Capturing Variation and Uncertainty in Human Judgment

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Abstract
The well-studied problem of statistical rank aggregation has been applied to comparing sports teams, information retrieval, and most recently to data generated by human judgment. Such human-generated rankings may be substantially different from traditional statistical ranking data. In this work, we show that a recently proposed generalized random utility model reveals distinctive patterns in human judgment across three different domains, and provides a succinct representation of variance in both population preferences and imperfect perception. In contrast, we also show that classical statistical ranking models fail to capture important features from human-generated input. Our work motivates the use of more flexible ranking models for representing and describing the collective preferences or decision-making of human participants.

1 Introduction
The problem of creating a ranking from noisy ranking or pairwise comparison data is widely studied across statistics, economics and machine learning. Statistical rank aggregation has been used to rank sports teams and racing drivers (Stern 1990), for information retrieval (Liu 2009), for collective ideation (Salganik and Levy 2012), in preference learning (Kamishima 2003), and for social choice (Conitzer, Rognlie, and Xia 2009).

Common to many current applications is that the input data comes from people, such as preferences in collaborative filtering or crowdsourcing and quality judgments in human computation. In crowdsourcing, a common problem is that of choosing the best or most desirable alternative from a large set of possibilities using ranking or voting (Chen et al. 2013). For example, Little et al. (2010) give an example of ranking a list of suggestions for things to do in New York.

Yet, human-generated ranking data can be fickle, encompassing both varying preferences between users and imperfect perception or judgment, and simply producing an aggregate ranking overlooks nuances in the data. Instead, we propose looking beyond simple rank aggregation to also understand the patterns of decision making that emerge. Specifically, we are interested in viewing human-generated rankings under one or both of the following settings:

- **Population preference**: Different rankings arise from a distribution over the population. By learning this distribution, we discover the most common preferences in the population as well as how uniform or varied preferences are across users.
- **Imperfect perception**: The variability in rankings arise from errors in the perception of an underlying truth. By learning this distribution, we discover which comparisons are noisy or cognitively difficult for users.

We consider three human-generated data sets demonstrating these two settings: preferences over types of sushi, decisions about ranking 8-puzzles by distance to the solution state, and decisions about ranking pictures of dots by number. Using several probabilistic models of rankings, we show that a generalized random utility model (RUM) based on the normal distribution (Azari Soufiani, Parkes, and Xia 2012), is able to better explain the variability in the data than the classical and often used Mallows (1957) and Plackett-Luce (1959, 1975) models.

In particular, we show that the Normal RUM is significantly better in matching the empirical pairwise comparison probabilities—the marginal probability that one alternative is ranked ahead of another—in the data, and reveals interesting patterns as a result. In data over preferences, we discover users’ ubiquitous affinity or dislike for certain alternatives as well as distinguishing between conventional and more controversial items. In data about decision making, we reveal the comparisons are that harder or easier to make, and how the difficulty of a ranking task affects these comparisons. In contrast, we also demonstrate why the Mallows and Plackett-Luce models have inherent limitations in capturing heterogeneity in human-generated data. We believe that these insights derived from more flexible models will prompt the use of new techniques for describing human preferences and perception beyond simple rank aggregation.

In the rest of the paper, Section 2 provides an overview of statistical ranking models. Sections 3 and 4 show how the Normal RUM allows for the interpretation of collective behavior on data where other ranking models do not (Appendix A analytically explains why the classical models fail to reveal this behavior.) Section 5 discusses implications and future work.
2 Ranking Models

We consider ranking problems with orderings over $m$ alternatives provided by $n$ agents. Let $A = \{a_1, a_2, \ldots, a_m\}$ denote the set of alternatives. Let $L$ denote the set of all total orders over $A$. Let $M$ with parameters $\theta$ denote a ranking model, generating an i.i.d. distribution on rankings: $\sigma_1 \sim M(\theta), \sigma_i \in L, i \in \{1 \ldots n\}$ denotes the $i^{th}$ ranking in the data.

In a random utility model (Thurstone 1927, McFadden 1974), each alternative $a_j$ has a random value (or utility) $x_j = \mu_j + \epsilon_j$, where $\epsilon_j$ is a zero-mean noise component, usually independent across alternatives, and $\mu_j \in \mathbb{R}$ is the mean. The realized values $(x_1, \ldots, x_m)$ induce a ranking $\sigma$ with $a_j > a_k \iff x_j > x_k$.

Different distributions for $\epsilon_j$ correspond to different random utility models (RUMs). In the Normal RUM (Azari Soufiani, Parkes, and Xia 2012), $\epsilon_j \sim \mathcal{N}(0, \sigma_j^2)$, where $\sigma_j$ can vary across alternatives. Although a straightforward model, it has been historically intractable to use; Azari et al. addressed this by adopting Monte Carlo Expectation Maximization (MC-EM) to estimate the parameters.

The classical Plackett-Luce model (Luce 1959, Plackett 1975) can be interpreted as a RUM in which the noise terms $\epsilon_j$ are independent Gumbel distributions with different means and the same variance (Yellott 1977). The Plackett-Luce model is popular due to its tractability. In particular, the likelihood function has a simple closed form and can be optimized efficiently with algorithms such as minorization-maximization (Hunter 2004).

Another popular ranking model is the Mallows model (Mallows 1957) in the statistical literature (also called Condorcet's model in social choice.) Mallows is not a random utility model. Rather, the parameters $\theta$ define a reference ranking $\sigma^* \in L$ and a noise parameter $p \in (0, 1]$. This model generates a random ranking by ordering all ordered pairs $(a_i, a_k)$ in agreement with reference $\sigma^*$ with probability $p$, and disagreement otherwise. If the result is an (acyclic) rank order then it is retained; otherwise, the process is repeated. A challenge with the Mallows model is that estimating the maximum likelihood parameters is NP-hard—in fact, the MLE of the reference ranking parameter is the social rank determined by the Kemeny voting rule (Young 1988).

Many extensions have been proposed to the above models in attempts to describe heterogeneity in data, such as by modeling agent-specific features or correlation between alternatives. The Mallows model has been extended to allow for mixtures (Lu and Boutilier 2011), and there are many extensions to the Plackett-Luce model, such as (Xia et al. 2008, Qin, Geng, and Liu 2010), which allow for a mixture of distributions or for the random utility parameters to depend on other features. The goal of these extensions has been largely to increase the models’ descriptiveness. In this work, we focus on simple models assuming that people are ex ante symmetric, and show that the Normal RUM already reveals interesting new observations about human judgment.

We implement maximum likelihood estimation for all the models previously described using multi-core parallelization, with numerical optimizations where possible. Our code is available as an integrated package\footnote{https://github.com/mizzao/libmao} with the aim of increasing the accessibility of using many different algorithms for modeling rank data, allowing for analysis and visualization using the methods we show in this work.

3 Sushi Ranking Dataset

Kamishima (2003) collected data on the rank order preferences of restaurant customers in Japan for different types of sushi. A particularly interesting subset of this data is a collection of 5000 rank orders on the same 10 pieces of sushi: ebi (shrimp), anago (sea eel), maguro (tuna), ika (squid), uni (sea urchin), sake (salmon roe), tamago (egg), toro (fatty tuna), tekkamaki (tuna roll), and kappa-maki (cucumber roll). These rankings, each provided by a unique customer, are an example of preference data. By accurately modeling the distribution of rankings, we can describe the distribution of preferences over the population, and ultimately produce a succinct depiction of collective preferences.

Azari et al. originally showed that the Normal RUM has better model fit on this dataset and others; we additionally demonstrate why this model can better describe collective preferences than the Plackett-Luce and Mallows models. We focus on the pairwise comparison probabilities—the marginal probability that one alternative is ranked above another alternative. Pairwise comparisons measure the first-order accuracy of the model and have been used since the earliest statistical ranking models (Mosteller 1951, Bradley and Terry 1952).

Figure 1 compares the aggregated pairwise comparison probabilities to the empirical data, with one model in each row, as surface plots of $m \times m$ matrices. The first column shows the model’s prediction according to maximum likelihood parameters. The second column shows the empirical probabilities, and the third column shows the element-wise difference. As each model implies a different modal (most likely) ordering of the alternatives, the plots in each row show the implied modal ordering along the axes. For the sake of clarity, and recognizing symmetry, we show only one comparison between each pair of items in the plots. A continuous color scale indicates the magnitude of each value, and in the third column, a flatter plot with more uniform color indicates a better fit between the model predictions and empirical comparisons.

The Mallows model, shown in the top row, can only fit a surface of probabilities derived from the single parameter $p$, which monotonically increases for alternatives that are further separated in its modal ranking (Figure 1a). Compared to empirically observed comparisons, the Mallows model suffers from both systematic over- and under-estimates of pairwise probabilities (Figure 1c). In the second row, the Plackett-Luce model, while more flexible than the Mallows model, predicts comparison probabilities that are constrained by monotone increases along its modal ordering (the axes of Figure 1d), and still shows significant deviations from the data (Figure 1f).
Figure 1: Comparison of different models for the sushi data: each row shows one model with axes of plots arranged in the model’s modal ordering. The first column shows the model’s predicted pairwise comparison probabilities (for an item on the left axis to be ranked above an item on the right axis). The second column shows the empirical pairwise comparison probabilities. The last column shows the difference of the two. For clarity and because of symmetry, we plot one probability for each pair of items.

The third row illustrates the predictions of the Normal RUM. Due to flexible variance parameters, one per alternative, the model is no longer subject to the monotonic behavior we observed previously (Figure 13), and bears a much closer resemblance to the empirical probabilities (Figure 11). The Normal RUM is able to fit the pairwise, empirical probabilities very closely compared to the other models (Figure 11 is much flatter).

Since these models attempt to capture a symmetric distribution over the population and not predictions for each person, the increased explanatory power of the Normal RUM is not due to overfitting: the model consists of only 20 parameters, while the data contains 5000 permutations of 10 alternatives.

Model Interpretation As the Normal RUM achieves a much more accurate estimate of marginal pairwise probabilities in the data, we now turn to interpretation of its distribution. Figure 2 plots the estimated, random utility distributions for each alternative. For identifiability in estimation of the model, an arbitrary distribution (magenta) is fixed to the standard normal distribution. Note that the x-axis is reversed, with larger values to the left. Under the model, each consumer’s preferences are represented by an independent draw of random values from each of the distributions, and ranked according to the realized values. The plot also shows the predicted, marginal pairwise probabilities that adjacent pairs in the modal ordering are ranked according to that ordering.
This interpretation yields a great deal of information about the preferences of sushi consumers. First, the most preferred (toro) and least preferred (kappa-maki) show a clear separation from the rest of the alternatives, and are separated from their immediate neighbors with a much greater probability than other adjacent pairs in the ranking. This shows greater universality in the like or dislike of these alternatives. In contrast, preferences over adjacent items in the middle of the ranking are more noisy and comparison probabilities are close to 0.5 for many pairs.

The variance of the distribution for each type of sushi is also informative. The greatest variance is for uni, while tekka-maki (tuna roll) has the lowest variance, revealing a dichotomy of preference for the former and a more consistent but average support for the latter. This illustrates how the Normal RUM allows a very large set of ranking data to be distilled into an intuitive explanation of the population. Notably, this interpretation would not be possible with the widely-used Mallows and Plackett-Luce models, whose parametrizations cannot capture the differing variance across items and are thus more inaccurate in representing the diversity of preferences. We discuss these model deficiencies in further detail in Appendix A.

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2Toro is a fatty cut from the belly of the bluefin tuna that is especially highly regarded, and invariably commands a premium price in sushi bars. Kappa-maki is a perhaps unremarkable sushi of cucumber and rice.

3In contrast to the conventional tekka-maki (tuna roll), uni is a rather unique type of sushi made from the gonads of the sea urchin, known to elicit delight or disgust depending on a person’s taste.

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**4 Ranking Decision Dataset**

Mao et al. (2013) collected data on human judgment in two different ranking problems that were designed for the level of difficulty to be systematically varied. We briefly describe the nature of the data below; for more details, please consult the original paper.

**Data set 1: Sliding 8-puzzles.** The 8-puzzle (Figure 3) consists of a square 3x3 board with tiles numbered from 1 to 8 and an empty space. From any legal board state, one solves the puzzle by sliding one tile after the next into the empty space, seeking to obtain a board state where the numbers are correctly ordered from top to bottom and left to right. Each movement of a single tile counts as one “move”; and one goal is to solve the puzzle in as few moves as possible.

In the 8-puzzle dataset, users ranked four instances of 8-puzzle game states by the least number of moves from the goal.
solution state, from closest to farthest away. Choosing a sequence of numbers, such as \(7, 10, 13, 16\), and generating a set of four random puzzles solvable in a corresponding number of moves as computed by \(A^*\) search produced ranking data at a specific, consistent difficulty. By fixing the difference between the numbers but varying the overall distance to the goal, the puzzles become harder or easier to rank relative to each other.

**Data set 2: Dot fields.** The problem of counting pseudo-randomly distributed dots in images has been suggested as a benchmark task for human computation (Horton 2010). Pfeiffer et al. (2012) used the task of comparing these dot fields as a proxy for noisy comparisons of items in ranking tasks. In the dots dataset, people ranked four pictures with many more dots than could be manually counted. For example, Figure 4 shows pictures with 200 and 208 dots, respectively. Varying the difference in the number of dots between pictures allows adjustment of the difficulty of the task.

In both domains, there are four levels of difficulty, each with 40 separate sets of four alternatives to rank. Around 20 ranking orders are elicited from people for each set, providing a total of 3,200 rankings for each of the two ranking problems. In contrast to the sushi data described in Section 3, these ranking datasets do not capture different preferences across individuals. Instead, every user has the same preferred ranking over the data, but variance arises from imperfect or noisy perception. Thus, by learning a distribution of rankings over the data, we can describe decision-making ability and cognitively difficult comparisons across the collective population.

**Model Interpretation** As in the sushi dataset, the Normal RUM provides the best approximation of empirical comparison probabilities on this data. We focus on the estimated parameters to gain an understanding of human perceptive judgment in these two domains.

Figures 5 and 6 show the properties of the estimated random utility distributions for each problem. The first graph shows the pairwise comparison probabilities for adjacent alternatives at each level of difficulty. The second and third plots show the estimated random utility distributions for the easiest and hardest ranking problems, respectively. In each case, the distributions are scaled so that the mean and variance of the highest-ranked item (closest puzzle to the goal or lowest number of dots) has the standard normal distribution.

In both problems, adjacent pairs of items become harder to rank as difficulty increases, with comparison probability decreasing toward 0.5—shown by the progression of successively lower line segments in Figures 5a and 6a. Yet, it is particularly interesting how the varying difficulty of the two different problems affects users’ judgments, as revealed from the estimated model. For the 8-puzzle, the probability of correctly ordering adjacent puzzles slopes downward from left to right, showing that it is harder to compare two puzzles that are further from the goal state than two that are closer. This is naturally explained by observing that it is easier, for example, to tell which of two puzzles that are 7 and 10 moves from the solution is closer, than two puzzles that are 13 and 16 moves from the solution. All levels of difficulty in the problem exhibit this property.
For the fields of dots, there is a different behavior as difficulty increases. At the easier levels of difficulty, the probability of ranking adjacent alternatives in the correct order does not significantly slope from left to right for the three adjacent pairs. This is quite different from the 8-puzzle setting, but is naturally explained considering that dot fields do not become harder to rank when the difference of dots is a similar percentage of the total (around 200 for all pictures). However, in the most difficult setting, with a difference of only 3 dots between pictures, there is a marked drop in accuracy for the intermediate two pairs of pictures.

In the distribution over rankings, we see a similar pattern to the sushi dataset: namely, there is more certainty about the pictures with the least and most number of dots (resp. stronger preference of the favorite and least favorite sushi) than among the intermediate choices. In the dots data, the lower certainty arises from increased perceptive error between the extremes, while in the sushi data this manifests from more varied preferences apart from the favorite and least favorite. The Normal RUM captures both of these cases, allowing it to describe both rankings of preference and rankings of imperfect perception.

As each set of parameters is generated from 800 rankings and the patterns we observe are consistent across the difficulty levels for both problems, we believe these results are robust. Compared with the parameters estimated for sushi data, the variance of the noise distributions for alternatives is more uniform. However, the expressiveness of the Normal RUM allows similar variances across alternatives to be interpreted as more or less uniform perceptive error in ranking, rather than a necessary limitation of the model (as is the case with Plackett-Luce: see Appendix A).

5 Discussion

Our results illustrate the importance of flexible and expressive models for fitting human ranking data, encompassing both variation across population preference as well as errors arising from imperfect perception. Commonly used but restrictive models such as Mallows and Plackett-Luce are insufficient to capture the various patterns encountered in human perception. However, an appropriately descriptive model such as the Normal RUM allows for intuitive interpretation of both types of data in a more nuanced way.

Such models allow us to gain new understanding and intuition about real data. We are able to see clear preferences for certain alternatives by users, as well as which choices present more contention or uncertainty and which are uncontroversial. We can also see how users’ perception and decision making is affected by harder and easier tasks, and for which alternatives they can make judgments that are either more certain or more ambiguous. Our approach also highlights the importance of detailed evaluation techniques that reveal the quality of a model’s fit to data in a natural way, such as looking at pairwise comparison probabilities. Our results clearly show how classical models fall short in representing human ranking data.

Our results motivate several areas of future work. The Normal RUM and other flexible random utility models are promising for discovering interesting patterns in preferences across a collective group of people. In our case, estimating such a model distilled several thousand rankings into a much more concise representation. We believe that this type of analysis facilitates a more natural understanding of collective preferences over simple rank aggregation. At the same time, we foresee that a better understanding of the difficulty and variance in human judgment problems can motivate the design of better user interfaces and crowdsourcing systems. Using any ranking data, one can explore errors and variance when human users are asked to make comparisons that are noisy or uncertain.

Our hope is that the analysis and methods in this work will motivate more insightful descriptions of collective human preferences as well as a better understanding of decision making in the design of systems involving human agents.
References

[2011] Ammar, A., and Shah, D. 2011. Ranking: Compare, dont score. In Proceedings of the 49th Annual Allerton Conference on Communication, Control, and Computing.

[2012] Azari Soufiani, H.; Parkes, D. C.; and Xia, L. 2012. Random utility theory for social choice: Theory and algorithms. In Proceedings of the 26th Annual Conference on Neural Information Processing Systems (NIPS).

[1952] Bradley, R. A., and Terry, M. E. 1952. Rank analysis of incomplete block designs: I. The method of paired comparisons. Biometrika 39(3/4):324–345.

[2013] Chen, X.; Bennett, P. N.; Collins-Thompson, K.; and Horvitz, E. 2013. Pairwise ranking aggregation in a crowdsourced setting. In Proceedings of the 6th ACM Conference on Web Search and Data Mining (WSDM), WSDM ’13, 193–202. New York, NY, USA: ACM.

[2000] Conitzer, V.; Rognlie, M.; and Xia, L. 2009. Preference functions that score rankings and maximum likelihood estimation. In Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI), 109–115.

[2012] Horton, J. J. 2010. The dot-guessing game: A fruit fly for human computation research. Available at SSRN: http://ssrn.com/abstract=1600372 or http://dx.doi.org/10.2139/ssrn.1600372.

[2004] Hunter, D. R. 2004. MM algorithms for generalized Bradley-Terry models. In The Annals of Statistics, volume 32, 384–406.

[2003] Kamishima, T. 2003. Nantonac collaborative filtering: Recommendation based on order responses. In Proceedings of the 9th International Conference on Knowledge Discovery and Data Mining (KDD).

[2010] Little, G.; Chilton, L. B.; Goldman, M.; and Miller, R. C. 2010. Turkit: Human computation algorithms on Mechanical Turk. In Proceedings of the 23rd Symposium on User Interface Software and Technology (UIST), 57–66.

[2009] Liu, T.-Y. 2009. Learning to rank for information retrieval. Foundations and Trends in Information Retrieval 3(3):225–331.

[2011] Lu, T., and Boutilier, C. 2011. Learning Mallows models with pairwise preferences. In Proceedings of the 28th International Conference on Machine Learning (ICML), 145–152.

[1959] Luce, R. D. 1959. Individual Choice Behavior: A Theoretical Analysis. Wiley.

[1957] Mallows, C. L. 1957. Non-null ranking model. Biometrika 44(1/2):114–130.

[2013] Mao, A.; Procaccia, A. D.; and Chen, Y. 2013. Better human computation through principled voting. In Proceedings of the 27th AAAI Conference on Artificial Intelligence (AAAI).

[1974] McFadden, D. 1974. Conditional Logit Analysis of Qualitative Choice Behavior. In Zarembka, P., ed., Frontiers in econometrics. New York: Academic Press. 105–142.

[1951] Mosteller, F. 1951. Remarks on the method of paired comparisons: I. the least squares solution assuming equal standard deviations and equal correlations. Psychometrika 16(1):3–9.

[2012] Pfeiffer, T.; Gao, X. A.; Mao, A.; Chen, Y.; and Rand, D. G. 2012. Adaptive polling and information aggregation. In Proceedings of the 26th AAAI Conference on Artificial Intelligence (AAAI).

[1975] Plackett, R. L. 1975. The analysis of permutations. Journal of the Royal Statistical Society. Series C (Applied Statistics) 24(2):193–202.

[2010] Qin, T.; Geng, X.; and Liu, T.-Y. 2010. A new probabilistic model for rank aggregation. In Proceedings of the 2010 Annual Conference on Neural Information Processing Systems (NIPS), 1948–1956.

[2012] Salganik, M. J., and Levy, K. E. 2012. Wiki surveys: Open and quantifiable social data collection. Technical Report arXiv:1202.0500.

[1990] Stern, H. 1990. Models for distributions on permutations. Journal of the American Statistical Association 85(410):pp. 558–564.

[1927] Thurstone, L. L. 1927. A law of comparative judgement. Psychological Review 34:273–286.

[2008] Xia, F.; Liu, T.-Y.; Wang, J.; Zhang, W.; and Li, H. 2008. Listwise approach to learning to rank: theory and algorithm. In Proceedings of the 25th International Conference on Machine Learning (ICML), 1192–1199. ACM.

[1977] Yellott, J. I. J. 1977. The relationship between Luce’s Choice Axiom, Thurstone’s Theory of Comparative Judgment, and the double exponential distribution. Journal of Mathematical Psychology 15(2):109–144.

[1988] Young, H. P. 1988. Condorcet’s theory of voting. The American Political Science Review 82(4):1231–1244.
A Model Properties
Sections 3 and 4 showed that the Normal RUM allows a better interpretation of several data sets than the classical Mallows and Plackett-Luce models, because of its advantages in correctly capturing comparison probabilities over all pairs of alternatives. Here, we show analytically why this generally can be true for any data set, due to inherent restrictions in the classical models.

By estimating a ranking model $\mathcal{M}$ (i.e., by using maximum likelihood), we obtain parameters $\hat{\theta}$ implying a distribution on rank orders as well as a marginal distribution over pairwise comparisons. We generally expect that the probability of users ranking one particular item above another will be only marginally affected by the presence of other choices, so we can evaluate the suitability of a model by how well it approximates all marginal pairwise probabilities. More generally, pairwise comparison data has a long history of use in learning rankings (Mosteller 1951; Bradley and Terry 1952) and is generally viewed as more robust than cardinal data (Ammar and Shah 2011).

Under the Normal RUM, the probability that a particular alternative $a_j$ is preferred to another item $a_k$ is

$$
\Pr_{\text{Normal}} (a_j \succ a_k \mid \mu, \sigma) = \Phi \left( \frac{(\mu_j - \mu_k)}{\sqrt{\sigma_j^2 + \sigma_k^2}} \right)
$$

where $\Phi$ is the CDF of the standard normal distribution. Figure 1 showed that this two-parameter model closely approximated all pairwise comparisons in a large set of empirical preference data.

Limitations of the Mallows Model Under the Mallows model, the noise parameter $p$ is also the probability that adjacent pairs in the reference ranking $\sigma^*$ are ranked in the same order. More generally, any two items separated by a fixed distance in $\sigma^*$ are ranked correctly with the same probability $\Pr(\sigma_{\sigma^*(k)} \succ \sigma_{\sigma^*(l)})$ for positive $z = l - k$, which can be shown to be

$$
\Pr_{\text{Mallows}} (a_{\sigma^*(k)} \succ a_{\sigma^*(l)} \mid \phi, \sigma^*) = \frac{\sum_{c=0}^{\infty} \frac{\phi^{c+1}}{(\sum_{z=1}^{\infty} \phi^{z+1}) (\sum_{z=0}^{\infty} \phi^{z})}}
$$

with $\phi = (1 - p)/p$. As shown in Figure 1a, this is monotone increasing in $c$ for a given $\phi$. In the context of rankings from human input, this rather strong assumption is easily violated, such as when agents have more certain comparisons over adjacent pairs in one part of the ranking than the other. In the case of the sushi data and the most difficult dots ranking problem, this occurred at endpoints of the ranking with more uncertainty in the middle. Hence, while popular and extended in many ways as outlined in Section 2, the Mallows model is inherently rather restrictive.

Limitations of The Plackett-Luce Model Under the Plackett-Luce model, a rather severe restriction is the fixed variance of the Gumbel distribution for the random utility across alternatives. The marginal probability that alternative $a_j$ ranked is higher than alternative $a_k$ is

$$
\Pr_{\text{PL}} (a_j \succ a_k \mid \mu) = 1/\left(1 + e^{-(\mu_j - \mu_k)}\right)
$$

Since the logistic sigmoid $g(x) = 1/(1 + e^{-x})$ is strictly monotonically increasing, there is a strict limitation in which pairwise probabilities for ordering items in a particular way are determined only by the difference in their strength values. Specifically, for any strictly monotone increasing function $f(x)$, we can show that for any fixed $\mu_j$,

$$
\mu_k > \mu'_k \iff f(\mu_j - \mu_k) < f(\mu_j - \mu'_k)
$$

This behavior is exemplified in Figure 1d, where the comparison probabilities monotonically increase along the axes, and emerges in all fixed-variance (one-parameter) random utility models.

As we have seen, this assumption is rather strong, and any RUM with this property cannot capture the notion that it may be less certain to compare some particular alternative (such as $\text{uni}$) versus others, and hence it would be sensible for the variance of utility for that alternative to change rather than the mean value.