The regular part of second-order differential sectorial forms with lower-order terms

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Abstract. We present a formula for the regular part of a sectorial form that represents a general linear second-order differential expression that may include lower-order terms. The formula is given in terms of the original coefficients. It shows that the regular part is again a differential sectorial form and allows to characterise when also the singular part is sectorial. While this generalises earlier results on pure second-order differential expressions, it also shows that lower-order terms truly introduce new behaviour.

1. Introduction

We study the regular part of a differential sectorial form that represents a general linear second-order differential expression that may include lower-order terms. Loosely speaking, such a differential expression has the form

\[- \sum_{k,l=1}^{d} \partial_l c_{kl} \partial_k + \sum_{k=1}^{d} b_k \partial_k - \sum_{k=1}^{d} \partial_k d_k + c_0.\]

We obtain a formula for the regular part of the corresponding differential sectorial form from which it follows that the regular part is again a differential sectorial form. Furthermore, this formula allows to characterise when the singular part is sectorial and when the regular part of the real part is equal to the real part of the regular part.

The regular part for positive symmetric forms was introduced in [Sim78] and recently generalised to sectorial forms in [AtE12]. In [Vog09] a formula was obtained for the regular part of a positive symmetric form representing a pure second-order differential expression. We make use of the techniques introduced there, as we already did in a previous paper [tES11].

A different approach to the regular part of a positive symmetric form based on parallel sums is given in [HSdS09]. Furthermore, we point out that in the context of nonlinear phenomena and discontinuous media the relaxation of a functional is a notion that in the linear setting corresponds to the regular part of positive symmetric forms [Mos94, Bra02, DM93].

The current article generalises our exposition for differential sectorial forms of pure second order as given in [tES11]. We note that here we impose the same mild conditions on the highest-order coefficients. In particular, we cover a large class of forms associated with important degenerate elliptic differential operators. The lower-order terms introduce new phenomena.
Even if $c_0 = 0$, then the regular part of the corresponding differential sectorial form may have a non-vanishing zeroth-order term. Moreover, this term can cause a shift in the vertex of the singular part of the form. We will present an example where this happens.

2. Preliminaries and definition of considered form

We will adopt the notation as in [tES11]. For the reader’s convenience, we recall the following definitions. Suppose $H$ is a Hilbert space, $D(b)$ is a linear subspace of $H$ and $b: D(b) \times D(b) \to C$ is a sesquilinear form with form domain $D(b)$. We say that the form $b$ is sectorial if there exist a vertex $\gamma \in \mathbb{R}$ and a semi-angle $\theta \in (0, \frac{\pi}{2})$ such that

$$b(u, u) - \gamma \|u\|^2_H \in \Sigma_{\theta}$$

for all $u \in D(b)$, where $\Sigma_{\theta} := \{z \in \mathbb{C} : z = 0 \text{ or } |\arg(z)| \leq \theta\}$. The real part $\Re b: D(b) \times D(b) \to C$ of a sesquilinear form $b$ is defined by $(\Re b)(u, v) = \frac{1}{2}(b(u, v) + \overline{b}(v, u))$, and similarly for the imaginary part.

For a sectorial form $b$ with vertex $\gamma$ we define the norm $\|\cdot\|_b$ on $D(b)$ by setting $\|u\|_b^2 := (\Re b)(u, u) + (1 - \gamma)\|u\|^2_H$. A sectorial form $b$ is called closed if $(D(b), \|\cdot\|_b)$ is a Hilbert space. A sectorial form is called closable if it has a closed extension. Our main reason to study sectorial forms is that with every densely defined sectorial form $b$ one can associate an $m$-sectorial operator $B$ in a natural way by [ATE12, Theorem 1.1].

By [tES11] Proposition 2.3 the following definition of the regular part makes sense.

**Definition 2.1.** Let $b$ be a densely defined sectorial form. Let $B$ be the $m$-sectorial operator associated with $b$. Then there exists a unique closable sectorial form $b_{\text{reg}}$ with form domain $D(b_{\text{reg}}) = D(b)$ such that $B$ is associated with $b_{\text{reg}}$. We call $b_{\text{reg}}$ the regular part of $b$. The singular part of $b$ is the form $b_s = b - b_{\text{reg}}$.

A useful formula for the regular part of a densely defined sectorial form is given in [tES11] Theorem 2.6.

We now introduce the form $a$ that is considered throughout this paper. Let $\Omega \subset \mathbb{R}^d$ be open. Let $H = L^2(\Omega)$ and suppose $D(a)$ is a vector subspace of $H$ that contains $C_0^\infty(\Omega)$. Suppose that $\partial_k u \in L^1_{\text{loc}}(\Omega)$ for all $k \in \{1, \ldots, d\}$ and $u \in D(a)$. For all $k, l \in \{1, \ldots, d\}$, let $c_{kl}, b_k, d_k$ and $c_0$ be measurable functions from $\Omega$ into $\mathbb{C}$. Suppose that $c_0 \in L^\infty(\Omega)$. Define $C: \Omega \to \mathbb{C}^{d \times d}$ by $C(x) = (c_{kl}(x))_{k,l=1}^d$ and $b, d: \Omega \to \mathbb{C}^d$ by $b(x) = (b_k(x))_{k=1}^d$ and $d(x) = (d_k(x))_{k=1}^d$. Suppose that $C\nabla u \cdot \nabla u \in L^1(\Omega)$ for all $u \in D(a)$, where $\zeta \cdot \eta$ denotes the Euclidean inner product of $\zeta, \eta \in \mathbb{C}^d$. Moreover, suppose there exists a $\theta \in [0, \pi/2)$ such that $C(x)\xi \cdot \xi \in \Sigma_{\theta}$ for all $\xi \in \mathbb{C}^d$ and $x \in \Omega$. Define the measurable function $A : \Omega \to \mathbb{C}^{d \times d}$ by $A(x) = \frac{1}{2}(C(x) + C(x)^*)$. Note that $A(x)$ is a positive semi-definite Hermitian matrix for all $x \in \Omega$. Hence $A(x)$ admits a unique positive semi-definite square root $A^{1/2}(x)$ for all $x \in \Omega$. Furthermore, suppose there exists a $K > 0$ such that

$$|\overline{b(x)} \cdot \xi| \leq K\|A^{1/2}(x)\xi\|_{\mathbb{C}^d} \quad \text{and} \quad |d(x) \cdot \xi| \leq K\|A^{1/2}(x)\xi\|_{\mathbb{C}^d}$$

for all $\xi \in \mathbb{C}^d$ and $x \in \Omega$. The map $x \mapsto A^{1/2}(x)$ is measurable from $\Omega$ into $\mathbb{C}^{d \times d}$ by [tES11] Lemma 4.1. [tES11]
Lemma 2.2. There exist measurable, bounded maps \(X, Y : \Omega \to \mathbb{C}^d\) such that \(A^{1/2}(x)X(x) = b(x)\) and \(A^{1/2}(x)Y(x) = d(x)\) for all \(x \in \Omega\).

Proof. Define \(g : [0, \infty) \to [0, \infty)\) by \(g(t) = t^{-1/2}\) if \(t > 0\) and \(g(0) = 0\). Then, as in the proof of [tES11] Lemma 4.1], the map \(x \mapsto g(A(x))\) is measurable from \(\Omega\) into \(\mathbb{C}^{d \times d}\). Therefore also the map \(X : \Omega \to \mathbb{C}^d\) defined by \(X(x) = g(A(x))b(x)\) is measurable. Moreover,
\[
|X(x) \cdot \xi| = |b(x) \cdot g(A(x))\xi| \leq K \|A^{1/2}(x)g(A(x))\xi\|_{\mathbb{C}^d} \leq K \|\xi\|_{\mathbb{C}^d}.
\]
This proves that \(X\) is bounded. Then arguing as in the proof of the first displayed formula on page 918 in [tES11], one deduces \(A^{1/2}(x)X(x) = b(x)\).

Existence and boundedness of \(Y\) is proved similarly. \(\square\)

Let \(Z : \Omega \to \mathbb{C}^{d \times d}\) be as in [tES11] Lemma 4.1]. Then \(Z\) is bounded, \(Z(x)\) is Hermitian and \(A^{1/2}(x)(I + iz(x))A^{1/2}(x) = C(x)\) for all \(x \in \Omega\). Finally, define the form \(a : D(a) \times D(a) \to \mathbb{C}\) by
\[
a(u, v) = \int_{\Omega} \sum_{k,l=1}^{d} c_{k,l}(\partial_k u)\overline{\partial_l v} + \int_{\Omega} \sum_{k=1}^{d} b_k(\partial_k u)\overline{v} + \int_{\Omega} \sum_{k=1}^{d} d_k u \partial_k v + \int_{\Omega} c_0 u \overline{v}.
\]
Let \(X\) and \(Y\) be as in Lemma \ref{2.2}. This allows us to write
\[
a(u, v) = (I + iz)A^{1/2}\nabla u \mid A^{1/2}\nabla v) + (A^{1/2}\nabla u \mid vX) + (uY \mid A^{1/2}\nabla v) + (c_0 u \mid v)
\]
for all \(u, v \in D(a)\). In particular, it follows that the first-order terms in (1) are indeed integrable.

Since \(X\), \(Y\), and \(Z\) are bounded, the next lemma follows from (2).

Lemma 2.3. The form \(a\) is a sectorial form in \(H\).

Let \(b\) be a sesquilinear form in \(L^2(\Omega)\). If \(b\) is equal to the form \(a\) for an appropriate choice of the coefficient functions \(c_{k,l}, b_k, d_k\) and \(c_0\) with \(k, l \in \{1, \ldots, d\}\), then we shall call \(b\) a differential sectorial form. The main result of this paper establishes that, under suitable mild conditions which we next introduce, the regular part of \(a\) is also a differential sectorial form.

As in [tES11], we introduce the following two conditions. We say that \(a\) satisfies Condition (L) if
\[
(i) \ D(a) \cap L^\infty(\Omega) \text{ is invariant under multiplication with } C_c^\infty(\Omega) \text{ functions,}
(ii) \text{there exists a } \psi \in C_b^1(R) \text{ such that } \psi(0) = 0, \ \psi'(0) = 1, \text{ and } \psi \circ (\text{Re} \ u) + i\psi \circ (\text{Im} \ u) \in D(a) \text{ for all } u \in D(a), \text{ and,}
(iii) \ c_{k,l} + c_{l,k}^\ast \text{ is real valued for all } k, l \in \{1, \ldots, d\}.
\]
This is the condition on the form \(a\) in [Vog09], adapted to complex vector spaces. Furthermore, we say that \(a\) satisfies Condition (B) if
\[
(i) \ D(a) \text{ is invariant under multiplication with } C_c^\infty(\Omega) \text{ functions, and,}
(ii) \ c_{k,l} \in L^\infty_{loc}(\Omega) \text{ for all } k, l \in \{1, \ldots, d\}.
\]
3. The Formula for the Regular Part

In this section we derive a formula for the regular part of the form $a$ in Section 2. To this end, we assume that $a$ satisfies Condition (L) or (B). It will be immediate from the obtained formula that both the regular and singular part of $a$ continue to be differential sectorial forms. Note that this is not at all clear as the definition of the regular part is rather abstract.

We will refine the methods used in [tES11, Section 4]. We first reformulate some of the results in [tES11, Section 2] in a convenient way for the current setting.

If $V_0$ is a vector space with a semi-inner product, then there exist a Hilbert space $V$ and an isometric linear map $\Phi: V_0 \to V$ such that $\Phi(V_0)$ is dense in $V$. Then $(V, \Phi)$ is unique up to unitary equivalence. We call $(V, \Phi)$ the completion of $V_0$. Moreover, every uniformly continuous map on $V_0$ has a unique continuous extension to $V$ in the obvious way.

Proposition 3.1. Let $H$ be a Hilbert space and $a$ be a densely defined sectorial form in $H$. Let $(V, \Phi)$ be the completion of $(D(a), \|\cdot\|_a)$. Let $\tilde{a}: V \times V \to C$ and $\tilde{j}: V \to H$ be the continuous extensions of $a$ and of the embedding of $D(a)$ into $H$, respectively. Let $\tilde{h}$ be the real part of $\tilde{a}$. Let $\pi_1$ be the orthogonal projection of $V$ onto $V_1 := \ker \tilde{j}$ and let $\pi_2 = I_V - \pi_1$. Then there exists a unique operator $T \in \mathcal{L}(V, V_1) \subset \mathcal{L}(V)$ such that

$$\langle \mathfrak{m} \tilde{a}(u, v) \rangle = \tilde{h}(Tu, v)$$

for all $u \in V$ and $v \in V_1$. Moreover, $T_{11} := T|_{V_1} \in \mathcal{L}(V_1)$ is self-adjoint. Define the operator $\Pi \in \mathcal{L}(V)$ by

$$(3) \quad \Pi u = \pi_2 u - i(I_{V_1} + iT_{11})^{-1}T\pi_2 u.$$ 

Then the regular part of $a$ is given by

$$(4) \quad a_{\text{reg}}(u, v) = \tilde{a}(\Pi \Phi(u), \Pi \Phi(v))$$

for all $u, v \in D(a)$.

Proof. In [tES11] the operator $\Pi$ was denoted by $P_{V(\tilde{a})}$. Now (3) and (4) follow from the displayed formula on the top of page 912 and (2) in [tES11]. It is easily verified that $\pi_2$ is the orthogonal projection in $V$ onto

$$\{u \in V : \tilde{h}(u, v) = 0 \text{ for all } v \in \ker \tilde{j}\}.$$ 

Then the remaining statements can be found in [tES11, Theorem 2.6].

We start by constructing a suitable completion of the pre-Hilbert space $(D(a), \|\cdot\|_a)$ that allows us to get hold of the corresponding continuous extensions of $a$ and of the embedding of $D(a)$ into $H$.

Let $h$ be the real part of $a$. Then

$$h(u, v) = \left( A^{1/2}v \mid A^{1/2}v \right) + \frac{1}{2} \left( A^{1/2}v \mid v(X + Y) \right)$$

$$+ \frac{1}{2} \left( u(X + Y) \mid A^{1/2}v \right) + \left( \Re c_0 \right) u \mid v$$
for all $u, v \in D(a)$. Let $\mathcal{H}$ be the Hilbert space $L^2(\Omega) \times (L^2(\Omega))^d$ with the usual inner product. Let $\gamma_0 \in \mathbb{R}$ be a vertex of the sectorial form $a$. Since $X$ and $Y$ are bounded, there exists a $\gamma \leq \gamma_0$ such that the sesquilinear form $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by

\[
\langle (u_1, w_1), (u_2, w_2) \rangle = (w_1 | w_2) + \frac{1}{2}(w_1 | u_2(X + Y)) + \frac{1}{2}(u_1(X + Y) | w_2) + ((1 - \gamma + \text{Re} \, c_0)u_1 | u_2)
\]

defines an equivalent inner product on $\mathcal{H}$. Note that $\gamma$ is also a vertex of $a$.

Let $\mathcal{H}'$ denote the space $L^2(\Omega) \times (L^2(\Omega))^d$ equipped with the inner product $\langle \cdot, \cdot \rangle$. Define the map $\Phi: (D(a), \| \cdot \|_a) \to \mathcal{H}'$ by

\[
\Phi(u) = (u, A^{1/2} \nabla u).
\]

Then $\Phi$ is an isometry. Hence the completion $V$ of $D(a)$ can be realised as the closure of $\Phi(D(a))$ in $\mathcal{H}'$ equipped with the inner product of $\mathcal{H}'$. Note that $V$ is also closed in $\mathcal{H}$. The embedding of $D(a)$ into $H$ has as continuous extension to the completion $V$ the map $\tilde{j}: V \to H$ given by $\tilde{j}(u, w) = u$. Furthermore, due to [2] the continuous extension of the form $a$ is the form $\tilde{a}: V \times V \to \mathbb{C}$ given by

\[
\tilde{a}((u_1, w_1), (u_2, w_2)) = ((I + iZ)w_1 | w_2) + (w_1 | u_2X) + (u_1Y | w_2) + (c_0u_1 | u_2).
\]

Then the real part $\tilde{h}$ of $\tilde{a}$ is given by

\[
\tilde{h}((u_1, w_1), (u_2, w_2)) = (w_1 | w_2) + \frac{1}{2}(w_1 | u_2(X + Y)) + \frac{1}{2}(u_1(X + Y) | w_2) + ((\text{Re} \, c_0)u_1 | u_2).
\]

Note that

\[
\langle (u_1, w_1), (u_2, w_2) \rangle = \langle (u_1, w_1) | (u_2, w_2) \rangle_{\tilde{h}}
\]

for all $(u_1, w_1), (u_2, w_2) \in V$.

Next, set $V_1 := \ker \tilde{j}$ and $V_2 := V_1^\perp$. Let $\pi_1$ and $\pi_2$ be the orthogonal projections in $V$ onto $V_1$ and $V_2$, respectively. We shall show that $\pi_1$ and $\pi_2$ can be represented by multiplication operators in $\mathcal{H}$. To this end, let $V_s$ be such that $V_s \subset (L^2(\Omega))^d$ and $V_1 = \{0\} \times V_s$. Clearly $V_s$ is a closed subspace of $(L^2(\Omega))^d$. Moreover, since $a$ satisfies Condition (L) or (B), the orthogonal projection in $(L^2(\Omega))^d$ onto $V_s$ is given by the multiplication operator associated with a measurable function $Q: \Omega \to \mathbb{C}^{d \times d}$ that has values in the orthogonal projection matrices. This result is based on [Vog09, Proof of Theorem 1], see also the discussion in [tES11, Section 4]. Now we are able to state the main result of this paper.

**Theorem 3.2.** Let $a$ be defined as in Section 2. Assume that $a$ satisfies Condition (L) or (B). Let $Q: \Omega \to \mathbb{C}^{d \times d}$ be as before and set $W := Q(I + iQZQ)^{-1}Q$ and $P := I - Q$. Then the regular part of $a$ is given by

\[
a_{\text{reg}}(u, v) = ((I + iZ + ZWZ)PA^{1/2} \nabla u | PA^{1/2} \nabla v)
\]

\[
+ (\bar{X}^i(I - iWZ)PA^{1/2} \nabla u | v) + (u | \bar{Y}^i(I + iW^*Z)PA^{1/2} \nabla v)
\]

\[
- ((X^1WY)u | v) + (c_0u | v).
\]
Proof. We will use and adopt the notation of Proposition 3.1. First we represent the operators \( \pi_2, T\pi_2 \) and \((I_{V_1} + iT_{11})^{-1}\) by multiplication operators.

Observe that \( (u, w) \mapsto (0, w + \frac{1}{2}u(X + Y)) \) is the orthogonal projection of \( \mathcal{H}' \) onto \( \{0\} \times (L^2(\Omega))^d \). Moreover, considering \( \{0\} \times (L^2(\Omega))^d \) as a (closed) subspace of \( \mathcal{H}' \), the map \( (0, w) \mapsto (0, Qw) \) is the orthogonal projection of \( \{0\} \times (L^2(\Omega))^d \) onto \( V_1 \). Since \( V_1 \subset V \subset \mathcal{H}' \), the map \( \pi_1 \) is given by

\[
\pi_1(u, w) = (0, Qw + \frac{1}{2}uQ(X + Y))
\]

for all \((u, w) \in V\). Therefore

\[
\pi_2(u, w) = (u, (I - Q)w - \frac{1}{2}uQ(X + Y))
\]

for all \((u, w) \in V\).

Let \((u_1, w_1), (u_2, w_2) \in V\). It follows from [5] that

\[
(\mathfrak{Im} \tilde{a})((u_1, w_1), (u_2, w_2)) = (Zw_1 \mid w_2) - \frac{1}{2}(u_1 \mid u_2(X - Y)) + \frac{1}{2}(u_1(X - Y) \mid w_2) + ((\text{Im} c_0)u_1 \mid u_2).
\]

So if \((u_2, w_2) \in V_1\), then \(u_2 = 0\) and

\[
(\mathfrak{Im} \tilde{a})((u_1, w_1), (0, w_2)) = (Zw_1 \mid w_2) + \frac{1}{2}(u_1(X - Y) \mid w_2)
\]

\[
= (QZw_1 + \frac{1}{2}u_1 Q(X - Y) \mid w_2)
\]

\[
= \tilde{h}((0, QZw_1 + \frac{1}{2}u_1 Q(X - Y)), (0, w_2)).
\]

Hence the operator \( T \in \mathcal{L}(V, V_1) \) in Proposition 3.1 is given by

\[
T(u, w) = (0, QZw + \frac{1}{2}uQ(X - Y))
\]

for all \((u, w) \in V\). Then \( T_{11} = T|_{V_1} \) is given by \( T_{11}(0, w) = (0, QZw) \) for all \((0, w) \in V_1\). So as in [TESII] proof of (13) we have

\[
(I_{V_1} + iT_{11})^{-1}(0, w) = (0, (I + iQZQ)^{-1}w) = (0, Ww)
\]

for all \((0, w) \in V_1\). Next note that

\[
(8) \quad T\pi_2(u, w) = (0, QZ(I - Q)w - \frac{1}{2}uQZQ(X + Y) + \frac{1}{2}uQ(X - Y))
\]

for all \((u, w) \in V\).

Now we plug the representations for \( \pi_2, T\pi_2 \) and \((I_{V_1} + iT_{11})^{-1}\) into (3). One easily verifies that \( WZQ = i(W - Q) \). Then by a straightforward computation it follows from (3) that

\[
\Pi(u, w) = (u, (I - iWZ)Pw - uWy)
\]
We shall see that for differential sectorial forms the presence of lower-order terms can lead to a
simplification of the differential calculus. Using the identity
\[ Q_I = Q(1 + iZ)(I - iWZ)P = Q(I + iZ)(I - iWZ)P, \]
This simplifies the first-order terms in \( Q_I \) that involve \( w_1 \) and \( w_2 \). Using \( Q_{I2} \) and \( PW = 0 \), one establishes that
\[ P(I + iZW^*)(I + iZ)(I - iWZ)P = P(I + iZ + ZWZ)P. \]
This simplifies the second-order terms in \( Q_I \). One readily verifies that \( 2W^*W = W^* + W \).
Then, also using \( QZW = i(W - Q) \) and \( W^*P = 0 \), it follows that
\[ -W^*(I - iZ)(I - iWZ)P + (I - iWZ)P = (I + i(2W^*W - W)Z)P = (I + iW^*Z)P. \]
This simplifies the first-order terms that involve \( u_1 \) and \( u_2 \). Finally, for the terms involving \( u_1 \) and \( u_2 \), observe that
\[ Q(I + iZ)W - Q = 0. \]
Now the theorem follows from \( Q_I \) and Proposition 3.1.

Theorem 3.2 shows that the regular part \( a_{\text{reg}} \) is indeed a differential sectorial form. The most remarkable aspect of \( \Pi \) is the appearance of the zeroth-order term involving \( X \) and \( Y \), even if \( c_0 = 0 \), i.e., if the original form \( a \) did not have a zeroth-order term. This new zeroth-order term can affect the vertex of the singular part. A simple concrete example where this happens is given in Example 4.7.

4. About the sectoriality of the singular part

In this section we first characterise in the setting of Proposition 3.1 when the singular part \( a_s = a - a_{\text{reg}} \) is sectorial. In [4ES11, Proposition 3.1] we proved that \( \Re(a_{\text{reg}}) = (\Re a)_{\text{reg}} \) if and only if \( T\pi_2 = 0 \). Moreover, if \( a \) is a pure second-order differential sectorial form satisfying Condition (L) or (B), then this is also equivalent to \( a_s \) being sectorial, cf. [4ES11, Corollary 4.3].

We shall see that for differential sectorial forms the presence of lower-order terms can lead to a more diverse behaviour than possible in the pure second-order case.

Lemma 4.1. Assume the notation and conditions of Proposition 3.1. Then \( a_s = a - a_{\text{reg}} \) is sectorial if and only if there exists an \( M > 0 \) such that
\[ \| T\pi_2\Phi(u) \|^2_V \leq M\| u \|^2_H \]
for all \( u \in D(a) \).
Proof. If $a_s$ is sectorial, then (11) follows from [tES11, (8)] and the identity \( \tilde{j}(\pi(A(u))) = u \) for all $u \in D(a)$.

For the converse, note that we may assume without loss of generality that $\|u\|_V^2 = \tilde{h}(u,u)$ for all $u \in V$. By the formula for $\mathfrak{Re}(a_s)$ in the proof of [tES11, Proposition 3.4] and (11), there exists an $\omega_s > 0$ such that

\[
\mathfrak{Re} a_s(u,u) \geq \|\pi_1 \Phi(u)\|_V^2 - \omega_s \|u\|_H^2
\]

for all $u \in D(a)$. By the formula for $\mathfrak{Im}(a_s)$ in the proof of [tES11, Proposition 3.4], we have

\[
|\mathfrak{Im} a_s(u,u)| \leq \|T \pi_1 \Phi(u)\|_V \|\pi_1 \Phi(u)\|_V + 2 T \pi_2 \Phi(u)\|_V \|\pi_1 \Phi(u)\|_V
\]

\[+ \left\| T_{11} (I_{V_1} + T_{11}^2)^{-1} \right\| T \pi_2 \Phi(u)\|_V^2
\]

for all $u \in D(a)$. Using (11) and taking $C > 0$ sufficiently large, we obtain

\[
|\mathfrak{Im} a_s(u,u)| \leq C \left( \|\pi_1 \Phi(u)\|_V^2 + \|u\|_H^2 \right).
\]

for all $u \in D(a)$. By (12) this shows that $a_s$ is sectorial. \qed

Remark 4.2. It is readily verified that, after obvious modifications, Lemma 4.1 holds in the general $j$-sectorial setting of [tES11, Section 3].

From now on we assume that $a$ is defined as in Section 2. Moreover, we assume that $a$ satisfies the conditions of Theorem 3.2, i.e., we assume that $a$ satisfies Condition (L) or (B). In the following, we shall use the notation of Theorem 3.2.

Let $a^p$ be the differential sectorial form that belongs to the pure second-order differential expression

\[
- \sum_{k,l=1}^d \partial_k c_{kl} \partial_l.
\]

We denote the regular part and the singular part of $a^p$ by $a^p_{\text{reg}}$ and $a^p_{\text{s}}$, respectively. Observe that

\[
a_{\text{reg}}(u,v) = a^p_{\text{reg}}(u,v) + \left( (I - iWZ)PA^{1/2} \nabla u \mid v X \right) + \left( uY \mid (I + iW^* Z)PA^{1/2} \nabla v \right)
\]

\[+ \left( uWY \mid vX \right) + \left( c_0 u \mid v \right)
\]

for all $u, v \in D(a)$. Therefore with (2) one deduces that

\[
a_s(u,v) = a^p(u,v) + \left( (Q + iWZP)A^{1/2} \nabla u \mid v X \right) + \left( uY \mid (Q - iW^* ZP)A^{1/2} \nabla v \right)
\]

\[+ \left( uWY \mid vX \right)
\]

for all $u, v \in D(a)$.

The next lemma is an intermediate result in the proof of [tES11, Corollary 4.3].

Lemma 4.3. If $QZ(I - Q)A^{1/2} = 0$ a.e., then $QZ = ZQ$ a.e.

The following result generalises [tES11, Corollary 4.3].

Proposition 4.4. The following statements are equivalent.

(i) $a_s$ is sectorial.
Proof.
(i) \( QZ = ZQ \) a.e.
(ii) For all \( u,v \in D(a) \) one has
\[
\begin{align*}
a_{\text{reg}}(u,v) &= \left( (I+iZ)(I-Q)A^{1/2}\nabla u \mid (I-Q)A^{1/2}\nabla v \right) \\
&\quad + \left( X^t(I-Q)A^{1/2}\nabla u \mid v \right) + \left( u \mid Y^t(I-Q)A^{1/2}\nabla v \right) \\
&\quad - \left( (X^tI + iZ)^{-1}QY \right) u \mid v + \left( c_0 u \mid v \right).
\end{align*}
\]
(iii) For all \( u,v \in D(a) \) one has
\[
T\pi_2\Phi(u) = \left( 0, -\frac{1}{2}uQZQ(X + Y) + \frac{1}{2}uQ(X-Y) \right).
\]
(iv) \( \mathfrak{a}_s^p \) is sectorial.

Remark 4.5. Suppose that \( \mathfrak{a}_s \) is sectorial. Then by Proposition 4.4 (i) \( (i) \Rightarrow (ii) \) and (14) it follows that
\[
\mathfrak{a}_s(u,v) = \mathfrak{a}_s^p(u,v) + \left( QA^{1/2}\nabla u \mid vX \right) + \left( uY \mid QA^{1/2}\nabla v \right) + \left( uWY \mid vX \right)
\]
for all \( u,v \in D(a) \).

Next we characterise when the regular part of the real part equals the real part of the regular part.

Lemma 4.6. We have \((\Re \mathfrak{a})_{\text{reg}} = \Re \mathfrak{a}_{\text{reg}}\) if and only if both \( QZ = ZQ \) and \( (I+iZ)QX = (I-iZ)QY \) a.e.

Proof. By [EST11 Proposition 3.1], we know that \((\Re \mathfrak{a})_{\text{reg}} = \Re \mathfrak{a}_{\text{reg}}\) if and only if \( T\pi_2 = 0 \).

\( \Rightarrow \): Suppose \( T\pi_2 = 0 \). By Lemma 4.1 the form \( \mathfrak{a}_s \) is sectorial. Therefore it follows from Proposition 4.4 (i) \( (i) \Rightarrow (ii) \) and (i) \( \Rightarrow (iv) \) that \( QZ = ZQ \) and \( iQ(X-Y) = QZQ(X+Y) \) a.e. Now the claim follows by rearranging the terms.

\( \Leftarrow \): After rearranging terms, we obtain \( iQ(X-Y) = QZQ(X+Y) \) a.e. and \( QZ = ZQ \) a.e. Hence it follows directly from (8) that \( T\pi_2 = 0 \).
We finish with an example that shows that $a_s$ can be sectorial while at the same time $(\Re a)_{\text{reg}} \neq \Re(a_{\text{reg}})$. Moreover, the example shows that if $\gamma$ is a vertex for $a$, then $\gamma$ needs not to be a vertex for $a_s$. Both phenomena do not occur for differential sectorial forms that are purely of second order.

**Example 4.7.** Let $K \subset [0, 1]$ be a compact set with empty interior and strictly positive Lebesgue measure $|K|$. Consider the form $a: H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ given by

$$a(u, v) = \int_\mathbb{R} 1_K u' v' + \int_\mathbb{R} 1_K u v' - \int_\mathbb{R} 1_K u' v + \int_\mathbb{R} 1_K u v.$$ 

Then $a$ is sectorial in $L^2(\mathbb{R})$. More precisely,

$$(\Re a)(u, v) = \int_\mathbb{R} 1_K u' v' + \int_\mathbb{R} 1_K u v$$

and

$$|\Im a(u, u)| \leq \Re a(u, u)$$

for all $u, v \in H^1(\mathbb{R})$. So $a$ has vertex 0. It follows from \cite[Lemma 4.4]{TEST} that we may take $Q = 1_K$. Clearly $Z = 0$, so $a_s$ is sectorial by Proposition 4.4. Using the formula in Proposition 4.4(iii), we obtain

$$a_{\text{reg}}(u, v) = 2 \int_\mathbb{R} 1_K u v$$

and hence

$$a_s(u, v) = \int_\mathbb{R} 1_K u' v' + \int_\mathbb{R} 1_K u v' - \int_\mathbb{R} 1_K u' v - \int_\mathbb{R} 1_K u v$$

for all $u, v \in H^1(\mathbb{R})$. It is easily seen that

$$\Re(a_{\text{reg}}) = a_{\text{reg}} \neq \frac{1}{2} a_{\text{reg}} = (\Re a)_{\text{reg}}.$$ 

Now let $u \in C^\infty_c(\mathbb{R})$ be such that $u|_{[0, 1]} = 1$. Then $\Re a_s(u, u) = -|K| < 0$. This shows that 0 is not a vertex of $a_s$.

Finally, if $b: H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ is the form without zeroth-order term given by

$$b(u, v) = \int_\mathbb{R} 1_K u' v' + \int_\mathbb{R} 1_K u v' - \int_\mathbb{R} 1_K u' v,$$

then

$$b_{\text{reg}}(u, v) = \int_\mathbb{R} 1_K u v$$

for all $u, v \in H^1(\mathbb{R})$ and $b_{\text{reg}}$ contains a non-trivial zeroth-order term.

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