Classification of nilpotent Lie superalgebras of multiplier-rank $\leq 2$

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Abstract: In this paper, we introduce the concept of (super-)multiplier-rank for Lie superalgebras and classify all the finite-dimensional nilpotent Lie superalgebras of multiplier-rank $\leq 2$ over an algebraically closed field of characteristic zero. In the process, we also determine the multipliers of Heisenberg superalgebras.

Keywords: Lie superalgebra; multipliers; (super-)multiplier-rank

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1. Introduction

As is well known, the notion of multipliers and covers for a group arose from Schur’s work on projective representations of groups. Analogous to the group theory case, for a finite-dimensional Lie algebra $L$ over a field, a cover is a central extension of the maximal possible dimension of $L$ with a kernel contained in the derived algebra of $L$; the corresponding kernel is a (Schur) multiplier of $L$. For a finite-dimensional Lie algebra, there exist uniquely a cover and a multiplier up to Lie algebra isomorphism, respectively. A typical fact analogous to the group theory case is that the multiplier of a finite-dimensional Lie algebra $L$ is isomorphic to the second cohomology group of $L$ with coefficients in the 1-dimensional trivial module $[1]$. The study on Schur multipliers of Lie algebras began in 1990’s (see [3, 9], for example) and the theory has seen a fruitful development (see [2, 5–7, 11, 14, 17], for example). Among the literatures, a main work is finding an upper bound for the multiplier dimension for a finite-dimensional nilpotent Lie algebra and classifying finite-dimensional nilpotent Lie algebras under certain conditions in terms of multipliers (see [2, 6, 7, 11, 14, 17], for example).

The notion of multipliers for Lie algebras may be naturally generalized to Lie superalgebra case. In this paper we first establish several lemmas for Lie superalgebras, which are parallel to the ones in non-super case. Then we introduce the notions of (super-)multiplier-ranks and (super-)derived-rank, which are analogous to two invariants in Lie algebra case. Our main result is classifying all the nilpotent Lie superalgebras of multiplier-rank $\leq 2$. As a byproduct, we also determine the multipliers of Heisenberg superalgebras.

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2. Basics

In this paper all (super)spaces, (super)algebras are over an algebraically closed field $F$ of characteristic zero. Let $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ the abelian group of order 2 and $V = V_\bar{0} \oplus V_\bar{1}$ a superspace. For a homogeneous element $x$ in $V$, write $|x|$ for the parity of $x$. The symbol $|x|$ implies that $x$ has been assumed to be a homogeneous element.

In $\mathbb{Z} \times \mathbb{Z}$, we define a partial order as follows:

$$(m, n) \leq (k, l) \iff m \leq k, n \leq l.$$ 

For $m, n \in \mathbb{Z}$, we write $|(m, n)| = m + n$. We also view $\mathbb{Z} \times \mathbb{Z}$ as the additive group in the usual way.

Write $\text{sdim} V$ for the superdimension of a superspace $V$ and $\dim V$ for the dimension of $V$ as an ordinary vector space. Note that

$$\dim V = |\text{sdim} V|.$$ 

Let $\Pi$ be the parity functor of superspaces. Then

$$\text{sdim} V + \text{sdim} \Pi(V) = (\dim V, \dim V).$$ 

Moreover, if $W$ is a subsuperspace of a superspace $V$, then

$$\text{sdim} V/W = \text{sdim} V - \text{sdim} W.$$ 

In this paper, we write $\text{Ab}(m, n)$ for the abelian Lie superalgebra of superdimension $(m, n)$.

As in the Lie algebra case [3, p. 4302], we introduce the following definition.

**Definition 2.1.** Let $L$ be a finite-dimensional Lie superalgebra. A Lie superalgebra pair $(K, M)$ is called a defining pair of $L$ provided that $L \cong K/M$ and $M \subset Z(K) \cap K^2$. A defining pair $(K, M)$ of $L$ is said to be maximal if among all the defining pairs of $L$, $M$ is of a maximal superdimension. In the case $(K, M)$ being a maximal defining pair of $L$, we also call $K$ a cover and $M$ a (Schur) multiplier of $L$.

The definition makes sense, since one may check as in Lie algebra case (see [3]) that for a finite-dimensional Lie superalgebra, covers and multipliers always exist and they are unique up to Lie superalgebra isomorphism, respectively. Write $\mathcal{C}(L)$ and $\mathcal{M}(L)$ for the cover and multiplier of Lie superalgebra $L$, respectively.

As in Lie algebra case [3], we will give an upper bound for the superdimension of the multiplier of a Lie superalgebra. To that aim, we first establish the following lemmas, which will be used in the sequel.

Recall that $Z(L) := \{x \in L \mid [x, L] = 0\}$ denotes the center of $L$ and $L^2 := [L, L]$ denotes the derived subalgebra of $L$.

**Lemma 2.2.** Let $L$ be a Lie superalgebra and suppose $\text{sdim} L/Z(L) = (m, n)$. Then

$$\text{sdim} L^2 \leq \left(\frac{1}{2} m(m - 1) + \frac{1}{2} n(n + 1), mn\right).$$

**Proof.** Let $\{u_1, \ldots, u_m \mid v_1, \ldots, v_n\}$ be a homogeneous cobasis of $Z(L)$ in $L$, where

$$|u_i| = \bar{0}, |v_j| = \bar{1}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$
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It follows that \( L^2 \) is spanned by the following elements:

\[
\begin{align*}
    x_{i,j} & := [u_i, u_j], \quad 1 \leq i < j \leq m; \\
    y_{i,j} & := [v_i, v_j], \quad 1 \leq i \leq j \leq n; \\
    z_{i,j} & := [u_i, v_j], \quad 1 \leq i \leq m, \quad 1 \leq n \leq l,
\end{align*}
\]

where \(|x_{i,j}| = |y_{i,j}| = 0\) and \(|z_{i,j}| = 1\). Then our conclusion follows.

**Lemma 2.3.** Let \( L \) be a Lie superalgebra with \( \text{sdim} L = (s, t) \). Then

\[
\text{sdim} \mathcal{M}(L) \leq \left( \frac{1}{2} s(s - 1) + \frac{1}{2} t(t + 1), st \right).
\]

**Proof.** Let \((K, M)\) be a defining pair of \( L \) and suppose \( \text{sdim} K / \text{Z}(K) = (k, l) \). Then \((k, l) \leq \text{sdim} K / \text{M} = (s, t)\) and by Lemma 2.2 we have

\[
\text{sdim} M \leq \text{sdim} K^2 \leq \left( \frac{1}{2} k(k - 1) + \frac{1}{2} l(l + 1), kl \right) \leq \left( \frac{1}{2} s(s - 1) + \frac{1}{2} t(t + 1), st \right).
\]

The proof is complete.

We should note that a non-super version of Lemmas 2.2 and 2.3 has been given in [10, Theorem 3.1 and Theorem 3.4].

For a Lie superalgebra \( L \) of superdimension \((s, t)\), define the super-multiplier-rank of \( L \) to be the number pair

\[
\text{smr}(L) = \left( \frac{1}{2} s(s - 1) + \frac{1}{2} t(t + 1), st \right) - \text{sdim} \mathcal{M}(L)
\]

and the multiplier-rank of \( L \) to be

\[
\text{mr}(L) = |\text{smr}(L)|.
\]

By Lemma 2.3, we have \( \text{smr}(L) \geq (0, 0) \).

The main purpose of the present paper is to determine all the nilpotent Lie superalgebras \( L \) satisfying that \( |\text{smr}(L)| \leq 2 \).

As in the Lie algebra case [2, Lemma 4 and Theorem 1], using the notion of free presentations for Lie superalgebras, one may prove the following two lemmas.

**Lemma 2.4.** Let \( L \) be a finite-dimensional Lie superalgebra. Then \( L^2 \cap \text{Z}(L) \) is a homomorphic image of \( \mathcal{M}(L / \text{Z}(L)) \).

**Lemma 2.5.** Let \( A \) and \( B \) be finite-dimensional Lie superalgebras. Then

\[
\text{sdim} \mathcal{M}(A \oplus B) = \text{sdim} \mathcal{M}(A) + \text{sdim} \mathcal{M}(B) + \text{sdim}(A/A^2 \otimes B/B^2).
\]
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3. Multiplier-rank 0 nilpotent Lie superalgebras

Using Lemma 2.2, we can determine all nilpotent Lie superalgebras of multiplier-rank 0. The multiplier-rank 0 case was also considered in [10, Theorem 3.5], where the multipliers was described in terms of non-super dimensions. For completeness, we give a proof, which is also somewhat different from the one in [10, Theorem 3.5].

**Proposition 3.1.** Let $L$ be a finite-dimensional Lie superalgebra. Then $\text{smr}(L) = (0, 0)$ if and only if $L$ is abelian.

**Proof.** Let $L$ be an of superdimension $(m, n)$. Suppose $L$ is abelian. Let $H$ be a superspace with a homogeneous basis
\[
\{ u_i, x_{k,l}, z_{s,t} | v_j, y_{p,q} \}
\]
where
\[
1 \leq i \leq m, \ 1 \leq k < l \leq m, \ 1 \leq s \leq t \leq n, \ 1 \leq j \leq n, \ 1 \leq p \leq m, \ 1 \leq q \leq n
\]
and
\[
|u_i| = |x_{k,l}| = |z_{s,t}| = 0, \ |v_j| = |y_{p,q}| = 1.
\]
Then $H$ becomes a Lie superalgebra by letting
\[
[u_k, u_l] = x_{k,l}, \ [u_p, v_q] = y_{p,q}, \ [v_s, v_t] = z_{s,t}
\]
and the other brackets of basis elements vanish. Clearly, $L \cong H/H^2$. Since $Z(H) = H^2$, one sees that $H^2 \subseteq Z(H) \cap H^2$. Hence $(H, H^2)$ is a defining pair of $L$ and $\text{smr}(L) = (0, 0)$.

Conversely, suppose $\text{smr}(L) = (0, 0)$. Then
\[
\text{sdim}_M(L) = \left( \frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn \right).
\]
Let $(K, M)$ be a maximal defining pair of $L$ and suppose $\text{sdim}_K/Z(K) = (k, l)$. Since $M \subseteq Z(K)$, it follows from Lemma 2.2 that
\[
\text{sdim}K^2 \leq \left( \frac{1}{2}k(k-1) + \frac{1}{2}(l+1)kl \right) \leq \left( \frac{1}{2}m(m-1) + \frac{1}{2}(n+1), mn \right).
\]
Since $M \subseteq K^2$, we have
\[
\left( \frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn \right) = \text{sdim}M \leq \text{sdim}K^2 \leq \left( \frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn \right).
\]
Therefore, $M = K^2$ and $L \cong K/M$ is abelian. \qed
4. Multiplier-rank 1 nilpotent Lie superalgebras

Throughout this section $L$ is a finite-dimensional non-abelian nilpotent Lie superalgebra and $sdimL/Z(L) = (m, n)$.

To determine all the nilpotent Lie superalgebras of multiplier-rank 1, we will establish several technical lemmas.

Recall that $Z(L)$ is the center of $L$ and $Z_2(L)$ is the ideal of $L$ such that $Z_2(L)/Z(L) = Z(L/Z(L))$. Suppose $z \in Z_2(L)\backslash Z(L)$ is a homogeneous element. Then $[L, z] \subseteq Z(L)$ is an ideal of $L$. For convenience, write

$$\lambda(z) = sdim[L, z], \quad \mu(z) = sdim((L/[L, z])/Z(L/[L, z])).$$

**Lemma 4.1.** Suppose $z \in Z_2(L)\backslash Z(L)$.

(1) If $|z| = 0$, then $\lambda(z) \leq (m - 1, n)$ and $\mu(z) \leq (m - 1, n)$.

(2) If $|z| = 1$, then $\mu(z) \leq (m, n - 1)$.

**Proof.** (1) In this case, it is clear that $L/Z_L(z) \cong [L, z]$ as superspaces. Since $z \notin Z(L)$, we have $Z(L) \subseteq Z_L(z)$ and $sdimZ(L) + (1, 0) \leq sdimZ_L(z)$. Hence

$$\lambda(z) = sdim[L, z] = sdimL - sdimZ_L(z) \leq sdimL - (sdimZ(L) + (1, 0)) = (m - 1, n)$$

Note that $z + [L, z] \in Z(L/[L, z])$ and $z + [L, z] \notin Z(L)/[L, z]$. Then $Z(L)/[L, z] \subseteq Z(L/[L, z])$ and

$$sdimZ(L)/[L, z] + (1, 0) \leq sdimZ(L/[L, z]).$$

Hence

$$\mu(z) = sdim((L/[L, z])/Z(L/[L, z])) = sdim((L/[L, z])/Z(L/[L, z])) \leq sdim((L/[L, z]) - (sdimZ(L)/[L, z] + (1, 0)) = sdimL - sdimZ(L) - (1, 0) = (m - 1, n)$$

(2) Note that $z + [L, z] \in Z(L/[L, z])$ and $z + [L, z] \notin Z(L)/[L, z]$. We have $Z(L)/[L, z] \subseteq Z(L/[L, z])$ and

$$sdimZ(L)/[L, z] + (0, 1) \leq sdimZ(L/[L, z]).$$

Hence

$$\mu(z) = sdim((L/[L, z])/Z(L/[L, z])) = sdim((L/[L, z])/Z(L/[L, z])) \leq sdim((L/[L, z]) - (sdimZ(L)/[L, z] + (0, 1)) = sdimL - sdimZ(L) - (0, 1) = (m, n - 1)$$

The proof is complete.
Recall that $\text{sdim}(L/Z(L)) = (m, n)$. Define the super-derived-rank of Lie superalgebra $L$ to be
\[
\text{sdr}(L) = \left( \frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn \right) - \text{sdim}L^2
\]
and the derived-rank to be
\[
\text{dr}(L) = |\text{sdr}(L)|.
\]
It follows from Lemma 2.2 that $\text{sdr}(L) \geq (0, 0)$. For our purpose, we will first determine all the nilpotent Lie superalgebras $L$ with $\text{dr}(L) \leq 1$.

Let $z \in Z_2(L) \setminus Z(L)$. Suppose $|z| = 1$. Then
\[
f : L \longrightarrow [L, z] \quad \quad x \longmapsto [x, z]
\]
is an odd linear epimorphism. Note that $\ker f = Z_L(z)$. Thus we have the following superspace isomorphism:
\[
\Pi(L/Z_L(z)) \cong [L, z].
\]

Lemma 4.2. The following statements hold.

1. If the center of $L/Z(L)$ has a nonzero even part, then
\[
\text{sdim}(L/Z(L))^2 \leq \text{sdr}(L) + (1, 0).
\]

2. If the center of $L/Z(L)$ has a nonzero odd part, then
\[
\text{sdim}(L/Z(L))^2 \leq (\text{dr}(L), \text{dr}(L)) - \text{sdr}(L).
\]

Proof. Suppose $z$ is a homogeneous element in $Z_2(L) \setminus Z(L)$ and $\mu(z) = (b_1, b_2)$. By Lemma 2.2 we have
\[
\text{sdim}(L/[L, z])^2 \leq \left( \frac{1}{2}b_1(b_1 - 1) + \frac{1}{2}b_2(b_2 + 1), b_1b_2 \right).
\]
Hence
\[
\text{sdim}L^2 \leq \left( \frac{1}{2}b_1(b_1 - 1) + \frac{1}{2}b_2(b_2 + 1), b_1b_2 \right) + \text{sdim}[L, z].
\]
Therefore,
\[
\left( \frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn \right) - \text{sdr}(L) \leq \left( \frac{1}{2}b_1(b_1 - 1) + \frac{1}{2}b_2(b_2 + 1), b_1b_2 \right) + \lambda(z).
\]
(1) Suppose $|z| = 0$. By Lemma 4.1, $\mu(z) \leq (m-1, n)$ and therefore
\[
\lambda(z) \geq (m, n) - (\text{sdr}(L) + (1, 0)). \tag{4.1}
\]
Since $L^2 \subset Z_L(z)$ and $Z(L) \subset Z_L(z)$, we have $L^2 + Z(L) \subset Z_L(z)$. Thus
\[
\text{sdim}L/(L^2 + Z(L)) \geq \text{sdim}L/Z_L(z)
\]
and
\[
\text{sdim}(L/Z(L))^2 = \text{sdim}L/Z(L) - \text{sdim}L/(L^2 + Z(L))
\leq \text{sdim}L/Z(L) - \text{sdim}L/Z_L(z)
= \text{sdim}L/Z(L) - \lambda(z)
\leq \text{sdr}(L) + (1, 0).
\]
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(2) Suppose \(|z| = \bar{1}\). By Lemma 4.1, we have \( \mu(z) \leq (m, n - 1) \) and therefore

\[
\lambda(z) \geq (n, m) - \text{sdr}(L).
\]

Since \( L^2 \subset Z_L(z) \) and \( Z(L) \subset Z_L(z) \), we have \( L^2 + Z(L) \subset Z_L(z) \). Thus

\[
s\dim(L/Z(L))^2 = \dim(L/Z(L) - \text{sdim}(L^2 + Z(L)) \\
\leq \dim(L/Z(L) - \text{sdim}(L/Z_L(z)) \\
= \dim(L/Z(L)) - \text{sdim}(L/Z_L(z)) \\
= \dim(L/Z(L)) - ((|\lambda(z)|, |\lambda(z)|) - \lambda(z)) \\
\leq (\text{dr}(L), \text{dr}(L)) - \text{sdr}(L).
\]

The proof is complete. \( \square \)

**Lemma 4.3.** Suppose \( z \in Z_2(L) \setminus Z(L) \). Suppose \(|z| = \bar{0}\) and \( \text{sdim}(L/Z(L))^2 = \text{sdr}(L) + (1, 0) \). Then \( Z_L(z) = L^2 + Z(L) \) and \( \lambda(z) = (m, n) - (\text{sdr}(L) + (1, 0)) \).

**Proof.** By \( \text{(1.1)} \), we have

\[
\text{sdim}(L/Z_L(z)) = \lambda(z) \geq (m, n) - (\text{sdr}(L) + (1, 0)).
\]

Hence

\[
\text{sdr}(L) + (1, 0) = \text{sdim}(L/Z(L))^2 \\
= \text{sdim}(L/Z(L)) - \text{sdim}(L^2 + Z(L)) \\
\leq \text{sdim}(L/Z(L)) - \text{sdim}(L/Z_L(z)) \\
\leq \text{sdr}(L) + (1, 0).
\]

Therefore, \( \text{sdim}(Z_L(z)) = \text{sdim}(L^2 + Z(L)) \). Since \( L^2 + Z(L) \subset Z_L(z) \), we have \( L^2 + Z(L) = Z_L(z) \). Then

\[
\lambda(z) = \dim(L/Z_L(z)) \\
= \dim(L/(L^2 + Z(L)) \\
= \dim(L/Z(L)) - \text{sdim}(L^2 + Z(L))/Z(L) \\
= (m, n) - (\text{sdr}(L) + (1, 0)).
\]

The proof is complete. \( \square \)

Recall that a finite-dimensional Lie superalgebra \( g \) is called a Heisenberg Lie superalgebra provided that \( g^2 = Z(g) \) and \( \text{sdim}(Z(g)) = (1, 0) \) or \( (0, 1) \). Heisenberg Lie superalgebras consist of two types according to the parity of the central elements (see \textsection 16). Suppose \( g \) is a Heisenberg Lie superalgebra and \( Z(g) = F_z \).

(1) If \(|z| = 0\), then \( g \) has a homogeneous basis (called a standard basis)

\[
\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p, z \mid w_1, w_2, \ldots, w_q\},
\]

where

\[
|u_i| = |v_j| = |z| = \bar{0}, \quad |w_k| = \bar{1}; \quad i = 1, \ldots, p, \quad j = 1, \ldots, p, \quad k = 1, \ldots, q.
\]
and the multiplication is given by
\[ [u_i, v_i] = -[v_i, u_i] = z, \quad [w_k, w_k] = z \]
and the other brackets of basis elements vanishing. Denote by \( H(p, q) \) the Heisenberg Lie superalgebra \( g \) of even center, where \( p + q \geq 1 \).

(2) If \( |z| = 1 \), then \( g \) has a homogeneous basis (called a standard basis)
\[ \{u_1, u_2, \ldots, u_k \mid z, w_1, w_2, \ldots, w_k \}, \]
where
\[ |u_i| = 0, \quad |w_j| = |z| = 1; \quad i = 1, \ldots, m, \quad j = 1, \ldots, k, \]
and the multiplication is given by
\[ [u_i, w_i] = -[w_i, u_i] = z \]
and the other brackets of basis elements vanishing. We write \( H(k) \) for the Heisenberg Lie superalgebra \( g \) of odd center, where \( k \geq 1 \).

**Proposition 4.4.** Let \( H(p, q) \) be a Heisenberg Lie superalgebra of even center. Then
\[
\text{sdim} \mathcal{M}(H(p, q)) = \begin{cases} 
(2p^2 - p + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq), & p + q \geq 2 \\
(0, 0), & p = 0, q = 1 \\
(2, 0), & p = 1, q = 0 
\end{cases}
\]

**Proof.** We consider only the case \( p = 0, q = 1 \), while the remaining cases may be argued as in [10, Theorem 4.3]. Suppose \( (K, M) \) is a defining pair of \( H(0, 1) \) and \( K/M \) has a standard basis \( \{a + M \mid b + M\} \), where \( a, b \in K \) with \( |a| = 0, \ |b| = 1 \). Then \( [b, b] \equiv a \pmod{M} \). Since \( [[b, b], b] = 0 \), one sees \( [a, b] = 0 \). It follows that \( K^2 \) is 1-dimensional and not contained in \( M \). Since \( M \subset K^2 \), we have \( M = 0 \). The proof is complete. \( \square \)

As in Lie algebra case [10, Theorem 4.3], one can determine the multiplier and cover for Heisenberg Lie superalgebras of odd center [3].

**Proposition 4.5.** Let \( H(k) \) be a Heisenberg Lie superalgebra of odd center. Then
\[
\text{sdim} \mathcal{M}(H(k)) = \begin{cases} 
(k^2, k^2 - 1), & k \geq 2 \\
(1, 1), & k = 1 
\end{cases}
\]

**Lemma 4.6.** If \( \text{sdr}(L) = (0, 0) \), then \( L/Z(L) \) is either abelian or isomorphic to \( H(1, 0) \).

**Proof.** If \( Z_2(L)/Z(L) \) has a nonzero odd part, then by Lemma [4.2], we have \( \text{sdim}(L/Z(L))^2 = (0, 0) \) and hence \( L/Z(L) \) is abelian.

Suppose the odd part of \( Z_2(L)/Z(L) \) is zero. Then by Lemma [4.2], we have \( \text{sdim}(L/Z(L))^2 = (0, 0) \) or \( (1, 0) \). If \( \text{sdim}(L/Z(L))^2 = (0, 0) \), then \( L/Z(L) \) is abelian. Thus we suppose \( \text{sdim}(L/Z(L))^2 = (1, 0) \). Then \( (L/Z(L))^3 = 0 \) and therefore \( (L/Z(L))^3 \subset Z(L/Z(L)) \). Assert that \( \text{sdim}(Z(L/Z(L))) = (1, 0) \). For any even element \( x \in Z_2(L) \), by Lemma [4.1], we have \( \mu(x) \leq (m - 1, n) \). Since \( \text{sdr}(L) = (0, 0) \), \( (L/Z(L))^2 = L^2/L, x \), and \( \mu(x) \leq (m - 1, n) \), it follows from Lemma [4.2] that \( \lambda(x) \geq (m - 1, n) \). By Lemma [4.1], we
also have \( \lambda(x) \leq (m - 1, n) \). Therefore, \( \lambda(x) = (m - 1, n) \). Then, since \([L, x] \cong L/Z_L(x)\), we have

\[
s\dim Z_L(x)/Z(L) = s\dim(L/Z(L)) - \lambda(x) = (1, 0).
\]

Let \( y \in Z_2(L)\backslash Z(L) \) be even. We have

\[
Z(L) \subset Z_L(x) \cap Z_L(y) \subset Z_L(x).
\]

If \( Z(L) = Z_L(x) \cap Z_L(y) \), since \( L^2 \subset Z_L(x) \cap Z_L(y) \), we have \( L^2 \subset Z(L) \) and \( s\dim(L/Z(L))^2 = (0, 0) \), a contradiction. Hence \( Z_L(x) \cap Z_L(y) = Z_L(y) \) and then \( Z_L(x) = Z_L(y) \). Since \( s\dim Z_L(x)/Z(L) = (1, 0) \), one sees that \( x + Z(L) \) and \( y + Z(L) \) are linearly dependent. Hence \( s\dim Z(L/Z(L)) = (1, 0) \) and \( L/Z(L) \) is a Heisenberg Lie superalgebra of even center. Suppose \( L/Z(L) \cong H(p, q) \), where \( p + q \geq 1 \). Assume that \( p + q \geq 2 \). By Proposition 4.4 we have

\[
s\dim M(L/Z(L)) = (2p^2 - p + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq)\]

and then by Lemma 2.4 we have

\[
s\dim L^2 = s\dim(L/Z(L))^2 + s\dim(L^2 \cap Z(L)) \\
\leq s\dim(L/Z(L))^2 + s\dim M(L/Z(L)) \\
< s\dim(L/Z(L)) + s\dim M(L/Z(L)) \\
= (2p + 1, q) + (2p - p + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq) \\
= (2p^2 + p + \frac{1}{2}q^2 + \frac{1}{2}q, 2pq + q).
\]

However, since \( s\dim(L) = (0, 0) \), we have

\[
s\dim L^2 = (2p^2 + p + \frac{1}{2}q^2 + \frac{1}{2}q, 2pq + q),
\]

a contradiction. Assume that \( p = 0 \) and \( q = 1 \). By Proposition 4.4 \( s\dim M(H(0, 1)) = (0, 0) \). Then by Lemma 2.4 and \( s\dim(L) = (0, 0) \), we have

\[
(1, 1) = s\dim L^2 = s\dim(L/Z(L))^2 + s\dim(L^2 \cap Z(L)) \\
\leq s\dim(L/Z(L))^2 + s\dim M(L/Z(L)) \\
= (1, 0),
\]

a contradiction. Summarizing, \( L/Z(L) \) is abelian or isomorphic to \( H(1, 0) \). \( \square \)

**Lemma 4.7.** Suppose \( L/Z(L) \cong g \), where \( g \) is a Heisenberg Lie superalgebra.

(1) If \( s\dim(L) = (1, 0) \), then \( g \cong H(1, 0) \).

(2) If \( s\dim(L) = (0, 1) \), then \( g \cong H(0, 1) \).

**Proof.** (1) Suppose \( L/Z(L) = H(p, q) \), where \( p + q \geq 1 \). Assume that \( p + q \geq 2 \). Then by Lemma 2.4 and Proposition 4.4 for \( p \neq 0 \),

\[
s\dim L^2 = s\dim(L/Z(L))^2 + s\dim(L^2 \cap Z(L)) \\
\leq s\dim(L/Z(L))^2 + s\dim M(L/Z(L)) \\
< s\dim L/Z(L) - (1, 0) + s\dim M(L/Z(L)) \\
= (2p^2 + p + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq + q).
\]
and for \( p = 0 \),

\[
\text{sdim}L^2 = \text{sdim}(L/Z(L))^2 + \text{sdim}(L^2 \cap Z(L)) \\
\leq \text{sdim}(L/Z(L))^2 + \text{sdim}M(L/Z(L)) \\
< \text{sdim}L/Z(L) - (0, 1) + \text{sdim}M(L/Z(L)) \\
= (2p^2 + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq + q - 1).
\]

However, since \( \text{sdr}(L) = (1, 0) \), we have \( \text{sdim}L^2 = (2p^2 + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq + q) \), a contradiction. Assume that \( p = 0 \) and \( q = 1 \). By Proposition 4.4, \( \text{sdim}M(H(0, 1)) = (0, 0) \).

Then by \( \text{sdr}(L) = (1, 0) \) and Lemma 2.4, we have

\[
(0, 1) = \text{sdim}L^2 = \text{sdim}(L/Z(L))^2 + \text{sdim}(L^2 \cap Z(L)) \\
\leq \text{sdim}(L/Z(L))^2 + \text{sdim}M(L/Z(L)) \\
= (1, 0),
\]

a contradiction.

Suppose \( L/Z(L) = H(k) \), where \( k \geq 1 \). Assume that \( k > 1 \). By Proposition 4.5, we have \( \text{sdim}M(L/Z(L)) = (k^2, k^2 - 1) \). Then by Lemma 2.4, we have

\[
\text{sdim}L^2 = \text{sdim}(L/Z(L))^2 + \text{sdim}(L^2 \cap Z(L)) \\
\leq \text{sdim}(L/Z(L))^2 + \text{sdim}M(L/Z(L)) \\
< \text{sdim}L/Z(L) - (0, 1) + \text{sdim}M(L/Z(L)) \\
= (k^2 + k, k^2 + k - 1).
\]

However, since \( \text{sdr}(L) = (1, 0) \), we have \( \text{sdim}L^2 = (k^2 + k, k^2 + k) \), a contradiction. Assume that \( k = 1 \). By Proposition 4.5, we have \( \text{sdim}M(H(1)) = (1, 1) \). Then by \( \text{sdr}(L) = (1, 0) \) and Lemma 2.4, we have

\[
(2, 2) = \text{sdim}L^2 = \text{sdim}(L/Z(L))^2 + \text{sdim}(L^2 \cap Z(L)) \\
\leq \text{sdim}(L/Z(L))^2 + \text{sdim}M(L/Z(L)) \\
< \text{sdim}L/Z(L) - (0, 1) + \text{sdim}M(L/Z(L)) \\
= (2, 2),
\]

a contradiction. Summarizing, we have \( L/Z(L) = H(1, 0) \).

(2) Suppose \( L/Z(L) = H(p, q) \), where \( p + q \geq 1 \). Assume that \( p + q \geq 2 \). Then by Lemma 2.4 and Proposition 4.4, we have

\[
\text{sdim}L^2 = \text{sdim}(L/Z(L))^2 + \text{sdim}(L^2 \cap Z(L)) \\
\leq \text{sdim}(L/Z(L))^2 + \text{sdim}M(L/Z(L)) \\
= \left( 2p^2 - p + \frac{1}{2}q^2 + \frac{1}{2}q, 2pq \right). \tag{4.3}
\]

However, since \( \text{sdr}(L) = (0, 1) \), we have

\[
\text{sdim}L^2 = \left( 2p^2 + p + \frac{1}{2}q^2 + \frac{1}{2}q, 2pq + q - 1 \right). \tag{4.4}
\]

By (4.3) and (4.4), we have \( p + q \leq 1 \), a contradiction. Assume that \( p = 1, q = 0 \). Since \( \text{sdr}(L) = (0, 1) \), we have \( \text{sdim}L^2 = (3, -1) \), contradicting the assumption that \( \text{sdim}L^2 > (0, 0) \).
Suppose $L/Z(L) = H(k)$, where $k \geq 1$. Assume that $k > 1$. Then by Lemma 2.4 and Proposition 4.5, we have

\[
sdim L^2 = sdim(L/Z(L))^2 + sdim(L^2 \cap Z(L)) \\
\leq sdim(L/Z(L))^2 + sdim(M/L/Z(L)) \\
< sdim L/Z(L) - (0, 1) + sdim M(L/Z(L)) = (k^2 + k, k^2 + k^2 - 1).
\]

However, since $sdr(L) = (0, 1)$, we have

\[
sdim L^2 = (\frac{1}{2}s(k - 1) + \frac{1}{2}(k + 1)(k + 2), k(k + 1) - 1) = (k^2 + k + 1, k^2 + k^2 + k - 1),
\]
a contradiction. Assume that $k = 1$. By Proposition 4.5, we have $sdim M(H(1)) = (1, 1)$. Since $sdr(L) = (0, 1)$, then by Lemma 2.4, we have

\[
(3, 1) = sdim L^2 = sdim(L/Z(L))^2 + sdim(L^2 \cap Z(L)) \\
\leq sdim(L/Z(L))^2 + sdim(M/L/Z(L)) \\
< sdim L/Z(L) + sdim M(L/Z(L)) = (2, 3),
\]
a contradiction. Summarizing, we have $L/Z(L) = H(0, 1)$.

The following proposition analogues to [2, Theorem 3].

**Proposition 4.8.** Suppose $L$ is a non-abelian nilpotent Lie superalgebra. Then

(1) $smr L \neq (0, 1)$.

(2) $smr(L) = (1, 0)$ if and only if $L \cong H(1, 0)$.

**Proof.** Let $sdim L = (s, t)$.

(1) Assume conversely that $smr(L) = (0, 1)$. By Proposition 4.1, $L$ is not abelian. Let $(K, M)$ be a maximal defining pair of $L$. Since $sdr(L) = (0, 1)$, we have

\[
sdim M = \left(\frac{1}{2}s(s - 1) + \frac{1}{2}t(t + 1), st\right) - (0, 1).
\]

Assert that $M = Z(K)$. If not, since $M \subseteq Z(K)$, we have $sdim(K/Z(K)) < sdim K/M = (s, t)$. Hence $sdim(K/Z(K)) \leq (s-1, t)$ or $sdim(K/Z(K)) \leq (s, t-1)$. Suppose $sdim(K/Z(K)) \leq (s, t-1)$. Then by Lemma 2.2, we have

\[
sdim K^2 \leq \left(\frac{1}{2}(s-1)(s-2) + \frac{1}{2}(t+1), (s-1)t\right).
\]

Since $M \subseteq K^2$, we have $sdim L \leq (1, 1)$. Since $L$ is not abelian, we must have $sdim L = (1, 1)$. It is easy to deduce that $L \cong H(0, 1)$. Consequently, $smr(L) = (1, 1)$, contradicting the assumption that $smr(L) = (0, 1)$.

Suppose $sdim K/Z(K) \leq (s, t-1)$. Then by Lemma 2.2, we have

\[
sdim K^2 \leq \left(\frac{1}{2}t(s-1) + \frac{1}{2}(t-1), s(t-1)\right).
\]
Since $M \subset K^2$, we have $\text{sdim}L \leq (1, 0)$, contradicting the assumption that $L$ is not abelian. Hence $M = Z(K)$ and $\text{sdim}K/Z(K) = (s, t)$. Consequently,

$$\text{sdim}K^2 = \left(\tfrac{1}{2}s(s-1) + \tfrac{1}{2}t(t+1), st\right) - \text{sdr}(K).$$

Since $L$ is not abelian, we have $M \not\subset K^2$. Noting that $\text{sdim}M = \left(\tfrac{1}{2}s(s-1) + \tfrac{1}{2}t(t+1), (s-1)t\right)$, we have $\text{sdim}K^2 \leq \left(\tfrac{1}{2}(s-1)(s-2) + \tfrac{1}{2}t(t+1), (s-1)t\right)$. 

Since $M \subset K^2$, we have $\text{sdim}L \leq (2, 0)$, contradicting the assumption that $L$ is nilpotent and not abelian.

Suppose $\text{sdim}K/Z(K) \leq (s, t-1)$. Then by Lemma 2.2, we have

$$\text{sdim}K^2 \leq \left(\tfrac{1}{2}s(s-1) + \tfrac{1}{2}t(t+1), st\right).$$

Since $M \subset K^2$, we have $\text{sdim}L \leq (0, 1)$, contradicting the assumption that $L$ is not abelian. Hence $M = Z(K)$ and $\text{sdim}K/Z(K) = (s, t)$. Consequently,

$$\text{sdim}K^2 = \left(\tfrac{1}{2}s(s-1) + \tfrac{1}{2}t(t+1), st\right) - \text{sdr}(K).$$

Since $L$ is not abelian, we have $M \not\subset K^2$. It follows that $\text{sdr}(K) = (0, 0)$. By Lemma 4.6, $L \cong K/Z(K) = H(1, 0)$.

### 5. Multiplier-rank 2 nilpotent Lie superalgebras

Throughout this section, suppose $L$ is a finite-dimensional non-abelian nilpotent Lie superalgebra and $\text{sdim}L/Z(L) = (m, n)$.

To determine the nilpotent Lie superalgebras of multiplier-rank 2, we will establish several technical lemmas.
Lemma 5.1. Suppose \( z \in Z_2(L) \backslash Z(L) \) is an even element and \( \lambda(z) = (m, n) - (\text{sdr}(L) + (1, 0)) \). Then \( \mu(z) = (m - 1, n) \) and \( L/[L, z]/Z(L/[L, z]) \) is either \( \text{Ab}(m - 1, n) \) or \( H(1, 0) \).

Proof. Suppose \( \mu(z) = (b_1, b_2) \). By Lemma 1.31 we have \( \mu(z) \leq (m - 1, n) \). By Lemma 2.2

\[
\left( \frac{1}{2} m(m - 1) + \frac{1}{2} n(n + 1), mn \right) - \text{sdr}(L)
\]

\[
= \text{sdim} L^2
\]

\[
= \text{sdim} L^2/[L, z] + \text{sdim}[L, z]
\]

\[
\leq \left( \frac{1}{2} b_1(b_1 - 1) + \frac{1}{2} b_2(b_2 + 1), b_1 b_2 \right) + \lambda(z)
\]

\[
= \left( \frac{1}{2} (m - 1)(m - 2) + \frac{1}{2} n(n + 1), (m - 1)n \right) + (m, n) - (\text{sdr}(L) + (1, 0))
\]

\[
= \left( \frac{1}{2} m(m - 1) + \frac{1}{2} n(n + 1), mn \right) - \text{sdr}(L).
\]

Therefore, \( \mu(z) = (m - 1, n) \) and \( \text{sdr}(L/[L, z]) = (0, 0) \). Then our lemma follows from Lemma 1.6.

Lemma 5.2. Let \( \text{sdr}(L) = (1, 0) \). Then \( L/Z(L) \) is isomorphic to one of the following Lie superalgebras:

1. an abelian Lie superalgebra;
2. \( H(1,0) \);
3. \( H(1,0) \oplus \text{Ab}(1,0) \);
4. a Lie algebra with basis \( \{x, y, z, t\} \) and multiplication given by

\[
[x, y] = -[y, x] = z, \quad [x, z] = -[z, x] = t
\]

and the other brackets of basis elements vanishing.

Proof. Our argument is divided into two parts.

(I) Suppose \( Z_2(L)/Z(L) \) has a nonzero odd part. Then by Lemma 1.22, we have \( \text{sdim}(L/Z(L))^2 = (0, 0) \) or \( (0, 1) \). If \( \text{sdim}(L/Z(L))^2 = (0, 0) \), then \( L/Z(L) \) is abelian. Thus we suppose \( \text{sdim}(L/Z(L))^2 = (0, 1) \). Then \( (L/Z(L))^2 = 0 \) and therefore \( (L/Z(L))^2 \subset Z(L/Z(L)) \). If \( \text{sdim}(L/Z(L)) = (0, 1) \), then \( L/Z(L) \) is a Heisenberg superalgebra of odd center, and then by Lemma 1.7.1, we have \( \text{sdr}(L) \neq (1, 0) \), contradicting the assumption. Then we can assume that \( \text{sdim}(L/Z(L)) = (k, l + 1) > (0, 1) \). Suppose \( S \) is a subsuperspace of \( L/Z(L) \) such that \( (L/Z(L))^2 \subset S = Z(L/Z(L)) \). Suppose \( T \) is a subsuperspace such that \( ((L/Z(L))^2 \oplus S) \oplus T = L/Z(L) \). Write \( H = (L/Z(L))^2 + T \). Then \( H < L/Z(L) \) and it is easy to deduce that \( H^2 = (L/Z(L))^2 + H \). Hence \( H \cong H(p) \), where \( p \geq 1 \). Then \( \text{sdim}\ S = (k, l) \) and \( (m, n) - (k, l) = (p, p + 1) \).

Assume that \( p > 1 \). Since \( \text{sdr}(L) = (1, 0) \), by Propositions 3.1 and 1.5, Lemmas 2.4 and
We have
\[
\left(\frac{1}{2}m(m-1) + \frac{1}{2}n(n+1) + mn\right) - (1,0) = \text{sdim}L^2
\]
\[
= \text{sdim}(L^2 \cap Z(L)) + \text{sdim}(L/Z(L))^2 \leq \text{sdim}\mathcal{M}(L/Z(L)) + \text{sdim}(L/Z(L))^2
\]
\[
= \text{sdim}\mathcal{M}(L/Z(L)) + (0,1)
\]
\[
= \text{sdim}\mathcal{M}(S) + \text{sdim}\mathcal{M}(H) + \text{sdim}(S \otimes H/H^2) + (0,1)
\]
\[
= \left(\frac{1}{2}k(k-1) + \frac{1}{2}(l+1), kl\right) + (p^2, p^2 - 1) + (pk + pl, pk + pl) + (0,1).
\]
Substituting \(m = p + k\) and \(n = p + l\), one may obtain that \(p + l \leq 0\), contradicting the assumption that \(p + l > 0\).

Assume that \(p = 1\). By Proposition 4.3, we have \(\text{sdim}\mathcal{M}(H(1)) = (1,1)\). Since \(\text{sd}(L) = (1,0)\), by Proposition 3.1, Lemmas 2.4 and 2.5, we have
\[
\left(\frac{1}{2}m(m-1) + \frac{1}{2}n(n+1) + mn\right) - (1,0) = \text{sdim}L^2
\]
\[
= \text{sdim}(L^2 \cap Z(L)) + \text{sdim}(L/Z(L))^2 \leq \text{sdim}\mathcal{M}(L/Z(L)) + \text{sdim}(L/Z(L))^2
\]
\[
= \text{sdim}\mathcal{M}(L/Z(L)) + (0,1)
\]
\[
= \text{sdim}\mathcal{M}(S) + \text{sdim}\mathcal{M}(H) + \text{sdim}(S \otimes H/H^2) + (0,1)
\]
\[
= \left(\frac{1}{2}k(k-1) + \frac{1}{2}(l+1), kl\right) + (1,1) + (k+l, k+l) + (0,1).
\]
Substituting \(m = k + 1\) and \(n = l + 2\), one may obtain that \(l \leq -1\), contradicting the assumption that \(l \geq 0\).

(II) Suppose the odd part of \(Z_2(L)/Z(L)\) is zero. By Lemma 4.2(1), we have \(\text{sdim}(L/Z(L))^2 = (0,0)\) or \((2,0)\). If \(\text{sdim}(L/Z(L))^2 = (0,0)\), then \(L/Z(L)\) is abelian. Suppose that \(\text{sdim}(L/Z(L))^2 = (1,0)\). If \(\text{sdim}(L/Z(L)) = (1,0)\), then by Lemma 4.7(1), we have \(L/Z(L) \cong H(1,0)\). Since \(\text{sdim}(L/Z(L))^2 = (1,0)\) and hence \((L/Z(L))^2 \subset Z(L/Z(L))\), we can assume that \(\text{sdim}(L/Z(L)) = (k+1,0) > (1,0)\). Let \(S\) be a subsuperspace of \(L/Z(L)\) such that \((L/Z(L))^2 \subset S = Z(L/Z(L))\). Suppose \(H\) is a subsuperspace containing \((L/Z(L))^2\) such that \(H \oplus S = L/Z(L)\). Then \(H\) is a subsuperalgebra of \(L/Z(L)\) and \(H^2 = (L/Z(L))^2 = Z(H)\). Since \(\text{sdim}(L/Z(L))^2 = (1,0)\), we have \(H \preceq H(p,q)\), where \(p + q \geq 1\). Then \(\text{sdim}S = (k,0)\) and \((m-k,n) = (2p+1,q)\).

Assume that \(p + q \geq 2\). Since \(\text{sd}(L) = (1,0)\), by Propositions 3.1 and 4.4, Lemmas 2.4 and 2.5, we have
and we have
\[
\left(\frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn\right) - (1,0)
\]
\[= \text{sdim}L^2
\]
\[= \text{sdim}(L^2 \cap \mathbb{Z}(L)) + \text{sdim}(L/\mathbb{Z}(L))^2
\]
\[\leq \text{sdim}\mathcal{M}(L/\mathbb{Z}(L)) + \text{sdim}(L/\mathbb{Z}(L))^2
\]
\[= \text{sdim}\mathcal{M}(L/\mathbb{Z}(L)) + (1,0)
\]
\[= \text{sdim}\mathcal{M}(S) + \text{sdim}\mathcal{M}(H) + \text{sdim}(S \otimes H/H^2) + (1,0)
\]
\[= \left(\frac{1}{2}k(k-1), 0\right) + \left(2p^2 - p + \frac{1}{2}q^2 + 1, 2pq\right) + (2pk, kq) + (1,0).
\]

Substituting \(m = 2p + 1 + k\) and \(n = q\), one may obtain that \(p + q = 0\), contradicting the assumption that \(p + q \geq 2\).

Assume that \(p = 1\) and \(q = 0\). By Proposition 4.4 we have \(\text{sdim}\mathcal{M}(H) = (2,0)\). Since \(\text{sdr}(L) = (1,0)\), by Proposition 3.1, Lemmas 2.4 and 2.5 we have
\[
\left(\frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn\right) - (1,0)
\]
\[= \text{sdim}L^2
\]
\[= \text{sdim}(L^2 \cap \mathbb{Z}(L)) + \text{sdim}(L/\mathbb{Z}(L))^2
\]
\[\leq \text{sdim}\mathcal{M}(L/\mathbb{Z}(L)) + \text{sdim}(L/\mathbb{Z}(L))^2
\]
\[= \text{sdim}\mathcal{M}(L/\mathbb{Z}(L)) + (1,0)
\]
\[= \text{sdim}\mathcal{M}(S) + \text{sdim}\mathcal{M}(H) + \text{sdim}(S \otimes H/H^2) + (1,0)
\]
\[= \left(\frac{1}{2}k(k-1), 0\right) + (2,0) + (2k, 0) + (1,0).
\]

Substituting \(m = 3 + k\) and \(n = 0\), one may obtain that \(k = 1\). Hence \(L/\mathbb{Z}(L) \cong H(1,0) \oplus \text{Ab}(1,0)\).

Assume that \(p = 0\) and \(q = 1\). By Proposition 4.4 we have \(\text{sdim}\mathcal{M}(H) = (0,0)\). Since \(\text{sdr}(L) = (1,0)\), by Proposition 3.1, Lemmas 2.4 and 2.5 we have
\[
\left(\frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn\right) - (1,0)
\]
\[= \text{sdim}L^2
\]
\[= \text{sdim}(L^2 \cap \mathbb{Z}(L)) + \text{sdim}(L/\mathbb{Z}(L))^2
\]
\[\leq \text{sdim}\mathcal{M}(L/\mathbb{Z}(L)) + \text{sdim}(L/\mathbb{Z}(L))^2
\]
\[= \text{sdim}\mathcal{M}(L/\mathbb{Z}(L)) + (1,0)
\]
\[= \text{sdim}\mathcal{M}(S) + \text{sdim}\mathcal{M}(H) + \text{sdim}(S \otimes H/H^2) + (1,0)
\]
\[= \left(\frac{1}{2}k(k-1), 0\right) + (0,0) + (0,k) + (1,0).
\]

Substituting \(m = 1 + k\) and \(n = 1\), one may obtain that \(1 \leq 0\), a contradiction.

Now suppose that \(\text{sdim}(L/\mathbb{Z}(L))^2 = (2,0)\). For any even element \(x \in \mathbb{Z}(L)\), by Lemma 4.3 we have \(L^2 + \mathbb{Z}(L) = L(x)\) and \(\lambda(x) = (m-2, n)\). Since \(x \in \mathbb{Z}(L) = L^2 + \mathbb{Z}(L)\),
we have $Z_2(L) \subset Z_L(x) = L^2 + Z(L)$. Assert that $\text{sdim}Z(L/Z(L)) = (1,0)$. If not, we have $\text{sdim}Z_2(L)/Z(L) = (2,0)$, since

$$\text{sdim}Z_2(L)/Z(L) \leq \text{sdim}Z_L(x)/Z(L)$$

$$= \text{sdim}(L^2 + Z(L))/Z(L)$$

$$= \text{sdim}(L/Z(L))^2 = (2,0).$$

Then $Z_L(x) = Z_2(L)$ for all $x \in Z_2(L) \setminus Z(L)$. Therefore,

$$(L/Z(L))^2 = Z_L(x)/Z(L) = Z_2(L)/Z(L) = Z(L/Z(L)).$$

Since $(x, Z(L))/(L, x) \subset Z(L/[L, x])$, by Lemma 5.1 we have

$$\text{sdim}(L/Z(L))/(x, Z(L))/Z(L)) = (m-1, n)$$

$$= \text{sdim}L/[L, Z(L)]$$

$$= \text{sdim}L/[L, x]/(x, Z(L))/[L, x]$$

$$\geq \text{sdim}(L/[L, x])/Z(L/[L, x])$$

$$= \mu(x)$$

$$= (m-1, n).$$

Then, as Lie superalgebras,

$$L/Z(L))/(x, Z(L))/Z(L) \cong L/[L, x]/Z(L/[L, x]).$$

Then by Lemma 5.1, $L/Z(L)/(x, Z(L))/Z(L)$ is abelian or isomorphic to $H(1,0)$. However, since $\text{sdim}(L/Z(L))^2 = (2,0)$, one sees that $L/Z(L)/(x, Z(L))/Z(L)$ is not abelian. Thus $L/Z(L)/(x, Z(L))/Z(L) \cong H(1,0)$. Then it is routine to deduce that $\text{sdim}Z(L/Z(L)) = (1,0)$, a contradiction.

Suppose $\text{sdim}Z(L/Z(L)) = (1,0)$. Let $x$ and $Z(L)$ generate $Z_2(L)$. By Lemma 4.3, $L^2 + Z(L) = Z_L(x) \supset Z_2(L)$, since $\text{sdim}Z_L(x)/Z(L) = (2,0)$. By Lemma 5.1 we have $\mu(x) = (m-1, n)$. Clearly, $Z_2(L)/[L, x] \subset Z(L/[L, x])$. Then

$$(m-1, n) = \text{sdim}L/Z(L)/Z_2(L)/Z(L)$$

$$= \text{sdim}L/[L, x]/Z_2(L)/[L, x]$$

$$\geq \text{sdim}L/[L, x]/(Z(L/[L, x]))$$

$$= \mu(x) = (m-1, n).$$

Therefore, $Z(L/[L, x]) = Z_2(L)/[L, x]$. By Lemma 5.1 we have

$$L/Z(L)/Z(L/Z(L)) \cong L/Z_2(L) \cong L/[L, x]/Z_2(L)/[L, x] = L/[L, x]/Z(L/[L, x])$$

is isomorphic to $H(1,0)$, since $L/Z_2(L)$ is not abelian. Hence $L/Z(L)$ is isomorphic to the Lie algebra in (4).

\textbf{Lemma 5.3.} Let $\text{sdr}(L) = (0, 1)$. Then $L/Z(L)$ is isomorphic to one of the following Lie superalgebras:

\begin{enumerate}
  \item An abelian Lie superalgebra;
  \item $H(0,1)$;
\end{enumerate}
(3) \( H(1, 0) \oplus \text{Ab}(0, 1) \).

**Proof.** Since sdr\((L) = (0, 1)\), by Lemma 3.2 we have sdim\((L/Z(L))^2 \leq (1, 1)\). If sdim\((L/Z(L))^2 = (0, 0)\), then \( L/Z(L) \) is abelian.

Suppose sdim\((L/Z(L))^2 = (1, 0)\). If sdim\(Z(L/Z(L)) = (1, 0)\), then by Lemma 3.7 we have \( L/Z(L) \cong H(0, 1) \). Suppose sdim\(Z(L/Z(L)) = (k + 1, l) > (1, 0)\). Since sdim\((L/Z(L))^2 = (1, 0)\), we have \( (L/Z(L))^2 \subset Z(L/Z(L)) \). Let \( S \) be a subsuperspace of \( L/Z(L) \) such that \( (L/Z(L))^2 \oplus S = Z(L/Z(L)) \). Suppose \( H \) is a subsuperspace containing \( (L/Z(L))^2 \) such that \( H \oplus S = L/Z(L) \). Then \( H \) is a subsuperalgebra of \( L/Z(L) \) and \( H^2 = (L/Z(L))^2 = Z(H) \). Since sdim\((L/Z(L))^2 = (1, 0)\), we have \( H \cong H(p, q) \), where \( p + q \geq 1 \). Then sdim\(S = (k, l)\) and \( (m - k, n - l) = (2p + 1, q) \).

Assume that \( p + q \geq 2 \). Since sdr\((L) = (0, 1)\), by Propositions 3.1, 4.4, Lemmas 2.4 and 2.5, we have

\[
\left(\frac{1}{2}m(m - 1) + \frac{1}{2}n(n + 1), mn\right) - (0, 1)
= \text{sdim}L^2
= \text{sdim}\left(L^2 \cap Z(L)\right) + \text{sdim}\left(L/Z(L)\right)^2
\leq \text{sdim}\left(L/Z(L)\right) + \text{sdim}\left(L/Z(L)\right)^2
= \text{sdim}\left(L/Z(L)\right) + (1, 0)
= \text{sdim}\left(S\right) + \text{sdim}\left(H\right) + \text{sdim}(S \oplus H/H^2)
= \left(\frac{1}{2}k(k - 1) + \frac{1}{2}l(l + 1), kl\right) + \left(2p^2 - p + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq\right)
+ (2pk +ql, kq + 2pl) + (1, 0).
\]

Substituting \( m = 2p + 1 + k \) and \( n = q + l \), one may obtain that \( p + q \leq 1 \), contradicting the assumption that \( p + q \geq 2 \).

Assume that \( p = 1 \), \( q = 0 \). By Proposition 4.4 we have sdim\(\left(H(1, 0)\right) = (2, 0)\). Since sdr\((L) = (0, 1)\), by Proposition 3.1, Lemmas 2.4 and 2.5, we have

\[
\left(\frac{1}{2}m(m - 1) + \frac{1}{2}n(n + 1), mn\right) - (0, 1)
= \text{sdim}L^2
= \text{sdim}\left(L^2 \cap Z(L)\right) + \text{sdim}\left(L/Z(L)\right)^2
\leq \text{sdim}\left(L/Z(L)\right) + \text{sdim}\left(L/Z(L)\right)^2
= \text{sdim}\left(L/Z(L)\right) + (1, 0)
= \text{sdim}\left(S\right) + \text{sdim}\left(H\right) + \text{sdim}(S \oplus H/H^2) + (1, 0)
= \left(\frac{1}{2}k(k - 1) + \frac{1}{2}l(l + 1), kl\right) + (2, 0) + (2k, 2l) + (1, 0).
\]

Substituting \( m = 3 + k \) and \( n = l \), one gets \( k = 0 \) and \( l = 1 \). Hence \( L/Z(L) \cong H(1, 0) \oplus \text{Ab}(0, 1) \).

Assume that \( p = 0 \), \( q = 1 \). By Proposition 4.4 we have sdim\(\left(H(0, 1)\right) = (0, 0)\). Since
dicting the assumption that $sdim(L/H)$ is isomorphic to $H$

We have $Z(L/2)$ is an element of $H$.

Hence, we have $Z(L) = 0$.

Now, $(k, m)$ by Proposition 5.4. A Lie superalgebra $L$ is called capable if there is a Lie superalgebra $H$ such that $L \cong H/Z(H)$. 

Definition 5.4. A Lie superalgebra $L$ is called capable if there is a Lie superalgebra $H$ such that $L \cong H/Z(H)$. 

Classification of nilpotent Lie superalgebras of multiplier-rank $\leq 2$
Lemma 5.5. Let L be a non-capable, nilpotent, non-abelian Lie superalgebra of superdimension (s, t). Then (s − 1, t) < sdr(L) or (t, s) < sdr(L).

Proof. Let (K, M) be a maximal defining pair of L. Since L is not abelian, we have \( L \cong K/M \) and \( M \subseteq K^2 \). Since L is not capable, we have \( M \subseteq Z(K) \) and \( \text{sdim}K/Z(K) \leq (s − 1, t) \) or \( (s, t − 1) \). Since \( M \subseteq K^2 \), by Lemma 2.2 one may easily obtain that \( \text{sdr}(L) > (s − 1, t) \) or \( \text{sdr}(L) > (t, s) \).

Lemma 5.6. Let L be a capable, nilpotent, non-abelian Lie superalgebra. Then \( \text{sdr}(C(L)) < \text{sdr}(L) \).

Proof. Let (K, M) be a maximal defining pair of L. We have \( L \cong K/M \) and \( M \subseteq Z(K) \cap K^2 \). Since L is capable, we have \( M = Z(K) \). Since L is not abelian, we have \( M \subseteq K^2 \). It follows that \( \text{sdr}(C(L)) < \text{sdr}(L) \).

Proposition 5.7. Let L be a finite-dimensional, non-abelian, nilpotent Lie superalgebra. Then

1. \( \text{smr}(L) \neq (0, 2) \).
2. \( \text{smr}(L) = (2, 0) \) if and only if \( L \cong H(1, 0) \oplus \text{Ab}(1, 0) \).
3. \( \text{smr}(L) = (1, 1) \) if and only if L is isomorphic to one of the following Lie superalgebras:
   - \( (3.1) \ H(1, 0) \oplus \text{Ab}(0, 1) \);
   - \( (3.2) \ H(0, 1) \).

Proof. Suppose \( \text{sdim}L = (s, t) \).

1. Assume conversely that \( \text{smr}(L) = (0, 2) \). Then by Proposition 3.4 L is not abelian. First suppose \( L \) is not capable. Then by Lemma 5.5 we have \( (s, t) < \text{smr}(L) + (1, 0) \) or \( (t, s) < \text{smr}(L) \). Since L is nilpotent and not abelian, we must have \( \text{sdim}L = (1, 1) \). It is easy to deduce that \( L \cong H(0, 1) \). Consequently, \( \text{smr}(L) = (1, 1) \), contradicting the assumption that \( \text{smr}(L) = (0, 2) \).

Next suppose \( L \) is capable. Suppose \( K \) is a cover of \( L \). Then we have \( L \cong K/Z(K) \). By Lemma 5.6 we have \( \text{sdr}(K) < \text{smr}(L) = (0, 2) \). If \( \text{sdr}(K) = (0, 0) \), then by Lemma 4.4 \( L \) is either abelian, a contradiction, or \( L = H(1, 0) \), which yields \( \text{smr}(L) = (1, 0) \), also a contradiction. Hence \( \text{sdr}(K) = (0, 1) \). Therefore, \( L \cong K/Z(K) \) is one of the Lie superalgebras listed in Lemma 5.3. Then \( \text{smr}(L) = (0, 0) \), \( (1, 1) \) or \( (1, 1) \), which is impossible. Hence, \( \text{smr}(L) \neq (0, 2) \).

2. Suppose \( L = H(1, 0) \oplus \text{Ab}(1, 0) \). Then by Propositions 3.1, 4.4 and Lemma 2.5 one may compute \( \text{sdim}M(L) = (4, 0) \). Then, since \( \text{sdim}L = (4, 0) \), we have \( \text{smr}(L) = (2, 0) \).

Conversely, suppose \( \text{smr}(L) = (2, 0) \). By Propositions 3.1 L is not abelian. First suppose \( L \) is not capable. Then by Lemma 5.5 we have \( (s, t) < \text{smr}(L) + (1, 0) \) or \( (t, s) < \text{smr}(L) \), contradicting the assumption that \( L \) is nilpotent and not abelian.

Next suppose \( L \) is capable. Suppose \( K \) is a cover of \( L \). Then we have \( L \cong K/Z(K) \). By Lemma 5.6 we have \( \text{sdr}(K) < \text{smr}(L) = (2, 0) \). If \( \text{sdr}(K) = (0, 0) \), then by Lemma 4.4 either \( L \) is abelian, a contradiction, or \( L = H(1, 0) \), which yields \( \text{smr}(L) = (1, 0) \), also a contradiction. Hence \( \text{sdr}(K) = (1, 0) \). Therefore, \( L \cong K/Z(K) \) is one of the superalgebras listed in Lemma 5.2. A direct verification shows that \( L \cong H(1, 0) \oplus \text{Ab}(1, 0) \).
(3) Suppose \( L = H(1,0) \oplus \text{Ab}(0,1) \). By Propositions 4.4, 3.1 and Lemma 2.5, one may compute \( \text{sdim} M(L) = (3,2) \). Then, since \( (s,t) = (3,1) \), we have \( \text{smr}(L) = (1,1) \).

Suppose \( L = H(0,1) \). Then by Proposition 4.4, we have \( \text{sdim} M(L) = (0,0) \). Then since \( (s,t) = (1,1) \), we have \( \text{smr}(L) = (1,1) \).

Conversely, suppose \( \text{smr}(L) = (1,1) \). By Proposition 3.1, \( L \) is not abelian. First suppose \( L \) is not capable. Then by Lemma 5.5, we have \( (s,t) < \text{smr}(L) + (1,0) \) or \( (t,s) < \text{smr}(L) \).

Since \( L \) is nilpotent and not abelian, we must have \( \text{sdim} L = (1,1) \). Then it is easy to deduce that \( L \cong H(0,1) \).

Next suppose \( L \) is capable. Suppose \( K \) is a cover of \( L \). Then we have \( L \cong K/\text{Z}(K) \). By Lemma 5.6, we have \( \text{sdr}(K) < \text{smr}(L) = (1,1) \). If \( \text{sdr}(K) = (0,0) \), then either \( L \) is abelian, a contradiction, or \( L = H(1,0) \), which yields \( \text{smr}(L) = (1,0) \), also a contradiction. If \( \text{sdr}(K) = (1,0) \), then \( L \cong K/\text{Z}(K) \) is one of the Lie superalgebras listed in Lemma 5.4 and then \( \text{smr}(L) = (0,0),(1,0),(2,0) \) or \( (4,0) \), a contradiction. Suppose \( \text{sdr}(K) = (0,1) \). Then \( L \cong K/\text{Z}(K) \) is one of the superalgebras listed in Lemma 5.3. A direct verification shows that \( L \cong H(1,0) \oplus \text{Ab}(0,1) \) or \( H(0,1) \).

We assemble Propositions 5.1, 5.4, 5.6 to be the following classification theorem. Recall that \( \text{Ab}(m,n) \) denotes the abelian Lie superalgebra of superdimension \( (m,n) \) and \( H(m,n) \) is the \( (2m+1,n) \)-dimensional Heisenberg Lie superalgebras of even center.

**Theorem 5.8.** Up to isomorphism, all the finite-dimensional nilpotent Lie superalgebras \( L \) of multiplier-rank \( \leq 2 \) are listed below:

| \( \text{smr}(L) \) | \( L \) |
|------------------|-------|
| (0,0) | any abelian Lie superalgebra |
| (1,0) | \( H(1,0) \) |
| (2,0) | \( H(1,0) \oplus \text{Ab}(1,0) \) |
| (1,1) | \( H(1,0) \oplus \text{Ab}(0,1) \) |
| (1,1) | \( H(0,1) \) |

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