WEGNER ESTIMATE AND LOCALISATION FOR ALLOY TYPE OPERATORS WITH MINIMAL SUPPORT ASSUMPTIONS ON THE SINGLE SITE POTENTIAL

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ABSTRACT. We prove a Wegner estimate for alloy type models merely assuming that the single site potential is lower bounded by a characteristic function of a thick set (a particular class of sets of positive measure). The proof exploits on one hand recently proven unique continuation principles or uncertainty relations for linear combinations of eigenfunctions of the Laplacian on cubes and on the other hand the well developed machinery for proving Wegner estimates.

We obtain a Wegner estimate with optimal volume dependence at all energies, and localisation near the minimum of the spectrum, even for some non-stationary random potentials.

We complement the result by showing that a lower bound on the potential by the characteristic function of a thick set is necessary for a (translation uniform) Wegner estimate to hold. Hence, we have identified a sharp condition on the size for the support of random potentials that is sufficient and necessary for the validity of Wegner estimates.

1. Model and results

We prove a Wegner estimate for continuum random Schrödinger operators with a very weak assumption on the supports of the single site potentials, which turns out to be optimal. The random potential needs not be stationary. Together with an initial scale estimate we conclude localisation for such models.

Until now, a fundamental assumption in this context has been that the potential either satisfies a covering condition, see e.g. [MH84, CH94, Kir96] or that it is uniformly positive on at least a non-empty open set, see e.g. [CHK03, CHK07, RMV13, Kle13, NTTV18]. We only ask the sum of potentials to be positive on a so-called thick set:

Definition 1. Let \( \gamma \in (0,1] \), \( a = (a_1, \ldots, a_d) \in (0,\infty)^d \), and set \( A_a := [0, a_1] \times \cdots \times [0, a_d] \subset \mathbb{R}^d \). A measurable set \( S \subset \mathbb{R}^d \) is called \((\gamma,a)\)-thick if

\[
\text{vol} (S \cap (x + A_a)) \geq \gamma \text{vol} (A_a) \quad \text{for all } x \in \mathbb{R}^d
\]

and simply thick if there exist \( \gamma \in (0,1] \) and \( a \in (0,\infty)^d \) such that (1) holds.

Thick sets are a generalization of periodic positive measure sets. Thickness is the minimal condition on the size of a characteristic function necessary for a uncertainty relation of spectral projectors to hold, cf. [EV20], reformulating a criterion of [Kov00].

We will study alloy type Hamiltonians on \( L^2(\mathbb{R}^d) \) satisfying the following assumption on the random variables:

(U) Let \([m_-, m_+] \subset \mathbb{R}\) be an interval and \( \Omega = \times_{j \in \mathbb{Z}^d} [m_-, m_+] \) a probability space equipped with a product measure \( \mathbb{P} = \otimes_{j \in \mathbb{Z}^d} \mu_j \). Denote by \( \pi_j : \Omega \rightarrow [m_-, m_+] \) the coordinate projections such that \( \{ \pi_j \}_{j \in \mathbb{Z}^d} \) are a family of independent and uniformly bounded random variables. We assume that all \( \pi_j \) are non-trivial.

While most of our results would hold also under more general conditions on the random variables \( \pi_j \), the spelled out condition is standard in the literature. In fact, we want to focus our attention on a different building block of the random potential and identify the

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Weakest possible condition on the support of the single site potentials, spelled out in the following hypothesis:

\((\Pi)\) There are \(C_a, R > 0\) and \(p \in [2, \infty)\) with \(p > d/2\) if \(d > 3\), such that for each \(j \in \mathbb{Z}^d\), there is a measurable function \(u_j : \mathbb{R}^d \to [0, \infty)\), supported in \(B_R(j)\), the ball of radius \(R\) around \(j\), and satisfying \(\|u_j\|_{L^p(B_R(j))} \leq C_a\). There exists a thick set \(S \subset \mathbb{R}^d\) such that

\[
\sum_{j \in \mathbb{Z}^d} u_j \geq 1_S,
\]

\(1_S\) being the characteristic function of \(S\).

We define the random potential \(V_\omega\), and the alloy type Hamiltonian \(H_\omega\) on \(L^2(\mathbb{R}^d)\) as

\[
V_\omega = \sum_{j \in \mathbb{Z}^d} \pi_j(\omega)u_j, \quad \text{and} \quad H_\omega = -\Delta + V_\omega, \quad \omega \in \Omega.
\]

The potential \(V_\omega\) describes the forces to which an electron in the interior (bulk) of a solid is exposed due to the atoms placed at lattice points \(j \in \mathbb{Z}^d\) with modulated interaction parameter \(\pi_j\). The assumptions \((U)\) and \((\Pi)\) imply self-adjointness of \(H_\omega\) for all \(\omega \in \Omega\). Indeed, there exists a \(C_V < \infty\) (independent of \(\omega\)) such that:

\[
\langle \varphi, V_\omega \varphi \rangle \leq |m_+|\langle \varphi, \sum_{j \in \mathbb{Z}^d} u_j \varphi \rangle \leq \frac{1}{2}\langle \varphi, -\Delta \varphi \rangle + C_V \|\varphi\|^2,
\]

\[
\|V_\omega \varphi\| \leq \frac{1}{2}\| - \Delta \varphi\| + C_V \|\varphi\|,
\]

see for instance Theorem XII.96 in [RS78] and its proof. Consequently by the Kato-Rellich theorem the operator \(H_\omega = -\Delta + V_\omega\) is self-adjoint on the domain of \(\Delta\) and essentially self-adjoint on \(C^\infty_c(\mathbb{R}^d)\).

Also note that one can without loss of generality put any positive constant \(c > 0\) in front of \(1_S\) in \((2)\) since this simply amounts to rescaling the distributions \(\mu_j\). In the special case where all \(u_j\) are translates of a single function, that is \(u_j(x) = u(x - j)\), it suffices to find one positive measure set \(T \subset \Lambda_1(0)\) with \(u \geq 1_T\) in order to satisfy \((2)\). This \(T\) could for instance be a Smith–Volterra–Cantor set, illustrating the novelty of the results in this article, even in the ergodic setting. Conversely, if \(0 \leq u \in L^p\) is not almost everywhere zero, there exist an \(\epsilon > 0\) and a positive measure set \(T\), such that \(u \geq \epsilon \chi_T\), implying \((2)\) for \(u_j(x) = u(x - j)\), up to the universal prefactor \(\epsilon\).

In order to state our first result, we define \(s: [0, \infty) \to [0, 1]\)

\[
s(\epsilon) = \sup_{j \in \mathbb{Z}^d} \sup_{E \in \mathbb{R}} \mu_j \left( [E - \epsilon/2, E + \epsilon/2] \right),
\]

the uniform modulus of continuity of the family of marginal probability measures \(\{\mu_j\}_{j \in \mathbb{N}}\).

For \(L > 0\) and \(x \in \mathbb{R}^d\) we denote by \(\Lambda_L(x) = x + (-L/2, L/2)^d\) the cube with side length \(L\), centered at \(x\), and simply write \(\Lambda_L = \Lambda_L(0)\). We write \(H_{\omega,L,x}\) for the self-adjoint restriction of the Schrödinger operator \(H_\omega\) to the cube \(\Lambda_L(x)\) with periodic, Dirichlet or Neumann boundary conditions. We also use the notation \(\chi_I(\Lambda)\) for the spectral projector of a self-adjoint operator \(\Lambda\) onto the set \(I \subset \mathbb{R}\).

**Theorem 2.** Let \(H_\omega\) satisfy \((U)\) and \((\Pi)\) above. Then, for every \(E_0 \in \mathbb{R}\) there exists \(C_W : = C_W(E_0) > 0\), such that, for all \(x \in \mathbb{R}^d\), all \(L \geq \max\{a_1, \ldots, a_d\}\), where \(a = (a_1, \ldots, a_d) \in (0, \infty)^d\) is from the definition of thickness of \(S\), and for all intervals \([E - \epsilon, E + \epsilon] \subset (-\infty, E_0]\), the following Wegner estimate holds

\[
\mathbb{E}\{\text{Tr} \left[ \chi_{[E-\epsilon, E+\epsilon]}(H_{\omega,L,x}) \right] \} \leq C_W s(\epsilon)L^d.
\]

We next establish that this statement is sharp by showing that a Wegner estimate cannot hold if condition \((\Pi)\) is omitted. Indeed, for the Wegner estimate, it is essential that a level set (to some positive value) of the overall potential contains a thick set. The contraposition is the following condition \((\text{No} - \Pi)\) which is a non-thick upper bound on (any
level set of) the potential.

(No – Π) The sequence \((u_j)_{j \in \mathbb{N}} \subset L^p(\mathbb{R}^d)\) of functions \(u_j: \mathbb{R}^d \to [0, \infty)\) entering (3) is such that

\[
U := \sum_{j \in \mathbb{Z}^d} u_j
\]

satisfies \(\|U\|_{L^\infty(\mathbb{R}^d)} \leq C_U\) for some \(C_U > 0\), and for no \(\kappa > 0\), the set \(S_\kappa := \{x \in \mathbb{R}^d: U(x) \geq \kappa\}\) is thick.

Note that in the ergodic setting \(u_j(x) = u(x - j)\), this conditions implies that the function \(u\) is zero almost everywhere. The following theorem shows that under Assumption (No – Π) no translation uniform Wegner estimate can hold. The reason is that in this case there is an increasing sequence of cubes with "stubborn" eigenvalues which are arbitrarily insensitive to the potential:

**Theorem 3.** Let \(H_\omega\) satisfy (U) and (No – Π). Denote by \(H_{\omega,L,x_j}\) the restriction of \(H_\omega\) onto \(\Lambda_L(x_j)\) with Dirichlet boundary conditions. Then, for every \(E \geq 0\) and all \(L \geq 1\) there are infinitely many mutually disjoint cubes \(\Lambda_L(x_j), j \in \mathbb{N}\) such that for all configurations \(\omega\)

\[
\sigma(H_{\omega,L,x_j}) \cap [E - \varepsilon, E + \varepsilon] \neq \emptyset \quad \text{with} \quad \varepsilon = 12\pi \sqrt{E + 1}/L. \quad (6)
\]

Note that the width \(2\varepsilon\) of the energy interval in (6) is proportional to \(1/L\). But in order to prove Anderson localisation using multiscale analysis it would suffice to consider interval lengths \(\varepsilon \sim \exp(-L^\beta)\) for some \(\beta \in (0,1)\) in a Wegner estimate. Hence, at a first glance, Theorem 3 might leave the possibility that with differently chosen interval lengths this phenomenon of stubborn eigenvalues might disappear. However, we can also rule this out. Note that Theorem 3 is about all energies \(E \geq 0\). If we focus on neighborhoods of eigenvalues of \(H_{0,L,0}\) we can strengthen the statement and find stubborn eigenvalues in smaller intervals. The following theorem makes this precise and shows that without Assumption (Π) there cannot be any Wegner estimate with interval lengths that are exponentially small in \(L\) and thus no Wegner estimate useful in the multiscale analysis.

**Theorem 4.** Let \(H_\omega\) satisfy (U) and (No – Π). Denote by \(H_{\omega,L,x_j}\) the restriction of \(H_\omega\) onto \(\Lambda_L(x_j)\) with Dirichlet boundary conditions. Then, for all \(L \geq 1\) and for any eigenvalue \(E\) of \(H_{0,L,0}\) there are infinitely many mutually disjoint cubes \(\Lambda_L(x_j), j \in \mathbb{N}\) such that for all configurations \(\omega\)

\[
\sigma(H_{\omega,L,x_j}) \cap [E - e^{-L}, E + e^{-L}] \neq \emptyset. \quad (7)
\]

The proofs of Theorem 3 and 4 can be found in Section 2.4.

**Remark 5.** Non-ergodic random Schrödinger operators have been investigated by several authors in the recent years [RM12, RMV13, Kle13, TTT17, NTTV18, ST20]. One underlying objective in these works has been the identification of minimal assumptions on random Schrödinger operators which still ensure localisation. This has not only served the purpose of expanding the class of models but it has also helped to better distinguish between necessary assumptions and technical assumptions arising from the particular method of proof.

While in the ergodic setting, it is intuitive that a minimal assumption should be that the single-site potential \(u\) is not identically zero, our contribution in this paper can be understood to firstly generalize this to Assumption (Π), which is also valid in the non-ergodic setting, and, secondly, to prove that this indeed still leads to Wegner estimates and localisation.

**Remark 6 (Sparse potentials).** Theorems 2 to 4 concern the Hamiltonian (3) governing the motion of an electron in the bulk of a solid. In the literature on random operators also surface interactions of the form

\[
V_\omega = \sum_{j \in \mathbb{Z}^d} \pi_j(\omega)u_j, \quad \text{with} \quad 0 < D < d \quad (8)
\]
exhibiting an (anisotropic) spacial decay (on average), are studied. In fact, they are a special case of so called sparse (random) potentials, which can be modeled in various ways, see for instance [Kr93, HK00, KS01, Kt02, KV02, Kt02, FGK07]. One way is to deterministically dilute the lattice sites to which the single site potentials are attached leading to a stochastic field of the type

\[
V_{\omega}^{\text{sprs}}(x) = \sum_{k \in \Gamma} \pi_k(\omega)u(x - k) \quad \text{with} \quad \lim_{L \to \infty} \frac{2\Gamma \cap \Lambda_L}{|\Lambda_L|} = 0.
\]

The random surface potential spelled out in (8) falls into this class. Another way would be to dilute the sites probabilistically

\[
V_{\omega}^{\text{sprs}}(x) = \sum_{k \in \mathbb{Z}^d} \pi_k(\omega)u(x - k) \quad \text{with} \quad \lim_{|k| \to \infty} \mathbb{P}\{\pi_k = 0\} = 1.
\]

Finally one can consider an arrangement of interactions \(u_k\) on a lattice which become weaker with increasing distance from the origin

\[
V_{\omega}^{\text{sprs}}(x) = \sum_{k \in \mathbb{Z}^d} \pi_k(\omega)u_k(x) \quad \text{with} \quad \text{supp } u_k \subset \Lambda_R(k) \text{ for some } R
\]

and \(\lim_{|k| \to \infty} u_k(\cdot + k) = 0\) in some appropriate sense.

Let us concentrate on the last model and ask how to distinguish between sparse potentials on one hand and macroscopic/bulk ones on the other. Our two complementary Theorems 2 and 4 show that (provided (U) holds) condition (II) distinguishes between the two cases: If (II) is satisfied one obtains translation uniform Wegner estimates, if not there is a sequence of larger and larger cubes escaping to infinity with eigenvalues of the local Hamiltonian insensitive to randomness. The latter means that the disorder present in the potential is too weak to efficiently influence eigenvalues in certain spatial sectors, which is a signature of sparse potentials.

Let us now turn back to Hamiltonians as in (3) satisfying (U) and (II). If the family \(\{H_\omega\}_{\omega \in \Omega}\) is even ergodic, cf. [CL90, PF92, Ves08], the thermodynamic limit

\[
N : \mathbb{R} \to [0, \infty), \quad N(\cdot) = \lim_{L \to \infty} \frac{\text{Tr} [\chi_{(-\infty,\cdot]}(H_\omega, L, x)\chi_{\omega, L}]}{L^d}
\]

of the normalized eigenvalue counting functions exists for all almost all \(\omega \in \Omega\) and \(x \in \mathbb{R}^d\), is a distribution function, and is called the integrated density of states (IDS). If the IDS exists and \(\lim_{\varepsilon \to 0} s(\varepsilon) = 0\), Theorem 2 implies continuity of the IDS:

**Corollary 7.** Let \(H_\omega\) satisfy (U) and (II) above. Assume furthermore that the random family \(\{H_\omega\}_{\omega \in \Omega}\) is ergodic. Then, the integrated density of states \(N \) exists almost surely. For every \(E_0 \in \mathbb{R}\) and \(\varepsilon \in (0, 1]\) one has

\[
0 \leq N(E_0) - N(E_0 - \varepsilon) \leq C_W(E_0)s(\varepsilon).
\]

If the random variables \(\pi_j\) have a Hölder continuous density, more precisely if for some \(\alpha > 0\)

\[
\sup_{\varepsilon > 0} s(\varepsilon)/\varepsilon^\alpha =: C_\alpha < \infty
\]

then the Wegner estimate of Theorem 2 can also be used to prove Anderson and dynamical localisation at the bottom of the spectrum via the multiscale analysis. Actually, Hölder continuity could be relaxed to sufficiently strong log-Hölder continuity, cf. e.g. Lemma 4.6.2. in [Ves08]. There exists an entire hierarchy of notions of localisation. We spell out only three of them and refer for more details to [GK01].

**Definition 8.** The random family of operators \(\{H_\omega\}_{\omega \in \Omega}\) exhibits

- **Anderson localisation** in \(I \subset \mathbb{R}\) if \(\mathbb{P}\)-almost surely the spectrum of \(H_\omega\) within \(I\) is only of pure point type with exponentially decaying eigenfunctions,
Let Assumptions requires the following (non-stationary) initial scale estimate:

$$\text{dist}(\mathop{\text{sup}}_{\mathcal{I}}(1 + |x|)^{n/2} \chi_J(H_\omega)e^{-iH_\omega \psi}^2 < \infty \quad \text{for all } n \geq 0,$$

- dynamical localisation in \(I\) if for \(\mathbb{P}\)-almost every \(\omega \in \Omega\), every compact interval \(J \subset I\), and every \(\psi \in L^2(\mathbb{R}^d)\) with compact support we have

$$\sup_{b \in \mathbb{R}} \| (1 + |x|)^{n/2} \chi_J(H_\omega)e^{-iH_\omega \psi}^2 < \infty \quad \text{for all } n \geq 0,$$

- and strong Hilbert–Schmidt dynamical localisation in \(I\) if for every compact interval \(J \subset I\), we have

$$\sup_{y \in \mathbb{R}^d} \mathbb{E} \left( \sup_{f \in \mathcal{L}^\infty(\mathbb{R})} \| (1 + |y|)^{n/2} f(H_\omega) \chi_J(H_\omega) 1_{\Lambda_\Omega(y)}^2 \right) < \infty$$

for all \(n \geq 0\). Here \((1 + |x - y|)\) is understood as the multiplication operator with the function \(x \mapsto (1 + |x - y|)\).

Strong Hilbert-Schmidt dynamical localisation implies dynamical localisation which implies Anderson localisation. In general, the reverse is not true, but for a natural class of random Hamiltonians these three properties turn out to be actually equivalent, see [GK01] for details.

If the (lower-bounded) random operator \(\{H_\omega\}_{\omega \in \Omega}\) is ergodic, it exhibits almost sure spectrum, and thus a well defined spectral minimum, which is in fact a fluctuation boundary in the sense of Lifschitz tails. Since we do not assume that the random variables \(\{\pi_j\}_{j \in \mathbb{Z}^d}\), are identically distributed, we are in a more general situation. To ensure that there is no spectrum below zero, that \(0 \in \sigma(H_\omega)\) with positive probability, and it is a (generalized) fluctuation boundary we need a further

\[ (O) \text{ For all } j \in \mathbb{Z}^d \text{ we have } \min \sup \mu_j = 0. \]

This implies that zero is the overall minimum of the spectrum in the following sense:

$$\sup \{ E \in \mathbb{R} \mid E \leq \min \sigma(H_\omega) \mathbb{P}\text{-almost surely} \} = 0, \quad (10)$$

see Section 2.2 for a proof.

Remark 9. If we require additionally to Assumption (\(O\)) that there exist a distribution \(\mu : \mathcal{B}(\mathbb{R}) \to [0, 1]\) and an \(\varepsilon_0 > 0\) such that

$$\min \sup \mu = 0 \quad \text{and} \quad \forall \varepsilon \in (0, \varepsilon_0), j \in \mathbb{Z}^d: \quad \mu_j([0, \varepsilon]) \geq \mu([0, \varepsilon])$$

then \(\min \sigma(H_\omega) = 0\) almost surely by a simple Borel-Cantelli argument.

**Theorem 10.** Let Assumptions (U), (II), and (O), as well as Hölder continuity (9) hold. Then there exists an \(E_+ > 0\) such that the operator \(H_\omega\) exhibits strong Hilbert–Schmidt dynamical localisation in \([0, E_+]\).

Theorem 10 is proved by the multiscale analysis, see for instance [Sto01, GK01, GK03]. More specifically, in our situation, Theorem 2.3 in [RM12] applies directly and yields strong Hilbert–Schmidt dynamical localisation. Apart from the Wegner estimate, Theorem 2 it requires the following (non-stationary) initial scale estimate:

**Theorem 11.** Let Assumptions (U), (II), and (O), as well as \(\lim_{\varepsilon \to 0} s(\varepsilon) = 0\) hold. Then there exist \(c_0, L_0 > 0\) such that for all \(L \geq L_0\), all \(x \in \mathbb{R}^d\), and all cubes \(A, B \subset \Lambda L(x)\) with \(\text{dist}(A, B) \geq L/3\) we have

$$\mathbb{P} \left\{ \| 1_A(H_\omega, L, x - L^{-1/2})^{-1} 1_B \| \leq \exp \left( -c_0 L^{1/2} \right) \right\} \geq 1 - \exp \left( -c_0 L^{d/4} \right).$$
2. Proofs

2.1. Spectral inequality and Wegner estimate. A core idea for the Wegner estimate is that modifying random variables by a fixed positive number $\delta > 0$ will make eigenvalues go up by an amount proportional to $\delta$, independently of the side length $L$ of the box. This will then imply that the probability of finding an eigenvalue in an interval is (at most) proportional to the uniform modulus of continuity of said interval – i.e. the Wegner estimate. In other words, one has to prove that the operator of multiplication by the potential difference $V$ is not only a non-negative operator, but that its effect on the part of the spectrum under consideration is the one of a strictly positive operator. This is the statement of quantitative unique continuation principles for spectral subspaces of Schrödinger operators. Typically, unique continuation principles for spectral subspaces of Schrödinger operators require that the multiplication operator is lower bounded by a characteristic function of an open set. One notable exception where a stronger statement is known, and which we build upon in this note, is the situation of spectral subspaces of the full Laplacian. In this case it suffices to have a multiplication by the characteristic function of a thick set. The following theorem makes this precise. It is implied by [EV20] and has been spelled out in Section 5 of [EV18].

Theorem 12. Assume that $S$ is a $(\gamma, a)$-thick set and that $L > 0$ satisfies $A_a \subset [0, L]^d$. Let $x \in \mathbb{R}^d$. Let $H_{0,L,x}$ denote the negative of the Laplace operator on $\Lambda L(x)$ with periodic, Dirichlet or Neumann boundary conditions. Then there is an absolute constant $K \geq 1$ such that for all $E \geq 0$, and all $f \in L^2(\Lambda L)$ we have

$$\|\chi(-\infty,E)(H_{0,L,x})f\|^2_{L^2(\Lambda L)} \leq \frac{(Kd)^{d\gamma}}{\gamma} K\sqrt{E}|a|_1^d \|\chi(-\infty,E)(H_{0,L,x})f\|^2_{L^2(\Lambda L \cap S)}.$$  

Theorem 12 is a finite volume variant of the Logvinenko-Sereda theorem [LS74, Kac73] in a version of Kovrijkine [Kov00, Kov01]. It can be interpreted as a quadratic form inequality

$$\chi(-\infty,E)(H_{0,L,x}) \leq \frac{(Kd)^{d\gamma}}{\gamma} \chi(-\infty,E)(H_{0,L,x})1_S \chi(-\infty,E)(H_{0,L,x}).$$  

(11)

Clearly, on can replace all spectral projectors in (11) by $\chi_J(H_{0,L,x})$ for any $J \subset (-\infty, E]$, since one can always multiply the inequality with these projectors from both sides.

Proof of Theorem 12. The statement is proved using the strategy of [CHK07]: We split the trace in (5) into two contributions according to spectral projectors of the unperturbed operator $H_{0,L,x}$, that is the pure negative Laplacian on $L^2(\Lambda L(x))$. For that purpose let $I := [E - \varepsilon, E + \varepsilon]$ and let $J \supset I$ be another interval with $\text{dist}(I, J^c) > 0$. Then

$$\text{Tr} [\chi_I(H_{0,L,x})] = \text{Tr} [\chi_I(H_{0,L,x})\chi_J(H_{0,L,x})] + \text{Tr} [\chi_I(H_{0,L,x})\chi_J^c(H_{0,L,x})].$$  

(12)

The strategy of [CHK07] relies on estimating the expectation of the two terms on the right hand side of (12) separately: The second term on the right hand side of (12) can be estimated as in (2.6) to (2.20) of [CHK07] by using the fact that $\text{supp} \ u_j \subset B_R(j)$ and Combes-Thomas bounds. This leads to

$$\mathbb{E} \{\text{Tr} [\chi_I(H_{0,L,x})\chi_J^c(H_{0,L,x})]\} \leq C_1 \mathbb{E} \{\text{Tr} [\chi_I(H_{0,L,x})]\} + C_2 s(\varepsilon)L^d,$$

where $C_1, C_2 > 0$ are constants and $C_1 < 1$, hence the corresponding term can be absorbed on the left hand side of (12).

The expectation of the first term on the right hand side of (12) is estimated in (2.21) to (2.32) of [CHK07] by using a scale-free quantitative unique continuation principle for spectral projectors. In their situation this relies on the fact that their sum $\sum_j u_j$ is periodic and uniformly positive on an open set of positive measure, cf. [CHK07, Theorem 2.1]. In our case, these assumptions of periodicity and positivity on an open set no longer hold, but
Theorem \cite{CHK07} yields an appropriate replacement and leads to the quadratic form inequality
\[
\chi_J(H_{0,L,x}) \leq C(J,S) \cdot \chi_J(H_{0,L,x}) \mathbf{1}_{S} \chi_J(H_{0,L,x})
\]
\[
\leq C(J,S) \sum_{j \in \mathbb{Z}^d} \chi_J(H_{0,L,x}) u_j \chi_J(H_{0,L,x})
\]
where it is implicitly understood that \( \sum_{j \in \mathbb{Z}^d} u_j \) is restricted to \( \Lambda_L(x) \). This replaces the inequality of \cite{CHK07} Theorem 2.1]. The rest of the proof follows along the lines of \cite{CHK07} Section 2]. \( \square \)

2.2. Spectral minimum. Let us prove Identity \eqref{eq:identity}. Assume \( \min \sup \mu_j = 0 \) for all \( j \in \mathbb{Z}^d \) and let \( \varepsilon \in (0,1) \). There exist \( r < \infty \) as well as an \( L^2 \)-normalized \( \varphi \in C_c^\infty(\mathbb{R}^d) \) with support contained in \( B_r(0) \) and such that
\[
\langle \varphi, -\Delta \varphi \rangle = \| \nabla \varphi \|^2 < \varepsilon/3.
\]
Set
\[
\Omega_\varepsilon := \pi_Q^{-1} \left( \times_{j \in Q} \left[ 0, \frac{\varepsilon}{3C_V} \right] \right)
\]
where \( Q = \mathbb{Z}^d \cap B_{2r+2R}(0) \) and \( \pi_Q : \Omega \to \times_{j \in Q} [m_-,m_+] \) is the canonical projection. Then by assumption \( \mathbb{P}(\Omega_\varepsilon) > 0 \) and we have for all \( \omega \in \Omega_\varepsilon \)
\[
\langle \varphi, V_\omega \varphi \rangle \leq \frac{\varepsilon}{3C_V} \left( \| \nabla \varphi \|^2 / 2 + C_V \| \varphi \|^2 \right) \leq \frac{\varepsilon}{3C_V} \left( \frac{\varepsilon}{6} + C_V \right) < \frac{\varepsilon}{2}
\]
where we assumed without loss of generality \( C_V \geq 1 \). Thus \( \langle \varphi, H_\omega \varphi \rangle \leq \varepsilon \) and by the variational characterization \( \min \sigma(H_\omega) < \varepsilon \) for all \( \omega \in \Omega_\varepsilon \). Since this holds for all \( \varepsilon \in (0,1) \), we obtain \eqref{eq:identity}.

2.3. Initial length scale estimate. We provide the proof of the initial length scale estimate for our non-stationary model, Theorem \ref{thm:initial_length_scale}. For this purpose, we show how our random potential can be lower bounded by a simpler one to which the results of \cite{SV} apply directly.

First we modify the random variables. Since \( \min \sup \mu_j = 0 \) for all \( j \in \mathbb{Z}^d \) and \( \lim_{\varepsilon \to 0} s(\varepsilon) = 0 \) there exist \( \varepsilon_1 > 0 \) such that \( 0 < s(\varepsilon_1) < 1 \). For all \( j \in \mathbb{Z}^d \) define \( \eta_j : \Omega \to \mathbb{R} \) by
\[
\eta_j = \varepsilon_1 \mathbf{1}_{\{ \pi_j \in [\varepsilon_1,m_+] \}},
\]
in particular \( 0 \leq \eta_j \leq \pi_j \). Let \( A_n \) be the window in the definition of the thick set \( S \).

Choose \( L \in \mathbb{N} \) with \( L \geq \max \{ a_1, \ldots, a_d \} \). Then
\[
| S \cap \Lambda_L(k) | \geq \gamma a_1 \cdots a_d = \frac{\gamma a_1 \cdots a_d}{L^d} \cdot L^d = : \tilde{\gamma} L^d,
\]
hence \( S \) is \( (\tilde{\gamma},(L,\ldots,L))-\)thick. For \( k \in \mathbb{Z}^d \) set \( \Lambda^+(k) = \Lambda_{L+2R}(k) \cap \mathbb{Z}^d \), \( N := \# \Lambda^+(k) \), and for \( j \in \Lambda^+(k) \) set
\[
S_j := \{ y \in \Lambda_L(k) \mid u_j(y) \geq 1/N \}.
\]
Note that by translation invariance \( N \) is \( k \)-independent. Then
\[
\bigcup_{j \in \Lambda^+(k)} S_j \supset S \cap \Lambda_L(k).
\]
Subadditivity gives
\[
\tilde{\gamma} L^d \leq \left| \bigcup_{j \in \Lambda^+(k)} S_j \right| \leq \sum_{j \in \Lambda^+(k)} |S_j|,
\]
hence there exist a \( j_k \in \Lambda^+(k) \) such that \( |S_{j_k}| \geq \tilde{\gamma} L^d / N \). It is possible to find a subset \( T_k \subset S_{j_k} \subset \Lambda_L(k) \) such that \( |T_k| = \tilde{\gamma} L^d / N \).

Thus we have identified for each \( k \in \mathbb{R}^d \) a single site potential \( u_{j_k} \) with the following properties:
\[
u_{j_k} \geq \frac{1}{N} \mathbf{1}_{T_k}, \quad |T_k| = \tilde{\gamma}.
\]
We choose the sublattice $\Gamma = ((L + 2R)\mathbb{Z})^d$ with periodicity cell $\Lambda_{L+2R}$. The lattice $\Gamma$ is sufficiently sparse, so that the map $\Gamma \ni k \mapsto j_k$ is injective, hence the random variables $\pi_{j_k}, k \in \Gamma$ are independent. Now compare the original operator with a diluted Anderson model

$$V_{\omega} \geq \sum_{k \in \Gamma} \pi_{j_k}(\omega) u_{j_k} \geq \sum_{k \in \Gamma} \eta_{j_k}(\omega) \frac{1}{N} 1_{T_k} =: W_{\omega}.$$ 

The initial scale estimate, Theorem 11 will follow from an application of Theorem 3.3 and its Corollary 4.2 in [SV]. For this purpose we need to verify the non-degeneracy condition in Definition 3.1 of [SV].

Since $W_{\omega} \leq V_{\omega}$, it is sufficient to establish this condition for $W_{\omega}$. For this purpose let us define

$$\eta_{j_k} : \Omega_{W} := \mathbb{R}^\Gamma \to \mathbb{R}, \quad \eta_{j_k}(\omega) = \eta_{j_k}(\pi(\omega)_{j_k}),$$

$$\lambda_k : \Omega_{W} \to \Omega_0 = \Gamma \times \Omega_{V}, \quad \lambda_k(\omega) = (k, \omega),$$

$$u : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}, \quad u((k, \omega), x) = \frac{1}{N} \eta_{j_k}(\omega) 1_{T_k}(x + k).$$

If we choose $m := \min\{\varepsilon_1/N, \gamma, 1 - s(\varepsilon_1)\} > 0$ it follows directly that $W_{\omega}$ is $m$-non-degenerate, more precisely

$$\inf_{k \in \mathbb{Z}^d} \mathbb{P}\{\text{vol} \{x \in \Lambda_{L+2R} \mid u_{\lambda_k}(x) \geq m\} \geq m\} \geq m$$

which is the condition formulated in Definition 3.1 of [SV]. Now Corollary 4.2 of [SV] immediately yields Theorem 11.

2.4. Proof of Theorems 3 and 4. In this section we prove Theorems 3 and 4, stating that (No - II), the contrapositive of Assumption (II), excludes a Wegner estimate.

Proof of Theorem 3. First, observe that (No - II) implies that for every choice $L, \delta, \kappa > 0$ there are infinitely many mutually disjoint cubes $\Lambda_L(x_j) \subset \mathbb{R}^d, j \in \mathbb{N},$ such that

$$\text{vol}(S_\kappa \cap \Lambda_L(x_j)) < \delta \quad \text{for all } j \in \mathbb{N}.$$ 

Indeed, if there were only finitely many such cubes, then they would all be contained in a larger cube $\Lambda_L(0)$. But then, for every $x \in \mathbb{R}^d$, the cube $\Lambda_{L+2L}(x)$ would have to contain at least one cube $\Lambda_L(y) \subset \Lambda_{L+2L}(x)$ with $\text{vol}(S_\kappa \cap \Lambda_L(y)) \geq \delta$. Hence, $\text{vol}(S_\kappa \cap \Lambda_{L+2L}(x)) \geq \delta$ for all $x \in \mathbb{R}^d$ and $S_\kappa$ would be thick, a contradiction.

Now, let $E \geq 0$. The maximal gap between two successive eigenvalues of $H_{0,L,0}$ below $E + 1$ tends to zero as $L \to \infty$. More precisely, the Dirichlet eigenvalues on the cube $\Lambda_L$ have the representation

$$E_n(L) = \frac{\pi^2}{L^2} \sum_{j=1}^d n_j^2, \quad n \in \mathbb{N}^d$$

from which it is easy to see that

$$\text{dist}(E, \sigma(H_{0,L,x_j})) = \text{dist}(E, \sigma(H_{0,L,0})) \leq 6\pi \sqrt{E + 1/L} =: \frac{\varepsilon}{2}$$

Take a normalized eigenfunction $\varphi$ of $H_{0,L,x_j}$ to an eigenvalue in an $\frac{\varepsilon}{2}$-neighborhood of $E$. Then

$$\| (H_{0,L,x_j} - E) \varphi \|_{L^2(\Lambda_L(x_j))} \leq \frac{\varepsilon}{2}.$$ 

Since $|V_{\omega}| \leq \max\{|m_-|, |m_+|\} U(x)$, we have for almost every $\omega \in \Omega$

$$\| V_{\omega} \varphi \|_{L^2(\Lambda_L(x_j))} \leq \max\{|m_-|, |m_+|\} \| U \varphi \|_{L^2(\Lambda_L(x_j))},$$

and estimate

$$\| U \varphi \|_{L^2(\Lambda_L(x_j))} \leq \| U \|_{L^4(\Lambda_L(x_j))} \| \varphi \|_{L^4(\Lambda_L(x_j))} = \| U \|_{L^4(\Lambda_L(0))} \| e^{-H_{0,L,x_j}} \|_{L^4(\Lambda_L(0))} \| e^{H_{0,L,x_j}} \varphi \|_{L^4(\Lambda_L(x_j))} \| \varphi \|_{L^4(\Lambda_L(x_j))}.$$. 
The exponential $e^{-tH_{0,L,x,j}}$ of the Dirichlet Laplacian (heat semigroup at time one) is a convolution operator with corresponding integral kernel bounded by the free kernel

$$p_t(x - y) = \frac{1}{(4\pi t)^{d/2}} \exp \left( \frac{|x - y|}{4t} \right)$$

see, for instance the proof of Lemma 2.2 in [TY13]. In particular, the function $\mathbb{R}^d \ni z \mapsto p_t(z)$ is in every $L^p$-space for $p \in [1, \infty)$. By Young’s convolution inequality, we can therefore estimate for every $g \in L^2(\Lambda_L(x_j))$

$$\|e^{-tH_{0,L,x,j}}g\|_{L^4(\Lambda_L(x_j))} \leq \|p_t * j_{\Lambda_L(x_j)}g\|_{L^4(\mathbb{R}^d)} \leq \|p_t\|_{L^{4/3}(\mathbb{R}^d)} \|g\|_{L^2(\Lambda_L(x_j))} = C_d \|g\|_{L^2(\Lambda_L(x_j))}$$

for an $L$-independent $C_d > 0$, where $j_{\Lambda_L(x_j)} : L^2(\Lambda_L(x_j)) \to L^2(\mathbb{R}^d)$ is the canonical embedding. Using $\|e^{H_0,L,x,j} \varphi\|_{L^2(\Lambda_L(x_j))} \leq e^{E + \frac{1}{4}}$, we conclude

$$\|V_\omega \varphi\|_{L^2(\Lambda_L(x_j))} \leq C_d \max\{|m_-|, |m_+|\} e^{E + \frac{1}{2}} \|U\|_{L^4(\Lambda_L(x_j))} \leq C \|U\|_{L^4(\Lambda_L(x_j))}$$

for some $C = C(d, m_+, m_-, E)$ if $L \geq 1$. We split $U = U_1 \mathbf{1}_{S_2} + U \mathbf{1}_{S_1}$, and find

$$\|U\|_{L^4(\Lambda_L(x_j))} \leq C_U \|\mathbf{1}_{S_1}\|_{L^4(\Lambda_L(x_j))} + \kappa L^{d/4} \leq C_U \delta^{1/4} + \kappa L^{d/4}.$$

Choosing

$$\delta \leq \left( \frac{6\pi \sqrt{E + 1}}{2CU CL} \right)^4 \quad \text{and} \quad \kappa \leq \frac{6\pi \sqrt{E + 1}}{2CU CL^{d/2}}$$

this sum can be made smaller than $\frac{\epsilon}{27\pi} = 3\pi \sqrt{E + 1}/(CL)$ whence

$$\|(H_{\omega,L,x,j} - E) \varphi\|_{L^2(\Lambda_L(x_j))} \leq \epsilon = 12\pi \sqrt{E + 1}/L.$$

This implies $\|(H_{\omega,L,x,j} - E)^{-1} \varphi\|_{L^2(\Lambda_L(x_j))} \geq \epsilon^{-1}$ which in turn shows

$$\text{dist}(E, \sigma(H_{\omega,L,x,j})) \leq \epsilon \quad \text{for all configurations } \omega \in \Omega. \quad \square$$

The proof of Theorem 4 proceeds along the lines of the proof of Theorem 3 with the difference that one chooses

$$\delta \leq \left( \frac{6\pi \sqrt{E + 1}}{2CU CL} \right)^4 \quad \text{and} \quad \kappa \leq \frac{6\pi \sqrt{E + 1}}{2CU CL^{d/2}L}.$$

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