R-NL: Fast and Robust Covariance Estimation for Elliptical Distributions in High Dimensions

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Abstract

We combine Tyler’s robust estimator of the dispersion matrix with nonlinear shrinkage. This approach delivers a simple and fast estimator of the dispersion matrix in elliptical models that is robust against both heavy tails and high dimensions. We prove convergence of the iterative part of our algorithm and demonstrate the favorable performance of the estimator in a wide range of simulation scenarios. Finally, an empirical application demonstrates its state-of-the-art performance on real data.

Keywords: Heavy Tails, Nonlinear Shrinkage, Portfolio Optimization

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1 Introduction

Many statistical applications rely on covariance matrix estimation. Two common challenges are (1) the presence of heavy tails or outliers and (2) the high-dimensional nature of the data. One promising line of research to address both problems is to extend (Maronna’s) $M$-estimators of scatter (Kent and Tyler, 1991) with a form of shrinkage for high dimensions. This approach is in particular popular with a specific example of $M$-estimators called “Tyler’s estimator” (Tyler, 1987a), which is derived for all elliptical distributions.

Several papers have studied this approach, using a convex combination of the base estimator and a target matrix, usually the (scaled) identity matrix. We generally refer to such approaches as robust linear shrinkage estimators. For instance, Ollila and Tyler (2014); Auguin et al. (2016); Ollila et al. (2021); Ashurbekova et al. (2021) combine the linear shrinkage with Maronna’s $M$-estimators, whereas Abramovich and Spencer (2007); Chen et al. (2011); Yang et al. (2014); Zhang and Wiesel (2016) do so with Tyler’s estimator. Since this approach of combining linear shrinkage with a robust estimator entails choosing a hyperparameter determining the amount of shrinkage, the second step often consists of deriving some (asymptotically) optimal parameter that then can be estimated from data. The approach results in estimation methods that are generally computationally inexpensive and it also enables strong theoretical results on the convergence of the underlying iterative algorithms.

Despite these advantages, several problems remain. First, the performance of these robust methods sometimes does not exceed the performance of the basic linear shrinkage estimator of Ledoit and Wolf (2004) in heavy-tailed models, except for very small sample sizes $n$ (say $n < 100$). In fact, the theoretical analysis of Couillet and McKay (2014); Auguin et al. (2016) shows that robust $M$-estimators using linear shrinkage are asymptotically equivalent to scaled versions of the linear shrinkage estimator of Ledoit and Wolf (2004). Depending on how the data-adaptive hyperparameter is chosen, the performance can even deteriorate quickly as the tails get lighter, as we demonstrate in our simulations in Section 4. Second, some robust methods cannot handle the case when the dimension $p$ is larger than the sample size $n$, such as Ollila et al. (2021). Third, some methods propose a choice of hyperparameter(s) through cross-validation, such as Yu et al. (2017); Yi and Tyler (2021), which can be computationally expensive.

In this paper, we address these problems by developing a simple algorithm based on nonlinear shrinkage (Ledoit and Wolf, 2012, 2015, 2020, 2022b), inspired by the above robust approaches and the work of Hediger and Nääf (2022). In essence, the algorithm applies the quadratic inverse shrinkage (QIS) method of Ledoit and Wolf (2022b) to appropriately standardized data, thereby greatly increasing its finite-sample performance in heavy-tailed models. Thus, we refer to the new method as “Robust Nonlinear Shrinkage” (R-NL). In particular, we extend the methodology of Hediger and Nääf (2022) from a parametric model to general elliptical mixture models. This approach includes an iteration over the space of orthogonal matrices, which we prove converges to a stationary point. We motivate our approach using properties of elliptical distributions along the lines of Chen et al. (2011); Zhang and Wiesel (2016); Ashurbekova et al. (2021) and demonstrate the favorable performance of our method in a wide range of settings. Notably, our approach (i) greatly improves the performance of (standard) nonlinear shrinkage in heavy-tailed settings and does not deteriorate when moving from heavy to Gaussian tails; (ii) can handle the case
\( p > n \); and (iii) does not require the choice of a tuning parameter.

The remainder of the article is organized as follows. Section 1.1 lists our contributions. Section 2 presents an example to motivate our methodology. Section 3 describes the proposed new methodology and provides results concerning the convergence of the new algorithm. Section 4 showcases the performance of our method in a simulation study using various settings for both \( p < n \) and \( p > n \). Section 5 applies our method to financial data, illustrating the superior performance of the method on real data. Finally, the proofs are contained in Appendix A, whereas Appendix B contains further empirical results.

1.1 Contributions

To the best of our knowledge, no paper has so far attempted to combine nonlinear shrinkage of (Ledoit and Wolf, 2012, 2015, 2020, 2022b) with Tyler’s method. As such, our approach differs markedly from previous ones. It is partly based on an \( M \)-estimator interpretation, but also adds the nonparametric nonlinear shrinkage approach. A downside of this approach is that theoretical convergence results are harder to come by. Nonetheless, we are able to show that the iterative part of our algorithm converges to a stationary point, a crucial result for the practical usefulness of the algorithm.

Closest in spirit to our proposal the parametric approach of Hediger and Näf (2022), and the present paper can be seen as a generalization of this approach to the nonparametric class of elliptical distributions, in line with the previous literature cited above. Whereas many of the aforementioned robust linear shrinkage papers have important theoretical results, the empirical examination of their estimators in simulations and real data applications is often limited. We attempt to give a more comprehensive empirical overview in this paper. Contrary to most of the previous papers, we also consider a comparatively large sample size of \( n = 300 \) in our simulation study. The larger sample size, as well as the wider range of scenarios considered, sometimes lead to results that contradict conclusions of previous work. For instance, Ollila and Raninen (2019) report that their robust linear shrinkage estimators perform better than linear shrinkage of Ledoit and Wolf (2004). However, their considered sample size is below \( n = 40 \) and in most scenarios there is no difference in performance compared to linear shrinkage for \( n \geq 30 \). Our simulation study suggests that for \( n \) in the hundreds, the method of Ledoit and Wolf (2004) performs similarly to or better than some of the robust methods even in heavy-tailed settings. Our new approach, on the other hand, displays a superior performance in most of the considered scenarios. We also provide a Matlab implementation of our method, as well as the code to replicate all simulations on https://github.com/hedigers/RNL_Code.

2 Motivational Example

Let for a collection of \( n \) independent and identically distributed (i.i.d.) random vectors with values in \( \mathbb{R}^p \), and let \( \hat{\mathbf{V}} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_p) \) be the matrix of eigenvectors of the sample covariance matrix \( \hat{\mathbf{S}} \). Nonlinear shrinkage, just as the linear shrinkage of Ledoit and Wolf (2004), only changes the eigenvalues of the sample covariance matrix, while keeping the eigenvectors \( \hat{\mathbf{V}} \). That is, nonlinear shrinkage is also in the class of estimators of the form \( \hat{\mathbf{V}} \Delta \hat{\mathbf{V}}^T \), with \( \Delta \) diagonal, a class that goes back to Stein (1975, 1986). It is well known
Table 1: Notation

| Symbol | Description |
|--------|-------------|
| \( n \) | Sample size |
| \( p \) | Dimensionality |
| \( \Sigma := \text{Var} (\mathbf{Y}) \) | The covariance matrix of the random vector \( \mathbf{Y} \). |
| \( \text{Tr}(\mathbf{A}) \) | Trace of a square matrix \( \mathbf{A} \) |
| \( \| \mathbf{A} \|_F \) | Frobenius norm \( \sqrt{\text{Tr}(\mathbf{A}^T \mathbf{A})} \) of a square matrix \( \mathbf{A} \) |
| \( \mathbf{H} \) | Dispersion matrix |
| \( \mathcal{O} \) | The orthogonal group |
| \( \mathcal{O}_0 \) | Equivalence class in \( \mathcal{O} \) |
| \( \mathcal{U} \) | Arbitrary element of \( \mathcal{O} \) |
| \( \mathbf{V} \) | Eigenvectors of \( \mathbf{H} = \mathbf{V} \Lambda \mathbf{V}^T \) |
| \( \mathbf{V}^{[\ell]} \) | \( \ell \)th iteration of the algorithm |
| \( \hat{\mathbf{V}} \) | Critical point/solution/estimate |
| \( \mathcal{V} \) | Subset of critical points of \( \mathcal{O} \) |
| \( \Lambda \) | True ordered eigenvalues of \( \mathbf{H} \), up to scaling |
| \( \Lambda_0 \) | Initial (shrunken) estimate of \( \Lambda \) |
| \( \Lambda_R \) | Final R-NL (shrunken) estimate of \( \Lambda \) |
| \( \hat{\Lambda} \) | Eigenvalues of \( F(\hat{\mathbf{V}}) \) |
| \( \hat{\Lambda}^{[\ell+1]} \) | Eigenvalues of \( F(\mathbf{V}^{[\ell]}) \) |
| \( \text{diag()} \) | Transforms a vector \( \mathbf{a} \in \mathbb{R}^p \) into an \( p \times p \) diagonal matrix \( \text{diag}(\mathbf{a}) \) |

that

\[
\arg \min_{\Delta \text{ diagonal}} \| \Sigma - \hat{\mathbf{V}} \Delta \hat{\mathbf{V}}^T \|_F = \text{diag}\left( (\delta_1 \ldots \delta_N)^T \right) \quad \text{with} \quad \delta_j := \hat{\mathbf{v}}_j^T \Sigma \hat{\mathbf{v}}_j ;
\]

for example, see Ledoit and Wolf (2022a, Section 3.1). Nonlinear shrinkage takes the sample covariance matrix \( \hat{\Sigma} \) as an input and outputs a shrunken estimate of \( \Sigma \) of the form \( \hat{\mathbf{V}} \hat{\Lambda}_0 \hat{\mathbf{V}}^T \), where \( \hat{\Lambda}_0 = \text{diag}(\hat{\delta}_1, \ldots, \hat{\delta}_N) \) is a diagonal matrix. Although there are different schemes to come with estimates \( \{ \hat{\delta}_j \} \), each scheme uses as the only inputs \( p, n \), and the set of eigenvalues of \( \hat{\Sigma} \). In this paper we derive a new estimator that is not in the class of Stein (1975, 1986) but applies nonlinear shrinkage to a transformation of the data. It thereby implicitly uses more information than just the sample covariance matrix (together with \( p \) and \( n \)). Since we focus in the following on the class of elliptical distributions, we will differentiate between the dispersion matrix \( \mathbf{H} \) and the covariance matrix \( \Sigma \). The main difference between the two population quantities is that \( \Sigma \) might not exist. However, if it does exist, \( \Sigma \) is simply given by \( c \cdot \mathbf{H} \), with \( c > 0 \) depending on the distribution.

To illustrate the advantage of our method, we now present a motivational toy example before moving on to the general methodology. We first consider a multivariate Gaussian distribution in dimension \( p = 200 \) with mean \( \mathbf{\mu} = \mathbf{0} \) and covariance matrix \( \Sigma = \mathbf{H} \), where the \( (i,j) \) element of \( \mathbf{H} \) is \( 0.7^{i-j} \), as in Chen et al. (2011). We simulate \( n = 300 \) i.i.d. observations from this distribution. For \( j = 1, \ldots, p \), the left panel of Figure 1 displays the theoretical optimum \( \delta_j, \hat{\mathbf{v}}_j^T \hat{\mathbf{V}} \hat{\Lambda}_0 \hat{\mathbf{V}} \hat{\mathbf{v}}_j =: \delta_j \), as well as \( \hat{\mathbf{v}}_j^T \hat{\mathbf{H}} \hat{\mathbf{v}}_j \). Importantly, the estimated values are very close to the theoretical optimum \( \delta_j, \ j = 1, \ldots, p \), for both nonlinear shrinkage and our proposed method.
We next consider the same setting, but instead simulate from a multivariate $t$ distribution with 4 degrees of freedom and dispersion matrix $\mathbf{H}$, such that the covariance matrix $\mathbf{\Sigma}$ is $\frac{4}{(4-2)} \cdot \mathbf{H}$. In particular the $\hat{\mathbf{v}}_j^T \tilde{\mathbf{H}} \hat{\mathbf{v}}_j$ are multiplied by $c = 2$ in this case to obtain an estimate of $\hat{\mathbf{v}}_j^T \mathbf{\Sigma} \hat{\mathbf{v}}_j$. (The value $c = 2$ would not be known in practice but is ‘fair’ to use it in this toy example, since doing so does not favor one estimation method over the other.)

The left panel of Figure 1 displays the results. It can be seen that nonlinear shrinkage overestimates large values of $\delta_j$ (by a lot) and underestimates small values of $\delta_j$; on the other hand, our new method does not have this problem and its performance (almost) matches the one from the Gaussian case.

Figure 1: Comparison of the estimated values of nonlinear shrinkage and R-NL. The theoretical optimal eigenvalues $\hat{\mathbf{v}}_j^T \mathbf{\Sigma} \hat{\mathbf{v}}_j$ are denoted by ‘Oracle’. In the left panel the sample is taken from a multivariate Gaussian distribution and in the right panel from a multivariate $t$-distribution with 4 degrees of freedom. For both distributions the $(i, j)$ element of $\tilde{\mathbf{H}}$ is $0.7|\text{i-j}|$, as in Chen et al. (2011). The number of observations is $n = 300$ and the dimension is $p = 200$. 

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3 Methodology

We assume to observe an i.i.d. sample $Y := \{Y_1, \ldots, Y_n\}$ from a $p$-dimensional elliptical distribution. If $Y$ has an elliptical distribution it can be represented as

$$Y \overset{D}{=} \mu + RH^{1/2}\xi,$$

(1)

where $R$ is a positive random variable, and $\xi$ is uniformly distributed on the $p$-dimensional unit sphere, independent of $R$, and where $\overset{D}{=}$ denotes equality in distribution (Cambanis et al., 1981). The dispersion matrix $H$ is assumed to be symmetric and positive-definite (pd), with eigendecomposition $H = V\Lambda V^\top$. If $Y$ meets (1), we write $Y \sim E_p(\mu, H, g)$, where $g$ is the “generator” which identifies the distribution of $R$, for example see McNeil et al. (2015, Section 3.3) or Ashurbekova et al. (2021). We assume this generator to exist, which is equivalent to $R$ having a density (Fang et al., 1990).

In the following we restrict ourselves to distributions of the form (1) with $\mu = 0$ and such that second moments exist. Then $\text{Var}(Y) = cH$ for some $c > 0$. Without stronger distributional assumptions, it is only possible to identify $H$ up to a constant, see, e.g., Ashurbekova et al. (2021). As such we will normalize our estimators of $H$ to have trace $p$, as in Chen et al. (2011). We note however that, to obtain an estimator of $\Sigma$, one could instead normalize the estimator to have the same trace as $\hat{S}$. As an illustration, in the example from the right panel of Figure 1, $\text{Tr}(\hat{S}) \approx 428$ whereas the true trace of $\Sigma$ is 400.

3.1 Robust Nonlinear Shrinkage

We start by outlining our main idea. Let $\| \cdot \|$ be the Euclidean norm on $\mathbb{R}^p$. As shown in Tyler (1987b), if $Y \sim E_p(\mu, H, g)$, $Z := \frac{Y}{|Y|}$ has a central angular Gaussian distribution with density:

$$p(z; H) \propto |H|^{-1/2} \cdot (z^\top H^{-1} z)^{-p/2},$$

(2)

where for $a, b \in \mathbb{R}$, $a \propto b$ means there exists $c > 0$ with $a = cb$. We will also write $A \propto B$, for two $p \times p$ matrices $A, B$ if $A = cB$. The likelihood in (2) is the starting point of the original Tyler’s method. Taking the derivative of (2), Tyler’s estimator $\hat{H}_T$ is implicitly given by the following condition:

$$\hat{H}_T = \frac{p}{n} \sum_{t=1}^n \frac{Z_t Z_t^\top}{Z_t^\top \hat{H}_T^{-1} Z_t}.$$  

(3)

This estimator is obtained as the limit of the iterations

$$\hat{H}^{[\ell+1]} \propto \frac{p}{n} \sum_{t=1}^n \frac{Z_t Z_t^\top}{Z_t^\top (\hat{H}^{[\ell]})^{-1} Z_t},$$

(4)

where $\propto$ indicates that $\hat{H}^{[\ell+1]}$ is actually obtained after an additional trace-normalization step, see, e.g., Tyler (1987a) or Chen et al. (2011). Robust linear shrinkage methods such as the method of Chen et al. (2011) augment (4) by shrinking towards the identity matrix in each iteration. That is, if for an $p \times p$ matrix $A$ and $\rho \in [0, 1]$, we define
Let \( \text{LS}(A, \rho) := (1 - \rho)A + \rho I \), then the robust linear shrinkage estimator is obtained from the iterations

\[
\hat{H}^{[\ell + 1]} \propto \text{LS} \left( \frac{p}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^\top (\hat{H}^{[\ell]})^{-1} Z_t}, \rho \right),
\]

(5)

where again \( \propto \) indicates a trace-normalization step.

Similarly, denote for any pd symmetric matrix \( A \) by \( \text{NL}(A) \) the matrix that is obtained when using nonlinear shrinkage on \( A \). A few clarifications are in order at this point. First, in the existing literature on nonlinear shrinkage, \( A \) is always the sample covariance matrix; but the ‘algorithm’ of nonlinear shrinkage allows for a more general input instead. Second, there (at least) three different nonlinear shrinkage schemes by now: the numerical scheme called QuEST of Ledoit and Wolf (2015), the analytical scheme of Ledoit and Wolf (2020), and the QIS method of Ledoit and Wolf (2022b), which is also of analytical nature; our methodology allows for the use of any such scheme, with our personal choice being the QIS method. Third, any ‘algorithm’ of nonlinear shrinkage needs as an additional input to \( A \), which of course determines the dimension \( p \), also the sample size \( n \), which we may treat as fixed and known in our methodology.

Applying nonlinear shrinkage to the matrix \( A \) leaves its eigenvectors unchanged and only changes its eigenvalues. The way the eigenvalue are changed depends on the particular nonlinear shrinkage scheme; see, e.g., Ledoit and Wolf (2022b, Section 4.5) for the details concerning the QIS method. In analogy to the case of linear shrinkage, we could now apply nonlinear shrinkage each time in the above iteration. That is, we could iterate

\[
\hat{H}^{[\ell + 1]} \propto \text{NL} \left( \frac{p}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^\top (\hat{H}^{[\ell]})^{-1} Z_t} \right),
\]

(6)

where the input to \( \text{NL} \) corresponds to the sample covariance matrix of the scaled data \( Z_t/(Z_t^\top (\hat{H}^{[\ell]})^{-1} Z_t)^{1/2} \). Unfortunately, contrary to the case of linear shrinkage, it is not clear how to ensure convergence for such an approach. However, we note that iteration (6) can be seen as a simultaneous iteration over the eigenvalues and eigenvectors, whereby only the former is changed by nonlinear shrinkage. Following the ideas in Hediger and Nāf (2022), we instead aim to iterate over the eigenvectors for fixed (shrunken) eigenvalues. That is, after the first iteration, we fix the eigenvalues obtained by nonlinear shrinkage, denoted \( \Lambda_0 \). Choosing \( \hat{H}^{[0]} = I \), this corresponds to using nonlinear shrinkage on the sample covariance matrix of \( Z := \{Z_1, \ldots, Z_T\} \), with \( Z_t := Y_t/\|Y_t\| \). It should be mentioned here that any nonlinear shrinkage scheme ensures that the elements on the diagonal of \( \Lambda_0 \), denoted \( \hat{\delta}_j \), \( j = 1, \ldots, p \), are all strictly positive.

We then optimize the likelihood of the central angular Gaussian distribution only with respect to the orthogonal matrix \( V \). That is, we solve,

\[
\hat{V} := \arg \max_{U \in \mathcal{O}} \sum_{t=1}^{n} \ln(p(Z_t; U, \Lambda_0)) = \arg \min_{U \in \mathcal{O}} \frac{1}{n} \sum_{t=1}^{n} \ln \left( Z_t^\top U \Lambda_0^{-1} U^\top Z_t \right),
\]

(7)

where \( \mathcal{O} := \{U : U^\top U = UU^\top = I\} \) is the orthogonal group. Finally, once \( \hat{V} \) is obtained, \( \Lambda_0 \) is updated. That is, we apply nonlinear shrinkage to the covariance matrix of the
standardized data
\[
\hat{Z}_t := \frac{Z_t}{\sqrt{Z_t^T \hat{\Lambda}_0^{-1} \hat{V} Z_t/p}}, \quad t = 1, \ldots, n, \tag{8}
\]
to obtain \( \Lambda_R \). The final estimate is then given as
\[
\hat{H} := p \cdot \hat{\Lambda}_R \hat{V}^\top / \text{Tr}(\hat{\Lambda}_R \hat{V}^\top). \tag{9}
\]
Since, as we will show below, the eigenvectors of the sample covariance matrix of \( \{\hat{Z}_1, \ldots, \hat{Z}_n\} \) are again given by \( \hat{V} \), it holds that,
\[
\hat{H} \propto \text{NL} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^T \hat{\Lambda}_0^{-1} \hat{V}^\top Z_t/p} \right). \tag{10}
\]
The whole procedure is summarized in Algorithm 2. We now detail how to solve (7).

For an \( p \times p \) symmetric pd matrix \( A \), let
\[
A = U_A \Lambda_A U_A^\top,
\]
be its eigendecomposition, where we assume the elements of \( \Lambda_A \) to be ordered from smallest to largest. We define \( \mathcal{E} \) to be the operator that returns all possible matrices of eigenvectors. That is, \( \mathcal{E}(A) \) is a subset of \( \mathcal{O} \) and for any \( U \in \mathcal{E}(A) \), \( U^\top A U \) is a diagonal matrix with elements ordered from smallest to largest. We also define in the following for \( U \in \mathcal{O} \),
\[
F(U) := \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^T U \Lambda_0^{-1} U^\top Z_t}, \tag{10}
\]
where the dependence on \( Z_1, \ldots, Z_n \) and \( \Lambda_0 \) is suppressed to keep notation compact.

**Lemma 1** A minimizer \( \hat{V} \) of (7) exists and meets the condition
\[
\hat{V}^\top F\left(\hat{V}\right) \hat{\Lambda}_0^{-1} = \Lambda_0^{-1} \hat{V}^\top F\left(\hat{V}\right) \hat{V}. \tag{11}
\]
The necessary condition in (11) is true in particular if \( \hat{V} \) diagonalizes \( F\left(\hat{V}\right) \), or
\[
\hat{V} \in \mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^T \hat{\Lambda}_0^{-1} \hat{V}^\top Z_t} \right), \tag{12}
\]
in analogy to (3). Thus given \( \Lambda_0 \), we propose the following iterations
\[
V^{[\ell+1]} \in \mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^T \hat{V}^{[\ell]} \Lambda_0^{-1} (\hat{V}^{[\ell]})^\top Z_t} \right), \tag{13}
\]
starting with
\[
V^{[1]} \in \mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{\|Z_t\|^2} \right) = \mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{\|Z_t\|^2} \right). \tag{14}
\]
We now proceed by showing that any sequence generated by these iterations has a limit $V^{[\alpha]}$ such that (11) holds. To this end we adapt the approach taken in Wiesel (2012); Razaviyayn et al. (2013); Sun et al. (2014) and define the surrogate function

$$g(U \mid V^{[\ell]}) := \frac{1}{n} \sum_{t=1}^{n} \ln (Z_t^t V^{[\ell]} \Lambda_0^{-1} (V^{[\ell]})^t Z_t) + \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t^t U \Lambda_0^{-1} U^t Z_t}{Z_t^t V^{[\ell]} \Lambda_0^{-1} (V^{[\ell]})^t Z_t} - 1 .$$  \hspace{1cm} (15)$$

Then for $f(U) := \frac{1}{n} \sum_{t=1}^{n} \ln (Z_t^t U \Lambda_0^{-1} U^t Z_t)$,

**Lemma 2** The surrogate function $g$ satisfies:

$$f(U) \leq g(U \mid V^{[\ell]}) \text{ for all } U, V^{[\ell]} \in \mathcal{O}$$  \hspace{1cm} (16)$$

$$f(V^{[\ell]}) = g(V^{[\ell]} \mid V^{[\ell]}) \ ,$$  \hspace{1cm} (17)$$

and for $V^{[\ell+1]}$ as in (13),

$$g(V^{[\ell+1]} \mid V^{[\ell]}) \leq g(U \mid V^{[\ell]}) \text{ for all } U \in \mathcal{O} .$$  \hspace{1cm} (18)$$

Define now the set of critical points as $V \subset \mathcal{O}$, i.e.,

$$V := \{ \hat{V} \in \mathcal{O} \text{ such that (11) holds}\} ,$$

and let for all $U \in \mathcal{O}$,

$$d(U, V) := \inf_{\hat{V} \in V} \| \hat{V} - U \|_F ,$$

as in Razaviyayn et al. (2013). Using Lemma 2 the following convergence result can be obtained.

**Theorem 1** For any sequence $(V^{[\ell]})_{\ell=1}^{\infty}$ generated by the above iterations,

$$f(V^{[\ell+1]}) \leq f(V^{[\ell]}) \ , \text{ for all } \ell ,$$  \hspace{1cm} (19)$$

and

$$\lim_{\ell \to \infty} d(V^{[\ell]}, V) = 0 .$$  \hspace{1cm} (20)$$

Thus, as $\ell \to \infty$, $V^{[\ell]}$ gets arbitrary close to a critical point. This leads to the following convergence criterion:

$$\| (V^{[\ell]})^t F (V^{[\ell]}) V^{[\ell]} \Lambda_0^{-1} - \Lambda_0^{-1} (V^{[\ell]})^t F (V^{[\ell]}) V^{[\ell]} \|_F \leq \epsilon ,$$

where $\epsilon > 0$ is some convergence tolerance. This criterion is used in Algorithm 1 and we set $\epsilon = 10^{-5}$ in Sections 4 and 5.
3.2 Practical Considerations

Although R-NL is no longer in the same class of estimators as nonlinear shrinkage, namely the class of Stein (1975, 1986), an interesting question is whether it is still rotation-equivariant. An estimator \( \hat{H} \) applied to \( Y = \{Y_1, \ldots, Y_T\} \) is rotation-equivariant if, for any rotation \( R \) and rotated data \( \tilde{Y}_t := RY_t, t = 1, \ldots, n \), the estimate of the rotated data, \( \hat{H}_R \), satisfies

\[
\hat{H}_R = R\hat{H}R^\top.
\]  

(21)

This is true for any estimator in the class of Stein (1975, 1986) and, therefore, in particular for nonlinear shrinkage. We now show that this is true for R-NL as well, using the following lemma:

**Lemma 3** Let \( R \) be an arbitrary rotation, \( \tilde{Z}_t := RZ_t \) and \( V^{[\ell]} \) be the \( \ell \)-th iteration of Algorithm 1 applied to \( \{\tilde{Z}_1, \ldots, \tilde{Z}_T\} \). Then

\[
\mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{\tilde{Z}_t\tilde{Z}_t^\top}{\tilde{Z}_t^\top V^{[\ell]} \Lambda_0^{-1}(V^{[\ell]}_t)^\top \tilde{Z}_t} \right) = \left\{ RV^{[\ell+1]} : V^{[\ell+1]} \in \mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_tZ_t^\top}{Z_t^\top V^{[\ell]} \Lambda_0^{-1}(V^{[\ell]}_t)^\top Z_t} \right) \right\}.
\]

(22)

We note that the rotation of the original data in \( Y \) corresponds to a rotation of \( Z \), since for all \( t \), \( RY_t/\|RY_t\| = RY_t/\|Y_t\| = RZ_t \). Thus if \( (V^{[\ell]}_{t\ell})_{\ell=1}^\infty \) is a sequence generated by Algorithm 1 for the data \( \{Y_1, \ldots, Y_T\} \), then \( (RV^{[\ell]}_{t\ell})_{\ell=1}^\infty \) is a sequence generated for the rotated data. Consequently, the corresponding estimate of \( H_R \) for the rotated data will be of the form (21).

The matrix \( \hat{V} \) in (12) (and consequently in (11)) is not unique in general. However, we are ultimately not interested in \( \hat{V} \), but in \( \hat{V} \Lambda_0 \hat{V}^\top \). A natural question is thus whether \( \hat{V} \Lambda_0 \hat{V}^\top \) is unique even if \( \hat{V} \) is not. This turns out to be true with probability 1, as we detail now. Let in the following \( \hat{\Lambda} \) be the diagonal matrix of (ordered) eigenvalues of \( F(\tilde{V}) \).

It has the form

\[
\hat{\Lambda} = \begin{cases} 
\hat{\Lambda}_{p-n}, \hat{\Lambda}_1, & \text{if } p > n \\
\hat{\Lambda}_1, & \text{if } p \leq n,
\end{cases}
\]

where \( \hat{\Lambda}_1 \) is a diagonal matrix with \( \min(n, p) \) eigenvalues and \( \hat{\Lambda}_{p-n} \) is a matrix of zeros. We denote the diagonal elements of \( \hat{\Lambda} \) as \( \hat{\lambda}_1, \ldots, \hat{\lambda}_p \). Define similarly the matrix of eigenvalues of \( F(V^{[\ell-1]}) \) as \( \hat{\Lambda}^{[\ell]} \), with elements \( \hat{\lambda}_1^{[\ell]}, \ldots, \hat{\lambda}_p^{[\ell]} \), and \( \hat{\lambda}_1^{[\ell]} \) as the largest \( \min(n, p) \) eigenvalues of \( \hat{\Lambda}^{[\ell]} \).

**Lemma 4** Assume that whenever \( \hat{\lambda}_i = \hat{\lambda}_j \), also \( \hat{\delta}_i = \hat{\delta}_j \), for \( j, i \in \{1, \ldots, p\} \). Then if \( \hat{V}_1 \) and \( \hat{V}_2 \) meet (12),

\[
\hat{V}_1 \Lambda_0 (\hat{V}_1)^\top = \hat{V}_2 \Lambda_0 (\hat{V}_2)^\top.
\]

Thus if the multiplicity of eigenvalues of \( \hat{\Lambda} \) implies the multiplicity of the corresponding eigenvalue in \( \Lambda_0 \), the resulting matrix will also be the same. This is true in particular if
\( \hat{\lambda}_i \neq \hat{\lambda}_j \) for all \( i, j \). Consequently, under the conditions of Lemma 4, \( \hat{V} \) in (12) is unique under the equivalence relation \( \sim_{\Lambda_0} \) with

\[ U_1 \sim_{\Lambda_0} U_2 \iff U_1 \Lambda_0 (U_1)^T = U_2 \Lambda_0 (U_2)^T. \]

More generally if we consider the space of equivalence classes \( \mathcal{O}_0 := \mathcal{O} \setminus \sim_{\Lambda_0} \) and define the metric

\[ \tilde{d}_{\Lambda_0}([U_1], [U_2]) := \| U_1 \Lambda_0 U_1^T - U_2 \Lambda_0 U_2^T \|_F, \]

where \([U] := \{ U_0 \in \mathcal{O} : U_0 \sim_{\Lambda_0} U \} \), then

**Lemma 5** Assume that

\[ \forall \ell, \quad \hat{\lambda}_i^{[\ell]} = \hat{\lambda}_j^{[\ell]} \implies \hat{\delta}_i = \hat{\delta}_j. \tag{23} \]

Then we can write iteration (13) in terms of equivalence classes:

\[ [V^{[\ell+1]}] = \mathcal{E}(F([V^{[\ell]}])). \tag{24} \]

Moreover there exists \([\hat{V}] \in \mathcal{O}_0 \) such that (12) holds and the generated sequence \(( [V^{[\ell]}] )_{\ell=1}^\infty \) satisfies

\[ \tilde{d}_{\Lambda_0}([V^{[\ell]}], [\hat{V}]) \to 0. \]

At first, it might seem unclear how to enforce the eigenvalue condition in Lemma 5. However, since \( \hat{\lambda}_i^{[\ell]}, i = 1, \ldots, p \) are eigenvalues of the sample covariance matrix of the standardized sample, Theorem 1 of Okamoto (1973) applies. This implies that the eigenvalues of \( \hat{\Lambda}_1^{[\ell]} \) are all nonzero and distinct with probability 1. Thus we only need to ensure that, for \( p > n \), the smallest \( p - n \) eigenvalues in \( \Lambda_0 \) are all the same. This is enforced in Algorithm 1 by simply setting the smallest \( p - n \) values of \( \Lambda_0 \) to the value with the highest multiplicity. Then

**Lemma 6** Condition (23) holds with probability 1.

We thus obtain uniqueness of \( \hat{V} \Lambda_0 \hat{V}^T \) and of \( \hat{H} \), up to scaling.

### 3.3 Robust Correlation-Based Nonlinear Shrinkage

In the context of covariance matrix estimation, an alternative approach is to use shrinkage estimation for the correlation matrix and to estimate the vector of variances separately, after which one combines the two estimators to obtain a ‘final’ estimator of the covariance matrix itself. Such an approach is used by Hediger and Naf (2022) in a static setting (that is, for i.i.d. data) and by Engle et al. (2019); De Nard et al. (2022) in a dynamic setting (that is, for time series data). It turns out that by adapting this approach for our method, a considerable boost in performance can be achieved in some settings.

In particular, we first calculate the sample variances \( \hat{\sigma}_1^2, \ldots, \hat{\sigma}_p^2 \) and obtain the scaled data as

\[ X_t := \hat{\sigma}^{-1} Y_t, \tag{25} \]
Algorithm 1: VIteration($Z, \Lambda_0$)

Inputs: centered data $Z$, eigenvalue matrix $\Lambda_0$;
Output: $\hat{V}$;
Hyper-parameters: Convergence Tolerance $\epsilon$;
Initiate $\ell = 0$, $c(0) = -\infty$, $c(1) = 0$, $V^{[0]} = I$;
while $c(\ell + 1) - c(\ell) > \epsilon$ do
  - Calculate $F(V^{[\ell]})$ as in (10) and its eigendecomposition $U\hat{A}^{[\ell+1]}U^\top$, with $\hat{A}^{[\ell+1]}$ ordered;
  - Take $V^{[\ell+1]} = U$;
  - $\ell = \ell + 1$;
  - $c(\ell + 1) = \| (V^{[\ell]} \top F(V^{[\ell]}) V^{[\ell]} \Lambda_0^{-1} - \Lambda_0^{-1}(V^{[\ell]} \top F(V^{[\ell]})) V^{[\ell]} \|_F$;
return $\hat{V} = V^{[\ell]}$

Algorithm 2: R-NL($Y$)

Inputs: centered data $Y$;
Output: $\hat{H}$;
- Calculate $Z := \{Z_1, \ldots, Z_T\}$ with $Z_t := Y_t / \| Y_t \|$, $t = 1, \ldots, n$;
- Obtain the sorted eigenvalues $\Lambda_0$ by applying NL to $Z$;
- If $p > n$: Ensure that the last $p - n$ elements of $\Lambda_0$ are equal;
- Obtain $\hat{V} := $ VIteration($Z, \Lambda_0$) using Algorithm 1;
- Calculate $\tilde{Z}_t, t = 1, \ldots, n$, as in (8);
- Apply NL to the sample $\tilde{Z}_1, \ldots, \tilde{Z}_n$ to obtain $\Lambda_R$;
- Calculate $\hat{H}$ as in (9);
return $\hat{H}$

Algorithm 3: R-C-NL($Y$)

Inputs: centered data $Y$;
Output: $\hat{H}$;
- Calculate the diagonal matrix of sample standard deviations $\hat{\sigma}$ of $Y$;
- Calculate $X := \{X_1, \ldots, X_T\}$ as in (25);
- Obtain $\hat{H}_0 := $ R-NL($X$) using Algorithm 2;
- Calculate $\hat{H} := p \cdot \hat{\sigma} \hat{H}_0 \hat{\sigma} / \text{Tr}(\hat{\sigma} \hat{H}_0 \hat{\sigma})$;
return $\hat{H}$
where $\hat{\sigma} := \text{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_p)$. Then R-NL is applied to $Z_t := \hat{\sigma}^{-1}Y_t/\|\hat{\sigma}^{-1}Y_t\|$ to obtain $H_0 := p\hat{V}A_R\hat{V}^\top/\text{Tr}(A_R)$. From these two inputs, we calculate the ‘final’ estimator of $H$ as

$$ \hat{H} := p \cdot \hat{\sigma}H_0\hat{\sigma}/\text{Tr}(\hat{\sigma}H_0\hat{\sigma}). \quad (26) $$

The approach is called “R-C-NL” and is summarized in Algorithm 3.

As we demonstrate in Section 4 this ‘variation’ on our methodology can have a substantial (beneficial) effect on the performance of the estimator. However, a potential disadvantage of this approach is that R-C-NL is no longer rotation-equivariant.

### 4 Simulation Study

We compare our proposed two methods to several competitors in various simulation scenarios. From the collection of approaches that use Tyler’s method together with (linear) shrinkage, we tried to pick the ones most appropriate for our analysis, without handpicking them to showcase the performance of our method. In particular, we did not pick methods that require the choice of a tuning parameter (such as Sun et al. (2014); Yu et al. (2017); Yi and Tyler (2021)) or require $p < n$ such as Ollila et al. (2021). This leads us to the following benchmarks:

- **S**: the sample covariance matrix $\hat{S}$.
- **LS**: the linear shrinkage estimator of Ledoit and Wolf (2004).
- **NL**: the quadratic inverse shrinkage (QIS) estimator of Ledoit and Wolf (2022b).
- **R-LS**: the robust linear shrinkage estimator of (Chen et al., 2011). An estimator that is “widely used and performs well in practice” (Sun et al., 2014).
- **R-GMV-LS**: the robust linear shrinkage estimator of (Zhang et al., 2014). This estimator is designed for global-minimum-variance portfolios.
- **R-A-LS**: the regularized estimator of Zhang and Wiesel (2016), a variation of the robust linear shrinkage approach.

Moreover, we consider the following six structures for the true dispersion matrix $H$:

- **(I)** Identity: The identity matrix $I$.
- **(A)** AR: the $(i, j)$ element of $H$ is $0.7^{|i-j|}$, as in Chen et al. (2011).
- **(F)** Full matrix: 1 on the diagonal and 0.5 on the off-diagonal.
- **(I')** Base: diagonal matrix, where 20% of the diagonal elements are equal to 1, 40% of the diagonal elements are equal to 3, and 40% of the diagonal elements are equal to 10, as in Ledoit and Wolf (2012, 2020, 2022b).
- **(A')** AR (non-constant diag): start with $H$ as in (A) and then pre- and post-multiply with the square-root of the diagonal matrix as in (I').
• (F’) Full Matrix (non-constant diag): start with $\mathbf{H}$ as in (F) and then pre- and post-multiply with the square-root of the diagonal matrix as in (I’).

Together these settings cover a wide range of structures for the dispersion matrix $\mathbf{H}$, from sparse to a “full” matrix with nonzero elements everywhere. Most papers related to our method simulate from a multivariate $t$-distribution with 3 or 4 degrees of freedom. In contrast, we let the degrees of freedom vary on a grid from 3 to “infinity”, i.e., to the Gaussian case. It appears the actual sample size does not matter as much as the concentration ratio in the relative performance of the methods. As such we choose two concentration ratios, $2/3$ and $4/3$, with a fixed sample size of $n = 300$. Appendix B presents the same analysis for $n = 150$ and $p \in \{100, 200\}$.

Finally, as a measure for comparing the different methods we consider the Percentage Relative Improvement in Average Loss (PRIAL) defined as

$$\text{PRIAL}(\hat{\mathbf{H}}) := 100 \times \left( 1 - \frac{\mathbb{E}[\|\hat{\mathbf{H}} - \mathbf{H}\|^2]}{\mathbb{E}[\|\hat{\mathbf{S}} - \mathbf{H}\|^2]} \right) \%,$$

where $\hat{\mathbf{H}}$ represents a generic estimator of $\mathbf{H}$. Note that our definition of PRIAL differs from the one in Ledoit and Wolf (2022b): For both definitions, the value of 0 corresponds to the (scaled) sample covariance matrix; but the value of 100 corresponds to the true matrix in our definition whereas it corresponds to the ‘oracle’ estimator in the class of Stein (1975, 1986) in the definition of Ledoit and Wolf (2022b). It does not make sense for us to use the definition of Ledoit and Wolf (2022b), since our estimator, unlike the their QIS estimator, is not in the Steinian class.

Figures 2 and 3 show the results. It is immediately visible that in the majority of considered settings both R-NL and R-C-NL outperform the other estimators. The only exceptions are the setting (I), for both $p = 200$ and $p = 400$, and (F), (F’) for $p = 200$. In the former, LS and R-A-LS recognize that shrinking maximally towards a multiple of the identity matrix is optimal, reaching a PRIAL of almost 100% through all $\nu$. Although R-NL and R-C-NL are close, they cannot quite match this strong performance. For the cases (F) and (F’), there is a performance drop of our methods compared to LS and NL for $\nu \geq 30$. However, the values of the methods again stay close. As one would expect, the PRIAL values of R-NL and R-C-NL are similar in the first row, where the true matrix $\mathbf{H}$ has constant diagonal elements. Moreover, although NL is greatly improved upon with both R-NL and R-C-NL for small to moderate $\nu$, both converge to the performance of NL as the degrees of freedom increase. In the case (I’), where the diagonal elements are non-constant, R-C-NL attains a strong boost compared to R-NL and NL, such that it outperforms all other benchmarks by a considerable margin. Although the performance of R-NL may not be as visually impressive as that of R-C-NL, we note that the consistently high performance through most settings is quite remarkable. Appendix B presents further simulations with similar findings. Given that other methods, such as that of Zhang and Wiesel (2016), perform quite well on balance, but can collapse in some cases, this consistent-throughout performance appears remarkable.

It is also worth noting how well the linear shrinkage methods perform in this setting of non-constant diagonal elements. This may seem counterintuitive, since the shrinkage target is a multiple of the identity matrix. Indeed, the diagonal elements of the linear shrinkage estimates are close to constant in these cases. However, at least for heavy-tails,
the errors the sample matrix admits on the off-diagonal elements far outweigh the errors of constant diagonal elements. Additionally, LS, to which most other papers compare their methods, is extremely competitive with the robust methods (even if \( \nu \) is very small). This is especially true for \( \mathbf{H} \) with non-constant diagonal elements, which most of the previous papers do not consider. The good performance of LS might also be due to the relatively high sample size used in this paper compared to others.

Figure 2: Percentage Relative Improvement in Average Loss (PRIAL) for various dispersion matrix structures, \( \nu \in \{3, 4, \ldots, 14, 15, 30, 60, 90, 120, 240, 500, \infty\} \), \( p = 200 \) and \( n = 300 \). The plots are scaled such that they extend to 100 and NL and R-C-NL are fully visible.
Figure 3: Percentage Relative Improvement in Average Loss (PRIAL) for various dispersion matrix structures, $\nu \in \{3, 4, \ldots, 14, 15, 30, 60, 90, 120, 240, 500, \infty\}$, $p = 400$ and $n = 300$. The plots are scaled such that they extend to 100 and NL and R-C-NL are fully visible.

5 Empirical Study

From the center for research in security prices (CRSP) we download daily simple percentage returns of the NYSE, AMEX, and NASDAQ stock exchanges, see https://www.crsp.org/node/1/activetab%3Ddocs for a documentation. The historical data ranges from 02.01.1976 until 31.12.2020 and contains $p^* = 23'131$ stocks in total. The outline of the empirical section is inspired by De Nard et al. (2019).
We conduct a rolling window type exercise, where we consider an estimation window of one year (252 days) and another one with five years (1260 days). In each rolling window we estimate the covariance matrix and perform a minimum variance portfolio optimization with no short selling limits. In the unconstrained case, the global minimum variance portfolio problem is formulated as

$$\min_{\tilde{w}} \tilde{w}^\top \tilde{\Sigma} \tilde{w}$$

subject to \( \tilde{w}^\top \mathbb{1} = 1 \),

where \( \mathbb{1} \) denotes a conformable vector of ones. In the absence of any short-sales constraints the problem has the analytical solution

$$w := \frac{\Sigma^{-1} \mathbb{1}}{\mathbb{1}^\top \Sigma^{-1} \mathbb{1}}.$$

The resulting portfolio weight is then kept fixed for the following 21 days (out-of-sample window). Afterwards the rolling window moves forward by 21 days, the covariance matrix is re-estimated and the weights are updated accordingly. In short, we rebalance the portfolio once a ‘month’, where our definition of a ‘month’ corresponds to 21 consecutive trading days rather than a calendar month. Depending on the size of the estimation window, the out-of-sample period starts on 14-Jan-1977 respectively 13-Jan-1981. This results in 528 respectively 480 out-of-sample months. In each month we only consider stocks which have no more than 32 days of missing values during the estimation window and a complete return in the out-of-sample window. The missing values in the remaining universe are set to 0. Further, every month, only the \( p \) stocks with the highest market capitalization are considered, where \( p \in \{100, 500, 1000\} \).

The solution of the minimum variance portfolio only depends on the second moment, i.e., the covariance matrix. Therefore, as a portfolio evaluation criterion, the out-of-sample standard deviation is the leading criterion of interest. Hence, in the main text we consider the following two portfolio performance measures:

- **SD**: annualized standard deviation of portfolio returns.
- **TO**: average monthly turnover given by

$$\text{TO} := \frac{1}{(\tau - 1)} \sum_{h=1}^{\tau - 1} \sum_{j=1}^{p^*} |w_{j,h+1} - w_{j,h}^{\text{hold}}|,$$

Here \( p^* \) denotes the size of the ‘combined’ investment universe over both months, \( h \) and \( h + 1 \); in general some stocks leave the universe, while the same number of new stocks enter the universe, as one advances from month \( h \) to month \( h + 1 \) such that \( p^* \geq p \). Furthermore, \( \tau \) represents the number of out-of-sample months (528 and 480, respectively) and

$$w_{j,h}^{\text{hold}} := \frac{w_{j,h}\alpha_{j,h}}{\sum_{j=1}^{p^*} w_{j,h}\alpha_{j,h}}.$$
with

\[ \alpha_{j,h} := \prod_{s=0}^{20} \left( 1 + r_{j,t+h+s} \right) \]

representing the return evolution in the days of month \( h \). Note: if stock \( j \) is not contained in the universe during month \( h \), we set \( w_{j,h} = 0 \) and \( r_{j,t+h+s} = 0 \ \forall \ s \in \{0, 1, \ldots, 20\} \).

Additionally, in order to test whether the difference of the out-of-sample standard deviation between NL and R-C-NL is significantly different from zero, we apply the HAC inference of Ledoit and Wolf (2011).

Table 2 presents the main results; for additional results pertaining to alternative performance measures, see Appendix B. The findings are as follows:

- R-C-NL has the lowest SD in every scenario.
- R-C-NL has a significantly lower SD than NL in every scenario.
- R-NL has a lower SD than NL, except for \( p = 1000 \), where they have a similar SD.
- The difference in SD between R-NL and R-C-NL increases as \( p \) increases.
- For both \( n = 252 \) and \( n = 1260 \), the robust estimators have a lower SD than the non-robust estimators when \( p \) is small. For large \( p \) the non-robust estimators perform similarly or better than the robust estimators. This holds for linear shrinkage and nonlinear shrinkage estimators; an exception is R-GMV-LS in the case \( n = 1260 \).
- For \( p = 500 \) and \( p = 1000 \), NL always outperforms the robust linear shrinkage estimators.
- For \( p = 100 \), NL and the robust linear shrinkage estimators perform similarly.
- Except for the case \( p = 100 \), R-C-NL has the lowest TO in every scenario.
- R-NL always has lower TO than NL.
Table 2: Out-of-sample portfolio statistics for the daily $p$ largest stocks on CRSP and an estimation window of $n = 252$ and $n = 1260$ days, respectively. Significant outperformance of the introduced R-C-NL estimator over NL in terms of SD is indicated by asterisks: ** indicates significance at the 0.05 level and *** indicates significance at the 0.01 level.



|     | S   | LS  | NL  | R-LS | R-GMV-LS | R-A-LS | R-NL | R-C-NL |
|-----|-----|-----|-----|------|----------|--------|------|--------|
|     |     |     |     |      |          |        |      |        |
| $n = 252$ |     |     |     |      |          |        |      |        |
| SD  | 13.41| 12.36| 11.91| 12.29| 11.91    | 12.20  | 11.86| 11.60***|
| TO  | 2.23 | 1.40 | 1.02 | 1.38 | 0.79      | 1.25   | 0.91 | 0.86   |
| $p = 500$ |     |     |     |      |          |        |      |        |
| SD  | –   | 9.65 | 9.10 | 10.01| 9.41      | 9.65   | 9.08 | 8.49***|
| TO  | –   | 2.29 | 1.16 | 2.72 | 1.59      | 2.18   | 1.04 | 1.03   |
| $p = 1000$ |     |     |     |      |          |        |      |        |
| SD  | –   | 8.33 | 8.23 | 8.38 | 8.45      | 8.34   | 8.23 | 7.04***|
| TO  | –   | 1.59 | 1.01 | 1.76 | 0.93      | 1.56   | 0.92 | 0.90   |
| $n = 1260$ |     |     |     |      |          |        |      |        |
|     |     |     |     |      |          |        |      |        |
| SD  | 12.83| 12.78| 12.68| 12.67| 12.69    | 12.68  | 12.59| 12.55**|
| TO  | 0.65 | 0.60 | 0.57 | 0.54 | 0.38      | 0.53   | 0.49 | 0.48   |
| $p = 500$ |     |     |     |      |          |        |      |        |
| SD  | 10.25| 9.82 | 9.35 | 9.72 | 9.44      | 9.69   | 9.32 | 9.12***|
| TO  | 1.97 | 1.53 | 0.92 | 1.45 | 0.72      | 1.41   | 0.79 | 0.67   |
| $p = 1000$ |     |     |     |      |          |        |      |        |
| SD  | 12.55| 8.91 | 8.04 | 9.04 | 8.12      | 8.91   | 8.05 | 7.46***|
| TO  | 6.56 | 2.38 | 0.94 | 2.52 | 0.84      | 2.35   | 0.80 | 0.64   |

6 Conclusion

This paper combines nonlinear shrinkage with Tyler’s method, thereby creating a fast and stable algorithm to estimate the dispersion matrix in elliptical models; the resulting estimator is robust against both heavy tails and high dimensions. We developed the algorithm by separating calculation of the eigenvalues and the eigenvectors and showed that eigenvectors could be obtained by an iterative procedure. We also showed that the resulting R-NL estimator is still rotation-equivariant, although it no longer is contained in the Steinian class of rotation-equivariant estimators that keeps the vectors of the sample covariance matrix and only shrinks the sample eigenvalues. We also compared our approach to existing methods from the literature using both extensive simulations and an application to real
data, showcasing its favorable performance. Last but not least, it turns out that a further performance boost can be obtained by using our method on scaled data, which basically amounts to separating the problem of estimating a covariance matrix into estimation of individual variances and estimation of the correlation matrix; the resulting estimator is called R-C-NL.
A Proofs

Lemma 1 A minimizer $\hat{V}$ of (7) exists and meets the condition

$$
\hat{V}^\top F(\hat{V}) \hat{V} \Lambda_0^{-1} = \Lambda_0^{-1} \hat{V}^\top F(\hat{V}) \hat{V} \ .
$$

(11)

Proof Since the orthogonal group is compact (Absil et al., 2007, Ch. 3) and

$$
U \mapsto f(U) := \frac{1}{n} \sum_{t=1}^{n} \ln (Z_t^\top U \Lambda_0^{-1} U^\top Z_t) \ ,
$$

(27)

is continuous, $f(U)$ takes its minimal and maximal value on $O$. Thus there exists $\hat{V} \in O$ such that $\hat{V}$ minimizes $f$.

On the other hand, according to Wen and Yin (2013), if $\hat{V}$ is a minimizer of $f$, it must satisfy the following first order conditions:

$$
G \hat{V}^\top - \hat{V} G^\top = 0 \ ,
$$

where $G$ is the unconstrained gradient of problem (7),

$$
G := \frac{1}{n} \sum_{t=1}^{n} \frac{1}{Z_t^\top \hat{V} \Lambda_0^{-1} \hat{V}^\top Z_t} Z_t Z_t^\top \hat{V} \Lambda_0^{-1} .
$$

Thus

$$
\sum_{t=1}^{n} \left( Z_t Z_t^\top \hat{V} \Lambda_0^{-1} \hat{V}^\top - \frac{1}{Z_t^\top \hat{V} \Lambda_0^{-1} \hat{V}^\top Z_t} Z_t Z_t^\top \hat{V} \Lambda_0^{-1} \hat{V}^\top - \Lambda_0^{-1} \hat{V}^\top \sum_{t=1}^{n} \frac{1}{Z_t^\top \hat{V} \Lambda_0^{-1} \hat{V}^\top Z_t} Z_t \right) \hat{V} = 0 .
$$

Thus any minimizer $\hat{V} \in O$ meets (11).

Lemma 2 The surrogate function $g$ satisfies:

$$
f(U) \leq g(U \mid V^{[\ell]}) \text{ for all } U, V^{[\ell]} \in O \quad (16)
$$

$$
f(V^{[\ell]}) = g(V^{[\ell]} \mid V^{[\ell]}) \ ,
$$

(17)

and for $V^{[\ell+1]}$ as in (13),

$$
g(V^{[\ell+1]} \mid V^{[\ell]}) \leq g(U \mid V^{[\ell]}) \text{ for all } U \in O \ .
$$

(18)

Proof The first inequality follows from the fact that $\log(x) \leq \log(a) + (\frac{x}{a} - 1)$ (Wiesel, 2012), while the second equality is trivial. For the last claim, we can write

$$
\arg\min_{U \in O} g(U \mid V^{[\ell]}) = \arg\min_{U \in O} \text{Tr} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^\top V^{[\ell]} \Lambda_0^{-1} (V^{[\ell]})^\top Z_t} U \Lambda_0^{-1} U^\top \right) .
$$

Since we assume $\Lambda_0$ has ordered values, this is globally minimized when $U$ is chosen to diagonalize $\frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^\top V^{[\ell]} \Lambda_0^{-1} (V^{[\ell]})^\top Z_t}$, see e.g., Hediger and Näf (2022).
Lemma 3 Let $R$ be an arbitrary rotation, $\tilde{Z}_t := RZ_t$ and $V^{[\ell]}_t$ be the $\ell$th iteration of Algorithm 1 applied to $\{\tilde{Z}_1, \ldots, \tilde{Z}_T\}$. Then

$$\mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{\tilde{Z}_t \tilde{Z}_t^\top}{\sum_{t=1}^{n} \tilde{Z}_t^\top V^{[\ell]}_t \Lambda_0^{-1}(V^{[\ell]}_t)^\top \tilde{Z}_t} \right) = \left\{ RV^{[\ell+1]}_t : V^{[\ell+1]}_t \in \mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^\top V^{[\ell]}_t \Lambda_0^{-1}(V^{[\ell]}_t)^\top Z_t} \right) \right\}.$$  \hfill (22)

Proof We first show this holds for $\ell = 0$. Since

$$\mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{\|Z_t\|} \right) = \mathcal{E} \left( R \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{\|Z_t\|} R^\top \right),$$

and

$$V^{[0]} \in \mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{\|Z_t\|} \right),$$

it clearly holds that (22) is true for $\ell = 0$.

Assume (22) holds for $\ell$, we show that it holds for $\ell + 1$: By assumption we can write $V^{[\ell]}_t = RV^{[\ell]}_t$. Thus

$$\mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{\tilde{Z}_t \tilde{Z}_t^\top}{\sum_{t=1}^{n} \tilde{Z}_t^\top V^{[\ell]}_t \Lambda_0^{-1}(V^{[\ell]}_t)^\top \tilde{Z}_t} \right) = \mathcal{E} \left( R \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^\top V^{[\ell]}_t \Lambda_0^{-1}(V^{[\ell]}_t)^\top Z_t} R^\top \right)$$

$$= R \mathcal{E} \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{Z_t^\top V^{[\ell]}_t \Lambda_0^{-1}(V^{[\ell]}_t)^\top Z_t} \right),$$

and thus (22) hold true.

Theorem 1 For any sequence $(V^{[\ell]})_{\ell=1}^\infty$ generated by the above iterations,

$$f(V^{[\ell+1]}) \leq f(V^{[\ell]}) \, , \text{ for all } \ell, \quad (19)$$

and

$$\lim_{\ell \to \infty} d(V^{[\ell]}, V) = 0 \, . \quad (20)$$

Proof The proof closely follows the argument in Razaviyayn et al. (2013, Theorem 1, Corollary 1). Using Lemma 2, we have that for all $\ell$,

$$f(V^{[\ell+1]}) \leq g(V^{[\ell+1]} \mid V^{[\ell]}) \leq g(V^{[\ell]} \mid V^{[\ell]}) = f(V^{[\ell]}),$$

proving the first part. For the second, since the orthogonal group $O$ is compact, there exists a subsequence $(V^{[\ell_k]})_{k=1}^\infty$ of $(V^{[\ell]})_{\ell=1}^\infty$ that converges to $V^{[\infty]} \in O$. Additionally, for all $U \in O$,

$$g(U \mid V^{[\ell_k]}) \geq g(V^{[\ell_k+1]} \mid V^{[\ell_k]}) \geq f(V^{[\ell_k+1]}) \geq f(V^{[\ell_{k+1}]}) = g(V^{[\ell_{k+1}]}) \mid V^{[\ell_{k+1}]},$$
since by the properties of subsequences $\ell_{k+1} \geq \ell_k + 1$. Letting $k \to \infty$, thanks to the joint continuity of $(U_1, U_2) \mapsto g(U_1 \mid U_2)$, this implies
\[
g(U \mid V^{[x]}) \geq g(V^{[x]} \mid V^{[x]}), \quad \text{for all } U \in \mathcal{O}.
\]
Thus $V^{[x]}$ is the global minimizer of the function $U \mapsto g(U \mid V^{[x]})$. In particular, the first order conditions must hold: Thus
\[
G(V^{[x]})^\top - V^{[x]}G^\top = 0,
\]
where $G$ is the unconstrained derivative at $V^{[x]}$:
\[
G := \frac{1}{n} \sum_{t=1}^{n} Z_t^\top V^{[x]} \Lambda_0^{-1}(V^{[x]})^\top Z_t Z_t^\top V^{[x]} \Lambda_0^{-1}.
\]
Thus it holds that
\[
(V^{[x]})^\top \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{(V^{[x]})^\top \Lambda_0^{-1} V^{[x]} \Lambda_0^{-1} Z_t} V^{[x]} \Lambda_0^{-1} - \Lambda_0^{-1}(V^{[x]})^\top \sum_{t=1}^{n} \frac{Z_t Z_t^\top}{(V^{[x]})^\top \Lambda_0^{-1} V^{[x]} \Lambda_0^{-1} Z_t} V^{[x]} = 0,
\]
which corresponds to the desired first order conditions for the minimization of $f$ and thus $V^{[x]} \in \mathcal{V}$, or $d(V^{[x]}, \mathcal{V}) = 0$.

Repeating this argument, it follows that any subsequence of $(V^{[\ell]}_\ell)_{\ell=1}^{\infty}$ has a further subsequence converging to some $V^{[x]}$ (depending on the subsequence) with $d(V^{[x]}, \mathcal{V}) = 0$. Now assume the overall sequence does not converge to a point in $\mathcal{V}$. Then there is a subsequence $(V^{[\ell]}_k)_{k=1}^{\infty}$ such that for all $k$
\[
d(V^{[\ell]}_k, \mathcal{V}) \geq \varepsilon,
\]
for some $\varepsilon > 0$. But then this will be true also for any subsequence, a contradiction.

**Lemma 4** Assume that whenever $\hat{\lambda}_i = \hat{\lambda}_j$, also $\hat{\delta}_i = \hat{\delta}_j$, for $j, i \in \{1, \ldots, p\}$. Then if $\hat{V}_1$ and $\hat{V}_2$ meet (12),
\[
\hat{V}_1 \Lambda_0 (\hat{V}_1)^\top = \hat{V}_2 \Lambda_0 (\hat{V}_2)^\top.
\]

**Proof** If $\hat{\lambda}_i$ is unique, the corresponding eigenvector $v_i$ is the basis of the one-dimensional space $\{u : (F(\hat{V}) - \hat{\lambda}_i I)u = 0\}$. As such if $v_i^1$, $v_i^2$ are the $i$th column of $\hat{V}_1$ and $\hat{V}_2$ respectively, it must hold that $v_i^1 = v_i^2$ or $v_i^1 = -v_i^2$. However as $\hat{V}_1 \Lambda_0 \hat{V}_1 = \sum_{i=1}^{p} \hat{\delta}_i v_i^1 (v_i^1)^\top$ this does not affect the overall matrix. This holds true whether or not $\hat{\delta}_i$ in $\Lambda_0$ is unique.

Now assume there is $\hat{\lambda}_i$ with multiplicity $p_0$, while all other $\hat{\lambda}_j$ are unique. By assumption, $\Lambda_0$ mimics this pattern and we can reorder their values such that:
\[
\hat{\Lambda} = \begin{pmatrix}
\hat{\Lambda}_1 & 0_{p-p_0, p_0} \\
0_{p_0, p-p_0} & \hat{\Lambda}_2
\end{pmatrix} \quad \text{and} \quad \Lambda_0 = \begin{pmatrix}
\Lambda_0^1 & 0_{p-p_0, p_0} \\
0_{p_0, p-p_0} & \Lambda_0^2
\end{pmatrix},
\]

where $\hat{\Lambda}_1$ contains unique ordered values and $\hat{\Lambda}_2$ of size $N_0 \times N_0$ contains one value with multiplicity. By assumption, $\Lambda_0^1$ might have values with multiplicity larger one, but $\Lambda_0^2$ also contains only copies of one value. We similarly decompose the newly ordered $\hat{V}_1$, $\hat{V}_2$:
\[
\hat{V}_1 = [\hat{V}_{11}, \hat{V}_{12}] \quad \text{and} \quad \hat{V}_2 = [\hat{V}_{21}, \hat{V}_{22}].
\]

23
First we note that, since the values in $\Lambda_0$ satisfy $U_p$, then for any two $f$ of an equivalence class is the same, such that we may again write $r$.

Then we can write iteration (13) in terms of equivalence classes:

$$[V^{[\ell+1]}] = E(F([V^{[\ell]}])) .$$

Moreover there exists $[\hat{V}] \in \mathcal{O}_0$ such that (12) holds and the generated sequence $([V^{[\ell]}])_{\ell=1}^\infty$ satisfies

$$\tilde{d}_{\Lambda_0}([V^{[\ell]}], [\hat{V}]) \to 0 .$$

**Proof**

First we note that, since the values in $\Lambda_0$ are all strictly larger than zero,

$$U_1 \sim_{\Lambda_0} U_2 \iff U_1 \Lambda_0^{-1}(U_1)^\top = U_2 \Lambda_0^{-1}(U_2)^\top .$$

Thus for any two $U_1 \in [U], U_2 \in [U], F(U_1) = F(U_2)$, such that we may write $F$ directly as a function of the equivalence class, $F([U])$. Moreover, since the eigenvalues of $F([V]^{[\ell]})$ meet the multiplicity condition, the same proof as in Lemma 4 gives that any $U_1 \in E(F([V]^{[\ell]})), U_2 \in E(F([V]^{[\ell]}))$ have $U_1 \sim_{\Lambda_0} U_2$. Thus (24) holds and we can write $[V]^{[\ell]} = [V^{[\ell]}]$. Moreover, by the same argument as above, the function value $f(U)$ of any member of an equivalence class is the same, such that we may again write $f([U])$. Finally $\mathcal{O}_0$ is still compact with the metric $\tilde{d}_{\Lambda_0}$. Indeed consider a sequence $([U_n])_n$ in $\mathcal{O}_0$. For each $n$ we choose an arbitrary representative $U_n \in \mathcal{O}$, to form the sequence $(U_n)_n$. Since $\mathcal{O}$ is compact, this sequence will have a convergent subsequence $(U_{n_k})_k$. We now show that the corresponding subsequence in $\mathcal{O}, ([U]_{n_k})_k$ converges in $\mathcal{O}_0$. Indeed notice that for any convergent sequence, i.e., $(U_n)_n$ such that $U_n \to U$, it follows by the continuity of the matrix product that

$$U_n \Lambda_0 U_n^\top \to UA_0 U^\top .$$

24
or
\[ \tilde{d}_{\Lambda_0}([U_n], [U]) = \|U_n \Lambda_0 U_n^T \rightarrow U \Lambda_0 U^T\|_F \rightarrow 0. \]

Applying this to \([U]_{nk}, \tilde{d}_{\Lambda_0}([U]_{nk}, [U]) \rightarrow 0. \) Since the sequence was arbitrary, every sequence in \(\mathcal{O}_0\) has a convergent subsequence in \(\mathcal{O}_0\) and thus \((\mathcal{O}_0, \tilde{d}_{\Lambda_0})\) is compact.

Finally we can trace the same steps as in Theorem 1 to show that
\[ \inf_{[\hat{V}] \in \mathcal{V}_0} \tilde{d}_{\Lambda_0}([V^{[\ell]}], [\hat{V}]) \rightarrow 0, \]

where now the set \(\mathcal{V}_0 \subset \mathcal{O}_0\) such that (12) holds has only one member \([\hat{V}].\)

### B  Further Empirical and Simulation Results

Figures 4 and 5 show the simulation results for \(n = 150\) and \(p = 100\) and \(p = 200\) respectively.

For our empirical application in Section 5, the following four additional performance measures are reported:

- **AV:** annualized average simple percentage portfolio return.
- **TR:** final cumulative simple percentage portfolio return.
- **MD:** percentage maximum drawdown given by

  \[ \text{MD} := \max_{t \in (0,n)} \left( \max_{t \in (0,t)} \left[ \frac{\tilde{R}_t - \tilde{R}_i}{\tilde{R}_t} \right] \right), \]

  where \(\tilde{R}_t\) is the cumulative simple percentage portfolio return at day \(t\).

- **IR:** annualized information ratio given by

  \[ IR := \frac{AV}{SD}. \]
Figure 4: Percentage Relative Improvement in Average Loss (PRIAL) for various dispersion matrix structures, $\nu \in \{3, 4, \ldots, 14, 15, 30, 60, 90, 120, 240, 500, \infty\}$, $p = 100$ and $n = 150$. The plots are scaled such that they extend to 100 and NL and R-C-NL are fully visible.
Figure 5: Percentage Relative Improvement in Average Loss (PRIAL) for various dispersion matrix structures, $\nu \in \{3, 4, \ldots, 14, 15, 30, 60, 90, 120, 240, 500, \infty\}$, $p = 200$ and $n = 150$. The plots are scaled such that they extend to 100 and NL and R-C-NL are fully visible.
Table 3: Additional out-of-sample portfolio statistics for the daily $p$ largest stocks on CRSP and an estimation window of $n = 252$ and $n = 1260$ days, respectively.

|       | S    | LS   | NL   | R-LS | R-GMV-LS | R-A-LS | R-NL  | R-C-NL |
|-------|------|------|------|------|----------|--------|-------|--------|
|       |      |      |      |      |          |        |       |        |
| $n = 252$ |      |      |      |      |          |        |       |        |
| $p = 100$ |      |      |      |      |          |        |       |        |
| IR    | 0.69 | 0.80 | 0.87 | 0.81 | 0.87     | 0.82   | 0.87  | 0.91   |
| AV    | 9.24 | 9.94 | 10.31| 9.97 | 10.37    | 9.97   | 10.32 | 10.55  |
| TR    | 3813.45 | 5548.60 | 6710.69 | 5641.21 | 6902.58 | 5677.55 | 6767.45 | 7607.62 |
| MD    | -46.56 | -35.20 | -38.48 | -37.94 | -33.40  | -37.05 | -35.48 | -33.26 |
| $p = 500$ |      |      |      |      |          |        |       |        |
| IR    | 1.17 | 1.21 | 1.16 | 1.20 | 1.20     | 1.22   | 1.35  |        |
| AV    | 11.27| 11.01| 11.57| 11.26| 11.56    | 11.07  | 11.50 |        |
| TR    | 11493.55 | 10447.08 | 12921.46 | 11521.84 | 13053.61 | 10776.80 | 13303.94 |        |
| MD    | -29.41| -29.24| -29.92| -30.33| -28.56  | -30.61 | -28.22|        |
| $p = 1000$ |      |      |      |      |          |        |       |        |
| IR    | 1.47 | 1.44 | 1.46 | 1.42 | 1.46     | 1.43   | 1.74  |        |
| AV    | 12.21| 11.87| 12.23| 12.04| 12.21    | 11.76  | 12.22 |        |
| TR    | 18342.65 | 15877.34 | 18502.15 | 16942.34 | 18347.06 | 15080.18 | 19254.29 |        |
| MD    | -33.88| -34.39| -34.52| -34.82| -33.63  | -35.19 | -26.75|        |
| $n = 1260$ |      |      |      |      |          |        |       |        |
| $p = 100$ |      |      |      |      |          |        |       |        |
| IR    | 0.93 | 0.94 | 0.94 | 0.94 | 0.95     | 0.94   | 0.93  | 0.95   |
| AV    | 11.95| 12.06| 11.97| 11.89| 12.01    | 11.90  | 11.74 | 11.86  |
| TR    | 8428.17 | 8863.69 | 8585.57 | 8323.75 | 8709.89 | 8341.21 | 7862.18 | 8267.73 |
| MD    | -39.08| -37.83| -38.08| -36.21| -34.00  | -36.12 | -35.31| -34.98 |
| $p = 500$ |      |      |      |      |          |        |       |        |
| IR    | 1.08 | 1.14 | 1.21 | 1.16 | 1.24     | 1.17   | 1.23  | 1.30   |
| AV    | 11.04| 11.23| 11.33| 11.23| 11.67    | 11.30  | 11.47 | 11.82  |
| TR    | 6599.24 | 7256.96 | 7703.75 | 7271.52 | 8779.64 | 7487.65 | 8162.71 | 9462.95 |
| MD    | -33.16| -31.64| -32.92| -30.85| -31.07  | -30.79 | -31.48| -31.56 |
| $p = 1000$ |      |      |      |      |          |        |       |        |
| IR    | 0.93 | 1.37 | 1.52 | 1.37 | 1.53     | 1.39   | 1.53  | 1.67   |
| AV    | 11.71| 12.23| 12.26| 12.38| 12.46    | 12.38  | 12.29 | 12.50  |
| TR    | 7799.17 | 11257.77 | 11731.10 | 11893.95 | 12676.34 | 11951.51 | 11857.04 | 13137.98 |
| MD    | -37.38| -33.98| -32.18| -33.42| -32.07  | -33.58 | -32.28| -27.65 |
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