Role of symmetries in the Kerr-Schild derivation of the Kerr black hole

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In this work we explore the consequences of considering from the very beginning the stationary and axisymmetric properties of the Kerr black hole as one attempts to derive this solution through the Kerr-Schild ansatz. The first consequence is kinematical and is based on a new stationary and axisymmetric version of the Kerr theorem that yields to the precise shear-free and geodesic null congruence of flat spacetime characterizing the Kerr solution. A straightforward advantage of this strategy is that now the parameter a appears naturally as associated to the conserved angular momentum of the geodesics due to axisymmetry. The second consequence is dynamical and takes into account the circularity theorem. In fact, a stationary-axisymmetric Kerr-Schild ansatz is in general incompatible with the circularity property warranted by vacuum Einstein equations unless the remaining angular dependence in the Kerr-Schild profile appears fixed in a precise way. Thanks to these two ingredients, the integration of the Einstein equations reduces to a simple ordinary differential equation on the radial dependence, whose integration constant is precisely the mass m. This derivation of the Kerr solution is simple but rigorous, and it may be suitable for any textbook.

I. INTRODUCTION

One of the most relevant and well-known predictions of general relativity is the existence of black holes. Nevertheless, an outstanding but less recognized prediction claims that if the 10^{20} black hole candidates in the observable Universe are indeed black holes then all of them, independent of their formation process, are described by a single exact solution of Einstein field equations found by Kerr a half-century ago [1]. The Kerr solution describes a stationary and axisymmetric black hole only characterized by its mass and angular momentum. The uniqueness proof of this solution was a tour de force of mathematical physics that took over 30 years and was dubbed in its beginning as the “no-hair” conjecture for black holes [2].

The highly nonlinear nature of the Einstein field equations makes difficult any attempt to provide a straightforward or elementary derivation of the Kerr solution. In spite of that, it will be desirable for methodological and pedagogical reasons to have in hand a derivation as close as possible to an elementary one. This is the motivation behind pursuing different criteria to understand from first principles this solution and its generalizations. One of these criteria is to look for a metric describing an “exact perturbation” of flat spacetime that propagates along a null direction. This is commonly known as the Kerr-Schild ansatz [3], and takes the following form

\[ g_{\mu\nu} = \eta_{\mu\nu} + 2S l_\mu l_\nu, \]

where \( \eta_{\mu\nu} \) is the flat Minkowski metric, \( S \) is a scalar profile function and \( l_\mu \) is the tangent vector to a shear-free and geodesic null congruence. The main merit and advantage of the Kerr-Schild ansatz lie in the fact that the Einstein equations reduce to a linear system of equations without any approximation. For vacuum and some specific types of matter, the shear-free and geodesic character of the null congruence has been widely discussed and justified in the current literature [3–6]. A relevant criterion is encoded by the so-called Kerr theorem [4, 5, 7], which provides the most general class of shear-free and geodesic null congruences in flat spacetime. Nevertheless, the resulting class being given implicitly as a root of an algebraic equation defined by an arbitrary complex function is quite general. In fact, only a specific election of such equation including information about the angular momentum leads to the stationary and axisymmetric configuration known as the Kerr solution [3]. However, this election and the specific way of introducing the angular momentum, apart from yielding the desired result, do not have any physical justifications.

As shown in the present work, this lack of justification can be circumvented by adopting a different perspective. This will be done assuming beforehand the relevant symmetries that the final state of the gravitational collapse is supposed to enjoy. As a consequence, the proof of the stationary and axisymmetric version of the Kerr theorem will be more involved than the classical one [7], but the advantage of such approach lies in the fact that the degeneracy produced by the classical derivation will be completely fixed. Indeed, in this case, it is possible to justify the uniqueness of the specific shear-free and geodesic null congruence of flat spacetime that characterizes the Kerr solution. In addition, the significance of the angular momentum as the conserved charge of the geodesic motion associated to the axial symmetry becomes evident and no artifact is needed. This will be shown explicitly in Sec. II. The second important consequence has to do with the incompatibility between the circularity theorem that must satisfy any vacuum stationary and axisymmet-
ric spacetime and the Kerr-Schild ansatz, which is manifestly noncircular by construction. In Sec. III we show that this incompatibility is cured only if the scalar profile $S$ of ansatz (11) has a precise dependence on the remaining angular coordinate. All these ingredients put together allow us to reduce the Einstein equations to a trivial first-order linear ordinary differential equation having the mass as the unique integration constant. These arguments are extended trivially in Sec. IV to the charged case, and provide as well an elementary derivation of the Kerr-Newman black hole.

II. STATIONARY AXISYMMETRIC KERR THEOREM

We start by revising the classical Kerr theorem [4,5,7]. It states that any shear-free and geodesic null congruence in Minkowski space

$$ds_0^2 = \eta_{\mu \nu} dx^\mu dx^\nu = -2dudv + 2d\zeta d\bar{\zeta},$$  \hspace{2cm} (2)

is given by $l = dv$ or by

$$l = du + YY dv + \bar{Y} d\zeta + Y d\bar{\zeta},$$  \hspace{2cm} (3)

where $Y = Y(u,v,\zeta,\bar{\zeta})$ is a complex function implicitly defined by

$$F(Y,\bar{Y} + u, vY + \zeta) = 0,$$  \hspace{2cm} (4)

with $F$ an arbitrary function in its three complex dependences.

The proof can be performed in two steps. The most general null vector field in Minkowski space with $l_u \neq 0$ is given by (3). The geodesic and shear-free conditions over $l$ are equivalent in this context to the equations [2]

$$(\partial_{\zeta} - Y \partial_u)Y = 0,$$ \hspace{2cm} (5a)

$$(\partial_v - Y \partial_{\bar{\zeta}})Y = 0.$$ \hspace{2cm} (5b)

Thus, $Y$ is at the same time an invariant of both operators in (5). First, the characteristic system related to Eq. (5a) is

$$\frac{d\bar{\zeta}}{l} = \frac{du}{-Y} = \frac{dv}{0} = \frac{d\zeta}{0},$$ \hspace{2cm} (6)

which implies that the general solution to Eq. (5a) must be necessarily a function of the three integration constants of this ordinary system, giving $Y = Y(\zeta + u,v,\zeta)$. The second step is taking into account the last functional dependence in the characteristic system of the second equation (5b)

$$\frac{dv}{l} = \frac{d\zeta}{-Y} = \frac{d(\bar{Y} + u)}{0},$$ \hspace{2cm} (7)

meaning that $Y = Y(\bar{Y} + u, vY + \zeta)$ or, equivalently, the implicit representation [4]. For $l_u = 0$, interchanging the roles of $u$ and $v$ in the above arguments we conclude that $l = dv$ in this case.

As said in the Introduction, it is a well-educated guess for the concrete form of the function $F$ in (3) that yields, after a highly nontrivial integration process of the vacuum Einstein equations, to the stationary and axisymmetric Kerr black hole [3,4]. Here, we are interested in exploring the consequences of assuming from the very beginning the symmetries of the involved physical system without appealing to any other assumption. This can be done by assuming that both the profile $S$ and the shear-free and geodesic null vector $l$ are stationary and axisymmetric. In other words, we are going to provide a stationary-axisymmetric version of the Kerr theorem. In order to achieve this task and facilitate as possible the proof, we first write the Minkowski flat spacetime in coordinates where the stationary-axisymmetric character is manifest, i.e. in cylindrical coordinates

$$ds_0^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2,$$ \hspace{2cm} (8)

where the stationary and axisymmetric isometries are realized by the Killing fields $k = \partial_t$ and $m = \partial_\rho$, respectively. The tangent vector to any congruence on flat Minkowski spacetime compatible with such symmetries is written in cylindrical coordinates as

$$l = l_t(\rho,z)dt + l_\rho(\rho,z)d\rho + l_\phi(\rho,z)d\phi + l_z(\rho,z)dz.$$ \hspace{2cm} (9)

Imposing the geodesic condition, $l^\mu \nabla_\mu l_\nu = 0$, we get

$$(l_\rho \partial_\rho + l_z \partial_z)l_\mu = \frac{l_\rho^2}{\rho^2} l_\mu.$$ \hspace{2cm} (10)

We notice from the identity $l^\mu \nabla_\mu (l_\mu l_\nu) = 2l^\nu l^\rho \nabla_\mu l_\nu$ that the geodesic equation for the component $l_\rho$ is satisfied for a null geodesic if the geodesic equations for the remaining components are satisfied. The remaining equations are just given by $l(l_t) = l(l_\phi) = l(l_z) = 0$, which means that these components are invariants over the integral curves of the vector field $l$ due to the translational isometries which are still present in cylindrical coordinates [3]. However, their determining equations have the common characteristic system

$$\frac{d\rho}{l_\rho} = \frac{dz}{l_z},$$ \hspace{2cm} (11)

having a single independent invariant. Consequently, all the other invariants must be functions of the independent one that we choose to be the component $l_z$. Additionally, since for null geodesics $l_t \neq 0$, we can use the scale invariance of the affine parametrizations of the geodesics to fix this invariant component as $l_t = 1$. In summary, the tangent vector to the most general stationary-axisymmetric null geodesic congruence in flat space can be written as

$$l = dt + l_\rho d\rho + l_\phi(l_z) d\phi + l_z dz,$$ \hspace{2cm} (12a)

where the component $l_\rho(l_\rho, \rho, z)$ is determined from the quadratic null condition

$$l_\rho^2 = 1 - \frac{l_\phi(l_z)^2}{\rho^2} - l_z^2,$$ \hspace{2cm} (12b)
and the component \( l_z = l_z(\rho, z) \) is an invariant of the geodesic motion, \( l(l_z) = 0 \).

We shall now impose the shear-free condition which means that the shear tensor of the congruence denoted by \( \sigma_{\mu\nu} \) vanishes. This latter is defined as the traceless contribution from the symmetric part of the covariant derivative, \( \nabla_{\mu}l_{\nu} \), previously projected into the two-dimensional spacelike sector orthogonal to the null vector \( l \). Hence, it is effectively a two-dimensional matrix having only two independent components. These two components can be read off from the positive definite quantity

\[
2\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu} = \nabla_{(\mu}l_{\nu)}\nabla^{\mu}l^{\nu} - \frac{1}{2}(\nabla_{\mu}l^{\mu})^2.
\]  

(13)

A direct calculation for the flat spacetime congruence \([12]\) gives the sum of squares

\[
2\sigma^2 = \frac{(1 - l_z^2)}{2\rho^2} \left( \frac{\rho}{\rho^2 - l_z^2} \right)^2 \left( \frac{\rho l_z}{\rho^2 - l_z^2} \right)^2 + \frac{1}{\rho^2} \left[ \partial_{\rho}l_z \frac{dl_\phi}{dz} + l_z l_\phi \left( \frac{\partial_{\rho}l_z}{\rho} - \frac{l_z}{\rho^2} \right)^2 \right]^2,
\]

(14)

where we have replaced \( \partial_{\rho}l_z \) in terms of \( \partial_{\rho}l_z \) using the invariant condition \( l(l_z) = 0 \). The shear-free condition is equivalent to impose \( \sigma^2 = 0 \), which implies the vanishing of the two squared quantities. Using both conditions, we get the following equation

\[
\frac{dl_\phi(l_z)}{dl_z} = -\frac{2l_\phi(l_z)}{1 - l_z^2},
\]

(15)

whose solution is given by

\[
l_\phi(l_z) = -a \left( 1 - l_z^2 \right),
\]

(16)

where \( a \) is an integration constant modulating the conservation of the angular momentum of the geodesic motion due to the presence of the axisymmetry. We shall see later that this constant is just the celebrated parameter describing the angular momentum of the Kerr black hole. We would like to emphasize the natural emergence of this constant through this approach in contrast with the standard derivation. Substituting the expression of the angular component in the vanishing of the first squared term in \([13]\) and using \( l(l_z) = 0 \), one yields to a system for the undetermined component \( l_z \) given by

\[
\partial_{\rho}l_z = -\frac{\rho l_z (1 - l_z^2)}{\rho^2 - a^2 (1 - l_z^2)^2},
\]

(17a)

\[
\partial_{l_z}l_z = \frac{\rho l_z (1 - l_z^2)}{\rho^2 - a^2 (1 - l_z^2)^2}.
\]

(17b)

It is easy to see that the integrability condition \( \partial_{\rho}\partial_{l_z}l_z = \partial_{\rho}\partial_{l_z}l_z \) is satisfied in this case. Consequently, Eq. (17a) can be cast into the form

\[
\partial_{\rho} \left( \frac{l_z^2 [\rho^2 - a^2 (1 - l_z^2)]}{1 - l_z^2} \right) = 0,
\]

(18)

which in turn implies that

\[
\frac{l_z^2 [\rho^2 - a^2 (1 - l_z^2)]}{1 - l_z^2} = f(z).
\]

(19)

Deriving this expression with respect to \( z \) and using Eq. (17b) restricts the above function to satisfy

\[
f'(z)^2 = 4f(z), \quad \Rightarrow \quad f(z) = (z - z_0)^2.
\]

(20)

Additionally, using the translation invariance of Minkowski spacetime \([3]\) along the symmetry axis \( z \) one can choose the integration constant to be zero, \( z_0 = 0 \). Hence, the relation \([13]\) completely determines the implicit dependence of the component \( l_z \) in terms of the coordinates \( \rho \) and \( z \). Nevertheless, it is more convenient to represent the component \( l_z = dz/dr \) in terms of the affine parameter \( r \). From the invariant character of \( l_z \), their definition is straightforwardly integrated as

\[
\frac{z}{l_z} = r - r_0,
\]

(21)

and, using the shift invariance of the affine parameter one can set \( r_0 = 0 \). Combining together Eqs. \([15]-[21]\), the affine parameter is implicitly given by

\[
\frac{\rho^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1,
\]

(22a)

and allows us to express the tangent vector to the most general stationary-axisymmetric shear-free and geodesic null congruences in flat Minkowski spacetime as

\[
l = dt + \frac{r\rho}{r^2 + a^2} d\rho - \frac{a\rho^2}{r^2 + a^2} d\phi + \frac{z}{r} dz.
\]

(22b)

This is precisely the expression for the familiar congruence characterizing the Kerr solution \([3-6]\). We emphasis that requiring stationarity and axisymmetry has allowed us to remove the infinite degeneracy involved in the election of the function \( F \) determining all the congruences allowed by the general Kerr theorem in \([1]\).

A better representation for the congruence is achieved by noticing that Eq. \([22a]\) describes a family of ellipsoids of revolution. They become spheres of radius \( r \) for \( a = 0 \); accordingly, the constant \( a \) denotes the departure from sphericity of the family and each ellipsoid is labeled by the affine parameter \( r \). It is natural to consider coordinates adapted to the ellipsoids by using the label \( r \) and a pair of angles parametrizing each ellipsoid. Since the angle \( \phi \) naturally defines the revolution around the symmetry axis, the second angle can be defined in order to satisfy the restriction \([22a]\), choosing \( \rho = \sqrt{r^2 + a^2} \sin \theta \) and \( z = r \cos \theta \). In these ellipsoidal coordinates, the tangent vector to the congruence is specified without any extra condition by

\[
l = dt + \frac{\sum dr}{r^2 + a^2} - a \sin^2 \theta d\phi,
\]

(23)
and the flat Minkowski spacetime now reads

\[ ds_0^2 = -dt^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + \frac{\Sigma dr^2}{r^2 + a^2} + \Sigma d\theta^2, \quad (24) \]

where \( \Sigma = r^2 + a^2 \cos^2 \theta \). The deduction of these celebrated ellipsoidal coordinates was the main motivation in looking for a stationary-axisymmetric version of the Kerr theorem.

The consequence for the Kerr-Schild ansatz (1) is that its general stationary and axisymmetric version is naturally written using ellipsoidal coordinates as

\[ ds^2 = ds_0^2 + 2S(r, \theta) \left( dt + \frac{\Sigma dr}{r^2 + a^2 - a \sin^2 \theta d\phi} \right)^2, \quad (25) \]

where only the profile \( S(r, \theta) \) is to be determined. This is the starting point to explore the existence of stationary-axisymmetric solutions of the Einstein equations by means of a Kerr-Schild transformation. We show in the next section that it is even possible to restrict more the ansatz by fixing the angular dependence of the profile using some circularity arguments.

### III. THE CIRCULARITY THEOREM: KERR BLACK HOLE

We start this section by reminding that any stationary-axisymmetric spacetime with commuting Killing fields \( k = \partial_t \) and \( m = \partial_\phi \) fulfills the following geometrical identities [2]

\[
\begin{align*}
\ast (k \wedge m \wedge dk) &= 2 \ast (k \wedge m \wedge R(k)), \\
\ast (k \wedge m \wedge dm) &= 2 \ast (k \wedge m \wedge R(m)),
\end{align*} \quad (26)
\]

where the Killing fields are expressed as one-forms, \( k = \gamma_a k^a dx^a \) and \( m = \gamma_a m^a dx^a \), and the Ricci one-forms are specified by \( R(k) = R_{\mu
u} k^\nu dx^\mu \) and \( R(m) = R_{\mu
nu} m^\nu dx^\mu \). These identities are the base of the so-called circularity theorem [3, 10]; for vacuum solutions, the left-hand sides of (26) vanish which in turn implies that the differentiated expressions are constants and must vanish at the symmetry axis where \( m = 0 \). For the stationary-axisymmetric Kerr-Schild ansatz (25), we get

\[
\begin{align*}
0 &= \ast (k \wedge m \wedge dk) = -\frac{2 \sin \theta}{\Sigma} \partial_\theta \left[ \Sigma S(r, \theta) \right], \\
0 &= \ast (k \wedge m \wedge dm) = \frac{2a \sin \theta}{\Sigma} \partial_\theta \left[ \Sigma S(r, \theta) \right].
\end{align*} \quad (27)
\]

The first equalities are the Frobenius integrability conditions defining the circularity property, which warrants that the planes orthogonal to the Killing vectors at any point are integrable to surfaces tangent to those planes in the whole spacetime. Consequently, by choosing coordinates along the Killing fields and on their orthogonal surfaces, the circular metrics become block diagonal.

The second equalities in (27) imply that the stationary-axisymmetric Kerr-Schild ansatz is not circular by construction unless the profile is appropriately restricted as \[ 3 \]

\[ S(r, \theta) = \frac{r M(r)}{\Sigma}, \quad (28) \]

where the angular dependence is dictated by the function \( \Sigma \). Only for the above family of profiles is it possible to transform to Boyer-Lindquist-like coordinates [11]

\[
\begin{align*}
\tilde{t} &= t - \int \frac{2r M(r)}{\Delta} dr, \\
\tilde{\phi} &= \phi - \int \frac{2a r M(r)}{\Delta} dr,
\end{align*} \quad (29)
\]

with \( \Delta = r^2 + a^2 - 2r M(r) \), where now the circularity is manifest

\[
\begin{align*}
\frac{ds^2}{\Sigma} &= -\frac{\Delta}{\Sigma} \left( d\tilde{t} - a \sin^2 \theta d\tilde{\phi} \right)^2 \\
&\quad + \frac{\sin^2 \theta}{\Sigma} \left( ad\tilde{t} - (r^2 + a^2)\tilde{\phi} \right)^2 + \frac{\Sigma dr^2}{\Delta} + \Sigma d\theta^2. 
\end{align*} \quad (30)
\]

It only remains to fix the radial dependence encoded in the function \( M(r) \). This is easily done by inspecting the radial component of the vacuum Einstein equations in the Boyer-Lindquist-like coordinates

\[ G_{rr} = -\frac{2r^2 M(r)}{\Sigma^2} \frac{dM(r)}{dr} = 0; \quad (31) \]

hence, \( M(r) = m \) where the constant \( m \) is identified with the mass. The remaining Einstein equations are identically satisfied. In summary, the unique vacuum spacetime that can be written as a stationary-axisymmetric Kerr-Schild transformation from flat Minkowski spacetime is given by the metric (30) with \( \Delta = r^2 + a^2 - 2m r \). This is precisely the Kerr black hole [11].

### IV. KERR-NEWMAN BLACK HOLE

In this section, we extend the previous analysis to the electrovac case following basically the same strategy. First, using Einstein equations with the Maxwell energymomentum tensor, the identities (26) are now given by

\[
\begin{align*}
-\frac{1}{4} \ast (k \wedge m \wedge dk) &= \ast F(k, m) E_k + F(k, m) B_k, \\
-\frac{1}{4} \ast (k \wedge m \wedge dm) &= \ast F(k, m) E_m + F(k, m) B_m, \quad (32)
\end{align*}
\]

where the electric and magnetic one-forms, \( E_X = -i_X F \) and \( B_X = i_X \ast F \), are defined in terms of the electromagnetic field strength \( F = dA = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu \) for any

1 We thank A. Anabalon for helping us to elucidate this interesting feature of the Kerr-Schild ansatz.
Killing field $X$. Second, for a stationary-axisymmetric electromagnetic field, $\mathcal{L}_K F = 0 = \mathcal{L}_m F$, it follows from the Maxwell equations that $F(k, m) = 0 = *F(k, m)$ \[2\]. Consequently, the right-hand sides of \[22\] vanish leading to the electrovac circularity \[11\], and in the case of a stationary-axisymmetric Kerr-Schild ansatz to the conditions \[27\] yielding as before to the metric profile \[28\]. The extension of the Kerr-Schild ansatz to the electromagnetic field is done by considering the vector potential $A$ proportional to the shear-free and geodesic null vector $l$. As it happens with the metric, a stationary-axisymmetric ansatz $A = S(r, \theta) l$ determined by \[23\] is not circular by construction, since $0 = *F(k, m) \propto \partial_\theta [\Sigma S(r, \theta)]$, unless the electromagnetic profile has an angular dependence dictated by $\hat{S}(r, \theta) = r Q(r)/\Sigma$. The remaining Maxwell equations impose that the radial charge is a constant $Q(r) = q$ identified with the electric charge. The radial dependence of the metric is again determined from the radial Einstein equation which is now given by $dM(r)/dr = q^2/2r^2$, and whose solution reads $M(r) = m - q^2/2r$ where the constant $m$ represents the mass as before. The full Einstein-Maxwell system is fulfilled in this way; consequently, the electrovac spacetime that can be written as a stationary-axisymmetric Kerr-Schild transformation from the flat Minkowski spacetime is given by the metric \[30\] with $\Delta = r^2 + q^2 - 2mr + q^2$ and the electromagnetic field $A = q/r l/\Sigma$. This is nothing but the so-called Kerr-Newman black hole \[12\].

\section{Conclusions}

The main objective of the present paper was to emphasize the crucial importance that the symmetries can play in the Kerr-Schild derivation of the Kerr solution in general relativity. Indeed, we have explicitly shown that the assumption of stationarity and axisymmetry allows us to define in a unique way the shear-free and geodesic null congruence of flat spacetime characterizing the Kerr black hole. The advantage of this symmetric version of the Kerr theorem lies in the fact that the infinite degeneracy present in the original Kerr theorem is no longer present. Another advantage of the process described here is that the meaning of the angular momentum as a conserved quantity of the geodesic motion is revealed without any artifact. The next step of the strategy consisted of using the circularity theorem from which we have managed to integrate out the angular dependence of the profile function. Finally, the Einstein equations reduce to an elementary first-order differential equation in the Boyer-Linquidst-like coordinates that is also trivially integrated. We believe that all these ingredients put together constitute a straightforward, elementary, but rigorous derivation of the Kerr black hole. A generalization in order to include the electric charge can be carried out following the same lines yielding the Kerr-Newman generalization.

As a natural extension of the present procedure, it will be interesting to establish the equivalent of the stationary and axisymmetric Kerr theorem in the (anti-)de Sitter spacetime, putting on a firmer ground the use of the Kerr-Schild ansatz in this context. Another nontrivial task to explore is the extension to higher dimensions where it is still not clear the role played by the shear-free property.

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