Article

Approximation of Real Functions by a Generalization of Ismail–May Operator

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Abstract: In this paper, we generalize a sequence of positive linear operators introduced by Ismail and May and we study some of their approximation properties for different classes of continuous functions. First, we estimate the error of approximation in terms of the usual modulus of continuity and the second-order modulus of Ditzian and Totik. Then, we characterize the bounded functions that can be approximated uniformly by these new operators. In the last section, we obtain the most important results of the paper. We give the complete asymptotic expansion for the operators and we deduce a Voronovskaya-type theorem, results that hold true for smooth functions with exponential growth.

Keywords: positive linear operators; Ismail–May operators; Jain operators; moduli of continuity; asymptotic expansion; Voronovskaya-type theorem

MSC: 41A36; 41A25

1. Introduction

In 1912, Bernstein [1] constructed the following sequence of positive linear operators

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), \quad n \in \mathbb{N}, \ x \in [0,1]$$

to give a proof that every continuous function $f \in C[0,1]$ can be uniformly approximated by a sequence of polynomial functions. Many papers were devoted to the study of the properties of these operators. In 1932, Voronovskaya [2] obtained the speed of convergence of Bernstein operators in terms of the usual modulus of continuity defined by

$$\omega(f, \delta) = \sup \{ |f(t) - f(x)| : t, x \in [0,1], |t-x| \leq \delta \}$$

for every $f \in C[0,1]$ and $\delta \geq 0$. Later, other moduli of smoothness were constructed, in order to obtain characterizations of the functions for which we have a certain error of approximation.

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For $\alpha > 0$ and $\beta \geq 0$, let us denote by $R_{\alpha, \beta}$ the operator defined for a real function $f \in C[0, \infty)$ by the equality

$$R_{\alpha, \beta}(f)(x) = \sum_{k=0}^{\infty} \omega_{\beta}(k, \alpha, z) \cdot f \left( \frac{k}{x} \right), \quad \text{for all } x \geq 0,$$

(1)

where $\omega_{\beta}(k, \alpha, z) = \frac{a(\alpha + \beta k)^{k-1}}{k!} e^{kz} \cdot e^{-a(\alpha + \beta k)z}$ for every nonnegative integer $k$ and real number $z$. Here, we assume that the function $f \in C[0, \infty)$ is such that the series on the right side of the equality (1) converges. In this article, we will obtain results for the operator $R_{\alpha, \beta}$ similar to the results presented above for Bernstein operators.

For $\alpha = n \in \mathbb{N}$, $\beta = 1$, the operator $R_{n, 1}$ represents Ismail and May’s operator $R_n$ (see [4] relation (3.14)). For $\beta = 0$, the operator $R_{n, 0}$ represents Szász–Mirakyan’s operator (see [4] relation (3.6)). Jain [5] considered the following positive linear operator $R_n(f)(x) = \sum_{k=0}^{\infty} \omega_{\beta_n}(k, nx, 1) \cdot f \left( \frac{k}{n} \right)$, where $n \in \mathbb{N}$, $x \geq 0$ and $(\beta_n)$ is a sequence of real numbers from $[0, 1]$ converging to 0, and proved some approximation properties.

In Section 2, we will present some basic approximation properties needed in the rest of the article. We include an evaluation of the error of the approximation in terms of the usual modulus of continuity.

In Section 3, we study the uniform approximation properties of $R_{\alpha, \beta}$, including an estimate of the rate of approximation in terms of the second-order modulus of Ditzian and Totik. We also obtain a characterization of the bounded functions which can be uniformly approximated by $R_{\alpha, \beta}$. To our knowledge, these kinds of results have not been obtained for the particular operators of Ismail and May and the operators of Jain. We must mention that Totik [6,7] solved the problem of uniform approximation for the classical exponential-type operators. The operators $R_{\alpha, \beta}$ are of exponential type, but a direct application of Totik’s results is not possible in our case (see the first paragraph of Section 3).

In the last section, we provide a pointwise complete asymptotic expansion for the newly introduced operators (1). Similar results were obtained in [8] for the Jain operators and functions with polynomial growth. We remark that our asymptotic results are valid for functions with exponential growth, as in the case of the operators of Ismail and May presented in [9]. However, our study presents, in addition to the previous articles, two aspects: first, an explicit computation of the images of exponential functions using the Lambert function (see Theorem 3), and second, a direct and detailed proof of the asymptotic result, which, in the previous papers, was based on a result of Sikkema [10] for functions with polynomial growth. As a corollary to our main result, we deduce a Voronovskaya-type theorem.

2. Basic Properties

**Lemma 1.** The operators $R_{\alpha, \beta}$ preserve the constant and the linear functions, i.e.,

$$R_{\alpha, \beta}(1)(x) = 1, \quad R_{\alpha, \beta}(t)(x) = x.$$

**Proof.** We use the following relation obtained by Jensen [11] as a consequence of the Lagrange inversion formula. Consider $\alpha, z \in \mathbb{R}$ and $\beta \geq 0$ such that $|\beta ze^{1-z\beta}| < 1$. In these conditions, we have

$$e^{\alpha z} = \sum_{k=0}^{\infty} \frac{a(\alpha + k\beta)^{k-1}}{k!} \cdot (ze^{-\beta z})^k.$$  

(2)

Let $z = \frac{x}{1 + \beta x}$. Because $\beta z \in [0, 1)$ and the expression $\beta ze^{1-z\beta}$ is an increasing function of $\beta z$, the conditions of (2) are satisfied. Thus, the equality $R_{\alpha, \beta}(1)(x) = 1$ is a direct consequence of (2).

To compute the first moment, we use another formula of Jensen [11]
\[ \frac{e^{az}}{1 - \beta z} = \sum_{k=0}^{\infty} \frac{(\alpha + k\beta)^k}{k!} \cdot (ze^{-\beta z})^k, \quad (3) \]

which is valid under the same conditions as (2) and which can be deduced from it by applying the derivative with respect to \( z \). Using (3) and (2), we obtain for \( \beta > 0 \)

\[
R_{a,\beta}(t)(x) = \sum_{k=0}^{\infty} a(\alpha + \beta k)^{k-1} \frac{k}{k!} e^{-(\alpha + \beta k)z} \cdot \frac{k}{\alpha} \\
= e^{-az} \sum_{k=0}^{\infty} (\alpha + \beta k)^{k-1} \frac{k}{k!} (ze^{-\beta z})^k \cdot \frac{\alpha + \beta k - \alpha}{\beta} \\
= e^{-az} \sum_{k=0}^{\infty} (\alpha + \beta k)^k \beta z^{-k} - e^{-az} \sum_{k=0}^{\infty} \frac{\alpha(\alpha + \beta k)^{k-1}}{k!} (ze^{-\beta z})^k \\
= \frac{1}{\beta(1 - \beta z)} - \frac{1}{\beta} = \frac{z}{1 - \beta z} = x.
\]

The last equality gives us the reason for the choice of \( z \). \( \square \)

For a given \( x \geq 0 \), let us denote by \( \psi_x \) the function defined on \([0, \infty)\) by \( \psi_x(t) = t - x \). Let us consider the central moments of order \( k \) of the operators \( R_{a,\beta} \):

\[
\mu_k(x) = R_{a,\beta}(\psi_x^k)(x), \quad k = 0, 1, 2, \ldots
\]

Based on Lemma 1, we have \( \mu_0(x) = 1, \mu_1(x) = 0 \) for every \( x \geq 0 \). For a simple method to obtain the second central moment, we need the following lemma.

**Lemma 2.** The operators \( R_{a,\beta} \) satisfy the following differential equation

\[
\frac{d}{dx} R_{a,\beta}(f)(x) = \frac{\alpha}{x(1 + \beta x)^2} \cdot R_{a,\beta}(\psi_x f)(x), \quad x > 0.
\]

**Proof.** We omit it, since it is trivial. \( \square \)

**Remark 1.** Relation (4) shows that \( R_{a,\beta} \) are a particular case of the exponential operators studied in [4]. Considering \( f = \psi_x^k \) in (4), we obtain the following recurrence for the central moments:

\[
\frac{d}{dx} (\mu_k(x)) + k \mu_{k-1}(x) = \frac{\alpha}{x(1 + \beta x)^2} \cdot \mu_{k+1}(x), \quad k = 1, 2, 3, \ldots
\]

In particular,

\[
\mu_2(x) = \frac{x(1 + \beta x)^2}{\alpha}.
\]

For higher-order moments, their expression can be obtained by other means (see Equation (18)).

**Remark 2.** Using the Shisha–Mond technique from [12], we can estimate the rate of convergence in terms of the usual modulus of continuity by

\[
|R_{a,\beta}(f)(x) - f(x)| \leq 2 \cdot \omega \left( f, \sqrt{\frac{x(1 + \beta x)^2}{\alpha}} \right).
\]

For a given \( x \geq 0 \), for a uniformly continuous function \( f \in C[0, \infty) \) and a fixed \( \beta \geq 0 \), the convergence of \( R_{a,\beta}(f)(x) \) toward \( f(x) \) is ensured by \( \alpha \to \infty \). Because \( \alpha \) can be chosen arbitrarily, the rate of convergence can be made as high as we want. We also notice that this convergence remains valid even if \( \beta \) varies, but in this case, we must also impose the condition that \( \beta^2 / \alpha \to 0 \). For a uniform approximation, we need a different approach, which will be presented in Section 3.
Remark 3. Another consequence of (4) is the inequality
\[ \left| \frac{d}{dx} R_{\alpha, \beta}(f)(x) \right| \leq \frac{\alpha}{x(1 + \beta x)^2} \cdot R_{\alpha, \beta}(|\psi_x f|)(x) \]
\[ \leq \frac{\alpha \|f\|}{x(1 + \beta x)^2} \cdot \sqrt{R_{\alpha, \beta}(\psi^2)(x)} \leq \frac{\sqrt{\alpha} \|f\|}{\sqrt{x(1 + \beta x)}}, \]
for every bounded function \( f \).

3. Uniform Approximation

In [6], Totik has given general results for the uniform approximation of functions by positive linear operators which reproduce constant and linear functions and satisfy certain mild assumptions. He obtained a characterization of functions which can be uniformly approximated by such operators and estimated the rate of convergence in terms of the Ditzian–Totik modulus of smoothness of order 2. Let us consider the function \( \phi : [0, \infty) \to [0, \infty) \) defined by \( \phi(x) = \sqrt{x(1 + \beta x)} \). The second-order modulus of Ditzian–Totik with respect to the weight function \( \phi \) (see also [13]) is defined by
\[ \omega^2_{\phi}(f, \delta) = \sup_{0 \leq t \leq \delta} \sup_{x \geq 0} |f(x + t\phi(x)) - 2f(x) + f(x - t\phi(x))|. \]

The results obtained by Totik are proven under some assumptions on the growth of the function \( \phi \), which are not verified in our case (condition (3) from page 164 of [6] is not valid at infinity, i.e., \( \lim_{x \to \infty} \phi(x)/x \) is not finite). This is why we have modified the method for obtaining such results.

We will present first an estimate of the rate of approximation in terms of the modulus \( \omega^2_{\phi} \). For this, we need the following auxiliary result.

Lemma 3. For every \( x \geq 0 \) and \( \alpha, \beta > 0 \) we have
\[ R_{\alpha, \beta}\left( \frac{t}{1 + \beta t} \right)(x) = \frac{\alpha}{\alpha + \beta} \cdot \frac{x}{1 + \beta x}. \]  
(7)

Proof. Let us denote \( z = \frac{x}{1 + \beta x} \). We have
\[ R_{\alpha, \beta}\left( \frac{t}{1 + \beta t} \right)(x) = \sum_{k=0}^{\infty} \frac{\alpha(\alpha + \beta k)^{k-1}}{k!} z^k e^{-\{(\alpha + \beta k)z\}} = \frac{k}{\alpha + \beta k} \]
\[ = \alpha z \sum_{k=1}^{\infty} \frac{\alpha + \beta + \beta(k - 1))^{k-2}}{(k - 1)!} z^{k-1} \cdot \left( e^{-\{(\alpha + \beta + \beta(k - 1))z\}} \right) \]
\[ = \alpha z \sum_{k=1}^{\infty} \frac{(\alpha + \beta + \beta(k - 1))^{k-2}}{(k - 1)!} \cdot \left( e^{-\{(\alpha + \beta + \beta(k - 1))z\}} \right) \]
\[ = \frac{\alpha z}{\alpha + \beta} \sum_{v=0}^{\infty} \frac{(\alpha + \beta + \beta v)^{v-1}}{v!} z^v e^{-\{(\alpha + \beta + \beta v)z\}}. \]

Using (2), we obtain
\[ R_{\alpha, \beta}\left( \frac{t}{1 + \beta t} \right)(x) = \frac{\alpha z}{\alpha + \beta}, \]
which proves (7). \( \Box \)

Theorem 1. Let \( f \in C[0, \infty) \) be a bounded function. Then,
\[ \| R_{\alpha, \beta}(f) - f \| \leq C \cdot \omega^2_{\phi}\left( f, \frac{1}{\sqrt{\alpha + \beta}} \right), \]  
(8)

for some constant \( C > 0 \) independent of \( \alpha \).
We know by (13) Theorem 2.1.1 that

\[ M_2^2(f, \delta^2) = \inf_{g \in D} \left( \| f - g \| + \delta^2 \| q^2 g'' \| \right). \]

We have by ([13] Theorem 2.1.1) that

\[ M^{-1} \cdot \omega_2^g(f, \delta) \leq K_2^g(f, \delta^2) \leq M \cdot \omega_2^g(f, \delta), \quad 0 \leq \delta \leq \delta_0, \]

for some constants \( M, \delta_0 > 0 \).

Let \( g \in D \). We have by a standard argument

\[ \| R_{\alpha, \beta}(f) - f \| \leq \| R_{\alpha, \beta}(f) - R_{\alpha, \beta}(g) \| + \| R_{\alpha, \beta}(g) - g \| + \| g - f \|. \]

It is easy to see that

\[ |g(t) - g(x) - (t - x)g'(x)| \leq \left| \int_x^t (t - u)g''(u)du \right| \leq \left| \int_x^t (t - u)|g''(u)|du \right| \]

\[ \leq \| q^2 g'' \| \cdot \left| \int_x^t \frac{(t - u)}{q^2(u)}du \right|. \]  \tag{9}

Let us prove that

\[ \left| \int_x^t \frac{(t - u)}{q^2(u)}du \right| \leq \frac{(t - x)^2}{x(1 + \beta x)(1 + \beta t)}. \]  \tag{10}

Consider first the case \( 0 < x < u < t \). Using Napier’s inequality \( \frac{b-a}{b} < \ln b - \ln a < \frac{b-a}{a} \), for \( b = \frac{t}{1 + \beta t} > a = \frac{x}{1 + \beta x} > 0 \), we obtain

\[ \left| \int_x^t \frac{(t - u)}{q^2(u)}du \right| < t \left( \frac{t}{1 + \beta t} - \frac{x}{1 + \beta x} \right) \cdot \frac{1 + \beta x}{x} - \frac{t - x}{1 + \beta x} \]

\[ = \frac{(t - x)^2}{x(1 + \beta x)(1 + \beta t)}. \]

Consider now the case \( 0 \leq t < u < x \). Similarly, we have

\[ \left| \int_x^t \frac{(t - u)}{q^2(u)}du \right| = \int_x^t \frac{u - t}{u(1 + \beta u)^2}du \]

\[ = - \int_x^t \frac{1}{u}du + \int_x^t \frac{t\beta}{1 + \beta u}du + \int_t^x \frac{1 + \beta t}{1 + \beta u}du \]

\[ = -t \left( \ln \frac{1 + \beta x}{1 + \beta t} - \frac{t}{1 + \beta t} \right) + \frac{x - t}{1 + \beta x} \]

\[ \leq -t \left( \frac{x}{1 + \beta x} - \frac{t}{1 + \beta t} \right) \cdot \frac{1 + \beta x}{x} + \frac{x - t}{1 + \beta x} \]

\[ = \frac{(t - x)^2}{x(1 + \beta x)(1 + \beta t)}. \]
We apply the operators $R_{a, \beta}$ to the relation (9) and we use inequality (10) to estimate the error for the approximation of $g$:

$$|R_{a, \beta}(g)(x) - g(x)| \leq \left\| \varphi^2 g'' \right\| \cdot \frac{1}{x(1 + \beta x)} \cdot R_{a, \beta}\left(\frac{(t - x)^2}{1 + \beta t}\right)(x).$$

We decompose the function to be evaluated by $R_{a, \beta}$ in

$$\frac{(t - x)^2}{1 + \beta t} = \frac{t^2}{1 + \beta t} - 2x \cdot \frac{t}{1 + \beta t} + x^2 \cdot \frac{1}{1 + \beta t}.$$  (11)

By simple computation, with $z = \frac{x}{1 + \beta x}$

$$R_{a, \beta}\left(\frac{t^2}{1 + \beta t}\right)(x) = \sum_{k=0}^{\infty} \frac{\alpha(\alpha + \beta k)^{k-1}}{k!} z^k e^{-(\alpha + \beta k)z} \cdot \frac{k^2}{\alpha(\alpha + \beta k)}$$

$$= z \sum_{k=1}^{\infty} \frac{(\alpha + \beta k)^{k-2}}{(k-1)!} z^{k-1} e^{-(\alpha + \beta k)z} \cdot k$$

$$= z \sum_{v=0}^{\infty} \frac{(\alpha + \beta)(\alpha + \beta + \beta v)^{v-1}}{v!} z^v e^{-(\alpha + \beta + \beta v)z} \cdot \frac{v + 1}{\alpha + \beta}$$

$$= z \left[ R_{a + \beta, \beta} (t) + \frac{1}{\alpha + \beta} R_{a + \beta, \beta} (1) \right].$$

We obtain

$$R_{a, \beta}\left(\frac{t^2}{1 + \beta t}\right)(x) = \frac{x^2}{1 + \beta x} + \frac{x}{(1 + \beta x)(\alpha + \beta)}.$$  Using relation (7), we have

$$R_{a, \beta}\left(\frac{1}{1 + \beta t}\right)(x) = R_{a, \beta}\left(\frac{1 + \beta t - \beta t}{1 + \beta t}\right)(x)$$

$$= 1 - \beta \cdot R_{a, \beta}\left(\frac{t}{1 + \beta t}\right)(x) = 1 - \frac{\alpha \beta x}{(1 + \beta x)(\alpha + \beta)}.$$  Using the equality (11) and applying the operators $R_{a, \beta}$, we deduce

$$R_{a, \beta}\left(\frac{(t - x)^2}{1 + \beta t}\right)(x) = \frac{x(1 + \beta x)}{\alpha + \beta}$$

and

$$\left\| R_{a, \beta}(f) - f \right\| \leq 2 \left\| f - g \right\| + \left\| \varphi^2 g'' \right\| \cdot \frac{1}{\alpha + \beta}.$$

Passing to the infimum with $g \in D$, we finally get

$$\left\| R_{a, \beta}(f) - f \right\| \leq 2 \cdot K_{\varphi}^\beta \left( f, \frac{1}{\alpha + \beta} \right) \leq 2M \cdot \omega^\varphi_2 \left( f, \frac{1}{\sqrt{\alpha + \beta}} \right).$$

We have estimated the rate of convergence in terms of the second-order modulus of Ditzian and Totik. Let us denote by $C^* [0, \infty)$ the space of continuous functions on $[0, \infty)$ that have a finite limit at infinity, endowed with the uniform norm, and let us prove that bounded functions which can be uniformly approximated are part of the space $C^* [0, \infty)$. Let us consider the function $\xi : [0, \infty) \to (0, 1]$ defined by $\xi(x) = \frac{1}{\sqrt{1 + \beta x}}$, $x \geq 0$ and the following modulus of continuity associated with the function $\xi$:

$$\omega_\xi^\varphi(f, \delta) = \sup \{ |f(t) - f(x)| \mid t, x \geq 0, |\xi(t) - \xi(x)| \leq \delta \},$$  (12)
where $\delta \geq 0$ and $f \in C^*[0, \infty)$.

The modulus $\omega_\zeta$ is a particular case of a more general modulus studied in [14,15]. This modulus is suitable for the estimation of the uniform approximation rate for functions of the space $C^*[0, \infty)$. Indeed, it can be expressed in terms of the usual modulus of continuity by $\omega_\zeta(f, \delta) = \omega(f \circ \zeta^{-1}, \delta)$ and the limit

$$
\lim_{\delta \to 0} \omega_\zeta(f, \delta) = 0
$$

is valid if and only if $f \circ \zeta^{-1}$ is uniformly continuous on $(0, 1]$. However, a function which is uniformly continuous on $(0, 1]$ has a finite limit at 0. In our case, the fact that $f \circ \zeta^{-1}$ has a finite limit at 0 means that $f$ has a finite limit at infinity.

**Theorem 2.** Let $f \in C[0, \infty)$ be a bounded function. Then, $R_{a,\beta} f$ converges uniformly to the function $f$ if and only if $f$ belongs to the space $C^*[0, \infty)$. Moreover, one has

$$
\| R_{a,\beta}(f) - f \| \leq 2 \cdot \omega_\zeta \left( f, \sqrt{\frac{\beta}{\alpha + \beta}} \right).
$$

**Proof.** Let $f \in C^*[0, \infty)$. We prove that $\| R_{a,\beta}(f) - f \| \to 0$. From the properties of the usual modulus of continuity, we have the following estimation

$$
|f(t) - f(x)| \leq \left( 1 + \frac{|\zeta(t) - \zeta(x)|}{\delta} \right) \cdot \omega_\zeta(f, \delta), \text{ for every } t, x \geq 0 \text{ and } \delta > 0.
$$

Applying the operators $R_{a,\beta}$, we get

$$
|R_{a,\beta}(f)(t) - f(x)| \leq \left( 1 + \frac{R_{a,\beta}(|\zeta(t) - \zeta(x)|)(x)}{\delta} \right) \cdot \omega_\zeta(f, \delta).
$$

For the evaluation of $R_{a,\beta}(|\zeta(t) - \zeta(x)|)(x)$, let us use the inequality

$$
|\zeta(t) - \zeta(x)| = \frac{|(1 + \beta t) - (1 + \beta x)|}{(\sqrt{1 + \beta t} + \sqrt{1 + \beta x})\sqrt{1 + \beta x} \sqrt{1 + \beta t}} \leq \frac{1}{\sqrt{1 + \beta x}} \sqrt{\frac{1 + \beta t}{1 + \beta x} - \frac{1 + \beta x}{1 + \beta t}}.
$$

We apply the Cauchy–Schwarz inequality

$$
R_{a,\beta}(|\zeta(t) - \zeta(x)|)(x) \leq \frac{1}{\sqrt{1 + \beta x}} \cdot \sqrt{R_{a,\beta} \left( \frac{1 + \beta t}{1 + \beta x} - \frac{1 + \beta x}{1 + \beta t} \right)^2}(x)
$$

$$
\leq \frac{1}{\sqrt{1 + \beta x}} \cdot \sqrt{R_{a,\beta} \left( \frac{1 + \beta t}{1 + \beta x} \right)(x) - 2 \cdot R_{a,\beta} \left( \frac{1 + \beta x}{1 + \beta t} \right)(x)}.
$$

In view of the linearity and the preservation properties of the operators $R_{a,\beta}$ presented in Lemmas 1 and 3, we obtain

$$
R_{a,\beta} \left( \frac{1 + \beta t}{1 + \beta x} \right)(x) = 1 \quad \text{and} \quad R_{a,\beta} \left( \frac{1 + \beta x}{1 + \beta t} \right)(x) = 1 + \beta x - \frac{\alpha \beta x}{\alpha + \beta}.
$$

Thus, we have proven that

$$
R_{a,\beta}(|\zeta(t) - \zeta(x)|)(x) \leq \frac{1}{\sqrt{1 + \beta x}} \cdot \sqrt{\beta x - \frac{\alpha \beta x}{\alpha + \beta}} \leq \sqrt{\frac{\beta}{\alpha + \beta}}.
$$
and this implies the inequality (13). This inequality and the properties of the modulus \(\omega_\zeta\) prove that \(R_{a,\beta}(f)\) converges uniformly to the function \(f\).

Suppose now that \(R_{a,\beta}(f)\) converges uniformly to the bounded function \(f\). We will prove that each term of the right-hand side of the above inequality tends to 0, when \(\theta \to 0\) with the property that \(\theta' = \frac{\sqrt{\beta}}{\sqrt{x(1+\beta x)}}, \) for \(x > 0\). Let us observe that

\[
(\theta \circ \zeta^{-1})(x) = 2 \arctan \sqrt{\frac{1}{x^2} - 1}
\]

is uniformly continuous on \([0, 1]\) and \(f \circ \zeta^{-1} = (f \circ \theta^{-1}) \circ (\theta \circ \zeta^{-1})\). Thus, it remains to prove that \(f \circ \theta^{-1}\) is uniformly continuous on \([0, \pi]\).

Using the properties of the usual modulus of continuity, we can write

\[
\omega \left(f \circ \theta^{-1}, \delta\right) \leq \omega \left((f - R_{a,\beta}(f)) \circ \theta^{-1}, \delta\right) + \omega \left(R_{a,\beta}(f) \circ \theta^{-1}, \delta\right).
\]

We will prove that each term of the right-hand side of the above inequality tends to 0, when \(\delta \to 0\). Firstly,

\[
\omega \left((f - R_{a,\beta}(f)) \circ \theta^{-1}, \delta\right) \leq 2\|f - R_{a,\beta}(f)\|.
\]

Since \(R_{a,\beta}(f)\) converges uniformly to \(f\), the left-hand side of the inequality from above tends to zero for all \(\delta > 0\).

Using the inequality from Remark 3, we obtain

\[
\omega \left(R_{a,\beta}(f) \circ \theta^{-1}, \delta\right) = \sup_{|\theta(u) - \theta(v)| \leq \delta} |R_{a,\beta}(f)(u) - R_{a,\beta}(f)(v)|
\]

\[
\leq \sup_{|\theta(u) - \theta(v)| \leq \delta} \left|\int_0^u \frac{d}{dx} R_{a,\beta}(f)(x)dx\right|
\]

\[
\leq \alpha \|f\| \sup_{|\theta(u) - \theta(v)| \leq \delta} \left|\int_0^u \theta'(x)dx\right| \leq \alpha \delta \|f\| \left|\frac{\alpha \delta}{\sqrt{\beta}}\right| = \frac{\alpha \delta}{\sqrt{\beta}}.
\]

We can choose an appropriate \(\delta > 0\) such that the right-hand side tends to zero. This proves that \(\omega(f \circ \theta^{-1}, \delta)\) tends to 0 as \(\delta \to 0\). Using the properties of the usual modulus of continuity, the function \(f \circ \theta^{-1}\) is uniformly continuous on \([0, \pi]\).

\section{4. A Voronovskaya-Type Result}

For a given \(u \geq -1/\epsilon\), the solution of the equation \(u = v e^v\) defines the Lambert function (also called product logarithm), denoted \(v = W(u)\). We need this function to express the image of exponential functions through the operators \(R_{a,\beta}\).

\textbf{Theorem 3.} Let \(\lambda \in \mathbb{R}\). Then, there is \(a_0 \geq 0\) such that

\[
R_{a,\beta}(e^{\lambda t})(x) = \exp \left(-\frac{\alpha x}{1 + \beta x} - \frac{\alpha}{\beta} \cdot W \left(-\frac{\beta x}{1 + \beta x} e^{-\frac{\beta x}{1 + \beta x} e^{\frac{\lambda}{\beta}}} \right)\right)
\]

(14)

for every \(x \geq 0\), \(\beta > 0\) and every \(\alpha > a_0\).

\textbf{Proof.} With the notation \(z = \frac{x}{1 + \beta x}\), we have

\[
R_{a,\beta}(e^{\lambda t})(x) = e^{-az} \sum_{k=0}^{\infty} \frac{a(\alpha + \beta k)^{k-1}}{k!} \cdot \left(ze^{-\beta z e^z}\right)^k.
\]
If \( \lambda \leq 0 \), then \( \beta z e^{-\beta z e^{\frac{1}{z}}} \leq \beta z e^{-\beta z} < \frac{1}{z} \) and the series converges for every \( a > 0 \). In this case, we can choose \( a_0 = 0 \).

Consider now the case \( \lambda > 0 \). Since \( \lim_{a \to \infty} \beta z e^{-\beta z e^{\frac{1}{z}}} = \beta z e^{-\beta z} < \frac{1}{z} \), there is \( a_0 > 0 \) such that \( \beta z e^{-\beta z e^{\frac{1}{z}}} < \frac{1}{z} \), for every \( a > a_0 \). This proves that \( R_{a,\beta}(e^{\lambda t}, x) \) is correctly defined for every \( x \geq 0, \beta > 0 \) and every \( a > a_0 \).

Since \( -\beta z e^{-\beta z e^{\frac{1}{z}}} > -\frac{1}{z} \), there is \( v \) such that \( -\beta z e^{-\beta z e^{\frac{1}{z}}} = -\beta v e^{-\beta v} \). Using the Lambert function, we have \( -\beta v = W(-\beta z e^{-\beta z e^{\frac{1}{z}}}) \). The image of the exponential function can be written

\[
R_{a,\beta}(e^{\lambda t})(x) = e^{-az} \sum_{k=0}^{\infty} \frac{a(a + \beta k)^{k-1}}{k!} \left( ve^{-\beta v} \right)^k = e^{-az} \cdot e^{\beta v}
\]

\[= \exp \left( -az - \frac{a}{\beta} \cdot W(-\beta z e^{-\beta z e^{\frac{1}{z}}}) \right).\]

\[\square\]

**Corollary 1.** For every \( \lambda \in \mathbb{R} \) and every \( x \in [0, \infty) \),

\[
\lim_{a \to \infty} R_{a,\beta}(e^{\lambda t})(x) = e^{\lambda x}.
\]

**Proof.** Using the relation (14) with the notation \( z = \frac{x}{1 + \beta z} \), we have

\[
R_{a,\beta}(e^{\lambda t})(x) = \exp \left( -az - \frac{a}{\beta} \cdot W(-\beta z e^{-\beta z e^{\frac{1}{z}}}) \right)
\]

\[= \exp \left( \frac{a}{\beta} \cdot \left[ -\beta z - W(-\beta z e^{-\beta z e^{\frac{1}{z}}} \right) \right] \]

\[= \exp \left( W(-\beta z e^{-\beta z e^{\frac{1}{z}}}) - W(-\beta z e^{-\beta z e^{\frac{1}{z}}} \right) \frac{p}{\beta}
\]

Since \( W \) is a differentiable function on \((-1/e, \infty)\) with \( W'(u) = \frac{W(u)}{u(1+W(u))} \), we apply the l'Hospital rule for \( u = \frac{1}{\beta} \to 0 \). Consequently,

\[
\lim_{a \to \infty} R_{a,\beta}(e^{\lambda t})(x) = \lim_{u \to 0} \exp \left( \frac{1}{\beta} \cdot W(-\beta z e^{-\beta z e^{\frac{1}{z}}} \right) = \exp \left( \frac{\lambda}{\beta} \cdot \frac{W(-\beta z e^{-\beta z e^{\frac{1}{z}}})}{1+W(-\beta z e^{-\beta z e^{\frac{1}{z}}})} \right)
\]

\[= \exp \left( \frac{\lambda}{\beta} \cdot \frac{-\beta z}{1-\beta z} \right) = e^{\lambda x}.
\]

\[\square\]

For \( \gamma \geq 0 \), let us denote by \( C_{\gamma}[0, \infty) \) the space of continuous functions on \([0, \infty)\) satisfying the growth condition \( f(t) = O(e^{\gamma t}) \) as \( t \to \infty \). As we have seen from (14), the operators \( R_{a,\beta} \) are correctly defined for every \( f \in C_{\gamma}[0, \infty) \). We can easily deduce from Corollary 1 that

\[R_{a,\beta}(e^{\lambda t})(x) \leq E(\beta, \gamma, x), \quad \text{for every } \beta, \gamma, x > 0,
\]

where \( E(\beta, \gamma, x) \) is independent of \( a \).

Based on the works of Abel and Agratini [8] and Abel and Gupta [9], we derive a complete asymptotic expansion for the operators \( R_{a,\beta} \). We need
**Lemma 4** (Lemma 3 of [8]). Let \( \alpha > 0, \beta \in [0,1) \) and \( p \in \mathbb{N} \). Then,
\[
\sum_{v=0}^{\infty} v(v-1) \cdots (v-p+1) \cdot \omega_{\beta}(v, \alpha, 1) = \sum_{j=0}^{p-1} \binom{p-1}{j} c_{p,j}(\beta) \alpha^{p-j},
\]
where \( c_{p,j}(\beta) \) are defined by
\[
c_{p,j}(\beta) = \lim_{w \to 0} \frac{\partial^j}{\partial w^j} \left( \frac{w e^{\beta w}}{1 + w - e^{\beta w}} \right)^p.
\]

**Theorem 4.** Let \( q \in \mathbb{N}, \gamma \geq 0 \) and \( x > 0 \). For every function \( f \in C_{\gamma}[0, \infty) \) possessing a derivative of order \( 2q \) at \( x \), we have
\[
R_{x,\beta}(f)(x) = f(x) + \sum_{k=1}^{q} \frac{1}{\alpha^k} \cdot r_k(f, x) + o(\alpha^{-q}) \quad (x \to \infty),
\]
where the coefficients \( r_k(f, x) \) are given by
\[
r_k(f, x) = \frac{2k}{s=k+1} \frac{\sigma(s)(x)}{s!} \cdot \frac{k}{r} \sum_{r=k+1}^{s} (-1)^{s-r} \binom{s}{r} \sigma(r, r - \ell) \left( \frac{r - \ell - 1}{k - \ell} \right) \times \frac{x^{s-k}}{(1 + \beta x)^{r-\ell}} c_{r-\ell,k-\ell} \left( \frac{\beta x}{1 + \beta x} \right).
\]

The coefficients \( c_{r-\ell,k-\ell} \left( \frac{\beta x}{1 + \beta x} \right) \) have been defined in Lemma 4 and \( \sigma(r, r - \ell) \) represent the Stirling numbers of the second kind.

**Proof.** Let us remark that
\[
\omega_{\beta}(v, \alpha, \frac{x}{1 + \beta x}) = \omega_{\frac{\beta x}{1 + \beta x}}(v, \frac{\alpha x}{1 + \beta x}, 1).
\]
We can apply Lemma 4 with \( \beta \) replaced by \( \frac{\beta x}{1 + \beta x} \in [0,1) \) and \( \alpha \) replaced by \( \frac{\alpha x}{1 + \beta x} > 0 \).

We can write the monomial function \( e_r \) (defined by \( e_r(t) = t^r, r \in \mathbb{N} \)) using Stirling numbers of the second kind by
\[
v^r = \sum_{p=1}^{r} \sigma(r, p) \cdot v(v-1) \cdots (v-p+1).
\]
The \( r \)th moment of the operators \( R_{x,\beta} \) will be expressed by
\[
R_{x,\beta}(e_r)(x) = \sum_{v=0}^{\infty} \frac{v^r}{\alpha^r} \cdot \omega_{\beta}(v, \alpha, \frac{x}{1 + \beta x}) = \frac{1}{\alpha^r} \sum_{p=1}^{r} \sigma(r, p) \binom{p-1}{j} c_{p,j} \left( \frac{\beta x}{1 + \beta x} \right) (\alpha x)^{p-j}.
\]
Exchanging the indexes of summation with \( k = r + j - p \) and \( \ell = r - p \), we can write
\[
R_{x,\beta}(e_r)(x) = \sum_{k=0}^{r-1} \frac{1}{\alpha^k} \cdot \frac{x^{r-k}}{(1 + \beta x)^{r-k}} \sum_{\ell=0}^{k} \sigma(r, r - \ell) \left( \frac{r - \ell - 1}{k - \ell} \right) c_{r-\ell,k-\ell} \left( \frac{\beta x}{1 + \beta x} \right).
\]
The term corresponding to \( k = 0 \) in the above representation is

\[
x^r \left( \frac{\beta x}{1 + \beta x} \right)^s = \frac{x^r}{(1 + \beta x)^s} \frac{\beta x}{1 + \beta x} \left( \lim_{w \to 0} \frac{\text{we}^{r \text{er}}}{1 + w - e^{r \text{er}}} \right)^r = x^r.
\]

Using the binomial formula, the central moment of order \( s \) of the operators \( R_{\alpha,\beta} \) is represented by

\[
\mu_s(x) = R_{\alpha,\beta}((t - x)^s)(x) = (-x)^s + \sum_{r=1}^{s} \binom{s}{r} (-x)^{s-r} R_{\alpha,\beta}(t^r)(x).
\]

The term independent of \( \alpha \) is obtained by considering only the term corresponding to \( k = 0 \) in the representation of \( R_{\alpha,\beta}(t^r)(x) \), i.e.,

\[
(-x)^s + \sum_{r=1}^{s} \binom{s}{r} (-x)^{s-r} \cdot x^r = 0.
\]

Replacing the other terms of \( R_{\alpha,\beta}(t^r)(x) \) and changing the order of summation, we obtain

\[
\mu_s(x) = \sum_{k=1}^{s-1} \frac{1}{k!} \sum_{\ell=0}^{k} \sum_{r=k+1}^{s} (-1)^k \binom{s}{r} \binom{r-\ell-1}{k-\ell} \left( \frac{\beta x}{1 + \beta x} \right)^{r-\ell-\ell} \sum_{r=1}^{s} \binom{s}{r} (-x)^{s-r} \frac{x^{\ell-\ell} \cdot \cdot x^r}{(1 + \beta x)^{r-\ell-\ell}}.
\]

(18)

We know from ([16] Lemma 2) that

\[
\mu_s(x) = O \left( \alpha^{-\left[ \frac{s+1}{2} \right]} \right) \quad (\alpha \to \infty).
\]

This means that the index \( k \) from the representation of \( \mu_s(x) \) grows actually only from \( \left[ \frac{s+1}{2} \right] \) to \( s-1 \).

Let us consider the Taylor expansion with the Peano remainder

\[
f(t) = f(x) + f'(x)(t-x) + \sum_{s=2}^{2q} \frac{f^{(s)}(x)}{s!} (t-x)^s + (t-x)^{2q} h(t,x),
\]

where \( h(t,x) \) is a function such that \( \lim_{t \to x} h(t,x) = 0 \). We apply the operators \( R_{\alpha,\beta} \) and use the representation of the central moments we have derived. We interchange the sums with indexes \( s \) and \( k \)

\[
\sum_{s=2}^{2q} \sum_{k=1}^{s-1} \frac{1}{k!} \sum_{\ell=0}^{k} \sum_{r=k+1}^{s} (-1)^k \binom{s}{r} \binom{r-\ell-1}{k-\ell} \left( \frac{\beta x}{1 + \beta x} \right)^{r-\ell-\ell} \sum_{r=1}^{s} \binom{s}{r} (-x)^{s-r} \frac{x^{\ell-\ell} \cdot \cdot x^r}{(1 + \beta x)^{r-\ell-\ell}} + o(\alpha^{-q}) \quad (\alpha \to \infty)
\]

and ignore the terms of higher order than \( \alpha^{-q} \). It remains to prove that

\[
R_{\alpha,\beta}((t-x)^{2q} h(t,x),x) = o(\alpha^{-q}) \quad (\alpha \to \infty).
\]

(19)

Let \( \varepsilon > 0 \). By the continuity of \( h \), there is \( \delta > 0 \) such that, for every \( t \geq 0 \) with the property \( |t-x| < \delta \), we have \( |h(t,x)| < \varepsilon \).

For the other values of \( t \), we have \( |t-x| \geq \delta \). Since \( f(t) \) and all \( (t-x)^s \), \( 0 \leq s \leq 2q \) belong to \( C_\gamma[0,\infty) \), the function \( (-x)^{2q} h(t,x) \) belongs to \( C_\gamma[0,\infty) \). Thus, for some constant \( C > 0 \) independent of \( t \), we have

\[
|h(t,x)| \leq \frac{C \varepsilon}{(t-x)^{2q}} \leq \frac{C}{\delta^{2q+1}} \cdot e^{\varepsilon t} \cdot |t-x|.
\]
We have proven that, for every \( t \geq 0 \), we have
\[
|h(t, x)| < \varepsilon + \frac{C}{\delta^{2q+1}} \cdot e^{\gamma t} \cdot |t - x|.
\]
We obtain
\[
\left| R_{\alpha, \beta}((t - x)^{2q}h(t, x))(x) \right| \leq \varepsilon \mu_{2q}(x) + \frac{C}{\delta^{2q+1}} \cdot R_{\alpha, \beta} \left( e^{\gamma t} \cdot |t - x|^{2q+1} \right)(x).
\]
Applying Hölder inequality for positive linear operators,
\[
R_{\alpha, \beta} \left( e^{\gamma t} |t - x|^{2q+1} \right)(x) \leq \left[ R_{\alpha, \beta} \left( e^{\gamma t} |t - x|^{2q+1} \right)(x) \right]^{2q+1} \cdot \left[ R_{\alpha, \beta} \left( |t - x|^{2q+2} \right)(x) \right]^{\frac{2q}{2q+1}}
\leq [E(\beta, \gamma(2q + 2), x)]^{\frac{1}{2q+2}} \cdot o \left( \alpha^{-\gamma - \frac{1}{2}} \right) = o(\alpha^{-\gamma}).
\]
Finally,
\[
\left| R_{\alpha, \beta}((t - x)^{2q}h(t, x))(x) \right| \leq \varepsilon O(\alpha^{-\gamma}) + o(\alpha^{-\gamma}).
\]
Since \( \varepsilon > 0 \) was chosen arbitrarily, letting \( \varepsilon \to 0 \), the relation (19) is proven.

**Corollary 2.** Let \( \gamma \geq 0 \) and \( x > 0 \). For every function \( f \in C_\gamma[0, \infty) \) having a derivative of second order at \( x \), it holds true
\[
\lim_{h \to 0} \alpha \cdot [R_{\alpha, \beta}(f)(x) - f(x)] = \frac{x(1 + \beta x)^2}{2} \cdot f''(x).
\]

**Remark 4.** The operators \( R_{\alpha, \beta} \) verify the conditions of ([17] Theorem 3.1), so the asymptotic relation (20) can be differentiated.

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