Some Compact Logics - Results in ZFC

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Dedicated to the memory of Alan by his friend, Saharon

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1 Preliminaries

While first order logic has many nice properties it lacks expressive power. On the other hand second order logic is so strong that it fails to have nice model theoretic properties such as compactness. It is desirable to find natural logics which are stronger than the first order logic but which still satisfy the compactness theorem. Particularly attractive are those logics which allow quantification over natural algebraic objects. One of the most natural choices is to quantify over automorphisms of a structure (or isomomorphisms between substructures). Generally compactness fails badly [16], but if we restrict ourself to certain concrete classes then we may be able to retain compactness. In this paper we will show that if we enrich first order logic by allowing quantification over isomorphisms between definable ordered fields the resulting logic, $L(Q_{Of})$, is fully compact. In this logic, we can give standard compactness proofs of various results. For example, to prove that there exists arbitrarily large rigid real closed fields, fix a cardinal $\kappa$ and form the $L(Q_{Of})$ theory in the language of ordered fields together with $\kappa$ constants which says that the constants are pairwise distinct and the field is a real closed field which is rigid. (To say the field is rigid we use the expressive power of $L(Q_{Of})$ to say that any automorphism is the identity.) This theory is consistent as the reals can be expanded to form a model of any finite subset of the theory. But a model of the theory must have cardinality at least $\kappa$. (Since we do not have the downward Löwenheim-Skolem theorem, we cannot assert that there is a model of cardinality $\kappa$.)

In [10] and [8], the compactness of two interesting logics is established under certain set-theoretic hypotheses. The logics are those obtained from first order logic by adding quantifiers which range over automorphisms of definable Boolean algebras or which range over automorphisms of definable ordered fields. Instead of the weaker version of dealing with automorphisms, it is also possible to deal with a quantifier which says that two Boolean algebras are isomorphic or that two ordered fields are isomorphic. The key step in proving these results lies in establishing the following theorems. (By definable we shall mean definable with parameters).

**Theorem 1.1** Suppose $\lambda$ is a regular cardinal and both $\diamondsuit(\lambda)$ and $\diamondsuit(\{\alpha < \lambda^+: \text{cf}\alpha = \lambda\})$ hold. Then if $T$ is any consistent theory and $|T| < \lambda$, there is a model $M$ of $T$ of cardinality $\lambda^+$ with the following properties:
(i) If $B$ is a Boolean algebra definable in $M$, then every automorphism of $B$ is definable.

(ii) $M$ is $\lambda$-saturated.

(iii) Every non-algebraic type of cardinality $<\lambda$ is realized in $M$ by $\lambda^+$ elements.

Theorem 1.2 Suppose $\lambda$ is a regular cardinal and both $\diamond(\lambda)$ and $\diamond(\{\alpha < \lambda^+: \text{cf} \alpha = \lambda\})$ hold. Then if $T$ is any consistent theory and $|T| < \lambda$, there is a model $M$ of $T$ of cardinality $\lambda^+$ with the following properties:

(i) If $F$ is an ordered field definable in $M$ then every automorphism of $F$ is definable and every isomorphism between definable ordered fields is definable.

(ii) $M$ is $\lambda$-saturated.

(iii) Every non-algebraic type of cardinality $<\lambda$ is realized in $M$ by $\lambda^+$ elements.

(iv) Every definable dense linear order is not the union of $\lambda$ nowhere dense sets.

These theorems are proved in [10] (in [10] section 9, Theorem 1.1 is proved from GCH) although there is not an explicit statement of them there. In order to show the desired compactness result (from the assumption that there are unboundedly many cardinals $\lambda$ as in the theorem statements) it is enough to use (i). However in our work on Boolean algebras we will need the more exact information above. Let us notice how the compactness of the various languages follow from these results. Since the idea is the same in all cases just consider the case of Boolean algebras.

First we will describe the logic $L(Q_{Ba})$. We add second order variables (to range over automorphisms of Boolean algebras) and a quantifier $Q_{Ba}$ whose intended interpretation is that there is an automorphism of the Boolean algebra. More formally if $\Theta(f)$, $\phi(x)$, $\psi(x, y)$, $\rho(x, y)$ are formulas (where $f$ is a second order variable, $x$ and $y$ are first order variables and the formulas may have other variables) then

$$Q_{Ba} f (\phi(x), \psi(x, y)) \Theta(f)$$
is a formula. In a model $M$ the tuple $(\phi(x), \psi(x,y))$ defines a Boolean algebra (where parameters from $M$ replace the hidden free variables of $\phi(x), \psi(x,y)$) if $\psi(x,y)$ defines a partial order $<$ on $B = \{ a \in M : M \models \phi[a] \}$ so that $(B; <)$ is a Boolean algebra. A model $M$ satisfies the formula $Q_{Ba} f (\phi(x), \psi(x,y)) \Theta(f)$ (where parameters from $M$ have been substituted for the free variables), if whenever $(\phi(x), \psi(x,y))$ define a Boolean algebra $(B; <)$ then there is an automorphism $f$ of $(B; <)$ such that $M \models \Theta(f)$. (It is easy to extend the treatment to look at Boolean algebras which are definable on equivalence classes, but we will avoid the extra complication.) We can give a more colloquial description of the quantifier $Q_{Ba}$ by saying the interpretation of $Q_{Ba}$ is that “$Q_{Ba} f(B) \ldots$” holds if there is an automorphism of the Boolean algebra $B$ so that $\ldots$. We will describe some of the other logics we deal with in this looser manner. For example, we will want to consider the quantifier $Q_{Of}$ where $Q_{Of} f(F_1, F_2) \ldots$ holds if there is is an isomorphism $f$ from the ordered field $F_1$ to the ordered field $F_2$ such that $\ldots$.

The proof of compactness for $L(Q_{Ba})$ follows easily from theorem 1.1. By expanding the language we can assume that there is a ternary relation $R(\ast, \ast, \ast)$ so that the theory says that any first order definable function is definable by $R$ and one parameter. By the ordinary compactness theorem if we are given a consistent theory in this logic then there is a model of the theory where all the sentences of the theory hold if we replace automorphisms by definable automorphisms in the interpretation of $Q_{Ba}$, since quantification over definable automorphisms can be replaced by first order quantification. Then we can apply the theorem to get a new model elementarily equivalent to the one given by the compactness theorem in which definable automorphisms and automorphisms are the same.

In the following we will make two assumptions about all our theories. First that all definable partial functions are in fact defined by a fixed formula (by varying the parameters). Second we will always assume that the language is countable except for the constant symbols.

In this paper we will attempt to get compactness results without recourse to $\diamondsuit$, i.e., all our results will be in ZFC. We will get the full result

\footnote{More exactly, for every model $M$ define a model $M^*$ with universe $M \cup \{ f : f$ a partial function from $|M|$ to $|M| \}$ with the relations of $M$ and the unary predicate $P$, $P^{M^*} = |M|$, and the ternary predicate $R$, $R = \{ (f, a, b) : f \in ^{|M|} M, a \in M, b = f(a) \}$. We shall similarly transform a theory $T$ to $T'$ and consider automorphism only of structures $\subseteq P$.}
for the language where we quantify over automorphisms (isomorphisms) of ordered fields in Theorem 6.4. Unfortunately we are not able to show that the language with quantification over automorphisms of Boolean algebras is compact, but will have to settle for a close relative of that logic. This is theorem 5.1. In section 4 we prove we can construct models in which all relevant automorphism are somewhat definable: 4.1, 4.8 for BA, 4.13 for ordered fields.

The reader may wonder why these results are being proved now, about 10 years after the results that preceeded them. The key technical innovation that has made these results possible is the discovery of ♦-like principles which are true in ZFC. These principles, which go under the common name of the Black Box, allow one to prove, with greater effort, many of the results which were previously known to follow from ♦ (see the discussion in [15] for more details). There have been previous applications of the Black Box to abelian groups, modules and Boolean Algebras — often building objects with specified endomorphism rings. This application goes deeper both in the sense that the proof is more involved and in the sense that the result is more surprising. The investigation is continued in [12], [13].

In this paper we will also give a new proof of the compactness of another logic — the one which is obtained when a quantifier $Q_{Brch}$ is added to first order logic which says that a level tree (definitions will be given later) has an infinite branch. This logic was previously shown to be compact — in fact it was the first logic shown in ZFC to be compact which is stronger than first order logic on countable structures — but our proof will yield a somewhat stronger result and provide a nice illustration of one of our methods. (The first logic stronger than first order logic which was shown to be compact was the logic which expresses that a linear order has cofinality greater than $\omega_1$ [14].) This logic, $L(Q_{Brch})$, has been used by Fuchs-Shelah [3] to prove the existence of nonstandard uniserial modules over (some) valuation domains. The proof uses the compactness of the tree logic to transfer results proved using ♦ to ZFC results. Eklof [2] has given an explicit version of this transfer method and was able to show that it settles other questions which had been raised. (Osofsky [1], [7] has found ZFC constructions which avoid using the model theory.)

Theorem’s 3.1 and 3.2 contain parallel results for Boolean algebra’s and fields. They assert the existence of a theory (of sets) $T_1$ such that in each model $M_1$ of $T_1$, $P(M_1)$ is a model $M$ of the first order theory $T$ such that for
every Boolean Algebra (respectively field) defined in $M$, every automorphism of the Boolean algebra (respectively field) that is definable in $M_1$ is definable in $M$. Moreover, each such $M_1$ has an elementary extension one of whose elements is a pseudofinite set $a$ with the universe of $M_1$ contained in $a$ and with $t(a/M_1)$ is definable over the empty set. This result depends on the earlier proof of our main result assuming $\diamondsuit$ and absoluteness. Theorem 4.1 uses the Black Box to construct a model $\mathfrak{C}$ of $T_1$ so that for any automorphism $f$ of a Boolean algebra $B = P(\mathfrak{c})$ there is a pseudofinite set $c$ such that for any atom $b \in B$, $f(b)$ is definable from $b$ and $c$. Theorem 4.13 is an analogous but stronger result for fields showing that for any $b$, $f(b)$ is definable from $b$ and $c$. In Lemma 4.7, this pointwise definability is extended by constructing a pseudo-finite partition of atoms of the Boolean algebra (respectively the elements of the field) such that $f$ is definable on each member of the partition. In Theorem 5.1 for Boolean algebras and 6.4 for fields this local definability is extended to global definability.

1.1 Outline of Proof

We want to build a model $\mathfrak{A}$ of a consistent $L(Q_{Of})$ theory $T$ which has only definable isomorphisms between definable ordered fields. By the ordinary compactness theorem, there is a non-standard model $\mathfrak{c}$ of an expansion of a weak set theory (say ZFC$^{-}$) which satisfies that there is a model $A_1$ of $T$. So $A_1$ would be a model of $T$ if the interpretation of the quantifier $Q_{Of}$ were taken to range over isomorphisms which are internal to $\mathfrak{c}$. We can arrange that $A_1$ will be the domain of a unary predicate $P$. Then our goal is to build our non-standard model $\mathfrak{c}$ of weak set theory in such a way that every external isomorphism between definable ordered subfields of $P(\mathfrak{c})$ is internal, i.e., definable in $\mathfrak{c}$.

The construction of $\mathfrak{c}$ is a typical construction with a prediction principle, in this case the Black Box, where we kill isomorphisms which are not pointwise definable over a set which is internally finite (or synonymously, pseudofinite). A predicted isomorphism is killed by adding an element which has no suitable candidate for its image. One common problem that is faced in such constructions is the question “how do we ensure no possible image of such an element exists?” To do this we need to omit some types. Much is known about omitting a type of size $\lambda$ in models of power $\lambda$ and even $\lambda^+$. But if say $2^\lambda > \lambda^{++}$, we cannot omit a dense set of types of power $\lambda$. So without
instances of GCH we are reduced to omitting small types, which is much harder. To omit the small types we will use techniques which originated in “Classification Theory”. In the construction we will have for some cardinal $\theta$ the type of any element does not split over a set of cardinality less than $\theta$ (see precise definitions below). This is analogous to saying the model is $\theta$-stable (of course we are working in a very non-stable context). The element we will add will have the property that its image (if one existed) would split over every set of cardinality $< \theta$.

The final problem is to go from pointwise definability to definability. The first ingredient is a general fact about $\aleph_0$-saturated models of set theory. We will show for any isomorphism $f$ that there is a large (internal) set $A$ and a pseudofinite sequence of one-one functions $(f_i: i < k^*)$ which cover $f|A$ in the sense that for every $a \in A$ there is $i$ so that $f_i(a) = f(a)$. Using this sequence of functions it is then possible to define $f$ on a large subset of $A$. Finally, using the algebraic structure, the definition extends to the entire ordered field.

In this paper we will need to use the following principle. In order to have the cleanest possible statement of our results (and to conform to the notation in [15]), we will state our results using slightly non-standard notation. To obtain the structure $H_\chi(\lambda)$, we first begin with a set of ordered urelements of order type $\lambda$ and then form the least set containing each urelement and closed under formation of sets of size less than $\chi$. In a context where we refer to $H_\chi(\lambda)$ by $\lambda$ we will mean the urelements and not the ordinals. In practice we believe that in a given context there will be no confusion.

**Theorem 1.3** Suppose $\lambda = \mu^+, \mu = \kappa^\theta = 2^\kappa$, $\chi$ is a regular cardinal, $\kappa$ is a strong limit cardinal, $\theta < \chi < \kappa$, $\kappa > \text{cf} \kappa = \theta \geq \aleph_0$ and $S \subseteq \{\delta < \lambda: \text{cf} \delta = \theta\}$ is stationary. Let $\rho$ be some cardinal greater than $\lambda$. Then we can find $W = \{\langle M^\alpha, \eta^\alpha \rangle: \alpha < \alpha(*)\}$ (actually a sequence) a function $\zeta: \alpha(*) \to S$ and $(C_\delta: \delta \in S)$ such that:

(a1) $M^\alpha = \langle M^\alpha_i: i \leq \theta \rangle$ is an increasing continuous elementary chain, each $M^\alpha_i$ is a model belonging to $H_\chi(\lambda)$ (and so necessarily has cardinality less than $\chi$), $M^\alpha_i \cap \chi$ is an ordinal, $\eta^\alpha \in \theta \lambda$ is increasing with limit $\zeta(\alpha) \in S$, for $i < \theta$, $\eta^\alpha_i \in M^\alpha_i$, $M^\alpha_i \in H_\chi(\eta^\alpha(i))$ and $(M^\alpha_j: j \leq i) \in M^\alpha_i$. 

\footnote{In our case $\alpha(*) = \lambda$ is fine}
(a2) For any set \( X \subseteq \lambda \) there is \( \alpha \) so that \( M^\alpha_\theta \equiv_{\lambda \cap M^\alpha_\theta} (H(\rho), \in, <, X) \), where < is a well ordering of \( H(\rho) \) and \( M \equiv_A N \) means \( (M, a)_{a \in A}, (N, a)_{a \in A} \) are elementarily equivalent.

(b0) If \( \alpha \neq \beta \) then \( \eta^\alpha \neq \eta^\beta \).

(b1) If \( \{ \eta^\alpha \upharpoonright i : i < \theta \} \subseteq M^\beta_\theta \) and \( \alpha \neq \beta \) then \( \zeta(\alpha) < \zeta(\beta) \).

(b2) If \( \eta^\alpha \upharpoonright (j + 1) \in M^\beta_\theta \) then \( M^\alpha_j \in M^\beta_\theta \).

(c2) \( \bar{C} = (C_\delta : \delta \in S) \) is such that each \( C_\delta \) is a club subset of \( \delta \) of the order type \( \theta \).

(c3) Let \( C_\delta = \{ \gamma_{\delta,i} : i < \theta \} \) be an increasing enumeration. For each \( \alpha < \alpha(*) \) there is \( \langle (\gamma^-_{\alpha,i}, \gamma^+_{\alpha,i}) : i < \theta \rangle \) such that: \( \gamma^-_{\alpha,i} \in M^\alpha_{i+1}, M^\alpha_{i+1} \cap \lambda \subseteq \gamma^+_\alpha,i, \gamma_{\zeta(\alpha),i} < \gamma^-_{\alpha,i} < \gamma^+_{\alpha,i} < \gamma_{\zeta(\alpha),i+1} \) and if \( \zeta(\alpha) = \zeta(\beta) \) and \( \alpha \neq \beta \) then for every large enough \( i < \theta \), \( \gamma^-_{\alpha,i}, \gamma^+_{\alpha,i} \cap [\gamma^-_{\beta,i}, \gamma^+_{\beta,i}] = \emptyset \). Furthermore for all \( i \), the sequence \( \langle \gamma^-_{\alpha,j} : j < i \rangle \) is in \( M^\alpha_{i+1} \).

This principle, which is one of the Black Box principles is a form of \( \diamond \) which is a theorem of ZFC. This particular principle is proved in [13] III 6.13(2). The numbering here is chosen to correspond with the numbering there. Roughly speaking clauses (a1) and (a2) say that there is a family of elementary substructures which predict every subset of \( \lambda \) as it sits in \( H(\lambda) \). (We will freely talk about a countable elementary substructure predicting isomorphisms and the like.) The existence of such a family would be trivial if we allowed all elementary substructures of cardinality less than \( \chi \). The rest of the clauses say that the structures are sufficiently disjoint that we can use the information that they provide without (too much) conflict.

The reader who wants to follow the main line of the arguments without getting involved (initially) in the complexities of the Black Box can substitute \( \diamond(\lambda) \) for the Black Box. Our proof of the compactness of \( L(Q_{Of}) \) does not depend on Theorem 1.2, so even this simplification gives a new proof of the consistency of the compactness of \( L(Q_{Of}) \). Our work on Boolean algebras does require Theorem 1.1.

The results in this paper were obtained while the first author was visiting the Hebrew University in Jerusalem. He wishes to thank the Institute of Mathematics for its hospitality.
2 Non-splitting extensions

In this section $\theta$ will be a fixed regular cardinal. Our treatment is self-contained but the reader can look at [14].

**Definition.** If $M$ is a model and $X,Y,Z \subseteq M$, then $X/Y$ does not split over $Z$ if and only if for every finite $d \subseteq Y$ the type of $X$ over $d$ (denoted either $\text{tp}(X/d)$ or $X/d$) depends only on the type of $d$ over $Z$.

We will use two constructions to guarantee that types will not split over small sets. The first is obvious by definition. (The type of $A/B$ is definable over $C$ if for any tuple $\bar{a} \in A$ and formula $\phi(\bar{x},\bar{y})$ there is a formula $\psi(\bar{y})$ with parameters from $C$ so that for any $\bar{b}$, $\phi(\bar{a},\bar{b})$ if and only if $\psi(\bar{b})$.)

**Proposition 2.1** If $X/Y$ is definable over $Z$ then $X/Y$ does not split over $Z$.

**Definition.** Suppose $M$ is a model and $X,Y \subseteq M$. Let $D$ be an ultrafilter on $X^\alpha$. Then the $\text{Av}(X,D,Y)$ (read the $(D)$-average type that $X^\alpha$ realizes over $Y$) is the type $p$ over $Y$ defined by: for $\bar{y} \subseteq Y, \phi(\bar{z},\bar{y}) \in p$ if and only if $\{x \in X^\alpha : \phi(\bar{x},\bar{y})\text{ holds}\} \in D$. We will omit $Y$ if it is clear from context. Similarly we will omit $\alpha$ and the “bar” for singletons, i.e., the case $\alpha = 1$.

The following two propositions are clear from the definitions.

**Proposition 2.2** If $\bar{a}$ realizes $\text{Av}(X,D,Y)$ then $\bar{a}/Y$ does not split over $X$. Also if there is $Z$ such that for $\bar{b} \in X$ the type of $\bar{b}/Y$ does not split over $Z$, then $\text{Av}(X,D,Y)$ does not split over $Z$.

**Proposition 2.3**

(i) Suppose $A/B$ does not split over $D$, $B \subseteq C$ and $C/B\cup A$ does not split over $D\cup A$ then $A\cup C/B$ does not split over $D$.

(ii) Suppose that $(A_i : i < \delta)$ is an increasing chain and for all $i$, $A_i/B$ does not split over $C$ then $\bigcup_{i<\delta} A_i/B$ does not split over $C$.

(iii) $X/Y$ does not split over $Z$ if and only if $X/\text{dcl}(Y\cup Z)$ does not split over $Z$. Here $\text{dcl}(Y\cup Z)$ denotes the definable closure of $Y\cup Z$.

(iv) If $X/Y$ does not split over $Z$ and $Z \subseteq W$, then $X/Y$ does not split over $W$.
**Definition.** Suppose $M_1 \prec M_2$ are models. Define $M_1 \prec^\otimes_\theta M_2$, if for every $X \subseteq M_2$ of cardinality less than $\theta$ there is $Y \subseteq M_1$ of cardinality less than $\theta$ so that $X/M_1$ does not split over $Y$. (If $\theta$ is regular, then we only need to consider the case where $X$ is finite.)

**Proposition 2.4** Assume that $\theta$ is a regular cardinal (needed for (2) only) and that all models are models of some fixed theory with Skolem functions (although this is needed for (3) only).

1. $\prec^\otimes_\theta$ is transitive and for all $M$, $M \prec^\otimes_\theta M$.
2. If $(M_i : i < \delta)$ is a $\prec^\otimes_\theta$-increasing chain, then for all $i$ $M_i \prec^\otimes_\theta \bigcup_{j<\delta} M_j$.
3. Suppose $M_2$ is generated by $M_1 \cup N_2$ and $N_1 = M_1 \cap N_2$. (Recall that we have Skolem functions.) If $|N_1| < \theta$ and $N_2/M_1$ does not split over $N_1$, then $M_1 \prec^\otimes_\theta M_2$.

An immediate consequence of these propositions is the following proposition.

**Proposition 2.5** Suppose $M \prec^\otimes_\theta N$, then there is a $\theta$-saturated model $M_1$ such that $N \prec M_1$ and $M \prec^\otimes_\theta M_1$.

**Proof** By the lemmas it is enough to show that given a set $X$ of cardinality $< \theta$ and a type $p$ over $X$ we can find a realization $a$ of that type so that $a/N$ does not split over $X$. Since we have Skolem functions every finite subset of $p$ is realized by an element whose type over $N$ does not split over $X$, namely an element of the Skolem hull of $X$. So we can take $a$ to realize an average type of these elements.

### 3 Building New Theories

The models we will eventually build will be particular non-standard models of an enriched version of ZFC$^-$. (Recall ZFC$^-$ is ZFC without the power set axiom and is true in the sets of hereditary cardinality $< \kappa$ for any regular uncountable cardinal $\kappa$.) The following two theorems state that appropriate theories exist.
Theorem 3.1  Suppose $T$ is a theory in a language which is countable except for constant symbols, $P_0$ a unary predicate so that in every model $M$ of $T$ every definable automorphism of a definable atomic Boolean algebra $\subseteq P_0^M$ is definable by a fixed formula (together with some parameters). Then there is $T_1$ an expansion of $ZFC^-$ in a language which is countable except for constant symbols with a unary predicate $P_0$ so that if $M_1$ is a model of $T_1$ then $P_0(M_1)$ is a model $M$ of $T$ (when restricted to the right vocabulary) and the following are satisfied to be true in $M_1$.

(i) Any automorphism of a definable (in $M$) atomic Boolean algebra contained in $P_0(M)$ which is definable in $M_1$ is definable in $M$.

(ii) $M_1$ (which is a model of $ZFC^-$) satisfies for some regular cardinal $\mu$ (of $M_1$), $|M| = \mu^+$, $M$ is $\mu$-saturated and every non-algebraic type (in the language of $M$) of cardinality $< \mu$ is realized in $M$ by $\mu^+$ elements of $M$.

(iii) $M \in M_1$.

(iv) $M_1$ satisfies the separation scheme for all formulas (not just those of the language of set theory).

(v) $M_1$ has Skolem functions.

(vi) For any $M_1$ there is an elementary extension $N_1$ so that the universe of $M_1$ is contained in $N_1$ in a pseudofinite set (i.e., one which is finite in $N_1$) whose type over the universe of $M_1$ is definable over the empty set.

Proof  We first consider a special case. Suppose that there is a cardinal $\lambda$ greater than the cardinality of $T$ satisfying the hypothesis of Theorem 1.1 (i.e., both $\Diamond(\lambda)$ and $\Diamond(\{\alpha < \lambda^+ : \text{cf}\alpha = \lambda\})$ hold). Then we could choose $\kappa$ a regular cardinal greater than $\lambda^+$. Our model $M_1$ will be taken to be a suitable expansion of $H(\kappa)$ where the interpretation of the unary predicate $P_0$ is the model $M$ guaranteed by Theorem 1.1 and $\mu = \lambda$, $\mu^+ = \lambda^+$. Since any formula in the enriched language is equivalent in $M_1$ to a formula of set theory together with parameters from $M_1$, $M_1$ will also satisfy (iv).

What remains is to ensure (v) and that appropriate elementary extensions always exist i.e. clause (vi). To achieve this we will expand the language by induction on $n$. Let $L_0$ be the language consisting of the language of $T$, $\{P\}$, and Skolem functions and $M_0^\rho$ be any expansion by Skolem functions of the structure on $H(\kappa)$ described above. Fix an index set $I$ and an ultrafilter $D$
on $I$ so that there is $a \in H(\kappa)^I/D$ such that $H(\kappa)^I/D \models \text{“} a \text{ is finite”}$ and for all $b \in H(\kappa)$, $H(\kappa)^I/D \models b \in a$. Let $N_0^I = M_0^I/D$.

Then for every formula $\phi(y, x_1, \ldots, x_n)$ of $L_0$ which does not involve constants, add a new $n$-ary relation $R_\phi$. Let $L_{0.5}$ by the language containing all the $R_\phi$. Let $M_{1.5}$ be the $L_{0.5}$ structure with universe $H(\kappa)$ obtained by letting for all $b_1, \ldots, b_n$, $M_{1.5}^I \models R_\phi[b_1, \ldots, b_n]$ if and only if $N_0^I \models \phi[a, b_1, \ldots, b_n]$. Let $L_1$ be an extension of $L_{0.5}$ by Skolem functions and let $M_1^I$ be an expansion of $M_{1.5}^I$ by Skolem functions. Condition (iv) still holds as it holds for any expansion of $(H(\kappa), \in)$. We now let $N_1^I = M_1^I/D$ and continue as before.

Let $L = \bigcup_{n<\omega} L_n$ and $M_1 = \bigcup_{n<\omega} M_1^n$ (i.e. the least common expansion; the universe stays the same). Let $T_1$ be the theory of $M_1$. As we have already argued $T_1$ has properties (i)–(iv). It remains to see that any model of $T_1$ has the desired extension property. First we consider $M_1$ and let $N_1 = M_1^I/D$. Then the type of $a$ over $M_1$ is definable over the empty set using the relations $R_\phi$ which we have added. Since $T_1$ has Skolem functions, for any model $A_1$ of $T_1$ there will be an extension $B_1$ of $A_1$ generated by $A_1$ and an element realizing the definable type over $A_1$.

In the general case, where we may not have the necessary hypotheses of Theorem 1.1, we can force with a notion of forcing which adds no new subsets of $|T|$ to get some $\lambda$ satisfying hypothesis of Theorem 1.1 (alternately, we can use $L[A]$ where $A$ is a large enough set of ordinals). Since the desired theory will exist in an extension it already must (as it can be coded by a subset of $|T|$) exist in the ground model.

Later on we will be juggling many different models of set theory. The ones which are given by the Black Box, and the non-standard ones which are models of $T_1$. When we want to refer to notions in models of $T_1$, we will use words like “pseudofinite” to refer to sets which are satisfied to be finite in the model of $T_1$.

In the same way as we proved the last theorem we can show the following theorem.

**Theorem 3.2** Suppose $T$ is a theory in a language with a unary predicate $P_0$ and which is countable except for constant symbols so that in every model $M$ of $T$ every definable isomorphism between definable ordered fields $\subseteq P_0^M$ is definable by a fixed formula (together with some parameters). Then there is
$T_1$ an expansion of ZFC$^-$ in a language which is countable except for constant symbols with a unary predicate $P_0$ so that if $M_1$ is a model of $T_1$ then $P_0(M_1)$ is a model $M$ of $T$ and the following are satisfied to be true in $M_1$.

(i) Any isomorphism of definable (in $M$) ordered fields contained in $P_0$ which is definable in $M_1$ is definable in $M$.

(ii) $M \in M_1$.

(iii) $M_1$ satisfies the separation scheme for all formulas (not just those of the language of set theory).

(iv) $M_1$ has Skolem functions.

(v) For any $M_1$ there is an elementary extension $N_1$ so that the universe of $M_1$ is contained in $N_1$ in a pseudofinite set (i.e., one which is finite in $N_1$) whose type over the universe of $M_1$ is definable over the empty set.

Since we have no internal saturation conditions this theorem can be proved without recourse to Theorem 1.2 (see the next section for an example of a similar construction).

### 3.1 A Digression

The method of expanding the language to get extensions which realize a definable type is quite powerful in itself. We can use the method to give a new proof of the compactness of a logic which extends first order logic and is stronger even for countable structures. This subsection is not needed in the rest of the paper.

**Lemma 3.3** Suppose that $N$ is a model. Then there is a consistent expansion of $N$ to a model of a theory $T_1$ with Skolem functions so that for every model $M$ of $T_1$ there is a so that $a/M$ is definable over the empty set and in $M(a)$ (the model generated by $M$ and $\{a\}$) for every definable directed partial ordering $<$ of $M$ without the last element there is an element greater than any element of $M$ in the domain of $<$. Furthermore the cardinality of the language of $T_1$ is no greater than that of $N$ plus $\aleph_0$.

**Proof** Fix a model $N$. Choose $\kappa$ and an ultrafilter $D$ so that for every directed partial ordering $<$ of $N$ without the last element there is an element
of $N^\kappa/D$ which is greater than every element of $N$ (e.g., let $\kappa = |N|$ and $D$ be any regular ultrafilter on $\kappa$). Fix an element $a \in N^\kappa/D \setminus N$. Abusing notation we will let $a : \kappa \to N$ be a function representing the element $a$. The new language is defined by induction on $\omega$. Let $N = N_0$. There are three tasks so we divide the construction of $N_{n+1}$ into three cases. If $n \equiv 0 \mod 3$, expand $N_n$ to $N_{n+1}$ by adding Skolem functions. If $n \equiv 1 \mod 3$, add a $k$-ary relation $R_\phi$ for every formula of arity $k + 1$ and let $R_\phi(b_0, \ldots, b_{k-1})$ hold if and only if $\phi(b_0, \ldots, b_{k-1}, a)$ holds in $N^\kappa/D$.

If $n \equiv 2 \mod 3$, we ensure that there is an upper bound to every definable directed partial orders without last element in $N_n$. For each $k+2$-ary formula $\phi(x_0, \ldots, x_{k-1}, y, z)$ we will add a $k + 1$-ary function $f_\phi$ so that for all $\bar{b}$ if $\phi(\bar{b}, y, z)$ defines a directed partial order without last element then $f_\phi(\bar{b}, a)$ is greater in that partial order than any element of $N$. Notice that there is something to do here since we must define $f_\phi$ on $N$ and then extend to $N^\kappa/D$ using the ultraproduct. For each such $\bar{b}$ choose a function $c : \kappa \to N$ so that $c/D$ is an upper bound (in the partial order) to all the elements of $N$. Now choose $f_\phi$ so that $f_\phi(\bar{b}, a(i)) = c(i)$. Let $T_1$ be the theory of the expanded model.

Suppose now that $M$ is a model of $T_1$. The type we want is the type $p$ defined by $\phi(c_1, \ldots, c_{k-1}, x) \in p$ if and only if $M \models R_\phi(c_1, \ldots, c_n)$.

In the process of building the new theory there are some choices made of the language. But these choices can be made uniformly for all models. We will in the sequel assume that such a uniform choice has been made.

¿From this lemma we can prove a stronger version of a theorem from [9], which says that the language which allows quantification over branches of level-trees is compact. A tree is a partial order in which the predecessors of any element are totally ordered. A level-tree is a tree together with a ranking function to a directed set. More exactly a level-tree is a model $(A : U, V, <_1, <_2, R)$ where:

1. $A$ is the union of $U$ and $V$;
2. $<_1$ is a partial order of $U$ such that for every $u \in U \{y \in U : y <_1 u\}$ is totally ordered by $<_1$;
3. $<_2$ is a directed partial order on $V$ with no last element;
4. $R$ is a function from $U$ to $V$ which is strictly order preserving.
The definition here is slightly more general than in [9]. In [9] the levels were required to be linearly ordered. Also what we have called a “level-tree” is called a “tree” in [9]. A branch $b$ of a level-tree is a maximal linearly-ordered subset of $U$ such that $\{R(t) : t \in b\}$ is unbounded in $V$. We will refer to $U$ as the tree and $V$ as the levels. For $t \in U$ the level of $t$ is $R(t)$. A tuple of formulas (which may use parameters from $M$), 

$$(\phi_1(x), \phi_2(x), \psi_1(x, y), \psi_2(x, y), \rho(x, y, z)),$$

defines a level-tree in a model $M$ if 

$$(\phi_1(M) \cup \phi_2(M); \phi_1(M), \phi_2(M), \psi_1(x, y)^M, \psi_2(x, y)^M, \rho(x, y, z)^M)$$

is a level tree. (There is no difficulty in extending the treatment to all level-trees which are definable using equivalence relations.)

Given the definition of a level-tree, we now define an extension of first order logic by adding second order variables (to range over branches of level-trees) and a quantifier $Q_{Brch}$, such that 

$$Q_{Brch} b(\phi_1(x), \phi_2(x), \psi_1(x, y), \psi_2(x, y), \rho(x, y, z))\Theta(b)$$

says that if $(\phi_1(x), \phi_2(x), \psi_1(x, y), \psi_2(x, y), \rho(x, y, z))$ defines a level-tree then there is a branch $b$ of the level-tree such that $\Theta(b)$ holds.

In [9], it is shown that (a first-order version of) this logic was compact. This is the first language to be shown (in ZFC) to be fully compact and stronger than first order logic for countable structures. In [9], the models are obtained at successors of regular cardinals.

**Theorem 3.4** The logic $L(Q_{Brch})$ is compact. Furthermore every consistent theory $T$ has a model in all uncountable cardinals $\kappa > |T|$.

**Proof** By expanding the language we can assume that any model of $T$ admits elimination of quantifiers. (I.e., add new relations for each formula and the appropriate defining axioms.)

For each finite $S \subseteq T$, choose a model $M_S$ of $S$. For each $S$ choose a cardinal $\mu$ so that $M_S \in H(\mu)$ and let $N_S$ be the model $(H(\mu^+), M_S, \in)$, where $M_S$ is the interpretation of a new unary predicate $P$ and the language of $N_S$ includes the language of $T$ (with the correct restriction to $M_S$). By
expanding the structure $N_S$ we can assume that the theory $T_S$ of $N_S$ satisfies the conclusion of Lemma 3.3. Furthermore we note that $T_S$ satisfies two additional properties. If a formula defines a branch in a definable level-tree contained in the domain of $P$ then this branch is an element of the model $N_S$. As well $N_S$ satisfies that $P(N_S)$ is a model of $S$.

Now let $D$ be an ultrafilter on the finite subsets of $T$ such that for all finite $S$, $\{S_1 : S \subseteq S_1\} \in D$. Finally let $T_1$ be the (first-order) theory of $\prod N_S/D$. If $N$ is any model of $T_1$ then for any sentence $\phi \in T$, $N$ satisfies “$P(N)$ satisfies $\phi$”. If we can arrange that the only branches of an $L(Q_{Brch})$-definable (in the language of $T$) level-tree of $P(N)$ are first order definable in $N$ then $P(N)$ satisfaction of an $L(Q_{Brch})$-formula will be the same in $N$ and the real world. Before constructing this model let us note that our task is a bit easier than it might seem.

**CLAIM 3.5** Suppose that $N$ is a model of $T_1$ and every branch of an first-order definable (in the language of $T$) level-tree of $P(N)$ is first order definable in $N$, then every branch of an $L(Q_{Brch})$-definable (in the language of $T$) level-tree of $P(N)$ is first order definable in $N$.

**Proof** (of the claim) Since we have quantifier elimination we can prove by induction on construction of formulas that satisfaction is the same in $N$ and the real world and so the quantifier-elimination holds for $P(N)$. In other words any $L(Q_{Brch})$-definable level-tree is first order definable.

It remains to do the construction and prove that it works. To begin let $N_0$ be any model of $T_1$ of cardinality $\kappa$. Let $\mu \leq \kappa$ be any regular cardinal. We will construct an increasing elementary chain of models $N_\alpha$ for $\alpha < \mu$ by induction. At limit ordinals we will take unions. If $N_\alpha$ has been defined, let $N_{\alpha+1} = N_\alpha(a_\alpha)$, where $a_\alpha$ is as guaranteed by Lemma 3.3. Now let $N = \bigcup_{\alpha<\mu} N_\alpha$.

**SUBCLAIM 3.6** Suppose $X$ is any subset of $N$ which is definable by parameters. Then for all $\alpha$, $X \cap N_\alpha$ is definable in $N_\alpha$.

**Proof** (of the subclaim) Suppose not and let $\beta$ be the least ordinal greater than $\alpha$ so that $X \cap N_\beta$ is definable in $N_\beta$. Such an ordinal must exist since for sufficiently large $\beta$ the parameters necessary to define $X$ are in $N_\beta$. Similarly there is $\gamma$ such that $\beta = \gamma + 1$. Since $N_\beta$ is the Skolem hull of
\( N_\gamma \cup \{a_\gamma\} \), there is \( \bar{b} \in N_\gamma \) and a formula \( \phi(x, \bar{y}, z) \) so that \( X \cap N_\beta \) is defined by \( \phi(x, \bar{b}, a_\gamma) \). But by the definability of the type of \( a_\gamma \) over \( N_\gamma \) there is a formula \( \psi(x, \bar{y}) \) so that for all \( a, \bar{c} \in N_\gamma, N_\beta \models \psi(a, \bar{c}) \) if and only if \( \phi(a, \bar{c}, a_\gamma) \). Hence \( \psi(x, \bar{b}) \) defines \( X \cap N_\gamma \) in \( N_\gamma \).

It remains to see that every branch of a definable level tree is definable. Suppose \((A; U, V, <_1, <_2, R)\) is a definable level-tree. Without loss of generality we can assume it is definable over the empty set. Let \( B \) be a branch. For any \( \alpha < \mu \) there is \( c \in V \) so that for all \( d \in V \cap N_\alpha, d <_2 c \). Since the levels of \( B \) are unbounded in \( V, B \cap N_\alpha \) is not cofinal in \( B \). Hence there is \( b \in B \) so that \( B \cap N_\alpha \subseteq \{a \in U : a <_1 b\} \). By the subclaim \( B \cap N_\alpha \) is definable in \( N_\alpha \).

Since \( \mu \) has the uncountable cofinality, by Fodor’s lemma, there is \( \alpha < \mu \) so that for unboundedly many (and hence all) \( \gamma < \mu, B \cap N_\gamma \) is definable by a formula with parameters from \( N_\alpha \). Fix a formula \( \phi(x) \) with parameters from \( N_\alpha \) which defines \( B \cap N_\alpha \). Then \( \phi(x) \) defines \( B \). To see this consider any \( \gamma < \alpha \) and a formula \( \psi(x) \) with parameters from \( N_\alpha \) which defines \( B \cap N_\gamma \). Since \( N_\alpha \) satisfies “for all \( x, \phi(x) \) if and only if \( \psi(x) \)” and \( N_\alpha \prec N_\gamma, \phi(x) \) also defines \( B \cap N_\gamma \). \( \blacksquare \)

**Remark.** The compactness result above is optimal as far as the cardinality of the model is concerned. Any countable level-tree has a branch and so there is no countable model which is \( L(Q_{\text{Brch}}) \)-equivalent to an Aronszajn tree. By the famous theorem of Lindstrom [4], this result is the best that can be obtained for any logic, since any compact logic which is at least as powerful as the first order logic and has countable models for all sentences is in fact the first order logic. The existence of a compact logic such that every consistent countable theory has a model in all uncountable cardinals was first proved by Shelah [11], who showed that the first order logic is compact if we add a quantifier \( Q^{\omega} \) which says of a linear order that its cofinality is \( \omega \). (Lindstrom’s theorem and the logic \( L(Q^{\omega}) \) are also discussed in [11]). The logic \( L(Q_{\text{Brch}}) \) has the advantage that it is stronger than the first order logic even for countable models [11].

Notice in the proof above in any definable level-tree the directed set of the levels has cofinality \( \mu \). Since we can obtain any uncountable cofinality this is also the best possible result. Also in the theorem above we can demand just \( \kappa \geq |T| + \aleph_1 \)
4 The Models

For the purposes of this section let $T$ and $T_1$ be theories as defined above in 3.1 or 3.2 (for Boolean algebras or ordered fields). In this section we will build models of our theory $T_1$. The case of Boolean algebras and the case of ordered fields are similar but there are enough differences that they have to be treated separately. We shall deal with Boolean algebras first.

We want to approximate the goal of having every automorphism of every definable atomic Boolean algebra in the domain of $P$ be definable. In this section, we will get that they are definable in a weak sense. In order to spare ourselves some notational complications we will make a simplifying assumption and prove a weaker result. It should be apparent at the end how to prove the same result for every definable atomic Boolean algebra in the domain of $P$.

**Assumption.** Assume $T$ is the theory of an atomic Boolean algebra on $P$ with some additional structure.

**Theorem 4.1** There is a model $C$ of $T_1$ so that if $B = P(C)$ and $f$ is any automorphism of $B$ as a Boolean algebra then there is a pseudofinite set $c$ so that for any atom $b \in B$, $f(b)$ is definable from $b$ and elements of $c$.

**Proof** We will use the notation from the Black Box. In particular we will use an ordered set of urelements of order type $\lambda$. We can assume that $\mu$ is larger than the cardinality of the language (including the constants). We shall build a chain of structures $(C_\varepsilon : \varepsilon < \lambda)$ such that the universe of $C_\varepsilon$ will be an ordinal $< \lambda$ and the universe of $c = \bigcup_{\varepsilon < \lambda} C_\varepsilon$ will be $\lambda$ (we can specify in a definable way what the universe of $C_\varepsilon$ is e.g. $\mu(1+\alpha)$ is o.k.). We choose $C_\varepsilon$ by induction on $\varepsilon$. Let $B_\varepsilon = P(C_\varepsilon)$. We will view the $(M^\alpha, \eta^\alpha) \in W$ in the Black Box as predicting a sequence of models of $T_1$ and an automorphism of the Boolean algebra. (See the following paragraphs for more details on what we mean by predicting.)

The construction will be done so that if $\varepsilon \not\in S$ and $\varepsilon < \zeta$ then $C_\varepsilon \prec_\theta C_\zeta$. (We will make further demands later.) The model $c$ will be $\bigcup_{\varepsilon < \lambda} c_\varepsilon$. When we are done, if $f$ is an automorphism of the Boolean algebra then we can choose $(M^\alpha, \eta^\alpha) \in W$ to code, in a definable way, $f$ and the sequence $(c_\varepsilon : \varepsilon < \lambda)$.
The limit stages of the construction are determined. The successor stage when \( \varepsilon \notin S \) is simple. We construct \( C_{\varepsilon+1} \) so that \( C_{\varepsilon} \preceq_{\delta} C_{\varepsilon+1} \), \( C_{\varepsilon+1} \) is \( \theta \)-saturated and there is a pseudofinite set in \( C_{\varepsilon+1} \) which contains \( C_{\varepsilon} \). By the construction of the theory \( T_1 \) there is \( c \), a pseudofinite set which contains \( C_{\varepsilon} \) such that the type of \( c \) over \( C_{\varepsilon} \) is definable over the empty set. Hence \( C_{\varepsilon} \preceq_{\delta} C_{\varepsilon}(c) \). By Proposition 2.3 there is \( C_{\varepsilon+1} \) which is \( \theta \)-saturated so that \( C_{\varepsilon}(c) \preceq_{\delta} C_{\varepsilon+1} \). Finally by transitivity (Proposition 2.3), \( C_{\varepsilon} \preceq_{\delta} C_{\varepsilon+1} \).

The difficult case occurs when \( \varepsilon \in S \) rename \( \varepsilon \) by \( \delta \). Consider \( \alpha \) so that \( \zeta(\alpha) = \delta \). We are interested mainly in \( \alpha \)'s which satisfy:

\((*)\) The model \( M^\alpha_\theta \) “thinks” it is of the form \( (H(\rho), \varepsilon, <, X) \) and by our coding yields (or predicts) a sequence of structures \( (D_\nu: \nu < \lambda) \) and a function \( f_\alpha \) from \( D = \bigcup_{\nu < \lambda} D_\nu \) to itself. (Of course all the urelements in \( M^\alpha_\theta \cap \lambda \) will all have order type less than \( \delta \).)

At the moment we will only need to use the function predicted by \( M^\alpha_\theta \).

We will say an obstruction occurs at \( \alpha \) if \((*)\) holds and we can make the following choices. If possible choose \( N_\alpha \subseteq C_\delta \) so that \( N_\alpha \in M^\alpha_\theta \) of cardinality less than \( \theta \) and a sequence of atoms \( \{a_i^\alpha: i \in C_\delta\} \) so that (naturally \( a_i^\alpha \in M^\alpha_{i+1} \)) \( a_i^\alpha/e_i^\alpha \), \( e_i^\alpha \), does not split over \( N_\alpha \) and \( a_0^\alpha \in [\gamma^-_{\alpha,i}, \gamma^+_{\alpha,i}] \) and \( f_\alpha(a_0^\alpha) \) is not definable over \( a \) and parameters from \( C_{\gamma^-_{\alpha,i}} \). At ordinals where an obstruction occurs we will take action to stop \( f_\alpha \) from extending to an automorphism of \( B = B^C \).

Notice that \( N_\alpha \) is contained in \( M^\alpha_\theta \subseteq M^\delta_\theta \).

Suppose an obstruction occurs at \( \alpha \). Let \( X_\alpha \) be the set of finite joins of the \( \{a_i^\alpha: i \in C_\delta\} \). In the obvious way, \( X_\alpha \) can be identified with the set of finite subsets of \( Y_\alpha = \{a_i^\alpha: i \in C_\delta\} \). Fix \( U_\alpha \) an ultrafilter on \( X_\alpha \) so that for all \( x \in X_\alpha \), \( \{y \in X_\alpha: x \subseteq y\} \in U_\alpha \). Now define by induction on \( \text{Ob}(\delta) = \{\alpha: \zeta(\alpha) = \delta \) and an obstruction occurs at \( \alpha \} \) an element \( x_\alpha \) so that \( x_\alpha \) realizes the \( U_\alpha \) average type of \( X_\alpha \) over \( C_\delta \cup \{x_\beta: \beta \in \text{Ob}(\delta), \beta < \alpha\} \). Then \( C_{\delta+1} \) is the Skolem hull of \( C_\delta \cup \{x_\alpha: \alpha \in \text{Ob}(\delta)\} \).

We now want to verify the inductive hypothesis and give a stronger property which we will use later in the proof. The key is the following claim.

**Claim 4.2** Suppose \( \alpha_0, \ldots, \alpha_{n-1} \in \text{Ob}(\delta) \), then for all but a bounded set of \( \gamma < \delta \), \( (\bigcup_{k<n} Y_{\alpha_k})/C_\gamma \) does not split over \( \bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n} (C_\gamma \cap Y_{\alpha_k}) \).

**Proof** (of the claim) Suppose \( \gamma \) is large enough so that for all \( m \neq k < n \), \( [\gamma^-_{\alpha_m,i}, \gamma^+_{\alpha_m,i}] \cap [\gamma^-_{\alpha_k,i}, \gamma^+_{\alpha_k,i}] = \emptyset \), whenever \( \gamma^-_{\alpha_m,i} \geq \gamma \) (recall clause (c3) of
the Black Box). It is enough to show by induction on $\gamma \leq \sigma < \delta$ that $(\bigcup_{k<n} Y_{\alpha_k}) \cap e_\sigma$ has the desired property. For $\sigma = \gamma$ there is nothing to prove. In the inductive proof we only need to look at a place where the set increases. By the hypothesis on $\gamma$ we can suppose the result is true up to $\sigma = \gamma^-_{\alpha_k,i}$ and try to prove the result for $\sigma = \gamma^+_{\alpha_k,i}$ (since new elements are added only in these intervals). The new element added is $a^\alpha_{\iota k,i}$. Denote this element by $a$. By hypothesis, $a/\bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n} (e_\gamma \cap Y_{\alpha_k})$ does not split over $\bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n} (e_\gamma \cap Y_{\alpha_k})$. Now we can apply the induction hypothesis and Proposition 2.3.

Notice that $X_{\alpha}^\gamma$ is contained in the definable closure of $Y_{\alpha}$ and vice versa so we also have $(\bigcup_{k<n} X_{\alpha_k})/e_\gamma$ does not split over $\bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n} (e_\gamma \cap Y_{\alpha_k})$. We can immediately verify the induction hypothesis that if $\gamma < \delta$, $\gamma \notin S$ then $e_\gamma \preceq^\theta e_{\delta+1}$. It is enough to verify for $\alpha_0, \ldots, \alpha_{n-1}$ and sufficiently large $\beta$ that $(x_{\alpha_0}, \ldots, x_{\alpha_{n-1}})/e_\beta$ does not split over $\bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n} (e_{\beta} \cap Y_{\alpha_k})$ (a set of size $< \theta$). But this sequence realizes the ultrafilter average of $X_{\alpha_0} \times \cdots \times X_{\alpha_{n-1}}$. So we are done by Proposition 2.2.

This completes the construction. Before continuing with the proof notice that we get the following from the claim.

**Claim 4.3** 1) For all $\alpha \in \text{Ob}(\delta)$, $D \subseteq e_{\delta+1}$, if $|D| < \theta$ then for all but a bounded set of $i < \theta$, $D/e_{\gamma^-_{\alpha_k,i}}$ does not split over $e_{\gamma^-_{\alpha_k,i}} \cup \{a^\alpha_i\}$. Moreover for all but a bounded set of $i < \theta$, $D/e_{\gamma^+_{\alpha_k,i}}$ does not split over a subset of $e_{\gamma^-_{\alpha_k,i}} \cup \{a^\alpha_i\}$ of size $< \theta$.

2) for every subset $D$ of $e_{\delta+1}$ of cardinality $< \theta$ there is a subset $w$ of $\text{Ob}(\delta)$ of cardinality $< \theta$ and subset $Z$ of $e_{\delta}$ of cardinality $< \theta$ such that: the type of $D$ over $e_{\delta}$ does not split over $Z \cup \bigcup_{j \in w} Y_j$.

3) In (2) for every large enough $i$, for every $\alpha \in w$ the type of $D$ over $e_{\gamma^+_{\alpha_k,i}}$ does not split over $Z \cup \bigcup_{j \in w} \{Y_j \cap e_{\gamma^-_{\alpha_k,i}} : j \in w\} \cup \{a_{\alpha,i}\}$.

4) In (2), (3) we can allow $D \subseteq e$.

We now have to verify that $e$ has the desired properties. Assume that $f$ is an automorphism of $B = B_e$. We must show that

**Claim 4.4** There is $\gamma$ so that for all atoms $a$, $f(a)$ is definable with parameters from $e_\gamma$ and $a$. 20
Proof (of the claim) Assume that $f$ is a counterexample. For every $\gamma \notin S$, choose an atom $a_\gamma$ which witnesses the claim is false with respect to $\mathfrak{c}_\gamma$. Since $\{\delta < \lambda : \delta \notin S, \text{cf}\delta \geq \theta\}$ is stationary, there is a set $N$ of cardinality less than $\theta$ so that for a stationary set of $\gamma$, $a_\gamma/\mathfrak{c}_\gamma$ does not split over $N$. In fact (since $(\forall \alpha < \lambda)\alpha^\theta < \lambda$ as $\lambda = \mu^+$, $\mu^{<\theta} = \mu$) for all but a bounded set of $\gamma$ we can use the same $N$. Let $X$ code the sequence $(\mathfrak{c}_\gamma : \gamma < \lambda)$ and the function $f$. Then, by the previous discussion, $(H(\rho), \in, \prec, X)$ satisfies “there exists $N \subseteq \mathfrak{c}$ so that $|N| < \theta$ and for all but a bounded set of ordinals $\gamma$, there is an atom $z$ so that $z/\mathfrak{c}_\gamma$ does not split over $N$”. Choose $\alpha$ so that

$$M^\alpha_\theta \equiv M^\alpha_\theta \cap (H(\rho), \in, \prec, X).$$

It is now easy to verify that an obstruction occurs at $\alpha$. Let $\delta = \zeta(\alpha)$. In this case, $f_\alpha$ is the restriction of $f$. We use the notation of the construction. By the construction there is $D \subseteq \mathfrak{c}_{\delta+1}$, $|D| < \theta$ such that the type of $f(a_\alpha)$ over $\mathfrak{c}_{\delta+1}$ does not split over $D$. Apply Claim 13 above, parts (2), (3) and get $Z, w$ and $i^* < \theta$ (the $i^*$ is just explicating the “for every large enough $i$ to ”for every $i \in [i^*, \theta]”$). Let $D^* = Z \cup \{Y_j \cap C_{\gamma_{\alpha,i}} : j \in w\}$ so for every $i \in [i^*, \theta]$ we have

1) $f(a_\alpha)/\mathfrak{c}_{\delta+1}$ does not fork over $D(\subseteq \mathfrak{c}_{\delta+1})$.
2) $D/\mathfrak{c}_{\gamma_{\alpha,i}}$ does not split over $D^* \cup a_{\alpha,i}$
3) $D^* \cup a_{\alpha,i} \subseteq \mathfrak{c}_{\gamma_{\alpha,i}} \subseteq \mathfrak{c}_{\delta+1}$

so by the basic properties of non splitting $f(x_\alpha)/\mathfrak{c}_{\gamma_{\alpha,i}}$ does not split over $D^* \cup \{a^\alpha_i\}$, and note that we have : $D^* \cup N_\alpha \subseteq \mathfrak{c}_{\gamma_{\alpha,i}}$ and $|D^* \cup N_\alpha| < \theta$ where $N_\alpha$ comes from the construction.

An important point is that by elementariness for all ordinals $\tau \in M^\alpha_i \cap \lambda$ and atoms $a \in M^\alpha_i$ there is an ordinal $\beta$ so that $\tau < \beta \in M^\alpha_i \cap \lambda$, $a \in B_\beta$, $\mathfrak{c}_\beta$ is $\theta$-saturated (just take $\text{cf}\beta > \theta$) and $f$ is an automorphism of $B_\beta$. Choose such a $\beta \in M^\alpha_i$ with respect to $a^\alpha_i$ and $\gamma_{\alpha,i}$.

Since $f(a^\alpha_i)$ is not definable from $a^\alpha_i$ and parameters from $\mathfrak{c}_{\gamma_{\alpha,i}}$ and $\mathfrak{c}_\beta$ is $\theta$-saturated there is $b \neq f(a^\alpha_i)$ realizing the same type over $D^* \cup \{a^\alpha_i\} \cup \{f(a^\alpha_j) : j < i\}$ with $b \in B_\beta$. Now for any atom $c \in B_\beta$ we have (by the definition of $x_\alpha$) that $c \leq x_\alpha$ if and only if $c = a^\alpha_j$ for some $j \leq i$. Since this property is preserved by $f$, we have that $f(a^\alpha_i) \leq f(x_\alpha)$ and $b \nleq f(x_\alpha)$. But $\beta < \gamma_{\alpha,i}^+$ and $f(x_\alpha)/\mathfrak{c}_{\gamma_{\alpha,i}}$ does not split over $\{a^\alpha_i\} \cup D^*$. So we have arrived at a contradiction. $\Box$
In the proof above if we take \( \theta \) to be uncountable then we can strengthen the theorem (although we will not have any current use for the stronger form).

**Theorem 4.5** In the Theorem above if \( \theta \) is uncountable then there is a finite set of formulas \( L' \) and a pseudofinite set \( c \) so that for every atom \( b \in B \) \( f(b) \) is \( L' \)-definable over \( \{b\} \cup c \).

**Proof** The argument so far has constructed a model in which every automorphism of \( B \) is pointwise definable on the atoms over some \( c_\gamma \) (i.e., for every atom \( b \in B \) \( f(b) \) is definable from \( b \) and parameters from \( c_\gamma \)). In the construction of the model we have that every \( c_\gamma \) is contained in some pseudofinite set so we are a long way towards our goal. To prove the theorem it remains to show that we can restrict ourselves to a finite sublanguage. (Since all the interpretations of the constants will be contained in \( c_1 \) we can ignore them.) Choose \( c_\gamma \) so that \( f \) and \( f^{-1} \) are pointwise definable over \( c_\gamma \). Let \( c \) be a pseudofinite set containing \( c_\gamma \). We can assume that \( f \) permutes the atoms of \( B_\gamma \).

Let the language \( L \) be the union of an increasing chain of finite sublanguages \( (L_n; n < \omega) \). Assume by way of contradiction that for all \( n \), \( f \) is not pointwise definable on the atoms over any pseudofinite set (and hence not over any \( c_\alpha \)) using formulas from \( L_n \). Choose a sequence \( d_n \) of atoms so that for all \( n \), \( e_n = f(d_n) \) is not \( L_n \)-definable over \( \{d_n\} \cup c \) and both \( d_{n+1} \) and \( e_{n+1} \) are not definable over \( \{d_k : k \leq n\} \cup \{e_k : k \leq n\} \cup c \). Furthermore \( d_{n+1} \) should not be \( L_n \)-definable over \( \{d_k : k \leq n\} \cup c \). The choice of \( d_n \) is possible by hypothesis, since only a pseudofinite set of possibilities has been eliminated from the choice.

Let \( \bar{d}, \bar{e} \) realize the average type (modulo some ultrafilter) over \( c_\gamma \cup \{c\} \) of \( \{(d_k : k \leq n), (e_k : k \leq n) : n \in \omega\} \). These are pseudofinite sequences which have \( (d_n : n < \omega) \) and \( (e_n : n < \omega) \) as initial segments. Say \( \bar{d} = (d_i : i < n^*) \) for some non-standard natural number \( n^* \). Now let \( x \in B \) be the join of \( \{d_i : i < n^*\} \). (This join exists since \( \bar{d} \) is a pseudofinite sequence.) For every \( i < n^* \) there is a (standard) \( n_i \) so that \( f(d_i) \) is \( L_{n_i} \)-definable over \( \{d_i\} \cup c_\gamma \) and \( d_i \) is \( L_{n_i} \)-definable over \( \{f(d_i)\} \cup c_\gamma \). Since \( c \) is \( \theta \)-saturated, the coinitiality of \( n^* \setminus \omega \) is greater than \( \omega \). So there is some \( n \) so that \( \{i : n_i = n\} \) is coinitial. (Notice that for non-standard \( i \), there is no connection between \( e_i \) and \( f(d_i) \).

Choose \( k \) so that the formulas in \( L_n \) have at most \( k \) free variables. Let \( Z \) be the set of subsets \( Y \) of \( c \) of size \( k \) so that for all \( i < n^* \), neither \( d_i \) nor \( e_i \)
is \( L_n \)-definable from \( Y \cup \{d_j, e_j : j < i\} \). By the choice of \( \tilde{d}, \tilde{e} \) every subset of \( \mathfrak{C} \gamma \) of size \( k \) is an element of \( Z \).

Consider \( \omega > m > n + 1 \). We claim there is no atom \( y < f(x) \) such that \( d_m \) and \( y \) are \( L_n \)-interdefinable over elements of \( Z \). Suppose there is one and \( y = f(d_i) \). Then \( i \) is non-standard, since, by the choice of \( Z \), \( y \neq e_j \) for all \( j \). Since \( d_i \) is definable from \( \{f(d_i)\} \cup c \), \( d_i \) is definable from \( \{d_m\} \cup c \). This contradicts the choice of the sequence. We can finally get our contradiction. For \( i < n^* \), we have \( i < \omega \) if and only if for all \( j > n + 1 \), if there is an atom \( y < f(x) \) so that \( d_j \) and \( y \) are \( L_n \)-interdefinable over elements of \( Z \) then \( i < j \). □

We will want to work a bit harder and get that the automorphisms in the model above is actually definable on a large set. To this end we prove an easy graph theoretic lemma.

**Lemma 4.6** Suppose \( G \) is a graph and there is \( 0 < k < \omega \) so that the valence of each vertex is at most \( k \). Then there is a partition of (the set of nodes of) \( G \) into \( k^2 \) pieces \( A_0, \ldots, A_{k^2-1} \) so that for any \( i \) and any node \( v \), \( v \) is adjacent to at most one element of \( A_i \). Furthermore if \( \lambda \) is an uncountable cardinal, each \( A_i \) can be chosen to meet any \( \lambda \) sets of cardinality \( \lambda \).

**Proof** Apply Zorn’s lemma to get a sequence \( \langle A_0, \ldots, A_{k^2-1} \rangle \) of pairwise disjoint set of nodes such that for \( i < k^2 \) and node \( v \), \( v \) is adjacent to at most one member of \( A_i \) and \( \bigcup_{i < k^2} A_i \) maximal [under those constrains]. Suppose there is \( v \) which is not in any of the \( A_i \). Since \( v \) is not in any of the \( A_i \) for each \( i \) there is \( u_i \) adjacent to \( v \) and \( w_i \neq u_i \) adjacent to \( v \) so that \( w_i \in A_i \). If no such \( u_i, w_i \) existed we could extend the partition by adding \( v \) to \( A_i \). But as the valency of every vertex of \( G \) is \( \leq k \) there are at most \( k(k-1) < k^2 \) such pairs.

As for the second statement. Since the valence is finite every connected component is countable. Hence we can partition the connected components and then put them together to get a partition meeting every one of the \( \lambda \) sets. □

Actually for infinite graphs we can get a sharper bound. Given \( G \) we form an associated graph by joining vertices if they have a common neighbour. This gives a graph whose valence is at most \( k^2 - k \). We want to vertex colour this new graph. Obviously (see the proof above) it can be vertex coloured
in \(k^2 + 1 - k\) colours. In fact, by a theorem of Brooks ([3], Theorem 6.5.1), the result can be sharpened further. In our work we will only need that the colouring is finite, so these sharpenings need not concern us.

We will want to form in \(C\) a graph whose vertices are the atoms of \(B\) with a pseudofinite bound so that any atom \(b\) is adjacent to \(f(b)\). This is easy to do in the case where all the definitions of \(f(b)\) from \(\{b\} \cup c\) use only a finite sublanguage. In the general case (i.e., when \(cf\theta\) may be \(\omega\)) we have to cover the possible definitions by a pseudofinite set.

**Lemma 4.7** Continue with the notation of the proof. Then there is a pseudofinite set \(D\) and a pseudofinite natural number \(k^*\) and a set of tuples \(Z\) of length at most \(k^*\) so that for every formula \(\phi(\bar{x}, \bar{y})\) there is \(d \in D\) so that for all \(\bar{a} \in c\) and \(\bar{b} \in B\), the tuple \((d, \bar{a}, \bar{b}) \in Z\) if and only if \(\phi(\bar{a}, \bar{b})\).

**Proof** Since \(C\) is \(\aleph_0\)-saturated the lemma just says that a certain type is consistent. Now \(B \in C\) and \(C\) satisfies the separation scheme for all formulas (not just those of set theory). Hence for any formula \(\phi\), \(\{(\bar{a}, \bar{b}) : \bar{a} \in c, \bar{b} \in B\text{ and }\phi(\bar{a}, \bar{b})\}\) exists in \(C\).

Fix such sets \(D\) and \(Z\) for \(c\). Say that an atom \(a\) is \(D, Z\)-definable from \(b\) over \(c\) if there are \(d \in D\) and a tuple (perhaps of non-standard length) \(\bar{x} \in c\) so that \(a\) is the unique atom of \(B\) so that \((d, \bar{x}, b, a) \in Z\). We say that \(a\) and \(b\) are \(D, Z\)-interdefinable over \(c\) if \(a\) is \(D, Z\)-definable from \(b\) over \(c\) and \(b\) is \(D, Z\)-definable from \(a\) over \(c\). Notice (and this is the content of the last lemma) that if \(a\) is definable from \(b\) over \(c\) then it is \(D, Z\)-definable from \(b\) over \(c\).

We continue now with the notation of the theorem and the model we have built. Suppose \(f\) is an automorphism which is pointwise definable over a pseudofinite set \(c\). Define the \(c\)-graph to be the graph whose vertices are the atoms of \(B\) where \(a, b\) are adjacent if \(a\) and \(b\) are \(D, Z\)-interdefinable over \(c\). Since \(D, k^*\) and \(c\) are pseudofinite this is a graph whose valency is bounded by some pseudofinite number. So in \(C\) we will be able to apply Lemma 4.6. We will say that \((A_i : i < n^*)\) is a good partition of the \(c\)-graph if it is an element of \(C\) which partitions the atoms of \(B\) into a pseudofinite number of pieces and for any \(i\) and any \(a, a\) is adjacent to at most one element of \(A_i\). If \((A_i : i < n^*)\) is a good partition then for all \(i, j\) let \(f_{i,j}\) be the partial function from \(A_i\) to \(A_j\) defined by letting \(f_{i,j}(a)\) be the unique element of \(A_j\) if any so that \(a\) is adjacent to \(f_{i,j}(a)\). Otherwise let \(f_{i,j}(a)\) be undefined.
We have proved the following lemma.

**Lemma 4.8** Use the notation above. For all \( a \in A_i \) there is a unique \( j \) so that \( f(a) = f_{i,j}(a) \)

The proof of the last lemma applies in a more general setting. Since we will want to use it later, we will formulate a more general result.

**Lemma 4.9** Suppose \( \mathfrak{C} \) is an \( \aleph_0 \)-saturated model of an expansion of ZFC\(^-\) which satisfies the separation scheme for all formulas. If \( A, B \) are sets in \( \mathfrak{C} \) and \( f \) is a bijection from \( A \) to \( B \) such that \( f \) and \( f^{-1} \) are pointwise definable over some pseudofinite set, then there is (in \( \mathfrak{C} \)) a partition of \( A \) into pseudofinitely many sets \( (A_i : i < k^*) \) and a pseudofinite collection \( (f_{i,j} : i, j < k^*) \) of partial one-to-one functions so that for all \( i \), the domain of \( f_{i,j} \) is contained in \( A_i \) and for all \( a \in A_i \) there exists \( j \) so that \( f_{i,j}(a) = f(a) \). Moreover if we are given (in \( \mathfrak{C} \)) a family \( P \) of \( |A| \) (in \( \mathfrak{C} \)'s sense) subsets of \( A \) of cardinality \( |A| \), we can demand that for every \( i < k^* \) and \( A^* \in P \) we have \( |A_i \cap A^*| = |A| \) (in \( \mathfrak{C} \)'s sense).

**Lemma 4.10** Use the notation and assumptions above. For any \( i < k^* \) we have: \( f|A_i \) is definable.

**Proof** For each \( \gamma < \lambda \) such that \( \gamma \notin S \) and \( \text{cf} \gamma \geq \theta \) choose \( y_\gamma \) so that \( y_\gamma \) is the join of a pseudofinite set of atoms contained in \( A_i \) and containing \( A_i \cap B_\gamma \).

**Claim 4.11** For all \( a \in A_i \) so that \( a \leq y_\gamma \), \( f(a) \) is definable from \( f(y_\gamma) \) and the set \( \{f_{i,j} : j < n^*\} \). In particular this claim applies to all \( a \in A_i \cap B_\gamma \).

**Proof** (of the claim) First note that \( f(a) < f(y_\gamma) \) and so there is \( j \) such that \( f_{i,j}(a) < f(y_\gamma) \). Suppose that there is \( k \neq j \) so that \( f_{i,k}(a) < f(y_\gamma) \). Choose \( b \in A_i \) (not necessarily in \( B_\gamma \)) so that \( f(b) = f_{i,k}(a) \) (such a \( b \) must exist since every atom below \( y_\gamma \) is in \( A_i \)). But since \( f_{i,k} \) is a partial one-to-one function, \( a = b \). But \( f(a) = f_{i,j}(a) \neq f_{i,k}(a) \). This is a contradiction, so \( f_i|A_i \cap B_\gamma \) is defined by "\( b = f(a) \) if there exists \( g \in \{f_{i,j} : j < n^*\} \) so that \( b = g(a) \) and \( g(a) < f(y_\gamma) \)."
Choose now $N_\gamma$ of cardinality $< \theta$ so that the type of $f(y_\gamma), (f_{i,j}: j < n^*)/\mathcal{C}_\gamma$ does not split over $N_\gamma$. Notice that $f\upharpoonright(A_i \cap B_\gamma)$ is definable as a disjunction of types over $N_\gamma$, namely the types satisfied by the pairs $(a, f(a))$. By Fodor’s lemma and the cardinal arithmetic, there is $N$ and a stationary set where all the $N_\gamma = N$ and all the definitions as a disjunction of types coincide. So we have that $f\upharpoonright A_i$ is defined as a disjunction of types over $N$, set of cardinality $< \theta$. We now want to improve the definability to definability by a formula. We show that for all $\gamma$, $f\upharpoonright(A_i \cap B_\gamma)$ is defined by a formula with parameters from $N$. This will suffice since some choice of parameters and formula will work for unboundedly many (and hence all) $\gamma$.

Suppose that $f\upharpoonright(A_i \cap B_\gamma)$ is not definable by a formula with parameters from $N$. Consider the following type in variables $x_1, x_2, z_1, z_2$.

\[
\{ \phi(x_1, x_2) \text{ if and only if } \phi(z_1, z_2) : \phi \text{ a formula with parameters from } N \}
\cup \{ x_1, z_1 \leq y_\gamma \} \cup \{ \psi(x_1, x_2), \neg \psi(z_1, z_2) \},
\]

where $\psi(u, v)$ is a formula saying “there exists $g \in \{ f_{i,j} : j < n^* \}$ so that $u = g(v)$ and $g(u) < f(y_\gamma)$”.

This type is consistent since by hypothesis it is finitely satisfiable in $A_i \cap B_\gamma$. Since $\mathcal{C}$ is $\theta$-saturated there are $a_1, a_2, b_1, b_2$ realizing this type. Since $a_1, b_1 \leq y_\gamma$, $\psi(a_1, a_2)$ implies that $f(a_1) = a_2$ and similarly $\neg \psi(b_1, b_2)$ implies that $f(b_1) \neq b_2$. On the other hand $a_1, a_2$ and $b_1, b_2$ realize the same type over $N$. This contradicts the choice of $N$.

The information we obtain above is a little less than we want since we want a definability requirement on every automorphism of every definable Boolean algebra. It is easy to modify the proof so that we can get the stronger result. Namely in the application of the Black Box we attempt to predict the definition of a definable Boolean algebra in the scope of $P$ as well as the construction of the model $\mathcal{C}$ and an automorphism. With this change the proof goes as before. So the following stronger theorem is true.

**Theorem 4.12** There is a model $\mathcal{C}$ of $T_1$ so that if $B \subseteq P(\mathcal{C})$ is a definable atomic Boolean algebra and $f$ is any automorphism of $B$ (as a Boolean algebra) then there is a pseudofinite set $c$ such that for any atom $b \in B$, $f(b)$ is definable from $\{ b \} \cup c$.

As before if $\theta$ is taken to be uncountable then we can find a finite sublanguage to use for all the definitions.
The proof of the analogue of the Theorem 4.1 for ordered fields is quite similar. Let $T_1$ denote the theory constructed in Theorem 3.2.

**Theorem 4.13** There is a model $C$ of $T_1$ so that if $F_1, F_2 \subseteq P(C)$ are definable ordered fields and $f$ is any isomorphism from $F_1$ to $F_2$ then there is a pseudofinite set $c$ so that for all $b \in F_1$ $f(b)$ is definable from $\{b\} \cup c$.

**Proof** To simplify the proof we will assume that $F = P(C)$ is an ordered field and $f$ is an automorphism of $F$. The construction is similar to the one for Boolean algebras, although we will distinguish between the case $\theta = \omega$ and the case that $\theta$ is uncountable. We will need to take a little more care in the case of uncountable cofinalities. The difference from the case of Boolean algebras occurs at stages $\delta \in S$. Again we predict the sequence of structures $(C_i: i < \lambda)$ and an automorphism $f_\alpha$ of the ordered field $F$. We say that an obstruction occurs at $\alpha$ if we can make the choices below. (The intuition behind the definition of an obstruction is that there are infinitesimals of arbitrarily high order for which the automorphism is not definable.) There are two cases, the one where $\theta = \omega$ and the one where $\theta$ is uncountable.

Since it is somewhat simpler we will first consider the case $\theta = \omega$. Suppose an obstruction occurs at $\alpha$. By this we mean that we can choose $N_\alpha \in M_0^\alpha$ such that for each $i < \omega$ we can choose $a_\alpha^i \in [\gamma_{\alpha,i}, \gamma_{\alpha,i}^+]$ so that $a_\alpha^i / C_{\gamma_{\alpha,i}}$ does not split over $N_\alpha$, $a_\alpha^i > 0$, $a_\alpha^i$ makes the same cut in $F_{\gamma_{\alpha,i}}$ as 0 does and $f(a_\alpha^i)$ is not definable over $a_\alpha^i$ and parameters from $C_{\gamma_{\alpha,i}}$.

Then we let $x_\alpha^i = \sum_{j \leq i} a_\alpha^i$. We choose $x_\alpha$ so that it realizes the average type over some non-principal ultrafilter of $\{x_\alpha^i : i < \omega\}$. As before we can show that both the inductive hypothesis and Claim 4.3 are satisfied. Notice in the construction that $f_\alpha(x_\alpha^i)$ is not definable over $\{a_\alpha^i\} \cup C_{\gamma_{\alpha,i}}$, since $x_\alpha^i = x_{i-1}^\alpha + a_\alpha^i$ and $f(x_\alpha^i) = f(x_{i-1}^\alpha) + f(a_\alpha^i)$ (here we conventionally let $x_0^\alpha = 0$ when it is undefined) and each $M_i^\alpha$ is closed under $f_\alpha$. Notice as well that for all $i$, $x_\alpha^i < x_\alpha$ and $x_i^\alpha$ and $x_\alpha$ make the same cut in $C_{\gamma_{\alpha,i}}$.

Suppose now that $f$ is an automorphism which is not pointwise definable over any $C_\gamma$. Since $F$ is an ordered field this is equivalent to saying that $f$ is not definable on any interval. Also since $C_\gamma$ is contained in a pseudofinite set (in $C$) there is a positive interval which makes the same cut in the pseudofinite set (and hence in $F_\gamma$) 0 does. So we can choose $a_\gamma$ in this interval so that $f(a_\gamma)$ is not definable over $C_\gamma \cup \{a_\gamma\}$. Arguing as before we can find a set
$N \subseteq \mathfrak{C}_\gamma$ of cardinality less than $\theta$ such that for all but boundedly many $\gamma$ $a_\gamma$ can be chosen so that $a_\gamma/\mathfrak{C}_\gamma$ does not split over $N$. Arguing as before we get an ordinal $\alpha$ so that an obstruction occurs at $\alpha$. Consider now $f(x_\alpha)$, $i$ and a finite set $N_\alpha \subseteq \mathfrak{C}_{\gamma_{a_\alpha,i}}$ so that $f(x_\alpha)/\mathfrak{C}_{\gamma_{a_\alpha,i}}$ does not split over $N_\alpha \cup \{a_\alpha^i\}$. Since $f(x_\alpha^i)$ is not definable over $\{a_\alpha^i\} \cup \mathfrak{C}_{\gamma_{a_\alpha,i}}$, there is $b > f(x_\alpha^i)$, $b \in F_{\gamma_{a_\alpha,i}}$ which realizes the same type over $N_\alpha \cup \{a_\alpha^i\}$. (Otherwise $f(x_\alpha^i)$ would be the rightmost element satisfying some formula with parameters in this set.) But $f(x_\alpha^i) < f(x_\alpha) < b$. This contradicts the choice of $N_\alpha$.

We now consider the case where $\theta$ is uncountable. Here we have to do something different at limit ordinals $i$. If $i$ is a limit ordinal, we choose $x_\alpha^i \in [\gamma_{\alpha,i}^-, \gamma_{\alpha,i}^+] \cap M_{\alpha+1}^i$ to realize the average type of the $\{x_j^i : j < i\}$. For the successor case we let we let $x_{i+1}^\alpha = x_i^\alpha + a_{i+1}^\alpha$ (where the $a_j^\alpha$ are chosen as before). In this construction for all $\gamma$ and all $i$ $x_i^\alpha/\mathfrak{C}_\gamma$ does not split over $N_\alpha \cup \{x_j^\alpha : \gamma_{a_j,j} < \gamma\}$. This is enough to verify the inductive hypothesis and Claim 4.3 if we restrict the statement to a successor ordinal $i$. The rest of the proof can be finished as above.

To apply the theorem we explicate the explanation in the introduction.

Observation 4.14 For proving the compactness of $L(Q_{\mathfrak{O}_f})$ (or $L(Q_{\mathfrak{B}_A})$ etc) it suffices to prove

(*) Let $T$ be a first order theory $T$ and let $P$, $R$ be unary predicates such that for every model $M$ of $T$ and an automorphism $f$ of a definable ordinal field (or Boolean algebra etc) $\subseteq P^M$ definable (with parameter), for some $c \in M$, $R(x,y,c)$ defines $f$.

Then $T$ has a model $M^*$ such that: every automorphism $f$ of definable ordered field (or Boolean algebra etc.) $\subseteq P^M$; $f$ is defined in $M^*$ by $R(x,y,c)$ for some $c \in M$.

Proof Assume (*) and let $T_0$ be a given theory in the stronger logic; without loss of generality all formulas are equivalent to relations. For simplicity ignore function symbols.

For every model $M$ let $M'$ be the model with the universe $|M| \cup \{f : f$ a partial function from $|M|$ to $|M|\}$, and relations those of $M$, $P^{M'} = |M|$, $R^{M'} = \{(f,a,b) : a,b \in M, f$ a partial function from $|M|$ to $|M|$ and $f(a) = b\}$. There is a parallel definition of $T'$ from $T$, and if we intersect with the first order logic, we get a theory to which it suffices to apply (*).
5 Augmented Boolean Algebras

For Boolean algebras we don’t have the full result we would like to have but we can define the notion of augmented Boolean algebras and then prove the compactness theorem for quantification over automorphisms of these structures.

**Definition.** An augmented Boolean algebra is a structure \((B, \leq, I, P)\) so that \((B, \leq)\) is an atomic Boolean algebra, \(I\) is an ideal of \(B\) which contains all the atoms, \(P \subseteq B, |P| > 1\), for \(x \neq y \in P\) the symmetric difference of \(x\) and \(y\) is not in \(I\), and for all atoms \(x \neq y\) there is \(z \in P\) such that either \(x \leq z\) and \(y \not\leq z\) or vice versa.

Notice that if we know the restriction of an automorphism \(f\) of an augmented Boolean algebra to \(P\) then we can recover \(f\). Since for any atom \(x\), \(f(x)\) is the unique \(z\) so that for all \(y \in P\) \(z \leq f(y)\) if and only if \(x \leq y\) and the action of \(f\) on the atoms determine its action on the whole Boolean algebra.

Let \(Q_{\text{Aug}}\) be the quantifier whose interpretation is “\(Q_{\text{Aug}} f(B_1, B_2)\ldots\)” holds if there is an isomorphism \(f\) of the augmented Boolean algebras \(B_1, B_2\) so that . . . .

**Theorem 5.1** The logic \(L(Q_{\text{Aug}})\) is compact.

**Proof** By 3.1 (and 4.9.1) it is enough to show that if \(T\) is a theory which says that every automorphism of a definable Boolean algebra \(\subseteq P\) is definable by a fixed formula and \(T_1\) and \(\mathfrak{c}\) are as above then every automorphism \(f\) of a definable augmented Boolean algebra \((B, \leq, I, P)\) of \(P(\mathfrak{c})\) is definable. We work for the moment in \(\mathfrak{c}\). First note that \(\mathfrak{I}\) contains the pseudofinite sets so \(\mathfrak{c}\) thinks that the cardinality of \(B\) is some \(\mu^*\) and for every \(x, y \in P\) the symmetric difference of \(x\) and \(y\) contains \(\mu^*\) atoms (since \(\mathfrak{c}\) thinks that \(B\) is \(\mu^*\) saturated).

So, by Lemma 4.9 and Lemma 4.10, we can find a set of atoms \(A\) so that \(f\mid A\) is definable and for every \(x\) and \(y\), \(A\) contains some element in the symmetric difference. Now we can define \(f(y)\) as the unique \(z \in B\) so that for all \(a \in A, a \leq y\) if and only if \(f(a) \leq z\). Clearly \(f(y)\) has this property. Suppose for the moment that there is some \(z \neq f(y)\) which also has this property. Since \(f\) is an automorphism there is \(x \in P\) so that \(f(x) = z\). Now
choose \( a \in A \) in the symmetric difference of \( x \) and \( y \). Then \( f(a) \) is in exactly one of \( f(y) \) and \( f(x) \).

### 6 Ordered Fields

Here we will prove the compactness of the quantifier \( Q_{O\ell} \). Let \( Q_{O\ell} \) be the quantifier whose interpretation is \( "Q_{O\ell} f \ (F_1, F_2) \ldots " \) holds if there is an isomorphism \( f \) of the ordered fields \( F_1, F_2 \) so that \( \ldots \). We will use various facts about dense linear orders. A subset of a dense linear order is somewhere dense if it is dense in some non-empty open interval. A subset which is not somewhere dense is nowhere dense. The first few properties are standard and follow easily from the fact that a finite union of nowhere dense sets is nowhere dense.

**Proposition 6.1** If a somewhere dense set is divided into finitely many pieces one of the pieces is somewhere dense.

**Proposition 6.2** If \( \{ f_k : k \in K \} \) is a finite set of partial functions defined on a somewhere dense set \( A \) then there is a somewhere dense subset \( A' \subseteq A \) so that each \( f_k \) is either total on \( A' \) or no element of the domain of \( f_k \) is in \( A' \).

We can define an equivalence relation on functions to a linearly ordered set by \( f \equiv g \) if for every non-empty open interval \( I \) the symmetric difference of \( f^{-1}(I) \) and \( g^{-1}(I) \) is nowhere dense.

**Proposition 6.3** Suppose \( \mathcal{F} \) is a finite collection of functions from a somewhere dense set \( A \) to a linearly ordered set. Then there is \( \mathcal{I} \) a collection of disjoint intervals (not necessarily open) and a somewhere dense set \( A' \subseteq A \) so that for all \( f, g \in \mathcal{F} \) either \( f|A' \equiv g|A' \) or there are disjoint intervals \( I, J \in \mathcal{I} \) so that \( f(A') \subseteq I \) and \( g(A') \subseteq J \).

**Proof** The proof is by induction on the cardinality of \( \mathcal{F} \). Suppose that \( \mathcal{F} = \mathcal{G} \cup \{ g \} \) and \( \mathcal{J} \) is a set of intervals and \( A'' \subseteq A \) is a somewhere dense set satisfying the conclusion of the theorem with respect to \( \mathcal{G} \). If there are some \( f \in \mathcal{G} \) and \( A''' \) a somewhere dense subset of \( A'' \) such that \( f|A''' \equiv g|A''' \) then
we can choose a somewhere dense $A' \subseteq A''$ such that for all $I \in \mathcal{J}$ we have $(f|A')^{-1}(I) = (g|A')^{-1}(I)$. Then $A'$ and $\mathcal{J}$ are as required in the theorem. So we can assume that no such $f$ and $A''$ exist.

Let $\{f_0, \ldots, f_n\}$ enumerate a set of $\equiv$ class representatives of $\mathcal{G}$ and let $\mathcal{J} = \{J_0, \ldots, J_n\}$ where $f_k(A'') \subseteq J_k$ for all $k$ (so for $\ell < m \leq n$ $J_\ell \cap J_m = \emptyset$).

By induction on $k$ we will define a descending sequence $A_k$ of somewhere dense subsets of $A''$ and intervals $I_k \subseteq J_k$ with the property that $f_k(A_k) \subseteq I_k$ and $g(A_k)$ is disjoint from $I_k$. Let $A_{-1} = A''$. Consider any $k$ and suppose $A_{k-1}$ has been defined. If possible, choose an interval $J'_k \subseteq J_k$ so that it is symmetric difference of $(g|A_{k-1})^{-1}(J'_k)$ is somewhere dense. There are two possibilities. If $(f_k|A_{k-1})^{-1}(J'_k) \cap (g|A_{k-1})^{-1}(J'_k)$ is somewhere dense, then let $A_k = (f_k|A_{k-1})^{-1}(J'_k) \cap (g|A_{k-1})^{-1}(J'_k)$ and let $I_k = J'_k$. Otherwise, we can choose $I_k$ as a subinterval of $J_k \setminus J'_k$ and $A_k$ as a somewhere dense subset of $(g|A_{k-1})^{-1}(J'_k)$ so that $f_k(A_k) \subseteq I_k$. If there was no respective $J'_k$ then we would have $f_k|A_{k-1} \equiv g|A_{k-1}$ - contrary to our assumption. Given $A_n$ and $I_0, \ldots, I_n$, we can choose $A_{n+1} \subseteq A_n$ somewhere dense and an interval $I_{n+1}$ disjoint from $I_k$, for all $k$ so that $g(A_{n+1}) \subseteq I_{n+1}$.

Finally we let $A' = A_{n+1} \cap \bigcap_{f \in \mathcal{G}} f^{-1}(\bigcup_{k \leq n} I_k)$ and $\mathcal{I} = \{I_0, \ldots, I_n\}$. 

\[\text{Theorem 6.4} \quad \text{The logic } L(Q_{O^l}) \text{ is compact.}\]

\[\text{Proof} \quad \text{We work in the model } \mathfrak{C} \text{ constructed before (suffic by 3.2 (and 4.9.1)). Suppose we have two definable ordered fields } F_1, F_2 \text{ contained in } P(\mathfrak{C}) \text{ and an isomorphism } f \text{ between them. Since } f \text{ and } f^{-1} \text{ are locally definable over a pseudofinite set there are } (A_i: i < k^*) \text{ and } (f_{i,j}: i, j < k^*) \text{ as in Lemma 13}. \text{ Since } \mathfrak{C} \text{ satisfies that } k^* \text{ is finite, there is some } i \text{ so that } A_i \text{ is somewhere dense. Fix such an } i \text{ and for notational simplicity drop the subscript } i. \text{ So } f_j \text{ denotes } f_{i,j}. \text{ By restricting to a somewhere dense subset (and perhaps eliminating some of the } f_j \text{ we can assume that each } f_j \text{ is a total function (we work in } \mathfrak{C} \text{ to make this choice) (use 6.2). Again choosing a somewhere dense subset we can find a somewhere dense set } A \text{ and a collection } \mathcal{I} \text{ of disjoint intervals of } F_2 \text{ so that the conclusion of Proposition 5.3 is satisfied. Since each } f_j \text{ is one-to-one we can assume that all the intervals in } \mathcal{I} \text{ are open. Choose now an interval } J \text{ contained in } F_1 \text{ so that } A \text{ is dense in } J. \text{ Next choose } a_1 < a_2 \in J \text{ and an interval } I \in \mathcal{I} \text{ so that } f(a_1), f(a_2) \in I. \text{ To see that such objects exist, first choose any } a_1 \in A \cap J. \text{ There is a unique}\]
interval $I \in \mathcal{I}$ so that $f(a_1) \in I$. Choose $b \in I$ so that $f(a_1) < b$ and let $a_2$ be any element of $J \cap A$ such that $a_1 < a_2 < f^{-1}(b)$. Of course this definition of $a_1, a_2$ cannot be made in $\mathfrak{c}$, but the triple $(a_1, a_2, I)$ exists in $\mathfrak{c}$. Without loss of generality we can assume that $A \subseteq (a_1, a_2) = J$. Consider now $\mathcal{F} = \{f_j : f_j(A) \subseteq I\}$. This is an equivalence class of functions and for every $a \in A$, there is some $g \in \mathcal{F}$ so that $g(a) = f(a)$.

The crucial fact is that for all $b \in J$ and $g \in \mathcal{F}$, $B = \{a \in A : b < a \text{ and } g(a) < f(b)\}$ is nowhere dense. Assume not. Since $\mathcal{F}$ is an equivalence class, for all $h \in \mathcal{F}$ \{a \in B : h(a) \geq f(b)\} is a nowhere dense set. But since $\mathcal{F}$ is a pseudofinite set, \{a \in B : \text{for some } h \in \mathcal{F}, h(a) \geq f(b)\} is nowhere dense. So there is some $a \in A$ so that $b < a$ and for all $g \in \mathcal{F}$ $g(a) < f(b)$. This gives a contradiction since there is some $g$ so that $f(a) = g(a)$. Similarly, \{a \in A : a < b \text{ and } g(a) > f(b)\} is nowhere dense.

With this fact in hand we can define $f$ on $J$, by $f(b)$ is the greatest $x$ so that for all $g \in \mathcal{F}$, both \{a \in A : b < a \text{ and } g(a) < x\} and \{a \in A : a < b \text{ and } g(a) > x\} are nowhere dense. Since an isomorphism between ordered fields is definable if and only if it is definable on an interval we are done.

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