COUNTING ELLIPTIC CURVES WITH BAD REDUCTION OVER A PRESCRIBED SET OF PRIMES

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Abstract. Let $p \geq 5$ be a prime and $T$ a Kodaira type of the special fiber of an elliptic curve. We estimate the number of elliptic curves over $\mathbb{Q}$ up to height $X$ with Kodaira type $T$ at $p$. This enables us find the proportion of elliptic curves over $\mathbb{Q}$, when ordered by height, with Kodaira type $T$ at a prime $p \geq 5$ inside the set of all elliptic curves. This proportion is a rational function in $p$. For instance, we show that $\frac{p^8(p-1)}{p^9-1}$ of all elliptic curves with bad reduction at $p$ are of multiplicative reduction. Furthermore, we prove that the prime-to-6 part of the conductors of a majority ($= \zeta(10)/\zeta(2) \approx 0.6$) of elliptic curves are squarefree, where $\zeta$ is the Riemann-zeta function.

1. Introduction

An elliptic curve $E$ defined over $\mathbb{Q}$ has good reduction everywhere except for finitely many primes. These are precisely the prime divisors of the minimal discriminant of $E$. If $p$ is a prime of good, multiplicative, or additive reduction, then $E$ is reduced modulo $p$ to an elliptic curve, a nodal curve, or a cuspidal curve, respectively.

Given a finite set $S$ of primes, one wants to identify the elliptic curves with bad reduction exactly at the primes in $S$. There are several approaches to study these curves. One may look for an algorithm that generates the elliptic curves with good reduction outside a prescribed set of primes, see [5] for an algorithm that finds these elliptic curves over $\mathbb{Q}$, and [11] for an algorithm that finds these curves over a number field. In [1], an algorithm was suggested to find all elliptic curves with bad reduction at exactly one prime.

Another approach is to list these elliptic curves explicitly. This was achieved for sets $S$ of size at most 2 and assuming certain conditions on the elliptic curves. One may consult [6, 7, 8, 14, 16, 17, 19] in which there are lists of elliptic curves with at most two bad primes assuming the existence of a rational torsion point. The main idea is to solve certain Diophantine equations obtained by equating the minimal discriminant to a product of at most two prime powers.

The prime divisors of the conductor are exactly those of the minimal discriminant. If $p \geq 5$ is a prime divisor of the conductor, then the power of $p$ dividing the conductor is at most 2. It is worth noting that there are certain positive integers that cannot appear as the conductor of an elliptic curve. For example, if $E$ is an elliptic curve over a number

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field $K$ with everywhere potential good reduction, i.e., it attains good reduction after a finite extension, then the minimal discriminant divides the fifth power of the conductor, see [12] and [15]. Furthermore, if the conductor of $E$ is a rational prime, then the minimal discriminant of $E$ divides the fifth power of the conductor, [13, 20]. In particular, this implies that there are no elliptic curves with bad reduction at exactly one prime $p$ such that this reduction is multiplicative and $p^6$ divides the discriminant.

There are finitely many elliptic curves having the same discriminant. In fact, there is only a finite number of elliptic curves that possess the same conductor. In [4], the number of elliptic curves with conductor $N$ is bounded by $O\left(N^{1/2+\epsilon}\right)$. This bound was improved in [18]. It was shown that the number of such elliptic curves is $O\left(N^{55/112+\epsilon}\right)$. The latter bound was improved further in [9] to be $O\left(N^{0.22377}\right)$. These results were performed by bounding the number of integral points on certain algebraic curves of genus at most 1.

If $E$ is an elliptic curve defined over the $p$-adic field $\mathbb{Q}_p$ with bad reduction over the finite field $\mathbb{F}_p$, then one may assign a symbol, Kodaira type, to the special fiber of the minimal proper regular model of $E$. This symbol describes the number, multiplicity, and intersection of the irreducible components of the special fiber. For instance, the Kodaira type $I_n$ indicates that the elliptic curve $E$ has multiplicative reduction modulo $p$, and that the irreducible components of the special fiber of the minimal proper regular model are of multiplicity 1, and they are arranged in an $n$-gon.

In this note, we consider the following question. Given a finite set $S$ of primes $5 \leq p_1 \leq \ldots \leq p_n$, how one may be able to count elliptic curves over $\mathbb{Q}$ with bad reduction described by the Kodaira type $T_i$ at $p_i$, $i = 1, \ldots, n$. These elliptic curves do not necessarily have good reduction outside $S$. Nevertheless, the set $S$ is contained in the set of bad primes of these elliptic curves.

The counting problem can be described as follows. Any elliptic curve can be described uniquely using a Weierstrass equation of the form $y^2 = x^3 + Ax + B$, $A, B \in \mathbb{Z}$, where if for a prime $q$ one has $q^4 \mid A$, then $q^6 \nmid B$. This implies that the elliptic curve is minimal except possibly at the primes 2 and 3. In fact the discriminant attached to this Weierstrass equation differs from the minimal discriminant by $d^{12}$ where $d$ is a divisor of 6. Elliptic curves can be ordered using the height function $H(E) = \max(|A|^3, |B|^2)$. There are only finitely many elliptic curves with height at most $X$. Now if $p \geq 5$ is a prime and $T$ is a Kodaira type, we set $\mathcal{E}_p^T(X)$ to be the set of elliptic curves with height at most $X$ and Kodaira type $T$ at $p$. The latter set can be described as a set of pairs $(A, B)$ of integers satisfying some congruence relations modulo powers of $p$. These congruences are implied by Tate’s algorithm that describes the special fiber of the minimal proper regular model of $E$.

Our main results rely on finding bounds for the set $\mathcal{E}_p^T(X)$. Given a Kodaira type $T$, we find the limit $\lim_{X \to \infty} \frac{\# \mathcal{E}_p^T(X)}{X^{5/6}}$. Moreover, if $\mathcal{E}_p(X)$ is the set of elliptic curves with bad
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reduction at \( p \) and height at most \( X \), we find the limit \( \lim_{X \to \infty} \frac{\# \mathcal{E}_p^T(X)}{\# \mathcal{E}_p(X)} \) for any Kodaira type \( T \) that describes bad reduction. We also evaluate the limit \( \lim_{X \to \infty} \frac{\# \mathcal{E}_p^T(X)}{\# \mathcal{E}(X)} \), where \( \mathcal{E}(X) \) is the set of all elliptic curves with height at most \( X \). The latter two limits are rational functions in \( p \). In particular, we prove that the majority of elliptic curves with bad reduction at \( p \) have multiplicative reduction. In fact, the proportion of elliptic curves with multiplicative reduction at \( p \) inside the set of elliptic curves with bad reduction at \( p \) is \( \frac{p^8(p - 1)}{p^9 - 1} \). This improves the result introduced in [2] that proves this proportion to be at least \((1-1/p)^2\). In fact, the latter lower bound is used in [2] to show that 100% of elliptic curves over \( \mathbb{Q} \), when ordered by height, have at least one prime of multiplicative reduction. We then display how one may compute the proportion of elliptic curves with prescribed Kodaira types at certain primes inside the set of elliptic curves (with bad reduction at these primes).

Semistable elliptic curves are elliptic curves with either good reduction or multiplicative reduction at any prime \( p \). In [23], it was proved that a positive proportion of elliptic curves consists of semistable elliptic curves. In fact, this proportion is 17.9%. We discuss the proportion of semistable elliptic curves except possibly at the primes 2 and 3. We show that the latter elliptic curves constitute \( \zeta(10)/\zeta(2) \approx 0.6 \) of all elliptic curves, when ordered by height, where \( \zeta \) is the Riemann-zeta function. In other words, more than 60% of all elliptic curves do not have additive reduction at any prime \( p \geq 5 \). This can be restated as follows: the prime-to-6 part of the conductor of 60% of all elliptic curves is squarefree.

2. Elliptic Curves with Bad Reduction at a Prime \( p \)

Let \( K \) be a discrete valuation field, \( O_K \) its ring of integers, \( \nu \) its normalized valuation, \( k \) the residue field of \( O_K \) with \( \text{char } k = p > 3 \). The reduction of an elliptic curve over \( K \) is classified according to its Kodaira type \( T \). In fact, assuming that \( k \) is algebraically closed, then \( T \) is described by one of the symbols in the set

\[
\mathcal{S} = \{ I_n, \ I_n^*, \ n \in \mathbb{Z}_{\geq 0}, \ II, \ II^*, \ III, \ III^*, \ IV, \ IV^* \}.
\]

Kodaira symbols are determined according to the number, arrangement, and multiplicity of the irreducible components of the special fiber of the minimal proper regular model of the elliptic curve. If \( k \) is not algebraically closed, then there exists a finite extension of \( k \) over which one of the above Kodaira types is realized. Furthermore, the valuation \( \nu(\Delta_E) \) of the discriminant \( \Delta_E := -4A^3 - 27B^2 \neq 0 \) of \( E \) is positive whenever the Kodaira type of \( E \) is not \( I_0 \), i.e., when \( E \) has bad reduction over \( k \).

Any elliptic curve \( E \) over \( \mathbb{Q} \) admits a unique Weierstrass equation of the form

\[
y^2 = x^3 + Ax + B
\]
where $A, B \in \mathbb{Z}$ and $\gcd(A^3, B^3)$ is not divisible by any twelfth power. Equivalently, for all primes $p$, $q^6 \nmid B$ whenever $p^4 \mid A$. In particular, the latter Weierstrass equation is minimal at any prime $p \geq 5$.

The naïve height of $E$ is defined by $\max (4|A|^3, 27|B|^2)$. In this note, by the height of $E$ we mean

$$H(E) := \max (|A|^3, |B|^2).$$

Results that we find using either the naïve height or the height of an elliptic curve are similar.

One sees now that there is a one-to-one correspondence between isomorphism classes of elliptic curves with height at most $X$ over $\mathbb{Q}$ and the following subset $\mathcal{E}(X)$ of $\mathbb{Z} \times \mathbb{Z}$

$$\mathcal{E}(X) := \left\{ (A, B) \in \mathbb{Z} \times \mathbb{Z} : \begin{array}{l}
|A| \leq \sqrt[3]{X}, |B| \leq \sqrt{X}, \\
4A^3 + 27B^2 \neq 0,
\end{array} \right\}.$$

We refer the reader to [3, Lemma 4.3] in which the size of the set of isomorphism classes of elliptic curves $\mathcal{E}(X)$ with height at most $X$ is given by $4X^{5/6}/\zeta(10) + O \left( \sqrt{X} \right)$.

Given $X > 0$, we write $I_X$ for the set $\{x \in \mathbb{Z} : -X \leq x \leq X\}$. We reproduce an estimate for the size of $\mathcal{E}(X)$ in the following proposition.

**Lemma 2.1.** Let $X > 0$ and $q_i$ the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_k := \prod_{i=1}^{k} q_i \leq \sqrt[12]{X}$. The size of $\mathcal{E}(X)$ is bounded above as follows

$$\#\mathcal{E}(X) \leq 4 \prod_{i=1}^{k} (q_i^{10} - 1) \lfloor \sqrt[3]{X}/Q_k^4 \rfloor \lfloor \sqrt[3]{X}/Q_k^6 \rfloor - 4 \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q \leq \sqrt[12]{X}} \lfloor \sqrt[3]{X}/q^4 Q_k^4 \rfloor \lfloor \sqrt[3]{X}/q^6 Q_k^6 \rfloor.$$

$$\#\mathcal{E}(X) \geq 4 \prod_{i=1}^{k} (q_i^{10} - 1) \lfloor \sqrt[3]{X}/Q_k^4 \rfloor \lfloor \sqrt[3]{X}/Q_k^6 \rfloor - 4 \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q \leq \sqrt[12]{X}} \lfloor \sqrt[3]{X}/q^4 Q_k^4 \rfloor \lfloor \sqrt[3]{X}/q^6 Q_k^6 \rfloor - (1 + 2\sqrt[3]{X}/\sqrt{2}).$$

**Proof:** The pair $(A, B)$ does not reduce to $(0, 0) \in \mathbb{Z}/q^4 \mathbb{Z} \times \mathbb{Z}/q^6 \mathbb{Z}$ for every prime $q \leq q_k$. Therefore, the pair $(A, B)$ can be a lift of one of $q^{10} - 1$ pairs of residue classes in $\mathbb{Z}/q^4 \mathbb{Z} \times \mathbb{Z}/q^6 \mathbb{Z}$. There might be still primes $q$ such that $q_k < q \leq \sqrt[12]{X}$ and $(A, B) = (0, 0) \in \mathbb{Z}/q^4 \mathbb{Z} \times \mathbb{Z}/q^6 \mathbb{Z}$. One then obtains the following estimate for the size of $\mathcal{E}(X)$

$$\#\mathcal{E}(X) \leq 4 \prod_{i=1}^{k} (q_i^{10} - 1) \lfloor \sqrt[3]{X}/Q_k^4 \rfloor \lfloor \sqrt[3]{X}/Q_k^6 \rfloor - 4 \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q \leq \sqrt[12]{X}} \lfloor \sqrt[3]{X}/q^4 Q_k^4 \rfloor \lfloor \sqrt[3]{X}/q^6 Q_k^6 \rfloor.$$
The bound obtained is not an exact value since we did not rule out the pairs which yield singular cubics, namely, the ones for which $4A^3 + 27B^2 = 0$.

The pairs $(A, B)$ for which $4A^3 + 27B^2 = 0$ can be parametrized as follows $(A, B) = (-3u^2, 2u^3)$ where $u \in \mathbb{Z}$. Therefore, the number of the latter pairs in $I_{\sqrt{X}} \times I_{\sqrt{X}}$ is $1 + 2\sqrt[3]{X}/\sqrt{2}$. However, when we rule these pairs out, we end up with the possibility that we ruled a singular cubic twice because we are disregarding the condition $q^6 \nmid B$ whenever $q^4 | B$. Thus we obtain the lower bound.

We are concerned about finding the size of subsets of $E(X)$. Namely, given an odd prime $p \geq 5$, we define the set $E_p(X)$ of elliptic curves with height at most $X$ and bad reduction at $p$. More precisely,

$$E_p(X) := \left\{ (A, B) \in \mathbb{Z} \times \mathbb{Z} : \begin{array}{l} |A| \leq \sqrt{X}, |B| \leq \sqrt{X}, \\ 4A^3 + 27B^2 \neq 0, \\ \text{for all primes } q \text{ if } q^4 \mid A \text{ then } q^6 \nmid B \\ 4A^3 + 27B^2 \equiv 0 \mod p \end{array} \right\}.$$

We define, moreover, the subset $E_p^T(X) \subset E_p(X)$ consisting of elliptic curves with Kodaira type $T$, $T \neq I_0$, at $p$ and height at most $X$.

We start with finding an upper bound for the size of $E_p(X)$ for an odd prime $p \geq 5$.

**Lemma 2.2.** Let $X > 0$ and $q_i$ the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_k := \prod_{i=1}^{k} q_i \leq \sqrt[3]{X}$. The size of $E_p(X)$ is bounded above by

$$\#E_p(X) \leq 4(p - 1) \prod_{i=1}^{k} (q_i^{10} - 1) [\sqrt{X}/pQ_k^6][\sqrt{X}/pQ_k^6] + 4(p^8 - 1) \prod_{i=1}^{k} (q_i^{10} - 1) [\sqrt[3]{X}/pq_i^4Q_k^4][\sqrt{X}/pq_i^6Q_k^6]$$

$$- 4(p - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q_i \leq \sqrt[3]{X}} [\sqrt[3]{X}/pq_i^4Q_k^4][\sqrt{X}/pq_i^6Q_k^6]$$

$$- 4(p^8 - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q_i \leq \sqrt[3]{X}} [\sqrt[3]{X}/pq_i^4Q_k^4][\sqrt{X}/pq_i^6Q_k^6].$$

**Proof:** One observes that the number of pairs of residue classes in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \setminus \{(0, 0)\}$ satisfying $4A^3 + 27B^2 = 0 \mod p$ is $p - 1$, see [21] Chapter III, §2, Proposition 2.5. We need to count the number of lifts of each such pair under the condition that $(A, B) \neq (0, 0) \in \mathbb{Z}/q_i^4\mathbb{Z} \times \mathbb{Z}/q_i^6\mathbb{Z}$ for each prime $q_i$. One notices that the number of such pairs in $\mathbb{Z}/q_i^4\mathbb{Z} \times \mathbb{Z}/q_i^6\mathbb{Z}$ is $1 + 2\sqrt[3]{X}/\sqrt{2}$.
Lemma 2.3. Let $q_i$ be the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_k := \prod_{i=1}^{k} q_i \leq \sqrt[3]{X}$. One has

$$4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left\lfloor \frac{\sqrt{X}}{pQ_k^4} \right\rfloor \left\lfloor \frac{\sqrt{X}}{pQ_k^6} \right\rfloor$$

pairs $(A, B)$ in $I_{\sqrt{X}} \times I_{\sqrt{X}}$ such that $(A, B) \neq (0, 0) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, $(A, B) \neq (0, 0) \in \mathbb{Z}/q_i^6\mathbb{Z} \times \mathbb{Z}/q_i^6\mathbb{Z}$, $i = 1, \ldots, k$, and $4A^2 + 27B^2 = 0 \mod p$.

For the point $(0, 0)$ in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, one has to be careful as there are lifts $(A, B)$ in $\mathbb{Z} \times \mathbb{Z}$ for which $p^4 \mid A$ and $p^6 \mid B$. In fact, if $(A, B) \in \mathbb{Z} \times \mathbb{Z}$ such that $(A, B) \neq (0, 0) \in \mathbb{Z}/p^4\mathbb{Z} \times \mathbb{Z}/p^6\mathbb{Z}$ whereas $(A, B) = (0, 0) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, then $(A, B)$ lies in one of $(p^3 \cdot p^5 - 1)$ pairs of residue classes in $\mathbb{Z}/p^4\mathbb{Z} \times \mathbb{Z}/p^6\mathbb{Z}$. Therefore, the number of pairs $(A, B)$ in $I_{\sqrt{X}} \times I_{\sqrt{X}}$ such that $(A, B) = (0, 0) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ but $(A, B) \neq (0, 0) \in \mathbb{Z}/p^4\mathbb{Z} \times \mathbb{Z}/p^6\mathbb{Z}$, and $(A, B) \neq (0, 0) \in \mathbb{Z}/q_i^4\mathbb{Z} \times \mathbb{Z}/q_i^6\mathbb{Z}$, $i = 1, \ldots, k$, is

$$4(p^8 - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \left\lfloor \frac{\sqrt{X}}{p^4Q_k^4} \right\rfloor \left\lfloor \frac{\sqrt{X}}{p^6Q_k^6} \right\rfloor.$$

Now for a prime $q$ such that $q_k < q \leq \sqrt[3]{X}$, one has to exclude the pairs $(A, B) \in \mathbb{Z} \times \mathbb{Z}$ for which $(A, B) = (0, 0) \in \mathbb{Z}/q^4\mathbb{Z} \times \mathbb{Z}/q^6\mathbb{Z}$. Therefore, the bound in the statement of the lemma is obtained.

In the above lemma we obtained an upper bound and not an exact value for the size of $\mathcal{E}_p(X)$ because we are disregarding the condition $4A^2 + 27B^2 \neq 0$.

Lemma 2.3. Let $X > 0$ and $q_i$ the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_k := \prod_{i=1}^{k} q_i \leq \sqrt[3]{X}$. One has

$$-\frac{X^{1/3}}{3d_1q_k^3} - \frac{X^{1/2}}{5d_2q_k^5} \leq \sum_{q_k < q \leq \sqrt[3]{X}} \left\lfloor \frac{\sqrt{X}/d_1q^4}{\sqrt{X}/d_2q^6} \right\rfloor \leq \frac{X^{5/6}}{9d_1d_2q_k^9}$$

where $d_1$ and $d_2$ are positive integers.

PROOF: The following inequalities hold

$$\sum_{q_k < q \leq \sqrt[3]{X}} \left(\sqrt{X}/d_1q^4 - 1\right) \left(\sqrt{X}/d_2q^6 - 1\right) \leq \sum_{q_k < q \leq \sqrt[3]{X}} \left\lfloor \sqrt{X}/d_1q^4 \right\rfloor \left\lfloor \sqrt{X}/d_2q^6 \right\rfloor \leq \frac{X^{5/6}}{d_1d_2} \sum_{q_k < q \leq \sqrt[3]{X}} 1/q^{10}.$$

The lower bound is

$$\frac{X^{5/6}}{d_1d_2} \sum_{q_k < q \leq \sqrt[3]{X}} 1/q^{10} - \frac{X^{1/3}}{d_1} \sum_{q_k < q \leq \sqrt[3]{X}} 1/q^4 - \frac{X^{1/2}}{d_2} \sum_{q_k < q \leq \sqrt[3]{X}} 1/q^6 + \sum_{q_k < q \leq \sqrt[3]{X}} \sum_{q_k < q \leq \sqrt[3]{X}} \left\lfloor \sqrt{X}/d_1q^4 \right\rfloor \left\lfloor \sqrt{X}/d_2q^6 \right\rfloor.$$

is $q_i^{10} - 1$. Therefore, according to the Chinese Remainder Theorem, there are
Thus,
\[-\frac{X^{1/3}}{d_1} \sum_{q_k < \sqrt[12]{X}} \frac{1}{q^4} - \frac{X^{1/2}}{d_2} \sum_{q_k < \sqrt[6]{X}} \frac{1}{q^6} \leq \sum_{q_k < \sqrt[12]{X}} \frac{\sqrt{X/d_1 q^4}}{2} \frac{\sqrt{X/d_2 q^6}}{2} \leq \frac{X^{5/6}}{d_1 d_2} \sum_{q_k < \sqrt[12]{X}} \frac{1}{q^{10}}.
\]

Now one concludes using the bound
\[
\sum_{q_k < \sqrt[12]{X}} \frac{1}{q^i} \leq \int_{q_k}^{\infty} \frac{1}{y^j} \, dy = \frac{1}{(j-1)q_k^{j-1}}, \quad j > 1.
\]

\[\square\]

The following is a direct consequence of Lemma 2.1, Lemma 2.2 and Lemma 2.3.

**Corollary 2.4.** Let $X > 0$ and $q_i$ the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_k := \prod_{i=1}^{k} q_i \leq \sqrt[12]{X}$. The size of $\mathcal{E}(X)$ is bounded as follows

\[
\#\mathcal{E}(X) \leq 4 \prod_{i=1}^{k} (q_i^{10} - 1) \frac{\sqrt{X/Q_k^4}}{3} \frac{\sqrt{X/Q_k^6}}{3} q_k Q_k^4 + \sqrt{X/5 q_k^5 Q_k^6}
\]

\[
\#\mathcal{E}(X) \geq 4 \prod_{i=1}^{k} (q_i^{10} - 1) \frac{\sqrt{X/Q_k^4}}{3} \frac{\sqrt{X/Q_k^6}}{3} - 4 \prod_{i=1}^{k} (q_i^{10} - 1) (X^{5/6}/9 q_k^3 Q_k^{10}) - (1 + 2 \sqrt{X/2}).
\]

In addition, the size of $\mathcal{E}_p(X)$ is bounded above by

\[
\#\mathcal{E}_p(X) \leq 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \frac{X^{5/6}}{p^2 Q_k^{10}} + 4(p^8 - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \frac{X^{5/6}}{p^1 Q_k^{10}}
\]

\[
+ \quad 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{X^{1/3}}{3 p Q_k^3 q_k^3} + \frac{X^{1/2}}{5 p Q_k^5 q_k^5} \right)
\]

\[
+ \quad 4(p^8 - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{X^{1/3}}{3 p^4 Q_k^3 q_k^3} + \frac{X^{1/2}}{5 p^6 Q_k^5 q_k^5} \right).
\]

3. **Elliptic curves with good reduction at a prime $p \geq 5$**

Given a prime $p \geq 5$, we find an estimate for the size of the set $\mathcal{E}_p^{lo}(X)$ containing isomorphism classes of elliptic curves with good reduction at $p$ and height at most $X$. We set

\[
\mathcal{E}_p^{lo}(X) = \left\{ (A, B) \in \mathbb{Z} \times \mathbb{Z} : \begin{array}{ccc} |A| \leq \sqrt[3]{X}, & |B| \leq \sqrt{X}, & 4A^3 + 27B^2 \neq 0 \mod p, \\ & & \text{for all primes } q \text{ if } q^4 \mid A \text{ then } q^6 \nmid B \end{array} \right\}
\]
The following result holds.

**Lemma 3.1.** Let $X > 0$ and $q_i$ the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_k := \prod_{i=1}^k q_i \leq \sqrt[p]{X}$. The size of $\mathcal{E}_p^l(X)$ is bounded as follows

$$
\#\mathcal{E}_p^l(X) \leq 4(p^2 - p) \prod_{i=1}^k (q_i^{10} - 1) \left\lfloor \frac{\sqrt[p]{X}}{pQ_k} \right\rfloor \left\lfloor \frac{\sqrt[p]{X}}{pQ_k^6} \right\rfloor + 4 \prod_{i=1}^k (q_i^{10} - 1) \left\lfloor \frac{\sqrt[p]{X}}{p^4Q_k^{10}} \right\rfloor \left\lfloor \frac{\sqrt[p]{X}}{p^6Q_k^{10}} \right\rfloor
$$

$$
- 4(p^2 - p) \prod_{i=1}^k (q_i^{10} - 1) \left( \frac{X^{5/6}}{9p^2q_k^{10}Q_k^{10}} \right) - 4 \prod_{i=1}^k (q_i^{10} - 1) \left( \frac{X^{5/6}}{9p^{10}q_k^{10}Q_k^{10}} \right).
$$

**Proof:** The number of pairs of residue classes in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ satisfying $4A^3 + 27B^2 \neq 0 \mod p$ is $p^2 - p$, see [21, Chapter III, §2, Proposition 2.5]. The number of lifts $(A, B)$ of each such pair such that $(A, B) \neq (0, 0) \in \mathbb{Z}/q^4\mathbb{Z} \times \mathbb{Z}/q^6\mathbb{Z}$ is $q^{10} - 1$ in $\mathbb{Z}/q^4\mathbb{Z} \times \mathbb{Z}/q^6\mathbb{Z}$. The Chinese Remainder Theorem implies that there are

$$
4(p^2 - p) \prod_{i=1}^k (q_i^{10} - 1) \left\lfloor \frac{\sqrt[p]{X}}{pQ_k} \right\rfloor \left\lfloor \frac{\sqrt[p]{X}}{pQ_k^6} \right\rfloor
$$

pairs $(A, B)$ in $I_{\sqrt[p]{X}} \times I_{\sqrt[p]{X}}$ such that $(A, B) \neq (0, 0) \in \mathbb{Z}/q^4\mathbb{Z} \times \mathbb{Z}/q^6\mathbb{Z}$, $i = 1, \ldots, k$, and $4A^2 + 27B^3 \neq 0 \mod p$.

Moreover, if $(A, B) = (0, 0) \in \mathbb{Z}/p^4\mathbb{Z} \times \mathbb{Z}/p^6\mathbb{Z}$, then the corresponding elliptic curve can be described using the following Weierstrass equation $y^2 = x^3 + A/p^4x + B/p^6$ which is minimal at $p$. This Weierstrass equation describes an elliptic curve whose reduction type is either good or multiplicative.

Now for a prime $q$ such that $q_k < q \leq \sqrt[p]{X}$, one has to exclude the pairs $(A, B) \in \mathbb{Z} \times \mathbb{Z}$ for which $(A, B) = (0, 0) \in \mathbb{Z}/q^4\mathbb{Z} \times \mathbb{Z}/q^6\mathbb{Z}$. Therefore, the bound follows as a direct consequence of Lemma 2.3.

One then obtains the following limits.

**Theorem 3.2.** The following limits hold

$$
\limsup_{X \to \infty} \frac{\#\mathcal{E}_p^l(X)}{X^{5/6}} \leq \frac{4(p - 1)}{p\zeta(10)} + \frac{4}{p^{10}\zeta(10)}
$$

$$
\limsup_{X \to \infty} \frac{\#\mathcal{E}_p^l(X)}{\#\mathcal{E}(X)} \leq 1 - \frac{1}{p} + \frac{1}{p^{10}},
$$

where $\zeta$ is the Riemann-zeta function.

**Proof:** Since $Y - 1 \leq \lfloor Y \rfloor \leq Y$, one sees that $\lim_{Y \to \infty} \lfloor Y \rfloor / Y = 1$. Moreover, one has $Q_k^{1/3} \leq 1$, $Q_k^{1/2} \leq 1$ and $q_k \to \infty$ as $X \to \infty$. Using the latter facts together with
Lemma 4.1. Let $Q$ in the definition of $E$. Theorem 3.2 implies that $\limsup E$. Unlike the definition of $E$, the second limit follows using the lower bound of $E$ in Lemma 2.1 and the lower bound for $E_p(X)$ in Lemma 3.1.

**Proof:** Using Lemmas 2.2 and 2.1, one deduces that $\limsup E$. Proof: Using Lemmas 2.2 and 2.1, one deduces that $\limsup E$. Theorem 3.2 implies that $\limsup E$. These two limits add up to 1, hence the result.

4. **Elliptic curves with multiplicative reduction at a prime $p \geq 5$**

In this section, given a prime $p \geq 5$, we find the size of the set $E_{p}^{I} \subset \mathcal{E}_{p}(X)$ where $n > 0$. We recall that the set $E_{p}^{I}(X)$ consists of elliptic curves with height at most $X$ and multiplicative reduction of Kodaira type $I_{n}$ at $p$. More precisely,

$$E_{p}^{I}(X) := \left\{ (A, B) \in \mathbb{Z} \times \mathbb{Z} : \begin{array}{l}
|A| \leq \sqrt[3]{X}, \ |B| \leq \sqrt[3]{X}, \\
p \nmid A, \ p \nmid B, \ p^{n} \parallel 4A^{3} + 27B^{2}, \\
(A, B) \neq (0, 0) \in \mathbb{Z}/q^{4}\mathbb{Z} \times \mathbb{Z}/q^{6}\mathbb{Z} \end{array} \right\}.$$ 

Unlike the definition of $E_{p}(X)$, one does not need to include the condition $4A^{3} + 27B^{2} \neq 0$ in the definition of $E_{p}^{I}(X)$, since this is implied by the fact that $p^{n+1} \nmid 4A^{3} + 27B^{2}$.

**Lemma 4.1.** Let $X > 0$ and $q_{i}$ the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_{k} := \prod_{i=1}^{k} q_{i} \leq \sqrt[3]{X}$. The size of $E_{p}^{I}(X)$, $n > 0$, is bounded as follows:

$$\#E_{p}^{I}(X) \leq 4p^{n}(p-1)^{2} \prod_{i=1}^{k} (q_{i}^{10} - 1) \left[ \frac{X^{5/6}}{9p^{2n+2}Q_{k}^{5/6}q_{k}^5} \right] - 4p^{n}(p-1)^{2} \prod_{i=1}^{k} (q_{i}^{10} - 1) \left( \frac{X^{1/3}}{3p^{n+1}Q_{k}^{4/3}q_{k}^3} + \frac{X^{1/2}}{5p^{n+1}Q_{k}^{5/2}q_{k}^{5/2}} \right)$$

**Proof:** The number of nonsingular points on the cubic curve $4A^{3} + 27B^{2} = 0 \mod p$ is $p - 1$. We disregard the singular point $(0, 0)$ as $p \nmid A$ and $p \nmid B$. The latter polynomial congruence has $p^{n-1}(p - 1)$ solution modulo $p^{n}$. Furthermore, Hensel’s Lemma states that
there are exactly $p^2 - p$ lifts of each such solution that do not satisfy the congruence modulo $p^{n+1}$. Therefore, one knows that the set $\mathcal{E}_p^m(X)$ has $p^n(p-1)^2$ representative pairs of residue classes in $\mathbb{Z}/p^{n+1}\mathbb{Z} \times \mathbb{Z}/p^{n+1}\mathbb{Z}$. One thus obtains the following value for the size of $\mathcal{E}_p^m(X)$ by excluding the pairs which reduce to $(0,0)$ in $\mathbb{Z}/q^4\mathbb{Z} \times \mathbb{Z}/q^6\mathbb{Z}$ where $q_k < q \leq \sqrt[5]{X}$

$$
\#\mathcal{E}_p^m(X) = 4p^n(p-1)^2 \prod_{i=1}^{k} (q_i^{10} - 1) \left[ \sqrt[p^{n+1}]{X/p^n Q_k^4} \right]\left[ \sqrt[p^{n+1}]{X/p^n Q_k^6} \right] - 4p^n(p-1)^2 \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q \leq \sqrt[5]{X}} \left[ \sqrt[p^{n+1}]{X/p^n q^4 Q_k^4} \right]\left[ \sqrt[p^{n+1}]{X/p^n q^6 Q_k^6} \right].
$$

Now using Lemma 2.3, one gets the bounds stated in the lemma.

\[\square\]

**Corollary 4.2.** One has \( \lim_{X \to \infty} \frac{\#\mathcal{E}_p^m(X)}{X^{5/6}} = 4 \frac{(p-1)^2}{p^{n+2} \zeta(10)} \), for \( n > 0 \).

**Proof:** The proof follows as a direct consequence of the bounds for $\#\mathcal{E}_p^m(X)$ in Lemma 4.1.

\[\square\]

**Theorem 4.3.** One has

$$
\liminf_{X \to \infty} \frac{\#\mathcal{E}_p^m(X)}{\#\mathcal{E}_p(X)} \geq \frac{p^8(p-1)^2}{p^{n+2}}, \quad \liminf_{X \to \infty} \frac{\#\mathcal{E}_p^m(X)}{\#\mathcal{E}(X)} \geq \frac{(p-1)^2}{p^{n+2}}.
$$

**Proof:** One uses the lower bound of $\#\mathcal{E}_p^m(X)$ in Lemma 4.1 and the upper bound of $\#\mathcal{E}_p(X)$ provided in Corollary 2.4 in order to bound liminf from below. The following inequality then follows

$$
\frac{\#\mathcal{E}_p^m(X)}{\#\mathcal{E}_p(X)} \geq \frac{4p^n(p-1)^2 [X^{1/3}/p^{n+1} Q_k^4] [X^{1/2}/p^{n+1} Q_k^6] - 4p^n(p-1)^2 \frac{X^{5/6}}{2p^{n+1} Q_k^4 Q_k^6}}{4(p-1) X^{5/6} Q_k^4 + 4(p^8 - 1) X^{5/6} Q_k^4 + 4(p-1) \left( \frac{X^{1/3}}{3p^{n+1} Q_k^4 Q_k^6} + \frac{X^{1/2}}{3p^{n+1} Q_k^4 Q_k^6} \right) + 4(p^8 - 1) \left( \frac{X^{1/3}}{3p^{n+1} Q_k^4 Q_k^6} + \frac{X^{1/2}}{3p^{n+1} Q_k^4 Q_k^6} \right)}.
$$

Since \( Y - 1 \leq [Y] \leq Y \), one sees that \( \lim_{Y \to \infty} [Y] / Y = 1 \). Moreover, the bounds \( Q_k^6 / X^{1/2} \leq 1 \) together with the fact that \( q_k \to \infty \) as \( X \to \infty \), imply that

$$
\liminf_{X \to \infty} \frac{\#\mathcal{E}_p^m(X)}{\#\mathcal{E}_p(X)} \geq \frac{4p^n(p-1)^2}{4(p-1)/p^2 + 4(p^8 - 1)/p^{10}} = \frac{p^8(p-1)^2}{p^{n+2}}.
$$

A similar argument yields the second limit.

\[\square\]

**Corollary 4.4.** Let $\mathcal{E}_p^m(X)$ be the set of isomorphism classes of elliptic curves over $\mathbb{Q}$ with multiplicative reduction at $p$ and height at most $X$. Then

$$
\liminf_{X \to \infty} \frac{\#\mathcal{E}_p^m(X)}{\#\mathcal{E}_p(X)} \geq \frac{p^8(p-1)}{p^n - 1}, \quad \liminf_{X \to \infty} \frac{\#\mathcal{E}_p^m(X)}{\#\mathcal{E}(X)} \geq \frac{1}{p} - \frac{1}{p^2}.
$$

**Proof:** This follows by summing over $n \geq 1$ in Theorem 4.3.

\[\square\]
5. The additive Kodaira types II, III, and IV

In this section, we measure the size of the set of isomorphism classes of elliptic curves with height at most $X$ and additive reduction of Kodaira type II, III, or IV at a prime $p \geq 5$.

In the following lemma we find bounds for the size of $\mathcal{E}_p^T$ where $T \in \{\text{II}, \text{III}, \text{IV}\}$.

**Lemma 5.1.** Let $X > 0$ and $q_i$ the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_k := \prod_{i=1}^{k} q_i \leq \frac{1}{2}X$. The size of $\mathcal{E}_p^T(X)$, $T \in \{\text{II}, \text{III}, \text{IV}\}$, is bounded as follows:

\[
\#\mathcal{E}_p^\text{II}(X) \geq 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) |\sqrt[p]{X}/pQ_k^4| |\sqrt[p]{X}/p^2Q_k^6| - 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \frac{X^{5/6}}{6p^4Q_k^6q_k^2} \\
\#\mathcal{E}_p^\text{III}(X) \geq 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) |\sqrt[p]{X}/p^2Q_k^6| |\sqrt[p]{X}/p^3Q_k^6| - 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \frac{X^{5/6}}{6p^4Q_k^6q_k^2} \\
\#\mathcal{E}_p^\text{IV}(X) \geq 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) |\sqrt[p]{X}/p^2Q_k^6| |\sqrt[p]{X}/p^3Q_k^6| - 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \frac{X^{5/6}}{6p^4Q_k^6q_k^2}
\]

**Proof:** Following Tate’s algorithm, [22] Chapter IV, §9, in case $E : y^2 = x^3 + Ax + B$ has multiplicative reduction of type II, then $p \mid A$ and $p \mid B$, hence $p^2 \mid \Delta$. It follows that the only pair of residue classes in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ that lifts to $(A, B)$ is $(0, 0)$. In fact, since $p \mid B$, one has that in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$ there are $p - 1$ elements that lift to $(A, B)$. They are the ones of the form $(0, \mu p)$, $\mu = 1, 2, \ldots, p - 1$. In order to check that $(A, B) \notin \mathbb{Z}/q^4\mathbb{Z} \times \mathbb{Z}/q^6\mathbb{Z}$ for every prime $q$, one has

\[
\#\mathcal{E}_p^\text{II}(X) = 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) |\sqrt[p]{X}/pQ_k^4| |\sqrt[p]{X}/p^2Q_k^6| \\
- 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q \leq \frac{1}{2}X} |\sqrt[p]{X}/pq^4Q_k^6| |\sqrt[p]{X}/p^2q^6Q_k^6|
\]

Now the bounds when the Kodaira type is II follow using Lemma [23].

For the Kodaira type III, the Weierstrass equation $y^2 = x^3 + Ax + B$ satisfies that $p \mid A$, $p^2 \mid B$, hence $p^3 \mid \Delta$. Therefore, in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$ one has one pair of residue classes that
lifts to \((A, B)\), namely \((0, 0)\). Consequently, in \(\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}\) one has \(p - 1\) such pairs. Therefore,

\[
\#\mathcal{E}_p^{III}(X) = 4(p - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \left[ \sqrt[3]{X/p^2Q_k^1} \right] \left[ \sqrt{X/p^2Q_k^6} \right]
- 4(p - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q \leq \sqrt[3]{X/p^2Q_k^4} \sqrt{X/p^2Q_k^6}} \left| \sqrt[3]{X/p^2q^4Q_k^4} \right| \left| \sqrt{X/p^2q^6Q_k^6} \right|.
\]

Now one uses Lemma 2.3 to get the bounds.

Finally, for the Kodaira type IV, the Weierstrass equation \(y^2 = x^3 + Ax + B\) satisfies that \(p^2 | A, p^2 || B\), hence \(p^4 || \Delta\). Therefore, there are \((p-1)\) pairs of residue classes in \(\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}\) that lift to such an \((A, B)\). Consequently, one gets

\[
\#\mathcal{E}_p^{IV}(X) = 4(p - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \left[ \sqrt[3]{X/p^2Q_k^1} \right] \left[ \sqrt{X/p^2Q_k^6} \right]
- 4(p - 1) \prod_{i=1}^{k} (q_i^{10} - 1) \sum_{q_k < q \leq \sqrt[3]{X/p^2Q_k^4} \sqrt{X/p^2Q_k^6}} \left| \sqrt[3]{X/p^2q^4Q_k^4} \right| \left| \sqrt{X/p^2q^6Q_k^6} \right|.
\]

Again we use Lemma 2.3 to conclude. \(\square\)

As a direct consequence of Lemma 5.1, one reaches the following corollary that displays the value of the limit \(\lim_{X \to \infty} \frac{\#\mathcal{E}_p^T(X)}{X^{5/6}}, T \in \{\text{II, III, IV}\}\).

**Corollary 5.2.** For \(X > 0\), one has

\[
\lim_{X \to \infty} \frac{\#\mathcal{E}_p^{II}(X)}{X^{5/6}} = 4 \frac{p - 1}{p^3 \zeta(10)}
\]
\[
\lim_{X \to \infty} \frac{\#\mathcal{E}_p^{III}(X)}{X^{5/6}} = 4 \frac{p - 1}{p^4 \zeta(10)}
\]
\[
\lim_{X \to \infty} \frac{\#\mathcal{E}_p^{IV}(X)}{X^{5/6}} = 4 \frac{p - 1}{p^5 \zeta(10)}
\]

The following result presents the proportion of elliptic curves with Kodaira type \(T, T \in \{\text{II, III, IV}\}\), at \(p \geq 5\) among elliptic curves with bad reduction at \(p\), when ordered by height.
Moreover, the following result.

Let \( \# \mathcal{E}_p^\| (X) \) the set of elliptic curves over \( \mathbb{Q} \) that admit multiplicative reduction after a finite extension. Those elliptic curves are the elliptic curves with potentially multiplicative reduction. We set \( \mathcal{E}_p^{pm}(X) \) to be the set of elliptic curves over \( \mathbb{Q} \) with potentially multiplicative reduction. One then obtains the following result.

**Theorem 5.3.** One has

\[
\liminf_{X \to \infty} \frac{\# \mathcal{E}_p^\| (X)}{\# \mathcal{E}_p(X)} \geq \frac{p^7(p-1)}{p^9-1}, \quad \liminf_{X \to \infty} \frac{\# \mathcal{E}_p^\| (X)}{\# \mathcal{E}(X)} \geq \frac{p-1}{p^3}.
\]

\[
\liminf_{X \to \infty} \frac{\# \mathcal{E}_p^\| (X)}{\# \mathcal{E}_p(X)} \geq \frac{p^6(p-1)}{p^5-1}, \quad \liminf_{X \to \infty} \frac{\# \mathcal{E}_p^\| (X)}{\# \mathcal{E}(X)} \geq \frac{p-1}{p^4}.
\]

\[
\liminf_{X \to \infty} \frac{\# \mathcal{E}_p^\| (X)}{\# \mathcal{E}_p(X)} \geq \frac{p^5(p-1)}{p^4-1}, \quad \liminf_{X \to \infty} \frac{\# \mathcal{E}_p^\| (X)}{\# \mathcal{E}(X)} \geq \frac{p-1}{p^5}.
\]

**Proof:** The proof is similar to the proof of Theorem 4.3. \( \square \)

6. **The additive Kodaira type \( \Gamma_n \), \( n \geq 0 \)**

An elliptic curve with Kodaira type \( \Gamma_n \), \( n > 0 \), is the quadratic twist of an elliptic curve with multiplicative reduction of Kodaira type \( \Gamma_n \). More precisely, given an elliptic curve \( y^2 = x^3 + Ax + B \) with Kodaira type \( \Gamma_n \), \( n > 0 \), at a prime \( p \geq 5 \), its quadratic twist \( y^2 = x^3 + p^2Ax + p^3B \) can be minimized to be given by \( y^2 = x^3 + (A/p^2)x + (B/p^3) \) whose Kodaira type is \( \Gamma_n \). For an elliptic curve with Kodaira type \( \Gamma_n \) described by the Weierstrass equation \( y^2 = x^3 + Ax + B \) where \( \gcd(A, B) \) is a 12-th power free, one has \( p^2 \| A \) and \( p^3 \| B \). Moreover, \( p^{n+6} \| \Delta \). Therefore, in order to count elliptic curves in \( \mathcal{E}_p^{\|} (X) \), we need to count the pairs \( (A, B) \in \mathbb{Z} \times \mathbb{Z} \) such that \( p \nmid A, p \nmid B, p^n|4A^3 + 27B^2 \) where \( A \leq \sqrt[3]{X}/p^2 \), \( B \leq \sqrt{X}/p^3 \). In view of Lemma 4.1 one has the following result.

**Lemma 6.1.** Let \( X > 0 \) and \( q_i \), the \( i \)-th prime. Let \( k > 0 \) be the largest positive integer such that \( Q_k := \prod_{i=1}^{k} q_i \leq \sqrt[3]{X} \). The size of \( \mathcal{E}_p^{\|} (X) \), \( n > 0 \), is bounded as follows:

\[
\# \mathcal{E}_p^{\|} (X) \geq 4p^n(p-1)^2 \prod_{i=1}^{k} (q_i^{10} - 1)[\sqrt{X}/p^{n+3}Q_k^4][\sqrt{X}/p^{n+4}Q_k^6] - 4p^n(p-1)^2 \prod_{i=1}^{k} (q_i^{10} - 1)\frac{X^{5/6}}{2p^{n+7}Q_k^{10}q_k}.
\]

\[
\# \mathcal{E}_p^{\|} (X) \leq 4p^n(p-1)^2 \prod_{i=1}^{k} (q_i^{10} - 1)[\sqrt{X}/p^{n+3}Q_k^4][\sqrt{X}/p^{n+4}Q_k^6] + 4(p^2 - p) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{X^{1/3}}{3p^{n+4}Q_k^4q_k} + \frac{X^{1/2}}{5p^{n+4}Q_k^6q_k} \right).
\]

In addition, \( \lim_{X \to \infty} \frac{\# \mathcal{E}_p^{\|} (X)}{X^{5/6}} = 4 \frac{(p-1)^2}{p^{n+7}\zeta(10)} \).

It is known that elliptic curves whose Kodaira type is \( \Gamma_n \), \( n > 0 \), at a prime \( p \geq 5 \) are those which admit multiplicative reduction after a finite extension. Those elliptic curves are the ones we call elliptic curves with potentially multiplicative reduction. We set \( \mathcal{E}_p^{pm}(X) \) to be the set of elliptic curves over \( \mathbb{Q} \) with potentially multiplicative reduction. One then obtains the following result.
Theorem 6.2. One has
\[ \liminf_{X \to \infty} \frac{\#E_p^{15}(X)}{\#E_p(X)} \geq \frac{p^3(p - 1)^2}{p^n(p^9 - 1)}, \quad \liminf_{X \to \infty} \frac{\#E_p^{15}(X)}{\#E(X)} \geq \frac{(p - 1)^2}{p^{n+7}}. \]

Moreover,
\[ \liminf_{X \to \infty} \frac{\#E_p^{pm}(X)}{\#E_p(X)} \geq \frac{p^3(p - 1)}{p^9 - 1}, \quad \liminf_{X \to \infty} \frac{\#E_p^{pm}(X)}{\#E(X)} \geq \frac{p - 1}{p^5}. \]

For an elliptic curve \( y^2 = x^3 + Ax + B \) with Kodaira type I\(_0^*\) at a prime \( p \geq 5 \), Tate’s algorithm implies that \( p^2 \mid A, p^3 \mid B \), and \( p^6 \parallel \Delta \). In addition, the quadratic twist \( y^2 = x^3 + p^2 Ax + p^3 B \) can be rewritten as \( y^2 = x^3 + (A/p^2)x + (B/p^3) \) which describes an elliptic curve with good reduction at \( p \). Therefore, in order to estimate the size of \( \#E_p^{15}(X) \), we need to count the pairs \( (A, B) \in \mathbb{Z} \times \mathbb{Z} \) such that \( A^3 - B^2 \neq 0 \) mod \( p \), \( p \nmid A \), \( p \nmid B \), and \( A \leq \sqrt[3]{X}/p^2 \), \( B \leq \sqrt[3]{X}/p^3 \). In \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \), the number of pairs of residue classes which do not lie on the curve \( 4A^3 + B^2 = 0 \) mod \( p \) is \( p^2 - p \). Therefore, one gets the following result.

Lemma 6.3. Let \( X > 0 \) and \( q_i \) the \( i \)-th prime. Let \( k > 0 \) be the largest positive integer such that \( Q_k := \prod_{i=1}^{k} q_i \leq \sqrt[3]{X} \). The size of \( E_p^{15}(X) \), \( n > 0 \), is bounded as follows:
\[ \#E_p^{15}(X) \geq 4(p^2 - p) \prod_{i=1}^{k} (q_i^{10} - 1)[\sqrt[3]{X}/p^3 Q_k^4]\frac{X^{5/6}}{9p^7 Q_k^{10} q_k^5} \]
\[ \#E_p^{15}(X) \leq 4(p^2 - p) \prod_{i=1}^{k} (q_i^{10} - 1)[\sqrt[3]{X}/p^3 Q_k^4]\frac{X^{1/3}}{3p^3 Q_k^{10} q_k^5} + 4p^n(p - 1)^2 \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{X^{1/3}}{3p^3 Q_k^{10} q_k^5} + \frac{X^{1/2}}{5p^4 Q_k^{10} q_k^5} \right) \]

In addition, \( \lim_{X \to \infty} \frac{\#E_p^{15}(X)}{X^{5/6}} = 4\frac{p - 1}{p^6 \zeta(10)} \).

Theorem 6.4. One has
\[ \liminf_{X \to \infty} \frac{\#E_p^{15}(X)}{\#E_p(X)} \geq \frac{p^4(p - 1)}{p^9 - 1}, \quad \liminf_{X \to \infty} \frac{\#E_p^{15}(X)}{\#E(X)} \geq \frac{p - 1}{p^6}. \]

7. The additive Kodaira types IV\(^*\), III\(^*\), and II\(^*\)

Assume that \( E \) is an elliptic curve defined over \( \mathbb{Q} \) by the Weierstrass equation \( y^2 = x^3 + Ax + B \), \( A, B \in \mathbb{Z} \), where \( \gcd(A, B) \) is 12-th power free. Assume, moreover, that \( E \) has additive reduction at a prime \( p \geq 5 \) with Kodaira type IV\(^*\), III\(^*\), or II\(^*\). Then the quadratic twist of \( E \) by \( p \) has additive reduction at \( p \) with respective Kodaira type II, III, or IV. The quadratic twist \( E_p \) of \( E \) by \( p \) may be described by \( y^2 = x^3 + (A/p^2)x + (B/p^3) \).

In case \( E \) has Kodaira type IV\(^*\) at \( p \), then \( p^3 \mid A \) and \( p^4 \parallel B \). Therefore, one needs to count the pairs \( (A, B) \in \mathbb{Z} \times \mathbb{Z} \) such that \( p \mid A, p \parallel B \), and \( A \leq \sqrt[3]{X}/p^2 \), \( B \leq \sqrt[3]{X}/p^3 \).
If \( E \) has Kodaira type III\(^*\) at \( p \), then \( p^3 \| A \) and \( p^5 \| B \). To find the size of \( E_{p}^{III*}(X) \), one counts the pairs \((A, B) \in \mathbb{Z} \times \mathbb{Z}\) such that \( p \| A \), \( p^2 \| B \), and \( A \leq \sqrt[3]{X}/p^2 \), \( B \leq \sqrt[5]{X}/p^3 \).

For the elliptic curve \( E \) to have Kodaira type II\(^*\) at \( p \), one has \( p^4 \| A \) and \( p^5 \| B \). To find the size of \( E_{p}^{II*}(X) \), one counts the pairs \((A, B) \in \mathbb{Z} \times \mathbb{Z}\) such that \( p^2 \| A \), \( p^3 \| B \), and \( A \leq \sqrt[3]{X}/p^2 \), \( B \leq \sqrt[5]{X}/p^3 \).

The following result is a direct consequence of the calculations above together with Lemma 5.1.

**Lemma 7.1.** Let \( X > 0 \) and \( q_i \) the \( i \)-th prime. Let \( k > 0 \) be the largest positive integer such that \( q_k := \prod_{i=1}^{k} q_i \leq \sqrt[5]{X} \). The size of \( E_{p}^{T}(X) \), \( T \in \{IV^*, III^*, II^*\} \), is bounded as follows:

\[
\#E_{p}^{IV^*}(X) \geq 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{1}{3q_i^{10}q_k^{10}q_k} \right) \left( \frac{X^{5/6}}{9p^{5/6}q_k^{10}q_k} \right) - 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{X^{1/3}}{3p^{3/4}q_k^{10}q_k} \right) + \frac{X^{1/2}}{5p^{5/6}q_k^{10}q_k}
\]

\[
\#E_{p}^{IV^*}(X) \leq 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{1}{3q_i^{10}q_k^{10}q_k} \right) \left( \frac{X^{5/6}}{9p^{5/6}q_k^{10}q_k} \right) + 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{X^{1/3}}{3p^{3/4}q_k^{10}q_k} \right) \left( \frac{X^{1/2}}{5p^{5/6}q_k^{10}q_k} \right)
\]

\[
\#E_{p}^{III^*}(X) \geq 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{1}{3q_i^{10}q_k^{10}q_k} \right) \left( \frac{X^{5/6}}{9p^{5/6}q_k^{10}q_k} \right) - 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{X^{1/3}}{3p^{3/4}q_k^{10}q_k} \right) \left( \frac{X^{1/2}}{5p^{5/6}q_k^{10}q_k} \right)
\]

\[
\#E_{p}^{III^*}(X) \leq 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{1}{3q_i^{10}q_k^{10}q_k} \right) \left( \frac{X^{5/6}}{9p^{5/6}q_k^{10}q_k} \right) + 4(p-1) \prod_{i=1}^{k} (q_i^{10} - 1) \left( \frac{X^{1/3}}{3p^{3/4}q_k^{10}q_k} \right) \left( \frac{X^{1/2}}{5p^{5/6}q_k^{10}q_k} \right)
\]

In addition, one has

\[
\lim_{X \to \infty} \frac{\#E_{p}^{IV^*}(X)}{X^{5/6}} = \frac{4}{p^5} \frac{p-1}{\zeta(10)}
\]

\[
\lim_{X \to \infty} \frac{\#E_{p}^{III^*}(X)}{X^{5/6}} = \frac{4}{p^5} \frac{p-1}{\zeta(10)}
\]

\[
\lim_{X \to \infty} \frac{\#E_{p}^{II^*}(X)}{X^{5/6}} = 4 \frac{p-1}{p^{10}}
\]
Theorem 7.2. One has

\[
\liminf_{X \to \infty} \frac{\#E_{IV}^*(X)}{\#E_p(X)} \geq \frac{p^2(p-1)}{p^9-1}, \quad \liminf_{X \to \infty} \frac{\#E_{IV}^*(X)}{\#E(X)} \geq \frac{p-1}{p^8},
\]

\[
\liminf_{X \to \infty} \frac{\#E_{III}^*(X)}{\#E_p(X)} \geq \frac{p(p-1)}{p^9-1}, \quad \liminf_{X \to \infty} \frac{\#E_{III}^*(X)}{\#E(X)} \geq \frac{p-1}{p^9},
\]

\[
\liminf_{X \to \infty} \frac{\#E_{II}^*(X)}{\#E_p(X)} \geq \frac{p-1}{p^9-1}, \quad \liminf_{X \to \infty} \frac{\#E_{II}^*(X)}{\#E(X)} \geq \frac{p-1}{p^{10}}.
\]

8. Exact proportions of elliptic curves with prescribed Kodaira types over a given set of primes

According to Theorems 4.3, 5.3, 6.2, 6.4 and 7.2, the lower bound of the sum of the proportions of elliptic curves with a certain Kodaira type describing a bad reduction at \( p \geq 5 \) among elliptic curves with bad reduction at \( p \) is 1. This implies that these lower bounds are the exact proportions. Thus, one concludes with the following result.

Theorem 8.1. Let \( p \geq 5 \) be a prime. The limit \( \lim_{X \to \infty} \frac{\#E_T(X)}{\#E_p(X)} \), where \( T \) is a Kodaira type describing a bad reduction, is determined according to the following table.

| \( T \) | \( \lim_{X \to \infty} \frac{\#E_T(X)}{\#E_p(X)} = \) |
|------|--------------------------------------------------|
| \( I_n, n > 0 \) | \( p^8(p-1)^2/p^n(p^9-1) \) |
| \( II \) | \( p^7(p-1)/(p^9-1) \) |
| \( III \) | \( p^6(p-1)/(p^9-1) \) |
| \( IV \) | \( p^5(p-1)/(p^9-1) \) |
| \( I_0 \) | \( p^4(p-1)/(p^9-1) \) |
| \( I_n, n > 0 \) | \( p^3(p-1)^2/p^n(p^9-1) \) |
| \( IV^* \) | \( p^2(p-1)/(p^9-1) \) |
| \( III^* \) | \( p(p-1)/(p^9-1) \) |
| \( II^* \) | \( (p-1)/(p^9-1) \) |

In a similar fashion, the limit \( \lim_{X \to \infty} \frac{\#E_T(X)}{\#E(X)} \) is determined as follows.
where \( \lim_{X \to \infty} \frac{\#\mathcal{E}_p^T(X)}{\#\mathcal{E}(X)} = \frac{(p-1)^2/p^{n+2}}{p^{10}} \) is the proportion of elliptic curves over \( \mathbb{Q} \) with height at most \( X \) and bad reduction described by the Kodaira type \( T_i \) at \( p_i \), \( i = 1, 2, \ldots, j \), and \( \mathcal{E}_p^{T_1,\ldots,T_j}(X) \) is the set of elliptic curves over \( \mathbb{Q} \) with height at most \( X \) and bad reduction described by the Kodaira type \( T_i \) at \( p_i \), \( i = 1, \ldots, j \). The proportion of elliptic curves over \( \mathbb{Q} \), when ordered by height, with bad reduction described by the Kodaira type \( T_i \) at \( p_i \), \( 1 \leq i \leq j \), is determined as follows:

\[
\lim_{X \to \infty} \frac{\#\mathcal{E}_p^{T_1,\ldots,T_j}(X)}{\#\mathcal{E}_p^{T_1}(X)} = \prod_{1 \leq i \leq j} P_{p_i}(T_i)
\]

whereas \( \lim_{X \to \infty} \frac{\#\mathcal{E}_p^0(X)}{\#\mathcal{E}(X)} = \frac{p^{10} - p^9 + 1}{p^{10}} \).
with height at most $X$ and bad reduction at $p_i$, $i = 1, \ldots, j$. In addition,
\[
\lim_{X \to \infty} \frac{\#E_{p_1^{\alpha_1} \ldots p_j^{\alpha_j}}(X)}{\#E(X)} = \prod_{1 \leq i \leq j} P'_{p_i}(T_i).
\]

9. Elliptic curves with squarefree conductors

Let $E$ be an elliptic curve over $\mathbb{Q}$. Let $q_i$ be the $i$th-prime. The conductor of $E$ is $N = \prod q_i^{\alpha_i}$ where $\alpha_i = 0$ if $E$ has good reduction at $q_i$, and $\alpha_i = 1$ if $E$ has multiplicative reduction at $q_i$. In case, $q_i$ is a prime of additive reduction for $E$, then $\alpha_i$ is 2 if $q_i \geq 5$; $2 \leq \alpha_i \leq 5$ if $q_i = 3$; or $2 \leq \alpha_i \leq 8$ if $q_i = 2$. We write $N^*$ for the greatest divisor of $N$ which is coprime-to-6, namely, $N/(2^{\alpha_1} 3^{\alpha_2})$.

Remark 9.1. Given an odd prime $p \geq 5$, in view of Theorem 8.1, among elliptic curves for which $p | N$, the majority of these curves, $p^8(p - 1)(p^9 - 1)$, satisfy that $p || N$.

In this section we find an estimate for the proportion of elliptic curves with square-free $N^*$. For this purpose we define the following two sets of isomorphism classes of elliptic curves.

\[
E^*(X) = \left\{ (A, B) \in \mathbb{Z} \times \mathbb{Z} : \begin{array}{l}
|A| \leq \sqrt[3]{X}, |B| \leq \sqrt{X}, \\
4A^3 + 27B^2 \neq 0, \\
\text{for all primes } q \text{ if } q^4 \mid A \text{ then } q^6 \nmid B, \\
A \neq 0 \text{ or } B \neq 0 \text{ in } \mathbb{Z}/q\mathbb{Z} \text{ for all primes } q
\end{array} \right\}
\]

The set above consists of elliptic curves with either good or multiplicative reduction at every prime $q \geq 5$ up to height $X$. This holds because if $A \neq 0 \mod q$, then the elliptic curve $E$ either have multiplicative reduction at $q$ if $\Delta = 0 \mod q$, hence $B \neq 0 \mod q$, or it has good reduction if $\Delta \neq 0 \mod q$. The same argument holds if $B \neq 0 \mod q$.

Lemma 9.2. Let $X > 0$ and $q_i$ the $i$-th prime. Let $k > 0$ be the largest positive integer such that $Q_k := \prod_{i=1}^{k} q_i \leq \sqrt[3]{X}$. The following bound holds

\[
\#E^*(X) \geq 4 \prod_{i=1}^{k} q_i^8(q_i^2 - 1)[\sqrt[3]{X}/Q_k^4][\sqrt{X}/Q_k^6] - 4 \prod_{i=1}^{k} q_i^8(q_i^2 - 1)\frac{X^{5/6}}{9q_kQ_k^1} - (1 + 2 \sqrt[3]{X}/3\sqrt{2}).
\]
Proof: The number of pairs of residue classes in \( \mathbb{Z}/q^4 \mathbb{Z} \times \mathbb{Z}/q^6 \mathbb{Z} \) that lift to pairs \((A, B) \in \mathbb{Z} \times \mathbb{Z}\) such that \( q \nmid A \) is given by \( q^6 \cdot q^3(q - 1) \). Now out of the remaining pairs of residue classes, we have to add those for which \( B \neq 0 \mod q \) whereas \( A = 0 \mod q \). These latter pairs are \([q^4 - q^3(q - 1)]q^5(q - 1)\). For primes \( q \) satisfying \( q_k \leq q \leq \frac{12}{\sqrt[6]{X}} \), we have to get rid of pairs of residue classes that lift to pairs for which \( q \mid A \) and \( q \mid B \). Furthermore, we have to rule out the pairs which yield singular cubics, i.e., those pairs \((A, B)\) for which \( 4A^3 + 27B^2 = 0 \). We obtain the following bound

\[
\#E^*(X) \geq 4 \prod_{i=1}^{k} q_i^8(q_i^2 - 1) \left[ \sqrt[4]{X}/Q_k^4 \right] \left[ \sqrt[6]{X}/Q_k^6 \right] - 4 \prod_{i=1}^{k} q_i^8(q_i^2 - 1) \sum_{q_k < q \leq \frac{12}{\sqrt[6]{X}}} [q^{10} - q^8(q^2 - 1)] \left[ \sqrt[4]{X}/q^4 Q_k^4 \right] \left[ \sqrt[6]{X}/q^6 Q_k^6 \right] - (1 + 2 \sqrt[6]{X}/\sqrt{2}).
\]

The bound above is not the exact size of the set because of the possibility that we ruled out the pairs \((A, B)\) for which \( 4A^3 + 27B^2 = 0 \) twice; once in the second term and once in the third. Now one uses Lemma 2.3 to conclude.

In the following theorem, we find the proportion of elliptic curves with nowhere additive reduction except possibly at 2 or 3 among all elliptic curves.

**Theorem 9.3.** The following limit holds

\[
\liminf_{X \to \infty} \frac{\#E^*(X)}{\#E(X)} \geq \frac{\zeta(10)}{\zeta(2)} \approx 0.608544.
\]

Proof: One uses Corollary 2.4 to bound the size of \( E(X) \) from above and Lemma 9.2 to bound the size of \( E^*(X) \) from below. this implies the following

\[
\frac{\#E^*(X)}{\#E(X)} \geq \frac{4 \prod_{i=1}^{k} q_i^8(q_i^2 - 1) \left[ \sqrt[4]{X}/Q_k^4 \right] \left[ \sqrt[6]{X}/Q_k^6 \right] - 4 \prod_{i=1}^{k} q_i^8(q_i^2 - 1) \frac{X^{1/6}}{q_k Q_k^4} - 2 \sqrt[6]{X}/\sqrt{2}}{4 \prod_{i=1}^{k} (q_i^{10} - 1) \left[ \sqrt[4]{X}/Q_k^4 \right] \left[ \sqrt[6]{X}/Q_k^6 \right] + 4 \prod_{i=1}^{k} (q_i^{10} - 1) \left( \sqrt[4]{X}/3q_k^3 Q_k^4 \right) \left( \sqrt[6]{X}/5q_k^5 Q_k^6 \right)}
\]

Since \( 1/q_k \to 0 \) as \( X \to \infty \), one has

\[
\liminf_{X \to \infty} \frac{\#E^*(X)}{\#E(X)} \geq \prod_{i=1}^{k} \left( 1 - \frac{1}{q_i^2} \right) = \frac{\zeta(10)}{\zeta(2)}.
\]

\( \square \)
References

[1] M. A. Bennett and A. Rechnitzer, Computing elliptic curves over $\mathbb{Q}$: Bad reduction at one prime, preprint.
[2] R. van Bommel, Almost all hyperelliptic Jacobians have a bad semi-abelian prime, Masters thesis, Algant, 2014.
[3] A. Bruner, The average rank of elliptic curves I, Invent. math., 109 (1992), 445–472.
[4] A. Bruner and J. H. Silverman, The number of elliptic curves over $\mathbb{Q}$ with conductor $N$, manuscripta mathematica, 91 (1996), no. 1, 95–102.
[5] J. Cremona and M. Lingham, Finding all elliptic curves with good reduction outside a given set of primes, Exp. Math., 16 (2007), no. 3, 303–312.
[6] B. Edixhoven, A. de Groot and J. Top, Elliptic curves over the rationals with bad reduction at only one prime, Math. comp., 54 (1990), no. 189, 413–419.
[7] T. Hadano, Remarks on the conductor of an elliptic curve, Proc. Japan Acad., 48 (1972), 166–167.
[8] T. Hadano, On the conductor of an elliptic curve with a rational point of order 2, Nagoya Math. J., 53 (1974), 199–210.
[9] H. A. Helfgott and A. Venkatesh, Integral points on elliptic curves and 3-torsion in class groups, J. Amer. Math. Soc., 19 (2006), no. 3, 527–550.
[10] S. Howe AND K. Joshi, Asymptotics of conductors of elliptic curves over $\mathbb{Q}$, preprint.
[11] A. Koutsianas, Computing all elliptic curves over an arbitrary number field with prescribed primes of bad reduction, preprint.
[12] Q. Liu, D. Lorenzini and M. Raynaud, Néron models, Lie algebras, and reduction of curves of genus one, Invent. Math., 157 (2004), no. 3, 455–518.
[13] J.-F. Mestre and J. Oesterlé, Courbes de Weil semi-stables de discriminant une puissance $m$-ième, J. Reine Angew. Math., 400 (1989), 173–184.
[14] I. Miyawaki, Elliptic curves of prime power conductor with $\mathbb{Q}$-rational points of finite order, Osaka J. Math., 10 (1973), no. 2, 309–323.
[15] A. Néron, Modèles minimaux des variétés abéliennes sur les corps locaux et globaux, Inst. Hautes Études Sci. Publ. Math., no. 2 (1964), 5–128.
[16] A. P. Ogg, Abelian curves of 2-power conductor, Proc. Cambridge Philos. Soc., 62 (1966), 143–148.
[17] A. P. Ogg, Abelian curves of small conductor, J. Reine Angew. Math., 226 (1967), 204–215.
[18] L. B. Pierce, The 3-part of class numbers of quadratic fields, J. London Math. Soc. (2), 71 (2005), no. 3, 579–598.
[19] M. Sadek, On elliptic curves whose conductor is a product of two prime powers, Math. Comp., 83 (2014), no. 285, 447–460.
[20] J.-P. Serre, Sur les représentations modulaires de degré 2 de $\text{Gal}(\mathbb{Q}/\mathbb{Q})$, Duke Math. J., 54 (1987), no. 1, 179–230.
[21] J. H. Silverman, The arithmetic of elliptic curves, GTM 106, Springer-Verlag, New York, 1986.
[22] J.H. Silverman, Advanced topics in the arithmetic of elliptic curves, GTM 151, Springer-Verlag, New York, 1994.
[23] S. Wong, On the density of elliptic curves, Compositio Mathematica, 127 (2001), no. 1, 23–54.

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