Wilson loop evaluations
in the stochastic vacuum model

Ken Williams

Department of Physical and Earth Sciences,
Jacksonville State University, Jacksonville, Alabama 36265

Abstract

The stochastic vacuum model description of a heavy meson is discussed in the context of a gauge-invariant approach where Wilson loop expectation values appear naturally in the $O(v^2)$ spin-orbit Hamiltonian. These expectation values have been derived elsewhere, however by a procedure whose legitimacy is now placed in question. Here they are derived by standard functional methods with a result that is identical to the previous one. In addition, a full spin-independent Hamiltonian reduction to $O(v^2)$ is carried out.
1 introduction

It was the work of Leutwyler and Voloshin some years ago in the context of the sum rule formalism that first suggested the fundamental nonlocality of nonperturbative interactions between hadronic quarks [1]. Leading effects were later shown proportional to a gluon condensate [2] thereby excluding the possibility of a purely local description. In a separate line of development Wilson’s lattice work [3] led to the well-known area law as a qualitative formulation of color confinement. There, gauge invariance of the hadronic state is the guiding principle. There too what is fundamental is the presence of a nonlocal operator - the Wilson loop.

In the stochastic vacuum model (svm) of Dosch and Simonov [4] we find these salient features mutually complementary, where area law asymptotics follow from a non-zero gluon condensate. Active gluon degrees via the field’s correlation length measured against $Q\bar{Q}$ correlations also come into the picture; this in such a way that the model also accounts for intermediate as well as short-range perturbative behavior.

The aim of the present article is to describe the reduction of the svm to an effective $O(v^2)$ spin-orbit interaction Hamiltonian for the heavy $Q\bar{Q}$ state. In fact, a general reduction in terms of Wilson loop expectation values (ev) has already been made [5], thus leaving as the main focus of this study only the ev derivations themselves. The ev too have been derived [6, 7, 8, 9, 10], however by a procedure whose legitimacy is here placed in question: From the functional variation of the Wilson-loop

$$\delta i\ln W = \int \delta \sigma_{\mu\nu}(z_1) \langle \langle F_{\mu\nu}(z_1) \rangle \rangle$$

is presumed the functional derivative relation

$$\frac{\delta}{\delta \sigma_{\mu\nu}(z_1)} i\ln W = \langle \langle F_{\mu\nu}(z_1) \rangle \rangle$$

resulting in a spin-orbit interaction that e.g. satisfies the Gromes relation.
We argue that procedure (1) → (2) lacks legitimacy by virtue of insufficient degrees of freedom characterizing the variational area element $\delta\sigma_{\mu\nu}$. In section 2 the familiar svm spin-orbit Hamiltonian is derived along the usual lines from assumption (1) → (2). We then demonstrate the general invalidity of the assumption and show that when omitted it leaves the Hamiltonian insufficiently specified. Additional constraints are derived in section 3 where the model statement itself is expanded to $O(m^{-2})$ and velocity coefficients matched term by term. This procedure leads again to the familiar svm spin-orbit Hamiltonian; i.e, our spin-orbit result agrees with the one derived from (2), although, as we point out, generally applied (2) may lead to incorrect results [11]. Finally, we lay out details of a full reduction to the $O(m^{-2})$ spin-independent Hamiltonian.

2 functional variation and the svm

From a Foldy-Wouthysen reduction of the gauge invariant $\bar{Q}Q$ 4-point function the $O(m^{-2})$ interaction Hamiltonian of [3] is given in terms of the Wilson loop

$$V = V_{so} + V_{si}$$

$$V_{so} = \sum_{i=1}^{2} (-1)^i \frac{1}{2m_i} \epsilon_{j\mu\nu k} s_i^j \hat{z}_i^\mu \langle \langle F_{\nu k}(z_i) \rangle \rangle$$

$$\equiv \left( \frac{1}{2m_1^2} \mathbf{L}_1 \cdot \mathbf{s}_1 - \frac{1}{2m_2^2} \mathbf{L}_2 \cdot \mathbf{s}_2 \right) \left[ V_0(r) + 2V_1(r) \right]' / r$$

$$+ \frac{1}{m_1 m_2} \left( \mathbf{L}_1 \cdot \mathbf{s}_2 - \mathbf{L}_2 \cdot \mathbf{s}_1 \right) V_2'(r) / r$$

(3)

$$\int dt V_{si} = i \ln W = i \ln \frac{1}{3} \langle \langle tr P \exp(ig \oint dt (A_0 - \hat{z}^i A^i)) \rangle \rangle$$

where Darwin and hyperfine terms, not relevant to the present discussion, are omitted. Key to this approach therefore is the evaluation of the six independent expectation values $\langle \langle F_{\mu\nu} \rangle \rangle$. The defining Euclidean svm statement in the present context
\[
\ln W = -\frac{\beta}{2} \int d\sigma_{\mu\nu}(u) \int d\sigma_{\lambda\rho}(v) \left\{ (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) D(w^2) \right\} + 1 \left[ \frac{\partial}{\partial u_{\mu}}(w_{\lambda}\delta_{\nu\rho} - w_{\rho}\delta_{\nu\lambda}) + \frac{\partial}{\partial u_{\nu}}(w_{\rho}\delta_{\mu\lambda} - w_{\lambda}\delta_{\mu\rho}) \right] D_1(w^2), w \equiv u - v
\]

where the integrals are evaluated over instantaneous straight-line surfaces: \((u_i = sz_i + (1 - s)z_{2i}, u_4 = t), 0 \leq s \leq 1\), with \(d\sigma_{\mu\nu}(u) \equiv dt ds(\partial u_{\mu}/dt)(\partial u_{\nu}/ds)\). \(\beta\) is the gluon condensate, and \(D\) and \(D_1\) are gluon correlation functions that fall off rapidly.

From the quark world-line variation, \(\delta z_{1\mu}\), of the Wilson loop

\[
\delta \ln W = \int \delta \sigma_{\mu\nu}(z_1) \langle \langle F_{\mu\nu}(z_1) \rangle \rangle = -i\beta \int \delta \sigma_{\mu\nu}(z_1) \int d\sigma_{\lambda\rho}(v) \left\{ (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) D(\bar{w}^2) \right\} + 1 \left[ \frac{\partial}{\partial z_{1\mu}}(\bar{w}_{\lambda}\delta_{\nu\rho} - \bar{w}_{\rho}\delta_{\nu\lambda}) + \frac{\partial}{\partial z_{1\nu}}(\bar{w}_{\rho}\delta_{\mu\lambda} - \bar{w}_{\lambda}\delta_{\mu\rho}) \right] D_1(\bar{w}^2), \bar{w} \equiv z_1 - v
\]

is proposed in references [6, 7, 8] the following the functional relation

\[
\frac{\delta}{\delta \sigma_{\mu\nu}(z_1)} \delta \ln W = \langle \langle F_{\mu\nu}(z_1) \rangle \rangle
\]

where \(\delta \sigma_{\mu\nu} \equiv (dz_{\mu}\delta z_{\nu} - dz_{\nu}\delta z_{\mu})/2\) is called the variational area element. From this follows

\[
\langle \langle F_{0l}(z_1) \rangle \rangle = \beta r_l \int d\tau \left\{ \int_0^r d\lambda \frac{1}{r} D(\tau^2 + \lambda^2) + \frac{1}{2} D_1(\tau^2 + r^2) \right\}
\]

\[
\langle \langle F_{il}(z_1) \rangle \rangle = \beta (\dot{z}_{1l}r_i - \dot{z}_{1i}r_l) \int d\tau \int_0^r d\lambda \frac{1}{r} \left( 1 - \frac{\lambda}{r} \right) D(\tau^2 + \lambda^2) + \beta (\dot{z}_{2l}r_i - \dot{z}_{2i}r_l) \int d\tau \left\{ \int_0^r d\lambda \frac{\lambda}{r^2} D(\tau^2 + \lambda^2) + \frac{1}{2} D_1(\tau^2 + r^2) \right\}
\]

yielding spin-orbit potentials

\[
V'_0(r) = \beta \int d\tau \left[ \int_0^r d\lambda D(\tau^2 + \lambda^2) + \frac{r}{2} D_1(\tau^2 + r^2) \right]
\]
\( V'_1(r) = -\beta \int d\tau \int_0^r d\lambda \left( 1 - \frac{\lambda}{r} \right) D(\tau^2 + \lambda^2) \)  
\( V'_2(r) = \beta \int d\tau \left[ \int_0^\tau d\lambda \frac{\lambda}{r} D(\tau^2 + \lambda^2) + \frac{r}{2} D_1(\tau^2 + r^2) \right] \)

which together e.g. satisfy the relation of Gromes \[12\]

\[ [V_0 + V_1 - V_2]' = 0. \]  

We show here that variational derivative (6) does indeed not follow from variation (5). One should notice that in passing from (5) to (6), (1) to (2), is the assumption that there are in the variational area element at least six quark and six antiquark coordinate degrees of freedom - an assumption clearly in error.

The Wilson loop is a functional of quark and antiquark world lines. With this in mind we rewrite the rhs of (5)

\[ \int dt \delta z_1^\nu \dot{z}_1^\mu \langle\langle F^\mu\nu(z_1) \rangle\rangle = \int dt \delta z_1^\nu \dot{z}_1^\mu \Theta_{\mu\nu} \]  

where

\[ \Theta_{\mu\nu} \equiv -i\beta \int d\sigma_{\lambda\rho}(v) \left\{ (\delta_{\lambda\nu} \delta_{\nu\rho} - \delta_{\mu\rho} \delta_{\nu\lambda}) D(\tilde{w}^2) \right\} \]

\[ \frac{1}{2} \left[ \frac{\partial}{\partial u_\mu}(\tilde{w}_\lambda \delta_{\nu\rho} - \tilde{w}_\rho \delta_{\nu\lambda}) + \frac{\partial}{\partial u_\nu}(\tilde{w}_\rho \delta_{\mu\lambda} - \tilde{w}_\lambda \delta_{\mu\rho}) \right] D_1(\tilde{w}^2) \right\}. \]

From this we should like to extract the six element \( \{\langle\langle F^\mu\nu\rangle\rangle\} \) for determination of the Hamiltonian (3). The set appears in (3) as the linearly transformed

\[ \{\delta_{ij} \delta_{\mu\nu} \tilde{z}_1^\mu \langle\langle F^\nu j(z_1) \rangle\rangle, \epsilon_{ij\mu\nu} \tilde{z}_1^\mu \langle\langle F^\nu j(z_1) \rangle\rangle\} \]

- a set of vectors spanning the space of field tensor elements. The first subset accounts for spin independent \( V_{si} \), by Stokes theorem, and the second for spin dependent \( V_{so} \) directly. It should be clear that the first subset appears in (3). Hence independence of \( V_{so} \) with respect to variation (3) follows from the linear independence
of vectors in the complete set (11). The independence is immediately established from the nonsingularity of the set’s coefficient matrix

$$\begin{vmatrix}
1 & 0 & 0 & -\dot{z}_2 & -\dot{z}_3 & 0 \\
0 & 1 & 0 & \dot{z}_1 & 0 & -\dot{z}_3 \\
0 & 0 & 1 & 0 & \dot{z}_1 & \dot{z}_2 \\
-\dot{z}_2 & \dot{z}_1 & 0 & 2 & 0 & 0 \\
-\dot{z}_3 & 0 & \dot{z}_1 & 0 & 2 & 0 \\
0 & -\dot{z}_3 & \dot{z}_2 & 0 & 0 & 2 \\
\end{vmatrix} \sim (2 - \dot{z}^2)^2$$

where the vectors have been arranged by row with field tensor elements in ascending order, left to right.

For concreteness, then, variation (5)

$$\int dt \delta z_{14} \dot{z}_{1j} \langle \langle F_{4j}(z_1) \rangle \rangle = \int dt \delta z_{14} \dot{z}_{1j} \Theta_{4j}$$

$$\int dt \delta z_{1j} (\langle \langle F_{4j}(z_1) \rangle \rangle + \dot{z}_{1i} \langle \langle F_{ij}(z_1) \rangle \rangle) = \int dt \delta z_{1j} (\Theta_{4j} + \dot{z}_{1i} \Theta_{ij})$$

leads to

$$\langle \langle F_{0j}(z_1) \rangle \rangle = \Theta_{0j} + (\dot{z}_{1i} \dot{r}_{2j} - \dot{z}_{2j} \dot{r}_{1j}) f'_1(r) + (\dot{z}_{1i} \dot{z}_{2j} - \dot{z}_{2j} \dot{z}_{1i}) f'_2(r)$$

$$\langle \langle F_{ij}(z_1) \rangle \rangle = \Theta_{ij} + (\dot{r}_{i} \dot{z}_{1j} - \dot{r}_{j} \dot{z}_{1i}) f'_1(r) + (\dot{r}_{i} \dot{z}_{2j} - \dot{r}_{j} \dot{z}_{2i}) f'_2(r)$$

yielding when installed into (3) the spin-orbit potentials

$$V'_1(r) = f'_1(r) - \beta \int d\tau \int_0^r d\lambda \left(1 - \frac{\lambda}{r}\right) D(\tau^2 + \lambda^2)$$

$$V'_2(r) = -f'_2(r) + \beta \int d\tau \left[ \int_0^r d\lambda \frac{\lambda}{r} D(\tau^2 + \lambda^2) + \frac{r}{2} D_1(\tau^2 + r^2) \right]$$

for arbitrary functions $(f'_1, f'_2)$. I.e., solutions (13) and (14) satisfy variation (4), yet leaves the spin-orbit Hamiltonian $V_{so}$ entirely unspecified. This claim may be verified by direct substitution of (13) into (3) and (13) into (8).
3 specification of the svm Wilson loop expectation values

We now revisit the spin-orbit Hamiltonian derivation beginning again from (5). As in ref\[11\], additional constraints are found from the functional expansion of the model's defining statement in orders of heavy quark velocity. Accordingly, definition (4) is expanded

\[
\iota \ln W \simeq \int dtdt' \left\{ V^0(r, r') + (\dot{z}_{1i} + \dot{z}_{2i})V^\alpha_i(r, r') \right.
\]

\[
+ (\dot{z}_{1i} \dot{z}'_{1j} + \dot{z}_{2i} \dot{z}'_{2j})V^\beta_{ij}(r, r') + (\dot{z}_{1i} \dot{z}'_{2j} + \dot{z}_{2i} \dot{z}'_{1j})V^\gamma_{ij}(r, r') \}
\]

which on the O(v^2) approximation

\[
r_i' \approx r_i - \dot{r}_i \tau + \dot{r}_i \tau^2 / 2, \quad \tau = t - t'
\]

yields

\[
\iota \ln W \simeq \int dt \left\{ V_0(r) + (\dot{z}_1^2 + \dot{z}_2^2)V_a(r) + \dot{z}_1 \cdot \dot{z}_2 V_b(r) \right.
\]

\[
+ [(\dot{z}_1 \cdot \dot{r})^2 + (\dot{z}_2 \cdot \dot{r})^2]V_c(r) + (\dot{z}_1 \cdot \dot{r})(\dot{z}_2 \cdot \dot{r})V_d(r) \}
\]

for functions V^\alpha, V^\beta, V^\gamma, V_0, V_a, V_b, V_c, V_d to be determined. Equation (15) leads to (see appendix)

\[
\langle \langle F_{ij}(z_1) \rangle \rangle = \beta(\dot{z}_{ij}r_i - \dot{z}_{1i}r_j) \int d\tau \int_0^\tau d\lambda \frac{1}{\tau} \left( 1 - \frac{\lambda}{r} \right) D(\tau^2 + \lambda^2)
\]

\[
+ \beta(\dot{z}_{2j}r_i - \dot{z}_{2i}r_j) \int d\tau \left\{ \int_0^\tau d\lambda \frac{\lambda}{r^2} D(\tau^2 + \lambda^2) + \frac{1}{2} D_1(\tau^2 + r^2) \right\}
\]

and for the spin-independent potentials of (17) we find upon application of (16) (see appendix)

\[
V_0(r) = \beta \int d\tau \int_0^\tau d\lambda \left[ (r - \lambda) D(\tau^2 + \lambda^2) + \frac{\lambda}{2} D_1(\tau^2 + \lambda^2) \right]
\]

7
\[ V_a(r) = -\beta \int d\tau \int_0^r d\lambda \left[ \left( \frac{r}{6} - \frac{\lambda}{4r} + \frac{\lambda^3}{12r^2} + \frac{\lambda \tau^2}{2r^2} + \frac{\tau^2}{2r} \right) D(\tau^2 + \lambda^2) \right. \\
\left. + \left( \frac{\lambda}{8} - \frac{\lambda^2}{4r^2} + \frac{\lambda^3}{8r^2} - \frac{\lambda \tau^2}{4r^2} + \frac{\tau^2}{4r} \right) D_1(\tau^2 + \lambda^2) \right] \] (20)

\[ V_b(r) = -\beta \int d\tau \int_0^r d\lambda \left[ \left( \frac{r}{6} - \frac{\lambda^3}{6r^2} - \frac{\lambda \tau^2}{r^2} + \frac{\tau^2}{r} \right) D(\tau^2 + \lambda^2) \right. \\
\left. + \left( \frac{\lambda^2}{2r} - \frac{\lambda^3}{4r^2} + \frac{\lambda \tau^2}{2r^2} - \frac{\tau^2}{2r} \right) D_1(\tau^2 + \lambda^2) \right] \] (21)

\[ V_c(r) = -\beta \int d\tau \int_0^r d\lambda \left[ \left( -\frac{r}{6} + \frac{\lambda^2}{2r} - \frac{\lambda^3}{3r^2} \right) D(\tau^2 + \lambda^2) \right] \] (22)

\[ V_d(r) = -\beta \int d\tau \left[ \int_0^r d\lambda \left( -\frac{r}{6} - \frac{\lambda^2}{r} + \frac{2\lambda^3}{3r^2} \right) D(\tau^2 + \lambda^2) - \frac{r^2}{4} D_1(\tau^2 + r^2) \right] \] (23)

A couple observations: The ev of (18) follow from strict independence of operators \((r, v)\) in the order expansion (15). The independence is relaxed in the \(O(v^2)\) evaluation (16) that leads to spin-independent potentials (19) - (23). This explains the discrepancy between ev (18) and the one obtained in an earlier effort by the present author [16] where instead the ev are derived improperly from expansion (17). We note that as (18) is identical with (7) it yields upon insertion into (8) the spin-orbit potentials (8) that satisfy relation (9). However, only incidentally [13]. We also note that in the long-range regime the spin-independent interaction, (19) - (23), agrees with minimum area or flux-tube asymptotics [3 [11]]

\[ V_{si} \rightarrow -\frac{1}{6} a_L^2 \frac{L^2}{r} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2} \right), \] (24)

\[ a \equiv \beta \int d\tau \int_0^r d\lambda D(\tau^2 + \lambda^2). \]

This too is a correction to the aforementioned earlier effort where in the corresponding equation (17) the relation

\[ \tilde{z}_{Euclidean}^2 = \tilde{z}_{Minkowskian}^2 \] (25)

is overlooked. In ref.[8] a spin independent Hamiltonian reduction of the svm to \(O(v^2)\) is carried out. The apparent disagreement with potentials (19) - (23) is due
to the use of a partial integration scheme slightly different from the one applied here. The two results are equivalent.

4 appendix

From the functional identities

$$\frac{\delta \xi(t)}{\delta \xi(t')} = \delta(t - t') \quad (26)$$

$$\delta f(\xi) = \delta \xi \left( \frac{\partial}{\partial \xi} f(\xi) \right) \quad (27)$$

the Taylor function and functional expansions are carried out to \(O(v^2)\)

$$f(\dot{z}_1, \dot{z}_2) = f(0) + \dot{z}_1^1 \int dt' \left( \frac{\delta}{\delta \dot{z}_1^1} f(\dot{z}_1', \dot{z}_2) \right)_0 + \dot{z}_2^i \int dt' \left( \frac{\delta}{\delta \dot{z}_2^i} f(\dot{z}_1, \dot{z}_2') \right)_0$$

$$+ \frac{1}{2} \dot{z}_1^1 \dot{z}_1^j \int dt'dt'' \left( \frac{\delta}{\delta \dot{z}_1^1} \frac{\delta}{\delta \dot{z}_1^j} f(\dot{z}_1'', \dot{z}_2') \right)_0$$

$$+ \frac{1}{2} \dot{z}_2^i \dot{z}_2^j \int dt'dt'' \left( \frac{\delta}{\delta \dot{z}_2^i} \frac{\delta}{\delta \dot{z}_2^j} f(\dot{z}_1, \dot{z}_2'') \right)_0$$

$$+ \dot{z}_1^1 \dot{z}_2^j \int dt'dt'' \left( \frac{\delta}{\delta \dot{z}_1^1} \frac{\delta}{\delta \dot{z}_2^j} f(\dot{z}_1', \dot{z}_2'') \right)_0 + h.o. \quad (28)$$

$$F[\dot{z}_1, \dot{z}_2] = F[0] + \int dt \dot{z}_1^1 \left( \frac{\delta}{\delta \dot{z}_1^1} F[\dot{z}_1', \dot{z}_2] \right)_0 + \int dt \dot{z}_2^i \left( \frac{\delta}{\delta \dot{z}_2^i} F[\dot{z}_1, \dot{z}_2'] \right)_0$$

$$+ \frac{1}{2} \int dt dt' \dot{z}_1^1 \dot{z}_1^j' \left( \frac{\delta}{\delta \dot{z}_1^1} \frac{\delta}{\delta \dot{z}_1^j} F[\dot{z}_1'', \dot{z}_2] \right)_0$$

$$+ \frac{1}{2} \int dt dt' \dot{z}_2^i \dot{z}_2^j' \left( \frac{\delta}{\delta \dot{z}_2^i} \frac{\delta}{\delta \dot{z}_2^j} F[\dot{z}_1, \dot{z}_2''] \right)_0$$

$$+ \int dt dt' \dot{z}_1^1 \dot{z}_2^j \left( \frac{\delta}{\delta \dot{z}_1^1} \frac{\delta}{\delta \dot{z}_2^j} F[\dot{z}_1', \dot{z}_2''] \right)_0 + h.o. \quad (29)$$

where primes indicate time dependence, e.g., \(\xi' = \xi(t')\), and subscript \(0\) means evaluation at \(\dot{z}_1 = \dot{z}_2 = 0\). The spatial field tensor Wilson loop expectation value to
first order is then
\[
\langle\langle F_{ij}(z_1)\rangle\rangle = \langle\langle F_{ij}\rangle\rangle_0 + \hat{z}_{1k} \int dt' \left( \frac{\delta}{\delta \hat{z}_{1k}} \langle\langle F_{ij}\rangle\rangle' \right)_0 + \hat{z}_{2k} \int dt' \left( \frac{\delta}{\delta \hat{z}_{2k}} \langle\langle F_{ij}\rangle\rangle' \right)_0
\]

To find the above rhs we expand both sides of the svm, (4), to O(\(v^2\))

\[
I \ln W \simeq \ln W_0 - \int dt (\hat{z}_{1i} \langle\langle A_i(z_1)\rangle\rangle_0 + \hat{z}_{2i} \langle\langle A_i(z_2)\rangle\rangle_0)
\]

\[
- \frac{1}{2} \int dt dt' \left[ \hat{z}_{1i} \hat{z}_{1j} \left( \frac{\delta}{\delta \hat{z}_{1j}} \langle\langle A_j(z_1)\rangle\rangle' \right)_0 - \hat{z}_{1i} \hat{z}_{2j} \left( \frac{\delta}{\delta \hat{z}_{2j}} \langle\langle A_j(z_2)\rangle\rangle' \right)_0 \right]
\]

\[
- \frac{1}{2} \int dt dt' \left[ \hat{z}_{1i} \hat{z}_{1j} \left( \frac{\delta}{\delta \hat{z}_{1j}} \langle\langle A_j(z_1)\rangle\rangle' \right)_0 - \hat{z}_{2i} \hat{z}_{2j} \left( \frac{\delta}{\delta \hat{z}_{2j}} \langle\langle A_j(z_2)\rangle\rangle' \right)_0 \right]
\]

\[
= \int dt dt' \left\{ V^\alpha_i(r, r') + (\hat{z}_{1i} + \hat{z}_{2i}) V^\alpha_i(r, r') \right\}
\]

where we find

\[
V^\alpha_i = -i \beta \int_0^1 ds ds' (\omega_i r_j r'_j - \omega_j r_j r'_i) \tau \frac{\partial}{\partial \tau^2} D_1
\]

\[
V^\gamma_{ij} = -\frac{i}{2} \beta \int_0^1 ds ds' (1 - s) s' \left\{ (\delta_{ij} r_k r'_k - r'_i r_j)(D + D_1) \right\}
\]

\[
\times [\omega_k \omega_l r_k r'_l \delta_{ij} + \omega_i \omega_j r_k r'_k - \omega_j \omega_i r_k r'_k - \omega_j \omega_k r_k r'_i] \frac{\partial}{\partial \tau^2} D_1
\]

\[
= -\frac{i}{2} \beta \int_0^1 ds ds' (1 - s) s' \left\{ (\delta_{ij} r_k r'_k - r'_i r_j)(D + D_1) \right\}
\]

\[
\times [\omega_k \omega_l r_k r'_l \delta_{ij} + \omega_i \omega_j r_k r'_k - \omega_j \omega_i r_k r'_k - \omega_j \omega_k r_k r'_i] \frac{\partial}{\partial \tau^2} D_1
\]

and where

\[
\frac{\delta}{\delta \hat{z}_{1j}} I \ln W = \frac{\delta}{\delta \hat{z}_{1j}} \left( \frac{1}{3} tr P \exp[\imath g \int dt' (A_0 + \hat{z}_i A_i)] \right)
\]
\[ = -\langle A_j(z_1) \rangle \] (35)

has been used. From this follows (equating velocity coefficients)

\[ \langle A_i(z_1) \rangle_0 = -\int dt' V_i^\alpha(r, r') \]
\[ \int dt' \left( \frac{\delta}{\delta z_{1i}} \langle A_j(z_1) \rangle'_0 \right) = -2 \int dt' V_{ij}^\beta(r, r') \]
\[ \int dt' \left( \frac{\delta}{\delta \dot{z}_{2i}} \langle A_j(z_1) \rangle'_0 \right) = -2 \int dt' V_{ij}^\gamma(r, r') . \] (36)

This result installed into (30) yields the ev (18) upon performing the indicated differentiations.

For the spin-independent Hamiltonian we make the further \( O(\nu^2) \) reduction on the rhs of (31)

\[ r'_i \approx r_i - \dot{v}_i \tau + \ddot{v}_i \tau^2 / 2 \]
\[ D(\omega^2) \approx \left( 1 + O_i^i \frac{\partial}{\partial \tau^2} + O_{ii}^{\frac{\partial}{\partial \tau^2}} \frac{\partial}{\partial \tau^2} / 2 \right) D(\tau^2 + \nu^2 r^2) \] (37)

where

\[ O_{ij} \equiv \lambda \tau (r_i \dot{v}_j + r_j \dot{v}_i) + \tau^2 \dot{v}_i \dot{v}_j - \lambda \nu^2 (r_i \ddot{v}_j + r_j \ddot{v}_i) / 2 \]
\[ u_i = s_z 1_i + (1 - s) z_{2i} \]
\[ v_i = s' z_{1i} + (1 - s') z_{2i} \]
\[ \lambda \equiv s - s', \tau \equiv t - t' \] (38)

which leads from (31) to the spin-independent potentials (19) - (23) upon performing the necessary summations and partial integrations.

References

[1] H. Leutwyler, Phys. Lett. B98, 447 (1981); M.B. Voloshin, Nucl. Phys. B187, 365 (1981).
[2] U. Marquard and H.G. Dosch, Phys. Rev. D35, 2238 (1987).

[3] K.G. Wilson, Phys. Rev. D10, 2445 (1974).

[4] H.G. Dosch and Yu. A. Simonov, Phys. Lett. B205, 339 (1988).

[5] N. Brambilla, P. Consoli, and G. M. Prosperi, Phys. Rev. D50, 5878 (1994).

[6] Yu. A. Simonov, Nucl. Phys. B324, 67 (1989).

[7] V. M. Kustov, Phys. Atom. Nucl. 60, 1754 (1997).

[8] N. Brambilla and A. Vario, Phys. Rev. D55, 3974 (1997).

[9] Yu. S. Kalashnikova and A. V. Nefediev, Phys. Lett. B414, 149 (1997).

[10] M. Schiestl and H.G. Dosch, Phys. Lett. B209, 85 (1988).

[11] K. Williams, The minimum area, the flux tube, and Thomas precession (hep-ph/9806269).

[12] D. Gromes, Z. Phys. C26, 401, (1984).

[13] K. Williams, Revisiting the Eichten-Feinberg-Gromes $Q\bar{Q}$ Spin-Orbit Interaction (hep-ph/9607211).

[14] M.G.Olsson, S.Veseli and K.Williams, Phys. Rev. D53, 4006 (1996).

[15] A. Barchielli, E. Montaldi and G. M. Prosperi, Nucl. Phys. B296, 625 (1988).

[16] See earlier version of present article.