THE EXISTENCE OF THE THERMODYNAMIC LIMIT FOR THE SYSTEM OF INTERACTING QUANTUM PARTICLES IN RANDOM MEDIA

NIKOLAJ A. VENIAMINOV

Abstract. The thermodynamic limit of the internal energy and the entropy of the system of quantum interacting particles in random medium is shown to exist under the crucial requirements of stability and temperedness of interactions. The energy turns out to be proportional to the number of particles and/or volume of the system in the thermodynamic limit. The obtained results require very general assumptions on the random one-particle model. The methods are mainly based on subadditive type inequalities.

1. Introduction

Since the fundamental work [And58] of P. W. Anderson, the theory of random Schrödinger operators has been an extensively studied field of mathematical physics. The greatest attention has been paid since now to the one-particle approximation and we do not try to list here even the major works on this topic.

There are relatively few papers where finitely many particles are considered. There is a series of papers by Michael Aizenman and Simone Warzel that generalize the techniques of fractional moments method (see, for example, [AW09]) and another series of articles by Victor Chulaevsky, Yuri Suhov and their collaborators that make use of the multiscale analysis [CS09, CBdMS11]. The common point of these works is that they consider the number of particles being fixed and study the infinite volume limit for such system.

The present paper is an attempt to give an insight of what happens if both the number of particles and the volume go to infinity together so that the number of particles per unit volume is kept constant.

The same question has already been addressed by various authors in case of absence of background potential, i.e., when one-particle propagation is given by pure Laplacian. In this paper, we will frequently follow the framework developed by David Ruelle in [Rue99], though the presence of random potential presents certain mathematical difficulty, which we will explain later. We would also like to refer to an outstanding article [LL72] of Elliot H. Lieb and J. L. Lebowitz, where Coulomb interactions (always in absence of background potential) are treated.

The idea that the number of particles grows with the volume looks natural in the context of condensed matter physics. As a reference real-world example consider a piece of metal or semiconductor. A bigger piece should contain proportionally more electrons. As macroscopic objects are composed of many atoms (Avogadro constant $N_A \approx 6 \times 10^{23}\text{mol}^{-1}$), and thus, ions and electrons, it turns out that the corresponding mathematical notion is the thermodynamic limit. Its existence for thermodynamic quantities, such as internal energy, free energy, calorific capacity, and so on, is
the mathematical verification of the fact that these quantities are extensive. The latter is barely assumed in physics but actually needs rigorous verification.

Let us briefly discuss the mathematical objects we study. All the notions will be introduced later in full regularity. Let

$$H_\omega = -\Delta_d + V_\omega$$  \hspace{1cm} (1.1)

be the random Schrödinger operator that describes a single quantum particle in random environment $V_\omega$. Kinetic part $\Delta_d$ is $d$-dimensional Laplacian. One may also consider magnetic Schrödinger operator or whatever, provided that a number of basic facts, such as Wegner estimate (see Proposition 5.9), from the theory of one particle random operators hold true. Actually, the whole ideology of this paper is that we take one-particle operators for known and deduce on this base properties for multiparticle operators.

The restriction of $H_\omega$ to the domain $\Lambda$ is denoted by $H_\omega(\Lambda)$. For one particle Hamiltonian as in (1.1), we define, with a slight abuse of notation, the $n$ particle operator (restricted in physical space to domain $\Lambda$) with pair interactions potential $U$ by

$$H_\omega(\Lambda, n) = -\Delta_{nd} + \sum_{i=1}^{n} V_\omega(x^i) + \sum_{i \neq j} U(x^i - x^j),$$

where $x^i \in \Lambda$, $i = 1, \ldots, n$, are particles’ coordinates.

Using the notations introduced above, the general question we want to understand is the behavior of $H_\omega(\Lambda, n)$ in the thermodynamic limit:

$$H_\omega(\Lambda, n) \rightarrow ?, \quad |\Lambda| \rightarrow \infty, \quad n \rightarrow \infty, \quad n/|\Lambda| \rightarrow \text{const}. \hspace{1cm} (1.2)$$

In this paper, we answer a much more modest question than (1.2). Namely, let $E_\omega(\Lambda, n)$ be the ground state energy of $H_\omega(\Lambda, n)$. In Theorem 3.5, we show, in particular, that the ground state energy per particle admits the thermodynamic limit:

$$\exists \lim \frac{E_\omega(\Lambda, n)}{n}, \quad |\Lambda| \rightarrow \infty, \quad n \rightarrow \infty, \quad n/|\Lambda| \rightarrow \text{const}. \hspace{1cm} (1.3)$$

Moreover, the same theorem gives a bit more general result that allows to scale on the eigenenergy number in the spectrum. Roughly, the eigenenergy number (counting function) should be of order of exponent of the number of particles to ensure the convergence.

Theorem 3.9 gives the reciprocal result interchanging roles of energy and the counting function in the spectrum (the theorem is stated in terms of entropy which is the logarithm of counting function).

The main tool we use to obtain our results is a modified version of subadditive ergodic theorem (see Proposition 4.6). For instance, one may show that the ground state energy $E_\omega(\Lambda, n)$ is additive with respect to the pair $(\Lambda, n)$ up to an error term that can be taken into account. To make use of subadditivity we follow the construction of D. Ruelle [Rue99]. Nevertheless, significant modifications are made in the proof because of the fact that instead of full translation invariance of free Laplacian, we have only the covariance property of the family of random operators. In general, we are only able to prove the convergence in $L^2$ with respect to randomness (see Theorem 3.5 case (a)). A stronger convergence in $L^1$ and almost surely is established for compactly supported interactions (Theorem 3.5 case (b)).

In the last part of the present paper, we consider the system of noninteracting fermions in random medium. We show that nontrivial effects arise due to Fermi-Dirac statistics even in absence of interactions. In particular, we give an exact expression for the limit (1.3) in terms of the one
particle density of states measure (see Theorem 5.13) and we find an interesting relation with the Fermi energy.

The rest of the paper is organized as follows. The model of interacting quantum particles in random media and the notion of thermodynamic limit are introduced in Section 2. The results (mainly on the existence of thermodynamic limit) constitute Section 3 followed by the proofs in Section 4. In addition, Section 4 uncovers some extra properties of the energy density (see Subsections 4.3 and 4.4). The proofs themselves may be instructive as well. In Section 5, simple calculations concerning the thermodynamic limit for vanishing interactions are provided.

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2. Model and Notations

2.1. Model of Interacting Quantum Particles in Random Media. We consider a system of \( n \) interacting quantum particles in a random medium. The discrete and continuum cases are treated simultaneously and an explicit indication is given if a result is valid only for one setting. In the discrete case, the configuration space is given by \( V = \mathbb{Z}^d \) and for the continuous case by \( V = \mathbb{R}^d \).

In uniform manner, the one-particle Hilbert space is given by

\[ \mathcal{H}_1 = L^2(V) \]

The \( n \)-particle Hilbert space definition depends on the statistics (physical nature of quantum particles). The following statistics are considered.

(a) **The Maxwell - Boltzmann statistics.** The particles are physically distinguishable and no restrictions are imposed on a multiparticle wavefunctions. This model is suitable, in particular, for the description of heavy atomic nuclei, i.e., for particles that exhibit classical properties. The corresponding Hilbert space is given by

\[ \mathcal{H}_n = \bigotimes_{j=1}^n \mathcal{H}_1 = L^2(V^n) \]

(b) **The Bose - Einstein statistics:** the particles are bosons. The wavefunction is necessarily symmetric with respect to the permutations of coordinates:

\[ \mathcal{H}_n^+ = \text{Sym}^n \mathcal{H}_1 = L^2_+ (V^n) \]

where Sym is the symmetrised tensor product.

(c) **The Fermi - Dirac statistics:** they describe fermions. Wavefunctions are restricted to the antisymmetric subspace

\[ \mathcal{H}_n^- = \bigwedge_{j=1}^n \mathcal{H}_1 = L^2_- (V^n) \]

where \( \bigwedge \) is the external product.

\( \mathcal{H}_n^+ \) and \( \mathcal{H}_n^- \) are proper subspaces of \( \mathcal{H}_n \). For \( \sharp \in \{ \varnothing, +, - \} \), we write \( P_\sharp \) to denote the orthogonal projector on \( \mathcal{H}_n^\sharp \), where \( \varnothing = \) Maxwell - Boltzmann statistics, \( + = \) Bose - Einstein statistics and \( - = \) for the Fermi - Dirac statistics. Obviously, \( P = P_\varnothing = 1_{\mathcal{H}_n} \) is the trivial projector.

One particle Hamiltonian is given by

\[ H_\omega = H_\omega(1) = -\Delta + V_\omega \]

and acts on \( \text{Dom}(H_\omega) \subset \mathcal{H}_1 \), where
• $\Delta$ is either discrete or continuous Laplacian,
• a random potential $V_\omega$ is (at least) $\mathbb{Z}^d$-ergodic and satisfies a decorrelation (independence at a distance) condition:

$$\exists R_0 > 0, \text{ such that if } \text{dist}(A,B) > R_0, \text{ then } \{V_\omega(x)\}_{x \in A} \text{ and } \{V_\omega(x)\}_{x \in B} \text{ are independent.} \quad \text{(IAD)}$$

**Remark 2.1.** We also take into account the classes of random potentials that have the ergodic group reacher than $\mathbb{Z}^d$-translations. For instance, everything what follows remains true for the Poisson model.

**Notation 2.2.** We write $\Omega, \mathbb{P}$ and $\mathbb{E}$ for the associated probability space, probability measure and expectation respectively. For $\gamma \in \mathbb{Z}^d$, we denote by $\tau_\gamma$ the corresponding translations (measure preserving transformations) in $\Omega$ and by $T_\gamma$ the corresponding unitary transformations (coordinate shifts) in $L^2(\mathcal{V})$. Namely,

$$H_{\tau_\gamma(\omega)} = T_\gamma^* H_\omega T_\gamma, \quad (2.1)$$

where $T_\gamma f(x) = f(x - \gamma), f \in L^2(\mathcal{V}), x \in \mathcal{V}$.

By $H^{(i)}_\omega$ we denote a corresponding operator in $\mathfrak{H}^n$ that acts only on the $i$-th particle. More precisely,

$$H^{(i)}_\omega = \mathbb{1}_0 \otimes \ldots \otimes \mathbb{1}_0 \otimes H_\omega \otimes \mathbb{1}_0 \otimes \ldots \otimes \mathbb{1}_0. \quad (2.2)$$

The $n$-particle Hamiltonian in random environment $V_\omega$ and with interactions $W$ is given by the following self-adjoint operator on $\mathfrak{H}^n_{\sharp}$:

$$H_{\omega,\sharp}(n) = P_{\sharp} \left[ \sum_{i=1}^n H^{(i)}_\omega + W_n \right]. \quad (2.3)$$

For each $n \in \mathbb{N}$, $W_n$ is an interaction potential given by a function of the $n$ particles coordinates $x = (x^1, \ldots, x^n), x_j \in \mathcal{V}$. We refer to the whole collection $W = \{W_n\}_{n \in \mathbb{N}}$ as *interactions* in general. Remark also that in this model interactions are deterministic and all particles live in the same random background potential $V_\omega$.

In (2.3), the free part

$$H^0_{\omega,\sharp}(n) = \sum_{i=1}^n H^{(i)}_\omega$$

is called the second quantization of $H_\omega$ in context of the Fock space (see, for example, [BR97]). Namely, we have to restrict the second quantization of $H_\omega$ to the $n$-particle subspace of the whole Fock space:

$$H^0_{\omega,\sharp}(n) = d\Gamma(H_\omega)|_{\mathfrak{H}^n_{\sharp}},$$

where $d\Gamma$ denotes second quantization procedure.

**Remark 2.3.** $H^0_{\omega,\sharp}(n)$ acts from $\mathfrak{H}^n_{\sharp}$ into itself for any choice of $\sharp$, whereas an arbitrary interaction potential $W_n$ does not necessarily preserve complete (anti)symmetry. That is why the projector $P_{\sharp}$ a-priori acts non trivially in this formula. However, potentials that we consider later are permutation symmetric (confer Section 3, property (PI)), so that the projector becomes obsolete in (2.3), i.e.,

$$H_{\omega,\sharp}(n) = H^0_{\omega,\sharp}(n) + W_n.$$
The Dirichlet and Neumann restrictions of $H_{\omega,\sharp}(n)$ to a finite box $\Lambda \subset V$ are denoted by $H_{\omega,\sharp}^\star(\Lambda, n)$, where $\star \in \{D, N\}$. $H_{\omega,\sharp}^\star(\Lambda, n)$ is a self-adjoint operator on $\mathcal{H}_2^n(\Lambda) = L_2^2(\Lambda^n)$. We omit $\star$ in notations frequently.

The operator $H_{\omega,\sharp}(\Lambda, n)$ has a discrete spectrum. We call the counting function associated to this operator
\[ \mathcal{N}_{\omega,\sharp}(E, \Lambda, n) = \text{card}\{E_k(\Lambda, n, \omega; \sharp) \leq E\}, \]
where $E_k(\Lambda, n, \omega; \sharp)$ are the eigenvalues of $H_{\omega,\sharp}(\Lambda, n)$. For the reasons that will become apparent later, the entropy is a more convenient quantity:
\[ S_{\omega,\sharp}(E, \Lambda, n) = \log \mathcal{N}_{\omega,\sharp}(E, \Lambda, n). \tag{2.4} \]

**Notation 2.4.** Sometimes we will drop some (if not all) of the indices and arguments of the counting function and the entropy. For example, if we are interested in the dependence on energy, we will write just:
\[ S(E) = S_{\omega,\sharp}(E, \Lambda, n), \quad \mathcal{N}(E) = \mathcal{N}_{\omega,\sharp}(E, \Lambda, n). \]

**Remark 2.5.** As the counting function $\mathcal{N}$ takes its values in $\mathbb{N} \cup \{0\}$, the entropy takes its values in $\log \mathbb{N} \cup \{-\infty\}$.

**Observation 2.6.** For fixed $\omega, \sharp, \Lambda$ and $n$, the entropy $E \mapsto S_{\omega,\sharp}(E, \Lambda, n)$ is a non-decreasing right-continuous step function.

The monotonicity of $S(E) = S_{\omega,\sharp}(E, \Lambda, n)$ allows to define a (quasi-)inverse function $E_{\omega,\sharp}(\Lambda, n, S)$. As $\mathcal{N}(\cdot, \Lambda, n)$ is not a local bijection at any point, the inverse function doesn’t exist in a canonical manner. Our choice of the inverse is the following.

For $S$ such that $e^S \in \mathbb{N}$ we define
\[ E_{\omega,\sharp}(\Lambda, n, S) = E_{\exp S}(\Lambda, n, \omega; \sharp). \tag{2.5} \]

The application $S \mapsto E(S)$ is a right inverse of the entropy $(2.4)$ in the following meaning. For $S \in \log \mathbb{N}$ one has
\[ S_{\omega,\sharp}(E_{\omega,\sharp}(\Lambda, n, S), \Lambda, n) = S. \tag{2.6} \]

Reciprocally, if $E \geq E_1(H_{\omega,\sharp}(\Lambda, n))$, then
\[ E_{\omega,\sharp}(\Lambda, n, S_{\omega,\sharp}(E, \Lambda, n)) = E^-, \tag{2.7} \]
where $E^-$ is the closest from below to $E$ eigenenergy of $H_{\omega,\sharp}(E, \Lambda, n)$.

The relations $(2.6)$ and $(2.7)$ motivate this choice of an inverse function.

**Definition 2.7.** We denote by $\mathcal{E} = \mathcal{E}_{\omega,\sharp}(\Lambda, n)$ the ground state energy of the operator $H_{\omega,\sharp}(\Lambda, n)$:
\[ \mathcal{E}_{\omega,\sharp}(\Lambda, n) = \inf_{\varphi \in \text{Dom}(H_{\omega,\sharp}(\Lambda, n)), \varphi \neq 0} \frac{\langle H_{\omega,\sharp}(\Lambda, n)\varphi, \varphi \rangle}{\|\varphi\|^2}. \]

Two characterizations of the ground state energy in terms of entropy are given below.

**Proposition 2.8.** $\mathcal{E}$ is the ground state energy if and only if $\mathcal{N}(\mathcal{E} - 0) = 0$ and $\mathcal{N}(\mathcal{E} + 0) > 0$ or, equivalently, if and only if $S(\mathcal{E} - 0) = -\infty$ and $S(\mathcal{E} + 0) \geq 0$.

**Proposition 2.9.** Alternatively, the ground state energy is given by the zero entropy:
\[ \mathcal{E}_{\omega,\sharp}(\Lambda, n) = E_{\omega,\sharp}(\Lambda, n, 0). \]

The latter characterization is essentially due to our choice of the inverse function $E(S)$ given by $(2.5)$ and would not be valid for another choice of the inverse, whereas the Proposition 2.8 is universal with respect to the particular choice of the function $E(S)$. 

2.2. Thermodynamic Limit. In this section we discuss the notion of thermodynamic limit, following the approach of [Rue99]. For sake of completeness and the ease of reading, we repeat here the basic definitions related to the notion of thermodynamic limit that can be found in various monographs and articles such as [Rue99, LSSY05, Gri65, LL72].

First of all, we give a precise meaning to the notion of a sequence of domains tending to infinity.

Definition 2.10. Let diam(Λ) be the diameter of Λ and ∂_hΛ be the h-neighborhood of ∂Λ, i.e.,

\[ \partial_h^h \Lambda = \partial \Lambda + B(0, h), \]

where \( B(0, h) \) is the open ball of center 0 and radius h.

Definition 2.11. The sets Λ tend to infinity in the sense of Fisher if

\[ \lim |\Lambda| = +\infty \]

and there exists a “shape function” \( \pi \) such that

\[ \lim_{\alpha \to 0} \pi(\alpha) = 0 \]

and for sufficiently small \( \alpha \) and all Λ

\[ |\partial_\alpha \text{diam}(\Lambda)\Lambda|/|\Lambda| \leq \pi(\alpha). \]

In what follows, we will always assume that \( \Lambda \to \infty \) in the sense of Fisher.

Remark 2.12. Consider a sequence of rectangular domains. The fact that they tend to infinity in the sense of Fisher is equivalent to say that all their sides tend to infinity at a comparable speed, i.e.,

\[
\prod_{j=1}^{d}[0, L_j] \to \infty \quad \Leftrightarrow \quad \{ \min_j L_j \to \infty, \quad 1 \geq \min_j L_j/\max_j L_j > 1/C. \}
\]

Definition 2.13. The limit \( \Lambda \to \infty, n/|\Lambda| \to \rho \), where \( \rho \) is a positive constant (density of particles), is called the thermodynamic limit.

Usually one is interested in extensive quantities per particle or per unit of volume (that is the same thing up to a multiplicative constant due to Definition 2.13) while considering the thermodynamic limit.

Definition 2.14. Let \( X_\omega(\Lambda, n; \mathcal{P}) \) be a random variable that depends on a domain Λ, a number of particles \( n \) and on a set of parameters \( \mathcal{P} \). We say that \( X_\omega(\Lambda, n, \mathcal{P}) \) admits the thermodynamic limit if the limit

\[
\lim_{n/|\Lambda| \to \rho, \mathcal{P}[\mathcal{P}] \to \rho} \frac{X_\omega(\Lambda, n, \mathcal{P})}{n}
\]

exists in some sense with respect to randomness \( \omega \) (almost sure, in probability, in \( L^2 \)). Here \( \mathcal{L}[\mathcal{P}] \) is a certain limiting procedure for the parameters \( \mathcal{P} \), i.e., it determines the way how the parameters \( \mathcal{P} \) evolve when \( \Lambda \) and \( n \) go to infinity in the thermodynamic limit. For example, see (3.3), where an extra parameter is entropy \( S \), and the limiting procedure for the entropy reads as it should tend to infinity linearly with the number of particles and/or the volume of the system.
In thermodynamics, some commonly used quantities (such as internal energy, for example) are assumed to be extensive, i.e., additive with respect to volume. The existence of the thermodynamic limit is the mathematically rigorous way of verifying the above assumption. Thus, it is one of the fundamental questions of statistical physics. Some authors go even further and refer to the question of existence of thermodynamic limit purely as “existence of thermodynamics” [LL72].

In what follows, we will be primarily concerned with the existence of the thermodynamic limit for the energy $E_\omega(\Lambda,n,S)$ with $S/n \to \sigma \geq 0$ and, in particular, the ground state energy $E_\omega(\Lambda,n)$, i.e., for $\sigma = 0$.

3. Main Results

Throughout this section we work with Dirichlet boundary conditions

$$H_\omega(\Lambda,n) = H^D_\omega(\Lambda,n)$$

and we omit the explicit indication $D$ in notations. We give a series of statements concerning the existence of the thermodynamic limit for the model of interacting quantum particles in random media, which was introduced in Section 2.1. Basic properties of the thus defined limits are discussed.

We shall need some assumptions on the model that we introduce now.

**Pair translation invariant interactions.** The interactions are by pairs and are invariant under translations if for all $n \in \mathbb{N}$

$$W_n(x) = \sum_{1 \leq i < j \leq n} U(x^i - x^j), \quad (\text{PI})$$

where $U$ is a function on $V$. We also assume that pair interactions are symmetric: $U(x) = U(-x)$, $x \in V$.

**Tempered interactions.** Assume (PI) and that there exist $R_0 > 0$, $A$ and $\lambda > d$ such that for all $|x| \geq R_0$

$$|U(x)| \leq A |x|^{-\lambda}. \quad (\text{PTI})$$

This condition (together with an additional assumption that $U$ is integrable in a neighborhood of zero) guarantees that interactions are of short range, i.e.,

$$\int_{\mathbb{R}^d} |U(x)| dx < +\infty.$$

The temperedness or similar conditions on the behavior of the interactions at the infinity have been used by various authors such as Léon van Hove, Joel L. Lebowitz, Robert B. Griffits and, in particular, Michael E. Fisher and David Ruelle. The reader is referred to [Fis64], [Rue99], [FR66], [Leb76], [Gri65].

**Remark 3.1.** The above assumption of temperedness of interactions can be physically motivated by the following argument. Consider electrons in metal or semiconductor as a reference system. Though electrons interact via Coulomb potential ($\sim 1/r$) in vacuum, the situation is different in metal where each electron is surrounded by a “cloud” of other electrons and lives in a grid of ions. This leads to what is called screening of Coulomb potential in metal (see [AM76, Zag98]) and results to the effective interaction potential of the form

$$U(r) = \frac{Q}{r} \exp(-r/\lambda). \quad (3.1)$$

The interaction is between quasiparticles “electron+cloud”, that are called plasmons.

\footnote{The potential (3.1) is called Yukawa potential, though it usually arises in a context of nuclear physics.}
Lower-bounded one particle Hamiltonian. The one-particle random operator is bounded from below uniformly with respect to randomness \( \omega \):
\[
\exists C > 0, \text{ such that } H_\omega \geq -C \text{ for all } \omega \in \Omega. \tag{LB}
\]

**Notation 3.2.** We write \( \mathbb{N}_n = \{1, \ldots, n\} \). For an index set \( I = \{i_1, \ldots, i_n\} \subset \mathbb{N} \) we write
\[
x^I = (x^{i_1}, \ldots, x^{i_n}) \in \mathbb{R}^{nd}
\]
for the vector of the coordinates of the particles enumerated by \( I \), where the elements are ordered in a nondecreasing fashion: \( i_p < i_q \) if \( p < q \).

**Definition 3.3.** Let \( I_1 \cup I_2 = \mathbb{N}_{n_1+n_2} \), \( |I_j| = n_j \), be a partition of \( n_1 + n_2 \) particles in two disjoint subsets. The term of interaction between the particles \( I_1 \) and \( I_2 \) is given by
\[
W_{I_1, I_2}(x^{N_{n_1+n_2}}) = W_{n_1+n_2}(x^{N_{n_1+n_2}}) - W_{n_1}(x^I) - W_{n_2}(x^I). \tag{3.3}
\]

**Repulsive interactions.** The interactions are repulsive, if for all \( I_1, I_2 \) as in Definition 3.3 it holds
\[
W_{I_1, I_2} \geq 0. \tag{Rep}
\]
If one assumes (\( \text{Rep} \)) and that there are no self-interactions: \( W_1 = 0 \), then for all \( n \in \mathbb{N} \)
\[
W_n(x^N) \geq W_{n-1}(x^{N_{n-1}}) + W_1(x^n) \geq W_{n-1}(x^{N_{n-1}}) \geq \ldots \geq 0, \quad x^N \in \mathbb{R}^{nd}. \tag{3.2}
\]
If one also assumes (\( \text{PI} \)), then (\( \text{Rep} \)) is equivalent to say that
\[
U \geq 0.
\]

**Stable interactions.** The interactions are stable if there exists \( B > 0 \), such that for all \( n \in \mathbb{N} \)
\[
W_n(x) \geq -nB. \tag{SI}
\]
By (3.2), repulsive interactions are stable with \( B = 0 \). The stability of interactions for various models is widely discussed, in particular, in [FR66].

**Compactly supported interactions.** Using the notations of Definition 3.3 the interactions \( W \) have compact support if there exists \( R_0 > 0 \) such that
\[
W_{I_1, I_2}(x^{N_{n_1+n_2}}) = 0 \tag{Comp}
\]
for all \( x^{N_{n_1+n_2}} \in \mathcal{V}_{n_1+n_2} \) such that \( \text{dist}(x^{I_1}, x^{I_2}) \geq R_0 \).

**Remark 3.4.** Obviously, for pair interactions, compact support is stronger than temperedness, i.e.,
\[
(\text{PI}) + (\text{Comp}) \implies (\text{PTI}) \text{ with } A = 0.
\]

Let us now discuss the physical validity of the above assumptions. For more details on classical electrodynamics, see, for example [Jac75] and for the electrodynamics of continuous media, see, for example [LL60].

- The model of pair translation invariant (\( \text{PI} \)) repulsive (\( \text{Rep} \)) interactions is natural for a description of identical quantum particles such as electrons.
- The condition of temperedness (\( \text{PTI} \)) might seem more restrictive at first glance, but is usually circumvented as described in Remark 3.1 by replacing actual interactions by screened interactions and bare electrons by quasiparticles.
- The condition of compactly supported interactions (\( \text{Comp} \)) is a technical one and allows us to treat interaction of higher order than pair (triple, etc.). However, even short range Yukawa interactions (3.1) are not compactly supported.
The repulsive nature of interactions between identical particles (Rep) is widely accepted. Though, mathematically only the condition of stability (SI) is needed. Further discussion of stability condition and examples of catastrophic, i.e., not stable, potentials may be found in [Rue99].

Finally, the lower boundedness of the one-particle operator (LB) seems a natural basic assumption.

The following theorem is the main result of this paper on the existence of thermodynamics for the model described in Section 2.

**Theorem 3.5 (existence of thermodynamic limit).** Suppose that the one particle operator is lower bounded (LB) and that the interactions are stable (SI). Let also any of the following two cases hold:

(a) interactions are translation invariant and by pairs, i.e., they satisfy (PTI)

(b) interactions are compactly supported, i.e., they satisfy (Comp).

Then, the energy per particle admits thermodynamic limit, namely

$$\frac{E_\omega(\Lambda, n, S)}{n} \rightarrow \mathcal{E}(\rho, \sigma) \quad \text{as} \quad \Lambda \rightarrow \infty, \quad n \left|\Lambda\right| \rightarrow \rho, \quad \sigma \rightarrow \sigma,$$

where $\rho > 0$ and $\sigma \geq 0$. The convergence takes place in $L^2(\Omega)$ in case (a) and in $L^1(\Omega)$ and $\omega$-almost sure in case (b). The limiting energy density $\mathcal{E}(\rho, \sigma)$ is defined by (3.3), is a non-random function (does not depend on $\omega$) and the limit is the same if both conditions (a) and (b) are satisfied.

The energy density has the following basic properties.

**Proposition 3.6 (critical density of particles).** There exists a critical density $\rho_c \in [0, +\infty]$ such that

$$\begin{align*}
\mathcal{E}(\rho, \sigma) &< +\infty, \quad \text{if} \quad \rho < \rho_c, \\
\mathcal{E}(\rho, \sigma) &= +\infty, \quad \text{if} \quad \rho > \rho_c,
\end{align*}$$

for all $\sigma \geq 0$.

**Proposition 3.7 (energy density properties).** The energy density $\mathcal{E}(\rho, \sigma)$ is

(a) a convex function of variables $(\rho^{-1}, \sigma)$;

(b) a nondecreasing function of $\rho$ and $\sigma$;

(c) a continuous function in the region $\{0 < \rho < \rho_c\} \times \{\sigma \geq 0\}$.

**Corollary 3.8.** The energy density $\mathcal{E}(\rho, \sigma)$ admits an inverse $\sigma(\rho, \mathcal{E})$. The latter is convex upwards with respect to $(\rho^{-1}, \mathcal{E})$ and is nondecreasing in $\mathcal{E}$ for any fixed $\rho$.

Next we state a reciprocal result exchanging the roles of energy and entropy (the proof follows [Gri65]).

**Theorem 3.9 (existence of thermodynamic limit for entropy).** Let the conditions of Theorem 3.5 be satisfied. Then for $0 < \rho < \rho_c$ and $\mathcal{E} \in \text{Ran} \mathcal{E}(\rho, \cdot)$

$$\frac{S_\omega(E, \Lambda, n)}{n} \rightarrow \sigma(\rho, \mathcal{E}) \quad \text{as} \quad \Lambda \rightarrow \infty, \quad n \left|\Lambda\right| \rightarrow \rho, \quad \frac{E}{n} \rightarrow \mathcal{E}.$$

The convergence takes place in the same sense as given by Theorem 3.5.

**Remark 3.10.** The condition that the energy belongs to the image of the function $\mathcal{E}(\rho, \cdot)$ is crucial. One might remark as well that due to monotonicity and convexity properties of $\mathcal{E}$, either $\mathcal{E}(\rho, \cdot) \equiv \text{const} \text{identically}$, or $\text{Ran} \mathcal{E}(\rho, \cdot) = [\inf \mathcal{E}(\rho, \cdot), +\infty)$. 
4. Proofs

This section is mainly devoted to the proof of $L^2$-convergence (case (3) of Theorem 3.5). The basic ideas were inspired by [Rue99] and [Gri65], though the crucial difference is that instead of translation invariance of one particle operator (which is free Laplacian for both of the above works) we have ergodicity, i.e., covariance with respect to a family of measure preserving transformations of the probability space.

We assume (LB), (PTI) and (SI) throughout this section, except for Subsection 4.6, where different assumptions will be made. We also recall that the Dirichlet boundary conditions are used, i.e., $H_\omega = H^D_\omega$.

4.1. Subadditive Inequalities. Subadditive inequalities play the key role in our proofs. The basic idea for all the proofs for existence theorems in this paper (and many others: see, for example, [Rue99, LL72, Gri65]) may be summarized as:

- find a subadditive type inequality,
- use the existing or prove an analog of subadditive ergodic theorem that guarantees the convergence.

Next is the core lemma that gives the subadditivity of energy.

**Lemma 4.1 (Test function construction).** Suppose (IAD) and (PTI) are satisfied. Let the statistics $\sharp \in \{\emptyset, +, -\}$ be fixed. Consider domains $\Lambda_1$, $\Lambda_2$ such that $\text{dist}(\Lambda_1, \Lambda_2) \geq r \geq R_0$ and functions $\varphi_j \in \mathcal{H}^\sharp_{n_j}(\Lambda_j)$, $j = 1, 2$, with energies below $E_j$:

$$\langle H_\omega, \varphi_j \rangle \leq E_j \|\varphi_j\|^2, \quad j = 1, 2. \quad (4.1)$$

Then, using $\varphi_1$, $\varphi_2$, one can construct explicitly $\zeta \in \mathcal{H}^{n_1+n_2}_\sharp(\Lambda_1 \cup \Lambda_2)$, a function of $n_1 + n_2$ particles defined of a unified box $\Lambda_1 \cup \Lambda_2$ with energy below $E_1 + E_2 + An_1n_2r^{-\lambda}$:

$$\langle H_\omega, \zeta \rangle \leq (E_1 + E_2 + An_1n_2r^{-\lambda})\|\zeta\|^2.$$

**Remark 4.2.** The construction of a test function is explicit in the proof that follows.

**Proof of the Lemma.** We consider the extensions of the functions $\varphi_j$, $j = 1, 2$, by zero on $(\Lambda_1 \cup \Lambda_2)^n_j$, which we also denote by $\varphi_j$. Remark that (4.1) implicitly contains the fact that $\varphi_j$ are zeros on the respective domains boundaries (due to Dirichlet condition) so that the zero extension is a natural operation. These extensions obviously preserve (anti)symmetry when $\sharp \in \{+,-\}$. Consequently, one has $\varphi_j \in \mathcal{H}^{n_j}_\sharp(\Lambda_1 \cup \Lambda_2)$ for any initial choice of $\sharp$.

We study each statistics separately now.

**Boltzmann statistics:** Take

$$\zeta = \varphi_1 \otimes \varphi_2.$$
Remark 4.3. The construction (4.4) is a generalization of the Slater determinant [Gre07].
Remark 4.4. The Dirichlet boundary conditions are crucial for the proof as they provide a zero cost (canonical) extension of functions from $\text{Dom} H_\omega(\Lambda_i, n_i)$ to a larger domain $\Lambda_1 \cup \Lambda_2$ without changing the norm. At this moment, we are not able to prove an analog of Theorem 3.3 (essentially, we need an analog of Lemma 4.1) for Neumann or periodic boundary conditions.

From now on, we omit the statistics sign $\sharp$ in the notations. Everything that follows is valid for all the statistics. However, one should be warned that quantities (such as limiting values) may depend on the statistics.

**Proposition 4.5.** Let the interactions $W$ be tempered (PTI). If $\text{dist}(\Lambda_1, \Lambda_2) \geq r \geq R_0$, then

\begin{align*}
N_\omega(\Lambda_1 \cup \Lambda_2, n_1 + n_2, E_1 + E_2 + A n_1 n_2 r^{-\lambda}) & \geq N_\omega(\Lambda_1, n_1, E_1) N_\omega, R(\Lambda_2, n_2, E_2), \tag{4.5} \\
S_\omega(\Lambda_1 \cup \Lambda_2, n_1 + n_2, E_1 + E_2 + A n_1 n_2 r^{-\lambda}) & \geq S_\omega(\Lambda_1, n_1, E_1) + S_\omega(\Lambda_2, n_2, E_2). \tag{4.6}
\end{align*}

**Proof.** The proof of (4.5) is done using the variational principle for eigenvalues of $H_\omega(\Lambda, n)$ and the function $\zeta$ from Lemma 4.1 as a test function. Taking the logarithm, one obtains (4.6). \qed}

**Proposition 4.6.** Let the interactions $W$ be tempered (PTI).

(a) Take $S_1, S_2$ such that $\exp S_i \in \mathbb{N}$, $i = 1, 2$. If $\text{dist}(\Lambda_1, \Lambda_2) \geq r \geq R_0$, then

$$E_\omega(\Lambda_1 \cup \Lambda_2, n_1 + n_2, S_1 + S_2) \leq E_\omega(\Lambda_1, n_1, S_1) + E_\omega(\Lambda_2, n_2, S_2) + A n_1 n_2 r^{-\lambda}. \tag{4.7}$$

(b) Take $S_i$ such that $\exp S_i \in \mathbb{N}$, $i = 1, \ldots, m$, and domains $\Lambda_1, \ldots, \Lambda_m$ at mutual distances greater than $r \geq R_0$. Then

$$E_\omega \left( \bigcup_{i=1}^{m} \Lambda_i, \sum_{i=1}^{m} n_i, \sum_{i=1}^{m} S_i \right) \leq \sum_{i=1}^{m} E_\omega(\Lambda_i, n_i, S_i) + A \left( \sum_{i=1}^{m} n_i \right)^2 r^{-\lambda}. \tag{4.8}$$

**Proof.** The inequality (4.8) is an immediate consequence of (4.7). The latter is obtained by taking $E_\omega(\Lambda_1 \cup \Lambda_2, n_1 + n_2, \cdot)$ of (4.6) and using (2.6). \qed}

4.2. $L^2$-convergence on a Special Sequence of Cubes. In this section, we will construct a special sequence of cubes $\Lambda_N$ in configuration space $\mathcal{V}$, on which the existence of thermodynamic limit will be proven. The idea is inspired by [Rue99].

Let $\theta$ be a number that satisfies

$$1 < 2^{d/\lambda} < \theta < 2$$

and let

$$\tilde{L} > R = \frac{R_0 + \delta}{2 - \theta}, \tag{4.9}$$

where $\delta > 0$ is a constant that will be fixed later. For an integer $N \geq 0$ put

$$L_N = 2 \left[ \frac{1}{2} \left( 2^N \tilde{L} - \theta^N R \right) \right], \tag{4.10}$$

so that $L_N \in 2\mathbb{Z}$, and define the cube $\Lambda_N$ by

$$\Lambda_N = [-L_N/2, L_N/2]^d \subset \mathcal{V}.$$  

Remark that the vertices of $\Lambda_N$ are at integer points. According to (4.10) it is possible to place $2^d$ translates of $\Lambda_N$ (cubes $\Lambda_N^{(i)}$) inside $\Lambda_{N+1}$ at mutual distances at least

$$R_N = L_{N+1} - 2L_N = \theta^N (2 - \theta) R + \varepsilon = \theta^N (R_0 + \delta) + \varepsilon \geq R_0,$$
where $\varepsilon \in [-4, 2]$ is the error due to the rounding procedure. The constant $\delta$ is chosen to compensate a possibly negative error term $\varepsilon$, so that the last inequality holds true. It suffices, for example, to choose $\delta = 4$.

We remark that cubes $\Lambda_N^{(i)}$ are explicitly given by

$$\Lambda_N^{(i)} = \Lambda_N + \gamma_i^{(N)},$$

where

$$\gamma_i^{(N)} = \frac{L_{N+1} - L_N}{2} \cdot e_i \in \mathbb{Z}^d, \quad e_i = (\pm 1, \ldots, \pm 1) \in \mathbb{R}^d, \quad i = 1, \ldots, 2^d.$$

**Remark 4.7.** It is important that the translation vectors $\gamma_i^{(N)}$ are integer, because it ensures that the restrictions of the random potential $V_\omega$ to $\Lambda_N^{(i)}$ for different $i$ are connected by the covariance relation (2.1).

The function $E_\omega(\Lambda, n, S)$ satisfies the following monotonicity properties.

**Lemma 4.8.** For fixed $\omega$ and $n$, the energy $E_\omega(\Lambda, n, S)$ is

(a) a nondecreasing function of $S$,

(b) a nonincreasing function of $\Lambda$.

By Lemma 4.8 and the almost-subadditivity condition (4.8) we obtain for $S_i$ in $\log N$ that

$$E_\omega \left( \Lambda_{N+1}, \sum_{i=1}^{2^d} n_i, \sum_{i=1}^{2^d} S_i \right) \leq E_\omega \left( \bigcup_{i=1}^{2^d} \Lambda_N^{(i)}, \sum_{i=1}^{2^d} n_i, \sum_{i=1}^{2^d} S_i \right) \leq \sum_{i=1}^{2^d} E_\omega(\Lambda_N^{(i)}, n_i, S_i) + \frac{A}{2} \left( \sum_{i=1}^{2^d} n_i \right)^2 R_N^{-\lambda} = \sum_{i=1}^{2^d} E_{\tau_\gamma^{(N)}}(\omega)(\Lambda_N, n_i, S_i) + \frac{A}{2} \left( \sum_{i=1}^{2^d} n_i \right)^2 R_N^{-\lambda},$$

where $\{\tau_\gamma\}_{\gamma \in \mathbb{Z}^d}$ is the family of ergodic transformations of $\Omega$, that were introduced in Notation 2.2

In particular, for $S \in \log N$

$$E_\omega \left( \Lambda_{N+1}, 2^d n, 2^d S \right) \leq \sum_{i=1}^{2^d} E_{\tau_\gamma^{(N)}}(\omega)(\Lambda_N, n, S) + \frac{A}{2} (2^d n)^2 R_N^{-\lambda}. \quad (4.11)$$

Let now $\rho$ and $\sigma$ be positive numbers such that $2^{N_0 d} \rho \tilde{L}^d$ and $\exp \left( 2^{N_0 d} \sigma \rho \tilde{L}^d \right)$ are integer, for a sufficiently large integer $N_0$. Plug in (4.11)

$$n_N = 2^{N_0 d} \rho \tilde{L}^d, \quad S_N = \sigma n_N = 2^{N_0 d} \rho \sigma \tilde{L}^d \quad (4.12)$$

for $N \geq N_0$. Remark that

$$n_N / |\Lambda_N| \to \rho, \quad S_N / n_N = \sigma, \quad 2^{N_0 d} \tilde{L}^d / |\Lambda_N| \to 1 \quad \text{as } N \to +\infty.$$

We introduce the following sequence of random variables

$$X_N(\omega) = 2^{-N_0} \left( E_\omega(\Lambda_N, n_N, S_N) + (B + C)n_N \right), \quad (4.13)$$
where $B$ is the constant from (SI) and $C$ is the constant from (LB). By (4.11) this sequence satisfies the inequality

$$X_{N+1}(\omega) \leq 2^{-d} \sum_{i=1}^{2^d} X_N(\tau_i^{(N)}(\omega)) + G_N$$

with

$$G_N = A \rho L^{2d+2(N+2)d} R_N^{-\lambda}.$$ 

In order to show the convergence of the sequence $X_N(\omega)$, we establish the following proposition.

**Proposition 4.9.** Let $X_N$ be a sequence of nonnegative random variables on a probability space $(\Omega, \mathbb{P})$, $X_N(\omega) \geq 0$, such that for each $N$ there exists a family of probability preserving transformations of $\Omega$, $\tau_i^{(N)}$, $i \in I_N$, with $\text{card} I_N < +\infty$, such that the variables $X_N \circ \tau_i^{(N)}$, $i \in I_N$ are i.i.d. (independent identically distributed). If the sequence $X_N$ satisfies

$$X_{N+1}(\omega) \leq \frac{1}{\text{card} I_N} \sum_{i \in I_N} X_N(\tau_i^{(N)}(\omega)) + G_N, \text{ where } \sum_{N=1}^{+\infty} |G_N| < +\infty, \quad (4.14)$$

then there exists a constant $\overline{X}$ (that does not depend on $\omega$) such that

$$X_N \xrightarrow{L^2} \overline{X}. \quad (4.15)$$

**Proof.** Since the terms at the r.h.s. of (4.14) are identically distributed, after taking the expectation one obtains:

$$\mathbb{E}X_{N+1} \leq \mathbb{E}X_N + G_N. \quad (4.16)$$

Consider the sequence

$$c_N = \mathbb{E}X_N - \sum_{i=1}^{N-1} G_i.$$

Obviously, (4.16) guarantees that $c_{N+1} \leq c_N$. Consequently, this sequence converges: $c_N \xrightarrow{N \to +\infty} c_0$.

Thus, as soon as the sum $\sum_{i=1}^{+\infty} G_i$ also converges, $\mathbb{E}X_N$ admits the limit that we denote by $\overline{X}$:

$$\mathbb{E}X_N \xrightarrow{N \to +\infty} \overline{X}.$$ 

Consider now the variance of $X_N$, $\mathbb{E}(X_N - \mathbb{E}X_N)^2 = \mathbb{E}(X_N^2) - (\mathbb{E}X_N)^2$. We will show that this variance tends to zero as $N$ goes to infinity. By (4.14)

$$(X_{N+1}(\omega))^2 \leq \left( \frac{1}{\text{card} I_N} \sum_{i \in I_N} X_N(\tau_i^{(N)}(\omega)) + G_N \right)^2$$

$$= \frac{1}{(\text{card} I_N)^2} \sum_{i \in I_N} \left( X_N(\tau_i^{(N)}(\omega)) \right)^2 + \frac{1}{(\text{card} I_N)^2} \sum_{i,j \in I_N} X_N(\tau_i^{(N)}(\omega))X_N(\tau_j^{(N)}(\omega))$$

$$+ \frac{2G_N}{\text{card} I_N} \sum_{i \in I_N} X_N(\tau_i^{(N)}(\omega)) + G_N^2.$$
Taking the expectation and using the fact that \( X_N(\tau_i^{(N)})_{i \in I_N} \) are i.i.d. for fixed \( N \), we find
\[
\mathbb{E}(X_{N+1}(\omega))^2 \leq \frac{1}{(\text{card } I_N)^2} \sum_{i \in I_N} \mathbb{E}(X_N(\tau_i^{(N)}))^2 + \frac{1}{(\text{card } I_N)^2} \sum_{i,j \in I_N \atop i \neq j} \mathbb{E}(X_N(\tau_i^{(N)}\omega)X_N(\tau_j^{(N)}\omega)) + 2G_N \sum_{i \in I_N} \mathbb{E}X_N(\tau_i^{(N)}\omega) + G_N^2
\]
\[
= \frac{1}{(\text{card } I_N)^2} \sum_{i \in I_N} \mathbb{E}(X_N^2) + \frac{1}{(\text{card } I_N)^2} \sum_{i,j \in I_N \atop i \neq j} (\mathbb{E}X_N)^2 + \frac{2G_N}{\text{card } I_N} \sum_{i \in I_N} \mathbb{E}X_N + G_N^2
\]
\[
= \frac{1}{\text{card } I_N} \mathbb{E}(X_N^2) + \frac{(\text{card } I_N)(\text{card } I_N - 1)}{(\text{card } I_N)^2} (\mathbb{E}X_N)^2 + 2G_N \cdot \mathbb{E}X_N + G_N^2.
\]
So, we get
\[
\mathbb{E}(X_{N+1}^2) \leq \frac{1}{\text{card } I_N} \mathbb{E}(X_N^2) + \left( 1 - \frac{1}{\text{card } I_N} \right) (\mathbb{E}X_N)^2 + 2G_N \cdot \mathbb{E}X_N + G_N^2. \quad (4.17)
\]
By the Schwarz inequality
\[
(\mathbb{E}X_N)^2 \leq \mathbb{E}(X_N^2). \quad (4.18)
\]
Hence, using (4.17), we obtain
\[
\mathbb{E}(X_{N+1}^2) \leq \mathbb{E}(X_N^2) + 2G_N\sqrt{\mathbb{E}(X_N^2)} + G_N^2 = \left( \sqrt{\mathbb{E}(X_N^2)} + G_N \right)^2.
\]
Finally,
\[
\sqrt{\mathbb{E}(X_{N+1}^2)} \leq \sqrt{\mathbb{E}(X_N^2)} + G_N.
\]
Arguing as for the expectation, this implies that the sequence \( \mathbb{E}(X_N^2) \) converges. Taking the limit in (4.17) and using the fact that \( G_N \to 0 \) as \( N \to +\infty \), we obtain
\[
\lim_{N \to +\infty} \mathbb{E}(X_N^2) \leq \mathbb{X}^2,
\]
but the Schwarz inequality (4.18) immediately gives the reciprocal estimate. We conclude that
\[
\lim_{N \to +\infty} \mathbb{E}(X_N^2) = \mathbb{X}^2
\]
and the variance \( \mathbb{E}(X_N - \mathbb{E}X_N)^2 \) tends to 0 as \( N \) goes to infinity. This proves (4.15). \( \square \)

As an immediate consequence of this general statement we obtain the existence of the thermodynamic limit on the special sequence of cubes \( \Lambda_N \).

**Corollary 4.10.** Suppose (LB), (PTI) and (SI). Then the thermodynamic limit for the energy \( E_\omega \) on the sequence of cubes \( \Lambda_N \) in the sense of \( L^2_\omega \) exists, i.e.,
\[
\frac{E_\omega(\Lambda_N, n_N, S_N)}{n_N} \xrightarrow{L^2} \mathcal{E}(\rho, \sigma), \quad N \to +\infty
\]
where \( \mathcal{E}(\rho, \sigma) \) is defined by this limit and is called the limiting energy per particle or the energy density.
Proof. We only need to prove that the random variable \(X_N(\omega)\) introduced in (4.13) is nonnegative. Recall that
\[
H_\omega(\Lambda_N, n_N) = \sum_{i=1}^{n_N} H_\omega^{(i)}(\Lambda_N, n_N) + W_{n_N}.
\]
Moreover (see also (2.2)),
\[
H_\omega^{(i)}(\Lambda_N, n_N) \geq -C, \quad i = 1, \ldots, n_N,
\]
by the lower boundedness of the one particle Hamiltonian (LB), and
\[
W_{n_N} \geq -B n_N
\]
by the stability of interactions (SI). Thus,
\[
H_\omega(\Lambda_N, n_N) \geq -(B + C)n_N,
\]
and consequently
\[
E_\omega(\Lambda_N, n_N, S_N) + (B + C)n_N \geq 0.
\]
\[\square\]

4.3. Critical Density of Particles. We now discuss the finiteness of the thermodynamic limit that was announced in Proposition 3.6.

Being essentially attained via a nonincreasing sequence, the limit is finite if and only if there are finite terms in the sequence \(E_{\omega}(\Lambda_N, n_N, S_N)\). In other words, if for a sufficiently big \(N\), \(E_{\omega}(\Lambda_N, n_N, S_N)\) is finite, then \(E(\rho, \sigma) < +\infty\).

The situation when the operator \(H_\omega(\Lambda_N, n_N)\) doesn’t possess an increasing sequence of eigenvalues may arise, according to variational principle, if and only if a subspace of functions \(\varphi\) such that \((H_\omega(\Lambda_N, n_N)\varphi, \varphi) < +\infty\) is of finite dimension. But this last condition is possible only if the interaction potential \(W_{n}\) takes the value \(+\infty\) and if there are too few configurations with a finite interaction term, i.e.,
\[
\text{meas}\left\{(x^1, \ldots, x^{n_N}) \in \Lambda_{n_N}^n \mid W_n(x^1, \ldots, x^{n_N}) < +\infty\right\} = 0. \tag{4.19}
\]

As a model case, suppose that the interactions are by pairs (PI) and that the pair potential \(U\) represents hard cores of radius \(r_0\) (see [Rue99]):
\[
U(x) \begin{cases} = +\infty, & |x| \leq r_0, \\ \neq +\infty, & |x| > r_0. \end{cases}
\]

In this case, the condition (4.19) is satisfied if there isn’t enough space for \(n_N\) balls of radius \(r_0/2\) with centers in the domain \(\Lambda_N\). In other words, define the set of denied spacial configurations of \(n\) particles by
\[
S_{r_0}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^{nd}, \text{ such that } |x^i - x^j| < r_0 \text{ for some } i \neq j\}.
\]

Then the Hamiltonian is defined on \(L^2_2(\mathcal{V}^n \setminus S_{r_0}^n),\) instead of \(L^2_2(\mathcal{V}^n)\) and it may happen that
\[
\text{meas}\left(\mathcal{V}^n \setminus S_{r_0}^n\right) = 0.
\]

The last observation suggests that there exists a critical density of particles \(\rho_c\) such that the energy density \(E(\rho, \sigma)\) is finite for \(\rho < \rho_c\) and infinite for \(\rho > \rho_c\). For example, for the case of hard cores, this is the closed packing density. Note that \(\rho_c = +\infty\) if the interaction potential takes only finite values.
4.4. Properties of the Energy Density. Before proceeding with the proof of the existence of thermodynamic limit for general domains, we establish some properties of the energy density $E$ (in particular, we prove Proposition 3.7).

From now on, we assume that $\rho < \rho_c$. Note that till now the function $E(\rho, \sigma)$ is defined (as a limit of a sequence) only for particle densities $\rho$ and entropy densities $\sigma$ of the form

$$\rho = \frac{m_1}{2^{N_0 d} \tilde{L}^d}, \quad \sigma = \frac{\log m_2}{m_1},$$

(4.20)

where $m_1$, $m_2$ and $N_0$ are positive integers. In all the statements that follow we implicitly assume that $\rho$ and $\sigma$ satisfy (4.20).

Proof of Proposition 3.7 (a). The convexity of the limiting function is an immediate consequence of the almost-subadditivity (4.7).

Proof of Proposition 3.7 (b). The monotonicity is given by Lemma 4.8.

Proposition 4.11. The energy density $E$ is locally bounded on the plane of parameters $0 < \rho < \rho_c$ and $\sigma \geq 0$.

Proof. Let $0 < \rho_1 < \rho_2 < \rho_c$ and $\sigma_0 > 0$ be of the form (4.20). We shall show that $E$ is bounded in the region $\Delta = \{\rho_1 \leq \rho \leq \rho_2\} \times \{0 \leq \sigma \leq \sigma_0\}$. First remark that the number of particles $n_N = 2^{Nd} \tilde{L}^d \rho$ with $\rho \leq \rho_2$ can be represented as

$$n_N = \sum_{j=1}^{2^{Nd}} n_0^{(j)},$$

(4.21)

where $n_0^{(j)} \in \left\{\left[\tilde{L}^d \rho\right], \left[\tilde{L}^d \rho\right] + 1\right\}$. Obviously, such a representation depends on $\rho$. On the other hand, the bound $n_0^{(j)} \leq \left[\tilde{L}^d \rho_2\right] + 1 =: n_0^{\max}$ depends only on $\rho_2$. This representation can be obtained as the result of a consecutive division of the domain $\Lambda_N$ in sub-domains (each time we divide the domain in $2^d$ parts) until one obtains the domains $\Lambda_0^{(j)}$. In (4.9), choose $\tilde{L}$ sufficiently large so that $\left[\tilde{L}^d \rho\right] \geq 1$, i.e., there is at least one particle in each sub-domain $\Lambda_0^{(j)}$.

Let us denote by $S^*$ the smallest number belonging to $\log N$ that is larger than $S$:

$$S^* = \inf\{Q \geq S, \exp Q \in \mathbb{N}\}. \quad (4.22)$$

Then one calculates:

$$E_\omega(\Lambda_N, n_N, S_N) \leq \sum_{j=1}^{2^{Nd}} E_\omega \left(\Lambda_0^{(j)}, n_0^{(j)}, (S_N/2^{Nd})^*\right) + \frac{A}{2} n_N n_0^{\max} \sum_{m=0}^{N-1} 2^{(m+1)d} R_m^{-2}. \quad (4.23)$$

Since $S_N/2^{Nd} \leq \tilde{L}^d \rho_2 \sigma_0$ and $n_0^{(j)} \leq n_0^{\max}$, we can deduce that $E_\omega \left(\Lambda_0^{(j)}, n_0^{(j)}, (S_N/2^{Nd})^*\right)$ is bounded uniformly with respect to $N$ and $\omega$. The proof is done by a trivial bounding of the potential (as the number of terms in the potential is bounded) and by the application of Weyl asymptotic. Dividing (4.23) by $2^{Nd}$, we finish the proof.

Proof of Proposition 3.7 (c). Having established Proposition 4.11 it is sufficient to apply a standard argument due to Jensen (see, for example, [PS98]).
Corollary 4.12. Suppose $\rho < \rho_c$ is of the form \(4.20\), $\sigma_1$ and $\sigma_2$ are fixed. Then the $L^2$-convergence
\[
n^{-1}_N E_\omega \left( \Lambda_N, 2^{N d} \tilde{L}^d \rho, 2^{N d} \tilde{L}^d \rho \sigma \right) \to \mathcal{E}(\rho, \sigma)
\]
is uniform in $\sigma \in [\sigma_1, \sigma_2]$ because the pointwise convergence of monotone functions to a continuous function on a compact interval implies the uniform convergence (Dini’s theorem).

From this corollary we deduce the following proposition, which weakens the restrictions on the way the entropy must go to infinity in the thermodynamic limit. Instead of a very specifically chosen sequence $S_N$ as in Corollary 4.11 we need only the linear dependence of entropy on the number of particles.

Proposition 4.13. Let $\rho > 0$ and $\sigma \geq 0$. Analogously to \(4.12\), construct a sequence of integers $n_N = 2^{N d} \tilde{L}^d \rho$. Let also $S_N$ be a sequence such that $S_N \xrightarrow{N \to +\infty} \sigma$. Then
\[
n^{-1}_N E_\omega (\Lambda_N, n_N, S_N) \xrightarrow{L^2, N \to +\infty} \mathcal{E}(\rho, \sigma).
\]

4.5. Proof of Theorem 3.5 (\textit{L}-Convergence for General Domains). Now we are ready to show the existence of the thermodynamic $L^2$-limit in full generality. First of all, we will establish that
\[
\limsup_{\Lambda \to \infty} \frac{E_\omega (\Lambda, n, S)}{n} \leq \mathcal{E}(\rho, \sigma).
\]
(4.24)

Let $\rho_0 > \rho$ of the form \(4.20\) be close to $\rho$:
\[
\rho_0 = \frac{n_0}{2^{N d} \tilde{L}^d}, \quad (n_0, N_0) \in \mathbb{N}^2.
\]
(4.25)

The representation \(4.25\) is not unique. Among all the representations for a fixed $\rho_0$ there exists the minimal one, i.e., where $N_0$ is minimal: $\rho_0 = n_{\min} 2^{-N_{\min} d} \tilde{L}^{-d}$. We write:
\[
n = m n_0 + r_0, \quad 0 \leq r_0 < n_0.
\]

For a fixed $\rho_0$ we choose $N_0$ in \(4.25\) as a function of $n$ such that
\[
\tilde{L}^d / n \to 0, \quad \tilde{L}^{N_0} / n \to +\infty.
\]
(4.26)

By Definition 2.11 a sufficiently large $\Lambda$ contains $m$ disjoint cubes with sides $\xi L_{N_0}$, where
\[
1 < \xi < \left(\frac{\rho_0}{\rho}\right)^{1/d}.
\]

Consequently, $\Lambda$ contains translated cubes $\Lambda_{N_0}^{(i)}$, $i = 1, \ldots, m$, at mutual distances at least $(\xi - 1) L_{N_0}$. By \(4.8\) one has
\[
E_\omega (\Lambda, n, S) \leq \sum_{i=1}^{m-1} E_\omega (\Lambda_{N_0}^{(i)}, n_0, S/(m - 1)) + E_\omega (\Lambda_{N_0}^{(m)} , n_0 + r_0, 0) + \frac{A}{2} n^2 (\xi - 1)^{-\lambda} L^{-\lambda}_{N_0}.
\]
(4.27)

We treat now the term $E_\omega (\Lambda_{N_0}^{(m)} , n_0 + r_0, 0)$. By the domain division procedure similar to that described in Section 4.2 we can reduce $\Lambda_{N_0}^{(m)}$ to the union of $2^{(N_0 - N_{\min}) d} = n_0 / n_{\min}$ (this is an integer) translates of $\Lambda_{N_{\min}}$. We obtain
\[
E_\omega (\Lambda_{N_0}^{(m)} , n_0 + r_0, 0) \leq \sum_{j=1}^{n_0/n_{\min}} E_\omega (\Lambda_{N_{\min}}^{(j)}, n_{\min} + r_j, 0) + \frac{A}{2} (2 n_0)^{N_0-1} \sum_{m=N_{\min}}^{2^{(m+1) d} R_m^{-2}}
\]
(4.28)
where $0 \leq r_j < n_{\text{min}}$ and $j \in \{1, \ldots, n_0/n_{\text{min}}\}$. Clearly,
\[ E_\omega(\Lambda_{N_{\text{min}}}^{(j)}, n_{\text{min}} + r_j, 0) \leq C_1 n_{\text{min}}^2 \]
where the constant $C_1$ is uniform in $j$ and $\omega$. By (4.28) one obtains
\[ E_\omega(\Lambda_{N_0}^{(m)}, n_0 + r_0, 0) \leq \frac{n_0}{n_{\text{min}}^2} C_1 n_{\text{min}}^2 + C_2 n_0 \leq C_3 n_{\text{min}} n_0. \]
Finally, from (4.27) one deduces that
\[ n^{-1} E_\omega(\Lambda, n, S) \leq \frac{n}{n_{\text{min}}} \sum_{i=1}^{n-1} \frac{1}{n_0} E_\omega(\Lambda_{N_0}^{(i)}, n_0, S/(m-1)) + \frac{A}{2} (\xi - 1)^{-\lambda} \frac{n}{L_{N_0}^\lambda}. \]
Note that $n_0/n \leq 1/m \to 0$ and $n/L_{N_0}^\lambda \to 0$ according to (4.26). Thus,
\[ \limsup_{\Lambda \to \infty} n^{-1} E_\omega(\Lambda, n, S) \leq \mathcal{E}(\rho_0, \sigma). \]
Approaching $\rho$ from above by $\rho_0$, one gets (4.24).

4.6. $L^1$ and Almost Sure Limits. In this section we assume (LB) and (Comp). We show that if the interactions are compactly supported, the convergence to the thermodynamic limit can be improved. The proof follows that of the $L^2$-convergence, so we only indicate the necessary modifications.

Let us introduce the following random variable. For all $n \in \mathbb{N}$, $S \in \mathbb{R}$ and domain $A \subset \mathbb{R}^d$ we set
\[ f_\omega(A, n, S) = E_\omega(\hat{A}, n, S^*), \]
where $\hat{A}$ is the $R_0/2$-interior of $A$, i.e.,
\[ \hat{A} = \{ x \in A, \text{dist}(x, \partial A) > R_0/2 \}, \]
and recall that $S^*$ is defined in (4.22). We make the following observations before giving a subadditivity condition.

Lemma 4.14. Let $A$ and $B$ be two domains in $\mathcal{V}$.
(a) If $A \subset B$, then $\hat{A} \subset \hat{B}$.
(b) If $A \cap B = \emptyset$, then $\text{dist}(\hat{A}, \hat{B}) \geq R_0$.

Let also $x, y \in \mathbb{R}$.
(a) If $x \leq y$, then $x^* \leq y^*$.
(b) $(x + y)^* \leq x^* + y^*$.

We modify now the subadditive inequality (4.7).

Proposition 4.15. Let the interactions $W$ be compactly supported (Comp). Let $n_1, n_2 \in \mathbb{N}$. If $A$ and $B$ are two disjoint domains in $\mathcal{V}$, $A \cap B = \emptyset$, then
\[ f_\omega(A \cup B, n_1 + n_2, S_1 + S_2) \leq f_\omega(A, n_1, S_1) + f_\omega(B, n_2, S_2). \]
(4.29)

Proof. As the interactions are compactly supported, we get (4.7) by the same manner as in Proposition 4.6 but without the interaction term (with $A = 0$). It remains to use Lemma 4.14 and the monotonicity of energy with respect to entropy in order to get (4.29). \qed
Thanks to the subadditivity (4.29), we prove the convergence in $L^1$ and the almost sure convergence.

*Proof of Theorem 3.5 (b).* In order to prove this type of convergence, which is stronger than that in the part (a) of the theorem, it is sufficient to modify Section 4.2. Everything what follows remains true without any modifications.

In Section 4.2 we change the definition of cubes $\Lambda_N$ by taking

$$L_N = 2 \left[ \frac{1}{2} \left( 2^N \tilde{L} - R_0 - \delta \right) \right]$$

in a place of (4.10), where $\delta$ is a fixed positive constant. This guarantees that one may put exactly $2^d$ translates of $\Lambda_N$ in a cube $\Lambda_{N+1}$ at distances at least $R_0$ for a properly chosen $\delta$. The lower boundedness is given by (LB) and (SI):

$$\frac{f_\omega(A,n,S)}{n} \geq -B - C.$$

Next, we apply the multidimensional subadditive ergodic theorem (see, for example, [Smy76]) and obtain the $L^1$- and almost sure convergence of the sequence. □

5. Free Particles

As a complement, we study the thermodynamic limit for the energy density $\mathcal{E}(\rho, \sigma)$ in the case of free (noninteracting) particles:

$$W \equiv 0. \quad (5.1)$$

We remark that the background potential $V_\omega$ remains present. Interestingly, even in this case the results are not as trivial as one could have expected. The obtained thermodynamic limits depend on quantum statistics.

5.1. Maxwell-Boltzmann Particles. For particles without statistics we establish the following theorem.

**Theorem 5.1.** Suppose that the interactions are absent (5.1) and that the particles are of Maxwell-Boltzmann statistics. Let $\Sigma$ be the almost sure spectrum of the one-particle Hamiltonian $H_{\omega}(1)$. If

$$\Sigma = \text{supp } dN, \quad (5.2)$$

then

$$\mathcal{E}(\rho, \sigma) = \inf \Sigma$$

for all $\rho > 0$ and $\sigma \geq 0$.

**Remark 5.2.** The condition (5.2) is satisfied under rather general assumptions on the random potential $V_\omega$ (see, for example, [Ves08]).

In order to prove Theorem 5.1 we will make use of two following lemmas. We assume that the conditions of this theorem are verified in the sequel.

**Lemma 5.3.** Let $\omega$ be such that $\text{Spec}(H_{\omega}(1)) = \Sigma$. Let $N \in \mathbb{N}$ be fixed. Then

$$E_N(H_{\omega}(\Lambda,1)) \to \inf \Sigma, \quad \Lambda \to \infty.$$
Proof. Consider, as usual, Dirichlet boundary conditions. Then for almost any \( E \in \mathbb{R} \),
\[
N_\Lambda(E) \nearrow N(E), \quad \Lambda \to \infty,
\]
where \( N_\Lambda \) is the pre-limit density of states, i.e., the counting function of the operator \( H_\omega(\Lambda,1) \) divided by \( |\Lambda| \), and \( N \) is the density of states of the one-particle operator \( H_\omega(1) \).

Just by the definition of the counting function
\[
N_\Lambda(E_N(H_\omega(\Lambda,1))) = \frac{N}{|\Lambda|} \to 0, \quad \Lambda \to \infty.
\]
Moreover, by the monotonicity of the Dirichlet eigenvalues, \( E_N(H_\omega(\Lambda,1)) \) decreases as \( \Lambda \to \infty \) and thus necessarily converges:
\[
E_N(H_\omega(\Lambda,1)) \searrow \tilde{E}, \quad \Lambda \to \infty.
\]
By combining the previous arguments, we find that
\[
N(\tilde{E}) \preceq \frac{N_\Lambda(\tilde{E})}{N_\Lambda(E_N(H_\omega(\Lambda,1)))} \to 0, \quad \Lambda \to \infty,
\]
which implies
\[
N(\tilde{E}) = 0.
\]
Consequently,
\[
\tilde{E} \leq \inf \Sigma.
\]
Finally, \( \tilde{E} < \inf \Sigma \) is impossible, because necessarily \( E_N(H_\omega(\Lambda,1)) \geq \inf \Sigma \). \( \square \)

The next lemma is a modification of Proposition 4.6 and expresses a subadditive-type condition that is even stronger than (4.3). It is exactly this lemma that is not valid for bosons and fermions.

**Lemma 5.4.** If \( W \equiv 0 \) and the particles under consideration are not restricted to any statistics, then
\[
E_\omega(\Lambda, n_1 + n_2, S_1 + S_2) \leq E_\omega(\Lambda, n_1, S_1) + E_\omega(\Lambda, n_2, S_2).
\]

Proof. The proof follows that of Proposition 4.6. The only modification is the construction of test functions in Lemma 4.1. When the interactions are absent, one can place two groups of particles in the same box and, consequently, one is not obliged to enlarge the size of a box together with the number of particles. \( \square \)

The last idea is not applicable to bosons or fermions, as one cannot guarantee the independence of constructed test functions and the orthogonality of terms in (4.3) or (4.4) is not assured. Moreover, a constructed test function may happen to be identically zero for fermions.

**Proof of Theorem 5.1.** Because of the subadditivity (5.3), the following limit exists:
\[
\exists \lim_{\substack{n \to \infty \\ \Lambda \text{ fixed}}} \frac{E_\omega(\Lambda, n, S)}{n} =: \zeta(\Lambda, \sigma)
\]
in the sense of \( L^1 \) and almost surely with respect to \( \omega \).

This is proved exactly in the same manner as the existence of \( \mathcal{E}(\rho, \sigma) \) (and in some aspect is even simpler). As Dirichlet eigenvalues are monotonic with respect to the domain, the function \( \zeta \) is nonincreasing in \( \Lambda \) and nondecreasing in \( \sigma \) (by obvious reasons). Due to the monotonicity in \( \Lambda \), we find also that for \( \rho > 0 \):
\[
\mathcal{E}(\rho, \sigma) \leq \lim_{\Lambda \to \infty} \zeta(\Lambda, \sigma).
\]

(5.4)
We remark as well that by (5.3) the function $E_\omega(\Lambda, n, \sigma n)/n$ is nonincreasing in $n$. So one can interchange the two limits in the r.h.s. of (5.4) to get

$$E(\rho, \sigma) \leqslant \lim_{\Lambda \to \infty} \lim_{S/n \to \sigma} \frac{E_\omega(\Lambda, n, S)}{n} = \lim_{S/n \to \sigma} \lim_{\Lambda \to \infty} \frac{E_\omega(\Lambda, n, S)}{n} \leqslant \lim_{S/n \to \sigma} \lim_{\Lambda \to \infty} E_\omega(\Lambda, 1, S/n).$$

Here (5.3) was used once more in the last inequality. Finally, by Lemma 5.3 we establish

$$\lim_{\Lambda \to \infty} E_\omega(\Lambda, 1, S/n) \to \inf \Sigma$$

with $S$ and $n$ being fixed.

Theorem 5.1 expresses the fact that the thermodynamic limit for Maxwell - Boltzmann particles is trivial in the absence of interactions. Thus, it is indeed the interactions that may possibly render the limit being nontrivial.

5.2. Bosons. We remark that the energy levels for a system of noninteracting bosons coincide with energies for Maxwell - Boltzmann particles (see, for example [LL77, Gre07]). On the contrary, for bosons the combinatorial degeneracy is lifted up by means of the symmetrization procedure (the degeneracy due to coincidences like $E_2 = E_3 = E_1 + E_4$ remains).

Nevertheless, the ground state energy for bosons is the same as in the previous section, and, consequently, the ground state energy per particle converges in the thermodynamic limit to the lower edge of the almost sure spectrum of the one-particle operator.

5.3. Fermions. The situation changes significantly if particles are fermions. For the basic properties of a system of noninteracting fermions, we refer the reader once more to [LL77, Gre07].

For fermions we know how only to obtain results on the ground state energy $E(\rho, 0)$. The arguments we use do not rely on subadditivity properties and, consequently, are valid for any boundary conditions.

The main difference between fermions and bosons is that the ground energy for $n$ noninteracting fermions is given by the sum of the first $n$ energies of a one-particle system

$$E_1(\Lambda, n) = \sum_{k=1}^{n} E_k(\Lambda, 1) \tag{5.5}$$

and not by $n$ times the one-particle ground energy. The ground state itself is given by the Slater determinant

$$\Omega_1(\Lambda, n) = \det (\psi_i(x_j))_{i,j},$$

where $\psi_i$ is the eigenfunction of $H_\omega(\Lambda, 1)$ corresponding to the energy $E_i(\Lambda, n, \omega)$.

A comparison with the Laplacian and the use of Weyl asymptotic provide a simple proof that the limit $E(\rho, 0)$ is strictly different from zero for $\rho > 0$ if the background potential is nonnegative.

**Proposition 5.5.** Suppose $V_\omega \geqslant 0$. Then there exists $\beta = \beta(d)$ such that

$$E(\rho, 0) \geqslant \beta \rho^{2/d} \tag{5.6}$$

**Proof.** By the variational principle, as the potential is positive, we obtain

$$E_j(\Lambda, 1) \geqslant E^0_j(\Lambda, 1),$$

where $E^0_j(\Lambda, 1)$ is the one-particle ground energy for the potential $V_\omega$. Then

$$E(\rho, 0) \geqslant \beta \rho^{2/d},$$

where $\beta = \inf \Sigma$. This completes the proof.
where $E^0_j$ are eigenvalues of $-\Delta$. Next, with the help of Weyl asymptotic for the Laplacian, by (5.5) we get
\[
\frac{E_1(\Lambda, n)}{n} = \frac{1}{n} \sum_{k=1}^{n} E_k(\Lambda, 1) \geq \frac{C_1}{n} \sum_{k=1}^{n} \left( \frac{k}{|\Lambda|} \right)^{2/d} \geq \frac{C_2}{n|\Lambda|^{2/d}} \cdot n^{2/d+1} \geq C_3 \rho^{2/d},
\]
which proves (5.6).

**Remark 5.6.** The generalization to the case of lower-bounded random potential is obvious.

We now compute an explicit expression for the limit
\[
\mathcal{E}(\rho, 0) = \lim_{\Lambda \to \infty} \frac{E_1(\Lambda, 1) + \ldots + E_n(\Lambda, n)}{n}
\]
in terms of the integrated density of states of the one-particle problem. Once more for simplicity, we suppose that background potential is nonnegative: $V_\omega \geq 0$. We need only rather general assumptions on the density of states, which we denote by $N(E)$.

**Condition 5.7.** The integrated density of states $N(E)$ is a continuous function and defines a positive measure $dN(E)$ such that the almost sure spectrum is equal to the support of $dN$:
\[
\text{supp } dN = \Sigma.
\]

**Remark 5.8.** The last condition is certainly verified, for example, if the Wegner estimate (W) holds for $H_\omega(\Lambda, 1)$.

**Proposition 5.9 (Wegner estimate).** Let $\Sigma$ be almost sure spectrum of $H_\omega$. There exists constant $C > 0$ such that, for any Borel subset $I \subset \mathbb{R}$,
\[
\mathbb{E} \left( \text{Tr} \left( \mathbf{1}_I(H_\omega(\Lambda)) \right) \right) \leq C |\Lambda| \cdot |I \cap \Sigma|.
\]

**Proof.** Wegner estimate is well known for both discrete and continuous Anderson model under the assumption that the random variables are i.i.d. and that their distribution is regular [CHK07, Ves08].

**Definition 5.10.** Fix a density of particles $\rho$. The Fermi energy $E_\rho$ is a solution of the equation
\[
N(E_\rho) = \rho. \quad (5.7)
\]

**Remark 5.11.** It may happen that $\sup_E N(E) < \rho$. For example, if one considers a discrete Anderson model, then $N(E) \leq 1$ and for $\rho > 1$ the equation (5.7) does not have any solutions. This is due to the fact that the density of particles is too big (in other words, there isn’t enough space for so many particles) to accommodate for $n$ fermions. This situation never arises in a continuous setting.

**Remark 5.12.** A solution of the equation (5.7) is not necessarily unique if the integrated density of states is flat on the level $\rho$. As $N(E)$ is a continuous nondecreasing function, the set of solutions is the closed interval $[E^\rho_{\text{min}}, E^\rho_{\text{max}}]$. From the spectral point of view the open interval $(E^\rho_{\text{min}}, E^\rho_{\text{max}})$ doesn’t play any role because its intersection with the almost sure spectrum $\Sigma$ is empty.

In this situation we will also use the notation introduced by the Definition 5.10, meaning $E_\rho = [E^\rho_{\text{min}}, E^\rho_{\text{max}}]$. As we will see, this convention is consistent with the results.

Next theorem is the main result of this section.
Theorem 5.13. Let $\rho > 0$. Then
\[ E(\rho, 0) = \frac{1}{\rho} \int_0^{E_\rho} EdN(E). \] (5.8)

To give a proof to this theorem, we will need the following crucial lemma, which explains why $E_\rho$ given by (5.7) corresponds exactly to the common physical notion of the Fermi energy.

Lemma 5.14.
\[ E_n(\Lambda, 1, \omega) \xrightarrow{\omega \to \infty} E_\rho. \] (5.9)

Proof. We denote by $N^\Lambda_\omega$ the density of states before taking the limit for a one-particle operator:
\[ N^\Lambda_\omega(E) = \frac{N(E, H_\omega(\Lambda, 1))}{|\Lambda|}. \]

Then by definition, in the thermodynamic limit
\[ N^\Lambda_\omega(E_n(\Lambda, 1, \omega)) = \frac{n}{|\Lambda|} \to \rho. \]

On the other hand, by the existence of the integrated density of states we get:
\[ N^\Lambda_\omega(\xi) \xrightarrow{\Lambda \to \infty} N(\xi) \quad \forall \xi \in \mathbb{R}. \]

We finish the proof by applying the monotonicity argument. Suppose that
\[ \liminf E_n(\Lambda, 1, \omega) < E^\text{min}_\rho. \]

By passing to a subsequence, we find that there exists $\delta > 0$ such that $E_n(\Lambda, 1, \omega) < E^\text{min}_\rho - \delta$. We arrive to a contradiction:
\[ N(E^\text{min}_\rho) \leq N^\Lambda_\omega(E_n(\Lambda, 1, \omega)) \leq N_\omega(\Lambda(E^\text{min}_\rho - \delta)) \to N(E^\text{min}_\rho - \delta) < N(E^\text{min}_\rho). \]

The last inequality is strict because $E^\text{min}_\rho$ is the minimal value of energy such that $N(E) = \rho$ and so for any $E$ above this level the density of states $N(E)$ is strictly smaller.

Similarly, we show that $\limsup E_n(\Lambda, 1, \omega) \leq E^\text{max}_\rho$, so
\[ E^\text{min}_\rho \leq \liminf E_n(\Lambda, 1, \omega) \leq \limsup E_n(\Lambda, 1, \omega) \leq E^\text{max}_\rho, \]
which is equivalent to (5.9). \hfill \square

Proof of Theorem 5.13. To show (5.8) we write
\[ \frac{E_1(\Lambda, n, \omega)}{n} = \frac{E_1(\Lambda, 1, \omega) + \ldots + E_n(\Lambda, 1, \omega)}{n} = \frac{1}{n} \text{Tr} \left[ H_\omega(\Lambda, 1) \cdot 1_{[0, E_n(\Lambda, 1, \omega)]}(H_\omega(\Lambda, 1)) \right] \]
\[ = \frac{|\Lambda|}{n} \int_0^{E_n(\Lambda, 1, \omega)} EdN^\Lambda_\omega(E) \xrightarrow{\omega \to \infty, n/|\Lambda| \to \rho} \frac{1}{\rho} \int_0^{E_\rho} EdN(E), \]
where the convergence is valid because the measure $dN^\Lambda_\omega$ converges weakly to $dN$, the integration limit converges to $E_\rho$ by Lemma 5.14 and the dominated convergence theorem can be applied. \hfill \square

Fermi energy is the energy of highest occupied quantum state in a system of fermions at absolute zero temperature. Alternatively, for non-interacting fermions, it is the increase in the ground state energy when one particle is added to the system. For more details on the concept on the Fermi energy, see [AM76].
Remark 5.15. The formula (5.8) admits an alternative form:
\[ \mathcal{E}(\rho, 0) = \int_0^{E_\rho} \frac{E dN(E)}{\int_0^{E_\rho} dN(E)}, \]
which reads as the ground state energy density is the energy averaged from zero to the Fermi energy with respect to the density of states.

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Laboratoire Analyse Géométrie et Applications, Université Paris 13 Nord, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

E-mail address: veniaminov@math.univ-paris13.fr