Modelling and analysis of prestress field in a thin plate with a non-uniform coating

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Abstract. In this paper, we describe a model of oscillations of a prestressed non-uniform body and study the problem of in-plane and out-of-plane vibrations of a plate with a functionally graded coating under the conditions of the initial stress-strain state. We build the numerical finite-element solution on the basis of the weak statement of the problem. We analyze the effect of the plate’s prestress levels on its frequency response functions and resonance frequencies. We estimate the prestress level in the coating by means of the measurement data on the plate’s frequency characteristics.

1. Introduction

Inhomogeneous plates are common structural elements, therefore, their deformation models are extremely relevant in solving problems arising in modern construction, production of military and civilian technical systems of wide use (e.g. cutting systems, membrane sensors, shielding elements, etc.).

In most construction applications, researchers confine themselves to restoring magnitudes of initial forces forming prestress fields. In [1], the techniques for determining prestress in concrete structures are developed. The paper [2] presents a method for identifying pre-tension in a prestressed concrete bridge deck by measuring its dynamic responses. To simulate the bridge deck, the authors used the Euler-Bernoulli beam model and FEM.

It is worth noting that in manufacturing, prestress is often intentionally embedded in structures in order to improve their mechanical properties, e.g. formability. The article [3] presents computational and full-scale experiments to study the effect of elastic prestress on the deformed shape of an aluminum alloy plate sample at the stage of laser hardening. The authors proposed a new technique for simulating the effect of prestress on bending deformation and residual stress formation based on the eigenstrain method [4] by A. Korsunsky. The numerical model allows predicting the deformed shape and 1D residual stress distributions in the plates under study.

We also note the importance of using variational and weak statements of direct and inverse problems allowing to describe operator relations and to provide efficient numerical solution. A new computational method for solving linear elasticity problems based on the combination of the Galerkin method and FEM is given in [5]. The idea behind the method lies in the use of weak statements for the corresponding differential operators.

As a matter of fact, among the works devoted to prestressed beams and plates, most of them relate to prestressed concrete. Problems on modern complex structural materials, for example, layered or...
functionally graded composites, in the presence of prestress fields, are scantily explored in the literature.

A review of various approaches to modeling prestressed elastic bodies is given in [6]. In the work [7], a number of ways to identify prestress fields in elastic bodies, like plates, are described in the framework of non-destructive acoustic method.

In [8–9], the problems of reconstructing an inhomogeneous prestressed state in non-uniform rods and thin plates are investigated, and some models of deforming prestressed plates within the classical approaches are proposed. Let us also mention the work [10] presenting the model of plate oscillations in the framework of non-classical Timoshenko’s deformation hypotheses and the technique for the inverse identification 2D prestressed state in the plate.

In this paper, we describe a model of a prestressed body which does not explicitly take into account the initial strain. On its basis, we consider the new particular problem of planar-flexural vibrations of a plate with a non-uniform functionally graded coating under the conditions of the initial stress-strain state. We give a weak formulation of the problem for five unknown functions — the components of the in-plane displacement vector, plate deflection, and the angles of rotation of the normals to the plate surface. We construct a numerical solution of the direct problem using FEM on the basis of the weak statement, estimate the effect of prestress levels in the plate on the frequency response and resonant frequencies. Finally, we investigate a problem of identifying the prestressed state level in the plate’s coating on the basis of the measurement data on the frequency characteristics of the plate.

2. Linearized problem on oscillations of a prestressed elastic body

One of the most widespread models of prestressed bodies that can be found in literature is a model in which the initial deformed state can be determined by a geometrically linear theory. This assumption corresponds to the fact that the gradients of initial displacements can be neglected in comparison with unity. The linearized statement of the problem on steady-state oscillations of a prestressed elastic body for small incremental quantities without explicitly taking into account the initial deformation has the form [6]

\[
\begin{align*}
(\sigma_{ij} + u_{ij,m} \sigma_{mj}^0)_{,j} + \rho \omega^2 u_{ij} &= 0, \\
u_i = 0, \\
(\sigma_{ij} + u_{ij,m} \sigma_{mj}^0) n^i_{,j} &= P_i,
\end{align*}
\]

Here \( \sigma_{ij} \) are the components of the linearized stress tensor, \( u_{ij} \) are the components of the small displacement vector, \( \sigma_{mj}^0 \) are the prestress tensor components, \( \rho \) is the body density, \( \omega \) is steady-state vibration frequency, \( P_i \) are the components of the surface load vector. The incremental stress and strain tensors components are determined according to

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}).
\]

Such a formulation is convenient for solving a large class of inverse coefficient problems in which it is required to determine the level or inhomogeneous fields of prestress on the basis of known information about the displacement field on a part of the body boundary. Further, in the present work we will use model (1)-(3) to model and identify prestress.

3. Weak statement of the problem for a prestressed non-uniform elastic plate

At present, various non-classical models of plates, in particular, based on the theories of Timoshenko, Reissner, Mindlin and others, are being actively developed and refined. These models allow simulating dynamic behavior of plates taking into account various factors.
Consider steady-state oscillations of an elastic isotropic thin plate with a plane cross section $S$, clamped at the boundary part $l_u$, under the action of periodic uniformly distributed force $P = Pe^{i\omega t}$, applied to the boundary part $l_p$ (figure 1). We assume that all the plate characteristics are given in the form of dependencies on the coordinates: $\rho = \rho(x_i)$, $h = h(x_i)$, $\lambda = \lambda(x_i)$, $\mu = \mu(x_i)$, $i, j, k = 1,2,3$. Here $\lambda = \frac{2\lambda'\mu}{\lambda + 2\mu}$ is the Lamé parameter for the plane stress state, $\lambda'$ is the standard Lamé parameter, $\mu$ is the shear modulus. Suppose that the plate contains non-uniform spatial prestress distribution $\sigma^0_i = \sigma^0_i(x_i)$, $i, j, k = 1,2,3$.

We consider coupled in-plane and out-of-plane vibrations. According to the plate theory in the framework of the Timoshenko model, the corresponding hypotheses are as follows

$$u_i = \theta_i x_i + \zeta_i, u_2 = \theta_2 x_i + \zeta_2, u_3 = w,$$

(4)

where $\theta_i = \theta_i(x_i)$ are the normal rotation angles along the axes $x_i$, $\zeta_i = \zeta_i(x_i)$ are in-plane displacements, $w = w(x_i)$ is the plate deflection, $\alpha, \beta = 1,2$. Denote the corresponding test functions by capital letters $\Theta_i$, $Z_i$, $W$ and assume they satisfy the same essential boundary as the functions $\theta_i$, $\zeta_i$, $w$ do:

$$\Theta_i|_{l_u} = 0, Z_i|_{l_u} = 0, W|_{l_u} = 0.$$

(5)

Based on hypotheses (4) and boundary conditions (5), the weak statement of the formulated problem will be rewritten the form

$$0 = \int_{l_u} PWdl + \int_{S} \left[ Q_{21} \Theta_{1,1} + Q_{22} \Theta_{1,2} + Q_{21} \Theta_{2,1} + Q_{22} \Theta_{2,2} +
\right.
$$

$$+ R_{11} Z_{1,1} + R_{12} Z_{1,2} + R_{11} Z_{2,1} + R_{12} Z_{2,2} + S_{11} \Theta_1 + S_{12} \Theta_2 + T_1 W_{1,1} + T_2 W_{2,2} -
$$

$$- \omega^2 \left[ P_{2} (\theta_1 \Theta_1 + \theta_2 \Theta_2) + P_{1} (\theta_1 Z_1 + \theta_2 Z_2 + \zeta_1 \Theta_1 + \zeta_2 \Theta_2) + P_{0} (\zeta_1 Z_1 + \zeta_2 Z_2 + wW) \right] ds,$$

(6)

where, considering the symmetry $\Sigma_{\alpha\beta} = \Sigma_{\beta\alpha}$, the introduced functions take form

$$Q_{11} = \Psi^2_{11} = \Lambda_1 \left( \theta_{1,1} + \theta_{2,2} \right) + \Lambda_1 \left( \zeta_{1,1} + \zeta_{2,2} \right) + 2M_1 \theta_{1,1} + 2M_1 \zeta_{1,1} +
$$

$$+ \Sigma_{12} \theta_1 + \Sigma_{12} \theta_2 + \Sigma_{11} \zeta_1 + \Sigma_{12} \zeta_2 + \Sigma_{13} \theta_1,$$

$$Q_{22} = \Psi^2_{22} = \Lambda_2 \left( \theta_{1,1} + \theta_{2,2} \right) + \Lambda_1 \left( \zeta_{1,1} + \zeta_{2,2} \right) + 2M_2 \theta_{1,2} + 2M_1 \zeta_{2,2} +
$$

$$+ \Sigma_{12} \theta_1 + \Sigma_{12} \theta_2 + \Sigma_{11} \zeta_1 + \Sigma_{12} \zeta_2 + \Sigma_{13} \theta_2,$$

$$Q_{12} = \Psi^2_{12} = M_2 (\theta_{1,2} + \theta_{2,1}) + M_1 (\zeta_{1,2} + \zeta_{2,1}) + \Sigma_{11} \theta_1 + \Sigma_{22} \theta_1 + \Sigma_{11} \zeta_1 + \Sigma_{22} \zeta_1 + \Sigma_{13} \theta_1,$$
The introduced functions $\Lambda_0, M_0, P_0, \Sigma_{\theta\theta}$ represent the averaged characteristics of the corresponding functions, depending on the coordinates $x_1, x_2$. The complete statement of the corresponding boundary value problem is given in [10]. Let us write in detail the averaged integral characteristics for the material modules:

\[
\Lambda_0 = \int_{-h/2}^{h/2} \lambda dx_3, \quad \Lambda_1 = \int_{-h/2}^{h/2} \lambda x_3 dx_3, \quad \Lambda_2 = \int_{-h/2}^{h/2} \lambda x_3^2 dx_3, \quad \Lambda_3 = \int_{-h/2}^{h/2} \mu dx_3, \quad \Lambda_4 = \int_{-h/2}^{h/2} \mu x_3 dx_3, \quad \Lambda_5 = \int_{-h/2}^{h/2} \mu x_3^2 dx_3,
\]

\[
M_2 = \int_{-h/2}^{h/2} \mu x_3^2 dx_3, \quad P_0 = \int_{-h/2}^{h/2} \rho dx_3, \quad P_1 = \int_{-h/2}^{h/2} \rho x_3 dx_3, \quad P_2 = \int_{-h/2}^{h/2} \rho x_3^2 dx_3.
\]

Similarly, the averaged prestress characteristics are defined as

\[
\Sigma_{11}^0 = \int_{-h/2}^{h/2} \sigma_{11}^0 dx_3, \quad \Sigma_{12}^0 = \int_{-h/2}^{h/2} \sigma_{12}^0 dx_3, \quad \Sigma_{22}^0 = \int_{-h/2}^{h/2} \sigma_{22}^0 dx_3, \quad \Sigma_{13}^0 = \int_{-h/2}^{h/2} \sigma_{13}^0 dx_3, \quad \Sigma_{12}^0 = \int_{-h/2}^{h/2} \sigma_{12}^0 dx_3,
\]

\[
\Sigma_{33}^0 = \int_{-h/2}^{h/2} \sigma_{33}^0 dx_3, \quad \Sigma_{11}^0 = \int_{-h/2}^{h/2} \sigma_{11}^0 x_3 dx_3, \quad \Sigma_{12}^0 = \int_{-h/2}^{h/2} \sigma_{12}^0 x_3 dx_3, \quad \Sigma_{22}^0 = \int_{-h/2}^{h/2} \sigma_{22}^0 x_3 dx_3, \quad \Sigma_{13}^0 = \int_{-h/2}^{h/2} \sigma_{13}^0 x_3 dx_3,
\]

\[
\Sigma_{23}^0 = \int_{-h/2}^{h/2} \sigma_{23}^0 x_3 dx_3, \quad \Sigma_{11}^0 = \int_{-h/2}^{h/2} \sigma_{11}^0 x_3 dx_3, \quad \Sigma_{12}^0 = \int_{-h/2}^{h/2} \sigma_{12}^0 x_3 dx_3, \quad \Sigma_{22}^0 = \int_{-h/2}^{h/2} \sigma_{22}^0 x_3 dx_3.
\]
4. Direct problem for a plate with a prestressed functionally graded coating

Consider the particular problem for a homogeneous plate with a functionally graded coating of thickness $\delta$, assuming that the plate and coating are in various homogeneous prestressed states (see figure 2).

The prestress tensor component $\sigma_{11}^0$ is the only nonzero one. Thus, on the segment $x_3 \in \left[ \frac{-h}{2}, \frac{h}{2} \right]$, the prestressed state is described by the following piecewise function (see figure 3):

$$\sigma_{11}^0(x_3) = \begin{cases} 
\sigma_p^0, & \text{for } x_3 \in \left[ \frac{-h}{2}, \frac{h}{2} - \delta \right]; \\
\sigma_c^0, & \text{for } x_3 \in \left[ \frac{h}{2} - \delta, \frac{h}{2} \right].
\end{cases}$$

(9)

The plate material is characterized by constant Lamé coefficients $\lambda_p$, $\mu_p$ and the density $\rho_p$, while the coating material is described by gradient linear laws; on the segment $x_3 \in \left[ \frac{-h}{2}, \frac{h}{2} \right]$ we have the following piecewise continuous functions:

$$\lambda(x_3) = \begin{cases} 
\lambda_p, & \text{for } x_3 \in \left[ \frac{-h}{2}, \frac{h}{2} - \delta \right]; \\
\lambda_p + \frac{\lambda_c - \lambda_p}{\delta} \left( x_3 - \frac{h}{2} + \delta \right), & \text{for } x_3 \in \left[ \frac{h}{2} - \delta, \frac{h}{2} \right].
\end{cases}$$

(10)

$$\mu(x_3) = \begin{cases} 
\mu_p, & \text{for } x_3 \in \left[ \frac{-h}{2}, \frac{h}{2} - \delta \right]; \\
\mu_p + \frac{\mu_c - \mu_p}{\delta} \left( x_3 - \frac{h}{2} + \delta \right), & \text{for } x_3 \in \left[ \frac{h}{2} - \delta, \frac{h}{2} \right].
\end{cases}$$

(11)

$$\rho(x_3) = \begin{cases} 
\rho_p, & \text{for } x_3 \in \left[ \frac{-h}{2}, \frac{h}{2} - \delta \right]; \\
\rho_p + \frac{\rho_c - \rho_p}{\delta} \left( x_3 - \frac{h}{2} + \delta \right), & \text{for } x_3 \in \left[ \frac{h}{2} - \delta, \frac{h}{2} \right].
\end{cases}$$

(12)

where $\lambda_c$, $\mu_c$, $\rho_c$ represent the corresponding values on the coating surface, for $x_3 = \frac{h}{2}$ (see figure 3).
Proceeding from this, some of the integral characteristics (8) will be nulled, and the expressions (7) included in the weak statement (6) will be simplified as follows:

\[ Q_{11} = \Lambda_2 \left( \theta_{1,1} + \theta_{2,2} \right) + \Lambda_1 \left( \zeta_{1,1} + \zeta_{2,2} \right) + 2M_1 \theta_{1,1} + 2M_1 \zeta_{1,1} + \Sigma_{11}^{\theta_{1,1}} + \Sigma_{11}^{\zeta_{1,1}}, \]
\[ Q_{22} = \Lambda_2 \left( \theta_{1,1} + \theta_{2,2} \right) + \Lambda_1 \left( \zeta_{1,1} + \zeta_{2,2} \right) + 2M_2 \theta_{2,2} + 2M_1 \zeta_{2,2}, \]
\[ Q_{12} = M_2 \left( \theta_{1,2} + \theta_{2,1} \right) + M_1 \left( \zeta_{1,2} + \zeta_{2,1} \right), \]
\[ Q_{21} = M_1 \left( \theta_{1,2} + \theta_{2,1} \right) + M_1 \left( \zeta_{1,2} + \zeta_{2,1} \right) + \Sigma_{11}^{\theta_{1,2}} + \Sigma_{11}^{\zeta_{1,2}}, \]
\[ R_{11} = \Lambda_1 \left( \theta_{1,1} + \theta_{2,2} \right) + \Lambda_0 \left( \zeta_{1,1} + \zeta_{2,2} \right) + 2M_1 \theta_{1,1} + 2M_0 \zeta_{1,1} + \Sigma_{11}^{\theta_{1,1}} + \Sigma_{11}^{\zeta_{1,1}}, \]
\[ R_{22} = \Lambda_1 \left( \theta_{1,1} + \theta_{2,2} \right) + \Lambda_0 \left( \zeta_{1,1} + \zeta_{2,2} \right) + 2M_1 \theta_{2,2} + 2M_0 \zeta_{2,2}, \]
\[ R_{12} = M_1 \left( \theta_{1,2} + \theta_{2,1} \right) + M_0 \left( \zeta_{1,2} + \zeta_{2,1} \right), \]
\[ R_{21} = M_1 \left( \theta_{1,2} + \theta_{2,1} \right) + M_0 \left( \zeta_{1,2} + \zeta_{2,1} \right) + \Sigma_{11}^{\theta_{1,2}} + \Sigma_{11}^{\zeta_{1,2}}, \]
\[ S_1 = M_0 \left( w_{1,1} + \theta_1 \right), \quad S_2 = M_0 \left( w_{1,2} + \theta_2 \right), \quad T_1 = M_0 \left( w_{1,2} + \theta_1 + \Sigma_{11}^{w_{1,1}}, T_2 = S_2, \right) \]

and the integral averaged characteristics for the of uniaxial prestress component will take the following form:

\[ \Sigma_{11}^0 = \int_{-h/2}^{h/2} \sigma_{11}^0 \mathrm{d}x_3 = \sigma_{r}^0 \left( h - \delta \right) + \sigma_{c}^0 \delta, \quad \Sigma_{11}^1 = \int_{-h/2}^{h/2} \sigma_{11}^0 x_3 \mathrm{d}x_3 = \frac{1}{2} \left( \sigma_{r}^0 - \sigma_{c}^0 \right) \left( \delta^2 - h\delta \right), \]
\[ \Sigma_{11}^2 = \int_{-h/2}^{h/2} \sigma_{11}^0 x_3^2 \mathrm{d}x_3 = \frac{1}{12} h^3 \sigma_{r}^0 + \left( \frac{1}{2} \frac{h^2 \delta^2}{4} - \frac{1}{4} h^2 \delta - \frac{1}{3} \frac{\delta^3}{3} \right) \left( \sigma_{r}^0 - \sigma_{c}^0 \right), \]

The averaged characteristics for the Lamé coefficients and density will be written out as

\[ \lambda_0 = \int_{-h/2}^{h/2} \lambda_0 \mathrm{d}x_3 = \int_{-h/2}^{h/2} \lambda_0 \mathrm{d}x_3 + \int_{h/2 - \delta}^{h/2} \lambda_0 + \frac{\lambda_0 - \lambda_r}{\delta} \left( x_3 - \frac{h}{2} + \delta \right) \mathrm{d}x_3 = \left( h - \frac{\delta}{2} \right) \lambda_0 + \frac{\delta}{2} \lambda_r, \]
\[ \lambda_1 = \int_{-h/2}^{h/2} \lambda_1 \mathrm{d}x_3 = \int_{-h/2}^{h/2} \lambda_1 \mathrm{d}x_3 + \frac{1}{12} \left( 2\delta - 3h \right) \left( \lambda_0 - \lambda_r \right), \]
\[ \lambda_2 = \int_{-h/2}^{h/2} \lambda_2 \mathrm{d}x_3 = \frac{1}{12} \left( 2\delta - 3h \right) \left( \lambda_0 - \lambda_r \right), \]
\[ M_0 = \left( h - \frac{\delta}{2} \right) \mu_p + \frac{\delta}{2} \mu_c, \quad M_1 = \frac{1}{12} \left( 2\delta - 3h \right) \left( \mu_p - \mu_c \right), \]

![Figure 3. Graphs of the prestress (on the left) and material modules (on the right) functions.](image-url)
\[ M_2 = \frac{1}{12} h^3 \mu_p - \left( \frac{1}{12} \delta^3 + \frac{1}{8} h^2 \delta - \frac{1}{6} h \delta^2 \right) (\mu_p - \mu_c), \]  
\[ P_0 = \left( h - \frac{\delta}{2} \right) \rho_p + \frac{\delta}{2} \rho_c, \quad P_1 = \frac{1}{12} \delta (2\delta - 3h) (\rho_p - \rho_c), \]  
\[ P_2 = \frac{1}{12} h^3 \rho_p - \left( \frac{1}{12} \delta^3 + \frac{1}{8} h^2 \delta - \frac{1}{6} h \delta^2 \right) (\rho_p - \rho_c). \]  
\[ (16) \]
\[ (17) \]

Thus, the problem in the specific formulation described above is reduced to the weak statement (6), taking into account the expressions (13) and the averaged integral characteristics (14)-(17).

Next we discuss a comparative analysis of two models of plate deformation:

1) Model 1: the new comprehensive model of a prestressed plate with a spatial distribution of inhomogeneous properties proposed in the present work;

2) Model 2: a simplified model of a homogeneous prestressed plate within the framework of the Timoshenko model, described in [11]. As material parameters in this model we use constant characteristics calculated by averaging according to the formula \[ g = \frac{1}{h} \int_{-h/2}^{h/2} g(x_i) dx_i, \] where \( g(x_i) \) is the averaged parameter (material modules, prestress components). The problem within the framework of the model 1 is solved for coupled planar-flexural vibrations.

The difference between the two models considered is that the model 1 contains more complete information on the inhomogeneous properties of the plate in thickness, contained in the integral characteristics \( \Sigma_{11}, \Sigma_{22}, \Lambda_1, \Lambda_2, M_1, M_2 \), in comparison to the model 2, where only the characteristics \( \Sigma_{11}^0, \Lambda_0, M_0 \) are taken into account. As well, the model 1 also includes the in-plane vibration component.

Next we discuss the computational experiments for the following problem parameters: \( l = 1 \) m (plate size along the axis \( x_1 \)), \( b = 0.4 \) m (plate size along the axis \( x_2 \)), \( h = 0.05 \) m, \( \delta = 0.1h \), \( p = 1 \) MPa, \( \nu_p = \nu_c = 0.29 \), \( E_p = 70 \) GPa, \( \rho_p = 2500 \) kg/m\(^3\), \( E_c = 210 \) GPa, \( \rho_c = 7700 \) kg/m\(^3\), \( \sigma_p^0 = 0 \), \( \sigma_0 = E_p \cdot 10^{-4} \). The transition from Young's modulus and Poisson's ratio to the Lamé coefficients was carried out according to standard formulas.

Below, in figures 4-6, the results of the comparison of the deflection fields and the angles of normals rotation for the two models, considered in the static case (for \( \omega = 0 \)), are presented. All the results are shown in the plate’s plane \( x_1, x_2 \), while the rigid clamping is on the left. For convenience, the rectangular area of the plate is projected onto the square in the figures.
Figure 4. Results of the finite element calculation: A) deflection field $w$ (mm), calculated in terms of model 1; B) deflection field $w$ (mm), calculated in terms of model 2; C) relative error field (%).

Figure 5. Results of the finite element calculation: A) field of normal rotation angle $\theta_1$ calculated in terms of model 1; B) field of normal rotation angle $\theta_1$ calculated in terms of model 2; C) relative error field (%).
Figure 6. Results of the finite element calculation: A) field of normal rotation angle $\theta_2$ calculated in terms of model 1; B) field of normal rotation angle $\theta_2$ calculated in terms of model 2; C) relative error field (%).

Figure 7 shows the corresponding fields of planar displacements obtained in the framework of the full model 1.

Figure 7. Results of the finite element calculation: in-plane displacement fields $\zeta_1, \zeta_2$, mm.

Below we present the results of calculation of the deflection fields, the normals rotation angles and planar displacements for the oscillation frequency located between the 3rd and 4th bending resonances, $f = \omega / 2\pi = 25$ Hz (figure 8).
Figure 8. Results of the finite element calculation: deflection field $w$, mm; fields of normal rotation angles $\theta_1, \theta_2$, in-plane displacement fields $\zeta_1, \zeta_2$, mm.

Figure 10 shows the amplitude-frequency characteristics (AFC) of the plate with the parameters described above for the two models considered, measured at the point $(l,0)$.

Based on the graphs presented in figure 9, one may conclude that with an increase in the oscillation frequency, the difference between the considered models increases.

Figures 10-11 depict the effect of the levels $\tau = \sigma_c^0 / E_p$ of compressive prestress in the coating on the AFC of the plate measured at $(l,0)$, within the model 1 for different coating thicknesses. In this case, two frequency ranges are considered – below the 1st resonance, and between the 1st and 2nd resonances. At the top of the figure there are graphs of AFC branches for different levels of $\tau$. The bottom of the figure shows the graphs of the relative influence of the greatest level $\tau_{\text{max}} = 10^{-3}$ on the AFC in percentage calculated by the formula $\varepsilon_{\text{max}} = \left| \frac{w^{\tau_{\text{max}}} - w^0}{w^{\tau_{\text{max}}}} \right| 100\%$, where $w^{\tau_{\text{max}}}$ is the deflection for $\tau = \tau_{\text{max}}$, $w^0$ is the deflection for the initially unstressed plate, for $\tau = 0$. The most significant influence of the compressive prestress component in the coating on AFC can be observed in close proximity to the resonant frequencies.
Figure 10. Effect of the coating prestress levels on the plate’s frequency response function $\delta = 0.1h$ for the frequency range below the 1$^{st}$ resonance (left) and between the 1$^{st}$ and 2$^{nd}$ resonances (right).

Figure 11. Effect of the coating prestress levels on the plate’s frequency response function for $\delta = 0.25h$ for the frequency range below the 1$^{st}$ resonance (left) and between the 1$^{st}$ and 2$^{nd}$ resonances (right).

The analysis of the effect of prestress levels on the flexural eigenfrequency offset is also conducted. Figure 12 provides the graphs of the relative change in the first three flexural eigenfrequencies depending on the prestress levels in the coating according to the formula $\delta f_i = \frac{|f_i - f_i^0|}{f_i^0} \times 100\%$, where $f_i^0$ are the corresponding eigenfrequencies of plates without prestress. The calculations were carried out for the coating thickness $\delta = 0.1h$. 

Figure 12. Analysis of bending eigenfrequencies when changing the prestress level in the coating.

From the above graph it can be seen that the 1st eigenfrequency is most sensitive to changes in the prestress level in the coating. Analysis has also shown that all eigenfrequencies of the plates, including planar and flexural ones, decrease with increasing compressive prestress in the coating.

Similarly, on the basis of the proposed model, it is possible to analyze oscillations of thin plates with an arbitrary law of inhomogeneity of material modules and prestress components in all the 3 spatial coordinates. By describing these laws, one can model and numerically calculate plates with prestressed coatings made of modern composite materials, like layered and functionally graded ones. We also note that the statement (6) has a number of advantages as compared to the standard ones: it takes into account the complete inhomogeneity of the material and prestress characteristics and considers both in-plane and out-of-plane oscillations components, and there is no explicit dependence on the initial plate deflection, which is convenient when solving a large class of inverse problems on identification of prestressed state.

5. Inverse problem of restoring the prestressed state of the coating

Consider a problem of identifying the level of uniform prestressed state of the coating for the considered rectangular plate. In this case, as the probing load, we choose the in-plane load applied to the free boundary part. The information on the plate geometry, including the thickness of the coating, is considered known. In this case, the weak formulation (6) takes the form

\[
0 = \int_{l_0} \mathcal{P} \mathcal{W} d l + \int_{s} \left( R_{11} Z_{1,1} + R_{12} Z_{1,2} + R_{21} Z_{2,1} + R_{22} Z_{2,2} - \omega^2 \left[ P_0 (\zeta_{1,1} Z_{1,1} + \zeta_{2,2} Z_{2,2}) \right] \right) d S,
\]

\[
R_{11} = \Lambda_0 (\zeta_{1,1} + \zeta_{2,2}) + 2 M_0 \zeta_{1,1} + \Sigma_{11} \zeta_{1,1}, \quad R_{22} = \Lambda_0 (\zeta_{1,1} + \zeta_{2,2}) + 2 M_0 \zeta_{2,2}
\]

\[
R_{12} = M_0 (\zeta_{1,2} + \zeta_{2,1}), \quad R_{21} = M_0 (\zeta_{1,2} + \zeta_{2,1}) + \Sigma_{11} \zeta_{2,1}.
\]

Note that in the particular case considered, formulation (18) includes only the averaged characteristics with the index “0”, corresponding to the common averaging according to the theory of the mean value of a function — characteristics of elastic modules \( \Lambda_0 \), \( M_0 \) and of prestress component \( \Sigma_{11}^0 \). In this case, based on the weak formulation of the direct problem (6) or from the similar formulation of the boundary value problem [10], we can express the integral characteristic \( \Sigma_{11}^0 \) through other known integral characteristics and the planar displacement field in the following form:
\[
\Sigma_{11}^0 = -h \left[ A_0 \left( \zeta_{1,1} + \zeta_{2,2} \right) + 2M_0 \zeta_{1,1} \right] + \left[ M_0 \left( \zeta_{1,2} + \zeta_{2,1} \right) \right].
\]

The formula (19) allows to restore the value \( \Sigma_{11}^0 \) based on the additional information on the given values of the components of planar displacements \( \zeta_{1,1}, \zeta_{2,2} \) measured in a set of points in the sectional area of the plate \( S \) for a given oscillation frequency. Using the spline interpolation, we calculate the functions of displacement \( \zeta_{1,1}, \zeta_{2,2} \) based on their values in the points set; the derivatives \( \zeta_{i,j} \) were found with the help of these functions at the required points of the plate. In view of the representation \( \Sigma_{11}^0 = \sigma_p^0 (h - \delta) + \sigma_c^0 \delta \) and the fact that the base plate is prestress free \( (\sigma_p^0 = 0) \), we find \( \sigma_c^0 = \Sigma_{11}^0 / \delta \).

A series of computational experiments was carried out to restore various values of the level of the prestressed state of the coating from the range \( \sigma_c^0 / E_p = 10^{-5} \div 10^{-2} \) for different frequencies from the frequency range below the 4th resonance. The plate parameters are as follows: \( l = 0.2 \text{ m}, \ b = 0.05 \text{ m}, \ h = 0.003 \text{ m}, \ \delta = 0.1h, \ \nu_p = \nu_c = 0.29, \ E_p = 70 \text{ GPa}, \ \rho_p = 2500 \text{ kg/m}^3, \ E_c = 210 \text{ GPa}, \ \rho_c = 7700 \text{ kg/m}^3, \ \sigma_p^0 = 0 \). A uniform finite element mesh of 80x50 was used. The analysis of numerical experiments on the level \( \sigma_c^0 \) reconstruction revealed that the accuracy of the inverse problem solution largely depends on the choice of the probing frequency and the loading mode. Table 1 shows some results of the reconstruction of \( \sigma_c^0 \) under the longitudinal uniformly distributed loading (along the axis \( x_1 \) ) applied to the free face of the plate, at a frequency of 220.5 Hz (in the vicinity of the first longitudinal resonance).

| Prestress level \( \tau = \sigma_c^0 / E_p \) | Exact value \( \sigma_c^0 \) | Reconstruction \( \tilde{\sigma}_c^0 \) | Relative error (\%) |
|---------------------------------|----------------|----------------|----------------|
| \( 10^{-5} \)                     | 21             | 18.7526        | 10.7019        |
| \( 10^{-4} \)                     | 210            | 205.206        | 2.28266        |
| \( 10^{-3} \)                     | 2100           | 2069.75        | 1.44051        |
| \( 10^{-2} \)                     | 21000          | 20715.4        | 1.3552         |

It can be seen from the table that the reconstruction is successful for the prestress levels exceeding the value \( \tau = \sigma_c^0 / E_p = 10^{-5} \); lower prestress levels do not have any significant frequency response, and therefore their recovery process is difficult. Note that for the series of experiments described above, the most advantageous ranges for selecting the probe frequency are located in a close vicinity of the first longitudinal resonance.

6. Conclusion
A model of a prestressed body in which the gradients of initial displacements can be neglected is described. On its basis, the problem of planar-flexural vibrations of a plate with a non-uniform functionally graded coating under the conditions of the initial stress-strain state is considered.
A numerical solution of the problem was constructed using the FEM based on the weak formulation of the original problem. The effects of prestress levels on the amplitude-frequency characteristics and the natural frequencies of the plate are analyzed.

The proposed plate model takes into account the complete inhomogeneity of the elastic material characteristics and prestress components, and it also considers relatedness of in-plane and out-of-plane vibration modes.

The level of the prestressed state in the coating is estimated on the basis of the measurements of the frequency characteristics of the plate.

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