ON CLASSIFICATION OF THE EXTREMAL
CONTRACTION FROM A SMOOTH FOURFOLD

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Abstract. We classify extremal divisorial contraction which contracts a divisor to a
curve from a smooth fourfold. We prove the exceptional divisor is $\mathbb{P}^2$ bundle or quadric
bundle over a smooth curve and the contraction is the blowing up along the curve.

0. Introduction

In his pioneer paper [M1] and [M2], Shigefumi Mori introduced the extremal
ray and classified completely the extremal contraction from a smooth 3-fold. In
dimension 4, we will consider the same problem, i.e., we want to classify the ex-
ternal contraction from a smooth 4-fold. In dimension 4, the situation is more
complicated.

First small (flipping) contraction appears. This case was completely classified
by Yujiro Kawamata in his ingenious paper [Ka1].

Secondly, in case the contraction is fibre type from 4-fold to 3-fold and divisorial
type which contracts a divisor to a surface, equidimensionality of the fibre is not
satisfied in general. (i.e., general fibres are 1 dimensional but some special fibres are
possibly 2 dimensional.) The special 2 dimensional fibre are classified by Yasuyuki
Kachi in [Kac] in case of fibre type, and by Marco Andreatta in case of divisorial
type.

Thirdly in case the contraction is divisorial type which contracts a divisor to a
point, the exceptional divisor is possibly nonnormal. In fact Mauro Beltrametti
listed up all the possibility of the exceptional divisor in [Be1], [Be2]. It include
nonnormal possibility. (But many cases are excluded by Takao Fujita in [F2].)

In this paper, we consider the contraction is divisorial type which contracts a
divisor to a curve. This case turns out to be very mild in contrast to the above
cases.

Main Theorem. Let $X$ be a smooth 4-fold and let $f: X \to Y$ be a divisorial
contraction which contracts a divisor to a curve. Let $E$ be the exceptional divisor
of $f$ and $C$ be $f(E)$. Then

1. $C$ is a smooth curve.
2. $f|_E: E \to C$ is $\mathbb{P}^2$ bundle or quadric bundle (see the definition 1.4 below) over
   $C$.
3. $f$ is the blowing up of $Y$ along $C$.

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1. Notations and Preliminaries

We cite the key theorems and make some definitions in this section.

**Notation 1.0.** The $\mathbb{P}^1$ bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ over $\mathbb{P}^1$ is called a Hirzebruch surface of degree $a$ and denoted by $\mathbb{F}_a$. The unique negative section of it is denoted by $C_0$ and a ruling is denoted by $f$.

The projective cone obtained from $\mathbb{F}_a$ by the contraction of $C_0$ is denoted by $\mathbb{F}_{a,0}$. A generating line on $\mathbb{F}_{a,0}$ is denoted by $l$.

**Theorem 1.1.** (cf. [TA], [Be1] and [Be2]) Let $X, Y, E$ and $C$ be as in Main Theorem. Let $F$ be a general fibre of $f|_E: E \to C$.

Then

$$(F, -K_X|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)), (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1)) \text{ or } (\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}})$$

We give the outline of the proof.

**Outline of the proof.** Once we prove the irreducibility of $F$, the results are follow from [TA] and [Be1] and [Be2], so we will prove only the irreducibility here. We assume that $F$ is reducible and get a contradiction. Let $H$ be a good supporting divisor of $f$. We may assume that $H$ is a smooth variety and at least locally $F = H \cap E$. By the adjunction formula, $-K_F = -K_X - E - H|_F$, but since $H|_F \sim 0$, $-K_F = -K_X - E|_F$. Note that $-K_X|_F$ is ample Cartier divisor on $F$. Since $-K_F$ is ample on $F$, $F$ is a generalized del Pezzo surface (i.e., a Gorenstein (possibly reducible) anti polarized surface). Write $F = \bigcup F_i$ where $F_i$ is an irreducible component of $F$. By [R3], $(F_i, -K_F|_{F_i})$ is one of the following:

(a) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(i))$ where $i$ is 1 or 2.
(b) $(\mathbb{F}_{a,0}, \mathcal{O}_{\mathbb{F}_{a,0}}(al))$
(c) $(\mathbb{F}_a, \mathcal{O}_{\mathbb{F}_a}(C_0 + (a + 1)f))$
(d) $(\mathbb{F}_a, \mathcal{O}_{\mathbb{F}_a}(C_0 + (a + 2)f))$

But since $-K_F$ is a sum of two ample Cartier divisor, (a) with $i = 1, (b),(c)$ and (d) is impossible. So we get $(F_i, -K_F|_{F_i}) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and $-K_X|_{F_i} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. Furthermore by [R3, Main Theorem and 1.3], $F$ is union of two $\mathbb{P}^2$’s which intersect line in $\mathbb{P}^2$. But this is impossible since $N_{F_i/H} \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$ and so the birational contraction which contracts only one $F_i$.

The next theorem of freeness is very useful for classification of low dimensional fibres of an extremal contraction.

**Theorem 1.2.** (see [A-W]) Let $X$ be a normal log terminal variety and $L$ be an ample line bundle on $X$. Let $f: X \to Y$ be the adjunction contraction supported by $K_X + rL$ and $F$ be a fibre of $f$. Assume that $\dim F < r + 1$ if $\dim Y < \dim X$ or $\dim E < a + 1$ if $\dim X < \dim Y$. 


Then $f^*f_*L \rightarrow L$ is surjective at every point of $F$. □

The next theorem was proved in [Wi1] and [Wi2] (see also [R1] and [R2]) but we will give the proof again here for readers’ convenience.

**Theorem 1.3.** Let $X$ be a smooth 3-fold and $Y$ be a canonical 3fold. Let $f: X \rightarrow Y$ be a crepant birational contraction which contracts a irreducible divisor to a curve. Let $E$ be the exceptional divisor and $C = f(E)$.

Then $C$ is a smooth curve.

**Proof.** Let $P$ be any point of $C$. The assertion is local, so we may replace $C$ and $Y$ with an affine (not analytic) neighborhood of $P$. We will keep this in mind below.

**Claim 1.** $P$ is a cDV point of $Y$.

**Proof.** Suppose that $P$ is not a cDV point. By [R1] and [R2], we have a birational morphism $g: X' \rightarrow X$ such that $X'$ is terminal, $g$ is crepant and $g$ has a exceptional divisor $E_0$ which contracts to $P$. Take a common resolution $\tilde{X}$.

$$
\begin{array}{ccc}
\tilde{X} & \longrightarrow & X' \\
\downarrow & & \downarrow g \\
X & \longrightarrow & Y
\end{array}
$$

Since $f$ and $g$ is crepant, strict transform of $E_0$ on $X$ is exceptional for $f$ but this contradicts to the irreducibility of the exceptional divisor of $f$. □

Let $H$ be the pull back of a very ample divisor on $Y$.

**Claim 2.** $|mH - E|$ is very ample for $m \gg 0$. In particular $f^*f_*O_X(-E) \rightarrow O_X(-E)$ is surjective.

**Proof.** Since $mH - 2E$ is ample for $m \gg 0$, it follow from the vanishing theorem (see [KMM]) and the exact sequence

$$0 \rightarrow O_X(mH - 2E) \rightarrow O_X(mH - E) \rightarrow O_E(mH - E) \rightarrow 0$$

that $H^0(O_X(mH - E)) \rightarrow H^0(O_E(mH - E))$ is surjective. Let $l$ be a fibre of $f|_E$. From the vanishing $H^1(X, O_X) = 0$, $l$ is a tree of $\mathbb{P}^1$, so $|mH - E|_l$ is very ample and so is $|mH - E|_E$ since we consider locally. From these, $|mH - E|$ is also very ample. □

**Claim 3.** $C$ can be embedded in a smooth surface.

**Proof.** In fact, let $S$ be a smooth general member of $|mH - E|$. Since $f|_S: S \rightarrow f(S)$ is etale, so $f(S)$ is smooth. $C$ is in $f(S)$ so we are done. □

**Claim 4.** $f$ is the blowing up of $Y$ along $C$.

**Proof.** Let $I_C$ be the ideal sheaf of $C$ in $Y$.

First we see that

$$f_*O_X(-E) = I_C$$

Let’s consider the exact sequence

$$0 \rightarrow O_X(-E) \rightarrow O_X \rightarrow O_E \rightarrow 0$$
From this and vanishing theorem, we get the exact sequence

$$0 \to f_* \mathcal{O}_X(-E) \to \mathcal{O}_Y \to f_* \mathcal{O}_E \to 0$$

Since except at finite points $f_* \mathcal{O}_E = \mathcal{O}_C$, we have $f_* \mathcal{O}_X(-E) = \mathcal{I}_C$ except at finite point. But they are reflexive(cf.[H2,Corollary 1.5,Proposition 1.6,Corollary 1.7]),so they are actually equal.

From this and claim2, we have $f^* \mathcal{I}_C \to \mathcal{O}_X(-E)$ is surjective, i.e., $\mathcal{I}_C \mathcal{O}_X = \mathcal{O}_X(-E)$. So by the universal property of blowing up(cf.[H1,II proposition7.14]), $f$ decomposes as $X \to X_1 \to Y$, where $X_1$ is the blowing up of $Y$ along $C$. $X_1$ is normal since $C$ is a $cDV$ curve in $Y$.(see also the calculations below.) So $X \simeq X_1$ since $f$ is a primitive contraction. This established claim4.

We suppose $P$ is a singular point of $C$ and get a contradiction. Recall that $P$ is a $cDV$ point and so $Y$ can be embedded(analytically locally)in $\mathbb{C}^4$. Let $x, y, z, t$ be its coordinate around $P$ and $g$ be the defining equation of $Y$ in $\mathbb{C}^4$. By claim3, We may assume $\mathcal{I}_C = \langle x, y, h \rangle$, where $h \in \mathbb{C}[[z,t]]$. Since we suppose $P$ is singular point of $C$, $h = 0$ is singular at $z = t = 0$ in $zt$-plane. $X$ is the strict transform of $Y$ in the blowing up $\tilde{\mathbb{C}}^4$ of $\mathbb{C}^4$ along $C$. $\tilde{\mathbb{C}}^4$ is in $\mathbb{C}^4 \times \mathbb{P}^2$ and given by

$$\text{rank} \begin{pmatrix} x & y & h \\ u & v & w \end{pmatrix} \leq 1$$

($u, v, w$ is the homogenous coordinate of $\mathbb{P}^2$). Take the affine piece given by $u = 1$. We can embed this affine piece of $\tilde{\mathbb{C}}^4$ in $\mathbb{C}^5$ with coordinate $(x, z, t, v, w)$ and equation $xw = h$. This affine variety is singular above $P$ along the line $L$ defined by $x = z = t = w = 0$. Write $g(x, xv, z, t) = x^m \tilde{g}(x, v, z, t)$,where $\tilde{g}$ can not devided by $x$. Then $X$ is defined by $xw = h$ and $\tilde{g} = 0$. This intersects $L$ and at the intersections, $X$ is singular, a contradiction. □

**Definition 1.4.** Let $E$ be a normal projective 3-fold and $C$ be a smooth curve. Let $f: E \to C$ be a projective surjective morphism.

We say $f: E \to C$ is quadric bundle if the following conditions are satisfied.

1. there exists a $f$-very ample line bundle $\mathcal{L}$ on $E$.
2. For any closed point $s$ of $C$, $h^0(E_s, \mathcal{L}_s) = 3$ and $E_s$ is a quadric in

$$\mathbb{P}(H^0(E_s, \mathcal{L}_s)^*) \simeq \mathbb{P}^3$$

If we can take such an $\mathcal{L}$, we say $f: E \to C$ is the quadric bundle associated to $\mathcal{L}$. □

**Remark.** We require $E$ is normal above so general fibres of $f$ are irreducible. □

2. **Proof of the Main theorem**

**Proof of (1).** Let $H$ be a good supporting divisor of $f$ and $L$ be $\mathcal{O}_X(mH - K_X)$ for $m \gg 0$. Since $L$ is ample, we apply Theorem 1.2 for this $f$ and $L$ with $r = 1$.(Remark that the dimension of fibres of $f$ is 2.) Then for any point $P$ of $C$ and a suitable affine neighborhood $U$ of $P$ in $Y$, $|L|_{f^{-1}(U)}$ has no base points on $E$. So furthermore if we replace $f^{-1}(U)$ with a suitable neighborhood $V$ of $E$ in $X$, $|L|_V$ is base point free on $V$. Let $X_0$ be a general smooth member of $|L|_V$, $Y_0 = f(X_0)$ and $E_0$ be $f|_{X_0}$. Then
Claim. $E_0$ is irreducible

Proof. Let $F$ be a general fibre of $f$ and $F_0$ be $F|_{X_0}$. It suffices to prove $F_0$, i.e., $X_0|_F$ is irreducible. For this it suffices to prove the surjectivity of $H^0(L|_V) \to H^0(L|_F)$. For by Theorem 1.1, general member of $|L|_F$ is irreducible. First from the exact sequence

$$0 \to L|_V \otimes \mathcal{O}_X(-E) \to L|_V \to L|_E \to 0$$

and the vanishing theorem, we have $H^0(L|_V) \to H^0(L|_E)$ is surjective. Secondly since we take $F$ to be a general fibre, $F$ is a Cartier of $E$ and since near $F$, $E \simeq \mathbb{P}^2 \times \mathbb{A}^1, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^1$ or $\mathbb{P}^2 \times \mathbb{A}^1, E$ has at worst canonical singularities near $F$ (cf. Theorem 1.1). So we can use the vanishing theorem for the exact sequence

$$0 \to L|_E \otimes \mathcal{O}_E(-F) \to L|_E \to L|_F \to 0$$

and we get the surjectivity of $H^0(L|_E) \to H^0(L|_F)$. This establishes the claim. □

Proof of (2). Let $F$ be a general fibre of $f$. If $(F, -K_F|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)), (2, \mathcal{O}(1,1)), \text{or } (\mathbb{P}^2 \times \mathbb{A}^1, \mathcal{O}(1,1)), \text{let } \mathcal{L} \text{ be } \mathcal{O}_E(-K_X)$.

If $(F, -K_F|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)), \text{let } \mathcal{L} \text{ be } \mathcal{O}_E(-E)$. Then we will prove

(i) If $F \simeq \mathbb{P}^2, f|_E: E \to C$ is $\mathbb{P}^2$-bundle.

(ii) If $F \simeq \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } \mathbb{P}^2 \times \mathbb{A}^1, f|_E: E \to C$ is the quadric bundle associated to $\mathcal{L}$.

If $\mathcal{L} \simeq \mathcal{O}_E(-K_X), \text{we can argue as follow.}$

First we see $f_* \mathcal{L}$ is locally free. The exact sequence

$$0 \to \mathcal{O}_X(-K_X - E) \to \mathcal{O}_X(-K_X) \to \mathcal{O}_E(-K_X) \to 0$$

and the vanishing theorem, we have $R^i f_* \mathcal{O}_E(-K_X) = 0 (i > 0)$. On the other hand, we have $H^i(E_s, \mathcal{L}_s) = 0$ for $i > 0$ and any $s \in C$. Furthermore by above (1), $C$ is smooth so $f|_E$ is flat. So by Cohomology and Base change theorem (cf. [H1, III Theorem 12.11]), $f_* \mathcal{L}$ is locally free.

Next we see $\mathcal{L}$ is $f$-free. Let’s consider the commutative diagram

$$\begin{array}{ccc}
f^* f_* \mathcal{O}_X(-K_X) & \longrightarrow & f^* f_* \mathcal{O}_E(-K_X) \\
\downarrow & & \downarrow \\
\mathcal{O}_X(-K_X) & \longrightarrow & \mathcal{O}_E(-K_X)
\end{array}$$

We see the left arrow is surjective by Theorem 1.2 and so is the bottom arrow by the above exact sequence and vanishing. So the right arrow must be surjective, i.e., $\mathcal{L}$ is $f$-free.

By these, we get the morphism $g: E \to \mathbb{P}(f_* \mathcal{L})$ defined by $\mathcal{L}$.

If we are in case (i), $g$ is birational since general fibre of $f$ is $\mathbb{P}^2$ and $\mathbb{P}(f_* \mathcal{L})$ is $\mathbb{P}^2$-bundle. And $g$ is finite since $\mathcal{L}$ is $f$-ample. So by Zariski Main Theorem, $g$ is isomorphism.

If we are in case (ii), then...
Claim. \( \mathcal{L} \) is \( f \)-very ample.

Proof. We will prove \( g \) is isomorphism onto \( g(E) \). \( g \) is birational because on the general fibre of \( f \), \( \mathcal{L} \) is very ample. \( g \) is finite because \( \mathcal{L} \) is \( f \)-ample. The dimension of singular locus of \( g(E) \) is not greater than 2 since general fibres of \( g(E) \to C \) are \( \mathbb{P}^1 \times \mathbb{P}^1 \) or \( \mathbb{F}_{2,0} \) and \( C \) is smooth. \( g(E) \) satisfies the Serre’s condition \( S_2 \) since \( E \) is a divisor of smooth 4-fold. So \( g(E) \) is normal. Then by Zariski main theorem, \( g \) is isomorphism.

From this claim, it is easy to see that \( E \) is quadric bundle associated to \( \mathcal{L} \).

If \( \mathcal{L} \simeq \mathcal{O}_E(-E) \), we can argue as follow. (cf.[F1,1.5])

Let \( F' \) be any fibre of \( f|_E \) and write \( F' = \bigcup F'_i \), where \( F'_i \) is an irreducible component of \( F' \). Since \( f|_E \) is flat, \( 1 = (-E)^2 F = \sum (-E)^2 F_i \). Since \(-E|_{F_i} \) is ample, \((-E)^2 F_i > 0 \). So \( F' \) must be irreducible. By the lower semicontinuity of \( \Delta \)-genus (cf.[H1,III,Theorem 12]), we have \( \Delta(F',-E|_{F'}) \leq \Delta(F,-E|_F) = 0 \). \( F' \) has no embedded points since \( E \) is Cohen-Macaulay and \( F' \) is Cartier divisor on \( E \). So \( \Delta(F',-E|_{F'}) \geq 0 \) by [F0] and so \( \Delta(F',-E|_{F'}) = 0 \). Since \((-E)^2 F' = 1 \), \((F',-E|_{F'}) \simeq (\mathbb{P}^2,\mathcal{O}(1)) \) by the classification of the varieties of \( \Delta \)-genus 0. So \( f:E \to C \) is \( \mathbb{P}^2 \)-bundle.

Remark. We cannot proceed in case \( \mathcal{L} \simeq \mathcal{O}_E(-E) \) similar to the case \( \mathcal{L} \simeq \mathcal{O}_E(-K_X) \) because we have not freeness of \( \mathcal{O}_E(-E) \) apriori.

As for (3), the proof is almost the same as [M2,Corollary(3.4)]. So we will give only the outline of the proof.

Outline of the proof of (3). We see that \( \mathcal{O}_E(-E) \) is \( f|_E \)-very ample by (2) and \( \mathcal{O}_X(-E) \) is \( f \)-ample. So we get the following.

Claim.

(a) \( R^i f_* \mathcal{O}_X(-jE) = 0 \) for \( i > 0 \) and \( j \geq 0 \).
(b) \( f_* \mathcal{O}_X(-jE) = \mathcal{L}_C^{-j}, \mathcal{L}_C^{-j} \mathcal{O}_X = \mathcal{O}_X(-jE) \) for \( j \geq 0 \).
(c) \( \oplus_{n \geq 0} \mathcal{L}_C^n/\mathcal{L}_C^{n+1} \simeq \oplus_{n \geq 0} f_* \mathcal{O}_E(-nE) \) as \( \mathcal{O}_C \) algebra.

By this claim, we can easily get the result.

Remarks and Examples. We can say the following about the local analytic structure of the contraction. Let \( F' \) be a fibre of \( f|_E \) and \( F \) is any general fibre of \( f|_E \) near \( F' \). We will give the description near \( F' \).

(1) If \( f|_E : E \to C \) is \( \mathbb{P}^2 \) bundle and \( \mathcal{O}_X(-E)|_F \simeq \mathcal{O}_{\mathbb{P}^2}(1), Y \) is smooth along \( C \) (cf.[SN])
(2) If \( f|_E : E \to C \) is \( \mathbb{P}^2 \) bundle and \( \mathcal{O}_X(-E)|_F \simeq \mathcal{O}_{\mathbb{P}^2}(2), Y \) can be considered as one parameter family of \( \frac{1}{2}(1,1,1) \) singularity. In fact, let \( P \) be any point of \( C \) and take a general very ample divisor \( A \) through \( P \). Let \( H \) be the pull back of \( A \). Then \( H \) is smooth along \( E|_H \simeq \mathbb{P}^2 \) since \( E|_H \) is smooth and a Cartier divisor of \( H \). So \( f|_H \) is the extremal contraction from a smooth 3-fold near the fibre over \( P \). Then by Mori’s classification, \( A \) has \( \frac{1}{2}(1,1,1) \) singularity at \( P \).
(3) If \( f|_E : E \to C \) is quadric bundle we see \( Y \) is locally hypersurface in \( \mathbb{C}^5 \) and \( C \) is locally complete intersection in the \( \mathbb{C}^5 \) because \( \mathcal{L}_C/\mathcal{L}_C^2 \) is locally free sheaf of rank 4 on \( C \) by the claim in the proof of Main Theorem (2). But the
type of singularity of $Y$ along $C$ is very various(case (c) and (e) below), so we will give some examples here. Below $\mathbb{C}^5$ has always coordinates $x, y, z, w, t$. $Y$ is given hypersurface in $\mathbb{C}^5$ and $X$ is the blow up of $Y$ along $C$. We assume $F'$ is the fibre over the origin.

(3a) $(F$ and $F'$ are $\mathbb{P}^1 \times \mathbb{P}^1)$ Let $Y$ be $(x^2 + y^2 + z^2 + w^2 = 0)$ and $C$ be $(x = y = z = w = 0)$. In this case above example is all.

(3b) $(F$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $F'$ is $\mathbb{F}_{2,0}$) Let $Y$ be $(x^2 + y^2 + z^2 + tw^2 = 0)$ or $(x^2 + y^2 + z^2 + t^m w^2 + w^3 = 0)$ and $C$ be $(x = y = z = w = 0)$. In this case above examples are all.

(3c) $(F$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $F'$ is union of two planes in $\mathbb{P}^3$) Let $Y$ be $(x^2 + y^2 + t^m z^2 + z^3 + t^m w^2 + w^3 = 0)$ and $(x = y = z = w = 0)$. In this case this example is all.

(3d) $(F$ and $F'$ are $\mathbb{F}_{2,0}$) Let $Y$ be $(x^2 + y^2 + z^2 + w^3 = 0)$ and $(x = y = z = w = 0)$.

(3e) $(F$ is $\mathbb{F}_{2,0}$ and $F'$ is union of two planes in $\mathbb{P}^3$) Let $Y$ be $(x^2 + y^2 + z^3 + t^m w^2 + w^3 = 0)$ and $(x = y = z = w = 0)$.

\[ \square \]

**Question.** Does any quadric bundle appear as the exceptional divisor of the contraction as in Main Theorem?

**References**

[TA] T. Ando, *On extremal rays of the higher dimensional varieties*, Invent.Math 81 (1985), 347–357.

[A-W] M. Andreata and J. Wiśniiewski, *A note on nonvanishing and applications*, Duke Math J 72 (1993), 739–755.

[Be1] M. Beltrametti, *On d-folds whose canonical bundle is not numerically effective, According to Mori and Kawamata*, Ann.Math.Pura.Appl 116 (1982), 133–176.

[Be2] —, *Contraction of non numerically effective extremal rays in dimension 4*, Teubner-Texte Math 92 (1986), 24–37.

[F0] T. Fujita, *Classification theories of polarized varieties*, London Math.Soc.Lecture Note Ser 115 (1990), Cambridge Univ.Press.

[F1] —, *On del Pezzo fibrations over curves*, Osaka.J.Math 27 (1990), 229–245.

[F2] —, *On singular Del Pezzo varieties*, Lecture Notes in Math., vol. 1417, Springer-Verlag, 1990, p. 117–128.

[H1] R. Hartshorne, *Algebraic Geometry*, GTM 52 (1977), Springer-Verlag.

[H2] —, *Stable reflexive sheaves*, Math.Ann 254 (1980), 121–176.

[Kac] Y. Kachi, *Extremal contractions from 4-dimensional manifolds to 3-folds*, preprint.

[Ka1] Y. Kawamata, *The cone of Curves of algebraic varieties*, Ann. of Math 119 (1984), 603–633.

[Ka2] —, *Small contractions of four dimensional algebraic manifolds*, Math.Ann 284 (1989), 595–600.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Adv.St.Pure Math 10 (1987), 287–360.

[M1] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math 110 (1979), 593–606.

[M2] —, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math 116 (1982), 133–176.

[SN] S. Nakano, *On the inverse of monoidal transformations*, Publ.RIMS Kyoto Univ 6 (1971), 483–502.

[R1] M. Reid, *Canonical 3-folds*, Journées de Géométrie Algébrique d’Angers, Sijthoff and Noordhoff, Alphen, 1980, p. 273–310.

[R2] —, *Minimal models of canonical 3-folds*, Adv.St.Pure Math 1 (1983), 131–180.

[R3] —, *Nonnormal del Pezzo surfaces*, Publ RIMS Kyoto Univ. (1984), 695–705.
[Wi1] P.M.H. Wilson, *The Kähler cone on Calabi Yau threefolds*, Invent.Math 107 (1992), 561–583.

[Wi2] , *Erratum The Kähler cone on Calabi Yau threefolds*, Invent.Math 114 (1993), 231–233.

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