Response of Laminated Composite Cylindrical Shell Using Higher Order Shear Deformation Theory

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Abstract. The present work proposes the response of composite laminated cylindrical shell using higher order shear deformation theory (HSDT). For the finite element formulation a displacement field having five degrees of freedom is considered and a nine noded isoparametric element is used. The finite element model is developed using MATLAB code. The numerical results are obtained for the deflection of cylindrical shell for arbitrary boundary condition, stacking angle and radius to thickness ratio. The results are validated using literature and it shows good agreement.

1. Introduction
Thin-walled composite structures have their use in aerospace, civil and automobile sectors. Light weight structures are helpful because of their properties like high stiffness-to-weight ratios and longer fatigue life. These properties are found in composite materials. Most of the researchers are focused on laminated composite plates whereas only few are working on laminated composite cylinder and spherical shells.

Most of the theories which were developed for elastic shells are based on Love-Kirchoff [1] assumptions in which transverse shear strains are neglected. This theory provided sufficiently accurate results and was known as Love’s first approximation. However, it gave large amount of errors in deflections when applied on layered composite shells.

Koiter [2] later said that Love’s approximation is not useful. Then improved theories were developed by Hildebrand, Reissner and Thomas [3], Reissner [4]. Thermal expansion for cylindrical shells was proposed by Zukas and Vinson [6], Gulati and Essenberg [5], Whitney and Sun [8]. Theories of higher order are dependent on displacement field. For higher order theories, displacements of a point on surface are linear functions of its thickness and transverse displacement have quadratic function of thickness. These higher-order shear deformation theories does not require shear correction factor as was required for first-order shear deformation theory.

The present study uses higher-order shear deformation theory for laminated shells where cubic functions of thickness are considered as function for displacement of middle surface.
2. General Formulation
Consider the kinematics for the arbitrary shell described by orthogonal curvilinear coordinates, the following displacement field is considered,

\[
\tilde{u}(\xi_1, \xi_2, \xi_3) = \left(1 + \frac{\varepsilon_1}{R_1}\right) u_0 + \xi_3 \phi_1 + \xi_3^2 \psi_1 + \xi_3^3 \varphi_1
\]

\[
\tilde{v}(\xi_1, \xi_2, \xi_3) = \left(1 + \frac{\varepsilon_2}{R_2}\right) v_0 + \xi_3 \phi_2 + \xi_3^2 \psi_2 + \xi_3^3 \varphi_2
\]

\[
\tilde{w}(\xi_1, \xi_2, \xi_3) = w_0
\]

(1)

where, \((\tilde{u}, \tilde{v}, \tilde{w})\) are displacements along \((\xi_1, \xi_2, \xi_3)\) coordinates, \((u_0, v_0, w_0)\) are displacements of a point on middle surface and \(\phi_1\) and \(\phi_2\) are rotations along \(\xi_3 = 0\) of normal to the mid-surface with respect to \(\xi_1\) and \(\xi_2\)-axes, respectively.

Corresponding to displacement field, the strain vectors are

\[
\{\varepsilon\} = \begin{pmatrix}
\frac{\partial \tilde{u}}{\partial r} + \frac{\tilde{w}}{R_1} \\
\frac{\partial \tilde{v}}{\partial s} + \frac{\tilde{w}}{R_2} \\
\frac{\partial \tilde{u}}{\partial s} + \frac{\partial \tilde{v}}{\partial r} \\
\frac{\partial \tilde{w}}{\partial \xi_3} + \frac{\partial w}{\partial r} \\
\frac{\partial \tilde{v}}{\partial \xi_3} + \frac{\partial \tilde{w}}{\partial s}
\end{pmatrix}
\]

(2)

The linear strain vectors are represented as

\[
\{\varepsilon_i\} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}_i^T
\]

(3)

The associated strains for the assumed displacement field are

\[
\varepsilon_1 = \varepsilon_1^0 + \xi_3 \left(\kappa_1^1 + \xi_3^3 \kappa_1^2\right),
\]

\[
\varepsilon_2 = \varepsilon_2^0 + \xi_3 \left(\kappa_2^1 + \xi_3^3 \kappa_2^2\right),
\]

\[
\varepsilon_3 = 0,
\]

\[
\varepsilon_4 = \varepsilon_4^0 + \xi_3^2 \kappa_4^1,
\]

\[
\varepsilon_5 = \varepsilon_5^0 + \xi_3^2 \kappa_5^2,
\]

\[
\varepsilon_6 = \varepsilon_6^0 + \xi_3^2 \left(\kappa_6^1 + \xi_3^2 \kappa_6^2\right)
\]

(4)

The strain displacement relations are
\varepsilon_1^0 = \frac{\partial u_0}{\partial r} + \frac{w_0}{R_1} \quad \kappa_1^1 = \frac{\partial \phi_r}{\partial r} \quad \kappa_1^2 = -c \left( \frac{\partial \phi_r}{\partial r} + \frac{\partial \theta_r}{\partial r} \right) \\
\varepsilon_2^0 = \frac{\partial v_0}{\partial s} + \frac{w_0}{R_2} \quad \kappa_2^1 = \frac{\partial \phi_s}{\partial s} \quad \kappa_2^2 = -c \left( \frac{\partial \phi_s}{\partial s} + \frac{\partial \theta_s}{\partial s} \right) \\
\varepsilon_6^0 = \frac{\partial u_0}{\partial s} + \frac{\partial w_0}{\partial r} \quad \kappa_6^1 = \frac{\partial \phi_r}{\partial r} + \frac{\partial \phi_s}{\partial s} \quad \kappa_6^2 = -c \left( \frac{\partial \phi_r}{\partial r} + \frac{\partial \phi_s}{\partial s} + \frac{\partial \theta_r}{\partial r} + \frac{\partial \theta_s}{\partial s} \right) \\
\varepsilon_4^0 = \phi_s + \frac{\partial w_0}{\partial r} \quad \kappa_4^2 = -3c \left( \phi_s + \theta_s \right) \\
\varepsilon_5^0 = \phi_r + \frac{\partial w_0}{\partial s} \quad \kappa_5^2 = -3c \left( \phi_r + \theta_r \right)

where,

c = \frac{4}{3h^2}

From equations (1.3) and (1.4),

\{ \varepsilon_i \} = [T'] \{ \bar{\varepsilon}_i \}

where,

\[ [T'] = \begin{bmatrix}
1 & 0 & 0 & \xi_3 & 0 & 0 & \xi_3^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \xi_3 & 0 & 0 & \xi_3^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \xi_3 & 0 & 0 & \xi_3^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \xi_3^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \xi_3^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[ \{ \bar{\varepsilon}_i \}^T = \{ \varepsilon_1^0, \varepsilon_2^0, \varepsilon_6^0, \kappa_1^1, \kappa_2^1, \kappa_6^1, \kappa_1^2, \kappa_2^2, \kappa_6^2, \varepsilon_4^0, \varepsilon_5^0, \kappa_4^2, \kappa_5^2 \} \]

The stress-strain relation with respect to the fibre-matrix coordinates for a given lamina is

\[ \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yz}
\end{bmatrix} =
\begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\
0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\
0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
\]

Here \( (\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}) \) are the stresses and their strain \( (\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \) components corresponding to laminate co-ordinates \( (x, y, z) \). Where \( \overline{Q}_{ij} \) is described in Reddy’s book [12].

3. Finite Element Formulation

In the present finite element analysis, the displacement vector \( \{ \Delta \} \) is expressed as

\[ \{ \Delta \} = \sum_{n=1}^{NN} N_n \{ \Delta_n \} \]

where, \( NN = \) total nodes per element, \( N_n = \) shape functions for \( n^{th} \) node and
The strain-displacement relation is given by matrix \( [B] \) for \( i \)th node
\[
[B_i] = [L'] N_i
\]
where \([L']\) is called as differential operator.

The stiffness matrix for an element is given as
\[
[k_{ij}] = \int_{A_e} [B]^T [D] [B] dA
\]

Material property matrix is expressed as
\[
[D] = \sum_{k=1}^{N elementos} \int [T]^T \left[ \bar{Q} \right][T] d\xi_3 = \begin{bmatrix}
[A_1] & [B] & [E] & 0 & 0 \\
[B] & [C_1] & [F_1] & 0 & 0 \\
[E] & [F_1] & [H] & 0 & 0 \\
0 & 0 & 0 & [A_2] & [C_2] \\
0 & 0 & 0 & [C_2] & [F_2] \\
\end{bmatrix}
\]

with,
\[
\left( A_{i1}, B_{i1}, C_{i1}, E_{i1}, F_{i1}, H_{i1} \right) = \sum_{k=1}^{N elementos} \int \bar{Q}_{ij}^{(k)} \left( 1, \xi_3, \xi_3^2, \xi_3^3, \xi_3^4, \xi_3^6 \right) d\xi_3
\]
for \( i,j = 1,2,6 \)

\[
\left( A_{i2}, C_{i2}, F_{i2} \right) = \sum_{k=1}^{N elementos} \int \bar{Q}_{ij}^{(k)} \left( 1, \xi_3^2, \xi_3^4 \right) d\xi_3
\]
for \( i,j = 4,5 \)

The stiffness matrix for an element in natural coordinate system \((\xi, \eta)\) can be represented as
\[
[k_{ij}] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \det [J] d\xi d\eta
\]

where the Jacobean \([J]\) is
\[
[J] = \begin{bmatrix}
\sum_{i=1}^{NN} dN_i d\xi d\eta x_i & \sum_{i=1}^{NN} dN_i d\xi d\eta y_i & \sum_{i=1}^{NN} dN_i d\xi d\eta z_i \\
\sum_{i=1}^{NN} dN_i d\eta d\xi x_i & \sum_{i=1}^{NN} dN_i d\eta d\xi y_i & \sum_{i=1}^{NN} dN_i d\eta d\xi z_i \\
\sum_{i=1}^{NN} dN_i d\eta d\eta x_i & \sum_{i=1}^{NN} dN_i d\eta d\eta y_i & \sum_{i=1}^{NN} dN_i d\eta d\eta z_i \\
\end{bmatrix}
\]
\[
\det [J] = \sqrt{\left| J_1 \right|^2 + \left| J_2 \right|^2 + \left| J_3 \right|^2}
\]
\[
\left| J_1 \right| = \det \begin{bmatrix}
\sum_{i=1}^{NN} dN_i d\xi y_i & \sum_{i=1}^{NN} dN_i d\xi z_i \\
\sum_{i=1}^{NN} dN_i d\eta y_i & \sum_{i=1}^{NN} dN_i d\eta z_i \\
\end{bmatrix}
\]
\[
\left| J_2 \right| = \det \begin{bmatrix}
\sum_{i=1}^{NN} dN_i d\xi x_i & \sum_{i=1}^{NN} dN_i d\xi z_i \\
\sum_{i=1}^{NN} dN_i d\eta x_i & \sum_{i=1}^{NN} dN_i d\eta z_i \\
\end{bmatrix}
\]
\[ |J_2| = - \det \begin{bmatrix} \sum_{i=1}^{NN} \frac{dN_i}{d\xi} x_i & \sum_{i=1}^{NN} \frac{dN_i}{d\xi} z_i \\ \sum_{i=1}^{NN} \frac{dN_i}{d\eta} x_i & \sum_{i=1}^{NN} \frac{dN_i}{d\eta} z_i \end{bmatrix} \]

\[ |J_1| = \det \begin{bmatrix} \sum_{i=1}^{NN} \frac{dN_i}{d\xi} x_i & \sum_{i=1}^{NN} \frac{dN_i}{d\xi} y_i \\ \sum_{i=1}^{NN} \frac{dN_i}{d\eta} x_i & \sum_{i=1}^{NN} \frac{dN_i}{d\eta} y_i \end{bmatrix} \]  \hspace{1cm} (23)

### 4. Solution Technique

If a continuous body is in equilibrium, the virtual work of all actual forces, external and internal, in moving through a virtual displacement is zero. The principle of virtual work can be expressed as

\[ \delta U - \delta W = 0 \]  \hspace{1cm} (24)

where \( \delta U \) is the variation in internal energy and \( \delta W \) is variation in external energy.

Therefore, the change in strain energy is written as

\[ \delta U = \int_V \delta \varepsilon^T \sigma dV \]  \hspace{1cm} (25)

The variation of external work for transverse static loading can be written as

\[ \delta W = \int_\Omega \delta \{\Delta\}^T \begin{bmatrix} 0 & 0 & f_w & 0 & 0 & 0 & 0 \end{bmatrix}^T d\Omega \]  \hspace{1cm} (26)

where \( \{F\} = \int_\Omega \begin{bmatrix} 0 & 0 & f_w & 0 & 0 & 0 & 0 \end{bmatrix}^T d\Omega \) is the force vector.

For the present problem, the equilibrium equation for a laminate is obtained by minimizing the potential energy with respect to generalized displacements \( \{q\} \).

\[ \begin{bmatrix} K_{ij} \end{bmatrix} \{q\} = \{F_i\} \]  \hspace{1cm} (27)

where \( \begin{bmatrix} K_{ij} \end{bmatrix} = \sum_{e=1}^{NE} \begin{bmatrix} k_{ij} \end{bmatrix} \) is global stiffness matrix, \( \{F_i\} = \sum_{e=1}^{NE} \{F\}^e \) is a global force vector,

\[ \{q\} = \sum_{e=1}^{NE} \{q\}^e \] is a global deflection vector.

### 5. Numerical Table Results

**Table 1.** Comparison of maximum radial deflection (mm) using two different theories for a clamped cylindrical shell subjected to an internal pressure

| Laminate | Reddy [11] | Reddy [11] | Present | Present |
|----------|------------|------------|---------|---------|
|          | FSDT       | FSDT       | HSDT    | HSDT    |
| 0/90     | 9.5352     | 9.4666     | 12.0    | 10.8732 |
| 4×4 Q9   | 9.5352     | 9.4666     | 12.0    | 10.8732 |

Reddy (2004) solved the problem using First Order Shear Deformation Theory (FSDT). In this work Higher Order Shear Deformation Theory is considered and nine noded element is used.
The same example was solved by Kant and Menon (1989) using HSDT for fiber angles of [45°/-45°/-45°/45°] and [0°/90°/90°/0°]. Table 2 gives the non-dimensional radial displacement, for cylindrical shell having R/h ratios of 20 and 100. The displacements calculated using the present model are in good agreement with Kant and Menon.

**Table 2.** Non-dimensional radial displacement of laminated cylindrical shell subjected to internal pressure

\[ w = w_0 \left( \frac{E_i h}{pR^2} \right) \]

| Laminate | R/h | Kant and Menon [9] | Present 2×2 Q9 | Present 4×4 Q9 |
|----------|-----|------------------|----------------|--------------|
| 45/-45/-45/45 | 20  | 2.21             | 1.8831         | 2.0076       |
|           | 100 | 1.95             | 1.8655         | 2.149        |
| 0/90/90/0  | 20  | 1.67             | 1.4687         | 1.5679       |
|           | 100 | 1.55             | 1.4716         | 1.6900       |

In table 3 results of cross ply laminated cylinder is considered having the same properties as in Reddy (2004).

**Table 3.** Deflection of cylindrical laminated shell subjected to an internal pressure providing simply supported boundary condition

| Laminate | R/h | Present |
|----------|-----|---------|
| 0/90/90/0 | 5  | 1.1736  |
|          | 10 | 2.3471  |
|          | 20 | 4.6943  |
|          | 30 | 7.0414  |
|          | 50 | 11.7357 |

From figure 1, we can observe that the displacement for a particular lamination angle decreases with the increase in aspect ratio for different boundary conditions. The second graph shows the variation of displacement at different boundary conditions having different lamination angle \([0°/θ/0°]\) at aspect ratio 5.

![Figure 1: Graph of various boundary condition at different aspect ratios and lamination schemes](image)

6. **Summary and Conclusion**

- The finite element code was developed in MATLAB and was used to find the deflection for laminated composite cylindrical shells.
The results obtained using this method matches with that from the literature. Hence it can be accepted as a successful tool in exploring the deflection under different parameters.

From the study we can also conclude that higher order shear deformation theory using nine noded isoparametric elements shows better results than first order shear deformation theory.

For deformation in angle ply laminates there are slight discrepancies in the results. So a certain multiplying factor needs to be introduced to match the results with accurate solution available in literatures.

As radius to thickness ratio (R/h) increases the deformation shows decrease in value of results.

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