Variational analysis of topological stationary barotropic MHD in the case of single-valued magnetic surfaces

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Abstract. Variational principles for magnetohydrodynamics have been introduced by previous authors both in Lagrangian and Eulerian form. Yahalom & Lynden-Bell (2008) have previously introduced simpler Eulerian variational principles from which all the relevant equations of barotropic magnetohydrodynamics can be derived. These variational principles were given in terms of six independent functions for non-stationary barotropic flows with given topologies and three independent functions for stationary barotropic flows. This is less than the seven variables which appear in the standard equations of barotropic magnetohydrodynamics which are the magnetic field $\vec{B}$, the velocity field $\vec{v}$ and the density $\rho$. Later, Yahalom (2010) introduced a simpler variational principle in terms of four functions for non-stationary barotropic magnetohydrodynamics.

It was shown that the above variational principles are also relevant for flows of non-trivial topologies and in fact using those variational variables one arrives at additional topological conservation laws in terms of cuts of variables which have close resemblance to the Aharonov-Bohm phase (Yahalom (2013)).

In previous examples (Yahalom & Lynden-Bell (2008); Yahalom (2013)) the magnetic field lines with non-trivial topology were at the intersection of two surfaces one of which was always multivalued; in this paper an example is introduced in which the magnetic helicity is not zero yet both surfaces are single-valued.

1. Introduction
Variational principles for magnetohydrodynamics have been introduced by previous authors both in Lagrangian and Eulerian form. Sturrock (1994) has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov & Moffatt (1994) in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of magnetohydrodynamics which are the magnetic field $\vec{B}$, the velocity field $\vec{v}$ and the density $\rho$. Kats (2003) has generalized Moffatt’s work for compressible non-barotropic flows but without reducing the number of functions and the computational load. Sakurai (1979) has introduced a two-function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme; his method is discussed by Sturrock (1994). Yahalom & Lynden-Bell (2008) have...
combined the Lagrangian of Sturrock (1994) with the Lagrangian of Sakurai (1979) to obtain an **Eulerian** variational principle which depends on only six functions for non-stationary flows. The variational derivative of this Lagrangian result in all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resemble the Hamiltonian equations of Frenkel et al. (1982). Later it was shown by Yahalom (2010) that only four functions will suffice for barotropic magnetohydrodynamic non-stationary flows. Furthermore, it was shown that for stationary flows three functions will suffice in order to describe a Lagrangian principle for barotropic magnetohydrodynamics.

The non-singlevaluedness of the functions appearing in the reduced representation of barotropic magnetohydrodynamics was shown in Yahalom (2013) to be an essential feature of the case of magnetohydrodynamics with non trivial topology. In particular the discontinuities of $\nu$ and $\zeta$ were shown to be conserved and equal to the cross helicity per unit magnetic flux and the magnetic helicity per unit magnetic flux respectively.

Still the question remains what about the single valuedness of the rest of the variational functions. In the variational formalism the magnetic field is represented by Sakurai (1979) representation:

$$\vec{B} = \vec{\nabla} \chi \times \vec{\nabla} \eta$$

In previous examples of fields of non trivial topology which could be represented that way given in Yahalom & Lynden-Bell (2008) the function $\chi$ was single valued and the function $\eta$ was multiple valued. Here we suggest an example in which both surfaces are single valued.

The plan of this paper is as follows: first we introduce the standard notations and equations of barotropic magnetohydrodynamics. Next we define the magnetic Load and Metage. This is followed by the derivation of a variational principle for stationary magnetohydrodynamics. Finally an example is given for a magnetic field of non-zero magnetic helicity which lies on the intersection of two single valued surfaces.

### 2. Basic equations

The standard set of equations solved for barotropic magnetohydrodynamics are given below:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}),$$

$$\vec{\nabla} \cdot \vec{B} = 0,$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0,$$

$$\rho \frac{d\vec{v}}{dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = -\vec{\nabla} p(\rho) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi}.$$  

The following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and $\vec{\nabla}$ has its standard meaning in vector calculus. $\vec{B}$ is the magnetic field vector, $\vec{v}$ is the velocity field vector and $\rho$ is the fluid density. Finally $p(\rho)$ is the pressure which we assume depends on the density alone (barotropic case). The justification for those equations and the conditions under which they apply can be found in standard books on magnetohydrodynamics (see for example Sturrock (1994)). Equation (2) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), equation (3) describes the fact that the magnetic field is solenoidal, equation (4) describes the conservation of mass and equation (5) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term:

$$\vec{j} = \frac{\vec{\nabla} \times \vec{B}}{4\pi},$$
is the electric current density which is not connected to any mass flow. The number of independent variables for which one needs to solve is seven \((\vec{v}, \vec{B}, \rho)\) and the number of equations \((2, 4, 5)\) is also seven. Notice that equation (3) is a condition on the initial \(\vec{B}\) field and is satisfied automatically for any other time due to equation (2). Also notice that \(p(\rho)\) is not a variable rather it is a given function of \(\rho\).

3. The Load and Metage
In this section we define the Load and Metage which are quantities which are useful for defining variational variables. The treatment in this section will follow closely an analogous treatment for non-magnetic fluid dynamics given by Lynden-Bell & Katz (1981). We assume here a cylindrical or spherical topology, for more general topologies see Yahalom & Lynden-Bell (2008). Consider a thin tube surrounding a magnetic field line as described in figure 1:

\[\Delta \Phi = \int \vec{B} \cdot d\vec{S},\]  

\[\Delta M = \int \rho dl \cdot d\vec{S},\]  

in which \(dl\) is a length element along the tube. Since the magnetic field lines move with the flow by virtue of equation (2) both the quantities \(\Delta \Phi\) and \(\Delta M\) are conserved and since the tube is thin we may define the conserved magnetic load:

\[\lambda = \frac{\Delta M}{\Delta \Phi} = \oint \frac{\rho}{B} dl,\]  

in which the above integral is performed along the field line. Obviously the parts of the line which go out of the flow to regions in which \(\rho = 0\) have a null contribution to the integral. Notice that \(\lambda\) is a \textbf{single valued} function that can be measured in principle. Since \(\lambda\) is conserved it satisfies the equation:

\[\frac{d\lambda}{dt} = 0.\]
By construction surfaces of constant magnetic load move with the flow and contain magnetic field lines. Hence the gradient to such surfaces must be orthogonal to the field line:

\[ \nabla \lambda \cdot \vec{B} = 0. \] (11)

Now consider an arbitrary comoving point on the magnetic field line and denote it by \( i \), and consider an additional comoving point on the magnetic field line and denote it by \( r \). The integral:

\[ \mu(r) = \int_{i}^{r} \frac{\rho}{B} dl + \mu(i), \] (12)

is also a conserved quantity which we may denote following Lynden-Bell & Katz (1981) as the magnetic metage. \( \mu(i) \) is an arbitrary number which can be chosen differently for each magnetic line. By construction:

\[ \frac{d\mu}{dt} = 0. \] (13)

Also it is easy to see that by differentiating along the magnetic field line we obtain:

\[ \nabla \mu \cdot \vec{B} = \rho. \] (14)

Notice that \( \mu \) will be generally a non single valued function. At this point we have two comoving coordinates of flow, namely \( \lambda, \mu \) obviously in a three dimensional flow we also have a third coordinate. However, before defining the third coordinate we will find it useful to work not directly with \( \lambda \) but with a function of \( \lambda \). Now consider the magnetic flux within a surface of constant load \( \Phi(\lambda) \) as described in figure 2 (the figure was given by Lynden-Bell & Katz (1981)). The magnetic flux is a conserved quantity and depends only on the load \( \lambda \) of the surrounding surface. Now we define the quantity:

\[ \chi = \frac{\Phi(\lambda)}{2\pi}. \] (15)

Obviously \( \chi \) satisfies the equations:

\[ \frac{d\chi}{dt} = 0, \quad \vec{B} \cdot \nabla \chi = 0, \] (16)

we will immediately show that this function is identical to Sakurai’s function defined in equation (1). Let us now define an additional comoving coordinate \( \eta^* \) since \( \nabla \mu \) is not orthogonal to the \( \vec{B} \) lines we can choose \( \nabla \eta^* \) to be orthogonal to the \( \vec{B} \) lines and not be in the direction of the \( \nabla \chi \) lines, that is we choose \( \eta^* \) not to depend only on \( \chi \). Since both \( \nabla \eta^* \) and \( \nabla \chi \) are orthogonal to \( \vec{B}, \vec{B} \) must take the form:

\[ \vec{B} = A \nabla \chi \times \nabla \eta^*. \] (17)

However, using equation (3) we have:

\[ \nabla \cdot \vec{B} = \nabla A \cdot (\nabla \chi \times \nabla \eta^*) = 0. \] (18)

Which implies that \( A \) is a function of \( \chi, \eta^* \). Now we can define a new comoving function \( \eta \) such that:

\[ \eta = \int_{0}^{\eta^*} A(\chi, \eta^*)d\eta^*, \quad \frac{d\eta}{dt} = 0. \] (19)

In terms of this function we recover the Sakurai presentation defined in equation (1):

\[ \vec{B} = \nabla \chi \times \nabla \eta. \] (20)
Hence we have shown how $\chi, \eta$ can be constructed for a known $\mathbf{B}, \rho$. Notice however, that $\eta$ is defined in a non unique way since one can redefine $\eta$ for example by performing the following transformation: $\eta \to \eta + f(\chi)$ in which $f(\chi)$ is an arbitrary function. The comoving coordinates $\chi, \eta$ serve as labels of the magnetic field lines. Moreover the magnetic flux can be calculated as:

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S} = \int d\chi d\eta. \quad (21)$$

In the case that the surface integral is performed inside a load contour we obtain:

$$\Phi(\lambda) = \int_\lambda d\chi d\eta = \chi \int_\lambda d\eta = \left\{ \chi(\eta_{\text{max}} - \eta_{\text{min}}) \right\}. \quad (22)$$

There are two cases involved; in one case the load surfaces are topological cylinders, in this case $\eta$ is not single valued and hence we obtain the upper value for $\Phi(\lambda)$. In a second case the load surfaces are topological spheres, in this case $\eta$ is single valued and has minimal $\eta_{\text{min}}$ and maximal $\eta_{\text{max}}$ values. Hence the lower value of $\Phi(\lambda)$ is obtained. For example in some cases $\eta$ is identical to twice the latitude angle $\theta$. In those cases $\eta_{\text{min}} = 0$ (value at the ”north pole”) and $\eta_{\text{max}} = 2\pi$ (value at the ”south pole”).

Comparing the above equation with equation (15) we derive that $\eta$ can be either single valued or not single valued and that its discontinuity across its cut in the non single valued case is $[\eta] = 2\pi$.

4. Simplified variational principle of stationary barotropic magnetohydrodynamics

Consider equation (16), for a stationary flow it takes the form:

$$\vec{v} \cdot \nabla \chi = 0. \quad (23)$$

Hence $\vec{v}$ can take the form:

$$\vec{v} = \frac{\nabla \chi \times \vec{K}}{\rho}. \quad (24)$$

However, the velocity field must satisfy the stationary mass conservation equation (4):

$$\nabla \cdot (\rho \vec{v}) = 0. \quad (25)$$

We see that a sufficient condition (although not necessary) for $\vec{v}$ to solve equation (25) is that $\vec{K}$ takes the form $\vec{K} = \nabla N$, where $N$ is an arbitrary function. Thus, $\vec{v}$ may take the form:

$$\vec{v} = \frac{\nabla \chi \times \nabla N}{\rho}. \quad (26)$$

Let us now calculate $\vec{v} \times \vec{B}$ in which $\vec{B}$ is given by Sakurai’s presentation equation (1):

$$\vec{v} \times \vec{B} = \left( \frac{\nabla \chi \times \nabla N}{\rho} \right) \times (\nabla \chi \times \nabla \eta)$$

$$= \frac{1}{\rho} \nabla \chi (\nabla \chi \times \nabla N) \cdot \nabla \eta. \quad (27)$$

Since the flow is stationary $N$ can be at most a function of the three comoving coordinates $\chi, \mu, \bar{\eta} = \eta + t^1$, hence:

$$\nabla N = \frac{\partial N}{\partial \chi} \nabla \chi + \frac{\partial N}{\partial \mu} \nabla \mu + \frac{\partial N}{\partial \bar{\eta}} \nabla \bar{\eta}. \quad (28)$$

1 While the physical variables are assumed to be stationary the potential need not be stationary, for more on this see Yahalom & Lynden-Bell (2008)
Inserting equation (28) into equation (27) will yield:

\[ \vec{v} \times \vec{B} = \frac{1}{\rho} \vec{\nabla} \chi \frac{\partial N}{\partial \mu} (\vec{\nabla} \chi \times \vec{\nabla} \mu) \cdot \vec{\nabla} \eta. \]  

(29)

Rearranging terms and using Sakurai’s presentation equation (1) we can simplify the above equation and obtain:

\[ \vec{v} \times \vec{B} = -\frac{1}{\rho} \vec{\nabla} \chi \frac{\partial N}{\partial \mu} (\vec{\nabla} \mu \cdot \vec{B}). \]  

(30)

However, using equation (14) this will simplify to the form:

\[ \vec{v} \times \vec{B} = -\frac{\vec{\nabla} \chi}{\partial N} \frac{\partial N}{\partial \mu}. \]  

(31)

Now let us consider equation (2); for stationary flows this will take the form:

\[ \vec{\nabla} \times (\vec{v} \times \vec{B}) = 0. \]  

(32)

Inserting equation (31) into equation (32) will lead to the equation:

\[ \vec{\nabla} \left( \frac{\partial N}{\partial \mu} \right) \times \vec{\nabla} \chi = 0. \]  

(33)

However, since \( N \) is at most a function of \( \chi, \mu, \bar{\eta} \) it follows that \( \frac{\partial N}{\partial \mu} \) is some function of \( \chi \):

\[ \frac{\partial N}{\partial \mu} = -F(\chi). \]  

(34)

This can be easily integrated to yield:

\[ N = -\mu F(\chi) + G(\chi, \bar{\eta}). \]  

(35)

Inserting this back into equation (26) will yield:

\[ \vec{v} = \frac{\vec{\nabla} \chi \times (-F(\chi) \vec{\nabla} \mu + \frac{\partial G}{\partial \bar{\eta}} \vec{\nabla} \bar{\eta})}{\rho}. \]  

(36)

Let us now replace the set of variables \( \chi, \bar{\eta} \) with a new set \( \chi', \bar{\eta}' \) such that:

\[ \chi' = \int F(\chi) d\chi, \quad \bar{\eta}' = \frac{\bar{\eta}}{F(\chi)}. \]  

(37)

This will not have any effect on the Sakurai representation given in equation (1) since:

\[ \vec{B} = \vec{\nabla} \chi \times \vec{\nabla} \eta = \vec{\nabla} \chi \times \vec{\nabla} \bar{\eta} = \vec{\nabla} \chi' \times \vec{\nabla} \bar{\eta}'. \]  

(38)

However, the velocity will have a simpler representation and will take the form:

\[ \vec{v} = \frac{\vec{\nabla} \chi' \times \vec{\nabla} (-\mu + G'(\chi', \bar{\eta}'))}{\rho}, \]  

(39)

in which \( G' = \frac{G}{F} \). At this point one should remember that \( \mu \) was defined in equation (12) up to an arbitrary constant which can vary between magnetic field lines. Since the lines are labelled by
their $\chi', \bar{\eta}'$ values it follows that we can add an arbitrary function of $\chi', \bar{\eta}'$ to $\mu$ without effecting its properties. Hence we can define a new $\mu'$ such that:

$$\mu' = \mu - G'(\chi', \bar{\eta}').$$  \hspace{1cm} (40)

Notice that $\mu'$ can be multi-valued. Inserting equation (40) into equation (39) will lead to a simplified equation for $\vec{v}$:

$$\vec{v} = \frac{\nabla \mu' \times \nabla \chi'}{\rho}.$$  \hspace{1cm} (41)

In the following the primes on $\chi, \mu, \bar{\eta}$ will be ignored. The above equation is analogous to Vladimirov & Moffatt (1994) equation 7.11 for incompressible flows, in which our $\mu$ and $\chi$ play the part of their $A$ and $\Psi$. It is obvious that $\vec{v}$ satisfies the following set of equations:

$$\vec{v} \cdot \nabla \mu = 0, \quad \vec{v} \cdot \nabla \chi = 0, \quad \vec{v} \cdot \nabla \bar{\eta} = 1,$$  \hspace{1cm} (42)

to derive the right hand equation we have used both equation (13) and equation (1). Hence $\mu, \chi$ are both comoving and stationary. It can be easily seen that if:

$$basis = (\nabla \chi, \nabla \bar{\eta}, \nabla \mu),$$  \hspace{1cm} (43)

is a local vector basis at any point in space than their exists a dual basis:

$$dual\ basis = \frac{1}{\rho}(\nabla \bar{\eta} \times \nabla \mu, \nabla \mu \times \nabla \chi, \nabla \chi \times \nabla \bar{\eta}) = (\frac{\nabla \bar{\eta} \times \nabla \mu}{\rho}, \vec{v}, \vec{B}/\rho).$$  \hspace{1cm} (44)

Such that:

$$basis_i \cdot dual\ basis_j = \delta_{ij}, \quad i, j \in [1, 2, 3],$$  \hspace{1cm} (45)

in which $\delta_{ij}$ is Kronecker’s delta. Hence while the surfaces $\chi, \mu, \bar{\eta}$ generate a local vector basis for space, the physical fields of interest $\vec{v}, \vec{B}$ are part of the dual basis. By vector multiplying $\vec{v}$ and $\vec{B}$ and using equations (41,1) we obtain:

$$\vec{v} \times \vec{B} = \nabla \chi,$$  \hspace{1cm} (46)

this means that both $\vec{v}$ and $\vec{B}$ lie on $\chi$ surfaces and provide a vector basis for this two dimensional surface. The above equation can be compared with Vladimirov & Moffatt (1994) equation 5.6 for incompressible flows in which their $J$ is analogue to our $\chi$.

5. The action principle

In the previous subsection we have shown that if the velocity field $\vec{v}$ is given by equation (41) and the magnetic field $\vec{B}$ is given by the Sakurai representation equation (1) than equations (2,3,4) are satisfied automatically for stationary flows. To complete the set of equations we will show how the Euler equations (5) can be derived from the action given in:

$$A \equiv \int \mathcal{L} d^3x dt,$$

$$\mathcal{L} \equiv \rho(\frac{1}{2} \dot{v}^2 - \varepsilon(\rho)) - \frac{\vec{B}^2}{8\pi},$$  \hspace{1cm} (47)

in which both $\vec{v}$ and $\vec{B}$ are given by equation (41) and equation (1) respectively and the density $\rho$ is given by equation (13):

$$\rho = \nabla \mu \cdot \vec{B} = \nabla \mu \cdot (\nabla \chi \times \nabla \bar{\eta}) = \frac{\partial (\chi, \eta, \mu)}{\partial (x, y, z)},$$  \hspace{1cm} (48)
In this case the Lagrangian density of equation (47) will take the form:

\[ L = \rho \left( \frac{1}{2} \left( \frac{\nabla \mu \times \nabla \chi}{\rho} \right)^2 - \varepsilon(\rho) \right) - \frac{(\nabla \chi \times \nabla \eta)^2}{8\pi} \]  

(49)

and can be seen explicitly to depend on only three functions. Let us make arbitrary small variations \( \delta \alpha_i = (\delta \chi, \delta \eta, \delta \mu) \) of the functions \( \alpha_i = (\chi, \eta, \mu) \). Let us define a \( \Delta \) variation that does not modify the \( \alpha_i \)'s, such that:

\[ \Delta \alpha_i = \delta \alpha_i + (\vec{\xi} \cdot \nabla) \alpha_i = 0, \]

(50)

in which \( \vec{\xi} \) is the Lagrangian displacement, thus:

\[ \delta \alpha_i = -\nabla \alpha_i \cdot \vec{\xi}. \]

(51)

Which will lead to the equation:

\[ \vec{\xi} \equiv -\frac{\partial \vec{r}}{\partial \alpha_i} \delta \alpha_i. \]

(52)

Making a variation of \( \rho \) given in equation (48) with respect to \( \alpha_i \) will yield:

\[ \delta \rho = -\nabla \cdot (\rho \vec{\xi}). \]

(53)

Furthermore, taking the variation of \( \vec{B} \) given by Sakurai’s representation (1) with respect to \( \alpha_i \) will yield:

\[ \delta \vec{B} = \nabla \times (\vec{\xi} \times \vec{B}). \]

(54)

It remains to calculate \( \delta \vec{v} \) by varying equation (41) this will yield:

\[ \delta \vec{v} = -\frac{\delta \rho}{\rho} \vec{v} + \frac{1}{\rho} \nabla \times (\rho \vec{\xi} \times \vec{v}). \]

(55)

Inserting equations (53,54,55) into:

\[ \delta A = \int \delta L d^3x dt, \]

\[ \delta \mathcal{L} = \delta \rho \left( \frac{1}{2} \vec{v}^2 - w(\rho) \right) + \rho \vec{v} \cdot \delta \vec{v} - \frac{\vec{B} \cdot \delta \vec{B}}{4\pi}, \]

will yield:

\[ \delta \mathcal{L} = \vec{v} \cdot \nabla \times (\rho \vec{\xi} \times \vec{v}) - \frac{\vec{B} \cdot \nabla \times (\vec{\xi} \times \vec{B})}{4\pi} - \delta \rho \left( \frac{1}{2} \vec{v}^2 + w \right) \]

\[ = \vec{v} \cdot \nabla \times (\rho \vec{\xi} \times \vec{v}) - \frac{\vec{B} \cdot \nabla \times (\vec{\xi} \times \vec{B})}{4\pi} + \nabla \cdot (\rho \vec{\xi}) \left( \frac{1}{2} \vec{v}^2 + w \right). \]

(57)

Using the well known vector identity:

\[ \vec{A} \cdot \nabla \times (\vec{C} \times \vec{A}) = \nabla \cdot ((\vec{C} \times \vec{A}) \times \vec{A}) + (\vec{C} \times \vec{A}) \cdot \nabla \times \vec{A} \]

(58)

and the theorem of Gauss we can write now equation (56) in the form:

\[ \delta A = \int dt \left\{ \int d\hat{S} \cdot [\rho (\vec{\xi} \times \vec{v}) \times \vec{v} - \frac{\vec{\xi} \times \vec{B}}{4\pi} \times \vec{B} - \frac{1}{2} \vec{v}^2 + w] \rho \vec{\xi} \right\} \]

\[ + \int d^3x \vec{v} \cdot [\rho \vec{\omega} + \vec{J} \times \vec{B} - \rho \nabla (\frac{1}{2} \vec{v}^2 + w)]. \]

(59)
The time integration is of course redundant in the above expression. Also notice that we have used the current definition equation (6) and the vorticity definition $\vec{\omega} = \vec{\nabla} \times \vec{v}$. Suppose now that $\delta A = 0$ for a $\vec{\xi}$ such that the boundary term (including both the boundary of the domain and relevant cuts) in the above equation is null but that $\vec{\xi}$ is otherwise arbitrary, then it entails the equation:

$$\rho \vec{v} \times \vec{\omega} + I \times \vec{B} - \rho \nabla \left( \frac{1}{2} \vec{v}^2 + w \right) = 0.$$  \hspace{1cm} (60)

Using the well known vector identity:

$$\frac{1}{2} \vec{\nabla} (\vec{v}^2) = (\vec{v} \cdot \vec{\nabla}) \vec{v} + \vec{v} \times (\vec{\nabla} \times \vec{v})$$ \hspace{1cm} (61)

and rearranging terms we recover the stationary Euler equation:

$$\rho (\vec{\nabla} \cdot \vec{v}) \vec{v} = -\vec{\nabla} p + I \times \vec{B}.$$ \hspace{1cm} (62)

6. The Azimuthal Magnetic Star

At this penultimate section let us consider a spherical star of radius $R$ and uniform density $\rho$, furthermore we assume that the star contains an azimuthal magnetic field:

$$\vec{B} = B_0 \sin(\theta) \hat{\phi}$$ \hspace{1cm} (63)

(which need not be azimuthal outside the star) in which $\hat{\phi}$ is a unit vector along the azimuthal direction, $\theta$ is the zenith angle and $B_0$ is constant. The vector potential $\vec{A}$ can be taken as:

$$\vec{A} = \frac{1}{2} B_0 r \sin(\theta) \hat{\theta} + \alpha \frac{r}{\sin \theta} \hat{\phi},$$ \hspace{1cm} (64)

in which $r$ is the radial coordinate, $\hat{\theta}$ is a unit vector along the zenith direction and $\alpha$ is constant inside the star and goes smoothly to zero on a infinitesimal shell outside the star. First let us calculate to load using equation (9) we obtain:

$$\lambda = 2\pi r \rho \frac{\rho}{B_0},$$ \hspace{1cm} (65)

hence the load surfaces are spheres. The $\chi$ function can now be calculated according to equation (15) to yield inside the star the value:

$$\chi = \frac{B_0 r^2}{2\pi}.$$ \hspace{1cm} (66)

Solving equation (20) for $\eta$ we obtain the following non unique solution:

$$\eta = -\pi \cos(\theta).$$ \hspace{1cm} (67)

Substituting equation (66) and equation (67) into:

$$\vec{A} = \chi \vec{\nabla} \eta + \vec{\nabla} \zeta,$$ \hspace{1cm} (68)

will result in the following equation:

$$\vec{A} = \frac{1}{2} B_0 r \sin(\theta) \hat{\theta} + \vec{\nabla} \zeta.$$ \hspace{1cm} (69)
Comparing equation (64) with equation (69) we can solve for $\zeta$ and obtain:

$$\zeta = \alpha \phi,$$

(70)

in which $\phi$ is the azimuthal angle. Hence $\zeta$ is non single valued and has a constant discontinuity value:

$$[\zeta] = 2\pi \alpha.$$

(71)

Thus we can easily integrate

$$\mathcal{H}_M = \int \zeta d\chi d\eta = \int [\zeta] d\Phi,$$

(72)

(Yahalom (2013)) in this simple case and obtain:

$$\mathcal{H}_M = [\zeta] \Phi = 2\pi \alpha \Phi.$$

(73)

In particular if are interested in calculating the magnetic helicity inside a surface of constant load (a sphere of any radius $r \leq R$) we can combine equation (15) with equation (73) to obtain:

$$\mathcal{H}_M = [\zeta] \Phi = (2\pi)^2 \alpha \chi = 2\pi \alpha \chi^2 B_0,$$

(74)

in which equation (66) was taken into account. This result coincides with a direct calculation using the magnetic helicity definition given by:

$$\mathcal{H}_M \equiv \int \vec{B} \cdot \vec{A} \, d^3 x,$$

(75)

The reader may wonder what is the reason that the second term in the vector potential that has a zero curl and hence produces a zero magnetic field contributes to the helicity. This has to do with the singularity of the term. Indeed this term produces a non zero ”magnetic flux”:

$$\oint \vec{A} \cdot d\vec{l} = 2\pi \alpha.$$

(76)

Thus introducing this term is equivalent to introducing an infinitesimal thin flux tube parallel to the z-axis in addition to the azimuthal magnetic field of equation (63), which is the reason for the non-zero helicity. Of course one may choose not to have such a thin flux tube in the model. If it does not exist than one should be careful in taking advantage of the vector potential gauge freedom since this freedom is only limited to regular functions. Singular gauges are not allowed as they have concrete physical effects as the above example clearly shows.

7. Conclusion

It was shown that unlike in previous examples (Yahalom & Lynden-Bell (2008)) a non-zero magnetic helicity can be obtained even if both $\chi$ and $\eta$ are single valued, the price to pay is having a singular vector potential.

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