A ring homomorphism $f : A \to B$ is a homological epimorphism if and only if $f$ is a universal localization.

Proof. Suppose first that $f : A \to B$ is a homological epimorphism. This gives rise to an Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ for Mod $A$, if we identify Mod $B$ with a full subcategory
Let $\mathcal{X}_{fp}$ denote the full subcategory that is formed by all finitely presented modules in $\mathcal{X}$. It follows from Theorem 5.1 that $\mathcal{X}_{fp}^\perp = \mathcal{Y}$. Now fix for each $X \in \mathcal{X}_{fp}$ an exact sequence

$$0 \to P_X \xrightarrow{\sigma_X} Q_X \to X \to 0$$

such that $P_X$ and $Q_X$ are finitely generated projective, and let $\Sigma = \{\sigma_X \mid X \in \mathcal{X}_{fp}\}$. Then

$$\text{Mod } B = \mathcal{X}_{fp}^\perp = \text{Mod } A_\Sigma.$$

Therefore, $f : A \to B$ is a universal localization, since $\mathcal{X}_{fp}^\perp$ determines the corresponding ring epimorphism uniquely up to isomorphism, see the proof of Proposition 3.2.

Now suppose $f : A \to B$ is a universal localization. Then restriction identifies the category of $B$-modules with a full extension closed subcategory of $\text{Mod } A$. Thus we have induced isomorphisms

$$\text{Ext}^*_B(X, Y) \xrightarrow{\sim} \text{Ext}^*_A(X, Y)$$

for all $B$-modules $X, Y$, since $A$ is hereditary. It follows that $f$ is a homological epimorphism. □

Remark 6.2. Neither implication in Theorem 6.1 is true if one drops the assumption on the ring $A$ to be hereditary, not even if the global dimension is 2. In [17], Keller gives an example of a Bézout domain $A$ and a non-zero ideal $I$ such that the canonical map $A \to A/I$ is a homological epimorphism, but any map $\sigma$ between finitely generated projective $A$-modules needs to be invertible if $\sigma \otimes_A A/I$ is invertible. We refine the construction so that $\text{gldim } A = 2$, see Example 7.8. On the other hand, Neeman, Ranicki, and Schofield use finite dimensional algebras to construct in [26] examples of universal localizations that are not homological epimorphisms. They are also able to construct such examples of global dimension 2, see [26, Remark 2.13].

7. The telescope conjecture

Now we are ready to state and prove an extended version of Theorem A after recalling the necessary notions.

Let $A$ be a ring. A complex of $A$-modules is called perfect if it is isomorphic to a bounded complex of finitely generated projective modules. Note that a complex $X$ is perfect if and only if the functor $\text{Hom}_{D(\text{Mod } A)}(X, -)$ preserves coproducts. One direction of this statement is easy to prove since $\text{Hom}_{D(\text{Mod } A)}(A, -)$ preserves coproducts and every perfect complex is finitely built from $A$. The converse follows from [24, Lemma 2.2] and [3 Proposition 3.4]. Recall also that a localizing subcategory $C$ of $D(\text{Mod } A)$ is generated by perfect complexes if $C$ admits no proper localizing subcategory containing all perfect complexes from $C$.

Theorem 7.1. Let $A$ be a hereditary ring. For a localizing subcategory $C$ of $D(\text{Mod } A)$ the following conditions are equivalent:

1. There exists a localization functor $L : D(\text{Mod } A) \to D(\text{Mod } A)$ that preserves coproducts and such that $C = \text{Ker } L$.
2. The localizing subcategory $C$ is generated by perfect complexes.
3. There exists a localizing subcategory $D$ of $D(\text{Mod } A)$ that is closed under products such that $C = \perp D$. 


Corollary 3.4 we have constructed a localization functor \( L \) that is closed under products and coproducts; see Proposition 2.4. In the proof of \( C \) that subcategory. \( C \) such that \( \text{Hom}_D \) such that \( L \). 7.2 Remark  \[ \text{Proof.} \quad (1) \Rightarrow (2): \text{The kernel Ker} L \text{ and the essential image Im}: L \text{ of a localization functor \( L \) form an Ext-orthogonal pair for \( D(\text{Mod} A) \); see [4, Lemma 3.3]. We obtain an Ext-orthogonal pair \( (\mathcal{X}, \mathcal{Y}) \) for \( \text{Mod} A \) by taking \( \mathcal{X} = \text{H}^0 \text{Ker} L \) and \( \mathcal{Y} = \text{H}^0 \text{Im} L \); see Proposition 2.4. The fact that \( L \) preserves coproducts implies that \( \mathcal{Y} \) is closed under taking coproducts. It follows from Theorem 5.1 that \( \mathcal{X} \) is generated by finitely presented modules. Each finitely presented module is isomorphic in \( D(\text{Mod} A) \) to a perfect complex, and therefore Ker \( L \) is generated by perfect complexes. \( (2) \Rightarrow (3): \text{Suppose that} \mathcal{C} \text{ is generated by perfect complexes. Then there exists a localization functor \( L: D(\text{Mod} A) \rightarrow D(\text{Mod} A) \) such that Ker \( L = \mathcal{C} \). Thus we have an Ext-orthogonal pair \( (\mathcal{C}, \mathcal{D}) \) for \( D(\text{Mod} A) \) with \( \mathcal{D} = \text{Im} L; \) see [4, Lemma 3.3]. Now observe that \( \mathcal{D} = \mathcal{C}^\perp \) is closed under coproducts, since for any perfect complex \( X \) the functor \( \text{Hom}_{D(\text{Mod} A)}(X, -) \) preserves coproducts. It follows that \( \mathcal{D} \) is a localizing subcategory. \( (3) \Rightarrow (1): \) Let \( \mathcal{D} \) be a localizing subcategory that is closed under products such that \( \mathcal{C} = \perp \mathcal{D} \). Then \( \mathcal{Y} = \text{H}^0 \mathcal{D} \) is an extension closed abelian subcategory of \( \text{Mod} A \) that is closed under products and coproducts; see Proposition 2.4. In the proof of Corollary 5.3 we have constructed a localization functor \( L: D(\text{Mod} A) \rightarrow D(\text{Mod} A) \) such that \( \mathcal{C} = \text{Ker} L \). More precisely, there exists a homological epimorphism \( A \rightarrow B \) such that \( L = - \otimes_A^L B \). It remains to notice that this functor preserves coproducts. \( \square \) Remark 7.2. The implication \( (1) \Rightarrow (2) \) is known as the telescope conjecture. Let us sketch the essential ingredients of the proof of this implication. In fact, the proof is not as involved as one might expect from the references to preceding results of this work. We need the 5-term exact sequence \( \varepsilon_M \) for each module \( M \) which one gets immediately from the the localization functor \( L \); see Proposition 2.4. The perfect complexes generating \( \mathcal{C} \) are constructed in the proof of Theorem 5.1 where the relevant implication is \( (1) \Rightarrow (2) \). For this proof, one uses Lemmas 5.2–5.4 but this is all. Remark 7.3. Let \( A \) be a hereditary ring and \( B \) a ring that is derived equivalent to \( A \), that is, there is an equivalence of triangulated categories \( D(\text{Mod} A) \overset{\sim}{\rightarrow} D(\text{Mod} B) \). Then the statement of Theorem 7.1 carries over from \( A \) to \( B \). In particular, the statement of Theorem 7.1 holds for every tilted algebra in the sense of Happel and Ringel [15]. Given the proof of the telescope conjecture for the derived categories of hereditary rings, one may be tempted to think that perhaps it is possible to get a similar result for rings of higher global dimension. Here we show that this is not the case. Namely, we construct a class of rings for which the conjecture fails for the derived category, and we will see that some of them have global dimension 2. To achieve this, we use the following result due to Keller [17].

Lemma 7.4. Let \( A \) be a ring and \( I \) a non-zero two-sided ideal of \( A \) such that

1. Tor\(^1\)\(_A\)\((A/I, A/I) = 0 \) for all \( i \geq 1 \) (that is, the surjection \( A \rightarrow A/I \) is a homological epimorphism), and
2. \( I \) is contained in the Jacobson radical of \( A \).

Then \( L = - \otimes_A^L A/I: D(\text{Mod} A) \rightarrow D(\text{Mod} A) \) is a coproduct preserving localization functor but Ker \( L \), which is the smallest localizing subcategory containing \( I \), contains no non-zero perfect complexes. In particular, the telescope conjecture fails for \( D(\text{Mod} A) \).

In order to find such \( A \) and \( I \) with (right) global dimension of \( A \) equal to 2, we restrict ourself to the case when \( A \) is a valuation domain. That is, \( A \) is a commutative domain
with the property that for each pair \( a, b \in A \), either \( a \) divides \( b \) or \( b \) divides \( a \). We refer to [10, Chapter II] for a discussion of such domains. Here, we mention only the properties which we need for our example:

**Lemma 7.5.** The following holds for a valuation domain \( A \) which is not a field.

1. The ring \( A \) is local and its weak global dimension equals 1.
2. The maximal ideal \( P \) of \( A \) is either principal or idempotent.
3. For any ideal \( I \) of \( A \) we have the isomorphism \( \text{Tor}^1_A(A/I, A/I) \cong I/I^2 \).

**Proof.** (1) The ring \( A \) is local since the ideals of \( A \) are totally ordered by inclusion. The second part of (1) follows from [10, VI.10.4].

(2) This is a direct consequence of results in [10, Section II.4]. For an ideal \( I \), one defines \( I' = \{ a \in A \mid aI \subseteq I \} \). It turns out that \( I' \) is always a prime ideal and \( I \) is naturally an \( R_{I'} \)-module. Moreover, \( I = I' \) if \( I \) itself is a prime ideal, [10, II.4.3 (iv)]. In particular we have \( P' = P \). On the other hand, [10, p. 69, item (d)] says that \( I' \cdot I \subseteq I \) if and only if \( I \) is a principal ideal of \( R_{I'} \). Specialized to \( P \), this precisely says that \( P^2 = P' \cdot P \subseteq P \) if and only if \( P \) is a principal ideal of \( R \).

(3) Tensoring the exact sequence \( 0 \to I \to A \to A/I \to 0 \) with \( A/I \) gives the exact sequence

\[
\frac{A/I \otimes_A I}{0} \to \frac{A/I}{\sim} A/I \otimes_A \frac{A/I}{0} \to 0.
\]

It follows that \( \text{Tor}^1_A(A/I, A/I) \cong A/I \otimes_A I \), and the right exactness of the tensor product yields \( A/I \otimes_A I \cong I/I^2 \). \( \square \)

The following result is a straightforward consequence.

**Proposition 7.6.** Let \( A \) be a valuation domain whose maximal ideal \( P \) is non-principal. Then the telescope conjecture fails for \( D(\text{Mod} \, A) \). More precisely, \( L = - \otimes_A^L A/P \) is a coproduct preserving localization functor on \( D(\text{Mod} \, A) \) whose kernel is non-trivial (it contains \( P \)) but not generated by perfect complexes.

**Proof.** It is enough to prove that the maximal ideal \( P \) meets the conditions of Lemma 7.4. As \( P \) is the Jacobson radical of \( A \), condition (2) is fulfilled. Condition (1) follows easily from Lemma 7.5. \( \square \)

What we are left with now is to construct a valuation domain whose maximal ideal is non-principal and whose global dimension is 2. To this end, we recall the basic tool to construct valuation domains with given properties: the value group. If \( A \) is a valuation domain, denote by \( Q \) its quotient field and by \( U \) the group of units of \( A \). Then \( U \) is a totally ordered abelian group. More precisely, \( G = Q^*/U \) is a totally ordered abelian group. More precisely, \( G \) is an abelian group, the relation \( \leq \) on \( G \) defined by \( aU \leq bU \) if \( ba^{-1} \in A \) gives a total order on \( G \), and we have the compatibility condition

\[ \alpha \leq \beta \quad \text{implies} \quad \alpha \cdot \gamma \leq \beta \cdot \gamma \quad \text{for all} \; \alpha, \beta, \gamma \in G. \]

The pair \((G, \leq)\) is called the value group of \( A \). We will use the following fundamental result [10, Theorem 3.8].
Proposition 7.7. Let \( k \) be a field and \((G, \leq)\) a totally ordered abelian group. Then there is a valuation domain \( A \) whose residue field \( A/P \) is isomorphic to \( k \), and whose value group is isomorphic to \( G \) as an ordered group.

Now, we can give the promised example.

Example 7.8. Let \( G \) be a free abelian group of countable rank. If we view \( G \) as the group \( \mathbb{Z}^{(\mathbb{N})} \) (with additive notation), then \( G \) is naturally equipped with the lexicographic ordering which makes it to a totally ordered group. Let \( A \) be a valuation domain whose value group is isomorphic to \( G \). In fact, looking closer at the particular construction in [10, Section II.3], we can construct \( A \) such that it is countable.

We claim that the maximal ideal \( P \) of \( A \) is non-principal and that \( \text{gldim} \ A = 2 \). Indeed, each ideal of \( A \) is flat and countably generated since the value group is countable. Thus, each ideal is of projective dimension at most 1 and \( \text{gldim} \ A \leq 2 \). On the other hand, it is easy to see that \( A \) has non-principal, hence non-projective, ideals and so is not hereditary. One of them is \( P \), which is generated by elements of \( A \) whose cosets in the value group \( Q^*/U \) correspond, under the isomorphism \( Q^*/U \cong \mathbb{Z}^{(\mathbb{N})} \), to the canonical basis elements \( e_1, e_2, e_3, \ldots \in \mathbb{Z}^{(\mathbb{N})} \).

This way, we obtain a countable valuation domain \( A \) of global dimension 2 such that the telescope conjecture fails for \( D(\text{Mod} \ A) \) by Proposition 7.6.

8. A bijective correspondence

In this final section we summarize our findings by stating explicitly the correspondence between various structures arising from Ext-orthogonal pairs for hereditary rings. In particular, this completes the proof of an extended version of Theorem B:

Theorem 8.1. For a hereditary ring \( A \) there are bijections between the following sets:

1. Ext-orthogonal pairs \((X, Y)\) for \( \text{Mod} \ A \) such that \( Y \) is closed under coproducts.
2. Ext-orthogonal pairs \((Y, Z)\) for \( \text{Mod} \ A \) such that \( Y \) is closed under products.
3. Extension closed abelian subcategories of \( \text{Mod} \ A \) that are closed under products and coproducts.
4. Extension closed abelian subcategories of \( \text{mod} \ A \).
5. Homological epimorphisms \( A \to B \) (up to isomorphism).
6. Universal localizations \( A \to B \) (up to isomorphism).
7. Localizing subcategories of \( D(\text{Mod} \ A) \) that are closed under products.
8. Localization functors \( D(\text{Mod} \ A) \to D(\text{Mod} \ A) \) preserving coproducts (up to natural isomorphism).
9. Thick subcategories of \( D^b(\text{mod} \ A) \).

Proof. We state the bijections explicitly in the following table and give the references to the places where these bijections are established.
For (3) \(\rightarrow\) (5), the functor \(F\) denotes a left adjoint of the inclusion \(\mathcal{Y} \rightarrow \text{Mod} A\). For (7) \(\rightarrow\) (8), the functor \(G\) denotes a left adjoint of the inclusion \(C \rightarrow \text{D(Mod} A)\).

Let us mention that this correspondence is related to recent work of some other authors. In [34], Schofield establishes for any hereditary ring the bijection (4) \(\leftrightarrow\) (6). In [27], Nicolás and Saorín establish for a differential graded algebra \(A\) a correspondence between recollements for the derived category \(\text{D}(A)\) and differential graded homological epimorphisms \(A \rightarrow B\). This correspondence specializes for a hereditary ring to the bijection (5) \(\leftrightarrow\) (8).

### A finiteness condition.

Given an Ext-orthogonal pair for the category of \(A\)-modules as in Theorem 8.1, it is a natural question to ask when its restriction to the category of finitely presented modules yields a complete Ext-orthogonal pair for \(\text{mod} A\). This is very important especially when considering relations of results from this paper to representation theory of finite dimensional algebras. For that setting, we characterize this finiteness condition in terms of finitely presented modules; see also Proposition 4.2.

**Proposition 8.2.** Let \(A\) be a finite dimensional hereditary algebra over a field and \(C\) an extension closed abelian subcategory of \(\text{mod} A\). Then the following are equivalent.

1. There exists a complete Ext-orthogonal pair \((C, D)\) for \(\text{mod} A\).
2. The inclusion \(C \rightarrow \text{mod} A\) admits a right adjoint.
3. There exists an exceptional object \(X \in C\) such that \(C\) is the smallest extension closed abelian subcategory of \(\text{mod} A\) containing \(X\).
4. Let \((X, \mathcal{Y})\) be the Ext-orthogonal pair for \(\text{Mod} A\) generated by \(C\). Then for each \(M \in \text{mod} A\) the 5-term exact sequence \(\varepsilon_M\) belongs to \(\text{mod} A\).

**Proof.** (1) \(\Rightarrow\) (2): For \(M \in \text{mod} A\) let \(0 \rightarrow D_M \rightarrow C_M \rightarrow M \rightarrow D^M \rightarrow C^M \rightarrow 0\) be its 5-term exact sequence. Sending a module \(M\) to \(C_M\) induces a right adjoint for the inclusion \(C \rightarrow \text{mod} A\); see Lemma 2.3.

(2) \(\Rightarrow\) (3): Choose an injective cogenerator \(Q\) in \(\text{mod} A\) and let \(X\) denote its image under the right adjoint of the inclusion of \(C\). A right adjoint of an exact functor preserves local structure.

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\(^2\)The first author is grateful to Manolo Saorín for pointing out this bijection.
injectivity. It follows that $X$ is an exceptional object and that $\mathcal{C}$ is the smallest extension closed abelian subcategory of $\text{mod } A$ containing $X$.

(3) $\Rightarrow$ (4): See Proposition 4.2.

(4) $\Rightarrow$ (1): The property of the pair $(\mathcal{X}, \mathcal{Y})$ implies that $(\mathcal{X} \cap \text{mod } A, \mathcal{Y} \cap \text{mod } A)$ is a complete Ext-orthogonal pair for $\text{mod } A$. An application of Proposition 5.5 yields the equality $\mathcal{X} \cap \text{mod } A = \mathcal{C}$. Thus there exists a complete Ext-orthogonal pair $(\mathcal{C}, \mathcal{D})$ for $\text{mod } A$.

□

Remark 8.3. There is a dual result which is obtained by applying the duality between modules over the algebra $A$ and its opposite $A^{\text{op}}$. Note that condition (3) is self-dual.

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