Electric quadrupole and magnetic octupole of the Δ

G. Ramalho\textsuperscript{1,2}, M.T. Peña\textsuperscript{2,3} and Franz Gross\textsuperscript{1,4}

\textsuperscript{1}Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA
\textsuperscript{2}Centro de Física Teórica e de Partículas, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
\textsuperscript{3}Department of Physics, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
\textsuperscript{4}College of William and Mary, Williamsburg, VA 23185, USA

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Using a covariant spectator constituent quark model we predict an electric quadrupole moment $Q_{\Delta^+} = -0.043$ fm$^2$ and a magnetic octupole moment $O_{\Delta^+} = -0.0035$ fm$^3$ for the $\Delta^+$ excited state of the nucleon.

Although it was the first nucleon resonance to be discovered, the properties of the $\Delta$ are almost completely unknown. Only the $\Delta^{++}$ and $\Delta^{+}$ magnetic moments have been measured, and these measurements have large error bars \cite{2, 3, 4}. Most of the information we have about the $\Delta$ comes from indirect information, such as the study of the $\gamma N \rightarrow \Delta$ transition \cite{4}.

The dominant $\Delta$ elastic form factors are the electric charge $G_{E0}$ and magnetic dipole $G_{M1}$. The subleading form factors are the electric quadrupole ($G_{E2}$) and magnetic octupole ($G_{M3}$). Those form factors measure the deviation of the charge and magnetic dipole distribution from a symmetric form \cite{3}. At $Q^2 = 0$ the form factors define the magnetic dipole $\mu_\Delta = G_{M1}(0)/2M_\Delta$, the electric quadrupole $Q_\Delta = G_{E2}(0)/2M_\Delta^2$, and the magnetic octupole $O_\Delta = G_{M3}(0)/2M_\Delta^3$ moments, where $\mu$ is the electric charge and $M_\Delta$ the $\Delta$ mass.

Until recently, there were essentially only theoretical predictions for $\mu_\Delta$ (see Ref. \cite{6} for details) and $Q_\Delta$ \cite{7, 8, 9, 10, 11, 12, 13, 14}. The exception was the pioneering work in lattice QCD \cite{13}, where all the form factors were estimated for low $Q^2$, although the statistics for $G_{E2}$ and $G_{M3}$ were very poor.

Recent lattice QCD calculations of all four form factors over a limited $Q^2$ range have revived interest in the $\Delta$ moments, especially the interesting quadrupole and octupole moments \cite{10, 11}. These results are obtained only for unpolarized pion masses in the range of 350-700 MeV so some extrapolation to the physical pion mass is required \cite{15, 16}. Still, in the absence of direct experimental information, lattice QCD provides the best reference for theoretical calculations. Stimulated by these new lattice results the covariant spectator quark model \cite{5} and chiral Quark-Soliton model ($\chi$QSM) \cite{20} have been used to estimate the $\Delta$ form factors. Simultaneously, a lattice technique based on the background-field method \cite{21} has been used to estimate the $\mu_\Delta$ with great precision \cite{22}.

The octupole moment $O_\Delta$ has also been evaluated by Buchmann \cite{2} using a deformed pion cloud model, and QCD sum rules (QCDSR) have been used to estimate both $Q_\Delta$ and $O_\Delta$ \cite{23}.

The size of the moments $Q_\Delta$ and $O_\Delta$ tells us if the $\Delta$ is deformed, and in which direction. The nucleon, as a spin $1/2$ particle, can have no electric quadrupole moment \cite{24} [although the possibility remains, as pointed out by Buchmann and Henley \cite{25}], that it might be a collective state with an intrinsic quadrupole moment. While the measurement of the quadrupole form factors for the $\gamma N \rightarrow \Delta$ transition gives some information about the deformation of the $\Delta$ \cite{26}, it is very important to obtain an independent estimate \cite{17, 27}. Motivated by these considerations, the Nicosia-MIT and the Adelaide groups are presently working on an evaluation of $G_{M3}$ using lattice QCD \cite{17, 28}. Also Ledwig and collaborators are working in the same subject \cite{20} using the $\chi$QSM.

In this Letter we use the covariant spectator formalism \cite{29} to evaluate $Q_\Delta$ and $O_\Delta$. Following previous work \cite{30, 31}, we describe the $\Delta$ as a quark-diquark system composed of a $S$-state with an admixture of two $D$ states

\begin{equation}
\Psi_\Delta(P,k) = N [\Psi_S + a\Psi_{D3} + b\Psi_{D1}] ,
\end{equation}

where $a$ is the mixture coefficient of the $D3$ state ($L = 2$, $S = 3/2$) and $b$ the mixture coefficient of the $D1$ state ($L = 2$, $S = 1/2$). Each of the states are separately normalized, so that $N = 1/\sqrt{1 + a^2 + b^2}$. The $S$, $D1$ and $D3$ wave functions are products of spin-isospin (and, for the $D$ states, $L = 2$) operators and an appropriate scalar wave function $\psi_S$, $\psi_{D1}$ and $\psi_{D3}$ which depends only the square of the momentum $(P - k)^2$ of the off-shell quark, where $k$ is the four-momentum of the on-shell diquark \cite{30}.

In this model \cite{5, 24, 30, 31, 32, 33} the $\Delta$ current can be written as

\begin{equation}
J^\mu = 3 \sum_\lambda \int_k \bar{\Psi}_\Delta(P_+, k) J_\lambda^\mu \Psi_\Delta(P_-, k) = N^2 J_S^\mu + aN^2 J_{D3}^\mu + bN^2 J_{D1}^\mu ,
\end{equation}

where $P_-$ ($P_+$) is the initial (final) $\Delta$ momentum, and the sum is over all polarizations ($\lambda$) of the diquark, and the covariant integral $\int_k \equiv \int \frac{d^4k}{(2\pi)^4}$. Where $E_s$ is the
diquark energy. Additional terms proportional to $a^2N^2$, $b^2N^2$ and $abN^2$ can be neglected if $a$ and $b$ are small. The quark current $j_{\tau}^\mu$ in Eq. (2) includes a dependence on the quark and $\tau$ charges and anomalous magnetic moments $\kappa_u$ and $\kappa_d$. See Refs. [6, 22] for details.

The current (2) can be written in a standard form involving four basic form factors, denoted $F_i^\tau$, $i = 1 - 4$. The electric and magnetic moments are linear combinations of these [6, 22, 34, 35], and at $Q^2 = 0$, to *first order in the mixing coefficients* $a$ and $b$, they become

$$
G_{E0}(0) = N^2 e \Delta T_S
$$
$$
G_{M1}(0) = N^2 f_\Delta T_S
$$
$$
G_{E2}(0) = 3(aN^2)e \Delta T_{D3}
$$
$$
G_{M3}(0) = f_\Delta N^2 [a I_{D3} + 2b I_{D1}],
$$
(3)

where $f_\Delta = \epsilon_\Delta + M_N N_\Delta/M_N$,

$$
e_\Delta = \frac{1}{2}(1 + \bar{T}_3), \quad \kappa_\Delta = \frac{1}{2}(\kappa_+ - \kappa_- \bar{T}_3),
$$

$$\kappa_+ = 2\kappa_u - \kappa_d, \quad \kappa_- = \frac{3}{2}\kappa_u + \frac{1}{2}\kappa_d,
$$
(4)

with $\bar{T}_3 = \text{diag}(3, 1, -1, -3)$, and

$$
I_{D3} = \lim_{\tau \to \infty} \frac{1}{\tau} \int k b(k, q, P_+ + k) \psi_{D3}(P_+, k) \psi_S(P_-, k)
$$

$$
I_{D1} = \lim_{\tau \to \infty} \frac{1}{\tau} \int k b(k, q, P_+ + k) \psi_{D1}(P_+, k) \psi_S(P_-, k),
$$

with $\tau = Q^2/(4M^2_\Delta)$ and $b(k, q, P_+) \approx Y_{20}(k)$ as defined in Ref. [30]. The S-state wave function is normalized to unity (so that $T_S = 1$), and to *first order in the mixing coefficients* $a$ and $b$, $N^2 \to 1$ so $G_{E0}(0) = e_\Delta$, giving the correct charge. The multipole moments E2 and M3 are fixed by the factors $I_{D3}$ and $I_{D3}$, and are zero if there are no D states. In particular, $G_{E2}(0)$ is determined only by $I_{D3}$, although $G_{M3}(0)$ can depend on a delicate balance between $I_{D3}$, $I_{D1}$ and the coefficients $a$ and $b$.

To illustrate how lattice data can be used to constrain models, we show results from two different parameterizations for the $\Delta$ wave functions. The first one, denoted by Spectator 1 (Sp 1), is model 4 of Ref. [31]. That model fixed the pion cloud contribution (using a simple parameterization) and adjusted the remaining valence contribution to fit the $\gamma N \to \Delta$ data. The second parameterization, from Ref. [31] and denoted Spectator 2 (Sp 2), uses the same functional form for the valence part of the D-state wave functions, but fits the valence part of the wave function directly to the lattice data [30]. Because the pion mass used in these lattice calculations is large, the pion cloud effects are negligible at the lattice “point” and provide a better determination of the valence quark contribution at that point. After the fit is made, the results are extrapolated to the physical “point” by replacing the masses of the nucleon, $\Delta$, and $\rho$ meson (all parameters that enter into the functional form of the wave functions and currents) to their physical masses. We believe that model Sp 2 gives a more reliable parameterization of the $\Delta$ wave function, but we compare it to model Sp 1 to show the impact of using the lattice data to constrain the fit. In the first model (Sp 1) there is a mixture of 0.88% of D3 state and 4.36% of D1 state; the second model (Sp 2) has a mixture of 0.72% for both the D3 and D1 states.

In this letter we restrict our discussion to the moments $Q_\Delta$ and $C_\Delta$, which are extracted from the values of the form factors $G_{E2}$ and $G_{M3}$ at $Q^2 = 0$. A more complete study will be presented in a future work [32]. Our results are true predictions; once the $\gamma N \to \Delta$ reaction has been described no additional parameters are adjusted. The results for $G_{E2}(0)$ are presented in Table I and for $G_{M3}(0)$ in Table II. These are obtained from the integrals $T_{D3} = \tau = -7.00$ and $I_{D1} = 1.59$ for Sp 1 and $T_{D3} = -6.65$ and $I_{D1} = 0.24$ for Sp 2.

Before comparing our results with other models note that Sp 1 and Sp 2 give similar predictions for the quadrupole moment but very different predictions for the octupole moment. Clearly the octupole moment is more sensitive to the details of the model, and it is only the strong constraint imposed by the lattice data that allows us to predict that $G_{M3}(0) \simeq -1.70$.

Both tables compare our results with predictions of other models. In Table I we include the classic nonrelativistic quark model (NRQM) from Isgur et al. [36], where the tensor color hyperfine interaction requires a mixture of D-state quarks with S-state quarks. This description
considers only the valence degrees of freedom, and the contribution for the electric quadrupole moment is determined by both the mixture coefficients and a confinement parameter [8, 12]. In these models the contribution for the electric quadrupole can be estimated in impulse approximation [10, 38] from

\[ Q_{\Delta}^{(imp)} = \frac{2}{3} e_{\Delta} r_n^2, \tag{5} \]

where \( r_n^2 \) is the neutron squared radius in fm\(^2\). Using a recent value of \( r_n^2 = -0.116 \) fm\(^2\), we obtain \( Q_{\Delta} \sim -0.0464 \) fm\(^2\), or \( G_{E2}^\Delta(0) \approx -1.81 \) in close agreement with the values from Ref. [10] quoted in the table. (For a review of the earlier results, see Ref. [10].) Similar results are obtained by Buchmann et al. [12] using a constituent quark model with a D-state admixture [8, 38] with a slightly different confinement parameterization and an impulse approximation to the one-body current.

In the same work [12], an estimate of the nonvalence contributions, based on a two-body exchange current representative of the nonvalence degrees of freedom, is obtained. These nonvalence contributions are the dominant ones, and assuming no D-state admixture, can be estimated from

\[ Q_{\Delta}^{(exc)} = e_{\Delta} r_n^2. \tag{6} \]

Although developed in the constituent quark formalism this relation is parameter independent [12]. The expression (6) has also been derived in the large \( N_c \) limit [13]. Later, the expression (6) was improved using a general parameterization (GP) of QCD [3, 39, 40], with the inclusion of higher order terms, and used to extract \( G_{E2}^\Delta(0) = -7.02 \pm 4.05 \) from the \( \gamma N \rightarrow \Delta \) electric quadrupole data [4]. All of these results seem to suggest that the contribution of the pion cloud to the quadrupole moment could be quite large. On the other hand, calculations based on \( \chi PT \) [11], and recent results derived in a \( \chi QSM \) [20] all of which include the pion cloud, suggest that the pion cloud effect might be smaller than estimates based on Eq. (6). From this we conclude that model calculations of the size of the pion cloud contribution to the quadrupole moment are inconclusive.

Finally, the tables show the lattice QCD simulations [17] based on three different approaches: a quenched calculation using a Wilson action with \( u \) and \( d \) quarks, a dynamical calculation using a Wilson action including \( u \) and \( d \) sea quarks, and a hybrid action which also includes strange sea quarks. The lattice data is however limited by the significant error bars that prevent an accurate extrapolation to \( Q^2 = 0 \) (assuming a dipole or an exponential dependence on \( Q^2 \)) [17] and by heavy pion masses (which require an extrapolation in \( m_f \)). Even so, the size of the hybrid calculation may be an indicator that the meson cloud contribution to \( G_{E2} \) is not negligible, although not comparable with (6). Quark models can be important for extrapolating the lattice data to \( Q^2 = 0 \) and to the physical pion mass. In any case, the predictions of our model should be compared to other calculations of the valence quark contributions to these moments.

The \( Q^2 \) dependence of the \( \Delta^+ \) form factors \( G_{E2} \) and \( G_{M3} \) are shown in figure 1. Our results are completely consistent with the \( Q^2 \) dependence of the lattice calculations [16, 17]. Future lattice QCD simulations would be important for a more precise constraint on \( O_{\Delta^+} \).

In conclusion, using our best model (Sp 2) we predict

\[ Q_{\Delta^+} = -0.043 \, \text{efm}^2 \quad O_{\Delta^+} = -0.0035 \, \text{efm}^3. \tag{7} \]

This estimate for \( O_{\Delta^+} \) lies between the negligible predictions of QCD sum rules and the high estimate of Buchmann [5] based on a pion cloud model and the GP formalism [8, 13, 40]. As we have previously emphasized, the small result for \( O_{\Delta^+} \) obtained from Sp 1 shows the importance of using the lattice data to constrain the model: without this constraint the uncertainty in our prediction of \( O_{\Delta^+} \) would be much larger.

Using the “minimal electromagnetic current” defined in the historical literature, these results imply an oblate form for both charge and magnetic distributions of the \( \Delta^+ \). However, the electromagnetic coupling recently de-
scribed by Alexandrou et al.\cite{Alexandrou:2007}, predicts that even a point-like $\Delta$ will have “natural” moments of $Q_{\Delta^+} = -0.077 (G_{E_2}^\Delta(0) = -3)$ and $O_{\Delta^+} = -0.0021 (G_{M_3}^\Delta(0) = -1)$ leading to a different interpretation of our results.

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