Examples of 4D, $\mathcal{N} = 2$ Holoraumy

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ABSTRACT

We provide an introduction to the concepts of Holoraumy tensors, Lorentz covariant four-dimensional “Gadgets”, and Gadget angles within the context of minimal off-shell 4D, $\mathcal{N} = 2$ supermultiplets. This is followed by the calculation of the Holoraumy tensors, Gadgets, and Gadget angles for minimal off-shell supermultiplets. Four tetrahedrons in four 3D subspaces of the Holoraumy lattice space are found.

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1 Introduction

Space-time supersymmetry theories in four dimensions can be dated to works of the early to middle seventies [1,2,3,4,5]. Their introduction to the physics literature also marked the inauguration of investigations into new mathematical subjects - “super Lie algebras” and “super Lie groups.” As the names suggest, these are extensions of the more well established subjects of Lie algebras and Lie groups. Currently is a time that is almost fifty years since the introduction of the space-time supersymmetry concept. A comparison of the former concepts with the latter shows a marked distinction.

Group theory is a much older subject having emerged from algebraic equations, geometry, and number theory in the seventeen hundreds. Thus the subject benefits from a long period of exploration and investigation. In particular, the representation theory of Lie algebras enjoys a highly developed status. This can partly be seen from the tight nexus of structures involving matrix algebra and elements of graph theory (roots, weights, Dynkin diagrams and Young Tableaux) which can be marshalled [6,7,8,9,10,11,12,13] to study issues that arise surrounding Lie algebras and groups. As Sophus Lie (1842-1899) was the pioneer who researched the issue of discovering all group actions infinitesimally acting on manifolds, the subject bares his name. However, an impressive and long list of other mathematicians drove the development of the subject to its current high level of sophistication. In particular it was Cartan [14] (1861-1959) who delivered a mathematically rigorous classification of all possible simple Lie algebras. Toward the end of the nineteen seventies, Kac [16] extended this result with a classification of simple super Lie algebras.

The concepts of weights and roots play an important role in this line of mathematical investigation. However, these two concepts rely on the fact it is possible to partition Lie algebras using the Jordan-Chevalley decomposition which splits all the generators of the algebra into two sets. In practice to accomplish this one must find the maximal commuting set of generators among all of the generators and construct a set of simultaneous eigenvectors (along with their corresponding eigenvalues) of this maximal commuting set. These eigenvalues then provide a basis for constructing vectors in the space of weights and the roots (as differences in weight vectors) lead to Dynkin diagrams.

It is here that the spacetime supersymmetry algebra presents a challenge to this tried and true method as there is no generally accepted definition about the concept of eigenvectors with respect to supersymmetry generators.

It has been a goal of some of our research to develop tools that can be used to fill in the gap left by this absence of concepts related to eigenvectors and eigenvalues. In a sense our quest has been to find what structures in spacetime SUSY hold the data that occurs in Lie algebra via use of eigenvectors and eigenvalues. This has occurred in a continuing series of works that began with a concept of $GR(d, N)$ algebras (or “garden algebras”) [17,18] as a foundation and subsequently led to the discover of “adinkras” [19]. Adinkras are one dimensional graphical representations [20,21,22,23,24] of the garden algebras and by their study has led to the concept of holoraumy [25,26] as a tool for this purpose. However, the concept of holoraumy possesses a natural extension from the context of 1D SUSY systems to higher dimensional ones and in particular to 4D SUSY representations [27,28]. So our proposition is the data normally accessed in Lie algebra (via use of eigenvectors and eigenvalues) within the context of spacetime SUSY theories is accessed via the holoraumy.

The purpose of this work is to continue the extension of this exploration by presenting the first report of holoraumy in the context of 4D, $N = 2$ minimal supersymmetrical representations.

The subsequent presentation unfolds in the following manner.

Chapter two is devoted to a quick review of the tools of representation theory as applied to the
very familiar example of the su(3) algebra. The usual Gell-Mann representation of $3 \times 3$ matrices is utilized and leads to the usual structure constants and “d-coefficients.” The traditional maximal set of commuting generators and the role of their simultaneous eigenvectors and associated eigenvalues are noted as the foundation that advances the understanding of the structure of the su(3) algebra. It is noted the partitioning of the generators into a semisimple portion (containing only the commuting generators) and a nilpotent portion achieves the Jordan-Chevalley decomposition of su(3). These are the basis for the considerations of the roots and weights of the algebra. By parallel transport of the roots, a lattice emerges and the vertices of the lattice are noted to be the weights of representations. Finally, the existence of Casimir operators leading to a classification of the representations in terms of two integers $p$ and $q$ is observed. The integers are then related to the structure of Young Tableaux.

Chapter three contains a discussion of the concept of “holoraumy” in the context of representations of the 4D, $\mathcal{N}$-extended supersymmetry algebra. The basic definition of holoraumy is followed by describing a set of conventions and the structures that emerge from the definition of the holoraumy operator are presented. For a general value of $\mathcal{N}$ it is noted that the irreducible (with respect to the covering algebra of the Dirac matrices, generators of so($\mathcal{N}$), and the symmetric tensors in the defining representation of so($\mathcal{N}$) appear in the holoraumy operator in precisely a manner that leads to a set of so(4$\mathcal{N}$) generators in the reduction to one dimensional supersymmetrical systems.

The fourth chapter contains the new results of this work by presenting the explicit form of holoraumy for the minimal off-shell supermultiplets that realize 4D, $\mathcal{N} = 2$ supersymmetrical systems with a finite number of auxiliary fields. The holoraumy operator is second order in the D-operators of SUSY and thus possesses an engineering dimension of one. It maps bosons to derivatives of bosons and separately fermions to derivatives of fermions. So evaluation of the holoraumy operator has two distinct parts. This chapter only presents the evaluation on the fermionic fields. The facts that holoraumy solely maps bosons to derivatives of bosons, separately fermions to derivatives of fermions, and possess an engineering dimension of one ensures that a set of dimensionless numbers emerge in these calculations. These numbers are specific to each supersymmetrical multiplet. It is the contention of this work that these dimensionless numbers are the analogs of eigenvalues seen in the Chevalley-Jordan decomposition of ordinary Lie algebras.

The chapter reviews the known off-shell 4D, $\mathcal{N} = 2$ supersymmetrical systems with a finite number of auxiliary fields. The systems consist of the vector, tensor, relaxed hypermultiplet, supergravity, hyperplet, and higher spin supermultiplets. The counting of component fields in each representation is given and the vector and tensor supermultiplets (together with their parity duals) are identified as the minimal ones. A brief discussion of some minimal on-shell representations is given to contrast with the off-shell constructions and notational conventions are set in place for the presentation of results.

The fifth chapter contains the information that is equivalent to that of the fourth chapter but with the distinction that these results describe the realization of the holoraumy operator solely on the bosonic fields. Owing the disparate spins among the bosons that appear in the minimal supermultiplets, the results do not present any obvious interpretation as the equivalent one presented on the spinor fields. For this reason, the spinorial ones have been and remain the prime focus of our study.

A sixth chapter sets into place an operator that takes pairs of the minimal supermultiplets and maps such pairs into a real number. In the past, this operator has been given the name of the “Gadget” and the corresponding real number is referred to as “the Gadget value” of the pair. The Gadget essentially defines a “dot product” on the space of supermultiplets. We show there are a priori a number of definitions of the Gadget that are consistent with Lorentz and so(2) covariance. All such definitions are found to lead to a possible matrix of dot product containing only three values which are denoted by $X_1$, $X_2$, and $X_3$. 3
In the seventh chapter, all the results in the previous ones are reduced to the case where all spatial dimensions are eliminated and thus links to adinkras can be directly studied. The corresponding “L-matrices” and “R-matrices” for the adinkras with four colors, four open nodes, and four closed nodes are obtained as substructures of adinkras with eight colors, eight open nodes, and eight closed nodes.

The eighth chapter builds upon the work of the seventh chapter and constructs the original 1D holoraumy matrices as conceptualized previously [25,26] but now for the first time explicitly in the context of adinkras with eight colors, eight open nodes, and eight closed nodes.

The ninth chapter is used to construct the 1D, $N = 8$ Gadget and Gadget values associated with the adinkras constructed in the previous chapter. It is shown that for one special choice of the parameters $\chi_1$, $\chi_2$, and $\chi_3$, the matrix of dot products for the 1D, $N = 8$ Gadget and the 4D, $N = 2$ Gadget agree and thus realize the concept of “SUSY holography”.

Chapter ten summarizes the results achieved in this work and examines the challenges ahead in this line of study.

This work contains three appendices. The first appendix simply lays out the explicit basis of matrices and vectors used in this work. The second chapter lays out the D-algebra results which are equivalent to SUSY transformation laws. These have been presented and validated in previous work and provide the basis for the calculations in chapter four. The final appendix contains the explicit expression for the 1D, $N = 8$ holoraumy matrices associated with the adinkras and used to calculate the values of the Gadget dot products.
2 Review of Representation Tools for the $\text{su}(3)$ Lie Algebra

For discussion of the $\text{su}(3)$ algebra, we use the standard Gell-Mann representation matrices $\lambda_i$ (see Appendix A) which satisfy the relations

$$[T_i, T_j] = i f_{ij}^k T_k, \quad (2.1)$$

where $T_i = \frac{1}{2} \lambda_i$ are the $\text{su}(3)$ generators. The only non-vanishing values of the totally anti-symmetric structure constant $f_{ij}^k$ are completely specified by giving the values shown in (2.2).

$$f_{123} = 1, \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \quad (2.2)$$

and we use Kronecker deltas to raise and lower indices. As $T_3$ and $T_8$ commute, we define three simultaneous eigenvectors of these generators in terms of their eigenvalues $t_3$, and $t_8$, and denoting the eigenvectors as $|t_3, t_8\rangle$, and thus

$$T_3|t_3, t_8\rangle = t_3|t_3, t_8\rangle, \quad T_8|t_3, t_8\rangle = t_8|t_3, t_8\rangle. \quad (2.3)$$

Given an arbitrary representation $(\mathcal{R})$ in this space of “3-tuples” we can write

$$|\langle\mathcal{R}\rangle\rangle = \sum_{t_3, t_8} c[(\mathcal{R}): t_3, t_8]|t_3, t_8\rangle \rightarrow T_i|\langle\mathcal{R}\rangle\rangle = \sum_{t_3, t_8} c[(\mathcal{R}): t_3, t_8]T_i|t_3, t_8\rangle, \quad (2.4)$$

where $c[(\mathcal{R}): t_3, t_8]$ are simply a set of constants.

A well-known result about the matrices in (A.1) involves their anti-commutator taking the form

$$\{T_i, T_j\} = \frac{1}{3} \delta_{ij} I_{3\times3} + d_{ij}^k T_k, \quad (2.5)$$

where the totally symmetric “d-coefficients” are completely specified by giving the values shown in (2.6)

$$d_{118} = d_{228} = d_{338} = d_{888} = \frac{1}{\sqrt{3}}, \quad d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \quad (2.6)$$

$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}.$$

It is of great importance to note that the result in (2.5) is not universal for all representations unlike (2.1).

The results in (2.3) also open the route to a deeper understanding of $\text{su}(3)$ as these lead to the roots and weights of the representations via interpreting the results of any explicit evaluation of (2.4) as describing “motions” in the space of the eigenvectors. We illustrate these in Figure 1 to follow.

![Figure 1: Illustration of “motions” of $\text{su}(3)$ generators.](image)

The angle shown above is equal to $2\pi/3$ and the arrows show the motions of the generators which are the “roots” of the $\text{su}(3)$ algebra. By parallel transport of the roots along one another, one can generate
a lattice-like structure (the root space) shown in Figure 2.

![Lattice diagram](image)

**Figure 2**: Illustrations of: (a) Lattice from su(3) motions and (b) three states on some vertices.

The weights of states of any representation can be visualized the coordinates of points occupying the vertices of the lattice as shown to the right in Figure 2.

It is well understood that the states of irreducible representations are not randomly distributed\(^3\), but instead fall into regular patterns as seen in Figure 3\(^4\).

![Representation diagrams](image)

**Figure 3**: Illustrations of: (a) octet representation and (b) decuplet representation.

Two integers \(p\) and \(q\) can be found by examining the weight of the state with the largest value of the eigenvalue \(t_3\) and simultaneously looking at the value of the eigenvalue of \(t_8\) for this same state. For this state, the relations

\[
t_3 = \frac{1}{2} (p + q) \quad , \quad t_8 = \frac{1}{2\sqrt{3}} (p - q)
\]

are valid and thus through these eigenvalues of the maximal commuting group, the integers \(p\) and \(q\) are determined quantities.

\(^3\)It is this fact, that led to the very discovery of the role that Lie algebras play in fundamental physics.

\(^4\)In the diagrams shown in Figure 3, the quantity \(s\) denotes strangeness while \(q\) denotes electrical charge.
All irreducible representations are characterized by the two integers \( p \) and \( q \) so that the number of states in any irreducible representations described by \( p \) and \( q \) is given by applying the Weyl dimension formula to \( \text{su}(3) \)

\[
d(p, q) = \frac{1}{2} (p + 1) (q + 1) (p + q + 2) \quad .
\]  

(2.8)

These same integers characterize the Quadratic Casimir via Freudenthal’s formula [29,30,31]

\[
C_2(p, q) \propto \frac{1}{4} \sum_{i=1}^{8} \text{Tr} (\lambda_i \lambda_i) = \frac{1}{3} \left( p^2 + q^2 + pq + 3p + 3q \right) \quad ,
\]  

(2.9)

as well as a Cubic Casimir [32]

\[
C_3(p, q) \propto \frac{1}{8} \sum_{i=1,j=1,k=1}^{8} d_{ijk} \text{Tr} (\lambda_i \lambda_j \lambda_k) = \frac{1}{18} (p - q) (3 + p + 2q) (3 + q + 2p) \quad .
\]  

(2.10)

The ubiquitous appearance of the two integers \( p \) and \( q \) point toward another aspect of the representation theory of \( \text{su}(3) \) as they are avatars for the existence of Young Tableaux. As shown in Figure 4, the integers \( p \) and \( q \) respectively describe the length of one-height rows and the length of two-height rows in any given irreducible representation.

Figure 4: Illustration of integers \( p \) and \( q \) within \( \text{su}(3) \) Young Tableau.

As the discussion in this chapter has shown, the matrix representation of \( \text{su}(3) \) leads, via eigenvectors and eigenvalues (2.3) to roots and weights. However, via the Casimir values the matrix representation also leads to the Young Tableaux representations. All are tightly linked together.

We wish to close this discussion by concentrating on one more Casimir, the one associated with a quartic Casimir value and given by

\[
C_4(p, q) = \frac{1}{16} \sum_{i=1,j=1}^{8} \text{Tr} (\{\lambda_i, \lambda_j\} \{\lambda^i, \lambda^j\}) = d(p, q) C_2(p, q) [4 C_2(p, q) - 3] \quad ,
\]  

(2.11)

and after reviewing the literature, we have not found a prior determination of the function to the far right in (2.11). In a separate work [33], an argument is given for how this conclusion was reached.

Whenever two representations \( (\mathcal{R}) \) and \( (\mathcal{R}') \) characterized respectively by \( (p, q) \) and \( (p', q') \) satisfy the condition

\[
d(p, q) = d(p', q') \quad ,
\]  

(2.12)

where \( d(p, q) \) is defined in Eq. (2.8), the matrices \( \lambda_i^{(\mathcal{R})} \) associated with the first representation have the same size as matrices \( \lambda_i^{(\mathcal{R}')} \) associated with the second representation. In this case, we can deform the quartic Casimir to define\(^5\)

\[
C_{4,\mathcal{G}}(\mathcal{R}, \mathcal{R}') = \frac{1}{16} \sum_{i=1,j=1}^{8} \text{Tr} \left( \{\lambda_i^{(\mathcal{R})}, \lambda_j^{(\mathcal{R})}\} \{\lambda^i^{(\mathcal{R}'}), \lambda^j^{(\mathcal{R}')}\} \right) \quad .
\]  

(2.13)

In the discussion of supersymmetry that follows a similar structure called “the Gadget” will play a role.

\(^{5}\)In appendix A, this concept is considered in the context of the C-K-M matrix.
3 General 4D, $\mathcal{N}$-extended SUSY Holoraumy Structure

On the basis of four dimensional Lorentz- and as well $\text{so}(\mathcal{N})$-covariance, the work of [25] asserted that the most general equation for the holoraumy operator of a supersymmetrical multiplet must take the form$^6$

\[
[D_{ai} , D_{bj}] = i C_{ab} \delta_{ij} \mathcal{H}^{(1)} + (\gamma^5)_{ab} \delta_{ij} \mathcal{H}^{(2)} + (\gamma^5 \gamma^\mu)_{ab} \delta_{ij} \mathcal{H}_\mu^{(3)} \\
+ i C_{ab} (\mathcal{S}^{(S)}_{ij} \cdot \mathcal{H}^{(4:S)}) + (\gamma^5)_{ab} (\mathcal{S}^{(S)}_{ij} \cdot \mathcal{H}^{(5:S)}) + (\gamma^5 \gamma^\mu)_{ab} (\mathcal{S}^{(S)}_{ij} \cdot \mathcal{H}_\mu^{(6:S)}) \\
+ i (\gamma^\mu)_{ab} (\mathcal{A}^{[A]}_{ij} \cdot \mathcal{H}_\mu^{(7:A)}) + i \frac{1}{2} ([\gamma^\mu , \gamma^\nu])_{ab} (\mathcal{A}^{[A]}_{ij} \cdot \mathcal{H}_{\mu\nu}^{(8:A)}) ,
\]

(3.1)

where in this expression, the quantity $C_{ab}$ is the usual “spinor metric” and $(\gamma^\mu)_{ab}$ denote the usual Dirac gamma matrices$^7$. The quantities $\delta_{ij}$, $\mathcal{S}^{(S)}_{ij}$ and $\mathcal{A}^{[A]}_{ij}$ are tensors in $\text{so}(\mathcal{N})$. All of these tensors possess indices $i$ and $j$ taking on values $1, \ldots, \mathcal{N}$. The first of these denotes the Kronecker delta tensor. The second collection of these objects denoted by $\mathcal{S}^{(S)}_{ij}$ satisfies $\mathcal{S}^{(S)}_{ij} = \mathcal{S}^{(S)}_{ji}$ and $\delta_{ij} \mathcal{S}^{(S)}_{ij} = 0$, where the index $(S)$ takes on values $1, \ldots, (\mathcal{N} + 2)(\mathcal{N} - 1)/2$. The final collection of such objects denoted by $\mathcal{A}^{[A]}_{ij}$ satisfies $\mathcal{A}^{[A]}_{ij} = -\mathcal{A}^{[A]}_{ji}$, where the index $[A]$ takes on values $1, \ldots, \mathcal{N}(\mathcal{N} - 1)/2$.

We may regard the quantities $\mathcal{A}^{[A]}_{ij}$ as the generators of $\text{so}(\mathcal{N})$. Under this interpretation, $\delta_{ij}$ and $\mathcal{S}^{(S)}_{ij}$ are respectively the $\text{so}(\mathcal{N})$ invariant and the second order traceless symmetric tensor representations, respectively, under the action of the $\text{so}(\mathcal{N})$ generators $\mathcal{A}^{[A]}_{ij}$. In the expression (3.1), the $4\mathcal{N}(4\mathcal{N} - 1)/2$ quantities

\[
i C_{ab} \delta_{ij} , (\gamma^5)_{ab} \delta_{ij} , (\gamma^5 \gamma^\mu)_{ab} \delta_{ij} , \\
i C_{ab} \mathcal{S}^{(S)}_{ij} , (\gamma^5)_{ab} \mathcal{S}^{(S)}_{ij} , (\gamma^5 \gamma^\mu)_{ab} \mathcal{S}^{(S)}_{ij} , \\
i (\gamma^\mu)_{ab} \mathcal{A}^{[A]}_{ij} , i \frac{1}{2} ([\gamma^\mu , \gamma^\nu])_{ab} \mathcal{A}^{[A]}_{ij}
\]

are equal in number to the generators of $\text{so}(4\mathcal{N})$. In fact, each of these quantities possesses an antisymmetry property under the pairwise exchange of indices $a i \leftrightarrow b j$. So the total collection (3.2) must provide a representation of $\text{so}(4\mathcal{N})$. When such a system is subjected to reduction on a torus to one dimension, this observation is the origin of the fact that an $\text{so}(4\mathcal{N})$ symmetry is present after the reduction.

Finally, in (3.1), the $4\mathcal{N}(4\mathcal{N} - 1)/2$ quantities

\[
\mathcal{H}^{(1)}_\mu , \mathcal{H}^{(2)}_\mu , \mathcal{H}^{(3)}_\mu , \\
\mathcal{H}^{(4:S)}_\mu , \mathcal{H}^{(5:S)}_\mu , \mathcal{H}^{(6:S)}_\mu , \\
\mathcal{H}^{(7:A)}_\mu , \mathcal{H}^{(8:A)}_{\mu\nu},
\]

(3.3)

correspond to Lie-algebra operators whose explicit forms depend on the supermultiplet on which the holoraumy operator is being evaluated.

A major purpose of this current work is to “flesh out” (i. e. present explicit results) the assertions made in writing (3.1). As our previous discussions of 4D holoraumy were in the context of $\mathcal{N} = 1$

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$^6$We have used a slightly different notation here in comparison to previous presentations.

$^7$See appendix A in the work of [34] for our conventions.
supermultiplets [27,28], the quantities $\mathcal{R}^{(S)}_{ij}$ and $\mathcal{R}^{[A]}_{ij}$ perforce vanish. Thus, the work that follows provides the first opportunity to realize these more general structures by working out details within the context of a specific set of examples. One such point of focus will be to examine the commutator algebra (appropriate for $\mathcal{N} = 2$) that follows from the generators shown in (3.2).
4 Structure of 4D, \( \mathcal{N} = 2 \) SUSY Fermionic Holoraumy For Minimal Multiplets

To begin the discussion of the 4D, \( \mathcal{N} = 2 \) fermionic holoraumy, we first need to specify the definition and notation of the 4D, \( \mathcal{N} = 2 \) supercovariant derivative operator via the equation

\[
D^i_a = \begin{bmatrix} D^1_a \\ D^2_a \end{bmatrix}, \quad D_{ai} = \delta_{ij} D^j_a, \quad D^i_a = \delta^{ij} D_{aj}, \quad (4.1)
\]

here \( i \) and \( j \) are isospin indices taking on two values. Above \( D^1_a \) and \( D^2_a \) are Majorana 4D operators. We use the convention that such isospin indices are raised and/or lowered via the Kronecker delta symbol.

The complete listing of known off-shell 4D, \( \mathcal{N} = 2 \) supermultiplets, containing a finite number of auxiliary fields to our knowledge, is shown in Table 1 where the initial literature presentation for each supermultiplet was given in:

- (a.) vector supermultiplet [35,36],
- (b.) tensor supermultiplet [35,36],
- (c.) relaxed supermultiplet [37],
- (d.) supergravity supermultiplet [38,39,40],
- (e.) “hyperplet” supermultiplet [41,42], and
- (f.) “HS”, higher spin supermultiplet [43,44].

Clearly, only the vector and the tensor 4D, \( \mathcal{N} = 2 \) supermultiplet are minimal\(^8\). We have also indicated the number of boson (alternately fermionic) degrees of freedom in Table 1 for each of these supermultiplets\(^9\).

| 4D, \( \mathcal{N} = 2 \) Supermultiplet | Multiplicity of Bosons or Fermions |
|----------------------------------------|----------------------------------|
| vector                                 | \{8\}                            |
| tensor                                 | \{8\}                            |
| relaxed                                | \{32\}                           |
| supergravity                           | \{40\}                           |
| hyperplet                              | \{96\}                           |
| HS (w. lowest spin of propagating field = integer \( s > 2 \)) | \{8s^2 + 8s + 4\}                |

Table 1: Degrees of Freedom In 4D, \( \mathcal{N} = 2 \) Supermultiplets.

In the work of [46], an exhaustive study was made of the most general construction of a 4D, \( \mathcal{N} = 2 \) supermultiplet based on pairings of the 4D, \( \mathcal{N} = 1 \) chiral, vector, or tensor supermultiplets. As this involved the three distinct 4D, \( \mathcal{N} = 1 \) supermultiplets, when taken in pairs, there might have existed as many as six combinations that form off-shell \( \mathcal{N} = 2 \) supermultiplets. For systems that only respect supersymmetry on-shell, any such pairing was found to work. However and in fact, it was found that only the pairings of 4D, \( \mathcal{N} = 1 \) chiral\( \oplus \)vector or the chiral\( \oplus \)tensor occur as

\(^8\)In the presentation of this table, we have ignored the possible existence of “variant representations” of these supermultiplets.

\(^9\)The astute reader will note that there is no result reported in the table for a 4D, \( \mathcal{N} = 2 \) supermultiplet with spins of \((3/2, 1, 1, 1/2)\). Although an on-shell description is available [45], no off-shell description has been presented.
reduce the notion of a “lattice variable”, denoted by Π, which is simply defined to be an ordered set
holoraumy on minimal representations. In order to achieve a maximally concise notation, we intro-

As minimal off-shell 4D, \( \mathcal{N} = 2 \) supermultiplets contain a doublet of Majorana spinors, we
introduce a notation of the form
\[
\Psi_{ci}^{(2VS)} = \begin{bmatrix} \psi_c^\lambda \\ \lambda_c \end{bmatrix}, \quad \Psi_{ci}^{(2TS)} = \begin{bmatrix} \psi_c^\lambda \\ \chi_c \end{bmatrix}, \quad \Psi_{ci}^{(2AVS)} = \begin{bmatrix} \psi_c^{\bar{\chi}} \\ \bar{\chi}_c \end{bmatrix}, \quad \Psi_{ci}^{(2ATS)} = \begin{bmatrix} \psi_c^{\bar{\chi}} \\ \bar{\chi}_c \end{bmatrix},
\]
where \( \psi_c \) denotes the spinor field of the off-shell 4D, \( \mathcal{N} = 1 \) chiral supermultiplet, \( \lambda_c \) denotes the spinor field of the off-shell 4D, \( \mathcal{N} = 1 \) vector supermultiplet, \( \chi_c \) denotes the spinor field of the off-shell 4D, \( \mathcal{N} = 1 \) tensor supermultiplet, \( \bar{\chi}_c \) denotes the spinor field of the off-shell 4D, \( \mathcal{N} = 1 \) axial-vector supermultiplet, and \( \bar{\chi}_c \) denotes the spinor field of the off-shell 4D, \( \mathcal{N} = 1 \) axial-tensor supermultiplet. The “isospin” index \( i \) takes on values 1 and 2. As in previous works [27,28] we
introduce a 4D supermultiplet “representation index” \((\hat{\mathcal{R}})\) that takes on the values of \((2VS), (2TS), (2AVS), \) and \((2ATS)\). This allows us to express the four spinors shown in (4.2) collectively as \( \Psi_{ci}^{(\hat{\mathcal{R}})} \) and we are now in position to state the main results of this investigation to calculate its fermionic holoraumy.

4.1 Results For Fermionic Holoraumy 4D, \( \mathcal{N} = 2 \) On Minimal Representations

Having dispensed with all the necessary preliminary set of review and statement of our problem,
conventions, etc., we now move towards the presentation of final results for the 4D, \( \mathcal{N} = 2 \) fermionic holoraumy on minimal representations. In order to achieve a maximally concise notation, we introduce the notion of a “lattice variable”, denoted by \( \Pi \), which is simply defined to be an ordered set of four integers \( (p, q, r, s) \).

After a set of calculation we find our answer can be expressed concisely by the equation,
\[
\left[ D^i_a, D^j_b \right] \Psi_{\bar{c}k}^{(\hat{\mathcal{R}})} = \left[ \hat{H}^{(\hat{\mathcal{R}})} \right]_{abck}^{ij} \partial_d \Psi_{\bar{c}k}^{(\hat{\mathcal{R}})} ,
\]
where
\[
\left[ \hat{H}^{(\hat{\mathcal{R}})} \right]_{abck}^{ij} \partial_d = - \delta^{ij} \delta_k^l \left[ \hat{h}^{(0)}(\Pi_0^{(\hat{\mathcal{R}})}) \right]_{ab}^{d} + \left( \sigma^3 \right)^{ij} \left( \sigma^3 \right)^k_l \left[ \hat{h}^{(3)}(\Pi_3^{(\hat{\mathcal{R}})}) \right]_{ab}^{d} + \left( \sigma^1 \right)^{ij} \left( \sigma^1 \right)^k_l \left[ \hat{h}^{(1)}(\Pi_1^{(\hat{\mathcal{R}})}) \right]_{ab}^{d} + \left( \sigma^2 \right)^{ij} \left( \sigma^2 \right)^k_l \left[ \hat{h}^{(2)}(\Pi_2^{(\hat{\mathcal{R}})}) \right]_{ab}^{d} ,
\]
and \( \left[ \hat{h}^{(0)}(\Pi) \right]_{ab}^{d} \), \( \left[ \hat{h}^{(3)}(\Pi) \right]_{ab}^{d} \), and \( \left[ \hat{h}^{(1)}(\Pi) \right]_{ab}^{d} \) satisfy the equations
\[
\left[ \hat{h}^{(0)}(\Pi) \right]_{ab}^{d} = \hat{h}^{(3)}(\Pi)_{ab}^{d} = \hat{h}^{(1)}(\Pi)_{ab}^{d} = i \left[ p C_{ab} (\gamma^\mu)_c^{d} + q (\gamma^5)_{ab} (\gamma^5 \gamma^\mu)_c^{d} \right. + r (\gamma^5 \gamma^\mu)_{ab} (\gamma^5 \gamma^\mu)_c^{d} + \frac{1}{2} s (\gamma^5 \gamma_{ab} (\gamma^5 \gamma^\mu)_c^{d} = \hat{h}^{(\mu)}(\Pi)_{ab}^{d}.
\]

The function \( \left[ \hat{h}^{(\mu)}(\Pi) \right]_{ab}^{d} \) was first presented in the work on 4D, \( \mathcal{N} = 1 \) minimal supermultiplets [28] previously and we see that the only distinction between \( \left[ \hat{h}^{(0)} \right]_{ab}^{d}, \left[ \hat{h}^{(3)} \right]_{ab}^{d}, \) and \( \left[ \hat{h}^{(1)} \right]_{ab}^{d} \), is that distinct lattice variables are utilized.
The quantity $\left[\hat{h}^{(2)\mu}(\Pi)\right]_{abc}^d$ is given by

$$\left[\hat{h}^{(2)\mu}(\Pi)\right]_{abc}^d = i \left[ \frac{1}{2} p (\gamma^5 [\gamma^\mu, \gamma^\nu])_{ab} (\gamma^5 \gamma^\nu)_{ce}^d + \frac{1}{2} q (\gamma^\mu, \gamma^\nu)_{ab} (\gamma^\nu)_{ce}^d + \frac{1}{2} r (\gamma^\nu)_{ab} (\gamma^\nu)_{ce}^d + s (\gamma^\mu)_{ab} \delta_e^d \right].$$

(4.6)

It can be seen there is a more substantial distinction between $\left[\hat{h}^\mu(\Pi)\right]_{abc}^d$ and $\left[\hat{h}^{(2)\mu}(\Pi)\right]_{abc}^d$ with regard to the symmetry on the exchange of the first two spinor indices on each. Namely, these respectively satisfy the identities

$$\left[\hat{h}^\mu(\Pi)\right]_{abc}^d = - \left[\hat{h}^\mu(\Pi)\right]_{bac}^d, \quad \left[\hat{h}^{(2)\mu}(\Pi)\right]_{abc}^d = + \left[\hat{h}^{(2)\mu}(\Pi)\right]_{bac}^d.$$  

(4.7)

In addition, we also see the following identities are valid.

$$\left[\hat{h}^\mu(\Pi)\right]_{ab}^{bc} = - (\gamma^5)_{a}^{k} (\gamma^5)_{c}^{m} \left[\hat{h}^\mu(\Pi)\right]_{k}^{l} \left[\hat{h}^\mu(\Pi)\right]_{l}^{n} (\gamma^5)_{l}^{b} (\gamma^5)_{n}^{d},$$

$$\left[\hat{h}^{(2)\mu}(\Pi)\right]_{ab}^{bc} = - (\gamma^5)_{a}^{k} (\gamma^5)_{c}^{m} \left[\hat{h}^{(2)\mu}(\Pi)\right]_{k}^{l} \left[\hat{h}^{(2)\mu}(\Pi)\right]_{l}^{n} (\gamma^5)_{l}^{b} (\gamma^5)_{n}^{d}. $$

(4.8)

In (4.4) the quantities denoted by $p_A^{(\vec{R}), q_A^{(\vec{R}), r_A^{(\vec{R}), s_A^{(\vec{R})}}, (\text{i.e. } \Pi_A^{(\vec{R})})}$ with $A = 0, \ldots , 3$ are integers listed in Table 2.

| $(\vec{R})$ | $p_0$ | $q_0$ | $r_0$ | $s_0$ | $p_3$ | $q_3$ | $r_3$ | $s_3$ | $p_1$ | $q_1$ | $r_1$ | $s_1$ | $p_2$ | $q_2$ | $r_2$ | $s_2$ |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $(2VS)$     | 1     | 1     | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| $(2TS)$     | -1    | 1     | -1    | -1    | -1    | 1     | -1    | -1    | 1     | -1    | -1    | -1    | 1     | -1    | -1    | -1    | -1    |
| $(2AVS)$    | -1    | -1    | 1     | -1    | -1    | 1     | -1    | 1     | -1    | 1     | -1    | 1     | -1    | 1     | -1    | 1     | -1    | 1     |
| $(2ATS)$    | 1     | -1    | -1    | -1    | 1     | -1    | 1     | -1    | 1     | -1    | 1     | -1    | 1     | -1    | 1     | -1    | 1     | -1    |

Table 2: Holoraumy Integers For $4D$, $\mathcal{N} = 2$ Vector, Tensor, Axial-Vector and Axial-Tensor Supermultiplets.

To be more explicit, we can substitute all the above information to re-write the result for each supermultiplet in the forms

$$\left[\hat{H}^{(2VS)}\right]_{ij}^{dl} = - \delta^{ij} \delta_k^l \left[\hat{h}^\mu(1, 1, 1, -1)\right]_{abc}^d + (\sigma^3)^{ij} (\sigma^3)_{k}^{l} \left[\hat{h}^\mu(1, 1, 1, 1)\right]_{abc}^d,$$

(4.9)

$$\left[\hat{H}^{(2TS)}\right]_{ij}^{dl} = - \delta^{ij} \delta_k^l \left[\hat{h}^\mu(-1, 1, 1, -1)\right]_{abc}^d + (\sigma^3)^{ij} (\sigma^3)_{k}^{l} \left[\hat{h}^\mu(-1, 1, 1, 1)\right]_{abc}^d,$$

(4.10)

$$\left[\hat{H}^{(2AVS)}\right]_{ij}^{dl} = - \delta^{ij} \delta_k^l \left[\hat{h}^\mu(-1, 1, 1, -1)\right]_{abc}^d + (\sigma^3)^{ij} (\sigma^3)_{k}^{l} \left[\hat{h}^\mu(-1, 1, 1, 1)\right]_{abc}^d,$$

(4.11)

$$\left[\hat{H}^{(2ATS)}\right]_{ij}^{dl} = - \delta^{ij} \delta_k^l \left[\hat{h}^\mu(1, -1, 1, -1)\right]_{abc}^d + (\sigma^3)^{ij} (\sigma^3)_{k}^{l} \left[\hat{h}^\mu(1, -1, 1, 1)\right]_{abc}^d,$$

(4.12)
and we note that

\[
\begin{align*}
\left[ \hat{H}^\mu(2\vec{V}\dot{S}) \right]_{ij}^{ab} dt &= (\gamma^5)_c^e \left[ \hat{H}^\mu(2\vec{V}\dot{S}) \right]_{ab}^{cd} f^l(\gamma^5)_f^d, \\
\left[ \hat{H}^\mu(2\vec{C}\dot{S}) \right]_{ij}^{ab} dt &= (\gamma^5)_c^e \left[ \hat{H}^\mu(2\vec{C}\dot{S}) \right]_{ab}^{cd} f^l(\gamma^5)_f^d,
\end{align*}
\]

(4.13)
such that the parity flip involves

\[
P_A(\vec{R}) \longrightarrow -P_A(\vec{R}), \quad q_A(\vec{R}) \longrightarrow -q_A(\vec{R}), \quad r_A(\vec{R}) \longrightarrow r_A(\vec{R}), \quad s_A(\vec{R}) \longrightarrow s_A(\vec{R}).
\]

(4.14)

for all \( A \).

It is useful at this point to review the results of the previous subsection within the context of our opening remarks. In particular, the calculations carried out that support the results shown in (3.2) and (3.3). The constant tensors \( \mathcal{A}_{ij}^{(S)} \) and \( \mathcal{A}_{ij}^{[A]} \) are clearly determined. Since \( N = 2 \), the (S) index on \( \mathcal{A}_{ij}^{(S)} \) takes on two values, which can be identified with

\[
\mathcal{A}_{ij}^{(3)} = (\sigma^3)_{ij}, \quad \mathcal{A}_{ij}^{(1)} = (\sigma^1)_{ij},
\]

(4.15)
while again since \( N = 2 \), the [A] index on \( \mathcal{A}_{ij}^{[A]} \) takes on a single value and we identify it as simply

\[
\mathcal{A}_{ij}^{[2]} = i (\sigma^2)_{ij}.
\]

(4.16)
Next the Lie-algebra valued quantities defined by (3.1) can now be explicitly demonstrated. One simply evaluates the RHS of Eq. (3.1) on the representation dependent fermion defined in (4.2)

This leads to four equations

\[
- \delta_k^l \left[ \hat{h}^\mu(\Pi_0^{(\vec{R})}) \right]_{abc}^d \partial_\mu \Psi_{dt}^{(\vec{R})} = \left[ i C_{ab} \mathcal{H}^{(1)} + (\gamma^5)_{ab} \mathcal{H}^{(2)} + (\gamma^5 \gamma^\mu)_{ab} \mathcal{H}^{(3)}_\mu \right] \Psi_{dt}^{(\vec{R})},
\]

(4.17)

\[
(\sigma^3)_k^l \left[ \hat{h}^\mu(\Pi_3^{(\vec{R})}) \right]_{abc}^d \partial_\mu \Psi_{dt}^{(\vec{R})} = \left[ i C_{ab} \left( \mathcal{H}^{(5;3)} \right) + (\gamma^5)_{ab} \left( \mathcal{H}^{(5;3)} \right) + (\gamma^5 \gamma^\mu)_{ab} \left( \mathcal{H}^{(5;3)}_\mu \right) \right] \Psi_{dt}^{(\vec{R})},
\]

(4.18)

\[
(\sigma^1)_k^l \left[ \hat{h}^\mu(\Pi_1^{(\vec{R})}) \right]_{abc}^d \partial_\mu \Psi_{dt}^{(\vec{R})} = \left[ i C_{ab} \left( \mathcal{H}^{(5;1)} \right) + (\gamma^5)_{ab} \left( \mathcal{H}^{(5;1)} \right) + (\gamma^5 \gamma^\mu)_{ab} \left( \mathcal{H}^{(5;1)}_\mu \right) \right] \Psi_{dt}^{(\vec{R})},
\]

(4.19)

\[
(\sigma^2)_k^l \left[ \hat{h}^{(2)\mu}(\Pi_2^{(\vec{R})}) \right]_{abc}^d \partial_\mu \Psi_{dt}^{(\vec{R})} = -\left[ (\gamma^\mu)_{ab} \mathcal{H}^{(7;2)}_\mu + \frac{1}{2} \left( (\gamma^\mu, \gamma^\nu) \right)_{ab} \mathcal{H}^{(8;2)}_\mu \right] \Psi_{dt}^{(\vec{R})},
\]

(4.20)
that can be used to extract explicit expressions for the action of each of the Lie-algebra generators.
5 4D, $\mathcal{N} = 2$ Bosonic Holoraumy Results For Minimal Multiplets

Up to this point as well as in the bulk of our previous discussions on the topic of holoraumy, the focus is on the results of the holoraumy calculations as evaluated on the fermions within supermultiplets. However, one can also carry out calculation on the bosons within any supermultiplet. The reason for focusing only on the fermions is one of convenience. Let us demonstrate some examples.

In the following, there will be presented the results for the evaluation of the holoraumy on the basis of the bosonic fields of each minimal off-shell 4D, $\mathcal{N} = 2$ supermultiplet. In particular, the evaluations allow for the comparison of the holoraumies of the pairs of supermultiplets (2VS) and (2TS) of (2AT). By use of Eq. (4.13) and (4.14) it was demonstrated that each member of these pairs are the “parity flipped” version of the other member. This is encoded by simply performing a “conjugation” of the holoraumy tensors by use of the $\gamma^5$-matrix.

In the remaining portions of this chapter, we simply present the results for the holoraumy operator in (3.1) explicitly evaluated on each bosonic component field in all the minimal 4D, $\mathcal{N} = 2$ off-shell supermultiplets.

5.1 Bosonic 4D, $\mathcal{N} = 2$ Vector Supermultiplet Holoraumy

$$
[D_a^i, D_b^j] A = -2\delta^{ij} [(\gamma^5 \gamma^\mu)_{ab} \partial_\mu B + iC_{ab} F] + 2(\sigma^3)^{ij} (\gamma^5)_{ab} G \\
+ 2(\sigma^1)^{ij} (\gamma^5)_{ab} d + (\sigma^2)^{ij} (\gamma^5 \gamma^\nu)_{ab} \partial_\mu A_\nu , \\
[D_a^i, D_b^j] B = 2\delta^{ij} [(\gamma^5 \gamma^\mu)_{ab} \partial_\mu A + (\gamma^5)_{ab} F] + i2(\sigma^3)^{ij} C_{ab} G \\
+ i2(\sigma^1)^{ij} C_{ab} d + i(\sigma^2)^{ij} (\gamma^5 [\gamma^\mu, \gamma^\nu])_{ab} \partial_\mu A_\nu , \\
[D_a^i, D_b^j] F = 2\delta^{ij} [-iC_{ab} \square A + (\gamma^5)_{ab} \square B] - 2(\sigma^3)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu G \\
- 2(\sigma^1)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu d - 2(\sigma^2)^{ij} (\gamma^5)_{ab} \partial_\mu F_{\mu \nu} , \\
[D_a^i, D_b^j] G = 2(\sigma^3)^{ij} [(\gamma^5)_{ab} \square A + iC_{ab} \square B + (\gamma^5 \gamma^\mu)_{ab} \partial_\mu F] \\
- 2(\sigma^1)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu F_{\mu \nu} + 2(\sigma^2)^{ij} (\gamma^5)_{ab} \partial_\mu d , \\
[D_a^i, D_b^j] A_\mu = -2\delta^{ij} \epsilon_{\mu \nu \alpha \beta} (\gamma^5 \gamma_\nu)_{ab} \partial_\alpha A_\beta - 2(\sigma^3)^{ij} (\gamma^5 \gamma^\mu)_{ab} d \\
+ 2(\sigma^1)^{ij} (\gamma^5 \gamma^\mu)_{ab} G \\
+ (\sigma^2)^{ij} [ (\gamma^5 \gamma^\mu)_{ab} \partial_\alpha \partial_\beta A + i(\gamma^5 \gamma^\mu)_{ab} \partial_\alpha \partial_\beta B + 2(\gamma^5 \gamma^\mu)_{ab} F] , \\
[D_a^i, D_b^j] d = 2(\sigma^3)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu F_{\mu \nu} \\
+ 2(\sigma^1)^{ij} [(\gamma^5)_{ab} \square A + iC_{ab} \square B + (\gamma^5 \gamma^\mu)_{ab} \partial_\mu F] \\
- 2(\sigma^2)^{ij} (\gamma^5)_{ab} \partial_\mu G .
$$

5.2 Bosonic 4D, $\mathcal{N} = 2$ Tensor Supermultiplet Holoraumy

$$
[D_a^i, D_b^j] A = 2(\sigma^3)^{ij} [-(\gamma^5 \gamma^\mu)_{ab} \partial_\mu B - iC_{ab} F + (\gamma^5)_{ab} G] \\
+ 2(\sigma^1)^{ij} \epsilon_\mu \nu \alpha \beta (\gamma^5 \gamma^\mu)_{ab} \partial_\nu B_{\alpha \beta} + 2(\sigma^2)^{ij} (\gamma^5)_{ab} \partial_\mu \varphi , \\
[D_a^i, D_b^j] B = 2\delta^{ij} [(\gamma^5)_{ab} F + iC_{ab} G] + 2(\sigma^3)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu A \\
+ 2(\sigma^1)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \varphi - 2(\sigma^2)^{ij} \epsilon_{\mu \nu \alpha \beta} (\gamma^5)_{ab} \partial_\nu B_{\alpha \beta} , \\
[D_a^i, D_b^j] F = 2\delta^{ij} [ -(\gamma^5)_{ab} \partial_\mu A + (\gamma^5)_{ab} \partial_\mu B] \\
+ 2(\sigma^3)^{ij} (\gamma^5 \gamma^\mu)_{ab} F_{\mu \nu} - 2(\sigma^1)^{ij} \epsilon_\mu \nu \alpha \beta (\gamma^5)_{ab} \partial_\nu B_{\alpha \beta} .
$$
\[ [D^i_a, D^j_b] F = 2\delta^{ij} \left( \gamma_5 \right)_{ab} \Box B - (\gamma_5 \gamma^\mu)_{ab} \partial_\mu G \right] - i2(\sigma^3)^{ij} C_{ab} \Box A \\
- i2(\sigma^1)^{ij} C_{ab} \Box \varphi + i(\sigma^2)^{ij} \epsilon^\mu_{\nu \alpha \beta} (\gamma_5 \gamma^\nu \gamma^\alpha)_{ab} \partial_\mu \partial_\nu B_{\alpha \beta} \] (5.9)
\[ [D^i_a, D^j_b] \tilde{\phi} = -2(\sigma^3)^{ij} \epsilon_{\mu}^{\alpha\beta}(\gamma^5\gamma^\mu)_{ab} \partial_{\mu} C_{\alpha\beta} \]
\[ + 2(\sigma^1)^{ij} \left[ (\gamma^5\gamma^\mu)_{ab} \partial_{\mu} A + (\gamma^5)_{ab} F + iC_{ab} G \right] - 2(\sigma^2)^{ij}(\gamma^\mu)_{ab} \partial_{\mu} B \quad , \]
\[ [D^i_a, D^j_b] C_{\mu\nu} = -\delta^{ij} \epsilon_{\mu}^{\rho\alpha\beta}(\gamma^5\gamma^\rho)_{ab} \partial_{\rho} C_{\alpha\beta} + (\sigma^3)^{ij} \epsilon_{\mu\nu}^{\alpha\beta}(\gamma^5\gamma^\alpha)_{ab} \partial_{\beta} \tilde{\phi} \]
\[ - (\sigma^1)^{ij} \epsilon_{\mu\nu}^{\alpha\beta}(\gamma^5\gamma^\alpha)_{ab} \partial_{\beta} B \]
\[ + i(\sigma^2)^{ij} \left[ i\epsilon_{\mu\nu}^{\alpha\beta}(\gamma^\alpha)_{ab} \partial_{\beta} A - \frac{1}{2}(\gamma^5[\gamma^\mu, \gamma^\nu])_{ab} F - i\frac{1}{2}([\gamma^\mu, \gamma^\nu])_{ab} G \right] \quad . \]

The actual derivation of all of these results follow from the forms of the 4D, \( \mathcal{N} = 2 \) supercovariant derivative operator when evaluated on each field. The explicit expressions for this evaluation were presented in \([46]\) and in order to streamline our presentation, we have given these in an appendix.
6 A Lorentz-Covariant & so(2)-Covariant 4D, $\mathcal{N} = 2$ Supermultiplet Gadget

In the works of [27,28], the concept of a spacetime Gadget function for supermultiplets in 4D was introduced. By definition, a spacetime Gadget is a bilinear function whose domain is pairs of 4D supermultiplet representations $(\hat{\mathcal{R}}_1)$ and $(\hat{\mathcal{R}}_2)$ such that its range is the real numbers. In a sense a spacetime Gadget introduces a metric on the space of supermultiplets. We typically use the notation $\hat{G}[(\hat{\mathcal{R}}_1), (\hat{\mathcal{R}}_2)]$ for the spacetime Gadget and in all previous known examples, this metric has two properties:

(a.) when $(\hat{\mathcal{R}}_1) = (\hat{\mathcal{R}}_2)$, the spacetime Gadget value is non-negative, and
(b.) it only maps to zero if either $(\hat{\mathcal{R}}_1)$ or $(\hat{\mathcal{R}}_2)$ is the zero supermultiplet.

Previously this result has been achieved via the spacetime Gadget being constructed from a quadratic that uses the holoraumy operator evaluated on the fermionic fields of a supermultiplet.

Following the spirit of the 4D, $\mathcal{N} = 1$ case, we define the 4D, $\mathcal{N} = 2$ Gadget as

$$\hat{G}[(\hat{\mathcal{R}}), (\hat{\mathcal{R}}')] = m_1[\hat{H}^{\mu}((\hat{\mathcal{R}}))]_{abj} {d^l}_{ic} [\hat{H}_\mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$$

+ $m_2(\gamma^5)_c e^{-\hat{H}^{\mu}((\hat{\mathcal{R}}))]_{abj} {d^l}_{ic} [\hat{H}_\mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$

+ $m_3(\gamma^5\gamma^\alpha)_c e^{-\hat{H}^{\mu}((\hat{\mathcal{R}}))]_{abj} {d^l}_{ic} [\hat{H}_\mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$

+ $m_4(\gamma^\alpha)_c e^{-\hat{H}^{\mu}((\hat{\mathcal{R}}))]_{abj} {d^l}_{ic} [\hat{H}_\mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$

+ $m_5([\gamma^\alpha, \gamma^\beta]) c e^{-\hat{H}^{\mu}((\hat{\mathcal{R}}))]_{abj} {d^l}_{ic} [\hat{H}_\mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$,

with the introduction (as of now) undetermined parameters $m_1, \ldots, m_5$. Let us note that it is possible to introduce more such parameters. The parameters in (6.1) are associated with performing all possible conjugations of the $a$ and $b$ spinor indices of the holoraumy with respect to the universal covering algebra of the $\gamma$-matrices. However, it should be recognized that such a set of conjugations may be taken with respect to the $i$ and $j$ isospinor indices of the holoraumy with respect to the $su(2)$ algebra.

To continue we note it is convenient to decompose the calculation into several parts that depend respectively on $[\hat{h}^{(0)}]_{abc} {d^d}_{f} [\hat{h}^{(3)}]_{abc} {d^d}_{f} [\hat{h}^{(1)}]_{abc} {d^d}_{f}$, and $[\hat{h}^{(2)}]_{abc} {d^d}_{f}$. So as an intermediate expression we write

$$\hat{G}[(\hat{\mathcal{R}}), (\hat{\mathcal{R}}')] \equiv 4 \{ \hat{G}_0[(\hat{\mathcal{R}}), (\hat{\mathcal{R}}')] + \hat{G}_3[(\hat{\mathcal{R}}), (\hat{\mathcal{R}}')] + \hat{G}_1[(\hat{\mathcal{R}}), (\hat{\mathcal{R}}')] + \hat{G}_2[(\hat{\mathcal{R}}), (\hat{\mathcal{R}}')] \}$$

where

$$\hat{G}_A[(\hat{\mathcal{R}}), (\hat{\mathcal{R}}')] = m_1[\hat{h}^{(A)}\mu(\hat{\mathcal{R}})]_{abc} {d^d}_{f} [\hat{h}^{(A)} \mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$$

+ $m_2(\gamma^5)_c e^{-\hat{h}^{(A)}\mu(\hat{\mathcal{R}})]_{abc} {d^d}_{f} [\hat{h}^{(A)} \mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$

+ $m_3(\gamma^5\gamma^\alpha)_c e^{-\hat{h}^{(A)}\mu(\hat{\mathcal{R}})]_{abc} {d^d}_{f} [\hat{h}^{(A)} \mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$

+ $m_4(\gamma^\alpha)_c e^{-\hat{h}^{(A)}\mu(\hat{\mathcal{R}})]_{abc} {d^d}_{f} [\hat{h}^{(A)} \mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$

+ $m_5([\gamma^\alpha, \gamma^\beta]) c e^{-\hat{h}^{(A)}\mu(\hat{\mathcal{R}})]_{abc} {d^d}_{f} [\hat{h}^{(A)} \mu(\hat{\mathcal{R}}')]^{abc} \right)_{d} \left( \right)_{d} \left( \right)_{c}$,

Recognizing the relations shown in (4.5), it becomes clear that in order to evaluate the first three terms is equivalent to only having to evaluate the first term explicitly and one can later make the appropriate substitutions for the lattice variables.
So we have

\[ \mathcal{G}_0[\hat{R}, (\hat{R}')] = 4^3 \left\{ p_0^0 (\hat{R}) (m_1 - m_2 + 2m_3 + 2m_4) \right. \\
+ q_0^0 q_0^0 (m_1 - m_2 + 2m_3 + 2m_4) \\
+ r_0^0 r_0^0 (m_1 + m_2 + 4m_3 - 4m_4 - 48m_5) \\
+ 3s_0^0 s_0^0 (m_1 - m_2 - 16m_5) \right\}, \tag{6.4} \]

and which also follow for \( \mathcal{G}_3 \) and \( \mathcal{G}_1 \). We continue by performing the evaluate of \( \mathcal{G}_2 \) and find

\[ \mathcal{G}_2[\hat{R}, (\hat{R}')] = 4^3 \left\{ 3p_2^0 (\hat{R}) (m_1 - m_2 + 2m_3 + 2m_4) \right. \\
+ 3q_2^0 q_2^0 (m_1 - m_2 + 2m_3 + 2m_4) \\
+ 3r_2^0 r_2^0 (m_1 - m_2 - 16m_5) \\
+ s_2^0 s_2^0 (m_1 + m_2 - 4m_3 + 4m_4 - 48m_5) \right\}. \tag{6.5} \]

Therefore, adding all the contributions, we obtain

\[ \mathcal{G}[\hat{R}, (\hat{R}')] = 4^4 \left\{ (p_0^0 (\hat{R}) + p_3^0 (\hat{R}')) + (p_1^0 (\hat{R}) + p_1^0 (\hat{R}')) (m_1 + m_2 + 2m_3 + 2m_4) \right. \\
+ (3p_2^0 p_2^0 (\hat{R}'))(m_1 - m_2 + 2m_3 + 2m_4) \\
+ (q_0^0 q_0^0 + q_3^0 q_3^0 + q_1^0 q_1^0) (m_1 - m_2 + 2m_3 + 2m_4) \\
+ (3q_2^0 q_2^0 (\hat{R}'))(m_1 + m_2 + 4m_3 - 4m_4 - 48m_5) \\
+ (3r_3^0 (\hat{R}'))(m_1 - m_2 - 16m_5) \\
+ (s_2^0 s_2^0 (\hat{R}'))(m_1 + m_2 - 4m_3 + 4m_4 - 48m_5) \right\}. \tag{6.6} \]

Let us also note for all of the supermultiplet representations \( (\hat{R}) \) in Table 2 we find

\[ \mathcal{G}[\hat{R}, (\hat{R}')] = 4^4 \{ 6 (-m_1 + m_2 + 2m_3 + 2m_4) + 6 (m_1 - m_2 + 2m_3 + 2m_4) \} \\
+ 3 (m_1 + m_2 + 4m_3 - 4m_4 - 48m_5) \\
+ (m_1 + m_2 + 4m_3 + 4m_4 - 48m_5) \right. \\
+ 12 (-m_1 - m_2 - 16m_5) \right\} \\
= - 4^4 \times 8 \left\{ m_1 + m_2 - 4m_3 + 2m_4 + 48m_5 \right\} \tag{6.7} .

Thus, this analysis shows the requirement that \( \mathcal{G}[\hat{R}, (\hat{R}')] \) should be independent of which minimal representation (which in principle could lead to four conditions) only leads to one condition. Continuing this analysis when \( (\hat{R}) \neq (\hat{R}') \) we find

\[ \mathcal{G}[(2VS), (2TS)] = \mathcal{G}[(2VS), (2ATS)] = \mathcal{G}[(2AVS), (2TS)] = \mathcal{G}[(2AVS), (2ATS)] = \]

\[ = - 4^4 \times 8 \left\{ m_1 + m_2 + 2m_3 - 2m_4 \right\}, \tag{6.8} \]

and

\[ \mathcal{G}[(2VS), (2AVS)] = \mathcal{G}[(2TS), (2ATS)] = - 4^4 \times 8 \left\{ m_1 + m_2 + 2m_3 + 4m_4 + 48m_5 \right\}, \tag{6.9} \]
So at this stage of the analysis we find

\[
\hat{G}[\hat{R}, (\hat{R}')] = \begin{bmatrix}
\mathcal{X}_1 & \mathcal{X}_2 & \mathcal{X}_3 & \mathcal{X}_2 \\
\mathcal{X}_2 & \mathcal{X}_1 & \mathcal{X}_2 & \mathcal{X}_3 \\
\mathcal{X}_3 & \mathcal{X}_2 & \mathcal{X}_1 & \mathcal{X}_2 \\
\mathcal{X}_2 & \mathcal{X}_3 & \mathcal{X}_2 & \mathcal{X}_1
\end{bmatrix}, \tag{6.10}
\]

where

\[
\begin{align*}
\mathcal{X}_1 &= -4^4 \times 8 \left\{ m_1 + m_2 - 4 m_3 - 2 m_4 + 48m_5 \right\}, \\
\mathcal{X}_2 &= -4^4 \times 8 \left\{ m_1 + m_2 + 2 m_3 - 2 m_4 \right\}, \\
\mathcal{X}_3 &= -4^4 \times 8 \left\{ m_1 + m_2 + 2 m_3 + 4 m_4 + 48m_5 \right\}.
\end{align*} \tag{6.11}
\]

Let us comment on role of the lattice variables in reaching this result. Looking back at (6.6), one would have said imposing a set of values for the diagonal and upper triangular entries would have led to ten equations on the five m-parameters, i.e. an over-constrained system. Instead and precisely due to how the lattice variables enter the Holoraumy, we are led to only three equations on the five m-parameters, i.e. an under-constrained system.

Taking advantage of this and as an analog to 4D, \(\mathcal{N} = 1\) case, we demand\(^{10}\)

\[
\hat{G}[\hat{R}, (\hat{R}')] = \frac{1}{28} \left\{ p_0(\hat{R}) p_0(\hat{R}') + q_0(\hat{R}) q_0(\hat{R}') + r_0(\hat{R}) r_0(\hat{R}') + 3s_0(\hat{R}) s_0(\hat{R}') + p_3(\hat{R}) p_3(\hat{R}') + q_3(\hat{R}) q_3(\hat{R}') + r_3(\hat{R}) r_3(\hat{R}') + 3s_3(\hat{R}) s_3(\hat{R}') + p_1(\hat{R}) p_1(\hat{R}') + q_1(\hat{R}) q_1(\hat{R}') + r_1(\hat{R}) r_1(\hat{R}') + 3s_1(\hat{R}) s_1(\hat{R}') + 3p_2(\hat{R}) p_2(\hat{R}') + 3q_2(\hat{R}) q_2(\hat{R}') + 3r_2(\hat{R}) r_2(\hat{R}') + 3s_2(\hat{R}) s_2(\hat{R}') \right\}, \tag{6.12}
\]

by choosing \(-m_1 = -m_2 = m_3 = m_4 = -8m_5 = \frac{1}{744}\). Then we can put all the 4D Gadget values in a matrix as follows:

\[
\hat{G}[\hat{R}, (\hat{R}')] = \begin{bmatrix}
1 & 1 \frac{1}{7} & 1 \frac{1}{7} & 1 \frac{1}{7} \\
1 \frac{1}{7} & 1 \frac{1}{7} & 1 \frac{1}{7} & 1 \frac{1}{7} \\
1 \frac{1}{7} & 1 \frac{1}{7} & 1 \frac{1}{7} & 1 \frac{1}{7} \\
1 \frac{1}{7} & 1 \frac{1}{7} & 1 \frac{1}{7} & 1 \frac{1}{7}
\end{bmatrix}, \tag{6.13}
\]

where the row and column indices run from \((2\text{VS}), (2\text{TS}), (2\text{AVS}), \) and \((2\text{ATS})\).

Given the metric defined by \(\hat{G}[\hat{R}, (\hat{R}')]\) in (6.12), we can explore the geometry of the hexadecimal dimensional space where the numbers \((\Pi_{\hat{R}}^{(\hat{R})})\) define a set of coordinates. The results shown in (Table 2) imply that each of the supermultiplets can be associated with a point in this space. As can be seen our choice of normalization implies that \(\hat{G}[\hat{R}, (\hat{R})] = 1\) for each of the representations. This means we may regard each supermultiplet as being a distance of one away from the origin. The fact that all the off diagonal values of \(\hat{G}[\hat{R}, (\hat{R}')]\) in (6.13) are equal informs us that the angles between lines drawn from the origin of this space to each of the points are all the same. By taking the differences between the hexadecimal coordinates and using the metric in (6.12), we find that the length of any side joining the points describing the location of any supermultiplet is given by \(\sqrt{12/7}\).

\(^{10}\) We will discuss the reason for this in our conclusion section.
The most important result of this chapter is that in (6.12) together with the specification of the $m$-parameters stated just below the equation and the expression shown in (6.1). Taken all together these show there exist a Lorentz covariant and so(2) covariant derivation from (6.12) to (6.1) or vice versa.
7 Reduction to 0-Brane

Thus far in this work, all our calculations have strictly been in the realm of 4D. However, these calculations are informed by structures related to adinkras [19]. The entire concept of Holoraumy arose from this source [25,26] and our proposal that there is a type of holography that connects the two seemingly separate domains. Under this circumstance, we think it is prudent to view the results shown in earlier sections of this work for the perspective of 1D, \( N = 8 \) adinkras.

In preparation for the discussion, we need to set in place some conventions.

There are two sets of the “general real algebra of dimension \( d \) and extension \( N \)”, or \( \mathcal{G}\mathcal{R}(d, N) \) or alternately “Garden Algebras” relevant in the following. The first of these corresponds to the values of \( d = N = 8 \) and when expressed as \( 8 \times 8 \) matrices \( \mathbf{L}_i^{(\mathcal{R})} \) and \( \mathbf{R}_i^{(\mathcal{R})} \) we require

\[
\begin{align*}
\mathbf{L}_i^{(\mathcal{R})} \cdot \mathbf{R}_j^{(\mathcal{R})} + \mathbf{L}_j^{(\mathcal{R})} \cdot \mathbf{R}_i^{(\mathcal{R})} &= 2 \delta_{ij} \mathbf{I}_{8\times8}, \\
\mathbf{R}_i^{(\mathcal{R})} \cdot \mathbf{L}_j^{(\mathcal{R})} + \mathbf{R}_j^{(\mathcal{R})} \cdot \mathbf{L}_i^{(\mathcal{R})} &= 2 \delta_{ij} \mathbf{I}_{8\times8}, \\
\mathbf{L}_i^{(\mathcal{R})} &= [\mathbf{R}_i^{(\mathcal{R})}]^t,
\end{align*}
\]

where the subscript indices \( i \) and \( j \) take on values of 1, \ldots, 8 since \( N = 8 \) and \( d = 8 \), these are eight \( 8 \times 8 \) matrices. Finally, there are different representations of these matrices. In order to indicate this, we use an “adinkra representation label” denoted by \((\mathcal{R})^{\dagger}\)\(^{11}\).

In order to take advantage of our previous extensive studies of \( \mathcal{G}\mathcal{R}(4, 4) \) algebras, we also introduce a set of \( 4 \times 4 \) “Garden Algebras” matrices \( \mathbf{L}_i^{(\mathcal{R})} \) and \( \mathbf{R}_i^{(\mathcal{R})} \) for the case where \( d = N = 4 \).

\[
\begin{align*}
\mathbf{L}_i^{(\mathcal{R})} \cdot \mathbf{R}_j^{(\mathcal{R})} + \mathbf{L}_j^{(\mathcal{R})} \cdot \mathbf{R}_i^{(\mathcal{R})} &= 2 \delta_{ij} \mathbf{I}_{4\times4}, \\
\mathbf{R}_i^{(\mathcal{R})} \cdot \mathbf{L}_j^{(\mathcal{R})} + \mathbf{R}_j^{(\mathcal{R})} \cdot \mathbf{L}_i^{(\mathcal{R})} &= 2 \delta_{ij} \mathbf{I}_{4\times4}, \\
\mathbf{L}_i^{(\mathcal{R})} &= [\mathbf{R}_i^{(\mathcal{R})}]^t,
\end{align*}
\]

where the underlined subscript indices \( \dot{i} \) and \( \dot{j} \) take on values of 1, \ldots, 4 since \( N = 4 \) and \( d = 4 \). The advantage afforded from this is it allows the explicit construction of the \( 8 \times 8 \) \( \mathbf{L}_i^{(\mathcal{R})} \) and \( \mathbf{R}_i^{(\mathcal{R})} \) matrices expressed in terms of \( 4 \times 4 \) \( \mathbf{L}_i^{(\mathcal{R})} \) and \( \mathbf{R}_i^{(\mathcal{R})} \) matrices . Careful attention should be noted that there is the additional underline as in \( \mathbf{L} \) and \( \mathbf{R} \) to distinguish the strictly \( 4 \times 4 \) matrices from the \( 8 \times 8 \) matrices.

The 1D, \( N = 8 \) supercovariant derivative operator \( \mathbf{D}_1 \) can be described as a pair of 1D, \( N = 4 \) supercovariant derivatives \( \mathbf{D}_1 = (\mathbf{D}_1^1, \mathbf{D}_1^2) \) where

\[
\mathbf{D}_1 = \begin{cases} 
\mathbf{D}_1^1, & \text{if } i = \dot{i} \\
\mathbf{D}_1^2, & \text{if } i = \dot{i} + 4
\end{cases}
\]

(7.3)

Using this definition, when acting on a set of 1D valise adinkra/superfields we can write,

\[
\begin{align*}
\mathbf{D}_1^1 \Phi_i &= i (\mathbf{L}_i^1)_{i\dot{k}} \Psi_{\dot{k}}, & \mathbf{D}_1^1 \Psi_{\dot{k}} &= (\mathbf{R}_1^1)_{\dot{k}i} \partial_0 \Phi_i, \\
\mathbf{D}_1^2 \Phi_i &= i (\mathbf{L}_{1+4}^1)_{i\dot{k}} \Psi_{\dot{k}}, & \mathbf{D}_1^2 \Psi_{\dot{k}} &= (\mathbf{R}_{1+4}^1)_{\dot{k}i} \partial_0 \Phi_i,
\end{align*}
\]

(7.4)

\(^{11}\)It should be noted that in order to distinguish the “supermultiplet representation label” from the “adinkra representation label”, we use the symbol \((\mathcal{R})\) for the former and \((\mathcal{R})^{\dagger}\) for the latter.
where \( I = 1, \ldots, 4 \) on the doublet of supercovariant derivative operators and the subscript of \( L_I \) runs from \( I = 1, \ldots, 8 \).

From how we construct the \( \mathcal{N} = 2 \) vector / tensor / axial-vector / axial-tensor supermultiplets from \( \mathcal{N} = 1 \) chiral supermultiplet plus \( \mathcal{N} = 1 \) vector / tensor / axial-vector / axial-tensor supermultiplets, we observe that

\[
L^{(2R)}_1 = \begin{bmatrix} L^{(CS)}_1 & 0 \\ 0 & L^{(R)}_1 \end{bmatrix}, \quad L^{(2R)}_{I+4} = \begin{bmatrix} 0 & S^{(2R)}_2 L^{(R)}_1 \\ L^{(R)}_1 & 0 \end{bmatrix}, \quad (7.5)
\]

\[
R^{(2R)}_1 = \begin{bmatrix} R^{(CS)}_1 & 0 \\ 0 & R^{(R)}_1 \end{bmatrix}, \quad R^{(2R)}_{I+4} = \begin{bmatrix} 0 & R^{(2R)}_1 S^{(R)}_2 \\ R^{(R)}_1 & 0 \end{bmatrix},
\]

where the \( N = 4 \) adinkra representation label, \((\mathcal{R})\), takes its values as \((\mathcal{VS})\), \((\mathcal{TS})\), \((\mathcal{AVS})\), and \((\mathcal{ATS})\); while \( N = 8 \) adinkra representation label, \((2\mathcal{R})\) takes its values as \((2\mathcal{VS})\), \((2\mathcal{TS})\), \((2\mathcal{AVS})\), \((2\mathcal{ATS})\).

Also in writing \((7.5)\), we have utilized another notational device (“Boolean Factors”) introduced previously in the work of [47]. We define Boolean Factors as real diagonal matrices whose square is the identity. Thus, any Boolean Factor has the form of the matrix shown in \((7.6)\)

\[
S_b = \begin{bmatrix} (-1)^{b_1} & 0 & 0 & \cdots & 0 \\ 0 & (-1)^{b_2} & 0 & \cdots & 0 \\ 0 & 0 & (-1)^{b_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & (-1)^{b_d} \end{bmatrix} \leftrightarrow [b_1 b_2 b_3 \cdots b_d]_2 = \left( \sum_{i=1}^{d} b_i 2^{i-1} \right)_b. \quad (7.6)
\]

where the diagonal entries may be expressed as exponentials of \((-1)\) to a set of bits, i.e. variables that only take on values of one or zero, indicated by \( b_1, \ldots, b_d \). On the RHS of \((7.6)\), we have shown how the bits that occur in a Boolean Factor can be written as an integer using base two. Thus any specific Boolean Factor can be denoted by such an integer as we show in \((7.7)\)

\[
S^{(2VS)}_1 = [0001]_2 = (8)_b, \quad S^{(2VS)}_2 = [1110]_2 = (7)_b, \quad \]

\[
S^{(2TS)}_1 = [1000]_2 = (1)_b, \quad S^{(2TS)}_2 = [0111]_2 = (14)_b, \quad \]

\[
S^{(2AVS)}_1 = [0010]_2 = (4)_b, \quad S^{(2AVS)}_2 = [1110]_2 = (7)_b, \quad \]

\[
S^{(2ATS)}_1 = [0100]_2 = (2)_b, \quad S^{(2ATS)}_2 = [0111]_2 = (14)_b. \quad (7.7)
\]

The L-matrices, themselves are simply signed permutation matrices, so we will express the as Boolean Factors times permutation factors of order four. We will make use of a notation \langle n_1 n_2 n_3 n_4 \rangle as in [47] to specify the permutation factors. Thus at the end of this discussion, we will be able to express the \( 8 \times 8 \) \( L_1^{(R)} \) matrices in terms of Boolean Factors times permutation elements denoted by \langle n_1 n_2 n_3 n_4 \rangle for some integers \( n_1, n_2, n_3, \) and \( n_4 \). These integers themselves are simply a reordering of the integers 1, 2, 3, and 4.

### 7.1 L-matrices from the 4D, \( \mathcal{N} = 2 \) Vector Supermultiplet

For the 4D, \( \mathcal{N} = 2 \) vector supermultiplet, the bosons are \( A, B, F, \) and \( G \) from the chiral 4D, \( \mathcal{N} = 1 \) chiral supermultiplet and the spatial components of \( A_\mu \) and \( d \) from the 4D, \( \mathcal{N} = 1 \) vector
supermultiplet. To define the bosons in the 1D, \( N = 8 \) \((2VS)\) adinkra representation we define the bosons \( \Phi_i \) via

\[
\begin{align*}
\Phi_1 &= A, & \Phi_2 &= B, & \partial_0 \Phi_3 &= F, & \partial_0 \Phi_4 &= G, \\
\Phi_5 &= A_1, & \Phi_6 &= A_2, & \Phi_7 &= A_3, & \partial_0 \Phi_8 &= d,
\end{align*}
\]

(7.8)

and the fermions \( \Psi_k \) in the 1D, \( N = 8 \) \((2VS)\) adinkra representation via

\[
\begin{align*}
i \Psi_1 &= \psi_1, & i \Psi_2 &= \psi_2, & i \Psi_3 &= \psi_3, & i \Psi_4 &= \psi_4, \\
i \Psi_5 &= \lambda_1, & i \Psi_6 &= \lambda_2, & i \Psi_7 &= \lambda_3, & i \Psi_8 &= \lambda_4.
\end{align*}
\]

(7.9)

The \( L \)-matrices of the \( \mathcal{N} = 2 \) vector supermultiplet are

\[
\begin{align*}
\mathbf{L}_{1}^{(2VS)} &= \begin{bmatrix} (10)_b \langle 1423 \rangle & 0 \\ 0 & (10)_b \langle 2413 \rangle \end{bmatrix}, & \mathbf{L}_{2}^{(2VS)} &= \begin{bmatrix} (12)_b \langle 2314 \rangle & 0 \\ 0 & (12)_b \langle 1324 \rangle \end{bmatrix}, \\
\mathbf{L}_{3}^{(2VS)} &= \begin{bmatrix} (6)_b \langle 3241 \rangle & 0 \\ 0 & (0)_b \langle 4231 \rangle \end{bmatrix}, & \mathbf{L}_{4}^{(2VS)} &= \begin{bmatrix} (0)_b \langle 4132 \rangle & 0 \\ 0 & (6)_b \langle 3142 \rangle \end{bmatrix}, \\
\mathbf{L}_{5}^{(2VS)} &= \begin{bmatrix} 0 & (2)_b \langle 1423 \rangle \\ (13)_b \langle 2413 \rangle & 0 \end{bmatrix}, & \mathbf{L}_{6}^{(2VS)} &= \begin{bmatrix} 0 & (4)_b \langle 2314 \rangle \\ (11)_b \langle 1324 \rangle & 0 \end{bmatrix}, \\
\mathbf{L}_{7}^{(2VS)} &= \begin{bmatrix} 0 & (14)_b \langle 3241 \rangle \\ (7)_b \langle 4231 \rangle & 0 \end{bmatrix}, & \mathbf{L}_{8}^{(2VS)} &= \begin{bmatrix} 0 & (8)_b \langle 4132 \rangle \\ (1)_b \langle 3142 \rangle & 0 \end{bmatrix}.
\end{align*}
\]

(7.10)

\subsection{L-matrices from 4D, \( \mathcal{N} = 2 \) Tensor Supermultiplet}

For the 4D, \( \mathcal{N} = 2 \) tensor supermultiplet, the bosons are \( A, B, F, \) and \( G \) from the chiral 4D, \( \mathcal{N} = 1 \) chiral supermultiplet and the spatial components of \( B_{\mu \nu} \) and \( \varphi \) from the 4D, \( \mathcal{N} = 1 \) tensor supermultiplet. To define the bosons in the 1D, \( N = 8 \) \((2TS)\) adinkra representation we define the bosons \( \Phi_i \) via

\[
\begin{align*}
\Phi_1 &= A, & \Phi_2 &= B, & \partial_0 \Phi_3 &= F, & \partial_0 \Phi_4 &= G, \\
\Phi_5 &= \varphi, & \Phi_6 &= 2B_{12}, & \Phi_7 &= 2B_{23}, & \Phi_8 &= 2B_{31},
\end{align*}
\]

(7.11)

and the fermions \( \Psi_k \) in the 1D, \( N = 8 \) \((2TS)\) adinkra representation via

\[
\begin{align*}
i \Psi_1 &= \psi_1, & i \Psi_2 &= \psi_2, & i \Psi_3 &= \psi_3, & i \Psi_4 &= \psi_4, \\
i \Psi_5 &= \chi_1, & i \Psi_6 &= \chi_2, & i \Psi_7 &= \chi_3, & i \Psi_8 &= \chi_4.
\end{align*}
\]

(7.12)

The \( L \)-matrices of the \( \mathcal{N} = 2 \) tensor supermultiplet are

\[
\begin{align*}
\mathbf{L}_{1}^{(2TS)} &= \begin{bmatrix} (10)_b \langle 1423 \rangle & 0 \\ 0 & (14)_b \langle 1342 \rangle \end{bmatrix}, & \mathbf{L}_{2}^{(2TS)} &= \begin{bmatrix} (12)_b \langle 2314 \rangle & 0 \\ 0 & (4)_b \langle 2431 \rangle \end{bmatrix}, \\
\mathbf{L}_{3}^{(2TS)} &= \begin{bmatrix} (6)_b \langle 3241 \rangle & 0 \\ 0 & (8)_b \langle 3124 \rangle \end{bmatrix}, & \mathbf{L}_{4}^{(2TS)} &= \begin{bmatrix} (0)_b \langle 4132 \rangle & 0 \\ 0 & (2)_b \langle 4213 \rangle \end{bmatrix}, \\
\mathbf{L}_{5}^{(2TS)} &= \begin{bmatrix} 0 & (11)_b \langle 1423 \rangle \\ (0)_b \langle 1342 \rangle & 0 \end{bmatrix}, & \mathbf{L}_{6}^{(2TS)} &= \begin{bmatrix} 0 & (13)_b \langle 2314 \rangle \\ (10)_b \langle 2431 \rangle & 0 \end{bmatrix}, \\
\mathbf{L}_{7}^{(2TS)} &= \begin{bmatrix} 0 & (7)_b \langle 3241 \rangle \\ (6)_b \langle 3124 \rangle & 0 \end{bmatrix}, & \mathbf{L}_{8}^{(2TS)} &= \begin{bmatrix} 0 & (1)_b \langle 4132 \rangle \\ (12)_b \langle 4213 \rangle & 0 \end{bmatrix}.
\end{align*}
\]

(7.13)
7.3 L-matrices from the 4D, $\mathcal{N} = 2$ Axial-Vector Supermultiplet

For the 4D, $\mathcal{N} = 2$ axial-vector supermultiplet, the bosons are $A$, $B$, $F$, and $G$ from the chiral 4D, $\mathcal{N} = 1$ chiral supermultiplet and the spatial components of $U_\mu$ and $\tilde{d}$ from the 4D, $\mathcal{N} = 1$ axial-vector supermultiplet. To define the bosons in the 1D, $N = 8$ (2AVS) adinkra representation we define the bosons $\Phi_i$ via

$$
\begin{align*}
\Phi_1 &= A, & \Phi_2 &= B, & \partial_0 \Phi_3 &= F, & \partial_0 \Phi_4 &= G, \\
\Phi_5 &= U_1, & \Phi_6 &= U_2, & \Phi_7 &= U_3, & \partial_0 \Phi_8 &= \tilde{d},
\end{align*}
$$

(7.14)

and the fermions $\psi_k$ in the 1D, $N = 8$ (2AVS) adinkra representation via

$$
\begin{align*}
i\psi_1 &= \psi_1, & i\psi_2 &= \psi_2, & i\psi_3 &= \psi_3, & i\psi_4 &= \psi_4, \\
i\psi_5 &= \tilde{\lambda}_1, & i\psi_6 &= \tilde{\lambda}_2, & i\psi_7 &= \tilde{\lambda}_3, & i\psi_8 &= \tilde{\lambda}_4.
\end{align*}
$$

(7.15)

The L-matrices of the $\mathcal{N} = 2$ axial-vector supermultiplet are

$$
\begin{align*}
L^{(2AVS)}_1 &= \begin{bmatrix} (10)_b \langle 1423 \rangle & 0 \\ 0 & (9)_b \langle 3142 \rangle \end{bmatrix}, & L^{(2AVS)}_2 &= \begin{bmatrix} (12)_b \langle 2314 \rangle & 0 \\ 0 & (0)_b \langle 4231 \rangle \end{bmatrix}, \\
L^{(2AVS)}_3 &= \begin{bmatrix} (6)_b \langle 3241 \rangle & 0 \\ 0 & (3)_b \langle 1324 \rangle \end{bmatrix}, & L^{(2AVS)}_4 &= \begin{bmatrix} (0)_b \langle 4132 \rangle & 0 \\ 0 & (10)_b \langle 2413 \rangle \end{bmatrix}, \\
L^{(2AVS)}_5 &= \begin{bmatrix} 0 & (14)_b \langle 1423 \rangle \\ (14)_b \langle 3142 \rangle & 0 \end{bmatrix}, & L^{(2AVS)}_6 &= \begin{bmatrix} 0 & (8)_b \langle 2314 \rangle \\ (7)_b \langle 4231 \rangle & 0 \end{bmatrix}, \\
L^{(2AVS)}_7 &= \begin{bmatrix} 0 & (2)_b \langle 3241 \rangle \\ (4)_b \langle 1324 \rangle & 0 \end{bmatrix}, & L^{(2AVS)}_8 &= \begin{bmatrix} 0 & (4)_b \langle 4132 \rangle \\ (13)_b \langle 2413 \rangle & 0 \end{bmatrix}.
\end{align*}
$$

(7.16)

7.4 L-matrices from the 4D, $\mathcal{N} = 2$ Axial-Tensor Supermultiplet

For the 4D, $\mathcal{N} = 2$ axial-tensor supermultiplet, the bosons are $A$, $B$, $F$, and $G$ from the chiral 4D, $\mathcal{N} = 1$ chiral supermultiplet and the spatial components of $C_{\mu\nu}$ and $\tilde{\varphi}$ from the 4D, $\mathcal{N} = 1$ axial-tensor supermultiplet. To define the bosons in the 1D, $N = 8$ (2ATS) adinkra representation we define the bosons $\Phi_i$ via

$$
\begin{align*}
\Phi_1 &= A, & \Phi_2 &= B, & \partial_0 \Phi_3 &= F, & \partial_0 \Phi_4 &= G, \\
\Phi_5 &= \tilde{\varphi}, & \Phi_6 &= 2C_{12}, & \Phi_7 &= 2C_{23}, & \Phi_8 &= 2C_{31},
\end{align*}
$$

(7.17)

and the fermions $\psi_k$ in the 1D, $N = 8$ (2ATS) adinkra representation via

$$
\begin{align*}
i\psi_1 &= \psi_1, & i\psi_2 &= \psi_2, & i\psi_3 &= \psi_3, & i\psi_4 &= \psi_4, \\
i\psi_5 &= \tilde{\chi}_1, & i\psi_6 &= \tilde{\chi}_2, & i\psi_7 &= \tilde{\chi}_3, & i\psi_8 &= \tilde{\chi}_4.
\end{align*}
$$

(7.18)

The L-matrices of the $\mathcal{N} = 2$ axial-tensor supermultiplet are

$$
\begin{align*}
L^{(2ATS)}_1 &= \begin{bmatrix} (10)_b \langle 1423 \rangle & 0 \\ 0 & (13)_b \langle 4213 \rangle \end{bmatrix}, & L^{(2ATS)}_2 &= \begin{bmatrix} (12)_b \langle 2314 \rangle & 0 \\ 0 & (8)_b \langle 3124 \rangle \end{bmatrix}, \\
L^{(2ATS)}_3 &= \begin{bmatrix} (6)_b \langle 3241 \rangle & 0 \\ 0 & (11)_b \langle 2431 \rangle \end{bmatrix}, & L^{(2ATS)}_4 &= \begin{bmatrix} (0)_b \langle 4132 \rangle & 0 \\ 0 & (14)_b \langle 1342 \rangle \end{bmatrix}, \\
L^{(2ATS)}_5 &= \begin{bmatrix} 0 & (8)_b \langle 1423 \rangle \\ (3)_b \langle 4213 \rangle & 0 \end{bmatrix}, & L^{(2ATS)}_6 &= \begin{bmatrix} 0 & (14)_b \langle 2314 \rangle \\ (6)_b \langle 3124 \rangle & 0 \end{bmatrix}, \\
L^{(2ATS)}_7 &= \begin{bmatrix} 0 & (4)_b \langle 3241 \rangle \\ (5)_b \langle 4231 \rangle & 0 \end{bmatrix}, & L^{(2ATS)}_8 &= \begin{bmatrix} 0 & (2)_b \langle 4132 \rangle \\ (0)_b \langle 1342 \rangle & 0 \end{bmatrix}.
\end{align*}
$$

(7.19)
8 Adinkra 1D, $\mathcal{N} = 8$ Holoraumy Matrices

We define the bosonic holoraumy matrix $V^{(R)}_{IJ}$ and the fermionic holoraumy matrix $\tilde{V}^{(R)}_{IJ}$ via the respective equations shown in (8.1)

\[
\begin{align*}
L^{(R)}_I R^{(R)}_J - L^{(R)}_J R^{(R)}_I &= i2V^{(R)}_{IJ}, \\
R^{(R)}_I L^{(R)}_J - R^{(R)}_J L^{(R)}_I &= i2\tilde{V}^{(R)}_{IJ},
\end{align*}
\]

in 1D, $\mathcal{N} = 8$ systems. Thus, if we assemble the bosonic and fermionic components into $\Phi^{(R)}$ and $\Psi^{(R)}$ according to the definitions

\[
\Phi^{(R)} = \begin{bmatrix} \Phi^{(R)}_1 \\ \vdots \\ \Phi^{(R)}_8 \end{bmatrix}, \quad \Psi^{(R)} = \begin{bmatrix} \Psi^{(R)}_1 \\ \vdots \\ \Psi^{(R)}_8 \end{bmatrix},
\]

then we have

\[
[D_I, D_J] \Phi^{(R)} = 2V^{(R)}_{IJ} \partial_0 \Phi^{(R)}, \quad [D_I, D_J] \Psi^{(R)} = 2\tilde{V}^{(R)}_{IJ} \partial_0 \Psi^{(R)}.
\]

The key point about the equations in (8.3) is that same quantities (i.e. $\Phi^{(R)}$ and $\Psi^{(R)}$) appear on both sides of each respective equation. So given differentiable functions $\Phi^{(R)}$ and $\Psi^{(R)}$, these equations allow the determination of the holoraumy quantities $V^{(R)}_{IJ}$ and $\tilde{V}^{(R)}_{IJ}$. Furthermore due to the engineering dimensions of the D-operators, $\partial_0$, and the functions $\Phi^{(R)}$ and $\Psi^{(R)}$, the holoraumy quantities are dimensionless. Aside from the fact that temporal derivatives appear on the RHS of each of these equations, they have exactly the form of equations that define eigenvalues within the context of simple Lie algebras. This is the observation that suggests a representation theory description of 1D supermultiplets can be constructed on the basis of the constants that appear in the holoraumy quantities $V^{(R)}_{IJ}$ and $\tilde{V}^{(R)}_{IJ}$.

Other implications of the equations in (8.3) is they can be used to derive the results

\[
\partial_0 \Phi^{(R)} = \frac{1}{28} V^{(R)}_{IJ} \left[ D_I D_J \Phi^{(R)} \right], \quad \partial_0 \Psi^{(R)} = \frac{1}{28} \tilde{V}^{(R)}_{IJ} \left[ D_I D_J \Psi^{(R)} \right].
\]

Written in this form, the holoraumy quantities $V^{(R)}_{IJ}$ and $\tilde{V}^{(R)}_{IJ}$ together with the D-operators generate time evolution of the bosonic and fermionic variables $\Phi^{(R)}$ and $\Psi^{(R)}$.

The bosonic holoraumy matrix $V^{(R)}_{IJ}$ and the fermionic holoraumy matrix $\tilde{V}^{(R)}_{IJ}$ are both defined in such a way as to be hermitian,

\[
[V^{(R)}_{IJ}]^\dagger = V^{(R)}_{IJ}, \quad [	ilde{V}^{(R)}_{IJ}]^\dagger = \tilde{V}^{(R)}_{IJ}.
\]

Now we focus on the fermionic holoraumy. From the definition (8.1), we have

\[
\tilde{V}^{(R)}_{IJ} = -i\frac{1}{2} (R_I L_J - R_J L_I)
\]

and we have “dropped” the adinkra representation label $(R)$ as the following will be valid for all such representations.
We can write
\[ \tilde{V}_{iJ} \tilde{V}_{KL} = - \frac{1}{4} \left( R_i L_j - R_j L_i \right) \left( R_K L_L - R_L L_K \right) \]
\[ = - \frac{1}{2} \left( R_i L_j R_K L_L - R_i L_j R_L L_K \right) - (I \leftrightarrow J) \]
\[ = - \frac{1}{2} \left( R_i L_i R_K L_i + 2 \delta_{IK} R_i L_i - 2 \delta_{JK} R_i L_j + 2 \delta_{IL} R_K L_i - 2 \delta_{KL} R_K L_j - R_i L_K R_i L_j - 2 \delta_{IL} R_i L_K - 2 \delta_{JK} R_i L_J - 2 \delta_{IK} R_i L_J \right) - (I \leftrightarrow J) \]
\[ = - \frac{1}{4} \left( R_K L_L - R_L L_K \right) (R_i L_J - R_J L_i) \]
\[ + \delta_{IK} (R_i L_i - R_L L_i) - \delta_{JK} (R_i L_L - R_L L_J) \]
\[ + \delta_{IL} (R_K L_L - R_L L_K) - \delta_{IL} (R_K L_J - R_L L_J) \]
\[ = \tilde{V}_{KL} \tilde{V}_{iJ} + i 2 \delta_{IK} \tilde{V}_{iJ} - i 2 \delta_{JK} \tilde{V}_{iL} + i 2 \delta_{IL} \tilde{V}_{KJ} - i 2 \delta_{IJ} \tilde{V}_{KL} \ . \] (8.7)

Therefore,
\[ [\tilde{V}_{iJ}, \tilde{V}_{KL}] = i 2 \delta_{IK} \tilde{V}_{iL} - i 2 \delta_{JK} \tilde{V}_{iL} + i 2 \delta_{IL} \tilde{V}_{KJ} - i 2 \delta_{IJ} \tilde{V}_{KL} \ , \] (8.8)
which shows that \( \tilde{V}_{iJ} \) belongs to the spinor representation of so\( (N) \) algebra. For the systems that we are considering in this paper, it is a so\( (8) \) algebra.

By using equation (7.5), the results seen in (8.9) and (8.10) below are derived.

\[ \tilde{V}^{(2r)}_{11} = \begin{bmatrix} \tilde{V}^{(CS)}_{11} & 0 \\ 0 & \tilde{V}^{(R)}_{11} \end{bmatrix} \]
\[ , \]
\[ \tilde{V}^{(2r)}_{1 + 4, 1 + 4} = \begin{bmatrix} \tilde{V}^{(R)}_{11} & 0 \\ 0 & \tilde{V}^{(CS)}_{11} \end{bmatrix} \] (8.9)

\[ \tilde{V}^{(R)}_{1 + 4} = - \tilde{V}^{(R)}_{1 + 4} \]
\[ = -i \frac{1}{2} \left( R_1^{(2R)} S_1^{(2R)} L_1^{(R)} - R_1^{(R)} S_1^{(2R)} L_1^{(CS)} \right) \]
\[ - i \frac{1}{2} \left( R_1^{(CS)} S_1^{(2R)} L_1^{(CS)} - R_1^{(R)} S_1^{(2R)} L_1^{(R)} \right) \] (8.10)

The “off diagonal” terms of the final expression (8.10) is very revealing in comparison to the two expressions for the “diagonal” terms on the first line of (8.9). The terms involving \( \tilde{V}^{(CS)}_{11} \) and \( \tilde{V}^{(R)}_{11} \) perforce all are elements of so\( (4) \). On the other hand, the “off diagonal” terms in (8.10) must lie in the coset so\( (8)/so(4) \), otherwise the quartet \( \tilde{V}^{(2R)}_{11}, \tilde{V}^{(2R)}_{1 + 4, 1 + 4}, \tilde{V}^{(2R)}_{1 + 4, 1 + 4}, \) and \( \tilde{V}^{(2R)}_{1 + 4, 1 + 4} \) cannot form a representation of so\( (8) \).

By this means, we can reinterpret the condition for when two 1D, \( N = 4 \) adinkras can be combined to form a single 1D, \( N = 8 \) adinkra. In the work of [46], it was shown that this condition was determined by the calculation of the quantity \( \chi_0 \) on the two 1D, \( N = 4 \) adinkras and demand that the some of this quantity for the two 1D, \( N = 4 \) adinkras must vanish. The discussion surrounding (8.9) and (8.10) show an equivalent condition that the chiral 1D, \( N = 4 \) adinkra can be combined with another 1D, \( N = 4 \) adinkra (\( \mathcal{R} \)) if the quantity at the end of the equations shown in (8.10) is an element of the so\( (8)/so(4) \) coset.

Complete sets of \( \tilde{V} \)-matrices for the 4D, \( \mathcal{N} = 2 \) vector, tensor, axial-vector and axial-tensor supermultiplets are explicitly listed in Appendix C.
9 Adinkra 1D, $N = 8$ Gadget

In the works [48,49], the 1D, $N = 4$ Gadget was defined. We need to extend this in the current discussion. So for our present purposes we define the 1D Gadget value as the following:

$$G[(\mathcal{R}), (\mathcal{R}')] = \frac{2}{N(N - 1)d_{\text{min}}(N)} \sum_{I,J} \text{Tr}[\tilde{V}^{(\mathcal{R})}_{1J} \tilde{V}^{(\mathcal{R}')}_{1J}]$$  \tag{9.1}

where the normalization factor has the term

$$d_{\text{min}}(N) = \begin{cases} 
2^{N-1}, & N \equiv 1, 7 \mod 8 \\
2^{N}, & N \equiv 2, 4, 6 \mod 8 \\
2^{N+1}, & N \equiv 3, 5 \mod 8 \\
2^{N-2}, & N \equiv 8 \mod 8 
\end{cases}$$  \tag{9.2}

and $N$ is the number of color, which is 8 in our case. Therefore, our normalization factor is $\frac{1}{224}$.

We can then build the matrix

$$\mathcal{G}[(\mathcal{R}), (\mathcal{R}')] = \begin{bmatrix} 1 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
\frac{1}{7} & 1 & \frac{1}{7} & \frac{1}{7} \\
\frac{1}{7} & \frac{1}{7} & 1 & \frac{1}{7} \\
\frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 1 
\end{bmatrix},$$  \tag{9.3}

where the row and column indices run from $(2VS), (2TS), (2AVS), (2ATS)$. Note that it agrees completely with the 4D Gadget, i.e.

$$\mathcal{G}[(\bar{\mathcal{R}}), (\bar{\mathcal{R}}')] = \mathcal{G}[(\mathcal{R}), (\mathcal{R}')]$$  \tag{9.4}

Define the angle between different representations as

$$\cos\{\theta[(\mathcal{R}), (\mathcal{R}')]\} = \frac{\mathcal{G}[(\mathcal{R}), (\mathcal{R}')] \cdot \mathcal{G}[(\bar{\mathcal{R}}), (\bar{\mathcal{R}}')] \cdot \mathcal{G}[(\mathcal{R}), (\mathcal{R}')]}{\sqrt{\mathcal{G}[(\mathcal{R}), (\mathcal{R})]} \sqrt{\mathcal{G}[(\mathcal{R}'), (\mathcal{R}')]}}$$  \tag{9.5}

and $\cos^{-1}\left(\frac{1}{7}\right) \approx 81.8^\circ$. Thus in the space of 1D, $N = 8$ minimal adinkras we are lead to propose that the four distinct adinkras that arise from the dimensional reduction to the corresponding 1D representations $(2VS), (2TS), (2AVS), (2ATS)$ should be regarded as four points in a hexadecimal dimensional space that has the topology of a tetrad, but with a metric defined by the 1D Gadget.

Let us close this chapter with the observation and emphasis that the derivation of the result in (9.3) is very different from that of (6.13). Due to the parameters $m_1, \ldots, m_5$ that appear in the definition of (6.1), there is a large parameter space that can be used to “engineer” the choice of a metric given a set of Holorama tensors for a set of 4D supermultiplets. There are no such parameters in the 1D construction. The metric in (9.3) follows from (9.1), which follows entirely from the form of the L-matrices and R-matrices and the definition of of the 1D Gadget.
10 Conclusion

In previous work [28], an observation was introduced into the literature about 4D, $\mathcal{N} = 1$ minimal off-shell supermultiplets. Namely, when one determines the lattice parameters $\Pi = (p, q, r, s)$ associated with each supermultiplet via a holoraumy calculation one is led to the function

$$
\hat{h}^\mu(\Pi)_{abc}^d = i \left[ p C_{ab} (\gamma^\mu)_c^d + q (\gamma^5)_{ab} (\gamma^5)_{cd}^e + r (\gamma^5)_{ab} (\gamma^5)_c^d + \frac{1}{2} s (\gamma^5)_ab (\gamma^5)_{cd}^e \right],
$$

and the values of the lattice parameters are shown in Figure 5.

| $\mathcal{R}$ | p | q | r | s |
|---------------|---|---|---|---|
| (CS)          | 0 | 0 | 0 | -1 |
| (VS)          | 1 | 1 | 1 | 0 |
| (TS)          | -1 | 1 | -1 | 0 |
| (AVS)         | -1 | -1 | 1 | 0 |
| (ATS)         | 1 | -1 | -1 | 0 |

Figure 5: Illustrations of tetrahedral geometry in $\mathcal{N} = 1$ minimal supermultiplets.

Taking a three dimensional projection of the four dimensional parameter space defined by “dropping” the last column, one is led to the tetrahedron shown above.

Having completed the holoraumy calculations in the case of the minimal 4D, $\mathcal{N} = 2$ supermultiplets, we can examine these results from the perspective of the tetrahedron geometry. The holoraumy tensors we have found for each of the minimal supermultiplets look as

$$
\hat{H}^\mu(\overrightarrow{CS})_{abc}^{ij} = -\delta^{ij} \delta_k^l \left[ \hat{h}^\mu(1, 1, 1, -1)_{abc}^d + (\sigma^3)^{ij}(\sigma^3)_k^l \left[ \hat{h}^\mu(1, 1, 1, 1)_{abc}^d \right. \right.
+ (\sigma^1)^{ij}(\sigma^1)_k^l \left[ \hat{h}^\mu(1, 1, 1, 1)_{abc}^d \right. \right.
+ (\sigma^2)^{ij}(\sigma^2)_k^l \left[ \hat{h}^\mu(1, 1, 1, 1)_{abc}^d \right. \right. ,
$$

$$
\hat{H}^\mu(\overrightarrow{VS})_{abc}^{ij} = -\delta^{ij} \delta_k^l \left[ \hat{h}^\mu(-1, 1, -1, 1)_{abc}^d + (\sigma^3)^{ij}(\sigma^3)_k^l \left[ \hat{h}^\mu(-1, 1, -1, 1)_{abc}^d \right. \right.
+ (\sigma^1)^{ij}(\sigma^1)_k^l \left[ \hat{h}^\mu(-1, 1, -1, 1)_{abc}^d \right. \right.
+ (\sigma^2)^{ij}(\sigma^2)_k^l \left[ \hat{h}^\mu(-1, 1, -1, 1)_{abc}^d \right. \right.,
$$

$$
\hat{H}^\mu(\overrightarrow{TS})_{abc}^{ij} = -\delta^{ij} \delta_k^l \left[ \hat{h}^\mu(-1, 1, -1, 1)_{abc}^d + (\sigma^3)^{ij}(\sigma^3)_k^l \left[ \hat{h}^\mu(-1, 1, -1, 1)_{abc}^d \right. \right.
+ (\sigma^1)^{ij}(\sigma^1)_k^l \left[ \hat{h}^\mu(-1, 1, -1, 1)_{abc}^d \right. \right.
+ (\sigma^2)^{ij}(\sigma^2)_k^l \left[ \hat{h}^\mu(-1, 1, -1, 1)_{abc}^d \right. \right. ,
$$

$$
\hat{H}^\mu(\overrightarrow{AVS})_{abc}^{ij} = -\delta^{ij} \delta_k^l \left[ \hat{h}^\mu(1, 1, -1, 1)_{abc}^d + (\sigma^3)^{ij}(\sigma^3)_k^l \left[ \hat{h}^\mu(1, 1, -1, 1)_{abc}^d \right. \right.
+ (\sigma^1)^{ij}(\sigma^1)_k^l \left[ \hat{h}^\mu(1, 1, -1, 1)_{abc}^d \right. \right.
+ (\sigma^2)^{ij}(\sigma^2)_k^l \left[ \hat{h}^\mu(1, 1, -1, 1)_{abc}^d \right. \right. ,
$$

It is immediately apparent that the numbers of lattice parameters for the $\mathcal{N} = 2$ case are four times more than for the $\mathcal{N} = 1$ case. The explicit values of the lattice parameters are shown in Figure 6. The reason for this is clear. In the $\mathcal{N} = 1$ case as there is no “isospin” space for the supercovariant
derivative, effectively one only has to consider a factor of $\delta_{11}$. Whereas in the $\mathcal{N} = 2$ case there are factors of $\delta_{ij}$, $\mathcal{S}_{ij}^{(S)}$, and $\mathcal{A}_{ij}^{[A]}$, which are four independent structures and each has an associated set of lattice parameters.

| $(\hat{\mathcal{R}})$ | $p_0$ | $q_0$ | $r_0$ | $s_0$ | $p_3$ | $q_3$ | $r_3$ | $s_3$ | $p_1$ | $q_1$ | $r_1$ | $s_1$ | $p_2$ | $q_2$ | $r_2$ | $s_2$ |
|--------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $(2VS)$            | 1     | 1     | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| $(2TS)$            | -1    | 1     | -1    | -1    | -1    | 1     | -1    | -1    | -1    | 1     | -1    | -1    | 1     |
| $(2AVS)$           | -1    | -1    | 1     | -1    | -1    | 1     | -1    | 1     | -1    | 1     | -1    | -1    | 1     |
| $(2ATS)$           | 1     | -1    | -1    | -1    | 1     | -1    | -1    | 1     | -1    | -1    | 1     | -1    | 1     |

Figure 6: Illustrations of tetrahedral geometry in $\mathcal{N} = 2$ minimal supermultiplets.

From the explicit values of the lattice parameters associated with each of the independent isospin structures, it can be seen that the tetrahedral structure persists in the case of the minimal $\mathcal{N} = 2$ supermultiplets. This is once more seen by neglecting respectively the $s_0$, $s_3$, $s_1$ and $s_2$ coordinate to obtain three dimensional subspaces in each of the corresponding isospin subsectors. In all subsectors, the tetrahedrons affiliated with each $\mathcal{N} = 2$ supermultiplets are identical to that of the $\mathcal{N} = 1$ case.

Using the facts that there should be $(\mathcal{N}+2)(\mathcal{N}-1)/2$ independent $\mathcal{S}_{ij}^{(S)}$ tensors and $\mathcal{N}(\mathcal{N}-1)/2$ independent $\mathcal{A}_{ij}^{[A]}$ tensors, in the case of $\mathcal{N} = 2$ there should exist two independent $\mathcal{S}_{ij}^{(S)}$ tensors and a single $\mathcal{A}_{ij}^{[A]}$ tensor. We can thus make the identifications

$$\mathcal{S}_{ij}^{(S)} = \{ (\sigma^3)_{ij}, (\sigma^1)_{ij} \}, \quad \mathcal{A}_{ij}^{[A]} = \{ i(\sigma^2)_{ij} \} .$$

This implies there should be a doublet of each of the Lie algebra-valued operators $\mathcal{H}^{(4)}$, $\mathcal{H}^{(5)}$, and $\mathcal{H}^{(6)}$ where the components of the doublets are associated with $(\sigma^3)_{ij}$, and $(\sigma^1)_{ij}$. In a similar manner since there is a single $\mathcal{A}_{ij}^{[A]}$, there must be singlets $\mathcal{H}^{(7)}$, and $\mathcal{H}^{(8)}$ operators. As we will see shortly, this will be born out by explicit calculations.

We now turn to conjectures about the $\mathcal{N} = 4$ extension of our results.

Although we have no explicit calculations in this current work for the case of $\mathcal{N} = 4$, having discussed the results of (3.1) for the case of $\mathcal{N} = 2$, we can make a conjecture about the case of $\mathcal{N} = 4$. For $\mathcal{N} = 4$, we note that $\mathcal{A}_{ij}^{[A]}$ tensors can be used to define $\mathcal{A}_{ij}^{[A] \pm}$ via

$$\mathcal{A}_{ij}^{[A] \pm} \equiv \frac{1}{2} \left[ \mathcal{A}_{ij}^{[A]}, \pm \frac{1}{2} \epsilon_{ijkl} \mathcal{A}_{kl}^{[A]} \right] .$$

As we began with only six $\mathcal{A}_{ij}^{[A] \pm}$ operators, this decomposition leads to three $\mathcal{A}_{ij}^{[A] +}$ and three $\mathcal{A}_{ij}^{[A] -}$ tensors. In fact, these triplets are precisely the $\alpha$-matrices and $\beta$-matrices seen in appendix.
A. With this recognition we are also able to make the identification

$$S_{ij} = \{ (\alpha^I)_{ij}, (\beta^K)_{ij} \} ,$$

where the $I$ type of index takes on three values.

In the case of $N = 2$, it was seen that there are three functions, one each associated with $\delta_{ij}$ and $S_{ij}$ such that

$$\hat{h}^{(0)}(\Pi)_{abc} = \hat{h}^{(3)}(\Pi)_{abc} = \hat{h}^{(1)}(\Pi)_{abc} .$$

We conjecture that for the case of $N = 4$, there are ten functions associated with $\delta_{ij}$ and $(\alpha^I)_{ij}$ that satisfy the same condition. An even bolder conjecture is to assert that all six of the functions associated with $(\alpha^I)_{ij}$ and $(\beta^K)_{ij}$ satisfy the same sort of conditions.

Let us be very clear about the result in (6.13), it is "engineered" ($X_1 = 7, X_2 = 7, X_3 = 1$) in (6.10) on the 4D side. In particular, the appearance of the factors of “1” and “3” in (6.12) is controlled by a rule related to how the integers in $\Pi$ are multiplied by $\gamma$-matrices in the functions $[\hat{h}^{\nu}(\Pi)]_{abc}^d$ and $[\hat{h}^{(2)\mu}(\Pi)]_{abc}^d$. If one of the integers $p, q, r, s$ is multiplied by a factor of $[\gamma_{\mu}, \gamma_{\nu}]$ in either of the functions $[\hat{h}^{\nu}(\Pi)]_{abc}^d$ or $[\hat{h}^{(2)\mu}(\Pi)]_{abc}^d$, then when the covariant Gadget is written, a corresponding factor of “3” should be engineered to appear multiplying that integer. Otherwise, only a factor of “1” should be present. On the adinkra side of (9.1), no engineering is necessary as there is no freedom permitted by our definitions. The validity of (9.4) is an example of what we have long called “SUSY holography” [50].

Though we make this as a “phenomenological observation” based on the study of minimal 4D, $N = 1$ and $N = 2$ supermultiplets, we have no deep understanding of why this must be enforced to lead from a direct relation of the covariant 4D Gadget to the corresponding 1D Gadget.

The reader who has been following our progress will not be surprised that on the basis of the results presented here we have a surmise to make about the lattice parameters. If we return to the formula for the dimension of the irreducible representations of su(3) given in (2.8), there it is clear the parameters $p$ and $q$ are of fundamental significance in order to be able to express the result. It is our assertion that the lattice parameters $p^{(R)}, q^{(R)}, r^{(R)},$ and $s^{(R)}$ in their entirety provide the analogs of these su(3) parameters in order to write for spacetime SUSY representations the analog of Freudenthal-type formulae. The implication of this surmise is that a covariant spacetime Gadget $\hat{G}[(\hat{R}), (\hat{R})]$, for any supersymmetrical representation $(\hat{R})$, provides a Casimir-like value for the representation. As our work has shown [11], there are classes of covariant spacetime Gadgets. However, the one that satisfies the condition

$$\hat{G}[(\hat{R}), (\hat{R}')] = G[(R), (R')] ,$$

is distinguished because the adinkra Gadget on the RHS appears parameter-free up to an overall normalization constant.

A future ambitious goal of our work is to explore whether a concept similar to the Gadget can be combined with the Cubic Casimir noted in (2.10) to find an expression for the spacetime SUSY anomaly written in terms of the lattice parameters.
“Up comes stream upon stream, hill upon hill,
When it appears there is no way ahead;
Beyond shady willows and bright flowers still —
Lies another quiet village instead.”

- Lu You

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A Matrix Representation Results

This appendix consists of two parts. The first provides a presentation of conventions and result for $3 \times 3$ matrices of our discussions and latter part does so for $4 \times 4$ matrices.

For the $3 \times 3$ matrix generators of $\text{su}(3)$ we use

\[
\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (A.1)
\]

\[
\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.
\]

As well with regard to explicit form of the eigenvectors we write them as $|\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle, |\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle,$ and $|0, -\frac{1}{\sqrt{3}}\rangle$ where

\[
|\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |0, -\frac{1}{\sqrt{3}}\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (A.2)
\]

Our final discussion of explicit matrices associated with $\text{su}(3)$ in this appendix involves the formula presented in (2.13).

It is well-known in the context of the Standard Model, the Cabibbo-Kobayashi-Maskawa matrix $[51,52]$ plays an important role with respect to CP-violation. The three generations of lowest isospin component quarks can be assembled into two triplets according to

\[
\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix} = \mathcal{V} \begin{bmatrix} d \\ s \\ b \end{bmatrix}, \quad (A.3)
\]

via use of the Cabibbo-Kobayashi-Maskawa matrix. As the triplets $d' - s' - b'$ and $d - s - b$ appear, the generators of an $\text{su}(3)$ can act upon these. However, this particular $\text{su}(3)$ is related to the distinct families of quark, not the usual flavor symmetry. Thus, it is possible to write currents related to this “family-su(3)” symmetry and the $\text{su}(3)$ matrices can be used to define such currents. Since the former triplet is related to the latter triplet via the $\mathcal{V}$ matrix, it follows that

\[
\lambda'_i = \mathcal{V} \lambda_i \mathcal{V}^{-1} \quad , \quad (A.4)
\]

So (2.11) takes the form

\[
C_{4,G}(R, R') = \frac{1}{16} \sum_{i=1,j=1}^{8} \text{Tr} \left( \{ \lambda'_i, \lambda'_j \} \{ \lambda_i, \lambda_j \} \right) 
= \frac{1}{288} \left( 33 + 15 \left| \text{Tr}(\mathcal{V}) \right|^2 \right) 
= \frac{1}{288} \left( 33 + 15 \left| c_1 + (c_1 - e^{i\delta})(c_2 c_3 + s_2 s_3) \right|^2 \right) 
= \frac{1}{288} \left( 33 + 15 \left| (c_{12} + c_{23}) c_{13} + c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} \right|^2 \right) . \quad (A.5)
\]
On the penultimate line of this equation, the value of $C_{4,G}(R, R')$ is expressed in terms of the “KM” parameters of the C-K-M matrix while the final line it is expressed in terms of the “standard” parameters of the C-K-M matrix.

Now we move on the relevant matrix structure for $4 \times 4$ matrix related to the isospin indices in our discussions.

The $\alpha$ and $\beta$ matrices used in this paper are:

\[
\begin{align*}
\alpha^1 &= \begin{bmatrix}
0 & 0 & 0 & -i \\
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{bmatrix}, & \alpha^2 &= \begin{bmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
i & 0 & 0 & 0
\end{bmatrix}, & \alpha^3 &= \begin{bmatrix}
0 & 0 & -i & 0 \\
0 & 0 & 0 & i \\
i & 0 & 0 & 0 \\
0 & -i & 0 & 0
\end{bmatrix}, \\
\beta^1 &= \begin{bmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
i & 0 & 0 & 0 \\
0 & -i & 0 & 0
\end{bmatrix}, & \beta^2 &= \begin{bmatrix}
0 & 0 & -i & 0 \\
0 & 0 & 0 & -i \\
i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{bmatrix}, & \beta^3 &= \begin{bmatrix}
0 & 0 & 0 & -i \\
0 & 0 & 0 & -i \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & i
\end{bmatrix}.
\end{align*}
\]  
(A.6)

In terms of tensor products of Pauli spin matrices $\sigma^i$ and the $2 \times 2$ identity matrix $I_{2\times2}$, this can be written as

\[
\begin{align*}
\alpha^1 &= \sigma^2 \otimes \sigma^1, & \alpha^2 &= I_{2\times2} \otimes \sigma^2, & \alpha^3 &= \sigma^2 \otimes \sigma^3, \\
\beta^1 &= \sigma^1 \otimes \sigma^2, & \beta^2 &= \sigma^2 \otimes I_{2\times2}, & \beta^3 &= \sigma^3 \otimes \sigma^2.
\end{align*}
\]  
(A.7) (A.8)

These matrices form two mutually commuting su(2) algebras

\[
[\alpha^{\hat{a}}, \alpha^{\hat{b}}] = i 2 \epsilon^{\hat{a}\hat{b}\hat{c}} \alpha^{\hat{c}}, \quad [\beta^{\hat{a}}, \beta^{\hat{b}}] = i 2 \epsilon^{\hat{a}\hat{b}\hat{c}} \beta^{\hat{c}}, \quad [\alpha^{\hat{a}}, \beta^{\hat{b}}] = 0.
\]  
(A.9)

Owing to the definitions above, the $\alpha$ and $\beta$ matrices satisfy the trace orthogonality relationships

\[
Tr(\alpha^{\hat{a}} \beta^{\hat{b}}) = 0, \quad Tr(\alpha^{\hat{a}} \alpha^{\hat{b}}) = Tr(\beta^{\hat{a}} \beta^{\hat{b}}) = 4 \delta^{\hat{a}\hat{b}}.
\]  
(A.10)
B 4D, \( \mathcal{N} = 2 \) Supermultiplets

In this paper, we consider 4D, \( \mathcal{N} = 2 \) supermultiplets with 8 bosons and 8 fermions as these are the minimal representation presented among the results shown in Table 1. In the work of [46] these results were presented. The transformation laws and anti-commutator algebra are reproduced here for the convenience of the reader.

B.1 4D, \( \mathcal{N} = 2 \) Vector Supermultiplet

We construct the 4D, \( \mathcal{N} = 2 \) vector supermultiplet from the 4D, \( \mathcal{N} = 1 \) chiral and vector supermultiplets. Let

\[
\Psi_{ck} = \begin{bmatrix} \psi_c \\ \lambda_c \end{bmatrix}, \tag{B.1}
\]

where \( k \) is the isospin index. The Lagrangian is

\[
\mathcal{L} = -\frac{1}{2} \partial_\mu A \partial^\mu A - \frac{1}{2} \partial_\mu B \partial^\mu B + \frac{1}{2} F^2 + \frac{1}{2} G^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} d^2 \tag{B.2}
\]

where

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{B.3}
\]

is the usual field strength. The corresponding transformation laws are

\[
\begin{align*}
D^i_a A &= \delta^{ij} \Psi_{aj} \\
D^i_a B &= i \delta^{ij} (\gamma^5)_a^b \Psi_{bj} \\
D^i_a F &= \delta^{ij} (\gamma^\mu)_a^b \partial_\mu \Psi_{bj} \\
D^i_a G &= i (\sigma^3)^{ij} (\gamma^5 \gamma^\mu)_a^b \partial_\mu \Psi_{bj} \\
D^i_a A_\mu &= i (\sigma^2)^{ij} (\gamma_\mu)_a^b \Psi_{bj} \\
D^i_a d &= i (\sigma^1)^{ij} (\gamma^5 \gamma^\mu)_a^b \partial_\mu \Psi_{bj} \\
D^i_a \Psi_{bj}^i &= \delta^{ij} \left[ i (\gamma^\nu)_a^b \partial_\mu A - (\gamma^5 \gamma^\nu)_{ab} \partial_\mu B - i C_{ab} F \right] + (\sigma^3)^{ij} (\gamma^5)_{ab} G \\
&\quad + (\sigma^1)^{ij} (\gamma^5)_{ab} d + \frac{1}{4} (\sigma^2)^{ij} \left[ [\gamma_\mu, \gamma^\nu] \right]_{ab} (\partial_\mu A_\nu - \partial_\nu A_\mu)
\end{align*}
\tag{B.4}
\]

The transformation laws satisfy the algebra

\[
\begin{align*}
\{D^i_a, D^j_b\} \chi &= i 2 \delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \chi \\
\{D^i_a, D^j_b\} A_\mu &= i 2 \delta^{ij} (\gamma^\nu)_{ab} F_{\nu\mu} + i (\sigma^2)^{ij} \left[ i 2 C_{ab} \partial_\mu A - 2 (\gamma^5)_{ab} \partial_\mu B \right]
\end{align*}
\tag{B.5}
\]

where

\[
\chi \in \{ A, B, F, G, d, \Psi_{ck} \} \tag{B.7}
\]

B.2 4D, \( \mathcal{N} = 2 \) Tensor Supermultiplet

We construct the 4D, \( \mathcal{N} = 2 \) tensor supermultiplet from the 4D, \( \mathcal{N} = 1 \) chiral and tensor supermultiplets. Let

\[
\Psi_{ck} = \begin{bmatrix} \psi_c \\ \chi_c \end{bmatrix} \tag{B.8}
\]
The Lagrangian is
\[ \mathcal{L} = -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B + \frac{1}{2}F^2 + \frac{1}{2}G^2 - \frac{1}{3}H_{\mu\nu\alpha}H^{\mu\nu\alpha} - \frac{1}{2}\partial_\mu \varphi\partial^\mu \varphi \]  
where
\[ H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu} \]  
The corresponding transformation laws are
\[ D^i_A = (\sigma^3)^{ij}\Psi_j \]  
\[ D^i_B = i\delta^{ij}(\gamma^5)^a_b\Psi_{bj} \]  
\[ D^i_F = \delta^{ij}(\gamma^\mu)^a_b\partial_\mu \Psi_{bj} \]  
\[ D^i_G = i\delta^{ij}(\gamma^5\gamma^\mu)^a_b\partial_\mu \Psi_{bj} \]  
\[ D^i_\varphi = (\sigma^1)^{ij}\Psi_j \]  
\[ D^i_{B_{\mu\nu}} = -i\frac{1}{4}(\sigma^2)^{ij}[(\gamma^{\mu},\gamma^\nu)]_a^b\Psi_{bj} \]  
The transformation laws satisfy the algebra
\[ \{D^i_A, D^j_B\} = i2\delta^{ij}(\gamma^\mu)^a_b\partial_\mu \chi \]  
\[ \{D^i_A, D^j_B\}B_{\mu\nu} = i2\delta^{ij}(\gamma^\mu)^a_bH_{\alpha\mu\nu} 
+ i(\gamma^{\mu})_{ac}\partial_\mu \]  
\[ \chi \in \{A, B, F, G, \varphi, \Psi_{ck}\} \]  

\section*{B.3 4D, $\mathcal{N}=2$ Axial-Vector Supermultiplet}

We construct the 4D, $\mathcal{N}=2$ axial-vector supermultiplet from the 4D, $\mathcal{N}=1$ chiral and axial-vector supermultiplets. Let
\[ \Psi_{ck} = \begin{bmatrix} \psi_c \\ \bar{\lambda}_c \end{bmatrix} \]  
The Lagrangian is
\[ \mathcal{L} = -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B + \frac{1}{2}F^2 + \frac{1}{2}G^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}d^2 \]  
where
\[ F_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu \]
The corresponding transformation laws are

\[ \begin{align*}
D^i_a A &= \delta^{ij} \Psi_{aj} \\
D^i_a B &= i \delta^{ij} (\gamma^5)_{ab} \Psi_{bj} \\
D^i_a F &= (\sigma^3)^{ij} (\gamma^\mu)_{ab} \partial_\mu \Psi_{bj} \\
D^i_a G &= i \delta^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \Psi_{bj} \\
D^i_a U_\mu &= - (\sigma^2)^{ij} (\gamma^5 \gamma^\mu)_{ab} \Psi_{bj} \\
D^i_a \tilde{d} &= - (\sigma^1)^{ij} (\gamma^\mu)_{ab} \partial_\mu \Psi_{bj} \\
D^i_b \Psi^j_b &= \delta^{ij} \left[ i (\gamma^\mu)_{ab} \partial_\mu A - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B + (\gamma^5)_{ab} G \right] - i (\sigma^3)^{ij} C_{ab} F \\
&+ i (\sigma^1)^{ij} C_{ab} \tilde{d} + i \frac{1}{2} (\sigma^2)^{ij} (\gamma^5 [\gamma^\mu, \gamma^\nu])_{ab} (\partial_\mu U_\nu - \partial_\nu U_\mu) \\
\end{align*} \]

The transformation laws satisfy the algebra

\[ \{ D^i_a, D^j_b \} \chi = i 2 \delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \chi \quad \text{,} \quad \{ D^i_a, D^j_b \} U_\mu = i 2 \delta^{ij} (\gamma^\nu)_{ab} F_{\mu \nu} - i (\sigma^2)^{ij} \left[ 2 (\gamma^5)_{ab} \partial_\mu A + i 2 C_{ab} \partial_\mu B \right] \quad \text{,} \]

where

\[ \chi \in \{ A, B, F, G, \tilde{d}, \Psi_{ck} \} \quad \text{.} \]

### B.4 4D, \( \mathcal{N}=2 \) Axial-Tensor Supermultiplet

Finally, we construct the 4D, \( \mathcal{N}=2 \) axial-tensor supermultiplet from the 4D, \( \mathcal{N}=1 \) chiral and axial-tensor supermultiplets. Let

\[ \Psi_{ck} = \begin{bmatrix} \psi_c \\ \tilde{\chi}_c \end{bmatrix} \quad \text{.} \]

The Lagrangian is

\[ \mathcal{L} = - \frac{1}{2} \partial_\mu A \partial^\mu A - \frac{1}{2} \partial_\mu B \partial^\mu B + \frac{1}{2} F^2 + \frac{1}{2} G^2 - \frac{1}{3} H_{\mu \nu \alpha} H^{\mu \nu \alpha} - \frac{1}{2} \partial_\mu \tilde{\psi} \partial^\mu \tilde{\psi} \\
+ i \frac{1}{2} \delta^{ij} (\gamma^\mu)_{bc} \Psi_{bi} \partial_\mu \Psi_{cj} \quad \text{,} \]

where

\[ H_{\mu \nu \alpha} = \partial_\mu C_{\nu \alpha} + \partial_\nu C_{\alpha \mu} + \partial_\alpha C_{\mu \nu} \quad \text{.} \]

The corresponding transformation laws are

\[ \begin{align*}
D^i_a A &= \delta^{ij} \Psi_{aj} \\
D^i_a B &= i (\sigma^3)^{ij} (\gamma^5)_{ab} \Psi_{bj} \\
D^i_a F &= \delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \Psi_{bj} \\
D^i_a G &= i \delta^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \Psi_{bj} \\
D^i_a \tilde{d} &= i (\sigma^1)^{ij} (\gamma^\mu)_{ab} \Psi_{bj} \\
D^i_a C_{\mu \nu} &= \frac{1}{4} (\sigma^2)^{ij} (\gamma^5 [\gamma^\mu, \gamma^\nu])_{ab} \Psi_{bj} \\
D^i_a \Psi^j_b &= \delta^{ij} \left[ i (\gamma^\mu)_{ab} \partial_\mu A - \partial_\mu F + (\gamma^5)_{ab} G \right] - (\sigma^3)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B \\
&- (\sigma^1)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \tilde{\psi} + (\sigma^2)^{ij} \epsilon^\mu_{\alpha \beta} (\gamma^\mu)_{ab} \partial_\beta \partial_\alpha \tilde{\psi} \quad \text{.} \\
\end{align*} \]

The transformation laws satisfy the algebra

\[ \{ D^i_a, D^j_b \} \chi = i 2 \delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \chi \quad \text{.} \]
\{D^i_a, D^j_b\} C_{\mu \nu} = i 2 \delta^{ij} (\gamma^\alpha)_{ab} H_{\alpha \mu \nu} \\
+ i (\gamma_{[\mu} \partial_{\nu]} [(\sigma^1)^{ij} \delta^c_b B - (\sigma^2)^{ij} (\gamma^5)_b^c A - (\sigma^3)^{ij} \delta^c_b \tilde{\phi}]) \\
\text{where} \\
\chi \in \{A, B, F, G, \tilde{\phi}, \Psi_{ck}\} .

\text{(B.27)}

\text{37}
C Fermionic Holoraumy Matrices of 1D, \( \mathcal{N} = 8 \) Supermultiplets

In this final appendix, explicit expressions for the fermionic holoraumy matrices are given in term of \( 2 \times 2 \) matrices whose elements are themselves signed \( 4 \times 4 \) permutations as this allows for a fairly compact notation.

C.1 Fermionic Holoraumy Matrices of 1D, \( \mathcal{N} = 8 \) Vector Supermultiplet

\[
\begin{align*}
\tilde{\mathbf{V}}_{12}^{(2VS)} &= i \begin{bmatrix} \langle 2143 \rangle & 0 \\ 0 & \langle 2143 \rangle \end{bmatrix}, & \tilde{\mathbf{V}}_{13}^{(2VS)} &= i \begin{bmatrix} \langle 3412 \rangle & 0 \\ 0 & \langle 3412 \rangle \end{bmatrix}, & \tilde{\mathbf{V}}_{14}^{(2VS)} &= i \begin{bmatrix} \langle 4321 \rangle & 0 \\ 0 & \langle 4321 \rangle \end{bmatrix}, \\
\tilde{\mathbf{V}}_{15}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 1234 \rangle \\ \langle 1234 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{16}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{17}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{bmatrix}, \\
\tilde{\mathbf{V}}_{18}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{23}^{(2VS)} &= i \begin{bmatrix} \langle 4321 \rangle & 0 \\ 0 & \langle 4321 \rangle \end{bmatrix}, & \tilde{\mathbf{V}}_{24}^{(2VS)} &= i \begin{bmatrix} \langle 3412 \rangle & 0 \\ 0 & \langle 3412 \rangle \end{bmatrix}, \\
\tilde{\mathbf{V}}_{25}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{26}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 1234 \rangle \\ \langle 1234 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{27}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{bmatrix}, \\
\tilde{\mathbf{V}}_{28}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{34}^{(2VS)} &= i \begin{bmatrix} \langle 2143 \rangle & 0 \\ 0 & \langle 2143 \rangle \end{bmatrix}, & \tilde{\mathbf{V}}_{35}^{(2VS)} &= i \begin{bmatrix} \langle 3412 \rangle & 0 \\ 0 & \langle 3412 \rangle \end{bmatrix}, \\
\tilde{\mathbf{V}}_{36}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{37}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 1234 \rangle \\ \langle 1234 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{38}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{bmatrix}, \\
\tilde{\mathbf{V}}_{45}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{46}^{(2VS)} &= i \begin{bmatrix} \langle 3412 \rangle & 0 \\ 0 & \langle 3412 \rangle \end{bmatrix}, & \tilde{\mathbf{V}}_{47}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{bmatrix}, \\
\tilde{\mathbf{V}}_{48}^{(2VS)} &= i \begin{bmatrix} 0 & \langle 1234 \rangle \\ \langle 1234 \rangle & 0 \end{bmatrix}, & \tilde{\mathbf{V}}_{56}^{(2VS)} &= i \begin{bmatrix} \langle 2143 \rangle & 0 \\ 0 & \langle 2143 \rangle \end{bmatrix}, & \tilde{\mathbf{V}}_{57}^{(2VS)} &= i \begin{bmatrix} \langle 3412 \rangle & 0 \\ 0 & \langle 3412 \rangle \end{bmatrix}, \\
\tilde{\mathbf{V}}_{58}^{(2VS)} &= i \begin{bmatrix} \langle 4321 \rangle & 0 \\ 0 & \langle 4321 \rangle \end{bmatrix}, & \tilde{\mathbf{V}}_{67}^{(2VS)} &= i \begin{bmatrix} \langle 4321 \rangle & 0 \\ 0 & \langle 4321 \rangle \end{bmatrix}, & \tilde{\mathbf{V}}_{68}^{(2VS)} &= i \begin{bmatrix} \langle 3412 \rangle & 0 \\ 0 & \langle 3412 \rangle \end{bmatrix}.
\end{align*}
\]
C.2 Fermionic Holomorphy Matrices of 1D, $N = 8$ Tensor Supermultiplets

\[
\begin{align*}
\tilde{V}_{12}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 2143 \\ 2143 & 0 \end{array} \right], & \tilde{V}_{13}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 3412 \\ 3412 & 0 \end{array} \right], & \tilde{V}_{14}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 4321 \\ 4321 & 0 \end{array} \right], \\
\tilde{V}_{15}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 1234 \\ 1234 & 0 \end{array} \right], & \tilde{V}_{16}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 2143 \\ 2143 & 0 \end{array} \right], & \tilde{V}_{17}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 3412 \\ 3412 & 0 \end{array} \right], \\
\tilde{V}_{18}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 4321 \\ 4321 & 0 \end{array} \right], & \tilde{V}_{23}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 3412 \\ 3412 & 0 \end{array} \right], & \tilde{V}_{24}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 4321 \\ 4321 & 0 \end{array} \right], \\
\tilde{V}_{25}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 2143 \\ 2143 & 0 \end{array} \right], & \tilde{V}_{26}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 1234 \\ 1234 & 0 \end{array} \right], & \tilde{V}_{27}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 4321 \\ 4321 & 0 \end{array} \right], \\
\tilde{V}_{28}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 3412 \\ 3412 & 0 \end{array} \right], & \tilde{V}_{34}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 2143 \\ 2143 & 0 \end{array} \right], & \tilde{V}_{35}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 3412 \\ 3412 & 0 \end{array} \right], \\
\tilde{V}_{36}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 4321 \\ 4321 & 0 \end{array} \right], & \tilde{V}_{37}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 1234 \\ 1234 & 0 \end{array} \right], & \tilde{V}_{38}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 2143 \\ 2143 & 0 \end{array} \right], \\
\tilde{V}_{45}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 4321 \\ 4321 & 0 \end{array} \right], & \tilde{V}_{46}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 3412 \\ 3412 & 0 \end{array} \right], & \tilde{V}_{47}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 2143 \\ 2143 & 0 \end{array} \right], \\
\tilde{V}_{48}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 1234 \\ 1234 & 0 \end{array} \right], & \tilde{V}_{56}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 2143 \\ 2143 & 0 \end{array} \right], & \tilde{V}_{57}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 3412 \\ 3412 & 0 \end{array} \right], \\
\tilde{V}_{58}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 4321 \\ 4321 & 0 \end{array} \right], & \tilde{V}_{67}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 3412 \\ 3412 & 0 \end{array} \right], & \tilde{V}_{68}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 4321 \\ 4321 & 0 \end{array} \right], \\
\tilde{V}_{78}^{(2TS)} &= i \left[ \begin{array}{cc} 0 & 2143 \\ 2143 & 0 \end{array} \right]. 
\end{align*}
\]
### C.3 Fermionic Holoraumy Matrices of 1D, $N = 8$ Axial-Vector Supermultiplet

\[
\tilde{V}^{(2AVS)}_{12} = i \begin{pmatrix} (2143) & 0 \\ 0 & (2143) \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{13} = i \begin{pmatrix} (3412) & 0 \\ 0 & (3412) \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{14} = i \begin{pmatrix} (4321) & 0 \\ 0 & (4321) \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{15} = i \begin{pmatrix} 0 & (1234) \\ (1234) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{16} = i \begin{pmatrix} 0 & (2143) \\ (2143) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{17} = i \begin{pmatrix} 0 & (3412) \\ (3412) & 0 \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{18} = i \begin{pmatrix} 0 & (4321) \\ (4321) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{23} = i \begin{pmatrix} (4321) & 0 \\ 0 & (4321) \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{24} = i \begin{pmatrix} (3412) & 0 \\ 0 & (3412) \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{25} = i \begin{pmatrix} (2143) & 0 \\ (2143) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{26} = i \begin{pmatrix} (1234) & 0 \\ (1234) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{27} = i \begin{pmatrix} (4321) & 0 \\ (4321) & 0 \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{28} = i \begin{pmatrix} (3412) & 0 \\ (3412) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{34} = i \begin{pmatrix} (2143) & 0 \\ (2143) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{35} = i \begin{pmatrix} (3412) & 0 \\ (3412) & 0 \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{36} = i \begin{pmatrix} (4321) & 0 \\ (4321) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{37} = i \begin{pmatrix} (1234) & 0 \\ (1234) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{38} = i \begin{pmatrix} (2143) & 0 \\ (2143) & 0 \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{45} = i \begin{pmatrix} (4321) & 0 \\ (4321) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{46} = i \begin{pmatrix} (3412) & 0 \\ (3412) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{47} = i \begin{pmatrix} (2143) & 0 \\ (2143) & 0 \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{48} = i \begin{pmatrix} (1234) & 0 \\ (1234) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{56} = i \begin{pmatrix} (2143) & 0 \\ (2143) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{57} = i \begin{pmatrix} (3412) & 0 \\ (3412) & 0 \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{58} = i \begin{pmatrix} (4321) & 0 \\ (4321) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{67} = i \begin{pmatrix} (4321) & 0 \\ (4321) & 0 \end{pmatrix}, \quad \tilde{V}^{(2AVS)}_{68} = i \begin{pmatrix} (3412) & 0 \\ (3412) & 0 \end{pmatrix}, \\
\tilde{V}^{(2AVS)}_{78} = i \begin{pmatrix} (2143) & 0 \\ (2143) & 0 \end{pmatrix}.
\]
\[ \tilde{V}^{(ATS)}_{12} = \begin{pmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{13} = \begin{pmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{14} = \begin{pmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{15} = \begin{pmatrix} 0 & \langle 1234 \rangle \\ \langle 1234 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{16} = \begin{pmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{17} = \begin{pmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{18} = \begin{pmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{23} = \begin{pmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{24} = \begin{pmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{25} = \begin{pmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{26} = \begin{pmatrix} 0 & \langle 1234 \rangle \\ \langle 1234 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{27} = \begin{pmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{28} = \begin{pmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{34} = \begin{pmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{35} = \begin{pmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{36} = \begin{pmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{37} = \begin{pmatrix} 0 & \langle 1234 \rangle \\ \langle 1234 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{38} = \begin{pmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{45} = \begin{pmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{46} = \begin{pmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{47} = \begin{pmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{48} = \begin{pmatrix} 0 & \langle 1234 \rangle \\ \langle 1234 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{56} = \begin{pmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{57} = \begin{pmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{58} = \begin{pmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{67} = \begin{pmatrix} 0 & \langle 4321 \rangle \\ \langle 4321 \rangle & 0 \end{pmatrix}, \quad \tilde{V}^{(ATS)}_{68} = \begin{pmatrix} 0 & \langle 3412 \rangle \\ \langle 3412 \rangle & 0 \end{pmatrix}, \]

\[ \tilde{V}^{(ATS)}_{78} = \begin{pmatrix} 0 & \langle 2143 \rangle \\ \langle 2143 \rangle & 0 \end{pmatrix}. \]

(C.4)
References

[1] Y. A. Gelfand and E. Likhtman, “Extension of the Algebra of Poincare Group Generators and Violation of p Invariance,” JETP Lett. 13 (1971) 323.

[2] D. V. Volkov and V. P. Akulov, “Possible universal neutrino interaction,” Pis’ Zh. Eksp. Teor. Fiz 16 (1972) 621.

[3] M. Shifman and A. Yung, “Extension of the Algebra of Poincare Group Generators and violation of P Invariance,” JETP Lett. 13 323'(1971).

[4] D. V. Volkov and V. P. Akulov, “Is the Neutrino a Goldstone Particle?,” Phys. Lett. 46B (1973) 109.

[5] J. Wess and B. Zumino, “Supergauge Transformations in Four-Dimensions,” Nucl. Phys. B70 (1974) 139.

[6] R. Horaud, “A Short Tutorial on Graph Laplacians, Laplacian Embedding, and Spectral Clustering,” 2012 (INRIA), https://csustan.csustan.edu/ tom/Clustering/GraphLaplacian-tutorial.pdf.

[7] B. Bollobás, “Modern Graph Theory,” (1998) Springer-Verlag, Heidelberg, eISBN-13: 978-1461206194, ISBN-13: 978-0387984889.

[8] J. A. Bondy, and U. S. R. Murty, “Graph Theory,” (2010) Springer-Verlag, Heidelberg, ISBN-13: 978-1849966900, ISBN-10: 1849966907.

[9] M. Bóna, “Walk Through Combinatorics,” (2011) World Scientific, Hong Kong, ISBN-13: 978-9814335232, ISBN-13: 978-9814460002.

[10] R. Diestel “Graph Theory,” (2016) Springer-Verlag, Heidelberg, ISBN-13: 978-3662536216, ISBN-13: eISBN 978-3961340057.

[11] J. Gross, and J. Yellen, “Graph Theory And Its Applications,” (2006) CRC Press, Taylor & Francis Group, LLCoca Raton, FL, ISBN-13: 978-1584885054, ISBN-10: 158488505X.

[12] A. Tannenbaum, C. Sander, R. Sandhu, L. Zhu, I. Kolesov, E. Reznik, Y. Senbabaoglu, and T. Georgiou, “Graph Curvature and the Robustness of Cancer Networks,”

[13] A. Barabási, “The network takeover,” Nature Physics 8 (2012), pp. 14-16.

[14] E. Cartan, “Oeuvres complétes. Partie I. Groups de Lie.” Second edition. Editions du Centre National de la Recherche Scientifique (CNRS), Paris, 1984, 1356 pp.

[15] H. Weyl, “Gesammelte Abhandlungen,” Bnde I, II, III, IV, Herausgegeben von K. Chandrasekharan Springer-Verlag, Berlin-New York 1968 Band I: 698 pp., Band II: 647 pp., Band III: 791 pp., Band IV: 694 pp.

[16] V. G, Kac, “Lie superalgebras,” Adv. in Math. 26 (1977), 896.
[17] S. J. Gates Jr., and L. Rana, “A Theory of Spinning Particles for Large N-extended Supersymmetry (I),” Phys. Lett. B352 (1995) 50, arXiv [hep-th:9504025].

[18] S. J. Gates Jr., and L. Rana, “A Theory of Spinning Particles for Large N-extended Supersymmetry (II),” ibid. Phys. Lett. B369 (1996) 262, arXiv [hep-th:9510151].

[19] M. Faux, S. J. Gates Jr. “Adinkras: A Graphical Technology for Supersymmetric Representation Theory,” Phys. Rev. D71 (2005) 065002, [hep-th/0408004].

[20] C. F. Doran, K. Iga, and G. Landweber, “An application of Cubical Cohomology to Adinkras and Supersymmetry Representations,” July 2012, 1207.6806, e-Print: arXiv:1207.6806 [hep-th], (unpublished).

[21] Yan X. Zhang, “Adinkras for Mathematicians,” Transactions of the American Mathematical Society, Vol. 366, No. 6, June 2014, Pages 3325355 S 0002-9947(2014)06031-5.

[22] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga, and G. D. Landweber, “On Graph-Theoretic Identifications of Adinkras, Supersymmetry Representations and Superfield,” Int. J. Mod. Phys. A22 (2007) 869-930, DOI: 10.1142/S0217751X07035112 e-Print: math-ph/0512016.

[23] C. Doran, K. Iga, J. Kostiuk, G. Landweber, and S. Mendez-Diez, “Geometrization of N-extended 1-dimensional supersymmetry algebras, I,” Adv. Theor. Math. Phys. 19 (2015) 1043-1113, DOI: 10.4310/ATMP.2015.v19.n5.a4. e-Print: arXiv:1311.3736 [hep-th].

[24] C. Doran, K. Iga, J. Kostiuk, G. Landweber, and S. Mendez-Diez, “Geometrization of N-Extended 1-Dimensional Supersymmetry Algebras II,” e-Print: arXiv:1610.09983 [hep-th].

[25] S. J. Gates Jr., T. Hübsch, and K. Stiffler, “On Clifford-algebraic ‘Holoraumy’, dimensional extension and SUSY holography,” Int. J. Mod. Phys. A30 (2015) no.09, 1550042,DOI: 10.1142/S0217751X15500426, e-Print: arXiv:1409.4445 [hep-th].

[26] M. Calkins, D. E. A. Gates, S. J. Gates Jr., and K. Stiffler, “Adinkras, 0-branes, Holoraumy and the SUSY QFT/QM Correspondence,” Int. J. Mod. Phys. A30 (2015) no.11, 1550050, DOI: 10.1142/S0217751X15500505, e-Print: arXiv:1501.00101 [hep-th].

[27] S. J. Gates, T. Grover, M. D. Miller-Dickson, B. A. Mondal, A. Oskoui, S. Regmi, E. Ross, and R. Shetty, “A Lorentz covariant holoraumy-induced Gadget from minimal off-shell 4D, N = 1 supermultiplets,” JHEP 1511 (2015) 113, e-Print: arXiv:1508.07546 [hep-th].

[28] W. Caldwell, A. N. Diaz, I. Friend, S. J. Gates, Jr., S. Harmarkar, T. Lambert-Brown, D. Lay, K. Martirosova, V. A. Meszaros, M. O'Nokanwaye, S. Rudman, D. Shin, and A. Vershov, “On the four-dimensional holoraumy of the 4D, N = 1 complex linear supermultiplet,” Int. J. Mod. Phys. A33 (2018) no.12, 1850072, DOI: 10.1142/S0217751X18500720. e-Print: arXiv:1702.05453 [hep-th].

[29] H. Freudenthal, “Zur Berechnung der Charaktere der halbeinfacher Liescher Gruppen I” Indag. Math., 16 (1954) pp. 369376.

[30] H. Freudenthal, “Zur Berechnung der Charaktere der halbeinfacher Liescher Gruppen II” Indag. Math., 16 (1954) pp. 487491.
[31] H. Freudenthal, “Zur Berechnung der Charaktere der halbeinfacher Liescher Gruppen III” Indag. Math., 18 (1956) pp. 511514.

[32] A. Pais, “Dynamical Symmetry in Particle Physics”. Rev. Mod. Phys. 38 (2): 215, (1966).

[33] S. J. Gates, Jr., S.-N. Mak, and X. Xiao, work in progress.

[34] S.J. Gates, Jr. J. Gonzales, B. MacGregor, J. Parker, R. Polo-Sherk, V.G.J. Rodgers and L. Wassink, “4D, N = 1 Supersymmetry Genomics (I),” JHEP 0912, 008 (2009), e-Print: arXiv:0902.3830 [hep-th].

[35] P. Fayet, “Fermi-Bose Hypersymmetry,” Nucl. Phys. B113 (1976) 135.

[36] J. Wess, Lectures given at the Bonn Summer School 1974.

[37] P. S. Howe, K. S. Stelle, and P. K. Townsend, “The Relaxed Hypermultiplet: An Unconstrained N=2 Superfield Theory,” Nucl. Phys. B214 519 (1983).

[38] E. S. Fradkin, and M. A. Vasiliev, “Minimal Set Of Auxiliary Fields In So(2) Extended Supergravity,” Phys. Lett. 85B (1979) 47-51, DOI: 10.1016/0370-2693(79)90774-3.

[39] E. S. Fradkin, and M. A. Vasiliev, “Minimal set of auxiliary fields and s-matrix for extended supergravity,” Lett. Al Nuovo Cimento, No. 25 79, (1979)/

[40] B. de Wit, and van Holten, “Multiplets of linearized SO(2) supergravity,” Nucl. Phys. B155 530, (1979).

[41] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hubsch, K. M. Iga, and G. D. Landweber, “On the Matter of N = 2 Matter,” Phys. Lett. B659 441 (2008), e-Print: arXiv:0710.5245 [hep-th].

[42] M. G. Faux, “The Conformal Hyperplet,” SUNY - Oneonta preprint # SUNY-O/1601, 30 June 2018, https://arxiv.org/abs/1610.07822, unpublished.

[43] S. Kuzenko, and A. G. Sibiryakov, “Massless gauge superfields of higher integer superspins,” JETP Lett. 57 (1993) 539-542, Pisma Zh. Eksp. Teor. Fiz. 57 (1993) 526-529.

[44] S. Kuzenko, and A. G. Sibiryakov, V. V. Postnikov, “Massless gauge superfields of higher half integer superspins,” JETP Lett. 57 (1993) 534-538, Pisma Zh. Eksp. Teor. Fiz. 57 (1993) 521-525.

[45] S. J. Gates, Jr., and V. A. Kostelecky, “Supersymmetric Matter Gravitino Multiplets,” Nucl. Phys. B248 570 (1984), DOI: 10.1016/0550-3213(84)90612-6.

[46] S. J. Gates, Jr., and K. Stiffler, “Adinkra Color Confinement In Exemplary Off-Shell Constructions Of 4D, N = 2 Supersymmetry Representations,” JHEP 1407 (2014) 051, e-Print: arXiv:1405.0048 [hep-th].

[47] I. Chappell, II, S. J. Gates, Jr, and T. Hübsch, “Adinkra (In)Equivalence From Coxeter Group Representations: A Case Study,” Int. J. Mod. Phys’ A29 (2014) 06, 1450029 e-Print: arXiv:1210.0478 [hep-th].
[48] S. J. Gates, Jr., F. Guyton, S. Harmalkar, D. S. Kessler, V. Korotkikh, and V. A. Meszaros, “Adinkras From Ordered Quartets of BC4 Coxeter Group Elements and Regarding 1,358,954,496 Matrix Elements of the Gadget,” JHEP 1706 (2017) 006 DOI: 10.1007/JHEP06(2017)006, e-Print arXiv:1701.00304v5 [hep-th].

[49] S. J. Gates, Jr., L. Kang, D. S. Kessler, and V. Korotkikh, “Adinkras from ordered quartets of BC4 Coxeter group elements and regarding another Gadgets 1,358,954,496 matrix elements,” Int. J. Mod. Phys. A33 (2018) no.12, 1850066, DOI: 10.1142/S0217751X18500665, e-Print: arXiv:1802.02890 [hep-th].

[50] S. J. Gates, Jr., W. D. Linch, III, J. Phillips, “When Superspace Is Not Enough,” Univ. of Md Preprint # UMDEPP-02-054, Caltech Preprint # CALT-68-2387, arXiv [hep-th:0211034], unpublished.

[51] N. Cabibbo, “Unitary Symmetry and Leptonic Decays”. Phys. Rev. Lett. 10 (12): 531 (1963), DOI:10.1103/PhysRevLett.10.531.

[52] M. Kobayashi, T. Maskawa. “CP Violation in the Renormalizable Theory of Weak Interaction,” Prog. Theor. Phys. 49 652 (1973), DOI: 10.1143/PTP.49.652.