Towards beating the probabilistic lower bound on $q$-perfect hashing
for all $q^*$

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Abstract

For an integer $q \geq 2$, a perfect $q$-hash code $C$ is a block code over $[q] := \{1, \ldots, q\}$ of length $n$ in which every subset $\{c_1, c_2, \ldots, c_q\}$ of $q$ elements is separated, i.e., there exists $i \in [n]$ such that $\{\text{proj}_i(c_1), \ldots, \text{proj}_i(c_q)\} = [q]$, where $\text{proj}_i(c_j)$ denotes the $i$th position of $c_j$. Finding the maximum size $M(n, q)$ of perfect $q$-hash codes of length $n$, for given $q$ and $n$, is a fundamental problem in combinatorics, information theory, and computer science. In this paper, we are interested in asymptotical behavior of this problem. More precisely speaking, we will focus on the quantity $R_q := \limsup_{n \to \infty} \frac{\log_2 M(n, q)}{n}$.

A well-known probabilistic argument shows an existence lower bound on $R_q$, namely $R_q \geq \frac{1}{q-1} \log_2 \left(\frac{1}{1-\frac{1}{q}}\right)$ [10, 12]. This is still the best-known lower bound till now except for the case $q = 3$ for which Körner and Matron [13] found that the concatenation technique could lead to perfect 3-hash codes beating this the probabilistic lower bound. The improvement on the lower bound on $R_3$ was discovered in 1988 and there has been no any progress on lower bound on $R_q$ for more than 30 years despite of some work on upper bounds on $R_q$. In this paper we show that this probabilistic lower bound can be improved for $q$ from 4 to 15 and all odd integers between 17 and 25 and all sufficiently large $q$. Although we are not able to prove that our construction can beat the probabilistic method for all $q$, the fact that our construction beat the probabilistic method for both small and large $q$ sheds light on that this construction might hold for all $q$. Our idea is based on a modified concatenation differing from the concatenation [10] where both the inner and outer codes are separated. In our concatenation, the inner code is not necessarily a perfect $q$-hash code. This gives a more flexible choice of inner codes and hence we are able to beat the probabilistic existence lower bound on $R_q$.

1 Introduction

Probabilistic method is one of the most powerful tools to derive lower bounds in theoretical computer science and extremal combinatorics [1]. Roughly speaking, to prove the existence of an object

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1 The case $q = 3$ is resolved in [13].
of a given size satisfying certain conditions, one shows that a random object of this size (maybe after being slightly modified) has a positive probability to satisfy these conditions. In many problems the lower bound given by this method is conjectured exact, at least asymptotically, and sometimes one can prove it is indeed so. This means that optimal solutions to such problems are rather common. On the other hand, when the probabilistic lower bound is not asymptotically exact, optimal solutions tend to be rare and have some particular structure. So, from a theoretical point of view, it is of great importance to know whether a problem belongs to one or the other of these two classes. Some exceptional examples where the probabilistic lower bounds are not asymptotically exact include the Gilbert-Varshamov bound in coding theory [16] and the probabilistic lower bound on perfect hash codes [13]. In this paper, we study lower bounds on perfect hash codes and compare them with the probabilistic lower bound.

A $q$-ary code $C \subseteq [q]^n$ is said to be a perfect $q$-hash code if for every subset of $C$ of $q$ elements, there exists an coordinate where the $q$ codewords in this subset have distinct values. The rate of a perfect $q$-hash code is defined as $R_C = \frac{\log_q |C|}{n}$.

The existence of a perfect $q$-hash code gives rise to a perfect $q$-hash family. To see this, let $C$ be the whole universe and the projection of each coordinate be a hash function. Then, for any $q$ elements of this universe, there exists a hash function mapping them to distinct values. Another application of perfect $q$-hash code is the zero-error list decoding on certain channels. A channel can be thought of as a bipartite graph $(V; W; E)$, where $V$ is the set of channel inputs, $W$ is the set of channel outputs, and $(w, v) \in E$ if on input $v$, the channel can output $w$. The $q/(q-1)$ channel then is the channel with $V = W = \{0, 1, \ldots, q-1\}$, and $(v, w) \in E$ if and only if $v \neq w$. If we want to ensure that the receiver can identify a subset of at most $q-1$ sequences that is guaranteed to contain the transmitted sequence, one can communicate via $n$ repeated uses of the channel using the perfect $q$-hash code. See [9, 7] for more details.

In this paper, we only consider the asymptotic behavior of rates of perfect $q$-hash codes, namely, we focus on the quantity $R_q := \limsup_{n \to \infty} \frac{\log_q M(n, q)}{n}$, where $M(n, q)$ stands for the maximum size of perfect $q$-hash codes of length $n$.

The study of $R_q$ could be dated back to 80s. There are a few works dedicated to the upper bound on $R_q$. Fredman and Komlós [10] showed a general upper bound: $R_q \leq \frac{q}{q^n}$ for all $q \geq 2$. Arikan [2] improved this bound for $q = 4$, and then Dalai, Guruswami and Radhakrishnan [7] further improved the upper bound on $R_q$. Recently, Guruswami and Riazanov [11] discovered a stronger bound for every $q \geq 4$. Costa and Dalai [6] further show that it is possible to explicitly compute this improvement over the previous upper bound.

Although there are some works towards tightening the upper bound on $R_q$. There are very few results about lower bounds on $R_q$. A plain probabilistic argument shows the existence of perfect $q$-hash code with rate $R_q \geq \frac{1}{q-1} \log_q \left( \frac{1}{1-q^{1/q}} \right) [10, 12]$. This is still the best-known lower bound till now except for the case $q = 3$ for which Körner and Matron [13] found that the concatenation technique could lead to perfect 3-hash codes beating this the probabilistic lower bound. The improvement on the lower bound on $R_3$ was discovered in 1988 and there has been no any progress on lower bounds on $R_q$ for more than 30 years. Körner and Matron’s idea is to concatenate an outer code, an 9-ary 3-hash code with an inner code, a perfect 3-hash code with size 9. They further posed an open problem whether there exist perfect $q$-hash codes beating the random argument for every $q$. In this paper, we provide a partial and affirmative answer to this open problem. We show that there exist perfect $q$-hash codes beating the random argument for all sufficiently large $q$ with.
q (mod 4) ̸= 2. To complement this result, we also prove the existence of perfect q-hash code that could beat random result for small q from 4 to 15 and odd q between 17 and 25, as well as many other odd integers between 27 and 155 (see Remark 4). Our computer search result together with asymptotic result suggests that our construction might beat the the probabilistic lower bound for every integer q.

The main technique of this paper is a modified version of concatenation. Unlike Körner and Matron’s concatenation where both inner and outer codes must be separated, we abandon this separateness of inner code at a cost of imposing a stronger requirement on the outer code. By relaxing the condition that the inner code is perfect q-hash code, we have more freedom to construct the inner code. As a result, we are able to improve the lower bound on \( R_q \).

Before explaining our technique in detail, let us recall the concatenation technique introduced by Körner and Matron. A plain probabilistic argument can prove the existence of an \( m \)-ary outer code \( C_1 \) of length \( n_1 \) that is \( q \)-separated with \( q \leq m \), i.e., for every \( q \)-element subset of \( C_1 \) (a \( q \)-element set is a set of size \( q \)), there exists \( i \in \{1, 2, \ldots, n_1\} \) such that elements of this \( q \)-subset are pairwise distinct at position \( i \). Then, they construct a perfect q-hash code \( C_2 \) of length \( n_2 \) as an inner code. By concatenating \( C_1 \) with \( C_2 \) (see Lemma 2 for detail), they obtain a perfect q-hash code of length \( n_1n_2 \).

In our concatenation, we make a trade-off between inner code and outer code by relaxing the condition on the inner code and imposing a stronger condition on the outer code. We first take a set \( \mathcal{A} \) consisting of some \( q \)-element subsets of \([m]\). We apply the probabilistic method to show the existence of an \( m \)-ary outer code \( C_1 \) such that, for every \( q \)-element subset \( \{c_1, c_2, \ldots, c_q\} \) of \( C_1 \), there exists \( i \) such that \( \{\text{proj}_i(c_1), \text{proj}_i(c_2), \ldots, \text{proj}_i(c_{n_1})\} \in \mathcal{A} \), where \( \text{proj}_i(c_j) \) stands for the \( i \)th coordinate of \( c_j \). Note that Körner and Matron’s concatenation only requires that there exists \( i \) such that \( \{\text{proj}_i(c_1), \text{proj}_i(c_2), \ldots, \text{proj}_i(c_{n_1})\} \) are pairwise distinct. In this sense, we extend their idea by confining \( \{\text{proj}_i(c_1), \text{proj}_i(c_2), \ldots, \text{proj}_i(c_{n_1})\} \) to be one of the subset in \( \mathcal{A} \). If we could find a \( q \)-ary inner code \( C_2 \) such that there are at least \(|\mathcal{A}| \) \( q \)-element subsets of \( C_1 \) that are separated, concatenating these two codes leads a perfect q-hash code. Now, it remains to look for suitable inner code \( C_2 \). One good candidate for the inner code is an MDS code. In this paper, we first choose an \( q \)-ary MDS code over an abelian group of length 3 and dimension 2 to be the inner code \( C_2 \). To compare our concatenation with the known lower bound, we have to estimate the number of separated \( q \)-element subsets of \( C_2 \). We then reduce determining the number of separated \( q \)-element subsets of \( C_2 \) to determining the number of \( q \)-element subsets of \( C_2 \) in which all three positions are separated. It turns out that the latter problem is equivalent to the following well-known combinatorial problem: determine the number \( s_q \) of pairs \((\pi_1, \pi_2)\) of bijections \([q] \rightarrow \mathbb{Z}_q\) such that \( \pi_1 + \pi_2 \) is a bijection of \( \mathbb{Z}_q \) as well. In literature, there is an asymptotic result on \( s_q \) for odd number \( q \) [15] which can be used to estimate the number of separated \( q \)-element subsets of \( C_2 \). As a result, we are able to improve \( R_q \) for large odd \( q \). Recently, this combinatorial problem is further extended to abelian group \( G \) with \( \sum_{x \in G} x = 0 \) [8]. In fact, they prove an even stronger result that holds for \((\pi_1, \pi_2)\) of bijections such that \( \pi_1 + \pi_2 + f \) is a bijection for some function \( f : [q] \rightarrow \mathbb{Z}_q \) with \( \sum_{i=1}^q f(i) = \sum_{x \in G} x \). Due to this result, we can also extend our result to improve \( R_q \) for every large \( q \).

We further extend this [3, 2]-MDS code result to a [4, 2]-MDS code. It turns out that an \( q \)-ary MDS code over an abelian group of length 4 and dimension 2 could lead to an even better lower bound on \( R_q \). Our main result is summarized below.
**Theorem 1.1.** For every integer \( q \) with \( q \pmod{4} \neq 2 \), one has a lower bound

\[
R_q \geq -\frac{1}{4(q-1)} \log_2 \left( \left( 1 - \frac{q!}{q^q} \right)^4 - \left( \frac{3q}{\sqrt{e}} + o(q) \right) \left( \frac{q!}{q^q} \right)^3 \right).
\]

For every integer \( q \) with \( q \pmod{4} = 2 \), one has a lower bound

\[
R_q \geq -\frac{1}{3(q-1)} \log_2 \left( \left( 1 - \frac{q!}{q^q} \right)^3 - \left( \frac{q}{2\sqrt{e}} + o(q) \right) \left( \frac{(q-1)!}{q^q} \right)^3 \right).
\]

This rate outperforms the probabilistic lower bound, \( R_q \geq -\frac{1}{(q-1)} \log_2(1 - \frac{q!}{q^q}) \), for all sufficiently large \( q \).

We note that the numerical results imply that the same construction also beat the probabilistic lower bound for small \( q \). This leads to the following conjecture.

**Conjecture 1.2.** For every integer \( q \), there exists a perfect \( q \)-hash code beating the probabilistic lower bound. Moreover, such construction can be obtained via a concatenation code defined in Theorem 3.14, Theorem 3.15 and Theorem 4.2.

This paper is organized as follows. In Section 2, we propose a new concatenation technique and derive a lower bound on \( R_q \) in terms of the number of separated \( q \)-element subsets of the inner code. In Section 3, we provide several candidates for the inner code of our concatenation technique and estimate the number of separated \( q \)-element subsets for these candidates. By plugging this number into the lower bound in Section 2, we manage to prove that the probabilistic lower bound on \( R_q \) with \( q \pmod{4} \neq 2 \) can be improved in many cases. In Section 4, we provide another candidate that can beat the probabilistic lower bound for \( q \pmod{4} = 2 \). In Section 5, we provide a construction that is not based on linear code which can further improve the lower bound on \( R_5 \) and \( R_7 \).

# 2 \( A \)-friendly codes and concatenation

## 2.1 Hash code

A set containing \( q \) elements is called a \( q \)-element set. Assume that \( m \geq q \), then a \( q \)-element subset \( \{c_1, c_2, \ldots, c_q\} \) of \([m]^N\) is called separated if there exists \( i \in [N] \) such that \( \text{proj}_i(c_1), \ldots, \text{proj}_i(c_q) \) are pairwise distinct. If \( q \) is a prime power, we denote by \( \mathbb{F}_q \) the finite field with \( q \) elements and let \( \mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} \) be a congruence class of integers modulo \( m \).

A subset \( C \) of \([m]^N\) is called an \( m \)-ary code of length \( N \). For an integer \( q \leq m \), an \( m \)-ary code \( C \) of length \( N \) is called an \( m \)-ary \( q \)-hash code if every \( q \)-element subset of \( C \) is separated. In particular, we say that \( C \) is a perfect \( q \)-hash code if \( m = q \).

Let us generalize the notion of an \( m \)-ary \( q \)-hash code. Let \( \binom{[m]}{q} \) denote the collection of all \( q \)-element subsets of \([m]\). Let \( \mathcal{A} \) be a subset of \( \binom{[m]}{q} \) and let \( C \) be a code in \([m]^N\). We say that a \( q \)-element subset \( \{c_1, \ldots, c_q\} \) of \([m]^N\) is \( \mathcal{A} \)-friendly if there exists \( i \in [N] \) such that \( \{\text{proj}_i(c_1), \text{proj}_i(c_2), \ldots, \text{proj}_i(c_q)\} \in \mathcal{A} \). Otherwise, we call \( \{c_1, \ldots, c_q\} \) an \( \mathcal{A} \)-unfriendly subset. If every \( q \)-element subset of \( C \) is \( \mathcal{A} \)-friendly, we say that \( C \) is an \( \mathcal{A} \)-friendly code. In particular, this definition coincides with an \( m \)-ary \( q \)-hash code when \( \mathcal{A} = \binom{[m]}{q} \).
2.2 Random $\mathcal{A}$-friendly codes

In this subsection, by applying a probabilistic argument, we prove the existence of $\mathcal{A}$-friendly codes.

**Lemma 2.1.** Let $\mathcal{A}$ be a nonempty subset of $[m]$. Then there exists an $m$-ary $\mathcal{A}$-friendly code $C$ of length $N$ and size at least $\left\lceil \frac{M}{3} \right\rceil$ as long as

$$
\left( \frac{M}{q} \right) \left( 1 - \frac{q^{|\mathcal{A}|}}{m^{|\mathcal{A}|}} \right)^N \leq \frac{M}{2q}.
$$

**Proof.** We sample $M$ codewords $c_1, \ldots, c_M$ uniformly at random in $[m]^N$ with replacement. The number of collisions is negligible compared to $M$. To see this, let $X_{i,j}$ be the 0,1-random variable such that $X_{i,j} = 1$ if $c_i = c_j$ and $X_{i,j} = 0$ otherwise. It is clear $P[X_{i,j} = 1] = m^{-N}$. It follows that $E[\sum_{1 \leq i < j \leq M} X_{i,j}] = \binom{M}{2} m^{-N} = o(M)$ due to the fact that $M = o(\sqrt{mN})$. Next, we bound the number of $\mathcal{A}$-friendly $q$-element sets from these $M$ codewords. Let us fix a $q$-element set $\{c_1, \ldots, c_q\}$ with $c_i = (c_{i,1}, \ldots, c_{i,N})$. For any $j \in [n]$, the probability that $\{c_{1,j}, \ldots, c_{q,j}\} \in \mathcal{A}$ is $\frac{q^{|\mathcal{A}|}}{m^{|\mathcal{A}|}}$ as $c_{i,j}$ is picked uniformly at random in $[m]$. It follows that the probability that $\{c_1, \ldots, c_q\}$ is not $\mathcal{A}$-friendly is $(1 - \frac{q^{|\mathcal{A}|}}{m^{|\mathcal{A}|}})^N$. There are at most $\binom{M}{q}$ $q$-element sets from $\{c_1, \ldots, c_M\}$. By union bound, the expected number of $\mathcal{A}$-unfriendly $q$-element sets is at most $\binom{M}{q} \left( 1 - \frac{q^{|\mathcal{A}|}}{m^{|\mathcal{A}|}} \right)^N \leq \frac{M}{2q}$. Remove all the codewords that lie in any of these $\mathcal{A}$-unfriendly $q$-element sets. Then, we remove at most $q \times \frac{M}{2q} = \frac{M}{2}$ codewords. According to our previous argument, there are $o(M)$ collisions among these $M$ codewords. Remove these $o(M)$ codewords and we obtain an $\mathcal{A}$-friendly code of size at least $\frac{M}{3}$. The desired result follows. \qed

**Remark 1.** Note that in [13], the set $\mathcal{A}$ is the collection of all $q$-element subsets of $[m]$. Thus, our random argument can be viewed as a generalization of the argument in [13]. This generalization allows us to relax the constraint on our inner code $C_1$, i.e., $C_1$ is not necessary a perfect $q$-hash code at a cost of imposing a stronger constraint on the outer code. That is, instead of requiring that $C_1$ is a perfect $q$-hash code, we only require that $|\mathcal{A}|/(\binom{m}{q})$ fraction of $q$-element sets of $C_1$ are separated.

If we choose $m = q$ in Lemma 2.1 then $|\mathcal{A}| = 1$. We obtain a random construction of perfect $q$-hash codes.

**Corollary 2.2.** Let $q \geq 2$. Then there exists $q$-hash code of length $N$ and size at least $\left\lceil \frac{M}{3} \right\rceil$ as long as

$$
\left( \frac{M}{q} \right) \left( 1 - \frac{q^1}{q^q} \right)^N \leq \frac{M}{2q}.
$$

In particular, we have a random $q$-hash code with rate

$$
R = \frac{\log_2 M}{N} = -\frac{1}{q - 1} \log_2 \left( 1 - \frac{q^1}{q^q} \right) + \frac{O(1)}{N}.
$$

Hence, we have a probabilistic lower bound

$$
R_q \geq \frac{1}{q - 1} \log_2 \left( \frac{1}{1 - q^1/q^q} \right).
$$
Proof. As \( (M \choose q) \leq \frac{M^q}{q!} \), the following inequality
\[
\frac{M^q}{q!} \left(1 - \frac{q!}{q^q}\right)^N \leq \frac{M}{2q}
\]  
implies the inequality (2). Choose \( M \) to be the largest integer satisfying the inequality (5) and consider the limit \( \lim_{N \to \infty} \frac{\log_q M}{N} \). The desired equality (3) follows. \( \square \)

### 2.3 A concatenation technique

Let \( C \) be a \( q \)-ary code of length \( n \) and size \( m \). Denote by \( S(C) \) the collection of all \( q \)-element subsets of \( C \) that are separated.

**Lemma 2.3.** Let \( C \) be a \( q \)-ary code of length \( n \) and size \( m \). Then one has
\[
R_q \geq -\frac{1}{(q-1)n} \log_2 \left(1 - \frac{q!|S(C)|}{m^q}\right). \tag{6}
\]

**Proof.** Let \( \pi \) be any bijection from \( C \) to \([m]\). Define \( A := \bigcup_{\{c_1, \ldots, c_q\} \in S(C)} \{\pi(c_1), \ldots, \pi(c_q)\} \). It is clear that \( A \subseteq \binom{[m]}{q} \) and \(|A| = |S(C)|\). Lemma 2.1 tells us that there exists an \( m \)-ary \( A \)-friendly code \( C_1 \) of length \( n_1 \) with rate
\[
R = -\frac{1}{(q-1)} \log_2 \left(1 - \frac{q!|A|}{m^q}\right) + O\left(\frac{1}{n_1}\right).
\]

Let \( C_2 \) be the concatenation of \( C_1 \) with \( C \), i.e.,
\[
C_2 := \{\pi^{-1}(c) = (\pi^{-1}(c_1), \pi^{-1}(c_2), \ldots, \pi^{-1}(c_{n_1})) : c = (c_1, c_2, \ldots, c_{n_1}) \in C_1\}.
\]

Clearly, the rate of \( C_2 \) is \( R = -\frac{1}{m(q-1)} \log_2 (1 - \frac{q!|A|}{m^q}) + O\left(\frac{1}{n_1n_2}\right) \). It remains to show that \( C_2 \) is a perfect \( q \)-hash code.

Choose any \( q \)-element subset \( \{\pi^{-1}(c_1), \pi^{-1}(c_2), \ldots, \pi^{-1}(c_q)\} \) from \( C_2 \) with \( \{c_1, c_2, \ldots, c_q\} \) being a \( q \)-element subset of \( C_1 \). Since \( C_1 \) is \( A \)-friendly, there exists \( i \in [N] \) such that \( \{\text{proj}_i(c_1), \text{proj}_i(c_2), \ldots, \text{proj}_i(c_q)\} \in A \). This implies that \( \{\pi^{-1}(\text{proj}_i(c_1)), \ldots, \pi^{-1}(\text{proj}_i(c_q))\} \in S(C) \) and thus \( \{\pi^{-1}(c_1), \pi^{-1}(c_2), \ldots, \pi^{-1}(c_q)\} \) is separated. The desired result follows from the definition of perfect \( q \)-hash codes. \( \square \)

**Remark 2.** Given a \( q \)-ary code \( C \) of length \( n \), Lemma 2.3 tells us there must exist an outer code whose concatenation with \( C \) yields a perfect \( q \)-hash code with rate \( -\frac{1}{m(q-1)} \log_2 (1 - \frac{q!|S(C)|}{m^q}) \). That means we only need to focus on finding good inner codes \( C \) with large subset \( S(C) \). In what follows, when we talk about concatenation, we only specify the inner code. The outer code is always given by Lemma 2.3.

### 3 A new lower bound for \( q \) (mod 4) \( \neq 2 \)

By Lemma 2.3 to have a good lower bound on \( R_q \), one needs to find a \( q \)-ary inner code \( C \) of length \( n \) such that \( S(C) \) has large size for fixed \( q, n \) and size \( |C| \). However, determining (or even estimating) the size of \( S(C) \) for a given inner code \( C \) with dimension at least 2 seems very difficult. In this section, we estimate the size of \( S(C) \) for some classes of codes and show that these inner codes give lower bounds on \( R_q \) better than the probabilistic lower bound (4).
3.1 Lower bounds from linear codes

To overcome the problem of estimating the size of $S(C)$, we resort to linearity and dual distance of linear codes and narrow our target to linear codes with simple structure. In this subsection, we investigate a promising candidate for the inner code.

Let us recall some facts about linear codes. Let $q$ be a prime power and let $C$ be a $q$-ary $[n,k]$-linear code. A subset $I$ of $[n]$ of size $k$ is called an information set of $C$ if every codeword $c \in C$ is uniquely determined by $c_I$, where $c_I$ is the projection of $c$ at $I$. In other words, let $G$ be a generator of $C$, then a subset $I$ of $[n]$ of size $k$ is an information set of $C$ if and only if $G_I$ is a $k \times k$ invertible matrix, where $G_I$ is the submatrix of $G$ consisting of those columns of $G$ indexed by $i \in I$.

**Lemma 3.1.** Let $C$ be a $q$-ary $[n,k]$-linear code with dual distance $d^\perp$. Then for any subset $J$ of $[n]$ with $|J| \leq d^\perp - 1$, there exists an information set $I$ such that $J \subseteq I$.

**Proof.** Let $G$ be a generator of $C$. As $C$ has dual distance $d^\perp$, any $d^\perp - 1$ columns of $G$ are linearly independent. Thus, the submatrix $G_J$ has rank $|J|$. Hence, one can find a subset $I$ of $[n]$ of size $k$ such that $J \subseteq I$ and $G_I$ has rank $k$. The proof is completed.

Let $F$ be a finite set of $q$ elements. Let $C$ be a $q$-ary code over $F$ of length $n$. For each $i \in [n]$, define the set

$$A_i = \{\{c_1, c_2, \ldots, c_q\} \subseteq C : \{\text{proj}_i(c_1), \ldots, \text{proj}_i(c_q)\} = F\}.$$  \hspace{1cm} (7)

Thus, we have $S(C) = \bigcup_{i=1}^n A_i$.

For any subset $T$ of $[n]$, we denote by $A_T$ the set $\cap_{i \in T} A_i$. Let $A_i$ denote the number

$$A_i = \sum_{T \subseteq [n], |T| = i} |A_T|.$$  \hspace{1cm} (8)

**Lemma 3.2.** Let $C$ be a $q$-ary $[n,k]$-linear code with dual distance $d^\perp$. Then

$$|S(C)| = \sum_{i=1}^{d^\perp - 1} (-1)^{i-1} \binom{n}{i} q^{n(i-k)} (q^i)^{k-1} + \sum_{i=d^\perp}^n (-1)^{i-1} A_i.$$  \hspace{1cm} (9)

**Proof.** First we claim that for any $j \in [d^\perp - 1]$ and subset $J$ of $[n]$ with $|J| = j$, we have $|A_J| = q^{(k-j-1)(q^j)^{j-1}}$.

By Lemma 3.1, we can choose an information set $I \subseteq [n]$ that includes $J$. For any matrix $M$ in $F_{q^{n-k}}^{q \times k}$, by the definition of the information set, it suffices to determine $M_I$ so as to fix $M$. Since $I$ is an information set, there is a unique $q$-tuple $(c_1, c_2, \ldots, c_q)$ such that

$$M = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_q \end{pmatrix}_I.$$  \hspace{1cm} (10)
It is clear that \(\{c_1, c_2, \ldots, c_q\} \in \mathcal{A}_j\) if and only if every column of \(M_j\) is a permutation of \((0, \ldots, q-1)\). There are \((q!)^{\lfloor j \rfloor} = \binom{q}{j}\) ways to pick \(M_j\) and \(q^{\lfloor |J| - |J| \rfloor} = q^{q(k-j)}\) ways to pick \(M_{j-1}\). This gives \((q!)^{j} q^{q(k-j)}\) different \(q\)-tuples \((c_1, c_2, \ldots, c_q)\) with \(\{c_1, c_2, \ldots, c_q\} \in \mathcal{A}_j\). It follows that the number of \(q\)-element sets in \(\mathcal{A}_j\) is \((q!)^{j-1} q^{q(k-j)}\).

By the inclusion-exclusion principle, we have

\[
|\mathcal{S}(C)| = \bigcup_{i=1}^{n} A_i = \sum_{i=1}^{d-1} (-1)^{i-1} \binom{n}{i} q^{q(k-i)}(q!)^{i-1} + \sum_{i=d-1}^{n} (-1)^{i-1} A_i.
\]

This completes the proof.

By the equality (11), we have

\[
|\mathcal{S}(C)| = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} q^{q(k-i)}(q!)^{i-1} - \sum_{i=d-1}^{n} (-1)^{i-1} \binom{n}{i} q^{q(k-i)}(q!)^{i-1} + \sum_{i=d-1}^{n} (-1)^{i-1} A_i
\]

\[
= \frac{-q^{qk}}{q!q^{qn}} (-q^{q^n} + (q^q - q!)^n) - \sum_{i=d-1}^{n} (-1)^{i-1} \binom{n}{i} q^{q(k-i)}(q!)^{i-1} + \sum_{i=d-1}^{n} (-1)^{i-1} A_i
\]

\[
= \frac{q^{qk}}{q!} \left(1 - \left(1 - \frac{q!}{q^q}\right)^n\right) - \sum_{i=d-1}^{n} (-1)^{i-1} \binom{n}{i} q^{q(k-i)}(q!)^{i-1} + \sum_{i=d-1}^{n} (-1)^{i-1} A_i.
\]

Thus, we have

\[
1 - \frac{q!}{q^{qn}} |\mathcal{S}(C)| = \left(1 - \frac{q!}{q^q}\right)^n + \sum_{i=d-1}^{n} (-1)^{i-1} \binom{n}{i} \left(\frac{q!}{q^q}\right)^i - \frac{q!}{q^{qn}} \sum_{i=d-1}^{n} (-1)^{i-1} A_i.
\]

Hence, in order to beat the probabilistic lower bound, we need to verify the following inequality for an inner code \(C = [n,k]_q\).

\[
\sum_{i=d-1}^{n} (-1)^{i-1} \binom{n}{i} \left(\frac{q!}{q^q}\right)^i < \frac{q!}{q^{qn}} \sum_{i=d-1}^{n} (-1)^{i-1} A_i
\] (11)

Lemma 3.2 shows that computing \(|\mathcal{S}(C)|\) is reduced to computing \(A_i\) for \(i = d-1, d-1 + 1, \ldots, n\). However, if \(d-1\) is too far from \(n\), we have to compute many \(A_i\) and this is rather difficult. The simplest case is \(d-1 = n\) where we need to compute only \(A_n\). In this case the dimension \(k\) is at least \(n-1\). Therefore, let us consider \([n,n-1] \) MDS codes. On the other hand, when \(C\) has dimension \(n-1\), we do not require that \(q\) is a prime power. Precisely speaking, we have the following result.

**Lemma 3.3.** Let \(q \geq 2\) be an integer and let \(F\) be an abelian group of order \(q\). Define the \(q\)-ary code \(C = \{(x_1, \ldots, x_{n-1}, \sum_{i=1}^{n-1} x_i) : x_1, \ldots, x_{n-1} \in F\}\). Let \(A_n\) denote the cardinality of the set

\[
\{\{c_1, c_2, \ldots, c_q\} \subseteq C : \{\text{proj}_i(c_1), \ldots, \text{proj}_i(c_q)\} = G\} \text{ for any } i \in [n]\}.
\]

Then \(|\mathcal{S}(C)| = \frac{q^{q(n-1)}}{q^{q}} \left(1 - \left(1 - \frac{q!}{q^q}\right)^n\right) - (-1)^{n-1} q^{q(n-1)} - (-1)^{n-1} A_n\).
Proof. One can show that in this case, we have $A_i = q^{q(n-i-1)}(q!)^{i-1}$ for any $1 \leq i \leq n - 1$. The desired result follows from the same arguments as in Lemma 3.2. \hfill \Box

Corollary 3.4. Let $q \geq 2$ be an integer and let $F$ be an abelian group of order $q$. Let $A_n$ be the number given in Lemma 3.3. If

$$(-1)^{n-1}A_n > (-1)^{n-1}\frac{(q!)^{n-1}}{q^n},$$

Then there exist families of perfect $q$-hash codes over $\mathbb{F}_q$ with rate better than the probabilistic lower bound (4).

Proof. Let $C$ be the $q$-ary code defined in Lemma 3.3. Then

$$1 - \frac{q^{|\mathcal{S}(C)|}}{q^{q(n-1)}} = \left(1 - \frac{q^1}{q^1}\right)^n + (-1)^{n-1}\left(\frac{q^1}{q^1}\right)^n - (-1)^{n-1}\frac{q^1}{q^{q(n-1)}}A_n < \left(1 - \frac{q^1}{q^1}\right)^n.$$

The desired result follows. \hfill \Box

If $C$ is the code of length 3 over $\mathbb{Z}_q$ given in Lemma 3.3 i.e., $C = \{(x, y, x+y) : x, y \in \mathbb{Z}_q\}$, then determining $A_3$ given in Lemma 3.3 is actually reduced to the following well-known combinatorial problem: determining the number $s_q$ of pairs $(\pi_1, \pi_2)$ of bijections $[q] \to \mathbb{Z}_q$ such that $\pi_1 + \pi_2$ is a bijection as well. The relation between $A_3$ and $s_q$ is $A_3 = \frac{s_q}{q^3}$.

The number $s_q$ has been studied somewhat extensively, but under a different guise 3, 5, 4, 17, 15. It is in general very difficult to determine the exact value of $s_q$ unless $q$ is an even number for which $s_q = 0$. It has been conjectured in 17 that there exists two positive constant $c_1$ and $c_2$ such that $c_1(q!)^2 < s_q < c_2(q!)^2$ for all odd $q$. Various upper bounds are given 15. To beat the probabilistic lower bound on $R_q$, we want to show $s_p > \left(\frac{2}{q^2}\right)$. That means, we are only interested in the lower bounds on $s_q$. A generic lower bound is $s_q \geq 3.246^q \times q!$ for all odd $q$. However, there is still a very big gap between this lower bound and the aforementioned conjecture. For sufficiently large $q$, we actually has some tight lower bound for $s_q$, we postpone this discussion to the next subsection. On the other hands, there are various algorithms to numerically approximate $s_q$ 14.

By taking exact value of $s_q$ for all odd $q$ between 3 and 25 from 14, we obtain the following result.

Corollary 3.5. There exists a family of perfect $q$-hash codes over $\mathbb{Z}_q$ with rate better than the probabilistic lower bound (4) for all odd $q$ between 3 and 25.

Proof. By Corollary 3.4 it is sufficient to verify the inequality

$$\frac{s_q}{q!} > \frac{(q!)^2}{q^q}$$

for all odd $q$ between 3 and 25. Taking the values of $s_q$ from Table I of 14 gives the desired claim. \hfill \Box

Remark 3. For completeness, we list the values of $A_3 = \frac{s_q}{q^3}$ and $\frac{(q!)^2}{q^q}$ for odd $q \in [3, 25]$ in Table I.

We observe that the ratio $A_3$ over $\frac{(q!)^2}{q^q}$ grows slowly but monotonically. In fact, we will see that this ratio is asymptotically equal to $\frac{s_q}{\sqrt{q}}$ in our following discussion.
Remark 4. In literature, various algorithms were proposed to compute $s_q$ for large odd $q$. Using these algorithms, for many odd $q$ in the interval $[27, 155]$, estimation on $s_q$ is given in [14]. One can verify from these estimation that the probabilistic lower bound (4) is improved for all odd integers $q$ for which available values of $s_q$ are given in [14].

For even $q$, we have $s_q = 0$. Therefore, we cannot use the codes defined in Lemma 3.3. Instead, we can replace $\mathbb{Z}_q$ by $\mathbb{F}_q$ if $q$ is a prime power.

Corollary 3.6. There exists a family of perfect $q$-hash code over $\mathbb{F}_q$ with rate better than the probabilistic lower bound (4) for $q = 4, 8, 9, 12$. Furthermore, the lower bound on $R_9$ given here is better than that in Corollary 3.5 and the probabilistic lower bound.

Proof. Let $C$ be a code with the form

$$C = \{(x, y, x + y) : x, y \in \mathbb{F}_q\}.$$ 

Let $A_3$ be defined in (8). With the help of computer search, we get the values $A_3$ of $C$: 8 for code over $\mathbb{F}_4$, 384 for code over $\mathbb{F}_8$ and 2241 for code over $\mathbb{F}_9$, respectively. We note the fact that $A_3$ from the code over $\mathbb{F}_9$ is 2241, while $A_3$ from the code over $\mathbb{Z}_9$ is 2025. It is straightforward to verify that the inequality $A_3 > \frac{(q!)^2}{q^3}$ is satisfied for $q = 4, 8$ and 9. For $q = 12$, we consider the code $C = \{(x, y, x + y) : x, y \in \mathbb{F}_4 \times \mathbb{F}_3\}$. Clearly, $C$ is still an MDS code. Moreover, we have $A_3 = 198144 > \frac{(12!)^2}{12^3}$. □

Remark 5. The lower bound on $R_3$ given in [13] is $R_3 \geq \frac{1}{4} \log_2 \frac{9}{7}$. Let $C$ be a ternary $[4, 2]$-MDS code. The computer search shows that $|S(C)| = 84$. By Lemma 2.3, we also obtain the same lower bound $R_3 \geq \frac{1}{3} \log_2 \frac{9}{7}$.

This remark indicates that $q$-ary MDS codes of larger length sometimes leads to a better lower bound on $R_q$ than $q$-ary MDS codes of length 3 and dimension 2. This is further confirmed by the following example for $q = 4$.

Corollary 3.7. There exists a family of perfect 4-hash code over $\mathbb{F}_4$ with rate at least 0.049586. This is better than both the lower bound given in Corollary 3.6 and the probabilistic lower bound.

| $\mathbb{Z}_q$ | $\mathbb{Z}_5$ | $\mathbb{Z}_7$ | $\mathbb{Z}_9$ | $\mathbb{Z}_{11}$ | $\mathbb{Z}_{13}$ | $\mathbb{Z}_{15}$ |
|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $A_3$        | 15          | 133         | 2025        | 37851       | 1.03×10^6   | 3.63×10^7   |
| $\frac{(q!)^2}{q^3}$ | 4.6         | 30.8        | 339.9       | 5584.6      | 1.28×10^5   | 3.90×10^6   |
| Ratio        | 3.26        | 4.32        | 5.96        | 6.78        | 8.04        | 9.30        |
| $\mathbb{Z}_q$ | $\mathbb{Z}_{17}$ | $\mathbb{Z}_{19}$ | $\mathbb{Z}_{21}$ | $\mathbb{Z}_{23}$ | $\mathbb{Z}_{25}$ |
| $A_3$        | 1.60×10^9   | 8.76×10^10  | 5.77×10^{12} | 4.52×10^{14} | 4.16×10^{16} |
| $\frac{(q!)^2}{q^3}$ | 1.52×10^8   | 7.47×10^9   | 4.47×10^{11} | 3.2×10^{13}  | 2.70×10^{15} |
| Ratio        | 10.53       | 11.71       | 12.93       | 14.12       | 15.4        |

Table 1: The comparison between $A_3$ and $\frac{(q!)^2}{q^3}$ for small odd $q$. 

| $\mathbb{Z}_q$ | $\mathbb{Z}_5$ | $\mathbb{Z}_7$ | $\mathbb{Z}_9$ | $\mathbb{Z}_{11}$ | $\mathbb{Z}_{13}$ | $\mathbb{Z}_{15}$ |
|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $A_3$        | 15          | 133         | 2025        | 37851       | 1.03×10^6   | 3.63×10^7   |
| $\frac{(q!)^2}{q^3}$ | 4.6         | 30.8        | 339.9       | 5584.6      | 1.28×10^5   | 3.90×10^6   |
| Ratio        | 3.26        | 4.32        | 5.96        | 6.78        | 8.04        | 9.30        |
| $\mathbb{Z}_q$ | $\mathbb{Z}_{17}$ | $\mathbb{Z}_{19}$ | $\mathbb{Z}_{21}$ | $\mathbb{Z}_{23}$ | $\mathbb{Z}_{25}$ |
| $A_3$        | 1.60×10^9   | 8.76×10^10  | 5.77×10^{12} | 4.52×10^{14} | 4.16×10^{16} |
| $\frac{(q!)^2}{q^3}$ | 1.52×10^8   | 7.47×10^9   | 4.47×10^{11} | 3.2×10^{13}  | 2.70×10^{15} |
| Ratio        | 10.53       | 11.71       | 12.93       | 14.12       | 15.4        |
Proof. Assume $\mathbb{F}_4 = \{0, 1, \alpha, \alpha + 1\}$. Consider a $[5, 2]$-MDS code:

$$C = \{(a, b, a + b, ab + b, a(\alpha + 1) + b) : a, b \in \mathbb{F}_4\}.$$  

By computer search, we find that there are $1100$ out of $\binom{32}{4}$ $4$-element subsets of $C$ that are separated. Plugging it parameters into Lemma 2.3, we obtain perfect $4$-hash code with rate $0.049586$.  

\hfill $\square$

3.2 A new lower bound for big $q \pmod{4} \not\equiv 2$

For some odd integers $q$ large than $25$, there are also some lower bounds on $s_q$ [14]. By these lower bounds, we can verify that the probabilistic lower bound on $R_q$ are improved for odd integers between $27$ and $155$. The computer search can only help for small values of $q$. To lower bound $s_q$ for large $q$, we have to look for a lower bound with rigorous mathematical proof. Fortunately, a recent progress on asymptotic behavior of $s_q$ is given in [15]. Recall that there is a conjecture saying that, for all odd $q$, the number $s_q$ lies in between $c_1^q n^2$ and $c_2^q n^2$ for some constants $c_1, c_2$. This conjecture is recently confirmed in [15]. Moreover, they even close the gap by showing $c_1 = c_2 = \frac{1}{\sqrt{e}} + o(1)$.

Proposition 3.8 ([15]). Let $q$ be an odd integer. Then, the number $s_q$ is $(\frac{1}{\sqrt{e}} + o(1))\frac{q^3}{q^2 - 1}$, and hence $A_3$ defined in Lemma 3.2 is $(\frac{1}{\sqrt{e}} + o(1))\frac{q^2}{q^2 - 1}$.

Plugging $A_3$ in Proposition 3.8 into (9) and (6) gives the following theorem.

Theorem 3.9. For every odd integer $q$, one has

$$R_q \geqslant -\frac{1}{3(q - 1)} \log_2 \left( 1 - 3 \frac{q^4}{q^2} + 3 \frac{(q!)^2}{q^q} - \left( \frac{1}{\sqrt{e}} + o(1) \right) \frac{(q!)^3}{q^{q - 1}} \right).$$

Moreover, for every sufficiently large odd $q$ this rate is bigger than that given by the probabilistic lower bound.

Proof. It remains to compare this rate with (3). It suffices to show that $A_3 > \frac{(q!)^2}{q^q}$. For large odd $q$, this inequality is reduced to prove $\left( \frac{1}{\sqrt{e}} + o(1) \right) \frac{(q!)^3}{q^{q - 1}} > \frac{(q!)^3}{q^{q - 1}}$. This holds as $\frac{1}{\sqrt{e}} + o(1) > \frac{1}{q}$ for sufficiently large $q$.  

As $s_q = 0$ for even $q$, we have to replace group $\mathbb{Z}_q$ by other abelian groups of order $q$. Recently, Eberhard [8] extends Proposition 3.8 to any abelian group $F$ with $\sum_{x \in F} x = 0$ and size $q$. In fact, they prove an even more general result.

Proposition 3.10 ([8]). Let $F$ be an abelian group of size $q$ and $f$ is a function from $[q]$ to $\mathbb{F}$ such that $\sum_{i=1}^{q} f(i) = \sum_{x \in F} x$. Let $S$ be the collection of bijections that maps $[q]$ to $\mathbb{F}$. Then, the set of $\{(\pi_1, \pi_2, \pi_3) \in S^3 : \pi_1 + \pi_2 + \pi_3 = f\}$ is of size $\left( \frac{1}{\sqrt{e}} + o(1) \right)\frac{q^3}{q^q - 1}$.  

Let $F$ be an abelian group of size $q \pmod{4} = 0$ and $f$ be a zero function, i.e., $f(i) = 0$ for all $i \in F$. We have the following corollary.

Corollary 3.11. Let $s_F$ be the number of pairs $(\pi_1, \pi_2)$ of bijections $[q] \rightarrow F$ such that $\pi_1 + \pi_2$ is a bijection as well. Then, $s_F$ is $\left( \frac{1}{\sqrt{e}} + o(1) \right)\frac{q^3}{q^q - 1}$.
Theorem 3.12. For every integer $q$ with $q \pmod 4 = 0$, one has

$$R = -\frac{1}{3(q-1)} \log_2 \left( 1 - 3q! + 3\frac{(q!)^2}{q^q} - \left( \frac{1}{\sqrt{e}} + o(1) \right) \frac{(q!)^3}{q^{q-1}} \right).$$

Moreover, for every sufficiently large $q$ with $q \pmod 4 = 0$, this rate is bigger than that given by the probabilistic lower bound.

Proof. Since $q \pmod 4 = 0$, let $q = 2^r p$ with an odd integer $p$ and $r \geq 2$. Let $F = F_{2^r} \times \mathbb{Z}_p$. It is clear that $F$ is an abelian group and $\sum_{x \in F} x = 0$. Define the code $C := \{(x, y, x+y) : x, y \in F\}$. Then, $C$ is an MDS code with dimension 2 and length 3. It remains to bound $A_3$. This is equivalent to counting the pair of bijections $(\pi_1, \pi_2) : [q] \to F$ such that $\pi_1 + \pi_2$ is a bijection as well. Corollary 3.11 says that the number $A_3$ of $C$ is $\frac{3q!}{q^q} = \left( \frac{1}{\sqrt{e}} + o(1) \right) \frac{(q!)^3}{q^{q-1}}$. Plugging $A_3$ into (9) and (6) gives the desired result. \hfill \Box

3.3 A better lower bound

The lower bounds given in Theorems 3.9 and 3.12 make use of linear codes over an abelian group of length 3 and dimension 2. As we have seen, this code does not always give the best lower bound. In this section, we show that a linear code over an abelian group of length 4 and dimension 2 provides a better lower bound than those given in Theorems 3.9 and 3.12.

Lemma 3.13. Let $q \geq 3$ be an odd integer. Consider the code

$$C = \{(x, y, x+y, x-y) : x, y \in \mathbb{Z}_q\}.$$

Then one has

$$|\mathcal{S}(C)| \geq \binom{4}{1} q^q - \binom{4}{2} q! + 3 \frac{3q!}{q^q}.$$

Proof. Similar to the arguments in Lemma 3.2 we have

$$|\mathcal{S}(C)| = \binom{4}{1} q^q - \binom{4}{2} q! + A_3 - A_4,$$

where $A_3$ is the number defined in (3). For any subset $T$ of $[4]$ with $|T| = 3$, we claim that $|A_T| = \frac{3q!}{q^q}$. To prove this claim, let us only consider the case where $T = \{1, 3, 4\}$. Note that $C$ can be rewritten as $C = (2^{-1}(w+z), 2^{-1}(w-z), w, z) : w, z \in \mathbb{Z}_q$. If the third and fourth positions of $\mathbb{Z}_q$ are associated with two permutations $\pi_1$ and $\pi_2$, respectively, then the first position forms a permutation of $\mathbb{Z}_q$ if and only if $2^{-1}(\pi_1 + \pi_2)$ is a permutation of $\mathbb{Z}_q$. This is equivalent to that $\pi_1 + \pi_2$ is a permutation of $\mathbb{Z}_q$. Hence, we have $|A_T| = \frac{3q!}{q^q}$. We can similarly prove the claim for other three cases.

Hence, we have $A_3 = 4 \frac{3q!}{q^q}$. As we have $A_4 \leq |A_{[3]}| = \frac{3q!}{q^q}$, the desired result follows. \hfill \Box

Theorem 3.14. For any odd integer $q \geq 3$, one has

$$R_q \geq -\frac{1}{4(q-1)} \log_2 \left( \left( 1 - \frac{q!}{q^q} \right)^4 - \left( \frac{3q!}{\sqrt{e}} + o(q) \right) \left( \frac{q!}{q^q} \right)^3 \right).$$

Moreover, for every sufficiently large odd $q$, this rate is bigger than that given in Theorem 3.9.
Proof. Let $C$ be the $q$-ary code given in Lemma 3.13. Then we have

$$1 - \frac{q! |S(C)|}{|C|^q} \leq 1 - \left( \frac{4}{q^q} + \left( \frac{q!}{q^q} \right)^2 \right) \frac{3q}{q^q} = \left( 1 - \frac{q^4}{q^q} \right) - \left( \frac{q^4}{q^q} + o(q) \right) \left( \frac{q^3}{q^4} \right) . \ (14)$$

The first claim is proved. To prove the second claim, it is sufficient to show that

$$\left( 1 - \frac{q^4}{q^q} \right)^4 - \left( \frac{3q}{q^q} + o(q) \right) \left( \frac{q^3}{q^q} \right) < \left( 1 - \frac{q^4}{q^q} \right)^3 - \left( \frac{q^4}{q^q} + o(q) \right) \left( \frac{q^3}{q^q} \right) , \ (15)$$

i.e.,

$$\left( 1 - \frac{q^4}{q^q} \right)^4 - \left( \frac{3q}{q^q} + o(q) \right) \left( \frac{q^3}{q^q} \right) < \left( 1 - \frac{q^4}{q^q} \right)^3 - \left( \frac{q^4}{q^q} + o(q) \right) \left( \frac{q^3}{q^q} \right) . \ (15)$$

The left-hand side of (15) is

$$\left( 1 - \frac{q^4}{q^q} \right)^4 - \left( \frac{9q}{q^q} + o(q) \right) \left( \frac{q^3}{q^q} \right) (1 + o(1)) = \left( 1 - \frac{q^4}{q^q} \right)^4 - \frac{9q}{q^q} \left( \frac{q^3}{q^q} \right) (1 + o(1)). \ (16)$$

Similarly, the right-hand side of (15) is

$$\left( 1 - \frac{q^4}{q^q} \right)^4 - \frac{9q}{q^q} \left( \frac{q^3}{q^q} \right) (1 + o(1)). \ (17)$$

As the number of (16) is less than the number of (17), the second claim follows. \(\square\)

Similar to the case where $q$ is odd, we can also improve the lower bound given in Theorem 3.12 if $q$ is divisible by 4.

**Theorem 3.15.** For any integer $q$ with $q \equiv 0 \pmod{4}$, one has

$$R_q \geq -\frac{1}{4(q-1)} \log_2 \left( \left( 1 - \frac{q^4}{q^q} \right)^4 - \left( \frac{3q}{q^q} + o(q) \right) \left( \frac{q^3}{q^q} \right) \right).$$

Moreover, for every sufficiently large $q$ with $q \equiv 0 \pmod{4}$, this rate is bigger than that given in Theorem 3.12.

**Proof.** Case 1: $q = 2^r$ for some integer $r \geq 2$. Choose an element $\alpha \in \mathbb{F}_q - \mathbb{F}_2$ and consider the code $C = \{(x, y, x + y, x + \alpha y) : x, y \in \mathbb{F}_q\}$. Then as in the proof of Lemma 3.13, one can show that $|S(C)| \geq (\frac{1}{2}) q^q - (\frac{3}{4}) q^q + 3^2 q^q$. By the same arguments in the proof of Theorem 3.14, we obtain the desired result.

Case 2: $q = 2^r p$ for some integer $r \geq 2$ and an odd prime $p \geq 3$. Choose an element $\alpha \in \mathbb{F}_{2^r} - \mathbb{F}_2$ and consider the ring $\mathbb{F}_{2^r} \times \mathbb{Z}_p$. Define the code $C = \{(x, y, x + y, x + (\alpha, -1)y) : x, y \in \mathbb{F}_{2^r} \times \mathbb{Z}_p\}$. $C$ is a $[4, 2]$-MDS code by observing that both $(\alpha, -1)$ and $(\alpha, -1) - (1, 1) = (\alpha - 1, -2)$ are invertible elements in $\mathbb{F}_{2^r} \times \mathbb{Z}_p$. The desired result then follows from the similar arguments in the proofs of Lemma 3.13 and Theorem 3.14. \(\square\)
4 A new lower bound for $q \pmod{4} = 2$

The previous section provides a construction of $q$-perfect hash code for any $q$. However, this construction can only beats the probabilistic lower bound for $q \pmod{4} = 2$. This is because if $\pi_1$ and $\pi_2$ are two bijections from $\mathbb{Z}_q$ with $q \pmod{4} = 2$, the sum $\pi_1 + \pi_2$ is not a bijection. To see this, any bijection $\pi$ satisfies that

$$\sum_{i \in \mathbb{Z}_q} \pi(i) = \sum_{i \in \mathbb{Z}_q} i = (q - 1) \times \frac{q}{2} \bmod q$$

which is not divisible by $q$ when $q \pmod{4} = 2$. However, the sum of two bijections satisfies that

$$\sum_{i \in \mathbb{Z}_q} \left( \pi_1(i) + \pi_2(i) \right) = q(q - 1) = 0 \bmod q.$$

It is clear that $s_q$ is 0 in this case. Therefore, we need to look for another candidate to beat the probabilistic lower bound. In the rest of this section, we assume that $q \pmod{4} = 2$. Let $C = \{(x, y, -(x + y)), x, y \in \mathbb{Z}_q\} \cup \{(x, y, -(x + y) + \frac{q}{4}), x, y \in \mathbb{Z}_q\}$. It is clear that $C$ is the union of two MDS codes with $C_1 = \{(x, y, -(x + y)), x, y \in \mathbb{Z}_q\}$ and $C_2 = C_1 + (0, 0, \frac{q}{4})$. Recall

$$\mathcal{A}_i = \{\{c_1, c_2, \ldots, c_q\} \subseteq C : \{\text{proj}_i(c_1), \ldots, \text{proj}_i(c_q)\} = \mathbb{Z}_q\}.$$  
We want to estimate the size of $S(C) = |\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3|$.

**Lemma 4.1.** Let $C$ be the code and $\mathcal{A}_i$ be the set defined above. Then, we have

$$|S(C)| = 3(2q)^q - 3 \times 2^q(q!) + |A_1 \cap A_2 \cap A_3|. \quad (18)$$

**Proof.** By the inclusion-exclusion principle, we have

$$|S(C)| = \sum_{i=1}^{3} |A_i| - (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|) + |A_1 \cap A_2 \cap A_3|.$$

The first two terms can be calculated precisely. Due to the symmetry and MDS property, it suffices to calculate $|A_1|$ and $|A_1 \cup A_2|$. Note that $C$ is the union of two MDS codes $C_1$ and $C_2$. This means, given any bijection $\pi = (x_1, \ldots, x_q)$ from $[q]$ to $\mathbb{Z}_q$, there are $2q$ codewords $c_i$ in $C$ such that $\text{proj}_i(c_i) = x_i$ for any $i \in [q]$. Note that $x_1, \ldots, x_q$ are all distinct, thus the number of tuples $(c_1, c_2, \ldots, c_q) \in C^q$ such that $(\text{proj}_1(c_1), \ldots, \text{proj}_1(c_q)) = \pi$ is $(2q)^q$. Since there are $q!$ bijections, we conclude that

$$\sum_{i=1}^{3} |A_i| = 3|A_1| = 3 \times \frac{(2q)^q(q!)}{q!} = 3 \times (2q)^q$$

as $S(C)$ is the collection of set instead of tuple. We proceed to calculate $|A_1 \cap A_2|$. Let $\pi_1 = (x_1, \ldots, x_q)$ and $\pi_2 = (y_1, \ldots, y_q)$ be any bijections from $[q]$ to $\mathbb{Z}_q$. Since $C_1$ and $C_2$ are MDS codes of dimension 2, there are exactly two codewords $c_i$, one from $C_1$ and another one from $C_2$ such that $(\text{proj}_1(c_i), \text{proj}_2(c_i)) = (x_i, y_i)$ for all $i \in [q]$. Since there are $(q!)^2$ pairs of bijections $(\pi_1, \pi_2)$, we conclude that

$$|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3| = 3|A_1 \cap A_2| = 3 \times \frac{2^q(q!)^2}{q!} = 3 \times 2^q q!$$

The proof is completed. \hfill \square
Plug Equation (18) into the Equation (6), the rate becomes

\[ R_q \geq -\frac{1}{(q-1)3} \log_2 \left( 1 - \frac{q!|S(C)|}{2q^2q^q} \right) \]

\[ = -\frac{1}{(q-1)3} \log_2 \left( 1 - 3 \times \frac{q^l}{q^q} + 3 \times \left( \frac{q^l}{q^q} \right)^2 - \frac{q!|A_1 \cap A_2 \cap A_3|}{2q^{q^q}} \right). \]  

We turn to lower bound the size of \( A_1 \cap A_2 \cap A_3 \).

**Theorem 4.2.** There exists a \( q \)-perfect hash code with rate at least

\[ R_q \geq -\frac{1}{3(q-1)} \log_2 \left( 1 - \frac{q^l}{q^q} + 3 \times \left( \frac{q^l}{q^q} \right)^2 - \frac{1}{2\sqrt{e} + o(1)} \left( \frac{q!}{q^{q-1}} \right)^3 \right). \]

**Proof.** We choose \( C = C_1 \cup C_2 \) as the inner code and the outer code is defined by Lemma 2.3 accordingly. Thanks to Lemma 4.1, it remains to lower bound the size of \( A_1 \cap A_2 \cap A_3 \). Let \( \{c_1, \ldots, c_q\} \) be any set belonging to \( A_1 \cap A_2 \cap A_3 \). Assume \( c_i = (x_i, y_i, z_i) \) and we have that \( \pi_1 := (x_1, \ldots, x_q), \pi_2 := (y_1, \ldots, y_q), \pi_3 := (z_1, \ldots, z_q) \) are three bijections by the definition of \( A_1 \cap A_2 \cap A_3 \). Assume that there are \( \ell \) codewords of \( \{c_1, \ldots, c_q\} \) from \( C_1 \) and \( q - \ell \) from \( C_2 \).

Without loss of generality, we assume \( c_1, c_2, \ldots, c_\ell \in C_1 \) and \( c_{\ell+1}, \ldots, c_q \in C_2 \). By the definition of \( C_1 \) and \( C_2 \), we have \( \pi_1(i) + \pi_2(i) + \pi_3(i) = 0 \) for \( i = 1, \ldots, \ell \) and \( \pi_1(i) + \pi_2(i) + \pi_3(i) = \frac{q}{2} \) for \( i = \ell + 1, \ldots, q \). Let \( f \) be a map from \([q]\) to \( Z_q \) such that \( f(i) = 0 \) for \( i = 1, \ldots, \ell \) and \( f(i) = \frac{q}{2} \) for \( i = \ell + 1, \ldots, q \). If \( \ell \) is odd number, then \( \sum_{i\in[q]} f(i) = \frac{q}{2} = \sum_{x\in Z_q} x \). By Proposition 3.10 when \( \ell \) is odd number, the number of triples of bijections \((\pi_1, \pi_2, \pi_3)\) with \( \pi_1 + \pi_2 + \pi_3 = f \) is \( \left( \frac{q!}{q^{q-1}} \right)^3 \). Since there are \( \frac{q}{2} \) ways to choose a subset of codewords of size \( \ell \) from \( C_1 \), the number of codewords \( \{c_1, \ldots, c_q\} \) belonging to \( A_1 \cap A_2 \cap A_3 \) is at least

\[ \sum_{i=0}^{\frac{q-2}{2i+1}} \left( \frac{q^i}{q^{q-1}} \right)^3 \times \frac{1}{\sqrt{e} + o(1)} \frac{(q!)^3 q^{i+1}}{q^q} = \frac{1}{\sqrt{e} + o(1)} \frac{2^{q-1}(q!)^2}{q^{q-1}}. \]

Plug this value into Equation (19) yields the desired result.

**Remark 6.** The probabilistic lower bound \( 41 \) can be written as

\[ -\frac{1}{(q-1)3} \log_2 \left( 1 - \frac{q^l}{q^q} + 3 \times \left( \frac{q^l}{q^q} \right)^2 - \frac{(q!)^3}{q^{q^q}} \right). \]

It is clear that the lower bound given by Theorem 4.2 is better as

\[ \frac{q}{2\sqrt{e}} \times \frac{(q!)^3}{q^{q^q}} > \frac{(q!)^3}{q^{q^q}}. \]

We note that our construction can be applied for any \( q \) (mod 4) = 2. For small \( q \), we do the calculation with the help of the computer. Our numerical result shows that our construction beats the probabilistic lower bound for \( q = 6, 10, 14 \). For large \( q \), we believe that such trend should keep as well. In conclusion, this construction is a very promising candidate to beat the probabilistic lower bound for all \( q \) (mod 4) = 2.

**Theorem 4.3.** There exists a 6-perfect hash code with rate 0.004488, a 10-perfect hash code with rate 5.8180030 \times 10^{-5} and a 14-perfect hash code with rate 8.7066030151 \times 10^{-7} while the probabilistic lower bound yields a 6-perfect hash code with rate 0.004487, a 10-perfect hash code with rate 5.8180021 \times 10^{-5} and a 14-perfect hash code with rate 8.706603140 \times 10^{-7}.
5 Lower bounds on $R_5$ and $R_7$

In previous section, we make use of linear codes $C$ and estimate the size $|S(C)|$ either numerically or asymptotically. However, linear codes do not always give the best lower bound on $R_q$. In this section we present a class of nonlinear inner code $C$ where many $q$-element subsets are separated.

**Lemma 5.1.** Assume $q$ is a prime. There exists a code $C$ over $\mathbb{Z}_q$ with length $q$ and size $2q$ such that $|S(C)| = 2^q q - 2(q - 1)$.

**Proof.** Let $C_1 = \{c_1 = (0, 1, \ldots, q - 1), c_2 = (1, 2, \ldots, q - 1, 0), \ldots, c_q = (q - 1, 0, \ldots, q - 2)\}$, i.e., $C_1$ consists of the codeword $(0, 1, \ldots, q - 1)$ and its $i$th shifts for $i = 1, \ldots, q - 1$. Let $C_2 = \{i \cdot 1 : 0 \leq i \leq q - 1\}$, where $1$ stands for all-one vector of length $q$. Let $C = C_1 \cup C_2$. Obviously, $C$ has length $q$ and size $2q$. It remains to show that $|S(C)| = 2^q q - 2(q - 1)$.

We pick any $0 < i < q$ codewords $c_1, \ldots, c_i$ from $C_1$. Denote by $c_j = (c_{j,1}, \ldots, c_{j,q})$ for $j \in [q]$. For coordinate $t \in [q]$, let $B_t := \{c_{1,t}, c_{2,t}, \ldots, c_{i,t}\}$ be the collection of $t$-th components of $c_1, \ldots, c_i$. It is clear that $|B_t| = i$ by observing that all codewords in $C_1$ have distinct values on each coordinate. Moreover, we will prove that $B_1, \ldots, B_q$ are distinct if $0 < i < q$. Assume not and we have $B_1 = B_a$ for some $a \in [q]$. The structure of code $C_1$ tells us that $c_{j,a} = c_{j,1} + a - 1$ for $j = 1, \ldots, i$. This coupled with $B_1 = B_a$ implies that $c_{1,1}, c_{1,1} + a - 1$ both belong to $B_1$. Continue this argument and we finally arrive at $c_{1,1}, c_{1,1} + a - 1, \ldots, c_{1,1} + (q - 1)(a - 1) \subseteq B_1$. It is clear that $c_{1,1}, c_{1,1} + a - 1, \ldots, c_{1,1} + (q - 1)(a - 1)$ are distinct which contradicts our assumption that $|B_t| = i < q$.

Now, we know that $B_1, \ldots, B_q$ are distinct. For each set $B_t = \{c_{1,t}, c_{2,t}, \ldots, c_{i,t}\}$, we choose a $(q - i)$-element set $A_t := \{i : i \notin B_t\} \subseteq C_2$. It is clear that $c_1, \ldots, c_i$ and the codewords in $B$ have distinct symbols on $i$-th coordinate. Moreover, for each value $t$, the set $A_t$ is distinct due to the fact that $B_1, \ldots, B_q$ are distinct. That means, for any $0 < i < q$-element set of $C_1$, we could obtain $q$ distinct $q$-element sets of $C$ that are separated. If $i = 0$ or $i = q$, it is clear that the only $q$-element sets that are separated is $C_1$ or $C_2$. Thus, the total number of $q$-element sets of $C$ that are separated is $\sum_{i=1}^{q-1} q(q) + 2 = 2q q - 2(q - 1)$. \hfill \Box

Combined this construction with Lemma 2.3 gives following lower bounds on $R_q$ for $q = 5$ and 7.

**Corollary 5.2.** One has $R_5 \geq 0.01452$ and $R_7 \geq 0.001483$. Furthermore, the lower bounds on $R_5$ and $R_7$ given in this corollary are better than those in Corollary 3.5 and the probabilistic lower bound.

**Proof.** Take the inner code to be the code in Lemma 5.1 for $q = 5$ and 7, respectively. The desired result follows from Lemma 5.1 and 2.3. \hfill \Box

Let us end this section by tabulating our best lower bound, denoted by $R_{new}$, obtained in this paper and the probabilistic lower bound denoted by $R_{ran}$ for some small $q$. We omit cases for $q \geq 12$. 


| $q$ | 4     | 5     | 6     | 7     |
|-----|-------|-------|-------|-------|
| $R_{\text{new}}$ | 0.0495 | 0.01452 | 0.004488 | 0.001483 |
| $R_{\text{ran}}$ | 0.0473 | 0.01412 | 0.004477 | 0.001476 |

| $q$ | 8     | 9     | 10    | 11    |
|-----|-------|-------|-------|-------|
| $R_{\text{new}}$ | $4.95909 \times 10^{-4}$ | $1.689931 \times 10^{-4}$ | $5.8180030 \times 10^{-5}$ | $2.01855746 \times 10^{-5}$ |
| $R_{\text{ran}}$ | $4.95905 \times 10^{-4}$ | $1.689929 \times 10^{-4}$ | $5.8180021 \times 10^{-5}$ | $2.01855739 \times 10^{-5}$ |

Table 2: New lower bounds versus the probabilistic lower bounds

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