THE RANK 8 CASE OF A CONJECTURE ON SQUARE-ZERO UPPER TRIANGULAR MATRICES

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Abstract. Let $A$ be the polynomial algebra in $r$ variables with coefficients in an algebraically closed field $k$. When the characteristic of $k$ is 2, Carlsson [8] conjectured that for any dg-$A$-module $M$, which has dimension $N$ as a free $A$-module, if the homology of $M$ is nontrivial and finite dimensional as a $k$-vector space, then $N \geq 2^r$. Here we examine a stronger conjecture concerning varieties of square-zero upper triangular $N \times N$ matrices with entries in $A$. Stratifying these varieties via Borel orbits, we show that the stronger conjecture holds when $N = 8$ without any restriction on the characteristic of $k$. This result also verifies that if $X$ is a product of 3 spheres of any dimensions, then the elementary abelian 2-group of order 4 cannot act freely on $X$.

1. Introduction

The long-standing Rank Conjecture states that if $(\mathbb{Z}/p\mathbb{Z})^r$ acts freely and cellulary on a finite CW-complex $X$ that is homotopy equivalent to $S^{n_1} \times \ldots \times S^{n_m}$, then $r$ is less than or equal to $m$. There are many partial results in several special cases. In the equidimensional case $n := n_1 = \ldots = n_m$, Carlsson [7], Browder [5], and Benson-Carlson [4] gave a proof under the assumption that the induced action on homology is trivial. Without the homology assumption, the equidimensional conjecture was proved by Conner [10] for $m = 2$, Adem-Browder [1] for $p \neq 2$ or $n \neq 1,3,7$, and Yalçın [22] for $p = 2$ and $n = 1$. In the non-equidimensional case, the conjecture was verified by Smith [21] for $m = 1$, Heller [14] for $m = 2$, Carlsson [9] for $p = 2$ and $r = 3$, Refai [18] for $p = 2$ and $r = 4$, Hanke [13] for $p$ large relative to the dimension of the product of spheres, and Okutan-Yalçın [17] for products in which the average of the dimensions is sufficiently large compared to the differences between them. The general case for $r \geq 5$ is still open.

A more general conjecture, known as Carlsson’s Rank Conjecture [8 Conjecture I.3] states that if $(\mathbb{Z}/p\mathbb{Z})^r$ acts freely on a finite CW-complex $X$, then $\sum_i \dim_{\mathbb{F}_p} H_i(X; \mathbb{Z}/p\mathbb{Z}) \geq 2^r$. Carlsson also states an algebraic analogue of the conjecture [8 Conjecture II.2], which is even stronger: If $C_*$ is a finite free $\mathbb{F}_p(\mathbb{Z}/p\mathbb{Z})^r$-chain complex with nonzero homology, then $\dim_{\mathbb{F}_p} H_\ast(C_*) \geq 2^r$. However, Iyengar-Walker [16] disproved the algebraic conjecture when $p \neq 2$ and $r \geq 8$. Even so, it remains open for $p = 2$. Moreover, the Iyengar-Walker counterexamples cannot be realized topologically [20], so the topological version of Carlsson’s Rank Conjecture is still open for all primes.

Let $R$ be a graded ring. A pair $(M, \partial)$ is a differential graded $R$-module, or simply dg-$R$-module, if $M$ is a graded $R$-module and $\partial$ is an $R$-linear endomorphism of $M$ of degree $-1$ that satisfies $\partial^2 = 0$. A dg-$R$-module is free if the underlying $R$-module is free.

When $k$ is an algebraically closed field of characteristic 2, Carlsson in [6] and [8] showed that the algebraic analogue of the conjecture is equivalent to a conjecture in commutative algebra:

2010 Mathematics Subject Classification. 13Dxx (Primary); 55M35 (Secondary).
Key words and phrases. rank conjecture, square-zero matrices, free flag.
The author is supported by TÜBİTAK 2219.
Conjecture 1. Let $k$ be an algebraically closed field and $A = k[y_1, \ldots, y_r]$ the polynomial ring over $k$ in the variables $y_1, \ldots, y_r$ of degree $-1$. If $M$ is a free, finitely generated dg-$A$-module, $H_s(M) \neq 0$, and $\dim_k H_s(M) < \infty$, then $N := \dim_A M \geq 2^r$.

For $r \leq 3$, the above conjecture was verified by Carlsson [9] when the characteristic of $k$ is 2, by Allday and Puppe [2] when $k$ is the rational numbers, and by Avramov, Buchweitz, and Iyengar [3] for regular rings without any restriction on the characteristic of $k$.

In [12], Ünlü and the author stated another conjecture from the perspective of algebraic geometry:

Conjecture 2. Let $k$ be an algebraically closed field, $r$ a positive integer, $N = 2n$ an even positive integer, and $d := (d_1, d_2, \ldots, d_N)$ $N$-tuple of nonincreasing integers. Then define $V(d, n)$ as the weighted quasi-projective variety of rank $n$ square-zero upper triangular $N \times N$ matrices $(p_{ij})$ with $\deg(p_{ij}) = d_i - d_j + 1$. Assume that there exists a nonconstant morphism $\psi$ from $\mathbb{P}^{r-1}_k$ to $V(d, n)$. Then $N \geq 2^r$.

We proved that Conjecture 2 implies Conjecture 1 in [12] Theorem 1]. The key point is that every free dg-$A$ module $(M, \partial)$ is quasi-isomorphic to a free dg-$A$ module $(\hat{M}, \hat{\partial})$ such that the differential of $\hat{M} \otimes_A k$ is the zero map. Let $N$ be the rank of $\hat{M}$ over $A$. Then there exists a homogeneous basis for $\hat{M}$ so that the boundary $\hat{\partial}$ can be represented by an upper triangular $N \times N$ matrix whose entries are homogeneous polynomials in $A$. If $m$ is any maximal ideal of $A$ other than $(y_1, \ldots, y_r)$, then $H_s(M \otimes_A A/m) = 0$ since $\dim_k H_s(M) < \infty$ (see [9] §1 Proposition 8). Moreover, for any finite-dimensional module $(M, \partial)$ over $k$, $H_s(M) = 0$ if and only if $\dim_k M = 2^\text{rank}_k \partial$. To work in a uniform setting, we define a new polynomial ring $S$ by replacing our indeterminates with $x_i$ such that $\deg(x_i) = 1$.

In our previous work [12], some of the results obtained by the computer algebra program GAP for small-dimensional cases of the conjecture led us to extend the conjecture for matrices of a certain form: Those with $C$ leading zero columns and $R$ ending zero rows. Hence the most general conjecture states that if the value of $(p_{ij})$ at every point in the image of $\psi$ is 0 whenever $i \geq N - R + 1$ or $j \leq C$, then $N \geq 2^{r-1}(R + C)$. Since we are concerned only with strictly upper triangular matrices, we have $R \geq 1$ and $C \geq 1$. Clearly, in this situation we have $N \geq 2^r$.

Now suppose that the morphism $\psi$ in the above conjecture is represented by the matrix $D$, so that $D$ is given $p_{ij}$ coordinate-wise. In this paper, we consider Conjecture 3 in terms of matrices in order to use the power of linear algebra:

Conjecture 3. Let $k$ be an algebraically closed field and $S := k[x_1, \ldots, x_r]$ the polynomial ring in $r$ variables with $\deg(x_i) = 1$. Assume that $n, r, R, C$ are positive integers, $N := 2n$, and $D = (p_{ij}) \in \text{Mat}_{N \times N}(S)$. If

1) $D$ is strictly upper triangular,
2) $D^2 = 0$,
3) for all $i$ and $j$, we have $p_{ij}(0, \ldots, 0) = 0$ (i.e., each constant term is 0),
4) for all $(a_1, \ldots, a_r) \in k^r - \{(0, \ldots, 0)\}$, we have $\text{rank}(D(a_1, \ldots, a_r)) = n$,
5) there exists an $N$-tuple of nonincreasing integers $(d_1, \ldots, d_N)$ such that for all $i$ and $j$, $p_{ij}$ is a homogeneous polynomial in $S$ of degree $d_i - d_j + 1$,
6) when $j \leq C$ or $i \geq N - R + 1$, we have $p_{ij} = 0$,

then $N \geq 2^{r-1}(R + C)$. 

When $N < 8$, Ünlü and the author have already proved Conjecture 3 in [12] Theorem 2]. Indeed, we also verified the conjecture for $r \leq 2$ in [12] Theorem 6]. However,
Iyengar and Walker’s construction in [16, Proposition 2.1] was used in [11, Example 3.2.4] to form an explicit counterexample to Conjecture 3 when \( p \neq 2 \) and \( r = 8 \). Even so, this method does not apply when \( r = 2 \), and we showed in this case that the conclusion of Conjecture 3 still holds in [11].

The main result of this paper holds without any restriction on the characteristic of the ground field \( k \).

**Theorem 1.** Conjecture 3 holds for \( N = 8 \).

The essential tool in the proof of Theorem 1 is the stratification of the varieties of square-zero upper-triangular matrices obtained by considering the conjugation action of a Borel subgroup of \( GL_N(k) \). As in our previous work [12, Section 3.4], this stratification can be applied to the subvarieties that contain the matrix \( D \) in Conjecture 3.

Note that this result means that if the conditions in Conjecture 3 are satisfied, then \( N = 8 > 2r - 1 (R + C) \geq 2^r \). Hence if \( r \geq 4 \), then \( N > 8 \). Now, consider a weaker version of Conjecture 1 for \( k = \mathbb{F}_2 \), that is, if \( M \) is a free, finitely generated \( \mathbb{F}_2 \)-module with \( 0 < \dim_k H_*(M) < \infty \), then \( N := \dim A M \geq 2^{r - 1} + 1 \). Note that by the Künneth Formula even the weaker version of Conjecture 1 implies Rank Conjecture for the \( p = 2 \) case. In other words, we have:

**Corollary 1.** Let \( k \) be an algebraically closed field of characteristic 2 and \( A = k[y_1, \ldots, y_r] \). If \( r \geq 4 \), and \( M \) is a free, finitely generated \( \mathbb{F}_2 \)-module with \( 0 < \dim_k H_*(M) < \infty \), then \( N := \dim A M > 8 \).

In particular, this result proves that the Rank Conjecture is true for product of three spheres of any dimensions:

**Corollary 2.** Let \( X \) be a finite CW-complex homotopy equivalent to \( S^{n_1} \times S^{n_2} \times \ldots \times S^{n_m} \). If \( m \leq 3 \), then \( \mathbb{Z}/2)^r \) cannot act freely and cellularly on \( X \) for \( r \geq 4 \).

Therefore, Theorem 1 shortens the proof already given by Refai in [18, Theorem 4.5].

A simplifying observation is:

**Remark 1.** Let \( D \) be a matrix satisfying all six conditions of Conjecture 3 with \( r \) variables. Then by evaluating some of variables at 0 (of course not letting all entries be identically zero), one can obtain a matrix that satisfies all six conditions in the conjecture with fewer variables. Hence if there is no matrix that satisfies all six conditions in Conjecture 3 for \( r \) variables, then the same is true for all matrices with more than \( r \) variables.

**Acknowledgements:** The author was supported by the Fellowship Program for Abroad Studies 2219 by the Scientific and Technological Research Council of Turkey (TÜBİTAK). The author is grateful to the Institut für Mathematik in Universität Augsburg and Bernhard Hanke for their hospitality and would like to thank Özgün Ünlü for many helpful discussions on this paper.

2. Free flags

The connection between various earlier algebraic versions of the conjecture was also explored by Avramov, Buchweitz, and Iyengar; and Conjecture 1 was extended to local rings in [3, Conjecture 5.3]. In this section, we recall some earlier results given by [3] in a special case in order to prove two special cases of our main result.

We are concerned with differential modules that admit a filtration compatible with their differentials:
Definition 1. [3, 2.1-2.2] A differential module over a ring $R$ is a pair $(F, d)$, where $F$ is an $R$-module and $d$ is an $R$-linear endomorphism of $F$ satisfying $d^2 = 0$. The homology of $F$ is the $R$-module $H(F) = \ker(d)/\im(d)$. A free flag of $F$ is a chain of $R$-submodules

$$0 = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq \ldots \subseteq F^l = F$$

such that $F^i/F^{i-1}$ is a free $R$-module of finite rank and $d(F^i) \subseteq F^{i-1}$ for all $i$. We say that $F$ has a free flag with $(l + 1)$-folds. If $F$ admits such a flag, we say that $F$ has free class at most $l$ and write $\text{FreeClass}_R F \leq l$. If there is no such filtration, we say $\text{FreeClass}_R F = \infty$.

Theorem 2. [3, Theorem 3.1] Let $R$ be a local ring, $F$ a differential $R$-module, and $D$ a retract of $F$ such that the $R$-module $H(D)$ has non-zero finite length. When $R$ has a big Cohen-Macaulay [3, Section 3.2] module one has:

$$\text{FreeClass}_R F \geq \dim R.$$

Theorem 3. Let $S := k[x_1, \ldots, x_r]$ be the polynomial ring in $r$ variables with coefficients in the algebraically closed field $k$. Suppose that $(F, d)$ is a differential graded $S$-module, and $H(F)$ is finite and non-zero. If $F$ has a free differential flag, i.e., $\text{FreeClass}_S F \leq l$, then $l \geq r$.

Proof. Let $R$ be the localization of $S = k[x_1, \ldots, x_r]$ at the ideal $(x_1, \ldots, x_r)$. Since $(x_1, \ldots, x_r)$ is one of the maximal ideals of $S$, we have the equality of Krull dimensions $\dim R = \dim S = r$. Note that big Cohen-Macaulay modules exist when our ring contains a field as a subring [15]. Then Theorem 2 implies that $l \geq \text{FreeClass}_R F \otimes_S R \geq r$ because $0 \subseteq F^0 \otimes_S R \subseteq \ldots \subseteq F^l \otimes_S R = F \otimes_S R$ is a flag. \hfill \Box

Note that Theorem 2 verifies Carlssson’s earlier result in [6, Theorem I.16]: If $S = \mathbb{F}_2[x_1, \ldots, x_r]$ and $M$ is a free, finitely generated dg-$S$-module with $H_*(M) \neq 0$ and $\dim_k H_*(M) < \infty$, then $\text{FreeClass}_S M \geq r$ (see [3, Remark 3.4]).

Lemma 1. [3, Lemma 5.8] Let $S := k[x_1, \ldots, x_r]$ and $(F, d)$ be a finitely generated differential graded $S$-module. If $F$ admits a free flag with $F^l = F$, then we have

(a) $d(F^i) \nsubseteq F^{i-2}$ for $i = 1, \ldots, l$ and

(b) $\text{rank}_S(F^i/F^{i-1}) \geq \begin{cases} 1 & i = 0 \text{ or } i = l \\ 2 & i = 1, \ldots, l - 1. \end{cases}$

Moreover, if $\text{FreeClass}_S F = l < \infty$, then we have $\text{rank}_S F \geq 2l$.

Remark 2. Given a matrix $D$ that satisfies all six conditions of Conjecture 3 there exists a corresponding free flag. More precisely, let $D$ be a nilpotent upper triangular block matrix of type $t = (t_0, t_1, \ldots, t_l)$, that is,
moves of the following form: to define a partial order on the corresponding orbits. For two given involutions A matrices. The one-to-one correspondence between σ contains a unique partial permutation matrix. Each of these partial permutation matrices (D free. This allows us to associate the matrix D by conjugation and there is a nice representative for each Borel orbit. So we have a finite increasing filtration of D by submodules F^i for i ∈ {0, ..., l} such that each successive quotient is free. This allows us to associate the matrix D of type t = (t_0, t_1, ..., t_l) with the flag F of (l + 1) folds, where t_i = rank_R(F^i/F^{i-1}).

3. Borel Orbits of Square Zero Matrices

We stratify the variety corresponding to the matrix D in Conjecture 3 by using Borel orbits. Much of the work in this section can be found in [12 Section 3.2 – 3.4] and [11 Chapter 2]. We recall it here for the reader’s convenience.

Let V_N be the variety of square-zero strictly upper triangular N × N matrices over k. We consider the action of Borel subgroup B_N of GL_N(k) on V_N. The group B_N acts on the variety V_N by conjugation and there is a nice representative for each Borel orbit. A partial permutation matrix is a matrix that has at most one entry of 1 in each row and column and 0s elsewhere. Rothbach [19 Theorem 1] verified that each B_N-orbit of V_N contains a unique partial permutation matrix. Each of these partial permutation matrices can be identified with an involution σ ∈ Sym(N). Let P be a partial permutation matrix in the set of non-zero N × N strictly upper triangular square zero partial permutation matrices. The one-to-one correspondence between P and σ is given by

\[(P)_{ij} = 1 \iff \sigma(i) = j \text{ and } i < j.\]

These involutions (or matrices) have a rich combinatorial structure, which allows us to define a partial order on the corresponding orbits. For two given involutions σ and σ' there correspond to two non-zero N × N strictly upper triangular square zero partial permutation matrices, and we have σ' ≤ σ if σ' can be obtained from σ by a sequence of moves of the following form:

- Type I replaces σ with σ(i, j) if σ(i) = j and i ≠ j,
- Type II replaces σ with σ(i, i') if σ(i) = i < i' < σ(i'),
- Type III replaces σ with σ(j, j') if σ(j) < σ(j') < j',
- Type IV replaces σ with σ(j, j') if σ(j') < j' = σ(j),
- Type V replaces σ with σ(i, j) if i < σ(i) < σ(j) < j,
where $\sigma(i, j)$ denotes the product of the permutations $\sigma$ and the transposition $(i, j)$, and $\sigma(i,j)$ denotes the right-conjugate of $\sigma$ by $(i, j)$.

Let $d := (d_1, d_2, \ldots, d_N)$ be $N$-tuple of nonincreasing integers and $n := N/2$. Let $V(d, n)$ denote the weighted quasi-projective variety of rank $n$ square-zero upper triangular $N \times N$ matrices $(p_{ij})$ with $\deg(p_{ij}) = d_i - d_j + 1$. Note that a similar stratification can be imposed on $V(d, n)$ by $B_N$; for details see [12] Section 3.4. Let $V(d, n)_{RC}$ be the subvariety of $V(d, n)$ given by the equations $p_{ij} = 0$ for $i \geq N - \mathcal{R} + 1$ or $j \leq C$. Then $V(d, n)_{RC}$ corresponds to exactly the variety contains the matrix $D$ that satisfies all six conditions of Conjecture 3.

In Conjecture 3 the matrix $D$ must have rank $n$, so our involutions corresponding to the variety that contains $D$ must be a product of $n$ distinct transpositions. Let $\text{RP}(N)$ denote the permutations in the set of involutions in $\text{Sym}(N)$ of rank $n$. We have already shown that the only possible moves are of types III or V between two permutations in $\text{RP}(N)$; see [12] Section 3.6.

Now, we consider a move of type III more closely. Let us represent a permutation $\sigma = (i_1 j_1) (i_2 j_2) \ldots (i_n j_n)$ in $\text{RP}(N)$ by

$$
\left( \begin{array}{ccc}
  i_1 & i_2 & \ldots & i_n \\
  j_1 & j_2 & \ldots & j_n
\end{array} \right).
$$

Then move of type III swaps $j_a$ with $j_b$ if $j_a > j_b$ for $1 \leq a < b \leq n$. Clearly, a move of type III preserves the rank of $\sigma$. Also observe that for a given $\sigma \in \text{RP}(N)$ if the corresponding partial permutation $P$ is of type $(t_0, \ldots, t_l)$, then moves of type III do not change the numbers $t_0 = C$ and $t_l = \mathcal{R}$.

In [12], we showed that when $N = 2, 4, \text{ or } 6$, there exists a unique a unique maximal $\sigma' \in \text{RP}(N)$ such that $\sigma$ can be obtained from $\sigma'$ by a sequence of moves of type III. In the Appendix of this paper we see the same holds for $N = 8$, see Figure II. Since moves of type III do not change the number of leading zero columns $C$ and ending zero rows $\mathcal{R}$ of the corresponding partial permutation matrices, the Borel orbit corresponding to $\sigma$ is contained in $V(d, n)_{RC}$ if and only if the Borel orbit corresponding to $\sigma'$ is contained in $V(d, n)_{RC}$ for all $\mathcal{R}, C$. Hence we restrict our attention to maximal elements in $\text{RP}(N)$ as it is enough to consider those in the proof of Theorem 5.

4. The Main Result

One of the main facts we will use to prove our main result, known as Multivariate Fundamental Theorem of Algebra, is stated as follows:

**Theorem 4.** [11, Theorem 2.1.3] Let $f_1, f_2, \ldots, f_s$ be nonconstant homogeneous polynomials in $k[x_1, \ldots, x_r]$. If $r > s$, then there exists $\gamma \in \mathbb{P}^{r-1}_k$ such that $f_1(\gamma) = 0, f_2(\gamma) = 0, \ldots, f_s(\gamma) = 0$.

**Theorem 5.** Let $k$ be an algebraically closed field and $S := k[x_1, \ldots, x_r]$ the polynomial ring over $k$ in the variables $x_1, \ldots, x_r$ of degree 1. Assume that $\mathcal{R}, C$ are positive integers, and $D = (p_{ij}) \in \text{Mat}_{8 \times 8}(S)$. If

1) $D$ is strictly upper triangular,
2) $D^2 = 0$,
3) for all $i$ and $j$, we have $p_{ij}(0, \ldots, 0) = 0$,
4) for all $(a_1, \ldots, a_r) \in k^r - \{(0, \ldots, 0)\}$, we have $\text{rank}(D(a_1, \ldots, a_r)) = 4$,
5) there exists an $N$-tuple of nonincreasing integers $(d_1, \ldots, d_N)$ such that for all $i$ and $j$, $p_{ij}$ is a homogeneous polynomial in $S$ of degree $d_i - d_j + 1$,
6) when $j \leq C$ or $i \geq N - \mathcal{R} + 1$, we have $p_{ij} = 0$,

then $8 \geq 2^{r-1}(\mathcal{R} + C)$.
As noted above, to prove Conjecture $\Box$ for $N = 8$, it suffices to consider the maximal elements in $\text{RP}(8)$, which are:

$$\text{max} \text{RP}(8) := \{(12)(34)(56)(78), (14)(23)(56)(78), (12)(34)(58)(67), (14)(23)(58)(67),
(12)(36)(45)(78), (12)(38)(47)(56), (16)(23)(45)(78), (16)(23)(45)(56), (12)(36)(45)(56),
(18)(23)(45)(78), (18)(25)(34)(67), (18)(23)(45)(67), (18)(27)(34)(56), (18)(27)(36)(45)\}.$$  

Note that we can reduce the number of cases from 14 to 10 because of symmetry with respect to the anti-diagonal of the corresponding partial permutation matrices. This symmetry takes a matrix of type $(t_0, \ldots, t_l)$ to one of type $(t_l, \ldots, t_0)$. Hence it is enough to check the list

$$\{(12)(34)(56)(78), (12)(34)(58)(67), (14)(23)(58)(67), (12)(36)(45)(78), (12)(38)(47)(56),
(16)(23)(45)(78), (18)(23)(47)(56), (18)(23)(45)(67), (18)(27)(34)(56), (18)(27)(36)(45)\}.$$  

We will also use the following notation in several cases: For a matrix $X$, the minor $m_{i_1 j_1 \cdots i_k j_k}(X)$ is the determinant of the $k \times k$ submatrix obtained by taking the $i_1$th, $i_2$th, $\ldots$, $i_k$th rows and $j_1$th, $j_2$th, $\ldots$, $j_k$th columns of $X$.

The proof is given by the following ten propositions:

**Proposition 1.** If $\sigma \leq (18)(27)(34)(56)$, then $r < 2$.

**Proof.** Suppose to the contrary that $\sigma \leq (18)(27)(34)(56)$ and $r \geq 2$. By Remark $\Box$ it is enough to consider $r = 2$. The matrix $D$ corresponding to the involution $\sigma \leq (18)(27)(34)(56)$ is

$$D = \begin{bmatrix}
0 & 0 & 0 & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & 0 & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & p_{46} & p_{47} & p_{48} & p_{49} \\
0 & 0 & 0 & 0 & 0 & p_{56} & p_{57} & p_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the $p_{ij}$ are homogeneous polynomials in $k[x_1, x_2]$. Suppose that

$$a := \text{gcd}(p_{14}, p_{15}), \quad p_{14} = \overline{p}_{14}a, \quad p_{15} = \overline{p}_{15}a,\$$

$$b := \text{gcd}(p_{24}, p_{25}), \quad p_{24} = \overline{p}_{24}b, \quad p_{25} = \overline{p}_{25}b,\$$

$$c := \text{gcd}(p_{34}, p_{35}), \quad p_{34} = \overline{p}_{34}c, \quad p_{35} = \overline{p}_{35}c,\$$

$$d := \text{gcd}(p_{46}, p_{56}), \quad p_{46} = \overline{p}_{46}d, \quad p_{56} = \overline{p}_{56}d,\$$

$$e := \text{gcd}(p_{47}, p_{57}), \quad p_{47} = \overline{p}_{47}e, \quad p_{57} = \overline{p}_{57}e,\$$

and $f := \text{gcd}(p_{48}, p_{58}), \quad p_{48} = \overline{p}_{48}f, \quad p_{58} = \overline{p}_{58}f.$

Since $D^2 = 0$, we have $p_{14}p_{46} + p_{15}p_{56} = 0$, which implies that $\overline{p}_{14} \overline{p}_{46} = -\overline{p}_{15} \overline{p}_{56}$. We may write $u \overline{p}_{14} = \overline{p}_{56}$ and $-u \overline{p}_{15} = \overline{p}_{46}$, where $u$ is an unit. Similarly, for some units $v, w, s$
and \( t \) we have
\[
D = \begin{bmatrix}
0 & 0 & 0 & \overline{p}_{14} a & \overline{p}_{15} a & p_{16} & p_{17} & p_{18} \\
0 & 0 & 0 & \overline{v}_{14} b & \overline{v}_{15} b & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & \overline{w}_{14} c & \overline{w}_{15} c & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & 0 & -u \overline{p}_{15} d & -s \overline{p}_{15} e & -\overline{p}_{15} f \\
0 & 0 & 0 & 0 & 0 & u \overline{p}_{14} d & s \overline{p}_{14} e & \overline{p}_{14} f \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Define
\[
T := \begin{bmatrix}
a & p_{16} & p_{17} & p_{18} \\
v b & p_{26} & p_{27} & p_{28} \\
w c & p_{36} & p_{37} & p_{38} \\
0 & u d & s e & t f
\end{bmatrix}.
\]

By Theorem 4 there exists \( \gamma \in \mathbb{P}_1 \) such that \( \det T(\gamma) = 0 \). Then we have
\[
\text{rank} \begin{bmatrix}
\overline{p}_{14} a(\gamma) & \overline{p}_{15} a(\gamma) & p_{16}(\gamma) & p_{17}(\gamma) & p_{18}(\gamma) \\
\overline{v}_{14} b(\gamma) & \overline{v}_{15} b(\gamma) & p_{26}(\gamma) & p_{27}(\gamma) & p_{28}(\gamma) \\
\overline{w}_{14} c(\gamma) & \overline{w}_{15} c(\gamma) & p_{36}(\gamma) & p_{37}(\gamma) & p_{38}(\gamma) \\
0 & 0 & -u \overline{p}_{15} d(\gamma) & -s \overline{p}_{15} e(\gamma) & -\overline{p}_{15} f(\gamma) \\
0 & 0 & u \overline{p}_{14} d(\gamma) & s \overline{p}_{14} e(\gamma) & \overline{p}_{14} f(\gamma)
\end{bmatrix} \leq 3,
\]
which is a contradiction. \( \square \)

**Proposition 2.** If \( \sigma \leq (16)(23)(45)(78) \) or \( \sigma \leq (12)(38)(45)(67) \), then \( r < 3 \).

Note that the matrix \( D \) corresponding to \( \sigma \leq (16)(23)(45)(78) \) is in the form of a nilpotent upper triangular block matrix of type \( t = (2, 2, 3, 1) \). The anti-diagonal flip of such a matrix of type \( (1, 3, 2, 2) \), which corresponds to a \( \sigma' \leq (12)(38)(45)(67) \).

**Proof.** Suppose to the contrary that \( r \geq 3 \). By Remark 1 we can take \( r = 3 \). Consider
\[
D = \begin{bmatrix}
0 & 0 & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & 0 & p_{35} & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & p_{45} & p_{46} & p_{47} & p_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{68} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
where the \( p_{ij} \) are homogeneous polynomials in \( k[x_1, x_2, x_3] \). Suppose that
\[
a := \gcd(p_{13}, p_{14}), \quad p_{13} = \overline{p}_{13} a, \quad p_{14} = \overline{p}_{14} a, \\
b := \gcd(p_{23}, p_{24}), \quad p_{23} = \overline{p}_{23} b, \quad p_{24} = \overline{p}_{24} b, \\
c := \gcd(p_{35}, p_{45}), \quad p_{35} = \overline{p}_{35} c, \quad p_{45} = \overline{p}_{45} c, \\
d := \gcd(p_{36}, p_{46}), \quad p_{36} = \overline{p}_{36} d, \quad p_{46} = \overline{p}_{46} d, \\
\text{and} \quad e := \gcd(p_{37}, p_{47}), \quad p_{37} = \overline{p}_{37} e, \quad p_{47} = \overline{p}_{47} e.
\]
Since \( D^2 = 0 \), we have \( p_{13} p_{35} + p_{14} p_{45} = 0 \), which implies that \( \overline{p}_{13} \overline{p}_{35} = -\overline{p}_{14} \overline{p}_{45} \). We may write \( u \overline{p}_{13} = \overline{p}_{35} \) and \( -u \overline{p}_{14} = \overline{p}_{35} \) where \( u \) is an unit. Similarly, for some units \( v, w \) and
s we have

\[
D = \begin{bmatrix}
0 & 0 & \overline{p_{13}}a & \overline{p_{14}}a & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & v\overline{p_{13}}b & v\overline{p_{14}}b & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & 0 & -uc\overline{p_{14}} & -wd\overline{p_{14}} & -se\overline{p_{14}} & p_{38} \\
0 & 0 & 0 & 0 & uc\overline{p_{13}} & wd\overline{p_{13}} & se\overline{p_{13}} & p_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{68} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Let \( \gamma \in \mathbb{P}_k^2 \) be the root of \( \overline{p_{13}} \) and \( \overline{p_{14}} \), then rank of the matrix \( D(\gamma) \) is at most 3, which is a contraction. \( \square \)

**Proposition 3.** If \( \sigma \leq (12)(34)(58)(67) \) or \( \sigma \leq (14)(23)(56)(78) \), then \( r < 3 \).

These cases are symmetric, so it is enough to prove the case \( \sigma \leq (12)(34)(58)(67) \).

**Proof.** Suppose to the contrary that \( r \geq 3 \). By Remark we can take \( r = 3 \). Consider

\[
D = \begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & 0 & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & 0 & 0 & p_{47} & p_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & p_{57} & p_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & p_{67} & p_{68} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Suppose that

\[
a := \gcd(p_{12}, p_{13}), \quad p_{12} = \overline{p_{12}a}, \quad p_{13} = \overline{p_{13}a},
\]

\[
b := \gcd(p_{24}, p_{34}), \quad p_{24} = \overline{p_{24}b}, \quad p_{34} = \overline{p_{34}b},
\]

\[
c := \gcd(p_{25}, p_{35}), \quad p_{25} = \overline{p_{25}c}, \quad p_{35} = \overline{p_{35}c},
\]

and \( d := \gcd(p_{26}, p_{36}), \quad p_{26} = \overline{p_{26}d}, \quad p_{36} = \overline{p_{36}d} \).

For some units \( u \) and \( w \), we have

\[
D = \begin{bmatrix}
0 & \overline{p_{12}a} & \overline{p_{13}a} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & 0 & -u\overline{p_{13}b} & v\overline{p_{14}b} & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & u\overline{p_{12}b} & v\overline{p_{12}b} & -w\overline{p_{13}c} & -u\overline{p_{14}d} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{47} & p_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{57} & p_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{67} & p_{68} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

By Theorem there exists \( \gamma \in \mathbb{P}_k^2 \) such that \( \overline{p_{12}}(\gamma) = 0 \) and \( \overline{p_{13}}(\gamma) = 0 \). Then the rank of matrix \( D(\gamma) \) can be at most 3. This is a contraction. \( \square \)

**Proposition 4.** If \( \sigma \leq (18)(27)(36)(45) \), then \( r < 2 \).
Proof. Suppose not. By Remark 1 we can take \( r = 2 \). Consider
\[
D = \begin{bmatrix}
0 & 0 & 0 & 0 & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & 0 & 0 & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & 0 & p_{35} & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & p_{45} & p_{46} & p_{47} & p_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
It is enough to consider a root of the minor \( m_{678}^{1234}(D) \) to prove this case. \( \square \)

**Proposition 5.** If \( \sigma \leq (12)(34)(56)(78) \), then \( r < 4 \).

Proof. Suppose to the contrary that \( r \geq 4 \). By Remark 1, we may take \( r = 4 \). Then we have
\[
D = \begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & 0 & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & 0 & p_{46} & p_{47} & p_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
where \( p_{ij} \)'s are homogeneous polynomials in \( k[x_1, x_2, x_3, x_4] \). Set
\[
\begin{align*}
a &:= \gcd(p_{24}, p_{25}), & p_{24} &= \overline{p}_{24}a, & p_{25} &= \overline{p}_{25}a, \\
b &:= \gcd(p_{34}, p_{35}), & p_{34} &= \overline{p}_{34}b, & p_{35} &= \overline{p}_{35}b, \\
c &:= \gcd(p_{46}, p_{56}), & p_{46} &= \overline{p}_{46}c, & p_{56} &= \overline{p}_{56}c, \\
d &:= \gcd(p_{47}, p_{57}), & p_{47} &= \overline{p}_{47}d, & p_{57} &= \overline{p}_{57}d.
\end{align*}
\]
Since \( D^2 = 0 \), we have \( p_{24}p_{46} + p_{25}p_{56} = 0 \), which implies that \( \overline{p}_{56} = up_{24} \) and \( \overline{p}_{46} = -up_{25} \) for some unit \( u \). Similarly, for some units \( v \) and \( w \), we have
\[
D = \begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & 0 & \overline{a}p_{24} & \overline{a}p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & \overline{v}p_{24} & \overline{v}p_{25} & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & 0 & -up_{25} & -wp_{25} & p_{48} \\
0 & 0 & 0 & 0 & 0 & up_{24} & wp_{24} & p_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{68} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
Let \( T \) be the boxed submatrix of \( D \) shown above. Since \( r = 4 \), by Theorem 4 there exists \( \gamma \in \mathbb{P}_k^3 \) such that \( \overline{p}_{24}(\gamma) = \overline{p}_{25}(\gamma) = 0 \) and \( m_{27}^{23}(D)(\gamma) = 0 \). Hence all \( 2 \times 2 \) minors of \( T \) at \( \gamma \) are zero. This implies that the rank of \( T(\gamma) \) is at most 1, so the total rank of \( D \) at \( \gamma \) is at most 3. This is a contradiction. \( \square \)

**Remark 3.** Proposition 5 itself also implies Corollary 1 as does Corollary 2.
Proof. Let \( k = \mathbb{F}_2 \) and \( r \geq 4 \). Suppose that \( A = k[y_1, \ldots, y_r] \) and \( M \) is a free, finitely generated \( dg\)-\( A \)-module with \( 0 < \dim_k H_* (M) < \infty \). Further assume that \( M \) has a free flag \( F \) with \((l + 1)\)-folds, so that \( \text{FreeClass}_g F \leq l \). By Theorem 3 we have \( l \geq 4 \). By Lemma 1 \( N \neq 2, 4, 6 \). We can also eliminate the case \( N = 8 \) easily. The only possibility for the sequence of integers \( t = (t_0, \ldots, t_l) \) corresponding to \( F \) is \((1, 2, 2, 2, 1)\). On the other hand, by Proposition 5 we see that if there exists a flag admits this sequence, then \( r < 4 \). That contradicts our assumption on \( r \), so \( N \) must be at least 10. \( \square \)

Note that Remark 3 gives an alternative proof of the result given by Refai in Theorem 4.2). Consequently, Proposition 5 itself verifies the Rank Conjecture when \( k = \mathbb{F}_2 \) and \( m = 3 \).

Proposition 6. If \( \sigma \leq (18)(23)(45)(67) \), then \( r < 3 \).

Proof. Suppose not; then \( r \geq 3 \), and by Remark 1 we may take \( r = 3 \). We have

\[
D = \begin{bmatrix}
0 & 0 & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & 0 & p_{35} & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Suppose that

\[
a := \gcd(p_{13}, p_{14}), \quad p_{13} = \overline{p}_{13} a, \quad p_{14} = \overline{p}_{14} a, \\
b := \gcd(p_{23}, p_{24}), \quad p_{23} = \overline{p}_{23} b, \quad p_{24} = \overline{p}_{24} b, \\
c := \gcd(p_{35}, p_{45}), \quad p_{35} = \overline{p}_{35} c, \quad p_{45} = \overline{p}_{45} c, \\
\text{and} \quad d := \gcd(p_{36}, p_{46}), \quad p_{36} = \overline{p}_{36} d, \quad p_{46} = \overline{p}_{46} d.
\]

Since \( D^2 = 0 \), we have \( p_{13} p_{35} + p_{14} p_{45} = 0 \), which implies that \( \overline{p}_{13} \overline{p}_{35} = -\overline{p}_{14} \overline{p}_{45} \). We may write \( u \overline{p}_{13} = \overline{p}_{45} \) and \( -v \overline{p}_{14} = \overline{p}_{36} \) for some unit \( v \). To simplify our notation, let \( \overline{p}_{13} = f_1 \) and \( \overline{p}_{14} = f_2 \), so that \( f_1 \) and \( f_2 \) are relatively prime homogeneous polynomials. Moreover, if we keep using \( b \) instead of \( bv \), \( c \) instead of \( uc \) etc., then we have

\[
D = \begin{bmatrix}
0 & 0 & a f_1 & a f_2 & A & B & p_{17} & p_{18} \\
0 & 0 & b f_1 & b f_2 & C & D & p_{27} & p_{28} \\
0 & 0 & 0 & -c f_2 & -d f_2 & E & F \\
0 & 0 & 0 & c f_1 & d f_1 & G & H \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Consider \( p_{57}, p_{58}, p_{67}, p_{68} \). Since \( D^2 = 0 \), \( c f_2 p_{57} + d f_2 p_{67} = 0 \) and \( c f_1 p_{57} + d f_1 p_{67} = 0 \). Note that \( f_1 \) and \( f_2 \) are not identically zero. Otherwise we do not have the matrix of type \((2, 2, 2, 2)\). Then \( c p_{57} = -d p_{67} \), so let \( p_{57} = -d v g_1 \) and \( p_{67} = c w g_1 \), where \( w \) is a unit.
Similarly we have
\[
D = \begin{bmatrix}
0 & 0 & af_1 & af_2 & A & B & p_{17} & p_{18} \\
0 & 0 & bf_1 & bf_2 & C & D & p_{27} & p_{28} \\
0 & 0 & 0 & 0 & -cf_2 & -df_2 & E & F \\
0 & 0 & 0 & 0 & cf_1 & df_1 & G & H \\
0 & 0 & 0 & 0 & 0 & 0 & -d wg_1 & -d w'_2 \\
0 & 0 & 0 & 0 & 0 & 0 & c w_1 & c w'_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Since \( D^2 = 0 \), we have
\[
(1) \quad af_1E + af_2G - dw_1A + cw_1B = 0,
\]
\[
(2) \quad bf_1E + bf_2G - dw_1C + cw_1D = 0,
\]
\[
(3) \quad af_1F + af_2H - dw'_2A + cw'_2B = 0,
\]
\[
(4) \quad bf_1F + bf_2H - dw'_2C + cw'_2D = 0.
\]
Since \( f_1 \) and \( g_1 \) are not identically zero, equations (1) and (2) imply that
\[
(5) \quad bdA - bcB - adC + acD = 0.
\]
Moreover, equations (1) and (3) imply that
\[
(6) \quad w'(f_1g_2E + f_2g_2G) = w(f_1g_1F + f_2g_1H).
\]
Let \( T_1 \) be the upper boxed submatrix of \( D \). One can check that all \( 3 \times 3 \) minors of \( T_1 \) are zero. For instance,
\[
m_{123}^{134}(D) = (bdA - bcB - adC + acD)f_1^2 = 0,
\]
\[
m_{123}^{135}(D) = (adC + bcB + bdA + acD)f_1f_2 = 0.
\]
Recall the definition of the rank of a matrix \( M \) in some quotient field \( Q \) of \( S \), that is,
\[
\text{rank}_Q M = \max\{t \in \mathbb{Z}^+ \mid t \times t \text{ minor of } M \text{ is non-zero}\}.
\]
Then by definition, \( \text{rank}_Q T_1 \leq 2 \).

Similarly, let \( T_2 \) be the lower boxed submatrix of \( D \). Then by Equation (6), it is clear that all \( 3 \times 3 \) minors in \( T_2 \) are zero. Hence \( \text{rank}_Q T_2 \leq 2 \).

Then in \( T_1 \),
\[
\text{rank}_Q \left( \begin{bmatrix} \alpha h_1 & \alpha h_2 \\ \beta h_1 & \beta h_2 \end{bmatrix} \right) = 1 \quad \text{rank}_Q \left( \begin{bmatrix} cf_1 & df_1 \\ af_2 & df_2 \end{bmatrix} \right) = 1, \quad \text{and so} \quad \text{rank}_Q \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \neq 2
\]
Therefore, \( AD - BC = 0 \) and similarly \( EH - FG = 0 \). Denote
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha h_1 & \alpha h_2 \\ \beta h_1 & \beta h_2 \end{bmatrix}
\]
Since we suppose that \( r \geq 3 \), there exists \( \gamma \in \mathbb{P}^2_k \) such that \( f_1(\gamma) = f_2(\gamma) = 0 \). We have

\[
D(\gamma) = \begin{bmatrix}
0 & 0 & 0 & \alpha h_1(\gamma) & \alpha h_2(\gamma) & p_{17}(\gamma) & p_{18}(\gamma) \\
0 & 0 & 0 & \beta h_1(\gamma) & \beta h_2(\gamma) & p_{27}(\gamma) & p_{28}(\gamma) \\
0 & 0 & 0 & 0 & 0 & E(\gamma) & F(\gamma) \\
0 & 0 & 0 & 0 & 0 & G(\gamma) & H(\gamma) \\
0 & 0 & 0 & 0 & 0 & p_{57}(\gamma) & p_{58}(\gamma) \\
0 & 0 & 0 & 0 & 0 & p_{67}(\gamma) & p_{68}(\gamma) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Then \( \text{rank}(D(\gamma)) \leq 3 \), which is a contradiction. \( \Box \)

**Proposition 7.** If \( \sigma \leq (12)(36)(45)(78) \), then \( r < 4 \).

**Proof.** Suppose to the contrary that \( \sigma \leq (12)(36)(45)(78) \), and by Remark 1 we have \( r = 4 \).

\[
D = \begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & 0 & 0 & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & 0 & p_{35} & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & p_{45} & p_{46} & p_{47} & p_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

First we will prove that any of two of the polynomials \( \{p_{12}, p_{13}, p_{14}\} \) are relatively prime. Suppose not, WLOG \( p_{12}, p_{13} \) are not relatively prime. Then by Theorem 1 there exists \( \gamma \in \mathbb{P}^2_k \) such that \( (p_{12}, p_{13})(\gamma) = 0 \), \( p_{14}(\gamma) = 0 \), and \( m_{12345678}(D)(\gamma) = 0 \).

Moreover, if \( p_{58}(\gamma) = 0 \), \( p_{68}(\gamma) = 0 \), and \( p_{78}(\gamma) = 0 \), then the rank of \( D(\gamma) \) is at most 3, which leads to a contradiction. Hence \( p_{58}(\gamma) \neq 0 \), \( p_{68}(\gamma) \neq 0 \), \( p_{78}(\gamma) \neq 0 \). Let

\[
c_5(\gamma) := \begin{bmatrix}
p_{15}(\gamma) \\
p_{25}(\gamma) \\
p_{35}(\gamma) \\
p_{45}(\gamma)
\end{bmatrix}, \quad c_6(\gamma) := \begin{bmatrix}
p_{16}(\gamma) \\
p_{26}(\gamma) \\
p_{36}(\gamma) \\
p_{46}(\gamma)
\end{bmatrix} \quad \text{and} \quad c_7(\gamma) := \begin{bmatrix}
p_{17}(\gamma) \\
p_{27}(\gamma) \\
p_{37}(\gamma) \\
p_{47}(\gamma)
\end{bmatrix}.
\]

Since \( D^2 = 0 \), we have

\[
p_{12}p_{28} + p_{13}p_{38} + p_{14}p_{48} + p_{15}p_{58} + p_{16}p_{68} + p_{17}p_{78} = 0,
\]
\[
p_{25}p_{58} + p_{26}p_{68} + p_{27}p_{78} = 0,
\]
\[
p_{35}p_{58} + p_{36}p_{68} + p_{37}p_{78} = 0,
\]
\[
\text{and} \quad p_{45}p_{58} + p_{46}p_{68} + p_{47}p_{78} = 0.
\]

Thus

\[
c_5(\gamma)p_{58}(\gamma) + c_6(\gamma)p_{68}(\gamma) + c_7(\gamma)p_{78}(\gamma) = 0.
\]

By the fact that \( p_{58}, p_{68}, \) and \( p_{78} \) are non-zero at \( \gamma \), \( c_5(\gamma), c_6(\gamma) \) and \( c_7(\gamma) \) are linearly dependent. Thus the rank of \( D(\gamma) \) is at most 3, which is a contradiction. Therefore we may assume that any two of the polynomials \( \{p_{12}, p_{13}, p_{14}\} \) are relatively prime.

Also, note that if \( p_{12}(\gamma) = 0, p_{13}(\gamma) = 0 \) and \( p_{14}(\gamma) = 0 \), then \( c_5(\gamma), c_6(\gamma) \) and \( c_7(\gamma) \) must be linearly independent. By Equation (7), it implies that \( p_{58}(\gamma) = 0, p_{68}(\gamma) = 0 \) and
By the argument above, 

\( p_{78}(\gamma) = 0. \)

Let us set \( f_1 := p_{12}, \ f_2 := p_{13}, \) and \( f_3 := p_{14} \) and write

\[
D = \begin{bmatrix}
0 & f_1 & f_2 & f_3 & p_{15} & p_{16} & p_{17} & p_{18} & p_{28} \\
0 & 0 & 0 & 0 & A & D & G & p_{28} \\
0 & 0 & 0 & 0 & B & E & H & p_{38} \\
0 & 0 & 0 & 0 & C & F & I & p_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{68} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Using \( D^2 = 0 \), we have

(8) \quad f_1A + f_2B + f_3C = 0,

(9) \quad f_1D + f_2E + f_3F = 0,

(10) \quad f_1G + f_2H + f_3I = 0.

Then if we multiply Equation (8) by \( E \) and Equation (9) by \( B \) and substract them, we have

(11) \quad (AE - BD)f_1 + (CE - BF)f_3 = 0.

Since \( f_1 \) and \( f_3 \) are relatively prime, \( (CE - BF) \) divides \( f_1 \) and \( (AE - BD) \) divides \( f_3 \). Moreover, by equations (9) and (10) we have

\[
(DH - EG)f_1 + (HF - EI)f_3 = 0
\]

Similarly we have

\[
\begin{align*}
(AH - BG)f_1 + (CH - BI)f_3 &= 0 \\
(BD - AE)f_2 + (CD - AF)f_3 &= 0 \\
(EG - DH)f_2 + (FG - DI)f_3 &= 0 \\
(BG - AH)f_2 + (CG - AI)f_3 &= 0 \\
(AF - CD)f_1 + (BF - CE)f_2 &= 0 \\
(DI - FG)f_1 + (EI - FH)f_2 &= 0 \\
(AI - CG)f_1 + (BI - CH)f_2 &= 0.
\end{align*}
\]

We obtain

\[
\begin{align*}
f_1 &= (CE - BF)u = (HF - EI)v = (CH - BI)w, \\
f_2 &= (CD - AF)u = (DI - FG)v = (AI - CG)w, \\
f_3 &= (AE - BD)u = (DH - EG)v = (AH - BG)w;
\end{align*}
\]

for some units \( u, v, \) and \( w \).

Since we supposed that \( r = 4 \), there exists \( \gamma \in \mathbb{P}_k^3 \) such that \( (HF - EI)(\gamma) = 0, \ (DI - FG)(\gamma) = 0 \) and \( (DH - EG)(\gamma) = 0 \). This implies that \( f_1(\gamma) = f_2(\gamma) = f_3(\gamma) = 0 \). By the argument above, \( p_{58}(\gamma) = p_{68}(\gamma) = p_{78}(\gamma) = 0 \). Then we have
Proposition 10. If \( l \) are zero. Hence \( \text{rank}(T(\gamma)) \) is at most 1, so the total rank of \( D \) at \( \gamma \) cannot be 4, which is a contradiction.

Note that Proposition 7 can be also proved by Theorem 2. More precisely, the type of \( D \) is \((1, 3, 3, 1)\), which means \( l = 3 \). Then by Theorem 2 we have \( r \leq 3 \).

Proposition 8. If \( \sigma \leq (12)(38)(47)(56) \) or \( \sigma \leq (16)(25)(34)(78) \), then \( r < 3 \).

These cases are symmetric, so it is enough to prove the case \( \sigma \leq (12)(38)(47)(56) \).

Proof. Suppose to the contrary that \( \sigma \leq (12)(38)(47)(56) \) and \( r \geq 3 \). By Remark 1 it is enough to consider \( r = 3 \).

The matrix \( D \) is of type \((1, 4, 3)\), and by Remark 2 there exists a free flag with \( l = 2 \). However, Theorem 2 implies that \( l \geq r \), so \( l \geq 3 \), which is a contradiction.

Proposition 9. If \( \sigma \leq (14)(23)(58)(67) \), then \( r < 3 \).

Proof. Suppose to the contrary that \( r \geq 3 \). By Remark 1 we may take \( r = 3 \) and consider

The matrix \( D \) is of type \((2, 4, 2)\), and by Remark 2 there exists a free flag with \( l = 2 \). However, Theorem 2 implies that \( l \geq 3 \), which is a contradiction.

Proposition 10. If \( \sigma \leq (18)(23)(47)(56) \) or \( \sigma \leq (18)(25)(34)(67) \), then \( r < 2 \).

Note that these cases are symmetric, so it is enough to prove the case \( \sigma \leq (18)(23)(47)(56) \).
Proof. Suppose not. By Remark 1, we may take \( r = 2 \).

\[
D = \begin{bmatrix}
0 & 0 & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\
0 & 0 & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & 0 & 0 & p_{36} & p_{37} & p_{38} \\
0 & 0 & 0 & 0 & 0 & p_{46} & p_{47} & p_{48} \\
0 & 0 & 0 & 0 & 0 & p_{56} & p_{57} & p_{58} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Define

\[
L := \begin{bmatrix}
p_{13} & p_{14} & p_{15} \\
p_{23} & p_{24} & p_{25}
\end{bmatrix} = \begin{bmatrix}
f_1 & f_2 & f_3 \\
g_1 & g_2 & g_3
\end{bmatrix}
\]

and

\[
M := \begin{bmatrix}
p_{36} & p_{37} & p_{38} \\
p_{46} & p_{47} & p_{48} \\
p_{56} & p_{57} & p_{58}
\end{bmatrix} = \begin{bmatrix}
A & D & G \\
B & E & H \\
C & F & I
\end{bmatrix}.
\]

Let \( Q \) denote the quotient field of \( S \). First observe that \( \text{rank}_Q(M) \neq 3 \). Otherwise, \( \det(M) \neq 0 \) in \( Q \) that would imply the rows of \( M \), say \( r_1, r_2, r_3 \) are linearly independent. Since \( D^2 = 0 \), we have

\[
\begin{align*}
f_1A + f_2B + f_3C &= 0, \\
f_1D + f_2E + f_3F &= 0, \\
f_1G + f_2H + f_3I &= 0.
\end{align*}
\]

In other words, \( f_1r_1 + f_2r_2 + f_3r_3 = 0 \). But the linearly independence of the rows of \( M \) implies that \( f_1 = f_2 = f_3 = 0 \) in \( Q \). Similarly \( g_1, g_2, \) and \( g_3 \) would vanish in \( Q \). Then \( \text{rank}_Q(D) \leq 3 \), which is a contradiction. Hence \( \det(M) = 0 \) in \( Q \).

Note that \( L.M = 0 \) and the composite

\[
Q^3 \xrightarrow{M} Q^3 \xrightarrow{L} Q^2
\]

is identically zero. Then \( \text{im}(M) \subseteq \ker(L) \), which means

\[
\dim_k \ker(L) \geq \dim_k \text{im}(M) = \text{rank}(M).
\]

Moreover, \( 3 = \dim_k \ker(L) + \text{rank}(L) \) and since \( \text{rank}(M) = 2 \), we have \( \text{rank}(L) \leq 1 \). Note that \( \text{rank}(L) \) cannot be zero, otherwise \( \text{rank}(D) \leq 3 \), that would be a contradiction. Hence \( \text{rank}(L) = 1 \). We can therefore write \( D \) as follows:

\[
D = \begin{bmatrix}
0 & 0 & \alpha f_1 & \alpha f_2 & \alpha f_3 & p_{16} & p_{17} & p_{18} \\
0 & 0 & \beta f_1 & \beta f_2 & \beta f_3 & p_{26} & p_{27} & p_{28} \\
0 & 0 & 0 & 0 & 0 & A & D & G \\
0 & 0 & 0 & 0 & 0 & B & E & H \\
0 & 0 & 0 & 0 & 0 & C & F & I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

WLOG suppose that the first and second rows of \( M \) are linearly dependent in the quotient field \( Q \). Assume that

\[
\begin{bmatrix}
A & D & G
\end{bmatrix} = p/q \begin{bmatrix}
B & E & H
\end{bmatrix},
\]

where \( p \) and \( q \) are rational
polynomials in $Q$. Then there exists a non-zero root $\gamma$ of the determinant of the following matrix:

$$T := \begin{bmatrix} \alpha & p_{16} & p_{17} & p_{18} \\ \beta & p_{26} & p_{27} & p_{28} \\ 0 & A & D & G \\ 0 & C & F & I \end{bmatrix}. $$

If there is no $\gamma$ such that $\det T(\gamma) = 0$ and $\{ [A \ D \ G] (\gamma), [B \ E \ H] (\gamma) \}$ is linearly independent, then we are done by the rank argument. Hence we need to show that if $\det T(\gamma) = 0$, then we cannot have $p(\gamma) = 0$ and $q(\gamma) = 0$. Suppose to the contrary $p(\gamma) = q(\gamma) = 0$. Then we have the following relations for the varieties:

$$V(\det T) \subseteq V(p) \quad \text{and} \quad V(\det T) \subseteq V(q)$$

Then by Hilbert’s Nullstellensatz, $p^m$ and $q^m$ are in the ideal generated by the polynomial $\det T$ for some $m$. This implies $\det T$ is a unit, which is a contradiction. \hfill \Box

With Propositions 1 through 10 we have verified Conjecture 3 for all maximal elements of $\text{RP}(8)$, and thus we have proved the conjecture when $N = 8$.

5. Appendix

The aim of this section is to show that every element in $\text{RP}(8)$ can be obtained from a unique maximal element by a sequence of moves of type III. When $N = 8$, the total number of elements of $\text{RP}(8)$ is 105 and 14 of them are maximal. The Hasse diagram of $\text{RP}(8)$ with representative permutations was computed using GAP 4.8.3, and is as follows:
Figure 1. The Hasse diagram of $\mathbb{RP}(8)$
Here we have abbreviated the entries according to the following table:

| Level | Involutions |
|-------|-------------|
| 1     | $1 = (12)(34)(56)(78), 2 = (12)(34)(58)(67), 3 = (12)(36)(45)(78), 4 = (12)(38)(45)(67), 5 = (12)(38)(47)(56), 6 = (14)(23)(56)(78), 7 = (14)(23)(58)(67), 8 = (16)(23)(45)(78), 9 = (16)(25)(34)(78), 10 = (18)(23)(45)(67), 11 = (18)(23)(47)(56), 12 = (18)(25)(34)(67), 13 = (18)(27)(34)(56), 14 = (18)(27)(36)(45)$ |
| 2     | $1 = (12)(34)(57)(68), 2 = (12)(37)(45)(68), 3 = (12)(35)(46)(78), 4 = (12)(38)(45)(67), 5 = (12)(37)(46)(58), 6 = (14)(23)(46)(58), 7 = (14)(23)(47)(56), 8 = (15)(23)(46)(78), 9 = (15)(24)(36)(78), 10 = (16)(24)(35)(78), 11 = (16)(25)(34)(78), 12 = (15)(26)(34)(78), 13 = (16)(27)(34)(56), 22 = (18)(24)(36)(57), 26 = (18)(24)(36)(57), 27 = (18)(24)(37)(56), 28 = (18)(27)(35)(46)$ |
| 3     | $1 = (12)(35)(47)(68), 2 = (12)(37)(46)(58), 3 = (12)(36)(48)(57), 4 = (12)(34)(48)(67), 5 = (13)(25)(46)(78), 6 = (15)(24)(36)(57), 7 = (14)(23)(45)(68), 8 = (15)(23)(46)(78), 9 = (17)(23)(46)(58), 10 = (16)(23)(48)(57), 11 = (13)(27)(45)(68), 12 = (13)(25)(48)(67), 13 = (13)(27)(48)(56), 14 = (13)(28)(46)(57), 15 = (15)(27)(34)(68), 16 = (17)(26)(34)(58), 17 = (16)(28)(34)(57), 18 = (17)(24)(35)(68), 19 = (17)(28)(47)(56), 20 = (15)(28)(34)(67), 21 = (17)(28)(34)(56), 22 = (18)(23)(46)(57), 23 = (18)(23)(56)(78), 24 = (18)(23)(46)(57), 25 = (18)(24)(35)(67), 26 = (18)(24)(37)(56), 27 = (18)(24)(37)(56), 28 = (18)(27)(35)(46)$ |
| 4     | $1 = (12)(36)(47)(58), 2 = (14)(25)(36)(78), 3 = (16)(23)(47)(58), 4 = (13)(24)(57)(68), 5 = (13)(27)(46)(58), 6 = (15)(26)(34)(57), 7 = (16)(27)(34)(58), 8 = (15)(24)(37)(58), 9 = (17)(24)(36)(58), 10 = (16)(24)(38)(57), 11 = (14)(27)(35)(68), 12 = (14)(25)(38)(67), 13 = (14)(27)(38)(56), 14 = (14)(28)(36)(57), 15 = (16)(27)(38)(45), 16 = (17)(26)(35)(48), 17 = (16)(28)(35)(47), 18 = (17)(25)(38)(46), 19 = (15)(28)(37)(46), 20 = (18)(25)(36)(47)$ |
| 5     | $1 = (13)(26)(47)(58), 2 = (16)(24)(37)(58), 3 = (14)(25)(37)(68), 4 = (14)(27)(36)(58), 5 = (14)(26)(38)(57), 6 = (16)(27)(35)(48), 7 = (17)(25)(36)(48), 8 = (16)(25)(38)(47), 9 = (15)(27)(38)(46), 10 = (15)(28)(36)(47)$ |
| 6     | $1 = (14)(26)(37)(58), 2 = (16)(25)(37)(48), 3 = (15)(27)(36)(48), 4 = (15)(26)(38)(47)$ |
| 7     | $1 = (15)(26)(37)(48)$ |

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