Fuzzy operators and the quantized electromagnetic field in the very-high-energy regime

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In this work, starting from commutation relations between phase-space operators (in “first quantization”) we define averaged creation and annihilation operators and show that they satisfy a simple, deformed commutation relation. By extending this relation to the quantized electromagnetic field, we determine the new vacuum state which has a non-zero component in standard occupied states. In addition we are led to non-linear de Broglie relations for photons, which appreciably depart from linearity only in the very-high-energy regime. The nonlinear Compton scattering that follows from these assumptions is discussed. We suggest that this hypothesis may be a way to deal with the transparency of the electromagnetic background light (EBL) and show that it may lead to an attenuation in the cosmological-constant problem of several orders of magnitude.

I. INTRODUCTION

Creation and annihilation operators play a central role in both, elementary quantum mechanics and quantum field theory (QFT). These operators represent the ideal processes of adding (subtracting) an exact amount of energy to (from) a physical system. What are the consequences of this “sharpness” to the description of physical systems, in particular, in the very-high-energy (VHE) domain? While QFT is at the heart of our understanding of a large variety of phenomena, there remains some discrepancies between theoretically predicted and observed quantities for VHE. The most known of these problems is related to the zero point energy that appears as a result of field quantization. On the one hand, it provides the conceptual framework to explain the finite life times of excited atoms and the Casimir effect, for instance, while, on the other hand, it leads to an abyss between observed and predicted values of the cosmological constant [1, 2]. The discrepancy is sometimes referred to as a “catastrophe”, since it amounts to something around 120 orders of magnitude, thus competing in severity with the ultra-high-energy domain [7–9, 11].

In this article we follow a somewhat complementary conceptual framework to explain the finite life times of excited atoms and the Casimir effect, for instance, while, on the other hand, it leads to an abyss between observed and predicted values of the cosmological constant [1, 2]. The discrepancy is sometimes referred to as a “catastrophe”, since it amounts to something around 120 orders of magnitude, thus competing in severity with the ultra-high-energy domain [7–9, 11].

In attempts to cope with these and other problems, non-linear dispersion relations have been proposed both, for photons and massive particles. Although some of the models in the literature are Lorentz invariant [6], most of them depart from relativistic covariance above the ultra-high-energy domain [7, 8, 11]. These non-linear energy-momentum relations are typically of the form \( E^2 = c^2 p^2 + m^2 c^4 + O(E^3) \), giving rise to dispersive relations: \( \omega = \epsilon k + O(k^2) \).

In this article we follow a somewhat complementary approach in the sense that we keep Lorentz invariance and the dispersion relation \( \omega = \epsilon k \), while changing the de Broglie relations in the VHE limit. The new relations are not imposed from the outset, but rather, obtained from a definition of fuzzy operators in the elementary context of one-body phase-space operators. The extension of the emerging algebraic structure of the creation and annihilation operators to quantum field theory requires non-linear de Broglie relations, which, in turn, may be helpful in discussing some of the aforementioned problems.

II. PRELIMINARIES

A. Phase-space operators

We start by addressing the elementary situation in which \( \mathcal{H} \) is the Hilbert space associated with a particle of mass \( m \) in one spatial dimension. Consider two operators \( \hat{A} = \alpha \hat{q} + i \beta \hat{p} \) and \( \hat{A}' = \alpha' \hat{q} + i \beta' \hat{p} \), where \( \hat{q} \) and \( \hat{p} \) are the canonically conjugated position and momentum operators acting on \( \mathcal{H} \). It is then immediate that

\[
[\hat{A}, \hat{A}'] = \hbar(\beta \alpha' - \alpha \beta').
\]

If we set

\[
\alpha = \frac{1}{\sqrt{2b}} \quad \text{and} \quad \beta = \frac{b}{\sqrt{2\hbar}}, \quad \text{with} \quad b = \sqrt{\frac{\hbar}{m\omega}},
\]

1/\( \omega \) being a constant with dimension of time, \( \hat{A} \) formally coincides with the annihilation operator

\[
\hat{A} \equiv \hat{a}_\omega = \frac{1}{\sqrt{2}} \left( \frac{\hat{q}}{b} + i \frac{\hat{p}}{\hbar} \right) = \sqrt{\frac{m\omega}{2\hbar}} \frac{\hat{q}}{\sqrt{2}} - \frac{i}{\sqrt{2m\omega}} \frac{\hat{p}}{\sqrt{2}}
\]

related to a harmonic potential of angular frequency \( \omega \). If we make analogous assignments for \( \alpha', \beta' \) and \( b' \), related to \( \omega' \), then we get the annihilation operator \( \hat{a}_{\omega'} \) associated with a harmonic potential of angular frequency \( \omega' \). It is important to note that we can make these definitions wether or not the system is, in fact, subjected to a harmonic potential. We can simply see \( \hat{a}_\omega \) and \( \hat{a}_{\omega'} \) as different linear combinations of \( \hat{q} \) and \( \hat{p} \). Importantly, in
this context, $\hat{a}_\omega$ and $\hat{a}_{\omega'}$ act on the same Hilbert space. It is then a trivial exercise to show that (1) becomes
\begin{equation}
[\hat{a}_\omega, \hat{a}_{\omega'}] = \frac{1}{2} \left( \frac{\sqrt{\omega}}{\sqrt{\omega'}} - \frac{\sqrt{\omega'}}{\sqrt{\omega}} \right), \tag{3}
\end{equation}
which is non-vanishing in general (except when $\omega' = \omega$).
This should be contrasted with the relation $[\hat{a}_\omega, \hat{a}_{\omega}] = 0$, for $\omega \neq \omega'$, when the operators refer to quantized bosonic fields, for which $\hat{a}_\omega$ and $\hat{a}_{\omega'}$ concern distinct modes, thus, acting on different Hilbert spaces.

Back to elementary quantum mechanics, by eliminating $\hat{q}$ and $\hat{p}$ in the expressions for $\hat{a}_\omega$ and $\hat{a}_{\omega'}$ one obtains
\begin{equation}
\hat{a}_{\omega+\Delta\omega} = \frac{1}{2\sqrt{1+\Delta\omega/\omega}} \left[ \left( \frac{\Delta\omega}{\omega} + 2 \right) \hat{a}_\omega + \frac{\Delta\omega}{\omega} \hat{a}_\omega^\dagger \right], \tag{4}
\end{equation}
where, for convenience, we set $\Delta\omega \equiv \omega' - \omega$.

Usually, we would restrict the validity of the previous relations (2), (3), and (4) to $\omega, \omega' > 0$ ($\Delta\omega \geq -\omega$). However, we note that the formal replacement $\omega \to -\omega$ leads to
\begin{equation}
\hat{a}_\omega \to \hat{a}_{-\omega} = i\hat{a}_\omega^\dagger,
\end{equation}
in definition (2). We must be careful about the branch of $\sqrt{-1}$ to be considered. Throughout this work we will adopt $\sqrt{-1} = i$ (accordingly $1/\sqrt{-1} = -i$), which maps annihilation into creation and vice-versa in (2), leading to $[\hat{a}_\omega, \hat{a}_{|\omega|}] = -i[\hat{a}_\omega, \hat{a}_{-\omega}]$. But, from (3) we get $[\hat{a}_\omega, \hat{a}_{-\omega}] = (i-1/i)/2 = i$, therefore $[\hat{a}_\omega, \hat{a}_{|\omega|}] = 1$, which reproduces the expected result for $\omega > 0$. With these extensions in mind, expression (4) becomes well defined for $\Delta\omega \in (-\infty, +\infty)$.

**B. Fuzzy operators**

Creation (annihilation) operators can be seen as the mathematical representations of the ideal process of adding (subtracting) the exact amount of energy $E_\omega = \hbar \omega$ to (from) a harmonic system. This is a “sharp” definition in the sense that no fluctuations around $E_\omega$ are possible at the operator level. Consider that some, yet unspecified factors make the physical processes inducing creation (annihilation) bear some level of variability. This naturally leads to the definition of averaged, or fuzzy operators
\begin{equation}
\bar{a}_\omega = \int_{-\infty}^{+\infty} f(\Delta\omega) \hat{a}_{\omega+\Delta\omega} d\Delta\omega, \tag{5}
\end{equation}
with an analogous definition being valid for $\bar{a}_{\omega'}^\dagger$. The function $f(\Delta\omega)$ is a normalized, real-valued distribution peaked at $\Delta\omega = 0$ and $\bar{a}_{\omega+\Delta\omega}$ is given by (4). In the case of a Dirac-delta distribution, $f(\Delta\omega) = \delta(\Delta\omega)$, we get the standard creation and annihilation operators. Although we will calculate the explicit expressions for the fuzzy operators, given a particular $f$, we will be primarily interested in the commutator $[\bar{a}_\omega, \bar{a}_{\omega'}^\dagger]$. This is so because, in addressing quantized fields in the next section, we will consider $\bar{a}_\omega$ and $\bar{a}_{\omega'}^\dagger$ to be fundamental elements (not derivable from more basic concepts). In addition, since we will associate the non-singular character of $f(\Delta\omega)$ to the effect of the vacuum, we will attribute a natural line shape to $f$, i.e., a Lorentzian distribution:
\begin{equation}
f(\Delta\omega) = \frac{1}{\pi} \frac{\Gamma/2}{\Delta\omega^2 + (\Gamma/2)^2}, \tag{6}
\end{equation}
where $\Gamma$ characterizes the width of the distribution, which may depend on the mode, $\Gamma = \Gamma(\omega)$. Using this expression in (5) and setting $x = \Delta\omega/\omega$ we get
\begin{equation}
\bar{a}_\omega = \frac{\zeta}{2\pi} \left[ (2I_0 + I_1)\bar{a}_\omega + I_1 \bar{a}_\omega^\dagger \right], \tag{7}
\end{equation}
where the single relevant parameter is given by the ratio $\zeta(\omega) = \Gamma(\omega)/2\omega$, and
\begin{equation}
I_k(\zeta) = \int_{-\infty}^{+\infty} \frac{x^k dx}{(x^2 + \zeta^2)^{k+1}} = \int_{-1}^{1} \frac{(-1)^k x^k dx}{\sqrt{1-x^2}} - i \int_{1}^{\infty} \frac{(-1)^k x^k dx}{(x^2 + \zeta^2)^{1/2}},
\end{equation}
for $k = 0, 1$, and $\sigma = \sqrt{\zeta^2 + 1}$. The desired commutator takes the fairly simple form:
\begin{equation}
[\bar{a}_\omega, \bar{a}_{\omega'}^\dagger] = \frac{\zeta^2}{\pi^2} \left[ |I_0|^2 + 2 \Re[I_0^* I_1] \right] = \frac{1}{\sqrt{\zeta^2(\omega) + 1}} \equiv C(\omega). \tag{8}
\end{equation}
The above, deformed commutation relation is arguably the simplest variation of the canonical relation. For instance, in the $q$-deformed bosonic realization of the quantum groups $SU(n)_q$ [12,14], the commutator is not proportional to the identity operator.

**C. Constraining the commutation function $C(\omega)$**

Consider a simple harmonic oscillator, whose Hamiltonian reads $\hat{H} = \hat{p}^2/2m + mu^2\hat{q}^2/2 = \hbar\omega(\hat{a}_{\omega}^\dagger \hat{a}_{\omega} + \hat{a}_{\omega'}^\dagger \hat{a}_{\omega'})$. This is as far as we can go without using the canonical relation $[\hat{a}_\omega, \hat{a}_{\omega'}^\dagger] = 1$, that allows us to write $\hat{H} = \hbar\omega(\hat{a}_{\omega}^\dagger \hat{a}_{\omega} + \hat{a}_{\omega'} \hat{a}_{\omega'})$. Let us make the working hypothesis that the harmonic oscillator has some builtin fuzziness, with $\hat{a}$ ($\hat{a}^\dagger$) replaced by $\bar{a}$ ($\bar{a}^\dagger$). In dealing with quantized fields in the next section we will attribute this assumption to the “presence” of the vacua associated with other fields. Thus, we write
\begin{equation}
\bar{H} = \hbar\omega(\bar{a}_{\omega}^\dagger \bar{a}_{\omega} + \bar{a}_{\omega'} \bar{a}_{\omega'}),
\end{equation}
that, due to (3), becomes
\[ \hat{H} = \hbar \omega \hat{a}^\dagger \hat{a} + \frac{\hbar \omega}{2} C(\omega). \]

Low energy quantum harmonic oscillators are routinely observed in atomic and condensed matter systems, for instance. No deviations are found in the ground state energies. So, our hypothesis would be tenable only if
\[ C(\omega) = \frac{1}{\sqrt{[\Gamma(\omega)/2\omega]^2 + 1}} \to 1 \quad (9) \]
as \( \omega \to 0. \) We assume the simple dependence compatible with this constraint, \( \Gamma(\omega) \propto \omega^\gamma \) or \( \Gamma(\omega) = \omega^\gamma/\omega_c^{\gamma-1}, \) with \( \gamma > 1, \) where the critical frequency \( \omega_c \) is the angular frequency above which the deviation from the linear regime becomes relevant.

III. QUANTIZED ELECTROMAGNETIC FIELD

We proceed by considering the free electromagnetic field, whose energy density can be written in terms of the quadrature fields as
\[ \mathcal{H} = \frac{1}{2} \sum_{k,x} (\hat{P}_{k,x}^2 + \omega^2 \hat{Q}_{k,x}^2), \]
where \( \chi \) refers to the polarization degree of freedom. By “free” we mean that no other fields are present or, at least, do not interact with the photons. However, this is always an approximation because, in quantum field theory, assuming that other fields are absent is equivalent to say that their vacua are present. Thus, rigorously speaking, one cannot consistently get rid of the vacuum related to other fields, even in principle.

We consider the possibility that these vacua have some tangible effect on the field under consideration, as is the case of the electron field in the electromagnetic vacuum. In trying to take this into account, without explicit mention to other fields, we will assume that the second-quantization creation and annihilation operators inherit the commutation relation (5). Loosely speaking, we will assume that the vacuum of other fields blur the electromagnetic field operators. Of course, this is not a rigorous argument and we may simply state that our goal is to investigate the consequences of the previous assumption, at least on formal grounds. We therefore write
\[ \hat{\mathcal{H}} = \sum_k \hat{H}_k = \sum_k \left[ \hbar \omega \hat{a}^\dagger_k \hat{a}_k + \frac{\hbar \omega}{2} C(\omega) \right]. \]

The commutation relation (3) induces the simple algebraic structure that we describe in what follows. Let us denote the excitations of a particular field mode as \( |\bar{n}\rangle, \) such that \( \hat{H}_k |\bar{n}\rangle_k = \hat{E}_n |\bar{n}\rangle_k \) (\( \hat{H}_\omega |\bar{n}\rangle_\omega = \hat{E}_n (\omega) |\bar{n}\rangle_\omega \)). We demand the field operators \( \hat{a}_\omega \) and \( \hat{a}^\dagger_\omega \) to be genuine annihilation and creation operators. This is consistent only if the number operator is defined by
\[ \hat{N}_\omega \equiv [C(\omega)]^{-1} \hat{a}^\dagger_\omega \hat{a}_\omega \quad (10) \]

Therefore, in terms of \( \hat{N}_\omega \), we get \( \hat{H}_k = C(\omega)(\hbar \omega \hat{N}_\omega + \hbar \omega/2) \). With this, we obtain the usual commutators \( [\hat{a}_\omega, \hat{N}_\omega] = \hat{a}_\omega \) and \( [\hat{a}^\dagger_\omega, \hat{N}_\omega] = -\hat{a}_\omega \), and thus,
\[ \hat{a}_\omega |\bar{n}\rangle = \sqrt{n} |n - 1\rangle , \quad \hat{a}^\dagger_\omega |\bar{n}\rangle = \sqrt{n+1} |n+1\rangle , \]
as intended. In the whole Fock space we have
\[ \prod_{n=q_1}^{q_j} \frac{(\hat{a}^\dagger_\omega)^{n_q} \sqrt{n_q}}{\sqrt{n_q!}|\text{vac}\rangle} = |\cdots, \bar{n}_{j-1}, \bar{n}_{q_1}, \bar{n}_{q_{j+1}}, \cdots\rangle , \]
where \( q \) is a collective index, including momentum and polarization degrees of freedom. It will be instructive to determine the deformed vacuum in terms of the usual, sharp Fock states. This can be done through relation \( \hat{a}_\omega |\text{vac}\rangle = 0 \) together with (7), leading to
\[ (2I_0 + I_1) \sum_{j=1}^{\infty} \alpha_j \sqrt{j} |j - 1\rangle + I_1 \sum_{j=0}^{\infty} \alpha_j \sqrt{j + 1} |j + 1\rangle = 0 , \]
which corresponds to
\[ \sum_{n=1}^{\infty} [2I_0 + I_1] \alpha_{n+1} \sqrt{n + 1} + I_1 \alpha_{n-1} \sqrt{n} |n\rangle = 0 \quad (11) \]
and \( \alpha_1 = 0. \) This leads to a recursion relation whose solution reads \( \alpha_n = 0 \) for \( n \) odd and
\[ \alpha_{2k} = (-1)^k \left( \frac{I_1}{2I_0 + I_1} \right)^k \prod_{\ell=1}^{k} \left( \frac{2\ell - 1}{2\ell} \right) \alpha_0 \]
\[ = \frac{1}{\pi^{1/4}} \left( \frac{i \zeta}{\zeta - 2i} \right)^k \sqrt{((k - 1)/2)!} \alpha_0 \quad (12) \]
with \( k = 0, 1, 2, \ldots \) and \( \zeta = \Gamma(\omega)/2\omega = (\omega/\omega_c)^{\gamma-1/2} . \) This result and the normalization condition imply
\[ |\text{vac}\rangle = \left( \frac{4}{\zeta^2 + 4} \right)^{1/4} \sum_{k=0}^{\infty} \left( \frac{i \zeta}{\zeta - 2i} \right)^k \sqrt{[k - 1/2]!} \sqrt{k!} |2k\rangle . \]
\[ (13) \]
So,
\[ \langle 0 |\text{vac}\rangle = \alpha_0 = \left( \frac{4}{\zeta^2 + 4} \right)^{1/4} , \quad (14) \]
which shows that part of the fuzzy vacuum does not correspond to the usual vacuum, since \( \alpha_0 \leq 1. \) In this exploratory scenario part of the energy that we would attribute to the vacuum would be distributed in the sharp-field excitations. This would have consequences to the total zero-point energy as we will see shortly. Any excitation of the fuzzy field can be obtained in terms of standard Fock states with the help of equations (11) and (13).
A. Nonlinear de Broglie relations

Due to \( \bar{E} \), the energy of a single excitation of a mode with angular frequency \( \omega \) is \( \langle \bar{n} \rangle \hat{H} |\bar{n}\rangle - \langle \bar{n} - 1 \rangle \hat{H} |\bar{n} - 1\rangle = \mathcal{C}(\omega)\hbar \omega \equiv \bar{E}(\omega) \). So the photon energy-frequency relation would read

\[
\bar{E}(\omega; \tau) = \frac{\hbar \omega}{\sqrt{1 + (\omega/2\omega_c)^{2\tau - 2}}.}
\]

The standard linear relation \( E = \hbar \omega \) can be obtained by setting \( \omega_c \to \infty \). We note that, for \( \tau > 2 \), the function \( \bar{E}(\omega; \tau) \) has the rather extravagant property of presenting a maximum at the finite frequency \( \bar{\omega} = [4/(\tau - 2)]^{2\tau - 2} \omega_c \). This would imply that, above the maximum energy, photons with higher frequencies would be less energetic. On the other hand, for \( \tau < 1 \), condition (19) is violated. For \( \tau = 1 \), we obtain a linear relation with a rescaling in the frequencies. The only non-trivial integer value would be \( \tau = 2 \), which would make \( \bar{\omega} \to \infty \). This reasoning leads to

\[
\bar{E}(\omega) = \frac{\hbar \omega}{\sqrt{1 + (\omega/2\omega_c)^{2}}},
\]

(15)

which has consistent properties in both low-energy and high-energy limits: \( \bar{E}(\omega) \to \hbar \omega \) for \( \omega \ll \omega_c \) and monotonically grows, saturating at

\[
2\hbar \omega_c = E_\infty
\]

as \( \omega \to \infty \). In fact, as is usual, we will assume that there is a cutoff frequency given by the Planck scale, \( \omega_{pl} = \sqrt{c^3/\hbar G} \), with \( G \) being the gravitational constant. So, the maximal physical energy for a photon would be

\[
E_{\text{max}} = \hbar \omega_{pl}/\sqrt{1 + (\omega_{pl}/2\omega_c)^{2}} < E_\infty
\]

Therefore, in this context, the energy of photons would be bounded from above. We should have that \( E_{\text{max}} > E_{\text{max}}^{\text{obs}} \), where the energy \( E_{\text{max}}^{\text{obs}} \) corresponds to the most energetic \( \gamma \)-rays ever observed.

We argue, however, that it would not be easy to determine the present value of such a maximal energy. Several measurements are spectroscopic and, therefore, what is often measured is the frequency, the energy being inferred from the standard de Broglie relation. To check the possibility of non-linear relations like (15) we should rely on direct energy estimates. Since the relation \( E = \hbar \omega \) is customarily taken for granted, it is presently hard to devise in the literature what would be \( E_{\text{max}}^{\text{obs}} \).

Even more direct measurements would have to be interpreted with care in the context of non-linear de Broglie relations. Consider, for instance, energy measurements based in the showers produced by VHE \( \gamma \)-rays, as those carried out by the Major Atmospheric Gamma Imaging Cherenkov (MAGIC) collaboration [13]. The point is that relation (15) and relation (17) below would modify the scattering of VHE radiation by material particles, as we will illustrate later in this manuscript. So, in principle, the secondary scattering events which give rise to the low energy radiation that is measured by the telescopes would have to be re-examined.

The general message is that, in this scenario, care must be taken with the terminology. For instance, we have to specify what is meant by an “1 TeV photon” in the sense that a photon with energy 1 TeV would be no longer equivalent to a photon with frequency 1/\( \hbar \) TeV. In spite of this difficulty, we will make some numeric estimations in the final part of this manuscript.

Despite the fact that many approaches to quantum gravity predict energy-momentum relations which deviate from Lorentz invariance [8, 11], here, we will assume the relativistic relation \( E^2 = c^2 p^2 + m^2 c^4 \) to be valid, \( c \) being the speed of light in vacuum. In this case, if we take any relation between \( E \) and \( p \) different from \( E = cp \), then we would have VHE photons with a seizure mass \( \bar{m} \). Although this possibility may be potentially interesting [10–19], for simplicity, we will not follow in this direction in the present work. Therefore, we simply adopt

\[
\bar{E} = c\bar{p}.
\]

In the standard quantum theory of light the relations between the pairs \( (E, \omega) \) and \( (p, k) \) are completely symmetric. Indeed, by using (16) together with (15) we see that this symmetry is kept when the usual dispersion relation \( \omega = ck \) is assumed

\[
\bar{p}(k) = \frac{\hbar k}{\sqrt{1 + (k/2k_c)^{2}}}.
\]

(17)

with \( k_c = \omega_c/c \). So, here, the nonlinear relations come from a modification in the quantum relations \( E = \hbar \omega \) and \( p = \hbar k \) while keeping the relativistic relation \( E = cp \). This is a complementary approach to the several aforementioned works that investigate the opposite situation.

Note carefully the implications of (15), (16), and (17). While frequency and wave number remain unbounded (or limited by the Planck scales only), energy and momentum are bounded by \( E_{\text{max}} \) and \( E_{\text{max}}/c \), respectively. In addition, since stringent bounds have been obtained for possible nonlinear dispersion relations [8, 10], the fact that we keep

\[
\omega = ck,
\]

may be a potential advantage of the present model.

It should be remarked that imposing relations (13) and (17) to massive particles would correspond to a bold contradiction with the theory of relativity, which states that the energy and momentum of a material particle diverge as the velocity approaches the speed of light.
IV. DISCUSSION

A. Nonlinear Compton scattering

As we remarked earlier, a consequence of Lorentz invariance in the context of equations (15) and (17), would be a nonlinear Compton scattering relation in the VHE regime. Nonlinearities in Compton scattering have been predicted long ago in the high-intensity regime and posteriorly observed in the laboratory, see for example [23].

We will use the classical collisional model to infer the first order correction to the standard Compton effect. Of course, one cannot go further with such a classical model above the Schwinger limit. In this case, full QED calculations should be employed.

Since our model is Lorentz invariant, we can immediately use energy and momentum conservation to write $mc^2(\bar{p}_i - \bar{p}_f) = \bar{p}_i\bar{p}_f(1 - \cos \theta)$ (in the reference frame where the massive particle is at rest). The variables $\bar{p}_i$ ($\bar{p}_f$) refer to the momentum of the incoming (outgoing) photon, $\theta$ is the scattering angle of the outgoing photon with respect to the direction of the incoming photon, and $m$ is the mass of the particle involved in the collision. If we use the linear relation $p = \omega$, the standard result follows:

$$\frac{\omega_f}{\omega_i} = \frac{1}{1 + (1 - \cos \theta)(\omega_i/w)} \equiv u(\theta),$$

where $w = mc^2/h$ is the Compton angular frequency. Therefore, $\omega_f$ is red shifted with respect to $\omega_i$.

If, instead, we employ (17), after some algebra, we get the lower order correction

$$\frac{\omega_f}{\omega_i} = u(\theta) - \frac{1}{2}(1 - \cos \theta)u(\theta)^3 \eta,$$

with $\eta = w^2/\omega^2$. So, this correction would represent an extra red shift in $\omega_f$ (regarding $\omega_f$).

For the sake of comparison, the first order expression for the nonlinear Compton scattering for high-intensities, as it can be obtained from [20, 22], reads

$$\frac{\omega_f}{\omega_i} = u(\theta) - \frac{1}{2}(1 - \cos \theta)u(\theta)^2 \eta',$$

also leading to an extra red shift, where the expansion parameter $\eta'$ is proportional to $\omega_i \theta$, with $\theta$ being the density of photons. Despite the similarity, due to the different powers of $u(\theta)$ in the previous expansions, the angular dependence would be distinct.

This example illustrates how high-energy scattering processes would be affected by nonlinear de Broglie relations.

B. Consequences on cosmological issues

The present model may provide a framework to discuss the transparency of the background infrared (IR) radiation with respect to VHE $\gamma$-rays. VHE photons propagating in the intergalactic medium are likely to experiment inelastic collisions with the electromagnetic background light (EBL), producing an electron-positron pair: $\gamma_{VHE} + \gamma_{IR} \rightarrow e^- + e^+$. The problem resides in the fact that the predicted mean-free path of these VHE photons may be much smaller than the distance between the earth and the emitting bodies. This problem is also referred to as the “pair production anomaly.”

Presently, it is acknowledged that the problem is more related with the general absorption level for VHE photons, without distinct properties in specific energy values. It is then clear that the presented model may be able to deal with this matter. Once the frequency is larger than $\omega_c$, $\omega \in [\omega_c, +\infty]$ all photons would have energies packed in the “short” interval $[E_{\infty}/\sqrt{3}, E_{max}]$, thus, leading to a globally similar behavior for VHE $\gamma$-rays. More specifically, while Gev photons have a large mean free path (about the size of the visible universe), the corresponding quantity for TeV photons is no more than 3.3 million light-years. However, TeV $\gamma$-rays have been detected from blazars which are as far as 5.3 billion light-years away, for instance. Therefore, if we had a low saturation energy, around $E_{max} \sim 100$ GeV $\sim E_{cuv}$ (in the electromagnetic energy range), this would help to explain the large mean free path of photons with very high frequencies (but with energies limited to $E_{max}$).

A more detailed analysis would have to address the photon-photon collision process, since relations (15) and (17) may have direct consequences in the cross section of high-energy collisions. This, however is beyond the scope of the present work.

Finally, we focus on the zero-point energy and its relation to the cosmological-constant problem. As is usual we will assume that space-time displays granularity at sufficiently large energy scales, leading to a cutoff frequency, which we assume to be Planck’s angular frequency. The ratio between the integrated zero-point energy associated with the linear energy-frequency relation and the one we are addressing here is

$$R = \frac{\int_0^{\omega_{pl}} \omega^2 d\omega}{\int_0^{\omega_{pl}} C(\omega) \omega^3 d\omega}.$$ 

Regarding the magnitude of the asymptotic energy $E_{max}$, to get a quantitative figure, let us use the approximate value of $E_{max}$ of 100 GeV, as we did previously.

The exact expression of $R$ is cumbersome and the important point here is the leading order result:

$$R = \frac{3}{4} \frac{E_{pl}}{E_{max}} + O \left( \frac{E_{max}}{E_{pl}} \right) \sim 10^{17}. \quad (19)$$

Although these figures are not sufficient to account for the entirety of the estimated discrepancy ($\sim 10^{120}$), they would represent a non negligible attenuation. Of course, the farther the energy $E_{max}$ is from the Planck energy $E_{pl} \sim 10^{16}$ TeV, the more relevant is the attenuation. It is worth mentioning that the gravitational properties
of vacuum fluctuations have been recently considered as a possible way out of the cosmological-constant problem [25, 26].

The same reasoning presented in this work could, in principle, be applied to other elementary bosonic fields and their vacua, leading to an attenuation which would not be restricted to the electromagnetic field. Whether or not it is possible to extend these results to fermionic fields is a matter to be investigated.

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