**APPENDIX SYMMETRIES OF LAGRANGIANS FOR PLANE SYMMETRIC GRAVITATIONAL WAVE-LIKE SPACETIMES**

**IBRAR HUSSAIN** and **ASGHAR QADIR**

†Centre for Advanced Mathematics and Physics  
National University of Sciences and Technology  
Campus of the College of Electrical and Mechanical Engineering  
Peshawar Road, Rawalpindi, Pakistan

E-mail: ihussain@camp.edu.pk, aqadirmath@yahoo.com

**Abstract.** Using Lie symmetry methods for differential equations we have investigated the symmetries of a Lagrangian for a plane symmetric static spacetime. Perturbing this Lagrangian we explore its approximate symmetries. It has a non-trivial first-order approximate symmetry.

1. Introduction

The problem of defining energy (or mass) in general relativity arises from the fact that arbitrary spacetimes can be non-static (or even non-stationary) and hence global (or even local) energy conservation may be lost. For static spacetimes there exists a timelike *isometry* or *Killing Vector* (KV) \[\mathbf{k}\], which can be used to define the energy, \(E\), of a test-particle by \(E = \mathbf{k} \cdot \mathbf{p}\), where \(\mathbf{k}\) is the timelike KV and \(\mathbf{p}\) is the momentum 4-vector of the test-particle. Further, energy conservation in the spacetime is guaranteed in the frame using \(\mathbf{k}\) to define the time direction. However if there does not exist a timelike KV, energy is not conserved and hence the energy of a test particle cannot be defined. Since gravitational waves *must* be given by non-static spacetimes, the problem of defining the energy content of gravitational waves is particularly severe.

Minkowski spacetime is maximally symmetric having 10 KVs which form the Poincaré, algebra \(so(1,3) \oplus_s \mathbb{R}^4\), where \(\oplus_s\) denotes a semi direct sum (the algebra is denoted by \(so(1,3) \oplus_s \mathbb{R}^4\) while the group is denoted by \(SO(1,3) \oplus_s \mathbb{R}^4\) \[1\]. The generators of this algebra give conservation laws for energy, spin angular momentum and linear momentum \[2\]. Going to the Schwarzschild and Reissnor-Nördstrom spacetimes we lose linear and spin angular momentum conservation while in the Kerr spacetime we lose two more conservation laws \[1\]. Approximate symmetries for differential equations (DEs) have been used to recover these lost conservation laws \[3, 4\].

Different people have tried to define “approximate symmetry” \[5, 6, 7, 8, 9\], in which the broken time-translation symmetry provides information about the energy content of
the gravitational wave spacetimes, but none of them have been unequivocally successful. Nevertheless, this approach of a slightly broken symmetry seems promising.

It was proposed [4] that the concept of “approximate symmetry” of DEs [10] could be extended and adapted for the purpose of defining energy in gravitational waves by using the connection found between isometries and symmetries of DEs through the geodesic equations [11, 12]. In this paper we will use this concept of approximate symmetries of DEs to look for a resolution of the problem of energy in gravitational wave spacetimes.

The plan of the paper is as follows. In the next section we briefly review the definitions of symmetries and approximate symmetries of a Lagrangian. In section 3 symmetries of a Lagrangian for plane symmetric static spacetime and approximate symmetries of a perturbed Lagrangian for a plane symmetric gravitational waves like spacetime are discussed. Finally a summary and discussion is given in section 4.

2. Symmetries and approximate symmetries of a Lagrangian

Noether’s theorem [13] relates the constants of motion of a given Lagrangian system to its symmetry transformation [10, 14]. Symmetry generators of a Lagrangian of a manifold form a Lie algebra [15] and from the geometric point of view symmetries of a manifold are characterized by its KVs, which always form a finite dimensional Lie algebra [16].

In general a manifold may not possess an exact symmetry but may approximately do so. It would be of interest to look at the approximate symmetries of the manifold. They form an approximate Lie algebra [17]. Methods for obtaining exact and approximate symmetries of a Lagrangian are available in the literature [10, 18, 19].

Symmetries of a Lagrangian, also known as Noether symmetries, are defined as follows. Consider a vector field [10]

\[
X = \xi(s, x^i) \partial / \partial s + \eta^j(s, x^i) \partial / \partial x^j \quad (i, j = 0, ..., 3).
\]  

(1)

Its first prolongation is

\[
X^{[1]} = X + (\eta^j_s + \eta^j_{ix} x^i'' - \xi_s x^j'' - \xi_{ix} x^j'') \partial / \partial x^j',
\]  

(2)

where “r” denote differentiation with respect to s. Now consider a set of second-order ordinary differential equations

\[
x'' = g(s, x^i, x'^i),
\]  

(3)

which has a Lagrangian \(L(s, x^i, x'^i)\). Then \(X\) is a Noether point symmetry of the Lagrangian \(L(s, x^i, x'^i)\) if there exists a function \(A(s, x^i)\) such that

\[
X^{[1]} L + (D_s \xi) L = D_s A,
\]  

(4)
where
\[ D_s = \partial/\partial s + x^i \partial/\partial x^i. \] (5)

For more general considerations see [10].

First-order approximate symmetries of the Lagrangian are defined as follows [19]. For a first-order perturbed system of equations, \[ E = E_0 + \epsilon E_1 = O(\epsilon^2) \] corresponding to the first-order perturbed Lagrangian, \[ L(s, x^i, x^i', \epsilon) = L_0(s, x^i, x^i') + \epsilon L_1(s, x^i, x^i') + O(\epsilon^2), \]
the functional \[ \int L ds \] is invariant under the one parameter group of transformations with approximate Lie symmetry generator \[ X = X_0 + \epsilon X_1 + O(\epsilon^2) \] up to the gauge \[ A = A_0 + \epsilon A_1, \]
where \[ X_0^{[1]} L_0 + (D_s \xi_0) L_0 = D_s A_0 \] and \[ X_1^{[1]} L_0 + X_0^{[1]} L_1 + (D_s \xi_1) L_0 + (D_s \xi_0) L_1 = D_s A_1. \] Here \( X_0 \) is the exact symmetry generator and \( X_1 \) the first-order approximate symmetry generator. The perturbed equations always have the trivial approximate symmetry generator \( \epsilon X_0 \) and \( X \) with \( X_0 \neq 0 \neq X_1 \neq kX_0 \), is called a non-trivial approximate symmetry.

3. Symmetries and approximate symmetries of a Lagrangian for a plane symmetric gravitational wave-like spacetime

To try to find a resolution of the problem of definition of energy in gravitational wave spacetimes, we consider a static spacetime and then perturb it time dependently (for definiteness by \( \epsilon t \)) to make it slightly non-static. For this purpose we take a plane symmetric static metric [20]
\[ ds^2 = e^{2\nu(x)} dt^2 - dx^2 - e^{2\mu(x)}(dy^2 + dz^2), \] (6)
with \( \mu(x) = \nu^2(x) = (x/X)^2 \), where \( X \) is a constant having the same dimensions as \( x \).

The Lagrangian defined by minimizing the arc length in (6) is
\[ L = e^{2x/X} \dot{t}^2 - \dot{x}^2 - e^{2x^2/X^2}(\dot{y}^2 + \dot{z}^2), \] (7)
where “.” denotes differentiation with respect to the geodetic parameter \( s \). Using (7) in (4) we obtain a set of 19 determining PDEs for 6 unknown functions \( \xi, \eta_i (i = 0, \ldots, 3) \) and \( A \), where each of these is a function of the 5 variables \( s \) and \( x^i \). Solving these equations we get the symmetry generators
\[ X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial z}, \quad X_3 = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \quad Y_0 = \frac{\partial}{\partial s}, \quad A = c, \] (8)
where \( c \) is a constant, \( X_0 \) corresponds to energy conservation, \( X_1 \) and \( X_2 \) correspond to linear momentum conservation along \( y \) and \( z \), while \( X_3 \) corresponds to angular momentum conservation in the \( yz \) plane.
For the approximate symmetries of a Lagrangian for this plane symmetric gravitational wave-like spacetime we consider \( \nu(x) = 2(x/X + ct/T) \) and \( \mu(x) = 2(x^2/X^2 + ct/T) \) in the metric (6), where \( T \) is a constant having dimensions of \( t \). Its first-order perturbed Lagrangian is

\[
L = e^{2x/X} \dot{t}^2 - \dot{x}^2 - e^{2x^2/X^2} (\dot{y}^2 + \dot{z}^2) + 2ct[e^{2x/X} \dot{t}^2 - e^{2x^2/X^2} (\dot{y}^2 + \dot{z}^2)]/T + O(\epsilon^2). \tag{9}
\]

Using the exact symmetry generators given by (8) and solving the system of determining equations for approximate symmetries, we get the non-trivial approximate symmetry \( X_a \), along with the trivial symmetries, and the gauge function \( A_1 \) is again a constant,

\[
X_a = \partial/\partial t - \epsilon(t\partial/\partial t + y\partial/\partial y + z\partial/\partial z)/T. \tag{10}
\]

The physical meaning of this non-trivial approximate symmetry found here is worth exploring.

4. Summary and Discussion

We addressed the problem of energy in gravitational wave spacetimes. For this purpose we first considered a plane symmetric static spacetime that has 4 KVs [20]. The Lagrangian for this metric has an additional symmetry \( \partial/\partial s \). Since gravitational waves must be given by non-static metrics, we perturbed the static metric here with a term \( ct \) and retained the terms containing \( \epsilon \), neglecting its higher powers. For the Lagrangian of this perturbed metric we found a non-trivial first-order approximate symmetry given by (10). The trivial first-order approximate timelike Noether symmetry, for the perturbed Lagrangian (9), can be interpreted to give the extent of energy non-conservation, and hence (possibly) the energy content, in the gravitational wave-like spacetime. Since the stress-energy tensor is non-zero [20], we need to consider the exact solution to understand how much energy is in the field and how much in its interaction with matter.

Acknowledgments

We thank NUST for support and Salento University for hospitality.

References

[1] Kramer D, Stephani H, MacCullum, M. A. H., and Herlt E, Exact Solutions of Einstein Field Equations, Cambridge University Press, Cambridge, 1980.
[2] Qadir A, Applications of Symmetry Methods, pp.45-73, eds. Qadir A and Saifullah K, National Centre for Physics, Islamabad, 2006.

[3] Kara, A. H., Mahomed, F. M., and Qadir A, “Approximate Symmetries and Conservation Laws of the Geodesic Equations for the Schwarzschild Metric”, Nonlinear Dynamics (to appear).

[4] Hussain, I. Mahomed, F. M., and Qadir A, SIGMA, 3 (2007) 9 pages, arXiv:0712.1089.

[5] Komar A, Phys. Rev., 127 (1962) 1411; 129 (1963) 1873.

[6] Matzner R, J. Math. Phys., 9 (1968) 1063; 10 (1968) 1657.

[7] Isaacson, R. A., Phys. Rev., 166 (1968) 1263; 1272.

[8] York, J. W., Ann. Inst. Henri Poincarè XXI, 4 (1974) 319.

[9] Spero A, and Baierlein R, J. Math. Phys., 18 (1977) 1330; 19 (1978) 1342.

[10] Ibragimov, N. H., Elementary Lie group Analysis and Ordinary Differential Equations, Wiely, Chichester, 1999.

[11] Feroze T, Mahomed, F. M., and Qadir A, Nonlinear Dynamics, 45 (2006) 65.

[12] Aminova A. V., Sobrink Mathematics, 186, (1995) 1711.

[13] Noether E, ”Invariant variations problems”, Nachr. Konig. Gissell. Wissen., Göttingen, Math.-Phys.Kl. 2 (1918) 235. (English translation in transport theory and Statistical Physics 1 (1971)) 186.

[14] Kara, A. H., Mahomed, F. M., J. Nonlinear Math. Phys., 9 (2002) 60.

[15] Bokhari, A.H., Kara, A. H., Kashif, A. R., and Zaman, F. D., International. J. Theoretical. Phys., 45 (2006) 1063.

[16] Hawking, S. W., and Ellis, G. F. R., The Large Scale Structure of Spacetime, Cambridge University Press, Cambridge, 1973.

[17] Gazizov, R. K., J. Nonlinear Math. Phys., 3 (1996) 96.

[18] Wafo Soh. C., and Mahomed, F. M., Class. Quantum Grav. 16 (1999) 3553.

[19] Feroze T. and Kara, A.H., International. J. Non-linear Mechanics 37 (2002) 275.

[20] Feroze T, Qadir A, and Ziad M, J. Math. Phys., 42 (2001) 4947.