SERIES EXPANSIONS FOR ANY REAL POWER OF THE SINC FUNCTION AND A CLOSED-FORM FORMULA FOR PARTIAL BELL POLYNOMIALS OF THE SINC FUNCTION IN TERMS OF WEIGHTED STIRLING NUMBERS OF THE SECOND KIND

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Abstract. In the paper, with the help of the Faá di Bruno formula and in terms of the (weighted) Stirling numbers of the second kind, the author derives two series expansions for any positive integer power of the sinc function, finds out a closed-form formula for partial Bell polynomials with relation to derivatives of the sinc function, establishes series expansions for any real powers of the sinc and hyperbolic sinc functions.

1. Motivations

In mathematical sciences, one usually and most possibly consider elementary functions
\[ e^x, \ln(1 + x), \sin x, \csc x, \cos x, \sec x, \tan x, \cot x, \]
\[ \arcsin x, \arccos x, \arctan x, \sinh x, \csc x, \cosh x, \sech x, \]
\[ \tanh x, \coth x, \arcsinh x, \arccosh x, \arctanh x \]
and their series expansions at \( x = 0 \). Their series expansions can be found in mathematical handbooks such as [1, 12, 22].

What are series expansions at \( x = 0 \) of positive integer powers or real powers of these functions?

2020 Mathematics Subject Classification. Primary 41A58; Secondary 05A19, 11B73, 11B83, 11C08, 33B10.

Key words and phrases. Faá di Bruno formula; series expansion; positive integer power; real power; sinc function; hyperbolic sinc function; partial Bell polynomial; Stirling numbers of the second kind; weighted Stirling numbers of the second kind; closed-form formula.

This paper was typeset using \textsc{AMSTeX}.
It is combinatorial knowledge [9, 11] that coefficients of the series expansion of the power function \((e^x - 1)^k\) for \(k \in \mathbb{N} = \{1, 2, \ldots\}\) are the Stirling numbers of the second kind, while coefficients of the series expansion of the power function \([\ln(1 + x)]^k\) for \(k \in \mathbb{N}\) are the Stirling numbers of the first kind. In other words, the power functions \((e^x - 1)^k\) and \([\ln(1 + x)]^k\) for \(k \in \mathbb{N}\) are generating functions of the Stirling numbers of the first and second kinds.

In the paper [8], among other things, Carlitz introduced the notion of weighted Stirling numbers of the second kind \(R(n, k, r)\). Carlitz also proved in [8] that the numbers \(R(n, k, r)\) can be generated by
\[
\frac{(e^z - 1)^k}{k!} e^{rz} = \sum_{n=k}^{\infty} R(n, k, r) \frac{z^n}{n!} \tag{1}
\]
and can be explicitly expressed by
\[
R(n, k, r) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (r+j)^n \tag{2}
\]
for \(r \in \mathbb{R}\) and \(n \geq k \geq 0\). Specially, when \(r = 0\), the quantities \(R(n, k, 0)\) become the Stirling numbers of the second kind \(S(n, k)\). By the way, the notion \(\{n\choose k\}_r\) is called the \(r\)-Stirling numbers of the second kind in [6] by Broder.

In the handbook [12], series expansions at \(x = 0\) of the functions \(\arcsin^m x\), \(\arcsinh^m x\), \(\arctan^m x\), \(\arctanh^m x\) for \(m \in \mathbb{N}\) have been established, applied, reviewed, and surveyed.

In the papers [5, 13, 14, 19, 26, 29] and plenty of references collected therein, the series expansions at \(x = 0\) of the functions \(\arcsin^m x\), \(\arcsinh^m x\), \(\arctan^m x\), \(\arctanh^m x\) for \(m \in \mathbb{N}\) were written down.

In the papers [2, 3, 16, 17, 20, 32, 33], series expansions of the functions \(I_\nu(x) I_\nu(x)\) and \([I_\nu(z)]^2\) were explicitly written out, while the series expansion of the power function \([I_\nu(z)]^r\) for \(\nu \in \mathbb{C} \setminus \{-1, -2, \ldots\}\) and \(z \in \mathbb{C}\) was recursively formulated, where \(I_\nu(z)\) denotes modified Bessel functions of the first kind.

In the paper [24], series expansions at \(x = 0\) of the functions \((\arccos x)^r\) and \((\arcsin x)^r\) were established for real \(r \in \mathbb{R}\). In [26], a series expansion at \(x = 1\) of \(\frac{\arccos x}{2(1-x)}\) was invented for real \(r \in \mathbb{R}\).

For \(x \in \mathbb{R}\), the functions
\[
\text{sinc } x = \begin{cases} 
1, & x = 0 \\
\frac{\sin x}{x}, & x \neq 0 
\end{cases}
\]
and
\[
\text{sinhc } x = \begin{cases} 
1, & x = 0 \\
\frac{\sinh x}{x}, & x \neq 0 
\end{cases}
\]
are called the sinc function and hyperbolic sinc function respectively. The function sinc\(x\) is also called the sine cardinal or sampling function, as well as the function sinh\(c\) is also called hyperbolic sine cardinal, see [31]. The sinc function sinc\(x\) arises frequently in signal processing, the theory of the Fourier transforms, and other areas in mathematics, physics, and engineering.
In [9, Theorem 11.4] and [11, p. 139, Theorem C], the Faà di Bruno formula is given for \( n \in \mathbb{N} \) by
\[
\frac{d^n}{dx^n} f \circ h(x) = \sum_{k=1}^{n} f^{(k)}(h(x)) B_{n,k}(h'(x), h''(x), \ldots, h^{(n-k+1)}(x)),
\]
where partial Bell polynomials \( B_{n,k} \) are defined for \( n \geq k \geq 0 \) by
\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{1 \leq i_1 \leq n-k+1} \prod_{i=1}^{n-k+1} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)^{\ell_i},
\]
in [9, Definition 11.2] and [11, p. 134, Theorem A].

In this paper, with the help of the Faà di Bruno formula (3) and in terms of weighted Stirling numbers of the second kind \( R(n, k, r) \) and the Stirling numbers of the second kind \( S(n, k) \), we will derive two series expansions at \( x = 0 \) of the positive integer power function \( \sin^\ell x \) for \( \ell \in \mathbb{N} \), we will find out a closed-form formula of specific partial Bell polynomials
\[
B_{n,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{(-1)^{n-k}}{n-k+2} \sin \left( \frac{(n-k)\pi}{2} \right) \right)
\]
for \( n \geq k \in \mathbb{N} \), we will discover series expansions at \( x = 0 \) of the power functions \( \sin^\ell x \) and \( \sinh^\ell x \) for \( r \in \mathbb{R} \), and we will deduce a closed-form formula for weighted Stirling numbers of the second kind \( R(2j + \ell, -\frac{\pi}{2}) \) for \( j, \ell \in \mathbb{N} \) in terms of the Stirling numbers of the second kind \( S(n, k) \).

2. Two series expansions of positive integer power

In this section, we derive two series expansions at \( x = 0 \) of the positive integer power function \( \sin^\ell x \) for \( \ell \in \mathbb{N} \) of the sine function \( \sin x \) in terms of weighted Stirling numbers of the second kind \( R(n, k, r) \) and the Stirling numbers of the second kind \( S(n, k) \). As a corollary, a closed-form formula for \( R(\ell + 2j, \ell, -\frac{\pi}{2}) \) for \( j, \ell \in \mathbb{N} \) in terms of the Stirling numbers of the second kind \( S(n, k) \) is deduced.

**Theorem 1.** For \( \ell \in \mathbb{N} \) and \( x \in \mathbb{R} \), we have
\[
\sin^\ell x = 1 + \sum_{j=1}^{\infty} (-1)^j \frac{R(\ell + 2j, \ell, -\frac{\pi}{2})}{(\ell + 2j)!} \left( \frac{2x}{2j+1} \right)^{2j}(2j)!.
\]
where \( R(\ell + 2j, \ell, -\frac{\pi}{2}) \) is given by (2).

**Proof.** For \( \ell \in \mathbb{N} \), the formula
\[
\sin^\ell x = \frac{(-1)^\ell}{2^\ell} \sum_{q=0}^{\ell} (-1)^q \left( \frac{\ell}{q} \right) \cos \left( 2q - \ell \right)x - \frac{\ell}{2} \pi \]
is given in [15, Corollary 2.1]. Applying the identity
\[
\cos(x - y) = \cos x \cos y + \sin x \sin y
\]
to the formula (5) leads to
\[
\sin^\ell x = \frac{(-1)^\ell}{2^\ell} \sum_{q=0}^{\ell} (-1)^q \left( \frac{\ell}{q} \right) \left[ \cos[(2q - \ell)x] \cos \left( \frac{\ell}{2} \pi \right) + \sin[(2q - \ell)x] \sin \left( \frac{\ell}{2} \pi \right) \right]
\]
\[
= \frac{(-1)^\ell}{2^\ell} \cos \left( \frac{\ell}{2} \pi \right) \sum_{q=0}^{\ell} (-1)^q \left( \frac{\ell}{q} \right) \left[ 1 + \sum_{j=1}^{\infty} (-1)^j (2q - \ell)^{2j} \frac{2j}{(2j)!} \right]
\]
Theorem 2. For \( \ell \in \mathbb{N} \) and \( x \in \mathbb{R} \), we have

\[
\sum_{k=0}^{2j-1} (-1)^k \binom{2j-1}{k} \left( \frac{2j}{\ell} \right)^k S(k + \ell, \ell) = 0
\]

and

\[
\text{sinc}^\ell x = 1 + \sum_{j=1}^{\infty} (-1)^j \left[ \sum_{k=0}^{2j} (-1)^k \binom{2j}{k} \left( \frac{2j}{\ell} \right)^k S(k + \ell, \ell) \right] (\ell x)^{2j} \frac{1}{(2j)!}.
\]

where \( S(k + \ell, \ell) \) denotes the Stirling numbers of the second kind.

Proof. Taking \( r = 0 \) in (1) and reformulating give

\[
\left( \frac{e^x - 1}{x} \right)^k = \sum_{n=0}^{\infty} S(n + k, k) \frac{x^n}{n!}.
\]

Since

\[
\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{2ix} - 1}{2i} = e^{-x^2} - e^{-x^2}
\]
and it is odd, by (9) and the Cauchy product of two series, we acquire

\[
\text{sinc}^\ell x = \left( \frac{\sin x}{x} \right)^\ell = \left( \frac{e^{2ix} - 1}{2i} \right)^\ell e^{-\ell x^2} = \left[ \sum_{n=0}^{\infty} S(n + \ell, \ell) (2x i)^n \frac{1}{n!} \right] \left[ \sum_{n=0}^{\infty} (-\ell x i)^n \frac{1}{n!} \right].
\]
From it follows that
\[ \square \]
The proof of Theorem 2 is complete.

\[ R(x) = \sum_{j=0}^{\infty} \sum_{n=0}^{j} \frac{S(n + \ell, \ell)}{(n+\ell)!} \frac{(2i)^n (-\ell i)^{n-j}}{n! (j-n)!} x^j \]
\[ = \sum_{j=0}^{\infty} \sum_{n=0}^{j} (-1)^j \frac{j!}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{2}{\ell} k S(k + \ell, \ell) \left( \cos \frac{j\pi}{2} + i \sin \frac{j\pi}{2} \right) x^j \]
\[ = \sum_{j=0}^{\infty} (-1)^j \frac{j!}{(2j)!} \sum_{k=0}^{2j} \frac{1}{k!} (-1)^k \binom{2j}{k} \binom{2j-1}{k-1} S(k + \ell, \ell) x^{2j} \]
\[ + i \sum_{j=1}^{\infty} (-1)^j \frac{j!}{(2j-1)!} \sum_{k=0}^{2j-1} (-1)^k \binom{2j-1}{k} \binom{2}{\ell} k S(k + \ell, \ell) x^{2j-1} \]
\[ = \sum_{j=0}^{\infty} (-1)^j \frac{j!}{(2j)!} \sum_{k=0}^{2j} (-1)^k \binom{2j}{k} \binom{2j}{k} S(k + \ell, \ell) x^{2j}. \]

The proof of Theorem 2 is complete.

Corollary 1. For \( j, \ell \in \mathbb{N} \), we have
\[ R \left( \frac{2j + \ell}{2} \right) = \left( \frac{2j + \ell}{\ell} \right) \sum_{m=0}^{2j} (-1)^m \binom{2j}{m} \left( \frac{\ell}{2} \right)^m S(2j + \ell - m, \ell). \]

Proof. This follows from comparing the series expansion (4) in Theorem 1 with the series expansion (8) in Theorem 2 and simplifying.

3. A closed-form formula for partial Bell polynomials

In this section, with the help of Theorem 1, we establish a closed-form formula for partial Bell polynomials \( B_{n,k} \) with relation to all derivatives at \( x = 0 \) of the sinc function sinc \( x \). This formula will be employed later.

Theorem 3. For \( n \geq k \geq 1 \) and \( m \in \mathbb{N} \), partial Bell polynomials \( B_{n,k} \) satisfy
\[ B_{2m-1,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{(-1)^m}{2m-k+1} \cos \frac{k\pi}{2} \right) = 0 \]
and
\[ B_{2m,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{(-1)^m}{2m-k+1} \sin \frac{k\pi}{2} \right) \]
\[ = (-1)^{m+k} \frac{2^{2m}}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} R(2m + j, j, -\frac{1}{2}) \]
where \( R(2m + j, j, -\frac{1}{2}) \) is given by (2).

Proof. From
\[ \text{sinc } x = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} x^{2j} \quad x \in \mathbb{R}, \]
it follows that
\[ (\text{sinc } x)^{(2j)} \big|_{x=0} = \frac{(-1)^j}{2j+1} \quad \text{and} \quad (\text{sinc } x)^{(2j-1)} \big|_{x=0} = 0 \quad (10) \]
for \( j \in \mathbb{N} \).

On [11, p. 133], the identity
\[ \frac{1}{m!} \left( \sum_{\ell=1}^{\infty} x^\ell \right)^m = \sum_{n=m}^{\infty} B_{n,m}(x_1, x_2, \ldots, x_{n-m+1}) \frac{t^n}{n!} \quad (11) \]
is given for \( m \geq 0 \). The formula (11) implies that

\[
B_{n+k,k}(x_1, x_2, \ldots, x_{n+1}) = \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left[ \sum_{\ell=0}^{\infty} \frac{x_{\ell+1}}{(\ell+1)!} t^{\ell} \right]^k
\]  

(12)

for \( n \geq k \geq 0 \). Substituting \( x_{2j} = \frac{(-1)^j}{2j+1} \) and \( x_{2j-1} = 0 \), that is, \( x_j = \frac{1}{2j+1} \cos\left(\frac{j}{2}\pi\right) \), for \( j \in \mathbb{N} \) into (12) results in

\[
B_{n+k,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{n+2} \cos\left(\frac{n+1}{2} \pi\right) \right)
= \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left( \frac{\sin t - 1}{t} \right)^k
= \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left( \frac{(-1)^k}{t^k} + \frac{(-1)^k}{t^k} \sum_{j=1}^{k} (-1)^{j} \binom{k}{j} (\sinh t)^j \right)
= \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left( \frac{(-1)^k}{t^k} \sum_{\ell=1}^{\infty} (-1)^\ell \sum_{j=1}^{k} \binom{k}{j} \frac{1}{2^\ell} \frac{1}{(j+2\ell)!} \right.
\times \sum_{q=0}^{j} (-1)^q \binom{j}{q} (2q-j)^{j+2\ell} \left. \right) t^{2\ell}
= (-1)^k \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \sum_{\ell=k}^{\infty} (-1)^\ell \left[ \sum_{j=1}^{k} \binom{k}{j} \frac{1}{2^\ell} \frac{1}{(j+2\ell)!} \right.
\times \sum_{q=0}^{j} (-1)^q \binom{j}{q} (2q-j)^{j+2\ell} \left. \right]
\times \sum_{q=0}^{j} (-1)^q \binom{j}{q} (2q-j)^{j+2\ell} \left[ 2\ell - k \right]_n t^{2\ell-k-n}
= \begin{cases} 
0, & n+k = 2m+1 \\
\left(-1\right)^{k+m} \frac{(2m)!}{k!} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{2^\ell} \frac{1}{(j+2\ell)!} \sum_{q=0}^{j} (-1)^q \binom{j}{q} (2q-j)^{j+2m} \right] \left( 2x \right)^{2m} \left( 2m! \right)
\end{cases}
\]

for \( m \in \mathbb{N} \) and \( n \geq k \geq 1 \), where we used the series expansion (4) in Theorem 1. The proof of Theorem 3 is complete.

4. Two series expansions of real powers

In this section, with the aid of Theorem 3, we establish series expansions at the point \( x = 0 \) of the power functions \( \sin^r x \) and \( \sinhc^r x \) for real \( r \in \mathbb{R} \).

Theorem 4. When \( r \geq 0 \), the series expansion

\[
\sin^r x = 1 + \sum_{m=1}^{\infty} (-1)^m \left[ \sum_{k=1}^{2m} \frac{(-r)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} R \left( \frac{2m+j-j}{2} \right) \right] \left( \frac{(2x)^{2m}}{(2m)!} \right)
\]

(13)
is convergent in \( x \in \mathbb{R} \), where the rising factorial \((r)_k\) is defined by
\[
(r)_k = \prod_{\ell=0}^{k-1} (r + \ell) = \begin{cases} 
 r(r+1) \cdots (r+k-1), & k \geq 1 \\
 1, & k = 0
\end{cases}
\]
and \(R(2m + j, j - \frac{1}{2})\) is given by (2).

When \( r < 0 \), the series expansion (13) is convergent in \( x \in (-\pi, \pi) \).

\[ \text{Proof.} \]
By virtue of the Faà di Bruno formula (3), we obtain
\[ \frac{d^j}{dx^j}(\sinh^r x) = \sum_{k=1}^j \frac{d^k}{du^k} B_{j,k} ((\sin x)^r, (\sin x)^{r-1}, \ldots, (\sin x)^{r-k+1}) \]
\[ = \sum_{k=1}^j (r)_k \sinh^{r-k} x B_{j,k} ((\sin x)^r, (\sin x)^{r-1}, \ldots, (\sin x)^{r-k+1}) \]
\[ \rightarrow \sum_{k=1}^j (r)_k B_{j,k} \left(0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{j-k+2} \sin \left(\frac{(j-k)\pi}{2}\right)\right), \ x \to 0 \]
\[ = \begin{cases} 
 0, & j = 2m - 1 \\
 \frac{2m}{(2m)_k} B_{2m,k} \left(0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{j-k+2} \sin \left(\frac{2m-k\pi}{2}\right)\right), & j = 2m
\end{cases} \]
for \( m \in \mathbb{N} \), where \( u = u(x) = \sin x \), the notation
\[ (r)_k = \prod_{\ell=0}^{k-1} (r - \ell) = \begin{cases} 
 r(r-1) \cdots (r-k+1), & k \geq 1 \\
 1, & k = 0
\end{cases} \]
for \( r \in \mathbb{R} \) is called the falling factorial, and we used derivatives in (10). Therefore, with the help of Theorem 3, we arrive at
\[
\sinh^r x = 1 + \sum_{j=1}^\infty \left[\lim_{x \to 0} \frac{d^j}{dx^j}(\sinh^r x) \right] \frac{x^j}{j!} \]
\[ = 1 + \sum_{m=1}^\infty \left[\lim_{x \to 0} \frac{d^{2m}}{dx^{2m}}(\sinh^r x) \right] \frac{x^{2m}}{(2m)!} \]
\[ = 1 + \sum_{m=1}^\infty \left[\sum_{k=1}^{2m} (-1)^{m+k} \frac{(r)_k}{k!} \sum_{j=1}^k \left(\frac{k}{j}\right) \frac{1}{2^j (2m+j)} \frac{1}{j!} \frac{1}{(2m-k)!} \sum_{q=0}^j (-1)^q \frac{1}{q!} \sin \left(\frac{(2m-k)\pi}{2}\right) \right] x^{2m} \]
\[ = 1 + \sum_{m=1}^\infty \left[\sum_{k=1}^{2m} (-1)^{m+k} \frac{(r)_k}{k!} \sum_{j=1}^k (-1)^j \frac{R(2m+j, j - \frac{1}{2})}{(2m+j)!} x^{2m} \right] \]
\[ \left(\frac{2x}{(2m)!}\right)^{2m} \]
The proof of Theorem 4 is thus complete. \( \square \)

**Corollary 2.** For \( x, r \in \mathbb{R} \), we have
\[
\sinh^r x = 1 + \sum_{m=1}^\infty \left[\sum_{k=1}^{2m} (-1)^{m+k} \frac{(r)_k}{k!} \sum_{j=1}^k (-1)^j \frac{R(2m+j, j - \frac{1}{2})}{(2m+j)!} x^{2m} \right] \]
\[ \frac{(2x)^{2m}}{(2m)!} \]
where \(R(2m + j, j - \frac{1}{2})\) is given by (2).

\[ \text{Proof.} \]
The series expansion (14) follows from replacing \( \sin x \) by \( \sinh(x) \) in (13) and then substituting \( x i \) for \( x \). \( \square \)
5. Remarks

Finally we list several remarks about our main results and related things.

Remark 1. From Theorem 1 and its proof, we can derive the following identities

\[ R(2j - 1, 2\ell - 1, -\frac{2\ell - 1}{2}) = \begin{cases} 0, & 1 \leq j \leq \ell - 1 \\ 1, & j = \ell \end{cases} \]

and

\[ R(2j, 2\ell, -\ell) = \begin{cases} 0, & 1 \leq j \leq \ell - 1 \\ 1, & j = \ell \end{cases} \]

for \( \ell \in \mathbb{N} \).

Remark 2. The formula (5) can also be found at the websites https://math.stackexchange.com/a/4331451/945479 and https://math.stackexchange.com/a/4332549/945479.

The formulation of the series expansion in Theorem 1 is better and simpler than corresponding ones in [7, Section 3, pp. 798–799].

Remark 3. After reading the preprint [25] of this paper, Jacques Gélinas pointed out that the series expansion (8) in Theorem 2 has been considered by John Blissard in [4, pp. 50–51] with different notations.

Remark 4. As long as the function \( f(u) \) is infinitely differentiable at the point \( u = 1 \), Theorem 3 can be utilized to compute series expansions at \( x = 0 \) of the functions \( f(\text{sinc} x) \) and \( f(\text{sinc}^2 x) \). For example, making use of the Faà di Bruno formula (3), the derivatives in (10), and Theorem 3, we acquire

\[ e^{\text{sinc} x} = \sum_{k=0}^{\infty} \left( \lim_{x \to 0} \frac{d^k e^{\text{sinc} x}}{dx^k} \right) \frac{x^k}{k!} \]

\[ = e + \sum_{k=1}^{\infty} \left[ \lim_{x \to 0} \sum_{j=1}^{k} e^{\text{sinc} x} B_{k,j} ((\text{sinc} x)', (\text{sinc} x)'', \ldots, (\text{sinc} x)^{(k-j+1)}) \right] \frac{x^k}{k!} \]

\[ = e + \sum_{k=1}^{\infty} \left[ \lim_{x \to 0} \sum_{j=1}^{k} B_{k,j} \left( (\text{sinc} x)' \big|_{x=0}, (\text{sinc} x)'' \big|_{x=0}, \ldots, (\text{sinc} x)^{(k-j+1)} \big|_{x=0} \right) \right] \frac{x^k}{k!} \]

\[ = e + \sum_{k=1}^{\infty} \left[ \lim_{x \to 0} \sum_{j=1}^{2k} B_{2k,j} \left( 0, -\frac{1}{3}, \ldots, \frac{1}{k-j+2} \cos \left( \frac{k-j+1}{2} \pi \right) \right) \right] \frac{x^{2k}}{(2k)!} \]

\[ = e + \sum_{k=1}^{\infty} \left[ \lim_{x \to 0} \sum_{j=1}^{2k} (-1)^{k+j} \frac{2k}{j!} \sum_{\ell=1}^{j} (-1)^{\ell} \binom{j}{\ell} R(2k + \ell, \ell, -\frac{4}{\ell}) \right] \frac{x^{2k}}{(2k)!} \]

that is,

\[ e^{\text{sinc} x - 1} = 1 + \sum_{k=1}^{\infty} (-1)^k \sum_{j=1}^{2k} \binom{2k}{j} \sum_{\ell=1}^{j} (-1)^{\ell} \binom{j}{\ell} R(2k + \ell, \ell, -\frac{4}{\ell}) \frac{x^{2k}}{(2k)!} \]

\[ = 1 - \frac{x^2}{6} + x^4 - \frac{107x^6}{45360} + \frac{1189x^8}{5443200} - \frac{1633x^{10}}{89812800} + \cdots, \]

where \( R(2k + \ell, \ell, -\frac{4}{\ell}) \) is given by (2).

Remark 5. The first identity in Theorem 3 is a special case of the following general conclusion in [14, Theorem 1.1].
For $k, n \geq 0$, $m \in \mathbb{N}$, and $x_m \in \mathbb{C}$, we have

$$B_{2n+1,k}(0, x_2, 0, x_4, \ldots, \frac{1+(-1)^k}{2}x_{2n-k+2}) = 0.$$ 

Remark 6. From the proof of Theorem 3, we derive the identity

$$\sum_{j=1}^{k} (-1)^j \frac{k!}{(2^m j)!} R(2\ell + j, j, -\frac{j}{2}) = 0$$

for $k \geq 2$ and $1 \leq \ell \leq k - 1$.

Remark 7. We guess that

$$\sum_{j=1}^{k} (-1)^j \frac{k!}{(2^m j)!} R(2m + j, j, -\frac{j}{2}) = 0, \quad k > m \geq 1 \tag{15}$$

and

$$\sum_{j=1}^{k} (-1)^j \frac{k!}{(2^m j)!} R(2m + j - 1, j, -\frac{j}{2}) = 0, \quad k, m \in \mathbb{N}. \tag{16}$$

Stronger but simpler than the guess (16), the identity

$$R(2m + j - 1, j, -\frac{j}{2}) = 0 \tag{17}$$

should be valid for $j, m \in \mathbb{N}$.

If these guesses were proved to be true, then we can reformulate Theorem 3 as that, for $n \geq k \geq 1$, partial Bell polynomials $B_{n,k}$ satisfy

$$B_{n,k}(0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{n-k+2} \cos \left(\frac{n-k+1}{2}\pi\right))$$

$$= (-1)^k \frac{2^m}{k!} \sum_{j=1}^{[n/2]} (-1)^j \frac{k!}{(n+j)!} R(n + j, j, -\frac{j}{2}), \tag{18}$$

where $R(n+j, j, -\frac{j}{2})$ is given by (2) and $[r]$ for $r \in \mathbb{R}$ denotes the floor function whose value is equal to the largest integer less than or equal to $r$.

These guesses (15), (16), and (17) have been posted as a question at the site https://mathoverflow.net/questions/420121/ for confirming or denying. At the site https://mathoverflow.net/a/420309/147732, Peter Taylor (https://mathoverflow.net/users/46140/peter-taylor, pjt33@cantab.net) has combinatorially described a confirmative answer to the question. Consequently, these three conjectures are true and Theorem 3 can be reformulated as (18).

Remark 8. The series expansion (4) in Theorem 1 has been applied to answer questions at https://math.stackexchange.com/a/4429078/945479, https://math.stackexchange.com/a/4332549/945479, and https://math.stackexchange.com/a/4331451/945479.

The series expansion (4) in Theorem 1 or the series expansion (13) in Theorem 4 can be used to answer questions at https://math.stackexchange.com/q/2267836 and https://math.stackexchange.com/q/3673133.

The series expansion (13) in Theorem 4 has been employed to answer questions at the sites https://math.stackexchange.com/a/4427504/945479, https://math.stackexchange.com/a/426821/945479, and https://math.stackexchange.com/a/4428010/945479.

The series expansion (13) in Theorem 4 has been utilized in a forthcoming paper to derive two closed-form formulas for the Bernoulli numbers $B_{2m}$ in terms of weighted Stirling numbers of the second kind $R(n, k, r)$. 
Remark 9. Let \( r > 0 \) and \( k \geq 0 \). Then, by virtue of the Faà di Bruno formula (3) and employing the formula
\[
B_{n,k}(x, 1, 0, \ldots, 0) = \frac{1}{2^{n-k} k!} \binom{k}{n-k} x^{2k-n}
\]
collected in [28, Section 1.4], we obtain
\[
\left( \frac{1}{(1 + x^2)^{r}} \right)^{(k)} = \sum_{j=0}^{k} \frac{d^j}{d u^j} \left( \frac{1}{u} \right) B_{k,j}(2x, 2, 0, \ldots, 0) = \sum_{j=0}^{k} \frac{(-r)^j}{u^{r+j}} 2^j B_{k,j}(x, 1, 0, \ldots, 0) = \sum_{j=0}^{k} \frac{(-r)^j}{(1 + x^2)^{r+j}} 2^j \frac{1}{2^{k-r} j!} \binom{j}{k-j} x^{2j-k} = \frac{k!}{2^k x^k (1 + x^2)^r} \sum_{j=0}^{k} \frac{(-r)^j 2^j j!}{j!} \binom{j}{k-j} x^{2j} \left( \frac{1}{1 + x^2} \right)^{j},
\]
where \( u = u(x) = 1 + x^2 \). See also texts at the site https://math.stackexchange.com/a/4418636.

Remark 10. We would like to mention the papers [10, 30, 34], in which the power function \( \text{sinc}^r x \) for some specific ranges of \( r, x \in \mathbb{R} \) is bounded from both sides, and to mention the papers [18, 21, 27], in which many bounds of the \( \text{sinc} \) function \( \text{sinc} x \) for \( x \in (0, \frac{\pi}{2}) \) are established, reviewed, and surveyed.

Remark 11. This paper is a revised version of the electronic preprint [25].

6. Declarations

6.1. Acknowledgements. The author thanks Jacques Gélinas, a retired mathematician at Ottawa in Canada, and Peter Taylor in Spain for their hard efforts to look up related references and for their valuable discussions mentioned in Remarks 3 and 7 of this paper.

6.2. Availability of data and material. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

6.3. Competing interests. The author declares that he has no conflict of competing interests.

6.4. Authors’ contributions. Not applicable.

6.5. Funding. Not applicable.

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