New Proof of the Generalized Second Law

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The generalized second law of black hole thermodynamics insists that an entropy of a black hole plus a thermodynamic entropy of fields outside the horizon does not decrease [1], where the black hole entropy is defined as a quarter of the area of the horizon. Namely it says that an entropy of the whole system does not decrease. It interests us in a quite physical sense since it links a world inside a black hole and our thermodynamic world. In particular it gives a physical meaning to black hole entropy indirectly since it concerns the sum of black hole entropy and ordinary thermodynamic entropy and physical meaning of the later is well-known by statistical mechanics.

Frolov and Page [2] proved the generalized second law for a quasi-stationary eternal black hole by assuming that a state of matter fields on the past horizon is thermal one and that a set of radiation modes on the past horizon and one on the past null infinity are quantum mechanically uncorrelated. The assumption is reasonable for the eternal case since a black hole emit a thermal radiation (the Hawking radiation). When we attempt to apply their proof to a non-eternal black hole which arises from a gravitational collapse, we might expect that things would go well by simply replacing the past horizon with a null surface at a moment of a formation of a horizon ($v = v_0$ surface in Figure 1). However, the expectation is disappointing since the assumption becomes ill in this case. The reason is that on a collapsing background the thermal radiation is observed not at the moment of the horizon formation but at the future null infinity and that any modes on the future null infinity have correlation with modes on the past null infinity located after the horizon formation. The correlation can be seen in the equation (2.5) of this paper explicitly. Thus, their proof does not hold for the case in which a black hole arises from a gravitational collapse. Since astrophysically a black hole is thought to arise from a gravitational collapse, we want to prove the generalized second law in this case.

In this paper we prove the generalized second law for a quasi-stationary black hole which arises from a gravitational collapse. For this purpose we concentrate on an inequality between functionals of a density matrix since the generalized second law can be rewritten as an inequality between functionals of a density matrix of matter fields as shown in the Sec. III. We seek a method to prove that a special functional of a density matrix cannot decrease under a physical evolution. (It is a generalization of a result by Sorkin [3].) To apply it to the system with a black hole and derive the generalized second law as its consequence we need to establish a property of physical evolution of matter fields around the black hole. Thus, for concreteness, we investigate a real massless scalar field semiclassically in a curved background which describes gravitational collapse and calculate conditional probabilities which, as a whole, have almost all information about behaviors of the scalar field after the formation of the horizon. (The probability we seek is a generalization of one calculated by Panangaden and Wald [4].) Using the result of the calculation, it is shown that a thermal density matrix of the scalar field at the past null infinity evolves to a thermal density matrix with the same temperature and the same chemical potential at the future null infinity, provided that the initial temperature and chemical potential are special values specified by the background geometry. Finally we prove the generalized second law by using these results.

The rest of the paper is organized as follows. In Sec. I we consider a real massless scalar field in a background of a gravitational collapse to show a thermodynamic property of it. A thermal state with special values of temperature and chemical potential evolves to a thermal state with the same temperature and the same chemical potential. These special values are determined by the background geometry. In Sec. II first the generalized second law is rewritten as an inequality which states that there is a non-decreasing functional of a density matrix of matter fields. After
that, we give a theorem which shows an inequality between functionals of density matrices. Finally we apply it to the scalar field investigated in Sec. IV to prove the generalized second law for the quasi-stationary background. In Sec. IV we summarize this paper.

II. MASSLESS SCALAR FIELD IN BLACK HOLE BACKGROUND

In this section we consider a real massless scalar field in a curved background which describes a formation of a quasi-stationary black hole. Let us denote a past null infinity by $I^-$, a future null infinity by $I^+$ and a future event horizon by $H^+$. Introduce usual null coordinate $u, v$ and suppose that the formation of the event horizon $H^+$ is at $v = v_0$ (see Figure 1). At $I^-$ and $I^+$, by virtue of the asymptotic flatness, there is a natural definition of Hilbert spaces $H_{I^-}$ and $H_{I^+}$ of mode functions with positive frequencies [3]. Hilbert spaces $F(H_{I^+})$ of all asymptotic states are defined as follows with a suitable completion (symmetric Fock spaces).

\[ F(H_{I^+}) \equiv C \oplus H_{I^+} \oplus (H_{I^+} \otimes H_{I^+})_{sym} \oplus \cdots, \]

where $\cdots_{sym}$ denotes the symmetrization $((\xi \otimes \eta)_{sym} = \frac{1}{2}(\xi \otimes \eta + \eta \otimes \xi), \text{etc.})$. Physically, $C$ denotes the vacuum state, $H_{I^+}$ one particle states, $(H_{I^+} \otimes H_{I^+})_{sym}$ two particle states, etc.. We suppose that all our observables are operators on $F(H_{I^+})$ since we observe a radiation of the scalar field radiated by the black hole at place far away from it. In this sense $F(H_{I^+})$ are quite physical. Next let us consider how to set an initial state of the scalar field. We want to see a response of the scalar field on the quasi-stationary black hole background which arises from a gravitational collapse of other materials (a dust, a fluid, etc.). Hence the initial state at $I^-$ must be such a state that it includes no excitations of modes located before the formation of the horizon (no excitation at $v < v_0$). A space of all such states is a subspace of $F(H_{I^-})$, and we denote it by $F_{I^-}(v > v_0)$. We like to derive a thermal property of a scattering process of the scalar field by the quasi-stationary black hole. Hence we consider density matrices on $F_{I^-}(v > v_0)$ and $F(H_{I^+})$.

Denote a space of all density matrices on $F_{I^-}(v > v_0)$ by $P$ and a space of all density matrices on $F(H_{I^+})$ by $\mathcal{P}$. Let us discuss an evolution of a state at $I^-$ to future. Since $I^+$ is not a Cauchy surface because of the existence of $H^+$, $F(H_{I^-})$ is mapped not to $F(H_{I^+})$ but to $F(H_{I^+}) \otimes F(H_{H^+})$ by a unitary evolution, where $H_{H^+}$ is a Hilbert space of mode functions on the horizon with a positive frequency and $F(H_{H^+})$ is a Hilbert space of all states on $H^+$ defined as a symmetric Fock space (see the definition of $F(H_{I^+})$). Although there is no natural principle to determine the positivity of the frequency (equivalently, there is no natural definition of the particle concept), the detailed definition of $H_{H^+}$ does not affect the result since we shall trace out the degrees of freedom of $F(H_{H^+})$ (see [24]). To describe an evolution of a quantum state of the scalar field from $F(H_{I^-})$ to $F(H_{I^+}) \otimes F(H_{H^+})$ a S-matrix is introduced [5]. For a given initial state $|\psi\rangle$ in $F(H_{I^-})$, the corresponding final state in $F(H_{I^+}) \otimes F(H_{H^+})$ is $S|\psi\rangle$. Then the corresponding evolution from $F_{I^-}(v > v_0)$ to $F(H_{I^+})$ is obtained by restricting $S$ to $F_{I^-}(v > v_0)$, and we denote it by $S$, too. In this section we show a thermal property of the scalar field in the background by using the S-matrix elements given by Wald [5].

A. Definition of $T$

Let the initial state of the scalar field be $|\phi\rangle \in F(H_{I^-})$, and observe the corresponding final state at $I^+$ (see the argument after the definition of $F(H_{I^+})$). Formally the observation corresponds to a calculation of a matrix element $\langle \phi| S^I O S|\phi\rangle$, where $S$ is the S-matrix which describes the evolution of the scalar field from $F_{I^-}(v > v_0)$ to $F(H_{I^+}) \otimes F(H_{H^+})$ and $O$ is a self-adjoint operator on $F(H_{I^+})$ corresponding to a quantity we want to observe. The matrix element can be rewritten in a convenient fashion as follows.

\[ \langle \phi| S^I O S|\phi\rangle = T_{R_{I^+}} [O \rho_{red}], \]

where

\[ \rho_{red} = T_{R_{H^+}} [S|\phi\rangle \langle \phi| S^I |], \]

$T_{R_{I^+}}, T_{R_{H^+}}$ denote partial trace over $F(H_{I^+}), F(H_{H^+})$ respectively. In viewing this expression we are lead to an interpretation that the corresponding final state at $I^+$ is represented by the reduced density matrix $\rho_{red}$. Next we generalize this argument to wider range of initial states, which includes all mixed states. For this case an initial state is represented not by an element of $F_{I^-}(v > v_0)$ but by an element of $P$ (a density matrix on $F_{I^-}(v > v_0)$). Its evolution
to $\mathcal{I}^+$ is represented as a map $T$ induced by $S$ followed by the partial trace $T r_{H^+}$: let $\rho (\in \mathcal{P})$ be an initial density matrix then the corresponding final density matrix $T(\rho ) (\in \hat{\mathcal{P}})$ is

$$T(\rho ) = T r_{H^+} [S \rho S^\dagger].$$

(2.1)

Note that $T$ is a linear map from $\mathcal{P}$ into $\hat{\mathcal{P}}$.

B. Thermodynamic property of $T$

In this subsection we show a thermal property of the map $T$, which is summarized as Theorem 3. First let us calculate a conditional probability defined as follows:

$$P(\{n_\rho\}|\{n_\gamma\}) \equiv \langle \{n_\rho\}|T (\{n_\gamma\})\{n_\gamma\}\rangle |\{n_\rho\}\rangle,$$

(2.2)

where

$$|\{n_\gamma\}) = \prod_{i} \frac{1}{\sqrt{n_\gamma}} (a^\dagger (A_{i_\gamma}))^{n_\gamma} |0\rangle,$$

(2.3)

$$|\{n_\rho\}) = \prod_{i} \frac{1}{\sqrt{n_\rho}} (a^\dagger (i_\rho))^{n_\rho} |0\rangle.$$

$|\{n_\gamma\})$ is a state in $\mathcal{F}_{\mathcal{I}^-} (v > v_0)$ characterized by a set of integers $n_\gamma (i = 1, 2, \cdots)$ and $|\{n_\rho\})$ is a state in $\mathcal{F}(\mathcal{H}^\gamma)$ characterized by a set of integers $n_\rho (i = 1, 2, \cdots)$. Therefore $P(\{n_\rho\}|\{n_\gamma\})$ is a conditional probability for a final state to be $|\{n_\rho\})$ when the initial state is specified to $|\{n_\gamma\})$. In the expressions $A$ is a representation of a Bogoliubov transformation from $\mathcal{H}^\gamma \oplus \mathcal{H}^\rho$ to $\mathcal{H}^\gamma$ and $\gamma$ is such a unit vector in $\mathcal{H}^\gamma \oplus \mathcal{H}^\rho$ that $A_{i_\gamma}$ corresponds to a wave packet whose peak is located at a point on $\mathcal{I}^-$ later than the formation of the horizon $\mathcal{I}^+$ (see Figure 3). On the other hand, $i_\rho$ is a unit vector in $\mathcal{H}^\rho$ and corresponds to a wave packet on $\mathcal{I}^+$ (see Figure 3). The probability (2.2) is a generalization of $P(k|j)$ investigated by Panangaden and Wald [4] (see Figure 3). It includes almost all information about a response of the scalar field on the quasi-stationary black hole which arises from the gravitational collapse while $P(k|j)$ does not, since any initial states on $\mathcal{I}^-$, which include no excitation before the formation of the horizon $(v < v_0)$, can be represented by using the basis $|\{n_\gamma\})$ and any final states on $\mathcal{I}^+$ can be expressed by the basis $|\{n_\rho\})$, i.e. a set of all $|\{n_\gamma\})$ generates $\mathcal{F}(\mathcal{H}(\mathcal{I}^-) (v > v_0))$ and a set of all $|\{n_\rho\})$ generates $\mathcal{F}(\mathcal{H}(\mathcal{I}^+))$. This is the very reason why we generalize $P(k|j)$ to $P(\{n_\rho\}|\{n_\gamma\})$.

By using the S-matrix elements given in [5], the conditional probability is rewritten as follows (see appendix A for its derivation).

$$P(\{n_\rho\}|\{n_\gamma\}) = \prod_i \left(1 - x_i \right) 2^{n_\gamma - n_\rho} (1 - |R_i|^2)^{n_\gamma + n_\rho} \left(\min(n_\rho, n_\gamma) \min(n_\rho, n_\gamma) \sum_{l_1=0}^{\min(n_\rho, n_\gamma)} \sum_{m_1=0}^{\min(n_\rho, n_\gamma)} \frac{[t_l(R_i^2)^{l_1 + m_1}]_{l_1}|n_\gamma\rangle_{n_\rho}|n_\rho\rangle_{n_\gamma}!}{l_1!(n_\gamma - l_1)!l_1!m_1!(n_\gamma - m_1)!} \right) \times \left(\sum_{n_i = n_\rho - \min(l_1, m_1)}^{\infty} \frac{n_i! (n_i - n_\rho + n_\gamma)!}{n_i! (n_i - n_\rho + l_1)! (n_i - n_\rho + m_1)!} (x_i^2 |R_i|^2)^{n_i - n_\rho} \right),$$

(2.4)

where $R_i$ is a reflection coefficient for the mode specified by the integer $i$ on the Schwarzschild metric (See and $x_i$ is a constant defined by $x_i = \exp(-\pi(\omega_i - \Omega_{BH} m_i)/\kappa)$. In the expression, $\omega_i$ and $m_i$ are a frequency and an azimuthal angular momentum quantum number of the mode specified by the integer $i$, $\Omega_{BH}$ and $\kappa$ are an angular velocity and a surface gravity of the black hole.

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* Their argument is restricted to the case when $n_\gamma = n_\rho = 0$ for all $i$ other than a particular value.

† All the information is included in $T^{(n_\gamma)}_{(n_\rho)}$ defined in Lemma 4.
Now, the expression in the squared bracket in (2.4) appears also in the calculation of $P(k|j)$. Using the result of [4], it is easily shown that

$$P(\{n_\rho\}|\{n_\gamma\}) = \prod_i \left[ K_i \sum_{s_i=0}^{\min(n_\rho,n_\gamma)} \frac{(n_\rho + n_\gamma - s_i)!v_i^{s_i}}{s_i!(n_\rho - s_i)!(n_\gamma - s_i)!} \right],$$  \hspace{1cm} (2.5)

where

$$K_i = \frac{(1-x_i)x_i^{2n_\rho} (1-|R_i|^2)^{n_\gamma+n_\rho}}{(1-|R_i|^2|x_i^2|)^n\gamma+n_\rho+1},$$

$$v_i = \frac{(|R_i|^2-x_i^2) (1-|R_i|^2|x_i^2|)}{(1-|R_i|^2|x_i^2|)^2}.$$  \hspace{1cm} (2.6)

This is a generalization of the result of [4], and the following lemma is easily derived by using this expression.

**Lemma 1** For the conditional probability defined by (2.2) the following equality holds:

$$P(\{n_\rho = k_i\}|\{n_\gamma = j_i\}) e^{-\beta_{BH}} \sum_{j_i} j_i (\omega_i - \Omega_{BH} m_i) = P(\{n_\rho = j_i\}|\{n_\gamma = k_i\}) e^{-\beta_{BH}} \sum_{k_i} k_i (\omega_i - \Omega_{BH} m_i),$$

where $\omega_i$ and $m_i$ are a frequency and an azimuthal angular momentum quantum number of the mode specified by $i$, $\Omega_{BH}$ is angular velocity of the horizon and

$$\beta_{BH} \equiv 2\pi/\kappa.$$  \hspace{1cm} (2.7)

In the expression $\kappa$ is a surface gravity of the black hole.

Note that $\beta_{BH}^{-1}$ is the Hawking temperature of the black hole. This lemma states that a detailed balance condition holds [4]. Summing up about all $k$'s, we expect that a thermal density matrix $\rho_{th}(\beta_{BH}, \Omega_{BH})$ in $\mathcal{P}$ with a temperature $\beta_{BH}^{-1}$ and a chemical potential $\Omega_{BH}$ for azimuthal angular momentum quantum number will be mapped by $T$ to a thermal density matrix $\tilde{\rho}_{th}(\beta_{BH}, \Omega_{BH})$ in $\tilde{\mathcal{P}}$ with the same temperature and the same chemical potential. To show that this expectation is true, we have to prove that all off-diagonal elements of $T(\rho_{th}(\beta_{BH}, \Omega_{BH}))$ are zero. For this purpose the following lemma is proved in appendix [4].

**Lemma 2** Denote a matrix element of $T$ as

$$T_{\{n_\rho\},\{n_\rho'\}}^{\{n_\gamma\},\{n_\gamma'\}} \equiv \langle \{n_\rho\}|T|\{n_\rho\} \rangle \langle \{n_\gamma\}|\{n_\gamma'\}\rangle \langle \{n_\gamma'\}|\{n_\gamma\}\rangle.$$  \hspace{1cm} (2.8)

Then

$$T_{\{n_\rho\},\{n_\rho'\}}^{\{n_\gamma\},\{n_\gamma'\}} = 0,$$

unless

$$n_\gamma - n_\gamma' = n_\rho - n_\rho'$$  \hspace{1cm} (2.9)

for $\forall i$.

Lemma 2 shows that all off-diagonal elements of $T(\rho)$ in the basis $\{|\{n_\rho\}\rangle\}$ vanish if all off-diagonal elements of $\rho$ in the basis $\{|\{n_\gamma\}\rangle\}$ is zero. Thus, combining it with Lemma 1, the following theorem is easily proved. Note that a set of all $\{|\{n_\gamma\}\rangle\langle n_\gamma'| \}$ generates $\mathcal{P}$ and a set of all $|\{n_\rho\}\rangle\langle n_\rho'| \}$ generates $\tilde{\mathcal{P}}$ (see the argument below (2.3)).

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\[\text{It guarantees that a thermal distribution of any temperature is mapped to a thermal distribution of some other temperature closer to the Hawking temperature, as far as the diagonal elements are concerned.}\]

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Theorem 3 Consider the linear map $T$ defined by (2.1) for a real, massless scalar field on a background geometry which describes a formation of a quasi-stationary black hole. Then

$$T(\rho_{\text{th}}(\beta_{\text{BH}},\Omega_{\text{BH}})) = \tilde{\rho}_{\text{th}}(\beta_{\text{BH}},\Omega_{\text{BH}}),$$

(2.10)

where

$$\rho_{\text{th}}(\beta_{\text{BH}},\Omega_{\text{BH}}) \equiv Z^{-1} \sum_{\{n_\gamma\}} e^{-\beta_{\text{BH}} \sum_i n_\gamma (\omega_i - \Omega_{\text{BH}} m_i)} |\{n_\gamma\}\rangle \langle \{n_\gamma\}|,$$

$$\tilde{\rho}_{\text{th}}(\beta_{\text{BH}},\Omega_{\text{BH}}) \equiv Z^{-1} \sum_{\{n_\rho\}} e^{-\beta_{\text{BH}} \sum_i n_\rho (\omega_i - \Omega_{\text{BH}} m_i)} |\{n_\rho\}\rangle \langle \{n_\rho\}|,$$

$$Z \equiv \sum_{\{n_i\}} e^{-\beta_{\text{BH}} \sum_i j_i (\omega_i - \Omega_{\text{BH}} m_i)}.$$

(2.11)

$\rho_{\text{th}}(\beta_{\text{BH}},\Omega_{\text{BH}})$ and $\tilde{\rho}_{\text{th}}(\beta_{\text{BH}},\Omega_{\text{BH}})$ can be regarded as 'grand canonical ensemble' in $\mathcal{P}$ and $\tilde{\mathcal{P}}$ respectively, which have a common temperature $\beta_{\text{BH}}^{-1}$ and a common chemical potential $\Omega_{\text{BH}}$ for azimuthal angular momentum quantum number. Thus the theorem says that the 'grand canonical ensemble' at $I^- (\nu > v_0)$ with special values of temperature and chemical potential evolves to a 'grand canonical ensemble' at $I^+$ with the same temperature and the same chemical potential. Note that the special values $\beta_{\text{BH}}^{-1}$ and $\Omega_{\text{BH}}$ are determined by the background geometry: $\beta_{\text{BH}}^{-1}$ is the Hawking temperature and $\Omega_{\text{BH}}$ is the angular velocity of the black hole formed. This result is used in subsection III B to prove the generalized second law for the quasi-stationary black hole.

III. THE GENERALIZED SECOND LAW

The generalized second law is one of the most interesting conjecture in black hole thermodynamics since it restricts ways of interaction between a black hole and ordinary thermodynamic matter. It can be regarded as a generalization both of the area law of black hole and of the second law of ordinary thermodynamics. The latter, which states that total entropy of a system cannot decrease under a physical evolution of a thermodynamic system, can be proved for a finite dimensional system if a microcanonical ensemble for the system does not change under the evolution.

In the previous section we have proved that the 'grand canonical ensemble' of the scalar field does not change under the physical evolution on a background which describes a formation of a quasi-stationary black hole. So we expect that the generalized second law may be proved in a way similar to the proof of the second law of ordinary thermodynamics. For the purpose of the proof we rewrite the generalized second law as an inequality between functionals of a density matrix of matter fields.

The generalized second law of black hole thermodynamics is

$$\Delta S_{\text{BH}} + \Delta S_{\text{matter}} \geq 0,$$

(3.1)

where $\Delta$ denotes a change of quantities under an evolution of the system, $S_{\text{BH}}$ and $S_{\text{matter}}$ are black hole entropy of the black hole and thermodynamic entropy of the matter fields respectively. For a quasi-stationary black hole, using the first law of the black hole thermodynamics

$$\Delta S_{\text{BH}} = \beta_{\text{BH}} (\Delta M_{\text{BH}} - \Omega_{\text{BH}} \Delta J_{\text{BH}}),$$

the conservation of total energy

$$\Delta M_{\text{BH}} + \Delta E_{\text{matter}} = 0$$

and the conservation of total angular momentum

$$\Delta J_{\text{BH}} + \Delta L_{\text{matter}} = 0,$$

it is easily shown that the generalized second law is equivalent to the following inequality:

$$\Delta S_{\text{matter}} - \beta_{\text{BH}} (\Delta E_{\text{matter}} - \Omega_{\text{BH}} \Delta L_{\text{matter}}) \geq 0,$$

(3.2)
where $\beta_{BH}, \Omega_{BH}, M_{BH}$ and $J_{BH}$ are inverse temperature, angular velocity, mass and angular momentum of the black hole; $E_{\text{matter}}$ and $L_{\text{matter}}$ are energy and azimuthal component of angular momentum of the matter fields. Thus this is of the form

$$U[\tilde{\rho}_0; \beta_{BH}, \Omega_{BH}] \geq U[\rho_0; \beta_{BH}, \Omega_{BH}], \quad (3.3)$$

where $U$ is a functional of a density matrix of the matter fields defined by

$$U[\rho; \beta_{BH}, \Omega_{BH}] \equiv -\text{Tr}[\rho \ln \rho] - \beta_{BH} (\text{Tr}[E \rho] - \Omega_{BH} \text{Tr}[L_z \rho]), \quad (3.4)$$

$\rho_0$ and $\tilde{\rho}_0$ are an initial density matrix and the corresponding final density matrix respectively. In the expression $E$ and $L_z$ are operators corresponding to the energy and the azimuthal component of the angular momentum. Note that $(3.3)$ is an inequality between functionals of a density matrix of matter fields.\footnote{Information about the background geometry appears in the inequality as variables which parameterize the functional.} We will prove the generalized second law by showing that this inequality holds. Actually we do it in subsection III B for a quasi-stationary black hole which arises from a gravitational collapse, using the results of Sec. II and a theorem given in the following subsection.

A. Non-decreasing functional

In this subsection a theorem which makes it possible to construct a functional which does not decrease by a physical evolution. It is a generalization of a result of [3]. In the next subsection we derive $(3.3)$ for a quasi-stationary black hole which arises from gravitational collapse, applying the theorem to the scalar field investigated in Sec. II.

Let us consider Hilbert spaces $\mathcal{F}$ and $\tilde{\mathcal{F}}$. First we give some definitions needed for the theorem.

**Definition 4** A linear bounded operator $\rho$ on $\mathcal{F}$ is called a density matrix, if it is self-adjoint, positive semi-definite and satisfies

$$\text{Tr} \rho = 1.$$ 

In the rest of this section we denote a space of all density matrices on $\mathcal{F}$ as $\mathcal{P}(\mathcal{F})$. Evidently $\mathcal{P}(\mathcal{F})$ is a linear convex set rather than a linear set.

**Definition 5** A map $\mathcal{T}$ of $\mathcal{P}(\mathcal{F})$ into $\mathcal{P}(\tilde{\mathcal{F}})$ is called linear, if

$$\mathcal{T}(a \rho_1 + (1-a) \rho_2) = a \mathcal{T}(\rho_1) + (1-a) \mathcal{T}(\rho_2)$$

for $0 \leq a \leq 1$ and $\forall \rho_1, \rho_2 \in \mathcal{P}(\mathcal{F})$.

By this definition it is easily proved by induction that

$$\mathcal{T} \left( \sum_{i=1}^{N} a_i \rho_i \right) = \sum_{i=1}^{N} a_i \mathcal{T}(\rho_i), \quad (3.5)$$

if $a_i \geq 0$, $\sum_{i=1}^{N} a_i = 1$ and $\rho_i \in \mathcal{P}(\mathcal{F})$.

Now we prove the following lemma which concerns the $N \to \infty$ limit of the left hand side of $(3.3)$. We use this lemma in the proof of theorem III B.

**Lemma 6** Consider a linear map $\mathcal{T}$ of $\mathcal{P}(\mathcal{F})$ into $\mathcal{P}(\tilde{\mathcal{F}})$ and an element $\rho_0$ of $\mathcal{P}(\mathcal{F})$. For a diagonal decomposition

$$\rho_0 = \sum_{i=1}^{\infty} p_i |i\rangle \langle i|,$$

define a series of density matrices of the form

6 Information about the background geometry appears in the inequality as variables which parameterize the functional.
\[ \rho_n = \sum_{i=1}^{n} p_i / a_n |i\rangle \langle i| \quad (n = N, N + 1, \cdots), \] (3.6)

where
\[ a_n \equiv \sum_{i=1}^{n} p_i \]

and \( N \) is large enough that \( a_N > 0 \). Then
\[ \lim_{n \to \infty} \langle \Phi | T(\rho_n) | \Psi \rangle = \langle \Phi | T(\rho_0) | \Psi \rangle \] (3.7)

for arbitrary elements \( |\Phi\rangle \) and \( |\Psi\rangle \) of \( \mathcal{F} \).

This lemma says that \( T(\rho_n) \) has a weak-operator-topology-limit \( T(\rho_0) \).

**Proof**

By definition,
\[ \rho_0 = a_n \rho_n + (1 - a_n) \rho'_n, \] (3.8)

where
\[ \rho'_n = \left\{ \begin{array}{ll}
\sum_{i=n+1}^{\infty} p_i / (1 - a_n) |i\rangle \langle i| & (a_n < 1) \\
\rho_n & (a_n = 1)
\end{array} \right. \]

Then the linearity of \( T \) shows
\[ \langle \Phi | T(\rho_0) | \Psi \rangle = a_n \langle \Phi | T(\rho_n) | \Psi \rangle + (1 - a_n) \langle \Phi | T(\rho'_n) | \Psi \rangle. \]

Thus, if \( \langle \Phi | T(\rho'_n) | \Psi \rangle \) is finite in \( n \to \infty \) limit, then the lemma is established since
\[ \lim_{n \to \infty} a_n = 1. \]

For the purpose of proving the finiteness of \( \langle \Phi | T(\rho'_n) | \Psi \rangle \), it is sufficient to show that \( |\langle \Phi | \tilde{\rho} | \Psi \rangle| \) is bounded from above by \( \|\Phi\| \|\Psi\| \) for an arbitrary element \( \tilde{\rho} \) of \( \mathcal{F} \).

This is easy to prove as follows.
\[ |\langle \Phi | \tilde{\rho} | \Psi \rangle| = \left| \sum_i \tilde{p}_i \langle \Phi | i \rangle \langle i | \Psi \rangle \right| \leq \sum_i |\langle \Phi | i \rangle \langle i | \Psi \rangle| \leq \|\Phi\| \|\Psi\|, \] (3.9)

where we have used a diagonal decomposition
\[ \tilde{\rho} = \sum_i \tilde{p}_i |i\rangle \langle i|. \]

□

**Theorem 7** Assume the following three assumptions: **a.** \( T \) is a linear map of \( \mathcal{P}(\mathcal{F}) \) into \( \mathcal{P}(\mathcal{F}) \),  
**b.** \( f \) is a continuous function convex to below and there are non-negative constants \( c_1, c_2 \) and \( c_3 \) such that \( |f((1 - \epsilon)x) - f(x)| \leq |\epsilon| (c_1 |f(x)| + c_2 |x| + c_3) \) for \( \epsilon \geq 0 \) and sufficiently small \( |\epsilon| \),  
**c.** there are positive definite density matrices \( \rho_{\infty} \) (\( \in \mathcal{P}(\mathcal{F}) \)) and \( \tilde{\rho}_{\infty} \) (\( \in \mathcal{P}(\mathcal{F}) \)) such that \( T(\rho_{\infty}) = \tilde{\rho}_{\infty} \).

If \( [\rho_{\infty}, \rho_0] = [\tilde{\rho}_{\infty}, T(\rho_0)] = 0 \) and \( \text{Tr}[\rho_{\infty} f(\rho_{\infty}^{-1})] \) \( < \infty \), then
\[ \tilde{U} [T(\rho_0)] \geq U [\rho_0], \] (3.10)

where
\[ U [\rho] \equiv -\text{Tr} [\rho_{\infty} f(\rho_{\infty}^{-1})], \]
\[ \tilde{U} [\rho] \equiv -\text{Tr} [\tilde{\rho}_{\infty} f(\tilde{\rho}_{\infty}^{-1})]. \] (3.11)
As stated in the first paragraph of this subsection, theorem 7 is used in subsection III B to prove the generalized second law for a quasi-stationary black hole which arises from gravitational collapse.

**Proof**

First let us decompose the density matrices diagonally as follows:

\[
\rho_0 = \sum_{i=1}^{\infty} p_i |i\rangle \langle i|, \quad \rho_{\infty} = \sum_{i=1}^{\infty} q_i |i\rangle \langle i|, \\
\mathcal{T}(\rho_0) = \sum_{i=1}^{\infty} \tilde{p}_i |\tilde{i}\rangle \langle \tilde{i}|, \quad \mathcal{T}(\rho_{\infty}) = \sum_{i=1}^{\infty} \tilde{q}_i |\tilde{i}\rangle \langle \tilde{i}|. 
\]

Then by lemma 6 and (3.5),

\[
\tilde{p}_i = \langle \tilde{i}| \mathcal{T}(\rho_0) |\tilde{i}\rangle = \lim_{n \to \infty} \sum_{j=1}^{n} A_{ij} p_j / a_n, \\
(3.13)
\]

where \(a_n = \sum_{i=1}^{n} p_i\) and \(A_{ij} = \langle \tilde{i}| \mathcal{T}(|j\rangle \langle j|) |\tilde{i}\rangle\). \(A_{ij}\) has the following properties:

\[\sum_{i=1}^{\infty} A_{ij} = 1, \quad 0 \leq A_{ij} \leq 1.\]

Similarly it is shown that

\[
\tilde{q}_i = \lim_{n \to \infty} \sum_{j=1}^{n} A_{ij} q_j / b_n, \\
(3.13)
\]

where \(b_n = \sum_{i=1}^{n} q_i\). By (3.13) and the continuity of \(f\), it is shown that

\[
f(\tilde{p}_i / \tilde{q}_i) = \lim_{n \to \infty} f \left( \sum_{j=1}^{n} A_{ij} p_j / a_n \right). \\
(3.14)
\]

Next define \(C^n_i\) and \(\tilde{C}^n_i\) by

\[
C^n_i \equiv \sum_{j=1}^{n} A_{ij} q_j / \tilde{q}_i, \quad C^n_i / a_n, \\
(3.15)
\]

then the convex property of \(f\) means

\[
f(\tilde{p}_i / \tilde{q}_i) \leq \lim_{n \to \infty} \sum_{j=1}^{n} A_{ij} q_j / C^n_i \tilde{q}_i f(\tilde{C}^n_i p_j / q_j) \\
\]

since

\[
\sum_{j=1}^{n} A_{ij} q_j / C^n_i \tilde{q}_i = 1, \quad A_{ij} q_i / C^n_i \tilde{q}_i \geq 0. 
\]

Hence

\[
-\tilde{U} \mathcal{T}(\rho_0) = \sum_{i=1}^{\infty} \tilde{q}_i f(\tilde{p}_i / \tilde{q}_i) \leq \sum_{i=1}^{\infty} \lim_{n \to \infty} \sum_{j=1}^{n} A_{ij} q_j / C^n_i \tilde{q}_i f(\tilde{C}^n_i p_j / q_j). \\
(3.16)
\]

Since \(C^n_i\) and \(\tilde{C}^n_i\) satisfy

\[\lim_{n \to \infty} C^n_i = \lim_{n \to \infty} \tilde{C}^n_i = 1,\]

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it is implied by the assumption about \( f \) that
\[
|f \left( \frac{\tilde{C}^n_i p_j}{q_j} \right) - f \left( \frac{p_j}{q_j} \right)| \leq |1 - \tilde{C}^n_i| (c_1|f(p_j/q_j)| + c_2 p_j/q_j + c_3)
\]
for sufficiently large \( n \). Therefore
\[
\left| \sum_{j=1}^{n} \frac{A_{ij} q_j}{C^n_i} \left( f(\tilde{C}^n_i p_j/q_j) - f(p_j/q_j) \right) \right| \leq |1 - \tilde{C}^n_i| \left( \sum_{j=1}^{n} A_{ij} q_j |f(p_j/q_j)| + c_2 \sum_{j=1}^{n} A_{ij} p_j + c_3 \sum_{j=1}^{n} A_{ij} q_j \right)
\]
\[
\leq |1 - \tilde{C}^n_i| \left( c_1 \sum_{j=1}^{n} q_j |f(p_j/q_j)| + c_2 \sum_{j=1}^{n} p_j + c_3 \sum_{j=1}^{n} q_j \right),
\]
where we have used \( 0 \leq A_{ij} \leq 1 \) to obtain the last inequality. Since the first term in the brace in the last expression is finite in \( n \to \infty \) limit by the assumption of the absolute convergence of \( \mathcal{U} \{\rho_0\} \) and all the other terms in the brace are finite,
\[
\lim_{n \to \infty} \left| \sum_{j=1}^{n} \frac{A_{ij} q_j}{C^n_i} \left( f(\tilde{C}^n_i p_j/q_j) - f(p_j/q_j) \right) \right| = 0.
\]
Moreover, by the absolute convergence of \( \mathcal{U} \{\rho_0\} \), it is easily shown that
\[
\lim_{n \to \infty} \left| \left( \frac{1}{C^n_i} - 1 \right) \sum_{j=1}^{n} A_{ij} q_j f(p_j/q_j) \right| = 0.
\]
Thus
\[
-\tilde{\mathcal{U}}[\mathcal{T}(\rho_0)] \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} A_{ij} q_j f(p_j/q_j).
\]
(3.17)
We can interchange sum over \( i \) and sum over \( j \) in the right hand side of (3.17) since it converges absolutely by the absolute convergence of \( \mathcal{U} \{\rho_0\} \). Hence
\[
-\tilde{\mathcal{U}}[\mathcal{T}(\rho_0)] \leq \sum_{j=1}^{\infty} q_j f(p_j/q_j) = -\mathcal{U}(\rho_0).
\]
\[
\square
\]

B. Proof of the generalized second law

Let us combine theorem \( \mathbb{B} \) with theorem \( \mathbb{H} \) to prove the generalized second law. In theorem \( \mathbb{H} \) set the linear map \( \mathcal{T} \), the convex function \( f(x) \) and the density matrices \( \rho_\infty \) and \( \tilde{\rho_\infty} \) as follows.
\[
\mathcal{T} = T, \\
f(x) = x \ln x, \\
\rho_\infty = \rho_\mathbb{H}(\beta_{\mathbb{BH}}, \Omega_{\mathbb{BH}}), \\
\tilde{\rho_\infty} = \tilde{\rho_\mathbb{H}(\beta_{\mathbb{BH}}, \Omega_{\mathbb{BH}})}.
\]
(3.18)
Note that it is theorem \( \mathbb{B} \) that makes such a setting possible. Hence, if an initial state \( \rho_0 \) and the corresponding final state \( \mathcal{T}(\rho_0) \) satisfy
\[
[\rho_0, \rho_\mathbb{H}(\beta_{\mathbb{BH}}, \Omega_{\mathbb{BH}})] = [\mathcal{T}(\rho_0), \rho_\mathbb{H}(\beta_{\mathbb{BH}}, \Omega_{\mathbb{BH}})] = 0
\]
(3.19)
and $\mathcal{U}[\rho_0]$ converges absolutely, theorem 7 can be applied to the system of the quasi-stationary black hole and the scalar field around it. Now

$$\mathcal{U}[\rho_0] = -Tr[\rho_0 \ln \rho_0] - \beta_{BH} (Tr[E\rho_0] - \Omega_{BH} Tr[L_z\rho_0]) - \ln Z$$

$$= U[\rho_0; \beta_{BH}, \Omega_{BH}] - \ln Z,$$

$$\tilde{\mathcal{U}}[T(\tilde{\rho}_0)] = -Tr[\tilde{\rho}_0 \ln \tilde{\rho}_0] - \beta_{BH} (Tr[\tilde{E}\tilde{\rho}_0] - \Omega_{BH} Tr[L_z\tilde{\rho}_0]) - \ln Z$$

$$= U[T(\tilde{\rho}_0); \beta_{BH}, \Omega_{BH}] - \ln Z,$$

(3.20)

where

$$E = \sum_{\{n_\gamma\}} \left( \sum_i n_\gamma \omega_i \right) |\{n_\gamma\}\rangle \langle \{n_\gamma\}|,$$

$$L_z = \sum_{\{n_\gamma\}} \left( \sum_i n_\gamma m_i \right) |\{n_\gamma\}\rangle \langle \{n_\gamma\}|,$$

and

$$\tilde{E} = \sum_{\{n_\rho\}} \left( \sum_i n_\rho \omega_i \right) |\{n_\rho\}\rangle \langle \{n_\rho\}|,$$

$$\tilde{L}_z = \sum_{\{n_\rho\}} \left( \sum_i n_\rho m_i \right) |\{n_\rho\}\rangle \langle \{n_\rho\}|.$$

Thus the inequality (3.10) in this case is (3.13) itself, which in turn is equivalent to the generalized second law. Finally, theorem 7 proves the generalized second law for a quasi-stationary black hole which arises from gravitational collapse, provided that an initial density matrix $\rho_0$ of the scalar field satisfies the above assumptions. For example, it is guaranteed by lemma 2 that if $\rho_0$ is diagonal in the basis $\{|\{n_\gamma\}\rangle\}$ then $T(\rho_0)$ is also diagonal in the basis $\{|\{n_\rho\}\rangle\}$ and (3.13) is satisfied. The assumption of the absolute convergence of $U[\rho_0; \beta_{BH}, \Omega_{BH}]$ holds whenever initial state $\rho_0$ at $\mathcal{I}^-$ contains at most finite number of excitations. Note that although $\rho_{0h}(\beta_{BH}, \Omega_{BH})$ contains infinite number of excitations by definition, $\rho_0$ has not to do. Therefore the assumptions are satisfied when $\rho_0$ is diagonal in the basis $\{|\{n_\gamma\}\rangle\}$ and contains at most finite number of excitations.

**IV. SUMMARY AND DISCUSSION**

In summary we have proved the generalized second law for a quasi-stationary black hole which arises from gravitational collapse. To prove it we have derived the thermal property of the semiclassical evolution of a real massless scalar field on the quasi-stationary black hole background and have given a method for searching a non-decreasing functional. These are generalizations of results of [4] and [3] respectively.

Now we make a comments on Frolov and Page’s statement that their proof of the generalized second law may be applied to the case of the black hole formed by a gravitational collapse [2]. Their proof for a quasi-stationary eternal black hole is based on the following two assumptions: (1) a state of matter fields on the past horizon is thermal one; (2) a set of radiation modes on the past horizon and one on the past null infinity are quantum mechanically uncorrelated. These two assumptions are reasonable for the eternal case since a black hole emit a thermal radiation. In the case of a black hole which arises from a gravitational collapse, we might expect that things would go well by simply replacing the past horizon with a null surface at a moment of a formation of a horizon ($v = v_0$ surface in Figure 4). However, a state of the matter fields on the past horizon is completely determined by a state of the fields before the horizon formation ($v < v_0$ in Figure 4), in which there is no causal effect of the existence of future horizon. Since the essential origin of the thermal radiation from a black hole is the existence of a horizon, the state of the fields on the null surface has not to be a thermal one. Hence the assumption (1) becomes ill in this case. Although the above replacement may be the most extreme one, an intermediate replacement causes an intermediate violation of the assumption (1) due to the correlation between modes on the future null infinity and modes on the past null infinity located after the horizon formation. The correlation can be seen in (2.9) explicitly. Thus we conclude that their proof can not be applied to the case of the black hole formed by a gravitational collapse.
Finally we discuss a generalization of our proof to a dynamical background. For the case of a dynamical background $\beta_{BH}$ and $\Omega_{BH}$ are changed from time to time by a possible backreaction. Thus, to prove the generalized second law for the dynamical background, we have to generalize theorem 3 to the dynamical case consistently with the backreaction. Once this can be achieved, theorem 3 seems useful to prove the generalized second law for the dynamical background.

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APPENDIX A: THE CONDITIONAL PROBABILITY

In this appendix we reduce (2.2) to (2.4). First the $S$-matrix obtained by [5] is

$$S(0) = N \sum_{n=0}^{\infty} \frac{\sqrt{2n}!}{2^n n!} \left( \frac{\nu}{\epsilon} \right)_{sym},$$

$$Sa^\dagger(A \gamma)S^{-1} = R_i a^\dagger(p) + T_i a^\dagger(\sigma), \quad (A1)$$

where $\epsilon$ and $N$ are a bivector and a normalization constant defined by

$$\epsilon = 2 \sum_i x_i (\lambda \otimes \tau)_{sym}, \quad N = \prod_i \sqrt{1 - x_i},$$

where

$$x_i = \exp \left( -\pi (\omega_i - \Omega_{BH} m_i) / \kappa \right).$$

In the expression, $\omega_i$ and $m_i$ are a frequency and an azimuthal angular momentum quantum number of a mode specified by integer $i$, $\Omega_{BH}$ and $\kappa$ are an angular velocity and a surface gravity of the black hole. $i\gamma, i\rho, i\sigma, i\lambda$ and $i\tau$ are unit vectors in $H_{L+} \oplus H_{H+}$ defined in [5], and the former four are related as follows:

$$i\gamma^a = T_i i\sigma^a + R_i i\rho^a,$$

$$i\lambda^a = t_i i\rho^a + r_i i\sigma^a, \quad (A2)$$

where $t_i, T_i$ are transmission coefficients for the mode specified by the integer $i$ on the Schwarzschild metric [5] and $r_i, R_i$ are reflection coefficients. They satisfy $**$

$$|t_i|^2 + |r_i|^2 = |T_i|^2 + |R_i|^2 = 1,$$

$$t_i = T_i, \quad r_i = -R_i^* T_i. \quad (A3)$$

By using the $S$-matrix, we obtain

$$S\{\{n_\gamma\} = N \left[ \prod_i \frac{1}{\sqrt{n_i \gamma}} \left[ R_i a^\dagger(p) + T_i a^\dagger(\sigma) \right]^{n_\gamma} \right] \sum_{n=0}^{\infty} \frac{\sqrt{2n}!}{2^n n!} \left( \frac{\nu}{\epsilon} \right)_{sym}$$

$$= N \sum_{n=0}^{\infty} \sum_i \left[ \prod_i \frac{1}{\sqrt{n_i \gamma}} \right] R_i^{m_i \gamma - m_i} \sum_{l=0}^{\infty} \left[ \prod_i \frac{x_i^{n_i}}{l_i^!} \right] \mu_{l_i}^{n_i} \nu_{l_i - l_i}^{n_i - m_i}$$

$$\times \sqrt{\frac{2(n + \sum_i n_i \gamma)!}{(2n)!}} \left( \prod_i \frac{n_i}{l_i} \otimes \tau \otimes \rho \otimes \sigma \right)_{sym}^{l_i} \otimes m_i \otimes \gamma \otimes m_i \otimes \sigma \otimes m_i \otimes \sigma$$

$**$The last two equations are consequences of the time reflection symmetry of the Schwarzschild metric.
Finally, by using (A3) and exchanging the order of the summation suitably, we can obtain

\[
P(\{n_\rho\}|\{n_\gamma\}) = \prod_i \left[ (1 - x_i) x_i^{2n_\rho} \frac{1 - |R_i|^2}{1 - |R_i|^2} n_\gamma + n_\rho \right. \\
\times \frac{\sum_{l_i=0}^{\min(n_\gamma, n_\rho)} \sum_{m_i=0}^{\min(n_\rho, n_\rho)} \frac{(-1)^{l_i + m_i}}{l_i! m_i! (n_\gamma - l_i)! (n_\rho - m_i)! (n_\gamma - m_i)!}}{\sum_{i=0}^{\infty} \frac{n_i! (n_i - n_\rho + n_\gamma)!}{(n_i - n_\rho + l_i)! (n_i - n_\rho + m_i)! (x_i^2 |R_i|^2)^{n_i - n_\rho}}}
\]

This is what we have to show.

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†† The number of the 'particle', \(\sigma\) in (A4) is \(n_i + n_\gamma - n_\rho\), setting the number of the 'particle', \(\rho\) to \(n_\rho\).

†† The range is obtained by inequalities \(n_i \geq 0\), \(0 \leq m_i \leq n_\gamma\), \(0 \leq l_i \leq n_i\), \(l_i + m_i = n_\rho\) and \(n_i + n_\gamma - n_\rho \geq 0\).

§§ (A4) is normalized to have unit norm.
In this appendix we give a proof of Lemma 2.

**Proof**

Since a set of all $i_\tau$ and $i_\sigma$ generates $\mathcal{H}_{H^+}$, the definition of $T_{\{n_{\rho}\}\{n'_{\rho}\}}^{\{n_{\gamma}\}\{n'_{\gamma}\}}$ leads

$$T_{\{n_{\rho}\}\{n'_{\rho}\}}^{\{n_{\gamma}\}\{n'_{\gamma}\}} = \sum_{\{n_{\sigma}\},\{n_{\tau}\}} \langle\{n_{\tau}, n_{\rho}, n_{\sigma}\}|S|\{n_{\gamma}\}\rangle \langle\{n_{\gamma}\}|S|\{n_{\tau}, n'_{\rho}, n_{\sigma}\}\rangle,$$

where

$$|\{n_{\tau}, n_{\rho}, n_{\sigma}\}\rangle \equiv \prod_i \left[ \frac{1}{\sqrt{n_{\tau}!n_{\rho}!n_{\sigma}!}} (a_{\tau}^\dagger)^{n_{\tau}} (a_{\rho}^\dagger)^{n_{\rho}} (a_{\sigma}^\dagger)^{n_{\sigma}} \right]|0\rangle.$$

In the expression, $S|\{n_{\gamma}\}\rangle$ is given by (A4) and $S|\{n'_{\gamma}\}\rangle$ is obtained by replacing $n_{\gamma}$ with $n'_{\gamma}$ in (A4). Now, those orthonormal basis vectors of the form $|\{n_{\tau}, n_{\rho}, n_{\sigma}\}\rangle$ that have a non-zero inner product with $S|\{n_{\gamma}\}\rangle$ must also be of the form (A5). Thus, $T_{\{n_{\rho}\}\{n'_{\rho}\}}^{\{n_{\gamma}\}\{n'_{\gamma}\}}$ vanishes unless there exist such a set of integers $\{n_i, n'_i\}$ ($i = 1, 2, \cdots$) that

$$n_i = n'_i,$$

$$n_i + n_{\gamma} - n_{\rho} = n'_i + n'_{\gamma} - n'_{\rho}$$

for $\forall i$. The existence of $\{n_i\}$ and $\{n'_i\}$ is equivalent to the condition $n_{\gamma} - n'_{\gamma} = n_{\rho} - n'_{\rho}$ for $\forall i$.

\[\square\]

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FIG. 1. A conformal diagram of a background geometry which describes a gravitational collapse. $I^-$ and $I^+$ are the past null infinity and the future null infinity, respectively and $H^+$ is the future event horizon.

Shaded region represents collapsing materials which forms the black hole. Besides the collapsing matter, we consider a real massless scalar field and investigate a scattering problem by the black hole after its formation ($v > v_0$). Thus we specify possible initial states at $I^-$ to those states which are excited from the vacuum by only modes whose support is within $v > v_0$ (elements of $\mathcal{F}_{I^- (v>v_0)}$), and possible mixed states constructed from them (elements of $\mathcal{P}$). In the diagram, $A_i \gamma$ ($i = 1, 2, \cdots$) is a mode function corresponding to a wave packet whose peak is at $v > v_0$ on $I^-$, $i\rho$ ($i = 1, 2, \cdots$) is a mode function corresponding to a wave packet on $I^+$. 