Two-dimensional spanning webs as (1, 2) logarithmic minimal model

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Abstract. A lattice model of critical spanning webs is considered for the finite cylinder geometry. Due to the presence of cycles, the model is a generalization of the known spanning tree model which belongs to the class of logarithmic theories with central charge $c = -2$. We show that in the scaling limit the universal part of the partition function for closed boundary conditions at both edges of the cylinder coincides with the character of symplectic fermions with periodic boundary conditions and for an open boundary at one edge and a closed one at the other coincides with the character of symplectic fermions with antiperiodic boundary conditions.

Keywords: conformal field theory, loop models and polymers, rigorous results in statistical mechanics, solvable lattice models

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1. Introduction

The mathematical problem of spanning trees on a connected graph can be considered as a model of statistical mechanics and, as such, it is the first non-trivial exactly solved multidimensional problem thanks to the famous Kirchhoff theorem [1]. In modern classification, the model belongs to the class of free fermion models [2] which admit determinant solutions. The spanning trees are associated with a variety of models, such as the Abelian sandpile [3], Hamiltonian walks on the Manhattan lattice and dense polymer models [4]–[6]. The enumeration of spanning trees on the two-dimensional square lattice is equivalent to the close packed dimer problem solved by Kasteleyn [7] and Temperley and Fisher [8]. In the scaling limit, correlation properties of the spanning trees can be described using the conformal field theory with central charge $c = -2$ [9]–[12].

The simplest generalization of spanning trees are the spanning webs, spanning subgraphs of a connected graph containing cycles together with tree branches attached to them [13]. There are two sources for the appearance of topologically different classes of cycles in models associated with spanning trees. First, periodic boundary conditions in at least one spatial dimension generate cycles. Such cycles appear in the exact solutions of the dimer problem on lattices wrapped on a cylinder or torus [7] and they are non-contractible to a point in the embedding surface. Second, lattice defects like monomers in a dense dimer packing give rise to a different kind of contractible cycles [14,15]. In general, the spanning web model does not belong to the free fermion class and, moreover, it is not exactly solvable. However, for particular geometries of the cycles and appropriate statistical weights of the configurations it retains the free fermion properties.

In this paper we calculate the partition function of a spanning web model on a finite cylinder by considering the number of cycles winding around the cylinder as a parameter. Our aim is to evaluate the leading finite-size corrections to the free energy in the limit of large perimeter of the cylinder. In the absence of cycles the finite-size effects of the spanning tree model confirm predictions of the logarithmic conformal field theory [16]. We show that the presence of cycles changes the Casimir effect in accordance
with conformal weights which appear in the Kac table [17]. An example of exactly solvable logarithmic models with conformal boundary conditions has been given recently by Pearce and Rasmussen [18]. They considered critical dense polymers with certain types of defects on a strip and reproduced the conformal weights in the first column of the extended Kac table. Their results were obtained by means of a functional equation for commuting transfer matrices, formulated in terms of the planar Temperley–Lieb algebra.

Despite the similarity between dense polymers at the free fermion point and spanning trees on an auxiliary sublattice, the classifications of conformal weights in these two models are quite different. The entries of the Kac table for the model of dense polymers are labeled with the number of ‘defect’ lines [18] under fixed boundary conditions on both sides of the strip. We will show that the cycles in the spanning webs play the role of pairs of defect lines in the model of dense polymers. However, in our case the boundary conditions are different for the odd and even entries in the first column of the Kac table. Some complication of the boundary conditions is the fee to be paid for simplicity of derivation of the partition function of the spanning web model. Analytical calculations in sections 2 and 3 are reduced to the standard determinant expressions for the free fermion model with subsequent analysis by use of the Euler–Maclaurin formula. In section 4 we show that the partition function calculated for a finite lattice with different boundary conditions coincides with the characters of coinvariants calculated in different modules for the algebra of symplectic fermions. This allows us to identify open and closed boundary conditions with modules generated by integer and half-integer modes of fermions respectively.

2. The spanning webs model

We consider the labeled graph \( G = (V, E) \) with vertex set \( V \) and set of bonds \( E \). The vertices are sites of the square lattice \( s_{x,y}, 1 \leq x \leq M, 1 \leq y \leq N \), from which we obtain a graph on a cylinder by identifying \( s_{x,y} \) and \( s_{x+M,y} \) for all \( x,y \). The graph \( G \) represents a finite square lattice embedded in a cylinder of height \( N \) and perimeter \( M \), with closed boundary conditions at the top and bottom edges. The term ‘closed’ means the absence of bonds connecting vertices of \( V \) with an exterior of \( G \). We shall consider also the case of open boundary conditions at the vertices \( B \subset V \) belonging to one of the edges \( \{s_{x,1}, 1 \leq x \leq M\} \) and \( \{s_{x,N}, 1 \leq x \leq M\} \) of the cylinder, or to both of them. These cases correspond to a graph \( G' = (V', E') \) with vertex set \( V' = V \cup g \) containing an additional vertex, the root \( g \), and the set of bonds \( E' = E \cup \{(j, g) : j \in B\} \) enlarged with the bonds connecting the vertices of \( B \) with the root \( g \). For convenience of notation we label the boundary conditions with the superscript \((\mu, \nu)\): \( \mu = 0 \) (\( \mu = 1 \)) denotes closed (open) top boundary and \( \nu = 0 \) (\( \nu = 1 \)) closed (open) bottom boundary, respectively. We find it convenient to construct the desired spanning web configurations on the above graphs by using the arrow representation; see e.g. [19]. Accordingly, to each vertex \( i \in V' \) we attach an arrow directed along one of the bonds \((i, i')\) incident to it. Each arrow defines a directed bond \((i \rightarrow i')\) and each configuration of arrows \(A\) on \( G \) defines a spanning directed graph (digraph) \( G_{sd}(A) \) with set of bonds \( E_{sd}(A) = \{(i \rightarrow i') : i, i' \in V\} \) depending on \( A \). Similarly, the arrow configurations on \( G' \) define a spanning digraph \( G'_{sd}(A) \) with set of bonds \( E'_{sd}(A) = \{(i \rightarrow i') : i \in V, i' \in V \cup g\} \). Note that no arrow is attached to vertex \( g \); thus it has out-degree zero. A cycle of length \( k \) is a sequence of directed bonds \((i_1, i_2), (i_2, i_3), (i_3, i_4), \ldots, (i_k, i_1)\) where all \( i_j, 1 \leq j \leq k \), are distinct. If both \((i \rightarrow i')\)
and \((i' \to i)\) belong to the spanning web we say that it contains a cycle of length 2. Our aim is to study sets of spanning digraphs with no cycles other than those which wrap the cylinder. The relevant configurations will be enumerated with the aid of a generating function defined as the determinant of an appropriately constructed weight matrix.

2.1. Cylinder with closed boundaries

We begin with the examination of the determinant expansion of the usual Laplace matrix \(\Delta\) for the graph \(G\). Let the vertices of the set \(V\) be labeled in arbitrary order from 1 to \(n = |V| = MN\). Then \(\Delta\) has the elements \((i, j \in \{1, \ldots, n\})\)

\[
\Delta_{ij} = \begin{cases} 
  z_i, & \text{if } i = j, \\
  -1, & \text{if } i \text{ and } j \text{ are adjacent}, \\
  0, & \text{otherwise},
\end{cases}
\]

where \(z_i\) is the order of vertex \(i\). Since the matrix \(\Delta\) has a zero eigenvalue, its determinant vanishes. On the other hand, the Leibniz formula expresses the determinant of \(\Delta\) as a sum over all permutations \(\sigma\) of the set \(\{1, 2, \ldots, n\}\):

\[
\det \Delta = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \Delta_{1, \sigma(1)} \Delta_{2, \sigma(2)} \cdots \Delta_{n, \sigma(n)} = 0,
\]

where \(S_n\) is the symmetric group, \(\mathrm{sgn}(\sigma) = \pm 1\) is the signature of the permutation \(\sigma\). The identity permutation \(\sigma = \sigma_{\text{id}}\) in (2) yields the term \(z_1 z_2 \cdots z_n\) equal to the number of all possible arrow configurations on \(G\).

In general, each permutation \(\sigma \in S_n\) can be factored into a product (composition) of disjoint cyclic permutations, say, \(\sigma = c_1 \circ c_2 \circ \cdots \circ c_k\). This representation partitions the set of vertices \(V\) into non-empty disjoint subsets—the orbits \(\mathcal{O}_i\) of the corresponding cycles \(c_i, i = 1, \ldots, k\). More precisely, if \(\mathcal{O}_i = \{v_{i,1}, v_{i,2}, \ldots , v_{i,l_i}\} \subset V\) is the orbit of \(c_i\), then \(\bigcup_{i = 1}^k \mathcal{O}_i = V\) and \(\sum_{i = 1}^k l_i = n\), where \(l_i\) is the cardinality of the orbit \(\mathcal{O}_i\) or, equivalently, the length of the cycle \(c_i\). The orbits consisting of just one element, if any, constitute the set \(S_{fp}(\sigma)\) of fixed points of the permutation: \(S_{fp}(\sigma) = \{v = \sigma(v), v \in V\}\). In the case of the identity permutation \(\sigma_{\text{id}} \in S_n\) all orbits consist of exactly one element, \(\mathcal{O}_{i}(\sigma_{\text{id}}) = \{v_i\} \subset V, i = 1, \ldots, n\), and \(S_{fp}(\sigma_{\text{id}}) = V\). A cycle \(c_i\) of length \(|c_i| = l_i \geq 2\) will be called a proper cycle. A proper cycle of length 2 corresponds to two oppositely directed edges which connect a pair of adjacent vertices: \((v_{i,1} \to v_{i,2}), (v_{i,2} \to v_{i,1})\). Note that the vertices of an orbit \(\mathcal{O}_i\) of cardinality \(l_i = |\mathcal{O}_i| \geq 3\) are connected by a closed path on \(G\) which can be traversed in two opposite directions: if \(c_i\) is the cycle defined by \(v_{i,1} \to \sigma(v_{i,1}) = v_{i,2}, \to \cdots \to \sigma(v_{i,l_i}) = v_{i,1}\), then the reverse cycle \(c'_i\) can be represented as \(v_{i,l_i} \to \sigma(v_{i,l_i}) = v_{i,l_i-1}, \to \cdots \to \sigma(v_{i,1}) = v_{i,1}\).

Now we take into account that the proper cycles on \(G\) are of even length only; hence, the signature of every permutation in the expansion of the determinant depends on the number of proper cycles in its factorization, i.e., if \(\sigma = c_1 \circ c_2 \circ \cdots \circ c_p\), where \(|c_j| \geq 2, i = 1, \ldots, p\), then \(\mathrm{sgn}(\sigma) = (-1)^p\). Thus, the terms in equation (2) can be rearranged according to the number \(p\) of disjoint proper cycles as follows:

\[
\prod_{i = 1}^n z_i = \sum_{p = 1}^{[n/2]} (-1)^{p+1} \sum_{\sigma = c_1 \circ \cdots \circ c_p} \prod_{i=1}^p \Delta_{c_i(v_i), c_i(v_i)} \Delta_{c_i^{-1}(v_i), c_i^{-1}(v_i)} \cdots \Delta_{c_i^{-1}(v_i), v_i} \prod_{j \in S_{fp}(\sigma)} z_j.
\]
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Figure 1. Possible spanning digraphs on a cylinder, generated by a single term in the determinant expansion of the corresponding Laplace matrix (see the text). Part (a) corresponds to closed boundary conditions, and part (b) to ones open at the bottom and closed at the top.

Here \(c^k_i\) is the \(k\)-fold composition of the cyclic permutation \(c_i\) of even length \(l_i\), \(v_i \in \mathcal{O}_i(\sigma)\), so that \(c^{k-1}_i(v_i) \neq c^k_i(v_i)\) and \(c^1_i(v_i) = v_i\). Note that all non-vanishing off-diagonal elements are equal to \(-1\).

The above expansion reveals the following features: (i) as expected, all spanning digraphs on \(G\) have at least one proper cycle; (ii) each term on the right-hand side with \(S_{fp}(\sigma) \neq \emptyset\) represents a set of \(\prod_{j \in S_{fp}} z_j\) distinct spanning digraphs which have in common the specified cycles \(c_1, \ldots, c_p\), and differ in the oriented edges outgoing from the vertices \(j \in S_{fp}(\sigma)\), and these oriented edges may form cycles on their own which do not enter into the list \(c_1, \ldots, c_p\); (iii) since the sets \(\bigcup_{p=1}^{p'} O_i\) and \(S_{fp}(c_1, \ldots, c_p)\) are disjoint, the proper cycles formed by the oriented edges incident to the fixed points of a given permutation \(\sigma = c_1 \circ c_2 \circ \cdots \circ c_p\) should enter into the enlarged list of cycles \(c_1, c_2, \ldots, c_p, \ldots, c_{p'}\), \(p' > p\), corresponding to the cycle decomposition of another permutation \(\sigma'\).

For example, consider the determinant of the Laplacian matrix of a cylinder of height 3 and perimeter 4. In the case of closed boundary conditions, the corresponding Leibniz expansion will contain the term

\[
(-1)^3(\Delta_{1,5}\Delta_{5,1})\Delta_{2,2}(\Delta_{3,7}\Delta_{7,8}\Delta_{8,4}\Delta_{4,3})\Delta_{6,6}(\Delta_{9,12}\Delta_{12,11}\Delta_{11,10}\Delta_{10,9}),
\]

which represents, up to the sign, \(z_2z_6 = 12\) spanning digraphs on \(G\) with three specified cycles and all possible oriented bonds outgoing from the vertices 2 and 6, as illustrated in figure 1(a). One of these 12 spanning digraphs will contain the additional cycle 2 \(\rightarrow\) 6 \(\rightarrow\) 2.

In the case of open boundary at the bottom and closed at the top, the term

\[
(-1)^2(\Delta_{1,5}\Delta_{5,1})\Delta_{2,2}(\Delta_{3,3}\Delta_{4,4}(\Delta_{6,10}\Delta_{10,9}\Delta_{9,12}\Delta_{12,11}\Delta_{11,7}\Delta_{7,6})\Delta_{8,8})
\]

represents \(z_2z_3z_4z_8 = 256\) spanning digraphs on \(G'\) with two specified cycles and all possible oriented bonds outgoing from the vertices 2, 3, 4 and 8, as illustrated in figure 1(b). The latter bonds will generate three digraphs with one additional cycle of length 2: 2 \(\rightarrow\) 3 \(\rightarrow\) 2, or 3 \(\rightarrow\) 4 \(\rightarrow\) 3, or 4 \(\rightarrow\) 8 \(\rightarrow\) 4, and one digraph with two additional cycles, 2 \(\rightarrow\) 3 \(\rightarrow\) 2 and 4 \(\rightarrow\) 8 \(\rightarrow\) 4.

As noticed first in [19], the expansion (3) parallels in form the inclusion–exclusion principle in combinatorial mathematics. Indeed, let \(c_1, c_2, \ldots, c_m\) be the list of all possible
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proper cycles on \( G \), labeled in an arbitrary order. Define \( A_i, i = 1, 2, \ldots, m \), as the set of all spanning digraphs on \( G \) containing the particular cycle \( c_i \). Then, expansion (3) can be written in the form of the inclusion–exclusion principle:

\[
|\bigcup_{i=1}^m A_i| = \sum_{i=1}^m |A_i| - \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{m+1} |A_1 \cap \cdots \cap A_m|,
\]

which holds for any finite sets \( A_1, A_2, \ldots, A_m \), where \( |A| \) is the cardinality of the set \( A \).

Now we are in a position to define a matrix \( D^{(0,0)} \), associated with the graph \( G \), such that \( \det D^{(0,0)} \) is the generating function of all spanning digraphs on \( G \) which have no contractible cycles. The elements \( D^{(0,0)}_{ij}, i, j = 1, \ldots, n = NM \), of \( D^{(0,0)} \) are explicitly given as

\[
D^{(0,0)}_{ij} = \begin{cases} 
  z_i, & \text{if } i = j, \\
  -1, & \text{if } i, j \text{ are vertical neighbors,} \\
  -a, & \text{if } i \text{ is left neighbor of } j, \\
  -a^{-1}, & \text{if } i \text{ is right neighbor of } j, \\
  0, & \text{otherwise.}
\end{cases}
\]

Here \( a = \omega^{1/M} e^{-i \pi/M} \), the terms ‘left’ and ‘right’ fix the opposite directions of the horizontal edges. Note that all closed paths which do not wrap the cylinder contain an equal number of horizontal edges with either orientation; hence, their weight in \( \det D^{(0,0)} \) remains the same as in \( \det \Delta \). Therefore, all the configurations which contain such closed paths (contractible cycles) cancel out in the expansion of \( \det D^{(0,0)} \). On the other hand, cycles generated by off-diagonal elements that wrap the cylinder change their sign, because they contain horizontal edges oriented in one direction exceeding by \( M \) the number of edges in the opposite direction. This amounts to the total factor of \( a^M = -\omega \) or \( a^{-M} = -\omega^{-1} \), depending on the orientation. Therefore, each non-contractible cycle with a given orientation is counted twice, however, with different weight—once it enters into the determinant expansion with unit weight, being generated by diagonal elements of the matrix \( D^{(0,0)} \), and the second time it enters with a factor \( \omega \) or \( \omega^{-1} \) (depending on the orientation) as generated by off-diagonal elements of that matrix. Thus, the total number of non-contractible cycles, irrespective of their origin and orientation, is given by the coefficient in front of the corresponding power of \( \omega + \omega^{-1} + 2 \equiv \xi \) in the series expansion of the partition function. The power of \( \xi \) is the ‘good quantum number’ which is a well defined and conserved quantity under the action of the transfer matrix along the cylinder. However, as we shall see below, more convenient expansions of the partition function, which can be directly compared with characters of the Virasoro modules, are given by the power series in \( \omega \) itself, or in terms of combinations like \((\omega + \omega^{-1})^s\) and \( \sum_{k=0}^s \omega^{s-2k} \). In general, besides the non-contractible cycles, the spanning digraph contains tree subgraphs connected to them. All branches of the trees can be generated only by the diagonal elements of \( D^{(0,0)} \) and, hence, carry unit weight.

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Figure 2. Spanning digraphs on a cylinder of height \(N = 6\) and perimeter \(M = 10\) under: (a) closed boundary conditions; the two closed paths \(x-x\) and \(y-y\) represent non-contractible cycles; (b) open boundary conditions at the bottom and closed ones at the top; the three bonds incident with the root give rise to boundary trees.

2.2. Cylinder with one closed and one open boundary

The Laplacian matrix \(\Delta^*\) for the graph \(G'\) corresponding to \((\mu, \nu) = (0, 1)\) boundary conditions is an \((n + 1) \times (n + 1)\) matrix of the same form (1) as long as the notions of degree of a vertex and adjacency are understood in the context of \(G'\). However, to make the similarities and dissimilarities with the former case apparent, we retain the notation \(z_i\) for the degree of vertex \(i, i = 1, \ldots, n,\) with respect to \(G',\) and explicitly introduce the label \(g\) for the root with degree \(M\) in \(G'\); see figure 2(b). Thus, for the matrix elements of \(\Delta^*\) we have

\[
\Delta^*_{ij} = \begin{cases} 
z_i, & \text{if } i = j \in V \setminus B, \\
z_i + 1, & \text{if } i = j \in B, \\
M, & \text{if } i = j = g, \\
-1, & \text{if } i, j \in V \text{ are adjacent in } G, \\
-1, & \text{if } i \in B, j = g \text{ or } i = g, j \in B, \\
0, & \text{otherwise.} \end{cases}
\]

(8)

Here \(B\) is the set of bottom boundary vertices adjacent to the root \(g\) in \(G'.\) Now we make use of the fact that by Kirchhoff’s theorem the number \(N_{st}\) of spanning trees on \(G'\) is equal to any cofactor of \(\Delta^*\) and chose for convenience the cofactor \(C_{gg}\) of the element \(\Delta^*_{gg}\).

Then \(N_{st} = C_{gg} = \det \Delta',\) where \(\Delta'\) is the \(n \times n\) matrix with elements \((i, j = 1, \ldots, n)\):

\[
\Delta'_{ij} = \begin{cases} 
z_i, & \text{if } i = j \in V \setminus B, \\
z_i + 1, & \text{if } i = j \in B, \\
-1, & \text{if } i, j \in V \text{ are adjacent,} \\
0, & \text{otherwise.} \end{cases}
\]

(9)

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By comparing the above expression with (1) one sees that the only difference is in the diagonal elements: the order of the vertices belonging to the open boundary has been increased by 1. Therefore, the same arguments as in the previous section lead us to the matrix $D^{(0,1)}$ with elements

$$D_{ij}^{(0,1)} = \begin{cases} 
  z_i, & \text{if } i = j \in V \setminus B, \\
  z_i + 1, & \text{if } i = j \in B, \\
  -1, & \text{if } i, j \text{ are vertical neighbors}, \\
  -a, & \text{if } i \text{ is left neighbor of } j, \\
  -a^{-1}, & \text{if } i \text{ is right neighbor of } j, \\
  0, & \text{otherwise}, 
\end{cases}$$

(10)

where, as before, $a = \omega^{1/M}e^{-i\pi/M}$. By construction, in the expansion of $\det D^{(0,1)}$ all the arrow configurations with contractible cycles cancel out. Thus $\det D^{(0,1)}$ is the generating function of all the spanning digraphs which are either spanning forests of trees rooted at the open bottom boundary, or contain non-contractible cycles wrapping the cylinder and tree subgraphs rooted at these cycles or at the open boundary. As in the case of closed boundaries, each non-contractible cycle with a given orientation is counted twice with different weights—once with unit weight, being generated by diagonal elements of the matrix $D^{(0,1)}$, and the second time with a factor $\omega$ or $\omega^{-1}$ (depending on the orientation) as generated by off-diagonal elements of that matrix.

The case of open–open boundary conditions, $(\mu, \nu) = (1, 1)$, can be considered analogously to the previous two cases and we do not describe it separately.

The partition function calculated as the determinant of the Laplacian, $Z_{NM}^{(\mu,\nu)} = \det D^{(\mu,\nu)}$, can be split into a product of two parts,

$$Z_{NM}^{(\mu,\nu)} = e^{F_{NM}^{(\mu,\nu)}} \tilde{Z}_{N}^{(\mu,\nu)}(q, \omega),$$

(11)

where $e^{F_{NM}^{(\mu,\nu)}}$ is the non-universal part of the partition function, including bulk and boundary free energy $F_{NM}^{(\mu,\nu)}$, and $\tilde{Z}_{N}^{(\mu,\nu)}(q, \omega)$ is the universal part which is a polynomial in the aspect ratio parameter $q = e^{-\pi M/N}$ and a Laurent polynomial in $\omega$. The universal part of the partition function can be decomposed into different combinations of $\omega$, for example,

$$\tilde{Z}_{N}^{(\mu,\nu)}(q, \omega) = \sum_{s=0}^{N} C_{1,s}^{(\mu,\nu)} [N](q)(\omega + \omega^{-1})^s,$$

(12)

$$\tilde{Z}_{N}^{(\mu,\nu)}(q, \omega) = \sum_{s=-N}^{N} C_{2,s}^{(\mu,\nu)} [N](q)\omega^s,$$

(13)

$$\tilde{Z}_{N}^{(\mu,\nu)}(q, \omega) = \sum_{s=0}^{N} C_{3,s}^{(\mu,\nu)} [N](q) \sum_{k=0}^{s} \omega^{s-2k},$$

(14)
valid for all boundary conditions under consideration, \( \mu, \nu = 0, 1 \). Note that the second and third polynomials can be obtained from the first one by using the relations

\[
C_{2,s}^{(\mu,\nu)}[N] = C_{2,-s}^{(\mu,\nu)}[N] = \sum_{p=0}^{([N-s]/2)} \left( \frac{s + 2p}{p} \right) C_{2,s+2p}^{(\mu,\nu)}[N], \quad s = 0, \ldots, N, \tag{15}
\]

\[
C_{3,s}^{(\mu,\nu)}[N] = C_{2,s}^{(\mu,\nu)}[N] - C_{2,s+2}^{(\mu,\nu)}[N], \quad s = 0, \ldots, N - 2, \tag{16}
\]

\[
C_{3,s}^{(\mu,\nu)}[N] = C_{2,s}^{(\mu,\nu)}[N], \quad s = N - 1, N. \tag{17}
\]

Each of the polynomials \( C_{i,s}^{(\mu,\nu)}[N](q) \), \( i = 1, 2, 3 \), has a well defined statistical meaning. For example, \( C_{i,s}^{(\mu,\nu)}[N](q) \) is the universal factor in the partition function which is proportional to the number of configurations with \( s \) non-contractible cycles of either orientation, generated by off-diagonal elements of the matrix \( D^{(\mu,\nu)} \); these configurations may have also any allowed (by the size of the cylinder) number of non-contractible cycles of both orientations, generated by diagonal elements of that matrix. On the other hand, the universal factor \( C_{2,s}^{(\mu,\nu)}[N](q) \) is proportional to the number of configurations with fixed to \( s \) difference in the numbers of non-contractible cycles with positive and negative orientation, generated by off-diagonal elements of the matrix \( D^{(\mu,\nu)} \); the total number of such cycles, as well as the number of non-contractible cycles of either orientation, generated by diagonal elements of the same matrix, may take any allowed values.

In the next section, we shall evaluate the asymptotic form of the coefficients \( C_{i,s}^{(\mu,\nu)}[N] \), \( i = 2, 3 \), for \( M \to \infty, N \to \infty \) and disclose their relation to the finitized characters of the logarithmic Virasoro modules.

### 3. Calculation of the partition function

The matrices of edge weights \( D^{(\mu,\nu)} \) can be written in the form of a sum of direct products of simple \( N \times N \) and \( M \times M \) matrices:

\[
D^{(\mu,\nu)} = (2E_N - Q_N^{(\mu,\nu)}) \otimes E_M + E_N \otimes (2E_M + aG_M + a^{-1}G_M^T), \tag{18}
\]

where \( E_N \) (\( E_M \)) is the unit \( N \times N \) (\( M \times M \)) matrix, \( Q_N^{(\mu,\nu)} = \{q_{i,j}^{(\mu,\nu)}\} \) is a tridiagonal matrix with elements dependent on the boundary conditions \( i, j = 1, \ldots, N \):

\[
q_{i,j}^{(0,0)} = \delta_{i,1}\delta_{j,1} + \delta_{i,-1,j} + \delta_{i,j,-1} + \delta_{i,N}\delta_{j,N}, \tag{19}
\]

\[
q_{i,j}^{(0,1)} = q_{i,j}^{(1,0)} = \delta_{i,1}\delta_{j,1} + \delta_{i,-1,j} + \delta_{i,j,-1}, \tag{20}
\]

\[
q_{i,j}^{(1,1)} = \delta_{i-1,j} + \delta_{i,j-1}. \tag{21}
\]

Note that \( 2E_N - Q_N^{(\mu,\nu)} \) differs only in sign from the one-dimensional discrete Laplacian on a chain of \( N \) sites with the analogues of Neumann–Neumann (\( \mu = \nu = 0 \)), Neumann–Dirichlet (\( \mu = 0, \nu = 1 \) or \( \mu = 1, \nu = 0 \)) and Dirichlet–Dirichlet (\( \mu = 1, \nu = 1 \)) boundary conditions. Hence, the eigenvalues of \( 2E_N - Q_N^{(\mu,\nu)} \) are \( \lambda_N^{(0,0)}(p) = 2 - 2\cos(\pi p/N) \), \( \lambda_N^{(0,1)}(p) = \lambda_N^{(1,0)}(p) = 2 - 2\cos(\pi(2p+1)/(2N+1)) \), and
The above exact partition function is a polynomial in $G$ and $\bar{G}$, and

$$Z^{(\mu,\nu)}_{NM} = \prod_{p=0}^{N-1} \prod_{k=0}^{M-1} \left[ \lambda_N^{(\mu,\nu)}(p) + 2 - \omega^{1/M} e^{i\pi(2k+1)/M} - \omega^{-1/M} e^{-i\pi(2k+1)/M} \right].$$

By analytic continuation of the identity

$$\prod_{k=0}^{M-1} \left[ Q^2 + Q^{-2} - 2 \cos \left( \frac{2\pi k}{M} + \alpha \right) \right] = Q^{2M} + Q^{-2M} - 2\cos(M\alpha),$$

with

$$Q \equiv Q_N^{(\mu,\nu)}(p) = \sqrt{1 + \sin^2 \phi_N^{(\mu,\nu)}(p) + \sin \phi_N^{(\mu,\nu)}(p)},$$

$$\phi_N^{(0,0)}(p) = \frac{\pi p}{2N}, \quad \phi_N^{(0,1)}(p) = \phi_N^{(1,0)}(p) = \frac{\pi(2p+1)}{2(2N+1)}, \quad \phi_N^{(1,1)}(p) = \frac{\pi(p+1)}{2(N+1)},$$

from real $\alpha$ to complex $\alpha = (\pi - i\ln \omega)/M$, we obtain

$$Z^{(\mu,\nu)}_{NM} = \prod_{p=0}^{N-1} \left[ (Q_N^{(\mu,\nu)}(p))^{2M} \left\{ 1 + (\omega + \omega^{-1})[Q_N^{(\mu,\nu)}(p)]^{-2M} + [Q_N^{(\mu,\nu)}(p)]^{-4M} \right\} \right].$$

The above exact partition function is a polynomial in $x = \omega + \omega^{-1}$ of degree $N$; see (12). When $M, N \to \infty$, and so $M/N = O(1)$, a standard finite-size analysis of the contribution to the free energy from the $\omega$-independent factor in the partition function yields the asymptotic expansion

$$\ln \prod_{p=0}^{N-1} (Q_N^{(\mu,\nu)}(p))^{2M} \simeq \frac{4G}{\pi} M N + M \left[ \frac{2G}{\pi} (\mu + \nu) - \ln(1 + \sqrt{2}) \right] - \frac{\pi M}{N} \left[ \frac{1}{12} - \frac{(\mu - \nu)^2}{8} \right],$$

where $G$ is Catalan’s constant.

The leading-order asymptotic form of the $\omega$-dependent factor in the partition function follows from the approximation $Q_N^{(\mu,\nu)}(p) \simeq 1 + \phi_N^{(\mu,\nu)}(p)$. Thus we obtain

$$\prod_{p=0}^{N-1} \left\{ 1 + x [Q_N^{(\mu,\nu)}(p)]^{-2M} + [Q_N^{(\mu,\nu)}(p)]^{-4M} \right\} \simeq \prod_{p=0}^{N-1} \left\{ 1 + xe^{-(\pi M/2N)(2p+\mu+\nu)} + e^{-(\pi M/N)(2p+\mu+\nu)} \right\},$$

where $x = \omega + \omega^{-1}$. doi:10.1088/1742-5468/2008/11/P11017
For $N$ and $M$ large and $N/M$ fixed, we rewrite (25) in the form (11) with the free energy

$$F_{MN}^{(\mu, \nu)} = \frac{4G}{\pi} MN + M \left[ \frac{2G}{\pi} (\mu + \nu) - \ln(1 + \sqrt{2}) \right],$$

(28)

and universal part of the partition function

$$\tilde{Z}_N^{(\mu, \nu)}(q, \omega) = q^{1/12 - (1/8)(\mu-\nu)^2} \prod_{j=0}^{N-1} \left[ 1 + (\omega + \omega^{-1})q^{j+(1/2)(\mu+\nu)} + q^{2j+\mu+\nu} \right].$$

(29)

In the next section, we show that (29) converges as $N \to \infty$ to the characters of symplectic fermions [21] or, equivalently, to the characters of the doublet algebra $A(2)$ [20] in the $p = 2$ logarithmic model. As is well known, the characters contain complete information about conformal dimensions and their multiplicities in the model.

Before considering the symplectic fermions, we calculate the coefficients $C_{2,s}^{(\mu, \nu)}[N](q)$ (16) and $C_{3,s}^{(\mu, \nu)}[N](q)$ (17) explicitly. The universal part of the partition function (29) can be rewritten by using the Newton $q$-binomial formula

$$\prod_{p=0}^{N-1} (1 + yq^p) = \sum_{s=0}^{N} q^{s(s-1)/2} \binom{N}{s}_q y^s,$$

(30)

with $q$-binomial coefficients

$$\binom{N}{s}_q = \frac{(1 - q^N) \cdots (1 - q^{N-s+1})}{(1 - q) \cdots (1 - q^s)}, \quad \text{when } 0 \leq s \leq N,$$

$$\binom{N}{s}_q = 0, \quad \text{when } s < 0 \text{ or } s > N,$$

(31)

in the form (13) with

$$C_{2,s}^{(\mu, \nu)}[N](q) = q^{1/12 - (1-\nu)^2/8 + s(1+\mu+\nu)/2} \sum_{k=0}^{N-s} q^{k^2+k(s-1+\mu+\nu)} \binom{N}{k}_q \binom{N}{s+k}_q,$$

(32)

and

$$C_{2,s}^{(\mu, \nu)}[N](q) = C_{2,-s}^{(\mu, \nu)}[N](q).$$

For closed–closed and open–closed boundary conditions the summation can be performed explicitly and the above expression simplifies to

$$C_{2,s}^{(0,0)}[N](q) = q^{1/12 + s(s-1)/2} \left[ \left( \binom{2N}{N+s-1}_q \right) + q^s \left( \binom{2N}{N+s}_q \right) \right]$$

(33)

and

$$C_{2,s}^{(0,1)}[N](q) = q^{-1/24 + s^2/2} \left( \binom{2N}{N+s}_q \right),$$

(34)

respectively.

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Now we give explicit expressions for $C_{3,s}^{(\mu,\nu)}[N](q)$, which are shown in the next section to converge to the Virasoro characters as $N \to \infty$. Using (16), we obtain for $0 \leq s \leq N-2$

$$C_{3,s}^{(\mu,\nu)}[N](q) = C_{2,s}^{(\mu,\nu)}[N](q) - C_{2,s+1}^{(\mu,\nu)}[N](q)$$

$$= q^{1/12-(\mu-\nu)/2} \sum_{k=0}^{N-s} q^{2k+k(s-1+\mu+\nu)/2} \left( \begin{array}{c} N \\ k \end{array} \right) q^{(N_k)} q^{(N_{k+1})} \left( \begin{array}{c} N \\ k \end{array} \right) q^{(N_{k+1})} \left( \begin{array}{c} N \\ k+1 \end{array} \right) q^{(N_{k+2})} \left( \begin{array}{c} N \\ k+2 \end{array} \right) q^{(N_{k+3})} \left( \begin{array}{c} N \\ k+3 \end{array} \right) .$$

We note that

$$C_{3,s}^{(0,0)}[N](q) = q^{1/12} \left[ q^{s(s-1)/2} \left( \begin{array}{c} 2N - 1 \\ N + s + 1 \end{array} \right) q^{(s+1)/2} \left( \begin{array}{c} 2N - 1 \\ N + s \end{array} \right) q^{s+1} \sum_{k=1}^{N-s} q^{2k+k(s-1+\mu+\nu)/2} \left( \begin{array}{c} N \\ k \end{array} \right) q^{(N_{k+1})} \left( \begin{array}{c} N \\ k+1 \end{array} \right) q^{(N_{k+2})} \left( \begin{array}{c} N \\ k+2 \end{array} \right) q^{(N_{k+3})} \left( \begin{array}{c} N \\ k+3 \end{array} \right) \right].$$

$$= \chi_{2s+1}^{(2N-1)}(q) + \chi_{2s+3}^{(2N-1)}(q), \quad (36)$$

where $\chi_{s}^{(N)}(q)$ are the finitized characters of the logarithmic Virasoro modules, given by equation (5.5) in [18] for $s$ odd. It should be mentioned that (36) resembles equation (7.25) in [18] for the finitized character of the logarithmic Virasoro module.

Similarly,

$$C_{3,s}^{(0,1)}[N](q) = q^{-1/24 + s^2/2} \left[ \left( \begin{array}{c} 2N \\ N + s \end{array} \right) q - q^{2s+2} \left( \begin{array}{c} 2N \\ N + s + 2 \end{array} \right) \right] = \chi_{2s+2}^{(2N+1)}(q), \quad (37)$$

coincides with the finitized Virasoro characters $\chi_{2s+2}^{(2N+1)}(q)$ given by equation (5.5) in [18] for $s$ even.

On the other hand, under the substitution $(N, s) \to (2N + 1, 2s + 1)$ in equation (5.5) in [18] for $s$ odd, we obtain the relationship

$$\chi_{2s+1}^{(2N+1)}(q) = C_{3,s-1}^{(1,1)}[N](q) + C_{3,s}^{(1,1)}[N](q), \quad (38)$$

This relation corresponds to the fact that $\chi_{2s+1}^{(2N+1)}(q)$ is the finitized character of the rank-1 indecomposable Virasoro module with two irreducible subquotients whose finitized characters are given by $C_{3,s-1}^{(1,1)}[N](q)$ and $C_{3,s}^{(1,1)}[N](q)$.

Next, equation (29) implies

$$\tilde{Z}_{N+1}^{(0,0)}(q, \omega) = (\omega + 2 + \omega^{-1}) \tilde{Z}_{N}^{(1,1)}(q, \omega). \quad (39)$$

Hence, the coefficients $C_{3,s}^{(0,0)}[N]$ and $C_{3,s}^{(1,1)}[N - 1]$ are related by the equality

$$C_{3,s}^{(0,0)}[N](q) = C_{3,s-1}^{(1,1)}[N - 1](q) + 2C_{3,s}^{(1,1)}[N - 1](q) + C_{3,s+1}^{(1,1)}[N - 1](q), \quad (40)$$

which repeats the relations between characters of the logarithmic and irreducible Virasoro modules. Finally, from equations (33) and (34) it follows that the conformal weights for

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Figure 3. Correspondence between spanning webs and dense polymers. (a) The cycle $x-x$ is surrounded by two polymers, 1–1 and 2–2, and the cycle $y-y$ by another two polymers, 3–3 and 4–4, which are considered as defect lines in the classification of [17,18]. (b) Besides the four defect polymer lines, there is a fifth line 5–5 separating the boundary trees from the rest of the lattice.

The open–open and open–closed boundary conditions are

$$\Delta_s^{(0,0)} = \frac{s(s-1)}{2}, \quad s = 0, 1, 2, \ldots$$

and

$$\Delta_s^{(0,1)} = \frac{4s^2 - 1}{8}, \quad s = 0, 1, 2, \ldots.$$  (42)

These two sequences can be arranged into the first column of the extended Kac table so that its odd entries correspond to (41) and the even ones to (42).

It is instructive to compare the results obtained with those of Pearce and Rasmussen [18]. The partition function of dense polymers evaluated in [18] is a function of the number of defect lines $l = 0, 1, 2, \ldots$ in the polymer system, so the extended Kac label $s = l+1$ runs over all entries in the first column of the table of conformal weights. It is easy to see that even cells differ from their odd counterparts by a fixed value $-1/8$. This value can be associated with the conformal weight $h_{\text{min}}$ of the operator with the smallest scaling dimension present in the spectrum of the Hamiltonian, which depends on the boundary conditions (see for instance [16] where it is shown that $h_{\text{min}} = 0$ for open–open boundary conditions and $h_{\text{min}} = -1/8$ for open–closed ones). However, for the dense polymers model the boundary conditions on either side of the infinite strip are equal for even and odd values of $s$. A comparison with the model of spanning webs resolves this illusory contradiction. The correspondence between our spanning webs model and the model of critical dense polymers is shown in figure 3. The left-hand figure shows a spanning web for closed boundary conditions. The polymer lines envelop branches and cycles of the spanning web so that each cycle is surrounded by two polymer lines. Then, for closed boundary conditions the number of defect lines corresponding to polymers surrounding the cycles is always even. The situation for open boundary conditions is shown in the right-hand figure. The branches of the spanning web going to the root can be separated.
from the rest of the web by an additional polymer line (marked by 5 in figure 3). Thus, the total number of defect lines becomes odd and one obtains the set of even entries of the Kac table.

4. Conformal field theory of spanning webs

The partition function (29) has an interpretation in terms of symplectic fermions [21]. The symplectic fermions are fermionic fields $\theta^\pm(z)$ with operator product expansion

$$\theta^+(z)\theta^-(w) \sim \log(z - w).$$  \hspace{1cm} (43)

These fields admit periodic and antiperiodic boundary conditions for which they decompose with integer $\theta^\pm_n$, $n \in \mathbb{Z}$, and half-integer $\theta^\pm_n$, $n \in \mathbb{Z} + \frac{1}{2}$, modes, respectively. These modes satisfy the anticommutation relations

$$[\theta^+_n, \theta^-_m] = n\delta_{n+m,0}. \hspace{1cm} (44)$$

Let $\mathcal{A}(2)$ denote this infinite-dimensional Clifford algebra. Strictly speaking, $\mathcal{A}(2)$ is not an algebra or a vertex-operator algebra because multiplication between integer and half-integer modes is not defined, but $\mathcal{A}(2)$ is very similar to ordinary vertex-operator algebras and many standard notions can be defined for it (see discussion on this subject in [22]).

The algebra $\mathcal{A}(2)$ contains the Virasoro subalgebra generated by the energy–momentum tensor

$$T(z) =: \partial \theta^+(z)\partial \theta^-(z):,$$  \hspace{1cm} (45)

with central charge $c = -2$.

The algebra $\mathcal{A}(2)$ has two irreducible modules $X_1$ and $X_2$ (see details in [20,22]). Modules $X_1$ and $X_2$ are cyclic with cyclic (vacuum) vectors $|11\rangle$ satisfying $\theta^\pm_n|11\rangle = 0$ for $n \geq 0$ and $|01\rangle$ satisfying $\theta^\pm_n|01\rangle = 0$ for $n \geq \frac{1}{2}$. Module $X_1$ is generated by integer modes $\theta^\pm_n$ with $n \leq -1$ from the vacuum vector $|11\rangle$. Module $X_2$ is generated by half-integer modes $\theta^\pm_n$ with $n \leq -\frac{1}{2}$ from the vacuum vector $|01\rangle$. The algebra $\mathcal{A}(2)$ has two projective modules $P_1$ and $P_2 = X_2$. The module $P_1$ contains four irreducible subquotients isomorphic to $X_1$ [22]. The module $P_1$ is cyclic with vacuum vector $|00\rangle$ satisfying $\theta^\pm_n|00\rangle = 0$ for $n \geq 1$ and is generated from $|00\rangle$ by integer modes $\theta^\pm_n$ with $n \leq 0$.

For an $\mathcal{A}(2)$-module $X$, we define the character

$$\chi(q,z) = \text{Tr}_X q^{L_0-c/24} z^h,$$  \hspace{1cm} (46)

where $L_0 = (1/2\pi i) \oint zT(z)dz$ and $h$ is the operator calculating the difference between the numbers of $\theta^+$ and $\theta^-$ modes in a state.

The character of $X_1$ is

$$\chi^{(1,1)}(q,\omega) = \frac{q^{1/12}}{\prod_{n=1}^\infty (1 - q^n)} \sum_{r \in \mathbb{N}} \sum_{j=0}^r \omega^{r-2j} q^{r(r-1)/2(1-q^r)}. \hspace{1cm} (47)$$

The character of $X_2$ is

$$\chi^{(0,1)}(q,\omega) = \frac{q^{-1/24}}{\prod_{n=1}^\infty (1 - q^n)} \sum_{r \in \mathbb{N}} \sum_{j=0}^r \omega^{r-2j} q^{r(r-1)/2(1-q^{2r})}. \hspace{1cm} (48)$$

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The character of $P_1$ is
\[
\chi^{(0,0)}(q, \omega) = (2 + \omega + \omega^{-1})\chi^{(1,1)}(q, \omega).
\] (49)

The same characters can be written in terms of Virasoro characters in the form
\[
\chi^{(1,1)}(q, \omega) = \sum_{r \in \mathbb{N}} \sum_{j=0}^{r} \omega^{r-2j} \chi_1(q)
\] (50)

and
\[
\chi^{(0,1)}(q, \omega) = \sum_{r \in \mathbb{N}} \sum_{j=0}^{r} \omega^{r-2j} \chi_2(q),
\] (51)

where
\[
\chi_{rs}(q) = q^{\Delta_{rs} - c/24} \frac{1 - q^{r+s}}{\prod_{n=1}^{\infty} (1 - q^n)}
\] (52)
are Rocha–Caridi characters of the irreducible Virasoro representations with conformal dimensions
\[
\Delta_{rs} = \frac{(2r - s)^2 - 1}{8}
\] (53)
and $c = -2$.

Now we intend to identify (29) with characters of some coinvariants calculated in $X_1$, $X_2$ and $P_1$. (By definition, the coinvariant in the $A$-module $X$ with respect to the subalgebra $B \subset A$ is the quotient $X/BX$.) We fix the subalgebra $A(2)[N]$ of $A(2)$ for $N \in \mathbb{N}$:
\[
A(2)[N] = \begin{cases} \{ \theta_{n}, n \geq N \}, & \text{periodic}, \\ \{ \theta_{-n+1/2}, n \geq N \}, & \text{antiperiodic}, \end{cases}
\] (54)
and consider the characters $\chi^{(1,1)}[N](q, \omega)$, $\chi^{(0,1)}[N](q, \omega)$ and $\chi^{(0,0)}[N](q, \omega)$ of coinvariants with respect to $A(2)[N]$ in the modules $X_1$, $X_2$ and $P_1$, respectively. These characters coincide with (29):
\[
\chi^{(1,1)}[N](q, \omega) = \tilde{Z}_{N}^{(1,1)}(q, \omega).
\] (55)

For $N \to \infty$ one has
\[
\chi^{(1,1)}[N](q, \omega) \to \chi^{(1,1)}(q, \omega).
\] (56)

The polynomials $C_{3,s}^{(1,1)}[N](q)$ and $C_{3,s}^{(0,1)}[N](q)$ converge to the characters of irreducible Virasoro modules
\[
C_{3,s}^{(1,1)}[N](q) \to \chi_{s1}(q), \quad C_{3,s}^{(0,1)}[N](q) \to \chi_{s2}(q),
\] (57)
and $C_{3,s}^{(0,0)}[N](q)$ converges to the characters of logarithmic Virasoro modules for the $\mathcal{LM}(1, 2)$ model [17] as $N$ tends to infinity. Note also that characters (55) can be expressed through the Kostka-like polynomials $\hat{K}_{L,N}^{(2)}(q, \omega)$ from [22] in the following way:
\[
\chi^{(1,1)}[N](q, \omega) = q^{1/12} \hat{K}_{1,2N}^{(2)}(q, \omega), \quad \chi^{(0,1)}[N](q, \omega) = q^{-1/24} \hat{K}_{2,2N+1}^{(2)}(q, \omega).
\] (58)
Now we can identify open and closed boundary conditions of the spanning webs model with irreducible modules of the algebra $\mathcal{A}(2)$. In conformal field theory, the boundary conditions are in one-to-one correspondence with irreducible modules of the chiral algebra [23]. For the partition function $\bar{Z}^{(\mu,\nu)}$ on a cylinder with boundary conditions $\mu$ and $\nu$ the conformal field theory predicts

$$\bar{Z}^{(\mu,\nu)}(q) = \sum_\gamma N^{\gamma}_{\mu\nu} \chi_\gamma(q),$$

where $N^{\gamma}_{\mu\nu}$ are structure constants in the fusion between modules labeled with $\mu$ and $\nu$ and $\chi_\gamma(q)$ are characters of modules appearing in this fusion. The fusion of $\mathcal{A}(2)$ modules is

$$X_1 \otimes X_1 = X_1, \quad X_1 \otimes X_2 = X_2, \quad X_2 \otimes X_2 = P_1.$$  

The fusion together with identification (55) leads to the correspondence

$$\text{open b.c.} \leftrightarrow X_1, \quad \text{closed b.c.} \leftrightarrow X_2.$$  

An identification of triplet $W$ algebra or Virasoro boundary conditions [24] is more subtle because of their non-local nature. We start the consideration with $W$ boundary conditions. Each $\mathcal{A}(2)$-module decomposes into direct sum of two $W$ modules, which are labeled with the same symbol with additional superscript $\pm$:

$$X_1 = X_1^+ \oplus X_1^-, \quad X_2 = X_2^+ \oplus X_2^-, \quad P_1 = P_1^+ \oplus P_1^-.$$  

The characters of $W$ modules are obtained from characters of $\mathcal{A}(2)$ modules by taking odd and even parts in $\omega$. For example

$$\chi_1^\pm(q) = \frac{1}{2} \left[ \chi^{(1,1)}(q,1) \pm \chi^{(1,1)}(q,-1) \right],$$

where $\chi_1^\pm(q)$ are the characters of $X_1^\pm$. We can interpret this in lattice terms as taking only configurations with even or odd differences between numbers of non-contractible loops of different orientations (see also the paragraph after equation (17)). However, it is not clear how to formulate such conditions as local boundary conditions without reference to the bulk.

In order to establish a connection with boundary conditions corresponding to Virasoro representations we note that the algebra $\mathcal{A}(2)$ admits an $sl(2)$ action such that $\theta^+(z)$ and $\theta^-(z)$ are the highest and lowest weight vectors of the doublet. This $sl(2)$ action commutes with the Virasoro subalgebra (45) and therefore irreducible $\mathcal{A}(2)$ modules decompose as

$$X_1 = \bigoplus_{r \in \mathbb{N}} \pi_r \otimes Y_{r,1}, \quad X_2 = \bigoplus_{r \in \mathbb{N}} \pi_r \otimes Y_{r,2},$$

where $\pi_r$ is an $r$-dimensional irreducible $sl(2)$ representation and $Y_{r,s}$ is the Virasoro irreducible representation with the conformal dimension (53). The character of $\pi_r$ is $\sum_{j=0}^r w^{r-2j}$, which explains the decompositions (50), (51) and our definition of polynomials $C_{3,s}^{(\mu,\nu)}[N](q)$ in (14). A lattice interpretation of $C_{3,s}^{(\mu,\nu)}[N](q)$ is very cumbersome and appeals to conditions on non-contractible loops in the bulk like in the $W$ case.
5. Conclusions

In this paper we have found the exact partition function for a model of spanning webs parameterized by the number of non-contractible cycles for the finite cylinder geometry. We have calculated the leading finite-size corrections and identified them with the finitized characters for the minimal logarithmic conformal field theory with $c = -2$.

The model considered here is similar in many aspects to the model of critical dense polymers solved by Pearce and Rasmussen [18] by using the planar Temperley–Lieb algebra and commuting double-row transfer matrices. There are, however, several features which differ in these models. First, the cylinder geometry admits classification of the web configurations in terms of numbers of non-contractible cycles which are well defined ‘quantum numbers’. Regarding the conservation law in the system of dense polymers, one notices that the transfer matrices used in [18] have a block triangular structure. This structure reflects the fact that defect lines can be annihilated in pairs and, therefore, the number of defects is not conserved. Embedding the system into the cylinder geometry, which is the case for our model, is equivalent to taking the trace of a transfer matrix; hence, the number of non-contractible cycles (or defect lines in the case of dense polymers) becomes automatically fixed. The simple geometry of the cylinder with closed or open boundary conditions on the edges allows an elementary evaluation of the partition function by using an extension of the Kirchhoff theorem.

The second and more important feature of the spanning webs model consists in the perfect coincidence of the universal part of its partition function $\bar{Z}_N^{(\mu,\nu)}(q, \omega)$ for different combinations of closed and open boundary conditions, $\mu = 0, 1$ and $\nu = 0, 1$, with the finitized characters of the symplectic fermions; see (55). This allows us to interpret the symplectic fermion model as a conformal field theory of spanning webs on a cylinder. The further identification of the triplet $W$ algebra in terms of spanning webs is an interesting open problem.

Another problem for future investigation is the explicit construction of Virasoro representations with finitized characters given by equations (36)–(38). Strictly speaking we have not proved the indecomposability of these Virasoro representations. However, the structure of characters (36)–(38) gives an indication that our model belongs to the universality class of the $c = -2$ logarithmic conformal field theory (LCFT). An additional argument in support of the above conjecture provides the one-to-one correspondence between the recurrent configurations of the Abelian sandpile model (ASM) and the spanning trees on the square lattice [25]. Under this mapping the height variables of the ASM correspond to non-local correlations in the spanning trees. Explicit calculations of height correlations in the ASM show logarithmic corrections [26] in complete agreement with the predictions of the $c = -2$ LCFT.

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