Hölder Equicontinuity of the Integrated Density of States at Weak Disorder

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Abstract. Hölder continuity, \(|N_\lambda(E) - N_\lambda(E')| \leq C|E - E'|^\alpha\), with a constant \(C\) independent of the disorder strength \(\lambda\) is proved for the integrated density of states \(N_\lambda(E)\) associated to a discrete random operator \(H = H_0 + \lambda V\) consisting of a translation invariant hopping matrix \(H_0\) and i.i.d. single site potentials \(V\) with an absolutely continuous distribution, under a regularity assumption for the hopping term.

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1. Introduction

Random operators on \(l^2(\mathbb{Z}^d)\) of the general form

\[ H_\omega = H_0 + \lambda V_\omega, \]  

play a central role in the theory of disordered materials, where:

1. \(V_\omega \psi(x) = \omega(x)\psi(x)\) with \(\omega(x), x \in \mathbb{Z}^d\), independent identically distributed random variables whose common distribution is \(\rho(\omega)d\omega\) with \(\rho\) a bounded function. The coupling \(\lambda \in \mathbb{R}\) is called the disorder strength.

2. \(H_0\) is a bounded translation invariant operator, i.e., \([S_\xi, H_0] = 0\) for each translation \(S_\xi \psi(x) = \psi(x - \xi), \xi \in \mathbb{Z}^d\).

The density of states measure for an operator \(H_\omega\) of the form Equation (1) is the (unique) Borel measure \(dN_\lambda(E)\) on the real line defined by

\[ \int f(E) dN_\lambda(E) = \lim_{L \to \infty} \frac{1}{\# \{x \in \mathbb{Z}^d: |x| < L\}} \sum_{x:|x|<L} \langle \delta_x, f(H_\omega)\delta_x \rangle, \]

and the integrated density of states \(N_\lambda(E)\) is

\[ N_\lambda(E) := \int_{(-\infty, E)} dN_\lambda(\varepsilon). \]
It is a well known consequence, e.g., reference [6], of the translation invariance of the distribution of $H_\omega$ that the density of states exists and equals

$$N_\lambda(E) = \int_\Omega (\delta_0, P(-\infty,E)(H_\omega)\delta_0) dP(\omega), \quad \text{every } E \in \mathbb{R};$$

for $P$ almost every $\omega$, where $P$ is the joint probability distribution for $\omega$ and $\Omega = \mathbb{R}^{2d}$ is the probability space.

The density of states measure is an object of fundamental physical interest. For example, the free energy $f$ per unit volume of a system of non-interacting identical Fermions, each governed by a Hamiltonian $H_\omega$ of the form Equation (1), is

$$f(\mu, \beta) = -\beta \int \ln(1 + e^{-\beta(E-\mu)}) dN_\lambda(E),$$

where $\beta$ is the inverse temperature and $\mu$ is the chemical potential. Certain other thermodynamic quantities (density, heat capacity, etc.) of the system can also be expressed in terms of $N_\lambda$.

Our main result is equicontinuity of the family $\{N_\lambda(\cdot), \lambda > 0\}$ within a class of Hölder continuous functions, that is

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq C_\alpha \delta^\alpha,
\quad \text{for all } \lambda > 0, \quad (2)$$

under appropriate hypotheses on $H_\omega$. The exponent $\alpha < 1$ depends on $H_\omega$ as well as the probability density, with $\alpha = \frac{1}{2}$ at generic $E$ for a large class of hopping terms if $\rho$ is compactly supported.

A bound of the form Equation (2) for the integrated density of states associated to a continuum random Schrödinger operator is implicit in Theorem 1.1 of reference [1], although uniformity in $\lambda$ is not explicitly noted there. The tools of reference [1] carry over easily to the discrete context to give an alternative proof of Equation (2). However the methods employed herein are in fact quite different from those of reference [1], and may be interesting in and of themselves.

The main point of Equation (2) is the uniformity of the bound as $\lambda \to 0$, since the well known Wegner estimate [9], see also [7, Theorem 8.2],

$$\frac{dN_\lambda(E)}{dE} \leq \frac{\|\rho\|_\infty}{\lambda}, \quad (3)$$

implies that $N_\lambda(E)$ is in fact Lipschitz continuous,

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq \frac{\|\rho\|_\infty}{\lambda} 2\delta. \quad (4)$$

However, the Lipschitz constant $\|\rho\|_\infty/\lambda$ in Equation (4) diverges as $\lambda \to 0$. Such a singularity is inevitable for a bound which makes no reference to the hopping term, since $dN_\lambda(E) = \lambda^{-1} \rho(E/\lambda) dE$ for $H_\omega = 0$, as may easily be verified. However if the background itself has an absolutely continuous density of states, the Wegner estimate is far from optimal at weak disorder.
The translation invariant operator $H_0$ may be written as a superposition of translations,

$$H_0 = \sum_{\xi} \tilde{\xi}(\xi) S_{\xi},$$

where

$$\tilde{\xi}(\xi) = \int_{T^d} \xi(q) e^{-i\xi \cdot q} \frac{dq}{(2\pi)^d},$$

is the inverse Fourier transform of a bounded real function $\xi$ on the torus $T^d = [0, 2\pi)^d$, called the symbol of $H_0$. For any bounded measurable function $f$,

$$f(H_0) = \sum_{\xi} \left[ \int_{T^d} f(\xi(q)) e^{-i\xi \cdot q} \frac{dq}{(2\pi)^d} \right] S_{\xi},$$

from which it follows that the density of states $N_0(E)$ for $H_0$ obeys

$$\int f(E) dN_0(E) = \int_{T^d} f(\xi(q)) \frac{dq}{(2\pi)^d}.$$

In particular,

$$N_0(E) = \int_{\xi(q) < E} \frac{dq}{(2\pi)^d}.$$

We define a regular point for $\xi$ to be a point $E \in \mathbb{R}$ at which

$$N_0(E + \delta) - N_0(E - \delta) \leq \Gamma(E) \delta,$$  \hfill (5)

for some $\Gamma(E) < \infty$. In particular if $\xi$ is $C^1$ and $\nabla \xi$ is non-zero on the level set $\{\xi(q) = E\}$, then $E$ is a regular point. For example, with $H_0$ the discrete Laplacian on $\ell^2(\mathbb{Z})$,

$$H_0 \psi(x) = \psi(x+1) + \psi(x-1),$$

we have the symbol $\xi(q) = 2 \cos(q)$ and every $E \in (-2, 2)$ is a regular point. However at the band edges, $E = \pm 2$, the difference on the left hand side of Equation (5) is only $O(\delta^2)$, and these points are not regular points. We consider the behavior of $N_0(E)$ at such ‘points of order $\alpha$’, here $\alpha = 1/2$, in Theorem 3 below.

Our main result involves the density of states of $H_\lambda$ at a regular point:

**THEOREM 1.** Suppose $\int |\omega|^q \rho(\omega) d\omega < \infty$ for some $2 < q < \infty$ or that $\rho$ is compactly supported, in which case set $q = \infty$. If $E$ is a regular point for $\xi$, then there is $C_\lambda = C_\lambda(\rho, \Gamma(E)) < \infty$ such that

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq \Gamma(E) \delta + C_\lambda \lambda^{\frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right)} \delta^{\frac{1}{2} \left( 1 - \frac{2}{q} \right)}$$  \hfill (6)

for all $\lambda, \delta \geq 0$.

For very small $\delta$, namely
\[
\frac{\delta}{\lambda} \lesssim \lambda^{\frac{1}{2}(1+\frac{2}{q})} \delta^{\frac{1}{2}(1-\frac{2}{q})},
\]
the Wegner bound Equation (3) is stronger than Equation (6).\(^1\) Thus Theorem 1 is useful only for
\[\delta \gtrsim \lambda^{\frac{2q+1}{2q+q+1}}.\]
Combining the Wegner estimate and Theorem 1 for these separate regions yields the following:

**Corollary 2.** Under the hypotheses of Theorem 1, there is \(C_q < \infty\), with \(C_q = C_q(\rho, \Gamma(E))\), such that
\[
N_\delta(E + \delta) - N_\delta(E - \delta) \leq C_q \delta^{\frac{1}{2}(1-\frac{1}{2q})}
\]
for all \(\lambda, \delta \geq 0\).

Thus, the integrated density of states is Hölder equi-continuous of order \(\frac{1}{2}\) as \(\lambda \to 0\) (if \(\rho\) is compactly supported).

The starting point for our analysis of the density of states is a well-known formula relating \(dN_\delta\) to the resolvent of \(H_\omega\),
\[
\frac{dN_\delta(E)}{dE} = \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{\Omega} \text{Im}(\langle \delta_0, (H_\omega - E - i\eta)^{-1}\delta_0 \rangle) dP(\omega).
\]

The general idea of the proof is to express \(\text{Im}(\langle \delta_0, (H_\omega - E - i\eta)^{-1}\delta_0 \rangle)\) using a finite resolvent expansion to second order
\[
(H_\omega - E - i\eta)^{-1}
= (H_\omega - E - i\eta)^{-1} - \lambda (H_\omega - E - i\eta)^{-1} V_\omega (H_\omega - E - i\eta)^{-1} + \\
+ \lambda^2 (H_\omega - E - i\eta)^{-1} V_\omega (H_\omega - E - i\eta)^{-1} V_\omega (H_\omega - E - i\eta)^{-1},
\]
and to use the Wegner bound Equation (3) to estimate the last term, with the resulting factor of \(1/\lambda\) controlled by the factor \(\lambda^2\).

Here is a simplified version of the argument which works if \(E\) falls outside the spectrum of \(H_\omega\) and \(\psi_E = (H_\omega - E)^{-1} \delta_0 \in l^1(\mathbb{Z}^d)\). The first two terms of Equation (8) are bounded and self-adjoint when \(\eta = 0\), so
\[
\lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{\Omega} \text{Im}(\langle \delta_0, (H_\omega - E - i\eta)^{-1}\delta_0 \rangle) dP(\omega)
= \lambda^2 \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{\Omega} \text{Im}(\langle \psi_E, V_\omega (H_\omega - E - i\eta)^{-1} V_\omega \psi_E \rangle) dP(\omega)
\leq \lambda^2 \lim_{\eta \downarrow 0} \sum_{x,y} |\psi_E(x)||\psi_E(y)| \times
\times \frac{\eta}{\pi} \int_{\Omega} \omega(x)\omega(y) \left(\delta_{x,y}, ((H_\omega - E)^2 + \eta^2)^{-1}\delta_{y}) \right) dP(\omega).
\]

\(^1\)We thank M. Disertori for this observation.
If $\rho$ is, say, compactly supported, then

$$\lim_{\eta \downarrow 0} \eta \pi \int_{\Omega} \left| \omega(x) \omega(y) \right| \delta_x \left( (H_\omega - E)^2 + \eta^2 \right)^{-1} \delta_y \, dP(\omega) \lesssim \lim_{\eta \downarrow 0} \eta \pi \int_{\Omega} \left| \delta_x \left( (H_\omega - E)^2 + \eta^2 \right)^{-1} \delta_y \right| \, dP(\omega) \lesssim \frac{1}{\lambda},$$

by the Wegner bound, and therefore

$$\frac{d N_\lambda(E)}{d E} \lesssim \lambda \| \psi_E \|_1^2, \quad \text{for } E \not\in \sigma(H_\omega).$$

We have used second order perturbation theory to ‘boot-strap’ the Wegner estimate and obtain an estimate of lower order in $\lambda$. Unfortunately, as $\rho$ was assumed compactly supported, $E$ is not in the spectrum of $H_\lambda$ for sufficiently small $\lambda$, and thus $d N_\lambda(E)/d E = 0$. So, in practice, Equation (9) is not a useful bound.

Nonetheless, in the cases covered by Theorem 1, $H_\lambda$ can have spectrum in a neighborhood of $E$, even for small $\lambda$, since $E$ may be in the interior of the spectrum of $H_\omega$. Although, the above argument does not go through, we shall exploit the translation invariance of the distribution of $H_\omega$ by introducing a Fourier transform on the Hilbert space of ‘random wave functions’, complex valued functions $\Psi(x, \omega)$ of $(x, \omega) \in \ell^2(\mathbb{Z}^d) \times \Omega$ with

$$\sum_x \int_{\Omega} |\Psi(x, \omega)|^2 \, dP(\omega) < \infty.$$

Under this Fourier transform an integral $\int_{\Omega}$ of a matrix element of $f(H_\omega)$ is replaced by an integral $\int_{T^d}$ over the $d$-torus of a matrix element of $f(\hat{H}_k)$, with $\hat{H}_k$ a certain family of operators on $L^2(\Omega)$ (see Equation (16)). Off the set $S_\varepsilon := \{ k \in T^d | |\varepsilon(k) - E| > \varepsilon \}$ with $\varepsilon \gg \delta$, we are able to carry out an argument similar to that which led to Equation (9). To prove Theorem 1, we shall directly estimate

$$N(E + \delta) - N(E - \delta) = \int_{\Omega} \langle \delta_0, P_\delta(H_\omega) \delta_0 \rangle \, dP(\omega),$$

with $P_\delta$ the characteristic function of the interval $[E - \delta, E + \delta]$, because the integrand on the r.h.s. is bounded by 1. Since $E$ is a regular point, the error in restricting to $S_\varepsilon$ will be bounded by $\Gamma(E) \varepsilon$. Choosing $\varepsilon$ optimally will lead to Theorem 1.

More generally, we say that $E$ is a point of order $\alpha$ for $\varepsilon$, if there exists $\Gamma(E; \alpha)$ such that

$$N_\alpha(E + \delta) - N_\alpha(E - \delta) \leq \Gamma(E; \alpha) \delta^\alpha.$$

If $E \not\in \sigma(H_\omega)$, we say that $E$ is a point of order $\infty$ and set $\Gamma(E; \infty) = 0$. For points of order $\alpha$, we have the following extension of Theorem 1.
THEOREM 3. Suppose $\int |\omega|^q \rho(\omega) d\omega < \infty$ for some $2 < q < \infty$ or that $\rho$ is compactly supported, in which case set $q = \infty$. If $E$ is a point of order $\alpha \leq \infty$ for $\epsilon$, then there is $C_{q,\alpha} = C_{q,\alpha}(\rho, \Gamma(E; \alpha)) < \infty$ such that

$$N_{\lambda}(E + \delta) - N_{\lambda}(E - \delta) \leq \Gamma(E; \alpha) \delta^\alpha + C_{q,\alpha} \left[ \lambda^{1 + \frac{2}{q}} \delta^{1 - \frac{2}{q}} \right]^{1 + \frac{1}{q}}$$

(10)

for all $\lambda, \delta \geq 0$.

When $\alpha = \infty$ and $q = \infty$, so $E \notin \sigma(H_\lambda)$ and $\rho$ is compactly supported, the result is technically true but uninteresting since $E \notin \sigma(H_\lambda)$ for small $\lambda$, as discussed above. However for $q < \infty$, we need not have that $\rho$ is compactly supported, and $E \notin \sigma(H_\lambda)$ may still be in the spectrum of $H_\lambda$ for arbitrarily small $\lambda$. In this case, Equation (10) signifies that

$$N_{\lambda}(E + \delta) - N_{\lambda}(E - \delta) \leq C_{q,\infty} \lambda^{1 + \frac{2}{q}} \delta^{1 - \frac{2}{q}},$$

which in fact improves on the Wegner bound for appropriate $\lambda, \delta$.

As above, we may use the Wegner bound for $\delta$ very small to improve on Equation (10):

COROLLARY 4. Under the hypotheses of Theorem 3, there is $C_{q,\alpha} = C_{q,\alpha}(\rho, \Gamma(E; \alpha)) < \infty$ such that

$$N_{\lambda}(E + \delta) - N_{\lambda}(E - \delta) \leq C_{q,\alpha} \delta \left[ \frac{1}{\frac{1}{q} \alpha + 1} \right]$$

for all $\lambda, \delta \geq 0$.

The inspiration for these results is the (non-rigorous) renormalized perturbation theory for $dN_\lambda$, which has appeared in the physics literature, e.g., reference [8] and references therein. If $\int \omega \rho(\omega) d\omega = 0$ and $\int \omega^2 \rho(\omega) d\omega = 1$, as can always be achieved by shifting the origin of energy and re-scaling $\lambda$, then the central result of that analysis is that

$$\frac{dN_\lambda(E)}{dE} \approx \frac{1}{\pi} \text{Im} \left\{ \delta_0 (H_\lambda - E - \lambda^2 \Gamma_\lambda(E))^{-1} \delta_0 \right\},$$

where $\Gamma_\lambda(E)$, the so-called ‘self energy’, satisfies $\text{Im} \Gamma_\lambda(E) > 0$ with

$$\lim_{\lambda \to 0} \text{Im} \Gamma_\lambda(E) \approx \lim_{\eta \to 0} \text{Im} \delta_0 (H_\lambda - E - i\eta)^{-1} \delta_0 = \pi \frac{dN_\lambda(E)}{dE}.$$
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PROPOSITION 1.1. If \( \int \omega \rho(\omega) d\omega = 0 \) and \( \int \omega^2 \rho(\omega) d\omega = 1 \), then for each \( \lambda > 0 \) there is a map \( \Gamma_\lambda \) from \( \{\text{Im} \, z > 0\} \) to the translation invariant operators with non-negative imaginary part on \( \ell^2(\mathbb{Z}^2) \) such that

\[
\int_\Omega (H_\omega - z)^{-1} dP(\omega) = (H_o - z - \lambda^2 \Gamma_\lambda(z))^{-1},
\]

and for fixed \( z \in \{\text{Im} \, z > 0\} \)

\[
\lim_{\lambda \to 0} \langle \delta_x, \Gamma_\lambda(z) \delta_y \rangle = \langle \delta_0, (H_o - z)^{-1} \delta_0 \rangle \delta_{x,y}.
\]

However there is \textit{a priori} no uniformity in \( z \) for the convergence in Equation (12), so for fixed \( \lambda \) we may conclude nothing about

\[
\lim_{\eta \downarrow 0} \frac{d}{dE} (H_o - E - i\eta - \lambda^2 \Gamma_\lambda(E + i\eta))^{-1}.
\]

Still, one is left feeling that Theorem 1 and Corollary 2 are not optimal, and the ‘standard wisdom’ is that something like the following is true.

CONJECTURE 5. Let \( \rho \) have moments of all orders, i.e., \( \int |\omega|^q \rho(\omega) < \infty \) for all \( q \geq 1 \). Given \( E_o \in \mathbb{R} \), if there is \( \delta > 0 \) such that on the set \( \{q : |\varepsilon(q) - E_o| < \delta\} \) the symbol \( \varepsilon \) is \( C^1 \) with \( \nabla \varepsilon(q) \neq 0 \), then there is \( C_\delta < \infty \) such that

\[
\frac{dN_\lambda(E)}{dE} \leq C_\delta
\]

for all \( \lambda \in \mathbb{R} \) and \( E \in [E_o - \frac{1}{2}\delta, E_o + \frac{1}{2}\delta] \).

Remark. The requirement that \( \rho \) have moments of all orders is simply the minimal requirement for the infinite perturbation series for \( (H_o - z - \lambda V_\omega)^{-1} \) to have finite expectation at each order (for \( \text{Im} \, z > 0 \)). In fact, this may be superfluous, as suggested by the example of Cauchy randomness, for which the density of states can be explicitly computed, see reference [7]:

\[
dN_\lambda(E) = \frac{1}{\pi} \int_{\mathbb{T}^d} \frac{\lambda}{(\varepsilon(q) - E)^2 + \lambda^2 (2\pi)^d} dq,
\]

for \( \rho(\omega) = \frac{1}{\pi} \frac{1}{1 + \omega^2} \),

although \( \int \rho(\omega)|\omega|^q = \infty \) for every \( q \geq 1 \).

2. Translation Invariance, Augmented Space, and a Fourier Transform

The joint probability measure \( P(\omega) \) for the random function \( \omega: \mathbb{Z}^d \to \mathbb{R} \) is

\[
dP(\omega) := \prod_{x \in \mathbb{Z}^d} \rho(\omega(x)) d\omega(x)
\]
on the probability space \( \Omega = \mathbb{R}^{2d} \). Clearly, \( \mathbb{P}(\omega) \) is invariant under the translations \( \tau_\xi : \Omega \to \Omega \) defined by
\[
\tau_\xi \omega(x) = \omega(x - \xi).
\]
In particular, since
\[
S_\xi H_\omega S_\xi^\dagger = H_\omega + V_{\tau_\xi} H_\omega = H_{\tau_\xi \omega},
\]
(13)
\( H_\omega \) and \( S_\xi H_\omega S_\xi^\dagger \) are identically distributed for any \( \xi \in \mathbb{Z}^d \).

To express this invariance in operator theoretic terms, we introduce the fibred action of \( H_\omega \) on the Hilbert space \( L^2(\Omega; \ell^2(\mathbb{Z}^d)) \) – the space of ‘random wave functions’ – namely,
\[
\Psi(\omega) \mapsto H_\omega \Psi(\omega).
\]

We identify \( L^2(\Omega; \ell^2(\mathbb{Z}^d)) \) with \( L^2(\Omega \times \mathbb{Z}^d) \) and denote the action of \( H_\omega \) on the latter space by \( H \), so
\[
[H \Psi](\omega, x) = \sum_\xi \delta(\xi) \Psi(\omega, x - \xi) + \lambda \omega(x) \Psi(\omega, x).
\]
The following elementary identity relates \( \int_\Omega f(H_\omega)d\mathbb{P}(\omega) \) to \( f(H) \), for any bounded measurable function \( f \),
\[
\int_\Omega d\mathbb{P}(\omega) \langle \delta_x, f(H_\omega) \delta_y \rangle = \langle \mathbb{E}^\dagger \delta_x, f(H) \mathbb{E}^\dagger \delta_y \rangle,
\]
(14)
where \( \mathbb{E}^\dagger \) is the adjoint of the linear expectation map \( \mathbb{E} : L^2(\Omega \times \mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \) defined by
\[
[\mathbb{E} \Psi](x) = \int_\Omega \Psi(\omega, x)d\mathbb{P}(\omega).
\]
Note that \( \mathbb{E}^\dagger \) is an isometry from \( \ell^2(\mathbb{Z}^d) \) onto the subspace of functions independent of \( \omega \) – ‘non-random functions’.

The general fact that averages of certain quantities depending on \( H_\omega \) can be represented as matrix elements of \( H \) is known, and is sometimes called the ‘augmented space representation’ (e.g., references [3–5]) where ‘augmented space’ refers to the Hilbert space \( L^2(\Omega \times \mathbb{Z}^d) \). There are ‘augmented space’ formulae other than Equation (14), such as
\[
\int_\Omega d\mathbb{P}(\omega) \omega(x) \omega(y) \langle \delta_x, f(H_\omega) \delta_y \rangle = \langle \mathbb{E}^\dagger \delta_x, V f(H) V \mathbb{E}^\dagger \delta_y \rangle,
\]
(15)
with \( V \) defined below, and
\[
\int_\Omega d\mathbb{P}(\omega) \langle \delta_x, f(H_\omega) \delta_0 \rangle \langle \delta_0, g(H_\omega) \delta_y \rangle = \langle \mathbb{E}^\dagger \delta_x, f(H) P_0 g(H) \mathbb{E}^\dagger \delta_y \rangle,
\]
where \( P_0 \) denotes the projection \( P_0 \Psi(\omega, x) = \Psi(\omega, 0) \) if \( x = 0 \) and 0 otherwise. The first of these (Equation (15)) will play a roll in the proof of Theorem 1.

There are two natural groups of unitary translations on \( L^2(\Omega \times \mathbb{Z}^d) \)
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\[ S_\xi \Psi(\omega, x) = \Psi(\omega, x - \xi), \]

and

\[ T_\xi \Psi(\omega, x) = \Psi(\tau_{-\xi} \omega, x). \]

Note that these groups commute: \([S_\xi, T_{\xi'}] = 0\) for every \(\xi, \xi' \in \mathbb{Z}^d\). A key observation is that the distributional invariance of \(H_\omega\), Equation (13), results in the invariance of \(H\) under the combined translations \(T_\xi S_\xi = S_\xi T_\xi\):

\[ S_\xi T_\xi H_T^\dag S_\xi^\dag = H. \]

In fact, let us define

\[ H_\omega = \sum_\xi \epsilon(\xi) S_\xi, \quad V \Psi(\omega, x) = \omega(x) \Psi(\omega, x). \]

Then

\[ H = H_\omega + \lambda V \]

where \(H_\omega\) commutes with \(S_\xi\) and \(T_\xi\) while for \(V\) we have

\[ V S_\xi = T_{-\xi} V. \]

To exploit this translation invariance of \(H\), we define a Fourier transform which diagonalizes the translations \(S_\xi T_\xi\) (and therefore partially diagonalizes \(H\)). The result is a unitary map \(F: L^2(\Omega \times \mathbb{Z}^d) \to L^2(\Omega \times T^d)\), with \(T^d\) the \(d\)-torus \([0, 2\pi)^d\).

Let us define \(F\) first on functions having finite support in \(\mathbb{Z}^d\) by

\[ F \psi(\omega, k) = \sum_\xi e^{-ik \cdot \xi} \psi(\tau_{-\xi} \omega, -\xi). \]

It is easy to verify, using well-known properties of the usual Fourier series mapping \(L^2(\mathbb{Z}^d) \to L^2(T^d)\), that \(F\) extends to a unitary isomorphism \(L^2(\Omega \times \mathbb{Z}^d) \to L^2(\Omega \times T^d)\), i.e. that \(FF^\dag = 1\) and \(F^\dag F = 1\) where \(FF^\dag\) is the adjoint map

\[ F^\dag \psi(\omega, x) = \int_{T^d} e^{-ik \cdot x} \tilde{\psi}(\tau_{-\xi} \omega, k) \frac{dk}{(2\pi)^d}. \]

Another way of looking at \(F\) is to define for each \(k \in T^d\) an operator \(F_k: L^2(\Omega \times \mathbb{Z}^d) \to L^2(\Omega)\) by

\[ F_k \Psi = \lim_{L \to \infty} \sum_{|\xi| < L} e^{-ik \cdot \xi} \mathcal{J} S_\xi T_\xi \Psi, \]

where \(\mathcal{J}\) is the evaluation map \(\mathcal{J} \Psi(\omega) = \Psi(\omega, 0)\). The maps \(F_k\) are not bounded, but are densely defined with \(F_k \Psi \in L^2(\Omega)\) for almost every \(k\), and

\[ F \psi(\omega, k) = F_k \Psi(\omega) \quad \text{a.e. } \omega, k. \]

If we look at \(L^2(\Omega \times T^d)\) as the direct integral \(\int d\Psi \in L^2(\Omega)\), then

\[ F = \int d\Psi \mathcal{F}_k. \]

This Fourier transform diagonalizes the combined translation \(S_\xi T_\xi\).
\[ F_k S T_k = e^{ik\xi} F_k, \]
as follows from the following identities for \( S \) and \( T \),

\[ F_k T_k = T_k F_k, \quad F_k S T_k = e^{ik\xi} T_k F_k, \]

where, on the right hand sides, \( T_k \) denotes the operator \( T_k \psi(\omega) = \psi(\tau_k \omega) \) on \( L^2(\Omega) \). Furthermore, explicit computation shows that

\[ F_k \psi(\omega) = \omega(0) F_k, \]

where \( \omega(0) \) denotes the operator of multiplication by the random variable \( \omega(0) \), \( \psi(\omega) \mapsto \omega(0) \psi(\omega) \).

Putting this all together yields:

**Proposition 2.1.** Under the natural identification of \( L^2(\Omega, T^d) \) with the direct integral

\[ \int d^d k L^2(\Omega), \]

the operator \( \hat{H}^d = F H F^d \) is partially diagonalized, \( \hat{H} = \int d^d k \hat{H}_k \),

with \( \hat{H}_k \) operators on \( L^2(\Omega) \) given by the following formula

\[ \hat{H}_k = \sum_{\xi} e^{-ik\xi} \hat{\epsilon}(\xi) T_k + \lambda \omega(0). \]

Let us introduce for each \( k \in T^d \),

\[ \hat{H}^o_k := \sum_{\xi} e^{-ik\xi} \hat{\epsilon}(\xi) T_k = \sum_{\xi} \left[ \int_{T^d} \hat{\epsilon}(q + k) e^{i\xi \cdot q} \frac{dq}{(2\pi)^d} \right] T_k, \]

so \( \hat{H}_k = \hat{H}^o_k + \lambda \omega(0) \). Note that

\[ \hat{H}_k \chi_{\Omega} = \epsilon(k) \chi_{\Omega}, \]

where \( \chi_{\Omega}(\omega) = 1 \) for every \( \omega \in \Omega \). That is, \( \chi_{\Omega} \) is an eigenvector for \( H^o_k \).

Applying the Fourier transform \( F \) to the right hand side of the “augmented space” formula Equation (14) we obtain the following beautiful identity, central to this work:

\[ \int d\omega(\omega) \langle \delta_x, f(H_\omega) \delta_y \rangle = \int_{T^d} \frac{dk}{(2\pi)^d} e^{ik \cdot (x-y)} \langle \chi_{\Omega}, f(\hat{H}_k) \chi_{\Omega} \rangle. \tag{16} \]

Similarly, we obtain

\[ \int_{\Omega} d\omega(\omega) \omega(\omega) \langle \chi(\omega) \delta_x, f(H_\omega) \delta_y \rangle \]

\[ = \int_{T^d} \frac{dk}{(2\pi)^d} e^{ik \cdot (x-y)} \langle \omega(0) \chi_{\Omega}, f(\hat{H}_k) \omega(0) \chi_{\Omega} \rangle \tag{17} \]

\footnote{In fact, if \( \epsilon \) is almost everywhere non-constant (so \( H_\omega \) has no eigenvalues) then \( \epsilon(k) \) is the unique eigenvalue for \( \hat{H}^o_k \) and the remaining spectrum of \( \hat{H}^o_k \) is infinitely degenerate absolutely continuous spectrum. One way to see this is to let \( \phi_\omega(v) \) be the orthonormal polynomials with respect to the weight \( \rho(v) \), and look at the action of \( \hat{H}^o_k \) on the basis for \( L^2(\Omega) \) consisting of products of the form \( \prod_{x \in \mathbb{Z}^d} \phi_\omega(x) \omega(x) \) with only finitely many \( n(x) \neq 0 \).}
from Equation (15). Related formulae have been used, for example, to derive the
Aubry duality between strong and weak disorder for the almost Mathieu equation,
see reference [2] and references therein.

As a first application of Equation (16), let us prove the existence of the self
energy (Proposition 1.1) starting from the identity

\[
\int_{\Omega} d\mathcal{P}(\omega) \left\{ \delta_0, (H_\omega - z)^{-1} \delta_0 \right\} = \int_{T^d} \frac{dk}{(2\pi)^d} \left\{ \chi_\Omega, (\tilde{H}_k - z)^{-1} \chi_\Omega \right\}.
\]

**Proof of Proposition 1.1.** Since \( \chi_\Omega \) is an eigenvector of \( \tilde{H}_k \) and
\[
\langle \chi_\Omega, \omega(0) \chi_\Omega \rangle = \int \omega \rho(\omega) d\omega = 0,
\]
the Feschbach mapping implies
\[
\left\langle \chi_\Omega, (\tilde{H}_k - z)^{-1} \chi_\Omega \right\rangle = \left( \varepsilon(k) - z - \lambda^2 \Gamma_\lambda(z; k) \right)^{-1},
\]
with
\[
\Gamma_\lambda(z; k) = \left\langle \omega(0) \chi_\Omega, (\tilde{H}_k P^\perp - z)^{-1} \omega(0) \chi_\Omega \right\rangle,
\]
where \( P^\perp \) denotes the projection onto the orthogonal complement of \( \chi_\Omega \) in \( L^2(\Omega) \).

Let the self energy \( \Gamma_\lambda(z) \) be the translation invariant operator with symbol \( \Gamma_\lambda(z; k) \), i.e.,
\[
\langle \delta_x, \Gamma_\lambda(z) \delta_y \rangle = \int_{T^d} e^{ik \cdot (x-y)} \Gamma_\lambda(z; k) \frac{dk}{(2\pi)^d}.
\]
Clearly \( \Gamma_\lambda(z) \) is bounded with non-negative imaginary part. Furthermore by Equations (16) and (18), the identity Equation (11) holds, namely
\[
\int_{\Omega} (H_\omega - z)^{-1} d\mathcal{P}(\omega) = (H_0 - z - \lambda^2 \Gamma_\lambda(z))^{-1}.
\]
It is clear that
\[
\lim_{\lambda \to 0} \Gamma_\lambda(z; k) = \left\langle \omega(0) \chi_\Omega, (\tilde{H}_k^\perp - z)^{-1} \omega(0) \chi_\Omega \right\rangle,
\]
from which Equation (12) follows easily.

**3. Proofs**

We first prove Theorem 1 and then describe modifications of the proof which imply Theorem 3.
3.1. PROOF OF THEOREM 1

Fix a regular point $E$ for $\varepsilon$, and for each $\delta > 0$ let

$$f_\delta(t) = \frac{1}{2} \left( \chi_{(E-\delta,E+\delta)}(t) + \chi_{[E-\delta,E+\delta]}(t) \right)$$

$$= \begin{cases} 
1 & \text{if } t \in (E-\delta,E+\delta), \\
\frac{1}{2} & t = E \pm \delta, \\
0 & t \notin [E-\delta,E+\delta]. 
\end{cases}$$

Since $N_\lambda(E)$ is continuous (see Equation (4)),

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) = \int_{\Omega} \langle \delta_0, f_\delta(H_\omega) \delta_0 \rangle d\Sigma(\omega).$$

Thus, in light of Equation (16), our task is to show that

$$\int_{T^d} \langle \chi_\Omega, f_\delta(\hat{\mathbf{H}}_k) \chi_\Omega \rangle \frac{dk}{(2\pi)^d} \leq \Gamma(\varepsilon) \delta + C_q \lambda \frac{1}{3} \left( 1 + \frac{2}{3} \right) \delta \frac{1}{3} \left( 1 - \frac{2}{3} \right),$$

with a constant $C_q$ independent of $\delta$ and $\lambda$. Note that for each $k \in T^d$

$$|\langle \chi_\Omega, f_\delta(\hat{\mathbf{H}}_k) \chi_\Omega \rangle| \leq 1,$$

so we can afford to neglect a set of Lebesgue measure $\lambda \frac{1}{3} \left( 1 + \frac{2}{3} \right) \delta \frac{1}{3} \left( 1 - \frac{2}{3} \right)$ on the left-hand side of Equation (19).

Consider $k \in T^d$ with $|\varepsilon(k) - E| > \delta$. Then

$$f_\delta(\hat{\mathbf{H}}^*_{\omega}) \chi_\Omega = f_\delta(\varepsilon(k)) \chi_\Omega = 0.$$

Thus

$$\langle \chi_\Omega, f_\delta(\hat{\mathbf{H}}_k) \chi_\Omega \rangle = \langle \chi_\Omega, (f_\delta(\hat{\mathbf{H}}_k) - f_\delta(\hat{\mathbf{H}}_k^*)) \chi_\Omega \rangle$$

$$= \lim_{\eta \to 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \text{Im} \left( \langle \chi_\Omega, \left( \frac{1}{H_k - t - i\eta} - \frac{1}{H_k^* - t - i\eta} \right) \chi_\Omega \rangle \right) dt$$

$$= \lambda \lim_{\eta \to 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \text{Im} \left( \frac{1}{H_k - t - i\eta} \chi_\Omega, -\frac{1}{H_k^* - t - i\eta} \omega(0) \chi_\Omega \right) dt$$

$$= \lambda \left( \langle \chi_\Omega, \frac{1}{H_k - \varepsilon(k)} f_\delta(\hat{\mathbf{H}}_k) \omega(0) \chi_\Omega \rangle \right),$$

since $(t - \varepsilon(k))^{-1}$ is continuous for $t \in [E - \delta, E + \delta]$. Using again that $f_\delta(\hat{\mathbf{H}}^*_{\omega}) \chi_\Omega = 0$, we find that the final term of Equation (20) equals
Putting Equations (20) and (21) together yields

\[
\langle \chi_\Omega, f_\delta(\hat H_0) \chi_\Omega \rangle = \lambda^2 \langle \omega(0) \chi_\Omega, \frac{f_\delta(\hat H_k)}{(\hat H_k - \epsilon(k))^2} \omega(0) \chi_\Omega \rangle \\
\leq \lambda^2 \frac{1}{(\epsilon - \delta)^2} \int_T |\omega(0)|^2 \langle \omega(0) \chi_\Omega, f_\delta(\hat H_k) \omega(0) \chi_\Omega \rangle,
\]

where in the last equality we have inverted the Fourier transform, using Equation (17). We may estimate the right hand side with Hölder’s inequality and the Wegner estimate:

\[
\int_\Omega |\omega(0)|^2 \langle \delta_0, f_\delta(H_0) \delta_0 \rangle d\mathcal{P}(\omega) \\
\leq \|\omega(0)\|_q^2 \int_\Omega \langle \delta_0, f_\delta(H_0) \delta_0 \rangle d\mathcal{P}(\omega) \\
\leq \|\omega(0)\|_q^2 \left( \frac{\|\rho\|_{\infty}}{\lambda} 2\delta \right)^{1 - \frac{3}{q}}.
\]

since \( \langle \delta_0, f_\delta(H_0) \delta_0 \rangle^p \leq \langle \delta_0, f_\delta(H_0) \delta_0 \rangle \) for \( p > 1 \) (because \( \langle \delta_0, f_\delta(H_0) \delta_0 \rangle \leq 1 \)). Here \( \|\omega(0)\|_q = \int \omega(0)^q d\mathcal{P}(\omega) \) for \( q < \infty \) and \( \|\omega(0)\|_{\infty} = \text{ess-sup}_\omega |\omega(0)| \).

Therefore

\[
\int_T \langle \chi_\Omega, f_\delta(\hat H_k) \chi_\Omega \rangle \leq \Gamma(E) e + \lambda^2 \frac{1}{(\epsilon - \delta)^2} \|\omega(0)\|_q^2 \left( \frac{\|\rho\|_{\infty}}{\lambda} 2\delta \right)^{1 - \frac{3}{q}},
\]

where the first term on the right hand side is an upper bound for

\[
\int_{|\epsilon(k) - E| \leq \epsilon} \langle \chi_\Omega, f_\delta(\hat H_k) \chi_\Omega \rangle \frac{dk}{(2\pi)^d} \leq \int_{|\epsilon(k) - E| \leq \epsilon} \frac{dk}{(2\pi)^d}.
\]
Upon optimizing over $\epsilon \in (\delta, \infty)$, this implies
\[
\int_{\Omega} \langle \delta_0, f_\delta(H_\omega)\delta_0 \rangle \leq \Gamma(E)\delta + C_{\rho, q, \Gamma} \lambda \frac{1}{2} \left(1 + \frac{\lambda}{\epsilon} \right) \delta \left(1 + \frac{\lambda}{\epsilon} \right),
\]
which completes the proof of Theorem 1. \hfill \Box

3.2. PROOF OF THEOREM 3

If instead of being a regular point, $E$ is a point of order $\alpha$ then the proof goes through up to Equation (22), in place of which we have
\[
\int_{\mathcal{R}} \langle \chi_{\Omega}, f_\delta(\tilde{H}_k)\chi_{\Omega} \rangle \leq \Gamma(E; \alpha)\epsilon^\alpha + \lambda^2 \left(1 + \frac{\lambda}{\epsilon - \delta} \right)^{2} \left(\frac{\|\rho\|_{\infty}}{\lambda} \delta \right)^{1 - \frac{\lambda}{\epsilon}}.
\]
Setting $\epsilon = \delta + \lambda^\gamma \delta^\beta$ and choosing $\gamma, \beta$ such that the two terms are of the same order yields
\[
\gamma = \frac{1}{2} + \alpha \left(1 + \frac{2}{q} \right), \quad \beta = \frac{1}{2} + \alpha \left(1 - \frac{2}{q} \right),
\]
which implies
\[
\int_{\Omega} \langle \delta_0, f_\delta(H_\omega)\delta_0 \rangle \leq \Gamma(E; \alpha)\delta^\alpha + C_{\rho, q, \Gamma} \lambda \frac{\epsilon^\alpha}{\delta^\alpha} \delta \frac{\epsilon^\alpha}{\delta^\alpha} \delta \left(1 + \frac{\lambda}{\epsilon} \right),
\]
completing the proof. \hfill \Box

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References

1. Combes, J.-M., Hislop, P. D. and Klopp, F.: H"{o}lder continuity of the integrated density of states for some random operators at all energies, Int. Math. Res. Not. 2003 (2003), 179–209.
2. Gordon, A. Y., Jitomirskaya, S., Last, Y. and Simon, B.: Duality and singular continuous spectrum in the almost Mathieu equation, Acta Math. 178 (1997), 169–183.
3. Kaplan, T. and Gray, L. J.: Elementary excitations in disordered systems with short range order, Phys. Rev. B 15 (1977), 3260–3266.
4. Mookerjee, A.: Averaged density of states in disordered systems. J. Phys. C 6 (1973), 1340–1349.
5. Mookerjee, A.: A new formalism for the study of configuration-averaged properties of disordered systems, J. Phys. C 6 (1973), L205–L208.
6. Pastur, L. and Figotin, A.: Spectra of Random and Almost-Periodic Operators, Springer-Verlag, Berlin, 1992.
7. Spencer, T. C.: The Schrödinger equation with a random potential – a mathematical review, In: Phénomènes critiques, systèmes aléatoires, théories de jauge. Part II (Les Houches, 1984), Elsevier, Amsterdam, 1986, pp. 895–942.
8. Thouless, D. J.: Introduction to disordered systems, In: Phénomènes critiques, systèmes aléatoires, théories de jauge. Part II (Les Houches, 1984), Elsevier, Amsterdam, 1986, pp. 681–722.
9. Wegner, F.: Bounds on the density of states in disordered systems, Z. Phys. B 44 (1981), 9–15.