Transport coefficients for relativistic gas mixtures of hard-sphere particles

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Abstract

In the present work, we calculate the transport coefficients for a relativistic binary mixture of diluted gases of hard-sphere particles. The gas mixture under consideration is studied within the relativistic Boltzmann equation in the presence of a gravitational field described by the isotropic Schwarzschild metric. We obtain the linear constitutive equations for the thermodynamic fluxes. The driving forces for the fluxes of particles and heat will appear with terms proportional to the gradient of gravitational potential. We discuss the consequences of the gravitational dependence on the driving forces. We obtain general integral expressions for the transport coefficients and evaluate them by assuming a hard-sphere interaction among the particles when they collide and not very disparate masses and diameters of the particles of each species. The obtained results are expressed in terms of their temperature dependence through the relativistic parameter which gives the ratio of the rest energy of the particles and the thermal energy of the gas mixture. Plots are given to analyze the behavior of the transport coefficients with respect to the temperature when small variations in masses and diameters of the particles of the species are present. We also analyze for each coefficient the corresponding limits to a single gas so the non-relativistic and ultra-relativistic limiting cases are recovered as well. Furthermore, we show that the transport coefficients have a dependence on the gravitational field.

1 Introduction

An inspection of the literature (see e.g. the references [1,2,3,4,5,6,7,8,9,10,11,12]) concerning the analysis of relativistic gas mixtures shows that in the majority of the works, only general expressions for the transport coefficients have been given. One interesting feature of the transport coefficients of relativistic gases is their dependence on a parameter $\zeta_a = m_ac^2/kT$ which gives the ratio of the rest energy of the particles of species $a$ and the thermal energy of the gas. This ratio is small for high temperatures so the gas is in the ultra-relativistic regime, for low temperatures such a ratio is large and then the gas is in the non-relativistic regime. For a single gas the expressions for the transport coefficients in these limits are well known (see e.g. [2,11,12]). For mixtures of relativistic gases there are only few works which analyze the transport coefficients. We quote: ref. [13] where the reaction rate coefficient was obtained by considering a relativistic reactive differential cross-section which takes into account the activation energy of the chemical reaction; ref. [14] where Grad’s moment method was employed for the determination of the transport coefficients for mixtures of Maxwellian particles; lastly ref. [15] where the diffusion coefficient was determined by using a BGK-type kinetic model for a mixture of hard-sphere particles.

Some researches of relativistic gases in the presence of gravitational fields have been recently performed, we quote for instance the works [15,16,17,18,19]. Among the results, it is interesting to note that relativistic effects arise in the form of the thermal and diffusion generalized forces. Such forces...
have a contribution due to the four-acceleration as originally presented by Eckart [20] for the thermal force and a gravitational potential gradient in accordance with Tolman’s law [21, 22].

This report focuses the evaluation of the transport coefficients for a binary mixture of relativistic ideal gases and represents a continuation of the research presented in ref. [19]. In such a reference we used a method of solution for the Boltzmann equation that combines the Chapman-Enskog and Grad representations (see e.g. [12, 25]). In ref. [19] we found the linear constitutive equations for the heat and particle fluxes, dynamic pressure and pressure deviator tensor, and gave the transport coefficients as general integrals for a gas mixture. In the present paper we rewrite such expressions in the particular case of a binary mixture and evaluate them by assuming two physical hypotheses: (1) a hard-sphere interaction of the particles when they collide and (2) not very disparate molecular masses and diameters for the species. The performance of the integrals for the determination of the transport coefficients represents long and tricky manipulations. For this reason we have added in the appendix A sufficient hints to reproduce all the calculations. In appendix B we have listed a number of integrals that appear along the process taking into account the two physical hypotheses mentioned before. We present the expressions for the transport coefficients when they depend on differences of masses and diameters of the particles, and on concentrations of the species. We explore their behavior by analyzing some graphics with respect to the relativistic parameter \( \zeta \).

Applications of these results can be done to some astrophysical situations like white dwarfs or clouds nearby a source of gravitational potential. An example of self-diffusion can be addressed to a situation in which different isotopes of the same gas diffuse through each other; in this case the mass difference between the components of the gas can be very small.

The structure of the work is as follows. In section 2 we recall the basic equations and definitions. In section 3 we analyze the shear and bulk viscosity coefficients for a binary mixture and the limiting cases for a simple fluid are given. Section 4 is devoted to the determination of the thermal conductivity, diffusion and thermal-diffusion rate coefficients; there the one-component limits are also given. The influence of the gravitational field on the transport coefficients is presented in section 5 and finally the main conclusions are stated in section 6.

2 Background

In this work we deal with a binary mixture of diluted ideal gases in the nearby of a gravitational potential produced by a spherical static source. We assume a curved space-time described with the isotropic Schwarzschild metric \( g^{\mu\nu} \):

\[
ds^2 = g_0(r) \left( dx^0 \right)^2 - g_1(r) \delta_{ij} dx^i dx^j, \quad g_0(r) = \left( \frac{1 + \frac{\Phi}{2c^2}}{1 - \frac{\Phi}{2c^2}} \right)^2, \quad g_1(r) = \left( 1 - \frac{\Phi}{2c^2} \right)^4.
\]  

(1)

The above equations contain the gravitational potential \( \Phi = -GM/r \), where \( G \) is the gravitational constant, \( M \) the total mass of the spherical source, \( r \) the corresponding radius and \( c \) the light speed and indexes \( \{i, j\} \) will run over spatial components.

The gases under consideration are composed by particles that do not have internal degrees of freedom and are characterized by their space-time coordinates \( x^\mu = (ct, \mathbf{x}) \). The four-momentum of each particle is \( p^\mu_a = (p^0_a, \mathbf{p}_a) \) where the Latin subindex \( a = 1, 2 \) denotes the species of the gas whereas the Greek index \( \mu \) denotes the tensor properties. The momentum of each particle represents a time-like four-vector that holds the so-called mass-shell condition, i.e., \( g_{\mu\nu} p^\mu_a p^\nu_a = m_a^2 c^2 \), where \( m_a \) is the rest mass of a particle of species \( a \). This last condition leads to a relation between the temporal and spatial
components of the four-momentum in the following way:

\[ p_a^0 = p_{a0}/g_0, \quad \text{and} \quad p_{a0} = \sqrt{g_0 (m_a^2 c^2 - g_1 |p_a|^2)} \]  

(2)

for the contravariant and covariant representations, respectively.

Statistical mechanics has as a basis the macroscopic description of a gas through a distribution function. As usual we will use the one-particle distribution function \( f_a (x, p_a^0) \) so that \( f_a (x, p_a^0) \, d^3 x \, d^3 p_a \) is the number of particles of the constituent \( a \) in the volume element between \( x, x + d^3 x \) and \( p_a, p_a + d^3 p_a \) at some instant of time \( t \). This last quantity \( f_a \) can be obtained by solving the Boltzmann equation, which for the case of a binary mixture in a Riemannian space and in an appropriate manifestly covariant language reads [11, 12]:

\[
p_a^\mu \frac{\partial f_a}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p_a^\rho p_a^\sigma \frac{\partial f_a}{\partial p_a^\mu} = \sum_{b=1}^2 (f'_a f'_b - f_a f_b) F_{ba} \sigma_{\rho\sigma} d\Omega \sqrt{-g} \frac{d^3 p_b}{p_{b0}}.
\]

(3)

Here we have introduced the Christoffel symbols \( \Gamma^\mu_{\rho\sigma} \). The invariant flux is \( F_{ba} = \sqrt{(p_a^0 p_{b0})^2 - m_a^2 m_b^2 c^2} \) and the invariant differential elastic cross section for collisions of species \( a \) and \( b \) is \( \sigma_{\rho\sigma} d\Omega \) where \( d\Omega \) is the corresponding solid angle element. We have the invariant differential element \( \sqrt{-g} \frac{d^3 p_b}{p_{b0}} \) with \( \sqrt{-g} = \det [g^{\mu\nu}] \). Quantities denoted with a prime are evaluated with the momentum of the particles after a binary collision occurs, that is, \( f'_a = f(x, p_a, t) \).

Following the standard processes [11, 12] it is possible to obtain balance equations from Boltzmann’s equation and therein some basic definitions are made. Here we remind the general setup. The four-flux of particles for the species \( a \) is

\[ N_a^\mu = c \int p_a^\rho f_a \sqrt{-g} \frac{d^3 p_a}{p_{a0}}, \quad \text{and for the mixture} \quad N^\mu = \sum_{a=1}^2 N_a^\mu. \]

(4)

Here we shall use the Eckart frame [20], in which the hydrodynamic four-velocity \( U^\mu \) is introduced with the following decomposition of \( N_a^\mu \) as

\[ N_a^\mu = n_a U^\mu + J_a^\mu, \quad \text{where} \quad n_a = \frac{N_a^\rho U_\rho}{c^2} \]

(5)

is the local number of particles of species \( a \). The definition of the diffusive particle four-flux \( J_a^\mu \) is

\[ J_a^\mu = \Delta^\rho U_\rho \int p_a^\rho f_a \sqrt{-g} \frac{d^3 p_a}{p_{a0}}, \quad \text{and it holds} \quad J^\mu_a U_\mu = 0. \]

(6)

The projector has been introduced as

\[ \Delta^{\mu\nu} = g^{\mu\nu} - \frac{1}{c^2} U^\mu U^\nu, \quad \text{with the property} \quad \Delta^{\mu\nu} U_\mu = 0. \]

(7)

The diffusive particle four-flow is constrained by \( J_1^\mu + J_2^\mu = 0 \), so that there exists only one linearly independent diffusive particle four-flow for a binary mixture.

Other important definitions must be reminded. The energy-momentum tensor for one of the species of the mixture is

\[ T_a^{\mu\nu} = c \int p_a^\rho p_a^\sigma f_a \sqrt{-g} \frac{d^3 p_a}{p_{a0}}, \quad \text{and for the mixture} \quad T^{\mu\nu} = \sum_{a=1}^2 T_a^{\mu\nu}. \]

(8)

Following the literature, for instance ref. [19] and references therein, the energy-momentum tensor for the mixture can be written as:

\[ T^{\mu\nu} = \frac{ne}{c^2} U^\mu U^\nu - (p + \infty) \Delta^{\mu\nu} + p^{(\mu\nu)} + \frac{1}{c^2} (U^{\mu} q^\nu + U^{\nu} q^\mu), \]

(9)
provided with definitions

\[
p^{(\mu \nu)} = \sum_{a=1}^{2} p^{(\mu \nu)}_a = \left( \Delta^\sigma \Delta^\gamma - \frac{1}{3} \Delta^\mu \Delta^\nu \Delta^\sigma \right) \sum_{a=1}^{2} T^a_{\sigma \gamma}, \quad q^\mu = \sum_{a=1}^{2} \left( q^\mu_a + h_a J^\mu_a \right) = \sum_{a=1}^{2} \Delta^\nu T^a_{\mu \nu} U_\nu, \quad (10)
\]

\[
p + \varpi = \sum_{a=1}^{2} \left( p_a + \varpi_a \right) = -\frac{1}{2} \sum_{a=1}^{2} \Delta^\mu T^a_{\mu \nu}, \quad \text{ne} = \sum_{a=1}^{2} n_a e_a = \sum_{a=1}^{2} \frac{1}{2} U_\mu T^a_{\mu \nu} U_\nu, \quad (11)
\]

Here we have introduced: the partial pressure deviator tensor \( p^{(\mu \nu)}_a \), the partial heat flux \( q^\mu_a \), the partial enthalpy \( h_a = e_a + p_a/n_a \), the internal energy per particle for the mixture \( ne = n_1 e_1 + n_2 e_2 \), the local partial pressure \( p_a = n_a kT \) and the partial dynamic pressure \( \varpi_a \). Of course quantities without Latin subindex refer to the mixture.

Equations (10) and (11) depend on the form of the solution of the Boltzmann equation for the distribution function \( f_a \) and in this work we have used a linear approximation that have been obtained in [19] with the thermodynamic local variables \( \{ n_1, n_2, U^\mu, T \} \), being \( T \) the local temperature. Details of such approximation to the solution can be found in the cited reference. Let us mention here, taking as a basis the results of ref. [19], the process to obtain the constitutive equations for the thermodynamical fluxes.

Firstly, to obtain the Fick and Fourier laws we address Eqs. (48) and (55) from ref. [19]. Such equations, when applied to a binary mixture and after some algebraic manipulations lead us to the following linear system for the heat and particle partial fluxes:

\[
- \frac{n_1 + n_2}{n_1 n_2} d^\mu_i p = \left( A_{11} - 2A_{12} + A_{22} \right) J^\mu_1 - \left( F_{11} - F_{21} \right) q^\mu_1 - \left( F_{12} - F_{22} \right) q^\mu_2,
\]

\[
\frac{\nabla^\mu T}{T} = \left( F_{11} - F_{21} \right) J^\mu_1 - H_{11} q^\mu_1 - H_{12} q^\mu_2, \quad (13)
\]

\[
\frac{\nabla^\mu T}{T} = \left( F_{12} - F_{22} \right) J^\mu_1 - H_{21} q^\mu_1 - H_{22} q^\mu_2. \quad (14)
\]

Here we remind that number indexes refer to a species of the mixture, the total pressure is \( p = (n_1 + n_2) kT \). The thermodynamic force directly related with diffusion arises as

\[
d^\mu_i = \nabla^\mu \chi_1 + (\chi_1 - 1) \nabla^\mu \ln p - \frac{n_1 h_1 - n b_2}{p c^2} \Delta^\mu J^\mu \left[ U^\tau \frac{\partial U^\tau}{\partial x^\tau} - \frac{1}{1 - \Phi^2/4e^2} \frac{\partial \Phi}{\partial x^\tau} \right], \quad (15)
\]

where \( \chi_1 = p_j/p = n_1/n \) is the concentration of species labeled by 1 and \( \nabla^\mu = \Delta^\mu \nabla \) is the gradient operator. Equations (13) and (14) refer to a generalized thermal force defined as

\[
\nabla^\mu T = \frac{\nabla^\mu T}{T} - \frac{T}{c^2} \Delta^\mu \left[ U^\tau \frac{\partial U^\tau}{\partial x^\tau} - \frac{1}{1 - \Phi^2/4e^2} \frac{\partial \Phi}{\partial x^\tau} \right]. \quad (16)
\]

As mentioned in the introduction, the generalized diffusive and thermal forces (13) and (14) contain terms of relativistic nature as well as gravitational contributions. There are some comments to be made about the significance and physical consequences of these new terms and we are leaving them to the conclusion section.

Secondly, a process described in [19] leads to the obtention of the constitutive equation for the viscosities. The bulk viscosity for a binary mixture emerges through

\[
- \left[ \frac{\rho_1 kT}{c^3} \frac{\partial \ln c_1^\alpha}{\partial \ln \zeta_1} \right] \nabla^\mu U^\mu = R_{11} \varpi_1 + R_{12} \varpi_2, \quad - \left[ \frac{\rho_2 kT}{c^3} \frac{\partial \ln c_2^\alpha}{\partial \ln \zeta_2} \right] \nabla^\mu U^\mu = R_{21} \varpi_1 + R_{22} \varpi_2. \quad (17)
\]

Above we have introduced the specific heat at constant volume \( c^a_\alpha = k \left( c_\alpha^2 + 5G_a \zeta_\alpha - G_2^2 \zeta_\alpha^2 - 1 \right) \) of species \( a = 1, 2 \) with \( G_a = K_1(\zeta_\alpha)/K_2(\zeta_\alpha) \). \( K_n \) are modified Bessel functions of the second kind and order \( n \) (see definition (70) of Appendix A).
The shear viscosity emerges from the definition of the partial pressure deviator tensors $p^{(\mu\nu)}_1$: 
\[ 2\nabla^{(\mu U^\nu)} = K_{11}p^{(\mu\nu)}_1 + K_{12}p^{(\mu\nu)}_2, \quad 2\nabla^{(\mu U^\nu)} = K_{21}p^{(\mu\nu)}_1 + K_{22}p^{(\mu\nu)}_2. \] (18)

The set (12) - (14), (17) and (18) constitutes an algebraic system for the constitutive equations. The functions $A'$s, $F'$s, $H'$s, $R'$s and $K'$s are general integrals that depend on the interaction of the particles when they collide. We have listed them in the appendix B where the first equality represents the general case whereas in the second one they are evaluated for the case of hard-sphere interaction and non disparate masses and diameters.

3 Navier-Stokes law

In this section we analyze the constitutive equations for the pressure deviator tensor and dynamical pressure for a diluted binary mixture. Such equations emerge by solving the two algebraic systems represented in Eqs. (17) and (18) and taking the sum over the species. We obtain
\[ p^{(\mu\nu)} = 2\mu \nabla^{(\mu U^\nu)} \quad \text{and} \quad \varpi = -\eta \nabla U^\mu, \] (19)
where the total shear and bulk viscosities are $\mu$ and $\eta$, respectively. The transport coefficients read
\[ \mu = \frac{K_{11} - 2K_{12} + K_{22}}{K_{11}K_{22} - K_{12}K_{21}}, \] (20)
\[ \eta = \frac{kT(R_{22} - R_{21})p_1}{c^2(R_{11}R_{22} - R_{12}R_{21})} \left[ \frac{\partial \ln c_a^1}{\partial \ln c_a^1} + \frac{(R_{11} - R_{12})n_2}{(R_{22} - R_{21})n_1} \frac{\partial \ln c_a^2}{\partial \ln c_a^1} \right], \] (21)
where the elements of the matrices $K_{ab}$ and $R_{ab}$ are integrals given in Appendix B (See Eqs. (59) – (60)). Furthermore, we have introduced the derivative of the heat capacity per particle at constant volume:
\[ \frac{\partial \ln c_a^b}{\partial \ln c_a^1} = \frac{\zeta_a(20G_a + 3\zeta_a - 13G_a^2\zeta_a - 2G_a\zeta_a^2 + 2G_a^3\zeta_a^2)}{(1 + G_a\zeta_a^2 - \zeta_a^2 - 5G_a\zeta_a)}, \quad a = 1, 2. \] (22)

If we consider that the rest masses, the particle number densities and the differential cross-sections of both constituents are the same – i.e., $m_1 = m_2 = m$, $n_1 = n_2 = n$ and $\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma$ – we get that the shear (20) and bulk (21) viscosities reduce to their expressions of a single constituent, namely
\[ \mu = \frac{15}{64\pi} \frac{kT}{c^2} \frac{\zeta^4K_3(\zeta)^2}{(2 + 15\zeta^2)K_2(2\zeta) + (3\zeta^3 + 49\zeta)K_3(2\zeta)^2}, \] (23)
\[ \eta = \frac{1}{64\pi} \frac{kT}{c^2} \frac{\zeta^4K_2(2\zeta)^2}{(20G + 3\zeta - 13G^2\zeta - 2G\zeta^2 + 2G^3\zeta^2)^2} \left[ (1 - 5G - \zeta^2 + G^2\zeta^2)^2 \right]. \] (24)

The limiting cases of low temperature (non-relativistic $\zeta \gg 1$) and high temperature (ultra-relativistic $\zeta \ll 1$) are well known and can be obtained from (23) and (24) as it is shown in the corresponding literature [12].

To determine the $K'$s and $R'$s functions contained in the the transport coefficients (20) and (21) we incorporate two physical hypotheses. Firstly we consider similar masses of the particles of different species, so we can write $m_2 = m_1(1 + \epsilon)$ with $\epsilon \ll 1$. Secondly we assume hard-spheres, so the diameters of the particles are constant. Therefore a small difference is assumed for the diameter of a particle of the species 2 with respect to the diameter of a particle of the species 1. That is $d_2 = d_1(1 + \xi)$ with $\xi \ll 1$. The hard-sphere differential cross sections can be written according to $\sigma_{11} = d_1^2/4$, $\sigma_{22} = d_2^2/4$ and $\sigma_{12} = (d_1 + d_2)^2/16$. Consequently we have the relations $\sigma_{11} = \sigma$, $\sigma_{12} = \sigma(1 + \xi)$ and $\sigma_{22} = \sigma(1 + 2\xi)$ with $\sigma$=constant. The evaluation of the integrals is a long task, we have depicted the principal steps in the appendix A whereas the second identity of the list of appendix B represents the expressions finally evaluated.
Figure 1: Dimensionless shear viscosity $\mu^*$ as function of the parameter $\zeta_1 = m_1 c^2 / kT$. The thick line represents the single fluid and the others are for the mixture with $\epsilon = 0.01$, $\xi = 0.1$ and different ratios $n_2/n_1$.

Figure 2: Dimensionless bulk viscosity $\eta^*$ as function of the parameter $\zeta_1 = m_1 c^2 / kT$. The thick line represents the single fluid and the others for the mixture with $\epsilon = 0.01$, $\xi = 0.1$ and different ratios $n_2/n_1$. 
By substituting the corresponding functions of appendix B into the shear and bulk viscosities and rewriting them as dimensionless quantities in the form \( \mu = 10\mu c\sigma_{11}/3kT \) and \( \eta = \eta c\sigma_{11}/kT \), we can plot them with respect to the relativistic parameter \( \zeta \). Figure 1 shows the behavior of \( \mu \) as function of \( \zeta \) when \( \epsilon = 0.01 \), \( \xi = 0.1 \) and different concentrations of the ratio \( n_2/n_1 \) are imposed. On the other hand we plot the dimensionless bulk viscosity in Figure 2 where the same conditions for the mass and concentration differences were assumed. We shall leave the comments of the figures to the conclusion section. Now we shall focus on the thermal conductivity, diffusion and thermal diffusion.

4 Fourier and Fick laws

In this section we analyze the constitutive equations for the heat and diffusive fluxes and the corresponding transport coefficients. We shall follow a similar process of the last section in which the general expressions are given, the one-component limit is recovered and a plot of the general expressions is shown. In the present article we are interested in the description of the Fick and Fourier laws. The Fick law describes the flux of particles due to the thermal and diffusive thermodynamic forces, see Eqs. (12) and (14). The Fourier law establishes that the thermal conductivity is the proportionality coefficient between the heat flux and the thermal force in the absence of diffusion. Given the objective of the present section we start by solving the system of equations (12)–(14) for the heat flux and neglecting the diffusion terms, then we obtain the Fourier law for the case of a binary mixture:

\[
q^\mu = \lambda \nabla^\nu T, \quad \text{where} \quad \lambda = -\frac{H_{11} + H_{22} - 2H_{12}}{T(H_{11}H_{22} - H_{12}^2)},
\]

where \( \lambda \) is the thermal conductivity. The \( H \)'s functions are given by Eqs. (82)–(84) in the Appendix B. To analyze the limit to a single component we again consider the conditions \( m_1 = m_2 = m, n_1 = n_2 = n \) and \( \sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma \). Then the thermal conductivity coefficient reduces to

\[
\lambda = \frac{3}{64\pi} \frac{c k (\zeta + 5G - G^2\zeta^2)\zeta^2K_2(\zeta)^2}{\sigma (\zeta^2 + 2)K_2(2\zeta) + 5\zeta K_3(2\zeta)},
\]

which is in accordance with the expression \([11, 12]\) obtained when the kinetic theory is applied to a single fluid. Of course this last expression guarantees the corresponding limiting cases for low temperatures (non-relativistic \( \zeta \gg 1 \)) and high temperatures (ultra-relativistic \( \zeta \ll 1 \)).

Figure 3 shows the dimensionless thermal conductivity \( \lambda_* = \lambda c\sigma_{11}m_1/kT \) as function of \( \zeta_1 \) with \( \epsilon = 0.01 \), \( \xi = 0.1 \) and different ratios \( n_2/n_1 \). Some comments about this graphic are given in the conclusion section.

Now we focus on the Fick law. For the case of the binary mixture we have only one diffusion flux as it can be appreciated from Eqs. (12)–(14). By solving for the particle flux we obtain:

\[
J^\mu = nD \left( d^\mu + \frac{kT}{T} \nabla^\nu T \right),
\]

where we have redefined \( J^\mu_i \) as \( J^\mu \) and \( d^\mu_i \) as \( d^\mu \). Here \( D \) and \( k_T \) are the coefficients of diffusion and thermal-diffusion ratio, respectively. They are given by

\[
D = \frac{kT/n}{A_{12} + \frac{F_{21}F_{22}}{H_{11}H_{22}H_{12}} \left[ \frac{2F_{21}}{n_1F_{12}}H_{22} + \frac{F_{21}}{n_2F_{21}}H_{11} + 2H_{12} \right]},
\]

\[
k_T = \frac{1}{p} \left[ \frac{n_2F_{21}(H_{12} - H_{22}) + n_1F_{12}(H_{11} - H_{12})}{H_{11}H_{22} - H_{12}^2} \right],
\]

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Figure 3: Dimensionless thermal conductivity $\lambda_*$ as function of the parameter $\zeta_1 = m_1 c^2 / kT$. The thick line represents the single fluid. Different scenarios for the mixture are plotted with $\epsilon = 0.01$, $\xi = 0.1$ and some ratios $n_2/n_1$. 
where the elements of the matrices $A_{ab}$, $F_{ab}$, $H_{ab}$ are given by the expressions (77) – (84) of the Appendix B.

Now we can obtain the one-species limit from the general Fick law Eq. (27) by setting $m_1 = m_2 = m$, $n_1 = n_2 = n$, and $\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma$. Equation (28) leads to the self-diffusion coefficient, but it is quite long to be shown here. Its low temperature (non-relativistic $\zeta \gg 1$) and high temperature (ultra-relativistic $\zeta \ll 1$) limiting cases read

$$D = \frac{3}{8n^2} \sqrt{\frac{kT}{\pi m_1}} \left( 1 - \frac{8}{15\zeta} + \ldots \right), \quad \zeta \gg 1, \quad (30)$$

$$D = \frac{6c}{13\pi n^2} \left( 1 - \frac{2\zeta^2}{13} + \ldots \right), \quad \zeta \ll 1. \quad (31)$$

Note that the low temperature self-diffusion coefficient (30) is proportional to $\sqrt{T/n}$ in accordance with the first non-relativistic approximation for a gas of hard-sphere particles [23, 24, 25]. On the other hand, the one-species limit for the thermal-diffusion coefficient $k_T$ goes to zero (note that $H_{11} = H_{22}$ and $F_{12} = F_{21}$), as expected from the kinetic theory when applied to a single component.

Figures 4 and 5 show the behavior of the dimensionless diffusion coefficient $D_*$ as function of the parameter $\zeta_1 = m_1 c^2 / kT$ for $\epsilon = 0.01$, $\xi = 0.1$ and different ratios $n_2/n_1$. The thick line represents the self-diffusion coefficient.

Figures 4 and 5 show the behavior of the dimensionless diffusion coefficient $D_*$ as function of the parameter $\zeta_1 = m_1 c^2 / kT$ for $\epsilon = 0.01$, $\xi = 0.1$ and different ratios $n_2/n_1$. The thick line represents the self-diffusion coefficient.

Comments are left to the conclusions. In the following section we are going to show the explicit dependence of the transport coefficients on the gravitational potential.
Figure 5: Thermal-diffusion ratio $k_T$ as function of the parameter $\zeta_1 = m_1 c^2 / kT$ for $\epsilon = 0.01$, $\xi = 0.1$ and different ratios $n_2/n_1$.

5 The gravitational field dependence on the transport coefficients

In this section we explore the dependence of the transport coefficients on the gravitational field. We quote two works [15, 18] where such a dependence arises naturally from the kinetic theory point of view. Then more discussion is needed and here we analyze the corresponding transport coefficients for a binary mixture.

Let us begin by noticing that the first term of the energy momentum tensor Eq. (9) can be written in the comoving frame, i.e. $U^{\mu} = (c/\sqrt{g_0}, 0)$, as:

$$\frac{ne}{c^2} U^\mu U^\nu = ne \left( 1 - \frac{\Phi}{c^2} \right) \equiv \bar{ne},$$

where the quantity $\bar{ne}$ is now playing the role of the internal energy density.

We can also write the projector $\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu/c^2$ with the Schwarzschild metric in the comoving frame, in this case the components of the projector become

$$\Delta^{00} = 0, \quad \Delta^{ij} = -\frac{\delta^{ij}}{(1 - \frac{\Phi}{c^2})^2}.$$ (33)

The pressure tensor $P^{\mu\nu}$ is defined as the second and third term of the energy-momentum tensor Eq. (9). When the constitutive equations for the bulk and shear viscosities (Eqs. (19)) are substituted in $P^{\mu\nu}$ we have

$$P^{\mu\nu} = -p\Delta^{\mu\nu} + \eta\nabla_\gamma U^\gamma \Delta^{\mu\nu} + 2\mu \left( \frac{\Delta^\mu_\gamma \Delta^\nu_\tau + \Delta^\nu_\gamma \Delta^\mu_\tau}{2} - \frac{\Delta^{\mu\nu} \Delta_{\sigma\tau}}{3} \right) \partial^\sigma U^\tau.$$ (34)
The evaluation of Eq. (34) as well as the Fourier (25) and Fick (27) laws in the comoving frame and in Cartesian coordinates leads to

\[ \mathbf{P}^{ij} = \left[ \tilde{p} - \tilde{\eta} \frac{\partial U^k}{\partial x^k} \right] \delta^{ij} - \tilde{\mu} \left[ \frac{\partial U^i}{\partial x_j} + \frac{\partial U^j}{\partial x_i} - \frac{2}{3} \frac{\partial U^k}{\partial x^k} \delta^{ij} \right], \]  
\[ q^i = -\tilde{\lambda} \frac{\partial T}{\partial x_i}, \quad J^i = -n \tilde{D} \left[ d^i + \frac{k_T}{T} \frac{\partial T}{\partial x_i} \right]. \]  

Here the generalized thermal and diffusion forces read

\[ \frac{\partial T}{\partial x_i} = \frac{\partial T}{\partial x_i} - \frac{T}{c^2} \left( \dot{U}^i - \frac{1}{1 - \frac{\Phi^2}{4c^4}} \frac{\partial \Phi}{\partial x_i} \right), \]  
\[ d^i = \frac{\partial x_1}{\partial x_i} + (x_1 - 1) \frac{\partial \ln p}{\partial x_i} + \frac{n_2 b_2}{pc^2} \left( \dot{U}^i - \frac{1}{1 - \frac{\Phi^2}{4c^4}} \frac{\partial \Phi}{\partial x_i} \right), \]  

respectively. In the above equations \( \dot{U}^i \) is the acceleration.

The coefficients with tildes in Eqs. (35), (36a) and (36b) are defined as follows:

\[ \tilde{p} = \frac{p}{(1 - \frac{\Phi^2}{2c^2})^4}, \quad \tilde{\eta} = \frac{\eta}{(1 - \frac{\Phi^2}{2c^2})^8}, \quad \tilde{\mu} = \frac{\mu}{(1 - \frac{\Phi^2}{2c^2})^8}, \]  
\[ \tilde{\lambda} = \frac{\lambda}{(1 - \frac{\Phi^2}{2c^2})^4}, \quad \tilde{D} = \frac{D}{(1 - \frac{\Phi^2}{2c^2})^4}. \]  

The above equations show a gravitational field dependence of the transport coefficients, we leave the analysis to the conclusion section.

6 Conclusions

In this work we have evaluated analytically the transport coefficients for a relativistic binary mixture of ideal gases in the presence of a gravitational field. This work represents the analysis of a particular case (binary mixture) of the \( r \)-mixture studied by the authors in ref. [19] in which, among other results, the transport coefficients were given in terms of general integrals. We applied the combined Chapman-Enskog linear method of solution to the Boltzmann equation and obtain the transport coefficients associated with the Navier-Stokes, Fourier and Fick laws. Such expressions are represented by general integrals for the shear (20) and bulk (21) viscosities, thermal conductivity (25b), diffusion (28) and thermal-diffusion ratio (29). The process for the evaluation of the integrals represents a hard task and in the appendix A such calculations are addressed. In the development of the integrals we assumed two physical hypotheses: Firstly, the masses of the particles of each species are similar i.e. \( m_2 = m_1(1 + \epsilon) \) with \( \epsilon \ll 1 \). Secondly, for the particles with hard-sphere interaction, we can assume that the diameters of the particles of each species are very similar, i.e. \( d_2 = d_1(1 + \xi) \) with \( \xi \ll 1 \). Once upon the transport coefficients were evaluated, we presented a figure for each one showing its dependence with respect to the relativistic parameter \( \zeta_1 \). Lastly we analyzed dependence of the transport coefficients on the gravitational potential and the corrections for the pressure and internal energy. Now we shall comment the results.

Following the theory of fluid mixtures (see e.g. [23, 24]) it is usual to express the diffusion force in terms of gradients of pressure and concentration. From the expression for the generalized diffusive force obtained in this article, see Eq. (15), we can identify four contributions to it: 1) a concentration gradient, 2) a pressure gradient, 3) terms proportional to the four-acceleration, and 4) terms proportional the gravitational potential gradient. The flux of particles or Fick’s law is proportional to the diffusive and thermal forces (see Eq. (27)). Then we can conclude that the nature of this transport is a consequence of:
• A concentration gradient that tends to reduce the non-homogeneity of the mixture.
• A pressure gradient, where heavy particles tend to diffuse to places with high pressures, e.g. in centrifuges.
• A temperature gradient, that transports the matter to a warmer regions.
• An acceleration which acts on different masses.
• A gravitational potential gradient.

There is a very interesting physical situation that we can underline. In the particular case where the acceleration is absent and the pressure and temperature are constant, the diffusion vanishes if the gradient of concentration counterbalances the gradient of gravitational potential. On the other hand, it is important to note that the last two terms of the diffusive force \(15\) are a combination of the acceleration and gravitational potential and are not of relativistic nature, since

\[
\frac{(n_a h_a - n h)}{c^2 \rho} \to \frac{(n_a m_a - \sum_{b=1}^{2} n_b m_b)}{\rho}
\]  

in the non-relativistic limiting case. Therefore we conclude that equation \(15\) is the generalization of the diffusion force originally written for the non-relativistic case \([23, 24]\).

Let us focus now on the generalized thermal force Eq. \(16\). There we identify that it has three contributions: 1) a pure gradient of temperature, 2) terms proportional to the four-acceleration and 3) terms proportional to the gravitational potential gradient. When the non-relativistic limit is analyzed, the term that is a combination of the acceleration and gravitational potential goes to zero (contrary to the case of the diffusive force) since it has a factor \(T/c^2\) of relativistic order. When the generalized thermal force \(16\) is substituted into the Fourier law \(25\), an acceleration term \(U^\nu \partial U_i/\partial x^\nu\) appears that coincides with the one presented in a phenomenological analysis made by Eckart \([20]\). Such a term represents an isothermal heat flux when matter is accelerated.

Let us now analyze a consequence of the gravitational potential gradient \(\partial \Phi/\partial x^i\) dependence on Eq. \(16\). First we analyze the corresponding factor to such gradient in the comoving frame \((U^\mu = (c/\sqrt{g_0}, 0))\), by expanding it with \(\Phi/c^2 \ll 1\) (weak field), so we have

\[
\frac{\Delta^{ij}}{1 - \Phi^2/4c^4} \simeq -\delta^{ij} \left[ 1 + \frac{2\Phi}{c^2} + \mathcal{O}\left(\frac{\Phi}{c^2}\right)^2 \right].
\]  

(42)

The Tolman law \([21, 22]\) comes from phenomenological arguments, and states that at equilibrium, a gradient of temperature is counterbalanced by a gravitational potential gradient. In a physical situation where the heat flux and the acceleration term vanish we get from equation \(16\) when \(42\) is inserted at lowest order

\[
\frac{\nabla T}{T} = -\frac{\nabla \Phi}{c^2},
\]  

(43)

i.e. we recover Tolman’s law. It is important to mention that, from the kinetic theory point of view, this fact was initially presented in ref. \([16, 15, 18, 19]\).

Now let us pay attention on the thermodynamic variables. Note that the quantity \(\tilde{\rho}\) from equation \(39i\) is playing the role of the pressure and it has a factor that depends on the gravitational potential. If we expand for a weak field, i.e. \(\Phi/c^2 \ll 1\), we obtain

\[
\tilde{\rho} = \rho \left( 1 + \frac{2\Phi}{c^2} + \mathcal{O}\left(\frac{\Phi}{c^2}\right)^2 \right).
\]  

(44)
In the same fashion, for $\Phi/c^2 \ll 1$, we expand Eq. (32) and obtain

$$\tilde{n}e = ne \left(1 - \frac{2\Phi}{c^2} + O\left(\frac{\Phi}{c^2}\right)^2\right). \quad (45)$$

Here at first order we have recovered the first post Newtonian approximation (1PN) for the pressure and energy density as given by Weinberg [26]. Recently, it has been shown the same dependence for non-relativistic, the pressure by using the 1PN formalism in a different research [27].

On the other hand, we can analyze the gravitational dependence of the one-component limit of Eqs. (39), (32), (40a) and (40b) for the bulk and shear viscosities, thermal conductivity and diffusion, respectively. We expand for a low temperature (non-relativistic, $\zeta \gg 1$) and weak field $\Phi/c^2 \ll 1$ case and we get:

$$\tilde{\eta} = \frac{25}{64d^2\zeta^2} \sqrt{\frac{mkT}{\pi}} \left(1 - \frac{183}{16} \frac{1}{\zeta^2} + \cdots\right) \left(1 - \frac{4GM}{c^2r} + \frac{9(GM)^2}{c^4r^2} - \cdots\right) \quad (46)$$

$$\tilde{\mu} = \frac{5}{16d^2} \sqrt{\frac{mkT}{\pi}} \left(1 + \frac{25}{16\zeta} + \cdots\right) \left(1 - \frac{4GM}{c^2r} + \frac{9(GM)^2}{c^4r^2} - \cdots\right) \quad (47)$$

$$\tilde{\lambda} = \frac{75}{64d^2} \frac{k}{m} \sqrt{\frac{mkT}{\pi}} \left(1 + \frac{13}{16\zeta} + \cdots\right) \left(1 - \frac{2GM}{c^2r} + \frac{5(GM)^2}{2c^4r^2} - \cdots\right) \quad (48)$$

$$D = \frac{3}{8nd^2} \sqrt{\frac{kT}{\pi m_1}} \left(1 + \frac{8}{15\zeta} + \cdots\right) \left(1 - \frac{2GM}{c^2r} + \frac{5(GM)^2}{2c^4r^2} - \cdots\right). \quad (49)$$

Here we conclude that the transport coefficients become smaller in the presence of a gravitational field. The dependence of the thermal conductivity on the gravitational field has been studied and reported in ref. 28 from a non-relativistic approach based on the Boltzmann equation. Other studies where the relativistic kinetic theory has been used in the presence of curved space-times are refs. 15–18. However, this dependence is small for non compact objects. We can list some values for the surface of some gravitational sources:

1. Earth: $M_\oplus \approx 5.97 \times 10^{24}$ kg; $R_\oplus \approx 6.38 \times 10^6$ m; $|\Phi(R_\oplus)|/c^2 \approx 7 \times 10^{-10};$
2. Sun: $M_\odot \approx 1.99 \times 10^{30}$ kg; $R_\odot \approx 6.96 \times 10^8$ m; $|\Phi(R_\odot)|/c^2 \approx 2.2 \times 10^{-6};$
3. White dwarf: $M \approx 1.02M_\odot$; $R \approx 5.4 \times 10^6$ m; $|\Phi(R)|/c^2 \approx 2.8 \times 10^{-4};$
4. Neutron star: $M \approx M_\odot$; $R \approx 2 \times 10^4$ m; $|\Phi(R)|/c^2 \approx 7.5 \times 10^{-2}.$

Lastly we comment the behavior of the transport coefficients with the help of the figures. We note that:

- All transport coefficients (shear and bulk viscosities, thermal conductivity, diffusion and thermal-diffusion ratio) have larger values in the non-relativistic limiting case $\zeta \gg 1$ (low temperatures) than those corresponding to the ultra-relativistic limiting case $\zeta \ll 1$ (high temperatures).

- Shear and bulk viscosities and thermal conductivity have smaller values in comparison with the corresponding ones for a single constituent. Setting particles of species 2 as the heavier ones, we note that the value of these transport coefficients decrease by increasing the concentration of species 2.

- The diffusion and thermal-diffusion coefficients increase when the concentration of the species with heavier particles grows.

- The thermal-diffusion rate is negative indicating that the constituent 1 tends to move into the warmer region and the constituent 2 towards the cooler region.
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A Evaluation of the integrals

In this appendix we show the main steps for the evaluation of the integrals listed in the appendix B. We start by introducing the definition of the total momentum $P^\mu$ and relative momentum $Q^\mu$ four-vectors through the relationships (see e.g. [2])

$$P^\mu \equiv p_a^\mu + p_b^\mu, \quad P^\mu \equiv p_a^\mu + p_b^\mu, \quad Q^\mu = p_a^\mu - p_b^\mu, \quad Q^\mu = p_a^\mu - p_b^\mu. \quad (50)$$

By using the energy-momentum conservation law i.e. $p_a^\mu + p_b^\mu = p_a^\mu + p_b^\mu$, above equations lead to

$$P^\mu = Q^\mu, \quad P^\mu Q_\mu = (m_a^2 - m_b^2)c^2, \quad Q^2 = P^2 - 2(m_a^2 + m_b^2)c^2, \quad (51)$$

where the magnitudes of the total and relative momentum four-vectors are given by $P^2 = P_\mu P^\mu$ and $Q^2 = -Q^\mu Q_\mu$, respectively. From (50) it follows the inverse transformations

$$p_a^\mu = \frac{P^\mu}{2} + \frac{Q^\mu}{2}, \quad p_b^\mu = \frac{P^\mu}{2} - \frac{Q^\mu}{2}, \quad p_a^\mu = \frac{P^\mu}{2} + \frac{Q^\mu}{2}, \quad p_b^\mu = \frac{P^\mu}{2} - \frac{Q^\mu}{2}. \quad (52)$$

The Jacobian of the transformation from $(p_a^\mu, p_b^\mu)$ to $(P^\mu, Q^\mu)$ is 1/8 so that $d^3p_a d^3p_b = d^3P d^3Q/8$.

If we introduce a space-like unit vector $k^a$ which is orthogonal to $P^\mu$ ($k^a P_\mu = 0$) the relative momentum four-vector can be decomposed as

$$Q^\mu = (m_a^2 - m_b^2)c^2\frac{P^\mu}{P^2} + \frac{k^\mu}{P} \sqrt{P^4 - 2P^2(m_a^2 + m_b^2)c^2 + (m_a^2 - m_b^2)^2c^4}. \quad (53)$$

Here we shall restrict ourselves to the case where the rest masses of the particles of the constituents are not too disparate, that is $m_b \approx m_a$. So we have $(m_a^2 + m_b^2) \approx 2m_a m_b$ and terms of higher order are neglected. In this approximation $P^\mu Q_\mu = 0$, and the relative momentum four-vector (50) and its modulus can be approximated by

$$Q^\mu = Q^a k^a, \quad Q^2 = P^2 - 4m_a m_b c^2. \quad (54)$$

This approximation transforms the invariant flux as

$$F_{ba} = \sqrt{(p_a^\mu p_b^\mu)^2 - m_a^2 m_b^2 c^4} = \frac{PQ}{2}. \quad (55)$$

Now the center-of-mass system is chosen where the spatial components of the total momentum four-vector vanish. Hence by introducing the representations $(P^\mu) = (P^0, 0)$ and $(Q^\mu) = (0, Q)$ it follows the relationship

$$F_{ba} \frac{d^3p_a}{p_{a0}} \frac{d^3p_b}{p_{b0}} = \frac{Q}{4 \sqrt{g_0}} \frac{d^3P}{P_0} d^3Q. \quad (56)$$

Furthermore, we write the element of solid angle as $d\Omega = \sin \Theta d\Theta d\Phi$, with $\Theta$ and $\Phi$ denoting the polar angles of the spatial components of $Q^a$ with respect to $Q^\alpha$ and such that $\Theta$ represents the scattering angle. The differential cross sections $\sigma_{ab}$ depend on the scattering angle $\Theta$ and on the modulus of the relative momentum four-vector $Q$ so that $\sigma_{ab} = \sigma_{ab}(Q, \Theta)$. In this work we are
interested in analyzing a mixture where the differential cross sections are constant, which in the non-relativistic case corresponds to a mixture of hard spheres particles. Besides we may assume without loss of generality that the spatial component of \( Q^\alpha \) is in the direction of the axis \( x^3 \), so that we can represent \( Q^\alpha \) and \( Q'^\alpha \) as:

\[
(Q^\mu) = Q \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (Q'^\mu) = Q \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix}.
\]  

(57)

By taking into account the above premisses we can obtain the following results

\[
\int (p^a_\mu - p^a_\nu)d\Omega = \frac{1}{2} \int (Q'^\mu - Q^\mu)d\Omega = -2\pi Q^\mu,
\]

(58)

\[
\int (p^a_\mu p^a_\nu - p^a_\nu p^a_\mu)d\Omega = \frac{\pi}{3} Q^2 \left( \frac{P^\mu P^\nu}{P^2} - g^{\mu\nu} - 3 \frac{Q^\mu Q^\nu}{Q^2} \right) - \pi(P^\mu Q^\nu + P^\nu Q^\mu).
\]

(59)

Furthermore, it is also possible to perform the integrations in the spherical angles of \( Q^\mu \), denoted by \( \theta \) and \( \phi \). We write

\[
(Q^\mu) = Q \begin{pmatrix} 0 \\ \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix},
\]

(60)

with \( d^3Q = Q^2 \sin \theta \, d\theta \, d\phi \, dQ = Q^2 \sin \theta \, d\Omega \, dQ \), where \( d\Omega = \sin \theta \, d\theta \, d\phi \) denotes an element of solid angle. From the above considerations it is easy to find the expressions for the integrals:

\[
\int d\Omega^* = 4\pi, \quad \int Q^\mu d\Omega^* = \int Q^\mu Q'^\nu d\Omega^* = 0, \quad \int Q^\mu Q'^\nu d\Omega^* = \frac{4\pi}{3} Q^2 \left( \frac{P^\mu P^\nu}{P^2} - g^{\mu\nu} \right),
\]

(61)

\[
\int Q^\mu Q'^\nu Q'^\rho d\Omega^* = \frac{4\pi}{15} Q^4 \left[ 3 \frac{P^\mu P^\nu P^\rho}{P^4} - \frac{1}{P^2} \left( g^{\mu\nu \rho} P^\tau + g^{\mu\rho \tau} P^\nu + g^{\nu \tau \rho} P^\mu \right) + g^{\mu\nu} g^{\rho \tau} + g^{\mu \rho} g^{\nu \tau} + g^{\nu \rho} g^{\mu \tau} \right].
\]

(62)

The integration in the total momentum four-vector can be performed and for that end we consider a comoving frame where \( (U^\mu) = (c/\sqrt{\gamma_0}, 0) \), a spherical coordinate system and write \( d^3P = |P|^2 d|P| \sin \theta \, d\phi \, d\theta \). We introduce a new variable of integration

\[
x = \sqrt{\zeta_a \zeta_b \left( \frac{Q^2}{m_a m_b c^2} + 4 \right)}, \quad Q \, dQ = \left( \frac{kT}{c} \right)^2 x \, dx,
\]

(63)

where the range of this new variable is \( 2\sqrt{\zeta_a \zeta_b} \leq x < \infty \). Hence, we can write the time and spatial coordinates of the total momentum four-vector as

\[
P_0 = \frac{kT}{c} \sqrt{\gamma_0} x y, \quad |P| = \frac{kT}{c} x \sqrt{y^2 - 1} \sqrt{\gamma_1}, \quad P = \frac{kT}{c} x.
\]

(64)

Hence, the element of integration becomes

\[
\sqrt{\gamma_1 \gamma_0} \frac{d^3P}{P_0} = \left( \frac{kT}{c} \right)^2 x^2 \sqrt{y^2 - 1} \sin \theta \, d\phi \, d\theta \, dy,
\]

(65)

while the range of integration of the new variable \( y \) is now \( 1 \leq y < \infty \).
The integration over the solid angle and over the variable $y$ can be performed and we obtain the following results:

\[
\sqrt{g_1^3} \int e^{-\frac{\rho^\mu \cdot \nu}{x^2}} \frac{d^3 P}{P^0} = 4\pi \left( \frac{kT}{c} \right)^2 x K_1(x), \quad \sqrt{g_1^3} \int e^{-\frac{\rho^\mu \cdot \nu}{x^2}} \frac{P^\mu d^3 P}{P^0} = 4\pi \left( \frac{kT}{c} \right) K_2(x) \frac{U^\mu}{c}, \quad (66)
\]

\[
\sqrt{g_1^3} \int e^{-\frac{\rho^\mu \cdot \nu}{x^2}} \frac{P^\mu P^\nu d^3 P}{P^0} = 4\pi \left( \frac{kT}{c} \right)^2 \left[ x K_3(x) \frac{U^\mu U^\nu}{c^2} - K_2(x) g^\mu g^\nu \right], \quad (67)
\]

\[
\sqrt{g_1^3} \int e^{-\frac{\rho^\mu \cdot \nu}{x^2}} \frac{P^\mu P^\nu P^\sigma d^3 P}{P^0} = 4\pi \left( \frac{kT}{c} \right)^3 \left[ x^2 K_4(x) \frac{U^\mu U^\nu U^\sigma}{c^3} + x K_5(x) \left( g^\mu \frac{U^\nu}{c} - g^\nu \frac{U^\mu}{c} \right) \right], \quad (68)
\]

\[
\sqrt{g_1^3} \int e^{-\frac{\rho^\mu \cdot \nu}{x^2}} \frac{P^\mu P^\nu P^\sigma P^\tau d^3 P}{P^0} = 4\pi \left( \frac{kT}{c} \right)^2 \left[ \frac{K_3(x)}{x} \left( g^\mu \frac{g^\sigma}{c^2} + g^\nu \frac{g^\tau}{c^2} + g^\tau \frac{g^\sigma}{c^2} \right) \right] + x K_5(x) \left( \frac{g^\mu U^\nu U^\sigma}{c^2} \right), \quad (69)
\]

Above $K_n(x)$ are modified Bessel functions of second kind

\[
K_n(x) = \left( \frac{x}{2} \right)^n \frac{\Gamma(1/2)}{\Gamma(n + 1/2)} \int_0^\infty e^{-x y} (y^2 - 1)^{n - \frac{1}{2}} dy, \quad (70)
\]

with $n = 0, 1, 2, \ldots$.

Furthermore, in order to perform the integrations in the variable $x$ we need the following integrals of Bessel functions, where $\chi = 2\sqrt{\alpha_1 \alpha_2}$:

\[
\int_x^\infty x^7 K_2(x) dx = \chi^5 \left[ \chi^2 + 48 \chi K_5(\chi) - 4 \chi K_6(\chi) \right], \quad \int_x^\infty x^5 K_2(x) dx = \chi^4 \left[ \chi K_5(\chi) - 6 K_4(\chi) \right], \quad (71)
\]

\[
\int_x^\infty x^3 K_2(x) dx = \chi^3 K_3(\chi), \quad \int_x^\infty x K_2(x) dx = \chi K_1(\chi) + 2 K_0(\chi), \quad \int_x^\infty x^4 K_3(x) dx = \chi^4 K_4(\chi), \quad (72)
\]

\[
\int_x^\infty x^8 K_3(x) dx = \chi^6 \left[ \chi^2 + 8 \chi K_6(\chi) - 6 \chi K_5(\chi) \right], \quad \int_x^\infty x^6 K_3(x) dx = \chi^5 \left[ \chi K_6(\chi) - 8 K_5(\chi) \right], \quad (73)
\]

\[
\int_x^\infty x^2 K_3(x) dx = \chi^2 K_2(\chi) + 4 \chi K_1(\chi) + 8 K_0(\chi), \quad \int_x^\infty K_3(x) dx = K_2(\chi) + \frac{2}{\chi} K_1(\chi). \quad (74)
\]

## B  Expressions for the elements of the matrices for binary mixtures

From the above considerations we can perform the integrals

\[
\int \psi_b \delta_{ab} \left[ \phi_a \right] \frac{d^3 p_a}{P^0} = \int \psi_b f_a^{(0)} (\phi_a - \phi_a) F_{ab \sigma \tau} d\Omega \sqrt{-g} \frac{d^3 p_b}{P^0} \frac{d^3 p_a}{P^0} \quad (75)
\]

where $\phi_a = \phi_a(p_a^\mu)$, $\psi_a = \psi_a(p_a^\mu)$ are functions of the momentum four-vector of the particles $p_a^\mu$ and $f_a^{(0)}$ is the Maxwell-Jüttner distribution function, which in a comoving frame reads

\[
f_a^{(0)} = \frac{n_a}{4\pi kT m_a^2 c K_2(\zeta_a)} e^{-\frac{\sqrt{m_a^2 + q_1^2 + q_2^2}}{c T}}, \quad (76)
\]

\[16\]
In the following list the first identity is a general integral and the second one is evaluated when similar masses of the species and hard-sphere interaction are considered. The resulting expressions for the elements of the matrices for a binary mixture are:

\[
A_{12} = \frac{c\Delta^{\mu\nu}}{3n_1n_2kT} \int p_{1\mu} T_{12}[p_{2\nu}] \sqrt{-g} \frac{d^3p_1}{p_{10}} - \frac{16\pi kT\sigma_{12}}{3cK_2(\zeta_1)K_2(\zeta_2)} h_{12}^I, \quad (77)
\]

\[
A_{11} = -\frac{c\Delta^{\mu\nu}}{3n_1n_2kT^2} \sum_{b=1}^2 \int p_{1\mu} T_{1b}[p_{1\nu}] + \int p_{1\mu} T_{11}[p_{1\nu}] \sqrt{-g} \frac{d^3p_1}{p_{10}} = -\frac{n_2}{n_1} A_{12}, \quad A_{22} = -\frac{n_1}{n_2} A_{12}, \quad (78)
\]

\[
F_{12} = -\frac{c\Delta^{\mu\nu}}{3n_1n_2kT^2} \int p_{1\mu} T_{12} \left[ \frac{\zeta_1}{c_p^2} \left( G_2 - \frac{U_p^2 c_2^2}{m_2 c^2} \right) p_{2\nu} \right] \sqrt{-g} \frac{d^3p_1}{p_{10}} = -\frac{16\pi kT\sigma_{12}}{3cK_2(\zeta_1)K_2(\zeta_2)c_p^2} \left[ 2h_{12}^{II} - \zeta_2 G_2 h_{12}^I \right], \quad (79)
\]

\[
F_{11} = -\frac{c\Delta^{\mu\nu}}{3n_1n_2kT^2} \sum_{b=1}^2 \int p_{1\mu} T_{1b} \left[ \frac{\zeta_1}{c_p^2} \left( G_1 - \frac{U_p^2 c_1^2}{m_1 c^2} \right) p_{1\nu} \right] + \int p_{1\mu} T_{11} \left[ \frac{\zeta_1}{c_p^2} \left( G_1 - \frac{U_p^2 c_1^2}{m_1 c^2} \right) p_{1\nu} \right] \sqrt{-g} \frac{d^3p_1}{p_{10}} = -\frac{n_2}{n_1} F_{21}, \quad (80)
\]

\[
F_{21} = -\frac{16\pi kT\sigma_{12}}{3cK_2(\zeta_1)K_2(\zeta_2)c_p^2} \left[ 2h_{12}^{II} - \zeta_1 G_1 h_{12}^I \right], \quad F_{22} = -\frac{n_1}{n_2} F_{12}, \quad (81)
\]

\[
\mathcal{H}_{12} = -\frac{c\Delta^{\mu\nu}}{3n_1n_2kT^3} \int \frac{\zeta_1}{c_p^2} \left( G_1 - \frac{U_p^2 c_1^2}{m_1 c^2} \right) p_{1\mu} T_{12} \left[ \frac{\zeta_1}{c_p^2} \left( G_2 - \frac{U_p^2 c_2^2}{m_2 c^2} \right) p_{2\nu} \right] \sqrt{-g} \frac{d^3p_1}{p_{10}} = -\frac{16\pi kT\sigma_{12}}{3cK_2(\zeta_1)K_2(\zeta_2)c_p^2} \left[ G_1 \zeta_1 G_2 h_{12}^I - 2 \left( G_1 \zeta_1 + G_2 \zeta_2 \right) h_{12}^{II} + 2 h_{12}^{IV} \right], \quad (82)
\]

\[
\mathcal{H}_{11} = -\frac{c\Delta^{\mu\nu}}{3n_1n_2kT^3} \sum_{b=1}^2 \int \frac{\zeta_1}{c_p^2} \left( G_1 - \frac{U_p^2 c_1^2}{m_1 c^2} \right) p_{1\mu} T_{1b} \left[ \frac{\zeta_1}{c_p^2} \left( G_1 - \frac{U_p^2 c_1^2}{m_1 c^2} \right) p_{1\nu} \right] + \int \frac{\zeta_1}{c_p^2} \left( G_1 - \frac{U_p^2 c_1^2}{m_1 c^2} \right) p_{1\mu} T_{11} \left[ \frac{\zeta_1}{c_p^2} \left( G_1 - \frac{U_p^2 c_1^2}{m_1 c^2} \right) p_{1\nu} \right] \sqrt{-g} \frac{d^3p_1}{p_{10}} = -\frac{n_2}{n_1} \mathcal{H}_{21}, \quad (83)
\]

\[
\mathcal{H}_{22} = -\frac{16\pi kT\sigma_{12}}{3cK_2(\zeta_1)K_2(\zeta_2)c_p^2} \left[ (G_1 \zeta_1) h_{12}^I - 4 G_1 \zeta_1 h_{12}^{II} + 2 h_{12}^{IV} \right] + \frac{4\sigma_{11}K_2(\zeta_2)}{\sigma_{12}K_2(\zeta_1)} \mathcal{H}_{11}, \quad \mathcal{H}_{22} = -\frac{16\pi kT\sigma_{12}}{3cK_2(\zeta_1)K_2(\zeta_2)c_p^2} \left[ (G_2 \zeta_2) h_{12}^I - 4 G_2 \zeta_2 h_{12}^{II} + 2 h_{12}^{IV} \right] + \frac{4\sigma_{22}K_2(\zeta_1)}{\sigma_{12}K_2(\zeta_2)} \mathcal{H}_{22}, \quad (84)
\]

\[
\mathcal{R}_{12} = \frac{U_{\mu} U_{\nu} U_{\sigma}}{c^2 P_2} \int p_{\mu} p_{\nu} T_{12} \left[ \frac{\partial \ln \zeta_2}{\partial \ln c_2^2} \left( \frac{U_p^2 c_2^2}{kT} - \frac{3(c_2^2 + h_2/T)}{c_2^2} \right) p_{\nu}^2 \right] \frac{kT}{c_2^2} \sqrt{-g} \frac{d^3p_1}{p_{10}} = \frac{32\pi P_{1\sigma_{12}}}{c^2 K_2(\zeta_1)K_2(\zeta_2) \partial \ln c_2^2} \left[ r_{12}^{II} - \frac{15(c_2^2 + h_2/T)}{c_2^2} r_{12}^{II} \right], \quad (85)
\]

\[
\mathcal{R}_{11} = \frac{U_{\mu} U_{\nu} U_{\sigma}}{c^2 P_2} \left\{ \sum_{b=1}^2 \int p_{\mu} p_{\nu} T_{1b} \left[ \frac{\partial \ln \zeta_1}{\partial \ln c_2^2} \left( \frac{U_p^2 c_2^2}{kT} - \frac{3(c_2^2 + h_1/T)}{c_2^2} \right) p_{\nu}^2 \right] \frac{kT}{c_2^2} \right\} \sqrt{-g} \frac{d^3p_1}{p_{10}} = -\frac{64\pi P_{1\sigma_{11}}}{c^2 K_2(\zeta_1)^2} \partial \ln c_1 \left[ r_{11}^{II} - \frac{32\pi P_{2\sigma_{12}}}{c^2 K_2(\zeta_1)K_2(\zeta_2)} \partial \ln c_2 \left[ r_{11}^{II} - \frac{15(c_1^2 + h_1/T)}{c_1^2} r_{11}^{II} \right] \right], \quad (86)
\]
\[
\begin{align*}
\mathcal{R}_{22} & = -\frac{64\pi p_2\sigma_{22}}{c^2\varepsilon_2^e K_2(\varepsilon_2^e)^2} \frac{\partial \ln \varepsilon_2}{\partial \ln c_0^e} r_{22} - \frac{32\pi p_1\sigma_{12}}{c^2 K_2(\varepsilon_1^e) K_2(\varepsilon_2^e)} \frac{\partial \ln \varepsilon_2}{\partial \ln c_0^e} \left[ r_{12}^{II} - \frac{15(c_p^2 + h_2/T)}{c^2} r_{12}^{II} \right], \\
\mathcal{K}_{12} & = -\frac{c^3 \Delta_{\mu(0)\Delta_{\nu}}}{{10p_1h_1p_2}} \int p_1^2 p_1^\nu T_{12} \left[ \frac{\zeta_2}{m_2h_2} p_2^\mu p_2^\nu \right] \sqrt{-g} \frac{d^3p_1}{p_1^{10}} = -\frac{64\pi c\sigma_{12}}{15kT G_1 G_2 \Gamma(\varepsilon_1^e) K_2(\varepsilon_2^e)} k_{12}, \\
\mathcal{K}_{11} & = -\frac{c^3 \Delta_{\mu(0)\Delta_{\nu}}}{{10p_1h_1p_1}} \left\{ \sum_{b=1}^2 \int p_1^2 p_1^\nu T_{1b} \left[ \frac{\zeta_1}{m_1h_1} p_1^\mu p_1^\nu \right] + \int p_1^2 p_1^\nu T_{11} \left[ \frac{\zeta_1}{m_1h_1} p_1^\mu p_1^\nu \right] \right\} \sqrt{-g} \frac{d^3p_1}{p_1^{10}} \\
& = \frac{64\pi c\sigma_{12}}{15kT \zeta_1^2 G_1^2 \Gamma(\varepsilon_1^e) K_2(\varepsilon_2^e)} \frac{n_2}{n_1} k_{12}^{II} + \frac{64\pi c\sigma_{11}}{15kT K_3(\zeta_1)^2} k_{11}, \\
\mathcal{K}_{22} & = \frac{64\pi c\sigma_{22}}{15kT \zeta_2^2 G_2^2 \Gamma(\varepsilon_1^e) K_2(\varepsilon_2^e)} \frac{n_2}{n_1} k_{22}^{II} + \frac{64\pi c\sigma_{22}}{15kT K_3(\zeta_2)^2} k_{22}.
\end{align*}
\]

Above we have introduced the following abbreviations with \( \chi = 2\sqrt{\zeta_1\zeta_2} \):

\[ h_{12}^{I} = \frac{14}{\chi} K_3(\chi) + \left( 2 + \frac{4}{\chi^2} \right) K_2(\chi), \]

\[ h_{12}^{II} = \left( \frac{\chi}{2} + \frac{34}{\chi} \right) K_3(\chi) + \left( 5 + \frac{8}{\chi^2} \right) K_2(\chi), \]

\[ h_{12}^{III} = \left( \frac{9\chi}{2} + \frac{184}{\chi} \right) K_3(\chi) + \left( 28 + \frac{\chi^2}{4} + \frac{32}{\chi^2} \right) K_2(\chi), \]

\[ f_{aa} = \left( \zeta_a + \frac{17}{\zeta_a} \right) K_3(2\zeta_a) + \left( 5 + \frac{2}{\zeta_a^2} \right) K_2(2\zeta_a), \]

\[ h_{12}^{IV} = \left( \frac{9\chi}{2} + \frac{204}{\chi} \right) K_3(\chi) + \left( 30 + \frac{\chi^2}{4} + \frac{48}{\chi^2} \right) K_2(\chi), \]

\[ f_{aa} = \frac{5}{\zeta_a} K_3(2\zeta_a) + \left( 1 + \frac{2}{\zeta_a^2} \right) K_2(2\zeta_a), \]

\[ k_{12}^{I} = \left( 2 + \frac{144}{\chi^2} \right) \frac{K_3(\chi)}{\chi} + \left( 20 + \frac{64}{\chi^2} \right) \frac{K_2(\chi)}{\chi^2}, \]

\[ k_{12}^{II} = \left( 2\chi + \frac{134}{\chi} \right) K_3(\chi) + \left( 20 + \frac{24}{\chi^2} \right) K_2(\chi), \]

\[ k_{aa} = \left( 3 + \frac{49}{\zeta_a} \right) \frac{K_3(2\zeta_a)}{\zeta_a} + \left( 15 + \frac{2}{\zeta_a^2} \right) \frac{K_2(2\zeta_a)}{\zeta_a}, \]

\[ r_{12}^{I} = \left( \chi + \frac{96}{\chi} \right) K_3(\chi) + \left( 13 + \frac{32}{\chi^2} \right) K_2(\chi), \]

\[ r_{12}^{II} = \frac{2K_3(\chi)}{\chi} + \frac{1}{5} \left( 1 + \frac{8}{\chi^2} \right) K_2(\chi), \]

\[ r_{12}^{III} = \left( \chi + \frac{100}{\chi} \right) K_3(\chi) + \left( 13 + \frac{48}{\chi^2} \right) K_2(\chi), \]

\[ r_{aa} = \zeta_a K_3(2\zeta_a) + 2K_2(2\zeta_a), \]

\[ \frac{c_p^2 + h_a/T}{c_a^2} \frac{\zeta_a^2 + 6G_a\zeta_a - G^2_a\zeta_a}{\zeta_a^2 + 5G_a\zeta_a - G^2_a\zeta_a - 1}. \]

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