SINGULAR VIRASORO VECTORS AND LIE ALGEBRA COHOMOLOGY

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Abstract. We present an explicit formula for a new family of Virasoro singular vectors. As a corrolary we get formulas for differentials of Feigin-Fuchs-Rocha-Carridi-Wallach resolution of the positive nilpotent part of Virasoro (or Witt) algebra $L_1$.

Introduction

The main goal of the paper is to present explicit formulae for all differentials $\delta_k$ of the Feigin-Fuchs-Rocha-Carridi-Wallach-resolution ([6], [13]), a free resolution of a one-dimensional $L_1$-module $\mathbb{C}$. By $L_1$ we denote the positive part of the Witt algebra. The differentials $\delta_k$ are $2 \times 2$-matrices whose elements belong to the universal enveloping algebra $U(L_1)$. More precisely, the matrix $\delta_k$ can be expressed by means of singular vectors $S_{p,q}(t)v$ in Verma modules over the Virasoro Lie algebra (Virasoro singular vectors $S_{p,q}(t) \in U(L_1)$), $t$ is a complex parameter:

$$\delta_1 = (S_{1,1}(-\frac{3}{2}), S_{1,2}(-\frac{3}{2})), \quad \delta_k = \begin{pmatrix} S_{1,3k+1}(-\frac{3}{2}) & S_{2k+1,2}(-\frac{3}{2}) \\ -S_{2k+1,1}(-\frac{3}{2}) & -S_{1,3k+2}(-\frac{3}{2}) \end{pmatrix},$$

It is easy to find the first few singular vectors, for instance

$$S_{1,1}(t) = e_1, \quad S_{1,2}(t) = e_1^2 + t^{-1}e_2, \quad S_{3,1}(t) = e_1^3 + 4te_2e_1 + (4t^2 + 2t)e_3.$$

However, further computational complexity starts to increase exponentially. Fuchs and Feigin proposed to look for the operators $S_{p,q}(t)$ as

$$S_{p,q}(t) = e_1^{pq} + \sum_{pq \geq i_1 \geq \cdots \geq i_s \geq 1} P_{p,q}^{i_1, \ldots, i_s} (t)e_{i_1} \cdots e_{i_s},$$

where

$$\sum_{i_1 + \cdots + i_s = pq} P_{p,q}^{i_1, \ldots, i_s} (t)e_{i_1} \cdots e_{i_s}$$
where $P_{i_1,\ldots,i_s}(t)$ are Laurent polynomials in the complex variable $t$. Feigin, Fuchs and later Astashkevich had found some properties of these Laurent polynomials, however no explicit general formula have been found.

For a long time no one could get to write an explicit formula for the $S_{p,q}(t)$. At the end of 80s Benoit and Saint-Aubin [3] found a beautiful explicit expression for one family of singular vectors $S_{p,1}(t)$. Hence three of the four matrix elements of $D_k$ we know (due to the property $S_{p,q}(t) = S_{q,p}(t^{-1})$), but the fourth element $S_{2k+1,2} \left(-\frac{3}{2}\right)$ remained unknown. The key idea of the approach by Benoit and Saint-Aubin was the idea to consider expansions of $S_{p,1}(t)$ in all monomials

$$e_{i_1}e_{i_2}\ldots e_{i_s}, i_1+i_2+\ldots+i_s=pq,$$

which correspond to all unordered partitions of $pq$, not only to ordered partitions $i_1 \geq i_2 \geq \cdots \geq i_s \geq 1$, as did Feigin and Fuchs. Of course this extends the formula for $S_{p,q}(t)$, a linear combination of this type is not unique, but in some cases (the calculation of the cohomology with different coefficients) it helps to get interesting combinatorial formulas [12].

Bauer, Di Francesco, Itzykson, Zuber [2] have found a very elegant proof of singularity of vectors $S_{p,1}(t)v$. Moreover, using the Benoit-Saint-Aubin formula as a starting point, Bauer, Di Francesco, Itzykson, Zuber [2] presented a complete and straightforward algorithm for finding all singular vectors. However, their algorithm encounters technical difficulties for $S_{p,q}(t), p, q \geq 3$ and it is not still clear whether it is possible to get an explicit formula in general case by means of it. The Benoit-Saint-Aubin formula, examples $S_{2,2}(t)$ and $S_{3,2}(t)$ considered in [2] has allowed us to guess the general explicit formula for $S_{2,p}(t)$. We prove the singularity of vectors $S_{2,p}(t)v$ following [2] the proof for another family $S_{p,1}(t)v$.

1. **Singular vectors of Verma modules over Virasoro Lie algebra**

   The Virasoro algebra Vir is infinite dimensional Lie algebra, defined by its basis $\{z,e_i,i \in \mathbb{Z}\}$ and commutator relations:

   $$[e_i,z] = 0, \forall i \in \mathbb{Z}, \quad [e_i,e_j] = (j-i)e_{i+j} + \frac{j^3-j}{12}\delta_{i,j}z.$$

   Vir is one-dimensional central extension of the Witt algebra $W$ (the one-dimensional center is spaned by $z$).
Remark. In \[5, 7, 8\] the symbol $L_1$ denotes the "positive part" of the Witt algebra $W$ (or the Virasoro algebra $Vir$), i.e. the algebra of polynomial vector fields on the line $\mathbb{R}$ which vanish at the origin together with their first derivatives. We will use further the symbol $L_1$ for the notation of both the algebras. This will not cause the confusion because the cohomology $H^*(L_1)$ and $H^*(W_+)$ are isomorphic \[4, 9\].

A $Vir$-module $V(h, c)$ is called a Verma module over the Virasoro algebra if it is free as a module over the universal enveloping algebra $U(L_1)$ of the subalgebra $L_1 \subset Vir$ and it is generated by some vector $v$ such that

$$zv = cv, \ e_0v = hv, \ e_i v = 0, \ i < 0,$$

where $c, h \in \mathbb{C}$. As a vector space $V(h, c)$ can be defined by its infinite basis

$$v, \ e_{i_1} \ldots e_{i_s} v, \ i_1 \geq i_2 \geq \cdots \geq i_s, \ s \geq 1.$$

A Verma module $V(h, c)$ is $\mathbb{Z}_+$-graded module:

$$V(h, c) = \bigoplus_{n=0}^{+\infty} V_n(h, c), \ V_n(h, c) = \langle e_{i_1} \ldots e_{i_s} v, \ i_1 + \cdots + i_s = n \rangle.$$

$V_n(h, c)$ is an eigen-subspace of the operator $e_0$ that corresponds to the eigenvalue $(h+n)$:

$$e_0(e_{i_1} \ldots e_{i_s} v) = (h + i_1 + \cdots + i_s)e_{i_1} \ldots e_{i_s} v.$$

In addition to this, $zw = cw$ for all $w \in V(h, c)$.

**Definition 1.1.** A nontrivial vector $w \in V(h, c)$ is called singular if $e_i w = 0$ for all $i < 0$.

**Remark.** A subalgebra $Vir^-$ which is spanned by $\{e_i, i < 0\}$ is multiplicatively generated by two elements $e_{-1}$ and $e_{-2}$. Hence a vector $w \in V(h, c)$ is singular if and only if

$$e_{-1} w = e_{-2} = 0.$$

A homogeneous singular vector $w \in V_n(h, c)$ with the grading equal to $n$ generates in $V(h, c)$ a submodule that is isomorphic to $V(h+n, c)$.

It is not difficult to present first examples of singular vectors. At the first level $n = 1$ there is a singular vector $w_1 \in V_1(h, c)$ if and only if $h = 0$. Indeed the subspace $V_1(h, c)$ is one-dimensional and it is spanned by $e_1 v$, but on the other hand

$$e_{-1} e_1 v = 2e_0 v = 2hv, \ e_{-2} e_1 v = 0.$$

A two-dimensional subspace coincides with the span of vectors $e_1^2 v$ and $e_2 v$. The condition that the vector $w_2 \in V_2(h, c)$ is annihilated by
the operator $e_{-1}$ is equivalent to the condition that (up to multiplication by a constant) the vector $w_2$ is equal to $e_1^2v - \frac{2}{3}(2h + 1)e_2v$. On the other hand $e_{-2}$ annihilates $w_2$ if and only if the parameters $h$ and $c$ of a Verma module $V(h, c)$ are related by

$$6h - \frac{2}{3}(2h + 1)\left(4h + \frac{c}{2}\right) = 0.$$  

It is easy to verify that the set of solutions of this equation can be parametrized in a following way

$$(2) \quad c(t) = 13 + 6t + 6t^{-1}, \quad h(t) = -\frac{3}{4}t - \frac{1}{2},$$

where $t \neq 0$ runs the complex numbers (or runs the reals, it depends on the task).

The last remark can be reformulated as follows: for any value of $t$ there is a unique (up to multiplication by a constant) singular vector $w_2 = e_1^2v + te_2v, w_2 \in V_2(h(t), c(t))$, where $h(t)$ and $c(t)$ are defined by equations (1).

**Theorem 1.2 ([11], [6, 7]).** In the Verma module $V(h, c)$ there is a singular vector $w \in V_n(h, c)$ with the grading not higher than $n$ if and only if, when two natural numbers $p$ and $q$ can be found and also a complex number $t$ such that

$$pq \leq n, \quad c = c(t) = 13 + 6t + 6t^{-1},$$

$$h = h_{p,q}(t) = \frac{1 - p^2}{4}t + \frac{1 - pq}{2} + \frac{1 - q^2}{4}t^{-1}. \quad (3)$$

In particular, the following assertion holds [10]: with the fixed natural numbers $p$ and $q$ and with an arbitrary complex number $t$ the Verma module $V(h_{p,q}(t), c(t))$ contains a singular vector $w_{p,q}(t)$ of degree $pq$, moreover the vector $w_{p,q}(t)$ is determined unambiguously up to a multiplication by some scalar:

$$w_{p,q}(t) = S_{p,q}(t)v = \sum_{|I|=pq} P^I_{p,q}(t) e_I v = \sum_{i_1 + \cdots + i_s = pq} P^{i_1,\ldots,i_s}_{p,q}(t) e_{i_1} \cdots e_{i_s} v,$$

where $S_{p,q}(t)$ denotes some element of the universal enveloping algebra $U(L_1)$. The coefficients $P^I_{p,q}(t)$ depend polynomially on $t$ and $t^{-1}$. We assume that the coefficient $P^{1,\ldots,1}_{p,q}(t)$ is equal to one. Obviously that $S_{p,q}(t) = S_{q,p}(t^{-1})$. 

$$\text{Theorem 1.2 ([11], [6, 7]).} \quad \text{In the Verma module } V(h, c) \text{ there is a singular vector } w \in V_n(h, c) \text{ with the grading not higher than } n \text{ if and only if, when two natural numbers } p \text{ and } q \text{ can be found and also a complex number } t \text{ such that}$$

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where $S_{p,q}(t)$ denotes some element of the universal enveloping algebra $U(L_1)$. The coefficients $P^I_{p,q}(t)$ depend polynomially on $t$ and $t^{-1}$. We assume that the coefficient $P^{1,\ldots,1}_{p,q}(t)$ is equal to one. Obviously that $S_{p,q}(t) = S_{q,p}(t^{-1})$. 

Theorem 1.3 (Benoit, Saint-Aubin).

\begin{equation}
S_{p,1}(t) = \sum_{i_1, \ldots, i_s} c_p(i_1, \ldots, i_s) t^{p-s} e_{i_1} \cdots e_{i_s},
\end{equation}

where the sums are all over all partitions of \( p \) by positive numbers without any ordering restriction, and the coefficients \( c_p(i_1, \ldots, i_s) \) are defined by the formulas

\begin{equation}
c_p(i_1, \ldots, i_s) = \frac{(p-1)!^2}{\prod_{l=1}^{s-1} ((\sum_{q=1}^l i_q)(p - \sum_{q=1}^l i_q))}
\end{equation}

Example 1.4.

\( S_{1,1}(t) = e_1, \quad S_{2,1}(t) = e_1^2 + te_2, \quad S_{3,1}(t) = e_1^3 + t(2e_1e_2 + 2e_2e_1) + 4t^2e_3 \).

Theorem 1.5. Let \( V \) be a Verma module over the Virasoro algebra \( \text{Vir}_V \). \( V \) generated by the vector \( v \) and such that \( V \) corresponds to the (complex) parameter \( t \): with

\( c = 13 + 6t + 6t^{-1}, \quad h = -\frac{(p-1+t)(t^{-1}(p+1)+3)}{4} \).

Let consider an element of universal enveloping algebra \( U(L_1) \) defined by the formula

\begin{equation}
S_{2,p}(t) = \sum_{i_1, \ldots, i_s} f_p(i_1, \ldots, i_s) e_{i_1} \cdots e_{i_s},
\end{equation}

where the sums are all over all partitions of \( 2p \) by positive numbers without any ordering restriction, and the coefficients (that are rational functions on \( t \)) \( f_p(i_1, \ldots, i_s) \) are defined by the formulas

\begin{equation}
f_p(i_1, \ldots, i_s) = \frac{(2p-1)!^2(2t)^{s-2p} \prod_{r=1}^{2p-1} (p-t-r) \prod_{m=1}^{s} (i_m(2t+1)+2(p-t- \sum_{n=1}^{m} i_n))}{\prod_{l=0}^{2p-1} (2p-1-2l) \prod_{l=1}^{s-1} ((\sum_{n=1}^{l} i_n)(2p- \sum_{n=1}^{l} i_n)(p-t- \sum_{n=1}^{l} i_n))}
\end{equation}

Then \( S_{2,p}(t)v \) is a singular vector of \( V \):

\begin{equation}
e_{-k}S_{2,p}v = 0, \quad k \in \mathbb{N}.
\end{equation}
For instance $S_{2,1} = e_1^2 + te_2$ and
\[
S_{2,2}(t) = e_1^4 + 4te_1e_2e_1 + \frac{(1-t^2)}{t}(e_1^2e_2 + e_2e_1^2) + \frac{(1-t^2)^2}{t^2}e_2 + \\
\frac{(1+t)(4t-1)}{t}e_1e_3 + \frac{(1-t)(4t+1)}{t}e_3e_1 + \frac{3(1-t^2)}{t}e_4.
\] (9)

Proof. We define the sequence of vectors $v^{(0)}, v^{(1)}, \ldots, v^{(2p-1)}$ by
\[
v^{(0)} = v
\]
and the following recursive relation for $k = 1, \ldots, 2p-1$:
\[
v^{(k)} = \frac{2t \sum_{j=1}^{k} ((j-1)(2t-1)+2k-2p-1) e_j v^{(k-j)}}{k(2p-k)(k-p-t)}.
\] (10)

For instance
\[
v^{(1)} = \frac{2t(1-2p)e_1 v^{(0)}}{(2p-1)(1-p-t)},
\] (11)
\[
v^{(2)} = \frac{2t (2(t+1-p)e_2 v^{(0)} + (3-2p)e_1 v^{(1)})}{2(2p-2)(2-p-t)}.
\]

After that we define the vector $w \in V_{2p}$ by the formula:
\[
w = 2t \sum_{j=1}^{2p} ((j-1)(2t-1)+2p-1) e_j v^{(2p-j)}.
\] (12)

Lemma 1.6.
\[
e_{-1} v^{(k)} = -(p+2-k+3t)v^{(k-1)}, \quad k = 1, \ldots, 2p-1.
\] (13)

and the vector $w$ annihilates the operator $e_{-1}$:
\[
e_{-1} w = e_{-1} \left( 2t \sum_{j=1}^{2p} ((j-1)(2t-1)+2p-1) e_j v^{(2p-j)} \right) = 0.
\]

Proof. We will prove the following formula for $k = 1, \ldots, 2p$:
\[
e_{-1} \left( 2t \sum_{j=1}^{k} ((j-1)(2t-1)+2k-2p-1) e_j v^{(k-j)} \right) = \]
\[
= -(p+2-k+3t)k(2p-k)(k-p-t) v^{(k-1)}.
\] (14)

We proceed by recursion on $k$. The recursion base is
\[
e_{-1} \left( 2t(2-2p-1)e_1 v^{(0)} \right) = 2t(1-2p)2h v^{(0)} = -(p+1+3t)(2p-1)(1-p-t)v^{(0)}.
\]
The recursive step is the following calculation

\[ e_{-1} \left( 2t \sum_{j=1}^{k} ((j-1)(2t-1) + 2k - 2p - 1) e_j v^{(k-j)} \right) = \]

\[ = 2t \sum_{j=1}^{k} ((j-1)(2t-1) + 2k - 2p - 1) ((j+1)e_{j-1} v^{(k-j)} + e_{j-1} v^{(k-j)}) = \]

\[ = 4t(2k - 2p - 1) e_0 v^{(k-1)} + 2t \sum_{j=2}^{k} ((j-1)(2t-1) + 2k - 2p - 1) (j+1)e_{j-1} v^{(k-j)} + \]

\[ - 2t \sum_{j=1}^{k-1} ((j-1)(2t-1) + 2k - 2p - 1) (p+2-k+j+3t)e_j v^{(k-j-1)} = \]

\[ = t(2k-2p-1)4(h+k-1)v^{(k-1)} + \]

\[ -(p-k+1+3t)2t \sum_{j=1}^{k-1} ((j-1)(2t-1) + 2k - 2p - 3) e_j v^{(k-j-1)}. \]

In this chain of equalities we replaced \( e_{-1} e_j \) by \((j+1)e_{j-1} + e_j e_{-1}\) and used the formula for \( e_{-1} v^{(k-j)} \) which we considered true for \( j = 1, \ldots, k \) by the induction hypothesis.

Then we shifted \( j = j'+1 \) the summation index in the sum

\[ \sum_{j=2}^{k} ((j-1)(2t-1) + 2k - 2p - 1) (j+1)e_{j-1} v^{(k-j)} \]

and used the following equality

\[ (j(2t-1) + 2k - 2p - 1)(j+2) - \]

\[ -((j-1)(2t-1) + 2k - 2p - 1)(p+2-k+j+3t) = \]

\[ = -(p-k+1+3t)((j-1)(2t-1) + 2k - 2p - 3). \]

It follows from the definition of \( v^{(k-1)} \) that

\[ 2t \sum_{j=1}^{k-1} ((j-1)(2t-1) + 2k - 2p - 3) e_j v^{(k-j-1)} = \]

\[ = (k-1)(2p-k+1)(k-1-p-t)v^{(k-1)}. \]

We remark that

\[ 4t(h+k-1) = (-(p-1+t)(p+1+3t)+4kt-4t) \]
and finish our calculations.

\[
e_{-1} \left( 2t \sum_{j=1}^{k} ((j-1)(2t-1)+2k-2p-1) e_j v^{(k-j)} \right) =
\]

(18) \quad = (2k-2p-1) \left( -(p-1+t)(n+1+3t) + 4kt - 4t \right) v^{(k-1)} - \\
\quad \quad \quad -(p-k+1+3t)(k-1)(2p-k+1)(k-1-p-t)v^{(k-1)} = \\
\quad \quad \quad = -k(2p-2)(k-p-t)(2p-k)(k-p-t)v^{(k-1)}.
\]

If we divide the resulting equality (18) by \( k(2p-2)(k-p-t) \), we obtain the required formula

\[
e_{-1} v^{(k)} = -(p+2-k+3t)v^{(k-1)}.
\]

Taking \( k = 2p \) we will get

\[
e_{-1} w = 0.
\]

Lemma 1.7.

(19) \quad e_{-2} v^{(1)} = 0, \quad e_{-2} v^{(k)} = -(p+4-k+5t)v^{(k-2)}, \quad k = 2, \ldots, 2p-1.

and the vector \( w \) annihilates the operator \( e_{-2} \):

\[
e_{-2} w = e_{-2} \left( 2t \sum_{j=1}^{2p} ((j-1)(2t-1)-1) e_j v^{(2p-j)} \right) = 0.
\]

Proof. We recursively prove this formula for \( k = 2, \ldots, 2p \):

(20) \quad e_{-2} \left( 2t \sum_{j=1}^{k} ((j-1)(2t-1)+2k-2p-1) e_j v^{(k-j)} \right) = \\
\quad \quad \quad = -(p+4-k+5t)k(2p-2)(k-p-t)v^{(k-2)}.

The starting point \( k=1 \) is evident

\[
e_{-2} \left( 2t(2-2p-1)e_1 v^{(0)} \right) = 2t(1-2p)(3e_{-1} v^{(0)} + e_1 e_{-2} v^{(0)}) = 0.
\]

Now we take \( k=2 \).

(21) \quad e_{-2} \left( 2t(2t+2k-2p-2)e_2 v^{(0)} + 2t(3-2p)e_1 v^{(1)} \right) = \\
\quad \quad \quad = 2t(3-2p)3e_{-1} v^{(1)} + 2t(2t + 2 - 2p)(4e_0 + \frac{1}{2} z)v^{(0)} = \\
\quad \quad \quad = 2t \left( -3(3-2p)(p+1+3t) + 2(t+1-p)(4h + \frac{13}{2} + 3t + 3t^{-1}) \right) v^{(0)} = \\
\quad \quad \quad = -(p+2+5t)2(2p-2)(2-p-t)v^{(0)}.
Now let consider the recursive step.

(22)
\[
e_{-2} \left( 2t \sum_{j=1}^{k} ((j-1)(2t-1)+2k-2p-1) e_j v^{(k-j)} \right) =
\]
\[
= 2t \sum_{j=1}^{k} ((j-1)(2t-1)+2k-2p-1) ( (j+2)e_{j-2} v^{(k-j)} + e_j e_{-2} v^{(k-j)} ) =
\]
\[
=6t(2k-2p-1)e_{-1} v^{(k-1)} + 4t(t+1-p)(4(h+k-2)+\frac{13}{2}+3t+3t^{-1})v^{(k-2)} +
\]
\[
+2t \sum_{j=3}^{k} ((j-1)(2t-1)+2k-2p-1) (j+2)e_{j-2} v^{(k-j)} -
\]
\[
-2t \sum_{j=1}^{k-2} ((j-1)(2t-1)+2k-2p-1) (p+4-j+k+5t)e_j v^{(k-j-2)} =
\]

We used the induction assumption for \( e_{-2} v^{(k-j)}, j = 1, \ldots, k-2 \).

Now we shift the summation index \( j' = j - 2 \) in the sum

\[
\sum_{j=3}^{k} ((j-1)(2t-1)+2k-2p-1) (j+2)e_{j-1} v^{(k-j)},
\]

replace \( 3e_{-1} v^{(k-1)} \) by \( -(p+3-k+3t) v^{(k-2)} \) and we have

(23)
\[
e_{-2} \left( 2t \sum_{j=1}^{k} ((j-1)(2t-1)+2k-2p-1) e_j v^{(k-j)} \right) =
\]
\[
= 2tQ v^{(k-2)} + 2t \sum_{j=1}^{k-2} R_j e_j v^{(k-j-2)}
\]

where

\[
Q = -3(2k-2p-1)(p+3-k+3t)+2(t+k-p-1) \left( 4h+4k-8+\frac{13}{2}+3t+3t^{-1} \right)
\]

and now we compute the coefficient \( R_j \)

(24)
\[
R_j = - ((j-1)(2t-1)+2k-2p-1) (p+4-k+j+5t) +
\]
\[
+ (4+j) ((j-1)(2t-1)+2k-2p-1) =
\]
\[
= -(p-k+2+5t) ((j-1)(2t-1)+2k-2p-5)
\]
The first factor of $R_j$ does not depend on $j$ and moreover

\[(25)\]
$$2t \sum_{j=1}^{k-2} R_j e_j v^{(k-j-2)} = -(p-k-2+5t)(k-2)(2p-k+2)(k-p-t) v^{(k-2)}$$

Hence

\[(26)\]
$$e_{-2} \left( 2t \sum_{j=1}^{k} ((j-1)(2t-1)+2k-2p-1) e_j v^{(k-j)} \right) =$$

\[= (2tQ-(p-k-2+5t)(k-2)(2p-k+2)(k-p-t)) v^{(k-2)} =

\[= -k(2p-k)(k-p-t)(p+4-k+5t) v^{(k-2)}.

We divide the resulting equality by $k(2p-k)(k-p-t)$ and obtain the required formula.

Taking $k = 2p$ in (26) we have

$$e_{-2} w = 0.$$  

This completes the proof of the lemma. \[\Box\]

Proof of the theorem follows from two lemmas. The only one thing we still have to do is to present explicit formula for $w$.

\[(27)\]
$$w = 2t \sum_{j_1=1}^{2p} ((j_1-1)(2t-1)+2p-1) e_{j_1} v^{(2p-j_1)},$$

$$v_{(2p-j_1)} = \sum_{j_2=1}^{2p-j_1} \frac{2t ((j_2-1)(2t-1)+2p-2j_1-1)}{(2p-j_1)j_1(p-j_1-t)} e_{j_2} v^{(2p-j_1-j_2)},$$

$$v_{(2p-j_1-j_2)} = \sum_{j_3=1}^{2p-j_1-j_2} \frac{2t ((j_3-1)(2t-1)+2p-2j_1-2j_2-1)}{(2p-j_1-j_2)(j_1+j_2)(p-j_1-j_2-t)} e_{j_3} v^{(2p-j_1-j_2-j_3)}.$$  

Proceeding further step by step we obtain the formula

\[(28)\]
$$w = \sum_{j_1, \ldots, j_s \atop j_1+\ldots+j_s=2p} \frac{(2t)^s \prod_{r=1}^{s} \left( (j_r-1)(2t-1)+2p-1-2 \sum_{q=1}^{r-1} j_q \right)}{\prod_{m=1}^{s-1} \left( (\sum_{q=1}^{m\phantom{-1}} j_q)(2p-\sum_{q=1}^{m} j_q)(p-t-\sum_{q=1}^{m} j_q) \right)} e_{j_1} \ldots e_{j_s} v.$$
We need to calculate the coefficient facing $e_1^{2p}$ in the expansion of $w$:

$$w = \frac{(2t)^{2p} \prod_{k=0}^{2p-1} (2p-1-2k)}{(2p-1)!^2 \prod_{q=1}^{2p-1} (p-t-q)} e_1^{2p} v + \ldots$$

Finally we get

$$S_{2,p}(t)v = \frac{(2p-1)!^2 \prod_{q=1}^{2p-1} (p-t-q)}{(2t)^{2p} \prod_{k=0}^{2p-1} (2p-1-2k)} w.$$

\[\square\]

**Example 1.8.**

(29)

$$S_{2,3}(t) = e_1^6 + \frac{(4-t^2)}{3t} (e_1^4 e_2 + e_2 e_1^4) + \frac{8(1-t^2)}{3t} (e_1^3 e_2 e_1 + e_1 e_2 e_1^3) + 9t e_1^2 e_2 e_1^2 +$$

$$+3(4-t^2)(e_1^2 e_2^2 + e_2 e_1^2) + \frac{64(1-t^2)^2}{9t^2} e_1^2 e_1 + \frac{(4-t^2)^2}{9t^2} e_1 e_2 e_1 +$$

$$+ \frac{8(1-t^2)(4-t^2)}{t} (e_1 e_2 e_1^2 + e_2 e_1 e_1^2) + \frac{(4-t^2)^2}{9t^2} e_1^2 e_2 +$$

$$+ \frac{4(4-t^2)(1-t^2)(9-16t^2)}{9t^4} e_1^2 e_3 + \frac{6(1+t)(4t-1)}{t} e_1^2 e_3 + \frac{6(1-t)(4t+1)}{t} e_2 e_1 e_3 +$$

$$+ \frac{2(1+t)(2t)(3-4t)}{3t^2} e_1^3 e_3 + \frac{2(1-t)(2t)(3+4t)}{3t^2} e_3 e_1^3 +$$

$$+ \frac{16(1-t^2)(1+t)(2t)(3-4t)}{9t^3} e_1 e_2 e_3 + \frac{16(1-t^2)(1-t)(2t)(3+4t)}{9t^3} e_2 e_1 e_3 +$$

$$+ \frac{2(4-t^2)(2+t)(1+t)(3-4t)}{9t^3} e_2 e_1 e_3 + \frac{2(4-t^2)(2-t)(1-t)(3+4t)}{9t^3} e_3 e_1 e_2 +$$

$$+ \frac{2(4-t^2)(1-t)(4t+1)}{t} e_1 e_2 e_3 + \frac{2(4-t^2)(1+t)(4t-1)}{t} e_2 e_3 e_1 +$$

$$+ \frac{6(1+t)(2+t)(3t-1)}{t^2} e_1^2 e_4 + \frac{48(1-t^2)}{t} e_1 e_4 e_1 + \frac{6(1-t)(2-t)(3t+1)}{t^2} e_2 e_4 +$$

$$+ \frac{(4-t^2)(2+t)(1+t)(3t-1)}{t^3} e_2 e_4 + \frac{(4-t^2)(2-t)(1-t)(3t+1)}{t^3} e_4 e_2 +$$

$$+ \frac{4(1-t^2)(2+t)(8t-1)}{t^3} e_1 e_5 + \frac{4(1-t^2)(2-t)(8t+1)}{t^3} e_5 e_1 + \frac{20(1-t^2)(4-t^2)}{t^3} e_6.$$
Obviously, coefficients $f_p(i_1, \ldots, i_s)$ facing monomials $e_{i_1} \cdots e_{i_s}$ (that correspond to unordered partitions of $2p$) in the expansion of $S_{2,p}(t)$ are not uniquely defined, because these monomials are linearly dependent in $V$. There are, however, two coefficients $f_p(1, \ldots, 1)$ and $f_p(2, \ldots, 2)$ facing $e_1^{2p}$ and $e_2^p$ respectively which, as it is easily seen, are uniquely determined. Let us calculate $f_p(2, \ldots, 2)$.

$$f_p(2, \ldots, 2) = \frac{\prod_{q=1}^{p} (t^2-(p+1-2q)^2)}{t^p}.$$ 

In particular, in our two previous examples we have

$$f_2(2, 2) = \frac{(t^2-1)^2}{t^2}, \quad f_3(2, 2, 2) = \frac{(t^2-4)^2t^2}{t^3}.$$ 

It was proved in [1] that

$$S_{k,l}(t) = (k-1)! e_k^l t^{k(l-1)} + \cdots + (l-1)! e_l^k t^{-(l-1)k},$$

where "\ldots" denotes intermediate degrees in $t$. we see that our coefficient $f_p(2, \ldots, 2)$ has prescribed asymptotic behavior with respect to $t$. Hence

$$S_{2,p}(t) = e_2^{p} t^p + \ldots$$

which is consistent with our calculations.

2. Singular vectors and cohomology

Now we are going to consider Verma modules over the Virasoro algebra with $c = 0$ that one can consider as Verma modules over the Witt algebra.

**Proposition 2.1** (Kac [11], Feigin and Fuchs [6, 7]). There is a singular vector $w_n$ in the homogeneous subspace $V_n(0, 0)$ of the Verma module $V(0, 0)$ then and only then when $n$ is equal to some pentagonal number $n = e_\pm(k) = \frac{3k^2 \pm k}{2}$.

It follows from the theorem [12] that if a Verma module $V(h, 0)$ (with $c=0$) has a singular vector $w_{p,q}(t)$ it implies that $t = -\frac{3}{2}$ or $t = -\frac{2}{3}$. We will fix the value $t = -\frac{2}{3}$ and we will write $S_{p,q}$ instead of $S_{p,q}(-\frac{2}{3})$ for convenience in the notations. Let us denote by $V\left(\frac{3k^2 \pm k}{2}\right)$ a submodule in the Verma module $V(0, 0)$ generated by a singular vector with the grading $\frac{3k^2 \pm k}{2}$. The submodule $V\left(\frac{3k^2 \pm k}{2}\right)$ is isomorphic to the Verma module $V\left(\frac{3k^2 \pm k}{2}, 0\right)$.
Proposition 2.2 ([13], [6]). The system of submodules $V\left(\frac{3k^2+k}{2}\right)$ has
the following important properties:
1) the sum $V(1) + V(2)$ is the subspace of codimension one in $V(0)$;
2) $V\left(\frac{3k^2-k}{2}\right) \cap V\left(\frac{3k^2+k}{2}\right) = V\left(\frac{3(k+1)^2-(k+1)}{2}\right) + V\left(\frac{3(k+1)^2+(k+1)}{2}\right)$, $k \geq 1$.

One can directly verify that the vectors $e_1v$ and $(e_1^2 - \frac{2}{3}e_2)v$ are
singular in the module $V(0, 0)$ with the gradings 1 and 2 respectively.
Let consider a submodule $V(1)$ generated by $w_1 = e_1v$. It is isomorphic
to the Verma module $V(1, 0)$ and it contains a singular vector $S_{1,4}w_1$.
A Verma module $V(2, 0) = V(2)$ (generated by the vector $w_2 = S_{1,2}v$)
contains a singular vector $S_{3,1}w_2$. Vectors $S_{1,4}w_1$ and $S_{3,1}w_2$ are both
singular and they at the level $n=5$ in the Verma module $V(0, 0)$. Hence
they coincide

$$w_5 = S_{1,4}w_1 = S_{3,1}w_2.$$ 

Similarly, one can check the other equality

$$w_7 = S_{3,2}w_1 = S_{1,5}.$$ 

We see that singular vectors $w_5, w_7 \in V(1) \cap V(2)$ as well as the sum
$V(5) + V(7)$ of submodules generated by $w_5$ and $w_7$ respectively:

$$V(5) + V(7) \subset V(1) \cap V(2).$$ 

The intersection $V(5) \cap V(7)$ contains two singular vectors

$$w_{12} = S_{1,7}w_5 = S_{5,1}w_7, \quad w_{15} = S_{5,2}w_5 = S_{1,8}w_7.$$ 

The inclusions of submodules $V\left(\frac{3k^2+k}{2}\right)$ provides us with an exact
sequence [13] [6] [7]:

\begin{equation}
\ldots \to V\left(\frac{3(k+1)^2-(k+1)}{2}\right) \oplus V\left(\frac{3(k+1)^2+(k+1)}{2}\right) \xrightarrow{\delta_{k+1}} V\left(\frac{3k^2-k}{2}\right) \oplus V\left(\frac{3k^2+k}{2}\right) \to \ldots
\end{equation}

\begin{equation}
\ldots \xrightarrow{\delta_3} V(5) \oplus V(7) \xrightarrow{\delta_2} V(1) \oplus V(2) \xrightarrow{\delta_1} V(0) \xrightarrow{\epsilon} \mathbb{C} \to 0,
\end{equation}

where $\delta_k$ are defined with the aid of operators $S_{p,q} \in U(L_1)$:

\begin{equation}
\delta_{k+1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_{1,3k+1} & S_{2k+1,2} \\ -S_{2k+1,1} & -S_{1,3k+2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad k \geq 1;
\end{equation}

\begin{equation}
\delta_1\begin{pmatrix} x \\ y \end{pmatrix} = (S_{1,1}, S_{1,2}) \begin{pmatrix} x \\ y \end{pmatrix},
\end{equation}

and $\epsilon$ is a projection to the one-dimensional $\mathbb{C}$-submodule generated
by the vector $v$. 

SINGULAR VIRASORO VECTORS AND COHOMOLOGY 13
Theorem 2.3 ([13], [6]). The exact sequence (30) considered as a sequence of $L_1$-modules is a free resolution of the one-dimensional trivial $L_1$-module $\mathbb{C}$.

Corollary 2.4 ([6]). Let $V$ be a $L_1$-module. Then the cohomology $H^*(L_1, V)$ is isomorphic the cohomology of the following complex:

\[ \cdots \xrightarrow{d_{k+1}} V \oplus V \xrightarrow{d_k} V \oplus V \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} V \oplus V \xrightarrow{d_0} V, \]

with the differentials

\[
d_k \left( \begin{array}{c} m_1 \\ m_2 \end{array} \right) = \left( \begin{array}{cc} S_{1,3k+1} & -S_{2k+1,1} \\ S_{2k+1,2} & -S_{1,3k+2} \end{array} \right) \left( \begin{array}{c} m_1 \\ m_2 \end{array} \right), \quad k \geq 1;
\]

\[
d_0(m) = \left( \begin{array}{c} S_{1,1m} \\ S_{1,2m} \end{array} \right), \quad m, m_1, m_2 \in V.
\]

Let consider the trivial one-dimensional module $V = \mathbb{C}$. All operators $d_k$ are trivial and we obtain the famous Goncharova theorem.

Theorem 2.5 ([4]). The space of $q$-cohomology $H^q(L_1, \mathbb{C})$ is two-dimensional for all $q \geq 1$, moreover it is the direct sum of its one-dimensional subspaces:

\[ H^q(L_1, \mathbb{C}) = H^q_{3q^2-q}(L_1, \mathbb{C}) \oplus H^q_{3q^2+q}(L_1, \mathbb{C}). \]

The numbers $e_{\pm}(q) = \frac{3q^2 \pm q}{2}$ are called Euler pentagonal numbers.

Remark. The original proof of the properties of the free resolution of one-dimensional $L_1$-module $\mathbb{C}$ by Rocha-Carridi and Wallach [13] seriously used the Goncharova theorem.

Fuchs and Feigin studied $L_1$-cohomology with coefficients in graded modules $V = \oplus_i V_i$ only with one-dimensional homogeneous components $V_i$. We will define them with the aid of the special basis $f_i, V_i = \langle f_i \rangle$ ($j \in \mathbb{Z}$ in the infinite dimensional case and $j \in \mathbb{Z}, m \leq j \leq n$ in the finite dimensional). For a given graded $L_1$-module $V = \oplus_i V_i$ let us introduce the numbers $\sigma_{p,q}(j) \in \mathbb{K}$ such that

\[ S_{p,q}f_j = \sigma_{p,q}(j)f_{j+pq}. \]

Example 2.6. The well-known $L_1$-module $F_{\lambda,\mu}$ of tensor densities [9]:

\[ e_i f_j = (j + \mu - \lambda(i + 1)) f_{i+j}, \forall i \in \mathbb{N}, j \in \mathbb{Z}, \]

where $\lambda, \mu \in \mathbb{K}$ are two complex parameters.
Corollary 2.7 ([6]). Let $V = \oplus_i V_i$ be a graded $L_1$-module over the field $\mathbb{K}$. Then the one-dimensional cohomology $H^*_s(L_1, V)$ is isomorphic to the cohomology of the following complex:

\[ \ldots \xleftarrow{D_{k+1}} K \oplus K \xleftarrow{D_k} K \oplus K \xleftarrow{D_{k-1}} \ldots \xleftarrow{D_1} K \oplus K \xleftarrow{D_0} K, \]

where the differentials $D_k$ are assigned by the numerical matrices

\[ D_k = \begin{pmatrix} \sigma_{1,3k+1} \left( s + \frac{3k^2-k}{2} \right) & -\sigma_{2k+1,1} \left( s + \frac{3k^2-k}{2} \right) \\ \sigma_{2k+1,2} \left( s + \frac{3k^2+k}{2} \right) & -\sigma_{1,3k+2} \left( s + \frac{3k^2+k}{2} \right) \end{pmatrix}, \quad D_0 = \begin{pmatrix} \sigma_{1,1}(s) \\ \sigma_{1,2}(s) \end{pmatrix}. \]

Fuchs and Feigin [6] have not found general explicit formulas for singular vectors of the type $S_{p,1}$ and $S_{p,2}$ but they managed however to find formulae for elements $\sigma_{p,q}(j)$ for modules $F_{\lambda,\mu}$ (using other arguments). But other graded $L_1$-modules are also can be important for applications [12] and require explicit formulas for singular vectors $S_{p,1}$ and $S_{p,2}$. The main result of this article, together with the Benoit-Saint-Aubin theorem provides us with the formulae useful for cohomology calculations.

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