Topological entropy of continuous self-maps on closed surfaces

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\textbf{ABSTRACT}
The objective of this work is to present sufficient conditions for having positive topological entropy for continuous self-maps defined on a closed surface by using the action of this map on the homological groups of the closed surface.

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1. Introduction

Along this work by a \textit{closed surface}, we denote a connected compact surface with or without boundary, orientable or not. More precisely, an \textit{orientable connected compact surface without boundary of genus $g \geq 0$, $M_g$}, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of $g$ copies of the torus if $g \geq 2$. An \textit{orientable connected compact surface with boundary of genus $g \geq 0$, $M_{g,b}$}, is homeomorphic to $M_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $M_{g,0} = M_g$.

A \textit{non-orientable connected compact surface without boundary of genus $g \geq 1$, $N_g$}, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of $g$ copies of the real projective plane if $g > 1$. A \textit{non-orientable connected compact surface with boundary of genus $g \geq 1$, $N_{g,b}$}, is homeomorphic to $N_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $N_{g,0} = N_g$.

Let $f : X \to X$ be a continuous map on a closed surface $X$. A point $x \in X$ is periodic of period $n$ if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n - 1$.

The \textit{topological entropy} of a continuous map $f : X \to X$ denoted by $h(f)$ is a non-negative real number (possibly infinite) which measures how much $f$ mixes up the phase
space of $X$. When $h(f)$ is positive the dynamics of the system is said to be complicated and the positivity of $h(f)$ is used as a measure of the so-called topological chaos.

Here we introduce the topological entropy using the definition of Bowen [4].

Since it is possible to embed any surface orientable or not in $\mathbb{R}^4$ by the Whitney immersion theorem, see [13], we consider the distance between two points of $X$ as the distance of these two points in $\mathbb{R}^4$. Now, we define the distance $d_n$ on $G$ by

$$d_n(x, y) = \max_{0 \leq i \leq n} d(f^i(x), f^i(y)), \quad \text{for all } x, y \in G.$$ 

A finite set $S$ is called $(n, \varepsilon)$-separated with respect to $f$ if for different points $x, y \in S$ we have $d_n(x, y) > \varepsilon$. We denote by $S_n$ the maximal cardinality of an $(n, \varepsilon)$-separated set. Define

$$h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S_n.$$

Then

$$h(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon)$$

is the topological entropy of $f$.

We have chosen the definition by Bowen because, probably it is the shorter one. The classical definition was due to Adler, Konheim and McAndrew [1]. See for instance the book of Hasselblatt and Katok [7] and [3] for other equivalent definitions and properties of the topological entropy. See [1, 2, 9, 10, 15] for more details on the topological entropy.

Let $f$ be a continuous self-map defined on $\mathbb{M}_{g, b}$ or $\mathbb{N}_{g, b}$, respectively. For a closed surface the homological groups with coefficients in $\mathbb{Q}$ are linear vector spaces over $\mathbb{Q}$. We recall the homological spaces of $\mathbb{M}_{g, b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g, b}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{N}_k \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{\ast k} : H_k(\mathbb{M}_{g, b}, \mathbb{Q}) \to H_k(\mathbb{M}_{g, b}, \mathbb{Q})$ by $f$ on the homological group $H_k(\mathbb{M}_{g, b}, \mathbb{Q})$ are $f_{\ast 0} = (1), f_{\ast 2} = (d)$ where $d$ is the degree of the map $f$ if $b = 0$, $f_{\ast 2} = (0)$ if $b > 0$, and $f_{\ast 1} = A$ where $A$ is an $n_1 \times n_1$ integral matrix (see for additional details [12, 14]).

We recall that the homological groups of $\mathbb{N}_{g, b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{N}_{g, b}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{N}_k \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear maps are $f_{\ast 0} = (1)$ and $f_{\ast 1} = A$ where $A$ is an $n_1 \times n_1$ integral matrix (see again for additional details [12, 14]).

Our main results are the following.

**Theorem 1.1:** Let $\mathbb{M}_g$ be an orientable connected compact surface without boundary of genus $g$. Then the following statements hold.

(a) If the degree $d \notin \{-1, 0, 1\}$, then the topological entropy of $f$ is positive.

(b) If the degree $d \in \{-1, 0, 1\}$ and the number of roots of the characteristic polynomial $f_{\ast 1}$ is equal to $\pm 1$ or $0$ taking into account their multiplicities is not even, then the topological entropy of $f$ is positive.
Theorem 1.2: Let $M_{g,b}$, $b > 0$, be an orientable connected compact surface with boundary of genus $g$. If the number $2g + b - 1$ and the number of roots of the characteristic polynomial of $f_{s1}$ equal to $\pm 1$ or $0$ taking into account their multiplicities have different parity, then the topological entropy of $f$ is positive.

Theorem 1.3: Let $N_{g,b}$, $b \geq 0$, be a non-orientable connected compact surface with boundary of genus $g$. If the number $g + b - 1$ and the number of roots of the characteristic polynomial of $f_{s1}$ equal to $\pm 1$ or $0$ taking into account their multiplicities have different parity, then the topological entropy of $f$ is positive.

2. Lefschetz zeta functions for surfaces

Let $f: \mathbb{X} \to \mathbb{X}$ be a continuous map and let $\mathbb{X}$ be either $M_{g,b}$ or $N_{g,b}$. Then the Lefschetz number of $f$ is defined by

$$L(f) = \text{trace}(f_{s0}) - \text{trace}(f_{s1}) + \text{trace}(f_{s2}).$$

We shall use the Lefschetz numbers of the iterates of $f$, i.e. $L(f^n)$. In order to study the whole sequence $\{L(f^n)\}_{n \geq 1}$ it is defined the formal Lefschetz zeta function of $f$ as

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n\right).$$

The Lefschetz zeta function is in fact a generating function for the sequence of the Lefschetz numbers $L(f^n)$.

From the work of Franks in [6], we have for a continuous self-map of a closed surface that its Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{s1})}{\det(I - tf_{s0}) \det(I - tf_{s2})},$$

where $I - tf_{sk}$ $I$ denotes the $n_k \times n_k$ identity matrix and $\det(I - tf_{s2}) = 1$ if $f_{s2} = (0)$. Then for a continuous map $f: M_{g,b} \to M_{g,b}$ we have

$$Z_f(t) = \begin{cases} \frac{\det(I - tA)}{(1 - t)(1 - dt)} & \text{if } b = 0, \\ \frac{\det(I - tA)}{1 - t} & \text{if } b > 0, \end{cases}$$

and for a continuous map $f: N_{g,b} \to N_{g,b}$ we have

$$Z_f(t) = \frac{\det(I - tA)}{1 - t}.$$

3. Basic results

In this section, we present the main result stated in Theorem 3.4 for proving Theorems 1.1–1.3. Since its proof is short and important for this work we provide it here.
For a polynomial $H(t)$, we define $H^*(t)$ by

$$H(t) = (1 - t)^{\alpha} (1 + t)^{\beta} t^\gamma H^*(t),$$

where $\alpha$, $\beta$ and $\gamma$ are non-negative integers such that $1 - t$, $1 + t$ and $t$ do not divide $H^*(t)$.

The spectral radii of the maps $f_{sk}$ are denoted $\text{sp}(f_{sk})$, and they are equal to the largest modulus of all the eigenvalues of the linear map $f_{sk}$. The spectral radius of $f_*$ is

$$\text{sp}(f_*) = \max_{k=0, \ldots, m} \text{sp}(f_{sk}).$$

The next result is due to Manning [11].

**Theorem 3.1:** Let $f : X \to X$ be a continuous map on a closed surface $X$. Then

$$\log \max\{1, \text{sp}(f_{s1})\} \leq h(f).$$

**Lemma 3.2:** Let $f : X \to X$ be a continuous map and let $X$ be a closed surface. If the topological entropy of $f$ is zero, then all the eigenvalues of the induced homomorphism $f_{s1}$ are zero or root of unity.

**Proof:** Since the topological entropy is zero, by Theorem 3.1 we have $\text{sp}(f_{s1}) = 1$. So, all the eigenvalues of $f_{s1}$ have modulus in the interval $[0, 1]$ and at least one of them is 1. Then the characteristic polynomial of $f_{s1}$ is of the form $t^m p(t)$, where $m$ is a non-negative integer, positive if the zero is an eigenvalue. And $p(t)$ is a polynomial with integer coefficients and whose independent term $a_0$ is non-zero. Since the product of all non-zeros eigenvalues of $f_{s1}$ is the integer $a_0$ and, these eigenvalues have modulus in $(0, 1]$, we have that any of these eigenvalues cannot have modulus smaller than one, otherwise we are in contradiction with the fact $a_0$ is an integer. In short, all the non-zero eigenvalues have modulus one, and consequently $a_0 = 1$.

Since if a polynomial has integer coefficients, constant term 1 and all of whose roots have modulus 1, then all of its roots are roots of unity, see [16], the lemma follows. $\blacksquare$

The $n$th cyclotomic polynomial is defined recursively by

$$c_n(t) = \frac{1 - t^n}{\prod_{d|n} c_d(t)},$$

for a positive integer $n > 1$ and $c_1(t) = 1 - t$. Note that all the zeros of $c_n(t)$ are roots of unity. See [8] for the properties of these polynomials.

For a positive integer $n$, the Euler function is $\varphi(n) = n \prod_{p|n, p\text{ prime}} (1 - 1/p)$. It is known that the degree of the polynomial $c_n(t)$ is $\varphi(n)$. Note that $\varphi(n)$ is even for $n > 2$.

A proof of the next result can be found in [8].

**Proposition 3.3:** Let $\xi$ be a primitive $n$th root of the unity and $P(t)$ a polynomial with rational coefficients. If $P(\xi) = 0$, then $c_n(t) | P(t)$.

The proofs of our results are strongly based in the next theorem originally proved in [5] in 1992. For completeness of this paper, we present its proof.
Theorem 3.4 (Theorem 3.2 of [5]): Let $X$ be a closed surface, $f : X \to X$ be a continuous self-map, and let $Z_f(t) = P(t)/Q(t)$ be its Lefschetz zeta function. If $P^*(t)$ or $Q^*(t)$ has odd degree, then the topological entropy of $f$ is positive.

Proof: From the definitions of a polynomial $H^*$ and of the Lefschetz zeta function, we have

$$Z_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a(1+t)^b ft^c \frac{P^*(t)}{Q^*(t)},$$

where $a$, $b$ and $c$ are integers.

Assume now that the topological entropy $h(f) = 0$. Then by Lemma 3.2, all the eigenvalues of the induced homomorphisms $f_*$'s are zero or roots of unity. Therefore, by (1) all the roots of the polynomials $P^*(t)$ and $Q^*(t)$ are roots of the unity different from $\pm 1$ and zero. Hence, by Proposition 3.3 the polynomials $P^*(t)$ and $Q^*(t)$ are product of cyclotomic polynomials different from $c_1(t) = 1 - t$ and $c_2(t) = 1 + t$. Consequently $P^*(t)$ and $Q^*(t)$ have even degree because all the cyclotomic polynomials which appear in them have even degree due to the fact that the Euler function $\varphi(n)$ for $n > 2$ only takes even values. But this is a contradiction with the assumption that $P^*(t)$ or $Q^*(t)$ has odd degree. \hfill \blacksquare

4. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1: Since $M_g$ is an orientable connected compact surface without boundary of genus $g$, then the Lefschetz zeta function of $f$ is equal to

$$Z_f(t) = \frac{\det(I - tA)}{(1-t)(1-dt)},$$

where $d$ is the degree of $f$ and $2g$ is the dimension of the characteristic polynomial $\det(I - tA)$ of $f_{s1} = A$. Note here that if $d \notin \{-1, 0, 1\}$, then $Q^*(t) = 1 - dt$ and therefore by Theorem 3.4 statement (a) of Theorem 1.1 is proved.

Assume now that $d \in \{-1, 0, 1\}$. Note that in this case $Q(t) = (1-t)(1-dt)$ and $Q^*(t) = 1$. So, by Theorem 3.4 the main role will be played by the $2g$ degree polynomial $P(t) = \det(I - tA)$ where $f_{s1} = A$. Since $2g$ is even and the number of roots of the characteristic polynomial of $f_{s1}$ equal to $\pm 1$ or 0 taking into account their multiplicities is not even, then $P^*(t)$ has odd degree. Therefore, statement (b) of Theorem 1.1 follows by the application of Theorem 3.4. \hfill \blacksquare

Proof of Theorem 1.2: Note now, since $M_{g,b}$ is an orientable connected compact surface with boundary ($b > 0$) of genus $g$, then the Lefschetz zeta function of $f$ is equal to

$$Z_f(t) = \frac{\det(I - tA)}{1-t},$$

being $2g + b - 1$ the degree of the characteristic polynomial $\det(I - tA)$ of $f_{s1} = A$. Now the proof is similar to the statements (a) and (b) of Theorem 1.1. \hfill \blacksquare

Proof of Theorem 1.3: This proof is exactly the same than the proof of Theorem 1.2. \hfill \blacksquare
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