Isolated circular orders of $PSL(2, \mathbb{Z})$

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Abstract. We give a bijection between the isolated circular orders of the group $G = PSL(2, \mathbb{Z}) \approx (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/3\mathbb{Z})$ and the equivalence classes of Markov systems associated to $G$. As applications, we present examples of isolated circular orders of the group $G$.

1. Introduction

Throughout this paper, $G$ always stands for the group $(\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/3\mathbb{Z})$. Our purpose is to give a bijection between the isolated circular orders of the group $G$ and the symbolic dynamics associated with $G$, called Markov systems. We also apply it for the construction of examples of isolated circular orders of $G$. The paper is divided into four parts. In Part I, after preparing necessary prerequisites, we define Markov systems of $G$ and state the main theorem. Part II is devoted to the proof of one half of the main theorem, and Part III of the other half. In Part IV, some examples of isolated circular orders are given. In [5], the space $LO(B_3)$ of the left orders (left invariant total orders) of the braid group $B_3$ of three strings are shown to be homeomorphis to the space $CO(G)$ of the circular orders of $G$. This provides examples of isolated left orders of $B_3$.

PART I

After preparing necessary prerequisites in Sections 2–4, we define Markov systems associated with $G$ and state the main result (Theorem I) in Section 5.

2. Circular orders

In this section, we provide preliminary facts about circular orders. Let $H$ be an arbitrary countable group.

Definition 2.1. A map $c : H^3 \to \{0, 1, -1\}$ is called a circular order of $H$ if it satisfies the following conditions

1. $c(g_1, g_2, g_3) = 0$ if and only if $g_i = g_j$ for some $i \neq j$.
2. For any $g_1, g_2, g_3, g_4 \in H$, we have $c(g_2, g_3, g_4) - c(g_1, g_3, g_4) + c(g_1, g_2, g_4) - c(g_1, g_2, g_3) = 0$.

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(3) For any \(g_1, g_2, g_3, g_4 \in H\), we have
\[c(g_1g_2, g_3g_4) = c(g_1, g_2, g_3, g_4)\]

**Definition 2.2.** Given a finite set \(F\) of \(H\), a configuration of \(F\) is an equivalence class of injections \(i : F \to S^1\), where two injections \(i\) and \(i'\) is said to be equivalent if there is an orientation preserving homeomorphism \(h\) of \(S^1\) such that \(i' = hi\).

Given a circular order \(c\) of \(H\), the configuration of the set \(\{g_1, g_2, g_3\}\) of three points is determined by the rule that \(g_1, g_2, g_3\) is ordered anticlockwise if \(c(g_1, g_2, g_3) = 1\), and clockwise if \(c(g_1, g_2, g_3) = -1\). By condition (2) of Definition 2.1, this is independent of the enumeration of the set. But (2) says more: an easy induction shows the following.

**Proposition 2.3.** A circular order of \(H\) determines the configuration of any finite subset \(F\) of \(H\).

Denote by \(CO(H)\) the set of all the circular orders. It is a closed subset of the space of maps from \(H^3\) to \(\{0, \pm 1\}\), and therefore equipped with a totally disconnected compact metrizable topology. A circular order is said to be isolated if it is an isolated point of \(CO(H)\). If \(c \in CO(H)\) is isolated, then there is a finite subset \(S\) of \(H\) such that any circular order which gives the same configuration of \(S\) as \(c\) is c, and vice versa. Such a set \(S\) is called a determining set of \(c\).

For an automorphism \(\sigma\) and \(c \in CO(H)\), we define \(\sigma_c \in CO(H)\) by
\[(\sigma_c)(g_1, g_2, g_3) = c(\sigma^{-1}g_1, \sigma^{-1}g_2, \sigma^{-1}g_3)\]
The order \(\sigma_c\) is called an automorphic image of \(c\). An automorphic image of an isolated circular order is isolated. We also say that \(c\) and \(\sigma_c\) belong to the same automorphism class.

Given \(c \in CO(H)\), we define an action of \(H\) on \(S^1\) as follows. Fix an enumeration of \(H\): \(H = \{g_i\mid i \in \mathbb{N}\}\) such that \(g_1 = e\) and a base point \(x_0 \in S^1\). Define an embedding \(\iota : H \to S^1\) inductively as follows. First, set \(\iota(g_1) = x_0\) and \(\iota(g_2) = x_0 + 1/2\). If \(i\) is defined on \(\{g_1, \cdots, g_n\}\), then there is a connected component of \(S^1 \setminus \{\iota(g_1), \cdots, \iota(g_n)\}\) where the point \(y_{n+1}\) should be embedded, by virtue of Proposition 2.3. Define \(\iota(g_{n+1})\) to be the midpoint of that interval. The left translation of \(H\) yields an action of \(H\) on \(\iota(H)\) which is shown to extend to a continuous action on \(\text{Cl}(\iota(H))\). Extend it further to an action on \(S^1\) by setting that the action on the gap \(x_0\) is linear. The action so obtained is called the dynamical realization of \(c\) based at \(x_0\).

**Definition 2.4.** An action \(\phi\) of the group \(H\) on \(S^1\) is called tight at \(x_0 \in S^1\) if it satisfies the following two conditions.
- It is free at \(x_0\), i.e., the stabilizer of \(x_0\) is trivial.
- If \(J\) is a gap of the orbit closure \(\text{Cl}(\phi(H)x_0)\), then \(\partial J\) is contained in the orbit \(\phi(H)x_0\).

We have three lemmas whose proofs are easy and omitted. (Lemma 2.5 is a consequence of the midpoint construction. In fact, it is used in the construction of the dynamical realization, where we extend the \(H\)-action from the orbit of \(x_0\) to the orbit closure.)

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\(^1\)A gap of a closed subset \(K\) of \(S^1\) means a connected component of \(S^1 \setminus K\).
Lemma 2.5. The dynamical realization is tight at the base point $x_0$.

Lemma 2.6. Two dynamical realizations obtained via different enumerations of $H$ are mutually conjugate by an orientation and base point preserving homeomorphism of $S^1$.

Lemma 2.7. An action $\phi$ of $H$ on $S^1$ which is tight at $x_0$ is topologically conjugate to the dynamical realization of some circular order $c$ based at $x_0$ by an orientation and base point preserving homeomorphism.

Henceforth any action $\phi$ as in the last lemma is referred to as a dynamical realization of $c$.

3. Preliminaries on $G$

Here we study properties necessary for us of the group

$$G = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = e \rangle.$$ 

As is well known, $G$ is isomorphic to $PSL(2, \mathbb{Z})$, by an isomorphism $\phi$ such that

$$\phi(\alpha) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \phi(\beta) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$ 

See Figure 1 for the action of $PSL(2, \mathbb{Z})$ on the Poincaré upper half plane $\mathbb{H}$.

![Figure 1](image-url)

**Figure 1.** The open disk bounded by the circle is the Poincaré upper half plane $\mathbb{H}$. The element $\phi(\alpha)$ is the $1/2$-rotation around $i$, and $\phi(\beta)$ the $1/3$-rotation around $\omega = (−1 + \sqrt{-3})/2$. The region $P$ bounded by the ideal triangle $\triangle 0 \infty \omega$ is a fundamental domain of $PSL(2, \mathbb{Z})$.

Proposition 3.1. (1) Any element of $G \setminus \{e\}$ is of order 2, 3 or infinite. Any element of order 2 is conjugate to $\alpha$, and any element of order 3 is conjugate either to $\beta$ or to $\beta^{-1}$.

(2) Any torsion free subgroup of $G$ is isomorphic to a free group, either finitely generated or not.

(3) The commutator subgroup is a free group freely generated by $\alpha \beta \alpha \beta^{-1}$ and $\alpha \beta^{-1} \alpha \beta$. 

(4) Any automorphism of \( G \) is the conjugation by an element of \( \text{PGL}(2, \mathbb{Z}) \) when we identify \( G \) with \( \text{PSL}(2, \mathbb{Z}) \) by \( \phi \). In other words, the outer automorphism group of \( G \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), generated by the involutin \( \sigma_0 \) defined by \( \sigma_0(\alpha) = \alpha \) and \( \sigma_0(\beta) = \beta^{-1} \).

**Proof.** (1) and (2) are well known, and can be obtained easily by considering the action on \( \mathbb{H} \). (3) can be shown, for example, by an induction on word length of elements in \([G, G] \), using the fact that the total exponents of \( \alpha \) (resp. \( \beta \)) in the word is even (resp. a multiple of 3). To show (4), notice that any automorphism \( \sigma \) of \( G \) sends \( \beta \) to an element which is conjugate either to \( \beta \) or to \( \beta^{-1} \). We only need to verify that \( \sigma \) is an inner automorphism in the former case. By composing with an appropriate inner automorphism if necessary, we may assume that \( \sigma(\alpha) = \alpha \). Let \( x \) be a fixed point of \( \phi(\sigma(\beta)) \) in \( \mathbb{H} \). Since \( \sigma(\beta) \) is a conjugate \( \gamma \beta \gamma^{-1} \), \( x \) is a translate of \( \omega \) in Figure 1 by an element \( \phi(\gamma) \in \text{PSL}(2, \mathbb{Z}) \). All such points \( x \), except \( \omega \) and \( \phi(\alpha)(\omega) \), satisfy \( \rho(x, i) > \rho(\omega, i) \) where \( \rho \) is the Poincaré distance, in which case \( \phi(\alpha) \) and \( \phi(\sigma(\beta)) \) generate a covolume infinite Fuchsian group.

Since \( \sigma \) is an automorphism, we have either \( x = \omega \) or \( \phi(\alpha) \omega \). Accordingly, \( \sigma(\beta) = \beta \) or \( \sigma(\beta) = \alpha \beta \alpha \). In either case, \( \sigma \) is an inner automorphism. \( \square \)

**Question.** Does the equality \( (\sigma_0)_*(c) = -c \) hold? Or at least for isolated order \( c \)? This is true for all the examples constructed in Sections 13 and 14.

4. Isolated circular orders of \( G \)

Let \( c \) be an isolated circular order of \( G \), and \( \rho \) a dynamical realization of \( c \) based at \( x_0 \in S^1 \).

**Proposition 4.1.** (1) There is a unique minimal set \( \mathcal{M} \) of the action \( \rho \) homeomorphic to a Cantor set. Moreover \( x_0 \notin \mathcal{M} \).

(2) Let \( I \) be a gap of \( \mathcal{M} \) which contains \( x_0 \). Then all the gaps of \( \mathcal{M} \) is a translate of \( I \) by the action \( \rho \). The stabilized \( G_I \) of \( I \) is infinite cyclic.

**Proof.** It is known by [3] Corollary 1.3 that a dynamical realization \( \rho \) of an isolated circular order cannot be minimal (for any countable group). Assume for contradiction that \( \rho \) admits a finite minimal set of cardinality \( n \). Considering the action of \( G \) on the minimal set, one obtains a surjective homomorphism \( \xi : G \to \mathbb{Z}/n\mathbb{Z} \). Since \( G/[G, G] \) is isomorphic to \( \mathbb{Z}/6\mathbb{Z} \), \( n \) is either 1, 2, 3 or 6. The circular order \( c \) of \( G \) induces a left order \( \lambda \) on \( \text{Ker}(\xi) \) by considering the action of \( \rho|_{\text{Ker}(\xi)} \) on the gap containing \( x_0 \). In particular, \( \text{Ker}(\xi) \) must be torsion free. This eliminates the case \( n = 1, 2, 3 \). To eliminate the case \( n = 6 \), we claim that the induced left order \( \lambda \) on \( \text{Ker}(\xi) \) must be isolated. Any left order \( \lambda' \) on \( \text{Ker}(\xi) \) together with the cyclic order on \( \mathbb{Z}/6\mathbb{Z} \) defines a cyclic order \( c' \) of \( G \) by the lexicographic construction. Moreover the map \( \lambda' \to c' \) is injective and continuous. Since \( c \) is isolated, \( \lambda \) must be isolated, showing the claim. But if \( n = 6 \), \( \text{Ker}(\xi) = [G, G] \) is a free group on 2 generators, and does not admit an isolated left order [2]. This shows that the minimal set cannot be a finite set. It must be a Cantor set. The uniqueness of a Cantor minimal set \( \mathcal{M} \) is easy and well known.

If \( x_0 \in \mathcal{M} \), the two boundary points of a gap of \( \mathcal{M} = \text{Cl}(\rho(G)x_0) \) cannot be from one orbit of the action, contradicting the tightness of the action \( \rho \) at \( x_0 \). This shows \( x_0 \notin \mathcal{M} \).

\(^2G_I = \{ g \in G \mid \rho(g)I = I \}\)
The same argument shows that there are no gaps of $\mathcal{M}$ which is not a translate of $I$. Such a gap might be a gap of $\text{Cl}(\rho(G)x_0)$.

Finally let us show that $G_I$ is infinite cyclic. First, $G_I$ is nontrivial. Assume it is trivial and consider an interval delimited by $x_0$ and a point $y \in \mathcal{M} \subset \text{Cl}(\rho(G)x_0)$. Then $y$ cannot be a translate of $x_0$, again contradicting the tightness. If $G_I$ is not infinite cyclic, $G_I$ is a free group on more than one generators, which does not admit an isolated left order (23). As before, the lexicographic construction gives a contradiction.

The subgroup $G_I$ in the last proposition is called the linear subgroup of the isolated circular order of $c$, and is denoted by $L_c$. For an automorphism $\sigma$ of $G$, we have $L_{\sigma,c} = \sigma(L_c)$. For a fixed isolated circular order $c$, consider the minimal word length of a generator of $\sigma(L_c)$ where $\sigma$ ranges over all the automorphisms of $G$. It is an even number, say $2k$.

**Definition 4.2.** The number $k$ is called the degree of the isolated circular order $c$, and is denoted by $\deg(c)$.

Clearly the degree is an automorphism class function. It is an odd number, as is remarked in Section 14.

5. Markov system

**Definition 5.1.** A Markov system $\mathcal{M} = (a,b,[a],[b],[b^{-1}])$ consists of an orientation preserving involution $a$, a period three homeomorphism $b$ of $S^1$ and subsets $[a],[b],[b^{-1}]$ of $S^1$ which satisfy the conditions (A)–(E) to be listed below.

(A) The sets $[a],[b]$ and $[b^{-1}]$ are disjoint, each consisting of $k$ closed intervals for some $k \in \mathbb{N}$. The number $k$ is called the multiplicity of the system $\mathcal{M}$.

A connected component of $[a]$ (resp. $[b]$ and $[b^{-1}]$) is called an $a$-interval (resp. $b$- and $b^{-1}$-interval). Denote $X = [a] \cup [b] \cup [b^{-1}]$.

(B) Two $a$-intervals are not adjacent in $X$. Likewise for $b$- and $b^{-1}$-intervals.

A gap of $X$ between an $a$-interval and a $b^{\pm 1}$-interval is called a principal gap. A gap between a $b$-interval and a $b^{-1}$-interval is called a complementary gap. A maximal interval which consists of $b$-intervals and complementary gaps is called a $b$-block. Any $b$-interval is contained in a unique $b$-block. The $a$-intervals and the $b$-blocks are alternating in $S^1$ and there are just $k$ $b$-blocks. The union of $b$-blocks is denoted by $[[b]]$. Let us continue the conditions, which are reminiscent of the action of $\alpha$ and $\beta$ on (the first letter of) the words representing elements of $G$.

(C) $a[a] = [[b]]$

This implies $a[[b]] = [a]$ and hence $a[b^{\pm 1}] \subset [a]$.

(D) $b[a] = [b], b[b] = [b^{-1}]$.

This implies $b[b^{-1}] = [a]$ since $b^3 = id$, and also $b^{-1}[a] = [b^{-1}], b^{-1}[b^{-1}] = [b]$ and $b^{-1}[b] = [a]$.

A principal gap $J$ is always mapped to a principal gap by $a$, and exactly one of $b$ and $b^{-1}$ maps $J$ to a principal gap, the other to a complementary gap. Therefore the principal gaps consist of several cycles. Our last condition is the following.

(E) The principal gaps are formed of one cycle.
**Definition 5.2.** Two Markov systems $M = (a, b, [a], [b], [b^{-1}])$ and $M' = (a', b', [a'], [b'], [(b')^{-1}])$ are said to be equivalent if there is an orientation preserving homeomorphism of $S^1$ which sends $[a]$, $[b]$ and $[b^{-1}]$ to $[a']$, $[b']$ and $[(b')^{-1}]$, respectively.

Our main result is the following.

**Theorem 1.** There is a bijection between the isolated circular orders of $G$ and the equivalence classes of the Markov systems.

We would like to emphasize that this is not a true classification theorem of isolated circular orders: it is too difficult to classify Markov systems. What we can do so far is to present examples of Markov systems, as in Part IV.

**Part II**

We construct Markov partitions associated to isolated circular orders, thereby showing one half of Theorem 1. The result of Part II is summarized as Theorem 6.1.

**6. Further properties of isolated circular orders of $G$**

This section is devoted to the preparation of basic facts needed for the proof of the following theorem. Let $\phi$ be a dynamical realization of an arbitrary isolated circular order of $G$, based at $x_0$, and $M$ the minimal set of the action $\phi$.

**Theorem 6.1.** There is a Markov partition $(a, b, [a], [b], [b^{-1}])$ such that $a = \phi(a)$, $b = \phi(b)$.

We denote $G = \phi(G)$. The group $G$ is isomorphic to $G$ (since $\phi$ is free at $x_0$) and is generated by $a$ and $b$.

**Definition 6.2.** A word of letters $a, b^{\pm 1}$ is called admissible if it is reduced and contains no consecutive same letters.

Any element $g \in G \setminus \{id\}$ is expressed uniquely by an admissible word denoted by $W(g)$, whose length by $\|g\|$. The first letter of $W(g)$ is called the prefix of $g$ and is denoted by $\text{pre}(g)$. We put $\text{pre}(id) = \emptyset$ for completeness. If $W(g) = t_1t_2\cdots t_r$ and $W(g') = t_is_{i+1}\cdots t_r$ for some $2 \leq i \leq r$, $g'$ is called a larva of $g$. The group $G$ acts freely at the base point $x_0$ and all the above terminologies about $G$ are carried over to the orbit $Gx_0$.

If $x = gx_0$ and $x' = g'x_0$ for some $g, g' \in G \setminus \{id\}$, the word $W(x)$ is to be the word $W(g)$, the length $\|x\|$ is to be $\|g\|$, the prefix $\text{pre}(x)$ is $\text{pre}(g)$, and $x'$ is said to be a larva of $x$ if $g'$ is a larva of $g$.

Choose a Riemannian metric on $S^1$ in such a way that the involution $a$ is an isometry. The distance of two points $x, y \in S^1$ is denoted by $|x - y|$. The length of an interval $J$ is denoted by $|J|$. As a corollary of Proposition 4.1, we get the following.

**Corollary 6.3.** There is $\epsilon_1 > 0$ such that if a closed interval $J$ satisfies $|J| < \epsilon_1$ and $x_0 \in J$, then $J \cap Gx_0 = \{x_0\}$.

**Proof.** Choose $\epsilon_1$ to be smaller than the distance of $x_0$ to the neighbouring points in $Gx_0$. \[\square\]
Lemma 6.4. There are finitely many gaps $I_1, \ldots, I_r$ of the minimal set $M$ with the following properties.

1. For any gap $J \neq I_i$, the prefixes of all the points of $Gx_0 \cap J$ are the same.
2. For $J = I_i$, there is an enumeration of points of $Gx_0 \cap J$:
$$Gx_0 \cap J = \{x, x^{-1}, \ldots, x_0, x_1, x_2, \ldots\}$$
in the anti-clockwise order of $S^1$ such that $\text{pre}(x_n) (n < 0)$ are the same, and $\text{pre}(x_n) (n > 0)$ are the same.

Proof. Denote by $I$ the gap of $M$ containing $x_0$ as before and by $G_I$ the stabilizer of $I$ by the action $\phi$. Given an arbitrary gap $J$ of $M$, choose $g \in G$ such that $J = gI$ and that $\|g\|$ is the smallest among such $g$. Let $h$ be a generator of $\phi(G_I)$. Then the word $W(g)$ cannot have $W(h)$ or $W(h^{-1})$ as a larva. This implies that if $\|g\| \geq \|h\|$, $W(g)$ is not completely cancelled out in the word $W gh^n$. In particular, $\text{pre}(gh^n)$ is the same for any $n \in \mathbb{Z}$, showing (1). In the remaining case $\|g\| < \|h\|$, $\text{pre}(gh^n) (n > 0)$ are the same, as well as $\text{pre}(gh^{-n}) (n > 0)$, finishing the proof of (2).

Corollary 6.5. There is $\varepsilon_2 > 0$ such that if a closed interval $J$ satisfies $|J| < \varepsilon_2$ and if there are points $x_1, x_2 \in J \cap Gx_0$ such that $\text{pre}(x_1) \neq \text{pre}(x_2)$, then $\text{Int}(J) \cap M \neq \emptyset$.

Proof. Choose $\varepsilon_2 > 0$ smaller than the distance of $x_0$ and $x_{\pm 1}$ for each of the intervals $I_1, \ldots, I_r$ inLemma 6.4.

For $t \in \{a, b^{\pm 1}\}$, let $G_t$ be the set of those elements $g \in G \setminus \{id\}$ such that the last letter of $W(g)$ is $t$.

Lemma 6.6. We have an inclusion $M \subset \text{Cl}(G_t x_0) \setminus G_t x_0$ for each $t \in \{a, b^{\pm 1}\}$.

Proof. It suffices to show that the closed set $X = \text{Cl}(G_t x_0) \setminus G_t x_0$ is invariant by $G_t$, since $M$ is the unique minimal set. Given $x \in X$, there is a sequence $\{g_n\}$ in $G_t$ such that $g_n x_0 \to x$ and $\|g_n\| \to \infty$. For any $f \in G_t$, we have $fg_n \in G_t$ if $n$ is sufficiently large, showing that $fx = \lim_{n \to \infty} fg_n x_0 \in X$.

Definition 6.7. An action $\psi : G \to \text{Homeo}_+(S^1)$ is said to be $\epsilon$-close to the dynamical realization $\phi$ if $\|\psi(\alpha) - \phi(\alpha)\|_0 < \epsilon$ and $\|\psi(\beta) - \phi(\beta)\|_0 < \epsilon$, where $\psi(g) - \phi(g)$ is a map from the abelian group $S^1$ to $S^1$, and $\|\cdot\|_0$ denotes the supremum norm.

Lemma 6.8. There is $\varepsilon_3 > 0$ with the following property: if $\psi : G \to \text{Homeo}_+(S^1)$ is $\varepsilon_3$-close to $\phi$, then $\psi$ is free at $x_0$ and the circular order of $G$ determined by the orbit $\psi(G)x_0 \subset S^1$ is the same as $c$.

Proof. Let $S \subset G$ be a finite determining set of $c$. One can choose $\varepsilon_3 > 0$ so that if $\psi$ is $\varepsilon_3$-near to $\phi$, then the circular order of $\psi(S)x_0$ is the same as $\phi(S)x_0$ and additionally $\psi(\alpha)$ and $\psi(\beta)$ is not the identity. Assume the isotropy group $H$ of $\psi$ at $x_0$ is nontrivial. Then $H$ is torsion free, since any torsion element $g$ of $G$ is conjugate to $\alpha$ or $\beta^{\pm 1}$ and $\psi(g)$ is fixed point free by the additional condition of $\varepsilon_3$. Therefore $H$ is a free group (Proposition 3.1) and admits a left order $\lambda$ and its reciprocal $-\lambda$. On the other hand, the quotient $G/H$ admits a left $G$-invariant circular order $c'$ determined by the orbit $\psi(G)x_0$. Now $\pm \lambda$ and $c'$ determines two distinct circular orders by the lexicographic constructions. This is contrary to the
definition of the determining set \( S \). We have shown that \( \psi \) is free at \( x_0 \). The rest of the assertion follows again by the definition of the determining set. \( \square \)

7. Continuity of \( W(x) \)

In this section, we show that the assignment \( \mathbb{G}x_0 \ni x \mapsto W(x) \) is continuous in some weak sense. The argument here follows closely the proof of [3], Proposition 4.7, but we need an elaboration since the group \( G \) is not a free group treated in [3].

Two letters from the set \( \{a,b^{±1}\} \) is said to be \emph{congruent} if either they are the same or one is \( b^{-1} \). Choose \( \epsilon \) so that \( 0 < \epsilon < \min\{\epsilon_1, \epsilon_2, \epsilon_3, 1/2\} \), where \( \epsilon_1, \epsilon_2, \epsilon_3 \) are the constants defined in the last section.

Proposition 7.1. If \( x,y \in \mathbb{G}x_0 \) satisfy \( |x - y| < \epsilon \), then \( \text{pre}(x) \) and \( \text{pre}(y) \) are congruent.

This section is devoted to the proof of the above proposition by contradiction. We assume the following conditions throughout this section, and we shall deduce a contradiction at the end of the section.

There is a closed interval \( J \) with the following properties.

\( (\text{(21)}) |J| < \epsilon. \)
\( (\text{(22)}) \partial J = \{x_1, x_2\} \subset \mathbb{G}x_0 \) and \( \text{pre}(x_1) \) and \( \text{pre}(x_2) \) are not congruent.

By Corollary 6.3, (21) and (22) imply the following (23).
\( (\text{(23)}) x_0 \notin J. \)

An interval which satisfies (21)-(23) is called a \( \xi \)-interval.

Lemma 7.2. There is a \( \xi \)-interval \( J \) such that no larva of a point in \( \partial J \) is contained in \( J \).

Proof. Let \( J = [x,y] \) be a \( \xi \)-interval such that \( N(J) = \|x\| + \|y\| \) is the smallest. Then \( x \) has no larva congruent to \( x \) and contained in \( J \). Likewise for \( y \). If there are no larvae of \( x \) and \( y \) contained in \( J \) at all, we are done. So assume one of them, say \( x \), has a larva (necessarily not congruent to \( x \)) contained in \( J \). Choose the larva \( z \) of \( x \) in \( J \) which is the nearest to \( x \). Then the interval \( J' = [x,z] \) is a \( \xi \)-interval. Moreover there are no larvae of \( x \) other than \( z \) contained in \( J' \).

Now since \( z \) is a larva of \( x \), \( x \) and \( z \) belong to the same \( \mathbb{G}_t \) for some \( t \in \{a,b^{±1}\} \). Choose \( s \in \{a,b^{±1}\} \setminus \{t\} \). By Corollary 6.5 and Lemma 6.6, \( \mathbb{G}_s x_0 \cap J' \) is nonempty since \( J' \) is a \( \xi \)-interval. Choose a point \( u \) of minimal length from \( \mathbb{G}_s x_0 \cap J' \). If \( \text{pre}(u) \) is congruent to \( \text{pre}(x) \), choose the interval \([u,z]\): if not, \([x,u]\). \( \square \)

Finally let us show that Lemma 7.2 leads to a contradiction. To fix the idea, let \( J \) in Lemma 7.2 be such that \( J = [x,y] \) and \( \text{pre}(x) = a \) and \( \text{pre}(y) = b^{±1} \). Choose an open interval \( U \supset J \) such that \( |U| < \epsilon \) and that all the larvae of \( x \) and \( y \) as well as \( x_0 \), are not contained in \( U \). Let \( h \) be a homeomorphism of \( S^1 \) supported on \( U \) such that \( h(x) \) to the opposite side of \( y \) in \( U \). Define an action \( \psi \) of \( G \) by setting \( \psi(\alpha) = h \phi(\alpha) h^{-1} \) and \( \psi(\beta) = \phi(\beta) \). Notice that \( \psi(\alpha) = \phi(\alpha) \) except on \( U \cup \phi(\alpha) U \). \( U \) and \( \phi(\alpha) \) is disjoint since \( \epsilon < 1/2 \) and \( \phi(\alpha) \) is an isometry.) By Lemma 6.8, the cyclic order of \( G \) obtained by the orbit \( \psi(G)x_0 \) must be the same as \( \epsilon \), i.e., the one obtained from \( \mathbb{G}x_0 \). However, inductions on the length of \( f \) and \( g \) show that

\[ (\psi(f)x_0, \psi(g)x_0, x_0) = (hx,y,x_0) \quad \text{and} \quad (\phi(f)x_0, \phi(g)x_0, x_0) = (x,y,x_0). \]

A contradiction. This completes the proof of Proposition 7.1.
8. Continuity of $W(x)$ -continued

Let $W_\infty$ be the set of infinite words of letters $a, b^{\pm1}$ and $\emptyset$ with the following properties:

(1) there is no consecutive appearence of $a, b$ and $b^{-1}$,
(2) $b^{\mp1}$ does not follow $b^{\pm1}$, and
(3) all the letters after $\emptyset$ are $\emptyset$.

By certain abuse, we denote $W(g) \in W_\infty$ to be a finite word $W(g)$ followed by a sequence of $\emptyset$. Thus for example $W(id) = \emptyset \emptyset \ldots$, and $W(ab) = ab \emptyset \emptyset \ldots$. For $n > 0$, the initial subword of $W(g)$ of length $n$ is denoted $W_n(g)$. We also define $W(x)$ and $W_n(x)$ for a point $x = gx_0 \in Gx_0$ by $W(x) = W(g)$ and $W_n(x) = W_n(g)$. As a consequence of Proposition 7.1 we get:

**Proposition 8.1.** For any $n > 0$, there is $\epsilon(n) > 0$ such that if two points $x, y \in Gx_0$ satisfy $|x - y| < \epsilon(n)$, then $W_n(x) = W_n(y)$.

**Proof.** For any $\eta > 0$, define $\delta(\eta) \in (0, \eta)$ so that if $|x - y| < \delta(\eta)$, then $|b^{\pm1}x - b^{\pm1}y| < \eta$. (Recall that $a$ is assumed to be an isometry.) Define $\epsilon(n) = \underbrace{\delta(\delta(\cdots \delta)}_{n}(\epsilon)$. An induction on $n$ shows the proposition. $\square$

9. Construction of Markov system

Given $x \in \mathcal{M}$, choose a point $x_i \in Gx_0$ from the $\epsilon(i)/2$-neighbourhood of $x$. Then $W_i(x_i)$ is independent of the choice of $x_i$. Moreover $W_i(x_i) = W_j(x_j)$ if $i < j$. Thus the sequece $\{W_i(x_i)\}$ stabilizes.

**Definition 9.1.** For any $x \in \mathcal{M}$, define a word $W(x) \in W_\infty$ as the limit of $W_i(x_i)$. Also define $W_n(x)$ to be the initial subword of length $n$ of $W(x)$.

Notice that $W(x)$ is a word of letters $a$ and $b^{\pm1}$: $\emptyset$ never shows up.

**Definition 9.2.** For an admissible word $w$ of length $n$ of letters $a, b^{\pm1}$, we define the subset $[w]$ of $S^1$ to be the union of the points $x \in \mathcal{M}$ such that $W_n(x) = w$ and the gaps $(x, y)$ of $\mathcal{M}$ such that $W_n(x) = W_n(y) = w$.

**Lemma 9.3.** (1) For any finite admissible word $w$, $[w]$ is a finite union of closed disjoint intervals.
(2) If $v \neq w$ are reduced words of the same length, then $[v] \cap [w] = \emptyset$.
(3) We have $a[ab^{\pm1}] = [b^{\pm1}]$.
(4) We have $b[a] = [b], b[b] = [b^{-1}]$, and $b[b^{-1}] = [a]$.
(5) The cardinalities of the components of $[a], [b]$ and $[b^{-1}]$ are the same.

**Proof.** (1) is a consequence of Proposition 8.1. (2)–(4) are clear from the definitions. (5) follows from (4). $\square$

**Proof of Theorem 9.1.** Define $\mathcal{M} = (a, b, [a], [b, [b^{-1}]]).$ It is obvious that $\mathcal{M}$ satisfies the conditions (A)–(E). $\square$

**Part III**

We define properties (*) and (**), and show that any Markov systems are equivalent to one with (*) and (**). Next, to Markov partitions with (*) and (**), we assign isolated circular orders, thereby showing the other half of Theorem 9.1. The result is summerized as Theorem 12.1.
10. Fundamental properties of Markov systems and modifications

Let \( M = (a, b, [a], [b], [b^{-1}]) \) be an arbitrary Markov system. Recall that the map \( a \) sends a principal gap \( J \) to a principal gap, and either one of \( b \) or \( b^{-1} \) sends \( J \) to a principal gap, the other to a complementary gap. By condition (E), the principal gaps \( I_i, I'_i \) and complementary gaps \( I''_i \), \( i \in \mathbb{Z}/k\mathbb{Z} \), are dynamically related as in (10.1) below, where \( b_i \) is either \( b \) or \( b^{-1} \). The gaps \( I'''_i \) are gaps between components of \([ab] \) and \([ab^{-1}] \) to be defined later.

\[
(10.1) \\
I_1 \xrightarrow{b_1} I'_1 \xrightarrow{a} I_2 \xrightarrow{b_2} I'_2 \xrightarrow{a} \cdots \xrightarrow{b_k} I'_k \xrightarrow{a} I_{k+1} = I_1
\]

In this section, we study fundamental properties of Markov systems. Besides, we show that any Markov system can be modified in its equivalence class to another one with good properties. First of all, consider the following property concerning diagram (10.1).

\((*)\) The map \( f_1 = ab \cdots ab_1 \) which leaves \( I_1 \) invariant admits no fixed point in the open interval \( I_1 \), and for any \( z \in I_1 \), \( \lim_{n \to \infty} f_1^n(z) \to x \), where \( x \) is a point in \( \partial J \cap [a] \).

Our first modification result is the following.

**Lemma 10.1.** Any Markov system is equivalent to a Markov system with \((*)\).

**Proof.** In the admissible word \( f_1 = ab \cdots ab_1 \), the last map \( a \) (first in the word) is a transposition of \( I'_k \) and \( I_1 \). If we change the map \( a \) by the conjugation by a homeomorphism supported in \( I_1 \) and leave \( b \) unchanged, the new maps still satisfy all the requirement for Markov systems. Since the modification is free, one can get \((*)\). \(\square\)

We introduce basic terminologies and notations.

**Definition 10.2.** For subsets \( P \) and \( Q \) consisting of finite disjoint closed intervals of \( S^1 \), the inclusion \( P \subset Q \) is called **precise** if any boundary point of \( Q \) is contained in \( P \).

The inclusion \([b] \cup [b^{-1}] \subset \{b\}\) is precise. The composite of precise inclusions is precise.

**Definition 10.3.** For the Markov system \( \mathcal{M} = (a, b, [a], [b], [b^{-1}]) \), denote by \( \mathcal{G} = \mathcal{G}(\mathcal{M}) \) the subgroup of Homeo\(_+\)(\( S^1 \)) generated by \( a \) and \( b \).

By \textit{word}, we always mean word of letters \( a, b^\pm 1 \). Any map of \( \mathcal{G} \setminus \{id\} \) is expressed uniquely as an admissible word, as can be shown by (4) and (7) of the next lemma.

**Definition 10.4.** For a admissible word \( w = vt \) where \( t \) is the last letter of \( w \), we define \([w] = v[t] \).

\(^{3}\)In \( f_1 = ab \cdots ab_1 \), \( b_1 \) is the first map.
Lemma 10.5. (1) For an admissible word \( vw \), we have \( w[v] = [vw] \).

(2) If \( vw, v^{-1}u \) and \( vu \) are admissible, then \( vw[v^{-1}u] = [wu] \).

(3) If \( wa \) is admissible, \( [wa] = [w] \).

(4) If \( vw \) is admissible, \( [vw] \subseteq [w] \).

(5) For an admissible word \( w \), \( [w] \) consists of \( k \) disjoint closed intervals.

(6) The inclusion \( [wb] \cup [wb^{-1}] \subseteq [w] \) is precise for an admissible word \( w \) which ends at \( a \).

(7) If \( w \) and \( w' \) are distinct admissible words of the same length, \( [w] \) and \( [w'] \) are disjoint.

Proof. (1) is obtained by an easy induction on the length of \( v \). For (2), \( wv[v^{-1}u] = wvu^{-1}[u] = w[u] = [wu] \). For (3), if \( w = vb^{\pm 1}, \ [wa] = vb^{\pm 1}[a] = v[b^{\pm 1}] = [vb^{\pm 1}] = [w] \). For (4), notice that if \( w = ua \) and \( v = b^{\pm 1}, \ [vw] = ua[b^{\pm 1}] \subseteq u[a] = [w] \). Together with (3), this implies \( [vw] \subseteq [w] \) if \( ||v|| = 1 \). The general case follows by an induction on \( ||v|| \). For (5), if \( w = vt \) where \( t \) is the last letter of \( v \), \( [vt] = [v[t]], [t] \) consists of disjoint \( k \) intervals and \( v \) is a homeomorphism. For (6), since the inclusion \( [b] \cup [b^{-1}] \subseteq [[b]] \) is precise, if we write \( w = va \), the inclusion

\[
[w] \cup [wb^{-1}] = w[b] (b^{-1}) \subseteq [w][b] = va[b] = [v[a]] = [w]
\]

is precise. (7) follows from an induction of word length.

For an infinite admissible word \( w = t_1t_2\ldots \), define \( [w] = \cap \{[t_1\cdots t_i]\} \). It is a nonempty set consisting of \( k \) closed intervals, some possibly degenerate to points. We have \( v[w] = [vw] \) for any finite admissible word \( v \) and any infinite admissible word \( w \).

Definition 10.6. Define \( X_\infty = \cap_{f \in G} f^{-1}X \) for \( X = [a] \cup [b] \cup [b^{-1}] \).

The set \( X_\infty \) is closed and \( G \)-invariant.

Lemma 10.7. We have \( X_\infty = \cup_{w} [w] \), where \( w \) runs over all the infinite admissible words.

Proof. The inclusion \( \supseteq \) is easy: for any \( f \in G \), we have \( f[w] = [fw] \subseteq X \), showing \( [w] \subseteq f^{-1}X \). Let us show \( \subseteq \). For any \( x \in X_\infty \), define \( t_1 \in \{a, b^{\pm 1}\} \) by the condition \( x \in [t_1] \), then \( t_2 \) by \( t_1^{-1}(x) \in [t_2] \), \( t_3 \) by \( t_2^{-1}t_1^{-1}(x) \in [t_3] \) \ldots \). The word \( w = t_1t_2\cdots \) we obtained is admissible and \( x \in [w] \).

For any \( x \in X_\infty \), define \( \hat{W}(x) = w \) if \( x \in [w] \). The inclusion \( X_\infty \subseteq X \) is "precise" in the sense that any boundary point of \( X \) is contained in \( X_\infty \). This stems from Lemma 10.5 (6) and Lemma 10.7. Therefore a gap of \( X \) is a gap of \( X_\infty \).

Lemma 10.8. The gaps of \( X_\infty \) form one orbit of the \( G \)-action.

Proof. Any gap of \( X_\infty \) other than principal or complementary gaps is a gap contained in \( [w] \) between components of \( [wb] \) and \( [wb^{-1}] \) for some admissible word \( w \) ending at \( a \), and hence is the image by \( w \) of a complementary gap.

Lemma 10.9. The set \( X_\infty \) admits no isolated component.

Proof. The proof is by contradiction. Let \( C \) be an isolated component contained in \( [w] \). Let \( w = t_1t_2t_3\cdots \) and for each \( m \in \mathbb{N} \), let \( C_m \) be the component of \( [t_1\cdots t_m] \) containing \( C \). We claim that the decreasing sequence \( C_1 \supseteq C_2 \supseteq \cdots \supseteq C \)
stabilizes, i.e., there is $m_0$ such that $C_{m_0+i} = C_{m_0}$ for any $i \in \mathbb{N}$. Assume not.
For any neighbourhood $U$ of $C$, some $C_{m}$ is contained in $U$. But if the sequence
does not stabilize, there is a component of $[w]$ distinct from $C$ contained in $C_m$ and
hence in $U$. This shows that $C$ is not isolated. The contradiction shows the claim.
Now the interval $t_{m_0-1}^{-1} \cdots t_{1}^{-1}C$ is at the same time a component of $X$ and of $X_{\infty}$.

It is no loss of generality to assume that $C$ itself is a component of both $X$ and $X_{\infty}$. Let $C(1) = C$, and $C(i) = t_{m_0-1}^{-1} \cdots t_{i-1}^{-1}C$ for any $i > 1$. Then $C(i)$ is also
a component of both $X$ and $X_{\infty}$. Since $X$ has only finitely many components, the
sequence $\{C(i)\}$ is eventually periodic.

Without losing generality, one may assume that $C(1)$ is in a periodic cycle:
$C(n + 1) = C(1)$ and $n$ such smallest. Notice that the intertwining arrows of the
cycle are $t_{i}^{-1} : C(i) \to C(i+1)$ and that $C(i)$ is a $t_{i}$-interval. In particular,

(1) if $C(i)$ is an $a$-interval, $t_{i}^{-1} = a$.

Let $J(i)$ be the gap of $X$ right to $C(i)$. It is also a gap of $X_{\infty}$, either principal
or complementary. They form a cycle with the same arrows $t_{i}^{-1} : J(i) \to J(i+1)$
as the arrows $t_{i}^{-1} : C(i) \to C(i+1)$ of the cycle $\{C(i)\}$. The cycle, consisting
of principal and complementary gaps, is contained in the first and second lines of
diagram (10.1). But since there is no consecutive $b$ or $b^{-1}$ in the arrows of
the cycle, it is the cycle in the first line or its reciprocal. In particular, $n = 2k$ and all
the $J(i)$’s are principal.

Now any $a$-interval appears in the cycle $\{C(i)\}$, since its right gap is principal
and any principal gap appears in the cycle $\{J(i)\}$. By (1), if $C(i)$ is an $a$-interval, it
is mapped by $a$ to a $b^{\pm 1}$-interval $C(i+1)$. That is, any $a$-interval must be mapped
by $a$ to a $b^{\pm 1}$-interval. But there are $k$ $a$-intervals and $2k$ $b^{\pm 1}$-intervals, and some
$a$-interval must be mapped by $a$ to a nontrivial $b$-block containing more than one
$b^{\pm 1}$-intervals. A contradiction. \hfill \Box

We consider the following property and the second modification.

(**) $\text{Int}X_{\infty} = \emptyset$.

LEMMA 10.10. Any Markov system is equivalent to a system with properties
(*) and (**).

PROOF. This is done by an anti-Denjoy modification: one collapses each com-
ponent $[w]$ of $X_{\infty}$ to a point, and define new maps $a$ and $b$ of the collapsed $S^1$.
The modification does not collapse $[a]$- or $[b^{\pm 1}]$-intervals to points, thanks to Lemma
10.9 and does not spoil property (*).
\hfill \Box

11. Circular order determined by Markov system

In this section, we shall show that any Markov system with (*) and (**) is a
dynamical realization of some circular order, i.e., that the associated action is tight
at some point. This is done by showing that the set $X_{\infty}$ is a minimal set with good
properties.

DEFINITION 11.1. For an admissible word $v$ of even length, $(v)$ denotes the
infinite admissible word which repeats $v$.

LEMMA 11.2. The boundary point $x \in \partial I_1 \cap [a]$ satisfies $\hat{W}(x) = (ab_k \cdots ab_1)$
and the other boundary point $y$ satisfies $\hat{W}(y) = (b_k^{-1}a \cdots b_1^{-1}a)$. 

PROOF. Since \( x \) is fixed by \( f_1 \), we have an equality of infinite words
\[
abk \cdots ab_1\hat{W}(x) = \hat{W}(x),
\]
where the LHS is before reducing. If there is no cancelation in the LHS, we get
\[
\hat{W}(x) = (abk \cdots ab_1). \quad \text{If } abk \cdots ab_1 \text{ is completely canceled out, we get } \hat{W}(x) = (b_{1}^{-1}a \cdots b_{k}^{-1}a).
\]
Otherwise there is an intermediate cancelation, and the LHS begins at \( a \) and the RHS begins at \( b_{1}^{-1} \). We thus obtained either \( \hat{W}(x) = (abk \cdots ab_1) \) or \( \hat{W}(x) = (b_{1}^{-1}a \cdots b_{k}^{-1}a) \). Like statement holds for \( \hat{W}(y) \). But since \( x \in [a] \), \( \hat{W}(x) = (abk \cdots ab_1) \), and since \( I_1 \) is a principal gap, \( \hat{W}(y) = (b_{1}^{-1}a \cdots b_{k}^{-1}a) \). \( \square \)

**Lemma 11.3.** The stabilizer \( G_{I_1} \) of \( I_1 \) is infinite cyclic generated by \( f_1 \).

**Proof.** Assume \( h \neq id \) stabilizes \( I_1 \). One may assume that \( h \) admits neither \( f_1 \) nor \( f_1^{-1} \) as an initial subword, by replacing \( h \) with a shorter word if necessary.
Since \( hx = x \) for \( x \) in the previous lemma, we have an equality of infinite word
\[
(11.1) \quad \hat{W}(a) \cdots \hat{W}(a) = \hat{W}(a) \cdots \hat{W}(a).
\]
If there is an intermediate cancelation in the LHS, then \( h \) begins and ends at \( a \).
But then since \( hy = y \), we have another equality
\[
(11.2) \quad h(b_{1}^{-1}a \cdots b_{k}^{-1}a) = (b_{1}^{-1}a \cdots b_{k}^{-1}a),
\]
where the LHS begins at \( a \) and the RHS at \( b_{1}^{-1} \), leading to a contradiction.

Consider the case where there is no cancelation at all in the LHS of \((11.1)\).
Then either \( h \) is \( f_1 \) or its initial subword of length \( 2\ell \), \( \ell < k \). In the latter case, the word \((abk \cdots ab_1)\) is periodic of period \( 2\ell \) and \( 2k \), hence of period \( 2(k, \ell) \). This shows that \( h = abk \cdots ab_1 \). But then in the diagram \((11.1)\), the principal gap \( I_{k+1} \) must be equal to \( I_1 \). A contradiction. In the remaining case where \( w \) is cancelled out in the LHS of \((11.1)\), there is no cancelation in the LHS of \((11.2)\), and one can show that \( h = f_{1}^{-1} \) by a like argument. \( \square \)

**Lemma 11.4.** For any Markov system with (**), \( X_\infty \) is a minimal set of \( G \).

**Proof.** First of all, notice that \( X_\infty \) is a Cantor set by Lemma \( [11.3] \) and (**).
This, together with Lemma \( [10.9] \) shows that the orbit of a boundary point of a gap is dense in \( X_\infty \). Assume there is a minimal set \( Y \) properly contained in \( X_\infty \). Then any boundary point of a gap of \( X_\infty \) cannot be contained in \( Y \). Therefore any gap \( K \) of \( Y \) contains infinitely many gaps \( J_i \) of \( X_\infty \). Since each \( J_i \) belongs to the orbit of \( I_1 \), it is left invariant by a map represented in an admissible word as \( h_i = v_i f_i v_i^{-1} \), where \( f_i \) is a cyclic permutation of \( f_1 \). Since all the \( h_i \) leaves a boundary point \( z \) of \( K \) invariant, we get \( v_i f_i v_i^{-1} \hat{W}(z) = \hat{W}(z) \). By the same argument as the proof of Lemma \( [11.2] \), \( \hat{W}(z) = v_i(f_i) \) or \( \hat{W}(z) = v_i(f_i^{-1}) \) for any \( i \). This contradicts the uniqueness of \( \hat{W}(z) \): the admissibility of the word \( v_i f_i v_i^{-1} \) implies that \( v_i \) contains neither \( f_i \) nor \( f_i^{-1} \) as the terminal subword. \( \square \)

**Definition 11.5.** Given a Markov system \( M = (a, b, [a], [b], [[b]]) \), define a homomorphism \( \phi_M : G \rightarrow \text{Homeo}_+(S^1) \) by \( \phi_M(\alpha) = a \) and \( \phi_M(\beta) = b \).

We have \( \phi_M(G) = G \).

**Lemma 11.6.** For any Markov system \( M \) with (**), \( \phi_M \) is a dynamical realization of a circular order, denoted by \( c_M \), based at some point \( x_0 \in I_1 \).
Proof. By Lemma 2.7, we only need to show that \( \phi_M \) is tight at \( x_0 \). Clearly \( \phi_M \) is free at \( x_0 \) by (\(*\)). To show the other condition, notice that \( X_{\infty} \) is a Cantor minimal set and that any gap of \( X_{\infty} \) is a translate of \( I_1 \). Now choose an arbitrary gap \( J \) of \( \text{Cl}(\phi_M(G)x_0) \) for \( x_0 \in I_1 \). If one end point of \( J \) belongs to the orbit \( \phi_M(G)x_0 \), so does the other end point. On the other hand, there is no gap of \( \text{Cl}(\phi_M(G)x_0) \) whose end points belong to the minimal set \( X_{\infty} \). \( \square \)

12. Isolation

The following theorem is the goal of Part II.

Theorem 12.1. For any Markov system \( M \) with (\(*\)) and (\(**\)), the circular order \( c_M \) given by Lemma 11.6 is isolated. Moreover \( \text{deg}(c_M) \) is equal to the multiplicity of \( M \).

The statement about the degree (Definition 11.2) is obvious. The rest of this section is devoted to the proof of the isolation of \( c_M \), by a pingpong argument. Let

\[
Y = [ab] \cup [ab^{-1}] \cup [b] \cup [b^{-1}], \quad h_1 = abab^{-1}, \quad h_2 = ab^{-1}ab,
\]

\[
\Omega(h_1) = [ab], \quad \Omega(h_2) = [ab^{-1}], \quad \Omega(h_1^{-1}) = [b], \quad \text{and} \quad \Omega(h_2^{-1}) = [b^{-1}].
\]

Notice that the “address” of \( \Omega(h_1^\pm 1) \) is just the first two letters of \( h_1^\pm 1 \), since \( [b^\pm 1] = [h^\pm 1a] \).

Lemma 12.2. For \( i = 1, 2 \), we have a precise inclusion

\[
h_i(Y - \Omega(h_i^{-1})) \subset \Omega(h_i) \quad \text{and} \quad h_i^{-1}(Y - \Omega(h_i)) \subset \Omega(h_i^{-1}).
\]

Proof. For example, the first inclusion for \( i = 1 \) follows from a sequence of precise inclusions;

\[
abab^{-1}([a] \cup [b^{-1}]) = aba([b^{-1}] \cup [b]) = ab([ab^{-1}] \cup [ab]) \subset ab([a]) = [ab].
\]

The other inclusions are similar. \( \square \)

However the pingpoing property on \( Y \) is not sufficient for the proof of the isolation of \( c_M \).

Lemma 12.3. There are open neighbourhoods \( N_i^\pm \) of \( \Omega(h_i^\pm 1) \) such that

1. The closures \( \text{Cl}(N_i^\pm) \) are mutually disjoint, and
2. \( h_i(S^1 - N_i^-) \subset N_i^+ \), or equivalently, \( h_i^{-1}(S^1 - N_i^+) \subset N_i^- \), \( i = 1, 2 \).

Proof. To fix the idea, we assume that the point \( x \in \partial I_1 \cap [a] \) is the right end point of \( I_1 \). Thus the map \( f_1 = ab_1 \cdots ab_1 \) is an increasing homeomorphism of \( I_1 \). One can choose two points in all the principal gaps \( I_i \) and \( I_i' \) in (10.1) which satisfy the relations in Figure 2, where the thin lines indicate the correspondences by the intertwining maps. This is possible simply because the first return map is increasing.

Then choose three points in \( I_i'' \) and \( I_i'' \) as the images of the previous points. See Figure 3. All these points are called distinguished points. Define the neighbourhood \( N_i^\pm \) of \( \Omega(h_i^\pm 1) \) by expanding each component until it reaches the first distinguished point. Condition (1) of the lemma is satisfied.

Let us check if (2) is satisfied. First of all, we only consider the right gap \( J \) of each component of \( \Omega(h_i^\pm 1) \) and check if \( J \setminus N_i^\mp \) is mapped by \( h_i^\pm 1 \) into \( h_i^\pm 1(J) \cap N_i^\pm \). We do not need to consider the left gaps, since the statement for them follows
automatically. We can see this by Figure 4. Our strategy is the following. We choose any gap \( J \in \{ I_i, I_i', I_i'', I_i''' \} \) from diagram 40.1. We consider the left end point \( \partial J \) and calculate to which class ([ab], [ab\(^{-1}\)], [b] or [b\(^{-1}\)]) it belongs. It tells us which one of \( \Omega(h_i^{\pm 1}) \) the left neighbour of \( J \) is, and thus which one of the maps \( h_i^{\pm 1} \) we should check. We explain it concretely with an example \( J = I_1 \). Recall that the left end point \( y \) of \( I_1 \) satisfies \( \hat{W}(y) = (b_1^{-1} a \cdots b_k^{-1} a) \) and it belongs to \( [b_1^{-1}] = \Omega(b_1^{-1} ab_1 a) \). Therefore the map we should check is \( ab_1^{-1} ab_1 \). For any gap \( J \),
we calculate its left end point and the map in concern this way. The actual proof is divided into four cases.

**Case 1.** $J = I_i$. Since

\[
\hat{W}(\partial_-I_i) = ab_{i-1}\cdots ab_1\hat{W}(y) = ab_{i-1}\cdots ab_1(b_1^{-1}a\cdots b_k^{-1}a) = b_i^{-1}ab_{i+1}^{-1}\cdots,
\]

we have $\partial_-I_i \in [b_i^{-1}] = \Omega(b_i^{-1}ab_i)$ and the map in concern is $h = ab_{i-1}ab_i$.

According as $b_{i+1} = b_i^{-1}$ or $b_{i+1} = b_i$, $h$ is either of the following composite.

\[
I_i \xrightarrow{b_i^{-1}} I'_i \xrightarrow{a} I_{i+1} \xrightarrow{b_{i+1}^{-1}} I'_{i+1} \xrightarrow{a} I_{i+2}
\]

\[
I''_i \xrightarrow{b_{i+1}^{-1}} I'_{i+1} \xrightarrow{a} I'''_{i+1}
\]

In any case, $h$ satisfies the required property. See Figure 5.

![Figure 5. $h$ maps $p$ above $q$ or $q'$](image)

**Case 2.** $J = I'_i$. Since $\hat{W}(\partial_-I'_i) = b_i\hat{W}(\partial_-I_i) = ab_{i+1}^{-1}\cdots$, we have $\partial_-I'_i \in \Omega(ab_{i+1}^{-1}ab_{i+1})$ and the map in concern is $h = ab_{i+1}^{-1}ab_i$. This case is analogous to Case 1, and is omitted. The point is that the map $b_i^{-1}ab_{i+1}$ defined on $I'_i$ also goes from left to right in (10.1).

**Case 3.** $J = I''_i$. Since $\hat{W}(\partial_-I''_i) = b_i^{-1}\hat{W}(\partial_-I_i) = b_i\cdots$, we have $\partial_-I''_i \in \Omega(b_iab_i^{-1})$ and the map in concern is $ab_iab_i^{-1}$.

It is either of the following composite.

\[
I'_i \xrightarrow{a} I_{i+1} \xrightarrow{b_{i+1}^{-1}} I'_{i+1} \xrightarrow{a} I_{i+2}
\]

\[
I''_i \xrightarrow{b_{i+1}^{-1}} I'_{i+1} \xrightarrow{a} I'''_{i+1}
\]

See Figure 6. In the bottom composite, the point $p$ is mapped exactly to $q'$. In this case, we enlarge the neighbourhood of $\Omega(b_iab_i^{-1})$ slightly so as to contain the point $p$. Then its complement in $J$ is mapped above $q'$.

**Case 4.** $J = I'''_i$. Since $\hat{W}(\partial_-I'''_i) = a\hat{W}(\partial_-I''_i) = ab_i\cdots$, we have $\partial_-I'''_i \in \Omega(ab_iab_i^{-1})$ and the map in concern is $b_iab_i^{-1}a$. This case is similar to Case 3 and is omitted.

For the proof of Theorem 12.1, we need Proposition 3.3 of [3]. The statement below is adapted for our purpose, and we shall give a complete proof in Appendix.
Proposition 12.4. Let $\phi$ be a dynamical realization of a circular order $c$ based at $x_0$. Given any neighbourhood $U$ of $\phi$ in $\text{Hom}(G, \text{Homeo}_+(S^1))$, there is a neighbourhood $V$ of $c$ in $CO(G)$ such that any order in $V$ has a dynamical realization based at $x_0$ contained in $U$.

We also need the following proposition whose proof is again contained in Appendix.

Proposition 12.5. The space $\text{Hom}(G, \text{Homeo}_+(S^1))$ is locally pathwise connected. In fact, more can be said: any neighbourhood of any point contains a pathwise connected neighbourhood.

Let us finish the proof of Theorem 12.1. Recall that $\phi_M$ is a dynamical realization of the circular order $c_M$ given by Lemma 11.6 based at a point $x_0 \in I_1$. One can choose $x_0 \in \text{Int}(S^1 \setminus N_1^\pm)$, where $N_1^\pm$ is from Lemma 12.3. Choose a pathwise connected neighbourhood $U$ of $\phi_M$ in $\text{Hom}(G, \text{Homeo}_+(S^1))$ such that any action $\psi \in U$ satisfies $h_1(\psi)(S^1 \setminus N_1^+) \subset N_1^+$ (and equivalently $h_1(\psi)^{-1}(S^1 \setminus N_1^+) \subset N_1^-$), where $h_1(\psi) = \psi(\alpha \beta \alpha^{-1} \beta^{-1})$ and $h_2(\psi) = \psi(\alpha \beta^{-1} \alpha \beta)$.

The pingpong lemma (Klein’s criterion) asserts that the subgroup $[G, G]$ generated by $\alpha \beta \alpha^{-1} \beta^{-1}$ and $\alpha \beta^{-1} \alpha \beta$ acts freely at $x_0$ by any $\psi \in U$. We claim that $G$ itself acts freely at $x_0$ by $\psi$. In fact, if $H$ is the stabilizer at $x_0$, then $H \cap [G, G] = \{e\}$. That is, the canonical projection $G \to G/[G, G] \cong \mathbb{Z}/6\mathbb{Z}$ is injective on $H$, and hence $H$, if nontrivial, contains an element $\gamma$ of finite order. But $\gamma$ is conjugate either to $\alpha$ or to $\beta \pm 1$, and therefore $\psi(\gamma)$ is fixed point free. A contradiction shows the claim.

Now let $V$ be a neighbourhood of $c_M$ in $CO(G)$ such that a dynamical realization $\psi$ of an arbitrary order $c'$ of $V$ belongs to $U$. Then there is a path $\psi_t$, $0 \leq t \leq 1$, joining $\phi_M$ and $\psi$ contained in $U$. Since all the $\psi_t$ act freely at $x_0$, the circular orders of $G$ obtained from the orbit $\psi_t(G)x_0$ are all the same, independent of $t$. (Recall that $CO(G)$ is totally disconnected.) This shows $c' = c_M$. The proof of Theorem 12.1 is now complete. □

Proof of Theorem 1. We have constructed a map from the set of the isolated circular orders to the set of the equivalence classes of the Markov systems in Part
II, and a map in the reverse direction in Part III. It is clear from the constructions that one is the inverse of the other. Thus the proof is done of Theorem 1. □

PART IV

We present some examples of isolated circular orders of $G$.

13. Primary examples

We revisit isolated circular orders of $G$ constructed in [5].

Standard example. Here is a Markov system $M_1$ with multiplicity 1. Just place intervals $[a], [b], [b^{-1}]$ in the anti-clockwise order on $S^1$. Clearly there are an involution $a$ of $S^1$ which interchanges $[a]$ and $[b]$ and a period 3 homeomorphism $b$ which circulates $[a], [b]$ and $[b^{-1}]$. Transitivity of gaps is also clear. Thus we obtain an isolated circular order $c_1 = c_{3b_1}$.

If the placement of $[a], [b], [b^{-1}]$ is clockwise, we get another order $c'$. But this is in the same automorphism class of $c_1$: $c' = (σ₀)ₜ c_1$ for $σ_0$ in Proposition 3.1 (4). By the homeomorphism $CO(G) ≈ LO(B_3)$, $c_1$ corresponds to the Dubrovin-Dubrovin order [1].

Finite lifts of the standard example. Let $φ₁$ be a dynamical realization of the previous example $c_1$. We shall construct more examples starting with finite lifts of $φ₁$. For $k > 1$, let $p_k : S^1 \to S^1$ be the $k$-fold covering map. A $G$-action $φ_k$ on $S^1$ is called a $k$-fold lift of $φ₁$ if it satisfies $p_k φ_k(g) = φ₁(g)$ for any $g \in G$.

Then the rotation numbers satisfy $k \cdot rot(φ_k(g)) = rot(φ₁(g))$. This shows that a lift of an involution is an involution if and only if $k$ is odd. Likewise a lift of a 3-periodic map is 3-periodic if and only if $k$ is coprime to 3. Therefore a $k$-fold lift $φ_k$ of $φ₁$ exists if and only if $k \equiv ±1 (6)$: moreover it is unique if it exists. The map $φ_k(β)$ is 3-periodic and has rotation number $±1/3$ according as $k \equiv ±1 (6)$, since $(6k ± 1) \cdot (±3/3) ≡ 1/3 (1)$ implies $± = ±1$.

We shall show that the lift $φ_k$ is associated with a Markov system $M_k$ for $k = 6ℓ + 1$. The case $k = 6ℓ - 1$ is similar and is left to the reader. Let $[a]_i, [b]_i$ and $[b^{-1}]_i$, $(i \in \mathbb{Z}/k \mathbb{Z})$ be the lifts of $[a]$, $[b]$ and $[b^{-1}]$ of the previous example $M_1$ by $p_k$, ordered anticlockwise in $S^1$ as

$$
\cdots [a]_i, [b]_i, [b^{-1}]_i, [a]_{i+1} \cdots.
$$

The maps $a, b$ of the system $M_k$ is to be $φ_k(α)$ and $φ_k(β)$. (13.1) has $3k = 18ℓ + 3$ terms, and $b$, being the lift of $φ₁(β)$, maps each term to the term $6ℓ + 1$ right to it. Therefore we have

$$
b[a]_i = [b]_{i+2ℓ}, \quad b[b]_i = [b^{-1}]_{i+2ℓ}, \quad b[b^{-1}]_i = [a]_{i+2ℓ+1}.
$$

The sequence (13.1) is contracted into a sequence

$$
\cdots [a]_i, [[b]_i], [a]_{i+1}, [[b]_{i+1}] \cdots
$$
of $12ℓ + 2$ terms. The map $a$ send each term to the term $6ℓ + 1$ right to it. That is,

$$
a[a]_i = [[b]_i]_{i+3ℓ}, \quad a[[b]_i] = [a]_{i+3ℓ+1}.
$$

In order to check (E), we shall classify all the principal gaps into two families $J_i$'s and $J_i'$'s $(i \in \mathbb{Z}/(6ℓ + 1)\mathbb{Z})$ by the following ordering:

$$
\cdots, [a]_i, J_i, [b]_i, [b^{-1}]_i, J_i', [a]_{i+1}, \cdots.
$$
By \([13.2]\) and \([13.3]\), we get

\[
(13.4) \quad b(J'_1) = J_{i+2\ell + 1} \quad \text{and} \\
(13.5) \quad ab(J'_1) = J'_{i+5\ell + 1}.
\]

Now \([13.5a]\) shows that the group generated by \(ab\) acts transitively on the family \(J'_1\)'s, since \((6\ell + 1, 5\ell + 1) = 1\), and \([13.4]\) shows that any \(J_i\) from the other family is mapped by \(b^{-1}\) to an element of this family. This shows (E).

The circular order defined by \(\mathbb{M}_k\) is denoted by \(c_k\). They are from distinct automorphism classes since \(\text{deg}(c_k) = k\).

### 14. Further example

Notice that the \(\text{deg}(c)\) of any isolated circular order \(c\) is odd, since the involution \(a\) transposes \([a]\) and \([b]\). According to our calculation, there is no new examples of isolated circular orders up to degree \(\leq 7\). But there is one in degree 9, which we shall present below.

This example is not well ordered as the previous one and it is no use to give an index to each component. Any component of \([a]\) is denoted by the same letter \(a\). Likewise we use notations \(b\) and \(b^{-1}\). Also a component of \([b]\) is denoted by \([b]\).

Consider the following ordering of 27 intervals in \(S^1\) which is grouped into three:

\[
(14.1) \quad a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}ab.
\]

Define a period 3 homeomorphism \(b\) of \(S^1\) by permuting the three groups cyclically to the right. Notice that \(b\) so defined satisfies \(ba = b, bb = b^{-1}\) and \(b^{-1} = a\). The sequence \((14.1)\) is contracted to

\[
(14.2) \quad a [b] a [b] a [b] a [b] a [b] a [b] a [b] a [b] a [b] a [b] a [b] a [b] a [b] a [b] a [b].
\]

Define an involution \(a\) by transposing the groups. The maps \(a\) and \(b\) satisfies (A)–(D). To show (E), indicate the principal gaps by \(\begin{bmatrix} i \\ j \end{bmatrix}\) and complementary gaps by \(\begin{bmatrix} * \\ * \end{bmatrix}\) as follows.

\[
\begin{array}{cccccccccccc}
1 & b^{-1} & 2 & a & 3 & b^{-1} & 4 & a & 5 & b^{-1} & 6 & a & 7 & b^{-1} & 8 & a & 9 \\
1 & 8 & 2 & 9 & 3 & a & 2 & 3 & b & 4 & a & 5 & b^{-1} & 6 & b^{-1} & 7 & a & 8 & b^{-1} & 9 & b^{-1} & 9
\end{array}
\]

Notice that either one of maps \(b^{\pm 1}\) sends \(\begin{bmatrix} i \\ j \end{bmatrix}\) to some \(\begin{bmatrix} i \\ j \end{bmatrix}\) and the other to \(\begin{bmatrix} * \\ * \end{bmatrix}\), and the map \(a\) sends \(\begin{bmatrix} i \\ j \end{bmatrix}\) to some \(\begin{bmatrix} k \\ j \end{bmatrix}\). We can find a cycle of principal gaps:

\[
\begin{array}{cccccccccccc}
1 & a & 3 & b^{-1} & 3 & a & 5 & b^{-1} & 5 & a & 8 & b^{-1} & 8 & a & 6 & b^{-1} & 6 & a \\
1 & 4 & 2 & a & 2 & 2 & 9 & a & 9 & b^{-1} & 9 & a & 7 & b^{-1} & 7 & a & 1 & b^{-1} & 1
\end{array}
\]
Therefore (E) is also satisfied. This yields an isolated circular order $c_9$ of degree 9. It is not in the automorphism classes of the previous examples. If we arrange the intervals in the clockwise order, we obtain another order $c'$. But again $c' = (\sigma_0)^2c_9$.

15. Appendix

We shall give proofs of Propositions 12.4 and 12.5. The former holds true for an arbitrary countable group $H$.

**Proposition 12.4.** Let $\phi$ be a dynamical realization of a circular order $c$ based at $x_0$. Given any neighbourhood $U$ of $\phi$ in $\text{Hom}(H, \text{Homeo}+(S^1))$, there is a neighbourhood $V$ of $c$ in $CO(H)$ such that any order in $V$ has a dynamical realization based at $x_0$ contained in $U$.

**Proof.** Given a finite set $F$ of $H$ and $\epsilon > 0$, let

$$U(F, \epsilon, \phi) = \{ \psi \in \text{Hom}(H, \text{Homeo}+(S^1)) \mid \|\psi(g) - \phi(g)\|_0 < \epsilon, \forall g \in F \}.$$ 

One may replace the given $U$ in the proposition by a smaller neighbourhood $U(F, \epsilon, \phi)$. One may also assume that $F$ is symmetric: $g \in F$ implies $g^{-1} \in F$. Given a finite subset $S$ of $H$, define a neighbourhood $V_S(c)$ of $c$ in $CO(H)$ by

$$V_S(c) = \{ c' \in CO(H) \mid c' \mid_{(S \cup \{\epsilon\})^3} = c \mid_{(S \cup \{\epsilon\})^3} \}.$$ 

Then any $c' \in V_S(c)$ admits a dynamical realization $\psi_{c', S}$ based at $x_0$ such that $\psi_{c', S}(g)x_0 = (\phi(g)x_0$ for any $g \in S$. Our aim is first to find good $S$ and then to alter $\psi_{c', S}$ by a $x_0$-preserving conjugacy to obtain a homomorphism contained in $U(F, \epsilon, \phi)$. Choose $\delta > 0$ so that whenever $f \in F$ and $|x - y| < \delta$, we have $|\phi(f)x - \phi(f)y| < \epsilon$.

**Case 1.** $\phi(H)x_0$ is dense in $S^1$. Choose a finite subset $S'$ of $H$ so that $X = \phi(S')x_0$, is $\delta/2$-dense in $S'$ and define $S = FS' \cup S'$. In this case there is no need for the alteration: we shall show that $\psi_{c', S} \in U(F, \epsilon, \phi)$. In fact, for any $x \in X$, we have $x = \psi_{c', S}(g)x_0 = \phi(g)x_0$ for any $g \in S$. Moreover $\psi_{c', S}(f)x = \phi(f)x$ for any $x \in X$ and $f \in F$, because $\phi(f)x = \phi(fg)x_0, \psi_{c', S}(f)x = \psi_{c', S}(fg)x_0$ and $fg \in S$. Choose any $f \in F$ and any $y \in S^1$. Then $y$ belongs to the closure $\text{Cl}(J)$ of some gap $J$ of $X$. Since $X$ is $\delta/2$-dense, we have $|J| < \delta$, and hence $|\phi(f)J| < \epsilon$. Since both points $\phi(f)y$ and $\psi_{c', S}(f)y$ belong to the same interval $\phi(f)\text{Cl}(J) = \psi_{c', S}(f)\text{Cl}(J)$, we have $|\phi(f)y - \psi_{c', S}(f)y| < \epsilon$. Since $f \in F$ and $y \in S^1$ are arbitrary, we obtain $\psi_{c', S} \in U(F, \epsilon, \phi)$, as is required.

**Case 2.** $\phi(H)x_0$ is not dense in $S^1$. Denote by $J_0$ the gap of $\text{Cl}(\phi(H)x_0)$ whose left end point is $x_0$. Define $h_{\text{min}} \in H$ so that $\phi(h_{\text{min}})x_0$ is the right end point of $J_0$. Since $\phi$ is tight at $x_0$, any gap $J$ of $\text{Cl}(\phi(H)x_0)$ is a translate of $J_0$. Denote by $V$ the set of the gaps $J$ such that $|J| \geq \delta$. Let

$$S_1 = \{ g \in H \mid \phi(g)(J_0) \in V \}$$

and let $S_2 = S_1 \cup S_1h_{\text{min}}$. Thus the end points of any interval of $V$ belongs to $\phi(S_2)x_0$. Add some more elements to $S_2$ to form a finite subset $S_3$ such that $\phi(S_3)x_0$ is $\delta/2$-dense in $\text{Cl}(\phi(H)x_0)$. Notice that any gap of $\phi(S_3)x_0$ either belongs to $V$ or is of length $< \delta$. Finally let $S = FS_3 \cup S_3$. We have the following property by the same argument as in Case 1.

1. For any point $x \in \phi(S_3)x_0$, and $f \in F$, we have $\psi_{c', S}(f)x = \phi(f)x$. 


Define a one-dimensional simplicial complex \( K \) as follows. The vertex set of \( K \) is \( V \). Two vertices \( J \) and \( J' \) are joined by an edge if there is \( f \in F \) such that \( \phi(f)J = J' \). Such \( f \) is unique since \( \phi(H) \) acts simply transitively on the gaps of \( \text{Cl}(\phi(H)x_0) \): \( \phi(g)J_0 = J_0 \) implies \( g = e \). We label the directed edge from \( J \) to \( J' \) by \( f \). For a directed edge \( J \xrightarrow{f} J' \) of \( K \), we have \( \psi_{e',S}(f)(J) = J' \) by \( (1) \). The simple transitivity on the gaps shows the following.

\( (2) \) For a directed cycle

\[
J_1 \xrightarrow{f_1} J_2 \xrightarrow{f_2} \cdots J_n \xrightarrow{f_n} J_{n+1} = J_1,
\]

we have \( f_n \cdots f_2 f_1 = e \).

Let us consider \( h \in \text{Homeo}_+(S^1) \) supported on the union of the closures of the intervals in \( V \) and leaving the end points of the intervals fixed. We claim that there is \( h \) such that for any directed edge \( J \xrightarrow{f} J' \) of \( K \), \( h\psi_{e',S}(f)h^{-1} = \phi(f) \) on \( J \). To show this, consider a spanning tree \( T_\nu \) of each component \( K_\nu \) of \( K \). Define \( h \) to be the identity on a prescribed base vertex \( J_1 \) of \( T_\nu \). For any directed edge \( J_1 \xrightarrow{f} J_2 \) of \( T_\nu \), define \( h \) on \( J_2 \) so that \( h\psi_{e',S}(f)h^{-1} = \phi(f) \) holds on \( J_1 \). We continue this process along directed paths in \( T_\nu \) issuing at \( J_1 \) until we define \( h \) on all the vertices of \( K_\nu \). Then for any directed edge \( J \xrightarrow{f} J' \) of \( T_\nu \), \( h\psi_{e',S}(f)h^{-1} = \phi(f) \) on \( J \) (even if the edge is directed toward the base vertex). Also for an edge of \( K_\nu \) not in \( T_\nu \), we have the same equality thanks to the relation \( (2) \). The proof of the claim is over.

Let us define a homomorphism \( \psi \) by \( \psi(g) = h\psi_{e',S}(g)h^{-1} \) for any \( g \in H \). The homomorphism \( \psi \) still satisfies \( (1) \) with \( \psi_{e',S} \) replaced by \( \psi \). Let \( J \) be any gap of \( \phi(S_3)x_0 = \psi(S_3)x_0 \) and \( f \) any element of \( F \). If either \( |J| < \delta \) or \( |\phi(f)J| < \delta \), then \( \psi(f) \) is \( \epsilon \)-near to \( \phi(f) \) on \( J \). If not, \( J \xrightarrow{f} \phi(f)J \) is an edge of \( K \), and \( \psi(f) = \phi(f) \) on \( J \). This shows \( \psi \in U(F, \epsilon, \phi) \), as is required.

\[\text{Proposition 12.5.} \text{ The space } \text{Hom}(G, \text{Homeo}_+(S^1)) \text{ is locally pathwise connected. In fact, any neighbourhood of any point contains a pathwise connected neighbourhood.}\]

\[\text{Proof.} \text{ The space in question is homeomorphic to } Q_2 \times Q_3, \text{ where}
\begin{align*}
Q_2 &= \{a \in \text{Homeo}_+(S^1) \mid a^2 = id\} \\
Q_3 &= \{b \in \text{Homeo}_+(S^1) \mid b^3 = id\}.
\end{align*}
\]

Choose a base point \( x_0 \in S^1 \). A map \( a \) in \( Q_2 \) is specified by a point \( a(x_0) \) and an orientation preserving homeomorphism from \( [x_0, a(x_0)] \) to \( [a(x_0), x_0] \). So \( Q_2 \) is homeomorphic to \( (0, 1) \times \text{Homeo}_+(0, 1) \). Likewise \( Q_3 \) is homeomorphic to \( (0, 1) \times \text{Homeo}_+(0, 1) \). It suffices to show that the space \( \text{Homeo}_+(0, 1) \) satisfies the claimed property. For any neighbourhood \( U \) of a point \( f \in \text{Homeo}_+(0, 1) \), choose \( \epsilon > 0 \) so that

\[
U(f, \epsilon) = \{g \in \text{Homeo}_+(0, 1) \mid \|g - f\| < \epsilon\}
\]

is contained in \( U \). Then for any \( g \in U(f, \epsilon) \), the path \( \{(1 - t)f + tg\}_{0 \leq t \leq 1} \) is contained in \( U(f, \epsilon) \).
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