Free Vibration Analysis of Moderately Thick Orthotropic Functionally Graded Plates with General Boundary Restraints

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Abstract: In this paper, a modified Fourier series method is presented for the free vibration of moderately thick orthotropic functionally graded plates with general boundary restraints based on the first-order shear deformation theory. Regardless of boundary restraints, displacements and rotations of each plate are described as an improved form of double Fourier cosine series and several closed-form auxiliary functions to eliminate all the boundary discontinuities and jumps. Exact solutions are obtained by the energy functions of the plates based on Rayleigh-Ritz method. The convergence and reliability of the current method and the corresponding theoretical formulations are verified by comparing the present results with those available in the literature, and numerous new results for orthotropic functionally graded (OFG) plates with general boundary restraints are presented. In addition, the effects of gradient index, volume fraction and geometric parameters on frequencies with general boundary restraints are illustrated.

Keywords: orthotropic functionally graded plates; modified Fourier series method; free vibration; general boundary restraints; gradient index; volume fraction

1. Introduction

As a kind of novel composite materials, functionally graded materials (FGMs) can be characterized by the variation in composition and structure gradually over volume, resulting in continuous changes along the desired directions. Compared to laminated plates, the continuity of FGMs properties eliminates interfacial stresses at the junctions of materials [1–4]. Therefore, FG plates have been widely used in various engineering fields, such as aircraft, nuclear and automobile manufacturing [5–9]. As we all known, dynamic load is unavoidable on the practical applications, and it may lead to fatigue damage and stability reduction of the structures. Thus, it is necessary to study the vibration characteristics of FG plate structures.

To deal with the vibration problem of FG plates, many accurate and efficient calculation methods have been developed in the last few decades, such as extended Kantorovich method [10], Ritz method [11–13], power series method [14], meshless method [15–17], wave propagation approach [18], finite element method [19,20], etc. Chi et al. [21,22] studied the bending problem of FG rectangular plates based on the classical plate theory. Qian et al. [23] applied high-order shear and normal deformable plate theory to study the static and dynamic deformations of FG plates. Liu et al. [24] used an element-free Galerkin method to study the dynamic response of FG plate containing distributed piezoelectric actuators and sensors. It should be emphasized that most of these methods were applied to isotropic structures which are only considering the change of the Young’s modulus in the thickness. However, due to the limitations of the process conditions, most of the FGMs are orthotropic. The vibration analysis of the orthotropic functionally
graded (OFG) plates has been the goal of intensive research, and many studies have been devoted to the OFG plates in the literature. Ramirez et al. [25] used the discrete layer theory in combination with the Ritz method to obtain an approximate solution for static analysis of OFG plates. Zhang et al. [26] adapted the third-order shear deformation theory to analyze chaotic vibrations of an OFG plate. Huang et al. [27] developed a discrete method for solving the vibration problem of orthotropic rectangular plates with variable thickness in one or two directions. Although these methods give sufficiently accurate results for thin plates, they are not valid for the vibration analysis of the moderately thick plates.

To eliminate the deficiency of the aforementioned methods, a Fourier series method was presented by Li [28,29]. This method has been subsequently transferred to the vibration analysis of more structures with various restraints [30–37]. From the review of the literature, most of the previous studies on the OFG plates are defined in a single volume distribution type. However, there are a variety of possible volume distributions in practical engineering applications, and these distributions have great influence on the vibration properties of the OFG plates. According to the effects of gradient index on the volume fraction in different distributions, the material properties change continuously through the thickness of the OFG plates.

Therefore, the objective of the present work is to provide an accurate and reliable method for the free vibration analysis of moderately thick OFG plates in various volume distribution types with general restraints. Displacements and rotations of each plate, regardless of boundary restraints, are described as an improved form of double Fourier cosine series and several closed-form auxiliary functions. Exact solutions are obtained by the energy functions of the plates based on Rayleigh–Ritz method. The excellent accuracy and reliability of the current results are verified by comparing the present solutions with those available in the literature. Studies focused on free vibration properties of OFG plates are presented, which may serve as a supplement of the material performance of OFG plates.

2. Theoretical Formulation

2.1. Model Description

A moderately thick OFG plate with length $a$, width $b$, and uniform thickness $h$ is depicted in Figure 1. The reference surface is taken to be its middle surface where the plate geometry and dimensions are arranged in a Cartesian coordinate system $(x, y, z)$. The displacements of the plate in the $x$, $y$, and $z$ directions are denoted by $u$, $v$, and $w$, respectively. The general boundary conditions are assumed to be restrained by three independent springs (translational, rotational and torsional springs) placed at the ends. Assigning the stiffness of the springs with various values from zero to infinity is equivalent to imposing different boundary forces on the plate. For example, a free boundary is obtained by setting the stiffness of springs to zero, and a clamped boundary is obtained by setting the stiffness of springs to infinity. For moderately thick plates, the Kirchhoff hypothesis is relaxed by assuming that the normal to the undeformed middle surface is not perpendicular to the deformed middle surface.

![Figure 1. Schematic diagram of a moderately thick OFG plate with the undeformed and deformed geometries of an edge including shear deformation.](image)
2.2. Material Properties

Typically, OFG plates are made from a mixture of two materials in different proportions, for example, the metal and ceramic used in the following analyses are listed in Table 1. Material parameters per unit volume are assumed to vary continuously through the plate thickness and can be obtained:

\[
\begin{align*}
E_1 &= E_f V_f + E_m V_m \\
E_2 &= \frac{E_f V_f + E_m V_m}{1 - V_f V_f + V_m V_m} \\
\rho &= \rho_f V_f + \rho_m V_m \\
\nu_{12} &= \nu_f V_f + \nu_m V_m
\end{align*}
\]

where \(E_1\) and \(E_2\) represent the horizontal and vertical Young’s modulus, respectively; \(\nu_{12}\) and \(\rho\) are the major Poisson ratio and density, respectively; \(E_f\) and \(E_m\) are the Young’s modulus of ceramic and metal, respectively; \(\nu_f\) and \(\rho_f\) are ceramic’s Poisson ratio and density, respectively; \(V_m\) and \(\rho_m\) are metal’s Poisson ratio and density, respectively; and \(V_f\) and \(V_m\) denote the volume fractions of ceramic and metal, respectively. The shear modulus of the material can be given by:

\[
\begin{align*}
G_f &= \frac{E_f}{2(1 + \nu_f)}, \\
G_m &= \frac{E_m}{2(1 + \nu_m)} \\
G_{12} &= G_{13} = G_{23} = \frac{G_f G_m}{G_f V_f + G_m V_m}
\end{align*}
\]

where \(G_f\) and \(G_m\) are the shear modulus of ceramic and metal. \(G_{12}\) is composite structure’s shear modulus. Furthermore, according to different ceramic-to-metal volume distributions in the thickness direction, the OFG plates are assumed as three types, P-, C- and S-OFG, respectively. \(V_f\) in the thickness direction \(z\) can be expressed as:

\[
\begin{align*}
P: \quad V_f &= V_1 + (V_2 - V_1) \left( \frac{1}{2} + \frac{z}{h} \right)^p \\
C: \quad V_f &= V_1 + (V_2 - V_1) \left( \frac{1}{2} + \frac{z}{h} + \left( \frac{1}{2} - \frac{z}{h} \right)^2 \right)^p \\
S: \quad V_f &= \begin{cases} \\
V_1 \left( \frac{1}{2} - \frac{2z}{h} \right)^p + V_2 \left( 1 - \frac{1}{2} \left( 1 - \frac{2z}{h} \right)^p \right), & 0 \leq z \leq \frac{h}{2} \\
V_1 \left( 1 - \frac{1}{2} \left( 1 + \frac{2z}{h} \right)^p \right) + V_2 \left( 1 + \frac{2z}{h} \right)^p, & -\frac{h}{2} \leq z \leq 0
\end{cases}
\end{align*}
\]

where \(V_1\) and \(V_2\) are available minimum and maximum values of \(V_f\). Especially for P- and C-OFG, \(V_1\) and \(V_2\) represent the ceramic volume fractions of bottom and top surfaces, respectively. In addition, \(p\) is the gradient index and only takes non-negative values. When the value of \(p\) varies between zero and infinity, non-homogeneous material properties can be obtained.

Table 1. Main material properties of the used OFG plates.

| Properties | \(E_m\) (GPa) | \(\rho_m\) (kg/m\(^3\)) | \(E_f\) (GPa) | \(\rho_f\) (kg/m\(^3\)) |
|------------|---------------|----------------------------|---------------|--------------------------|
| Metal (Al) | 70            | 0.3                        | 200           | 0.3                      |
| Ceramic (ZrO\(_2\)) | 2702        |                            | 5700          |                          |
2.3. Stress–Strain Relations and Stress Resultants

Based on the assumptions of the first-order shear deformation theory (FSDT) [38,39], the displacement components of moderately thick OFG plates are:

\[
\begin{align*}
U(x, y, z) &= u(x, y) + z\phi_x(x, y) \\
V(x, y, z) &= v(x, y) + z\phi_y(x, y) \\
W(x, y, z) &= w(x, y)
\end{align*}
\]

where \(u, v\) and \(w\) denote the middle surface displacements of the plate in the \(x\), \(y\) and \(z\) directions, respectively. \(\phi_x\) and \(\phi_y\) represent the transverse normal rotations of the reference surface respect to the \(y\) and \(x\) directions. Under the assumption of linear and small deformation, the strains and curvature can be defined in terms of displacements as:

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + z\frac{\partial w}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} + z\frac{\partial \phi_x}{\partial x} \\
\gamma_{xz} &= \frac{\partial w}{\partial z} \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{align*}
\]

where \(\varepsilon_x, \varepsilon_y\) and \(\gamma_{xy}\) are the normal and shear strains in the \(x\), \(y\) and \(z\) directions. \(\gamma_{xz}\) and \(\gamma_{yz}\) indicate the transverse shear strains, which are assumed to be constants through the thickness. The matrix can be denoted as:

\[
\varepsilon = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \end{bmatrix}^T
\]

\[
X = \begin{bmatrix} \frac{\partial \phi_x}{\partial x} & \frac{\partial \phi_y}{\partial y} & \frac{\partial \phi_x}{\partial y} & \frac{\partial \phi_y}{\partial x} \end{bmatrix}^T
\]

\[
\gamma = \begin{bmatrix} \frac{\partial w}{\partial z} & \frac{\partial v}{\partial x} & \frac{\partial \phi_y}{\partial x} & \frac{\partial \phi_y}{\partial y} & \frac{\partial \phi_x}{\partial x} \end{bmatrix}^T.
\]

According to the generalized Hooke’s law [40], the corresponding stress–strain relations of a moderately thick OFG plate can be expressed as follows:

\[
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11}(z) & Q_{12}(z) & 0 & 0 & 0 \\ Q_{12}(z) & Q_{22}(z) & 0 & 0 & 0 \\ 0 & 0 & Q_{44}(z) & 0 & 0 \\ 0 & 0 & 0 & Q_{55}(z) & 0 \\ 0 & 0 & 0 & 0 & Q_{66}(z) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xz} \\ \gamma_{yz} \\ \gamma_{xy} \end{bmatrix}
\]

where the elastic constants \(Q_{ij}(z)\) are defined in terms of the material properties as:

\[
\begin{align*}
Q_{11}(z) &= \frac{E_1(z)}{1 - \nu_{12} \nu_{21}} \\
Q_{12}(z) &= \frac{E_2(z)}{1 - \nu_{12} \nu_{21}} \\
Q_{22}(z) &= \frac{E_1(z) \nu_{21}}{1 - \nu_{12} \nu_{21}} \\
Q_{44}(z) &= G_{23}(z) \\
Q_{55}(z) &= G_{13}(z) \\
Q_{66}(z) &= G_{12}(z)
\end{align*}
\]
The force and moment resultants are obtained by integrating the stresses over the plate thickness:

\[
(N_x, N_y, N_{xy})^T = \int_{-h/2}^{h/2} [\sigma_x, \sigma_y, \tau_{xy}] dz
\]

(10)

\[
(M_x, M_y, M_{xy})^T = \int_{-h/2}^{h/2} [\sigma_x, \sigma_y, \tau_{xy}] zdz
\]

(11)

\[
(Q_x, Q_y)^T = \int_{-h/2}^{h/2} [\tau_{xz}, \tau_{yz}] dz
\]

(12)

where \(N_x, N_y\) and \(N_{xy}\) are the force resultants. \(M_x, M_y\) and \(M_{xy}\) are the moment resultants. The transverse shear force resultants are denoted as \(Q_x\) and \(Q_y\), respectively. Performing the integration operation in Equations (10)–(12), the force and moment resultants can be written as:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{bmatrix} \varepsilon + \begin{bmatrix}
B_{11} & B_{12} & 0 \\
B_{12} & B_{22} & 0 \\
0 & 0 & B_{66}
\end{bmatrix} \chi
\]

(13)

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \begin{bmatrix}
B_{11} & B_{12} & 0 \\
B_{12} & B_{22} & 0 \\
0 & 0 & B_{66}
\end{bmatrix} \varepsilon + \begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{66}
\end{bmatrix} \chi
\]

(14)

\[
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix}
= \begin{bmatrix}
A_{44} & 0 \\
0 & A_{55}
\end{bmatrix} \gamma
\]

(15)

The stiffness coefficients \(A_{ij}, B_{ij}\) and \(D_{ij}\) are expressed as:

\[
(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(z)(1, z, z^2) dz
\]

(16)

2.4. Energy Functions

In this subsection, the modified Fourier series version of Rayleigh–Ritz method is presented. In the Rayleigh–Ritz method, a displacement field associated with undetermined coefficients is assumed firstly, and substituted into the Lagrangian energy function [41]. Then, the undetermined coefficients in the displacement field can be obtained by finding the stationary value of the energy function, namely, minimizing the energy function with respect to the undetermined coefficients and making them equal to zero. Finally, a series of equations related to corresponding coefficients can be achieved and summed up in matrix form as a standard characteristic equation. The desired frequencies of the structure can be determined easily by solving the standard characteristic equation.

For free vibration analysis, the Lagrangian energy function of the plates can be simplified and written in terms of the strain energy and kinetic energy functions as:

\[
L = T - U_s - U_{sp}
\]

(17)
The strain energy $U_s$ of the moderately thick OFG plates during vibration can be defined in terms of displacements ($u, v, w$), curvature changes and stress resultants as:

$$U_s = \frac{1}{2} \int_{0}^{a} \int_{0}^{b} \left\{ N_{x}e_{x}^{0} + N_{y}e_{y}^{0} + N_{xy}e_{xy}^{0} + M_{x}\chi_{x} + M_{y}\chi_{y} + M_{xy}\chi_{xy} + Q_{x}\gamma_{x}^{0} + Q_{y}\gamma_{y}^{0} \right\} dydx \quad (18)$$

Substituting Equations (5), (6), and (13)–(15) into Equation (18), the strain energy can be expressed in terms of displacements ($u, v, w$) and rotations components ($\phi_x, \phi_y$) as:

$$U_s = \frac{1}{2} \int_{0}^{a} \int_{0}^{b} \left\{ A_{11} \left( \frac{\partial u}{\partial x} \right)^2 + A_{22} \left( \frac{\partial v}{\partial y} \right)^2 + A_{44} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + A_{55} \left( \frac{\partial w}{\partial x} \right)^2 + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \right)^2 \right\} dydx \quad (19)$$

The kinetic energy $T$ of the vibrating OFG plate is given by:

$$T = \frac{1}{2} \int_{0}^{a} \int_{0}^{b} \left\{ I_{0} \left( \frac{\partial u}{\partial t} \right)^2 + I_{0} \left( \frac{\partial v}{\partial t} \right)^2 + I_{0} \left( \frac{\partial w}{\partial t} \right)^2 + 2I_{1} \left( \frac{\partial u}{\partial t} \right) \left( \frac{\partial \phi_x}{\partial t} \right) + 2I_{1} \left( \frac{\partial v}{\partial t} \right) \left( \frac{\partial \phi_y}{\partial t} \right) \right\} dydx \quad (20)$$

Assuming the distributed external forces $q_x, q_y$ and $q_z$ are in the $x, y$ and $z$ directions, respectively. $m_x$ and $m_y$ are the external couples in the middle surface. Thus, the work $W_e$ done by the forces and moments is:

$$W_e = \int_{0}^{a} \int_{0}^{b} \left\{ q_xu + q_yv + q_zw + m_x\phi_x + m_y\phi_y \right\} dydx \quad (21)$$

$k_u, k_v, k_w, K_{\phi x}, K_{\phi y}$ and $K_{\phi z}$ ($q = x_0, x_1, y_0, y_1$) are used to indicate the rigidities (per unit length) of the boundary springs at the $x = 0, x = a, y = 0$ and $y = b$, respectively (see Figure 2). Therefore, the deformation strain energy ($U_{de}$) stored in the boundary springs can be expressed as:

$$U_{de} = \int_{0}^{a} \int_{0}^{b} \left\{ \left[ k_u^0 u^2 + k_v^0 v^2 + k_w^0 w^2 + K_{\phi x}^0 \phi_x^2 + K_{\phi y}^0 \phi_y^2 \right]_{y = 0} + \left[ k_u^0 u^2 + k_v^0 v^2 + k_w^0 w^2 + K_{\phi x}^0 \phi_x^2 + K_{\phi y}^0 \phi_y^2 \right]_{y = 0} \right\} dx + \int_{0}^{a} \int_{0}^{b} \left\{ \left[ k_u^0 u^2 + k_v^0 v^2 + k_w^0 w^2 + K_{\phi x}^0 \phi_x^2 + K_{\phi y}^0 \phi_y^2 \right]_{x = a} + \left[ k_u^0 u^2 + k_v^0 v^2 + k_w^0 w^2 + K_{\phi x}^0 \phi_x^2 + K_{\phi y}^0 \phi_y^2 \right]_{x = a} \right\} dy \quad (22)$$

**Figure 2.** Boundary restraints of a moderately thick OFG plate.
2.5. Governing Equations and Boundary Restraints

By applying Hamilton’s principle, the governing equations of moderately thick OFG plates can be obtained:

\[
\frac{\partial N_{x}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_{x} = I_{0} \frac{\partial^{2} u}{\partial t^{2}} + I_{1} \frac{\partial^{2} \phi_{x}}{\partial t^{2}}
\]

\[
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{y}}{\partial y} + q_{y} = I_{0} \frac{\partial^{2} v}{\partial t^{2}} + I_{1} \frac{\partial^{2} \phi_{y}}{\partial t^{2}}
\]

\[
\frac{\partial Q_{x}}{\partial x} + \frac{\partial Q_{y}}{\partial y} + q_{z} = I_{0} \frac{\partial^{2} w}{\partial t^{2}}
\]

\[
\frac{\partial M_{x}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{x} + m_{x} = I_{1} \frac{\partial^{2} u}{\partial t^{2}} + I_{2} \frac{\partial^{2} \phi_{x}}{\partial t^{2}}
\]

\[
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{y}}{\partial y} - Q_{y} + m_{y} = I_{1} \frac{\partial^{2} v}{\partial t^{2}} + I_{2} \frac{\partial^{2} \phi_{y}}{\partial t^{2}}
\]

The general boundary restraints for moderately thick OFG plates can be expressed as the following forms:

On \( x = 0 \)

\[
N_{x} = k_{x0}^{u} u, \quad N_{xy} = k_{x0}^{v} v, \quad M_{x} = K_{x0}^{x} \phi_{x}, \quad M_{xy} = K_{x0}^{y} \phi_{y}, \quad Q_{x} = k_{x0}^{w} w
\]

On \( x = a \)

\[
N_{x} = -k_{x1}^{u} u, \quad N_{xy} = -k_{x1}^{v} v, \quad M_{x} = -K_{x1}^{x} \phi_{x}, \quad M_{xy} = K_{x1}^{y} \phi_{y}, \quad Q_{x} = -k_{x1}^{w} w
\]

On \( y = 0 \)

\[
N_{xy} = k_{y0}^{u} u, \quad N_{y} = k_{y0}^{v} v, \quad M_{xy} = K_{y0}^{x} \phi_{x}, \quad M_{y} = K_{y0}^{y} \phi_{y}, \quad Q_{y} = k_{y0}^{w} w
\]

On \( y = b \)

\[
N_{xy} = -k_{y1}^{u} u, \quad N_{y} = -k_{y1}^{v} v, \quad M_{xy} = -K_{y1}^{x} \phi_{x}, \quad M_{y} = -K_{y1}^{y} \phi_{y}, \quad Q_{y} = -k_{y1}^{w} w
\]

2.6. Admissible Displacement Functions

In this subsection, we consider free vibration of moderately thick OFG plates with general boundary restraints. Although the Fourier functions exhibit an excellent numerical stability, conventional Fourier series expression will have a convergence problem along the boundary edges except for a few simple boundary restraints.

This article proposes a modified Fourier series method for the displacement and rotation components of the OFG plates, by an improved form of double Fourier cosine series and several closed-form auxiliary functions. Regardless of boundary conditions, each displacement and rotation component of the OFG plate is expanded as a modified Fourier series as:

\[
u(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} A_{mn} \cos \lambda_{m} x \cos \lambda_{n} y + \sum_{l=1}^{2} \sum_{n=0}^{N} b_{l}^{n} \zeta_{l}^{n}(x) \cos \lambda_{n} y + \sum_{l=1}^{2} \sum_{m=0}^{M} b_{l}^{m} \zeta_{l}^{m}(y) \cos \lambda_{m} x
\]

\[
w(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} B_{mn} \cos \lambda_{m} x \cos \lambda_{n} y + \sum_{l=1}^{2} \sum_{n=0}^{N} c_{l}^{n} \zeta_{l}^{n}(x) \cos \lambda_{n} y + \sum_{l=1}^{2} \sum_{m=0}^{M} d_{l}^{m} \zeta_{l}^{m}(y) \cos \lambda_{m} x
\]

\[
\phi_{x}(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} C_{mn} \cos \lambda_{m} x \cos \lambda_{n} y + \sum_{l=1}^{2} \sum_{n=0}^{N} e_{l}^{n} \zeta_{l}^{n}(x) \cos \lambda_{n} y + \sum_{l=1}^{2} \sum_{m=0}^{M} f_{l}^{m} \zeta_{l}^{m}(y) \cos \lambda_{m} x
\]

\[
\phi_{y}(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} D_{mn} \cos \lambda_{m} x \cos \lambda_{n} y + \sum_{l=1}^{2} \sum_{n=0}^{N} g_{l}^{n} \zeta_{l}^{n}(x) \cos \lambda_{n} y + \sum_{l=1}^{2} \sum_{m=0}^{M} h_{l}^{m} \zeta_{l}^{m}(y) \cos \lambda_{m} x
\]
where $\lambda_m = m\pi/a$ and $\lambda_n = n\pi/b$. $M$ and $N$ denote the truncation numbers with respect to variables $x$ and $y$, respectively. $A_{mn}$, $B_{mn}$, $C_{mn}$, $D_{mn}$, and $E_{mn}$ are the Fourier expansion coefficients of the cosine Fourier series. $a_m^n$, $b_m^n$, $c_m^n$, $d_m^n$, $e_m^n$, $f_m^n$, $g_m^n$, $h_m^n$, $i_m^n$ and $j_m^n$ are the corresponding supplement coefficients. $\zeta^a_m(x)$ and $\zeta^b_m(y)$ denote the auxiliary polynomial functions introduced to remove all the discontinuities potentially associated with the first-order derivatives at the boundaries. The auxiliary functions are expressed as follows:

$$
\zeta^a_1(x) = x\left(\frac{x}{a} - 1\right)^2, \quad \zeta^b_1(x) = \frac{x^2}{a^2}\left(\frac{x}{a} - 1\right) \tag{37}
$$

$$
\zeta^a_1(y) = y\left(\frac{y}{b} - 1\right)^2, \quad \zeta^b_1(y) = \frac{y^2}{b^2}\left(\frac{y}{b} - 1\right)
$$

It is easy to verify that,

$$
\begin{align*}
\zeta^a_1(0) &= \zeta^a_1(a) = \zeta^a_1(b) = 0, & \zeta^b_1(0) &= 1 \\
\zeta^a_2(0) &= \zeta^a_2(a) = \zeta^a_2(b) = 0, & \zeta^b_2(0) &= 1 \\
\zeta^b_1(0) &= \zeta^b_1(a) = \zeta^b_1(b) = 0, & \zeta^a_1(0) &= 1 \\
\zeta^b_2(0) &= \zeta^b_2(a) = \zeta^b_2(b) = 0, & \zeta^a_2(0) &= 1
\end{align*} \tag{38}
$$

All the expansion coefficients in Equation (25) can be treated independently and equally as the generalized coordinates and solved directly from the Ritz method. The method can be summed up in a matrix form as:

$$
(K - \omega^2 M)G = 0 \tag{39}
$$

where $K$ and $M$ is the stiffness matrix and mass matrix of the OFG plate, respectively. Both are symmetric matrices and can be written as:

$$
K = \begin{bmatrix}
K_{uu} & K_{uv} & K_{uw} & K_{ux} & K_{uy} \\
K_{vu} & K_{vv} & K_{vw} & K_{vx} & K_{vy} \\
K_{wu} & K_{wv} & K_{ww} & K_{wx} & K_{wy} \\
K_{ux} & K_{vx} & K_{wx} & K_{ux} & K_{uy} \\
K_{uy} & K_{vy} & K_{wy} & K_{uy} & K_{vy} \\
\end{bmatrix} \tag{40}
$$

$$
M = \begin{bmatrix}
M_{uu} & 0 & 0 & M_{ux} & M_{uy} \\
0 & M_{uu} & 0 & 0 & M_{uy} \\
0 & 0 & M_{ww} & 0 & 0 \\
M_{ux} & 0 & 0 & M_{ux} & M_{uy} \\
0 & M_{uy} & 0 & 0 & M_{uy} \\
\end{bmatrix} \tag{41}
$$

The coefficient eigenvector $G$ is the unknown expansion coefficient in the series expansions, and determined for a given frequency, namely:

$$
G = [G^u, G^v, G^w, G^\phi, G^\psi]^T \tag{42}
$$

The Fourier coefficient eigenvector $G$, stiffness matrix $K$ and mass matrix $M$ are given in the Appendix A.

3. Numerical Results and Discussion

In this section, numerical examples for the free vibration analysis of moderately thick OFG plates with various gradient indexes and general boundary restraints are presented. Firstly, the convergence and reliability of the proposed modified Fourier series method is validated by comparing the current solutions with those results published in the literature under the distribution of P-OFG. Secondly, the free
vibration behavior of OFG plates with general boundary restraints is studied. Then, the effects of gradient index \( p \) on the volume fraction in the thickness direction, in various distributions (P-, C- and S-OFG), are discussed. Finally, the relations between fundamental frequencies \( f \) (Hz) and gradient index \( p \) in the various volume fractions with general boundary restraints are contrasted and analyzed as well.

### 3.1. OFG Plates with General Boundary Restraints

The aforementioned general boundary restraints can be readily realized by assigning the stiffness of the boundary springs at proper values. Taking edge \( x = 0 \) as an example, the frequently encountered boundary restraints F (free edge), C (clamped edge) and S (simply-supported edge) can be defined as follows:

\[
F: \quad k_{u,0}^{x} = k_{v,0}^{x} = k_{w,0}^{x} = K_{x,0}^{y} = K_{y,0}^{x} = 0
\]

\[
C: \quad k_{u,0}^{x} = k_{v,0}^{x} = k_{w,0}^{x} = K_{x,0}^{y} = K_{y,0}^{x} = 10^7 D
\]

\[
S: \quad k_{u,0}^{x} = k_{v,0}^{x} = k_{w,0}^{x} = K_{y,0}^{x} = 10^7 D, \quad K_{x,0}^{y} = 0 \tag{43}
\]

where \( D = E_1 h^3 / 12(1 - \nu_1^2) \) is the flexural stiffness of the plate. The accuracy and convergence of the present solution is demonstrated in Tables 2 and 3. Table 2 compares the first five frequency parameters \( \Omega = \omega a^2 \sqrt{\rho h / D} \) of OFG plates with CCCC, SSSS and FFFF boundary restraints and thickness–length ratio \( a/b = 0.5 \). Three different thickness–length ratios, \( h/a = 0.1, 0.2 \) and 0.3, corresponding to the moderately thick plates, are considered in the comparison. The solution given by Jin et al. [35] by the three-dimensional elasticity method is provided for a direct comparison. The difference does not exceed 0.044% for the worst case, which is acceptable.

| \( h/a \) | Method | Mode Number |
|----------|--------|-------------|
|          | Present | 1 2 3 4 5   |
| CCCC     |        |             |
| 0.1      | Present | 12.767 13.242 14.454 16.649 19.937 |
|          | Ref. [35] | 12.767 13.243 14.451 16.647 19.938 |
| 0.2      | Present | 7.5324 8.0879 9.3818 10.206 11.430 |
|          | Ref. [35] | 7.5325 8.0882 9.3822 10.210 11.435 |
| 0.3      | Present | 5.2981 5.8807 6.8086 7.0848 8.7976 |
|          | Ref. [35] | 5.2982 5.8807 6.8086 7.0848 8.7975 |
| SSSS     |        |             |
| 0.1      | Present | 8.2283 8.3304 8.8058 10.181 12.615 |
|          | Ref. [35] | 8.2286 8.3304 8.8058 10.182 12.616 |
| 0.2      | Present | 4.1653 6.0783 6.5920 7.8472 8.3304 |
|          | Ref. [35] | 4.1652 6.0783 6.5922 7.8472 8.3304 |
| 0.3      | Present | 2.7768 4.6194 5.1094 5.5535 5.5535 |
|          | Ref. [35] | 2.7768 4.6197 5.1096 5.5536 5.5536 |
| FFFF     |        |             |
| 0.1      | Present | 1.2016 1.6451 3.2673 3.5279 5.8824 |
|          | Ref. [35] | 1.2016 1.6450 3.2673 3.5278 5.8822 |
| 0.2      | Present | 1.1741 1.5491 3.0822 3.2398 3.9205 |
|          | Ref. [35] | 1.1742 1.5490 3.0822 3.2398 3.9204 |
| 0.3      | Present | 1.1337 1.4422 2.6131 2.8420 2.9385 |
|          | Ref. [35] | 1.1337 1.4422 2.6132 2.8421 2.9387 |

The non-dimensional frequency parameters \( \bar{\omega} = \omega h \sqrt{\rho_m / E_m} \) for square plates with CCCC, SSSS, CFCE, and SCSC boundary restraints are shown in Table 3. The thickness–length ratio used for the analysis is \( h/a = 0.2 \) and the truncation number is \( M = N = 9 \) and 11. It can be seen that the present solutions are in close agreement with the results obtained from TSDT method [15]. In general, a
consistent agreement of the present results is seen from the tables by comparing with those available in the literature.

Table 3. The comparison of non-dimensional frequency parameters $\omega = \omega h \sqrt{\rho m / E m}$ for square plates with CCCC, SSSS, CFCC and SCSC boundary restraints, respectively ($h/a = 0.2$).

| Table 3 | The comparison of non-dimensional frequency parameters $\omega = \omega h \sqrt{\rho m / E m}$ for square plates with CCCC, SSSS, CFCC and SCSC boundary restraints, respectively ($h/a = 0.2$). |
|---------|----------------------------------------------------------------------------------|

Numerous new results of fundamental frequencies $f$ (Hz) are presented in Tables 4 and 5 for moderately thick OFG plates with a variety of general boundary restraints. In the case of Tables 4 and 5, the gradient indexes and geometrical parameters of the OFG plates are taken to be $p = 0.5, 2,$ and 10, and $a/b = 1.5$ and $h/a = 0.2, 0.3,$ and 0.5. The boundary restraints, including SSSS, SSCF, SSSF, CFCC, CFCS, CFCC, CFSS and CFSC, are considered. It can be seen from the tables that the solutions of the fundamental frequencies $f$ (Hz) corresponding to different boundary restraints have obvious difference. The frequencies of the moderately thick OFG plates with SSSS, SSCF and SSSF are significantly lower the other restraints, this is due to that the smaller restraints at the edges decrease the flexural rigidity of the plate, resulting in smaller frequency response. The frequencies of the OFG plates decrease as thickness–length ratio ($h/a$) and length–width ratio ($a/b$) increase. The first four mode shapes with SSSS, SSCF, SSSF, CFCC, CFCS and CFCC boundary restraints are depicted in Figure 3 to further enrich the vibration results of OFG plates. The gradient index and geometrical parameters are set as $p = 1, a/b = 1.5,$ and $h/a = 0.3.$ Next, the effects of gradient index and volume fraction on frequencies in various volume distributions with general boundary restraints are illustrated.

Table 4. The first four frequencies $f$ (Hz) of moderately thick OFG plates with various boundary restraints, gradient indexes and thickness–length ratios ($a/b = 1.5$).
| P   | $h/a$ | Mode | Boundary Restraints |
|-----|------|------|---------------------|
|     |      |      | SSSS | SSSC | SSSF | CFCC | CFCS | CFCF | CFSS | CFSF |
| 0.5 | 0.3  |      | 73.82 | 73.08 | 73.436 | 89.413 | 93.036 | 94.231 | 89.782 | 94.232 |
|     | 0.5  |      | 103.90 | 103.90 | 103.31 | 140.17 | 131.40 | 131.22 | 138.48 | 131.22 |
|     | 0.5  |      | 148.69 | 150.15 | 149.87 | 157.01 | 188.64 | 189.83 | 158.70 | 189.83 |
|     | 0.5  |      | 158.85 | 158.85 | 158.81 | 200.97 | 201.05 | 194.98 | 201.41 | 201.98 |
| 0.5 | 0.5  |      | 22.872 | 22.871 | 23.486 | 25.853 | 28.699 | 29.260 | 26.981 | 38.988 |
|     | 0.5  |      | 36.633 | 36.056 | 35.493 | 50.398 | 46.378 | 45.522 | 51.837 | 45.508 |
|     | 0.5  |      | 57.082 | 57.082 | 56.925 | 57.403 | 71.642 | 71.353 | 57.263 | 71.345 |
|     | 0.5  |      | 60.526 | 60.526 | 60.548 | 75.468 | 76.637 | 75.184 | 77.240 |
| 0.5 | 0.2  |      | 139.82 | 138.54 | 137.19 | 175.35 | 173.10 | 173.92 | 177.47 | 173.92 |
|     | 0.2  |      | 176.40 | 176.40 | 175.66 | 218.98 | 224.11 | 222.62 | 219.83 | 222.62 |
|     | 0.2  |      | 219.76 | 220.94 | 219.01 | 289.21 | 277.88 | 275.82 | 289.21 | 279.37 |
|     | 0.2  |      | 235.79 | 235.79 | 235.47 | 289.58 | 299.38 | 300.53 | 289.34 | 288.57 |
| 0.5 | 0.3  |      | 65.998 | 65.998 | 65.164 | 62.771 | 83.441 | 83.373 | 84.466 | 83.809 |
|     | 0.3  |      | 89.699 | 89.699 | 89.101 | 112.91 | 113.12 | 113.57 | 112.24 | 113.57 |
|     | 0.3  |      | 117.20 | 117.20 | 117.80 | 156.24 | 156.17 | 146.97 | 152.31 | 146.97 |
|     | 0.3  |      | 126.79 | 126.79 | 128.58 | 157.38 | 160.19 | 156.18 | 156.24 | 160.19 |
| 0.5 | 0.5  |      | 19.385 | 19.385 | 19.526 | 25.401 | 24.863 | 24.626 | 25.130 | 24.603 |
|     | 0.5  |      | 29.511 | 29.510 | 29.624 | 37.273 | 36.784 | 36.578 | 37.560 | 36.582 |
|     | 0.5  |      | 41.343 | 41.343 | 41.060 | 54.783 | 52.479 | 52.129 | 54.783 | 52.125 |
|     | 0.5  |      | 47.861 | 45.759 | 45.111 | 65.442 | 57.663 | 57.311 | 61.643 | 61.643 |
| 0.5 | 0.2  |      | 126.24 | 126.24 | 124.07 | 170.09 | 159.65 | 159.72 | 172.19 | 159.72 |
|     | 0.2  |      | 155.05 | 155.70 | 155.24 | 195.56 | 197.08 | 197.04 | 195.59 | 197.09 |
|     | 0.2  |      | 181.92 | 181.92 | 181.99 | 234.65 | 230.57 | 230.56 | 234.50 | 245.77 |
|     | 0.2  |      | 213.04 | 195.00 | 196.87 | 287.83 | 248.14 | 247.57 | 285.93 | 268.55 |
| 0.5 | 0.3  |      | 59.038 | 59.039 | 59.980 | 81.646 | 72.792 | 74.173 | 80.591 | 74.541 |
|     | 0.3  |      | 76.164 | 76.052 | 77.799 | 99.184 | 96.254 | 96.253 | 97.498 | 95.994 |
|     | 0.3  |      | 91.501 | 91.798 | 91.059 | 122.82 | 116.05 | 115.18 | 121.14 | 114.86 |
|     | 0.3  |      | 100.93 | 92.095 | 92.095 | 153.18 | 124.46 | 127.23 | 152.02 | 127.10 |
| 0.5 | 0.5  |      | 16.608 | 16.608 | 16.210 | 21.258 | 20.341 | 20.629 | 24.613 | 20.501 |
|     | 0.5  |      | 24.392 | 23.587 | 23.378 | 30.328 | 29.454 | 29.473 | 31.454 | 29.251 |
|     | 0.5  |      | 28.855 | 28.855 | 28.908 | 42.315 | 36.206 | 36.074 | 41.992 | 36.187 |
|     | 0.5  |      | 33.175 | 33.175 | 33.255 | 54.897 | 41.721 | 42.218 | 54.944 | 42.077 |

Table 5. The first four frequencies $f$ (Hz) of moderately thick OFG plates with various boundary restraints, gradient indexes and thickness–length ratios ($a/b = 2$).
### Table 5. Cont.

| P  | h/a  | Mode | Boundary Restraints |
|----|------|------|----------------------|
|    |      |      | SSSS | SSSC | SSSF | CFCC | CFCS | CFCF | CFSS | CFSSF |
| 0.2| 1    | 121.99 | 121.78 | 121.99 | 164.35 | 153.67 | 153.22 | 164.34 | 153.22 |
| 0.3| 2    | 139.33 | 139.32 | 138.89 | 179.97 | 177.43 | 175.93 | 180.63 | 175.97 |
| 0.5| 3    | 163.88 | 163.88 | 164.16 | 203.80 | 207.55 | 208.46 | 204.18 | 208.41 |
| 10 | 4    | 176.79 | 177.01 | 177.53 | 234.65 | 222.20 | 223.76 | 235.40 | 224.36 |

**Figure 3.** The first four mode shapes of the moderately thick OFG plates with various boundary restraints: (a) SSSS; (b) SSSF; (c) SSSC; (d) CFCC; (e) CFCS; and (f) CFCF.

### 3.2. Volume Fraction Analysis

In this study, the material properties are assumed as three types (P-OFG, C-OFG, and S-OFG), which are realized by different ceramic-to-metal volume distributions in the thickness direction. The material properties are assumed to vary through the thickness of the plate with ceramic-to-metal...
volume distribution between the two surfaces. Specifically, the horizontal and vertical Young’s modulus \((E_1 \text{ and } E_2)\), density \(\rho\) and main Poisson ratio \(\nu_{12}\) are assumed to vary continuously through the plate thickness.

According to Equations (1)–(3), the variation of the ceramic volume fraction through coordinate-thickness ratio in various types is presented in Figure 4 \((V_1 = 0, \ V_2 = 1)\). In Figure 4a, the volume fraction of ceramic varies quickly near the lowest surface for \(p < 1\) and increases quickly near the top surface for \(p > 1\) under the case of P-OFG. In Figure 4b, the distribution of the ceramic volume fraction is symmetric about the middle surface for C-OFG. In Figure 4c, the distribution is inverse symmetric about the middle surface for S-OFG, and each part is similar to P-OFG.

![Figure 4](image_url)

**Figure 4.** Variation of ceramic volume fraction \(V_f\) through the coordinate-thickness ratio in various types: (a) P-OFG; (b) C-OFG; and (c) S-OFG.
3.3. Fundamental Frequencies Analysis

In this subsection, the relations between fundamental frequencies $f$ (Hz) and gradient index $p$ with general boundary restraints for P-, C- and S-OFG plates are contrasted and analyzed. The variation of the fundamental frequencies $f$ (Hz) through gradient index $p$ for OFG plates with $h/a = 0.3$ is shown in Figure 5. SSSS, SSSF, SSSC, CFCC, CFCS and CFCF boundary restraints are studied for P-, C- and S-OFG plates.

![Figure 5](image)

**Figure 5.** Variation of the frequencies through $p$ for OFG plates with $h/a = 0.5$: (a) OFG plates with SSSS; (b) OFG plates with SSSF; (c) OFG plates with SSSC; (d) OFG plates with CFCC; (e) OFG plates with CFCS; and (f) OFG plates with CFCF.

The six sets of curves show a similar behavior. For P- and C-OFG, the frequency parameters are considerably decreased by increasing the gradient index $p$. This is because $E_m$ is much smaller than $E_f$ and the stiffness of the plate decreases with the increased distribution range of metal, thus the frequency parameter declines. When the value of $p$ equals to zero, a complete ceramic plate is obtained, whereas infinity $p$ indicates a complete metal plate. For the case of S-OFG, although there is a small range of fluctuations as $p$ changes, the interval of fundamental frequencies $f$ (Hz) is not large.
Without loss of generality, the ceramic volume fraction always exhibits the opposite changes and the effect of $p$ on the stiffness is small. Therefore, $p$ has little influence on the fundamental frequencies $f$ (Hz) with different boundary restraints overall.

Based on the above analysis, the fundamental frequencies $f$ (Hz) for OFG plates with respect to gradient index $p$ and volume fractions with simply supported boundary restraint are presented in Figure 6. It can be seen that, with the increase of $V_2-V_1$, the variation of fundamental frequencies $f$ (Hz) through the gradient index $p$ is more obvious for OFG plates. The results in Figure 6 show that the effects of the gradient index $p$ on the fundamental frequencies $f$ (Hz) vary with ceramic volume fractions between bottom and top surfaces.

**Figure 6.** Variation of the fundamental frequencies $f$ (Hz) through $p$ with simply supported boundary restraint in various types: (a) P-OFG; (b) C-OFG; and (c) S-OFG.
4. Conclusions

In this investigation, a modified Fourier series method has been applied to solve the free vibrations of moderately thick OFG plates with general boundary restraints. Displacements and rotations of each plate, regardless of boundary restraints, are described as an improved form of double Fourier cosine series and several auxiliary functions to effectively eliminate any possible jumps with the original displacement function and its relevant derivatives at the boundaries. Exact solutions are obtained by the energy functions of the plates based on Rayleigh–Ritz method. The general boundary restraints are achieved by setting the stiffness of springs without requiring any special procedures or schemes. It is shown that the present method has high reliability and accuracy. Numerous new results for moderately thick OFG plates with general boundary restraints are presented, which may serve as benchmark solutions for future research in this field.

A comprehensive investigation focused on free vibration properties of OFG materials is given, which serves as a supplement of the properties of FGMs. It is shown that vibration frequencies of the OFG plates are strongly influenced by the ceramic-to-metal volume distribution, gradient index, geometric parameter and boundary restraint. With the interval of ceramic volume fraction increases, the variation of fundamental frequencies through the gradient index is more obvious. When the interval of volume fraction is certain, S-OFG is less affected by the gradient index, while the vibration frequencies of P- and C-OFG are significantly influenced by the gradient index. The effects of the gradient index on the volume fraction are also discussed as well.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Fourier coefficients eigenvector $\mathbf{G}$, stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$ of the moderately thick plates:

$$
\mathbf{A} = [A_{00}, \ldots, A_{mn}, \ldots, A_{MN}]^T \mathbf{a} = [a_0^0, \ldots, a_n^0, \ldots, a_2^N]^T \mathbf{b} = [b_1^0, \ldots, b_1^m, \ldots, b_2^M]^T
$$

$$
\mathbf{B} = [B_{00}, \ldots, B_{mn}, \ldots, B_{MN}]^T \mathbf{c} = [c_0^0, \ldots, c_n^0, \ldots, c_2^N]^T \mathbf{d} = [d_1^0, \ldots, d_1^m, \ldots, d_2^M]^T
$$

$$
\mathbf{C} = [C_{00}, \ldots, C_{mn}, \ldots, C_{MN}]^T \mathbf{e} = [e_0^0, \ldots, e_n^0, \ldots, e_2^N]^T \mathbf{f} = [f_1^0, \ldots, f_1^m, \ldots, f_2^M]^T
$$

$$
\mathbf{D} = [D_{00}, \ldots, D_{mn}, \ldots, D_{MN}]^T \mathbf{g} = [g_1^0, \ldots, g_1^n, \ldots, g_2^N]^T \mathbf{h} = [h_1^0, \ldots, h_1^m, \ldots, h_2^M]^T
$$

$$
\mathbf{E} = [E_{00}, \ldots, E_{mn}, \ldots, E_{MN}]^T \mathbf{i} = [i_1^0, \ldots, i_1^n, \ldots, i_2^N]^T \mathbf{j} = [j_1^0, \ldots, j_1^m, \ldots, j_2^M]^T
$$

The unknown Fourier coefficients eigenvector $\mathbf{G}$ in the displacement expressions is divided into five parts: $\mathbf{G}^u$, $\mathbf{G}^p$, $\mathbf{G}^w$, $\mathbf{G}^\phi$, $\mathbf{G}^\psi$.

$$\mathbf{G}^u = [\mathbf{A}, \mathbf{a}, \mathbf{b}]^T = \begin{bmatrix} A_{00}, A_{01}, \ldots, A_{mn}, A_{MN}; a_0^0, \ldots, a_n^0, \ldots, a_2^N; b_1^0, \ldots, b_1^m, \ldots, b_2^M \end{bmatrix}$$

$$\mathbf{G}^p = [\mathbf{B}, \mathbf{c}, \mathbf{d}]^T = \begin{bmatrix} B_{00}, B_{01}, \ldots, B_{mn}, B_{MN}; c_0^0, \ldots, c_n^0, \ldots, c_2^N; d_1^0, \ldots, d_1^m, \ldots, d_2^M \end{bmatrix}$$

$$\mathbf{G}^w = [\mathbf{C}, \mathbf{e}, \mathbf{f}]^T = \begin{bmatrix} C_{00}, C_{01}, \ldots, C_{mn}, C_{MN}; e_0^0, \ldots, e_n^0, \ldots, e_2^N; f_1^0, \ldots, f_1^m, \ldots, f_2^M \end{bmatrix}$$

$$\mathbf{G}^\phi = [\mathbf{D}, \mathbf{g}, \mathbf{h}]^T = \begin{bmatrix} D_{00}, D_{01}, \ldots, D_{mn}, D_{MN}; g_1^0, \ldots, g_1^n, \ldots, g_2^N; h_1^0, \ldots, h_1^m, \ldots, h_2^M \end{bmatrix}$$

$$\mathbf{G}^\psi = [\mathbf{E}, \mathbf{i}, \mathbf{j}]^T = \begin{bmatrix} E_{00}, E_{01}, \ldots, E_{mn}, E_{MN}; i_1^0, \ldots, i_1^n, \ldots, i_2^N; j_1^0, \ldots, j_1^m, \ldots, j_2^M \end{bmatrix}$$
\[ G^y = [E, i, j]^T = \begin{bmatrix} E_{00}, E_{01}, \ldots, E_{m0}, E_{m1}, \ldots, E_{mnr}, i_0^1, \ldots, i_{r1}^1, \ldots, i_0^N, i_{r1}^N, \ldots, i_0^m, \ldots, i_{r1}^m \end{bmatrix} \]

Sub-matrices in the K and M are listed as follows:

\[ H = [H_{xy}, H_x, H_y] \]

\[ H_{xy} = \begin{bmatrix} \cos \lambda_0 x \cos \lambda_0 y, \ldots, \cos \lambda_m x \cos \lambda_n y, \ldots, \cos \lambda_N x \cos \lambda_N y \end{bmatrix} \]

\[ H_x = \begin{bmatrix} \zeta_0^2(x) \cos \lambda_0 y, \ldots, \zeta_0^2(x) \cos \lambda_n y, \ldots, \zeta_0^2(x) \cos \lambda_N y \end{bmatrix} \]

\[ H_y = \begin{bmatrix} \zeta_0^2(y) \cos \lambda_0 x, \ldots, \zeta_0^2(y) \cos \lambda_m x, \ldots, \zeta_0^2(y) \cos \lambda_N x \end{bmatrix} \]

\[ \{K_{uu}\} = \int_{a}^{b} \int_{0}^{b} A_{11} \frac{\partial H^T}{\partial x} \frac{\partial H}{\partial x} + A_{66} \frac{\partial H^T}{\partial y} \frac{\partial H}{\partial y} \, dy \, dx + \int_{a}^{b} \int_{0}^{b} k_{x0}^u H^T H |_{x=0} + k_{x1}^u H^T H |_{x=a} \, dy \, dx \]

\[ \{K_{vv}\} = \int_{a}^{b} \int_{0}^{b} A_{22} \frac{\partial H^T}{\partial y} \frac{\partial H}{\partial y} + A_{66} \frac{\partial H^T}{\partial x} \frac{\partial H}{\partial x} \, dy \, dx + \int_{a}^{b} \int_{0}^{b} k_{y0}^v H^T H |_{y=0} + k_{y1}^v H^T H |_{y=b} \, dx \, dy \]

\[ \{K_{ww}\} = \int_{a}^{b} \int_{0}^{b} A_{44} \frac{\partial H^T}{\partial y} \frac{\partial H}{\partial y} + A_{55} \frac{\partial H^T}{\partial x} \frac{\partial H}{\partial x} \, dy \, dx + \int_{a}^{b} \int_{0}^{b} k_{x0}^w H^T H |_{x=0} + k_{x1}^w H^T H |_{x=a} \, dy \, dx \]

\[ \{K_{xy}\} = \int_{a}^{b} \int_{0}^{b} D_{11} \frac{\partial H^T}{\partial \phi_x} \frac{\partial \phi_x}{\partial x} + D_{66} \frac{\partial H^T}{\partial \phi_y} \frac{\partial \phi_y}{\partial x} + A_{33} H^T H \, dy \, dx + \int_{a}^{b} \int_{0}^{b} k_{x0}^y H^T H |_{x=0} + k_{x1}^y H^T H |_{x=a} \, dy \, dx \]

\[ \{K_{yy}\} = \int_{a}^{b} \int_{0}^{b} D_{22} \frac{\partial H^T}{\partial \phi_y} \frac{\partial \phi_y}{\partial y} + D_{66} \frac{\partial H^T}{\partial \phi_x} \frac{\partial \phi_x}{\partial y} + A_{44} H^T H \, dy \, dx + \int_{a}^{b} \int_{0}^{b} k_{y0}^y H^T H |_{y=0} + k_{y1}^y H^T H |_{y=b} \, dx \, dy \]
\[
\left\{ K_{\psi_{x}\psi_{y}} \right\} = \int_{0}^{b} \int_{0}^{b} \left\{ D_{12} \frac{\partial H^{T}}{\partial \psi_{x}} \frac{\partial H}{\partial \psi_{y}} + D_{66} \frac{\partial H^{T}}{\partial \psi_{y}} \frac{\partial H}{\partial \psi_{x}} \right\} dydx
\]

\[
\left\{ K_{\iota_{x}\psi_{y}} \right\} = \int_{0}^{b} \int_{0}^{b} \left\{ A_{35} \frac{\partial H^{T}}{\partial \psi_{x}} H \right\} dydx
\]

\[
\left\{ K_{\iota_{y}\psi_{x}} \right\} = \int_{0}^{b} \int_{0}^{b} \left\{ A_{44} \frac{\partial H^{T}}{\partial \psi_{y}} H \right\} dydx
\]

\[
\left\{ K_{\iota_{x}\iota_{y}} \right\} = \int_{0}^{b} \int_{0}^{b} \left\{ A_{55} \frac{\partial H^{T}}{\partial \iota_{x}} \frac{\partial H}{\partial \iota_{y}} \right\} dydx
\]

\[
\left\{ K_{\iota_{x}\iota_{y}} \right\} = \int_{0}^{b} \int_{0}^{b} \left\{ A_{44} \frac{\partial H^{T}}{\partial \iota_{y}} \frac{\partial H}{\partial \iota_{x}} \right\} dydx
\]

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