ATTRACTORS FOR FITZHUGH-NAGUMO LATTICE SYSTEMS WITH ALMOST PERIODIC NONLINEAR PARTS

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Abstract. For FitzHugh-Nagumo lattice dynamical systems (LDSs) many authors studied the existence of global attractors for deterministic systems [4, 34, 41, 43] and the existence of global random attractors for stochastic systems [23, 24, 27, 48, 49], where for non-autonomous cases, the nonlinear parts are considered of the form \( f(u) \). Here we study the existence of the uniform global attractor for a new family of non-autonomous FitzHugh-Nagumo LDSs with nonlinear parts of the form \( f(u, t) \), where we introduce a suitable Banach space of functions \( W \) and we assume that \( f \) is an element of the hull of an almost periodic function \( f_0(\cdot, t) \) with values in \( W \).

1. Introduction. Lattice dynamical systems (LDSs) are infinite systems of ordinary differential equations or of difference equations, indexed by points in a lattice such as the \( n \)-dimensional integer lattice \( \mathbb{Z}^n \). They occur in a wide variety of applications in science and engineering, for instance, in propagation of nerve pulses in myelinated axons [7, 8, 31, 32], electrical engineering [11], pattern recognition [15, 16, 37], image processing [17, 18, 19], chemical reaction theory [21, 30], etc. In each case, they have their own form, but in some cases, they appear as spatial discretizations of corresponding continuous partial differential equations (PDEs) on unbounded domains [12, 14].

The existence of global attractors for different types of LDSs have been studied, cf. [1, 2, 3, 4, 5, 6, 9, 10, 25, 26, 28, 29, 35, 39, 42, 44, 45, 46, 47, 50, 51, 52, 53, 55, 56]. In fact the global attractor is a natural object for studying the long time behavior of a dissipative dynamical system since it is the smallest compact set, with respect to inclusion, that is invariant, attracts all the trajectories originated from the whole phase space and sometimes it has finite dimension. For continuous PDEs on unbounded domains, it is not easy to introduce the global attractors because some difficulties appear such as well-posedness and lack of compactness of Sobolev embeddings. Therefore it is important to study the existence of global attractors for LDSs because of the importance of such systems and they can be regarded as spatial discretizations of such PDEs.

For FitzHugh-Nagumo autonomous and non-autonomous LDSs, many authors studied the existence of global attractors for deterministic systems [4, 34, 41, 43] and the existence of global random attractors for stochastic systems [23, 24, 27, 48, 49].
where for non-autonomous cases, the nonlinear parts are considered of the form \( f(u) \).

In [1], we studied the existence of the uniform global attractor for a family of first order non-autonomous LDSs with nonlinear part of the form \( f(u, t) \). Here the existence of the uniform global attractor for a new family of non-autonomous deterministic FitzHugh-Nagumo LDSs with nonlinear parts of the form \( f(u, t) \) is carefully studied, where a suitable Banach space is endowed with the inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). In fact we introduce the following FitzHugh-Nagumo system defined on the infinite lattices \( \mathbb{Z}^n \), such that for \( i \in \mathbb{Z}^n, t > \tau \), and \( \tau \in \mathbb{R} \),

\[
\begin{align*}
\dot{u}_i + (Au)_i + f_{1i}(u_i, t) + \alpha_1 v_i &= \xi_{1i}(t), \\
\dot{v}_i + f_{2i}(v_i, t) - \alpha_2 u_i &= \xi_{2i}(t),
\end{align*}
\]

with the initial conditions

\[
u_i(\tau) = u_{i, \tau}, \quad v_i(\tau) = v_{i, \tau}.
\]

The FitzHugh-Nagumo system appears in many applications in biology and circuit models. For instance it describes the signal transmission across axons in neurobiology [7, 8, 22, 31, 32, 38]. In particular, the lattice of the FitzHugh-Nagumo system models. For instance it describes the signal transmission across axons in neurobiology [7, 8, 22, 31, 32, 38]. In particular, the lattice of the FitzHugh-Nagumo system is used to stimulate the propagation of action potentials in myelinated nerve axons [20], where the membrane is excitable only at spatially discrete sites and in such a case \( u_i \) represents the potential at the \( i \)th active site and \( v_i \) is the recovery variable.

This work is organized as follows. In section 2, the assumptions on the linear and nonlinear parts of the family of non-autonomous LDSs (1)-(3) are presented and some introductory results are introduced. In section 3, the family of LDSs (1)-(3) is written in the abstract form (43)-(44) in the Hilbert space \( l^2 \times l^2 \), where the well-posedness of the system (43)-(44) is established, a family of processes associated with this system is defined, and the existence of a uniform absorbing set and the continuity for this family of process is verified. In section 4, the uniform estimates on the tails of solutions with respect to initial data from the uniform absorbing set and the translations of the time symbol are introduced, where such estimates are needed to obtain the asymptotic compactness of the solution semigroup, then by the semigroup theory, the uniform global attractor for the family of processes is presented.

2. Preliminaries. For \( n \in \mathbb{N} \), consider the Hilbert space

\[
l^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}^n} : i = (i_1, i_2, ..., i_n) \in \mathbb{Z}^n, u_i \in \mathbb{R}, \sum_{i \in \mathbb{Z}^n} u_i^2 < \infty \right\},
\]

endowed with the inner product \( \langle \cdot, \cdot \rangle_{l^2} \) and norm \( \| \cdot \|_{l^2} \) as follows:

\[
\langle u, v \rangle_{l^2} = \sum_{i \in \mathbb{Z}^n} u_i v_i, \quad \| u \|_{l^2} = \sqrt{\langle u, u \rangle_{l^2}}, \forall u = (u_i)_{i \in \mathbb{Z}^n}, \quad v = (v_i)_{i \in \mathbb{Z}^n} \in l^2.
\]

In the Hilbert space \( H = l^2 \times l^2 \), we consider the following inner product \( \langle \cdot, \cdot \rangle_H \) and norm \( \| \cdot \|_H \) for \( U_1 = \left( \begin{array}{c} u_1 \\ v_1 \end{array} \right) \), \( U_2 = \left( \begin{array}{c} u_2 \\ v_2 \end{array} \right) \in H \),

\[
\langle U_1, U_2 \rangle_H = |\alpha_2| \langle u_1, u_2 \rangle + |\alpha_1| \langle v_1, v_2 \rangle, \quad \| U_1 \|_H = \sqrt{\langle U_1, U_1 \rangle_H},
\]

where \( \alpha_1 \) and \( \alpha_2 \) are the nonzero real constants given by (1)-(2) and satisfy (14).
Let $p$ and $q$ be two given functions such that
\begin{equation}
    p(u) = (p_i(u_i))_{i \in \mathbb{Z}^n}, \quad q(u) = (q_i(u_i))_{i \in \mathbb{Z}^n}, \quad \forall u = (u_i)_{i \in \mathbb{Z}^n}, u_i \in \mathbb{R},
\end{equation}
\begin{equation}
    p_i, q_i \in C(\mathbb{R}, \mathbb{R}_+), \quad \forall i \in \mathbb{Z}^n,
\end{equation}
and there exist functions $K_1, K_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that for $r \in \mathbb{R}_+$,
\begin{equation}
    ||p(u)||_{L^2} \leq K_1(r), \quad \forall ||u||_{L^2} \leq r,
\end{equation}
\begin{equation}
    q_i(s) \leq K_2(r), \quad \forall i \in \mathbb{Z}^n, |s| \leq r,
\end{equation}
where $\mathbb{R}_+ = (0, +\infty)$.

**Lemma 2.1.** Let $W$ be the set of functions $\varphi$ such that
\begin{equation}
    \varphi(u) = (\varphi_i(u_i))_{i \in \mathbb{Z}^n}, \quad \forall u = (u_i)_{i \in \mathbb{Z}^n}, u_i \in \mathbb{R},
\end{equation}
\begin{equation}
    \varphi_i \in C^1(\mathbb{R}, \mathbb{R}), \quad \forall i \in \mathbb{Z}^n,
\end{equation}
\begin{equation}
    \varphi_i(0) = 0, \quad \forall i \in \mathbb{Z}^n,
\end{equation}
and
\begin{equation}
    \sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left( \frac{\varphi_i(s)}{p_i(s)} + \frac{\varphi_i'(s)}{q_i(s)} \right) < \infty,
\end{equation}
where $p$ and $q$ are the functions given by (5)-(8). Then $W$ is a real Banach space whose norm is given by:
\begin{equation}
    \|\varphi\|_W = \sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left( \frac{\varphi_i(s)}{p_i(s)} + \frac{\varphi_i'(s)}{q_i(s)} \right), \quad \forall \varphi \in W.
\end{equation}

**Proof.** The proof is similar to Lemma 5.2 [1].

Note that if we consider the following particular choice of the functions $p$ and $q$ given by (5)-(8):
\begin{equation}
    p_i(s) = |s|^{x+1} + a_i, \quad q_i(s) = |s|^x + a_i, \quad \forall i \in \mathbb{Z}^n,
\end{equation}
where
\begin{equation}
    x \geq 0, \quad 0 < a_i < 1, \quad a = (a_i)_{i \in \mathbb{Z}^n} \in l^2,
\end{equation}
then the Banach space $W$ given by Lemma 2.1 is that given by (3)-(8) in [1].

In this work we assume that:
\begin{enumerate}
    \item[(A0)] $\alpha_1$ and $\alpha_2$ are nonzero real constants with $\alpha_1 \alpha_2 > 0$.
    \item[(A1)] For $k = 1, 2$, $\eta_k : \mathbb{R} \to l^2$ with $\eta_k(t) = (\eta_{ki}(t))_{i \in \mathbb{Z}^n}$ is an almost periodic function of $t$ with values in $l^2$.
    \item[(A2)] For $k = 1, 2$, $g_k(u, t) = (g_{ki}(u_i, t))_{i \in \mathbb{Z}^n}$ is a nonlinear function of $t \in \mathbb{R}$ and $u = (u_i)_{i \in \mathbb{Z}^n}, u_i \in \mathbb{R}$, with $g_{ki} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g_k(\cdot, t)$ is an almost periodic function of $t$ with values in $W$ and there exists $\beta_k > 0$ such that for $s, t \in \mathbb{R}$ and $i \in \mathbb{Z}^n$,
\begin{equation}
    g_{ki}(s, t) s \geq \beta_k s^2.
\end{equation}
    \item[(A3)] $A : l^2 \to l^2$ is a bounded linear operator of the following form
\begin{equation}
    A = A_1 + A_2 + \ldots + A_n,
\end{equation}
and for $j = 1, 2, \ldots, n$, there exist bounded linear operators $D_j : l^2 \to l^2$ given by
\begin{equation}
    (D_j u)_i = \sum_{l=-m_0}^{m_0} d_{j,l} u_{i+l}, \quad u = (u_i)_{i \in \mathbb{Z}^n} \in l^2.
\end{equation}
\end{enumerate}
From (17)-(18), it follows that
\[ \|D_j\|_G \leq c_0, \]  
where \( d_{j,l} \in \mathbb{R}, m_0 \) is a positive integer, \( i_{j,l} = (i_1, ..., i_{j-1}, i_j + l, i_{j+1}, ..., i_n) \in \mathbb{Z}^n \), and \( \|\cdot\|_G \) is the norm of a bounded linear operator from \( l^2 \) into \( l^2 \), such that
\[ A_j = D_j^*D_j = D_jD_j^*, \]  
where \( D_j^* \) is the adjoint operator of \( D_j \), that is,
\[ (D_j^*u)_i = \sum_{l=-m_0}^{m_0} d_{j,-l}u_{i_{j,l}}, \]  
\[ \langle D_ju, v \rangle_{l^2} = \langle u, D_j^*v \rangle_{l^2}, \quad u = (u_i)_{i \in \mathbb{Z}^n}, \quad v = (v_i)_{i \in \mathbb{Z}^n} \in l^2. \]

From (17)-(18), it follows that
\[ |d_{j,l}| \leq c_0, \quad j = 1, ..., n, \quad l = -m_0, ..., m_0. \]  

Let \( C_b(\mathbb{R}, Y) \) be the Banach space of bounded continuous functions on \( \mathbb{R} \) with values in a Banach space \( Y \) whose norm is given by
\[ \|\rho\|_{C_b(\mathbb{R}, Y)} = \sup_{t \in \mathbb{R}} \|\rho(t)\|_Y, \quad \rho \in C_b(\mathbb{R}, Y), \]
and let \( \rho_0 : \mathbb{R} \to Y \) be an almost periodic function in time \( t \) with values in \( Y \). Along the lines of the Bochner’s criterion [33], the set of translations \( \{\rho_0(\cdot + h) : h \in \mathbb{R}\} \) is precompact in \( C_b(\mathbb{R}, Y) \). The closure of this set in \( C_b(\mathbb{R}, Y) \) is said to be the hull \( \mathcal{H}(\rho_0) \) of the function \( \rho_0 \):
\[ \mathcal{H}(\rho_0) = \{\rho_0(\cdot + h) : h \in \mathbb{R}\} \subset C_b(\mathbb{R}, Y). \]  
Moreover, for any \( \rho \in \mathcal{H}(\rho_0), \rho : \mathbb{R} \to Y \) is an almost periodic function in time \( t \) with values in \( Y \) and \( \mathcal{H}(\rho) = \mathcal{H}(\rho_0) \).

In this work we consider the Banach space \( X = l^2 \times l^2 \times W \times W \) and the time symbol
\[ \sigma_0 = (t, \eta_1(t), g_1(u, t), g_2(v, t)), \]  
where for \( k = 1, 2, \eta_k \) and \( g_k \) are given by assumptions (A1) and (A2), respectively. Then \( \sigma_0 : \mathbb{R} \to X \) is an almost periodic function in time \( t \) with values in \( X \). For \( k = 1, 2 \), let \( \mathcal{H}(\eta_k) \) be the hull of \( \eta_k \) in \( C_b(\mathbb{R}, l^2) \) and \( \mathcal{H}(g_k) \) be the hull of \( g_k \) in \( C_b(\mathbb{R}, W) \). We consider the symbol space
\[ \Omega = \mathcal{H}(\eta_1) \times \mathcal{H}(\eta_2) \times \mathcal{H}(g_1) \times \mathcal{H}(g_2). \]

In such a case, for \( \sigma \in \Omega, \sigma : \mathbb{R} \to X \) is an almost periodic function in time \( t \) with values in \( X \).

Taking into account the LDS (1)-(3), we shall study the existence of the uniform global attractor with respect to \( \sigma \in \Omega \) with
\[ \sigma = (t, \xi_1(t), \xi_2(t), f_1(u, t), f_2(v, t)), \]
for the family of FitzHugh-Nagumo non-autonomous LDSs of the following form for \( t > \tau \) and \( \tau \in \mathbb{R} \):
\[ \dot{u} + Au + f_1(u, t) + \alpha_1 v = \xi_1(t), \]
\[ \dot{v} + f_2(v, t) - \alpha_2 u = \xi_2(t), \]  
with the initial conditions
\[ u(\tau) = (u_{i,\tau})_{i \in \mathbb{Z}^n} = u_{\tau}, \quad v(\tau) = (v_{i,\tau})_{i \in \mathbb{Z}^n} = v_{\tau}. \]

where \( u = (u_i)_{i \in \mathbb{Z}^n}, \quad v = (v_i)_{i \in \mathbb{Z}^n}, \quad Au = ((Au)_i)_{i \in \mathbb{Z}^n}, \quad f_1(u, t) = (f_{1i}(u_i, t))_{i \in \mathbb{Z}^n}, \quad f_2(v, t) = (f_{2i}(v_i, t))_{i \in \mathbb{Z}^n}, \) and \( \xi_k(t) = (\xi_{ki}(t))_{i \in \mathbb{Z}^n}, \) \( k = 1, 2. \)
3. Global solutions and absorbing sets.

**Lemma 3.1.** Considering (25)-(26), for \( \sigma \in \Omega \) and \( k = 1, 2 \), there exist constants \( \gamma_k (\eta_k) \geq 0 \) and \( \delta_k (g_k) \geq 0 \) depending on \( \eta_k \) and \( g_k \) such that

\[
\| \xi_k \|_{C^0(\mathbb{R}^2)} = \sup_{t \in \mathbb{R}} \| \xi_k (t) \|_{L^2} = \gamma_k (\eta_k),
\]

and for \( s, t \in \mathbb{R} \) and \( i \in \mathbb{Z}^n \),

\[
f_{ki} (s, t) s \geq \beta_k s^2,
\]

\[
|f_{ki} (s, t)| \leq \delta_k (g_k) p_i (s),
\]

and

\[
\left| \frac{\partial f_{ki}}{\partial s} (s, t) \right| \leq \delta_k (g_k) q_i (s).
\]

**Proof.** From assumptions (A1) and (A2), we know that for \( k = 1, 2 \), \( \eta_k \) and \( g_k \) are almost periodic functions of \( t \) with values in \( L^2 \) and \( W \), respectively. In such a case \( \eta_k \in C_b (\mathbb{R}, L^2) \) and \( g_k \in C_b (\mathbb{R}, W) \), that is, there exist constants \( \gamma_k (\eta_k) \geq 0 \) and \( \delta_k (g_k) \geq 0 \) depending on \( \eta_k \) and \( g_k \), respectively, such that

\[
\| \eta_k \|_{C^0(\mathbb{R}, L^2)} = \sup_{t \in \mathbb{R}} \| \eta_k (t) \|_{L^2} = \gamma_k (\eta_k),
\]

and

\[
g_k \in \sup_{t \in \mathbb{R}} \sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left( \frac{|g_{ki} (s, t)|}{p_i (s)} + \frac{\left| \frac{\partial g_{ki}}{\partial s} (s, t) \right|}{q_i (s)} \right) = \delta_k (g_k).
\]

From (35), we find that for \( s, t \in \mathbb{R} \), and \( i \in \mathbb{Z}^n \),

\[
|g_{ki} (s, t)| \leq \delta_k (g_k) p_i (s),
\]

\[
\left| \frac{\partial g_{ki}}{\partial s} (s, t) \right| \leq \delta_k (g_k) q_i (s).
\]

Recalling (23)-(26), for \( \sigma \in \Omega \) and \( k = 1, 2 \), there exist sequences of functions \( \{ \eta_k (\cdot + h_{km}) \}_{m \in \mathbb{N}} \) and \( \{ g_k (\cdot + r_{km}) \}_{m \in \mathbb{N}} \) with \( h_{km}, r_{km} \in \mathbb{R} \) such that as \( m \to \infty \),

\[
\sup_{t \in \mathbb{R}} \| \xi_k (t) - \eta_k (t + h_{km}) \|_{L^2} \to 0,
\]

and

\[
\sup_{t \in \mathbb{R}} \| f_k (\cdot, t) - g_k (\cdot, t + r_{km}) \|_{W} \to 0,
\]

that is,

\[
\sup_{t \in \mathbb{R}} \sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left( \frac{|f_{ki} (s, t) - g_{ki} (s, t + r_{km})|}{p_i (s)} + \frac{\left| \frac{\partial f_{ki}}{\partial s} (s, t) - \frac{\partial g_{ki}}{\partial s} (s, t + r_{km}) \right|}{q_i (s)} \right) \to 0.
\]

From (34) and (38), it is clear that (30) is satisfied. Following (39), we find that (31), (32), and (33) are satisfied since (15), (36), and (37) are satisfied. \( \square \)

**Lemma 3.2.** Recalling (25)-(26), for \( \sigma \in \Omega \) and \( k = 1, 2 \), the following are satisfied:

(a) \( f_k (u, t) \) maps \( L^2 \times \mathbb{R} \) into \( L^2 \) and for \( R > 0 \), \( u, v \in L^2 \), with \( \| u \|_{L^2} \leq R \), \( \| v \|_{L^2} \leq R \), and \( t \in \mathbb{R} \),

\[
\| f_k (u, t) - f_k (v, t) \|_{L^2} \leq \delta_k (g_k) \| u - v \|_{L^2},
\]

that is, \( f_k : L^2 \times \mathbb{R} \to L^2 \) is a locally Lipschitz function of \( u \) uniformly on \( t \).

(b) \( \xi_k : \mathbb{R} \to L^2 \) and \( f_k : L^2 \times \mathbb{R} \to L^2 \) are continuous functions of \( t \).
Proof. (a) Using (7) and (32), it follows that $f_k(u,t)$ maps $l^2 \times \mathbb{R}$ into $l^2$. Recalling (8) and (33) we get (40).

(b) $\xi_k : \mathbb{R} \to l^2$ is continuous since $\xi_k \in H(\eta_k) \subset C_b(\mathbb{R},l^2)$, and $f_k(\cdot,t) : \mathbb{R} \to W$ is continuous since $f_k(u,t) \in H(g_k) \subset C_b(\mathbb{R},W)$. Here we show that $f_k : l^2 \times \mathbb{R} \to l^2$ is a continuous function of $t$. Indeed, since $f_k(\cdot,t) : \mathbb{R} \to W$ is continuous, then for $t \in \mathbb{R}$, given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$ with $|x - t| < \delta$ we have
\[
\|f_k(\cdot,x) - f_k(\cdot,t)\|_W < \frac{\varepsilon}{c},
\]
where $c$ is any positive constant. In such a case we have
\[
\sup_{i \in \mathbb{Z}^n} \sup_{s \in \mathbb{R}} \left( \frac{|f_{k_i}(s,x) - f_{k_i}(s,t)|}{p_i(s)} \right) < \frac{\varepsilon}{c},
\]
and for $u = (u_i)_{i \in \mathbb{Z}^n} \in l^2$,
\[
|f_{k_i}(u_i,x) - f_{k_i}(u_i,t)| < \frac{\varepsilon}{c}p_i(u_i), \ i \in \mathbb{Z}^n. \tag{41}
\]
For any fixed $w = (w_i)_{i \in \mathbb{Z}^n} \in l^2$, taking into account (7), we can assume that $c$ is sufficiently large such that
\[
\sum_{i \in \mathbb{Z}^n} (p_i(w_i))^2 \leq c^2, \ i \in \mathbb{Z}^n. \tag{42}
\]
Using (41)-(42), it follows that
\[
\|f_k(w,x) - f_k(w,t)\|_{l^2} < \varepsilon.
\]
That is $f_k : l^2 \times \mathbb{R} \to l^2$ is a continuous function of $t$. \(\square\)

The family of FitzHugh-Nagumo non-autonomous LDSs (27)-(29) can be written in the following form in $H = l^2 \times l^2$:
\[
\dot{\varphi} + C\varphi = F(\varphi,t), \ \sigma \in \Omega, t > \tau, \tau \in \mathbb{R}, \tag{43}
\]
\[
\varphi(\tau) = \varphi_\tau = \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix}, \tag{44}
\]
where $\Omega$ and $\sigma$ are given by (25) and (26), respectively, $\varphi = \begin{pmatrix} u \\ v \end{pmatrix}$, $C : H \to H$ is the bounded linear operator
\[
C = \begin{pmatrix} A & \alpha_1 I \\ -\alpha_2 I & 0 \end{pmatrix}, \tag{45}
\]
$I : l^2 \to l^2$ is the identity operator, and $F : H \times \mathbb{R} \to H$ is the nonlinear operator
\[
F(\varphi,t) = \begin{pmatrix} \xi_1(t) - f_1(u,t) \\ \xi_2(t) - f_2(v,t) \end{pmatrix}. \tag{46}
\]

Lemma 3.3. Given $\tau \in \mathbb{R}$, $\varphi_\tau \in H$, and $\sigma \in \Omega$, there exists a unique local maximal classical solution $\varphi$ satisfying (43)-(44) in $H$ such that $\varphi \in C^1([\tau,T),H)$, for some $T > \tau$. Moreover, if $T < +\infty$, then
\[
\lim_{t \to T^-} \|\varphi(t)\|_H = +\infty. \tag{47}
\]
Proof. Taking into account the linear operator $C$ and the nonlinear operator $F$, given by (45) and (46), respectively. From Lemma 3.2, it is clear that $F(\varphi, t) : H \times \mathbb{R} \to H$ is a continuous function of $t$ and a locally Lipschitz function of $\varphi$ uniformly on $t$. Since $C : H \to H$ is a bounded linear operator, $-C$ is the infinitesimal generator of a $C_0$ semigroup on $H$. Moreover, the domain of $C$ is $H$. In such a case, following [40], specifically Theorems 1.4 and 1.7 of chapter 6, the proof is completed.

Lemma 3.4. Given $\tau \in \mathbb{R}$, $\varphi_\tau \in H$, and $\sigma \in \Omega$, the solution $\varphi$ of (43)-(44) in $H$ satisfies

$$\|\varphi(t)\|_H^2 \leq e^{\mu_1 (\tau-t)} \|\varphi(\tau)\|_H^2 + \frac{\mu_2}{\mu_1} \left(1 - e^{\mu_1 (\tau-t)}\right), \ t \geq \tau,$$

where

$$\mu_1 = \min \{\beta_1, \beta_2\}, \ \mu_2 = \frac{|\alpha_2|}{\beta_1} (\gamma_1 (\eta_1))^2 + \frac{|\alpha_1|}{\beta_2} (\gamma_2 (\eta_2))^2.$$

Proof. Considering the inner product of (43) with $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ in $H$, taking into account (4) and (14), we obtain that

$$\frac{1}{2} \frac{d}{dt} \left( |\alpha_2| \|u(t)\|_2^2 + |\alpha_1| \|v(t)\|_2^2 \right)$$

$$+ |\alpha_2| \sum_{j=1}^n \|D_j u(t)\|_2^2 + |\alpha_2| \langle f_1 (u(t), t), u(t) \rangle_2 + |\alpha_1| \langle f_2 (v(t), t), v(t) \rangle_2$$

$$= |\alpha_2| \langle \xi_1(t), u(t) \rangle_2 + |\alpha_1| \langle \xi_2(t), v(t) \rangle_2, \ t > \tau.$$  (50)

From (30), we get

$$|\langle \xi_1(t), u(t) \rangle_2| \leq \frac{\beta_1}{2} \|u(t)\|_2^2 + \frac{1}{2\beta_1} \|\xi_1(t)\|_2^2$$

$$\leq \frac{\beta_1}{2} \|u(t)\|_2^2 + \frac{1}{2\beta_1} (\gamma_1 (\eta_1))^2, \ t > \tau,$$  (51)

$$|\langle \xi_2(t), v(t) \rangle_2| \leq \frac{\beta_2}{2} \|v(t)\|_2^2 + \frac{1}{2\beta_2} \|\xi_2(t)\|_2^2 \leq \frac{\beta_2}{2} \|v(t)\|_2^2 + \frac{1}{2\beta_2} (\gamma_2 (\eta_2))^2, \ t > \tau.$$  (52)

Putting (31), (51), and (52), into (50), we find

$$\frac{d}{dt} \left( |\alpha_2| \|u(t)\|_2^2 + |\alpha_1| \|v(t)\|_2^2 \right) + \beta_1 |\alpha_2| \|u(t)\|_2^2 + \beta_2 |\alpha_1| \|v(t)\|_2^2$$

$$\leq \frac{|\alpha_2|}{\beta_1} (\gamma_1 (\eta_1))^2 + \frac{|\alpha_1|}{\beta_2} (\gamma_2 (\eta_2))^2, \ t > \tau,$$  (53)

and from (4) and (49), we get

$$\frac{d}{dt} \|\varphi(t)\|_H^2 + \mu_1 \|\varphi(t)\|_H^2 \leq \mu_2, \ t > \tau,$$

that is,

$$\frac{d}{dt} \left( e^{\mu_1 t} \|\varphi(t)\|_H^2 \right) \leq \mu_2 e^{\mu_1 t}, \ t > \tau.$$  (54)

Integrating the last inequality from $\tau$ into $t$, we get (48).

Following (47) and (48), we find that the solution $\varphi$ of (43)-(44) is defined globally for $t \geq \tau$. In such a case we can introduce a family of processes $\{\Phi^\sigma(t, \tau) : t \geq \tau, \ \sigma \in \Omega\}$ on $H$ such that for $\tau \in \mathbb{R}, t \geq \tau$, and $\varphi_\tau \in H$, we have $\Phi^\sigma(t, \tau) \varphi_\tau =$
and taking the inner product of (61) with \( v \), where \( \varphi \) is the solution of (43)-(44). By the unique solvability of (43)-(44), the family of processes \( \{ \Phi^\sigma (t, \tau) \}_{\sigma \in \Omega} \) satisfies the multiplicative properties:

\[
\Phi^\sigma (t, s) \Phi^\sigma (s, \tau) = \Phi^\sigma (t, \tau), \quad t \geq s \geq \tau, \tau \in \mathbb{R}, \quad \Phi^\sigma (\tau, \tau) = I, \quad \tau \in \mathbb{R},
\]

where \( I \) is the identity operator. Moreover, the following translation identity holds for the processes and the translation group \( \{ T(h) \}_{h \in \mathbb{R}} \):

\[
\Phi^\sigma (t + h, \tau + h) = \Phi^{T(h)\sigma} (t, \tau), \quad h, \tau \in \mathbb{R}, t \geq \tau,
\]

where

\[
T(h) \sigma = \sigma (\cdot + h), \quad \sigma \in \Omega.
\]

Along the lines of (48), we get the following lemma.

**Lemma 3.5.** In \( H \), the closed bounded ball \( B = B(0, R_0) \) with center 0 and radius

\[
R_0 > \sqrt{\frac{\mu_2}{\mu_1}},
\]

where \( \mu_1 \) and \( \mu_2 \) are given by (49), is a uniform absorbing set for the family of processes \( \{ \Phi^\sigma (t, \tau) : t \geq \tau, \tau \in \mathbb{R} \}_{\sigma \in \Omega} \) corresponding to the LDS (43)-(44). That is, for \( \tau \in \mathbb{R} \) and \( R_1 \geq 0 \), if \( \| \varphi_\tau \|_H \leq R_1 \), then there exists a constant \( T_0 = T_0 (R_1) \geq 0 \) such that

\[
\| \Phi^\sigma (t, \tau) \varphi_\tau \|_H \leq R_0, \quad \sigma \in \Omega, t - \tau \geq T_0.
\]

Moreover, there exists a constant \( R_2 = R_2 (R_1) \) such that

\[
\| \Phi^\sigma (t, \tau) \varphi_\tau \|_H \leq R_2, \quad \sigma \in \Omega, t - \tau \in [0, T_0).
\]

The following lemma is necessary to investigate the uniform asymptotic behavior of the family of processes.

**Lemma 3.6.** The family of processes \( \{ \Phi^\sigma (t, \tau) : t \geq \tau, \tau \in \mathbb{R} \}_{\sigma \in \Omega} \) corresponding to the LDS (43)-(44) is \( (H \times \Omega, H) \)-continuous, that is, for any fixed \( t \) and \( \tau \), \( t \geq \tau, \tau \in \mathbb{R} \), the mapping \( (\phi, \sigma) \mapsto \Phi^\sigma (t, \tau) \phi \) is continuous from \( H \times \Omega \) into \( H \).

**Proof.** For \( \tau \in \mathbb{R} \) and \( k = 1, 2 \), let \( \varphi_k (t) = \left( \begin{array}{c} u_k (t) \\ v_k (t) \end{array} \right) = \Phi^{\sigma_k} (t, \tau) \varphi_{k\tau} \) be the solution of the LDS (43)-(44) with symbol \( \sigma_k (t) = (\xi_{1k} (t), \xi_{2k} (t), f_{1k} (u_k, t), f_{2k} (v_k, t)) \) \( \in \Omega \) and initial data \( \varphi_{k\tau} = \varphi_{k\tau} = \left( \begin{array}{c} u_{k\tau} \\ v_{k\tau} \end{array} \right) \). Let

\[
\varphi_d (t) = \varphi_1 (t) - \varphi_2 (t) = \left( \begin{array}{c} u_d (t) \\ v_d (t) \end{array} \right) = \left( \begin{array}{c} u_1 (t) - u_2 (t) \\ v_1 (t) - v_2 (t) \end{array} \right).
\]

From (27)-(28) we have, for \( t > \tau \),

\[
\dot{u}_d + Au_d + f_{11} (u_1, t) - f_{12} (u_2, t) + \alpha_1 v_d = \xi_{11} - \xi_{12}, \quad (60)
\]

\[
\dot{v}_d + f_{21} (v_1, t) - f_{22} (v_2, t) - \alpha_2 u_d = \xi_{21} - \xi_{22}. \quad (61)
\]

Considering the inner product of (60) with \( u_d \) in \( l^2 \), we get

\[
\frac{1}{2} \frac{d}{dt} \| u_d \|_l^2 + \sum_{j=1}^n \| D_j u_d \|_l^2 + \left\langle f_{11} (u_1, t) - f_{12} (u_2, t), u_d \right\rangle_{l^2} + \alpha_1 \left\langle v_d, u_d \right\rangle_{l^2} = \left\langle \xi_{11} - \xi_{12}, u_d \right\rangle_{l^2}, \quad t > \tau,
\]

and taking the inner product of (61) with \( v_d \) in \( l^2 \), we find

\[
\frac{1}{2} \frac{d}{dt} \| v_d \|_l^2 + (f_{21} (v_1, t) - f_{22} (v_2, t), v_d)_{l^2} - \alpha_2 (u_d, v_d)_{l^2} = \left\langle \xi_{21} - \xi_{22}, v_d \right\rangle_{l^2}, \quad t > \tau.
\]
If we multiply (62) by $|\alpha_2|$, (63) by $|\alpha_1|$, and then we sum up the results, taking into account (14), it follows that

\[
\frac{1}{2} \frac{d}{dt} \left( |\alpha_2| \| u_d \|_2^2 + |\alpha_1| \| v_d \|_2^2 \right) \\
+ |\alpha_2| \langle f_{11} (u_1, t) - f_{12} (u_2, t), u_d \rangle_{l2} \\
+ |\alpha_1| \langle f_{21} (v_1, t) - f_{22} (v_2, t), v_d \rangle_{l2} \\
\leq |\alpha_2| \langle \xi_{11} - \xi_{12}, u_d \rangle_{l2} \\
+ |\alpha_1| \langle \xi_{21} - \xi_{22}, v_d \rangle_{l2}, \quad t > \tau. \tag{64}
\]

Following Lemma 3.5, since $\varphi_1$, and $\varphi_2$, are bounded in $H$, there exists a constant $R_3 > 0$ such that

\[
\| u_1 \|_{l2} \leq R_3, \| u_2 \|_{l2} \leq R_3, \| u_d \|_{l2} \leq R_3, \quad t \geq \tau, \tag{65}
\]

\[
\| v_1 \|_{l2} \leq R_3, \| v_2 \|_{l2} \leq R_3, \| v_d \|_{l2} \leq R_3, \quad t \geq \tau. \tag{66}
\]

Using (65) and (66), we obtain

\[
\langle \xi_{11} - \xi_{12}, u_d \rangle_{l2} \leq \| \xi_{11} - \xi_{12} \|_{l2} \| u_d \|_{l2} \leq R_3 \| \xi_{11} - \xi_{12} \|_{l2}, \quad t \geq \tau, \tag{67}
\]

\[
\langle \xi_{21} - \xi_{22}, v_d \rangle_{l2} \leq \| \xi_{21} - \xi_{22} \|_{l2} \| v_d \|_{l2} \leq R_3 \| \xi_{21} - \xi_{22} \|_{l2}, \quad t \geq \tau, \tag{68}
\]

and following (40) and (65), we get

\[
\langle f_{11} (u_1, t) - f_{12} (u_2, t), u_d \rangle_{l2} \\
= \langle f_{11} (u_1, t) - f_{11} (u_2, t), u_d \rangle_{l2} + \langle f_{11} (u_2, t) - f_{12} (u_2, t), u_d \rangle_{l2} \\
\leq \delta_1 (g_1) K_2 (R_3) \| u_d \|_{l2}^2 + \| f_{11} (u_2, t) - f_{12} (u_2, t) \|_{l2} \| u_d \|_{l2} \\
\leq \left[ \frac{1}{2} + \delta_1 (g_1) K_2 (R_3) \right] \| u_d \|_{l2}^2 + \frac{1}{2} \| f_{11} (u_2, t) - f_{12} (u_2, t) \|_{l2}^2, \quad t \geq \tau. \tag{69}
\]

Similarly, we find

\[
\langle f_{21} (v_1, t) - f_{22} (v_2, t), v_d \rangle_{l2} \\
\leq \left[ \frac{1}{2} + \delta_2 (g_2) K_2 (R_3) \right] \| v_d \|_{l2}^2 + \frac{1}{2} \| f_{21} (v_2, t) - f_{22} (v_2, t) \|_{l2}^2, \quad t \geq \tau. \tag{70}
\]

Substituting (67)-(70) into (64), it follows that, for $t > \tau$,

\[
\frac{d}{dt} \left( |\alpha_2| \| u_d \|_2^2 + |\alpha_1| \| v_d \|_2^2 \right) \\
- |\alpha_2| \left[ 1 + 2 \delta_1 (g_1) K_2 (R_3) \right] \| u_d \|_2^2 + \| f_{11} (u_2, t) - f_{12} (u_2, t) \|_2^2 \\
- |\alpha_1| \left[ 1 + 2 \delta_2 (g_2) K_2 (R_3) \right] \| v_d \|_2^2 + \| f_{21} (v_2, t) - f_{22} (v_2, t) \|_2^2 \\
\leq 2 |\alpha_2| R_3 \| \xi_{11} - \xi_{12} \|_{l2} + 2 |\alpha_1| R_3 \| \xi_{21} - \xi_{22} \|_{l2}. 
\]

Taking into account (4), we find that for $t > \tau$,

\[
\frac{d}{dt} \| \varphi_d \|_H^2 - \mu_3 \| \varphi_d \|_H^2 \leq \mu_4 \left( \| \xi_{11} - \xi_{12} \|_{l2} + \| \xi_{21} - \xi_{22} \|_{l2} \right) \\
+ \mu_4 \left( \| f_{11} (u_2, t) - f_{12} (u_2, t) \|_{l2}^2 + \| f_{21} (v_2, t) - f_{22} (v_2, t) \|_{l2}^2 \right), \tag{71}
\]

where

\[
\mu_3 = \max \left\{ 1 + 2 \delta_1 (g_1) K_2 (R_3), 1 + 2 \delta_2 (g_2) K_2 (R_3) \right\},
\]

and

\[
\mu_4 = \max \left\{ |\alpha_1|, 2 |\alpha_1| R_3, |\alpha_2|, 2 |\alpha_2| R_3 \right\}.
\]

From (71), it is clear that for $t > \tau$,

\[
\frac{d}{dt} \left( e^{-\mu_3 t} \| \varphi_d \|_H^2 \right) \leq \mu_4 e^{-\mu_3 t} \left( \| \xi_{11} - \xi_{12} \|_{l2} + \| \xi_{21} - \xi_{22} \|_{l2} \right) \\
+ \mu_4 e^{-\mu_3 t} \left( \| f_{11} (u_2, t) - f_{12} (u_2, t) \|_{l2}^2 + \| f_{21} (v_2, t) - f_{22} (v_2, t) \|_{l2}^2 \right). 
\]
Integrating both sides of the last inequality from $\tau$ into $t$, we get for $t \geq \tau$,
\[
\|\varphi_d(t)\|_H^2 \leq e^{\mu_3 (t-\tau)} \|\varphi_d(\tau)\|_H^2 + \frac{\mu_4}{\mu_3} e^{\mu_3(t-\tau)} \max_{x \in [\tau,t]} \left( \left\| \xi_{11}(x) - \xi_{12}(x) \right\|_2 + \left\| \xi_{21}(x) - \xi_{22}(x) \right\|_2 
+ \left\| f_{11}(u_2, x) - f_{12}(u_2, x) \right\|_2^2 + \left\| f_{21}(v_2, x) - f_{22}(v_2, x) \right\|_2^2 \right).
\]
But
\[
\max_{x \in [\tau,t]} \left( \left\| \xi_{11}(x) - \xi_{12}(x) \right\|_2 + \left\| \xi_{21}(x) - \xi_{22}(x) \right\|_2 \right) \leq \left\| \xi_{11} - \xi_{12} \right\|_{C_b(\mathbb{R}, H^2)} + \left\| \xi_{21} - \xi_{22} \right\|_{C_b(\mathbb{R}, H^2)},
\]
and
\[
\max_{x \in [\tau,t]} \| f_{11}(u_2(x), x) - f_{12}(u_2(x), x) \|_2^2 
= \max_{x \in [\tau,t]} \sum_{\xi \in \mathbb{Z}^n} \sup_{x, s} \left( (p_1(u_2(x)))^2 \right)^{\frac{1}{2}} \sup_{x, s} \left( (p_1(u_2(x)))^2 \right)^{\frac{1}{2}} 
\leq \left\| f_{11} - f_{12} \right\|_{C_b(\mathbb{R}, H^2)} \sup_{x \in [\tau,t]} \| p(u_2(x)) \|_2.
\]
From (7), (65), and (74), we find
\[
\max_{x \in [\tau,t]} \| f_{11}(u_2(x), x) - f_{12}(u_2(x), x) \|_2^2 \leq K_1(R_3) \| f_{11} - f_{12} \|_{C_b(\mathbb{R}, H^2)},
\]
and similarly, taking into account (7) and (66), we get
\[
\max_{x \in [\tau,t]} \| f_{21}(v_2(x), x) - f_{22}(v_2(x), x) \|_2^2 \leq K_1(R_3) \| f_{21} - f_{22} \|_{C_b(\mathbb{R}, H^2)}.
\]
Putting (73), (75), and (76) into (72), we obtain
\[
\left( \| \varphi_d(t) \|_H^2 \leq e^{\mu_3 (t-\tau)} \|\varphi_d(\tau)\|_H^2 
+ K_1(R_3) \left( \| f_{11} - f_{12} \|_{C_b(\mathbb{R}, H^2)}^2 + \| f_{21} - f_{22} \|_{C_b(\mathbb{R}, H^2)}^2 \right) \right).
\]
From the last inequality, it is clear that for fixed $t$ and $\tau$, $t \geq \tau$, $\tau, t \in \mathbb{R}$, $\| \varphi_1(t) - \varphi_2(t) \|_H \to 0$ as $\| \varphi_1 - \varphi_2 \|_H \to 0$, $\| \xi_{11} - \xi_{12} \|_{C_b(\mathbb{R}, H^2)} \to 0$, and $\| f_{k1} - f_{k2} \|_{C_b(\mathbb{R}, H^2)} \to 0$, for $k = 1, 2$. The proof is completed. \hfill \square

4. Uniform global attractors.

**Lemma 4.1.** Given $\varepsilon > 0$, there exist constants $I = I(\varepsilon)$ and $T = T(\varepsilon)$ such that for $\sigma \in \Omega, \tau \in \mathbb{R}, t \geq \tau$, and $\varphi_\tau = \left( \begin{array}{c} u_\tau \\ v_\tau \end{array} \right) \in \mathcal{B}$, the solution $\varphi(t) = (\varphi_i(t))_{1 \leq i \leq \mathbb{Z}^n} = \left( \begin{array}{c} u(t) \\ v(t) \end{array} \right)$ is in $H$ satisfies
\[
\sum_{i \in \mathbb{Z}^n} \| \varphi_i(t) \|_H^2 \leq \varepsilon, \forall t - \tau \geq T,
\]
where $\mathcal{B}$ is the uniform absorbing ball given by Lemma 3.5 and
\[
\| \varphi_i(t) \|_H^2 = |\alpha_2| \| u_i(t) \|^2 + |\alpha_1| \| v_i(t) \|^2.
\]

**Proof.** Consider a smooth increasing function $\theta \in C^1(\mathbb{R}^+, \mathbb{R})$ such that
\[
\begin{cases}
\theta(s) = 0, & 0 \leq s < 1, \\
0 \leq \theta(s) \leq 1, & 1 \leq s < 2, \\
\theta(s) = 1, & 2 \leq s,
\end{cases}
\]

and there exists a constant $M_0$ such that

$$|\theta'(s)| \leq M_0, \forall s \geq 0.$$  

(79)

For $k = 1, 2$, since $\xi_k$ is almost periodic with values in $I^2$, the set $\{(\xi_k(t))_{t \in \mathbb{R}} : t \in \mathbb{R}\}$ is precompact in $I^2$. Hence for $\varepsilon > 0$ there exists a constant $K_3 (\xi_1, \xi_2, \varepsilon)$ depending on $\xi_1, \xi_2,$ and $\varepsilon$, such that

$$\sum_{||i||_m \geq K_3} \xi_{ki}^2 (t) \leq \frac{\mu_1^2 \varepsilon}{\delta \mu_5}, \forall t \in \mathbb{R},$$  

(80)

where $\mu_1$ is given by (49),

$$\mu_5 = \max \{||\alpha_1||, ||\alpha_2||\}.$$  

(81)

But $\xi_k \in \mathcal{H}(\eta_k)$ and $\mathcal{H}(\eta_k)$ is compact in $C_0(\mathbb{R}, I^2)$. In such a case, taking into account (80), there exists a constant $K_4 (\varepsilon)$ depends on $\varepsilon$ but it is independent of $\xi_1$ and $\xi_2$ such that

$$\sum_{||i||_m \geq K_4} \xi_{ki}^2 (t) \leq \frac{\mu_3^2 \varepsilon}{\delta \mu_5}, \forall t \in \mathbb{R}.$$  

(82)

Let $K_5 = K_5 (\varepsilon)$ be a positive constant such that

$$K_5 = \max \left\{ K_4, \frac{8 ||\alpha_2|| n R_0^2 M_0 m_0 (2 m_0 + 1)^2 \varepsilon^2}{\mu_1 \varepsilon} \right\},$$  

(83)

and for $i \in \mathbb{Z}^n$, let $w_i = \theta \left( \frac{||i||_m}{K_5} \right) u_i$, $z_i = \theta \left( \frac{||i||_m}{K_5} \right) v_i$, $w = (w_i)_{i \in \mathbb{Z}^n}$ and $z = (z_i)_{i \in \mathbb{Z}^n}$. Taking into account (14), if we consider the inner product of (43) with

$$\left( \frac{||\alpha_2|| w}{||\alpha_1|| z} \right) \text{ in } H,$$

we get

$$\frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) \left( ||\alpha_2|| u_i^2 + ||\alpha_1|| v_i^2 \right) + ||\alpha_2|| \sum_{j=1}^n \langle D_j u, D_j w \rangle + ||\alpha_2|| \sum_{j=1}^n \langle D_j v, D_j z \rangle = ||\alpha_2|| \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) u_i + ||\alpha_1|| \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) z_i, \forall t - \tau \geq 0.$$  

In such a case, using (31), (49), and (77), we obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) |\varphi_i|^2_H + ||\alpha_2|| \sum_{j=1}^n \langle D_j u, D_j w \rangle + ||\alpha_1|| \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) |\varphi_i|^2_H \leq ||\alpha_2|| \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) u_i + ||\alpha_1|| \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) z_i, \forall t - \tau \geq 0.$$  

(84)

By (78) and (82)-(83), we find

$$\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) u_i \leq \frac{1}{2 \mu_1} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) u_i^2 + \frac{\mu_1}{2} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) u_i^2 + \frac{\mu_1}{2} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) u_i^2, \forall t - \tau \geq 0,$$

$$\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) z_i \leq \frac{1}{2 \mu_5} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) z_i^2 + \frac{\mu_5}{2} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) z_i^2, \forall t - \tau \geq 0,$$

and

$$\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) v_i \leq \frac{1}{16 \mu_5} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) v_i^2 + \frac{\mu_5}{2} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{||i||_m}{K_5} \right) v_i^2, \forall t - \tau \geq 0.$$  

(85)

(86)
Integrating the last inequality from $\tau$ and from (83), we find

\[ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|}{K_5} \right) (D_j u(t), D_j w(t)) \] 

\[ \geq -\frac{R_5^2 M a_m (2m a_1 + 1)^2 e_3}{K_5}, \quad \forall j = 1, 2, \ldots, n, t - \tau \geq T_1, \]

and from (83), we find

\[ |\alpha_2| (D_j u(t), D_j w(t)) \geq -\frac{\mu_1 e}{8 n}, \quad \forall j = 1, 2, \ldots, n, t - \tau \geq T_1. \quad (87) \]

Putting (85)-(87) into (84) and taking into account (81), it follows that

\[ \frac{d}{dt} \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|}{K_5} \right) |\varphi_i|^2_H + \mu_1 \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|}{K_5} \right) |\varphi_i|^2_H \leq \frac{\mu_1 e}{2} \varepsilon, \forall t - \tau \geq T_1. \]

Integrating the last inequality from $\tau$ into $t$, we get

\[ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|}{K_5} \right) |\varphi_i(t)|^2_H \leq e^{\mu_1 (\tau - t)} \left( \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|}{K_5} \right) |\varphi_i(t)|^2_H \right) + \frac{\varepsilon}{2}. \]

Thus for $I = I(\varepsilon) = 2K_5(\varepsilon)$ and $T = T(\varepsilon) = \max \left\{ T_1, \frac{1}{\mu_1} \ln \frac{2M e}{\varepsilon} \right\}$, we find

\[ \sum_{\|i\| \geq t} |\varphi_i(t)|^2_H \leq \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|}{K_5} \right) |\varphi_i(t)|^2_H \leq \varepsilon, \forall t - \tau \geq T. \]

In $H$, let us connect the family of processes $\{ \Phi^\sigma (t) : t \geq \tau, \tau \in \mathbb{R} \}_{\sigma \in \Omega}$ corresponding to the LDS (43)-(44) with an associated semigroup of nonlinear operators, where we use the semigroup theory to investigate the existence of the uniform global attractor of the processes. Along the lines of [13], we define the nonlinear semigroup $\{ S(t) \}_{t \geq 0}$ associated with the LDS (43)-(44) acting on the extended phase space $H \times \Omega$ by

\[ S(t) (\varphi, \sigma) = (\Phi^\sigma (t), 0) \varphi, T(t) \sigma), \quad t \geq 0, \varphi \in H, \sigma \in \Omega. \quad (88) \]

In such a case, we find that $\{ S(t) \}_{t \geq 0}$ satisfies the semigroup identities:

\[ S(t) S(s) = S(t + s), S(0) = I, \forall t \geq s \geq 0. \]

**Lemma 4.2.** The solution semigroup $\{ S(t) \}_{t \geq 0}$ associated with the LDS (43)-(44) is asymptotically compact, that is, if $\{ (\varphi_n, \sigma_n) \}_{n \in \mathbb{N}}$ is bounded in $H \times \Omega$, and $t_n \to \infty$, then $\{ S(t_n) (\varphi_n, \sigma_n) \}_{n \in \mathbb{N}}$ is precompact in $H \times \Omega$.

**Proof.** Using Lemmas 3.5 and 4.1, above, the proof is similar to that of Lemma 5.4 [42].

**Definition 4.3.** In $H$, a closed set $A$ is called the uniform attractor for the family of processes $\{ \Phi^\sigma (t) : t \geq \tau, \tau \in \mathbb{R} \}_{\sigma \in \Omega}$ with respect to $\sigma \in \Omega$ if:

(a) For any bounded set $G \subset H$,

\[ \lim_{t \to \infty} \sup_{\sigma \in \Omega} \text{dist} (\Phi^\sigma (t), G, A) = 0, \forall t \in \mathbb{R}. \]
(b) (Minimal property) If $\overline{A}$ is any closed subset of $H$ satisfying property (a), then $A \subseteq \overline{A}$.

**Definition 4.4.** Given $\sigma \in \Omega$, a curve $t \rightarrow \varphi(t) \in H$ is said to be a complete solution for the process $\Phi^\sigma(t, \tau)$, if it satisfies

$$\Phi^\sigma(t, \tau) \varphi(\tau) = \varphi(t), \forall \tau \in \mathbb{R}, t \geq \tau.$$  \hspace{1cm} (89)

The kernel of the process $\Phi^\sigma(t, \tau)$ is the collection $\mathcal{K}_\sigma$ of all its bounded complete solutions. The kernel section of the process $\Phi^\sigma(t, \tau)$ at time $s \in \mathbb{R}$ is the set

$$\mathcal{K}_\sigma(s) = \{ \varphi(s) : \varphi(\cdot) \in \mathcal{K}_\sigma \}.$$  

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the projectors of $H \times \Omega$ onto $H$ and $\Omega$, respectively. Following the uniform attractor theory [13], we have the following proposition.

**Proposition 1.** In $H \times \Omega$, if the semigroup $\{S(t)\}_{t \geq 0}$ is continuous, point dissipative, and asymptotically compact, then it has a compact global attractor $\mathcal{A}_S$. Furthermore, in $H, \mathcal{A} = \mathcal{F}_1 \mathcal{A}_S$ is the compact uniform attractor for the family of processes $\{\Phi^\sigma(t, \tau)\}_{\sigma \in \Omega}$. In addition,

(a) $\mathcal{A}_S = \bigcup_{\sigma \in \Omega} \mathcal{K}_\sigma(0) \times \{\sigma\}$,

(b) $A = \bigcup_{\sigma \in \Omega} \mathcal{K}_{g, f}(0)$,

(c) $\mathcal{F}_2 \mathcal{A}_S = \Omega$.

**Theorem 4.5.** In $H$, the family of processes $\{\Phi^\sigma(t, \tau)\}_{\sigma \in \Omega}$ associated with the LDS (43)-(44) has a compact uniform attractor $\mathcal{A}$ with respect to $\sigma \in \Omega$.

**Proof.** Following Lemma 3.6, we find that the family of processes are continuous from $H \times \Omega$ into $H$. In such a case, using the continuity of translation group $\{T(t)\}_{t \in \mathbb{R}}$, it follows that the solution semigroup $\{S(t)\}_{t \geq 0}$ associated with the LDS (43)-(44) is continuous in $H \times \Omega$. Let $B_S = B \times \Omega$, where $B$ is the uniform absorbing set of $H$, $A = \mathcal{F}_1 A_S$ is the compact uniform global attractor for the family of processes $\{\Phi^\sigma(t, \tau)\}_{\sigma \in \Omega}$ in $H$ with respect to $\sigma \in \Omega$. \hfill $\Box$

**REFERENCES**

[1] A. Y. Abdallah, Attractors for first order lattice systems with almost periodic nonlinear part, *Discrete Contin. Dyn. Syst. Ser. B*, 25 (2020), 1241–1255.

[2] A. Y. Abdallah, Long-time behavior for second order lattice dynamical systems, *Acta Appl. Math.*, 106 (2009), 47–59.

[3] A. Y. Abdallah, Uniform global attractors for first order non-autonomous lattice dynamical systems, *Proc. Amer. Math. Soc.*, 138 (2010), 3219–3228.

[4] A. Y. Abdallah, Upper semicontinuity of the attractor for lattice dynamical systems of partly dissipative reaction diffusion systems, *J. Appl. Math.*, 2005 (2005), 273–288.

[5] A. Y. Abdallah and R. T. Wannan, Second order non-autonomous lattice systems and their uniform attractors, *Commun. Pure Appl. Anal.*, 18 (2019), 1827–1846.

[6] P. W. Bates, K. Lu and B. Wang, Attractors for lattice dynamical systems, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 11 (2001), 143–153.

[7] J. Bell, Some threshold results for models of myelinated nerves, *Math. Biosci.*, 54 (1981), 181–190.

[8] J. Bell and C. Cosner, Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons, *Quart. Appl. Math.*, 42 (1984), 1–14.

[9] T. Caraballo, F. Morillas and J. Valero, Attractors of stochastic lattice dynamical systems with a multiplicative noise and non-Lipschitz nonlinearities, *J. Differential Equations*, 253 (2012), 667–693.
10. T. Caraballo, F. Morillas and J. Valero, Random attractors for stochastic lattice systems with non-Lipschitz nonlinearity, *J. Difference Equ. Appl.*, 17 (2011), 161–184.

11. T. L. Carrol and L. M. Pecora, Synchronization in chaotic systems, *Phys. Rev. Lett.*, 64 (1990), 821–824.

12. H. Chate and M. Courbage, Lattice systems, *Phys. D.*, 103 (1997), 1–611.

13. V. V. Chepyzhov and M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, *J. Math. Pures Appl. (9)*, 73 (1994), 279–333.

14. S.-N. Chow, Lattice dynamical systems, in *Dynamical Systems*, Lecture Notes in Math., 1822, Springer, Berlin, 2003, 1–102.

15. S.-N. Chow and J. Mallet-Paret, Pattern formation and spatial chaos in lattice dynamical systems. I, II, *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 42 (1995), 746–751, 752–756.

16. S.-N. Chow, J. Mallet-Paret and E. S. Van Vleck, Pattern formation and spatial chaos in spatially discrete evolution equations, *Random Comput. Dynam.*, 4 (1996), 109–178.

17. L. O. Chua and T. Roska, The CNN paradigm, *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 40 (1993), 147–156.

18. L. O. Chua and L. Yang, Cellular neural networks: Theory, *IEEE Trans. Circuits and Systems*, 35 (1988), 1257–1272.

19. L. O. Chua and Y. Yang, Cellular neural networks: Applications, *IEEE Trans. Circuits and Systems*, 35 (1988), 1273–1290.

20. C. E. Elmer and E. S. Van Vleck, Spatially discrete FitzHugh-Nagumo equations, *SIAM J. Appl. Math.*, 65 (2005), 1153–1174.

21. T. Erneux and G. Nicolis, Propagating waves in discrete bistable reaction-diffusion systems, *Phys. D.*, 67 (1993), 237–244.

22. R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, *Biophys. J.*, 1 (1961), 445–466.

23. A. Gu and Y. Li, Singleton sets random attractor for stochastic FitzHugh-Nagumo lattice equations driven by fractional Brownian motions, *Commun. Nonlinear Sci. Numer. Simul.*, 19 (2014), 3929–3937.

24. A. Gu, Y. Li and J. Li, Random attractors on lattice of stochastic FitzHugh-Nagumo systems driven by α-stable Lévy noises, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 24 (2014), 9pp.

25. X. Han and P. E. Kloeden, Asymptotic behavior of a neural field lattice model with a Heaviside operator, *Phys. D.*, 389 (2019), 1–12.

26. X. Han, W. Shen and S. Zhou, Random attractors for stochastic lattice dynamical systems in weighted spaces, *J. Differential Equations*, 250 (2011), 1235–1266.

27. J. Huang, The random attractor of stochastic FitzHugh-Nagumo equations in an infinite lattice with white noises, *Phys. D.*, 233 (2007), 83–94.

28. J. Huang, X. Han and S. Zhou, Uniform attractors for non-autonomous Klein-Gordon-Schrödinger lattice systems, *Appl. Math. Mech. (English Ed.)*, 30 (2009), 1597–1607.

29. X. Jia, C. Zhao and X. Yang, Global attractor and Kolmogorov entropy of three component reversible Gray-Scott model on infinite lattices, *Appl. Math. Comput.*, 218 (2012), 9781–9789.

30. R. Kapral, Discrete models for chemically reacting systems, *J. Math. Chem.*, 6 (1991), 113–163.

31. J. P. Keener, Propagation and its failure in coupled systems of discrete excitable cells, *SIAM J. Appl. Math.*, 47 (1987), 556–572.

32. J. P. Keener, The effects of global gap junction coupling on propagation in myocardium, *J. Theor. Biol.*, 148 (1991), 49–82.

33. B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge-New York, 1982.

34. X.-J. Li and D.-B. Wang, Attractors for partly dissipative lattice dynamic systems in weighted spaces, *J. Math. Anal. Appl.*, 325 (2007), 141–156.

35. X. Liao, C. Zhao and S. Zhou, Compact attractors for dissipative non-autonomous lattice dynamical systems, *Commun. Pure Appl. Anal.*, 6 (2007), 1087–1111.

36. Q. Ma, S. Wang and C. Zhong, Necessary and sufficient conditions for the existence of global attractors for semigroups and applications, *Indiana Univ. Math. J.*, 51 (2002), 1541–1559.

37. J. Mallet-Paret and S.-N. Chow, Pattern formation and spatial chaos in lattice dynamical systems. I, II, *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 42 (1995), 752–756.

38. J. Nagumo, S. Arimoto and S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proc. IRE*, 50 (1964), 2061–2070.
[39] J. C. Oliveira, J. M. Pereira and G. Perla Menzala, Attractors for second order periodic lattices with nonlinear damping, *J. Difference Equ. Appl.*, 14 (2008), 899–921.

[40] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983.

[41] E. Van Vleck and B. Wang, Attractors for lattice FitzHugh-Nagumo systems, *Phys. D*, 212 (2005), 317–336.

[42] B. Wang, Asymptotic behavior of non-autonomous lattice systems, *J. Math. Anal. Appl.*, 331 (2007), 121–136.

[43] B. Wang, Dynamical behavior of the almost-periodic discrete FitzHugh-Nagumo systems, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 17 (2007), 1673–1685.

[44] B. Wang, Dynamics of stochastic reaction-diffusion lattice system driven by nonlinear noise, *J. Math. Anal. Appl.*, 477 (2019), 104–132.

[45] C. Wang, G. Xue and C. Zhao, Invariant Borel probability measures for discrete long-wave-short-wave resonance equations, *Appl. Math. Comput.*, 339 (2018), 853–865.

[46] R. Wang and Y. Li, Regularity and backward compactness of attractors for non-autonomous lattice systems with random coefficients, *Appl. Math. Comput.*, 354 (2019), 86–102.

[47] R. Wang and B. Wang, Random dynamics of lattice wave equations driven by infinite-dimensional nonlinear noise, *Discrete Contin. Dyn. Syst. Ser. B*, 25 (2020), 2461–2493.

[48] Y. Wang, Y. Liu and Z. Wang, Random attractors for partly dissipative stochastic lattice dynamical systems, *J. Difference. Equ. Appl.*, 14 (2008), 799–817.

[49] Z. Wang and S. Zhou, Random attractors for non-autonomous stochastic lattice FitzHugh-Nagumo systems with random coupled coefficients, *Taiwanese J. Math.*, 20 (2016), 589–616.

[50] X. Yang, C. Zhao and J. Cao, Dynamics of the discrete coupled nonlinear Schrödinger-Boussinesq equations, *Appl. Math. Comput.*, 219 (2013), 8508–8524.

[51] C. Zhao and S. Zhou, Compact uniform attractors for dissipative lattice dynamical systems with delays, *Discrete Contin. Dyn. Syst.*, 21 (2008), 643–663.

[52] C. Zhao, G. Xue and G. Łukaszewicz, Pullback attractors and invariant measures for discrete Klein-Gordon-Schrödinger equations, *Discrete Contin. Dyn. Syst. Ser. B*, 23 (2018), 4021–4044.

[53] S. Zhou, Attractors for second order lattice dynamical systems, *J. Differential Equations*, 179 (2002), 605–624.

[54] S. Zhou, Attractors for first order dissipative lattice dynamical systems, *Phys. D*, 178 (2003), 51–61.

[55] S. Zhou, Attractors and approximations for lattice dynamical systems, *J. Differential Equations*, 200 (2004), 342–368.

[56] S. Zhou and M. Zhao, Uniform exponential attractor for second order lattice system with quasi–periodic external forces in weighted space, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 24 (2014), 9pp.

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