Manifolds of Differentiable Densities

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Abstract

We develop a family of infinite-dimensional (i.e. non-parametric) manifolds of probability measures. The latter are defined on underlying Banach spaces, and have densities of class $C^k_b$ with respect to appropriate reference measures. The case $k = \infty$, in which the manifolds are modelled on Fréchet spaces, is included. The manifolds admit the Fisher-Rao metric and the dually flat geometry of Amari’s $\alpha$-covariant derivatives, for all $\alpha \in \mathbb{R}$. By construction, they are $C^\infty$-embedded submanifolds of particular manifolds of finite measures. Unusually for the non-parametric case, the likelihood function associated with a finite sample is a continuous function on each of the manifolds.

Keywords: Fisher-Rao Metric; Banach Manifold; Fréchet Manifold; Information Geometry; Non-parametric Statistics.

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1 Introduction

Information Geometry is the study of differential-geometric structures arising in the theory of statistical estimation, and has a history going back (at least) to the work of C.R. Rao [23]. It is finding increasing application in many fields including asymptotic statistics, machine learning, signal processing and statistical mechanics. (See, for example, [19] [20] for some recent developments.) The theory in finite dimensions (the parametric case) is well developed, and treated pedagogically in a number of texts [11] [3] [7] [12] [14].

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A classical example is the finite-dimensional \textit{exponential model}, in which linear combinations of a finite number of real-valued random variables (defined on an underlying probability space \((\mathbb{X}, \mathcal{X}, \mu)\)) are exponentiated to produce probability density functions with respect to the reference measure \(\mu\). The topology induced on the set of probability measures, thus defined, is consistent with the important statistical divergences of estimation theory, and derivatives of the latter can be used to define geometric objects such as a Riemannian metric (the Fisher-Rao metric) and a family of covariant derivatives.

Central to any infinite-dimensional extension of these ideas, is the use of charts with respect to which statistical divergences are sufficiently smooth. For example, the \textit{Kullback-Leibler divergence} between two probability measures \(P \ll Q\) is defined as follows:

\[
\mathcal{D}_{KL}(P \mid Q) := E_Q(dP/dQ) \log(dP/dQ),
\]

where \(E_Q\) represents expectation (integration) with respect to \(Q\). As is clear from (1), the regularity of \(\mathcal{D}_{KL}\) is closely connected with that of the density, \(dP/dQ\), and its log (considered as elements of dual spaces of real-valued functions on \(\mathbb{X}\)). In fact, much of information geometry concerns the interplay between these two representations of \(P\), and the exponential map that connects them. The two associated affine structures form the basis of a Fenchel-Legendre transform underpinning the subject, and so manifolds that fully accommodate these structures are particularly amenable to analysis.

In the series of papers [6, 11, 21, 22], G. Pistone and his co-workers developed an infinite-dimensional variant of the exponential model outlined above. Probability measures in the manifold are mutually absolutely continuous with respect to the reference measure \(\mu\), and the manifold is covered by the charts \(s_Q(P) = \log dP/dQ - E_Q \log dP/dQ\) for different “patch-centric” probability measures \(Q\). These readily give \(\log dP/dQ\) the desired regularity, but require exponential Orlicz model spaces in order to do the same for \(dP/dQ\). The exponential Orlicz manifold has a strong topology, under which \(\mathcal{D}_{KL}\) is of class \(C^\infty\). The author’s own papers [16, 18] use, instead, the “balanced” global chart \(\phi(P) = dP/d\mu - 1 + \log dP/d\mu - E_\mu \log dP/d\mu\), thereby enabling the use of model spaces with weaker topologies. (In order for \(\mathcal{D}_{KL}\) to be of class \(C^k\), it suffices to use the Lebesgue model space \(L^p(\mu)\) with \(p = k + 1\).) The Hilbert case, in which \(p = 2\), is developed in detail in [16].

The exponential Orlicz and balanced \(L^p\) manifolds (for \(p \geq 2\)) all support the infinite-dimensional variant of the Fisher-Rao metric, and (for \(p \geq 3\)}
the infinite-dimensional variant of the Amari-Chentsov tensor. The latter can be used to define \(\alpha\)-derivatives on particular statistical bundles. (See, for example, \[11\].) However, with the exception of the case \(\alpha = 1\) on the exponential Orlicz manifold, these bundles differ from the tangent bundle, and so the \(\alpha\)-derivatives do not constitute covariant derivatives in the usual sense. The problem is that (with the exception of special cases, such as that in which \(X\) is finite) the model spaces do not support a multiplication operator.

In \[2\], the authors define a very general notion of statistical model. This is a manifold equipped with a metric and symmetric 3-tensor, together with an embedding into a space of finite measures, such that these become the Fisher-Rao metric and Amari-Chentsov tensor. They extend a result of Chentsov (on the uniqueness of these tensors as invariants under sufficient statistics) to this much wider class of statistical models. The exponential Orlicz and balanced \(L^p\) manifolds (for \(p \geq 3\)) all fit within this framework.

The topologies of these manifolds (like those of all manifolds of “pure” information geometry) have no direct connection with any topology that the underlying space \((X, \mathcal{X}, \mu)\) may possess. They concern statistical inference in its barest form – statistical divergences measure dependency between random variables without recourse to structures in their range spaces any richer than a \(\sigma\)-algebra of events. Nevertheless, metrics, topologies and linear structures on \(X\) play important roles in many applications. In maximum likelihood estimation, for example, it is desirable for the likelihood function associated with a finite sample to be continuous, which is not so on these manifolds. It is, therefore, of interest to develop statistical manifolds that embrace both topologies. This is a central aim here; we incorporate the topology of \(X\) by using model space norms that explicitly include derivatives of the densities. A different approach is pursued in \[10\]. The exponential manifolds developed there admit, by construction, continuous evaluation maps (such as the likelihood function) since they are based on reproducing kernel Hilbert space methods. However, they do not fully accommodate the affine structure associated with the density.

The paper is structured as follows. Sections \[2\] and \[3\] construct \(M\), a smooth manifold of finite measures on a Banach space \(X\), whose densities with respect to a reference measure are of class \(C_b^k\). \(M\) is covered by each chart in a one-parameter family \((\phi_\alpha, \alpha \in \mathbb{R})\). The charts \(\phi_\alpha\) and \(\phi_{-\alpha}\) map to open subsets of the dual affine spaces of a Fenchel-Legendre transform involving the \(\alpha\)-divergences \(D_\alpha\) and \(D_{-\alpha}\). As such, they define a metric and
dual notions of parallel transport on the tangent bundle (together with the associated covariant derivatives) for each $\alpha \in \mathbb{R}$. Section 4 considers the subset of probability measures, $\mathcal{N}$. This is a $C^\infty$-embedded submanifold of $M$, from which it inherits all its important properties. In particular, the projection of the metric and covariant derivatives of $M$ onto $\mathcal{N}$ yields the Fisher-Rao metric, and the $\alpha$-covariant derivatives on $\mathcal{N}$. In contrast with the manifolds of [6, 11, 16, 18, 21, 22], the latter are all defined on the tangent bundle of $\mathcal{N}$. Of course, this extra regularity is gained at the cost of inclusiveness. (See Remark 2.1(i).) $\mathcal{N}$ is (dually) flat in the $\alpha = \pm 1$-covariant derivatives. Finally, section 5 uses the method of projective limits to extend these results to manifolds of smooth densities.

In recent work [4, 5], the authors construct a manifold of smooth densities on an underlying finite-dimensional manifold by considering such densities to be smooth sections of the associated volume bundle. (This is a vector bundle of dimension 1 that endows the underlying manifold with an intrinsic notion of volume.) They consider a property of invariance of Riemannian metrics under the diffeomorphism group of the underlying manifold, and construct the class of all metrics with this property. When restricted to the submanifold of probability measures, these all coincide (modulo scaling) with the Fisher-Rao metric. In [5], they develop the Levi-Civita covariant derivative and carry out a number of extensions and completions of the manifold in order to study its geometry. The approach taken here is more extrinsic, in the sense that the geometry of $\mathcal{N}$ is constructed through its embedding in the dually $\alpha$-flat manifold $M$.

2 The exponential map

Let $B$ be an open subset of a Banach space $\mathbb{X}$, on which is defined a probability measure $\mu$ with the following properties: (i) $\mu(B) = 1$; (ii) for any open subset $A \subset B$, $\mu(A) > 0$. (For example, $\mathbb{X} = \mathbb{R}^d$, $B$ is a bounded open rectangle, and $\mu$ is normalised Lebesgue measure.) Let $G := C^k_b(B; \mathbb{R})$ be the space of continuous and bounded Lebesgue measure $\mu$ is normalised Lebesgue measure of $B \to \mathbb{R}$, that have continuous and bounded (Fréchet) derivatives of all orders up to some $k \in \mathbb{N}_0$. $G$ is a Banach space over $\mathbb{R}$ when endowed with the norm:

$$
\|a\|_G = \sup_{x \in B} |a(x)| + \sum_{i=1}^k \sup_{x \in B} \|a^{(i)}_x\|_{L(\mathbb{X}; \mathbb{R})},
$$

(2)
where \(a^{(i)}: B \to L(\mathbb{X}^i; \mathbb{R})\) is the \(i\)’th derivative of \(a\), and \(L(\mathbb{X}^i; \mathbb{R})\) is the space of continuous multilinear functions from \(\mathbb{X}^i\) to \(\mathbb{R}\), topologised by the operator norm. The (continuous bilinear) multiplication operator \(\pi: \mathbb{G} \times \mathbb{G} \to \mathbb{G}\), and the (continuous linear) expectation operator \(E_\mu: \mathbb{G} \to \mathbb{R}\), are as follows

\[
(a \cdot b)(x) = \pi(a, b)(x) = a(x)b(x) \quad \text{and} \quad E_\mu a = \int_B a(x)\mu(dx). \tag{3}
\]

**Proposition 2.1.** The Nemytskii (superposition) operator, \(\exp_G: \mathbb{G} \to \mathbb{G}^+\), defined by \(\exp_G(a)(x) := \exp_\mathbb{R}(a(x))\), is diffeomorphic and has first derivative \(\exp_G^{(1)} u = \exp_G(a) \cdot u\). (Here, \(\mathbb{G}^+ := \{ a \in \mathbb{G} : \inf_{x \in B} a(x) > 0 \}\).)

**Proof.** Let \(F(a, b) := \exp(b) - \exp(a) - \exp(a) \cdot (b - a)\). (We drop the domain subscript from the exponential map where no confusion can arise.) In order to establish that \(\exp\) is differentiable (with the stated derivative) it suffices to show that, for any \(a \in \mathbb{G}\), there exists a \(K < \infty\) such that

\[
\|F(a, b)\|_G \leq K\|b - a\|^2_G \quad \text{for all } b \in B(a, 1), \tag{4}
\]

where \(B(a, 1)\) is the open unit ball centered at \(a\). That this is true when \(k = 0\) follows from Taylor’s theorem applied to \(\exp_\mathbb{R}\). Suppose, then, that \(k \geq 1\). For any \(1 \leq i \leq k\), let \(S_i\) be the set of all permutations of the integers 1 to \(i\), and let \(y \in \mathbb{X}^i\). An induction argument, starting from the definition of \(F\), shows that there exist constants \(\gamma_{i,\rho, j} \in \mathbb{R}\) such that

\[
F(a, b)^{(i)}_x y = \sum_{\rho \in S_i} \sum_{j=1}^i \gamma_{i,\rho, j} b_x^{(j)} y_{\rho j}^{\rho_1} F(a, b)_x^{(i-j)} y_{\rho_{j+1}}^{\rho_i} + H(a, b, \cdot)^{(i-1)}_x y, \tag{5}
\]

where \(y_{\rho_0}^m := (y_m, \ldots, y_n)\) and \(H: \mathbb{G} \times \mathbb{G} \times B \to L(\mathbb{X}^i; \mathbb{R})\) is defined by

\[
H(a, b, x)y = \exp(a)(x)(b(x) - a(x))(b_x^{(1)} - a_x^{(1)})y.
\]

For any \(a \in \mathbb{G}\), there exists a \(K < \infty\) such that

\[
\sup_{x \in B} \|H(a, b, \cdot)^{(i-1)}_x\|_{L(\mathbb{X}^i; \mathbb{R})} \leq K\|b - a\|^2_G \quad \text{for all } b \in B(a, 1). \tag{6}
\]

An induction argument on \(i\) thus establishes (5). A further induction argument readily shows that \(\exp \in C^\infty(\mathbb{G}; \mathbb{G})\).

For any \(a \in \mathbb{G}\), the linear map \(\exp_a^{(1)}: \mathbb{G} \to \mathbb{G}\) is clearly a toplinear isomorphism, and so the statement of the proposition follows from the inverse mapping theorem.
Remark 2.1. (i) Boundedness is required of members of $G$ so that $\exp(G)$ is open. This is a significant restriction if, for example, $B = X = \mathbb{R}^d$. On the other hand, if $\mu$ has compact support (and $B$ is its interior) then boundedness is a very natural condition.

(ii) The results that follow hold true in other scenarios. For example, that in which $X = (-\pi, \pi)^d$, and $G$ is the subspace of $C^k_b(X; \mathbb{R})$ whose members satisfy a suitable periodic boundary condition. The manifolds constructed then comprise measures defined on the $d$-dimensional torus.

$G$ can also be replaced by $L^\infty(\mu)$, but no account is then taken of the topology of $X$.

3 The manifold of finite measures

Let $M$ be the set of finite measures on $B$ that are mutually absolutely continuous with respect to $\mu$, and have densities of the form,

$$p = dP/d\mu = \exp(a) \quad \text{for some } a \in G. \quad (7)$$

$M$ is covered by the single chart $\phi_1 : M \to G$, defined by $\phi_1(P) = \exp^{-1}(p)$.

For any $\alpha \in \mathbb{R} \setminus \{1\}$, let $\phi_\alpha : M \to G$ be defined as follows:

$$\phi_\alpha(P) = \frac{2}{1-\alpha} \left( \exp \left( \frac{1-\alpha}{2} \phi_1(P) \right) - 1 \right). \quad (8)$$

Proposition 2.1 shows that the map $\phi_\alpha \circ \phi_1^{-1}$ is diffeomorphic, and so $(\phi_\alpha, \alpha \in \mathbb{R})$ is a smooth atlas, each chart of which covers $M$.

Remark 3.1. The maps $\phi_\alpha$ are derived from Amari’s $\alpha$-embedding maps. (See section 2.6 in [1].) The offset $-1$ is included in (8) so that $\phi_\alpha(\mu) = 0$. This also ensures that $\phi_\alpha \circ \phi_{-1}^{-1} \circ (\text{id}_{G^+}, -1) : G^+ \to G$ is Naudts’ $q$-deformed logarithm (as defined in chapter 7 of [13]) with $q = (1 + \alpha)/2$.

A tangent vector $U$ at $P \in M$ is an equivalence class of smooth curves passing through $P$: two curves $(P(t) \in M, t \in (-\epsilon, \epsilon))$ and $(Q(t) \in M, t \in (-\epsilon, \epsilon))$ being equivalent at $P$ if $P(0) = Q(0) = P$ and $\phi_1(P)'(0) = \phi_1(Q)'(0)$. We denote the tangent space at $P$ by $T_PM$ and the tangent bundle by $TM := \bigcup_{P \in M}(P, T_PM)$. The latter admits the global charts $(\Phi_\alpha : TM \to G \times G, \alpha \in \mathbb{R})$, where $\Phi_\alpha(P, U) = (\phi_\alpha(P), U\phi_\alpha)$ and, for any differentiable Banach-space-valued map $f : M \to Y$,

$$Uf := f(P)'(0) \quad \text{for any } P \in U. \quad (9)$$
For any $\alpha, \beta \in \mathbb{R}$, the derivative of the transition map $\phi_\alpha \circ \phi_\beta^{-1}$ is,

$$
(\phi_\alpha \circ \phi_\beta^{-1})_a^{(1)} u = \exp \left( \frac{\beta - \alpha}{2} a_1 \right) \cdot u \quad \text{where} \quad a_1 = \phi_1 \circ \phi_\beta^{-1}(a).
$$

(10)

Remark 3.2.  (i) Each chart, $\phi_\alpha$, covers $M$, and so induces its own global trivialisation of the tangent bundle; we introduce multiple charts to enable the definition of different notions of parallel transport on $TM$.

(ii) The charts $\phi_{-1}$ and $\phi_1$ are particularly important. $\phi_{-1}$ reflects the inherent linear structure of a set of measures—tangent vectors can be interpreted in this chart as signed measures. On the other hand, $\phi_1$ is surjective, and so trivially introduces a Lie group structure on $M$. For $P, Q \in M$, the product $(PQ)_M$ and inverse $(P^{-1})_M$ are defined as follows:

$$
d(PQ)_M = d(QP)_M = (p \cdot q) d\mu \quad \text{and} \quad d(P^{-1})_M = p^{-1} d\mu,
$$

(11)

and the identity element is $\mu$.

Let $\Gamma TM$ be the space of smooth sections of $TM$ (i.e. smooth vector fields). Each chart $\Phi_\alpha$ induces a notion of parallel transport on $TM$; tangent vectors in different fibres of $TM$, $U \in T_P M$ and $\tilde{U} \in T_Q M$, are $\alpha$-parallel transports of each other if $U \phi_\alpha = \tilde{U} \phi_\alpha$. The associated covariant derivative, $\nabla^\alpha : \Gamma TM \times \Gamma TM \to \Gamma TM$, is that for which $\phi_\alpha$ is an affine chart:

$$
\nabla^\alpha_U V \phi_\alpha = UV \phi_\alpha.
$$

(12)

$M$ is $\nabla^\alpha$-flat (or simply $\alpha$-flat) for all $\alpha \in \mathbb{R}$. $\alpha$-geodesics are curves of $M$ whose $\phi_\alpha$-representations are straight lines in $G$.

We define a weak Riemannian metric on $M$ via the inclusion $\Phi_1(P, T_P M) \subset L^2(P)$: for any $U, V \in T_P M$,

$$
\langle U, V \rangle_P := \langle U \phi_1, V \phi_1 \rangle_{L^2(P)} = \langle U \phi_\alpha, V \phi_{-\alpha} \rangle_{L^2(\mu)} \quad \text{for all} \quad \alpha \in \mathbb{R},
$$

(13)

where we have used (10) in the second step. (This is positive definite since, for any open $A \subset B$, $P(A) > 0$.) As is clear from (13), if $\tilde{U}, \tilde{V} \in T_Q M$ are obtained by parallel transport of $U, V \in T_P M$, one according $\nabla^\alpha$ and the other according to $\nabla^{-\alpha}$, then

$$
\langle \tilde{U}, \tilde{V} \rangle_Q = \langle U, V \rangle_P.
$$

(14)
In this sense $\nabla^\alpha$ and $\nabla^{-\alpha}$ are dual with respect to the metric. Being self-dual (and torsion free), $\nabla^0$ is the Levi-Civita covariant derivative associated with the metric. As in the finite-dimensional case [1], this relation can be expressed in differential form: for any $U, V, W \in \Gamma T M$,

$$U \langle V, W \rangle = U \langle V \phi_\alpha, W \phi_{-\alpha} \rangle_{L^2(\mu)} + \langle V \phi_\alpha, U W \phi_{-\alpha} \rangle_{L^2(\mu)} = \langle \nabla^\alpha_U V, W \rangle + \langle V, \nabla^\alpha_U W \rangle. \quad (15)$$

The linear relation between the $\alpha$-covariant derivatives is also retained:

$$\nabla^\alpha = \frac{1 - \alpha}{2} \nabla^{-1} + \frac{1 + \alpha}{2} \nabla^1. \quad (16)$$

This follows from (10), which shows that

$$\nabla^\pm_U \phi_\alpha = \phi_\alpha \circ \phi_{\pm 1}(P) \nabla^\pm_U \phi_{\pm 1}$$

and, for $\alpha \neq \pm 1$,

$$\nabla^\pm_U \phi_{-\alpha} = \phi_{-\alpha} \circ \phi_{\pm 1}(Q) \nabla^\pm_U \phi_{-\alpha}$$

The geometry developed herein is a particular instance of Hessian geometry, in which a metric and dual covariant derivatives are derived from convex functions that are dual in the Fenchel-Legendre sense. The latter are expressed in terms of the so-called $\alpha$-divergences.

### 3.1 The $\alpha$-divergences

These are defined on $M$ as follows. (See section 3.6 in [1].)

$$D_{-1}(P|Q) = D_1(Q|P) = E_\mu(\phi_{-1}(Q) - \phi_{-1}(P)) + \langle \phi_{-1}(P) + 1, \phi_1(P) - \phi_1(Q) \rangle_{L^2(\mu)}, \quad (17)$$

and, for $\alpha \neq \pm 1$,

$$D_\alpha(P|Q) = \frac{2}{1+\alpha} E_\mu(\phi_{-1}(P) - \phi_\alpha(P)) + \frac{2}{1-\alpha} E_\mu(\phi_{-1}(Q) - \phi_{-\alpha}(Q)) - \langle \phi_\alpha(P), \phi_{-\alpha}(Q) \rangle_{L^2(\mu)}. \quad (18)$$

Together with Proposition 2.1, these show that $D_\alpha \in C^\infty(M \times M; \mathbb{R})$. The following proposition summarises some other properties.
Proposition 3.1. For any $\alpha \in \mathbb{R}$:

(i) $\mathcal{D}_{-\alpha}(P \mid Q) = \mathcal{D}_{\alpha}(Q \mid P) \geq 0$, with equality if and only if $P = Q$;

(ii) the following generalised cosine rule applies

\[ \mathcal{D}_{\alpha}(P \mid R) = \mathcal{D}_{\alpha}(P \mid Q) + \mathcal{D}_{\alpha}(Q \mid R) - \langle \phi_{\alpha}(P) - \phi_{\alpha}(Q), \phi_{-\alpha}(R) - \phi_{-\alpha}(Q) \rangle_{L^2(\mu)}; \]  

(iii) the set $\phi_{\alpha}(M)$ is convex;

(iv) for any $Q \in M$, the function $\mathcal{D}_{\alpha}(\phi_{\alpha}^{-1} \mid Q) : \phi_{\alpha}(M) \to \mathbb{R}$ admits the following derivatives

\[ \mathcal{D}_{\alpha}(\phi_{\alpha}^{-1} \mid Q)_{a}^{(1)} u = \langle \phi_{-\alpha} \circ \phi_{\alpha}^{-1}(a) - \phi_{-\alpha}(Q), u \rangle_{L^2(\mu)} \]  

\[ \mathcal{D}_{\alpha}(\phi_{\alpha}^{-1} \mid Q)_{a}^{(2)}(u, v) = \mathbb{E}_\mu \exp(\alpha \phi_{1} \circ \phi_{\alpha}^{-1}(a)) \cdot u \cdot v; \]

in particular $\mathcal{D}_{\alpha}(\phi_{\alpha}^{-1} \mid Q)$ is strictly convex;

(v) for any $a \in \phi_{-\alpha}(M)$,

\[ \mathcal{D}_{-\alpha}(\phi_{-\alpha}^{-1}(a) \mid Q) = \max_{b \in \phi_{\alpha}(M)} \{ \langle a - \phi_{-\alpha}(Q), b - \phi_{\alpha}(Q) \rangle_{L^2(\mu)} \} \]  

\[ -\mathcal{D}_{\alpha}(\phi_{\alpha}^{-1}(b) \mid Q), \]

and the unique maximiser is $\phi_{\alpha} \circ \phi_{-\alpha}^{-1}(a)$.

Proof. Parts (i), (ii) and (iv) can be proven by straightforward calculations. Part (iii) is trivial when $\alpha = 1$, since $\phi_1(M) = \mathbb{G}$. Suppose, then, that $\alpha \in \mathbb{R} \setminus \{1\}$. For any distinct $P_0, P_1 \in M$, and any $t \in (0, 1)$, let

\[ a_t := (1 - t)\phi_{\alpha}(P_0) + t\phi_{\alpha}(P_1); \]

then we can define

\[ p_t = \left(1 + \frac{1-\alpha}{2}a_t\right)^{2/(1-\alpha)} = \left((1 - t)p_0^{(1-\alpha)/2} + tp_1^{(1-\alpha)/2}\right)^{2/(1-\alpha)}. \]

Since the infimum (over $x \in B$) of the term in brackets on the right-hand side here is strictly positive, log $p_t$ is well defined and bounded. $p_t$ is thus the density of a measure $P_t \in M$, and $\phi_{\alpha}(P_t) = a_t$, which completes the proof of part (iii).
Let \( a \in \phi_{-\alpha}(M) \), and let \( f : \phi_{\alpha}(M) \to \mathbb{R} \) be defined as follows:

## 4 The manifold of probability measures

Let \( G_0 := \{ a \in G : E_\mu a = 0 \} \), let \( N := \{ P \in M : P(B) = 1 \} \), and let \( \phi_m : N \to G_0 \) be the restriction of \( \phi^{-1} \) to \( N \). \( N \) is a statistical manifold modelled on \( G_0 \), with global mixture chart \( \phi_m \). It is trivially a \( C^\infty \)-embedded submanifold of \( M \). The tangent bundle, \( TN \), admits the global chart \( \Phi_m : TN \to G_0 \times G_0 \), defined by

\[
\Phi_m(P, U) = (\phi_m(P), U\phi_m). \tag{25}
\]

We can also define an exponential chart, \( \Phi_e : TN \to G_0 \times G_0 \), as follows:

\[
\Phi_e(P, U) = (\phi_e(P), U\phi_e) \quad \text{where} \quad \phi_e := \phi_1 - E_\mu \phi_1. \tag{26}
\]

(That \( \phi_e \) is bijective follows since, for any \( a \in G_0 \), \( d\phi_e^{-1}(a)/d\mu = \exp(a)/E_\mu \exp(a) \); that \( \phi_e \circ \phi_m^{-1} \) is diffeomorphic follows from Proposition 2.4.)
\(M\) and \(N\) are connected by the normalisation map \(\nu: M \to N\) (\(\nu(P) := P/P(B)\)) and the inclusion map, \(\iota: N \to M\). The associated tangent maps, \(T\nu\) and \(T\iota\), have particularly simple representations in terms of the charts \(\Phi_1\) and \(\Phi_e\):

\[
\Phi_e \circ \nu \circ \Phi_1^{-1}(a, u) = (a - E\mu a, u - E\mu u),
\]

\[
\Phi_1 \circ \iota \circ \Phi_e^{-1}(b, v) = (b - \log E\mu \exp(b), v - E_P v),
\]

where \(E_P\) is expectation with respect to \(P = \phi^{-1}_e(b)\). So

\[
\Phi_1 \circ \iota \circ \nu \circ \Phi_1^{-1}(a, u) = (a - \log E\mu \exp(a), u - E_P u),
\]

where \(P = \nu \circ \phi^{-1}_1(a)\). As this shows, a tangent vector \(V \in T_PM\) at \(P \in N\) is in \(T_PM\) if and only if \(E_P V = 0\) (or equivalently \(E_{\mu} V \phi_{-1} = 0\)). So, for any \(P \in N, U \in T_PM\) and \(V \in T_PM\),

\[
\langle U, V \rangle_P = \langle U\phi_1, V\phi_1 \rangle_{L^2(P)} = \langle U\phi_1 - E_P U\phi_1, V\phi_1 \rangle_{L^2(P)} = \langle T_P\nu U\phi_1, V\phi_1 \rangle_{L^2(P)} = \langle T_P\nu U, V \rangle_P,
\]

which shows that \(T_P\nu U\) is the projection of \(U\) onto \(T_PM\) in the metric of \((13)\). (This corresponds, in the \(\phi_1\)-representation, to projection from \(L^2(P)\) onto the subspace of functions with \(P\)-mean zero.) More generally, \(T\nu\) effects 1-parallel transport of tangent vectors from \(P \in M\) to \(\nu(P) \in N\), followed by projection onto \(T_\nu(P)N\).

The Fisher-Rao metric on \(TN\) is the restriction of the metric of \((13)\) to \(TN\):

\[
\langle U, V \rangle_P = \langle U\phi_1, V\phi_{-1} \rangle_{L^2(\mu)} = \langle U\phi_1 - E_{\mu} U\phi_1, V\phi_{-1} \rangle_{L^2(\mu)} = \langle U\phi_e, V\phi_m \rangle_{L^2(\mu)}.
\]

The \(\alpha\)-covariant derivative on \(TN\) is the projection of that defined in \((12)\); for any \(U, V \in \Gamma TN\),

\[
\nabla^\alpha_U V = T\nu \circ \Phi^{-1}_\alpha(\phi_\alpha, UV\phi_\alpha).
\]

**Proposition 4.1.** For any \(\alpha \in \mathbb{R}\) and any \(U, V \in \Gamma TN\),

\[
\nabla^\alpha_U V\phi_e = UV\phi_e + \frac{1-\alpha}{2} \left[ (U\phi_e - E_P U\phi_e) \cdot (V\phi_e - E_P V\phi_e) - E\mu (U\phi_e - E_P U\phi_e) \cdot (V\phi_e - E_P V\phi_e) \right].
\]
Proof. Let \( W_\alpha \in \Gamma T M \) be defined by \( W_\alpha \phi_\alpha = U V \phi_\alpha \). According to (10) and (21), for any \( P \in N \),

\[
W_\alpha(P)\phi_1 = (\phi_1 \circ \phi^{-1}_\alpha)^{(1)} \phi_\alpha(P)^{(1)} U \left[ (\phi_\alpha \circ \phi^{-1}_1)^{(1)} (\phi_1 \circ \phi^{-1}_\alpha)^{(1)} v_e \right] P
\]

\[
= \exp \left( \frac{a-1}{2} a_1 \right) U \left[ \exp \left( \frac{1-a}{2} a_2 \right) \cdot (v_e - E_P v_e) \right] 
\]

\[
= U(v_e - E_P v_e) + \frac{1-a}{2} U \phi_1 \cdot (v_e - E_P v_e) 
\]

\[
= U(v_e - E_P v_e) + \frac{1-a}{2} (u_e - E_P u_e) \cdot (v_e - E_P v_e),
\]

where \( u_e := U \phi_e, v_e := V \phi_e \) and \( a_1 = \phi_1(P) \). Now \( \nabla^\alpha_U V = T v W_\alpha \), and so \( \nabla^\alpha_U V \phi_e = W_\alpha \phi_1 - E_\mu W_\alpha \phi_1 \), which completes the proof. \( \square \)

Remark 4.1. The Amari-Chentsov tensor on \( N \) is the symmetric covariant 3-tensor \( \tau \), defined as follows: for any \( P \in N \) and any \( U, V, W \in T_P N \)

\[
\tau_P(U, V, W) = E_P (u - E_P u) \cdot (v - E_P v) \cdot (w - E_P w),
\]

(33)

where \( u = U \phi_e, v = V \phi_e \) and \( w = W \phi_e \). As in the finite-dimensional case,

\[
D_\alpha(\phi^{-1}_e|\phi^{-1}_a)(u, v, w) = -\frac{1-a}{2} \tau_P(U, V, W),
\]

(34)

where \( a = \phi_a(P) \), and this could be used to define the \( \alpha \)-covariant derivative directly on \( N \).

Setting \( \alpha = 1 \) in (32), we see that \( N \) is 1-flat and that \( \phi_e \) is an affine chart for \( \nabla^{-1} \). Furthermore, since \( \phi_m(N) = \phi_{-1}(N) = \phi_{-1}(M) \cap G_0 \), it is clear that \( N \) is also \( -1 \)-flat and that \( \phi_m \) is an affine chart for \( \nabla^{-1} \). \( N \) is thus dually flat (\( \alpha = \pm 1 \)). Its \( -1 \)-flatness arises from the trivial nature of its embedding in \( M \) when expressed in terms of the chart \( \phi_{-1} \); this is the natural linear embedding of a set of probability measures in a set of finite measures. Its 1-flatness is associated with its Lie group structure: for any \( P, Q \in N \), the product \( (PQ)_N \) and inverse \( (P^{-1})_N \) are defined as follows:

\[
(PQ)_N = (QP)_N = \nu((PQ)_M) \quad \text{and} \quad (P^{-1})_N = \nu((P^{-1})_M),
\]

(35)

and the identity element is \( \mu \). The product here has important practical significance, in that it is the “data fusion” operator of Bayesian estimation. To see this, let \( X : \Omega \to X \) be a random variable with distribution \( \mu \), defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). For \( i = 1, 2 \), let \( Y_i : \Omega \to \mathbb{Y}_i \) be \( X \)-conditionally independent random variables taking values in measure spaces
\((Y_i, Y_i, \lambda_i)\), such that \(P_{XY_i} \ll \mu \otimes \lambda_i\) (where \(P_{XY_i}\) is the joint distribution of \(X\) and \(Y_i\)). In this scenario, it is possible to construct regular conditional probability distributions for \(X\) given \(Y_1, Y_2\) and \((Y_1, Y_2)\). (See section 1 in \[17\].) Denoting these \(P_1(\omega), P_2(\omega)\) and \(P_{1,2}(\omega)\) (and assuming that \(Y_1\) and \(Y_2\) are such that they lie in \(N\)), it can be shown that \(\mathbb{P}(P_{1,2} = (P_1 P_2)_N) = 1\).

It follows from (32) that an \(\alpha\)-geodesic of \(N\) is a smooth curve \(\mathbf{P}\) satisfying the differential equation

\[
\phi_e(\mathbf{P})'' = -\frac{1-\alpha}{2} \left[ (\phi_e(\mathbf{P})' - \mathbf{E}_P \phi_e(\mathbf{P})')^2 - \mathbf{E}_\mu (\phi_e(\mathbf{P})' - \mathbf{E}_P \phi_e(\mathbf{P})')^2 \right].
\]

(36)

The Fenchel-Legendre transform of Proposition 3.1 is preserved on \(N\) when \(\alpha = \pm 1\); the role of the dual variables \(\phi_1\) and \(\phi_{-1}\) is then played by \(\phi_e\) and \(\phi_{m}\). Finally, straightforward calculations show that, for any \(\alpha \in \mathbb{R}\) and any \(U, V, W \in \Gamma T N\),

\[
\nabla^{\alpha}_U \nabla^{\alpha}_V W - \nabla^{\alpha}_V \nabla^{\alpha}_U W - \nabla_{[U, V]}^{\alpha} W = R_\alpha(U, V, W),
\]

(37)

where \(R_\alpha : T N^3 \to T N\) is the following Riemann curvature tensor:

\[
R_\alpha(U, V, W) = \frac{1 - \alpha^2}{4} (\langle V, W \rangle_{P U} - \langle U, W \rangle_{P V}).
\]

(38)

5 Manifolds of smooth densities

In this section we consider the sequences of manifolds \((M^k, k \in \mathbb{N}_0)\) and \((N^k, k \in \mathbb{N}_0)\), as developed in sections 3 and 4, making explicit their dependence on the number of derivatives in the definition of \(\mathbb{G} (= \mathbb{G}^k)\). By developing projective limits of these sequences, we define Fréchet manifolds of measures having smooth densities with respect to \(\mu\). The manifold of finite measures in this context, and its model space, are as follows:

\[
\bar{M} := \bigcap_{k \in \mathbb{N}_0} M^k \quad \text{and} \quad \bar{\mathbb{G}} := \bigcap_{k \in \mathbb{N}_0} \mathbb{G}^k.
\]

(39)

Let \(\rho^k : \bar{\mathbb{G}} \to \mathbb{G}^k\) be the inclusion map. \(\bar{\mathbb{G}}\) is a Fréchet space, whose topology is generated by the sequence of norms \(||\rho^k||_{\mathbb{G}^k}, k \in \mathbb{N}_0\).

We denote the exponential map of section 2 by \(\exp^k\), and its restriction to \(\bar{\mathbb{G}}\) by \(\exp\). The latter is Leslie differentiable \[13\], with derivative \(d\exp^k a = \exp(a) \cdot u\), in the sense that, for any \(a \in \bar{\mathbb{G}}\), the map \(R : \mathbb{R} \times \bar{\mathbb{G}} \to \bar{\mathbb{G}}\), with

\[
R(t, a) := \begin{cases} 
 t^{-1}(\exp(a + tu) - \exp(a)) - \exp(a) \cdot u & \text{if } t \neq 0 \\
 0 & \text{if } t = 0,
\end{cases}
\]

(40)
is continuous at \((0, u)\) for every \(u \in \mathbb{G}\). The study of the Leslie differentiability properties of a map between Fréchet spaces (including the regularity of its derivatives, considered as maps into spaces of continuous linear maps) becomes substantially easier if the map in question is the projective limit of a system of maps between Banach spaces \([8]\), as is the case with \(\exp\).

For any \(0 \leq j \leq k < \infty\), let \(\rho^{k j} : \mathbb{G}^k \to \mathbb{G}^j\) be the (continuous linear) inclusion map. The system \((\mathbb{G}^k, \rho^{k j}, 0 \leq j \leq k < \infty)\) is a projective system with factor spaces \(\mathbb{G}^k\) and connecting morphisms \(\rho^{k j}\). The projective limit of this system is the following subset of the cartesian product \(\Pi := \prod_{k=0}^{\infty} \mathbb{G}^k:\)

\[
\lim_{\leftarrow} \mathbb{G}^k := \{(a^0, a^1, \ldots) \in \Pi : \rho^{k j} a^k = a^j \text{ for all } 0 \leq j \leq k < \infty\}.
\]

(41)

In this particular example, the map \(\lim_{\leftarrow} \mathbb{G}^k \ni (\rho^{0 \bar{a}}, \rho^{1 \bar{a}}, \ldots) \mapsto \bar{a} \in \mathbb{G}\) is a toplinear isomorphism, and so we can identify \(\lim_{\leftarrow} \mathbb{G}^k\) with \(\mathbb{G}\). The inclusion map \(\rho^k : \mathbb{G} \to \mathbb{G}^k\) then plays the role of the canonical projection \([8]\).

Suppose that \((\mathbb{F}^k, \sigma^{k j}, 0 \leq j \leq k < \infty)\) is another projective system of Banach spaces with projective limit \(\mathbb{F}\). The sequence \((f^k : \mathbb{G}^k \to \mathbb{F}^k, k \in \mathbb{N}_0)\) is a projective system of maps if

\[
\sigma^{k j} f^k = f^j \rho^{k j} \quad \text{for all } 0 \leq j \leq k < \infty.
\]

(42)

The projective limit of this system is \(\bar{f} : \mathbb{G} \to \mathbb{F}\), defined by \(\bar{f}(\bar{a}) = (f^0(\bar{a}), f^1(\bar{a}), \ldots)\). If each \(f^k\) is (Fréchet) differentiable then \(\bar{f}\) is Leslie differentiable, and its derivative can be associated with a projective limit of those of \(f^k\). (See Proposition 2.3.11 in \([8]\).) The appropriate projective system of derivatives is \((\Delta f^k : \mathbb{G}^k \to H^k(\mathbb{G}; \mathbb{F}), k \in \mathbb{N}_0)\), where

\[
\Delta f^k := \left( f^{0(1)}_{\rho^{00}}, f^{1(1)}_{\rho^{11}}, \ldots, f^{k(1)} \right),
\]

(43)

and

\[
H^k(\mathbb{G}; \mathbb{F}) = \left\{ (\lambda^0, \ldots, \lambda^k) \in \prod_{i=0}^{k} L(\mathbb{G}^i; \mathbb{F}^i) : \sigma^{ji} \lambda^i = \lambda^i \rho^{ji}, i \leq j \right\}.
\]

(44)

The factor spaces \(H^k(\mathbb{G}; \mathbb{F})\) are connected by the morphisms \(h^{k j} : H^k(\mathbb{G}; \mathbb{F}) \to H^j(\mathbb{G}; \mathbb{F})\),

\[
h^{k j}(\lambda^0, \ldots, \lambda^k) = (\lambda^0, \ldots, \lambda^j), \quad j \leq k,
\]

and so constitute a projective system of Banach spaces. The associated projective limit is toplinear isomorphic with \(\bar{H}(\mathbb{G}; \mathbb{F})\) (defined by the obvious variant of (44)), and the map \(\epsilon : \bar{H}(\mathbb{G}; \mathbb{F}) \to L(\mathbb{G}; \mathbb{F})\), defined by
\( \exp(\lambda^0, \lambda^1, \ldots) = \lim \lambda^k = (\lambda^0 \rho^0, \lambda^1 \rho^1, \ldots) \), is continuous linear, with respect to the topology of uniform convergence on bounded sets. (See Theorem 2.3.10 in [8].) That \( H(\mathbb{G}; \bar{\mathbb{F}}) \) is a projective limit of Banach spaces is central to the regularity of the Leslie derivative \( df \). If each \( f^k \) is smooth then \( df : \bar{\mathbb{G}} \to L(\mathbb{G}; \bar{\mathbb{F}}) \) is Leslie smooth. (See Propositions 2.3.11, 2.3.12 in [8].)

Applying these ideas to the exponential maps \( \exp^k \) and their inverses, we see that the projective limit \( \exp \) is Leslie diffeomorphic, and all its derivatives (together with those of its inverse) are smooth maps from \( \bar{\mathbb{G}} \) to appropriate spaces of continuous linear maps.

\( \bar{M} \) is a manifold of finite measures on \( B \) with smooth densities (with respect to \( \mu \)) of the form \( \exp(\bar{a}) \) where \( \bar{a} \in \bar{\mathbb{G}} \). It is covered by the single chart \( \bar{\phi}_1(P) := \exp^{-1}(\bar{p}) \), where \( \bar{p} = d\bar{P}/d\mu \). For any \( \alpha \in \mathbb{R} \setminus \{1\} \), we can define a chart \( \bar{\phi}_\alpha : \bar{M} \to \bar{\mathbb{G}} \) as in [8]. The transition maps \( \bar{\phi}_\beta \circ \bar{\phi}_\alpha^{-1} \) are Leslie diffeomorphic, and their derivatives \( d(\bar{\phi}_\beta \circ \bar{\phi}_\alpha^{-1}) : \bar{\phi}_\alpha(\bar{M}) \to L(\bar{\mathbb{G}}; \bar{\mathbb{F}}) \) are smooth maps. The tangent space at base point \( P \in \bar{M} \) can be defined in the usual way: \( \bar{U} \subset T_P \bar{M} \) is an equivalence class of smooth curves \( (P(t) \in \bar{M}, t \in (-\epsilon, \epsilon)) \) passing through \( P \) at \( t = 0 \). (Such a curve is the projective limit of the sequence of Fréchet smooth curves \((\bar{t}^k P, k \in \mathbb{N}_0)\), where \( \bar{t}^k : \bar{M} \to \bar{M}^k \) is the inclusion map. See Remark 3.1.9 in [8].) The tangent bundle, \( T\bar{M} \), admits the global charts \( \bar{\Phi}_\alpha(P, \bar{U}) = (\bar{\phi}_\alpha(P), \bar{U} \bar{\phi}_\alpha) \), where for any Leslie differentiable, Fréchet-space-valued map \( \bar{f} : \bar{M} \to Y \),

\[
\bar{U}\bar{f} := \bar{f}(P)'(0) = d(\bar{f} \circ \bar{\phi}_\alpha^{-1})\bar{U} \bar{\phi}_\alpha \quad \text{for any } P \in \bar{U}, \alpha \in \mathbb{R}.
\]

A metric can now be defined as in [13].

We can now define a special class of smooth vector fields of \( \bar{M} \) — those whose \( \bar{\Phi}_\alpha \)-representations are projective limits of smooth maps between the Banach spaces \( \mathbb{G}^k \). Let \( S \) be the following set of sequences:

\[
S = \{(n_k \in \mathbb{N}_0, k \in \mathbb{N}_0) : n_k \leq n_{k+1}, \sup n_k = +\infty\}, \quad (45)
\]

and note that, for any \( n \in S \), \((\mathbb{G}^{n_k}, \rho^{n_k}) \), \( 0 \leq j \leq k < \infty \) is a projective system of Banach spaces with projective limit \( \bar{\mathbb{G}} \). For some \( n \in S \), let \((u^k : \mathbb{G}^k \to \mathbb{G}^{n_k}, k \in \mathbb{N}_0)\) be a projective system of smooth maps, with projective limit \( u : \bar{\mathbb{G}} \to \bar{\mathbb{G}} \). We regard \( u \circ \bar{\phi}_1 \) as being the \( \bar{\Phi}_1 \)-representation of a smooth vector field \( \bar{U} \) (\( \bar{\Phi}_1 := u \circ \bar{\phi}_1 \)). We denote the set of all such projective-limit smooth vector fields \( \Gamma_{pl}T\bar{M} \). This has a linear structure, in which the sum of \((u^k : \mathbb{G}^k \to \mathbb{G}^{n_k})\) and \((v^k : \mathbb{G}^k \to \mathbb{G}^{m_k})\), for \( m, n \in S \), is the projective system \((w^k := \rho^{n_k}u^k + \rho^{m_k}v^k : \mathbb{G}^k \to \mathbb{G}^l)\), where \( l_k := \min\{m_k, n_k\} \).
Remark 5.1. \(\Gamma_{pl}T\bar{M}\) is strictly smaller than \(\Gamma TM\) — it does not contain the vector field with \(\Phi_1\)-representation \(\bar{u}(\bar{a}) = \bar{a}r(\bar{a}, 0)\), for example, where \(r\) is the usual metric on \(\mathbb{G}\). However, it does contain many useful vector fields occurring in the theory of partial differential equations. For example, if \(\mathbb{X} = \mathbb{R}^d\) then the second-order differential operator \(\partial^2/\partial x_i \partial x_j\) lifts to a vector field in \(\Gamma_{pl}T\bar{M}\).

**Proposition 5.1.** Let \(\bar{u} : \mathbb{G} \to \mathbb{G}\) be as defined above, and let \((f^k : \mathbb{G}^k \to \mathbb{F}^k)\) be a projective system of smooth maps, as described in (42). Then the sequence of maps

\[
\left( f_{\rho_{kl}}^{l,(1)} \rho^{n_{kl}} u^k : \mathbb{G}^k \to \mathbb{F}^l, \quad l = \min\{k, n_k\}, \quad k \in \mathbb{N}_0 \right)
\]

(46)
is a projective system, with projective limit \(d\bar{f}\bar{u}\), and \(\bar{U}(\bar{f} \circ \bar{\phi}_1) = d\bar{f}\bar{u} \circ \bar{\phi}_1\).

**Proof.** For any \(j \leq k\), let \(l := \min\{k, n_k\}\) and \(m := \min\{j, n_j\}\). Differentiating the projective relation \(s^l_{lm} = f^m \rho^m_{\rho^m_{lm}}\), we obtain \(s^l_{lm} = f^m \rho^m_{\rho^m_{lm}}\).

Restricting the base-point from \(\mathbb{G}^l\) to \(\mathbb{G}^k\), and applying the resulting linear map to \(\rho^{n_{kl}} u^k\), we obtain

\[
s^l_{lm}(\rho^{n_{kl}} u^k) = f^m \rho^m_{\rho^m_{lm}} \rho^{n_{kl}} u^k = \left( f^m_{\rho^m_{lm}} \rho^{n_{kl}} u^k \right) \rho^j_{kl},
\]

which establishes the projective property. The projective limit is

\[
\left( f_{\rho_{kl}}^{l,(1)} \rho^{n_{kl}} u^k, f_{\rho_{kl}}^{l,(1)} \rho^{n_{kl}} u^k, \ldots \right) \equiv d\bar{f}\bar{u},
\]

where \(l_k := \min\{k, n_k\}\). Let \(P \in \bar{U}(\bar{P})\); then

\[
\bar{U}(\bar{P})(\bar{f} \circ \bar{\phi}_1) = (\bar{f} \circ \bar{\phi}_1(\bar{P}))'(0) = d\bar{f}\bar{U}(\bar{P})\bar{\phi}_1 = d\bar{f}\bar{u} \circ \bar{\phi}_1(\bar{P}),
\]

which completes the proof. \(\square\)

Suppose that \(\bar{V} \in \Gamma_{pl}T\bar{M}\) is defined by the projective system of smooth maps \((v^k : \mathbb{G}^k \to \mathbb{G}^m_k, \quad k \in \mathbb{N}_0)\) for some \(m \in S\). By applying Proposition 5.1 to the projective system \((\phi_{\alpha}^{m_k} \circ (\phi_1^{m_k})^{-1} \circ v^k : \mathbb{G}^k \to \mathbb{G}^{m_k} (=: \mathbb{F}^k))\), we can define the \(\alpha\)-covariant derivative on \(\bar{M}: \nabla^\alpha_U \bar{V} \bar{\phi}_1 = \bar{w} \circ \bar{\phi}_1\), where

\[
\bar{w} = d(\bar{\phi}_1 \circ \bar{\phi}_1^{-1})d(\bar{\phi}_1 \circ \bar{\phi}_1^{-1})\bar{v}u = d\bar{v}\bar{u} + \frac{1}{2}u \bar{\nabla} \cdot \bar{u}.
\]

(47)

The \(\Phi_1\)-representation \(\bar{w}\) is the projective limit of the system \((w^k : \mathbb{G}^k \to \mathbb{G}^k, \quad k \in \mathbb{N}_0)\), where \((i_k = \min\{m_k, n_k, m_n\}, \quad k \in \mathbb{N}_0) \in S\), and so \(\nabla^\alpha_U \bar{V} \in \Gamma_{pl}T\bar{M}\). The remaining constructions in sections 3 and 4 carry over to \(\bar{M}\) without difficulty. Key points to note are as follows.
The smoothness of the $\alpha$-divergences on $\bar{M}$ follows from their smoothness on $M^k$, and that of the inclusion map $i^k$. The metric and covariant derivatives could be derived directly from $D_\alpha$ as in sections 3 and 4.

The statistical manifold $\bar{N}$ is defined in the obvious way. It is a Leslie $C^\infty$-embedded submanifold of $M$ since its $\phi_{-1}$-representation is a subspace of that of $\bar{M}$.

An $\alpha$-geodesic of $\bar{N}$ is a smooth curve $P$ whose projection $i^kP$ satisfies (36) for all $k$. ($\alpha$-geodesics of $\bar{M}$, and $\pm 1$-geodesics of $\bar{N}$ are, of course, straight lines in appropriate charts.)

References

[1] S.-I. Amari, H. Nagaoka, *Methods of Information Geometry*, Translations of Mathematical Monographs, 191, American Mathematical Society, Providence, 2000.

[2] N. Ay, J. Jost, H. V. Lê and L. Schwachhofer, Information geometry and sufficient statistics, *Probab. Theory Related Fields*, 162 (2015) 327–364.

[3] O.E. Barndorff-Nielsen, *Information and Exponential Families in Statistical Theory*, Wiley, 1978.

[4] M. Bauer, M. Bruveris, P.W. Michor, Uniqueness of the Fisher-Rao metric on the space of smooth densities, *Bull. London Math. Soc.*, 48 (2016) 499–506.

[5] M. Bruveris and P.W. Michor, Geometry of the Fisher-Rao metric on the space of smooth densities on a compact manifold, arXiv:1607.04550 (2016).

[6] A. Cena, G. Pistone, Exponential statistical manifold, *Ann. Inst. Statist. Math.*, 59 (2007) 27–56.

[7] N.N. Chentsov, *Statistical Decision Rules and Optimal Inference*, Translations of Mathematical Monographs, 53, American Mathematical Society, Providence, 1982.
[8] C.T.J. Dodson, G. Galanis, E. Vassiliou, *Geometry in a Fréchet Context: A Projective Limit Approach*, London Mathematical Society Lecture Note Series, 428, Cambridge University Press, 2016.

[9] S. Eguchi, Second order efficiency of minimum contrast estimators in a curved exponential family, *Ann. Statist.*, **11** (1983) 793–803.

[10] K. Fukumizu, Exponential manifold by reproducing kernel Hilbert spaces, in: P. Gibilisco, E. Riccomagno, M.P. Rogantin, H. Winn (eds.), *Algebraic and Geometric Methods in Statistics*, Cambridge University Press, (2009) 291–306.

[11] P. Gibilisco, G. Pistone, Connections on non-parametric statistical manifolds by Orlicz space geometry, *Infinite-dimensional analysis, Quantum Probability and Related Topics*, **1** (1998) 325–347.

[12] S.L. Lauritzen, *Statistical Manifolds*, IMS Lecture Notes Series, **10**, Institute of Mathematical Statistics, 1987.

[13] J.A. Leslie, On a differential structure for the group of diffeomorphisms, *Topology*, **46** (1967) 263–271.

[14] M.K. Murray, J.W. Rice, *Differential Geometry and Statistics*, Monographs in Statistics and Applied Probability, **48**, Chapman Hall, 1993.

[15] J. Naudts, *Generalised Thermostatistics*, Springer, London, 2011.

[16] N.J. Newton, An infinite-dimensional statistical manifold modelled on Hilbert space, *J. Functional Analysis*, **263** (2012) 1661–1681.

[17] N.J. Newton, Information geometric nonlinear filtering, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, **18** (2015) 1550014.

[18] N.J. Newton, Infinite-dimensional statistical manifolds based on a balanced chart, *Bernoulli*, **22** (2016) 711–731.

[19] F. Nielsen, F. Barbaresco (Eds.), Proceedings of GSI 2013 Conference, *Lecture Notes in Computer Science*, 8085, Springer, Berlin, 2013.

[20] F. Nielsen, F. Barbaresco (Eds.), Proceedings of GSI 2015 Conference, *Lecture Notes in Computer Science*, 9389, Springer, Berlin, 2015.
[21] G. Pistone, M.P. Rogantin, The exponential statistical manifold: mean parameters, orthogonality and space transformations, *Bernoulli*, 5 (1999) 721-760.

[22] G. Pistone, C. Sempi, An infinite-dimensional geometric structure on the space of all the probability measures equivalent to a given one, *Annals of Statistics*, 23 (1995) 1543–1561.

[23] C.R. Rao, Information and accuracy obtainable in the estimation of statistical parameters, *Bulletin of the Calcutta Mathematical Society*, 37 (1945) 81–91.