Dispersive and hyperbolic models for non-hydrostatic shallow water flows and their application to steep forced waves modelling

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Abstract. We propose a hyperbolic system of first-order equations that approximates the 1D Nwogu model of the shallow water theory for non-hydrostatic unsteady flows. Solitary waves in the framework of these models are constructed and studied. The evolution of solitary waves on a mildly sloping beach is considered. We show that the solution of the hyperbolic system practically coincides with the corresponding solution of the Nwogu dispersive equations. Steep forced water waves generated by a harmonically oscillating rectangular tank are studied both experimentally and numerically. A comparison of the solutions of the modified Green–Naghdi and Nwogu equations with the obtained experimental data is made.

1. Introduction
Forced water free-surface motion under gravity are a concern for many different human activities. For instance, the impact of steep water waves can result in damage or collapse of coastal structures. Both experimental [1] and theoretical [2] studies have emphasized the fundamental role of the large impact pressures which are impulsively exerted on sea walls. For industrial applications, the importance of the stability of a vehicle transporting liquids in partially filled tanks has led to the need for understanding the forced fluid motion inside confined spaces. An important feature of sloshing motions in a tank is the generation of patterns of steep standing waves. For example, when a container of liquid is subject to vertical sinusoidal oscillations, the free surface becomes unstable and gives rise to standing waves. Experimental observations and numerical modelling of sloshing and steep Faraday waves generated in a rectangular tank were performed in [3]. For numerical simulation of such flows, Boussinesq-type equations taking into account the non-hydrostatic pressure distribution are often used [4]. Boussinesq-type models and the non-hydrostatic free surface shallow flow theory are of wide applicability in geophysical hydrodynamics [5, 6] and coastal engineering [7]. To describe the wave motion of a fluid in open channels, the dispersive Green–Naghdi equations [8] are widely used, as well as their various modifications. In particular, an alternative form of Boussinesq equations for nearshore wave propagation was proposed in [9]. This model has improved dispersion properties, making it applicable to a wide range of water depth.
All the non-linear models of the theory of long waves mentioned above are dispersive. Alternative formulations of these models within the framework of hyperbolic systems of equations have been proposed in [10, 11]. The main advantage of such hyperbolic approximation of the dispersive equations is essential simplification of the algorithms of numerical calculation and formulation of the boundary conditions. In particular, the inversion of an elliptic operator is needed at each time step when the Green–Naghdi equations are numerically solved [12]. Another important numerical problem is how to impose non-reflecting conditions at the boundary of the calculation interval for dispersive equations.

In this paper we derive a hyperbolic approximation of the Nwogu equations. We show that the obtained hyperbolic system approximates solutions of the original dispersive model. As an example, the evolution of solitary waves on a mildly sloping beach is considered. We also construct a solitary wave solution in the framework of both the dispersive and hyperbolic versions of the Nwogu equations. Substantial attention is paid to the study of forced water waves generated by a harmonically oscillating reservoir. It is shown that both the Green–Naghdi and Nwogu models describe well enough the moderate-amplitude forced waves.

2. Mathematical models of shallow water theory for non-hydrostatic flows

The one-dimensional Green–Naghdi equations for flows over a flat bottom have the form [10]

\[ h_t + (uh)_x = 0, \quad (uh)_t + \left( u^2h + \frac{gh^2}{2} + \frac{h^2d^2h}{3 dt^2} \right)_x = 0, \]  

(1)

where \( h \) is the fluid depth, \( u \) is the velocity in the horizontal direction \( Ox \), constant \( g \) is the gravitational acceleration in the vertical direction \( Oz \), and \( d/dt = \partial_t + u \partial_x \) stands for material derivative. To solve the Green–Naghdi equations describing dispersive shallow water waves, a hybrid numerical method using a Godunov type scheme was proposed in [12]. The method is based on the application of the following formulation of equations (1)

\[ h_t + (uh)_x = 0, \quad K_t + \left( uK - \frac{u^2}{2} + gh - \frac{h^2u_x}{2} \right)_x = 0, \quad K = u - \frac{1}{3h}(h^3u_x)_x. \]  

(2)

The numerical resolution of system (2) is divided in two successive steps: 1) the time evolution of the conservative variables \( h \) and \( K \) using a Godunov type scheme; 2) the resolution of an ordinary differential equation to obtain the values of velocity \( u \) from variables \( h \) and \( K \).

As it was shown in [10, 11], dispersive equations (1) can be approximated by a hyperbolic system of the form

\[ h_t + (uh)_x = 0, \quad (uh)_t + \left( u^2h + \frac{gh^2}{2} + \frac{\alpha gh}{3}(h - \zeta) \right)_x = 0, \]  

(3)

\[ (h\zeta)_t + (uh\zeta)_x = Vh, \quad (hV)_t + (uhV)_x = \alpha g(h - \zeta). \]

Here \( \zeta \) and \( V \) are the instantaneous depth and velocity, \( \alpha \) is the relaxation parameter. This system is hyperbolic, the characteristic slopes are:

\[ \lambda_{1,2} = u \pm \sqrt{1 + \frac{\alpha}{3}(2 - \frac{\zeta}{h})} gh, \quad \lambda_{3,4} = u. \]

Thus, system (3) has two sonic characteristics \( \lambda_{1,2} \) and one contact characteristic \( \lambda_{3,4} \) (multiplicity two). The transition from equations (2) to system (3) is as follows:

\[ \frac{d^2h}{dt^2} \to \frac{\alpha g}{h}(h - \zeta), \quad \frac{d\zeta}{dt} = V, \quad \frac{dV}{dt} = \frac{\alpha g}{h}(h - \zeta). \]
It is known that solutions of hyperbolic equations (3) tend to solutions of dispersive system (1) as \( \alpha \to \infty \).

An alternative formulation of the Boussinesq equations proposed by Nwogu for one-dimensional unsteady flows over a flat topography has the form \([9, 6]\)

\[
h_t + \left( uh + \left( h_0 + \frac{1}{3} h_0^2 u_{xx} \right) \right)_x = 0, \quad (u + \alpha_1 h_0^2 u_{xx})_t + \left( \frac{u^2}{2} + gh \right)_x = 0,
\]

where \( h_0 \) is the unperturbed fluid depth, and \( \alpha_1 = -0.39 \). Obviously, if we introduce the function \( K_1 = u + \alpha_1 h_0^2 u_{xx} \), then equations (4) can be numerically solved in two steps, similarly to system (2).

The dispersion equations (4) can be approximated by the following hyperbolic system

\[
h_t + \left( uh + h_0 \Phi \right)_x = 0, \quad \Phi_t + \left( \frac{u^2}{2} + cu + gh \right)_x = -v,
\]

\[
u_t - cu_x = v, \quad v_t + cv_x = -\frac{c^2 \Phi}{\alpha_1 h_0^2} \quad \left( \frac{\Phi}{h} = \frac{\alpha_1}{\alpha_1 + 1/3} > 0 \right).
\]

Here positive constant \( c \) is the ‘hyperbolisation’ parameter. It is easy to see that this system is unconditionally hyperbolic and has four sonic characteristics. The characteristic slopes are:

\[
\lambda_{1,2} = \frac{u}{2} \pm \sqrt{\frac{u^2}{4} + \frac{gh_0}{\kappa}}, \quad \lambda_{3,4} = \pm c.
\]

The third and fourth equations in (5) yield

\[
u_{xx} - \frac{1}{c^2} \nu_{tt} = \frac{\Phi}{\alpha_1 h_0^2}.
\]

This means that \( \Phi \to \alpha_1 h_0^2 u_{xx} \) for \( c^2 \to \infty \) and, consequently, in this case system (5) reduces to the Nwogu equations (4).

### 3. Numerical results for the dispersive and hyperbolic Nwogu equations

Let us present the results of calculations using dispersive and hyperbolic Nwogu equations. First, we compare solutions in the form of travelling waves. Then equations (4) can be rewritten as

\[
u' = u_1, \quad \nu'_1 = \frac{-Dh_0 + (u - D)h}{(\alpha_1 + 1/3)h_0^3}, \quad \frac{h}{h_0} = \frac{Du - u^2/2 + gh_0 - \kappa D^2}{gh_0 + \kappa D(u - D)},
\]

where ‘prime’ denotes the derivative with respect to the variable \( \xi = x - Dt \) and \( D \) is the constant velocity of the travelling wave. We assume that \( h \to h_0 \) and \( u \to 0 \) as \( \xi \to -\infty \). To construct such a solution, it is necessary to understand the asymptotic behaviour of the travelling wave solution at negative infinity.

Let us consider small perturbations of the constant solution: \( (h, u, u_1) = (h_0, 0, 0) + \varepsilon (\tilde{h}, \tilde{u}, \tilde{u}_1) \).

Substitution this representation into system (6) and linearisation yield

\[
\tilde{\nu}' = \tilde{u}_1, \quad \tilde{u}'_1 = \frac{D\tilde{h} - h_0 \tilde{u}}{(\alpha_1 + 1/3)h_0^3}, \quad \tilde{h} = \frac{(1 - \kappa) D \tilde{u}}{g - \kappa D^2/h_0}.
\]

Further, we are looking for the solutions that vanish at negative infinity in the form \( (\tilde{h}, \tilde{u}, \tilde{u}_1) = (\tilde{h}, \tilde{u}, \tilde{u}_1) \exp(\nu \xi) \). Substituting this representation of the solution into the previous equations we get

\[
\hat{\nu} = \frac{g - \kappa D^2/h_0}{(1 - \kappa) D} \hat{h}, \quad \hat{u}_1 = \nu \hat{u}, \quad \nu = \sqrt{\frac{\kappa}{\alpha_1 h_0^2} \frac{D^2 - gh_0}{gh_0 - \kappa D^2}}.
\]
Figure 1. A solitary wave: solid curve — solution of the Nwogu equations, dashed and dot-dashed curves — solutions of the hyperbolised Nwogu equations for \( c = 30 \) and \( c = 8 \), respectively.

Figure 2. The evolution of a solitary wave on a mildly sloping beach: 1, 2 — free surface at \( t = 0 \) and \( t = 9 \) (solid curve — Nwogu equations, dashed curve — hyperbolised system), 3 — bottom topography.

We use these asymptotic expressions when the conditions \( h = h_0 + \hat{h} \), \( u = \hat{u} \) and \( u_1 = \hat{u}_1 \) are imposed at \( \xi = \xi_0 \) in the numerical treatment of ODE (6).

The construction of solutions in the class of travelling waves for equations (5) is carried out similarly. By obvious transformations, system (5) reduces to the following ODEs

\[
\begin{align*}
    u' &= -\frac{v}{c + D}, \\
    v' &= -\frac{c^2\Phi}{(c - D)\alpha_1 h_0^2}, \\
    h' &= \frac{F}{\Delta},
\end{align*}
\]

(7)

where

\[
\Phi = -\frac{\kappa}{h_0} (Dh_0 + (u - D)h), \quad F = \frac{v}{c + D} (u - D)h_0 + \kappa Dh, \quad \Delta = gh_0 + \kappa D(u - D).
\]

To construct a solution satisfying the condition \( h = h_0 \), \( u = v = \Phi = 0 \) for \( \xi \to \infty \), it is necessary to carry out an asymptotic analysis by analogy with the previous one. Omitting the intermediate calculations, we give the perturbation amplitudes at \( \xi = \xi_0 \) in order to construct the desired solution to ODEs (7)

\[
\hat{u} = \frac{\kappa D}{h_0} \left( 1 - \frac{D^2}{c^2} \right) \alpha_1 h_0^2 \nu^2 + \kappa \hat{h}, \quad \hat{v} = -(c + D)\nu \hat{u},
\]

\[
\nu = \sqrt{\frac{1}{\alpha_1 h_0^2} \left( 1 - \frac{gh_0}{D^2} \right) \left( 1 - \frac{gh_0}{\kappa D^2} \right)^{-1} \left( \frac{D^2}{c^2} - 1 \right)^{-1}}.
\]

Figure 1 (solid curve) shows the solution of equations (6) with the following parameters: \( g = 1 \), \( h_0 = 1 \), \( D = 1.5 \), and \( \hat{h} = 0.001 \). This solitary wave can also be obtained in the framework of a hyperbolic model. The corresponding solution of equations (7) is shown in figure 1 by dashed \( (c = 30) \) and dash-dotted \( (c = 8) \) curves. As can be seen from the figure, an increase in the ‘hyperbolisation’ parameter \( c \) ensures the convergence of the solution of equations (6) to the solution of system (7).
We consider the evolution of solitary waves on a mildly sloping beach and carry out calculations using models (4) and (5). In this case, for both models (4) and (5), the term $-gZ_x$ is added to the right-hand side of the momentum equation. Here $z = Z(x)$ is the bottom topography. The results of the calculation using non-stationary equations (4) and (5) in dimensionless variables ($g = 1$) are shown in figure 2. For the hyperbolic model, we additionally set $c = 8$. The calculations are performed using the Nessyahu–Tadmor second-order central scheme [13] on a uniform grid with $N = 1000$ nodes. As can be seen from the figure, the calculations for the dispersive and hyperbolic models almost coincide. As it is known, the Green–Naghdi equations (as well as the Nwogu equations) in the considered problem give an excess increase in the wave amplitude. More accurate dispersive models that take into account the vortex nature of the flow during wave breaking were recently proposed in [14, 15]. The hyperbolic approximation of these advanced models was derived in [16].

4. Steep forced waves generated by an oscillating tank

We compared solutions of the developed mathematical models against the results obtained in laboratory experiments on surface waves [17]. The experiments were carried out in a rectangular tank with dimensions of 1.98 m in length, 0.08 m in width and 0.2 m in height (figure 3). The tank generated surface gravity waves by oscillating harmonically about a horizontal axis which was at the centre of the tank bottom and perpendicular to the side walls. The rotation of the tank was performed using a slider crank mechanism. The design of the gear guaranteed 1.2% accuracy of the harmonic oscillations. The amplitude of the surface waves was measured by four water level meters located at both ends of the tank, in the middle and at the distance of a third of the length of the tank. Video registration was performed at a framerate of 125 frames per second.

To describe the surface waves in an oscillating tank we use a coordinate system associated with the tank, which resulted in a time depended harmonic external force in the governing equations. We introduced also friction and viscosity terms into the momentum equations for both models to account for real physical effects.

The modified Green–Naghdi equations (2) now read

$$ K_t + \left( uK - \frac{u^2}{2} + gh \cos \varphi - \frac{h^2u^2}{2} - \nu u_x \right) = -g \sin \varphi - cf \frac{|u|}{h}, $$

$$ h_t + (uh)_x = 0, \quad K = u - \frac{1}{3h} (h^3 u_x)_x. $$

(8)

While the modified Nwogu equations (4) have the form

$$ (u + \alpha_1 h^2_0 u_{xx})_t + \left( \frac{u^2}{2} + gh \cos \varphi - \nu u_x \right) = -g \sin \varphi - cf \frac{|u|}{h}, $$

$$ h_t + (uh + (\alpha_1 + \frac{1}{3}) h^3_0 u_{xx})_x = 0, \quad \alpha_1 = -0.39. $$

(9)

Figure 3. Free surface at $t = 2.5T_1$ where $T_1 = 4.3$ s is the tank oscillation period. Solid curve — solution of the Green–Naghdi equations, dashed curve — Nwogu equations.
Figure 4. Time history of the non-dimensional water surface elevation \( \eta = (h - h_0)/h_0 \) at \( x = 0 \) (curves 1) and \( x = L/3 \) (curves 2): a — experimental data (solid lines) and solution of the Green–Naghdi equations (dashed lines); b — solution of the Green–Naghdi equations (solid lines) and the Nwogu equations (dashed lines).

In numerical simulations following parameters were used: gravity acceleration \( g = 9.8 \text{ m/s}^2 \), tank length \( L = 1.79 \text{ m} \), friction coefficient \( c_f = 0.004 \), kinematic viscosity \( \nu = 0.009 \text{ m}^2/\text{s} \), initial water layer height at rest \( h_0 = 0.072 \text{ m} \) and tank rotation \( \varphi = \varphi_0 \sin(2\pi t/T_1) \) with initial angle \( \varphi_0 = 0.008 \) and period \( T_1 = 4.3 \text{ s} \).

Figure 3 shows free surface obtained in numerical simulations and compared against the result of the laboratory experiment at time \( t = 2.5T_1 \) with \( T_1 \) being the oscillation period. The blue solid line corresponds to the solution of the modified Green–Naghdi equations (8), while the red dashed line represents the solution of the modified Nwogu system (9). The results obtained by modified dispersive Nwogu model are in good agreement with the laboratory experiment compared to the simulations by the dispersive Green–Naghdi model.

Evolution in time of the surface level at two fixed points along the tank are shown in figure 4. The left-hand panel (a) shows comparison between the experiment data (solid lines) and the solution of the Green–Naghdi equations (dashed lines). A comparison between the simulations by the Green–Naghdi (solid lines) and Nwogu (dashed lines) models is presented in the right-hand panel (b). In both panels, plots 1 correspond to the measurements at the left end of the tank \( (x = 0) \), and plots 2 correspond to the measurements at one third of the tank length \( (x = L/3, L \text{ is the tank length}) \). The results of the simulations are in good agreement with experiment data and both models can describe characteristic features of the flow.

5. Conclusion

Geophysical applications and coastal engineering widely employ Boussinesq-type models to describe free surface gravity waves. We showed that the hyperbolic approximation of the 1D dispersive Boussinesq-type Nwogu model of the shallow water theory is applicable for describing non-hydrostatic unsteady flows. The derived hyperbolic system and the original Nwogu dispersive equations result in almost identical evolution of solitary waves on a mildly sloping beach. One of the advantages of hyperbolicity of the system in use is that we can implement robust computational algorithms developed for hyperbolic systems to perform numerical simulations.

We also compared performance of the Nwogu equations and Green–Naghdi equations in governing steep forced water waves generated by a harmonically oscillating rectangular tank. It
was shown that both the GreenNaghdi and Nwogu models describe well enough the moderate-amplitude forced waves.

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