HERMITIAN AND NON-HERMITIAN PERTURBATIONS OF
CHIRAL GAUSSIAN $\beta$-ENSEMBLES

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ABSTRACT. We compute the joint eigenvalue distribution for the rank one Hermitian and non-Hermitian perturbations of chiral Gaussian $\beta$-ensembles ($\beta > 0$) of random matrices.

1. Introduction

Let $X$ be an $m \times n$ matrix with entries being i.i.d. real ($\beta = 1$), complex ($\beta = 2$), or quaternionic ($\beta = 4$) centered normal random variables with $\mathbb{E}(|X_{11}|^2) = \beta$. Then we say that the $(m+n) \times (m+n)$ Hermitian matrix

$$H = \begin{pmatrix} 0_{m \times m} & X \\ X^* & 0_{n \times n} \end{pmatrix}.$$  

belongs to the chiral Gaussian orthogonal ($\beta = 1$), unitary ($\beta = 2$), symplectic ($\beta = 4$) random matrix ensemble (chGOE, chGUE, chGSE, respectively).

In this paper we find explicitly the joint eigenvalue distribution of rank one Hermitian and non-Hermitian perturbations of chiral ensembles:

$$\tilde{H} = \begin{pmatrix} \Gamma & X \\ X^* & 0_{n \times n} \end{pmatrix}.$$  

Here $\Gamma$ is an $m \times m$ matrix with rank $\Gamma = 1$ and either $\Gamma = \Gamma^*$ (Hermitian perturbation) or $\Gamma = -\Gamma^*$ (anti-Hermitian perturbation). The matrix $\Gamma$ can be either deterministic or random but independent from $X$. We will also allow arbitrary $\beta > 0$ different from $\beta = 1, 2, 4$ (see Section 2 for details).

The main results are Theorems 6.1 and 7.1 for Hermitian and non-Hermitian perturbations, respectively. We use methods developed in [25, 26, 27]. Namely, first, we develop sparse (Jacobi) matrix models for chiral ensembles and their perturbations in the spirit of Dumitriu–Edelman [12] (see Section 2). This allows us to use the theory of orthogonal polynomials and Jacobi matrices to compute a Jacobian of a certain change of variables (Section 5) which leads to the desired distribution (Sections 6 and 7).

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By multiplying matrices in (1.1) and (1.2) by $i$, and letting $Y = iX$, $\Lambda = i\Gamma$, we can equivalently work with the chiral Gaussian anti-Hermitian model
\[
\begin{pmatrix}
0_{m \times m} & Y \\
-Y^* & 0_{n \times n}
\end{pmatrix}
\]
and its Hermitian $\Lambda = \Lambda^*$ and anti-Hermitian $\Lambda = -\Lambda^*$ perturbations
\[
\begin{pmatrix}
\Lambda & Y \\
-Y^* & 0_{n \times n}
\end{pmatrix}.
\]
All the results in this paper can be trivially restated for this case: all the matrix models and eigenvalues simply get a factor of $i$. The benefit of this would be that the characteristic polynomial of (1.3) in the case $\Lambda = \Lambda^*$ has real coefficients (instead of alternating between purely imaginary and purely real as in Section 7), so its zeros belong to \(\{z : \text{Re}z < 0\}\) and are symmetric with respect to $\mathbb{R}$.

Chiral random matrix theory has been an important instrument in quantum chromodynamics (QCD), going back to works [31, 35, 38], see [1, 9, 36, 37] for overviews, lecture notes, and further references.

There is a vast literature on low rank non-Hermitian perturbations of Hermitian random matrices, owing to its physical applications in quantum chaotic scattering. For an overview, physical applications, and references, we refer readers to the papers [16, 19, 20, 28]. The exact eigenvalue distribution of low rank non-Hermitian perturbations of Gaussian and Laguerre $\beta$-ensembles was the topic of [18, 26, 27, 32, 33, 34] in particular.

The low rank non-Hermitian perturbations of chiral ensembles that we study here do not seem to have been studied in the literature before. A different type of non-Hermitian perturbations (of full rank) have been studied recently in [24].

Literature that studies Hermitian perturbations of Gaussian and Laguerre random matrix ensembles is huge.

The additive model $H + \Gamma$ for perturbations of Gaussian random matrices $H$ bears the name Gaussian with an external source or shifted mean Gaussian ensemble, see [5, 8, 29, 39, 40] among many others.

The usual model for perturbations of Laguerre ensembles is $(I + \Gamma)^{1/2}X^*X(I + \Gamma)^{1/2}$ with $\Gamma = \Gamma^*$ of low rank. This is typically referred to as the spiked Wishart ensembles, see, e.g., [3, 4, 10, 23]. Clearly this corresponds to perturbation $X \mapsto X(I + \Gamma)^{1/2}$ and $X^* \mapsto (I + \Gamma)^{1/2}X^*$ in the chiral model (1.1).

Another type of perturbation of Laguerre/Wishart ensembles actively studied in the literature is $(X + \Gamma)^*(X + \Gamma)$. This corresponds to $X \mapsto X + \Gamma$ and $X^* \mapsto (X + \Gamma)^*$ in (1.1) which bear the name chiral Gaussian ensembles with a source, see e.g. [11, 15, 17, 30] and [14, Sect 11.2.2].

We stress that eigenvalues of our Hermitian perturbed model (1.2), however, do not correspond to a change of variables applied to eigenvalues of a simple perturbation of the Laguerre random matrix.

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2. Jacobi matrix models

2.1. Jacobification: case \( m \leq n \). As was shown by Dumitriu–Edelman [12], \( X \) can be bidiagonalized in the following sense: there are \( m \times m \) and \( n \times n \) unitary matrices \( L \) and \( R \) such that

\[
B := LXR = \begin{pmatrix}
x_1 & & & \\
y_1 & x_2 & & \\
& y_2 & \ddots & \\
& & \ddots & \ddots \\
& & & y_{m-1} & x_m
\end{pmatrix} \quad (2.1)
\]

with

\[
Le_1 = L^*e_1 = e_1,
\]

where \( e_1 \) is the vector with 1 in the first entry and 0 in all others. Here the \( x_j \)'s and \( y_j \)'s are independent random variables with the distributions

\[
x_j \sim \chi_{\beta(n-j+1)}, \quad 1 \leq j \leq m,
\]

\[
y_j \sim \chi_{\beta(m-j)}, \quad 1 \leq j \leq m-1,
\]

where \( \chi_{\alpha} \) stands for the chi-distributed random variable with parameter \( \alpha > 0 \) given by the p.d.f.

\[
\frac{1}{2^{\alpha/2-1}\Gamma(\alpha/2)}x^{\alpha-1}e^{-x^2/2} \text{ for } x > 0.
\]

Trivially, (2.1) implies \( B^* := R^*X^*L^* \) which means that our chiral matrix \( H \) from (1.1) can be unitarily reduced to

\[
\begin{pmatrix}
L & 0_{m \times n} \\
0_{n \times m} & R^*
\end{pmatrix}
\begin{pmatrix}
L^* & 0_{m \times n} \\
0_{n \times m} & R
\end{pmatrix}
= \begin{pmatrix}
0_{m \times m} & B \\
B^* & 0_{n \times n}
\end{pmatrix}
\]

(2.5)

The right-hand side is a sparse matrix with independent entries. However we want a Jacobi (tridiagonal) form in order to employ theory of orthogonal polynomials. To this end, we introduce the \((m+n) \times (m+n)\) permutation matrix \( P \) corresponding to the permutation

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & \cdots & 2m-1 & 2m \\
1 & m+1 & 2 & m+2 & \cdots & m & 2m+1 & \cdots & m+n
\end{pmatrix}.
\]

(2.6)

This produces

\[
P\begin{pmatrix}
0_{m \times m} & B \\
B^* & 0_{n \times n}
\end{pmatrix}P^* = \begin{pmatrix}
0 & x_1 & & & & \\
x_1 & 0 & y_1 & & & \\
y_1 & 0 & x_2 & & & \\
& \ddots & \ddots & \ddots & \\
& & y_{m-1} & 0 & x_m \\
& & & & & y_m
\end{pmatrix} \quad (2.7)
\]

\[
\begin{pmatrix}
0_{(n-m) \times 2m} & 0_{2m \times (n-m)}
\end{pmatrix} =: J.
\]

Observe also that

\[
P e_1 = P^* e_1 = e_1, \quad PI_{1 \times 1}P^* = I_{1 \times 1},
\]

(2.8)
where \( I_{1 \times 1} \) is the diagonal matrix with 1 in \((1, 1)\)-entry and 0 everywhere else. We will use these properties later in the text.

This ensemble already appeared earlier in [22], see also [13].

2.2. Jacobification: case \( m \geq n + 1 \). Arguments of Dumitriu–Edelman work for the case \( m \geq n + 1 \) with the following modifications: (2.1) becomes

\[
B := LXR = \begin{pmatrix}
x_1 & x_2 & & & \\
y_1 & y_2 & & & \\
 & & \ddots & & \\
 & & & y_{n-1} & x_n \\
0_{m-n-1,n} & & & y_n
\end{pmatrix};
\]

(2.9)

distributions of \( x_j \)'s and \( y_j \)'s are now

\[
x_j \sim \chi_{\beta(n-j+1)}, \quad 1 \leq j \leq n,
\]

(2.10)
\[
y_j \sim \chi_{\beta(m-j)}, \quad 1 \leq j \leq n;
\]

(2.11)
equation (2.5) remains unchanged; the permutation matrix \( P \) in (2.6) is now

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n \\
1 & m+1 & 2 & m+2 & \cdots & n & n+m \\
2n+1 & \cdots & m+n
\end{pmatrix};
\]

(2.12)
finally, (2.7) becomes

\[
P \begin{pmatrix}
0_{m \times m} & B \\
B^* & 0_{n \times n}
\end{pmatrix} P^* = \begin{pmatrix}
0 & x_1 & & & \\
x_1 & 0 & y_1 & & \\
 & & \ddots & & \\
 & & & y_{n-1} & x_n \\
0 & & & y_n & 0
\end{pmatrix}
\begin{pmatrix}
0_{(2n+1) \times (m-n-1)} \\
0_{(m-n-1) \times (2n+1)} & 0_{(m-n-1) \times (m-n-1)}
\end{pmatrix} =: J.
\]

(2.13)

2.3. Chiral Gaussian \( \beta \)-ensembles. In the previous two subsections we have obtained that \( H \) from (1.1) is unitarily equivalent to the Jacobi matrix \( J \) in (2.7) with (2.3)–(2.4).

It will occasionally be convenient to have a notation for the same Jacobi matrix but without the last zero block. So let us introduce the matrix \( J \) which is obtained by removing the last \( n - m \) of zero rows and columns of \( J \) in (2.7) \((m \leq n)\) or the last \( m - n + 1 \) of zero rows and columns in (2.13) \((m \geq n + 1)\). We obtain the \( N \times N \)
Jacobi matrix

\[ J := \begin{pmatrix}
0 & a_1 & a_2 & \cdots & a_{N-1} \\
a_1 & 0 & a_2 & \cdots & a_{N-1} \\
a_2 & 0 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{N-1} & 0 & \cdots & \cdots & 0
\end{pmatrix}, \quad (2.14) \]

where \( a_{2j-1} = x_j, a_{2j} = y_j \) and either \( N = 2m, \) \( (2.3)-(2.4) \) \((m \leq n)\) or \( N = 2n+1, \) \( (2.10)-(2.11) \) \((m \geq n+1)\).

We will say that \( J \) belongs to the \textbf{chiral Gaussian} \( \beta \)-ensemble, \( \text{chG}_\beta \text{E} \) for short. This ensemble makes sense for arbitrary \( \beta > 0 \), not just \( \beta = 1, 2, 4 \).

2.4. \textbf{Rank one Hermitian perturbations}. Now we consider the perturbed model \( (1.2) \) with Hermitian \( \Gamma \). Since \( \Gamma \) has rank 1, we can choose \( \Gamma \) to be positive semi-definite. Let

\[ l = \|\Gamma\|_{HS} := \left( \sum_{j,k=1}^{m} |\Gamma_{jk}|^2 \right)^{1/2}. \quad (2.15) \]

be the Hilbert–Schmidt norm of the perturbation.

\textbf{Proposition 2.1.} Let \( \widetilde{H} \) be as in \((1.2)\). Assume that \( \Gamma = \Gamma^* \geq 0_{m \times m} \) has rank \( \Gamma = 1 \) and \( \|\Gamma\|_{HS} = l \). Further assume that \( \Gamma \) has real, complex, quaternionic entries for \( \beta = 1, 2, 4 \), respectively, that are either deterministic or random but independent from \( X \). Then \( \widetilde{H} \) is unitarily equivalent to

\[ J + lI_{1 \times 1}, \quad (2.16) \]

where \( J \) is \( (2.7) \) or \( (2.13) \).

\textbf{Remarks.} 1. We will consider \((2.16)\) for general \( \beta > 0 \) and view it as the rank one Hermitian perturbation of the chiral Gaussian \( \beta \)-ensemble from Subsection 2.3. In fact, we will remove the zero block and will be working with \( J + lI_{1 \times 1} \).

2. The trick in the proof with reducing rank one perturbation to \((1, 1)\)-entry which carries through to the Jacobi matrix model is well known: it has been used in \([25, 26, 27]\), and even earlier by Bloemendal–Virág \([6]\) in their study of spiked Laguerre ensembles.

\textbf{Proof.} \( \Gamma \) can be represented as \( \Gamma = U(1_{1 \times 1})U^* \) for some \( m \times m \) matrix \( U \) which is orthogonal, unitary, or unitary symplectic for \( \beta = 1, 2, 4 \), respectively.

Then the matrix \( \widetilde{H} \) (see \((1.2)\)) satisfies

\[ \begin{pmatrix}
U^* & 0_{m \times n} \\
0_{n \times m} & 1_{n \times n}
\end{pmatrix} \widetilde{H} \begin{pmatrix}
U & 0_{m \times n} \\
0_{n \times m} & 1_{n \times n}
\end{pmatrix} = \begin{pmatrix}
0_{m \times m} & U^*X \\
(U^*X)^* & 0_{n \times n}
\end{pmatrix} + lI_{1 \times 1}. \quad (2.17) \]

Here \( 1_{n \times n} \) stands for the \( n \times n \) identity matrix. Now, note that \( U \) is independent of \( X \), so the joint distribution of the elements of \( Y = U^*X \) is identical to the distribution
of $X$ by Gaussianity. Hence we can apply the arguments from Subsection 2.1 but to $Y$ instead of $X$ to arrive at

$$P \left( \begin{pmatrix} L & 0_{m \times n} \\ 0_{n \times m} & R^* \end{pmatrix} \left( \begin{pmatrix} 0_{m \times m} & Y \\ Y^* & 0_{n \times n} \end{pmatrix} + lI_{1 \times 1} \right) \begin{pmatrix} L^* & 0_{n \times m} \\ 0_{m \times n} & R \end{pmatrix} \right) P^* = P \begin{pmatrix} 0_{m \times m} & B \\ B^* & 0_{n \times n} \end{pmatrix} P^* + lP I_{1 \times 1} P^* = J + lI_{1 \times 1}, \quad (2.18)$$

where we have used (2.2) and (2.8). \hfill \Box

2.5. **Rank one non-Hermitian perturbations.** In the exact same way, we can consider the perturbed model (1.2) with anti-Hermitian $\Gamma$.

**Proposition 2.2.** Let $\tilde{\mathcal{H}}$ be as in (1.2). Assume $\Gamma = -\Gamma^*$, $-i\Gamma \geq 0_{m \times m}$, rank $\Gamma = 1$ and $||\Gamma||_{HS} = l$. Further assume that $\Gamma$ has real, complex, quaternionic entries for $\beta = 1, 2, 4$, respectively, that are either deterministic or random but independent from $X$. Then $\tilde{\mathcal{H}}$ is unitarily equivalent to

$$\tilde{\mathcal{H}} = J + il I_{1 \times 1}, \quad (2.19)$$

where $J$ is (2.7) or (2.13).

**Proof.** Notice that $-i\Gamma$ is Hermitian positive semi-definite and of rank one, so $-i\Gamma = U(lI_{1 \times 1})U^*$ for some $m \times m$ matrix $U$ which is orthogonal, unitary, or unitary symplectic for $\beta = 1, 2, 4$, respectively. The rest of the arguments go through without any changes. \hfill \Box

2.6. **Anti-bidiagonal models.** Matrix model $\mathcal{J} + lI_{1 \times 1}$ (as well as $\mathcal{J} + il I_{1 \times 1}$, of course) can be also represented in the so-called anti-bidiagonal form. To do so, we introduce another permutation matrix

$$Q = \begin{pmatrix} 1 & 2 & 3 & \cdots & N-2 & N-1 & N \\ N & N-2 & N-4 & \cdots & N-5 & N-3 & N-1 \end{pmatrix}.$$ 

Then

$$Q (\mathcal{J} + lI_{1 \times 1}) Q^* = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_{N-1} \\ 0 & 0 & \cdots & a_{N-3} & a_{N-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & a_{N-3} & \cdots & 0 & 0 \\ a_{N-1} & a_{N-2} & \cdots & 0 & 0 \end{pmatrix} \quad (2.20)$$

This matrix has two anti-diagonals with the perturbation term $l$ being now “in the middle” at the position $(\lfloor \frac{N}{2} \rfloor + 1, \lfloor \frac{N}{2} \rfloor + 1)$.

3. **Location of the eigenvalues**

In the next two statements, we find all the possible configurations of eigenvalues for our perturbed Jacobi ensembles (2.16), (2.19). Even more is true: every possible configuration of eigenvalues occurs exactly once.
Proposition 3.1. Let \( N > 1 \). Then there is a one-to-one correspondence between \( N \) points \( z_1, z_2, \ldots, z_N \) with \( z_1 > -z_2 > z_3 > \cdots > (-1)^{N-1}z_N \) and the matrices \( J + llI_{1\times 1} \) where \( J \) is of the form (2.14) and \( a_1, \ldots, a_{N-1}, l > 0 \).

Proof. This was shown by Holtz [21, Corollary 2] who classified eigenvalues of matrices (2.20).

Proposition 3.2. Let \( N > 1 \). Then there is a one-to-one correspondence between \( N \) points \( z_1, z_2, \ldots, z_N \) in \( \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Im} z > 0 \} \) (counting multiplicity) that are symmetric with respect to the imaginary axis and the matrices \( J + ilI_{1\times 1} \) where \( J \) is of the form (2.14) and \( a_1, \ldots, a_{N-1}, l > 0 \).

Proof. Let \( z_1, \ldots, z_N \) be \( N \) points in \( \mathbb{C}_+ \) that are symmetric with respect to the imaginary axis. By the results of Arlinski˘ı–Tsekanovski˘ı [2, Theorem 5.1, Corollary 6.5], there is a \( J \) of the form (2.14) and \( l > 0 \) such that \( z_1, \ldots, z_N \) are the eigenvalues of \( J + ilI_{1\times 1} \).

Conversely, let \( J \) be a Jacobi matrix as in (2.14) and \( l > 0 \). By [2, Prop 4.1] eigenvalues of \( J + ilI_{1\times 1} \) belong to \( \mathbb{C}_+ \). Since \( -(J + ilI_{1\times 1})^* = W(J + ilI_{1\times 1})W^* \), where \( W \) is the diagonal unitary matrix with diagonal \( \{1, -1, 1, -1, \ldots\} \), we obtain the symmetry of the eigenvalues with respect to the imaginary axis.

4. Spectral measures of chiral Gaussian \( \beta \)-ensembles

Given an \( k \times k \) Hermitian matrix \( H \), define its spectral measure with respect to \( e_1 \) to be the probability measure \( \mu \) satisfying

\[
\langle e_1, H^j e_1 \rangle = \int_{\mathbb{R}} x^j d\mu(x), \quad \text{for all } j \in \mathbb{Z}_{\geq 0}.
\]

We will refer to it as simply “the spectral measure” from now on. By diagonalizing \( H \) and assuming \( e_1 \) is cyclic, one can see that

\[
\mu = \sum_{j=1}^{k} w_j \delta_{\lambda_j}
\]

with \( \sum_{j=1}^{k} w_j = 1 \) and \( w_j > 0 \). Here \( \{\lambda_j\}_{j=1}^{k} \) are the eigenvalues of \( H \) (which are distinct by cyclicity), and \( w_j = |\langle v_j, e_1 \rangle|^2 \), where \( v_j \) is the corresponding eigenvector.

Now let us assume that \( H \) is from the chGOE, chGUE, or chGSE. As we show in Subsections 2.1 and 2.2, \( H \) and \( J \) are unitarily equivalent \( H = UJU^* \). Moreover, \( Ue_1 = U^* e_1 = e_1 \) implies that they have identical spectral measures. Finally, spectral measures of \( J \) and \( J \) coincide, which can be trivially seen from (4.1). In the next theorem, we compute this common spectral measure. The result works for any \( \beta > 0 \).

Theorem 4.1. For \( \beta > 0 \) let \( J \) belong to chG\( \beta \)E (see Subsection 2.3). Let \( a = |n - m| + 1 - 2/\beta \).
(i) If \( m \leq n \) (that is, \( J \) is \( 2m \times 2m \)), then with probability 1 the spectral measure of \( J \) is:

\[
\mu = \sum_{j=1}^{m} \frac{1}{2} w_j (\delta_{\lambda_j} + \delta_{-\lambda_j})
\]

with the joint distribution of \( \lambda_1, \ldots, \lambda_m, w_1, \ldots, w_{m-1} \) given by

\[
\frac{2^m}{h_{\beta,m,a}} \prod_{j=1}^{m} \lambda_{j}^{\beta a + 1} e^{-\lambda_{j}^{2}/2} \prod_{1 \leq j < k \leq m} |\lambda_{k}^{2} - \lambda_{j}^{2}|^{\beta} d\lambda_1 \ldots d\lambda_m
\]

\[
\times \frac{\Gamma(\beta m/2)}{\Gamma(\beta/2)^{m}} \prod_{j=1}^{m} w_{j}^{\beta/2-1} dw_1 \ldots dw_{m-1}.
\]

(ii) If \( m \geq n + 1 \) (that is, \( J \) is \( (2n + 1) \times (2n + 1) \)), then with probability 1 the spectral measure of \( J \) is:

\[
\mu = w_0 \delta_0 + \sum_{j=1}^{n} \frac{1}{2} w_j (\delta_{\lambda_j} + \delta_{-\lambda_j})
\]

with the joint distribution of \( \lambda_1, \ldots, \lambda_n, w_1, \ldots, w_n \) given by

\[
\frac{2^n}{h_{\beta,n,a}} \prod_{j=1}^{n} \lambda_{j}^{\beta a + 1} e^{-\lambda_{j}^{2}/2} \prod_{1 \leq j < k \leq n} |\lambda_{k}^{2} - \lambda_{j}^{2}|^{\beta} d\lambda_1 \ldots d\lambda_n
\]

\[
\times \frac{\Gamma(\beta m/2)}{\Gamma(\beta/2)^{n}} \Gamma(\beta(m - n)/2) w_0^{\beta(m-n)/2-1} \prod_{j=1}^{n} w_{j}^{\beta/2-1} dw_1 \ldots dw_n.
\]

Here the normalization constant is

\[
h_{\beta,s,a} = 2^{s(a \beta/2 + 1 + (s-1) \beta/2)} \prod_{j=1}^{s} \frac{\Gamma(1 + j \beta/2)\Gamma(1 + \beta a/2 + (j - 1) \beta/2)}{\Gamma(1 + \beta/2)}. \tag{4.8}
\]

Proof. Jacobi matrices \((2.14)\) with non-zero \( a_j \)'s have simple spectrum. From this and symmetry, we then get that for \( m \leq n \), \( J \) has \( m \) distinct positive eigenvalues \( \lambda_1, \ldots, \lambda_m \) and \( m \) distinct negative eigenvalues \( -\lambda_1, \ldots, -\lambda_m \), so the spectral measure of \( J \) has form \((4.2)\).

Similarly, if \( m \geq n + 1 \) then \( J \) has \( n \) distinct positive eigenvalues \( \lambda_1, \ldots, \lambda_n \), \( n \) distinct negative eigenvalues \( -\lambda_1, \ldots, -\lambda_n \) and a simple eigenvalue at \( \lambda_0 := 0 \). Consequently, the spectral measure of \( J \) has form \((4.5)\).

Notice that the matrix

\[
G = \begin{pmatrix} 0_{m \times m} & B \\ B^* & 0_{n \times n} \end{pmatrix}
\]

is unitarily equivalent to \( J \): see \((2.7)\) and \((2.13)\). Moreover, because of \((2.8)\), \( G \) has the same spectral measure as \( J, J \).
For $k \neq 0$, we can write a normalized eigenvector of $G$ corresponding to $\lambda_k$ in the form

\[
\begin{pmatrix}
u^{(k)} \\
u^{(k)}
\end{pmatrix}
\]

so that

\[
BB^* u^{(k)} = \lambda_k^2 u^{(k)},
\]

\[
B^* B v^{(k)} = \lambda_k^2 v^{(k)}
\]

are satisfied. Note that

\[
\begin{pmatrix}
u^{(k)} \\
-v^{(k)}
\end{pmatrix}
\]

is a normalized eigenvector of $G$ associated with $-\lambda_k$. By orthononormality of the eigenvectors we have

\[
\|u^{(k)}\|^2 + \|v^{(k)}\|^2 = 1,
\]

\[
\|u^{(k)}\|^2 - \|v^{(k)}\|^2 = 0,
\]

and thus $\|u^{(k)}\|^2 = 1/2$. Recall (4.2), (4.5) that denoted the eigenweight on $\lambda_k$ by $w_k$. Then $w_k = 2|\langle u^{(k)}, e_1 \rangle|^2$.

For $k > 0$, let

\[
\lambda'_k := \lambda_k^2
\]

and $w'_k$ be the eigenweight for $BB^*$ at $\lambda'_k$. Then $\sqrt{2}u^{(k)}$ is a normalized eigenvector for $BB^*$ corresponding to $\lambda'_k$. Thus,

\[
w'_k = 2|\langle u^{(k)}, e_1 \rangle|^2 = w_k.
\]

For the case (ii) we also have $w'_0 = 1 - \sum_{k=1}^\infty w'_k = 1 - \sum_{k=1}^\infty w_k = w_0$.

Finally, recall that the joint distribution of $\{\lambda'_k\}$ and $\{w'_k\}$ of $BB^*$ and of the $\beta$-Laguerre random matrix coincide ([12], [26, Lemma 4], [26, Proposition 1]). Using (4.16), (4.17) we can therefore write the joint distribution of the $\lambda_k$’s and $w_k$’s. □

5. Jacobians

We fix $l > 0$ and for $\mathcal{J}$ as in (2.14) let

\[
\mathcal{J}_l = \mathcal{J} + lI_{1 \times 1},
\]

\[
\mathcal{J}_{il} = \mathcal{J} + ilI_{1 \times 1}.
\]

In this section, we compute the Jacobian(s) of the change of variables from the spectral parameters (that is, $\lambda_j$’s and $w_j$’s) to the Maclaurin coefficients $\kappa_j$’s of the characteristic polynomial $\kappa(z)$ of $\mathcal{J}_l$ or $\mathcal{J}_{il}$.

**Theorem 5.1.** Let $l > 0$. 

(i) Let $\mathcal{J}$ be a $2m \times 2m$ Jacobi matrix of the form \((2.14)\) with $a_1, \ldots, a_{2m-1} > 0$ and $m > 0$. Denote $\mu$ to be its spectral measure \((4.2)\). Let $\det(z - \mathcal{J}_l) = \sum_{j=0}^{2m} \kappa_j z^j$. (5.3) Then

$$
\left| \frac{\partial (\kappa_0, \ldots, \kappa_{2m-2})}{\partial (\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_{m-1})} \right| = 2^{m^2-1} \prod_{j=1}^{m} \lambda_j \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|^2. \tag{5.4}
$$

(ii) Let $\mathcal{J}$ be a $(2n+1) \times (2n+1)$ Jacobi matrix of the form \((2.14)\) with $a_1, \ldots, a_{2n} > 0$ and $n > 0$. Denote $\mu$ to be its spectral measure \((4.5)\). Let $\det(z - \mathcal{J}_l) = \sum_{j=0}^{2n+1} \kappa_j z^j$. (5.5) Then

$$
\left| \frac{\partial (\kappa_0, \ldots, \kappa_{2n-1})}{\partial (\lambda_1, \ldots, \lambda_n, w_1, \ldots, w_n)} \right| = 2^{n^2} \prod_{j=1}^{n} \lambda_j^3 \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|^2. \tag{5.6}
$$

Proof. (i) Note that $\kappa_{2m} = 1$ and $\kappa_{2m-1} = -l$ are fixed constants here. Let $m(z) = \langle e_1, (\mathcal{J} - z)^{-1} e_1 \rangle$. Then

$$
m(z) = \sum_{j=1}^{m} \frac{w_j}{2} \left( \frac{1}{\lambda_j - z} + \frac{1}{-\lambda_j - z} \right) = z \sum_{j=1}^{m} \frac{w_j}{\lambda_j^2 - z^2}. \tag{5.7}
$$

First, we observe that

$$
\sum_{j=0}^{2m} \kappa_j z^j = \det(z - \mathcal{J}_l) \tag{5.8}
$$

$$
= \det(z - \mathcal{J}) \det(I - (\mathcal{J} - z)^{-1} l I_{1 \times 1}) \tag{5.9}
$$

$$
= (1 + l m(z)) \prod_{j=1}^{m} (z^2 - \lambda_j^2) \tag{5.10}
$$

and

$$
l m(z) \prod_{j=1}^{m} (z^2 - \lambda_j^2) = -l z \sum_{j=1}^{m} w_j \prod_{1 \leq k \leq m \atop k \neq j} (z^2 - \lambda_k^2). \tag{5.11}
$$

Let

$$
c_j = \kappa_{2j}, \quad j = 0, \ldots, m, \tag{5.12}
$$

$$
d_j = \kappa_{2j+1}, \quad j = 0, \ldots, m-1, \tag{5.13}
$$

where $c_m = 1$, $d_{m-1} = -l$. 


Letting $u = z^2$ and $\lambda'_j = \lambda_j^2$ we get from (5.10) and (5.11) that

$$\sum_{j=0}^{m} c_j u^j = \prod_{j=1}^{m} (u - \lambda'_j),$$

(5.14)

$$\sum_{j=0}^{m-1} d_j u^j = -l \sum_{j=1}^{m} w_j \prod_{1 \leq k \leq m} (u - \lambda'_k).$$

(5.15)

From (5.14) we get

$$\left| \det \frac{\partial (c_0, \ldots, c_{m-1})}{\partial (\lambda'_1, \ldots, \lambda'_m)} \right| = \prod_{1 \leq j < k \leq m} |\lambda'_j - \lambda'_k| = \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|.$$  

(5.16)

Since

$$\left| \det \frac{\partial (\lambda'_1, \ldots, \lambda'_m)}{\partial (\lambda_1, \ldots, \lambda_m)} \right| = 2^m \prod_{j=1}^{m} \lambda_j,$$  

(5.17)

(5.13) yields

$$\left| \det \frac{\partial (c_0, \ldots, c_{m-1})}{\partial (\lambda_1, \ldots, \lambda_m)} \right| = 2^m \prod_{j=1}^{m} \lambda_j \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|.$$  

(5.18)

By (5.14),

$$\frac{\partial (c_0, \ldots, c_{m-1})}{\partial (w_1, \ldots, w_{m-1})} = (0_{m \times (m-1)}).$$  

(5.19)

Now we consider (5.15). In view of [26, eq.(5.9), eq.(5.14)], (5.15) implies that

$$\left| \det \frac{\partial (d_0, \ldots, d_{m-2})}{\partial (w_1, \ldots, w_{m-1})} \right| = l^{m-1} \prod_{1 \leq j < k \leq m} |\lambda'_j - \lambda'_k|$$

(5.20)

$$= l^{m-1} \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|.$$  

(5.21)

Combining (5.18), (5.19), (5.21) we get

$$\left| \det \frac{\partial (\kappa_0, \ldots, \kappa_{m-2})}{\partial (\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_{m-1})} \right| = \left| \det \frac{\partial (c_0, \ldots, c_{m-1}, d_0, \ldots, d_{m-2})}{\partial (\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_{m-1})} \right|$$

(5.22)

$$= 2^m l^{m-1} \prod_{j=1}^{m} \lambda_j \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|^2.$$  

(5.23)

(ii) Note that $\kappa_{2n+1} = 1$ and $\kappa_{2n} = -l$ are constants. We again start with $m(z) = \langle e_1, (\mathcal{J} - z)^{-1} e_1 \rangle$ which becomes

$$m(z) = \sum_{j=1}^{n} \frac{w_j}{2} \left( \frac{1}{\lambda_j - z} + \frac{1}{-\lambda_j - z} \right) - w_0 = z \sum_{j=1}^{n} \frac{w_j}{\lambda_j^2 - z^2} - \frac{w_0}{z}.$$  

(5.24)
Now,
\[ \sum_{j=0}^{2n+1} \kappa_j z^j = \det(z - \mathcal{J}i) \] (5.25)
\[ = \det(z - \mathcal{J}) \det(I - (z - \mathcal{J})^{-1}I_{1 \times 1}) \] (5.26)
\[ = (1 + \text{Im}(z)) \prod_{j=1}^{n} (z^2 - \lambda_j^2) \] (5.27)

and
\[ l \text{Im}(z) \prod_{j=1}^{n} (z^2 - \lambda_j^2) = -l \sum_{j=0}^{n} w_j \prod_{0 \leq k \leq n, k \neq j} (z^2 - \lambda_k^2) \] (5.28)

Define
\[ c_j = \kappa_{2j+1}, \quad j = 0, \ldots, n, \] (5.29)
\[ d_j = \kappa_{2j}, \quad j = 0, \ldots, n, \] (5.30)

with \( c_n = 1, \) \( d_n = -l. \) Taking \( u = z^2 \) and \( \lambda_j' = \lambda_j^2 \) we get from (5.27) and (5.28) that
\[ \sum_{j=0}^{n} c_j u^j = \prod_{j=1}^{n} (u - \lambda_j'), \] (5.31)
\[ \sum_{j=0}^{n} d_j u^j = -l \sum_{j=0}^{n} w_j \prod_{0 \leq k \leq n, k \neq j} (u - \lambda_k'). \] (5.32)

Using (5.31) we get
\[ \left| \det \frac{\partial (c_0, \ldots, c_{n-1})}{\partial (\lambda_1, \ldots, \lambda_n)} \right| = 2^n \prod_{j=1}^{n} \lambda_j \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|. \] (5.33)

Using (5.32),
\[ \left| \det \frac{\partial (d_0, \ldots, d_{n-1})}{\partial (w_1, \ldots, w_n)} \right| = l^n \prod_{0 \leq j < k \leq n} |\lambda_j' - \lambda_k'| \] (5.34)
\[ = l^n \prod_{j=1}^{n} \lambda_j^2 \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|. \] (5.35)
Combining (5.33), (5.35) we get

\[
\left| \det \frac{\partial (\kappa_0, \ldots, \kappa_{2n-1})}{\partial (\lambda_1, \ldots, \lambda_n, w_1, \ldots, w_n)} \right| = \left| \det \frac{\partial (c_0, \ldots, c_{n-1}, d_0, \ldots, d_{n-1})}{\partial (\lambda_1, \ldots, \lambda_n, w_1, \ldots, w_n)} \right| = 2^n l^n \prod_{j=1}^n \lambda_j^3 \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|^2. \quad (5.36)
\]

\[
= 2^n l^n \prod_{j=1}^n \lambda_j^3 \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|^2. \quad (5.37)
\]

Notice that in the case (5.1) coefficients of \( \kappa \) were real, while in the case (5.2) they are real or purely imaginary. Indeed, for a monic polynomial \( \kappa(z) = \sum_{j=0}^k \kappa_j z^j \) of degree \( k \) whose zeros are symmetric with respect to imaginary axis,

\[
Q(z) = i^k \kappa(z/i)
\]
is a monic polynomial with real coefficients. This means \( \kappa(z) = Q(iz)i^{-k} \), and therefore \( \text{Im} \kappa_{k-2} = \text{Im} \kappa_{k-4} = \cdots = 0 \) and \( \text{Re} \kappa_{k-1} = \text{Re} \kappa_{k-3} = \cdots = 0 \).

**Theorem 5.2.** Let \( l > 0 \).

(i) Let \( J \) be a \( 2m \times 2m \) Jacobi matrix of the form (2.14) with \( a_1, \ldots, a_{2m-1} > 0 \) and \( m > 0 \). Denote \( \mu \) to be its spectral measure (4.2). Let

\[
\det(z - J_{il}) = \sum_{j=0}^{2m} \kappa_j z^j. \quad (5.38)
\]

Then

\[
\left| \det \frac{\partial (\text{Re} \kappa_0, \text{Im} \kappa_1, \ldots, \text{Re} \kappa_{2m-4}, \text{Im} \kappa_{2m-3}, \text{Re} \kappa_{2m-2})}{\partial (\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_{m-1})} \right| = 2^n l^{m-1} \prod_{j=1}^m \lambda_j \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|^2. \quad (5.39)
\]

(ii) Let \( J \) be a \( (2n+1) \times (2n+1) \) Jacobi matrix of the form (2.14) with \( a_1, \ldots, a_{2n} > 0 \) and \( n > 0 \). Denote \( \mu \) to be its spectral measure (4.5). Let

\[
\sum_{j=0}^{2n+1} \kappa_j z^j = \det(z - J_{il}). \quad (5.40)
\]

Then

\[
\left| \det \frac{\partial (\text{Im} \kappa_0, \text{Re} \kappa_1, \ldots, \text{Im} \kappa_{2n-2}, \text{Re} \kappa_{2n-1})}{\partial (\lambda_1, \ldots, \lambda_n, w_1, \ldots, w_n)} \right| = 2^n l^n \prod_{j=1}^n \lambda_j^3 \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|^2. \quad (5.41)
\]

**Proof.** The only difference from the setting in the previous theorem is that \( l \) gets an extra factor of \( i \), and the same happens with the coefficients \( \kappa_{j-1} \)'s in (i) or \( \kappa_{j} \)'s in (ii). The modulus of the Jacobian in (5.39) and (5.41) is therefore the same as in (5.4) and (5.6), respectively. \( \square \)
6. Eigenvalues for rank one Hermitian perturbations

**Theorem 6.1.** Let $\mathcal{J}$ belong to $\text{chG}\beta E$ (see Section 2.3), $l > 0$, $a = |n-m| + 1 - 2/\beta$, and $\mathcal{J}_l := \mathcal{J} + lI_{1\times 1}$. (6.1)

(i) Let $m \leq n$. The eigenvalues of $\mathcal{J}_l$ are distributed on

\[ \left\{ (z_j)_{j=1}^{2m} : \sum_{j=1}^{2m} z_j = l, \; z_1 > -z_2 > z_3 > \cdots > z_{2m-1} > -z_{2m} > 0 \right\} \] (6.2)

according to

\[ \frac{1}{Z_{\beta,m,a}} l^{1-\frac{ma}{2}} e^{l^2/4} \prod_{j=1}^{2m} |z_j|^{2} e^{-z_j^2/4} \prod_{1 \leq j < k \leq 2m} |z_j - z_k|^{2m-2} \prod_{j=1}^{2m-1} dz_j. \] (6.3)

Here

\[ Z_{\beta,m,a} = \frac{2^{m(\beta-2)/2} h_{\beta,m,a} \Gamma(\beta/2)^m}{m! \Gamma(\beta m/2)}. \] (6.4)

(ii) Let $m \geq n + 1$. The eigenvalues of $\mathcal{J}_l$ are distributed on

\[ \left\{ (z_j)_{j=1}^{2n+1} : \sum_{j=1}^{2n+1} z_j = l, \; z_1 > -z_2 > z_3 > \cdots > -z_{2n} > z_{2n+1} > 0 \right\} \] (6.5)

according to

\[ \frac{1}{W_{\beta,m,n,a}} l^{1-\frac{ma}{2}} e^{l^2/4} \prod_{j=1}^{2n+1} |z_j|^{2} e^{-z_j^2/4} \prod_{1 \leq j < k \leq 2n+1} |z_j - z_k|^{2n+1} \prod_{j=1}^{2n} dz_j. \] (6.6)

Here

\[ W_{\beta,m,n,a} = \frac{2^{(2n+1)(\beta-2)/4} h_{\beta,n,a} \Gamma(\beta/2)^n \Gamma(\beta(m-n)/2)}{n! \Gamma(\beta m/2)}. \] (6.7)

**Remarks.** 1. As a corollary, eigenvalues of Hermitian perturbations of $\text{chGOE}$, $\text{chGUE}$, $\text{chGSE}$ (see Proposition 2.1) are (6.3) together with $z = 0$ of algebraic multiplicity $n - m$ (for the case $m \leq n$), and (6.6) together with $z = 0$ of algebraic multiplicity $m - n - 1$ (for the case $m \geq n + 1$).

2. See the end of this section for the case when $l$ is not deterministic but random.

**Proof.** (i) Let $\sum_{j=0}^{2m} \kappa_j z^j = \det(z - \mathcal{J}_l)$. Then
\begin{align*}
\prod_{1 \leq j < k \leq 2m} |z_j - z_k| &= \left| \det \frac{\partial (\kappa_0, \ldots, \kappa_{2m-1})}{\partial (z_1, \ldots, z_{2m})} \right| \quad (6.8) \\
&= \left| \det \frac{\partial (\kappa_0, \ldots, \kappa_{2m-1})}{\partial (z_1, \ldots, z_{2m-1}, \kappa_{2m-1})} \right| \quad (6.9) \\
&= \left| \det \frac{\partial (\kappa_0, \ldots, \kappa_{2m-2})}{\partial (z_1, \ldots, z_{2m-1})} \right| \quad (6.10)
\end{align*}

The equality (6.8) is well known, (6.9) is a result of \( \sum_{j=1}^{2m} z_j = -\kappa_{2m-1} \) and (6.10) follows by removing the last row and column from the determinant (6.9).

Combining part (i) of Theorem 4.1, (6.10) and (5.4) we get the density of \( dz_1 \cdots dz_{2m-1} \):

\begin{align*}
l^{1-m} \prod_{1 \leq j < k \leq 2m} |z_j - z_k| \\
m! h_{\beta, m, a} \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2| \\
\prod_{j=1}^{m} \lambda_j^{\alpha} e^{-\lambda_j^2/2} \prod_{1 \leq j < k \leq m} |\lambda_k^2 - \lambda_j^2|^\beta \\
\times \Gamma(\beta m/2) \prod_{j=1}^{m} \frac{w_j^{\beta/2-1}}{\Gamma(\beta/2)}. \quad (6.11)
\end{align*}

Notice the extra factor of \( m! \) that comes from the fact that \( \lambda_j \)'s were not ordered while \( z_j \)'s are.

It follows from (5.8), (5.14) that

\begin{align*}
\sum_{j=1}^{m} \lambda_j^2 &= -c_{m-1} = -\kappa_{2m-2} = - \sum_{1 \leq i < j \leq 2m} z_i z_j. \quad (6.12)
\end{align*}

Since \( \sum_{j=1}^{2m} z_j = l \), we have

\begin{align*}
l^2 = \sum_{j=1}^{2m} z_j^2 + 2 \sum_{1 \leq i < j \leq 2m} z_i z_j \\
= \sum_{j=1}^{2m} z_j^2 - 2 \sum_{j=1}^{m} \lambda_j^2. \quad (6.13)
\end{align*}

Thus

\begin{align*}
\sum_{j=1}^{m} \lambda_j^2 &= \frac{-l^2 + \sum_{j=1}^{2m} z_j^2}{2}. \quad (6.15)
\end{align*}

It follows from (5.8), (5.14) that

\begin{align*}
\prod_{j=1}^{m} \lambda_j^2 &= |c_0| = |\kappa_0| = \prod_{j=1}^{2m} |z_j|. \quad (6.16)
\end{align*}
By (5.10), we have
\[
\frac{w_j}{2} = |\text{Res}_{z=\lambda_j} m(z)| = \left| \text{Res}_{z=\lambda_j} \prod_{k=1}^{2m} (z - z_k) \right| = \left| \prod_{k=1}^{2m} (\lambda_j - z_k) \right| \over 2l \lambda_j \prod_{1 \leq k \leq m, k \neq j} (\lambda_k^2 - \lambda_j^2) .
\]
(6.17)

Similarly,
\[
\frac{w_j}{2} = |\text{Res}_{z=-\lambda_j} m(z)| = \left| \text{Res}_{z=-\lambda_j} \prod_{k=1}^{2m} (z - z_k) \right| = \left| \prod_{k=1}^{2m} (\lambda_j + z_k) \right| \over 2l \lambda_j \prod_{1 \leq k \leq m, k \neq j} (\lambda_k^2 - \lambda_j^2) .
\]
(6.18)

By (5.14)
\[
\prod_{k=1}^{m} (z^2 - \lambda_k^2) = \sum_{j=0}^{m} \kappa_{2j} z^{2j} = \frac{1}{2} \prod_{k=1}^{2m} (z - z_k) + \frac{1}{2} \prod_{k=1}^{2m} (z + z_k)
\]
(6.19)
is satisfied.

Letting \( z = z_1, \ldots, z_{2m} \) in (6.19) yields
\[
\prod_{k=1, j=1, m}^{2m} |z_k^2 - \lambda_j^2| = \frac{1}{4^m} \prod_{k,j=1}^{2m} |z_j + z_k| .
\]
(6.20)

Combining (6.17), (6.18), and (6.20), and we get
\[
\prod_{j=1}^{m} w_j^2 = \frac{1}{2^{2m} 4^m} \prod_{k,j=1}^{2m} |z_j + z_k| \prod_{j=1}^{m} \lambda_j^2 \prod_{1 \leq k \leq m} |\lambda_k^2 - \lambda_j^2| .
\]
(6.21)

Substituting (6.21), (6.15), (6.16) into (6.11) we obtain (6.3).

\((ii)\) Let \( \sum_{j=0}^{2n+1} \kappa_j z^j = \det(z - J_l) \). By a similar argument as in (i), we see that
\[
\prod_{1 \leq j < k \leq 2n+1} |z_j - z_k| = \left| \det \frac{\partial (\kappa_0, \ldots, \kappa_{2n-1})}{\partial(z_1, \ldots, z_{2n})} \right|
\]
(6.22)

and
\[
\sum_{j=1}^{n} \lambda_j^2 = -l^2 + \sum_{j=1}^{2n+1} z_j^2 .
\]
(6.23)

Using part \((ii)\) in Theorem 4.1, (6.22) and (5.6), we find the distribution of the \( z_j \)'s:
Combining (6.29), (6.26), (6.27), we obtain

\[ \frac{n! 2^n}{l^n} \frac{\prod_{1 \leq j < k < 2n+1} |z_j - z_k|}{\prod_{j=1}^{n} \lambda_j^2 \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|^2} \frac{2^n \prod_{j=1}^{n} \lambda_j}{h_{\beta, n, a}} \prod_{j=1}^{n} \lambda_j^\beta e^{-\lambda_j^2/2} \]

\[ \times \prod_{1 \leq j < k \leq n} |\lambda_k^2 - \lambda_j^2|^{\beta} \times \frac{w_0^{\beta(m-n)/2-1}}{\Gamma(\beta(m-n)/2)} \times \Gamma(\beta m/2) \prod_{j=1}^{n} w_j^{\beta/2-1} \]

\[ \times dz_1 \ldots dz_{2n}. \]  

(6.24)

It follows from (5.27) that

\[ w_0 = |\text{Res}_{z=0} m(z)| = \frac{\prod_{k=1}^{2n+1} (z - z_k)}{l z \prod_{k=1}^{n} (z^2 - \lambda_k^2)} = \frac{\prod_{k=1}^{2n+1} z_k}{l \prod_{k=1}^{n} \lambda_k^2}. \]  

(6.25)

Similarly,

\[ \frac{w_j}{2} = |\text{Res}_{z=\lambda_j} m(z)| = \frac{\prod_{k=1}^{2n+1} (z - z_k)}{l z \prod_{k=1}^{n} (z^2 - \lambda_k^2)} = \frac{\prod_{k=1}^{2n+1} (\lambda_j - z_k)}{2l \lambda_j^2 \prod_{1 \leq k \leq n \setminus \{k\neq j\}} (\lambda_k^2 - \lambda_j^2)} \]

(6.26)

\[ = |\text{Res}_{z=-\lambda_j} m(z)| = \frac{\prod_{k=1}^{2n+1} (\lambda_j + z_k)}{2l \lambda_j^2 \prod_{1 \leq k \leq n \setminus \{k\neq j\}} (\lambda_k^2 - \lambda_j^2)}. \]  

(6.27)

By (5.27) and (5.28)

\[ z \prod_{k=1}^{n} (z^2 - \lambda_k^2) = \frac{1}{2} \prod_{k=1}^{2n+1} (z - z_k) + \frac{1}{2} \prod_{k=1}^{2n+1} (z + z_k). \]  

(6.28)

Letting \( z = z_1, \ldots, z_{2n+1} \) in (6.28) implies

\[ \prod_{k=1, \ldots, 2n+1 \setminus \{j\}}^{n} |z_k^2 - \lambda_j^2| = \prod_{j=1, \ldots, 2n+1}^{n} |z_j + z_k| \frac{2^{2n+1} \prod_{k=1}^{2n+1} |z_k|}{2^{2n+1} \prod_{k=1}^{2n+1} |z_k|}. \]  

(6.29)

Combining (6.29), (6.26), (6.27), we obtain

\[ \prod_{j=1}^{n} w_j^2 = \frac{\prod_{k, j=1}^{2n} |z_j + z_k|}{l^{2n+1} \prod_{j=1}^{2n+1} |z_j| \prod_{j=1}^{n} \lambda_j^4 \prod_{1 \leq j < k \leq n} |\lambda_k^2 - \lambda_j^2|^4}. \]  

(6.30)

Substituting (6.23), (6.25), (6.30) into (6.24), we get (6.6). 

□
It is natural to choose \( l \) to be random and independent of \( J \). For example, let \( l = \sqrt{2\chi_{\beta m/2}} \)-distributed, i.e., with probability distribution
\[
F(l) \, dl = \frac{1}{2^{\beta m/2 - 1}\Gamma(\beta m/4)} \frac{m^\beta}{\Gamma(\beta m/4)} l^{m\beta - 1} e^{-l^2/4} \, dl
\]
on \((0, \infty)\). Then making an extra change of variables from \( \{z_1, \ldots, z_k, l\} \) to \( \{z_1, \ldots, z_k\} \), we arrive at the following joint distribution of eigenvalues:

(i) If \( m \leq n \), then eigenvalues of \( J_l \) are distributed on
\[
\{(z_j)_{j=1}^{2m} : z_1 > -z_2 > z_3 > \cdots > z_{2m-1} > -z_{2m} > 0\}
\]
according to
\[
\frac{1}{Z_{\beta, m, a}} \prod_{j=1}^{2m} |z_j|^{2\beta a - 3/4} e^{-z_j^2/4} \prod_{1 \leq j < k \leq 2m} |z_j - z_k|^{2m} \prod_{j,k=1}^{2m} |z_j + z_k|^{\beta - 2} \prod_{j=1}^{2m} dz_j.
\]  
Here
\[
\tilde{Z}_{\beta, m, a} = \frac{2^{m\beta - m - 1} h_{\beta, m, a} \Gamma(\beta m/4)\Gamma(\beta/2)^m}{m!\Gamma(\beta m/2)}.
\]  
For \( \beta = 2 \) this takes an especially simple form
\[
\frac{1}{Z_{2, m, n-m}} \prod_{j=1}^{2m} |z_j|^{2n-m} e^{-z_j^2/4} \prod_{1 \leq j < k \leq 2m} |z_j - z_k|^{2n} \prod_{j=1}^{2n} dz_j.
\]  
At first sight one might expect that (6.34) has a Pfaffian structure but recall the configuration space is (6.31) which complicates analysis substantially.

(ii) If \( m \geq n + 1 \), then eigenvalues of \( J_l \) are distributed on
\[
\{(z_j)_{j=1}^{2n+1} : z_1 > -z_2 > z_3 > \cdots > -z_{2n} > z_{2n+1} > 0\}
\]
according to
\[
\frac{1}{W_{\beta, m, n, a}} \prod_{j=1}^{2n+1} |z_j|^{2\beta - 3/4} e^{-z_j^2/4} \prod_{1 \leq j < k \leq 2n+1} |z_j - z_k|^{2n+1} \prod_{j,k=1}^{2n+1} |z_j + z_k|^{\beta - 2} \prod_{j=1}^{2n+1} dz_j.
\]  
Here
\[
\tilde{W}_{\beta, m, n, a} = \frac{2^{(2n+1)(\beta-2)} h_{\beta, m, a} \Gamma(\beta m/4)\Gamma(\beta/2)^n \Gamma(\beta(m-n)/2)}{n!\Gamma(\beta m/2)}.
\]  
For \( \beta = 2 \) this becomes
\[
\frac{1}{W_{2, m, n, |m-n|}} \prod_{j=1}^{2n+1} |z_j|^{m-n-1} e^{-z_j^2/4} \prod_{1 \leq j < k \leq 2n+1} |z_j - z_k|^{2n+1} \prod_{j=1}^{2n+1} dz_j.
\]
7. Eigenvalues for rank one non-Hermitian perturbations

Let $\mathcal{J}$ be an $N \times N$ random matrix from chG$\beta$E, and consider

$$\mathcal{J}_l := \mathcal{J} + ilI_{1 \times 1}$$

for some $l > 0$.

In order to simplify the final answer we will assume $l$ to be random, independent from $\mathcal{J}$ (or $H$ for $\beta = 1, 2, 4$) with absolutely continuous distribution $F(l) \, dl$ with $F(l) > 0$ for $l > 0$ and 0 otherwise. Other distributions of $l$ (or the deterministic case) can also be treated in the exact same manner, and we leave it as an exercise to an interested reader.

As we discussed in Proposition 3.2, eigenvalues of (7.1) belong to $\mathbb{C}_+$, and they are symmetric with respect to the imaginary axis. The set of all possible configurations $\{z_j\}_{j=1}^N$ of these eigenvalues, therefore, decomposes as the disjoint union

$$X_N := \bigcup_{L \geq 0, M \geq 0} X_{L,M},$$

where

$$X_{L,M} := \left\{ \{z_j\}_{j=1}^N \in \mathbb{C}_+^N : z_1, \ldots, z_L \in i\mathbb{R}_+; \right. \left. z_{L+1} = -\bar{z}_{L+1}, \ldots, z_{L+M} = -\bar{z}_{L+2M} \right\}. \quad (7.2)$$

For each $z_j$, let $z_j = x_j + iy_j, x_j, y_j \in \mathbb{R}$.

We will say that $\{z_j\}_{j=1}^N$ on $X_N$ have joint distribution $f(z_1, \ldots, z_N) \left| \bigwedge_{j=1}^N z_j \right|$ (with $f$ being invariant under permutation of its arguments), if conditionally on the event $\{z_j\}_{j=1}^N \in X_{L,M}$ the distribution becomes

$$2^M \frac{1}{L!M!2^M} f(iy_1, \ldots, iy_L, \pm x_{L+1} + iy_{L+1}, \pm x_{L+2} - iy_{L+2}, \ldots, \pm x_{L+M} + iy_{L+M}) \times \prod_{j=1}^L dy_j \prod_{j=L+1}^{L+M} (dx_j dy_j). \quad (7.3)$$

Here the factor $\frac{1}{L!M!2^M}$ corresponds to the number of permutations on $X_{L,M}$ that preserve the configuration, and $2^M$ comes from $|dz \wedge d(-z)| = 2dx \, dy$.

For a more formal introduction to such point processes, we refer the reader to [7].

**Theorem 7.1.** Let $\mathcal{J}$ belong to chG$\beta$E (see Section 2.3), $a = |n - m| + 1 - 2/\beta$,

$$\mathcal{J}_l := \mathcal{J} + ilI_{1 \times 1}, \quad (7.4)$$

where $l$ is independent of $\mathcal{J}$ with distribution $F(l) \, dl$, $F(l) > 0$ for $l > 0$ and 0 otherwise.
(i) Let \( m \leq n \). Then \( \{z_j\}_{j=1}^{2m} \) are jointly distributed on \( X_{2m} \) according to

\[
\frac{1}{Z_{\beta,m,n}} F(l) l^{1-\frac{m\alpha}{2}} e^{-l^2/4} \prod_{j=1}^{2m} |z_j|^{2\beta} e^{-z_j^2/4} \times \prod_{1 \leq j < k \leq 2m} |z_j - z_k| \prod_{j,k=1}^{2m} |z_j - \bar{z}_k|^{\beta-2} \bigwedge_{j=1}^{2m} dz_j, \tag{7.5}
\]

where \( l = | \sum_{j=1}^{2m} z_j | \) and \( Z_{\beta,m,n} \) is \( (6.4) \).

(ii) Let \( m \geq n + 1 \). Then \( \{z_j\}_{j=1}^{2n+1} \) are jointly distributed on \( X_{2n+1} \) according to

\[
\frac{1}{W_{\beta,m,n,a}} F(l) l^{1-\frac{m\alpha}{2}} e^{-l^2/4} \prod_{j=1}^{2n+1} |z_j|^{2\beta} e^{-z_j^2/4} \times \prod_{1 \leq j < k \leq 2n+1} |z_j - z_k| \prod_{j,k=1}^{2n+1} |z_j - \bar{z}_k|^{\beta-2} \bigwedge_{j=1}^{2n+1} dz_j, \tag{7.6}
\]

where \( l = | \sum_{j=1}^{2n+1} z_j | \) and \( W_{\beta,m,n,a} \) is \( (6.7) \).

Remarks. 1. As a corollary, eigenvalues of non-Hermitian perturbations of chGOE, chGUE, chGSE (see Proposition 2.2) are \( (7.5) \) together with \( z = 0 \) of algebraic multiplicity \( n - m \) (for the case \( m \leq n \)), and \( (7.6) \) together with \( z = 0 \) of algebraic multiplicity \( m - n - 1 \) (for the case \( m \geq n + 1 \)).

2. Even though \( z_j \)'s are in \( \mathbb{C}_+ \), because of the symmetry \( \sum z_j^2 = \sum \text{Re}(z_j^2) \) is a real quantity.

**Proof.** (i) Recall the characteristic polynomial \( \kappa(z) \) in \( (3.38) \) and that

\[
Q(z) = i^N \kappa(z/i)
\]

is a monic polynomial with real coefficients and zeros at \( \{iz_j\}_{j=1}^{2m} \).

Let us assume that \( z_j \)'s belong to \( X_{L,M} \subset X_{2m} \). Using [25, Lemma 6.5] (if one applies it to \( Q \)), we get

\[
\left| \det \frac{\partial (\text{Re}\kappa_0, \text{Im}\kappa_1, \ldots, \text{Im}\kappa_{2m-3}, \text{Re}\kappa_{2m-2}, \text{Im}\kappa_{2m-1})}{\partial (y_1, \ldots, y_L, x_{L+1}, y_{L+1}, \ldots, x_{L+M}, y_{L+M})} \right| = 2^M \prod_{1 \leq j < k \leq 2m} |z_j - z_k|. \tag{7.7}
\]

Combining \( (7.7) \) with \( (5.39) \) and \( \kappa_{2m-1} = -il \) we obtain

\[
\left| \det \frac{\partial (\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_{m-1}, l)}{\partial (y_1, \ldots, y_L, x_{L+1}, y_{L+1}, \ldots, x_{L+M}, y_{L+M})} \right| = 2^{M-m} \prod_{1 \leq j < k \leq 2m} |z_j - z_k| \prod_{j=1}^{m-1} \lambda_j \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|^2. \tag{7.8}
\]

\[
\left| \det \frac{\partial (\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_{m-1}, l)}{\partial (y_1, \ldots, y_L, x_{L+1}, y_{L+1}, \ldots, x_{L+M}, y_{L+M})} \right| = 2^{M-m} \prod_{1 \leq j < k \leq 2m} |z_j - z_k| \prod_{j=1}^{m-1} \lambda_j \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|^2. \tag{7.9}
\]
Now we use this Jacobian together with Theorem 4.1(i) we obtain the joint density of $x_j$’s and $y_j$’s:

$$
\begin{align*}
\frac{m!}{2^M M! L!} h_{\beta,m,a} &\frac{2^M}{\Gamma(\beta/2)} \prod_{1 \leq j < k \leq 2m} |z_j - z_k| \prod_{j=1}^m \lambda_j^\beta e^{-\lambda_j^2/2} \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|^{\beta-2} \\
&\times \Gamma(\beta m/2) \prod_{j=1}^m \frac{w_j}{\Gamma(\beta/2)} l^{1-m} F(l) \times \prod_{j=1}^L dy_j \prod_{j=L+1}^{L+M} (dx_j dy_j).
\end{align*}
$$

(7.10)

Notice the extra factor of $\frac{1}{2^M M! L!}$ since we do not impose ordering on our $z_j$’s so each configuration appears $2^M M! L!$ times. Similarly, $m!$ comes from the absence of ordering in $\lambda_j$’s.

Using (5.10),

$$
\sum_{j=0}^{2m} \kappa_j z^j = (1 + il m(z)) \prod_{j=1}^m (z^2 - \lambda_j^2).
$$

(7.11)

Here

$$
\begin{align*}
\kappa_{2m} &= 1, \\
\text{Im}\kappa_{2j} &= 0, \quad j = 0, \ldots, m-1, \\
\text{Re}\kappa_{2j+1} &= 0, \quad j = 0, \ldots, m-1,
\end{align*}
$$

(7.12)-(7.14)

and

$$
\sum_{j=0}^{2m} \text{Re}\kappa_j z^j = \prod_{j=1}^m (z^2 - \lambda_j^2).
$$

(7.15)

It follows that

$$
\prod_{j=1}^m \lambda_j^2 = |\text{Re}\kappa_0| = |\kappa_0| = \prod_{j=1}^{2m} |z_j|.
$$

(7.16)

and

$$
\sum_{j=1}^{m} \lambda_j^2 = -\kappa_{2m-2} = - \sum_{1 \leq i < j \leq 2m} z_i z_j.
$$

(7.17)

Since $\sum_{j=1}^{2m} z_j = \text{Tr}(J_{il}) = il$, we have

$$
\begin{align*}
-l^2 &= \sum_{j=1}^{2m} z_j^2 + 2 \sum_{1 \leq i < j \leq 2m} z_i z_j, \\
&= \sum_{j=1}^{2m} z_j^2 + 2 \sum_{j=1}^{m} \lambda_j^2.
\end{align*}
$$

(7.18)-(7.19)
Thus
\[ \sum_{j=1}^{m} \lambda_j^2 = \frac{t^2 + \sum_{j=1}^{2m} \lambda_j^2}{2}. \] (7.20)

Using (7.11), (7.15), we obtain
\[ \frac{1}{2} \prod_{j=1}^{2m} (z - j) + \frac{1}{2} \prod_{j=1}^{2m} (z - \bar{j}) = \prod_{j=1}^{m} (z^2 - \lambda_j^2). \] (7.21)

Letting \( z = z_1, \ldots, z_{2m} \) in (7.21), we get
\[ \prod_{k=1}^{m} \prod_{j=1}^{2m} \left| z_k^2 - \lambda_j^2 \right| = \frac{1}{4^m} \prod_{k,j=1}^{2m} \left| z_j - \bar{z}_k \right|. \] (7.22)

By (7.11),
\[ \frac{w_j}{2} = \left| \text{Res}_{z=\lambda_j} m(z) \right| = \left| \text{Res}_{z=\lambda_j} \prod_{k=1}^{2m} (z - z_k) \prod_{k=1}^{2m} (z^2 - \lambda_k^2) \right| = \left| \prod_{k=1}^{2m} (\lambda_j - \bar{z}_k) \right| \left( 2l \lambda_j \prod_{1 \leq k < m, k \neq j} (\lambda_k^2 - \lambda_j^2) \right). \] (7.23)

Equality (7.24) yield
\[ \frac{1}{4^m} \prod_{j=1}^{m} \frac{w_j}{2} = \frac{1}{(2l)^{2m} 4^m} \prod_{j,k=1}^{2m} \left| z_j - \bar{z}_k \right| \prod_{1 \leq k \leq m} \lambda_k^2 \prod_{1 \leq k < m, k \neq j} \left| \lambda_k^2 - \lambda_j^2 \right|^4. \] (7.25)

Substituting (7.16), (7.20), (7.25) into (7.10) we get (7.5).

(ii) We follow similar line of reasoning as in (i). Suppose \( z_j \)'s belong to \( X_L, M \subset X_{2n+1} \). By [25, Lemma 6.5]
\[ \left| \text{det} \frac{\partial (\text{Im} \kappa_0, \text{Re} \kappa_1, \ldots, \text{Im} \kappa_{2n-2}, \text{Re} \kappa_{2n-1}, \text{Im} \kappa_{2n})}{\partial (y_1, \ldots, y_L, x_L+1, y_{L+1}, \ldots, x_{L+M}, y_{L+M})} \right| = 2^M \prod_{1 \leq j < k \leq 2n+1} \left| z_j - z_k \right|. \] (7.26)

and then from (5.41) we get
\[ \left| \text{det} \frac{\partial (\lambda_1, \ldots, \lambda_n, w_1, \ldots, w_n, l)}{\partial (y_1, \ldots, y_L, x_L+1, y_{L+1}, \ldots, x_{L+M}, y_{L+M})} \right| = 2^{M-n} \prod_{1 \leq j < k \leq 2n+1} \left| z_j - z_k \right|. \] (7.27)
Combining part \((ii)\) of Theorem \([1,1]\) and \((7.26)\) we obtain the joint density of \(z_j\)'s:

$$
\frac{n!}{2^M M! L! L^n \prod_{j=1}^{n} \lambda_j} \frac{2^{M-n} \prod_{1 \leq j < k \leq 2n+1} |z_j - z_k|}{\prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^2} \frac{2^n \prod_{j=1}^{n} \lambda_j}{h_{\beta, n, a}} \prod_{j=1}^{n} \lambda_j^{\beta a} e^{-\lambda_j^2/2} \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|^\beta \\
\times \frac{w_0^{\beta(m-n)/2-1}}{\Gamma(\beta(m-n)/2)} \times \Gamma(\beta m/2) \prod_{j=1}^{n} \frac{w_j^{\beta/2-1}}{\Gamma(\beta/2)} \times \prod_{j=1}^{L} (\prod_{j=L+1}^{L+M} (dx_jdy_j)).
$$

(7.29)

Substituting

$$
\sum_{j=1}^{n} \lambda_j^2 = \frac{1}{2} \left( l^2 + \sum_{j=1}^{2n+1} z_j^2 \right),
$$

(7.30)

$$
w_0 = \frac{\prod_{j=1}^{2n+1} |z_j|}{l \prod_{j=1}^{n} |\lambda_j|^2},
$$

(7.31)

$$
\prod_{j=1}^{n} w_j^2 = \frac{\prod_{j=k=1}^{2n+1} |z_j - z_k|}{l^{2n+1} \prod_{j=1}^{2n+1} \prod_{j=1}^{n} \lambda_j^4 \prod_{1 \leq j < k \leq n} |\lambda_j^2 - \lambda_k^2|^{4}},
$$

(7.32)

into \((7.29)\) we get \((7.30)\).

\[\square\]

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