Planar Yang-Mills theory: Hamiltonian, regulators and mass gap

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Abstract
We carry out the Hamiltonian analysis of non-Abelian gauge theories in (2+1) dimensions in a gauge-invariant matrix parametrization of the fields. A detailed discussion of regularization issues and the construction of the renormalized Laplace operator on the configuration space, which is proportional to the kinetic energy, are given. The origin of the mass gap is analyzed and the lowest eigenstates of the kinetic energy are explicitly obtained; these have zero charge and exhibit a mass gap. The nature of the corrections due to the potential energy, the possibility of an improved perturbation theory and a Schrodinger-like equation for the states are also discussed.
1. Introduction

Non-Abelian gauge theories are central to our current understanding of physical phenomena. The perturbative analysis of such theories is fairly well understood by now, having been extensively developed over the last three decades. Many of the nonperturbative aspects are also more or less understood at a qualitative level. However, it is fair to say that, as of now, we do not have calculational techniques or detailed understanding regarding nonperturbative phenomena in non-Abelian gauge theories, eventhough there has recently been significant progress regarding the nonperturbative aspects of supersymmetric gauge theories [1]. Recently we have analyzed Yang-Mills theories in two spatial dimensions, in particular the question of how the mass gap could be generated [2]. The motivation for considering the case of two spatial dimensions is that it may capture some features of the more realistic case of three dimensions (in this connection, see also refs.[3]), yet, at the same time, it would be mathematically simpler to analyze since there are many known exact results about two-dimensional field theories. Indeed, in our approach, in a Hamiltonian analysis we are able to use a number of results from two-dimensional conformal field theory. An additional motivation is that there is at least one interesting physical situation, viz., the high temperature phase of Chromodynamics and associated magnetic screening effects, to which the (2+1)-dimensional theory can be directly applied.

Our approach was to carry out a Hamiltonian analysis utilising the geometrical properties of the space of gauge-invariant field configurations $C$. This configuration space $C$ is infinite-dimensional and the construction of a metric, volume element, Laplacian, etc. requires appropriate regularization. Regulators were used in arriving at the various results presented in [2], although not all calculations were done within a single regularization scheme. In this paper, regularization issues are treated in much greater detail; all calculations are done with essentially the same regulator eliminating the possibilities of conflicts among different regulators used for different calculations. The basic results are, of course, unchanged. We also carry out the construction of the first excited eigenstate of the kinetic energy in detail. (This was only briefly indicated in [2].) The nature of the corrections due to the potential energy term is also analyzed. Comparison of some results with the
Abelian case and the possibility of doing an “improved” perturbation theory are additional new results in this paper.

In the next section we give an outline of the main argument. The purpose of this is to identify pieces of the calculations which need more careful regularized treatment. This also serves to give a perspective on the fairly technical regularization issues discussed in the subsequent sections. We introduce the matrix parametrization of fields and obtain the volume element of $\mathcal{C}$ in terms of a Wess-Zumino-Witten (WZW) action for a hermitian matrix. The wavefunctions can be taken as functions of the current of this WZW theory and the arguments from conformal field theory which lead to this conclusion are reviewed. The kinetic energy term of the Hamiltonian is proportional to the Laplacian on the configuration space $\mathcal{C}$. We discuss the construction of this operator and give the arguments for understanding how a mass gap arises.

In section 3 we define the basic regularization scheme. The volume element of $\mathcal{C}$, adjointness properties of certain operators with the Haar measure for hermitian matrices, self-adjointness of the kinetic energy and consistency with the Yang-Mills (YM) equations are analyzed with this regulator. This is a fairly technical section with detailed calculations, justifying many of the steps in our arguments. However, the later sections can be read somewhat independently of this section.

An expression for the kinetic energy is obtained as an operator on functionals of currents in section 4. In section 5 we discuss some aspects of the Abelian theory; this is needed to clarify the interpretation of some of the results in section 6. The construction of the ground state and the first excited state of the kinetic energy operator, utilising the regulated expressions of the previous sections is carried out in section 6. The question of how self-energy subtractions can be done at any chosen scale and some issues related to the choices of energy scales are also discussed. The effects of adding the potential energy term are discussed in section 7.

In section 8, we show that one can do an “improved” perturbation theory where some of the terms in the measure for integration over the configuration space are treated exactly while other terms are expanded perturbatively. We show that this is consistent with the self-adjointness of the Hamiltonian and also incorporates the mass gap. The picture which
emerges is as follows. One has “constituent bosons” which carry non-Abelian charge and behave as massive particles but which are interacting and get bound into states of zero charge. The expansion scheme of section 8 can potentially be used for a systematic analysis of higher excited states. This is currently under investigation.

We conclude with a discussion comparing our results to the electric field representation as well as estimating the significance of the potential energy term.

2. Outline of the main argument

In this section we give an outline of the main argument before zeroing in on specific pieces of the calculations which need more elaborate analysis using regulators.

We shall discuss an $SU(N)$-gauge theory. As is convenient for a Hamiltonian formulation, we shall work in the $A_0 = 0$ gauge. The gauge potentials can be written as $A_i = -it^a A^a_i$, $i = 1, 2$, where $t^a$ are hermitian $(N \times N)$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = if^{abc} t^c$, $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$. $\mathbb{A}$ will denote the set of all gauge potentials $A^a_i$. Gauge transformations act on $A_i$ in the standard way, $A_i \rightarrow A_{i}^{(g)}$, where

$$A_{i}^{(g)} = gA_{i}g^{-1} - \partial_{i}gg^{-1} \quad (2.1)$$

and $g(\vec{x}) \in SU(N)$ The gauge group $\mathbb{G}_*$ is defined by

$$\mathbb{G}_* = \{ \text{set of all } g(\vec{x}) : \mathbb{R}^2 \rightarrow SU(N), \ g \rightarrow 1 \text{ as } |\vec{x}| \rightarrow \infty \} \quad (2.2)$$

The space of gauge-invariant field configurations is

$$\mathbb{C} = \mathbb{A}/\mathbb{G}_* \quad (2.3)$$

The basic strategy adopted in papers [2] was to formulate the calculation, as much as possible, in terms of the geometry of $\mathbb{C}$. (For discussions on the geometry of $\mathbb{C}$, see ref. [4].) Specifically we shall need a metric and volume element on $\mathbb{C}$ and eventually also the Laplace operator $\Delta$ on $\mathbb{C}$. The YM Lagrangian in the $A_0 = 0$ gauge is given by

$$L = \int d^2x \left[ \frac{e^2}{2} \frac{\partial A^a_i}{\partial t} \frac{\partial A^a_i}{\partial t} - \frac{1}{2e^2} B^a B^a \right] \quad (2.4)$$
where \( B^a = \frac{1}{2} \epsilon_{ij} (\partial_i A_j^a - \partial_j A_i^a + f^{abc} A_i^b A_j^c) \). By comparison of the kinetic term with the standard point-particle Lagrangian \( L = \frac{1}{2} g_{\mu\nu} \frac{dq^\mu}{dt} \frac{dq^\nu}{dt} \), we see that the metric on the space of potentials \( A \) which is appropriate for YM theory is

\[
\begin{align*}
ds_A^2 &= \int d^2 x \, \delta A_i^a \delta A_i^a
\end{align*}
\] (2.5)

This is the starting point for calculations on \( C \). A good parametrization of the fields \( A_i^a \) which allows for explicit calculations is a first step in the reduction of this metric to the gauge-invariant configuration space \( C \). (There are many different parametrizations which have been studied; for some other parametrizations, see ref. [5].) We shall combine the spatial coordinates \( x_1, x_2 \) into the complex combinations \( z = x_1 - ix_2, \bar{z} = x_1 + ix_2 \); correspondingly we have \( A \equiv A_z = \frac{1}{2} (A_1 + iA_2), \bar{A} \equiv A_{\bar{z}} = \frac{1}{2} (A_1 - iA_2) = -(A_z)\). The parametrization we use is given by

\[
A_z = -\partial_z MM^{-1}, \quad A_{\bar{z}} = M^{-1} \partial_{\bar{z}} M^\dagger
\] (2.6)

Here \( M, M^\dagger \) are complex \( SL(N, \mathbb{C}) \)-matrices (for an \( SU(N) \)-gauge theory; for group \( G \), \( M, M^\dagger \) belong to \( G^C \), the complexification of \( G \)). Such a parametrization is standard in many discussions of two-dimensional gauge fields. We can define Green’s functions \( G, \bar{G} \) for \( \partial, \bar{\partial} \) by

\[
\begin{align*}
\partial_z G(\vec{x}, \vec{y}) &= \bar{\partial}_{\bar{z}} \bar{G}(\vec{x}, \vec{y}) = \delta^{(2)}(\vec{x} - \vec{y}) \\
G(\vec{x}, \vec{x}^\prime) &= \frac{1}{\pi (z - z')} \quad \bar{G}(\vec{x}, \vec{x}^\prime) = \frac{1}{\pi (\bar{z} - \bar{z}')}
\end{align*}
\] (2.7)

We have chosen the boundary condition \( G, \bar{G} \to 0 \) as \( |\vec{x} - \vec{x}^\prime| \to \infty \). For any gauge potentials \( A, \bar{A} \), one can easily check that a choice of \( M, M^\dagger \) is given by

\[
\begin{align*}
M(\vec{x}) &= 1 - \int G(\vec{x}, \vec{z}_1) A(\vec{z}_1) + \int G(\vec{x}, \vec{z}_1) A(\vec{z}_1) G(\vec{z}_1, \vec{z}_2) A(\vec{z}_2) - ... \\
&= 1 - \int_y D^{-1}(\vec{x}, \vec{y}) A(\vec{y})
\end{align*}
\] (2.8)

\[
\begin{align*}
M^\dagger(\vec{x}) &= 1 - \int_y A(\vec{y}) D^{-1}(\vec{y}, \vec{x})
\end{align*}
\]

Here \( D = \partial + A, \bar{D} = \bar{\partial} + \bar{A} \) are covariant derivatives. (There may be many choices for \( M, M^\dagger \); we shall discuss this a little later.) From the definition (2.6), it is clear that the gauge transformation (2.1) is expressed in terms of \( M, M^\dagger \) by

\[
\begin{align*}
M \to M^{(g)} &= gM, \\
M^\dagger^{(g)} &= M^\dagger g^{-1}
\end{align*}
\] (2.9)
for \( g(\vec{x}) \in SU(N) \). In particular, if we split \( M \) into a unitary part \( U \) and a hermitian part \( \rho \) as \( M = U\rho \), then \( U \) is the ‘gauge part’, so to speak; it can be removed by a gauge transformation and \( \rho \) represents the gauge-invariant degrees of freedom. Alternatively, we can use \( H = M^\dagger M = \rho^2 \) as the gauge-invariant field parametrizing \( \mathcal{C} \). Since \( M \in SL(N,\mathbb{C}) \), \( \rho \), and hence \( H \), belong to \( SL(N,\mathbb{C})/SU(N) \).

In terms of the parametrization (2.6), the metric (2.5) can be written as

\[
\begin{align*}
\nonumber ds^2_{\mathcal{A}} &= \int d^2 x \, \delta A^a_i \delta A^a_i = -8 \int \text{Tr}(\delta A_z \delta A_{\bar{z}}) \\
&= 8 \int \text{Tr} [D(\delta MM^{-1}) \bar{D}(M^\dagger^{-1} \delta M^\dagger)] 
\end{align*}
\]

(2.10)

where the covariant derivatives \( D, \bar{D} \) are in the adjoint representation. Two remarks about this metric are in order. This is a standard Euclidean metric in terms of \( \mathcal{A}'s \) and hence the corresponding volume element \( d\mu(\mathcal{A}) \) for \( \mathcal{A} \) is the standard Euclidean one, i.e.,

\[
d\mu(\mathcal{A}) = [dA d\bar{A}] = \prod_{x,a} dA^a(\vec{x})dA^a(\vec{x}).
\]

Secondly, this is a Kähler metric. Evidently

\[
\begin{align*}
\nonumber ds^2_{\mathcal{A}} &= \delta \mathcal{A} \delta \bar{\mathcal{A}} W \\
W &= -8 \int Tr(A \bar{A}) + f(A) + \bar{f}(\bar{A}) 
\end{align*}
\]

(2.11)

The Kähler potential \( W \) is defined, as usual, only up to the addition of a purely \( \mathcal{A} \)-dependent function \( f(A) \) and a purely \( \bar{\mathcal{A}} \)-dependent function \( \bar{f}(\bar{A}) \). (It is possible to choose \( f, \bar{f} \) such that \( W \) is gauge-invariant which is nice but not particularly relevant to our discussion.)

The matrices \( M, M^\dagger \) are elements of \( SL(N,\mathbb{C}) \) and we have the Cartan-Killing metric for \( SL(N,\mathbb{C}) \), viz.,

\[
ds^2 = 8 \text{Tr}(\delta MM^{-1} M^\dagger^{-1} \delta M^\dagger).
\]

For \( SL(N,\mathbb{C}) \)-valued fields, we thus have

\[
ds^2_{SL(N,\mathbb{C})} = 8 \int \text{Tr}[(\delta MM^{-1})(M^\dagger^{-1} \delta M^\dagger)]
\]

(2.12)

We denote the corresponding volume element, the Haar measure, by \( d\mu(M, M^\dagger) \). From Eq.(2.10) we can see that \( d\mu(\mathcal{A}) = \det(D \bar{D}) d\mu(M, M^\dagger) \).

The volume element for \( SL(N,\mathbb{C}) \) is of the form

\[
dV(M, M^\dagger) \propto \epsilon_{a_1...a_n}(dMM^{-1})_{a_1} \wedge ... \wedge (dMM^{-1})_{a_n} \\
\times \epsilon_{b_1...b_n}(M^\dagger^{-1}dM^\dagger)_{b_1} \wedge ... \wedge (M^\dagger^{-1}dM^\dagger)_{b_n}
\]

(2.13)
where \( n = \dim G = N^2 - 1 \). (We use proportionality relationship, there are some constant numerical factors which are irrelevant for our discussion.) By direct substitution of \( M = U \rho \), Eq.(2.13) becomes

\[
\begin{align*}
&dV(M, M^\dagger) \propto \epsilon_{a_1 \ldots a_n} (d\rho \rho^{-1} + \rho^{-1}d\rho)_{a_1} \wedge \ldots \wedge (d\rho \rho^{-1} + \rho^{-1}d\rho)_{a_n} \\
&\times \epsilon_{b_1 \ldots b_n} (U^{-1}dU)_{b_1} \wedge \ldots \wedge (U^{-1}dU)_{b_n} \\
&\propto \epsilon_{a_1 \ldots a_n} (H^{-1}dH)_{a_1} \wedge \ldots \wedge (H^{-1}dH)_{a_n} d\mu(U)
\end{align*}
\]

Here \( d\mu(U) \) is the Haar measure for \( SU(N) \). Notice that

\[
\epsilon_{a_1 \ldots a_n} (H^{-1}dH)_{a_1} \ldots (H^{-1}dH)_{a_n} = \det r \prod_x d\varphi^a
\]

where \( H^{-1}dH = d\varphi^a r_{ak}(\varphi) t_k \). We parametrize \( H \) in terms of the real parameters \( \varphi^a \).

Upon taking the product over all points, \( d\mu(U) \) gives the volume of \( G \) and thus

\[
\begin{align*}
d\mu(M, M^\dagger) = = \prod_x dV(M, M^\dagger) = \prod \det r [d\varphi] \ vol(G_*)
\end{align*}
\]

\[
= d\mu(H) \ vol(G_*)
\]

\[
d\mu(H) = \prod_x \det r [d\varphi^a]
\]

is the Haar measure for hermitian matrix-valued fields and is the volume element associated with

\[
ds^2 = 2 \int \ Tr(H^{-1} \delta H)^2
\]

(This metric can also be derived directly by “reduction” of Eq.(2.12) to the coset space \( SL(N, \mathbb{C})/SU(N) \).) From Eq.(2.16) we have

\[
\frac{d\mu(M, M^\dagger)}{vol(G_*)} = d\mu(H)
\]

The volume element for \( \mathcal{C} \) is now obtained as

\[
d\mu(\mathcal{C}) = \frac{d\mu(A)}{vol(G_*)} = \frac{[dA_x dA_{\bar{x}}]}{vol(G_*)}
\]

\[
= (\det D_x D_{\bar{x}}) \frac{d\mu(M, M^\dagger)}{vol(G_*)} = (\det D \bar{D}) d\mu(H)
\]

The problem is thus reduced to the calculation of the determinant of the two-dimensional operator \( D \bar{D} \). This is well known [6]. We get

\[
(\det D \bar{D}) = \left[ \frac{\det' \partial \tilde{\phi}}{\int \ d^2x} \right]^{\dim G} \exp [2c_A \ S(H)]
\]
where \( c_A \delta^{ab} = f^{amn} f^{bmn} \) and \( S(H) \) is the Wess-Zumino-Witten (WZW) action for the hermitian matrix field \( H \) given by [7]

\[
S(H) = \frac{1}{2\pi} \int \text{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi} \int \epsilon^{\mu\nu\alpha} \text{Tr}(H^{-1} \partial_{\mu} HH^{-1} \partial_{\nu} HH^{-1} \partial_{\alpha} H) \tag{2.21}
\]

The calculation of the \((\det D\bar{D})\) is most easily done as follows. Defining \( \Gamma = \log \det D\bar{D} \), we find

\[
\frac{\delta \Gamma}{\delta A^a} = -i \text{Tr} \left[ \bar{D}^{-1}(\vec{x}, \vec{y}) T^a \right]_{\vec{y} \to \vec{x}} \tag{2.22}
\]

\((T^a)_{mn} = -if^a_{mn}\) are the generators of the Lie algebra in the adjoint representation. The coincident-point limit of \( \bar{D}^{-1}(\vec{x}, \vec{y}) \) is, of course, singular and needs regularization. Since \( d\mu(C) \) must be gauge-invariant, a gauge-invariant regularization is appropriate here. With a gauge-invariant regulator, as we shall see in the next section,

\[
\text{Tr} \left[ \bar{D}_{\text{reg}}^{-1}(\vec{x}, \vec{y}) T^a \right]_{\vec{y} \to \vec{x}} = \frac{2c_A}{\pi} \text{Tr} \left[ (A - M^{\dagger} \partial M^{\dagger}) t^a \right] \tag{2.23}
\]

Using this result in Eq.(2.22), and with a similar result for the variation of \( \Gamma \) with respect to \( A^a \), and integrating we get \( \Gamma = 2c_A S(H) \).

The calculation in Eq.(2.23) is essentially the anomaly calculation in two dimensions and the result is quite robust; different regulators, such as covariant point-splitting, Pauli-Villars, etc., lead to the same result so long as gauge invariance is preserved.

We now have the result, upto an irrelevant constant factor [8,9],

\[
d\mu(C) = d\mu(H) e^{2c_A S(H)} \tag{2.24}
\]

The inner product for physical states is given by

\[
\langle 1 \mid 2 \rangle = \int d\mu(H) e^{2c_A S(H)} \Psi_1^*(H)\Psi_2(H) \tag{2.25}
\]

This formula shows that all matrix elements in \((2+1)\)-dimensional \( SU(N) \)-gauge theory can be evaluated as correlators of the \( SL(N,\mathbb{C})/SU(N) \)- WZW model. (For a general gauge group \( G \), we will have a \( G^C/G \)-WZW model, \( G^C \) being the complexification of \( G \).)

There is an interesting point of comparison between Eq.(2.24) and two-dimensional Euclidean YM theory. First of all, notice that the “total volume” of \( C \) as given by \( \int d\mu(C) \)
is the partition function for a Euclidean two-dimensional hermitian WZW model. This can be explicitly evaluated as \[8,9\]

\[
\int d\mu(H) \exp [2c_A S(H)] = \left[ \frac{\det' \partial \bar{\partial}}{\int d^2x} \right]^{-\dim G} \quad (2.26)
\]

The “total volume” of \(\mathcal{C}\), so defined, is finite. Indeed, if we retain the determinantal factors from Eq.(2.20), we find that \(\int d\mu(\mathcal{C}) = 1\). (We should keep in mind that there is still a regularization implicit in this statement.)

Since we are dealing only with \(A_x, A_{\bar{x}}\), the gauge-invariant measure \(d\mu(\mathcal{C})\) is identical to what is needed for the functional integral of two-dimensional Euclidean YM theory. The partition function for that theory has been evaluated on arbitrary Riemann surfaces by different methods and is given by \[10\]

\[
Z(g) = \int d\mu(\mathcal{C}) \exp \left[ -\frac{1}{4g^2} \int d^2x \, F^a_{\mu\nu} F^a_{\mu\nu} \right] = \sum_{R} d_R^{2-2G} \exp \left( -\frac{1}{2} g^2 c_R \, A \right) \quad (2.27)
\]

Here \(g\) is the two-dimensional coupling constant, \(G\) is the genus of the Riemann surface and \(A\) is its area. The summation is over all the irreducible representations of \(SU(N)\); \(d_R\) is the dimension and \(c_R\) is the quadratic Casimir of the representation \(R\). Given this result, we may define \(\int d\mu(\mathcal{C})\) as the limit of \(Z(g)\) for \(g \to \infty\). We see that only the identity representation survives on the right hand side of Eq.(2.27) giving \(\int d\mu(\mathcal{C}) = \lim_{g \to \infty} Z(g) = 1\), which is consistent with what we found.

Some properties of the hermitian WZW-model will be relevant to our discussion. These can be obtained by comparison with the \(SU(N)\)-model defined by \(e^{kS(U)}, \ U(\vec{x}) \in SU(N)\). The quantity which corresponds to \(e^{kS(U)}\) for the hermitian model is \(e^{(k+2c_A)S(H)}\). The hermitian analogue of the renormalized level \(\kappa = (k + c_A)\) of the \(SU(N)\)-model is \(-(k + c_A)\). Correlators can be calculated from the Knizhnik-Zamolodchikov equation. Since the latter involves only the renormalized level \(\kappa\), we see that the correlators of the hermitian model (of level \((k+2c_A)\)) can be obtained from the correlators of the \(SU(N)\)-model (of level \(k\)) by the analytic continuation \(\kappa \to -\kappa\). For the \(SU(N)_k\)-model there are the so-called integrable representations whose highest weights are limited by \(k\) (spin \(\leq k/2\)
for $SU(2)$, for example). Correlators involving the nonintegrable representations vanish.

For the hermitian model the corresponding statement is that the correlators involving nonintegrable representations are infinite [8,9].

In our case, $k = 0$, and we have only one integrable representation corresponding to the identity operator (and its current algebra descendents). The matrix elements of the gauge theory being correlators of the hermitian WZW-model, we have the result that all wavefunctions of finite inner product and norm are functions of the current

$$J_a(\vec{x}) = \frac{c_A}{\pi} (\partial H H^{-1})_a(\vec{x}) = \frac{c_A}{\pi} \left[ iM^\dagger_{ab}(\vec{x})A_b(\vec{x}) + (\partial M^\dagger M^\dagger)^{-1}_a(\vec{x}) \right]$$

(2.28)

where $M^\dagger_{ab} = 2\text{Tr}(t^aM^\dagger t^bM^\dagger)^{-1}$ is the adjoint representation of $M^\dagger$. (This restriction from conformal field theory can be evaded if wavefunctions are so chosen as to have a growing exponential which compensates for the factor $e^{2c_A S(H)}$ in (2.25). However such wavefunctions will have infinite expectation values for the potential energy term $\int B^2$ and can therefore be ruled out.)

It is instructive to consider how this infinite value arises for correlators of nonintegrable representations. Consider the four-point function for four $H$’s taken in the fundamental representation, i.e., $(N \times N)$-matrices, for the hermitian model of level number $(k + 2c_A)$.

This is given by [9,11]

$$< H_{i_1j_1}(1)H^{-1}_{i_2j_2}(2)H_{i_3j_3}(3)H^{-1}_{i_4j_4}(4) > = \sum_{p=0,1} \lambda_p (\tilde{M}_p^{ij} \delta_{i_1i_2} \delta_{i_3i_4} + \tilde{M}_p^{ij} \delta_{i_1i_4} \delta_{i_2i_3})$$

$$\times (M_p^{\dagger ij} \delta_{j_1j_2} \delta_{j_3j_4} + M_p^{\dagger ij} \delta_{j_1j_4} \delta_{j_2j_3})$$

(2.29)

$$M_A^p = (z_{13}z_{24})^\frac{N^2 - 1}{N(N + k)} F_A^{(p)}(x), \quad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad z_{ij} \equiv z_i - z_j$$

$$\lambda_0 = 1 \quad \text{and} \quad \lambda_1 = h(-N - k)$$

where

$$h(\kappa) = \frac{1}{N^2} \frac{\Gamma(\frac{N-1}{\kappa})\Gamma(\frac{N+1}{\kappa})\Gamma^2(\frac{k}{\kappa})}{\Gamma(\frac{k+1}{\kappa})\Gamma(\frac{k-1}{\kappa})\Gamma^2(\frac{N}{\kappa})}$$

(2.30)

(We have used the fact that $c_A = N$ for $SU(N)$.) As $k \to 0$, we find $\lambda_1 = \frac{1}{N^2 - 1}$: The chiral blocks $F_A^{(0)}$ are nonsingular as $k \to 0$. The $F_A^{(1)}$ are given by

$$F_A^{(1)} = [x(1-x)]^\frac{1}{N(k+N)} F(-\frac{(N-1)}{k+N}, -\frac{(N+1)}{k+N}, \frac{k}{k+N}, x)$$

$$F_A^{(1)} = -N[x(1-x)]^\frac{1}{N(k+N)} F(-\frac{(N-1)}{k+N}, -\frac{(N+1)}{k+N}, -\frac{N}{k+N}, x)$$

(2.31)
where $F(\alpha, \beta, \gamma, x)$ is the hypergeometric function. The hypergeometric function has simple poles at $\gamma = 0, -1, -2, \ldots$ (which are the same as for the Eulerian gamma function [12]) and so, as $k \to 0$, these chiral blocks become infinite for any value of $x$. Notice that this is not a spacetime singularity, or a regularization problem, it is a singularity in the coupling constant $k$. This is in agreement with our statements since the fundamental representation is nonintegrable for $k \to 0$.

Wavefunctions, as we have argued, are functions of the current (2.28). From the physical point of view of accounting for all the gauge-invariant degrees of freedom this is not a limitation since the Wilson loop operator can be written in terms of the current as

$$W(C) = \text{Tr} \ P \ e^{-\oint C (Adz + \tilde{A}d\bar{z})} = \text{Tr} \ P \ e^{\frac{\pi}{cA}} \oint C J \ (2.32)$$

and, at least in principle, all gauge-invariant functions of $(A, \tilde{A})$ can be constructed from $W(C)$.

We now turn to the construction of the kinetic energy term which is proportional to the Laplacian on $C$. First consider the change of variables from $A, \tilde{A}$ to $M, M^\dagger$. Parametrizing $M, M^\dagger$ in terms of $\theta^a(\vec{x}), \tilde{\theta}^a(\vec{x})$ respectively, we can write

$$M^{-1} \delta M = \delta \theta^a R_{ab}(\theta) t_b, \quad \delta M^\dagger M^{-1} = \delta \tilde{\theta}^a R^*_{ab}(\tilde{\theta}) t_b \quad (2.33)$$

(These define $R_{ab}, R^*_{ab}$.) One can now write the electric field operators as

$$E_k(\vec{x}) = -\frac{i}{2} \frac{\delta}{\delta A_k(\vec{x})} = \frac{i}{2} M^\dagger_{ak}(\vec{x}) \int_y \bar{G}(\vec{x}, \vec{y}) \bar{p}_a(\vec{y})$$

$$\bar{E}_k(\vec{x}) = -\frac{i}{2} \frac{\delta}{\delta \bar{A}_k(\vec{x})} = -\frac{i}{2} M_{ka}(\vec{x}) \int_y G(\vec{x}, \vec{y}) p_a(\vec{y}) \quad (2.34)$$

where $M_{ab} = 2\text{Tr}(t^a M^b M^{-1})$ is the adjoint representation of $M$. $p_a$ is the right-translation operator on $M$ and $\bar{p}_a$ is the left-translation operator on $M^\dagger$, i.e.,

$$[p_a(\vec{x}), M(\vec{y})] = M(\vec{y})(-it_a) \delta(\vec{y} - \vec{x})$$

$$[\bar{p}_a(\vec{x}), M^\dagger(\vec{y})] = (-it_a) M^\dagger(\vec{y}) \delta(\vec{y} - \vec{x}) \quad (2.35)$$

Explicitly

$$p_a(\vec{x}) = -i R^{-1}_{ab}(\vec{x}) \frac{\delta}{\delta \theta^b(\vec{x})}, \quad \bar{p}_a(\vec{x}) = -i R^*_{ab}(\vec{x}) \frac{\delta}{\delta \bar{\theta}^b(\vec{x})} \quad (2.36)$$
The kinetic energy operator is given by

\[
T = -\frac{e^2}{2} \int_x \delta^2 \frac{\delta}{\delta A_k(x)} \delta A_k(x) = \frac{e^2}{2} \int_x K_{ab}(x)(\hat{G}\bar{p}_a)(x)(Gp_b)(x) \tag{2.37}
\]

where \( K_{ab} = M_{ak}^t M_{kb} = 2\text{Tr}(t^a H t^b H^{-1}) \) and \( Gp_b(x) \equiv \int_y G(x, y)p_b(y) \), etc. This expression is still defined on \( \mathcal{A} \). With a splitting \( M = U \rho \) we can write

\[
p_a = (1 + \rho)^{-1}_a (\alpha_b + I_b) \\
\bar{p}_a = [\rho^2(1 + \rho)^{-1}]_a [\alpha_b - [\rho(1 + \rho)^{-1}]_b I_b] \tag{2.38}
\]

where \( \alpha_a \) generates right translations on \( \rho \) and \( I_a \) generates right translations on \( U \), i.e., \( [\alpha_a(x), \rho(y)] = \rho(y)(-it^a)\delta(y - x) \), \( [I_a(x), U(y)] = U(y)(-it^a)\delta(y - x) \) and \( \rho_{ab} = 2\text{Tr}(t^a t^b \rho^{-1}) \). On functions which are gauge-invariant and hence independent of \( \Theta \), \( \rho \) is set to zero. When \( I_a \) is set to zero, one can easily check that

\[
[p_a(x), H(y)] = H(y)(-it^a)\delta(y - x) \\
p_a(x) = -ir_{ab}^{-1} \frac{\delta}{\delta \varphi^b} \\
\bar{p}_a(x) = K_{ab}(x)p_b(x) \tag{2.39}
\]

(\( \bar{p}_a(x) \) generates left-translations on \( H \).) This gives the construction of \( T \) as an operator on functions on \( \mathcal{C} \). The expression (2.37) however needs regularization. It is not manifestly self-adjoint. In checking self-adjointness one encounters singular commutators such as \( [\hat{G}\bar{p}_a(x), K_{ab}(x)] \). Further, \( p_a, \bar{p}_a \) are formally adjoints of each other with respect to the Haar measure \( d\mu(H) \). One needs to check whether this holds in an appropriately regularized version.

\( T \) is proportional to the Laplacian on \( \mathcal{C} \). One may, starting from the metric on \( \mathcal{A} \), directly construct it as well. We have

\[
ds_A^2 = -8 \int \text{Tr}(\delta A \delta \dot{A}) = \int g_{ab}(\bar{x}, y)\delta \theta^a(\bar{x})\delta \bar{\theta}^b(y) + \text{h.c.} \tag{2.40a}
\]

\[
g_{ab}(\bar{x}, y) = 2 \int_{\bar{u}, v} \partial_a [\delta(\bar{u} - \bar{x}) R_{ar} (\bar{u})] M_{kr}(u) M_{sk}^\dagger (\bar{v}) \partial_v [\delta(v - \bar{y}) R_{bs}(\bar{v})] \tag{2.40b}
\]

The Laplacian \( \Delta \) on a complex manifold has the general form

\[
\Delta = g^{-1}(\partial_a g^{\bar{a}a} g \partial_a + \partial_a g^{a \bar{a}} g \partial_{\bar{a}}) \tag{2.41}
\]
where \( g = \det(g_{a\bar{a}}) \). Using the metric components (2.40b) we find

\[
T = -\frac{e^2}{2} \Delta = \frac{e^2}{4} \int_x e^{-2c_A S(H)} \left[ \tilde{G}\bar{p}_a(x)K_{ab}(\bar{x})e^{2c_A S(H)}Gp_b(\bar{x}) + Gp_a(x)K_{ba}(\bar{x})e^{2c_A S(H)}\tilde{G}\bar{p}_b(\bar{x}) \right]
\]

(2.42)

Once again we obtain \( T \) on \( \mathcal{C} \) by setting \( I \)'s to zero in Eq.(2.38) for \( p_a, \bar{p}_a \). This expression has manifest self-adjointness since

\[
\langle 1 | T | 2 \rangle = \frac{e^2}{4} \int d\mu(H) e^{2c_A S(H)} \left[ Gp_a\psi_1 K_{ab}(Gp_b\psi_2) + \bar{G}\bar{p}_a\psi_1 K_{ba}(\bar{G}\bar{p}_b\psi_2) \right]
\]

(2.43)

provided \( p, \bar{p} \) are adjoints of each other with respect to \( d\mu(H) \). Notice that if we attempt to move \( \tilde{G}\bar{p}_a \) through \( K_{ab}e^{2c_A S} \) to the right end we encounter the singular commutator \([\tilde{G}\bar{p}_a(x), K_{ab}(\bar{x})]\). Again expression (2.42) needs to be regularized to show that it is the same as expression (2.37), thereby proving self-adjointness of the latter form of \( T \).

The identity of the two expressions (2.37, 2.42) is obtained if, in terms of metric components, we have \( \partial_{\bar{a}}(g^{\bar{a}a}g) = 0 \), so that all derivatives in Eq.(2.41) can be moved to the right. As we have noted in Eq.(2.11), the metric (2.40a) is a Kähler metric. For a finite-dimensional Kähler metric, \( \partial_{\bar{a}}(g^{\bar{a}a}g) = 0 \), as can be checked directly. Showing the identity of expressions (2.37, 2.42) is thus equivalent to proving \( \partial_{\bar{a}}(g^{\bar{a}a}g) = 0 \) for the infinite-dimensional case we have. (This interpretation in terms of a Kähler property is clearly on \( \mathcal{A} \), i.e., before we set \( I_a \)'s to zero in \( p_a, \bar{p}_a \). There is no reason why \( \mathcal{C} \) should be a Kähler manifold.)

The regularization questions we have isolated so far have to do with the adjointness of \( p_a, \bar{p}_a \) with respect to \( d\mu(H) \) and the equality of expressions (2.37, 2.42). For the sake of completeness, we shall also recheck Eq.(2.23) in the calculation of the determinant in the next section even though this is essentially the anomaly calculation.

As we have discussed before, it suffices to consider wavefunctions which are functions of the current \( J_a \). Therefore, before leaving this section, we shall evaluate the action of \( T \) on \( J_a(\bar{x}) \), which is the simplest case to consider. Using the expression (2.28) for the current in terms of \( M^\dagger \) and \( \bar{A} \), we find

\[
T J_a(\bar{x}) = -\frac{e^2}{2} \int d^2y \frac{\delta^2 J_a(\bar{x})}{\delta A^b(y)\delta A^b(\bar{y})} = \frac{e^2 c_A}{2\pi} M_{am}^\dagger \text{Tr} \left[ T^m \tilde{D}^{-1}(\bar{y}, \bar{x}) \right]_{\bar{y} \to \bar{x}}
\]

(2.44a)

\[
= m J_a(\bar{x})
\]

(2.44b)
We encounter the same expression with the coincident-point limit as in the calculation of \((\det D \bar{D})\). Using the same result as in that calculation, viz., Eq.(2.23), we get the result (2.44b). The validity of this particular result is thus on the same footing as the calculation of the volume element \(d\mu(C)\).

The Hamiltonian \(\mathcal{H} = T + V\) can also be expressed entirely in terms of \(H\) or the current \(J_a\) since the potential energy \(V\) can be written as

\[
V = \int \frac{B^a B^a}{2e^2} = \frac{\pi}{mc_A} \int \bar{\partial} J_a \partial J_a
\]

(2.45)

3. Regularization

We now consider the choice of a regulator. Any regulator we choose must, of course, be gauge-invariant. It should also respect the “holomorphic invariance”. This arises as follows. We have discussed the construction of \(M, M^\dagger\) for given potentials \(A, \bar{A}\). The boundary condition \(G, \bar{G} \to 0\) as \(|\vec{x} - \vec{x}'| \to \infty\) leads to the choice (2.7) for the Green’s functions. Even so, there are many possible choices for \(M, M^\dagger\). Eq.(2.6) may be inverted as

\[
M(\vec{x}) = \bar{V}(\bar{x}) + \int A(\vec{z}_1) M(\vec{z}_1) G(\vec{z}_1, \vec{x}) = \bar{V}(\bar{x}) + \int A(\vec{z}_1) \bar{V}(\bar{z}_1) G(\vec{z}_1, \vec{x}) + \ldots = (1 + \int A(\vec{z}_1) \bar{G}(\vec{z}_1, \vec{x}) + \ldots) \bar{V}(\bar{x})
\]

(3.1)

where \(\bar{G}(\bar{z}, \bar{x}) = \bar{V}(\bar{z}) G(\bar{x}, \bar{x})\bar{V}^{-1}(\bar{x})\). Thus a different choice of the starting point of the iteration, viz., \(\bar{V}(\bar{x})\) rather than 1, corresponds to \(M \to M \bar{V}(\bar{x})\) and \(G(\bar{x}, \bar{y}) \to \bar{V}(\bar{x}) G(\bar{x}, \bar{y}) \bar{V}^{-1}(\bar{y})\). Since we have the same \(A, \bar{A}\), clearly physical results must be insensitive to this redundancy in the parametrization in terms of \(M, M^\dagger\); we must require invariance under \(M \to M \bar{V}(\bar{x})\), \(M^\dagger \to V(x) M^\dagger\), \(H \to V H \bar{V}\) (and \(G(\bar{x}, \bar{y}) \to \bar{V}(\bar{x}) G(\bar{x}, \bar{y}) \bar{V}^{-1}(\bar{y})\), \(\bar{G}(\bar{x}, \bar{y}) \to V(x) \bar{G}(\bar{x}, \bar{y}) V^{-1}(y)\) appropriately). We shall refer to this requirement as holomorphic invariance. (To clarify the notation in the above discussion, \(V, \bar{V}\) depend only on the holomorphic coordinate \(x\) and the antiholomorphic coordinate \(\bar{x}\) respectively while the fields \(M, M^\dagger, A, \bar{A}\) in general depend on both \(x, \bar{x}\), indicated by the vector notation for the arguments.)
Of course there are no antiholomorphic functions \( \bar{V}(\bar{x}) \), except for \( \bar{V} \) being a constant matrix, if we do not allow singularities in \( \bar{V} \) anywhere including spatial infinity. Generally what happens in such a case is that \( H \) can have singularities. The location of these singularities can be changed by the transformation \( H \to VH\bar{V} \). One can eliminate such singularities by defining \( M, M^\dagger \) and hence \( H \) separately on coordinate patches and using (anti) holomorphic matrices as transition functions, i.e., \( M_1 = M_2\bar{V}_{12}, \) etc., or \( H_1 = V_{12}H_2\bar{V}_{12} \) in terms of \( H = M^\dagger \). Since this is an ambiguity of choice of field variables, the wavefunctions must be invariant under this. (The ambiguity in the choice of \( M \) or \( H \) and the need for (anti)holomorphic transition functions are related to the geometry of \( \mathcal{A} \) as a \( \mathcal{G}_s \)-bundle over \( \mathcal{C} \) and the Gribov problem [13]. For a discussion of these issues, see reference [2].)

We can easily check the holomorphic invariance of the various expressions we have. From the Polyakov-Wiegman formula [6]

\[
S(h_1 h_2) = S(h_1) + S(h_2) - \frac{i}{\pi} \int \text{Tr}(\partial h_1 h_1^{-1} h_2^{-1} \partial h_2)
\]

we can see that \( S(V H \bar{V}) = S(H) \); this is the Kac-Moody symmetry of the WZW-model. \( d\mu(H) \) is easily checked to be invariant. When \( M \to M \bar{V}, p_a \to \bar{V}_{ab} p_b \) and, similarly, \( \bar{p}_a \to V_{ab} \bar{p}_b \) for \( M^\dagger \to V M^\dagger \), as may be checked from Eq.(2.35). With the change \( G \to \bar{V} \bar{G} \bar{V}^{-1}, \bar{G} \to V \bar{G} V^{-1} \), we find

\[
(Gp)_a \to V_{ab}(Gp)_b \quad (\bar{Gp})_a \to \bar{V}_{ab}(\bar{Gp})_b
\]

The electric fields of Eq.(2.34) and expressions (2.37, 2.42) for \( T \) are thus seen to be invariant.

Any regulator we choose must preserve this “holomorphic invariance”. We shall use a version of point-splitting which preserves this invariance. Explicitly we take

\[
p_a(\bar{x}) \to (p_{\text{reg}})_a(\bar{x}) = \int_y \sigma(\bar{x}, \bar{y}; \epsilon) \left(K^{-1}(y, \bar{x})K(y, \bar{y})\right)_{ab} p_b(\bar{y})
\]

\[
\bar{p}_a(\bar{x}) \to (\bar{p}_{\text{reg}})_a(\bar{x}) = \int_y \sigma(\bar{x}, \bar{y}; \epsilon) \left(K(x, \bar{y})K^{-1}(y, \bar{y})\right)_{ab} \bar{p}_b(\bar{y})
\]

where

\[
\sigma(\bar{x}, \bar{y}; \epsilon) = \frac{e^{-|\bar{x} - \bar{y}|^2/\epsilon}}{\pi \epsilon}
\]
\( \epsilon \) is the regulator parameter. As \( \epsilon \to 0 \), \( \sigma(\vec{x}, \vec{y}; \epsilon) \to \delta(\vec{x} - \vec{y}) \) and we obtain \( p, \bar{p} \) from the integrals in Eq.(3.4). The factors \( K^{-1}(y, \bar{x})K(y, \bar{y}) \) and \( K(x, y)K^{-1}(y, \bar{y}) \) are needed so that \( p_{\text{reg}}(\vec{x}) \) transforms as \( \bar{V}(\vec{x})p_{\text{reg}}(\vec{x}) \) and \( p_{\text{reg}}(\vec{x}) \) transforms as \( V(\vec{x})p_{\text{reg}}(\vec{x}) \). An expression like \( K(x, \bar{y})K^{-1}(y, \bar{y}) \) may be interpreted by the power series expansion \( \sum \frac{(x-y)^n}{n!}(\partial^n KK^{-1})(\bar{y}) \). All terms in this expansion can be expressed in terms of the current \( J \) and its derivatives. (Similar statements hold for \( K^{-1}(y, \bar{x})K(y, \bar{y}) \)). The regularization is thus purely a function of the currents. The regularization (3.4) leads to

\[
\bar{G}\bar{p}_a \to (\bar{G}\bar{p}_a)_{\text{reg}} = \int_y \bar{G}_{ab}(\vec{x}, \bar{y})\bar{p}_b(\bar{y})
\]

\[
Gp_a \to (Gp_a)_{\text{reg}} = \int_y G_{ab}(\vec{x}, \bar{y})p_b(\bar{y})
\]

where

\[
\bar{G}(\vec{x}, \bar{y}) = \int_u \bar{G}(\vec{x}, \vec{u})\sigma(\vec{u}, \vec{y}; \epsilon)K(u, \bar{y})K^{-1}(y, \bar{y})
\]

\[
G(\vec{x}, \bar{y}) = \int_u G(\vec{x}, \vec{u})\sigma(\vec{u}, \vec{y}; \epsilon)K^{-1}(y, \vec{u})K(y, \bar{y})
\]

Under “holomorphic” transformations \( \mathcal{G} \), \( \bar{G} \) transform the same way as \( G, \bar{G} \), i.e., \( \mathcal{G} \to \bar{V}\mathcal{G}\bar{V}^{-1} \) and \( \bar{G} \to V\mathcal{G}V^{-1} \).

Expanding the \( K \)'s and carrying out the integrations in Eq.(3.7) we find

\[
\mathcal{G}_{ma}(\vec{x}, \bar{y}) = \bar{G}(\vec{x}, \bar{y})[\delta_{ma} - e^{-|\vec{x}-\bar{y}|^2/\epsilon}(K(x, \bar{y})K^{-1}(y, \bar{y}))_{ma}]
\]

\[
\mathcal{G}_{ma}(\vec{x}, \bar{y}) = G(\vec{x}, \bar{y})[\delta_{ma} - e^{-|\vec{x}-\bar{y}|^2/\epsilon}(K^{-1}(y, \vec{x})K(y, \bar{y}))_{ma}]
\]

As \( \epsilon \to 0 \), for finite \( |\vec{x} - \bar{y}| \), \( \mathcal{G}, \bar{G} \to G, \bar{G} \).

a) Calculation of \( \text{Tr}[T^a \bar{D}^{-1}(\bar{y}, \vec{x})] \)

Since \( \bar{D} = \bar{\partial} + \bar{A} = \bar{\partial} + M^{\dagger - 1}\bar{\partial}M^{\dagger} \), we have \( \bar{D}^{-1}(\bar{y}, \vec{x}) = M^{\dagger - 1}(\bar{y})\bar{G}(\bar{y}, \vec{x})M^{\dagger}(\vec{x}) \). Replacing \( G \) by \( \mathcal{G} \) we have

\[
\bar{D}^{-1}(\bar{y}, \vec{x})_{\text{reg}} = M^{\dagger - 1}(\bar{y})\bar{G}(\bar{y}, \vec{x})M^{\dagger}(\vec{x})
\]

As \( \bar{y} \to \vec{x} \), we see by power series expansion,

\[
\bar{G}(\bar{y}, \vec{x}) = \bar{G}(\vec{y}, \vec{x})(1 - e^{-|\vec{x}-\bar{y}|^2/\epsilon}) - e^{-|\vec{x}-\bar{y}|^2/\epsilon}(\partial KK^{-1})(\vec{x}) + \ldots
\]

\[
\bar{G}(\vec{x}, \vec{x}) = -\frac{(\partial KK^{-1})(\vec{x})}{\pi}
\]
Thus
\[ \bar{D}^{-1}(\bar{x}, \bar{\xi})_{\text{reg}} = -\frac{1}{\pi} M^{\dagger -1}(\bar{x})(\partial KK^{-1})M^{\dagger}(\bar{x}) \]
\[ = -\frac{1}{\pi} (M^{\dagger -1} \partial M^{\dagger} + \partial MM^{-1})(\bar{x}) \]
\[ = \frac{1}{\pi} (A - M^{\dagger -1} \partial M^{\dagger})(\bar{x}) \]  
(3.11)

This leads to Eq.(2.23).

b) \( \bar{p} \), \( \bar{\bar{p}} \) as adjoints for \( d\mu(H) \)

We have the representation
\[ \bar{p}_a = -ir^{-1}_a \frac{\delta}{\delta \varphi_k} \quad \quad p_a = -ir^{-1}_a \frac{\delta}{\delta \varphi_k} \]  
(3.12)

By direct partial integration we find
\[ \int d\mu(H) \bar{p}_a \psi_1 \psi_2 = \int d\mu(H) \psi^*_1 \bar{p}_a \psi_2 + \int [d\varphi] O_a \psi_1^* \psi_2 \]  
(3.13a)
\[ O_a = -i \frac{\delta}{\delta \varphi^k(\bar{x})} [r^{*-1}_a(\bar{x}) \det r] \]  
(3.13b)

From the definition \( \delta HH^{-1} = \delta \varphi^a r^{*a}_b \), we have
\[ r^{*-1}_a(\bar{x}) \frac{\delta r^{*-1}_b(\bar{\gamma})}{\delta \varphi^k(\bar{x})} - r^{*-1}_b(\bar{\gamma}) \frac{\delta r^{*-1}_a(\bar{x})}{\delta \varphi^k(\bar{\gamma})} = -if^{abc}_c r^{*-1}_c(\bar{x}) \delta(\bar{x} - \bar{\gamma}) \]  
(3.14)

We want to multiply this by \( r^{*}_a(\bar{x}) \) and take \( \bar{\gamma} \rightarrow \bar{x} \). Regularizing as discussed before we get, with \( \Sigma_{sa}(\bar{\gamma}, \bar{x}) = \sigma(\bar{\gamma}, \bar{x}; \epsilon)(K(\bar{y}, \bar{x})K^{-1}(x, \bar{x}))_{sa} \),
\[ \int \Sigma_{sa}(\bar{y}, \bar{x}) r^{*}_a(\bar{x}) \frac{\delta r^{*-1}_b(\bar{y})}{\delta \varphi^k(\bar{x})} - \int \Sigma_{sa}(\bar{y}, \bar{x}) r^{*}_a(\bar{x}) \frac{\delta r^{*-1}_b(\bar{x})}{\delta \varphi^k(\bar{y})} \]
\[ = -if^{abc}_c r^{*-1}_c(\bar{y}) r^{*}_a(\bar{y}) \Sigma_{sa}(\bar{y}, \bar{y}) \]  
(3.15)

The right hand side is seen to be zero. The first term on the left hand side is the regularized meaning of \( (\delta/\delta \varphi^k(\bar{y})) r^{*-1}_a(\bar{y}) \), viz.,
\[ \int \Sigma_{sa}(\bar{y}, \bar{x}) r^{*}_a(\bar{x}) \frac{\delta r^{*-1}_b(\bar{y})}{\delta \varphi^k(\bar{x})} = \left[ \frac{\delta r^{*-1}_b(\bar{y})}{\delta \varphi^k(\bar{y})} \right]_{\text{reg}} \]  
(3.16)

The second term can be written as
\[ -\int \Sigma_{sa}(\bar{y}, \bar{x}) r^{*}_a(\bar{x}) \frac{\delta r^{*-1}_b(\bar{y})}{\delta \varphi^k(\bar{y})} = -i \int \Sigma_{sa}(\bar{y}, \bar{x}) \bar{p}_b(\bar{y}) r^{*-1}_a(\bar{x}) \]
\[ = -i \text{Tr}[ \bar{p}_b(\bar{y}) r^{*-1}_a ]_{\text{reg}} \]  
(3.17a)
\[ = i \bar{p}_b(\bar{y})(\log \det r^*)_{\text{reg}} \]  
(3.17b)
Eq.(3.17a) gives the regularized meaning of \((\log\det r^*)\). Eq.(3.15) can now be written as

\[
\left( \frac{\delta r^k_{bk} - 1}{\delta \varphi^k(y)} \right)_{\text{reg}} + i \bar{p}_b(y) (\log\det r^* \)_{\text{reg}} = 0 \tag{3.18}
\]

This tells us that \(O_a = 0\) and hence that \(p_a, \bar{p}_a\) are adjoints of each other with the Haar measure for \(H\).

There is another way to obtain the result. The regularized meaning of the measure \(d\mu(H)\) is implicitly given by the formulae for correlators such as \(< J_a(\vec{x}) J_b(\vec{y}) >\). For example,

\[
<J_a(\vec{x}) J_b(\vec{y})> = N \int d\mu(H) e^{2c_A S} J_a(\vec{x}) J_b(\vec{y}) \tag{3.19}
\]

where \(N^{-1} = \int d\mu(H) e^{2c_A S}\). From the Polyakov-Wiegmann formula, Eq.(3.2),

\[
\bar{p}_a(\vec{x}) (2c_A S(H)) = -i \bar{\partial} J_a(\vec{x}) \tag{3.20}
\]

Thus we can write

\[
<J_a(\vec{x}) J_b(\vec{y})> = -i \int d\mu(H) J_a(\vec{x}) \bar{G}(\vec{y}, \vec{z}) (\bar{p}_b(\vec{z}) e^{2c_A S})
\]

\[
= -i \int d\mu(H) e^{2c_A S} \bar{G}(\vec{y}, \vec{z}) (\bar{p}_b(\vec{z}) J_a(\vec{x}))
\]

\[
= - <D^a_{\vec{x}} \bar{G}(\vec{y}, \vec{x})> = - \frac{c_A \delta_{ab}}{\pi} \partial_{\vec{x}} \bar{G}(\vec{y}, \vec{x}) \tag{3.21a}
\]

where we assumed \(p_a = \bar{p}_a^\dagger\) on going to Eq.(3.21a) and used the result \(\bar{p}_b(\vec{z}) J_a(\vec{x}) = -iD^a_{\vec{x}} \delta(\vec{x} - \vec{z}) = -i[\frac{c_A}{\pi} \partial_{\vec{x}} \delta_{ab} + i f_{abc} J_c(\vec{x})] \delta(\vec{x} - \vec{z})\). The result (3.21b) agrees with the standard conformal field theory result for the correlator \(< J_a(\vec{x}) J_b(\vec{y}) >\), which, for example, may be evaluated independently by operator product expansions. This consistency requires \(p_a = \bar{p}_a^\dagger\) for the Haar measure.

c) Kähler property, self-adjointness of \(T\)

The regularized form of the Laplacian can be written as \(\Delta = \Delta_1 + \Delta_2\) where

\[
-\Delta_1 = - \int e^{-\mathcal{F}} \frac{\delta}{\delta \theta_p(\vec{u})} r^{*,-1}_p(\vec{u}) \bar{G}_{ar}(\vec{x}, \vec{u}) K_{ab}(\vec{x}) e^{\mathcal{F}} \bar{G}_{bs}(\vec{x}, \vec{v}) r^{-1}_q(\vec{v}) \frac{\delta}{\delta \theta_q(\vec{v})} \tag{3.22}
\]
\[ e^F = (\det r)e^{2c_A S(H, \epsilon)} \] where \( S(H, \epsilon) \) is the WZW-action plus possibly \( O(\epsilon) \)-terms and \( \Delta_2 \) is given by \( \Delta_1 \) with \( \theta, \bar{\theta} \) exchanged. By using Eq. (3.18) we can rewrite this as

\[
-\Delta_1 = - \int e^{-2c_A S(H, \epsilon)} \tilde{p}_r(\bar{u}) \tilde{G}_{ar}(\bar{x}, \bar{u}) K_{ab}(\bar{x}) e^{2c_A S(H, \epsilon)} \tilde{G}_{bs}(\bar{x}, \bar{v}) p_s(\bar{v})
\]  

(3.23)

We want to move \( \tilde{p}_r(\bar{u}) \) to the right end. In moving \( \tilde{p}_r(\bar{u}) \) through \( \tilde{G}_{ar}(\bar{x}, \bar{u}) \) we encounter the potentially singular term \([\tilde{p}_r(\bar{u}), (K(x, \bar{u}) K^{-1}(u, \bar{u}))_{ar}]\). By writing this as

\[
[\tilde{p}_r(\bar{u}), (K(x, \bar{u}) K^{-1}(u, \bar{u}))_{ar}]_{\text{reg}} = \int_z (K(u, \bar{z}) K^{-1}(\bar{z}))_{rs} \sigma(\bar{u}, \bar{z}; \epsilon) [\tilde{p}_s(\bar{z}), (K(x, \bar{u}) K^{-1}(u, \bar{u}))_{ar}]
\]

(3.24)

and evaluating the commutator, we see that this is indeed zero. The vanishing of at least part of this expression may be seen from the Gauss law. On gauge-invariant functions we have \( \tilde{p}_a = K_{ab} \tilde{p}_b \); this is essentially the Gauss law condition on wavefunctions. Taking conjugates and writing \( \tilde{p}_a^\dagger = p_a \) we get \( p_a = \tilde{p}_b K_{ba} \). However, directly from \( \tilde{p}_a = K_{ab} \tilde{p}_b \) we get \( p_a = K_{ba} \tilde{p}_b \) using \( (KK^T)_{ab} = \delta_{ab} \). The consistency of these expressions requires that in functional form \([\tilde{p}_b(\bar{x}), K_{ba}(\bar{x})] = 0 \); the chosen regulator must give this result for consistency of the Gauss law condition.

With this result we can write Eq. (3.23) as

\[
-\Delta_1 = - \int e^{-2c_A S(H, \epsilon)} (\tilde{G}_{ar} \tilde{p}_r) K_{ab} e^{2c_A S(H, \epsilon)} (\tilde{G}_{bs} p_s)
\]

(3.25)

The Kähler property and the equivalence of the regularized form of Eq. (2.37) and Eq. (2.42) follow if \( Q = 0 \) where

\[
Q = \int e^{-2c_A S(H, \epsilon)} \tilde{G}_{ar}(\bar{x}, \bar{u}) [\tilde{p}_r(\bar{u}), K_{ab}(\bar{x}) e^{2c_A S(H, \epsilon)} \tilde{G}_{bs}(\bar{x}, \bar{v})] p_s(\bar{v})
\]

(3.26)

Using the expansion of \( \tilde{G}(\bar{x}, \bar{u}) \) as in Eq. (3.10), it is easy to see that the \( \epsilon \)-independent part of \( e^{2c_A S(H)} \tilde{G}_{ar} [\tilde{p}_r, K_{ab}] \) cancels the contribution \( \tilde{G}_{ar} K_{ab} [\tilde{p}_r, e^{2c_A S(H)}] \). Writing \( 2c_A S(H, \epsilon) = 2c_A S(H) + \tilde{S} \), we then find

\[
Q = \int \left[ \int_{x,u} e^{-|x-u|^2/\epsilon} \frac{1}{\pi(x-u)} \tilde{G}_{ar}(\bar{x}, \bar{u}) K_{ab}(\bar{x}) K_{an}(x, \bar{u}) K_{rn}(u, \bar{u}) F_{bs}(\bar{x}, \bar{v}) \right. \\
- \int \frac{e^{-2|x-\bar{v}|^2/\epsilon}}{\pi(x-v)} f_{mr1} K_{ab}(\bar{x}) K_{an}(x, \bar{v}) K_{rn}(\bar{v}) K_{sb}(\bar{v}, v) K_{ms}(v, \bar{v}) \\
+ \int_{x,u} \tilde{G}_{ar}(\bar{x}, \bar{u}) K_{ab}(\bar{x}) [\tilde{p}_r(\bar{u}), \tilde{S}] F_{bs}(\bar{x}, \bar{v}) G(\bar{x}, \bar{v}) p_s(\bar{v})
\]

(3.27)
where \( F_{bs}(\vec{x}, \vec{v}) = \delta_{bs} - e^{-|\vec{x}-\vec{u}|^2/\epsilon}(K^{-1}(\vec{x}, v)K(v, \vec{v}))_{bs} \). For the first term in \( Q \), because of the exponential \( e^{-|\vec{x}-\vec{u}|^2/\epsilon} \), the contribution to the integral for small \( \epsilon \) comes from \(|\vec{x} - \vec{u}| \lesssim \sqrt{\epsilon} \) and we can expand in powers of \((u - x)\). Likewise, for the second term, we can expand the product of the \( K \)'s around \( v \). In this case we find

\[
Q = -\epsilon \int i[\partial^2 \bar{J}_b(\vec{x})F_{bs}(\vec{x}, \vec{v}) + \frac{1}{2}\delta_{bs}\partial^2 \bar{J}_b(\vec{x})]G(\vec{x}, \vec{v})p_s(\vec{v})
+ \int \bar{G}_{ar}(\vec{x}, \vec{u})K_{ab}(\vec{x})[\bar{p}_r(\vec{u}), \tilde{S}]G_{bs}(\vec{x}, \vec{v})p_s(\vec{v})
\]

(3.28)

It is consistent to set \( \tilde{S} = 0 \), and we find that \( Q = 0 \) up to \( O(\epsilon) \)-terms. It is also possible to choose \( \tilde{S} \) of order \( \epsilon \) (which is consistent with our evaluation of the volume element \( d\mu(C) \)) so that \( Q = 0 + O(\epsilon^2) \). In fact such choice is given by

\[
\tilde{S} = -\epsilon \frac{3\pi}{4c_A} \int \partial_a \bar{J}_a \partial_a + O(\epsilon^2)
\]

(3.29)

In any case, this checks the identity of Eq.(2.37) and Eq.(2.42). (A similar result can be shown directly on \( \mathcal{A} \) without restricting to \( \mathcal{C} \); in other words the Kähler property \( \partial_a (g^{aa}g) = 0 \) is obtained on \( \mathcal{A} \).) The regularized version of Eq.(2.37) is given by

\[
T = \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v})\bar{p}_r(\vec{u})p_s(\vec{v})
\]

(3.30)

where

\[
\Pi_{rs}(\vec{u}, \vec{v}) = \int_x \bar{G}_{ar}(\vec{x}, \vec{u})K_{ab}(\vec{x})G_{bs}(\vec{x}, \vec{v})
\]

(3.31)

Notice also that from the above calculation \( \Delta_1 = \Delta_2 \). The property \( Q = 0 \) is equivalent to a check of the self-adjointness of the expression (3.30) for \( T \).

d) Checking equations of motion

The original Yang-Mills equations, in \( A_0 = 0 \) gauge, are

\[
\dot{A} = i[T, A] = E
\]

(3.32a)

\[
\dot{E} = i[\mathcal{H}, E] = i[V, E]
\]

(3.32b)

There is no contribution from \( T \) in Eq.(3.32b), i.e.,

\[
[T, [T, A]] = 0
\]

(3.33)
This equation is straightforward in $\mathcal{A}$, in terms of the original gauge variables. However, it is highly nontrivial in terms of the matrix parametrization of the theory. Its validity provides an indirect check of the consistency of the matrix reformulation of the theory and the corresponding ordering and regularization procedures.

From Eqs. (3.30, 3.31) and the fact that

$$[p_s(\vec{v}), A_l(\vec{z})] = -M_{ls}(\vec{z})\partial_z \delta(\vec{v} - \vec{z})$$  \hspace{1cm} (3.34a)$$

$$[\bar{p}_r(\vec{u}), A_l(\vec{z})] = 0$$  \hspace{1cm} (3.34b)$$

we find that

$$[T, A_l(\vec{z})] = \int_u C_{rl}(\vec{u}, \vec{z})\bar{p}_r(\vec{u})$$

$$C_{rl}(\vec{u}, \vec{z}) = M_{ls}(\vec{z})\int_x \tilde{G}_{ar}(\vec{x}, \vec{u})K_{ab}(\vec{x}, \vec{z}; \epsilon)(1 + \frac{\epsilon}{\vec{x} - \vec{z}}\partial_z)[K^{-1}(\vec{x}, z)K(z, \vec{z})]_{bs}$$  \hspace{1cm} (3.35)$$

If we take the $\epsilon \to 0$ limit of Eq. (3.35) we find, as expected, that

$$\lim_{\epsilon \to 0}[T, A_l(\vec{z})] = \int_u \tilde{G}(\vec{z}, \vec{u})M^r_l(\vec{z})\bar{p}_r(\vec{u}) = -iE_l(\vec{z})$$  \hspace{1cm} (3.36)$$

The evaluation of $[T, [T, A_l(\vec{z})]]$ produces three kind of terms.

$$[T, [T, A_l(\vec{z})]] = \int_{\vec{\omega}, \vec{u}, \vec{v}} \Pi_{mn}(\vec{\omega}, \vec{v})[\bar{p}_m(\vec{\omega}), [p_n(\vec{v}), C_{rl}(\vec{u}, \vec{z})]]\bar{p}_r(\vec{u})$$  \hspace{1cm} (3.37a)$$

$$+ \int_{\vec{\omega}, \vec{u}, \vec{v}} \Pi_{mn}(\vec{\omega}, \vec{v})[p_n(\vec{v}), C_{rl}(\vec{u}, \vec{z})]\bar{p}_m(\vec{\omega})\bar{p}_r(\vec{u})$$  \hspace{1cm} (3.37b)$$

$$+ \left(\int_{\vec{\omega}, \vec{u}, \vec{v}} \Pi_{mn}(\vec{\omega}, \vec{v})[\bar{p}_m(\vec{\omega}), C_{rl}(\vec{u}, \vec{z})] - f_{mkv} \int_{\vec{u}, \vec{v}} \Pi_{mn}(\vec{u}, \vec{v})C_{kl}(\vec{u}, \vec{z})\right)\bar{p}_r(\vec{\omega})p_n(\vec{v})$$  \hspace{1cm} (3.37c)$$

Evaluation of the coefficients of $\bar{p}, \bar{p}\bar{p}, \bar{p}p$ terms are quite tedious; we eventually find that in the $\epsilon \to 0$ limit they vanish, thus confirming Eq.(3.33).

4. **An expression for $T$ in terms of currents**

We have obtained a regularized construction of $T$ as an operator on functions on $\mathcal{C}$. In this section we shall obtain an expression for $T$ in terms of currents which can be useful in evaluating the action of $T$ on wavefunctions, which are functions of currents.
Using expressions (3.30, 3.31) for \( T \) and the chain rule of differentiation, we can obtain the action of \( T \) on a function of the currents as

\[
T \Psi(J) = m \left[ \int_z \omega_a(z) \frac{\delta}{\delta J_a(z)} + \int_{z,w} \Omega_{ab}(z,w) \frac{\delta}{\delta J_a(z)} \frac{\delta}{\delta J_b(w)} \right] \Psi(J) \quad (4.1)
\]

\[
\omega_a(z) = -if_{arm} \left[ \partial_z \Pi_{rs}(\vec{u}, \vec{z}) \right]_{\vec{u} \rightarrow \vec{z}} K_{sm}^{-1}(\vec{z}) \\
= if_{arm} \Lambda_{rm}(\vec{u}, \vec{z}) \quad (4.2a)
\]

\[
\Omega_{ab}(z, \vec{w}) = - \left[ \left[ \frac{CA}{\pi} \partial_w \delta_{br} + if_{brm} J_m(\vec{w}) \right] \partial_z \Pi_{rs}(\vec{w}, \vec{z}) \right] K_{sa}^{-1}(\vec{z}) \\
= D_{wbr} \Lambda_{ra}(\vec{w}, \vec{z}) \quad (4.2b)
\]

where

\[
\Lambda_{ra}(\vec{w}, \vec{z}) = - (\partial_z \Pi_{rs}(\vec{w}, \vec{z})) K_{sa}^{-1}(\vec{z}) \quad (4.3a)
\]

\[
D_{wab} = \frac{CA}{\pi} \partial_w \delta_{ab} + if_{abc} J_c(\vec{w}) \quad (4.3b)
\]

We have also used the commutation rules

\[
[p_s(\vec{v}), J_a(\vec{z})] = -i \frac{CA}{\pi} K_{as}(\vec{z}) \partial_z \delta(\vec{z}, \vec{v}) \\
[p_r(\vec{u}), J_b(\vec{w})] = -i (D_w)_{br} \delta(\vec{w} - \vec{u}) \quad (4.4)
\]

From the definition of \( \Pi_{rs}(\vec{u}, \vec{v}) \) we find

\[
\Lambda_{ra}(\vec{w}, \vec{z}) = \int_x \hat{G}_{mr}(\vec{x}, \vec{w}) G(\vec{x}, \vec{z}) e^{-|\vec{x} - \vec{z}|^2/\epsilon} \left[ \frac{\vec{x} - \vec{z}}{\epsilon} K(x, \vec{x}) K^{-1}(z, \vec{x}) \right. \\
\left. + K(x, \vec{x}) \partial_z K^{-1}(z, \vec{z}) K(z, \vec{z}) K^{-1}(z, \vec{z}) \right]_{ma} \quad (4.5)
\]

For \( \omega_a(\vec{z}) \), we need the \( \vec{w} \rightarrow \vec{z} \) limit of \( \Lambda \). The exponential \( e^{-|\vec{x} - \vec{z}|^2/\epsilon} \) assures us that the contribution to the \( x \)-integral is mostly from the region \( |\vec{x} - \vec{z}|^2 \lesssim \epsilon \). Expanding around \( z \), we then find

\[
\omega_a(\vec{z}) = J_a(\vec{z}) + O(\epsilon) \quad (4.6)
\]

Since \( \hat{G} \) has two terms, the expression (4.5) for \( \Lambda \) splits into four terms.

\[
\Lambda = I + II + III + IV \quad (4.7a)
\]
\[ I = \frac{1}{\pi} \int_x K(\vec{x})K^{-1}(z, \vec{x}) \frac{\sigma(\vec{x}, \vec{z}; \epsilon)}{x - w} \]  
\[ II = \frac{\epsilon}{\pi} \int_x K(\vec{x})\partial_z \left( K^{-1}(z, \vec{x})K(z) \right) K^{-1}(\vec{z}) \frac{\sigma(\vec{x}, \vec{z}; \epsilon)}{(x - \bar{z})(x - w)} \]  
\[ III = -\frac{1}{\pi}K(\vec{w}) \int_x K^{-1}(x, \vec{w})K(\vec{x})K^{-1}(z, \vec{x}) \frac{\sigma(\vec{x}, \vec{z}; \epsilon)}{x - w} e^{-|x - w|^2/\epsilon} \]  
\[ IV = -\frac{\epsilon}{\pi}K(\vec{w}) \int_x K^{-1}(x, \vec{w})K(\vec{x})\partial_z \left( K^{-1}(z, \vec{x})K(z) \right) K^{-1}(\bar{z}) \frac{\sigma(\vec{x}, \vec{z}; \epsilon)}{(x - \bar{z})(x - w)} e^{-|x - w|^2/\epsilon} \]  

We can write
\[
\sigma(\vec{x}, \vec{z}; \epsilon) = \sigma(\vec{x}, \vec{w}; \epsilon) \exp \frac{(x - w)(z - \bar{w}) + (\bar{x} - \bar{w})(z - w)}{\epsilon} \exp -\frac{(z - w)(\bar{z} - \bar{w})}{\epsilon} \]  

Expanding the integrands in powers of \((x - w)\), \((\bar{x} - \bar{w})\) and performing the \(x\)-integration we derive a systematic \(\epsilon\)-expansion for the expressions (4.7). The calculation is straightforward and we find

\[ I = \frac{1}{\pi(z - w)} \left[ 1 - K(\vec{w})K^{-1}(z, \vec{w})e^{-\alpha} \right] - \frac{\epsilon}{\pi} \left[ \frac{1}{z - w} \partial_z (K(\vec{z})\partial_z K^{-1}(\vec{z})) - \frac{1}{(z - w)^2} \partial_w (K(\vec{w})K^{-1}(z, \vec{w}))e^{-\alpha} \right] + \mathcal{O}(\epsilon^2) \]  
\[ II = \frac{\epsilon}{\pi(z - w)} \left[ (K\partial(\bar{\partial} K^{-1} K^{-1})(\vec{z}) + K(\vec{w})\partial_z (K^{-1}(z, \vec{w})K(z))K^{-1}(\bar{z}) \frac{e^{-\alpha}}{\bar{z} - \bar{w}} \right] + \mathcal{O}(\epsilon^2) \]  
\[ III = -\frac{1}{\pi(z - w)} \left[ K(\vec{w})K^{-1}(u, \bar{w})K(\vec{u})K^{-1}(z, \bar{u})e^{-\alpha/2} - K(\vec{w})K^{-1}(z, \bar{w})e^{-\alpha} \right] - \frac{\epsilon}{\pi} \left[ \frac{1}{z - w} \partial_z (K(\vec{w})K^{-1}(z, \bar{w}))e^{-\alpha/2} \right] + 2\partial_z \frac{1}{z - w} \partial_z (K(\vec{w})K^{-1}(u, \bar{w})K(\vec{u})K^{-1}(z, \bar{u}))e^{-\alpha/2} - 2\partial_z \partial_z (K(\vec{w})K^{-1}(u, \bar{w})K(\vec{u})K^{-1}(z, \bar{u}))e^{-\alpha/2} + \mathcal{O}(\epsilon^2) \]  
\[ IV = \frac{\epsilon}{\pi(z - w)(\bar{z} - \bar{w})} \left[ 2K(\vec{w})K^{-1}(u, \bar{w})K(u, \bar{u})\partial_z (K^{-1}(z, \bar{u})K(z))K^{-1}(\bar{z})e^{-\alpha/2} - K(\vec{w})\partial_z (K^{-1}(z, \bar{w})K(\bar{w}))K^{-1}(\bar{z})e^{-\alpha/2} \right] + \mathcal{O}(\epsilon^2) \]  

where \(u = \frac{1}{2}(z + w)\), \(\bar{u} = \frac{1}{2}(\bar{z} + \bar{w})\) and \(\alpha = (z - w)(\bar{z} - \bar{w})/\epsilon\). Putting everything together
we then find
\[ \Lambda_{ra}(\bar{w}, \bar{z}) = \frac{1}{\pi(z-w)} \left[ \delta_{ra} - \left( K(\bar{w})K^{-1}(u, \bar{w})K(\bar{u})K^{-1}(z, \bar{u}) \right)_{ra} e^{-\alpha/2} \right] 
+ \frac{\epsilon}{\pi} e^{-\alpha/2} \left[ -2\partial_{z} \frac{1}{z-w} \partial_{\bar{z}} \left( K(\bar{w})K^{-1}(u, \bar{w})K(\bar{u})K^{-1}(z, \bar{u}) \right) \right] 
+ \frac{2}{z-w} \partial_{\bar{z}} \left( K(\bar{w})K^{-1}(u, \bar{w})K(\bar{u})\partial_{z} K^{-1}(z, \bar{u}) \right) 
+ \frac{2}{(z-w)(\bar{z}-\bar{w})} K(\bar{w})K^{-1}(u, \bar{w})K(\bar{u})\partial_{z} \left( K^{-1}(z, \bar{u})K(\bar{z}) \right) K^{-1}(\bar{z}) \right]_{ra} 
+ O(\epsilon^2) \]

(4.10)

We can further expand the functions with arguments \( u = \frac{1}{2}(z+w), \bar{u} = \frac{1}{2}(\bar{z}+\bar{w}) \) around \((z, \bar{z})\) to obtain
\[ \Lambda_{ra}(\bar{w}, \bar{z}) = \frac{1}{\pi(z-w)} \left[ \delta_{ra} - \left( K(\bar{w})K^{-1}(z, \bar{w}) \right)_{ra} e^{-\alpha/2} \right] 
+ \text{(terms of higher order in } \epsilon \text{ or } (z-w), (\bar{z}-\bar{w}) \text{)} \]

(4.11)

\[ \equiv \bar{\mathcal{G}}'_{ra}(\bar{z}, \bar{w}) + ... \]

Notice that \( \bar{\mathcal{G}}' \) is the transpose of \( \bar{\mathcal{G}} \) as defined in Eq.(3.8), with \( \epsilon \) replaced by \( 2\epsilon \). As \( \epsilon \to 0 \), \( \Lambda(\bar{w}, \bar{z}) \to \bar{\mathcal{G}}'(\bar{z}, \bar{w}) \). For the action of \( T \) on products of currents at the same point one has to be careful. If we have only terms of the form \( \partial^{n}J(\bar{y})J(\bar{y}) \) (for \( n = 0, 1, \ldots \)) then one can check that terms in Eq.(4.11) other than \( \bar{\mathcal{G}}'(\bar{z}, \bar{w}) \) do not contribute. We may thus write
\[ \Omega_{ab}(\bar{z}, \bar{w}) = (\mathcal{D}_{w}\bar{\mathcal{G}}'(\bar{z}, \bar{w}))_{ba} + ... \]

(4.12)

where the ellipsis refers to terms which do not contribute either for \( z \neq w \), or for the action on terms like \( \partial^{n}J(\bar{y})J(\bar{y}) \). They may contribute to the action of \( T \) on a product like \( \bar{\partial}J(\bar{y})\bar{\partial}J(\bar{y}) \). We will not encounter products like \( \bar{\partial}J(\bar{y})\bar{\partial}J(\bar{y}) \) since we shall point-separate products of \( \bar{\partial}J \)'s. We shall however encounter terms like \( \partial^{n}J(\bar{y})J(\bar{y}) \) and for these the expression (4.12) suffices.

The kinetic energy term of Eq.(4.1) now becomes
\[ T\Psi(J) = m \left[ \int J_{a}(\bar{z}) \frac{\delta}{\delta J_{a}(\bar{z})} + \int (\mathcal{D}_{w}\bar{\mathcal{G}}'(\bar{z}, \bar{w}))_{ab} \frac{\delta}{\delta J_{a}(w)} \frac{\delta}{\delta J_{b}(\bar{z})} \right] \Psi(J) + O(\epsilon) \]

(4.13)

In arriving at the expression (4.6) for \( \omega_{a} \), we have cancelled powers of \( (z-w) \) against \( \bar{G}(\bar{z}, \bar{w}) \). Keeping track of these more carefully one finds
\[ T\Psi(J) = m \int_{z,w} \left[ \bar{\partial}J_{a}(\bar{w})\bar{G}(\bar{z}, \bar{w}) \frac{\delta}{\delta J_{a}(\bar{z})} + (\mathcal{D}_{w}\bar{\mathcal{G}}'(\bar{z}, \bar{w}))_{ab} \frac{\delta}{\delta J_{a}(w)} \frac{\delta}{\delta J_{b}(\bar{z})} \right] \Psi(J) + O(\epsilon) \]

(4.14)
A partial integration in the first term takes us back to Eq. (4.13). Under a holomorphic transformation $J_a(\vec{z}) \rightarrow V_{ab}J_b(\vec{z}) + (c_A/\pi)(\partial VV^{-1})_a$ and $\delta J_a \rightarrow V_{ab}\delta J_b$. Expression (4.14) has manifest holomorphic invariance.

5. A digression on the Abelian case

The next step in our discussion is naturally to consider eigenstates of $T$. However, to clarify the nature of some of the terms which arise, we shall, in this section, consider the Abelian case with an added charge density due to matter fields. In the Abelian case we have $A = -\partial \theta$, $\bar{A} = \bar{\partial} \bar{\theta}$. (Recall that we are using antihermitian components for the potentials.) Writing $\theta = \chi + i\phi$ with $\chi$, $\phi$ real, we see that $\phi$ corresponds to the gauge part of $A$. With this splitting

$$E = \frac{1}{4} \int G(\frac{\delta}{\delta \chi} + i\frac{\delta}{\delta \phi}), \quad \bar{E} = -\frac{1}{4} \int G(\frac{\delta}{\delta \chi} - i\frac{\delta}{\delta \phi})$$

This gives $\partial \bar{E}_i = \frac{i}{\delta \phi}$ and the Gauss law condition for physical states becomes

$$(i\frac{\delta}{\delta \phi} - \rho)\Psi = 0$$

This has the solution

$$\Psi(\theta) = e^{-i\int \rho \phi} \Phi(\chi)$$

The kinetic energy operator becomes

$$T = \frac{e^2}{8} \int \Pi(\vec{u}, \vec{v})(\frac{\delta}{\delta \chi(\vec{u})} + i\frac{\delta}{\delta \phi(\vec{u})})(\frac{\delta}{\delta \chi(\vec{v})} - i\frac{\delta}{\delta \phi(\vec{v})})$$

where, in this Abelian limit, $\Pi(\vec{u}, \vec{v})$ is given by

$$\Pi(\vec{x}, \vec{y}) = \int_u \bar{G}(\vec{u}, \vec{x})(1 - e^{-|\vec{u} - \vec{x}|^2/\epsilon})G(\vec{u}, \vec{y})(1 - e^{-|\vec{u} - \vec{y}|^2/\epsilon})$$

The regulated Green’s functions in the Abelian limit are given by

$$\mathcal{G}(\vec{x}, \vec{y}) \equiv \frac{1}{\pi\xi}(1 - e^{-\xi^2/\epsilon}) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik\xi}e^{-ik^2/4}}{ik}$$

$$\bar{\mathcal{G}}(\vec{x}, \vec{y}) \equiv \frac{1}{\pi\xi}(1 - e^{-\xi^2/\epsilon}) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik\xi}e^{-ik^2/4}}{ik}$$

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where $\xi = x - y$. Using these expressions in Eq.(5.5), we may write

$$\Pi(\vec{x}, \vec{y}) = \int u \bar{G}(\vec{u}, \vec{x})G(\vec{u}, \vec{y}) = \int \frac{d^2 k}{(2\pi)^2} \frac{e^{i \vec{k} \cdot (\vec{x} - \vec{y})} e^{-\epsilon \vec{k}^2 / 2}}{kk}$$

(5.7)

This integral can be evaluated after introducing an infrared cutoff $R$ as

$$\Pi(\vec{x}, \vec{y}) = \frac{1}{\pi} \left[ \text{Ein}(s/2R^2) - \text{Ein}(s/2\epsilon) + \log(R^2/\epsilon) \right]$$

(5.8)

where $s = |\vec{x} - \vec{y}|^2$ and

$$\text{Ein}(z) = \int_0^z \frac{dt}{t} (1 - e^{-t})$$

(5.9)

We have used the fact that $R^2 \gg s$, so $\text{Ein}(s/2R^2) \approx 0$. As $\epsilon \to 0$ for fixed $s$, $\text{Ein}(s/2\epsilon) \approx \log(s/2\epsilon)$ [12].

Using Eq.(5.3) we can write

$$T \Phi(\chi) = \frac{e^2}{8} \int \Pi(\vec{u}, \vec{v}) \left( \frac{\delta}{\delta \chi(\vec{u})} + \rho(\vec{u}) \frac{\delta}{\delta \chi(\vec{v})} - \rho(\vec{v}) \right) \Phi(\chi)$$

$$= \left[ \frac{e^2}{8} \int \Pi(\vec{u}, \vec{v}) \frac{\delta}{\delta \chi(\vec{u})} \frac{\delta}{\delta \chi(\vec{v})} - \frac{e^2}{8} \int \Pi(\vec{u}, \vec{v}) \rho(\vec{u}) \rho(\vec{v}) \right] \Phi(\chi)$$

(5.10)

The term quadratic in $\rho$ is the Coulomb interaction,

$$T_{Coul} = -\frac{e^2}{8} \int \Pi(\vec{u}, \vec{v}) \rho(\vec{u}) \rho(\vec{v})$$

(5.11)

For a two-body state with $\rho(\vec{u}) = \delta(\vec{u} - \vec{x}) - \delta(\vec{u} - \vec{y})$, we get

$$T_{Coul} = -\frac{e^2}{8} \left[ \Pi(\vec{x}, \vec{x}) + \Pi(\vec{y}, \vec{y}) - \Pi(\vec{x}, \vec{y}) - \Pi(\vec{y}, \vec{x}) \right]$$

$$\approx -\frac{e^2}{4\pi} \log(|\vec{x} - \vec{y}|^2/2\epsilon)$$

(5.12)

This is indeed the expected logarithmic Coulomb interaction. However, its dependence on the short distance cut-off $\epsilon$ deserves comment. Going back to expression (5.11), we see that a change of scale in $\Pi(\vec{u}, \vec{v})$, say, $R \to \alpha R$ produces a correction of the form

$$\log \alpha \int \rho(\vec{u}) \int \rho(\vec{v})$$

in $T$, which is zero for states with total charge zero, as for the two-body state we are considering. The Coulomb interaction is thus expected to be independent of
the cut-off scales; some physical scale \( \lambda \) should appear in the logarithmic term. The reason why \( \epsilon \) appears in Eq.(5.12) is that, with our regulator, the self-energy subtractions are also automatically done at scale \( \epsilon \). This can be clarified by considering two matter fields, say, \( \psi \) and \( \zeta \) of positive and negative unit charges respectively. Thus \( \rho = \psi \dagger \psi - \zeta \dagger \zeta \). The two-body state of zero total charge is given by

\[
|x, y\rangle = \psi \dagger(x) \zeta \dagger(y) \langle 0 | (5.13)
\]

This is an eigenstate of \( \rho \) with eigenvalue \( [\delta(\vec{u} - \vec{x}) - \delta(\vec{u} - \vec{y})] \) and leads to the result (5.12).

We can write the product of the charge densities as

\[
\rho(\vec{u}) \rho(\vec{v}) =: \rho(\vec{u}) \rho(\vec{v}) : + \delta(\vec{u}, \vec{v}) \left( \psi \dagger \psi + \zeta \dagger \zeta \right) (5.14)
\]

where the colons indicate normal ordering. Thus

\[
T_{Coul} = -\frac{e^2}{8} \int \Pi(\vec{u}, \vec{v}) : \rho(\vec{u}) \rho(\vec{v}) : - \frac{e^2}{8} \Pi(\vec{u}, \vec{u}) \int \left( \psi \dagger \psi + \zeta \dagger \zeta \right) (5.15)
\]

The second term is a correction to the mass of the matter fields. Indeed, if we have a mass term \( \mathcal{H}_{mass} = m \int (\psi \dagger \psi + \zeta \dagger \zeta) \), we see that the correction \( -(e^2/8)\Pi(\vec{u}, \vec{u}) \) can be absorbed into the definition of mass. Alternatively, we can introduce a renormalized mass \( m_{ren} \) defined at scale \( \lambda \) by

\[
m = m_{ren} - \frac{e^2}{8\pi} \log(2\epsilon/\lambda) (5.16)
\]

The energy of the two-body state now becomes

\[
(T_{Coul} + \mathcal{H}_{mass}) |x, y\rangle = \left[ 2m_{ren} - \frac{e^2}{4\pi} \log(|\vec{x} - \vec{y}|^2/\lambda) \right] |x, y\rangle (5.17)
\]

As expected, the subtraction scale \( \lambda \) appears in the Coulomb interaction.

We can also phrase this as follows. We do not need to introduce a mass term or, equivalently, we can set \( m_{ren} = 0 \). Instead, the properly regularized \( T_{Coul} \) is defined as

\[
T_{Coul}(\lambda) = -\frac{e^2}{8} \int \Pi(\vec{u}, \vec{v}) \rho(\vec{u}) \rho(\vec{v}) + \frac{e^2}{2} \log(2\epsilon/\lambda) \ Q
\]

\[
Q = -\frac{1}{4\pi} \int \left( \psi \dagger \psi + \zeta \dagger \zeta \right)
\]
We introduce a new operator $Q$ which gives self-energy subtractions at the desired scale. Obviously, for $\lambda = 2\epsilon$, viz., subtractions at scale $\epsilon$, we go back to the expression (5.12).

This latter point of view of adding an operator $Q$ is more appropriate for the non-Abelian case, where the mass is dynamically generated.

6. Construction of eigenstates of $T$

We now turn to the construction of eigenstates of $T$. The lowest eigenstate is given by $\Psi_0 = \text{constant}$, since $T$ involves derivatives. We may take the normalized state as $\Psi_0 = 1$ since $\int d\mu(C) = 1$. The state with the lowest number of $J$’s we can construct, which also has holomorphic invariance, is

$$\Psi_2 = \int_{x,y} f(\vec{x}, \vec{y}) \left[ \partial J_a(\vec{x}) K(x, \vec{y}) K^{-1}(y, \vec{y}) \right]_{ab} \partial J_b(\vec{y})$$

(6.1)

The term $K(x, \vec{y}) K^{-1}(y, \vec{y})$ ensures the holomorphic invariance of the product of two currents in the above expression. The term $K(x, \vec{y}) K^{-1}(y, \vec{y})$ can also be written in terms of currents and derivatives of currents by a Taylor expansion

$$K(x, \vec{y}) K^{-1}(y, \vec{y}) = \sum_{0}^{\infty} \frac{(x-y)^n}{n!} (\partial^n K K^{-1})(y, \vec{y})$$

(6.2)

The lowest order term in $\Psi_2$ has two currents (two $\partial J$’s). $\Psi_2$ is in general not an eigenstate of $T$; the action of $T$ can generate terms which have at least three currents, four currents and so on. These terms generally come with powers of $(x - y)$. By taking $(x - y)$ small we can avoid such terms and obtain an eigenstate. It is instructive to keep the separation $(x - y)$ arbitrary for the moment and evaluate the action of $T$ on $\Psi_2$. We find

$$T \Psi_2 = 2m \int_{x,y} f(\vec{x}, \vec{y}) \left\{ \partial J_c(\vec{x}) \left( K(x, \vec{y}) K^{-1}(y, \vec{y}) \right)_{ab} \partial J_b(\vec{y}) \left[ \delta_{ca} + V_{ca}(\vec{x}, \vec{y}) \right] ight\}$$

$$+ \frac{c_A \text{dim} G}{\pi} \partial_x \partial_x \sigma(\vec{x}, \vec{y}; \epsilon) + \int \mathcal{O}((x - y) J^3) f(\vec{x}, \vec{y})$$

(6.3)

where $\mathcal{O}((x - y) J^3)$ refers to terms which have at least three currents and one power of $(x - y)$. Also $V(\vec{x}, \vec{y})$ is defined by

$$V_{ca}(\vec{x}, \vec{y}) = \frac{\pi}{2c_A} (T^k T^l)_{ca} \left\{ [\Pi(x, \vec{x}, x, \vec{y}) - \Pi(x, \vec{x}, y, \vec{y})] K^{-1}(x, \vec{y}) \right\}_{kl}$$

(6.4)
where \( \Pi(u, \bar{v}, v, \bar{u}) = \Pi(u, \bar{v}) \) is defined by Eq.(3.31). From the transformation properties of \( \Pi(u, \bar{v}) \) and hence of \( \mathcal{V}_{ca}(\bar{x}, \bar{y}) \), the holomorphic covariance of Eq.(6.3) can be verified.

We can think of the value \( 2m \) as arising from one factor of \( m \) for each \( \bar{\partial}J \) in \( \Psi_2 \) which is in accord with Eq.(2.44). It is like \( \Psi_2 \) has two constituent particles each represented by \( \bar{\partial}J \). \( \mathcal{V}_{ca}(\bar{x}, \bar{y}) \) is thus an interaction potential for the two currents.

For most of the terms in the above calculation of \( T\Psi_2 \), the naive replacement of \( \bar{G}' \) by \( \bar{G} \) suffices. Only the terms involving \( \mathcal{V}_{ca} \) in Eq.(6.3) require more careful treatment. This arises from \( \bar{\partial}J_a(\bar{x}) \left( K(x, \bar{y})K^{-1}(y, \bar{y}) \right)_{ab} \bar{\partial}J_b(\bar{y}) \) when one of the \( \delta/\delta J \)'s in Eq.(4.13) acts on a \( \bar{\partial}J \) and the other on \( K(x, \bar{y})K^{-1}(y, \bar{y}) \). One can evaluate this by using the power series expansion of Eq.(6.2). A simpler method is to use the expression for \( T \) in terms \( p_a, \bar{p}_a \) as in Eq.(3.30) for this particular term. This is what we have done and leads directly to the result of Eq.(6.3).

There are a number of interesting points to be made regarding Eq.(6.3). First of all, it is easy to see that the leading term in an expansion around the Abelian limit is given by

\[
\mathcal{V}_{ca}(\bar{x}, \bar{y}) \approx \delta_{ca} \frac{\pi}{2} \left\{ \Pi(x, \bar{x}, x, \bar{y}) - \Pi(x, \bar{x}, y, \bar{y}) \right\} \approx \frac{\delta_{ca}}{2} \text{Ein}(s/2\epsilon) \approx \frac{\delta_{ca}}{2} \log(s/2\epsilon)
\]

(6.5)

where we have used Eq.(5.8). Comparison with the Abelian limit shows that this is indeed the logarithmic Coulomb potential between the two constituent particles of \( \Psi_2 \).

Consider now the \( \epsilon \)-dependence of Eq.(6.5). \( \epsilon \) is a short distance cut-off and we should expect physical results to be independent of \( \epsilon \). \( \mathcal{V}(\bar{x}, \bar{y}) \) is properly regulated at short distances so that \( \mathcal{V}(\bar{x}, \bar{x}) = 0 \). In analogy with the Abelian case, we see that this corresponds to the subtraction of Coulomb self-interactions at the scale \( \epsilon \). In order to obtain subtractions of self-energy at some other desired scale \( \lambda \), we must introduce the operator \( Q \). For the non-Abelian theory, this can be defined as

\[
Q = \epsilon \int \Pi'_{rs}(\bar{u}, \bar{v})p^\dagger_r(\bar{u})p_s(\bar{v})
\]

\[
\Pi'_{rs}(\bar{u}, \bar{v}) = \sigma(\bar{u}, \bar{v}; \epsilon)K_{rs}(u, \bar{v})
\]

(6.6)

where \( p^\dagger_r \) is the adjoint of \( p_r \) including the \( \exp(2c_A S(H)) \) term in the measure of integration, i.e., \( p^\dagger_r = (\bar{p}_r - i\bar{\partial}J_r) \). \( Q \) is a self-adjoint operator. The action of \( Q \) on \( J_a \) is proportional
to \[ \partial_x \Pi'(\vec{u}, \vec{x}) \] which is easily checked to be zero. Thus adding a term proportional to \( Q \) to \( T \) would not change the result of Eq.(2.44).

We now calculate the action of \( Q \) on \( \Psi_2 \). Because of the prefactor \( \epsilon \) in the definition, most of the terms in \( Q \Psi_2 \) are zero, at least as \( \epsilon \to 0 \); only one term
\[
\epsilon \int \Pi'_r \left[ \bar{p}_r(\vec{u}), \quad \bar{\partial} J_a(\vec{x}) \right] | p_s(\vec{v}) \rangle \left( K(x, \vec{y}) K^{-1}(y, \vec{y}) \right)_{ab} | \bar{\partial} J_b(\vec{y}) \rangle
\]
gives a nonzero contribution. We get
\[
Q \Psi_2 = \frac{c A}{\pi} \Psi_2 + \ldots \quad (6.7)
\]
where the ellipsis refers to terms which vanish as \( \epsilon \to 0 \); such terms are of the order of \( \epsilon \) and still vanish if we multiply \( Q \) by a factor proportional to \( \log \epsilon \).

We now define the regularized expression for \( T \), with self-energy subtractions at scale \( \lambda \) as
\[
T(\lambda) = T + \frac{e^2}{2} \log(2\epsilon/\lambda)^2 \quad (6.8)
\]
Using Eq.(6.7), the action of \( T(\lambda) \) on \( \Psi_2 \) is easily evaluated as
\[
T(\lambda) \Psi_2 = 2m \int_{x,y} f(\vec{x}, \vec{y}) \left\{ \bar{\partial} J_c(\vec{x}) \left( K(x, \vec{y}) K^{-1}(y, \vec{y}) \right)_{ab} \bar{\partial} J_b(\vec{y}) \right\} \left[ \delta_{ca} + \nabla_{ca}(\vec{x}, \vec{y}) + \frac{1}{2} \delta_{ca} \log(2\epsilon/\lambda) \right] \]
\[+ \frac{c A \text{dim} G}{\pi} \partial_x \bar{\partial}_{\vec{x}} \sigma(\vec{x}, \vec{y}; \epsilon) \right\} + \int \mathcal{O}((x - y) J^3) f(\vec{x}, \vec{y}) \]
\[= 6.9 \]
The new potential is given by
\[
\nabla_{ca}(\vec{x}, \vec{y}) + \frac{1}{2} \delta_{ca} \log(2\epsilon/\lambda) \approx \frac{1}{2} \delta_{ca} \log(|\vec{x} - \vec{y}|^2/\lambda) \quad (6.10)
\]
The result for the potential (at finite nonzero separation \( |\vec{x} - \vec{y}| \)) is independent of \( \epsilon \) as expected; the limit \( \epsilon \to 0 \) can now be taken without difficulty. The scale factor \( \lambda \) enters the expression for the energy. The former expression (6.3) is also seen to be the special case of \( \lambda = 2\epsilon \).

\( \lambda \) is a physical scale parameter. However, since \( T/m \) is a scale-invariant operator, the numerical value of \( \lambda \) cannot be determined by consideration of \( T \) alone; it can be freely chosen as far as eigenstates of \( T \) alone are concerned. The inclusion of the potential energy term will determine what \( \lambda \) should be; we expect it to be of the order of \( (1/m^2) \) itself.

Generally speaking, \( \Psi_2 \) cannot be an eigenstate because of the Coulomb-like interaction and because of \( \mathcal{O}((x - y) J^3) \) terms. However, if we are only interested in constructing
an eigenstate of $T$, we can use an appropriate $f(\vec{x}, \vec{y})$ which gives a specific value to the interaction energy and take a limit where the terms $O((x - y)J^3)$ in Eq.(6.9) can be neglected. We do this by first taking $|\vec{x} - \vec{y}| \approx \sqrt{\lambda'}$, which can be achieved by choosing $f(\vec{x}, \vec{y})$ to be

$$f(\vec{x}, \vec{y}) = e^{-|\vec{x} - \vec{y}|^2/\lambda'/\pi} f(\vec{X}) = \sigma(\vec{x}, \vec{y}; \lambda') f(\vec{X})$$

(6.11)

where $\vec{X} = \frac{1}{2}(\vec{x} + \vec{y})$ is the center of momentum coordinate. Using the above form of $f(\vec{x}, \vec{y})$ and carrying out the integration over the relative coordinate, we find the leading term of $T(\lambda)\Psi_2$ in the Abelian limit to be

$$T(\lambda)\Psi_2 = \int \sigma(x, y; \lambda') 2m \left( 1 + \frac{1}{2} \text{Ein}(s/2\epsilon) + \frac{1}{2} \log(2\epsilon/\lambda) \right) f(\vec{X})$$

$$\left[ \partial_a J_a(\vec{x})(K(x, \vec{y})K^{-1}(y, \vec{y}))_{ab}\partial_j J_b(\vec{y}) \right] + ...$$

(6.12)

(The term involving $(c_A \text{dim} G/\pi)\partial \bar{\partial} \sigma(\vec{x}, \vec{y}; \epsilon)$ is not included in this expression; this term is discussed below.)

In order to eliminate the $O((x - y)J^3)$ terms in Eq.(6.9) which are of the order of $\lambda'$, we shall take $\lambda'$ very small. As we have already mentioned, $\lambda$ is not determined by $T(\lambda)$ alone. For obtaining an eigenstate of $T(\lambda)$, we may thus take $\lambda'$ small, but with a fixed value for $(\lambda'/\lambda)$. (Of course, we must also have $\lambda, \lambda' \gg \epsilon$.) As will be clear from the next section, the perturbative inclusion of the potential energy term is valid only for the low momentum modes, $\lambda$ giving the scale for the distinction between low and high momentum. Thus for consistency, we must also have $\lambda' \overset{\sim}{\ll} \lambda$. Again, as far as $T$ alone is concerned, the numerical value of the ratio $(\lambda'/\lambda)$ is undetermined; $\lambda$ will be fixed by inclusion of the potential energy term in $\mathcal{H}$ and $\lambda'$ will be determined by balance of kinetic and potential terms via the uncertainty principle or equivalently by solving a Schrodinger-like equation.

The remaining term

$$\int \frac{c_A \text{dim} G}{\pi} f(\vec{x}, \vec{y}) \partial_\vec{x} \partial_\vec{x} \sigma(\vec{x}, \vec{y}; \epsilon) = - \int_X \frac{c_A \text{dim} G}{\pi^2 \lambda'^2} f(\vec{X})$$

(6.13)

is a constant normal-ordering correction for $\bar{\partial} J \bar{\partial} J$. Combining the above equation with
Eq. (6.12) we see that the state

$$\tilde{\Psi}_2 = \int_X f(\vec{X}) \left[ \sigma(\vec{x}, \vec{y}; \lambda'; \lambda) \partial J_a(\vec{x})(K(x, y)K^{-1}(y, \vec{y}))_{ab} \partial J_b(\vec{y}) - \frac{c_A \text{dimG}}{\pi^2 \lambda^2} \frac{1}{(1 + \frac{1}{2} \log(\lambda'/\lambda))} \right]$$

$$\equiv \int_X f(\vec{X}) : \partial J \partial J : (\vec{X})$$

(6.14a)

is an eigenstate of $T$ with eigenvalue $2m(1 + \log(\lambda'/\lambda))$, as $\lambda' \to 0$, i.e.,

$$T(\lambda) \tilde{\Psi}_2 = 2m_\ast \tilde{\Psi}_2$$

$$m_\ast = m(1 + \frac{1}{2} \log(\lambda'/\lambda))$$

(6.14b)

A special choice is to take $\lambda' = \lambda$, in which case, it is easily checked by direct computation that $\tilde{\Psi}_2$ is orthogonal to the ground state we have obtained. In the limit of very small $\lambda$, $\lambda'$ with $(\lambda'/\lambda) = 1$ we clearly have an excited eigenstate of $T(\lambda)$ with eigenvalue $2m$.

Since $\tilde{\Psi}_2$ is a function of the currents, the normalization presents no difficulties. The normalization condition becomes *

$$\frac{c_A^2 \text{dimG}^2}{6\pi^3} \int \partial^3 f \partial^3 f^* = 1$$

(6.15)

7. Corrections due to the potential term

So far we have considered the kinetic term $T$ by itself and obtained $\Psi_0$, $\tilde{\Psi}_2$ as eigenstates of $T$. Concerning the diagonalization and construction of eigenstates of $T$ there are two different points of view. Mathematically $T$ is proportional to the Laplacian on the configuration space $C$ and one can ask what the eigenstates are, independently of the Yang-Mills Hamiltonian. The question of how good an approximation $T$ is to $(T + V)$ is irrelevant for this and the discussion of section 6 is directly applicable. However if we regard the diagonalization of $T$ as an approximation to the diagonalization of $(T + V)$, we see that this is a meaningful starting point only for modes of momenta $k \ll m$. For, as will be clear soon, the potential energy term gives contributions of the order $\vec{k}^2/m^2$ where $k$ is a typical momentum. Part of the potential term pertaining to modes of momenta $k \ll m$ can be treated in an expansion in $1/m$.

* We thank G. Alexanian for the computation of this condition.
We write the potential term as

\[ V = \frac{\pi}{mc_A} \int_x \partial J_a(\vec{x}) \partial J_a(\vec{x}) : \]

\[ = \frac{\pi}{mc_A} \left[ \int_{x,y} \sigma(\vec{x}, \vec{y}, \lambda) \partial J_a(\vec{x})(K(x, \vec{y})K^{-1}(y, \vec{y}))_{ab} \partial J_b(\vec{y}) - \frac{c_A \dim G}{\pi^2 \lambda^2} \right] \]  

Since the kinetic term has been defined with a subtraction scale \( \lambda \), we are using the same value in defining the potential term as well. \( \Psi_0 = 1 \) is the lowest order result for the vacuum wavefunction. To include the correction due to \( V \), we consider \( e^P \) where \( P \) can be expanded in powers of \( 1/m \) with

\[ P = \beta V + \mathcal{O}(1/m^3) \]  

and \( \beta \approx 1/m \). We find

\[ e^{-P} \mathcal{H} e^P = e^{-P}(T + V)e^P \equiv T + [T, P] + V + \mathcal{O}(1/m^2) \]

\[ = T + (2m \beta + 1)V + \frac{4\pi \beta}{c_A} \int (D \partial J) \delta \delta J + \mathcal{O}(1/m^2) \]  

We have used the result (6.14). Choosing \( \beta = -1/(2m) \) we find

\[ \mathcal{H} e^P \Psi_0 = e^P[T - \frac{2\pi}{mc_A} \int D \partial J \delta \delta J + ...] \Psi_0 \]

\[ = 0 + \mathcal{O}(1/m^2) \]  

Thus \( e^P = e^{-V/2m} \) gives the corrected vacuum wavefunction to order \( 1/m \).

The expectation value of an operator \( \mathcal{O} \) in this corrected vacuum can be written as

\[ \langle 0 | \mathcal{O} | 0 \rangle = \int d\mu(C) \exp \left[ -\frac{V}{m} \right] \mathcal{O} \]  

This is the functional integral for two-dimensional (Euclidean) YM theory of coupling constant \( g^2 = me^2 = e^4c_A/2\pi \). For the Wilson loop operator \( W_R(C) \) in the representation \( R \), we can use the results of references [10] to obtain

\[ < W_R(C) > \sim \exp \left[ -e^4c_A c_R A_C \right] \]  

where \( A_C \) is the area of the curve \( C \). This result pertains only to the contribution of modes of momenta \( k \ll m \). The high-momentum modes can give a contribution which
goes like the perimeter due to the correlation of currents at nearby points on \( C \) and this can dominate for large loops.

The action of \( \mathcal{H} \) on a perturbed state \( e^P J_a \) gives

\[
\mathcal{H}(e^P J_a(\vec{x})) \simeq (m - \frac{\nabla^2}{2m})(e^P J_a(\vec{x})) + \frac{2i\pi}{mc_A} f_{abc} e^P J_b(\vec{x}) \partial J_c(\vec{x}) + \ldots \tag{7.7}
\]

This is not quite an eigenstate; however the corrected energy starts off as \( m + \frac{k^2}{2m} \) for momentum \( k \). A \((1/m)\)-expansion is necessarily a nonrelativistic expansion and this result is just what we expect. This is similar to what happens with solitons and one must sum up a sequence of terms to obtain the relativistic result \( \sqrt{k^2 + m^2} \) [14] (see next section).

The perturbative inclusion of the potential energy applies to low momentum modes. For the other modes, one must seek a diagonalization of the high momentum part of \((T + V)\), perhaps along the lines of the next section, and match with the low momentum expansion. This matching, among other things, will determine the scale \( \lambda \) introduced in section 6.

Notice also that if we include the potential energy \( V \) as above, the action of \( \mathcal{H} \) on \( \Psi_2 \) gives a result of the form

\[
\mathcal{H} e^P \Psi_2 = e^P \int_{x,y} \bar{\partial} J_a(\vec{x}) (K(x, \vec{y})K^{-1}(y, \vec{y}))_{ab} \bar{\partial} J_b(\vec{y}) \times
\left[ 2m \left( 1 + \frac{1}{2} \log[|\vec{x} - \vec{y}|^2/\lambda] \right) - \frac{\nabla^2_x}{2m} - \frac{\nabla^2_y}{2m} \right] f(\vec{x}, \vec{y}) + \ldots \tag{7.8}
\]

We see that eigenstates can be constructed by taking \( f(\vec{x}, \vec{y}) \) to be solutions of the two-body Schrödinger equation

\[
\left[ -\frac{\nabla^2_x}{2m} - \frac{\nabla^2_y}{2m} + 2m \left( 1 + \frac{1}{2} \log[|\vec{x} - \vec{y}|^2/\lambda] \right) \right] f(\vec{x}, \vec{y}) = Ef(\vec{x}, \vec{y}) \tag{7.9}
\]

The states so obtained will be the orbital excitations of the basic two-body state. Of course, to do this properly one must go beyond the nonrelativistic approximation and the lowest order logarithmic potential.

8. A consistent truncation

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Qualitatively, the emergence of the mass gap is the most interesting nonperturbative effect. As we have argued, this has to do with the $e^{2c_A S(H)}$ factor in $d\mu(C)$. A perturbation theory around the Abelian limit (which is an expansion in powers of the structure constants $f_{abc}$) would not see this effect; however having obtained the factor $e^{2c_A S(H)}$, an improved perturbative expansion can be done.

We write $H = e^{i_a \phi_a}$ and do an expansion in powers of $\phi$ for $T, J_a$ and the WZW action $S(H)$. This is equivalent to an expansion around the Abelian limit for these terms. For example,

$$2c_A S(H) \equiv -\frac{c_A}{2\pi} \int \partial \varphi_a \partial \varphi_a + \ldots \quad (8.1)$$

We will however retain the factor $e^{2c_A S(H)} \simeq e^{-\frac{c_A}{2\pi} \int \partial \varphi \partial \varphi}$ rather than expanding this as $(1 - \frac{c_A}{2\pi} \int \partial \varphi \partial \varphi + \ldots)$. This expansion is thus not the same as expansion around the Abelian limit for the full theory. To the lowest order in the $\varphi$’s we find

$$d\mu(C) \simeq [d\varphi] e^{-\frac{c_A}{2\pi} \int \partial \varphi_a \partial \varphi_a} \quad (8.2a)$$

$$T = T_1 + T_2 \quad (8.2b)$$

$$T_1 \simeq m \int \varphi_a \frac{\delta}{\delta \varphi_a} ; \quad T_2 \simeq \frac{m\pi}{c_A} \int C(\vec{x}, \vec{y}) \frac{\delta}{\delta \varphi_a(\vec{x})} \frac{\delta}{\delta \varphi_a(\vec{y})} \quad (8.2c)$$

$$C(\vec{x}, \vec{y}) = \int_G \bar{G}(\vec{x}, \vec{z}) G(\vec{z}, \vec{y}) = - \int \frac{d^2 k}{(2\pi)^2} \frac{e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{kk} \quad (8.2d)$$

$$V \simeq \frac{c_A}{m\pi} \int \partial \varphi_a (-\partial \bar{\partial}) \partial \varphi_a \quad (8.2e)$$

This expansion is consistent in the sense that the self-adjointness of $T$ and $V$ is respected. It is in fact instructive to consider the self-adjointness of $T$ as given above. We find

$$\langle \psi_1 | T_1 | \psi_2 \rangle = - \langle T_1 \psi_1 | \psi_2 \rangle - m\delta(0) \int d^2x \ dimG \langle \psi_1 | \psi_2 \rangle$$

$$+ \frac{mc_A}{\pi} \langle \psi_1 \partial \varphi \partial \varphi | \psi_2 \rangle \quad (8.3a)$$

$$\langle \psi_1 | T_2 \psi_2 \rangle = \langle T_2 \psi_1 | \psi_2 \rangle + m\delta(0) \int d^2x \ dimG \langle \psi_1 | \psi_2 \rangle$$

$$+ 2\langle T_1 \psi_1 | \psi_2 \rangle - \frac{mc_A}{\pi} \langle \psi_1 \partial \varphi \partial \varphi | \psi_2 \rangle \quad (8.3b)$$

$$\langle \psi_1 | (T_1 + T_2) \psi_2 \rangle = \langle (T_1 + T_2) \psi_1 | \psi_2 \rangle \quad (8.3c)$$
Eventhough formal expressions like $\delta(0)$ and $\int d^2x$ occur here, these equations illustrate the main point, viz., that $T_2$ is not self-adjoint by itself; $T_1$, which is the crucial term for the mas gap, is needed for self-adjointness so long as we have the factor $\exp\left[-\frac{cA}{2\pi} \int \partial \phi \bar{\partial} \phi\right]$ in $d\mu(C)$. (The cancellation of formal expressions involving $\delta(0)$ and $\int d^2x$ need not worry us at this stage; these arise from the truncations. We have already checked that $T^\dagger = T$ in the regulated version.)

We now absorb the factor $\exp\left[-\frac{cA}{2\pi} \int \partial \phi \bar{\partial} \phi\right]$ into the wavefunctions, defining $\Phi = e^{-\frac{cA}{4\pi} \int \partial \phi \bar{\partial} \phi} \Psi$, so that

$$\langle 1 | 2 \rangle \simeq \int [d\varphi] \Phi_1^* \Phi_2$$ (8.4)

For the wavefunctions $\Phi$ we get, up to an additive constant,

$$H = \frac{m\pi}{cA} \int C(\vec{x}, \vec{y}) \frac{\delta}{\delta \varphi_a(\vec{x})} \frac{\delta}{\delta \varphi_a(\vec{y})} + \frac{mcA}{4\pi} \int \partial \varphi_a \bar{\partial} \varphi_a + \frac{cA}{m\pi} \int \partial \varphi_a (-\partial \bar{\partial}) \bar{\partial} \varphi_a + ...$$ (8.5)

Defining $\phi_a(\vec{k}) = \sqrt{cAkk/(2\pi m)} \varphi_a(\vec{k})$, we have

$$H \simeq \frac{1}{2} \int_x \left[ -\frac{\delta^2}{\delta \phi^2_a(\vec{x})} + \phi_a(\vec{x})(m^2 - \nabla^2) \phi_a(\vec{x}) \right] + ...$$ (8.6)

We see that $\phi_a(\vec{x})$ behaves like a particle of mass $m$. (We also obtain the relativistic energies $\sqrt{\vec{k}^2 + m^2}$ as mentioned at the end of the last section.) We are currently investigating how $O(\varphi^3)$-terms can correct these results.

The picture which emerges from our discussions is as follows. We can think of $\phi_a(\vec{x})$ as massive particles carrying non-Abelian charge. When higher order terms are included, clearly we will get an interacting theory of these massive particles. Although in the interest of finding an eigenstate for $T$, we considered the special choice of $f(\vec{x}, \vec{y})$ in section 6, with $\lambda' \to 0$, $\lambda' \gg \epsilon$, we can, in Eq.(6.9), keep the separation $|\vec{x} - \vec{y}|$ finite and nonzero, which gives the interaction $V_{ca}(\vec{x}, \vec{y})$ between the massive particles (and some other corrections as well). We thus get a picture of the states being formed of massive constituents which are interacting, the interaction binding them into states of zero charge. The $\phi_a$’s are the “constituents” of the state. This is all in accord with the Schrodinger equation we obtained at the end of the last section. It should be possible to develop this constituent picture further, leading to a sequence of states as bound states of the constituents with some interaction potential. This is under investigation.
9. Discussion

The mass was obtained by the action of the kinetic energy $T$ on $J^a$ and in this context we consider the following potential counterargument to obtaining a mass at the level of $T$ alone. The electric fields are the canonical momenta for $A^a(\vec{x})$ and commute among themselves; so $T$ being $\int \vec{E}^2/2$, we have a field theoretic analogue of the free particle and would expect a continuous spectrum for $T$ alone. In particular we could use an $\vec{E}$-diagonal representation with $E^a|f\rangle = f^a(\vec{x})|f\rangle$, where $f$ is arbitrary, and hence $T$ can be made equal to any positive number by choice of $f^a(\vec{x})$. Furthermore, Feynman has argued that one needs the potential energy term to cut off possible “escaping directions” in $A/A^*_s$, so that plane waves along such directions, which may have a continuous spectrum, can be eliminated. We shall reexamine the ingredients which have gone into the mass for $J^a(\vec{x})$ to see how these arguments are reconciled with our calculation.

In analyzing the $\vec{E}$-representation, it is useful to consider the following parametrization of the electric fields. The complex component $E = \frac{1}{2}(E_1 + iE_2)$ is an element of the Lie algebra of $SL(N, \mathbb{C})$ and therefore, except for a set of matrices of measure zero, it can be diagonalized by a complex $SL(N, \mathbb{C})$-transformation $X$ [15]. Thus

$$E = X \Lambda X^{-1}, \quad \vec{E} = X^\dagger \Lambda \dagger X^\dagger$$

(9.1)

where $\Lambda$ is a complex diagonal matrix. In the $\vec{E}$-representation, $\Lambda$ and $X$ are $c$-numbers and the gauge potentials ($A, \vec{A}$) become functional differential operators as given by

$$\vec{A}^a = \frac{i}{2} D_{ab}(X) \left[ \sqrt{2} T^b_{ii} \frac{\partial}{\partial \lambda^i} - R^{bk} \bar{I}_k \right]$$

$$A^a = \frac{i}{2} D_{ba}(X^\dagger) \left[ \sqrt{2} T^a_{ii} \frac{\partial}{\partial \bar{\lambda}^i} - R^{k*} \bar{I}_k \right]$$

$$D_{ab}(X) = 2 \text{Tr}(T^a T^b X^{-1})$$

$$R^{ak} = 2 \sum_{i \neq j} \frac{T^a_{ij} T^b_{ji}}{\lambda_i - \lambda_j}$$

(9.2)

$$[I^k, X] = X T^k, \quad [\bar{I}^k, X^\dagger] = T^k X^\dagger$$

where $I, \bar{I}$ represent left and right translations on $X, X^\dagger$ respectively. In evaluating the action of ($A, \vec{A}$) or the magnetic field $B$ on a wave function in the $\vec{E}$-representation, the
$R^b_k I^k$ and $R^{*bk}I^k$ terms can bring in potential singularities when we have coincidence of eigenvalues of $E$ due to the $(\lambda_i - \lambda_j)^{-1}$-factor. (Notice that this factor and its contribution to $B$ via the commutator term are purely non-Abelian effects; they vanish for the Abelian theory.) In particular, as $\vec{E} \to 0$, all eigenvalues tend to zero and the action of the potential term on the wave function can become very large. Although very explicit in the parametrization (9.1), this property is simply a reflection of the uncertainty principle for $(\vec{E}, B)$ and is not restricted to the specific parametrization.

Consider now a state with low values for the kinetic energy, say, $\Psi(\vec{E}) \sim \delta(\int \vec{E}^2/2 - \epsilon)$ with $\epsilon \to 0$. (There is also an additional phase factor required by the implementation of the Gauss law in the $\vec{E}$-representation. We have not displayed this since it does not affect our arguments [16,17].) As $\epsilon \to 0$, we need $\vec{E} \to 0$ since $\int \vec{E}^2$ has a positive integrand. In this case, the contribution of $B^2$ to the energy can become very large. Thus $T$ cannot be made arbitrarily small keeping finiteness of the expectation value for $B^2$. Lowering the total energy requires some sort of balance between the kinetic and potential energies and this could lead to a gap. In particular for states with finite total energy, $B^2$ will have a finite expectation value. This argument is still far from giving an understanding of our results in the $\vec{E}$-representation, but it does, we believe, carry the essential physics of the problem. (The potential singularity for $\vec{E} \to 0$ can be avoided for states for which the wave functions vanish near $\vec{E} = 0$. The probability density for such states will be very small for small values of $\vec{E}$ and hence there will be significant probability for finite nonzero values of $\vec{E}^2$. Therefore they can actually contribute a noninfinitesimal value to $T$. Such states are not relevant to the potential counterargument which needs infinitesimal values for $T$.)

It may seem somewhat puzzling in this regard that we find a gap by considering $T$ alone, rather than $T + V$ as in the above argument. Actually, we do have finite values for $\langle B^2 \rangle$. We are considering states with finite norm where the inner product carries the factor $e^{2cA^S(H)}$. Such states have finite expectation values for $B^2 \sim: \bar{\partial} J_a(\vec{x}) \partial J_a(\vec{x}) :$. In other words we are looking for eigenstates of $T$ within the set of states of finite norm (and finite $\langle B^2 \rangle$) and hence it is consistent with the previous argument to get a mass gap even if the potential term is not included as part of the Hamiltonian. In other words, it is not necessary to consider the spectrum of $\int (e^2 \frac{\vec{E}^2}{2} + \frac{B^2}{\epsilon^2})$ as a whole. It would be possible to
see a mass gap with $T$ alone provided the restriction to states of finite $\langle B^2 \rangle$ arises via the inner product as in our case. It may indeed be possible to obtain a continuous spectrum with no gap for $T$ if we give up finiteness of $\langle B^2 \rangle$. Even if this may be a mathematical possibility, it is clearly unphysical since we do eventually have to include the potential term in the Hamiltonian and would need it to be finite.

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