On Singular Vortex Patches, I: Well-posedness Issues

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Abstract

The purpose of this work is to discuss the well-posedness theory of singular vortex patches. Our main results are of two types: well-posedness and ill-posedness. On the well-posedness side, we show that globally $m$–fold symmetric vortex patches with corners emanating from the origin are globally well-posed in natural regularity classes as long as $m \geq 3$. In this case, all of the angles involved solve a closed ODE system which dictates the global-in-time dynamics of the corners and only depends on the initial locations and sizes of the corners. On the ill-posedness side, we show that any other type of corner singularity in a vortex patch cannot evolve continuously in time except possibly when all corners involved have precisely the angle $\frac{\pi}{2}$ for all time. Even in the case of vortex patches with corners of angle $\frac{\pi}{2}$ or with corners which are only locally $m$–fold symmetric, we prove that they are generically ill-posed. We expect that in these cases of ill-posedness, the vortex patches actually cusp immediately in a self-similar way and we derive some asymptotic models which may be useful in giving a more precise description of the dynamics. In a companion work \cite{11}, we discuss the long-time behavior of symmetric vortex patches with corners and use them to construct patches on $\mathbb{R}^2$ with interesting dynamical behavior such as cusping and spiral formation in infinite time.

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1 Introduction

1.1 The notion of vortex patches

In this paper, we investigate the dynamics of singular vortex patches, which are patch-like solutions to the 2D Euler equations with non-smooth boundaries. We first recall that the 2D Euler equations on \( \mathbb{R}^2 \), in vorticity form, are given by

\[
\partial_t \omega + (u \cdot \nabla) \omega = 0,
\]

where \( u \) is determined from \( \omega \) by

\[
u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y) dy.
\]

The transport nature of (1.1) suggests that if the initial vorticity \( \omega_0(x) \) is given by the characteristic function of a domain \( \Omega_0 \subset \mathbb{R}^2 \), the solution should take the form of the characteristic function of a domain that moves with time. We shall refer to such a solution as a vortex patch. Indeed, the theorem of Yudovich in [91] gives that for any \( \omega_0 \in L^1 \cap L^\infty (\mathbb{R}^2) \), there exists a unique solution to (1.1) in the class \( \omega(t,x) \in C^0_t (\mathbb{R}; L^1 \cap L^\infty (\mathbb{R}^2)) \) with \( \omega(0,x) = \omega_0 \), where \( C^0_t \) denotes that \( \omega(t, \cdot) \) is weak-star continuous in time. It turns out that this regularity is just sufficient to make sense of the flow maps \( \Phi(t, \cdot) \) as homeomorphisms of \( \mathbb{R}^2 \) for all \( t \in \mathbb{R} \): the velocity vector field satisfies the following log-Lipschitz estimate

\[
|u(t,x) - u(t,x')| \leq C \|\omega_0\|_{L^\infty \cap L^1} |x-x'| \log \left(1 + \frac{1}{|x-x'|}\right)
\]

which gives rise to a unique solution to the following ordinary differential equation

\[
\frac{d}{dt} \Phi(t,x) = u(t, \Phi(t,x)), \quad \Phi(0,x) = x.
\]
As a particular case, if the initial data is given by \( \omega_0(x) = \chi_{\Omega_0} \) for some bounded measurable set \( \Omega_0 \), the associated unique solution to (1.1) takes the form

\[
\omega(t, x) = \chi_{\Omega(t)}, \quad \Omega(t) = \Phi_t^{-1}(\Omega_0)
\]

where \( \Phi_t^{-1} \) is the inverse of \( \Phi(t, \cdot) \). Therefore, the following vortex patch problem is well-defined:

Given a bounded measurable set \( \Omega_0 \), what can be said about the sets \( \Omega(t) \) for \( t \neq 0 \)?

Before we proceed further, let us point out a simple consequence of the following Yudovich estimate:

\[
|x - x'|e^{ct\|\omega_0\|_{L^\infty} \cap L^1} \leq |\Phi(t, x) - \Phi(t, x')| \leq |x - x'|e^{-ct\|\omega_0\|_{L^\infty} \cap L^1}
\]

for all \( x, x' \in \mathbb{R}^2 \) with \( |x - x'| < 1/2 \) where \( c > 0 \) is an absolute constant. It guarantees that, if the boundary of \( \Omega_0 \) is given by a Jordan curve, this property holds for all of the domains \( \Omega(t) \). However, since the estimate deteriorates with time, in general no uniform regularity can be obtained for all \( \partial \Omega(t) \).

Often, a vortex patch could mean the following more general object: a solution of the 2D Euler equations in the form

\[
\omega(t, x) = \sum_{i=1}^{N} f_i(t, x) \chi_{\Omega_i(t)}
\]

where \( N \geq 1 \) is an integer, \( \Omega_i(t) \) are mutually disjoint bounded measurable sets that move with time, and \( f_i(t, x) \) are functions describing the profiles of vorticity. In this case, it is reasonable to require that \( f_i(t, \cdot) \) is at least continuous on \( \Omega_i(t) \). Moreover, the fluid domain could be a bounded domain in \( \mathbb{R}^2 \), the 2D torus, or some other surface. Unless otherwise stated, we shall restrict ourselves to simple \((N = 1)\) patches on \( \mathbb{R}^2 \), with the normalization \( f_1 \equiv 1 \).

### 1.2 Smooth versus singular patches

Given Yudovich’s theorem, it is natural to ask the smooth version of the above vortex problem: that is, if \( \partial \Omega_0 \) is given by a smooth curve, does this property hold for all \( \partial \Omega(t) \)? It turns out that the answer is positive: precisely, if \( \partial \Omega_0 \) is a \( C^{k,\alpha} \) Hölder continuous curve for some \( k \geq 1 \) and \( 0 < \alpha < 1 \), then \( \partial \Omega(t) \) is \( C^{k,\alpha} \)-regular for all \( t \). In particular, the boundary remains a \( C^\infty \)-curve for all times if it is so initially. There are two separate issues for this smooth vortex patch problem, namely propagation of smoothness locally and globally in time.

Note that even local propagation is non-trivial as \( \omega(t) \in L^1 \cap L^\infty(\mathbb{R}^2) \) does not give that the corresponding velocity \( u(t) \) is Lipschitz in space, which is necessary to keep the boundary smooth. What saves us is the following special property of the double Riesz transforms (stated somewhat roughly):

If \( \omega \) is \( C^{k-1,\alpha} \)-smooth along a \( C^{k-1,\alpha} \) vector field, then \( R_i R_j \omega \) has the same property.

Here we need \( k \geq 1 \) and \( 0 < \alpha < 1 \), and \( R_i \) denote the Riesz transform with \( i, j \in \{1, 2\} \). Applying this fact to the case \( \omega = \chi_{\Omega} \), we obtain that if the boundary is \( C^{k,\alpha} \)-smooth, then the velocity field belongs to \( C^{k,\alpha}(\Omega) \) and also to \( C^{k,\alpha}(\mathbb{R}^2 \setminus \Omega) \). Here we have taken the closures to emphasize that the \( C^{k,\alpha} \)-regularity is valid uniformly up to the boundary. This “frozen-time” fact alone suffices to show local propagation of the boundary regularity. Note also that as long as the smooth solution exists, the flow maps \( \Phi(t, \cdot) : \Omega_0 \to \Omega(t) \) are actually \( C^{k,\alpha} \)-regular diffeomorphisms in this case.

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1 Actually, \( u(t) \) is never \( C^1 \)-smooth across the boundary of the patch simply because \( \omega = \partial_y u^2 - \partial_x u^1 \).
The issue of global regularity is much more subtle and really hinges on the vectorial nature of the velocity field defined by the 2D Biot-Savart kernel. Still, it is relatively straightforward to obtain the following statement on the propagation of regularity:

If $\partial \Omega_0$ is $C^{k,\alpha}$-smooth and somehow $\|\nabla u(t,x)\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} < \infty$, then $\Omega(t)$ is $C^{k,\alpha}$ up to time $T$.

Of course, this is reminiscent of the classical estimate for smooth solutions to the Euler equations:

$$\frac{d}{dt}\|\omega(t)\|_{C^{k,\alpha}} \lesssim \|\nabla u(t)\|_{L^\infty} \|\omega(t)\|_{C^{k,\alpha}},$$

which guarantees that the vorticity retains its initial Hölder regularity as long as the velocity remains Lipschitz. Indeed, in several respects, the regularity theory for smooth patches is parallel to the one for smooth vorticities.

At this point, it is worth emphasizing that the Yudovich theory is not relevant (probably even misleading) for the smooth vortex patch problem (both local and global); the latter is really about the anisotropic regularity statement for certain singular integral transforms. Hence it should not be surprising that even for systems such as the surface quasi-geostrophic equations and the 3D Euler equations, smooth patches can be solved locally in time. The Yudovich theorem only guarantees unique existence of a solution after the potential blow-up time (which does not happen for the 2D Euler equations, anyways).

The story is completely different for patches without smooth boundaries. Let us even imagine an initial patch whose boundary is completely smooth except at a point where it is no better than $C^1$ (e.g. a slice of pizza). Then in general the corresponding initial velocity will fail to be Lipschitz (which is necessary to propagate regularity), and we are in the Yudovich regime, where the velocity is only log-Lipschitz. Here, let us clarify a theorem of Danchin which shows that for an initial patch with isolated singularities in the boundary (and otherwise smooth), the patch boundary remains smooth away from the trajectories of the singular points by the flow. However, it does not show propagation of piecewise smoothness uniform up to each singularity, which may be valid for the initial patch as a slice of pizza does. Indeed, one of our results here shows that any uniform regularity strictly better than $C^1$ is instantaneously lost for such a data. Then, of course, the right question is to ask what exactly happens, and this is what this work makes progress on.

### 1.3 Motivations for vortex patches

Before we show some explicit computations on vortex patches, let us give a few motivations towards the vortex patch problem in general, with some emphasis on its singular version. The following items are indeed deeply related with each other.

- **Vortex patches as idealized physical objects:** It is reasonable to use vortex patches to model physical situations where a strong eddy-like motion is observed, e.g. a hurricane. In particular, a motivation for studying patches with corner singularities in aerodynamics is discussed in the introduction of [20]. For more information, one may consult classical textbooks on vortex dynamics ([60, 77]). It in particular motivates the study of vortex patches on the 2-sphere ([87, 88, 79, 83]).

- **Long-time behavior of smooth solutions:** Regarding the 2D Euler equations, one of the most important problems is to understand the asymptotic behavior of smooth solutions as time goes to infinity. The strongest conservation law is the $L^\infty$-norm for the vorticity, and it is possible that any higher regularity blows up for $T = +\infty$ (this explicitly happens near the so-called Bahouri-Chemin solution; see [59, 89] and Subsection 2.2 below). Hence $L^\infty$ is the natural space to study the long-time behavior.
\begin{itemize}
  \item Critical phenomena: The space $L^\infty$ in terms of the vorticity is a critical space, in the sense that the associated velocity field barely fails to be a Lipschitz function in space. This leads to interesting phenomena such as instantaneous cusp/spiral formation which is impossible with Lipschitz velocity fields. Moreover, recently there have been significant progress on understanding the Cauchy problem with critical initial data \cite{10,17,43,44,69,70}. For instance it has been shown that the incompressible Euler equations are \textit{ill-posed} in critical Sobolev and Hölder spaces. The corresponding problem for patches is a (folklore) open problem: what happens to the initial patch whose boundary is exactly $C^k$ or $C^{k-1,1}$ with some $k \geq 1$? Note that, as in \cite{44}, the case $k = 1$ seems to be much more difficult than the case $k \geq 2$. This is because there is much better control on the velocity field in the latter case.
  \item Construction of special solutions: There has been a lot of interest in constructing solutions of the Euler equations with certain dynamical behavior. In this context, the class of vortex patches provides a whole variety of interesting solutions to the 2D Euler equations. Even in situations where one needs smooth solutions, a strategy that has proven useful is to consider patch solutions with the same dynamics and then try to “smooth out” the patch. (See a recent work \cite{22} where the authors constructed compactly supported and smooth rotating solutions to the 2D Euler equations.)
    \begin{itemize}
        \item V-states: Patches which simply rotate with some constant angular speed are called V-states \cite{18,21,22,31,32,28,50,78,84}. One may bifurcate from radial profiles to obtain $m$-fold symmetric V-states, and it is expected that in the limit one obtains V-states with either $90^\circ$ corners or cusps \cite{72,88,90}. See \cite{47} for recent rigorous progress on this problem.
        \item Solutions with infinite norm growth: In two dimensions, Sobolev and Hölder norms of smooth Euler solutions can grow at most double exponentially in time. This sharp rate was achieved in the presence of a physical boundary in \cite{59} by smoothing out the Bahouri-Chemin solution. In terms of vortex patches, the relevant question is whether two disjoint patches can approach each other double exponentially in time as $t \to +\infty$ (see \cite{34}).
        \item Instantaneous instability: On the other hand, one may ask for initial vorticity configurations which maximize a certain functional (such as palenstrophy); see \cite{4,6} and references therein. It seems that in certain cases the maximizer takes the form of a (slightly regularized) vortex patch; the work \cite{43} shows this for the case of the $H^1$-norm in terms of the vorticity.
    \end{itemize}
\end{itemize}

In the opposite direction, one may consider patches as smoother alternatives for even more singular constructs, such as vortex sheets or point vortices. The study of singular vortex patches becomes relevant in this regard; for instance, one may take the vanishing angle limit of the patch supported on a sector, keeping the $L^1$-norm. In the limit one obtains a sheet with linearly growing intensity from the corner which was numerically studied by Pullin \cite{74,76}.

\subsection{1.4 Main results and ideas of the proof}

As we have mentioned earlier, the primary goal in this paper is to understand the dynamics of patches initially supported on either a corner or a union of corners meeting at a point. In one sentence, our conclusion is that such a corner structure propagates continuously in time if and only if the initial patch satisfies an appropriate rotational symmetry condition at the origin, namely $m$-fold symmetry with some $m \geq 3^\circ$. We actually show that when such a symmetry condition is satisfied, then the propagation is global in time.

Our main well-posedness result concerns symmetric patches which have corners meeting at a point, which can be set as the origin without loss of generality. The result shows that the regularity of the patch boundary which is uniform up to the corner propagates for all time. For the economy of

\footnote{This is with the exception of special angles $0, \pi/2, \pi$, and $3\pi/2$, which we discuss separately.}
presentation, we present a somewhat rough statement here; detailed statements are given in Theorem 2 and Corollary 4.5.

**Theorem A.** Consider \( \omega_0 = \chi_{\Omega_0} \) where \( \Omega_0 \) is \( m \)-fold rotationally symmetric around the origin with \( m \geq 3 \), \( \partial \Omega_0 \) is \( C^{1,\alpha} \)-smooth away from the origin, and can be mapped by a \( C^{1,\alpha} \)-diffeomorphism \( \Psi_0 \) of \( \mathbb{R}^2 \) to a union of non-intersecting sectors meeting at the origin locally with some \( 0 < \alpha < 1 \); that is,

\[
\Psi_0(\Omega_0) \cap B_0(r_0) = \bigcup_{k=0}^{m-1} \bigcup_{i=1}^{N} \{(r,\theta) : 0 < r < r_0, a_i,0 + 2\pi k/m < \theta < b_i,0 + 2\pi k/m\}
\]

(1.3)

for some \( r_0 > 0 \). Then, the corresponding patch solution \( \Omega(t) \) has the same property for all \( t > 0 \), with some \( C^{1,\alpha} \)-diffeomorphism \( \Psi(t) \) and \( r(t) > 0 \). Moreover, the corner angles of \( \Omega(t) \) evolve according to a closed system of ordinary differential equations; in the simplest case of \( N = 1 \) in (1.3), the corners rotate with a constant angular speed for all time, which is determined only by the initial angle and \( m \).

In the statement, \( C^{1,\alpha} \) can be replaced by \( C^{k,\alpha} \) throughout, for any integer \( k \geq 1 \). In particular if the initial boundary is uniformly \( C^\infty \)-smooth up to the corner, the boundary will remain so for all time. A prototypical example of a patch satisfying the assumption above is given by the region

\[
\{(r,\theta) : 0 < r < \sin(m\theta)\}
\]

with some \( m \geq 3 \); see Figure 1 for the case \( m = 3 \). Since \( N = 1 \) in this case, our result dictates that near the corner, the motion of the patch is given by a uniform rotation for all time. This is completely consistent with the existence of \( V \)-states which take a similar form as in Figure 1 reported by numerical analysts [1, 2]. It turns out that the angular speed of rotation is a monotonic function of the initial angle. Therefore, if we perturb the circular patch in \( L^1 \) so that locally it looks as in Figure 1, there is a discrepancy between the speeds of rotation near the corner and at the bulk for all time, from a well-known stability result for the circular patch. Combining this with some topological and measure-theoretic arguments, we conclude infinite in time spiral formation in the companion work [3].

Our analysis is not limited to the case of \( N = 1 \), but also covers the case when there are multiple corners in a fundamental domain of the \( m \)-fold rotation. For an example, one can consider the domain

\[
\text{Figure 1: Symmetric corners with smooth boundary}
\]

\[
\text{Figure 1: Symmetric corners with smooth boundary}
\]

3Interestingly such \( V \)-states can be found numerically by carefully bifurcating from a \( V \)-state consisting of three chunks of vorticity arranged symmetrically around the origin.
Figure 2: Global in time propagation of Hölder regularity of vorticity

obtained by the 3-fold symmetrization of

\[ \{(r, \theta) : 0 < r < \sin(6\theta), 0 < \theta < \pi/6\} \cup \{(r, \theta) : 0 < r < 2 \sin(6\theta), \pi/3 < \theta < \pi/2\}. \]

In such cases, the corner angles satisfy an interesting system of ODEs which we briefly study in Subsection 4.4. We emphasize that this system is completely closed by itself, so that the local asymptotic shape of the patch for any \( t > 0 \) is determined from the initial corner angles.

The statement regarding the angles might be counter-intuitive; after all, strong non-locality in the Biot-Savart kernel of the incompressible Euler equations is its main difficulty. However, consider for instance a radial vorticity \( \omega = f(r) \) which is supported away from the origin. Then the velocity near the origin is identically zero; that is, symmetry introduces cancellations. For our purpose, which is to localize the dynamics of the angles, it suffices to guarantee that \( u_{far}(x) = o(|x|) \) for \( |x| \ll 1 \) where \( u_{far} \) is the non-local contribution to the velocity. As we will show in this work, it suffices to assume \( m \)-fold symmetry with \( m \geq 3 \).

It turns out that the proof for the local in time statement is rather straightforward, and follows readily from the explicit computations that we shall demonstrate in the next section. Let us give the main points here: For local propagation of regularity, it suffices to establish that the velocity restricted onto the patch boundary is \( C^{1,\alpha} \)-smooth. However, for a patch given in the statement of Theorem A, the corresponding velocity can be considered as a sum of main part coming from exact sectors and remainder associated with cusp regions. The latter component of the velocity is smooth on the boundary. On the other hand, the velocity generated by a symmetric union of exact sectors takes the form \( \nabla \perp (r^2 H(\theta)) \), with \( H \in W^{2,\infty}([0,2\pi]) \). The log-Lipschitz part vanishes by symmetry, and it is not hard to see using this explicit expression that it is \( C^{1,\alpha} \) along any \( C^{1,\alpha} \)-curve emanating from the origin. Essentially, this concludes the proof for local well-posedness.

Unfortunately, the global well-posedness statement for such patches does not seem to follow from a simple adaptation of any of the existing arguments showing global well-posedness for smooth patches. For instance, let us explain the difficulty with respect to the “geometric” approach of Bertozzi and Constantin (see Subsection 2.3 below for a brief review of their approach). In this framework, the patch boundary regularity is encoded by a level set function \( \phi : \mathbb{R}^2 \to \mathbb{R} \), characterized by the property that \( \phi > 0 \) exactly in the interior of the patch. Then, the \( C^{1,\alpha} \) norm of \( \phi \) is (roughly) associated with the \( C^{1,\alpha} \)-regularity of the patch boundary, under the condition that \( \nabla \phi \) is non-degenerate. Note that if we want such a level set function for the domain in Figure 4, \( \phi \) certainly cannot be better than Lipschitz. To encode the information that the patch boundary is uniformly piecewise \( C^{1,\alpha} \) up to the corner, we need to either give up that \( \nabla \phi \) is non-degenerate, or use multiple level set functions to characterize the boundary. None of these variations seemed to work out well.

However, see a recent work Kiselev, Ryzhik, Yao, and Zlatos [58] where the authors overcome a similar type of difficulty on the upper half-plane with brute force estimates.

\[ \text{Figure 2: Global in time propagation of Hölder regularity of vorticity} \]

\( \text{\{r, \theta\} : 0 < r < \sin(6\theta), 0 < \theta < \pi/6\} \cup \{r, \theta\} : 0 < r < 2 \sin(6\theta), \pi/3 < \theta < \pi/2\} \).
Our approach was to go around this problem by first “completing the square” (see Figure 2) and extract the global-in-time bound on the Lipschitz norm of the velocity from it. This piece of information combined with a Beale-Kato-Majda type argument was sufficient to conclude Theorem A. Let us now briefly explain Figure 2; on the top left side, the classical result on the global well-posedness of $C^\alpha$ vorticity is placed. Then, the vertical and horizontal arrows correspond to the properties of the Euler equations which propagate anisotropic and scale-invariant Hölder regularity of the vorticity, respectively. The latter holds only in the presence of $m$-fold rotational symmetry with $m \geq 3$. The notation $C^\alpha$ was introduced in [42] and encodes scale-invariant $C^\alpha$-regularity; roughly, “homogeneous” derivatives $\partial_\theta^\alpha \omega$ and $r^\alpha \partial_r^\alpha \omega$ should be bounded, where $\partial_\theta$ and $\partial_r$ denote the $\alpha$-fractional derivative in the angle and radius, respectively. The global well-posedness of $C^\alpha$-vorticity under symmetry was established in [42], and it is natural to consider the patch version of this result. On the other hand, one can equivalently consider the scale-invariant version of the $C^\alpha$-patch result. This is the content of the following result:

**Theorem B.** Consider a patch $\Omega_0$ which is $m$-fold symmetric for some $m \geq 3$ and the piece of boundary at distance $O(r)$ from the origin is $C^{1,\alpha}$-smooth with Lipschitz norm bounded uniformly in $r$ and $C^1$-norm bounded by $Cr^{-\alpha}$ for some $C > 0$ and $0 < \alpha < 1$. Then the patch solution $\Omega(t)$ retains this property for all $t > 0$.

It is easy to see that the patches considered in Theorem A satisfy this condition. Note that the logarithmic spirals considered in the above satisfy this assumption as well, so that Theorem B establishes global-in-time regularity propagation for them. The uniform Lipschitz assumption in Theorem B in particular requires that the patch domain is weakly Lipschitz, in the sense that near every point $p \in \partial \Omega_0$, there is a bi-Lipschitz map of $\mathbb{R}^2$ sending a neighborhood of $p$ intersected with $\Omega_0$ and $\partial \Omega_0$ to the upper half-plane and the boundary of the upper half-plane, respectively. Indeed, the logarithmic spirals are well-known examples of weakly Lipschitz domains which are not strongly Lipschitz (near every point on the boundary, the boundary of the domain is given by the graph of a Lipschitz function); see [3, 28]. Hence, this result shows that even weakly Lipschitz domains propagate its regularity if we assume symmetry and scale-invariant Hölder condition. We shall give more details on the ideas of the proofs in the beginning of Sections 3 and 4.

We now state our main ill-posedness result, which states roughly that when the symmetry condition in the above well-posedness statements are not satisfied, then the corner structure is lost immediately.

**Theorem C.** Assume that $\omega(t) = \chi_{\Omega(t)}$ is a patch-type solution to the 2D Euler equation with a corner singularity whose initial angle is less than $180^\circ$ and propagates continuously in time on some interval $[0, \delta]$. Then, either the corner has angle $90^\circ$ for all $t \in [0, \delta]$ or the vortex patch is locally $m$-fold symmetric with respect to the corner for some $m \geq 3$ for all $t \in [0, \delta]$. Moreover, there exist initially locally $m$-fold symmetric patches and patches with a single $90^\circ$ corner which do not propagate continuously in time.

In addition to this, we shall show in Theorem ?? that the exact $m$-fold symmetry condition is essential even for local well-posedness: for an initial vortex patch which is $m$-fold symmetric with $m \geq 3$ only locally at the origin, it is possible for the velocity to lose Lipschitz continuity immediately.

Lastly, we discuss the important question of what is the actual dynamics of a corner without any symmetries. In Subsection 2.2 below, we shall carry out some computations for vortex patches supported on cusps and spirals, as possible candidates for describing the evolution of the corner. Let us explain here why we expect the corner to immediately cusp or spiral: To begin with, the passive transport by the initial velocity indicates that the corner rotates $45^\circ$ instantaneously and form a cusp there. However, as soon as this happens, if the vorticity near the point of singularity is “thick” enough in the angle, then the new velocity can make the patch rotate even further, up to another $45^\circ$. Then, either this process can go on indefinitely so that the resulting patch has formed an (infinite) spiral, or stop at some point that the patch is just a cusp. The difficulty is that this entire process is supposed
to happen exactly at $t = 0$. Therefore, it makes sense to define a new variable incorporating both time and length scales, which rescales the instantaneous behavior of the patch to occur on a non-zero interval in this variable. It turns out that the natural change of variables is to introduce new time variable $\tau = t \ln \frac{1}{r}$. With this variable, we derive a formal evolution equation (a second order system of ordinary differential equations in terms of $\tau$) which is supposed to describe the boundary evolution near the corner at least for a short period of time. This procedure is comparable with introducing a self-similar variable in the study of vortex sheets supported on algebraic spirals.

### 1.5 Historical background

The celebrated 1963 theorem of Yudovich [91] made it possible to pose the vortex problem, without any regularity assumptions on the patch boundary. Later this well-posedness result was extended in various directions (see e.g. [2, 11, 13, 39, 42, 56, 71, 81, 82, 85–87, 92]). We just note that when the path satisfies $m$-fold symmetry with some $m \geq 3$, global existence and uniqueness can be proved with just $L^\infty$ of vorticity, which makes it possible to treat patches with non-compact support in $\mathbb{R}^2$ ([42]).

The dynamics of vortex patches, either numerically or theoretically, are usually considered using the contour dynamics equation (CDE), which reduce the 2D dynamics to a 1D evolution equation in terms of the boundary parametrization. It is required that the patch boundary is at least piecewise $C^1$. In the context of 2D Euler patches, the corresponding CDE seems to have first appeared in the work [93] published in 1979, in the context of providing reliable numerical scheme for the 2D Euler equations. In the thesis of Bertozzi [15], a local well-posedness theorem for smooth vortex patches was proved based on the CDE. It is worth noting that at the time of this work (which was 1991), it was still an open problem whether smooth vortex patches can become singular in finite time, and corners were investigated as a possible candidate for the profile of the patch at the (hypothetical) blow-up time in [15]. In the late 80s and early 90s there have been a lot of numerical and theoretical works investigating the possibility of finite time singularity, which seemed highly likely ([1, 19, 27, 36, 37, 65]).

This issue was settled by Chemin [23, 24] in 1991 who showed global well-posedness using paradifferential calculus. Then several other proofs, based on different arguments, followed [9, 82]. See also more recent works [7, 52] as well as textbooks [10, 25] which cover the proof of global well-posedness. The works of Danchin [29, 30] cover global well-posedness for (regular) cusps as well as propagation of patch boundary regularity away from singularities. In the case when the physical domain has a boundary, it is more delicate to propagate regularity globally in time for smooth patches touching the boundary (see [35, 58] and references therein). In [51], it was shown that a corner supported on the boundary of the Half-plane cusps immediately as $t > 0$. Here, the physical boundary significantly simplifies the analysis – we revisit this result in the section on illposedness (Section 5). The works [20, 26] numerically investigate the dynamics of a corner; the pictures suggest that initial angles smaller than 90° shrink and those larger than 90° expand for $t > 0$.

Many interesting dynamic problems regarding vortex patches are wide open. For patches in 3D and higher, it is certainly a challenging problem to prove whether smooth vortex patches can become singular in finite time. Regarding 2D patches, it is not known whether a (signed) patch can initially have finite diameter and the diameter grows without a uniform bound as $t \to \infty$. Non-trivial upper bounds on the diameter growth are known (and they are polynomial in time; see e.g. [53, 54, 67]). In the case when both signs are allowed, [54] shows that the patch diameter can grow at least linearly in time. A similar question can be asked for the perimeter. In contrast with the diameter case, there is a possibility for a patch with rectifiable boundary to instantaneously lose this property for $t > 0$. However, the result [57] which gives upper bounds on the growth of the Dirichlet eigenvalues for the Laplacian with little assumption on the boundary regularity suggests that such a behavior is unlikely.

The study of patches with 90° corners are left out in this work. As we have seen in the above, the difficulty is that the log-Lipschitz part of velocity only exists in the direction tangent to the patch boundary. It would be interesting to rigorously show existence of not only rotating patches with 90° corners but also translating ones with an odd symmetry (see figures from [62] and references therein).
1.6 Outline of the paper

The rest of this paper is organized as follows. The notations that we use throughout the paper are collected in the first subsection of Section 2. Then in Section 3 we prove Theorem B which is global well-posedness for symmetric patches whose boundaries are $C^{\alpha}$-smooth in the angle. Using this result together with a tedious local calculation, we conclude Theorem A in Section 4. Possible extensions to this main result are sketched at the end of that section. Ill-posedness results, including Theorem C, are proved in Section 5. Finally in Section 6, we formally write down the effective system which describes the dynamics of the patch with a single corner. The necessary local well-posedness results for symmetric patches that we consider in Sections 3 and 4 are proved in the Appendix for completeness. We emphasize that the work consists of two different results whose proofs are independent of each other: well-posedness and ill-posedness. As such, a reader interested in the well-posedness results may focus solely on Sections 2, 3, and 4 while a reader interested mainly in the ill-posedness results may read Sections 2, 5, and 6.

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2 Background Material

This section goes through some useful background material for the benefit of the reader. We begin by going through a few simple computations which give the reader a sense of the difficulties associated with vortex patches in general and singular vortex patches in particular. Then we discuss two prior works which are important to know: Chemin’s global well-posedness result for smooth vortex patches, particularly the proof of Bertozzi-Constantin and our previous result on scale-invariant Hölder regularity for $m$-fold symmetric solutions to 2D Euler.

2.1 Notations and definitions

Let us collect a few definitions and conventions that will be used throughout.

- For $\theta \in [0, 2\pi)$, we let $R_\theta$ be the matrix of counterclockwise rotation around the origin by the angle $\theta$. Using this notation, we say that a scalar-valued function $f : \mathbb{R}^2 \to \mathbb{R}$ (e.g. vorticity, level-set function, stream function) is $m$-fold symmetric if $f(x) = f(R_{2\pi/m}x)$ for any $x \in \mathbb{R}^2$. On the other hand, a vector field $v : \mathbb{R}^2 \to \mathbb{R}^2$ (e.g. velocity, flow maps) is $m$-fold symmetric if $v(R_{2\pi/m}x) = R_{2\pi/m}v(x)$.

- Given a vector $f = (f_1, f_2)$, we denote the counterclockwise $90^\circ$ rotation by $f^\perp = (-f_2, f_1)$. Similarly, $\nabla^\perp \phi = (-\partial_2 \phi, \partial_1 \phi)$ for a scalar function $\phi : \mathbb{R}^2 \to \mathbb{R}$.

- The classical Hölder spaces are defined as follows: for $0 < \alpha \leq 1$,

$$
\|f\|_{C^\alpha(U)} = \|f\|_{L^\infty(U)} + \|f\|_{C^\alpha(U)}
= \sup_{x \in U} |f(x)| + \sup_{x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|^\alpha}.
$$
We shall often use the “inf norm”, defined by
\[ \|f\|_{\text{inf}(F)} = \inf_{x \in F} |f(x)|. \]

- We say that \( \omega \) is a patch if it is a characteristic function on some (open) set \( \Omega \in \mathbb{R}^2 \). It will be assumed that the boundary \( \partial \Omega \) is either a Jordan curve, or a union of a few Jordan curves intersecting only at the origin. We often identify the function \( \omega \) with the set \( \Omega \).

- We denote the Biot-Savart kernel as
  \[ K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \]
  and \( \nabla K \) as its gradient. Convolution against \( \nabla K \) is defined in the sense of principal value integration.

- For functions depending on time and space, we write \( f(t, \cdot) = f_t(\cdot) \). The latter notation is not to be confused with the partial derivative in time, which we always denote as \( \partial_t \).

- The flow \( \Phi \) is defined as a map \([0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2 \). For each fixed \( t \geq 0 \), \( \Phi(t, \cdot) = \Phi_t \) is a homeomorphism of \( \mathbb{R}^2 \) whose inverse is denoted by \( \Phi_t^{-1} \).

- A point in \( \mathbb{R}^2 \) is denoted by \( x = (x_1, x_2) \) or by \( y = (y_1, y_2) \). Often we slightly abuse notation and consider polar coordinates \((r, \theta)\), where \( r = |x| \) and \( \theta = \arctan(x_2/x_1) \).

- Given \( x \in \mathbb{R}^2 \) and \( r > 0 \), we define \( B_x(r) = \{y \in \mathbb{R}^2 : |x - y| < r\} \).

- Given two angles \( 0 \leq \theta_1 < \theta_2 < 2\pi \), we define the sector
  \[ S_{\theta_1, \theta_2} = \{(r, \theta) : \theta_1 < \theta < \theta_2\}. \]

As it is usual, we use letters \( C, c, \cdots \) to denote various positive absolute constants whose values may vary from a line to another (and even within a line). Moreover, we write \( A \lesssim B \) if there is an absolute constant \( C > 0 \) such that \( A \leq CB \). We also use \( A \sim B \) when we have \( A \lesssim B \) and \( A \gtrsim B \). We fix some value of \( 0 < \alpha < 1 \) throughout the paper, and the constants \( C, c \) may depend on \( \alpha \) as well.

### 2.2 A few explicit computations

In this subsection, we perform some simple computations which already illustrate key issues related to the vortex patch problem.

#### Case of the disc

Consider the patch supported on the unit disc. Then, using the Biot-Savart law (it is much easier to use its radial version), one can explicitly compute that the corresponding velocity is given by

\[ u(x) = \begin{cases} \frac{1}{2} x^\perp & \text{if } |x| \leq 1, \\ \frac{1}{2} \frac{x^\perp}{|x|^2} & \text{if } |x| \geq 1. \end{cases} \]

Note that in the regions \( \{ x : |x| \leq 1 \} \) and \( \{ x : |x| \geq 1 \} \), the velocity is \( C^\infty \)-smooth, respectively.

This simple computation can be used as a basis for the following general result mentioned earlier in the introduction: for a patch \( U \) bounded by a \( C^{k,\alpha} \)-curve, the velocity is a \( C^{k,\alpha} \) function inside the patch. The point is that there exists a \( C^{k,\alpha} \)-diffeomorphism of the plane \( \Psi \) which maps the unit disc onto \( U \). Then after a change of variables, each component of \( \nabla^k \nabla^k \Delta^{-1} \chi_U \) has an explicit integral representation involving derivatives of \( \Psi \) in the unit disc. Working directly with this expression, the Hölder estimate can be achieved. We leave the details of this (tedious) computation to the interested reader.
Bahouri-Chemin solution

On the torus $\mathbb{T}^2 = [-1, 1)^2$, $\omega(x) = \text{sgn}(x_1)\text{sgn}(x_2)$ defines a stationary patch solution, which is often called the Bahouri-Chemin solution after the work [9]. Stationarity follows since the particle trajectories cannot cross the axes by odd symmetry. Here we consider a configuration in $\mathbb{R}^2$ which is odd with respect to both variables $x_1, x_2$ and given by $\text{sgn}(x_1)\text{sgn}(x_2)$ near the origin, say on the unit disc for concreteness. It is well-known that the associated velocity field is only log-Lipschitz at the origin. For a computation based on Fourier series, see [33]. Here we present a simpler way to see it by working in polar coordinates. To begin with, observe that the Bahouri-Chemin solution can be locally written as

$$
\omega(x) = \sum_{k \geq 0} \frac{\sin(2(2k + 1)\theta)}{2k + 1}, \quad |x| \leq 1
$$

where $\tan(\theta) = x_2/x_1$ and therefore to compute $u = \nabla \perp \Delta^{-1}\omega$, it suffices to invert the Laplace operator for functions $\sin(m\theta)$. Note that

$$
\Delta(r^2 \sin(m\theta)) = \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta}^2\right)(r^2 \sin(m\theta)) = (4 - m^2) \sin(m\theta).
$$

From this computation, it can be argued that for $m \geq 3$,

$$
\Delta^{-1}(\sin(m\theta)) = -\frac{1}{m^2 - 4} r^2 \sin(m\theta).
$$

(Strictly speaking $\sin(m\theta)$ on both sides needs to be appropriately truncated for $r \geq 1$.) Therefore, from straightforward estimates one can show that

$$
\nabla^2 \Delta^{-1}\left(\sum_{k \geq 1} \frac{\sin(2(2k + 1)\theta)}{2k + 1}\right) = -\nabla^2 \left(r^2 \sum_{k \geq 1} \frac{1}{4(2k + 1)^2 - 4} \frac{\sin(2(2k + 1)\theta)}{2k + 1}\right)
$$

is summable; that is, the corresponding velocity field is Lipschitz continuous. On the other hand, for $m = 2$ we have instead

$$
\Delta\left(r^2 \ln \frac{1}{r} \sin(2\theta)\right) = -4 \sin(2\theta),
$$

so that

$$
\partial_{x_1} \partial_{x_2} \Delta^{-1}(\sin(2\theta)) = -\frac{1}{4} \ln \frac{1}{|x|} + \text{bounded}.
$$

We conclude that $\partial_{x_1} u_1 = -\partial_{x_2} u_2$ are divergent logarithmically at the origin. In particular on the separatrices $\{x_1 = 0\}$ and $\{x_2 = 0\}$, this stationary velocity produces double exponential in time contraction and expansion, respectively.

Patches supported on sectors

Generalizing the previous computation, we perform a similar calculation for patches that are locally supported on a union of sectors, which are the main object of study in this work. Explicit computations have appeared in several places (e.g. [15, 20]) but again we provide a shortcut using polar coordinates. The arguments here which might seem formal can be justified either using directly the Biot-Savart kernel or arguments based on the uniqueness of $\Delta^{-1}$. 

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We consider vorticity which takes the form \( \omega(x) = h(\theta) \) for \( |x| \leq 1 \), with some bounded function \( h \) of the angle. Taking in particular \( h \) to be the characteristic function on a union of intervals in \([0, 2\pi)\), we obtain a vortex patch supported on a union of sectors meeting at the origin. The computations below go through for any function \( h \). In view of the above, we know that the inverse Laplacian of a bounded function of \( \theta \) may involve a logarithm. This suggests us to prepare an ansatz

\[
\Delta^{-1} h = r^2 H(\theta) + r^2 \ln \frac{1}{r} G(\theta),
\]

where \( H \) and \( G \) are functions to be determined. Here, we are neglecting possible constant and linear terms on the right hand side (this can be justified for instance when \( h \) has some symmetries), which does not affect the velocity gradient in any essential way. Then,

\[
h = \Delta \left( r^2 H(\theta) + r^2 \ln \frac{1}{r} G(\theta) \right) = 4H + H'' + (4G + G'') \ln \frac{1}{r} - 4G.
\]

This forces \( 4G + G'' = 0 \), or \( G = c \cos(2\theta) + s \sin(2\theta) \) for some constants \( c, s \), which are determined by multiplying both sides of the above equation by \( \cos(2\theta) \) and \( \sin(2\theta) \), respectively and integrating on \([0, 2\pi)\):

\[
\left( \begin{array}{c} c \\ s \end{array} \right) = -\frac{1}{4\pi} \int_0^{2\pi} h(\theta) \left( \begin{array}{c} \cos(2\theta) \\ \sin(2\theta) \end{array} \right) d\theta.
\]

Then \( H \) is determined uniquely by

\[
(I + \partial_{\theta\theta})^{-1}(h + 4G) = H,
\]

which is well-defined since \( h + 4G \) is orthogonal to \( \cos(2\theta) \) and \( \sin(2\theta) \) (see [42] for a proof). An explicit kernel expression for \( H \) have been derived in [42]; see also Section 2. Note that the velocity gradient coming from \( r^2 H \) is bounded. Therefore, we conclude that

\[
\nabla u(x) = \left( \begin{array}{c} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} \end{array} \right) = \ln \frac{1}{|x|} \left( \begin{array}{cc} -2s & 2c \\ 2c & 2s \end{array} \right) + \text{bounded.} \tag{2.1}
\]

Interestingly, the non-Lipschitz part of the velocity is simply a constant multiple of the log-linear function, where the constant is determined only by the second-order Fourier coefficients of the vorticity profile. In particular, if \( h \) has zero second-order coefficients, then the corresponding velocity gradient is bounded! This happens when \( h \) is \( m \)-fold symmetric with some \( m \geq 3 \), but this is certainly not a necessary condition; for example one can take \( h = \chi_{[-\theta_0, \theta_0]} + \chi_{[\pi/2 - \theta_0, \pi/2 + \theta_0]} \) (see [40] for a necessary and sufficient condition).

We also compute the eigenvectors for the gradient matrix, which correspond to the separatrices generated by the flow:

\[
\left( \begin{array}{c} \frac{c}{s + \sqrt{s^2 + c^2}} \\ \frac{s}{s + \sqrt{s^2 + c^2}} \end{array} \right), \quad \left( \begin{array}{c} s + \sqrt{s^2 + c^2} \\ c \end{array} \right),
\]

which are orthogonal to each other.

Now, let us take the concrete case of \( h(\theta) = \chi_{[-\theta_0, \theta_0]} \) for some \( 0 < \theta_0 < \pi/2 \). Then, \( s = 0 \) and \( c = \frac{1}{2\pi} \sin(2\theta_0) \). The separatrices are always given by the diagonals \( \{ x_1 = x_2 \} \) and \( \{ x_1 = -x_2 \} \), independent of \( \theta \). On the other hand, if we 2-fold symmetrize \( h \), that is, take \( \chi_{[-\theta_0, \theta_0]} + \chi_{[\pi - \theta_0, \pi + \theta_0]} \), then \( s = 0 \) again and \( c \) is simply multiplied by 2. This shows that the effect of 2-fold symmetrization is just rescaling time by 2, modulo the effect of the bounded term, which is negligible for \( |t|, |x| \ll 1 \).

Note that for \( 0 < \theta_0 < \pi/2 \), the log-Lipschitz part of the velocity which is normal to the patch boundary vanishes only for \( \theta_0 = \pi/4 \), which corresponds to the 90°-corner. It gives some possibility for a patch with 90°-corners to retain its shape, which happens explicitly for the Bahouri-Chemin solution and conjecturally happens for certain \( V \)-states ([72] [88]).
formally by taking the following ansatz for the stream function:

$$h$$

for some region concreteness we take a patch $$\Omega$$ whose intersection with a small square $$\Omega \cap \Omega$$ immediately become a logarithmic spiral. If one consider the passive transport of the patch supported on a corner (whose angle is between 0 and $$\pi/2$$) by the flow associated with the initial velocity, then the corner immediately becomes a cusp tangent to one of the separatrices. On the other hand, one may transport the corner with the time-independent velocity $$v(x) = x^+ \log |x|$$, and this would cause the corner to immediately become a logarithmic spiral.

An elementary but important fact regarding cusps is that the associated velocity is smooth as long as two pieces of the boundary curves meeting at the cusping point are smooth. Indeed, instantaneous cusping or (logarithmic) spiraling of a corner is possible under the flow by a log-Lipschitz velocity. If one consider the passive transport of the patch supported on a corner with the time-independent velocity $$v$$ immediately becomes a cusp tangent to one of the separatrices. On the other hand, one may transport the corner with the time-independent velocity $$v(x) = x^+ \log |x|$$, and this would cause the corner to immediately become a logarithmic spiral.

Lastly, we consider vortex patches supported on cusps and logarithmic spirals. For us, a cusp (naively) refers to the region bounded by two $$C^1$$ curves which meet at a point with the same tangent vectors. By a logarithmic spiral, we mean a spiral where the distances between the turns are related by a geometric progression. These objects are not only of significant interest by themselves, but they are particularly relevant for our study as main candidates which would describe the evolution of a single corner. Indeed, instantaneous cusping or (logarithmic) spiraling of a corner is possible under the flow by a log-Lipschitz velocity. If one consider the passive transport of the patch supported on a corner (whose angle is between 0 and $$\pi/2$$) by the flow associated with the initial velocity, then the corner immediately becomes a cusp tangent to one of the separatrices. On the other hand, one may transport the corner with the time-independent velocity $$v(x) = x^+ \log |x|$$, and this would cause the corner to immediately become a logarithmic spiral.

Cusps and logarithmic spirals

By a logarithmic spiral, we mean a spiral where the distances between the turns are related by a geometric progression. These objects are not only of significant interest by themselves, but they are particularly relevant for our study as main candidates which would describe the evolution of a single corner. Indeed, instantaneous cusping or (logarithmic) spiraling of a corner is possible under the flow by a log-Lipschitz velocity. If one consider the passive transport of the patch supported on a corner (whose angle is between 0 and $$\pi/2$$) by the flow associated with the initial velocity, then the corner immediately becomes a cusp tangent to one of the separatrices. On the other hand, one may transport the corner with the time-independent velocity $$v(x) = x^+ \log |x|$$, and this would cause the corner to immediately become a logarithmic spiral.

An elementary but important fact regarding cusps is that the associated velocity is smooth as long as two pieces of the boundary curves meeting at the cusping point are smooth. To see this, for concreteness we take a patch $$\Omega$$ whose intersection with a small square $$\Omega \cap \Omega$$ is given by the region

$$\{(x_1, x_2) : 0 < x_1 < \delta, g(x_1) < x_2 < f(x_1)\}$$

with some $$C^{1,\alpha}$$ functions $$g, f$$ on $$[0, \delta]$$ satisfying $$g'(0) = f'(0) = 0$$. Then, locally near the origin, $$\Omega \cap (-\delta, \delta)^2 = A \setminus B$$, where

$$A = \{(x_1, x_2) : -\delta < x_1 \leq 0, x_2 < 0\} \cup \{(x_1, x_2) : 0 < x_1 \leq \delta, x_2 < f(x_1)\}$$

and

$$B = \{(x_1, x_2) : -\delta < x_1 \leq 0, x_2 \leq 0\} \cup \{(x_1, x_2) : 0 < x_1 \leq \delta, x_2 \leq g(x_1)\}.$$  

Then, the domains $$A$$ and $$B$$ have $$C^{1,\alpha}$$ boundaries on $$[-\delta, \delta]^2$$, so that the velocities associated with $$\chi_A$$ and $$\chi_B$$ are $$C^{1,\alpha}$$ near the origin in the interior of their respective domains. Hence the velocity of $$\chi_\Omega$$ is $$C^{1,\alpha}$$ in the interior of $$\Omega \cap [-\delta/2, \delta/2]^2$$, uniformly up to the boundary.

Similarly, it follows that the velocity $$\nabla^1 \Delta^{-1} \chi_U$$ is $$C^{k,\alpha}(\bar{U})$$ near the origin if $$g, f \in C^{k,\alpha}$$ for some $$k \geq 1$$ and $$0 < \alpha < 1$$. If the preceding argument is somehow not convincing, one can compute explicitly the Biot-Savart kernel for $$\nabla^1 \Delta^{-1} \chi_U$$ by first integrating out the second coordinate variable and check directly that the resulting function is $$C^{1,\alpha}$$. Indeed, we carry out such a computation (in a more complicated setting of a patch consisting of a corner with cusps attached to its sides) in Section 4 in the course of proving our main well-posedness result.

In the meanwhile, this frozen-time argument already shows that the $$C^{1,\alpha}$$-cusps should be (at least) locally well-posed: in particular, there is no hope of starting from a corner and immediately becoming a $$C^{1,\alpha}$$-cusp. Actually such cusps are globally well-posed, and we sketch the argument below in Section 2. Therefore, while a cusp may be considered as a singularity, as long as the boundary curves are smooth, the corresponding vortex patch will not lose any regularity in time.

We now turn to spirals. For simplicity we consider (locally) self-similar spirals with some bounded profile $$h$$: using polar coordinates,

$$\omega(r, \theta) = h(c \ln \frac{1}{r} + \theta), \quad r \leq 1$$

for some $$h \in L^\infty([0, 2\pi])$$ and $$c > 0$$. Let us compute the velocity associated with it. We proceed formally by taking the following ansatz for the stream function:

$$\Psi = r^2 H(c \ln \frac{1}{r} + \theta).$$
Then the relation \( \Delta \Psi = \omega \) gives that
\[
4H - 4cH' + (1 + c^2)H'' = h.
\]

It is not difficult to show that there exists a unique solution \( H \in L^2([0, 2\pi]) \) (e.g. using Fourier series) and it actually belongs to \( W^{2,\infty}([0, 2\pi]) \). Using \( u_r := u \cdot e_r = -\frac{1}{r} \partial_r \Psi \) and \( u_\theta = u \cdot e_\theta = -\partial_\theta \Psi \), we deduce that
\[
u_r(r, \theta) = \frac{rH'}{c} (c \ln \frac{1}{r} + \theta), \quad u_\theta(r, \theta) = \frac{2rH}{c} - crH'(c \ln \frac{1}{r} + \theta)
\]

near \( r = 0 \). Taking another \( \partial_r \) and \( r^{-1} \partial_\theta \), it follows in particular that \( u \) is indeed Lipschitz continuous.

Moreover, using the above ansatz for \( \Psi \), we may write down a closed evolution equation in terms of \( h \):
\[
\partial_t h + 2H \partial_\theta h = 0, \quad 4H - 4cH' + (1 + c^2)H'' = h.
\]

The conservation of \( h(t) \in L^\infty \) ensures that this is globally well-posed. Note that in stark contrast to the radially homogeneous case, whose evolution equation for \( h \) requires rotational symmetry, no such assumption is needed for the spiral case. While showing global well-posedness for the logarithmic spirals rigorously may take some work (we achieve this in the presence of rotational symmetry in Section \( \text{3} \)), this is very plausible as the above ansatz for the stream function is correct modulo perturbation which is \( C^\infty \) near the origin. In the end, it suggests that a corner cannot become a logarithmic spiral, and even if it spirals, the turns should be sparser than those of a logarithmic spiral.

### 2.3 Smooth vortex patches: approach by Bertozzi and Constantin

In this subsection, let us provide a brief outline of the elegant proof of Bertozzi and Constantin [14] on global regularity of smooth vortex patches. We restrict ourselves to domains \( \Omega \) (bounded open set in \( \mathbb{R}^2 \)) which has a level set \( \phi : \mathbb{R}^2 \to \mathbb{R} \) such that:

- We have \( \phi(x) > 0 \) if and only if \( x \in \Omega \) (hence \( \phi \) vanishes precisely on \( \partial \Omega \)).
- The tangent vector field of \( \phi \) satisfies \( \nabla^\perp \phi \in C^\alpha(\mathbb{R}^2) \).
- The function \( \phi \) is non-degenerate near \( \partial \Omega \), i.e., \( \| \nabla^\perp \phi \|_{\inf(\partial \Omega)} := \inf_{x \in \partial \Omega} |\nabla^\perp \phi| \geq c > 0 \).

Then we say that the patch \( \Omega \) is \( C^{1,\alpha} \)-regular, or a \( C^{1,\alpha} \)-patch. Given such a \( \phi \), we associate the following characteristic quantity:
\[
\Gamma = \left( \frac{\| \nabla^\perp \phi \|_{C^\alpha(\mathbb{R}^2)}}{\| \nabla^\perp \phi \|_{\inf(\partial \Omega)}} \right)^{1/\alpha},
\]
which quantifies the \( C^{1,\alpha} \)-regularity of \( \Omega \). Note that it has units of inverse length, so that \( \Gamma^{-1} \) provides a \( C^{1,\alpha} \)-characteristic length scale for \( \Omega \). An alternative way of defining \( C^{1,\alpha} \) patches is to require that, for any point \( x \in \partial \Omega \), there exists a ball \( B_x(r) \) with some radius \( r > 0 \) uniform over \( x \) such that the intersection \( B_x(r) \cap \partial \Omega \) is given by the graph of a \( C^{1,\alpha} \) function, after rotating the patch if necessary. Indeed, given \( \Gamma \), one may take \( r \) to be \( 1/(10\Gamma) \) and vice versa; given \( r > 0 \) for each \( x \in \partial \Omega \), one may construct a level set function \( \phi \).
Taking the initial vorticity to be the characteristic function \( \omega_0 = \chi_\Omega \), we may denote its unique solution by \( \chi_{\Omega t} \). Since the vorticity is simply being transported by the flow, once we define the evolution of \( \phi \) via
\[
\partial_t \phi + (u \cdot \nabla)\phi = 0 ,
\] (2.2)
then it follows that
\[
\phi(t,x) > 0 \quad \text{if and only if} \quad x \in \Omega_t .
\]
To show that \( \Omega_t \) stays as a \( C^1,\alpha \)-patch for all times, it suffices to establish an a priori bound on \( \Gamma_t \).

In Bertozzi-Constantin [14], the authors have provided a proof that \( \Gamma_t \) remains bounded for all time, based on the following two “frozen-time” lemmas:

**Lemma** (\( L^\infty \)-bound on \( \nabla u \)). Consider the velocity \( u(x) = K * \chi_\Omega(x) \), where \( \Omega \) is a \( C^1,\alpha \)-patch with a level set \( \phi \). Then, we have a bound
\[
\| \nabla u \|_{L^\infty(\mathbb{R}^2)} \leq C \left( 1 + \log \left( 1 + \frac{\| \nabla \perp \phi \|_{C^\alpha(\mathbb{R}^2)}}{\| \nabla \perp \phi \|_{\text{inf}(\partial \Omega)}} \right) \right) ,
\] (2.3)

**Lemma** (Directional \( C^\alpha \)-bound on \( \nabla u \)). We have a pointwise identity
\[
\nabla u \nabla \perp \phi(x) = \frac{1}{2\pi} \int_{\Omega} \nabla K(x-y) \left( \nabla \perp \phi(x) - \nabla \perp \phi(y) \right) dy ,
\] (2.4)
and in particular, this gives a bound
\[
\| \nabla u \nabla \perp \phi \|_{C^\alpha(\mathbb{R}^2)} \leq C \| \nabla u \|_{L^\infty(\mathbb{R}^2)} \| \nabla \perp \phi \|_{C^\alpha(\mathbb{R}^2)} .
\] (2.5)

The point of (2.5) is that we do not need to take the \( C^\alpha \)-norm of the velocity gradient.

Given these lemmas, one can finish the global well-posedness proof with a simple Gronwall estimate (details of this argument can be found in [14]). We differentiate (2.2) to obtain
\[
\partial_t \nabla \perp \phi + (u \cdot \nabla) \nabla \perp \phi = \nabla u \nabla \perp \phi .
\]

Working on the Lagrangian coordinates, and using the bound (2.5) and then the logarithmic estimate (2.3) allows one to close the estimates in terms of \( \| \nabla \perp \phi \|_{C^\alpha} \) to show the bound
\[
\| \nabla \perp \phi(t) \|_{C^\alpha(\mathbb{R}^2)} \leq C \exp(C \exp(Ct))
\]
as well as
\[
\| \nabla \perp \phi(t) \|_{\text{inf}(\partial \Omega)} \geq c \exp(-ct)
\]
with positive constants depending only on the initial data \( \nabla \perp \phi_0 \) (and \( 0 < \alpha < 1 \)).

We would like to point out that, although it was not necessary in the above global well-posedness argument, the velocity gradient is indeed uniformly \( C^\alpha \) inside the patch, up to the boundary. There are a number of ways to obtain this piece of information. One approach, due to Serfati [80], is that from the directional Hölder regularity \( \nabla \perp \phi \cdot \nabla u \in C^\alpha \) that we already have, one can "invert" this using \( \nabla \cdot u = 0 \) and \( \nabla \times u = 1 \) (inside the patch) to recover \( u \in C^\alpha \). We exploited this idea in our proof of local well posedness (see Lemma A.6). Alternatively, Friedman and Velazquez [46] have shown, directly working with the Biot-Savart kernel, the following estimate:
**Lemma** (Friedman and Velazquez [46]). Assume that a $C^{1,\alpha}$-patch $\Omega$ is tangent to the horizontal axis at the origin, and that near the origin, $\partial \Omega$ is described as the graph of a $C^{1,\alpha}$-function:

$$\partial \Omega \cap [-\delta, \delta]^2 = \{(x_1, x_2) : x_2 = f(x_1), \quad f \in C^\alpha([-\delta, \delta]), \quad \sup_{[-\delta, \delta]} |f'| \leq 1\}.$$

Then, the velocity $u = K \ast \chi_\Omega$ is $C^{1,\alpha}$ along this portion of the boundary:

$$\|\nabla u(x_1, f(x_1))\|_{C^\alpha([-\delta/10, \delta/10])} \leq C \|f\|_{C^{1,\alpha}[-\delta, \delta]} \log \left(1 + \frac{1}{\delta}\right).$$

With elliptic regularity, the above lemma immediately implies that for $C^{1,\alpha}$-patches, the velocity gradient is uniformly $C^\alpha$ up to the boundary.

The above lemma of Friedman and Velazquez actually gives $C^{1,\alpha}$-regularity for velocity coming from a $C^{1,\alpha}$-cusp: consider the domain $\Omega$ satisfying

$$\Omega \cap [-\delta, \delta]^2 = \{(x_1, x_2) \subset [0, \delta] \times [-\delta, \delta] : g(x_1) < x_2 < f(x_1)\}$$

where $g < f$ are $C^{1,\alpha}[0, \delta]$-functions with $g(0) = f(0) = 0$ and $g'(0) = f'(0) = 0$. Then, applying the lemma first with a $C^{1,\alpha}$ domain obtained by taking $(x_1, f(x_1))$ and the semi-axis $\{(x_1, 0) : x_1 \leq 0\}$ as a portion of its boundary, and then using the lemma another time with a domain using $(x_1, g(x_1))$ instead of $f$ establishes that $\nabla u$ is uniformly $C^\alpha$ in $[0, \delta/10] \times [-\delta, \delta] \cap \Omega$. We shall use this bound a few times in our arguments.

### 2.4 Euler equations in critical spaces under symmetry

In this subsection, let us provide a brief review of some of the results from [12]. The contents of Sections 3 and 4 may be viewed as generalizations of the results below to the class of vortex patch solutions.

**Well-posedness of the 2D Euler equations in critical spaces**

The following result shows that in the $L^1 \cap L^\infty(\mathbb{R}^2)$-theory of Yudovich, one can actually drop the $L^1$ assumption under $m$-fold rotational symmetry for some $m \geq 3$.

**Theorem.** Assume that $\omega_0 \in L^\infty(\mathbb{R}^2)$ and $m$-fold symmetric for some $m \geq 3$. Then, there is a unique solution to the 2D Euler equation $\omega \in L^\infty((0, \infty); L^\infty(\mathbb{R}^2))$ and $m$-fold symmetric. Here, $u$ is the unique solution to the system

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0.$$

under the assumptions $|u(x)| \leq C|x|$ and $m$-fold symmetric. It is well-defined pointwise by

$$u(t, x) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{|y| \leq R} \frac{(x - y) \perp \omega(t, y)}{|x - y|^2} dy.$$

Under the assumption of the above theorem, the velocity is only log-Lipschitz, just as in the case of Yudovich theory, but now one has the following scale-invariant log-Lipschitz estimate, which is a key step in the proof.

**Lemma.** Under the $m$-fold symmetry assumption for $m \geq 3$, we have

$$|u(x) - u(x')| \leq C\|\omega\|_{L^\infty} |x - x'| \log \left(\frac{c \max(|x|, |x'|)}{|x - x'|}\right).$$
In particular, under symmetry, vortex patches can have infinite mass and the evolution is still well-defined. This allows us to treat infinite patches in the setup of Sections 3 and 1 (assuming that the boundary regularity of the initial patch as \( |x| \to +\infty \) satisfies suitable bounds), but we shall not pursue this generalization.

It turns out that under the symmetry assumption, one can prove higher regularity in the angular direction. A model situation is when the vorticity takes the form \( \omega = h(\theta) + \bar{\omega} \), where \( h(\cdot) : S^1 \to \mathbb{R} \) defines a radially homogeneous function on \( \mathbb{R}^2 \), and \( \bar{\omega} \) is smooth on \( \mathbb{R}^2 \). Then, one sees that while \( \omega \) cannot be better than \( L^\infty(\mathbb{R}^2) \) in the \( C^{k,\alpha} \)-scale (unless \( h \) is trivial), but one can take as many angular derivatives \( \partial_\theta \) as \( h \) allows. In this setup, one would like to say that the Euler dynamics propagates this regularity. To this end, we have introduced the scale-invariant spaces \( \dot{C}^\alpha(\mathbb{R}^2) \): for any \( 0 < \alpha \leq 1 \), consider the norm
\[
\|f\|_{\dot{C}^\alpha(\mathbb{R}^2)} := \|f\|_{L^\infty(\mathbb{R}^2)} + \| |x|^\alpha f(x)\|_{\dot{C}^\alpha(\mathbb{R}^2)} = \sup_x |f(x)| + \sup_{x \neq x'} \frac{| |x|^\alpha f(x) - |x'|^\alpha f(x')|}{|x - x'|^\alpha}.
\]
Note that if \( f \) is a function of the angle, \( f(x) = h(\theta) \), then
\[
\|f\|_{\dot{C}^\alpha(\mathbb{R}^2)} \approx \|h(\theta)\|_{C^\alpha(S^1)}.
\]

**Theorem.** Assume that \( \omega_0 \in \dot{C}^\alpha(\mathbb{R}^2) \) is \( m \)-fold symmetric for some \( m \geq 3 \). Then, the unique solution in \( L^\infty([0, \infty); L^\infty(\mathbb{R}^2)) \) actually belongs to \( L^\infty_{\text{loc}} \dot{C}^\alpha \) with a bound
\[
\|\omega(t)\|_{\dot{C}^\alpha} \leq C \exp(c_1 \exp(c_2 t)),
\]
with constants depending only on \( 0 < \alpha \leq 1 \) and the initial data.

A key ingredient is the following scale-invariant bounds on the velocity gradient:

**Lemma.** The velocity gradient satisfies
\[
\|\nabla u\|_{L^\infty} \leq C_\alpha \|\omega\|_{L^\infty} \left( 1 + \log \left( 1 + c_\alpha \frac{\|\omega\|_{\dot{C}^\alpha}}{\|\omega\|_{L^\infty}} \right) \right)
\]
and
\[
\|\nabla u\|_{\dot{C}^\alpha} \leq C_\alpha \|\omega\|_{\dot{C}^\alpha}.
\]

It is important to keep in mind the following Bahouri-Chemin [1] counterexample, which is only 2-fold rotationally symmetric. Take \( \omega(x_1, x_2) = \text{sign}(x_1)\text{sign}(x_2)\chi_R \), where \( \chi_R \) is some smooth radial cutoff. This belongs to \( L^\infty \) but near the origin, it can be computed that \( u(x_1, 0) \approx Cx_1 \log x_1 \), so that in particular the estimate \( |u(x)| \leq C|x| \) fails. Moreover, even if we smooth it out in the angular direction, for instance by putting \( \omega(x_1, x_2) = \cos(2\theta)\chi_R \), then \( \omega \) belongs to \( \dot{C}^\alpha \) but still one has \( u(x_1, 0) \approx C'x_1 \log x_1 \).

**The 1D system for radially homogeneous vorticity**

The \( L^\infty \)-theorem described above gives rise to a class of (infinite) vortex patch solutions to the 2D Euler equation, by taking vorticity which is radially homogeneous.

Indeed, when the initial data is of the form \( \omega_0 = h_0(\theta) \) with \( h_0 \in L^\infty(S^1) \), then the unique solution must stay radially homogeneous for all time, and therefore the dynamics reduces to a one-dimensional equation on \( h(t) \). We have derived this evolution equation in [42, Section 3]:

\[\text{\footnotesize{Here, we are using } \approx \text{ to say that both sides coincide up to a smooth function vanishing at the origin.}}\]
**Theorem.** Consider the following transport equation on \( S^1 = [-\pi, \pi] \)

\[
\partial_t h + 2H \partial_\theta h = 0,
\]

where the initial data \( h_0 \) is \( m \)-fold rotationally symmetric on \( S^1 \) for some \( m \geq 3 \). Here, \( H \) is the unique solution of

\[
h = 4H + H'' - \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\theta) \exp(\pm 2i\theta) d\theta = 0.
\]

Alternatively,

\[
H(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_{S^1}(\theta - \theta') h(\theta') d\theta',
\]

with

\[
K_{S^1}(\theta) := \frac{\pi}{2} \sin(2\theta) \theta \bigg/ |\theta| - \frac{1}{2} \sin(2\theta) \theta - \frac{1}{8} \cos(2\theta).
\]

The system is globally well-posed for either \( h_0 \in L^\infty \) or \( h_0 \in C^\alpha \) for \( 0 < \alpha \leq 1 \).

By taking

\[
\omega(t,x) = h(t,\theta),
\]

\[
u(t,x) = 2H(t,\theta) \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} - \partial_\theta H(t,\theta) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]

we obtain the unique solution to the 2D Euler equation with initial data \( \omega_0(x) = h_0(\theta) \).

Indeed, one may check with direct computations that the velocity defined in the above formula satisfies \( \nabla \times u = 4H + H'' = \omega \) and \( \nabla \cdot u = 0 \), which characterizes the velocity.

The kernel \( K_{S^1} \) is simply the Biot-Savart kernel, restricted to the case of radially homogeneous vorticity. Since the vorticity has \( m \)-fold symmetry, it is more efficient to symmetrize the kernel as well: we have

\[
K_1^{(4)}(\theta) := \frac{1}{4} \sum_{j=0}^{3} K_{S^1}(\theta + j\pi/2) = \frac{\pi}{8} |\sin(2\theta)|.
\]

In general,

\[
K_1^{(m)}(\theta) := \frac{1}{m} \sum_{j=0}^{m-1} K_{S^1}(\theta + 2j\pi/m) = c_1^{(m)} |\sin(m\theta/2)| + c_2^{(m)}
\]

for some constants \( c_1 > 0 \) and \( c_2 \). We shall use these expressions in Subsection 4.4.

In the special case when \( h_0 \) is the \( (m \)-fold symmetric) characteristic function of a disjoint union of intervals in \( S^1 \), we obtain a vortex patch solution on the plane, which is a union of sectors and whose boundary is a union of straight lines passing through the origin. The dynamics of these lines determine the evolution of the patch, and it takes the form of a system of ODEs, which we derive and briefly study in Subsection 4.4.

Lastly, consider the situation where the initial vorticity is the sum of a radially homogeneous function and a smooth function vanishing at the origin. Then, the next result says that near the origin, the dynamics is determined by the 1D evolution of the radially homogeneous part.
Theorem. Assume that the initial vorticity \( \omega_0 \in \dot{C}^\alpha(\mathbb{R}^2) \) is \( m \)-fold symmetric for some \( m \geq 3 \) and satisfies
\[
\omega_0(x) = h_0(\theta) + \tilde{\omega}_0(x),
\]
where \( h_0 \in \dot{C}^\alpha(S^1) \) and \( \tilde{\omega}_0 \in C^{1,\alpha}(\mathbb{R}^2) \) with \( \tilde{\omega}_0(0) = 0 \). Then, the solution satisfies
\[
\omega(t, x) = h(t, \theta) + \tilde{\omega}(t, x)
\]
where \( h(t, \cdot) \) is the unique solution to the 1D equation with initial data \( h_0 \), and \( \tilde{\omega}(t, \cdot) \in C^{1,\alpha}(\mathbb{R}^2) \) with \( \tilde{\omega}(t, 0) = 0 \).

In particular, \( \sup_{t \in [0,T]} |\tilde{\omega}(t, x)| \leq C(T)|x|^{1+\alpha} \) for some constant \( C(T) \) depending on \( T \) and initial data, and therefore, it is negligible relative to \( h(t, \theta) \) in the regime \( |x| \ll 1 \) (unless \( h_0 \) were trivial to begin with).

3 Global well-posedness for symmetric patches in an intermediate space

In this section, we show that if a vortex patch admits a level set whose gradient is, roughly speaking, \( C^\alpha \) in the angle and non-degenerate, then the corresponding Yudovich solution retains this property for all time. As a consequence, we shall have that the velocity, and hence the flow map and its inverse, are Lipschitz functions in space for all finite. In this setup, it is necessary to impose that the patch is \( m \)-fold rotationally symmetric for some \( m \geq 3 \).

Definition 3.1. Let us say that a domain \( \Omega \) is a \( \dot{C}^{1,\alpha} \)-patch, if it admits a level set \( \phi : \mathbb{R}^2 \to \mathbb{R} \) such that:

- We have \( \phi(x) > 0 \) if and only if \( x \in \Omega \).
- The tangent vector field of \( \phi \) satisfies \( \nabla \perp \phi \in \dot{C}^\alpha(\mathbb{R}^2) \) (In particular \( \phi \) is Lipschitz).
- The function \( \phi \) is non-degenerate near \( \partial \Omega \), i.e., \( \|\nabla \perp \phi\|_{\text{inf}(\partial \Omega)} := \inf_{x \in \partial \Omega} |\nabla \perp \phi| \geq c > 0 \).

We are ready to state our main result of this section.

Theorem 1. Assume that the initial patch \( \Omega_0 \) is \( m \)-fold symmetric for some \( m \geq 3 \) and admits a level set \( \phi_0 \) described in Definition 3.1. Then, the Yudovich solution \( \Omega_t \) continues to have this property; more specifically, by defining \( \phi(t) \) as the solution of (2.2), we have a global-in-time bounds
\[
\|\nabla \perp \phi(t)\|_{\dot{C}^{\alpha}(\mathbb{R}^2)} \leq C \exp(C \exp(\alpha t)),
\]
(3.1)
\[
\|\nabla \perp \phi(t)\|_{\text{inf}(\partial \Omega_t)} \geq c \exp(-ct),
\]
(3.2)
and
\[
\|\nabla u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C \exp(Ct),
\]
(3.3)
with constants \( C, c > 0 \) depending only on \( \nabla \perp \phi_0 \) and \( 0 < \alpha < 1 \).

Remark 3.2. Note that, in the above theorem, we do not require the initial patch \( \Omega_0 \) to have compact support. However, we do require that the gradient \( \nabla \perp \phi_0 \) to have \( C^\alpha \)-norm uniformly bounded on all of \( \mathbb{R}^2 \).
Recall from Subsection 2.4 that the 2D Euler equation is globally well-posed with \( \omega_0 \in \dot{C}^\alpha \) under symmetry. Therefore, the global well-posedness of the patch admitting a level set (under the same symmetry assumption) with \( \nabla^\perp \phi_0 \in C^\alpha \) is a natural analogue of the classical global well-posedness result of \( C^{1,\alpha} \)-patches. As an immediate consequence of the above theorem, we have that,

**Corollary 3.3.** Under the assumptions of Theorem 1, the flow map \( \Phi_t \) is a Lipschitz bijection of the plane with a Lipschitz inverse for all times \( t \geq 0 \).

Before we proceed to the proof, let us describe a few classes of vortex patches satisfying the requirements of Definition 3.1.

**Examples and Remarks.** Theorem 1 establishes global well-posedness for each of the following classes of example, under the assumption of \( C^m \) classes of example, under the assumption of \( \phi_0 \) is a \( C^1,\alpha \)-smooth in the complement of \( B_0(r) \).

(i) **Sectors:** Assume that for some ball \( B_0(r) \), the intersection \( \Omega_0 \cap B_0(r) \) is a union of sectors meeting at the origin (see Figures 1 for symmetric examples). In addition, assume that \( \partial \Omega_0 \) is \( C^{1,\alpha} \)-smooth in the complement of \( B_0(r) \). Then, one may take a level set locally by \( \phi_0(x) = rh_0(\theta) \) in polar coordinates with some \( h_0(\cdot) \in C^{1,\alpha}(S^1) \), where \( h_0 \) can be appropriately chosen that \( \phi_0 \) satisfies Definition 3.1. Moreover, the same holds for the image \( \Psi(\Omega_0) \) of such a patch \( \Omega_0 \) under a global \( C^{1,\alpha} \)-diffeomorphism of the plane \( \Psi \) satisfying \( |\Psi(x)| \leq C|\frac{1}{x}|^{1+\alpha} \) for some \( C > 0 \). These facts are proved in Lemma 4.3 of the next section, where we study in detail the evolution of such vortex patches under the assumption of \( m \)-fold symmetry.

This class of vortex patches (which are locally the \( C^{1,\alpha} \)-diffeomorphic image of a union of sectors meeting at the origin) are studied in great detail in Section 4. Unfortunately, the fact that \( \nabla^\perp \phi \) stays in \( \dot{C}^\alpha \) for all time is not sufficient to conclude that the evolved patch is still given by the image of some \( C^{1,\alpha} \)-diffeomorphism. Therefore, a careful local analysis should be supplemented to recover this information (see Subsection 4.2).

(ii) **Logarithmic spirals:** Take some indicator function \( \chi_I \) where \( I \) is some interval of \( S^1 = [0, 2\pi] \) and consider a patch \( \Omega_0 \) which is locally given by

\[
\omega_0(r, \theta) = \chi_I (\theta - c \log r + \theta), \quad r < 1/2
\]

where \( c > 0 \) is some constant. Taking \( h_0 \in C^{1,\alpha}(S^1) \) vanishing precisely on the endpoints of the interval \( I \) with non-zero derivatives, and then by setting \( \phi_0 = rh_0(\theta - c \log r + \theta) \), one may check that this function satisfies the requirements of Definition 3.1 (assuming for instance \( \Omega_0 \) is a \( C^{1,\alpha} \)-patch in \( r \geq 1/2 \)). This boils down to checking that, for a given function \( \zeta \in C^\alpha(S^1) \) with \( 0 < \alpha \leq 1 \), \( \zeta (\theta - c \log r + \theta) \in C^\alpha(\mathbb{R}^2) \). For simplicity, take the case \( \alpha = 1 \), and then

\[
\frac{1}{r} \partial_r \zeta = \frac{1}{r} \zeta', \quad \partial_\theta \zeta = -\frac{c}{r} \zeta',
\]

so that switching to rectangular coordinates, \( |x| \nabla \zeta(x) | \in L^\infty(\mathbb{R}^2) \), or equivalently \( \zeta(x) \in \dot{C}^1(\mathbb{R}^2) \). Similarly as in the case of (i), one can treat patches which are given as the image of an exact spiral by a \( C^{1,\alpha} \)-diffeomorphism of the plane fixing the origin.

In the special case when the initial vorticity is given exactly by \( \omega_0 = h_0(-c \log r + \theta) \), then as we have seen in the introduction, a 1D evolution equation satisfied by \( h(t, \cdot) \) can be derived, so that \( \omega(t, x) := h(-c \log r + \theta, t) \) solves the 2D Euler equations. This remark is due to Julien Guillod (private communication).

It is interesting question to see if one can start with a patch which locally looks like a union of sectors (as in the case (i)) and converges to a logarithmic spiral when \( t \to +\infty \).

The patch corresponding to the case \( c = 5 \) and \( I = [0, 5\pi/24] \), with 3-fold symmetrization, is given in Figure 3.
(iii) **Cusps:** Consider the (infinite) region bounded by two tangent $C^{1,\alpha}$-functions $f_0, g_0 : [0, \infty) \to \mathbb{R}$:

$$
\Omega_0 = \{(x_1, x_2) : g_0(x_1) < x_2 < f_0(x_1)\}, \quad f_0(0) = g_0(0) = 0, \quad g_0 < f_0 \text{ on } (0, \infty).
$$

Here, we require that $f_0$ and $g_0$ are uniformly $C^{1,\alpha}$ in all of $\mathbb{R}$. A model case is provided by taking $f_0(x_1) = x_1^{1+\alpha}$ and $g_0(x_1) = -x_1^{1+\alpha}$ (locally for $x_1$ near 0). One may take a number of such cusps (possibly with different boundary profiles for each of them) and rotate each of them around the origin to make them disjoint. In particular, the resulting union of cusps can be $m$-fold symmetric for any $m \geq 3$. In this setting, it is convenient to consider the complement $\mathbb{R}^2 \setminus \Omega$, which is more-or-less a union of corners. Then one may take some $\phi_0$ with $\nabla^\perp \phi_0 \in C^\alpha$ defined on $\mathbb{R}^2 \setminus \Omega_0$. It can be taken to be $C^{1,\alpha}$ smooth when one “crosses” each of the cusps (see Figure 6). We discuss them in some detail in Subsection 4.5.

Danchin has shown in [30] that the cusp-like singularities in a smooth vortex patch propagates globally in time. We are also aware of works of Serfati in this direction. It is likely that the following alternative argument for the global well-posedness would go through: first apply Theorem 1 to obtain global propagation in the intermediate class $C^\alpha$, and supply an additional local argument to recover $C^{1,\alpha}$-regularity up to the point of singularity.

(iv) **Bubbles accumulating at the origin:** Take a sequence of smooth $C^{1,\alpha}$-patches $\{U_n\}_{n \geq 0}$, which for simplicity are assumed to have comparable diameters (say less than $1/2$) and $C^{1,\alpha}$-characteristic scales. Now rescale the $n$-th patch $U_n$ by a factor of $2^{-n}$, denote it by $\tilde{U}_n$, and place it inside the annulus $A_n = \{x : 2^{-n} < |x| < 2^{-n+1}\}$. Then define $\Omega_n$ as the union of rescaled patches $\cup_{n \geq 0} \tilde{U}_n$. It can be easily arranged that, by placing several disjoint patches in each annulus region, the entire set $\Omega_0$ is $m$-fold symmetric for some $m \geq 3$.

Assuming $m$-fold symmetry, Theorem 1 applies to show that the evolution of the (rescaled) $n$-th patch $\tilde{U}_n$ has boundary in $C^{1,\alpha}$ with its characteristic satisfying

$$
c(T)2^n \leq \Gamma_n(t) \leq C(T)2^n
$$

for any $T > 0$ and $t \in [0, T]$. In particular, by rescaling each of $\tilde{U}_n$ back to a patch of diameter $O(1)$, we have that their $C^{1,\alpha}$-characteristics are uniformly bounded from above and below. Even without the symmetry, it can be shown that the boundary of each $\tilde{U}_n$ stays in $C^{1,\alpha}$ for all time. However, a uniform bound (after rescaling) cannot hold in general. Indeed, such a non-uniform growth was utilized in the work of Bourgain-Li [16, 17] (see also [43, 55]), after smoothing out the patches appropriately, to produce examples of $\omega_0 \in H^1(\mathbb{R}^2)$ which escapes $H^1(\mathbb{R}^2)$ instantaneously for $t > 0$.

The proof of Theorem 1 is parallel to the one given in [14] and based on two “frozen time” estimates, except that the $m$-fold rotational symmetry gets involved in the current setup. We first observe that in this setting, an identity of the form (2.4) still holds:

$$
\nabla u \nabla^\perp \phi(x) = \int_{\Omega} \nabla K(x - y) \left( \nabla^\perp \phi(x) - \nabla^\perp \phi(y) \right) dy,
$$

(3.4)

since all that was necessary to establish the above formula is to have the vector field $\nabla^\perp \phi$ divergence free and tangent to the boundary of the patch. Given the identity (3.4), we can prove the following estimate:

**Lemma 3.4.** Assume that a domain $\Omega$ admits a level set $\phi$ satisfying Definition 3.1. Then, we have a bound

$$
\|\nabla u \nabla^\perp \phi\|_{C^\alpha(\mathbb{R}^2)} \leq C \left( 1 + \|\nabla u\|_{L^\infty(\mathbb{R}^2)} \right) \|\nabla^\perp \phi\|_{C^\alpha(\mathbb{R}^2)}.
$$

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This is just a particular case of a general estimate about the space $\dot{C}^\alpha$, which works in the setting of convolution against classical Calderon-Zygmund kernels. It is worth noting that the symmetry is not necessary for this particular lemma. Next,

**Lemma 3.5.** Under the assumptions of Lemma 3.4 we have the following logarithmic bound:

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq C_\alpha \left(1 + \log \left(1 + \frac{\|\nabla^\perp \phi\|_{\dot{C}^\alpha(\mathbb{R}^2)}}{\|\nabla^\perp \phi\|_{\inf(\partial \Omega)}}\right)\right) \tag{3.5}$$

The symmetry assumption is essential here; basically, the information that $\nabla^\perp \phi$ belongs to $\dot{C}^\alpha$ gives an effective $C^{1,\alpha}$ bound on $\partial \Omega$ only in a region of $O(|x|)$ at a given point $x$, and the procedure of "zooming out" it to a region of size $O(1)$ will in general bring the logarithmic loss, unless the $m$-fold rotational symmetry for some $m \geq 3$ is imposed on the set $\Omega$.

Given these lemmas, let us give a sketch of the proof.

**Proof of Theorem** We assume that the local-in-time existence in the desired class is given, so that as long as the $\dot{C}^\alpha$-characteristic for $\nabla^\perp \phi_t$ remains finite, the solution can be extended further. (This part is deferred to the Appendix.)

It suffices to obtain a global-in-time a priori estimate for the characteristic quantity:

$${\dot{\Gamma}_t} = \left(\frac{\|\nabla^\perp \phi_t\|_{\dot{C}^\alpha(\mathbb{R}^2)}}{\|\nabla^\perp \phi_t\|_{\inf(\partial \Omega)}}\right)^{1/\alpha}.$$  

As we have mentioned earlier, this proof is completely parallel to the arguments of Bertozzi and Constantin [14]. We start with $W := \nabla^\perp \phi$, which satisfies

$$\partial_t W + (u \cdot \nabla) W = \nabla u W.$$  

Note that, unlike the $C^{1,\alpha}$-characteristic quantity that appeared earlier in the case of smooth patches, this quantity is non-dimensional. We use the notation $\dot{\Gamma}_t$ to emphasize this fact from now on.
Then, solving this equation along the flow,

\[ \frac{d}{dt} W(t, \Phi(t, x)) = \nabla u(t, \Phi(t, x)) W(t, \Phi(t, x)). \]

Integrating in time and then changing variables gives

\[ W(t, x) = W_0(\Phi^{-1}_t(x)) + \int_0^t (\nabla u W) \Phi^{-1}_s(x, s) ds. \]

Using the bound on \( \nabla \Phi^{-1}_t \) in terms of the velocity gradient, this implies, for points \( x \neq x' \) satisfying \( |x'| \leq |x| \) and \( |x - x'| \leq |x|/2, \)

\[ |W(t, x) - W(x', t)| \leq \| W_0 \|_{\dot{C}^\alpha} \exp \left( c \int_0^t \| \nabla u_s \|_{L^\infty} ds \right) \cdot \frac{|x - x'|^\alpha}{|x|^\alpha} + \int_0^t \| \nabla u_s W_s \|_{\dot{C}^\alpha} \exp \left( c \int_s^t \| \nabla u_s' \|_{L^\infty} ds' \right) ds \cdot \frac{|x - x'|^\alpha}{|x|^\alpha}. \]

Introducing \( Q(s) = \| \nabla u_s \|_{L^\infty} \) and using Lemma 3.4

\[ \| W_t \|_{\dot{C}^\alpha} \leq \| W_0 \|_{\dot{C}^\alpha} \exp \left( c \int_0^t Q(s) ds \right) + C \int_0^t Q(s) \| W_s \|_{\dot{C}^\alpha} \exp \left( c \int_s^t Q(s') ds' \right) ds. \]

(For a pair of points \( x \neq x' \) and \( |x'| \leq |x| \) not satisfying \( |x - x'| \leq |x|/2 \), we can simply use the \( L^\infty \)-bound \( \partial_t \| W_t \|_{L^\infty} \leq Q_t \| W_t \|_{L^\infty} \).) Then, writing

\[ G(t) := \| W_t \|_{\dot{C}^\alpha} \exp \left( -c \int_0^t Q(s) ds \right), \]

we have, after a little bit of manipulation,

\[ G(t) \leq \| W_0 \|_{\dot{C}^\alpha} + C \int_0^t Q(s) G(s) ds, \]

so that by Gronwall’s Lemma,

\[ \| W_t \|_{\dot{C}^\alpha} \leq \| W_0 \|_{\dot{C}^\alpha} \exp \left( (C + c) \int_0^t \| \nabla u_s \|_{L^\infty} ds \right). \]

On the other hand, we have trivially

\[ \| W_t \|_{\inf(\partial \Omega)} \geq \| W_0 \|_{\inf(\partial \Omega)} \exp \left( -c \int_0^t \| \nabla u_s \|_{L^\infty} ds \right). \]

Combining these estimates, and then applying Lemma 3.5 finishes the proof. \( \square \)

Proof of Lemma 3.4. Let us set

\[ G(x) = \nabla u \nabla^\perp \phi(x). \]

Then, we have trivially an \( L^\infty \) bound: \( |G(x)| \leq \| \nabla u \|_{L^\infty} \| \nabla^\perp \phi \|_{L^\infty} \). Now the proof of the \( C^\alpha \)-estimate for \( |x|^\alpha G(x) \) is strictly analogous to the proof of (2.5) given in [14, Proof of Corollary 1]. To see this, fix some \( x, h \) and consider the difference

\[ |x|^\alpha G(x) - |x + h|^\alpha G(x + h). \]
First, in the case $|h| > |x|/2$, after a rewriting the above expression is bounded in absolute value by

$$\|x\|^\alpha (G(x) - G(x + h)) + G(x + h)(|x|^\alpha - |x + h|^\alpha) \leq C|h|\|G\|_{L^\infty} + |h|^\alpha \|G\|_{L^\infty}.$$ 

Therefore, we may assume that $|h| \leq |x|/2$. Then, we write with $f := \nabla^\perp \phi$

$$|x|^\alpha G(x) - |x + h|^\alpha G(x + h)$$

$$= |x|^\alpha \int_\Omega \nabla K(x - y)(f(x) - f(y))dy - |x + h|^\alpha \int_\Omega \nabla K(x + h - y)(f(x) - f(y))dy$$

$$= |x|^\alpha \int_{\{|x-y|<2|h|\} \cap \Omega} \nabla K(x - y)(f(x) - f(y))dy$$

$$- |x + h|^\alpha \int_{\{|x-y|<2|h|\} \cap \Omega} \nabla K(x + h - y)(f(x + h) - f(y))dy$$

$$+ \int_{\{|x-y|\geq2|h|\} \cap \Omega} \nabla K(x - y)(|x|^\alpha f(x) - |x + h|^\alpha f(x + h))dy$$

$$+ \int_{\{|x-y|\geq2|h|\} \cap \Omega} (\nabla K(x - y) - \nabla K(x + h - y))(|x + h|^\alpha f(x + h) - |x|^\alpha f(y))dy$$

$$= I + II + III + IV.$$ 

Then,

$$|I| \leq C|x|\int_0^{2|h|} \|f\|_{C_0} |x|^{\alpha-1} \leq C\|f\|_{C_0} |x|^{\alpha}$$

and similarly $|II| \leq C\|f\|_{C_0} |x|^\alpha$. For $III$, we note that

$$|III| \leq \|x|^\alpha f(x) - |x + h|^\alpha f(x + h)\| \int_{\{|x-y|\geq2|h|\} \cap \Omega} \nabla K(x - y)dy \leq C\|f\|_{C_0} h^\alpha (1 + \|\nabla u\|_{L^\infty}),$$

and finally for $IV$, simply rewrite

$$|x + h|^\alpha f(x + h) - |x|^\alpha f(x) = (|x + h|^\alpha f(x + h) - |y|^\alpha f(y)) - f(y)(|x|^\alpha - |y|^\alpha)$$

we use the decay of $\nabla \nabla K$ to bound both of them by

$$|IV| \leq C\left(\|f\|_{C_0} + \|f\|_{L^\infty}\right) \int_{\{|x-y|\geq2|h|\} \cap \Omega} h \frac{1}{|x-y|^{3-\alpha}}dy \leq C\|f\|_{C_0} h^\alpha.$$ 

This concludes the proof.

**Proof of Lemma 3.5.** Let us take a (non-dimensional) parameter

$$\delta = \frac{1}{10} \min \left\{ 1, \left( \frac{\|\nabla^\perp \phi\|_{L^\infty}}{\|\nabla^\perp \phi\|_{C_0}} \right)^{1/\alpha} \right\}. $$

We then split the integral

$$I(x) = \int_\Omega \nabla K(x - y)dy, $$

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(assuming that $x \neq 0$) as follows:

$$\left[ \int_{\Omega \cap \{|x-y|<\delta|x|\}} + \int_{\Omega \cap \{\delta|x|\leq|x-y|<10|x|\}} + \int_{\Omega \cap \{|y|\leq|x-y|<\delta|x|\}} \right] \nabla K(x-y)dy =: I_1 + I_2 + I_3.$$ 

The bound for $I_1$ follows from “geometric lemma” of Bertozzi and Constantin [14]. To see this, first note that in the region $|x-y| < \delta|x|$, $\nabla^+ \phi$ is a uniformly $C^\alpha$-function with norm $\approx |x|^{-\alpha}$. Then, we have

**Geometric Lemma.** For each $\rho > 0$, consider the total angle $R_\rho(x)$ of deviation of $\Omega \cap \partial B_x(\rho)$ from being a half-circle. Formally, $R_\rho(x) := S_\rho(x) \triangle \Sigma(x)$, where $\triangle$ denotes the symmetric deference) with

$$S_\rho(x) := \{w : |w| = 1, x + \rho w \in \Omega\}, \quad \Sigma(x) := \{w : |w| = 1, \nabla \rho(\tilde{x}) \cdot w \geq 0\},$$

where $\tilde{x}$ is a point on $\partial \Omega$ which achieves the minimum distance between $x$ and $\partial \Omega$.

Then,

$$|R_\rho(x)| \leq C \left( \frac{d(x, \partial \Omega)}{\rho} + \left( \frac{\rho}{\delta|x|} \right)^{\alpha} \right)$$

as long as we take $\rho < \delta|x|$.

Given the above lemma, after using the fact that the averages of $\nabla K$ along half-circles vanish, we bound

$$|I_1| = \left| \int_{\Omega \cap \{|x-y|<\delta|x|\}} \nabla K(x-y)dy \right| \leq C \int_{d(x, \partial \Omega)} |R_\rho(x)| d\rho \leq C.$$ 

Next, the bound on $I_2$ is straightforward; just taking absolute values,

$$|I_2| \leq \int_{\delta|x|}^{10|x|} \frac{C}{\rho} d\rho \leq C \log(\delta^{-1}).$$

Finally, we use symmetry of the domain for $I_3$: note that

$$|I_3| \approx \frac{1}{m} \left| \int_{\Omega \cap \{|y|\geq10|x|\}} \sum_{j=0}^{m-1} \nabla K(x - R_{2\pi j/m}y)dy \right|$$

(Strictly speaking, the region $\Omega \cap \{|x-y| \geq 10|x|\}$ is not really $m$-fold symmetric but the extra terms coming from the difference of the symmetrization of this set and $\Omega \cap \{|y| \geq 10|x|\}$ can be bounded as in $I_2$) and then we use the fact that (see [12, Lemma 2.17]) for $|y| \gtrsim |x|$, 

$$\frac{1}{m} \left| \nabla K(x - R_{2\pi j/m}y) \right| \leq C \frac{|x|}{|y|^3}.$$ 

This gives $|I_3| \leq C$, finishing the proof. 

---

1Indeed, one may imagine that $x \in \partial \Omega$ and hence $x = \tilde{x}$, as $\nabla u$ is potentially most singular at such points.
4 Global well-posedness for symmetric $C^{1,\alpha}$-patches with corners

4.1 The geometric setup and the main statement

In this section, we show global well-posedness of $C^{1,\alpha}$ vortex patches with corners meeting symmetrically at a point. Here we make this notion precise. For the convenience of the reader, let us recall some notations:

- For a given angle $\theta$, we denote $R_{\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ to be the counter-clockwise rotation of the plane by $\theta$ around the origin.
- We define sectors using polar coordinates:

$$S_{\beta, \beta + \zeta} := \{(r, \theta) : \beta < \theta < \beta + \zeta\}.$$

**Definition 4.1 ($C^{1,\alpha}$-patches with symmetric corners).** We deal with patches $\Omega$ enjoying the following properties:

- (Symmetry) There exists an open domain $\Omega_1$, and some $m \geq 3$, such that

$$\Omega = \bigcup_{j=0}^{m-1} R_{2\pi j/m}(\Omega_1)$$

where the open sets $R_{2\pi j/m}(\Omega_1)$ are disjoint from each other and their closures intersect only at the origin.

- ($C^{1,\alpha}$ away from the origin) For any $\epsilon > 0$, and for any point $x \in \partial \Omega$ with $|x| > \epsilon$, there exists a small ball $B$ around $x$ such that $B \cap \partial \Omega$ is described by the graph of a $C^{1,\alpha}$-function (after rotating the patch if necessary).

- ($C^{1,\alpha}$ corner) There exists a $C^{1,\alpha}$-diffeomorphism $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane with $\Psi(0) = 0$ and $\nabla \Psi|_{x=0} = I$, such that for some $\delta > 0$, the image $\Psi(\Omega_1)$ is an exact sector of angle less than $2\pi/m$:

$$\Psi(\Omega_1) \cap B_0(\delta) = S_{\beta}(\zeta) \cap B_0(\delta)$$

with some $0 < \zeta < 2\pi/m$ and $\beta \in [0, 2\pi]$.

In the following, let us call such a patch by a “symmetric $C^{1,\alpha}$-patch with corners”, or symmetric patch with corners for short.

**Example 4.2.** For each $m \geq 3$, the domain bounded by the set $\{(r, \theta) : r = 1 + \cos(m\theta)\}$ (in polar coordinates) give an explicit example satisfying Definition 4.1.

Quantifying $C^{1,\alpha}$-regularity of such a patch is a simple matter; one may use directly the $C^{1,\alpha}$-norm of the diffeomorphism $\Psi$, but we shall work with the following alternative description. By rotating the plane if necessary, we may assume that the boundary $\partial \Omega_1$ is locally described by the graph of two $C^{1,\alpha}$ functions $g < f$; that is,

$$\partial \Omega_1 \cap [-\delta, \delta]^2 = \{(x_1, f(x_1)) : 0 \leq x_1 \leq \delta\} \cup \{(x_1, g(x_1)) : 0 \leq x_1 \leq \delta\}.$$

Then, the last condition of Definition 4.1 is equivalent to saying that $f$ and $g$ are $C^{1,\alpha}$-regular up to the boundary of the interval $[0, \delta]$. Then, the regularity of $\Omega$ may be quantified with the characteristic (note that it has the unit of inverse length)

$$\Gamma(\Omega) := \|\nabla f\|^{1/\alpha}_{C^0[0, \delta]} + \|\nabla g\|^{1/\alpha}_{C^0[0, \delta]} + \Gamma(\Omega \setminus B_0(\delta/2))$$

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where $\Gamma(\Omega \setminus B_0(\delta))$ is the (usual) $C^{1,\alpha}$ characteristic of $\partial \Omega$ away from $B_0(\delta)$, where it is uniformly $C^{1,\alpha}$.

Our main statement of this section states that if $\Omega_0$ is a symmetric $C^{1,\alpha}$-patch with corners, then the unique Yudovich solution $\Omega(t, \cdot)$ remains so for all times $t > 0$. We state it formally as follows:

**Theorem 2.** Let us assume that $\Omega_0$ is a vortex patch satisfying Definition 4.1. Then the Yudovich solution $\Omega_t$, associated with $\Omega_0$, satisfies the same properties for all $t > 0$; that is, $\Gamma(\Omega_t) < +\infty$. Moreover, the angles simply rotate with a constant angular speed for all time which is determined only by the size of the angle. In particular, the value of the angle does not change with time.

To be clear, part of the statement is that for any $t > 0$, one can find $\delta = \delta(t) > 0$ such that in the ball $B_0(\delta)$, the boundary of $\Omega(t)$ is given by two $C^{1,\alpha}$-curves $f_t$ and $g_t$, and in the complement of the ball $B_0(\delta/2)$, the boundary of $\Omega_t$ is uniformly $C^{1,\alpha}$.

At this point, let us note that all the hard work necessary in establishing the above result is to establish the last property in Definition 4.1, the first is trivial in view of the uniqueness and non-collision of particle trajectories. The second property is well-known; more generally, if a vortex patch is $C^{1,\alpha}$ away from some closed set, the solution remains smooth away from the image of the closed set under the flow $[29]$. Indeed, this is an immediate consequence of Theorem 1 as soon as we prove the following

**Lemma 4.3.** Assume that $\Omega$ is a $C^{1,\alpha}$-patch with a symmetric corner. Then it admits a level set $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying conditions of Definition 3.1.

**Proof.** We first check the statement in the case when $\Omega$ is given by an exact symmetric corner near the origin, that is,

$$\Omega \cap B_0(\delta) = \bigcup_{j=0}^{m-1} R^j_0(S_{\beta, \beta + \zeta}) \cap B_0(\delta)$$

for some $\beta \in [0, 2\pi)$ and $\zeta < 2\pi/m$. In this case, one may take a function $\phi$ which is $C^{1,\alpha}$ outside of the ball $B_0(\delta)$ and then locally

$$\phi(x) = |x| \cdot g(\theta(x)), \quad \theta(x) = \tan^{-1}(x_2/x_1).$$

Here, one may take a smooth function $g \in C^{1,\alpha}(S^1)$ so that $\phi(x)$ strictly positive inside the patch and negative outside, and also $g'$ is non-vanishing for angles which correspond to $\partial \Omega$. Then, taking the gradient one obtains for $|x| < \delta$

$$\nabla^\perp \phi(x) = \frac{x^\perp}{|x|} g(\theta) - \frac{x}{|x|} g'(\theta)$$

and note that for $x \in \partial \Omega \cap B_0(\delta)$,

$$|\nabla^\perp \phi(x)| = |g'(\theta)| > 0$$

by our choice of $g$ (here $\theta = \theta(x)$) and also

$$\left| |x|^\alpha \cdot \left(\frac{x}{|x|} g(\theta) + \frac{x^\perp}{|x|} g'(\theta)\right)\right|_{C^\alpha(\mathbb{R}^2)} \leq C\|g\|_{C^{1,\alpha}(S^1)}.$$

In the general case, recall that there is a map $\Psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that locally maps the path $\Omega$ to a union of exact symmetric corners. Then one just take the level set

$$\phi(x) = \tilde{\phi} \circ \Psi(x), \quad \tilde{\phi}(z) = |z| \cdot g(\theta(z)).$$

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Then taking the gradient gives
\[ \nabla \phi = (\nabla \tilde{\phi}) \circ \Psi \cdot \nabla \Psi, \]
and recalling that \( \nabla \Psi = I + M \) with a matrix \( M \in C^{1,\alpha}(\mathbb{R}^2) \) and \( |M| \leq C|x|^{1+\alpha} \), it is direct to show that, using bounds given in Lemma A.5 in the Appendix, \( |\nabla \perp \phi| \) has a lower bound on \( \partial \Omega \) as well as \( \nabla \perp \phi \in \mathring{C}^\alpha \).

A brief outline of the proof. The proof of Theorem 2 will be completed in the following two subsections. In Subsection 4.2, we prove a frozen-time \( C^{1,\alpha} \)-estimate pertaining to the boundary of the patch near the corner. After that, we conclude the proof in Subsection 4.3 by combining the local estimate together with the \( C^\alpha \)-result. In the following subsections, we explore some consequences of Theorem 2 and several possible extensions.

4.2 Local \( C^{1,\alpha} \)-estimate near the corner

To complete the proof of the main statement, it needs to be argued that right at the origin, the \( C^{1,\alpha} \) norms of the boundary curves does not blow up at any finite time. Near the corner, it does not seem appropriate to use a level set function which is uniformly \( C^{1,\alpha} \). Instead, we will show via a direct computation that the velocity is uniformly \( C^{1,\alpha} \) on the boundary, up to the origin.

To get an idea of how such a statement could be true, one may first take the case of exact (either infinite or localized) sectors meeting symmetrically at the origin. While the velocity gradient associated with a single sector diverges logarithmically at the origin, with coefficient depending on the angle, it was established in the previous work of the first author [40, Section 6] that those logarithmic terms precisely cancel out when the sectors are arranged in an \( m \)-fold symmetric fashion. Even after cancellations of the divergent terms, the velocity gradient has a part which is a smooth function of the angle only (that is, a \( C^\infty \)-function of the variable \( \tan^{-1}(x_2/x_1) \)) and hence it only belongs to \( L^\infty \) and not better.

However, a key observation we make is that a smooth function of the angle on the plane is actually \( C^{k,\alpha} \)-smooth when restricted onto any \( C^{k,\alpha} \)-curve passing through the origin.

Next, one can consider the case where the patch \( \Omega_1 \) is given (locally) by an exact sector with two \( C^{1,\alpha} \)-cusps attached at its sides. Then, from the above, we know that the velocity gradient coming from the sector is \( C^{1,\alpha} \)-smooth along the boundary curves of \( \Omega_1 \). Moreover, as a consequence of the work of Friedman and Velázquez [46], we also have that the velocity gradient coming from a \( C^{1,\alpha} \)-cusp is actually \( C^\alpha \) along each piece of the boundary.

In the general setting, though, it may happen that a boundary curve of \( \Omega_1 \) oscillates infinitely often around its tangent line at the origin. Therefore, we have simply chosen to estimate the \( C^\alpha \)-norm of \( \nabla u \) with brute force by directly integrating the kernel.

Before we begin the estimate, let us recall an explicit representation formula for the velocity gradient associated with vorticity \( \chi_\Omega \) [14]:

\[ \nabla u(x) = \frac{1}{2\pi} p.v. \int_\Omega \frac{\sigma(x-y)}{|x-y|^2} dy + \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi_\Omega, \]

where the characteristic function \( \chi_\Omega \) is defined to be \( 1/2 \) on \( \partial \Omega \). The \( 2 \times 2 \) symmetric matrix \( \sigma(z) \) is

\[ \frac{1}{|z|^2} \begin{pmatrix} 2z_1z_2 & z_1^2 - z_2^2 \\ z_2^2 - z_1^2 & -2z_1z_2 \end{pmatrix}. \]

In particular, note that this formula provides a decomposition of \( \nabla u \) into its symmetric and antisymmetric parts. The antisymmetric part is completely smooth on the patch, so it is only necessary to deal with the symmetric part, given by a principal value integration against a \( 0 \)-homogeneous kernel.

\[ \text{Here, by a } C^{1,\alpha} \text{-cusp, we mean a region bounded between two } C^{1,\alpha} \text{-curves meeting at the origin with the same slope.} \]
Figure 4: Description of the patch near the corner.

For the convenience of the reader, let us briefly recall the geometric setup for $\Omega$ and $\Omega_1$. We assume that a patch $\Omega_1$ is locally given by the region between two $C^{1,\alpha}$ curves meeting at the origin: to be precise, there exists some $\gamma > 0$, so that the boundary of $\Omega_1$ in the region $[-\gamma, \gamma]^2$ is given by

$$\{(y_1, y_2) : g(y_1) < y_2 < f(y_1)\}$$

where $g, f$ belong to $C^{1,\alpha}[0, \gamma]$ with $f(0) = 0 = g(0)$ and $f'(0) > 0 > g'(0)$. Then, we consider the disjoint union $\Omega = \bigcup_{j=1}^{4} \Omega_j$ where $\Omega_j = R_{\pi(j-1)/2}(\Omega_1)$. Here, we are assuming that the patch is 4-fold symmetric ($m = 4$) just for the simplicity of notation. The other cases can be treated similarly, using the results of [10].

The value of $\delta > 0$ is chosen in a way that

$$\delta^\alpha = \frac{1}{10} \min \left\{ \left( \frac{\gamma}{2} \right)^\alpha, \|f\|_{C^{1,\alpha}[0,\gamma]}^{-1}, \|g\|_{C^{1,\alpha}[0,\gamma]}^{-1} \right\}.$$

Moreover, without loss of generality it can be assumed that on the interval $x \in [0, \delta]$, $\frac{1}{2}|x| < |f(x)|, |g(x)| \leq 2|x|, \quad |f'(0)|, |g'(0)| \leq 2$

(the specific values 1/2 and 2 will not play any essential role).

Lemma 4.4 ($C^\alpha$-estimate on the velocity gradient). In the above setting, we have a bound

$$\left\| \frac{d}{dx} u(x, f(x)) \right\|_{C^\alpha[0, \delta/10]} \leq C (\|f\|_{C^{1,\alpha}} + \|g\|_{C^{1,\alpha}}) (1 + \|\nabla u\|_{L^\infty}) + C\delta^{-\alpha}. \quad (4.2)$$

Proof. Let us write down explicitly the expression for $\nabla u$ along a $C^{1,\alpha}$-curve $(x, h(x))$, which lies between two boundary curves of $\Omega_1$:

$$g(x) < h(x) \leq f(x), \quad 0 < x \leq \delta.$$
We begin with $\frac{d}{dx}u_2$:

\[
\frac{d}{dx}u_2(x, h(x)) \equiv \frac{1}{2\pi} \frac{d}{dx} \int_{\Omega} \frac{x - y_1}{(x - y_1)^2 + (h(x) - y_2)^2} dy \\
= \frac{1}{2\pi} \int_{\Omega} \frac{(x - y_1)^2 - (h(x) - y_2)^2 + 2f'(x)(x - y_1)(h(x) - y_2)}{(x - y_1)^2 + (h(x) - y_2)^2} dy \\
= \frac{1}{2\pi} \left[ \int_{\Omega \cap [-\delta, \delta]^2} + \int_{\Omega \setminus [-\delta, \delta]^2} \right] \frac{(x - y_1)^2 - (h(x) - y_2)^2 + 2f'(x)(x - y_1)(h(x) - y_2)}{(x - y_1)^2 + (h(x) - y_2)^2} dy
\]

where we have separated contribution from the bulk of the patch. The contribution from the bulk can be trivially bounded in $C^0$ using the decay of the kernel:

\[
\frac{1}{|x - x'|^{1+\alpha}} \left| \int_{|y| > \delta} (\nabla K(x) - \nabla K(x')) \chi_{\Omega} dy \right| \\
\leq C |x - x'|^{1-\alpha} \int_{|y| > \delta} \frac{1}{|x - y|^3} + \frac{1}{|x' - y|^3} dy \leq C \delta^{-\alpha}.
\]

Now, let us separately consider two integrals

\[
I_1(x) := \int_{\Omega \cap [-\delta, \delta]^2} \frac{(x - y_1)^2 - (h(x) - y_2)^2}{((x - y_1)^2 + (h(x) - y_2)^2)^2} dy \\
I_2(x) := f'(x) \int_{\Omega \setminus [-\delta, \delta]^2} \frac{2(x - y_1)(h(x) - y_2)}{((x - y_1)^2 + (h(x) - y_2)^2)^2} dy.
\]

We shall only consider \( I_1 \), and just briefly comment on the other term \( I_2 \) below. One can further write:

\[
I_1(x) = \sum_{j=1}^{4} I_1^j(x),
\]

where

\[
I_1^j(x) := \int_{\Omega \cap [-\delta, \delta]^2} \frac{(x - y_1)^2 - (h(x) - y_2)^2}{((x - y_1)^2 + (h(x) - y_2)^2)^2} dy
\]

(recall that $\Omega_j := R_{\pi(j-1)/2}((\Omega_1)$).

We have, after integrating in \( y_2 \),

\[
I_1^j(x) = \int_{0 \leq y_1 \leq \delta} \int_{g(y_1) \leq y_2 \leq f(y_1)} \frac{(x - y_1)^2 - (h(x) - y_2)^2}{((x - y_1)^2 + (h(x) - y_2)^2)^2} dy_2 dy_1 \\
= \int_{0}^{\delta} \left[ \frac{h(x) - f(z)}{(x - z)^2 + (h(x) - f(z))^2} - \frac{h(x) - g(z)}{(x - z)^2 + (h(x) - g(z))^2} \right] dz
\]
(we have renamed \( y_1 \) by \( z \) for simplicity). Similarly,

\[
I_1^2(x) = \int_{0 \leq y_2 \leq \delta} \int_{-f(y_2) \leq y_1 \leq -g(y_2)} \frac{(x - y_1)^2 - (h(x) - y_2)^2}{((x - y_1)^2 + (h(x) - y_2)^2)^2} dy_1 dy_2
\]

\[
= \int_0^\delta \left[ \frac{x + f(z)}{(x + f(z))^2 + (h(x) - z)^2} - \frac{x + g(z)}{(x + g(z))^2 + (h(x) - z)^2} \right] dz
\]

**Claim.** The integral

\[
\int_0^\delta \left[ \frac{h(x) - f(z)}{(x - z)^2 + (h(x) - f(z))^2} + \frac{x + f(z)}{(x + f(z))^2 + (h(x) - z)^2} \right] dz
\]

defines a \( C^\alpha \)-function of \( 0 \leq x \leq \delta/10 \) with \( C^\alpha \)-norm bounded by the right hand side of [4.2].

Once we show the **Claim** (together with the upper bound stated in [4.2]) for \( h = f \) and \( h = g \), this concludes the proof that \( I_1(x) \) belongs to \( C^\alpha \), since each of \( I_1^1 + I_1^2 \) and \( I_1^2 + I_1^4 \) belongs to \( C^\alpha \), by symmetry. A similar argument can be given for the other term \( I_2(x) \): we write it as

\[
I_2(x) = f'(x) \sum_{j=1}^4 I_2^j(x),
\]

where

\[
I_2^2(x) = \int \frac{2(x - y_1)(h(x) - y_2)}{((x - y_1)^2 + (h(x) - y_2)^2)^2} dy.
\]

Then, we can integrate each of \( I_2^j \) once with respect to either \( y_1 \) or \( y_2 \), resulting in similar expressions as above.

Let us consider the case \( h(z) = f(z) \), which is actually the most difficult case. In this specific case, we rewrite the integrand in [4.3] as

\[
\left[ \frac{f(x) - f(z)}{(x - z)^2 + (f(x) - f(z))^2} - \frac{f'(x)}{1 + (f(x))^2} \cdot \frac{1}{x - z} \right]
\]

\[
+ \left[ \frac{x + f(z)}{(x + f(z))^2 + (f(x) - z)^2} - \frac{f'(x)}{1 + (f(x))^2} \cdot \frac{F}{x + f(z)} \right] + \frac{f'(x)}{1 + (f(x))^2} \left( \frac{1}{x - z} + \frac{F}{x + f(z)} \right)
\]

where \( F := f'(0) > 0 \). Let us first estimate in \( C^\alpha \) the last term, which we further rewrite as:

\[
\frac{f'(x)}{1 + (f(x))^2} \left[ \left( \frac{1}{x - z} + \frac{F}{x + Fz} \right) + F \cdot \frac{Fz - f(z)}{(f(x))(x + Fz)} \right].
\]

Since \( f' \in C^\alpha \), it suffices to estimate in \( C^\alpha \) the integrals of two terms in the large brackets. Regarding the first term, one just explicitly evaluate that

\[
\int_0^\delta \frac{1}{x - z} + \frac{F}{x + Fz} dz = \log \left( \frac{F\delta + x}{\delta - x} \right),
\]

(defined by the principal value) which is clearly bounded in \( C^\alpha \) by the right hand side of [4.3]. Note that the logarithmically divergent terms (as \( x \to 0^+ \)) present in each of the integrals cancel each other exactly. Regarding the second term, we first note that it is uniformly bounded:

\[
\left| \int_0^\delta \frac{Fz - f(z)}{(x + f(z))(x + Fz)} dz \right| \leq CF \int_0^\delta \frac{z}{x^2 + (Fz)^2} dz \leq C \frac{F}{1 + F^2}.
\]

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To bound the $C^\alpha$-norm, we need to estimate for $0 \leq x < x'$

$$
\int_0^\delta \frac{1}{|x-x'|^\alpha} \cdot |Fz - f(z)| \cdot \left| \frac{1}{(x+f(z))(x+Fz)} - \frac{1}{(x'+f(z))(x'+Fz)} \right| \, dz,
$$

and simply using that $|Fz - f(z)| \leq \|f\|_{C^{1,\alpha}} |z|^{1+\alpha}$, we bound the above by

$$
C\|f\|_{C^{1,\alpha}} \cdot \int_0^\delta \frac{|x-x'|^{-\alpha}z^{1+\alpha}(x+x'+Fz)}{(x+f(z))(x+Fz)(x'+f(z))(x'+Fz)} \, dz
\leq C\|f\|_{C^{1,\alpha}} \cdot \int_0^\delta \frac{|x-x'|^{-\alpha}z^{1+\alpha}(x+x'+z)}{(x+z)^2(x'+z)^2} \, dz
\leq C\|f\|_{C^{1,\alpha}} \cdot \int_0^\delta \frac{x^{1-\alpha}z^{1+\alpha}(x+z)}{(x+z)^2z^2} \, dz \leq C\|f\|_{C^{1,\alpha}}.
$$

It remains to estimate

$$
T_1(x) = \int_0^\delta \left[ \frac{f(x) - f(z)}{(x-z)^2 + (f(x) - f(z))^2} - \frac{f'(x)}{1 + (f'(x))^2} \cdot \frac{1}{x-z} \right] \, dz
$$

and

$$
T_2(x) = \int_0^\delta \left[ \frac{x + f(z)}{(x + f(z))^2 + (f(x) - f(z))^2} - \frac{f'(x)}{1 + (f'(x))^2} \cdot \frac{F}{x + f(z)} \right] \, dz.
$$

We begin with $T_1(x)$. After a bit of re-arranging, we have

$$
T_1(x) = \int_0^\delta \frac{1}{x-z} \left[ \frac{f(z) - f(x)}{z-x} - \frac{f'(x)}{1 + (f'(x))^2} \right] \, dz
= \int_0^\delta \frac{1}{x-z} \left[ \left( \frac{1}{1 + (f(x) - f(z))^2} - \frac{1}{1 + (f'(x))^2} \right) \frac{f(x) - f(z)}{x - z} \right.
\left. + \frac{1}{1 + (f'(x))^2} \left( \frac{f(x) - f(z)}{x - z} - f'(x) \right) \right] \, dz.
$$

Let us first estimate the latter term: dropping the multiplicative factor which belongs to $C^\alpha$,

$$
\int_0^\delta \frac{1}{x-z} \left( \frac{f(x) - f(z)}{x - z} - f'(x) \right) \, dz.
$$

Take two points $0 \leq x < x' < \delta/10$, and let us further assume that $|x| \geq |x' - x|$ (the other case is simpler). We need to take

$$
\int_0^\delta \frac{1}{|x-x'|^\alpha} \int_0^\delta \frac{1}{x-z} \left( \frac{f(x) - f(z)}{x - z} - f'(x) \right) - \frac{1}{x'-z} \left( \frac{f(x') - f(z)}{x' - z} - f'(x') \right) \, dz
= \int_0^\delta \frac{1}{|x-x'|^\alpha} \left[ \int_0^\delta \int_{x-|x-x'|/2}^{x+|x-x'|/2} \int_{x'-|x-x'|/2}^{x'+|x-x'|/2} \, dz \right] \cdots dz
$$

(4.4)

To begin with, we treat the second integral of (4.4): we simply use the bound

$$
|f(x) - f(z) - f'(x)(x-z)| \leq \|f\|_{C^{1,\alpha}} |x-z|^{1+\alpha}
$$
(and similarly for \( x \) replaced by \( x' \)) to bound it in absolute value by
\[
\|f\|_{C^{1,\alpha}} \int_0^{x-|x-x'|/2} \frac{|x-z|^n + (|x-x'| + |x-z|)^{n-1} dz} \leq C\|f\|_{C^{1,\alpha}}.
\]
The third integral from (4.4) can be treated in a parallel way. Turning to the first integral, we rewrite as
\[
\frac{1}{|x-x'|^\alpha} \int_0^{x-|x-x'|/2} \frac{(f(x) - f(z)) - (f(x') - f'(x)(x' - z)}{(x-z)^2} - \left( \frac{1}{(x-z)^2} - \frac{1}{(x'-z)^2} \right) (f(x') - f(z) - f'(x')(x' - z)) dz
\]
Note that the numerator of the first term equals
\[
(f(x) - f(x') - f'(x)(x - x')) + ((x' - x) + (x - z)) (f(x') - f'(x)),
\]
and simply using the bounds
\[
|f(x) - f(x')|, |x - x'|, |f'(x')|, \quad |f(x) - f'(x')| \leq C\|f\|_{C^{1,\alpha}} |x - x'|^\alpha,
\]
we bound the first term by \( C\|f\|_{C^{1,\alpha}}. \) The second one can be bounded by
\[
\|f\|_{C^{1,\alpha}} |x - x'|^{1-\alpha} \int_0^{x-|x-x'|/2} \frac{|x-z| + |x-x'|}{(x-z)^2}|x'-z|^{\alpha-1} \alpha \int_0^{x-|x-x'|/2} \frac{1+|v|^\alpha}{v^2} dv \leq C\|f\|_{C^{1,\alpha}}.
\]
Now the last integral of (4.4) can be treated in an analogous fashion. To finish the estimate of \( T_1 \), we still need to consider the expression
\[
\int_0^\delta \frac{1}{x-z} \cdot \frac{f(x) - f(z)}{x-z} \cdot \left[ \frac{1}{1 + \left( \frac{f(x) - f(z)}{x-z} \right)^2} - \frac{1}{1 + (f'(x))^2} \right] dz
\]
\[
= - \frac{1}{1 + (f'(x))^2} \int_0^\delta \frac{1}{x-z} \cdot \frac{f(x) - f(z)}{x-z} \cdot \left( \frac{f(x) - f(z)}{x-z} - f'(x) \cdot \frac{f(x) - f(z)}{x-z} + f'(x) \right) \cdot \frac{1}{1 + \left( \frac{f(x) - f(z)}{x-z} \right)^2} dz,
\]
where \( F = f'(0) \) and \( \tilde{f}(x) = f(x) - Fx \). Consider the expansion
\[
\frac{1}{1 + \left( F + \frac{f(x) - f(z)}{x-z} \right)^2} = \frac{1}{1 + F^2 + \left( F + \frac{f(x) - f(z)}{x-z} \right)^2} - F^2
\]
\[
= \frac{1}{1 + F^2} \cdot \sum_{m \geq 0} (-1)^m \left( \frac{1}{1 + F^2} \right)^m \cdot \left( \frac{F + \tilde{f}(x) - f(z)}{x-z} \right)^2 - F^2 \right)^m,
\]
which is convergent simply because \( \|\tilde{f}'\|_{L^\infty[0,\delta]} \leq 1/10 \) from our choice of \( \delta \). Inspecting the terms, it suffices to estimate in \( C^n \) the following integrals:
\[
\int_0^\delta \frac{1}{x-z} \left( \frac{f(x) - f(z)}{x-z} \right)^m \left( \frac{f(x) - f(z)}{x-z} - f'(x) \right) dz.
\]
We have already treated the case \( m = 0 \) in the above. For any \( m \geq 1 \), the same proof carries over, resulting in a bound (using that \( \| f \|_{L^\infty[0, \delta]} \leq 1/10 \))

\[
\left\| \int_0^\delta \frac{1}{x - z} \left( \frac{\hat{f}(x) - \hat{f}(z)}{x - z} \right)^m \left( \frac{f(x) - f(z)}{x - z} - f'(x) \right) dz \right\|_{C^\infty[0, \delta/10]} \leq Cm^{2m/10^m} \| f \|_{C^{1, \alpha}[0, \delta]},
\]

which is clearly summable in \( m \). This concludes the argument for \( T_1(x) \). The other term \( T_2(x) \) can be treated similarly, and it is simpler since the corresponding integral is less singular than that of \( T_1 \).

We now sketch a proof that **Claim** holds in the case \( h(x) \equiv g(x) \). In this case, the arguments are simpler since we have a gap \(| h(x) - g(x) | \geq | x | \). It suffices to show that the differences

\[
\int_0^\delta \left[ \frac{g(x) - f(z)}{(x - z)^2 + (g(x) - f(z))^2} - \frac{Gx - Fz}{(x - z)^2 + (Gx - Fz)^2} \right] dz,
\]

\[
\int_0^\delta \left[ \frac{x + f(z)}{(x + f(z))^2 + (g(x) - z)^2} - \frac{x + Fz}{(x + Fz)^2 + (Gx - z)^2} \right] dz,
\]

and

\[
\int_0^\delta \left[ \frac{Gx - Fz}{(x - z)^2 + (Gx - Fz)^2} + \frac{x + Fz}{(x + Fz)^2 + (Gx - z)^2} \right] dz
\]

belong to \( C^\alpha \) with appropriate bounds, where \( G := g'(0) < 0 \). To begin with, the last integral can be evaluated directly,

\[
- \frac{1}{1 + F^2} \left[ \tan^{-1} \left( \frac{-F + G}{-(1 + FG)x + (1 + F^2)z} \right) + \frac{F}{2} \log \left( (1 + G^2)x^2 + (1 + F^2)x^2 + 2xz(1 + FG) \right) \right.
\]

\[
+ \tan^{-1} \left( \frac{x(1 + FG)}{(F - G)x + (1 + F^2)z} \right) - \frac{F}{2} \log \left( (1 + G^2)x^2 + (F - G)xz + (1 + F^2)x^2 \right) \right]_0^\delta,
\]

which gives a \( C^\alpha \)-function of \( x \): two logarithmic terms cancel each other, and a simple computation shows that

\[
\left\| \tan^{-1} \left( \frac{Ax}{Bx + \delta} \right) \right\|_{C^\alpha[0, \delta/10]} \leq C(A, B) \delta^{-\alpha}
\]

for nonzero constants \( A \) and \( B \). Now we turn to the first integral, which equals

\[
\int_0^\delta \left[ \frac{(x - z)^2 ((Gx - g(x)) - (Fz - f(z)))}{((x - z)^2 + (g(x) - f(z))^2)((x - z)^2 + (Gx - Fz)^2)} + \frac{(g(x) - f(z))(Gx - Fz)((g(x) - f(z)) - (Gx - Fz))}{((x - z)^2 + (g(x) - f(z))^2)((x - z)^2 + (Gx - Fz)^2)} \right] dz
\]

(4.5)

Here, the key points are:

- On the numerator, we gain an extra power of \( | x |^\alpha \) or \( | z |^\alpha \), from Hölder continuity of \( f' \) and \( g' \).

- The denominator is uniformly bounded from above and below by constant multiples of \( x^2 + z^2 \).

We sketch the proof of \( C^\alpha \)-continuity for the first term only, since the second one can be treated similarly. We need to estimate

\[
\frac{1}{| x - x' |^\alpha} \int_0^\delta \left[ \frac{(x - z)^2 ((Gx - g(x)) - (Fz - f(z)))}{((x - z)^2 + (g(x) - f(z))^2)((x - z)^2 + (Gx - Fz)^2)} \right.
\]

\[
- \frac{(x' - z)^2 ((Gx' - g(x')) - (Fz - f(z)))}{((x' - z)^2 + (g(x') - f(z))^2)((x' - z)^2 + (Gx' - Fz)^2)} \bigg] dz
\]

(4.6)
and we may assume \(|x-x'| \leq |x|\). Let us even further assume that the denominators in (4.6) are the same, as they are roughly of the same size (and bounded uniformly from below by a constant multiple of \(x^2+z^2\) and \(x'^2+z^2\), respectively). Then, the resulting difference is bounded by:

\[
C(\|f\|_{C^{1,0}} + \|g\|_{C^{1,0}}) \frac{1}{|x-x'|^{\alpha}} \int_0^1 \frac{|x-x'| ((|x| + |x-x'|) + |z|) (|x|^{1+\alpha} + |z|^{1+\alpha})}{(x^2+z^2)^2} dz
\]

and at this point, the \(C^{\alpha}\)-bound simply follows from rescaling the variable \(z = xv\). The actual proof can be done for instance by expanding one of the denominators in (4.6) around the other denominator in a power series as we have done earlier.

The argument for the other component \(\frac{dx}{dt} u_1\) is completely analogous. We just note that along a curve \((x, h(x))\), it has the form:

\[
\frac{d}{dx} u_1(x, h(x)) = \frac{1}{2\pi} \int_{\Omega} -\frac{(h(x)-y_2)}{(x-y_1)^2 + (h(x)-y_2)^2} dy
\]

This finishes the proof.

### 4.3 Proof of the main result

We are now in a position to complete the proof of Theorem 2. Let us recall that as a consequence of Theorem 1, for any \(T > 0\), we have \(L^\infty\)-bounds

\[
\sup_{t \in [0,T]} (\|\nabla \Phi_t\|_{L^\infty} + \|\nabla \Phi_t^{-1}\|_{L^\infty} + \|\nabla u_t\|_{L^\infty}) \leq C(T),
\]

(4.7)

and moreover, for any \(r > 0\),

\[
\sup_{t \in [0,T]} \|\nabla u_t\|_{C^{\alpha}(\mathbb{R}^2 \setminus B_0(r))} \leq C(T)r^{-\alpha}.
\]

(4.8)

As in the case of Theorem 1, the issue of local well-posedness is deferred to the Appendix (Proposition A.2), hence, we shall assume that at least for some short time interval \([0, T_1]\), each piece of the boundary of \(\Omega_1(t)\) remains uniformly \(C^{1,\alpha}\) up to the origin.

**Proof of Theorem 2** We just need to show an a priori estimate which is sufficient to guarantee that for all \(t > 0\), the boundary of \(\Omega_1(t)\) is given by two \(C^{1,\alpha}\)-curves \(f_t\) and \(g_t\) (after rotating the plane if necessary) on some nonempty interval \([0, \delta_t]\). Let us consider only the evolution of the upper boundary \((x, f_0(x))\), since the other part can be treated similarly. We shall fix some \(T > 0\), and use the bounds (4.7) and (4.8).

Since \(f_t\) itself does not obey a simple evolution equation, let us work directly with the particle trajectories

\[
\eta^1(t, x) := \Phi^1(t, x, f_0(x)), \quad \eta^2(t, x) := \Phi^2(t, x, f_0(x)),
\]

which is well-defined on \(x \in [0, \delta_0]\). Then, since \(\dot{\eta}(t, x) = u(t, \eta(t, x))\), we have upon differentiating

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} \eta(t, x) \right) = \nabla u(t, \eta(t, x)) \left( \frac{\partial}{\partial x} \eta(t, x) \right).
\]

First, from (4.7) we have

\[
\sup_{x \in [0, \delta_0]} |\partial_x \eta_t| \leq C(T) < +\infty, \quad \inf_{x \in [0, \delta_0]} |\partial_x \eta_t| \geq c(T) > 0.
\]

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Then, from Lemma 4.4 and 4.8, we have a bound
\[ \| \nabla u_t \circ \eta \|_{C^{\alpha}[0,\delta_0]} \leq C(T) \left( \| \partial_x \eta \|_{C^{\alpha}[0,\delta_0]} + \delta_t^{-\alpha} \right) \]
for some \( \delta_t > 0 \), as long as the boundary of the \( \Omega_1(t) \) is given by the graph of two \( C^{1,\alpha} \) curves \( f_t \) and \( g_t \) defined on \([0,c\delta_t]\) for some constant \( c > 0 \). However, from the inverse function theorem, we can say that\(^{10}\)
\[ c(T) \| \partial_x \eta \|_{C^{\alpha}[0,\delta_0]}^{1/\alpha} \leq \delta_t \]
as well as
\[ \| f_t \|_{C^{1,\alpha}[0,\delta_0]} + \| g_t \|_{C^{1,\alpha}[0,\delta_0]} \leq C(T) \| \partial_x \eta \|_{C^{\alpha}[0,\delta_0]} \]
for all \( t \in [0,T] \), with some constants \( c(T), C(T) > 0 \) depending only on \( T \). Then simply from the algebra property of the space \( C^\alpha \), we deduce an a priori bound
\[
\frac{d}{dt} \| \partial_x \eta(t) \|_{C^{\alpha}[0,\delta_0]} \leq \| \nabla u_t \|_{L^\infty} \| \partial_x \eta(t) \|_{C^{\alpha}[0,\delta_0]} + \| \nabla u_t \circ \eta_t \|_{C^{\alpha}[0,\delta_0]} \| \partial_x \eta(t) \|_{L^\infty}
\leq C(T) \| \partial_x \eta(t) \|_{C^{\alpha}[0,\delta_0]}.
\]
This shows that for all \( t > 0 \), \( \| \partial_x \eta \|_{C^\alpha} \) as well as \( \delta_t \) remains finite and non-zero, respectively.

It remains to show the statement regarding the dynamics of the angles. For this purpose, let us decompose
\[ \omega = \omega^{\text{homog}} + \omega^{\text{cusp}} + \omega^{\text{far}}, \]
where \( \omega^{\text{homog}} \) is the \( 0 \)-homogeneous vorticity which is the characteristic function of the \( m \)-fold symmetrization of the infinite sector
\[ \{(x_1,x_2): 0 < x_1, G_t x_1 < x_2 < F_t x_1\}, \]
and \( \omega^{\text{cusp}} \) is simply \( \chi_{B_0(\delta_t)} \cdot (\chi_{\Omega_1} - \omega^{\text{homog}}) \). To be concrete, modulo \( m \)-fold symmetry,
\[ \omega^{\text{cusp}}(x_1,x_2) = \begin{cases} +1 & \text{if } F_t x_1 < x_2 < f_t(x_1) \text{ or } g_t(x_1) < x_2 < G_t x_1, \\ -1 & \text{if } F_t x_1 > x_2 > f_t(x_1) \text{ or } g_t(x_1) > x_2 > G_t x_1. \end{cases} \]
inside the ball \( B_0(\delta_t) \). Then \( \omega^{\text{far}} \) is defined as \( \omega - \omega^{\text{homog}} - \omega^{\text{cusp}} \), and one note that it is supported outside the ball \( B_0(\delta_t) \). Then, accordingly, we obtain a decomposition of the velocity
\[ u = u^{\text{homog}} + u^{\text{cusp}} + u^{\text{far}}, \]
and we claim that \( u^{\text{cusp}} \) and \( u^{\text{far}} \) does not effect the dynamics of the tangent lines to the boundary curves \( (x_1,f_t(x_1)) \) and \( (x_1,g_t(x_1)) \) at the origin for all times. This clearly follows once we establish that \( |u^{\text{cusp}}(x)|, |u^{\text{far}}(x)| \lesssim |x|^{1+\alpha} \).

To begin with, the radially homogeneous component \( u^{\text{homog}} \) induces the same rotation speed on the tangent lines \( (x_1,F_t x_1) \) and \( (x_1,G_t x_1) \). However, since \( \nabla u \) is bounded for all time, the angle between \( (x_1,F_t x_1) \) and \( (x_1,g_t(x_1)) \), and also between \( (x_1,G_t x_1) \) and \( (x_1,g_t(x_1)) \) stays zero. Next, we

\(^{10}\)Strictly speaking, to apply the inverse function theorem, we need to estimate the \( C^{1,\alpha} \)-norm of the lower piece of the boundary at the same time.
know that \( u^{far} \) is \( C^{1,\alpha} \) (indeed, \( C^\infty \)) inside \( B_0(\delta/2) \). Therefore, the associated stream function \( \psi^{far} \) is \( C^{2,\alpha} \) in the ball\(^{11}\). Taylor expansion gives

\[
\psi^{far}(x_1, x_2) = A + Bx_1 + Cx_2 + D(x_1^2 + x_2^2) + Ex_1x_2 + O(|x|^{2+\alpha}).
\]

However, \( A = 0 \) by assumption and \( B = C = E = 0 \) is forced under the \( m \)-fold rotational symmetry. Furthermore, \( \Delta \psi^{far} \equiv 0 \) near 0 so that \( D = 0 \). In particular,

\[
\psi^{far}(x_1, x_2) = \nabla^2 \psi^{far}(x_1, x_2) = O(|x|^{1+\alpha})
\]
as \( |x| \to 0 \). Lastly, it is known that within each connected component of the complement of the (closure of the) support of \( \omega^{cusp} \), the associated velocity \( u^{cusp} \) is uniformly \( C^{1,\alpha} \) up to the boundary. It follows from our computations in Subsection \( 4.2 \) but also directly from the arguments of Friedman and Velázquez \( [46] \) (see the statement of their Lemma in Subsection \( 2.3 \)). Then, an identical argument as in the case of \( u^{far} \) shows that, this time, \( u^{cusp} \) is of order \( |x|^{1+\alpha} \) in the complement of the support of \( \omega^{cusp} \). The proof is now complete. \( \square \)

### 4.4 Multiple corners

In the above main result, we have only dealt with the case when there is a single corner in a sector of angle \( 2\pi/m \) (which serves as a fundamental domain for rotations by multiples of \( 2\pi/m \)). In this case, we have seen that the angle of the patch is preserved for all time. However, one may consider the case when there are several corners (separated from each other by some angle; see Figure \( 5 \)) in each case, we have seen that the angle of the patch is preserved for all time. However, one may consider the case when there are several corners (separated from each other by some angle; see Figure \( 5 \)) in each fundamental domain, and then some interesting dynamics for the angles can be observed.

We just note that an essentially identical proof carries over to this case to establish global wellposedness of such patches, and also the fact that the angles evolve exactly as in the case of infinite sectors, up to a constant overall rotation.

We just modify the last item from the Definition \( 4.1 \) to allow such patches:

- (multiple \( C^{1,\alpha} \) corners) There exists a \( C^{1,\alpha} \) diffeomorphism \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) of the plane with \( \Psi(0) = 0 \) and \( \nabla \Psi|_{x=0} = I \), such that for some \( \delta > 0 \), the image \( \Psi(\Omega_1) \) is a union of exact sectors with total angle less than \( 2\pi/m \):

\[
\Psi(\Omega_1) \cap B_0(\delta) = \bigcup_{j=1}^k S_{\beta_k + \zeta_k} \cap B_0(\delta)
\]

with some \( 0 < \zeta_j \) and \( -\pi \leq \beta_j < \pi \) satisfying

\[
\beta_j + \zeta_j < \beta_{j+1} \quad \text{and} \quad \beta_{j+1} + \zeta_{j+1} - \beta_1 < 2\pi/m \quad \text{for all} \quad j = 1, \ldots, k-1,
\]

(the ordering is well-defined on the interval \([ -\pi, \pi ] \), assuming without loss of generality that \( -\pi \leq \beta_1 < 0 \)).

Then as before, we define \( \Omega = \bigcup_{i=0}^{m-1} R_{2\pi i/m}(\Omega_1) \).

Alternatively, we may describe the patch locally as a union of approximate sectors with angles \( \zeta_1, \ldots, \zeta_k \), in counter-clockwise order, with gaps between them \( \gamma_1, \ldots, \gamma_k \) where \( \gamma_{j+1} := \beta_{j+1} - \beta_j - \zeta_j \). Note that given some value of \( m \geq 3 \), the values \( \zeta_1, \ldots, \zeta_k \) together with \( \gamma_1, \ldots, \gamma_k \) determine the local shape of the patch, up to a rotation of the plane.

\(^{11}\)Here, although \( \omega^{far} \) may have non-compact support, the Poisson problem \( \Delta \psi^{far} = \omega^{far} \) has a unique solution with \( \psi^{far} \in W^{2,\infty}(\mathbb{R}^2) \) under \( m \)-fold rotational symmetry with \( m \geq 3 \) and \( \psi^{far}(0) = 0 \); see \( [42] \) Lemma 2.6.
Corollary 4.5 (Dynamics of the angles). Assume that $\Omega_0$ is a symmetric $C^{1,\alpha}$-patch with multiple corners as defined in the above, with corner angles $\zeta_1(0), \ldots, \zeta_k(0)$ with separation angles $\gamma_{j+1/2}(0), \ldots, \gamma_{k-1/2}(0)$. Then, the angles evolve according to the following system of ordinary differential equations for all $t \in \mathbb{R}$:

$$
\frac{d\zeta_j(t)}{dt} = C_m \sin\left(\frac{m}{4} \zeta_j\right) \left[ \sum_{l=1}^{j-1} \sin\left(\frac{m}{4} \zeta_l\right) \cos\left(\frac{m}{4} (2(\beta_j - \beta_l) + (\zeta_j - \zeta_l))\right) - \sum_{l=j+1}^{k} \sin\left(\frac{m}{4} \zeta_l\right) \cos\left(\frac{m}{4} (2(\beta_j - \beta_l) + (\zeta_j - \zeta_l))\right) \right] \quad (4.10)
$$

and

$$
\frac{d\gamma_{j+1/2}(t)}{dt} = C_m \sin\left(\frac{m}{4} \gamma_{j+1/2}\right) \left[ \sum_{l=1}^{j} \sin\left(\frac{m}{4} \zeta_l\right) \cos\left(\frac{m}{4} ((\beta_{j+1} - \beta_l) + (\beta_j - \beta_l) + (\zeta_j - \zeta_l))\right) - \sum_{l=j+1}^{k} \sin\left(\frac{m}{4} \zeta_l\right) \cos\left(\frac{m}{4} ((\beta_{j+1} - \beta_l) + (\beta_j - \beta_l) + (\zeta_j - \zeta_l))\right) \right] \quad (4.11)
$$

with

$$\beta_j - \beta_l = (\gamma_{j-1/2} + \cdots + \gamma_{l+1/2}) + (\zeta_{j-1} + \cdots + \zeta_l), \quad j > l$$

for some constant $C_m > 0$ depending only on $m$.

Proof. It suffices to recall the 1D system describing the evolution of 0-homogeneous vorticities. On the unit circle, we are given initial vorticity

$$h_0(\theta) = \sum_{i=0}^{m-1} \sum_{j=1}^{k} S_{\beta_j + 2\pi i/m, \beta_i + \zeta_j + 2\pi i/m}. $$

Moreover, given $h$, the corresponding angular velocity (counter-clockwise rotation) on the circle is defined explicitly by

$$v(\theta) = \int_{-\pi/m}^{\pi/m} \left( c_m^1 \sin\left(\frac{m}{2} |\theta - \theta'|\right) - c_m^2 \right) h(\theta') d\theta' $$

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for some constants $c_m^1 > 0$ and $c_m^2$ depending only on $m \geq 3$. Since the integral of $h$ over the circle is conserved in time, one may redefine the angular velocity to be

$$\dot{v}(\theta) = c_m^1 \int_{-\pi/m}^{\pi/m} \sin \left( \frac{m}{2} (\theta - \theta') \right) h(\theta') \, d\theta'$$

up to an overall rotation. Therefore,

$$\frac{d}{dt} \gamma_j(t) = \dot{v}(\beta_j + \zeta_j) - \dot{v}(\beta_j)$$

$$= \sum_{l=1}^{j-1} c_m^1 \int_{\beta_l}^{\beta_l + \zeta_l} \left[ \sin \left( \frac{m}{2} (\beta_j + \zeta_j - \theta) \right) - \sin \left( \frac{m}{2} (\beta_j - \theta) \right) \right] \, d\theta$$

$$+ \sum_{l=j+1}^{k} c_m^1 \int_{\beta_{l-1}}^{\beta_l + \zeta_{l-1}} \left[ \sin \left( \frac{m}{2} (\theta - \beta_j - \zeta_j) \right) - \sin \left( \frac{m}{2} (\theta - \beta_j) \right) \right] \, d\theta$$

$$= \sum_{l=1}^{j-1} c_m^1 \sin \left( \frac{m}{4} \gamma_j \right) \sin \left( \frac{m}{4} \zeta_l \right) \cos \left( \frac{m}{4} (2(\beta_j - \beta_l) + (\zeta_j - \zeta_l)) \right)$$

$$- \sum_{l=j+1}^{k} c_m^1 \sin \left( \frac{m}{4} \gamma_{j+1/2} \right) \sin \left( \frac{m}{4} \zeta_l \right) \cos \left( \frac{m}{4} (\beta_j + (\zeta_j - \zeta_l)) \right)$$

(note that the contribution from the $j$-th sector cancels out) and the relations

$$\beta_j - \beta_l = (\gamma_{j-1/2} + \cdots + \gamma_{l+1/2}) + (\zeta_{j-1} + \cdots + \zeta_l), \quad j > l$$

enables us to express the right hand side in terms of $\gamma$'s and $\zeta$'s. Similarly,

$$\frac{d}{dt} \gamma_{j+1/2}(t) = \dot{v}(\beta_{j+1}) - \dot{v}(\beta_j + \zeta_j)$$

$$= \sum_{l=1}^{j} c_m^1 \sin \left( \frac{m}{4} \gamma_{j+1/2} \right) \sin \left( \frac{m}{4} \zeta_l \right) \cos \left( \frac{m}{4} ((\beta_{j+1} - \beta_l) + (\beta_j - \beta_l) + (\zeta_j - \zeta_l)) \right)$$

$$- \sum_{l=j+1}^{k} c_m^1 \sin \left( \frac{m}{4} \gamma_{j+1/2} \right) \sin \left( \frac{m}{4} \zeta_l \right) \cos \left( \frac{m}{4} ((\beta_{j+1} - \beta_l) + (\beta_j - \beta_l) + (\zeta_j - \zeta_l)) \right)$$

This finishes the proof. \[\square\]

### 4.5 Extensions

#### Generality of Serfati and Chemin

The results of Serfati [80], [81] and Chemin [23], [25], [24] demonstrates that propagation of boundary regularity for smooth patches is just a special instance – the Euler equations indeed propagates “striated” regularity of vorticity. Here we present the version given by Bae and Kelliher [8], [7].

To formally state the general result, assume that a family of $C^\alpha(\mathbb{R}^2)$ vector fields $\{Y^\lambda_0\}_{\lambda \in \Lambda}$ is given, and satisfies the following properties:

$$\inf_{x \in \mathbb{R}^2} \left( \sup_{\lambda} |Y^\lambda_0(x)| \right) \geq c_0 > 0$$

and

$$\sup_{\lambda} (\|Y^\lambda_0\|_{C^\alpha} + \|\nabla \cdot Y^\lambda_0\|_{C^\alpha}) < +\infty.$$
Moreover, assume that the initial vorticity satisfies

\[ \omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2), \quad \sup_{\lambda} \| (Y^\lambda_0 \cdot \nabla) \omega_0 \|_{C^{\alpha-1}} < +\infty. \]

The latter condition says that \( \omega_0 \) is \( C^\alpha \)-regular in the direction of \( Y^\lambda_0 \). The negative index Hölder spaces may be defined in terms of the Littlewood-Paley decomposition, but it can be avoided as the above condition is equivalent to \( K \ast ((Y^\lambda_0 \cdot \nabla) \omega_0) \in C^\alpha \) (see \[8\]), where \( K \) is the usual Biot-Savart kernel.

We evolve the family of vector fields by

\[ Y^\lambda_t(\Phi(t,x)) := (Y^\lambda_0(x) \cdot \nabla) \Phi(t,x). \]

**Theorem** (See Theorem 8.1 of \[8\]). In the above setting, the Yudovich solution \( \omega_t \) and the vector fields \( Y^\lambda_t \) satisfy the global-in-time bounds

\[ \sup_{\lambda} \| (Y^\lambda_t \cdot \nabla) \omega_t \|_{C^{\alpha-1}} \leq C \exp(\exp( ct )) \]  

(4.12)

and

\[ \sup_{\lambda} ( \| Y^\lambda_t \|_{C^\alpha} + \| \nabla \cdot Y^\lambda_t \|_{C^\alpha} ) \leq C \exp(\exp( ct )). \]  

(4.13)

The associated velocity is Lipschitz in space and indeed uniformly \( C^{1,\alpha} \) after being corrected by a smooth multiple of the vorticity. That is, there is a matrix \( A_t \) with \( \| A_t \|_{C^\alpha} \leq C \exp(\exp( ct )) \) such that

\[ \| \nabla u_t \|_{L^\infty} \leq C \exp(\exp( ct )). \]

\[ \| \nabla u_t - \omega_t A_t \|_{C^\alpha} \leq C \exp(\exp( ct )) \]  

(4.14)

holds. Here, the constants \( C, c > 0 \) depend only on \( 0 < \alpha < 1 \) and the initial data.

**Example.** Let us present two examples from \[8\] Section 10,.

(i) \( C^{1,\alpha} \) Patches with \( C^\alpha \) vortex profile: Take some \( C^{1,\alpha} \)-domain \( \Omega_0 \) and \( C^\alpha \)-function \( f_0 \), and then define \( \omega_0 = \chi_{\Omega_0} f_0 \). Then, we can take \( Y^1_0 := \nabla^\perp \phi_0 \) where \( \phi_0 \) is a \( C^{1,\alpha} \) level set function for \( \Omega_0 \). In addition, we can choose a vector field \( Y^2_0 \) so that \( \{ Y^1_0, Y^2_0 \} \) satisfy all the requirements described above (most importantly, \( Y^2_0 \) should not be vanishing whenever \( Y^1_0 \) vanishes).

We recover the usual vortex patch when the profile \( f_0 \) is a constant function. This result shows that the vorticity can actually have a \( C^\alpha \)-profile on the patch. This particular statement also follows directly from the main result of Huang \[52\], which we discuss in the Appendix.

(ii) Vorticity smooth along leaves of a \( C^{1,\alpha} \)-foliation: Consider \( \phi_0 \in C^{1,\alpha} \) with \( |\nabla^\perp \phi_0| \geq c > 0 \) on \( \mathbb{R}^2 \), such that each level curve of \( \phi_0 \) crosses any vertical line exactly once. Under these assumptions, we define \( \xi_{x_2}(x_2) \) so that \( \phi_0(x_1, \xi_{x_1}(x_2)) = \phi_0(0, x_2) \).

Take some bounded measurable function \( W : \mathbb{R} \to \mathbb{R} \) supported on some bounded interval \([c, d]\). Then, fix some \( L > 0 \) and define

\[ \omega_0(x_1, x_2) := \chi_{[-L, L]}(x_1) W(\xi_{x_1}(x_2)). \]

The above theorem applies to this case, simply with \( Y_0 = \nabla^\perp \phi_0 \). It follows that for all time, all the level curves of \( \omega \) remain (uniformly in \( \mathbb{R}^2 \)) \( C^{1,\alpha} \). In the words of Bae and Kelliher, “extreme lack of regularity of \( \omega_0 \) transversal to \( Y_0 \) does not disrupt the regularity of the flow lines.”
The generalization described in the above theorem can be easily adapted to our setting. Let us only described the necessary modifications in the assumptions. To begin with, we require that \( \omega_0 \in L^1 \cap L^\infty \) is \( m \)-fold symmetric for some \( m \geq 3 \), as usual. We need in addition that there is a distinguished vector field in the family, say \( Y_0^c \), which is \( m \)-fold symmetric and satisfies
\[
\inf_{x \in B_0(r_0)} |Y_0^c(x)| \geq c_0 > 0, \quad \text{for some } r_0 > 0,
\]
and
\[
\|Y_0^c\|_{C^\kappa(\mathbb{R}^2)} + \|\nabla Y_0^c\|_{C^\kappa(\mathbb{R}^2)} + \|K * ((Y_0^c \cdot \nabla)\omega_0)\|_{C^\kappa(\mathbb{R}^2)} < +\infty.
\]
Then, we claim that the bounds (4.12) and (4.13) hold, with \( \hat{C}_\kappa^\alpha \) instead of \( C^\alpha \) when \( \lambda = c \). Moreover, the velocity will be Lipschitz in space for all time, and its gradient will belong to \( \hat{C}_\kappa^\alpha \) after being corrected by a \( C^\alpha \)-matrix multiple of the vorticity.

**Symmetric Cusps**

Consider a \( m \)-fold symmetric set \( \Omega_0 \) which is a union of \( C^{1,\alpha} \)-cusps for some \( m \geq 3 \) in some ball \( B_0(r_0) \) and has \( C^{1,\alpha} \) boundary outside \( B_0(r_0) \).

It is possible to show that, using the methods of this paper (and the generalization described in the above), the boundaries of the cusp remain as \( C^{1,\alpha} \) (uniformly up to the origin) curves for all time. This can be done as a two-step procedure – the same strategy we have utilized to prove the propagation of \( C^{1,\alpha} \)-corners.

Note that the complementary region \( B_0(r_0) \backslash \Omega_0 \) is a disjoint union of regions, each of which can be given as the image of an exact sector under a \( C^{1,\alpha} \)-diffeomorphism of the plane fixing the origin. Therefore, in each of these regions, we can place a (divergence-free) vector field \( Y_0^c \) just as in the case of \( C^{1,\alpha} \)-corners (see Figure 6). This vector field can be extended to the interior of the cusp, so that \( Y_0^c \) is non-vanishing in \( B_0(r_0) \), \( Y_0^c \in C^\alpha \), and finally \( \nabla Y_0^c \in C^\alpha \). After that, one takes a complementary vector field \( Y_0^b \) which is \( C^\alpha \)(\( \mathbb{R}^2 \)), tangent to the boundary of the patch, with divergence in \( C^\alpha \)(\( \mathbb{R}^2 \)) and supported outside the ball \( B_0(r_0/2) \). This construction of vector fields \{\( Y_0^c, Y_0^b \)\} gives global-in-time propagation of the \( C^\alpha \)-regularity of the patch. Moreover, the velocity is Lipschitz in space for all time.

After that, to recover the extra information that the boundary of the cusp stays in \( C^{1,\alpha} \), one performs a local analysis which is parallel to the one given in [4.2]. Indeed, the velocity generated by the cusps is uniformly \( C^{1,\alpha} \) in the interior of the patch, up to the boundary. This finishes the argument.

### 5 Ill-posedness results for vortex patches with corners

In this section we will give several results which show that vortex patches with corners which do not fall in the well-posedness results cannot retain a corner structure continuously in time. These are based on a general local expansion result for the velocity field associated to a bounded vorticity profile. As is well known, boundedness of the vorticity does not imply Lipschitz continuity of the velocity field; however, it turns out to be possible to give a first-order expansion of the velocity field near the origin (or any point) which isolates the non-Lipschitzian part in an explicit way. This expansion is reminiscent of the Key Lemma of Kiselev and Šerák [59] but it is without any symmetry assumptions on the vorticity and it is valid for all \( x \in \mathbb{R}^2 \). After giving this expansion, we use it to show ill-posedness for vortex patches with corners. In the case where the vortex patch satisfies an odd symmetry, we can actually prove immediate cusp formation. When there is just a single corner, we just show discontinuity though

---

\( x_1 = 0 \) and have \( |\nabla Y(\bar{x})| \in L^\infty \), which is equivalent to saying that \( Y \in \hat{C} \). Finally, \( e \cdot \nabla Y = 2/x_1 \) and hence \( \nabla(\nabla \cdot Y) = (2/x_1, 0) \), and since \( |x| \leq 2, |x| \nabla(\nabla \cdot Y) | \in L^\infty \) as well.

---

To see this, consider the simple case of the \( C^{1,1} \)-cusp given by the region \( \{ -x_1^2 \leq x_2 \leq x_1^2, x_1 \geq 0 \} \). Then, define \( Y(x_1, x_2) = (1, x_2/x_1) \) in the interior of the cusp. Then, \( \partial_{x_1} Y = (0, 2/x_1) \) and \( \partial_{x_2} Y = -2x_2/x_1^2 \) so that \( |\nabla Y(\bar{x})| \in L^\infty \), which is equivalent to saying that \( Y \in \hat{C} \). Finally, \( \nabla \cdot Y = 2/x_1 \) and hence \( \nabla(\nabla \cdot Y) = (2/x_1, 0) \), and since \( |x_2| \geq x_1^2, |x| \nabla(\nabla \cdot Y) | \in L^\infty \) as well.
we believe that there is actually cusp formation and further investigation into this question is given in the next section. For corners which are only locally \( m \)–fold symmetric we also show ill-posedness by applying the results of the first author and Masmoudi [44] to show that the Lipschitz bound on the velocity field of a locally \( m \)–fold symmetric patch may be lost immediately. This shows that the global symmetry assumptions which give the propagation of regularity proven in the previous two sections cannot be replaced by local symmetry assumptions.

5.1 An expansion for the velocity field associated to a bounded vorticity profile

We now state the first and most important lemma toward the ill-posedness result which shows how one can expand the velocity field associated to a bounded vorticity profile near a point (the origin) up to “Lipschitz” error terms.

**Lemma 5.1.** Assume that \( \omega \in L^{\infty}(\mathbb{R}^2) \). Then, with polar coordinates, the corresponding velocity \( u = \nabla^\perp \Delta^{-1} \omega \) satisfies the estimate

\[
\left| u(r, \theta) - u(0) - \frac{1}{2\pi} \left( \frac{\cos \theta}{-\sin \theta} \right) r I^s(r) + \frac{1}{2\pi} \left( \frac{\sin \theta}{\cos \theta} \right) r I^c(r) \right| \leq C r \| \omega \|_{L^\infty} \tag{5.1}
\]

with some absolute constant \( C > 0 \) independent on the size of the support of \( \omega \). Here,

\[
u(0) = \left( -\frac{1}{2\pi} \int_0^\infty \sin(\theta) \omega(r, \theta) d\theta, \frac{1}{2\pi} \int_0^\infty \cos(\theta) \omega(r, \theta) d\theta \right)^T,
\]

\[
I^s(r) := \int_r^\infty \int_0^{2\pi} \sin(2\theta) \omega(s, \theta) \frac{s}{s} d\theta ds,
\]

and

\[
I^c(r) := \int_r^\infty \int_0^{2\pi} \cos(2\theta) \omega(s, \theta) \frac{s}{s} d\theta ds.
\]

**Remark 5.2.** The idea of the proof is to simply decompose \( \omega \) as

\[
\omega(r, \theta) = \sum_{m \geq 0} \left( \sin(m\theta) f^{m,s}(r) + \cos(m\theta) f^{m,c}(r) \right),
\]
and compute, more or less explicitly, the velocity vector field corresponding to each term on the right hand side.

**Proof.** Using polar coordinates, we write down the following decomposition of the vorticity:

\[
\omega = \omega^0 + \omega^1 + \omega^2 + \omega^r,
\]

where

\[
\omega^0(r) := \frac{1}{2\pi} \int_0^{2\pi} \omega(r, \theta) d\theta
\]

is the radial component,

\[
\omega^m(r, \theta) := \sin(m\theta) \frac{1}{\pi} \int_0^{2\pi} \sin(m\theta') \omega(r, \theta') d\theta' + \cos(m\theta) \frac{1}{\pi} \int_0^{2\pi} \cos(m\theta') \omega(r, \theta') d\theta'
\]

is the \(m\)-fold symmetric component for \(m = 1, 2\), and finally \(\omega^r := \omega - \omega^0 - \omega^1 - \omega^2\). Then, we can accordingly write

\[
u = \nu^0 + \nu^1 + \nu^2 + \nu^r,
\]

where \(\nu^m := \nabla^\perp \Delta^{-1} \omega^m\) for \(m \in \{0, 1, 2, r\}\). We estimate each component of velocity separately.

**Radial part**

We first solve \(\Delta \Psi = \omega\) assuming that \(\omega = \omega^0(r)\), i.e. when the vorticity is a radial function. In this case, it is well-known that the stream function is given by

\[
\Psi^0(r) = \int_0^r \frac{1}{s} \int_0^s \tau \omega^0(r) d\tau ds
\]

and the velocity is then

\[
u^0(r, \theta) = \frac{1}{r} \int_0^r s \omega^0(s) ds \left( \frac{-\sin \theta}{\cos \theta} \right).
\]

In particular,

\[
\left| \frac{\nu^0(r, \theta)}{r} \right| \leq C \|\omega^0\|_{L^\infty} \leq C \|\omega\|_{L^\infty}. \tag{5.2}
\]

**1-fold symmetric part**

Next, we assume that \(\omega = \omega^1(r, \theta) = \sin \theta f^{1,s}(r) + \cos \theta f^{1,c}(r)\). In this case, we write \(\Psi^1(r, \theta) = \sin \theta \psi^{1,s}(r) + \cos \theta \psi^{1,c}(r)\), and consider the equations

\[
\partial_{rr} \psi^{1,i} + \frac{1}{r} \partial_r \psi^{1,i} - \frac{1}{r^2} \psi^{1,i} = f^{1,i}, \quad i = s, c.
\]

We then have

\[
\partial_r \left( \partial_r \psi^{1,i} + \frac{1}{r} \psi^{1,i} \right) = f^{1,i}
\]

and from \(r^{-1} \psi^{1,s}(r), \partial_r \psi^{1,i}(r) \to 0\) as \(r \to \infty\),

\[
\frac{1}{r} \partial_r (r \psi^{1,i}) = \partial_r \psi^{1,i} + \frac{1}{r} \psi^{1,i} = - \int_{r}^{\infty} f^{1,i}(s) ds
\]
and hence
\[
\psi^{1,i}(r) = -\frac{1}{r} \int_{0}^{r} s \int_{s}^{\infty} f^{1,i}(\tau)d\tau ds.
\]
From the formula
\[
u^{1}(r, \theta) = \partial_{\tau} \psi^{1}(r, \theta) \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} - \frac{1}{r} \partial_{\theta} \psi^{1}(r, \theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},
\]
taking the first component, one obtains
\[
u_{1}^{1}(r, \theta) = \sin^{2} \theta \left( \frac{\psi^{1,s}}{r} + \int_{r}^{\infty} f^{1,s}(s) ds \right) - \cos^{2} \theta \frac{\psi^{1,c}(r)}{r} + \sin \theta \cos \theta \left( \frac{2\psi^{1,c}}{r} + \int_{r}^{\infty} f^{1,c}(s) ds \right).
\]
Observing the bound
\[
\left| \int_{r}^{\infty} f^{1,i}(s) ds \right| \leq Cr\|f^{1,i}\|_{L^{\infty}} \leq C\|\omega\|_{L^{\infty}}
\]
we rewrite \( u_{1}^{1} \) in the form
\[
u_{1}^{1}(r, \theta) = \frac{1}{2} \int_{r}^{\infty} f^{1,s}(s) ds - \left( \frac{1}{2} - \sin^{2} \theta \right) \left( \frac{2\psi^{1,s}}{r} + \int_{r}^{\infty} f^{1,s}(s) ds \right)
+ \sin \theta \cos \theta \left( \frac{2\psi^{1,c}}{r} + \int_{r}^{\infty} f^{1,c}(s) ds \right),
\]
and finally arrive at the following bound:
\[
\left| u_{1}^{1}(r, \theta) - \frac{1}{2} \int_{0}^{\infty} f^{1,s}(s) ds \right| \leq C\|\omega\|_{L^{\infty}}. \tag{5.3}
\]
On the other hand, for the second component of velocity we obtain
\[
\left| u_{1}^{2}(r, \theta) + \frac{1}{2} \int_{0}^{\infty} f^{1,c}(s) ds \right| \leq C\|\omega\|_{L^{\infty}}. \tag{5.4}
\]

2-fold symmetric part

We now need to solve \( \Delta \psi = \omega \) in the case when \( \omega(r, \theta) = \sin(2\theta)f^{2,s}(r) \) and \( \cos(2\theta)f^{2,c}(r) \), respectively. Setting \( \Psi(r, \theta) = \sin(2\theta)\psi^{2,s}(r, \theta) \) and \( \cos(2\theta)\psi^{2,c}(r, \theta) \) respectively gives the relations
\[
\partial_{rr}\psi^{2,i} + \frac{1}{r} \partial_{r}\psi^{2,i} - \frac{4}{r^{2}} \psi^{2,i} = f^{2,i}, \quad i = s, c. \tag{5.5}
\]
Then, one may rewrite it as
\[
\partial_{r} \left( \frac{1}{r} \partial_{r}\psi^{2,i} + \frac{2}{r^{2}} \psi^{2,i} \right) = \frac{f^{2,i}}{r},
\]
and since we are looking for a solution with bounds \( |\psi^{2,i}(r)| \leq C\ln(1+r) \) and \( |\partial_{r}\psi^{2,i}(r)| \leq C r^{-1} \), we obtain
\[
\frac{1}{r} \partial_{r}\psi^{2,i} + \frac{2}{r^{2}} \psi^{2,i} = - \int_{r}^{\infty} \frac{f^{2,i}(s)}{s} ds.
\]
Similarly as in the case of $u^2,s$, we rearrange it to obtain
\[
\psi^{2,i}(r) = -\frac{1}{r^2} \int_0^r s^3 \int_s^\infty \frac{f^{2,i}(\tau)}{\tau} d\tau ds
\]

Now, we compute $u^{2,s} = \nabla \perp (\sin(2\theta)\psi^{2,s})$ as well as $u^{2,c} = \nabla \perp (\cos(2\theta)\psi^{2,c})$. In the case of $u^{2,s}$, we have
\[
u^{2,s}(r, \theta) = \partial_r (\sin(2\theta)\psi^{2,s}) \left( \begin{array}{c} -\sin \theta \\ \cos \theta \\ \sin \theta \end{array} \right) - \frac{1}{r} \partial_\theta (\sin(2\theta)\psi^{2,s}) \left( \begin{array}{c} \cos \theta \\ \sin \theta \end{array} \right).
\]

Taking the first component, after a bit of rearranging we get:
\[
u^{1,s}(r, \theta) = -\sin(2\theta) \sin \theta \partial_r \psi^{2,s} - 2 \cos(2\theta) \cos \theta \frac{\psi^{2,s}}{r} = \sin(2\theta) \sin \theta \left( \frac{4}{r} \psi^{2,s} + r \int_r^\infty \frac{f^{2,s}}{s} ds \right) - \left( \frac{r}{2} \int_r^\infty \frac{f^{2,s}}{s} ds - \frac{r}{2} \int_r^\infty \frac{f^{2,s}}{s} ds + \frac{2}{r} \psi^{2,s} \right) \cos \theta
\]

We note that
\[
\left| \frac{4}{r} \psi^{2,s} + r \int_r^\infty \frac{f^{2,s}}{s} ds \right| \leq C \| f^{2,s} \|_{L^\infty}
\]
uniformly in $r \geq 0$, with some absolute constant $C > 0$. Hence, we obtain
\[
u^{1,s}(r, \theta) = \frac{r \cos \theta}{2} \int_r^\infty \frac{f^{2,s}}{s} ds + 2 \sin(2\theta) \sin \theta \cos \theta \frac{\psi^{2,s}}{2} + \left( \frac{4}{r} \psi^{2,s} + r \int_r^\infty \frac{f^{2,s}}{s} ds \right)
\]

and
\[
\left| \frac{\nu^{1,s}(r, \theta)}{r \cos \theta} - \frac{1}{2} \int_r^\infty \frac{f^{2,s}}{s} ds \right| \leq C \| f^{2,s} \|_{L^\infty} \leq C \| \omega \|_{L^\infty}.
\] (5.6)

Similarly, for the second component of velocity, we obtain
\[
\left| \frac{\nu^{2,s}(r, \theta)}{r \sin \theta} + \frac{1}{2} \int_r^\infty \frac{f^{2,s}}{s} ds \right| \leq C \| f^{2,s} \|_{L^\infty} \leq C \| \omega \|_{L^\infty}.
\] (5.7)

Next, in the case of $u^{2,c}$,
\[
u^{2,c}(r, \theta) = \partial_r (\cos(2\theta)\psi^{2,c}) \left( \begin{array}{c} -\sin \theta \\ \cos \theta \\ \sin \theta \end{array} \right) - \frac{1}{r} \partial_\theta (\cos(2\theta)\psi^{2,c}) \left( \begin{array}{c} \cos \theta \\ \sin \theta \end{array} \right).
\]

Similarly as in the case of $u^{2,s}$, we rearrange it to obtain
\[
u^{2,c}(r, \theta) = -\left( \frac{4}{r} \psi^{2,c} + r \int_r^\infty \frac{f^{2,c}}{s} ds \right) \left( \begin{array}{c} \cos(2\theta) \sin \theta \\ -\cos(2\theta) \cos \theta \end{array} \right)
\]

\[+ \left( \frac{r}{2} \int_r^\infty \frac{f^{2,c}}{s} ds - \frac{r}{2} \int_r^\infty \frac{f^{2,c}}{s} ds + \frac{2}{r} \psi^{2,c} \right) \left( \begin{array}{c} \sin \theta \\ \cos \theta \end{array} \right),
\]

and in particular,
\[
\left| \frac{\nu^{1,c}(r, \theta)}{r \sin \theta} + \frac{1}{2} \int_r^\infty \frac{f^{2,c}}{s} ds \right|, \quad \left| \frac{\nu^{2,c}(r, \theta)}{r \cos \theta} + \frac{1}{2} \int_r^\infty \frac{f^{2,c}}{s} ds \right| \leq C \| \omega \|_{L^\infty}.
\] (5.8)
Remainder

We shall assume that \(\omega(r, \theta) = \omega^r(r, \theta)\), the point being that
\[
\int_0^{2\pi} \omega^r(r, \theta) \cos(m\theta) d\theta = 0 = \int_0^{2\pi} \omega^r(r, \theta) \sin(m\theta) d\theta
\]
for all \(r \geq 0\) and \(m = 0, 1, 2\). In this case, we simply use the Biot-Savart kernel:
\[
u^1(x) = \frac{1}{2\pi} \int_{|y| \leq 2|x|} \frac{(x - y)^4}{|x - y|^2} \omega^r(y) dy.
\]
Taking the first component, we write
\[
u^1_1(x) = \frac{1}{2\pi} \int_{|y| \leq 2|x|} \frac{-(x_2 - y_2)}{|x - y|^2} \omega^r(y) dy + \frac{1}{2\pi} \int_{|y| > 2|x|} \frac{-(x_2 - y_2)}{|x - y|^2} \omega^r(y) dy,
\]
and the first piece is bounded by
\[
\left| \frac{1}{2\pi} \int_{|y| \leq 2|x|} \frac{-(x_2 - y_2)}{|x - y|^2} \omega^r(y) dy \right| \leq C\|\omega^r\|_{L^\infty} \int_{|y| \leq 2|x|} \frac{1}{|x - y|} dy \leq C|x|\|\omega\|_{L^\infty}.
\]
Regarding the second piece, we first use that
\[
\int_{|y| > 2|x|} \frac{y_2}{|y|^2} \omega^r(y) dy = 0
\]
to rewrite it as
\[
-\frac{1}{2\pi} \int_{|y| > 2|x|} \left[ \frac{x_2 - y_2}{|x - y|^2} + \frac{y_2}{|y|^2} \right] \omega^r(y) dy,
\]
and note that
\[
\frac{x_2 - y_2}{|x - y|^2} + \frac{y_2}{|y|^2} = -x_1(2y_1 y_2) + x_2(y_1^2 - y_2^2) + y_2|x|^2.
\]
Then, using that
\[
\int_{|y| > 2|x|} \frac{2y_1 y_2}{|y|^4} \omega^r(y) dy = 0 = \int_{|y| > 2|x|} \frac{y_1^2 - y_2^2}{|y|^4} \omega^r(y) dy,
\]
we have that
\[
\frac{1}{2\pi} \int_{|y| > 2|x|} \frac{-(x_2 - y_2)}{|x - y|^2} \omega^r(y) dy = -\frac{1}{2\pi} \int_{|y| > 2|x|} \left[ \frac{x_2 - y_2}{|x - y|^2} + \frac{y_2}{|y|^2} + x_1 \frac{2y_1 y_2}{|y|^4} - x_2 \frac{y_1^2 - y_2^2}{|y|^4} \right] \omega^r(y) dy,
\]
and the expression in the large brackets equals
\[
y_2|x|^2|y|^2 + (x_2(y_1^2 - y_2^2) - 2x_1 y_1 y_2)(2x \cdot y - |x|^2)
\]
\[
|x - y|^2|y|^4
\]
is bounded in absolute value by \(C|x|^2|y|^{-3}\) for some uniform constant \(C > 0\) in the region \(|y| > 2|x|\). Therefore,
\[
\left| \frac{1}{2\pi} \int_{|y| > 2|x|} \frac{-(x_2 - y_2)}{|x - y|^2} \omega^r(y) dy \right| \leq C|x|^2\|\omega\|_{L^\infty} \int_{|y| > 2|x|} \frac{1}{|y|^3} dy \leq C|x|\|\omega\|_{L^\infty}.
\]
We have shown the desired bound
\[ |u_1^*(x)| \leq C|x|\|\omega\|_{L^\infty}. \quad (5.9) \]
A completely parallel argument establishes that
\[ |u_2^*(x)| \leq C|x|\|\omega\|_{L^\infty}. \quad (5.10) \]
Combining the estimates (5.2), (5.3), (5.4), (5.6), (5.7), (5.8), (5.9), and (5.10) finishes the proof.

5.2 Loss of boundary regularity for odd-odd patches
In this subsection, we demonstrate that under the odd-odd symmetry, vortex patches with a corner may continuously lose boundary regularity in time. By the odd-odd symmetry, we mean that the vorticity satisfies
\[ \omega(x_1, x_2) = -\omega(-x_1, x_2) = -\omega(x_1, -x_2) \]
on \([\mathbb{R}^2] \) (or on \( \mathbb{T}^2 = [-1, 1]^2 \)). Equivalently, one may consider vorticities which is odd in \( x_1 \) on the upper half-plane \( \mathbb{R} \times \mathbb{R}_+ \), with the slip boundary condition.

In the result below, we consider an odd-odd patch with four corners meeting at the origin and also tangent to the \( x_1 \)-axis (see Figure 7). It shows that the “angle” of each corner at the origin immediately becomes \( \pi/2 \) for \( t > 0 \), and 0 for \( t < 0 \).

\textbf{Theorem 3.} Consider an odd-odd vortex patch supported on \( \Omega_0 \subset \mathbb{R}^2 \) such that
\[ \Omega_0 \cap \{(x_1, x_2) : 0 \leq x_1, x_2 \leq 1/2\} = \{(x_1, x_2) : 0 \leq x_1 \leq x_2 \leq 1/2\} . \]
Consider the trajectories of points which initially lie on the diagonal \( \Phi(t, (x, x)) =: z(t, x) \). Then, there exist constants \( T^*, \delta > 0 \), such that in the ball \([0, \delta]^2\), we have bounds
\[ 2z_1(t, x)^{\beta(t)} \geq z_2(t, x) \geq \frac{1}{2} z_1(t, x)^{\alpha(t)}, \]
for some strictly decreasing, positive, and continuous functions defined on \([0, T^*]\) with \( \alpha(0) = \beta(0) = 1 \).
In particular, the angle of the patch at the origin becomes immediately \( \pi/2 \). On the other hand, if one considers the backwards in time evolution, there exists some time interval \([-T', 0)\) with \( T' > 0 \), during which the angle of the patch at the origin is zero.

One may formally define the (cosine of the) angle as follows: given a domain \( U \subset [0, 1]^2 \) (which is assumed to intersect any small square \([0, \delta]^2\)),
\[ \cos \theta_U := \lim_{\delta \to 0^+} \sup_{x', x'' \in U \cap [0, \delta]^2} \frac{x \cdot x'}{||x|| ||x'||} . \]

We may consider these data on the upper half-plane, and in this case, the boundary of the initial patch \( \partial \Omega_0 \) is given as the graph of a \( C^{0,1} \) and \( C^{1,0} \)-function, respectively, near the origin. This property is not maintained for any small time \( t > 0 \). On the other hand, if initially one considers odd-odd patch given by the region below the graph of a \( C^{1,0} \)-function whose derivative vanish at the origin (see Figure 8), it can be shown using the ideas of previous sections that the solution continues to satisfy these properties. See also the recent work of Kiselev, Ryzhik, Yao, and Zlatoš [58] where they show (among other things) well-posedness for cusps touching the boundary. In this sense, these two results show that the vortex patch problem is ill-posed, when its boundary on the upper half-plane is only \( C^{0,1} \) or \( C^1 \).
Remark 5.3. The second part of the theorem was established in an earlier work by Hoff and Per- pelitsa \[51\], with a similar patch initial data but having just one odd symmetry with respect to the \(x_1\)-axis. It is expected that the dynamics in that case is equivalent to the odd-odd symmetry case, up to a translation of the corner point. Here we offer a simplified proof using the Key Lemma.

We now recall the key lemma of Kiselev-Šverák \[59\] and Zlatoš \[94\]:

**Lemma 5.4.** Let \(\omega(t, \cdot) \in L^\infty(\mathbb{R}^2)\) be odd-odd. For \(x_1, x_2 \in (0, 1/2]\), we have

\[
(-1)^j u_j(t, x) = \frac{4}{\pi} \int_{Q(2x)} \frac{y_1 y_2}{|y|^4} \omega(t, y)dy + B_j(t, x)
\]

where \(Q(2x) := [2x_1, 1] \times [2x_2, 1]\) and

\[
|B_j(t, x)| \leq C \|\omega(t, \cdot)\|_{L^\infty} \left(1 + \ln \left(1 + \frac{x_{3-j}}{x_j}\right)\right)
\]

for \(j \in \{1, 2\}\).

**Remark 5.5.** We note that this actually follows from the more general expansion given in Lemma 5.1.

For simplicity of notation, we will denote the integral in (5.13) as

\[
I(t, x) := \frac{4}{\pi} \int_{Q(2x)} \frac{y_1 y_2}{|y|^4} \omega(t, y)dy .
\]

On the other hand, we have the well-known log-Lipschitz bound for velocity: for any \(x, x'\) with \(|x - x'| < 1/2\),

\[
|u(t, x) - u(t, x')| \leq C \|\omega(t, \cdot)\|_{L^\infty} |x - x'| \ln \frac{1}{|x - x'|} .
\]
Proof of Theorem 3. We consider the case \( t \geq 0 \). We shall work within a short time interval \([0, T^*]\) for some \( T^* > 0 \), and in several places, the value of \( T^* \) will be taken to be sufficiently small for the arguments to work.

We begin with a simple observation. Note that on the diagonal \( x = (x', x') \) with \( 0 < x' \), we have

\[
\frac{u_1(t, x)}{u_2(t, x)} = -\frac{I(t, x) + B_1(t, x)}{I(t, x) + B_2(t, x)}
\]

and \( |B_2(t, x)| \leq C \) for all time. Clearly, one can find some small \( \delta_1 > 0 \) and \( T^* > 0 \) such that \( I(t, (\delta_1, \delta_1)) \geq 10C \) for all \( 0 \leq t \leq T^* \), simply because \( I(t, (\delta_1, \delta_1)) \) is continuous in \( t, \delta_1 \) and \( I(0, (\delta_1, \delta_1)) \to +\infty \) as \( \delta_1 \to 0^+ \). Therefore, we take \( \delta_1 \geq \delta_2 > 0 \) such that the triangle \( \{0 \leq x_1 \leq x_2 \leq \delta_2\} \) is contained in \( \Phi(t, \Omega_0) \) for all \( 0 < t < T^* \).

Consider a point on the diagonal \((x, x)\) with \( 0 < x < \delta \ll \delta_2 \) (the value of \( \delta > 0 \) will be specified later) and denote its trajectory by \( z(t, x) = (z_1(t, x), z_2(t, x)) := \Phi(t, (x, x)) \). From the basic log-Lipschitz estimate on \( u_2 \),

\[
\left| \frac{d}{dt} z_2(t, x) \right| \leq C z_2(t, x) \ln \frac{c}{z_2(t, x)},
\]

(since \( u_2(t, (z_1(t, x), 0)) = 0 \) by odd symmetry) and upon integration, we deduce that \( z_2(t, x) \leq c e^{\exp(-Ct)} \). Proceeding analogously for \( z_1(t) \), we obtain \( z_1(t, x) \geq c x^{\exp(Ct)} \) this time. Inserting these crude bounds,

\[
|B_1(t, z(t, x))| \leq C \left( 1 + \exp(2CT^*) \ln \frac{1}{x} \right)
\]  

(5.14)

for \( 0 \leq t \leq T^* \). Moreover, with \( \beta(t) := \exp(-2Ct) \), we obtain

\[
\frac{z_1(t, x)^\beta(t)}{2} \geq z_2(t, x),
\]

for \( 0 \leq t \leq T^* \) by choosing \( T^* \) smaller if necessary. A lower bound for the integral \( I(t, z(t, x)) \) comes from the fact that \( \Phi(t, \Omega_0) \) contains a triangle.

We could have chosen \( \delta > 0 \) small so that for all \( x < \delta \), its trajectory satisfies the bound \( z_2(t) \leq \delta_2 \). In particular, the region \( Q(z(t, x)) \) contains the triangle with vertices \((z_2(t, x), z_2(t, x)), (\delta_2, z_2(t, x)), (\delta_2, \delta_2)\). Hence

\[
I(t, z(t, x)) \geq c \ln \frac{\delta_2}{z_2(t, x)} \geq c \exp(-Ct) \ln \frac{\delta_2}{x}
\]

and comparing this with (5.14), we could have chosen \( \delta_1, T^* > 0 \) smaller so that for \( 0 < t < T^* \) and \( 0 < x < \delta \),

\[
I(t, z(t, x)) \geq \frac{1}{10} |B_1(t, z(t, x))|.
\]

Therefore, we may neglect the \( B_1 \)-term in (5.13) at the cost of changing the multiplicative constant, and deduce

\[
- \frac{u_1(t, z(t, x))}{z_1(t, x)} \geq c \ln \frac{\delta_1}{x}.
\]

In turn, this ensures that

\[
z_1(t, x) \leq c' x^{1+ct}
\]

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with $c' \to 1$ as $T^* \to 0^+$. From the trivial bound $z_2(t, x) \geq x$, we obtain
\[ z_2(t, x) \geq \frac{1}{2} z_1(t, x)^{\alpha(t)}, \]
with $\alpha(t) := (1 + c t)^{-1}$. This finishes the proof of the first part.

We now consider the backwards in time dynamics. Instead of reversing time, we revert the sign of vorticity, so that now initially the direction of velocity is southeast on the diagonal segment. As in the above, we set $z(t, x) := \Phi(t, (x, x))$, and restrict our attention to $0 < x < \delta$ and $0 \leq t \leq T'$, for small $\delta, T' > 0$ to be chosen below. Similarly as before, using either the Key Lemma or the log-Lipschitz estimate on velocity gives
\[ z_2(t, x) \leq z_1(t, x) \leq 2 z_2(t, x)^{\gamma(t)} \leq 2 x^{\gamma(t)} \]
for all $0 < x < \delta$ and $0 \leq t < T'$ with some sufficiently small $\delta, T' > 0$. Here $\gamma(t) > 0$ is some continuous monotonically decreasing function with $\gamma(0) = 1$ and $\gamma(t) < 1 - c t$. This gives a lower bound on the integral
\[ I(t, z(t, x)) = I(t, (z_1(t, x), z_1(t, x))) \geq c \frac{\delta^{\eta(t)} - z_1^{\eta(t)}}{\eta(t)} \geq c \frac{\delta^{\eta} - x^{\eta}}{ct} \]
where $\eta(t) = 2(1/\gamma(t) - 1) \geq t$ satisfies $\eta(0) = 0$ and is monotonically increasing with $t$. Applying the Key Lemma to each of $z_1(t, x)$ and $z_2(t, x)$, we see that
\[ \frac{d}{dt} \frac{z_1(t, x)}{z_2(t, x)} \geq z_1(t, x) \left( 2\int(t, z(t, x)) - C \ln \left( 1 + \frac{z_1(t, x)}{z_2(t, x)} \right) \right), \]
or equivalently,
\[ \frac{d}{dt} \ln \left( 1 + \frac{z_1(t, x)}{z_2(t, x)} \right) \geq c \frac{\delta^{\eta} - x^{\eta}}{ct} - C \ln \left( 1 + \frac{z_1(t, x)}{z_2(t, x)} \right). \]
Fix some small $x > 0$ and consider the ODE
\[ \frac{d}{dt} f(x)(t) = \frac{\delta^{\eta} - x^{\eta}}{ct} - C f(x)(t), \quad f(0) = \ln 2. \]
It is straightforward to show that, for all sufficiently small $0 < x < T'$, we have
\[ \lim_{x \to 0^+} \frac{f(x)(t)}{x} = +\infty. \]
Then, this implies that
\[ \lim_{x \to 0^+} \frac{z_1(t, x)}{z_2(t, x)} = +\infty \]
for all $0 < t < T'$, which shows that the angle of the patch is zero in the same time interval. \[ \Box \]

**Remark 5.6.** In the result above, the initial corner angle of the patch can be an arbitrary number strictly between $0$ and $\pi/2$. Moreover, a straightforward modification of the proof shows that there is an initial patch with corner angle zero whose angle immediately becomes $\pi/2$ for $t > 0$. This can be done for instance using a patch of the form
\[ cl(\Omega_0) \cap [0, 1/2]^2 = \left\{ (x_1, x_2) : 0 \leq x_2 \leq x_1 \left( \ln \ln \frac{1}{x_1} \right)^{-1} \right\} \cap [0, 1/2]^2. \]

**Remark 5.7.** We note that very recently, the key lemma was utilized to obtain double exponential rate of growth in time for the curvature of smooth vortex patches touching the horizontal axis – see [61].
5.3 Non-continuity of the angle in a vortex patch with a corner

In this subsection, we prove two non-continuity results. The first is descriptive in that it gives a lower bound on the angular movement of particles close to the corner. The second one asserts that if at any time a good portion of the mass of a vortex patch (or sequence of vortex patches) asymptotically is in a sector, then the location of that sector cannot be continuous in time. We now state and prove our first theorem in this direction. The reader should take note that the first theorem is easier to prove and contains two of the ideas contained in the proof of the second theorem.

**Theorem 4.** Assume that $\Omega_0 \subset B_{1/2}(0)$ is a 2-fold symmetric open set and $\Omega_0 \cap B_{1/4}(0) = S_{-\theta_1, \theta_1} \cup S_{\pi - \theta_1, \pi + \theta_1}$ with $\theta_1 \in (0, \pi/2) \cup (\pi/2, \pi)$. Then, there exists a fixed constant $c > 0$, a sequence of radii $\epsilon_n \to 0$, and a sequence of times $t_n \to 0$ so that for all $n \in \mathbb{N}$ we have

$$\theta_1 + c \leq \arctan \left( \frac{\Phi_2(t_n, x)}{\Phi_1(t_n, x)} \right) \leq \theta_1 + 10c$$

for all $x$ with $\epsilon_n \leq |x| \leq 2\epsilon_n$.

**Remark 5.8.** The theorem indicates that the patch “jumps” up at least by the angle $c$ once $t > 0$.

**Proof.** First, it is not difficult to show that since $\Omega_0 \subset B_{1/2}(0)$, we have that for all $t \geq 0$

$$|u(x) - u(y)| \leq 100|x - y||\log |x - y||$$

for all $|x - y| < \frac{1}{4}$. To see this, we need only observe that $||\omega||_{L^\infty} \leq 1$ and $||\omega||_{L^1} \leq 1$ and run a standard potential theory argument. Moreover, it suffices to consider the case $\theta_1 \in (0, \pi/2)$, since otherwise one can argue with the complement of $\Omega_0$ instead, which has acute angles.

Next, we define $\Phi(t, x_0)$ to be the position of a particle initially at $x_0$ at time $t$. Using the log-Lipschitz bound on the velocity field above and the (generalized) Gronwall lemma we have the following bounds on a small interval of time:

$$|x|^{1+2Ct} \leq |x|^e \leq |\Phi(t, x)| \leq |x|^{e^{-Ct}} \leq |x|^{1-2Ct}.$$

Now we take $\gamma_1 = \min\left\{ \frac{1}{40C}, \frac{1}{30} \right\}$ so that if $|t| \leq t_0 := \frac{\gamma}{\log |x_0|}$ we get that

$$\frac{1}{2} \leq \frac{|\Phi(t, x)|}{|x|} \leq 2$$

if $|x| \geq \frac{1}{10}|x_0|$ where we used that $|x_0| \frac{1}{\log |x_0|} = \sqrt{e}$. Next, we claim that if $x \in \partial \Omega_0 \cap B_{\frac{1}{4}}(0) \cap \{x_1, x_2 \geq 0\}$ is so that $|x| \geq \frac{1}{10}|x_0|$ and if $t < \frac{\gamma}{\log |x_0|}$ with $\gamma \leq \gamma_1$ small enough we have:

$$\theta_1 - \epsilon \leq \arctan \left( \frac{\Phi_2(t_0, x)}{\Phi_1(t_0, x)} \right) \leq \theta_1 + \epsilon < \frac{\pi}{2}$$

for given $\epsilon > 0$ small. Indeed, for all such $x$ and $t$ we have that

$$|u(t, x)| \leq 2C|x||\log |x_0||.$$

Consequently,

$$|\Phi(t, x_0) - x_0| \leq C\gamma|x|$$

from which the claim follows. The reader should notice that $\gamma$ depends linearly on $\epsilon$. In particular, if $|x| \geq |x_0|$ and $t < \frac{\gamma}{\log |x_0|}$, the “bulk” of the vortex patch does not move too much.
Let $A_0 = \Omega_0 \cap \{\frac{t}{\gamma} | x_0 | \leq x \leq 4 | x_0 | \}$. From Lemma 5.1 we now see that we have the following bound for all $t \in [0, \frac{\gamma}{\log |x_0|}]$:

$$|u(t, x) - u(0, x)| \leq 10|c| \log |x| + C|x| \leq 20|c| \log |x|$$

if $|x|$ is sufficiently small. However, invoking Lemma 5.1 again we see that $u_0(x) = \sin(2\theta_1) \log \frac{1}{|x|} (x_2, x_1) + C|x|$ for $C$ a fixed constant. Now we notice, putting together the preceding considerations, that

$$\frac{d}{dt}\left(\frac{\Phi_2(t, x)}{\Phi_1(t, x)}\right) \geq \frac{\sin(2\theta_1)}{2} \log |x_0| \left(1 - \frac{\Phi_2(t, x)^2}{\Phi_1(t, x)^2} - C\epsilon \right).$$

It is now easy to see that $\frac{\Phi_2(t, x)}{\Phi_1(t, x)}$ grows by a fixed constant depending only on $\gamma$ over the time interval $[0, \frac{\gamma}{\log |x_0|}]$ by rescaling time by $\log |x_0|$. This concludes the proof.

It is possible that this proof can be strengthened to actually give cusp formation for such an initial patch. Essentially, for points of size $\epsilon$, we can only track the evolution of the point for time $\epsilon \log(\epsilon)^{-1}$ where $c$ is a small universal constant. (Hence the variable $t \ln \frac{1}{\epsilon}$ is natural, which is used explicitly in Section 6). Any improvement in this time-scale would be a step forward. Our next result is slightly more general in that we make no assumption on the initial configuration of the patch except to say that the patch is asymptotically close to a sector as $r \to 0$.

**Theorem 5.** There exists an absolute constant $M > 0$ such that there are no angles $\theta_1(t)$ and $\theta_2(t)$ which depend continuously on any time interval $[0, \delta]$ with the property that

$$\limsup_{r \to 0} \sup_{t \in [0, \delta]} \frac{|(\Omega(t)) \Delta S_{\theta_1(t), \theta_2(t)} \cap B_r(0)|}{|B_r(0)|} < \frac{\theta_2(0) - \theta_1(0)}{M}, \quad (5.15)$$

if $0 < \theta_2(0) - \theta_1(0) < \frac{\pi}{2}$.

**Remark 5.9.** This theorem says that the vortex patch $\Omega(t)$ near $x = 0$ cannot asymptotically be approximated by a sector even with a small (but non-zero) error depending on the initial size of the angle. Of course, this implies that the corner could never remain a regular corner continuously in time since that would imply that the limit in the statement of the theorem vanishes. Note also that this theorem also applies to the case of several (or infinitely many) vortex patches. One way to interpret the result is to say that acute or obtuse corners cannot be formed dynamically in time.

**Remark 5.10.** We give the statement and proof for 2-fold symmetric patches but the proof remains the same for single corners. The only difference is that we have to factor out translation with respect to the velocity at the corner. Otherwise, the expansion of the velocity field and all other arguments are identical.

**Proof.** Toward a contradiction, assume such $\theta_1(t), \theta_2(t)$ and $\delta > 0$ exist. By rotation invariance and continuity, we may assume that $\theta_1(0) = -\theta_2(0)$ while $-\pi/4 < \theta_1(t) < 0 < \theta_2(t) < \pi/4$ for all $t \in [0, \delta]$. Now let us expand the velocity field $u$. It is easy to see that (see explicit computations in Subsection 2.2)

$$u_0(x) = \frac{1}{2\pi} \sin(2\theta_2(0)) \log \frac{1}{|x|} \left(\frac{x_2}{x_1}\right) + O(|x|)$$

as $|x| \to 0$. Now define $\alpha := \frac{1}{2\pi} \sin(2\theta_2(0)) > 0$. Then we have

$$u(t, x) = \alpha \log \frac{1}{|x|} \left(\frac{x_2}{x_1}\right) + O((\epsilon_1 + \frac{\alpha}{M}) |x| \log \frac{1}{|x|})$$
as \( t, |x| \to 0 \). In particular, if \( |x| \) and \( t \) are small enough (depending only on \( \alpha \)), we can essentially neglect the second term in regions where the sizes of \( x_1 \) and \( x_2 \) are comparable. To see this, by continuity of \( \theta_i \) there exists \( t_1 < \delta \) so that if \( t \in [0, t_1] \),

\[
|\sin(2\theta_2(t)) - \sin(2\theta_1(t)) - 2\sin(2\theta_2(0))| + |\cos(2\theta_2) - \cos(2\theta_1(t))| < \epsilon_1
\]

for given \( \epsilon_1 > 0 \). Next, from the assumption \( 5.15 \) there exists \( \delta_1 > 0 \) so that if \( x \in B_{\delta_1}(0) \) we have that

\[
|u(t, x) - u_{\theta_1(t), \theta_2(t)}(x)| \leq \frac{C\alpha}{M} \log \frac{1}{|x|}.
\]

Here \( u_{\theta_1(t), \theta_2(t)}(x) \) is the velocity associated with \( S_{\theta_1(t), \theta_2(t)} \cup S_{\theta_1(t) + \pi, \theta_2(t) + \pi} \).

Next, we claim that there exist \( \delta_2 > 0 \) and \( t_2 < t_1 \) so that for all \( x_0 \in B_{\delta_2}(0) \) we have that the solution to the ODE:

\[
\Phi(t) = u(t, \Phi(t))
\]

\[
\Phi(0) = x_0
\]

satisfies that \( \Phi(t) \in B_{\delta_1}(0) \) for all \( t \in [0, t_2] \). This is due to the trivial estimate:

\[
|\Phi(t)| \leq |x_0| \exp(-Ct)
\]

which we know from the Yudovich theory. In particular, we may take \( \delta_2 = \delta_1^2 \) and let \( t_2 < t_1 \) small independent of \( \delta_1 \).

Summing up the preceding considerations, given \( \epsilon > 0 \), we can find a radius \( \delta_2 > 0 \) and a time interval \( [0, t_2] \) so that for all \( x_0 \in B_{\delta_2}(0) \) the associated trajectory \( \Phi(t, x) \) remains in the ball \( B_{\delta_1}(0) \) for all \( t \in [0, t_2] \) where we know:

\[
|u(t, x) - x_0 - \alpha \log \frac{1}{|x|} \binom{x_2}{x_1}| \leq \epsilon|x| \log \frac{1}{|x|}.
\]

Notice that the vector field \( (x_2, x_1) \) is tangent to the lines \( z_2 = z_1 \) and \( z_2 = -z_1 \) and that the flow associated to this vector field is hyperbolic near 0. We now want to observe that \( u \) is “almost” hyperbolic: indeed, consider the region \( V_1 = \{ 0 \leq (1 - \eta)z_1 \leq z_2 \leq (1 + \eta)z_1 \} \) for \( \eta = 3\epsilon < \frac{1}{2} \). We claim that a particle \( X(0) \) starting in \( B_{\delta_1}(0) \cap V_1 \) never escapes \( \cap B_{\delta_1}(0) \cap V_1 \) for \( t \in [0, t_2] \). Indeed, by construction, we know that \( X(t) \in B_{\delta_1}(0) \) for all \( t \in [0, t_2] \). Thus, to conclude, it suffices to show that \( u(z) \cdot n(z) < 0 \) for all \( z \in \partial V_1 \cap B_{\delta_1}(0) \setminus \{ 0 \} \) where \( n(z) \) is the unique outer normal to \( V_1 \) at such \( z \). Now we compute: If \( z_2 = (1 + \eta)z_1 \) we have

\[
\sqrt{(1 + \eta)^2 + 1(u(z) \cdot n(z))} = -\alpha(1 + \eta)u_1(z) + u_2(z) \leq \alpha(-(1 + \eta)^2 + 1)z_1 \log \frac{1}{|z|} + \epsilon\alpha|z| \log \frac{1}{|z|} < 0.
\]

Similarly, if \( z_2 = (1 - \eta)z_1 \)

\[
\sqrt{(1 - \eta)^2 + 1(u(z) \cdot n(z))} = (1 - \eta)u_1(z) - u_2(z) \leq \alpha((1 - \eta)^2 - 1)z_1 \log \frac{1}{|z|} + \epsilon\alpha|z| \log \frac{1}{|z|} < 0.
\]

Thus, any particle starting in \( B_{\delta_2} \cap V_1 \) stays in \( V_1 \) for all \( t \in [0, \delta_2] \). Since we are only concerned with \( t \in [0, t_2] \) and \( x \in B_{\delta_1}(0) \), \( V_1 \) is an invariant region. It can be shown similarly that \( V_3 := -V_1 \)

is an invariant region. Now consider \( V_2 = \{ z^+ : z \in V_1 \} \) and \( V_4 = \{ z^+ : z \in V_3 \} \). A similar calculation shows that \( u(z) \cdot n(z) < 0 \) for points on \( \partial V_2 \cap B_{\delta_1}(0) \) and \( \partial V_4 \cap B_{\delta_1}(0) \).

Now we are ready to show that \( \theta_i \) cannot be continuous. We will do this by showing that most of the vortex patch is immediately pushed up close to the line \( z_1 = z_2 \). This will contradict the continuity.
of $\theta_i$. Take $X(0) \in T := \{ |z_2| < (1 - \eta)z_1 \} \cap B_\delta(0)$ for some $\delta \leq \delta_2$. By our choice of $t_2$ and the Yudovich bound, we know that $X(t) \in B_\gamma(0)$ for all $t \in [0, t_2]$. If for some $t \in [0, t_2]$, $X(t)$ hits $\partial T$, it must be that it hits the line $z_2 = (1 - \eta)z_1$ since the velocity field is pushing into $T$ on the lower boundary. Thereafter, $X(t)$ does not exit $V_1$ until after time $t_2$. Now let us study what happens for $t \in [0, t_2]$. By using the expansion of the velocity again, we find for $t \in [0, t_2]$:

$$\frac{d}{dt} \left( \frac{X_2}{X_1} \right) \geq \alpha \log \frac{1}{|X(t)|} \left( 1 - \frac{X_2^2}{X_1^2} \right) - \epsilon \frac{|X|}{|X_1|} \log \frac{1}{|X|} \geq \alpha \log \frac{1}{|X|} \left( 1 - \frac{2\epsilon}{\alpha} - \frac{X_2^2}{X_1^2} \right).$$

Thus,

$$\frac{d}{dt} \left( \frac{X_2}{X_1} \right) \geq \alpha \log \frac{1}{\sqrt{\delta}} \left( 1 - \frac{2\epsilon}{\alpha} - \frac{X_2^2}{X_1^2} \right).$$

Letting $\lambda = \frac{X_2}{X_1}$, we see:

$$\lambda'(t) \geq \frac{1}{2} \alpha \log \frac{1}{|X(0)|} \left( 1 - \frac{2\epsilon}{\alpha} - \lambda^2 \right).$$

Consequently, $\lambda$ is increasing, so long as $-\sqrt{1 - \frac{2\epsilon}{\alpha}} < \lambda < \sqrt{1 - \frac{2\epsilon}{\alpha}}$. In fact, so long as $\lambda < \frac{1}{2}$ we see:

$$\lambda(t) \geq \frac{1}{2} \alpha \log \frac{1}{|X(0)|} \left( 1 - \frac{2\epsilon}{\alpha} + \lambda(0) \right) \exp(c \log \frac{1}{|X(0)|})$$

for $c = \frac{1}{4} \alpha$. In particular, if $\lambda(0) < 0$, we have that $\lambda(t)$ will hit 0 before the time $\frac{1}{c \log |X(0)|}$. Now let $\tilde{T} = \{ |z_2| \leq \sin(\theta_1(0))z_1 \}$. It is easy to see that if $X(0) \in B_{\delta_0}(0) \cap \tilde{T}$, then there exists $t_2 \leq \frac{C}{|\log \delta|}$, with $C$ independent of $\epsilon$ so that $X_2(t) \geq 0$ for all $t \in [t_3, t_2]$. In particular, all particles initially in $\tilde{T} \cap B_{\delta_0}(0)$ are transported to the region $z_2 \geq 0$ by the time $t_2$ and they never leave this region for $t \in [0, t_2]$. By the Yudovich bound

$$|X(0)|^{\exp(C\epsilon)} \leq |X(t)| \leq |X(0)|^{\exp(-Ct)}$$

we have that at time $t_2$, there are no trajectories in $B_{\delta_0}(0) \cap \{ z_2 \leq 0 \leq z_1 \}$ which began in $\tilde{T}$. Thus, all particles which lie in $B_{\delta_0}(0) \cap \{ z_2 \leq 0 \leq z_1 \}$ at time $t_2$ must have come from the set $B_{C\delta_0}(0) \cap \{ \sin(2\theta_1(0))z_1 \leq z_2 \leq 2z_1 \}$. By assumption, the total area of this arbitrarily small relative to the size of $|B_{\delta_0}(0)|$ as $\delta \to 0$. Now let's estimate the measure of $\Omega(t) \cap S_{-\theta_1(0)}, \theta_1(0) \cap B_{\delta}(0)$ at time $t_2$. According to the assumption, the measure of this set should be approximately $2|\theta_1(0)|/|B_{\delta}(0)|$. We will show that it is bounded by $(\theta_1(0) + o(1))|B_{\delta}(0)|$ as $\delta \to 0$, which is a contradiction. Indeed,

$$|\Omega(t_3) \cap S_{-\theta_1(0)}, \theta_1(0) \cap B_{\delta}(0)| \leq |B_{\delta}(0)| \theta_1(0) + |\Omega(t_3) \cap S_{-\theta_1(0)}, 0 \cap B_{\delta}(0)|$$

$$\leq |B_{\delta}(0)| \theta_1(0) + |\Omega(0) \cap B_{C\delta_0}(0) \cap S_{-2\theta_1(0)}, -\theta_1(0)| \leq |B_{\delta}(0)| \theta_1(0) + \frac{C}{M} |B_{\delta}(0)| 2\theta_1(0).$$

This finishes the proof. \(\square\)

### 5.4 Ill-posedness for vortex patches with corners of size $\pi/2$

The purpose of this subsection is to establish the following proposition.

**Proposition 5.11.** For generic initial vortex patches $\Omega_0$ with $\Omega_0 \cap B_1(0) = S_{-\frac{\pi}{4}, \frac{\pi}{4}} \cup S_{\frac{3\pi}{4}, \frac{5\pi}{4}}$ and $\Omega_0$ smooth and compactly supported outside of $B_\frac{1}{2}(0)$ we have that the associated unique vortex patch solution $\Omega(t)$ does not keep a pair of regular corners of size $\pi/2$ for positive time. That is, $\partial \Omega(t) \cap B_1(0)$ cannot be written as the intersection of two $C^{1,\alpha}$ curves lying in $C([0, \delta]; C^{1,\alpha})$ for any $\delta > 0$ and $\alpha > 0$. 

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**Remark 5.12.** “Generic” means that given any vortex patch $\Omega_0$ with a 90-degree corner at the origin, it can be perturbed very slightly by a smooth perturbation very far away from the origin so that the corner does not remain regular for positive time.

**Remark 5.13.** It will be apparent from the proof that we actually only need the two curves to be uniformly $C^1$ continuously in time.

**Proof.** By our previous results, for such curves to exist (for any vortex patch with a right angle at time zero), a necessary condition is that the curves intersect at a right angle for all $t \in [0, \delta]$. Let $\alpha(t)$ and $\beta(t)$ denote the tangent vectors to the two curves. We claim that:

$$\alpha'(t) = \lim_{\lambda \to 0^+} \frac{u(t, \lambda \alpha(t)) \cdot \alpha(t) \perp}{\lambda},$$

and

$$\beta'(t) = \lim_{\lambda \to 0^+} \frac{u(t, \lambda \beta(t)) \cdot \beta(t) \perp}{\lambda}.$$

To establish the claim, we just need to show that so long as $\Omega$ can be locally written as an intersection of two $C^1, \alpha$ curves for some $\alpha > 0$, then the limits above exist. The definition of tangent vector then implies that the tangent vector actually evolves as claimed. The existence of the limit follows since $\frac{1}{|x|} u(t, x) \cdot x \perp$ can be written as a smooth function of $\frac{1}{|x|}$ plus a logarithmic term which vanishes when $x = \alpha(t)$ and $x = \beta(t)$ plus a term which vanishes as $|x| \to 0$. In fact, the assumption implies that the limits are continuous in time. As a consequence, if we have that

$$\lim_{\lambda \to 0^+} \frac{u_0(\lambda \alpha(0)) \cdot \alpha(0) \perp}{\lambda} \neq \lim_{\lambda \to 0^+} \frac{u_0(\lambda \beta(0)) \cdot \beta(0) \perp}{\lambda},$$

then we will have a contradiction. If equality holds for $\Omega_0$, we can just choose a small perturbation so that equality does not hold at time zero.

**5.5 Loss of Lipschitz continuity for locally symmetric patches**

In this subsection we give an example of a vortex patch $\Omega_0$ with corners which is locally four-fold symmetric about the origin for which the velocity field satisfies:

$$\sup_{0 < t < \delta} \| \nabla u(t) \|_{L^\infty} = +\infty$$

for all $\delta > 0$ despite the initial velocity field being Lipschitz continuous. This shows that the global symmetry assumption in Theorem A cannot be replaced by a local symmetry assumption. This is done using the framework introduced in [44].

**Proof.** Let $\hat{\Omega}_0$ be a compactly supported vortex patch which near the origin is equal to $\theta \in \bigcup[-\pi/8 + k\pi/2, \pi/8 + k\pi/2], k = 0, 1, 2, 3]$. In particular, we assume that $\hat{\Omega}_0$ satisfies the conditions of Theorem A. Now we define $\Omega_0$ by $\Omega_0 = \hat{\Omega}_0 \cup \Omega_p$, where $0 \notin \Omega_p$ and $\Omega_p$ is compactly supported and 2-fold symmetric. The index $p$ in $\Omega_p$ stands for perturbation. Note that $\Omega_0$ is 2-fold symmetric but need not be 4-fold symmetric. We further require that $\Omega_0$ be infinitely smooth away from 0. We now claim that for some special choices of $\Omega_p$ we have that for all $\delta > 0$

$$\sup_{0 < t < \delta} \| \nabla u(t) \|_{L^\infty} = +\infty.$$

Following Section 8 of [44], to prove this, all we need to show is that the associated velocity field $u_0$ satisfies the following estimates:

$$\nabla u_0 \in L^\infty,$$
\[ \|D^2 p_0\|_{L^q} \geq C q, \]

for some fixed constant \( C > 0 \) and all \( q \geq 2 \), where \( p_0 \) is the initial Eulerian pressure given by:

\[ \Delta p_0 = 2 \det(\nabla u_0). \]

Now we write \( u_0 = \tilde{u}_0 + u_{0,p} \) where \( \tilde{u}_0 = \nabla^2 \Delta^{-1} \chi_{\Omega_0} \) while \( u_{0,p} = \nabla^2 \Delta^{-1} \chi_{\Omega_p} \). Note that \( u_{0,p} \) is infinitely smooth in some neighborhood of 0. In particular,

\[ u_{0,p}(x) = \nabla u_{0,p}(0) x + O(|x|^2) \]

as \( |x| \to 0 \). On the other hand,

\[ \tilde{u}_0(x) = G(\theta) x + 2G'(\theta)x^\perp + O(|x|^2) \]

as \( |x| \to 0 \), with \( G \) the unique \( \frac{\pi}{2} \)-periodic solution to

\[ 4G + G'' = \chi[-\frac{\pi}{2},\frac{\pi}{2}]. \]

Note that \( G \) is even with respect to \( \theta \). It is then easy to see that the \( O(|x|^2) \) terms in the expansions of \( u_{0,p} \) and \( \tilde{u}_0 \) are negligible and that, in a neighborhood of 0,

\[ \Delta p_0 = 2 \det(\nabla u_{0,p}(0) + \nabla(G(\theta)x + 2G'(\theta)x^\perp)) + f, \]

with \( f \in C^\alpha \) for all \( \alpha < 1 \). Now let’s choose \( u_{0,p} \) so that

\[ \nabla u_{0,p}(0) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} \]

for some large constant \( K \). Then we see that

\[ \Delta p_0 = 2K \partial_1(G(\theta)x_2 - 2G'(\theta)x_1) + F \]

with \( \|F\|_{L^\infty} \leq C \) for some fixed constant \( C \) independent of \( K \). In particular,

\[ \|D^2 \Delta^{-1}(F \chi)\|_{L^q} \leq C q \]

for all \( q \in [1, \infty) \) where \( \chi \) is a smooth cut-off function which is identically 1 near zero and \( C \) again is a fixed constant independent of \( K \). Now let us consider \( |D^2 \Delta^{-1}(\partial_1(G(\theta)x_2 - 2G'(\theta)x_1))|_{L^q} \). It is easy to see that it suffices to show that there exists a small constant \( c > 0 \) so that

\[ |\nabla \Delta^{-1}(\partial_1(G(\theta)x_2 - 2G'(\theta)x_1))| \geq c|x| \log \frac{1}{|x|}. \]

As we have shown in our expansion of the velocity field in Lemma 5.1, it suffices to show that the quantity \( I \) below is non-zero:

\[ I := \int_0^{2\pi} \int_{|x|}^1 \frac{\sin(2\theta)}{r^2} (\partial_1(G(\theta)x_2 - 2G'(\theta)x_1)) r dr d\theta \]

\[ = \log \frac{1}{|x|} \int_0^{2\pi} \sin(2\theta)(-G'(\theta)\sin^2(\theta) + 2G''(\theta)\sin(\theta)\cos(\theta) - 2G'(\theta)) d\theta. \]

Noting that \( G \) is \( \pi/2 \) periodic, we see that \( \int_0^{2\pi} G'(\theta)\sin(2\theta) = 0 \). Moreover, we note that

\[ \sin(2\theta)\sin^2(\theta) = \frac{1}{2} \sin(2\theta) - \frac{1}{4} \sin(4\theta) \]

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and
\[ \sin(2\theta) \sin(\theta) \cos(\theta) = \frac{1}{2} \sin^2(2\theta) = \frac{1}{4} (1 - \cos(4\theta)). \]

Now we see:
\[ \int_0^{2\pi} G'(\theta) \sin(2\theta) = 0, \]
\[ -\int_0^{2\pi} G'(\theta) \sin(2\theta) \sin^2(\theta) = \frac{1}{4} \int_0^{2\pi} G'(\theta) \sin(4\theta), \]
\[ 2 \int_0^{2\pi} G''(\theta) \sin(\theta) \cos(\theta) \sin(2\theta) = -\frac{1}{2} \int_0^{2\pi} G''(\theta) \cos(4\theta). \]

Thus,
\[ I = 7 \log \left| \frac{1}{|x|} \right| \int_0^{2\pi} G(\theta) \cos(4\theta) \, d\theta \neq 0 \]
as can be easily seen from the relation
\[ G''(\theta) + 4G(\theta) = \chi_{[-\frac{\pi}{8}, \frac{\pi}{8}]} \]
by multiplying by \( \cos(4\theta) \). Thus by choosing \( K \) sufficiently large, we can ensure that
\[ \| \nabla u_0 \|_{L^\infty} < \infty \]
while
\[ \| D^2 p_0 \|_{L^q} \geq q \]
for all \( q \geq 2 \). Now we may apply the arguments of Section 7 of [44] to conclude. \( \square \)

6 Effective system for the boundary evolution near the corner

In the previous section we established that a non-right angle in a vortex patch does not propagate continuously in time. Then the question may be raised: what exactly happens to the vortex patch near the corner? Using the expansion of the velocity field in the previous section, it is easy to see that the purpose of this section is to propose some asymptotic models which we believe describe the behavior of a vortex patch.

6.1 The formal evolution equation near the corner

From now on, we shall assume that the vorticity is two-fold symmetric around the origin. Here, this symmetry assumption is just for simplicity and does not seem to alter the qualitative dynamics near the origin, except for the translation of the patch which can be fixed using the Galilean invariance. Moreover, assume that the vorticity is supported in a small ball of radius less than 1, which is guaranteed for some nonempty time interval if the initial vorticity has this property. Then, we have from Lemma [5.1] that
\[ u(r, \theta) = \frac{1}{2\pi} \left( \frac{\cos(\theta)}{-\sin(\theta)} \right) r I^s(r) - \frac{1}{2\pi} \left( \frac{\sin(\theta)}{\cos(\theta)} \right) r I^c(r) + O(r) \]  
(6.1)
where
\[ I^s(r) = \int_r^1 \int_0^{2\pi} \sin(2\theta') \frac{\omega(s, \theta')}{s} \, d\theta' \, ds, \quad I^c(r) = \int_r^1 \int_0^{2\pi} \cos(2\theta') \frac{\omega(s, \theta')}{s} \, d\theta' \, ds. \]  
(6.2)
From (6.2), note that in the limit \( r \to 0^+ \), if for majority of \( s \in [r, 1] \) we have a lower bound on the integral of \( \omega(s, \cdot) \) against either \( \sin(2\theta) \) or \( \cos(2\theta) \) on the circle, then either \( |I^s(r)| \gg 1 \) or \( |I^c(r)| \gg 1 \), and it is reasonable to believe that the behavior of the vorticity at the origin is determined only by the first two terms in the right hand side of (6.1). Moreover, the term of order \( r \) cannot account for a sudden change of angle, such as instantaneous cusp or spiral formation. Then, we may formally consider the following modified Euler equation:

\[
\partial_t \omega + \frac{1}{2\pi} \left[ \left( \frac{\cos(\theta)}{-\sin(\theta)} \right) I^s(r) - \left( \frac{\sin(\theta)}{\cos(\theta)} \right) I^c(r) \right] r \cdot \nabla \omega = 0. \tag{6.3}
\]

The measure \( \frac{ds}{s} \) in the expression (6.2) suggests that we write the vorticity as

\[
\omega(t, r, \theta) = g(t \ln \frac{1}{r}, \theta) + \text{remainder}, \tag{6.4}
\]

where we formally assume that the remainder term is negligible for some time interval \([0, t^*]\) in the limit \( r \to 0^+ \) compared to the \( g \)-term. Using (6.4) and neglecting the terms involving the remainder, we obtain from

\[
I^s(r) = \frac{1}{t} \int_0^{t \ln \frac{1}{r}} \int_0^{2\pi} \sin(2\theta') g(t \ln \frac{1}{s}, \theta') d\theta' d(t \ln \frac{1}{s}),
\]

\[
I^c(r) = \frac{1}{t} \int_0^{t \ln \frac{1}{r}} \int_0^{2\pi} \cos(2\theta') g(t \ln \frac{1}{s}, \theta') d\theta' d(t \ln \frac{1}{s})
\]

that after introducing the variable \( \tau = t \ln \frac{1}{r} \),

\[
\ln \frac{1}{r} \partial_t g + \frac{1}{2\pi r} \left[ \left( \frac{\cos(\theta)}{-\sin(\theta)} \right) \int_0^\tau \int_0^{2\pi} \sin(2\theta') g(\tau', \theta') d\theta' d\tau' \right.
\]

\[
\left. - \left( \frac{\sin(\theta)}{\cos(\theta)} \right) \int_0^\tau \int_0^{2\pi} \cos(2\theta') g(\tau', \theta') d\theta' d\tau' \right]
\]

\[
\cdot \left[ \frac{1}{r} \partial_\theta g \left( \frac{-\sin(\theta)}{\cos(\theta)} \right) - \frac{t}{r} \partial_{\tau'} \left( \frac{\cos(\theta)}{-\sin(\theta)} \right) \right] = 0.
\]

Dividing by \( \ln \frac{1}{r} \),

\[
\partial_\tau g - \frac{1}{2\pi r} \left[ \sin(2\theta) \left( \int_0^\tau \int_0^{2\pi} \sin(2\theta') g(\tau', \theta') d\theta' d\tau' \right) + \cos(2\theta) \left( \int_0^\tau \int_0^{2\pi} \cos(2\theta') g(\tau', \theta') d\theta' d\tau' \right) \right] \partial_\theta g
\]

\[
= \frac{t}{r} \left[ \cos(2\theta) \left( \int_0^\tau \int_0^{2\pi} \sin(2\theta') g(\tau', \theta') d\theta' d\tau' \right) + \sin(2\theta) \left( \int_0^\tau \int_0^{2\pi} \cos(2\theta') g(\tau', \theta') d\theta' d\tau' \right) \right] \partial_{\tau'} g.
\]

For \( 0 \leq t \ll 1 \), again formally we may drop the entire right hand side, which results in the following transport system for \( g \):

\[
\partial_\tau g - \frac{1}{2\pi r} \left[ \sin(2\theta) \left( \int_0^\tau \int_0^{2\pi} \sin(2\theta') g(\tau', \theta') d\theta' d\tau' \right) + \cos(2\theta) \left( \int_0^\tau \int_0^{2\pi} \cos(2\theta') g(\tau', \theta') d\theta' d\tau' \right) \right] \partial_\theta g = 0. \tag{6.5}
\]

We now investigate the system (6.5) in a number of concrete situations.

### 6.2 Evolution of a corner under the odd symmetry

We now consider the simpler case of vorticity which is odd in the \( x_1 \)-axis. Together with the two-fold symmetry assumption, we have that the vorticity is odd also in the \( x_2 \)-axis. Then, in (6.5), the term involving integration against \( \cos(2\theta') \) vanishes, and we are left with

\[
0 = \partial_\gamma g \left( \frac{1}{2\pi r} \int_0^\tau \int_0^{2\pi} \sin(2\theta') g(\tau', \theta') d\theta' d\tau' \right) \sin(2\theta) \partial_\theta g. \tag{6.6}
\]
Using the Fourier expansion
\[ g(\tau, \theta) = \sum_{k \geq 1} g_k(\tau) \sin(2k\theta), \] (6.7)
we may rewrite (6.6) in the following equivalent form:
\[ \dot{g}_k(\tau) = \left( \frac{1}{2\pi\tau} \int_{0}^{\tau} g_1(\tau') \, d\tau' \right) \left( (k-1)g_{k-1}(\tau) - (k+1)g_{k+1}(\tau) \right), \] (6.8)
where we have used the convention that \( g_0 \equiv 0 \). From (6.8), it is straightforward to see that the Bahouri-Chemin solution is characterized as the unique stationary solution to (6.6).

We now consider the case where the initial data is locally a union of corners attached on the \( x_1 \)-axis: that is,
\[ g(0, \theta) = \pm \left( 1_{[0,A_0] \cup [\pi,A_0+\pi]} - 1_{[-A_0,0] \cup [-A_0+\pi,\pi]} \right) \]
for some \( 0 \leq A_0 < \pi/2 \). Then, we have that \( g(\tau, \theta) = \pm \left( 1_{[0,A(\tau)] \cup [\pi,A(\tau)+\pi]} - 1_{[-A(\tau),0] \cup [-A(\tau)+\pi,\pi]} \right) \)
with \( A(0) = A_0 \) from the transport nature of the system (6.6). In the case of the negative sign (i.e. when the vorticity is negative on the positive quadrant), we obtain from (6.6) that \( A(\cdot) \) satisfies
\[ \dot{A}(\tau) = 4 \sin(2A(\tau)) \int_{0}^{\tau} \int_{0}^{A(\tau')} \sin(2\theta') \, d\theta' \, d\tau', \] (6.9)
and in the opposite sign case,
\[ \dot{A}(\tau) = -\sin(2A(\tau)) \int_{0}^{\tau} 1 - \cos(2A(\tau')) \, d\tau'. \] (6.10)
Note that (6.9), (6.10) can be rewritten into the form of a second order ordinary differential equation; using \( a := 2A \), we have
\[ \ddot{a}(\tau) = \cos(a(\tau)) \frac{\dot{a}(\tau)}{\sin(a(\tau))} \left( \frac{1}{\tau} \right) \left( \dot{a}(\tau) \pm \frac{2}{\pi} \sin(a(\tau))(1 - \cos(a(\tau))) \right), \] (6.11)
with initial data
\[ a(0) = 2A_0, \quad \dot{a}(0) = \pm \frac{2}{\pi} \sin(2A_0)(1 - \cos(2A_0)). \] (6.12)
Using (6.9), it is direct to see that for all \( 0 \leq \tau, A_0 \leq A(\tau) < \frac{\pi}{2} \). Since \( \cos(2 \cdot) \) is a decreasing function on \([0, \pi/2]\), we have that
\[ \dot{A}(\tau) \geq \frac{\sin(2A(\tau))}{\pi} (1 - \cos(2A)), \]
and this guarantees that \( A(\tau) \to \frac{\pi}{2} \) at exponential speed as \( \tau \to +\infty \). The dynamics is more delicate in the case of (6.10); while it is not difficult to show that the solution decays to 0 as \( \tau \) goes to infinity with bounds
\[ \frac{c}{1 + \tau} \leq A(\tau) \leq \frac{C}{1 + \tau^{1/2}}, \quad \tau \geq 0, \] (6.13)
obtaining the rate of decay is an interesting problem. Direct numerical simulations show that \( A \) decays as \( \tau^{-1} \). Let us show that the integral
\[ \int_{0}^{\infty} 1 - \cos(2A(\tau)) \, d\tau \] (6.14)
is bounded by a constant depending only on $A_0$. Otherwise, for any large $M > 0$, one can find 
$	au^* = \tau^*(M) > 0$ such that

$$\forall \tau \geq \tau^*, \quad \int_0^\tau 1 - \cos(2A(\tau'))d\tau' \geq M.$$ 

Then

$$\forall \tau \geq \max(\tau^*, \tau'), \quad  \dot{A}(\tau) \leq -\frac{M}{10\pi \tau}A(\tau)$$

where $\tau' = \tau'(A_0) > 0$ is chosen that $\sin(2A(\tau')) \geq \frac{A(\tau')}{10}$, and

$$\forall \tau \geq \max(\tau^*, \tau'), \quad \frac{c}{1 + \tau} \leq A(\tau) \leq C(M)\tau^{-\frac{M}{10}},$$

where $C(M) > 0$ is a constant depending only on $M$. Taking $\tau \to +\infty$, we obtain a contradiction. If the ansatz (6.4) were correct, the bound on the integral in (6.14) implies that for small $t > 0$, we have

$$\lim_{t \to 0^+} \frac{|u(t,r,\theta)|}{r} \leq \frac{C}{t},$$

where $u(t, \cdot)$ is the solution associated with initial vorticity given by $\omega_0(r, \theta) = g(0, \theta)1_{\{r \leq \frac{\pi}{2}\}}$.

### 6.3 Evolution of a single corner

We now consider initial data of the form

$$g(0, \theta) = 1_{[A_0-B_0, A_0+B_0]} + 1_{[A_0-B_0+\pi, A_0+B_0+\pi]}.$$ 

The initial corner is centered at $A_0$ and has angle $2B_0$, with two-fold symmetry. Without loss of generality, we may set $A_0 = 0$. From

$$g(\tau, \theta) = 1_{[A-B, A+B]} + 1_{[A-B+\pi, A+B+\pi]},$$

we obtain, by evaluating (6.5) at $A + B$ and $A - B$, that

$$(A \pm B)'(\tau) = -\frac{1}{2\pi \tau} \left[ \sin(2A(\tau') \pm 2B(\tau')) \int_0^\tau \cos(2A(\tau') - 2B(\tau')) - \cos(2A(\tau') + 2B(\tau'))d\tau' 
+ \cos(2A(\tau') \pm 2B(\tau')) \int_0^\tau \sin(2A(\tau') + 2B(\tau')) - \sin(2A(\tau') - 2B(\tau'))d\tau' \right].$$

It follows that

$$\tau B'(\tau) = -\frac{1}{\pi} \left( \sin(2B) \cos(2A) \int_0^\tau \sin(2B(\tau')) \sin(2A(\tau'))d\tau' 
- \sin(2B) \sin(2A) \int_0^\tau \sin(2B(\tau')) \cos(2A(\tau'))d\tau' \right) \quad (6.15)$$

and

$$\tau A'(\tau) = -\frac{1}{\pi} \left( \cos(2B) \sin(2A) \int_0^\tau \sin(2B(\tau')) \sin(2A(\tau'))d\tau' 
+ \cos(2B) \cos(2A) \int_0^\tau \sin(2B(\tau')) \cos(2A(\tau'))d\tau' \right). \quad (6.16)$$
We may rewrite (6.15)–(6.16) in the form

\[ B'(\tau) = -\frac{\sin(2B(\tau))}{\pi \tau} \int_0^\tau \sin(2B(\tau')) \sin(2A(\tau') - 2A(\tau)) d\tau' \]

and

\[ A'(\tau) = -\frac{\cos(2B(\tau))}{\pi \tau} \int_0^\tau \sin(2B(\tau')) \cos(2A(\tau') - 2A(\tau)) d\tau'. \]

This implies that if \(-\pi/4 \leq A \leq 0\) in an interval \([0, \tau']\), then \(A'(\tau) \leq 0\), which in turn implies \(B'(\tau) \geq 0\).

Similarly as in the case of odd symmetry, it is possible to turn the above system into a system of second order ordinary differential equations. Differentiating both sides of (6.15)–(6.16) gives

\[ \tau \left( B'' \right) = -\frac{1}{\pi} \left( \begin{array}{c} 0 \\ \cos(2B) \sin(2B) \end{array} \right) + M'^{-1} \tau \left( \begin{array}{c} B' \\ A' \end{array} \right) \]

with

\[ M = \left( \begin{array}{cc} \sin(2B) \cos(2A) & -\sin(2B) \sin(2A) \\ \cos(2B) \sin(2A) & \cos(2B) \cos(2A) \end{array} \right). \]

Then

\[ M' = 2B' \left( \begin{array}{cc} \cos(2B) \cos(2A) & -\cos(2B) \sin(2A) \\ -\sin(2B) \sin(2A) & -\sin(2B) \cos(2A) \end{array} \right) + 2A' \left( \begin{array}{cc} -\sin(2B) \sin(2A) & -\sin(2B) \cos(2A) \\ \cos(2B) \cos(2A) & \cos(2B) \sin(2A) \end{array} \right) \]

and

\[ M'^{-1} = \frac{1}{\sin(2B) \cos(2B)} \left( \begin{array}{cc} \cos(2B) \cos(2A) & \sin(2B) \sin(2A) \\ -\cos(2B) \sin(2A) & \sin(2B) \cos(2A) \end{array} \right). \]

Then finally, we arrive at the system

\[ \tau \left( \begin{array}{c} B'' \\ A'' \end{array} \right) = -\frac{1}{\pi} \left( \begin{array}{c} 0 \\ \cos(2B) \sin(2B) \end{array} \right) + \frac{2\tau}{\sin(2B) \cos(2B)} \left( \begin{array}{c} \cos^2(2B)(B')^2 - \sin^2(2B)(A')^2 \\ -\sin^2(2B)B'A' + \cos^2(2B)B'A' \end{array} \right). \]

The initial condition is given by

\[ B(0) = B_0, B'(0) = 0, A(0) = 0, A'(0) = -\frac{1}{\pi} \cos(2B_0) \sin(2B_0). \]

(6.17)

Numerical simulations suggest that \(A \rightarrow A_\infty\) and \(B \rightarrow 0\) for some constant \(A_\infty\) depending on \(B_0\), as \(\tau \rightarrow +\infty\). This suggests instantaneous cusping without spiral formation, which is comparable with the direct numerical study on vortex patches by Carrillo and Soler [20].

Appendix

A Local well-posedness for symmetric patches

The local well-posedness results for smooth vortex patches is usually obtained via an iteration scheme, using the contour dynamics equation (see for instance [15, 66]). An alternative approach which works directly with the flow maps restricted to the patch was described in an illuminating work of Huang
This method originates from a previous work of Friedman and Huang \[45\], and it seems to be applicable for a wide variety of situations. We shall adopt this approach to show local well-posedness (as well as continuation criteria) in the setting of Section \[3\], i.e., patches admitting a level set function \( \phi \) with \( \nabla^\perp \phi \in C^\alpha \).

The starting point of this method is to write the 2D Euler equation purely in terms of the flow maps:

\[
\Phi(t, x) = x + \int_0^t \int_{\mathbb{R}^2} K(\Phi(x, s) - y)\omega_0(\Phi^{-1}_t(y))dyds,
\]

\[
= x + \int_0^t \int_{\Omega_0} K(\Phi(s, x) - \Phi(s, z))dzds.
\]

At this point, note that we only need to know \( \Phi(t, \cdot) \) on \( \Omega_0 \) to determine the velocity of the Euler equation everywhere in \( \mathbb{R}^2 \). It is easy to show that the above formulation is equivalent to the (usual) weak formulation of the 2D Euler equations under \( \omega \in L^\infty \cap L^1 \), and the Yudovich theorem gives that there is a unique solution \( \Phi \) satisfying (A.1).

The formulation (A.1) suggests one to build an iteration scheme; all that is necessary to appropriately define the space of functions. Following \[52\], we consider

\[
B(M, T) = \left\{ \Phi(t, x) \in X : \Phi(0, x) = x, \Phi(t, 0) = 0, \|\Phi\|_X \leq M, \sup_{t \in [0, T]} |\nabla \Phi(t, x) - I| \leq 1/2 \right\}
\]

where the space \( X \) is defined for functions \( \Phi : \overline{\Omega}_0 \times [0, T] \rightarrow \mathbb{R}^2 \) with \( \det(\nabla_x \Phi) \equiv 1 \) by the norm

\[
\|\Phi\|_X = \sup_{t \in [0, T]} \left( \|\nabla_x \Phi\|_{C^\alpha(\overline{\Omega}_0)} + \|\partial_t \Phi\|_{L^\infty(\overline{\Omega}_0)} \right).
\]

That is, we have simply replaced the assumption in \[52\] that \( \nabla \Phi(t, \cdot) \) is uniformly \( C^\alpha \) (up to the boundary of \( \overline{\Omega}_0 \)) by \( C^\alpha \). The extra assumption that \( \Phi(t, 0) = 0 \) holds will be guaranteed by symmetry. Under the assumption \( |\nabla \Phi(t, x) - I| \leq 1/2 \), it follows that the inverse map \( \Phi^{-1}_t : \overline{\Omega}_t \rightarrow \overline{\Omega}_0 \) is Lipschitz with \( |\nabla \Phi^{-1}_t| \leq 2 \). Moreover, it is elementary to verify that for \( \Phi \in B(M, T) \), \( \nabla \Phi^{-1}_t \) belongs to \( C^\alpha(\overline{\Omega}_t) \) with norm depending only on \( M \) (see below Lemma \[A.5\]).

Then, we define a mapping \( F \),

\[
F(\Phi)(t, x) := x + \int_0^t \int_{\Omega_0} K(\Phi(s, x) - \Phi(s, z))dzds,
\]

so that a fixed point of \( F \) provides a solution to the 2D Euler equation on \([0, T]\) with initial data \( \omega_0 = \chi_{\Omega_0} \).

We need to propagate the regularity of \( \phi \) in time, where the level set function \( \phi_0 \) is given together with the initial data \( \Omega_0 \). We observe that, as long as \( \Phi \in B(M, T) \), by defining

\[
\phi(t, x) := \phi_0(\Phi^{-1}_t(x)), \quad x \in \overline{\Omega}_t,
\]

we have

\[
\sup_{t \in [0, T]} \Gamma_t := \sup_{t \in [0, T]} \left( \frac{\|\nabla^\perp \phi_t(\cdot)\|_{C^\alpha(\Omega_t)}}{\|\nabla^\perp \phi_t(\cdot)\|_{L^\infty(\partial \Omega_t)}} \right)^{1/\alpha} \leq C(M),
\]

where, here and in the following, we use the notation \( C(M) \) to denote a positive and increasing function of \( M > 0 \) depending on \( \Gamma_0 \). This function may change from a line to another.

We are now in a position to formally state the local well-posedness results:

---

\[13\]The main result of this work is that \( C^{1,\alpha} \)-patches in 3D is locally well-posed under the Euler equations (see also an earlier work of Serfati \[79\]). In the three-dimensional case, the vorticity does not remain a constant inside the patch even if initially so, and therefore the contour dynamics approach is not available.
Proposition A.1. Assume that $\Omega_0$ is $m$-fold symmetric for some $m \geq 3$ admitting a level set $\phi_0$ satisfying Definition 3.1. Then there exists some $T > 0$, depending only on $\Gamma_0$, such that there is a unique local solution $\Phi \in X$ of (A.1). In particular, we can extend the solution beyond some $T^*$ as long as $\sup_{t \in [0,T^*)} \Gamma_t < +\infty$.

In the case of $C^{1,\alpha}$-patch with symmetric corners, we have:

Proposition A.2. Assume that $\Omega_0$ is a $C^{1,\alpha}$-patch with symmetric corners satisfying Definition 4.1. Then there exists some $T > 0$, depending only on its initial $C^{1,\alpha}$-characteristic $\Gamma_0$, such that $\sup_{t \in [0,T]} \Gamma_t < +\infty$: that is, the associated flow map $\Phi \in X$ on the time interval $[0,T]$ provided by Proposition A.1 satisfies $\Phi_t(x,f_0(x)), \Phi_t(x,g_0(x)) \in C^{1,\alpha}_{x}[0,\delta_0]$ uniformly in $t \in [0,T]$. In particular, we can extend the solution beyond some $T^* > 0$ as long as $\sup_{t \in [0,T^*)} \Gamma_t < +\infty$.

The proof is a direct consequence of the following estimates:

Lemma A.3. For any initial data $\Omega_0$ satisfying Definition 3.1, there exists some $M,T > 0$ depending only on $\Gamma_0$ so that $F$ maps the space $B(M,T)$ to itself.

Lemma A.4. Assume that we are in the situation where Lemma A.3 holds. Then, there exists some $0 < T_1 \leq T$, depending only on $M$ and $\Gamma_0$, so that for any $\Phi, \tilde{\Phi} \in B(M,T)$,

\[
\|F(\Phi)(t) - F(\tilde{\Phi})(t)\|_{L^\infty(\Omega_0)} \leq C(M) \int_0^t \|\Phi_s - \tilde{\Phi}_s\|_{L^\infty(\Omega_0)} \left( 1 + \log \left( 1 + \|\Phi_s - \tilde{\Phi}_s\|_{L^\infty(\Omega_0)} \right) \right) ds
\]

and

\[
\|\nabla F(\Phi)(t) - \nabla F(\tilde{\Phi})(t)\|_{L^\infty(\Omega_0)} \leq C(M) \int_0^t \|\nabla\Phi_s - \nabla\tilde{\Phi}_s\|_{L^\infty(\Omega_0)} \left( 1 + \log \left( 1 + \|\nabla\Phi_s - \nabla\tilde{\Phi}_s\|_{L^\infty(\Omega_0)} \right) \right) ds
\]

hold for any $t \in [0,T_1]$.

Assuming the statements of Lemmas A.3 and A.4 let us just provide a sketch of the proof, as the argument is parallel to [52, Proof of Theorem 4.1].

Proof of Proposition A.1. Take $M$ and $T_1$ such that the map $F$ sends $B(M,T_1)$ to itself, and moreover, for any $\Phi, \tilde{\Phi} \in B(M,T_1)$,

\[
\|F(\Phi)(t) - F(\tilde{\Phi})(t)\|_{W^{1,\infty}(\Omega_0)} \leq C(M) \int_0^t \|\Phi(s) - \tilde{\Phi}(s)\|_{W^{1,\infty}(\Omega_0)} \left( 1 + \log \left( 1 + \|\Phi(s) - \tilde{\Phi}(s)\|_{W^{1,\infty}(\Omega_0)} \right) \right) ds
\]

for $t \in [0,T_1]$. Here, $M$ and $T_1$ depends only on $\Gamma_0$. Define a sequence $\{\Phi_n\}_{n \geq 0}$ in $B(M,T_1)$ by

$\Phi_0(t,x) = x, \quad \Phi_{n+1}(t,x) = F(\Phi_n)(t,x), \quad n \geq 0.$

It is straightforward to see that at each step of the iteration, the flow is $m$-fold symmetric around the origin and therefore $\Phi_n(t,0) = 0$ for all $n \geq 0$. Setting

$\rho_n(t) := \|\Phi_{n+1}(t) - \Phi_n(t)\|_{W^{1,\infty}(\Omega_0)},$

we have

$\rho_n(t) \leq C(M) \int_0^t \rho_{n-1}(s) (1 + \log(1 + \rho_{n-1}(s))) ds.$
This is sufficient to deduce that, taking a smaller value of $T_1$ depending only on $M$ if necessary (see [68] Chapter 2 for instance), there exists a function $\Phi : [0, T_1] \times \Omega_0 \to \mathbb{R}^2$ such that

$$\|\Phi_n - \Phi\|_{L^\infty((0, T_1); W^{1, \infty}(\Omega_0))} \to 0.$$ 

At this point, it is easy to see that $\Phi$ actually belongs to $B(M, T_1)$ and $F(\Phi) = \Phi$. Therefore, we have constructed a solution, which belongs to the desired class, to the 2D Euler equation with initial data $\Omega_0$ on the time interval $[0, T_1]$.

We briefly comment on the issue of continuing the solution past $T_1$. (All the details can be found in [52].) Take $\Omega_{T_1}$ as the new initial data, which has associated level set function $\phi_{T_1}$ with its characteristic $\Gamma_{T_1}$. Going through the exact same iteration scheme again with this new data, one obtains a unique solution on some time interval $[0, T_2]$, with $T_2 = T_2(\Gamma_{T_1}) > 0$. Then, by putting this solution together with the previous one, we obtain a patch solution, admitting a $C^{1,\alpha}$-level set, to the 2D Euler equation on the time interval $[0, T_1 + T_2]$ with initial data $\Omega_0$. This procedure can go on as long as we have a bound on $\Gamma_1$. This finishes the proof.  

Proof of Proposition A.2. The assumptions given in Definition 4.1 are strictly stronger than the ones in Definition 3.1, so we may work inside the time interval within which we have available the iterates $\Phi_n$ and the limit $\Phi$ belonging to the class $X$, defined in the above proof of Proposition A.1. It suffices to carry the information that, by shrinking $T$ if necessary in a way only depending on $T_0$, for some time interval $[0, T]$, each of $\Phi_n$ satisfies following the H"{o}lder estimate uniformly in $n$:

$$\|\Phi_n(t, (x, f_0(x)))\|_{C^{1,\alpha}(\Omega_0)} + \|\Phi_n(t, (x, g_0(x)))\|_{C^{1,\alpha}(\Omega_0)} \leq C(\Gamma_0) < \infty.$$ 

This follows directly from the a priori estimates given in the proof of Theorem 2. It is not difficult to see that $\Phi$ inherits the same H"{o}lder estimate.  

The lemmas A.3, A.4, and the bound (A.3) are direct consequences of the following simple lemmas. The first one provides substitutes for the usual calculus inequalities on $C^\alpha$-spaces.

Lemma A.5. Let $f$ and $g$ be $C^\alpha$ functions on some domain $\Omega \subset \mathbb{R}^2$. Then we have

$$\|fg\|_{C^\alpha} \leq C \left( \|f\|_{C^\alpha} \cdot \|g\|_{L^\infty} + \|f\|_{L^\infty} \cdot \|g\|_{C^\alpha} \right)$$

(A.4)

and if we assume further that $|f| > 0$ on $\Omega$,

$$\|1/f\|_{C^\alpha} \leq C(\|f\|_{\inf(\Omega)}) \|f\|_{C^\alpha}.$$  

(A.5)

Moreover, if $\Psi$ is a Lipschitz diffeomorphism of $\mathbb{R}^2$ with $\Psi(0) = 0$, then

$$\|f \circ \Psi\|_{C^\alpha} \leq C \left( \|\nabla \Psi\|_{L^\infty}, \|\nabla \Psi\|_{\inf} \right) \|f\|_{C^\alpha}.$$  

(A.6)

Proof. Let us note first that for two points at comparable distance, i.e. if $x \neq x'$ satisfy $c_1|x'| \leq |x| \leq c_2|x'|$,

$$\frac{|f(x) - f(x')|}{|x - x'|\alpha} \leq C \frac{\|f\|_{C^\alpha}}{|x|\alpha}$$

with $C$ depending on $c_1, c_2$.

We begin with (A.4). First, we have an $L^\infty$-bound $\|fg\|_{L^\infty} \leq \|f\|_{L^\infty} \cdot \|g\|_{L^\infty}$. Now take two points $x \neq x' \in \Omega$ and assume without loss of generality that $|x| \geq |x'|$. Consider two cases, (i) $|x - x'| \leq |x|/2$ and (ii) $|x - x'| > |x|/2$. In the latter case,

$$\frac{|x|^\alpha f(x)g(x) - |x'|^\alpha f(x')g(x')}{|x - x'|^\alpha} = \frac{|x|^\alpha (f(x)g(x) - f(x')g(x')) + (|x|^\alpha - |x'|^\alpha) f(x')g(x')}{|x - x'|^\alpha} \leq C \|f\|_{L^\infty} \cdot \|g\|_{L^\infty}.$$
Next, when (i) holds, we rewrite
\[
\frac{|x|^\alpha (f(x)g(x) - f(x')g(x')) + (|x|^\alpha - |x'|^\alpha) f(x')g(x')}{|x - x'|^\alpha} = \frac{|x|^\alpha (f(x) - f(x'))g(x)}{|x - x'|^\alpha} + \frac{|x|^\alpha f(x')(g(x) - g(x'))}{|x - x'|^\alpha} + \frac{|x|^\alpha - |x'|^\alpha}{|x - x'|^\alpha} f(x')g(x'),
\]
which is bounded in absolute value by the right hand side of (A.4), noting that
\[
|f(x) - f(x')| \leq C\|f\|_{\mathcal{C}^\alpha} \frac{|x - x'|^\alpha}{|x|^\alpha}
\]
whenever |x - x'| \leq |x|/2. The proof of (A.5) is strictly analogous, so let us omit it.

To show the last statement (A.6), it suffices to treat the case when |x'| \leq |x| and |x - x'| \leq |x|/2. Moreover, it suffices to bound the quantity
\[
|x|^\alpha \frac{|f(\Psi(x)) - f(\Psi(x'))|}{|x - x'|^\alpha} = |x|^\alpha \frac{|f(\Psi(x)) - f(\Psi(x'))|}{|\Psi(x) - \Psi(x')|^\alpha} \cdot \frac{|\Psi(x) - \Psi(x')|^\alpha}{|x - x'|^\alpha}.
\]

Note that since \( \Psi(0) = 0 \),
\[
\|\nabla \Psi\|_{\text{inf}} \leq \frac{|\Psi(z)|}{|z|} \leq \|\nabla \Psi\|_{L^\infty}
\]
for any \( z \), and since we have |x'| \leq |x| \leq 2|x'|, there exists some constants \( c_1, c_2 > 0 \) so that
\[
c_1|\Psi(x')| \leq |\Psi(x)| \leq c_2|\Psi(x')|.
\]

This allows us to bound
\[
|x|^\alpha \frac{|f(\Psi(x)) - f(\Psi(x'))|}{|\Psi(x) - \Psi(x')|^\alpha} \cdot \frac{|\Psi(x) - \Psi(x')|^\alpha}{|x - x'|^\alpha} \leq C\|f\|_{\mathcal{C}^\alpha} \cdot \frac{|x|^\alpha}{|\Psi(x)|^\alpha} \cdot (\|\nabla \Psi\|_{L^\infty})^\alpha.
\]
This finishes the proof. \( \square \)

Next, we shall need the piece of information that in the setting of Proposition (A.1) for each fixed time \( t \), the velocity gradient \( \nabla u_t \) actually belongs to \( \mathcal{C}^\alpha(\bar{\Omega}_t) \). In the case of \( C^{1,\alpha} \)-patches, this is a direct consequence of velocity being \( C^{1,\alpha} \) on the boundary, since then \( \Delta u_t = 0 \) in \( \bar{\Omega} \) and hence an elliptic regularity statement applies. It is likely that such an argument could be used here, but let us adopt the approach of Serfati [80] (see also recent papers by Bae and Kelliher [3, 7]):

**Lemma A.6.** Let \( W \) be a vector field on a domain \( \Omega \) with components in \( \mathcal{C}^\alpha(\bar{\Omega}) \). Assume further that \( |W| \geq c_0 > 0 \) on \( \Omega \). Then, for \( \omega = \chi_\Omega \), the associated velocity satisfies
\[
\|\nabla u\|_{\mathcal{C}^\alpha(\bar{\Omega})} \leq C(c_0)\|W\cdot \nabla u\|_{\mathcal{C}^\alpha(\bar{\Omega})}.
\]

**Proof.** With \( W = (W_1, W_2) \) and \( u = (u_1, u_2) \), one computes that
\[
\begin{pmatrix} \partial_1 u_1 \\ \partial_2 u_1 \end{pmatrix} = \frac{1}{|W|^2} \begin{pmatrix} W_1 & -W_2 \\ W_2 & W_1 \end{pmatrix} \begin{pmatrix} W_1 \partial_1 u_1 + W_2 \partial_2 u_1 \\ W_1 \partial_2 u_1 - W_2 \partial_1 u_1 \end{pmatrix}
\]
and note that using \( \partial_1 u_1 + \partial_2 u_2 = 0 \) as well as \( \partial_1 u_2 - \partial_2 u_1 = \omega \equiv \text{constant} \),
\[
W_1 \partial_2 u_1 - W_2 \partial_1 u_1 = W \cdot \nabla u_2 - W_1 \omega,
\]
so that using (A.4) and (A.5), we conclude that \( \nabla u_1 \in \mathcal{C}^\alpha \). It follows that \( \nabla u_2 \in \mathcal{C}^\alpha \) as well. \( \square \)
Remark A.7. To apply the above lemma to the setting of Proposition A.1, taking \( W_0 := \nabla^\perp \phi_0 \) is strictly speaking not allowed since it may vanish at some points in the interior of the initial patch \( \Omega_0 \). This can be simply fixed as follows (see [8, Section 10]). First, we know that for points \( x \in \Omega \) with \( d(x, \partial \Omega_0) < \delta|x|, |\nabla^\perp \phi_0| \) is bounded from below with a constant uniform in \( |x| \), where \( \delta \) can be taken as \( 1/(10 \Gamma_0) \), for instance. Then it suffices to take a vector field \( \tilde{W}_0 \) which does not vanish for points \( x \in \Omega_0 \) with \( d(x, \partial \Omega_0) \geq \delta|x| \). It is easy to require in addition that \( \tilde{W}_0 \) vanishes on \( \partial \Omega_0 \) and \( \nabla \cdot \tilde{W}_0 \in C^\alpha(\mathbb{R}^2) \).\(^{14}\) Then, we evolve the vector field by

\[
\dot{W}(t, \Phi(t, x)) := (\tilde{W}_0(x) \cdot \nabla)\Phi(t, x),
\]

which is consistent with the evolution of vector fields having the form \( \nabla^\perp \phi \) for some scalar function \( \phi \) advected by the flow.

Proof of Lemma A.3. Given an initial vortex patch \( \Omega_0 \) satisfying conditions of Proposition A.1 we fix a vector field \( \tilde{W}_0 \) described in the remark following Lemma A.6 as well as the level set \( \phi_0 \). Then, one may fix a vector field \( W_0 \) which coincides with \( \nabla^\perp \phi_0 \) near \( \partial \Omega_0 \) and with \( \tilde{W}_0 \) in a region where \( \nabla^\perp \phi_0 \) vanishes. We have \( \nabla \cdot W_0 \in C^\alpha \).

We have

\[
F(\Phi)(t, x) = x + \int_0^t u(s, \Phi(s, x))ds
\]
as well as

\[
(\nabla F(\Phi))(t, x) = I + \int_0^t \nabla u(s, \Phi(s, x)) \nabla \Phi(s, x) ds.
\]

We claim that the push-forward of the vector field \( W_0 \) (recall that \( W(t, \Phi(t, x)) := (W_0(x) \cdot \nabla) \Phi(t, x) \)) satisfies \( \sup_{t \in [0,T]} ||W_t||_{C^\alpha(\overline{\Omega}_t)} \leq C(M) \) as well as \( \inf_{t \in [0,T]} ||W_t||_{\text{inf}(\overline{\Omega}_t)} \geq (C(M))^{-1} > 0 \) (see [8], [7] for complete details of this proof in the context of \( C^\alpha \) vector fields – the proof can be adapted to our setting with straightforward modifications). It then follows from Lemma A.6 that

\[
||\nabla u||_{C^\alpha(\overline{\Omega}_t)} \leq C(M).
\]

Then, using the inequalities from \(^{14}\) we immediately obtain

\[
||\nabla F(\Phi)||_{C^\alpha} \leq C(M)T
\]

and also

\[
\sup_{\overline{\Omega}_0 \times [0,T]} |\nabla F(\Phi) - I| \leq C(M)T.
\]

Taking \( T \) sufficiently small, we see that \( F(\Phi) \in B(M, T) \).

Finally, we give a sketch of the proof of Lemma A.4

Proof of Lemma A.4. Fix some \( x \in \Omega_0 \) and \( t \in [0, T_1] \), and let us first obtain a bound on \( |F(\Phi)(t, x) - F(\Phi)(t, x)| \). We need to estimate

\[
\int_{\Omega_0} \left| K(\Phi(s, x) - \Phi(s, z)) - K(\tilde{\Phi}(s, x) - \tilde{\Phi}(s, z)) \right| dz
\]

(\text{A.7})

\(^{14}\)To construct such a vector field, one first considers the family of annuli \( A_n = \{ x \in \mathbb{R}^2 : 2^{-n-1} < |x| < 2^{-n+1} \} \). By rescaling the region \( A_\delta \cap \Omega \) to a domain of size \( O(1) \), we obtain a region with boundary in \( C^{1,\alpha} \). Then in this rescaled subset of the annulus one constructs easily a vector field in \( C^\alpha \) with desired properties. Rescaling it back, and patching all the vector fields together finishes the construction of \( \tilde{W}_0 \).
for each \( s \in [0, \epsilon] \). We split the integral: when \(|z - x| > \epsilon\), we have
\[
\int_{\Omega_0 \setminus B_\epsilon(x)} \left| K(\Phi(s, x) - \Phi(s, z)) - K(\tilde{\Phi}(s, x) - \tilde{\Phi}(s, z)) \right| dz \\
\leq \mathcal{C}(M) \int_{\Omega_0 \setminus B_\epsilon(x)} \|\Phi(s) - \tilde{\Phi}(s)\|_{L^\infty} \cdot \frac{1}{|x - z|^2} dz \\
\leq \mathcal{C}(M)\|\Phi(s) - \tilde{\Phi}(s)\|_{L^\infty} (1 + |\log(\epsilon)|),
\]
whereas
\[
\int_{\Omega_0 \cap B_\epsilon(x)} \left| K(\Phi(s, x) - \Phi(s, z)) - K(\tilde{\Phi}(s, x) - \tilde{\Phi}(s, z)) \right| dz \leq \mathcal{C}(M) \int_{\Omega_0 \cap B_\epsilon(x)} \frac{1}{|x - z|^2} dz \leq \mathcal{C}(M)\epsilon.
\]
We have used the following elementary inequality:
\[
|K(a) - K(b)| \leq C|a - b| \left( \frac{1}{|a|^2} + \frac{1}{|b|^2} \right).
\]
Choosing \( \epsilon = \|\Phi(s) - \tilde{\Phi}(s)\|_{L^\infty} \) establishes the desired inequality (assuming that the latter quantity is non-zero – otherwise the result is trivial).

Turning to the next inequality, one sees that the key is to obtain a bound on the following integral:
\[
\int_{\Omega_0} \left| \nabla K(\Phi(s, x) - \Phi(s, z)) - \nabla K(\tilde{\Phi}(s, x) - \tilde{\Phi}(s, z)) \right| dz,
\]
modulo the terms which are trivially bounded by \( \mathcal{C}(M)\|\nabla \Phi - \nabla \tilde{\Phi}\|_{L^\infty} \).

To begin with, take some constant \( \epsilon_0 > 0 \) (depending only on \( \Omega_0 \)) with the property that, for any \( x \in \partial \Omega_0 \), there is an open ball of radius \( 4\epsilon_0 |x| \) contained in \( \Omega_0 \) and whose boundary contains \( x \). Now let us take some \( \epsilon < \epsilon_0 \), whose value will be determined later. We shall consider two cases: (i) \( d(x, \partial \Omega_0) > 2\epsilon |x| \), (ii) \( d(x, \partial \Omega_0) \leq 2\epsilon |x| \).

When (i) holds, let us split the integral as
\[
\int_{\Omega_0 \setminus B_{\epsilon|x|}(x)} + \int_{\Omega_0 \cap B_{\epsilon|x|}(x)} \left| \nabla K(\Phi(s, x) - \Phi(s, z)) - \nabla K(\tilde{\Phi}(s, x) - \tilde{\Phi}(s, z)) \right| dz,
\]
and in the former region, we further decompose into regions where \( \epsilon |x| < |z - x| \leq 10|x| \) and \( 10|x| < |z - x| \). Then, in the case \( \epsilon |x| < |z - x| \leq 10|x| \), using the mean value theorem with the decay of \( \nabla \nabla K \) gives a bound
\[
\mathcal{C}(M)\|\nabla \Phi - \nabla \tilde{\Phi}\|_{L^\infty} (1 + |\log(\epsilon)|).
\]
Then, when \( 10|x| < |z - x| \) holds, one first symmetrizes the kernel to gain extra decay and then use the mean value theorem to obtain
\[
\mathcal{C}(M)\|\nabla \Phi - \nabla \tilde{\Phi}\|_{L^\infty}.
\]
In the latter region, the integral is bounded by
\[
\int_{\Sigma} |\nabla K(z)| dz \leq C \int_{\Sigma} |z|^{-2} dz,
\]
where
\[
\Sigma = (\Phi(s, B_{\epsilon|x|}(x)) - \Phi(s, x)) \Delta(\tilde{\Phi}(s, B_{\epsilon|x|}(x)) - \tilde{\Phi}(s, x)) \\
:= \{y - \Phi(s, x) : y \in \Phi(s, B_{\epsilon|x|}(x))\} \Delta\{y - \tilde{\Phi}(s, x) : y \in \tilde{\Phi}(s, B_{\epsilon|x|}(x))\}.
\]
For any unit vector \( \omega \), define
\[
r_1(\omega) = \min \{ r > 0 : r \omega \in \Sigma \}, \quad r_2(\omega) = \max \{ r > 0 : r \omega \in \Sigma \}.
\]
Then, the claim of Huang [52, (4.20) on p. 531] translates in our setting to give that (after the usual scaling argument in \( |x| \))
\[
r_1(\omega) \geq (C(M))^{-1} \epsilon |x|, \quad r_2(\omega) \leq C(M) \epsilon |x| \left( \epsilon^\alpha + \| \nabla \Phi - \nabla \tilde{\Phi} \|_{L^\infty} \right).
\]
Using these bounds, we integrate
\[
\int_\Sigma |\nabla K(z) dz| \leq C(M) \int_{\partial B_1(0)} \int_{r_1(\omega)}^{r_2(\omega)} \frac{1}{r} dr d\omega \leq C(M) \int_{\partial B_1(0)} \log \left( 1 + \frac{r_2(\omega) - r_1(\omega)}{r_1(\omega)} \right) d\omega
\]
\[
\leq C(M) \int_{\partial B_1(0)} \frac{r_2(\omega) - r_1(\omega)}{r_1(\omega)} d\omega \leq C(M)(\epsilon^\alpha + \| \nabla \Phi - \nabla \tilde{\Phi} \|_{L^\infty}).
\]
We have established the desired bound on \((A.7)\), and it follows immediately that
\[
|\nabla u(s, \Phi(s, x)) - \nabla \tilde{u}(s, \tilde{\Phi}(s, x))| \leq C(M)(\epsilon^\alpha + \| \nabla \Phi - \nabla \tilde{\Phi} \|_{L^\infty} (1 + |\log(\epsilon)|))
\]
when (i) holds, and with \( \epsilon < \epsilon_0 \).

Now, when (ii) holds for \( x \in \Omega_0 \), we can select (by the assumption on \( \epsilon_0 \)) a point \( y \in \Omega_0 \), such that
\[
d(y, \partial \Omega_0) \geq 2\epsilon |x| \text{ and } |x - y| \leq 2\epsilon |x|.
\]
Then,
\[
|\nabla u(s, \Phi(s, x)) - \nabla \tilde{u}(s, \tilde{\Phi}(s, x))| \leq |\nabla u(s, \Phi(s, x)) - \nabla u(s, \Phi(s, y))|
\]
\[
+ |\nabla u(s, \Phi(s, y)) - \nabla \tilde{u}(s, \tilde{\Phi}(s, y))|
\]
\[
+ |\nabla \tilde{u}(s, \tilde{\Phi}(s, y)) - \nabla \tilde{u}(s, \tilde{\Phi}(s, x))|
\]
\[
\leq C(M)(\epsilon^\alpha + \| \nabla \Phi - \nabla \tilde{\Phi} \|_{L^\infty} (1 + |\log(\epsilon)|)) + C(M)\epsilon^\alpha,
\]
where we have used that \( \nabla u, \nabla \tilde{u} \in C^\alpha \):
\[
|\nabla u(s, \Phi(s, x)) - \nabla u(s, \Phi(s, y))| \leq C(M)\frac{|\Phi(s, x) - \Phi(s, y)|^\alpha}{|\Phi(s, x)|^\alpha} \leq C(M)\| \nabla \Phi \|_{L^\infty} \cdot \frac{|x - y|^\alpha}{|x|^\alpha} \leq C(M)\epsilon^\alpha,
\]
and similarly for the other term.

At this point, observe that
\[
\frac{d}{dt} |\nabla \Phi(t, x) - \nabla \tilde{\Phi}(t, x)| \leq C(M),
\]
so that
\[
\| \nabla \Phi(t, \cdot) - \nabla \tilde{\Phi}(t, \cdot) \|_{L^\infty} \leq C(M)t,
\]
and therefore by taking \( T_1 \) sufficiently small, relative to \( M \) and \( \Omega_0 \), it can be assumed that
\[
\sup_{t \in [0, T_1]} \| \nabla \Phi - \nabla \tilde{\Phi} \|_{L^\infty} \leq \frac{1}{10} \epsilon_0^\alpha.
\]
Now we may take \( \epsilon^\alpha = \| \nabla \Phi(\cdot, s) - \nabla \tilde{\Phi}(\cdot, s) \|_{L^\infty} \) for each \( s \in [0, T_1] \) (or just a sufficiently small constant when the latter is zero). This finishes the proof. \( \square \)
In the course of the above local well-posedness proof, we needed to prove that the flow maps having regularity $\nabla \Phi \in \mathcal{C}^{\alpha}$ implies that the corresponding velocity gradient satisfies $\nabla u \in \mathcal{C}^{\alpha}$. For completeness we show that the converse also holds.

**Proposition A.8.** Let $u$ be a vector field with regularity $\nabla u \in L^\infty([0,T); \mathcal{C}^{\alpha}(\mathbb{R}^2))$ for some $0 < \alpha \leq 1$ and satisfy $u(t,0) = 0$ for all $t \in [0,T)$. Then the associated flow map $\Phi$ satisfies

$$\|\nabla \Phi(t)\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)} \leq M = M(t, \sup_{s \in [0,t]} \|\nabla u(s)\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}).$$

**Proof.** Since the velocity is Lipschitz, there is a unique solution to

$$\frac{d}{dt} \Phi(t,a) = u(t, \Phi(t,a)), \quad \Phi(0,a) = a,$$

which defines the flow map $\Phi(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ for each $t \in [0,T)$. Clearly $\Phi(t,0) = 0$.

Taking two points $a \neq b$, we obtain that

$$\left| \frac{d}{dt} \left( \Phi(t,a) - \Phi(t,b) \right) \right| \leq \|\nabla u(t, \cdot)\|_{L^\infty(D)} \frac{|\Phi(t,a) - \Phi(t,b)|}{|a - b|},$$

and therefore, by integrating in time, we obtain

$$\exp\left(-\int_0^t \|\nabla u(s, \cdot)\|_{L^\infty(D)} ds\right) \leq \frac{|\Phi(t,a) - \Phi(t,b)|}{|a - b|} \leq \exp\left(\int_0^t \|\nabla u(s, \cdot)\|_{L^\infty(D)} ds\right).$$

We now proceed to obtain $\mathcal{C}^{\alpha}$-estimates for the gradient of the flow. We know that $\Phi(t, \cdot)$ is differentiable almost everywhere, and it is straightforward to show that the gradient (defined almost everywhere) satisfies

$$\frac{d}{dt} \nabla \Phi(t,a) = \nabla u(t, \Phi(t,a)) \nabla \Phi(t,a)$$

for almost every $a \in D$. For two points $a, b \in D$, we write

$$\frac{d}{dt} (|a|^\alpha \nabla \Phi(t,a) - |b|^\alpha \nabla \Phi(t,b)) = \left( \frac{|a|}{|\Phi(t,a)|} \right)^\alpha (|\Phi(t,a)|^\alpha \nabla u(t, \Phi(t,a)) - |\Phi(t,b)|^\alpha \nabla u(t, \Phi(t,b)))$$

$$+ (|a|^\alpha - |b|^\alpha) \nabla u(t, \Phi(t,b)) \nabla \Phi(t,a).$$

We further write

$$|a|^\alpha \nabla u(t, \Phi(t,a)) - |b|^\alpha \nabla u(t, \Phi(t,b)) = \left( \frac{|a|}{|\Phi(t,a)|} \right)^\alpha (|\Phi(t,a)|^\alpha \nabla u(t, \Phi(t,a)) - |\Phi(t,b)|^\alpha \nabla u(t, \Phi(t,b)))$$

$$+ \left[ |a|^\alpha - |b|^\alpha + \frac{(|\Phi(t,b)|^\alpha - |\Phi(t,a)|^\alpha)|a|^\alpha}{|\Phi(t,a)|^\alpha} \right] \nabla u(t, \Phi(t,b)).$$

Hence,

$$\frac{d}{dt} \left( \frac{|a|^\alpha \nabla \Phi(t,a) - |b|^\alpha \nabla \Phi(t,b)|}{|a - b|^\alpha} \right)$$

$$\leq \left( \frac{|a|}{|\Phi(t,a)|} \right)^\alpha \left( |\Phi(t,a)|^\alpha \nabla u(t, \Phi(t,a)) - |\Phi(t,b)|^\alpha \nabla u(t, \Phi(t,b)) \right) \|\nabla \Phi(t, \cdot)\|_{L^\infty}^{1+\alpha}$$

$$+ \left( 1 + \|\nabla \Phi(t, \cdot)\|_{L^\infty} \left( \frac{|a|}{|\Phi(t,a)|} \right)^\alpha \right) \|\nabla u(t, \cdot)\|_{L^\infty}$$

$$+ \|\nabla u(t, \cdot)\|_{L^\infty} \left( \frac{|a|^\alpha \nabla \Phi(t,a) - |b|^\alpha \nabla \Phi(t,b)|}{|a - b|^\alpha} \right)$$

$$+ \|\nabla u(t, \cdot)\|_{L^\infty} \|\nabla \Phi(t, \cdot)\|_{L^\infty}.$$
Note that
\[
\frac{|a|}{\Phi(t,a)} \leq \exp \left( \int_0^t \| \nabla u(s,\cdot) \|_{L^\infty(D)} ds \right),
\]
and therefore we have the bound
\[
\left| \frac{d}{dt} \frac{|a|^\alpha \nabla \Phi(t,a) - |b|^\alpha \nabla \Phi(t,b)}{|a-b|^\alpha} \right|
\leq \exp \left( C \int_0^t \| \nabla u(s,\cdot) \|_{L^\infty(D)} ds \right) \left( 1 + \| \nabla u(t,\cdot) \|_{C^0} + \frac{|a|^\alpha \nabla \Phi(t,a) - |b|^\alpha \nabla \Phi(t,b)}{|a-b|^\alpha} \right).
\]
Integrating in time using the Gronwall inequality, the proof is complete. \qed

References

[1] Serge Alinhac. Remarques sur l’instabilité du problème des poches de tourbillon. *J. Funct. Anal.*, 98(2):361–379, 1991.

[2] David M. Ambrose, James P. Kelliher, Milton C. Lopes Filho, and Helena J. Nussenzveig Lopes. Serfati solutions to the 2D Euler equations on exterior domains. *J. Differential Equations*, 259(9):4509–4560, 2015.

[3] Andreas Axelsson and Alan McIntosh. Hodge decompositions on weakly Lipschitz domains. In *Advances in analysis and geometry*, Trends Math., pages 3–29. Birkhäuser, Basel, 2004.

[4] Diego Ayala and Bartosz Protas. Maximum palinstrophy growth in 2D incompressible flows. *J. Fluid Mech.*, 742:340–367, 2014.

[5] Diego Ayala and Bartosz Protas. Vortices, maximum growth and the problem of finite-time singularity formation. *Fluid Dyn. Res.*, 46(3):031404, 14, 2014.

[6] Diego Ayala and Bartosz Protas. Extreme vortex states and the growth of enstrophy in threedimensional incompressible flows. *J. Fluid Mech.*, 818:772–806, 2017.

[7] Hantaek Bae and James P. Kelliher. Propagation of striated regularity of velocity for the Euler equations. *arXiv:1508.01915*.

[8] Hantaek Bae and James P. Kelliher. The vortex patches of Serfati. *arXiv:1409.5169*.

[9] H. Bahouri and J.-Y. Chemin. Équations de transport relatives à des champs de vecteurs non-lipschitziens et mécanique des fluides. *Arch. Rational Mech. Anal.*, 127(2):159–181, 1994.

[10] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[11] D. Benedetto, C. Marchioro, and M. Pulvirenti. On the Euler flow in \( \mathbb{R}^2 \). *Arch. Rational Mech. Anal.*, 123(4):377–386, 1993.

[12] Frédéric Bernicot and Taoufik Hmidi. On the global well-posedness for Euler equations with unbounded vorticity. *Dyn. Partial Differ. Equ.*, 12(2):127–155, 2015.

[13] Frédéric Bernicot and Sahbi Keraani. On the global well-posedness of the 2D Euler equations for a large class of Yudovich type data. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(3):559–576, 2014.
[14] A. L. Bertozzi and P. Constantin. Global regularity for vortex patches. *Comm. Math. Phys.*, 152(1):19–28, 1993.

[15] Andrea Louise Bertozzi. *Existence, uniqueness, and a characterization of solutions to the contour dynamics equation*. ProQuest LLC, Ann Arbor, MI, 1991. Thesis (Ph.D.)–Princeton University.

[16] Jean Bourgain and Dong Li. Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces. *Invent. Math.*, 201(1):97–157, 2015.

[17] Jean Bourgain and Dong Li. Strong illposedness of the incompressible Euler equation in integer $C^m$ spaces. *Geom. Funct. Anal.*, 25(1):1–86, 2015.

[18] Jacob Burbea. Motions of vortex patches. *Lett. Math. Phys.*, 6(1):1–16, 1982.

[19] T. F. Buttke. The observation of singularities in the boundary of patches of constant vorticity. *Phys. Fluids A*, 1:1283–1285, 1989.

[20] J. A. Carrillo and J. Soler. On the evolution of an angle in a vortex patch. *J. Nonlinear Sci.*, 10(1):23–47, 2000.

[21] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations. *Duke Math. J.*, 165(5):935–984, 2016.

[22] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Uniformly rotating analytic global patch solutions for active scalars. *Ann. PDE*, 2(1):Art. 1, 34, 2016.

[23] J.-Y. Chemin. Persistance des structures géométriques liées aux poches de tourbillon. In *Séminaire sur les Équations aux Dérivées Partielles, 1990–1991*, pages Exp. No. XIII, 11. École Polytech., Palaiseau, 1991.

[24] Jean-Yves Chemin. Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. *Ann. Sci. École Norm. Sup. (4)*, 26(4):517–542, 1993.

[25] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.

[26] Albert Cohen and Raphael Danchin. Multiscale approximation of vortex patches. *SIAM J. Appl. Math.*, 60(2):477–502, 2000.

[27] P. Constantin and E. S. Titi. On the evolution of nearly circular vortex patches. *Comm. Math. Phys.*, 119(2):177–198, 1988.

[28] Bernard Dacorogna. *Introduction to the calculus of variations*. Imperial College Press, London, third edition, 2015.

[29] Raphaël Danchin. évolution temporelle d’une poche de tourbillon singulière. *Comm. Partial Differential Equations*, 22(5-6):685–721, 1997.

[30] Raphaël Danchin. évolution d’une singularité de type cusp dans une poche de tourbillon. *Rev. Mat. Iberoamericana*, 16(2):281–329, 2000.

[31] Francisco de la Hoz, Zineb Hassainia, Taoufik Hmidi, and Joan Mateu. An analytical and numerical study of steady patches in the disc. *Anal. PDE*, 9(7):1609–1670, 2016.
[32] G. S. Deem and N. J. Zabusky. Stationary “v-states,” interactions, recurrence, and breaking. *Phys. Rev. Lett.*, 40:859–862, 1978.

[33] Sergey A. Denisov. Double exponential growth of the vorticity gradient for the two-dimensional Euler equation. *Proc. Amer. Math. Soc.*, 143(3):1199–1210, 2015.

[34] Sergey A. Denisov. The sharp corner formation in 2D Euler dynamics of patches: infinite double exponential rate of merging. *Arch. Ration. Mech. Anal.*, 215(2):675–705, 2015.

[35] Nicolas Depauw. Poche de tourbillon pour Euler 2D dans un ouvert à bord. *J. Math. Pures Appl. (9)*, 78(3):313–351, 1999.

[36] D. G. Dritschel and M. E. McIntyre. Does contour dynamics go singular? *Phys. Fluids A*, 2(5):748–753, 1990.

[37] David G. Dritschel. Contour surgery: a topological reconnection scheme for extended integrations using contour dynamics. *J. Comput. Phys.*, 77(1):240–266, 1988.

[38] David G. Dritschel and Lorenzo M. Polvani. The roll-up of vorticity strips on the surface of a sphere. *J. Fluid Mech.*, 234:47–69, 1992.

[39] Tarek M. Elgindi. Propagation of Singularities for the 2d incompressible Euler equations. *In Preparation.*

[40] Tarek M. Elgindi. Remarks on functions with bounded Laplacian. *arXiv:1605.05266.*

[41] Tarek M. Elgindi and In-Jee Jeong. On singular vortex patches, ii: Long-time dynamics. *in preparation.*

[42] Tarek M. Elgindi and In-Jee Jeong. Symmetries and critical phenomena in fluids. *arxiv:1610.09701.*

[43] Tarek M. Elgindi and In-Jee Jeong. Ill-posedness for the Incompressible Euler Equations in Critical Sobolev Spaces. *Ann. PDE*, 3(1):3:7, 2017.

[44] Tarek M. Elgindi and Nader Masmoudi. Ill-posedness results in critical spaces for some equations arising in hydrodynamics. *arXiv:1405.2478*, 2014.

[45] Avner Friedman and Chao Cheng Huang. Averaged motion of charged particles under their self-induced electric field. *Indiana Univ. Math. J.*, 43(4):1167–1225, 1994.

[46] Avner Friedman and Juan J. L. Velázquez. A time-dependent free boundary problem modeling the visual image in electrophotography. *Arch. Rational Mech. Anal.*, 123(3):259–303, 1993.

[47] Zineb Hassainia, Nader Masmoudi, and Miles Wheeler. Global bifurcation of rotating vortex patches. *arXiv:1712.03085.*

[48] Taoufik HMidi and Joan Mateu. Bifurcation of rotating patches from Kirchhoff vortices. *Discrete Contin. Dyn. Syst.*, 36(10):5401–5422, 2016.

[49] Taoufik HMidi and Joan Mateu. Degenerate bifurcation of the rotating patches. *Adv. Math.*, 302:799–850, 2016.

[50] Taoufik HMidi, Joan Mateu, and Joan Verdera. Boundary regularity of rotating vortex patches. *Arch. Ration. Mech. Anal.*, 209(1):171–208, 2013.

[51] David Hoff and Misha Perepelitsa. Instantaneous boundary tangency and cusp formation in two-dimensional fluid flow. *SIAM J. Math. Anal.*, 41(2):753–780, 2009.
[52] Chaocheng Huang. Singular integral system approach to regularity of 3D vortex patches. *Indiana Univ. Math. J.*, 50(1):509–552, 2001.

[53] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. On the large-time behavior of two-dimensional vortex dynamics. *Phys. D*, 179(3-4):153–160, 2003.

[54] Dragos Iftimie, Thomas C. Sideris, and Pascal Gamblin. On the evolution of compactly supported planar vorticity. *Comm. Partial Differential Equations*, 24(9-10):1709–1730, 1999.

[55] In-Jee Jeong and Tsuyoshi Yoneda. A remark on the zeroth law and instantaneous vortex stretching on the incompressible 3D Euler equations. *arXiv:1902.02032*.

[56] James P. Kelliher. A characterization at infinity of bounded vorticity, bounded velocity solutions to the 2D Euler equations. *Indiana Univ. Math. J.*, 64(6):1643–1666, 2015.

[57] Namkwon Kim. Eigenvalues associated with the vortex patch in 2-D Euler equations. *Math. Ann.*, 330(4):747–758, 2004.

[58] Alexander Kiselev, Lenya Ryzhik, Yao Yao, and Andrej Zlatos. Finite time singularity for the modified SQG patch equation. *Ann. of Math. (2)*, 184(3):909–948, 2016.

[59] Alexander Kiselev and Vladimir Šverák. Small scale creation for solutions of the incompressible two-dimensional Euler equation. *Ann. of Math. (2)*, 180(3):1205–1220, 2014.

[60] Horace Lamb. *Hydrodynamics*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, sixth edition, 1993. With a foreword by R. A. Caflisch [Russel E. Caflisch].

[61] Chao Li. Global regularity and fast small scale formation for euler patch equation in a disk. *arXiv:1703.09674*.

[62] Paolo Luzzatto-Fegiz. Bifurcation structure and stability in models of opposite-signed vortex pairs. *Fluid Dyn. Res.*, 46(3):031408, 14, 2014.

[63] Paolo Luzzatto-Fegiz and Charles H. K. Williamson. An efficient and general numerical method to compute steady uniform vortices. *J. Comput. Phys.*, 230(17):6495–6511, 2011.

[64] Paolo Luzzatto-Fegiz and Charles H. K. Williamson. Investigating stability and finding new solutions in conservative fluid flows through bifurcation approaches. In *Nonlinear physical systems*, Mech. Eng. Solid Mech. Ser., pages 203–221. Wiley, Hoboken, NJ, 2014.

[65] Andrew Majda. Vorticity and the mathematical theory of incompressible fluid flow. *Comm. Pure Appl. Math.*, 39(S, suppl.):S187–S220, 1986. Frontiers of the mathematical sciences: 1985 (New York, 1985).

[66] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[67] Carlo Marchioro. Bounds on the growth of the support of a vortex patch. *Comm. Math. Phys.*, 164(3):507–524, 1994.

[68] Carlo Marchioro and Mario Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[69] Gerard Misiolek and Tsuyoshi Yoneda. Local ill-posedness of the incompressible Euler equations in $C^1$ and $B^1_{\infty,1}$. *Math. Ann.*, 364(1-2):243–268, 2016.
Gerard Misiolek and Tsuyoshi Yoneda. Continuity of the solution map of the Euler equations in Hölder spaces and weak norm inflation in Besov spaces. Trans. Amer. Math. Soc., 370(7):4709–4730, 2018.

Piotr Bogusław Mucha and Walter M. Rusin. Zygmund spaces, inviscid limit and uniqueness of Euler flows. Comm. Math. Phys., 280(3):831–841, 2008.

Edward A. Overman, II. Steady-state solutions of the Euler equations in two dimensions. II. Local analysis of limiting V-states. SIAM J. Appl. Math., 46(5):765–800, 1986.

Lorenzo M. Polvani and David G. Dritschel. Wave and vortex dynamics on the surface of a sphere. J. Fluid Mech., 255:35–64, 1993.

D. I. Pullin. The large-scale structure of unsteady self-similar rolled-up vortex sheets. J. Fluid Mech., 88(3):401–430, 1978.

D. I. Pullin. On similarity flows containing two-branched vortex sheets. In Mathematical aspects of vortex dynamics (Leesburg, VA, 1988), pages 97–106. SIAM, Philadelphia, PA, 1989.

D. I. Pullin. Vortex tubes, spirals, and large-eddy simulation of turbulence. In Tubes, sheets and singularities in fluid dynamics (Zakopane, 2001), volume 71 of Fluid Mech. Appl., pages 171–180. Kluwer Acad. Publ., Dordrecht, 2002.

P. G. Saffman. Vortex dynamics. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1992.

G. Saffman P. and R. Szeto. Equilibrium shapes of a pair of equal uniform vortices. Phys. Fluids, 23:2339, 1980.

Philippe Serfati. Régularité stratifiée et équation d’Euler 3D à temps grand. C. R. Acad. Sci. Paris Sér. I Math., 318(10):925–928, 1994.

Philippe Serfati. Une preuve directe d’existence globale des vortex patches 2D. C. R. Acad. Sci. Paris Sér. I Math., 318(6):515–518, 1994.

Philippe Serfati. Solutions $C^\infty$ en temps, n-log Lipschitz bornées en espace et équation d’Euler. C. R. Acad. Sci. Paris Sér. I Math., 320(5):555–558, 1995.

Philippe Serfati. Structures holomorphes à faible régularité spatiale en mécanique des fluides. J. Math. Pures Appl. (9), 74(2):95–104, 1995.

Sung-Ik Sohn, Takashi Sakajo, and Sun-Chul Kim. Stability of barotropic vortex strip on a rotating sphere. Proc. A., 474(2210):20170883, 25, 2018.

J. T. Stuart. Nonlinear Euler partial differential equations: singularities in their solution. In Applied mathematics, fluid mechanics, astrophysics (Cambridge, MA, 1987), pages 81–95. World Sci. Publishing, Singapore, 1988.

Yasushi Taniuchi. Uniformly local $L^p$ estimate for 2-D vorticity equation and its application to Euler equations with initial vorticity in bmo. Comm. Math. Phys., 248(1):169–186, 2004.

Yasushi Taniuchi, Tomoya Tashiro, and Tsuyoshi Yoneda. On the two-dimensional Euler equations with spatially almost periodic initial data. J. Math. Fluid Mech., 12(4):594–612, 2010.

Misha Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. Ann. Sci. École Norm. Sup. (4), 32(6):769–812, 1999.
[88] H. M. Wu, E. A. Overman, II, and N. J. Zabusky. Steady-state solutions of the Euler equations in two dimensions: rotating and translating V-states with limiting cases. I. Numerical algorithms and results. *J. Comput. Phys.*, 53(1):42–71, 1984.

[89] Xiaoqian Xu. Fast growth of the vorticity gradient in symmetric smooth domains for 2D incompressible ideal flow. *J. Math. Anal. Appl.*, 439(2):594–607, 2016.

[90] B. B. Xue, E. R. Johnson, and N. R. McDonald. New families of vortex patch equilibria for the two-dimensional Euler equations. *Physics of Fluids*, 29(12):123602, 2017.

[91] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Z. Vycisl. Mat. i Mat. Fiz.*, 3:1032–1066, 1963.

[92] V. I. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Lett.*, 2(1):27–38, 1995.

[93] Norman J. Zabusky, M. H. Hughes, and K. V. Roberts. Contour dynamics for the Euler equations in two dimensions. *J. Comput. Phys.*, 30(1):96–106, 1979.

[94] Andrej Zlatoš. Exponential growth of the vorticity gradient for the Euler equation on the torus. *Adv. Math.*, 268:396–403, 2015.