Interaction-around-a-face and consistency-around-a-face-centered-cube

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Abstract

There is a correspondence between integrable lattice models of statistical mechanics, and discrete integrable equations which satisfy multi-dimensional consistency, where the latter may be found in a quasi-classical expansion of the former. This paper extends this correspondence to interaction-around-a-face (IRF) models, resulting in a new formulation of the consistency-around-a-cube (CAC) integrability condition applicable to 5-point lattice equations. For the latter equations, multi-dimensional consistency is formulated as consistency-around-a-face-centered-cube (CAFCC), which namely involves satisfying an over-determined system of 14 5-point lattice equations for 8 unknown variables, on the face-centered cubic unit cell. From the quasi-classical limit of IRF models, which are constructed from the continuous spin solutions of the star-triangle relations associated to the Adler-Bobenko-Suris (ABS) list, 15 sets of equations are obtained which satisfy CAFCC.

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1
1 Introduction

The property of multi-dimensional consistency, or consistency-around-a-cube (CAC), is widely accepted as a definition for integrability of partial difference equations defined on faces of the square lattice (e.g., [1, Ch. 3] and references therein). This property essentially implies that the equations may be consistently extended into 2-dimensional sublattices of n-dimensional lattices depending on n associated lattice parameters, which may be regarded as the discrete analogue of the existence of hierarchies of compatible equations that are found for integrable partial differential equations. Perhaps the most interesting result following the emergence of CAC as an integrability condition, was a classification of scalar CAC equations given by Adler, Bobenko, and Suris (ABS) [2,3], now commonly referred to as the ABS list.

In recent works [4–7], it has been found how integrable lattice (or quad) equations in the latter ABS list, arise as part of a more general integrable structure based on a special form of the Yang-Baxter equation (YBE), known as the star-triangle relation (STR). Basically, the ABS equations are equivalent to the equations for the critical points in an asymptotic, or quasi-classical expansion of the STR. Through this connection, the STR’s themselves may be naturally interpreted as quantum counterparts (in a path-integral sense) of the equations in the
ABS list, and the entire ABS list may be systematically generated through the degenerations and quasi-classical expansions of the STR’s [7].

Motivated by such connections, this paper considers the classical equations that arise from interaction-round-a-face (IRF) models of statistical mechanics. The form of the IRF Yang-Baxter equation will motivate a new formulation of multi-dimensional consistency, applicable to 5-point equations coming from the classical limit of the IRF Boltzmann weights. Such 5-point equations can be interpreted as equations defined on a face of the face-centered cube\(^1\) (which will be a central idea for their multi-dimensional consistency), as shown below, and will be referred to in this paper as face-centered quad equations, to distinguish them from the regular quad equations which satisfy CAC.

\[\begin{array}{cc}
\alpha & \beta \\
\alpha & \beta \\
\end{array}\]

The face-centered quad equations are expressed in terms of polynomials in five variables \(x, x_a, x_b, x_c, x_d\), with linear dependence on the four corner variables \(x_a, x_b, x_c, x_d\), respectively (i.e. they are affine-linear in \(x_a, x_b, x_c, x_d\)). They also depend on the two 2-component parameters \(\alpha = \{\alpha_1, \alpha_2\}, \beta = \{\beta_1, \beta_2\}\), assigned to the edges. Such face-centered quad equations may be written in the following form

\[\sum_{i=0}^{n} P_i(x_a, x_b, x_c, x_d; \alpha, \beta)x^i = 0,\]

where under a parameter specialisation, the coefficients \(P_i(x_a, x_b, x_c, x_d; \alpha, \beta), i = 0, \ldots, n\), are the usual expressions for affine-linear quad polynomials in the variables \(x_a, x_b, x_c, x_d\).

The above face-centered quad equations cannot satisfy a usual form of multi-dimensional consistency formulated as CAC, particularly due to the appearance of the additional variable \(x\) associated to the face vertex, which should also be involved in determining the consistency of the equations. To address this, an analogue of the property of CAC will be formulated as consistency-around-a-face-centered-cube (CAFCC). For CAFCC, the six face-centered quad equations are defined on faces of the face-centered cube, with the three parameters \(\alpha, \beta, \gamma\), associated to three orthogonal lattice directions, analogously to CAC. However, in order to properly define an evolution around the face-centered cube, 8 additional equations centered at corners are needed, which will provide the relations between the different face variables. Then setting 6 initial variables on the face-centered cube, the CAFCC property is satisfied.

\(^1\)Throughout this paper, face-centered cube refers to the face-centered cubic unit cell.
if the 14 face-centered quad equations give consistent solutions for the remaining 8 variables. In comparison, CAC involves consistency of 6 quad equations on the cube for 4 unknown variables, so there is an increase in complexity in solving for CAFCC.

A main result of this paper, is the derivation of 15 explicit sets of equations which satisfy CAFCC, from new types of IRF Boltzmann weights associated to continuous spin solutions of the STR. The equations are grouped as one of types-A, -B, or -C, where the type-A and -B equations are centered at faces, and the type-C equations are centered at corners, of the face centered-cube respectively. The type-A equations satisfy CAFCC on their own, as well as in combination with the type-B and -C equations. The expressions for the type-A equations have also previously appeared as expressions for discrete Toda-type (or Laplace-type) equations, associated to type-Q equations in the ABS list [8–10], while some degenerate cases of the type-B equations may be identified with equations from the $H^6$ list of Boll [11]. For type-A and type-C equations, the ABS equations also appear as a parameter specialisation of the coefficient $P_1(x_a, x_b, x_c, x_d; \alpha, \beta)$, of their affine-linear forms. The latter connections to ABS equations should not be too surprising, since both ABS equations and face-centered quad equations are constructed from the same Boltzmann weights/Lagrangian functions coming from the star-triangle relations.

The paper is organised as follows. In Section 2, new expressions for the IRF forms of the YBE will be given, which are derived from the continuous-spin solutions of the STR’s. In Section 3, the face-centered quad equations and CAFCC property are introduced, where the latter is also depicted graphically in Appendix B. Note that Section 3 is more or less self-contained, describing the formulation of the face-centered quad equations and CAFCC property, without requiring knowledge of the connection to IRF models of statistical mechanics from Section 2. Section 3.3 contains a list of equations which satisfy CAFCC, which also are given in affine-linear form in Appendix A. In Section 4, it is shown how to derive the face-centered quad equations from the IRF equations of Section 2, with the use of explicit solutions of the STR’s given in Appendices D, and E.

2 Star-triangle relations to IRF Yang-Baxter equations

In this section, it will be shown how new expressions for the interaction-round-a-face (IRF) form of the Yang-Baxter equation, may be derived from continuous spin solutions of the star-triangle relations. The form of the classical IRF equations, will motivate the formulation of the new multi-dimensional consistency condition for 5-point lattice equations, in Section 3.

2.1 Boltzmann weights and star-triangle relations

The star-triangle relation is a particular form of the Yang-Baxter equation, for integrability of Ising-type lattice models of statistical mechanics [12]. The star-triangle relation is written in terms of Boltzmann weights, which for this paper will be denoted by

$$W_{p-q}(\sigma_a, \sigma_b), \quad W_{p-q}(\sigma_a, \sigma_b), \quad V_{p-q}(\sigma_a, \sigma_b), \quad V_{p-q}(\sigma_a, \sigma_b),$$

and depend on the two spin variables $\sigma_a, \sigma_b$, and the difference of two rapidity variables, $p - q$. For the considerations here, the Boltzmann weights (1) can be assumed to be complex-
valued, however for some special cases they can be positive-valued, depending on their explicit expressions and the chosen values of the spin variables. The Boltzmann weights (1) are assigned to edges \((ab)\) of a graph, which connect two vertices \(a, b\), and such a graphical representation of these Boltzmann weights is shown in Figure 1.

Here the two Boltzmann weights \(W_{p-q}(\sigma_a, \sigma_b)\), \(\overline{W}_{p-q}(\sigma_a, \sigma_b)\), will be assumed to satisfy
\[
W_{p-q}(\sigma_a, \sigma_b) = W_{p-q}(\sigma_b, \sigma_a), \quad \overline{W}_{p-q}(\sigma_a, \sigma_b) = \overline{W}_{p-q}(\sigma_b, \sigma_a),
\]
\[
W_{p-q}(\sigma_a, \sigma_b)\overline{W}_{q-p}(\sigma_a, \sigma_b) = 1,
\]
while the Boltzmann weights \(V_{p-q}(\sigma_a, \sigma_b)\), and \(\overline{V}_{p-q}(\sigma_a, \sigma_b)\), aren’t assumed to satisfy such symmetries, unless otherwise stated.

Two more Boltzmann weights will be denoted by
\[
\hat{W}_{p-q}(\sigma_a, \sigma_b), \quad \hat{\overline{W}}_{p-q}(\sigma_a, \sigma_b),
\]
which differ from \(W_{p-q}(\sigma_a, \sigma_b)\), and \(\overline{W}_{p-q}(\sigma_a, \sigma_b)\) respectively, but also satisfy the respective symmetries (2).

Finally, there is also a Boltzmann weight denoted by
\[
S(\sigma_a),
\]
(as well as its hatted counterpart \(\hat{S}(\sigma_a)\)) which depends on a single spin variable \(\sigma_a\), and is independent of any rapidity variables. The Boltzmann weight (4) may be associated to a vertex \(a\), for the spin variable \(\sigma_a\).

Note that in the expressions for the star-triangle relations and related integral formulas that will follow, the choice of integration contour depends on the explicit form of the Boltzmann weights, as well as the values of the spin variables. For explicit cases, there is typically a choice of spin variables for which the integration can be taken over a subset of \(\mathbb{R}\), with the most common case being an integration over \(\mathbb{R}\) itself. Then if desired, the variables of the star-triangle relations as well as the contour of integration may be analytically continued into the
complex plane. Such an analytic continuation of the continuous spin star-triangle relations, and their interpretation as hypergeometric integrals, was previously considered in [7].

2.1.1 Star-triangle relations

As was described in [13], the continuous spin star-triangle relations can be considered as particular cases of the following star-triangle relations

\[
\int d\sigma_d S(\sigma_d) \nabla_{p-r}(\sigma_d, \sigma_a) V_{p-q}(\sigma_d, \sigma_b) \nabla_{p-r}(\sigma_c, \sigma_a) = V_{q-r}(\sigma_c, \sigma_b),
\]

(5)

\[
\int d\sigma_d \hat{S}(\sigma_d) \nabla_{p-r}(\sigma_a, \sigma_d) V_{p-r}(\sigma_b, \sigma_d) \nabla_{p-r}(\sigma_c, \sigma_d) = \hat{V}_{q-r}(\sigma_c, \sigma_b),
\]

(6)

where the \( W_{p-q}(\sigma_a, \sigma_b) \), \( \nabla_{p-q}(\sigma_a, \sigma_b) \), \( \hat{W}_{p-q}(\sigma_a, \sigma_b) \), and \( \hat{\nabla}_{p-q}(\sigma_a, \sigma_b) \), are also assumed to satisfy the following two star-triangle relations

\[
\int d\sigma_d S(\sigma_d) \nabla_{p-r}(\sigma_a, \sigma_d) W_{p-r}(\sigma_b, \sigma_d) \nabla_{p-q}(\sigma_d, \sigma_c) = W_{q-r}(\sigma_c, \sigma_b),
\]

(7)

\[
\int d\sigma_d \hat{S}(\sigma_d) \nabla_{p-r}(\sigma_a, \sigma_d) \hat{W}_{p-r}(\sigma_b, \sigma_d) \nabla_{p-q}(\sigma_d, \sigma_c) = \hat{W}_{q-r}(\sigma_c, \sigma_b),
\]

(8)

Using the graphical representation of Boltzmann weights of Figure 1, the star-triangle relations (5), (6), are pictured graphically in Figure 2. The star-triangle relations (7), (8), have a similar graphical representation, but involving only the first two types of edges that appear on the left of Figure 1. In [13], the above were referred to as the non-symmetric mixed cases of the star-triangle relations. Essentially all equations considered in this paper can be obtained from the above four forms of the star-triangle relations (5)–(8).

For the particular case when the Boltzmann weights are symmetric, satisfying

\[
V_{p-q}(\sigma_a, \sigma_b) = V_{p-q}(\sigma_b, \sigma_a), \quad \nabla_{p-q}(\sigma_a, \sigma_b) = \nabla_{p-q}(\sigma_b, \sigma_a),
\]

(9)

the two star-triangle relations (5), (6), will have the same form, and may be written as a single star-triangle relation

\[
\int d\sigma_d S(\sigma_d) \nabla_{q-r}(\sigma_a, \sigma_d) V_{p-r}(\sigma_b, \sigma_d) \nabla_{p-r}(\sigma_c, \sigma_d) = V_{q-r}(\sigma_c, \sigma_b),
\]

(10)

with \( W_{p-q}(\sigma_a, \sigma_b) \), \( \nabla_{p-q}(\sigma_a, \sigma_b) \), also still satisfying the star-triangle relation (7). In [13], the star-triangle relations (10), and (7), were referred to as the symmetric mixed case, and symmetric case, respectively.
\[ \begin{align*}
q - r & = p - q \\
p - r & = p - q \\
p - q & = p - q
\end{align*} \]

\[ \begin{align*}
q - r & = p - q \\
p - r & = p - q \\
p - q & = p - q
\end{align*} \]

Figure 2: Expressions for the star-triangle relations (5) (top), and (6) (bottom), or classical star-triangle relations (15) (top), and (16) (bottom), where for the latter \((p, q, r) \rightarrow (u, v, w)\). The filled vertex \(d\) corresponds to \(S(\sigma_d)\) (or \(C(x_d)\)), while the unfilled vertex corresponds to \(\hat{S}(\sigma_d)\) (or \(\hat{C}(x_d)\)). The vertices \(a, b, c\) could be either filled or unfilled, since any factors associated to these vertices would drop out of both sides of the star-triangle relations.

The explicit solutions of the above star-triangle relations given in Appendix D, are grouped into one of the four cases: elliptic, hyperbolic, rational, or algebraic. There is only a symmetric case of the star-triangle relation at the elliptic level, while there can be found all of the above-mentioned cases of the star-triangle relations at the hyperbolic level. However, due to some symmetry breaking that occurs when going to the rational and algebraic levels, not all of the symmetries of Boltzmann weights (2) are satisfied, and typically only one of the two star-triangle relations (5), (6), or (7), (8), are satisfied for the latter cases.

2.2 Lagrangian functions and classical star-triangle relations

For integrable quad equations, the counterparts of the Boltzmann weights in (1), are the Lagrangian functions, which are denoted here by

\[ L_{u-v}(x_a, x_b), \quad \overline{L}_{u-v}(x_a, x_b), \quad \Lambda_{u-v}(x_a, x_b), \quad \overline{\Lambda}_{u-v}(x_a, x_b). \] (11)

Similarly to Boltzmann weights, these Lagrangian functions are associated to edges of a graph which connect two vertices, and have the same graphical representation that is pictured in Figure 1. The variables \(x_a, x_b\) are the counterparts of the spin variables \(\sigma_a, \sigma_b\), and the parameters \(u, v\), are the counterparts of the rapidity variables \(p, q\). Similarly to the Boltzmann
weights (1), the Lagrangian functions (11) will generally be complex valued. These Lagrangian functions arise as the leading order asymptotic term in a quasi-classical expansion of the Boltzmann weights (1), the details of which have been given in previous works [4–7], and are not essential for the purposes of this paper.

The Lagrangian functions \( \mathcal{L}_{u-v}(x_a, x_b) \), and \( \overline{\mathcal{L}}_{u-v}(x_a, x_b) \), are assumed to satisfy the following analogues of (2)

\[
\begin{align*}
\mathcal{L}_{u-v}(x_a, x_b) &= \mathcal{L}_{u-v}(x_b, x_a), \\
\overline{\mathcal{L}}_{u-v}(x_a, x_b) &= \overline{\mathcal{L}}_{u-v}(x_b, x_a), \\
\mathcal{L}_{u-v}(x_a, x_b) + \mathcal{L}_{v-u}(x_a, x_b) &= 0.
\end{align*}
\]

(12)

The counterparts of the hatted Boltzmann weights (3), are denoted by

\[
\hat{\mathcal{L}}_{u-v}(x_a, x_b), \quad \hat{\overline{\mathcal{L}}}_{u-v}(x_a, x_b).
\]

(13)

where the latter are understood to be different from \( \mathcal{L}_{u-v}(x_a, x_b) \), and \( \overline{\mathcal{L}}_{u-v}(x_a, x_b) \), respectively, but satisfy the same symmetries given in (12).

Finally, the counterpart for the vertex Boltzmann weight (4), is a complex-valued Lagrangian function, denoted by

\[
C(x_a),
\]

(14)

along with the hatted counterpart \( \hat{C}(x_a) \).

### 2.2.1 Classical star-triangle relations

The counterparts of the star-triangle relations that appeared in the previous subsection, are the classical star-triangle relations, which are written in terms of sums of the Lagrangian functions. For the non-symmetric mixed cases of the star-triangle relations (5), (6), the classical counterparts are respectively given by

\[
\begin{align*}
C(x_d) + \overline{\Lambda}_{v-w}(x_d, x_a) + \Lambda_{u-w}(x_d, x_b) + \overline{\mathcal{L}}_{u-v}(x_c, x_d) &= \Lambda_{v-w}(x_c, x_b) + \overline{\Lambda}_{u-w}(x_c, x_a) + \hat{\mathcal{L}}_{u-v}(x_a, x_b), \\
\hat{C}(x_d) + \overline{\Lambda}_{v-w}(x_d, x_a) + \Lambda_{u-w}(x_d, x_b) + \overline{\mathcal{L}}_{u-v}(x_c, x_d) &= \Lambda_{v-w}(x_b, x_c) + \overline{\Lambda}_{u-w}(x_a, x_c) + \mathcal{L}_{u-v}(x_b, x_a),
\end{align*}
\]

(15)

(16)

where the variables for (15), and (16), are to satisfy the respective three-leg equations

\[
\begin{align*}
\frac{\partial}{\partial x} \left( \hat{C}(x) + \overline{\Lambda}_{v-w}(x_a, x) + \Lambda_{u-w}(x_b, x) + \overline{\mathcal{L}}_{u-v}(x_c, x) \right)_{x=x_d} &= 0, \\
\frac{\partial}{\partial x} \left( C(x) + \overline{\Lambda}_{v-w}(x_a, x) + \Lambda_{u-w}(x_b, x) + \overline{\mathcal{L}}_{u-v}(x_c, x) \right)_{x=x_d} &= 0.
\end{align*}
\]

(17)

(18)

The Lagrangian functions \( \mathcal{L}_{u-v}(x_a, x_b) \), and \( \overline{\mathcal{L}}_{u-v}(x_a, x_b) \), are to also satisfy the classical counterparts of the star-triangle relations (7), (8), given by

\[
\begin{align*}
C(x_d) + \overline{\mathcal{L}}_{v-w}(x_a, x_d) + \mathcal{L}_{u-w}(x_b, x_d) + \overline{\mathcal{L}}_{u-v}(x_d, x_c) &= \mathcal{L}_{v-w}(x_b, x_c) + \overline{\mathcal{L}}_{u-w}(x_a, x_c) + \mathcal{L}_{u-v}(x_b, x_a), \\
\hat{C}(x_d) + \overline{\mathcal{L}}_{v-w}(x_a, x_d) + \hat{\mathcal{L}}_{u-w}(x_b, x_d) + \overline{\mathcal{L}}_{u-v}(x_d, x_c) &= \hat{\mathcal{L}}_{v-w}(x_b, x_c) + \overline{\mathcal{L}}_{u-w}(x_a, x_c) + \hat{\mathcal{L}}_{u-v}(x_b, x_a),
\end{align*}
\]

(19)

(20)
where the variables for (19), and (20), satisfy the respective three-leg equations

\[
\frac{\partial}{\partial x} \left( C(x) + \mathcal{L}_{v-w}(x_a, x) + \mathcal{L}_{u-w}(x_b, x) + \mathcal{L}_{u-v}(x, x_c) \right)_{x=x_d} = 0, \quad (21)
\]

\[
\frac{\partial}{\partial x} \left( \hat{C}(x) + \hat{\mathcal{L}}_{v-w}(x_a, x) + \hat{\mathcal{L}}_{u-w}(x_b, x) + \hat{\mathcal{L}}_{u-v}(x, x_c) \right)_{x=x_d} = 0. \quad (22)
\]

The above expressions for the classical star-triangle relations (15), (16), have the same graphical representations that appear in Figure 2. Such classical star-triangle relations, may be found in the leading order quasi-classical expansion of the star-triangle relations of the form (5)-(8), where the three-leg equations are equivalent to the saddle-point equations. Classical star-triangle relations have also been studied for the ABS equations independently of the quasi-classical expansion [14,15].

If the Lagrangians are symmetric, satisfying

\[
\Lambda_{u-v}(x_a, x_b) = \Lambda_{u-v}(x_b, x_a), \quad \Lambda_{u-v}(x_a, x_b) = \Lambda_{u-v}(x_b, x_a), \quad (23)
\]

then the above star-triangle relations reduce to the following star-triangle relation for the symmetric mixed case

\[
C(x_d) + \Lambda_{v-w}(x_a, x_d) + \Lambda_{u-w}(x_b, x_d) + \mathcal{L}_{u-v}(x_d, x_c) = 0,
\]

\[
\Lambda_{v-w}(x_b, x_c) + \Lambda_{u-w}(x_a, x_c) + \mathcal{L}_{u-v}(x_b, x_a) = 0,
\]

where the variables satisfy the three-leg equation

\[
\frac{\partial}{\partial x} \left( C(x) + \Lambda_{v-w}(x_a, x) + \Lambda_{u-w}(x_b, x) + \mathcal{L}_{u-v}(x, x_c) \right)_{x=x_d} = 0. \quad (25)
\]

and the Lagrangian functions \( \mathcal{L}_{p-q}(x_a, x_b), \mathcal{L}_{p-q}(x_a, x_b) \), also satisfy the star-triangle relation (19) for the symmetric case.

### 2.3 Star-star relations

The above star-triangle relations imply another relation for integrability of lattice models of statistical mechanics, known as the star-star relation [12,16,17]. To give the expressions for these star-star relations, it is first useful to define four quantities in terms of the Boltzmann weights (1), and (4), as

\[
V_{pq}^{(1)}(\sigma_a \sigma_b \sigma_c \sigma_d) = \int d\sigma e S(\sigma) V_{p_2-q_1}(\sigma_a, \sigma_e) V_{p_1-q_2}(\sigma_b, \sigma_e) W_{p_1-q_1}(\sigma_c, \sigma_e) W_{p_2-q_1}(\sigma_d, \sigma_e),
\]

\[
V_{pq}^{(2)}(\sigma_a \sigma_b \sigma_c \sigma_d) = \int d\sigma f S(\sigma) V_{p_1-q_2}(\sigma_a, \sigma_f) W_{p_1-q_1}(\sigma_f, \sigma_b) V_{p_2-q_2}(\sigma_e, \sigma_f) W_{p_2-q_1}(\sigma_f, \sigma_d),
\]

\[
\hat{V}_{pq}^{(1)}(\sigma_a \sigma_b \sigma_c \sigma_d) = \int d\sigma e S(\sigma) \hat{W}_{q_1-p_2}(\sigma_e, \sigma_a) V_{q_2-p_2}(\sigma_e, \sigma_b) W_{p_1-q_1}(\sigma_c, \sigma_e) W_{p_1-q_2}(\sigma_d, \sigma_e),
\]

\[
\hat{V}_{pq}^{(2)}(\sigma_a \sigma_b \sigma_c \sigma_d) = \int d\sigma f S(\sigma) \hat{W}_{p_1-q_2}(\sigma_f, \sigma_a) \hat{W}_{p_1-q_1}(\sigma_f, \sigma_b) V_{q_2-p_2}(\sigma_e, \sigma_f) \hat{W}_{q_1-p_2}(\sigma_d, \sigma_f).
\]
On the left of the above expressions, the subscripts \( p, q \), are 2-component rapidity variables
\[
p = \{p_1, p_2\}, \quad q = \{q_1, q_2\}.
\] (27)

In terms of (26), there are two forms of the star-star relations which will be used, given by the following two expressions
\[
V_{q_1-q_2}(\sigma_a, \sigma_b)W_{p_1-p_2}(\sigma_a, \sigma_c)\hat{V}_{pq}^{(1)}(\sigma_a \sigma_b) = V_{q_1-q_2}(\sigma_c, \sigma_d)W_{p_1-p_2}(\sigma_b, \sigma_d)\hat{V}_{pq}^{(2)}(\sigma_a \sigma_b), \quad (28)
\]
\[
\nabla_{p_1-p_2}(\sigma_d, \sigma_b)W_{q_2-q_1}(\sigma_c, \sigma_d)\hat{V}_{pq}^{(1)}(\sigma_a \sigma_b) = \nabla_{p_1-p_2}(\sigma_c, \sigma_a)W_{q_2-q_1}(\sigma_a, \sigma_b)\hat{V}_{pq}^{(2)}(\sigma_a \sigma_b). \quad (29)
\]

The expressions (28), and (29), are in fact new forms of the star-star relations which have not been considered previously. Using the graphical representation of Boltzmann weights in Figure 1, the star-star relations (28), and (29) are pictured in Figure 3. The star-star relations (28), (29), may be straightforwardly derived through the use the star-triangle relations (5)–(8).

![Figure 3: Expressions for the star-star relations (28) (top), and (29) (bottom), or classical star-star relations (35) (top), and (36) (bottom), where for the latter \((p_1, p_2, q_1, q_2) \rightarrow (u_1, u_2, v_1, v_2)\).](image.png)

For the symmetric case of the star-triangle relation (7), the following substitutions
\[
V \rightarrow W, \quad \nabla \rightarrow \hat{W}, \quad S \rightarrow S, \quad \hat{W} \rightarrow W, \quad \hat{W} \rightarrow \hat{W},
\] (30)
should be made in the star-star relations (28), (29), which results in the usual form of the star-star relation [16, 17]
\[
W_{q_1-q_2}(\sigma_c, \sigma_a)W_{p_2-p_1}(\sigma_c, \sigma_d)\hat{W}_{pq}^{(1)}(\sigma_a \sigma_b) = W_{q_1-q_2}(\sigma_d, \sigma_b)W_{p_2-p_1}(\sigma_a, \sigma_b)\hat{W}_{pq}^{(2)}(\sigma_a \sigma_b), \quad (31)
\]
where
\[
W^{(1)}_{pq}(\sigma_a \sigma_b | \sigma_c \sigma_d) = \int d\sigma e S(\sigma e) \mathcal{W}_{p_1-q_2}(\sigma_e, \sigma_a) \mathcal{W}_{p_2-q_2}(\sigma_e, \sigma_b) \mathcal{W}_{p_1-q_1}(\sigma_c, \sigma_e) \mathcal{W}_{p_2-q_1}(\sigma_d, \sigma_e),
\]
\[
W^{(2)}_{pq}(\sigma_a \sigma_b | \sigma_c \sigma_d) = \int d\sigma f S(\sigma f) \mathcal{W}_{p_2-q_1}(\sigma_f, \sigma_a) \mathcal{W}_{p_1-q_1}(\sigma_f, \sigma_b) \mathcal{W}_{p_2-q_2}(\sigma_c, \sigma_f) \mathcal{W}_{p_1-q_2}(\sigma_d, \sigma_f).
\]

2.3.1 Classical star-star relations

The classical counterparts of (27), are the 2-component parameters
\[
\mathbf{u} = \{u_1, u_2\}, \quad \mathbf{v} = \{v_1, v_2\}.
\]

The expressions in (26), have the following Lagrangian counterparts
\[
\Lambda^{(1)}_{uv}(x_e | x_a x_b, x_c x_d) = C(x_e) + \mathcal{L}_{u_2-v_1}(x_a, x_e) + \Lambda_{u_2-v_2}(x_e, x_b) + \mathcal{L}_{u_1-v_1}(x_c, x_e) + \Lambda_{u_1-v_2}(x_e, x_d),
\]
\[
\Lambda^{(2)}_{uv}(x_f | x_a x_b, x_c x_d) = \dot{C}(x_f) + \Lambda_{u_1-v_2}(x_a, x_f) + \dot{\mathcal{L}}_{u_1-v_1}(x_f, x_b) + \Lambda_{u_2-v_2}(x_c, x_f) + \dot{\mathcal{L}}_{u_2-v_1}(x_f, x_d),
\]
\[
\hat{\Lambda}^{(1)}_{uv}(x_e | x_a x_b, x_c x_d) = C(x_e) + \Lambda_{v_1-u_2}(x_e, x_a) + \Lambda_{v_2-u_2}(x_e, x_b) + \mathcal{L}_{u_1-v_1}(x_c, x_e) + \Lambda_{u_1-v_2}(x_d, x_e),
\]
\[
\hat{\Lambda}^{(2)}_{uv}(x_f | x_a x_b, x_c x_d) = \dot{C}(x_f) + \dot{\Lambda}_{u_1-v_2}(x_f, x_a) + \dot{\mathcal{L}}_{u_1-v_1}(x_f, x_b) + \Lambda_{v_2-u_2}(x_c, x_f) + \dot{\mathcal{L}}_{u_1-u_2}(x_d, x_f).
\]

Note that for the classical expressions (34), there is a dependence on the face variables $x_e, x_f$ (which in the classical star-star relation will be fixed to a four-leg equation of motion), whereas in (26) the corresponding spin variables were the variables of integration.

In terms of (34), there are two expressions for the classical star-star relations
\[
\Lambda_{v_1-u_2}(x_a, x_b) + \mathcal{L}_{u_1-u_2}(x_a, x_c) + \Lambda^{(1)}_{uv}(x_e | x_a x_b, x_c x_d)
\]
\[
= \Lambda_{v_1-u_2}(x_c, x_d) + \mathcal{L}_{u_1-u_2}(x_b, x_d) + \Lambda^{(2)}_{uv}(x_f | x_a x_b, x_c x_d),
\]
\[
\Lambda_{u_1-u_2}(x_d, x_b) + \mathcal{L}_{v_2-v_1}(x_c, x_d) + \hat{\Lambda}^{(1)}_{uv}(x_e | x_a x_b, x_c x_d)
\]
\[
= \Lambda_{u_1-u_2}(x_c, x_a) + \mathcal{L}_{v_2-v_1}(x_a, x_b) + \hat{\Lambda}^{(2)}_{uv}(x_f | x_a x_b, x_c x_d),
\]

where the variables of (35), and (36), satisfy the respective four-leg equations
\[
\frac{\partial}{\partial x_e} \Lambda^{(1)}_{uv}(x_e | x_a x_b, x_c x_d) = 0, \quad \frac{\partial}{\partial x_f} \Lambda^{(2)}_{uv}(x_f | x_a x_b, x_c x_d) = 0,
\]
\[
\frac{\partial}{\partial x_e} \hat{\Lambda}^{(1)}_{uv}(x_e | x_a x_b, x_c x_d) = 0, \quad \frac{\partial}{\partial x_f} \hat{\Lambda}^{(2)}_{uv}(x_f | x_a x_b, x_c x_d) = 0.
\]
The classical star-star relations (35), (36), may be derived directly through the use of the classical star-triangle relations (15), (16), (19), (20), and they also arise as the leading order quasi-classical expansion of the star-star relations (28), (29).

For the symmetric case of the star-triangle relations (19), the following substitutions should be made in the classical star-star relations (35), (36),

\[ \Lambda \rightarrow \mathcal{L}, \quad \Lambda \rightarrow \mathcal{E}, \quad \hat{C} \rightarrow C, \quad \hat{\mathcal{L}} \rightarrow \mathcal{L}, \quad \hat{\mathcal{E}} \rightarrow \mathcal{E}, \]

which results in the following expression for the classical star-star relation

\[ \mathcal{L}_{v_1-u_2}(x_c, x_a) + \mathcal{L}_{u_2-u_1}(x_c, x_d) + \mathcal{L}_{uv}^{(1)}(x_e | x_a x_e x_c x_d) \]

\[ = \mathcal{L}_{v_1-v_2}(x_d, x_b) + \mathcal{L}_{u_2-u_1}(x_a, x_b) + \mathcal{L}_{uv}^{(2)}(x_f | x_a x_e x_c x_d), \]

where

\[ \mathcal{L}_{uv}^{(1)}(x_e | x_a x_d) = C(x_e) + \mathcal{L}_{u_1-u_2}(x_e, x_a) + \mathcal{L}_{u_2-u_1}(x_e, x_b) + \mathcal{L}_{u_1-v_1}(x_c, x_e) + \mathcal{L}_{u_2-v_1}(x_d, x_e), \]

\[ \mathcal{L}_{uv}^{(2)}(x_f | x_a x_d) = C(x_f) + \mathcal{L}_{u_2-v_1}(x_f, x_a) + \mathcal{L}_{u_1-v_1}(x_f, x_b) + \mathcal{L}_{u_2-v_2}(x_c, x_f) + \mathcal{L}_{u_1-v_2}(x_d, x_f), \]

and the variables of (40), satisfy the four-leg equation

\[ \frac{\partial}{\partial x_e} \mathcal{L}_{uv}^{(1)}(x_e | x_a x_e x_c x_d) = 0, \quad \frac{\partial}{\partial x_f} \mathcal{L}_{uv}^{(2)}(x_f | x_a x_e x_c x_d) = 0. \]

The classical star-star relation (40) may be obtained through the use of the classical star-triangle relation (19), or in the quasi-classical expansion of the star-star relation (31).

2.4 IRF Yang-Baxter equations

The above star-star relations will be seen to imply a form of the Yang-Baxter equation for IRF Boltzmann weights and associated classical Lagrangian functions. Here the 2-component parameters are given by

\[ p = \{p_1, p_2\}, \quad q = \{q_1, q_2\}, \quad r = \{r_1, r_2\}. \]

First, the following four quantities are defined in terms of the Boltzmann weights (1), (4),

\[ W_{pq}(\sigma_a \sigma_b \sigma_c \sigma_d) = \int d\sigma_e S(\sigma_e)W_{p_2-q_1}(\sigma_a, \sigma_e)\Omega_{p_2-q_2}(\sigma_e)\Omega_{p_1-q_1}(\sigma_c, \sigma_e)\Omega_{p_1-q_2}(\sigma_d, \sigma_e), \]

\[ \hat{W}_{pq}(\sigma_a \sigma_b \sigma_c \sigma_d) = \int d\sigma_e \hat{S}(\sigma_e)W_{p_2-q_1}(\sigma_a, \sigma_e)\Omega_{p_2-q_2}(\sigma_e)\Omega_{p_1-q_1}(\sigma_c, \sigma_e)\Omega_{p_1-q_2}(\sigma_d, \sigma_e), \]

\[ V_{pq}(\sigma_a \sigma_b \sigma_c \sigma_d) = \int d\sigma_e \hat{V}(\sigma_e)V_{p_2-q_1}(\sigma_a, \sigma_e)\Omega_{p_2-q_2}(\sigma_e)\Omega_{p_1-q_1}(\sigma_c, \sigma_e)\Omega_{p_1-q_2}(\sigma_d, \sigma_e), \]

\[ \hat{V}_{pq}(\sigma_a \sigma_b \sigma_c \sigma_d) = \int d\sigma_e S(\sigma_e)V_{p_2-q_1}(\sigma_a, \sigma_e)\Omega_{p_2-q_2}(\sigma_e)\Omega_{p_1-q_1}(\sigma_c, \sigma_e)\Omega_{p_1-q_2}(\sigma_d, \sigma_e). \]
The above expressions may be interpreted as IRF Boltzmann weights, which relate the four spin variables $\sigma_a, \sigma_b, \sigma_c, \sigma_d$, associated to the four vertices of a square. Such Boltzmann weights may be translated in orthogonal lattice directions to define an IRF model of statistical mechanics \cite{12, 17}. In terms of Figure 1, the IRF Boltzmann weights (44), and (46), are depicted in Figure 4. The IRF Boltzmann weight (45) has the same graphical representation as (44), but with an unfilled vertex $e$, while (47) should have the orientation of each arrow reversed in comparison to (46), along with a filled vertex $e$.

![Figure 4: The IRF Boltzmann weights](image)

To give an expression for the IRF Yang-Baxter equation, the following slightly different forms of the IRF Boltzmann weights (44), and (45), will be defined

\[
\prod_{pq} \left( \frac{\sigma_a \sigma_b}{\sigma_c \sigma_d} \right) = W_{q_1 - q_2}(\sigma_a, \sigma_b) W_{q_2 - q_1}(\sigma_c, \sigma_d) W_{pq} \left( \frac{\sigma_a \sigma_b}{\sigma_c \sigma_d} \right), \quad (48)
\]

\[
\prod_{pq} \left( \frac{\sigma_a \sigma_b}{\sigma_c \sigma_d} \right) = \dot{W}_{q_1 - q_2}(\sigma_a, \sigma_b) \dot{W}_{q_2 - q_1}(\sigma_c, \sigma_d) \dot{W}_{pq} \left( \frac{\sigma_a \sigma_b}{\sigma_c \sigma_d} \right). \quad (49)
\]

In terms of (46), and (48), the first expression for the IRF YBE, for the non-symmetric mixed case, is given by

\[
\int d\sigma_h \dot{W}_{pq} \left( \frac{\sigma_f \sigma_h}{\sigma_a \sigma_b} \right) V_{pr} \left( \frac{\sigma_h \sigma_d}{\sigma_b \sigma_c} \right) V_{qr} \left( \frac{\sigma_f \sigma_e}{\sigma_h \sigma_d} \right) = \int d\sigma_h V_{qr} \left( \frac{\sigma_a \sigma_h'}{\sigma_b \sigma_c} \right) V_{pr} \left( \frac{\sigma_f \sigma_e'}{\sigma_h \sigma_d} \right) \dot{W}_{pq} \left( \frac{\sigma_e \sigma_d}{\sigma_h' \sigma_c} \right). \quad (50)
\]

while in terms of (47), and (49), the second expression for the IRF YBE for the non-symmetric mixed case, is given by

\[
\int d\sigma_h \dot{W}_{pq} \left( \frac{\sigma_f \sigma_h}{\sigma_a \sigma_b} \right) V_{pr} \left( \frac{\sigma_h \sigma_d}{\sigma_b \sigma_c} \right) V_{qr} \left( \frac{\sigma_f \sigma_e}{\sigma_h \sigma_d} \right) = \int d\sigma_h' V_{qr} \left( \frac{\sigma_a \sigma_h'}{\sigma_b \sigma_c} \right) V_{pr} \left( \frac{\sigma_f \sigma_e'}{\sigma_h' \sigma_d} \right) \dot{W}_{pq} \left( \frac{\sigma_e \sigma_d}{\sigma_h' \sigma_c} \right). \quad (51)
\]

Equations (50), (51), are new expressions for the Yang-Baxter equations which have not appeared before. The first of the above expressions (50), has the graphical representation shown in Figure 5, while the second expression (51) has the same graphical representation, but with all the arrows having the reverse orientation, and the filled and unfilled vertices being exchanged. The Yang-Baxter equation (50) may be derived with the use of the star-star relations (28), (29), through the sequence of deformations which appear in Figure 6, while the Yang-Baxter equation (51) follows from a similar sequence of transformations.
Figure 5: The Yang-Baxter equation (50), or classical Yang-Baxter equation (61).

Figure 6: Deformations of Yang-Baxter equation of Figure 5, with the use of the star-star relations (28), (29), or classical star-star relations (35), (36), of Figure 3.
In addition to (50), (51), the IRF Boltzmann weights given in (44), (48), can be shown to satisfy the following IRF YBE for the symmetric case

\[
\int d\sigma_h \overline{W}_{pq}(\sigma_f \sigma_h) W_{pr}(\sigma_h \sigma_d) W_{qr}(\sigma_f \sigma_e) = \int d\sigma'_h \overline{W}_{qr}(\sigma_a \sigma'_h) W_{pr}(\sigma_f \sigma_e) \overline{W}_{pq}(\sigma_e \sigma_d),
\]

and similarly, the IRF Boltzmann weights (45), (49), will satisfy

\[
\int d\sigma_h \overline{W}_{pq}(\sigma_f \sigma_h) \hat{W}_{pr}(\sigma_h \sigma_d) \hat{W}_{qr}(\sigma_f \sigma_e) = \int d\sigma'_h \overline{W}_{qr}(\sigma_a \sigma'_h) \hat{W}_{pr}(\sigma_f \sigma_e) \overline{W}_{pq}(\sigma_e \sigma_d).
\]

The equations (52), (53), are the more usual expressions for the IRF Yang-Baxter equation constructed from edge Boltzmann weights [16, 17]. The Yang-Baxter equations (52), (53) follow from analogous sequences of transformations shown in Figure 6, with the appropriate form of the star-star relation given in (31).

### 2.4.1 Classical IRF Yang-Baxter equations

As expected, there are Lagrangian counterparts of each of the expressions of the previous subsection, resulting in classical forms of the IRF Yang-Baxter equations.

The IRF Lagrangian functions corresponding to (44)-(47), are given by the expressions

\[
\mathcal{L}_{uv}(x_e | x_a x_b, x_c x_d) = C(x_e) + \mathcal{L}_{u_2-v_1}(x_a, x_e) + \mathcal{L}_{u_2-v_2}(x_b, x_e) + \mathcal{L}_{u_1-v_1}(x_c, x_e) + \mathcal{L}_{u_1-v_2}(x_d, x_e),
\]

\[
\hat{\mathcal{L}}_{uv}(x_e | x_a x_b, x_c x_d) = \hat{C}(x_e) + \hat{\mathcal{L}}_{u_2-v_1}(x_a, x_e) + \hat{\mathcal{L}}_{u_2-v_2}(x_b, x_e) + \hat{\mathcal{L}}_{u_1-v_1}(x_c, x_e) + \hat{\mathcal{L}}_{u_1-v_2}(x_d, x_e),
\]

\[
\Lambda_{uv}(x_e | x_a x_b, x_c x_d) = \hat{C}(x_e) + \Lambda_{u_2-v_1}(x_a, x_e) + \Lambda_{u_2-v_2}(x_b, x_e) + \Lambda_{u_1-v_1}(x_c, x_e) + \Lambda_{u_1-v_2}(x_d, x_e),
\]

\[
\hat{\Lambda}_{uv}(x_e | x_a x_b, x_c x_d) = C(x_e) + \Lambda_{u_2-v_1}(x_a, x_e) + \Lambda_{u_2-v_2}(x_b, x_e) + \Lambda_{u_1-v_1}(x_c, x_e) + \Lambda_{u_1-v_2}(x_d, x_e),
\]

and the counterparts of (48), (49), are given by

\[
\mathcal{L}_{uv}(x_e | x_a x_b, x_c x_d) = \mathcal{L}_{v_1-v_2}(x_a, x_b) + \mathcal{L}_{v_2-v_1}(x_c, x_d) + \mathcal{L}_{uv}(x_e | x_a x_b, x_c x_d),
\]

\[
\hat{\mathcal{L}}_{uv}(x_e | x_a x_b, x_c x_d) = \hat{\mathcal{L}}_{v_1-v_2}(x_a, x_b) + \hat{\mathcal{L}}_{v_2-v_1}(x_c, x_d) + \hat{\mathcal{L}}_{uv}(x_e | x_a x_b, x_c x_d).
\]

It is also useful to introduce the following additional variables, which are associated to the interior vertices in the Yang-Baxter equation,

\[
x_h, x_i, x_j, x_k, \quad x'_h, x'_i, x'_j, x'_k.
\]
Specifically, the four variables \( x_h, x_i, x_j, x_k \), are associated to interior vertices \( h, i, j, k \), on the left hand side of Figure 5, and the four variables \( x'_h, x'_i, x'_j, x'_k \), are associated to interior vertices \( h', i', j', k' \), on the right hand side of Figure 5. Note that for the case of the star-star relations (50)–(53), it was not necessary to keep track of the corresponding spin variables at these vertices, which were the variables of integration.

In terms of the IRF Lagrangian functions (54)–(59), the first expression for the classical IRF YBE for the non-symmetric mixed case is given by

\[
\mathcal{L}_{uv}(x_i | x_f x_h x_a x_b) + \Lambda_{uw}(x_j | x_h x_d x_b x_c) + \Lambda_{vw}(x_k | x_f x_e x_h x_d) = 0, \quad (61)
\]

and the second expression for the classical IRF YBE for the non-symmetric mixed case is given by

\[
\mathcal{L}_{uv}(x_i | x_f x_h x_a x_b) + \hat{\Lambda}_{uw}(x_j | x_h x_d x_b x_c) + \hat{\Lambda}_{vw}(x_k | x_f x_e x_h x_d) = 0, \quad (62)
\]

For (61) the variables on the left and right hand sides are required to satisfy the following 8 equations

\[
\frac{\partial}{\partial x_i} \mathcal{L}_{uv}(x_i | x_f x_h x_a x_b) = 0, \quad \frac{\partial}{\partial x_j} \Lambda_{uw}(x_j | x_h x_d x_b x_c) = 0, \quad \frac{\partial}{\partial x_k} \Lambda_{vw}(x_k | x_f x_e x_h x_d) = 0,
\]

\[
\frac{\partial}{\partial x'_j} \mathcal{L}_{uv}(x_i | x_f x_h x_a x_b) = 0, \quad \frac{\partial}{\partial x'_k} \Lambda_{uw}(x_j | x_h x_d x_b x_c) = 0, \quad \frac{\partial}{\partial x'_l} \Lambda_{vw}(x_k | x_f x_e x_h x_d) = 0,
\]

\[
\frac{\partial}{\partial x_h} \Lambda^{(1)}_{\phi_1 \phi_2}(x_h | x_k x_j x_f x_i) = 0, \quad \frac{\partial}{\partial x'_h} \Lambda^{(1)}_{\phi_1 \phi_2}(x'_h | x'_k x'_j x'_f x'_i) = 0,
\]

where in the last two equations \( \theta_1 = \{u_2, v_1\} \), \( \theta_2 = \{v_2, w_1\} \), and \( \phi_1 = \{v_2, u_1\} \), \( \phi_2 = \{w_2, v_1\} \). These 8 equations essentially mean that the derivatives of the classical IRF YBE (61), with respect to the 8 variables (60), associated to the 8 interior vertices \( h, i, j, k, h', i', j', k' \) of Figure 5, should vanish, and there are a similar set of 8 equations to be satisfied for (36). The first of the above expressions for the classical IRF YBE (61), has the same graphical representation that appeared in Figure 5, and it may be derived with the use of the classical star-star relations (35), (36), through the same sequence of deformations which appear in Figure 6.

Note also that the IRF Lagrangian functions (54), (58), satisfy the following form of the classical IRF YBE

\[
\mathcal{L}_{uv}(x_i | x_f x_h x_a x_b) + \mathcal{L}_{uw}(x_j | x_h x_d x_b x_c) + \mathcal{L}_{vw}(x_k | x_f x_e x_h x_d) = 0, \quad (64)
\]
while (55), (59), satisfy
\[
\hat{\mathcal{L}}_{uv}(x_i \mid x_f x_h x_a x_b) + \hat{\mathcal{L}}_{uw}(x_j \mid x_h x_d x_b x_c) + \hat{\mathcal{L}}_{vw}(x_k \mid x_f x_e x_h x_d) = \hat{\mathcal{L}}_{vw}(x' \mid x' a x' h x' c),
\]
with similar equations to (63), to be satisfied for the variables of (64), and (65), respectively, i.e., the derivatives of (64), (65), with respect to each of the 8 variables (60), associated to the 8 interior vertices, must vanish.

The equations in (63), are known in the integrable systems literature as discrete Laplace-type equations \[18\]. There are also six other discrete Laplace-type equations that will be satisfied as a consequence of the Yang-Baxter equation (61), which are obtained by taking the derivatives on both sides with respect to each of the six variables \(x_a, x_b, x_c, x_d, x_e, x_f\). This gives a total of 14 discrete Laplace-type equations which should be satisfied for the derivatives of the Yang-Baxter equation (61). The 14 equations are given explicitly in terms of Lagrangian functions in Appendix C, and will be utilised in the coming sections in the derivation of the multi-dimensionally consistent 5-point lattice equations.

3 Face-centered quad equations and CAFCC

A main goal of this paper, is to show how the classical equations coming from the IRF equations of Section 2, may be interpreted in terms of lattice equations which are multi-dimensionally consistent. For this purpose, the new concept of face-centered quad equations will be introduced here, which are basically 5-point evolution equations in the square lattice. The multi-dimensional consistency for face-centered quad equations is proposed as consistency-around-a-face-centered-cube (CAFCC), which namely involves satisfying an over-determined system of 14 face-centered quad equations for 8 unknown variables, on the face-centered cube. The CAFCC condition for the face-centered quad equations, may be regarded as the analogue for the consistency-around-the-cube (CAC) integrability condition for the usual quad equations.

3.1 Face-centered quad equations

Consider a face of the face-centered cube, as is shown in Figure 7, where the four variables \(x_a, x_b, x_c, x_d\), are associated to the four corner vertices, and the variable \(x\) is associated to the face vertex.\(^2\) The two 2-component parameters
\[
\alpha = (\alpha_1, \alpha_2), \quad \beta = (\beta_1, \beta_2),
\]
are associated to the edges of the face, where two opposite edges that are parallel should be assigned the same parameters, as shown in Figure 7.

\(^2\)Note that in comparison to a previous work \[19\], the face variable \(x\) will be treated here as a variable on the same level as the corner variables \(x_a, x_b, x_c, x_d\), rather than as a parameter.
A face-centered quad equation associated to Figure 7, will be denoted by

$$A(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0,$$  \hspace{1cm} (67)

where $A$ is a polynomial in the five variables $x, x_a, x_b, x_c, x_d$, with linear dependence on the four corner variables $x_a, x_b, x_c, x_d$ (i.e. $A$ is affine-linear in $x_a, x_b, x_c, x_d$).

Such an equation may be written in the following form

$$A(x; x_a, x_b, x_c, x_d; \alpha, \beta) = \sum_{i=0}^{n} P_i(x_a, x_b, x_c, x_d; \alpha, \beta)x^i,$$  \hspace{1cm} (68)

where under a parameter specialisation, the coefficients $P_i(x_a, x_b, x_c, x_d; \alpha, \beta), i = 0, \ldots, n,$ are the usual expressions for affine-linear quad polynomials in the variables $x_a, x_b, x_c, x_d$.

The face-centered quad equations (67) that will be considered here, may also be expressed in the equivalent form

$$\frac{a(x; x_a; \alpha_2, \beta_1)a(x; x_d; \alpha_1, \beta_2)}{a(x; x_b; \alpha_2, \beta_2)a(x; x_c; \alpha_1, \beta_1)} = 1,$$  \hspace{1cm} (69)

where $a(x; y; \alpha, \beta)$ is a ratio of two polynomials of degree 1 in the corner variable $y$, and satisfies

$$a(x; y; \alpha, \beta)a(x; y; \beta, \alpha) = 1.$$  \hspace{1cm} (70)

The face-centered quad equation (67), (68), is then recovered by simply multiplying both sides of (69) by the denominator, bringing all terms to one side, and simplifying the resulting expression. The expression (69) may be regarded as a four-leg form of a face-centered quad equation (67), in analogy to the three-leg forms that are found for the regular integrable quad equations [2].

The form of the equation given in (69), along with the symmetry (70), implies that the following symmetries are satisfied for (67)

$$-A(x; x_a, x_b, x_c, x_d; \alpha, \beta) = A(x; x_d, x_a, x_b; \beta, \alpha)$$

$$= A(x; x_c, x_d, x_a; x_b; \alpha, \beta)$$

$$= A(x; x_b, x_a, x_d; x_c; \alpha, \beta),$$  \hspace{1cm} (71)
where \( \hat{\alpha} \), and \( \hat{\beta} \), respectively represent \( \alpha \), and \( \beta \), with their components exchanged i.e.

\[
\hat{\alpha} = \{\alpha_2, \alpha_1\}, \quad \hat{\beta} = \{\beta_2, \beta_1\}.
\] (72)

The above may be regarded as the analogues of the square symmetries that are satisfied by the regular integrable quad equations.

Due to the affine-linear property of the face-centered quad equations (67), (68), under appropriate initial conditions, they should have a unique evolution in the square lattice. For regular quad equations which satisfy CAC, two of the standard initial conditions on the square lattice, are the corner-type initial condition, and the staircase-type initial condition. For the face-centered quad equations (67), the analogues of the latter initial conditions on the rotated square lattice, are shown in Figure 8. Note that the evolutions are unique, because the variables of the square lattice are always determined by solving for the linear corner variables, and not the face variable.

![Figure 8: The corner-type (left) and staircase-type (right) initial conditions on the rotated square lattice, for the face-centered quad equations (67) defined on Figure 7. Filled vertices are the initial values, and the unfilled vertices are values that may be uniquely determined through evolution of (67).](image)

Although the CAFCC property can be presented in terms of the single equation (67), a more general form of CAFCC will be described with the use of two more types of face-centered quad equations, denoted respectively by

\[
B(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0, \quad C(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0.
\] (73)

Here the face-centered quad equation (67), will be denoted as type-A, while the equations (73), will be denoted as type-B, and type-C, respectively. Similarly to the type-A equation (67), the type-B and type-C equations (73) are polynomials in the five variables \( x, x_a, x_b, x_c, x_d \), with linear dependence on the four corner variables \( x_a, x_b, x_c, x_d \), and hence may be written in the same form as (68).

The equations (73), are also taken to have the following equivalent four-leg forms

\[
\frac{b(x; x_a, \alpha_2, \beta_1)b(x; x_d, \alpha_1, \beta_2)}{b(x; x_b, \alpha_2, \beta_2)b(x; x_c, \alpha_1, \beta_1)} = 1, \quad \frac{a(x; x_a, \alpha_2, \beta_1)c(x; x_d, \alpha_1, \beta_2)}{a(x; x_b, \alpha_2, \beta_2)c(x; x_c, \alpha_1, \beta_1)} = 1,
\] (74) (75)
for type-B, and type-C respectively. Here $a(x; y; \alpha, \beta)$ is the same function from (69), associated to a type-A equation (67), while the two new edge functions $b(x; y; \alpha, \beta)$ and $c(x; y; \alpha, \beta)$, are either polynomials, or ratios of polynomials, of degree 1 in the corner variable $y$, but do not satisfy the same reflection symmetry (70), that was satisfied by $a(x; y; \alpha, \beta)$.

Thus the expression (74), implies just the two symmetries for type-B equations

$$-B(x; x_a, x_b, x_c, x_d; \alpha, \beta) = B(x; x_c, x_d, x_a, x_b; \alpha, \beta) = B(x; x_b, x_a, x_d, x_c; \alpha, \beta),$$

while the expression (75), implies the single symmetry for type-C equations

$$-C(x; x_a, x_b, x_c, x_d; \alpha, \beta) = C(x; x_b, x_a, x_d, x_c; \alpha, \beta).$$

Each of the three types of four-leg expressions for the face-centered quad equations (69), (74), (75), may be associated to the faces of the face-centered cube, shown in Figure 9. Note that the edge function $a(x; y; \alpha, \beta)$ for type-A equations is associated to single-line edges, and the edge function $b(x; y; \alpha, \beta)$ for type-B equations is associated to double-line edges. The two edge functions for type-C equations are also associated to the same single- and double-line edges, because the type-C equations will be seen to effectively be defined from the combination of type-A and -B equations that meet at a corner vertex, when considered in terms of multi-dimensional consistency on the face-centered cube.

![Figure 9](image)

Figure 9: Graphical representation of four-leg forms (69), (74), (75), for the face-centered quad equations of type-A, type-B, and type-C, respectively. Type-C equations share the same types of single solid edges with type-A equations, and double solid edges with type-B equations (but not the same edge functions $c(x; y; \alpha, \beta)$).

### 3.2 Consistency-around-a-face-centered-cube

A multi-dimensional consistency property can be formulated for the face-centered quad equations introduced above, whereby under an appropriate initial condition, they are required to have a consistent evolution around the vertices and edges of the face-centered cube. This property will be referred to as consistency-around-a-face-centered-cube (CAFCC), and may be regarded as an analogue of the consistency-around-a-cube (CAC) property, that is used as an integrability condition for the usual integrable quad equations.
The CAFCC property has a convenient graphical representation in terms of the four-leg expressions that are pictured in Figure 9. The two relevant halves of the face-centered cube are pictured in Figure 10, where the following six type-A and -B equations from Figure 9

\[
A(y_1; x_f, y_0, x_a, x_b; \alpha, \gamma) = 0, \quad B(y_2; y_0, x_d, x_b, x_c; \alpha, \beta) = 0, \quad B(y_3; x_f, x_e, y_0, x_d; \gamma, \beta) = 0, \\
A(z_1; x_e, x_d, z_0, x_c; \alpha, \gamma) = 0, \quad B(z_2; x_f, x_e, x_a, z_0; \alpha, \beta) = 0, \quad B(z_3; x_a, z_0, x_b, x_c; \gamma, \beta) = 0,
\]

are centered at the six face vertices \(y_1, y_2, y_3, z_1, z_2, z_3\), respectively.

![Figure 10](image)

Figure 10: The 6 type-A and type-B equations (78), and 8 type-C equations (79), centered at the 14 vertices of the face-centered cube. Note that the edges on the right hand side, appear with reverse orientation with respect to the type-C equations (79) pictured in Figure 9.

The six equations (78) may be regarded as the analogues of the six equations on the faces of a cube for CAC, however they are not enough to describe an evolution around a face-centered cube, because the face variables \(y_1, y_2, y_3, z_1, z_2, z_3\), are not shared by any two of the equations in (78). This may be addressed by also taking into account the following 8 equations of type-C

\[
C(y_0; y_1, x_f, y_2, y_3; \{\beta_1, \gamma_2\}, \{\alpha_2, \gamma_1\}) = 0, \quad C(z_0; z_1, x_e, z_2, z_3; \{\beta_2, \gamma_1\}, \{\alpha_1, \gamma_2\}) = 0, \\
C(x_a; x_1, x_b, z_2, z_3; \{\beta_1, \gamma_1\}, \{\alpha_1, \gamma_2\}) = 0, \quad C(x_b; x_a, y_1, z_3, y_2; \{\beta_1, \gamma_2\}, \{\alpha_1, \gamma_1\}) = 0, \\
C(x_c; z_1, z_0, y_2, z_3; \{\beta_2, \gamma_2\}, \{\alpha_1, \gamma_1\}) = 0, \quad C(x_d; x_e, z_1, y_3, y_2; \{\beta_2, \gamma_1\}, \{\alpha_2, \gamma_1\}) = 0, \\
C(x_e; z_1, x_d, z_2, y_3; \{\beta_2, \gamma_1\}, \{\alpha_2, \gamma_2\}) = 0, \quad C(x_f; y_0, y_1, x_3, z_2; \{\beta_1, \gamma_1\}, \{\alpha_2, \gamma_2\}) = 0,
\]

which appear in Figure 10 centered at the 8 corner vertices \(y_0, z_0, x_a, x_b, x_c, x_d, x_e, x_f\), respectively. Note however that the edges of the type-C equations on the right hand side of Figure 10, are pictured with a reversed orientation with respect to the type-C equation of Figure 9. The type-C equations (79) always share one edge with a type-A equation, and two edges with type-B equations from (78) (but the respective edge functions \(b(x; y; \alpha, \beta)\), \(c(x; y; \alpha, \beta)\) are different for the same edge). The 4th edge for the type-C equation, is one of the four vertical edges which connect the four pairs of corner vertices \((x_a, x_b), (x_f, y_0), (x_c, x_d), (z_0, x_c)\),
respectively, and these are only associated to the type-C equations. The latter vertical edges have the effect of exchanging the components of the parameter $\gamma$, for the type-A equations, as is seen from Figure 10.

3.2.1 CAFCC algorithm

CAFCC for the 14 equations (78), (79), on the two halves of the face-centered cube of Figure 10, can be formulated as follows.

In Figure 10, there are six components of the parameters $\alpha$, $\beta$, $\gamma$,

$$\alpha = \{\alpha_1, \alpha_2\}, \quad \beta = \{\beta_1, \beta_2\}, \quad \gamma = \{\gamma_1, \gamma_2\},$$

which take some fixed values. Also the following six variables

$$y_0, y_1, y_2, y_3, x_b, x_d,$$

are chosen to take some fixed initial values. There remain a total of 8 undetermined variables associated to the vertices of the face-centered cube, and the 14 equations (78), (79), to be satisfied.

For the above initial conditions, the CAFCC property can be checked with the following 6 steps (also depicted in Appendix B).

1. The following two equations centered at $y_0$, and $y_2$

$$C(y_0; y_1, x_f, y_2, y_3; \{\beta_1, \gamma_2\}, \{\alpha_2, \gamma_1\}) = 0,$$

$$B(y_2; y_0, x_d, x_b; \alpha, \beta) = 0,$$

may be solved respectively, to uniquely determine the two variables

$$x_f, \quad x_e.$$ (83)

2. The following three equations centered at $y_1, y_3, x_f$,

$$A(y_1; x_f, y_0, x_a, x_b; \alpha, \gamma) = 0,$$

$$B(y_3; x_f, x_e, y_0, x_d; \gamma, \beta) = 0,$$

$$C(x_f; y_0, y_1, y_3, z_2; \{\beta_1, \gamma_1\}, \{\alpha_2, \gamma_2\}) = 0,$$

may be solved respectively, to uniquely determine the three variables

$$x_a, \quad x_e, \quad z_2.$$ (85)

3. Consider the following two equations centered at $x_d, x_e$,

$$C(x_d; x_e, z_1, y_3, y_2; \{\beta_2, \gamma_2\}, \{\alpha_2, \gamma_1\}) = 0,$$

$$C(x_e; z_1, x_d, z_2, y_3; \{\beta_2, \gamma_1\}, \{\alpha_2, \gamma_2\}) = 0.$$ (86)

For the first consistency check, both of the latter two equations may be used to solve for the variable

$$z_1,$$ (87)

and the two solutions must be in agreement.
4. Consider the following two equations centered at \( x_a, x_b \),

\[
C(x_a; y_1, x_b, z_2, z_3; \{\beta_1, \gamma_1\}, \{\alpha_1, \gamma_2\}) = 0, \\
C(x_b; x_a, y_1, z_3, y_2; \{\beta_1, \gamma_2\}, \{\alpha_1, \gamma_1\}) = 0.
\]  

(88)

For the second consistency check, both of the latter two equations may be used to solve for the variable \( z_3 \),

(89)

and the two solutions must be in agreement.

5. Consider the following four equations centered at \( z_1, z_2, z_3, x_c \),

\[
A(z_1; x_e, x_d, z_0, x_c; \alpha, \gamma) = 0, \\
B(z_2; x_f, x_e, x_a, z_0; \alpha, \beta) = 0, \\
B(z_3; x_a, z_0, x_b, x_c; \gamma, \beta) = 0, \\
C(x_c; z_1, z_0, y_2, z_3; \{\beta_2, \gamma_2\}, \{\alpha_1, \gamma_1\}) = 0.
\]  

(90)

For the third consistency check, each of the latter four equations may be used to solve for the final variable \( z_0 \),

(91)

and the four solutions must be in agreement.

6. For the final consistency check, the remaining equation centered at \( z_0 \),

\[
C(z_0; z_1, x_c, z_2, z_3; \{\beta_2, \gamma_1\}, \{\alpha_1, \gamma_2\}) = 0,
\]  

(92)

must be satisfied by the variables that have been determined in the previous steps.

3.3 Examples of equations that satisfy CAFCC

Examples of combinations of face-centered quad equations which satisfy CAFCC are given in Table 1, with the explicit expressions for type-A equations given in Table 2, and type-B and type-C equations given in Table 3. The latter tables give the expressions \( a(x; y; \alpha, \beta) \), \( b(x; y; \alpha, \beta) \), \( c(x; y; \alpha, \beta) \), for the four-leg forms (69), (74), (75), for type-A, B, and C equations respectively, while each of the equations are also given in their affine-linear forms in Appendix A. The derivation of these equations from the IRF equations of Section 2, will be given in Section 4.

In both Tables 2, and 3, the notation \( \overline{x} \), is used to denote

\[
\overline{x} = x + \sqrt{x^2 - 1},
\]  

(93)

for a variable \( x \). Furthermore, in Table 2, the functions for \( A4 \) are defined by

\[
F(x_a, x_b, \alpha, \beta) = (4(x_a + x_b)(\alpha - \beta)^2 + Q(\alpha, \beta))(4x_a(\alpha - \beta)^2 - Q(\alpha, \beta))^2, \\
G_{\pm}(x, \alpha, \beta) = (4(x - \alpha)^3 \pm R(\alpha, \beta))^2,
\]  

(94)

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and the notation \( \dot{\to} \) indicates an additive form of the equations (see (95)).

Table 1: Examples of different combinations of type-A, -B, and -C equations, listed in Tables 2, and 3, that satisfy CAFCC. The type-A equations also satisfy CAFCC on their own. \( P_1 \) is the CAC equation from the ABS list, that arises as the coefficient of \( x \) for the affine-linear form of the type-C equation.

| Type-A  | Type-B  | Type-C  | \( P_1(x_a, x_b, x_c, x_d) \) |
|---------|---------|---------|-------------------------------|
| \( A_3(\delta=1) \) | \( B_3(\delta_1=\frac{1}{3}; \delta_2=\frac{1}{3}, \delta_3=0) \) | \( C_3(\delta_1=\frac{1}{4}; \delta_2=\frac{1}{4}, \delta_3=0) \) | \( H_3(\delta=1; \varepsilon=1) \) |
| \( A_2(\delta=0) \) | \( B_3(\delta_1=\frac{1}{3}; \delta_2=0, \delta_3=\frac{1}{3}) \) | \( C_3(\delta_1=\frac{1}{4}; \delta_2=0; \delta_3=\frac{1}{4}) \) | \( H_3(\delta=1; \varepsilon=1) \) |
| \( A_3(\delta=0) \) | \( B_3(\delta_1=1; \delta_2=0, \delta_3=0) \) | \( C_3(\delta_1=1; \delta_2=0; \delta_3=0) \) | \( H_3(\delta=0; \varepsilon=0) \) |
| \( A_3(\delta=0) \) | \( B_3(\delta_1=0; \delta_2=0, \delta_3=0) \) (\( D_4 \)) | \( C_3(\delta_1=0; \delta_2=0; \delta_3=0) \) | \( H_3(\delta=0; \varepsilon=0) \) |
| \( A_2(\delta_1=1; \delta_2=1) \) | \( D_1 \) | \( C_1 \) | \( H_1(\varepsilon=1) \) |

Table 2: The list of functions \( a(x; y; \alpha, \beta) \) for (69), for the type-A equations listed in Table 1, as well as for \( A_4 \), which satisfies CAFCC on its own. \( P_1 \) is the CAC equation from the ABS list, that arises as the coefficient of \( x \), for the affine-linear form of the type-A equation (68). The abbreviation “add.” indicates an additive form of the equations (see (97) or (171)).

\[
\begin{align*}
\text{Type-A} & \quad a(x; y; \alpha, \beta) & \quad P_1(x_a, x_b, x_c, x_d) \\
A_4 & \quad \frac{(G_+(x, \alpha, \beta) - F(x, y, \alpha, \beta))S_-(x, \alpha, \beta)}{(G_-(x, \alpha, \beta) - F(x, y, \alpha, \beta))S_+(x, \alpha, \beta)} & \quad Q_4 \\
A_3(\delta=1) & \quad \frac{\alpha^2 + \beta^2 \bar{x}^2 - 2\alpha \beta \bar{x}y}{\beta^2 + \alpha^2 \bar{x}^2 - 2\alpha \beta \bar{x}y} & \quad Q_3(\delta=1) \\
A_3(\delta=0) & \quad \frac{\beta x - \alpha y}{\alpha x - \beta y} & \quad Q_3(\delta=0) \\
A_2(\delta_1=1; \delta_2=1) & \quad \frac{(\sqrt{x} + \alpha - \beta)^2 - y}{(\sqrt{x} - \alpha + \beta)^2 - y} & \quad Q_2 \\
A_2(\delta_1=1; \delta_2=0) & \quad \frac{-x + y + \alpha - \beta}{x - y + \alpha - \beta} & \quad Q_1(\delta=1) \\
A_2(\delta_1=0; \delta_2=0) & \quad \frac{\alpha - \beta}{x - y} \quad \text{(add.)} & \quad Q_1(\delta=0)
\end{align*}
\]

where
\[
S_\pm(x, \alpha, \beta) = \hat{x}(\beta - \alpha) \pm x(\hat{\alpha} + \hat{\beta}) \mp (\hat{\beta} \alpha + \hat{\alpha} \beta),
\]
\[
Q(\alpha, \beta) = (\hat{\alpha} + \hat{\beta})^2 - 4(\alpha + \beta)(\alpha - \beta)^2,
\]
\[
R(\alpha, \beta) = 4(\hat{\alpha} \beta + \hat{\beta} \alpha)(\alpha - \beta)^2 - (\hat{\alpha} + \hat{\beta})Q(\alpha, \beta),
\]
and the notation \( \hat{x} \), is used to denote
\[
\dot{x} = 4x^3 - g_2x - g_3,
\]

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### Table 3: The list of functions $b(x; y; \alpha, \beta)$ for (74) (left), and $c(x; y; \alpha, \beta)$ for (75) (right), for the type-B and type-C equations listed in Table 1. The function $a(x; y; \alpha, \beta)$ for the type-C equation (75), should be taken from the corresponding type-A entry in Tables 1 and 2. The abbreviation “add.”, indicates an additive form of the equations (see (172) and (173)).

| Type-B          | $b(x; y; \alpha, \beta)$                                           | Type-C          | $c(x; y; \alpha, \beta)$                                           |
|-----------------|--------------------------------------------------------------------|-----------------|--------------------------------------------------------------------|
| $B3(\delta_1=\frac{1}{2}; \delta_2=\frac{1}{2}; \delta_3=0)$ | $\beta^2 + \alpha^2 x^2 - 2\alpha\beta xy$                        | $C3(\delta_1=\frac{1}{2}; \delta_2=\frac{1}{2}; \delta_3=0)$ | $\frac{\alpha - \beta xy}{\alpha x - \beta y}$                        |
| $B3(\delta_1=\frac{1}{2}; \delta_2=0; \delta_3=\frac{1}{2})$ | $\frac{\alpha y - \beta x}{\alpha x y - \beta}$                   | $C3(\delta_1=\frac{1}{2}; \delta_2=0; \delta_3=\frac{1}{2})$ | $(\frac{x^2}{\beta} + \beta x^2 - 2\alpha xy)$                        |
| $B3(\delta_1=1; \delta_2=0; \delta_3=0)$                   | $\beta - \alpha xy$                                                 | $C3(\delta_1=1; \delta_2=0; \delta_3=0)$                   | $xy - \frac{\alpha}{\beta}$                                             |
| $B3(\delta_1=0; \delta_2=0; \delta_3=0)$                   | $(x + \alpha - \beta)^2 - y$                                        | $C3(\delta_1=0; \delta_2=0; \delta_3=0)$                   | $y$                                                                   |
| $B2(\delta_1=1; \delta_2=1; \delta_3=0)$                   | $\sqrt{x} + y + \alpha - \beta$                                    | $C2(\delta_1=1; \delta_2=1; \delta_3=0)$                   | $\sqrt{x} + y - \alpha + \beta$                                       |
| $B2(\delta_1=1; \delta_2=0; \delta_3=1)$                   | $\sqrt{x} + y + \alpha - \beta$                                    | $C2(\delta_1=1; \delta_2=0; \delta_3=1)$                   | $(x - \alpha + \beta)^2 - y$                                           |
| $B2(\delta_1=1; \delta_2=0; \delta_3=0)$                   | $x + y + \alpha - \beta$                                            | $C2(\delta_1=1; \delta_2=0; \delta_3=0)$                   | $x + y - \alpha + \beta$                                               |
| $B2(\delta_1=0; \delta_2=0; \delta_3=0)$                   | $y$ (add.)                                                          | $C2(\delta_1=0; \delta_2=0; \delta_3=0)$                   | $-\frac{y + \beta}{2x}$ (add.)                                         |

for a variable $x$, where $g_2$, $g_3$, are Weierstrass elliptic invariants [20].

Table 1 also lists the ABS quad equation $P_1(x_a, x_b, x_c, x_d)$, which is found to arise as the coefficient of $x$, in the affine-linear form of the type-C equations (see also Appendix A), while Table 2, similarly lists the ABS quad equation for the affine-linear from of the type-A equations (68). On the other hand, the type-B equations don’t seem to have an interesting quad equation appear for this same coefficient.

Note also that in Table 2, the abbreviation “add.”, is used to indicate an edge function for an additive form of a type-A equation (69), which is given by

$$a(x; x_a; \alpha_2, \beta_1) + a(x; x_d; \alpha_1, \beta_2) - a(x; x_b; \alpha_2, \beta_2) - a(x; x_c; \alpha_1, \beta_1) = 0. \quad (97)$$

Similarly the appearance of “add.” in Table (3), indicates that the type-B and type-C equations (74), and (75), should be replaced by their respective additive forms (see (172), (173)).

### 3.3.1 Remarks

1. In addition to the combinations of equations given in Table 1, the type-A equations of Table 2 also satisfy CAFCC on their own. That is, CAFCC will also be satisfied with the use of an type-A equation (69) of Table 2, replacing

$$B \rightarrow A, \quad C \rightarrow A, \quad (98)$$

for each type-B and -C equation appearing in (78), (79), and the 6 steps of CAFCC. Furthermore, the equation $A4$ in Table 2, is only known to satisfy CAFCC on its own.

2. For the equations in Tables 1–3, there are up to three parameters $\delta_1, \delta_2, \delta_3$ in the subscripts, which appear for their affine-linear expressions given in Appendix A. For each case, the indicated values should be chosen for the equations to satisfy CAFCC.
3. Although there appear square roots in the expressions given in Tables 2, and 3, any square roots cancel out of the respective affine-linear forms (see Appendix A).

4. The affine-linear form (68), for A4, is the only case where the degree \( n \) of the face variable \( x \) appears to be \( n > 2 \). Since all other cases of equations have degree \( n \leq 2 \), it is expected that a simpler expression for A4 may be found, for which \( n = 2 \).

5. Some of the equations in Tables 2, and 3, have appeared previously in contexts different from CAFCC. The equations \( D1 \), and \( D4 \), are independent of the face variable \( x \), and may be identified with two of the \( H^6 \)-type quad equations classified by Boll [11]. The expressions for the four-leg forms (69) for A-type equations, may also be identified with expressions for discrete Toda-type equations associated to type-Q ABS equations [8–10]. The equation \( A2(\delta_1=0,\delta_2=0) \) also has a different interpretation as the simplest \( n = 1 \) case of a set of \( n \)-component CAC quad equations [19].

4 CAFCC from IRF

In this section, it will be seen how the face-centered quad equations given in Section 3, may be derived from the equations for IRF models of statistical mechanics, given in Section 2.

4.1 IRF Boltzmann weights to face-centered quad equations

In Section 2, it was shown how star-triangle relations imply the Yang-Baxter equation for the IRF Boltzmann weights. For the classical IRF Yang-Baxter equations (61), (62), the equations of motion and derivatives with respect to its 14 variables, lead to the 14 expressions for the discrete Laplace-type equations, which are listed in Appendix C. An appropriate transformation of variables will be applied to the latter Laplace-type equation, which gives a linear dependence on the four corner variables, to finally arrive at the form of the face-centered quad equations that were listed in Tables 1–3 (and Appendix A). The steps in going from the IRF Boltzmann weights to the face-centered quad equations, are outlined in Figure 11. The main focus of this section will be the last step of Figure 11, in going from the discrete Laplace-type equations, to the face-centered quad equations.

The 14 Laplace-type equations of Appendix C can be grouped into three different types, which will correspond to one of the face-centered quad equations of types-A, -B, or -C, respectively, pictured in Figure 9. Type-A equations correspond to the discrete Laplace-type equations which involve only derivatives of \( L \) and \( \overline{L} \), appearing in (C.1), and (C.5). Type-B equations correspond to the discrete Laplace-type equations which involve only derivatives of \( \Lambda \) and \( \overline{\Lambda} \), appearing in (C.2), (C.3), (C.6), (C.7). Type-C equations correspond to the discrete Laplace-type equations which involve derivatives of two of \( L \) and \( \overline{L} \), along with two of \( \Lambda \) and \( \overline{\Lambda} \), appearing in the 8 equations (C.4), (C.8), and (C.9)–(C.14).

There are 14 variables involved in the discrete Laplace-type equations of Appendix C, which are associated to the 14 vertices of the Yang-Baxter equation in Figure 5. For convenience, these 14 variables are divided into the following two sets

\[
X_A = \{x_a, x_b, x_c, x_d, x_e, x_f, x_h, x'_h, x_i, x'_i\}, \quad X_B = \{x_j, x'_j, x_k, x'_k\}.
\]
1) IRF Boltzmann weights (Equations (44)–(47))
(Quasi-classical expansion)

2) Lagrangian IRF functions (Equations (54)–(57))
(Equations of critical point/derivatives of YBE)

3) Discrete Laplace-type equations (Appendix C)
(Linearising change of variables)

4) Face-centered quad equations (Tables 1–3, and Appendix A)

Figure 11: Descending from the IRF Boltzmann weights of Section 2, to the face-centered quad equations of Section 3. The details of going from 1) to 3), were given in Section 2, and this section is concerned with going from 3) to 4).

according to whether or not they appear as one of the face variables $X_B$, of a discrete Laplace-type equation of type-B. The latter two sets of variables are distinguished, since in some cases it will be necessary to have different transformations for variables in $X_A$ and $X_B$ respectively, in order to arrive at the desired expressions for the face-centered quad equations, which have linear dependence on the four corner variables.

For convenience, let also $Z$ denote the set of components of the parameters $u, v, w$, as

$$Z = \{u_1, u_2, v_1, v_2, w_1, w_2\}.$$  \hfill (100)

Then there will be three different types of changes of variables, denoted by $f(x), g(x), h(x)$, for the different variables and parameters in $X_A, X_B, Z$, respectively, indicated as follows

$$x \rightarrow \left\{ \begin{array}{ll}
y = f(x), & x \in X_A, \\
y = g(x), & x \in X_B, \\
y = h(x), & x \in Z.
\end{array} \right. \hfill (101)$$

The transformed variables/parameters denoted by $y$ in (101), then become the actual variables/parameters of the face-centered quad equations. The specific forms of $f(x), g(x), h(x)$, depend on the types of functions involved in the specific expressions for the Lagrangian functions from Appendix E. The latter functions are always one of elliptic-, hyperbolic-, rational-, or algebraic-types, and the associated changes of variables for these four types is indicated below, in Table 4.

There is another symmetry that will be used, to simplify the expressions of the 14 Laplace-type equations in Appendix C, and the resulting face-centered quad equations. Namely, the derivatives of the Lagrangian functions in Appendix E, satisfy (except for the elliptic case)

$$\frac{\partial \mathcal{L}_{u-v}(x, y)}{\partial x} = -\frac{\partial \mathcal{L}_{\pm \eta_0 + u-v}(x, y)}{\partial x} + k_1 \pi i, \quad \frac{\partial \Lambda_{u-v}(x, y)}{\partial x} = -\frac{\partial \Lambda_{\pm \eta_0 + u-v}(x, y)}{\partial x} + k_2 \pi i,$$  \hfill (102)
for some constant \(\eta_0\), where for the algebraic additive cases \(k_1 = k_2 = 0\), otherwise \(k_1 + k_2\) is an even integer. For the rational and algebraic cases, \(\eta_0 = 0\), while for the hyperbolic cases \(\eta_0 = \pi\). For the elliptic case, \(2\eta_0\) is a quasi-period of the Weierstrass \(\wp\) function, and there will appear an additional \(x\) dependent term for the difference of derivatives of Lagrangian functions in (102). However, these additional terms always cancel out of the combinations of Lagrangian functions in the expressions for the Laplace-type equations of Appendix C, and thus have no contribution to the derivation of the face-centered quad equations.

Motivated by (102), the first components of the parameters of the 14 Laplace-type equations of Appendix C, will be shifted by

\[
\begin{align*}
\mathbf{u} \rightarrow \mathbf{u}' &= \{u_1 + \eta_0, u_2\}, & \mathbf{v} \rightarrow \mathbf{v}' &= \{v_1 + \eta_0, v_2\}, & \mathbf{w} \rightarrow \mathbf{w}' &= \{w_1 + \eta_0, w_2\}.
\end{align*}
\] (103)

The shift of the form (103), may simply be regarded as a part of the change of variables given in (101). Observe that for the Lagrangian functions \(L\), and \(\Lambda\), in the equations given in Appendix C, the components \(u_1, v_1,\) and \(w_1\), only appear as a difference with one of \(u_2, v_2,\) or \(w_2\), while for the Lagrangian functions \(\overline{L}\), and \(\overline{\Lambda}\), the components \(u_1, v_1,\) and \(w_1\), only appear as a difference with one of \(u_1, v_1,\) or \(w_1\). Then by the above symmetries (102), the shifts (103) effectively take \(L \rightarrow \overline{L}\), and \(\Lambda \rightarrow \overline{\Lambda}\), and leave \(\overline{L}\), and \(\overline{\Lambda}\), unchanged, for all equations in Appendix C. Thus applying this shift in combination with the change of variables (101), will give a simpler form of the equations to work with, in terms of just two Lagrangian functions.

In the next section, the expressions for the IRF Lagrangian functions \(L_{\mathbf{u}'\mathbf{v}'}\) in (54), and \(\Lambda_{\mathbf{u}'\mathbf{v}'}\) in (56), will be used with the transformation of variables described above, to derive the face-centered quad equations of types A, and B, respectively. For the remaining face-centered quad equation of type-C, the following additional IRF Lagrangian function will be defined

\[
\Xi_{\mathbf{uv}} \left( x \begin{array}{c} x_a \ x_b \\ x_c \ x_d \end{array} \right) = \overline{\Xi}_{v_1 - u_2}(x, x_a) + \overline{L}_{v_1 - u_2}(x, x_b) - \overline{\Lambda}_{v_1 - u_1}(x, x_c) - \Lambda_{v_2 - u_1}(x, x_d).
\] (104)

Then with the following relabelling of variables

\[
\begin{align*}
x_a &\rightarrow x, & x_i &\rightarrow x_a, & x_j' &\rightarrow x_c, & x_k' &\rightarrow x_d, \\
w_1 &\rightarrow u_1, & v_1 &\rightarrow u_2, & u_1 &\rightarrow v_1
\end{align*}
\] (105)
the expression for the type-C discrete Laplace-type equation (C.9), is equivalent to

\[ \frac{\partial}{\partial x} \Xi_{uv}(x| x_a \ x_b \ x_c \ x_d) = 0. \]  
(106)

Thus the expression (104), may be used to derive the face-centered quad equations of type-C. Note that any of the 8 equations (C.4), (C.8), (C.9)-(C.14), may be used for this same purpose, and under the appropriate transformations, would each lead to the same expressions for the type-C face-centered quad equations.

In summary, according to the prescription given in (101), the expressions for the face-centered quad equation of type-A, will be obtained by using (101), on (54), as\(^3\)

\[ A(y_e; y_a, y_b, y_c, y_d; \alpha, \beta) = \exp \left\{ \frac{\partial}{\partial x_e} \mathcal{L}_{uv}(x_e| x_a \ x_b) \right\}, \]  
(107)

where \( y_I = f(x_I), \) \((I = a, b, c, d, e), \) the expressions for the face-centered quad equation of type-B, will be obtained by using (101), on (56), as

\[ B(y_e; y_a, y_b, y_c, y_d; \alpha, \beta) = \exp \left\{ \frac{\partial}{\partial x_e} \Lambda_{uv}(x_e| x_a \ x_b) \right\}, \]  
(108)

where \( y_I = f(x_I), \) \( y_e = g(x_e), \) \((I = a, b, c, d), \) and the expressions for the face-centered quad equation of type-C, will be obtained by using (101), on (104), as

\[ C(y_e; y_a, y_b, y_c, y_d; \alpha, \beta) = \exp \left\{ \frac{\partial}{\partial x_e} \Xi_{uv}(x_e| x_a \ x_b) \right\}, \]  
(109)

where \( y_I = f(x_I), \) \((I = a, b, e), \) \( y_J = g(x_J), \) \((J = c, d), \) and for each of the above three expressions

\[ \alpha_1 = h(u_1), \ \alpha_2 = h(u_2), \ \beta_1 = h(v_1), \ \beta_2 = h(v_2). \]  
(110)

### 4.2 CAFCC and the IRF YBE

The face-centered quad equations which are derived using the above approach, with the use of explicit solutions of the star-triangle relations in Appendices D, and E, are found to satisfy the property of CAFCC given in Section 3.2. The underlying reason for this is that the resulting equations which satisfy CAFCC, are effectively a reinterpretation of the 14 discrete Laplace-type equations of Appendix C, which are implied by the classical IRF YBE (61). Then it can be seen that CAFCC will be satisfied if the IRF YBE is satisfied, by comparing the 6 steps of CAFCC in Section 3.2, with the discrete Laplace-type equations of Appendix C, as follows.

First consider the CAFCC algorithm of Section 3.2, and note that the initial conditions along with equations centered at \( y_0, \ y_1, \ y_2, \ y_3, \) uniquely determine all of the variables that appear on the left hand side of the face-centered cube of Figure 10 (steps 1 and 2). These

\(^3\)For additive type equations of Tables 2, and 3, the exponentials on the right hand sides are not required.
correspond, up to the linearising change of variables (101), to the variables that appear on the left hand side of the classical IRF YBE (61). Since the four equations centered at \(y_0, y_1, y_2, y_3\) are satisfied, which correspond to the four discrete Laplace-type equations in (C.1)-(C.4), this implies that there is a classical IRF YBE (61), that is satisfied for the same (transformed) variables \(x_a, x_b, x_c, x_d, x_e, x_f\), which have been determined through CAFCC, where the YBE variables corresponding to the undetermined variables \(z_0, z_1, z_2, z_3\) satisfy (C.5)-(C.8).

For the CAFCC algorithm of Section 3.2, it remains to show that the face-centered quad equations are consistent in steps 3–6. Consider first the two equations centered at \(x_d, x_e\), either of which may be used to determine the variable \(z_1\) (step 3). Let the different expressions for \(z_1\), resulting from the latter equations, be denoted by \(z_1^A\) and \(z_1^B\), respectively, and assume that \(z_1^A \neq z_1^B\). This implies that there are two solutions for the right hand side of the classical IRF YBE (61); one solution with \(z_1 = z_1^A\), and the other solution with \(z_1 = z_1^B\). Now the classical IRF YBE with \(z_1 = z_1^A\), implies a set of 14 discrete Laplace-type equations, for which the equation centered at \(x_e\), must be satisfied by \(z_1 = z_1^A\) (with the variable transformation (101)). However this same equation was determined through the CAFCC method to have the solution \(z_1 = z_1^B\), where \(z_1^B \neq z_1^A\), which is a contradiction. Then it must be the case that \(z_1^A = z_1^B\), and thus the two equations centered at \(x_d, x_e\), are consistent (i.e they determine the same expression for \(z_1\)). Similar arguments may be used to show that the two equations centered at \(x_a, x_b\), used to determine the variable \(z_3\) (step 4), must be consistent.

For step 5 of the CAFCC algorithm, any of the four face-centered quad equations centered at \(z_1, z_2, z_3, x_c\), may next be used to determine the final variable \(z_0\). The consistency of these four equations can be shown using similar arguments to the above. Let any two distinct solutions of the latter four equations, be denoted by \(z_0^A, z_0^B\), and assume that \(z_0^A \neq z_0^B\). The latter means that there are at least two solutions for the right hand side of the classical IRF YBE (61); one solution with \(z_0 = z_0^A\), and the other solution with \(z_0 = z_0^B\). Now the classical IRF YBE with \(z_0 = z_0^A\), implies its own set of 14 Laplace-type equations, for which the same face-centered quad equation that was used to determine \(z_0 = z_0^B\), will be solved with \(z_0 = z_0^A\) (with the variable transformation (101)). This is impossible as the corresponding face-centered quad equation is linear in \(z_0\), and it was assumed that \(z_0^A \neq z_0^B\). Then it must be the case that \(z_0^A = z_0^B\), and thus the two equations that were used to determine the latter two expressions for \(z_0\) must be consistent. Consequently the four face-centered quad equations centered at \(z_1, z_2, z_3, x_c\) are consistent.

The final step of the CAFCC algorithm (step 6), requires that the equation centered at \(z_0\) is automatically satisfied by the variables that were obtained in previous steps. Since the variables that have been determined through CAFCC, correspond to the variables that are satisfied by the classical IRF YBE, and this equation centered at \(z_0\) is obtained with the change of variables (101), from the Laplace-type equation (C.8) implied by the latter IRF YBE, this equation must also be satisfied.

It should be emphasised that the above arguments do not apply to all cases of face-centered quad equations that will be derived here, since a classical IRF YBE is not known for every case at the rational and algebraic levels. Nevertheless, even for the latter cases, the method of derivation described in this section can still be used, and will be found to result in equations that satisfy CAFCC. Since CAFCC is satisfied, the associated classical IRF YBE’s for the ra-
tional and algebraic cases are also expected to be satisfied, independently of the star-triangle relations of Section 2. This is because if the 14 equations (78), (79), on the face-centered cube are satisfied through CAFC, then the discrete Laplace-type equations of Appendix C are satisfied up to the variable transformation (101), which in turn implies that the classical IRF YBE (61) is satisfied up to a term that depends only on the parameters $u, v, w$. Presumably the latter term may be absorbed into the expressions for the IRF Lagrangian functions (56), (58). Furthermore, the associated quantum IRF YBE is expected to arise through degenerations of the IRF YBE’s at the elliptic or hyperbolic levels.

4.3 Explicit cases

The approach of Section 4.1, will be used here to derive the face-centered quad equations which satisfy CAFC, listed in Tables 1–3. For each case, the derivation starts with the expressions for the IRF Lagrangian functions (54), (56), (104), using the explicit Lagrangian functions that appear in Appendix E. All face-centered quad equations may be derived by considering just the mixed cases of the Yang-Baxter equations (61), (62), corresponding to the equations of Table 1, together with an elliptic symmetric case for (64), corresponding to A4 in Table 2 (there is no elliptic mixed case), and this is the approach that is taken below.

Throughout this section, for a function $f(z)$, the notation $f(z_1 \pm z_2)$ is used to denote

$$f(z_1 \pm z_2) = f(z_1 + z_2) + f(z_1 - z_2).$$

(111)

Also, $\simeq$ will be used to indicate that the expressions for the IRF Lagrangian functions, are given up to terms that are constant with respect to the face variable $x$. This is because the face-centered quad equations are obtained from derivatives with respect to the latter variable, so extra terms are not needed.

4.3.1 Elliptic case

Here $\wp(z)$, and $\sigma(z)$, denote the Weierstrass elliptic function, and the Weierstrass sigma function respectively, with associated elliptic invariants $g_2, g_3$, or half-periods $\omega_1, \omega_2$ [20]. For this case, $\eta_0 = \frac{\pi \omega_2}{2 \omega_1}$, for (103). Also the notation $\dot{x}$, will be used to denote

$$\dot{x} = 4x^3 - g_2x - g_3.$$  

(112)

4.3.1.1 A4.

This is an example of a symmetric case described in Section 2.4.1. The IRF Lagrangian function (54), is written in terms of edge Lagrangian functions (E.5), for Q4. The derivative of the latter IRF Lagrangian function with respect to the face variable $z$, is given by

$$\frac{\partial}{\partial x} \mathcal{L}_{uv} \left(\begin{array}{c} x_a \\ x_b \\ x_c \\ x_d \end{array} \right) = \varphi(x, x_a, u_2 - v_1) + \varphi(x, x_d, u_1 - v_2)$$

$$+ \varphi(x, x_b, v_2 - u_2) + \varphi(x, x_c, v_1 - u_1),$$

(113)

where

$$\varphi(x_i, x_j, t) = \log \frac{\sigma(\frac{2\omega_1}{\pi}(x_i - x_j + t)) \sigma(\frac{2\omega_1}{\pi}(x_i + x_j + t))}{\sigma(\frac{2\omega_1}{\pi}(x_i - x_j - t)) \sigma(\frac{2\omega_1}{\pi}(x_i + x_j - t))}.$$ 

(114)
A linearising transformation of variables of the form (101), is given by
\[ f(x) = \varphi(2\omega_1 x \pi^{-1}), \quad g(x) = \varphi(2\omega_1 x \pi^{-1}), \quad h(x) = \varphi(2\omega_1 x \pi^{-1}). \] (115)

Using the latter transformation on the above expression for the derivative of the IRF Lagrangian, results in the type-A equation (69) which satisfies CAFCC on its own, where (A4 case)
\[ a(x; y; \alpha, \beta) = \frac{(G_+(x, \alpha, \beta) - F(x, y, \alpha, \beta))(\dot{x}(\beta - \alpha) - x(\dot{\alpha} + \dot{\beta}) + (\dot{\beta} \alpha + \dot{\alpha} \beta))}{(G_-(x, \alpha, \beta) - F(x, y, \alpha, \beta))(\dot{x}(\beta - \alpha) + x(\dot{\alpha} + \dot{\beta}) - (\beta \alpha + \dot{\beta} \alpha))}, \] (116)

and
\[ F(x_a, x_b, \alpha, \beta) = (4(x_a + x_b)(\alpha - \beta)^2 + Q(\alpha, \beta))(4x_a(\alpha - \beta)^2 - Q(\alpha, \beta))^2, \]
\[ G_\pm(x, \alpha, \beta) = (4\dot{x}(\alpha - \beta)^3 \pm R(\alpha, \beta))^2; \]
\[ Q(\alpha, \beta) = (\dot{\alpha} + \dot{\beta})^2 - 4(\alpha + \beta)(\alpha - \beta)^2, \]
\[ R(\alpha, \beta) = 4(\dot{\alpha} \beta + \dot{\beta} \alpha)(\alpha - \beta)^2 - (\dot{\alpha} + \dot{\beta})Q(\alpha, \beta). \] (117)

It is worth remarking, that there are several rather different looking (but equivalent) forms of (116) that may be found, due to the wide variety of identities that are known for the Weierstrass (or Jacobi) elliptic functions. It is likely that there could be found a simpler form of (116), particularly one that exhibits the expected quadratic dependence on the face variable \( x \), that is found for all other equations in the affine-linear forms of Appendix A.

As was noted in Section 3.3, the CAC polynomials in the ABS list arise as the coefficient \( P_1(x_a, x_b, x_c, x_d) \) of the linear term in \( x \) in (68). For the equation (69) with (116), \( P_1(x_a, x_b, x_c, x_d) \) will correspond to Q4, although it appears difficult to see this explicitly, due to the complicated form of the equations. One way is to observe that the discriminants
\[ r_k(x_k) = \left( \frac{\partial P_{ij}(x_k)}{\partial x_l} \right)^2 - 2P_{ij}(x_k) \frac{\partial^2 P_{ij}(x_k)}{\partial x_l^2}, \] (118)
of \( P_1(x_a, x_b, x_c, x_d) \), where \( i, j, k, l \), are distinct elements of \( \{a, b, c, d\} \), and \( P_{ij} \) is (the bi-quadratic polynomial)
\[ P_{ij} = \frac{\partial P_1}{\partial x_i} \frac{\partial P_1}{\partial x_j} - P_1 \frac{\partial^2 P_1}{\partial x_i \partial x_j}, \quad i, j = a, b, c, d, \quad i \neq j, \] (119)
are quartic polynomials in \( x_k \), for \( k = a, b, c, d \). Then according to the classification result of [2,3], this implies that \( P_1(x_a, x_b, x_c, x_d) \) is M"obius equivalent to the polynomial for Q4 from the ABS list.

### 4.3.2 Hyperbolic cases

For the hyperbolic cases, the Lagrangian functions are given in terms of the dilogarithm function, defined by
\[ \text{Li}_2(z) = -\int_0^z \frac{\log(1 - t)}{t} dt, \quad z \in \mathbb{C} - [1, \infty). \] (120)

For these cases \( \eta_0 = \pi \), for (103). Also the notation \( \varpi \), will be used to denote
\[ \varpi = x + \sqrt{x^2 - 1}. \] (121)
4.3.2.1  \( A3(\delta=1) \),  \( B3(\delta_1=\frac{1}{2}, \delta_2=\frac{1}{2}, \delta_3=0) \),  \( C3(\delta_1=\frac{1}{2}, \delta_2=\frac{1}{2}, \delta_3=0) \)

This is an example of the non-symmetric mixed case of Section 2.4.1, and will result in two sets of equations, given here and in the next case. The IRF Lagrangian functions (55), (57), (104), are given in terms of edge Lagrangian functions (E.9) for \( H3(\delta=1; \varepsilon=1) \), by

\[
\mathcal{L}_{u'v'}(x) \approx 4x^2 + 2i(1 + u_2 - v_1 - v_2 - \pi)x + 2x^2 + Li_2(e^{i(u_2-v_1)+x\pm x_\alpha}) + Li_2(e^{i(u_1-v_2)+x\pm x_\beta}),
\]

\[
\Lambda_{u'v'}(x) \approx 2i(u_1 + u_2 - v_1 - v_2 + \pi)x + 2x^2 + Li_2(e^{i(u_2-v_1)+x\pm x_\alpha}) + Li_2(e^{i(u_1-v_2)+x\pm x_\beta}),
\]

\[
\Xi_{u'v'}(x) \approx -x^2 - Li_2(e^{i(u_2-v_1)+x\pm x_\alpha}) - Li_2(e^{i(u_2-v_2)+x\pm x_\beta}) - Li_2(e^{i(u_1-v_1)+x\pm x_\beta}).
\]

Note that the first of the above Lagrangian functions (122), corresponds to an IRF Lagrangian function for the edge Lagrangian functions (E.7) for \( Q3(\delta=1) \).

For this case, a linearising transformation of variables of the form (101), is given by

\[
f(x) = \cosh(x), \quad g(x) = e^x, \quad h(x) = e^{ix}.
\]

Applying the latter transformation on the derivative of (122) with respect to \( x \), results in the type-A equation (69) which satisfies CAFCC on its own, where \( a(x; y; \alpha, \beta) \) is given by \( (A3(\delta=1) \) case)

\[
a(x; y; \alpha, \beta) = \frac{\alpha^2 + \beta^2 x^2 - 2\alpha \beta xy}{\beta^2 + \alpha^2 x^2 - 2\alpha \beta xy}.
\]

Applying the same transformation on the derivatives of (123), (124), with respect to \( x \), results in the type-B and -C equations (74), and (75), which satisfy CAFCC in combination with (69) and (126), where \( b(x; y; \alpha, \beta) \), and \( c(x; y; \alpha, \beta) \), are given by

\[
b(x; y; \alpha, \beta) = \beta^2 + \alpha^2 x^2 - 2\alpha \beta xy, \quad c(x; y; \alpha, \beta) = \frac{\alpha - \beta \bar{y}}{\alpha x - \beta y}.
\]

The coefficient of \( x \) in the affine-linear form of (69), with \( a(x; y; \alpha, \beta) \) from (126), is given by (A.4) with \( \delta = 1 \). Setting \( x_b \rightarrow -x_b \), \( \alpha_1 \rightarrow \alpha \), \( \beta_1 \rightarrow \beta \), \( \alpha_2 \rightarrow 1 \), \( \beta_2 \rightarrow 1 \), in the latter gives

\[
4\beta(\alpha^2 - 1)(x_a x_c + x_b x_d) - 4\alpha(\beta^2 - 1)(x_a x_b + x_c x_d) - 4(\alpha^2 - \beta^2)(x_b x_c + x_a x_d) - (\alpha^2 - 1)(\beta^2 - 1)(\alpha \beta^{-1} - \beta \alpha^{-1} - 1).
\]

This is a polynomial for \( Q3(\delta=1) \) from the ABS list [2].

The coefficient of \( x \) in the affine-linear form of (75), with \( a(x; y; \alpha, \beta) \) from (126), and \( c(x; y; \alpha, \beta) \) from (127), is given by (A.11) with \( (\delta_1, \delta_2, \delta_3) = (\frac{1}{2}, \frac{1}{2}, 0) \). Setting \( x_a \leftrightarrow x_c \), \( x_b \rightarrow -x_b \), \( \alpha_1 \rightarrow 1 \), \( \beta_1 \rightarrow \beta \), \( \alpha_2 \rightarrow \alpha \), \( \beta_2 \rightarrow \beta \), in the latter gives

\[
2\alpha(x_a x_c + x_b x_d) - 2\beta(x_a x_b + x_c x_d) - (\alpha \beta^{-1} - \beta \alpha^{-1})(1 + \alpha \beta x_a x_d).
\]

This is a polynomial for \( H3(\delta=1; \varepsilon=1) \) from the ABS list [3].
4.3.2.2  \( A_3(\delta=0), B_3(\delta_1=\frac{1}{4}, \delta_2=0; \delta_3=\frac{1}{4}), C_3(\delta_1=\frac{1}{4}, \delta_2=0; \delta_3=\frac{1}{4}) \)

This is an example of the non-symmetric mixed case of Section 2.4.1, where the first set of equations were given in (126), (127). The IRF Lagrangian functions (54), (56), (104), are given in terms of edge Lagrangian functions (E.10) for \( H_3(\delta=1; \varepsilon=1) \), by

\[
\mathcal{L}_{u'v'}(x | x_a x_b x_c x_d) \simeq \frac{(x-x_a)^2 + (x-x_b)^2 + (x-x_c)^2 + (x-x_d)^2}{2} + \text{Li}_2(e^{(u_2-u_1)(x-x_a)}) + \text{Li}_2(e^{(v_2-v_1)(x-x_b)}) + \text{Li}_2(e^{(v_1-u_1)(x-x_c)}) + \text{Li}_2(e^{(u_1-v_2)(x-x_d)}),
\]

(130)

\[
\Lambda_{u'v'}(x | x_a x_b x_c x_d) \simeq 2x^2 + \text{Li}_2(e^{(u_2-u_1)(x+x_0)}) + \text{Li}_2(e^{(v_2-v_1)(x+x_0)}) + \text{Li}_2(e^{(v_1-u_1)(x+x_0)}) + \text{Li}_2(e^{(u_1-v_2)(x+x_0)}),
\]

(131)

\[
\Xi_{u'v'}(x | x_a x_b x_c x_d) \simeq (i(2\pi - v_1 - v_2 + 2u_1) - x + x_b - x_a)x + \text{Li}_2(e^{(u_2-u_1)(x-x_a)}) - \text{Li}_2(e^{(v_2-v_1)(x-x_b)}) - \text{Li}_2(e^{(v_1-u_1)(x-x_c)}) - \text{Li}_2(e^{(u_1-v_2)(x-x_d)}).
\]

(132)

Note that the first of the above Lagrangian functions (130), corresponds to an IRF Lagrangian function for the edge Lagrangian functions (E.8) for \( Q_3(\delta=0) \).

For this case, a linearising transformation of variables of the form (101), is given by

\[
f(x) = e^x, \quad g(x) = \cosh(x), \quad h(x) = e^{ix}.
\]

(133)

Applying the latter transformation on the derivative of (130) with respect to \( x \), results in the type-A equation (69) which satisfies CAFCC on its own, where \( a(x; y; \alpha, \beta) \) is given by (A3(\( \delta=0 \)) case)

\[
a(x; y; \alpha, \beta) = \frac{\beta x - \alpha y}{\alpha x - \beta y}.
\]

(134)

Applying the same transformation on the derivatives of (131), (132), with respect to \( x \), results in the type-B and -C equations (74), and (75), which satisfy CAFCC in combination with (69) and (134), where \( b(x; y; \alpha, \beta) \), and \( c(x; y; \alpha, \beta) \), are given by

\[
b(x; y; \alpha, \beta) = \frac{\alpha y - \beta x}{\alpha x - \beta y}, \quad c(x; y; \alpha, \beta) = \beta^{-1}(\alpha^2 + \beta^2 x^2 - 2\alpha\beta xy).
\]

(135)

The coefficient of \( x \) in the affine-linear form of (69), with \( a(x; y; \alpha, \beta) \) from (134), is given by (A.4) with \( \delta = 0 \). Setting \( x_b \to -x_b, \alpha_1 \to \alpha, \beta_1 \to \beta, \alpha_2 \to 1, \beta_2 \to 1 \), in the latter gives

\[
\beta(\alpha^2 - 1)(x_a x_c + x_b x_d) - \alpha(\beta^2 - 1)(x_a x_b + x_c x_d) - (\alpha^2 - \beta^2)(x_b x_c + x_a x_d).
\]

(136)

This is a polynomial for \( Q_3(\delta=0) \) from the ABS list [2].

The coefficient of \( x \) in the affine-linear form of (75), with \( a(x; y; \alpha, \beta) \) from (134), and \( c(x; y; \alpha, \beta) \) from (135), is given by (A.11) with \( (\delta_1, \delta_2, \delta_3) = (\frac{1}{2}, 0, \frac{1}{2}) \). Setting \( x_a \leftrightarrow x_c, x_b \leftrightarrow -x_b, \alpha_1 \to 1, \beta_1 \to \beta, \alpha_2 \to \alpha, \beta_2 \to \beta \), in the latter gives

\[
2\alpha(x_a x_c + x_b x_d) - 2\beta(x_a x_b + x_c x_d) - (\alpha\beta^{-1} - \beta\alpha^{-1})(1 + \alpha\beta x_b x_c).
\]

(137)

This is a polynomial for \( H_3(\delta=1; \varepsilon=1) \) from the ABS list [3].
This is an example of the symmetric mixed case of Section 2.4.1. The IRF Lagrangian functions (54), (56), (104), are given in terms of edge Lagrangian functions (E.11), for $H3(\delta=0; \varepsilon=1-\delta)$. This gives the IRF Lagrangian function $\mathcal{L}_{u'v'}$ for the $A3(\delta=0)$ case already given in (130), as well as

\[
\begin{split}
\Lambda_{u'v'}(x | x_a x_b x_c x_d) &\simeq 2\pi ix + \text{Li}_2(e^{i(u_2-v_1)+x+x_a}) - \text{Li}_2(e^{i(u_2-v_2)+x+x_c}) \\
&- \text{Li}_2(e^{i(u_1-v_1)+x+x_e}) + \text{Li}_2(e^{i(u_1-v_2)+x+x_d}), \\
\Xi_{u'v'}(x | x_a x_b x_c x_d) &\simeq \frac{(x_a-x)^2 - (x_b-x)^2}{2} - i(v_2(x_d + x) - v_1(x_c + x)) \\
&+ \text{Li}_2(e^{i(u_2-v_1)\pm(x-x_a)}) - \text{Li}_2(e^{i(u_2-v_2)\pm(x-x_b)}) \\
&+ \text{Li}_2(e^{i(v_1-u_1)+x+x_e}) - \text{Li}_2(e^{i(v_2-u_1)+x+x_d}).
\end{split}
\]

(138)

(139)

For this case, a linearising transformation of variables of the form (101), is given by

\[
\begin{align*}
  f(x) &= e^x, \\
  g(x) &= e^x, \\
  h(x) &= e^{ix}.
\end{align*}
\]

(140)

Applying this transformation on the derivatives of (130), (138), (139), with respect to $x$, results in the type-A, -B, and -C equations (69), (74), (75), which satisfy CAFCC with $a(x; y; \alpha, \beta)$ given by (134), and

\[
\begin{align*}
  b(x; y; \alpha, \beta) &= \beta - \alpha xy, \\
  c(x; y; \alpha, \beta) &= xy - \frac{\alpha}{\beta}.
\end{align*}
\]

(141)

The coefficient of $x$ in the affine-linear form of (75), with $a(x; y; \alpha, \beta)$ from (134), and $c(x; y; \alpha, \beta)$ from (141), is given by (A.11) with $(\delta_1, \delta_2, \delta_3) = (1, 0, 0)$. Setting $x_a \leftrightarrow x_c$, $x_b \rightarrow -x_b$, $\alpha_1 \rightarrow 1$, $\beta_1 \rightarrow \beta$, $\alpha_2 \rightarrow \alpha$, $\beta_2 \rightarrow \beta$, in the latter gives

\[
\alpha^2 \beta(x_ax_c + x_b x_d) - \alpha \beta^2(x_ax_b + x_c x_d) + \beta^2 - \alpha^2.
\]

(142)

This is a polynomial for $H3(\delta=0; \varepsilon=1-\delta)$ from the ABS list [3].

This is an example of the symmetric mixed case of Section 2.4.1. The IRF Lagrangian functions (54), (56), (104), are given in terms of edge Lagrangian functions (E.12) for $H3(\delta=0; \varepsilon=0)$. This gives the Lagrangian function $\mathcal{L}_{u'v'}$ for the $A3(\delta=0)$ case already given in (130), as well as

\[
\begin{align*}
\Lambda_{u'v'}(x | x_a x_b x_c x_d) &= x(x_b + x_c - x_a - x_d), \\
\Xi_{u'v'}(x | x_a x_b x_c x_d) &\simeq x(x_b + x_c - x_a - x_d) + \text{Li}_2(e^{i(u_2-v_1)\pm(x-x_a)}) - \text{Li}_2(e^{i(u_2-v_2)\pm(x-x_b)}).
\end{align*}
\]

(143)

(144)

For this case, a linearising transformation of variables of the form (101), is given by

\[
\begin{align*}
  f(x) &= e^x, \\
  g(x) &= e^x, \\
  h(x) &= e^{ix}.
\end{align*}
\]

(145)
Applying this transformation on the derivatives of (130), (143), (144), with respect to \( x \), results in the type-A, -B, and -C equations (69), (74), (75), which satisfy CAFCC, with \( a(x; y; \alpha, \beta) \) given by (134), and

\[
b(x; y; \alpha, \beta) = y, \quad c(x; y; \alpha, \beta) = y. \tag{146}\]

The coefficient of \( x \) in the affine-linear form of (75), with \( a(x; y; \alpha, \beta) \) from (134), and \( c(x; y; \alpha, \beta) \) from (146), is given by (A.11) with \( (\delta_1, \delta_2, \delta_3) = (0, 0, 0) \). Setting \( x_a \leftrightarrow x_c, x_b \rightarrow -x_b, \alpha_1 \rightarrow 1, \beta_1 \rightarrow \beta, \alpha_2 \rightarrow \alpha, \beta_2 \rightarrow \beta \), in the latter gives

\[
\alpha(x_a x_c + x_b x_d) - \beta(x_a x_b + x_c x_d). \tag{147}\]

This is a polynomial for \( H^{3(\delta=0; \varepsilon=0)} \) from the ABS list [2,3].

### 4.3.3 Rational cases

For the rational cases, the Lagrangian functions are given in terms of a function \( \gamma(z) \), which is defined in terms of the complex logarithm by

\[
\gamma(z) = iz \log(iz), \quad iz \in \mathbb{C} - (-\infty, 0]. \tag{148}\]

Note that for these cases, since \( \eta = 0 \), there is no shift of the form (103) that will be applied to the parameters \( u, v, w \).

#### 4.3.3.1 \( A2(\delta_1=1; \delta_2=1), B2(\delta_1=1; \delta_2=1; \delta_3=0), C2(\delta_1=1; \delta_2=1; \delta_3=0) \)

This case was not covered in Section 2.4.1, because for the Lagrangian functions of (E.16), only one of the two star-triangle relations (15), (16), are found to be satisfied. However, for the purpose of deriving the face-centered quad equations, this case may be treated as a non-symmetric mixed case, analogously to the \( A3(\delta=1), B3(\delta_1=\frac{1}{2}; \delta_2=\frac{1}{2}; \delta_3=0), C3(\delta_1=\frac{1}{2}; \delta_2=\frac{1}{2}; \delta_3=0) \) case given above, since this case is the rational limit of the latter. This will lead to two sets of face-centered quad equations, which will be given here and in the next subsection.

The IRF Lagrangian functions (54), (56), (104), are given in terms of edge Lagrangian functions (E.16) for \( H^{2(\varepsilon=1)} \) by (note that \( \mathcal{E}_{u-v}(x, y) \) is taken to be \( \mathcal{L}_{v-u}(x, y) \) from (E.16), in order to be consistent with the \( A3(\delta=1), B3(\delta_1=\frac{1}{2}; \delta_2=\frac{1}{2}; \delta_3=0); C3(\delta_1=\frac{1}{2}; \delta_2=\frac{1}{2}; \delta_3=0) \) case)

\[
\mathcal{L}_{uv}(x | x_a x_b, x_c x_d) = \gamma(i(u_2 - v_1) \pm x + x_a) - \gamma(i(v_1 - u_2) \pm x + x_a) \\
+ \gamma(i(u_2 - v_1) \pm x \pm x_b) - \gamma(i(u_2 - v_1) \pm x \pm x_b) + \gamma(i(v_1 - u_2) \pm x \pm x_c) \\
- \gamma(i(u_1 - v_1) \pm x \pm x_c) + \gamma(i(u_1 - v_1) \pm x \pm x_d) - \gamma(i(v_2 - u_1) \pm x \pm x_d), \tag{149}\]

\[
\Lambda_{uv}(x | x_a x_b, x_c x_d) = \gamma(i(u_2 - v_1) \pm x \pm x_a) + \gamma(i(v_2 - u_2) \pm x \pm x_b) \\
+ \gamma(i(v_1 - u_1) \pm x \pm x_c) + \gamma(i(u_1 - v_2) \pm x \pm x_d), \tag{150}\]

\[
\Xi_{uv}(x | x_a x_b, x_c x_d) = \gamma(i(u_2 - v_1) \pm x \pm x_a) - \gamma(i(v_1 - u_2) \pm x \pm x_a) \\
+ \gamma(i(v_2 - u_2) \pm x \pm x_b) - \gamma(i(u_2 - v_2) \pm x \pm x_b) \\
- \gamma(i(u_1 - v_1) \pm x \pm x_c) - \gamma(i(v_2 - u_1) \pm x \pm x_d). \tag{151}\]
Note that the first of the above Lagrangian functions (149), corresponds to an IRF Lagrangian function for the edge Lagrangian functions for $Q_2$.

For this case, a linearising transformation of variables of the form (101), is given by

$$f(x) = x^2, \quad g(x) = x, \quad h(x) = ix. \quad (152)$$

Applying the latter transformation on the derivative of (149), with respect to $x$, results in the type-A equation (69) which satisfies CAFCC on its own, where $a(x; y; \alpha, \beta)$ is given by

$$(A2_{(\delta_1=1; \delta_2=1)} \text{ case})$$

$$a(x; y; \alpha, \beta) = \frac{(\sqrt{x} + \alpha - \beta)^2 - y}{(\sqrt{x} - \alpha + \beta)^2 - y}, \quad (153)$$

Applying the same transformation on the derivatives of (150), (151), with respect to $x$, results in the type-B and -C equations (74), and (75), which satisfy CAFCC in combination with (69) and (153), where $b(x; y; \alpha, \beta)$, and $c(x; y; \alpha, \beta)$, are given by

$$b(x; y; \alpha, \beta) = (x + \alpha - \beta)^2 - y, \quad c(x; y; \alpha, \beta) = \frac{y - \sqrt{x} - \alpha + \beta}{y + \sqrt{x} + \alpha - \beta}. \quad (154)$$

The coefficient of $x$ in the affine-linear form of (69), with $a(x; y; \alpha, \beta)$ from (153), is given by (A.5) with $(\delta_1, \delta_2) = (1, 1)$. Setting $x_b \to -x_b$, $\alpha_1 \to \alpha$, $\beta_1 \to \beta$, $\alpha_2 \to 0$, $\beta_2 \to 0$, in the latter gives

$$a(x_a - x_b)(x_c - x_d) - \beta(x_a - x_c)(x_b - x_d) + \alpha\beta(\alpha - \beta)(x_a + x_b + x_c + x_d - \alpha^2 + \alpha\beta - \beta^2). \quad (155)$$

This is a polynomial for $Q2$ from the ABS list [2].

The coefficient of $x$ in the affine-linear form of (75), with $a(x; y; \alpha, \beta)$ from (153), and $c(x; y; \alpha, \beta)$ from (154), is given by (A.12) with $(\delta_1, \delta_2, \delta_3) = (1, 1, 0)$. Setting $x_a \leftrightarrow x_c$, $x_b \to -x_b$, $\alpha_1 \to 0$, $\beta_1 \to \beta$, $\alpha_2 \to \alpha$, $\beta_2 \to \beta$, in the latter gives

$$(x_b - x_c)(x_a - x_d) + (\alpha^2 - \beta^2)(x_a + x_d) - (\alpha - \beta)(x_b + x_c - 2x_a x_d - \alpha^2 - \beta^2). \quad (156)$$

This is a polynomial for $H2_{(\varepsilon=1)}$ from the ABS list [3].

### 4.3.3.2 $A2_{(\delta_1=1; \delta_2=0)}$, $B2_{(\delta_1=1; \delta_2=0; \delta_3=1)}$, $C2_{(\delta_1=1; \delta_2=0; \delta_3=1)}$

This is a continuation of the previous case, where a second set of face-centered quad equations will be derived. To be consistent with the $A3_{(\delta=0)}$, $B3_{(\delta_1=\frac{1}{2}; \delta_2=0; \delta_3=\frac{1}{2})}$, $C3_{(\delta_1=\frac{1}{2}; \delta_2=0; \delta_3=\frac{1}{2})}$ case from above, the Lagrangian functions $\Lambda_{u-v}(x, y)$, $\overline{\Lambda}_{u-v}(x, y)$, should be taken from (E.16), while the Lagrangian functions $\hat{\Lambda}_{u-v}(x, y)$, $\overline{\hat{\Lambda}}_{u-v}(x, y)$, should be taken as $\mathcal{L}_{u-v}(x, y)$, $\mathcal{L}_{v-u}(x, y)$, respectively, from (E.17). The IRF Lagrangian functions (54), (56), (104), are then given by

$$\mathcal{L}_{uv} \begin{pmatrix} x_a x_b \\ x_c x_d \end{pmatrix} = \gamma(i(u_2 - v_1) - x + x_a) - \gamma(i(v_1 - u_2) - x + x_a)$$

$$- \gamma(i(u_2 - v_2) - x + x_b) + \gamma(i(v_2 - u_2) - x + x_b) - \gamma(i(u_1 - v_1) - x + x_c)$$

$$+ \gamma(i(v_1 - u_1) - x + x_c) + \gamma(i(u_1 - v_2) - x + x_d) - \gamma(i(v_2 - u_1) - x + x_d), \quad (157)$$

37
\[ \Lambda_{uv}(x) \bigg| \begin{array} {c|c} x_a x_b & x_c x_d \\ \hline x_c x_d & x_a x_b \end{array} = \gamma(i(u_2 - v_1) \pm x + x_a) + \gamma(i(v_2 - u_2) \pm x - x_b) + \gamma(i(v_1 - u_1) \pm x - x_c) + \gamma(i(u_1 - v_2) \pm x + x_d), \]  
\[ \Xi_{uv}(x) \bigg| \begin{array} {c|c} x_a x_b & x_c x_d \\ \hline x_c x_d & x_a x_b \end{array} = \gamma(i(u_2 - v_1) + x - x_a) - \gamma(i(v_2 - u_2) + x - x_a) + \gamma(i(v_1 - u_1) + x - x_b) - \gamma(i(u_1 - v_2) + x - x_b) - \gamma(i(v_1 - u_1) - x \pm x_c) - \gamma(i(v_2 - u_1) + x \pm x_d). \]  
Note that the first of the above Lagrangian functions (157), corresponds to an IRF Lagrangian function for the edge Lagrangian functions for \( Q_{1(\delta=1)} \).

For this case, a linearising transformation of variables of the form (101), is given by

\[ f(x) = x, \quad g(x) = x^2, \quad h(x) = ix. \]  

Applying the latter transformation on the derivative of (157), with respect to \( x \), results in the type-A equation (69) which satisfies CAFCC on its own, where \( a(x; y; \alpha, \beta) \) is given by \( (A_2(\delta_1=1; \delta_2=0) \) case

\[ a(x; y; \alpha, \beta) = \frac{y - x + \alpha - \beta}{x - y + \alpha - \beta}. \]  

Applying the same transformation on the derivatives of (158), (159), with respect to \( x \), results in the type-B and -C equations (74), and (75), which satisfy CAFCC in combination with (69) and (161), where \( b(x; y; \alpha, \beta) \), and \( c(x; y; \alpha, \beta) \), are given by

\[ b(x; y; \alpha, \beta) = \frac{y + \sqrt{x + \alpha - \beta}}{y - \sqrt{x + \alpha - \beta}}, \quad c(x; y; \alpha, \beta) = (x - \alpha + \beta)^2 - y. \]  

The coefficient of \( x \) in the affine-linear form of (69), with \( a(x; y; \alpha, \beta) \) from (161), is given by (A.5) with \( (\delta_1, \delta_2) = (1, 0) \). Setting \( x_b \to -x_b, \quad \alpha_1 \to \alpha, \quad \beta_1 \to \beta, \quad \alpha_2 \to 0, \quad \beta_2 \to 0 \), in the latter gives

\[ \alpha(x_a - x_b)(x_c - x_d) - \beta(x_a - x_c)(x_b - x_d) + \alpha\beta(\alpha - \beta). \]  

This is a polynomial for \( Q_{1(\delta=1)} \) from the ABS list [2].

The coefficient of \( x \) in the affine-linear form of (75), with \( a(x; y; \alpha, \beta) \) from (161), and \( c(x; y; \alpha, \beta) \) from (162), is given by (A.12) with \( (\delta_1, \delta_2, \delta_3) = (1, 0, 1) \). Setting \( x_a \leftrightarrow x_c, \quad x_b \to -x_b, \quad \alpha_1 \to 0, \quad \beta_1 \to \beta, \quad \alpha_2 \to \alpha, \quad \beta_2 \to \beta \), in the latter gives

\[ (x_b - x_c)(x_a - x_d) + (\alpha^2 - \beta^2)(x_b + x_c) - (\alpha - \beta)(x_a + x_d - 2x_bx_c - \alpha^2 - \beta^2). \]  

This is a polynomial for \( H_{2(\varepsilon=1)} \) from the ABS list [3].

4.3.3.3 \( A_2(\delta_1=1; \delta_2=0), \quad B_2(\delta_1=1; \delta_2=0; \delta_3=0), \quad C_2(\delta_1=1; \delta_2=0; \delta_3=0) \)

This case was not covered in Section 2.4.1, because for the Lagrangian functions of (E.18), \( \mathcal{L}_{u-v}(x, y) \neq \mathcal{L}_{v-u}(y, x) \). However, it may still be treated as a symmetric mixed case, which will lead to a set of face-centered quad equations that satisfy CAFCC.
The IRF Lagrangian functions (54), (56), (104), are given in terms of edge Lagrangian functions (E.18) for \( H_{2(\varepsilon=0)} \), by

\[
\begin{align*}
\mathcal{L}_{uv}(x \mid x_a x_b & \ x_c x_d) \simeq \gamma(i(u_2 - v_1) - x + x_a) - \gamma(i(v_1 - u_2) - x + x_d) \\
& + \gamma(i(v_2 - u_2) \pm (x - x_b)) + \gamma(i(v_1 - u_1) \pm (x - x_c)) \\
& + \gamma(i(u_1 - v_2) - x + x_d) - \gamma(i(v_2 - u_1) - x + x_d), \\
\Lambda_{uv}(x \mid x_a x_b & \ x_c x_d) = \gamma(i(u_2 - v_1) + x + x_a) + \gamma(i(v_2 - u_2) - x - x_b) \\
& + \gamma(i(v_1 - u_1) - x - x_c) + \gamma(i(u_1 - v_2) + x + x_d), \\
\Xi_{uv}(x \mid x_a x_b & \ x_c x_d) \simeq \gamma(i(u_2 - v_1) \pm (x - x_a)) + \gamma(i(v_2 - u_2) + x - x_b) \\
& - \gamma(i(u_2 - v_2) + x - x_b) - \gamma(i(u_1 - v_1) - x - x_c) + \gamma(i(v_2 - u_1) + x + x_d).
\end{align*}
\]

Note that the first of the above Lagrangian functions (165), corresponds to an IRF Lagrangian function for edge Lagrangian functions of \( Q_{1(\beta=1)} \), which is slightly different from (157).

For this case, a linearising transformation of variables of the form (101), is given by

\[
f(x) = x, \quad g(x) = x, \quad h(x) = ix.
\]

Applying this transformation on the derivatives of (165), (166), (167), with respect to \( x \), results in the type-A, -B, and -C equations (69), (74), (75), which satisfy CAFCC, with \( a(x; y; \alpha, \beta) \) given by (161) \((A2(\delta_1=1; \delta_2=0) \) case), and

\[
b(x; y; \alpha, \beta) = x + y + \alpha - \beta, \quad c(x; y; \alpha, \beta) = x + y - \alpha + \beta.
\]

The coefficient of \( x \) in the affine-linear form of (75), with \( a(x; y; \alpha, \beta) \) from (161), and \( c(x; y; \alpha, \beta) \) from (169), is given by (A.12) with \((\delta_1, \delta_2, \delta_3) = (1, 0, 0)\). Setting \( x_a \leftrightarrow x_c \), \( x_b \rightarrow -x_b \), \( \alpha_1 \rightarrow 0 \), \( \beta_1 \rightarrow \beta \), \( \alpha_2 \rightarrow \alpha \), \( \beta_2 \rightarrow \beta \), in the latter gives

\[
(x_b - x_c)(x_a - x_d) - (\alpha - \beta)(x_a + x_b + x_c + x_d) - \alpha^2 + \beta^2.
\]

This is a polynomial for \( H_{2(\varepsilon=0)} \) from the ABS list [2, 3].

4.3.4 Algebraic cases

The face-centered quad equations at the algebraic level are typically written in an additive form. That is, the CAFCC property will be satisfied by equations of the form\(^4\)

\[
A(x; x_a, x_b, x_c, x_d; \alpha, \beta)
= a(x, x_a, \alpha_2, \beta_1) + a(x, x_d, \alpha_1, \beta_2) - a(x, x_b, \alpha_2, \beta_2) - a(x, x_c, \alpha_1, \beta_1) = 0,
\]

\[
B(x; x_a, x_b, x_c, x_d; \alpha, \beta)
= b(x, x_a, \alpha_2, \beta_1) + b(x, x_d, \alpha_1, \beta_2) - b(x, x_b, \alpha_2, \beta_2) - b(x, x_c, \alpha_1, \beta_1) = 0,
\]

\[
C(x; x_a, x_b, x_c, x_d; \alpha, \beta)
= a(x, x_a, \alpha_2, \beta_1) + c(x, x_d, \alpha_1, \beta_2) - a(x, x_b, \alpha_2, \beta_2) - c(x, x_c, \alpha_1, \beta_1) = 0.
\]

\(^4\)As an exception, the first case below involves a combination of both multiplicative and additive equations.
As for the rational cases, for the cases \( \eta = 0 \), there is no shift of the form (103) that will be applied to the parameters \( u, v, w \).

4.3.4.1 \( A2_{(\delta_{1}=0; \delta_{2}=0)}, B2_{(\delta_{1}=0; \delta_{2}=0; \delta_{3}=0)}, C2_{(\delta_{1}=0; \delta_{2}=0; \delta_{3}=0)} \)

This case was not covered in Section 2.4.1, because for the Lagrangian functions of (E.21), only one of the two star-triangle relations (15), (16), are found to be satisfied. However, it may still be treated as a symmetric mixed case, which will lead to a set of face-centered quad equations that satisfy CAFCC.

The IRF Lagrangian functions (54), (56), (104), are given in terms of edge Lagrangian functions (E.21) for \( H1(\varepsilon=1) \), by

\[
\begin{align*}
\mathcal{L}_{uv}(x|a b x_c x_d) &= (u_2 - v_1) \log(|x + |a|) + (v_2 - u_2) \log(|x + |b|) \\
&\quad + (v_1 - u_1) \log(|x + |c|) + (u_1 - v_2) \log(|x + |d|), \\
\Lambda_{uv}(x|a b x_c x_d) &\approx ix(\log|x_b| + \log|x_c| - \log|x_a| - \log|x_d|), \\
Ξ_{uv}(x|a b x_c x_d) &= (v_1 - v_2 - i(x_c - x_d)) \log| |x| + 2(u_2 - v_1) \log(|x + |a|) \\
&\quad - 2(u_2 - v_2) \log(|x + |b|).
\end{align*}
\]  

Note that (174) is not quite in the desired form of the IRF Lagrangian function for the edge Lagrangian functions (E.19) for \( Q1(\delta=0) \), since the arguments of the logarithms should be a difference of two variables, rather than a sum of two variables. The resulting face-centered quad equation given below will then need to be modified accordingly to satisfy CAFCC.

For this case, a linearising transformation of variables of the form (101), is given by

\[
f(x) = |x|, \quad g(x) = x, \quad h(x) = ix.
\]

Using the latter transformation on the derivative of (174), results in the type-A equation (69), where \( a(x; y; \alpha, \beta) \) is given by

\[
a(x; y; \alpha, \beta) = \frac{\alpha - \beta}{x + y}.
\]

Note that in this form, (69) with (178), does not satisfy CAFCC on its own.

Applying the above transformation on the derivatives of (166), (167), with respect to \( x \), results in the type-B and -C equations (74), and (75), with \( a(x; y; \alpha, \beta) \) given by (178), and

\[
\begin{align*}
b(x; y; \alpha, \beta) &= \log|y|, \\
c(x; y; \alpha, \beta) &= \frac{y + \beta}{2x}.
\end{align*}
\]

To have an expression for \( b(x; y; \alpha, \beta) \) which is linear in \( y \), requires taking an exponential of (172) with (179), resulting in the type-B equation in multiplicative form (74), with

\[
b(x; y; \alpha, \beta) = y.
\]
The multiplicative type-B equation (74) with (181), along with the additive type-A and -C equations (171) with (178), and (173) with (180), satisfy the CAFCC property. However, as mentioned above, (178) does not take the expected form of the type-A equation (171) for $A2(\delta_1=0;\delta_2=0)$, and the latter equation doesn’t satisfy CAFCC on its own. To arrive at the equation which will satisfy CAFCC requires simply negating a variable, which leads to the type-A equation (171), with $(A2(\delta_1=0;\delta_2=0)$ case)

$$a(x; y; \alpha, \beta) = \frac{\alpha - \beta}{x - y}. \quad (182)$$

This gives the desired form of the face-centered quad equation (171) for $A2(\delta_1=0;\delta_2=0)$, which satisfies the CAFCC property both on its own, and in combination with the type-B and -C equations (74), (173), with (181) and (180), respectively.

The coefficient of $x$ in the affine-linear form of (171), with $a(x; y; \alpha, \beta)$ from (182), is given by (A.5) with $(\delta_1, \delta_2) = (0, 0)$. Setting $x_b \rightarrow -x_b$, $\alpha_1 \rightarrow \alpha$, $\beta_1 \rightarrow \beta$, $\alpha_2 \rightarrow 0$, $\beta_2 \rightarrow 0$, in the latter gives

$$\alpha(x_a - x_b)(x_c - x_d) - \beta(x_a - x_c)(x_b - x_d). \quad (183)$$

This is a polynomial for $Q1(\delta=0)$ from the ABS list [2].

The coefficient of $x$ in the affine-linear form of (173), with $a(x; y; \alpha, \beta)$ from (182), and $c(x; y; \alpha, \beta)$ from (180), is given by (A.12) with $(\delta_1, \delta_2, \delta_3) = (0, 0, 0)$. Setting $x_a \leftrightarrow x_c$, $x_b \rightarrow -x_b$, $\alpha_1 \rightarrow 0$, $\beta_1 \rightarrow \beta$, $\alpha_2 \rightarrow \alpha$, $\beta_2 \rightarrow \beta$, in the latter gives

$$(x_b - x_c)(x_a - x_d) - (\alpha - \beta)(x_b + x_c). \quad (184)$$

This is a polynomial for $H1(\varepsilon=1)$ from the ABS list [3].

4.3.4.2  $A2(\delta_1=0;\delta_2=0)$, $D1$, $C1$

This is an example of the symmetric mixed case of Section 2.4.1, however care needs to be taken, since the corresponding Lagrangian functions (E.22) for $H1(\varepsilon)$, don’t satisfy the symmetries (102) in their present form. This is rectified by simply negating the $x_1$ variable of the classical star-triangle relation (24), which is equivalent to replacing (E.22) with

$$\Lambda(x_i, x_j) = ix_i x_j, \quad \overline{\Lambda}(x_i, x_j) = -\Lambda(x_i, x_j), \quad \overline{\Lambda}_\alpha(x_i, x_j) = 2i\alpha \log |x_i - x_j|. \quad (185)$$

Then the IRF Lagrangian functions (54), (56), (104), are given in terms of edge Lagrangian functions (185) for $H1(\varepsilon)$, by

$$\mathcal{L}_{uv}(x | x_a x_b \atop x_c x_d) \simeq 2(u_2 - v_2) \log |x - x_b| + 2(u_1 - v_1) \log |x - x_c|$$
$$+ (v_1 - u_2) \log (\pm i(x - x_a)) + (v_2 - u_1) \log (\pm i(x - x_d)), \quad (186)$$

$$\Lambda_{uv}(x | x_a x_b \atop x_c x_d) = x(x_a - x_b - x_c + x_d), \quad (187)$$

$$\Xi_{uv}(x | x_a x_b \atop x_c x_d) \simeq x(x_c - x_d) + 2(v_1 - u_2) \log |x - x_a| + (u_2 - v_2) \log (\pm i(x - x_b)). \quad (188)$$
Note that the first of the above Lagrangian functions (186), corresponds to an IRF Lagrangian function for the edge Lagrangian functions for $Q_{1(\delta=0)}$.

For this case the change of variables (101), is trivial
$$f(x) = x, \quad g(x) = x, \quad h(x) = x. \tag{189}$$

The derivatives of the equations (186), (187), (188), with respect to $x$, may be written in the form of the type-A, -B, and -C equations (171), (172), (173), which satisfy CAFCC, with $a(x; y; \alpha, \beta)$ given by (182) ($A2(\delta_1=0; \delta_2=0)$ case), and
$$b(x; y; \alpha, \beta) = y, \quad c(x; y; \alpha, \beta) = -\frac{y}{2}. \tag{190}$$

The coefficient of $x$ in the affine-linear form of (173), with $a(x; y; \alpha, \beta)$ from (182), and $c(x; y; \alpha, \beta)$ from (190), is given by (A.13). Setting $x_a \leftrightarrow x_c$, $x_b \rightarrow -x_b$, $x_c \rightarrow -x_c$, $\alpha_1 \rightarrow 0$, $\beta_1 \rightarrow \beta$, $\alpha_2 \rightarrow \alpha$, $\beta_2 \rightarrow \beta$, in the latter gives
$$(x_b - x_c)(x_a - x_d) - 2(\alpha - \beta). \tag{191}$$
This is a polynomial for $H_{1(\varepsilon=0)}$ from the ABS list [2, 3].

5 Conclusion

This paper considers the discrete integrability of 5-point lattice equations that arise in a quasi-classical expansion of interaction-round-a-face (IRF) type equations of statistical mechanics. The main results of this paper, were a new formulation of the multi-dimensional consistency integrability condition applicable to such 5-point equations, called consistency-around-a-face-centered-cube (CAFCC), as well as 15 sets of equations which satisfy CAFCC, which were derived from a new form of IRF Boltzmann weights and Yang-Baxter equations.

It is not known if other integrability characteristics associated to multi-dimensional consistency can be derived from CAFCC, such as Lax pairs and Bäcklund transformations, as it is not yet clear how the evolution around the face-centered cube can be reinterpreted in terms of such quantities. It is also an open problem to classify the face-centered quad equations, and doing so may lead to new equations, or possibly reveal different combinations of the type-A, -B, or -C equations, that will satisfy CAFCC.

One other direction where equations should arise that satisfy CAFCC, is for classical equations obtained from multi-component solutions of the Yang-Baxter equation [17]. The latter equations would correspond to $n$-component extensions of $A4$, while degenerations should lead to multi-component extensions of other equations which satisfy CAFCC. This potential application to multi-component equations was one of the main motivations for determining the multi-dimensional consistency of the classical 5-point equations in this paper.

Appendix A  Affine-linear expressions

Here it is useful to first define the following quad polynomials:
$$D_1(x_a, x_b, x_c, x_d) = x_a - x_b - x_c + x_d, \quad D_4(x_a, x_b, x_c, x_d) = x_a x_d - x_b x_c, \tag{A.1}$$
$$L(x_a, x_b, x_c, x_d, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 x_a + \alpha_2 x_b + \alpha_3 x_c + \alpha_4 x_d.$$
For affine-linear forms of the type-A equations given in Table A.1, Symmetric cases (type-A equations)

It is also useful to define the following combinations of components of the parameters \( \alpha = \{\alpha_1, \alpha_2\}, \beta = \{\beta_1, \beta_2\} \),

\[
\theta(\alpha, \beta) = (\alpha_1 - \alpha_2)(\beta_1 - \beta_2), \quad \phi(\alpha, \beta) = \alpha_1 + \alpha_2 - \beta_1 - \beta_2. \quad (A.2)
\]
The following notations for the parameter \( \alpha \) will also be used (and similarly for \( \beta \))

\[
-\alpha = \{-\alpha_1, -\alpha_2\}, \quad \alpha^2 = \{\alpha_1^2, \alpha_2^2\}, \quad \alpha^* = \{\alpha_1, -\alpha_2\}, \quad \alpha_i = \{\alpha_i, \alpha_i\}, \quad (i = 1, 2). \quad (A.3)
\]

A.1 Symmetric cases (type-A equations)

For affine-linear forms of the type-A equations given in Table 2, there will appear the following 4-parameter versions of the type-Q quad polynomials in the ABS list [2]:

\[
Q_3(\delta)(x_a, x_b, x_c, x_d; \alpha, \beta) = \beta_1\beta_2(\alpha_1^2 - \alpha_2^2)D_4(x_a, x_b, x_c) + \alpha_1\alpha_2(\beta_1^2 - \beta_2^2)D_4(x_a, x_d, x_b) - (\alpha_1^2\alpha_2^2 - \beta_1^2\beta_2^2)D_4(x_a, x_b, x_c, x_d) - \frac{4}{3}\theta(\alpha^2, \beta^2)(\frac{\alpha_{12}}{\beta_1\beta_2} - \frac{\beta_{12}}{\alpha_1\alpha_2}), \quad (A.4)
\]

\[
Q_2(\delta_1, \delta_2)(x_a, x_b, x_c, x_d; \alpha, \beta) = (\alpha_1 - \alpha_2)D_4(x_a, x_b, x_c) + (\beta_1 - \beta_2)D_4(x_a, x_d, x_b) - \phi(\alpha, \beta)D_4(x_a, x_b, x_c, x_d) + \delta_1\theta(\alpha, \beta)\phi(\alpha, \beta)(\theta(\alpha^*, \beta^*) - \phi(\alpha^2, \beta^2))^{\delta_2}, \quad (A.5)
\]

where

\[
\begin{align*}
\rho_1 &= \phi(\alpha, \beta)(\alpha_2(\beta_2 - \beta_1) + \alpha_1(\beta_1 + \beta_2)) - 2(\alpha_1\alpha_2(\alpha_1 - \beta_1) + \beta_1\beta_2(\alpha_2 - \beta_2)), \\
\rho_2 &= \phi(\alpha, \beta)(\alpha_2(\beta_2 - \beta_1) + \alpha_1(\beta_1 + \beta_2)) - 2(\alpha_1\alpha_2(\alpha_1 - \beta_1) + \beta_1\beta_2(\alpha_2 - \beta_1)), \\
\rho_3 &= \phi(\alpha, \beta)(\alpha_1(\beta_1 - \beta_2) - \alpha_2(\beta_1 + \beta_2)) + 2(\alpha_1\alpha_2(\alpha_2 - \beta_1) + \beta_1\beta_2(\alpha_1 - \beta_2)), \\
\rho_4 &= \phi(\alpha, \beta)(\alpha_1(\beta_1 - \beta_2) - \alpha_2(\beta_1 + \beta_2)) - 2(\alpha_1\alpha_2(\alpha_2 - \beta_2) + \beta_1\beta_2(\alpha_1 - \beta_1)). \quad (A.6)
\end{align*}
\]

As was given in Section 4.3, for \( Q_3(\delta) \) in (A.4), the case \( \delta = 1 \), is associated to the ABS polynomial \( Q_3(\delta=1) \), and the case \( \delta = 0 \), is associated to the ABS polynomial \( Q_3(\delta=0) \).

For \( Q_2(\delta_1, \delta_2) \) in (A.5), the case \( (\delta_1, \delta_2) = (1, 1) \), is associated to the ABS polynomial \( Q_2 \), the case \( (\delta_1, \delta_2) = (1, 0) \), is associated to the ABS polynomial \( Q_1(\delta=1) \), and the case \( (\delta_1, \delta_2) = (0, 0) \), is associated to the ABS polynomial \( Q_1(\delta=0) \).

The elliptic case A4, is written here in terms of

\[
\begin{align*}
F(x_a, x_b, \alpha, \beta) &= (4(x_a + x_b)(\alpha - \beta)^2 + Q(\alpha, \beta))(4x_a(\alpha - \beta)^2 - Q(\alpha, \beta))^2, \\
G_{\pm}(x, \alpha, \beta) &= (4\dot{\alpha}(\alpha - \beta)^3 \pm R(\alpha, \beta))^2, \\
S_{\pm}(x, \alpha, \beta) &= \dot{x}(\alpha - \beta) \pm x(\dot{\alpha} + \dot{\beta}) \mp (\dot{\beta} \alpha + \dot{\alpha} \beta), \\
Q(\alpha, \beta) &= (\dot{\alpha} + \dot{\beta})^2 - 4(\alpha + \beta)(\alpha - \beta)^2, \\
R(\alpha, \beta) &= 4(\dot{\alpha} + \dot{\beta})(\alpha - \beta)^2 - (\dot{\alpha} + \dot{\beta})Q(\alpha, \beta), \quad (A.7)
\end{align*}
\]
The affine-linear expressions of type-A equations given in Table 2, are as follows:

\[ A4 : \]
\[
(G_+(x, \alpha_2, \beta_2) - F(x, x_b, \alpha_2, \beta_2))(G_+(x, \alpha_1, \beta_1) - F(x, x_c, \alpha_1, \beta_1))
\times
(G_-(x, \alpha_2, \beta_1) - F(x, x_a, \alpha_2, \beta_1))(G_-(x, \alpha_1, \beta_2) - F(x, x_d, \alpha_1, \beta_2))
\times
(S_-(x, \alpha_2, \beta_2)S_-(x, \alpha_1, \beta_1)S_+(x, \alpha_2, \beta_1)S_+(x, \alpha_1, \beta_2))^2
\]
- \( (G_-(x, \alpha_2, \beta_2) - F(x, x_b, \alpha_2, \beta_2))(G_-(x, \alpha_1, \beta_1) - F(x, x_c, \alpha_1, \beta_1))
\times
(G_+(x, \alpha_2, \beta_1) - F(x, x_a, \alpha_2, \beta_1))(G_+(x, \alpha_1, \beta_2) - F(x, x_d, \alpha_1, \beta_2))
\times
(S_+(x, \alpha_2, \beta_2)S_+(x, \alpha_1, \beta_1)S_-(x, \alpha_2, \beta_1)S_-(x, \alpha_1, \beta_2))^2 = 0.

\[ A3(\delta) : \]
\[
L(x_a, x_b, x_c, x_d, \rho_1, \rho_2, \rho_3, \rho_4)x^2 + Q3(\delta)(x_a, x_b, x_c, x_d, \alpha, \beta)x
- x_a x_b x_c x_d L(x_a^{-1}, x_c^{-1}, x_b^{-1}, x_a^{-1}, \rho_4, \rho_3, \rho_2, \rho_1) + \frac{1}{4}L(x_a, x_b, x_c, x_d, \gamma_1, \gamma_2, \gamma_3, \gamma_4) = 0.
\]

\[ A2(\delta_1; \delta_2) : \]
\[
(L(x_a, x_b, x_c, x_d, \alpha_2 - \beta_1, \beta_2 - \alpha_2, \beta_1 - \alpha_1, \alpha_1 - \beta_2) + \delta_2(\alpha, \beta)\phi(\alpha, \beta))x^2
+ Q2(\delta_1; \delta_2)(x_a, x_b, x_c, x_d, \alpha, \beta)x
+ \delta_1 L(x_a, x_b, x_c, x_d, \rho_1 \gamma_1^2 \beta_2^2, \rho_2 \gamma_2^2 \beta_2, \rho_3 \gamma_2 \beta_2, \rho_4 \gamma_2)
\]
\[
+ \delta_2 \left( (\alpha_1 - \alpha_2)(\alpha_1 - \beta_1)(\alpha_2 - \beta_1)x_b x_d - (\alpha_1 - \beta_2)(\alpha_2 - \beta_2)x_a x_c \right)
+ (\beta_1 - \beta_2)(\alpha_2 - \beta_2)x_c x_d - (\alpha_1 - \beta_1)(\alpha_1 - \beta_2)x_a x_b
+ \phi(\alpha, \beta)((\alpha_2 - \beta_1)(\alpha_1 - \beta_2)x_b x_c - (\alpha_1 - \beta_1)(\alpha_1 - \beta_2)x_a x_d)
\]
\[
+ (\alpha_1 - \beta_1)(\alpha_2 - \beta_1)(\alpha_2 - \beta_2)\phi(\alpha, \beta)\theta(\alpha, \beta) \right)
+ x_a x_b x_c x_d L(x_a^{-1}, x_c^{-1}, x_b^{-1}, x_a^{-1}, \beta_2 - \alpha_1, \alpha_1 - \beta_1, \alpha_2 - \beta_2, \beta_1 - \alpha_2) = 0.
\]

For \( A3(\delta) \), the \( \rho_i, \gamma_i, (i = 1, \ldots, 4) \), are given by:

\[
\rho_1 = \alpha_1 \beta_2 (\alpha_2^2 - \beta_1^2), \quad \gamma_1 = (\alpha_1^2 - \beta_1^2)(\alpha_2^2 - \beta_2^2)(\alpha_1 \beta_2 - \beta_2 \alpha_1),
\]
\[
\rho_2 = \alpha_1 \beta_1 (\beta_2^2 - \alpha_2^2), \quad \gamma_2 = (\alpha_2^2 - \beta_2^2)(\alpha_2^2 - \beta_1^2)(\alpha_1 \beta_2 - \beta_2 \alpha_1),
\]
\[
\rho_3 = \alpha_2 \beta_2 (\beta_1^2 - \alpha_2^2), \quad \gamma_3 = (\alpha_1^2 - \beta_1^2)(\alpha_2^2 - \beta_2^2)(\beta_2 \alpha_1 - \alpha_2 \alpha_2),
\]
\[
\rho_4 = \alpha_2 \beta_1 (\alpha_2^2 - \beta_2^2), \quad \gamma_4 = (\alpha_1^2 - \beta_1^2)(\alpha_2^2 - \beta_2^2)(\alpha_2 \alpha_1 - \beta_1 \alpha_2).
\]

For \( A2(\delta_1; \delta_2) \), the \( \rho_i, \gamma_i, (i = 1, \ldots, 4) \), are given by:

\[
\rho_1 = (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)(\beta_2 - \alpha_1), \quad \gamma_1 = (\alpha_1(\beta_1 + \beta_2) - \alpha_2(\beta_1 - \beta_2) - \alpha_1^2 - \beta_2^2),
\]
\[
\rho_2 = (\alpha_1 - \beta_2)(\alpha_2 - \beta_1)(\alpha_1 - \beta_1), \quad \gamma_2 = \alpha_1(\beta_1 + \beta_2) - \alpha_2(\beta_2 - \beta_1) - \alpha_1^2 - \beta_1^2,
\]
\[
\rho_3 = (\alpha_1 - \beta_2)(\alpha_2 - \beta_1)(\alpha_2 - \beta_2), \quad \gamma_3 = (\alpha_2(\beta_1 + \beta_2) - \alpha_1(\beta_1 - \beta_2) - \alpha_2^2 - \beta_2^2),
\]
\[
\rho_4 = (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)(\beta_1 - \alpha_2), \quad \gamma_4 = \alpha_2(\beta_1 + \beta_2) - \alpha_1(\beta_2 - \beta_1) - \alpha_2^2 - \beta_1^2.
\]
A.2 Mixed cases (type-B and -C equations)

For the type-C face-centered quad equations in Table 3, there will appear the following 4-parameter versions of the type-H quad polynomials in the ABS list [3]:

\[
H_3(\delta_1; \delta_2; \delta_3)(x_a, x_b, x_c, x_d; \alpha, \beta) = \frac{2}{\beta_1} x_a \left( x_b - x_a \right) \left( x_c - x_a \right)
\]

\[+ \left( \frac{\beta_2}{\beta_1} - \frac{\beta_1}{\beta_2} \right) \left( \frac{\alpha_1 \alpha_2}{\alpha_1} + \frac{\alpha_2 \beta_1 \beta_2}{\alpha_1} \right) \left( x_a x_b - \frac{\beta_1}{\beta_2} \right)
\]

\[
H_2(\delta_1; \delta_2; \delta_3)(x_a, x_b, x_c, x_d; \alpha, \beta) = \left( x_b - x_a \right) \phi(\alpha_2, \beta) \phi(\alpha_1, \beta) \delta_3 + \left( x_a + x_b \right) \left( x_c - x_d \right)
\]

\[- \delta_1(\beta_1 - \beta_2) \left( \phi(\alpha_1, \beta)(-\beta_1 - \beta_2)^{\delta_2 + \delta_3} - (x_c + x_d) \phi(\alpha_1, \beta)^{\delta_2} \right)
\]

\[+ 2\delta_2 \left( \beta_2 - \beta_1 \right) \left( x_c x_d - \beta_1 \beta_2 + \alpha_1^2 \right) + \left( \alpha_2 - \beta_2 \right)(\alpha_2 - \beta_2)(x_c - x_d)
\]

\[+ 2\delta_3(\beta_2 - \beta_1)(x_a x_b - \alpha_1^2 - \alpha_2(\alpha_2 - \beta_2 - \beta_1)) \right)
\]

\[
H_1(x_a, x_b, x_c, x_d; \alpha, \beta) = 2\phi(\beta, \alpha_2) - (x_a + x_b)(x_c + x_d).
\]

As was given in Section 4.3, for \(H_3(\delta_1; \delta_2; \delta_3)\) in (A.11), the cases \((\delta_1, \delta_2, \delta_3) = \left( \frac{1}{2}, \frac{1}{2}, 0 \right), \left( \frac{1}{2}, 0, \frac{1}{2} \right)\), are associated to the ABS polynomial \(H_3(\delta_1; \delta_2; \delta_3)\), the case \((\delta_1, \delta_2, \delta_3) = (1, 0, 0)\), is associated to the ABS polynomial \(H_3(\delta_1; \delta_2; \delta_3)\), and the case \((\delta_1, \delta_2, \delta_3) = (0, 0, 0)\), is associated to the ABS polynomial \(H_3(\delta_1; \delta_2; \delta_3)\).

For \(H_2(\delta_1; \delta_2; \delta_3)\) in (A.12), the cases \((\delta_1, \delta_2, \delta_3) = (1, 1, 0), (1, 0, 1)\), are associated to the ABS polynomial \(H_2(\delta_1; \delta_2; \delta_3)\), the case \((\delta_1, \delta_2, \delta_3) = (1, 0, 0)\), is associated to the ABS polynomial \(H_2(\delta_1; \delta_2; \delta_3)\), and the case \((\delta_1, \delta_2, \delta_3) = (0, 0, 0)\), is associated to the ABS polynomial \(H_1(\delta_1; \delta_2; \delta_3)\).

\[H_1(\delta_1; \delta_2; \delta_3) = (\alpha_1 - \alpha_2)(\alpha_1 - \beta_2) = 0.
\]

\[D_1 : D_1(x_a, x_b, x_c, x_d) = 0.
\]

A.2.1 Type-B equations

The affine-linear expressions of type-B equations given in Table 3, are as follows:

\[\Delta_3(\delta_1; \delta_2; \delta_3) : \]
\[
\delta_2 L(x_a, x_b, x_c, x_d; \alpha_1, \beta_1, \alpha_2, \beta_2) = -\frac{\alpha_1 \alpha_2}{\alpha_1} x - \frac{\alpha_2 \beta_1 \beta_2}{\alpha_1} x - \frac{\alpha_1 \alpha_2}{\alpha_1} \theta(\alpha_1 \alpha_2 \beta_1 \beta_2)^{-1}
\]
\[+ \delta_3 (x_a x_b \alpha_2 \delta_2 \beta_2 \beta_3 - \delta_2 \beta_1 \alpha_1 x_c x_d - \delta_2 \beta_1 \alpha_1 \beta_2 \beta_3 - \delta_2 \beta_1 \alpha_1 x_c x_d) x^{-1}
\]
\[+ \delta_1 L(x_a, x_b, x_c, x_d; \alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{\alpha_1 \alpha_2}{\alpha_1} x - \frac{\alpha_2 \beta_1 \beta_2}{\alpha_1} x - \frac{\alpha_1 \alpha_2}{\alpha_1} \theta(\alpha_1 \alpha_2 \beta_1 \beta_2)^{-1}
\]

\[\Delta_2(\delta_1; \delta_2; \delta_3) : \]
\[\delta_1 D_1(x_a, x_b, x_c, x_d; x = \left( \frac{1}{2} \right) \left( -x \right) \delta_2 + \delta_3 - 2\delta_2 \phi(\alpha_1, \beta_1, \alpha_2, \beta_2) x \delta_2
\]
\[\delta_1 \left( \left( 1 - \alpha_1 \right) \left( 1 - \alpha_2 \right) \right) \left( x_a x_b x_c x_d; \alpha_1, \beta_1, \alpha_2, \beta_2 \right) \left( x_a x_b x_c x_d; \alpha_1, \beta_1, \alpha_2, \beta_2 \right)
\]
\[- \theta(\alpha_1, \beta_1, \alpha_2, \beta_2) \left( x_a x_b x_c x_d; \alpha_1, \beta_1, \alpha_2, \beta_2 \right) \phi(\alpha_1, \beta_1, \alpha_2, \beta_2) x \delta_2
\]
\[- \delta_2 + \delta_3 \Delta_2(x_a, x_b, x_c, x_d; \alpha_1, \alpha_2, \alpha_1 - \beta_1, \beta_1 - \alpha_2) x = \left( \frac{1}{2} \right) \left( -x \right) \delta_2 + \delta_3
\]
\[\Delta_2(x_a, x_b, x_c, x_d; \alpha_1, \alpha_2, \alpha_1 - \beta_1, \beta_1 - \alpha_2) x = \left( \frac{1}{2} \right) \left( -x \right) \delta_2 + \delta_3
\]
\[- D_4(x_a, x_b, x_c, x_d) \phi(\beta_1, \beta_2) + D_4(x_a, x_b, x_c, x_d) \phi(\beta_1, \beta_2) = 0.
\]

\[D_1 : D_1(x_a, x_b, x_c, x_d) = 0.
\]
A.2.2 Type-C equations

The affine-linear expressions of type-C equations given in Table 3, are as follows:

\[ C3(\delta_1; \delta_2; \delta_3) : \]
\[ \left( \alpha_2 (\beta_1 x_d - \beta_2 x_c) - \delta_3 \alpha_1^{-1} (\alpha_2^2 (\beta_1 x_b - \beta_2 x_a) + \beta_1 \beta_2 (\beta_1 x_a - \beta_2 x_b)) \right) x^2 \]
\[ + H3(\delta_1; \delta_2; \delta_3) (x_a, x_b, x_c, x_d; \alpha, \beta) x + \alpha_2 x_a x_b (\beta_2 x_d - \beta_1 x_c) \]
\[ + \delta_1 (\alpha_1 (\beta_1 x_b - \beta_2 x_a) + \alpha_1 \alpha_2^2 (x_a \beta_2^{-1} - x_b \beta_1^{-1})) \]
\[ + \frac{\delta_3}{\alpha_1} \left( \frac{\alpha_2}{\alpha_1} - \frac{\delta_1}{\alpha_2} \right) \left( \frac{\alpha_2}{\alpha_2} - \frac{\delta_1}{\alpha_1} \right) (\beta_2 x_d - \beta_1 x_c) \]
\[ + 2 \delta_3 \left( \beta_1 \beta_2 (\beta_1 x_b - \beta_2 x_a) + \alpha_2^2 (\beta_1 x_a - \beta_2 x_b) \right) = 0. \]

\[ C2(\delta_1; \delta_2; \delta_3) : \]
\[ \left( (\beta_1 - \beta_2) \phi(\alpha_1, \beta) \delta_3 - x_c + x_d + 2 \delta_3 (\alpha_2 - \beta_1) x_a - (\alpha_2 - \beta_2) x_b \right) x^2 \]
\[ + H2(\delta_1; \delta_2; \delta_3) (x_a, x_b, x_c, x_d; \alpha, \beta) x + x_a x_b ((\beta_2 - \beta_1) \phi(\alpha_1, \beta) \delta_3 + x_d - x_c) \]
\[ + \delta_1 \left( (\alpha_2 - \beta_1) x_b x_d (2(\alpha_1 - x_c) - \alpha_2 - \beta_2) \delta^2 - (\alpha_2 - \beta_2) x_a x_c (2(\alpha_1 - x_d) - \alpha_2 - \beta_2) \delta^2 \right) \]
\[ - (\alpha_2 - \beta_2) x_a x_d (2(\alpha_1 - \beta_1) + \alpha_2 - \beta_2) \delta^2 + (\alpha_2 - \beta_1) x_b x_c (2(\alpha_1 - \beta_2) + \alpha_2 - \beta_1) \delta^2 \]
\[ + \phi(\alpha_1, \beta) (\beta_1 x_b - \beta_2 x_a + \alpha_2 (x_a - x_b)) (-\beta_1 - \beta_2) \delta^2 + \delta_3 \]
\[ + (\alpha_2 - \beta_1) (\alpha_2 - \beta_2) (x_c - x_d) ((\alpha_2 - \beta_1) (\beta_2 - \alpha_2) - (\beta_1 - \beta_2)^2) \delta^2 \]
\[ + (\alpha_2 - \beta_1) (\alpha_2 - \beta_2) (\beta_1 \beta_2 - \alpha_1 \alpha_2 + (2 \alpha_1 - \alpha_2) \phi(\alpha_1, \beta) \delta^2 \phi(\alpha_1, \beta) \delta_3) \]
\[ + \delta_3 (2 \alpha_1 - \beta_1 \beta_2) x_a (x_a - x_b) - (\alpha_2 - \beta_1) (\alpha_2 - \beta_2) (\beta_1 \beta_2 - \alpha_1 \alpha_2 + (2 \alpha_1 - \alpha_2) \phi(\alpha_1, \beta) (x_c + x_d)) \]
\[ + 2 \delta_3 (\alpha_1^2 - \beta_1 \beta_2) (\beta_1 x_b - \beta_2 x_a + \alpha_2 (x_a - x_b)) = 0. \]

\[ C1 : \]
\[ (x_c + x_d) x^2 + H1(x_a, x_b, x_c, x_d; \alpha, \beta) x + (2(\alpha_2 - \beta_2) + x_b x_c) x_a + (2(\alpha_2 - \beta_1) + x_a x_d) x_b = 0. \]

Appendix B  Graphical sequence of CAFCC

In Section 3.2, six steps were given to check the property of CAFCC, which are shown graphically in Figure 12. The unfilled vertices indicate the variables that remain to be determined by the steps of CAFCC, while the edges are used to indicate the equations that have been used from (78), (79), according to the representation of each of the equations shown in Figures 9, and 10.
Step 0.  \(x_f\)

Step 1.  \(x_f\)

Step 2.  \(x_f\)

Steps 3 & 4.  \(x_f\)

Steps 5 & 6.  \(x_f\)

Figure 12: CAFCC algorithm of Section 3.2. Step 0 is the choice of initial variables. In steps 1 and 2, five equations are used to uniquely determine five respective variables \(x_c, x_f, x_a, x_e, z_2\). Steps 3 and 4 require consistency of two equations solving for \(z_1\), and consistency of two equations solving for \(z_3\), respectively. Step 5 requires consistency of four equations solving for \(z_0\). Steps 1–5 have determined all 14 variables, and step 6 requires the remaining equation centered at \(z_0\) to be automatically satisfied.

Appendix C 14 discrete Laplace-type equations on the face-centered cube

In Section 2, it was seen how the interaction-round-a-face form of the Yang-Baxter equation, implies 14 discrete Laplace-type equations, which are obtained from the derivatives taken
with respect to the 14 variables on vertices of the face-centered cube. These equations are listed explicitly here, using the notations of Section 2. The equations here are also referred to as one of types-A, -B, or C, according to which type of face-centered quad equation they correspond to through the method of Section 4.

There are four equations obtained from derivatives with respect to the variables at interior vertices on the left hand side of Figure 5

\[
\frac{\partial}{\partial x_i} (C(x_i) + \mathcal{L}_{u_2-v_1}(x_f, x_i) + \mathcal{L}_{u_2-v_2}(x_h, x_i) + \mathcal{L}_{u_1-v_1}(x_a, x_i) + \mathcal{L}_{u_1-v_2}(x_b, x_i)) = 0, \quad (C.1)
\]

\[
\frac{\partial}{\partial x_j} (\hat{C}(x_j) + \Lambda_{u_2-w_1}(x_h, x_j) + \mathcal{L}_{u_2-w_2}(x_d, x_j) + \mathcal{L}_{u_1-w_1}(x_b, x_j) + \Lambda_{u_1-w_2}(x_c, x_j)) = 0, \quad (C.2)
\]

\[
\frac{\partial}{\partial x_k} (\hat{C}(x_k) + \Lambda_{v_2-w_1}(x_f, x_k) + \mathcal{L}_{v_2-w_2}(x_e, x_k) + \mathcal{L}_{v_1-w_1}(x_h, x_k) + \Lambda_{v_1-w_2}(x_d, x_k)) = 0, \quad (C.3)
\]

\[
\frac{\partial}{\partial x_h} (C(x_h) + \mathcal{L}_{v_1-v_2}(x_f, x_h) + \mathcal{L}_{u_2-v_2}(x_h, x_i) + \Lambda_{u_2-w_1}(x_h, x_j) + \mathcal{L}_{v_1-w_1}(x_h, x_k)) = 0. \quad (C.4)
\]

The equation (C.1) is of type-A, the equations (C.2), (C.3) are of type-B, and the equation (C.4) is of type-C.

There are four equations obtained from derivatives with respect to the variables at interior vertices on the right hand side of Figure 5

\[
\frac{\partial}{\partial x_i'} (C(x_i') + \mathcal{L}_{u_2-v_1}(x_e, x_i') + \mathcal{L}_{u_2-v_2}(x_d, x_i') + \mathcal{L}_{u_1-v_1}(x_h, x_i') + \mathcal{L}_{u_1-v_2}(x_c, x_i')) = 0, \quad (C.5)
\]

\[
\frac{\partial}{\partial x_j'} (\hat{C}(x_j') + \Lambda_{u_2-w_1}(x_f, x_j') + \mathcal{L}_{u_2-w_2}(x_e, x_j') + \Lambda_{u_1-w_1}(x_a, x_j') + \Lambda_{u_1-w_2}(x_h, x_j')) = 0, \quad (C.6)
\]

\[
\frac{\partial}{\partial x_k'} (\hat{C}(x_k') + \Lambda_{v_2-w_1}(x_f, x_k') + \mathcal{L}_{v_2-w_2}(x_e, x_k') + \Lambda_{v_1-w_1}(x_h, x_k') + \Lambda_{v_1-w_2}(x_d, x_k')) = 0, \quad (C.7)
\]

\[
\frac{\partial}{\partial x_h'} (C(x_h') + \mathcal{L}_{v_1-v_2}(x_f, x_h') + \mathcal{L}_{u_1-v_2}(x_h, x_i') + \Lambda_{u_1-w_2}(x_h, x_j') + \mathcal{L}_{v_2-w_2}(x_h, x_k')) = 0. \quad (C.8)
\]

The equation (C.5) is of type-A, the equations (C.6), (C.7) are of type-B, and the equation (C.8) is of type-C.

There are six equations obtained from derivatives with respect to the variables at boundary vertices of Figure 5

\[
\frac{\partial}{\partial x_a} (\mathcal{L}_{u_1-v_1}(x_a, x_i) + \mathcal{L}_{v_2-v_1}(x_a, x_b) - \Lambda_{u_1-w_1}(x_a, x_j') - \Lambda_{v_2-w_1}(x_a, x_k')) = 0, \quad (C.9)
\]

\[
\frac{\partial}{\partial x_b} (\mathcal{L}_{u_1-v_1}(x_b, x_i) + \mathcal{L}_{v_2-v_1}(x_a, x_b) + \Lambda_{u_1-w_1}(x_b, x_j) - \Lambda_{v_1-w_1}(x_b, x_k)) = 0, \quad (C.10)
\]

\[
\frac{\partial}{\partial x_c} (\mathcal{L}_{u_1-v_1}(x_c, x_i') + \mathcal{L}_{v_2-v_1}(x_h, x_c) + \Lambda_{v_1-w_2}(x_c, x_k') - \Lambda_{u_1-w_2}(x_c, x_j)) = 0, \quad (C.11)
\]

\[
\frac{\partial}{\partial x_d} (\mathcal{L}_{u_2-v_2}(x_d, x_i') + \mathcal{L}_{v_1-v_2}(x_d, x_e) - \Lambda_{v_1-w_2}(x_d, x_j) - \mathcal{L}_{u_2-w_2}(x_d, x_j)) = 0, \quad (C.12)
\]

\[
\frac{\partial}{\partial x_e} (\mathcal{L}_{u_2-v_1}(x_e, x_i') + \mathcal{L}_{v_1-v_2}(x_d, x_e) + \Lambda_{v_2-w_2}(x_e, x_j') - \Lambda_{v_2-w_2}(x_e, x_k)) = 0, \quad (C.13)
\]

\[
\frac{\partial}{\partial x_e} (\mathcal{L}_{u_2-v_1}(x_f, x_i) + \mathcal{L}_{v_1-v_2}(x_f, x_h) + \Lambda_{v_2-w_1}(x_f, x_k) - \Lambda_{u_2-w_1}(x_f, x_j')) = 0. \quad (C.14)
\]

These are all equations of type-C.
Appendix D  List of Boltzmann weights for the star-triangle relations

The list of Boltzmann weights which satisfy the continuous spin forms of the star-triangle relations (7), and (10), are given below. The Boltzmann weights appearing here, have been obtained in previous works [4–7, 22, 23]. As was done in [13], the Boltzmann weights are labelled by

\[
\text{Quad equation (Hypergeometric integral) [Lattice model]} \quad (D.1)
\]

according to the integrable quad equation they correspond to from the ABS list (in a quasi-classical expansion), the hypergeometric integral corresponding to the star-triangle relation, and also the associated lattice model, if it has appeared previously.

In the following, the Boltzmann weights are written with the notation

\[
W_\theta(\sigma_i, \sigma_j), \quad \overline{W}_\theta(\sigma_i, \sigma_j), \quad V_\theta(\sigma_i, \sigma_j), \quad \overline{V}_\theta(\sigma_i, \sigma_j), \quad (D.2)
\]

where \(\theta\) represents a difference of the rapidity parameters \(p - q\), from Section 2. Also if an expression for \(S(\sigma_i)\) is not given here, it is assumed that \(S(\sigma_i) = 1\).

D.1 Elliptic case

The Boltzmann weights for the elliptic case are given in terms of the elliptic gamma function

\[
\Gamma_e(z; \tau_1, \tau_2) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2i(\pi\tau_1(j+\frac{1}{2})+\pi\tau_2(k+\frac{1}{2})+z)}}{1 - e^{2i(\pi\tau_1(j+\frac{1}{2})+\pi\tau_2(k+\frac{1}{2})-z)}}, \quad (D.3)
\]

The following renormalised Jacobi theta function is also used

\[
\vartheta_1(z|\tau) = 2e^{\frac{2\pi i z^2}{\tau}} \sin(z) \prod_{j=0}^{\infty} \left(1 - e^{2i(\pi\tau(j+1)+z)}\right) \left(1 - e^{2i(\pi\tau(j+1)-z)}\right). \quad (D.4)
\]

For this case, the variables take the values

\[
\sigma_i \in [0, 2\pi], \quad 0 < \theta_i < \eta, \quad \eta = \frac{-i\tau(\tau_1 + \tau_2)}{2}, \quad (D.5)
\]

and the elliptic parameters are chosen as

\[
\text{Re}(\tau_i) = 0, \quad \text{Im}(\tau_i) > 0, \quad (D.6)
\]

such that the crossing parameter \(\eta\) is real valued and positive. If necessary, the star-triangle relation can also be analytically continued from (D.5), (D.6), to complex valued variables.

At the elliptic level, there is one continuous spin solution of the star-triangle relation, which was introduced by Bazhanov and Sergeev [5] as the master solution of the star-triangle relation. As a mathematical formula, it is equivalent to the elliptic beta integral that was introduced by Spiridonov [25].
D.1.1 Quantum $Q4$ (Elliptic beta integral) [Master solution model]

The Boltzmann weights satisfying (7), are given by [5]

$$W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma_e(\sigma_i + \sigma_j + i\theta; \tau_1, \tau_2)}{\Gamma_e(\sigma_i + \sigma_j - i\theta; \tau_1, \tau_2)} \frac{\Gamma_e(\sigma_i - \sigma_j + i\theta; \tau_1, \tau_2)}{\Gamma_e(\sigma_i - \sigma_j - i\theta; \tau_1, \tau_2)},$$

$$\overline{W}_\theta(\sigma_i, \sigma_j) = \frac{W_{\eta-\theta}(\sigma_i, \sigma_j)}{\Gamma_e(i(\eta - 2\theta); \tau_1, \tau_2)}$$

$$(D.7)$$

$$S(\sigma_i) = \frac{e^{\frac{\eta}{2}}}{4\pi} \vartheta_1(2\sigma_i | \tau_1) \vartheta_1(2\sigma_i | \tau_2).$$

D.2 Hyperbolic cases

The Boltzmann weights at the hyperbolic level are given in terms of the hyperbolic gamma function (also non-compact quantum dilogarithm, or double sine function), defined here by

$$\Gamma_h(z; b) = \exp\left\{ \int_{[0, \infty)} \frac{dx}{x} \left( \frac{iz}{x} - \frac{\sinh(2izx)}{2\sinh(xb)\sinh(x/b)} \right) \right\}, \quad \text{Im}(z) < \text{Re}(\eta),$$

$$(D.8)$$

where

$$\eta = \frac{b + b^{-1}}{2}, \quad b > 0.$$  

$$\text{(D.9)}$$

The variables take the values

$$\sigma_i \in \mathbb{R}, \quad 0 < \theta_i < \eta.$$  

$$\text{(D.10)}$$

If necessary, the star-triangle relations for the hyperbolic cases can also be analytically continued from (D.9), (D.10), to complex valued variables.

D.2.1 Quantum $Q3(\delta=1)$ (Hyperbolic beta integral) [Generalised Faddeev-Volkov model]

The Boltzmann weights satisfying (7), are given by [23]

$$W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i + \sigma_j + i\theta; b)}{\Gamma_h(\sigma_i + \sigma_j - i\theta; b)} \frac{\Gamma_h(\sigma_i - \sigma_j + i\theta; b)}{\Gamma_h(\sigma_i - \sigma_j - i\theta; b)}, \quad \overline{W}_\theta(\sigma_i, \sigma_j) = \frac{W_{\eta-\theta}(\sigma_i, \sigma_j)}{\Gamma_h(i(\eta - 2\theta); b)}.$$  

$$S(\sigma_i) = \frac{1}{2} \Gamma_h(2\sigma_i - i\eta; b) \Gamma_h(-2\sigma_i - i\eta; b) = 2\sinh(2\pi\sigma_i b) \sinh(2\pi\sigma_i b^{-1}).$$  

$$\text{(D.11)}$$

$$\text{(D.12)}$$

D.2.2 Quantum $Q3(\delta=0)$ (Hyperbolic Saalschütz integral) [Faddeev-Volkov model]

The Boltzmann weights satisfying (7), are given by [4]

$$W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i - \sigma_j + i\theta; b)}{\Gamma_h(\sigma_i - \sigma_j - i\theta; b)}, \quad \overline{W}_\theta(\sigma_i, \sigma_j) = \frac{W_{\eta-\theta}(\sigma_i, \sigma_j)}{\Gamma_h(i(\eta - 2\theta); b)}.$$  

$$\text{(D.13)}$$

$^5$The name typically refers to the convention used for the function (D.8). The convention used here appeared in [24], and the different conventions are related by a change of variables, e.g., as outlined in [23].
The Boltzmann weights satisfying (D.2.5 Quantum H3(δ=1;ε=1) (Hyperbolic Askey-Wilson integral))
The Boltzmann weights satisfying (10), are given by
\[ V_{\theta}(\sigma_i, \sigma_j) = \Gamma_h(\sigma_i + \sigma_j + i\theta; b) \Gamma_h(\sigma_i - \sigma_j + i\theta; b), \quad V_{\theta}(\sigma_i, \sigma_j) = \Gamma_h(\sigma_i - \sigma_j, \tau; b), \]
\[ W_{\theta}(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i - \sigma_j + i\theta; b)}{\Gamma_h(\sigma_i - \sigma_j - i\theta; b)}, \quad \overline{W}_{\theta}(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i + \sigma_j + i(\eta - \theta); b) \Gamma_h(\sigma_i - \sigma_j + i(\eta - \theta); b)}{\Gamma_h(i(\eta - 2\theta); b) \Gamma_h(\sigma_i - \sigma_j - i(\eta - \theta); b)}, \]
\[ S(\sigma_i) = 2 \sinh(2\pi \sigma_i b) \sinh(2\pi \sigma_i b^{-1}). \]

D.2.4 Quantum H3(δ=1;ε=1) (Hyperbolic Saalschütz integral)
The Boltzmann weights satisfying (10), are given by
\[ V_{\theta}(\sigma_i, \sigma_j) = \Gamma_h(\sigma_i + \sigma_j + i\theta; b) \Gamma_h(\sigma_i - \sigma_j + i\theta; b), \quad \overline{V}_{\theta}(\sigma_i, \sigma_j) = \Gamma_h(\sigma_i - \sigma_j, \tau; b), \]
\[ W_{\theta}(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i + \sigma_j + i\theta; b) \Gamma_h(\sigma_i - \sigma_j + i\theta; b)}{\Gamma_h(\sigma_i + \sigma_j - i\theta; b) \Gamma_h(\sigma_i - \sigma_j - i\theta; b)}, \quad \overline{W}_{\theta}(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i - \sigma_j + i(\eta - \theta); b)}{\Gamma_h(i(\eta - 2\theta); b) \Gamma_h(\sigma_i - \sigma_j - i(\eta - \theta); b)}, \]
\[ S(\sigma_i) = 2 \sinh(2\pi \sigma_i b) \sinh(2\pi \sigma_i b^{-1}). \]

D.2.5 Quantum H3(δ=0;ε=1−δ) (Hyperbolic Barnes’s first lemma)
The Boltzmann weights satisfying (10), are given by
\[ V_{\theta}(\sigma_i, \sigma_j) = e^{2\pi i \sigma_i \sigma_j} \Gamma_h(\sigma_i + \sigma_j; \tau), \quad \overline{V}_{\theta}(\sigma_i, \sigma_j) = \frac{1}{V_{\theta}(\sigma_i, \sigma_j)}, \]
\[ W_{\theta}(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i - \sigma_j + i\theta; b)}{\Gamma_h(\sigma_i - \sigma_j - i\theta; b)}, \quad \overline{W}_{\theta}(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i - \sigma_j + i(\eta - \theta); b)}{\Gamma_h(i(\eta - 2\theta); b)}, \]
\[ S(\sigma_i) = 2 \sinh(2\pi \sigma_i b) \sinh(2\pi \sigma_i b^{-1}). \]

D.2.6 Quantum H3(δ=0;ε=0) (Hyperbolic Barnes’s 2F1 integral)
The Boltzmann weights satisfying (10), are given by
\[ V_{\theta}(\sigma_i, \sigma_j) = e^{2\pi i \sigma_i \sigma_j}, \quad \overline{V}_{\theta}(\sigma_i, \sigma_j) = \frac{1}{V_{\theta}(\sigma_i, \sigma_j)}, \]
\[ W_{\theta}(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i - \sigma_j + i\theta; b)}{\Gamma_h(\sigma_i - \sigma_j - i\theta; b)}, \quad \overline{W}_{\theta}(\sigma_i, \sigma_j) = \frac{\Gamma_h(\sigma_i - \sigma_j + i(\eta - \theta); b)}{\Gamma_h(i(\eta - 2\theta); b)}, \]
\[ S(\sigma_i) = 2 \sinh(2\pi \sigma_i b) \sinh(2\pi \sigma_i b^{-1}). \]

D.3 Rational cases
At the rational level, the Boltzmann weights are given in terms of the regular gamma function
\[ \Gamma(z) = \int_{0}^{\infty} dt t^{z-1} e^{-t}, \quad \text{Re}(z) > 0. \]
The variables for the rational cases take the values
\[ \sigma_i \in \mathbb{R}, \quad 0 < \theta_i < \zeta, \quad \zeta > 0. \] (D.20)

If necessary, the star-triangle relations for the rational cases can also be analytically continued from (D.20), to complex valued variables.

### D.3.1 Quantum Q2 (Rational beta integral)

The Boltzmann weights satisfying (7), are given by
\[
W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\zeta + i(\sigma_i + \sigma_j) - \theta) \Gamma(\zeta + i(\sigma_i - \sigma_j) - \theta)}{\Gamma(\zeta + i(\sigma_i + \sigma_j) + \theta) \Gamma(\zeta + i(\sigma_i - \sigma_j) + \theta)}, \quad \mathcal{W}_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\theta + i(\sigma_i + \sigma_j)) \Gamma(\theta + i(\sigma_i - \sigma_j)) \Gamma(\theta - i(\sigma_i + \sigma_j)) \Gamma(\theta - i(\sigma_i - \sigma_j))}{2\pi \Gamma(2\theta)},
\]
(D.21)
\[
S(\sigma_i) = (2 \Gamma(2\sigma_i) \Gamma(-2i\sigma_i))^{-1} = \pi^{-1} \sigma_i \sin(2\pi \sigma_i). \quad (D.22)
\]

### D.3.2 Quantum Q1(δ=1) (Barnes’s 2nd lemma)

The Boltzmann weights satisfying (7), are given by
\[
W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\zeta - i(\sigma_i + \sigma_j) - \theta)}{\Gamma(\zeta - i(\sigma_i + \sigma_j) + \theta)}, \quad \mathcal{W}_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\theta + i(\sigma_i - \sigma_j)) \Gamma(\theta - i(\sigma_i - \sigma_j))}{2\pi \Gamma(2\theta)}.
\]
(D.23)

### D.3.3 Quantum H2(ε=1) (De-Branges Wilson integral)

The Boltzmann weights satisfying (10), are given by
\[
V_\theta(\sigma_i, \sigma_j) = \Gamma(\zeta + i(\sigma_i + \sigma_j) - \theta) \Gamma(\zeta + i(\sigma_i - \sigma_j) - \theta), \quad \mathcal{V}_\theta(\sigma_i, \sigma_j) = V_{-\theta}(-\sigma_i, \sigma_j), \quad \mathcal{W}_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\theta + i(\sigma_i + \sigma_j)) \Gamma(\theta + i(\sigma_i - \sigma_j)) \Gamma(\theta - i(\sigma_i + \sigma_j)) \Gamma(\theta - i(\sigma_i - \sigma_j))}{2\pi \Gamma(2\theta)},
\]
(D.24)
\[
S(\sigma_i) = \pi^{-1} \sigma_i \sin(2\pi \sigma_i), \quad W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\zeta + i(\sigma_i - \sigma_j) - \theta)}{\Gamma(\zeta + i(\sigma_i - \sigma_j) + \theta)}.
\]

### D.3.4 Quantum H2(ε=1) (Barnes’s 2nd lemma)

The Boltzmann weights satisfying (10), are given by
\[
V_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\zeta + i(\sigma_i + \sigma_j) - \theta)}{\Gamma(\zeta + i(\sigma_i + \sigma_j) + \theta)}, \quad \mathcal{V}_\theta(\sigma_i, \sigma_j) = \Gamma(\theta + i(\sigma_i - \sigma_j)) \Gamma(\theta - i(\sigma_i + \sigma_j)), \quad \mathcal{W}_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\theta + i(\sigma_i - \sigma_j)) \Gamma(\theta - i(\sigma_i - \sigma_j))}{2\pi \Gamma(2\theta)}.
\]
(D.25)
D.3.5 Quantum $H2_{(\varepsilon=0)}$ (Barnes’s 1st lemma)

The Boltzmann weights satisfying (10), are given by

\[
V_\theta(\sigma_i, \sigma_j) = \Gamma(\zeta + i(\sigma_i + \sigma_j) - \theta), \quad \nabla V_\theta(\sigma_i, \sigma_j) = V_{\zeta-\theta}(-\sigma_i, -\sigma_j),
\]

\[
W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\zeta + i(\sigma_i - \sigma_j) - \theta)}{\Gamma(\zeta + i(\sigma_i - \sigma_j) + \theta)}, \quad \nabla W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\theta + i(\sigma_i - \sigma_j)) \Gamma(\theta - i(\sigma_i - \sigma_j))}{2\pi \Gamma(2\theta)}.
\]

(D.26)

D.4 Algebraic cases

Unless otherwise stated, the variables for the algebraic cases take values

\[
\sigma_i \in \mathbb{R}, \quad 0 < \theta_i < \frac{1}{2}.
\]

(D.27)

D.4.1 Quantum $Q1_{(\delta=0)}$ (Selberg-type integral) [Zamolodchikov fish-net model]

The Boltzmann weights satisfying (7), are given by [22]

\[
W_\theta(\sigma_i, \sigma_j) = |\sigma_i - \sigma_j|^{-2\theta}, \quad \nabla W_\theta(\sigma_i, \sigma_j) = \frac{\sqrt{\pi} \Gamma(\frac{1}{2} - \theta)}{\Gamma(\theta)} W_{\frac{1}{2} - \theta}(\sigma_i, \sigma_j).
\]

(D.28)

D.4.2 Quantum $H1_{(\varepsilon)}$ (Euler beta function)

The Boltzmann weights satisfying (10), are given by

\[
V_\theta(\sigma_i, \sigma_j) = e^{i\sigma_i \sigma_j}, \quad \nabla V_\theta(\sigma_i, \sigma_j) = V_\theta(\sigma_i, \sigma_j),
\]

\[
W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\frac{1}{2} + i(\sigma_i + \sigma_j) - \theta) \Gamma(\frac{1}{2} - i(\sigma_i + \sigma_j) - \theta)}{\Gamma(\frac{1}{2} + i(\sigma_i + \sigma_j)) \Gamma(\frac{1}{2} - i(\sigma_i + \sigma_j))},
\]

\[
\nabla W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\frac{1}{2} + \theta) \Gamma(\frac{1}{2} - \theta)}{2\pi \Gamma(2\theta)} |2\sinh \frac{\sigma_i - \sigma_j}{2}|^{2\theta-1}.
\]

(D.29)

D.4.3 Quantum $H1_{(\varepsilon=1)}$ (Barnes’s $2F_1$ integral)

For this case, the variables are

\[
\sigma_i \in \mathbb{R}, \quad 0 < \theta_i.
\]

(D.30)

The Boltzmann weights satisfying (10), are given by

\[
V_\theta(\sigma_i, \sigma_j) = |\sigma_i|^{i\sigma_j - \theta}, \quad \nabla V_\theta(\sigma_i, \sigma_j) = \frac{1}{V_\theta(\sigma_i, \sigma_j)},
\]

\[
W_\theta(\sigma_i, \sigma_j) = (|\sigma_i| + |\sigma_j|)^{-2\theta}, \quad \nabla W_\theta(\sigma_i, \sigma_j) = \frac{\Gamma(\theta + i(\sigma_i - \sigma_j)) \Gamma(\theta - i(\sigma_i - \sigma_j))}{2\pi \Gamma(2\theta)}.
\]

(D.31)
Appendix E  List of Lagrangian functions for classical star-triangle relations

In this appendix, a list of Lagrangian functions which satisfy the expressions for the classical star-triangle relations (19), and (24), are given. The Lagrangian functions of this section, have previously been found from the Boltzmann weights of the previous section through the quasi-classical limit [4–7]. Particularly, the labelling of the Lagrangian functions will follow the labelling of the Boltzmann weights given in the previous appendix.

In the following, the Lagrangian functions are written with the notation

\[ L_\alpha(x_i, x_j), \quad \overline{L}_\alpha(x_i, x_j), \quad \Lambda_\alpha(x_i, x_j), \quad \overline{\Lambda}_\alpha(x_i, x_j), \]

(E.1)

where \( \alpha \) represents a difference of the parameters \( u - v \), introduced in Section 2.2. Also if an expression for \( C(x_i) \) is not given here, it is assumed that \( C(x_i) = 0 \).

E.1 Elliptic case

For the elliptic case, \( \sigma(z) \) denotes the Weierstrass sigma function, and \( \zeta(z) \) denotes the Weierstrass zeta function, which both depend on the elliptic invariants \( g_2, g_3 \), or associated half-periods \( \omega_1, \omega_2 \) [20]. These functions are related to each other, and to the Weierstrass elliptic function \( \wp(z) \), by

\[ \frac{\partial}{\partial z} \log \sigma(z) = \zeta(z), \quad \frac{\partial}{\partial z} \zeta(z) = -\wp(z). \]

(E.2)

In the following \( \omega_1, \omega_2 \), are related to \( \tau_1 \) from (D.7), by

\[ \tau_1 = \frac{\omega_2}{\omega_1}. \]

(E.3)

and the Weierstrass sigma function \( \sigma(z) \), is related to the Jacobi theta function \( \vartheta_1(z \mid \tau) \) in (D.4), by

\[ \sigma(z) = \frac{2\omega_1}{\pi} \exp \left( \frac{z^2 \zeta(\omega_1)}{2\omega_1} \right) \vartheta_1 \left( \frac{\pi z}{2\omega_1} \mid \frac{\omega_2}{\omega_1} \right). \]

(E.4)

E.1.1 Q4 (Elliptic beta integral) [Master solution model]

The Lagrangian functions satisfying (19), are given by\(^6\)

\[ L_\alpha(x_i, x_j) = i\alpha (\pi - 4x_i) \zeta(\omega_2) + \frac{i\alpha}{2\pi} \left( \pi^2 - 8(x_i^2 + x_j^2) \right) \zeta(\omega_1) + \frac{\pi i}{2\omega_1} \left( \int_{\omega_2}^{2\omega_1(x_i-x_j) + \omega_2} + \int_{\omega_1}^{2\omega_1(x_i+x_j) + \omega_2} \right) \frac{\sigma(w + \frac{2\omega_1\alpha}{\pi})}{\sigma(w - \frac{2\omega_1\alpha}{\pi})}, \]

(E.5)

\[ \overline{L}_\alpha(x_i, x_j) = \frac{\sigma_{\omega_1 \omega_2}}{\sigma_{\omega_1} - \alpha(x_i, x_j)}, \quad C(x_i) = -\frac{(\pi - 4x_i)^2}{4}, \]

\( ^6 \)The Jacobi form of these equations appeared in [5].
E.2 Hyperbolic cases

For the hyperbolic cases, the Lagrangian functions are given in terms of the dilogarithm function \( \text{Li}_2(z) \), defined by

\[
\text{Li}_2(z) = -\int_0^z \frac{\log(1 - t)}{t} dt, \quad z \in \mathbb{C} - [1, \infty).
\]  

(E.6)

E.2.1 \( Q_3(\delta=1) \) (Hyperbolic beta integral) [Generalised Faddeev-Volkov model]

The Lagrangian functions satisfying (19), are given by

\[
\mathcal{L}_\alpha(x_i, x_j) = \text{Li}_2(-e^{x_i + x_j + i\alpha}) + \text{Li}_2(-e^{x_i - x_j + i\alpha}) + \text{Li}_2(-e^{-x_i + x_j + i\alpha}) + \text{Li}_2(-e^{-x_i - x_j + i\alpha})
\]

\[
+ \text{Li}_2(-e^{-x_i - x_j + i\alpha}) - 2 \text{Li}_2(-e^{i\alpha}) + x_i^2 + x_j^2 - \frac{\alpha^2}{2} + \frac{\pi^2}{6},
\]  

(E.7)

\[
\overline{\mathcal{L}}_\alpha(x_i, x_j) = \mathcal{L}_{\pi - \alpha}(x_i, x_j), \quad C(x_i) = 2\pi i x_i \text{ sgn}(\text{Re}(x_i)).
\]

E.2.2 \( Q_3(\delta=0) \) (Hyperbolic Saalschütz integral) [Faddeev-Volkov model]

The Lagrangian functions satisfying (19), are given by [4]

\[
\mathcal{L}_\alpha(x_i, x_j) = \text{Li}_2(-e^{x_i - x_j + i\alpha}) + \text{Li}_2(-e^{x_j - x_i + i\alpha}) - 2 \text{Li}_2(-e^{i\alpha}) + \frac{(x_i - x_j)^2}{2},
\]  

(E.8)

\[
\overline{\mathcal{L}}_\alpha(x_i, x_j) = \mathcal{L}_{\pi - \alpha}(x_i, x_j).
\]

E.2.3 \( H_3(\delta=1; \epsilon=1) \) (Hyperbolic Askey-Wilson integral)

The Lagrangian functions satisfying (24), are given by

\[
\mathcal{L}_\alpha(x_i, x_j) = \text{Li}_2(-e^{x_i - x_j + i\alpha}) + \text{Li}_2(-e^{x_j - x_i + i\alpha}) - 2 \text{Li}_2(-e^{i\alpha}) + \frac{(x_i - x_j)^2}{2},
\]  

(E.9)

\[
\overline{\mathcal{L}}_\alpha(x_i, x_j) = \text{Li}_2(e^{x_i + x_j - i\alpha}) + \text{Li}_2(e^{x_i - x_j - i\alpha}) + \text{Li}_2(e^{-x_i + x_j - i\alpha}) + \text{Li}_2(e^{-x_i - x_j + i\alpha})
\]

\[
- 2 \text{Li}_2(e^{-i\alpha}) + x_i^2 + x_j^2 - \frac{(\pi - \alpha)^2}{2} + \frac{\pi^2}{6},
\]

\[
\Lambda_\alpha(x_i, x_j) = \text{Li}_2(-e^{x_i + x_j + i\alpha}) + \text{Li}_2(-e^{x_j - x_i + i\alpha}) + \frac{(x_i + i\alpha)^2 + x_j^2}{2},
\]

\[
\overline{\Lambda}_\alpha(x_i, x_j) = \Lambda_{\pi - \alpha}(-x_i, -x_j), \quad C(x_i) = 2\pi i x_i \text{ sgn}(\text{Re}(x_i)).
\]

E.2.4 \( H_3(\delta=1; \epsilon=1) \) (Hyperbolic Saalschütz integral)

The Lagrangian functions satisfying (24), are given by

\[
\mathcal{L}_\alpha(x_i, x_j) = \text{Li}_2(-e^{x_i + x_j + i\alpha}) + \text{Li}_2(-e^{x_i - x_j + i\alpha}) + \text{Li}_2(-e^{-x_i + x_j + i\alpha}) + \text{Li}_2(-e^{-x_i - x_j + i\alpha})
\]

\[
+ \text{Li}_2(-e^{-x_i - x_j + i\alpha}) - 2 \text{Li}_2(-e^{i\alpha}) + x_i^2 + x_j^2 - \frac{\alpha^2}{2} + \frac{\pi^2}{6},
\]  

(E.10)

\[
\overline{\mathcal{L}}_\alpha(x_i, x_j) = \text{Li}_2(e^{x_i - x_j - i\alpha}) + \text{Li}_2(e^{x_j - x_i - i\alpha}) - 2 \text{Li}_2(e^{-i\alpha}) + \frac{(x_i - x_j)^2}{2},
\]

\[
\Lambda_\alpha(x_i, x_j) = \text{Li}_2(-e^{x_i + x_j + i\alpha}) + \text{Li}_2(-e^{x_j - x_i + i\alpha}) + \frac{(x_i + i\alpha)^2 + x_j^2}{2},
\]

\[
\overline{\Lambda}_\alpha(x_i, x_j) = \Lambda_{\pi - \alpha}(x_i, -x_j)
\]

55
E.2.5 \( H_{3(\delta=0,1; \varepsilon=1-\delta)} \) (Hyperbolic Barnes’s 1st lemma)

The Lagrangian functions satisfying (24), are given by
\[
\begin{align*}
\mathcal{L}_\alpha(x_i, x_j) &= \text{Li}_2(-e^{x_i-x_j+\text{i}\alpha}) + \text{Li}_2(-e^{x_j-x_i+\text{i}\alpha}) - 2\text{Li}_2(-\text{e}^{\text{i}\alpha}) + \frac{(x_i-x_j)^2}{2}, \\
\Lambda_\alpha(x_i, x_j) &= \text{Li}_2(-e^{x_i+x_j+\text{i}\alpha}) + i(x_i + x_j)\alpha + x_i^2 + x_j^2 - \frac{\alpha^2}{2} + \frac{\pi^2}{6}, \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= \mathcal{L}_{-\alpha}(x_i, x_j), \quad \overline{\Lambda}_\alpha(x_i, x_j) = -\Lambda_{-\alpha}(x_i, x_j).
\end{align*}
\] (E.11)

E.2.6 \( H_{3(\delta=0; \varepsilon=0)} \) (Hyperbolic Barnes’s \( _2F_1 \) integral)

The Lagrangian functions satisfying (24), are given by
\[
\begin{align*}
\mathcal{L}_\alpha(x_i, x_j) &= \text{Li}_2(-e^{x_i-x_j+\text{i}\alpha}) + \text{Li}_2(-e^{x_j-x_i+\text{i}\alpha}) - 2\text{Li}_2(-\text{e}^{\text{i}\alpha}) + \frac{(x_i-x_j)^2}{2}, \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= \mathcal{L}_{-\alpha}(x_i, x_j), \quad \Lambda(x_i, x_j) = -x_ix_j, \quad \overline{\Lambda}(x_i, x_j) = -\Lambda(x_i, x_j).
\end{align*}
\] (E.12)

E.3 Rational cases

In the following, \( \gamma(z) \) is a function defined in terms of the complex logarithm, by
\[
\gamma(z) = iz\text{Log}(iz), \quad iz \in \mathbb{C} - (-\infty, 0].
\] (E.13)

E.3.1 \( Q2 \) (Rational beta integral)

The Lagrangian functions satisfying (19), are given by
\[
\begin{align*}
C(x_i) &= 2ix_i(\text{Log}(-2ix_i) - \text{Log}(2ix_i)), \\
\mathcal{L}_\alpha(x_i, x_j) &= \gamma(x_i + x_j + \text{i}\alpha) + \gamma(x_i - x_j + \text{i}\alpha) - \gamma(x_i + x_j - \text{i}\alpha) - \gamma(x_i - x_j - \text{i}\alpha), \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= \gamma(x_i + x_j - \text{i}\alpha) + \gamma(x_i - x_j - \text{i}\alpha) + \gamma(-x_i + x_j - \text{i}\alpha) \\
&\quad + \gamma(-x_i - x_j - \text{i}\alpha) - \gamma(-2\text{i}\alpha).
\end{align*}
\] (E.14)

E.3.2 \( Q1(\delta=1) \) (Barnes’s 2nd lemma)

The Lagrangian functions satisfying (19), are given by
\[
\begin{align*}
\mathcal{L}_\alpha(x_i, x_j) &= \gamma(-x_i - x_j + \text{i}\alpha) - \gamma(-x_i - x_j - \text{i}\alpha), \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= \gamma(x_i - x_j + \text{i}\alpha) + \gamma(x_i - x_j - \text{i}\alpha) - \gamma(-2\text{i}\alpha).
\end{align*}
\] (E.15)

E.3.3 \( H_{2(\varepsilon=1)} \) (De-Branges Wilson integral)

The Lagrangian functions satisfying (24), are given by
\[
\begin{align*}
C(x_i) &= 2ix_i(\text{Log}(-2ix_i) - \text{Log}(2ix_i)), \\
\mathcal{L}_\alpha(x_i, x_j) &= \gamma(x_i + x_j + \text{i}\alpha) + \gamma(x_i - x_j + \text{i}\alpha), \quad \overline{\mathcal{L}}_\alpha(x_i, x_j) = \Lambda_{-\alpha}(-x_i, x_j), \\
\Lambda_\alpha(x_i, x_j) &= \gamma(x_i - x_j + \text{i}\alpha) - \gamma(x_i - x_j - \text{i}\alpha), \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= \gamma(x_i + x_j - \text{i}\alpha) + \gamma(x_i + x_j - \text{i}\alpha) + \gamma(-x_i + x_j - \text{i}\alpha) \\
&\quad + \gamma(-x_i - x_j - \text{i}\alpha) - \gamma(-2\text{i}\alpha).
\end{align*}
\] (E.16)
E.3.4 \( H^2_{\epsilon=1} \) (Barnes’s 2nd lemma)

The Lagrangian functions satisfying (24), are given by

\[
\begin{align*}
\Lambda_\alpha(x_i, x_j) &= \gamma(x_i + x_j + i\alpha) - \gamma(x_i - x_j - i\alpha), \\
\overline{\Lambda}_\alpha(x_i, x_j) &= \Lambda(-\alpha | -x_i, -x_j), \\
\mathcal{L}_\alpha(x_i, x_j) &= \gamma(x_i - x_j + i\alpha) + \gamma(-x_i - x_j - i\alpha) - \gamma(x_i - x_j - i\alpha), \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= \gamma(x_i - x_j - i\alpha) + \gamma(x_j - x_i - i\alpha) - \gamma(-2i\alpha).
\end{align*}
\] (E.17)

E.3.5 \( H^2_{\epsilon=0} \) (Barnes’s 1st lemma)

The Lagrangian functions satisfying (24), are given by

\[
\begin{align*}
\Lambda_\alpha(x_i, x_j) &= \gamma(x_i + x_j + i\alpha), \\
\overline{\Lambda}_\alpha(x_i, x_j) &= \Lambda(-\alpha | -x_i, -x_j), \\
\mathcal{L}_\alpha(x_i, x_j) &= \gamma(x_i - x_j + i\alpha) - \gamma(x_i - x_j - i\alpha), \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= \gamma(x_i - x_j - i\alpha) + \gamma(x_j - x_i - i\alpha) - \gamma(-2i\alpha).
\end{align*}
\] (E.18)

E.4 Algebraic cases

E.4.1 \( Q^1_{\delta=1} \) (Selberg-type) [(Zamolodchikov fish-net model]

The Lagrangian functions satisfying (19), are given by

\[
\mathcal{L}_\alpha(x_i, x_j) = \alpha \text{Log}|x_i - x_j| - \frac{\alpha}{2}\text{Log}|\alpha|, \quad \overline{\mathcal{L}}_\alpha(x_i, x_j) = \mathcal{L}_{-\alpha}(x_i, x_j).
\] (E.19)

E.4.2 \( H^1_{\epsilon=1} \) (Euler beta function)

The Lagrangian functions satisfying (24), are given by

\[
\begin{align*}
\Lambda(x_i, x_j) &= ix_i x_j, \quad \overline{\Lambda}(x_i, x_j) = \Lambda(x_i, x_j), \\
\mathcal{L}(x_i, x_j) &= i(\gamma(x_i + x_j - \alpha) + \gamma(-x_i - x_j - \alpha) - \gamma(x_i + x_j - i\alpha) - \gamma(x_i - x_j - i\alpha)) + \gamma(2\alpha), \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= 2i\alpha \text{Log}\left|\sinh\left(\frac{x_i - x_j}{2}\right)\right|.
\end{align*}
\] (E.20)

E.4.3 \( H^1_{\epsilon=1} \) (Barnes’s \( _2F_1 \) integral)

The Lagrangian functions satisfying (24), are given by

\[
\begin{align*}
\Lambda_\alpha(x_i, x_j) &= (ix_j - \alpha) \text{Log}|x_i|, \quad \overline{\Lambda}_\alpha(x_i, x_j) = -\Lambda_\alpha(x_i, x_j), \\
\mathcal{L}_\alpha(x_i, x_j) &= -2\alpha \text{Log}(|x_i| + |x_j|), \\
\overline{\mathcal{L}}_\alpha(x_i, x_j) &= \gamma(x_i - x_j - i\alpha) + \gamma(x_j - x_i - i\alpha) - \gamma(-2i\alpha).
\end{align*}
\] (E.21)
E.4.4  \( H_{1(\varepsilon=0)} \) (Euler beta function)

The Lagrangian functions satisfying (24), are given by

\[
\Lambda(x_i, x_j) = i x_i x_j, \quad \bar{\Lambda}(x_i, x_j) = \Lambda_\alpha(x_i, x_j), \\
\mathcal{L}_\alpha(x_i, x_j) = i \alpha \left( \log(-i\alpha) + \log(i\alpha) - \log(-i(x_i + x_j)) - \log(i(x_i + x_j)) \right), \quad (E.22)
\]

\[
\bar{\mathcal{L}}_\alpha(x_i, x_j) = 2i \alpha \log|x_i - x_j|.
\]

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