Dissipative dynamics of few-photons superposition states: A dynamical invariant

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Abstract

By numerically calculating the time-evolved Wigner functions, we investigate the dynamics of a few-photon superposed (e.g., up to two ones) state in a dissipating cavity. It is shown that, the negativity of the Wigner function of the photonic state unquestionably vanishes with the cavity’s dissipation. As a consequence, the nonclassical effects related to the negativity of the Wigner function should be weakened gradually. However, it is found that the value of the second-order correlation function $g^{(2)}(0)$ (which serves usually as the standard criterion of a typical nonclassical effect, i.e., $g^{(2)}(0) < 1$ implies that the photon is anti-bunching) is a dynamical invariant during the dissipative process of the cavity. This feature is also proven analytically and suggests that $g^{(2)}(0)$ might not be a good physical parameter to describe the photonic decays. Alternatively, we find that the anti-normal-order correlation function $g^{(2A)}(0)$ changes with the cavity’s dissipation and thus is more suitable to describe the dissipative-dependent cavity. Finally, we propose an experimental approach to test the above arguments with a practically-existing cavity QED system.

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I. INTRODUCTION

It is well-known that the Wigner function, introduced 70 years ago by Wigner to describe the quasi-probability distribution of a quantum particle in its phase space, is a very popular tool to study the statistical properties of various quantum states [1]. Basically, once the Wigner function has been determined, all the knowable information on the quantum state (such as its nonclassical statistical properties) can be extracted [2-5]. Typically, differing from the standard probability distribution, such a quasi-probability distribution can be assigned by a negative value [6]. Therefore, a quantum state with the negative Wigner function should be nonclassical and thus certain nonclassical effects (such as the photon anti-bunchings) [7-11] demonstrate. This indicates that, determining the Wigner function of a selected quantum state plays an important role both fundamentally and practically in quantum-state engineerings.

Usually, any selected quantum system is always surrounded by the classical environments. Thus, dissipation of the artificially-prepared quantum state is one of the central topics in quantum coherence science. Roughly, due to the existence of various dissipation and fluctuations from the environments, any excited quantum state will decay to the ground state and the relevant system finally becomes classical. Under the standard logic, people pay the most attention to calculate either decoherence or decay time of a superposition quantum state, rather than cares on the process of the decoherence or decay [12-15]. Alternatively, in the present work we investigate exactly the dissipative dynamics for a prepared quantum state by calculating its dissipative-dependent Wigner function. Our discussions are based on the typical few-photon quantum state in a cavity, but can be directly generalized to other quantum systems such as qubits and qutrits.

The paper is organized as: in Sec. 2, we describe how the Wigner function for a few-photon superposed state changes with the cavity’s dissipation. Our numerical results show naturally that the negativity of the Wigner function weakens gradually with the dissipation and the final state of the cavity should be “classical” with positive Wigner function. With the calculated Wigner function we investigate how the nonclassical properties, such as the anti-bunching effect of photons, changes with the cavity dissipation. It is surprised that the value of the second-order correlation function $g^{(2)}(0)$ (which serves usually as the standard criterion of a nonclassical effect, i.e., $g^{(2)}(0) < 1$ implies that the photon is anti-bunching) is an invariant during the dissipative process of the cavity. We prove such an argument analytically by directly solving the relevant master equation and suggests that $g^{(2)}(0)$ is not a good parameter to describe dissipative-dependent non-
classicality of the photonic decays. Alternatively, we find that the anti-normal-order correlation function $g^{(2,A)}(0)$ changes with the cavity’s dissipation and thus could be more suitable to describe the dissipative-dependent cavity. With an experimentally-demonstrated cavity QED system we propose an approach to test our results, including how to prepare the investigated few-photon superposed state of the cavity and measure its Wigner function. Finally, our conclusions and discussions are given in Sec. 4.

II. DISSIPATIVE DYNAMICS OF WIGNER FUNCTIONS FOR FEW-PHOTONS SUPERPOSITION STATES

Generally, the quasi-probability distribution $W(\alpha, \alpha^*)$ can be defined by the Fourier transform of the symmetrical-ordered characteristic function $C(\lambda, \lambda^*)$ [16], i.e.,

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2 \lambda \ C(\lambda, \lambda^*) e^{\alpha \lambda^* - \alpha^* \lambda},$$  

with $\lambda$ and $\alpha$ being the complex parameters in phase space. The expression of the symmetrical-ordered characteristic function is defined as

$$C(\lambda, \lambda^*) = Tr[\rho e^{\lambda \hat{a}^\dagger - \alpha^* \hat{a}}],$$

where $\rho$ is the density matrix of the cavity state $|\psi\rangle$, and $\hat{a}$ and $\hat{a}^\dagger$ the usual annihilation and creation operators of the photons, respectively.

For the simplicity and without loss of the generality, let us assume that the cavity is initially prepared in the following few-photon superposition state

$$|\psi(0)\rangle = C_0|0\rangle + C_1|1\rangle + C_2|2\rangle,$$

with the complex amplitudes: $C_0 = |C_0|e^{i\phi}$, $C_1 = |C_1|$, and $C_2 = |C_2|e^{i\phi}$. Then, with the matrix elements of Wigner operator: $\Delta(\alpha, \alpha^*) = \int d^2 z e^{[z(\hat{a}^\dagger - \alpha^*) - z^* (\hat{a} - \alpha)]}/2\pi^2$, in the Fock representation [17]

$$\langle n| \Delta(\alpha, \alpha^*)|m\rangle = \frac{(-1)^m}{\pi} \sqrt{\frac{m!}{n!}} (2\alpha)^{n-m} e^{(-2|\alpha|^2)} L_m^{(n-m)}(4|\alpha|^2), n, m = 0, 1, 2, ...$$
here \( n > m, \alpha_0 = |\alpha_0| e^{i\theta} \), one can easily obtain the Wigner function of the initial state

\[
W(\alpha_0, \alpha_0^*, 0) = \frac{2}{\pi} [ |C_0|^2 - |C_1|^2 L_0^0(4|\alpha_0|^2) + |C_2|^2 L_0^0(4|\alpha_0|^2) ] e^{(-2|\alpha_0|^2)}
+ \frac{8\sqrt{2}}{\pi} e^{(-2|\alpha_0|^2)} |C_0C_2||\alpha_0|^2 \cos(2\theta - \varphi + \phi)
- \frac{4\sqrt{2}}{\pi} e^{(-2|\alpha_0|^2)} |C_1C_2||\alpha_0|^2 \cos(\theta - \varphi) L_1^1(4|\alpha_0|^2)
+ \frac{8}{\pi} e^{(-2|\alpha_0|^2)} |C_0C_1||\alpha_0|^2 \cos(\theta + \phi).
\]

(5)

for the above superposition initial state. Above, \( L_n^J(x) \) is an associated Laguerre polynomial defined by [18]

\[
L_n^J(x) = \sum_{\kappa=0}^{n} (-1)^{\kappa} \frac{(n + J)!}{(n - \kappa)! (J + \kappa)!} \kappa! x^\kappa.
\]

(6)

In what follows we discuss how such a state decay in a loss cavity by investigating the time-evolutions of the above initial Wigner function.

A. Dissipative dynamics for the Wigner function

We now consider how the above few-photons superposition state dissipates in a loss cavity without any thermal photon (i.e., \( \langle n \rangle_{th} = 1/[\exp(\hbar \nu / k_B T) - 1] \to 0 \), for the present optical frequency photons and at the room temperature: \( \hbar \nu / k_B T \gg 1 \)), which is described simply by the following master equation [19-20]

\[
\frac{d\rho}{dt} = -\kappa (\hat{a}^\dagger \hat{a} \rho + \rho \hat{a}^\dagger \hat{a} - 2\hat{a} \hat{a}^\dagger \rho),
\]

(7)

with \( k \) being the loss coefficient. Our discussion is based on the time-evolutions of the Wigner function, i.e.,

\[
\frac{d}{dt} W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2 \lambda \frac{d C(\lambda, \lambda^*)}{dt} e^{\alpha^* \lambda - \alpha \lambda^*},
\]

(8)

with

\[
\frac{d C(\lambda, \lambda^*)}{dt} = Tr[\frac{d}{dt} e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}] = \kappa |\lambda|^2 Tr[(2\hat{a} \rho \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \rho - \rho \hat{a}^\dagger \hat{a}) e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}].
\]

(9)

Formally, Eq. (8) can be rewritten as

\[
\frac{d}{dt} W(\alpha, \alpha^*) = 2\kappa W^{[\hat{a} \rho \hat{a}^\dagger]}(\alpha, \alpha^*) - \kappa W^{[\hat{a}^\dagger \hat{a} \rho]}(\alpha, \alpha^*) - \kappa W^{[\rho \hat{a}^\dagger \hat{a}]}(\alpha, \alpha^*),
\]

(10)

where the symbol \( W^{[x]}(\alpha, \alpha^*) \) is defined as

\[
W^{[x]}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2 \lambda C^{[x]}(\lambda, \lambda^*, t) e^{\alpha^* \lambda - \alpha \lambda^*},
\]

(11)

\[
C^{[x]}(\lambda, \lambda^*) = Tr[x e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}],
\]

(11)
with \( W^\rho(\alpha, \alpha^*) = W(\alpha, \alpha^*) \), and \( C^\rho(\lambda, \lambda^*) = C(\lambda, \lambda^*) \). Note that
\[
C^{\rho \tilde{a}}(\lambda, \lambda^*) = \left[ \frac{1}{2} + \frac{\partial}{\partial \lambda} \left( -\frac{\partial}{\partial \lambda^*} \right) \right] C(\lambda, \lambda^*) = \left( \frac{\partial}{\partial \lambda} + \frac{\lambda^*}{2} \right) \left( \frac{\lambda}{2} - \frac{\partial}{\partial \lambda^*} \right) C(\lambda, \lambda^*),
\] (12)
and
\[
\frac{1}{\pi^2} \int d^2 \lambda \ e^{(\alpha \lambda - \alpha^* \lambda)} \frac{\partial}{\partial \lambda} C(\lambda, \lambda^*) = \alpha^* W(\alpha, \alpha^*),
\]
\[
\frac{1}{\pi^2} \int d^2 \lambda \ e^{(\alpha \lambda - \alpha^* \lambda)} \frac{\partial}{\partial \lambda^*} C(\lambda, \lambda^*) = -\alpha W(\alpha, \alpha^*),
\]
\[
\frac{1}{\pi^2} \int d^2 \lambda \ e^{(\alpha \lambda - \alpha^* \lambda)} \lambda C(\lambda, \lambda^*) = \frac{\partial}{\partial \alpha} W(\alpha, \alpha^*),
\]
\[
\frac{1}{\pi^2} \int d^2 \lambda \ (-\lambda) e^{(\alpha \lambda - \alpha^* \lambda)} C(\lambda, \lambda^*) = \frac{\partial}{\partial \alpha^*} W(\alpha, \alpha^*),
\] (13)
we then have
\[
W^{\rho \tilde{a}}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2 \lambda \ C^{\rho \tilde{a}}(\lambda, \lambda^*) e^{(\alpha \lambda - \alpha^* \lambda)}
\]
\[
= \frac{1}{\pi^2} \int d^2 \lambda \ [\alpha \alpha^* + \frac{1}{2} - \alpha^* \frac{\partial}{\partial \alpha} - \frac{1}{4} \frac{\partial}{\partial \alpha^*} + \frac{\alpha}{2} \frac{\partial}{\partial \alpha^*}] C(\lambda, \lambda^*) e^{(\alpha \lambda - \alpha^* \lambda)}
\]
\[
= \left[ \frac{\alpha^* + 1}{2} \frac{\partial}{\partial \alpha} \right] [\alpha - \frac{1}{2} \frac{\partial}{\partial \alpha^*}] W(\alpha, \alpha^*),
\] (14)
Similarly,
\[
W^{\rho \tilde{a}}(\alpha, \alpha^*) = [\alpha + \frac{1}{2} \frac{\partial}{\partial \alpha^*}] [\alpha + \frac{1}{2} \frac{\partial}{\partial \alpha}] W^{\rho}(\alpha, \alpha^*)
\]
\[
W^{\tilde{a} \rho}(\alpha, \alpha^*) = [\alpha^* - \frac{1}{2} \frac{\partial}{\partial \alpha}] [\alpha + \frac{1}{2} \frac{\partial}{\partial \alpha^*}] W^{\rho}(\alpha, \alpha^*).
\] (15)
As a consequence, Eq. (10) reduces to
\[
\frac{dW(\alpha, \alpha^*)}{dt} = k \left[ \frac{\partial^2}{\partial \alpha \partial \alpha^*} + \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha^*} \alpha^* \right] W(\alpha, \alpha^*),
\] (16)
whose solution reads [12]
\[
W(\alpha, \alpha^*, t) = \frac{2}{1 - e^{-2\kappa t}} \int \frac{d^2 \alpha_0}{\pi} e^{-\frac{1}{1 - e^{-2\kappa t}} |\alpha - \alpha_0 e^{-\kappa t}|^2} W(\alpha_0, \alpha_0^*, 0),
\] (17)
For the cavity initial state $|\psi(0)\rangle$ we substitute Eq. (5) into Eq. (17) and get

$$W(\alpha, \alpha^*, t) = \frac{2}{\pi} e^{(-2|\alpha|^2)} |C_0|^2 - |C_1|^2 (2e^{-2\kappa t} - 1) L_1^0[-\frac{|2e^{2\kappa t}|^2}{1 - 2e^{-2\kappa t}}] + \frac{2}{\pi} e^{-2|\alpha|^2} |C_2|^2 (2e^{-2\kappa t} - 1)^2 L_2^0[-\frac{|2e^{2\kappa t}|^2}{1 - 2e^{-2\kappa t}}]$$

$$+ \frac{8\sqrt{2}}{\pi} |C_0C_2| e^{-|2|\alpha|^2-2\kappa t}|\alpha|^2 \cos(2\theta - \varphi + \phi)$$

$$+ \frac{8}{\pi} |C_0C_1| e^{-|2|\alpha|^2-\kappa t}|\alpha| \cos(\theta + \phi)$$

$$+ \frac{8\sqrt{2}}{\pi} |C_1C_2| e^{-|2|\alpha|^2-k\kappa t}|\alpha| \cos(\theta - \varphi) [2(|\alpha|^2 - 1)e^{-2\kappa t} + 1]. \quad (18)$$

Above, an integral formula [21]

$$\int \frac{d^2 z}{\pi} z^n z^* m e^{(x_1z^2 + x_2z + x_3 z^*)}$$

$$= e^{(-x_2 x_3)} \sum_{\kappa=0}^{\min(m,n)} \frac{n!m!}{\kappa!(n-\kappa)!(m-\kappa)!} (-x_1)^{m-n+\kappa+1} x_2^{m-k} x_3^{n-k}, \quad Re(x_1) < 0, \quad (19)$$

has been used and the unassociated Laguerre Polynomial $L_m(x, y)$:

$$L_m(x, y) = \frac{(-1)^m}{m!} H_{m,n}(x, y), \quad H_{m,n} = \frac{\partial^{m+n}}{\partial T^m \partial T'^n} e^{-TT' + Tx + T'y} |_{T = T' = 0}, \quad (20)$$

was introduced [22-23] with $H_{m,n}(x, y)$ being the generating function of two-variable Hermite polynomial.

**B. Time-dependent negativity of the Wigner function**

With the above time-evolution Wigner function, we next check how its negativity changes with the cavity loss. Fig. 1 numerically shows these changes with the effective time $\kappa t$ for the parameters: $|C_1| = 1/3, |C_2| = \sqrt{2}/2, \theta = \varphi = \pi, \phi = 0$. Here, for convenience we define the Wigner function $W(x, p, t)$ in the $(x, p)$-space with $x = (\alpha + \alpha^*)/2$ and $p = (\alpha - \alpha^*)/(2i)$. One can see:

(i) Initially, the Wigner function shows obviously a negativity, i.e., at certain phase space points, $W(x, p) < 0$. This means that certain nonclassical effects (such as the anti-bunching of photons) can be revealed in this initial cavity state.

(ii) With the cavity dissipation, the state of the cavity decays and the negativity of its time-dependent Wigner function vanishes gradually. This implies that the nonclassical properties possessed initially would be weakened with the dissipation of the cavity.
FIG. 1: Wigner functions versus phase space points, \((x, p)\) (upper line) and \((x, p = 0)\) (lower line), of the few-photon superposed state (3) for different decay times, i.e., \(\kappa t = 0, 0.2, 0.35\) in Fig. 1(c). The values of the Wigner functions reveal the expected non-negativity, i.e., \(W(x, p) \geq 0\). In this evolved state the decayed cavity should be classical and the corresponding Wigner functions could be explained as the usual probabilistic distributions.

(iii) After certain times, e.g., \(\kappa t \geq 0.35\) in Fig. 1(c), the values of the Wigner functions reveal the expected non-negativity, i.e., \(W(x, p) \geq 0\). In this evolved state the decayed cavity should be classical and the corresponding Wigner functions could be explained as the usual probabilistic distributions.

(iv) After the sufficiently-long dissipative time, the cavity state will decay to the expectable vacuum state or thermal state with the mean photon number being zero (i.e., \(\bar{n} = 0\)). The Wigner function for such a dissipated final state should be a Gaussian distribution. Indeed, from Eq. (18),
we have

\[
W(\alpha, \alpha^*, \infty) = \frac{2}{\pi} e^{(-2|\alpha|^2)} \left[ |C_0|^2 + |C_1|^2 L_1^0(0) + |C_2|^2 L_2^0(0) \right] \\
= \frac{2}{\pi} e^{(-2|\alpha|^2)} \left[ |C_0|^2 + |C_1|^2 + |C_2|^2 \right] \\
= \frac{2}{\pi} e^{(-2|\alpha|^2)}.
\]  

(21)

III. DISSIPATIVE-DEPENDENT QUANTUM STATISTICAL PROPERTIES OF THE FEW-PHOTONS CAVITY INITIAL STATE

Various nonclassical effects, e.g., squeezings on quantum fluctuations and sub-Poisson photon statistics, in quantum optical states have attracted considerable and continuing interests[24-26]. Many attentions have been paid to find various non-classical optical states, while how these non-classical effects change with the decays of the selected non-classical states is a relatively-new topic. Recently, Biswas and Agarwal discussed how the Mandel Q-factor decreases with the decays of the photon-subtracted squeezed states[12]. Their numerical results showed that the Q-factor vanishes at the long dissipative times (i.e., \( \kappa t \to \infty \)) and the initial cavity state will decay to the vacuum. With the dissipative-dependent Wigner functions obtained in the previous section, we can investigate how the photonic anti-bunching effect changes with the decay of the few-photon superposition state \( |\psi(0)\rangle \) defined in Eq. (3).

It is well-known that, if the second-order correlation function

\[
g^{(2)}(0) = \frac{\langle \hat{a}^{12}\hat{a}^2 \rangle}{\langle \hat{a}^1\hat{a} \rangle^2}
\]  

(22)

is less than 1, then the photonic distribution in the state \( |\psi\rangle \) is anti-bunching; otherwise, it is bunching. The symbol \( \langle \hat{O} \rangle \) represents the expectation value of the operator \( \hat{O} \) in a quantum state \( \rho \). For the present case we need to calculate the time-dependent expectation values of the operators \( \hat{a}^{12}\hat{a}^2 \) and \( \hat{a}^\dagger\hat{a} \) for the decaying cavity state with time-dependent Wigner function \( W(\alpha, \alpha^*, t) \).

Formally, for an operator function [16]

\[
O(\hat{a}, \hat{a}^\dagger)(t) = \sum_{n,m} C_{n,m}(t) \hat{a}^\dagger n(t) \hat{a}^m(t),
\]  

(23)

we have

\[
\langle O(\hat{a}, \hat{a}^\dagger) \rangle(t) = Tr[O(\hat{a}, \hat{a}^\dagger)\rho(t)] = \int d^2\alpha_{S(\alpha, \alpha^*)} W(\alpha, \alpha^*, t).
\]  

(24)
On the other hand, from
\[ \langle \hat{a}^\dagger \rangle(t) = \left[ \frac{\partial}{\partial \lambda} + \frac{\lambda^*}{2} \right] C(\lambda, \lambda^*, t) \bigg|_{\lambda = \lambda^* = 0} \]
\[ \langle \hat{a} \rangle(t) = -\frac{\partial}{\partial \lambda^*} - \frac{\lambda}{2} \right] C(\lambda, \lambda^*, t) \bigg|_{\lambda = \lambda^* = 0}, \]  
we can find that
\[ \langle O(\hat{a}, \hat{a}^\dagger) \rangle(t) = \sum_{n,m} C_{n,m} \left[ \frac{\partial}{\partial \lambda} + \frac{\lambda^*}{2} \right]^n \left[ -\frac{\partial}{\partial \lambda^*} - \frac{\lambda}{2} \right]^m C(\lambda, \lambda^*, t) \bigg|_{\lambda = \lambda^* = 0} \]
\[ = \int d^2 \alpha \sum_{n,m} C_{n,m} \left[ \frac{\partial}{\partial \lambda} + \frac{\lambda^*}{2} \right]^n \left[ -\frac{\partial}{\partial \lambda^*} - \frac{\lambda}{2} \right]^m e^{-\alpha \lambda^* + \alpha^* \lambda} \bigg|_{\lambda = \lambda^* = 0} \sqrt{W(\alpha, \alpha^*, t)} \]
\[ = \int d^2 \alpha O_S(\alpha, \alpha^*) W(\alpha, \alpha^*, t). \]  
Comparing (24) and (26), we obtain
\[ O_S(\alpha, \alpha^*) = \sum_{n,m} C_{n,m} \left[ \frac{\partial}{\partial \lambda} + \frac{\lambda^*}{2} \right]^n \left[ -\frac{\partial}{\partial \lambda^*} - \frac{\lambda}{2} \right]^m e^{-\alpha \lambda^* + \alpha^* \lambda} \bigg|_{\lambda = \lambda^* = 0}. \]  
Specifically, if \( \hat{O} = \hat{a}^\dagger \hat{a} \), then
\[ O_S(\alpha, \alpha^*) \big|_{\hat{O} = \hat{a}^\dagger \hat{a}} = |\alpha|^2 - \frac{1}{2}, \]
and thus
\[ \langle \hat{a}^\dagger \hat{a} \rangle(t) = \int d^2 \alpha W(\alpha, \alpha^*, t) O_S(\alpha, \alpha^*) \big|_{\hat{O} = \hat{a}^\dagger \hat{a}^2} \]
\[ = 4 |C_2|^2 e^{-4\kappa t} + 2 |C_2|^2 e^{-2\kappa t} [1 - 2 e^{-2\kappa t}] + |C_1|^2 e^{-2\kappa t}, \] \[ \text{(29)} \]
Also, if \( \hat{O} = \hat{a}^\dagger \hat{a}^2 \), then
\[ O_S(\alpha, \alpha^*) \big|_{\hat{O} = \hat{a}^\dagger \hat{a}^2} = \frac{1}{2} - 2 |\alpha|^2 + |\alpha|^4, \] \[ \text{(30)} \]
and thus
\[ \langle \hat{a}^\dagger \hat{a}^2 \rangle(t) = \int d^2 \alpha W(\alpha, \alpha^*, t) O_S(\alpha, \alpha^*) \big|_{\hat{O} = \hat{a}^\dagger \hat{a}} \]
\[ = 2 |C_2|^2 e^{-4\kappa t}. \] \[ \text{(31)} \]
Above, the dissipative-dependent Wigner function shown in Eq. (18) was used. Consequently, we have
\[ g^{(2)}(0; t) = \frac{\langle \hat{a}^\dagger \hat{a}^2 \rangle(t)}{[\langle \hat{a}^\dagger \hat{a} \rangle(t)]^2} \]
\[ = \frac{2 |C_2|^2 e^{-4\kappa t}}{\left\{ 4 |C_2|^2 e^{-4\kappa t} + 2 |C_2|^2 e^{-2\kappa t} [1 - 2 e^{-2\kappa t}] + |C_1|^2 e^{-2\kappa t} \right\}^2} \]
\[ = \frac{2 |C_2|^2}{[|C_1|^2 + 2 |C_2|^2]^2} = g^{(2)}(0). \] \[ \text{(32)} \]
FIG. 2: (a): Normal-ordered correlation function $g^{(2)}(t)$ is unchanged with the decay of the few-photons cavity state; (b): Anti-normally-order correlation function $g^{(2,A)}(t)$ versus the effective decay time of the cavity. Here, the relevant parameters are taken as: $\theta = \varphi = \pi, \phi = 0$, and $|C_1| = \sqrt{6}/6, |C_2| = \sqrt{6}/3$ (blue line), $|C_1| = 2/9, |C_2| = 2/3$ (red line), $|C_1| = 1/3, |C_2| = 1/3$ (gray line), and $|C_1| = 1/5, |C_2| = 1/3$ (green line), respectively.

This indicates that the normally-order correlation function $g^{(2)}(0; t)$ is cavity-loss-invariant; its value depends only on the initial cavity state. This is a surprise argument; imagining that the photons in the initial cavity state is anti-bunching (i.e., $g^{(2)}(0; t) < 1$), then such a non-classical feature is kept unchanged even the state approached finally to the vacuum with non-negative Wigner function. This argument is verified numerically by Fig. 2(a), which really shows that the value of $g^{(2)}(0; t)$ is really unchanged with the decay. It is noted that, at the exact vacuum $|0\rangle$ the expected value of operator $\langle a^\dagger a \rangle$ is zero and thus the definition of $g^{(2)}(0)$ for this state is bizarre and insignificant. Therefore, our discussion always works for the dissipative process approaching to (but not arriving at) the exact vacuum.

The dissipative-independence of the normally-correlation function $g^{(2)}$ can also be proven analytically from the master equation (7). In fact, at any time $t$ we have

$$
\langle a^\dagger a \rangle(t) = Tr[a^\dagger a \rho(t)]
= \langle 0|a^\dagger a \rho(t)|0 \rangle + \langle 1|a^\dagger a \rho(t)|1 \rangle + \langle 2|a^\dagger a \rho(t)|2 \rangle + \ldots + \langle n|a^\dagger a \rho(t)|n \rangle + \ldots
= 0 + \langle 1|\rho(t)|1 \rangle + 2\langle 2|\rho(t)|2 \rangle + \ldots + n\langle n|\rho(t)|n \rangle + \ldots
= \rho_{11}(t) + 2\rho_{22}(t) + \ldots + n\rho_{nn}(t) + \ldots
= \sum_{n=0}^{\infty} n\rho_{n,n}(t),
$$

(33)
\[ \langle \hat{a}^2 \hat{a}^2 \rangle (t) = Tr[\hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rho(t)] = Tr[\hat{a}^\dagger (\hat{a} \hat{a}^\dagger - 1) \hat{a} \rho(t)] \]
\[ = [\langle 0 | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rho(t) | 0 \rangle + \langle 1 | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rho(t) | 1 \rangle + \ldots + \langle n | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rho(t) | n \rangle + \ldots] - \langle \hat{a}^\dagger \hat{a} \rangle (t) \]
\[ = [\rho_{11}(t) + 2^2 \rho_{22}(t) + \ldots + n^2 \rho_{nn}(t) + \ldots] - \langle \hat{a}^\dagger \hat{a} \rangle (t) \]
\[ = \sum_{n=0}^{\infty} n(n - 1) \rho_{n,n}(t), \quad (34) \]

and thus
\[ g^{(2)}(0; t) = \frac{\langle \hat{a}^2 \hat{a}^2 \rangle (t)}{[\langle \hat{a}^\dagger \hat{a} \rangle (t)]^2} \]
\[ = \frac{\langle n^2 \rangle (t) - \langle n \rangle (t)}{[\langle n \rangle (t)]^2} \]
\[ = \frac{[\rho_{11}(t) + 2^2 \rho_{22}(t) + \ldots + n^2 \rho_{nn}(t) + \ldots] - [\rho_{11}(t) + 2\rho_{22}(t) + \ldots + n\rho_{nn}(t) + \ldots]}{[\rho_{11}(t) + 2\rho_{22}(t) + \ldots + n\rho_{nn}(t) + \ldots]^2} \]
\[ = \frac{\sum_{n=0}^{\infty} (n^2 - n) \rho_{nn}(t)}{[\sum_{n=0}^{\infty} n\rho_{nn}(t)]^2}. \quad (35) \]

Above, \( \rho_{n,n}(t) \) is the diagonal elements of the density matrix \( \rho(t) \) in the Fock space. For the loss cavity initially prepared in the few-photon superposition state (3), one can easily see that \( \rho_{n,n} = 0 \), for \( n > 2 \), and the other non-zero diagonal elements are determined by the following equation (from Eq. (7)),

\[ \dot{\rho}_{00}(t) = 2\kappa \rho_{11}(t), \]
\[ \dot{\rho}_{11}(t) = -2\kappa \rho_{11}(t) + 4\kappa \rho_{22}(t), \]
\[ \dot{\rho}_{22}(t) = -4\kappa \rho_{22}(t). \quad (36) \]

The solutions to these equations are
\[ \rho_{11}(t) = \rho_{11}(0) + 2\rho_{22}(0)e^{-2\kappa t} - 2\rho_{22}(0)e^{-4\kappa t} \]
\[ \rho_{22}(t) = \rho_{22}(0)e^{-4\kappa t}. \quad (37) \]

Consequently,
\[ g^{(2)}(0; t) = \frac{2\rho_{22}(t)}{[\rho_{11}(t) + 2\rho_{22}(t)]^2} \]
\[ = \frac{2\rho_{22}(0)e^{-4\kappa t}}{[\rho_{11}(0)e^{-2\kappa t} + 2\rho_{22}(0)e^{-2\kappa t}]^2} \]
\[ = \frac{2\rho_{22}(0)}{[\rho_{11}(0) + 2\rho_{22}(0)]^2} = g^{(2)}(0). \quad (38) \]
Suppose that any non-classical effect should vanish due to the dissipation, the dissipative-independence of the normally-correlation function implies that such a parameter should not be a good physical quantity to describe the cavity loss. Alternatively, the anti-normal ordered correlation function, defined as

\[ g(2A) \equiv \langle \hat{a}^2 \hat{a}^\dagger \rangle = \frac{\langle \hat{a}^2 \hat{a}^\dagger \rangle + 4\langle \hat{a}^\dagger \hat{a} \rangle + 2}{\langle \hat{a}^\dagger \hat{a} \rangle + 1} \],

(39)

could be utilized to describe the dissipative process of the few-photon cavity. Indeed, with Eqs. (29) and (31) we have

\[ g(2A)(0; t) = \langle \hat{a}^2 \hat{a}^\dagger \rangle(t) + 4\langle \hat{a}^\dagger \hat{a} \rangle(t) + 2 \]

\[ \left[ \langle \hat{a}^\dagger \hat{a} \rangle(t) + 1 \right]^2 \]

\[ = \frac{2|C_2|^2 e^{-4\kappa t} + 4\{4|C_2|^2 e^{-4\kappa t} + 2|C_2|^2 e^{-(2\kappa t)}[1 - 2e^{-(2\kappa t)}] + |C_1|^2 e^{-(2\kappa t)}\} + 2}{\{4|C_2|^2 e^{-4\kappa t} + 2|C_2|^2 e^{-(2\kappa t)}[1 - 2e^{-(2\kappa t)}] + |C_1|^2 e^{-(2\kappa t)} + 1\}^2} \]

\[ = 4|C_1|^2 e^{-2\kappa t} + 8|C_2|^2 e^{-(4\kappa t)} + 2|C_2|^2 e^{-(2\kappa t)} + 2 \]

\[ \left[ |C_1|^2 e^{-(2\kappa t)} + 2|C_2|^2 e^{-(2\kappa t)} + 1\right]^2, \]

(40)

which is not an invariant during the cavity dissipation. One can see also from Fig. 2(b) that, the value of the anti-normal correlation function changes with the cavity loss. After a sufficiently-long decay time the value of \( g(2A)(0; t) \) should limit to 2, whatever its initial value is less than 2 or not. Certainly, such a dissipative-dependent behavior of the \( g(2A)(0; t) \)-parameter can also be exactly verified by using the analytic solutions, i.e., Eq. (37). In fact, we can see that

\[ g(2A)(0; t) = \frac{\langle \hat{a}^2 \hat{a}^\dagger \rangle(t) + 4\langle \hat{a}^\dagger \hat{a} \rangle(t) + 2}{\langle \hat{a}^\dagger \hat{a} \rangle(t) + 1} \]

\[ = \frac{4\rho_{11}(0) e^{-2\kappa t} + 8\rho_{22}(0) e^{-2\kappa t} + 2\rho_{22}(0) e^{-4\kappa t} + 2}{\rho_{11}(0) e^{-2\kappa t} + 2\rho_{22}(0) e^{-2\kappa t} + 1} \]

(41)

It is consistent with the Eq. (41), as if \( t \to \infty \), Eq.(42) can be shown

\[ g(2A)(0; t \to \infty) = 2. \]

IV. POSSIBLE EXPERIMENTAL VERIFICATION: THE PREPARATION OF FEW-PHOTON SUPERPOSED STATES AND MEASUREMENT OF ITS WIGNER FUNCTION

We now discuss how to test the above arguments with a typical cavity QED system, i.e., highly excited Rydberg atoms in a high-Q microwave cavity [28]. An ideal setup is schematized in Fig. 4, wherein an atom is emitted from the source O and then flies across sequentially a quantized cavity,
FIG. 3: An experimental setup for preparing the superposition states of $|0\rangle$, $|1\rangle$ and $|2\rangle$. Here, an atom is emitted from the source O, then it flies sequentially across the J-C cavity, the classical microwave field, and at last is detected in the detector I.

A classical microwave field, and finally is detected in the detector I. When the atom passes through the quantized cavity, the usual Jaynes-Cummings model with the Hamiltonian ($\hbar = 1$)

$$H = \omega_a S_z + \omega_c \hat{a}^\dagger \hat{a} + g(\hat{a}S_+ + \hat{a}^+ S_-),$$

works. Here, $\omega_a$, $\omega_c$ are the atomic transition frequency and the cavity field frequency, respectively. $S_z$, $S_\pm$ are the atomic operators, such that $[S_+, S_-] = 2S_z$, $[S_z, S_\pm] = \pm S_\pm$. $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the cavity field, respectively. And, $g$ is the atom-field coupling strength.

Initially, the atom is in the ground state $|e_1\rangle$ and the cavity mode in the vacuum state, i.e., the wave function of the atom-cavity system is $|\psi(0)\rangle = |0, e_1\rangle$. Next, the atom is injected into the cavity and the state of the atom-cavity system evolves to

$$|\psi(t)\rangle_1 = \cos(gt_1)|0, e_1\rangle - i \sin(gt_1)|1, g_1\rangle,$$

after the passage time $t_1$. Then, we let the atom continuously across a classical microwave field for evolving the atomic states as: $|e_1\rangle \rightarrow \cos(\theta_1/2)|e_1\rangle + ie^{-i\varphi_1} \sin(\theta_1/2)|g_1\rangle$ and $|g_1\rangle \rightarrow \cos(\theta_1/2)|g_1\rangle + ie^{i\varphi_1} \sin(\theta_1/2)|e_1\rangle$. Here, the values of $\theta_1$ and $\varphi_1$ are adjustable. Therefore, before arriving at the atomic detector I, the state of the atom-cavity system reads

$$|\psi(t)\rangle_1 = [\cos(gt_1) \cos(\frac{\theta_1}{2})|0\rangle + e^{i\varphi_1} \sin(gt_1) \sin(\frac{\theta_1}{2})|1\rangle]|e_1\rangle$$

$$+ [ie^{-i\varphi_1} \cos(gt_1) \sin(\frac{\theta_1}{2})|0\rangle - i \sin(gt_1) \cos(\frac{\theta_1}{2})|1\rangle]|g\rangle.$$ (45)

In order to generate the desirable superposition of the states $|0\rangle$, $|1\rangle$ and $|2\rangle$, we must let another atom (as the same of the former one) pass sequentially across the cavity and the microwave field.
Finally, the state of the whole system including two atoms and a cavity mode can be expressed as:

\[
|\psi(t)\rangle_2 = |0\rangle \{ \cos(gt_1) \cos(gt_2) \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |e_1\rangle |e_2\rangle \\
+ i e^{-i\varphi_2} \cos(gt_1) \cos(gt_2) \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} |e_1\rangle |g_2\rangle \}
+ |1\rangle \{ \sin(gt_1) \cos(gt_2) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\varphi_1} |e_1\rangle |e_2\rangle \\
+ i e^{-i\varphi_1} e^{i\varphi_2} \sin(gt_1) \cos(gt_2) \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} |e_1\rangle |g_2\rangle \\
- i \cos(gt_1) \sin(gt_2) \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |e_1\rangle |g_2\rangle \\
+ e^{i\varphi_2} \cos(gt_1) \sin(gt_2) \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} |e_1\rangle |e_2\rangle \}
+ |2\rangle \{ e^{i\varphi_1} e^{i\varphi_2} \sin(gt_1) \sin(gt_2) \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} |e_1\rangle |e_2\rangle \\
- i e^{i\varphi_1} \sin(gt_1) \sin(gt_2) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |e_1\rangle |g_2\rangle \}
\]

As a consequence, the desirable few-photon superposed state can be generated by the state-selective measurements on the atoms. For example, if the atoms are detected at the state \( |e_1\rangle |e_2\rangle \), then the cavity mode collapses into

\[
|\psi(t)\rangle_2 = \frac{1}{\sqrt{N}} \{ |0\rangle [\cos(gt_1) \cos(gt_2) \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2}] \\
+ |1\rangle [e^{i\varphi_1} \sin(gt_1) \cos(gt_2) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + e^{i\varphi_2} \cos(gt_1) \sin(gt_2) \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2}] \\
+ |2\rangle [e^{i\varphi_1} e^{i\varphi_2} \sin(gt_1) \sin(gt_2) \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}] |e_1\rangle |e_2\rangle \},
\]

with

\[
N = [\cos gt_1 \cos gt_2 \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2}]^2 + [\sin gt_1 \sin gt_2 \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}]^2 \\
+ [\sin gt_1 \cos gt_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2}]^2 + [\cos gt_1 \sin gt_2 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2}]^2 \\
+ \frac{1}{8} \cos(\varphi_1 - \varphi_2) \sin 2gt_1 \sin 2gt_2 \sin \theta_1 \sin \theta_2
\]

being the normalized coefficient. If the relevant parameters are set properly as: \( \varphi_1 = \pi, \varphi_2 = 0, gt_1 = gt_2 = \theta_2/2 = \pi/4, \theta_1/2 = 7\pi/4 \), then a typical few-photon state discussed above

\[
|\psi(t)\rangle_2 = \frac{\sqrt{6}}{6} |0\rangle + \frac{\sqrt{6}}{3} |1\rangle + \frac{\sqrt{6}}{6} |2\rangle
\]

can be obtained.
The method to measure the Wigner function for a given cavity state is relative standard. Here, we recommend the approach proposed by Lutterbach and Davidovich [28] by directly detecting the negativity of Wigner function via the atomic Ramsey interferometries. In fact, at a phase space point $\alpha$, Wigner function for the cavity state with the density matrix $\rho$ can be simply expressed by [29]

$$W(\alpha) = 2\text{Tr}[D(-\alpha)\rho D(\alpha)P] = 2\langle P \rangle. \quad (50)$$

Here, $P = \exp(i\pi\hat{a}^+\hat{a})$ and $D(\alpha) = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a})$. Furthermore, the quantity $\langle P \rangle$ can be determined by measuring the probability $P_e$ (or $P_g$) of the atom is detected at its excited state $|e\rangle$ (or $|g\rangle$), i.e.,

$$P_e(\phi, \alpha) = \frac{1}{2}[1 + \langle P \rangle \cos \phi]. \quad (51)$$

Therefore, the Wigner function is determined by

$$W(\alpha) = 2[P_e(0, \alpha) - P_e(\pi, \alpha)]. \quad (52)$$

Consequently, if we have

$$P_e(0, \alpha) < P_e(\pi, \alpha), \quad (53)$$

then the Wigner function attains a negative value. With these preparations and measurements, the dissipative dynamics presented above could be tested experimentally.

V. DISCUSSIONS AND CONCLUSIONS

With the few-photon superposed state, in this paper we have investigated the dissipative dynamics of the quantized mode without any thermal photon. By numerical method, we discuss how the Wigner function of the cavity state changes with the dissipation of the cavity. Our results show clearly that the initial negativity of the Wigner function vanishes with the cavity dissipation. With the dissipative-dependent Wigner function, we further discuss how a typical quantum statistical property, the second-order correlation function $g^{(2)}(0)$, changes with the dissipation of the cavity. It is surprised that such a quantity is an invariant during the dissipation of the cavity. This argument was also verified by analytical method directly solving the master equation of the dissipative cavity. This implies that the $g^{(2)}(0)$ should not be a good physical quantity to describe the dissipative dynamics of the cavity, at least for the few-photon state.
The discussion in the present work is limited to the photons in optical cavity, and thus the mean thermal photons at room temperature can be really negligible. This implies that the final state of the dissipative optical cavity is exactly vacuum, at which the standard definition of the second-order correlation function is bizarre and insignificant. The generalization to the dissipative cavity with non-zero thermal photons is in progress.

Given the few-photon superposed state of the cavity is not difficult to be prepared and its relevant Wigner function can also be easily measured in the usual cavity QED system, we believe that our arguments are testable with the current experimental technique.

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