Infinite Mode Quantum Gaussian States

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Abstract

Quantum Gaussian states on Bosonic Fock spaces are quantum versions of Gaussian distributions. Here infinite mode quantum Gaussian states have been explored. They are type I quasi-free states. We extend many of the results of Parthasarathy [Par10] and [Par13] to the infinite mode case. This include various characterizations, convexity and symmetry properties.

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1 Introduction

Finite mode quantum Gaussian states was initiated back in early 1970’s (for example [Hol75]). It is well studied both theoretically and experimentally ([KLM02]) in the literature. Recently finite mode Gaussian states have been getting more attention in the context of its importance in quantum information theory. Some references are [FOP05], [WGC06], [WHTH07], [ARL14] etc. We define, characterize and study properties of an infinite mode analog of quantum Gaussian states described in [Par10]. A systematic study of the quantum Gaussian states in the infinite mode setting is initiated in this work. This naturally leads to quasifree states on the CCR–algebra and Hilbert-Schmidt, Trace class restrictions on the covariance operators. Nevertheless it is observed that the symmetry properties of the finite mode Gaussian states ([Par13]) are preserved in this general setting also. The scheme is as follows.

In Section 2, we collect some terminology and known results. For notation and basics on Boson Fock spaces we follow mostly Parthasarathy [Par12]. We need some refinements regarding results existed on Shale Operators ([Sha62, Par12, BS05]), and this has been carried out in Section § 2.3.1. We crucially make use of the
notion of quasi-free states on CCR Algebra and their classification theory. For this we are depending on Petz [Pet90], Holevo ([Hol71a], [Hol71b]) and van Daele [vD71]. Finally we need suitable extension of Williamson’s normal form to infinite dimensions ([BJ18]). We quote the result we need in Section 2.4.

Section 3, has the formal definition of quantum Gaussian states in infinite dimensions, by first introducing quantum characteristic function or quantum Fourier transform of states on Boson Fock space. Every quantum Gaussian state comes with a ‘covariance matrix’, which now is a symmetric and invertible real linear operator. We begin with deriving some necessary conditions this operator has to satisfy (Theorem 3.12) and connect them with existence of a quasifree state in Theorem 3.17. Some further analysis finally leads to a complete characterization of covariance operators of quantum Gaussian states in infinite mode (Theorem 3.28).

Now we have the proper setting to extend the results of K R Parthasarathy [Par13] to infinite mode Gaussian states. We characterize extreme points of quantum Gaussian state covariance operators and express every interior point as midpoint of two extreme points (Theorem 4.5). Theorem 5.1 provides explicit description for every quantum Gaussian state, in terms of symplectic eigenvalues of the covariance operator. We see that every mixed Gaussian state can be purified to a pure Gaussian state.

A unitary operator in the Boson Fock space is called a Gaussian symmetry if it preserves Gaussianity of states under conjugation. The last result in this article (Theorem 6.5) is a complete characterization of Gaussian unitaries.

2 Preliminaries

§ 2.1 Symmetric Fock space \(\Gamma_s(\mathcal{H})\)

Let \(\mathcal{H}\) be a complex Hilbert space with inner product \(\langle \cdot, \cdot \rangle\), which is anti-linear in the first variable. For \(n \in \mathbb{N}\), let \(S_n\) denote the group of all permutations of the set \(\{1, 2, \ldots, n\}\). Thus any \(\sigma \in S_n\) is a one-to-one map of \(\{1, 2, \ldots, n\}\) onto itself. For each \(\sigma \in S_n\), let \(U_\sigma\) be defined on the product vectors in \(\mathcal{H}^\otimes n\) by

\[U_\sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)}\]

where \(\sigma^{-1}\) is the inverse of \(\sigma\). Then \(U_\sigma\) is a scalar product preserving map of the total set of product vectors in \(\mathcal{H}^\otimes n\) onto itself. Hence \(U_\sigma\) extends uniquely to a unitary operator on \(\mathcal{H}^\otimes n\), which we shall denote by \(U_\sigma\) itself. Clearly \(\sigma \mapsto U_\sigma\) is a unitary representation of the group \(S_n\). The closed subspace of fixed points,

\[\mathcal{H}^{\otimes n}_{\text{fix}} = \{f \in \mathcal{H}^\otimes n | U_\sigma f = f, \forall \sigma \in S_n\}\]

of \(\mathcal{H}^\otimes n\) is called the \(n\)-fold symmetric tensor product of \(\mathcal{H}\). The symmetric Fock space (also known as Boson Fock space) over \(\mathcal{H}\) is defined as

\[\Gamma_s(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}\]

where we take \(\mathcal{H}^{\otimes 0} := \mathbb{C}\). The \(n\)-th direct summand is called the \(n\)-particle subspace. Any element in the \(n\)-particle subspace is called an \(n\)-particle vector. When \(n = 0\)
we call it as the vacuum space. The vector \( \Phi := 1 \oplus 0 \oplus 0 \oplus \cdots \) is called the vacuum vector. We denote by \( \Gamma^0(\mathcal{H}) \) the dense linear subspace generated by all \( n \)-particle vectors, \( n = 0, 1, 2, \ldots \) and we call them as finite particle spaces. For \( f \in \mathcal{H} \), define the exponential vector

\[
e(f) = 1 \oplus f \oplus \frac{f \otimes f}{\sqrt{2!}} \oplus \cdots \oplus \frac{f \otimes f \cdots f}{\sqrt{n!}} \otimes \cdots
\]

then \( e(f) \in \Gamma_s(\mathcal{H}) \). Notice that

\[
\langle e(f), e(g) \rangle = \exp \langle f, g \rangle
\]

for all \( f, g \in \mathcal{H} \). The set \( E := \{e(f) | f \in \mathcal{H} \} \) of all exponential vectors is linearly independent and total in \( \Gamma_s(\mathcal{H}) \). Further if \( A \) is a dense set in \( \mathcal{H} \) then the linear span of the set \( \{e(f) | f \in A \} \) is dense in \( \Gamma_s(\mathcal{H}) \).

\section{Basic operators in quantum theory}

For any fixed \( f \in \mathcal{H} \) consider the map defined on the set of exponential vectors \( E = \{e(g) : g \in \mathcal{H} \} \), by \( e(g) \mapsto \{\exp(-\frac{1}{2\|f\|^2 - \langle f, g \rangle})e(f + g)\} \). This yields an inner product preserving map of \( E \) onto itself. As \( E \) is total, there exists a unique unitary operator \( W(f) \in \mathcal{B}(\Gamma_s(\mathcal{H})) \) satisfying

\[
W(f)e(g) = \{\exp\left(-\frac{1}{2}\|f\|^2 - \langle f, g \rangle\right)e(f + g)
\]

\( W(f) \) is called the Weyl operator associated with \( f \in \mathcal{H} \).

\begin{proposition}

The mapping \( f \mapsto W(f) \) from \( \mathcal{H} \) into \( \mathcal{B}(\Gamma_s(\mathcal{H})) \) is strongly continuous. Further,

\[
W(-f) = W(f)^*, \forall f \in \mathcal{H}
\]

\[
W(f)W(g) = \exp(-i\text{Im} \langle f, g \rangle)W(f + g).
\]

By Proposition 2.1 every \( f \in \mathcal{H} \) yields a strongly continuous one parameter unitary group \( \{W(tf) | t \in \mathbb{R} \} \). Let us denote by \( p(f) \), the observable obtained as the Stone generator of this group. Then

\[
W(tf) = e^{-itp(f)}, t \in \mathbb{R}, f \in \mathcal{H}.
\]

Recall the fact that the exponential domain \( \mathcal{E} \) (which is the dense subspace spanned by exponential vectors in \( \Gamma_s(\mathcal{H}) \)) is a core for \( p(f) \) for all \( f \in \mathcal{H} \). The space of all finite particle vectors, \( \Gamma^0_s(\mathcal{H}) \) is also a core for \( p(f) \) for all \( f \). Let us fix a basis \( \{e_j\} \) for \( \mathcal{H} \) and let

\[
p_j = 2^{-1/2}p(e_j), \quad q_j = -2^{-1/2}p(ie_j)
\]

\[
a_j = 2^{-1/2}(q_j + ip_j), \quad a_j^\dagger = 2^{-1/2}(q_j - ip_j)
\]

for each \( j \in \mathbb{N} \). Then we have the Lie brackets

\[
[\cdot, \cdot]_r = i\delta_{rs}I, \quad [a_r, a_s^\dagger] = \delta_{rs}
\]

\( \forall r, s \in \mathbb{N} \). Further \( \{a_r, r \in \mathbb{N} \} \) and \( \{a_r^\dagger, r \in \mathbb{N} \} \) commute among themselves. We call \( p_j \) and \( q_j \) as the \( j \)-th momentum and position operator, \( a_j \) and \( a_j^\dagger \) as the \( j \)-th annihilation and creation operator for all \( j \in \mathbb{N} \). We refer to Section 20 of [Par12] for more details on these operators.
2.2 Proposition. Let \( z \in \mathcal{H} \) be such that \( z = \sum_{j=1}^{n} \alpha_j e_j \), where \( \alpha_j = x_j + iy_j, x_j, y_j \in \mathbb{R}, \forall j \), then

\[
W(z) = e^{-i\sqrt{2} \sum_{j=1}^{n} (x_j p_j - y_j q_j)}; \quad p(z) = \sqrt{2} \sum_{j=1}^{n} (x_j p_j - y_j q_j).
\]

2.3 Definition. If \( T \) is any contraction on \( \mathcal{H} \) then define \( \Gamma_s(T) \) on \( \Gamma_s(\mathcal{H}) \) by

\[
\Gamma_s(T)(e(f)) = e(Tf) \quad (2.11)
\]

These are called the second quantization homomorphisms.

Note that if \( U \) is a unitary then \( \Gamma_s(U) \) is a unitary. Further we have

\[
\Gamma_s(U)^{-1} = \Gamma_s(U^{-1}) \quad (2.12)
\]

\[
\Gamma_s(U)W(u)\Gamma_s(U)^{-1} = W(Uu) \quad (2.13)
\]

Also, it is possible to define \( \Gamma_s(U) \) via (2.11) even if \( U \) is unitary mapping \( \mathcal{H} \) to a different Hilbert space \( \mathcal{K} \), where the exponential vector on the left is in \( \Gamma_s(\mathcal{H}) \) and that on the right is in \( \Gamma_s(\mathcal{K}) \).

§ 2.1.2 Exponential Property of \( \Gamma_s(\mathcal{H}) \)

2.4 Proposition (Section 20, [Par12]). If \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), then there is a unique unitary isomorphism between \( \Gamma_s(\mathcal{H}) \) and \( \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2) \) satisfying \( e(f \oplus g) \leftrightarrow e(f) \otimes e(g) \). Further, under this isomorphism we have \( W(f \oplus g) = W(f) \otimes W(g) \).

Recall the countable tensor product of Hilbert spaces (Exercise 15.10 in [Par12]). We summarize some properties of infinite tensor product of Fock spaces in the following proposition.

2.5 Proposition. Let \( \mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n \), where \( \mathcal{H}_n, n = 1, 2, 3 \ldots \) is a sequence of Hilbert spaces. Consider the infinite tensor product \( \bigotimes_{n=1}^{\infty} \Gamma_s(\mathcal{H}_n) \) constructed using the stabilizing sequence \( \{\Phi_n\} \), where \( \Phi_n \in \Gamma_s(\mathcal{H}_n) \) is the vacuum vector for every \( n \). Then

\[
\Gamma_s(\mathcal{H}) = \bigotimes_{n=1}^{\infty} \Gamma_s(\mathcal{H}_n) \quad (2.14)
\]

under the natural isomorphism. In this identification, for \( \bigoplus_{n=1}^{\infty} x_n \in \mathcal{H} \), and contractions \( A_n \in \mathcal{B}(\mathcal{H}_n), n \geq 1 \),

\[
e(\bigoplus_{n=1}^{\infty} x_n) = \bigotimes_{n=1}^{\infty} e(x_n) := \lim_{N \to \infty} \bigotimes_{j=1}^{N} e(x_j) \otimes e(0) \otimes e(0) \otimes \cdots \quad (2.15)
\]

\[
W(\bigoplus_{n=1}^{\infty} x_n) = \bigotimes_{n=1}^{\infty} W(x_n) := \lim_{N \to \infty} \bigotimes_{j=1}^{N} W(x_j) \otimes I \otimes I \otimes \cdots \quad (2.16)
\]

\[
\Gamma_s(\bigoplus_{n=1}^{\infty} A_n) = \bigotimes_{n=1}^{\infty} \Gamma_s(A_n) := \lim_{N \to \infty} \bigotimes_{j=1}^{N} \Gamma_s(A_j) \otimes I \otimes I \otimes \cdots \quad (2.17)
\]

§ 2.2 CCR Algebra

In this Section we list some basic facts about symplectic spaces and quasifree states of CCR algebras. For more on these notions see ([Pet90], [Hol71a], [Hol71b], [Hol75], [vD71]).
§ 2.2.1 Symplectic Space

2.6 Definition. Let $H$ be a real linear space. A bilinear form $\sigma : H \times H \to \mathbb{R}$ is called a symplectic form if $\sigma(f, g) = -\sigma(g, f)$, for every $f, g \in H$. The pair $(H, \sigma)$ is called a symplectic space. A symplectic form $\sigma$ on $H$ is called nondegenerate if $\sigma(f, g) = 0, \forall g \in H$ implies $f = 0$. A symplectic space $(H, \sigma)$ is called a standard (symplectic) space if $H$ is a Hilbert space over $\mathbb{C}$ with respect to some inner product $\langle \cdot, \cdot \rangle$ and $\sigma(\cdot, \cdot) = \text{Im} \langle \cdot, \cdot \rangle$. It is called separable if there exists in $H$ a countable family of vectors $\{f_k\}$ such that $\sigma(f, f_k) = 0$ for all $k$ implies $f = 0$. Note that standard symplectic spaces are separable.

2.7 Definition. Let $(H, \sigma)$ be a symplectic space. The $C^*$-algebra of the canonical commutation relation over $(H, \sigma)$, written as $CCR(H, \sigma)$, is by definition a $C^*$-algebra generated by elements $\{W(f) : f \in H\}$ such that

\begin{align*}
W(-f) &= W(f)^* \quad \forall f \in H \quad (2.18) \\
W(f)W(g) &= \exp(i\sigma(f, g))W(f + g) \quad (2.19)
\end{align*}

2.8 Theorem. [Pet90] For any nondegenerate symplectic space $(H, \sigma)$, the $C^*$-algebra $CCR(H, \sigma)$ exists and unique up to isomorphism. Further, the linear hull of $\{W(f) : f \in H\}$ is dense in $CCR(H, \sigma)$.

2.9 Definition. A representation $\Pi$ of $CCR(H, \sigma)$ in $\mathcal{B}(H)$ is called regular if the mapping

\[ t \mapsto \langle \Pi(W(tf))\zeta, \eta \rangle, \]

is continuous for all $\zeta, \eta \in H$, for every $f \in H$.

All the representations of $CCR(H, \sigma)$ considered in this note will be regular.

§ 2.2.2 Quasifree States on $CCR(H, \sigma)$

2.10 Definition. A linear functional $\phi$ on a unital $C^*$-algebra $A$ is a called state if $\phi(x^*x) \geq 0$ for every $x \in A$, and $\phi(I) = 1$, where $I$ is the identity in $A$.

Let $A$ be a $C^*$-algebra and $\phi$ be a state on it, then by the GNS construction there exists a unique cyclic representation of $A$. We denote the corresponding GNS triple by $(H_\phi, \Pi_\phi, \Omega_\phi)$, where $H_\phi$ is a Hilbert space, $\Pi_\phi$ is the representation and $\Omega_\phi$ is the cyclic vector.

2.11 Definition. A state $\phi$ on $A$ is called primary if the von Neumann algebra $(\Pi_\phi(A))''$ corresponding to the GNS-representation is a factor. It is called type I if the von Neumann algebra $(\Pi_\phi(A))''$ corresponding to the GNS-representation is a Type 1 factor.

2.12 Definition. Two states $\phi$ and $\psi$ on $A$ are called quasi-equivalent if $(\Pi_\phi(A))''$ and $(\Pi_\psi(A))''$ are isomorphic von Neumann algebras.

2.13 Theorem. [Pet90] Let $(H, \sigma)$ be a symplectic space and $\alpha : H \times H \to \mathbb{R}$ be a real inner product on $H$. Then there exists a state $\phi$ on $CCR(H, \sigma)$ such that

\[ \phi(W(f)) = \exp \left( -\frac{1}{2} \alpha(f, f) \right) \quad \forall f \in H. \quad (2.20) \]
if and only if
\[ \sigma(f, g)^2 \leq \alpha(f, f)\alpha(g, g), \quad \forall f, g \in H. \] (2.21)

2.14 Definition. A state \( \phi \) on \( CCR(H, \sigma) \) determined in the form of \( (2.20) \) is called a quasifree state. A \( CCR \)-algebra corresponding to a standard symplectic space \( (H, \sigma) \) will be called a standard \( C^* \)-algebra of the \( CCR \) or standard \( CCR(H, \sigma) \).

Notation. If \( A \) is real linear operator on a real Hilbert space \( H \) we use the notation \( A^\tau \) to denote the transpose of the operator defined by the equation \( \langle x, Ay \rangle = \langle A^\tau x, y \rangle \) for all \( x, y \in H \).

2.15 Proposition (Proposition 1 and 2 from [Hol71a]). A quasifree state of a standard \( CCR(H, \sigma) \), (Take \( \sigma(\cdot, \cdot) = -\operatorname{Im} \langle \cdot, \cdot \rangle \) here) is primary (c.f 2.11) if and only if \( \alpha \) in \( (2.20) \) satisfies one of the following equivalent conditions.

1. The space \( H \) is complete with respect to the norm coming from \( \alpha \). In other words, \( (H, \alpha(\cdot, \cdot)) \) is a real Hilbert space.

2. There exists a bounded, invertible real linear operator \( A \) on \( (H, \alpha(\cdot, \cdot)) \) such that \( \alpha(f, g) = \sigma(Af, g) \), \( \forall f, g \in H \).

Further in this case, \( A^\tau = -A \); \( -A^2 - I \geq 0 \) (2.23) on \( (H, \alpha(\cdot, \cdot)) \).

Since \( (H, \sigma) \) is standard \( (2.22) \) can also be written as
\[ \alpha(f, g) = -\operatorname{Im} \langle Af, g \rangle, \quad \forall f, g \in H. \]

Sometimes we write \( \phi_A \) to denote the primary quasifree state obtained by \( (2.22) \) on a standard \( CCR(H, \sigma) \), also we write \( H_A \) to denote the real Hilbert space \( (H, \alpha(\cdot, \cdot)) \) in this case.

2.16 Theorem (Theorem in [Hol71a]). Two primary quasifree states \( \phi_A \) and \( \phi_B \) on a standard \( CCR(H, \sigma) \) are quasi equivalent if and only if \( A - B \) and \( \sqrt{-A^2 - I} - \sqrt{-B^2 - I} \) are Hilbert-Schmidt operators on \( H_A \).

§ 2.2.3 Representation of standard CCR algebra in Fock Space

Let \( \mathcal{H} \) be a complex Hilbert space with inner product \( \langle \cdot, \cdot \rangle \). Let \( \bar{\mathcal{H}} \) denote \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \). Take \( \sigma(\cdot, \cdot) = -\operatorname{Im} \langle \cdot, \cdot \rangle \), Since \( \bar{\mathcal{H}} \) is a Hilbert space, \( (\mathcal{H}, \sigma) \) is the standard symplectic space associated with \( \bar{\mathcal{H}} \). Consider the symmetric Fock space \( \Gamma_s(\mathcal{H}) \) associated with \( \mathcal{H} \), then Proposition 2.1 provides a regular representation of \( CCR(\mathcal{H}, \sigma) \) in \( \Gamma_s(\mathcal{H}) \). It should be noted that this representation is also the GNS representation corresponding to the quasifree state \( \phi_i \), where \( i \) denote the operator of scalar multiplication by the complex number \( i \) considered as a real linear operator on \( \mathcal{H} \). We may call \( \phi_i \) as vacuum state. The name vacuum state will have a precise meaning when we consider the "quantum characteristic function" (see Definition 3.1) of the vacuum state \( |e(0)\rangle \langle e(0)| \) on \( \Gamma_s(\mathcal{H}) \).
§ 2.3  Symplectic automorphisms and transformations

Our basic set up is as in § 2.2.3. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Often we consider $\mathcal{H}$ as real Hilbert space with $\langle \cdot, \cdot \rangle_R = \text{Re} \langle \cdot, \cdot \rangle$. Let $H \subset \mathcal{H}$ be a real subspace such that $H = \{ x + iy | x, y \in H \} = \mathcal{H} + iH$ (i.e., $H$ is the complexification of the real Hilbert Space $(H, \langle \cdot, \cdot \rangle_R)$, where $\langle \cdot, \cdot \rangle_R := \text{Re} \langle \cdot, \cdot \rangle$).

Now consider $H$ as real Hilbert space with the inner product $\text{Re} \langle \cdot, \cdot \rangle$. Then
\[
\text{Re} \langle x + iy, u + iv \rangle = \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_R
\]
(2.24)

where $\langle \cdot, \cdot \rangle_R$ on right is the canonical inner product on $H \oplus H$ inherited from $H$. Thus the real Hilbert space $H$ is isomorphic to $H \oplus H$ via the map $U$ which takes $x + iy \mapsto \left( \begin{smallmatrix} x \\ y \end{smallmatrix} \right)$.

For any real linear operator $S$ on $\mathcal{H}$, define operators $S_{ij}$ on $H$ such that
\[
S(x + iy) = S_{11}x + iS_{21}x + S_{12}y + iS_{22}y.
\]
(2.25)

Define the operator $S_0$ on $H \oplus H$ by
\[
S_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]
(2.26)

Then (2.24) and (2.25) implies
\[
\text{Re} \langle S(x + iy), S(u + iv) \rangle = \left\langle \begin{pmatrix} S_{11}x + S_{12}y \\ S_{21}x + S_{22}y \end{pmatrix}, \begin{pmatrix} S_{11}u + S_{12}v \\ S_{21}u + S_{22}v \end{pmatrix} \right\rangle_R = \left\langle S_0 \begin{pmatrix} x \\ y \end{pmatrix}, S_0 \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_R.
\]
(2.27)

Thus
\[
S = U^t S_0 U.
\]
(2.28)

Therefore, we identify $S$ with $S_0$ as a real linear operator and often switch between them freely. We also note here that if $S$ is a complex linear operator then $S_{11} = S_{22}(= S_1, \text{say})$ and $S_{12} = -S_{21}(= S_2, \text{say})$, then we can write $S_0 = \begin{bmatrix} -S_2 & S_1 \\ S_1 & S_2 \end{bmatrix}$.

If $\mathcal{K}$ is another Hilbert space and $S : \mathcal{H} \to \mathcal{K}$ is real linear, then also the same analysis hold. When we talk about a real linear operator $S$ on $\mathcal{H}$, we reserve the notation $S_0$ to mean the operator we constructed as above.

Let $J$ be the operator of multiplication by $-i$ on $\mathcal{H}$ considered as a real linear map then
\[
J_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.
\]
We have $J_0^* = J_0^{-1} = -J_0$, (same is true for $J$ also) and thus $J_0$ (and $J$) are orthogonal transformations.

A real linear bijective map $L : \mathcal{H} \to \mathcal{H}$ is said to be a symplectic automorphism if it satisfies (i) $L$ and $L^{-1}$ are continuous (bounded) (ii) $\text{Im} \langle Lz, Lw \rangle = \text{Im} \langle z, w \rangle$. 


for all \( z, w \in \mathcal{H} \). If \( L \) from \( \mathcal{H} \) to \( \mathcal{K} \) satisfies the same conditions then we say \( L \) is a symplectic transformation. Correspondingly \( L_0 \) will also be called as symplectic automorphism (or transformation).

2.17 Proposition (Section 22 in [Par12]). \( L : \mathcal{H} \to \mathcal{K} \) is symplectic if and only if

\[
L_0^* J_0 L_0 = J_0,
\]

where \( J_0 \) on left side is the involution operator on \( K \oplus K \) and that on the right side is the involution operator on \( H \oplus H \).

2.18 Example. Let \( A \in \mathcal{B}(H) \) be any invertible operator on \( H \) then the operator \( T \) defined on \( \mathcal{H} \) by \( T(u + iv) = Au + iA^{-1}v \) is a symplectic automorphism of \( \mathcal{H} \). Further note that

\[
T_0 = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}.
\]

We will have occasions to deal with the complexification of Hilbert Spaces. We saw above that \( \mathcal{H} \) has a canonical isomorphism to \( H \oplus H \) as a real Hilbert space. Let \( \mathcal{H} \) denote its complexification. If \( A \) is a real linear operator on \( \mathcal{H} \) then let \( \hat{A} \) denote the complexification of \( A \) defined by \( \hat{A}(z + iw) = Az + iAw \). The following holds.

2.19 Proposition. \( \hat{\mathcal{H}} \) is canonically isomorphic to \( \mathcal{H} \oplus \mathcal{H} \) as a complex Hilbert space. Further, if a real linear operator \( S \) on \( \mathcal{H} \) corresponds to \( S_0 = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \) on \( H \oplus H \), then under this isomorphism \( \hat{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \) on \( \hat{\mathcal{H}} \).

Proof. As an element of the real Hilbert space \( \mathcal{H} \), the vector \( x + iy \) is identified with \( \begin{bmatrix} x \\ y \end{bmatrix} \). Consider the mapping \( (\begin{bmatrix} x \\ y \end{bmatrix}) + i(\begin{bmatrix} u \\ v \end{bmatrix}) \mapsto (\begin{bmatrix} x + iu \\ y + iv \end{bmatrix}) \) from \( \hat{\mathcal{H}} \) to \( \mathcal{H} \oplus \mathcal{H} \). Let us denote the inner product in \( \mathcal{H} \) by \( \langle \cdot, \cdot \rangle \) and that in \( \mathcal{H} \oplus \mathcal{H} \) by \( \langle \cdot, \cdot \rangle \). Then

\[
\langle \begin{bmatrix} x_1 + iu_1 \\ y_1 + iv_1 \end{bmatrix}, \begin{bmatrix} x_2 + iu_2 \\ y_2 + iv_2 \end{bmatrix} \rangle = \langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \rangle + \langle \begin{bmatrix} iu_1 \\ iv_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \rangle + \langle \begin{bmatrix} iu_2 \\ iv_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \rangle = \langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \rangle + i \langle \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \rangle + i \langle \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \rangle \%
\]

Therefore, the isomorphism is proved. Now we proceed to prove the second statement. We know that \( S \) and \( S_0 \) are identified.

\[
\hat{S}_0 \left( \begin{bmatrix} x \\ y \end{bmatrix} + i \begin{bmatrix} u \\ v \end{bmatrix} \right) = S_0 \left( \begin{bmatrix} x \\ y \end{bmatrix} + i \begin{bmatrix} u \\ v \end{bmatrix} \right) = \begin{bmatrix} S_{11}x + S_{12}y + i(S_{11}u + S_{12}v) \\ S_{21}x + S_{22}y + i(S_{21}u + S_{22}v) \end{bmatrix} \]

\[
= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \hat{S}_0 \left( \begin{bmatrix} x + iu \\ y + iv \end{bmatrix} \right) \]

\[
= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} x + iu \\ y + iv \end{bmatrix}. \]

\[ \square \]
2.20 Corollary. \( \hat{j} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \) on \( \mathcal{H} \oplus \mathcal{H} \).

2.21 Proposition. [Generalization of Proposition 22.1 in [Par12]] Let \( \mathcal{H}, \mathcal{K} \) be complex Hilbert spaces and let \( S : \mathcal{H} \to \mathcal{K} \) be symplectic. Then it admits a decomposition:

\[
S = UTV
\]

where \( U : \mathcal{H} \to \mathcal{K} \) and \( V : \mathcal{H} \to \mathcal{H} \) are unitaries and \( T : \mathcal{H} \to \mathcal{H} \) has the form

\[
T(u + iv) = Au + iA^{-1}v,
\]

where \( A \in B(H) \) is a positive and invertible operator.

Proof. Apply polar decomposition to \( S \) and do the same analysis as in Proposition 22.1 in [Par12]. \( \square \)

§ 2.3.1 Shale Unitaries

Shale’s theorem was proved in [Sha62]. It was further generalized in [BS05] for the case of operators of the form \( T \) above (but between two different Hilbert spaces) in Proposition 2.21. In the work done later, we need a generalization of this (Theorem 2.1 in [BS05]) to the case of general symplectic operators. Let \( \mathcal{H}, \mathcal{K} \) be two Hilbert spaces, define \( S(\mathcal{H}, \mathcal{K}) \) by

\[
S(\mathcal{H}, \mathcal{K}) = \{ L \in B_R(\mathcal{H}, \mathcal{K}) : L \text{ is symplectic and } L^*L - I \text{ is Hilbert-Schmidt.} \}
\]

We denote \( S(\mathcal{H}) := S(\mathcal{H}, \mathcal{H}) \).

2.22 Theorem. 1. Let \( L \in S(\mathcal{H}, \mathcal{K}) \) then there exists a unique unitary operator \( \Gamma_s(L) : \Gamma_s(\mathcal{H}) \to \Gamma_s(\mathcal{K}) \) such that

\[
\Gamma_s(L)W(u)\Gamma_s(L)^* = W(Lu), \forall u \in \mathcal{H}
\]

\[
\langle \Gamma_s(L)\Phi_\mathcal{H}, \Phi_\mathcal{K} \rangle \in \mathbb{R}^+ \tag{2.31}
\]

where \( \Phi_\mathcal{H} \) and \( \Phi_\mathcal{K} \) are vacuum vectors in \( \Gamma_s(\mathcal{H}) \) and \( \Gamma_s(\mathcal{K}) \) respectively. Further,

\[
\Gamma_s(L^{-1}) = \Gamma_s(L)^* \tag{2.32}
\]

2. Let \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \) be three Hilbert spaces and \( L_1 \in S(\mathcal{H}_1, \mathcal{H}_2), L_2 \in S(\mathcal{H}_2, \mathcal{H}_3) \). Then

\[
\Gamma_s(L_2L_1) = \Gamma_s(L_2)\Gamma_s(L_1). \tag{2.33}
\]

Proof. \( \square \) We will prove the existence first. By Proposition 2.21 there exist unitaries \( U : \mathcal{K} \to \mathcal{K}, V : \mathcal{H} \to \mathcal{H} \) such that \( L = UTV \) where \( T \) is a symplectic automorphism of \( H \) such that

\[
T(u + iv) = Au + iA^{-1}v
\]

where \( A \in B(H) \) is positive and invertible. It can be seen from the proof of Proposition 2.21 that

\[
L_0 = U_0 \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} V_0
\]
for some orthogonal transformations \( U_0 \in \mathcal{B}(H,K) \) and \( V_0 \in \mathcal{B}(H) \). Now it can be seen that
\[
L_0^* L_0 = V_0^{-1} \begin{bmatrix} A^2 & 0 \\ 0 & A^{-2} \end{bmatrix} V_0.
\]
Therefore \( L_0^* L_0 - I = V_0^{-1} \begin{bmatrix} A^2 & 0 \\ 0 & A^{-2} \end{bmatrix} - I \) \( V_0 \). Hence we get that \( A^2 - I \) is Hilbert-Schmidt and since \( A \) is positive, Theorem 2.1 of [BS05] applies. Thus there exists \( \Gamma_s(T) \) such that
\[
\Gamma_s(T) W(u) \Gamma_s(T)^* = W(Tu), \forall u \in \mathcal{H}, \quad (2.34)
\]
\[
(\Gamma_s(T) \Phi_H, \Phi_K) \in \mathbb{R}^+.
\]
Define
\[
\Gamma_s(L) := \Gamma_s(U) \Gamma_s(T) \Gamma_s(V), \quad (2.36)
\]
where \( \Gamma_s(U) \) and \( \Gamma_s(V) \) are the second quantization associated with the unitary \( U \) and \( V \). A direct computation shows that \( \Gamma_s(L) \) satisfies the (2.30) (because of properties of \( \Gamma_s(U) \), \( \Gamma_s(V) \)) and equation 2.35 and 2.31 (because second quantizations \( \Gamma_s(U_j) \) acts as identity on vacuum vector).

To see the uniqueness let \( \Gamma_s^1(L) \) and \( \Gamma_s^2(L) \) satisfy 2.30 and 2.31. Therefore we get \( \Gamma_s^1(L)^* \Gamma_s^1(L) W(u) = W(u) \Gamma_s^2(L)^* \Gamma_s^2(L) \), \( \forall u \in \mathcal{H} \). Therefore by irreducibility of Weyl operators (Proposition 20.9 in [Par12]), \( \Gamma_s^2(L)^* \Gamma_s^1(L) = c I \) for some complex scalar of unit modulus. But now by 2.31 we get \( \Gamma_s^2(L) = \Gamma_s^1(L) \).

To prove 2.32, note that \( \langle \Gamma_s(L)^* \Phi_K, \Phi_H \rangle = \langle \Gamma_s(U) \Phi_H, \Phi_K \rangle \in \mathbb{R}^+ \) if we show that \( \Gamma_s(L)^* W(u) \Gamma_s(L) = W(L^{-1}u) \) then by the uniqueness of \( \Gamma_s(L^{-1}) \) we get 2.32. Recall from Theorem 2.1 of [BS05] that \( \Gamma_s(T^{-1}) = \Gamma_s(T)^* \) and by (2.12) \( \Gamma_s(U_j^*) = \Gamma_s(U_j)^* \). Further by (2.36), and (2.13) we have
\[
\Gamma_s(L)^* W(u) \Gamma_s(L) = \Gamma_s(U_2)^* \Gamma_s(T)^* \Gamma_s(U_1)^* W(u) \Gamma_s(U_1) \Gamma_s(T) \Gamma_s(U_2)
\]
\[
= W(U_2^* T^{-1} U_1^* u)
\]
\[
= W(L^{-1} u)
\]

It is easy to see that both \( \Gamma_s(L_2 L_1) \) and \( \Gamma_s(L_2) \Gamma_s(L_1) \) are two unitaries which satisfy (2.30) with \( L = L_2 L_1 \) on the right. Therefore the uniqueness of Shale operators shows that (2.33) is satisfied if \( \langle \Gamma_s(L_2) \Gamma_s(L_1) \Phi_{H_1}, \Phi_{H_2} \rangle \in \mathbb{R}^+ \). Let \( L_j = U_j T_j V_j, j = 1, 2 \) as in Proposition 2.21. Further assume that the decomposition \( \mathcal{H}_2 = H_2 + i H_2 \) is chosen in such a way that \( U_1 \) maps the real subspace \( H_1 \) to the real subspace \( H_2 \). This is possible by choosing a basis \( \{ e_j \} \) of \( H_1 \) and taking \( H_2 = \text{span} \{ U_1 e_j \} \) and finding the decomposition \( L_2 = U_2 T_2 V_2 \) with respect to \( H_2 = H_2 + i H_2 \). Then \( T_2 = U_1 T_1 U_1^* \) where \( T_1 : H_1 \rightarrow H_1 \) and \( T_1(x + iy) = A_1 + i (A_1)^{-1} \)

\[
\langle \Gamma_s(L_2) \Gamma_s(L_1) \Phi_{H_1}, \Phi_{H_2} \rangle = \langle \Gamma_s(U_2) \Gamma_s(T_2) \Gamma_s(V_2) \Gamma_s(U_1) \Gamma_s(T_1) \Gamma_s(V_1) \Phi_{H_1}, \Phi_{H_2} \rangle
\]
\[
= \langle \Gamma_s(T_2) \Gamma_s(V_2) \Gamma_s(U_1) \Gamma_s(T_1) \Phi_{H_1}, \Phi_{H_2} \rangle
\]
\[
= \langle \Gamma_s(U_1) \Gamma_s(T_1) \Gamma_s(U_1)^* \Gamma_s(V_2) \Gamma_s(U_1) \Gamma_s(T_1) \Phi_{H_1}, \Phi_{H_2} \rangle
\]
\[
= \langle \Gamma_s(T_1) \Gamma_s(U_1)^* \Gamma_s(V_2) \Gamma_s(U_1) \Gamma_s(T_1) \Phi_{H_1}, \Phi_{H_1} \rangle
\]
\[
= \langle \Gamma_s(T_1) \Gamma_s(U_1)^* V_2 U_1 \Gamma_s(T_1) \Phi_{H_1}, \Phi_{H_1} \rangle \quad (2.37)
\]
\[ \begin{aligned}
&= \langle \Gamma_s(T'_1 U'_1 V_2 U_1 T_1) \Phi_{H_1}, \Phi_{H_1} \rangle \\
&> 0
\end{aligned} \]  

(2.38)

where (2.37) follows because \( U_1 \) and \( V_2 \) are unitaries and (2.38) because \( T'_1 U'_1 V_2 U_1 T_1 \) is a symplectic automorphism of \( H_1 \).

2.23 Definition. We call the \( \Gamma_s(L) \) obtained in Theorem 2.22 as the Shale unitary corresponding to an \( L \in S(\mathcal{H}, \mathcal{K}) \).

2.24 Example. Let \( V \in \mathcal{B}(\mathcal{H}) \) be a unitary operator, since \( V \) preserves the real part of the inner product, it is an orthogonal transformation on \( (\mathcal{H}, \text{Re} \langle \cdot, \cdot \rangle) \). (Hence if \( V_0 = \begin{bmatrix} V_1 & V_2 \\ -V_2 & V_1 \end{bmatrix} \), then \( V_1^2 + V_2^2 = I \) on \( H \) and the commutator \( [V_1, V_2] = 0 \). Therefore \( V^* V - I = 0 \) and by the uniqueness of Shale unitary the second quantization of \( V \) coincides with the Shale unitary corresponding to \( V \).

§ 2.4 Williamson’s Normal Form

2.25 Definition. Let \( H \) be a real Hilbert space and \( I \) be the identity operator on \( H \). Define the involution operator \( J_0 \) on \( H \oplus H \) by

\[ J_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \]

2.26 Theorem. [Bhat-John, BJ81] Let \( H \) be a real Hilbert space and \( S_0 \) be a strictly positive invertible operator on \( H \oplus H \) then there exists a real Hilbert space \( K \), a positive invertible operator \( P \) on \( K \) and a symplectic transformation \( L_0 : H \oplus H \to K \oplus K \) such that

\[ S_0 = L_0^* \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} L_0 \]  

(2.39)

Keeping the notations in § 2.3, note that if \( P_0 = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \) then \( P \) is a positive invertible complex linear operator on \( K \). We have

2.27 Corollary. Let \( S \) be a real linear positive, invertible operator on a complex Hilbert space \( \mathcal{H} \). Then there exists a complex Hilbert space \( \mathcal{K} \), a complex linear positive invertible operator \( \mathcal{P} \) and a symplectic transformation \( L : \mathcal{H} \to \mathcal{K} \) such that

\[ S = L^* \mathcal{P} L. \]  

(2.40)

Further, \( \mathcal{P} \) has the property that \( \mathcal{P}_0 = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \).

3 Quantum Gaussian States

By a state (or density matrix) \( \rho \) on a Hilbert space \( \mathcal{H} \) we mean a positive operator of unit trace i.e. \( \rho \geq 0 \) and \( \text{Tr} \rho = 1 \). Note that a state \( \rho \) on \( \mathcal{H} \) gives rise to a unique state on the \( C^* \)-algebra \( \mathcal{B}(\mathcal{H}) \) (as in definition 2.10) as the functional \( Y \mapsto \text{Tr} \rho Y \), \( Y \in \mathcal{B}(\mathcal{H}) \).

3.1 Definition. Let \( \rho \in \mathcal{B}(\Gamma_s(\mathcal{H})) \) be a density matrix. Then a (possibly) complex valued function \( \hat{\rho} \) on \( \mathcal{H} \) defined by

\[ \hat{\rho}(z) = \text{Tr} \rho W(z), \quad z \in \mathcal{H} \]  

(3.1)

is called the quantum characteristic function (or quantum Fourier transform) of \( \rho \).
We would like to observe at this point that the mapping \( \rho \rightarrow \hat{\rho} \) is a one-one mapping, proof of this fact is essentially the same as that of Proposition 2.4 in [Par10]. Further, it may be noted that \( W(z) \rightarrow \hat{\rho}(z) \) defines a state on the CCR-algebra generated by the Weyl operators (actually \( \hat{\rho} \) corresponds a state on \( B(\Gamma_s(\mathcal{H})) \) itself!).

3.2 Proposition. Let \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), we have \( \Gamma_s(\mathcal{H}) = \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2) \). Let \( \rho_1 \) and \( \rho_2 \) be states on \( \Gamma_s(\mathcal{H}_1) \) and \( \Gamma_s(\mathcal{H}_2) \) respectively. Then the quantum characteristic function of the state \( \rho_1 \otimes \rho_2 \) is given by

\[
(\rho_1 \otimes \rho_2)^\wedge (f \oplus g) = \hat{\rho}_1(f)\hat{\rho}_2(g).
\]

(3.2)

Further, if \( \rho \) is any state on \( \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2) \) then the marginal state \( \rho_1 \) obtained by

\[
\rho_1 = \text{Tr}_2 \rho
\]

(3.3)

where \( \text{Tr}_2 \) denotes the relative trace over the second factor \( \Gamma_s(\mathcal{H}_2) \) is

\[
\hat{\rho}_1(f) = \hat{\rho}(f \oplus 0).
\]

(3.4)

Proof. These follow because \( W(f \oplus g) = W(f) \otimes W(g) \).

If \( \rho \) is a state so is any unitary conjugation of it. It is important to understand how the quantum characteristic function changes when a unitary conjugation is applied to \( \rho \). We will explore this now with the fundamental unitaries, Weyl operators and second quantizations, recall Definition 2.3. By using Theorem 2.22, proof of the following proposition follows in the same way as that of Proposition 2.5 in [Par10].

3.3 Proposition. If \( \rho \) is a state on \( \Gamma_s(\mathcal{K}) \) and \( L \in \mathcal{S}(\mathcal{H}, \mathcal{K}) \) then

\[
\{\Gamma_s(L)^* \rho \Gamma_s(L)\}^\wedge (\beta) = \hat{\rho}(L\beta).
\]

Further, for every \( \alpha \in \mathcal{H} \),

\[
\{W(\alpha) \rho W(\alpha)^{-1}\}^\wedge (\beta) = \hat{\rho}(\beta) e^{2i \text{Im}(\alpha, \beta)}.
\]

3.4 Definition. Let \( \rho \in B(\Gamma_s(\mathcal{H})) \) be a state, \( \rho \) is said to be Gaussian if there exists \( w \in \mathcal{H} \) and \( S \in B_{\mathbb{R}}(\mathcal{H}) \) such that

\[
\hat{\rho}(z) = \exp \left\{ -i \text{Re} \langle w, z \rangle - \frac{1}{2} \text{Re} \langle z, Sz \rangle \right\}, \forall z \in \mathcal{H}.
\]

(3.5)

In such a case we write \( \rho = \rho_g(w, S) \).

Note that this definition determines a real linear functional \( z \mapsto \text{Re} \langle w, z \rangle \) and a bounded quadratic form \( z \mapsto \text{Re} \langle z, Sz \rangle \) on the real Hilbert space \( \mathcal{H} \). Hence \( w \) and \( S \) are uniquely determined by the definition.

We call \( w \) the mean vector and \( S \) the covariance operator associated with \( \rho \). Suppose \( \mathcal{H} = H + iH \), where \( H \) is a completely real subspace and let \( w = \sqrt{2}(l - im) \), then we call \( l \) and \( m \) as mean momentum vector and mean position vector respectively. Further \( S_0 \) corresponding to \( S \) (§ 2.3) will be called as the momentum-position covariance operator.
Notation. Let $G$ denote the set of all Gaussian states on $\Gamma_s(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the set of all Gaussian covariance operators on $\mathcal{H}$.

We will characterize the elements of $\mathcal{K}(\mathcal{H})$ in Theorem 3.28.

3.5 Examples. 1. For $f \in \mathcal{H}$ consider the normalized exponential vector $\psi(f) := e^{-\frac{1}{2}\|f\|^2}e(f)$. Let the pure state $|\psi(f)\rangle \langle \psi(f)|$ be called the coherent state. Same proof as that of Proposition 2.9 in [Par10] proves that the coherent state is a pure Gaussian state on $\Gamma_s(\mathcal{H})$ and

$$\langle |\psi(f)\rangle \langle \psi(f)| \rangle \hat{\wedge} (z) = \exp \left\{ -2i \text{Im} \langle z, f \rangle - \frac{1}{2}\|z\|^2 \right\}$$ (3.6)

In particular,

$$|e(0)\rangle \langle e(0)| = \rho_g(0, I)$$ (3.7)

Notice at this point that the quantum characteristic function of the density matrix $|e(0)\rangle \langle e(0)|$ corresponds to the vacuum state defined in § 2.2.3.

2. Let $L$ be a symplectic automorphism on $\mathcal{H}$ such that $L^{\tau}L - I$ is Hilbert-Schmidt. Define $\psi_L = \Gamma_s(L)^* |e(0)\rangle$. Then

$$\langle |\psi_L\rangle \langle \psi_L| \rangle \hat{\wedge} (z) = \text{Tr} |\psi_L\rangle \langle \psi_L| W(z)$$

$$= \text{Tr} |\psi_L\rangle \langle W(z)^*\psi_L|$$

$$= \langle \psi_L, W(z)\psi_L \rangle$$

$$= \langle e(0), \Gamma_s(L)W(z)\Gamma_s(L)^*e(0) \rangle$$

$$= \langle e(0), W(Lz)e(0) \rangle$$

$$= e^{-\frac{1}{2}\langle z, L^{\tau}\ell z \rangle}.$$ 

Therefore, $|\psi_L\rangle \langle \psi_L| = \rho_g(0, L^{\tau}L)$

3.6 Proposition. Let $\alpha \in \mathcal{H}$. Then

$$W(\alpha)\rho_g(w, S)W(\alpha)^{-1} = \rho_g(w - 2i\alpha, S).$$

In particular,

$$W(-\frac{1}{2}i w)\rho_g(w, S)W(-\frac{1}{2}i w)^{-1} = \rho_g(0, S).$$

Proof. This is a direct consequence of the definition of $\rho_g(\cdot, \cdot)$ and Proposition 3.3.

3.7 Proposition. Let $\rho_1 = \rho_g(w, S_1)$ and $\rho_2 = \rho_g(w_2, S_2)$ be Gaussian states on $\Gamma_s(\mathcal{H}_1)$ and $\Gamma_s(\mathcal{H}_2)$ respectively. Then $\rho_1 \otimes \rho_2 = \rho_g(w_1 \oplus w_2, S_1 \oplus S_2)$.

Proof. This follows directly from Proposition 3.2.

3.8 Proposition. If $\rho = \rho_g(w, S)$ on $\Gamma_s(\mathcal{K})$ and $L \in S(\mathcal{H}, \mathcal{K})$ then

$$\Gamma_s(L)^* \rho \Gamma_s(L) = \rho_g(L^{\tau}w, L^{\tau}SL)$$

Proof. Follows from Proposition 3.3.
§ 3.1 Necessary conditions on the covariance operator

3.9 Lemma. If $\rho$ is any density matrix then the kernel $K_\rho$ on $\mathcal{H}$ defined by $K_\rho(z, w) = e^{i \text{Im} \langle z, w \rangle} \hat{\rho}(w - z)$ is positive definite.

Proof.

\begin{align*}
\sum_{j,k=1}^{n} c_j c_k K_\rho(z_j, z_k) &= \sum_{j,k=1}^{n} c_j c_k e^{i \text{Im} \langle z_j, z_k \rangle} \hat{\rho}(z_k - z_j) \\
&= \sum_{j,k=1}^{n} c_j c_k e^{i \text{Im} \langle z_j, z_k \rangle} \text{Tr} \rho W(z_k - z_j) \\
&= \sum_{j,k=1}^{n} c_j c_k \text{Tr} \rho (-z_j) W(z_k) \\
&= \text{Tr} \rho X^* X \\
&\geq 0
\end{align*}

where $X = \sum_{j=1}^{n} c_j W(z_j) \quad \Box$

Recall from § 2.2.3 that $\text{CCR}(\mathcal{H}, \sigma) \hookrightarrow B(\Gamma_s(\mathcal{H}))$ as a standard space, if we take $\sigma(\cdot, \cdot) = -\text{Im} \langle \cdot, \cdot \rangle$. Also we will use the work done in § 2.3 in what follows.

3.10 Lemma. Let $S$ be a real linear, invertible operator on $\mathcal{H}$ and $\hat{S} - i \hat{J} \geq 0$ on $\hat{\mathcal{H}}$. Then

1. $S \geq 0$.

2. If $S = L^T P L$ is the Williamson’s normal form associated with $S$ (as in Corollary 2.27), then $P - I \geq 0$ on $K$.

3. There exists a primary quasifree state $\phi$ on $\text{CCR}(\mathcal{H}, \sigma)$ such that

$$\phi(W(z)) = e^{-\frac{1}{2} \text{Re} \langle z, Sz \rangle}. \quad (3.8)$$

Further, $\phi = \phi_A$, where $A = -JS$ (the notation $\phi_A$ is as in § 2.2.2).

Proof. Note that $\hat{S} - i \hat{J} \geq 0$ implies $\hat{S}$ is symmetric, hence we have $S$ is also symmetric. Let us denote the complex inner product in both $\mathcal{H}$ and $\hat{\mathcal{H}}$ by $\langle \cdot, \cdot \rangle$. Let $z, w \in \mathcal{H}$ then $z + iw \in \hat{\mathcal{H}}$ and

\begin{align*}
0 &\leq \langle z + iw, (\hat{S} - i \hat{J})z + iw \rangle \\
&= \text{Re} \langle z, Sz \rangle + i \text{Re} \langle z, Sw \rangle - i \text{Re} \langle w, Sz \rangle + \text{Re} \langle w, Sw \rangle \\
&\quad - i \text{Re} \langle z, Jz \rangle + \text{Re} \langle z, Jw \rangle - \text{Re} \langle w, Jz \rangle - \text{Re} \langle w, Jw \rangle \\
&= \text{Re} \langle z, Sz \rangle + \text{Re} \langle w, Sw \rangle + 2 \text{Re} \langle z, Jw \rangle \quad (3.9)
\end{align*}

where we used the facts $S$ is symmetric, the real inner product is symmetric and $\text{Re} \langle \cdot, Jz \rangle = 0$ for all $z$ to obtain (3.9). If we take $z = w$ in the above computation then we get $S \geq 0$, since it is already symmetric. Note that the invertibility of $S$ is not used to prove this.
Let $\mathcal{P}_0 = [\hat{P} \ 0 \ 0 \ \hat{P}]$. Then $\hat{P} = \begin{bmatrix} \hat{P} & 0 \\ 0 & \hat{P} \end{bmatrix}$ and $\hat{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ on $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ by Proposition 2.19. $\hat{S} - i\hat{J} \geq 0$ implies $\hat{L}^T\begin{bmatrix} \hat{P} & 0 \\ 0 & \hat{P} \end{bmatrix}\hat{L} - i\hat{J} \geq 0$. By a conjugation with $\hat{L}^{-1}$ and using the fact that $L^{-1}$ is symplectic we get $\begin{bmatrix} \hat{P} & -\hat{P} \\ \hat{P} & \hat{P} \end{bmatrix} \geq 0$ on $\hat{K} = \mathcal{K} \oplus \mathcal{K}$. Hence (by Proposition 2.19) we get $\begin{bmatrix} \hat{P} & -\hat{P} \\ \hat{P} & \hat{P} \end{bmatrix} \geq 0$ on $\mathcal{K} \oplus \mathcal{K}$. But this means $P \geq I$ on $K$ and correspondingly $\mathcal{P} \geq I$ on $\mathcal{K}$.

Since the CCR($\mathcal{H}, \sigma$) is standard we will use 1) of Proposition 2.15. Since $S$ is positive and invertible, $\alpha(z,w) := \text{Re} \langle z, Sw \rangle$ defines a complete real inner product on $\mathcal{H}$. Therefore by Proposition 2.15 $\phi$ as in (3.8) exists if $\sigma(z,w)^2 \leq \alpha(z,z)\alpha(w,w)$, for all $f,g \in \mathcal{H}$. This is same as

$$\text{Im} \langle z, w \rangle^2 \leq \text{Re} \langle z, Sz \rangle \text{Re} \langle w, Sw \rangle$$

(3.10)

Thus the only thing left to prove for the existence of $\phi$ is (3.10). To keep track of the inner product in $\mathcal{H}$ and $\mathcal{K}$ we put a subscript, (for eg. $\langle \cdot, \cdot \rangle_\mathcal{H}$ will denote the inner product in $\mathcal{H}$). Now

$$\text{Im} \langle z, w \rangle^2_\mathcal{K} = \text{Im} \langle Lz, Lw \rangle^2_\mathcal{K}$$

$$\leq \langle Lz, Lw \rangle^2_\mathcal{K}$$

$$\leq \langle Lz, Lz \rangle_\mathcal{K} \langle Lw, Lw \rangle_\mathcal{K}$$

$$\leq \langle Lz, \mathcal{P}Lz \rangle_\mathcal{K} \langle Lw, \mathcal{P}Lw \rangle_\mathcal{K}$$

$$= \text{Re} \langle Lz, \mathcal{P}Lz \rangle_\mathcal{K} \text{Re} \langle Lw, \mathcal{P}Lw \rangle_\mathcal{K}$$

$$= \langle z, L^T\mathcal{P}Lz \rangle_\mathcal{H} \langle w, L^T\mathcal{P}Lw \rangle_\mathcal{H},$$

(3.11)

where (3.11) follows from (3.2). Thus we proved (3.10). Hence first part of (3.1) is proved. Further, $\phi = \phi_A$ because $\text{Re} \langle \cdot, S(\cdot) \rangle_\mathcal{H} = -\text{Im} \langle A(\cdot), \cdot \rangle$. 

3.11 Lemma. Let $\mathcal{H}$ be a real Hilbert space and $\hat{\mathcal{H}} = \mathcal{H} \oplus i\mathcal{H}$ be its complexification. Let $A \in \mathcal{B}(\mathcal{H})$ be self adjoint. Define a hermitian kernel, $K$ on $\mathcal{H}$ by

$$K(x,y) := \langle x, Ay \rangle \quad \text{for all } x,y \in \mathcal{H}.$$ 

Then $K$ is positive definite if and only if $A \geq 0$ in the sense of positive definiteness of operators in $\mathcal{B}(\mathcal{H})$.

3.12 Theorem. Let $S$ be a real linear symmetric and invertible operator on $\mathcal{H}$, the function $f : \mathcal{H} \to \mathbb{R}$ defined by $f(z) = e^{-\frac{1}{2} \text{Re} \langle z, Sz \rangle}$ be the quantum characteristic function of a density matrix $\rho$ ie, $S \in \mathcal{K}(\mathcal{H})$ then

1. On $\hat{\mathcal{H}}$ we have,

$$\hat{S} - i\hat{J} \geq 0.$$  

(3.12)

2. $S - I$ is Hilbert-Schmidt on $(\mathcal{H}, \text{Re} \langle \cdot, \cdot \rangle)$.

3. $(\sqrt{S}J\sqrt{S})^*(\sqrt{S}J\sqrt{S}) - I$ is trace class on $(\mathcal{H}, \text{Re} \langle \cdot, \cdot \rangle)$.

Proof. 1. Proof of (3.12) will follow in similar lines to the proof the corresponding theorem in [Par10] for the finite mode case, we will give a proof here because there are slight changes to be noticed in the infinite mode case. Define the kernel

$$K_\rho(\alpha, \beta) = e^{i\text{Im} \langle \alpha, \beta \rangle} f(\beta - \alpha), \quad \alpha, \beta \in \mathcal{H}.$$  

(3.13)
By Lemma 3.9, $K_{\rho}$ is a positive definite kernel on $H$. If $\alpha = x + iy, \beta = u + iv$ where $x, y, u, v \in H$ then $\text{Im} \langle \alpha, \beta \rangle = \langle \frac{\alpha}{\bar{\alpha}}, J_0(\frac{u}{v}) \rangle$ on $H \oplus H$ (c.f. §2.3) we can rewrite the definition of $K_{\rho}$ as

$$K_{\rho}(\alpha, \beta) = \exp \left\{ i \left( \langle \frac{\alpha}{\bar{\alpha}}, J_0(\frac{u}{v}) \rangle - \left\langle \left( \frac{u-x}{v-y} \right), \frac{1}{2} S_0(\frac{u-x}{v-y}) \right\rangle \right\}$$

(3.14)

Now positive definiteness of $K_{\rho}$ in $H$ reduces to that of $L$ in $H \oplus H$ where

$$L ((x, y), (u, v)) = \exp \left\{ i \left( \langle \frac{\alpha}{\bar{\alpha}}, J_0(\frac{u}{v}) \rangle - \left\langle \left( \frac{u-x}{v-y} \right), \frac{1}{2} S_0(\frac{u-x}{v-y}) \right\rangle \right\}$$

(3.15)

This is equivalent to the positive definiteness of

$$L_t ((x, y), (u, v)) = L \left( \sqrt{t}(x, y), \sqrt{t}(u, v) \right)$$

for all $t \geq 0$. But $\{L_t\}$ is a one parameter multiplicative semigroup of kernels on $H \oplus H$. By elementary properties of positive definite kernels as described in Section 1 of [PS72], positive definiteness of $L_t, t \geq 0$ is equivalent to the conditional positive definiteness of

$$N ((x, y), (u, v)) = i \langle \frac{\alpha}{\bar{\alpha}}, J_0(\frac{u}{v}) \rangle - \left\langle \left( \frac{u-x}{v-y} \right), \frac{1}{2} S_0(\frac{u-x}{v-y}) \right\rangle$$

or equivalently (by the same Proposition), the positive definiteness of

$$N ((x, y), (u, v)) = N ((x, y), (0, 0)) - N ((0, 0), (u, v)) - N ((0, 0), (0, 0))$$

$$= i \langle \frac{\alpha}{\bar{\alpha}}, J_0(\frac{u}{v}) \rangle - \left\langle \left( \frac{u-x}{v-y} \right), \frac{1}{2} S_0(\frac{u-x}{v-y}) \right\rangle + \left\langle \left( \frac{v}{u} \right), \frac{1}{2} S_0(\frac{v}{u}) \right\rangle$$

$$= i \langle \frac{\alpha}{\bar{\alpha}}, J_0(\frac{u}{v}) \rangle + \left\langle \left( \frac{v}{u} \right), \frac{1}{2} S_0(\frac{v}{u}) \right\rangle$$

$$= i \langle \frac{\alpha}{\bar{\alpha}}, J_0(\frac{u}{v}) \rangle + \langle \frac{v}{u}, 0 \rangle, S_0(\frac{v}{u}) \rangle$$

(3.16)

$$= i \langle \frac{\alpha}{\bar{\alpha}}, J_0(\frac{u}{v}) \rangle + \langle \frac{v}{u}, -0, J_0(\frac{\alpha}{\bar{\alpha}}) \rangle$$

(3.17)

(3.18)

where (3.16) follows because the real innerproduct is symmetric and (3.17) because $S_0$ is symmetric, and (3.18) for the same reasons. But $H \oplus H \subset \hat{H} = (H \oplus H) + i(H \oplus H)$, the positive definiteness of (3.18) lifts to the positive definiteness of

$$M(w, z) := \langle w, \left\{ \hat{S} - i\hat{J} \right\} z \rangle = \langle \frac{w}{\bar{w}}, -iJ_0(\frac{\alpha}{\bar{\alpha}}) \rangle + \langle \frac{z}{\bar{z}}, \hat{S}(\frac{\alpha}{\bar{\alpha}}) \rangle$$

(3.19)

where $M$ is a kernel defined (as above) in $H \subset \hat{H}$. Now by Lemma 3.11 positive definiteness of $M$ in (3.19) is equivalent to (3.12).

Now we set out to prove that $\hat{S} - I$ is Hilbert-Schmidt on the real Hilbert space $H$. We are given that there exists a density matrix $\rho$ such that $\hat{\rho}(z) = e^{-\frac{1}{2} \text{Re}(z, Sz)}$. Since $\hat{S} - \hat{J} \geq 0$, by Lemma 3.10 there exists a primary quasifree state $\phi$ on $CCR(H, \sigma)$ such that

$$\phi(W(z)) = e^{-\frac{1}{2} \text{Re}(z, Sz)}$$

Claim : $\phi_A$ and $\phi_{-J}$ are quasi equivalent, where $A = -JS$. 

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Proof (of Claim). Consider the state $\psi$ on $B(\Gamma_s(H))$ given by $X \mapsto \text{Tr} \rho X$. The quasifree state $\phi_A$ is the restriction of $\psi$ to $A := CCR(H, \sigma) \hookrightarrow \Gamma_s(H)$.

Let $(H_\psi, \Pi_\psi, \Omega_\psi)$ be the GNS triple for $B(H)$ with respect to $\psi$. Then $(H_\psi, \Pi_\psi|_A, \Omega_\psi)$ is the GNS triple for $A$ with respect to $\phi_A$. To see this, only thing to be noticed is $\Omega_\psi$ is cyclic for $\Pi_\psi(A)$, which is clear since $A$ is strongly dense in $B(\Gamma_s(H))$.

We further note that the inclusion $A \subseteq B(\Gamma_s(H))$ is the GNS representation with respect to the vacuum state which is the quasi-free state given by $\phi_{-J}$. It can be seen that the association

$$W(x) \mapsto \Pi_\psi(W(x))$$

can be extended as an isomorphism between $B(\Gamma_s(H)) = A''$ and $\Pi_\psi(B(\Gamma_s(H)))$.

Thus the claim is proved.

Since $\phi_{(-J)}$ and $\phi_A$ are quasi equivalent, by Theorem 2.16 we get $A + J$ is Hilbert-Schmidt on $H_{-J}$ which is same as $H$ with the real inner product $\text{Re} \langle \cdot, \cdot \rangle_H$.

This follows due to the same reason as that of 2) because of Theorem 2.16 itself. We get $\sqrt{-A^2 - I}$ is Hilbert-Schmidt on $(H, \text{Re} \langle \cdot, \cdot \rangle)$. This is same as $-A^2 - I$ is trace class on the same Hilbert space. Hence we have $-JSJS - I$ is trace class. By multiplying with $\sqrt{S}$ on the left and $(\sqrt{S})^{-1}$ on the right we see that $-\sqrt{S}JSJ\sqrt{S} - I$ is trace class. The result follows because $J^r = -J$.

$$\square$$

Note. It may be noted at this point that the operator $\sqrt{S}JS\sqrt{S}$ in 3) of the above theorem is the skew symmetric operator $B$ appearing in the proof of Williamson’s normal form in [BJ18]. Proof of Williamson’s normal form was obtained there by applying spectral theorem (as proved in [BJ18]) to $B$,

$$\Gamma^T B \Gamma = \begin{bmatrix} 0 & -P \\ P & 0 \end{bmatrix}$$

where $\Gamma$ is an orthogonal transformation. $L$ was obtained by taking

$$L = \begin{bmatrix} P^{-1/2} & 0 \\ 0 & P^{-1/2} \end{bmatrix} \Gamma^r S^{1/2}.$$ 

This choice of $L$ provides $S = L^r P L$, where $P_0 = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$.

3.13 Corollary. $(\sqrt{S}JS\sqrt{S})^r(\sqrt{S}JS\sqrt{S}) - I$ is trace class if and only if $-JSJS - I$ is trace class.

Proof. This is the content of the proof of 3. in Theorem 3.12. $\square$

3.14 Corollary. Assuming the hypothesis of the previous Theorem 3.12 we have

1. If $S - I \geq 0$ then $S - I$ is trace class on $(H, \text{Re} \langle \cdot, \cdot \rangle)$

2. If $S$ is complex linear then $S - I \geq 0$ and $S - I$ is trace class on $(H, \text{Re} \langle \cdot, \cdot \rangle)$.

Proof. [11] We have $-\sqrt{S}JSJ\sqrt{S} - I$ is trace class on $(H, \text{Re} \langle \cdot, \cdot \rangle)$. Hence by multiplying with $(\sqrt{S})^{-1}$ on both sides $(J)SJ - S^{-1}$ is trace class. Since $S - I \geq 0$, $(J)SJ - I \geq 0$ and $S^{-1} \leq I$ therefore we have

$$0 \leq (J)SJ - I \leq (J)SJ - S^{-1}$$
and we conclude that \((-J)SJ - I\) is trace class on \((\mathcal{H}, \text{Re} \langle \cdot, \cdot \rangle)\). Thus Claim (6) is proved by a conjugation with \(J\).

2. By (k) in Lemma 3.10 and (2) of Proposition 2.15 we have

\[ -A^2 - I \geq 0 \]  

(3.20)

with respect to the real inner product \(\text{Re} \langle \cdot, S(\cdot) \rangle\). We have \(A^2 = JSJS\) but since \(S\) is complex linear it commutes with \(J\), thus \(A^2 = -S^2\) and we see that \(S^2 - I \geq 0\), consequently \(S \geq I\) on \((\mathcal{H}, \text{Re} \langle \cdot, S(\cdot) \rangle)\). But this implies \(S \geq I\) on \((\mathcal{H}, \text{Re} \langle \cdot, \cdot \rangle)\) since \(S\) is positive. Since \(S\) commutes with \(J\), by (3) of Theorem 3.12 we see that \(S^2 - I\) is Hilbert-Schmidt on \((\mathcal{H}, \text{Re} \langle \cdot, \cdot \rangle)\). Now the Claim follows because \(0 \leq S - I \leq S^2 - I\).

We just notice the following which follows from the fact that \(\phi_A\) and \(\phi_{-J}\) are quasiequivalent

3.15 Corollary. \(\phi_A\) is a Type 1 quasifree state.

Note. By (2) of Example 3.5 we have seen that for a symplectic automorphism \(L\), \(L^*L\) is a covariance operator whenever \(L^*L - I\) is Hilbert-Schmidt. Now by Theorem 3.12 we get that \(L^*L\) satisfies the conditions (1), (2) and (3) there. This is true also for any such symplectic transformation. But since \(\sqrt{L^*L}\) is symplectic whenever \(L\) is so, the condition (3) is just void. Also it can be proved independently that for any symplectic transformation the positivity condition (1) on \(L^*L\) is true. Therefore, \(L^*L - I\) is Hilbert-Schmidt is the only non-trivial condition here.

§ 3.1.1 What is the meaning of the condition \(\hat{S} - i\hat{J} \geq 0\)?

3.16 Lemma. Let \(S\) be a real linear operator on \(\mathcal{H}\) then \(\hat{S} - i\hat{J} \geq 0\) if and only if there exists a state \(\phi\) on \(CCR(\mathcal{H}, \sigma)\) such that \(\phi(W(z)) = e^{-\frac{i}{2} \text{Re}(z, Sz)}\).

Proof. We saw in the proof of Theorem 3.12 that the condition \(\hat{S} - i\hat{J} \geq 0\) is equivalent to the positive definiteness of the kernel \(K_\rho\) in 3.13, where \(f(z) = e^{-\frac{i}{2} \text{Re}(z, Sz)}\). Since \(f(0) = 1\) by Proposition 3.1 in [Pet90] \(K_\rho\) is positive definite if and only if there exists a state \(\phi\) on \(CCR(\mathcal{H}, \sigma)\) such that \(\phi(W(z)) = f(z)\).  

By Lemma 3.10 if \(S\) is real linear, invertible and \(\hat{S} - i\hat{J} \geq 0\) then there exists a primary quasifree state \(\phi\) such that (3.8) holds. On the otherhand if there is a primary quasifree state \(\phi\) such that (3.8) holds, by Lemma 3.16 above we have \(\hat{S} - i\hat{J} \geq 0\). Thus we have

3.17 Theorem. Let \(S\) be a real linear, invertible operator on \(\mathcal{H}\). Then \(\hat{S} - i\hat{J} \geq 0\) on \(\hat{\mathcal{H}}\) if and only if there exists a primary quasifree state \(\phi\) on \(CCR(\mathcal{H}, \sigma)\) such that

\[ \phi(W(z)) = e^{-\frac{i}{2} \text{Re}(z, Sz)}. \]  

(3.21)

3.18 Corollary. Let \(S\) be a real linear, invertible operator on \(\mathcal{H}\). Then \(\hat{S} - i\hat{J} \geq 0\) on \(\hat{\mathcal{H}}\) if and only if \(\text{Im} \langle z, w \rangle^2 \leq \text{Re} \langle z, S\rangle \text{Re} \langle w, Sw \rangle\).
§ 3.2 Positivity and Trace class conditions imply Gaussian state

Now we proceed to prove the converse of Theorem 3.12.

3.19 Lemma. If $s_j > 0$ then $\sum_{j=1}^{\infty} \left( \frac{e^{-s_j}}{1-e^{-s_j}} \right) < \infty$ if and only if $\sum_{j=1}^{\infty} e^{-s_j}$ is convergent.

Proof. Assume $\sum_{j=1}^{\infty} \left( \frac{e^{-s_j}}{1-e^{-s_j}} \right) < \infty$. Since $e^{-s_j} > 0$ and $\frac{1}{1-e^{-s_j}} > 0$, we have $0 < \sum_{j=1}^{\infty} e^{-s_j} < \sum_{j=1}^{\infty} \left( \frac{e^{-s_j}}{1-e^{-s_j}} \right) < \infty$. Now assume that $\sum_{j=1}^{\infty} e^{-s_j} < \infty$. Then $s_j \to \infty$ and hence $\frac{1}{1-e^{-s_j}} \to 1$. This means we have $0 < \frac{1}{1-e^{-s_j}} < M, \forall j$, for some $M > 1$. Therefore, $\sum_{j=1}^{\infty} \left( \frac{e^{-s_j}}{1-e^{-s_j}} \right) < \infty$. □

Let $\mathcal{H} = H + iH$ and $\{e_1,e_2,e_3,\cdots\}$ be an orthonormal basis for $H$. Note that $\{e_j\}$ is also a basis for $\mathcal{H}$ as a complex Hilbert space. Let $D = \text{Diag}(d_j)$ be a bounded diagonal operator on $\mathcal{H}$, with $d_j > 1$, $j = 1, 2, 3, \ldots$ in the given basis. Since $d_j > 1$ there exists $s_j > 0$ such that $d_j = \coth\left(\frac{s_j}{2}\right)$ for all $j$. If we consider $D$ as a real linear operator on $\mathcal{H}$, then $D_0 = [D \ 0 \ 0]$ on $H \oplus H$.

3.20 Lemma. Let $D = \text{Diag}(d_j)$ be a bounded diagonal operator on $\mathcal{H}$, with $d_j > 1$, $j = 1, 2, 3, \ldots$ with respect to a basis. Write $d_j = \coth\left(\frac{s_j}{2}\right)$ for all $j$. Then $D - I$ is trace class if and only if $\sum_{j=1}^{\infty} e^{-s_j}$ is convergent.

Proof. Observe,

\[
D - I \text{ is in trace class} \Leftrightarrow \sum_{j=1}^{\infty} (d_j - 1) < \infty
\]

\[
\Leftrightarrow \sum_{j=1}^{\infty} \left( \coth\left(\frac{s_j}{2}\right) - 1 \right) < \infty
\]

\[
\Leftrightarrow \sum_{j=1}^{\infty} \left( \frac{1 + e^{-s_j}}{1 - e^{-s_j}} - 1 \right)
\]

\[
\Leftrightarrow \sum_{j=1}^{\infty} \left( \frac{e^{-s_j}}{1 - e^{-s_j}} \right) < \infty
\]

\[
\Leftrightarrow \sum_{j=1}^{\infty} e^{-s_j} < \infty \quad (3.22)
\]

where (3.22) follows from Lemma 3.19. □

3.21 Proposition. Let $D$ be as described in Lemma 3.20. Then there exists a state $\rho_D$ on $\Gamma_s(\mathcal{H})$ such that $\hat{\rho}_D(x) = e^{-\frac{1}{2}(x,Dx)}$. 

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Proof. Consider the diagonal operator $T = \text{Diag}(e^{-s_j})$ with respect to the same basis in which $D$ is diagonal then the second quantization $\Gamma_s(T)$ is a trace class operator on the symmetric Fock space, $\Gamma_s(H)$. This is because of the following reasoning. $T$ is positive and by Lemma 3.20 it is a trace class operator. Thus we have $s_j > 0$ and $s_j \to \infty$, which implies $e^{-s_j} < 1$, for all $j$. Since $e^{-s_j}$ is maximum when $s_j$ is minimum, we get $\sup_j (e^{-s_j}) < 1$. Now by Exercise 20.22 (iv) in [Par10], $\Gamma_s(T)$ exists and is trace class with

$$\text{Tr} \Gamma_s(T) = \prod_{j=1}^\infty (1 - e^{-s_j})^{-1} \tag{3.23}$$

Define $\rho_D = \prod_{j=1}^\infty (1 - e^{-s_j})\Gamma_s(T)$, then $\rho$ is a density matrix on $\Gamma_s(H)$. We have $H = \oplus_j C e_j$. Since $\Gamma_s(e^{-s_j}) = e^{-s_j}a_j^\dagger a_j$ on $\Gamma_s(C e_j)$, under the isomorphisms described in Proposition 2.25 $\rho_D = \prod_{j=1}^\infty (1 - e^{-s_j})\Gamma_s(\oplus_j e^{-s_j}) = \otimes_j \rho_j$, where $\rho_j = (1 - e^{-s_j})e^{-s_j}a_j^\dagger a_j$. Let $x = \oplus_j x_j e_j$ then

$$\hat{\rho}_D(x) = \text{Tr} \rho W(x)$$

$$= \text{Tr} \left( \prod_{j=1}^\infty (1 - e^{-s_j})\Gamma_s(\otimes_j e^{-s_j}) W(\oplus_j x_j) \right)$$

$$= \text{Tr} \left( \otimes_j (1 - e^{-s_j})\Gamma_s(e^{-s_j}) W(x_j) \right)$$

$$= \text{Tr} \left( \otimes_j (1 - e^{-s_j})e^{-s_j}a_j^\dagger a_j W(x_j) \right)$$

$$= \prod_{j=1}^\infty \text{Tr} \left( (1 - e^{-s_j})e^{-s_j}a_j^\dagger a_j W(x_j) \right)$$

$$= \prod_{j=1}^\infty e^{-\langle x_j, \frac{1}{2} \coth(\frac{1}{2} s_j) x_j \rangle} \tag{3.24}$$

where (3.24) follows from Proposition 2.12 in [Par10].

Recall from Example 1 that the vacuum state $\langle e(0) \rangle \langle e(0) \rangle$ on $\Gamma_s(H)$ is a Gaussian state with covariance operator $I$.

**3.22 Theorem.** If $\mathcal{P}$ is any complex linear operator on $H$ such that $\mathcal{P} - I$ is positive and trace class then there exists a state $\rho$ on $\Gamma_s(H)$ such that the quantum characteristic function $\hat{\rho}$ associated with $\rho$ is given by

$$\hat{\rho}(x) = e^{-\frac{1}{2} \langle x, \mathcal{P} x \rangle}$$

for every $x \in H$.

**Proof.** Let $U$ be a unitary operator such that $\mathcal{P} = U^* D U$. Such a $U$ exists by applying spectral theorem to the compact positive operator $\mathcal{P} - I$. Since $\mathcal{P} \geq I$ assume without loss of generality that $H = H_1 \oplus H_2$ is such that $D = \begin{bmatrix} D_1 & 0 \\ 0 & 1 \end{bmatrix}$, where we separated all the diagonal entries of $D$ which are equal to one and not equal to one. Then $D_1$ satisfies the assumptions in Proposition 3.21 and $\rho_{D_1}$ exists as a Gaussian state on $\Gamma_s(H_1)$. Let $\rho_0$ denote the vacuum state $\langle e(0) \rangle \langle e(0) \rangle$ on $\Gamma_s(H_2)$, which is Gaussian by Example 1. Then by Proposition 3.7 $\rho_{D_1} \otimes \rho_0 = \rho_j(0, D)$. Define $\rho = \Gamma_s(U^*) \rho_{D_1} \otimes \rho_0 \Gamma_s(U)$ and the result follows from Proposition 3.3. \qed

**3.23 Lemma.** Let $C - I$ is Hilbert-Schmidt (trace class) then

1. If $C \geq 0$ then $\sqrt{C} - I$ is Hilbert-Schmidt (trace class).
2. If $C$ is invertible then $C^{-1} - I$ is Hilbert-Schmidt (trace class).

3.24 Lemma. Let $S$ be a real linear, positive and invertible operator on $\mathcal{H}$. Then $L$ and $\mathcal{P}$ as in Corollary 2.27 can be chosen such that

1. If $S - I$ is Hilbert-Schmidt then $L^T L - I$ is Hilbert Schmidt, i.e $L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

2. If $(\sqrt{S} JS\sqrt{S})^*(\sqrt{S} JS\sqrt{S}) - I$ is trace class then $\mathcal{P} - I$ is a trace class operator on $\mathcal{K}$.

Proof. \[1\] It can be seen from the proof of Williamson’s normal form in [BJ18] that $L$ can be chosen as $L = \mathcal{P}^{-1/2} \Gamma^* S^{1/2}$, where $\Gamma_0: K \oplus K \to H \oplus H$ is an orthogonal transformation such that the skew symmetric operator

$$B_0 := S_0^{1/2} J_0 S_0^{1/2} = \Gamma_0^* \begin{bmatrix} 0 & -P \\ P & 0 \end{bmatrix} \Gamma_0.$$ \hfill (3.25)

and $\mathcal{P}_0 = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$. Then

$$L^T L = S^{1/2} \Gamma \mathcal{P}^{-1} \Gamma^* S^{1/2}. \hfill (3.26)$$

But

$$\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 0 & -P \\ P & 0 \end{bmatrix} = \begin{bmatrix} P^2 & 0 \\ 0 & P^2 \end{bmatrix}, \hfill (3.27)$$

therefore if we write $\mathcal{P}_0 = \begin{bmatrix} 0 & -P \\ P & 0 \end{bmatrix}$, we see that

$$\mathcal{P}^{-1} = (\sqrt{\mathcal{P}^T \mathcal{P}})^{-1} \hfill (3.28)$$

Since $\Gamma$ is orthogonal, by (3.25) and (3.28) we get $(\sqrt{B^T B})^{-1} = \Gamma \mathcal{P}^{-1} \Gamma^*$. Now by (3.26) we get

$$L^T L = S^{1/2} (\sqrt{B^T B})^{-1} S^{1/2}. \hfill (3.29)$$

We have $S - I$ is Hilbert-Schmidt. Therefore, so is $J^* S J - I$. Hence $S^{1/2} J^* S J S^{1/2} = S$ is Hilbert-Schmidt. By adding and substracting $I$ and using the fact the $S - I$ is Hilbert-schmidt we get $S^{1/2} J^* S J S^{1/2} - I$ is also so. In otherwords, we just got $B^* B - I$ is Hilbert-Schmidt. Now by Lemma 3.23 we get $(\sqrt{B^T B})^{-1} - I$ is Hilbert-Schmidt. This along with (3.29) finally allows us to conclude that $L^T L - I$ is Hilbert-Schmidt.

2 By keeping the notations above and using Lemma 3.23 we have $(\sqrt{B^T B})^{-1} - I$ is trace class and thus $S^{1/2} (\sqrt{B^T B})^{-1} S^{1/2} - S = L^T L - S$ is trace class. Since $S = L^T \mathcal{P} L$ we get $L^T (\mathcal{P} - I) L$ is trace class. Since $L$ is invertible we see that $\mathcal{P} - I$ is trace class.

3.25 Theorem. Let $S$ be a real linear invertible operator on $\mathcal{H}$ such that

1. $\hat{S} - i\hat{J} \geq 0$ on $\mathcal{H}$.

2. $S - I$ is Hilbert-Schmidt on $(\mathcal{H}, \text{Re } \langle \cdot, \cdot \rangle)$.

3. $(\sqrt{S} JS\sqrt{S})^*(\sqrt{S} JS\sqrt{S}) - I$ is trace class on $(\mathcal{H}, \text{Re } \langle \cdot, \cdot \rangle)$.

Then there exists a density matrix $\rho$ on $\Gamma_*(\mathcal{H})$ such that the quantum characteristic function $\hat{\rho}(z) = e^{-\frac{1}{2} \text{Re}(z:S)}$ i.e $\rho \in \mathcal{K}(\mathcal{H})$. 

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Proof. Since $\hat{S} - i\hat{J} \geq 0$, $S \geq 0$. Since $S$ is invertible we apply Williamson’s normal form to it. Thus there exists a Hilbert space $K$ and a symplectic transformation $L : \mathcal{H} \to K$ such that $S = L^*\mathcal{P}L$ (Corollary 2.27). By Lemma 3.24 $\mathcal{P} - I$ is trace class and $L^*L - I$ is Hilbert-Schmidt. Now by Theorem 2.22 there exists a unique unitary operator $\Gamma_s(L) : \Gamma_s(\mathcal{H}) \to \Gamma_s(K)$ such that

$$\Gamma_s(L)W(u)\Gamma_s(L)^* = W(Lu)$$

(3.30)

It is understood that $W(\cdot)$ on either side of the above equality are considered in the corresponding Fock spaces.

Since $\mathcal{P} - I$ is trace class and positive, by Theorem 3.22 there exists a density matrix $\rho_{\mathcal{P}}$ such that $\hat{\rho}_{\mathcal{P}}(y) = e^{-\frac{1}{2}Re(y^*y)}$ for every $y \in K$. Define

$$\rho = \Gamma_s(L)^*\rho_{\mathcal{P}}\Gamma_s(L)$$

(3.31)

Claim. $\hat{\rho}(z) = e^{-\frac{1}{2}Re(z^*z)}$ for every $z \in \mathcal{H}$.

Proof (of Claim). By Proposition 3.3 we have

$$\hat{\rho}(z) = \hat{\rho}_{\mathcal{P}}(Lz)$$

$$= e^{-\frac{1}{2}(Lz^*\mathcal{P}Lz)}$$

$$= e^{-\frac{1}{2}Re(Lz^*\mathcal{P}Lz)}$$

$$= e^{-\frac{1}{2}Re(z^*L^*\mathcal{P}Lz)}$$

$$= e^{-\frac{1}{2}Re(z,Sz)}.$$

\[\square\]

3.26 Corollary. Let $S$ be a complex linear positive and invertible operator on $\mathcal{H}$, then $S \in K(\mathcal{H})$ if and only if $\hat{S} - i\hat{J} \geq 0$ and $S - I$ is trace class.

Note that by Theorem 3.17, the condition $\hat{S} - i\hat{J} \geq 0$ above is equivalent to the existence of a primary quasifree state $\phi$ on $CCR(\mathcal{H},\sigma)$ such that $\phi(W(z)) = e^{-\frac{1}{2}Re(z^*z)}$. Further, by Theorem 2.16 along with 2 and 3 we infer that this $\phi$ is quasiequivalent to the vacuum state. So a restatement of Theorem 3.25 is

3.27 Theorem. A primary quasifree state which is quasi equivalent to the vacuum state is a normal state in the GNS representation corresponding to the vacuum state (§ 2.2.3) and the extension of this normal state to $\mathcal{B}(\Gamma_s(\mathcal{H}))$ can be explicitly constructed.

Combining Theorem 3.12 and Theorem 3.25 we have

3.28 Theorem. Let $S$ be a real linear, bounded, symmetric and invertible operator on $\mathcal{H}$, then $S$ is the covariance operator of a quantum Gaussian state if and only if the following holds

1. $\hat{S} - i\hat{J} \geq 0$ on $\mathcal{H}$.
2. $S - I$ is Hilbert-Schmidt on $(\mathcal{H}, Re(\cdot, \cdot))$.
3. $(\sqrt{S}J\sqrt{S})^*(\sqrt{S}J\sqrt{S}) - I$ is trace class on $(\mathcal{H}, Re(\cdot, \cdot))$. 22
3.29 Corollary. Let $S$ be a complex linear, selfadjoint and invertible operator on $\mathcal{H}$. Then $S$ is the covariance operator of a quantum Gaussian state on $\Gamma_s(\mathcal{H})$ if and only if $\hat{S} - i\hat{J} \geq 0$ and $S - I$ is trace class.

3.30 Corollary. Let $S \geq I$ be real linear then $S$ is the covariance operator of a quantum Gaussian state on $\Gamma_s(\mathcal{H})$ if and only if $S - I$ is trace class.

3.31 Theorem. There exists a quantum Gaussian state $\rho$ with covariance matrix $S$ if and only if $\rho|_{CCR(\mathcal{H},\sigma)}$ is a primary quasifree state $\phi_A$ quasiequivalent to the vacuum state $\phi_{-J}$ on $CCR(\mathcal{H},\sigma)$, where $A = -JS$.

Proof. If $S$ is a covariance operator, by Theorem 3.17 and proof of 2 and 3 in Theorem 3.12 we see the existence of the required quasifree state. It can be seen from Theorem 2.16, the same proofs mentioned above and Theorem 3.28 that existence of a quasifree state as in the statement give rise to the existence of required quantum Gaussian state.

Note. By the Theorem 3.31 it is established that quantum Gaussian states are nothing but a subclass of quasifree states on $CCR(\mathcal{H},\sigma)$, more clearly, quantum Gaussian states are precisely those quasifree states on $CCR(\mathcal{H},\sigma)$ which are quasi equivalent to the vacuum state $\phi_{-J}$.

4 Convexity Properties of Covariance Operators

The previous section described and characterized infinite mode Gaussian states. Now we extend some beautiful symmetry properties of Gaussian states proved by Parthasarathy ([Par10], [Par13]) in the finite mode case to this setting. The methods are similar.

In the following $\mathcal{H}, \mathcal{K}$ are complex Hilbert spaces, $\mathcal{K}(\mathcal{H})$ denotes the collection of covariance operators for Gaussian states on $\Gamma_s(\mathcal{H})$ and $S(\mathcal{H}, \mathcal{K})$ are Shale operators from $\mathcal{H}$ to $\mathcal{K}$.

4.1 Proposition. Consider two mean zero Gaussian states

$$\rho_i = \rho_g(0, S_i), i = 1, 2$$

on $\Gamma_s(\mathcal{H})$. For $\theta \in \mathbb{R}$, let $U_\theta$ be the unitary operator $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. Then

$$\text{Tr}_2 (\Gamma_s(U_\theta)(\rho_1 \otimes \rho_2)\Gamma_s(U_\theta)^*) = \rho_g(0, (\cos^2 \theta)S_1 + (\sin^2 \theta)S_2)$$

where $\text{Tr}_2$ denotes the relative trace over the second factor of $\Gamma_s(\mathcal{H}) \otimes \Gamma_s(\mathcal{H})$ and $\Gamma_s(U_\theta)$ is considered under the identification between $\Gamma_s(\mathcal{H} \oplus \mathcal{H})$ and $\Gamma_s(\mathcal{H}) \otimes \Gamma_s(\mathcal{H})$.

Proof. Easy consequence of Proposition 3.2 and Proposition 3.8 and the definition of Gaussian states. We note that $(\cos^2 \theta)S_1 + (\sin^2 \theta)S_2$ is a convex combination of two positive and invertible operators and hence it is positive and invertible.

As a consequence we have the following result.

4.2 Corollary. $\mathcal{K}(\mathcal{H})$ is a convex set.
4.3 Lemma. Let $P \geq I$ be a positive operator then there exists invertible positive operators $P_1$ and $P_2$ such that

$$P = \frac{1}{2}(P_1 + P_2) = \frac{1}{2}(P_1^{-1} + P_2^{-1})$$ (4.1)

Proof. Take $P_1 = P + \sqrt{P^2 - I}$ and $P_2 = P - \sqrt{P^2 - I}$. Then $P_1P_2 = P_2P_1 = I$ and (4.1) is satisfied.

4.4 Lemma. Let $S = H + iH$ and $P \in B(H)$ be such that $P - I$ is positive and trace class, further let $P_0 = [P_0^* P_0]$ on $H \oplus H$ (§ 2.3). Then $P = \frac{1}{2}(P_1 + P_2)$, for some $P_j \in S(H, K)$, and $P_j \geq 0, j = 1, 2$.

Proof. Take $P_1 = P + \sqrt{P^2 - I}$ and $P_2 = P - \sqrt{P^2 - I}$, then by (4.1)

$$P_0 = \frac{1}{2} \left\{ \begin{bmatrix} P_1 & 0 \\ 0 & P_1^{-1} \end{bmatrix} + \begin{bmatrix} P_2 & 0 \\ 0 & P_2^{-1} \end{bmatrix} \right\}$$

Define $P_j$ such that $P_j(x + iy) = P_jx + P_j^{-1}y, \forall x, y \in H, j = 1, 2$. Then $P_j$ is symplectic and positive. Since $P - I$ is trace class, $P_j - I$ is trace class and hence $P_j^2 - I$ is Hilbert-Schmidt, therefore $P_j \in S(H, K)$ for $j = 1, 2$.

4.5 Theorem. $S \in \mathcal{K}(H)$ if and only if

$$S = \frac{1}{2}(N^*N + M^*M)$$ (4.2)

for some $N, M \in S(H, K)$, for some Hilbert space $K$. Further, $S$ is an extreme point of $\mathcal{K}(H)$ if and only if $S = N^*N$ for some $N \in S(H, K)$.

Proof. Note that if $N \in S(H, K)$ for some $K$ then $N^*N$ is a covariance operator by taking $S = I$ and $L = N$ in Proposition 3.8. Therefore by convexity of $\mathcal{K}(H)$, if $S$ is on the form (4.2) then $S \in \mathcal{K}(H)$.

Now let $S \in \mathcal{K}(H)$, let $S = L^*P_L$ be the Williamson’s normal form as in Corollary 2.27. Then by Lemma 3.23 and Lemma 3.10 we have $L \in S(H, K)$ and $P - I$ is trace class and positive. By Corollary 2.27 we have $P_0 = [P_0^* P_0]$. By Lemma 4.4 $P = \frac{1}{2}(P_1 + P_2)$ with $P_j \geq 0, j = 1, 2$. Therefore we have

$$S = \frac{1}{2}L^*(P_1 + P_2)L.$$’

By taking $N = P_1^{1/2}L$ and $M = P_2^{1/2}L$ we get (4.2). An easy computation shows $N, M \in S(H, K)$.

Proof of second part of the Theorem goes in similar lines to the proof of the similar statement in the finite mode case, Theorem 3 in Par13. □

4.6 Remark. (4.2) is called the circle property because every point is the midpoint of a line joining two extreme points.

4.7 Corollary. Let $S_1, S_2$ be extreme points of $\mathcal{K}(H)$ such that $S_1 \geq S_2$. Then $S_1 = S_2$. 

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Proof. By Theorem 4.5 let $S_1 = L_1^*L_1$ and $S_2 = L_2^*L_2$ for some $L_1 \in \mathcal{S}(\mathcal{H}, \mathcal{K}_1)$ and $L_2 \in \mathcal{S}(\mathcal{H}, \mathcal{K}_2)$ but without lose of generality we assume that $\mathcal{K}_1 = \mathcal{K}_2$. This can be done by going through the proof of previous theorem and identifying $\mathcal{K}_1$ and $\mathcal{K}_2$ in such a way that the Williamson’s normal form of both $S_1$ and $S_2$ are obtained in the same Hilbert space $\mathcal{K}$ (by a possibly different real and complex decomposition of $\mathcal{K}$). $L_1^*L_1 \geq L_2^*L_2$ implies that the symplectic transformation $M := L_2L_1^{-1}$ (well defined because $\mathcal{K}_1 = \mathcal{K}_2$) has the property $M^*M \leq I$. But since $M^*M$ is a positive symplectic automorphism $M^*M = VTV^*$ for some unitary $V$, where $\mathcal{H} = H + iH$ and $T(x + iy) = Ax + iA^{-1}y$ for some positive invertible operator $A$ on $H$. This can be understood from the the proof of Proposition 20.1 in [Par12]. But such a $T \leq I$ if and only if $A = I$. This proves $M^*M = I$. But this implies $L_2^*L_2 = L_1^*L_1$ from the definition of $M$.

5 Structure of Quantum Gaussian States

If $S$ is a Gaussian covariance matrix it satisfies the properties listed in Theorem 5.28 then by combining Lemma 5.10 and Lemma 5.24 we get a Williamson’s normal form (Corollary 2.27), $S = L^*PL$ such that $P - I$ is positive and trace class. By applying spectral theorem to $P - I$ we see that there exists a unitary $U$ such that $P = U^*DU$, where $D$ is diagonal and positive. Recall that a unitary is already symplectic. Therefore, whenever $S$ is a covariance operator we can assume without loss of generality that the $P \in \mathcal{B}(\mathcal{K})$ occurring in the Williamson’s normal form is of the form $P = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ on a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, with $D = \text{Diag}(d_1, d_2, \ldots)$, $d_1 \geq d_2 \geq \cdots > 1$. But now we fixed a basis of $\mathcal{K}$, therefore by considering the identifican of $\Gamma_s(\mathbb{C})$ with $L^2(\mathbb{R})$ (Lebesgue measure), where $e(z) \in \Gamma_s(\mathbb{C})$ is identified with the $L^2$-function $x \mapsto (2\pi)^{-1/4} \exp\{-4^{-1}x^2 + zx - 2^{-1}z^2\}$ (refer Example 19.8 and Exercise 20.20 in [Par12]), we can assume without loss of generality that $\Gamma_s(K) = \otimes_j L^2(\mathbb{R})$, with respect to the stabilizing vector $e(0)$. It may be noted that these identifications does not alter the existence of $\Gamma_s(L)$.

5.1 Theorem. Let $\rho_g(w, S)$ be a Gaussian state in $\Gamma_s(\mathcal{H})$. Let $S = L^*PL$ be a Williamson’s normal form of $S$, where $L : \mathcal{H} \to \mathcal{K}$, with $L^*L - I$ is Hilbert-Schmidt and $P = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, on a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, with $D = \text{Diag}(d_1, d_2, \ldots)$, $d_1 \geq d_2 \geq \cdots > 1$, $d_j = \coth(\frac{a_j}{2})$, $\forall j$. Then

$$\rho_g(w, S) = W(\frac{-i}{2}w)^*\Gamma_s(L)^*[\otimes_j(1 - e^{-s_j})e^{-s_ja_j^*} \otimes \rho_0]\Gamma_s(L)W(\frac{i}{2}w).$$

(5.1)

where $\rho_0 = |e(0)|\langle e(0)|$ is the the vacuum state on $\Gamma_s(\mathcal{K}_2)$.

Proof. By Proposition 3.6 $\rho_g(w, S) = W(\frac{-i}{2}w)^{-1}\rho_g(0, S)W(\frac{i}{2}w)$. Since $S = L^*PL$, by Proposition 3.8 $\rho_g(0, S) = \Gamma_s(L)^*\rho_g(0, P)\Gamma_s(L)$. Since $P = D \oplus I$, by Proposition 3.7 $\rho_g(0, P) = \rho_g(0, D) \otimes \rho_g(0, I)$. But $\rho_g(0, D) = \otimes_j(1 - e^{-s_j})e^{-s_ja_j^*}$ since both on left and right hand sides have same quantum characteristic function by proof of Proposition 3.21 and it is obvious that $\rho_g(0, I) = \rho_0$.

5.2 Corollary. If $\{e_j\}$ is a basis of $\mathcal{H}$, consider $\Gamma_s(\mathcal{H}) = \otimes_j L^2(\mathbb{R})$ then the wave function of a general pure quantum Gaussian state is of the form

$$|\psi\rangle = W(\alpha)^{-1}\Gamma_s(U)(\otimes_j |e_{\lambda_j}\rangle)$$

(5.2)
where $e_\lambda \in L^2(\mathbb{R})$ and
\[
e_\lambda(x) = (2\pi)^{-1/4}\lambda^{-1/2}\exp\{-4^{-1}\lambda^{-2}x^2\}, \quad x \in \mathbb{R}, \lambda > 0
\]
$\alpha \in \mathcal{H}$, $U$ is a unitary operator on $\mathcal{H}$, $\Gamma_s(U)$ is the second quantization unitary operator associated with $U$ and $\lambda_j$, $j \in \mathbb{N}$ are positive scalars.

\textbf{Proof.} The proof is essentially similar to the proof of Corollary 2 in [Par13] because of Theorem 5.1 and Proposition 2.21. \hfill \Box

Now we show that all Gaussian states can be purified to get pure Gaussian states.

\textbf{5.3 Theorem (Purification).} Let $\rho$ be a mixed Gaussian state in $\Gamma_s(\mathcal{H})$. Then there exists a pure Gaussian state $|\psi\rangle$ in $\Gamma_s(\mathcal{H}) \otimes \Gamma_s(\mathcal{H})$ such that
\[
\rho = \text{Tr}_2 U |\psi\rangle \langle \psi| U^*
\]
where $U$ is a unitary and $\text{Tr}_2$ is the relative trace over the second factor.

\textbf{Proof.} Proof is same as that of Theorem 5 in [Par13]. \hfill \Box

6 Symmetry group of Gaussian states

Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space and let $G$ denote the set of all Gaussian states on $\Gamma_s(\mathcal{H})$.

\textbf{6.1 Definition.} A unitary operator $U$ on $\Gamma_s(\mathcal{H})$ is called a \textit{Gaussian symmetry} if $U \rho U^* \in G$ for every $\rho \in G$.

We use $\mathbb{Z}_+$ to denote the set $\{0, 1, 2, 3, \ldots\}$ and take $\mathbb{Z}_+^\infty := \{(k_1, k_2, \ldots, k_n, 0, 0, \ldots)^\tau| k_j \in \mathbb{Z}_+, j, n \in \mathbb{N}\}$. Let $\{e_j\}_{j \in \mathbb{N}}$ denote the standard orthonormal basis for $\ell^2(\mathbb{N})$, where $e_j$ is the column vector with 1 at the $j^{th}$ position and zero elsewhere. An infinite order matrix $A$ is said to be a permutation matrix if $A$ is the matrix with respect to the standard orthonormal basis, corresponding to a unitary operator which maps $\{e_j\}$ to itself.

\textbf{6.2 Lemma.} Let $\{s_j\}_{j \in \mathbb{N}}$ and $\{t_j\}_{j \in \mathbb{N}}$ be two sets consisting of positive numbers such that
\[
\left\{\sum_{j=1}^n s_j k_j |k_j \in \mathbb{Z}_+ \forall j, n \in \mathbb{N}\right\} = \left\{\sum_{j=1}^n t_j k_j |k_j \in \mathbb{Z}_+ \forall j, n \in \mathbb{N}\right\}.
\] (6.1)

If $\{s_j\}$ and $\{t_j\}$ are linearly independent over the field $\mathbb{Q}$, then $\{s_j\} = \{t_j\}$.

\textbf{Proof.} Consider $s = (s_1, s_2, s_3, \ldots)^\tau$ and $t = (t_1, t_2, t_3, \ldots)^\tau$ they need not be $\ell^2$ vectors therefore we consider them as formal vectors only. Set $A = ((a_{ij}))$ and $B = ((b_{ij}))$. Note that, by construction, each row of $A, B, AB$ and $BA$ has only finitely many non zero entries. Clearly $As = t, Bt = s$ and hence $BAs = s$. Since each row of $BA$ has only finitely many non zero entries, by the rational linear independence of $\{s_j\}$, we get $BA = I$. Similarly since $ABt = t$, we get $AB = I$. Now $BA = I$ implies that $\sum_{j=1}^\infty b_{ij}a_{j1} = 1$ and $\sum_{j=1}^\infty b_{ij}a_{j1} = 0$ for all $i \neq 1$. Then there

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exists $k_1 \in \mathbb{N}$ such that $b_{1k_1} = a_{k_11} = 1$. Since $b_{ik_1}a_{k_11} = 0$ we have $b_{ik_1} = 0$ for all $i \neq 1$. Thus $k_1$-th column of $B$ is $e_1$. Similarly, if $k \neq 1$, since $\sum_{j=1}^{\infty} b_{1j}a_{jk} = 0$ we get $a_{jk} = 0, \forall j \neq 1$ or row $k_1$ of $A$ is $e_1$.

Suppose $k_1, k_2, \ldots k_{n-1} \in \mathbb{N}$ are obtained such that $k_i \neq k_j$ for $i \neq j$, column $k_i$ of $B$ (and row $k_i$ of $A$) is $e_i$, $1 \leq i \leq n - 1$. We prove that there exist $k_n \in \mathbb{N}$ such that $k_n \neq k_i$ for $i < n$ and column $k_n$ of $B$ (and row $k_n$ of $A$) is $e_n$. Since $\sum_{j=1}^{\infty} b_{nj}a_{jn} = 1$ there exists $k_n \in \mathbb{N}$ such that $b_{nk_n} = a_{k_n1} = 1$. If $k_n = k_i$ for some $i < n$ then column $k_i$ of $B$ cannot be $e_i$ thus $k_n \neq k_i$ for $i < n$. Now a similar argument as above concludes that column $k_n$ of $B$ (and row $k_n$ of $A$) is $e_n$. Thus every $e_n$ occurs at least once in the columns of $B$ and rows of $A$. Similarly, by considering $AB = I$ we see that every $e_n$ occurs at least once in the columns of $A$ and rows of $B$. Now to see that $A$ and $B$ are permutation matrices, first note that because $AB = BA = I$, none of the rows (or columns) of $B$ or $A$ can be zero. Further if there exists a row of $B$ where there are two non zero entries, say at the positions $l$ and $m$ then because of the presence of $e_l$ and $e_m$ in the columns of $A$ we see that the product $BA$ cannot be $I$. Continuing similar arguments it is seen that $A$ and $B$ are permutation matrices.

Let us fix some notations and conventions before we proceed further. Recall from Exercise 20.18(b) in [Par12] that on $\Gamma_s(\mathbb{C})$, the spectrum of the number operator $\sigma(a^\dagger a) = \mathbb{Z}_+$, where each $k \in \mathbb{Z}_+$ is an eigenvalue with multiplicity one. Let us denote by $|k\rangle$ the eigenvector corresponding to the eigenvalue $k$. It is also true that $|0\rangle = e(0)$, the vacuum vector. Further, $\{|k\rangle | k \in \mathbb{Z}_+\}$ forms an orthonormal basis for $\Gamma_s(\mathbb{C})$. Now consider $\Gamma_s(\mathcal{H}) = \bigotimes_{j=1}^{\infty} \Gamma_s(\mathbb{C}e_j)$, where $\{e_j\}$ is an orthonormal basis for $\mathcal{H}$ (recall Proposition 2.3). If $E$ denote the orthogonal projection of $\Gamma_{fr}(\mathcal{H})$ (the free Fock space which we didn’t define but a standard object in the literature) onto $\Gamma_s(\mathcal{H})$, we define

$$|k\rangle = E(|k_1\rangle \otimes |k_2\rangle \otimes \cdots \otimes |k_N\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots) = |k_1\rangle |k_2\rangle \cdots |k_N\rangle$$

(6.2) corresponding to an element $k = (k_1, k_2, k_3, \ldots)^{\tau} \in \mathbb{Z}_+^\infty$ where $k_j = 0, \forall j > N$. It can be seen that $\{|k\rangle | k \in \mathbb{Z}_+^\infty\}$ forms an orthonormal basis for $\Gamma_s(\mathcal{H})$. We have

$$(I \otimes I \otimes \cdots \otimes I \otimes a_j^\dagger a_j \otimes I \otimes I \otimes \cdots) (|k\rangle) = \begin{cases} k_j |k\rangle, & \text{if } j \leq N \\ 0, & \text{otherwise} \end{cases}$$

(6.3) where $a_j^\dagger a_j$ is the number operator on $\Gamma_s(\mathbb{C}e_j), j \in \mathbb{N}$.

Consider $\mathcal{H} = \bigoplus_{j=1}^{\infty} \mathcal{H}_j$ where $\mathcal{H}_j$’s are all one dimensional. For a sequence of positive numbers $\{s_j\}_{j \in \mathbb{N}}$ such that $d_j = \coth(\frac{s_j}{2}) > 1$ and $\sum_{j}(d_j - 1)$ is finite, we know from Theorem 5.1 that, there exists a Gaussian state $\rho_s = \Pi_{j=1}^{\infty}(1 - e^{-s_j}) \otimes_{j=1}^{\infty} e^{-s_j a_j^\dagger a_j} \in \mathcal{B}(\Gamma_s(\mathcal{H}))$. Then we have the

6.3 Lemma. The spectrum of the Gaussian state $\rho_s$ is the closure of the set,

$$\sigma_{\rho} (\rho_s) = \left\{ p e^{-\sum_{j=1}^{N} s_j k_j} | k_j \in \mathbb{Z}_+, N \in \mathbb{N} \right\},$$

(6.4)
where \( p := \Pi_{j=1}^{\infty}(1 - e^{-s_j}) \). Further, if \( \{s_j\}_{j \in \mathbb{N}} \) is a sequence of (distinct) irrational numbers which are linearly independent over the field \( \mathbb{Q} \) then each number \( p e^{-\sum_{j=1}^{N} s_j k_j} \) is an eigenvalue with multiplicity one.

Proof. Without loss of generality we assume \( \mathcal{H} = \ell^2(\mathbb{N}) \) and \( \mathcal{H}_j = \mathbb{C}e_j \), where \( \{e_j\} \) is the standard orthonormal basis of \( \ell^2(\mathbb{N}) \). We have \( \otimes_{j=1}^{\infty} e^{-s_j a_j} = \text{s-lim}_{N \to \infty} \sum_{j=1}^{N} e^{-s_j a_j} \otimes I \otimes I \otimes \cdots \). Therefore, \( \otimes_{j=1}^{\infty} e^{-s_j a_j} \sum_{j=1}^{N} e(0) = e(0) \otimes e(0) \otimes e(0) \otimes \cdots = e(s_j a_j) e(0) \otimes e(0) \otimes \cdots, \forall u \in \mathbb{C}^N \). Thus \( \Gamma_s(\mathbb{C}^N) \) is a reducing subspace for \( \rho_s \) and \( \rho_s|_{\Gamma_s(\mathbb{C}^N)} = \otimes_{j=1}^{\infty} e^{-s_j a_j}, \forall N \). Therefore,

\[
\rho_s(|k\rangle) = (p \otimes_{j=1}^{\infty} e^{-s_j a_j}) |k\rangle = p e^{-\sum_{j=1}^{N} s_j k_j} |k\rangle, \forall k \in \mathbb{Z}_+^\infty. \quad (6.5)
\]

Since \( \{|k\rangle | k \in \mathbb{Z}_+^\infty\} \) forms a complete orthonormal basis for \( \Gamma_s(\mathcal{H}) \), \( \{p e^{-\sum_{j=1}^{N} s_j k_j} |k_j \in \mathbb{Z}_+, N \in \mathbb{N}\} \) is the complete set of eigenvalues for \( \rho_s \). If \( \{s_j\} \) is linearly independent over \( \mathbb{Q} \), then we see that the eigenvalues corresponding to \( |k_1\rangle \neq |k_2\rangle \) are not same. Thus the multiplicity of each of these eigenvalues is one. \( \square \)

6.4 Theorem. Let \( \rho_s \) be as in Lemma 6.3 where \( \{s_j\}_{j \in \mathbb{N}} \) is a sequence of (distinct) irrational numbers which are linearly independent over the field \( \mathbb{Q} \). Then a unitary operator \( U \) in \( \Gamma_s(\mathcal{H}) \) is such that \( U \rho_s U^* \) is a Gaussian state if and only if for some \( \alpha \in \mathcal{H}, L \in \mathcal{S}(\mathcal{H}) \) and a complex valued function \( \beta \) of modulus one on \( \mathbb{Z}_+^\infty \)

\[
U = W(\alpha) \Gamma_s(L) \beta(a_1^{\dagger} a_1, a_2^{\dagger} a_2, \ldots),
\]

where \( \beta(a_1^{\dagger} a_1, a_2^{\dagger} a_2, \ldots) \) is the unique unitary which satisfies

\[
\beta(a_1^{\dagger} a_1, a_2^{\dagger} a_2, \ldots) |k\rangle = \beta(k) |k\rangle, \forall k \in \mathbb{Z}_+^\infty.
\]

Proof. Since \( \beta(a_1^{\dagger} a_1, a_2^{\dagger} a_2, \ldots) \) commutes with \( \rho_s \) the sufficiency is immediate from Proposition 5.8 and Proposition 5.6. To prove the necessity let

\[
U \rho_s U^* = \rho_g(w, \mathcal{F}). \quad (6.6)
\]

The eigenvalues and multiplicities of \( \rho_s \) and \( U \rho_s U^* \) are same. Therefore by Theorem 5.1 there exists \( z \in \mathcal{H}, \) a Hilbert space \( \mathcal{K}, M \in \mathcal{S}(\mathcal{H}, \mathcal{K}) \) and \( \rho_t := \Pi_{j=1}^{\infty}(1 - e^{-t_j}) \otimes_{j=1}^{\infty} e^{-t_j a_j} \in \mathcal{B}(\Gamma_s(\mathcal{K})) \) such that

\[
U \rho_s U^* = W(z)^* \Gamma_s(M)^* \rho_t \Gamma_s(M) W(z). \quad (6.7)
\]

By Lemma 6.3 \( \rho_s \) has a complete orthonormal eigenbasis with corresponding eigenvalues distinct. By 6.7 \( \rho_s \) and \( \rho_t \) are unitarily equivalent and thus their eigenvalues and multiplicities are same. Therefore by applying Lemma 6.3 to \( \rho_t, \Pi_{j=1}^{\infty}(1 - e^{-t_j}) = p \) (since \( p \) is the maximum eigen value of \( \rho_s \)) and \( \rho_t \) has a set of distinct eigenvalues \( p e^{-\sum_{j=1}^{N} t_j k_j} \) corresponding to the eigenvectors \( |k\rangle, k = (k_1, k_2, \ldots, k_N, 0, 0, \ldots)^T \in \mathbb{Z}_+^\infty, N \in \mathbb{N} \).
Claim. The sequence \( \{t_j\}_{j \in \mathbb{N}} \) consists of (distinct) numbers which are linearly independent over the field \( \mathbb{Q} \).

Proof (of Claim). If \( t_i = t_k \) for some \( i \neq k \) then it is possible to choose distinct \( k, k' \in \mathbb{Z}_+^\infty \) such that the eigenvalues of \( \rho_t \) corresponding to \( |k\rangle \) and \( |k'\rangle \) are same. This will imply that the corresponding eigenspace is at least two dimensional which is not possible. To see the rational independence note that for any two finite subsets \( I, J \subset \mathbb{N}, \sum_{j \in I} t_j k_j \neq \sum_{j \in J} t_j k'_j \) where \( k_j, k'_j \in \mathbb{Z}_+, \forall j \). Now if

\[
\sum_{j=1}^{N} t_j q_j = 0 \tag{6.8}
\]

for a finite collection of rational numbers \( q_j \)'s, since \( t_j > 0, \forall j \) then there must be negative rational numbers in the set \( \{q_1, q_2, \ldots, q_N\} \) (unless \( q_j = 0, \forall j \)). Then \( 6.8 \) can be written in the form \( \sum_{j \in I} t_j k_j = \sum_{j \in J} t_j k'_j \) for two finite sets \( I, J \), which is not possible. Thus the claim is proved.

We have \( \{pe^{-\sum_{j=1}^{N} s_j k_j} | k_j \in \mathbb{Z}_+, N \in \mathbb{N} \} = \{pe^{-\sum_{j=1}^{N} t_j k_j} | k_j \in \mathbb{Z}_+, N \in \mathbb{N} \} \). Therefore \( \{\sum_{j=1}^{N} s_j k_j | k_j \in \mathbb{Z}_+, \forall j, n \in \mathbb{N} \} = \{\sum_{j=1}^{N} t_j k_j | k_j \in \mathbb{Z}_+, \forall j, n \in \mathbb{N} \} \). Now by the Lemma \( 6.2 \) \( \{s_j\} = \{t_j\} \) as sets and there is an infinite permutation matrix \( A \) such that \( As = t \), where we consider \( s = (s_1, s_2, s_3, \ldots)^\tau \) and \( t = (t_1, t_2, t_3, \ldots)^\tau \) as formal vectors (they need not be \( \ell^2 \) vectors) and we use matrix multiplication as a notation to mean the corresponding (finite) linear combinations. We do the same in rest of this proof also.

By \( 6.7 \) there exists a unitary \( V \) such that

\[
V \rho_a V^* = \rho_t. \tag{6.9}
\]

where \( V = \Gamma_a(M)W(z)U \). Let \( k = (k_1, k_2, \ldots, k_N, 0, 0, \ldots)^\tau \in \mathbb{Z}_+^\infty \) be arbitrary, by \( 6.9 \) if \( |k\rangle \) is an eigenvector for \( \rho_a \) then \( V |k\rangle \) is an eigenvector for \( \rho_t \) with the same eigenvalue. Therefore, \( V |k\rangle \) is an eigenvector for \( \rho_t \) with eigenvalue \( pe^{-\sum_{j=1}^{N} s_j k_j} = pe^{-\sum_{j=1}^{N} s^\tau k} \). If we write \( B := A^\tau \) then \( Bt = s \) and

\[
pe^{-\sum_{j=1}^{N} s^\tau k} = pe^{-\sum_{j=1}^{N} t^\tau B^\tau k} = pe^{-\sum_{j=1}^{N} t^\tau Ak}. \tag{6.10}
\]

But \( pe^{-\sum_{j=1}^{N} t^\tau Ak} \) is the eigenvalue of \( \rho_t \) corresponding to the eigenvector \( |Ak\rangle \) where \( A \) is now considered as a unitary operator between \( \mathcal{H} \) and \( \mathcal{K} \) in the following way. Let \( \{f_1, f_2, \ldots\} \) be the orthonormal basis of \( \mathcal{K} \) with respect to which \( \mathcal{K} = \bigoplus_{j=1}^{\infty} \mathbb{C} f_j \) and \( \rho_t \) is constructed on \( \otimes_{j=1}^{\infty} \Gamma_a(\mathbb{C} f_j) \). Corresponding to a \( m = (m_1, m_2, \ldots, m_N, 0, 0, \ldots)^\tau \in \mathbb{Z}_+^\infty \mapsto \bigoplus_{j=1}^{\infty} \mathbb{C} f_j \) we have \( |m\rangle \) as an eigenvector of \( \rho_t \) by Lemma \( 6.3 \). Therefore for
each \( k \in \mathbb{Z}^\infty_+ \rightarrow \ell^2(\mathbb{N}) \), \( A_k \) can be considered as a vector in \( \bigoplus_{j=1}^\infty \mathbb{C}f_j \). Thus \( A \) is seen as a unitary between \( \mathcal{H} \) and \( \mathcal{K} \).

Since the eigenspace is one dimensional we see that there exists a complex number \( \beta(k) \) of unit modulus such that

\[
V |k\rangle = \beta(k) A_k
\]

\[
= \Gamma_s(A) \beta(k) |k\rangle
\]

\[
= \Gamma_s(A) \beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots) |k\rangle
\]

where \( \Gamma_s(A) : \Gamma_s(\mathcal{H}) \rightarrow \Gamma_s(\mathcal{K}) \) is the second quantization of the unitary operator corresponding to the unitary \( A \). Now by (6.9) \( U = W(z)^* \Gamma_s(M)^* \Gamma_s(A) \beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots) \).

Now the proof is complete due to Theorem 2.22.

Proof of the following Theorem follows similar to that of Theorem 7 in [Par13] but we will write it here to clear the technical difficulties encountered in the infinite mode case.

**6.5 Theorem.** A unitary operator \( U \in \mathcal{B}(\Gamma_s(\mathcal{H})) \) is a Gaussian symmetry if and only if

\[
U = \lambda W(\alpha) \Gamma_s(L)
\]

for some \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \), \( \alpha \in \mathcal{H} \), and \( L \) is a Shale operator (\( L \in S(\mathcal{H}) \)).

**Proof.** The sufficiency is immediate from Proposition 3.8 and Proposition 3.6. To prove the necessity, let us consider \( \mathcal{H} = \bigoplus_j \mathbb{C}e_j \) with respect to some orthonormal basis \( \{e_j\} \), if \( U \) is a Gaussian symmetry then in particular \( U \rho U^* \) is a Gaussian state for \( \rho_s \) as in Theorem 6.4. Therefore we can assume without loss of generality that \( U = \beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots) \). We will show that \( U = \Gamma_s(D) \) for some unitary operator \( D \) and this will prove the theorem because of (2) of Theorem 2.22.

Let \( \psi \in \Gamma_s(\mathcal{H}) \) be such that \( |\psi\rangle \langle \psi| \) is a pure Gaussian state. Then by assumption \( |U\psi\rangle \langle U\psi| \) is also a Gaussian state (pure state because it is obtained from the wave function \( |U\psi\rangle \)). We choose the coherent state (Example 1)

\[
\psi = e^{-\frac{1}{2}\|u\|^2} |e(u)\rangle = W(u) |e(0)\rangle,
\]

where \( u = (u_1, u_2, \ldots)^t \in \bigoplus_j \mathbb{C}e_j \). Now

\[
|U\psi\rangle = e^{-\frac{1}{2}\|u\|^2} \beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots) |e(u)\rangle
\]

(6.11)

By Corollary 5.2 there exists a unitary \( A \) and an \( \alpha \in \mathcal{H} \) such that

\[
|U\psi\rangle = W(\alpha) \Gamma_s(A) \otimes_j |e_{\chi_j}\rangle,
\]

(6.12)

where \( \chi(x) = (2\pi)^{-1/4}\lambda^{-1/2} \exp\{-4^{-1}\lambda^{-2}x^2\}, \ x \in \mathbb{R}, \lambda > 0 \) on \( L^2(\mathbb{R}) \).
We have by Proposition 2.5 $e(u) = \lim_{M \to \infty} \sum_{j=1}^{M} e(u_j) \otimes e(0) \otimes e(0) \otimes \cdots$. Let $k \in \mathbb{Z}_+^\infty$, with $k = (k_1,k_2,\ldots,k_n,0,0,\ldots)^r$. Since $\langle e(u_j)|e(0)\rangle = 1$,
\[
\langle e(u)|k \rangle = \lim_{M \to \infty} \sum_{j=1}^{M} \langle e(u_j)|e(0) \otimes e(0) \otimes e(0) \otimes \cdots|k \rangle
= \prod_{j=1}^{n} \langle e(u_j)|k_j \rangle
= \prod_{j=1}^{n} \left( \sum_{m=0}^{\infty} \frac{u_j^m}{\sqrt{m!}} |m\rangle \right) |k_j \rangle
= \prod_{j=1}^{n} \frac{u_j^{k_j}}{\sqrt{k_j!}} := \frac{u^k}{\sqrt{k!}},
\]
where the last line defines the multi index notation and also we take $0^0 = 1$. Therefore we write,
\[
e(u) = \sum_{k \in \mathbb{Z}_+^\infty} \frac{u^k}{\sqrt{k!}} |k\rangle. \quad (6.13)
\]
Now for each finite vector $z = (z_1, z_2, \ldots, z_N, 0, 0, \ldots)^T \in \oplus_j \mathbb{C}e_j$, $N \in \mathbb{N}$,
\[
e(z) = \sum_{m \in \mathbb{Z}_+^N} \frac{z^m}{\sqrt{k!}} |m\rangle, \quad (6.14)
\]
where $m \in \mathbb{Z}_+^N$ is considered as the vector $(m_1, m_2, \ldots, m_N, 0, 0, \ldots)^t \in \mathbb{Z}_+^\infty$ and $|m\rangle = |m_1\rangle |m_2\rangle \cdots |m_n\rangle \in \Gamma(H)$ as in the notation of (6.2).

We will evaluate the function $f(z) = \langle U\psi, e(z) \rangle$ using (6.11) and (6.12). From (6.11), (6.13), (6.14) and continuity of $\beta(a_1^t a_1, a_2^t a_2, \ldots)$ we have
\[
f(z) = e^{-\frac{1}{2}||u||^2} \left\langle \beta(a_1^t a_1, a_2^t a_2, \ldots) e(u), e(z) \right\rangle
= e^{-\frac{1}{2}||u||^2} \left\langle \beta(a_1^t a_1, a_2^t a_2, \ldots) \sum_{k \in \mathbb{Z}_+^\infty} \frac{u^k}{\sqrt{k!}} |k\rangle, \sum_{m \in \mathbb{Z}_+^N} \frac{z^m}{\sqrt{k!}} |m\rangle \right\rangle
= e^{-\frac{1}{2}||u||^2} \left\langle \sum_{k \in \mathbb{Z}_+^\infty} \frac{u^k}{\sqrt{k!}} \beta(k) |k\rangle, \sum_{m \in \mathbb{Z}_+^N} \frac{z^m}{\sqrt{k!}} |m\rangle \right\rangle
= e^{-\frac{1}{2}||u||^2} \sum_{k \in \mathbb{Z}_+^N} \frac{(\bar{u}z)^k}{k!}
\]
where $\bar{u} := (\bar{u}_1, \bar{u}_2, \ldots)^r$, $\bar{u}z := (\bar{u}_1z_1)(\bar{u}_2z_2)\cdots(\bar{u}_Nz_N)$ and the last line follows because the second term in the innerproduct of (6.15) has summation in $m \in \mathbb{Z}_+^N$.

Thus
\[
f(z) = e^{-\frac{1}{2}||u||^2} \sum_{k \in \mathbb{Z}_+^N} \frac{(\bar{u}_1z_1)^{k_1}(\bar{u}_2z_2)^{k_2} \cdots (\bar{u}_Nz_N)^{k_N}}{k_1!k_2!\cdots k_N!} \beta(k). \quad (6.16)
\]

Since $|\beta(k)| = 1$, from (6.16) we see that
\[
|f(z)| \leq \exp \left\{ -\frac{1}{2}||u||^2 + \sum_{j=1}^{N} |u_j||z_j| \right\}. \quad (6.17)
\]
From the definition of \(e(w)\) and \(e_\lambda\) in \(L^2(\mathbb{R})\) we have
\[
\langle e_\lambda, e(w) \rangle = \sqrt{\frac{2\lambda}{1 + \lambda^2}} \exp \frac{1}{2} \left( \frac{\lambda^2 - 1}{\lambda^2 + 1} \right) w^2, \lambda > 0, w \in \mathbb{C} \quad (6.18)
\]

Using (6.12),
\[
f(z) = \langle W(\alpha)\Gamma_s(A) \otimes_j e_\lambda, e(z) \rangle
\]
\[
= \langle \otimes_j e_\lambda, \Gamma_s(A^*)W(-\alpha)e(z) \rangle
\]
\[
= e^{(\alpha,z)} - \frac{1}{2} ||\alpha||^2 \langle \otimes_j e_\lambda, e(A^*(z - \alpha)) \rangle \quad (6.19)
\]

Since \(z\) is a finite vector and \(\alpha\) is fixed, each coordinate of \(A^*(z - \alpha)\) is a first degree polynomial in \(z_j\)'s. Therefore \(e(A^*(z - \alpha)) = \otimes_j e(w_j)\) where each \(w_j\) is a first degree polynomial in \(z_j\)'s. Therefore from (6.18) and property of infinite tensor products
\[
f(z) = e^{(\alpha,z)} - \frac{1}{2} ||\alpha||^2 \lim_{n \to \infty} \Pi_j \sqrt{\frac{2\lambda_j}{1 + \lambda_j^2}} \exp \left( \frac{1}{2} \left( \frac{\lambda_j^2 - 1}{\lambda_j^2 + 1} \right) w_j^2 \right)
\]

Since each \(w_j^2\) is a second degree polynomial in \(z_1, z_2, \ldots, z_N\). This contradicts (6.17) unless \(\lambda_j = 1\) for all \(j\). Now (6.12) implies
\[
|U\psi\rangle = W(\alpha)\Gamma_s(A)|e(0)\rangle
\]
\[
= e^{-\frac{1}{2} ||\alpha||^2} |e(\alpha)\rangle
\]

Now from (6.11) we get
\[
e^{-\frac{1}{2} ||\alpha||^2} |e(\alpha)\rangle = e^{-\frac{1}{2} ||\alpha||^2} |e(\alpha)\rangle \quad (6.20)
\]

Thus \(\beta(a^\dagger_1 a_1, a^\dagger_2 a_2, \ldots)\) is a unitary with the following properties:

1. \(\beta(a^\dagger_1 a_1, a^\dagger_2 a_2, \ldots)|k\rangle = \beta(k)|k\rangle\) for every \(k \in \mathbb{Z}_+^\infty\).

2. It maps coherent vectors to coherent vectors.

We will prove that \(\beta(a^\dagger_1 a_1, a^\dagger_2 a_2, \ldots) = \Gamma_s(D)\) for a diagonal unitary \(D\). To this end we fix a \(u = (u_1, u_2, \ldots)^T \in \mathbb{C} e_j\) with \(u_j \neq 0, \forall j\). We have \(\beta(a^\dagger_1 a_1, a^\dagger_2 a_2, \ldots)|e(u)\rangle = e^{\frac{1}{2} ||\alpha||^2} |e(\alpha)\rangle\). Therefore if \(\alpha = (\alpha_1, \alpha_2, \ldots)^T\) from (6.20) and (6.13) we get,
\[
\sum_{k \in \mathbb{Z}_+^\infty} \frac{u_k^k}{\sqrt{k!}} \beta(k)|k\rangle = e^{\frac{1}{2} ||\alpha||^2 - ||\alpha||^2} \sum_{k \in \mathbb{Z}_+^\infty} \frac{\alpha_k^k}{\sqrt{k!}} |k\rangle
\]

Therefore,
\[
u_k^k \beta(k) = e^{\frac{1}{2} ||\alpha||^2 - ||\alpha||^2} \alpha_k^k, \forall k \in \mathbb{Z}_+^\infty
\]

Since \(u_j \neq 0\) for all \(j\), we see that if \(k = (k_1, k_2, \ldots, k_m, 0, 0, \ldots) \in \mathbb{Z}_+^\infty\),
\[
\beta(k) = e^{\frac{1}{2} ||\alpha||^2 - ||\alpha||^2} \left( \frac{\alpha_1}{u_1} \right)^{k_1} \left( \frac{\alpha_2}{u_2} \right)^{k_2} \cdots \left( \frac{\alpha_m}{u_m} \right)^{k_m}, \forall k \in \mathbb{Z}_+^\infty
\]
Since $|\beta(k)| = 1$, we get $\left| \frac{\alpha_j}{u_j} \right| = 1$ for all $j$. If we write $\frac{\alpha_j}{u_j} = e^{i\theta_j}$, then from (6.20) we get

$$\beta(a_1^a a_1, a_2^a a_2, \ldots) |e(u)\rangle = |e(Du)\rangle,$$

where $D$ is the unitary Diag$(e^{i\theta_1}, e^{i\theta_2}, \ldots)$, for every $u = (u_1, u_2, \ldots)^T \in \oplus_j \mathbb{C}e_j$ with $u_j \neq 0, \forall j$. Now it is easy to see that (6.21) holds for all $u \in \mathcal{H}$. We conclude that $\beta(a_1^a a_1, a_2^a a_2, \ldots) = \Gamma_s(D)$.

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