ELEMENTARY EVALUATION OF CONVOLUTION SUMS INVOLVING PRIMITIVE DIRICHLET CHARACTERS FOR A CLASS OF POSITIVE INTEGERS

EBÉNÉZER NTIENJEM

ABSTRACT. We extend the results obtained by E. Ntienjem [23] to all positive integers. Let \( \mathbb{N} \) be the subset of \( \mathbb{N} \) consisting of \( 2^v \mathbb{N} \), where \( v \) is in \{0, 1, 2, 3\} and \( \mathbb{U} \) is a squarefree finite product of distinct odd primes. We discuss the evaluation of the convolution sum,

\[
\sum_{\substack{l,m \in \mathbb{N}^2 \\ \alpha \cdot l + \beta \cdot m = n}} \sigma(l) \sigma(m), \quad \text{when } \alpha \beta \text{ is in } \mathbb{N} \setminus \mathbb{N}.
\]

The evaluation of convolution sums belonging to this class is achieved by applying modular forms and primitive Dirichlet characters. In addition, we revisit the evaluation of the convolution sums for \( \alpha \beta = 9, 16, 18, 25, 36 \). If \( \alpha \beta \equiv 0 \pmod{4} \), we determine natural numbers \( a, b \) and use the evaluated convolution sums together with other known convolution sums to carry out the number of representations of \( n \) by the octonary quadratic forms \( \alpha \left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right) + b \left( x_5^2 + x_6^2 + x_7^2 + x_8^2 \right) \). Similarly, if \( \alpha \beta \equiv 0 \pmod{3} \), we compute natural numbers \( c, d \) and make use of the evaluated convolution sums together with other known convolution sums to determine the number of representations of \( n \) by the octonary quadratic forms \( c \left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right) + d \left( x_5^2 + x_6^2 + x_7^2 + x_8^2 \right) \). We illustrate our method with the explicit examples \( \alpha \beta = 3^2 \cdot 5, \alpha \beta = 2^4 \cdot 3, \alpha \beta = 2^4 \cdot 5^2 \), and \( \alpha \beta = 2^6 \).

1. INTRODUCTION

In this work, we denote by \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) the sets of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. Let \( i, j, k, l, m, n \in \mathbb{N} \) in the sequel. The sum of positive divisors of \( n \) to the power of \( k \), \( \sigma_k(n) \), is defined by

\[(1.1) \quad \sigma_k(n) = \sum_{0 \leq d \mid n} d^k.
\]

We let \( \sigma(n) \) stand as a synonym for \( \sigma_1(n) \). For \( m \notin \mathbb{N} \) we set \( \sigma_k(m) = 0 \).

Let \( \alpha, \beta \in \mathbb{N} \) be such that \( \alpha \leq \beta \). We define the convolution sum, \( W_{(\alpha,\beta)}(n) \), as follows:

\[(1.2) \quad W_{(\alpha,\beta)}(n) = \sum_{(l,m) \in \mathbb{N}^2, \alpha \cdot l + \beta \cdot m = n} \sigma(l) \sigma(m).
\]

We write \( W_{\beta}(n) \) as a synonym for \( W_{(1,\beta)}(n) \). If for all \( (l,m) \in \mathbb{N}^2 \) it holds that \( \alpha l + \beta m \neq n \), then we set \( W_{(\alpha,\beta)}(n) = 0 \).

So far known convolution sums are displayed on Table 1.

| Level \( \alpha \beta \) | Authors | References |
|-----------------------|---------|------------|
| 1                     | M. Besge, J. W. L. Glaisher, S. Ramanujan | [7][11][27] |

2010 Mathematics Subject Classification. 11A25, 11F11, 11F20, 11E20, 11E25, 11F27.

Key words and phrases. Sums of Divisors; Dedekind eta function; Convolution Sums; Modular Forms; Dirichlet Characters; Eisenstein forms; Cusp Forms; Octonary quadratic Forms; Number of Representations.
\[ \alpha \beta = 2^\nu \prod_{j \geq 2} p_j, \quad \gcd(\alpha, \beta) = 1, \quad 0 \leq \nu \leq 3, \quad \kappa \in \mathbb{N}, \quad p_j > 2 \text{ distinct primes} \]

Table 1: Known convolution sums \( W_{(\alpha, \beta)}(n) \) of level \( \alpha \beta \)

| Level \( \alpha \beta \) | Authors |
|------------------------|---------|
| 2, 3, 4                | J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams |
| 5, 7                   | M. Lemire & K. S. Williams, S. Cooper & P. C. Toh |
| 6                      | S. Alaca & K. S. Williams |
| 8, 9                   | K. S. Williams |
| 10, 11, 13, 14         | E. Royer |
| 12, 16, 18, 24         | A. Alaca & S. Alaca & K. S. Williams |
| 15                     | B. Ramakrishnan & B. Sahu |
| 10, 20                 | S. Cooper & D. Ye |
| 23                     | H. H. Chan & S. Cooper |
| 25                     | E. X. W. Xia & X. L. Tian & O. X. M. Yao |
| 27, 32                 | S. Alaca & Y. Kesicioğlu |
| 36                     | D. Ye |
| 14, 26, 28, 30         | E. Ntienjem |
| 22, 44, 52             | E. Ntienjem |
| 33, 40, 56             | E. Ntienjem |

Let \( \mathcal{N} = \{ 2^\nu \mathcal{U} | \nu \in \{0, 1, 2, 3\} \text{ and } \mathcal{U} \text{ is a squarefree finite product of distinct odd primes} \} \) be a subset of \( \mathbb{N} \).

We evaluate the convolution sum, \( W_{(\alpha, \beta)}(n) \), for the class of natural numbers \( \alpha \beta \) such that \( \alpha \beta \in (\mathbb{N} \setminus \mathcal{N}) \). We use Dirichlet characters and modular forms to evaluate these convolution sums.

We observe that the positive integers \( \alpha \beta = 9, 16, 18, 25, 32, 36 \) from Table 1 belong to the class of integers for which the evaluation of the convolution sum is discussed in this paper. From these integers, the convolution sums for \( \alpha \beta = 27, 32 \) are evaluated using the approach that we are generalizing in the sequel. We revisit the evaluation of the convolution sums for \( \alpha \beta = 9, 16, 18, 25, 36 \) using our method.

We use the result from the above general case to obtain the evaluation of the convolution sum for \( \alpha \beta = 3^2 \cdot 5 \), \( \alpha \beta = 2^4 \cdot 3 \), \( \alpha \beta = 2 \cdot 5^2 \) and \( \alpha \beta = 2^6 \). These convolution sums have not been evaluated as yet.

As an application, convolution sums are used to determine explicit formulae for the number of representations of a positive integer \( n \) by the octonary quadratic forms

(1.3) \[ a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2) \]

and

(1.4) \[ c(x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2) + d(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2), \]
respectively, where \( a, b, c, d \in \mathbb{N} \).

So far known explicit formulae for the number of representations of \( n \) by the octonary form Equation 1.3 are displayed in Table 2.

| (a, b)          | Authors                              | References |
|-----------------|--------------------------------------|------------|
| (1,1),(1,3),    | E. Ntienjem                          | [23]       |
| (1,9),(2,3)     |                                      |            |
| (1,2)           | K. S. Williams                        | [31]       |
| (1,4)           | A. Alaca & Ş. Alaca & K. S. Williams | [2]        |
| (1,5)           | S. Cooper & D. Ye                    | [10]       |
| (1,6)           | B. Ramakrishnan & B. Sahu            | [26]       |
| (1,7)           | E. Ntienjem                           | [22]       |
| (1,8)           | Ş. Alaca & Y. Kesicioğlu             | [5]        |
| (1,11),(1,13)   | E. Ntienjem                           | [24]       |
| (1,10),(1,14),  |                                      |            |
| (2,5),(2,7),    |                                      |            |

\[
ab = 2^\nu \prod_{j \geq 2} p_j, \quad \text{where} \quad \gcd(a, b) = 1, \ 0 \leq \nu \leq 1, \quad \kappa \in \mathbb{N}, \ p_j > 2 \text{ distinct primes}
\]

Table 2: Known representations of \( n \) by the form Equation 1.3

Similarly, so far known explicit formulae for the number of representations of \( n \) by the octonary form Equation 1.4 are referenced in Table 3.

| (c, d) | Authors                              | References |
|--------|--------------------------------------|------------|
| (1,1)  | G. A. Lomadze                        | [18]       |
| (1,2)  | Ş. Alaca & K. S. Williams            | [6]        |
| (1,3)  | K. S. Williams                       | [30]       |
| (1,4),(1,6), | A. Alaca & Ş. Alaca & K. S. Williams | [1,2,3]   |
| (1,8),(2,3) |                                      |            |
| (1,5)  | B. Ramakrishnan & B. Sahu           | [26]       |
| (1,9)  | Ş. Alaca & Y. Kesicioğlu            | [4]        |
| (1,10), (2,5) |                                      |            |
| (1,12),(3,4) |                                      |            |
| (1,11), |                                      |            |

\[
cd = 2^\nu \prod_{j \geq 3} p_j, \quad \text{where} \quad \gcd(c, d) = 1, \ 0 \leq \nu \leq 3, \quad \kappa \in \mathbb{N}, \ p_j > 3 \text{ distinct primes}
\]

Table 3: Known representations of \( n \) by the form Equation 1.4

We also determine explicit formulae for the number of representations of a positive integer \( n \) by such octonary quadratic forms whenever \( \alpha \beta \equiv 0 \pmod{4} \) or \( \alpha \beta \equiv 0 \pmod{3} \).
We then use the convolution sums, \( W_{(\alpha, \beta)}(n) \), where \( \alpha \beta = 3^2 \cdot 5 \cdot 2^4 \cdot 3.2^5 \), to give examples of explicit formulae for the number of representations of a positive integer \( n \) by the octonary quadratic forms \( \text{Equation 1.3} \) and \( \text{Equation 1.4} \).

This paper is organized as follows. In Section 2 we discuss modular forms and briefly define \( \eta \)-functions and convolution sums. We assume that \( \alpha \beta \) has the above form and then discuss the evaluation of the convolution sum, \( W_{(\alpha, \beta)}(n) \), in Section 3. In Section 4 and Section 5 we discuss a technique for computing all pairs of natural numbers \( (a, b) \) and \( (c, d) \), and then determine explicit formulae for the number of representations of \( n \) by the octonary form \( \text{Equation 1.3} \) and \( \text{Equation 1.4} \) when \( \alpha \beta \equiv 0 \pmod{4} \) or \( \alpha \beta \equiv 0 \pmod{3} \). In Section 6, we evaluate the convolution sums \( W_{(1,45)}(n), W_{(5,9)}(n), W_{(1,48)}(n), W_{(3,16)}(n), W_{(1,35)}(n), W_{(2,25)}(n) \) and \( W_{(1,64)}(n) \); then in Section 8, we make use of these convolution sums and other known convolution sums to determine an explicit formula for the number of representations of a positive integer \( n \) by the octonary quadratic form

- \( \text{Equation 1.3} \) where \( (a, b) \) stands for \( (1, 12) \), \( (3, 4) \), \( (1, 16) \).
- \( \text{Equation 1.4} \) where \( (c, d) \) stands for \( (1, 15) \), \( (3, 5) \), \( (1, 16) \).

The evaluation of the convolution sums for \( \alpha \beta = 9, 16, 18, 25, 36 \) is revisited in Section 9. Outlook and concluding remarks are made in Section 10.

Software for symbolic scientific computation is used to obtain the results of this paper. This software comprises the open source software packages \textit{GiNaC, Maxima, Reduce, Sage} and the commercial software package \textit{MAPLE}.

2. Essentials to the Understanding of the Problem

2.1. Modular Forms. Let \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \), be the upper half-plane and let \( \Gamma = G = \text{SL}_2(\mathbb{R}) = \{ (a \ b) \quad (c \ d) \in \mathbb{R} \mid ad - bc = 1 \} \) be the group of \( 2 \times 2 \)-matrices. Let \( N \in \mathbb{N} \), then

\[ \Gamma(N) = \{ (a \ b) \in \text{SL}_2(\mathbb{Z}) \mid (a \ b) \equiv (1 \ 0) \pmod{N} \} \]

is a subgroup of \( G \) and is called a \textit{principal congruence subgroup of level} \( N \). A subgroup \( H \) of \( G \) is called a \textit{congruence subgroup of level} \( N \) if it contains \( \Gamma(N) \).

For our purposes the following congruence subgroup is relevant:

\[ \Gamma_0(N) = \{ (a \ b) \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \} \].

Let \( k, N \in \mathbb{N} \) and let \( \Gamma' \subset \Gamma \) be a congruence subgroup of level \( N \in \mathbb{N} \). Let \( k \in \mathbb{Z}, \gamma \in \text{SL}_2(\mathbb{Z}) \) and \( f : \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \to \mathbb{C} \cup \{ \infty \} \). We denote by \( f^{[\delta \kappa]}(z) \) the function whose value at \( z \) is \( (cz + d)^{-k}f(\gamma(z)) \), i.e., \( f^{[\delta \kappa]}(z) = (cz + d)^{-k}f(\gamma(z)) \). The following definition is according to N. Koblitz [14, p. 108].

**Definition 2.1.** Let \( N \in \mathbb{Z}, k \in \mathbb{Z}, f \) be a meromorphic function on \( \mathbb{H} \) and \( \Gamma' \subset \Gamma \) a congruence subgroup of level \( N \).

- (a) \( f \) is called a \textit{modular function of weight} \( k \) for \( \Gamma' \) if
  - (a1) for all \( \gamma \in \Gamma' \) it holds that \( f^{[\delta \kappa]} = f \).
  - (a2) for any \( \delta \in \Gamma \) it holds that \( f^{[\delta \kappa]}(z) \) has the form \( \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \delta z} \) and \( a_n \neq 0 \) for finitely many \( n \in \mathbb{Z} \) such that \( n < 0 \).
- (b) \( f \) is called a \textit{modular form of weight} \( k \) for \( \Gamma' \) if
  - (b1) \( f \) is a modular function of weight \( k \) for \( \Gamma' \).
  - (b2) \( f \) is holomorphic on \( \mathbb{H} \).
  - (b3) for all \( \delta \in \Gamma \) and for all \( n \in \mathbb{Z} \) such that \( n < 0 \) it holds that \( a_n = 0 \).
- (c) \( f \) is called a \textit{cusp form of weight} \( k \) for \( \Gamma' \) if
(c1) \( f \) is a modular form of weight \( k \) for \( \Gamma' \),
(c2) for all \( \delta \in \Gamma \) it holds that \( a_0 = 0 \).

Let us denote by \( \mathcal{M}_k(\Gamma') \) the set of modular forms of weight \( k \) for \( \Gamma' \), by \( \mathcal{S}_k(\Gamma') \) the set of cusp forms of weight \( k \) for \( \Gamma' \) and by \( \mathcal{E}_k(\Gamma') \) the set of Eisenstein forms. The sets \( \mathcal{M}_k(\Gamma') \), \( \mathcal{S}_k(\Gamma') \) and \( \mathcal{E}_k(\Gamma') \) are vector spaces over \( \mathbb{C} \). Therefore, \( \mathcal{M}_k(\Gamma_0(N)) \) is the space of modular forms of weight \( k \) for \( \Gamma_0(N) \), \( \mathcal{S}_k(\Gamma_0(N)) \) is the space of cusp forms of weight \( k \) for \( \Gamma_0(N) \), and \( \mathcal{E}_k(\Gamma_0(N)) \) is the space of Eisenstein forms. Consequently, W. A. Stein [29, p. 81] has shown that \( \mathcal{M}_k(\Gamma_0(N)) = \mathcal{E}_k(\Gamma_0(N)) \oplus \mathcal{S}_k(\Gamma_0(N)). \)

We assume in this paper that \( 4 \leq k \) is even and that \( \chi \) and \( \psi \) are primitive Dirichlet characters with conductors \( L \) and \( R \), respectively. W. A. Stein [29, p. 86] has noted that

\[
E_{k,\chi,\psi}(q) = C_0 + \sum_{n=1}^{\infty} \left( \sum_{d|n} \psi(d)\chi\left(\frac{n}{d}\right) d^{k-1} \right) q^n,
\]

where

\[
C_0 = \begin{cases} 
0 & \text{if } L > 1 \\
-\frac{B_{k,\chi}}{2\pi} & \text{if } L = 1
\end{cases}
\]

and \( B_{k,\chi} \) are the generalized Bernoulli numbers. Theorems 5.8 and 5.9 in Section 5.3 of W. A. Stein [29, p. 86] are then applicable.

2.2. \( \eta \)-Quotients. On the upper half-plane \( \mathbb{H} \), the Dedekind \( \eta \)-function, \( \eta(z) \), is defined by \( \eta(z) = e^{\frac{2\pi iz}{24}} \prod_{n=1}^{\infty} \left( 1 - e^{2\pi inz} \right) \). Let us set \( q = e^{2\pi iz} \). Then it follows that

\[
\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} F(q), \quad \text{where } F(q) = \prod_{n=1}^{\infty} (1 - q^n).
\]

We will use eta function, eta quotient and eta product interchangeably as synonyms.

M. Newman [20] applied the Dedekind \( \eta \)-function to systematically construct modular forms for \( \Gamma_0(N) \). Newman then establishes conditions (i)-(iv) in the following theorem. G. Ligozat [17] determined the order of vanishing of an \( \eta \)-function at all cusps of \( \Gamma_0(N) \), which is condition (v) or (\( \nu' \)) in the following theorem.

L. J. P. Kilford [13, p. 99] and G. Köhler [15, p. 37] have formulated the following theorem; it will be used to exhaustively determine \( \eta \)-quotients, \( f(z) \), which belong to \( \mathcal{M}_k(\Gamma_0(N)) \), and especially those \( \eta \)-quotients which are in \( \mathcal{S}_k(\Gamma_0(N)) \).

**Theorem 2.2** (M. Newman and G. Ligozat). Let \( \mathbb{N}, \mathbb{D}(N) \) be the set of all positive divisors of \( N \), \( \delta \in \mathbb{D}(N) \) and \( r_\delta \in \mathbb{Z} \). Let furthermore \( f(z) = \prod_{\delta \in \mathbb{D}(N)} \eta^{r_\delta}(\delta z) \) be an \( \eta \)-quotient.

If the following five conditions are satisfied

(i) \( \sum_{\delta \in \mathbb{D}(N)} \delta r_\delta \equiv 0 \pmod{24} \), \hspace{1cm} (ii) \( 0 \leq \sum_{\delta \in \mathbb{D}(N)} \delta r_\delta \equiv 0 \pmod{24} \),

(iii) \( \prod_{\delta \in \mathbb{D}(N)} \delta^{r_\delta} \) is a square in \( \mathbb{Q} \), \hspace{1cm} (iv) \( 0 < \sum_{\delta \in \mathbb{D}(N)} r_\delta \equiv 0 \pmod{4} \),

(v) for each \( d \in \mathbb{D}(N) \) it holds that \( \sum_{\delta \in \mathbb{D}(N)} \frac{\gcd(\delta,d)^2}{\delta} r_\delta \geq 0 \),

then

\( f(z) \in \mathcal{M}_k(\Gamma_0(N)) \), where \( k = \frac{1}{2} \sum_{\delta \in \mathbb{D}(N)} r_\delta \).

Moreover, the \( \eta \)-quotient \( f(z) \) is an element of \( \mathcal{S}_k(\Gamma_0(N)) \) if (v) is replaced by

(v') for each \( d \in \mathbb{D}(N) \) it holds that \( \sum_{\delta \in \mathbb{D}(N)} \frac{\gcd(\delta,d)^2}{\delta} r_\delta > 0 \).
2.3. **Convolution Sums** \( W_{(\alpha, \beta)}(n) \). Given \( \alpha, \beta \in \mathbb{N} \) such that \( \alpha \leq \beta \), let the convolution sum be defined by [Equation 1.2](#).

E. Ntienjem [22, 23] has shown and A. Alaca et al. [11] has remarked that one can simply assume that \( \gcd(\alpha, \beta) = 1 \).

Let \( q \in \mathbb{C} \) be such that \( |q| < 1 \). Let furthermore \( \chi \) and \( \psi \) be primitive Dirichlet characters with conductors \( L \) and \( R \), respectively. We assume that \( \chi = \psi \) and that \( \chi \) is a Kronecker symbol in the following. Then the following Eisenstein series hold.

\[
L(q) = E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n, \\
M(q) = E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \\
M_\chi(q) = E_4(q) = C_0 + \sum_{n=1}^{\infty} \chi \sigma_3(n)q^n.
\]

Note that \( M(q) \) is a special case of [Equation 2.1](#) or [Equation 2.4](#). We state two relevant results for the sequel of this work.

**Lemma 2.3.** Let \( \alpha, \beta \in \mathbb{N} \). Then

\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in \mathfrak{M}_4(\Gamma_0(\alpha \beta)).
\]

**Proof.** See E. Ntienjem [23]. \( \square \)

**Theorem 2.4.** Let \( \alpha, \beta, N \in \mathbb{N} \) be such that \( N = \alpha \beta, \alpha < \beta, \) and \( \alpha \) and \( \beta \) are relatively prime. Then

\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = (\alpha - \beta)^2 + 240 \alpha^2 \sigma_3(\frac{n}{\alpha}) + 240 \beta^2 \sigma_3(\frac{n}{\beta}) + 1152 \alpha \beta W_{(\alpha, \beta)}(n)q^n.
\]

**Proof.** See E. Ntienjem [23]. \( \square \)

3. **Evaluating** \( W_{(\alpha, \beta)}(n) \), **where** \( \alpha \beta \in \mathbb{N} \setminus \mathbb{N} \)

We carry out an explicit formula for the convolution sum \( W_{(\alpha, \beta)}(n) \).

3.1. **Bases of** \( \mathfrak{E}_4(\Gamma_0(\alpha \beta)) \) **and** \( \mathfrak{S}_4(\Gamma_0(\alpha \beta)) \). Let \( \mathcal{D}(\alpha \beta) \) denote the set of all positive divisors of \( \alpha \beta \).

A. Pizer [25] has discussed the existence of a basis of the space of cusp forms of weight \( k \geq 2 \) for \( \Gamma_0(\alpha \beta) \) when \( \alpha \beta \) is not a perfect square. We apply the dimension formulae in T. Miyake’s book [19, Thrm 2.5.2, p. 60] or W. A. Stein’s book [29, Prop. 6.1, p. 91] to conclude that

- for the space of Eisenstein forms

\[
\dim(\mathfrak{E}_4(\Gamma_0(\alpha \beta))) = \sum_{d|\alpha \beta} \varphi(\gcd(d, \frac{\alpha \beta}{d})) = m_E,
\]

where \( m_E \in \mathbb{N} \) and \( \varphi \) is the Euler’s totient function.

- for the space of cusp forms \( \dim(\mathfrak{S}_4(\Gamma_0(\alpha \beta))) = m_S, \) where \( m_S \in \mathbb{N} \).
We use Theorem 2.2 (ii) – (vi') to exhaustively determine as many elements of the space \( \mathcal{S}_4(\Gamma_0(\alpha\beta)) \) as possible. From these elements of the space \( \mathcal{S}_4(\Gamma_0(\alpha\beta)) \) we select relevant ones for the purpose of the determination of a basis of this space.

Let \( \mathcal{C} \) denote the set of Dirichlet characters \( \chi = \left( \frac{m}{n} \right) \) as assumed in \( \text{Equation 2.4} \), where \( m, n \in \mathbb{Z} \) and \( \left( \frac{m}{n} \right) \) is the Kronecker symbol. Let furthermore \( D(\chi) \subseteq D(\alpha\beta) \) denote the subset of \( D(\alpha\beta) \) associated with the character \( \chi \).

**Theorem 3.1.**

(a) The set \( \mathcal{B}_E = \{ M(q') \mid t \in D(\alpha\beta) \} \cup \bigcup_{\chi \in \mathcal{C}} \{ M_\chi(q') \mid t \in D(\chi) \} \) is a basis of \( \mathcal{S}_4(\Gamma_0(\alpha\beta)) \).

(b) Let \( 1 \leq i \leq m_s \) be positive integers, \( \delta \in D(\alpha\beta) \) and \( (r(i, \delta))_{i, \delta} \) be a table of the powers of \( \eta(\delta z) \). Let furthermore \( \mathcal{B}_{\alpha\beta,i}(q) = \prod_{\delta \in \alpha\beta} \eta^{(r(i, \delta))}(\delta z) \) be selected elements of \( \mathcal{S}_4(\Gamma_0(\alpha\beta)) \). Then the set \( \mathcal{B}_S = \{ \mathcal{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq m_s \} \) is a basis of \( \mathcal{S}_4(\Gamma_0(\alpha\beta)) \).

(c) The set \( \mathcal{B}_E = \mathcal{B}_E \cup \mathcal{B}_S \) constitutes a basis of \( \mathcal{M}_4(\Gamma_0(\alpha\beta)) \).

**Remark 3.2.**

r1) Each eta quotient \( \mathcal{B}_{\alpha\beta,i}(q) \) is expressible in the form \( \sum_{n=1}^{\infty} b_{\alpha\beta,i}(n)q^n \), where for each \( n \geq 1 \) the coefficient \( b_{\alpha\beta,i}(n) \) is an integer.

r2) When we divide the sum that results from \( \text{Theorem 2.2} (vi') \), when \( d = N \), by 24, then we obtain the smallest positive degree \( q \) in \( \mathcal{B}_{\alpha\beta,i}(q) \).

**Proof.**

(a) W. A. Stein [29] Thms 5.8 and 5.9, p. 86] has shown that for each \( t \) positive divisor of \( \alpha\beta \) it holds that \( M(q') \) is in \( \mathcal{M}_4(\Gamma_0(t)) \). Since \( \mathcal{M}_4(\Gamma_0(t)) \) is a vector space, it also holds for each Kronecker symbol \( \chi \in \mathcal{C} \) and \( s \in D(\chi) \) that \( M_\chi(q') \) is in \( \mathcal{M}_4(\Gamma_0(s)) \).

Since the dimension of \( \mathcal{E}_4(\Gamma_0(\alpha\beta)) \) is finite, it suffices to show that \( \mathcal{B}_E \) is linearly independent. Suppose that for each \( \chi \in \mathcal{C}, s \in D(\chi) \) we have \( z(\chi)_s \in \mathbb{C} \) and that for each \( t | \alpha\beta \) we have \( x_t \in \mathbb{C} \). Then

\[ \sum_{t | \alpha\beta} x_t M(q') + \sum_{\chi \in \mathcal{C}} \left( \sum_{s \in D(\chi)} z(\chi)_s M_\chi(q') \right) = 0. \]

We recall that \( \chi \) is a Kronecker symbol; therefore, for all \( 0 \neq a \in \mathbb{Z} \) it holds that \( \left( \frac{a}{n} \right) = 0 \). Then we equate the coefficients of \( q^n \) for \( n \in D(\alpha\beta) \cup \bigcup_{\chi \in \mathcal{C}} s \in D(\chi) \) to obtain the homogeneous system of linear equations in \( m_E \) unknowns:

\[ \sum_{n | \alpha\beta} \sigma_3(\frac{t}{n}) x_n + \sum_{\chi \in \mathcal{C}} \sum_{s \in D(\chi)} \chi \sigma_3(\frac{t}{s}) Z(\chi)_s = 0, \quad t \in D(\alpha\beta). \]

The determinant of the matrix of this homogeneous system of linear equations is not zero. Hence, the unique solution is \( x_t = z(\chi)_s = 0 \) for all \( t \in D(\alpha\beta) \) and for all \( \chi \in \mathcal{C}, s \in D(\chi) \). So, the set \( \mathcal{B}_E \) is linearly independent and hence is a basis of \( \mathcal{S}_4(\Gamma_0(\alpha\beta)) \).

(b) We show that each \( \mathcal{B}_{\alpha\beta,i}(q) \), where \( 1 \leq i \leq m_s \), is in the space \( \mathcal{S}_4(\Gamma_0(\alpha\beta)) \). This is obviously the case since \( \mathcal{B}_{\alpha\beta,i}(q) \), \( 1 \leq i \leq m_s \), are obtained using an exhaustive search which applies items (i) – (vi) in \( \text{Theorem 2.2} \).

Since the dimension of \( \mathcal{S}_4(\Gamma_0(\alpha\beta)) \) is finite, it suffices to show that the set \( \mathcal{B}_S \) is linearly independent. Suppose that \( x_t \in \mathbb{C} \) and \( \sum_{i=1}^{m_s} x_i \mathcal{B}_{\alpha\beta,i}(q) = 0 \). Then

\[ \sum_{i=1}^{m_s} x_i \mathcal{B}_{\alpha\beta,i}(q) = \sum_{i=1}^{m_s} \left( \sum_{n=1}^{m_s} x_i b_{\alpha\beta,i}(n) \right) q^n = 0 \]

which gives the homogeneous system.
of \( m_S \) linear equations in \( m_S \) unknowns:

\[
\sum_{i=1}^{m_S} b_{\alpha\beta}(n)x_i = 0, \quad 1 \leq n \leq m_S.
\]

Two cases arise:

The smallest degree of \( \mathcal{B}_{\alpha\beta}(q) \) is \( i \) for each \( 1 \leq i \leq m_S \): Then the square matrix which corresponds to this homogeneous system of \( m_S \) linear equations is triangular with 1’s on the diagonal. Hence, the determinant of that matrix is 1 and so the unique solution is \( x_i = 0 \) for all \( 1 \leq i \leq m_S \).

The smallest degree of \( \mathcal{B}_{\alpha\beta}(q) \) is \( i \) for \( 1 \leq i < m_S \): Let \( n' \) be the largest positive integer such that \( 1 \leq i \leq n' < m_S \). Let \( \mathcal{B}'_S = \{ \mathcal{B}_{\alpha\beta}(q) \mid 1 \leq i \leq n' \} \) and \( \mathcal{B}''_S = \{ \mathcal{B}_{\alpha\beta}(q) \mid n' < i \leq m_S \} \). Then \( \mathcal{B}_S = \mathcal{B}'_S \cup \mathcal{B}''_S \) and we may consider \( \mathcal{B}_S \) as an ordered set. By the case above, the set \( \mathcal{B}'_S \) is linearly independent. Hence, the linear independence of the set \( \mathcal{B}_S \) depends on that of the set \( \mathcal{B}''_S \).

Let \( A = (b_{\alpha\beta}(n)) \) be the \( m_S \times m_S \) matrix in Equation 3.2. If \( \det(A) \neq 0 \), then \( x_i = 0 \) for all \( 1 \leq i \leq m_S \) and we are done. Suppose that \( \det(A) = 0 \). Then for some \( n' < k \leq m_S \) there exists \( \mathcal{B}_{\alpha\beta,k}(q) \) which is causing the system of equations to be inconsistent. We substitute \( \mathcal{B}_{\alpha\beta,k}(q) \) with, say \( \mathcal{B}_{\alpha\beta,k}(q) \), which does not occur in \( \mathcal{B}_S \) and compute the determinant of the new matrix \( A \).

Since there are finitely many \( \mathcal{B}_{\alpha\beta,k}(q) \) with \( n' < k \leq m_S \) that may cause the system of linear equations to be inconsistent and finitely many elements of \( \mathcal{M}_4(\Gamma_0(\alpha\beta)) \setminus \mathcal{B}_S \), the procedure will terminate with a consistent system of linear equations. Since A. Pizer \[25\] has proved the existence of a basis for the space of cusps, we will find a basis of \( \mathcal{M}_4(\Gamma_0(\alpha\beta)) \).

Therefore, the set \( \{ \mathcal{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq m_S \} \) is linearly independent and hence is a basis of \( \mathcal{M}_4(\Gamma_0(\alpha\beta)) \).

(c) Since \( \mathcal{M}_4(\Gamma_0(\alpha\beta)) = \mathcal{E}_4(\Gamma_0(\alpha\beta)) \oplus \mathcal{E}_4(\Gamma_0(\alpha\beta)) \), the result follows from (a) and (b).

Note that if \( \mathcal{C} = \emptyset \), that means that the Dirichlet character is trivially one, then Theorem 3.1 proved by E. Ntienjem \[23\] is obtained as an immediate corollary.

3.2. Evaluating the convolution sum \( W_{(\alpha,\beta)}(n) \). We recall the assumption that \( \chi \neq 1 \) since the case \( \chi = 1 \) has been discussed by E. Ntienjem \[23\].

**Lemma 3.3.** Let \( \alpha, \beta \in \mathbb{N} \) be such that \( \gcd(\alpha, \beta) = 1 \). Let furthermore \( \mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S \) be a basis of \( \mathcal{M}(\Gamma_0(\alpha\beta)) \). Then there exist \( X_{\delta}, Z(\chi), Y_j \in \mathbb{C}, 1 \leq j \leq m_S, \chi \in \mathbb{C}, s \in D(\chi) \) and \( \delta|\alpha\beta \) such that

\[
(\alpha L(q^n) - \beta L(q^\delta))^2 = \sum_{\delta|\alpha\beta} X_{\delta} + \sum_{\chi \in \mathbb{C}, s \in D(\chi)} C_{\delta}(\chi) Z(\chi) + \sum_{n=1}^{\infty} \left( 240 \sum_{\delta|\alpha\beta} \sigma_{1}(\delta) X_{\delta} + \sum_{\chi \in \mathbb{C}, s \in D(\chi)} \sigma_{1}(\delta) \chi Z(\chi) + \sum_{j=1}^{m_S} b_{j}(n) Y_j \right) q^n.
\]
Proof. That \((\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in \mathbb{M}_4(\Gamma_0(\alpha\beta))\) follows from Lemma 2.3. Hence, by Theorem 3.1 (c), there exist \(X_\delta, Z(\chi)_s, Y_j \in \mathbb{C}, 1 \leq j \leq m_S, \chi \in \mathbb{C}, s \in D(\chi)\) and \(\delta, \alpha, \beta\), such that

\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = \sum_{\delta, \alpha, \beta} X_\delta M(q^\delta) + \sum_{\chi \in \mathbb{C}} \sum_{s \in D(\chi)} Z(\chi)_s M_X(q^\delta)
\]

\[
+ \sum_{j=1}^{m_S} Y_j M(q^\delta) = \sum_{\chi \in \mathbb{C}} \sum_{s \in D(\chi)} C_0 Z(\chi)_s + \sum_{n=1}^\infty \left( 240 \sum_{\delta, \alpha, \beta} \sigma_3 \left( \frac{n}{\delta} \right) X_\delta \right.
\]

\[
+ \sum_{\chi \in \mathbb{C}} \sum_{s \in D(\chi)} \chi \sigma_3 \left( \frac{n}{s} \right) Z(\chi)_s + \sum_{j=1}^{m_S} b_j(n) Y_j \bigg) q^n.
\]

We equate the right hand side of Equation 3.3 with that of Equation 2.5 to obtain

\[
\sum_{n=1}^\infty \left( 240 \sum_{\alpha, \beta} \sigma_3 \left( \frac{n}{\delta} \right) X_\delta \right) + \sum_{n=1}^\infty \left( \sum_{\chi \in \mathbb{C}} \sum_{s \in D(\chi)} \chi \sigma_3 \left( \frac{n}{s} \right) Z(\chi)_s \right) + \sum_{j=1}^{m_S} Y_j b_j(n) q^n
\]

\[
= \sum_{n=1}^\infty \left( 240 \alpha^2 \sigma_3 \left( \frac{n}{\alpha} \right) + 240 \beta^2 \sigma_3 \left( \frac{n}{\beta} \right) + 48 \alpha (\beta - 6n) \sigma \left( \frac{n}{\alpha} \right)
\]

\[
+ 48 \beta (\alpha - 6n) \sigma \left( \frac{n}{\beta} \right) - 1152 \alpha \beta W_{(\alpha, \beta)}(n) \right) q^n.
\]

We then take the coefficients of \(q^n\) such that \(n\) is in \(D(\alpha \beta)\) and \(1 \leq n \leq m_S\), but as many as the unknown \(X_\delta, Z(\chi)_s, Y_j\), for all \(\chi \in \mathbb{C}, s \in D(\chi)\), and \(Y_1, Y_2, \ldots, Y_{m_S}\), to obtain a system of \(m_E + m_S\) linear equations whose unique solution determines the values of the unknowns. Hence, we obtain the result. \(\square\)

For the following theorem, let for the sake of simplicity \(X_\delta, Z(\chi)_s\) and \(Y_j\) stand for their values obtained in the previous theorem.

Theorem 3.4. Let \(n\) be a positive integer. Then

\[
W_{(\alpha, \beta)}(n) = -\frac{5}{24} \alpha \beta \sum_{\delta, \alpha, \beta} \sigma_3 \left( \frac{n}{\delta} \right) X_\delta - \frac{1}{1152} \alpha \beta \sum_{\chi \in \mathbb{C}} \sum_{s \in D(\chi)} Z(\chi)_s \sigma_3 \left( \frac{n}{s} \right)
\]

\[
+ \frac{5}{24} \alpha \beta \left( \alpha^2 - X_\alpha \right) \sigma_3 \left( \frac{n}{\alpha} \right) + \frac{5}{24} \beta \left( \alpha^2 - X_\beta \right) \sigma_3 \left( \frac{n}{\beta} \right)
\]

\[
- \sum_{j=1}^{m_S} \frac{1}{1152} \alpha \beta b_j(n) Y_j + \left( \frac{1}{24} - \frac{1}{4} \alpha \beta \right) \sigma \left( \frac{n}{\alpha} \right) + \left( \frac{1}{24} - \frac{1}{4} \alpha \beta \right) \sigma \left( \frac{n}{\beta} \right)
\]

Proof. We equate the right hand side of Equation 3.3 with that of Equation 2.5 to yield

\[
1152 \alpha \beta W_{(\alpha, \beta)}(n) = -240 \sum_{\delta, \alpha, \beta} \sigma_3 \left( \frac{n}{\delta} \right) X_\delta - \sum_{\chi \in \mathbb{C}} \sum_{s \in D(\chi)} Z(\chi)_s \sigma_3 \left( \frac{n}{s} \right)
\]

\[
- \sum_{j=1}^{m_S} b_j(n) Y_j + 240 \alpha^2 \sigma_3 \left( \frac{n}{\alpha} \right) + 240 \beta^2 \sigma_3 \left( \frac{n}{\beta} \right)
\]

\[
+ 48 \alpha (\beta - 6n) \sigma \left( \frac{n}{\alpha} \right) + 48 \beta (\alpha - 6n) \sigma \left( \frac{n}{\beta} \right).
\]

We then solve for \(W_{(\alpha, \beta)}(n)\) to obtain the stated result. \(\square\)
Remark 3.5. (a) As observed by E. Ntienjem [23], the following part of Theorem 3.4 depends only on \( n, \alpha \) and \( \beta \) but not on the basis of the modular space \( \mathcal{M}_4(\Gamma_0(\alpha\beta)) \):
\[
\left( \frac{1}{24} - \frac{1}{4\beta} n \right) \sigma(n) + \left( \frac{1}{24} - \frac{1}{4\alpha} n \right) \sigma(n) \beta.
\]
(b) If \( \mathcal{C} = \emptyset \), that means that the Dirichlet character is trivially one, then Theorem 3.2 proved by E. Ntienjem [23] is obtained as an immediate corollary of Theorem 3.4.
(c) For all \( \chi \in \mathcal{C} \) and for all \( s \in D(\chi) \) the value of \( Z(\chi)_s \) appears to be zero in all explicit examples evaluated as yet. Will the value of \( Z(\chi)_s \) always vanish for all \( \alpha\beta \) belonging to this class?

We now have the prerequisite to determine a formula for the number of representations of a positive integer \( n \) by the octonary quadratic form.

4. Number of Representations of a Positive Integer \( n \) by the Octonary Quadratic Form

In this section, we only consider those \( \alpha\beta \in \mathbb{N} \setminus \mathfrak{N} \) such that \( \alpha\beta \equiv 0 \pmod{4} \). That means, for a given \( \kappa \in \mathbb{N} \), we restrict the form of \( \alpha\beta \) to
\[
\alpha\beta = 2^{e_1} \prod_{j=1}^{\kappa} p_j^{e_j}, \quad \text{where } e_1 \geq 2 \text{ and } e_j \geq 2 \text{ for at least one } 2 \leq j \leq \kappa.
\]

4.1. Determining \( (a, b) \in \mathbb{N}^2 \). This approach is similar to the one given by E. Ntienjem [23].

Let \( \Lambda = 2^{e_1-2} \prod_{j=1}^{\kappa} p_j^{e_j} \), the set \( \mathcal{P} = \{ p_1 = 2^{e_1-2} \} \cup \{ p_j^{e_j} \mid 1 < j \leq \kappa \} \), and \( \mathcal{P}(\mathcal{P}) \) be the power set of \( \mathcal{P} \). Then for each \( Q \in \mathcal{P}(\mathcal{P}) \) we define \( \mu(Q) = \prod_{q \in Q} q \). We set \( \mu(\emptyset) = 1 \) if \( Q = \emptyset \).

\[\Omega_4 = \{ (\mu(Q_1), \mu(Q_2)) \mid \text{there exist } Q_1, Q_2 \in \mathcal{P}(\mathcal{P}) \text{ such that } \gcd(\mu(Q_1), \mu(Q_2)) = 1 \text{ and } \mu(Q_1) \mu(Q_2) = \Lambda \}.\]

Observe that \( \Omega_4 \neq \emptyset \) since \( (1, \Lambda) \in \Omega_4 \).

Proposition 4.1. Suppose that \( \alpha\beta \) has the above form and suppose that \( \Omega_4 \) is defined as above. Then for all \( n \in \mathbb{N} \) the set \( \Omega_4 \) contains all pairs \( (a, b) \in \mathbb{N}^2 \) such that \( N(a,b)(n) \) can be obtained by applying \( W_{(a,b)}(n) \).

Proof. Similar to the proof given by E. Ntienjem [23]. \( \square \)

4.2. Number of Representations of a positive Integer. As an immediate application of Theorem 3.4, the number of representations of a positive integer \( n \) by the octonary quadratic form \( a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2) \) is determined.

Let \( n \in \mathbb{N} \) and the number of representations of \( n \) by the quaternary quadratic form \( x_1^2 + x_2^2 + x_3^2 + x_4^2 \) be denoted by \( r_4(n) \). That means,
\[r_4(n) = \operatorname{card}\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid m = x_1^2 + x_2^2 + x_3^2 + x_4^2 \} \].

We set \( r_4(0) = 1 \). K. S. Williams [22] has shown that for all \( n \in \mathbb{N} \)
\[
r_4(n) = 8\sigma(n) - 32\sigma(n/4).
\]
Now, let the number of representations of \( n \) by the octonary quadratic form
\[
a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2)
\]
be \( N_{(a,b)}(n) \). That means,
\[
N_{(a,b)}(n) = \text{card}\{ (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) \in \mathbb{Z}_{\geq 0}^8 \mid n = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2) \}.
\]

We then derive the following result:

**Theorem 4.2.** Let \( n \in \mathbb{N} \) and \((a,b) \in \Omega_4\). Then
\[
N_{(a,b)}(n) = 8\sigma\left(\frac{n}{a}\right) - 32\sigma\left(\frac{n}{4a}\right) + 8\sigma\left(\frac{n}{b}\right) - 32\sigma\left(\frac{n}{4b}\right) + 64W_{(a,b)}(n)
+ 1024W_{(a,b)}\left(\frac{n}{4}\right) - 256\left(W_{(4a,b)}(n) + W_{(a,4b)}(n)\right).
\]

**Proof.** It holds that
\[
N_{(a,b)}(n) = \sum_{(l,m) \in \mathbb{N}^2, a l + bm = n} r_4(l)r_4(m) = r_4\left(\frac{n}{a}\right)r_4(0) + r_4(0)r_4\left(\frac{n}{b}\right) + \sum_{(l,m) \in \mathbb{N}^2, a l + bm = n} r_4(l)r_4(m)
\]
We use Equation 4.2 to derive
\[
N_{(a,b)}(n) = 8\sigma\left(\frac{n}{a}\right) - 32\sigma\left(\frac{n}{4a}\right) + 8\sigma\left(\frac{n}{b}\right) - 32\sigma\left(\frac{n}{4b}\right)
+ \sum_{(l,m) \in \mathbb{N}^2, a l + bm = n} (8\sigma(l) - 32\sigma\left(\frac{l}{4}\right))(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)).
\]
We observe that
\[
(8\sigma(l) - 32\sigma\left(\frac{l}{4}\right))(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)) = 64\sigma(l)\sigma(m) - 256\sigma\left(\frac{l}{4}\right)\sigma(m)
- 256\sigma(l)\sigma\left(\frac{m}{4}\right) + 1024\sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right).
\]
We assume in the sequel of this proof that the evaluation of
\[
W_{(a,b)}(n) = \sum_{(l,m) \in \mathbb{N}^2, a l + bm = n} \sigma(l)\sigma(m),
\]
\( W_{(4a,b)}(n) \) and \( W_{(a,4b)}(n) \) are known. We map \( l \) to \( 4l \) and \( m \) to \( 4m \) to derive
\[
W_{(4a,b)}(n) = \sum_{(l,m) \in \mathbb{N}^2, a l + bm = n} \sigma\left(\frac{l}{4}\right)\sigma(m) = \sum_{(l,m) \in \mathbb{N}^2, 4a l + bm = n} \sigma(l)\sigma(m)
\]
and
\[
W_{(a,4b)}(n) = \sum_{(l,m) \in \mathbb{N}^2, a l + bm = n} \sigma(l)\sigma\left(\frac{m}{4}\right) = \sum_{(l,m) \in \mathbb{N}^2, a l + 4b m = n} \sigma(l)\sigma(m),
\]
respectively. We simultaneously map \( l \) to \( 4l \) and \( m \) to \( 4m \) to infer
\[
\sum_{(l,m) \in \mathbb{N}^2, a l + bm = 2} \sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right) = \sum_{(l,m) \in \mathbb{N}^2, a l + bm = n} \sigma(l)\sigma(m) = W_{(a,b)}\left(\frac{n}{4}\right).
\]
We put these evaluations together to obtain the stated result for $N_{(a,b)}(n)$. \qed

5. Number of Representations of a Positive Integer $n$ by the Octonary Quadratic Form \textbf{Equation 1.4}

We now consider those $\alpha \beta \in \mathbb{N} \setminus \mathbb{R}$ for which $\alpha \beta \equiv 0 \pmod{3}$. That means, for a given $\kappa \in \mathbb{N}$, we consider the restricted form of $\alpha \beta$

\begin{equation}
\alpha \beta = 2^{e_1} \cdot 3^{e_2} \prod_{j>2}^{\kappa} p_j^{e_j}, \text{ where } p_j \geq 5 \text{ and at least for one } 2 \leq j \leq \kappa \text{ we have } e_j \geq 2.
\end{equation}

5.1. Determining $(c,d) \in \mathbb{N}^2$. This method is similar to the one given by E. Ntienjem \cite{23}. The construction is almost the same as the one given in Subsection 4.1.\footnote{127x690}

Let $\Delta = 2^{e_1} \cdot 3^{e_2-1} \prod_{j>2}^{\kappa} p_j^{e_j}$, the set $P = \{p_1 = 2^{e_1}, p_2 = 3^{e_2-1}\} \cup \{p_j^{e_j} | 2 \leq j \leq \kappa\}$, and $\mathcal{P}(P)$ be the power set of $P$. Then for each $Q \in \mathcal{P}(P)$ we define $\mu(Q) = \prod q$. We set $\mu(\emptyset) = 1$ if $Q = \{\emptyset\}$. Let now $\Omega_3$ be defined as in Subsection 4.1 with $\Delta$ instead of $\Lambda$, i.e.,

$\Omega_3 = \{\mu(Q_1), \mu(Q_2)) | \text{ there exist } Q_1, Q_2 \in \mathcal{P}(P) \text{ such that } \gcd(\mu(Q_1), \mu(Q_2)) = 1 \text{ and } \mu(Q_1) \mu(Q_2) = \Delta\}.

Note that $\Omega_3 \neq \emptyset$ since $(1, \Delta) \in \Omega_3$.

**Proposition 5.1.** Suppose that $\alpha \beta$ has the above form and Suppose that $\Omega_3$ be defined as above. Then for all $n \in \mathbb{N}$ the set $\Omega_3$ contains all pairs $(c,d) \in \mathbb{N}^2$ such that $R_{(c,d)}(n)$ can be obtained by applying $W_{(a,b)}(n)$.

**Proof.** Similar to the proof of Proposition 4.1. \qed

5.2. Number of Representations of a Positive Integer. The number of representations of a positive integer $n$ by the octonary quadratic form $c(x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2) + d(x_5x_6 + x_6^2 + x_7x_8 + x_8^2)$ is determined as an immediate application of Theorem 3.4.

Let $n \in \mathbb{N}$ and let $s_4(n)$ denote the number of representations of $n$ by the quaternary quadratic form $x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2$, that is

$s_4(n) = \text{card}(\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 | n = x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2\}).$

We set $s_4(0) = 1$. J. G. Huard et al. \cite{12} and G. A. Lomadze \cite{18} have proved that for all $n \in \mathbb{N}$

\begin{equation}
s_4(n) = 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right).
\end{equation}

Now, let the number of representations of $n$ by the octonary quadratic form

$c(x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2) + d(x_5x_6 + x_6^2 + x_7x_8 + x_8^2)

by R_{(c,d)}(n)$. That is,

$R_{(c,d)}(n) = \text{card}(\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 | n = c(x_1^2 + x_1x_2 + x_2^2

\begin{equation}
\begin{aligned}
+ x_3x_4 + x_4^2) + d(x_5^2 + x_5x_6 + x_6^2 + x_7x_8 + x_8^2\})\}.
\end{aligned}
\end{equation}

We infer the following result.
Theorem 5.2. Let \( n \in \mathbb{N} \) and \((c, d) \in \Omega_3\). Then
\[
R_{(c,d)}(n) = 12\sigma\left(\frac{n}{c}\right) - 36\sigma\left(\frac{n}{3c}\right) + 12\sigma\left(\frac{n}{d}\right) - 36\sigma\left(\frac{n}{3d}\right) + 144W_{(c,d)}(n) + 1296W_{(3c,d)}(n) - 432\left(W_{(3c,d)}(n) + W_{(c,3d)}(n)\right).
\]

Proof. It is obvious that
\[
R_{(c,d)}(n) = \sum_{(l,m)\in\mathbb{N}^2, \quad cl+dm=n} s_4(l)s_4(m) = s_4\left(\frac{n}{c}\right)s_4(0) + s_4(0)s_4\left(\frac{n}{d}\right) + \sum_{(l,m)\in\mathbb{N}^2, \quad cl+dm=n} s_4(l)s_4(m).
\]

We make use of Equation 5.2 to deduce
\[
R_{(c,d)}(n) = 12\sigma\left(\frac{n}{c}\right) - 36\sigma\left(\frac{n}{3c}\right) + 12\sigma\left(\frac{n}{d}\right) - 36\sigma\left(\frac{n}{3d}\right) + \sum_{(l,m)\in\mathbb{N}^2, \quad cl+dm=n} (12\sigma(l) - 36\sigma\left(\frac{l}{3}\right))(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)).
\]

We observe that
\[
(12\sigma(l) - 36\sigma\left(\frac{l}{3}\right))(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)) = 144\sigma(l)\sigma(m) - 432\sigma\left(\frac{l}{3}\right)\sigma(m) - 432\sigma(l)\sigma\left(\frac{m}{3}\right) + 1296\sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right).
\]

We assume that the evaluation of
\[
W_{(c,d)}(n) = \sum_{(l,m)\in\mathbb{N}^2, \quad cl+dm=n} \sigma(l)\sigma(m),
\]

\(W_{(3c,d)}(n)\) and \(W_{(c,3d)}(n)\) are known. We apply the transformations \(m\) to \(3m\) and \(l\) to \(3l\) to infer
\[
\sum_{(l,m)\in\mathbb{N}^2, \quad cl+dm=n} \sigma(l)\sigma\left(\frac{m}{3}\right) = \sum_{(l,m)\in\mathbb{N}^2, \quad cl+3dm=n} \sigma(l)\sigma(m) = W_{(3c,d)}(n)
\]

and
\[
\sum_{(l,m)\in\mathbb{N}^2, \quad 3cl+dm=n} \sigma(m)\sigma\left(\frac{l}{3}\right) = \sum_{(l,m)\in\mathbb{N}^2, \quad 3cl+dm=n} \sigma(l)\sigma(m) = W_{(3c,d)}(n),
\]

respectively. We simultaneously map \(l\) to \(3l\) and \(m\) to \(3m\) deduce
\[
\sum_{(l,m)\in\mathbb{N}^2, \quad cl+dm=n} \sigma\left(\frac{m}{3}\right)\sigma\left(\frac{l}{3}\right) = \sum_{(l,m)\in\mathbb{N}^2, \quad cl+3dm=n} \sigma(l)\sigma(m) = W_{(c,d)}\left(\frac{n}{3}\right).
\]

We put these evaluations together to obtain the stated result for \(R_{(c,d)}(n)\). \(\square\)

6. EVALUATION OF THE CONVOLUTION SUMS \(W_{(\alpha,\beta)}(n)\) WHEN \(\alpha\beta = 45, 48, 50, 64\)

In this Section, we give explicit formulae for the convolution sums \(W_{(1,45)}(n)\), \(W_{(5,9)}(n)\), \(W_{(1,48)}(n)\), \(W_{(3,16)}(n)\), \(W_{(1,50)}(n)\), \(W_{(2,25)}(n)\) and \(W_{(1,64)}(n)\).

The two convolution sums \(W_{(1,50)}(n)\) and \(W_{(2,25)}(n)\) are worth mentioning due to the fact that the set of divisors of 50 which are associated with the Dirichlet character for the formation of a basis of the space of Eisenstein forms is the whole set of divisors of 50.
6.1. **Bases for** $\mathcal{E}_4(\Gamma_0(\alpha\beta))$ **and** $\mathcal{E}_4(\Gamma_0(\alpha\beta))$ **when** $\alpha\beta = 45, 48, 50, 64$. The dimension formulae for the space of cusp forms as given in T. Miyake’s book [19 Thm 2.5.2, p. 60] or W. A. Stein’s book [29 Prop. 6.1, p. 91] and **Equation 3.1** are applied to compute

$$\dim(\mathcal{E}_4(\Gamma_0(45))) = 8, \quad \dim(\mathcal{E}_4(\Gamma_0(48))) = 14,$$
$$\dim(\mathcal{E}_4(\Gamma_0(50))) = \dim(\mathcal{E}_4(\Gamma_0(64))) = 12,$$
$$\dim(\mathcal{E}_4(\Gamma_0(48))) = \dim(\mathcal{E}_4(\Gamma_0(50))) = \dim(\mathcal{E}_4(\Gamma_0(64))) = 18.$$

We use **Theorem 2.2** to determine $\eta$-quotients which are elements of $\mathcal{E}_4(\Gamma_0(45)), \mathcal{E}_4(\Gamma_0(48)), \mathcal{E}_4(\Gamma_0(50))$ and $\mathcal{E}_4(\Gamma_0(64))$, respectively.

Let $D(45), D(48), D(50)$ and $D(64)$ denote the sets of positive divisors of 45, 48, 50 and 64, respectively.

We observe that

\begin{align*}
(6.1) \quad & \mathcal{M}_4(\Gamma_0(5)) \subset \mathcal{M}_4(\Gamma_0(15)) \subset \mathcal{M}_4(\Gamma_0(45)) \\
(6.2) \quad & \mathcal{M}_4(\Gamma_0(9)) \subset \mathcal{M}_4(\Gamma_0(45)) \\
(6.3) \quad & \mathcal{M}_4(\Gamma_0(6)) \subset \mathcal{M}_4(\Gamma_0(12)) \subset \mathcal{M}_4(\Gamma_0(24)) \subset \mathcal{M}_4(\Gamma_0(48)) \\
(6.4) \quad & \mathcal{M}_4(\Gamma_0(8)) \subset \mathcal{M}_4(\Gamma_0(24)) \subset \mathcal{M}_4(\Gamma_0(48)) \\
(6.5) \quad & \mathcal{M}_4(\Gamma_0(8)) \subset \mathcal{M}_4(\Gamma_0(16)) \subset \mathcal{M}_4(\Gamma_0(48)) \\
(6.6) \quad & \mathcal{M}_4(\Gamma_0(5)) \subset \mathcal{M}_4(\Gamma_0(25)) \subset \mathcal{M}_4(\Gamma_0(50)) \\
(6.7) \quad & \mathcal{M}_4(\Gamma_0(5)) \subset \mathcal{M}_4(\Gamma_0(10)) \subset \mathcal{M}_4(\Gamma_0(50)) \\
(6.8) \quad & \mathcal{M}_4(\Gamma_0(8)) \subset \mathcal{M}_4(\Gamma_0(16)) \subset \mathcal{M}_4(\Gamma_0(32)) \subset \mathcal{M}_4(\Gamma_0(64)).
\end{align*}

A graphical illustration of the inclusion relation represented by **Equation 6.1**, **Equation 6.2**, **Equation 6.6** and **Equation 6.7** are given in Figure 1, and that represented by **Equation 6.3**, **Equation 6.4** and **Equation 6.5** in Figure 2.

**Corollary 6.1.**

(a) The sets

\[
\mathcal{B}_E,45 = \{ M(q') \mid t \mid 45 \} \cup \{ M_{(\frac{\alpha q}{t\beta})}(q') \mid s = 1,3 \},
\]
\[
\mathcal{B}_E,48 = \{ M(q') \mid t \in D(48) \} \cup \{ M_{(\frac{\alpha q}{t\beta})}(q') \mid s = 1,2 \},
\]
\[
\mathcal{B}_E,50 = \{ M(q') \mid t \mid 50 \} \cup \{ M_{(\frac{\alpha q}{t\beta})}(q') \mid s \in D(50) \} 	ext{ and}
\]
\[
\mathcal{B}_E,64 = \{ M(q') \mid t \in D(64) \} \cup \{ M_{(\frac{\alpha q}{t\beta})}(q') \mid s = 1,2,4,8,16 \}
\]

constitute bases of $\mathcal{E}_4(\Gamma_0(45)), \mathcal{E}_4(\Gamma_0(48)), \mathcal{E}_4(\Gamma_0(50))$ and $\mathcal{E}_4(\Gamma_0(64))$, respectively.

(b) Let $1 \leq i \leq 14$, $1 \leq j \leq 17$, $1 \leq k, l \leq 18$ be positive integers.

Let $\delta_1 \in D(45)$ and $(r(i, \delta_1))_{i, \delta_1}$ be the Table 4 of the powers of $\eta(\delta_1)z$.

Let $\delta_3 \in D(48)$ and $(r(k, \delta_3))_{k, \delta_3}$ be the Table 5 of the powers of $\eta(\delta_3)z$.

Let $\delta_5 \in D(50)$ and $(r(j, \delta_5))_{j, \delta_5}$ be the Table 6 of the powers of $\eta(\delta_5)z$.

Let $\delta_4 \in D(64)$ and $(r(l, \delta_4))_{l, \delta_4}$ be the Table 7 of the powers of $\eta(\delta_4)z$.

Let furthermore

\[
\mathcal{B}_{45, i}(q) = \prod_{\delta_1 \mid 45} \eta^{(r,i,\delta_1)}(\delta_1 z), \quad \mathcal{B}_{48, k}(q) = \prod_{\delta_3 \mid 48} \eta^{(r,k,\delta_3)}(\delta_3 z),
\]
\[
\mathcal{B}_{50, j}(q) = \prod_{\delta_5 \mid 50} \eta^{(r,j,\delta_5)}(\delta_5 z), \quad \mathcal{B}_{64, l}(q) = \prod_{\delta_4 \mid 64} \eta^{(r,l,\delta_4)}(\delta_4 z).
\]
be selected elements of $\mathcal{S}_4(\Gamma_0(45))$, $\mathcal{S}_4(\Gamma_0(48))$, $\mathcal{S}_4(\Gamma_0(50))$ and $\mathcal{S}_4(\Gamma_0(64))$, respectively.

The sets

\[
\mathcal{B}_{S,45} = \{ \mathcal{B}_{45,i}(q) \mid 1 \leq i \leq 14 \}, \quad \mathcal{B}_{S,48} = \{ \mathcal{B}_{48,k}(q) \mid 1 \leq k \leq 18 \}, \\
\mathcal{B}_{S,50} = \{ \mathcal{B}_{50,j}(q) \mid 1 \leq j \leq 17 \}, \quad \mathcal{B}_{S,64} = \{ \mathcal{B}_{64,l}(q) \mid 1 \leq l \leq 18 \}
\]

are bases of $\mathcal{S}_4(\Gamma_0(45))$, $\mathcal{S}_4(\Gamma_0(48))$, $\mathcal{S}_4(\Gamma_0(50))$ and $\mathcal{S}_4(\Gamma_0(64))$, respectively.

(c) The sets

\[
\mathcal{B}_{M,45} = \mathcal{B}_{E,45} \cup \mathcal{B}_{S,45}, \quad \mathcal{B}_{M,48} = \mathcal{B}_{E,48} \cup \mathcal{B}_{S,48}, \\
\mathcal{B}_{M,50} = \mathcal{B}_{E,50} \cup \mathcal{B}_{S,50}, \quad \mathcal{B}_{M,64} = \mathcal{B}_{E,64} \cup \mathcal{B}_{S,64}
\]

constitute bases of $\mathcal{M}_4(\Gamma_0(45))$, $\mathcal{M}_4(\Gamma_0(48))$, $\mathcal{M}_4(\Gamma_0(50))$ and $\mathcal{M}_4(\Gamma_0(64))$, respectively.

By Remark 3.2 (r1), each $\mathcal{B}_{qB,i}(q)$ is expressible in the form $\sum_{n=1}^{\infty} b_{qB,i}(n) q^n$.

**Proof.** We only give the proof for $\mathcal{B}_{M,45} = \mathcal{B}_{E,45} \cup \mathcal{B}_{S,45}$ since the other cases are done similarly. In the case of $\mathcal{B}_{E,48}$, $\mathcal{B}_{E,50}$, $\mathcal{B}_{E,64}$ an applicable primitive Dirichlet character is

\[
\left( \frac{-3}{n} \right) = \begin{cases} 
-1 & \text{if } n \equiv 2 \pmod{3}, \\
0 & \text{if } \gcd(3,n) \neq 1, \\
1 & \text{if } n \equiv 1 \pmod{3}.
\end{cases}
\]

(a) Suppose that $x_5, z_1, z_3 \in \mathbb{C}$ with $\delta \mid 45$. Let

\[
\sum_{\delta \mid 45} x_5 M(q^\delta) + z_1 M\left(\frac{q}{\sqrt{z}}\right)(q) + z_3 M\left(\frac{q}{\sqrt{z^3}}\right)(q^3) = 0.
\]

We observe that

\[
\left( \frac{-4}{n} \right) = \begin{cases} 
-1 & \text{if } n \equiv 3 \pmod{4}, \\
0 & \text{if } \gcd(4,n) \neq 1, \\
1 & \text{if } n \equiv 1 \pmod{4}.
\end{cases}
\]

and recall that for all $0 \neq a \in \mathbb{Z}$ it holds that $\left( \frac{a}{n} \right) = 0$. Since the conductor of the Dirichlet character $\left( \frac{-4}{n} \right)$ is 4, we infer from [Equation 2.1] that $C_0 = 0$. We then deduce

\[
\sum_{\delta \mid 45} x_5 + \sum_{\delta \mid 45} \left( 240 \sum_{\delta \mid 45} \sigma_3\left(\frac{n}{\delta}\right)x_5 + \left( \frac{-4}{n} \right) \sigma_3(n)z_1 + \left( \frac{-4}{n} \right) \sigma_3(n)z_3 \right) q^n = 0.
\]

Then we equate the coefficients of $q^n$ for $n \in D(45)$ plus for example $n = 2, 7$ to obtain a system of 8 linear equations whose unique solution is $x_5 = z_1 = z_3 = 0$ with $\delta \in D(45)$. So, the set $\mathcal{B}_E$ is linearly independent. Hence, the set $\mathcal{B}_E$ is a basis of $\mathcal{C}_4(\Gamma_0(45))$.

(b) Suppose that $x_i \in \mathbb{C}$ with $1 \leq i \leq 14$. Let $\sum_{i=1}^{14} x_i \mathcal{B}_{45,i}(q) = 0$. Then

\[
\sum_{i=1}^{14} x_i \sum_{n=1}^{\infty} b_{45,i}(n) q^n = \sum_{n=1}^{\infty} \left( \sum_{i=1}^{14} b_{45,i}(n) x_i \right) q^n = 0.
\]
So, we equate the coefficients of \( q^n \) for \( 1 \leq n \leq 14 \) to obtain a system of 14 linear equations whose unique solution is \( x_i = 0 \) for all \( 1 \leq i \leq 14 \). It follows that the set \( B_S \) is linearly independent. Hence, the set \( B_S \) is a basis of \( \mathcal{S}_4(\Gamma_0(45)) \).

(c) Since \( \mathcal{M}_4(\Gamma_0(45)) = \mathcal{E}_4(\Gamma_0(45)) \oplus \mathcal{S}_4(\Gamma_0(45)) \), the result follows from (a) and (b).

\[ \square \]

6.2. **Evaluation of** \( W_{(\alpha, \beta)}(n) \) **for** \( \alpha \beta = 45, 48, 50, 64 \). We evaluate the convolution sums \( W_{(\alpha, \beta)}(n) \) for \( (\alpha, \beta) = (1, 45), (5, 9), (1, 48), (3, 16), (1, 50), (1, 64) \).

**Corollary 6.2.** It holds that

\[
(6.11) \quad (5L(q^5) - 9L(q^9)) = 16 + \sum_{n=1}^{\infty} \left( -\frac{120}{13} \sigma_3(n) - \frac{51960}{923} \sigma_3\left(\frac{n}{3}\right) - \frac{5089800}{923} \sigma_3\left(\frac{n}{45}\right) + \frac{5184000}{71} \sigma_3\left(\frac{n}{45}\right) \right) q^n.
\]

\[
(6.12) \quad (L(q) - 45L(q^{45})) = 1936 + \sum_{n=1}^{\infty} \left( -\frac{120}{13} \sigma_3(n) - \frac{51960}{923} \sigma_3\left(\frac{n}{3}\right) + \frac{75000}{13} \sigma_3\left(\frac{n}{5}\right) \right) q^n.
\]
\[(6.13) \quad (L(q) - 48L(q^{48}))^2 = 2209 + \sum_{n=1}^{\infty} \left( \frac{1164}{5} \sigma_3(n) - \frac{20412}{65} \sigma_3\left(\frac{n}{2}\right) - \frac{324}{5} \sigma_3\left(\frac{n}{3}\right) \right) q^n \]

\[\quad - \frac{290736}{65} \sigma_3\left(\frac{n}{4}\right) + \frac{6372}{65} \sigma_3\left(\frac{n}{6}\right) + \frac{281664}{65} \sigma_3\left(\frac{n}{8}\right) + \frac{234576}{65} \sigma_3\left(\frac{n}{12}\right) - \frac{9216}{5} \sigma_3\left(\frac{n}{16}\right) \]

\[\quad - \frac{506304}{65} \sigma_3\left(\frac{n}{24}\right) + \frac{2681856}{65} \sigma_3\left(\frac{n}{48}\right) + \frac{38772}{65} b_{48,1}(n) + \frac{639792}{65} b_{48,2}(n) \]

\[\quad + \frac{546804}{65} b_{48,3}(n) + \frac{661824}{65} b_{48,4}(n) + \frac{195264}{65} b_{48,5}(n) + \frac{3729456}{65} b_{48,6}(n) \]

\[\quad + \frac{67968}{5} b_{48,7}(n) + \frac{5422464}{65} b_{48,8}(n) + \frac{4144464}{65} b_{48,9}(n) + \frac{3151872}{65} b_{48,10}(n) \]

\[\quad + \frac{1426176}{65} b_{48,11}(n) + \frac{14145408}{65} b_{48,12}(n) - \frac{3907584}{65} b_{48,13}(n) - \frac{1693440}{13} b_{48,14}(n) \]

\[\quad + \frac{313344}{65} b_{48,15}(n) - \frac{7299072}{13} b_{48,16}(n) + \frac{1032192}{13} b_{48,17}(n) + \frac{92736}{65} b_{48,18}(n) \] \(q^n\),

\[(6.14) \quad (3L(q^3) - 16L(q^{16}))^2 = 169 + \sum_{n=1}^{\infty} \left( -\frac{36}{5} \sigma_3(n) + \frac{43092}{1885} \sigma_3\left(\frac{n}{2}\right) \right) q^n \]

\[\quad + \frac{10476}{5} \sigma_3\left(\frac{n}{3}\right) + \frac{1094256}{1885} \sigma_3\left(\frac{n}{4}\right) - \frac{450252}{1885} \sigma_3\left(\frac{n}{6}\right) - \frac{1992384}{1885} \sigma_3\left(\frac{n}{8}\right) \]

\[\quad - \frac{2722896}{1885} \sigma_3\left(\frac{n}{12}\right) + \frac{297984}{5} \sigma_3\left(\frac{n}{16}\right) - \frac{140076}{5} \sigma_3\left(\frac{n}{24}\right) - \frac{4567104}{1885} \sigma_3\left(\frac{n}{48}\right) \]

\[\quad - \frac{34668}{65} b_{48,1}(n) + \frac{3135888}{1885} b_{48,2}(n) - \frac{140076}{65} b_{48,3}(n) - \frac{527104}{1885} b_{48,4}(n) \]

\[\quad - \frac{115776}{65} b_{48,5}(n) + \frac{20304}{1885} b_{48,6}(n) + \frac{91008}{5} b_{48,7}(n) - \frac{12196224}{1885} b_{48,8}(n) \]

\[\quad + \frac{589824}{65} b_{48,9}(n) + \frac{10119168}{1885} b_{48,10}(n) - \frac{140544}{65} b_{48,11}(n) - \frac{118171008}{1885} b_{48,12}(n) \]

\[\quad - \frac{3032064}{65} b_{48,13}(n) - \frac{7167744}{377} b_{48,14}(n) - \frac{562176}{65} b_{48,15}(n) \]

\[\quad - \frac{3218272}{377} b_{48,16}(n) + \frac{2313216}{13} b_{48,17}(n) + \frac{35136}{65} b_{48,18}(n) \] \(q^n\),

\[(6.15) \quad (2L(q^2) - 25L(q^{25}))^2 = 529 + \sum_{n=1}^{\infty} \left( \frac{810}{13} \sigma_3(n) + \frac{11460}{13} \sigma_3\left(\frac{n}{2}\right) \right) q^n \]

\[\quad - \frac{3210}{13} \sigma_3\left(\frac{n}{5}\right) - \frac{660 \sigma_3\left(\frac{n}{10}\right)}{13} + \frac{1890000}{13} \sigma_3\left(\frac{n}{25}\right) - \frac{240000}{13} \sigma_3\left(\frac{n}{50}\right) \]

\[\quad - \frac{810}{13} b_{50,1}(n) + \frac{6714}{13} b_{50,2}(n) - \frac{620 b_{50,3}(n) - 4230 b_{50,4}(n) - 178950}{13} b_{50,5}(n) \]

\[\quad - \frac{20250}{13} b_{50,6}(n) + \frac{810 b_{50,7}(n) - 13050 b_{50,8}(n) + 12420 b_{50,9}(n)}{13} \]

\[\quad - \frac{68400}{13} b_{50,10}(n) - \frac{4500 b_{50,11}(n) - 36000 b_{50,12}(n) - 21150 b_{50,13}(n)}{13} \]

\[\quad + \frac{1800 b_{50,14}(n) - 15000 b_{50,15}(n) + 20700 b_{50,16}(n) + 28800 b_{50,17}(n)}{13} \] \(q^n\).
Proof. It follows immediately from \[\text{Lemma 3.3}\] when one sets \(\alpha = 5\) and \(\beta = 9\). However, we briefly show the proof for \((5L(q^5) - 9L(q^9))^2\) as an example. One obtains

\[
(5L(q^5) - 9L(q^9))^2 = \sum \left( 5q^5 + 9q^9 \right) + \sum \left( 5q^5 + 9q^9 \right) + \sum \left( 5q^5 + 9q^9 \right)
\]

since the conductor of the Dirichlet character \(\left(\frac{-4}{n}\right)\) is 4, and hence from \[\text{Equation 2.1}\] we have \(C_0 = 0\). Now when we equate the right hand side of \[\text{Equation 6.18}\] with that of \[\text{Equation 2.5}\] and when we take the coefficients of \(q^n\) for which \(1 \leq n \leq 15\) and \(n = 17, 19, 21, 23, 25, 27, 45\) for example, we obtain a system of linear equations with a unique solution. Hence, we obtain the stated result. □

Now we state and prove our main result of this subsection.
Corollary 6.3. Let $n$ be a positive integer. Then

$$W_{(5,9)}(n) = \frac{1}{5616}\sigma_3(n) + \frac{433}{398736}\sigma_3(n) + \frac{25}{5616}\sigma_3(n) + \frac{13}{568}\sigma_3(n) \quad (6.20)$$

$$W_{(1,45)}(n) = \frac{217}{1872}\sigma_3(n) + \frac{433}{398736}\sigma_3(n) \quad (6.20)$$

$$W_{(1,48)}(n) = \frac{1}{7680}\sigma_3(n) + \frac{189}{32380}\sigma_3(n) + \frac{3}{2560}\sigma_3(n) + \frac{673}{8320}\sigma_3(n) \quad (6.21)$$
\[
W_{(3,16)}(n) = \frac{1}{7680} \sigma_3(n) - \frac{399}{965120} \sigma_3\left(\frac{n}{2}\right) + \frac{3}{2560} \sigma_3\left(\frac{n}{3}\right) - \frac{2533}{241280} \sigma_3\left(\frac{n}{4}\right) + \frac{4169}{965120} \sigma_3\left(\frac{n}{6}\right) + \frac{1153}{60320} \sigma_3\left(\frac{n}{8}\right) + \frac{6303}{241280} \sigma_3\left(\frac{n}{12}\right) + \frac{1}{30} \sigma_3\left(\frac{n}{16}\right) + \frac{2617}{60320} \sigma_3\left(\frac{n}{24}\right) + \frac{3}{10} \sigma_3\left(\frac{n}{48}\right) + \left(\frac{1}{24} - \frac{1}{64}n\right)\sigma(n)
\]
\[
+ \frac{321}{33280} b_{48,1}(n) - \frac{7259}{241280} b_{48,2}(n) + \frac{1297}{33280} b_{48,3}(n) + \frac{2643}{60320} b_{48,4}(n) + \frac{67}{2080} b_{48,5}(n) - \frac{47}{241280} b_{48,6}(n) - \frac{79}{240} b_{48,7}(n) + \frac{3529}{30160} b_{48,8}(n) - \frac{32}{195} b_{48,9}(n) - \frac{183}{1885} b_{48,10}(n) + \frac{61}{1560} b_{48,11}(n) + \frac{34193}{30160} b_{48,12}(n) + \frac{329}{390} b_{48,13}(n) + \frac{1037}{3016} b_{48,14}(n) + \frac{61}{390} b_{48,15}(n) + \frac{582}{377} b_{48,16}(n) - \frac{251}{78} b_{48,17}(n) - \frac{61}{6240} b_{48,18}(n).
\]
\[(6.22)\]

\[
W_{(2,25)}(n) = -\frac{9}{8320} \sigma_3(n) + \frac{17}{12480} \sigma_3\left(\frac{n}{2}\right) + \frac{107}{24960} \sigma_3\left(\frac{n}{4}\right) + \frac{11}{960} \sigma_3\left(\frac{n}{10}\right) + \frac{25}{312} \sigma_3\left(\frac{n}{25}\right) + \frac{25}{78} \sigma_3\left(\frac{n}{50}\right) + \left(\frac{1}{24} - \frac{1}{100}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{8}n\right)\sigma\left(\frac{n}{25}\right)
\]
\[
+ \frac{9}{8320} b_{50,1}(n) - \frac{373}{41600} b_{50,2}(n) + \frac{9}{320} b_{50,3}(n) + \frac{47}{640} b_{50,4}(n) + \frac{1193}{4992} b_{50,5}(n) + \frac{45}{128} b_{50,6}(n) - \frac{9}{640} b_{50,7}(n) + \frac{29}{128} b_{50,8}(n) - \frac{69}{320} b_{50,9}(n) + \frac{19}{208} b_{50,10}(n) - \frac{5}{64} b_{50,11}(n) + \frac{5}{8} b_{50,12}(n) + \frac{47}{128} b_{50,13}(n) - \frac{1}{32} b_{50,14}(n) + \frac{25}{96} b_{50,15}(n) - \frac{23}{64} b_{50,16}(n) + \frac{1}{2} b_{50,17}(n)
\]
\[(6.23)\]

\[
W_{(1,50)}(n) = -\frac{149}{24960} \sigma_3(n) - \frac{15}{832} \sigma_3\left(\frac{n}{2}\right) + \frac{229}{24960} \sigma_3\left(\frac{n}{5}\right) + \frac{77}{2496} \sigma_3\left(\frac{n}{10}\right) + \frac{25}{312} \sigma_3\left(\frac{n}{25}\right) + \frac{25}{78} \sigma_3\left(\frac{n}{50}\right) + \left(\frac{1}{24} - \frac{1}{200}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{50}\right)
\]
\[
+ \frac{1277}{4807} b_{50,1}(n) - \frac{243}{1664} b_{50,2}(n) + \frac{39}{320} b_{50,3}(n) + \frac{45}{128} b_{50,4}(n) + \frac{4907}{4992} b_{50,5}(n) + \frac{259}{128} b_{50,6}(n) + \frac{5}{128} b_{50,7}(n) + \frac{275}{128} b_{50,8}(n) - \frac{111}{64} b_{50,9}(n) + \frac{253}{208} b_{50,10}(n) + \frac{43}{64} b_{50,11}(n) + \frac{43}{8} b_{50,12}(n) + \frac{31}{128} b_{50,13}(n) + \frac{49}{32} b_{50,14}(n) + \frac{215}{96} b_{50,15}(n) + \frac{7}{64} b_{50,16}(n) - \frac{1}{2} b_{50,17}(n)
\]
\[(6.24)\]
\[
W_{(1,64)}(n) = \frac{1}{12288} \sigma_3(n) + \frac{1}{4096} \sigma_3\left(\frac{n}{2}\right) - \frac{967}{13312} \sigma_3\left(\frac{n}{4}\right) + \frac{129}{1664} \sigma_3\left(\frac{n}{8}\right) \\
+ \frac{1}{64} \sigma_3\left(\frac{n}{16}\right) + \frac{1}{16} \sigma_3\left(\frac{n}{32}\right) + \frac{1}{3} \sigma_3\left(\frac{n}{64}\right) + \left(\frac{1}{24} - \frac{1}{208}\right) \sigma(n) \\
+ \left(\frac{1}{24} - \frac{1}{4}\right) \sigma\left(\frac{n}{64}\right) - \frac{155}{4096} b_{64,1}(n) - \frac{105}{1024} b_{64,2}(n) - \frac{35}{128} b_{64,3}(n) \\
- \frac{49}{416} b_{64,4}(n) - \frac{21}{16} b_{64,5}(n) - \frac{21}{32} b_{64,6}(n) - \frac{21}{16} b_{64,7}(n) \\
- \frac{15}{64} b_{64,8}(n) - b_{64,9}(n) - 3 b_{64,10}(n) - \frac{9}{2} b_{64,11}(n) - \frac{7}{8} b_{64,12}(n) \\
+ 3 b_{64,13}(n) + \frac{15}{4} b_{64,14}(n) - 5 b_{64,15}(n) + \frac{25}{416} b_{64,16}(n) \\
- \frac{35}{2} b_{64,17}(n) - \frac{3}{2} b_{64,18}(n).
\]

**Proof.** It follows immediately from Theorem 3.4 when we set \((\alpha, \beta) = (1, 16), (1, 25), (5, 9), (1, 45), (2, 25), (1, 50), (1, 64). \square

7. Number of Representations of a Positive Integer \(n\) by the Octonary Quadratic Form \[\text{Equation 1.3}\]

We apply the convolution sums \(W_{(1,48)}(n), W_{(3,16)}(n), W_{(1,64)}(n)\) and other known evaluated convolution sums to determine explicit formulae for the number of representations of a positive integer \(n\) by the octonary quadratic form \[\text{Equation 1.3}\].

Since \(64 = 2^6\) and \(48 = 2^4 \cdot 3\) it follows from \[\text{Equation 4.1}\] that \(\Omega_4 = \{(1,16)\} \cup \{(3,4),(1,12)\}\).

The following result is then deduced.

**Corollary 7.1.** Let \(n \in \mathbb{N}\) and \((a, b) = (1, 12), (1, 16), (3, 4)\). Then

\[
N_{(1,12)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{12}\right) - 32\sigma\left(\frac{n}{48}\right) + 64W_{(1,12)}(n) \\
+ 1024W_{(1,12)}\left(\frac{n}{4}\right) - 256 \left(W_{(1,3)}\left(\frac{n}{4}\right) + W_{(1,48)}(n)\right).
\]

\[
N_{(1,16)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{16}\right) - 32\sigma\left(\frac{n}{64}\right) + 64W_{(1,16)}(n) \\
+ 1024W_{(1,16)}\left(\frac{n}{4}\right) - 256 \left(W_{(1,4)}\left(\frac{n}{4}\right) + W_{(1,64)}(n)\right).
\]

\[
N_{(3,4)}(n) = 8\sigma\left(\frac{n}{3}\right) - 32\sigma\left(\frac{n}{12}\right) + 8\sigma\left(\frac{n}{4}\right) - 32\sigma\left(\frac{n}{16}\right) + 64W_{(3,4)}(n) \\
+ 1024W_{(3,4)}\left(\frac{n}{4}\right) - 256 \left(W_{(1,3)}\left(\frac{n}{4}\right) + W_{(3,16)}(n)\right).
\]

**Proof.** These identities follow immediately from \[\text{Theorem 4.2}\]. We can make use of the results obtained by A. Alaca et al. \[\text{[1]}\] and J. G. Huard et al. \[\text{[12]}\] Thrm 3, p. 20, and \[\text{Equation 6.22}\] and \[\text{Equation 6.21}\] to simplify for example \(N_{(1,12)}(n)\) and \(N_{(3,4)}(n)\). \square

8. Number of Representations of a Positive Integer \(n\) by the Octonary Quadratic Form \[\text{Equation 1.4}\]

We make use of the convolution sums \(W_{(5,9)}(n), W_{(1,45)}(n), W_{(3,16)}(n), W_{(1,48)}(n)\) and other well-known convolution sums to determine explicit formulae for the number of representations of a positive integer \(n\) by the octonary quadratic form \[\text{Equation 1.4}\].
Since $45 = 3^2 \cdot 5$ and $48 = 2^4 \cdot 3$ it follows from Equation 5.1 that $\Omega_3 = \{(3, 5), (1, 15)\} \cup \{(1, 16)\}$. We revisit the evaluation of the convolution sums for $\alpha \beta = 5, 15$ using modular forms. The result for $\alpha \beta = 5$ was obtained by M. Lemire and K. S. Williams [16], and S. Cooper and P. C. Toh [9], that for $\alpha \beta = 15$ was achieved by B. Ramakrishnan and B. Sahu [26] when using a basis which contains one cusp form of weight 2. We note that $\alpha \beta = 5, 15$ belong to the class of positive integers $\alpha \beta$ discussed by E. Ntienjem [23]. Therefore, it suffices to determine a basis of the space of cusp forms for $\Gamma_0(\alpha \beta)$ and apply [23] Thrm 3.4. Because of Equation 6.1 $B_{45,2}(q)$, $B_{45,3}(q)$ and in addition

$$B_{45,1}(q) = \eta^4(z)\eta^4(5z) = \sum_{n \geq 1} b_{45,1}(n)q^n,$$

$$B_{45,4}(q) = \frac{\eta^3(z)\eta^3(2z)\eta^7(15z)}{\eta^5(5z)} = \sum_{n \geq 1} b_{45,4}(n)q^n,$$

are basis elements of $\mathcal{S}_4(\Gamma_0(15))$. We note that $B_{45,1}(q)$ is the basis element of $\mathcal{S}_4(\Gamma_0(5))$.

**Theorem 8.1.** Let $n$ be a positive integer. Then

$$W_{(1,5)}(n) = \frac{5}{312} \sigma_3(n) + \frac{125}{132} \sigma_3(\frac{n}{3}) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(\frac{n}{2})$$

(8.1)

$$- \frac{1}{130} b_{45,1}(n),$$

$$W_{(3,5)}(n) = \frac{1}{390} \sigma_3(n) + \frac{7}{520} \sigma_3(\frac{n}{3}) - \frac{175}{312} \sigma_3(\frac{n}{5}) + \frac{25}{26} \sigma_3(\frac{n}{15}) + \left(\frac{1}{24} - \frac{n}{20}\right)\sigma(\frac{n}{3}) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma(\frac{n}{5}) - \frac{1}{390} b_{45,1}(n) - \frac{1}{30} b_{45,2}(n)$$

(8.2)

$$W_{(1,15)}(n) = \frac{1}{1560} \sigma_3(n) + \frac{1}{65} \sigma_3(\frac{n}{3}) + \frac{25}{39} \sigma_3(\frac{n}{5}) - \frac{25}{104} \sigma_3(\frac{n}{15}) + \left(\frac{1}{24} - \frac{n}{60}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(\frac{n}{15}) - \frac{1}{39} b_{45,1}(n) - \frac{2}{15} b_{45,2}(n)$$

(8.3)

$$- \frac{1}{65} b_{45,3}(n) + \frac{1}{5} b_{45,4}(n).$$

We make use of these results to deduce the following.

**Corollary 8.2.** Let $n \in \mathbb{N}$ and $c, d = (1, 15), (3, 5), (1, 16)$. Then

$$R_{(1,15)}(n) = 12\sigma(n) - 36\sigma(\frac{n}{3}) + 12\sigma(\frac{n}{15}) - 36\sigma(\frac{n}{45}) + 144 W_{(1,15)}(n) + 1296 W_{(1,15)}(\frac{n}{3}) - 432 \left( W_{(1,5)}(\frac{n}{3}) + W_{(1,45)}(n) \right),$$

$$R_{(3,5)}(n) = 12\sigma(\frac{n}{3}) - 36\sigma(\frac{n}{9}) + 12\sigma(\frac{n}{5}) - 36\sigma(\frac{n}{15}) + 144 W_{(3,5)}(n) + 1296 W_{(3,5)}(\frac{n}{3}) - 432 \left( W_{(1,5)}(\frac{n}{3}) + W_{(3,9)}(n) \right).$$
Proof. It follows immediately from Theorem 5.2. We can make use of Theorem 8.1, Equation 6.19 and Equation 6.20 to simplify \( R_{(1,15)}(n) \) and \( R_{(3,5)}(n) \) for example.

9. Revisited evaluation of the convolution sums for \( \alpha \beta = 9, 16, 18, 25, 36 \)

We revisit the evaluation of the convolution sums for \( W_{(1,9)}(n), W_{(1,16)}(n), W_{(1,18)}(n), W_{(2,9)}(n), W_{(1,25)}(n), W_{(1,36)}(n) \) and \( W_{(4,9)}(n) \) obtained by K. S. Williams [30], A. Alaca et al. [4, 2], E. X. W. Xia et al. [33] and D. Ye [34], respectively. These convolution sums have been evaluated using a different technique.

Due to Equation 6.2 using \( \mathcal{B}_{45,1}(q) \) as basis element of \( \mathcal{E}_4(\Gamma_0(9)) \) and applying the same primitive Dirichlet character as for \( \mathcal{E}_4(\Gamma_0(45)) \), one easily replicates the result for the convolution sum \( W_{(1,9)}(n) \) obtained by K. S. Williams [30].

Observe that

\[
\dim(\mathcal{E}_4(\Gamma_0(16))) = 6, \quad \dim(\mathcal{E}_4(\Gamma_0(16))) = 3, \\
\dim(\mathcal{E}_4(\Gamma_0(25))) = 6, \quad \dim(\mathcal{E}_4(\Gamma_0(25))) = 5.
\]

These convolution sums are improved using our method since we apply the right number of basis elements of the space of cusp forms corresponding to level 16, 18 and 25. In case of the evaluation of \( W_{(1,16)}(n) \), we will use \( \mathcal{B}_{64,3}(q) = \frac{\eta^8(48\eta^8(16z))}{\eta^8(8z)} = \sum_{n \geq 1} b'_{64,3}(n)q^n \) instead of \( \mathcal{B}_{64,3}(q) \) given in Table 7, the primitive Dirichlet character Equation 6.9 is applicable. For the evaluation of \( W_{(1,25)}(n) \) we will use \( \mathcal{B}_{50,2}(q) = \eta(4z)\eta(25z) = \sum_{n \geq 1} b'_{50,2}(n)q^n \) instead of \( \mathcal{B}_{50,2}(q) \) given in Table 6, we apply the primitive Dirichlet character Equation 6.9.

Since

\begin{align}
(9.1) & \quad \mathcal{M}_4(\Gamma_0(6)) \subset \mathcal{M}_4(\Gamma_0(12)) \subset \mathcal{M}_4(\Gamma_0(36)), \\
(9.2) & \quad \mathcal{M}_4(\Gamma_0(9)) \subset \mathcal{M}_4(\Gamma_0(18)) \subset \mathcal{M}_4(\Gamma_0(36)),
\end{align}

it suffices to consider the basis of \( \mathcal{E}_4(\Gamma_0(36)) \), whose table of the exponent of the \( \eta \)-quotients is given in Table 8. Note that

\[
\dim(\mathcal{E}_4(\Gamma_0(18))) = 8, \quad \dim(\mathcal{E}_4(\Gamma_0(18))) = 5, \\
\dim(\mathcal{E}_4(\Gamma_0(36))) = 12, \quad \dim(\mathcal{E}_4(\Gamma_0(36))) = 12.
\]

The primitive Dirichlet character Equation 6.10 is applicable in case of \( \mathcal{E}_4(\Gamma_0(18)) \) and \( \mathcal{E}_4(\Gamma_0(36)) \).

**Corollary 9.1.** It holds that

\[
(9.3) \quad (L(q) - 9L(q^9))^2 = 64 + \sum_{n=1}^\infty \left( 192\sigma_3(n) - 384\sigma_3\left(\frac{n}{3}\right) + 15552\sigma_3\left(\frac{n}{9}\right) \\
+ 192b_{45,1}(n) \right)q^n.
\]
(9.4) \((L(q) - 16L(q^{16}))^2 = 225 + \sum_{n=1}^{\infty} \left( 216 \sigma_3(n) - 72 \sigma_3(n/2) - 288 \sigma_3(n/4) \right.
- 1152 \sigma_3(n/8) + 55296 \sigma_3(n/16) + 504 b_{64,1}(n) + 864 b_{64,2}(n) + 2304 b'_{64,3}(n) \left) q^n. \right.

(9.5) \((L(q) - 18L(q^{18}))^2 = 289 + \sum_{n=1}^{\infty} \left( 1104 \frac{5}{5} \sigma_3(n) - 384 \frac{5}{3} \sigma_3(n/2) - 768 \frac{3}{3} \sigma_3(n/3) \right.
- 3072 \frac{5}{5} \sigma_3(n/6) - 7776 \frac{5}{5} \sigma_3(n/9) + 357696 \frac{5}{5} \sigma_3(n/18) + 2976 \frac{5}{5} b_{36,1}(n) + 8544 \frac{5}{5} b_{36,2}(n)
+ 17952 \frac{5}{5} b_{36,3}(n) + 53376 \frac{5}{5} b'_{36,4}(n) - 52992 \frac{5}{5} b_{36,5}(n) \left) q^n. \right.

(9.6) \((2L(q) - 9L(q^9))^2 = 49 + \sum_{n=1}^{\infty} \left( -\frac{96}{5} \sigma_3(n) + \frac{4416}{5} \sigma_3(n/2) - \frac{768}{3} \sigma_3(n/3) \right.
- 3072 \frac{5}{5} \sigma_3(n/6) + 89424 \frac{5}{5} \sigma_3(n/9) - 31104 \frac{5}{5} \sigma_3(n/18) - \frac{96}{5} \sigma_3(n/3) + \frac{96}{5} b_{36,1}(n)
+ 3552 \frac{5}{5} b_{36,3}(n) - 4224 \frac{5}{5} b'_{36,4}(n) + 16128 \frac{5}{5} b_{36,5}(n) \left) q^n. \right.

(9.7) \((L(q) - 25L(q^{25}))^2 = 576 + \sum_{n=1}^{\infty} \left( \frac{2880}{13} \frac{5}{5} \sigma_3(n) - \frac{5760}{13} \sigma_3(n/5) \right.
+ \frac{1800000}{13} \frac{5}{5} \sigma_3(n/25) + \frac{12096}{13} \frac{5}{5} b_{50,1}(n) + 5760 b'_{50,2}(n) + 17280 b_{50,3}(n)
+ 28800 b_{50,4}(n) + \frac{302400}{13} b_{50,5}(n) \left) q^n. \right.

(9.8) \((L(q) - 36L(q^{36}))^2 = 1225 + \sum_{n=1}^{\infty} \left( \frac{1152}{5} \frac{5}{5} \sigma_3(n) - \frac{1008}{5} \sigma_3(n/2) - \frac{384}{3} \sigma_3(n/3) \right.
+ \frac{13056}{5} \frac{5}{5} \sigma_3(n/4) - \frac{288}{5} \sigma_3(n/6) - \frac{3888}{5} \sigma_3(n/9) - \frac{19968}{5} \sigma_3(n/12) - \frac{11664}{5} \sigma_3(n/18)
+ \frac{1492992}{5} \frac{5}{5} \sigma_3(n/36) + \frac{7248}{5} \frac{5}{5} b_{36,1}(n) + 3744 b_{36,2}(n) + \frac{19008}{5} \frac{5}{5} b_{36,3}(n) + \frac{77664}{5} \frac{5}{5} b_{36,4}(n)
+ \frac{14688}{5} \frac{5}{5} b_{36,5}(n) + 16416 b_{36,6}(n) + \frac{88064}{5} \frac{5}{5} b_{36,7}(n) + \frac{84672}{5} \frac{5}{5} b_{36,8}(n)
+ \frac{90624}{5} \frac{5}{5} b_{36,9}(n) + 90624 b_{36,10}(n) + 5184 b_{36,11}(n) + 2592 b_{36,12}(n) \left) q^n. \right.

Let \( n \) be a positive integer. Then

\[
(4L(q) - 9L(q^9))^2 = 25 + \sum_{n=1}^{\infty} \left( \frac{1152}{5} \sigma_3(n) - \frac{1008}{5} \sigma_3(n/2) - \frac{384}{5} \sigma_3(n/3) \\
+ \frac{13056}{5} \sigma_3(n/4) + \frac{80928}{5} \sigma_3(n/6) - \frac{388}{5} \sigma_3(n/9) - \frac{913344}{5} \sigma_3(n/12) - \frac{50872}{5} \sigma_3(n/18) \\
+ \frac{29187648}{5} \sigma_3(n/36) + \frac{7248}{5} b_{36,1}(n) + 3744 b_{36,2}(n) + \frac{19008}{5} b_{36,3}(n) + \frac{77664}{5} b_{36,4}(n) \\
+ \frac{14688}{5} b_{36,5}(n) + \frac{864}{5} b_{36,6}(n) + \frac{80604}{5} b_{36,7}(n) + \frac{84672}{5} b_{36,8}(n) - 12960 b_{36,9}(n) \\
+ \frac{90624}{5} b_{36,10}(n) + 5184 b_{36,11}(n) + 2592 b_{36,12}(n) \right) q^n.
\]

\[\begin{align*}
W_{1,9}(n) &= \frac{1}{216} \sigma_3(n) + \frac{1}{27} \sigma_3(n/3) + \frac{3}{8} \sigma_3(n/9) - \frac{1}{54} b_{45,1}(n) \\
W_{1,16}(n) &= \frac{1}{768} \sigma_3(n) + \frac{1}{256} \sigma_3(n/2) + \frac{1}{64} \sigma_3(n/4) + \frac{1}{16} \sigma_3(n/8) + \frac{1}{3} \sigma_3(n/16) \\
&\quad + \frac{1}{24} - \frac{1}{64} n)\sigma(n) + (\frac{1}{24} - \frac{1}{4} n)\sigma(n/16) - \frac{7}{256} b_{64,1}(n) \\
&\quad - \frac{3}{64} b_{64,3}(n) - \frac{1}{8} b_{64,3}^* (n) \\
W_{1,18}(n) &= \frac{1}{1080} \sigma_3(n) + \frac{1}{270} \sigma_3(n/2) + \frac{1}{135} \sigma_3(n/3) + \frac{4}{135} \sigma_3(n/6) + \frac{3}{40} \sigma_3(n/9) \\
&\quad + \frac{3}{10} \sigma_3(n/18) + (\frac{1}{24} - \frac{1}{72} n)\sigma(n) + (\frac{1}{24} - \frac{1}{4} n)\sigma(n/18) - \frac{31}{1080} b_{36,1}(n) \\
&\quad - \frac{89}{1080} b_{36,2}(n) - \frac{187}{1080} b_{36,3}(n) - \frac{139}{270} b_{36,4}(n) + \frac{23}{45} b_{36,5}(n) \\
W_{2,9}(n) &= \frac{1}{1080} \sigma_3(n) + \frac{1}{270} \sigma_3(n/2) + \frac{1}{135} \sigma_3(n/3) + \frac{4}{135} \sigma_3(n/6) + \frac{3}{40} \sigma_3(n/9) \\
&\quad + \frac{3}{10} \sigma_3(n/18) + (\frac{1}{24} - \frac{1}{72} n)\sigma(n) + (\frac{1}{24} - \frac{1}{8} n)\sigma(n/18) - \frac{1}{1080} b_{36,1}(n) \\
&\quad + \frac{1}{1080} b_{36,2}(n) - \frac{37}{1080} b_{36,3}(n) + \frac{11}{270} b_{36,4}(n) - \frac{7}{45} b_{36,5}(n) \\
W_{1,25}(n) &= \frac{1}{1560} \sigma_3(n) + \frac{1}{65} \sigma_3(n/5) + \frac{125}{312} \sigma_3(n/25) + (\frac{1}{24} - \frac{1}{100} n)\sigma(n) \\
&\quad + (\frac{1}{24} - \frac{1}{4} n)\sigma(n/25) - \frac{21}{650} b_{50,1}(n) - \frac{1}{5} b_{50,2}(n) - \frac{3}{5} b_{50,3}(n) \\
&\quad - b_{50,4}(n) - \frac{21}{26} b_{50,5}(n)
\end{align*}\]

**Proof.** Similar to that of Theorem 6.2. \(\Box\)

**Corollary 9.2.** Let \( n \) be a positive integer. Then

\[
W_{1,9}(n) = \frac{1}{216} \sigma_3(n) + \frac{1}{27} \sigma_3(n/3) + \frac{3}{8} \sigma_3(n/9) - \frac{1}{54} b_{45,1}(n)
\]

\[
W_{1,16}(n) = \frac{1}{768} \sigma_3(n) + \frac{1}{256} \sigma_3(n/2) + \frac{1}{64} \sigma_3(n/4) + \frac{1}{16} \sigma_3(n/8) + \frac{1}{3} \sigma_3(n/16)
\]

\[
+ \frac{1}{24} - \frac{1}{64} n)\sigma(n) + (\frac{1}{24} - \frac{1}{4} n)\sigma(n/16) - \frac{7}{256} b_{64,1}(n)
\]

\[
- \frac{3}{64} b_{64,3}(n) - \frac{1}{8} b_{64,3}^* (n)
\]

\[
W_{1,18}(n) = \frac{1}{1080} \sigma_3(n) + \frac{1}{270} \sigma_3(n/2) + \frac{1}{135} \sigma_3(n/3) + \frac{4}{135} \sigma_3(n/6) + \frac{3}{40} \sigma_3(n/9)
\]

\[
+ \frac{3}{10} \sigma_3(n/18) + (\frac{1}{24} - \frac{1}{72} n)\sigma(n) + (\frac{1}{24} - \frac{1}{4} n)\sigma(n/18) - \frac{31}{1080} b_{36,1}(n)
\]

\[
- \frac{89}{1080} b_{36,2}(n) - \frac{187}{1080} b_{36,3}(n) - \frac{139}{270} b_{36,4}(n) + \frac{23}{45} b_{36,5}(n)
\]

\[
W_{2,9}(n) = \frac{1}{1080} \sigma_3(n) + \frac{1}{270} \sigma_3(n/2) + \frac{1}{135} \sigma_3(n/3) + \frac{4}{135} \sigma_3(n/6) + \frac{3}{40} \sigma_3(n/9)
\]

\[
+ \frac{3}{10} \sigma_3(n/18) + (\frac{1}{24} - \frac{1}{72} n)\sigma(n) + (\frac{1}{24} - \frac{1}{8} n)\sigma(n/18) - \frac{1}{1080} b_{36,1}(n)
\]

\[
+ \frac{1}{1080} b_{36,2}(n) - \frac{37}{1080} b_{36,3}(n) + \frac{11}{270} b_{36,4}(n) - \frac{7}{45} b_{36,5}(n)
\]

\[
W_{1,25}(n) = \frac{1}{1560} \sigma_3(n) + \frac{1}{65} \sigma_3(n/5) + \frac{125}{312} \sigma_3(n/25) + (\frac{1}{24} - \frac{1}{100} n)\sigma(n)
\]

\[
+ (\frac{1}{24} - \frac{1}{4} n)\sigma(n/25) - \frac{21}{650} b_{50,1}(n) - \frac{1}{5} b_{50,2}(n) - \frac{3}{5} b_{50,3}(n)
\]

\[
- b_{50,4}(n) - \frac{21}{26} b_{50,5}(n)
\]
\[ W_{1,36}(n) = \frac{1}{4320} \sigma_3(n) + \frac{7}{1440} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{540} \sigma_3\left(\frac{n}{3}\right) - \frac{17}{270} \sigma_3\left(\frac{n}{4}\right) + \frac{1}{720} \sigma_3\left(\frac{n}{6}\right) \\
+ \frac{3}{160} \sigma_3\left(\frac{n}{5}\right) + \frac{13}{135} \sigma_3\left(\frac{n}{12}\right) + \frac{9}{160} \sigma_3\left(\frac{n}{18}\right) + \frac{3}{10} \sigma_3\left(\frac{n}{36}\right) \\
+ \left(\frac{1}{24} - \frac{1}{144}n\right) \sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right) \sigma\left(\frac{n}{36}\right) - \frac{151}{4320} b_{36,1}(n) - \frac{13}{144} b_{36,2}(n) \\
- \frac{11}{120} b_{36,3}(n) - \frac{2160}{809} b_{36,4}(n) - \frac{17}{240} b_{36,5}(n) - \frac{19}{48} b_{36,6}(n) - \frac{139}{360} b_{36,7}(n) \\
\frac{1}{49} b_{36,8}(n) + \frac{5}{16} b_{36,9}(n) - \frac{59}{135} b_{36,10}(n) - \frac{1}{8} b_{36,11}(n) - \frac{1}{16} b_{36,12}(n) \] (9.15)

\[ W_{4,9}(n) = \frac{1}{4320} \sigma_3(n) + \frac{7}{1440} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{540} \sigma_3\left(\frac{n}{3}\right) - \frac{17}{270} \sigma_3\left(\frac{n}{4}\right) - \frac{281}{720} \sigma_3\left(\frac{n}{6}\right) \\
+ \frac{3}{160} \sigma_3\left(\frac{n}{5}\right) + \frac{4757}{1080} \sigma_3\left(\frac{n}{12}\right) + \frac{1171}{96} \sigma_3\left(\frac{n}{18}\right) - \frac{15991}{120} \sigma_3\left(\frac{n}{36}\right) \\
+ \left(\frac{1}{24} - \frac{1}{36}n\right) \sigma\left(\frac{n}{4}\right) + \left(\frac{1}{24} - \frac{1}{16}n\right) \sigma\left(\frac{n}{9}\right) - \frac{151}{4320} b_{36,1}(n) - \frac{13}{144} b_{36,2}(n) \\
- \frac{11}{120} b_{36,3}(n) - \frac{2160}{809} b_{36,4}(n) - \frac{17}{240} b_{36,5}(n) - \frac{1}{240} b_{36,6}(n) - \frac{139}{360} b_{36,7}(n) \\
\frac{1}{49} b_{36,8}(n) + \frac{5}{16} b_{36,9}(n) - \frac{59}{135} b_{36,10}(n) - \frac{1}{8} b_{36,11}(n) - \frac{1}{16} b_{36,12}(n) \] (9.16)

**Proof.** Similar to that of Theorem 6.3

### 10. Concluding Remark

The set of natural number \( \mathbb{N} \) can be expressed as the disjoint union of the sets \( \mathcal{R} \) and \( \mathbb{N} \setminus \mathcal{R} \). When assuming that a basis of the space of cusp forms is determined, E. Ntienjem [23] has evaluated convolution sums for natural numbers which belong to \( \mathcal{R} \). In this paper we have evaluated convolution sums for natural numbers which are in \( \mathbb{N} \setminus \mathcal{R} \) making the same assumption. When we put altogether, we can say that for all natural numbers \( \alpha \) and \( \beta \), the convolution sums for \( \alpha \beta \) are evaluated.

The determination of a basis of the space of cusp forms is tedious, especially when \( \alpha \beta \) is large and has a large number of divisors. An effective and efficient approach to build a basis of the space of cusp forms of weight 4 for \( \Gamma_0(\alpha \beta) \) is a work in progress.

### References

[1] A. Alaca, S. Alaca, and K. S. Williams. Evaluation of the convolution sums \( \sum_{l+4m=n} \sigma(l)\sigma(m) \) and \( \sum_{3l+4m=n} \sigma(l)\sigma(m) \). *Adv. Theor. Appl. Math.*, 1(1):27–48, 2006.

[2] A. Alaca, S. Alaca, and K. S. Williams. Evaluation of the convolution sums \( \sum_{l+12m=n} \sigma(l)\sigma(m) \) and \( \sum_{2l+12m=n} \sigma(l)\sigma(m) \). *Int. Math. Forum*, 2(2):45–68, 2007.

[3] A. Alaca, S. Alaca, and K. S. Williams. Evaluation of the convolution sums \( \sum_{l+2m=n} \sigma(l)\sigma(m) \) and \( \sum_{3l+2m=n} \sigma(l)\sigma(m) \). *Math J Okayama Univ.*, 49:93–111, 2007.

[4] A. Alaca, S. Alaca, and K. S. Williams. The convolution sum \( \sum_{m<\frac{n}{10}} \sigma(m)\sigma(n-16m) \). *Canad. Math. Bull.*, 51(1):3–14, 2008.

[5] S. Alaca and Y. Kesicioğlu. Evaluation of the convolution sums \( \sum_{l+27m=n} \sigma(l)\sigma(m) \) and \( \sum_{l+32m=n} \sigma(l)\sigma(m) \). *Int. J. Number Theory*, 12(1):1–13, 2016.
[6] Ş. Alaca and K. S. Williams. Evaluation of the convolution sums \( \sum_{l=3m=n} \sigma(l)\sigma(m) \) and \( \sum_{l=2m=n} \sigma(l)\sigma(m) \). \textit{J Number Theory}, 124(2):490–510, 2007.

[7] M. Besge. Extrait d’une lettre de M Besge à M Liouville. \textit{J Math Pure Appl}, 7:256, 1885.

[8] H. H. Chan and S. Cooper. Powers of theta functions. \textit{Pac J Math}, 235:1–14, 2008.

[9] S. Cooper and P. C. Toh. Quintic and septic Eisenstein series. \textit{Ramanujan J}, 19:163–181, 2009.

[10] S. Cooper and D. Ye. Evaluation of the convolution sums \( \sum_{l=20m=n} \sigma(l)\sigma(m) \). \textit{Int J Number Theory}, 10(6):1386–1394, 2014.

[11] J. W. L. Glashier. On the square of the series in which the coefficients are the sums of the divisors of the exponents. \textit{Messenger Math}, 14:156–163, 1862.

[12] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams. Elementary evaluation of certain convolution sums involving divisor functions. \textit{Number Theory Millenium}, 7:229–274, 2002. A K Peters, Natick, MA.

[13] L. J. P. Kilford. \textit{Modular forms: A classical and computational introduction}. Imperial College Press, London, 2008.

[14] N. Koblitz. \textit{Introduction to Elliptic Curves and Modular Forms}, volume 97 of \textit{Graduate Texts in Mathematics}. Springer Verlag, New York, 2 edition, 1993.

[15] G. Köhler. \textit{Eta Products and Theta Series Identities}, volume 3733 of \textit{Springer Monographs in Mathematics}. Springer Verlag, Berlin Heidelberg, 2011.

[16] M. Lemire and K. S. Williams. Evaluation of two convolution sums involving the sum of divisors function. \textit{Bull Aust Math Soc}, 73:107–115, 2006.

[17] G. Ligozat. Courbes modulaires de genre 1. \textit{Bull Soc Math France}, 43:5–80, 1975.

[18] G. A. Lomadze. Representation of numbers by sums of the quadratic forms \( x_1^2 + x_2^2 + x_3^2 \). \textit{Acta Arith}, 54(1):9–36, 1989.

[19] T. Miyake. \textit{Modular Forms}. Springer monographs in Mathematics. Springer Verlag, New York, 1989.

[20] M. Newman. Construction and application of a class of modular functions. \textit{Proc Lond Math Soc}, 7(3):334–350, 1957.

[21] M. Newman. Construction and application of a class of modular functions II. \textit{Proc Lond Math Soc}, 9(3):373–387, 1959.

[22] E. Ntienjem. Evaluation of the convolution sums \( \sum_{\alpha \beta | m=n} \sigma(\alpha)\sigma(\beta) \), where \( (\alpha, \beta) \) is in \( \{(1, 14), (2, 7), (1, 26), (2, 13), (1, 28), (4, 7), (1, 30), (2, 15), (3, 10), (5, 6)\} \). Master’s thesis, School of Mathematics and Statistics, Carleton University, 2015.

[23] E. Ntienjem. Elementary Evaluation of Convolution Sums Involving the Sum of Divisors Function for a Class of Positive Integers. \textit{ArXiv e-prints}, July 2016.

[24] E. Ntienjem. Evaluation of the Convolution Sums \( \sum_{\alpha \beta | m=n} \sigma(\alpha)\sigma(\beta) \), where \( \alpha \beta = 44, 52 \). \textit{ArXiv e-prints}, June 2016.

[25] A. Pizer. The representability of modular forms by theta series. \textit{J Math Soc Japan}, 28(4):689–698, 10 1976.

[26] B. Ramakrishnan and B. Sahu. Evaluation of the convolution sums \( \sum_{l=15m=n} \sigma(l)\sigma(m) \) and \( \sum_{3l=5m=n} \sigma(l)\sigma(m) \). \textit{Int J Number Theory}, 9(3):799–809, 2013.

[27] S. Ramanujan. On certain arithmetical functions. \textit{T Cambridge Phil Soc}, 22:159–184, 1916.

[28] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. \textit{Int J Number Theory}, 3(2):231–261, 2007.

[29] W. A. Stein. \textit{Modular Forms, A Computational Approach}, volume 79. American Mathematical Society, Graduate Studies in Mathematics, 2011. http://wwwstein.org/books/modform/modform/.

[30] K. S. Williams. The convolution sum \( \sum_{n \leq x} \sigma(n)\sigma(n - 9m) \). \textit{Int J Number Theory}, 1(2):193–205, 2005.

[31] K. S. Williams. The convolution sum \( \sum_{m \leq x} \sigma(m)\sigma(n - 8m) \). \textit{Pac J Math}, 228:387–396, 2006.

[32] K. S. Williams. \textit{Number Theory in the Spirit of Liouville}, volume 76 of \textit{London Mathematical Society Student Texts}. Cambridge University Press, Cambridge, 2011.

[33] E. X. W. Xia, X. L. Tian, and O. X. M. Yao. Evaluation of the convolution sums \( \sum_{l=25m=n} \sigma(l)\sigma(m) \). \textit{Int J Number Theory}, 10(6):1421–1430, 2014.

[34] D. Ye. Evaluation of the convolution sums \( \sum_{l=36m=n} \sigma(l)\sigma(m) \) and \( \sum_{4l=9m=n} \sigma(l)\sigma(m) \). \textit{Int J Number Theory}, 11(1):171–183, 2015.
FIGURE 1. Inclusion relationship of the modular space of weight 4 for $\Gamma_0(45)$ and $\Gamma_0(50)$

```
F V 1. Inclusion relationship of the modular space of weight 4 for
$\Gamma_0(45)$ and $\Gamma_0(50)$
```

```
F V 2. Inclusion relation of the modular space of weight 4 for $\Gamma_0(48)$.
```

```
TABLES

|   | 1 | 3 | 5 | 9 | 15 | 45 |
|---|---|---|---|---|----|----|
| 1 | 0 | 8 | 0 | 0 | 0  | 0  |
| 2 | 2 | 2 | 0 | 2 | 0  | 0  |
| 3 | 0 | 4 | 0 | 0 | 4  | 0  |
| 4 | 3 | 4 | 0 | -1| 0  | 2  |
| 5 | 2 | 0 | 2 | 0 | 2  | 0  |
| 6 | 4 | 0 | 1 | 0 | 0  | 3  |
| 7 | 1 | 0 | 1 | 3 | 0  | 3  |
| 8 | 3 | 0 | 1 | 0 | 4  |    |
| 9 | 5 | 0 | -1| -1| 0  | 5  |
| 10| 0| 3 | 0 | -1| 1  | 5  |
| 11| 1| 1 | 0 | -1| 6  |    |
| 12| 4| 0 | 0 | 0 | 0  |    |
```

Table 4: Power of $\eta$-quotients being basis elements for $S_4(\Gamma_0(45))$

| 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 48 |
|---|---|---|---|---|---|----|----|----|----|
| 1 | 0 | 4 | 0 | 4 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 4 | 0 | 4 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 4 | 0 | 4 | 0 | 0 | 0 |
| 4 | 0 | 0 | 2 | 0 | 2 | 2 | 0 | 2 | 0 |
| 5 | 0 | 2 | 0 | -2 | -2 | 2 | 6 | 0 | 2 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 4 | 0 | 0 |
| 7 | 0 | 2 | 0 | 0 | -2 | -2 | 4 | 0 | 6 | 0 |
| 8 | 0 | 0 | 0 | 0 | 2 | 0 | 2 | 2 | 2 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 6 | 0 | -2 | 4 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 1 | -3 | 5 |
| 11 | 0 | 0 | 0 | 0 | 0 | 3 | 2 | -3 | 1 | 5 |
| 12 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | -2 | 0 | 6 |
| 13 | 0 | 0 | 0 | -1 | 0 | 7 | 1 | -4 | -3 | 8 |
| 14 | 0 | 0 | 0 | 0 | 3 | 4 | -3 | -5 | 9 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 4 | 2 | -2 | -6 | 10 |
| 16 | 0 | -1 | 0 | -1 | 3 | 3 | -1 | -3 | -1 | 9 |
| 17 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | -2 | -2 | 10 |
| 18 | 0 | 0 | 1 | 0 | -1 | 0 | 0 | 3 | 5 | -3 |

Table 5: Power of $\eta$-functions being basis elements of $S_4(\Gamma_0(48))$

| 1 | 2 | 5 | 10 | 25 | 50 |
|---|---|---|----|----|----|
| 1 | 4 | 0 | 4 | 0 | 0 |
| 2 | 0 | 4 | 0 | 4 | 0 |
| 3 | 2 | 0 | 4 | 0 | 2 |
| 4 | 1 | 0 | 4 | 0 | 3 |
| 5 | 0 | 0 | 4 | 0 | 4 |
| 6 | 0 | 2 | 0 | 4 | 0 |
| 7 | 0 | 4 | 2 | 0 | -2 |
| 8 | 0 | 1 | 0 | 4 | 0 |
| 9 | 1 | 0 | 0 | 4 | -1 |
| 10 | 0 | 0 | 0 | 4 | 0 |
| 11 | 0 | 2 | 2 | 0 | -2 |
| 12 | 0 | -1 | 0 | 4 | 0 |
| 13 | 0 | 1 | 2 | 0 | -2 |
| 14 | 1 | 0 | 2 | 0 | -3 |
| 15 | 0 | 0 | 2 | 0 | -2 |
| 16 | -1 | 0 | 6 | -2 | -5 |
| 17 | 0 | -1 | 1 | 3 | -5 |

Table 6: Power of $\eta$-quotients being basis elements for $S_4(\Gamma_0(50))$
Table 7: Power of $\eta$-functions being basis elements of $S_4(\Gamma_0(64))$

|   | 1   | 2   | 3   | 4   | 8   | 16  | 32  | 64  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | 0   | 4   | 4   | 0   | 0   | 0   | 0   | 0   |
| 2 | 0   | 0   | 4   | 0   | 0   | 0   | 0   | 0   |
| 3 | 0   | 4   | 0   | 0   | 4   | 0   | 0   | 0   |
| 4 | 0   | 0   | 0   | 4   | 0   | 0   | 0   | 0   |
| 5 | 0   | 2   | 1   | 0   | 3   | 2   | 0   |     |
| 6 | 0   | 0   | 4   | 0   | 0   | 4   | 0   |     |
| 7 | 0   | 0   | 2   | 0   | 2   | 4   | 0   |     |
| 8 | 0   | 0   | 0   | 0   | 4   | 0   | 4   | 0   |
| 9 | 0   | 0   | 2   | -4 | 8   | 0   |     |     |
|10 | 0   | 0   | 0   | 2   | -2 | 8   | 0   |     |
|11 | 0   | 0   | 0   | 1   | 2   | 3   | 2   |     |
|12 | 0   | 0   | 0   | 0   | 6   | -2 | 4   |     |
|13 | 0   | -4 | 10  | -1 | 0   | -3 | 6   |     |
|14 | 0   | 0   | 0   | 2   | 0   | 2   | 4   |     |
|15 | 0   | 0   | 0   | 1   | 4   | -3 | 6   |     |
|16 | 0   | 0   | -4 | 6   | 2   | 4   | 0   |     |
|17 | 0   | 0   | -2 | 8   | -2 | -4 | 8   |     |
|18 | 0   | 0   | 0   | 2   | 2   | -4 | 8   |     |

Table 8: Power of $\eta$-functions being basis elements of $S_4(\Gamma_0(36))$

|   | 1   | 2   | 3   | 4   | 6   | 9   | 12  | 18  | 36  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | 0   | 0   | 8   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2 | 0   | 0   | 0   | 8   | 0   | 0   | 0   | 0   | 0   |
| 3 | 0   | 0   | 2   | 0   | 2   | 0   | 2   | 0   | 0   |
| 4 | 0   | 0   | 3   | 0   | 1   | -1 | 0   | 5   | 0   |
| 5 | 0   | 0   | 4   | 0   | -4 | 0   | 8   | 0   | 0   |
| 6 | 0   | 0   | 0   | 0   | 2   | 0   | 2   | 2   | 2   |
| 7 | 0   | 0   | 0   | 3   | 0   | -1 | 3   | 3   | 3   |
| 8 | 0   | 0   | 0   | 3   | 0   | 1   | -1 | 5   |     |
| 9 | 0   | 0   | 0   | 0   | 4   | 0   | -2 | 0   | 6   |
|10 | 0   | 0   | 0   | 0   | 4   | 0   | 0   | -4 | 8   |
|11 | 0   | 0   | 0   | 0   | 5   | 0   | -3 | -3 | 9   |
|12 | -5 | 11  | 5   | -5 | -1 | -2 | 0   | 0   | 5   |