I review some ways in which spacetime dimensionality enters explicitly in gravitation. In particular, I recall some unusual geometrical gravity models that are constructible in dimensions different from four, especially in D=3 where even ordinary Einstein theory is “different”, e.g., fully Machian.

It is a pleasure to dedicate this little travelogue/catalogue of exotic gravity models to John Stachel, whose loyalty to the D=4 Einstein cause is too steadfast to be subverted by reading it.

Once unleashed by general relativity, dynamical geometry has become a fertile playground for generalization in many directions beyond Einstein’s D=4 Ricci-flat choice. This trend has intensified with string theory, where D=10 is normal as are (higher curvature power) corrections to the Einstein action. There are many other reasons to study different dimensions; here is one: As I became aware, thanks to John, Einstein already foresaw [1] the potential danger of letting geometry be at the mercy of field equations, in particular worrying about spaces with closed timelike curves, but also optimistically hoping that they would be forbidden in “physically acceptable” matter contexts (this is not a tautology since acceptable means having decent stress tensor). Although the best-known examples, such as Gödel’s universes [2], fall in this class, it is in the simpler setting of D=3 (“planar”) gravity that they have recently been studied on an industrial scale [3], and have yielded Einstein’s hoped-for taboo in a clear way. More generally, one can learn about D=4 Einstein’s virtues from studying different D’s, and the different sorts of models they support. What is more, we are very likely to be embedded in a world, which, if it has any classical geometry at all, is likely to have as many as eleven dimensions! In this short excursion, I can only point out some recent examples of theories that I have been involved with directly; equations and further references will have to be found in the citations.

Let us begin with some remarks about ordinary Einstein theory in the smaller worlds of D<4. One does not normally think of the curvature components as being dimension-dependent, but we all know that in D=3, Einstein and Riemann tensors have the same number of components and indeed are equivalent, since \( G^\mu_\nu = \frac{1}{4} \epsilon^\mu\nu\alpha\beta \epsilon_{\alpha\beta\lambda\sigma} R^{\lambda\sigma} \). Strangely, it was a long time before the import of this was appreciated: that outside sources, spacetime is flat! More precisely it is locally, but not globally, flat. Philosophically, D=3 Einstein theory presents the Machian dream in its purest form: there are no gravitational excitations, so geometry is entirely – and locally – determined by matter. There is a field-current identity: Riemann (being Einstein) equals stress tensor. So the picture that emerges is that this planar world consists of patches of Minkowski space glued together at the sources (most simply discrete point particles, representing parallel strings in a D=4 Einstein world). The 1-particle conical space solution is amusing enough [4] but things really get to be fun for two or more stationary or, better still, moving ones [5]. If a cosmological constant is present, it’s even more fun as the patches are constant curvature spacetimes [6]. In that case (for negative cosmological constant) it is also possible to have black holes by suitable identifications of points, [7]. Time-helical structures, requiring identification of times in a periodic way (as well as the space gluings)arise for stationary, rotating, solutions and lead to the whole gamut of possible closed timelike curves and, as mentioned, a clear arena to examine whether they can be physically generated. But D=3 can be more amusing still, for it permits (as does any odd-dimensional space) the construction of different invariants, the Chern–Simons (CS) terms. These are the gravitational analogs of the simple electrodynamics (or Yang–Mills) \( \int A \wedge F \) structures that in turn arise from the next higher dimensional topological invariants such as \( F_{\mu\nu} \star F^{\mu\nu} \equiv \partial_\mu (\epsilon^{\mu\nu\alpha\beta} A_\nu F_{\alpha\beta}) \) in the abelian
context. Here we have the Pontryagin invariant $R^* R$ instead. Varying these gravitational CS terms with respect to the metric leads to a tensor, because the integral (if not the CS integrand) is gauge invariant. In D=3, this is the famous Cotton tensor, $C^{\mu\nu} = \epsilon^{\mu\alpha\beta} D_{\alpha}(R_{\nu} - \frac{1}{4} \delta^{\nu}_{\beta} R)$, discovered long before general relativity [3]; $C^{\mu\nu}$ is the conformal tensor in D=3, replacing the (identically vanishing) Weyl curvature. It is a symmetric, traceless, identically conserved quantity, although it superficially seems to be none of these. Its interest lies not so much for generating a theory of gravity in its own right (it could at best only couple to traceless sources) but as an added term to the Einstein one. Being of third derivative order, it has a coefficient with relative inverse length or mass dimension (in Planck units) to that of the Einstein action. This mass is in fact that of small excitations (of helicity $\pm 2$) of the metric about flat space: adding CS has restored a degree of freedom absent in either $R$ or CS alone. This combined theory [4], called topologically massive gravity (TMG) for obvious reasons, has many other wondrous properties and unsolved aspects. First, despite being a higher derivative theory, it has no unitarity or ghost problems; it may even be finite as a quantum theory, although that is still an open mathematical problem [11]. If so, it might really have some lessons for us, for it would be unique in this respect amongst truly dynamical gravity models without ghosts (unlike four-derivative theories) but with a dimensional coupling constant; pure Einstein D=3 theory is renormalizable but that doesn’t count – it is non-dynamical [11]. Second, TMG, at least in its linearized guise [12] acts to turn its sources into anyons; that is, a particle can acquire any desired spin simply by coupling to TMG. But, thirdly, no-one has succeeded as yet in finding the simplest possible, “Schwarzschild” solution to the nonlinear model, i.e., a circularly symmetric time-independent (we don’t even know if there’s a Birkhoff theorem) exterior geometry that obeys the $G_{\mu\nu} + m^{-1} C_{\mu\nu} = 0$ equations. Although CS-like terms can be constructed for higher odd-D spaces, they have not been studied much because they have no linearized, kinematical, effects beyond D=3 because they are of higher powers in an expansion about flat space. There are both strong similarities and differences between TMG and its spin 1 counterpart, topologically massive Yang–Mills theory. The most striking difference is that in the quantum theory, the coefficient of the CS term in “TM–YM” must necessarily be quantized [9], but not that of TMG [13].

But the twists in D=3 gravity do not stop there: there is yet another “CS-ness” present. Once it is noted that, in Einstein gravity, spacetime is flat outside sources, one realizes that this is just the same as what happens to abelian or nonabelian vector fields in their pure CS models: the field equations are just $* F^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta} = 0$, so that the field “curvature” also vanishes here [in both cases, the full “curvatures” are proportional to currents, $* F^\mu = J^\mu$]. Indeed, there is an equivalence (except for some interesting find print) between the two models and one can formally recast the Einstein action and equations into non-abelian vector field CS form in terms of the dreibein and spin connections. So this is yet another vast subject straddling two ostensibly different types of theory; for a review see e.g. [14].

We will not descend much to D=2, another well-studied subject [15], because there is no Einstein gravity there at all: only the Ricci scalar is non-vanishing, being the “double-dual” of Riemann, while the Einstein tensor vanishes identically. As usual, D=2 is different from all other dimensions in this respect (it is also here that Maxwell theory ceases to have excitations); some sort of additional scalar field is required to assure the Hilbert action from just being a dull Euler topological invariant and this departs from the realm of pure geometry.

What about dimensions beyond D=4? This becomes a generic area where the differences from D=4 are more quantitative than qualitative. Still, there are some amusing points to be noted. For example, consider the Gauss–Bonnet invariant $R^* R^*$, defined in D=4. There, it is a total divergence and hence irrelevant to field equations. However, in higher dimensions, it can still be
defined by writing it out in terms of metrics; for example we all know it is proportional to the combination \( R_{\mu\nu\alpha\beta}^2 - 4R_{\mu\nu}^2 + R^2 \). As a Lagrangian, it is seemingly dangerous to unitarity of excitations because of its fourth derivative order. In fact, there is no danger, because (say about flat space) this combination is a total divergence in its leading quadratic order in \( h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu}) \) in any \( D \). Thus, this is a “safe” class of alternative actions, say when added to Einstein’s. Some of their solutions, e.g., Schwarzschild-like ones, have been studied to see whether they are better or still unique. For example, there can be cosmological solutions without an explicit cosmological constant \( \Lambda \), which is not necessarily a good thing, physically.

In quantum field theory, a powerful tool has been the “large \( N \) limit” of Yang–Mills theory in which the number of flavors (internal degrees of freedom: \( A_{\mu}^a, a = 1 \ldots N \)) is sent to infinity. The equivalent in gravity, rather naturally, the dimensionality \( D \) of spacetime, over which the “internal” index \( a \) of the vielbein \( e_{\mu}^a \) ranges. As far as I know, the literature here consists of but one brave paper \([14]\), which however did not get far. This seems to me a worthy subject of study also by classical relativists, who have instead mostly considered what is in some ways the opposite, ultra-local, limit \([18]\). There must be some simplifying aspects as the number of degrees of freedom \( \frac{1}{2}D(D-3) \), rises quadratically, and the Newtonian potential that enters in the Schwarzschild metric behaves as \( r^{-(D-3)} \).

There are also auxiliary quantities that are interestingly dimension-dependent; we encountered some of them in current work on \( D=11 \) supergravity \([14]\). I will not transgress further into the superworld here, except to say that it is absolutely amazing that a) Einstein gravity always has a “Dirac square root” for all \( D \leq 11 \), i.e., can always be consistently supersymmetrized without the need for higher spins or more than one graviton, and b) that this possibility stops \([21]\) at \( D=11 \) and c) that cosmological terms are allowed for all \( D<11 \), but forbidden \([21]\) at \( D=11 \). I certainly do not know of any “Riemannian” differences between \( D=11 \) and \( D=12 \) for example! In our case, we needed to find a basis of monomial local invariants made up from Riemann or Weyl tensors at a given quartic order. Their number had a natural “generic” bound \([22]\) at \( D=8 \), whereas \( D=4 \) is always the most degenerate dimension (I drop \( D<4 \) here). This sort of property is also of importance when studying gravitational trace anomalies \([23]\) in which identities arising from antisymmetrizing an expression over more indices than there are dimensions essentially generate all the strange looking identities between tensors, such as \( C^{\mu\nu\alpha\beta}C_{\lambda\alpha\beta}^{\mu\nu} = \frac{1}{4}g^{\mu\nu}C^2 \) in \( D=4 \), from antisymmetrizing expressions such as \( C^{\mu\nu}C_{\mu\nu}^{\alpha\beta}X^\lambda \), where \( X \) is arbitrary and brackets indicate antisymmetrizations over 5 indices. In the anomaly context we are actually interested, using so-called dimensional regularization, in spaces of dimension differing from an integer by an infinitesimal parameter, still another unlikely departure from \( D=4 \), but one that has its own unlikely set of geometrical rules. In conclusion, I have tried to indicate that the list of interesting, useful and even important consequences to be drawn from excursions away from our favorite Einstein action in its \( D=4 \) world is substantial and by no means complete.

This work was supported by NSF grant PHY 93-15811. I am grateful to my collaborators in our explorations of the areas discussed here.

References

[1] A. Einstein, in Albert Einstein: Philosopher–Scientist, ed. P. Schilpp (Tudor, New York, 1957); Berliner Berichte (1914) 1030–1085, p. 1079 (and in letters).

[2] K. Gödel, Rev. Mod. Phys. 21 (1949) 447; see also W.J. van Stockum, Proc. R. Soc. Edin. 57 (1937) 13.
[3] J.R. Gott, Phys. Rev. Lett. 66 (1991) 1126; S. Deser, R. Jackiw, and ’t Hooft, Phys. Rev. Lett. 68 (1992) 267; S. Carroll, E. Farhi, and A. Guth, Phys. Rev. Lett. 68 (1992) 263, (E) 3368; G. ’t Hooft, Class. Quant. Grav. 9 (1992) 1335; for further references see for example S. Deser and R. Jackiw, Comments Nucl. Part. Phys. 20 (1992) 337.

[4] A. Staruszkiewicz, Acta Phys. Pol. 24 (1963) 735.

[5] S. Deser, R. Jackiw, ’t Hooft, Ann. Phys. 152 (1984) 220.

[6] S. Deser and R. Jackiw, Ann. Phys. 153 (1984) 405.

[7] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48 (1993) 1506.

[8] E. Cotton, C.R. Acad. Sci. Paris 127 (1898) 349.

[9] S. Deser and R. Jackiw, and Templeton, Phys. Rev. Lett. 48 (1982) 975; Ann. Phys. 140 (1982) 372.

[10] S. Deser and Z. Yang, Class. Quant. Grav. 7 (1990) 1603.

[11] E. Witten, Nucl. Phys. B311 (1988) 46; S. Deser, J. McCarthy, and Z. Yang, Phys. Lett. B222 (1989) 61.

[12] S. Deser, Phys. Rev. Lett. 64 (1990) 611.

[13] R. Percacci, Ann. Phys. 177 (1987) 27.

[14] A. Achucarro and P. Townsend, Phys. Lett. B180 (1986) 89.

[15] R. Jackiw; C. Teiltelboim in Quantum Theory of Gravity (S. Christensen, ed.) Adam Hilger, Bristol 1984.

[16] D. Boulware and S. Deser, Phys. Rev. Lett. 55 (1985) 2656; S. Deser and Z. Yang, Class. Quant. Grav. 6 (1989) L83.

[17] A. Strominger, Phys. Rev. D24 (1981) 3082.

[18] C.J. Isham, Proc. Roy. Soc. A351 (1976) 209; C.J. Isham and A. Kakas, Class. Quant. Grav. 1 (1984) 633; C. Teitelboim in Einstein Centenary Volume, (A. Held, ed.) Plenum, NY 1980.

[19] D. Seminara and S. Deser “Counterterms/M-theory Corrections for D=11 Supergravity,” (submitted to Phys. Rev. Lett.).

[20] W. Nahm, Nucl. Phys. B135 (1978) 145.

[21] K. Bautier, S. Deser, M. Henneaux, and D. Seminara, Phys. Lett. B406 (1997) 49.

[22] S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummins, Class. Quant. Grav. 9 (1992) 1151.

[23] S. Deser and A. Schwimmer, Phys. Lett. B309 (1993) 279; S. Deser, Helv. Phys. Acta 69 (1996) 570.