A NEW PROOF OF A VANISHING RESULT DUE TO BERTHELOT, ESNAULT, AND RÜLLING

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Abstract. The goal of this small note is to give a more concise proof of a result due to Berthelot, Esnault, and Rülling in [4]. For a regular, proper, and flat scheme \( X \) over a discrete valuation ring of mixed characteristic \((0, p)\), it relates the vanishing of the cohomology of the structure sheaf of the generic fibre of \( X \) with the vanishing of the Witt vector cohomology of its special fibre. We use as a critical ingredient results and constructions by Beilinson [2] and Nekovář–Nizioł [13] related to the h-topos over a \( p \)-adic field.

Résultat. Le but de cette brève note est de donner une démonstration plus courte d’un résultat de Berthelot, Esnault et Rülling dans [4]. Pour un schéma régulier, propre et plat \( X \) sur un anneau de valuation discrète de caractéristique \((0, p)\), il lie la disparition de la cohomologie du faisceau structural de la fibre générique de \( X \) à la disparition de la cohomologie de Witt de sa fibre spéciale. On utilise de manière critique des résultats et des constructions de Beilinson [2] et Nekovář–Nizioł [13] concernant le h-topos sur un corps \( p \)-adique.

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According to [1] schemes of semistable reduction over a complete discrete valuation ring \( \mathcal{O}_K \) with perfect residue field form a basis of the h-topology on the category \( \text{Var}(K) \) of varieties over the fraction field \( K \) of \( \mathcal{O}_K \). As a consequence, h-sheafification makes it sometimes possible to generalise constructions or results from schemes of semistable reduction to varieties over a \( p \)-adic field.

In this small note we want to illustrate the advantages of this technique and give a shorter and, as we hope, more conceptual proof of the following vanishing result due to Berthelot, Esnault, and Rülling in [4, Thm. 1.3].

\textbf{Theorem} (P. Berthelot, H. Esnault, K. Rülling). \textit{Let} \( R \) \textit{be a discrete valuation ring of mixed characteristic with fraction field} \( K \) \textit{and perfect residue field} \( k \), \textit{and let} \( X \) \textit{be a regular proper flat scheme over} \( R \). \textit{Assume that} \( H^q(X_K, \mathcal{O}) = 0 \) \textit{for some} \( q \geq 0 \). \textit{Then} \( H^q(X_0, W\mathcal{O})_{\mathbb{Q}} = 0 \) \textit{as well}.

As a consequence of this result the authors obtain, under the additional assumption that \( k \) is finite, a congruence on the number of rational points of \( X \) with values in finite extensions of \( k \). As explained in [4] this fits into the general analogy between the vanishing of Hodge numbers for varieties over a field of characteristic 0 and congruences on the number of rational points with values in finite extensions for varieties over a finite field.

The above theorem itself is an application of \( p \)-adic Hodge theory. In [4] Thm. 2.1] the semistable case is discussed which we recall here briefly as it provides a guideline for our proof.

Thus in the situation of the theorem, let \( X/R \) be of semistable reduction. Without loss of generality one can assume that \( R \) is a complete discrete valuation ring. Endow \( R \) and \( X \) with the canonical log structure denoted by \( R^\times \) and \( X^\times \), the special fibre \( X_0 \) with the pull-back log structure denoted by \( X_0^\times \),

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and the Witt vectors $W(k)$ with the log-structure associated to $(1 \to 0)$ denoted by $W(k)^0$. Consider the log crystalline cohomology groups $H^q_{\mathrm{cris}}(X_0^\times / W(k)^0)$. This cohomology is sometimes called the Hyodo–Kato cohomology. In the case at hand, they can be computed by the logarithmic de Rham–Witt complex $W\omega^*$ which leads to a spectral sequence

$$E_{ij}^q = H^i(X_0^\times, W\omega_Q) \Rightarrow H^{i+j}_{\mathrm{cris}}(X_0^\times / W(k)^0)_Q$$

endowed with a Frobenius action. On the left this Frobenius action is induced by $p^i F$, where $F$ is the Witt vector Frobenius, whereas on the right the Frobenius action $\varphi$ is induced by the absolute Frobenii of $X_0$ and $W(k)$. Similarly to the classical case, it follows that this spectral sequence degenerates at $E_1$ and that $H^i(X_0^\times, W\omega_Q)$ corresponds to the part of $H^q_{\mathrm{cris}}(X_0^\times / W(k)^0)_Q$ where Frobenius has slope in $[i, i+1]$. Hence one obtains a canonical quasi-isomorphism

$$H^q_{\mathrm{cris}}(X_0^\times / W(k)^0)^{\leq 1} \sim H^q(X_0, W\Theta)_Q.$$
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1. The logarithmic de Rham–Witt complex

It is common when one studies non-smooth objects to consider log versions of the usual complexes appearing in the different cohomology theories. In [10] Sec. 2] Hyodo and Kato describe the log crystalline site for schemes with fine log structure as a generalisation of the usual crystalline site. One case which is relatively well studied, is the semistable case in positive characteristic p, and the case of semistable reduction in mixed characteristic (0, p). This is a special case of fine log schemes of Cartier type over $k^0$ or $\mathcal{O}_K^\times$.

The Hyodo–Kato complex computes log crystalline cohomology over $W(k)^0$. There are different quasi-isomorphic presentations of this complex. One that is particularly useful to us is the de Rham–Witt presentation from [10 Sec. 4] (see also [12 Sec. 1]).

For a fine log scheme $Y$ of Cartier type over $k^0$, let $(Y/W_n(k)^0)_{\text{cris}}$ be its log crystalline site over $W_n(k)$ endowed with the $(1 \mapsto 0)$-log structure, let $\mathcal{O}_{Y/W_n(k)^0}$ be its crystalline structure sheaf, and let $u_{Y/W_n(k)^0}: (Y/W_n(k)^0)_{\text{cris}} \to Y_{\text{ét}}$ be the canonical morphism of sites.

Definition 1.1. The logarithmic Witt differentials of $Y$ of degree $i$ and level $n \geq 1$ are defined as

$$W_n^i \omega_Y := R^iu_{Y/W_n(k)^0}^*, \mathcal{O}_{Y/W_n(k)^0}.$$

According to [12 Cor. 1.17] $W_n^i \omega_Y$ is a coherent $W_n \mathcal{O}_Y$-module. For $n$ fixed $W_n^i \omega_Y$ is a complex called the logarithmic de Rham–Witt complex of level $n$. By [10 Prop. (4.6)] there is a canonical isomorphism of $W_n(k)$-algebras

$$W_n^i \omega_Y \cong W_n \mathcal{O}_Y.$$

The Frobenius $F$, Verschiebung $V$ and projection operators extend from the Witt vectors to the de Rham–Witt complex, and are subject to certain relations. In particular, $W_n^i \omega_Y$ is a differentially graded $W_n \mathcal{O}_Y$-algebra, which computes log crystalline cohomology, i.e. there is by [10 Thm. (4.19)] a canonical quasi-isomorphism

$$W_n^i \omega_Y \xrightarrow{\sim} R u_{Y/W_n(k)^0}^*, \mathcal{O}_{Y/W_n(k)^0}$$

compatible with Frobenius and the canonical projections as $n$ varies. Here the Frobenius on the left hand side is given by $p^i F$ in degree $i$, while on the right hand side, it is induced by the absolute Frobenius of $Y$ and the endomorphism $\sigma$ of $W_n(k)$.

Finally, for $n$ varying we obtain a projective system $\{W_n^i \omega_Y\}_{n \geq 1}$ differentially graded algebras.

Definition 1.2. We call the limit

$$W^i \omega_Y := \varprojlim W_n^i \omega_Y$$

the logarithmic de Rham–Witt complex of $Y$ over $W(k)^0$. It is equipped with operators called Frobenius and Verschiebung induced by $F$ and $V$ which satisfy the usual relations.

The log crystalline cohomology of $Y$ over $W(k)^0$ is

$$R \Gamma_{\text{cris}}(Y/W(k)^0) := \varinjlim R \Gamma_{\text{ét}}(Y, R u_{Y/W_n(k)^0}^*, \mathcal{O}_{Y/W_n(k)^0}).$$

and we have by [12 Cor. 1.23] the following statement.

Lemma 1.3 (P. Lorenzon). If $Y$ is a proper fine log scheme of Cartier type over $k^0$, then $R \Gamma_{\text{ét}}(Y, W^i \omega_Y)$ is a perfect complex and the canonical morphism $\{\} \text{ induces a natural isomorphism}$

$$\lambda: R \Gamma_{\text{ét}}(Y, W^i \omega_Y) \xrightarrow{\sim} R \Gamma_{\text{cris}}(Y/W(k)^0).$$

In case $Y$ is the special fibre of a fine log scheme $X$ of Cartier type over $\mathcal{O}_K^\times$, we write $R \Gamma_{\text{HK}}(X) := R \Gamma_{\text{cris}}(X_0/W(k)^0)$ and call it the Hyodo–Kato complex of $X$. In this case, there is yet another presentation of the rational Hyodo–Kato complex due to Beilinson [2 1.16.1]. Without going into details we
remark that it depends on the reduction of $X$ modulo $p$ and not just on the special fibre $X_0$ and that there is a natural quasi-isomorphism \cite[(28)]{13}

$$\kappa : R\Gamma^B_{HK}(X) \xrightarrow{\sim} R\Gamma_{HK}(X)_Q$$

compatible with Frobenius.

2. HYODO–KATO COMPLEXES FOR $K$-VARIETIES

In \cite{13} Nekovář and Nizioł describe how to extend several $p$-adic cohomology theories for log schemes over $k$ or $\Theta_K$ to $K$-varieties. This technique is based on observations due to Beilinson in \cite{1} and we start by recalling some of the relevant notions.

**Definition 2.1.** For a field $K$ of characteristic 0 a geometric pair is an open embedding $i : U \hookrightarrow \overline{U}$ of $K$-varieties such that $U$ is dense in $\overline{U}$ and $\overline{U}$ is proper. A geometric pair is called a semistable pair if $U$ is regular, $\overline{U} \setminus U$ is a divisor with normal crossings, and the closed fibre $\overline{U}_0$ is reduced. It is said to be strict, if the irreducible components of $\overline{U} \setminus U$ are regular.

This definition can be adjusted to the arithmetic setting as follows.

**Definition 2.2.** An arithmetic $K$-pair is an open embedding $i : U \hookrightarrow \overline{U}$ of a $K$-variety $U$ with dense image in a reduced proper flat $\Theta_K$-scheme $\overline{U}$. Such a pair is called a semistable pair if $U$ is regular, $\overline{U} \setminus U$ is a divisor with normal crossings, and the closed fibre $\overline{U}_0$ is reduced. It is said to be strict, if the irreducible components of $\overline{U} \setminus U$ are regular.

As explained in \cite[2.3]{13} one can regard a semistable pair $(U, \overline{U})$ as a log scheme $\overline{U}$ over $\Theta_K^\times$ equipped with the log structure associated to the divisor $\overline{U} \setminus U$. Such as it is a proper log smooth fine log scheme of Cartier type over $\Theta_K^\times$. In particular, its special fibre $\overline{U}_0$ equipped with the pull back log structure is a proper log smooth fine log scheme of Cartier type over $k^\text{h}$. Following \cite{13} we denote by $\mathcal{P}^\text{ar}_K$, $\mathcal{P}^\text{ss}_K$, and $\mathcal{P}^\text{nc}_K$ respectively the category of arithmetic, semistable and normal crossings pairs over $K$ respectively. Moreover, we denote by $\mathcal{P}^\text{log}_K$ the subcategory of $\mathcal{P}^\text{ar}_K$ of proper log smooth fine log schemes of Cartier type over $\Theta_K^\times$.

A key point in Beilinson’s work is that the categories $\mathcal{P}^\text{ar}_K$, $\mathcal{P}^\text{ss}_K$, $\mathcal{P}^\text{log}_K$ and $\mathcal{P}^\text{nc}_K$ respectively form a base for the $h$-site $(\text{Var}/K)_h$ of $K$-varieties in the sense that there is an equivalence between the associated $h$-sites \cite[2.5 Prop.]{1}. Keeping in mind that alterations are $h$-morphisms, Beilinson uses the following formulation of de Jong’s alteration theorem \cite[2.3 Thm.]{1}.

**Theorem 2.3** (A.J. de Jong). Every geometric pair admits a strict normal crossings alteration. Every arithmetic pair over $K$ or $\overline{K}$ admits a strict semistable alteration. Alterations can be chosen in such a way that $\overline{V}$ is projective.

Nekovář and Nizioł explain how to $h$-sheafify the rational Hyodo–Kato and the Beilinson–Hyodo–Kato complexes on $\text{Var}(K)$ \cite[Sec. 3.3]{13}. Denote the resulting sheaves by $\mathcal{A}_{HK}$ and $\mathcal{A}^B_{HK}$. The same procedure can be done to the logarithmic de Rham–Witt complex $W\omega^*$ defined above.

**Definition 2.4.** Let $\mathcal{A}_{dhrw}$ be the $h$-sheafification of the presheaf

$$(U, \overline{U}) \mapsto R\Gamma_{\text{et}}((U, \overline{U})_0, W\omega^*)_Q$$
on the category $\mathcal{P}^\text{log}_K$ of proper log smooth fine log schemes of Cartier type over $\Theta_K^\times$. This results in an $h$-sheaf of commutative dg $K_0$-algebras on $\text{Var}(K)$.

The canonical maps $\kappa : R\Gamma^B_{HK}(U, \overline{U}) \to R\Gamma_{HK}(U, \overline{U})_Q$ \cite[Sec. 3.3]{13} and $\lambda : R\Gamma_{\text{et}}((U, \overline{U})_0, W\omega^*)_Q \to R\Gamma_{HK}(U, \overline{U})_Q$ \cite[Thm. (4.19)]{1} $h$-sheafify and we obtain a functorial quasi-isomorphism of $h$-sheaves

$$\mathcal{A}_{dhrw} \cong \mathcal{A}_{HK} \cong \mathcal{A}^B_{HK}.$$ 

**Lemma 2.5.** For any proper log smooth fine log scheme of Cartier type $(U, \overline{U}) \in \mathcal{P}^\text{log}_K$ over $\Theta_K^\times$, the canonical map

$$R\Gamma_{\text{et}}((U, \overline{U})_0, W\omega^*)_Q \to R\Gamma_{h}(U, \mathcal{A}_{dhrw})$$
is a quasi-isomorphism.
Proof. According to [13, Prop. 3.18] analogous statements are true for the Hyodo–Kato and Beilinson–Hyodo–Kato complexes. Because of the canonical quasi-isomorphism the statement follows. □

3. Admissible filtered \((\varphi,N,G_K)\)-modules

The Beilinson–Hyodo–Kato complex has additional structure and as such fits into the theory of \(p\)-adic Galois representations [8]. Indeed, it’s cohomology groups are admissible filtered \((\phi,N,G_K)\)-modules. Let us explain this.

**Definition 3.1.**

(i) A filtered \(\varphi\)-module is a triple \((M_0,\varphi,Fil^*)\), where \(M_0\) is a finite dimensional \(K_0\)-vector space, with a \(\sigma\)-semi-linear isomorphism \(\varphi : M_0 \rightarrow M_0\), called the Frobenius map and a decreasing, separated, exhaustive filtration \(Fil^*\) on \(M = M_0 \otimes K_0\) called the Hodge filtration.

(ii) A filtered \((\phi,N)\)-module \((M_0,\phi,N,Fil^*)\) consists of a filtered \(\phi\)-module \((M_0,\varphi,Fil^*)\) together with a \(K_0\)-linear monodromy operator \(N\) on \(M_0\) satisfying the relation \(N\varphi = p\varphi N\).

(iii) A filtered \((\phi,N,G_K)\)-module is a tuple \((M_0,\varphi,N,\rho,Fil^*)\) where \(M_0\) is a finite dimensional \(K_0^\text{nr}\)-vector space, \(\varphi : M_0 \rightarrow M_0\) is a Frobenius map, \(N : M_0 \rightarrow M_0\) is a \(K_0^\text{nr}\)-linear monodromy operator such that \(N\varphi = p\varphi N\), \(\rho\) is a \(K_0^\text{nr}\)-semi-linear action of \(G_K\) on \(M_0\) factoring through a quotient of the inertia group and commuting with \(\varphi\) and \(N\), and \(Fil^*\) is a decreasing, separate, exhaustive filtration of \(M = (M_0 \otimes K_0^\text{nr})^G_K\).

Denote by

\[
MF_K^\text{ad}(\phi) \subset MF_K^\text{fil}(\phi,N) \subset MF_K^\text{ad}(\phi,N,G_K)
\]

the categories of admissible filtered \(\phi\)-modules, \((\phi,N)\)-modules and \((\phi,N,G_K)\)-modules, where admissible is meant in the sense of [8]. They are equivalent to crystalline, semistable, and potentially semistable Galois representations.

The categories of admissible filtered \(\phi\), \((\phi,N)\), and \((\phi,N,G_K)\)-modules are known to be Tannakian. Thus it makes sense to consider their respective bounded derived dg categories denoted by \(\mathcal{D}^b(MF_K^\text{ad}(\phi))\), \(\mathcal{D}^b(MF_K^\text{fil}(\phi,N))\) and \(\mathcal{D}^b(MF_K^\text{ad}(\phi,N,G_K))\) respectively.

A filtered \(\phi\), \((\phi,N)\), or \((\phi,N,G_K)\)-module \(M_0\) has both, a Newton polygon, associated to the eigenvalues of the Frobenius morphism \(\varphi\) on \(M_0\), and a Hodge polygon, associated to the filtration \(Fil^*\) on \(M\). If the Newton polygon lies above the Hodge polygon, \(M_0\) is weakly admissible (c.f. [8, 4.4.6 Rem.]).

It turns out that this is the critical piece of information that will allow us to relate \(H^q(X,K_\ell)\) and \(H^q(X,\mathcal{O})\) of a \(K\)-variety. By definition it is equipped with a Frobenius \(\varphi\), and a nilpotent monodromy operator \(N\). Note that this is a difference from the usual Hyodo–Kato complex, where the monodromy is at best homotopically nilpotent. This is crucial to h-sheafify the Hyodo–Kato morphism, relating the (Beilinson–)Hyodo–Kato complex to the de Rham complex, which provides the filtration.

For a proper log smooth fine log scheme of Cartier type \((U,\overline{U}) \in \mathcal{D}^\log_K\) over \(\mathcal{O}_K^\text{nr}\) there is a morphism (c.f. [13, 22], [10, Sec. 5])

\[
\iota_{\text{dr},\pi} : R\Gamma_{HK}(U,\overline{U})_{\mathbb{Q}} \rightarrow R\Gamma_{\text{ad}}(U,\overline{U}_K)
\]

called the Hyodo–Kato morphism which becomes a \(K\)-linear functorial quasi-isomorphism after tensoring with \(K\). However, it depends on the choice of a uniformiser \(\pi\) of \(\mathcal{O}_K\) and is therefore not suitable for h-sheafification. By contrast, there is a morphism [2, 1.6.3]

\[
\iota_{\text{ad}}^B : R\Gamma_{HK}^B(U,\overline{U}) \rightarrow R\Gamma_{\text{ad}}(U,\overline{U}_K)
\]

independent of the choice of a uniformiser which is also a \(K\)-linear functorial quasi-isomorphism after tensoring with \(K\). For more details see [13, Ex. 3.5(1)]. The two morphisms are compatible with the comparison map [2]
as explained in [13] after Ex. 3.5.

The map $i_{dR}^*$ can be h-sheafified. For this we also have to h-sheafify the de Rham complex on $\text{Var}(K)$. Thus, consider the presheaf

$$(U, \overline{U}) \mapsto R\Gamma((U, \overline{U}), \Omega^*)$$

doing filtered $K$-algebras on $\mathcal{P}^\text{mc}_K$ and let $\mathcal{A}_{dR}$ be its h-sheafification. This results in an h-sheaf of commutative filtered dg $K$-algebras on $\text{Var}(K)$. It can be identified with Deligne’s de Rham complex equipped with Deligne’s Hodge filtration $\text{Fil}^*$ (c.f. [11 Prop. 7.4 and Thm. 7.7]). Moreover, Beilinson showed in [11 2.4] that for $(U, \overline{U}) \in \mathcal{P}^\text{mc}_K$ the canonical map

$$R\Gamma_{dR}(U, \overline{U}) \xrightarrow{\sim} R\Gamma_{dR}(U, \mathcal{A}_{dR})$$

is a quasi-isomorphism.

There are analogous statements over $\overline{K}$, and for $Z \in \text{Var}(K)$ the projection $\varepsilon : Z_{\overline{K}, h} \to Z_{h}$ of sites induces pull-back maps

$$\varepsilon^* : R\Gamma_{h}(Z, \mathcal{A}_{hK}) \to R\Gamma_{h}(Z_{\overline{K}, h}, \mathcal{A}_{hK})^{G_K}$$

$$\varepsilon^* : R\Gamma_{h}(Z, \mathcal{A}_{dR}) \to R\Gamma_{h}(Z_{\overline{K}, \mathcal{A}_{dR}})^{G_K}$$

where by [13 Prop. 3.22] the first one is a quasi-isomorphism and the second one is a filtered quasi-isomorphism. The Beilinson–Hyodo–Kato map extends to

$$R\Gamma_{h}(Z_{\overline{K}, \mathcal{A}_{HK}}) \to R\Gamma_{h}(Z_{\overline{K}, \mathcal{A}_{dR}}),$$

which induces a quasi-isomorphism

$$R\Gamma_{h}(Z_{\overline{K}, \mathcal{A}_{dR}})^{G_K} \to R\Gamma_{h}(Z_{\overline{K}, \mathcal{A}_{dR}}).$$

This identification provides the last piece of data necessary and by [6 2.20] we have the following statement.

**Lemma 3.2 (F. Dégile, W. Niziol).** Let $Z \in \text{Var}(K)$. Then $R\Gamma_{h}(Z_{\overline{K}, \mathcal{A}_{hK}})$ with the Frobenius $\varphi$, the monodromy operator $N$, the canonical $G_K$-action, and the Hodge filtration on $R\Gamma_{h}(Z, \mathcal{A}_{dR})$ is an object in $D^b(M\text{F}^{\text{adm}}(\varphi, N, G_K))$.

As a consequence we obtain the promised statement which relates the Hodge and the Newton polygon of $H^q_{\text{h}}(Z_{\overline{K}, \mathcal{A}_{hK}})$.

**Lemma 3.3.** Let $Z \in \text{Var}(K)$. For any $q \in \mathbb{N}_0$, the Newton polygon of $H^q_{\text{h}}(Z_{\overline{K}, \mathcal{A}_{hK}})$ lies above its Hodge polygon.

**Proof.** By the previous lemma $H^q_{\text{h}}(Z_{\overline{K}, \mathcal{A}_{hK}})$ is an admissible filtered $(\varphi, N, G_K)$-module. Therefore it is weakly admissible in the sense of Fontaine [8 5.6.7 Thm. (vi)]. This means as remarked in [8 4.4.6 Rem.] that for each $q$ the Newton polygon of $H^q_{\text{h}}(Z_{\overline{K}, \mathcal{A}_{hK}})$ lies above its Hodge polygon.

\[\square\]

4. Two spectral sequences

In this section we consider two spectral sequences, one for de Rham cohomology and one for Hyodo–Kato cohomology, which are very similar in spirit for their are related to the Hodge and Newton slope of a $(\varphi, N, G_K)$-module in a geometric situation. The first one is an h-sheafified version of the Hodge-to-de Rham spectral sequence.

For this we introduce the h-sheafifications $\mathcal{A}'_{\text{dR}}$ of the differential sheaves $\Omega^i, i \geq 0$, on $\text{Var}(K)$. To be consistent with the constructions in the previous sections, we can think of $\mathcal{A}'_{\text{dR}}$ as the h-sheafification of the presheaf

$$(U, \overline{U}) \mapsto \Gamma((U, \overline{U}), \Omega^i)$$

on the category $\mathcal{P}^\text{mc}_K$ of normal crossing pairs. It gives a coherent h-sheaf on $\text{Var}(K)$.

Now the Hodge filtration of $\mathcal{A}_{dR}$ induces a spectral sequence for which Huber and Jörder in [9 Thm. 7.7] prove the following.
Lemma 4.1 (A. Huber, C. Jörder). Let \( Z \in \text{Var}(K) \) be proper. Then the Hodge-to-de Rham spectral sequence
\[
E_1^{rs} = H^s_h(Z, \mathcal{A}_{\text{dR}}^r) \Rightarrow H^{r+s}_h(Z, \mathcal{A}_{\text{dR}})
\]
degenerates at \( E_1 \).

It becomes immediately clear that the Hodge filtration on \( \mathcal{A}_{\text{dR}} \) induces Deligne’s Hodge filtration as mentioned above.

Corollary 4.2. Let \( Z \in \text{Var}(K) \). Then for any \( q \geq 0 \) the Hodge-to-de Rham spectral sequence yields an isomorphism
\[
H^r_h(Z, \mathcal{A}_{\text{dR}}) \stackrel{\sim}{\rightarrow} H^0_h(Z, \mathcal{A}_{\text{dR}}),
\]
where on the left hand side we mean the part of \( H^r_h(Z, \mathcal{A}_{\text{dR}}) \) with Hodge slope \( < 1 \).

We come now to an analogous statement for the Hyodo–Kato cohomology. Thus for \( i \geq 0 \) consider the h-sheafification \( \mathcal{A}_{\text{dRW}} \) of the presheaf
\[
(U, \overline{U}) \mapsto \Gamma((U, \overline{U})_0, W\omega^i),
\]
on the category \( \mathcal{O}_K \) of proper log smooth fine log scheme of Cartier type over \( \mathcal{O}_K \). This is a quasi-coherent h-sheaf on \( \text{Var}(K) \).

Lemma 4.3. Let \( Z \in \text{Var}(K) \). There is a spectral sequence
\[
E_1^{rs} = H^s_h(Z, \mathcal{A}_{\text{dRW}}) \Rightarrow H^{r+s}_h(Z, \mathcal{A}_{\text{dRW}})
\]
which is Frobenius equivariant and degenerates at \( E_1 \).

Proof. The existence of the sequence follows as in the classical case, namely it is induced from the naive filtration of the complex \( \mathcal{A}_{\text{dRW}} \). As mentioned above Nekovář and Nizioł show (c.f. [13, p. 5]) that the cohomology groups \( H^{r+s}_h(Z, \mathcal{A}_{\text{dRW}}) \) are finite rank \( K_0 \)-vector spaces with an action of Frobenius \( \varphi \). Similarly the action of the Witt vector Frobenius \( F \) induces a Frobenius action on \( \mathcal{A}_{\text{dRW}} \) and therefore on the associated cohomology groups. For fixed \( r \geq 0 \), we consider the action on \( H^0_h(Z, \mathcal{A}_{\text{dRW}}) \) given by \( p^r F \). For all \( r, s \), we have to show that \( (H^0_h(Z, \mathcal{A}_{\text{dRW}}, p^r F) \) is canonically isomorphic to the part of \( (H^{r+s}_h(Z, \mathcal{A}_{\text{dRW}}), \varphi) \) which has slopes in \( [r, r+1] \), denoted by \( (H^{r+s}_h(Z, \mathcal{A}_{\text{dRW}}), \varphi)[r,r+1] \). From this it follows immediately that \( H^0_h(Z, \mathcal{A}_{\text{dRW}}) \) is a finite rank \( K_0 \)-vector space for all \( r, s \) because that is the case for \( H^{r+s}_h(Z, \mathcal{A}_{\text{dRW}}) \).

Indeed for each h-hypercover \( (U, \overline{U}) \) of \( Z \) by semistable pairs the naive filtration on \( W\omega^* \) induces a spectral sequence
\[
E_1^{rs} = H^*((U, \overline{U}), W\omega^i_Q) \Rightarrow H^{r+s}((U, \overline{U}), W\omega^*),
\]
which has the desired properties. To see this we can work with truncated hypercovers, which is possible because on the one hand \( \tau_{<s} R\Gamma_{et}((U, \overline{U}), W\omega^i_Q) \cong \tau_{<s} R\Gamma_{et}((U, \overline{U}), W\omega^i_Q) \) and on the other hand \( \tau_{<t} R\Gamma_{et}((U, \overline{U}), W\omega^i_Q) \cong \tau_{<t} R\Gamma_{et}((U, \overline{U}), W\omega^i_Q) \), so that we can choose a suitable common truncation. Since the claim is true for a single log scheme by [12, Thm. 3.1] it is then also true for each truncated h-hypercover as above.

Therefore, \( (H^*((U, \overline{U}), W\omega^i_Q), p^r F) \) and \( (H^{r+s}((U, \overline{U}), W\omega^*), \varphi)[r,r+1] \) are canonically isomorphic. But \( \varphi \) and \( F \) sheafify well with respect to the h-topology so that we may take the limit over all h-hypercovers and obtain a canonical isomorphism
\[
(H^*(Z, \mathcal{A}_{\text{dRW}}^r, p^r F) \cong (H^{r+s}_h(Z, \mathcal{A}_{\text{dRW}}), \varphi)[r,r+1]
\]
where the right hand side is the part of \( (H^{r+s}_h(Z, \mathcal{A}_{\text{dRW}}), \varphi) \) with slope in the interval \( [r, r+1] \). In particular, \( H^*(Z, \mathcal{A}_{\text{dRW}}) \) are finite rank \( K_0 \)-vector spaces and the spectral sequence from the statement degenerates. \( \Box \)

For obvious reasons this spectral sequence is called the slope spectral sequence.
Corollary 4.4. Let $Z \in \text{Var}(K)$. For any $q \geq 0$ the slope spectral sequence yields an isomorphism

$$H^q_h(X, \mathcal{A}_{\text{dR}}^0)^{<1} \cong H^q_h(X, \mathcal{A}_{\text{dR}}^0),$$

where on the left hand side we mean the part of $H^q_h(Z, \mathcal{A}_{\text{dR}})$ where Frobenius acts with slope $< 1$.

This allows us to transfer the descent statement from Lemma 2.5 to differentials of degree zero.

Corollary 4.5. For any proper log smooth fine log scheme of Cartier type $(U, \mathcal{T}_{\text{log}})$ over $\mathcal{O}_K^\times$, the canonical map

$$R\Gamma_\ell((U, \mathcal{T}_{\text{log}})_0, W\omega^0)_\mathbb{Q} \to R\Gamma_h(U, \mathcal{A}_{\text{dR}}^0)$$

is a quasi-isomorphism.

Proof. For any $q \geq 0$ we have a commutative diagram

$$\begin{array}{ccc}
H^q((U, \mathcal{T}_{\text{log}})_0, W\omega^*)^<_1 \cong H^q((U, \mathcal{T}_{\text{log}})_0, W\omega^0)_\mathbb{Q} \\
\sim \downarrow \sim \\
H^q_h(U, \mathcal{A}_{\text{dR}}^{<1}) \cong H^q_h(U, \mathcal{A}_{\text{dR}}^0)
\end{array}$$

where the horizontal isomorphisms are induced from the classical and the h-sheafified slope spectral sequence respectively and the vertical maps are the canonical morphisms. Since the left vertical map is an isomorphism by Lemma 2.5, the right one is as well. □

We can take this a bit further.

Lemma 4.6. Let $X$ be a reduced, proper and flat $\mathcal{O}_K$-scheme of finite type. Then there is a quasi-isomorphism

$$R\Gamma_h(X, \mathcal{A}_{\text{dR}}^0) \cong R\Gamma_\ell(X_0, W\mathcal{O})_\mathbb{Q}.$$  

Proof. Since $(X_K, X)$ forms an arithmetic pair in the sense of Beilinson (see Definition 2.2), it has by de Jong’s theorem a strictly semistable alteration $(U, \mathcal{T}) \to (X_K, X)$. This gives rise to an h-cover $U \to X_K$ of the generic fibre, and an h-cover $\mathcal{T}_0 \to X_0$ of the special fibre. Denote by $(U, \mathcal{T}_0) \to (X_K, X)$ its Čech nerve. In particular $U \to X_K$ is an h-cover of the generic fibre of $X$ and $\mathcal{T}_0 \to X_0$ is an h-cover of its special fibre.

Using Corollary 4.5 we have

$$R\Gamma_h(X, \mathcal{A}_{\text{dR}}^0) \cong R\Gamma_\ell(U, \mathcal{A}_{\text{dR}}^0) \cong R\Gamma_h((U, \mathcal{T}_0)_0, W\omega^0)_\mathbb{Q} \cong R\Gamma_\ell((U, \mathcal{T}_0)_0, W\mathcal{O})_\mathbb{Q}.$$  

However, by [5] Prop. 11.41] rational Witt cohomology satisfies cohomological h-descent and hence the right most expression is just $R\Gamma_\ell(X_0, W\mathcal{O})_\mathbb{Q}$. □

Therefore we can rewrite Corollary 4.4.

Corollary 4.7. Let $X$ be a reduced, proper and flat $\mathcal{O}_K$-scheme of finite type. For any $q \geq 0$ the slope spectral sequence yields an isomorphism

$$H^q_h(X, \mathcal{A}_{\text{dR}}^0)^{<1} \cong H^q_h(X, W\mathcal{O})_\mathbb{Q}.$$  

5. A VANISHING THEOREM

We can now put the pieces together to give a simplified proof of the vanishing theorem due to Pierre Berthelot, Hélène Esnault and Kay Rülling. We will use the following statement about de Rham cohomology groups. Note that h-sheafifying the de Rham complex results in Deligne’s de Rham complex [9] Thm. 7.4, i.e. $H^q_h(Z, \Omega^q_\mathbb{Q}) = H^q_{\text{dR}}(Z)$.

Lemma 5.1. Let $Z \in \text{Var}(K)$ be proper with only Du Bois singularities, and assume that $H^q(Z, \mathcal{O}) = 0$ for some $q \geq 0$. Then the smallest Hodge slope of $H^q_{\text{dR}}(Z)$ is at least 1.

Proof. If $Z$ has only Du Bois singularities $H^q_h(Z, \Omega^q_\mathbb{Q}) \cong H^q(Z, \mathcal{O})$ by [9] Cor. 7.17. As the h-sheafified Hodge-to-de Rham spectral sequence from Lemma 4.4 degenerates for a proper $K$-variety at $E_1$, the hypothesis $H^q(Z, \mathcal{O}) = 0$ implies that the smallest Hodge slope of $H^q_h(Z, \Omega^q_\mathbb{Q}) \cong H^q_{\text{dR}}(Z)$ is at least 1. □
Remark 5.2. In general we can say that for a proper $K$-variety $Z \in \text{Var}(K)$ such that $H^q_{\text{dR}}(Z, \mathcal{O}_Z) = 0$ for some $q$ the smallest Hodge slope of $H^q_{\text{dR}}(Z)$ is at least 1.

We obtain now the desired vanishing theorem in a slightly more general form than originally stated.

**Theorem 5.3.** Let $X$ be a proper, reduced and flat scheme over $\mathcal{O}_K$, such that $X_K$ has at most Du Bois singularities. Fix $q \in \mathbb{N}_0$. If $H^q(X_K, \mathcal{O}_X) = 0$, then $H^q(X_0, W \mathcal{O})_Q = 0$.

**Proof.** Consider the cohomology group $H^q_{\text{dR}}(X_K, \mathcal{A}^B_{\text{HK}})$. By Lemma 3.3 its Newton polygon lies above its Hodge polygon. Because $X_K$ has only Du Bois singularities Lemma 5.1 applies, which means that the smallest Hodge slope of $H^q_{\text{dR}}(X_K)$ is $\geq 1$. Therefore the part of $H^q_{\text{dR}}(X_K, \mathcal{A}^B_{\text{HK}})$, where the Newton slope is $< 1$ vanishes. By definition this is exactly the part where Frobenius acts with slope $< 1$ and hence by Corollary 4.4

$$H^q_{\text{dR}}(X_K, \mathcal{A}^B_{\text{dR}}) = 0.$$ 

But according to Lemma 4.6 this means that $H^q(X_0, W \mathcal{O})_Q = 0$ as well. $\square$

**Remark 5.4.** As pointed out in [4] one easily generalises this result to a scheme $X$ as in the theorem, but over a discrete valuation ring $V$ which is not necessarily complete.

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