It is known that the bounded Geodesic Length Problem in free metabelian groups is NP-complete. We construct a 2-approximation polynomial time deterministic algorithm for the Geodesic Problem. We show that the Geodesic Problem in the restricted wreath product of a finitely generated non-trivial group with a finitely generated abelian group containing $\mathbb{Z}_2$ is NP-hard and there exists a Polynomial Time Approximation Scheme for this problem. We also show that the Geodesic Problem in the restricted wreath product of two finitely generated non-trivial abelian groups is NP-hard if and only if the second abelian group contains $\mathbb{Z}_2$.

Keywords: Keyword1; keyword2; keyword3.

1. Introduction

Studying algorithmic problems in group theory could be traced back to Max Dehn. In 1911, Dehn highlighted the word problem alongside two other problems: the Conjugacy Problem and the Group Isomorphism Problem. Since then, studying the algorithmic aspects of different problems has been an active field in combinatorial and geometric group theory. Algorithmic results on solvable groups were for a long
time the real jems of combinatorial group theory, revealing interesting relations with computational commutative algebra and number theory. Researchers started looking again at the general classes of solvable groups, but this time from asymptotic, geometric, and computational perspectives.

In a recent work Myasnikov et al. [8] consider the following three algorithmic problems related to Geodesic Problem in free solvable groups.

**The Geodesic Problem (GP):** Let \( X = \{x_1, \ldots, x_m\} \) be a set of generators for a finitely generated group \( G \) and \( F(X) \) be the free group generated by \( X \). Let \( \mu : F(x) \to G \) be the canonical epimorphism. For a word \( w \in F(X) \) represented in the alphabet \( X^{\pm1} \) we denote by \( |w| \) the length of \( w \) (number of generators used in a reduced presentation of \( w \)). We define the Geodesic Length \( l_X(g) \) of \( g \in G \) with respect to the set of generators \( X \) as follows

\[
l_X(g) := \min\{|w| \mid w \in F(X), \mu(w) = g\}
\]

A word \( w \in F(x) \) is called a geodesic word if \( l_X(\mu(w)) = |w| \). The Geodesic Problem relative to \( X \) is the following problem: given a word \( g \in G \) find a geodesic word \( w \in F(X) \) such that \( g = \mu(w) \) in \( G \).

**The Geodesic Length Problem (GLP):** Let \( G \) be a finitely generated group and \( X \) be a set of generators of \( G \). Given a word \( w \in F(X) \) find \( l_X(\mu(w)) \).

**Bounded Geodesic Length Problem (BGLP):** Let \( G \) be a finitely generated group with a set of generators \( X \). Given a word \( w \in F(X) \) and \( k \in \mathbb{N} \) determine if \( l_X(\mu(w)) \leq k \).

The hardness of BGLP, GLP and GP in a given group \( G \) is discussed in [8]. It is shown that each problem in the list is polynomial time Turing reducible to the next one. In the same paper \( NP \)-completeness of BGLP in free metabelian groups of arbitrary rank is proved.

We need to remind some definitions from Complexity Theory.

**Polynomial-time Reducibility** Problem \( P \) is Polynomial-time reducible to problem \( Q \) if whenever \( P \) has a solution, such a solution can be obtained from a solution of \( Q \) in polynomial time.

**Class NP** Class \( NP \) is the class of all decision problems for which the "yes"-instances are recognizable in polynomial time by a non-deterministic Turing machine.

**NP-complete** A problem \( P \) is \( NP \)-complete if it is in class \( NP \) and any other problem in class \( NP \) is reducible to \( P \) in polynomial-time. In other words, \( P \) is at least as hard as any other problem in the class \( NP \).

**NP-hard** A problem \( P \) is \( NP \)-hard if any \( NP \)-complete problem is polynomial time Turing reducible to \( P \).

In a more recent paper [4] Elder and Rechnitzer show that for a finitely generated group \( G \) these three problems are equivalent, i.e., they are reducible to each other in polynomial time.

Motivated by some applications in group-based cryptography (see [9]), we con-
struct a 2-approximation polynomial time deterministic algorithm for the Geodesic Problem in free metabelian groups. (Theorem 3.1). To the best of our knowledge, this is the first time that the idea of finding a good approximation to the optimal solution of computationally hard problem is used in infinite group theory. We show that the Geodesic Problem in the restricted wreath product of a finitely generated group with a finitely generated abelian group containing \( \mathbb{Z}^2 \) is \( NP \)-hard and there exists a Polynomial Time Approximation Scheme for this problem. These approximation algorithms have an important role in solving the Conjugacy Problem in the so-called length-based attack in group-based cryptography (\cite{9}). We also show that the Geodesic Problem in the restricted wreath product of two finitely generated non-trivial abelian groups is \( NP \)-hard if and only if the second group contains \( \mathbb{Z}^2 \) (Theorems 2.3 and 2.4). This answers a question from \cite{8}.

We wish to thank A. Myasnikov and B. Shepherd for making many very helpful suggestions. We also thank L. Babai for showing that the TSP in virtually cyclic groups is in P, and S. Arora for giving a hint how to extend his polynomial time approximation schemes for the construction of Euclidean Traveling Salesman Tour to the construction of Traveling Salesman Path.

2. Geodesic Problem In Restricted Wreath Product

The results in this section are obtained by combining known results from group theory and computer science. Theorems 2.3, 2.4 and 2.6 are not hard to prove but the statements are interesting since they introduce the idea of Polynomial time Approximation Algorithm into infinite group theory. Furthermore, they answer open questions and have never been formulated elsewhere.

Let us first agree about notations. Let \( A \) and \( B \) be groups. Let

\[
D := \{ f : B \to A ||\text{supp}(f)|| < \infty \}.
\]

For \( b \in B \) we define the following action on \( D \):

\[
\hat{b} : D \to D
\]

\[
f(.) \mapsto f(b^{-1}.)
\]

We have an embedding \( B \hookrightarrow \text{Aut}(D) \) by identifying \( b \) with \( \hat{b} \).

The restricted wreath product \( G = A \wr B \) is the restricted semidirect product \( D \times B \) consisting of all pairs \( (f, b) \), \( f \in D \) and \( b \in B \) with the operation

\[
(f, b)(f', b') := (fb(f'), bb').
\]

For any \( a \in A \), let \( f_a \in D \) be the function that maps identity to \( a \) and everything else to identity. The map \( a \mapsto (f_a, 1) \) gives the inclusion \( A \hookrightarrow G \). We identify \( a \in A \) with \((f_a, 1) \in G \). We also have the natural inclusion of \( B \) in \( G \) by \( b \mapsto (1, b) \).
We denote by $f^b = b^{-1}fb$ the conjugation by elements of $B$. Then

$$f_a^b = b^{-1}fa = (1, b^{-1})(f_a, 1)(1, b) = (b^{-1}(f_a), b^{-1})(1, b) = (b^{-1}(f_a), 1).$$

(1)

Thus, $D \vartriangleleft G$. We also notice that $b^{-1}(f_a)$ is the function that maps $b^{-1}$ to $a$ and everything else to 1.

Now let $A$ be a finitely generated group and $B$ be a finitely generated abelian group. Let $S_A = \{a_1, \ldots, a_t\}$ and $S_B = \{b_1, \ldots, b_r\}$ be the sets of generators of $A$ and $B$ respectively. Then $S_G = S_A \cup S_B$ generates $G$ (see [10]).

Let $F = F(S_G)$ and $g \in F$. $g$ can be expressed as follows

$$g = b_1^{r_1} \cdots b_r^{r_r} c_1 b_1^{s_1} \cdots b_r^{s_r} c_2 \cdots b_1^{s_1} \cdots b_r^{s_r} c_k b_1^{s_1} \cdots b_r^{s_r},$$

where $c_1, \ldots, c_k \in A$.

We note that $b_1^{r_1} \cdots b_r^{r_r}$ and $b_1^{s_1} \cdots b_r^{s_r}$ might be identity.

Let $h_i = b_1^{r_i} \cdots b_r^{r_i}$ for $1 \leq i \leq k$ and $h' = b_1^{s_1} \cdots b_r^{s_r}$. Then, $g = h_1 \cdots h_k c_k h'$.

We can rewrite $g$ as follows (using the fact that $B$ is abelian)

$$g = (h_1 c_1 h_1^{-1})(h_1 h_2 c_2 h_2^{-1} h_1^{-1})(h_1 h_2 h_3 c_3 h_3^{-1} h_2^{-1} h_1^{-1}) \cdots (h_1 h_2 \cdots h_k c_k h_k^{-1} \cdots h_2^{-1} h_1^{-1}) h_1 \cdots h_k h'$$

$$= c_1 h_1^{-1} c_2 h_2^{-1} \cdots c_k h_k^{-1} h' h_1 \cdots h_k.$$  

(3)

Let $w_i = h_1^{-1} \cdots h_{i-1}^{-1}$ for $1 \leq i \leq k$ and $b = h' h_1 \cdots h_k$. Then, $g = c_1^{w_1} \cdots c_k^{w_k} b$.

Since $c_i^{w_i} \in A$ is the function that maps $w_i^{-1}$ to $c_i$ and any other element of $B$ to the identity, if $w_i \neq w_j$, $c_i^{w_i} c_j^{w_j} = c_j^{w_j} c_i^{w_i}$. On the other hand, we can assume that $w_i \neq w_j$ for $i \neq j$. To see this suppose that there exist $w_i$ and $w_j$ such that $w_i = w_j$ and $i < j$. Also assume that $j$ is the smallest index satisfying this property. Then, we can flip $c_i^{w_i}$ over $c_i^{w_{i+1}}, \ldots, c_{j-1}^{w_{j-1}}$:

$$g = c_1^{w_1} \cdots c_i^{w_{i+1}} c_j^{w_j} \cdots c_k^{w_k} b$$

$$= c_1^{w_1} \cdots c_i^{w_{i+1}} c_j^{w_j} \cdots c_k^{w_k} b$$

$$\vdots$$

$$= c_1^{w_1} \cdots c_i^{w_{i+1}} c_j^{w_j} \cdots c_k^{w_k} b$$

$$= c_1^{w_1} \cdots c_i^{w_{i+1}} (c_i c_j)^{w_j} \cdots c_k^{w_k} b.$$  

(4)

since we are looking for the minimal number of the generators needed to present $g$ and $|c_i c_j|_A \leq |c_i|_A + |c_j|_A$, we don’t lose anything in terms of the minimality of $|c_i|_A$. Therefore, we can assume that $w_i \neq w_j$, for all $i \neq j$.

Now, we consider the Geodesic Problem for $g$ with respect to the set of generators $S_G$. Suppose that we know how to solve the Geodesic Problem in $A$ with respect to $S_A$, i.e. we know the Geodesic Length of $c_i$ with respect to $S_A$ denoted by $|c_i|_A$.  

4 O. Kharchenko, A. Mohajeri
Approximation of Geodesics in Metabelian Groups

5

For \( g = b_{i_1}^{n_1} \cdots b_{i_r}^{n_r} c_1 b_{j_1}^{n_{j_1}} \cdots b_{j_m}^{n_{j_m}} c_2 \cdots b_{r}^{n_r} c_k b_{1}^{n_1} \cdots b_{m}^{n_m} \) we denote by \( l \) the following sum:

\[
    l = \sum_{i=1}^{r} n_{i_1} + \cdots + \sum_{i=1}^{r} n_{i_r} + \sum_{i=1}^{k} n_{i} + \sum_{i=1}^{k} |c_i|_A. \tag{5}
\]

We want to relate this sum to a path in the Cayley graph of \( B \). For the moment let assume that \( B \) is free and consider \( C = \mathbb{Z}^r \) viewed as the Cayley graph of \( B \) with respect to the set of generators \( S_B \). Let \( v_0 \) be the vertex of \( C \) corresponding to identity. Let \( x = b_{i_1}^{r_1} \cdots b_{i_r}^{r_r} \) and \( y = b_{i_1}^{r_1} \cdots b_{i_r}^{r_r} \) be two vertices in \( C \). The Manhattan distance between \( x \) and \( y \) is defined as

\[
    d(x, y) = \sum_{i=1}^{r} |x_i - y_i|. \tag{6}
\]

In other words, \( d(\ldots) \) is the restriction of the distance induced by \( L_1 \)-norm on \( \mathbb{R}^r \) to \( \mathbb{Z}^r \).

Definition 1. Let \( S = \{w_1, \ldots, w_m, b\} \) be a set of elements of \( B \) such that \( w_i \neq w_j \) for \( i \neq j \). We say \( l \) is a walk on \( S \) with end point \( b \) if \( l \) is a path of the form \( v_0, w_{i_1}, \ldots, w_{i_m}, b \), where \( w_{i_1}, \ldots, w_{i_m} \) is a permutation of \( w_1, \ldots, w_m \). We call such \( l \) minimum if it has the minimum length with respect to \( d(\ldots) \) among all such walks.

Lemma 2. \([10]\) Let \( g = a_{w_{i_1}}^{a_{w_{i_2}}} \cdots a_{w_{i_k}}^{b} \in F \) where \( a_{w_{i_1}}, \ldots, a_{w_{i_k}} \in A, w_1, \ldots, w_k \in B \) and \( w_i \) are all different. Any geodesic word \( g' \) such that \( g' = \mu(g) \) corresponds to a minimum walk in \( C \) on \( S = \{w_1, \ldots, w_m, b\} \) with end point \( b \).

Corollary 3. If we know how to solve Geodesic Problem in \( A \), solving Geodesic Problem in \( G = A \wr B \) is equivalent to finding a minimum walk on a finite subgraph of the Cayley graph of \( B \).

The Traveling Salesman Path Problem in \( \mathbb{Z}^r \) (TSPP) \([5]\) : Given a set of points \( \{s, w_1, \ldots, w_n, t\} \subset \mathbb{Z}^r \) with two distinguished points \( s, t \) and a metric on \( \mathbb{Z}^r \), find a minimum path of the form \( s, w_{i_1}, \ldots, w_{i_n}, t \) where \( w_{i_1}, \ldots, w_{i_n} \) is a permutation of \( w_1, \ldots, w_n \).

It has been shown that the Traveling Salesman Problem in \( \mathbb{Z}^r \) with Manhattan distance is \( NP \)-hard for \( n \geq 2 \) (see \([5]\) ). From that, one can easily conclude the \( NP \)-hardness of TSPP in \( \mathbb{Z}^2 \).

Theorem 4. Geodesic Problem in \( G = A \wr B \) is \( NP \)-hard where \( A \) is a finitely generated non-trivial group and \( B \) is a finitely generated abelian group containing \( \mathbb{Z}^2 \).

Proof. Without loss of generality we can assume that \( B \) is free. In order to show \( NP \)-hardness of the Geodesic Problem in \( G \), we will reduce the problem of finding a minimum walk on a finite set of \( C = \mathbb{Z}^r \), viewed as the Cayley graph of \( B \), to the Geodesic Problem in \( G \). Let \( v_0, w_1, \ldots, w_k, b \) be \( k + 2 \) different vertices in \( C \) (we
can always assume that \( v_0 \) is one the points). We want to find a minimum walk on \( S = \{ w_1, \ldots, w_k, b \} \) with endpoint \( b \). Now consider the following Geodesic Problem in \( G \). Fix \( a \in S_A \) and Consider \( g = a^{w_1} a^{w_2} \cdots a^{w_k} b \). Assume that we know how to solve the Geodesic Problem for \( g \) Thus, we can find a \textit{geodesic word} \( g' \) representing \( g \). By lemma \( 2 \) \( g \) corresponds to a minimum walk \( v_0, w_{i_1}, w_{i_2}, \ldots, w_{i_k}, b \) on \( S \). Thus, if we know how to solve Geodesic Problem in \( G \), we will be able to solve the minimum walk problem in \( C \) in polynomial time which is a contradiction. This proves the \( NP \)-hardness of the Geodesic Problem in \( G \).

**Remark 5.** The Bounded Geodesic Length Problem (BGLP) is the decision version of the Geodesic Problem. An immediate consequence of the previous theorem is that if \( BGLP \) for \( A \) is in class \( NP \) then it is \( NP \)-complete for \( G \).

**Theorem 6.** If Geodesic Problem in \( A \) is polynomial and \( B = \mathbb{Z} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_t} \), then Geodesic Problem in \( G = A \wr B \) is polynomial.

**Proof.** A polynomial time algorithm to solve TSP in the Cayley graph of \( B = \mathbb{Z} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_t} \) with the degree of the polynomial equal to \( k_1 \cdots k_t \) can be constructed as in \[11\]. The statement of the theorem follows now from Lemma 3.1. There exists even better algorithm for TSP in \( B \) with the degree of the polynomial independent of \( k_1 \cdots k_t \) \[4\].

Theorems 2.3 and 2.4 imply the following result that answers an open question from \[8\].

**Corollary 7.** Let \( G = A \wr B \) where \( A, B \) are finitely generated non-trivial abelian groups. Then the Geodesic Problem is \( NP \)-hard in \( G \) if and only if \( B \) contains \( \mathbb{Z}^2 \).

We proved that Geodesic Problem in \( G = A \wr B \) is \( NP \)-hard if \( B \) contains \( \mathbb{Z}^2 \). So trying to get an approximative solution would be natural as the next step. Indeed, for applications a solution that is close to an optimal one very often is as good as the optimal solution.

**Approximation Algorithm.** Let denote by \( OPT \) the optimal solution of a minimization problem \( P \). An algorithm \( A \) approximates problem \( P \) within a factor of \( \rho \geq 1 \) if \( f(A)/OPT \leq \rho \), where \( f(A) \) is the solution given by \( A \).

**Polynomial-Time Approximation Scheme.** A PTAS or Polynomial-Time Approximation Scheme for a problem is a family of polynomial-time algorithms such that for any given constant \( c \), there is an algorithm in the family that approximates the problem within a factor of \( (1 + 1/c) \). The running time of the algorithm might depend on \( c \), but for each fixed \( c \), it is polynomial in the size of the input.

We state the main result of this section in the next theorem.

**Theorem 8.** Assuming that the Geodesic Problem in \( A \) is solvable in polynomial time, there exists a PTAS for the Geodesic Problem in \( G \).
Approximation of Geodesics in Metabelian Groups

In order to prove the theorem we need the following result.

**Proposition 9.** [3] There exists a PTAS for TSP in $R^d$ with Euclidean norm which generalizes to other $L^p$ norms for $p \geq 1$. A randomized version of the algorithm gives an approximation within a factor of $(1 + 1/c)$ of the optimal tour in $O(n \log n)^{(C/2)^{d-1}}$. If we derandomize the algorithm, we multiply the running time by $O(n^d)$.

The same is true if instead of a TSP tour we consider a TSP path [2].

The theorem is a consequence of corollary 3 and the previous proposition. Let $|S_B| = r$ and $g = a_1^{w_1} \cdots a_k^{w_k}b$. We need to solve the Traveling Salesman Problem in the subgraph of $C$ induced by $\{v_0, w_1, \ldots, w_k, b\}$, we consider C with Manhattan distance. The PTAS finds a word $g' = a_{i_1}^{w_{i_1}} \cdots a_{i_k}^{w_{i_k}}b$ such that

$$\sum_{j=1}^{k} d(w_{i_j}, w_{i_{j-1}}) + \sum_{j=1}^{k} |a_{i_j}| \leq (1 + 1/c)|S_B, S_A(g)|,$$

where $l_S, S_B(g)$ is the Geodesic Length of $g$ with respect to $S_B$ and $S_A$. If we use a randomized version of the algorithm the total running time is

$$O(k(\log k)^{(C/2)^{d-1}}) + \text{(time needed to find geodesic words representing } a_i)$$

3. Geodesic Problem In Free Metabelian groups

Let $F = F(X)$ be a free group of rank $r$. Denote by $F' = [F, F]$ the derived subgroup of $F$ and by $F'' = [F', F']$ the second derived subgroup of $F$. Let $G = F / F'$ and $H = F / F''$. $G$ is a free abelian group of rank $r$ and $H$ is a free metabelian group of rank $r$.

In [8] the authors show that $BGLP$ is $NP$-complete for free metabelian groups of arbitrary rank. Since $BGLP$ is polytime reducible to the $GLP$, $GLP$ is $NP$-hard in free metabelian groups. In this section we show that there is a 2-approximation for the Geodesic Problem in $H$. First, we need to remind some definitions.

**Flow.** Let $\gamma = (V, E)$ be a directed graph with two distinguished vertices: a source $s$ and a sink $t$. A flow on $\gamma$ is a function $f : E \rightarrow \mathbb{R}$ such that

$$\sum_{e : o(e) = v} f(e) - \sum_{e : t(e) = v} f(e) = 0 \quad \forall v \in V - \{s, t\}. \quad (7)$$

The number $f^*(v) := \sum_{e : o(e) = v} f(e) - \sum_{e : t(e) = v} f(e)$ is called the net flow at $v$. The condition above is the same as $f^*(v) = 0$ for $v \neq s, t$ and is usually referred to as Kirchhoff law. We call a flow $f$ circulation if $f^*(v) = 0$ also for $s$ and $t$. In this discussion we only consider $\mathbb{Z}$-flows, i.e., $f : E \rightarrow \mathbb{Z}$.

Let $p$ be a path in $\gamma$ from $v_1$ to $v_2$ and let $\pi_p : E \rightarrow \mathbb{Z}$ be such that $\pi_p(e)$ is equal to the number of times that $p$ passes through $e$ counted $-1$ when $p$ takes $e$ backward. $\pi_p$ satisfies the flow condition for $v_1$ as source and $v_2$ as sink. Thus, any path in $\gamma$ induces a $\mathbb{Z}$-flow on $\gamma$.

Let $w \in F$ and let $p_w$ be the path labeled by $w$ in $\gamma$ Cayley graph of $G$. We denote by $\pi_w$ the flow induced by $p_w$ in $\gamma$. 
Let $\mu' : F(X) \to H$ be the canonical epimorphism. We identify $X$ with its image under $\mu'(\mu')$ is one to one on $X$). An expression of $l_X(\mu'(w))$ for an arbitrary word $w \in F$ has been given in [8]. Before presenting our result we need to remind the definition of two problems.

**Minimum Steiner Tree Problem:** Given a graph $\gamma = (V, E)$ and a subset $V_1 \subset V$, find a minimum subgraph (subtree) such that covers all vertices in $V_1$.

**Minimum group Steiner Tree Problem:** Given connected components $C_i$ of $\gamma$, find a minimum subgraph that makes the subgraph $\cup C_i$ connected.

Let $w \in F$ and $\pi_w$ be the flow induced by $w$ in $\gamma$. We define $\text{supp}(\pi_w) = \{e \in E | \pi(e) \neq 0\}$. A minimum Group Steiner Tree for $w$ is a minimal Group Steiner Tree for the connected components of the subgraph induced by $\text{supp}(\pi_w)$ in $\gamma$.

**Proposition 10.** [8] Let $\gamma$ be the cayley graph of $G$ with respect to the set of generators $X$. then for $w \in F$ we have

$$l_X(\mu'(w)) = \sum_{e \in \text{supp}(\pi_w)} \pi_w(e) + 2|E(Q)|,$$

where $\pi_w$ is the path induced by $w$ in $\gamma$ and $Q$ is a minimum group steiner tree for $w$ in $\gamma$.

In the expression for $l_X(\mu'(w))$ we can evaluate $\sum_{e \in \text{supp}(\pi_w)} \pi_w(e)$ in polynomial time. The time-consuming part is finding $Q$. Unfortunately, there has been no PTAS known so far for the Group Steiner Tree problem. Nevertheless, in the same paper a similar statement to the PTAS result for Euclidean Traveling Salesman Problem is proved for Minimum Steiner Tree Problem.

**Proposition 11.** [8] There exists a PTAS for the Euclidean Minimum Steiner Tree in $R^d$ which generalizes to other $L^p$ norms for $p \geq 1$. A randomized version of the algorithm gives an approximation within a factor of $(1 + 1/c)$ of the optimal tree in $O(n(\log n)(\sqrt{d}c)^{d-1})$. If we derandomize the algorithm, we multiply the running time by $O(n^d).

**Theorem 12.** There is a 2-approximation algorithm for the Geodesic Problem in the free metabelian group of arbitrary rank $r$.

**Proof.** Let $C_i$ be the connected components of the subgraph induced by $\text{supp}(\pi_w)$ in $\gamma$. We choose an arbitrary point $y_i$ in each $C_i$. Let $Q_1$ be the $(1 + 1/c)$-approximation of the Minimum Steiner Tree Problem for $\{y_i\}$. So $|Q_1|/(1 + 1/c)$ is the optimal value for this problem. We denote by $Q^*$ the optimal solution for the Minimum Group Steiner Tree on $C_i$. $Q^* \cup (\cup C_i)$ is a connected subgraph of $\gamma$ containing $\{y_i\}$ (see Fig. 1). Thus

$$\frac{|Q_1|}{(1 + 1/c)} \leq |Q^* \cup (\cup C_i)| = |Q^*| + \sum |C_i|.$$
Approximation of Geodesics in Metabelian Groups

If we use $Q_1$ instead of $Q^*$ in formula (8), we get an approximation $l'(w)$ for $l(w)$ such that

$$
\frac{l'(w) - l(w)}{l(w)} = \frac{\left(\sum_{e \in \text{supp}(\pi_w)} \pi_w(e) + 2 |Q_1|\right) - \left(\sum_{e \in \text{supp}(\pi_w)} \pi_w(e) + 2 |Q^*|\right)}{\sum_{e \in \text{supp}(\pi_w)} \pi_w(e) + 2 |Q^*|} \\
= \frac{2(|Q_1| - |Q^*|)}{\sum_i |C_i| + 2 |Q^*|} \\
\leq \frac{2(1/c|Q^*| + (1 + 1/c) \sum_i |C_i|)}{\sum_i |C_i| + 2 |Q^*|}.
$$

(11)

If we choose $c$ such that $1/c < \frac{1}{\sum_i |C_i|}$, then

$$
1/c|Q^*| + 1/c \sum_i |C_i| < 1/c|Q^*| + 1 \\
\leq 2|Q^*|.
$$

(12)
The last inequality is satisfied since $|Q^*| > 0$, otherwise there is nothing to prove. This implies

$$\frac{1/c|Q^*| + (1 + 1/c) \sum |C_i|}{\sum |C_i| + 2|Q^*|} < 1.$$  

(13)

So

$$\frac{l'(w) - l(w)}{l(w)} \leq \frac{2(1/c|Q^*| + (1 + 1/c) \sum |C_i|)}{\sum |C_i| + 2|Q^*|} < 2.$$  

(14)

which proves that $l'(w)$ is a 2-approximation of $l(w)$. 

References

[1] S. Arora, Polynomial time approximation schemes for Euclidean TSP and other geometric problems, *37th Annual Symposium on Foundations of Computer Science* (Burlington, VT, 1996), 2-11, IEEE Comput. Soc. Press, Los Alamitos, CA, 1996.

[2] S. Arora, Private communication.

[3] L. Babai, Private communication.

[4] M. Elder, Some Geodesic Problems in groups, *Groups, Complexity, Cryptology*. 2 (2010) Issue 2, 223-229.

[5] M. R. Garey, R. L. Graham, D. S. Johnson, Some NP-complete geometric problems, *Eighth Annual ACM Symposium on Theory of Computing* (Hershey, Pa., 1976), pp. 102-22. Assoc. Comput. Mach., New York, 1976.

[6] R. Karp, Reducibility among combinatorial problems, *Complexity of computer computations* (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972), pp. 85-103. Plenum, New York, 1972.

[7] Dehn, Max (1911), Ber unendlich diskontinuierliche Gruppen, *Mathematische Annalen* 71 (1): 116-144, doi:10.1007/BF01456932, MR1511645, ISSN 0025-5831.

[8] A. Myasnikov, V. Roman’kov, A. Ushakov, and A. Vershik, The word and Geodesic Problems in free solvable groups, *Trans. Amer. Math. Soc.* 362 (2010), no. 9, 4655-4682.

[9] A. Myasnikov, V. Shpilrain, A. Ushakov, Group-based Cryptography, in *Advanced Courses in Mathematics*. CRM Barcelona. Birkhäuser Verlag, Basel (2008).

[10] W. Parry, Growth series of some wreath products, *Trans. Amer. Math. Soc.* 331 (1992), no. 2, 751-759.

[11] G. Rothe, (1988), Two solvable cases of the Traveling Salesman Problem. Thesis (Dr.Tech.)Technische Universitaet Graz (Austria). 55 pp, ProQuest LLC.