Barrier-based MPC was proposed in [19] where stability of state feedback MPC can be established using gradient recentering; its implementation on an edible oil refining plant was reported in [20]. Fast implementations of MPC using barriers are reported in [21]. Recently, in a series of papers [22], [23], [24] barrier-based MPC is developed to establish stability results for numerically efficient and practical MPC implementations. Innovations of these works include weight-recentered barriers, analysis of relaxed barriers for soft constraints and stability results for anytime algorithms (where the number of Newton steps in the associated optimization algorithm may be small).

In this paper, we use recentered barriers for hard constraints and a relaxed centered barrier for soft constraints. This complements the work in [24], [25] which considers stability of state feedback MPC for soft input and state constraints. Here, stability analysis allows hard input constraints which may be time variant or nonlinear. In the case of time invariant constraints, ZF multipliers can be used. Hence, the search for such multipliers becomes crucial. In this paper we construct static and dynamic multipliers, exploiting convex searches, for barrier-based MPC, which are then used for input-to-output stability and robust analysis, illustrating the advantages that barrier MPC can provide.

In section II the basic notations used in the paper are presented. In section III the formulation of barrier MPC is discussed and in section IV some basic results are presented. In section V and VI our main results are presented. The properties of barrier MPC are investigated, and the existence of static/dynamic multipliers is shown. In section VII a convex search methodology for efficiently computing multipliers is presented. In section VIII an illustrative numerical example is shown, while conclusions are given in IX.

II. Notation

Let \( \ell^m \) be the space of all real-valued sequences. \( \mathbb{RH}_\infty \) is the set of rational matrix transfer function matrices without poles outside the unit circle. Let \( x_j \in \mathbb{R}^{n_j} \) be the value of \( x \in \ell^m \) at sample \( k \). Let \( A^* \) be the complex conjugate transpose of the matrix \( A \) and let \( G^* \) be the \( l_2 \)-adjoint operator of \( G \). \( \langle f, g \rangle \) is the inner product of real-valued sequences \( f \) and \( g \), defined as \( \sum_{k=-\infty}^{\infty} f_k g_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{f}(e^{j\omega}) \hat{g}(e^{j\omega}) d\omega \), \( \hat{f} \) being the Fourier transform of \( f \). \( \|f\|_2 \) is the \( l_2 \) norm \( \|f\|_2 \). The discrete convolution at time \( i \) is notated as \( (f * g)_i = \sum_{k=-\infty}^{\infty} f_k g_{i-k} \).

The size of signal \( x \) is \( n_x \), while \( I \) denotes the identity matrix. For a matrix \( \tilde{A} \in \mathbb{R}^{m \times n} \) with rank \( r \) we define \( A^* \in \mathbb{R}^{n \times r \times n} \) such that \( A^* A^T = 0 \), \( A^T A^* = I \), and \( \tilde{A} \in \mathbb{R}^{r \times n} \) such that

\[
\tilde{A} = \begin{cases} (A^*)^T, & \text{when } r < n \\ I, & \text{when } r = n \end{cases}
\]
Hence, $\tilde{A}A^T = I$ and the rows of $\tilde{A}$ form an orthonormal basis of the space spanned by the rows of $A$. Furthermore, the set of all sub-gradients of a function $f$ at $x$ is called the subdifferential of $f$ at $x$ and it is denoted by $\partial f = \partial f(x)$.

**III. Formulation of Barrier MPC**

The controller is designed using the nominal LTI model (without any information for the disturbances). The nominal system is modelled as

\[
x_{k+1} = Ax_k + Bu_k
\]
\[
y_k = Cx_k
\]

where $x_k \in \mathbb{R}^{n_x}$ is the vector of states, $u_k \in \mathbb{R}^{n_u}$ the vector of manipulated variables, and $y_k \in \mathbb{R}^{n_y}$ the vector of measured output variables. Additionally, we assume that $A \in \mathbb{R}^{n_x \times n_x}$ is Schur stable. The formulation of barrier MPC is deduced from the nominal constrained problem, hence nominal MPC is presented first. The control action for the nominal problem is given as:

\[
U_k = \arg \min_{\hat{u}} \frac{1}{2} \sum_{i=1}^{N} x_{k+i|k}^T Q x_{k+i|k} + \sum_{i=0}^{N-1} \hat{u}_{k+i|k}^T R \hat{u}_{k+i|k}
\]
\[
s.t. \quad \hat{x}_{k+i|k} = x_k
\]
\[
\hat{x}_{k+i|k} = A \hat{x}_{k+i-1|k} + B u_{k+i-1|k}
\]
\[
\hat{y}_{k+i|k} = C \hat{x}_{k+i|k}
\]
\[
\hat{u}_{k+i|k} = [\hat{u}_{k+i|k} \ldots \hat{u}_{k+N-1|k}]^T \in \mathcal{U}
\]

with $N$ the prediction horizon. The non-empty compact convex set $\mathcal{U}$ represents the input constraints, $F_i : \mathbb{R}^{n_x \times n_u} \to \mathbb{R}$:

\[
\mathcal{U} : \{ F_i(\hat{u}) \leq W_i | i \in \{1, \ldots, n\} \}
\]

where $F$ is a convex function. For the specific case of linear inequalities, the constraints can be written as $F(\hat{u}) = L \hat{u} \leq W$, assuming $F_i(0) = 0$ and $W_i \geq 0$. If we add the equality constraints (3b) to the objective function (3a) we obtain:

\[
U_k = \arg \min_{\hat{u}} \frac{1}{2} \hat{u}^T H \hat{u} - \theta_k^T \hat{u}
\]
\[
\hat{u} \in \mathcal{U}
\]

where $H, R, Q$ can be defined as in [26] and $\theta$ is a linear function of states ($\theta_k = -Sx_k$). The constrained problem is then transformed to an unconstrained problem according to [22], [19] using barrier functions and/or penalty (relaxed barrier).

**Definition 1.** Let $U$ be an open (strictly) convex set which contains the origin, defined as $U = \cap_{\mathcal{U}_i}$, where $\mathcal{U}_i$ is $\{ \hat{u} : F_i(\hat{u}) \leq W_i \}$ for the hard constraints or $\mathbb{R}^{n_u}$ for the soft constraints. Then $\mathcal{B}$ is the set of all twice continuous differentiable $\theta$-self-concordant (strictly) convex barrier functions over $U$, $B : U \to \mathbb{R}$ with $B(0) = 0$ and $\nabla B(0) = 0$.

In the literature there are two popular barrier functions that are utilized. The gradient recentered log-barrier [19]

\[
B(U) = \sum_{i} B_i = \sum_{i} \left( -\ln(W_i - F_i(U)) + \ln(W_i) - \frac{\nabla F_i(U)^T}{W_i - F_i(0)} U \right)
\]

and the weighted recentered log-barrier [22].

\[
B(U) = \sum_{i} B_i = \sum_{i} (1 + w_i) \left( -\ln(W_i - F_i(U)) + \ln(W_i) \right)
\]

with $w_i > 0$. In addition, Feller and Ebenbauer [22] have proposed a relaxed barrier function to be employed, substituting the natural logarithm in (6) & (7) with a quadratic function $\tilde{B}$ when $W_i - F_i(U) \leq \delta_i$ with $\delta_i > 0$. The quadratic function is defined so that the properties from Definition 1 are maintained but the domain of the function $B_i$ is $\mathbb{R}^{n_u}$. The use of a barrier function allows the elimination of the inequality constraints.

\[
U_k = \phi(\theta_k) = \arg \min_{\hat{u}} \frac{1}{2} \hat{u}^T H \hat{u} - \theta_k^T \hat{u} + \mu B(\hat{u})
\]

**IV. Background**

**A. Properties of Nonlinear Functions**

A multi-valued map $\phi$ is sector-bounded in the sense that there exists some $K \in \mathbb{R}^{n \times n}$, $K > 0$ (or equivalently belongs to the sector $[0, K]$) such that

\[
\phi(\theta)^T(K^{-1} \phi(\theta) - \theta) \leq 0
\]

for all $\theta \in \mathbb{R}^n$ and it is additionally slope-restricted, if there is $S \in \mathbb{R}^{n \times n}$, $S > 0$ such that for all $\theta_k, \theta_j \in \mathbb{R}^n$ and $\phi_k = \phi(\theta_k)$:

\[
(\phi_k - \phi_j)^T(S^{-1}(\phi_k - \phi_j) - (\theta_k - \theta_j)) \leq 0
\]

Additionally, if $\phi(0) = 0$ then a slope-restricted non-linearity is also sector bounded. Another property that is exploited for our main results is cyclic monotonicity. A $n$-cyclic monotone increasing multi-valued map $\phi$ is defined as follows:

**Definition 2.** If $\phi$ is a $n$-cyclic monotone increasing map and $\phi_k = \phi(\theta_k)$ then $\forall n$

\[
\langle \theta_0 - \theta_1, \phi_0 \rangle + \langle \theta_1 - \theta_2, \phi_1 \rangle + \ldots + \langle \theta_n - \theta_0, \phi_n \rangle \geq 0
\]

The $n$-cyclic monotone is an extension of the monotone property. Namely for $n = 1$, inequality (11) turns into the monotone increasing property and the existence of a convex gradient function is summarized in [27].

**Theorem IV.1.** [27] Let $\phi$ be a multi-valued mapping from $\mathbb{R}^n \to \mathbb{R}^n$. In order for a closed proper convex function $P$ on $\mathbb{R}^n$ such that $\phi(\theta) \subset D\phi$ for every $\theta$ to exist, it is necessary and sufficient that $\phi$ is cyclically monotone.
B. Integral Quadratic Constraints

IQC{s} provide a way of conveniently representing associations between nonlinear or possibly unknown processes [3]. Two signals $w \in l^2_n$ and $v \in l^2_n$ are said to satisfy the IQC defined by a multiplier $\Pi(z)$, which is measurable, bounded and Hermitian, if

$$\int_{-\pi}^{\pi} \Pi(e^{j\omega}) d\omega \geq 0$$  \hfill (12)

The classic stability theorem presented in [3], assumes that the interconnection between the system transfer function $G$ and the augmented nonlinearity $\Delta$ is well-posed. In addition the feedback interconnection between $G$ and $\Delta$ is stable if there exists $\epsilon > 0$ such that

$$\Pi(e^{j\omega}) \geq -\epsilon I$$  \hfill (13)

C. Zames-Falb Multipliers

Definition 3. The class of discrete-time rational Zames-Flab multipliers $\mathcal{M}$ contains all MIMO rational transfer functions $M_{ZF} \in \mathbb{RL}^{n_{x} \times m}$ such that $M_{ZF}(z) = H_{e} - H_{ZF}(z)$, where $||H_{ZF}||_{1} < H_{e}$ are symmetric doubly hyper-dominant [28]:

$$H_{ij} \geq \sum_{j \neq i} |H_{ij}| + \sum_{j} ||H_{ZF_{ij}}||_{1}$$  \hfill (14)

with entries $H_{ZF} \in l^{1}$. Additionally, the subclass $\mathcal{M}_{+} \subset \mathcal{M}$ requires the following:

$$H_{ij} \leq 0, H_{ZF_{ij}} \geq 0$$  \hfill (15)

V. Properties for Barrier MPC

In this section the properties related to the barrier MPC are explored to further derive the IQCs.

The next two lemmas are required to show that $\phi : \theta_k \mapsto U_k$ is slope-restricted, sector-bounded and cyclic monotone.

Lemma V.1.

1) If $B \in \mathbb{R}$, then $\nabla B$ is monotone increasing and there exists $m \geq 0$ such that $\nabla^2 B \geq m I$ and $U^{T} \nabla B - mU^{T} U \geq 0$.

2) If $B$ is also strongly convex then we can find $m > 0$.

Proof:

1) This is trivial with $m = 0$ through convexity.

2) If $B$ is strongly convex then we can find $m > 0$ such that $\nabla^2 B - mL \geq 0$. Define $\tilde{B} = B - \frac{1}{2}ml^{T} U$. Then $\tilde{B}$ is convex and the result follows since $U^{T}(\nabla B - mL) = U^{T} \nabla \tilde{B} \geq 0$.

Lemma V.2. If $\mathcal{U}$ is a set of bounded linear inequalities, $B \in \mathbb{R}$ and the constraints are either hard or $B$ is relaxed by a quadratic function $b_{c}$, then the recentered barrier is strongly convex with $m > 0$.

Proof: For linear and bounded constraints, there is a finite positive number $\delta_{e}$ such that $b_{i} - L_{i}U \leq \delta_{e} \leq \delta_{e}$. For a given $U \in \mathcal{U}$, the following holds:

$$\nabla^2 B_{i} = \frac{L_{i}^{T} L_{i}}{(b_{i} - L_{i}U)^{2}} \leq \frac{L_{i}^{T} L_{i}}{\delta_{e}^{2}} \geq 0$$  \hfill (16)

It is trivial to show that there exists a $B_{j}$ such that $\nabla^2 B_{j} > 0$.

Theorem V.3. The nonlinearity $\phi : \mathbb{R}^{n_{u}} \mapsto \mathbb{R}^{m_{u}}$ belongs to the sector $[0, H^{-1}]$, where $H = H + \mu mL$, with $m \geq 0$ from lemma V.1 and V.2 $B \in \mathcal{B}$ and $\mathcal{U}$ is a convex set, if $\phi(0) = 0$.

Proof: Using the KKT conditions of (8) we have:

$$H \phi - \theta + \mu B \phi = 0$$  \hfill (17)

Since $\mathcal{U}$ is convex, multiplying (17) by $U^{T}$, using lemma V.1 we get:

$$\phi^{T}(H + \mu mL)\phi - \phi^{T} \theta \leq 0$$  \hfill (18)

Theorem V.4. The nonlinearity $\phi : \mathbb{R}^{n_{u}} \mapsto \mathbb{R}^{m_{u}}$ is additionally slope-restricted on $[0, H^{-1}]$ with $m \geq 0$, $B \in \mathcal{B}$ and $\mathcal{U}$ is a time invariant convex set.

Proof: Using the KKT conditions of (8) we have the following for $\phi_{i} = \phi(\theta_{i})$ and $\phi_{j} = \phi(\theta_{j})$

$$H\phi_{i} - \theta_{i} + \mu BV_{i} = 0$$  \hfill (19a)

$$H\phi_{j} - \theta_{j} + \mu BV_{j} = 0$$  \hfill (19b)

Subtract (19b) from (19a) and multiply by $(\phi_{i} - \phi_{j})^{T}$ to get:

$$(\phi_{i} - \phi_{j})^{T}(H(\phi_{i} - \phi_{j}) - (\theta_{i} - \theta_{j})) + (\phi_{i} - \phi_{j})^{T}(\mu BV_{i} - \mu BV_{j}) = 0$$  \hfill (20)

Applying Lemma V.1 and V.2:

$$(\phi_{i} - \phi_{j})^{T}(H(\phi_{i} - \phi_{j}) - (\theta_{i} - \theta_{j})) = - (\phi_{i} - \phi_{j})^{T}(\mu BV_{i} - \mu BV_{j}) \leq \mu m ||\phi_{i} - \phi_{j}||^{2}$$  \hfill (21)

Then

$$(\phi_{i} - \phi_{j})^{T}(H + \mu mL)(\phi_{i} - \phi_{j}) - (\theta_{i} - \theta_{j}) \leq 0$$  \hfill (22)

Therefore, $\phi$ is slope-restricted on $[0, H^{-1}]$.

Remark 1. This result shows that the inclusion of a barrier can change the maximum slope of the input-output map of the controller. This will widen the stability region of the closed-loop system. It should be noted that $m$ depends only on the set of constraints and not on the design parameter $\mu$. In the numerical examples below, it will be shown that such a formulation can reduce the conservatism significantly in comparison to $[8]$.

A simple example is utilized to illustrate the effect of the barrier-based MPC on the maximum slope. The nonlinearity is given by $U = (\arg \min_{u} 0.25u^{2} - \theta u + \mu (-In(1-u) - In(2+u) - 0.5u))$. Fig. 2 depicts the solution for different values of $\mu$. In this case the value of $m$ (and the maximum slope) can be computed analytically (See Appendix), namely $m = 0.889$ and $Slope_{max} = (0.5 + 8/9\mu)^{-1}$. Fig. 2 shows that the maximum slope decreases as $\mu$ increases. If $m$ is not computed (and is e.g. assumed to be $m = 0$) then the maximum slope will be overestimated compared to the actual value of the slope, hence increasing the conservatism of the stability analysis.
VI. MULTIPLIERS FOR BARRIER MPC

IQC for the barrier MPC will be derived in this section using the results from Section V.

A. Static Multipliers

**Corollary 1.** For $\mathcal{U}$ being a convex set and $B \in \mathcal{B}$, the nonlinearity $\phi : \mathbb{R}^{\mathcal{N}} \rightarrow \mathbb{R}^{\mathcal{N}}$ admit IQC with the following multiplier:

$$
\begin{bmatrix}
0 & I \\
I & -2\hat{H}
\end{bmatrix}
$$

**Proof:** Immediate from Theorem VI.3.

**Remark 3.** Theorem VI.3 introduces an IQC for the barrier MPC where the constraints are generally convex constraints (as long as the optimization problem is always feasible). This case will be referred to as general. Nevertheless, by tightening the class of constraints, less conservative results can be found.

B. Dynamic Multipliers

Time-invariant constraints can be used next to derive dynamic multipliers for less conservative stability analysis. We prove the existence of ZF multipliers for the case of time-invariant convex constraints in Lemma VI.1 and of less conservative multipliers for the case of box and staged constraints. According to [12], [14], the existence of ZF multipliers additionally requires the line integral $\int_{\theta}^{\tilde{\theta}} \phi(x)^T dx$ to be independent of the path. This property is equivalent to the condition that $\phi$ is the gradient of some convex potential function. When, however, the nonlinearity is not explicitly given, the above properties are difficult to be found. Here we propose the use of the property of cyclic monotonicity in order to prove the existence of the potential convex function. From Theorem VI.1, the conditions in [12], [14], can be substituted by the condition of $\phi$ being cyclically monotone:

**Lemma VI.1.** Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be bounded and cyclic monotone increasing. Then for any $\theta \in l^2_2$ we have

$$
\sum_{t=-\infty}^{\infty} \theta(t+\tau)^T \phi(\theta(t)) \leq \sum_{t=-\infty}^{\infty} \theta(t)^T \phi(\theta(t))
$$

and if $\phi$ is odd then

$$
\sum_{t=-\infty}^{\infty} \theta(t+\tau)^T \phi(\theta(t)) \leq \sum_{t=-\infty}^{\infty} \theta(t)^T \phi(\theta(t))
$$

**Proof:** From Theorem IV.1, it is necessary and sufficient that the mapping is cyclically monotone, for a closed proper convex function $P$ on $\mathbb{R}^n$ to exist, such that $\phi \subset \partial P$. Now the results from [12] can be employed in order to complete the proof.

Next, the theorem for the existence of ZF multipliers is given:

**Theorem VI.2.** Let the nonlinearity $\phi : l^2_2 \rightarrow l^2_2$ be bounded, cyclic monotone increasing and slope-restricted, with slope $\tilde{H}$. Let the SISO multiplier be $M \in \mathcal{M}_+$ (or $M \in \mathcal{M}$ and $\phi$ is additionally odd), then $M_{ZF} = M$, and $\forall \theta \in l^2_2$, $\phi$ admits IQCs with the following multipliers:
Lemma VI.3. \(\phi\) is a direct consequence of lemma VI.3 is the following:

The key idea is to represent the nonlinear function \(\psi\) to the case of (possibly relaxed) recentered Barrier MPC.

Lemma VI.6. Each \(u_i\) can equivalently be written as \(u_i = \bar{L}_i \psi(\theta')\) and \(L_i\) having the following property:

\[ L_i L_i^T = 0 \]

for \(i \neq j\). Hence, \(U = \psi(\theta')\) can now be written as a set of parallel convex programs \((u_i)\) for the case of staged and box constraints.

Remark 4. If there is no special structure, e.g. repeated nonlinearities, then SISO multipliers should be used in the form of \(M_{ZF} = M1\) with \(M \in \mathcal{M}_{SISO}\). \(\mathcal{M}_{SISO}\) (or \(\mathcal{M}_{SISO}^{-}\)) contains all SISO rational transfer functions \(M_{ZF} \in \mathcal{R}L_{\infty}\) that maintain the properties of Definition 3.

Corollary 2. The nonlinearity \(\phi : \mathcal{R}^n \to \mathcal{R}^n\) admits IQC with multiplier \(\Pi(z)\) (equation 29), \(\forall \theta \in \mathcal{P}\) with \(\psi\) a time-invariant convex set and \(B \in \mathcal{B}\).

C. Multipliers for Box/Staged Constraints

The conservatism can be dropped even further when a special structure of constraints (linear set) is used and more general multipliers than SISO ZF can be utilized. The following analysis generalizes the results from 31, extending them to the case of (possibly relaxed) recentered Barrier MPC. The key idea is to represent the nonlinear function \(\phi\) as an equivalent feedback structure. This structure is then modified to a nonlinear program \(\psi\) together with a linear feedback term.

The new nonlinear program \(\psi\) can be separated into several smaller parallel nonlinear programs \(\psi_i\). Multipliers can then be associated with each \(\psi_i\) in our results, we show that there is a class of MIMO ZF multipliers \(M_{ZF}\) for a special structure of staged and box constraints.

Let \(\psi : \mathcal{R}^n \to \mathcal{R}^n\) be the following convex program:

\[ U = \psi(\theta') = \arg \min_u \frac{1}{2} u^T u - u^T \theta' + \mu B(u) \]

with \(B \in \mathcal{B}\)

Lemma VI.3. If \(\theta' = \theta + (I - \bar{H}) \phi(\theta)\), then \(U = \phi(\theta)\) and \(U = \psi(\theta')\) are equivalent.

Proof: Substituting \(\theta' = \theta + (I - \bar{H}) \phi(\theta)\) in the KKT conditions of (30), our result follows immediately.

This equivalent feedback structure is depicted in Fig. 3 and a direct consequence of lemma VI.3 is the following:

\[ \begin{bmatrix} \theta' \\ U \end{bmatrix} = \begin{bmatrix} I & I - \bar{H} \\ I \end{bmatrix} \begin{bmatrix} \theta' \\ U \end{bmatrix} \]

\[ \begin{bmatrix} \theta' \\ U \end{bmatrix} = \begin{bmatrix} I & I - \bar{H} \\ I \end{bmatrix} \begin{bmatrix} \theta' \\ U \end{bmatrix} \]

(31)

The rest of the analysis is based on the fact that \(U = \psi(\theta')\) can be written as many parallel convex programs with \(U = \sum_{i=0}^{N-1} u_i\). To do so, special structures of the constraints are considered such as limitations between adjacent actuators' movement (so-called staged constraints) 31 as well as box constraints. Both cases can be written as:

\[ L = [L_0^T \cdots \ L_{N-1}^T]^T = \text{diag}(\tilde{L}_0, \cdots, \tilde{L}_{N_L-1}) \]

with \(L_i\) having the following property:

\[ L_i L_i^T = 0 \]

for \(i \neq j\). Hence, \(U = \psi(\theta')\) can now be written as a set of parallel convex programs \((u_i)\) for the case of staged and box constraints.

Lemma VI.4. The nonlinear convex program \(U = \psi(\theta')\) given by (30), can equivalently be transformed to \(U = \sum_{i=0}^{N} u_i\) with \(u_i\) being parallel convex programs:

\[ u_i = \arg \min_u \frac{1}{2} u^T u - u^T \theta' + \mu \left( \sum_{j=1}^{N} \bar{B}_{ij}(u) + \ln(b_{ij}) - \frac{L_i^T L_j^T}{b_{ij}} \right) \]

s.t.

\[ L_i^T u = 0 \]

\[ \bar{B}_{ij}(u) = \begin{cases} -\ln(b_{ij}) - L_{ij} u & \text{for } L_{ij} u + b_{ij} \geq \delta \\ \beta_{ij}(u) & \text{elsewhere} \end{cases} \]

(34)

Proof: See Appendix B

Now, let \(L_i\) be an orthonormal basis of the space spanned by the rows of \(L_i\) and \(v_i(p)\) be the convex program

\[ v_i(p) = \arg \min_q \frac{1}{2} q^T q - q^T p + \mu \left( \sum_{j=1}^{N} \bar{B}_{ij}(L_j^T q) - \frac{L_i^T L_j^T}{b_{ij}} q \right) \]

(35)

A direct consequence is that each \(v_i\) is bounded, \(n\)-cyclic monotone and slope restricted with slope \(I\). Theorem VI.2 can, therefore, be applied:

Lemma VI.5. Let \(M_{ZF} \in \mathcal{M}_{SISO}\) be a SISO rational strictly proper transfer function. Then \(\forall p \in \mathcal{P}\), \(v_i\) admits IQC with the following multiplier:

\[ \Pi_{\psi}(z) = \begin{bmatrix} 0 & M_{ZF} \\ M_{ZF}^T & -M_{ZF} - M_{ZF}^T \end{bmatrix} \]

(36)

Proof: \(v_i\) is bounded, \(n\)-cyclic monotone and slope restricted with slope \(I\). Theorem VI.2 provides the result.

The next lemma shows that each \(u_i\) can be written as a function of \(v_i\).

Lemma VI.6. Each \(u_i\) can equivalently be written as

\[ u_i = \bar{L}_i v_i(\bar{L}_i \theta') \]

(37)

Proof: See Appendix C

Consequently an IQC for the nonlinear system \(\psi(\theta')\) with the following multiplier can be formulated:

\[ \Pi_{\psi}(z) = \sum_{i=0}^{N-1} \begin{bmatrix} L_i & L_i^T \end{bmatrix} \Pi_{\psi}(z) \begin{bmatrix} L_i \\ L_i \end{bmatrix} \]

(38)

Theorem VI.7. Let \(\phi : \mathcal{R}^n \to \mathcal{R}^n\) be bound, \(n\)-cyclic monotone

\[ \begin{bmatrix} \theta' \\ U \end{bmatrix} = \begin{bmatrix} I & I - \bar{H} \\ I \end{bmatrix} \begin{bmatrix} \theta' \\ U \end{bmatrix} \]

(31)
increasing and slope-restricted, with slope $\bar{H}$ under box or staged constraints. Let the multiplier $M_{ZF} \in M_+$ be a SISO rational strictly proper transfer function,

$$M_{ZF}(z) = \text{diag}(M_{ZF1}, \ldots, M_{ZF(N_{ZF}-1)}, I)$$

then $\forall \theta \in l^u_2$, $\phi$ admits IQC with the following multiplier:

$$\Pi(z) = \begin{bmatrix} I & I - \bar{H}^T \Pi(z) & I & I - \bar{H} \\ 0 & M_{ZF} \bar{H}M_{ZF} - M_{ZF}^T \bar{H} \end{bmatrix} = \begin{bmatrix} I & 0 \\ M_{ZF} & 0 \end{bmatrix} \left( I - \bar{H} \right)$$

(39)

Proof: See Appendix D

This theorem can be extended further for an even tighter class of box constraints where symmetric bounds are employed. That is, a wider class of multipliers can be employed as it can be proven that $\psi$ can be written as a linear transformation of repeated nonlinearities, hence full-block doubly hyper-dominant multipliers [28] can be utilized.

VII. CONVEX SEARCH FOR MULTIPLIERS

In this section the convex search applied to the stability analysis is presented. The results in this work allow the use of static multipliers for a wide class of constraints or dynamic multipliers for a tighter class of constraints. We revisit [16], [17] and expand the results from [17], in order to incorporate a larger class of problems, where the slopes are given by a full-block matrix. Dynamic multipliers may be non-causal and a factorization is required since they do not have a state-space representation. For the $i^{th}$ IQC its multiplier can be written as:

$$\Pi_i(z) = \Psi^\tau(z)K_i \Psi(z) = \Psi^\tau(z) \begin{bmatrix} M_{i1} & M_{i2} \\ M_{i2}^T & M_{i2}^T \end{bmatrix} \Psi(z)$$

For $N$ IQCs then we can then write:

$$\Pi(z) = \Psi^\tau(z)K \Psi(z)$$

(40)

$$K = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1j} \\ \vdots & \ddots & \ddots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{nj} \\ M_{N2}^T & \cdots & \cdots & M_{Nj}^T \end{bmatrix}$$

(41)

In this work, finite impulse response (FIR) type multipliers [42] are used. Their use can be justified using the phase equivalence argument [32],

$$M_{ZF} = \sum_{j=-N_{ZF}}^{N_{ZF}} R_j (1-z^j)$$

(42)

Using the largest $N_{ZF} = \text{max}(N_{ZF}, N_{ZF}^-)$ of all multipliers, $\Psi$ can be defined as:

$$\Psi_{11} = \begin{bmatrix} I & (1-z^{-1})I & \cdots & (1-z^{-N_{ZF}})I \end{bmatrix}^T$$

(43a)

$$\Psi = \text{diag}(\Psi_{11}, \Psi_{11})$$

(43b)

The slope-restricted nonlinearities described in this work admit IQC multipliers as in [39]. As a result for the nonlinearity $\phi$, we have the following: $M_{\phi}^{11} = 0$,

$$M_{\phi}^{12} = \begin{bmatrix} R_0 & \cdots & R_{N_{ZF}} \\ \vdots & \ddots & \vdots \\ R_{-N_{ZF}} & \cdots & (R_{N_{ZF}} + R_{-N_{ZF}}) \end{bmatrix}$$

and

$$M_{\phi}^{22} = \begin{bmatrix} R_0 \bar{H} + \bar{H}R_0 & \cdots & (R_{N_{ZF}} + R_{-N_{ZF}}) \bar{H} \\ \vdots & \ddots & \vdots \\ \bar{H}(R_{N_{ZF}} + R_{-N_{ZF}}) & \cdots & 0 \end{bmatrix}$$

Therefore, the dynamic system $G_\Psi(z)$ has non-singular state-space representation and the LMI conditions can be constructed:

$$G_\Psi(z) = \Psi(z) \begin{bmatrix} G(z) \\ I \end{bmatrix}$$

(44)

with:

$$G_\Psi \sim \begin{bmatrix} A_\Psi & B_\Psi \\ C_\Psi & D_\Psi \end{bmatrix}$$

(45)

By the KYP lemma [33], inequality [33] can be transformed into the following LMI optimization:

$$\min_{\lambda, K} \text{ s.t. } \begin{bmatrix} A_\Psi & B_\Psi \\ C_\Psi & D_\Psi \end{bmatrix} + \begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix} \leq -\lambda I,$$

and additional constraints for the multipliers

The additional constraints depend on the class of the multipliers (if $M \in M_+, \mathcal{A}$). For the case of static multipliers, $R_j$ for $j \neq 0$ can be set equal to zero. When diagonal multipliers are utilized (for example when asymmetric box or stage constraints are applied, $C-ZF$), then $R_j \geq 0$ for $j \neq 0$ and $R_0 > 0$. The condition of doubly hyper-dominance can be similarly expressed.

VIII. ROBUSTNESS OF BARRIER MPC

In this section the robustness of the barrier MPC is considered through an illustrative numerical example. If the transfer function of the open-loop LTI plant in [2] is $G_{22}$, the system under the unstructured uncertainty ($\Delta$) is given by Fig. 4 with $\Delta : l^u_2 \rightarrow l^u_2$. Then, the barrier MPC ($\phi$) can be included in the analysis as in [21] with $\Delta = \begin{bmatrix} \Delta_1 \\ \phi \end{bmatrix}$. The robustness of the MPC can be analyzed in terms of input-to-output stability given that the uncertainty admits an IQC.

A. Numerical Example

The illustrative example consists of a nominal plant under a norm-bounded unstructured uncertainty $\Delta_1 : l \rightarrow l$ which is
bounded as $||\Delta|| \leq b^2$. The nominal dynamic system is given by the following equation:

$$x_{k+1} = \begin{bmatrix} 0.7 & 0.3 \\ 0.8 & 0.01 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k$$  

$$y_k = \begin{bmatrix} 1 & 1.5 \end{bmatrix} x_k$$  

For this example the LTI plant has eigenvalues 0.9542, -0.2442 and zero equal to -1.19. As a result, the system is non-minimum phase. In addition to the nominal plant, the state observer is given by

$$\hat{x}(t) = J_u(z)u(t) + J_y(z)y(t)$$  

Here a steady-state Kalman filter is used with $J_u(z) = (zI - A + ALC)^{-1}B$ and $J_y = (zI - A + ALC)^{-1}AL$. For the numerical example the observer gain $L$ was calculated via the discrete algebraic Riccati equation with weighting matrices set equal to the identity matrix.

The linear part of Fig. 5 can be transformed into augmented linear system $M_d$

$$M_d(z) = \begin{bmatrix} \sqrt{b} I & G \\ -S & J_y - J_y J_d G \end{bmatrix} \begin{bmatrix} \sqrt{b} I \\ E \end{bmatrix}$$  

with $I$ and $0$ the identity and zero matrix, respectively. Only the control action is applied and hence $E = [I \ 0 \ \cdots \ 0]$. Additionally, the scaled uncertainty is defined as $||\Delta|| \leq 1$ and $\phi$ the input-output map of the barrier MPC. The control action is given by $\Phi$, where $\theta = -S x$ and $B$ the gradient re-centered barrier function.

As a result two IQCs can be written: one for the controller and one for the given unstructured uncertainty. For $\Delta$ we have

$$\Pi_\Delta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

For the controller different multipliers are utilized depending on the case. All the algorithms presented in this work have been implemented for $\mu = 0.8$ and sufficiently large $N_{ZF}$. The constraints added to the manipulated variables are $-0.5 \leq u_k \leq 1.0$. The control and prediction horizon are both set equal to 2 and $Q = I$. Two problems are investigated: Task 1. A positive gain $\kappa > 1$ is applied to the output of the dynamical system for $b = 0$ and the goal is to compute the maximum stable gain. Task 2 is aimed at finding the smallest positive parameter $r$ of the objective function for $b = 0.25$ and also the largest positive $b$ for $r = 0.001$ so that the system is guaranteed stable. The results for task 1 are shown in Table I and the case of a nominal MPC is conducted for comparison purposes. From Table II the advantage of barrier MPC compared to the nominal MPC becomes obvious. The maximum gain for the case of barrier MPC is 2.913 compared to 1.130 for the nominal MPC.

Additionally, after trial and error, we found that for $\kappa = 3.4$, barrier MPC is destabilized, which is very close to the computed value. Additionally, for $\kappa = 2.9$, some simulations have been conducted for various $\mu$. From Fig. 6 (a) the advantage of the barrier MPC is clear, since the nominal MPC is unstable as expected since the maximum computed $\kappa$ is 1.130.

The analysis shows that barrier MPC is more robust than the nominal MPC, for all the different methods applied. Additionally, C-ZF seems to produce the least conservative results, predicting the system is stable for all possible $r$ even for $N_{ZF} = 1$.

For the next task, the results are depicted in Table II. The analysis shows that barrier MPC is more robust than the nominal MPC, for all the different methods applied. Additionally, C-ZF seems to produce the least conservative results, predicting the system is stable for all possible $r$ even for $N_{ZF} = 1$.
To confirm that the closed loop system is stable a simulation is conducted for design parameter $r = 0.001$ and $b = 0.25$ for several initial conditions. All of them produced stable results. These simulation results are depicted in Fig. 6(b). The lines represent the average behavior while the shaded areas correspond to the different initial values.

![Simulation Results](image)

Fig. 6. Simulations of numerical example for (a) task 1 and (b) task 2

### IX. Conclusion

In this work input-to-output stability results are developed for the barrier MPC. The barrier can improve the robustness of the MPC due to the change in the slope. Additionally, general convex constraints can be employed. Tighter time-invariant convex constraints as well as staged constraints are considered. The tighter the constrained case the less conservative the analysis can become, through the use of dynamic multipliers. A convex search is presented in order to be able to apply the stability criteria using multipliers. The results of this paper can be further generalized for the case of PWA models extending our recent work [34].

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APPENDIX A

The parameter $m$ that makes the slope of the nonlinearity tighter seems to be able to significantly reduce conservatism. The corresponding optimization problem is not convex and this may complicate the algorithm. Nevertheless, here we demonstrate that even though the problem is not convex it can have only one solution in the feasible region. The parameter $m$ can be calculated by minimizing the smallest eigenvalue of the Hessian of the barrier inside the feasible region. This problem can be formulated as:

$$m = \min_{u,x} x^T \nabla^2 B(u)x$$

s.t. $x^T x = 1$

$$L u \leq b$$

where

$$\nabla^2 B(u) = \sum_{i} \frac{1}{(b_i - Lu)^2} L_i^T L_i$$

The KKT conditions of this problem can be written as the following set of equations:

$$(51a) \quad \sum_{i} \frac{1}{(b_i - Lu)^2} L_i^T L_i x - \lambda x = 0$$

$$(51b) \quad 1 - x^T x = 0$$

$$(51c) \quad \sum_{i} L_i^T x L_i^T L_i x - \sum_{i} \frac{\lambda^i_{in}}{2} L_i^T = 0$$

$$(51d) \quad \lambda^i_{in}(L_i^T u - b) = 0$$

This set of equations corresponds to the minimum of the smallest eigenvalue. The optimum objective function will be equivalent to the parameter $m$ used in our analysis. This optimization, however, as is non-convex in the general case can be a bottleneck for the conservatism of the proposed analysis. Here a result regarding the box constraints is provided as well as a relaxation for the general case of bounded constraints. Box constraints can be seen as

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

of the KKT conditions assuming the solution can never be in the bound of the polyhedral, can be written as:

$$-\frac{1}{(b_i + u_i)^3} x_i^2 + \frac{1}{(b_i - u_i)^3} x_i^2 = 0$$

The solution of $x^T \nabla^2 B(u)x$ is unique and independent of $x$. Precisely, the value of $m$ is calculated as:

$$m = \min \left[ \frac{(b_1 + b_1)^3}{(b_1 - b_1)^3}, \frac{(b_2 + b_1)^3}{(b_2 - b_1)^3}, \ldots, \frac{(b_N + b_1)^3}{(b_N - b_1)^3} \right].$$

Nevertheless, the previous case accounts only box constraints, and a decomposition for the staged constraints is next considered. The Hessian can be decomposed using $L_i$ as:

$$\nabla^2 B(u) = \begin{bmatrix} \sum_{i=1}^{n_u} L_i^T L_i & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^{n_u} L_i^T L_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Therefore, since $L_i$ has block diagonal structure, the eigenvalues can be found separately for each block using (50), and the smallest one can be selected as $m$.

APPENDIX B

The KKT conditions for $u_i$ with $i = 0, \ldots, N_L - 1$ are

$$u_i - \theta' + \mu \left( \sum_{j=1}^{N_0} \nabla \tilde{B}_{ij} - \frac{L_i^T}{b_{ij}} \right) + L_i^T z_i = 0$$

$$L_i^T u_i = 0$$

with $z_i = -L_i^T \theta'$

for $i = 0, \ldots, N_L - 1$. Summing (56) over $i$ together with (57) gives

$$u - (N_L - 1) \theta' + \sum_{i=0}^{N_L-1} \mu \left( \sum_{j=1}^{N_0} \nabla \tilde{B}_{ij}(u_i) - \frac{L_i^T}{b_{ij}} \right) = \sum_{i=0}^{N_L-1} L_i^T \theta' = (N_L - 1) \theta'$$

Therefore

$$u - \theta' + \mu \nabla B(u) = 0$$

APPENDIX C

$v_i$ is bounded, $n$-cyclic monotone and slope restricted with slope $(I)$. Corollary VI.2 gives the result.

APPENDIX D

$\phi(\theta)$ can be expressed with respect to $\psi(\theta')$, using (31), as a result

$$\Pi(z) = \begin{bmatrix} I & I - \bar{H} \end{bmatrix} \Pi_\psi(z) \begin{bmatrix} I \\ I - \bar{H} \end{bmatrix}$$

To complete the proof, we use the fact that

$$L_i^T L_i = \begin{bmatrix} \cdots & I \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

$\hat{h}$ row