Spinorial coordinates for Lorentzian 4-metrics

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Abstract:
Lorentzian 4-metrics are expressed in spinorial coordinates. In these coordinates the metrics components can be factorized into a product of complex conjugate quantities. The linearized theory and Einstein’s vacuum field equations are studied using these coordinates. The relationship between Lorentzian and complex 4-metrics is discussed.
1 Introduction

In this paper a spinorial form for Lorentzian 4-metrics on real four dimensional manifolds is presented and discussed. By using the two-component spinor formalism [1] it is shown that any Lorentzian 4-metric can be locally expressed in terms of a conformal factor, a two index symmetric spinor field and its complex conjugate. It is demonstrated that in this coordinate system such a metric admits a factorization into a product of complex conjugate quantities. These spinorial coordinates are used to study the metric, and Einstein’s vacuum field equations. The linear approximation about flat space is also discussed.

The two-component spinor formalism that is needed is reviewed in the next two sections. Here Cartan’s structure equations for metric geometries and Einstein’s vacuum field equations for signature (1, 3) metrics are presented and a linearized version of these equations is recalled. Anti-self dual (and self-dual) solutions of Cartan’s equations on real four dimensional manifolds are defined. In the fourth and fifth sections spinor coordinates for Lorentzian four-metrics are introduced and used to factorize the metric into a product of complex conjugate quantities. These coordinates, and the linearized versions of Cartan’s structure equations and Einstein’s vacuum field equations, are used in the sixth section to derive the metric associated with Roger Penrose’s Hertz potentials for spin two fields in Minkowski space-time [2]. Similar coordinate systems arise in investigations of holomorphic metrics, particularly in the approaches initiated by Jerzy Plebański and others, [3], [4]. In the final section spinorial coordinates and complex anti self-dual systems are considered and their possible use in the construction of Ricci flat Lorentzian 4-metrics is briefly discussed.

All considerations in this paper are local. Upper case Latin indices range and sum over 0 to 1 and are raised and lowered with the antisymmetric spinors $\epsilon^{AB}$ and $\epsilon_{AB}$ as in reference [1].

2 Two component spinor formalism

The spinor forms of Cartan’s structure equations and Einstein’s gravitational equations used in this paper are as follows.

When the metric is written in terms of two component spinors so that

$$ds^2 = \epsilon_{AB} \epsilon_{A'B'} \theta^{AA'} \otimes \theta^{BB'},$$

(1)
the first Cartan equations are

\[ D\theta^{AA'} \equiv d\theta^{AA'} + \omega^A_B \theta^{BA'} + \omega^{A'}_{B'} \theta^{AB'} = 0, \tag{2} \]

where the co-frame \( \theta^{AA'} \) is a hermitian matrix-valued one-form, the complex conjugate \( \bar{\omega}^{A'}_{B'} \) and \( \omega^A_B \) correspond, respectively, to the \( sl(2,C) \)-valued, self-dual and anti self-dual parts of the torsion-free and metric connection one-form. The second Cartan equations are

\[ \Omega^A_B = d\omega^A_C \omega^C_B = \frac{1}{2} R^A_BCC'DD' \theta^{CD} \theta^{D'}, \tag{3} \]
\[ \bar{\Omega}^{A'}_{B'} = d\omega^{A'}_{C'} \omega^{C'}_{B'} = \frac{1}{2} R^A_{B'C'D'D'} \theta^{C'D} \theta^{D'}. \]

Here \( \Omega^A_B \) and its complex conjugate \( \bar{\Omega}^{A'}_{B'} \) are \( sl(2,C) \)-valued two-forms and \( R^A_BCC'DD' \) and its complex conjugate \( R^A_{B'C'D'D'} \) are, respectively, the anti-self dual and self-dual parts of the Riemann tensor. Furthermore

\[ \Omega^A_B = \Psi^{A}_{BCD} \Sigma^{CD} + 2\Lambda \Sigma^A_B + \Phi^A_{BCD} \Sigma^{CD}; \tag{4} \]
\[ \bar{\Omega}^{A'}_{B'} = \bar{\Psi}^{A'}_{B'C'D'} \Sigma^{C'D'} + 2\Lambda \Sigma^{A'}_{B'} + \bar{\Phi}^{A'}_{B'CD} \Sigma^{CD}; \]
\[ \Sigma^{CD} = \frac{1}{2} \theta^C_B \theta^{DB'}, \Sigma^{C'D'} = \frac{1}{2} \theta^C_{B'} \theta^{DB}. \]

The Weyl spinors \( \Psi^{ABCD} = \Psi_{(ABCD)} \) and \( \bar{\Psi}^{A'B'C'D'} = \bar{\Psi}_{(A'B'C'D')} \) correspond respectively to the anti-self dual and self-dual parts of the Weyl tensor, \(-2\Phi^A_{BCD'}\) corresponds to the trace-free part of the Ricci tensor and \(24\Lambda\) corresponds to the Ricci scalar.

The first and second Bianchi identities are

\[ \Omega^A_B \theta^{BA'} + \bar{\Omega}^{A'}_{B'} \theta^{AB'} = 0, \tag{5} \]
\[ D\Omega^A_B = D\bar{\Omega}^{A'}_{B'} = 0, \]

and \( D \) always denotes the relevant covariant exterior derivative.

Under a change of co-frame

\[ \theta^{AA'} \rightarrow (L^{-1})^A_B (L^{-1})^{A'}_{B'} \theta^{BB'}; \quad \omega^A_B \rightarrow (L^{-1})^A_B dL^B_C + (L^{-1})^A_B \omega^C_B L^D_C; \tag{6} \]
\[ \Omega^A_B \rightarrow (L^{-1})^A_C \Omega^C_D L^D_B \]
where \( L^A_B \in SL(2, C) \) and \( \overline{L}^{A'}_{B'} \) is its complex conjugate. Similar results hold for the complex conjugates \( \overline{w}^{A'}_{B'} \) and \( \overline{\Omega}^{A'}_{B'} \).

Einstein’s vacuum field equations with zero cosmological constant, \( \Phi^A_{BC'D'} = \Lambda = 0 \), can also be written either as

\[
\Omega^A_B = \Psi^A_{BCD} \Sigma^{CD},
\]

or as

\[
\Omega^A_B \theta^{BA'} = 0.
\]

Cartan’s equations can be extended to complex ones permitting complex solutions and having as structure group \( SO(4, C) \sim SL(2, C)_L \times SL(2, C)_R / \mathbb{Z}_2 \). A (complex) anti-self dual solution of Cartan’s structure equations on a real four dimensional manifold is a complex co-frame which satisfies the first Cartan structure equations above with flat self-dual connection \( \text{\textbf{w}}^{A'}_{B'} \). The anti-self dual curvature consequently satisfies Eqs.(7) and (8), that is \( \Omega^A_B = \Psi^A_{BCD} \Sigma^{CD} \) and \( \Omega^A_B \theta^{BA'} = 0 \). By using Eq.(6) with \( L^C_B \in SL(2, C)_R \) the self-dual connection forms can be set equal to zero. Self-dual solutions are defined in an analogous way. Anti-self dual solutions of Cartan’s equations do not define real four-metrics but they can be combined with their complex conjugates (that is self-dual solutions) to construct Lorentzian 4-metrics. Examples of combinations which are Ricci flat are discussed in the final section.

### 3 Linearized equations

Consider a metric linearized about the flat metric \( ds^2 = \epsilon_{AB} \epsilon_{A'B'} dx^{AA'} \otimes dx^{BB'} \) so that the linearized metric takes the form

\[
ds_{lin}^2 = (\epsilon_{AB} \epsilon_{A'B'} + \gamma_{AA'BB'}) dx^{AA'} \otimes dx^{BB'}.
\]

When a linearized co-frame is chosen to be

\[
\theta_{lin}^{AA'} = (\delta^A_B \delta^{A'}_{B'} + \mu^{AA'}_{BB'}) dx^{BB'},
\]

so that

\[
ds_{lin}^2 = \epsilon_{AB} \epsilon_{A'B'} \theta_{lin}^{AA'} \otimes \theta_{lin}^{BB'}
\]

\[
= (\epsilon_{AB} \epsilon_{A'B'} + \mu_{AA'BB'} + \mu_{BB'AA'}) dx^{AA'} \otimes dx^{BB'},
\]
then
\[ \gamma_{AA'BB'} = \mu_{AA'BB'} + \mu_{BB'AA'}. \] (12)

The linearized Cartan equations for such a co-frame are [5]
\[ d\theta_{lin}^{AA'} + \omega_{linB}^{A} dx^{BA'} + \overline{\omega}_{linB}^{A} dx^{AB'} = 0; \] (13)
\[ \Omega_{linB}^{A} = d\omega_{linB}^{A}, \]

where
\[ \Omega_{linB}^{A} = \frac{1}{2} R_{linBCC'DD'}^{A} \epsilon^{CD} \sum_{lin}^{C'D'} + \frac{1}{2} R_{linBCC'DD'}^{A} \epsilon^{C'D'} \sum_{lin}^{CD}, \] (14)
\[ d\epsilon^{CC'} dx^{DD'} = \epsilon^{CD} \sum_{lin}^{C'D'} + \epsilon^{C'D'} \sum_{lin}^{CD}, \]

and similarly for the complex conjugates \( \overline{\Omega}_{linB'}^{A'} \) and \( d\overline{\omega}_{linB'}^{A'}. \) The linearized first and second Bianchi identities are
\[ d\omega_{linB}^{A} dx^{BA'} + d\overline{\omega}_{linB}^{A'} dx^{AB'} = 0, \] (15)
\[ d\Omega_{linB}^{A} = \frac{1}{2} \partial_{EE'} R_{linBCC'DD'}^{A} dx^{CC'} dx^{DD'} dx^{EE'} = 0, \]
\[ d\overline{\Omega}_{linB'}^{A'} = \frac{1}{2} \partial_{EE'} \overline{R}_{linB'CC'DD'}^{A'} dx^{CC'} dx^{DD'} dx^{EE'} = 0. \]

Under a linearized (first order) diffeomorphism
\[ x^{AA'} \mapsto x^{AA'} + \xi^{AA'}, \] (16)

\[ \mu_{AA'BB'} \mapsto \mu_{AA'BB'} + \partial_{BB'} \xi_{AA'}, \]
\[ \gamma_{AA'BB'} \mapsto \gamma_{AA'BB'} + \partial_{BB'} \xi_{AA'} + \partial_{BB'} \xi_{AA'}, \]

where \( \partial_{BB'} \) denotes partial differentiation with respect to \( x^{BB'}. \) Under a linearized change of co-frame Eq.(6), linearized about the identity, gives
\[ \theta_{lin}^{AA'} \mapsto (\delta_{B}^{A} + l_{B}^{A})((\delta_{B'}^{A'} + \overline{l}_{B'}^{A'}) \theta_{lin}^{BB'}, \] (17)
\[ l_{AB} = l_{BA}, \overline{l}_{AB'} = \overline{l}_{B'A'}, \]

and
\[ \omega_{linB}^{A} \mapsto \omega_{linB}^{A} + dl_{B}^{A}, \omega_{linB'}^{A'} \mapsto \overline{\omega}_{linB'}^{A'} + d\overline{l}_{B'}^{A'}. \] (18)
It follows that a co-frame can be chosen for which $\mu_{AA'BB'} = \mu_{BB'AA'}$ and hence, with this choice,
\[
\gamma_{AA'BB'} = 2\mu_{AA'BB'}. \tag{19}
\]
From Eqs. (7) and (8) the linearized Einstein vacuum field equations are
\[
d\omega^A_{\text{lin}B} = \Omega^A_{\text{lin}B} = \frac{1}{2} \Psi^A_{\text{lin}BCD} dx^C_{D'} dx^{DD'}, \tag{20}
\]
or
\[
\Omega^A_{\text{lin}B} dx^{BB'} = d(\omega^A_{\text{lin}B} dx^{BB'}) = 0, \tag{21}
\]
where $\Psi^A_{\text{lin}BCD}$ denotes the components of the totally symmetric, linearized anti-self dual Weyl spinor (and similarly for the linearized self-dual quantities).

4 Spinorial coordinates and Lorentzian 4-metrics

Locally any Lorentzian 4-metric can always be written in terms of null coordinates $(u, r, \zeta, \bar{\zeta})$
\[
ds^2 = adu^2 + 2dudr + 2bud\zeta + 2 \bar{\zeta}d \bar{u} + cd\zeta^2 + c\bar{\zeta}^2 - 2pd\zeta d\bar{\zeta}. \tag{22}
\]
where here the hypersurfaces given by constant $u$ are chosen to be retarded null hypersurfaces, $r$ is an affine parameter along the null geodesics ruling such hypersurfaces and $\zeta$ is a complex (angular) coordinate labelling such null geodesics. The latter can always be chosen, as will be done here, so that $p > 0$. Under a change of coordinates $r \to v$, where the inverse transformation $r = r(u, v, \zeta, \bar{\zeta})$ is determined by the equation
\[
\frac{\partial r}{\partial v} = p(u, r, \zeta, \bar{\zeta}), \tag{23}
\]
the metric takes the form
\[
ds^2 = \exp 2\sigma(2dudv - 2d\zeta d\bar{\zeta} + Adu^2 + 2Bdud\zeta + 2\bar{\zeta}dud\bar{\zeta} + C\zeta^2 + \bar{\zeta}^2) \tag{24}
\]
where
\[
\exp 2\sigma = p; \quad A = p^{-1}(a + 2 \frac{\partial r}{\partial u}); \tag{25}
\]
\[
B = p^{-1}(b + \frac{\partial r}{\partial \zeta}); \quad C = cp^{-1}.
\]
The metric in Eq.(24) can be written in spinorial form by introducing spinorial coordinates \( x^{AA'} \) and a spin dyad \((o^A, \iota^A)\) where

\[
x^{AA'} = \begin{bmatrix} v & \zeta \\ \zeta & u \end{bmatrix}; \quad o^A = \delta^A_0, \quad \iota^A = \delta^A_1.
\]

(26)

Then Eq.(24) takes the form

\[
ds^2 = g_{AA'BB'} dx^{AA'} \otimes dx^{BB'}
\]

(27)

\[
= \exp(2\sigma)\left[\epsilon_{AB}\psi_{A'B'} + 2o_Ao_B\overline{\psi}_{A'B'} + 2\overline{\sigma}_{A'}\sigma_{B'}\psi_{AB}\right] dx^{AA'} \otimes dx^{BB'},
\]

where the symmetric spinor \( \psi_{AB} \) is related to \( A, B \) and \( C \) by

\[
\psi_{00} = \frac{C}{2}, \quad \psi_{01} = \frac{B}{2}, \quad \psi_{11} + \overline{\psi}_{11} = \frac{A}{2}, \quad \psi_{1A} = \psi_{00}o_A + \psi_{01}(o_Ao_B + \iota_Ao_B) + \psi_{11}o_Ao_B.
\]

(28)

This metric form is preserved by the global coordinate transformations

\[
x^{AA'} \mapsto e^{-\lambda}L_A^AL_B^BB'x^{BB'} + p^{AA'},
\]

\[
L_C^AL_D^B\epsilon_{AB} = \epsilon_{CD},
\]

(29)

where \( \lambda, L_A^A \) and \( p^{AA'} \) are constants, and

\[
\sigma \mapsto \sigma + \lambda, \quad o_A \mapsto (L^{-1})_A^B o_B, \quad \psi_{AB} \mapsto (L^{-1})_A^C(L^{-1})_B^D\psi_{CD},
\]

(30)

and similarly for the complex conjugates. Furthermore \( o^A \mapsto \delta^A_0 \) when \( L_0^A = \delta^A_0 \). In addition this form of the metric is preserved under the transformation \( x^{AA'} \mapsto x^{AA'} + 2o^A\overline{\sigma}^A f(u) \), \( \psi_{AB} \mapsto \psi_{AB} + o_Ao_B\frac{df}{du} \), for any real-valued function \( f \).

The conformal geometry is determined by the symmetric spinor \( \psi_{AB} \) but it should be noted that \( \text{Im} \psi_{11} \) does not appear in the conformal metric.

The inverse \( g^{-1} \) of the metric \( g \) with components \( g_{AA'BB'} \) has components

\[
(g^{-1})_{AA'BB'} = \exp(-2\sigma)[1 - 4\psi_{00}\overline{\psi}_{00}]^{-1}[\epsilon^{AB}\epsilon^{A'B'} - 2o^A\overline{o}^B\overline{\psi}_{A'B'} - 2\overline{\sigma}^A\sigma^B\psi_{AB}
\]

\[
- 4o^A\overline{\sigma}^A\overline{\psi}_{B'}\psi^B - 4o^B\overline{\sigma}^B\psi_{A'}\overline{\psi}^A + 4o^A\overline{\sigma}^A o^B\overline{\sigma}^B(\Delta\overline{\psi}_{00} + \overline{\Delta}\psi_{00})]
\]

(31)
where
\[ \psi^A = \psi^A_0 = \psi^A \]
\[ \Delta = \psi_{AB}\psi^{AB}, \]
and similarly for the complex conjugate quantities. Consequently regularity of the inverse requires
\[ 4\psi_{00}^0\psi_{00'}^0 \neq 1. \]

A co-frame for the metric given in Eq.(27) is
\[ \theta^{AA'} = \exp(\sigma - \zeta)[\delta^A_B\delta^{A'}_{B'} + o^A_Bo_B\overline{\varphi}_{B'} + \overline{\sigma}^{A'}_{B'}\varphi^A_B + o^A_0\overline{\varphi}_{0B}\overline{\varphi}_{0'B'}]dx^{BB'} \]  
(33)
where
\[ \varphi_{AB} = \psi_{AB} \exp 2\zeta, \]
\[ \exp 2\zeta = 1 + \varphi_{00}\overline{\varphi}_{00'}. \]
and
\[ ds^2 = \epsilon_{AB}\epsilon_{A'B'}\theta^{AA'} \otimes \theta^{BB'}. \]

The dual frame is
\[ E_{BB'} = [\exp(\zeta - \sigma)][(1 - \varphi\overline{\varphi})^{-1}][\delta^A_B\delta^{A'}_{B'} - o^A_0o_B\overline{\varphi}_{B'} - \overline{\sigma}^{A'}_{B'}\varphi^A_B - o^A_0\overline{\varphi}_{0B}\overline{\varphi}_{0'B'}] \]  
(35)
where
\[ \varphi = \varphi_{00}, \Delta_{\varphi} = \varphi_{AB}\varphi^{AB} \] and similarly for the complex conjugate quantities.

## 5 Factorization of co-frame and metric

An interesting feature of the metric in spinor coordinates as presented in the previous section is that the expressions for the co-frame and metric given in Eqs.(33) and (27) admit factorizations into the products of complex conjugate terms as follows. The co-frame factorizes as
\[ \theta^{AA'} = \chi_{BP'}^A\chi_{PB}^{A'}dx^{BB'}. \]  
(36)
where
\[ \chi_{BP'}^A = \exp \alpha(\delta^P_{B'}\delta^{A'}_{P'} + \overline{\sigma}^{A'}_{P'}\overline{\varphi}_{P'B'}), \]

8
\[
\alpha + \overline{\alpha} = \exp(\sigma - \zeta).
\]

(38)

It follows that the metric components given in Eq.(27) can also be written as the product of complex conjugate terms. If this metric’s components are written as \(g_{CC' DD'}\) so that Eq.(27) is

\[
d s^2 = g_{CC' DD'} dx^{CC'} \otimes dx^{DD'},
\]

(39)

then it follows that

\[
g_{CC' DD'} = k_{PCP' QDQ'}^{Q' D' Q}
\]

(40)

where

\[
k_{PCP' QDQ'} = \epsilon_{A'B'}\chi_{PCP'} \chi_{QDQ'}^{Q' D' Q},
\]

(41)

so

\[
k_{PCP' QDQ'} = \frac{1}{2} \exp 2\alpha[\epsilon_{PC} h_{QP' DQ'} + \epsilon_{DQ} h_{CP' PQ'}],
\]

(42)

where

\[
h_{QP' DQ'} = \epsilon_{QD} \epsilon_{P' Q'} + 2\sigma_{P'} \epsilon_{Q' Q} \varphi_{QP},
\]

(43)

and similarly for the complex conjugate of \(k_{PCP' QDQ'}\). The imaginary part of \(\alpha\) does not appear in the metric, Eq.(27), which has components \(g_{CC' DD'}\) equal to

\[
\frac{1}{4} \exp(2\alpha + 2\overline{\alpha})[h_{QP' DQ'}^{Q' D' Q} - h_{QP' DQ'}^{Q D' Q} \overline{h}_{PP' DQ'}^{P' DD'} h_{CC' Q} h_{QDD'}^{Q' D Q'} - h_{CC' Q}^{Q' DD'} h_{QDD'}^{Q D' Q'} + h_{CC' Q}^{Q' DQ} \overline{h}_{QDD'}^{Q D' Q'}].
\]

(44)

6 Linearization of the 4-metric using spinorial coordinates

Now consider again the metric in Eq.(27) but now linearized about the Minkowski metric so that

\[
ds_{lin}^2 = (\epsilon_{AB} \epsilon_{A'B'} + \gamma_{AA' BB'}) dx^{AA'} \otimes dx^{BB'},
\]

(45)

\[
\gamma_{AA' BB'} = 2\sigma \epsilon_{AB} \epsilon_{A'B'} + 2o_{A0B} \psi_{A'B'} + 2\overline{\sigma} \overline{A'B'} \psi_{AB},
\]
where in this section only \( \sigma \) and \( \psi_{AB} = \varphi_{AB} \) are first order terms. Hence a linearized co-frame and frame for the linearized metric are

\[
\theta_{lin}^{AA'} = [\delta_B^A \delta_{B'}^{A'} (1 + \sigma) + o^A o_B \psi_{B'}^{A'} + \sigma^{A' \theta_{B'}^{A'}}] dx^{BB'},
\]

\[
E_{linBB'} = (\delta_B^{A'} \delta_{B'}^{A'} (1 - \sigma) - o^A o_B \psi_{B'}^{A'} - \sigma^{A' \theta_{B'}^{A'}}) \frac{\partial}{\partial x^{AA'}}.
\]

It follows from Eq.(13) that the corresponding anti-self dual linearized connection one-form is given by

\[
\omega_{linAB} = \frac{1}{2} \left[ -\partial AC \sigma \epsilon_{BC} - \partial_{BC} \sigma \epsilon_{AC} + o_C (o_A \partial_{BA} \psi_{C'}^{A'} + o_B \partial_{AA'} \psi_{C'}^{A'}) + \sigma^{C'} \sigma^{A'} (\partial_{BA} \psi_{AC} + \partial_{AA'} \psi_{BC}) \right] dx^{CC'}
\]

and similarly for the complex conjugate self-dual connection one-form. The anti-self dual curvature two-forms is given by

\[
R_{linABCC'DD'} dx_{CC'} dx_{DD'}
\]

\[
= \left[ -\partial_{[B[C]} \sigma o_{D]} \partial_{A)] \sigma - \partial_{[B[D]} \sigma o_{A]} \partial_{C] \sigma} + o_C (o_A \partial_{BA} \psi_{C'}^{A'} + o_B \partial_{AA'} \psi_{C'}^{A'}) - \sigma^{A'} \sigma^{C'} (\partial_{BA} \psi_{AC} + \partial_{AA'} \psi_{BC}) \right] \Omega_{lin}^{CD'}
\]

\[
+ \left[ \frac{1}{2} (\epsilon_{AC} \epsilon_{BD} + \epsilon_{BC} \epsilon_{AD}) \right] \square + o_A o_C \partial_{D} \partial_{B} \psi_{A'}^{A'} + o_B o_A \partial_{D} \partial_{B} \psi_{A'}^{A'} + \sigma^{A'} \sigma^{C'} \partial_{DC} \partial_{A(B} \psi_{A')C} + \sigma^{A'} \sigma^{C'} \partial_{CC'} \partial_{A(B} \psi_{A')D} \right] \Omega_{lin}^{CD}.
\]

where \( \partial_{AA'} \partial_{B}^{A} = \frac{1}{2} \epsilon_{AB} \square \), and similarly for the complex conjugate self-dual curvature. While this expression is more complicated than the expression for the linearized curvature obtained in the standard way a number of interesting conclusions can be drawn from it. For instance, comparing it with the equations of Sec.3 it can immediately be seen that the linearized curvature has vanishing Ricci tensor when \( \sigma = 0 \) and \( \psi_{AB} \) satisfies the source-free Maxwell equations in Minkowski space-time, that is

\[
\partial_{AA'} \psi^{AC} = 0.
\]

\(^1\)Only a restricted set of the linearized diffeomorphisms of Eq.(16) satisfying \( \partial_{BB'} \xi_{AA'} = \partial_{BB'} \partial_{AA'} \xi = \frac{1}{2} \square \xi_{AA'} \xi_{A'B'} + \sigma_{A'B'} \xi_{AA'} + \sigma_{A'B'} \xi_{AA'} \xi_{A'B'} \), for a real function \( \xi \), and the complex conjugates \( \zeta_{AB} = \zeta_{BA} \) and \( \zeta_{A'B'} = \zeta_{B'A'} \), preserve the form of both this metric and these bases.
In this case the anti-self dual linearized Weyl spinor is given by $\Psi_{linABCD} = \frac{1}{2}[\partial_D\partial(B\psi_{A})C + \partial_C\partial(B\psi_{A})D]$, where $\sigma^{A'}\partial_{AA'\equiv\partial_{A}}$, and similarly for the complex conjugate self-dual Weyl spinor.

Many years ago Penrose, [1], showed locally, and globally subject to topological conditions, that a totally symmetric spinor, $\Xi_{ABCD..P}$, satisfies the spin $s$ zero rest-mass field equation in Minkowski space-time

$$\partial_{AA'}\Xi_{BCD..P} = 0 \quad (50)$$

if and only if, for any choice of a constant spinor $\kappa^{A'}$, there exists a complex function $\xi$ such that

$$\Xi_{ABCD..P} = \kappa^{A'}\kappa^{B'}\kappa^{C'}\kappa^{D'}..\partial_{AA'}\partial_{BB'}\partial_{CC'}\partial_{DD'}...\partial_{PP'}\xi \quad (51)$$

where $\xi$ satisfies the wave equation,

$$\Box \xi = 0. \quad (52)$$

When $s = 2$, with $\Xi_{ABCD} = \Psi_{linABCD}$, Eq.(50) corresponds to the linearized second Bianchi identity when the linearized Einstein vacuum field equations are satisfied. In this case, with $\kappa^{A'} = \sigma^{A'}$, Eqs.(51) and (52) become

$$\Psi_{linABCD} = \partial_{A}\partial_{B}\partial_{C}\partial_{D}\psi_{lin}, \quad (53)$$

$$\Box \psi_{lin} = 0.$$ 

By using these results and the expression $\Psi_{linABCD} = \partial_{A}\partial_{B}\partial_{C}\partial_{D}\psi_{lin}$ in the second linearized Cartan equation of Eq.(13), the corresponding linearized anti self-dual connection one-form can be shown to be

$$\omega_{linAB} = \partial_{A}\partial_{B}\partial_{C}\partial_{CC'}\sigma_{C'} + d\alpha_{AB} \quad (54)$$

Here $\alpha_{AB}$ are arbitrary functions which can be removed by using the linearized gauge transformation Eq.(18). It then follows, by using this result and its complex conjugate in the first linearized Cartan equation, Eq.(13), that

$$\theta_{lin}^{AA'} = (\delta_{B}^{A}\delta_{B'}^{A'} + \sigma^{A}_{B'}\overline{\psi}_{B'}^{A'} + \sigma^{A'}\overline{\sigma}_{B'}\overline{\psi}_{B}^{A} + \partial_{BB'}\beta^{AA'})dx^{BB'}, \quad (55)$$

$$\psi_{AB} = \partial_{A}\partial_{B}\psi; \quad \overline{\psi}_{A'B'}^{A'} = \partial_{A'}\partial_{B'}\overline{\psi},$$
where $\beta^{AA'}$ are arbitrary functions which can be set equal to zero by using the linearized diffeomorphisms of Eq.(16).

Hence Penrose’s result implies that the linearized Einstein vacuum field equations are satisfied by the linearized metric in Eq.(45) when $\sigma = 0$ and the complex function $\psi$ satisfies the Minkowski space-time wave equation. The linearized vacuum solutions are then given by

$$ds^2_{\text{lin}} = (\epsilon_{AB}\epsilon_{A'B'} + 2o_Ao_B\partial_A\partial_B\overline{\psi} + 2\overline{\sigma}_{AB}\partial_A\partial_B\psi)dx^{AA'} \otimes dx^{BB'}, \quad (56)$$

$$\Box \psi = \partial^{AA'}\partial_{AA'}\psi = 0.$$

Similar linearized solutions were identified by Jerzy Plebański and Ivor Robinson working in the complex domain [6]. A useful discussion of other work on Hertz potentials is included in [7].

Finally in this section it should be noted that, in the factorization, considered in the previous section, $\alpha + \overline{\alpha} = 1 + \sigma$ and $\varphi = \psi$, when only zeroth and first order terms are retained in these expressions.

## 7 Complex and real solutions

Holomorphic 4-metrics on complex four dimensional manifolds have been extensively investigated, particularly in the context of half-flat metrics as in the approach of Newman [9], the use of twistors [10] and the work of Plebański [3]. A selection of reviews of this research can be found in [11], [12], [13]. The aim of this section is to discuss the relationship between complex and real solutions of Cartan’s and Einstein’s equations using work on holomorphic metrics, spinorial coordinates, and the results of Sec. 5.

As far as this paper is concerned certain complex solutions of Cartan’s structure equations on a real four dimensional manifold can be simply obtained by re-interpreting formulae obtained by Plebański and co-workers in their research on holomorphic half-flat metrics [3, 6, 4]. In particular certain complex anti self-dual (or self-dual) solutions to Eqs.(2-4) and Eqs.(7-8) can be so obtained. These solutions are given, in spinorial coordinates $x^{AA'}$, by the complex one-forms $\chi^{AA'} = \chi_{BB'}dx^{BB'}$, constructed using Eq.(37) but with $\alpha = 0$, $\overline{\varphi}_{A'B'} = 0$ and $\varphi_{AB} = \psi_{AB} = \partial_A\partial_B\psi$. These complex one-forms satisfy the first set of Cartan’s equations, Eq.(2), when $\psi$ satisfies the

\footnote{Subsequently it was realized that research on Wave Geometry in Hiroshima in the 1930’s predates some of this work. It is reviewed in [8].}
The corresponding spinorial quantities $h_{QP'DQ'}$ and $\overline{r}_{QP'DQ'}$ of Sec. (5) are no longer complex conjugates and are given by

$$h_{QP'DQ'} = \epsilon_{QD}\epsilon_{P'Q'} + 2\overline{\sigma}_{P'}\overline{\sigma}_{Q'}\partial_{A}\partial_{B}\psi,$$

$$\overline{r}_{QP'DQ'} = \epsilon_{QD}\epsilon_{P'Q'}.$$

The relationship, mentioned above, of these equations to those formulated in the holomorphic context is the following. Introducing certain complex coordinates, denoted here as complex spinorial coordinates $z^{AA'}$, Plebański showed that all half-flat holomorphic four-metrics on complex four dimensional manifolds could be locally expressed as

$$ds^2 = h_{AA'BB'} dz^{AA'} \otimes dz^{BB'},$$

where the holomorphic metric components $h_{AA'BB'}$ are given by the holomorphic version of Eq.(43) with $\varphi_{AB} = \psi_{AB} = \overline{\sigma}^{A'}\overline{\sigma}^{B'}\partial/z_{AA'}\partial/z_{BB'}\psi$. The holomorphic version of Eq.(57) is Plebański’s second heavenly equation \[3\]. If $z^{AA'} = x^{AA'} + iy^{AA'}$, with $x^{AA'}$ and $y^{AA'}$ the spinor correspondents of real coordinates $x^a$ and $y^a$, the pullbacks of Plebański’s holomorphic forms to the real four manifold $M$ given by $y^a = 0$ gives a class of anti self-dual solutions of the complex Cartan equations on $M$ (and similarly for self-dual solutions).

It has been demonstrated in other papers that, by using spinorial coordinates and the results of Sec.5, certain real solutions of Einstein’s vacuum field equations can be constructed \[14\], \[15\]. These satisfy the calculation-ally simplifying condition that $o^A$ is a principal spinor of $\psi_{AB}$. When anti self-dual solutions, satisfying both this condition and Eq.(57) are combined
with their complex conjugates, using the co-frame $\chi^{AA'}$ and its complex con-
jugate as in Eqs.(36) and (37) in Sec.5, Ricci flat Lorentzian metrics result. 
A co-frame for these real metrics is given by Eq.(33), or equivalently the 
combination of self-dual and anti self-dual expressions in Eqs.(36) and (37), 
with $\varphi_{AB} = \psi_{AB}$ and $\sigma = 0$. The Lorentzian line elements of these solu-
tions are given by Eq.(27), with $\sigma = 0$ and $\psi_{AB} = \partial_A \partial_B \psi$ (plus its complex 
conjugate). These vacuum solutions are Petrov type III or N.

Can further interesting Lorentzian metrics be constructed by combi-
ning complex solutions, either by using spinorial coordinates and the approach of 
Sec.5 or in some other way? To date this question has received only limited 
and partial answers. A discussion of some other answers can be found in 
[16].

In conclusion it should be noted that spinorial coordinates may have other 
uses. For instance they may be a useful tool in the analysis of asymptotically 
flat metrics and radiating systems.

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