DEGREE OF RATIONAL MAPS VIA SPECIALIZATION

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ABSTRACT. One considers the behavior of the degree of a rational map under specialization of the coefficients of the defining linear system. The method rests on the classical idea of Kronecker as applied to the context of projective schemes and their specializations. For the theory to work one is led to develop the details of rational maps and their graphs when the ground ring of coefficients is a Noetherian integral domain.

1. INTRODUCTION

The overall goal of this paper is to obtain bounds for the degree of a rational map in terms of the main features of its base ideal (i.e., the ideal generated by a linear system defining the map). In order that this objective stay within a reasonable limitation, one focuses on rational maps whose source and target are projective varieties. Although there is some recent progress in the multi-projective environment (see [2] and [5]), it is the present authors’ believe that a thorough examination of the projective case is a definite priority.

Now, to become more precise one should rather talk about projective schemes as source and target of the envisaged rational maps. The commonly sought interest is the case of projective schemes over a field (typically, but not necessarily, algebraically closed). After all, this is what core classical projective geometry is all about. Alas, even this classical setup makes it hard to look at the degree of a rational map since one has no solid grip on any general theory that commutative algebra lends, other than the rough skeleton of field extension degree theory.

One tactic that has often worked is to go all the way up to a generic case and then find sufficient conditions for the specialization to keep some of the main features of the former. The procedure depends on taking a dramatic number of variables to allow modifying the given data into a generic shape. The method is seemingly due to Kronecker and was quite successful in the hands of Hurwitz ([20]) in establishing a new elegant theory of elimination and resultants.

Of a more recent crop, one has, e.g., [18], [19], [28], [26].

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In a related way, one has the notion of when an ideal specializes modulo a regular sequence: given an ideal $I \subset R$ in a ring, one says that $I$ specializes with respect to a sequence of elements $\{a_1, \ldots, a_n\} \subset R$ if the latter is a regular sequence both on $R$ and on $R/I$. A tall question in this regard is to find conditions under which the defining ideal of some well-known rings – such as the Rees ring or the associated graded ring of an ideal (see, e.g., [12], [22]) – specialize with respect to a given sequence of elements. Often, at best one can only describe some obstructions to this sort of procedure, normally in terms of the kernel of the specialization map.

The core of the paper can be said to lie in between the two ideas of specialization as applied to the situation of rational maps between projective schemes and their related ideal-theoretic objects.

It so happens that at the level of the generic situation the coefficients live in a polynomial ring $A$ over a field, not anymore on a field. This entails the need to consider rational maps defined by linear systems over the ring $A$, that is, rational maps with source $\mathbb{P}^r_A$. As it turns out, it is not exceedingly more complicated to consider rational maps with source an integral closed subscheme of $\mathbb{P}^r_A$.

Much to our surprise a complete such theory, with all the required details that include the ideal-theoretic transcription, is not easily available. For this reason, the first part of the paper deals with such details with an eye for the ideal-theoretic behavior concealed in or related to the geometric facts. A tall order in these considerations will be a so-called relative fiber cone that mimics the notion of a fiber cone in the classical environment over a field – this terminology is slightly misleading as the notion is introduced in algebraic language, associated to the concept of a Rees algebra rather than to the geometric version (blowup); however, one will draw on both the algebraic and the geometric versions.

Another concept dealt with is the saturated fiber cone, an object perhaps better understood in terms of global sections of a suitable sheaf of rings. The concept has been introduced in [5] in the coefficient field environment and is presently extended to the case when the coefficients belong to a Noetherian integral domain of finite Krull dimension. It contains the relative fiber cone as a subalgebra and plays a role in rational maps, mainly as an obstruction to birationality in terms of this containment. Also, its multiplicity is equal to the product of the degree of a rational and the degree of the corresponding image.

With the introduction of these considerations, one will be equipped to tackle the problem of the specialization, which is the main objective of this paper. The neat application so far is to the multiplicity of the saturated fiber cone and to the degree of a rational map defined by the maximal minors of a homogeneous $(r + 1) \times r$ matrix, when in both situations one assumes that the coefficient ring $A$ is a polynomial ring over a field of characteristic zero.

Next is a summary of the contents in each section.

For the time being, let $A$ be a Noetherian integral domain of finite Krull dimension.

In Section 2, some of the terminologies and notations that will be used throughout the paper are fixed.

In Section 3, the basics of rational maps between irreducible projective varieties over $A$ are developed. Particular emphasis is set towards a description of geometric concepts
in terms of the algebraic analogs. For instance, and as expected, the image and the graph of a rational are described in terms of the fiber cone and the Rees algebra, respectively. In the last part of this section, the notion of a saturated fiber cone is introduced and studied in the relative environment over \( A \).

Section 4 is devoted to a few algebraic tools that will be used later. Of particular interest are the upper bounds for the dimension of certain graded parts of local cohomology modules of a finitely generated module over a bigraded algebra.

The core of the paper is Section 5.

Here one assumes that the ground ring is a polynomial ring \( A := \mathbb{F}[z_1, \ldots, z_m] \) over a field \( \mathbb{F} \) and specializes these variables to elements of \( \mathbb{F} \). Thus, one considers a maximal ideal of the form \( \mathfrak{n} := (z_1 - \alpha_1, \ldots, z_m - \alpha_m) \) and lets \( k := A/\mathfrak{n} \) denote the residue field thereof. One takes a standard graded polynomial ring \( R := A[x_0, \ldots, x_r] ([R]_0 = A) \) and a tuple of forms \( \{g_0, \ldots, g_s\} \subset R \) of the same positive degree. Let \( \{\overline{g_0}, \ldots, \overline{g_s}\} \subset R/nR \) denote the corresponding tuple of forms in \( R/nR \cong k[x_0, \ldots, x_r] \) where \( \overline{g_i} \) is the image of \( g_i \) under the canonical homomorphism \( R \to R/nR \).

Consider the rational maps
\[
\mathcal{G} : \mathbb{P}^r_A \dashrightarrow \mathbb{P}^s_A \quad \text{and} \quad g : \mathbb{P}^r_k \dashrightarrow \mathbb{P}^s_k
\]
determined by the tuples of forms \( \{g_0, \ldots, g_s\} \) and \( \{\overline{g_0}, \ldots, \overline{g_s}\} \), respectively.

The main target is finding conditions under which the degree \( \text{deg}(g) \) of \( g \) can be bounded above or below by the degree \( \text{deg}(\mathcal{G}) \) of \( \mathcal{G} \). The main result in this line is Theorem 5.12. In addition, set \( \mathcal{I} := \langle g_0, \ldots, g_s \rangle \subset R \) and \( I := \langle \overline{g_0}, \ldots, \overline{g_s} \rangle \subset R/nR \). Let \( \mathbb{E}(\mathcal{I}) \) be the exceptional divisor of the blow-up of \( \mathbb{P}^r_A \) along \( \mathcal{I} \). A bit surprisingly, having a grip on the dimension of the scheme \( \mathbb{E}(\mathcal{I}) \times_A k \) is the main condition to determine whether \( \text{deg}(g) \leq \text{deg}(\mathcal{G}) \) or \( \text{deg}(g) \geq \text{deg}(\mathcal{G}) \). In order to control \( \dim(\mathbb{E}(\mathcal{I}) \times_A k) \) one can impose some constraints on the analytic spread of \( \mathcal{I} \) localized at certain primes, mimicking an idea in \([12]\) and \([22]\).

An additional interest in this section is the specialization of the saturated fiber cone of \( \mathcal{I} \). By assuming a condition on \( \dim(\mathbb{E}(\mathcal{I}) \times_A k) \) and letting the specialization be suitably general, it is proved in Theorem 5.15 that the multiplicity of the saturated fiber cone of \( I \) is at most the one of the saturated fiber cone of \( \mathcal{I} \otimes_A \text{Quot}(A) \), where \( \text{Quot}(A) \) denotes the field of fractions of \( A \). As a consequence, when the coefficients of the forms \( \{\overline{g_0}, \ldots, \overline{g_s}\} \) are general, one obtains an upper bound for the product of the degree of \( g \) and the degree of the image of \( g \).

Section 6 focuses on the case of a codimension 2 perfect ideal. The main idea is that in this situation the ideal \( \mathcal{I} \) will satisfy certain generic condition that allows one to compute the degree of \( \mathcal{G} \) by only considering the degrees of the syzygies of \( \mathcal{I} \) (see \([9]\)). Then, in Theorem 6.3, an easy application of Theorem 5.12 provides upper bounds for the degree of certain rational maps.

For the reader interested in the main results, here is a pointer to those: Theorem 4.4, Proposition 5.3, Proposition 5.11, Theorem 5.12, Theorem 5.15 and Theorem 6.3.

2. TERMINOLOGY AND NOTATION

Let \( R \) be a Noetherian ring and \( I \subset R \) be an ideal.

**Definition 2.1.** Let \( m \geq 0 \) be an integer (one allows \( m = \infty \)).
(G) $I$ satisfies the condition $G_m$ if $\mu(I_p) \leq \operatorname{ht}(p)$ for all $p \in \operatorname{Spec}(R)$ such that $\operatorname{ht}(p) \leq m - 1$.

(F) In addition, suppose that $I$ has a regular element. $I$ satisfies the condition $F_m$ if $\mu(I_p) \leq \operatorname{ht}(p) + 1 - m$ for all $p \in \operatorname{Spec}(R)$ such that $I_p$ is not principal. Provided $I$ is further assumed to be principal locally in codimension at most $m - 1$, the condition is equivalent to requiring that $\mu(I_p) \leq \operatorname{ht}(p) + 1 - m$ for all $p \in \operatorname{Spec}(R)$ containing $I$ such that $\operatorname{ht}(p) \geq m$.

In terms of Fitting ideals, $I$ satisfies $G_m$ if and only if $\operatorname{ht}({\text{Fitt}}_i(I)) > i$ for all $i < m$, whereas $I$ satisfies $F_m$ if and only if $\operatorname{ht}({\text{Fitt}}_i(I)) \geq m + i$ for all $i \geq 1$. These conditions were originally introduced in [1, Section 2, Definition] and [16, Lemma 8.2, Remark 8.3], respectively. Both conditions are more interesting when the cardinality of a global set of generators of $I$ is large and $m$ stays low. Thus, $F_m$ is typically considered for $m = 0, 1$, while $G_m$ gets its way when $m \leq \dim R$.

**Definition 2.2.** The Rees algebra of $I$ is defined as the $R$-subalgebra

$$\mathcal{R}_R(I) := R[It] = \bigoplus_{n \geq 0} I^n t^n \subset R[t],$$

and the associated graded ring of $I$ is given by

$$\operatorname{gr}_I(R) := \mathcal{R}_R(I)/\mathcal{R}_R(I) \simeq \bigoplus_{n \geq 0} I^n/I^{n+1}.$$

If, moreover, $R$ is local, with maximal ideal $m$, one defines the fiber cone of $I$ to be

$$\mathfrak{F}_R(I) := \mathcal{R}_R(I)/m\mathcal{R}_R(I) \simeq \operatorname{gr}_I(R)/m\operatorname{gr}_I(R),$$

and the analytic spread of $I$, denoted by $\ell(I)$, to be the (Krull) dimension of $\mathfrak{F}_R(I)$.

The following notation will prevail throughout most of the paper.

**Notation 2.3.** Let $A$ be a Noetherian ring of finite Krull dimension. Let $(R, m)$ denote a standard graded algebra over a $A = \langle R \rangle_0$ and its graded irrelevant ideal $m = (\langle R \rangle_1)$. Let $S := A[y_0, \ldots, y_s]$ denote a standard graded polynomial ring over $A$.

Let $I \subset R$ be a homogeneous ideal generated by $s + 1$ polynomials $\{f_0, \ldots, f_s\} \subset R$ of the same degree $d > 0$ – in particular, $I = ([I]_d)$. Consider the bigraded $A$-algebra

$$\mathcal{A} := R \otimes_A S = R[y_0, \ldots, y_s],$$

where $\operatorname{bideg}([R]_1) = (1, 0)$ and $\operatorname{bideg}(y_i) = (0, 1)$. By setting $\operatorname{bideg}(t) = (-d, 1)$, then $\mathcal{R}_R(I) = R[It]$ inherits a bigraded structure over $A$. One has a bihomogeneous (of degree zero) $R$-homomorphism

$$\theta: \mathcal{A} \rightarrow \mathcal{R}_R(I) \subset R[t], \quad y_i \mapsto f_i t.$$

Thus, the bigraded structure of $\mathcal{R}_R(I)$ is given by

$$\mathcal{R}_R(I) = \bigoplus_{c, n \in \mathbb{Z}} [\mathcal{R}_R(I)]_{c, n} \quad \text{and} \quad [\mathcal{R}_R(I)]_{c, n} = [I^n]_{c+nd} t^n.$$
One is primarily interested in the $R$-grading of the Rees algebra, namely, $[\mathcal{R}_R(I)]_c = \bigoplus_{n=0}^{\infty} [\mathcal{R}_R(I)]_{c,n}$, and of particular interest is

$$[\mathcal{R}_R(I)]_0 = \bigoplus_{n=0}^{\infty} [I^n]_{nd} = A[[I]] = \bigoplus_{n=0}^{\infty} [I^n]_{nd} \subset R.$$ 

Clearly, $\mathcal{R}_R(I) = [\mathcal{R}_R(I)]_0 \oplus \left( \bigoplus_{c \geq 1} [\mathcal{R}_R(I)]_c \right) = [\mathcal{R}_R(I)]_0 \oplus m\mathcal{R}_R(I)$. Therefore, one gets

$$(2) \quad A[[I]] \simeq [\mathcal{R}_R(I)]_0 \simeq \mathcal{R}_R(I)/m\mathcal{R}_R(I)$$

as graded $A$-algebras.

**Definition 2.4.** Because of its resemblance to the fiber cone in the case of a local ring, one here refers to the right-most algebra above as the (relative) fiber cone of $I$, and often identify it with the $A$-subalgebra $A[[I]] \subset R$ by the above natural isomorphism. It will also be denoted by $\mathcal{F}_R(I)$.

**Remark 2.5.** If $R$ has a distinguished or special maximal ideal $m$ (that is, if $R$ is graded with graded irrelevant ideal $m$ or if $R$ is local with maximal ideal $m$), then the fiber cone also receives the name of special fiber ring.

3. **Rational maps over an integral domain**

In this part one develops the main points of the theory of rational maps with source and target projective varieties defined over an arbitrary Noetherian integral domain of finite Krull dimension. Some of these results will take place in the case the source is a biprojective (more generally, a multi-projective) variety, perhaps with some extra work in the sleeve. From now on assume that $R$ is an integral domain, which in particular implies that $A = [R]_0$ is also an integral domain. Some of the subsequent results will also work assuming that $R$ is reduced, but additional technology would be required.

3.1. **Dimension.** In this subsection one considers a simple way of constructing chains of relevant graded prime ideals and draw upon it to algebraically describe the dimension of projective schemes. These results are possibly well-known, but one includes them anyway for the sake of completeness.

The following easy fact seems to be sufficiently known.

**Lemma 3.1.** Let $B$ be a commutative ring and $A \subset B$ a subring. Then, for any minimal prime $p \in \text{Spec}(A)$ there exists a minimal prime $\mathfrak{p} \in \text{Spec}(B)$ such that $p = \mathfrak{p} \cap A$.

**Proof.** First, there is some prime of $B$ lying over $p$. Indeed, any prime ideal of the ring of fractions $B_0 = B \otimes_A A_0$ is the image of a prime ideal $P \subset B$ not meeting $A \setminus p$, hence contracting to $p$.

For any descending chain of prime ideals $P = P_0 \supset P_1 \supset \cdots$ such that $P_i \cap A \subset p$ for every $i$, their intersection $Q$ is prime and obviously $Q \cap A \subset p$. Since $p$ is minimal, then $Q \cap A = p$.

Therefore, Zorn’s lemma yields the existence of a minimal prime in $B$ contracting to $p$. \qed
Lemma 3.1, there is a minimal prime $\mathfrak{p}$ such that $\dim(\mathfrak{p}) \geq 1$. Proceed by induction on $k = \dim(A)$.

Proof. Proceed by induction on $k = \dim(A)$.

Let $m$ be a maximal ideal of $A$ with $\dim(m) = k$. By [24, Theorem 13.6] one can choose $0 \neq a \in m \subset A$ such that $\dim(m/aA) = \dim(m) - 1$. Let $q$ be a minimal prime of $aA$ such that $\dim(m/q) = \dim(m) - 1$. From the ring inclusion $A/aA \hookrightarrow R/aR$ (because $A/aA$ is injected as a graded summand) and Lemma 3.1, there is a minimal prime $\mathfrak{q}$ of $aR$ such that $\dim(\mathfrak{q}) = \dim(\mathfrak{m})$.

Clearly, $m \not\subseteq \mathfrak{q}$. Indeed, otherwise $(q, m) \subseteq \mathfrak{q}$ and since $m$ is a prime ideal of $R$ of height at least 1 then $(q, m)$ has height at least 2; this contradicts Krull’s Principal Ideal Theorem since $\mathfrak{q}$ is a minimal prime of a principal ideal.

Let $R' = R/\mathfrak{q}$ and $A' = A/q$. Then $R'$ is a finitely generated graded algebra over $A'$ with $[R']_0 = A'$ and $m' := ([R']_+ = mR'$. Since $\mathfrak{q} \not\subseteq m$, it follows $\dim(mR') \geq 1$ and by construction, $\dim(A') = \dim(A) - 1$. So by the inductive hypothesis there is a chain of graded primes $0 = \mathfrak{p}'_0 \subset \cdots \subset \mathfrak{p}'_{k-1} \subset \mathfrak{p}'_k$ in $R'$ such that $\mathfrak{p}'_{k-1} \not\supseteq mR'$. Finally, for $j \geq 1$ define $\mathfrak{p}'_j$ as the inverse image of $\mathfrak{p}'_{j-1}$ via the surjection $R \rightarrow R'$.

Recall that $X := \text{Proj}(R)$ is a closed subscheme of $\mathbb{P}_A^r$, for suitable $r$ (= relative embedding dimension of $X$) whose underlying topological space is the set of all homogeneous prime ideals of $R$ not containing $m$ and it has a basis given by the open sets of the form $D_+(f) := \{\varnothing \in X \mid f \notin \varnothing\}$, where $f \in R_+$. Here, the sheaf structure is given by the degree zero part of the homogeneous localizations

$$\Gamma \left(D_+(f), \mathcal{O}_X \big|_{D_+(f)} \right) := R(f) = \left\{ \frac{g}{f} \mid g, f \in R, \deg(g) = k \deg(f) \right\}.$$

Let $K(X) := R(0)$ denote the field of rational functions of $X$, where

$$R(0) = \left\{ \frac{f}{g} \mid f, g \in R, \deg(f) = \deg(g), g \neq 0 \right\},$$

the degree zero part of the homogeneous localization of $R$ at the null ideal.

Likewise, denote $\mathbb{P}_A^s = \text{Proj}(S') = \text{Proj}(A[y_0, \ldots, y_n])$.

The dimension $\dim(X)$ of the closed subscheme $X$ is defined to be the supremum of the lengths of chains of irreducible closed subsets (see, e.g., [15, Definition, p. 5 and p. 86]). The next result is possibly part of the dimensional folklore (cf. [21, Lemma 1.2]).

For any integral domain $D$, let $\text{Quot}(D)$ denote its field of fractions.

Corollary 3.3. If $X = \text{Proj}(R) \subset \mathbb{P}_A^r$ is an integral subscheme then

$$\dim(X) = \dim(R) - 1 = \dim(A) + \text{trdeg}_{\text{Quot}(A)}(K(X)).$$

Proof. For any prime $\mathfrak{p} \in X$, the ideal $(\mathfrak{p}, m) \neq R$ is an ideal properly containing $\mathfrak{p}$, hence the latter is not a maximal ideal. Therefore $\dim(\mathfrak{p}) \leq \dim(R) - 1$ for any $\mathfrak{p} \in X$, which clearly implies that $\dim(X) \leq \dim(R) - 1$. 

From [27, Lemma 1.1.2] one gets the equalities
\[ \dim (R) = \dim (A) + \text{ht}(m) = \dim (A) + \text{trdeg}_{\text{Quot}(A)}(\text{Quot}(R)). \]
There exists a chain of graded prime ideals \( 0 = \mathfrak{P}_0 \subseteq \cdots \subseteq \mathfrak{P}_{h-1} \subseteq \mathfrak{P}_h = m \) such that \( h = \text{ht}(m) \) (see, e.g., [24, Theorem 13.7], [4, Theorem 1.5.8]). Let \( T = R/\mathfrak{P}_{h-1} \).
Since \( \text{ht}(mT) = 1 \), Proposition 3.2 yields the existence of chain of graded prime ideals \( 0 = \mathfrak{Q}_0 \subseteq \cdots \subseteq \mathfrak{Q}_k \in T \), where \( k = \dim (A) \) and \( \mathfrak{Q}_k \not\supseteq mT \). By taking inverse images along the surjection \( R \to T \), one obtains a chain of graded prime ideals not containing \( m \) of length \( h - 1 + k = \dim (R) - 1 \). Thus, one has the reverse inequality \( \dim (X) \geq \dim (R) - 1 \).

Now, for any \( f \in [R]_1 \), one has
\[ \text{Quot}(R) = R_{(0)}(f) \]
with \( f \) transcendental over \( K(X) = R_{(0)} \). Therefore
\[ \dim (X) = \dim (A) + \text{trdeg}_{\text{Quot}(A)}(\text{Quot}(R)) - 1 = \dim (A) + \text{trdeg}_{\text{Quot}(A)}(K(X)), \]
as was to be shown. \( \square \)

3.2. Main definitions. One restates the following known concept.

**Definition 3.4.** Let \( \mathcal{R}(X, \mathbb{P}^s_A) \) denote the set of pairs \((U, \varphi)\) where \( U \) is an open dense subscheme of \( X \) and where \( \varphi : U \to \mathbb{P}^s_A \) is a morphism of \( A \)-schemes. Two pairs \((U_1, \varphi_1), (U_2, \varphi_2) \in \mathcal{R}(X, \mathbb{P}^s_A)\) are said to be *equivalent* if there exists an open dense subscheme \( W \subseteq U_1 \cap U_2 \) such that \( \varphi_1 \mid_W = \varphi_2 \mid_W \). This gives an equivalence relation on \( \mathcal{R}(X, \mathbb{P}^s_A) \). A rational map is defined to be an equivalence class in \( \mathcal{R}(X, \mathbb{P}^s_A) \) and any element of this equivalence class is said to define the rational map.

A rational map as above is denoted \( \mathcal{F} : X \dashrightarrow \mathbb{P}^s_A \), where the dotted arrow reminds one that typically it will not be defined everywhere as a map. In [14, Lecture 7] (see also [10]) it is explained that, in the case where \( A \) is a field the above definition is equivalent to a more usual notion of a rational map in terms of homogeneous coordinate functions. Next, one proceeds to show that the same is valid in the relative environment over \( A \).

First it follows from the definition that any morphism \( U \to \mathbb{P}^s_A \) as above from a open dense subset defines a unique rational map \( X \dashrightarrow \mathbb{P}^s_A \). Now, let there be given \( s + 1 \) forms \( f = \{f_0, f_1, \ldots, f_s\} \subseteq R \) of the same degree \( d > 0 \). Let \( h : S \to R \) be the graded homomorphism of \( A \)-algebras given by
\[ h : S = A[y_0, y_1, \ldots, y_s] \longrightarrow R \]
\[ y_i \mapsto f_i. \]
There corresponds to it a morphism of \( A \)-schemes
\[ \Phi(f) = \text{Proj}(h) : D_+(f) \longrightarrow \text{Proj}(S) = \mathbb{P}^s_A \]
where \( D_+(f) \subseteq \text{Proj}(R) = X \) is the open subscheme given by
\[ D_+(f) = \bigcup_{i=0}^{s} D_+(f_i). \]
Therefore, a set of \( s + 1 \) forms \( f = \{f_0, f_1, \ldots, f_s\} \subseteq R \) of the same positive degree determines a unique rational map given by the equivalence class of \( (D_+(f), \Phi(f)) \) in \( \mathcal{R}(X, \mathbb{P}^s_A) \).
Definition 3.5. Call $\Phi(f)$ the $f$-coordinate morphism and denote the corresponding rational map by $F_f$.

Conversely:

Lemma 3.6. Any rational map $F : X = \text{Proj}(R) \dashrightarrow \mathbb{P}_A^s$ is of the form $F_f$, where $f$ are forms of the same positive degree.

Proof. Let $U$ be an open dense subset in $X$ and $\varphi : U \rightarrow \mathbb{P}_A^s$ be a morphism, such that the equivalence class of the pair $(U, \varphi)$ in $\mathfrak{M}(X, \mathbb{P}_A^s)$ is equal to $F$.

Consider $V = D_+(y_0)$ and $W = \varphi^{-1}(V)$ and restrict to an affine open subset, $W' = \text{Spec}(R(\ell)) \subset W$, where $\ell \in R$ is a homogeneous element of positive degree. It yields a morphism $\varphi |_W : W' \rightarrow V$, that corresponds to a ring homomorphism

$$\tau : S_{(y_0)} \rightarrow R(\ell),$$

where $T_{(h)}$ stands for the degree zero part of the homogeneous localization of a graded ring $T$ at the powers of a homogeneous element $h \in T$.

For each $0 < i \leq s$ one has

$$\tau \left( \frac{y_i}{y_0} \right) = \frac{g_i}{\ell^{\alpha_i}}$$

where $\deg(g_i) = \alpha_i \deg(\ell)$. Setting $\alpha := \max_{1 \leq i \leq s} \{ \alpha_i \}$, one writes

$$f_0 := \ell^\alpha \quad \text{and} \quad f_i := \ell^{\alpha} \frac{g_i}{\ell^{\alpha_i}} = \ell^{\alpha - \alpha_i} g_i \quad \text{for} \quad 1 \leq i \leq s.$$ 

By construction, $\varphi |_W = \Phi(f)|_W$, where $\Phi(f)$ denotes the $f$-coordinate morphism determined by $f = \{f_0, \ldots, f_s\}$, as in definition Definition 3.5, hence $F = F_f$ where $f = \{f_0, \ldots, f_s\}$.

Given a rational map $F : X \dashrightarrow \mathbb{P}_A^s$, any ordered $(s + 1)$-tuple $f = \{f_0, f_1, \ldots, f_s\}$ of forms of the same positive degree such that $F = F_f$ is called a representative of the rational map $F$.

The following result explains the flexibility of representatives of the same rational map.

Lemma 3.7. Let $f = \{f_0, \ldots, f_s\}$ and $f' = \{f'_0, \ldots, f'_s\}$ stand for representatives of a rational map $F : X \dashrightarrow \mathbb{P}_A^s$. Then $(f_0 : \cdots : f_s)$ and $(f'_0 : \cdots : f'_s)$ are proportional coordinate sets in the sense that there exits homogeneous forms $h, h'$ of positive degree such that $hf'_i = h'f_i$ for $i = 0, \ldots, s$.

Proof. Proceed similarly to Lemma 3.6. Let $\Phi(f) : D_+(f) \rightarrow \mathbb{P}_A^s$ and $\Phi(f') : D_+(f') \rightarrow \mathbb{P}_A^s$ be morphisms as in Definition 3.5. Let $V = \text{Spec}(D_+(y_0))$ and choose $W = \text{Spec}(R(\ell))$ such that $W \subset \Phi(f)^{-1}(V) \cap \Phi(f')^{-1}(V)$ and $\Phi(f)|_W = \Phi(f')|_W$.

The morphisms $\Phi(f)|_W : W \rightarrow V$ and $\Phi(f')|_W : W \rightarrow V$ correspond with the ring homomorphisms $\tau : S_{(y_0)} \rightarrow R(\ell)$ and $\tau' : S_{(y_0)} \rightarrow R(\ell)$ such that

$$\tau \left( \frac{y_i}{y_0} \right) = \frac{f_i}{f_0} \quad \text{and} \quad \tau' \left( \frac{y_i}{y_0} \right) = \frac{f'_i}{f'_0},$$

respectively. Since this is now an affine setting, the ring homomorphisms $\tau$ and $\tau'$ are the same (see e.g. [13, Theorem 2.35], [15, Proposition II.2.3]). It follows that, for every $i = 0, \ldots, s$, $f'_i/f'_0 = f_i/f_0$ as elements of the homogeneous total ring of quotients of
Lemma 3.7. There exist homogeneous elements \( h, h' \in R \) \((h = f_0, h' = f'_0)\) such that \( hf'_i = h'f_i \) for \( i = 0, \ldots, s \). The claim now follows. \( \square \)

In the above notation, one often denotes \( \mathcal{F}_f \) simply by \((f_0 : \cdots : f_s)\) and use this symbol for a representative of \( \mathcal{F} \).

**Remark 3.8.** Note that the identity morphism of \( \mathbb{P}^r_A \) is a rational map of \( \mathbb{P}^r_A \) to itself with natural representative \((x_0 : \cdots : x_r)\) where \( \mathbb{P}^r_A = \text{Proj}(A[x_0, \ldots, x_r]) \). Similarly, the identity morphism of \( X = \text{Proj}(R) \) is a rational map represented by \((x_0 : \cdots : x_r)\), where now \( x_0, \ldots, x_r \) generate the \( A \)-module \([R]_1\), and it is denoted by \( \text{Id}_X \).

The following sums up a version of [25, Proposition 1.1] over an integral domain. Due to Lemma 3.7, the proof is a literal transcription of the proof in loc. cit.

**Proposition 3.9.** Let \( \mathcal{F} : X \dashrightarrow \mathbb{P}^s_A \) be a rational map with representative \( f \). Set \( I = (f) \). Then, the following statements hold:

(i) The set of representatives of \( \mathcal{F} \) correspond bijectively to the non-zero homogeneous vectors in the rank one graded \( R \)-module \( \text{Hom}_R(I, R) \).

(ii) If \( \text{grade}(I) \geq 2 \), any representative of \( \mathcal{F} \) is a multiple of \( f \) by a homogeneous element in \( R \).

**Remark 3.10.** If \( R \) is in addition an UFD then any rational map has a unique representative up to a multiplier – this is the case, e.g., when \( A \) is a UFD and \( R \) is a polynomial ring over \( A \).

One more notational convention: if \( f = \{f_0, \ldots, f_s\} \) are forms of the same degree, \( A[f] \) will denote the \( A \)-subalgebra of \( R \) generated by these forms.

An important immediate consequence is as follows:

**Corollary 3.11.** Let \( f = (f_0 : \cdots : f_s) \) and \( f' = (f'_0 : \cdots : f'_s) \) stand for representatives of the same rational map \( \mathcal{F} : X = \text{Proj}(R) \dashrightarrow \mathbb{P}^s_A \). Then \( A[f] \simeq A[f'] \) as graded \( A \)-algebras and \( \mathcal{R}_R(I) \simeq \mathcal{R}_R(I') \) as bigraded \( A \)-algebras, where \( I = (f) \) and \( I' = (f') \).

**Proof.** Let \( \mathcal{J} \) and \( \mathcal{J}' \) respectively denote the ideals of defining equations of \( \mathcal{R}_R(I) \) and \( \mathcal{R}_R(I') \), as given in (1). From Lemma 3.7, there exist homogeneous elements \( h, h' \in R \) such that \( hf'_i = h'f_i \) for \( i = 0, \ldots, s \). Clearly, then \( I \simeq I' \) have the same syzygies, hence the defining ideals \( \mathcal{L} \) and \( \mathcal{L}' \) of the respective symmetric algebras coincide. Since \( R \) is a domain and \( I \) and \( I' \) are nonzero, then

\[
\mathcal{J} = \mathcal{L} : I^\infty = \mathcal{L}' : I'^\infty = \mathcal{J}'.
\]

Therefore, \( \mathcal{R}_R(I) \simeq A/\mathcal{J} = A/\mathcal{J}' \simeq \mathcal{R}_R(I') \) as bigraded \( A \)-algebras. Consequently,

\[
A[f] \simeq \mathcal{R}_R(I)/m\mathcal{R}_R(I) \simeq \mathcal{R}_R(I')/m\mathcal{R}_R(I') \simeq A[f']
\]

as graded \( A \)-algebras. \( \square \)

### 3.3. Image, degree and birational maps.

This part is essentially a recap on the algebraic description of the image, the degree and the birationality of a rational map in the relative case. Most of the material here has been considered in a way or another as a previsible extension of the base field situation (see, e.g., [6, Theorem 2.1]).

**Definition-Proposition 3.12.** Let \( \mathcal{F} : X \dashrightarrow \mathbb{P}^s_A \) be a rational map. The image of \( \mathcal{F} \) is equivalently defined as:
(I1) The closure of the image of a morphism \( U \to \mathbb{P}_A^s \) defining \( \mathcal{F} \), for some (any) open dense subset.

(I2) The closure of the image of the \( f \)-coordinate morphism \( \Phi(f) \), for some (any) representative \( f \) of \( \mathcal{F} \).

(I3) \( \text{Proj} \left( A\{f\} \right) \), for some (any) representative \( f \) of \( \mathcal{F} \), up to degree normalization of \( A\{f\} \).

**Proof.** The equivalence of (I1) and (I2) is clear by the previous developments. To check that (I2) and (I3) are equivalent, consider the the ideal sheaf \( \mathcal{J} \) given as the kernel of the canonical homomorphism

\[
\mathcal{O}_{\mathbb{P}_A^s} \to \Phi(f)_s \mathcal{O}_{D_+(f)}
\]

It defines a closed subscheme \( Y \subset \mathbb{P}_A^s \) which corresponds with the schematic image of \( \Phi(f) \) (see, e.g., [13, Proposition 10.30]). The underlying topological space of \( Y \) coincides with the closure of the image of \( \Phi(f) \). Now, for any \( 0 \leq i \leq s \), \( \mathcal{O}_{\mathbb{P}_A^s}(D_+(y_i)) = S(y_i) \) and \( (\Phi(f)_s \mathcal{O}_{D_+(f)}) (D_+(y_i)) = R(f_i) \). Then, for \( 0 \leq i \leq s \), there is an exact sequence

\[
0 \to \mathcal{J}(D_+(y_i)) \to S(y_i) \to R(f_i).
\]

Thus, \( \mathcal{J}(D_+(y_i)) = J(y_i) \) for any \( 0 \leq i \leq s \), where \( J \) is the kernel of the \( A \)-algebra homomorphism \( \alpha : S \to A\{f\} \subset R \) given by \( y_i \mapsto f_i \). This implies that \( \mathcal{J} \) is the sheafification of \( J \). Therefore, \( Y \simeq \text{Proj}(S/J) \simeq \text{Proj}(A\{f\}) \).

Now one considers the degree of a rational map \( \mathcal{F} : X \dashrightarrow \mathbb{P}_A^s \). By **Definition-Proposition 3.12** , the field of rational functions of the image \( Y \) of \( \mathcal{F} \) is

\[
K(Y) = A\{f\}_{(0)},
\]

where \( f = (f_0 : \cdots : f_s) \) is a representative of \( \mathcal{F} \). Here \( A\{f\} \) is naturally \( A \)-graded as an \( A \)-subalgebra of \( R \), but one may also consider it as a standard graded \( A \)-graded algebra by a degree normalization.

One gets a natural field extension \( K(Y) = A\{f\}_{(0)} \hookrightarrow R_{(0)} = K(X) \).

**Definition 3.13.** The **degree** of \( \mathcal{F} : X \dashrightarrow \mathbb{P}_A^s \) is

\[
\deg(\mathcal{F}) := [K(X) : K(Y)].
\]

One says that \( \mathcal{F} \) is **generically finite** if \( [K(X) : K(Y)] < \infty \). If the field extension \( K(X)|K(Y) \) is infinite, one agrees to say that \( \mathcal{F} \) has no well-defined degree (also, in this case, one often says that \( \deg(\mathcal{F}) = 0 \)).

The following properties are well-known over a coefficient field. Its restatement in the relative case is for the reader’s convenience.

**Proposition 3.14.** Let \( \mathcal{F} : X \dashrightarrow \mathbb{P}_A^s \) be a rational map with image \( Y \subset \mathbb{P}_A^s \).

(i) Let \( f \) denote a representative of \( \mathcal{F} \) and let \( \Phi(f) \) be the associated \( f \)-coordinate morphism. Then, \( \mathcal{F} \) is generically finite if and only if there exists an open dense subset \( U \subset Y \) such that \( \Phi(f)^{-1}(U) \to U \) is a finite morphism.

(ii) \( \mathcal{F} \) is generically finite if and only if \( \dim(X) = \dim(Y) \).

**Proof.** (i) Let \( \Phi(f) : D_+(f) \subset X \to Y \subset \mathbb{P}_A^s \) be the \( f \)-coordinate morphism of \( \mathcal{F} \). One has an equality of fields of rational functions \( K(X) = K(D_+(f)) \). But on \( D_+(f) \) the
rational map \( \mathcal{F} \) is defined by a morphism, in which case the result is given in [15, Exercise II.3.7].

(ii) By Corollary 3.3 one has \( \dim(X) = \dim(A) + \text{trdeg}_{\text{Quot}(A)}(K(X)) \) and by the same token, \( \dim(Y) = \dim(A) + \text{trdeg}_{\text{Quot}(A)}(K(Y)) \). It follows that

\[
\dim(X) = \dim(Y) \iff \text{trdeg}_{\text{Quot}(A)}(K(X)) = \text{trdeg}_{\text{Quot}(A)}(K(Y)).
\]

Since the later condition is equivalent to \( \text{trdeg}_{K(Y)}(K(X)) = 0 \), one is through. \( \square \)

Next one defines birational maps in the relative environment over \( A \). While any of the three alternatives below sounds equally fit as a candidate (as a deja vu of the classical coefficient field setup), showing that they are in fact mutually equivalent requires a small bit of work.

**Definition-Proposition 3.15.** Let \( \mathcal{F} : X \subset \mathbb{P}_A^r \rightarrow \mathbb{P}_A^s \) be a rational map with image \( Y \subset \mathbb{P}_A^s \). The map \( \mathcal{F} \) is said to be **birational onto its image** if one of the following equivalent conditions is satisfied:

- (B1) \( \deg(\mathcal{F}) = 1 \), that is \( K(X) = K(Y) \).
- (B2) There exists some open dense subset \( U \subset X \) and a morphism \( \varphi : U \rightarrow \mathbb{P}_A^s \) such that the pair \((U, \varphi)\) defines \( \mathcal{F} \) and such that \( \varphi \) is an isomorphism onto an open dense subset \( V \subset Y \).
- (B3) There exists a rational map \( \mathcal{G} : Y \subset \mathbb{P}_A^s \rightarrow \mathcal{F} : X \subset \mathbb{P}_A^r \) such that, for some (any) representative \( f \) of \( \mathcal{F} \) and some (any) representative \( g = (g_0 : \cdots : g_r) \) of \( \mathcal{G} \), the composite

\[
g(f) = (g_0(f) : \cdots : g_r(f))
\]

is a representative of the identity rational map on \( X \).

**Proof.** (B1) \( \Rightarrow \) (B2). Let \( \varphi' : U' \rightarrow \mathbb{P}_A^s \) be a morphism from an open dense subset \( U' \subset X \) such that \((U', \varphi')\) defines \( \mathcal{F} \). Let \( \eta \) denote the generic point of \( X \) and \( \xi \) that of \( Y \). The field inclusion \( \mathcal{O}_{Y, \xi} \backsimeq K(Y) \hookrightarrow K(X) \simeq \mathcal{O}_{X, \eta} \) coincides with the induced local ring homomorphism

\[
(\varphi')^\sharp_\eta : \mathcal{O}_{Y, \xi} \rightarrow \mathcal{O}_{X, \eta}.
\]

Since by assumption \( \deg(\mathcal{F}) = 1 \), then \((\varphi')^\sharp_\eta\) is an isomorphism. Then, by [13, Proposition 10.52] \((\varphi')^\sharp_\eta\) "extends" to an isomorphism from an open neighborhood \( U \) of \( \eta \) in \( X \) onto an open neighborhood \( V \) of \( \xi \) in \( Y \). Now, take the restriction \( \varphi = \varphi'|_U : U \rightarrow V \) as the required isomorphism.

(B2) \( \Rightarrow \) (B3) Let \( \varphi : U \subset X \rightarrow V \subset Y \) be a morphism defining \( \mathcal{F} \), which is an isomorphism from an open dense subset \( U \subset X \) onto an open dense subset \( V \subset Y \). Let \( \psi = \varphi^{-1} : V \subset Y \rightarrow U \subset X \) be the inverse of \( \varphi \). Let \( \mathcal{G} : Y \subset \mathbb{P}_A^s \rightarrow \mathcal{F} : X \subset \mathbb{P}_A^r \) be the rational map defined by \((V, \psi)\).

Let \( \text{Id}_X \) be the identity rational map on \( X \) (Remark 3.8). Take any representatives \( f = (f_0 : \cdots : f_s) \) of \( \mathcal{F} \) and \( g = (g_0 : \cdots : g_r) \) of \( \mathcal{G} \). Let \( \mathcal{G} \circ \mathcal{F} \) be the composition of \( \mathcal{F} \) and \( \mathcal{G} \), i.e. the rational map defined by \((U, \psi \circ \varphi)\). Since \( \psi \circ \varphi \) is the identity morphism on \( U \), **Definition 3.4** implies that the pair \((U, \psi \circ \varphi)\) gives the equivalence class of \( \text{Id}_X \). Thus, one has \( \text{Id}_X = \mathcal{G} \circ \mathcal{F} \), and by construction \( g(f) \) is a representative of \( \mathcal{G} \circ \mathcal{F} \).

(B3) \( \Rightarrow \) (B1) This is quite clear: take a representative \( (f_0 : \cdots : f_s) \) of \( \mathcal{F} \) and let \( \mathcal{G} \) and \((g_0 : \cdots : g_r)\) be as in the assumption. Since the identity map of \( X \) is defined
by the representative \((x_0 : \cdots : x_r)\), where \([R]_1 = Ax_0 + \cdots + Ax_r\) (see Remark 3.8), then Lemma 3.7 yields the existence of nonzero (homogeneous) \(h, h' \in R\) such that 
\[h \cdot g_i(f) = h' \cdot x_i, \text{ for } i = 0, \ldots, r.\]
Then, for suitable \(e \geq 0,
\[\frac{x_i}{x_0} = \frac{g_i(f)}{g_0(f)} = \frac{f_0^e (g_i(f_1/f_0, \ldots, f_m/f_0))}{f_0^e (g_0(f_1/f_0, \ldots, f_m/f_0))} = \frac{g_i(f_1/f_0, \ldots, f_m/f_0)}{g_0(f_1/f_0, \ldots, f_m/f_0)}, \quad i = 0, \ldots, r\]
This shows the reverse inclusion \(K(X) \subset K(Y)\). □

3.4. The graph of a rational map. The tensor product \(A := R \otimes_A A[y] \simeq R[y]\) has a natural structure of a standard bigraded \(A\)-algebra. Accordingly, the fiber product 
\(\text{Proj}(R) \times_A \mathbb{P}^s_A\) has a natural structure of a biprojective scheme over \(\text{Spec}(A)\). Thus, 
\(\text{Proj}(R) \times_A \mathbb{P}^s_A = \text{BiProj}(A)\).
The graph of a rational map \(F : X = \text{Proj}(R) \dashrightarrow \mathbb{P}^s_A\) is a subscheme of this structure, in the following way:

**Definition-Proposition 3.16.** The *graph* of \(F\) is equivalently defined as:

1. **(G1)** The closure of the image of the morphism \((\iota, \varphi) : U \to X \times_A \mathbb{P}^s_A\), where \(\iota : U \hookrightarrow X\) is the natural inclusion and \(\varphi : U \to \mathbb{P}^s_A\) is a morphism from some (any) open dense subset defining \(F\).
2. **(G2)** For some (any) representative \(f\) of \(F\), the closure of the image of the morphism 
\((\iota, \Phi(f)) : D_+(f) \to X \times_A \mathbb{P}^s_A\), where \(\iota : D_+(f) \hookrightarrow X\) is the natural inclusion and \(\Phi(f) : D_+(f) \to \mathbb{P}^s_A\) is the \(f\)-coordinate morphism.
3. **(G3)** \(\text{BiProj}(R)(I))\), where \(I = (f)\) for some (any) representative \(f\) of \(F\).

**Proof.** The equivalence of (G1) and (G2) is clear, so one proceeds to show that (G2) and (G3) give the same scheme. Recall that, as in (1), the Rees algebra of an ideal such as \(I\) is a bigraded \(A\)-algebra. The proof follows the same steps of the argument for the equivalence of (I2) and (I3) in the definition of the image of \(F\) (cf. Definition-Proposition 3.12).

Let \(\Gamma(f)\) denote the morphism as in (G2) and let \(\mathcal{G} \subset X \times_A \mathbb{P}^s_A\) denote its schematic image. The underlying topological space of \(\mathcal{G}\) coincides with the closure of the image of \(\Gamma(f)\). Then the ideal sheaf of \(\mathcal{G}\) is the kernel \(\mathcal{J}\) of the corresponding homomorphism of ring sheaves

\[\mathcal{O}_{X \times_A \mathbb{P}^s_A} \to \Gamma(f)_* \mathcal{O}_{D_+(f)}\]

Since the irrelevant ideal of \(A\) is \(([R]_1) \cap (y)\), by letting \([R]_1 = Ax_0 + \cdots + Ax_r\) one can see that an affine open cover is given by \(\text{Spec} \left(A_{(x_iy_j)}\right)\) for \(0 \leq i \leq r\) and \(0 \leq j \leq s\), where \(A_{(x_iy_j)}\) denotes the degree zero part of the bihomogeneous localization at powers of \(x_iy_j\), to wit

\[A_{(x_iy_j)} = \left\{ \frac{g}{(x_iy_j)_{\alpha}} \mid g \in A \text{ and } \text{bideg}(g) = (\alpha, \alpha) \right\}.
\]

One has
\[\mathcal{O}_{X \times_A \mathbb{P}^s_A} (D_+(x_iy_j)) = A_{(x_iy_j)} \text{ and } \left(\Gamma(f)_* \mathcal{O}_{D_+(f)}\right) (D_+(x_iy_j)) = R_{(x_iy_j)}, \text{ for } 0 \leq i \leq r \text{ and } 0 \leq j \leq s.
\]
Then (3) yields the exact sequence
\[0 \to \mathcal{J}(D_+(x_iy_j)) \to A_{(x_iy_j)} \to R_{(x_iy_j)}.
\]
Let \(\mathcal{J}\) be the kernel of the homomorphism of bigraded \(A\)-algebras \(A \to R_R(I) \subset R[t]\) given by \(y_i \mapsto f_it\). The fact that \(R_R(I)_{(x_iy_j)} \simeq R_{(x_iy_j)}\), yields the equality

This shows the reverse inclusion \(K(X) \subset K(Y)\). □
\( \mathfrak{J}(D^+(x_iy_j)) = \mathfrak{J}(x_iy_j) \). It follows that \( \mathfrak{J} \) is the sheafification of \( \mathcal{J} \). Therefore, \( \mathfrak{S} \simeq \text{BiProj}(A/\mathcal{J}) \simeq \text{BiProj}(\mathcal{R}_R(I)) \). \( \square \)

3.5. Saturated fiber cones over an integral domain. In this part one introduces the notion of a saturated fiber cone over an integral domain, by closely lifting from the ideas in [5]. As will be seen, the notion is strongly related to the degree and birationality of rational maps.

For simplicity, assume that \( R = A[x] = A[x_0, \ldots, x_r] \), a standard graded polynomial ring over \( A \) and set \( K := \text{Quot}(A), m = (x_0, \ldots, x_r) \).

The central object is the following graded \( A \)-algebra

\[
\widehat{\mathfrak{F}_R(I)} := \bigoplus_{n=0}^{\infty} \left[ (I^n : m^\infty) \right]_{nd},
\]

which one calls the saturated fiber cone of \( I \).

Note the natural inclusion of graded \( A \)-algebras \( \mathfrak{S}_R(I) \subset \widehat{\mathfrak{F}_R(I)} \).

For any \( i \geq 0 \), the local cohomology module \( H^i_m(\mathcal{R}_R(I)) \) has a natural structure of bi-graded \( \mathcal{R}_R(I) \)-module, which comes out of the fact that \( H^i_m(\mathcal{R}_R(I)) = H^i_m(\mathcal{R}_R(I)) \) (see also [8, Lemma 2.1]). In particular, each \( R \)-graded part

\[
[H^i_m(\mathcal{R}_R(I))]_j
\]

has a natural structure of graded \( \widehat{\mathfrak{F}_R(I)} \)-module.

Let \( \text{Proj}_{R-\text{gr}}(\mathcal{R}_R(I)) \) denote the Rees algebra \( \mathcal{R}_R(I) \) viewed as a “one-sided” graded \( R \)-algebra.

Lemma 3.17. With the above notation, one has:

(i) There is an isomorphism of graded \( A \)-algebras

\[
\widehat{\mathfrak{F}_R(I)} \simeq H^0 \left( \text{Proj}_{R-\text{gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\text{Proj}_{R-\text{gr}}(\mathcal{R}_R(I))} \right).
\]

(ii) \( \widehat{\mathfrak{F}_R(I)} \) is a finitely generated graded \( \mathfrak{F}_R(I) \)-module.

(iii) There is an exact sequence

\[
0 \to \mathfrak{F}_R(I) \to \widehat{\mathfrak{F}_R(I)} \to \left[ H^1_m(\mathcal{R}_R(I)) \right]_0 \to 0
\]

of finitely generated graded \( \mathfrak{F}_R(I) \)-modules.

(iv) If \( A \to A' \) is a flat ring homomorphism, then there is an isomorphism of graded \( A' \)-algebras

\[
\widehat{\mathfrak{F}_R(I)} \otimes_A A' \simeq \widehat{\mathfrak{F}_R(R')}.
\]

where \( R' = R \otimes_A A' \).
Proof. (i) Since $\mathcal{R}_R(I) \cong \bigoplus_{n=0}^{\infty} I^n(nd)$, by computing Čech cohomology with respect to the affine open covering $(\text{Spec}(\mathcal{R}_R(I)(x_i)))_{0 \leq i \leq r}$ of $\text{Proj}_{\text{gr}}(\mathcal{R}_R(I))$, one obtains

$$H^0\left(\text{Proj}_{\text{gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\text{Proj}_{\text{gr}}(\mathcal{R}_R(I))}\right) \cong \bigoplus_{n \geq 0} H^0(\text{Proj}(R), (I^n)^{\sim}(nd)) = \bigoplus_{n=0}^{\infty} \left[(I^n : m^{\infty})\right]_{nd} \quad ([15, \text{Exercise II.5.10}]).$$

(ii) and (iii) From [21, Corollary 1.5] (see also [11, Theorem A4.1]) and the fact that $H^0_m(\mathcal{R}_R(I)) = 0$, there is an exact sequence

$$0 \to [\mathcal{R}_R(I)]_0 \to H^0(\text{Proj}_{\text{gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\text{Proj}_{\text{gr}}(\mathcal{R}_R(I))}) \cong \mathfrak{F}_R(I) \to [H^1_m(\mathcal{R}_R(I))]_0 \to 0$$

of $\mathfrak{F}_R(I)$-modules. Now $[H^1_m(\mathcal{R}_R(I))]_0$ is a finitely generated module over $\mathfrak{F}_R(I)$ (see, e.g., [5, Proposition 2.7], [7, Theorem 2.1]), thus implying that $\mathfrak{F}_R(I)$ is also finitely generated over $\mathfrak{F}_R(I)$.

(iv) Since $A \to A'$ is flat, base change yields

$$H^0(B, \mathcal{O}_B) \simeq H^0\left(\text{Proj}_{\text{gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\text{Proj}_{\text{gr}}(\mathcal{R}_R(I))}\right) \otimes_A A',$$

where $B = \text{Proj}_{\text{gr}}(\mathcal{R}_R(I)) \times_A A' = \text{Proj}_{\text{gr}}(\mathcal{R}_R(I) \otimes_A A')$. Also $\mathcal{R}_R(I) \otimes_A A' \simeq \mathcal{R}_R(IR')$, by flatness, hence the result follows. \qed

Let $\mathcal{F} : \mathbb{P}^s_A \dashrightarrow \mathbb{P}^s_A$ be a rational map with representative $f = (f_0 : \cdots : f_s)$. Let $\mathcal{G} : \mathbb{P}^s_K \dashrightarrow \mathbb{P}^s_K$ denote a rational map with representative $f$, where each $f_i$ is considered as an element of $K[x]$. Set $I = (f) \subset R$.

Remark 3.18. The rational map $\mathcal{F}$ is generically finite if and only if the rational map $\mathcal{G}$ is so, and one has the equality $\text{deg}(\mathcal{F}) = \text{deg}(\mathcal{G})$. In fact, let $Y$ and $Z$ be the images of $\mathcal{F}$ and $\mathcal{G}$, respectively. Since $K(\mathbb{P}^s_A) = R(0) = K[x](0) = K(\mathbb{P}^s_K)$ and $K(Y) = A[f]_{(0)} = K[f]_{(0)} = K(Z)$, then the statement follows from Definition 3.13 and Proposition 3.14.

The following result is a simple consequence of [5, Theorem 2.4].

Theorem 3.19. Suppose that $\mathcal{F}$ is generically finite. Then, the following statements hold:

(i) $\text{deg}(\mathcal{F}) = \left[\mathfrak{F}_R(I) : \mathfrak{F}_R(I)\right]$.

(ii) $e\left(\mathfrak{F}_R(I) \otimes_A K\right) = \text{deg}(\mathcal{F}) \cdot e\left(\mathfrak{F}_R(I) \otimes_A K\right)$, where $e(-)$ stands for Hilbert-Samuel multiplicity.

(iii) Under the additional condition of $\mathfrak{F}_R(I)$ being integrally closed, then $\mathcal{F}$ is birational if and only if $\mathfrak{F}_R(I) = \mathfrak{F}_R(I)$.

Proof. (i) Let $\mathcal{G} : \mathbb{P}^s_K \dashrightarrow \mathbb{P}^s_K$ be the rational map as above. Since $A \hookrightarrow K$ is flat, $\mathfrak{F}_R(I) \otimes_A K \cong \mathfrak{F}_K[\mathfrak{m}](IK[\mathfrak{x}])$ and $\mathfrak{F}_R(I) \otimes_A K \cong \mathfrak{F}_K[\mathfrak{x}](IK[\mathfrak{x}])$ (Lemma 3.17 (iv)).
Thus from [5, Theorem 2.4], one obtains \( \deg(G) = \left[ \hat{\mathcal{F}}_R(I) \otimes_A K : \hat{\mathcal{F}}_R(I) \otimes_A K \right] \). It is clear that \( \left[ \hat{\mathcal{F}}_R(I) \otimes_A K : \hat{\mathcal{F}}_R(I) \otimes_A K \right] = \left[ \hat{\mathcal{F}}_R(I) : \hat{\mathcal{F}}_R(I) \right] \). Finally, Remark 3.18 yields the equality \( \deg(F) = \deg(G) \).

(ii) It follows from the associative formula for multiplicity (see, e.g., [4, Corollary 4.7.9]).

(iii) It suffices to show that, assuming that \( \mathcal{F} \) is birational onto the image and that \( \hat{\mathcal{F}}_R(I) \) is integrally closed, then \( \hat{\mathcal{F}}_R(I) = \mathcal{F}_R(I) \). Since \( \deg(F) = 1 \), part (i) gives

\[
\text{Quot} \left( \frac{\hat{\mathcal{F}}_R(I)}{\mathcal{F}_R(I)} \right) = \text{Quot} \left( \frac{\hat{\mathcal{F}}_R(I)}{\mathcal{F}_R(I)} \right).
\]

Since \( \mathcal{F}_R(I) \) is integrally closed and \( \mathcal{F}_R(I) \rightarrow \hat{\mathcal{F}}_R(I) \) is an integral extension (see Lemma 3.17(ii)), then \( \hat{\mathcal{F}}_R(I) = \mathcal{F}_R(I) \). □

4. Additional algebraic tools

In this section one gathers a few algebraic tools to be used in the specialization of rational maps. The section is divided in two subsections, and each subsection deals with a different theme that is important on its own.

4.1. Grade of certain generic determinantal ideals. One provides lower bounds for the grade of certain generic determinantal ideals. As a consequence, one derives that the base ideal of a certain generic rational maps satisfies the \( G_m \) type condition (see Definition 2.1).

In this subsection one agrees to change the previous notation, by letting \( R \) denote an arbitrary Noetherian ring.

The next lemma deals with generic ideals deforming ideals in \( R \) (see, e.g., [26, Proposition 3.2] for a similar setup).

**Lemma 4.1.** Let \( z = (z_{i,j}) \) be a new set of variables with \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \) and \( S \) be the polynomial ring \( S = R[z] \). Let \( I = (f_1, \ldots, f_m) \subset R \) be an ideal. Let \( J \) be the ideal \( J = (p_1, p_2, \ldots, p_n) \subset R[z] \) such that

\[
p_i = f_1 z_{i,1} + f_2 z_{i,2} + \cdots + f_m z_{i,m}.
\]

Then \( \text{grade}(J) \geq \min\{n, \text{grade}(I)\} \).

**Proof.** Let \( Q \) be a prime ideal containing \( J \). If \( Q \) contains all the \( f_i \)'s, then \( \text{depth}(S_Q) \geq \text{grade}(I) \). Otherwise, say, \( f_1 \notin Q \). Then one can write

\[
\frac{p_i}{f_1} = \frac{f_2}{f_1} z_{i,2} + \cdots + \frac{f_m}{f_1} z_{i,m} \in R_{f_1}[z]
\]

as elements of the localization \( S_{f_1} = R_{f_1}[z] \). Since \( \{z_{1,1}, \ldots, z_{n,1}\} \) is a regular sequence in \( R_{f_1}[z] \), then so is the sequence \( \{p_1/f_1, \ldots, p_n/f_1\} \). Then, clearly \( \{p_1, \ldots, p_n\} \) is a regular sequence in \( R_{f_1}[z] \), hence \( \text{depth}(S_Q) \geq \text{grade}(JR_{f_1}[z]) \geq n \). □

The next proposition is now an easy routine procedure of inverting-localizing at a suitable entry. One gives the proof for the sake of completeness.
**Proposition 4.2.** Let \( I_j = (f_{j,1}, \ldots, f_{j,m_j}) \subset R \) be ideals for \( 1 \leq j \leq s \). Set \( g = \min_{1 \leq j \leq s}\{\text{grade}(I_j)\} \). Let \( z = (z_{i,j,k}) \) be a new set of variables with \( 1 \leq i \leq r, 1 \leq j \leq s \) and \( 1 \leq k \leq m_j \). Let \( S \) be the polynomial ring \( S = R[z] \). Let \( M \) be the \( r \times s \) matrix with entries in \( S \) given by

\[
M = \begin{pmatrix}
    p_{1,1} & p_{1,2} & \cdots & p_{1,s} \\
p_{2,1} & p_{2,2} & \cdots & p_{2,s} \\
    \vdots & \vdots & & \vdots \\
p_{r,1} & p_{r,2} & \cdots & p_{r,s}
\end{pmatrix}
\]

where each polynomial \( p_{i,j} \in S \) is given by

\[
p_{i,j} = f_{j,1}z_{i,j,1} + f_{j,2}z_{i,j,2} + \cdots + f_{j,m_j}z_{i,j,m_j}.
\]

Then

\[
\text{grade}(I_t(M)) \geq \min\{r - t + 1, g\}.
\]

for \( 1 \leq t \leq \min\{r, s\} \).

**Proof.** Proceed by induction on \( t \). The case \( t = 1 \) follows from Lemma 4.1 since \( I_1(M) \) is generated by the \( p_{i,j} \)'s themselves.

Now suppose that \( 1 < t \leq \min\{r, s\} \). Let \( Q \) be a prime ideal containing \( I_t(M) \). If \( Q \) contains all the polynomials \( p_{i,j} \), then again Lemma 4.1 yields \( \text{depth}(S_Q) \geq \min\{r, g\} \geq \min\{r - t + 1, g\} \). Otherwise, say, \( p_{r,s} \notin Q \).

Let \( M' \) denote the \((r - 1) \times (s - 1)\) submatrix of \( M \) of the first \( r - 1 \) rows and first \( s - 1 \) columns. Clearly,

\[
I_{t-1}(M')S_{pr,s} \subset I_t(M)S_{pr,s}
\]

in the localization \( S_{pr,s} \). The inductive hypothesis gives

\[
\text{depth}(S_Q) \geq \text{grade}(I_t(M)S_{pr,s})
\]

\[
\geq \text{grade}(I_{t-1}(M')S_{pr,s})
\]

\[
\geq \text{grade}(I_{t-1}(M'))
\]

\[
\geq \min\{(r - 1) - (t - 1) + 1, g\}.
\]

Therefore, \( \text{depth}(S_Q) \geq \min\{r - t + 1, g\} \) as was to be shown. \( \square \)

**4.2. Local cohomology of bigraded algebras.** One studies the dimension of certain graded parts of local cohomology modules of a finitely generated module over a bigraded algebra. It will come out as a far reaching generalization of [5, Proposition 3.1], a result that has proven to be useful under various situations (see, e.g., [5, Proof of Theorem 3.3], [9, Proof of Theorem A]).

The following notation will prevail along this subsection only.

**Notation 4.3.** Let \( k \) be a field. Let \( (R, m) \) and \( (S, n) \) be standard graded algebras over \( k \), i.e. \( m = R_+ = ([R]_1), n = S_+ = ([S]_1) \) and \( R_0 = S_0 = k \). Let \( A \) be the bigraded \( k \)-algebra \( A = R \otimes_k S \) with \( \text{bideg}([R]_1) = (1, 0) \) and \( \text{bideg}([S]_1) = (0, 1) \).

Let \( M \) be a bigraded module over \( A \). Denote by \( [M]_j \) the “one-sided” \( R \)-graded part

\[
[M]_j = \bigoplus_{k \in \mathbb{Z}} [M]_{j, k}.
\]
Note that, for any $i \geq 0$, the local cohomology module $H^i_m(\mathcal{M})$ has a natural structure of bigraded $A$-module, and this can be seen from the fact that $H^i_m(\mathcal{M}) = H^i_{m,A}(\mathcal{M})$ (also, see [8, Lemma 2.1]). In particular, each $R$-graded part

$$[H^i_m(\mathcal{M})]_j$$

has a natural structure of graded $S$-module.

For any finitely generated bigraded $A$-module $\mathcal{M}$, $[H^i_m(\mathcal{M})]_j$ is a finitely generated graded $S$-module for any $i \geq 0, j \in \mathbb{Z}$ (see, e.g., [5, Proposition 2.7], [7, Theorem 2.1]).

The next theorem contains the main result of this subsection.

**Theorem 4.4.** Let $\mathcal{M}$ be a finitely generated bigraded $A$-module. Then

$$\dim \left( [H^i_m(\mathcal{M})]_j \right) \leq \min \{ \dim(\mathcal{M}) - i, \dim(S) \}$$

for any $i \geq 0, j \in \mathbb{Z}$.

**Proof.** Let $d = \dim(\mathcal{M})$. By the well-known Grothendieck Vanishing Theorem (see, e.g., [3, Theorem 6.1.2]), $H^i_m(\mathcal{M}) = 0$ for $i > d$, so that one takes $i \leq d$. Since $[H^i_m(\mathcal{M})]_j$ is a finitely generated $S$-module, it is clear that $\dim \left( [H^i_m(\mathcal{M})]_j \right) \leq \dim(S)$.

Proceed by induction on $d$.

Suppose that $d = 0$. Then $[\mathcal{M}]_{j,k} = 0$ for $k \gg 0$. Since $[H^0_m(\mathcal{M})]_j \subset [\mathcal{M}]_j$, one has $\left( [H^0_m(\mathcal{M})]_j \right)_k = 0$ for $k \gg 0$. Thus, $\dim \left( [H^0_m(\mathcal{M})]_j \right) = 0$.

Suppose that $d > 0$. There exists a finite filtration

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n = \mathcal{M}$$

of $\mathcal{M}$ such that $\mathcal{M}_l/\mathcal{M}_{l-1} \cong (A/p_l)(a_l, b_l)$ where $p_l \subset A$ is a bigraded prime ideal with dimension $\dim(A/p_l) \leq d$ and $a_l, b_l \in \mathbb{Z}$. The short exact sequences

$$0 \to \mathcal{M}_{l-1} \to \mathcal{M}_l \to (A/p_l)(a_l, b_l) \to 0$$

induce the following long exact sequences in local cohomology

$$[H^i_m(\mathcal{M}_{l-1})]_j \to [H^i_m(\mathcal{M}_l)]_j \to [H^i_m((A/p_l)(a_l, b_l))]_j.$$

By iterating on $l$, one gets

$$\dim \left( [H^i_m(\mathcal{M})]_j \right) \leq \max_{1 \leq i \leq n} \left\{ \dim \left( [H^i_m((A/p_l)(a_l, b_l))]_j \right) \right\}.$$

If $p_l \supseteq nA$ then $A/p_l$ is a quotient of $A/nA \cong R$ and this implies that $\left( [H^i_m(A/p_l)]_j \right)_k = 0$ for $k \neq 0$. Thus, one assumes that $p_l \not\supseteq nA$.

Alongside with the previous reductions, one can then assume that $\mathcal{M} = A/p$ where $p$ is a bigraded prime ideal and $p \not\supseteq nA$. In this case there exists an homogeneous element $y \in S_1$ such that $y \not\in p$. The short exact sequence

$$0 \to (A/p)(0, -1) \xrightarrow{y} A/p \to A/(y, p) \to 0$$

yields the long exact sequence in local cohomology

$$[H^{i-1}_m(A/(y, p))]_j \to \left( [H^i_m(A/p)]_j \right)_j (-1) \xrightarrow{y} [H^i_m(A/p)]_j \to [H^i_m(A/(y, p))]_j.$$
Therefore, it follows that
$$\dim \left( [H^i_m(\mathcal{A}/\mathfrak{p})]_j \right) \leq \max \left\{ \dim \left( [H^{i-1}_m(\mathcal{A}/(y, \mathfrak{p}))]_j \right), 1 + \dim \left( [H^i_m(\mathcal{A}/(y, \mathfrak{p}))]_j \right) \right\}.$$  
Since \( \dim(\mathcal{A}/(y, \mathfrak{p})) \leq d - 1 \), the induction hypothesis gives
$$\dim \left( [H^{i-1}_m(\mathcal{A}/(y, \mathfrak{p}))]_j \right) \leq (d - 1) - (i - 1) = d - i$$
and
$$1 + \dim \left( [H^i_m(\mathcal{A}/(y, \mathfrak{p}))]_j \right) \leq 1 + (d - 1) - i = d - i.$$  
Therefore, \( \dim \left( [H^i_m(\mathcal{A}/\mathfrak{p})]_j \right) \leq d - i \), as meant to be shown.  

Of particular interest is the following corollary that generalizes [5, Proposition 3.1].

**Corollary 4.5.** Let \((R, \mathfrak{m})\) be a standard graded algebra over a field with graded irrelevant ideal \(\mathfrak{m}\). For any ideal \(\mathfrak{I} \subset \mathfrak{m}\) one has
$$\dim \left( [H^i_m(R_\mathfrak{I})]_j \right) \leq \dim(R) + 1 - i$$
for any \(i \geq 0, j \in \mathbb{Z}\).

**Proof.** It follows from Theorem 4.4 and the fact that \(\dim(R_\mathfrak{I}) \leq \dim(R) + 1\).  

5. **Specialization**

In this section one studies how the process of specializing Rees algebras and saturated fiber cones affects the degree of specialized rational maps, where the latter is understood in terms of coefficient specialization.

The following notation will take over throughout this section.

**Setup 5.1.** Essentially keep the basic notation as in the previous section, but this time around take \(A = \mathbb{F}[z_1, \ldots, z_m]\) to be a polynomial ring over a field \(\mathbb{F}\) (for the present purpose forget any grading). Consider a rational map \(G : \mathbb{P}^r_\mathbb{A} \dashrightarrow \mathbb{P}^s_\mathbb{A}\) given by a representative \(g = (g_0 : \cdots : g_s)\), where \(\mathbb{P}^r_\mathbb{A} = \Proj(R)\) with \(R = A[x] = A[x_0, \ldots, x_r]\). Fix a maximal ideal \(\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)\) of \(A\), where \(\alpha_i \in \mathbb{F}\), and set \(k := A/\mathfrak{n}\). Then \(R/\mathfrak{n}R \simeq k[x_0, \ldots, x_r]\). Let \(g\) denote the rational map \(g : \mathbb{P}^r_k \dashrightarrow \mathbb{P}^s_k\) with representative \(\overline{g} = (\overline{g}_0 : \cdots : \overline{g}_s)\), where \(\overline{g}_i\) is the image of \(g_i\) under the canonical map \(R \to R/\mathfrak{n}R\). Further assume that \(\overline{g}_i \neq 0\) for all \(0 \leq i \leq s\).

Finally, denote \(\mathcal{I} := (g_0, \ldots, g_s) \subset R\) and \(\mathcal{I} := (\mathcal{I}, \mathfrak{n})/\mathfrak{n} = (\overline{g}_0, \ldots, \overline{g}_s) \subset R/\mathfrak{n}R\).

5.1. **Algebraic lemmata.**

**Lemma 5.2.** With the notation introduced above, one has a commutative diagram

$$
\begin{array}{ccc}
A[g] & \rightarrow & R_R(I) \\
\downarrow & & \downarrow \\
A[g] \otimes_A k & \rightarrow & R_R(I) \otimes_A k \\
\downarrow & & \downarrow \\
k[\overline{g}] & \rightarrow & R_R/\mathfrak{n}R(I).
\end{array}
$$
where $A[g]$ (respectively, $k[G]$) is identified with $\mathfrak{g}(I)$ (respectively, $\mathfrak{g}(R/nR(I))$).

**Proof.** The upper vertical maps are obvious surjections as $k = A/n$, hence the upper square is commutative – the lower horizontal map of this square is injective because in the upper horizontal map $A[g]$ is injected as a direct summand. The right lower vertical map is naturally induced by the natural maps

$$R[t] \rightarrow R[t] \otimes_A k = R[t]/nR[t] = A[x][t]/nA[x][t] \simeq A/n[x][t] = k[x][t],$$

where $t$ is a new indeterminate. The left lower vertical map is obtained by restriction thereof. □

**Proposition 5.3.** Consider the naturally induced homomorphism of bigraded algebras $s : R(R/I) \otimes_A k \rightarrow R(R/nR(I))$. If $I$ is not the null ideal, one has:

(i) $\ker(s)$ is a minimal prime ideal of $R(R/I) \otimes_A k$ and, for any minimal prime $\varpi$ of $R(R/I) \otimes_A k$ other than $\ker(s)$, one has

$$\dim ((R(R/I) \otimes_A k)/\varpi) \leq \dim(\text{gr}_{T}(R) \otimes_A k).$$

In particular,

$$\dim (R(R/I) \otimes_A k) = \max\{\dim(R/nR) + 1, \dim(\text{gr}_{T}(R) \otimes_A k)\} = \max\{r + 2, \dim(\text{gr}_{T}(R) \otimes_A k)\}.$$

(ii) Let $k \geq 0$ be an integer such that $t(I_{\varpi}) \leq \text{ht}(\varpi/nR) + k$ for every prime ideal $\varpi \in \text{Spec}(R)$ containing $(I, n)$. Then

$$\dim(\text{gr}_{T}(R) \otimes_A k) \leq \dim(R/nR) + k.$$

In particular,

$$\dim (R(R/I) \otimes_A k) \leq \max\{r + 2, \dim(R/nR) + k\}.$$

**Proof.** (i) Let $P \in \text{Spec}(R)$ be a prime ideal not containing $I$. Localizing the surjection $s : R(R/I) \otimes_A k \rightarrow R(R/nR(I))$ at $R \setminus P$, one easily sees that it becomes an isomorphism. It follows that some power of $I$ annihilates $\ker(s)$, that is

$$\mathcal{I}^c \cdot \ker(s) = 0$$

for some $c > 0$. Therefore, any prime ideal of $R(R/I) \otimes_A k$ contains either the prime ideal $\ker(s)$ or the ideal $I$. Thus, $\ker(s)$ is a minimal prime and any other minimal prime $\varpi$ of $R(R/I) \otimes_A k$ contains $I$. Clearly, then any such $\varpi$ is a minimal prime of $(R(R/I) \otimes_A k)/I(R(R/I) \otimes_A k)$. But the latter has the same dimension as $\text{gr}_{T}(R) \otimes_A k$. Since $\dim (R(R/nR(I))) = \dim(R/nR) + 1$, the claim follows.

(ii) For this, let $\mathfrak{M}$ be a minimal prime of $\text{gr}_{T}(R) \otimes_A k$ of maximal dimension, i.e.:

$$\dim(\text{gr}_{T}(R) \otimes_A k) = \dim((\text{gr}_{T}(R) \otimes_A k)/\mathfrak{M}),$$
and let \( \mathfrak{P} = \mathfrak{M} \cap R \) be its contraction to \( R \). Clearly, \( \mathfrak{P} \supseteq (\mathcal{I}, n) \). By [27, Lemma 1.1.2] and the hypothesis,

\[
\dim (\text{gr}_R(R) \otimes_A k) = \dim((\text{gr}_R(R) \otimes_A k)/\mathfrak{M}) \\
= \dim(R/\mathfrak{P}) + \text{trdeg}_{R/\mathfrak{P}}(\text{gr}_R(R) \otimes_A k)/\mathfrak{M}) \\
= \dim(R/\mathfrak{P}) + \dim \left( (\text{gr}_R(R) \otimes_A k)/\mathfrak{M} \otimes_{R/\mathfrak{P}} R_\mathfrak{P}/\mathfrak{P}R_\mathfrak{P} \right) \\
\leq \dim(R/\mathfrak{P}) + \dim \left( (\text{gr}_R(R) \otimes_R R_\mathfrak{P}/\mathfrak{P}R_\mathfrak{P} \right) \\
= \dim(R/\mathfrak{P}) + \ell(\mathcal{I}_\mathfrak{P}) \\
\leq \dim(R/\mathfrak{P}) + \text{ht}(\mathfrak{P}/n) + k \\
\leq \dim(R/nR) + k,
\]

as required.

The supplementary assertion on \( \dim (\mathcal{R}_R(\mathcal{I}) \otimes_A k) \) is now clear.

\[\Box\]

The next lemma is a consequence of the Primitive Element Theorem and will be useful to study how the degree of rational maps varies under specialization.

**Lemma 5.4.** Let \( \mathbb{F} \) denote a field of characteristic zero and let \( C \subset B \) stand for a finite extension of finitely generated \( \mathbb{F} \)-domains. Let \( b \subset B \) be a prime ideal and set \( c := b \cap C \subset C \) for its contraction. If \( C \) is integrally closed then one has

\[
[\text{Quot}(B/b) : \text{Quot}(C/c)] \leq [\text{Quot}(B) : \text{Quot}(C)].
\]

**Proof.** Let \( \{b_1, \ldots, b_c\} \) generate \( B \) as a \( C \)-module. Setting \( \overline{C} := C/c \subset \overline{B} = B/b \) then the images \( \{\overline{b_1}, \ldots, \overline{b_c}\} \) generate then \( \overline{B} \) as a \( \overline{C} \)-module. Since the field extensions \( \text{Quot}(B)/\text{Quot}(C) \) and \( \text{Quot}(\overline{B})/\text{Quot}(\overline{C}) \) are separable, and since \( \mathbb{F} \) is moreover infinite, one can find elements \( \lambda_1, \ldots, \lambda_c \in \mathbb{F} \) such that \( L := \sum_{i=1}^c \lambda_i b_i \in B \) and \( \ell := \sum_{i=1}^c \lambda_i \overline{b_i} \in \overline{B} \) are respective primitive elements of the above extensions. Let \( X^u + a_1 X^{u-1} + \cdots + a_u = 0 \) denote the minimal polynomial of \( L \) over \( \text{Quot}(C) \). Since \( C \) is integrally closed, then \( a_i \in C \) for all \( 1 \leq i \leq u \) (see, e.g., [24, Theorem 9.2]). Reducing modulo \( b \), one gets \( \overline{\ell}^u + \overline{a_1} \overline{\ell}^{u-1} + \cdots + \overline{a_u} = 0 \). Then the degree of the minimal polynomial of \( \ell \) over \( \text{Quot}(\overline{C}) \) is at most \( u \), as was to be shown.

\[\Box\]

### 5.2. Geometric picture

Introduce some additional notation for the geometric environment:

**Setup 5.5.** First, recall from **Setup 5.1** that \( R = A[x_0, \ldots, x_r] \) is a standard polynomial ring over \( A \). One sets \( \mathbb{P}_A = \text{Proj}(R) \) as before. In addition, one had \( A = \mathbb{F}[z_1, \ldots, z_m] \) and \( n \subset A \) a given (rational) maximal ideal, with \( k := A/n \).

Let \( \mathcal{G} \) and \( g \) be as in **Setup 5.1**. Denote by \( \text{Proj}(A[g]) \) and \( \text{Proj}(k[\overline{g}]) \) the images of \( \mathcal{G} \) and \( g \), respectively (see **Definition-Proposition 3.12**). Let \( \mathbb{E}(\mathcal{I}) := \text{BiProj}(\mathcal{R}_R(\mathcal{I})) \) and \( \mathbb{E}(I) := \text{BiProj}(\mathcal{R}_{R/nR}(I)) \) be the graphs of \( \mathcal{G} \) and \( g \), respectively (see **Definition-Proposition 3.16**).

Let \( \mathbb{E}(\mathcal{I}) := \text{BiProj}(\text{gr}_R(R)) \) be the exceptional divisor of \( \mathbb{E}(\mathcal{I}) \).
Consider the commutative diagrams

\[
\begin{array}{ccc}
\mathbb{B}(I) & \to & \text{Proj}(A[g]) \\
\Pi' & \downarrow & \\
P^r_A & \stackrel{G}{\longrightarrow} & \text{Proj}(k[G])
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathbb{B}(I) & \to & \text{Proj}(k[G]) \\
\pi' & \downarrow & \\
P^r_k & \stackrel{g}{\longrightarrow} & \text{Proj}(k[G])
\end{array}
\]

where \(\Pi'\) and \(\pi'\) are the blowing-up structural maps, which are well-known to be birational (see, e.g., [15, Section II.7]) – note that, had one taken care of a full development of rational/birational maps in the biprojective situation, this fact would be routinely verified.

One sees that \(\Pi\) and \(\pi\) fall within the general notion of rational maps with source a biprojective scheme. Most of the presently needed material in the biprojective situation is more or less a straightforward extension of the projective one. Thus, for example, the field of rational functions of the biprojective scheme \(\mathbb{B}(I)\) is given by the bihomogeneous localization of \(R_T(I)\) at the null ideal, that is

\[
K(\mathbb{B}(I)) := R_T(I)_{(0)} = \left\{ \frac{f}{g} \mid f, g \in R_T(I), \text{bideg}(f) = \text{bideg}(g), g \neq 0 \right\}.
\]

Then, the degree of the morphism \(\Pi\) (respectively, \(\Pi'\)) is given by

\[
[K(\mathbb{B}(I)) : K(\text{Proj}(A[g]))] \quad (\text{respectively, } [K(\mathbb{B}(I)) : K(P^r_A)])
\]

Likewise, one has:

**Lemma 5.6.** The following statements hold:

(i) \(K(\mathbb{B}(I)) = K(P^r_k)\).
(ii) \(\Pi'\) is a birational morphism.
(iii) \(\text{deg}(\Pi) = \text{deg}(G)\).

**Proof.** (i) It is clear that \(K(P^r_k) \subset K(\mathbb{B}(I))\). Let \(f/g \in K(\mathbb{B}(I))\) with \(f, g \in [R_T(I)]_{(0)}\), then it follows that \(f = pt^\beta\) and \(g = p't^\beta\) where \(p, p' \in R_{(0)}\). Thus, \(f/g = p/p' \in R_{(0)}\) and so \(K(\mathbb{B}(I)) \subset K(P^r_k)\).

(ii) Use essentially the same argument of the implication (B1) \(\Rightarrow\) (B2) in Definition-Proposition 3.15.

Let \(\eta\) denote the generic point of \(\mathbb{B}(I)\) and \(\xi\) that of \(P^r_A\). From part (i), \((\Pi')_\eta^\xi : O_{P^r_A, \xi} \to O_{\mathbb{B}(I), \eta}\) is an isomorphism. Therefore, [13, Proposition 10.52] yields the existence of dense open subsets \(U \subset \mathbb{B}(I)\) and \(V \subset P^r_A\) such that the restriction \(\Pi' |_U : U \to V\) is an isomorphism.

(iii) It follows from (i). \(\square\)
Thus, one has as expected: \( \Pi \) and \( \pi \) are generically finite morphisms if and only if \( G \) and \( g \) are so, in which case one has
\[
\deg(G) = \deg(\Pi) \quad \text{and} \quad \deg(g) = \deg(\pi).
\]

**Lemma 5.7.** There is a commutative diagram
\[
\begin{array}{ccc}
\mathbb{B}(I) & \xrightarrow{\pi} & \text{Proj}(k[G]) \\
p_1 \downarrow & & \downarrow q_1 \\
\mathbb{B}(I) \times_A k & \xrightarrow{\Pi \times_A k} & \text{Proj}(A[g]) \times_A k \\
p_2 \downarrow & & \downarrow q_2 \\
\mathbb{B}(I) & \xrightarrow{\Pi} & \text{Proj}(A[g])
\end{array}
\]

where the statements below are satisfied:
(i) \( p_1 \) and \( q_1 \) are closed immersions.
(ii) \( p_2 \) and \( q_2 \) are the natural projections from the fiber products.

**Proof.** It is an immediate consequence of Lemma 5.2 by taking the respective associated Proj and BiProj schemes. \( \square \)

In the following lemma one relates the dimension of an irreducible closed subscheme of \( \mathbb{P}^r_A \times_A \mathbb{P}^s_A \) to the Krull dimension of its bihomogeneous coordinate ring, when \( A \) is polynomial ring over a field \( F \). This is possibly well-known, but the proof is included for the sake of easy reference and completeness. Note that a full analog of the statement in Corollary 3.3 would face at least the difficulty of that proof. Recall that throughout this part one is assuming that \( A \) is polynomial ring over a field \( F \). In addition, one lets \( A = A[\ldots, x_r, y_0, \ldots, y_s] \) stand for a standard bigraded polynomial ring over \( A \).

**Lemma 5.8.** Let \( C \) denote the quotient \( A/P \) of \( A \) by a bigraded prime ideal \( P \in \text{BiProj}(A) \). Then
\[
\dim(\text{BiProj}(C)) = \dim(C) - 2.
\]

**Proof.** Fix \( 0 \leq i \leq r \) and \( 0 \leq j \leq s \) such that \( x_i y_j \notin P \). There is a \( C_{(x_i y_j)} \)-algebra isomorphism
\[
C_{(x_i y_j)}[u, u^{-1}, v, v^{-1}] \xrightarrow{\sim} C_{x_i y_j}
\]
which sends \( u \) to \( x_i \) and \( v \) to \( y_j \). Hence it follows that \( \dim(C_{x_i y_j}) = \dim(C_{(x_i y_j)}) + 2 \). Since \( \text{BiProj}(C) \) is a scheme of finite type over the field \( F \), then \( \dim(\text{BiProj}(C)) = \dim(\text{Spec}(C_{(x_i y_j)})) \) (see, e.g., [13, Theorem 5.22], [15, Exercise II.3.20]).

Summing up,
\[
\dim(\text{BiProj}(C)) = \dim(\text{Spec}(C_{(x_i y_j)})) = \dim(C) - 2
\]
as required. \( \square \)

**Corollary 5.9.** The following statements hold:
(i) \( \dim(\mathbb{B}(I)) = \dim(A) + r \).
(ii) \( \dim(B(I)) = r. \)

**Proof.** It follows from Lemma 5.8 and the equalities \( \dim(R_R(I)) = \dim(R) + 1 \) and \( \dim(R_R/nR(I)) = \dim(R/nR) + 1. \)

The next result is an immediate consequence of Proposition 5.3 and Lemma 5.8.

**Lemma 5.10.** Assuming that \( I \not\subset nR \), the following statements hold:

(i) \( B(I) \) is an irreducible component of \( B(I) \times_A k \) and, for any irreducible component \( Z \) of \( B(I) \times_A k \) other than \( B(I) \), one has
   \[
   \dim(Z) \leq \dim(E(I) \times_A k).
   \]

(ii) Let \( k \geq 0 \) be an integer such that \( \ell(IP) \leq \text{ht}(\mathfrak{P}/nR) + k \) for every prime ideal \( \mathfrak{P} \in \text{Spec}(R) \) containing \( (I, n) \). Then
   \[
   \dim(E(I) \times_A k) \leq \dim(B(I)) + k - 1.
   \]

In particular,
   \[
   \dim(B(I) \times_A k) \leq \max\{\dim(B(I)), \dim(B(I)) + k - 1\}.
   \]

### 5.3. Main specialization result.

**Proposition 5.11.** Under Setup 5.5, assume that both \( G \) and \( g \) are generically finite. Then, the following statements are satisfied:

(i) \( U = \{y \in \text{Proj}(A[g]) \mid \Pi^{-1}(y) \text{ is a finite set}\} \) is a nonempty open set in \( \text{Proj}(A[g]) \) and the restriction \( \Pi^{-1}(U) \rightarrow U \) is a finite morphism.

(ii) If \( \dim(E(I) \times_A k) \leq \dim(B(I)) \) then
   \[
   q_1^{-1}(q_2^{-1}(U)) \neq \emptyset.
   \]

**Proof.** (i) Clearly, \( \Pi \) is a projective morphism, hence is a proper morphism. Thus, as a consequence of Zariski’s Main Theorem (see [13, Corollary 12.90]), the set \( U = \{y \in \text{Proj}(A[g]) \mid \Pi^{-1}(y) \text{ is a finite set}\} \) is open in \( \text{Proj}(A[g]) \) and the restriction \( \Pi^{-1}(U) \rightarrow U \) is a finite morphism. Since \( \Pi \) is generically finite, \( U \) is nonempty (see, e.g., [15, Exercise II.3.7]).

(ii) In notation of (8), considering \( \text{Proj}(k[\mathfrak{g}]) \) as a closed subscheme of \( \text{Proj}(A[\mathfrak{g}]) \times_A k \) via \( q_1 \), take the restriction
   \[
   \Psi : W = (\Pi \times_A k)^{-1}(\text{Proj}(k[\mathfrak{g}])) \rightarrow \text{Proj}(k[\mathfrak{g}]).
   \]

From Lemma 5.10 and the fact that \( g \) is generically finite, it follows that
   \[
   \dim(B(I) \times_A k) = \dim(B(I)) = \dim(\text{Proj}(k[\mathfrak{g}])).
   \]

Let \( \xi \) be the generic point of \( \text{Proj}(k[\mathfrak{g}]) \). So the map \( \Psi \) is also generically finite, and the fiber \( \Psi^{-1}(\xi) = W_\xi = W \times_{\text{Proj}(k[\mathfrak{g}])} k(\xi) \) of \( \Psi \) over \( \xi \) is finite.
Letting \( w = q_2(q_1(\xi)) \), one has the following canonical scheme isomorphisms
\[
\Psi^{-1}(\xi) = W \times_{\text{Proj}(k[\mathbf{g}])} k(\xi)
\]
\[
\simeq \left( (\mathcal{B}(I) \times_A k) \times_{\text{Proj}(A[\mathbf{g}])} \text{Proj}(k[\mathbf{g}]) \right) \times_{\text{Proj}(k[\mathbf{g}])} k(\xi)
\]
\[
\simeq (\mathcal{B}(I) \times_A k) \times_{\text{Proj}(A[\mathbf{g}])} k(\xi)
\]
\[
(9)
\]
\[
\simeq (\mathcal{B}(I) \times_{\text{Proj}(A[\mathbf{g}])} k(w)) \times_{k(w)} k(\xi)
\]
\[
\simeq \Pi^{-1}(w) \times_{k(w)} k(\xi),
\]
where \( \Pi^{-1}(w) = \mathcal{B}(I)_w = \mathcal{B}(I) \times_{\text{Proj}(A[\mathbf{g}])} k(w) \) denotes the fiber of \( \Pi \) over \( w \). Thus, it follows that \( \dim(\Pi^{-1}(w)) = \dim(\Psi^{-1}(\xi)) = 0 \) (see, e.g., [13, Proposition 5.38]) and so \( \Pi^{-1}(w) \) is also a finite fiber. Therefore, \( w \in U \) and \( \xi \in q_1^{-1}(q_2^{-1}(U)) \), which clearly implies \( q_1^{-1}(q_2^{-1}(U)) \neq \emptyset \). □

Next is the main result of this part.

**Theorem 5.12.** Under **Setup 5.5**, suppose that both \( \mathcal{G} \) and \( \mathfrak{g} \) are generically finite.

(i) Assume that the following conditions hold:

(a) \( \text{Proj}(A[\mathbf{g}]) \) is a normal scheme.

(b) \( \dim(\mathcal{B}(I) \times_A k) \leq \dim(\mathcal{B}(I)) \).

(c) \( R \) is a field of characteristic zero.

Then
\[
\deg(g) \leq \deg(\mathcal{G}).
\]

(ii) If \( \dim(\mathcal{B}(I) \times_A k) < \dim(\mathcal{B}(I)) \), then
\[
\deg(g) \geq \deg(\mathcal{G}).
\]

(iii) Consider the following condition:

\( (\mathcal{K}) \) \( k \geq 0 \) is a given integer such that \( \ell(\mathcal{I}_q) \leq \text{ht}(\mathcal{Q}/\mathfrak{n}R) + k \) for every prime ideal \( \mathcal{Q} \in \text{Spec}(R) \) containing \((I, \mathfrak{n})\).

Then:

(\( \mathcal{K}1 \)) If \( (\mathcal{K}) \) holds with \( k \leq 1 \), then condition (b) of part (i) is satisfied.

(\( \mathcal{K}2 \)) If \( (\mathcal{K}) \) holds with \( k = 0 \), then the assumption of (ii) is satisfied.

**Proof.** (i) Using condition (b), take an open set \( U \) as provided by Proposition 5.11 and shrink it down to an affine open subset \( U' := \text{Spec}(\mathcal{C}) \subset U \) such that \( q_1^{-1}(q_2^{-1}(U')) \neq \emptyset \). The scheme \( q_1^{-1}(q_2^{-1}(U')) \) is also affine because \( q_1 \) and \( q_2 \) are affine morphisms (Lemma 5.7). Then set
\[
q_1^{-1}(q_2^{-1}(U')) =: \text{Spec}(C).
\]
Since the restriction \( \Pi^{-1}(U') \to U' \) is a finite morphism, \( \Pi^{-1}(U') \) is also affine (see, e.g., [13, Remark 12.10], [15, Exercise 5.17]). Set \( \Pi^{-1}(U') =: \text{Spec}(\mathcal{B}) \). Similarly,
\[
p_1^{-1}(p_2^{-1}(\Pi^{-1}(U'))) =: \text{Spec}(B)
\]
is also affine.
The following commutative diagram of scheme maps stems from these considerations:

\[
\begin{array}{ccc}
\text{Spec}(B) & \xrightarrow{\pi} & \text{Spec}(C) \\
\downarrow & & \downarrow \\
\text{Spec}(\mathcal{B}) & \xrightarrow{\Pi} & \text{Spec}(\mathcal{C})
\end{array}
\]

(10)

where \(\pi \mid_{\text{Spec}(B)}\) and \(\Pi \mid_{\text{Spec}(B)}\) are finite morphisms. It corresponds to the following commutative diagram of ring homomorphisms

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\pi} & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xleftarrow{\Pi} & \mathcal{B}
\end{array}
\]

with finite horizontal maps, which are injective because \(\pi \mid_{\text{Spec}(B)}\) and \(\Pi \mid_{\text{Spec}(B)}\) are dominant morphisms and \(B\) and \(\mathcal{B}\) are integral domains (see [13, Corollary 2.11]). Since \(\text{Proj}(A[g])\) is given to be a normal scheme then \(C\) is integrally closed. By Lemma 5.4,

\[
\text{deg}(g) = \text{deg}(\pi) = \text{deg}(\pi \mid_{\text{Spec}(B)}) \leq \text{deg}(\Pi \mid_{\text{Spec}(B)}) = \text{deg}(\Pi) = \text{deg}(\mathcal{G}).
\]

(ii) By the hypothesis and Lemma 5.10, one has the set-theoretic equality

\[
\mathcal{B}(I) \times_A k = \mathcal{B}(I) \cup \mathcal{V}
\]

where \(\mathcal{V}\) is the union of the irreducible components of \(\mathcal{B}(I) \times_A k\) other than \(\mathcal{B}(I)\), and \(\text{dim}(\mathcal{Z}) < \text{dim}(\mathcal{B}(I)) = \text{dim}(\text{Proj}(k[\mathcal{G}]))\) for each irreducible component \(\mathcal{Z} \subset \mathcal{V}\). With notation as in (8), considering \(\text{Proj}(k[\mathcal{G}])\) as a closed subscheme of \(\text{Proj}(A[g]) \times_A k\) via \(q_1\), take the restriction

\[
\Psi : W = (\Pi \times_A k)^{-1}(\text{Proj}(k[\mathcal{G}])) \longrightarrow \text{Proj}(k[\mathcal{G}]).
\]

Let \(\xi\) be the generic point of \(\text{Proj}(k[\mathcal{G}])\) and denote \(w = q_2(q_1(\xi))\). If \(\mathcal{Z}\) is any irreducible component of \(\mathcal{B}(I) \times_A k\) other than \(\mathcal{B}(I)\), one has \(\Psi^{-1}(\xi) \cap \mathcal{Z} = \emptyset\), since otherwise the restriction

\[
\Psi \mid_{(W \cap \mathcal{Z})} : (W \cap \mathcal{Z}) \to \text{Proj}(k[\mathcal{G}])
\]

gives a dominant morphism, thus implying that \(\text{dim}(\mathcal{Z}) \geq \text{dim}(\text{Proj}(k[\mathcal{G}]))\), which is a contradiction. Therefore, \(\Psi^{-1}(\xi) \subset \mathcal{B}(I)\), and so \(\Psi^{-1}(\xi)\) and \(\pi^{-1}(\xi)\) have the same cardinality. Since \(\pi\) is assumed to be generically finite, the generic point \(u\) of \(\mathcal{B}(I)\) is the only point of \(\pi^{-1}(\xi)\). Thus, set-theoretically \(\pi^{-1}(\xi) = \{u\}\) and \(\Psi^{-1}(\xi) = \{p_1(u)\}\).

Referring to (10), one takes the affine open subsets \(\text{Spec}(D) := p_2^{-1}(\text{Spec}(B)) \subset \mathcal{B}(I) \times_A k\) and \(\text{Spec}(E) := q_2^{-1}(\text{Spec}(C)) \subset \text{Proj}(A[g]) \times_A k\). Then, there is an induced commutative diagram of scheme maps...
one gets the exact sequence
\[ (\Pi \times_A k) \mid_{\text{Spec}(D)} \rightarrow \text{Spec}(E) \]

with corresponding commutative diagram of ring homomorphisms

\[ \begin{array}{ccc}
\text{Spec}(B) & \xrightarrow{\pi \mid_{\text{Spec}(B)}} & \text{Spec}(C) \\
\downarrow & & \downarrow \\
\text{Spec}(D) & \xrightarrow{(\Pi \times_A k) \mid_{\text{Spec}(D)}} & \text{Spec}(E)
\end{array} \]

where \( B \) and \( C \) are integral domains, while \( D \) and \( E \) may not be. Also, the homomorphism \( E \rightarrow D \) is not necessarily injective (see [13, Corollary 2.11]).

From (5), one obtains \( \mathcal{I} \cdot \ker(s) = 0 \) where \( s : \mathcal{R}_R(\mathcal{I}) \otimes_A k \rightarrow \mathcal{R}_{R/R}(\mathcal{I}) \). Since \( I \neq 0 \), it follows that \( \mathcal{I} \not\subseteq \ker(s) \) and so \( \ker(s) \not\subseteq V(\mathcal{I} \cdot (\mathcal{R}_R(\mathcal{I}) \otimes_A k)) \supset \text{Supp}_{R/R}(\mathcal{I}) \otimes_A k (\ker(s)) \).

In terms of sheaves, the closed immersion \( p_1 \) in (8) gives the short exact sequence
\[ 0 \rightarrow \mathfrak{J} \rightarrow \mathcal{O}_{E \times_A k} \rightarrow p_1 \mathcal{O}_{E(I)} \rightarrow 0 \]
where \( \mathfrak{J} \) is the sheafification of the ideal \( \ker(s) \). Then it follows that \( p_1(u) \not\in \text{Supp}(\mathfrak{J}) \).

Restricting (11) to \( \text{Spec}(D) \) yields the exact sequence
\[ 0 \rightarrow \Omega \rightarrow \mathcal{O}_D \rightarrow B \rightarrow 0 \]
where \( \Omega \) is the ideal associated with the restriction \( \mathfrak{J} \mid_{\text{Spec}(D)} \). Since \( B \cong D/\Omega \), the ideal \( \Omega \subset \text{Spec}(D) \) is the prime ideal of the point \( p_1(u) \), and therefore \( \Omega \) is not in the support of \( \mathfrak{J} \) as a \( D \)-module (i.e., \( \mathfrak{J} \not\subseteq \text{Supp}_D(\Omega) \)).

Now, after these reductions one has \( \Psi^{-1}(\xi) \cong \text{Spec}(D \otimes_E \text{Quot}(C)) \) and \( \pi^{-1}(\xi) \cong \text{Spec}(B \otimes_C \text{Quot}(C)) \) as schemes. Since \( E \rightarrow C \) is surjective, \( C \cong E/J \) for some ideal \( J \subset E \). Since \( B \) is a \( C \)-module, then \( B \otimes_E C \cong B/JB = B \) and \( JD \subset \Omega \). By applying the tensor product \( - \otimes_E C \) to (12) one gets the exact sequence
\[ 0 \rightarrow \Omega/JD \rightarrow D/JD \rightarrow B \rightarrow 0 \]

One also has that \( \Omega/JD \not\in \text{Supp}_{D/JD}(\Omega/JD) \). From the fact that \( B \otimes_C \text{Quot}(C) = \text{Quot}(B) \neq 0 \), then \( (D/JD) \otimes_C \text{Quot}(C) \neq 0 \) and so one has an injection \( C \hookrightarrow D/JD \). Tensoring (13) with \( \text{Quot}(C) \) over \( C \), one obtains the exact sequence
\[ 0 \rightarrow q \rightarrow (D/JD) \otimes_C \text{Quot}(C) \rightarrow B \otimes_C \text{Quot}(C) \rightarrow 0 \]
where \( q = (\Omega/JD) \otimes_C \text{Quot}(C) \) and \( q \not\subseteq \text{Supp}_{(D/JD)\otimes_C\text{Quot}(C)}(q) \). Since \( \Psi^{-1}(\xi) \) has only one point then \( q \) is the unique prime ideal of \((D/JD)\otimes_C\text{Quot}(C) \cong D\otimes_E\text{Quot}(C)\), and this necessarily implies that \( q = \{0\} \).

Therefore, there is actually an isomorphism \( \Psi^{-1}(\xi) \cong \pi^{-1}(\xi) \) of schemes.

By (9), \( \Psi^{-1}(\xi) \cong \Pi^{-1}(w) \times_{k(w)} k(\xi) \), from which follows that
\[ \dim_{k(\xi)}(\mathcal{O}(\Psi^{-1}(\xi))) = \dim_{k(w)}(\mathcal{O}(\Pi^{-1}(w))) \].
Let $U$ be the open set of Proposition 5.11 and $\eta$ be the generic point of $\text{Proj}(A[\mathfrak{g}])$. Consider the finite morphism

$$
\Pi^{-1}(U) \to U.
$$

Then one has

$$
\text{deg}(g) = \dim_{k(\xi)}(\mathcal{O}(\pi^{-1}(\xi)))
= \dim_{k(\xi)}(\mathcal{O}(\Psi^{-1}(\xi)))
= \dim_{k(w)}(\mathcal{O}(\Pi^{-1}(w)))
\geq \dim_{k(\eta)}(\mathcal{O}(\Pi^{-1}(\eta))) = \text{deg}(\mathcal{G}),
$$

where the inequality follows by the upper semi-continuity of the degree of the fibers of a dominant finite morphism between integral schemes (see, e.g., [23, Exercise 5.1.25], [13, Corollary 7.30]).

(iii) Both (K1) and (K2) follow from Lemma 5.10.

5.4. Specialization of the saturated fiber cone. This part deals with the problem of specializing saturated fiber cones. Under certain general conditions it will turn out that the multiplicity of the saturated fiber cone decreases under specialization.

The reader is referred to the notation of §3.5.

Setup 5.13. Keep the notation introduced in Setup 5.1 and Setup 5.5. Let $\mathbb{K} = \text{Quot}(A)$ denote the field of fractions of $A$ and let $\mathbb{T} := \mathbb{K}[x_0, \ldots, x_r]$ denote the standard polynomial ring over $\mathbb{K}$ obtained from $R = A[x_0, \ldots, x_r]$ by base change (i.e., considering the A-coefficients of a polynomial as $\mathbb{K}$-coefficients). Let $\mathcal{G}$ and $g$ be as in Setup 5.1.

In addition, let $G$ denote the rational map $G : \mathbb{P}_R^r \to \mathbb{P}_K^r$ with representative $G = (G_0 : \cdots : G_s)$, where $G_i$ is the image of $g_i$ along the canonical inclusion $R \hookrightarrow \mathbb{T}$. Finally, set $\mathfrak{g} := (G_0, \ldots, G_s) \subset \mathbb{T}$.

As in Remark 3.18, the rational map $\mathcal{G}$ is generically finite if and only if the rational map $G$ is so, and one has the equality $\text{deg}(\mathcal{G}) = \text{deg}(G)$.

Consider the projective $R$-scheme $\text{Proj}_{R[\mathfrak{g}]}(\mathcal{R}_R(I))$, where $\mathcal{R}_R(I)$ is viewed as a “one-sided” $R$-graded algebra.

For any $p \in \text{Spec}(A)$, let $k(p) = A_p/pA_p$. The fiber $\mathcal{R}_R(I) \otimes_A k(p)$ inherits a one-sided structure of a graded $R(p)$-algebra, where $R(p) := R_p/pR_p = k(p)[x_0, \ldots, x_r]$. Moreover, it has a natural structure as a bigraded algebra over $R(p)[y_0, \ldots, y_s] = R(p) \otimes_A A[y_0, \ldots, y_s]$.

Therefore, for $0 \leq i \leq r$ the sheaf cohomology

$$
\mathcal{M}(p)^i := H^i\left(\text{Proj}_{R(p)-gr}(\mathcal{R}_R(I) \otimes_A k(p)), \mathcal{O}_{\text{Proj}_{R(p)-gr}(\mathcal{R}_R(I) \otimes_A k(p))}\right)
$$

has a natural structure as a finitely generated graded $k(p)[y_0, \ldots, y_s]$-module (see, e.g., [5, Proposition 2.7]). In particular, one can consider its Hilbert function $H\left(\mathcal{M}(p)^i, t\right) := \dim_{k(p)}\left(\left[\mathcal{M}(p)^i\right]_t\right)$.

Lemma 5.14. For any given $p \in \text{Spec}(A)$, consider the function $\chi_p : \mathbb{N} \to \mathbb{N}$ defined by

$$
\chi_p(t) := \sum_{i=0}^{r} (-1)^i H\left(\mathcal{M}(p)^i, t\right)
$$
Then, there exists an open dense subset $U \subset \text{Spec}(A)$, such that $\chi_p$ is the same for all $p \in U$.

**Proof.** Consider the affine open covering

$$W := \left( \text{Spec} \left( \mathcal{R}_R(I)(x_i) \right) \right)_{0 \leq i \leq r}$$

of $\text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I))$, with corresponding Čech complex

$$C^\bullet(W) : 0 \to \bigoplus_i \mathcal{R}_R(I)(x_i) \to \bigoplus_{i < j} \mathcal{R}_R(I)(x_ix_j) \to \cdots \to \mathcal{R}_R(I)(x_0 \cdots x_r) \to 0.$$

Note that each $C^i(W)$ has a natural structure of finitely generated graded algebra over $A$, and its grading comes from the graded structure of $A[y]$. By using the Generic Freeness Lemma (see, e.g., [11, Theorem 14.4]), there exist elements $a_i \in A$ such that each graded component of the localization $C^i(W)_{a_i}$ is a free module over $A_{a_i}$.

Let $D^\bullet$ be the complex given by $D^i = C^i(W)_{a}$, where $a = a_0a_1 \cdots a_r$. Hence, now $D^\bullet$ is a complex of graded $A_{a}$-modules and each graded strand $[D^\bullet ]_t$ is a complex of free $A_{a}$-modules. Notice that each of the free $A_{a}$-modules $[D^i ]_t$ is almost never finitely generated.

The $i$-th cohomology of a (co-)complex $F^\bullet$ is denoted by $H^i(F^\bullet)$. Since each $[D^\bullet ]_t$ is a complex of free $A_{a}$-modules (in particular, flat), [15, Lemma III.12.3] yields the existence of complexes $L^\bullet _t$ of finitely generated free $A_{a}$-modules such that

\[ H^i \left( [D^\bullet ]_t \otimes_{A_{a}} k(p) \right) \simeq H^i \left( L^\bullet _t \otimes_{A_{a}} k(p) \right) \]

for all $p \in \text{Spec}(A_{a}) \subset \text{Spec}(A)$. Let $U := \text{Spec}(A_{a}) \subset \text{Spec}(A)$.

**CLAIM.** $\chi_p$ is independent of $p$ on $U$; in other words, for any $p \in U$ and any $q \in U$, one has $\chi_p(t) = \chi_q(t)$ for every $t \in \mathbb{N}$.

Consider an arbitrary $p \in U$. Since $\mathcal{R}_R(I) \otimes_A k(p) \simeq \mathcal{R}_R(I)_{a} \otimes_{A_{a}} k(p)$, then $D^\bullet \otimes_{A_{a}} k(p)$ coincides with the Čech complex corresponding with the affine open covering

$$\left( \text{Spec} \left( \left( \mathcal{R}_R(I) \otimes_A k(p) \right)(x_i) \right) \right)_{0 \leq i \leq r}$$

of $\text{Proj}_{R(p)\text{-gr}}(\mathcal{R}_R(I) \otimes_A k(p))$. Hence, from (14) and (15), for any $t \in \mathbb{N}$ there is an isomorphism

$$\left[ M(p) \right]_t \simeq H^i \left( L^\bullet _t \otimes_{A_{a}} k(p) \right).$$

But since each $L^i_t$ is a finitely generated free $A_{a}$-module, it follows that

$$\sum_{i=0}^{r} (-1)^i \text{dim}_{k(p)} \left( \left[ M(p) \right]_t \right) = \sum_{i=0}^{r} (-1)^i \text{rank}_{A_{a}} (L^i_t).$$

Therefore, for every $t \in \mathbb{N}$, $\chi_p(t) = \sum_{i=0}^{r} (-1)^i H \left( M(p)^t, i \right)$ does not depend on $p$. □

The following theorem contains the main result of this part. By considering saturated fiber cones, one asks how the product of the degrees of the map and of its image behave under specialization.

**Theorem 5.15.** Under Setup 5.13, suppose that both $\mathcal{G}$ and $\mathcal{G}$ are generically finite. Let $U \subset \text{Spec}(A)$ be an open dense subset as in Lemma 5.14.
(i) Assume that the following conditions are satisfied:

(a) \( n \in \mathcal{U} \),
(b) \( \dim (\mathbb{E}(I) \times_A \mathbb{k}) \leq \dim (\mathbb{B}(I)) \).

Then

\[
\deg(g) \cdot \deg_{\mathbb{k}}(Y) = e \left( \widetilde{\delta_{\mathbb{k}}}(I) \right) \leq e \left( \widetilde{\delta_{\mathbb{k}}}(\mathbb{C}) \right) = \deg(g) \cdot \deg_{\mathbb{k}}(\mathbb{Y}),
\]

where \( Y := \text{Proj}(k[\mathbb{G}]) \) and \( \mathbb{Y} := \text{Proj}(k[\mathbb{G}]) \).

(ii) If \( \ell(I_\mathcal{Q}) \leq \text{ht}(\mathcal{Q}/nR) + 1 \) for every prime ideal \( \mathcal{Q} \in \text{Spec}(R) \) containing \( (I, n) \), then condition (b) of part (i) is satisfied.

**Proof.** (i) Let \( W := \text{Proj}_{_R}(\mathcal{R}(\mathcal{I}) \otimes_A \mathbb{k}) \), as in (14), \( H^i(W, \mathcal{O}_W) = M(n)^i \). By a similar token, \( H^i(\mathcal{W}, \mathcal{O}_W) = M((0))^i \), where \( \mathcal{W} := \text{Proj}_{_R}(\mathcal{R}(\mathcal{I}) \otimes_A \mathbb{k}) \), with \( R(0) := R \otimes_A \mathbb{k} = k[x_0, \ldots, x_r] \) and (0) denotes the null ideal of \( A \).

Now, clearly \( (0) \in \mathcal{U} \) and \( n \in \mathcal{U} \). Therefore, Lemma 5.14 yields the equalities

\[
(16) \quad \sum_{i=0}^{r} (-1)^i \mathcal{H} \left( H^i(W, \mathcal{O}_W), t \right) = \sum_{i=0}^{r} (-1)^i \mathcal{H} \left( H^i(\mathcal{W}, \mathcal{O}_W), t \right)
\]

for all \( t \in \mathbb{N} \).

For any \( i \geq 1 \), one has the known isomorphisms

\[
H^i(\mathcal{W}, \mathcal{O}_W) \simeq [H^{i+1}_m(\mathcal{R}(\mathcal{I}) \otimes_A \mathbb{k})]_0 \quad \text{and} \quad H^i(\mathcal{W}, \mathcal{O}_W) \simeq [H^{i+1}_m(\mathcal{R}(\mathcal{I}) \otimes_A \mathbb{k})]_0.
\]

Together with Theorem 4.4 they imply the inequalities

\[
\dim \left( H^i(\mathcal{W}, \mathcal{O}_W) \right) \leq \dim \left( R/nR \right) - 1 \quad \text{and} \quad \dim \left( H^i(\mathcal{W}, \mathcal{O}_W) \right) \leq \dim(\mathcal{T}) - 1.
\]

Therefore, (16) gives that

\[
\dim \left( H^0(\mathcal{W}, \mathcal{O}_W) \right) = \dim \left( H^0(\mathcal{W}, \mathcal{O}_W) \right) = \dim(\mathcal{T}) = \dim (R/nR),
\]

and that the leading coefficients of the Hilbert polynomials of \( H^0(W, \mathcal{O}_W) \) and \( H^0(\mathcal{W}, \mathcal{O}_W) \) coincide, and so \( e \left( H^0(\mathcal{W}, \mathcal{O}_W) \right) = e \left( H^0(\mathcal{W}, \mathcal{O}_W) \right) \).

Consider the exact sequence of finitely generated graded \((\mathcal{R}(\mathcal{I}) \otimes_A \mathbb{k})\)-modules

\[
0 \rightarrow \mathcal{Q} \rightarrow \mathcal{R}(\mathcal{I}) \otimes_A \mathbb{k} \rightarrow \mathcal{R}(\mathcal{R}/nR) \rightarrow 0.
\]

Sheafifying and taking the long exact sequence in cohomology yield an exact sequence of finitely generated graded \( \mathbb{k} \)-modules

\[
0 \rightarrow H^0(W, \mathcal{Q}^\sim) \rightarrow H^0(W, \mathcal{O}_W) \rightarrow H^0(W, \mathcal{R}(\mathcal{R}/nR)(I^\sim)) \rightarrow H^1(W, \mathcal{Q}^\sim).
\]

Note that

\[
\mathcal{F}_{\mathcal{R}(\mathcal{R}/nR)}(I) \simeq H^0 \left( \text{Proj}_{_R}(\mathcal{R}(\mathcal{R}/nR)(I)), \mathcal{O}_{\text{Proj}_{_R}(\mathcal{R}(\mathcal{R}/nR)(I))} \right)
\]

\[
\simeq H^0 \left( W, \mathcal{R}(\mathcal{R}/nR)(I)^{\sim} \right)
\]

(see, e.g., [15, Lemma III.2.10]).

From Lemma 5.10, it follows that \( \dim (\mathcal{R}(\mathcal{I}) \otimes_A \mathbb{k}) = \dim(R/nR) + 1 \). Since \( H^1(W, \mathcal{Q}^\sim) \simeq [H^1_m(\mathcal{Q})]_0 \), Theorem 4.4 gives that \( \dim (H^1(W, \mathcal{Q}^\sim)) \leq \dim(R/nR) - 1 \). Therefore, one gets the inequality

\[
e \left( \mathcal{F}_{\mathcal{R}(\mathcal{R}/nR)}(I) \right) \leq e \left( H^0(W, \mathcal{O}_W) \right).
\]
Since \( \widehat{\mathcal{F}}_T(l) \simeq H^0(\mathfrak{w}, \mathcal{O}_\mathcal{W}) \), summing up yields
\[
e \left( \frac{e(\mathcal{F} R/n R)}{e(\mathcal{F} R/n R)} \right) \leq e \left( H^0(W, \mathcal{O}_W) \right) = e \left( H^0(\mathfrak{w}, \mathcal{O}_\mathcal{W}) \right) = e \left( \widehat{\mathcal{F}}_T(l) \right).
\]
Finally, by [5, Theorem 2.4] it follows that
\[
e \left( \frac{e(\mathcal{F} R/n R)}{e(\mathcal{F} R/n R)} \right) = \deg(\mathfrak{g}) \cdot \deg_{P_k}(Y) \quad \text{and} \quad e \left( \widehat{\mathcal{F}}_T(l) \right) = \deg(\mathfrak{g}) \cdot \deg_{P_k}(Y).
\]
(ii) It follows from Lemma 5.10.

6. Perfect ideals of height two

In this section one deals with the case of a rational map \( \mathcal{F} : \mathbb{P}_R^r \dashrightarrow \mathbb{P}_R^r \) with a perfect base ideal of height 2, where \( \mathbb{P} \) is a field of characteristic zero. Note that the condition \( G_{r+1} \) is satisfied for the generic perfect ideal of height 2.

The main idea is that one can compute the degree of the rational map ([9, Corollary 3.2]) when the condition \( G_{r+1} \) is satisfied, then a suitable application of Theorem 5.12 gives an upper bound for all the rational maps that satisfy the weaker condition \( F_0 \).

Below Setup 5.1 is adapted to the particular case of a perfect ideal of height 2.

**Notation 6.1.** Let \( \mathbb{P} \) be a field of characteristic zero. Let \( 1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_r \) be integers with \( \mu_1 + \mu_2 + \cdots + \mu_r = d \). Given integers \( 1 \leq i \leq r+1 \) and \( 1 \leq j \leq r \), let
\[
z_{i,j} = \{z_{i,j,1}, z_{i,j,2}, \ldots, z_{i,j,m_j}\}
\]
denote a set of variables over \( \mathbb{P} \), of cardinality \( m_j := (\mu_j+1)^r \) – the number of coefficients of a polynomial of degree \( \mu_j \) in \( r+1 \) variables.

Let \( z \) be the set of mutually independent variables \( z = \bigcup_{i,j} z_{i,j}, A \) be the polynomial ring \( A = \mathbb{P}[z] \), and \( R \) be the polynomial ring \( R = A[x_0, \ldots, x_r] \). Let \( M \) be the \((r+1) \times r\) matrix with entries in \( R \) given by
\[
M = \\
\begin{pmatrix}
p_{1,1} & p_{1,2} & \cdots & p_{1,r} \\
p_{2,1} & p_{2,2} & \cdots & p_{2,r} \\
\vdots & \vdots & \ddots & \vdots \\
p_{r+1,1} & p_{r+1,2} & \cdots & p_{r+1,r}
\end{pmatrix}
\]
where each polynomial \( p_{i,j} \in R \) is given by
\[
p_{i,j} = z_{i,j,1}x_0^{\mu_j} + z_{i,j,2}x_0^{\mu_j-1}x_1 + \cdots + z_{i,j,m_j-1}x^{\mu_j-1}x_r + z_{i,j,m_j}x^\mu_j.
\]

Fix a (rational) maximal ideal \( \mathfrak{n} := (z_{i,j,k} - \alpha_{i,j,k}) \subset A \) of \( A \), with \( \alpha_{i,j,k} \in \mathbb{P} \).

Set \( K := \text{Quot}(A) \) and \( k = A/\mathfrak{n} \) and denote \( \mathbb{T} := R \otimes_A K = K[x_0, \ldots, x_r] \) and \( R/\mathfrak{n} R = k[x_0, \ldots, x_r] \).

Let \( \mathbb{M} \) and \( M \) denote respectively the matrix \( M \) viewed as a matrix with entries over \( \mathbb{T} \) and \( R/\mathfrak{n} R \). Let \( \{g_0, g_1, \ldots, g_r\} \subset R \) be the ordered signed minors of the matrix \( M \). Then, the ordered signed minors of \( \mathbb{M} \) and \( M \) are given by \( \{G_0, G_1, \ldots, G_r\} \subset \mathbb{T} \) and \( \{\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_r\} \subset R/\mathfrak{n} R \), respectively, where \( G_i = g_i \otimes_R \mathbb{T} \) and \( \mathfrak{g}_i = g_i \otimes_R (R/\mathfrak{n} R) \).

Let \( \mathcal{G} : \mathbb{P}_A^r \dashrightarrow \mathbb{P}_A^r \), \( \mathcal{G} : \mathbb{P}_K^r \dashrightarrow \mathbb{P}_K^r \) and \( \mathfrak{g} : \mathbb{P}_K^r \dashrightarrow \mathbb{P}_K^r \) be the rational maps given by the representatives \( (g_0 : \cdots : g_r), (G_0 : \cdots : G_r) \) and \( (\mathfrak{g}_0 : \cdots : \mathfrak{g}_r) \), respectively.

**Lemma 6.2.** The following statements hold:

(i) The ideal \( I_r(\mathbb{M}) \) is perfect of height two and satisfies the condition \( G_{r+1} \).
(ii) The rational map $G : \mathbb{P}^r_k \rightarrow \mathbb{P}^r_k$ is generically finite.

Proof. Let $\mathfrak{l} = I_r(\mathfrak{m})$.

(i) The claim that $\mathfrak{l}$ is perfect of height two is clear from Hilbert-Burch Theorem (see, e.g., [11, Theorem 20.15]). From Proposition 4.2, $\text{ht}(I_r(\mathfrak{m})) \geq \text{ht}(I_i(M)) \geq r + 2 - i$ for $1 \leq i \leq r$. Since the $G_{r+1}$ condition on $\mathfrak{l}$ (see Definition 2.1) is equivalent to

$$\text{ht}(I_{r+1-i}(\mathfrak{m})) = \text{ht}(\text{Fitt}_i(\mathfrak{l})) > i$$

for $1 \leq i \leq r$, one is through.

(ii) Note that $\mathfrak{l}$ is generated in fixed degree $d = \mu_1 + \cdots + \mu_r$. Thus, the image of the rational map $G$ is Proj($K[[\mathfrak{l}]_d]) \subset \mathbb{P}^r_k$. Then the generic finiteness of $G$ is equivalent to having $\dim(K[[\mathfrak{l}]_d]) = r + 1$. On the other hand, $\mathfrak{f}_I(\mathfrak{l}) \cong K[[\mathfrak{l}]_d]$. Now, as is well-known, $\dim(\mathfrak{f}_I(\mathfrak{l})) = \dim(\mathfrak{f}_I(\mathfrak{m})) = \ell(\mathfrak{m}_\mathfrak{g})$, where $\mathfrak{m} = (x_0, \ldots, x_r)\mathfrak{m}$ and $\ell$ stands for the analytic spread.

The ideal $\mathfrak{m}_\mathfrak{g} \subset \mathfrak{m}_{\mathfrak{g}}$ of height 2 being a strongly Cohen–Macaulay ideal ([17, Theorem 0.2 and Proposition 0.3]), hence in particular satisfies the sliding-depth property ([29, Definitions 1.2 and 1.3]).

Then part (i) and [29, Corollary 4.3] imply that the analytic spread of $\mathfrak{l}$ is equal to $\ell(\mathfrak{m}_\mathfrak{g}) = r + 1$, as wished. \qed

The main result of this section is a straightforward application of the previous developments.

**Theorem 6.3.** Let $\mathbb{F}$ be a field of characteristic zero and let $D = \mathbb{F}[x_0, \ldots, x_r]$ denote a polynomial ring over $\mathbb{F}$. Let $I \subset D$ be a perfect ideal of height two minimally generated by $r + 1$ forms $\{f_0, f_1, \ldots, f_r\}$ of the same degree $d$ and Hilbert-Burch resolution of the form

$$0 \rightarrow \bigoplus_{i=1}^r D(-d - \mu_i) \xrightarrow{\varphi} D(-d)^{r+1} \rightarrow I \rightarrow 0.$$ 

Consider the rational map $\mathcal{F} : \mathbb{P}^r_\mathbb{F} \rightarrow \mathbb{P}^r_\mathbb{F}$ given by

$$(x_0 : \cdots : x_r) \mapsto (f_0(x_0, \ldots, x_r) : \cdots : f_r(x_0, \ldots, x_r)).$$

When $\mathcal{F}$ is generically finite and $I$ satisfies the property $F_0$, one has

$$\deg(\mathcal{F}) \leq \mu_1 \mu_2 \cdots \mu_r.$$ 

In addition, if $I$ satisfies the condition $G_{r+1}$ then

$$\deg(\mathcal{F}) = \mu_1 \mu_2 \cdots \mu_r.$$ 

Proof. Let the $\alpha_{i,j,k}$’s introduced in Notation 6.1 stand for the coefficients of the polynomials in the entries of the presentation matrix $\varphi$. Then, under Notation 6.1, there is a canonical isomorphism

$$\Phi : (A/n)[x_0, \ldots, x_r] \cong D = \mathbb{F}[x_0, \ldots, x_r]$$

which, when applied to the entries of the matrix $M$, yields the respective entries of the matrix $\varphi$. Thus it is equivalent to consider the rational map $g : \mathbb{P}^r_k \rightarrow \mathbb{P}^r_k$ determined by the representative $(\varphi_0 : \cdots : \varphi_r)$ where $\Phi(\varphi_i) = f_i$. 

Since $I_r(Y)$ satisfies the $G_{r+1}$ condition (Lemma 6.2), then [9, Corollary 3.2] gives us that $\deg(G) = \mu_1\mu_2\cdots\mu_r$.

Since $G$ is generically finite by Lemma 6.2(ii) and Remark 3.18, its image is the whole of $\mathbb{P}^r$, the latter obviously being a normal scheme. In addition, since $I$ satisfies $F_0$, the conditions of Theorem 5.12(i) are satisfied, hence

$$\deg(F) = \deg(g) \leq \deg(G) = \mu_1\mu_2\cdots\mu_r.$$  

When $I$ satisfies $G_{r+1}$, then the equality $\deg(F) = \mu_1\mu_2\cdots\mu_r$ follows directly from [9, Corollary 3.2]. □

A particular satisfying case is when $F$ is a plane rational map. In this case $F_0$ is not a constraint at all, and one recovers the result of [5, Proposition 5.2].

Corollary 6.4. Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be a rational map defined by a perfect base ideal $I$ of height two. Then,

$$\deg(F) \leq \mu_1\mu_2$$

and an equality is attained if $I$ is locally a complete intersection at its minimal primes.

Proof. In this case property $F_0$ comes for free because $\text{ht}(I_1(\varphi)) \geq \text{ht}(I_2(\varphi)) = 2$ is always the case. Also, here l.c.i. at its minimal primes is equivalent to $G_3$. □

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