FRÉCHET DIFFERENTIABILITY OF $S^p$ NORMS

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ABSTRACT. One of the long standing questions in the theory of Schatten-von Neumann ideals of compact operators is whether their norms have the same differentiability properties as the norms of their commutative counterparts. We answer this question in the affirmative. A key technical observation underlying our proof is a discovery of connection between this question and recent affirmative resolution of L.S. Koplienko’s conjecture concerning existence of higher order spectral shift functions.

It was conjectured in [21] and [2, Remark, p.35], that the norm of the Schatten-von Neumann class $S^p$ on an arbitrary real Hilbert space $\mathcal{H}$ is $[p]$-times Fréchet differentiable for any $1 < p < \infty$, $p \notin \mathbb{N}$. That is,

**Theorem 1.** The function $H \mapsto \|H\|_p^p$, $H \in S^p$, $1 < p < \infty$ is $m$-times Fréchet differentiable away from zero, where $m = [p]$ and $p \notin \mathbb{N}$.

This manuscript gives a proof to this conjecture. Let $\mathcal{H}$ be an arbitrary complex Hilbert space and let us consider the Schatten-von Neumann class $S^p$ associated with $\mathcal{H}$ as a Banach space over the field of real numbers. We prove (in Theorem below) the following Taylor expansion result (for all relevant definitions and terminology concerning differentials of abstract functions we refer to [10]).

**Theorem.** If $H \in S^p$, $\|H\|_p \leq 1$, $1 < p < \infty$ and if $m \in \mathbb{N}$ is such that $m < p \leq m + 1$, then there are bounded symmetric polylinear forms $\delta_H^{(1)} : S^p \mapsto \mathbb{R}$, $\delta_H^{(2)} : S^p \times S^p \mapsto \mathbb{R}$, . . . , $\delta_H^{(m)} : \underbrace{S^p \times \ldots \times S^p}_{m\text{-times}} \mapsto \mathbb{R}$

such that

$$\|H + V\|_p^p - \|H\|_p^p - \sum_{k=1}^{m} \delta_H^{(k)} \left( \underbrace{V, \ldots, V}_{k\text{-times}} \right) = O\left(\|V\|_p^p\right),$$

where $V \in S^p$ and $\|V\|_p \to 0$.

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1Symbol $\lfloor \cdot \rfloor$ stands for the integral part function.

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Taking into account that for an even \( p \in \mathbb{N} \) the norm of \( S^p \) is obviously infinitely many times differentiable, we indeed confirm the conjecture that the norm of \( S^p \) and that of their classical counterpart \( \ell^p \) share the same differentiability properties. For the proofs of corresponding commutative results see [5] and [20].

Our techniques are based on a new approach to and results from the theory of multiple operator integration presented in [15]. In that paper the authors (jointly with A. Skripka) applied that theory to resolve L.S. Koplienko’s conjecture that a spectral shift function exists for every integral \( p > 2 \). Suitably enhanced and strengthened technical estimates from [15] and its companion paper [16] are crucially used in the proofs below.

Finally, we mention that a closely related problem concerning differentiability properties of the norm of general non-commutative \( L_p \)-spaces was also stated in [13]. Our methods allow a further extension to cover also a case of \( L_p \)-spaces associated with an arbitrary semifinite von Neumann algebra \( M \). This extension will be presented in a separate article, however in the last section we resolve this problem for a special case when von Neumann algebra \( M \) is of type I.

1. MULTIPLE OPERATOR INTEGRALS

The proof of Theorem [15] is based on methods drawn from the theory of multiple operator integrals. A brief account of that theory is given below together with some new results.

Multiple operator integrals from [12] and [3]. Let \( \mathcal{A}_m \) be the class of functions \( \phi : \mathbb{R}^{m+1} \mapsto \mathbb{C} \) admitting the representation

\[
\phi(x_0, \ldots, x_m) = \int_{\Omega} \prod_{j=0}^{m} a_j(x_j, \omega) \, d\mu(\omega),
\]

for some finite measure space \( (\Omega, \mu) \) and bounded Borel functions

\[
a_j(\cdot, \omega) : \mathbb{R} \mapsto \mathbb{C}.
\]

The class \( \mathcal{A}_m \) is in fact an algebra with respect to the operations of pointwise addition and multiplication [3, Proposition 4.10]. The formula

\[
\|\phi\|_{\mathcal{A}_m} = \inf \int_{\Omega} \prod_{j=0}^{m} \|a_j(\cdot, \omega)\|_{\infty} \, d|\mu|(\omega),
\]
where the infimum is taken over all possible representations \( \mathfrak{A}_m \) (see [11]).

For every \( \phi \in \mathfrak{A}_m \), and a fixed \((m+1)\)-tuple of self-adjoint operators \( \vec{H} := (H_0, \ldots, H_m) \), the multiple operator integral

\[
T_\phi : S^{p_1} \times \cdots \times S^{p_m} \mapsto S^p, \quad \text{where } \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}. \tag{3}
\]

is defined as follows

\[
T_\phi (V_1, \ldots, V_m) = \int_{\Omega} a_0(H_0, \omega) V_1 a_1(H_1, \omega) \cdots V_m a_m(H_m, \omega) d\mu(\omega),
V_j \in S^{p_j}, \ j = 1, \ldots, m.
\]

Here \( a_i \)'s and \((\Omega, \mu)\) are taken from the representation \( \mathfrak{A}_m \) and one of the main results of this theory is that the value \( T_\phi (V_1, \ldots, V_m) \) does not depend on that representation [12, Lemma 3.1], [3, Lemma 4.3]. If it is necessary to specify the \((m+1)\)-tuple \( \vec{H} \) used in the definition of the multiple operator integral \( T_\phi \), we write \( T_{\vec{H}} \). Observe that \( T_\phi \) is a multilinear operator and, if \( 1 \leq p \leq \infty \), \( T_\phi \) is bounded, i.e.,

\[
\| T_\phi \|_{\vec{p} \rightarrow p} \leq \| \phi \|_{\mathfrak{A}_m}, \tag{4}
\]

where the norm \( \| T_\phi \|_{\vec{p} \rightarrow p} \) is the norm of multilinear operator, that is

\[
\| T_\phi \|_{\vec{p} \rightarrow p} := \sup \| T_\phi (V_1, \ldots, V_m) \|_p,
\]

where \( \vec{p} = (p_1, \ldots, p_m) \) and the supremum is taken over all \( m \)-tuples \((V_1, \ldots, V_m)\) such that \( \| V_j \|_{p_j} \leq 1, j = 1, \ldots, m \). The proof of this assertion follows along the same line of thought as in [3, Lemma 4.6] (see also [1, Section 4.1] and [15, Lemma 3.5]).

The transformation \( T_\phi \) with \( \phi \in \mathfrak{A}_m \) defined above has the following simple algebraic property [3, Proposition 4.10(ii)]. If \( b_j, j = 0, \ldots, m \) are bounded Borel functions, then \( \psi \in \mathfrak{A}_m \), where

\[
\psi(x_0, \ldots, x_m) = b_0(x_0) \cdots b_m(x_m) \phi(x_0, \ldots, x_m)
\]

and

\[
T_\psi(V_1, \ldots, V_m) = T_\phi(V'_1, \ldots, V'_m) \tag{5}
\]

where

\[
V'_1 = b_0(H_0) V_1 b_1(H_1) \quad \text{and} \quad V'_j = V_j b_j(H_j), \ j = 2, \ldots, m.
\]
In particular, if \( \psi(x_0, \ldots, x_m) = x_0^{s_0} \cdots x_m^{s_m} \phi(x_0, \ldots, x_m) \), where \( s_0, \ldots, s_m \) are non-negative integers and \( \hat{H} \) consists of bounded operators, then

\[
T_\psi(V_1, \ldots, V_m) = T_\psi(H_0^{s_0}V_1 H_1^{s_1}, V_2 H_2^{s_2}, \ldots, V_m H_m^{s_m}).
\]  

(6)

**A version of multiple operator integrals from [15].** We shall also need a closely related but distinct version of operators \( T_\psi \) introduced recently in [15].

Let \( m \in \mathbb{N} \). Let \( dE_\lambda, \lambda \in \mathbb{R} \) be the spectral measure corresponding to the self-adjoint operator \( H_j \) from the \((m+1)\)-tuple \( \hat{H} \). We set \( E_j^l = E_j \big[ \frac{l}{n}, \frac{l+1}{n} \big) \), for every \( n \in \mathbb{N} \) and \( l \in \mathbb{Z} \), where \( E^l[a, b) \) is the spectral projection of the operator \( H_j \) corresponding to the semi-interval \([a, b)\).

Let \( 1 \leq p_j \leq \infty \), with \( 1 \leq j \leq m \), be such that \( 0 \leq \frac{1}{p_1} + \ldots + \frac{1}{p_m} \leq 1 \). Let \( V_j \in S^{p_j} \) and denote \( V = (V_1, \ldots, V_m) \). Fix a bounded Borel function \( \phi : \mathbb{R}^{m+1} \rightarrow \mathbb{C} \).

Suppose that for every \( n \in \mathbb{N} \) the series

\[
S_{\phi,n}(V) := \sum_{l_0, \ldots, l_m \in \mathbb{Z}} \phi \left( \frac{l_0}{n}, \ldots, \frac{l_m}{n} \right) E_{l_0,n}^0 V_1 E_{l_1,n}^1 V_2 \cdots V_m E_{l_m,n}^m
\]

converges in the norm of \( S^p \), where \( \frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m} \) and

\[
V \mapsto S_{\phi,n}(V), \quad n \in \mathbb{N},
\]

is a sequence of bounded multilinear operators \( S^{p_1} \times \ldots \times S^{p_m} \mapsto S^p \). If the sequence of operators \( \{S_{\phi,n}\}_{n \geq 1} \) converges strongly to some multilinear operator \( \hat{T}_\phi \), then, according to the Banach-Steinhaus theorem, \( \{S_{\phi,n}\}_{n \geq 1} \) is uniformly bounded and the operator \( \hat{T}_\phi \) is also bounded. In this case the operator \( \hat{T}_\phi \) is called a modified multiple operator integral. If it is necessary to specify the \((m+1)\)-tuple \( \hat{H} \) used in the definition of the multiple operator integral \( \hat{T}_\phi \), we write \( \hat{T}_\phi^{\hat{H}} \).

Let \( \mathcal{C}_m \) be the class of functions \( \phi : \mathbb{R}^{m+1} \mapsto \mathbb{C} \) admitting the representation (2) with bounded continuous functions

\[
a_j(\cdot, s) : \mathbb{R} \mapsto \mathbb{C}
\]

for which there is a growing sequence of measurable subsets \( \{\Omega^{(k)}\}_{k \geq 1} \), with \( \Omega^{(k)} \subseteq \Omega \) and \( \cup_{k \geq 1} \Omega^{(k)} = \Omega \) such that the families

\[
\{a_j(\cdot, s)\}_{s \in \Omega^{(k)}}, \quad 0 \leq j \leq m,
\]
are uniformly bounded and uniformly equicontinuous. The class $C_m$ has the norm

$$\|\phi\|_{C_m} = \inf \int_{\Omega} \prod_{j=0}^{m} \|a_j(\cdot, s)\|_\infty \, d|\mu|(s),$$

where the infimum is taken over all possible representations as specified above.

Hence, we have

$$\|\phi\|_{A_m} \leq \|\phi\|_{C_m}, \quad \forall \phi \in C_m. \quad (7)$$

The following lemma demonstrates a connection between two types of operator integrals $\hat{T}_\phi$ and $T_\phi$.

**Lemma 2** ([15, Lemma 3.5]). Let $1 \leq p_j \leq \infty$, with $1 \leq j \leq m$, be such that $0 \leq \frac{1}{p_1} + \ldots + \frac{1}{p_m} \leq 1$. For every $\phi \in C_m$, the operator $\hat{T}_\phi$ exists and is bounded on $S^{p_1} \times \ldots \times S^{p_m}$, with

$$\|\hat{T}_\phi\|_{p \to p} \leq \|\phi\|_{C_m}. \quad (8)$$

Moreover, $\hat{T}_\phi = T_\phi$.

The result above is stated in [15] under the additional assumption that

$$\hat{H} = (H, H, \ldots, H),$$

however, it is straightforward to see that the latter restriction is redundant.

It is important to observe that the class of functions to which the definition from [12] and [3] is applicable is distinct from the class of functions for which the definition from [15] makes sense. Observe also that the algebraic relations from [5] and [6] continue to hold for the modified operators $\hat{T}_\phi$ (see [15, Lemma 3.2]).

**Besov spaces.** For the function $f \in L_1$ by $\hat{f}$ we denote its Fourier transform, i. e.,

$$\hat{f}(t) = \int f(x)e^{-ixt}dx.$$

We shall also sometimes use the same symbol for Fourier transform of a tempered distribution.

For a given $s \in \mathbb{R}$, the homogeneous Besov space $\dot{B}^{s}_{\infty 1}$ is the collection of all generalized functions on $\mathbb{R}$ satisfying the inequality

$$\|f\|_{\dot{B}^{s}_{\infty 1}} := \sum_{n \in \mathbb{Z}} 2^{sn} \|f * W_n\|_\infty < +\infty,$$

where

$$W_n(x) = 2^n W_0(2^n x), \quad x \in \mathbb{R}, \quad n \in \mathbb{Z} \quad (9)$$
and $W_0$ is a smooth function whose Fourier transform is like on fig. 1. We also require that

$$\hat{W}_0(y) + \hat{W}_0\left(\frac{y}{2}\right) = 1, \quad 1 \leq y \leq 2.$$  

![Figure 1. The Fourier transform $\hat{W}_0$](image)

Observe that if $f$ is a polynomial, then its Fourier transform is supported at $x = 0$, and since the functions $W_n$ are not supported at 0, we have $\|f\|_{\tilde{B}^s_{\infty1}} = 0$. The modified homogeneous Besov class $\tilde{B}^s_{\infty1}$ is given by

$$\tilde{B}^s_{\infty1} = \left\{ f \in B^s_{\infty1}, f^{(|s|)} \in L^\infty := L_\infty(\mathbb{R}) \right\},$$

where $[s]$ is the integral part of $s$. The value of the norm $\|\cdot\|_{\tilde{B}^s_{\infty1}}$ on the elements of $\tilde{B}^s_{\infty1}$ will be denoted by $\|\cdot\|_{\tilde{B}^s_{\infty1}}$. Note also that the norms $\|f\|_{\tilde{B}^s_{\infty1}}$ and $\|f^{(|s|)}\|_{\tilde{B}^{s-|s|}_{\infty1}}$ are equivalent on the space $\tilde{B}^s_{\infty1}$. Indeed, this equivalence may be easily inferred from [23, (36)] (all what one needs to recall is that the Poisson integral used in [23, (36)] commutes with the differentiation, that is $P(t)f^{(k)} = (P(t)f)^{(k)}$).

The elements of $\tilde{B}^s_{\infty1}$ can also be described as follows

$$f \in \tilde{B}^s_{\infty1} \iff f(x) = c_0 + c_1x + \ldots + c_mx^m + f_0(x),$$

$$c_j \in \mathbb{C}, \quad j = 0, \ldots, m, \quad f_0 \in B^s_{\infty1}, \quad \text{supp } f_0 \subseteq \mathbb{R} \setminus \{0\}. $$

Recall that $\Lambda_\alpha$ is the class of all Hölder functions of exponent $0 < \alpha < 1$, that is the functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{\Lambda_\alpha} := \sup_{t_1, t_2} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\alpha} < +\infty.$$  

We also need the following simple criterion.

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2This condition ensures that

$$\sum_{n \in \mathbb{Z}} \hat{W}_n(x) = 1, \quad x \neq 0.$$
Lemma 3. If \( f^{(m-1)} \in \Lambda_{1-\varepsilon} \) and \( f^{(m)} \in \Lambda_\varepsilon \) for \( 0 < \varepsilon < 1 \) for some \( m \in \mathbb{N} \), then \( f \in \tilde{B}_n^{m} \).

The case \( m = 1 \) is proved in \([14]\) proof of Theorem 4, the proof of the general case is identical to the case \( m = 1 \). We leave details to the reader.

Polynomial integral momenta. Let \( \mathcal{P}_m \) be the class of polynomials of \( m \) variables with real coefficients. Let \( S_m \) be the simplex

\[
S_m := \left\{ (s_0, \ldots, s_m) \in \mathbb{R}^{m+1} : \sum_{j=0}^{m} s_j = 1, \ s_j \geq 0, \ j = 0, \ldots, m \right\}
\]

and let

\[
R_m := \left\{ (s_1, \ldots, s_m) \in \mathbb{R}^{m} : \sum_{j=1}^{m} s_j \leq 1, \ s_j \geq 0, \ j = 1, \ldots, m \right\}.
\]

We equip the simplex \( S_m \) with the finite measure \( d\sigma_m \) defined by the requirement that the equality

\[
\int_{S_m} \phi(s_0, \ldots, s_m) \, d\sigma_m = \int_{R_m} \phi \left( 1 - \sum_{j=1}^{m} s_j, s_1, \ldots, s_m \right) \, dv_m,
\]

holds for every continuous function \( \phi : \mathbb{R}^{m+1} \to \mathbb{C} \), where \( dv_m \) is the Lebesgue measure on \( \mathbb{R}^m \). It can be seen via a straightforward change of variables in (10) that the measure \( d\sigma_m \) is invariant under any permutation of the variables \( s_0, \ldots, s_m \).

Let \( \tilde{s} = (s_1, \ldots, s_m) \in R_m \) and let \( (s_0, \tilde{s}) \in S_m \), that is \( s_0 = 1 - \sum_{j=1}^{m} s_j \). Given \( h \in L^\infty \), and \( Q \in \mathcal{P}_m \), we set

\[
\phi_{m,h,Q}(\tilde{x}) = \int_{S_m} Q(\tilde{s}) \ h \left( \sum_{j=0}^{m} s_j x_j \right) \, d\sigma_m,
\]

where \( \tilde{x} = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \). Following the terminology set in \([15]\) we shall call the function \( \phi_{m,h,Q} \) a polynomial integral momentum. This notion plays a crucial role in our present approach. Indeed, the functions \( \phi : \mathbb{R}^{m+1} \to \mathbb{C} \) for which we shall be considering multiple operator integrals \( T_\phi \) and \( \hat{T}_\phi \) are in fact of the form \( \phi_{m,h,Q} \) for suitable choice of \( h \) and \( Q \).

Multiple operator integral of a polynomial integral momentum. In this subsection, we describe the connection between the norm \( \|\phi_{m,h,Q}\|_{C_m} \) and a norm of the function \( h \) and thereby connect the latter with the norm \( \|T_\phi\|_{p \to p} \) (see (7) and (4)). The following result extends \([12]\) Theorem 5.1. It also improves \([15]\) Lemma 5.2.

Theorem 4. Let \( \phi = \phi_{m,h,Q} \) be a polynomial integral momentum.
(i) If $\text{supp} \hat{h} \subseteq [2^{N-1}, 2^{N+1}]$, for some $N \in \mathbb{Z}$, then $\phi \in \mathcal{E}_m$ and

$$\|\phi\|_{\mathcal{E}_m} \leq \text{const} \|h\|_{\infty}.$$ 

(ii) If $h \in \tilde{B}_0^{0,1}$, then $\phi \in \mathcal{E}_m$ and

$$\|\phi\|_{\mathcal{E}_m} \leq \text{const} \|h\|_{\tilde{B}_0^{0,1}}.$$ 

The constants above do not depend on $N$ or $h$.

The proof is based on the following lemma

**Lemma 5.** (i) A tempered distribution $r_a$, $a > 0$ defined via Fourier transform by

$$\hat{r}_a(y) = 1 - \frac{a}{|y|}, \text{ if } |y| > a \text{ and } \hat{r}_a(y) = 0, \text{ otherwise}$$

is a finite measure whose total variation satisfies

$$c_0 := \sup_{a > 0} \|r_a\|_1 < +\infty.$$ 

(ii) In particular, if $h \in L^\infty$ such that $\text{supp} \hat{h} \subseteq \mathbb{R}_+$, then

$$\|h_{a,\gamma}\|_{\infty} \leq c_0 \|h\|_{\infty}, \forall a > 0, \gamma \geq 1,$$

where

$$h_{a,\gamma}(x) := \int_0^\infty \left[\frac{y}{y+a}\right]^\gamma \hat{h}(y+a) e^{i xy} dy.$$ 

(iii) If $\text{supp} \hat{h} \subseteq [2^{N-1}, 2^{N+1}]$, then

$$\|h_{m,N}\|_{\infty} \leq \text{const} \cdot 2^{mN} \|h\|_{\infty},$$

where

$$h_{m,N}(x) = \int_0^\infty y^m \hat{h}(y) e^{i xy} dy.$$ 

The constant above does not depend on $N$ and $h$. 
Proof of Lemma 5 Combining [14, Lemma 7] and the assumptions
\[ 1 - r_1 \in L^2(\mathbb{R}) \text{ and } \frac{d}{dy} (1 - r_1) \in L^2(\mathbb{R}), \]
we see that the function \( 1 - r_1 \) is a Fourier transform of an \( L^1 \)-function. Thus, \( r_1 \) is a finite measure as a combination of former \( L^1 \)-function and the Dirac delta function.

Observe further that
\[ r_a(x) = ar_1(ax), \quad x \in \mathbb{R}, \]
so
\[ \|r_a\|_1 = \|r_1\|_1. \]
This completes the proof of (i).

The part (ii) follows from Young’s inequality and the observation that
\[
\mathcal{I}_{a,\gamma}(x) = \int_{a}^{\infty} \left[ \frac{y - a}{y} \right]^{\gamma} \mathcal{I}(y) e^{ixy} e^{-ix\gamma a} dy = e^{-ix\gamma a} r_a \ast \cdots \ast r_a \ast h(x). \tag{12}
\]
For the part (iii), we consider the function \( \delta_{m,0} \) such that its Fourier transform is smooth and is as follows
\[
\supp \hat{\delta}_{m,0} \subseteq \left[ \frac{1}{4}, 4 \right] \quad \text{and} \quad \hat{\delta}_{m,0}(y) = \begin{cases} y^m, & \text{if } \frac{1}{2} \leq y \leq 2. \end{cases}
\]
By, e.g., [14, Lemma 7], \( \delta_{m,0} \in L^1(\mathbb{R}) \). We also set
\[
\delta_{m,N}(x) = 2^{(m+1)N} \delta_{m,0} \left( x2^N \right).
\]
Clearly, \( \delta_{m,N} \in L^1(\mathbb{R}) \) and
\[
\|\delta_{m,N}\|_1 = 2^{mN} \|\delta_{m,0}\|_1.
\]
Observe also that on the Fourier side
\[
\hat{\delta}_{m,N}(y) = 2^{mN} \hat{\delta}_{m,0}(2^{-N}y).
\]
In particular,
\[
\hat{\delta}_{m,N}(y) = \begin{cases} y^m, & \text{if } 2^{N-1} \leq y \leq 2^{N+1}. \end{cases}
\]
The claim now follows from
\[
\mathcal{I}_{m,N}(x) = \delta_{m,N} \ast h(x) \tag{13}
\]
and Young’s inequality. \( \square \)
Proof of Theorem 4. To prove part (i), we fix a function \( h \) such that \( \text{supp} \hat{h} \subseteq [2^{N-1}, 2^{N+1}] \). Using the definition of the integral momentum \( \phi_{m,h,Q} \) and the Fourier expansion

\[
h(x) = \int_0^\infty \hat{h}(y) e^{ixy} dy,
\]

we obtain

\[
\phi_{m,h,Q}(\tilde{x}) = \int_{S_m} Q(\tilde{s}) \hat{h}(s_0x_0 + \ldots + s_mx_m) \, d\sigma_m
\]

\[
= \int_0^\infty dy \int_{S_m} Q(\tilde{s}) \hat{h}(y) e^{iyy_0} \ldots e^{iyy_m} \, d\sigma_m.
\]

We shall now make a substitution in the latter integration via replacing the current integration variables \( y \) and \( s_j, j = 1, \ldots, m \) with the variables \( y_j, j = 0, \ldots, m \) such that

\[
y_j = y s_j, \quad \text{if } j = 1, \ldots, m \quad \text{and} \quad y_0 = y s_0 = y \cdot \left( 1 - \sum_{j=1}^m s_j \right).
\]

This substitution transforms the domain of integration

\[
y \geq 0 \quad \text{and} \quad s_j \geq 0 \quad \text{and} \quad s_1 + \ldots + s_m \leq 1
\]

into the first octant

\[
y_j \geq 0, \quad j = 0, \ldots, m.
\]

Introducing the notation

\[
J_k(a) := \begin{bmatrix}
a & -y & -y & \cdots & -y & -y \\
s_1 & y & 0 & \cdots & 0 & 0 \\
s_2 & 0 & y & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_{k-1} & 0 & 0 & \cdots & y & 0 \\
s_k & 0 & 0 & \cdots & 0 & y
\end{bmatrix}, \quad k \leq m, \quad a \in \mathbb{R},
\]

we observe that the Jacobian of our substitution is given by

\[
J_m(1 - s_1 - \ldots - s_m).
\]

Using the last column decomposition of the latter determinant, we obtain

\[
J_k(a) = (-1)^k (-y) (-1)^{k-1} s_k y^{k-1} + (-1)^{2k} y J_{k-1}(a) = (-1)^{2k} s_k y^k + y J_{k-1}(a) = s_k y^k + y J_{k-1}(a), \quad \text{for every } k \leq m, \quad a \in \mathbb{R}.
\]
Thus,
\[ f_k(a) = y f_{k-1}(a) + s_k y^k \]
\[ = y (y f_{k-2}(a) + s_{k-1} y^{k-1}) + s_k y^k \]
\[ = y^2 f_{k-2}(a) + y^k (s_{k-1} + s_k) \ldots \]
\[ = y^k f_0(a) + y^k (s_1 + \ldots + s_k), \quad \text{for every } k \leq m. \]

Since \( f_0(1 - s_1 - \ldots - s_m) = 1 - s_1 - \ldots - s_m \), the Jacobian of the substitution above is
\[ J_m(1 - s_1 - \ldots - s_m) = y^m. \]

Therefore, the integral momentum \( \phi_{m,h,Q} \) takes the form
\[ \phi_{m,h,Q}(\tilde{x}) = \int_{\mathbb{R}^{m+1}} Q(\tilde{y}) y^{-m} \hat{h}(y) e^{i y_0 x_0} \ldots e^{i y_m x_m} \, dy_0 \ldots dy_m, \]

where
\[ y := y_0 + \ldots + y_m \text{ and } \tilde{s} = (s_1, \ldots, s_m), \quad s_j := \frac{y_j}{y}, \quad j = 1, \ldots, m. \]

Observe that, since \( \text{supp } \hat{h} \subseteq [2^{N-1}, 2^{N+1}] \), the integration over \( \mathbb{R}^{m+1} \) is in fact only taken over the strip
\[ y_j \geq 0, \quad j = 0, \ldots, m \text{ and } 2^{N-1} \leq y_0 + \ldots + y_m \leq 2^{N+1}. \]

For the rest of the proof observe that it suffices to show the claim of the theorem for a monomial
\[ Q(\tilde{s}) = s_0^{\gamma_0} \cdot s_1^{\gamma_1} \ldots \cdot s_m^{\gamma_m}. \]

For such monomial, we shall consider two different scenarios.

Assume first that the monomial \( Q \equiv 1 \), i.e., \( \gamma_j = 0, \quad j = 0, \ldots, m \). In this case, using the fact that
\[ 1 = s_0 + \ldots + s_m, \]
we can split the monomial \( Q \equiv 1 \) into \( m + 1 \) sum of monomials where not every \( \gamma_j \) vanishes, i.e. \( Q(\tilde{s}) = s_0 + s_1 + \ldots + s_m \).

So we have arrived at the second scenario. Assume now that not all of \( \gamma_j, \quad j = 0, \ldots, m \) vanish. For simplicity, assume \( \gamma_0 \geq 1 \). In this case, we shall show that \( \phi_{m,h,Q} \) admits a representation \( \Omega_N \) with
\[ \Omega_N = \{ \tilde{y} = (y_1, \ldots, y_m) : \quad y_j \geq 0, \quad j = 1, \ldots, m \text{ and } y_1 + \ldots + y_m \leq 2^{N+1} \}, \]
equipped with the (scalar multiple of) Lebesgue measure \( d\mu_N = \frac{\|h\|_\infty}{2^m} d\mu \) on \( \mathbb{R}^m \) and
\[ a_j(x, \tilde{y}) = \frac{y_j^{\gamma_j}}{y^{\gamma_j}} e^{ixy_j}, \quad j = 1, \ldots, m \] and
\[ a_0(x, \tilde{y}) = \frac{2^m N}{\|h\|_{\infty}} \int_0^\infty \frac{y_0^{\gamma_0}}{y^{\gamma_0}} y^{-m} \tilde{h}(y_0 + y_1 + \ldots + y_m) e^{ixy_0} dy_0, \]
where \( y := y_0 + \ldots + y_m, \tilde{y} = (y_1, \ldots, y_m) \in \Omega_N. \)

It is obvious that
\[
\|a_j(\cdot, \tilde{y})\|_{\infty} = \left( \frac{y_j}{y} \right)^{\gamma_j} \leq 1, \quad \|a'_j(\cdot, \tilde{y})\|_{\infty} = y_j (\frac{y_j}{y})^{\gamma_j} \leq 2^{N+1}
\]
for all \( j = 1, \ldots, m. \) Hence, the functions \( a_j(\cdot, \tilde{y}), j = 1, \ldots, m, \tilde{y} \in \Omega_N \) are uniformly bounded and uniformly equicontinuous. We claim that the same conclusion also holds for the functions \( a_0(\cdot, \tilde{y}), \tilde{y} \in \Omega_N. \)

Firstly, we check that
\[
\|a_0(\cdot, \tilde{y})\|_{\infty} \leq \text{const}.
\]
Indeed, using the notation of Lemma 5, we see that
\[
a_0(x, \tilde{y}) = \frac{2^m N}{\|h\|_{\infty}} \tilde{h}_{a,\gamma_0}(x), \quad \text{where} \quad a = y_1 + \ldots + y_m, \quad \tilde{h}(x) = h_{-m,N}(x).
\]

Thus, by Lemma 5(b) and (iii) we have
\[
\|a_0(\cdot, \tilde{y})\|_{\infty} \leq \frac{2^m N}{\|h\|_{\infty}} \|\tilde{h}_{a,\gamma_0}\|_{\infty} \leq \frac{2^m N}{\|h\|_{\infty}} \|h_{-m,N}\|_{\infty} \leq \text{const}.
\]

Secondly, we claim that the derivative \( \frac{d}{dx}a_0(x, \tilde{y}) \) is a uniformly bounded function. Indeed, writing this derivative as
\[
\frac{d}{dx}a_0(x, \tilde{y}) = \frac{2^m N}{\|h\|_{\infty}} \int_0^\infty \frac{y_0^{\gamma_0+1}}{y^{\gamma_0+1}} y^{-m+1} \tilde{h}(y_0 + y_1 + \ldots + y_m) e^{ixy_0} dy_0,
\]
and repeating the argument used above with \( (\gamma_0 + 1) \) proves the uniform boundedness of \( a_0(\cdot, \tilde{y}) \).
Now observing that
\[
\int_{\Omega} \prod_{j=0}^{m} a_j(x_j, \omega) d\mu_N
\]
\[
= \int_{\Omega} \left[ \prod_{j=0}^{\infty} \frac{y_j(y)}{y_0(y)} y^{-m} \tilde{h}(y) e^{ix_0y_0} dy_0 \right] \prod_{j=1}^{m} y_j(y) e^{ix_jy_j} dy_1 \ldots dy_m
\]
\[
= \int_{R^{m+1}} \prod_{j=0}^{m} s_j(y) y^{-m} \tilde{h}(y) e^{ix_0y_0} e^{ix_1y_1} \ldots e^{ix_my_m} dy_0 dy_1 \ldots dy_m
\]
\[
= \int_{R^{m+1}} Q(s) y^{-m} \tilde{h}(y) e^{ix_0y_0} e^{ix_1y_1} \ldots e^{ix_my_m} dy_0 dy_1 \ldots dy_m
\]
\[
= \Phi_{m,h,Q}(x_0, x_1, \ldots, x_m),
\]
we see that \(\Phi_{m,h,Q}\) admits a representation (4) with are uniformly bounded and uniformly equicontinuous functions \(\{a_j(\cdot, \omega)\}_{\omega \in \Omega} \), 0 ≤ j ≤ m (in this case, we have \(\Omega^{(k)} = \Omega_N\) for all \(k \geq 1\)).

The proof of part (iii) is based on the estimates obtained in the proof of part (i) and the approach from the proof of [12, Theorem 5.1]. Observe that every element \(h \in B_{0,1}^0\) may be represented as a uniformly convergent infinite sum \(h = \sum_{N \in \mathbb{Z}} h_N\), where \(h_N = h * W_N\), of uniformly bounded functions \(h_N\) such that \(\text{supp} \tilde{h}_N\) is contained in \([2^{N-1}, 2^{N+1}]\), for every \(N \in \mathbb{Z}\). Now, let \((\Omega, \mu)\) be the direct sum of the measure spaces \((\Omega_N, \mu_N)\), \(N \in \mathbb{Z}\) (so \(\Omega = \cdot \cdot \cdot \cup \Omega_1 \cup \Omega_2 \cup \cdots\) is given by the disjoint union of \(\Omega_N\)'s). Recalling from the proof above that \(\mu_N(\Omega_N) = \frac{\|h_N\|_{\infty}}{2^N}\mu(\Omega_N) = \text{const} \|h_N\|_{\infty}\) (here the constant does not depend on \(N\)), we see that the assumption \(\sum_{N \in \mathbb{Z}} \|h_N\|_{\infty} < \infty\) guarantees that \((\Omega, \mu)\) is a measurable space with finite (\(\sigma\)-additive) measure.

The definition of the functions \(\{a_j(\cdot, \omega)\}_{\omega \in \Omega}\), 0 ≤ j ≤ m is now straightforward: the value of any such function for \(\omega \in \Omega_N\) is given by the value of the corresponding function defined in the proof of part (i). It remains to verify that there exists a growing sequence \(\{\Omega^{(k)}\}_{k \geq 1}\) of measurable subsets of \(\Omega\) such that for every \(\omega \in \Omega\) there exists \(k\) so that \(\omega \in \Omega^{(k)}\) and such that the families \(\{a_j(\cdot, \omega)\}_{\omega \in \Omega^{(k)}}\), 0 ≤ j ≤ m consist of uniformly bounded and uniformly equicontinuous functions. To this end, it is sufficient to set \(\Omega^{(k)} := \cup_{N \leq k} \Omega_N\) and refer to the results from part (i). This completes the proof of the theorem. □
A modified multiple operator integral of a polynomial integral momentum. We continue discussing the polynomial integral momentum \( \phi_{m,h,Q} \) of order \( m \) associated with the function \( h \in L^\infty \) and a polynomial \( Q \in \mathcal{P}_m \).

If \( h \in \tilde{B}_{\infty,1}^0 \), then it follows from Theorem 4 and (7) that \( T_\phi \) is well defined. However, in the case when \( h \notin \tilde{B}_{\infty,1}^0 \), it is not generally true that \( \phi = \phi_{m,h,Q} \in \mathcal{A}_m \) and therefore the definition (5) of the operator \( T_\phi \) associated with \( \phi \) no longer makes any sense. In this latter case, we have to resort to the modified operator integral \( \hat{T}_\phi \).

An important result established in [15], which we shall exploit here is that the concept of multiple operator integral \( \hat{T}_{\phi_{m,h,Q}} \) can be successfully defined under the assumptions that \( h \in C_b \) (continuous and bounded) and that the index \( p \) in (3) satisfies \( 1 < p < \infty \). The former assumption is rather auxiliary and can likely be further relaxed, whereas the later is principal.

**Theorem 6 ([15, Theorem 5.3]).** Let \( h \in C_b, \ m \geq 1, \ Q \in \mathcal{P}_m \). Let \( \hat{T}_\phi = \hat{T}_{\phi_{m,h,Q}} \) be the modified multiple operator integral associated with a polynomial integral momentum \( \phi = \phi_{m,h,Q} \) and an arbitrary \((m+1)\)-tuple of bounded self-adjoint operators \( H = (H_0, \ldots, H_m) \). If \( 1 < p < \infty \) and \( \tilde{p} = (p_1, \ldots, p_m) \), \( 1 < p_j < \infty \), \( 1 \leq j \leq m \) satisfies the equality \( \frac{1}{\tilde{p}} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \), then

\[
\|\hat{T}_\phi\|_{\tilde{p} \rightarrow p} \leq \text{const} \|h\|_{\infty}. \tag{14}
\]

We briefly explain why the estimate (14) is far more superior than any previously available estimates (in particular (4) and (8)) of the norm of a multiple operator integral. For example, for special integral polynomial momenta given by divided differences (we explain this notion below in some detail), the best earlier available estimate follows from a combination of (8) and [3, Lemma 2.3] yielding

\[
\|T_\phi\|_{\tilde{p} \rightarrow p} \leq \text{const} \int\limits_{\mathbb{R}} \left| h^{(m)}(s) \right| ds.
\]

Of course, the condition that the function \( h \) (or its derivatives) has an absolutely integrable Fourier transform is very restrictive. Even in the case when \( h \in \tilde{B}_{\infty,1}^0 \) and when we deal with the ‘classical’ multiple operator integral \( T_\phi \) (defined via Theorem 4), the best estimate available from a combination of (4) and Theorem 4(ii)

\[
\|T_\phi\|_{\tilde{p} \rightarrow p} \leq \text{const} \|h\|_{\tilde{B}_{\infty,1}^0},
\]
is still much weaker than the estimate [14]. To see that Theorem 6 is applicable here, observe that the assumption \( h \in B_{01}^0 \) guarantees that the corresponding integral momentum \( \phi \in C_m \) (the latter assertion is proved in Theorem 3) and hence, by Lemma 4 we may replace \( \hat{T}_\phi \) on the right hand side of (14) with the operator \( T_\phi \).

In the special case when \( m = 1 \), the result of Theorem 6 may be found in [17]. For an arbitrary \( m \in \mathbb{N} \), this result was proved in [15] under an additional assumption that the \((m+1)\)-tuple \( \tilde{H} \) consists of identical operators. The proof of Theorem 6 follows by a careful inspection of the proof of [15, Theorem 5.3], which shows that the argument there continues to stand if this additional assumption is omitted. We leave further details to the reader.

We shall need a small addendum to Theorem 6, which may be viewed as a variant of (Weak) Dominated Convergence Lemma for modified operator integrals of polynomial integral momenta.

**Lemma 7.** Let \( h_n,h \in C_b \) be compactly supported functions such that

\[
\lim_{n \to \infty} h_n(x) = h(x), \quad \forall x \in \mathbb{R}.
\]

Let also \( \phi_n = \phi_{m,h_n,Q} \) and \( \phi = \phi_{m,h,Q} \) be the polynomial integral momenta associated with \( Q \in P_m \) and the functions \( h_n \) and \( h \) respectively. If \( \sup_n \|h_n\|_{\infty} < +\infty \), then the sequence of operators \( \{T_{\phi_n}\} \) converges to \( T_\phi \) weakly, i.e.,

\[
\lim_{n \to \infty} \text{tr} (V_0 T_{\phi_n} (\tilde{V})) = \text{tr} (V_0 T_\phi (\tilde{V})), \quad \forall \tilde{V} = (V_1, \ldots, V_m),
\]

for every \( V_j \in S^{p_j} \), where \( 1 < p_j < \infty \) for every \( j = 0, \ldots, m \) and \( \sum_{j=0}^{m} \frac{1}{p_j} = 1 \). In particular,

\[
\|T_{\phi}\|_{\rho \to \rho'} \leq \lim_{n \to \infty} \inf \|T_{\phi_n}\|_{\rho \to \rho'}, \quad \text{where} \quad \rho = (p_1, \ldots, p_m), \quad \text{and} \quad \frac{1}{p_0} + \frac{1}{p_0'} = 1.
\]

**Proof of Lemma 7.** Fix \( V_j \) as in the statement of the lemma. According to Theorem 6 the mapping

\[
h \in C_b \mapsto \psi(h) := \text{tr} (V_0 T_\phi (\tilde{V}))
\]

is a continuous linear functional on \( C_b \). By the Riesz-Markov theorem [18, Theorem IV.18], there is a finite measure \( m \) such that

\[
\psi(f) = \int_{\mathbb{R}} f(x) \, dm(x)
\]
for any continuous function $f$ of compact support. Under such terms, the weak convergence claimed in the lemma, turns into the convergence
\[
\lim_{n \to \infty} \psi(h_n) = \psi(h)
\]
or rather
\[
\lim_{n \to \infty} \int_{\mathbb{R}} h_n(x) \, dm(x) = \int_{\mathbb{R}} h(x) \, dm(x).
\]
The latter can be seen via the classical dominated convergence theorem for Lebesgue

**Divided differences.** Let $x_0, x_1, \ldots \in \mathbb{R}$ and let $f$ be a tempered distribution such that $f^{(k)} \in L^\infty$ for every $1 \leq k \leq m$. The divided difference $f^{[k]}$ is defined recursively as follows.

The divided difference of the zeroth order $f^{[0]}$ is the function $f$ itself. The divided difference of order $k = 1, \ldots, m$ is defined by
\[
f^{[k]}(x_0, x_1, \ldots, x_k) := \begin{cases} 
\frac{f^{(k-1)}(x_0, \ldots, x_k, x_{k-1}) - f^{(k-1)}(x_0, \ldots, x_{k-1}, x_k)}{x_{k-1} - x_k}, & \text{if } x_0 \neq x_k, \\
\frac{d}{dx_1} f^{(k-1)}(x_1, \ldots, x_k), & \text{if } x_0 = x_k,
\end{cases}
\]
where $\hat{x} = (x_2, \ldots, x_k) \in \mathbb{R}^{k-1}$. Note that $f^{[k+1]} = (f^{[k]})^{[1]}$. We claim that the function $f^{[k]}$ admits the following integral representation
\[
f^{[k]}(x_0, \ldots, x_k) = \int_{S_k} f^{(k)}(s_0 x_0 + \cdots + s_k x_k) \, d\sigma_k, \quad \text{for every } k \leq m. \tag{15}
\]
In other words, the function $f^{[k]}$ is an $k$-th order polynomial integral momentum associated with the polynomial $Q \equiv 1$ and the $k$-th derivative $h = f^{(k)}$.

If $k = 1$, then the claim (15) is a simple restatement of the fundamental theorem of calculus with the substitution $t = x_0 - s_1 x_0 + s_1 x_1$ as follows
\[
\int_{S_1} f' (s_0 x_0 + s_1 x_1) \, d\sigma_1 = \int_0^1 f' ((1 - s_1) x_0 + s_1 x_1) \, ds_1 = \begin{cases} 
\int_{S_1} f' (t) \, dt, & x_0 \neq x_1,
\int_0^1 f' (x_0) \, ds_1, & x_0 = x_1.
\end{cases}
\]

For $1 < k \leq m$, we prove (15) via the method of mathematical induction. Suppose that we have already established that
\[
f^{[k]}(x_0, \ldots, x_k) = \int_{S_k} f^{(k)} (s_0 x_0 + \cdots + s_k x_k) \, d\sigma_k, \quad \text{for all } k \leq n < m.
\]
Let us prove the statement for \( n + 1 \). For \( \tilde{x} = (x_2, \ldots, x_{n+1}) \) denote

\[
f_{\tilde{x}}(x) := f^{(n)}(x, x_2, \ldots, x_{n+1}) = \int_{S_n} f^{(n)}(s_0 x + s_1 x_2 + \ldots + s_n x_{n+1}) \, ds_n,
\]

which is an \( n \)-th order integral momentum with the function \( h := f^{(n)} \) and \( Q \equiv 1 \).

Now, it follows from Lemma \( 9 \) below that

\[
\psi(x_0, \ldots, x_{n+1}) := f^{[1]}_{\tilde{x}}(x_0, x_1) = f^{(n+1)}(x_0, \ldots, x_{n+1})
\]
is an \((n + 1)\)-th order integral momentum associated with the function \( h' = f^{(n+1)} \) and \( Q \equiv 1 \), that is

\[
f^{(n+1)}(x_0, \ldots, x_{n+1}) = \int_{S_{n+1}} f^{(n+1)}(s_0 x_0 + \ldots + s_{n+1} x_{n+1}) \, ds_{n+1}.
\]

In other words, the claim \( 15 \) also holds for \( k = n + 1 \).

Immediate implications of Theorems \( 4 \) and \( 6 \) for divided differences are as follows.

**Theorem 8.**

(i) If \( f \in \tilde{B}_b^m \), then the operator \( \tilde{T}_f \) is bounded and

\[
\| T_f^{[m]} \|_{\tilde{p} \to p} \leq \| f \|_{\tilde{B}_b^m}.
\]

(ii) If \( f^{(m)} \in C_b \) and if \( 1 < p < \infty \), then the (modified) operator \( \tilde{T}_f^{[m]} \) is bounded and

\[
\| \tilde{T}_f^{[m]} \|_{\tilde{p} \to p} \leq \| f^{(m)} \|_{C_b}.
\]

When \( f^{(m)} \in C_b \), we shall also consider the function \( \tilde{f}^{[m]} \) defined by setting

\[
\tilde{f}^{[m]}(x_0, \ldots, x_{m-1}) := g^{[m-1]}(x_0, \ldots, x_{m-1}), \text{ where } g := f'.
\]

It follows from \( 15 \) and definition \( 11 \) that the function \( \tilde{f}^{[m]} \) is an \((m - 1)\)-th order polynomial integral momentum associated with the function \( h = f^{(m)} \) and the polynomial \( Q = 1 \).

**Perturbation of multiple operator integrals.** Let \( \phi = \phi_{m,h,Q} \) be a polynomial integral momentum associated with a function \( h \) such that \( h' \in L^\infty \) and \( Q \in \mathcal{P}_m \).

For \( \tilde{x} = (x_2, \ldots, x_{m+1}) \) we set \( f_{\tilde{x}}(x) = \phi(x, \tilde{x}) \). We now consider the divided difference

\[
\psi(x_0, \ldots, x_{m+1}) := f^{[1]}_{\tilde{x}}(x_0, x_1).
\]
Lemma 9. The function $\psi$ is the $(m + 1)$-th order integral momentum $\phi_{m+1}h', Q_1$ associated with the function $h'$ and the polynomial $Q_1 \in P_{m+1}$ given by

$$Q_1(s_1, \ldots, s_{m+1}) = Q(s_2, \ldots, s_{m+1}).$$

Proof of Lemma 9. By definition (11) and taking the derivative,

$$f'_{\tilde{s}}(x) = \int_{s_2 + \ldots + s_{m+1} \leq 1} Q(\tilde{s}) s_0 h'(s_0 x + s_2 x_2 + \ldots + s_{m+1} x_{m+1}) \, ds_2 \ldots ds_{m+1},$$

where $s_0 = 1 - s_2 - \ldots - s_{m+1}$ and $\tilde{s} = (s_2, \ldots, s_{m+1}).$

On the other hand, via the representation (15) for $m = 1,$

$$f'_{\tilde{s}}(x_0, x_1) = \int_0^1 f'_{\tilde{s}}(l_0 x_0 + l_1 x_1) \, dl_1,$$

where $l_0 = 1 - l_1.$

Combining the two,

$$\psi(x_0, \ldots, x_{m+1}) = \int_0^1 \int_{s_2 + \ldots + s_{m+1} \leq 1} Q(\tilde{s}) s_0 h'(s_0 (l_0 x_0 + l_1 x_1) + s_2 x_2 + \ldots + s_{m+1} x_{m+1}) \, ds_2 \ldots ds_{m+1}.$$

We next substitute the integration $(m + 1)$-tuple $(l_1, s_2, \ldots, s_{m+1})$ with $(m + 1)$-tuple $(s'_1, \ldots, s'_{m+1})$ as follows

$$s'_1 = s_0 l_1 \text{ and } s'_j = s_j, \; j = 2, \ldots, m + 1.$$

Under such substitution, the integration domain

$$0 \leq l_1 \leq 1 \text{ and } s_2 + \ldots + s_{m+1} \leq 1, \; s_2, \ldots, s_{m+1} \geq 0$$

becomes the domain

$$s'_1 + \ldots + s'_{m+1} \leq 1 \text{ and } s'_2, \ldots, s'_{m+1} \geq 0;$$

$$s'_1 + \ldots + s'_{m+1} = s_0 l_1 + s_2 + \ldots + s_{m+1} = (1 - s_2 - \ldots - s_{m+1}) l_1 + s_2 + \ldots + s_{m+1}$$

$$= l_1 + (1 - l_1)(s_2 + \ldots + s_{m+1}) \leq l_1 + 1 - l_1 = 1.$$

Computing the Jacobian $J$ of the substitution, we have
\[ I := \begin{bmatrix} s_0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = s_0. \]

Observe also that if
\[ s'_0 := 1 - s'_1 - \ldots - s'_{m+1}, \]
then \( s'_0 = 1 - s_0l_1 - s_2 - \ldots - s_{m+1} = s_0 - s_0l_1 = s_0l_0. \) Thus, we obtain that
\[
\psi(x_0, \ldots, x_{m+1}) = \int_{s'_1 + \ldots + s'_{m+1} \leq 1}^{s'_1 + \ldots + s'_{m+1} \geq 0} Q(\tilde{s}) h'(s'_0x_0 + \ldots + s'_{m+1}x_{m+1}) \, ds'_1 \ldots ds'_{m+1} \\
= \int_{s_{m+1}} Q_1(\tilde{s}') h'(s'_0x_0 + \ldots + s'_{m+1}x_{m+1}) \, ds_{m+1},
\]
where \( \tilde{s}' = (s'_1, \ldots, s'_{m+1}). \)

That is, function \( \psi \) is a polynomial integral momentum. \( \square \)

**Theorem 10.** Let \( \phi = \phi_{m,h,Q} \) and \( \psi = \phi_{m+1,h',Q_1} \) be from Lemma \( \square \). Let \( A, B \) are bounded self-adjoint operators. If \( h \in \mathcal{B}^1_{\infty 1} \) and if \( \mathcal{H} = (H_1, \ldots, H_m) \), then
\[
T_{\phi}^{A,B}(V_1, \ldots, V_m) - T_{\psi}^{B,A}(V_1, \ldots, V_m) = T_{\varphi}^{A,B,A}(A - B, V_1, \ldots, V_m).
\]

**Proof of Theorem \[ \square \]** Let us denote
\[
\psi_1(x_0, \ldots, x_{m+1}) := x_0 \psi(x_0, \ldots, x_{m+1}); \ 
\psi_2(x_0, \ldots, x_{m+1}) := x_1 \psi(x_0, \ldots, x_{m+1}); \\
\phi_1(x_0, \ldots, x_{m+1}) := \phi(x_0, x_2, \ldots, x_{m+1}); \ 
\phi_2(x_0, \ldots, x_{m+1}) := \phi(x_1, \ldots, x_{m+1}).
\]

We claim that
\[
\psi_1 - \psi_2 = \phi_1 - \phi_2.
\]
To see the claim simply set \( \tilde{x} := (x_2, \ldots, x_{m+1}) \) and recall from the definitions that
\[
\psi(x_0, \ldots, x_{m+1})(x_0 - x_1) = f(x_0, \tilde{x}) - f(x_1, \tilde{x}) = \phi(x_0, \tilde{x}) - \phi(x_1, \tilde{x}).
\]

Note also that, since \( h \in \mathcal{B}^1_{\infty 1} \), by Theorem \[ \square \] and Lemma \( \square \) the operator \( T_{\phi} \)
is well-defined as well as the operators \( T_{\psi_1 - \psi_2} \) and \( T_{\phi_1 - \phi_2} \), which are well-defined and satisfy \( T_{\psi_1 - \psi_2} = T_{\phi_1 - \phi_2} \) and \( T_{\phi_1 - \phi_2} = T_{\psi_1 - \psi_2} \) due to \( \exists \) Proposition 4.10.\]
Letting \( \tilde{V} = (V_1, \ldots, V_m) \) and using (5), we then have
\[
T^{A,B,\tilde{R}}_\psi(A - B, \tilde{V}) = T^{A,B,\tilde{R}}_\psi(A, \tilde{V}) - T^{A,B,\tilde{R}}_\psi(B, \tilde{V})
\]
\[
= T^{A,B,\tilde{R}}_\psi(1, \tilde{V}) - T^{A,B,\tilde{R}}_\psi(1, \tilde{V}) = T^{A,B,\tilde{R}}_\psi(1, \tilde{V}) - T^{A,B,\tilde{R}}_\psi(1, \tilde{V})
\]
\[
= T^{\tilde{A},\tilde{R}}_\phi(\tilde{V}) - T^{\tilde{B},\tilde{R}}_\phi(\tilde{V}).
\]
\[\square\]

Hölder type estimates for polynomial integral momenta. In this section, we fix a polynomial integral momentum \( \phi = \phi_{m,h,Q} \) associated with a polynomial \( Q \in \mathcal{P}_m \) and a function \( h \in L^\infty \). Let also \( \tilde{H} = (H_1, \ldots, H_m) \) and let \( V_j \in S^{p_j}, 1 \leq p_j \leq \infty, j = 1, \ldots, m \) be fixed. For a self-adjoint bounded operator \( A \), we shall consider the mapping
\[
F : A \mapsto F(A) := T^{A,\tilde{H}}(V_1, \ldots, V_m).
\]

In this section we shall establish Hölder estimates for the mapping \( F \). In the special case \( h = f, m = 0, Q \equiv 1 \) the Hölder properties of the mapping \( F \) were studied in [1] §5. This section extends the technique of [1] §5 to the general mappings \( F \). It should be pointed out that a vital ingredient in our extension (even when \( m = 1 \)) is supplied by Theorem 6 (see the estimate concerning the element \( Q_0 \) in the proof of Theorem 11 below).

Recall that \( s_k(U) \) stands for the \( k \)-th singular number associated with a compact operator \( U \). The symbol \( s(U) \) stands for the sequence \( \{s_k(U)\}_{k \geq 1} \). For the purposes of this section, we introduce the following truncated norm
\[
\|U\|_{p,v} = \left( \sum_{k=1}^{N} (s_k(U))^p \right)^{\frac{1}{p}}.
\]

The theorem below estimates the singular values of the operator \( F(A) - F(B) \).

**Theorem 11.** Assume that \( A - B \in S^{p_{0,1}} \), that \( \sum_{j=0}^{m} \frac{1}{p_j} \leq 1 \) and set
\[
U := F(A) - F(B) = T^{A,\tilde{H}}(V_1, \ldots, V_m) - T^{B,\tilde{H}}(V_1, \ldots, V_m).
\]
If \( h \in \Lambda_\alpha \cap B_{\infty,1}^0 \) for some \( 0 < \alpha < 1 \), then
\[
s_k(U) \leq \text{const} \ k^{-\frac{1}{p}} \|h\|_{\Lambda_\alpha} \|A - B\|^\alpha_{p_{0,v}} \|V_1\|_{p_1} \cdots \|V_m\|_{p_m},
\]
where \( \frac{1}{p} = \frac{\alpha}{p_0} + \sum_{j=1}^{m} \frac{1}{p_j} \) and \( v \geq \frac{1}{2} \).
Proof of Theorem 11. Assume for simplicity that
\[ \|V_1\|_{p_1} = \ldots = \|V_m\|_{p_m} = 1. \]

Let \(W_n\) be the Schwartz function from the definition of the Besov spaces (see (9)). For every \(n \in \mathbb{Z}\), we set
\[
h_n := W_n \ast h, \quad \phi_n := \phi_{m,h_n,Q_r}, \quad U_n := T^{\tilde{A},\tilde{R}}_{\phi_n}(V_1, \ldots, V_m) - T^{\tilde{B},\tilde{R}}_{\phi_n}(V_1, \ldots, V_m).
\]

Here, we justify the existence of the operator \(T_{\phi_n}\) by appealing to Theorem 4(i). We fix \(N \in \mathbb{Z}\) (the choice of \(N\) will be specified later) and set
\[
R_N := \sum_{n \leq N} U_n \quad \text{and} \quad Q_N := \sum_{n > N} U_n.
\]

We claim that \(U = R_N + Q_N\).

Since \(h \in \tilde{B}^{0}_{1,\infty}\), it follows from the definition of the norm of the Besov space \(\tilde{B}^{0}_{1,\infty}\) that the series \(\sum_{n \in \mathbb{Z}} h_n\) converges uniformly. Noting that the latter series consists of continuous (in fact smooth and rapidly decreasing at \(\infty\)) functions, we conclude that it also converges in the space of all continuous functions on \(\mathbb{R}\). It follows (see also (11)) that \(\sum_{n \in \mathbb{Z}} (h \ast W_n) = h\) and so for \(\tilde{s} = (s_1, \ldots, s_m)\) we have
\[
\phi(x_0, \ldots, x_m) = \int_{S_m} Q(\tilde{s}) h \left( \sum_{j=0}^{m} s_j x_j \right) d\sigma_m = \int_{S_m} Q(\tilde{s}) \sum_{n \in \mathbb{Z}} (h \ast W_n) \left( \sum_{j=0}^{m} s_j x_j \right) d\sigma_m
\]
\[
= \int_{S_m} Q(\tilde{s}) \sum_{n \in \mathbb{Z}} h_n \left( \sum_{j=0}^{m} s_j x_j \right) d\sigma_m = \sum_{n \in \mathbb{Z}} \int_{S_m} Q(\tilde{s}) h_n \left( \sum_{j=0}^{m} s_j x_j \right) d\sigma_m
\]
\[
= \sum_{n \in \mathbb{Z}} \phi_n(x_0, \ldots, x_m).
\]

Now, we arrive at the claim as follows
\[
U = T^{\tilde{A},\tilde{R}}_{\phi}(V_1, \ldots, V_m) - T^{\tilde{B},\tilde{R}}_{\phi}(V_1, \ldots, V_m)
\]
\[
= T^{\tilde{A},\tilde{R}}_{\sum_{n \in \mathbb{Z}} \phi_n}(V_1, \ldots, V_m) - T^{\tilde{B},\tilde{R}}_{\sum_{n \in \mathbb{Z}} \phi_n}(V_1, \ldots, V_m)
\]
\[
= \sum_{n \in \mathbb{Z}} \left( T^{\tilde{A},\tilde{R}}_{\phi_n}(V_1, \ldots, V_m) - T^{\tilde{B},\tilde{R}}_{\phi_n}(V_1, \ldots, V_m) \right)
\]
\[
= \sum_{n \in \mathbb{Z}} U_n = R_N + Q_N.
\]
Here, the step from the second to the third line above is justified as follows. Firstly, we note that the series \( \phi = \sum_{n \in \mathbb{Z}} \phi_n \) converges also in the norm \( \| \cdot \|_{\mathcal{A}_c} \). This convergence follows from the already used above fact that \( \sum_{n \in \mathbb{Z}} \| h_n \|_{\mathcal{A}} < \infty \). Combined with Theorem 4(i). Hence, appealing to (4), we infer that \( T \phi = \sum_{n \in \mathbb{Z}} T \phi_n \) (in the sense of the strong operator topology).

Observing the following elementary properties of singular values

\[
    s_k(U + V) \leq s_k(U) + s_k(V) \quad \text{and} \quad s_k(U) \leq k^{-\frac{1}{p}} \|U\|_{p,v}, \quad k \leq v,
\]

we see

\[
    s_k(U) \leq s_k(R_N) + s_k(Q_N) \leq \left( k \frac{2}{7} \right) R_N \|_{r,v} + \left( k \frac{2}{7} \right) Q_N \|_{r,v}, \quad k \leq v, \tag{17}
\]

where \( r^{-1}_0 = \sum_{j=1}^m p_j^{-1} \) and \( r^{-1} = p_0^{-1} + r^{-1}_0 \). We now estimate \( R_N \) and \( Q_N \) separately.

We estimate \( R_N \) as follows. Observe that by Theorem 10

\[
    U_h = T^{A,B,\hat{\Gamma}}_{\psi_n} (A - B, V_1, \ldots, V_m),
\]

where \( \psi_n = \phi_{m+1,h_{n,Q}} \) is the polynomial integral momentum of order \( m + 1 \) associated with the function \( h'_n = h' * W_n \), where we view \( h' \) as a generalized function (the preceding equality follows immediately from the definition \( h_h := W_h * h \)). Since \( \hat{h}_n' (\xi) = 2 \pi i \xi \hat{h}_n(\xi) \), we readily infer from Lemma 3 part (iii) with \( N = n \) and \( m = 1 \) that

\[
    \| h'_n \|_{\infty} \leq \text{const} \ 2^n \| h_n \|_{\infty}. \tag{18}
\]

It is also known as a combination of [22] Proposition 7 and [23] Corollary 2 that

\[
    2^\text{const} \ 2^n \| h'_n \|_{\infty} \leq \| h \|_{\Lambda_a}. \tag{19}
\]

Combining (19) with (4) and Theorem 4 part (i) we see that

\[
    \|U_h\|_{r,v} = \|T^{A,B,\hat{\Gamma}}_{\psi_n} (A - B, V_1, \ldots, V_m)\|_{r,v}
\]

\[
\leq \|T^{A,B,\hat{\Gamma}}_{\psi_n}\|_{p \to r} \|A - B\|_{p_0,v} \|V_1\|_{p_1} \ldots \|V_m\|_{p_m}
\]

\[
\leq \text{const} \ 2^n \| h_n \|_{\infty} \|A - B\|_{p_0,v}
\]

\[
\leq \text{const} \ 2^{(1 - \alpha)n} \| h \|_{\Lambda_a} \|A - B\|_{p_0,v}.
\]

Noting that \( \sum_{n \leq N} 2^{(1 - \alpha)n} = \text{const} \ 2^{(1 - \alpha) N} \), we obtain

\[
\|R_N\|_{r,v} \leq \sum_{n \leq N} \|U_h\|_{r,v} \leq \text{const} \ 2^{(1 - \alpha) N} \| h \|_{\Lambda_a} \|A - B\|_{p_0,v}.
\]
In order to estimate $Q_N$, we combine Theorem 6 (see the comments following the statement of Theorem 6 which explain why we are in a position to identify operators $T_\phi$ and $\hat{T}_\phi$) and (19) as follows

$$
\|U_n\|_{r_0} \leq \left\| T_{\phi_n}^A \left( V_1, \ldots, V_m \right) \right\|_{r_0} + \left\| T_{\phi_n}^B \left( V_1, \ldots, V_m \right) \right\|_{r_0}
\leq \text{const} \|h\|_\infty \leq \text{const} 2^{-an} \|h\|_{\Lambda_a}.
$$

Consequently,

$$
\|Q_N\|_{r_0} \leq \sum_{n>N} \|U_n\|_{r_0} \leq \text{const} 2^{-aN} \|h\|_{\Lambda_a}.
$$

Returning back to (17), we arrive at

$$s_k(U) \leq \text{const} 2^{-aN} \|h\|_{\Lambda_a} \left( \frac{k}{2} \right)^{-\frac{1}{p_0}} \left( \frac{k}{2} \right)^{-\frac{1}{p_0}} \|A - B\|_{p_0, \nu} + 1
$$

The proof can now be finished by choosing $N \in \mathbb{Z}$ such that

$$2^{-N-1} \leq \left( \frac{k}{2} \right)^{-\frac{1}{p_0}} \|A - B\|_{p_0, \nu} < 2^{-N}.
$$

Indeed, suppose $N$ is such as above. Then rewriting the preceding estimate, we have

$$s_k(U) \leq \text{const} 2^{-aN} \|h\|_{\Lambda_a} \left( \frac{k}{2} \right)^{-\frac{1}{p_0}} \left( \frac{k}{2} \right)^{-\frac{1}{p_0}} \|A - B\|_{p_0, \nu} 2^N + 1
$$

$$= \text{const} 2^{1+a} 2^a \left( -N - 1 \right) \|h\|_{\Lambda_a} \left( \frac{k}{2} \right)^{-\frac{1}{p_0}} \left( 2^{-N} \|A - B\|_{p_0, \nu} \right) + 1
$$

$$\leq \text{const} 2^{1+a} \left( \frac{k}{2} \right)^{-\frac{1}{p_0}} \|A - B\|_{p_0, \nu} \|h\|_{\Lambda_a} \left( \frac{k}{2} \right)^{-\frac{1}{p_0}}
$$

$$= \text{const} k^{-\frac{1}{p}} \|A - B\|_{p_0, \nu} \|h\|_{\Lambda_a}.
$$

where we used firstly the right hand side from (20) and then its left hand side, and, in the last step, the equalities $\frac{1}{p_0} = \frac{1}{p_0} + \sum_{j=1}^{m} \frac{1}{p_j}$ and $\frac{1}{\prod_{j=1}^{m} p_j} = \sum_{j=1}^{m} \frac{1}{p_j}$.

Recall that $S_p^{\infty}$, $1 \leq p < \infty$ stands for the weak Schatten-von Neumann quasi-normed ideal defined by the relation

$$\|U\|_{p, \infty} := \sup_{k \geq 1} k^{\frac{1}{p}} s_k(U) < +\infty.
$$

Letting $\nu \to \infty$, we also have the corollary.
Corollary 12. In the setting of Theorem 11, we have $U \in S^{p,\infty}$ and
\[
\|U\|_{p,\infty} \leq \text{const} \|h\|_{\Lambda_{\alpha}} \|A - B\|_{p_0}^\alpha \|V_1\|_{p_1} \cdot \ldots \cdot \|V_m\|_{p_m},
\]
where $\frac{1}{p} = \frac{\alpha}{p_0} + \sum_{j=1}^m \frac{1}{p_j}.$

Finally, the Hölder estimate for the mapping $F$ is given below.

Theorem 13. In the setting of Theorem 11, if $\sum_{j=1}^m \frac{1}{p_j} < 1$, then $U \in S^p$ and
\[
\|U\|_p \leq \text{const} \|h\|_{\Lambda_{\alpha}} \|A - B\|_{p_0}^\alpha \|V_1\|_{p_1} \cdot \ldots \cdot \|V_m\|_{p_m},
\]
where $\frac{1}{p} = \frac{\alpha}{p_0} + \sum_{j=1}^m \frac{1}{p_j}.$

Proof of Theorem 13. We shall consider two mutually exclusive situations. Firstly, we assume that there is $p_j$ ($1 \leq j \leq m$) such that $p_j < \infty$. In this case the claim follows from the real interpolation method directly. Indeed, assume for simplicity that $p_1 < \infty$. Let $A - B \in S^{p_0}$ and $V_j \in S^{p_j}$, $j = 2, \ldots, m$ be fixed such that
\[
\|V_j\|_{p_j} = 1, \quad j = 2, \ldots, m.
\]
Let
\[
T(V) = T_{\phi}^{A, B}(V, V_2, \ldots, V_m) - T_{\phi}^{B, \overline{A}}(V, V_2, \ldots, V_m).
\]
Applying Corollary 12 with $\frac{1}{p} = \frac{\alpha}{p_0} + \sum_{j=2}^m \frac{1}{p_j}$ and with $\frac{1}{r} = \frac{1}{\theta} + \frac{1}{p}, \frac{1}{p} := 1 - \sum_{j=2}^m \frac{1}{p_j}$, we have respectively
\[
\|T(V)\|_{r_0, \infty} \leq \text{const} \|h\|_{\Lambda_{\alpha}} \|A - B\|_{p_0}^\alpha \|V\|_{\infty}
\]
and
\[
\|T(V)\|_{r_1, \infty} \leq \text{const} \|h\|_{\Lambda_{\alpha}} \|A - B\|_{p_0}^\alpha \|V\|_{p_1}.
\]
Observe that $\frac{1}{p_1} < \frac{1}{p}$ and hence, $0 < \theta := \frac{\alpha}{p_1} < 1$. Applying the real interpolation method $[\cdot, \cdot]_{\theta, p}$ to the quasi-Banach pair $(S^{r_0, \infty}, S^{r_1, \infty})$, we conclude the proof.

Now, we assume now that $p_j = \infty$ for every $j = 1, \ldots, m$. In this case, the proof is similar to the argument used in [1] Theorem 5.8. Assume for simplicity that
\[
\|V_j\|_{\infty} = 1, \quad j = 1, \ldots, m.
\]
Applying Theorem 11 with $p_0 = 1$ (so $\frac{1}{p} = \alpha$) and $p_j = \infty$ for all $1 \leq j \leq m$, we have
\[
s_k(U) \leq \text{const} \|h\|_{\Lambda_{\alpha}} \left(\frac{1}{k}\right)^\alpha \|A - B\|_{1,\nu}^\alpha \forall \nu \geq \frac{k}{2},
\]
or equivalently,
\[
s_k^\frac{1}{k} (U) \leq \text{const} \|h\|_{A_k}^\frac{1}{k} \|A - B\|_{1,\nu}, \forall \nu \geq k \frac{1}{2}.
\]

In particular, setting \(\nu = k\), we obtain
\[
s_k(|U|^\frac{1}{k}) \leq \text{const} \|h\|_{A_k}^\frac{1}{k} \sum_{n=1}^{k} s_n(A - B).
\]

Considering Cesaro operator \(C\) on the space \(l_\infty\) of all bounded sequences \(x = \{x_n\}_{n \geq 1}\) given by the formula
\[
(Cx)_k := \frac{1}{k} \sum_{n=1}^{k} x_n, \quad k \geq 1,
\]
we may interpret the preceding estimate as
\[
s(|U|^\frac{1}{k}) \leq \text{const} \|h\|_{A_k}^\frac{1}{k} Cs(A - B).
\]

Recalling that the operator \(C\) maps the space \(l_{p_0}\) into itself (for every \(1 < p_0 \leq \infty\)) and that, by the assumption, \(A - B \in \mathcal{S}_{p_0}\), we obtain that \(Cs(A - B) \in l_{p_0}\) and therefore \(|U|^\frac{1}{k} \in \mathcal{S}_{p_0}\), or equivalently \(|U|^{\frac{p}{p_0}} = |U|^p \in \mathcal{S}_1\) (indeed, in our current setting we have \(\alpha p = p_0\)) and furthermore
\[
\||U|^p\|_1 \leq \text{const} \|h\|_{A_k}^\frac{p}{p_0} \|Cs(A - B)\|_{p_0} \leq \text{const} \|h\|_{A_k}^\frac{p}{p_0} \|A - B\|_{p_0},
\]
which is equivalent to the claim. \(\square\)

**Remark 14.** We observe that the assertion of Theorem 13 also holds when \(\alpha = 1\) (in this case, we speak of Lipschitz functions rather than Hölder functions with exponent \(\alpha\)). However, the proof of this case is based on totally different ideas. In fact, this case is justified by Theorem 6.

2. **Proof of the main result**

In this section, we consider \(\mathcal{S}_p, 1 \leq p \leq \infty\) as a Banach space over the field \(\mathbb{R}\) of real numbers.

**Theorem 15.** If \(1 < p < \infty\) and if \(m \in \mathbb{N}\) is such that \(m < p \leq m + 1\), then for every \(H \in \mathcal{S}_p, \|H\|_p \leq 1\), there exist bounded symmetric polylinear forms
\[
\delta_H^{(1)} : \mathcal{S}_p \to \mathbb{R}, \quad \delta_H^{(2)} : \mathcal{S}_p \times \mathcal{S}_p \to \mathbb{R}, \quad \ldots, \quad \delta_H^{(m)} : \underbrace{\mathcal{S}_p \times \ldots \times \mathcal{S}_p}_{\text{m-times}} \to \mathbb{R}
\]
such that
\[ \| H + V \|_p^p - \| H \|_p^p - \sum_{k=1}^{m} \delta_H^{(k)}(V, \ldots, V) = O(\| V \|_p^p), \]

(21)

where $V \in S^p$ and $\| V \|_p \to 0$.

Observe that, without loss of generality, the above theorem needs only a proof for the special case when $H$ and $V$ are self-adjoint operators. Indeed, let us assume that the theorem is proved in the self-adjoint case, that is for every self-adjoint operators $H$ and $V$ from $S^p$ the existence of $\delta^{(k)}_H$'s satisfying (21) is established. Fixing an infinite projection on $\mathcal{H}$ with the infinite orthocomplement, we may represent an arbitrary element $X \in S^p$ as

\[
X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}
\]

with $X_{ij} \in S^p, 1 \leq i, j \leq 2$. Furthermore, setting for an arbitrary $X \in S^p$

\[
\alpha(X) = \frac{1}{2^{1/p}} \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}
\]

we see that $\alpha$ is an isometrical embedding of $S^p$ into itself (in fact, into a (real) Banach subspace of $S^p$ consisting of self-adjoint operators). Finally, for arbitrary operators $H, V \in S^p$, we set

\[
\delta^{(k)}_H(V, \ldots, V) := \frac{1}{2} \delta^{(k)}_{\alpha(H)}(\alpha(V), \ldots, \alpha(V)), 1 \leq k \leq m.
\]

It is trivial that $\delta^{(k)}_H$'s are bounded symmetric polylinear forms satisfying (21).

So, from now and until the end of the proof, we assume that $H$ and $V$ are self-adjoint operators such that

\[ \| H \|_\infty \leq 1 \quad \text{and} \quad \| V \|_\infty \leq 1. \]

Let $f_p$ be the “smoothed” function $|\cdot|^p$, that is $f_p$ is a $C^\infty$ compactly supported function on $\mathbb{R} \setminus \{0\}$ such that $f_p(x) = |x|^p$ for all $|x| \leq 2$. Clearly,

\[ \| H \|_p^p = \text{tr} \left( f_p(H) \right) \quad \text{and} \quad \| H + V \|_p^p = \text{tr} \left( f_p(H + V) \right). \]
The definition of functionals $\delta^{(k)}_H$, $1 \leq k \leq m$. We shall explicitly define the functionals $\delta^{(k)}_H$ from (21) in (24) below. However, given that the definition in (24) is rather complex, we shall first give some guiding explanations.

We observe first that if $\delta^{(k)}_H$ is a set of functionals from the expansion (21), then it is readily seen that

$$\frac{d^k}{dt^k} \left[ \text{tr} \left( f_p(H_t) \right) \right]_{t=0} = k! \delta^{(k)}_H \left( V, \ldots, V \right),$$

where $H_t = H + tV$. On the other hand, it is known from [12, Theorem 5.6] and [3, Theorem 5.7] that

$$\frac{d^k}{dt^k} \left[ f_p(H_t) \right] = k! \ T^H_{f_p} \left( V, \ldots, V \right),$$

where, for a function $\phi \in C_m$, we used the abbreviation

$$T^H_{\phi} = \ T^H_{\tilde{\phi}}, \ \tilde{H} = \left( H_r, \ldots, H \right). \ \ \ \ (22)$$

Comparing the two identities above, it seems natural to suggest the following definition for the functionals $\delta^{(k)}_H$:

$$\delta^{(k)}_H \left( V_1, \ldots, V_k \right) = \text{tr} \left( T^H_{f_p} \left( V_1, \ldots, V_k \right) \right), \ \ V_1, \ldots, V_k \in S^p.$$

However, this suggestion is flawed since a combination of Lemma 3 and Theorem 8(i) yields only that $\|T^H_{f_p}\|_{\tilde{p} \to \tilde{p}} \leq \text{const} \|f_p\|_{\tilde{B}^k_{\infty}}$, where $\tilde{p} = (p, p, \ldots, p)$, in particular

$$U_k := T^H_{f_p} \left( V_1, \ldots, V_k \right) \in S^k, \ \ k = 1, \ldots, m.$$

In other words, it is not known (and not clear) whether $U_k \in S^1$.

To circumvent this difficulty, we use the approach implicitly suggested in [16, Lemma 2.2]. This approach is based on the identity.

$$\text{tr} \left( T^H_{f_p} \left( V, \ldots, V \right) \right) = \frac{1}{k} \text{tr} \left( V \cdot T^H_{f_p} \left( V, \ldots, V \right) \right),$$

where $f^{(k)}$ is the polylinear integral momentum defined in [16], that is for $f^{(m)} \in C_b$,

$$f^{(m)}(x_0, \ldots, x_{m-1}) = g^{(m-1)}(x_0, \ldots, x_{m-1}), \ \ g = f'.$
Using the identity above as a guidance and setting aside for a moment the question why the operator $V_1 \cdot T_{f_\theta}^H(V_2, \ldots, V_k)$ (see below) belongs to the trace class $S^1$ for every $k = 2, \ldots, m$, we now explicitly define the functional $\delta_H^{[k]}$ as follows

$$
\delta_H^{[k]}(V_1, \ldots, V_k) = \begin{cases} 
\text{tr} \left( V_1 f_p'(H) \right), & k = 1 \\
\frac{1}{k} \text{tr} \left( V_1 \cdot T_{f_\theta}^H(V_2, \ldots, V_k) \right), & 1 < k \leq m 
\end{cases}
$$

The definition above is crucially important for the proof. In the next two subsections we shall confirm that for every $1 \leq k \leq m$ the functional $\delta_H^{[k]}$ is well defined and satisfies all the properties required in Theorem 15 (excepting the symmetricity). However, the functionals $\delta_H^{[k]}$ are not symmetric. To obtain symmetric functionals satisfying all the requirements of Theorem 15 we resort to the standard symmetrisation trick (see e.g. [10, Section 40]) by setting

$$
\delta_{\theta}^{(k)}(V_1, \ldots, V_k) := \frac{1}{k!} \sum_{\sigma} \delta_H^{[k]}(V_{\sigma(1)}, V_{\sigma(2)}, \ldots, V_{\sigma(k)})
$$

where the sum is taken over all permutations $\sigma(1), \sigma(2), \ldots, \sigma(k)$ of the indices $1, 2, \ldots, k$. It is trivial to verify that the functionals $\delta_{\theta}^{(k)}$’s satisfy already all the requirements of Theorem 15 as soon as such a verification is firstly performed for the functionals $\delta_H^{[k]}$’s.

Such a verification for functionals $\delta_H^{[k]}$’s (including their continuity) is presented in Theorem 17 below. The proof is partly based on our improvement of the method of complex interpolation explained below.

**Complex method of interpolation.** We shall now briefly recall the complex method of interpolation. For a compatible pair of Banach spaces $(A_0, A_1)$, and $0 < \theta < 1$, the complex interpolation Banach space $A_{\theta} = (A_0, A_1)_{\theta}$ is defined as follows (see e.g. [8, Section 4.1]):

$$
A_{\theta} := \{ x \in A_0 + A_1 : \exists f \in \mathcal{F}(A_0, A_1) \text{ such that } x = f(\theta) \}.
$$

Here the class $\mathcal{F}(A_0, A_1)$ consists of all bounded and continuous functions $f : \tilde{S} \mapsto A_0 + A_1$ defined on the closed strip

$$
\tilde{S} := \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1 \}
$$

such that $f$ is analytic on the open strip $S := \{ z \in \mathbb{C} : 0 < \text{Re} z < 1 \}$ and such that $t \mapsto f(j + it) \in A_j$, $j = 0, 1$ are continuous functions on the real line. We provide
\[ \mathcal{F}(A_0, A_1) \] with the norm
\[ \|f\|_{\mathcal{F}(A_0, A_1)} := \max_{j=0,1} \{c_0(f), c_1(f)\}, \]
where \( c_j(f) := \sup_{t \in \mathbb{R}} \|f(j+it)\|_{A_j}, j = 0, 1. \)

Setting
\[ \|x\|_{A_\theta} := \inf \{ \|f\|_{\mathcal{F}(A_0, A_1)} : f(\theta) = x, f \in \mathcal{F}(A_0, A_1) \} \]
we obtain a Banach space \((A_\theta, \| \cdot \|_{A_\theta})\). It is well known that
\[ \|x\|_{A_\theta} \leq c_0^{-\theta} c_1^\theta \|f\|_{A_0, A_1} \]
where \( f(\theta) = x, f \in \mathcal{F}(A_0, A_1) \).

**Lemma 16.** Let \( F_z \) be the multilinear operator
\[ F_z : \mathcal{S}^\infty \times \ldots \times \mathcal{S}^\infty \rightarrow \mathcal{S}^\infty, \forall z \in S, \]
such that \( z \mapsto F_z \) is analytic in \( S \). If the constants
\[ c_j = \sup_{t \in \mathbb{R}} \|F_{j+it}\|_{\mathcal{S}_0^\infty \mapsto \mathcal{S}_0^\infty, j \rightarrow j}, \]
where \( \tilde{q}^{(i)} = (q_1^{(i)}, \ldots, q_m^{(i)}) \) and \( j = 0, 1 \)
are finite, then
\[ \|F_\theta\|_{\tilde{q} \rightarrow r} \leq c_0^{1-\theta} c_1^\theta, \text{ where } \tilde{q} = (q_1, \ldots, q_m) \]
and \( \frac{1}{q_k} = \frac{1 - \theta}{q_k^{(0)}} + \frac{\theta}{q_k^{(1)}}, k = 1, \ldots, m \)
and \( \frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1} \).

**Proof.** Fix \( \varepsilon > 0 \). For every \( 1 \leq k \leq m \), there exists a function \( g_k \in \mathcal{F}(\mathcal{S}_{q_k^{(0)}}^0, \mathcal{S}_{q_k^{(1)}}^1) \)
such that
\[ g_k(\theta) = V_k \text{ and} \]
\[ \|g_k\|_{q_k} \leq \sup_{t \in \mathbb{R}} \max_{j} \left\{ \|g_k(it)\|_{q_k^{(0)}}, \|g_k(1+it)\|_{q_k^{(1)}} \right\} \leq (1 + \varepsilon) \|V_k\|_{q_k}. \]

Define an analytic function \( h \) in the strip \( S \) by setting
\[ h(z) = F_z(g_1(z), \ldots, g_m(z)). \]

By the assumption,
\[ \|h(it)\|_{r_0} \leq \|F_{it}\|_{\tilde{q}^{(0)} \rightarrow r_0} \|g_1(it)\|_{q_1^{(0)}} \cdots \|g_m(it)\|_{q_m^{(0)}} \leq (1 + \varepsilon)^m c_0 \|V_1\|_{q_1} \cdots \|V_m\|_{q_m}. \]
Similarly,

$$\|h(1+it)\|_{r_0} \leq (1+\epsilon)^m \epsilon_1 \|V_1\|_{q_1} \cdots \|V_m\|_{q_m}.$$ 

It follows from the definition of complex interpolation method combined with the fact that $$(S^{r_0}, S^{r_1})_{\theta} = S^{r},$$ that

$$\|F_{\theta}(V_1, \cdots, V_m)\|_r \leq \|h(it)\|^\theta_0 \|h(1+it)\|^{1-\theta}_1.$$ 

Since $\epsilon$ is arbitrarily small, it follows that

$$\|F_{\theta}(V_1, \cdots, V_m)\|_r \leq c_{\theta_0} c_1^{1-\theta} \|V_1\|_{q_1} \cdots \|V_m\|_{q_m}.$$ 

□

The functionals $\delta^{[k]}_H$ are well-defined. The following theorem is the key to showing that the functionals $\delta^{[k]}_H$’s are well-defined and continuous.

**Theorem 17.** If $H \in S^p$, then the operator $T^H_{f_p^k}$ maps $S^p \times \cdots \times S^p \rightarrow S^{p'}$, for every integral $2 \leq k < p$. Moreover,

$$\left\| T^H_{f_p^k}(V_1, \cdots, V_{k-1}) \right\|_{p'} \leq \text{const} \|H\|_{p-k} \|V_1\|_p \cdots \|V_{k-1}\|_p,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

**Proof of Theorem 17** We fix $V_1, \ldots, V_{k-1} \in S^p$ and assume that $\|V_j\|_p = 1$, $j = 1, \ldots, k-1$. By Lemma 3 applied to the smoothed function $f_p$, we have $f_p \in \tilde{B}^k_{0,1}$ for every positive integral $k < p$. Hence the function $h = f_p^{(k)}$ belongs to $\tilde{B}^0_{0,1}$ and since $f_p^{(k)}$ is a $(k-1)$-th order polynomial integral momentum associated with the function $h = f_p^{(k)}$ (see (16)) we infer from Theorem 4 that $f_p^{(k)} \in \mathcal{C}_{k-1}$. Now, by Lemma 2, we have $T^H_{f_p^{(k)}} = T^H_{f_p^k}$ and applying Theorem 6 we obtain

$$\left\| T^H_{f_p^k} \right\|_{\tilde{p} \rightarrow p} \leq \text{const} \|f_p^{(k)}\|_{\infty},$$

where $\tilde{p} = (p, p, \ldots, p)$. In particular,

$$k - 1 \text{-times} \quad T^H_{f_p^k}(V) \in S^{\tilde{p}}, \quad \text{where} \quad V = (V_1, \ldots, V_{k-1}).$$

However, the estimate above is weaker than the claim of Theorem 17. To achieve the claim, we need a rather delicate application of the complex interpolation Lemma 16.

For the rest of the proof, we fix an integral $n \geq 0$ such that

$$2n < p - k \leq 2n + 2.$$
In order to use Lemma 16 we will construct a family of analytic operator valued functions
\[ z \in \mathbb{C}, \epsilon > 0 \mapsto F_{z,\epsilon} : S^{\infty} \times \ldots \times S^{\infty} \mapsto S^{\infty}, \]
such that
\[ F_{j+it,\epsilon} : S^{q_j} \times \ldots \times S^{q_j} \mapsto S^{r_j}, \quad j = 0, 1, \]
where the exponents \( q_j \) and \( r_j \) are given by
\[
1_{q_0} = \frac{p - 2n}{kp} \quad \text{and} \quad 1_{q_1} = \frac{p - 2n - 2}{kp},
\]
\[
1_{r_0} = \frac{2n + k - 1}{p} + \frac{1}{q_0} \quad \text{and} \quad 1_{r_1} = \frac{2n + 2}{p} + \frac{k - 1}{q_1}.
\]
We observe right away that due to the assumption \( 2 \leq k < p \) and the choice of \( n \), the indices \( r_j, q_j \) are non-trivial, that is
\[ 1 < r_j, q_j < \infty, \quad j = 0, 1. \]
In addition, the family \( F_{z,\epsilon} \) will have also satisfied the boundary estimates
\[ \left\| F_{j+it,\epsilon} (\mathcal{V}) \right\|_{r_j} \leq \text{const} \left\| H \right\|_{p}^{2n+2} \left\| V_1 \right\|_{q_j} \cdots \left\| V_{k-1} \right\|_{q_j}, \quad (25) \]
with the constant in (25) being independent of \( \epsilon > 0 \) and such that
\[ F_{\theta,\epsilon} (\mathcal{V}) = T_{f_{p+\epsilon}}^H (\mathcal{V}), \quad \text{where} \quad \theta = \frac{p - k}{2} - n. \]
Given the family \( F_{z,\epsilon} \) as above and using Lemma 16 we readily arrive at the estimate
\[ \left\| F_{\theta,\epsilon} (\mathcal{V}) \right\|_{p'} = \left\| T_{f_{p+\epsilon}}^H (\mathcal{V}) \right\|_{p'} \leq \text{const} \left\| H \right\|_{p}^{p-k} \left\| V_1 \right\|_{p} \cdots \left\| V_{k-1} \right\|_{p}, \]
where the constant is independent of \( \epsilon > 0 \). The claim of the theorem
\[ \left\| T_{f_{p}}^H (\mathcal{V}) \right\|_{p'} \leq \text{const} \left\| H \right\|_{p}^{p-k} \left\| V_1 \right\|_{p} \cdots \left\| V_{k-1} \right\|_{p}, \]
now follows from Lemma 7 which is applicable due to pointwise convergence
\[ \lim_{\epsilon \to 0} f_{p+\epsilon}^{(k)} (x) = f_{p}^{(k)} (x), \quad x \in \mathbb{R}, \]
of compactly supported continuous functions.
We now focus on the construction of the family \( \{ F_{z,\epsilon} \}_{\epsilon > 0} \). The construction is based on the following auxiliary lemma.
Lemma 18. Let \( f_z(x) := [f_1(x)]^z \), \( z \in \mathbb{C} \) be the analytic continuation of the mapping \( p \to f_p \) to \( \mathbb{C} \), and let
\[
z \mapsto \hat{F}_z : S^\infty \times \cdots \times S^\infty \rightarrow S^\infty
\]
be the analytic (in \( \mathbb{C} \)) operator valued function given by
\[
\hat{F}_z (\tilde{V}) = T_{\tilde{F}_z}^H (\tilde{V}) \, , \quad \tilde{V} = (V_1, \ldots, V_{k-1}).
\]
Let \( m \geq 0 \) be an integer such that \( \Re z > 2m + k \). If \( 1 < r, q < \infty \) are such that
\[
1/r = 2m/p + k - 1/q,
\]
then, we have the following estimate
\[
\| \hat{F}_z (\tilde{V}) \|_r \leq \text{const} \, (1 + |\text{Im } z|)^k \| H \|^{2m}_p \| V_1 \|_q \cdots \| V_{k-1} \|_q
\]
with constant being independent of \( z \).

The proof of the lemma will follow momentarily. However, we shall first finish the proof of the theorem. Given the lemma above it is now straightforward. Indeed, we choose the family \( \{ F_{z,\epsilon} \}_{\epsilon > 0} \) as follows
\[
F_{z,\epsilon} = e^{z^2 - \theta^2} \hat{F}_{z+2n+k+\epsilon}.
\]
Clearly,
\[
F_{\theta,\epsilon} = F_{p+\epsilon} = T_{\tilde{F}_{p+\epsilon}}^H.
\]
Also, the boundary estimates (25), both follow from the lemma with \( r = r_j, q = q_j, m = n + j, z = 2j + 2n + k + \epsilon, j = 0, 1 \). Observe that the polynomial growth with respect to \( |\text{Im } z| \) in the lemma is controlled by the exponential decay of the function \( e^{z^2 - \theta^2} \) on the boundary of the strip \( S \). Thus, the theorem is completely proved.

Proof of Lemma\textsuperscript{18}. We write the \( k \)-th derivative as
\[
f_z^{(k)}(x) = \frac{z(z-1)(z-2)\ldots(z-k+1)}{(1 + |\text{Im } z|)^k} (\text{sgn } x)^k (1 + |\text{Im } z|)^k |x|^{z-k}
= w(z) (1 + |\text{Im } z|)^k |x|^{z-k}, \quad |x| \leq 2,
\]
where
\[
\sup_{z \in S} |w(z)| \leq \text{const}, \quad w(z) = (\text{sgn } x)^k \frac{z(z-1)(z-2)\ldots(z-k+1)}{(1 + |\text{Im } z|)^k}.
\]
The derivative above is continuous, since \( \Re z > k \). Moreover, by Lemma\textsuperscript{3} the function \( h = f_z^{(k)} \) belongs to \( \hat{B}^0_{\omega,1} \).
We assume for simplicity that
\[ \|V_1\|_q = \ldots = \|V_{k-1}\|_q = 1. \]

We then have to show that
\[ \left\| T_{f_z^{(k)}}^{H} (\vec{V}) \right\|_r \leq \text{const} (1 + |\text{Im } z|)^k \|H\|_{p}^{2m}. \]

(27)

Recall (see (15)) that \( f_z^{(k)} \) is the polynomial integral momentum of order \( k-1 \) associated with the polynomial \( Q = 1 \) and the function \( h = f_z^{(k)}. \) Using the polynomial expansion
\[
|s_0 x_0 + \ldots + s_{k-1} x_{k-1}|^{2m} = \sum_{m_0, \ldots, m_{k-1} = 2m} C_{m_0, \ldots, m_{k-1}} (s_0 x_0)^{m_0} \cdots (s_{k-1} x_{k-1})^{m_{k-1}}
\]

where \( C_{m_0, \ldots, m_{k-1}} = \frac{(2m)!}{m_0! \cdots m_{k-1}!}, \)

we represent the momentum \( f_z^{(k)} \) via (15) as follows
\[
f_z^{(k)} (x_0, \ldots, x_{k-1}) := \int_{S_{k-1}} f_z^{(k)} (s_0 x_0 + \ldots + s_{k-1} x_{k-1}) \, d\sigma_{k-1}
\]
\[
= w(z) \, (1 + |\text{Im } z|)^k \int_{S_{k-1}} |s_0 x_0 + \ldots + s_{k-1} x_{k-1}|^{-k} \, d\sigma_{k-1}
\]
\[
= w(z) \, (1 + |\text{Im } z|)^k \int_{S_{k-1}} |s_0 x_0 + \ldots + s_{k-1} x_{k-1}|^{-k+2m} \, d\sigma_{k-1}
\]
\[
= w(z) \, (1 + |\text{Im } z|)^k \int_{S_{k-1}} \tilde{h}_z (s_0 x_0 + \ldots + s_{k-1} x_{k-1}) \, d\sigma_{k-1}
\]
\[
\times \sum_{m_0, \ldots, m_{k-1} = 2m} C_{m_0, \ldots, m_{k-1}} (s_0 x_0)^{m_0} \cdots (s_{k-1} x_{k-1})^{m_{k-1}} \, d\sigma_{k-1}
\]
\[
= w(z) \, (1 + |\text{Im } z|)^k \sum_{m_0, \ldots, m_{k-1} = 2m} C_{m_0, \ldots, m_{k-1}} x_0^{m_0} \cdots x_{k-1}^{m_{k-1}}
\]
\[
\times \int_{S_{k-1}} s_0^{m_0} \cdots s_{k-1}^{m_{k-1}} \tilde{h}_z (s_0 x_0 + \ldots + s_{k-1} x_{k-1}) \, d\sigma_{k-1}
\]
\[
= w(z) \, (1 + |\text{Im } z|)^k \sum_{m_0, \ldots, m_{k-1} = 2m} C_{m_0, \ldots, m_{k-1}} x_0^{m_0} \cdots x_{k-1}^{m_{k-1}} \phi_{z, m_0, \ldots, m_{k-1}} (x_0, \ldots, x_{k-1})
\]
\[
= w(z) \, (1 + |\text{Im } z|)^k \sum_{m_0, \ldots, m_{k-1} = 2m} C_{m_0, \ldots, m_{k-1}} f_{z, m_0, \ldots, m_{k-1}} (x_0, \ldots, x_{k-1})
\]

where
\[
f_{z, m_0, \ldots, m_{k-1}} (x_0, \ldots, x_{k-1}) = x_0^{m_0} \cdots x_{k-1}^{m_{k-1}} \phi_{z, m_0, \ldots, m_{k-1}} (x_0, \ldots, x_{k-1})
and where the \((k-1)\)-th polynomial integral momentum \(\phi_{z,m_0,...,m_{k-1}}\) is associated with the function
\[
\tilde{h}_z(x) = |x|^{2-k-2m}
\]
and the polynomial
\[
Q_{m_0,...,m_{k-1}}(s_1, \ldots, s_{k-1}) = s_0^{m_0} \cdots s_{k-2}^{m_{k-2}} s_{k-1}^{m_{k-1}}.
\]
Thus, to see (27) it is sufficient to estimate the individual summand in the multiple operator integral
\[
T^H_{f_z,m_0,...,m_{k-1}}(V_1, \ldots, V_{k-1}) = \phi_{f_z,m_0,...,m_{k-1}}(H^{m_0} V_1 H^{m_1}, V_2 H^{m_2}, \ldots, V_{k-1} H^{m_{k-1}}).
\]
(28)

Looking at the integrals associated with individual functions \(f_z,m_0,...,m_{k-1}\) and appealing to (6), we obtain

\[
\left\| T^H_{f_z,m_0,...,m_{k-1}}(V_1, \ldots, V_{k-1}) \right\|_p \leq \text{const} \left\| H^{m_0} V_1 H^{m_1} \right\|_{\alpha_1} \left\| V_2 H^{m_2} \right\|_{\alpha_2} \cdots \left\| V_{k-1} H^{m_{k-1}} \right\|_{\alpha_{k-1}}
\]
\[
\leq \text{const} \left\| H \right\|_{\alpha_1}^{m_0} \left\| V_1 \right\|_q \left\| H \right\|_{\alpha_2}^{m_1} \left\| V_2 \right\|_q \left\| H \right\|_{\alpha_3}^{m_2} \cdots \left\| V_{k-1} \right\|_q \left\| H \right\|_{\alpha_{k-1}}^{m_{k-1}}
\]
\[
\leq \text{const} \left\| H \right\|_p^{2m} \left\| V_1 \right\|_q \cdots \left\| V_{k-1} \right\|_q
\]
\[
= \text{const} \left\| H \right\|_p^{2m'} \text{ where }
\]
\[
\frac{1}{\alpha_1} = \frac{m_0 + m_1}{p} + \frac{1}{q}, \quad \frac{1}{\alpha_j} = \frac{m_j}{p} + \frac{1}{q}, \quad j = 2, \ldots, k-1.
\]

Thus, the estimate (27) is shown. The proof of the lemma is finished. \(\Box\)

Remark 19. The assumption \(\Re z > 2m + k\) is used because the RHS of (28) does not make sense when \(\Re z = 2m + k\). However, the LHS in (26) does make sense even when \(\Re z = 2m + k\), unless \(m = 0\). Consequently, the assertion of Lemma 18 holds for \(\Re z \geq 2m + k\) when \(m > 0\).
Taylor expansion for $t \mapsto \text{tr} \left( f_p(H_t) \right)$. Here, we deal with the final step of the proof of Theorem 15. Let $1 < p < \infty$. Fix self-adjoint elements $H_1, H_0 \in \mathcal{S}^p$ such that $\|H_j\|_p \leq 1$, $j = 0, 1$ and set $H_t := (1-t)H_0 + tH_1$, $V := H_1 - H_0$.

We begin our discussion of Taylor expansion of $t \mapsto \text{tr} \left( f_p(H_t) \right)$ with the simplest case, when $m = 1$. Firstly, by the fundamental theorem of the calculus, we write

$$
\text{tr} \left( f_p(H_1) \right) - \text{tr} \left( f_p(H_0) \right) = \int_0^1 \frac{d}{dt} \text{tr} \left( f_p(H_t) \right) \, dt.
$$

Now, we use well-known formulae (see e.g. [4] or [11, Corollary 6.8] together with [6, Lemma 20]) and rewrite the preceding formula as

$$
\text{tr} \left( f_p(H_1) \right) - \text{tr} \left( f_p(H_0) \right) = \int_0^1 \text{tr} \left( V f_p' (H_t) \right) \, dt.
$$

Now, we claim that the following formula holds for all $m \in \mathbb{N} : 1 < m < p$.

$$
\text{tr} \left( f_p (H_1) \right) = \text{tr} \left( f_p (H_0) \right) + \sum_{k=1}^{m-1} \frac{1}{k} \text{tr} \left( V T^{H_0}_{f_p^k} (\underbrace{V, \ldots, V}_{k-1\text{-times}}) \right)
$$

$$
+ \int_0^1 t^{m-1} \text{tr} \left( V T^{H_0}_{f_p^m} (\underbrace{V, \ldots, V}_{m-1\text{-times}}) \right) \, dt. \quad (29)
$$

In writing the multiple operator integral in the first line of (29) we used convention (22), whereas in the second line we used a similar convention by replacing $(\underbrace{H_t, H_0, \ldots, H_0}_{m-1\text{-times}})$ with just $(H_t, H_0)$.

We prove (29) by the method of mathematical induction. The induction step is justified as follows. Assuming that (29) holds for $m - 1$, we add and subtract

$$
\frac{1}{m-1} \text{tr} \left( V T^{H_0}_{f_p^{m-1}} (\underbrace{V, \ldots, V}_{m-2\text{-times}}) \right), \quad \left[ \pm \text{tr} \left( V f_p' (H_0) \right) \text{ if } m = 2 \right]
$$

We then have

$$
\text{tr} \left( f_p (H_1) \right) = \text{tr} \left( f_p (H_0) \right) + \sum_{k=1}^{m-2} \frac{1}{k} \text{tr} \left( V T^{H_0}_{f_p^k} (\underbrace{V, \ldots, V}_{k-1\text{-times}}) \right)
$$

$$
+ \frac{1}{m-1} \text{tr} \left( V T^{H_0}_{f_p^{m-1}} (\underbrace{V, \ldots, V}_{m-2\text{-times}}) \right)
$$

$$
+ \int_0^1 t^{m-2} \text{tr} \left( V T^{H_0}_{f_p^m} (\underbrace{V, \ldots, V}_{m-2\text{-times}}) \right) \, dt
$$

$$
- \frac{1}{m-1} \text{tr} \left( V T^{H_0}_{f_p^{m-1}} (\underbrace{V, \ldots, V}_{m-2\text{-times}}) \right),
$$

$$
+ \int_0^1 t^{m-1} \text{tr} \left( V T^{H_0}_{f_p^m} (\underbrace{V, \ldots, V}_{m-1\text{-times}}) \right) \, dt.
$$

For the last two terms we now observe that

$$
\int_0^1 t^{m-2} \left( \text{tr} \left( V T_{f_p}^{H_0, H_0} \left( V, \ldots, V \right) \right) - \frac{1}{m-1} \text{tr} \left( V T_{f_p}^{H_0} \left( V, \ldots, V \right) \right) \right) dt
$$

Finally, by Theorem 10 (with \( \phi = f^{[m-1]}_p \), \( \psi = f^{[m]}_p \) and \( h = f^{(m-1)}_p \), which belongs to the space \( \tilde{B}^1_{\infty, 1} \) by Lemma 3, we see that

$$
T_{f_p}^{H_0} \left( V, \ldots, V \right) = T_{f_p}^{H_0} \left( H_1 - H_0, V, \ldots, V \right)
$$

This completes the proof of formula (29). Now, we are in a position to prove the expansion (21). By adding and subtracting to (29)

we obtain that

$$
\frac{1}{m} \text{tr} \left( V T_{f_p}^{H_0} \left( V, \ldots, V \right) \right),
$$

we have

$$
\text{tr} \left( f_p \left( H_1 \right) \right) = \text{tr} \left( f_p \left( H_0 \right) \right) + \sum_{k=1}^{m-1} \frac{1}{k} \text{tr} \left( V T_{f_p}^{H_0} \left( V, \ldots, V \right) \right)
$$

Observe that the first line above, given the definition of the functionals \( \delta_{H}^{[k]} \), is the complete left hand side of (21). In other words, we have

$$
||H_0 + V||^p_p - ||H_0||^p_p - \sum_{k=1}^{m} \delta_{H}^{(k)} \left( V, \ldots, V \right)
$$

Setting in Theorem 13 (and Remark 14) \( A = H_1, B = H_0, V_1 = \cdots = V_{m-1} = V, \)

\( p_0 = p_1 = \cdots = p_{m-1} = p, \) \( \alpha = p - m \) and applying that theorem with \( m - 1 \)
instead of $m$, we have $\frac{1}{p} + \frac{1}{p'} = 1$ and

$$\left\| T^{H_1, H_0}_{p,m}(V_1, \ldots, V_m) - T^{H_1, H_0}_{p',m}(V_1, \ldots, V_m) \right\|_{p'} = O \left( \|V\|^{p-1}_p \right).$$

Thus, the expansion (21) follows immediately via Hölder inequality. Theorem 15 is completely proved.

3. CONCLUDING REMARKS

As we noted in the Introduction, our methods are also suitable to resolve a similar problem concerning differentiability properties of general non-commutative $L_p$-spaces associated with general semifinite von Neumann algebras $\mathcal{M}$ stated in [13]. In this paper we demonstrate such a resolution for the special case when $\mathcal{M}$ is an arbitrary type $I$ von Neumann algebra acting on a separable Hilbert space. Using well-known structural results for such algebras, it is easy to see that it is sufficient to deal with $L_p$-spaces associated with the von Neumann tensor product $L_\infty(0, 1) \bar{\otimes} B(H)$, or equivalently, it is sufficient to deal with Lebesgue-Bochner spaces $L_p(S^p) := L_p([0, 1], S^p)$ (see e.g. [19] and references therein). The case $1 \leq p \leq 2$ is easy and in fact has been dealt with in full generality in [7, Lemma 4.1]. In order to deal with the case $2 \leq p < \infty$ and thus, with higher order derivatives, it is convenient to cite the following result from [9].

Theorem 20. [9, Theorem 3.5] Let $E$ be a Banach space, $(T, \Sigma, \mu)$ a measure space, and $k$ a positive integer with $p > k$. If the norm $\| \cdot \| : E \rightarrow \mathbb{R}$ is $k$-times continuously differentiable away from zero and the $k$-th derivative of the norm in $E$ is uniformly bounded on the unit sphere in $E$, then the norm $\| \cdot \| : L_p(E, \mu) \rightarrow \mathbb{R}$ is $k$-times continuously differentiable away from zero.

In view of this result, we shall obtain the result similar to Theorem 1 as soon as we verify the assumptions of the preceding theorem concerning the norm $\| \cdot \| : S^p \rightarrow \mathbb{R}$. We shall verify these assumptions on the (open) unit ball $S^p_1$ of $S^p$. To this end, we firstly need to verify the mapping $\| \cdot \|_p : S^p \rightarrow \mathbb{R}$ is $k$-times continuously differentiable that is (see [9, p.232]) it is $k$-times differentiable and the $k$-th derivative $\delta^{(k)}$ is continuous on $S^p_1$. Secondly, we need to ascertain that the derivative $\delta^{(k)}$ is uniformly bounded on $S^p_1$. Now, the existence of the derivative
$\delta^{(k)}$ is of course our main result Theorem 15. The continuity of $\delta^{(k)}$ on the unit sphere of $S^p$ follows from the definitions (23) and (24) together with the estimate obtained in Theorem 17 and Hölder inequality. Finally, the same estimate also yields uniform boundedness of $\delta^{(k)}$ on $S^p_1$. This completes the proof of analogue of Theorem 1 for the space $L_p(S^p)$. The general case of an arbitrary semifinite von Neumann algebra $\mathcal{M}$ of type $II$ depends on substantial technical preparations needed to extend definitions (23) and (24) to the setting of unbounded operators from $L_p(\mathcal{M})$ and will be dealt separately.

References

[1] A. B. Aleksandrov and V. V. Peller, Functions of operators under perturbations of class $S_p$, J. Funct. Anal. 258 (2010), no. 11, 3675–3724.
[2] J. Arazy and Y. Friedman, Contractive projections in $C_p$. Mem. Amer. Math. Soc. 95 (1992), no. 459.
[3] N. A. Azamov, A. L. Carey, P. G. Dodds, F. A. Sukochev, Operator integrals, spectral shift, and spectral flow, Canad. J. Math. 61 (2009), No. 2, 241 – 263.
[4] M. S. Birman and M. Z. Solomyak, Double Stieltjes operator integrals III(Russian), Probl. Math. Phys., Leningrad Univ., 6 (1973), 27-53.
[5] R. Bonic and J. Frampton, Smooth functions on Banach manifolds, J. Math. Mech. 15 (1966), No.5, 877-898.
[6] A. L. Carey, D. S. Potapov, F. A. Sukochev, Spectral flow is the integral of one forms on Banach manifolds of self adjoint Fredholm operators, Adv. Math, 222 (2009), 1809–1849.
[7] P.G. Dodds and T.K. Dodds, On a singular value inequality of Ky Fan and Hoffman, Proc. Amer. Math. Soc. 117 (1993), no. 1, 115–124.
[8] S. Krein, Ju. Petunin, E. Semenov, Interpolation of linear operators, Nauka, Moscow, 1978 (in Russian); English translation in Translations of Math. Monographs, Vol. 54, Amer. Math. Soc., Providence, RI, 1982.
[9] I.E. Leonard and K. Sundaresan, Geometry of Lebesgue-Bochner function spaces–smoothness, Trans. Amer. Math. Soc. 198 (1974), 229–251.
[10] L. A. Lusternik and V. L. Sobolev, Elements of functional analysis, Translated from the Russian. Frederick Ungar Publishing Co., New York 1961.
[11] B. de Pagter and F. A. Sukochev, Differentiation of operator functions in non-commutative $L_p$-spaces, J. Funct. Anal. 212 (2004), no. 1, 28–75.
[12] V. V. Peller, Multiple operator integrals and higher operator derivatives, J. Funct. Anal. 233 (2006), no. 2, 515–544.
[13] G. Pisier and Q. Xu, Non-commutative $L_p$-spaces, Handbook of the geometry of Banach spaces, Vol. 2, 1459-1517, North-Holland, Amsterdam, 2003.
[14] D. Potapov and F. Sukochev, Unbounded Fredholm modules and double operator integrals, J. Reine Angew. Math. 626 (2009), 159–185.
[15] D. Potapov, A. Skripka, and F. Sukochev, *Spectral shift function of higher order*, to appear in Invent. Math. DOI 10.1007/s00222-012-0431-2.

[16] D. Potapov, A. Skripka, and F. Sukochev, *Higher order spectral shift for contractions*, to appear in Proc. London Math. Soc.

[17] D. Potapov and F. Sukochev, *Operator-Lipschitz functions in Schatten-von Neumann classes*, Acta Math. 207 (2011), no. 2, 375–389.

[18] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Second edition. Academic Press, Inc., New York, 1980.

[19] F. Sukochev, *Linear-topological classification of separable \( L_p \)-spaces associated with von Neumann algebras of type I*, Israel J. Math. 115 (2000), 137–156.

[20] K. Sundaresan, *Smooth Banach spaces*, Math. Ann. 173 (1967), 191–199.

[21] N. Tomczak-Jaegermann, *On the differentiability of the norm in trace classes \( S_p \)*, Séminaire Maurey-Schwartz 1974–1975: Espaces \( L_p \), applications radonifiantes et géométrie des espaces de Banach, Exp. No. XXII, Centre Math., École Polytech., Paris, 1975, p. 9.

[22] E. M. Stein, *Singular integrals and differentiability properties of functions*, New Jersey, Princeton Univ. Press, 1970.

[23] H. Triebel, *Characterizations of Besov-Hardy-Sobolev spaces via harmonic functions, temperatures, and related means*, J. Approx. Theory 35 (1982), no. 3, 275–297.

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