Description of partial actions

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Abstract
In this paper, we study partial actions of groups on $R$-algebras, where $R$ is a commutative ring. We describe the partial actions of groups on the indecomposable algebras with enveloping actions. Then we work on algebras that can be decomposed as product of indecomposable algebras and we give a description of the partial actions of groups on these algebras in terms of global actions.

Keywords Partial actions · Enveloping actions

Mathematics Subject Classification 16W22

1 Introduction

Partial actions of groups were first proposed by R. Exel and other authors in the context of $C^*$-algebras, see for example, [13], and afterwards they appeared in a pure algebraic setting, see [10]. A different way of looking at partial actions, appeared earlier in the work of Green and Marcos, see [16].

Partial actions of groups became an important tool to characterize algebras as partial crossed products. In particular, several aspects of Galois theory can be generalized to partial group actions, see [11] (at least under the additional assumption that the associated ideals are generated by central idempotents). Soon after the initial
definition, the theory of partial actions was extended to the Hopf algebraic setting, see [4] and [2].

An interesting class of partial actions is the one which can be obtained by a restriction of a global action. In this class, we have the possibility to transfer properties of the global action to the partial action. The problem of existence of globalization was firstly studied in [1] and many other results concerning globalization were obtained later on (see, for instance, [6, 10, 12, 14]). This was also studied in the setting of categories by the authors in [7].

In this paper, $R$ will always denote a commutative ring with identity and all algebras are $R$-algebras. In this work, all $R$-algebras do not necessarily have an identity, unless otherwise stated. Moreover, for an $R$-algebra $\Lambda$ there is a $R$-module structure compatible with the multiplications, that is, $r(m \times n) = (rm) \times n = m \times (rn)$ for all $r \in R$, $m$ and $n \in \Lambda$.

The idea underlying this article is to consider partial actions of groups on algebras, which have a globalization.

The contents of the article are as follows.

In Sect. 2, we present some preliminary results and definitions which will be used in this paper.

In Sect. 3, we present our principal results. We begin with the notion of extension by zero of a partial action and we explicitly give the description of the enveloping actions of partial actions that are extension by zero of the global actions. In a certain way, which is clear in the paper, we give a characterization of all partial actions with enveloping actions in indecomposable algebras and product of them. We give an explicit characterization of enveloping actions of partial actions on certain algebras that are product of indecomposable algebras. In this case, Proposition 3.13 and Proposition 3.17 are refinements about the facts on enveloping actions proved in [9] in some particular situations.

2 Preliminaries

In this section, we present some definitions and results that will be used in the rest of the paper.

In [10], the authors introduced the following definition.

**Definition 2.1** Let $G$ be a group and $\Lambda$ an $R$-algebra. A partial action $\alpha$ of $G$ on $\Lambda$ is a collection of ideals $S_g$ of $\Lambda$, $g \in G$, and isomorphisms of (non-necessarily unital) $R$-algebras $\alpha_g : S_{g^{-1}} \to S_g$ such that:

1. $S_e = \Lambda$ and $\alpha_e$ is the identity map of $\Lambda$, where $e$ is the identity of $G$;
2. $S_{(gh)^{-1}} \supseteq \alpha_g^{-1}(S_h \cap S_{g^{-1}})$;
3. $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for every $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

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It is convenient to point out that the property (ii) of the definition above easily implies that \( \alpha_g(S_{g^{-1}} \cap S_h) = S_g \cap S_{gh} \), for all \( g, h \in G \). Also \( \alpha_{g^{-1}} = \alpha_g^{-1} \), for every \( g \in G \).

Next, we see a natural example of a partial action where we restrict a global action to an ideal generated by a central idempotent.

**Example 2.2** Let \( \beta \) be a global action of a group \( G \) on a (non-necessarily unital) ring \( L \) and \( \Lambda \) an ideal of \( L \) generated by a central idempotent \( 1_\Lambda \). We can restrict \( \beta \) to \( \Lambda \) as follows: putting \( S_g = \Lambda \cap \beta_g(\Lambda) = \Lambda \beta_g(\Lambda) \), each \( S_g \) has an identity element \( 1_\Lambda \beta_g(1_\Lambda) \). If \( \alpha_g = \beta_g|_{S_{g^{-1}}} \), for all \( g \in G \), then the items (i), (ii) and (iii) of Definition 2.1 are satisfied. So, \( \alpha = \{ \alpha_g : S_{g^{-1}} \to S_g : g \in G \} \) is a partial action of \( G \) on \( \Lambda \).

The following definition appears in ([10], p. 9).

**Definition 2.3** A global action \((L, \{ \beta_g \}_{g \in G})\) of a group \( G \) on an associative (non-necessarily unital) ring \( L \) is said to be an enveloping action (or a globalization) for a partial action \( \alpha \) of \( G \) on a ring \( \Lambda \) if there exists a ring monomorphism \( \varphi : \Lambda \to L \) such that the following properties hold.

1. \( \varphi(\Lambda) \) is an ideal of \( L \);
2. \( L = \sum_{g \in G} \beta_g(\varphi(\Lambda)) \);
3. \( \varphi(S_g) = \varphi(\Lambda) \cap \beta_g(\varphi(\Lambda)) \), for all \( g \in G \);
4. \( \varphi \circ \alpha_g(a) = \beta_g \circ \varphi(a) \), for all \( a \in S_{g^{-1}} \), for all \( g \in G \).

In this case the algebra \( L \) is called the **enveloping algebra**.

The following theorem appears in ([10], Theorem 4.5) and it is central on the study of the partial actions of groups on algebras.

**Theorem 2.4** A partial action \( \alpha \) of a group \( G \) on a unital \( R \)-algebra \( \Lambda \) admits an enveloping action \((L, \beta)\) if and only if each ideal \( S_g, g \in G \), is generated by a central idempotent. Moreover, if the enveloping action exists it is unique up to equivalence. In particular, if there exists an \( R \)-algebra \( L' \) with a global action \( \beta' \) of \( G \) such that the properties of Definition 2.3 are satisfied, then \( L' \simeq L \).

When \((\Lambda, \alpha)\) has an enveloping action \((L, \beta)\) we may consider that \( \Lambda \) is an ideal of \( L \) and the following properties hold:

1. The subalgebra of \( L \) generated by \( \bigcup_{g \in G} \beta_g(\Lambda) \) coincides with \( L \) and we have \( L = \sum_{g \in G} \beta_g(\Lambda) \);
2. \( S_g = \Lambda \cap \beta_g(\Lambda) \), for every \( g \in G \);
3. \( \alpha_g(x) = \beta_g(x) \), for every \( g \in G \) and \( x \in S_g^{-1} \).
3 Description of the partial actions

In this section, we present our main results and we start with the following definition which plays a role on this work.

Definition 3.1 Extension by zero

a) Let $H$ be a subgroup of a group $G$ and $\alpha$ a partial action of $H$ on an algebra $\Lambda$. The extension by zero to $G$ of $(\Lambda, \alpha, H)$, is the partial action $(\gamma, S^\gamma, \alpha^\gamma)$ of $G$ defined as follows:

\[
\begin{align*}
S^\gamma_g &= 0 \text{ if } g \notin H \\
S^\gamma_g &= S_g \text{ if } g \in H \\
\alpha^\gamma_g &= 0 \text{ if } g \notin H \\
\alpha^\gamma_g &= \alpha_g \text{ if } g \in H
\end{align*}
\]

Remark 3.2 It is convenient to point out that in the case $H$ globally acts on $\Lambda$, we say that the partial action defined above is an extension by zero of the $H$-action. We also say that a partial action $\alpha$ of a group $G$ is an extension by zero of an action if it is a extension by zero of an $H$-action for some subgroup $H$ of $G$. Observe that a global action is a particular case where $H = G$.

It is convenient to point out that the definition above shows an advantage of partial actions, i.e., partial actions of subgroups always can be extended to the all group, which is not the case for global actions.

We are ready to show our first lemma. We thank an anonymous reader for correcting the statement which we had before.

Lemma 3.3 Let $\alpha$ be a partial action of a group $G$ on a unital $R$-algebra $\Lambda$. Then the set $H = \{ h \in G | S_h = \Lambda \}$ is a submonoid of $G$.

Proof Let $h, t \in H$. Then, $S_h = S_t = \Lambda$ and we have that $\alpha_h(S_h^{-1} \cap S_t) = S_h \cap S_h = \Lambda$. Thus, $S_{ht} = \Lambda$. \qed

Remark 3.4 1. We thank an anonymous reader for correcting the statement of the former proposition, we had written group instead of monoid. The next example was made by the anonymous reader.

Let $K$ be a field, $\Lambda = K^\infty = (K, K, \ldots, K, \ldots)$, a direct product of countable many copies of $K$, $G = \mathbb{Z}$. Consider the following partial action $\alpha$ of $G$ on $\Lambda$: define the ideals

$S_i = \Lambda, S_{-i} = (0, \ldots, 0, K, K, \ldots),$

and the partial isomorphisms

$\alpha_{-i}(x_1, x_2, \ldots) = (0, \ldots, 0, x_1, x_2, \ldots), \alpha_i(0, \ldots, 0, x_1, x_2, \ldots) = (x_1, x_2, \ldots),$

for all $i \geq 0$. The maps $\alpha_i : S_{-i} \rightarrow S_i, i \in \mathbb{Z}$, form a partial action of $\mathbb{Z}$ on $\Lambda$, and $H = \{ i \in \mathbb{Z} : S_i = \Lambda \} = \{ i \geq 0 \}$ is a monoid which is not a group.
2. We recall that an algebra with unity is indecomposable if and only if it has exactly two central idempotents 0 and 1. Now, if the algebra is indecomposable then the monoid defined above is a group. Moreover, if the algebra is a finite dimensional algebra, then the monoid above is a group.

In the next proposition, it is explained why we introduced the notion of extension by zero.

**Proposition 3.5** Let \(\alpha\) be a partial action of the group \(G\) on a unital indecomposable algebra \(\Lambda\). Then \(\alpha\) has an enveloping action if and only if there exists a subgroup \(H\) of \(G\) and an action of \(H\) on \(\Lambda\) such that \(\alpha\) is the extension by zero of this action.

**Proof** Let \(\alpha = \{\alpha_g : S_g^{-1} \rightarrow S_g | g \in G\}\) be a partial action of a group \(G\) on an indecomposable algebra \(\Lambda\) and suppose that \(\alpha\) has enveloping action. According to Theorem 2.4, we have that the partial action \(\alpha\) has a globalization if and only if each ideal \(S_g, g \in G\), is generated by a central idempotent. Since \(\Lambda\) is indecomposable then the unique central idempotents are 0 and 1 and it follows that each \(S_g, g \in G\), is either the zero ideal or the algebra \(\Lambda\). So, the result follows from Lemma 3.3 and Remark 3.4 (2).

Conversely, by assumption we have that all the ideals \(S_g\) are either the zero ideal or the algebra \(\Lambda\). Thus, by Theorem 2.4, we have the result.

The next remark will be useful in our first main result.

**Remark 3.6** Let \(H\) be a subgroup of \(G\) and \(T = \{g_i\}_{i \in I}\) a left transversal set of the congruence defined by \(H\), i.e., \(G = \bigcup_{i \in I} g_iH\) and we additionally require that the element that represents \(H\) is the identity \(e\). The functions \(j : G \times T \rightarrow T\) and \(h : G \times T \rightarrow H\) are defined by the equality \(gg_i = j(g, g_i)h(g, g_i)\). Moreover, note that for all \(t, g \in G\) we get \(tg_i = j(t, g_i)h(t, g_i)\) and \(gtg_i = j(gt, g_i)h(gt, g_i)\).

From now on given any subgroup \(H\) of a group \(G\) we fix a left transversal \(T\). So, the maps \(j\) and \(h\) above are well defined.

A bit about notation: recall that given a family of abelian groups \(\{A_i\}_{i \in I}\), the direct sum \(\bigoplus_{i \in I} A_i\) is the set of all functions from \(I\) to \(\Lambda\) which are almost zero, i.e., it is not zero for only a finite number of values of \(i \in I\).

In the next theorem, we completely describe the enveloping action of the extensions by zero of global actions of the algebra \(\Lambda\). Moreover, in the next result, given \(a \in \bigoplus_{i \in I} A_i\), we denote by \(a|_i\) for \(a(i)\)

**Theorem 3.7** Let \(\Lambda\) be a unital \(R\)-algebra, \(H\) a subgroup of \(G\) that acts globally on \(\Lambda\) by the action \(h \cdot a\), where \(h \in H\) and \(a \in \Lambda\), \(\alpha\) the extension by zero of the global action of \(H\) on \(\Lambda\) to \(G\) and \(\{g_i\}\) a transversal of \(G/H\). Then we can describe the enveloping action \((\Gamma, \beta)\) of \((\Lambda, \alpha)\) as follows:

\[(a) \quad \Gamma = \bigoplus_{i \in I} \Lambda_{g_i} \text{ with } \Lambda_{g_i} = \Lambda;\]
(b) For each $g \in G$, $\beta_g(\lambda)|_{j(g, g_i)} = h(g, g_i).\lambda|_{g_i}$.

**Proof** Since $e g_i = g_i e$, we have that $j(e, g_i) = g_i$ and $h(e, g_i) = e$, where $e$ is the identity element of $G$. Thus, $\beta_i(\lambda)|_{j(g, g_i)} = (e \cdot \lambda)|_{g_i} = \lambda|_{g_i}$ and we obtain that $\beta_e = id$.

We claim that $\beta_g \beta_t = \beta_{gt}$, for all $g, t \in G$. In fact, note that

$$
\beta_i(\lambda)|_{j(g, g_i)} = \beta_i(\lambda)|_{j(t, g_i)(j(g, t))} = \left( h(g, j(t, g_i))(h(t, g_i)).\lambda|_{j(g, t, g_i)}(j(g, t)) \right)
$$

and $\beta_{gt}(\lambda)|_{j(g, t, g_i)} = h(g, t, g_i).\lambda|_{j(g, t, g_i)}$. From Remark 3.6, we have that

$$
tg_i = j(t, g_i)h(t, g_i)(1)
$$

and

$$
tg_i = j(g, t, g_i)h(g, t, g_i)(2)
$$

From (1) and (2) we have that

$$
gtg_i = g(j(t, g_i)h(t, g_i)) = j(g, j(t, g_i))h(g, j(t, g_i))h(t, g_i).
$$

Consequently, $j(g, t, g_i) = j(g, j(t, g_i))$ and $h(g, t, g_i) = h(g, j(t, g_i))h(t, g_i)$. Hence, $\beta_i \beta_t = \beta_{gt}$. One can see also that $\beta_t$, for each $g \in G$, is an isomorphism of algebras. We claim that $(\Gamma, \beta)$ is an enveloping action of $(\Lambda, a)$, i.e., we need to show the four items of Definition 2.3 hold. In fact, note that by the construction of $\Gamma$ we have that $\Lambda$ is an ideal of $\Gamma$ and $\Gamma = \sum_{g \in G} \beta_g(\Lambda)$.

The next step is to show that $\beta_g(\alpha) = \alpha_g(\alpha)$ for each $\alpha \in S_{g^{-1}}$. To do this, let $g \in G$. If $g \notin H$, then for each $\alpha \in \beta_g(\Lambda) \cap \Lambda$ we have that

$$
a = \beta_g(\lambda)|_{j(g, e)} = (h(g, e).\lambda)|_{j(g, e)} = \lambda|_g
$$

Since the sum $\oplus_{\alpha \in \Lambda} \beta_g(\alpha)$ is direct, then $a = \lambda|_g = 0$ and we have that $\beta_g(\Lambda) \cap \Lambda = 0$. Consequently, $\alpha_g(y) = \beta_g(\alpha)$ for each $y \in S_{g^{-1}}$. Next, suppose that $g \in H$. Then $S_{g^{-1}} = \Lambda$ and we have, for each $\lambda \in \Lambda$, that

$$
\beta_g(\lambda) = \beta_g(\lambda)|_e = h(g, e).\lambda|_{j(g, e)} = g.\lambda|_e = g.\lambda = \alpha_g(\lambda)
$$

Finally, let $g \in G$ and we claim that $\beta_g(\Lambda) \cap \Lambda = S_g$. In fact, we have the following two cases:

Case 1: Suppose that $g \in H$. In this case, $S_g = \Lambda$ and we have that $\beta_g(\Lambda) \cap \Lambda \subseteq \Lambda$. Now, for each $\alpha \in \Lambda$, we have that $\alpha = \alpha_g(\alpha') = \beta_g(\alpha') \in \beta_g(\Lambda) \cap \Lambda$.

Case 2: Suppose that $g \notin H$. In this case, $S_g = 0$, and we obtain as before that $\beta_g(\Lambda) \cap \Lambda = 0$.

We now show that partial action with indecomposable enveloping algebras are in reality global actions.

**Corollary 3.8** Let $\Lambda$ be a unital indecomposable algebra and $\alpha$ a partial action of a group $G$ on $\Lambda$ with enveloping action $(S, \beta)$. Then $S$ is an indecomposable algebra if and only if the partial action $\alpha$ is a global action.
Proof Suppose that $S$ is indecomposable. Then by Proposition 3.5 there exists a subgroup $H$ of $G$ such that the partial action $\alpha$ is the extension by zero of the action of $H$ on $\Lambda$. Thus, by Theorem 3.7, the algebra $\bigoplus_{g \in T} \Lambda_g$, where $\Lambda_g = \Lambda$ and $T = \{g_i : i \in I\}$ a transversal of $G/H$ is the enveloping algebra of $(\Lambda, \alpha)$. Hence, by ([10], Theorem 4.5) we have that $\bigoplus_{g \in T} \Lambda_g \simeq S$. So, $G/H$ is trivial and we have that the partial action $\alpha$ is a global action.

The other implication is clear.

Now, we give an example to illustrate our Theorem 3.7.

Example 3.9 Let $\Lambda$ be the $K$-algebra, where $K$ is a field, given by the quiver of type $A_2$ with cyclic orientation.

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\circ 1 \quad \circ 2 \\
\beta
\end{array}
\]

We define an action of $\mathbb{Z}_2 = \{1, \sigma\}$ on $\Lambda$ as follows: \[
\begin{cases}
1 \to 2 \\
2 \to 1
\end{cases} \quad \text{and} \quad \begin{cases}
\alpha \to \beta \\
\beta \to \alpha
\end{cases}
\]

Note that $\mathbb{Z}_2$ is isomorphic to a subgroup of order of 2 of $S_3$ and we can consider the subgroup $L = \{(1)(12)\}$ of order 2 of $S_3$. We have the left cosets $\{(1), (12)\}, \{(132), (23)\}$ and $\{(13), (123)\}$ where a set of representatives of left cosets modulo $L$ is $\{1, (2, 3), (1, 3)\}$. Thus, we have the following table where we compute the functions $h$ and $j$ defined as before:

|      | $j$ | $h$ |
|------|-----|-----|
| $(1,1)$ | 1   | 1   |
| $(1,(23))$ | (23) | 1   |
| $(1,(13))$ | (13) | 1   |
| $(12),(1)$ | 1   | (12) |
| $(12),(23))$ | (13) | (12) |
| $(23),(1)$ | (23) | 1   |
| $(23),(23))$ | 1   | 1   |
| $(23),(13))$ | (13) | (12) |
| $(123),(1)$ | (13) | (12) |
| $(123),(23))$ | 1   | (12) |
| $(123),(13))$ | (23) | 1   |
| $(13),(1)$ | (13) | 1   |
| $(13),(23))$ | (23) | (12) |
| $(13),(13))$ | 1   | 1   |
| $(132),(1)$ | (23) | (12) |
| $(132),(23))$ | (13) | 1   |
| $(132),(13))$ | 1   | (12) |
| $(12),(13))$ | (23) | (12) |
By Theorem 3.7, we have that the enveloping action is \((S, \beta)\), where \(S = \Lambda_1 \times \Lambda_{(13)} \times \Lambda_{(23)} = \Lambda \times \Lambda \times \Lambda\) and the global action \(\beta\) of \(S_3\) is defined as follows:

- \(\beta_1 = \text{id}_T\)
- \(\beta_{12}(x, y, z) = ((12)x, (12)z, (12)y)\)
- \(\beta_{13}(x, y, z) = (y, x, (12)z)\)
- \(\beta_{23}(x, y, z) = ((12)z, (12)x, y)\)
- \(\beta_{123}(x, y, z) = ((12)y, z, (12)x)\)

Definition 3.10 Let \(A, B, C\) be \(R\)-algebras such that \(A = B \times C\), \(G\) a group,

\[
\alpha = (A, \{A_g\}_{g \in G}, \{a_g\}_{g \in G}), \quad \omega = (B, \{B_g\}_{g \in G}, \{\omega_g\}_{g \in G}), \quad \eta = (C, \{C_g\}_{g \in G}, \{\eta_g\}_{g \in G})
\]

partial actions of \(G\) on \(A, B\) and \(C\), respectively. We say that \(\alpha\) is a product of the partial actions \(\omega\) and \(\eta\) if, for each \(g \in G\), \(A_g = B_g \times C_g\) and \(\alpha_g = \omega_g \times \eta_g\).

The following result is a direct consequence presented in ([9, p. 4144]) and we put its proof for the reader’s convenience.

Proposition 3.11 Let \(\Lambda_1, \ldots, \Lambda_k\) be unital indecomposable algebras such that \(\Lambda_i\) is not isomorphic to \(\Lambda_j\) for \(i \neq j\), \(\Lambda = \Lambda_1^{n_1} \times \cdots \times \Lambda_k^{n_k}\) and \(\alpha\) a partial action of a group \(G\) on \(\Lambda\) with an enveloping action. Then each component \(\Lambda_i^{n_i}\) of \(\alpha\) is \(\alpha\)-invariant and the restriction of \(\alpha\) to the component \(\Lambda_i^{n_i}\) defines a partial action on it and the original partial action is a product of these partial actions. Conversely, given a set of partial actions \((\Lambda_i^{n_i}, \alpha_i)\) we can define the partial action \(\alpha\) of \(G\) on \(\Lambda\), as the product of the given partial actions.

Proof Let \(e\) be a central idempotent of \(\Lambda\). Then we have a decomposition \(e = e_1 + \cdots + e_k\) with \(e_i \in \Lambda_i^{n_i}\), it follows that each \(e_i\) is a central idempotent of \(\Lambda_i^{n_i}\). Note that all central idempotents of \(\Lambda\) are of this form. We easily have that any ideal \(I\) of \(\Lambda\) generated by a central idempotent is \(I = I_1 \times \cdots \times I_k\) where \(I_i = \Lambda e_i = \Lambda_i^{n_i} e_i\), with \(e_i\) being a central idempotent in \(\Lambda_i^{n_i}\), for each \(t \in \{1, \ldots, k\}\).

Now, let \(I_i\) be an ideal of \(\Lambda_i^{n_i}\) and \(I_j\) an ideal of \(\Lambda_j^{n_j}\) generated by central idempotents with \(i \neq j\). Then there is no isomorphism between \(I_i\) and \(I_j\) because of \(\Lambda_i\) is not isomorphic to \(\Lambda_j\). Thus, \(\alpha_i((\Lambda_i^{n_i} \cap S_{k-1}) \subseteq \Lambda_i^{n_i} \cap S_k\), for each \(g \in G\) and it follows that \(\Lambda_i^{n_i}\) is \(\alpha\)-invariant. Hence, for each \(i \in \{1, \ldots, k\}\), we have a partial action \(\alpha_i\) of \(G\) on \(\Lambda_i^{n_i}\) and we have that the partial action \(\alpha\) is of the form \(\alpha = \alpha_1 \times \cdots \times \alpha_k\), where \(\alpha_i\) is a partial action of \(G\) on \(\Lambda_i^{n_i}\), for \(i \in \{1, \ldots, k\}\).

For the converse, we observe that if an algebra \(A\) is of the form \(A = B \times C\), then the product of a partial action \(\alpha_1\) of \(G\) on \(B\) and a partial action \(\alpha_2\) of \(G\) on \(C\) is a partial action of \(G\) on \(A\).

The former proposition tells us that in order to describe all the partial actions on an algebra \(\Lambda\) which is a finite product of indecomposable algebras, we only
need to describe the partial actions on algebras of the form $\Lambda^n$, where $\Lambda$ is a unital indecomposable algebra.

The following proposition goes in this direction and it is probably well known. We give a proof here, for the sake of completeness.

**Proposition 3.12** Let $B = \Lambda^n$, where $\Lambda$ is a unital indecomposable algebra. Then $\text{Aut}(B) \cong S_n \times \text{Aut}(\Lambda)^n$, where $S_n$ is the group of permutations on $n$ elements, $\text{Aut}(\Lambda)$ is the group of automorphisms of $\Lambda$ and $\text{Aut}(B)$ the group of automorphisms of $B$.

**Proof** Define the map $\psi : S_n \times \text{Aut}(\Lambda)^n \to \text{Aut}(B)$ by $\psi(n,f_1,\ldots,f_n)(\sum \lambda_i e_i) = \sum f_i(\lambda_i)e_{\eta(i)}$. We easily have that $\psi$ is a group isomorphism. \qed

The next result is the second important result of this article.

**Theorem 3.13** Let $G$ be any group, $\Lambda$ a unital indecomposable algebra and $(\Lambda^n, \alpha)$ a partial action of $G$ on $\Lambda^n$ that is $\Lambda$-linear with an enveloping action $(T, \psi)$. Then the set of the primitive central idempotents $Y = \{e_1, \ldots, e_n\}$ of $\Lambda^n$ is $\alpha$-invariant. Moreover, if $X$ is the enveloping set of the induced partial action on $Y$, then $S \cong \bigoplus_{i \in I} \Lambda_i$, where $\Lambda_i \cong \Lambda$, for all $i \in I$ and $I$ is an index set such that $\#I = \#X$. In this case we have that $\#I \leq n|G|$, because of $\#X \leq n|G|$.

**Proof** First, we show that $Y$ is $\alpha$-invariant. In fact, let $g \in G$ and $A = Y \cap S_{g^{-1}}$. If $S_{g^{-1}}$ is the zero ideal, then $\alpha_g(Y \cap S_{g^{-1}}) = Y \cap S_g$. Suppose that $S_{g^{-1}} \neq 0$ and in this case $Y \cap S_{g^{-1}} \neq \emptyset$, since $S_{g^{-1}}$ is a direct sum of ideals of $\Lambda^n$. Thus, $1_g = e_{i_1} + \cdots + e_{i_k}$ and we clearly have that $e_{i_1}, \ldots, e_{i_k} \in D_{g^{-1}}$. Hence, $1_g = \alpha_g(e_{i_1}) + \cdots + \alpha_g(e_{i_k})$. By the fact that $1_g = e_{j_1} + \cdots + e_{j_k}$ and the elements of the set $\{\alpha_g(e_{i_1}), \ldots, \alpha_g(e_{i_k})\}$ are primitive central idempotents of $D_g$, we have that $k = s$ and $\alpha_g(e_{p_i}) = e_{p_i}$. Consequently, $\alpha_g(e_{p_i})$ is a primitive central idempotent of $\Lambda^n$, for all $p \in \{1, \ldots, n\}$. So, $\alpha_g(Y \cap S_{g^{-1}}) \subseteq Y \cap S_g$ and it follows that $Y$ is $\alpha$-invariant.

We define the partial action $\gamma$ of $G$ on $\{e_1, \ldots, e_n\}$ by the restriction of the partial action $\alpha$ to the set $\{e_1, \ldots, e_n\}$.

Next, we consider $(X, \theta)$ be the enveloping action of the partial action $\gamma$ on $Y = \{e_1, \ldots, e_n\}$, where $X$ is the $\psi$-orbit of $Y$ and $\theta$ is the restriction of $\psi$ to $X$. We consider $L = \bigoplus_{i \in I} \Lambda_i$, where $\Lambda_i = \Lambda f_i$, for all $i \in I$ with $X = \{f_i : i \in I\}$. Choice $j_1 \in I$ and the $j_1$-inclusion, $i_{j_1} : \Lambda \to L$ given by $i_{j_1}(\alpha) = \sum_{i \in I} x_i = \begin{cases} x_k = a, & k = j_1 \\ x_k = 0, & k \neq j_1 \end{cases}$ is an injective morphism. Thus, we have an inclusion $i : \Lambda^n \to \bigoplus_{i \in I} \Lambda_i$.

We define a global action $\beta$ of $G$ on $L$ by $\beta_g(\sum \lambda_i f_i) = \sum \lambda_i \theta_g(f_i)$. Note that $\beta|_{\{e_1, \ldots, e_n\}}$ gives the partial action $\gamma$. Then the restriction of the action $\beta$ to $\Lambda^n$ is the partial action $\alpha$ and it follows that $(L, \beta)$ is the enveloping action of the $(\Lambda^n, \alpha)$. So, by ([10], Theorem 4.5) we have that $S \cong \bigoplus_{i \in I} \Lambda_i$.

Moreover, all the partial actions which restricts to $\gamma$ are of this form. \qed
According to [15] a partial action \( \gamma = \{ \gamma_g : S_{g^{-1}} \to S_g, \ g \in G \} \) of a group \( G \) on a ring \( S \) is of finite type if there exists a finite subset \( \{ g_1, \ldots, g_n \} \) of \( G \) such that \( \sum_{1 \geq i \geq n} S_{g_i} = S \), for any \( g \in G \). Now, according to ([15], Proposition 1.2) a partial action \( \gamma = \{ \gamma_g : S_{g^{-1}} \to S_g, \ g \in G \} \) of a group \( G \) on a ring \( S \) with enveloping action \( (U, \beta) \) is of finite type if and only if \( U \) is a unital algebra.

As a consequence of the last theorem, we have the following result.

**Corollary 3.14** Let \( \alpha \) and \( \alpha' \) be globalizable partial actions of finite type of \( G \) on \( \Lambda^n \), where \( \Lambda \) is a unital indecomposable algebra, \( (T, \beta) \) and \( (T', \beta') \) the respective enveloping actions. Assume that both partial actions induce the same partial action on the set of primitive central idempotents \( \{ e_1, \ldots, e_n \} \). Then, \( T = \Lambda^m \) and \( T' = \Lambda^{m'} \). In this case, \( m = m' \).

In the literature, there are a lot of examples of partial actions of groups on algebras of type \( \Lambda = \bigoplus_{i=1}^n K e_i \), where \( \{ e_i : 1 \leq i \leq n \} \) is the set of primitive central idempotents and \( K \) is a field, see [5, 8] and the references therein. In the rest of this section, let \( \Lambda = \bigoplus_{i=1}^n \Omega e_i \), where \( \Omega \) is an indecomposable algebra, \( \{ e_i \} \) is a set of primitive central idempotents whose sum is the identity element.

**Definition 3.15** Let \( \Lambda \) as above and \( \alpha \) a partial action of a group \( G \) on \( \Lambda \) that is \( \Omega \)-linear. We say that the partial action \( \alpha \) satisfies property \( \ast \) if all the ideals associated to the partial action \( \alpha \) are of the form \( \bigoplus_{j=1}^k \Omega e_{i_j} \), where \( \{ e_{i_1}, \ldots, e_{i_k} \} \subseteq \{ e_1, \ldots, e_n \} \).

The next proposition was proved in ([3], Proposition 3.6) and we put it here in our context for the sake of completeness. Moreover, it is convenient to point out that for a partial action of \( G \) on \( \Lambda \) is the same as giving a partial action on the set \( \{ e_i : 1 \leq i \leq n \} \).

**Proposition 3.16** Let \( \Lambda = \bigoplus_{i=1}^n \Omega e_i \) and \( G \) be a group. Then, all the partial actions of \( G \) on \( \Lambda \) that are \( \Omega \)-linear can be restricted to the set \( \{ e_i \}_{i=1}^n \). Moreover, we obtain a bijection between the sets of the partial actions of \( G \) on \( \Lambda \) that are \( \Omega \)-linear and satisfy the property \( \ast \) and all the partial actions of \( G \) on \( \{ e_i \}_{i=1}^n \).

In ([9], Proposition 8.4), the authors proved that given a twisted partial action of a group on algebras decomposed by infinite or finite blocks which has enveloping algebras and these algebras are decomposed by blocks. The next result has an easy proof, but we put it here for the reader’s convenience.

**Proposition 3.17** Let \( \Lambda = \bigoplus_{i=1}^n \Omega e_i \) be as before and \( \alpha \) a partial action of \( G \) on \( \Lambda \) that is \( \Omega \)-linear and satisfy the property \( \ast \). Suppose that \( \alpha \) has enveloping action. Then the enveloping algebra is \( B = \bigoplus_{e \in X} \Omega e \), where \( X \) is the enveloping algebra of the induced partial action \( \bar{\alpha} \) of \( \alpha \) on \( \{ e_i : 1 \leq i \leq n \} \).

**Proof** We define the global action \( \beta^1 \) of \( G \) on \( B \) by
where $\beta$ is the global action of $G$ on $X$. We leave to the reader to show that $(B, \beta^{1})$ is the enveloping action of $(\Lambda, \alpha)$.

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