Note on 4-Coloring 6-Regular Triangulations on the Torus

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Abstract. Altshuler (Discrete Math 4(3):201–217, 1973) characterized the 6-regular triangulations on the torus to be precisely those that are obtained from a regular triangulation of the $r \times s$ toroidal grid where the vertices in the first and last column are connected by a shift of $t$ vertices. Such a graph is denoted $T(r, s, t)$. Collins and Hutchinson (Graph colouring and applications. CRM proceedings and lecture notes, vol 23. American Mathematical Society, Providence, pp 21–34, 1999) classified the 4-colorable graphs $T(r, s, t)$ with $r, s \geq 3$. In this paper, we point out a gap in their classification and show how it can be fixed. Combined with the classification of the 4-colorable graphs $T(1, s, t)$ by Yeh and Zhu (Discrete Math 273(1–3):261–274, 2003), this completes the characterization of the colorability of all the 6-regular triangulations on the torus.

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1. Introduction

A classical result due to Heawood [8] states that the chromatic number $\chi(G)$ of any graph $G$ embeddable on an orientable surface of genus $g > 0$ is bounded above by the so-called Heawood number $H(g) := \lfloor (7 + \sqrt{1 + 48g})/2 \rfloor$. Heawood also showed that this upper bound is tight for $g = 1$ by exhibiting an embedding of $K_7$ in the torus. In fact, $K_7$ embeds in the torus as a triangulation, that is, an embedding in which each face is homeomorphic to a disc and is bounded by three edges. It follows from Euler’s formula that every regular triangulation on the torus (of which $K_7$ is one example) is necessarily 6-regular, and that every 6-regular graph embeddable on the torus necessarily embeds as a triangulation.

Altshuler [3] classified the 6-regular triangulations of the torus as follows (we follow the notation in [4]). For integers $r \geq 1$, $s \geq 1$ and $0 \leq t \leq s − 1$,
take $V = \{(i,j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ to be the vertex set of the graph $T(r,s,t)$ equipped with the following edges:

- For each $1 < i < r$, $(i,j)$ is adjacent to $(i,j+1)$, $(i+1,j)$ and $(i+1,j-1)$.
- If $r > 1$, $(1,j)$ is adjacent to $(1,j+1)$, $(2,j)$, $(2,j-1)$, $(r,j+t+1)$ and $(r,j+t)$.
- If $r > 1$, $(r,j)$ is adjacent to $(r,j+1)$, $(r-1,j+1)$, $(r-1,j)$, $(1,j-t)$ and $(1,j-t-1)$.
- If $r = 1$, $(1,j)$ is adjacent to $(1,j+1)$, $(1,j+t)$ and $(1,j+(t+1))$.

Here, addition in the first coordinate is taken modulo $r$ and in the second coordinate is taken modulo $s$. Figure 1 depicts the graph $G = T(5,6,2)$; note that the edges between the top and bottom rows are not shown.

It is clear that each $T(r,s,t)$ is a 6-regular triangulation of the torus. Altshuler’s theorem says that these are all the 6-regular triangulations on the torus up to isomorphism (similar constructions also appear in [9,10]).

**Theorem 1.** (Altshuler [3]) Every 6-regular triangulation on the torus is isomorphic to $T(r,s,t)$ for some integers $r \geq 1$, $s \geq 1$, and $0 \leq t < s$.

**Remark 1.** As shown by Altshuler in [2,3], through every vertex $v$ of $T(r,s,t)$ there are three so-called normal circuits, which are the simple cycles obtained by traversing through $v$ along each of the three directions—vertical, horizontal, and diagonal—in the natural fashion. These normal circuits have lengths $s$, \( \frac{n}{\gcd(s,t)} \), and \( \frac{n}{\gcd(s, r+t)} \), respectively, where $n = rs$ is the order of $T(r,s,t)$.

By picking a different normal circuit to be represented as the vertical cycle, one can see that there exist $0 \leq t_1 < \frac{n}{\gcd(s,t)}$ and $0 \leq t_2 < \frac{n}{\gcd(s, r+t)}$ such that $T(r,s,t)$ is isomorphic to $T(\gcd(s,t), \frac{n}{\gcd(s,t)}, t_1)$ as well as to $T(\gcd(s, r+t), \frac{n}{\gcd(s, r+t)}, t_2)$. By swapping the horizontal and diagonal normal circuits, one can see that $T(r,s,t)$ is isomorphic to $T(r,s,t')$ for $0 \leq t' < s$ such that $t' \equiv -r - t \pmod{s}$.

| (1,6) | (2,6) | (3,6) | (4,6) | (5,6) |
| (1,5) |
| (1,4) |
| (1,3) |
| (1,2) |
| (1,1) |
| (1,1) |
| (2,1) | (3,1) | (4,1) | (5,1) |
| (1,6) |
| (1,5) |
| (1,4) |
| (1,3) |
| (1,2) |
| (1,1) |
| (1,1) |

**Figure 1.** $G = T(5,6,2)$
Now, Dirac’s map color theorem [7] states that the only connected graph
$G$ with $\chi(G) = H(g)$ that is embeddable on a surface of genus $g > 0$ is
$K_{H(g)}$. So, $K_7$ is the only 7-chromatic 6-regular triangulation on the torus.
Albertson and Hutchinson [1] showed that there is a unique 6-chromatic 6-
regular simple triangulation on the torus, which has 11 vertices. Then, Collins
and Hutchinson [6] gave a characterization of the 4-colorable triangulations
$T(r, s, t)$ with $r, s \geq 3$ as follows:

**Theorem 2.** (Collins–Hutchinson [6, Theorem 1.2]) Let $G = T(r, s, t)$. If $r, s \geq 3$, then $G$ can be 4-colored, with a finite number of exceptions.

Note that a 6-regular right-diagonal (unshifted) $m \times n$ grid in the notation
of Collins–Hutchinson is what we call $T(n, m, 0)$, and that a 6-regular right-
diagonal $(m \times n; k)$ grid for $k > 1$ in their notation is what we call $T(n, m, m – k + 1)$. In particular, the $(m \times n; 1)$ grid is the same as the unshifted $m \times n$
grid, which is $T(n, m, 0)$ in our notation.

In this paper, we point out a gap in the proof of Theorem 2 that makes
the statement incorrect, and we provide a patch to the statement and proof.
In Sect. 2, we locate the error in the proof of Theorem 2, and provide explicit
counterexamples to its statement. In Sect. 3, we prove the following modification
of the above theorem:

**Theorem 3.** Let $G = T(r, s, t)$ be a simple 6-regular triangulation having nor-
mal circuits of lengths $a \geq b \geq c$. Suppose that $(\frac{n}{a}, \frac{n}{b}) \neq (1, 1), (1, 2)$, where
$n = rs$ is the order of $G$. Then $G$ can be 4-colored.

Combined with the earlier results [1,6–8] as well as the classification
of the 4-colorable triangulations $T(1, s, t)$ by Yeh–Zhu [11], we complete the
characterization of the colorability of all the 6-regular triangulations on the
torus in Theorem 8 in Sect. 4.

2. Examining the Statement of Theorem 2 and Its Proof

2.1. Constructing Counterexamples to Theorem 2

Collins and Hutchinson identify that $T(3, 3, 1)$, $T(3, 3, 2)$, $T(3, 5, 3)$, and
$T(3, 5, 4)$ are not 4-colorable, but state that there are no others of the form
$T(3, s, t)$ for $s \geq 3$ [6, Theorem 3.7]. However, as we show below, the graphs
of the form $T(3, s, s – 2)$ and $T(3, s, s – 1)$ are not 4-colorable for all $s \neq 0$
(mod 4). Note that the four graphs mentioned in the beginning are obtained
by plugging in $s = 3, 5$ in these expressions.

Now, consider the triangulations $T(1, s, 2)$ for $s \geq 7$. These are simple
graphs, and, as noted in [6, Section 3], every four successive vertices of $T(1, s, 2)$
duce a $K_4$. Thus, $T(1, s, 2)$ is 4-colorable for $s \geq 7$ if and only if $s \equiv 0$
(mod 4).

Therefore, we consider the graphs $T(1, 3s, 2)$ for $s \geq 3$ and $s \neq 0$ (mod 4).
These are all 5-chromatic graphs by [1,7,8]. The normal circuits in $T(1, 3s, 2)$
have lengths $3s$, $3s$, and $s$, respectively, so $T(1, 3s, 2)$ is isomorphic to $T(3, s, t)$
for some $0 \leq t \leq s – 1$. Since there are infinitely many $s \geq 3$ such that $s \neq 0$
(mod 4), there are infinitely many graphs of the form \(T(3, s, t)\) that are not 4-colorable, contradicting the statement of Theorem 2.

In fact, one can check by a careful computation that \(t = s - 2, s - 1\) in this case. For simplicity, we label the vertex \((1, j)\) with the integer \(j\) (recall that \(j\) is taken modulo \(3s\)). Map the vertical, horizontal, and diagonal normal circuits of \(T(1, 3s, 2)\) to the horizontal, diagonal, and vertical normal circuits of \(T(3, s, 2)\), respectively. Then, the vertical normal circuit of \(T(3, s, t)\) has labels \(3s, 3s - 3, 3s - 6, 3s - 9, \ldots\) from top to bottom when drawn as in Fig. 1. The horizontal normal circuit through the vertex with label \(3s\) has the first four labels in the right-to-left direction as \(3s, 3s - 1, 3s - 2,\) and \(3s - 3\). Thus, the shift is \(t = s - 1\). Since \(T(r, s, t)\) is isomorphic to \(T(r, s, t')\) for \(0 \leq t' < s\) such that \(t' \equiv -r - t \pmod{s}\), the graph \(T(3, s, s - 1)\) is isomorphic to \(T(3, s, s - 2)\). Thus, the graphs \(T(3, s, s - 1)\) and \(T(3, s, s - 2)\) are not 4-colorable for all \(s \geq 3\) such that \(s \not\equiv 0 \pmod{4}\).

### 2.2. Gap in the Proof of Theorem 2

The proof of Theorem 2 in [6] is broken up into a sequence of results, first for the unshifted triangulations \(T(r, s, 0)\) [6, Lemma 3.2, Theorem 3.3, Lemma 3.4], and then for the shifted triangulations \(T(r, s, t)\) with \(t \neq 0\) [6, Theorems 3.6 and 3.7]. We identify the following theorem as the source of the contradiction:

**Theorem 4.** (Collins–Hutchinson [6, Theorem 3.6]) Let \(G = T(r, s, t)\) for some \(0 < t \leq s - 1\). Then if \(3 \leq s, r, G\) can be 4-colored except possibly in the case when \(r = 5\), or when \(t = s - 1\) and \(r = s\) or \(s + 1\), or when \(t = s - 2\) and \(r = s\).

The proof of this theorem proceeds as follows. Let \(C_i\) denote the \(i\)th column of \(T(r, s, t)\), for \(i = 1, \ldots, r\). First, a proper 4-coloring of \(T(y, s, 0)\) is used to color \(C_1, \ldots, C_y\), where \(y \geq 3\) is to be determined. (Note that \(y \geq 3\) is needed to ensure that a proper 4-coloring of \(T(y, s, 0)\) can be found using [6, Theorem 3.3].) Then, the coloring on \(C_1\) is repeated on \(C_{y+1}\). Then, the coloring on \(C_{y+1}\) is repeated on \(C_{y+2}\) after an upward shift by one vertex. As shown in [6, Lemma 3.1], this coloring on \(C_{y+2}\) is compatible with the coloring on \(C_{y+1}\) as long as \(r, s \neq 5\). Repeat this process of repeating the coloring on successive columns with an upward shift to color the \(t\) columns \(C_{y+2}, \ldots, C_{y+t+1}\). Now, note that the coloring on \(C_{y+t+1}\) is identical to the coloring on \(C_1\), except that it is shifted upwards by \(t\) vertices. Thus, this gives a valid coloring of \(T(r, s, t)\) provided that \(r = y + t\).

At this point, Collins and Hutchinson state that the inequality \(y \geq 3\) fails only when \(t = s - 1\) and \(r = s\) or \(s + 1\), or when \(t = s - 2\) and \(r = s\), so this concludes their proof. However, this conclusion can only be drawn under the additional hypothesis that \(r \geq s\). Thus, their proof holds only under the additional hypothesis that \(r \geq s\).

Hence, the statements of Theorem 4 and [6, Theorem 3.7] need to be modified by adding the hypothesis that \(r \geq s\). However, the statement of [6, Theorem 3.8] is now weakened, since the colorability of the triangulations \(T(2, s, t)\) (for odd \(s\)) that are not isomorphic to \(T(1, 2s, t')\) for any \(0 \leq t' < 2s\) is no longer completely settled by their previous results.
Furthermore, the above method of proof does not seem to easily lend itself to the cases when \( r < s \). The above argument does extend to the graphs \( T(r, s, t) \), \( 3 \leq r, s, r \neq 5, s \neq 5, \) with \( r \geq t + 3 \) or \( r > s - \lceil t/2 \rceil \), the latter by extending the coloring on \( C_y \) using downward shifts by two vertices instead of upward shifts by one vertex, but it is not clear, for instance, how one can 4-color the graph \( T(10, 990, 100) \) by an argument along the above lines.

In the next section, we provide a different argument to color the shifted triangulations.

3. Main Result

We shall use Collins and Hutchinson’s coloring of the unshifted triangulations [6, Theorem 3.3] to color the shifted triangulations.

**Theorem 5.** Let \( G = T(r, s, t) \) be a simple triangulation with \( r \neq 1 \) and \( s \) as the maximal length of a normal circuit in \( G \). Then \( G \) is 4-colorable.

**Proof.** Note that the conditions on \( r \) and \( s \) imply that \( \gcd(r, s), \gcd(s, r + t) \geq r \geq 2 \).

Suppose that \( r \geq 3 \). Now, use a proper coloring of \( T(r, \gcd(s, t), 0) \) to color the first \( \gcd(s, t) \)-many rows of \( T(r, s, t) \), and repeat this coloring block \( (s/\gcd(s, t)) \)-many times to get a coloring of \( T(r, s, t) \). For this coloring to be proper, the coloring on the column \( C_r \) is required to be compatible with an upward shift by \( t \) vertices of the coloring on the column \( C_1 \). But, since the coloring on the column \( C_1 \) is periodic with period \( \gcd(s, t) \), an upward shift by \( t \) vertices of the coloring on \( C_1 \) is the same as no shift. Thus, we only need to check that the coloring on \( C_r \) is compatible with that on \( C_1 \), and this holds since it is obtained from a periodic coloring of the unshifted triangulation \( T(r, \gcd(s, t), 0) \).

Next, suppose that \( r = 2 \). As observed in [6, Theorem 3.8], if \( s \) is even, then \( G \) is 4-colorable, simply by 2-coloring the columns \( C_1 \) and \( C_2 \) with the colors \( \{1, 2\} \) and \( \{3, 4\} \), respectively.

So, suppose that \( r = 2 \) and that \( s \) is odd, which imply that \( \gcd(s, t), \gcd(s, t + 2) \geq 3 \). Thus, \( G \) is isomorphic to \( T(r', s', t') \) for \( r' = \gcd(s, t), s' = 2s/\gcd(s, t), \) and some \( 0 \leq t' < s' \) such that \( \gcd(s', t') = \gcd(s, t + 2) \geq 3 \). This is possible by Remark 1. Thus, we can repeat the previous algorithm to color \( T(r', s', t') \) as follows. First, use a proper coloring of \( T(r', \gcd(s', t'), 0) \) to color the first \( \gcd(s', t') \)-many rows of \( T(r', s', t') \). Then, repeat this coloring block \( (s'/\gcd(s', t')) \)-many times to get a coloring of \( T(r', s', t') \). This is verified to be a proper coloring for the same reason as in the case \( r \geq 3 \), so this completes the case \( r = 2 \) as well.

In fact, the proof in the case \( r = 2 \) and \( s \) odd shows that the following theorem is also true.

**Theorem 6.** Let \( G = (V, E) \) be a simple 6-regular triangulation on the torus with normal circuits of lengths \( a \geq b \geq c \) such that \( \frac{n}{c} \geq \frac{n}{b} \geq 3 \), where \( |V| = n \). Then, \( G \) is 4-colorable.
Specifically, a coloring of $G$ can be found by viewing $G$ as $T\left(\frac{n}{c}, c, t\right)$, where $0 \leq t < c$ is such that $\gcd(c, t) = \frac{a}{b}$, and then coloring $G$ by repeating $(a/\gcd(c, t))$-many times a proper coloring of $T\left(\frac{n}{c}, \frac{n}{b}, 0\right)$. This can be done since it is assumed that $\frac{n}{c} \geq \frac{n}{b} \geq 3$.

3.1. Proof of Theorem 3
Suppose that $G = T(r, s, t)$ is a simple 6-regular triangulation on the torus with normal circuits of lengths $a \geq b \geq c$ such that $(\frac{n}{a}, \frac{n}{b}) \neq (1, 1), (1, 2)$, where $n = rs$.

If $\frac{n}{a} = 1$, then $3 \leq \frac{n}{b} \leq \frac{n}{c}$, so $G$ is 4-colorable by Theorem 6.
If $\frac{n}{a} = 2$, then $G$ is isomorphic to $T(2, a, t)$ for some $0 \leq t < a$. If $\frac{n}{b} = 2$, then $a$ is even, since $b = n/\gcd(a, t)$ or $n/\gcd(a, t + 2)$. So $G$ is 4-colorable by Theorem 5. If $\frac{n}{b} \geq 3$, then $\frac{n}{c} \geq 3$, so again we are done by Theorem 6.
If $\frac{n}{a} \geq 3$, then $G$ is isomorphic to $T(r, a, t)$ for some $0 \leq t < a$, where $r \geq 3$. So, we are done by Theorem 5. This completes the proof of Theorem 3.

4. Summary of the Colorability of 6-Regular Toroidal Triangulations
In this section, we shall present a complete picture of the colorings of 6-regular triangulations on the torus.

By the results in Sect. 3, we are left to classify the colorability of those 6-regular triangulations $G$ that are either loopless multigraphs, or isomorphic to some simple $T(r, s, t)$ having normal circuits of lengths $a \geq b \geq c$ such that $(\frac{n}{a}, \frac{n}{b}) = (1, 1)$ or $(1, 2)$.

4.1. The Loopless Multigraphs of the Form $T(1, s, t)$
Note that the graphs $T(1, s, t)$ and $T(1, s, s-t-1)$ are isomorphic by Remark 1. So, when $r = 1$ we shall only focus on the values of $t$ in the range $0 \leq t \leq \lfloor (s-1)/2 \rfloor$.

Now, it is easy to check that $T(1, s, t)$ has loops if and only if either $s \leq 2$ or $t = 0$, and that $T(1, s, t)$ is a loopless multigraph if and only if $s \geq 3$ and $t = 1, \lfloor (s-1)/2 \rfloor$.

So, we start by considering the graph $T(1, s, 1)$ for $s \geq 3$. Collins and Hutchinson [6] gave explicit 4-colorings of $T(1, s, 1)$ for $s > 5$. Furthermore, Yeh and Zhu [11, Theorem 6] observed that $T(1, s, 1)$ is 3-chromatic if and only if $s \equiv 0 \pmod{3}$ (after deleting the duplicated edges in $T(1, s, 1)$, this graph is isomorphic to $G_s[1, 2]$ in their notation). Lastly, one can see that the graph $T(1, 5, 1)$ is isomorphic to $K_5$ after deleting the duplicated edges, so it is 5-chromatic.

Next, we consider the graph $T(1, s, \lfloor (s-1)/2 \rfloor)$ for $s \geq 5$. For $s = 2k+1$ ($k \geq 2$), Yeh and Zhu [11, Theorem 6] have shown that this graph is isomorphic to $T(1, s, 1)$ (after deleting the duplicated edges in $T(1, s, k)$ for $s = 2k+1$, this graph is isomorphic to the graph $G_s[k, 1]$ in their notation). Hence, $T(1, 2k+1, 1, k)$ is 4-colorable for all $k \geq 3$, and is 3-chromatic if and only if $s \equiv 0 \equiv k - 1$
(mod 3), and \( T(1, 5, 2) \) is 5-chromatic since it is isomorphic to \( K_5 \) after deleting the duplicated edges.

For \( s = 2k + 2 \ (k \geq 2) \), Yeh and Zhu [11, Theorem 5] have shown that \( T(1, s, [(s - 1)]/2) \) is 4-colorable (and in fact 4-chromatic) if and only if \( s \equiv 0 \) (mod 4) (this graph is isomorphic to \( G_s[1, k, k + 1] \) in their notation). In this case, by removing the duplicated edges we get a 5-regular graph on the torus. So, by Brooks’s theorem [5], when \( k \geq 4 \) is even the graph \( T(1, 2k + 2, k) \) is 5-chromatic, and when \( k = 2 \) the graph \( T(1, 6, 2) \) is isomorphic to \( K_6 \) after deleting the duplicated edges, and hence is 6-chromatic.

4.2. The Loopless Multigraphs of the Form \( T(2, s, t) \)

The graph \( T(2, s, t) \) has loops if and only if \( s = 1 \), so we assume that \( s \geq 2 \). One can check that \( T(2, s, t) \) is a loopless multigraph if and only if \( t = 0, s - 2, \) or \( s - 1 \). Furthermore, \( T(2, s, 0) \) and \( T(2, s, s - 2) \) are isomorphic by Remark 1, so there are only two cases to consider.

As observed in [6, Theorem 3.8], \( T(2, s, 0) \) is 4-colorable (and in fact 4-chromatic) if and only if \( s \geq 2 \) is even. When \( s \geq 3 \) is odd, \( T(2, s, 0) \) is isomorphic to \( T(1, 2s, [(s - 1)/2]) \), which was discussed earlier. Next, we look at \( T(2, s, s - 1) \). This graph is isomorphic to \( T(1, 2s, 1) \) for all \( s \geq 2 \), which we have discussed earlier. So, this completes the case \( r = 2 \).

4.3. The Loopless Multigraphs of the Form \( T(r, s, t) \) for \( r \geq 3 \)

The graph \( T(r, s, t) \) for \( r \geq 3 \) has loops if and only if \( s = 1 \), and it is a loopless multigraph if and only if \( s = 2 \). When \( t = 0 \), the graph \( T(r, 2, 0) \) is isomorphic to \( T(2, r, 0) \), which we have discussed earlier. When \( t = 1 \), the graph \( T(r, 2, 1) \) is isomorphic to \( T(1, 2r, [(r - 1)/2]) \), which was also discussed earlier.

Thus, the colorability of all the loopless multigraphs \( T(r, s, t) \) is known. Next, we need to consider the colorability of the simple graphs \( T(r, s, t) \). Theorem 3 covers the 4-colorability of those \( T(r, s, t) \) that have normal circuits of lengths \( a \geq b \geq c \) such that \( (\frac{a}{r}, \frac{b}{s}) \) \( \neq (1, 1), (1, 2) \), where \( n = rs \). So, we are only left to consider the remaining cases, namely when \( (\frac{a}{r}, \frac{b}{s}) = (1, 1) \) or \( (1, 2) \). As a step towards that, let us first consider the colorability of the simple graphs of the form \( T(1, s, t) \).

4.4. The Simple Graphs \( T(1, s, t) \)

From the previous discussions, it suffices to consider the graphs \( T(1, s, t) \) for those values of \( t \) in the range \( 2 \leq t \leq [(s - 1)/2] - 1 \). In particular, we assume that \( s \geq 7 \) in what follows.

Now, as shown in [6, Theorem 3.8] and discussed above in Sect. 2.1, the graphs \( T(1, s, 2) \) are simple triangulations that are 4-colorable (and in fact 4-chromatic) if and only if \( s \equiv 0 \) (mod 4) since every four consecutive vertices in \( T(1, s, 2) \) induce a \( K_4 \). Collins and Hutchinson [6] observe that these grids are all easily seen to be 5-chromatic when \( s \geq 15 \). Explicit 5-colorings for all \( s \geq 8, s \neq 11 \), in the spirit of Collins and Hutchinson’s work, can be given as follows: write \( s = 4u + 5v \) for \( u \geq 0 \) and \( v \in \{0, 1, 2, 3, 4\} \) (which can be done for all \( s \geq 8, s \neq 11 \)), and color \( T(1, s, 2) \) using \( u \) sets of 1234 followed by \( v \) sets of 12345. This is easily seen to be a proper coloring of \( T(1, s, 2) \).
When \( s = 11 \), the coloring \( 12345123456 \) is seen to work: this is the 6-chromatic graph on 11 vertices found by Albertson and Hutchinson [1], which is also the unique simple 6-regular triangulation on the torus having 11 vertices, up to isomorphism.

When \( s = 7 \), \( T(1, 7, 2) \) is 7-chromatic since it is isomorphic to \( K_7 \).

Next, for each \( t \geq 3 \), Collins and Hutchinson [6, Theorem 3.9] exhibited 4-colorings for all but finitely many of the graphs \( T(1, s, t) \) with \( s \) such that \( t \leq \lfloor (s - 1)/2 \rfloor - 1 \). The remaining cases were handled by Yeh and Zhu [11, Theorem 5]:

**Theorem 7.** (Yeh–Zhu [11, Theorem 5]) Let \( G = T(1, s, t) \) be a simple triangulation on the torus, for \( 3 \leq t \leq \lfloor (s - 1)/2 \rfloor - 1 \). Then \( G \) is 4-colorable, unless \( G \) satisfies one of the following conditions:

1. \( s \in \{2t + 3, 3t + 1, 3t + 2\} \) and \( s \not\equiv 0 \pmod{4} \); or
2. \( (s, t) \in \{(13, 3), (17, 3), (17, 4), (17, 6), (18, 3), (19, 7), (25, 3), (25, 6), (25, 7), (25, 9), (25, 10), (26, 7), (26, 10), (33, 6), (33, 14), (37, 10)\} \).

Yeh and Zhu have also shown that the graphs \( T(1, s, t) \) for \( s \in \{2t + 3, 3t + 1, 3t + 2\} \) are in fact isomorphic to \( T(1, s, 2) \). Note that \( T(1, s, t) \) is isomorphic to the graph \( G_s[1, t, t + 1] \) in their notation.

**4.5. The Simple Graphs** \( T(r, s, t) \) with \( (\frac{n}{a}, \frac{n}{b}) = (1, 1) \) or \( (1, 2) \)

Let \( G = (V, E) \) be a simple 6-regular triangulation on the torus with \( |V| = n \). Suppose that \( G \) has normal circuits of lengths \( a \geq b \geq c \) such that \( (\frac{n}{a}, \frac{n}{b}) = (1, 1) \) or \( (1, 2) \). Then, \( G \) can be represented as \( T(1, s, t) \), and by the discussion in Sect. 4.4 we know exactly what values \( t \) can take if \( G \) is 5-chromatic. Thus, to classify the 5-chromatic graphs \( G \) satisfying \( (\frac{n}{a}, \frac{n}{b}) = (1, 1) \) or \( (1, 2) \), it suffices to consider the 5-chromatic graphs of the form \( T(1, s, t) \) discussed in Sect. 4.4 and see whether and how they can be represented as \( T(r', s', t') \) with \( r' > 1 \).

First, consider the graphs \( T(1, s, 2) \) for \( s \not\equiv 0 \pmod{4}, s \geq 9, s \neq 11 \). Since its normal circuits have lengths \( s, s/\gcd(s, 2), \) and \( s/\gcd(s, 3) \), it can be be represented as \( T(r', s', t') \) with \( r' > 1 \) only if \( s \) is a multiple of 2 or 3. The chromaticity of the graphs \( T(1, 3s, 2) \) was discussed in Sect. 2, and a similar analysis can be done for the graphs \( T(1, 2s, 2) \) with \( s \not\equiv 0 \pmod{2} \) to show that \( T(2, s, 1) \) and \( T(2, s, s - 3) \) are 5-chromatic for all odd \( s \geq 5 \).

Next, we consider the exceptional graphs listed in Theorem 7. The graphs listed in the first point in Theorem 7 are already covered by the above analysis, since Yeh and Zhu have shown that the graphs \( T(1, s, t) \) for \( s \in \{2t + 3, 3t + 1, 3t + 2\}, s \not\equiv 0 \pmod{4}, t \geq 3, \) are all isomorphic to \( T(1, s, 2) \).

Thus, it only remains to consider the finitely many exceptional graphs listed in the second point in Theorem 7 that have composite order. A similar analysis can be done for these graphs as was done in Sect. 2 for \( T(1, 3s, 2) \). We omit the details and only state the final results in the next theorem. Just one observation needs to be added before we do so: it is easy to show that a simple graph \( T(r, s, t) \) is 3-chromatic if and only if \( s \equiv 0 \equiv r - t \pmod{3} \).
We conclude this paper with a compilation of the known results from the previous work of [1,6–8,11] as well as the present work, which characterizes the colorability of all the 6-regular toroidal triangulations. We follow the convention as adopted in [6,11] to specify the classification by the parameters \( r, s, \) and \( t \), instead of only listing isomorphism classes of graphs.

**Theorem 8.** Let \( G = T(r,s,t) \) for \( r \geq 1, s \geq 1, 0 \leq t \leq s − 1 \) be a 6-regular triangulation on the torus. If \( r = 1 \), then \( T(1,s,t) \) is isomorphic to \( T(1,s−t−1) \), so in this case consider \( t \) only in the range \( 0 \leq t \leq \lfloor (s−1)/2 \rfloor \).

1. \( G \) contains loops if and only if either \( s = 1 \), or \( r = 1 \) and \( s = 2 \), or \( r = 1 \) and \( t = 0 \).
2. \( G \) is 7-chromatic if and only if \( G \) is isomorphic to \( K_7 \), and this happens only when \( G = T(1,7,2) \).
3. \( G \) is 6-chromatic if and only if \( G \) is isomorphic either to \( K_6 \) (after deleting duplicated edges), or to the graph of Albertson and Hutchinson [1] on 11 vertices. The former happens only when \( G \in \{T(1,6,2), T(2,3,0), T(2,3,1), T(3,2,0), T(3,2,1)\} \) and the latter only when \( G \in \{T(1,11,2), T(1,11,3), T(1,11,4)\} \).
4. \( G \) is 5-chromatic if and only if \( G \) is one of the following graphs:
   (a) \( T(1,5,1), T(1,5,2) \) (these are isomorphic to \( K_5 \) after deleting duplicated edges);
   (b) \( T(1,2,s) \) for \( s \geq 9, s \not\equiv 11, s \not\equiv 0 \pmod{4} \);
   (c) \( T(1,s,t) \) for \( s \in \{2t+2, 2t+3, 3t+1, 3t+2\}, s \geq 9, s \not\equiv 0 \pmod{4} \);
   (d) \( T(2,s,0), T(2,s,1), T(2,s,s−3), T(2,s,s−2) \) for odd \( s \geq 5 \);
   (e) \( T(3,s,s−2), T(3,s,s−1) \) for \( s \geq 3, s \not\equiv 0 \pmod{4} \);
   (f) \( T(r,2,0), T(r,2,1) \) for odd \( r \geq 5 \);
   (g) \( T(1,s,t) \) for \( (s,t) \in \{(13,3), (17,3), (17,4), (17,6), (18,3), (19,7), (25,3), (25,6), (25,7), (25,9), (25,10), (26,7), (26,10), (33,6), (33,14), (37,10)\} \);
   (h) \( T(2,s,t) \) for \( (s,t) \in \{(9,3), (9,4), (13,3), (13,8)\} \);
   (i) \( T(3,s,t) \) for \( (s,t) \in \{(6,1), (6,2), (11,2), (11,6)\} \);
   (j) \( T(5,s,t) \) for \( (s,t) \in \{(5,2), (5,3)\} \).
5. \( G \) is 4-colorable in all other cases.
6. In particular, \( G \) is 3-chromatic if and only if \( s \equiv 0 \equiv r−t \pmod{3} \).

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