Radion Stabilization by Stringy Effects in General Relativity

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We consider the effects of a gas of closed strings (treated quantum mechanically) on a background where one dimension is compactified on a circle. After we address the effects of a time dependent background on aspects of the string spectrum that concern us, we derive the energy-momentum tensor for a string gas and investigate the resulting space-time dynamics. We show that a variety of trajectories are possible for the radius of the compactified dimension, depending on the nature of the string gas, including a demonstration within the context of General Relativity (i.e. without a dilaton) of a solution where the radius of the extra dimension oscillates about the self-dual radius, without invoking matter that violates the various energy conditions. In particular, we find that in the case where the string gas is in thermal equilibrium, the radius of the compactified dimension dynamically stabilizes at the self-dual radius, after which a period of usual Friedmann-Robertson-Walker cosmology of the three uncompactified dimensions can set in. We show that our radion stabilization mechanism requires a stringy realization of inflation as scalar field driven inflation invalidates our mechanism. We also show that our stabilization mechanism is consistent with observational bounds.

I. INTRODUCTION

In the early days of string cosmology, it was realized that superstrings had an effect on space-time dynamics that was qualitatively quite different from that of particles or fields. In particular, it was realized that string winding modes could provide a confining mechanism for certain compact directions in such a way as to allow only three spatial dimensions to grow large \cite{1}. Key to this realization are the T-duality of the spectrum of string states, and the fact that the background is described by Dilaton Gravity, and not by General Relativity with a fixed dilaton (this is crucial in order that the background equations obey the T-duality symmetry). The arguments of \cite{1} were put on a firmer basis by the analysis of \cite{2} (see also \cite{3}).

Starting point of the considerations of \cite{1} is the assumption that all spatial dimensions begin at close to the self-dual radius (the string scale), and that matter consists of a hot gas of string states. The considerations of \cite{1} were more recently applied to “brane gas cosmology” \cite{4,5}, a scenario in which the initial string gas is generalized to be a gas of all brane modes. It was shown that given the hot dense initial conditions assumed in \cite{1}, the string winding modes are the last modes to fall out of equilibrium and thus dominate the late time dynamics. Hence \cite{4}, the inclusion of brane degrees of freedom does not change the prediction that only three dimensions to grow large. The dynamical equations describing the growth of the three dimensions which can become large were solved in \cite{5} (see also \cite{6}). In \cite{7}, it was shown that isotropy in these large dimensions is a consequence of the dynamics. In \cite{8}, it was found that if both the momentum and winding modes of the strings are included in the dynamical equations, the radius of the compactified dimensions is stabilized at the self-dual radius. More precisely, the expansion of the three large dimensions leads to damped oscillations in the “radion” about the self-dual value. Thus, in the context of a background described by Dilaton Gravity, radion stabilization is a natural consequence of brane gas cosmology \cite{26}.

At the present time, however, the dilaton is most likely fixed (see, however, \cite{11} for an alternate scenario). Thus, it is of interest to explore how the inclusion of string (and brane) winding and momentum modes influences the dynamical evolution of the radion in a background space-time described by General Relativity (GR). There is another motivation for studying this issue. Another corner of the M-theory moduli space is 11-d supergravity. In \cite{12} it was found that a brane gas in this background also admits a region in the phase space of initial conditions in which only three spatial dimensions can become large, although this corner may not be consistent with holographic entropy bounds \cite{13} (see also \cite{14} where the considerations in this corner of M-theory moduli space was extended to spaces with more general topologies). Motivated by these considerations, we in this paper study a simplified problem, namely the questions of how a gas of winding and momentum modes of strings winding one compactified spatial dimension (taken to be a circle) effects the evolution of the radius (the radion). We start with initial conditions in which the three spatially non-compact dimensions are expanding. We find that the gas of string winding and momentum modes gives a natural radion stabilization mechanism. Our approach is to consider the effect of strings on 5D space-time dynamics (with the extra spatial dimension compactified to a circle) by adding the appropriate matter term to the standard Einstein-Hilbert action. We will derive this term shortly (see also \cite{15} for a similar derivation). The result-
ing energy-momentum tensor leads to a novel behavior when inserted into the Einstein equations. We will find that we can generate a non-singular bouncing solution for the radius of the compactified dimension in the context of GR (without a dilaton) while respecting the Dominant Energy Condition for the matter content. Specifically, the radion performs damped oscillations about the self-dual radius. Initially, we study a pure state of matter with specific quantum numbers obeying the T-duality symmetry. However, we will find that we can rather naturally extend the analysis to a gas of these strings in thermal equilibrium (with a bath of gravitons and photons), with the result that the radius of the compact dimension is dynamically stabilized at the self-dual radius \( R = \sqrt{\alpha'} \), where \( 2\pi\alpha' \) is the inverse of the string tension (see also [10] for a study of string gases in thermal equilibrium).

In addition, we find that our model evolves according to standard Friedmann-Robertson-Walker (FRW) cosmology after the compact dimension has been stabilized, and that the resultant stabilization is incompatible with any subsequent inflationary epoch driven by a bulk scalar field (for string-specific ideas on how to generate inflation in brane gas cosmology see [17]). However this conclusion can be avoided if some form of stringy inflation is produced world-sheet metric (where the unmatched indices \( g^{\mu\nu} \) the space-time metric \( g^{\mu\nu} \)).

Before we can turn to any of this, we will have to address a question of principle concerning the string spectrum in a cosmological context (this issue is also being studied in [16]). The question of formulating String Theory in a time-dependent background is a current and active area of research. However, we are primarily interested in the behavior of strings in a background that evolves on a cosmological time scale. As can be seen from the FRW equations, the cosmological time scale \( H^{-1} \) (where \( H \) is the Hubble expansion rate) is larger than the characteristic microscopic time \( \sigma^{-1} \) (where \( \sigma^4 \) is the matter energy density) by a factor of \( m_{pl}/\sigma \), where \( m_{pl} \) is the Planck mass. Thus, away from singular epochs in the history of the Universe, the cosmological time scale is going to be many, many orders of magnitude longer than the characteristic time scale of the string dynamics, and hence we should be able to inherit many of the features of the string spectrum in a static space-time (with some obvious modifications). We justify this intuition more rigorously in the Appendix, but we feel that it might suffice at this point to remind the reader of the approximate irrelevance of a time dependent background for a much more familiar theory: Quantum Field Theory (QFT). Although quantum fields in curved spaces exhibit several qualitatively different features from quantum fields in flat spaces [21], we still manage to do a lot of sensible (and spectacularly successful) flat space-time QFT calculations despite the persistent Hubble expansion of space-time. The reason for this is easy to see: the contributions to masses, to scattering amplitudes, to the structure of the Hilbert space of our theory, etc., that come from terms that depend on derivatives of the metric are in the present epoch highly suppressed and irrelevant. This is partly captured by the Adiabatic Theorem, which is the statement that given two systems with Hamiltonians that can be continuously interpolated, then in a precise sense, the eigenstates of the initial system will evolve into the eigenstates of the final system if this interpolation takes place slowly enough. Slow enough in simple quantum systems usually means that the variation happens over much longer time scales than the characteristic time of the system (by which we mean the time associated with the typical energy of the system: \( \tau \sim \frac{1}{T} \)). Having said this, were we to study QFT in places where the metric varies a lot more rapidly (at the edges of black holes or in the very early Universe) we invariably have to account for the curvature of space. Thus, we can hope that the effects of a time dependent background on the closed string spectrum only require minor modifications to the flat space spectrum, if this time dependence is slow compared to the characteristic time of the string dynamics. We show in the Appendix that this is indeed the case, and in what follows we will stay within this regime.

The Outline of this paper is as follows: we first derive the energy-momentum tensor of a string gas (the derivation here is more general than the one given in [17]). Then we insert this tensor into the Einstein equations and study the dynamics of the radius of the compact dimension, assuming that the three large spatial dimensions are in the expanding phase. First, we consider a pure state of matter. Next, we extend the discussion to a thermal state. In Section IV, we discuss the late time dynamics and show that the stabilization of the radion is not compatible with inflation in the three large spatial dimensions, assuming the simplified description of matter which we are using.

A few words on our notation: Greek indices typically stand for 5-dimensional space-time indices, Roman indices \( i, j, ... \) are associated with the non-compact spatial dimensions, and Roman indices \( a, b, ... \) are string world-sheet coordinates. The 5-dimensional Planck mass is denoted by \( M_{pl} \) (or \( M_5 \) in abbreviated form). We also work in natural units \( (\hbar = c = k_B = 1) \) where we pick energy to be measured in electron volts.

### II. THE ENERGY-MOMENTUM TENSOR

To study how a gas of strings affects space-time dynamics, we need to derive the energy-momentum tensor of such a gas. We begin by studying the energy-momentum tensor of a single closed string. Starting with the Nambu-Goto action

\[
S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h},
\]

(1)

where \( h_{ab} \) denotes the world sheet metric

\[
h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)
\]

(2)

(and \( h \) is its determinant), we see that any variation in the space-time metric \( g_{\mu\nu} \) induces a variation in the induced world-sheet metric (where the unmatched indices
indicate that we perturb only the $\lambda \beta$ component of the metric:

\[
g_{\mu\nu}(X) \rightarrow g_{\mu\nu}(X) + \delta_{\mu\nu} \delta^D(X^\tau - y^\tau) \delta g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}}
\]

\[
h_{ab}(\sigma) \rightarrow h_{ab} + \partial_\sigma X^\lambda \partial_b X^\beta \delta^D(X^\tau - y^\tau) \delta h_{ab}
\]

Now, varying the Nambu-Goto action with respect to the space-time metric (performing a perturbation $\delta g_{\mu\nu}$ which acts on the metric as given above) will give us the space-time energy-momentum tensor of a single string:

\[
\delta S_{NG} \frac{\delta}{\delta g_{\mu\beta}(y)} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h_{ab} \delta h_{ab} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h_{ab} \partial_a X^\lambda \partial_b X^\beta \delta^D(X^\tau - y^\tau).
\]

We must first discuss the meaning of the expression

\[
\int d^2\sigma \delta^D(X^\tau - y^\tau)
\]

\[
= \int d^2\sigma \delta(X^0 - y^0) \delta(X^1 - y^1) \ldots \delta(X^D - y^D).
\]

In order to change the variable of integration, we need to apply $d\sigma = dX^\lambda / \sqrt{|g_{00}|}$ and sum over all the zeroes of $X^\lambda[\sigma] - y^\lambda$ when performing the integration. However, since we are considering modes winding one particular spatial direction, there are precisely two coordinates that are monotonic functions of the world-sheet parameters: $X^0$ being a monotonic function of $\sigma^0$ and $X^D$ being a monotonic function of $\sigma^1$ (the $D^{th}$ direction is taken to be compact). Thus,

\[
d^2\sigma \delta^D(X^\tau - y^\tau)
\]

\[
= \int dX^0 dX^D \sqrt{-g_{00}g_{DD}}
\]

\[
\times \delta(X^0 - y^0) \delta(X^D - y^D) \delta^D(X^\tau - y^\tau),
\]

where we include the metric factors in the last line so that we can take the delta functions in $X^0$ and $X^D$ to be properly normalized. With this result, we get:

\[
\delta S_{NG} \frac{\delta}{\delta g_{\lambda\beta}} = -\frac{\delta^D(X^\tau - y^\tau)}{4\pi\alpha' \sqrt{-g_{00}g_{DD}}}
\]

\[
\times \int \frac{dX^0}{|X^0|} \frac{dX^D}{|X^D|} \delta(X^0 - y^0) \delta(X^D - y^D) \sqrt{-h} h_{ab} \partial_a X^\lambda \partial_b X^\beta
\]

\[
= -\frac{1}{4\pi\alpha' \sqrt{-g_{00}g_{DD}}}
\]

\[
\times \delta^D(X^\tau - y^\tau) \sqrt{-h} h_{ab} \partial_a X^\lambda \partial_b X^\beta \bigg|_{X^0 = y^0, X^D = y^D},
\]

where we use the inverse metric to write the metric contributions in the denominator. Thus, the single string space-time energy-momentum tensor becomes

\[
T^{\lambda\beta} = \frac{-2}{\sqrt{-g}} \delta S \frac{\delta}{\delta g_{\lambda\beta}}
\]

\[
= \frac{1}{2\pi\alpha'} \frac{\delta^D(X^\tau - y^\tau) \sqrt{-h} h_{ab} \partial_a X^\lambda \partial_b X^\beta}{X^0 X^D |X^0| \sqrt{-g} \sqrt{-g_{DD}}).
\]

Inserting the explicit form of the inverse world-sheet metric

\[
h_{ab} = \frac{1}{h} \begin{pmatrix} h_{12} & h_{11} \end{pmatrix} = \frac{1}{h} \left( \begin{array}{cc} X^\mu X'^\mu & \dot{X}^\mu \dot{X}'^\mu \\ \dot{X}^\mu X'^\mu & X^\mu X'^\mu \end{array} \right)
\]

and using the constraints on the world-sheet fields

\[
P_\mu X'^\mu = 0,
\]

we can write \( T^{\lambda\beta} \) as

\[
T^{\lambda\beta} = -\frac{1}{2\pi\alpha'} \frac{\delta^D(X^\tau - y^\tau)}{X^0 X^D |X^0| \sqrt{-\det g_{ij}}} [X^\lambda X'^\beta - X^\beta X'^\lambda] \]

Next, we solve for $\dot{X}^0$ using the constraint \( \dot{X}^0 = \dot{X}^1 X_i + \dot{X}^D X^D $ \]

\[
-\dot{X}^0 X'^0 + X'^1 X_i + X'^D X^D,$

where we have explicitly used the background metric

\[
g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t), b^2(t)).
\]

It is consistent with the equations of motion in a (slow enough) time varying background to set $X'^0 = 0$, so that

\[
\dot{X}^0 = P^i P_i + X'^i X_i + P^D P_D + X'^D X^D.
\]

In addition (in a slowly varying background) the right hand side can be expressed in terms of the familiar oscillator expansion. Accounting for the zero mode operators explicitly, we get the center of mass momentum from the spatial zero modes and the winding energy from the zero mode terms in the compactified direction. The other modes give us the left and right moving oscillator terms (see \cite{ details):}

\[
\dot{X}^0 = \sqrt{g^{ij} p_i p_j + \frac{2}{\alpha'} (N + \tilde{N} - 2) + \left( \frac{n}{b} \right)^2 + \left( \frac{ab}{\alpha'} \right)^2},
\]

where $n$ and $w$ are the quantum numbers for momentum and winding in the compact direction, respectively, and
\(N\) and \(\tilde{N}\) are the levels of the left- and right-moving oscillator modes of the string, respectively. The expression above is none other than the energy of the string. Using the level matching constraints
\[
N + \nu w = \tilde{N} = 0, \tag{11}
\]
we finally end up with
\[
X^0 = \sqrt{g^{ij}p_ip_j + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}. \tag{12}
\]

Now, we are ready to evaluate (17) for a single wound string. We have an explicit expression for \(X^0\) and we know that \(|X^5| = |w|b\) in units of \(\alpha\) for a wound string, the factor of \(|w|\) being canceled by the summation over all \((w\) in total) zeroes of the argument of the delta function. Thus, component by component, we get:

\[
T_0^0 = -\rho = -\frac{1}{2\pi} \frac{\delta^3(X^i - y^i)}{a^{3}b} \sqrt{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}, \tag{13}
\]

\[
T_i^j = \rho = \frac{1}{2\pi} \frac{\delta^3(X^i - y^i)}{a^{3}b} \frac{p^i p_j}{\sqrt{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}}, \tag{14}
\]

\[
T_5^5 = r = \frac{1}{2\pi} \frac{\delta^3(X^i - y^i)}{a^{3}b} \frac{n^2 - w^2}{\sqrt{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}}, \tag{15}
\]

(note that we label the extra spatial coordinate by \(\tilde{5}\)) where we ignore off-diagonal components since we are about to apply these expressions to an isotropic gas of strings.

However, we wish to present at this point another derivation of this result which is rather more direct. Consider the energy of a single wound string:
\[
E^2 = p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2. \tag{16}
\]

A spatially uniform gas of such strings with the same quantum numbers would have a 5-dimensional energy density
\[
\epsilon = \frac{\mu(t)}{2\pi b} \sqrt{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}, \tag{17}
\]
where \(\mu(t)\) is the number density of strings. We divide by \(2\pi b\) since this energy will be uniformly distributed over the length of the string. The momentum that appears in this expression is now the momentum squared of a gas of strings whose momenta have identical magnitudes, but whose directions are distributed isotropically. To fully account for the metric factors in this expression, we write \(\mu(t)\) as \(\mu_0(t)/a^3(t)\) since this is how a number density explicitly depends on the metric. Now, realizing that this is an energy density, we can introduce this gas of strings as matter interacting with the gravitational field by just adding the following term to the gravitational part of the action:
\[
S_{\text{int}} = -\int d^5x \sqrt{-g\epsilon}\tag{18}
\]
(see e.g. Section 10.2 of [20]).

Realizing now that the metric factors in the denominator of the expression for the energy density can be written as \(a^3 = \sqrt{\text{det}(g_{ij})}\) and \(b = \sqrt{g_{55}}\), we can write the above equation as:

\[
S_{\text{int}} = -\int d^5x \sqrt{-g} \frac{\mu_0(t)}{2\pi} \times \frac{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}{\sqrt{\text{det}(g_{ij})}} \tag{19}
\]

\[
S_{\text{int}} = -\int d^5x \frac{\rho_0(t)}{2\pi} \times \frac{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}{\sqrt{\text{det}(g_{ij})}} \tag{20}
\]

By our metric ansatz and the isotropy of the distribution of the momenta, we have that \(p^i p_i = a^{-2}(p_x^2 + p_y^2 + p_z^2)\). Using this fact, it is straightforward to show that the energy-momentum tensor derived from this interaction term is:

\[
T_0^0 = -\rho = -\frac{1}{2\pi a^{3}b} \sqrt{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}, \tag{19}
\]

\[
T_i^j = \rho = \frac{1}{2\pi a^{3}b} \frac{\rho_0(t)}{2\pi} \frac{p^2/3}{\sqrt{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}}, \tag{20}
\]

\[
T_5^5 = r = \frac{1}{2\pi a^{3}b} \frac{\rho_0(t)}{2\pi} \frac{\frac{w^2 - w^2}{\alpha^2}}{\sqrt{p^i p_i + \frac{4}{\alpha}(N - 1) + \left(\frac{n}{b} + \frac{w}{\alpha}\right)^2}}, \tag{21}
\]

which is exactly what we would get from \(\text{[13, 14]}\) and \(\text{[15]}\). We were to construct a hydrodynamical average with an isotropic momentum distribution.

We now investigate some simple aspects of our result. The first thing to note is that \(T_5^5\), which is the pressure along the compact direction, gets a negative contribution from the winding of our strings and a positive contribution from the momentum along this direction. The spatial pressure is always positive, and for the simple case \(n = w = 0, N = 1\), which describes a gas of gravitons moving in the non compact directions, we obtain \(r = 0, p = \rho/3\).
Since we are about to study the effects of this energy-momentum tensor on space-time, we should make sure that the energy-momentum tensor is covariantly conserved, or else it will not be consistent to equate it to the covariantly conserved Einstein tensor. The covariant conservation of $T^\mu_\nu$

$$0 = \nabla_\mu T^\mu_\nu,$$

where $\nabla_\mu$ is the covariant derivative operator, implies

$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) + \frac{\dot{b}}{b}(\rho + r)$$

$$0 = \partial_\mu p$$

$$0 = \partial_\nu r.$$

It is straightforward to check that our energy-momentum tensor satisfies this as an identity. In the continuity equation, this is due to the metric factors contained in the energy density, which upon differentiation produce terms that exactly cancel the terms proportional to the Hubble factors. The remaining equations are trivially satisfied by our setup, which assumed an axis of symmetry along the compactified dimension (the Kaluza-Klein setup) with homogeneous and isotropic spatial sections.

One final point to note is that we have derived an energy-momentum tensor that exhibits positive pressures along the non-compact directions and positive or negative pressures along the compactified direction. We need to ensure that this negative pressure has a bounded equation of state as otherwise our theory would be unstable. The Dominant Energy Condition (DEC) of General Relativity [21] ensures the stability of the vacuum, and requires that the equation of state parameter $\omega = p/\rho$ be greater than or equal to -1 (see e.g. [22] for a recent discussion). Since the spatial pressures are always positive, we only need to check our equation of state for the pressure along the compact direction:

$$r = \frac{1}{2\pi a^2 b} \frac{n^2 - \omega^2 b^2}{\alpha^2 \nu^2} \frac{p^2 p_1 + \frac{4}{a} (N - 1) + (\frac{2}{b} + w\frac{b}{\alpha})^2}{p^2 p_1 + \frac{4}{a} (N - 1) + (\frac{2}{b} + w\frac{b}{\alpha})^2},$$

where the co-efficient of $\rho$ in the above is our equation of state parameter. Were we to consider states described by $n = \pm 1$, $w = -n$, $N = 1$ (which as we will see further on, turn out to be the relevant states that give us stabilization), this parameter remains bounded as $b$ varies [33].

$$-1 \leq \omega \leq 1$$

Thus, we have verified that the spectrum of string states satisfies the DEC, and in doing so ensured ourselves of sensible space-time dynamics arising from the string gas, the topic we will turn our attention to next.

### III. SPACE-TIME DYNAMICS

We start with the Einstein tensor derived from the metric (9):

$$G^0_0 = -3\frac{\dot{a}}{a} \left[ \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right]$$

$$G^i_j = -\delta^i_j \left[ \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{\alpha}}{a} \right]$$

$$G^0_i = -3\frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2.$$ 

Equating this to $\frac{1}{M^2_{pl}} T^\mu_\nu$ will give us the Einstein equations. However, let us focus on the equation that governs the evolution of the scale factor $b$. Starting with $G^0_i$ and eliminating $a$ and $\dot{a}^2$ by adding the appropriate combinations of $G^0_0$ and $G^0_i$, we get:

$$\ddot{b} + 3H \dot{b} + \frac{b}{M^2_{pl}} \left( p - \frac{2r}{3} - \frac{\rho}{3} \right) = 0,$$  \hspace{1cm} (23)

where $H$ is the 3-dimensional Hubble factor. This is a second order, nonlinear (because of the $b$ dependence in the matter terms) differential equation with a damping term and a driving term. We will demonstrate further on that the Einstein equations admit expanding solutions for the non-compact dimensions ($H > 0$), and take it as a given for what follows. Thus, in spite of its non-linearity, we easily see that $\dot{b}$ describes an expanding or a contracting scale factor depending on the sign of the driving term.

The first thing to notice from this equation is that matter for which the quantity $p - \frac{2r}{3} - \frac{\rho}{3}$ vanishes will not contribute to the dynamics of the compact dimension. Thus, recalling that a gas of gravitons ($n = w = 0$, $N = 1$) has an equation of state $p = \frac{\rho}{3}$, $r = 0$, as does a gas of ordinary 4-dimensional photons, we see that such matter will not affect the dynamics of the scale factor $b$. In fact, such a background 4-dimensional gas provides an excellent candidate for a thermal bath which will eventually couple our gas of winding modes to.

First, however, we will study this driving term as it is, for a gas consisting of strings with identical quantum numbers. Upon evaluating the driving term, we find that:

$$\frac{b}{M^2_{pl}} \left( p - \frac{2r}{3} - \frac{\rho}{3} \right) = \frac{\mu_0}{M^2_{pl} a^2 2\pi}$$

$$\times \frac{-n^2 \nu^2 - 2n w \nu^2 + w^2 b^2 \alpha^2 - 4(N-1) \frac{\alpha^2}{3\nu^2}}{\sqrt{p^2 p_1 + \frac{4}{a} (N - 1) + (\frac{2}{b} + w\frac{b}{\alpha})^2}},$$

from which we infer that momentum modes and oscillator modes lead to expansion of the scale factor, whereas the winding modes produce contraction. Exactly what happens, of course, depends on the values of the quantum numbers. It should be recalled that the quantum numbers are subject to the constraint $nw + N \geq 0$ coming from the level matching conditions (see Eq. [14]).
Let us pick a particular set of quantum numbers. As we shall see later, the most interesting case is when $a = -w = \pm 1$, $N = 1$, in which case the driving term becomes:

$$b \frac{p - \frac{2r}{3} - \frac{\rho}{3}}{M^3_{pl}} = 2\mu_0 \frac{2\mu_0}{M^3_{pl} a^3 2\pi \sqrt{\alpha'}} \frac{-\frac{1}{2} + \frac{2}{3} + \frac{\rho}{3}}{\sqrt{\alpha'}^{\rho} \rho_1 + (\frac{1}{\beta} - \beta)^2},$$

(25)

where $\beta$ is the scale factor in units of $\sqrt{\alpha'}$. Quite generically, we can explore features of the “potential” energy that will yield this driving term. We see that, since the denominator is strictly positive, and the driving term changes sign at $\beta = 1$, this value will be a minimum of the potential energy, and hence a point of equilibrium. Numerical integration of the driving term yields the potential energy curve of Figure 1 [13], where the potential is plotted in units of $\frac{2\mu_0}{M^3_{pl} a^3 2\pi \sqrt{\alpha'}}$ as a function of $\beta$.

Because of the Hubble damping term in the equation of motion for $\beta$ (which is obtained by dividing through by a factor of $\sqrt{\alpha'}$), the scale factor will perform damped oscillations about the minimum of the potential to which it will evolve with rapidity depending on the value of the “spring constant” multiplying the driving term:

$$k = \frac{2\mu_0}{M^3_{pl} a^3 2\pi \alpha'}. $$

(26)

Thus, we have established that a gas of string modes with non zero winding and momentum numbers in the compact direction will provide a dynamical stabilization mechanism for the radion, provided that the three non-compact dimensions are expanding (such a behavior was already found in an early study [10] of the dynamics of string winding modes - we thank Scott Watson for drawing our attention to this paper). We will address the phenomenology of this stabilization mechanism further on, simply stating for now that we can obtain a robust stabilization mechanism which is consistent with observational bounds.

At this point, we wish to mention that the “Quantum Gravity Effects” required to stabilize the extra dimensions in earlier attempts [23, 21] at Kaluza-Klein cosmology find a stringy realization here, in that all that was required for radius stabilization was matter that depended on the size of the extra spatial dimensions in a non-trivial way.

To round off the discussion, we wish to demonstrate that our assumption of an expanding scale factor $a(t)$ is consistent with an oscillating scale factor $b(t)$. Consider the two Einstein equations that do not contain second derivatives of $b$:

$$\frac{\rho}{3M^3_{pl}} = H^2 + \frac{\dot{b}}{b},$$

$$\frac{-r}{3M^3_{pl}} = \dot{H} + 2H^2.$$  

These equations imply that

$$\dot{H} - 2H \frac{\dot{b}}{b} = -\frac{1}{3M^3_{pl}} (2\rho + r).$$

(27)

The resulting equation for $H$ has the integrating factor $1/b^2$, and hence the solution:

$$H(t) = H_0 \left( \frac{b(t)}{b_0} \right)^2 - \frac{b^2(t)}{3M^3_{pl}} \int_0^t dt' \left( \frac{2\rho + r}{b^2(t')} \right).$$

(28)

Now, from the discussion surrounding (22), we see that $\rho \geq r \geq -\rho$. Thus, the contribution to the integral in the above is strictly positive, and the second term on the right hand side of (28) can at most take on the value:

$$\frac{b^2(t)}{M^3_{pl}} \int_0^t dt' \frac{\rho(t')}{b^2(t')}.$$

(29)

Thus, we see that if we pick the initial conditions for $H$ appropriately, the scale factor $a$ can be taken to be expanding ($H > 0$) regardless of the detailed motion of $b$. In fact, if we assume that $H_0$ starts out positive, then $H(t)$ will remain so if

$$H_0 \geq \frac{1}{3M^3_{pl}} \int_0^t dt' \frac{2\rho + r}{b^2(t')/b_0^2},$$

(30)

where the eventual stabilization of $b$ and the $1/Vol$ dependence of $\rho$ and $r$ will bound the integral, which implicitly depends on $H$ itself. This implicit dependence works in our favor in that the larger we make $H_0$, the smaller the integral becomes and so we can imagine picking an initial $H_0$ such that a persistent expansion of the non-compact dimensions results. Note, however that if in the spirit of brane gas cosmology, we assume that all spatial dimensions are starting out with the same size and instantaneously static, then it may not be possible to evolve to a situation in which three large spatial dimensions are expanding. This is, in fact, the result that emerges from the work of [13], at least in a certain region of phase space.
IV. THERMAL STRING GASES

In what we have done so far, we have just considered the behavior of the size of the extra dimension in a rather artificial setting, namely imposing a gas of strings with a fixed set of quantum numbers. One expects the early Universe to be in a state of thermal agitation, and it is inevitable that transitions between different energy levels will be induced in the string gas. Thus, to have any hope of realistically applying our setup to cosmology, we need to study the effects of placing the string gas in a thermal bath. Referring to our expression for the energy density of a string level reaction\[\text{reaction}\]

\[ p = \sum_{n,w,N,p^2} \frac{\mu_{n,w,N,p^2}}{a^3 b^2 \pi} \sqrt{p^2} p^4 + \frac{4}{\alpha^2}(N-1) + \left( \frac{n}{b} + \frac{wb}{\alpha} \right)^2 \]  

(31)

with densities \( \mu_{n,w,N,p^2} \) for each given set of quantum numbers. The expressions for the pressure terms \( p \) and \( r \) are similarly modified. If we are in thermal equilibrium, the densities are given by the Boltzmann weight

\[ \mu_{n,w,N,p^2} = e^{\beta E_{\text{ref}}} e^{-\beta E_{n,w,N,p^2} \mu_{\text{ref}}} \]  

(32)

where the subscript “\( \text{ref} \)” refers to some arbitrary reference energy level.

What constitutes the thermal bath to which the string gas is coupled to? We know from the discussion at the end of Section II that gravitons described by unwound strings propagate in the non-compact directions with an equation of state \( p = \rho/3 \). Introducing a gas of ordinary photons will also add a 4-dimensional component to the energy-momentum tensor with the same equation of state. Such particles offer us an ideal candidate for a thermal bath, for two reasons. Firstly, thermal equilibrium demands a coupling of some kind between the gas of winding modes and the gas of gravitons and photons. Such a coupling is readily provided by the tree-level reaction \( w + \gamma \rightarrow h_{\mu \nu} \) via which winding modes of equal and opposite winding scatter to produce 4-d gravitons. This thermalization mechanism will, at non-zero temperatures, create an equilibrium where there will be an ever-present non-zero winding mode density due to gravitons scattering into winding modes (and vice-versa). This thermal bath has the further property that it does not affect the dynamics of the extra dimension other than through the Hubble factor (which it influences), since the driving term is only sensitive to the combination \( p - \rho/3 - 2r/3 \) which vanishes for the graviton and photon components of the energy-momentum tensor.

With the above in mind, the driving term for the scale factor \( b \) becomes:

\[ \frac{b}{M^3_{\text{pl}}} \left( p - \frac{2}{3} \rho - \frac{2}{3} r \right) = \frac{\mu_{\text{ref}}}{} \times \sum_{n,w,N,p^2} e^{-\beta E} \frac{\mu_{n,w,N,p^2}}{M^3_{\text{pl}} a^2 \pi} \sqrt{p^2} p^4 + \frac{4}{\alpha^2}(N-1) + \left( \frac{n}{b} + \frac{wb}{\alpha} \right)^2 \]  

(33)

where the Boltzmann weight in the summation depends on the values of the quantum numbers. We remind the reader that the sum is restricted by the level matching condition \( N + nw \geq 0 \). For completeness we also remind the reader of the resulting equation of motion for \( b \):

\[ 0 = \ddot{b} + 3H \dot{b} + \frac{\mu_{\text{ref}} e^{\beta E_{\text{ref}}}}{M^3_{\text{pl}} a^2 \pi} \times \sum_{n,w,N,p^2} e^{-\beta E} \left( \frac{n^2}{b^2} - \frac{2nw}{3\alpha'} + \frac{w^2 b^2}{3\alpha'^2} - \frac{4(N-1)}{3\alpha'} \right) \]  

(34)

The summation which has to be performed in order to obtain the driving term is quite formidable, were it not for a rather special feature of string thermodynamics. Consider the argument of the exponential in the Boltzmann factor:

\[ \beta E_{n,w,N,p^2} = \beta \sqrt{\frac{p^2}{\alpha^2} + \frac{4}{\alpha^2}(N-1) + \left( \frac{n}{b} + \frac{wb}{\alpha} \right)^2} \]

\[ = \frac{\beta}{\sqrt{\alpha'}} \sqrt{\alpha' P^2 + 4(N-1) + \left( \frac{n}{b} + \frac{wb}{\alpha} \right)^2} \]

We see that when the energy is expressed in terms of dimensionless variables, we pull out a factor of \( \sqrt{\alpha'} \). Thus, the argument of the exponential in the Boltzmann weight is \( \beta/\sqrt{\alpha'} \) times a term of order unity. To be able to neglect all but the first few terms in the summation, we need the Boltzmann factor to be considerably less than unity, i.e. that

\[ e^{-\frac{\beta}{\sqrt{\alpha'}}} \ll 1 \]

Thus, if this condition is satisfied, then the terms which dominate the sum will be those whose quantum numbers render them nearly massless, since any state with even one of its quantum numbers being different from the nearly massless combination will produce a term of order unity times \( \beta/\sqrt{\alpha'} \).

Let us take a closer look at the above condition. We know from string thermodynamics that there exists a limiting temperature – the Hagedorn temperature \( T_H \). Thus, for us to even be able to apply thermodynamics, we need to be well below this temperature, which for all the string theories is of the order of \( T_H \sim 1/\sqrt{\alpha'} \). Thus, \( \beta_H \sim \sqrt{\alpha'} \), and so if we are at a temperature much lower than the Hagedorn temperature, i.e. \( T \ll T_H \) or equivalently \( \beta_H \ll \beta \), then

\[ \sqrt{\alpha'} \ll \beta \]  

(35)

which is exactly what we need for the Boltzmann weights of higher mass states to be negligible. So, even if the thermal bath has a temperature of only one order of magnitude below the Hagedorn temperature, then \( e^{-\frac{\beta}{\sqrt{\alpha'}}} \sim 10^{-5} \) which clearly lets us ignore any term whose energy in dimensionless units \( \sqrt{\alpha' P^2 + 4(N-1) + \left( \frac{n}{b} + \frac{wb}{\alpha} \right)^2} \) is anything other than zero. This translates into us being...
able to \textit{neglect all states other than those that are massless}. Thus, the summation now becomes very tractable, and we can also have faith in our truncation of the string spectrum to the lightest states all the way up to very high temperatures ($T \sim T_H/10$). Before we carry on we should remark that exactly massless states have a non-zero momentum given by the thermal expectation value of $E = |p| = \beta/3$.

Let us then proceed to evaluate (33), so that we can evolve $b$ in time using (34), recalling that now we only sum over the massless and near massless states subject to the level matching constraint. Let us begin near $\tilde{b}_0 = 1$, i.e. $\tilde{b}_0 = 1 + \Gamma$. Then for the case that $\Gamma \neq 1$, we only have one truly massless state: $n = w = 0$, $N = 1$. This term will not contribute to the driving force for $b$ since

$$\frac{\left(-\frac{n^2}{2} - \frac{2nw}{3\alpha} + \frac{w^2}{3\alpha} - \frac{4(N-1)}{3\alpha}\right)}{|p|} = 0.$$ \hspace{1cm} (36)

Thus, the next lightest state which has quantum numbers $N = 1, n = -w = \pm 1$ will dominate the evolution of $b$. The level matching constraints $N + nw \geq 0$ ensure that there are no more nearly massless states (Note we only consider states with positive mass squared - any tachyonic states are posited to be absent from our spectrum). Such states will contribute:

$$\frac{e^{-\beta|p|}}{\sqrt{\alpha'|p|}} \sqrt{\frac{8\Gamma}{3}},$$ \hspace{1cm} (37)

Expanding $\tilde{b}$ as $1 + \Gamma$ and ignoring terms higher than quadratic in $\Gamma$ results in a contribution to (33) of:

$$\frac{e^{-\beta|p|}}{\sqrt{\alpha'|p|}} \sqrt{\frac{8\Gamma}{3}}.$$

Since there are two such terms which yield identical contributions, the sum total of the contributions from the near massless states yields the equation of motion

$$\ddot{\tilde{b}} + 3H \dot{\tilde{b}} + \frac{\mu}{M_5^{3\alpha} a^{3\alpha/2}} \left(\frac{8\Gamma}{3}\right) = 0,$$ \hspace{1cm} (38)

where the exponential factor gets cancelled if we use this massless state as our reference state, as in (34). The form of this equation clearly shows that $\Gamma$ will tend to zero if it starts out on either side of this value.

However, to confirm that $\Gamma = 0$ is a genuine point of equilibrium, we need to confirm that the extra massless states that appear at this radius ($8$ in all) contribute in such a way so that their sum vanishes. This can be verified by a straightforward calculation (35).

However, we wish to point out that as long as we stay in thermal equilibrium with the graviton gas, this equilibrium is actually metastable. The reason for this is easy to see from the formula for the mass of a winding mode:

$$\alpha' m^2 = \left(\frac{n}{b} + w \tilde{b}\right)^2 + 4(N-1).$$ \hspace{1cm} (39)

In addition to the massless state given by $n = w = 0$, $N = 1$ (the graviton), and the $8$ other massless states that appear at the self-dual radius (which are given by $N = 1, n = -w = \pm 1$; $N = 0, w = n = \pm 1$; $N = 0, w = 0, n = \pm 2$ and $N = 0, w = \pm 2, n = 0$), there are additional massless states at further special radii:

$$\tilde{b} = \frac{2}{|m|}; \quad w = \pm m, n = 0, N = 0$$

$$\tilde{b} = \frac{|m|}{2}; \quad n = \pm m, w = 0, N = 0$$

$$m \in \mathbb{Z}.$$ \hspace{1cm}

Thus, at half-integer multiples and and half integer fractions of the self-dual radius, two massless modes appear and will thus yield the dominant contributions to the driving term. These contributions again exactly cancel at twice the self dual radius, and at half the self dual radius. However in general, we will get a driving term that yields expansion at half integer points above twice the self dual radius and similarly, contraction at half integer fractions below half the self dual radius. However since we posit that we begin at or near the self-dual radius, we are guaranteed to stay locked near it if our initial conditions satisfy

$$b(0) \sim \sqrt{\frac{\alpha}{\beta}},$$

where the last condition constrains the initial “velocity” of the scale factor to be such that it cannot roll over the “hump” in the potential energy surrounding the metastable equilibrium at $b \sim \sqrt{\alpha'}$.

Thus, we have demonstrated in the context of GR how a string gas in thermal equilibrium with a bath of gravitons and photons will \textit{dynamically stabilize the scale factor of the compact direction} if we begin close to that radius. Thermal equilibrium with the graviton bath ensures a persistent non-zero density of such winding modes. One can now imagine that, at some point, the winding mode gas becomes decoupled from the graviton gas, i.e. falls out of thermal equilibrium. In this situation, we are left with an unchanging driving term of the form (24), which yields the potential in Fig. 1, which will guarantee radion stabilization at the self dual radius for the remainder of the Universe’s dynamics. We now turn our attention to the possible connection between this mechanism and inflationary and standard Big Bang cosmology.
V. LATE TIME EVOLUTION

Recall the Einstein tensor for our metric setup:

\[ G_{00} = -3\dot{a} \left( \frac{\dot{a}}{a} + \frac{\ddot{b}}{b} \right) \]
\[ G_{ij} = -\delta_{ij} \left[ 2\frac{\dot{a}}{a} + \frac{\ddot{b}}{b} + \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{b}}{b} \frac{\dot{a}}{a} \right] \]
\[ G_{b5} = -3\frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \].

We know that the dynamics of the scale factor \( b \) in the situations we studied above cause it to undergo damped oscillations around the self dual radius. We demonstrated in a previous section how the “spring constant” of this evolution will lock in to this equilibrium fairly rapidly.

We can then study the evolution of the Universe after radius stabilization, which implies that \( \dot{b} = \ddot{b} = 0 \) and \( p - 2r/3 - \rho/3 = 0 \). The resulting Einstein equations are:

\[ G_{00} = \frac{1}{M_5^2} T_{00} \rightarrow H^2 = \frac{\rho}{3} \]
\[ G_{ij} = \frac{1}{M_5^2} T_{ii} \rightarrow \dot{H} = -\frac{1}{2}(\rho + p) \]
\[ G_{b5} = \frac{1}{M_5^2} T_{b5} \rightarrow p - \frac{2r}{3} - \frac{\rho}{2} = 0 \]
\[ \nabla_\mu T^\mu = 0 \rightarrow \rho + 3H(\rho + p) = 0 , \]

where the 55 equation is precisely the equilibrium condition. Thus, we recognize in the above the basic equations of FRW cosmology. We now consider how one achieves the two important epochs of late time FRW cosmology, namely the radiation dominated era and the matter dominated era.

A. Radiation Dominated Evolution

If we assume that the density of 4-d matter gas is far greater than the density of the winding mode gas, then this will be the dominant component that drives the evolution of the macroscopic dimensions. If the 4-d matter has an equation of state parameter \( w \), then the solutions to the Einstein equations become:

\[ \rho(t) \propto a^{-3(1+w)} \quad (40) \]
\[ a(t) \propto t^{\frac{2}{3(1+w)}} . \quad (41) \]

Thus, for a 4-d graviton and photon background, we get that \( a(t) \propto t^{1/2} \), and so we reproduce a late time FRW evolution that is consistent with standard Big-Bang cosmology immediately after the end of inflation, whilst maintaining radius stabilization.

B. Matter Dominated Evolution

Reconsidering (24):

\[ \dot{b} + 3H \dot{b} + \frac{b}{M_{pl}^2} \left( p - \frac{2r}{3} - \frac{\rho}{3} \right) = 0 , \]

We see that any matter with the equation of state of non-relativistic dust \( (p = 0) \), can only drive the expansion of the radion if it is of a 3-dimensional nature (i.e. \( r \equiv 0 \)). This is surely to be a cause for concern when considering that at late times, one (naively) expects 3-dimensional non-relativistic dust to be the major driving component of the expansion of the universe, which would normally invalidate our stabilization mechanism in the present epoch.

However we wish to remind the reader that present day observations demand that a significant fraction of the energy density of the universe, which also drives the present day matter dominated expansion, be in the form of cold dark matter – whose nature is as of yet completely unknown. There is a significant amount of interest is the prospect that extra-dimensional matter or extra dimensional effects might account for the ‘missing’ matter in the universe. In what follows, we find that the only way to make our stabilization mechanism consistent with a ‘matter dominated’ epoch is to introduce extra dimensional cold dark matter. We propose a candidate for this cold dark matter which is naturally contained in our framework, and discuss other possibilities which might have a natural realisation within the general brane gas framework (note that a similar proposal was made in [25]).

We see that in order for the driving term in (24) to correspond to a stable minimum at the self dual radius for matter which obeys the equation of state for non-relativistic dust \( (p = 0) \), we need to consider matter for which:

\[ r = -\rho/2 \big|_{b=\sqrt{x}} \]. \quad (42) \]

That is, we require the dominant component of the energy density which is driving the expansion of the universe be such that it preserves the stability of the radion at the self dual radius. Matter which exhibits such an equation of state will surely have to be massive (else there will be a non-zero pressure along the non-compact directions for any non-zero energy density). In addition, such matter will have to be something beyond presently supposed dark matter candidates (WIMPS, supersymmetric relics etc.) as it will necessarily have to be ‘extra-dimensional’ in nature. We now show that our model naturally contains such a candidate. Recalling the discussion surrounding (24), we see that the equation of state parameter for a gas of strings with a particular set of quantum numbers is obtained from the following equa-
tion:
\[ r = \rho \times \frac{\frac{\dot{a}^2}{a^2} - \frac{\dot{a}^2}{a^2}}{p^2 p_1 + \frac{1}{\alpha'} (N - 1) + \left( \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2} \right)^2} \]  
(43)

where the momentum will be set to zero (or is vanishingly small) in order to satisfy \( p = 0 \) (c.f. (20)). In particular, since we are looking for states which can satisfy (47) at the self-dual radius, we need to find the appropriate quantum numbers which have an equation of state parameter \( w = -1/2 \) at \( b = \sqrt{\alpha'} \), which reduces to the following condition:
\[ 3n^2 - w^2 + 4(N - 1) + 2nw = 0, \]
(44)

and we have to be mindful of the level matching constraint: \( N + nw \geq 0 \). As expected, it turns out that the massless states that satisfy these conditions have an energy density proportional to \( |p| \), whereas the pressure is proportional to \( |p|^2/3 \), and hence one cannot have a non-zero pressure without having a vanishing energy density.

The first massive states which satisfy (14) are represented by the quantum numbers:
\[ |p| = 0, N = 2; n = 0, w = \pm 2. \]  
(45)

These states contribute to the energy momentum tensor as follows (c.f. (19) - (21)):
\[ p = 0, \rho = \frac{2\mu\sqrt{2}}{2\pi a^3\alpha'}, \quad r = -\rho/2, \]  
(46)

where the factor of \( \alpha' \) in the denominator comes from two factors of \( \sqrt{\alpha'} \) - one from the metric factor \( b \) which has stabilized at the self dual radius, and the other as the pre-factor of the non zero rest mass of this string state. As we will see in the next section, we will look at fluctuations of the radion around the self dual radius, these states also provide a stable equilibrium at \( b = \sqrt{\alpha'} \) in a phenomenologically acceptable manner. Thus, taking questions of consistency with observation for granted for the moment, we see if these particular string states are taken to dominate the present energy density of the universe, then by the Einstein equations derived at the start of this section, we can admit an epoch of dust driven FRW expansion \( (p = 0, a \propto t^{2/3}) \) at late times, consistent with radius stabilization.

However, there are many further issues that will have to be resolved if we are to take this idea of stringy dark matter seriously, which we postpone to a future report. At present, however, we wish to state that there are indications that such stringy dark matter might have the right clustering properties at the level of first order perturbation theory, in that local overdensities of this stringy dark matter induces gravitational clustering in the remaining 3-dimensions.

We also wish to point out that in certain situations, non-relativistic \( p \)-branes might also be able to provide us with a matter content with satisfies (12) [15]. In the context of brane gas cosmology, this is an appealing idea as one might need higher dimensional branes to stabilize compact sub-manifolds that do not admit topological one-cycles (and hence do not admit winding modes). We will investigate this possibility further in a future report. Finally, we wish to address the effects of an intermediary epoch of scalar field driven inflation on our stabilization mechanism.

### C. Intermediate (Non-Stringy) Inflation

We find that our mechanism for radius stabilization might not be compatible with an intermediate epoch of bulk scalar field driven inflation. To investigate this, we first consider the energy-momentum tensor of an almost homogeneous inflaton field:
\[ T_{\mu}^{\nu} = diag \left( -\frac{\dot{\phi}^2}{2} + V(\phi), \frac{\dot{\phi}^2}{2} - V(\phi), \ldots, \frac{\dot{\phi}^2}{2} - V(\phi) \right) \]  
(47)

Adding this energy-momentum tensor to our string gas yields a non-trivial contribution to the driving term in the equation of motion for \( b \). This driving term then takes the form
\[ -\frac{2b}{3M_5^2} V(\phi) + \frac{\mu_{ref} e^{\beta E_{ref}}}{M_{pl}^2 a^2 2\pi} \]  
(48)

\[ \sum_{n,w,N,p^2} e^{-\beta E} \left( -\frac{n^2}{b^2} - \frac{2nw}{3\alpha'} + \frac{w^2b^2}{3\alpha'^2} + \frac{4(N - 1)}{3\alpha'} \right) \]

from which it is easy to see that the inflaton contribution will drive expansion in the extra dimension. In general, this term will compete with the string gas contribution which, as we have seen, drives contraction, if we are above the self-dual radius. However, this competition is short lived, as the factor of \( a^3 \) in the denominator of the string gas driving term will quickly render it irrelevant and the scale factor will then expand according to
\[ \dot{b} + 3H \dot{b} = -\frac{2b}{3M_5^2} V(\phi) = 0. \]  
(49)

Recalling that during this (slow roll) inflation \( H \), and by the time-time Einstein equation, also \( V(\phi) \) are almost constant, we can solve the above equation, with the resulting two solutions:

\[ b(t) \propto \exp^{-\frac{\mu}{H^3} \left( 1 + \sqrt{1 + \frac{8V(\phi)}{4H^2 + 3M_5^2}} \right) t} \]

\[ b(t) \propto \exp^{\frac{\mu}{H^3} \left( 1 - \sqrt{1 + \frac{8V(\phi)}{4H^2 + 3M_5^2}} \right) t}. \]

Substituting in the Einstein equation \( H^2 = V(\phi)/3M_5^3 \) gives us

\[ b(t) \propto \exp^{-\frac{\mu}{H^3} \left( 1 + \sqrt{1 + \frac{8V(\phi)}{4H^2 + 3M_5^2}} \right) t} \propto e^{-3.56 H t} \]

\[ b(t) \propto \exp^{\frac{\mu}{H^3} \left( 1 - \sqrt{1 + \frac{8V(\phi)}{4H^2 + 3M_5^2}} \right) t} \propto e^{0.56 H t}. \]
Except for very special initial conditions, the growing mode will rapidly come to dominate. Thus, we conclude that $b$ expands exponentially (though not as fast as $a$). After inflation has finished (and after reheating to a temperature smaller than the one required for the pair production of string winding modes), $a$ expands as $t^{1/2}$. The energy density in the string gas will have been exponentially suppressed by the inflationary evolution, and thus to good approximation the equation of motion for $b$ will take the form

$$\dot{b} + 3H b = 0$$

leading to

$$\dot{b} = C a^{-3}$$

which implies

$$b(t) = b(1) + 2 \dot{b}(1) \left(1 - \frac{1}{\sqrt{2}}\right),$$

where $(1 \leq t)$. Thus, $b$ asymptotically expands to some limiting, and very large (due to the initial conditions that result at the end of inflation) value.

In conclusion, we have seen how our radion stabilization mechanism is consistent with the FRW evolution of the non-compact dimensions, but not with an intermediate inflationary period, with inflation driven by a bulk scalar matter field. Thus, in order for brane gas cosmology to make contact with the present cosmological observations, one either needs a different (maybe stringy) realization of inflation (see [17] for some ideas) where strings are produced in reheating, or a non-inflationary mechanism to solve the cosmological problems of standard Big Bang cosmology and to produce a spectrum of almost scale-invariant cosmological perturbations. Finally, we turn to various phenomenological issues pertaining to this model.

VI. PHENOMENOLOGY

There are two potential phases of applicability of our considerations. The first is to the early phase of stringy cosmology before a period of inflation. In this case, there are no phenomenological constraints on the model since the number density of the particles (from the four dimensional perspective) which correspond to the string states wrapping the extra dimension are diluted exponentially during inflation. However, in this case our considerations would no longer be relevant for the late-time stabilization of the extra dimension.

The second phase of potential applicability of our considerations is to the universe after inflation of our three spatial dimensions. We then need to assume that winding and momentum modes about the extra dimension can be regenerated in sufficient number, as discussed in [20]. In this case, there are important constraints on our model.

We must ensure that the particles corresponding to our string states do not overclose the universe. In addition, there is a radion mass constraint. Since the radion corresponds to a scalar particle from our four-dimensional perspective, we must make sure that its mass is consistent with the experimental constraints (we thank the Referee for stressing this point to us).

An additional constraint comes from the string theoretical aspect of our model: we must ensure that the derivatives of the metric remain several orders of magnitude smaller than the worldsheet derivatives (see Appendix). This is to ensure that we can inherit aspects of the string spectrum and constraint algebra that we have used so far. From [25], which has the form:

$$\ddot{\Gamma} + 3H \dot{\Gamma} + k \Gamma = 0$$

we see that the ‘spring constant’ which sets the scale for the how fast the metric factor $b$ varies, is given by

$$k = \frac{8\mu_0}{3M_5^3 \alpha'^{3/2} |p| \alpha'^3}$$

where the subscript on $\mu$ is to remind the reader that any explicit metric dependence has already been factored out (see section II). We have taken the stabilization to be provided by the massless states discussed earlier with $|p|$ denoting the momentum in the non compact directions, $N = 1$, and $n = -w = \pm 1$. In order that our metric factors evolve much slower than the string scale, we require that $\partial_{\mu} \mu \ll 1/\sqrt{2\pi\alpha'}$. Since (49) is a second order ODE, this implies that $k \ll 1/2\pi\alpha'$. As discussed in the Appendix, we choose to be quite conservative and demand that $k \leq 10^{-6}/2\pi\alpha'$.

A second constraint comes from requiring that the stabilization mechanism be effective at all times. This leads to a lower bound on $k$. We take this bound to be given by the ‘critically damped’ value for $k$:

$$k_{\text{crit}} = 9H^2.$$

The two above constraints yield the following bounds:

$$9H^2 \leq \frac{8\mu_0}{3M_5^3 \alpha'^{3/2} |p| \alpha'^3} \leq \frac{10^{-6}}{2\pi\alpha'}.$$

Since the winding states that stabilize the extra dimensions are massless at the self-dual radius, they will behave as hot dark matter – dark matter because they only interact gravitationally (through the tree level interactions $w+ w \rightarrow h_{\mu\nu}$) with other fields, hot because they are massless and have a radiative equation of state. We have to ensure that we do not introduce too many of these objects so that we can ensure consistency with observational bounds.

Next, we have to ensure that the massive string states that we propose as a candidate for the cold dark matter that is presently driving the ‘dust dominated’ expansion
of the universe do not violate any observational bounds while preserving our stabilization mechanism.

Let us begin by considering the massless states which are presently stabilizing the radion. From (17) we see that post stabilization, for stringy matter with \( N = 1, \ n = -w = \pm 1 \), we must have:

\[
\rho = \frac{\mu_0 |p|}{a^3} \leq 10^{-4} \rho_{\text{crit}}
\]  

(56)

in order to ensure consistency with the nucleosynthesis bounds. The critical density of the universe is \( \rho_{\text{crit}} = 10^{-29} \text{g/cm}^3 \). We then find that (56) becomes

\[
\mu_0 \lesssim 10^{-41} 10^{-37} \text{GeV}^4 |p|^{-1}.
\]  

(57)

Let us now parametrize the present value of \( |p| \) as

\[
|p| = 10^{-\gamma} \text{eV}
\]  

(58)

where \( \gamma \) is some constant determined by the initial conditions and the history of the universe, and this parametrization being motivated by the fact that \( |p| \) is likely to be of the order of a few eV in the present epoch if it corresponds to an initial \( |p| \) of the order of the Planck energy. Then, the above bound takes the form

\[
\mu_0 \lesssim 10^{-41} \text{GeV}^3 10^\gamma.
\]  

(59)

On the other hand, the first inequality in (58) becomes:

\[
\mu_0 \geq H^2(t_0) |p| \sim 10^{-93} 10^{-\gamma} \text{GeV}^3
\]  

(60)

which is consistent with (59). In the above, we have made use of

\[
M_5^3 = 8 \pi G_5 = 8 \pi G_4 \sqrt{\alpha}'.
\]  

(61)

Using (61), it can easily be checked that the upper bound on \( \mu_0 \) which follows from the second inequality in (58) is much weaker than the bound (59). We conclude that one can easily arrange the number of string modes such that stabilization of the extra dimension is ensured and at the same time the massless modes do not conflict with the nucleosynthesis bound.

Let us next turn to the radion mass constraint: Since the radion appears in four dimensions as a scalar field, its mass must be larger than

\[
m_{\text{crit}} = 10^{-12} \text{GeV}
\]  

(62)

in order to avoid fifth force type constraints. Since the square of the radion mass is given by \( k \), this constraint becomes

\[
\mu_0 \geq M_5^3 \alpha'^{3/2} |p|m_{\text{crit}}^2 \sim 10^{-35-\gamma} \text{GeV}^3
\]  

(63)

which is consistent with the upper bound (59) on \( \mu_0 \) if \( \gamma \geq 3 \). Such a value of \( \gamma \) is not at all unreasonable and could easily arise from an additional suppression of the momentum during a short period of inflation.

It turns out to be crucial that we use massless modes to stabilize the extra dimensions, as more massive string states would bring down the upper bound, and as we are about to see, do not provide as effective a spring constant and hence bring up the lower bound, to the net effect that it is phenomenologically inconsistent to have these as the only strings that are stabilizing the radion. We arrive at this observation by considering the second aspect of our phenomenology—namely, that we would like the cold dark matter content of our present universe to consist of massive string modes (which satisfy (19), which as we have seen is require in order to maintain stabilization at the self dual radius).

Considering the contribution to the energy density by a string gas with quantum numbers \( |p| = 0, N = 2, n = 0, w = \pm 2 \) (66), and equating this to the critical density of the universe, we see that:

\[
\frac{2 \sqrt{2} \mu_{dm}}{2 \pi a^3 \alpha'^{1/2}} \approx \rho_{\text{crit}}
\]  

(64)

Where now the subscript on \( \mu \) serves to indicate that this is our dark matter candidate. This requires

\[
\mu_{dm} \sim 10^{-67} \text{GeV}^3
\]  

(65)

Upon perturbing around a stabilized radius, we find that these string modes contribute to the stability of the radion with the spring constant:

\[
k_{dm} = \frac{8 \pi G_5 \mu_{dm} \sqrt{8}}{2 \pi a^3 \alpha'}.
\]  

(66)

Demanding that this value of \( k \) is consistent with the radion mass bound yield a lower bound on \( \mu_{dm} \)

\[
\mu_{dm} \geq m_{\text{crit}}^2 M_4 \sim 10^{-6} \text{GeV}
\]  

(67)

which is clearly inconsistent with (66) for values of \( \gamma \) which are extremely large.

Thus we see that if we introduce the correct amount of our dark matter candidate, it contributes too feebly to the dynamics of the radion (even though it does provide its own contribution to stabilization). However, this is of no concern to us, as we have already shown that the massless string states provide a robust stabilization mechanism that is consistent with observational bounds. Thus, if our string gas has a massive component that serves as cold dark matter, and a massless component that stabilizes the radion (and behaves like hot dark matter) in the right proportions, which as we have shown is quite easy to achieve, we can be assured of the phenomenological consistency of our stabilization mechanism with late time FRW cosmology.

VII. CONCLUSIONS

The analysis in this paper is motivated by brane gas cosmology [1, 4]. As a simplified problem, we have stud-
ied the effects of a gas of strings with non-vanishing momentum and winding modes about a single compact extra dimension taken to be a circle (the three large dimensions are taken to be spatially flat and isotropic) on the evolution of the radius of that dimension, assuming that the background space-time satisfies the equations of motion of General Relativity. We discovered that such a string gas leads to a dynamical stabilization mechanism for the radius of this dimension, the radion. Assuming initial conditions in which the three large dimensions are expanding, we found that the radion performs damped oscillations about the self-dual radius.

In a first step, we studied the effects of a gas of non-interacting strings, each string having the identical momentum and winding numbers. Key to the stabilization mechanism is the fact that winding modes and momentum modes contribute with opposite sign to the pressure of the string gas in the direction of the compact dimension, and that the winding modes generate a potential for the radion which favors contraction, whereas the momentum modes generate a potential favoring expansion. We then showed that the stabilization mechanism also holds for a gas of strings in thermal equilibrium.

We also showed that, after radion stabilization, the scale factor for the three large spatial dimensions obeys the usual FRW equations of standard Big Bang cosmology. Thus, our scenario leads in a natural way to a late time FRW Universe. However, we have also shown that the radion stabilization mechanism is not compatible with a period of scalar field driven bulk inflation. Thus, in order for brane gas cosmology to make successful contact with present cosmological observations, one either needs to find a stringy mechanism for driving inflation where strings are produced in the re-heating epoch, or else one must provide an alternative to inflationary cosmology, both for solving the mysteries of standard Big Bang cosmology, and for explaining the origin of the observed large-scale fluctuations.

Note that we start with the assumption that three spatial dimensions are already much larger than the other ones (one other dimension in our case). Whether or not the dynamics of strings in the initial stages will indeed lead to this situation may depend on the corner of M-theory one is working in, i.e. on the specific form of the background equations of motion and initial conditions (see \[13\] for different angles on this issue). However, it should not be hard to generalize our analysis to a situation with more compact dimensions or different topologies and geometries of the extra dimensions, which will be the focus of our future work.

In future work we also plan to study the annihilation rate of the string modes which are central to this work, namely modes which have both winding and momentum in the compact direction. Since these modes interact only gravitationally just like gravitons, they will be out of thermal equilibrium at late times and hence will not decay.

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**APPENDIX A: THE STRING SPECTRUM IN A TIME DEPENDENT BACKGROUND**

Let us then begin with the Polyakov action for a closed string:

\[ S = \frac{-1}{4\pi\alpha'} \int d^2 \sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X). \] (A1)

Varying this action with respect to the world sheet metric gives us the equation of motion

\[ \gamma_{ab} = \lambda h_{ab} = \lambda \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X), \] (A2)

where we can exploit the world-sheet diffeomorphism and Weyl invariance to make the world-sheet metric flat (conformal gauge):

\[ \gamma_{ab} = \gamma^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \] (A3)

Varying the action with respect to the world-sheet fields, and imposing the gauge choice yields the equations

\[ \partial_a (\partial^\rho X^\lambda g_{\lambda\mu}(X)) = \frac{1}{2} \partial^\rho X^\lambda \partial_a X^\nu g_{\lambda\nu,\mu}(X), \] (A4)

where the meaning of the derivatives of the metric should be clear. This equation translates into

\[ \partial_a \partial^\rho X^\tau + \Gamma^\tau_{\lambda\nu} \partial_a X^\nu \partial^\rho X^\lambda = 0. \] (A5)

Now, we consider the case when our metric depends only on time, that is \( g_{\mu\nu}(X) = g_{\mu\nu}(X^0) \), is diagonal and has -1 as its 00 entry (This last point is not essential to the argument, it only serves to simplify the equations). Unpacking the above equation yields the equations

\[ -\partial_\tau^2 X^0 + \partial_\tau^2 X^0 = -\frac{1}{2} \partial_{\lambda\nu,0} (-\partial_\tau X^\nu \partial_\tau X^\lambda + \partial_\sigma X^\nu \partial_\sigma X^\lambda), \] (A6)

and

\[ -\partial_\tau^2 X^i + \partial_\tau^2 X^i = -g^{ij} g_{ij,0} (-\partial_\tau X^i \partial_\tau X^0 + \partial_\sigma X^i \partial_\sigma X^0). \] (A7)

On the right hand sides of the equations there is an overall factor containing the time derivative of the metric. To estimate the magnitude of either side of the equations, we realize that world-sheet time derivatives will be of the order of the energy of the string, which is of the order of the square root of the string tension: \( \partial_\tau \sim \frac{1}{\sqrt{\alpha'}} \).
Similarly, the world-sheet spatial derivative will be of the order of the inverse of the string length $l_s$, $\partial_\tau \sim 1/l_s$, which is of the order of the square root of the string tension. On the right hand sides, the terms are of the same order as the terms on the left, except for the factors containing derivatives of the metric. As long as these metric derivatives are several orders of magnitude smaller than the string tension scale, we can safely ignore them [36]. Assuming such a background, (A6) and (A7) reduce to

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), b^2(t)),$$  \hfill (A13)

where the 5'th dimension is taken to be compact with radius $2\pi b$. With this as our background metric, (A12) becomes (for a string wound along the 5'th dimension):

$$- E^2 + g^{ij} p_i p_j + \frac{2}{\alpha'} (N + \tilde{N} - 2) + g^{55} P_5 P_5 + g_{55} X_5^5 X_5^5 = 0,$$  \hfill (A14)

where all we have done is expanded out (A12), and realized that the terms coming from the non-compact $X^\mu$ and $P_\mu$ give us the center of mass momenta and the left and right oscillator pieces, and the terms coming from $X^5$ have been accounted for explicitly. We know that this part of the energy contributes [19]:

$$P_5 P_5 + X_5^i X_5^i = \frac{n^2}{b^2} + \frac{w^2 b^2}{\alpha'^2},$$  \hfill (A15)

so that combined with the level matching conditions

$$nw + N - \tilde{N} = 0,$$  \hfill (A16)

we get

$$E = \sqrt{g^{ij} p_i p_j + \frac{4 \alpha'}{\alpha} (N-1) + \left(\frac{n}{b} + \frac{w}{\alpha^2}\right)^2},$$  \hfill (A17)

where the only remnant of the level matching condition is the requirement that

$$N + nw \geq 0.$$  \hfill (A18)

Thus, we see that the only effect of working in a slowly varying background is to introduce time-dependent metric factors in the obvious places in (A17) which is otherwise identical to the result we would obtain in a static background.

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Which come from working in conformal gauge. Examples are particle creation, non-uniqueness of the vacuum, non-trivial issues concerning existence of asymptotic states.

Which come from working in conformal gauge $h_{ab} = \text{diag}(-1, 1)$.

See the Appendix for all statements made in this section concerning results that are valid in a time dependent background.

See the second footnote in the Appendix where we remind the reader why the momenta must be contracted with the inverse metric.

Where key to this is the observation that as we approach the value $b = \sqrt{\alpha'}$, these states become massless, and acquire a non-zero momentum along the non-compact directions, which depends on the ambient temperature. If this momentum is large enough, we can be assured that $\delta N$ is satisfied.

We model the momentum squared as a smooth function of $\tilde{b}$ such that it takes on some non zero value at $\tilde{b} = 1$ and falls off on either side. This is because at $\tilde{b} = 1$ the state described by $n = -w = 1$, $N = 1$ becomes massless and should have a finite non zero momentum, but as the scale factor increases or decreases the state becomes more and more massive and hence the momentum becomes negligible in comparison. The generic feature of a minimum at $\tilde{b} = 1$ is robust, however, since as we mentioned earlier, the force term will always change sign at $\tilde{b} = 1$. One finds that the minimum is always fairly concave independent of the nature of the $b$ dependence of the momentum.

The only non-zero contribution to the driving term comes from the states $n = 0, w = \pm 2, N = 0$ and $n = \pm 2, w = 0, N = 0$ which make equal and opposite contributions and hence cancel.

For instance, we are tempted to be conservative and to ask for them to be roughly 6-8 orders of magnitude smaller, in order to be certain of a consistent treatment.

Recall that $\partial_n X^\nu := P_n$ is the canonical momentum, where we must view $P_n$ as having its index lowered which is to be raised with the inverse metric. Not realizing this will produce nonsensical results elsewhere (such as in the computation of the energy-momentum tensor of the string) in addition to making quantization very awkward. This fact is easier to understand if we recall that $P_n$ and $X^\nu$ are canonically conjugate world sheet fields and not 4-vectors. With this in mind, canonical quantization means imposing $[P_n, \tau, \sigma] = \pm \hat{\delta}^\nu_\sigma (\sigma - \sigma')$ which, as is, does not involve the metric. Hence we do not have to do anything different at this stage of the analysis.

The results in our paper easily generalize to backgrounds of any number of non-compact dimensions so long as precisely one dimension is compactified on a circle. For consistency, we could state that our background is a compactified 10 dimensional space with six dimensions compactified on a Calabi-Yau (CY)space and one dimension compactified on a circle. Since in the prototypical compactification scenarios such as the Horava-Witten model the radius of the CY is smaller than the radius of the circle, since we will be interested in string winding modes but Calabi-Yau spaces do not admit one-cycles, we can ignore the presence of the Calabi-Yau space if we work in an effective Lagrangian description valid at energies smaller than the energy scale of CY compactification. We could always go back and work in a $9 + 1 \times S^1$ or $24 + 1 \times S^1$ space-time where we would derive the same conclusions as we do here.

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