ON ORBITS IN DOUBLE FLAG VARIETIES FOR SYMMETRIC PAIRS

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Dedicated to Jiro Sekiguchi on the occasion of his sixtieth birthday.

Abstract. Let $G$ be a connected, simply connected semisimple algebraic group over the complex number field, and let $K$ be the fixed point subgroup of an involutive automorphism of $G$ so that $(G, K)$ is a symmetric pair.

We take parabolic subgroups $P$ of $G$ and $Q$ of $K$ respectively and consider the product of partial flag varieties $G/P$ and $K/Q$ with diagonal $K$-action, which we call a double flag variety for symmetric pair. It is said to be of finite type if there are only finitely many $K$-orbits on it.

In this paper, we give a parametrization of $K$-orbits on $G/P \times K/Q$ in terms of quotient spaces of unipotent groups without assuming the finiteness of orbits. If one of $P \subset G$ or $Q \subset K$ is a Borel subgroup, the finiteness of orbits is closely related to spherical actions. In such cases, we give a complete classification of double flag varieties of finite type, namely, we obtain classifications of $K$-spherical flag varieties $G/P$ and $G$-spherical homogeneous spaces $G/Q$.

INTRODUCTION

Let $G$ be a connected, simply connected semisimple algebraic group over the complex number field $\mathbb{C}$.

Let $P_i$ $(i = 1, 2, \ldots, k)$ be parabolic subgroups of $G$ and consider partial flag varieties $\mathfrak{F}_{P_i} := G/P_i$ of $G$. We are interested in the product of flag varieties $\mathfrak{F}_{P_1} \times \mathfrak{F}_{P_2} \times \cdots \times \mathfrak{F}_{P_k}$, on which $G$ acts diagonally. We say a multiple flag varieties $\mathfrak{F}_{P_1} \times \mathfrak{F}_{P_2} \times \cdots \times \mathfrak{F}_{P_k}$ is of finite type if it admits only finitely many $G$-orbits. It is an interesting problem to classify multiple flag varieties of finite type. According to a result of Magyar-Weyman-Zelevinsky (MWZ99 [MWZ00]), $k$ must be less than or equal to 3 if a multiple flag variety is of finite type and if $G$ is of classical type.

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Let us consider a triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$. If $P_3 = B$ is a Borel subgroup, then the triple flag variety is of finite type if and only if $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2}$ is a spherical $G$-variety. For maximal parabolic subgroups $P_1$ and $P_2$, Littelmann classified such spherical double flag varieties ([Lit94], see also [Pan93]). For general parabolic subgroups $P_1$ and $P_2$, Stembridge [Ste03] classified them completely. In [MWZ99] [MWZ00], they classified the triple flag varieties $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_3} \times \mathfrak{X}_{P_3}$ of finite type for $G = \text{SL}_n$ and $G = \text{Sp}_n$. They also gave a complete enumeration of the $G$-orbits on such triple flag varieties.

Let $K$ be the subgroup of fixed points of a non-trivial involution $\theta$ of $G$. Take parabolic subgroups $P$ of $G$ and $Q$ of $K$. Then we call $\mathfrak{X}_P \times Z_Q(= G/P \times K/Q)$ a double flag variety for a symmetric pair $(G, K)$, where $Z_Q := K/Q$ (see [NO11]). The subgroup $K$ naturally acts on $\mathfrak{X}_P \times Z_Q$ diagonally. This notion is a generalization of the triple flag varieties (see §4.3), and we say it is of finite type if there exist only finitely many $K$-orbits.

In some cases, finiteness of $G$-orbits on a triple flag variety implies finiteness of $K$-orbits on a double flag variety for $(G, K)$. In [NO11], we investigated such situations and got two sufficient conditions for $\mathfrak{X}_P \times Z_Q$ to be of finite type:

**Theorem 1** ([NO11] Theorem 3.1]). Let $P'$ be a $\theta$-stable parabolic subgroup of $G$ such that $P' \cap K = Q$. If the number of $G$-orbits on $\mathfrak{X}_P \times \mathfrak{X}_{\theta(P)} \times \mathfrak{X}_{P'}$ is finite, then there are only finitely many $K$-orbits on the double flag variety $\mathfrak{X}_P \times Z_Q$.

**Theorem 2** ([NO11] Theorem 3.4]). Let $P_i$ ($i = 1, 2, 3$) be a parabolic subgroup of $G$. Suppose that $\mathfrak{X}_{P_i} \times \mathfrak{X}_{P_j} \times \mathfrak{X}_{P_i}$ has finitely many $G$-orbits and that $Q := P_1 \cap P_2$ is a parabolic subgroup of $K$. Then $\mathfrak{X}_{P_i} \times Z_Q$ has finitely many $K$-orbits.

Moreover, if $P_3$ is a Borel subgroup $B$ and the product $P_1P_2$ is open in $G$, then the converse is also true, i.e., the double flag variety $\mathfrak{X}_B \times Z_Q$ is of finite type if and only if the triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_B$ is of finite type.

Using these two theorems, we can produce many examples of double flag varieties of finite type. However, a complete classification is not known yet.

In this paper, we study $K$-orbit structure on an arbitrary double flag variety $\mathfrak{X}_P \times Z_Q$ which is not necessarily of finite type and, as a result, we get some criteria for the finiteness of orbits. Our method relies on the Bruhat decomposition and “KGB-decomposition” (i.e., the $K$-orbit decomposition of flag varieties; see §2.2). The set of $K$-orbits in $\mathfrak{X}_P \times Z_Q$ is decomposed into a finite disjoint union of some quotient spaces parametrized by elements of Weyl groups (or “Bruhat parameters”) and “KGB-parameters”. For each of these parameters, we construct a certain double coset space of unipotent subgroups related to $P$ and $Q$, which admits an action of a subgroup of Levi component of $Q$. The quotient spaces are obtained from this action.

Though general description of the orbit space structure of $K \backslash (\mathfrak{X}_P \times Z_Q)$ is much complicated, it becomes considerably simpler if $Q$ is a Borel subgroup of $K$. So let us give a parametrization of orbits in this special case here, and for general case we refer to Theorem 2.7.
Let $B$ be a $\theta$-stable Borel subgroup of $G$ which contains a $\theta$-stable maximal torus $T$, and $W$ the Weyl group of $G$. We denote by $U_B$ the unipotent radical of $B$ so that $B = TU_B$. We write $B_K = B \cap K = T_K U_{B_K}$, a Borel subgroup of $K$.

**Theorem 3.** Let $P$ be a standard parabolic subgroup of $G$ containing $B$, and $W_P$ the subgroup of $W$ corresponding to the Levi component of $P$. Then the $K$-orbits on the double flag variety are parametrized as follows:

$$K\backslash(\mathfrak{x}_P \times Z_{B_K}) \simeq \coprod_{w \in W_P \backslash W} \left( (w^{-1} P \cap U_B) \backslash U_B / U_{B_K} \right) / T_K,$$

where the maximal torus $T_K$ of $K$ acts on the double coset space via conjugation.

Let us discuss finiteness of $K$-orbits on the double flag varieties. If $P$ is a Borel subgroup of $G$ or $Q$ is a Borel subgroup of $K$, then we will see that the finiteness of the double flag variety is reduced to the sphericity of a certain linear action. The classification of spherical linear actions were established by Kac [Kac80] for irreducible case and independently by Benson-Ratcliff [BR96] and Leahy [Lea98] for reducible case. We therefore obtain a classification of double flag varieties of finite type in such cases.

If $Q = B_K$, we can apply Panyushev’s theorem to obtain that the conormal bundle $T_O \mathfrak{x}_P$ (or the normal bundle $T_O \mathfrak{x}_P$) is $K$-spherical for any $K$-orbit $O$ in $\mathfrak{x}_P$ if and only if the flag variety $\mathfrak{x}_P$ is $K$-spherical (see [Pan99]). As a consequence, taking $O$ as the closed orbit through the base point $e \cdot P$, we see that $\mathfrak{x}_P \times Z_Q$ is of finite type if and only if the action of Levi component of $P \cap K$ on the fiber of $T_O \mathfrak{x}_P$ at $e \cdot P$ is spherical. We refer to Theorem 4.2 for the details. Using the tables of multiplicity free actions by Benson-Ratcliff [BR96], we obtain a complete classification of the double flag varieties $\mathfrak{x}_P \times Z_{B_K}$ of finite type in Theorem 5.2.

**Theorem 4.** Let $G$ be a connected simple algebraic group and $(G, K)$ a symmetric pair. Let $P$ be a parabolic subgroup of $G$. Then the double flag variety $G/P \times K/B_K$ is of finite type if and only if it appears in Table 2. The table also serves as a complete list of $K$-spherical partial flag varieties $G/P$.

Recently, for $G = SL_n$, Petukhov classified reductive subgroups $H$ of $G$ and parabolic subgroups $P$, for which a partial flag variety $G/P$ is $H$-spherical [Pet11]. We thank the referee for pointing out the reference.

On the other hand, if $P = B$, a double flag variety $\mathfrak{x}_B \times Z_Q$ is of finite type if and only if $G/Q$ is a $G$-spherical variety. In this case, Theorem 2.7 implies that a double flag variety is of finite type if and only if a certain linear action of a reductive subgroup of $K$ is a spherical action (Theorem 4.5). So we can again use tables in [BR96] to get a classification of such double flag varieties of finite type. See Theorem 6.2 and Table 3 for details.

Another motivation to study double flag varieties $\mathfrak{x}_B \times Z_Q$ of finite type comes from the theory of character sheaves. Character sheaves were first introduced by Lusztig [Lus].
There are certain $G$-equivariant simple perverse sheaves on $G$, which provide a geometric theory of characters of a connected reductive group over an arbitrary algebraically closed field.

Recently, some generalizations of character sheaves have been studied. Finkelberg, Ginzburg and Travkin developed the theory of mirabolic character sheaves in [FG10] and [FGT09]. Following the work of Kato [Kat09], Henderson and Trapa suggested the theory of exotic character sheaves in [HT12]. These character sheaves are certain $K$-equivariant simple perverse sheaves on $V \times G/K$, where $V$ is some $K$-module. Here in the mirabolic case, $(G, K, V) = (GL_n \times GL_n, (GL_n)_{\text{diag}}, \mathbb{C}^n)$ and in the exotic case, $(G, K, V) = (GL_{2n}, Sp_{2n}, \mathbb{C}^{2n})$. A key ingredient is that there are only finitely many $K$-orbits on the generalized flag $V \times G/B$.

One may hope that there is a generalization of character sheaves on $K/Q \times G/K$, which generalizes both the mirabolic character sheaves and exotic character sheaves. In order to do this, one first need to know when a double flag variety $X_B \times Z_Q$ has only finitely many $K$-orbits. We believe that the classification of double flag varieties $X_B \times Z_Q$ of finite type is a necessary ingredient for establishing the (conjectural) generalization of character sheaf theory.

The Robinson-Schensted correspondence is a bijection correspondence between permutations and pairs of standard Young tableaux of the same shape. Steinberg gave a geometric interpretation of this correspondence, by showing that both sides naturally parametrize the irreducible components of the Steinberg variety, which is by definition the conormal variety of the product of flag varieties. It is interesting to study a similar question for conormal variety of double flag of finite type. The mirabolic case was obtained by Travkin [Tra09], Finkelberg-Ginzburg-Travkin [FGT09] and the exotic case was obtained by Henderson-Trapa [HT12], in which they also made some conjectures relating the exotic Robinson-Schensted correspondence to exotic character sheaves.

This paper is organized as follows. We fix basic notation and terminology in §1. In Theorem 2.7, we give a parametrization of $K$-orbits in $X_P \times Z_Q$. When $Q \subset K$ is a Borel subgroup, we see in Proposition 3.6 that a part of the $K$-orbit decomposition in $X_P \times Z_Q$ is reduced to a linear action. Classification of the double flag varieties $X_P \times Z_Q$ of finite type in the extreme case, namely, the case where $Q = B_K$ or $P = B$ is our main result in this paper. In §4, we reduce the finiteness of the double flag variety $X_P \times Z_Q$ to the sphericity of a linear action for $Q = B_K$ (Theorem 4.2) and for $P = B$ (Theorem 4.5). We also recall one of Stembridge’s results in Theorem 4.7 which classifies the triple flag varieties $G/P_1 \times G/P_2 \times G/B$ of finite type. Classifications of double flag varieties $X_P \times Z_Q$ of finite type are given in Theorem 5.2 with Table 2 for $Q = B_K$ and Theorem 6.2 with Table 3 for $P = B$.

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1. Preliminaries

1.1. Let $G$ be a connected, simply connected semisimple algebraic group over the complex number field $\mathbb{C}$ and $\theta$ a non-trivial involutive automorphism of $G$. We put $K = G^\theta = \{g \in G : \theta(g) = g\}$, the subgroup of fixed elements of $\theta$, which is connected and reductive by our assumption on $G$ (see [Ste68, Theorem 8.1]). We denote the Lie algebra of $G$ (resp. $K$) by $\mathfrak{g}$ (resp. $\mathfrak{k}$). In the following, we use similar notation; for an algebraic group we use a Roman capital letter, and for its Lie algebra the corresponding German small letter.

Let $B \subset G$ be a $\theta$-stable Borel subgroup and take a $\theta$-stable maximal torus $T$ in $B$. We consider the root system $\Delta = \Delta(\mathfrak{g}, T)$, the Weyl group $W = W_G = N_G(T)/Z_G(T)$ with respect to $T$, and the positive system $\Delta^+$ corresponding to $B$. Then $\Delta^+$ determines a set of simple roots $\Pi$. Since $B$ and $T$ are $\theta$-stable, $\theta$ naturally acts on $W_G$ and $\Delta$, and preserves $\Delta^+$ and $\Pi$.

We say that a parabolic subgroup $P$ of $G$ is standard if $P \supset B$. There exists a one-to-one correspondence between the standard parabolic subgroups $P$ and the subsets $J \subset \Pi$; the root subsystem $\Delta_J$ generated by $J$ is the root system of the standard Levi component $L$ of $P$. Notice that the $\theta$-stable parabolic subgroups correspond exactly to the $\theta$-stable subsets in $\Pi$. If $P$ corresponds to $J$, then we will write $P = P_J$ and write $P_J = L_J U_J$ for the Levi decomposition, where $L_J$ is the standard Levi factor and $U_J$ is the unipotent radical. We denote the Weyl group of $\Delta_J$ by $W_J$ and put $W_J = W/W_J$. For two subsets $J, J' \subset \Pi$, put $W_J = W_J \setminus W/W_J$. In the following, we often take representatives of elements of $W$ in $N_G(T)$ and regard $W$ as a subset of $N_G(T)$ or of $G$. Similarly, we take representatives of elements of $W_J, W_J'$ in $W$ and then in $N_G(T)$ so that we have $W_J, W_J' \subset N_G(T)$.

1.2. In this subsection, we assume that $G$ is a connected reductive algebraic group over $\mathbb{C}$. Let $X$ be an irreducible normal $G$-variety. If $X$ has an open $B$-orbit for a Borel subgroup $B$ of $G$, it is called a spherical variety and the $G$-action is called a spherical action. It is well-known that $X$ is spherical if and only if there are only finitely many $B$-orbits in $X$ ([Bri86, Vin86]). The $G$-action on $X$ induces a $G$-action on the ring of regular functions $\mathbb{C}[X]$. It is easy to see that if $X$ is spherical, then the decomposition of $\mathbb{C}[X]$ into irreducible $G$-modules is multiplicity-free. The converse is also true if $X$ is an affine variety ([VK78]). If $X$ is isomorphic to a vector space and the $G$-action is linear, $\mathbb{C}[X]$ can be identified with the symmetric power $S(X^*)$ of the dual space $X^*$. Hence a linear $G$-action is spherical if and only if the decomposition of $S(X^*)$ into irreducible $G$-modules is multiplicity-free.
2. Parametrization of $K$-orbits in the double flag variety

Suppose that $G$ is a connected, simply connected semisimple algebraic group over $\mathbb{C}$ with an involutive automorphism $\theta$ and let $K := G^\theta$. Let $P$ be a parabolic subgroup of $G$, and $Q$ a parabolic subgroup of $K$. We denote the partial flag varieties $G/P$ and $K/Q$ by $X_P$ and $Z_Q$, respectively. The product $X_P \times Z_Q$ is called a double flag variety for symmetric pair $(G, K)$. We say that a double flag variety $X_P \times Z_Q$ is of finite type if there are only finitely many orbits on the product $X_P \times Z_Q$ with respect to the diagonal $K$-action (see [NO11]).

In this paper, we study the structure of the orbit space $K \setminus (X_P \times Z_Q)$, and give a parametrization of orbits. As a consequence of the parametrization, we get a criterion to determine if the double flag variety is of finite type.

It is known that there exists a $\theta$-stable parabolic subgroup $P'$ of $G$ such that $Q = P' \cap K$ ([BH00, Theorem 2]). Then by replacing $P$ with its conjugate subgroup, we may assume that $P$ and $P'$ are standard parabolic subgroups for a $\theta$-stable Borel subgroup $B$. We use notations in §1.1 for our $G$, $K$, and $B$. Write $J, J' \subset \Pi$ for the subsets such that $P = P_J$ and $P' = P_{J'}$. Let $P = LU$ and $P' = L'U'$ be the standard Levi decompositions. This means that $L$ is the Levi subgroup of $P$ such that $L \supset T$ and $U$ is the unipotent radical of $P$. Similarly for $L'$ and $U'$.

We parametrize $K$-orbits on $X_P \times Z_Q$ using reduction by two well-known decompositions: the Bruhat decomposition and the KGB decomposition.

First we reduce the orbit space by the Bruhat decomposition.

2.1. Reduction by Bruhat decomposition. Notice that there is a bijection

$$K \setminus (X_P \times Z_Q) \simeq P \setminus G/Q, \quad K \cdot (gP, kQ) \mapsto P g^{-1} k Q \quad (g \in G, k \in K).$$

Since $Q = P' \cap K$, we have the following reduction map $\Phi$.

$$K \setminus (X_P \times Z_Q) \sim \mbox{proj} \quad P \setminus G/Q \quad \Phi \quad P \setminus G/P' = \coprod_{w \in J' W J'} \quad P_w P'$$

Thus we can reduce the determination of the orbit structure $K \setminus (X_P \times Z_Q)$ to the analysis of the fiber

$$\Phi^{-1}(w) \simeq P \setminus P_w P'/Q \quad (w \in J' W J'). \quad (2.1)$$

Let us fix $w \in J' W J'$ in the following. We put

$$P^w := w^{-1} P_w.$$
More generally, we write $H^g = g^{-1}Hg$ for any subgroup $H \subset G$ and $g \in G$.

**Lemma 2.1.** The map $(P^w \cap P') \backslash P'/Q \to P \backslash P'\backslash P'/Q$ given by $(P^w \cap P')aQ \mapsto PwaQ$ for $a \in P'$ is bijective.

This is a consequence of a more general lemma below. In the lemma, $G, H, H'$ refer arbitrary groups, which are different from the present notation.

**Lemma 2.2.** Let $G$ be a group and $H, H'$ its subgroups. Let $A \subset H'$ be a subset and $g \in G$ an element. Then the map $(H^g \cap H') \backslash (H^g \cap H')A \to H \backslash HgA$ given by $(H^g \cap H')a \mapsto H ga$ for $a \in A$ is bijective.

**Proof.** The surjectivity is clear. If $Hga_1 = Hga_2$ for $a_1, a_2 \in A$, then $a_2a_1^{-1} \in H^g \cap H'$. Hence $(H^g \cap H')a_1 = (H^g \cap H')a_2$ and the map is injective. \(\square\)

We apply Lemma 2.2 in the setting where $G = G, H = P, H' = P', A = P'$ and $g = w$. Taking quotients by right $Q$-action, we get Lemma 2.1.

2.2. Reduction by smaller symmetric spaces. The orbit structure $K \backslash (X \times Z_Q)$ is reduced to the structure of fibers (2.1) of $\Phi$. In this subsection, we further reduce it by the KGB decomposition, or by KGP decomposition we should say, for smaller symmetric spaces. For KGB decomposition, we refer the readers to [RS90, RS94, RS93] and [LV83].

Put $L'_K := L' \cap K$ and consider $P^w \cap L'$, which is a parabolic subgroup of $L'$ by [Car85, Proposition 2.8.9]. Then $L'_K$ is a symmetric subgroup of $L'$, and it is known that $(P^w \cap L') \backslash L'/L'_K$ is a finite set. Let us denote this finite set by $\mathcal{V}(w)$ for $w \in JW^{J'}$.

**Lemma 2.3.** The map

$$\Psi_w : (P^w \cap P') \backslash P'/Q \to (P^w \cap L') \backslash L'/L'_K = \mathcal{V}(w)$$

given by $(P^w \cap P')aQ \mapsto (P^w \cap L')bL'_K$ is well-defined, where $a = bu \in L'U' = P'$ is the Levi decomposition.

**Proof.** We put $U'_K := U' \cap K$ so that $Q = L'_KU'_K$ and consider the following diagram.

$$
\begin{array}{ccc}
(P^w \cap P') \backslash P'/Q & \xleftarrow{=} & (P^w \cap P') \backslash L'/L'_K U'_K \\
\Psi_w \downarrow \quad & & \quad \downarrow \text{proj.}
\end{array}
$$

Here a bijective map

$$\iota : (P^w \cap L') \backslash L'/L'_K \to (P^w \cap P') \backslash L'/L'_K U'$$

is induced by the inclusion $L' \hookrightarrow L'U'$ and the second vertical arrow in the diagram is the inverse of $\iota$. The bijectivity of $\iota$ is deduced from the following general lemma.
Lemma 2.4. Let $L_1 \ltimes U_1$ be a semidirect product group of two groups $L_1$ and $U_1$. Let $L_2, L_3 \subseteq L_1$ and $U_2 \subseteq U_1$ be subgroups and assume that $L_2$ normalizes $U_2$ so that $L_2 \ltimes U_2$ is a subgroup of $L_1 \ltimes U_1$. Then the natural inclusion map induces the following bijections:

$$L_2 \backslash L_1 / L_3 \sim L_2 / (L_1 U_1) / (L_3 U_1) \sim (L_2 U_2) / (L_1 U_1) / (L_3 U_1).$$

Proof. It is easy to see that the both maps are well-defined and surjective. So it is enough to see that the composite map is injective.

Suppose $l, l' \in L_1$ satisfy $L_2 U_2 l L_3 U_1 = L_2 U_2 l' L_3 U_1$. This means that there exist $l_2 \in L_2$, $l_3 \in L_3$, $w_2 \in U_2$, $u_1 \in U_1$ such that $l' = (l_2 u_2) l (l_3 u_1)$, or equivalently $l' = (l_2 l_3)(\((l_3)\)^{-1} u_2 ((l_3) u_1)) \in L_1 \ltimes U_1$. By the uniqueness of the semidirect product decomposition, we have $l' = l_2 l_3$ and hence $l' \in L_2 l L_3$. \hfill \Box

To see that the map $\iota$ in (2.2) is bijective, we use Lemma 2.4 in the setting where $L_1 = L'$, $U_1 = U'$, $L_2 = P^w \cap L'$, $U_2 = P^w \cap U'$, and $L_3 = L'_K$. Note that

$$P^w \cap P' = (P^w \cap L') \ltimes (P^w \cap U')$$

holds (see [Car85, Theorem 2.8.7 and Proposition 2.8.9]). \hfill \Box

Let us summarize the above situation into a diagram:

$$\begin{array}{ccc}
P^w a Q & \hookrightarrow & P \backslash P^w P' / Q \\
& \Psi_w \downarrow & \downarrow \text{projection} \\
(P^w \cap L') b L'_K & \sim & (P^w \cap P') \backslash P' / Q' \\
& \Psi_w^{-1} \downarrow & \downarrow \text{projection} \\
(P^w \cap L')' U'_K / L' & \sim & (P^w \cap P') \backslash P' / Q' / Q'/Q
\end{array}$$

where $a = bu \in L' U'$ is the Levi decomposition. Let us take representatives of $\mathcal{V}(w) = (P^w \cap L') \backslash L' / L'_K$ from $L'$ and identify them with $\mathcal{V}(w)$ in the following.

2.3. Parametrization of orbits in the double flag variety. Now we get a rough parametrization of orbits, first by the Bruhat decomposition $P \backslash G / P' \simeq J W \backslash J'$, then next by the KGB decomposition for the smaller symmetric space $L' / L'_K$.

The following lemma describes the fiber $\Psi^{-1}_w(v) = P \backslash P^w v U' Q / Q$ for $v \in \mathcal{V}(w)$.

**Lemma 2.5.** Let us fix $w \in J W \backslash J'$ and $v \in \mathcal{V}(w)$.

1. $\Psi^{-1}_w(v) = P \backslash P^w v U' Q / Q \simeq (P^w \cap P') \backslash (P^w \cap P') U' Q / Q$.

2. We can define the following surjective map:

$$\begin{array}{ccc}
(P^w \cap U') \backslash U'/U'_K & \longrightarrow & (P^w \cap P') \backslash (P^w \cap P') U'/Q/Q \\
(P^w \cap U') u U'_K & \longrightarrow & (P^w \cap P') u Q.
\end{array}$$

3. The above surjection factors through to a bijection

$$\left((P^w \cap U') / U' / U'_K\right) \sim (P^w \cap U') / (P^w \cap P') U' Q / Q.$$
Notice that $P^{uw} \cap L'_K$ normalizes $P^{uw} \cap U'$ and $U'_K$, hence the conjugation action of $P^{uw} \cap L'_K$ on $U'$ induces an action on $(P^{uw} \cap U') \setminus U'/U'_K$. The corresponding quotient space is the one we considered above.

\textbf{Proof.} (1) This follows from Lemma 2.2.

(2) Since $P^{uw} \cap U' \subset P^{uw} \cap P'$ and $U'_K \subset Q$, our map is just a projection.

(3) We use the following general lemma, in which the notations are independent of the rest of the arguments.

\textbf{Lemma 2.6.} Let $L_1 \ltimes U_1$ be a semidirect product group of two groups $L_1$ and $U_1$. Let $L_2, L_3 \subset L_1$ and $U_2, U_3 \subset U_1$ be subgroups and assume that $L_i$ normalizes $U_i$ for $i = 2, 3$ so that $L_2 \ltimes U_2$ and $L_3 \ltimes U_3$ are subgroups of $L_1 \ltimes U_1$.

(1) The conjugation action of the group $L_2 \cap L_3$ on $U_1$ by $u \mapsto l ul^{-1}$ induces a well-defined action of $L_2 \cap L_3$ on $U_2 \setminus U_1/U_3$.

(2) The natural map $U_2 \setminus U_1/U_3 \to (L_2 U_2) \setminus (L_2 U_1 L_3)/(L_3 U_3)$, $u_2 u_3 \mapsto (L_2 U_2) u_2 (L_3 U_3)$ induces a bijective map

$$\varphi : (U_2 \setminus U_1/U_3)/(L_2 \cap L_3) \xrightarrow{\sim} (L_2 U_2) \setminus (L_2 U_1 L_3)/(L_3 U_3).$$

\textbf{Proof.} The claim (1) is obvious.

Let us prove (2). The surjectivity of $\varphi$ is clear since we can always take a representative of the right-hand side in $U_1$. We give a proof of injectivity. For $u, u' \in U_1$, let us assume that $(L_2 U_2) u (L_3 U_3) = (L_2 U_2) u' (L_3 U_3)$. Then there exist $l_2 u_2 \in L_2 U_2$ and $l_3 u_3 \in L_3 U_3$ such that $u' = (l_2 u_2) u (l_3 u_3)$. We rewrite it as

$$u' = (l_2 l_3)((l_3^{-1} u_2 u l_3) u_3) \in L_1 U_1.$$

By the uniqueness of the semidirect product, $l_2 l_3 = e$ and $u' = (l_3^{-1} u_2 u l_3) u_3$. Therefore, we have $l_2 = l_3^{-1} \in L_2 \cap L_3$ and

$$u' = (l_2 u_2 l_2^{-1}) (l_2 u l_2^{-1}) u_3 \in U_2 (l_2 u l_2^{-1}) U_3.$$

This shows $u' \in (L_2 \cap L_3) \cdot (U_2 u U_3)$, where $\cdot$ denotes the conjugation action. \hfill $\Box$

To prove Lemma 2.5 (3), we apply Lemma 2.6 (2) in the setting where

$$L_1 = L', \quad L_2 = P^{uw} \cap L', \quad L_3 = L'_K, \quad U_1 = U', \quad U_2 = P^{uw} \cap U', \quad U_3 = U'_K.$$ 

We need to use again $P^{uw} \cap P' = (P^{uw} \cap L') \ltimes (P^{uw} \cap U')$ (Car85, Theorem 2.8.7 and Proposition 2.8.9], and $Q = L'_K U'_K$. \hfill $\Box$

Lemmas 2.5 with two reductions ($\S$ 2.1 and \S 2.2] gives us a parametrization of $K$-orbits in the double flag variety $\mathcal{X}_P \times Z_Q$.

\textbf{Theorem 2.7.} Let $P = P_j$ and $P' = P_j'$ be standard parabolic subgroups of $G$ and assume that $P'$ is $\theta$-stable with the standard ($\theta$-stable) Levi decomposition $P' = L' U'$. Define $Q := P' \cap K$, which is a parabolic subgroup of $K$, and put $L'_K := L' \cap K$, $U'_K := U' \cap K$. 

(1) The $K$-orbits in the double flag variety $\mathfrak{X}_P \times Z_Q = G/P \times K/Q$ are parametrized as follows:

$$K\backslash(\mathfrak{X}_P \times Z_Q) \simeq \bigsqcup_{w \in J'W'} \bigsqcup_{v \in \mathcal{V}(w)} \left( (P^{uw} \cap U') \backslash U'/U'_K \right) / P^{uv} \cap L'_K,$$

Here we write $J'W' := WJW/WJ'$ and $\mathcal{V}(w) := (P^w \cap L') \backslash L'/L'_K$ and identify them with their representatives.

(2) The double flag variety $\mathfrak{X}_P \times Z_Q$ is of finite type if and only if for any $w \in J'W'$ and $v \in \mathcal{V}(w)$, the conjugation action of $P^{uw} \cap L'_K$ on the double coset space $(P^{uw} \cap U') \backslash U'/U'_K$ has only finitely many orbits.

**Proof.** The claim (1) was already proved. Since $J'W'$ is a finite set and $\mathcal{V}(w)$ is also finite for any $w \in J'W'$, the claim (2) follows. □

**Corollary 2.8.** The double flag variety $\mathfrak{X}_P \times Z_Q$ is of finite type if and only if for any $g \in P'W'L'$, the conjugation action of $P^g \cap L'_K$ on $(P^g \cap U') \backslash U'/U'_K$ has only finitely many orbits.

**Proof.** This follows directly from Theorem 2.7 once one knows

$$\bigcup_{w \in J'W'} \bigcup_{v \in \mathcal{V}(w)} P^{uv}L'_K = \bigcup_{w \in J'W'} PwL' = P'WL'.$$

□

### 3. Reduction to linear actions

Under the setting of §2, we now assume that $P' = B$ so that $Q = B \cap K = : B_K$ is a Borel subgroup of $K$ and we consider the double flag variety $\mathfrak{X}_P \times Z_{B_K} = G/P \times K/B_K$.

We take a $\theta$-stable maximal torus $T$ as in §1 and denote by $B = TU_B$ a Levi decomposition of $B$ ($U_B$ denotes the unipotent radical of $B$). In our former notation,

- $P' = B = TU_B = L'U'$,
- $Q = B_K = T_KU_{B_K} = L'_KU'_K$,
- $J'W' = J'W = : JW \ni w$,
- $P^w \cap L' = P^w \cap T = T$ (for any $w \in JW$),
- $\mathcal{V}(w) = (P^w \cap L') \backslash L'/L'_K = T \backslash T/K = \{e\}$,
- $P^{uw} \cap L'_K = T_K$, $P^w \cap U' = P^w \cap U_B$.

Then Theorem 2.7 (1) in this case can be rewritten as follows.
Lemma 3.4. Let \( u = u_1 \oplus \cdots \oplus u_n \) be a decomposition of \( u \) as a \( T^1 \)-module and \( U_i = \exp(u_i) \). Then the multiplication map
\[
\varphi : U_1 \times \cdots \times U_n \longrightarrow U
\]
\[
(g_1, \ldots, g_n) \longmapsto g_1 \cdots g_n
\]
is an isomorphism.

Proof. The argument goes by the induction of the dimension of \( u \). Let \( m \) be the maximum weight appearing in \( u \). We may assume that the weight \( m \) appears in \( u_i \) for some \( 1 \leq i \leq n \). Take a non-zero weight vector \( X \in u_i \) with weight \( m \). Let \( \mathfrak{z} = \mathfrak{c}X \). Then \( \mathfrak{z} \) is contained in the center of the Lie algebra \( u \).

Now we set up the induction. Let \( \bar{u}_i := u_i/\mathfrak{z} \), \( \bar{u}_j := u_j \) for \( j \neq i \), \( \bar{u} := \bigoplus_{j=1}^{n} \bar{u}_j \), \( Z := \exp(\mathfrak{z}) \), \( \bar{U} := U/Z \). Then the map \( \varphi : \bigoplus_{j=1}^{n} \bar{u}_j \to \bar{U} \) is defined by \( (Y_1, \ldots, Y_n) \mapsto \exp Y_1 \cdots \exp Y_n \). We assume \( \varphi \) is bijective by induction hypothesis.

Let us prove the surjectivity of \( \varphi \). Take \( u \in U \) and write \( \bar{u} \in \bar{U} \) for its image by the quotient map. Since \( \varphi \) is surjective, there exists \( (Y_1, \ldots, Y_n) \in u \) such that \( \varphi(Y_1, \ldots, Y_n) = \bar{u} \). If we take a lift \( X_i \in u_i \) of \( Y_i \in u_i/\mathfrak{z} \) and put \( \bar{X}_j := Y_j \) for \( j \neq i \), then the image of \( \varphi(X_1, \ldots, X_n) \in U \) and that of \( u \in U \) in \( \bar{U} \) are equal. Hence there exists \( z_i \in Z \) such that \( u = \varphi(X_1, \ldots, X_n)z_i \). Take \( Z_i \in \mathfrak{z} \) such that \( \exp(Z_i) = z_i \). Then \( u = \varphi(X_1, \ldots, X_{i-1}, X_i + Z_i, X_{i+1}, \ldots, X_n) \), showing the surjectivity of \( \varphi \).
The injectivity of \( \varphi \) is similarly proved. Suppose \( (X_1, \ldots, X_n), (X'_1, \ldots, X'_n) \in u \) has the same image by \( \varphi \) in \( U \). Put \( (Y_1, \ldots, Y_n) \in u \) the image of \( (X_1, \ldots, X_n) \). Similarly for \( (Y'_1, \ldots, Y'_n) \). Since \( \tilde{\varphi}(Y_1, \ldots, Y_n) = \tilde{\varphi}(Y'_1, \ldots, Y'_n) \), we have \( (Y_1, \ldots, Y_n) = (Y'_1, \ldots, Y'_n) \) by the injectivity assumption of \( \tilde{\varphi} \). Hence \( X_j = X'_j \) for \( j \neq i \). Then

\[
\exp(X_1) \cdots \exp(X_{i-1}) \exp(X_i) \exp(X_{i+1}) \cdots \exp(X_n) = \exp(X_1) \cdots \exp(X_{i-1}) \exp(X'_i) \exp(X_{i+1}) \cdots \exp(X_n)
\]

and this implies \( \exp(X_i) = \exp(X'_i) \). Since the exponential map is bijective for a unipotent group, we have \( X_i = X'_i \).

The lemma follows from the fact that a bijective morphism \( f : X \to Y \) between algebraic varieties over an algebraically closed field of characteristic zero is an isomorphism if \( Y \) is normal (see [Spr98, Theorem 5.1.6 (iii) and Theorem 5.2.8]). \( \square \)

Let \( U \) and \( T^1 \) be as above. Let \( U_1, U_2 \subset U \) be subgroups of \( U \) which are stable under the action of \( T^1 \). Take a decomposition \( u = (u_1 + u_2) \oplus \mathfrak{W} \) as a \( T^1 \)-module. We further assume that \( \mathfrak{W} \) is stable under the adjoint action of \( U_1 \cap U_2 \).

**Lemma 3.5.** Under the above notations and assumptions, the map

\[
\Phi : \mathfrak{W}/(U_1 \cap U_2) \to U_1 \backslash U/U_2
\]

given by \((U_1 \cap U_2) \cdot Z \mapsto U_1(\exp Z)U_2\) is bijective. Here \( U_1 \cap U_2 \) acts on \( \mathfrak{W} \) by the adjoint action.

**Proof.** We take a \( T^1 \)-stable complementary subspace \( \mathfrak{w}_i \) \((i = 1, 2)\) of \( u_1 \cap u_2 \) in \( u_i \) so that \( u_i = \mathfrak{w}_i \oplus (u_1 \cap u_2) \) is a decomposition of a \( T^1 \)-module. Then we have a decomposition of \( u \) as

\[
u = \mathfrak{w}_1 \oplus \mathfrak{W} \oplus (u_1 \cap u_2) \oplus \mathfrak{w}_2.
\]

By applying Lemma 3.4 to this decomposition, we see that every element \( u \in U \) is uniquely written as \( u = u_1(\exp Z)u_3u_2 \) where \( u_1 \in \exp \mathfrak{w}_1, Z \in \mathfrak{W}, u_3 \in U_1 \cap U_2, \) and \( u_2 \in \exp \mathfrak{w}_2 \). Therefore, the map \( \Phi \) is surjective. To prove the injectivity, suppose that \( \exp Z_1 = u_1(\exp Z_2)u_2 \) for \( Z_1, Z_2 \in \mathfrak{W} \) and \( u_i \in U_i \) \((i = 1, 2)\). By Lemma 3.4 again, we have \( u_1 = u''_1 u'_1 \) where \( u''_1 \in \exp(\mathfrak{w}_1) \) and \( u'_1 \in \exp(u_1 \cap u_2) = U_1 \cap U_2 \). Also we can write \( u_2 = u''_2 u'_2 \) where \( u'_2 \in U_1 \cap U_2 \) and \( u''_2 \in \exp(\mathfrak{w}_2) \). Then we can compute as

\[
u_1(\exp Z_2)u_2 = u''_1 u'_1(\exp Z_2) u'_2 u''_2 = u''_1(\exp \text{Ad}(u'_1) Z_2) (u'_1 u'_2) u''_2 \in \exp \mathfrak{W} \exp(u_1 \cap u_2) \exp \mathfrak{w}_2.
\]

Since the decomposition is unique, we get \( Z_1 = \text{Ad}(u'_1) Z_2 \), which shows \( Z_1 \) and \( Z_2 \) are in the same \( \text{Ad}(U_1 \cap U_2) \)-orbit. \( \square \)
3.2. Parametrization of open stratum via linear actions. In general, we cannot apply Lemma 3.3 to Proposition 3.1 because our assumptions do not hold for general $w \in J^W = W_J \setminus W$. However, we can apply it to the fiber of the longest element of $J^W$, which corresponds to the open stratum of the Bruhat decomposition $P \setminus G/B$. Note that this stratum gives an open $K$-stable set in $\mathfrak{X}_P \times \mathbb{Z}_{B_K}$.

Let us denote by $w_0 \in W$ the minimal representative of a coset $W_J w_0$ containing the longest element. We put

$$J^* := -w_0(J) \subset \Pi, \quad (3.2)$$

$$P_J^* = L_J, U_J^* : \text{Levi decomposition,}$$

$$Pw_0 \cap U_B = L_J^* \cap U_B = U_{L_J^*} : \text{a maximal unipotent subgroup of } L_J^*.$$

With this notation, we can state the following proposition which linearizes the unipotent and reductive.

**Proposition 3.6.** (1) The intersection $L_J \cap K = L_J \cap \theta(J^*) \cap K$ is a connected reductive group and $B_K \cap L_J$ is a Borel subgroup of $L_J \cap K$.

(2) The reductive subgroup $L_J \cap K$ acts on $u_{P_J^*} \cap \mathfrak{g}^{-\theta}$ by the adjoint action and the exponential map induces a bijective map

$$\frac{(u_{P_J^*} \cap \mathfrak{g}^{-\theta})/(B_K \cap L_J^*)}{(Pw_0 \cap U_B) \setminus U_B/U_{B_K})} \sim T_K. \quad (3.3)$$

*Note that the target space is the term for $w = w_0$ in Equation (3.1).*

**Proof.** (1) It follows that $L_J \cap K = L_J \cap \theta(J^*) \cap K = L_J \cap \theta(J^*) \cap K$.

Then $L_J \cap \theta(J^*)$ is a Levi component of the parabolic subgroup $P \cap \theta(J^*) \cap K$ of $K$. Hence it is connected and reductive.

We have $B_K \cap L_J = (B \cap L_J \cap \theta(J^*)) \cap K$. Since $B \cap L_J \cap \theta(J^*)$ is a $\theta$-stable Borel subgroup of $L_J \cap \theta(J^*)$, it cuts out a Borel subgroup of $(L_J \cap \theta(J^*))^\theta = L_J \cap \theta(J^*) \cap K = L_J \cap K$.

(2) Since $L_J$ normalizes $u_{P_J^*}$ and $K$ normalizes $\mathfrak{g}^{-\theta}$, the intersection $L_J \cap K$ acts on $u_{P_J^*} \cap \mathfrak{g}^{-\theta}$. We use Lemma 3.3 by taking $U = U_B$, $U_1 = U_B \cap L_J$, and $U_2 = U_{B_K}$. Since $T_K$ contains a regular element of $G$, we can take a subtorus $T^1 = \mathbb{C}^\times$ in $T_K$ so that the weights in $u = u_B$ are all positive. Then $u_B \cap l_{J^*}$ and $u_{B_K} = u_B^\theta$ are stable under $T^1$. We have

$$u_1 = u_B \cap l_{J^*}, \quad u_2 = u_{B_K} = u_B^\theta,$$

$$u_1 \cap u_2 = (u_B \cap l_{J^*}) \cap u_{B_K} = u_B^\theta \cap l_{J^*} = u_B^\theta \cap (l_{J^*} \cap \theta(J^*)),$$

$$u_1 + u_2 = (u_B \cap l_{J^*}) + u_B^\theta = (u_B \cap (l_{J^*} + \theta(J^*))) + u_B^\theta,$$

$$u_{P_J^*} \cap \mathfrak{g}^{-\theta} = u_{P_J^*} \cap u_{P_{\theta(J^*)}} \cap \mathfrak{g}^{-\theta},$$

$$u = u_B = (u_B \cap (l_{J^*} + \theta(J^*))) \oplus (u_{P_J^*} \cap u_{P_{\theta(J^*)}}) = (u_1 + u_2) \oplus (u_{P_J^*} \cap \mathfrak{g}^{-\theta}).$$
Here $u_{P_J^*}$ is the nilradical of $p_{J^*} = \text{Lie } P_{J^*}$. Since $U_1 \cap U_2 = U_{B_K} \cap L_{J^*}$ is contained in $L_{J^*} \cap K$, it acts on $u_{P_J^*} \cap g^{-\theta}$ via the adjoint action. Now we take $\mathfrak{U} = u_{P_J^*} \cap g^{-\theta}$ in Lemma 3.5 and conclude that the exponential map induces a bijective map

$$(u_{P_J^*} \cap g^{-\theta})/(U_{B^*} \cap L_{J^*}) \sim (U_B \cap L_J^*) \setminus U_B/U_{B_K}.$$ 

The torus $T_K$ acts on both hand sides by the adjoint (or conjugation) action and we see that

$$(u_{P_J^*} \cap g^{-\theta})/(B_K \cap L_{J^*}) \sim ((U_B \cap L_J^*) \setminus U_B/U_{B_K}) / T_K.$$ 

□

4. Spherical actions

From the next section, we will give a classification of double flag varieties $X_P \times Z_Q$ of finite type when $P = B$ is a Borel subgroup of $G$ or $Q = B_K$ is a Borel subgroup of $K$. For this purpose, in this section, we summarize some known properties and consequences of our results in §§1–3.

4.1. Spherical action of a symmetric subgroup on a partial flag variety. Let us consider a double flag variety $X_P \times Z_{B_K}$, where $B_K$ is a Borel subgroup of $K$. Since $X_P \times Z_{B_K}/K \simeq B_K \setminus G/P$, we get several equivalent conditions for finiteness of orbits:

$X_P \times Z_{B_K}$ is of finite type
$\iff X_P = G/P$ has finitely many $B_K$-orbits
$\iff X_P$ is $K$-spherical, i.e., there exists an open dense $B_K$-orbit on $X_P$
$\iff$ there exists an open dense $K$-orbit on $X_P \times Z_{B_K}$.

In this situation, we can apply the following theorem by Panyushev.

**Theorem 4.1 (Pan99).** Let $H$ be a connected reductive group which acts on a smooth variety $X$ and $M \subset X$ a smooth locally closed $H$-stable subvariety. Let $B$ be a Borel subgroup of $H$. Then generic stabilizers for the actions of $B$ on $X$, the normal bundle $T_M X$, and the conormal bundle $T^*_M X$ are isomorphic. In particular, the following are equivalent: $X$ is $H$-spherical; $T_M X$ is $H$-spherical; $T^*_M X$ is $H$-spherical.

We now get a criterion for the finiteness of the double flag variety $G/P \times K/B_K$.

**Theorem 4.2.** Let $B_K$ be a Borel subgroup of $K$ and $P = P_J$ a parabolic subgroup of $G$. Then the following are all equivalent.

1. The double flag variety $X_P \times Z_{B_K} = G/P \times K/B_K$ is of finite type.
2. $X_P = G/P$ is $K$-spherical.
3. For some $K$-orbit $O \subset X_P$ the normal bundle $T_O X_P$ is $K$-spherical. (Hence so is for any $K$-orbit.)
(4) For some $K$-orbit $O \subset X_P$ the conormal bundle $T_O^*X_P$ is $K$-spherical. (Hence so is for any $K$-orbit.)

(5) The adjoint action of $L \cap K$ on $u_P \cap g^{-\theta}$ is spherical.

Proof. We have already seen the equivalence between (1) and (2). The equivalence of (2), (3) and (4) follows from Theorem 4.1.

To see that (5) is equivalent to others, let us consider the $K$-orbit $O$ through the base point $P$. Then $O \simeq K/(P \cap K)$ and the fiber of the conormal bundle $T_O^*X_P$ at $P$ is isomorphic to $(g/(p + t))^* \simeq u_P \cap g^{-\theta}$. Hence $T_O^*X_P \simeq K \times_{P \cap K} (u_P \cap g^{-\theta})$. Let us take the opposite Borel subgroup $B_K$ of $B_K$. Since $B_K(P \cap K)$ is open in $K$, the conormal bundle has an open $B_K$-orbit if and only if there is an open $(P \cap B_K)$-orbit in the fiber $u_P \cap g^{-\theta}$. Notice that $P \cap B_K = L \cap B_K$ is a Borel subgroup of $L \cap K$. Thus (5) is equivalent to (4).

Remark 4.3. Note that Condition (5) is equivalent to requiring the orbit space \([3,3]\) in Proposition 3.6 to be finite for $J^*$ instead of $J$. In fact, the conditions for $J^*$ and $J$ are equivalent in view of the equivalence of Conditions (3) and (4) above. This can be also seen directly by taking a Weyl involution of $G$ which sends $P_J$ to $P_{J^*}$ and stabilizes $K$.

4.2. Spherical fiber bundle over symmetric space. In this subsection, we consider the case where $P = B$ is a Borel subgroup so $J = \emptyset$ in our previous notation and the double flag variety is $X_B \times Z_Q = G/B \times K/Q$. Recall that we have assumed that $B$ is $\theta$-stable.

We summarize the notation here, which is adapted to the present situation.

$$JW^J = W^{J^*} \supseteq w,$$

$$P^w \cap L' = w^{-1}Bw \cap L': a \text{ Borel subgroup of } L',$$

$$\forall(w) = (P^w \cap L')\backslash L'/L'_K = (w^{-1}Bw \cap L')\backslash L'/L'_K \ni v: \text{ a representative in } L',$$

$$P^w \cap L'_K = (L' \cap (wv)^{-1}Bwv) \cap K,$$

where $L' \cap (wv)^{-1}Bwv$ runs over all the Borel subgroups in $L'$ up to $L'_K$-conjugacy,

$$P^w \cap U' = (wv)^{-1}Bwv \cap U' = v^{-1}(w^{-1}Bw \cap U')v = v^{-1}(U' \cap w^{-1}Bw)wv.$$

Thus we have

$$\left( (P^w \cap U') \backslash U'/U'_K \right) / (P^w \cap L'_K)$$

$$\simeq \left( (U' \cap (wv)^{-1}Bwv) \backslash U'/U'_K \right) / (L' \cap (wv)^{-1}Bwv \cap K),$$

where $U'_K := U' \cap K$. So Theorem 2.7 becomes

$$K \backslash (X_B \times Z_Q) \simeq \coprod_{w \in W^J} \coprod_{v \in \forall(w)} \left( (U' \cap (wv)^{-1}Bwv) \backslash U'/U'_K \right) / (L' \cap (wv)^{-1}Bwv \cap K).$$
Since \( \mathfrak{x}_B \times Z_Q \) is of finite type if and only if \( G/Q \) is \( G \)-spherical, we can concentrate on the existence of an open \( B \)-orbit in \( G/Q \). Take the longest element \( w_0 \in W \) so that
\[
U' \cap w_0^{-1} U_B w_0 = \{ e \} \quad \text{and} \quad L' \cap w_0^{-1} B w_0 = L' \cap B^-,
\]
where \( B^- \) denotes the opposite Borel subgroup corresponding to \( \Delta^\sim = -\Delta^+ \). Therefore, we get
\[
(U' \cap (w_0 v)^{-1} U_B w_0 v) \setminus U'/U'_K = U'/U'_K
\]
and
\[
L' \cap (w_0 v)^{-1} B w_0 v \cap K = v^{-1}(L' \cap B^-) v \cap K.
\]
Since \( \mathcal{Y}(w_0) = (L' \cap B^-) \setminus L'/L'_K \) and \( (L' \cap B^-) \setminus L' \) is the full flag variety of \( L' \), the subgroup \( v^{-1}(L' \cap B^-) v (v \in \mathcal{Y}(w_0)) \) runs over all the Borel subgroups of \( L' \) up to \( L'_K \)-conjugacy. However, for the existence of an open orbit, it is enough to consider a Borel subgroup \( v^{-1}(L' \cap B^-) v \) of \( L' \) such that \( v^{-1}(L' \cap B^-) v L'_K \) is open in \( L' \). Let us denote such a Borel subgroup by \( B_{L'}^1 = v^{-1}(L' \cap B^-) v \). Then we conclude that \( \mathfrak{x}_B \times Z_Q \) is of finite type if and only if \( U'/U'_K \) has an open \( B_{L'}^1 \)-orbit.

To describe such Borel subgroups, let us briefly recall some general facts about the double coset space \( B' \setminus G/K \) and its relation to minimal \( \theta \)-split parabolic subgroups. For these facts, we refer the readers to [Vus74, § 1]. Also references [Spr86, § 2] and [HW93] will be useful.

A parabolic subgroup \( P \) of \( G \) is called \( \theta \)-split if \( P \) and \( \theta(P) \) are opposite to each other. In other words, the intersection \( P \cap \theta(P) \) is a Levi component of \( P \), which is a \( \theta \)-stable reductive subgroup of the same rank as \( G \). It is known that there exists uniquely a minimal \( \theta \)-split parabolic subgroup of \( G \) up to \( K \)-conjugacy ([Vus74, Proposition 5]).

Let \( P_{\min} \) be a minimal \( \theta \)-split parabolic subgroup of \( G \). Then \( P_{\min} \) has a \( \theta \)-stable Levi subgroup which contains a \( \theta \)-stable maximal torus \( H \). Put \( A := \exp h^{-d} \) (maximal \( \theta \)-split torus) and \( M := Z_K(A) \), the centralizer of \( A \) in \( K \). Then it follows that \( P_{\min} \cap K = M \) and \( P_{\min} = MAN \), where \( N \) is the unipotent radical of \( P_{\min} \) ([Vus74, Proposition 2]).

**Lemma 4.4.** (1) For \( v \in G \), the set \( BvK \) is open in \( G \) if and only if there exists a minimal \( \theta \)-split parabolic subgroup \( P_{\min} \) that contains \( v^{-1}Bv \).

(2) If \( v^{-1}Bv \subset P_{\min} \) for \( v \in G \), then \( v^{-1}Bv \cap K = v^{-1}Bv \cap M \) and the identity component of \( v^{-1}Bv \cap M \) is a Borel subgroup of the identity component of \( M \).

**Proof.** (1) This follows from the remark after Corollary to Theorem 1 in [Vus74].

(2) The equation \( v^{-1}Bv \cap K = v^{-1}Bv \cap M \) follows from
\[
v^{-1}Bv \cap K = v^{-1}Bv \cap \theta(v^{-1}Bv) \cap K \subset P_{\min} \cap \theta(P_{\min}) \cap K = MA \cap K = M.
\]
For the last assertion, we work on the Lie algebra level. Since \( \text{ad} (v^{-1})b \) is a Borel subalgebra (i.e. a maximal solvable subalgebra) of \( p_{\min} \), it contains the nilradical \( n \) of \( p_{\min} \). Hence we have \( \text{ad} (v^{-1})b = (\text{ad} (v^{-1})b \cap (m + a)) \oplus n \) and \( \text{ad} (v^{-1})b \cap (m + a) \) is a Borel
subalgebra of $\mathfrak{m} + \mathfrak{a}$. Moreover since $\mathfrak{a}$ is contained in the center of $\mathfrak{m} + \mathfrak{a}$, we have $\text{ad}(v^{-1})\mathfrak{b} = (\text{ad}(v^{-1})\mathfrak{b} \cap \mathfrak{m}) + \mathfrak{a} + \mathfrak{n}$ and $\text{ad}(v^{-1})\mathfrak{b} \cap \mathfrak{m}$ is a Borel subalgebra of $\mathfrak{m}$. \hfill \square

Now we return to the situation in the former paragraph. Since $\mathfrak{u}'$ is $\theta$-stable, we have a decomposition $\mathfrak{u}' = \mathfrak{u}'_K \oplus (\mathfrak{u}')^{-\theta}$. Therefore, we get an $L'_K$-equivariant isomorphism $(\mathfrak{u}')^{-\theta} \simeq U'/U'_K$ via the exponential map.

**Theorem 4.5.** Let $P'$ be a $\theta$-stable parabolic subgroup of $G$ such that $P' \cap K = Q$. We denote by $P' = L'U'$ the standard Levi decomposition. Let $P'_\text{min}$ be a minimal $\theta$-split parabolic subgroup of $L'$ and put $M' := P'_\text{min} \cap K$. Then the following three conditions are all equivalent.

1. The double flag variety $\mathfrak{X}_B \times \mathcal{Z}_Q = G/B \times K/Q$ is of finite type.
2. The homogeneous space $G/Q$ is $G$-spherical.
3. The adjoint action of the identity component $M'_0$ of $M'$ on $(\mathfrak{u}')^{-\theta}$ is spherical.

**Proof.** The equivalence of (1) and (2) is clear.

Let $B'_{L'}$ be a Borel subgroup of $L'$ such that $B'_{L'} \cdot L'_K$ is open in $L'$. By Lemma 4.4, we may assume that $B'_{L'}$ is contained in $P'_\text{min}$ and then the connected component of $B'_{L'} \cap K$ is a Borel subgroup of $M'_0$. Since we have already seen above that $\mathfrak{X}_B \times \mathcal{Z}_Q$ is of finite type if and only if $U'/U'_K$ has an open $(B'_{L'} \cap K)$-orbit, the equivalence of (1) and (3) follows from the isomorphism $U'/U'_K \simeq (\mathfrak{u}')^{-\theta}$. \hfill \square

**Corollary 4.6.** If the double flag variety $\mathfrak{X}_B \times \mathcal{Z}_Q$ is of finite type, then the double flag variety $\mathfrak{X}_{P'} \times \mathcal{Z}_{B_K}$ is also of finite type.

**Proof.** By Theorem 4.5, the adjoint action of $M'_0$ on $(\mathfrak{u}')^{-\theta}$ is spherical. Since $M' \subseteq L'_K$, we conclude that $\mathfrak{X}_{P'} \times \mathcal{Z}_{B_K}$ is of finite type by Theorem 4.2. \hfill \square

### 4.3. Triple flag varieties

Let us take three parabolic subgroups $P_1, P_2$ and $P_3$ of $G$. If one considers $\mathbb{G} = G \times G$ and an involution $\theta(g_1, g_2) = (g_2, g_1)$ of $\mathbb{G}$, the symmetric subgroup $K = G^\theta$ is the diagonal subgroup $\text{diag}(G) \subset \mathbb{G}$. Thus $(G \times G, \text{diag}(G))$ is a symmetric pair. Then $\mathbb{P} = P_1 \times P_2$ is a parabolic subgroup of $\mathbb{G}$ and $Q = \text{diag}(P_3)$ is a parabolic subgroup of $K$. Therefore our double flag variety becomes

$$\mathbb{G}/\mathbb{P} \times K/Q = (G \times G)/(P_1 \times P_2) \times (\text{diag}(G)/\text{diag}(P_3)) \simeq \mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3},$$

which is a triple flag variety for $G$. So the double flag variety for symmetric pair is a generalization of the triple flag variety. We can take a parabolic subgroup $\mathbb{P}' = P_3 \times P_3$ of $\mathbb{G}$ so that $\mathbb{P}' \cap K = \text{diag}(P_3) = Q$ holds.

A triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$ is said to be of finite type if there are only finitely many $G$-orbits in it. Note that this terminology agrees with that of the double flag variety for symmetric pair.

If $T$ is a maximal torus of $G$, then $\mathbb{T} = T \times T$ is a maximal torus of $\mathbb{G}$. The root system of $\mathbb{G}$ with respect to this $\mathbb{T}$ is decomposed as $\Delta_{\mathbb{G}} = \Delta^{(1)} \sqcup \Delta^{(2)}$, where $\Delta^{(1)}$ denotes the roots of the first factor and $\Delta^{(2)}$ denotes the roots of the second factor. For a parabolic
subgroup $P_i$ of $G$, there corresponds to a subset $J_i$ of the set of simple roots $\Pi$. We put $J = J_1^{(i)} \cup J_2^{(i)}$. Here $J_i^{(i)}$ is a copy of $J_i$ in $\Delta_i$ for $i = 1, 2$.

Let us specialize what we have already proved to the case of triple flag variety. In order to do this we will give a list of notations, which tells the correspondence of concepts.

$$J \equiv J_1^{(i)} W J_3 \times J_2 W J_3 \ni w = (w_1, w_2),$$

$$\mathbb{P} \backslash G / P' = (P_1 \backslash G / P_3) \times (P_2 \backslash G / P_3) \simeq \mathbb{P}^3 = \mathbb{P} W^3 = J_1 W J_3 \times J_2 W J_3 \ni w = (w_1, w_2),$$

$$L' = L_3 \times L_3 \supset L'_3 = \text{diag}(L_3),$$

$$U' = U_3 \times U_3,$$

$$\mathbb{P}^w \cap P' = w^{-1} \mathbb{P} \cap P' = (w_1^{-1} P_1 w_1 \cap P_3) \times (w_2^{-1} P_2 w_2 \cap P_3),$$

$$\mathbb{P}^w \cap L' = w^{-1} \mathbb{P} \cap L' = (w_1^{-1} P_1 w_1 \cap L_3) \times (w_2^{-1} P_2 w_2 \cap L_3),$$

$$((P_1^{w_1} \cap L_3) \times (P_2^{w_2} \cap L_3)) \cdot (v, e) \cdot \text{diag}(L_3) \longmapsto (P_1^{w_1} \cap L_3) \cdot v \cdot (P_2^{w_2} \cap L_3) \quad \text{for } v \in W_{J_3}.$$ 

Put $v = (v, e) \in \mathbb{L}_3 \times L_3$.

Let us continue reinterpreting the notations:

$$\mathbb{P}^w \cap U' = v^{-1} w^{-1} \mathbb{P}^w \cap U' = (v^{-1} w_1^{-1} P_1 w_1 v \cap U_3) \times (w_2^{-1} P_2 w_2 \cap U_3),$$

$$\mathbb{P}^{w_1} \cap L'_3 = v^{-1} (\mathbb{P}^w \cap L') v \cap L'_3,$$

$$= \text{diag}(L_3) \cap (v^{-1} w_1^{-1} P_1 w_1 v \cap L_3) \times (w_2^{-1} P_2 w_2 \cap L_3),$$

$$= \text{diag}(P_1^{w_1} \cap P_2^{w_2} \cap L_3) \supset \text{diag}(T).$$

We therefore get

$$((P_1^{w_1} \cap U_3) \times (P_2^{w_2} \cap U_3)) \backslash (U_3 \times U_3) / \text{diag}(U_3) \simeq (P_1^{w_1} \cap U_3) \backslash U_3 / (P_2^{w_2} \cap U_3).$$

On this last double coset space, the group $P_1^{w_1} \cap P_2^{w_2} \cap L_3 \simeq \mathbb{P}^w \cap L'_3$ acts by conjugation.

**Theorem 4.7.** Let $P_1, P_2, P_3$ be three parabolic subgroups of $G$. Then the diagonal $G$-orbits on the triple flag variety can be described as

$$G \backslash (G / P_1 \times G / P_2 \times G / P_3) \simeq \coprod_{P_1, P_2} \left( (P_1 \cap U_3) \backslash U_3 / (P_2 \cap U_3) \right) / (P_1 \cap P_2 \cap L_3),$$

$$\text{where } P_1, P_2 \subset \mathbb{P} \backslash G / P_3 \text{ and } P_1, P_2 \subset \mathbb{P} \backslash G / P_3 \text{ are parabolic subgroups of } G.$$
where \((\tilde{P}_1, \tilde{P}_2)\) runs over all pairs \((P^1w_1, P^2w_2)\) for \(w_1 \in J_1WJ_3\), \(w_2 \in J_2WJ_3\), and \(v \in \mathcal{V}(\langle w_1, w_2 \rangle)\).

Let us consider the special case where \(P_3 = B\) is a Borel subgroup. In this case \(\tilde{P}_1 \cap \tilde{P}_2 \cap L_3 = T\) is a maximal torus of \(G\) and we have

\[
G \setminus (G/P_1 \times G/P_2 \times G/B) \simeq \bigsqcup_{\tilde{P}_1, \tilde{P}_2} ((\tilde{P}_1 \cap U_B)\setminus U_B/(\tilde{P}_2 \cap U_B)) / T.
\]

Also by applying Theorem 4.2 to the present case, we conclude that:

**Corollary 4.8.** A triple flag variety \(G/P_1 \times G/P_2 \times G/B\) is of finite type if and only if \(u_{J_1} \cap u_{J_2}\) is \(L_{J_1 \cap J_2}\)-spherical.

**Remark 4.9.** This corollary can be deduced from [Pan93, Theorem 3].

The triple flag varieties \(G/P_1 \times G/P_2 \times G/B\) of finite type (or equivalently the \(G\)-spherical double flag varieties \(G/P_1 \times G/P_2\)) were classified by Stembridge [Ste03] in connection with multiplicity-free tensor product of two irreducible \(G\)-modules.

**Theorem 4.10 (Ste03).** Let \(G\) be a connected simple algebraic group. Let \(P_1, P_2\) be parabolic subgroups of \(G\) corresponding to sets of simple roots \(J_1, J_2 \subseteq \Pi\), respectively. Then the triple flag variety \(G/P_1 \times G/P_2 \times G/B\) is of finite type if and only if the pair \((\Pi \setminus J_1, \Pi \setminus J_2)\) appears in Table 4 up to switching \(J_1\) and \(J_2\).

If we assume further that \(P_2\) is a Borel subgroup, we see from Table 4 that:

**Corollary 4.11.** Let \(G\) be a connected simple algebraic group and \(P\) a parabolic subgroup. Then the triple flag variety \(G/P \times G/B \times G/B\) is of finite type if and only if \(G = SL_n\) and \(G/P\) is isomorphic to the projective space of dimension \(n - 1\) (i.e. \(P\) is a mirabolic subgroup of \(G\)).

**Remark 4.12.** Recently Tanaka [Tan12] proved that \(G/P_1 \times G/P_2\) is \(G\)-spherical if and only if the action of a compact real form of \(G\) on \(G/P_1 \times G/P_2\) is strongly visible.
In this section, we give a classification of the triples \((G, K, P)\) such that \(G/P \times K/B_K\) is of finite type, where \(B_K\) is a Borel subgroup of \(K\). It is known that any symmetric pair \((G, K)\) with \(G\) connected, simply connected, and semisimple is a direct product of symmetric pairs \((G, K)\) such that

- \(G\) is simple, or
- \(G = G' \times G', K = \text{diag} G',\) and \(G'\) is simple.

### Table 1: \(G/P_1 \times G/P_2 \times G/B\) of Finite Type

| \(\mathfrak{so}_{2n}\) |
|-----------------|
| \(\alpha_1\) ──\(\alpha_2\) ──\(\alpha_{n-2}\) ──\(\alpha_n\) |
| \(\alpha_1\) \(\alpha_2\) \(\alpha_{n-1}\) \(\alpha_n\) |
| \(n \geq 4\) |
| \(\{\alpha_1\}, \{\alpha_i\}\)(\forall i), \(\{\alpha_i\}, \{\alpha_j\}\)(\forall i), \(\{\alpha_{n-1}\}, \{i, \alpha_2\}\), \(\{\alpha_n\}, \{\alpha_1, \alpha_2\}\), \(\{\alpha_i\}, \{\alpha_j, \alpha_k\}\)(\forall i), \(\{\alpha_3\}, \{\alpha_2, \alpha_4\}\) if \(n = 4\), \(\{\alpha_4\}, \{\alpha_2, \alpha_3\}\) if \(n = 4\) |

| \(\mathfrak{sp}_n\) |
|-----------------|
| \(\alpha_1\) ──\(\alpha_2\) ──\(\alpha_{n-1}\) ──\(\alpha_n\) |
| \(\{\alpha_1\}, \{\alpha_i\}\)(\forall i), \(\{\alpha_i\}, \{\alpha_n\}\) |

| \(\mathfrak{e}_6\) |
|-----------------|
| \(\alpha_1\) ──\(\alpha_3\) ──\(\alpha_5\) ──\(\alpha_6\) ──\(\alpha_7\) |
| \(\{\alpha_i\}, \{\alpha_j\}\)(\forall i, j \neq 4), \(\{\alpha_1\}, \{\alpha_1, \alpha_6\}\), \(\{\alpha_6\}, \{\alpha_1, \alpha_6\}\) |

| \(\mathfrak{e}_7\) |
|-----------------|
| \(\alpha_1\) ──\(\alpha_3\) ──\(\alpha_5\) ──\(\alpha_6\) ──\(\alpha_7\) ──\(\alpha_4\) |
| \(\{\alpha_i\}, \{\alpha_j\}\)(\forall i, j \neq 4), \(\{\alpha_1\}, \{\alpha_1, \alpha_6\}\), \(\{\alpha_6\}, \{\alpha_1, \alpha_6\}\) |

Table 1: \(G/P_1 \times G/P_2 \times G/B\) of Finite Type
For the latter case, $G/P \times K/B_K$ can be written as the triple flag variety $G'/P'_i \times G'/P'_j \times G'/B_{G'}$ and the classification was already given (see Theorem 4.10 and [Ste03]).

In the rest of this section we assume that $G$ is simple.

We first consider the case where $(G,K)$ is the complexification of a Hermitian symmetric pair, or equivalently the center of $K$ is one-dimensional. In this case, $K$ equals the Levi component of a maximal parabolic subgroup of $G$. Therefore we can choose a $\theta$-stable Borel subgroup $B$ of $G$ and a simple root $\alpha_i \in \Pi$ such that $K = L_{\Pi \setminus \{\alpha_i\}}$. Then $K = P_{\Pi \setminus \{\alpha_i\}} \cap P_{\Pi \setminus \{\alpha_i\}}^-$, where $P_{\Pi \setminus \{\alpha_i\}}^-$ is the opposite parabolic subgroup of $P_{\Pi \setminus \{\alpha_i\}}$.

Lemma 5.1. Suppose that $(G,K)$ is the complexification of a Hermitian symmetric pair and $P$ is a parabolic subgroup of $G$. Choose a $\theta$-stable Borel subgroup $B$ and a simple root $\alpha_i$ such that $K = L_{\Pi \setminus \{\alpha_i\}}$. Then $G/P \times K/B_K$ is of finite type if and only if $G/P \times G/P_{\Pi \setminus \{\alpha_i\}}^* \times G/B$ is of finite type. Here $\Pi \setminus \{\alpha_i\}^* := -w_0(\Pi \setminus \{\alpha_i\})$ for the longest element $w_0 \in W$.

Proof. We follow an argument of [NO11] Theorem 2. The opposite parabolic subgroup $P_{\Pi \setminus \{\alpha_i\}}^-$ of $P_{\Pi \setminus \{\alpha_i\}}$ is conjugate to $P_{\Pi \setminus \{\alpha_i\}}^*$. Hence $G/P \times G/P_{\Pi \setminus \{\alpha_i\}}^* \times G/B$ is of finite type if and only if $G/P \times G/P_{\Pi \setminus \{\alpha_i\}}^* \times G/B$ is of finite type. Since $BP_{\Pi \setminus \{\alpha_i\}}^-$ is open in $G$ and since $B \cap P_{\Pi \setminus \{\alpha_i\}}^- = B \cap L_{\Pi \setminus \{\alpha_i\}} = B_K$, a natural morphism $G/B_K \to G/B \times G/P_{\Pi \setminus \{\alpha_i\}}^-$ is an open immersion. Hence the conditions above are also equivalent to that $P$ has an open orbit in $G/B_K$.

By Theorem 4.10 and Lemma 5.1 we get a list of $G/P \times K/B_K$ of finite type if $(G,K)$ is a Hermitian symmetric pair.

For the remaining pairs $(G,K)$, we use Theorem 4.2 and the classification of spherical linear actions by Benson and Ratcliffe [BR96]. We carry out a classification according to the following procedure.

1. For each symmetric pair $(G,K)$ we fix a $\theta$-stable Borel subgroup $B$ of $G$ and a $\theta$-stable Cartan subgroup $T$ in $B$.
2. Take a standard parabolic subgroup $P$ and determine $\mathfrak{l} \cap \mathfrak{t}$ and $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$.
3. Check whether the $(L \cap K)$-action on $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$ is spherical using the list of [BR96].

In addition, the obvious dimension condition $\dim G/P + \dim K/B_K \leq \dim K$ is helpful in some cases. This is equivalent to $\dim L \geq \dim G - \dim K - \text{rank } K$. We note that the choice of a $\theta$-stable Borel subgroup $B$ is not unique up to $K$-conjugacy in general and the Lie algebras $\mathfrak{l} \cap \mathfrak{t}$ and $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$ depend on this choice. For our purpose, it is enough to check Theorem 4.2 (3) for one choice of $B$.

Let us explain how the procedure above will be done in more detail.

For step (1) we describe $B$ and $T$ by using a Vogan diagram (see [Kna02], § VI.8) for details). The Vogan diagram is defined by incorporating the involution in the Dynkin diagram. If $B$ is $\theta$-stable, then $\theta$ permutes simple roots. The simple roots in the same
\( \theta \)-orbit are connected by arrows. If a simple root \( \alpha \) is fixed by \( \theta \), then it is painted or not according to \( g_\alpha \subset g^{-\theta} \) or \( g_\alpha \subset \mathfrak{t} = g^{\theta} \). We label the simple roots for \( g \) and \( \mathfrak{t} \) as \( \Pi = \{ \alpha_1, \alpha_2, \ldots \} \) and \( \Pi_K = \{ \beta_1, \beta_2, \ldots \} \), respectively. Then a simple root \( \beta_i \) for \( \mathfrak{t} \) is written as a restriction \( \alpha|_{\theta} \) for some \( \alpha \in \Delta^+ = \Delta^+(g, \mathfrak{t}) \). We note that if \( \alpha, \alpha' \in \Delta^+ \) and \( \alpha|_{\theta} = \alpha'|_{\theta} \), then \( \alpha = \alpha' \) or \( \theta \alpha = \alpha' \) holds.

For step (2) take a standard parabolic subgroup \( P = P_J \) with \( J \subset \Pi \). Put

\[
J_K = \{ \beta_i : \exists \alpha \in \Delta^+ \text{ such that } \alpha|_{\theta} = \beta_i \text{ and } \alpha, \theta \alpha \in \Delta(I, \mathfrak{t}) \}
\]

so that \( L_J \cap K = L_{K,J_K} \), which is a Levi subgroup of \( K \) corresponding to \( J_K \subset \Pi_K \). We then describe the \((L \cap K)\)-module \( u \cap g^{-\theta} \) by giving the highest weights of its irreducible constituents. For \( \alpha \in \Delta^+ \), the restriction \( \alpha|_{\theta} \) is a weight in \( u \cap g^{-\theta} \) if and only if one of the following two conditions holds:

- \( \alpha = \theta \alpha \in \Delta(u, \mathfrak{t}) \) and \( g_\alpha \subset g^{-\theta} \);
- \( \alpha \neq \theta \alpha \) and \( \alpha, \theta \alpha \in \Delta(u, \mathfrak{t}) \).

We write \( \Lambda^+(u \cap g^{-\theta}) \) for the set of highest weights of irreducible constituents of the \((L \cap K)\)-module \( u \cap g^{-\theta} \). Denote by \( \omega_i \in (t^\theta)^* \) the fundamental weight corresponding to \( \beta_i \). Then \( \Lambda^+(u \cap g^{-\theta}) \) can be given in terms of \( \omega_i \) and step (3) can be carried out.

We give computations for each case in the following. We abbreviate \( \alpha_k + \alpha_{k+1} + \cdots + \alpha_l \) to \( \alpha_{[k, l]} \).

5.1. \((\mathfrak{sl}_n, \mathfrak{so}_n)\).

Let \( (g, \mathfrak{t}) = (\mathfrak{sl}_n, \mathfrak{so}_n) \). We assume \( n \) is even and put \( m = \frac{n}{2} \). The case where \( n \) is odd can be treated similarly. We fix a \( \theta \)-stable Borel subgroup \( B \), a \( \theta \)-stable Cartan subgroup \( T \) and a labeling \( \alpha_1, \alpha_2, \ldots, \alpha_n \) of simple roots corresponding to the following Vogan diagram.

![Vogan Diagram](image)

Let \( \beta_i := \alpha_i|_{\theta} \) for \( 1 \leq i \leq m - 1 \) and \( \beta_m := (\alpha_{m-1} + \alpha_m)|_{\theta} \). Then \( \Pi_K = \{ \beta_1, \ldots, \beta_m \} \) is a set of simple roots for \( K \) and the corresponding Dynkin diagram is given as above.

Suppose first that \( J = \Pi \setminus \{ \alpha_i \} \) with \( 1 \leq i \leq m \). Then \( J_K = \Pi_K \setminus \{ \beta_i \} \) if \( i \neq m - 1 \) and \( J_K = \Pi_K \setminus \{ \beta_{m-1}, \beta_m \} \) if \( i = m - 1 \). We have \( \mathfrak{l} \cap \mathfrak{k} = \mathfrak{l}_{K,J_K} \cong \mathfrak{gl}_1 \oplus \mathfrak{so}_{n-2i} \),

\[
\Delta(u \cap g^{-\theta}, t^\theta) = \{ \alpha_{[k, l]}|_{\theta} : k \leq i \text{ and } n - i \leq l \}, \quad \Lambda^+(u \cap g^{-\theta}) = \{ \alpha_{[1, n-1]}|_{\theta} \} = \{ 2\omega_1 \}.
\]

Hence \( u \cap g^{-\theta} \cong S^2(\mathbb{C}^i) \), on which \( L \cap K \) acts spherically. The case \( i > m \) is similar.

Suppose next that \( J = \Pi \setminus \{ \alpha_i, \alpha_j \} \) with \( 1 \leq i < j \leq n \). We may assume that \( i < m \) and \( i \leq j' \), where \( j' := \min \{ j, n - j \} \). Then \( J_K = \Pi_K \setminus \{ \beta_i, \beta_{j'} \} \) and \( \mathfrak{l} \cap \mathfrak{k} = \mathfrak{l}_{K,J_K} \cong \mathfrak{gl}_1 \oplus \mathfrak{gl}_{j'-i} \oplus \mathfrak{so}_{n-2j'} \). We have \( \Delta(u \cap g^{-\theta}, t^\theta) = \{ \alpha_{[k, l]}|_{\theta} : k \leq j' \text{ and } n - j' \leq l \} \) if \( j \leq m \).
and $Δ(u \cap g^{-θ}, t^θ) = \{α_{[k,l]}|_v : k ≤ j' \text{ and } n - j' ≤ l\} \cup \{α_{[k,l]}|_v : k ≤ i \text{ and } j' ≤ l\}$ if $j > m$. Hence

$$Λ^+(u \cap g^{-θ}) = \{α_{[1,n-1]}|_v, α_{[1,n-i-1]}|_v, α_{[i+1,n-i-1]}|_v, α_{[i,j-1]}|_v\}$$

$$= \{2ω_1, ω_1 - ω_i + ω_{i+1}, -2ω_i + 2ω_{i+1}\},$$

$$u \cap g^{-θ} \simeq S^2(C^i) \oplus (C^i \otimes C^{j'-i}) \oplus S^2(C^{j'-i})$$

if $j ≤ m$,

$$Λ^+(u \cap g^{-θ}) = \{α_{[1,n-1]}|_v, α_{[1,n-i-1]}|_v, α_{[i+1,n-i-1]}|_v, α_{[i,j-1]}|_v\}$$

$$= \{2ω_1, ω_1 - ω_i + ω_{i+1}, -2ω_i + 2ω_{i+1}, ω_1 - ω_{j'} + ω_{j'+1}\},$$

$$u \cap g^{-θ} \simeq S^2(C^i) \oplus (C^i \otimes C^{j'-i}) \oplus S^2(C^{j'-i}) \oplus (C^i \otimes C^{n-2j'})$$

if $m < j < n - i$, and

$$Λ^+(u \cap g^{-θ}) = \{α_{[1,n-1]}|_v, α_{[1,j-1]}|_v\} = \{2ω_1, ω_1 - ω_{j'} + ω_{j'+1}\},$$

$$u \cap g^{-θ} \simeq S^2(C^i) \oplus (C^i \otimes C^{n-2j'})$$

if $j = n - i$. We can see that none of them are $(L \cap K)$-spherical.

We therefore conclude that $G/P_J × K/B_K$ is of finite type if and only if $|Π \setminus J| = 1$.

5.2. $(\mathfrak{sl}_{2n}, \mathfrak{sp}_n)$.

Let $(g, ℱ) = (\mathfrak{sl}_{2n}, \mathfrak{sp}_n)$. We fix $B$, $T$ and simple roots $α_1, \ldots, α_{2n-1}$ corresponding to the following Vogan diagram.

Let $β_i := α_i|_v$ for $1 ≤ i ≤ n$. Then $Π_K = \{β_1, \ldots, β_n\}$ is a set of simple roots for $K$ and the corresponding Dynkin diagram is given as above.

Suppose that $J = Π \setminus \{α_i, α_j\}$ with $1 ≤ i < j ≤ 2n - 1$. We assume that $i < n$ and $i ≤ j'$, where $j' := \min\{j, 2n - j\}$. Then $J_K = Π_K \setminus \{β_i, β_j\}$ and $l \cap ℱ = I_{K,J_K} \simeq \mathfrak{gl}_l \oplus \mathfrak{gl}_{j'-i} \oplus \mathfrak{sp}_{n-j'}$.

We have $Δ(u \cap g^{-θ}, t^θ) = \{α_{[k,l]}|_v : k + l < 2n \text{ and } 2n - j' ≤ l\}$ if $j ≤ n$ and $Δ(u \cap g^{-θ}, t^θ) = \{α_{[k,l]}|_v : k + l < 2n \text{ and } 2n - j' ≤ l\} \cup \{α_{[k,l]}|_v : k + l < 2n, k ≤ i \text{ and } j' ≤ l\}$ if $j > n$.

As in the case $(\mathfrak{sl}_n, \mathfrak{so}_n)$ above, we have

$$u \cap g^{-θ} \simeq \begin{cases} \Lambda^2 C^i \oplus \Lambda^2 C^{j'-i} \oplus (C^i \otimes C^{j'-i}) & \text{if } j ≤ n, \\ \Lambda^2 C^i \oplus \Lambda^2 C^{j'-i} \oplus (C^i \otimes C^{j'-i}) \oplus (C^i \otimes C^{2n-2j'}) & \text{if } n < j < 2n - i, \\ \Lambda^2 C^i \oplus (C^i \otimes C^{2n-2j'}) & \text{if } j = 2n - i. \end{cases}$$

This is $(L \cap K)$-spherical if and only if $i = 1$ or $i + 1 = j$.

Suppose that $J = Π \setminus \{α_i, α_{i+1}\}$ with $2 ≤ i ≤ 2n - 3$. Put $i' := \min\{i, 2n - i - 1\}$. Then $J_K = Π_K \setminus \{β_i, β_{i'}, β_{i'+1}\}$ and $l \cap ℱ = I_{K,J_K} \simeq \mathfrak{gl}_l \oplus \mathfrak{gl}_{i'-1} \oplus \mathfrak{gl}_i \oplus \mathfrak{sp}_{n-i'-1}$. We
have \( \Delta(u \cap g^{-\theta}, t^\theta) = \{ \alpha_{[k,l]}|_\theta : k + l < 2n \text{ and } 2n - i' - 1 \leq l \} \) if \( i \leq n - 1 \) and 
\( \Delta(u \cap g^{-\theta}, t^\theta) = \{ \alpha_{[k,l]}|_\theta : k + l < 2n \text{ and } 2n - i' - 1 \leq l \} \cup \{ \alpha_{[1,l]}|_\theta : i' \leq l < 2n - 1 \} \) if \( i \geq n \). Then
\[
\Lambda^+(u \cap g^{-\theta}) = \{ \alpha_{[1,2n-2]}|_\theta, \alpha_{[1,2n-3]}|_\theta, \alpha_{[2,2n-3]}|_\theta \} = \{ \omega_2, \omega_1 - \omega_2 + \omega_3, -\omega_1 + \omega_3 \},
\]
\[
u \cap g^{-\theta} \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}
\]
if \( i = 2 \),
\[
\Lambda^+(u \cap g^{-\theta}) \supset \{ \alpha_{[1,2n-2]}|_\theta, \alpha_{[1,2n-i'-1]}|_\theta, \alpha_{[2,2n-3]}|_\theta, \alpha_{[2,2n-i'-1]}|_\theta \}
\]
\[
= \{ \omega_2, \omega_1 - \omega_2 + \omega_3, -\omega_1 + \omega_2 + \omega_i' + \omega_{i'+1} \},
\]
\[
u \cap g^{-\theta} \supset \mathbb{C}^{i'-1} \oplus \mathbb{C} \oplus \bigwedge^2 \mathbb{C}^{i'-1} \oplus \mathbb{C}^{i'-1}
\]
if \( 3 \leq i \leq 2n - 4 \), and
\[
\Lambda^+(u \cap g^{-\theta}) = \{ \alpha_{[1,2n-2]}|_\theta, \alpha_{[2,2n-3]}|_\theta, \alpha_{[1,2]}|_\theta \} = \{ \omega_2, -\omega_1 + \omega_3, \omega_1 + \omega_2 - \omega_3 \},
\]
\[
u \cap g^{-\theta} \supset \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}
\]
if \( i = 2n - 3 \). We can see that \( u \cap g^{-\theta} \) is \((L \cap K)\)-spherical if and only if \( i = 2 \).

Suppose that \( J = \Pi \setminus \{ \alpha_1, \alpha_i, \alpha_{2n-1} \} \) with \( 2 \leq i \leq n \). Then \( J_K = \Pi_K \setminus \{ \beta_1, \beta_i \} \) and 
\( l \cap t = l_{K,J_K} \simeq g_{l_1} \oplus g_{l_{i-1}} \oplus sp_{n-i} \). We have
\[
\Lambda^+(u \cap g^{-\theta}) = \{ \alpha_{[1,2n-2]}|_\theta, \alpha_{[1,2n-i'-1]}|_\theta, \alpha_{[2,2n-3]}|_\theta, \alpha_{[1,i-1]}|_\theta \}
\]
\[
= \{ \omega_2, \omega_1 - \omega_i + \omega_{i+1}, -\omega_1 + \omega_3, \omega_1 + \omega_{i-1} - \omega_i \},
\]
\[
u \cap g^{-\theta} \simeq \mathbb{C}^{i-1} \oplus \mathbb{C}^{2n-2i} \oplus \bigwedge^2 \mathbb{C}^{i-1} \oplus (\mathbb{C}^{i-1})^*
\]
if \( i < n \),
\[
\Lambda^+(u \cap g^{-\theta}) = \{ \alpha_{[1,2n-2]}|_\theta, \alpha_{[2,2n-3]}|_\theta, \alpha_{[1,n-1]}|_\theta \} = \{ \omega_2, -\omega_1 + \omega_3, \omega_1 + \omega_{n-1} - \omega_n \},
\]
\[
u \cap g^{-\theta} \simeq \mathbb{C}^{n-1} \oplus \bigwedge^2 \mathbb{C}^{n-1} \oplus (\mathbb{C}^{n-1})^*
\]
if \( i = n \geq 3 \), and
\[
\Lambda^+(u \cap g^{-\theta}) = \{ \alpha_{[1,2]}|_\theta, \alpha_{1}|_\theta \} = \{ \omega_2, 2\omega_1 - \omega_2 \}, \quad \nu \cap g^{-\theta} \simeq \mathbb{C} \oplus \mathbb{C}^*
\]
if \( i = n = 2 \). Note that in both cases the factor \( g_{l_1} \) (central torus) acts on \( \mathbb{C}^{i-1} \oplus (\mathbb{C}^{i-1})^* \) by the same scalar, which can be read off from the explicit description of \( \Lambda^+(u \cap g^{-\theta}) \). Therefore \( u \cap g^{-\theta} \) is \((L \cap K)\)-spherical if and only if \( i = 2 \).
These observations imply that $G/P_j \times K/B_K$ is of finite type if and only if

$$\Pi \setminus J = \{\alpha_i\} (1 \leq i \leq 2n - 1), \ \{\alpha_1, \alpha_i\} (2 \leq i \leq 2n - 1),$$

$$\{\alpha_i, \alpha_{2n-1}\} (1 \leq i \leq 2n - 2), \ \{\alpha_i, \alpha_{i+1}\} (1 \leq i \leq 2n - 2),$$

$$\{\alpha_1, 2, \alpha_3\}, \ \{\alpha_{2n-3}, \alpha_{2n-2}, \alpha_{2n-1}\}, \ \{\alpha_1, \alpha_2, \alpha_{2n-1}\}, \ \{\alpha_1, \alpha_{2n-2}, \alpha_{2n-1}\}.$$

5.3. $(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)$.

Let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)$ with $p + q = 2n + 1$. We may assume that $p$ is even and $q$ is odd. Put $p' = \frac{p}{2}$ and $q' = \frac{q-1}{2}$. We fix $B$, $T$ and simple roots $\alpha_1, \ldots, \alpha_n$ corresponding to the following Vogan diagram.

Let $\beta_i := \alpha_i$ for $i \neq p'$ and $\beta_{p'} := \alpha_{p'} + 2(\alpha_{p'} + \cdots + \alpha_n)$. Then $\Pi_K = \{\beta_1, \ldots, \beta_n\}$ is a set of simple roots for $K$ and the corresponding Dynkin diagram is

We first consider the case $p, q \geq 3$. Suppose that $J = \Pi \setminus \{\alpha_i\}$ with $1 \leq i \leq p'$. Then $J_K = \Pi_K \setminus \{\beta_i\}$ if $i \neq p' - 1$ and $J_K = \Pi_K \setminus \{\beta_{p' - 1}, \beta_{p'}\}$ if $i = p' - 1$. Hence $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{l}_{K,J_K} \simeq \mathfrak{gl}_l \oplus \mathfrak{so}_{p-2l} \oplus \mathfrak{so}_q$. We have $\Delta(\mathfrak{u} \cap \mathfrak{g}^{-\theta}, \mathfrak{t}^\theta) = \{\alpha_{[k,l]} : k \leq i < p' \leq l\} \cup \{\alpha_{[k,l]} + 2\alpha_{[l+1,n]} : k \leq i < p' \leq l\}$. Hence $\mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq \mathbb{C}^i \otimes \mathbb{C}^\theta$, which is $(L \cap K)$-spherical if and only if $i = 1$.

Suppose that $J = \Pi \setminus \{\alpha_i\}$ with $p' < i \leq n$. Then $J_K = \Pi_K \setminus \{\beta_{p'}, \beta_i\}$ and $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{l}_{K,J_K} \simeq \mathfrak{gl}_{p'} \oplus \mathfrak{gl}_{p'-l} \oplus \mathfrak{so}_{2n-2l+1}$. We have $\Delta(\mathfrak{u} \cap \mathfrak{g}^{-\theta}, \mathfrak{t}^\theta) = \{\alpha_{[k,l]} : k \leq p' \leq l\} \cup \{\alpha_{[k,l]} + 2\alpha_{[l+1,n]} : k \leq p' \leq l\}$. Hence $\mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq (\mathbb{C}^{p'} \otimes \mathbb{C}^{i-p'}) \oplus (\mathbb{C}^{p'} \otimes \mathbb{C}^{2n-2l+1})$, which is $(L \cap K)$-spherical if and only if $i = n$.

Suppose that $J = \Pi \setminus \{\alpha_1, \alpha_n\}$. Then $J_K = \Pi_K \setminus \{\beta_1, \beta_{p'}, \beta_n\}$ and $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{l}_{K,J_K} \simeq \mathfrak{gl}_1 \oplus \mathfrak{gl}_{p'-1} \oplus \mathfrak{gl}_{p'}$. We have

$$\Delta(\mathfrak{u} \cap \mathfrak{g}^{-\theta}, \mathfrak{t}^\theta) = \{\alpha_{[1,k]} : p' \leq k\} \cup \{\alpha_{[k,n]} : k \leq p'\} \cup \{\alpha_{[k,l]} + 2\alpha_{[l+1,n]} : k \leq p' \leq l\}.$$ 

Hence

$$\Lambda^+(\mathfrak{u} \cap \mathfrak{g}^{-\theta}) = \{\alpha_{[1,p']} + 2\alpha_{[p'+1,n]}, \alpha_{[1,n]}, \alpha_{[1,n-1]}, \alpha_{[2,p']} + 2\alpha_{[p'+1,n]}, \alpha_{[2,n]}\},$$

$$\{\omega_1 + \omega_{p'+1}, \omega_1 + \omega_{n-1} - 2\omega_n, -\omega_1 + \omega_2 + \omega_{p'+1}, -\omega_1 + \omega_2\}$$

$$\mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq \mathbb{C}^{p'} \oplus \mathbb{C} \oplus (\mathbb{C}^{p'})^* \oplus (\mathbb{C}^{p'-1} \otimes \mathbb{C}^{p'}) \oplus \mathbb{C}^{p'-1},$$

which is not $(L \cap K)$-spherical.

Hence $G/P_j \times K/B_K$ is of finite type if and only if $\Pi \setminus J = \{\alpha_1\}$ or $\{\alpha_n\}$ for $p, q \geq 3$. 


Let \( \beta_i := \alpha_i \) for \( i \neq p \) and \( \beta_p := 2(\alpha_p + \cdots + \alpha_{n-1}) + \alpha_n \). Then \( \Pi_K = \{ \beta_1, \ldots, \beta_n \} \) is a set of simple roots for \( K \) and the corresponding Dynkin diagram is

![Dynkin diagram](image)

Suppose that \( J = \Pi \setminus \{ \alpha_i \} \) with \( 1 \leq i \leq p \). Then \( J_K = \Pi_K \setminus \{ \beta_i \} \) and \( \mathfrak{t} \cap \mathfrak{k} = \mathfrak{t}_{K,J_K} \simeq \mathfrak{gl}_i \oplus \mathfrak{sp}_{p-i} \oplus \mathfrak{sp}_q \). We have \( \mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq \mathbb{C}^i \otimes \mathbb{C}^{2q} \), which is \((L \cap K)\)-spherical if and only if \( i \leq 3 \) or \( q \leq 2 \).

Suppose that \( J = \Pi \setminus \{ \alpha_i \} \) with \( p < i \leq n \). Then \( J_K = \Pi_K \setminus \{ \beta_p, \beta_i \} \) and \( \mathfrak{t} \cap \mathfrak{k} = \mathfrak{t}_{K,J_K} \simeq \mathfrak{gl}_p \oplus \mathfrak{gl}_{i-p} \oplus \mathfrak{sp}_{n-i} \). We have \( \mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq (\mathbb{C}^p \otimes \mathbb{C}^{i-p}) \oplus (\mathbb{C}^p \otimes \mathbb{C}^{2(n-i)}) \), which is \((L \cap K)\)-spherical if and only if \( i-j = 1 \) or \( q = 1 \).

Suppose that \( J = \Pi \setminus \{ \alpha_i, \alpha_j \} \) with \( 1 \leq i < j \leq p \). Then \( J_K = \Pi_K \setminus \{ \beta_i, \beta_j \} \) and \( \mathfrak{t} \cap \mathfrak{k} = \mathfrak{t}_{K,J_K} \simeq \mathfrak{gl}_i \oplus \mathfrak{gl}_{i-j} \oplus \mathfrak{sp}_{n-j} \). We have \( \mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq (\mathbb{C}^i \otimes \mathbb{C}^{j-p}) \oplus (\mathbb{C}^i \otimes \mathbb{C}^{2(n-j)}) \oplus (\mathbb{C}^i \otimes (\mathbb{C}^{j-p})^*) \oplus (\mathbb{C}^{p-i} \otimes \mathbb{C}^{j-p}) \oplus (\mathbb{C}^{p-i} \otimes \mathbb{C}^{2(n-j)}) \), which is \((L \cap K)\)-spherical if and only if \( i = p = 1 \) or \( j = n = p+1 \).

Suppose that \( J = \Pi \setminus \{ \alpha_i, \alpha_j \} \) with \( p < i < j \leq n \). Then \( J_K = \Pi_K \setminus \{ \beta_p, \beta_i, \beta_j \} \) and \( \mathfrak{t} \cap \mathfrak{k} = \mathfrak{t}_{K,J_K} \simeq \mathfrak{gl}_p \oplus \mathfrak{gl}_{i-p} \oplus \mathfrak{gl}_{j-i} \oplus \mathfrak{sp}_{n-j} \). We have \( \mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq (\mathbb{C}^p \otimes \mathbb{C}^{j-i}) \oplus (\mathbb{C}^p \otimes (\mathbb{C}^{j-i})^*) \oplus (\mathbb{C}^p \otimes \mathbb{C}^{2(n-j)}) \), which is \((L \cap K)\)-spherical if and only if \( p = 1 \).

From these observations we see that for \( p, q \geq 3 \), \( G/P_J \times K/B_K \) is of finite type if and only if \( \Pi \setminus J = \{ \alpha_1 \}, \{ \alpha_2 \}, \{ \alpha_3 \}, \{ \alpha_n \}, \) or \( \{ \alpha_1, \alpha_2 \} \). Add to this, \( \Pi \setminus J = \{ \alpha_i \} \) for \( 1 \leq i \leq n \)
are also the cases if \( \min\{p, q\} \leq 2 \). For \( \min\{p, q\} = 1 \), \( u \cap g^{-\theta} \) is \((L \cap K)\)-spherical if \( |\Pi \setminus J| = 2 \). To prove that these are all the cases we need to check the case \( p = 1 \) and \( |\Pi \setminus J| = 3 \).

Suppose that \( p = 1 \) and \( J = \Pi \setminus \{\alpha_i, \alpha_j, \alpha_k\} \) with \( 1 \leq i < j < k \leq n \). Then \( J_K = \Pi_K \setminus \{\beta_1, \beta_i, \beta_j, \beta_k\} \) and \( I \cap \mathfrak{t} = I_{K, J_K} \simeq gl_1 \oplus gl_{l-1} \oplus gl_{j-i} \oplus gl_{k-j} \oplus sp_{n-j} \). We have

\[
\begin{aligned}
  u \cap g^{-\theta} &\simeq \mathbb{C}^{i-1} \oplus \mathbb{C}^{j-i} \oplus \mathbb{C}^{k-j} \oplus \mathbb{C}^{2(n-k)} \oplus (\mathbb{C}^{k-j})^* \oplus (\mathbb{C}^{j-i})^*,
\end{aligned}
\]

which is not \((L \cap K)\)-spherical.

5.6. \((g_2, sl_2 \oplus sl_2)\).

Let \((g, \mathfrak{t}) = (g_2, sl_2 \oplus sl_2)\). Then the dimension condition is \( \dim L \geq \dim G - \dim K - \text{rank } K = 14 - 6 - 2 = 6 \). But this does not hold for a proper parabolic subgroup \( P \subset G \).

5.7. \((f_4, sp_3 \oplus sp_1)\).

Let \((g, \mathfrak{t}) = (f_4, sp_3 \oplus sp_1)\). Then the dimension condition is \( \dim L \geq \dim G - \dim K - \text{rank } K = 52 - 24 - 4 = 24 \). But this does not hold for a proper parabolic subgroup \( P \subset G \).

5.8. \((f_4, so_9)\).

Let \((g, \mathfrak{t}) = (f_4, so_9)\). We fix \( B, T \) and simple roots \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) corresponding to the following Vogan diagram.

\[
\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \beta_1 & \beta_2 \\
\circ & \circ & \circ & \circ & \circ & \circ
\end{array}
\]

Let \( \beta_1 := 2\alpha_2 + 2\alpha_3 + 2\alpha_4 \), \( \beta_2 := \alpha_1 \), \( \beta_3 := \alpha_3 \), \( \beta_4 := \alpha_3 \). Then \( \Pi_K = \{\beta_1, \beta_2, \beta_3, \beta_4\} \) is a set of simple roots for \( K \) and the corresponding Dynkin diagram is given as above.

The dimension condition is \( \dim L \geq \dim G - \dim K - \text{rank } K = 52 - 36 - 4 = 12 \). This holds for \( J = \Pi \setminus \{\alpha_1\} \) with \( 1 \leq i \leq 4 \) or \( J = \Pi \setminus \{\alpha_1, \alpha_4\} \).

Suppose that \( J = \Pi \setminus \{\alpha_1, \alpha_4\} \). Then \( J_K = \Pi_K \setminus \{\beta_1, \beta_2\} \) and \( I \cap \mathfrak{t} = I_{K, J_K} \simeq gl_1 \oplus gl_1 \oplus so_5 \simeq gl_1 \oplus gl_1 \oplus sp_2 \). We have \( \Delta(u \cap g^{-\theta}, \mathfrak{t}^\theta) = \{\sum_{i=1}^4 m_i \alpha_i \in \Delta^+ : m_4 = 1\} \). Hence

\[
\Lambda^+(u \cap g^{-\theta}) = \{2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4\} = \{\omega_1 - \omega_2 + \omega_4, \omega_4\}
\]

and \( u \cap g^{-\theta} \simeq \mathbb{C}^4 \oplus \mathbb{C}^4 \), which is \((L \cap K)\)-spherical.

Suppose that \( J = \Pi \setminus \{\alpha_2\} \). Then \( J_K = \Pi_K \setminus \{\beta_1, \beta_3\} \) and \( I \cap \mathfrak{t} = I_{K, J_K} \simeq gl_2 \oplus gl_2 \). We have \( \Delta(u \cap g^{-\theta}, \mathfrak{t}^\theta) = \{\sum_{i=1}^4 m_i \alpha_i \in \Delta^+ : m_2 > 0, m_4 = 1\} \). Hence

\[
\Lambda^+(u \cap g^{-\theta}) = \{2\alpha_1 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4\} = \{\omega_2 - \omega_3 + \omega_4, \omega_4\}
\]

and \( u \cap g^{-\theta} \simeq (\mathbb{C}^2 \otimes \mathbb{C}^2) \oplus \mathbb{C}^2 \), which is \((L \cap K)\)-spherical.
Suppose that $J = \Pi \setminus \{\alpha_3\}$. Then $J_K = \Pi_K \setminus \{\beta_1, \beta_4\}$ and $\mathfrak{l} \cap \mathfrak{t} = \mathfrak{l}_{K,J_K} \simeq \mathfrak{gl}_1 \oplus \mathfrak{gl}_3$. We have $\Delta(u \cap \mathfrak{g}^{-\theta}, \mathfrak{t}^\theta) = \{\sum_{i=1}^4 m_i \alpha_i \in \Delta^+ : m_3 > 0, m_4 = 1\}$. Hence

$$\Lambda^+(u \cap \mathfrak{g}^{-\theta}) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4\}$$

and $u \cap \mathfrak{g}^{-\theta} \simeq \mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus \mathbb{C}$, which is $(L \cap K)$-spherical.

Hence $G/P_J \times K/B_K$ is of finite type if and only if $|\Pi \setminus J| = 1$ or $\Pi \setminus J = \{\alpha_1, \alpha_4\}$.

5.9. $(\mathfrak{g}, \mathfrak{k})$.

Let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}_6, \mathfrak{sp}_4)$. We fix $B$, $T$ and simple roots $\alpha_1, \ldots, \alpha_6$ corresponding to the following Vogan diagram.

Let $\beta_1 := (\alpha_2 + \alpha_3 + \alpha_4)|_\mathfrak{v}$, $\beta_2 := \alpha_1|_\mathfrak{v}$, $\beta_3 := \alpha_3|_\mathfrak{v}$, and $\beta_4 := \alpha_4|_\mathfrak{v}$. Then $\Pi_K = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a set of simple roots for $K$ and the corresponding Dynkin diagram is given as above.

The dimension condition is $\dim L \geq \dim G - \dim K - \text{rank } K = 78 - 36 - 4 = 38$. This holds only if $\Pi \setminus J = \{\alpha_1\}$ or $\{\alpha_6\}$.

Suppose that $J = \Pi \setminus \{\alpha_1\}$. Then $J_K = \Pi_K \setminus \{\beta_2\}$ and $\mathfrak{l} \cap \mathfrak{t} = \mathfrak{l}_{K,J_K} \simeq \mathfrak{gl}_2 \oplus \mathfrak{sp}_2(\simeq \mathfrak{gl}_2 \oplus \mathfrak{so}_5)$. We have

$$\Delta(u \cap \mathfrak{g}^{-\theta}, \mathfrak{t}^\theta) = \{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)|_\mathfrak{v}, (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)|_\mathfrak{v},$$

$$(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)|_\mathfrak{v}, (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6)|_\mathfrak{v},$$

$$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)|_\mathfrak{v}\}.$$

Hence $\Lambda^+(u \cap \mathfrak{g}^{-\theta}) = \{(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)|_\mathfrak{v}\} = \{\omega_4\}$ and $u \cap \mathfrak{g}^{-\theta} \simeq \mathbb{C}^5$, which is $(L \cap K)$-spherical.

Therefore, $G/P_J \times K/B_K$ is of finite type if and only if $\Pi \setminus J = \{\alpha_1\}$ or $\{\alpha_6\}$.

5.10. $(\mathfrak{g}, \mathfrak{sl}_6 \oplus \mathfrak{sl}_2)$.

Let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}_6, \mathfrak{sl}_6 \oplus \mathfrak{sl}_2)$. We fix $B$, $T$ and simple roots $\alpha_1, \ldots, \alpha_6$ corresponding to the following Vogan diagram.
Let $\beta_1 := \alpha_1$, $\beta_2 := \alpha_3$, $\beta_3 := \alpha_4$, $\beta_4 := \alpha_5$, $\beta_5 := \alpha_6$, and $\beta_6 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. Then $\Pi_K = \{\beta_1, \ldots, \beta_6\}$ is a set of simple roots for $K$ and the corresponding Dynkin diagram is given as above.

The dimension condition is $\dim L \geq \dim G - \dim K - \text{rank } K = 78 - 38 - 6 = 34$. This holds only if $\Pi \setminus J = \{\alpha_1\}, \{\alpha_2\}$ or $\{\alpha_6\}$.

Suppose that $J = \Pi \setminus \{\alpha_1\}$. Then $J_K = \Pi_K \setminus \{\beta_1, \beta_6\}$ and $\mathfrak{t} \cap \mathfrak{k} = \mathfrak{sl}_6 \oplus \mathfrak{gl}_1$. We have $\Delta(u \cap g^{-\theta}, t^\theta) = \{\sum_{i=1}^6 m_i \alpha_i \in \Delta^+: m_1 = m_2 = 1\}$. Hence $\Lambda^+(u \cap g^{-\theta}) = \{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\} = \{\omega_3 + \omega_6\}$ and $u \cap g^{-\theta} \simeq \bigwedge^2 \mathbb{C}^5$, which is $(L \cap K)$-spherical.

Suppose that $J = \Pi \setminus \{\alpha_2\}$. Then $J_K = \Pi_K \setminus \{\beta_6\}$ and $\mathfrak{t} \cap \mathfrak{k} = \mathfrak{sl}_6 \oplus \mathfrak{gl}_1$. We have $\Delta(u \cap g^{-\theta}, t^\theta) = \{\sum_{i=1}^6 m_i \alpha_i \in \Delta^+: m_2 = 1\}$. Hence $\Lambda^+(u \cap g^{-\theta}) = \{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\} = \{\omega_3 + \omega_6\}$ and $u \cap g^{-\theta} \simeq \bigwedge^3 \mathbb{C}^6$, which is not $(L \cap K)$-spherical.

Therefore, $G/P_J \times K/B_K$ is of finite type if and only if $\Pi \setminus J = \{\alpha_1\}$ or $\{\alpha_6\}$.

5.11. $(\mathfrak{c}_6, \mathfrak{f}_4)$.

Let $(\mathfrak{g}, \mathfrak{t}) = (\mathfrak{c}_6, \mathfrak{f}_4)$. We fix $B$, $T$ and simple roots $\alpha_1, \ldots, \alpha_6$ corresponding to the following Vogan diagram.

Let $\beta_1 := \alpha_2|\varphi$, $\beta_2 := \alpha_4|\varphi$, $\beta_3 := \alpha_3|\varphi$, and $\beta_4 := \alpha_1|\varphi$. Then $\Pi_K = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a set of simple roots for $K$ and the corresponding Dynkin diagram is given as above.

The dimension condition is $\dim L \geq \dim G - \dim K - \text{rank } K = 78 - 52 - 4 = 22$. This is satisfied for $\Pi \setminus J = \{\alpha_i\}(i \neq 4), \{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_6\}, \{\alpha_1, \alpha_3\}, \{\alpha_5, \alpha_6\}, \{\alpha_1, \alpha_6\}$.

Suppose that $J = \Pi \setminus \{\alpha_1, \alpha_2\}$. Then $J_K = \Pi_K \setminus \{\beta_1, \beta_4\}$ and $\mathfrak{t} \cap \mathfrak{k} = \mathfrak{sl}_6 \oplus \mathfrak{gl}_1$. We have

$$\Delta(u \cap g^{-\theta}, t^\theta) = \{(\alpha_2 + \alpha_3 + \alpha_4)|\varphi, (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)|\varphi,$$

$$\quad (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)|\varphi, (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)|\varphi,$$

$$\quad (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)|\varphi, (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)|\varphi\}.$$

Hence

$$\Lambda^+(u \cap g^{-\theta}) = \{(\alpha_2 + \alpha_3 + \alpha_4)|\varphi, (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)|\varphi,$$

$$\quad (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)|\varphi\}$$

$$\quad = \{\omega_1 - \omega_4, \omega_3 - \omega_4, \omega_1\}$$

and $u \cap g^{-\theta} \simeq \mathbb{C} \oplus \mathbb{C}^4 \oplus \mathbb{C}$, which is $(L \cap K)$-spherical.
Suppose that \( J = \Pi \setminus \{ \alpha_1, \alpha_3 \} \). Then \( J_K = \Pi_K \setminus \{ \beta_3, \beta_4 \} \) and \( I \cap t = I_{K,J_K} \simeq \mathfrak{gl}_3 \oplus \mathfrak{g}l_1 \).

We have
\[
\Delta(u \cap g^{-\theta}, t^\theta) = \{(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)_{\psi}, (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)_{\psi}, (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)_{\psi}, (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)_{\psi}, (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)_{\psi}\}.
\]

Hence
\[
\Lambda^+(u \cap g^{-\theta}) = \{(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)_{\psi}, (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)_{\psi}, (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)_{\psi}\}
\]

and \( u \cap g^{-\theta} \simeq (\mathbb{C}^3)^* \oplus \mathbb{C} \oplus \mathbb{C} \), which is \((L \cap K)\)-spherical.

Suppose that \( J = \Pi \setminus \{ \alpha_1, \alpha_6 \} \). Then \( J_K = \Pi_K \setminus \{ \beta_4 \} \) and \( I \cap t = I_{K,J_K} \simeq \mathfrak{so}_7 \oplus \mathfrak{g}l_1 \).

We have
\[
\Lambda^+(u \cap g^{-\theta}) = \{(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)_{\psi}, (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)_{\psi}\}
\]

and \( u \cap g^{-\theta} \simeq \mathbb{C}^8 \oplus \mathbb{C} \), which is not \((L \cap K)\)-spherical.

Hence \( G/P_J \times K/B_K \) is of finite type if and only if \( \Pi \setminus J = \{ \alpha_i \} (i \neq 4), \{ \alpha_1, \alpha_2 \}, \{ \alpha_2, \alpha_6 \}, \{ \alpha_1, \alpha_3 \}, \{ \alpha_5, \alpha_6 \} \).

5.12. \((\mathfrak{e}_7, \mathfrak{s}l_8)\).

Let \((\mathfrak{g}, \mathfrak{t}) = (\mathfrak{e}_7, \mathfrak{s}l_8)\). We fix \( B, T \) and simple roots \( \alpha_1, \ldots, \alpha_7 \) corresponding to the following Vogan diagram.

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_5 & \alpha_6 & \alpha_7 & \beta_1 & \beta_2 & \beta_7 \\
& \alpha_4 & & & & & & \\
\end{array}
\]

Let \( \beta_1 := \alpha_1, \beta_i := \alpha_{i+1} \) for \( 2 \leq i \leq 6 \), and \( \beta_7 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \). Then \( \Pi_K = \{ \beta_1, \ldots, \beta_7 \} \) is a set of simple roots for \( K \) and the corresponding Dynkin diagram is given as above.

The dimension condition is \( \dim L \geq \dim G - \dim K - \text{rank } K = 133 - 63 - 7 = 63 \), which implies \( \Pi \setminus J = \{ \alpha_1 \} \) or \( \{ \alpha_7 \} \).

Suppose that \( J = \Pi \setminus \{ \alpha_1 \} \). Then \( J_K = \Pi_K \setminus \{ \beta_1, \beta_7 \} \) and \( I \cap t = I_{K,J_K} \simeq \mathfrak{gl}_1 \oplus \mathfrak{sl}_6 \oplus \mathfrak{g}l_1 \).

We have \( \Delta(u \cap g^{-\theta}, t^\theta) = \{ \sum_{i=1}^7 m_i \alpha_i \in \Delta^+: m_1 = m_2 = 1 \} \). Hence
\[
\Lambda^+(u \cap g^{-\theta}) = \{ \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \} = \{ \omega_4 \}
\]

and \( u \cap g^{-\theta} \simeq \wedge^3 \mathbb{C}^6 \), which is not \((L \cap K)\)-spherical.
Suppose that \( J = \Pi \setminus \{ \alpha_7 \} \). Then \( J_K = \Pi_K \setminus \{ \beta_6 \} \) and \( \mathfrak{t} \cap \mathfrak{k} = \mathfrak{k}_{K,J_K} \simeq \mathfrak{sl}_6 \oplus \mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \). We have \( \Delta(\mathfrak{u} \cap \mathfrak{g}^{-\theta}, t^\theta) = \{ \sum_{i=1}^{7} m_i \alpha_i \in \Delta^+ : m_2 = m_7 = 1 \} \). Hence \( \Lambda^+(\mathfrak{u} \cap \mathfrak{g}^{-\theta}) = \{ \omega_4 \} \) and \( \mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq \mathbb{C}^2 \mathbb{C}^6 \), which is \( (L \cap K) \)-spherical.

Therefore, \( G/P_J \times K/B_K \) is of finite type if and only if \( \Pi \setminus J = \{ \alpha_7 \} \).

5.13. \((\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)\).

Let \((\mathfrak{g}, \mathfrak{t}) = (\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)\). We fix \( B, T \) and simple roots \( \alpha_1, \ldots, \alpha_7 \) corresponding to the following Vogan diagram.

Let \( \beta_i := \alpha_{8-i} \) for \( 1 \leq i \leq 6 \), and \( \beta_7 := 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \). Then \( \Pi_K = \{ \beta_1, \ldots, \beta_7 \} \) is a set of simple roots for \( K \) and the corresponding Dynkin diagram is given as above.

The dimension condition is \( \dim \mathfrak{L} \geq \dim \mathfrak{G} - \dim \mathfrak{K} - \text{rank} \mathfrak{K} = 133 - 69 - 7 = 57 \), which implies \( \Pi \setminus J = \{ \alpha_1 \} \) or \( \{ \alpha_7 \} \).

Suppose that \( J = \Pi \setminus \{ \alpha_1 \} \). Then \( J_K = \Pi_K \setminus \{ \beta_7 \} \) and \( \mathfrak{t} \cap \mathfrak{k} = \mathfrak{k}_{K,J_K} \simeq \mathfrak{so}_{12} \oplus \mathfrak{gl}_1 \). We have \( \Delta(\mathfrak{u} \cap \mathfrak{g}^{-\theta}, t^\theta) = \{ \sum_{i=1}^{7} m_i \alpha_i \in \Delta^+ : m_1 = 1 \} \). Hence

\[
\Lambda^+(\mathfrak{u} \cap \mathfrak{g}^{-\theta}) = \{ \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \} = \{ \omega_5 + \omega_7 \}
\]

and \( \mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq \mathbb{C}^{32} \), which is not \( (L \cap K) \)-spherical.

Suppose that \( J = \Pi \setminus \{ \alpha_7 \} \). Then \( J_K = \Pi_K \setminus \{ \beta_1, \beta_7 \} \) and \( \mathfrak{t} \cap \mathfrak{k} = \mathfrak{k}_{K,J_K} \simeq \mathfrak{gl}_1 \oplus \mathfrak{so}_{10} \oplus \mathfrak{gl}_1 \). We have \( \Delta(\mathfrak{u} \cap \mathfrak{g}^{-\theta}, t^\theta) = \{ \sum_{i=1}^{7} m_i \alpha_i \in \Delta^+ : m_1 = m_7 = 1 \} \). Hence \( \Lambda^+(\mathfrak{u} \cap \mathfrak{g}^{-\theta}) = \{ \omega_5 + \omega_7 \} \) and \( \mathfrak{u} \cap \mathfrak{g}^{-\theta} \simeq \mathbb{C}^{16} \), which is \( (L \cap K) \)-spherical.

Therefore, \( G/P_J \times K/B_K \) is of finite type if and only if \( \Pi \setminus J = \{ \alpha_7 \} \).

5.14. \((\mathfrak{e}_8, \mathfrak{so}_{16})\).

Let \((\mathfrak{g}, \mathfrak{t}) = (\mathfrak{e}_8, \mathfrak{so}_{16})\). We fix \( B, T \) and simple roots \( \alpha_1, \ldots, \alpha_8 \) corresponding to the following Vogan diagram.
Let $\beta_1 := 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ and $\beta_i := \alpha_{10 - i}$ for $2 \leq i \leq 8$. Then $\Pi_K = \{\beta_1, \ldots, \beta_8\}$ is a set of simple roots for $K$ and the corresponding Dynkin diagram is given as above.

The dimension condition is $\dim L \geq \dim G - \dim K - \rank K = 248 - 120 - 8 = 120$, which implies $\ dim \Pi \setminus J = \{\alpha_8\}$.

Suppose that $J = \Pi \setminus \{\alpha_8\}$. Then $J_K = \Pi_K \setminus \{\beta_8\}$ and $T \setminus \mathfrak{t} = \mathfrak{t}_{K,JK} \simeq \mathfrak{gl}_2 \oplus \mathfrak{so}_{12}$. We have $\Delta(u \cap \mathfrak{g}^\theta, \theta_\phi) = \{\sum_{i=1}^8 m_i \alpha_i \in \Delta^+ : m_1 = m_8 = 1\}$,

$\Lambda^+(u \cap \mathfrak{g}^\theta) = \{\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8\} = \{\omega_8\}$,

and $u \cap \mathfrak{g}^\theta \simeq \mathbb{C}^{32}$, which is not $(L \cap K)$-spherical.

Hence there is no $G/P_J \times K/B_K$ of finite type.

5.15. $(e_8, e_7 \oplus \mathfrak{sl}_2)$.

Let $(\mathfrak{g}, \mathfrak{t}) = (e_8, e_7 \oplus \mathfrak{sl}_2)$. We fix $B, T$ and simple roots $\alpha_1, \ldots, \alpha_8$ corresponding to the following Vogan diagram.

Let $\beta_i := \alpha_i$ for $1 \leq i \leq 7$ and $\beta_8 := 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$. Then $\Pi_K = \{\beta_1, \ldots, \beta_8\}$ is a set of simple roots for $K$ and the corresponding Dynkin diagram is given as above.

The dimension condition is $\dim L \geq \dim G - \dim K - \rank K = 248 - 136 - 8 = 104$, which implies $\ dim \Pi \setminus J = \{\alpha_8\}$.

Suppose that $J = \Pi \setminus \{\alpha_8\}$. Then $J_K = \Pi_K \setminus \{\beta_8\}$ and $T \setminus \mathfrak{t} = \mathfrak{t}_{K,JK} \simeq \mathfrak{e}_7 \oplus \mathfrak{g}_1$. We have $\Delta(u \cap \mathfrak{g}^\theta, \theta_\phi) = \{\sum_{i=1}^8 m_i \alpha_i \in \Delta^+ : m_8 = 1\}$,

$\Lambda^+(u \cap \mathfrak{g}^\theta) = \{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8\} = \{\omega_7 + \omega_8\}$,

and $u \cap \mathfrak{g}^\theta \simeq \mathbb{C}^{56}$, which is not $(L \cap K)$-spherical.

Hence there is no $G/P_J \times K/B_K$ of finite type.

We thus conclude that:

**Theorem 5.2.** Let $G$ be a connected simple algebraic group and $(G, K)$ a symmetric pair. Let $P$ be a parabolic subgroup of $G$ corresponding to $J \subseteq \Pi$. Then the double flag variety $G/P \times K/B_K$ is of finite type if and only if the triple $(\mathfrak{g}, \mathfrak{t}, \Pi \setminus J)$ appears in Table 2.

Remark 5.3. For $g \simeq \mathfrak{so}_{10}$, a symmetric subalgebra $\mathfrak{t}$ that is isomorphic to $\mathfrak{sl}_{2n} \oplus \mathbb{C}$ is not unique up to inner automorphisms of $g$. For $(\mathfrak{g}, \mathfrak{t}) = (\mathfrak{so}_{2n}, \mathfrak{sl}_n \oplus \mathbb{C})$ in Table 2 we take
(g, ℱ) and a positive system Δ⁺ in such a way that the Vogan diagram becomes

\[
\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n
\]

In particular, K is the Levi component \( L_{\Pi \{\alpha_n\}} \) of the parabolic subgroup \( P_{\Pi \{\alpha_n\}} \).

Similarly, the subalgebra ℱ of \( g \) is not unique for \((g, ℱ) \simeq (\mathfrak{so}_8, \mathfrak{so}_7), (\mathfrak{so}_8, \mathfrak{so}_6 \oplus \mathbb{C}), (\mathfrak{so}_8, \mathfrak{so}_5 \oplus \mathfrak{so}_3)\). In Table 2, we take \((g, ℱ)\) and positive systems Δ⁺ in such a way that the Vogan diagrams become

\[
\begin{align*}
\alpha_1 & \circ \alpha_2 \circ \cdots \circ \alpha_n \\
\text{α₁} & \circ \text{α₂} \circ \cdots \circ \text{αₙ} \\
\end{align*}
\]

for \((g, ℱ) = (\mathfrak{so}_8, \mathfrak{so}_7), (\mathfrak{so}_8, \mathfrak{so}_6 \oplus \mathbb{C}), (\mathfrak{so}_8, \mathfrak{so}_5 \oplus \mathfrak{so}_3)\), respectively.
\[
\begin{array}{|c|c|c|}
\hline
\text{sO}_{p+q} & 1 \leq p \leq q & \{\alpha_1\}, \{\alpha_{n-1}\}, \{\alpha_n\}, \\
p + q = 2n & & \{\alpha_i\}(\forall i) \text{ if } p = 2, \\
n \geq 4 & & \{\alpha_i, \alpha_{n-1}\}(\forall i) \text{ if } p = 2, \\
& & \{\alpha_i, \alpha_n\}(\forall i) \text{ if } p = 1, \\
& & \text{any subset of } \Pi \text{ if } p = 1 \\
\hline \end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{so}_{2n} & \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_{n-1}\}, \{\alpha_n\}, \\
n \geq 4 & \{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_{n-1}\}, \{\alpha_1, \alpha_n\}, \{\alpha_{n-1}, \alpha_n\}, \\
& \{\alpha_2, \alpha_3\} \text{ if } n = 4 \\
\hline \end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\alpha_1 & \alpha_2 & \alpha_{n-1} & \alpha_n \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{sp}_n & \{\alpha_1\}, \{\alpha_n\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{so}_9 & \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_4\}, \{\alpha_1, \alpha_4\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\alpha_1 & \alpha_3 & \alpha_5 & \alpha_6 \\
\alpha_4 & \hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{sp}_4 & \{\alpha_1\}, \{\alpha_6\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{sp}_7 & \{\alpha_1\}, \{\alpha_6\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{sl}_6 \oplus \text{sl}_2 & \{\alpha_1\}, \{\alpha_6\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{so}_{10} \oplus \mathbb{C} & \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_5\}, \{\alpha_6\}, \{\alpha_1, \alpha_6\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{sp}_{p+q} & \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_n\}, \{\alpha_1, \alpha_2\}, \\
p + q = n & \{\alpha_i\}(\forall i) \text{ if } p \leq 2, \\
1 \leq p \leq q & \{\alpha_i, \alpha_j\}(\forall i, j) \text{ if } p = 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{sp}_{p} \oplus \text{sp}_{q} & \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_n\}, \{\alpha_1, \alpha_2\}, \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{so}_{9} & \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_4\}, \{\alpha_1, \alpha_4\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\alpha_1 & \alpha_3 & \alpha_5 & \alpha_6 \\
\alpha_4 & \alpha_2 & \hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{sp}_4 & \{\alpha_1\}, \{\alpha_6\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{sl}_6 \oplus \text{sl}_2 & \{\alpha_1\}, \{\alpha_6\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{so}_{10} \oplus \mathbb{C} & \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_5\}, \{\alpha_6\}, \{\alpha_1, \alpha_6\} \\
\hline
\end{array}
\]

\[\text{1See Remark 5.3}\]
In this section, we give a classification of the triples \((G, K, Q)\) such that \(G/B \times K/Q\) is of finite type. As in the previous section, it is enough to consider the two cases:

- \(G\) is simple;
- \(G = G' \times G'\), \(K = \text{diag } G'\), and \(G'\) is simple.

In the latter case, \(G/B \times K/Q\) can be written as the triple flag variety \(G'/B_{G'} \times G'/B_{G'} \times G'/Q\) and the classification was already given (see Corollary 4.11).

In what follows we assume that \(G\) is simple.

We first consider the case where \((G, K)\) is the complexification of a Hermitian symmetric pair. We choose a \(\theta\)-stable Borel subgroup \(B\) of \(G\) and a simple root \(\beta \in \Pi\) such that \(K = L_{\Pi \setminus \{\beta\}}\). Since \(\text{rank } G = \text{rank } K\), we have \(T = T_K\). Therefore the set of simple roots \(\Pi_K\) for \(K\) can be regarded as a subset of \(\Pi\) and then \(\Pi = \Pi_K \cup \{\beta\}\).

**Lemma 6.1.** Suppose that \((G, K)\) is the complexification of a Hermitian symmetric pair and \(Q_{J_K}\) is the parabolic subgroup of \(K\) corresponding to a subset \(J_K \subset \Pi_K (\subset \Pi)\). Choose a \(\theta\)-stable Borel subgroup \(B\) and a simple root \(\beta \in \Pi\) such that \(K = L_{\Pi \setminus \{\beta\}}\). Then \(G/B \times K/Q_{J_K}\) is of finite type if and only if \(G/P_{\Pi \setminus \{\beta\}} \times G/P_{\Pi \setminus \{\beta\}}\) is of finite type.

**Proof.** The opposite parabolic subgroup \(P_{\Pi \setminus \{\beta\}}^-\) is conjugate to \(P_{\Pi \setminus \{\beta\}}\) and hence \(G/P_{\Pi \setminus \{\beta\}}^- \simeq G/P_{\Pi \setminus \{\beta\}}\). Since \(P_{\Pi \setminus \{\beta\}}^- \cdot P_{J_K}\) is open in \(G\) and

\[
P_{\Pi \setminus \{\beta\}}^- \cap P_{J_K} = P_{\Pi \setminus \{\beta\}}^- \cap P_{\Pi \setminus \{\beta\}} \cap P_{J_K} = K \cap P_{J_K} = Q_{J_K},
\]

\(G/Q_{J_K}\) is openly embedded into \(G/P_{\Pi \setminus \{\beta\}}^- \times G/P_{J_K}\). Hence \(G/P_{J_K} \times G/P_{\Pi \setminus \{\beta\}}^-\) is \(G\)-spherical if and only if \(G/Q_{J_K}\) is \(G\)-spherical. (See also [KNOT13].) \(\square\)

By Theorem 4.10 and Lemma 6.1, we get a list of \(G/B \times K/Q\) of finite type.

For the remaining pairs \((G, K)\), we use Theorem 4.5 and the classification of spherical linear actions in [BR96]. We carry out a classification according to the following procedure.

1. For each triple \((G, K, Q)\) we choose a \(\theta\)-stable parabolic subgroup \(P'\) of \(G\) such that \(P' \cap K = Q\).
2. Determine the Lie algebras \(\mathfrak{g}', \mathfrak{f} \cap \mathfrak{k}\) and \((\mathfrak{u}')^{-\theta}\).
3. Determine \(\mathfrak{m}'\) (see Theorem 4.5) and check whether the \(M_0\)-action on \((\mathfrak{u}')^{-\theta}\) is spherical using the list of [BR96].
The dimension condition \( \dim G/B + \dim K/Q \leq \dim K \) is helpful in some cases. This is equivalent to \( \dim L'_K \geq \dim G - \dim K - \text{rank} G \), where \( L'_K \) is a Levi component of \( Q \). Also we use Corollary 4.6 to exclude unsuitable cases.

For step (1) we choose a Borel subgroup \( B_K \) of \( K \) and a Cartan subgroup \( T_K \) in \( B_K \). We label the simple roots as \( \Pi_K = \{ \beta_1, \beta_2, \ldots \} \) by describing the Dynkin diagram for \( K \) and choose a parabolic subgroup \( Q \supset B_K \) in terms of \( \Pi_K \). Then we take a \( \theta \)-stable Borel subgroup \( B \) of \( G \) and a \( \theta \)-stable Cartan subgroup \( T \) in \( B \) satisfying:

- \( B \cap K = B_K \),
- \( T \cap K = T_K \), and
- there exists a \( \theta \)-stable parabolic subgroup \( P' \) containing \( B \) such that \( P' \cap K = Q \).

Because of the third condition here, we need to choose different \( B \) depending on \( Q \). We write the Vogan diagram corresponding to \((G, B, K)\) and label the simple roots of \( G \) as \( \Pi = \{ \alpha_1, \alpha_2, \ldots \} \). The standard parabolic subgroup \( P' \) is given as \( P' = P'_{J'} \) for \( J' \subset \Pi \).

For step (2) we have \( L' = L_{J'} \) and \( L' \cap K = L'_K \). The determination of \( (u')^{-\theta} \) will be done as in §5.

For step (3) the ‘\( M \)-part’ of a symmetric pair is well-known and found for example in [Kna02, Appendix C]. We restrict the \((L' \cap K)\)-module \( (u')^{-\theta} \) to the subgroup \( M'_0 \) and see whether it is spherical.

6.1. \((\mathfrak{sl}_n, \mathfrak{so}_n)\).

Let \((\mathfrak{g}, \mathfrak{t}) = (\mathfrak{sl}_n, \mathfrak{so}_n)\). Then the dimension condition is

\[
\dim L'_K \geq \dim G - \dim K - \text{rank} G = (n^2 - 1) - \frac{n(n-1)}{2} - (n-1) = \frac{n(n-1)}{2} = \dim K.
\]

This does not hold for a proper parabolic subgroup \( Q \subset K \).

6.2. \((\mathfrak{sl}_{2n}, \mathfrak{sp}_n)\).

Let \((\mathfrak{g}, \mathfrak{t}) = (\mathfrak{sl}_{2n}, \mathfrak{sp}_n)\) for \( n \geq 2 \). We fix the numbering \( \beta_1, \ldots, \beta_n \in \Pi_K \) in such a way that the Dynkin diagram of \( K \) becomes

\[
\begin{array}{c}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_{n-1} \\
\beta_n
\end{array}
\]

Suppose that \( J_K = \Pi_K \setminus \{ \beta_i \} \) with \( 1 \leq i \leq n \) and \( Q = Q_{J_K} \). We take \( B \) and \( T \) as in §5.2 so the Vogan diagram is

\[
\begin{array}{c}
\alpha_1 \\
\alpha_{n-1} \\
\alpha_n \\
\alpha_{n+1} \\
\alpha_{2n-1}
\end{array}
\]

and \( \beta_j = \alpha_j |_{\theta^j} \). By putting \( J' := \Pi \setminus \{ \alpha_i, \alpha_{2n-i} \} \) and \( P' = P_{J'} \), we get \( P' \cap K = Q \). In §5.2 we saw that \( (u')^{-\theta} \) is \((L' \cap K)\)-spherical only if \( i = 1 \) or \( n \).
If \( i = 1 \), we have \( (\mathfrak{l}', \mathfrak{l}' \cap \mathfrak{k}) \simeq (\mathfrak{gl}_1 \oplus \mathfrak{sl}_{2m-2} \oplus \mathfrak{gl}_1 \oplus \mathfrak{sp}_{n-1}) \) and hence \( \mathfrak{m}' = \mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2 \). Since \( (\mathfrak{u}')^{-\theta} \simeq \mathbb{C}^{2n-2} \) is a natural representation of the \( \mathfrak{sp}_{n-1} \)-component in \( \mathfrak{l}' \cap \mathfrak{k} \), its restriction to \( \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2 \) decomposes as \( (\mathfrak{u}')^{-\theta} \simeq \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2 \). Here each \( \mathfrak{sl}_2 \) acts naturally on one \( \mathbb{C}^2 \) and trivially on the others. Hence this is \( M_0' \)-spherical.

If \( i = n \), we have \( (\mathfrak{l}', \mathfrak{l}' \cap \mathfrak{k}) = (\mathfrak{sl}_n \oplus \mathfrak{gl}_1 \oplus \mathfrak{sl}_n, \mathfrak{gl}_1 \oplus \mathfrak{sl}_n) \) and hence \( \mathfrak{m}' = \mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \cdots \oplus \mathfrak{gl}_1 \).

Therefore, \( \mathfrak{m}' = \mathfrak{v}^\theta \) and \( M_0' \)-module is spherical if and only if the weights are linearly independent. Since \( (\mathfrak{u}')^{-\theta} \simeq \bigwedge^2 \mathbb{C}^n \) as an \((\mathfrak{l}' \cap \mathfrak{g})\)-module, it is spherical as an \( M_0' \)-module if and only if \( n \leq 3 \).

Suppose that \( J_K = \Pi_K \setminus \{\beta_1, \beta_n\} \). Then we can take \( B \) as above and then putting \( J' = \{\alpha_1, \alpha_n, \alpha_{m-1}\} \), we have \( \mathfrak{p}' \cap \mathfrak{g} = \mathfrak{q} \). According to §5.2, \( (\mathfrak{u}')^{-\theta} \) is \((\mathfrak{l}' \cap \mathfrak{g})\)-spherical only if \( n = 2 \). For \( n = 2 \), since \( (\mathfrak{sl}_1, \mathfrak{sp}_2) \simeq (\mathfrak{so}_6, \mathfrak{so}_5) \), any double flag variety is of finite type by Theorem 5.2.

Consequently, \( G / B \times K / Q_{J_K} \) is of finite type if and only if at least one of the following three conditions holds:

- \( J_K = \Pi_K \setminus \{\beta_1\} \),
- \( n = 3 \) and \( J_K = \Pi_K \setminus \{\beta_3\} \),
- \( n = 2 \).

6.3. \((\mathfrak{so}_n, \mathfrak{so}_p \oplus \mathfrak{so}_q)\).

Let \((\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}_n, \mathfrak{so}_p \oplus \mathfrak{so}_q)\) with \( p + q = n \). If \( p = 1 \) or \( q = 1 \), then Theorem 5.2 implies that any double flag variety is of finite type. If \( p = 2 \) or \( q = 2 \), the pair \((\mathfrak{g}, \mathfrak{k})\) is of Hermitian type. Let \( p, q \geq 3 \). We assume moreover that \( n \) is odd as the case where \( n \) is even can be treated similarly. Then we may assume that \( p \) is even and \( q \) is odd. Put \( m := \frac{n-1}{2}, p' = \frac{p}{2}, \) and \( q' = \frac{q-1}{2} \). We fix the numbering \( \beta_1, \ldots, \beta_m \in \Pi_K \) in such a way that the Dynkin diagram of \( K \) becomes

\[
\begin{align*}
\beta_1 & \quad \cdots \quad \beta_{p'-2} \quad \beta_{p'-1} \quad \beta_{p'} \quad \beta_{p'+1} \quad \beta_{m-1} \quad \beta_m \\
\end{align*}
\]

Suppose that \( J_K = \Pi_K \setminus \{\beta_i\} \) with \( 1 \leq i \leq p' \) and \( Q = Q_{J_K} \). We take \( B \) and \( T \) as in §5.3 so the Vogan diagram is

\[
\begin{align*}
\alpha_1 & \quad \cdots \quad \alpha_{p'-1} \quad \alpha_{p'} \quad \alpha_{p'+1} \quad \alpha_{m-1} \quad \alpha_m \\
\end{align*}
\]
and we have $\beta_j = \alpha_j$ for $j \neq p'$ and $\beta_{p'} = \alpha_{p'-1} + 2(\alpha_{p'} + \cdots + \alpha_m)$. By putting $J' := \Pi \setminus \{\alpha_i\}$ and $P' = P_{p'}$, we get $P' \cap K = Q$. In §5.3 we saw that $(u')^{-\theta}$ is $(L' \cap K)$-spherical only if $i = 1$. For $i = 1$, we have $(t', \ell' \cap t) \simeq (\mathfrak{gl}_1 \oplus \mathfrak{so}_{n-2}, \mathfrak{gl}_1 \oplus \mathfrak{so}_{p-2} \oplus \mathfrak{so}_q)$ and hence $\mathfrak{m}' \simeq \mathfrak{gl}_1 \oplus \mathfrak{so}_{p-q-2}$. Here $\mathfrak{so}_{p-q-2}$ is contained in $\mathfrak{so}_{p-2}$-component of $t' \cap t$ if $p - 2 \geq q$ and in the $\mathfrak{so}_p$-component if $p - 2 < q$. Since $(u')^{-\theta} \simeq \mathbb{C}^q$ is a natural representation of the $\mathfrak{so}_p$-component in $t' \cap t$, its restriction to $\mathfrak{so}_{p-q-2}$ decomposes as

$$(u')^{-\theta} \simeq \begin{cases} \mathbb{C} \oplus \cdots \oplus \mathbb{C} & \text{if } p - 2 \geq q, \\ \mathbb{C} \oplus \cdots \oplus \mathbb{C} & \text{if } p - 2 < q, \end{cases}$$

which is not $M'_0$-spherical.

Suppose that $J_K = \Pi_K \setminus \{\beta_j\}$ with $p' + 1 \leq i \leq m$ and $Q = Q_{J_K}$. We take $B$ and $T$ corresponding to the following Vogan diagram

![Vogan Diagram](image)

and then $\beta_j = \alpha_{i+j-p'}$ for $1 \leq j \leq p'-1$, $\beta_{p'} = \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_m)$, $\beta_j = \alpha_{j-p'}$ for $p' + 1 \leq j \leq i - 1$, $\beta_i = \alpha_{i-p'} + \cdots + \alpha_i$, and $\beta_j = \alpha_j$ for $i + 1 \leq j \leq m$. By putting $J' := \Pi \setminus \{\alpha_{i-p'}\}$ and $P' = P_{p'}$, we get $P' \cap K = Q$. Theorem 5.2 implies that $(u')^{-\theta}$ is $(L' \cap K)$-spherical only if $i - p' = 1$. For $i - p' = 1$, we have $(t', \ell' \cap t) \simeq (\mathfrak{gl}_1 \oplus \mathfrak{so}_{n-2}, \mathfrak{gl}_1 \oplus \mathfrak{so}_p \oplus \mathfrak{so}_{q-2})$ and hence $\mathfrak{m}' \simeq \mathfrak{gl}_1 \oplus \mathfrak{so}_{p-q+2}$. As in the previous case, we see that $(u')^{-\theta}$ is not $M'_0$-spherical.

Hence there is no $G/B \times K/Q_{J_K}$ of finite type if $p, q \geq 3$.  

6.4. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_q)$.

Let $(\mathfrak{g}, \mathfrak{t}) = (\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_q)$ with $p + q = n$. We fix the numbering $\beta_1, \ldots, \beta_n \in \Pi_K$ in such a way that the Dynkin diagram of $K$ becomes

![Dynkin Diagram](image)

Suppose that $J_K = \Pi_K \setminus \{\beta_j\}$ with $1 \leq i \leq p$ and $Q = Q_{J_K}$. We take $B$ and $T$ as in §5.3 so the Vogan diagram is

![Vogan Diagram](image)

and we have $\beta_j = \alpha_j$ for $j \neq p$ and $\beta_p = 2(\alpha_p + \cdots + \alpha_{n-1}) + \alpha_n$. By putting $J' := \Pi \setminus \{\alpha_i\}$ and $P' = P_{p'}$, we get $P' \cap K = Q$. Put $r := \min\{p - i, q\}$. We have $(t', \ell' \cap t) \simeq (\mathfrak{gl}_1 \oplus \mathfrak{sp}_{n-1}, \mathfrak{gl}_1 \oplus \mathfrak{sp}_{p-i} \oplus \mathfrak{sp}_q)$ and hence $\mathfrak{m}' \simeq \mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2 \oplus \mathfrak{sp}_{p-q-i}$. Here $\mathfrak{sl}_2$s are diagonally embedded in $\mathfrak{sp}_{p-i} \oplus \mathfrak{sp}_q$ and $\mathfrak{sp}_{p-q-i}$ is contained in $\mathfrak{sp}_{p-i}$ if $p - i \geq q$ and in
$sp$ if $p - i < q$. According to §5.3, $(u')^{-\theta} \simeq (C^i \otimes C^2 \oplus \cdots \oplus (C^i \otimes C^2)$ as an $(L' \cap K')$-module, where $C^i$ and $C^2q$ are natural representation of $gl_i$ and $sp_q$, respectively. Suppose $p - i \geq q$. Then

$$(u')^{-\theta} \simeq \left( \bigoplus_{i=0}^{q} (C^i \otimes C^2) \right)$$

as an $m'$-module, which is spherical if and only if $i = 1$ or $q = 1$. Suppose $p - i < q$. Then

$$(u')^{-\theta} \simeq \left( \bigoplus_{i=0}^{p-i} (C^i \otimes C^2) \oplus \bigoplus_{i=0}^{p-i} (C^i \otimes C^2) \oplus (C^i \otimes C^2(-p+q+i)) \right)$$

as an $m'$-module. This is spherical if and only if at least one of the following three conditions holds:

- $i = 1$,
- $p - i = 0$ and $q \leq 2$,
- $p - i \leq 3$.

Suppose that $J_K = \Pi_K \setminus \{\beta_i, \beta_j\}$ with $1 \leq i < j \leq p$ and $Q = Q_{J_K}$. We take $B$ and $T$ as above. By putting $J' := \Pi \setminus \{\alpha_i, \alpha_j\}$ and $P' = P_{\nu}$, we get $P' \cap K = Q$. Put $r := \min\{p - j, q\}$. We have $(\nu', \nu \cap t) \simeq (gl_i \oplus gl_{j-i} \oplus sp_{p-n} \oplus gl_i \oplus gl_{j-i} \oplus sp_{p-j} \oplus sp_q)$ and hence $m' \simeq gl_i \oplus gl_{j-i} \oplus sl \oplus \cdots \oplus (\nu') \cap (\nu)$. According to §5.3, $(u')^{-\theta} \simeq (C^i \otimes C^2 \oplus (C^i \otimes C^2 \oplus (C^{j-i} \otimes C^2)$ as an $(L' \cap K')$-module. Suppose $p - j \geq q$. Then

$$(u')^{-\theta} \simeq \left( \bigoplus_{i=0}^{q} (C^i \otimes C^2) \oplus \cdots \oplus (C^i \otimes C^2) \oplus (C^{j-i} \otimes C^2) \oplus \cdots \oplus (C^{j-i} \otimes C^2) \right)$$

as an $m'$-module, which is spherical if and only if $q = 1$. Suppose $p - j < q$. Then

$$(u')^{-\theta} \simeq \left( \bigoplus_{i=0}^{p-j} (C^i \otimes C^2) \oplus \bigoplus_{i=0}^{p-j} (C^i \otimes C^2) \oplus (C^{j-i} \otimes C^2) \oplus (C^{j-i} \otimes C^2(-p+q+j)) \right)$$

as an $m'$-module, which is spherical if and only if $q = 1$ or $(i, j, p) = (1, 2, 2)$.

Suppose that $J_K = \Pi_K \setminus \{\beta_i, \beta_j\}$ with $1 \leq i < j \leq p$ and $p - i \leq n - j$. We take $B$ and $T$ corresponding to the following Vogan diagram

\[ \begin{array}{cccccc}
\alpha_1 & \alpha_{j-p} & \alpha_j & \alpha_{n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array} \]

and we have $\beta_k = \alpha_{k+j-p}$ for $1 \leq k \leq p - 1$, $\beta_p = 2(\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n$, $\beta_k = \alpha_{k-p}$ for $p + 1 \leq k \leq j - 1$, $\beta_j = \alpha_{j-p} + \cdots + \alpha_j$, and $\beta_k = \alpha_{k}$ for $j + 1 \leq k \leq n$. By putting $J' := \Pi \setminus \{\alpha_{j-p}, \alpha_{i+j-p}\}$ and $P' = P_{\nu}$, we get $P' \cap K = Q$. Then $(\nu', \nu \cap t) \simeq (gl_i \oplus gl_{j-p} \oplus \cdots \oplus (\nu') \cap (\nu)$.
\( \sp_{n-(i+j-p)} \oplus \gl_i \oplus \gl_{j-p} \oplus \sp_p \oplus \sp_{n-j} \) and \( \mathfrak{m}' \simeq \gl_i \oplus \gl_{j-p} \oplus \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2 \oplus \sp_{q+i-j} \).

Then we can compute that
\[
(\mathfrak{m}')^{-\theta} \simeq (\mathbb{C}^i \otimes \mathbb{C}^{j-p}) \oplus ((\mathbb{C}^i)^* \otimes \mathbb{C}^{j-p}) \oplus (\mathbb{C}^i \otimes \mathbb{C}^{2(n-j)}) \oplus (\mathbb{C}^{j-p} \otimes \mathbb{C}^{2(p-i)})
\]
as an \((L' \cap K)\)-module. When restricted to \( \mathfrak{m}' \) it decomposes as
\[
(\mathbb{C}^i \otimes \mathbb{C}^{j-p}) \oplus ((\mathbb{C}^i)^* \otimes \mathbb{C}^{j-p}) \oplus (\mathbb{C}^i \otimes \mathbb{C}^{2}) \oplus \cdots \oplus (\mathbb{C}^i \otimes \mathbb{C}^{2}) \nonumber
\]
\[
\oplus (\mathbb{C}^i \otimes \mathbb{C}^{2(q+i-j)}) \oplus (\mathbb{C}^{j-p} \otimes \mathbb{C}^{2}) \oplus \cdots \oplus (\mathbb{C}^{j-p} \otimes \mathbb{C}^{2}) \nonumber
\]
which is spherical if and only if
- \( p - i = 0 \) and \( i = 1 \), or
- \( p - i = 0 \), \( q + i - j = 0 \), and \( j - p = 1 \).

This is equivalent to \( \min\{p, q\} = 1 \) under our assumption.

Consequently, \( G/B \times K/Q_JK \) is of finite type if and only if at least one of the following holds:
- \( \Pi_K \setminus J_K = \{\beta_1\} \),
- \( \Pi_K \setminus J_K = \{\beta_p\} \) and \( q \leq 2 \),
- \( \Pi_K \setminus J_K = \{\beta_p\} \) and \( p \leq 3 \),
- \( \Pi_K \setminus J_K = \{\beta_q\} \) and \( p \leq 2 \),
- \( \Pi_K \setminus J_K = \{\beta_q\} \) and \( q \leq 3 \),
- \( \Pi_K \setminus J_K = \{\beta_1, \beta_2\} \) and \( p = 2 \),
- \( \Pi_K \setminus J_K = \{\beta_{p+1}, \beta_{p+2}\} \) and \( q = 2 \),
- \( |\Pi_K \setminus J_K| \leq 2 \) and \( \min\{p, q\} = 1 \).

6.5. \( \mathfrak{g}_2, \mathfrak{e}_8 \) and \((\mathfrak{f}_4, \sp_3 \oplus \sp_1)\).

For \((\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}_2, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2), (\mathfrak{f}_4, \sp_3 \oplus \sp_1), (\mathfrak{e}_8, \mathfrak{so}_{16}), \) and \((\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{sl}_2), \) there does not exist the double flag variety \( G/P \times K/B_K \) of finite type for \( P \subsetneq G \). Therefore, \( G/B \times K/Q \) cannot be of finite type for \( Q \subsetneq K \) by Corollary 4.6.

6.6. \( \mathfrak{f}_4, \mathfrak{so}_9 \).

Let \((\mathfrak{g}, \mathfrak{t}) = (\mathfrak{f}_4, \mathfrak{so}_9)\). We fix the numbering \( \beta_1, \beta_2, \beta_3, \beta_4 \in \Pi_K \) in such a way that the Dynkin diagram of \( K \) becomes

\[
\begin{array}{cccc}
\beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\end{array}
\]
Suppose that \( J_K = \Pi_K \setminus \{ \beta_1, \beta_2 \} \) and \( Q = Q_{J_K} \). We take \( B \) and \( T \) as in §5.8 so the Vogan diagram is

\[
\begin{array}{ccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\end{array}
\]

and we have \( \beta_1 = \alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1, \beta_3 = \alpha_2, \beta_4 = \alpha_3 \). By putting \( J' := \Pi \setminus \{ \alpha_1, \alpha_4 \} \) and \( P' = P_{J'}, \) we get \( P' \cap K = Q \). We saw in §5.8 that \( (u')^{-\theta} \) is \((L' \cap K)\)-spherical. Since \( \ell' = \ell' \cap \mathfrak{k} \simeq \mathfrak{gl}_1 + \mathfrak{sp}_2 + \mathfrak{gl}_1 \) we have \( \ell' \cap \mathfrak{k} = \mathfrak{m}'. \) Hence \( (u')^{-\theta} \) is \( M'_0 \)-spherical as well.

Suppose that \( J_K = \Pi_K \setminus \{ \beta_3 \} \) and \( Q = Q_{J_K} \). We take \( B \) and \( T \) corresponding to the following Vogan diagram

\[
\begin{array}{ccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\end{array}
\]

and we have \( \beta_1 = \alpha_2, \beta_2 = \alpha_1, \beta_3 = \alpha_2 + 2\alpha_3, \beta_4 = \alpha_4 \). By putting \( J' := \Pi \setminus \{ \alpha_3 \} \) and \( P' = P_{J'}, \) we get \( P' \cap K = Q \). Theorem 5.2 implies that \( (u')^{-\theta} \) is \((L' \cap K)\)-spherical. Since \( \ell' = \ell' \cap \mathfrak{k} \simeq \mathfrak{gl}_3 + \mathfrak{sl}_2 \) we have \( \ell' \cap \mathfrak{k} = \mathfrak{m}' \) and \( (u')^{-\theta} \) is \( M'_0 \)-spherical.

Suppose that \( J_K = \Pi_K \setminus \{ \beta_4 \} \) and \( Q = Q_{J_K} \). We take \( B \) and \( T \) corresponding to the following Vogan diagram

\[
\begin{array}{ccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\end{array}
\]

and we have \( \beta_1 = \alpha_2 + 2\alpha_3, \beta_2 = \alpha_1, \beta_3 = \alpha_2 + 2\alpha_3 + \alpha_4 \). By putting \( J' := \Pi \setminus \{ \alpha_4 \} \) and \( P' = P_{J'}, \) we get \( P' \cap K = Q \). We have \((\ell', \ell' \cap \mathfrak{k}) \simeq (\mathfrak{so}_7 + \mathfrak{gl}_1, \mathfrak{so}_6 + \mathfrak{gl}_1)\) and hence \( \mathfrak{m}' \simeq \mathfrak{so}_5 + \mathfrak{gl}_1 \). We can compute that \( (u')^{-\theta} \simeq \mathbb{C}^4 \oplus \mathbb{C} \) as an \((\ell' \cap \mathfrak{k})\)-module, where \( \mathfrak{so}_6 \) acts on the first factor \( \mathbb{C}^4 \) as a spin representation and \( \mathfrak{gl}_1 \) acts as non-zero scalar on the second factor \( \mathbb{C} \). Therefore, \( (u')^{-\theta} \) is \( M'_0 \)-spherical.

If \( J_K = \Pi_K \setminus \{ \beta_i, \beta_j \} \) and \( \{ i, j \} \neq \{1, 2\} \), then \( \dim L'_{K} \leq 10 \). Hence the dimension condition \( \dim L'_{K} \geq \dim G - \dim K - \text{rank } G = 52 - 36 - 4 = 12 \) does not hold.

We conclude that \( G/B \times K/Q_{J_K} \) is of finite type if and only if \( |\Pi_K \setminus J_K| = 1 \) or \( \Pi_K \setminus J_K = \{ \beta_1, \beta_2 \} \).

6.7. (\( \mathfrak{e}_6, \mathfrak{sp}_4 \)).

Let \((\mathfrak{g}, \mathfrak{k}) = (\mathfrak{e}_6, \mathfrak{sp}_4)\). Then the dimension condition is \( \dim L'_{K} \geq \dim G - \dim K - \text{rank } G = 78 - 36 - 6 = 36 = \dim K \). This does not hold for \( Q \not\subset K \).

6.8. (\( \mathfrak{e}_6, \mathfrak{sl}_6 \oplus \mathfrak{sl}_2 \)).

Let \((\mathfrak{g}, \mathfrak{k}) = (\mathfrak{e}_6, \mathfrak{sl}_6 \oplus \mathfrak{sl}_2)\). We fix the numbering \( \beta_1, \ldots, \beta_6 \in \Pi_K \) in such a way that the Dynkin diagram of \( K \) becomes

\[
\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 \beta_6
\]
Suppose that $J_K = \Pi_K \setminus \{\beta_6\}$ and $Q = Q_{J_K}$. We take $B$ and $T$ as in §5.10 so the Vogan diagram is

![Vogan Diagram](image)

and we have $\beta_1 = \alpha_1$, $\beta_2 = \alpha_3$, $\beta_3 = \alpha_4$, $\beta_4 = \alpha_5$, $\beta_5 = \alpha_6$, and $\beta_6 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. By putting $J' := \Pi \setminus \{\alpha_2\}$ and $P' = P_{J'}$, we get $P' \cap K = Q$. We saw in §5.10 that $(u')^{-\theta}$ is not $(L' \cap K)$-spherical.

If $\Pi_K \setminus J_K = \{\beta_i\}$ for $1 \leq i \leq 5$, then $\dim L'_K \leq 28$. It does not satisfy the dimension condition $\dim L'_K \geq \dim G - \dim K - \rank G = 78 - 38 - 6 = 34$.

Hence there is no $G/B \times K/Q_{J_K}$ of finite type.

6.9. $(\epsilon_6, \mathfrak{f}_4)$.

Let $(\mathfrak{g}, \mathfrak{k}) = (\epsilon_6, \mathfrak{f}_4)$. We fix the numbering $\beta_1, \beta_2, \beta_3, \beta_4 \in \Pi_K$ in such a way that the Dynkin diagram of $K$ becomes

![Dynkin Diagram](image)

Suppose that $J_K = \Pi_K \setminus \{\beta_1\}$ and $Q = Q_{J_K}$. We take $B$ and $T$ as in §5.11 so the Vogan diagram is

![Vogan Diagram](image)

and we have $\beta_1 = \alpha_2|\epsilon$, $\beta_2 = \alpha_4|\epsilon$, $\beta_3 = \alpha_3|\epsilon$, and $\beta_4 = \alpha_1|\epsilon$. By putting $J' := \Pi \setminus \{\alpha_2\}$ and $P' = P_{J'}$, we get $P' \cap K = Q$. We have $(\mathfrak{k}', \mathfrak{k}' \cap \mathfrak{t}) \simeq (\mathfrak{sl}_6 \oplus \mathfrak{gl}_1, \mathfrak{sp}_3 \oplus \mathfrak{gl}_1)$ and hence $\mathfrak{m}' \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{gl}_1$. We can compute $(u')^{-\theta} \simeq \mathbb{C}^6$ is a natural representation of $\mathfrak{sp}_3$ in $\mathfrak{k}' \cap \mathfrak{t}$. Hence $(u')^{-\theta}$ decomposes as $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$ as an $\mathfrak{m}'$-module, where each $\mathfrak{sl}_2$ in $\mathfrak{m}'$ acts naturally on one $\mathbb{C}^2$ and trivially on the other two. This is a spherical action.

Suppose that $J_K = \Pi_K \setminus \{\beta_4\}$ and $Q = Q_{J_K}$. We take $B$ and $T$ as above. Putting $J' := \Pi \setminus \{\alpha_1, \alpha_6\}$ and $P' = P_{J'}$, we get $P' \cap K = Q$. We saw in §5.11 that $(u')^{-\theta}$ is not $(L' \cap K)$-spherical.

If $\Pi_K \setminus J_K = \{\beta_i\}$ for $i = 2$ or $3$, then $\dim L'_K = 12$. It does not satisfy the dimension condition $\dim L'_K \geq \dim G - \dim K - \rank G = 78 - 52 - 6 = 20$.

Hence $G/B \times K/Q_{J_K}$ is of finite type if and only if $J_K = \Pi_K \setminus \{\beta_1\}$. 
6.10. \((\mathfrak{e}_7, \mathfrak{sl}_8)\).

Let \((\mathfrak{g}, \mathfrak{k}) = (\mathfrak{e}_7, \mathfrak{sl}_8)\). Then the dimension condition is \(\dim L'_K \geq \dim G - \dim K - \text{rank } G = 133 - 63 - 7 = 63 = \dim K\). This does not hold for \(Q \subseteq K\).

6.11. \((\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)\).

Let \((\mathfrak{g}, \mathfrak{k}) = (\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)\). We fix the numbering \(\beta_1, \ldots, \beta_7 \in \Pi_K\) in such a way that the Dynkin diagram of \(K\) becomes

\[
\begin{array}{ccccccc}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\
\end{array}
\]

Suppose that \(J_K = \Pi_K \setminus \{\beta_7\}\) and \(Q = Q_{J_K}\). We take \(B\) and \(T\) as in §5.13 so the Vogan diagram is

\[
\begin{array}{ccccccc}
\alpha_1 & \alpha_3 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array}
\]

and we have \(\beta_i = \alpha_{8-i}\) for \(1 \leq i \leq 6\), \(\beta_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\). By putting \(J' := \Pi \setminus \{\alpha_1\}\) and \(P' = P'_\mu\), we get \(P' \cap K = Q\). We saw in §5.13 that \((u')^{-\theta}\) is not \((L' \cap K)\)-spherical.

If \(\Pi_K \setminus J_K = \{\beta_i\}\) for \(i \neq 7\), then \(\dim L'_K \leq 49\). It does not satisfy the dimension condition \(\dim L'_K \geq \dim G - \dim K - \text{rank } G = 133 - 69 - 7 = 57\).

Hence there is no \(G/B \times K/Q_{J_K}\) of finite type.

We thus conclude that:

**Theorem 6.2.** Let \(G\) be a connected simple algebraic group and \((G, K)\) a symmetric pair. Let \(Q\) be a parabolic subgroup of \(K\) corresponding to \(J_K \subseteq \Pi_K\). Then the double flag variety \(G/B \times K/Q\) is of finite type if and only if the triple \((\mathfrak{g}, \mathfrak{k}, \Pi_K \setminus J_K)\) appears in Table 3.

Remark 6.3. In Table 3 the labeling \(\beta_1, \beta_2, \ldots\) of the simple roots for \(\mathfrak{k}\) is not unique up to isomorphisms in the cases where \((\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_{p+q+2}, \mathfrak{sl}_{p+1} \oplus \mathfrak{sl}_{q+1} \oplus \mathbb{C})\) with \(1 \leq p \leq q\), \((\mathfrak{so}_{2n+2}, \mathfrak{so}_{2n} \oplus \mathbb{C})\) with \(n = 4\), and \((\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})\) with \(n = 4\). In order to fix this, we describe the \([\mathfrak{k}, \mathfrak{k}]\)-module \(g^{-\theta}\) in terms of \(\beta_i\). We denote by \(\omega_i \in (\mathfrak{k}^\theta \cap [\mathfrak{k}, \mathfrak{k}])^*\) the fundamental weight for \([\mathfrak{k}, \mathfrak{k}]\) corresponding to \(\beta_i\). Write \(V(\lambda)\) for the irreducible \([\mathfrak{k}, \mathfrak{k}]\)-module with highest weight \(\lambda\).

For \((\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_{p+q+2}, \mathfrak{sl}_{p+1} \oplus \mathfrak{sl}_{q+1} \oplus \mathbb{C})\) with \(0 \leq p \leq q\) in Table 3 we label \(\beta_i\) so that \(g^{-\theta} \simeq V(\omega_1 + \omega_{p+q}) \oplus V(\omega_p + \omega_{p+1})\) if \(p > 0\) and \(g^{-\theta} \simeq V(\omega_1) \oplus V(\omega_q)\) if \(p = 0\).
For \((g, \mathfrak{k}) = (\mathfrak{so}_{2n+2}, \mathfrak{so}_{2n} \oplus \mathbb{C})\) in Table 3, we label \(\beta_i\) so that \(g^{\theta} \simeq V(\omega_1) \oplus V(\omega_1)\). For \((g, \mathfrak{k}) = (\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})\) in Table 3 we label \(\beta_i\) so that \(g^{\theta} \simeq V(\omega_1)\).

| \(g\)         | \(\mathfrak{k}\)          | \(\Pi_K \setminus J_K (Q = Q_{JK})\)                                                                 |
|--------------|-----------------|--------------------------------------------------------------------------------------------------|
| \(\mathfrak{sl}_{2n}\) \(n \geq 2\) | \(\mathfrak{sp}_n\)  | \(\beta_1 \quad \beta_2 \quad \beta_{n-1} \quad \beta_n\)                                        |
| \(\mathfrak{sl}_{p+q+2}\) \(p + q \geq 1\) \(0 \leq p \leq q\) | \(\mathfrak{sl}_{p+1} \oplus \mathfrak{sl}_{q+1} \oplus \mathbb{C}\) | \(\beta_1 \quad \beta_2 \quad \beta_p\) \(\beta_{p+1} \quad \beta_{p+2} \quad \beta_{p+q}\) \(\{\beta_1\}, \{\beta_p\}, \{\beta_{p+1}\}, \{\beta_{p+q}\}, \{\beta_i\} (\forall i) \) if \(p = 1\), \(\text{any subset of } \Pi_K\) if \(p = 0\) |
| \(\mathfrak{so}_{2n+2}\) \(n \geq 3\) | \(\mathfrak{so}_{2n} \oplus \mathbb{C}\)                  | \(\beta_1 \quad \beta_{n-2} \quad \beta_{n-1} \quad \beta_n\)                                        |
| \(\mathfrak{so}_{2n+1}\) \(n \geq 3\) | \(\mathfrak{so}_{2n}\)                                      | \(\beta_1 \quad \beta_{n-2} \quad \beta_{n-1} \quad \beta_n\)                                        |
| \(\mathfrak{so}_{2n+2}\) \(n \geq 3\) | \(\mathfrak{so}_{2n+1}\)                                      | \(\beta_1 \quad \beta_{n-1} \quad \beta_n\)                                                                 |
| \(\mathfrak{so}_{2n+2}\) \(n \geq 3\) | \(\mathfrak{sl}_{n+1} \oplus \mathbb{C}\)                  | \(\beta_1 \quad \beta_2 \quad \beta_n\)                                                             |
| \(\mathfrak{sp}_{p+q}\) | \(\mathfrak{sp}_p \oplus \mathfrak{sp}_q\)                  | \(\beta_1 \quad \beta_{p-1} \quad \beta_p\)                                                             |
\[ 1 \leq p \leq q \]

\[ \beta_{p+1} \beta_{p+q-1} \beta_{p+q} \]

\[ \{ \beta_1 \}, \{ \beta_{p+1} \}, \{ \beta_p \} \text{ if } p \leq 3, \{ \beta_{p+q} \} \text{ if } p \leq 2, \{ \beta_{p+q} \} \text{ if } q \leq 3, \]

\[ \{ \beta_1, \beta_2 \} \text{ if } p = 2, \{ \beta_{p+1}, \beta_{p+2} \} \text{ if } q = 2, \]

\[ \{ \beta_i \} (\forall i) \text{ if } p = 1, \{ \beta_i, \beta_j \} (\forall i, j) \text{ if } p = 1 \]

\[ \begin{array}{c|c}
 f_4 & so_9 \\
 e_6 & so_{10} \oplus \mathbb{C} \\
 e_6 & f_4 \\
\end{array} \]

\[ \begin{array}{cccc}
 \beta_1 & \beta_2 & \beta_3 & \beta_4 \\
 \{ \beta_1 \} (\forall i), \{ \beta_1, \beta_2 \} \\
 \beta_1 & \beta_2 & \beta_3 & \beta_4 \\
 \{ \beta_1 \} \\
\end{array} \]

Table 3: $G$-spherical $G/Q$

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