On the boundedness of infinite matrix products with alternating factors from two sets of matrices

Victor Kozyakin

Abstract

We consider the question of boundedness of matrix products $A_n B_n \cdots A_1 B_1$, with factors from two sets of matrices, $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$, due to an appropriate choice of matrices $\{B_i\}$. It is assumed that for every sequence of matrices $\{A_i\}$ there is a sequence of matrices $\{B_i\}$ for which the sequence of matrix products $\{A_n B_n \cdots A_1 B_1\}_{n=1}^\infty$ is norm bounded. Some situations are described where in this case the norms of the matrix products $A_n B_n \cdots A_1 B_1$ are uniformly bounded, that is, $\|A_n B_n \cdots A_1 B_1\| \leq C$ for all natural numbers $n$, where $C > 0$ is a constant independent of the sequence $\{A_i\}$ and the corresponding sequence $\{B_i\}$. For the general case, the question of the validity of the corresponding statement remains open.

Key words and phrases: infinite matrix products, alternating factors, boundedness of matrix products

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1 Introduction

Let $\mathcal{M}$ be a set of matrices. By $\mathcal{M}^\infty$ we denote the set of infinite sequences of matrices $\{M_i \in \mathcal{M}: i = 1, 2, \ldots\}$, and by $\mathcal{M}^n$, where $n = 1, 2, \ldots$, the set of finite ordered sets $\{M_i \in \mathcal{M}: i = 1, 2, \ldots, n\}$ consisting of $n$ matrices. By $\mathcal{M}(p,q)$ we denote the space of real $(p \times q)$-matrices with the topology of element-wise convergence.

In the theory of matrix products the following statement is well known (see for example, [1, 2, 5, 6, 8]):

Proposition 1. Let $\mathcal{A} \subset \mathcal{M}(N,N)$ be a finite set of matrices in which for each sequence of matrices $\{A_n \} \in \mathcal{A}^\infty$ the sequence of norms $\{\|A_n \cdots A_1\|\}_{n=1}^\infty$ is bounded. Then all sequences of norms $\{\|A_n \cdots A_1\|\}_{n=1}^\infty$ are uniformly bounded, i.e., there is a constant $C > 0$ such that for any sequence of matrices $\{A_n \} \in \mathcal{A}^\infty$ for all $n = 1, 2, \ldots$ the inequalities $\|A_n \cdots A_1\| \leq C$ hold.

Some time ago an attempt was made in [9] to extend the analysis of the convergence of matrix products to the matrix products with alternating factors from two sets of matrices. The need for such an analysis was motivated by the problem of stabilization of switching linear systems with discrete time. More specifically, in [9] the following question was considered: Let $\mathcal{A} \subset \mathcal{M}(N,M)$ and $\mathcal{B} \subset \mathcal{M}(M,N)$ be finite sets of matrices, what can be said about the convergence rate to zero of the matrix products

$$A_n B_n \cdots A_1 B_1, \quad A_i \in \mathcal{A}, \ B_i \in \mathcal{B},$$

provided that convergence to zero $A_n B_n \cdots A_1 B_1 \to 0$ for any sequence of matrices $\{A_i\}$ can be guaranteed by an appropriate choice of a sequence of matrices $\{B_i\}$? As it turned out (see [9, Theorem 2]), the following statement is true:

$^{*}$Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoj Karetny lane 19, Moscow 127051, Russia, e-mail: kozyakin@iitp.ru
Proposition 2. Let $\mathcal{A} \subset \mathcal{M}(N, M)$ and $\mathcal{B} \subset \mathcal{M}(M, N)$ be sets of matrices with the property, that for any sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$, one can specify a sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$ such that $A_nB_n \cdots A_1B_1 \to 0$. Then there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ there is a sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$, for which $\|A_nB_n \cdots A_1B_1\| \leq C\lambda^n$ for $n = 1, 2, \ldots$.

The purpose of this paper is to continue the study of the properties of matrix products with alternating factors from two sets of matrices. We say that the sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is $\mathcal{B}$-right-bounded\(^1\) if there is a constant $C = C(\{A_n\})$ such that

$$\exists \{B_n\} \in \mathcal{B}^\infty : \|A_nB_n \cdots A_1B_1\| \leq C, \quad n = 1, 2, \ldots,$$

that is, if there exists a sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$ for which the sequence of norms

$$\\left(\|A_nB_n \cdots A_1B_1\|\right)_{n=1}^\infty$$

is bounded. We shall be interested in the question whether Proposition \(^1\) (or any of its analogues) applies to matrix products with alternating factors from two sets of matrices, apparently first formulated in \([9, \text{Question 2}]\);

\textbf{Question 1.} Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets of matrices such that every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is $\mathcal{B}$-right-bounded. In this case, is there a universal constant $C > 0$ such that for any sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ the inequality

$$\inf_{\{B_n\} \in \mathcal{B}^\infty} \nu_{\infty}(\{A_n\}, \{B_n\}) < \infty$$

always holds.

\textbf{Question 1.} Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets of matrices such that every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is $\mathcal{B}$-right-bounded. In this case, is there a universal constant $C > 0$ such that

$$\inf_{\{B_n\} \in \mathcal{B}^\infty} \nu_{\infty}(\{A_n\}, \{B_n\}) \leq C$$

for any sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$?

\(^1\)Here the prefix ‘right-’ means that in the corresponding matrix products the factors $B_i$ are applied to the factors $A_i$ on the right.
Question 1 can also be rephrased as follows:

**Question 1′.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be finite sets of matrices such that every sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) is \( \mathcal{B} \)-right-bounded. In this case, if the inequality

\[
\mu_\infty(\mathcal{A}, \mathcal{B}) := \sup_{\{A_i\} \in \mathcal{A}^\infty} \inf_{\{B_i\} \in \mathcal{B}^\infty} \nu_\infty(\{A_i\}, \{B_i\}) < \infty \tag{2}
\]

hold?

We note that similar issues of pointwise stabilizability of switching systems have been considered in [3, 7, 10].

## 2 Main Lemma

Let us introduce finite analogs of the quantity \( \mu_\infty(\mathcal{A}, \mathcal{B}) \), by setting

\[
\mu_n(\mathcal{A}, \mathcal{B}) = \max_{\{A_i\} \in \mathcal{A}^n} \min_{\{B_i\} \in \mathcal{B}^n} \nu_n(\{A_i\}, \{B_i\}), \quad n = 1, 2, \ldots. \tag{3}
\]

**Lemma 1.** Inequality (2) is satisfied if and only if there is a constant \( C > 0 \) such that

\[
\sup_{n \geq 1} \mu_n(\mathcal{A}, \mathcal{B}) \leq C. \tag{4}
\]

**Proof.** Let us show that, assuming that (2) is satisfied, there exists a constant \( C > 0 \) for which inequality (4) holds. Note first that (2) implies the existence of a constant \( C' > 0 \), so that

\[
\mu_\infty(\mathcal{A}, \mathcal{B}) := \sup_{\{A_i\} \in \mathcal{A}^\infty} \inf_{\{B_i\} \in \mathcal{B}^\infty} \nu_\infty(\{A_i\}, \{B_i\}) \leq C'. \tag{5}
\]

Let us take an arbitrary natural number \( n \) and choose such sets of matrices \( \{A_i\} \in \mathcal{A}^n \) and \( \{B_i\} \in \mathcal{B}^n \) for which the equality

\[
\mu_n(\mathcal{A}, \mathcal{B}) = \nu_n(\{A_i\}, \{B_i\})
\]

holds (this is possible because the sets \( \mathcal{A} \) and \( \mathcal{B} \) are finite, which implies that the corresponding minima and maxima are reached in (3)).

Further, we extend the set of matrices \( \{A_i\} \in \mathcal{A}^n \) arbitrarily to the right to the infinite sequence \( \{\bar{A}_i\} \in \mathcal{A}^\infty \) and choose an infinite sequence of matrices \( \{\bar{B}_i\} \in \mathcal{B}^\infty \) such that the inequality

\[
\nu_\infty(\{\bar{A}_i\}, \{\bar{B}_i\}) \leq 2 \inf_{\{\bar{B}_i\} \in \mathcal{B}^\infty} \nu_\infty(\{\bar{A}_i\}, \{\bar{B}_i\})
\]

is valid (this is possible due to the definition of the value \( \nu_\infty(\cdot, \cdot) \)). Then, setting \( C = 2C' \), we get by definition (5):

\[
\nu_\infty(\{\bar{A}_i\}, \{\bar{B}_i\}) \leq 2 \mu_\infty(\mathcal{A}, \mathcal{B}) \leq 2C' = C.
\]

Thus, since the estimate

\[
\nu_n(\{\bar{A}_i\}_{i=1}^n, \{\bar{B}_i\}_{i=1}^n) \leq \nu_\infty(\{\bar{A}_i\}, \{\bar{B}_i\})
\]

holds for any natural number \( n \), we obtain that

\[
\nu_n(\{\bar{A}_i\}_{i=1}^n, \{\bar{B}_i\}_{i=1}^n) \leq C,
\]

whence, by the definition of sets of matrices \( \{A_i\} = \{\bar{A}_i\}_{i=1}^n \in \mathcal{A}^n \) and \( \{\bar{B}_i\} \in \mathcal{B}^\infty \), we obtain:

\[
\mu_n(\mathcal{A}, \mathcal{B}) = \min_{\{\bar{B}_i\} \in \mathcal{B}^n} \nu_n(\{A_i\}, \{\bar{B}_i\}_{i=1}^n) \leq \nu_n(\{A_i\}, \{\bar{B}_i\}_{i=1}^n) = \nu_n(\{\bar{A}_i\}_{i=1}^n, \{\bar{B}_i\}_{i=1}^n) \leq C.
\]
In one direction, the assertion of the lemma is proved.

Let us prove the assertion of the lemma in the opposite direction: Suppose that inequalities (4) hold, and prove that in this case the inequality
\[
\mu_\infty(\mathcal{A}, \mathcal{B}) := \sup_{\{A_i\} \in \mathcal{A}^\infty} \inf_{\{B_i\} \in \mathcal{B}^\infty} \nu_\infty(\{A_i\}, \{B_i\}) \leq C
\]
also holds. To do this, choose an arbitrary sequence of matrices \(\{A_i\} \in \mathcal{A}^\infty\), and then for each \(n = 1, 2, \ldots\) choose a set of matrices \(\{B_i^{(n)}\}_{i=1}^n \in \mathcal{B}^n\) that satisfy the inequalities
\[
\nu_n(\{A_i\}_{i=1}^n, \{B_i^{(n)}\}_{i=1}^n) \leq \mu_n(\mathcal{A}, \mathcal{B}) \leq C.
\]
Then, for each \(k \geq n\), due to the definition of the quantities \(\nu_n(\cdot, 
\max_{1 \leq j \leq n} \|A_j B_j^{(k)} \cdots A_1 B_1^{(k)}\| = \nu_n(\{A_i\}_{i=1}^n, \{B_i^{(k)}\}_{i=1}^n) \leq \mu_n(\mathcal{A}, \mathcal{B}) \leq C, \quad n = 1, 2, \ldots,
\]
will be valid. Further, since for each natural number \(i\) the elements of the sequence of matrices \(\{B_i^{(k)}\}_{k=1}^\infty\) are taken from a finite set, then without loss of generality we can assume that there exist limits
\[
\lim_{k \to \infty} B_i^{(k)} = \bar{B}_i \in \mathcal{B}, \quad i = 1, 2, \ldots.
\]
Passing to the limit in (7) as \(k \to \infty\), we obtain
\[
\max_{1 \leq j \leq n} \|A_j \bar{B}_j \cdots A_1 \bar{B}_1\| = \nu_n(\{A_i\}_{i=1}^n, \{\bar{B}_i\}_{i=1}^n) = \sup_{n \geq 1} \|A_n \bar{B}_n \cdots A_1 \bar{B}_1\| \leq C.
\]
Therefore, for the chosen sequence of matrices \(\{A_i\} \in \mathcal{A}^\infty\) and the constructed sequence of matrices \(\{\bar{B}_i\} \in \mathcal{B}^\infty\), the relations
\[
\nu_\infty(\{A_i\}, \{\bar{B}_i\}) = \sup_{n \geq 1} \nu_n(\{A_i\}_{i=1}^n, \{\bar{B}_i\}_{i=1}^n) = \sup_{n \geq 1} \|A_n \bar{B}_n \cdots A_1 \bar{B}_1\| \leq C
\]
hold. Since the sequence of matrices \(\{A_i\} \in \mathcal{A}^\infty\) is arbitrary, this implies inequality (6). The lemma is proved.

\[\square\]

3  Right- and Left-Bounded Matrix Products

When considering matrix products with alternating factors from two sets of matrices, instead of products of the form
\[
A_n B_n \cdots A_1 B_1, \quad \text{where} \quad A_i \in \mathcal{A}, \quad B_i \in \mathcal{B},
\]
in which the factors \(B_i\) are ‘to the right’ of the corresponding factors \(A_i\), one could consider matrix products of the form
\[
B_n A_n \cdots B_1 A_1, \quad \text{where} \quad A_i \in \mathcal{A}, \quad B_i \in \mathcal{B},
\]
where the factors \(B_i\) are ‘to the left’ of the corresponding factors \(A_i\). It is more or less clear that the difference between these two cases should have no bearing on the question of convergence or boundedness of the respective products. However, for the sake of completeness, we present more formalized statements and arguments below.

We say that a sequence of matrices \(\{A_n\} \in \mathcal{A}^\infty\) is \(\mathcal{B}\)-left-bounded, if there exists a sequence of matrices \(\{B_n\} \in \mathcal{B}^\infty\) for which the sequence of norms \(\{\|B_n A_n \cdots B_1 A_1\|\}_{n=1}^\infty\) is bounded, i.e., there exists a constant \(\tilde{C} = \tilde{C}(\{A_n\})\) such that
\[
\exists \{B_n\} \in \mathcal{B}^\infty : \quad \|B_n A_n \cdots B_1 A_1\| \leq \tilde{C}, \quad n = 1, 2, \ldots
\]
Lemma 2. Let the sets of matrices $\mathcal{A}$ and $\mathcal{B}$ be bounded. Then if every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is $\mathcal{B}$-left-bounded, then every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is also $\mathcal{B}$-right-bounded. And if every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is $\mathcal{B}$-right-bounded, then every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is also $\mathcal{B}$-left-bounded.

Proof. Since by the condition of the lemma the sets of matrices $\mathcal{A}$ and $\mathcal{B}$ are bounded, there are constants $a$ and $b$ such that

$$\sup\{\|A\| : A \in \mathcal{A}\} \leq a, \quad \sup\{\|B\| : B \in \mathcal{B}\} \leq b.$$ 

Let us first consider the case where any sequence of matrices with values in $\mathcal{A}$ is $\mathcal{B}$-right-bounded. We take an arbitrary sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ and prove that in this case it is $\mathcal{B}$-left-bounded.

By assumption, the ‘shifted’ sequence of matrices $\{\tilde{A}_n\} \in \mathcal{A}^\infty$, defined by the equalities

$$\tilde{A}_n = A_{n+1}, \quad n = 1, 2, \ldots,$$

is $\mathcal{B}$-right-bounded. Then due to (1) there is a sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$ and a constant $C = C(\{\tilde{A}_n\})$ such that

$$\|\tilde{A}_nB_n \cdots \tilde{A}_1B_1\| \leq C, \quad n = 1, 2, \ldots.$$ 

From this and from the definition (8)

$$\|B_nA_n \cdots B_1A_1\| = \|B_n (A_nB_{n-1} \cdots A_2B_1) A_1\| = \|B_n \left(\tilde{A}_{n-1}B_{n-1} \cdots \tilde{A}_1B_1\right) A_1\| \leq \|B_n\|\|\tilde{A}_{n-1}B_{n-1} \cdots \tilde{A}_1B_1\||A_1\| \leq bCa, \quad n = 2, 3, \ldots.$$ 

This proves the $\mathcal{B}$-left-boundedness of the sequence $\{A_n\} \in \mathcal{A}^\infty$. The $\mathcal{B}$-right-boundedness of any sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is proved in a similar way, provided that any sequence of matrices with values in $\mathcal{A}$ is $\mathcal{B}$-left-bounded.

From Lemma 2 it follows that when considering Question 1, instead of $\mathcal{B}$-right-boundedness one can assume $\mathcal{B}$-left-boundedness of the corresponding matrix products.

4 Nondegenerate Matrices

In this section we will assume\(^2\) that

$$\det(AB) \neq 0, \quad \forall A \in \mathcal{A}, \ B \in \mathcal{B}.$$ 

The matrix $AB$ has dimension $N \times N$. Moreover, its rank is not greater than the minimum of the ranks of the matrices $A$ and $B$, that is, not greater than $\min\{M, N\}$. Therefore, the relation (9) in which the matrix $B$ is applied to the matrix $A$ on the right-hand side can hold only if $N \leq M$.

In the case where $N > M$, there may be valid the relation

$$\det(BA) \neq 0, \quad \forall A \in \mathcal{A}, \ B \in \mathcal{B},$$ 

in which the matrix $B$ is applied to the matrix $A$ on the left-hand side. Accordingly, in this case, further reasoning in this section should be done with $\mathcal{B}$-left-bounded matrix products.

Let us show that under condition (9) the answer to Question 1 is affirmative:

Theorem 1. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets of matrices satisfying condition (9), and let every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ be $\mathcal{B}$-right-bounded. Then there exists a constant $C > 0$ such that for any sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ inequalities (1) hold.

\(^2\)Recall that the matrix sets $\mathcal{A}$ and $\mathcal{B}$ do not necessarily consist of square matrices, so it is natural to define the non-degeneracy of these sets by the products $AB$ or $BA$. 

5
Proof. Let us prove the required assertion by contradiction: Show that the absence of the required constant $C$ implies the existence of such a sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$, for which the sequence of norms $\left\|\left\|A_nB_n \cdots A_1B_1\right\|\right\|_{n=1}^\infty$ is unbounded for any choice of the sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$.

So suppose that the required constant $C$ does not exist, then inequalities (4) are true for no $C$ and therefore

$$\limsup_{n \to \infty} \mu_n(\mathcal{A}, \mathcal{B}) = \infty. \quad (10)$$

Let us take an arbitrary positive number $\varepsilon_1$, for example $\varepsilon_1 = 1$. Then, due to (10), there is an integer $n_1 \geq 1$ such that

$$\mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_1 = 1. \quad (11)$$

Let us denote by $\bar{A}_1, \ldots, \bar{A}_{n_1} \in \mathcal{A}$ the set of matrices for which the right side in (3) for $n = n_1 \geq 1$ reaches the maximum over $\{A_i\} \in \mathcal{A}^{n_1}$. Then, given (11) and the definition (3) of the quantities $\mu_n(\mathcal{A}, \mathcal{B})$, we have

$$\min_{\{B_i\} \in \mathcal{B}^{n_1}} \nu_{n_1}(\{\bar{A}_i\}_{i=1}^{n_1}, \{B_i\}) = \mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_1 = 1$$

and therefore by the definition of $\nu_n(\cdot, \cdot)$ for any sequence of matrices $\{B_i\} \in \mathcal{B}^{n_1}$ the following relations hold:

$$\max_{1 \leq k \leq n_1} \|\bar{A}_k B_k \cdots \bar{A}_1 B_1\| = \nu_{n_1}(\{\bar{A}_i\}_{i=1}^{n_1}, \{B_i\}) = \mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_1 = 1. \quad (12)$$

Now note that due to the assumption that the sets $\mathcal{A}$ and $\mathcal{B}$ are finite, the set of matrices $\bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1$ in (12) is also finite and all these matrices are invertible due to the assumption (9). Then one can specify a constant $\eta_1 > 0$ such that

$$\|(\bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1)^{-1}\| \leq \eta_1, \quad \forall \{B_i\} \in \mathcal{B}^{n_1}. \quad (13)$$

Let us now choose an arbitrary number $\varepsilon_2 \geq 2\eta_1$. Then, again due to (10), there is an integer $n_2 \geq 1$ such that

$$\mu_{n_2}(\mathcal{A}, \mathcal{B}) \geq \varepsilon_2. \quad (14)$$

Let us denote by $\bar{A}_{n_1+1}, \ldots, \bar{A}_{n_1+n_2}$ the set of matrices from $\mathcal{A}$ for which the right-hand side in (3) for $n = n_2 \geq 1$ reaches its maximum over $\{A_i\} \in \mathcal{A}^{n_2}$. Then, due to (14)

$$\min_{\{B_i\} \in \mathcal{B}^{n_2}} \nu_{n_2}(\{\bar{A}_i\}_{i=n_1+1}^{n_1+n_2}, \{B_i\}) = \mu_{n_2}(\mathcal{A}, \mathcal{B}) \geq \varepsilon_2$$

and therefore by the definition of $\nu_n(\cdot, \cdot)$ the relations

$$\max_{1 \leq k \leq n_2} \|\bar{A}_{n_1+k}B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1}\| = \nu_{n_2}(\{\bar{A}_i\}_{i=n_1+1}^{n_1+n_2}, \{B_i\}) = \mu_{n_2}(\mathcal{A}, \mathcal{B}) \geq \varepsilon_2 \quad (15)$$

hold for any sequence of matrices $\{B_i\} \in \mathcal{B}^{n_2}$.

Now consider for $k = 1, 2, \ldots, n_2$ the product of the matrices

$$W_k = \bar{A}_{n_1+k}B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1} \cdot \bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1$$

and estimate its norm from below. To this end, we note that for any matrices $P$ and $Q$ of which the matrix $Q$ is invertible, the relations $\|P\| = \|PQ \cdot Q^{-1}\| \leq \|PQ\| \cdot \|Q^{-1}\|$ hold, whence

$$\|PQ\| \geq \|P\| \cdot \|Q^{-1}\|^{-1}.$$ 

So for the matrix $W_k = P_kQ$, where

$$P_k = \bar{A}_{n_1+k}B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1}, \quad Q = \bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1,$$
we obtain for each \( k = 1, 2, \ldots, n_2 \) the inequalities
\[
\| \bar{A}_{n_1+k} B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1} \cdot \bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1 \| = \|W_k\| = \|P_kQ\| = \|P_k\| \cdot \|Q^{-1}\| \geq \\
\geq \| \bar{A}_{n_1+k} B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1} \cdot (\bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1)^{-1} \|^{-1} \quad (16)
\]
Because of (15), here the first factor on the right-hand side satisfies the estimate
\[
\max_{1 \leq k \leq n_2} \| \bar{A}_{n_1+k} B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1} \| \geq \kappa_2 \geq 2\eta_1, \quad \forall \{B_1+i\} \in \mathcal{B}^{n_2},
\]
and for the second factor, due to (13), the estimate
\[
\| (\bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1)^{-1} \|^{-1} \geq \eta_1^{-1}, \quad \forall \{B_i\} \in \mathcal{B}^{n_1}
\]
takes place. Consequently, due to (16)
\[
\max_{1 \leq k \leq n_2} \| \bar{A}_{n_1+n_2} B_{n_1+n_2} \cdots \bar{A}_{n_1+1}B_{n_1+1} \cdot \bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1 \| \geq 2, \quad \forall \{B_i\} \in \mathcal{B}^{n_1+n_2},
\]
and therefore, even more,
\[
\max_{1 \leq k \leq n_1+n_2} \| \bar{A}_k B_k \cdots \bar{A}_1B_1 \| \geq \max_{1 \leq k \leq n_2} \| \bar{A}_{n_1+k} B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1} \cdot \bar{A}_{n_1}B_{n_1} \cdots \bar{A}_1B_1 \| \geq 2
\]
for any sequence of matrices \( \{B_i\} \in \mathcal{B}^{n_1+n_2} \).

Similarly, we define a constant \( \eta_2 > 0 \) to satisfy the inequality
\[
\| (\bar{A}_{n_1+n_2} B_{n_1+n_2} \cdots \bar{A}_1B_1)^{-1} \| \leq \eta_2, \quad \forall \{B_i\} \in \mathcal{B}^{n_1+n_2},
\]
and choose an arbitrary number \( \kappa_3 \geq 3\eta_2 \). Then, as in the previous case, one can specify such an integer \( n_3 \geq 1 \) and a set of matrices \( \bar{A}_{n_1+n_2+1}, \ldots, \bar{A}_{n_1+n_2+n_3} \) for which
\[
\max_{1 \leq k \leq n_1+n_2+n_3} \| \bar{A}_k B_k \cdots \bar{A}_1B_1 \| \geq 3, \quad \forall \{B_i\} \in \mathcal{B}^{n_1+n_2+n_3}.
\]

Acting similarly, for each \( m = 1, 2, \ldots \) we can specify an integer \( n_m \geq 1 \) and have a set of matrices
\[
\bar{A}_{n_1+\cdots+n_{m-1}+1}, \ldots, \bar{A}_{n_1+\cdots+n_{m-1}+n_m}
\]
such that
\[
\max_{1 \leq k \leq n_1+\cdots+n_m} \| \bar{A}_k B_k \cdots \bar{A}_1B_1 \| \geq m, \quad \forall \{B_i\} \in \mathcal{B}^{n_1+\cdots+n_m}. \quad (17)
\]

The resulting estimate contradicts the assumption from Question 1 that for every sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) there is a sequence of matrices \( \{B_n\} \in \mathcal{B}^\infty \) for which the sequence of norms \( \{\|A_n B_n \cdots A_1B_1\|, n = 1, 2, \ldots\} \) is bounded. In fact, due to inequalities (17) for the constructed sequence of matrices \( \{\bar{A}_i\} \in \mathcal{A}^\infty \) and any sequence of matrices \( \{B_i\} \in \mathcal{B}^\infty \), the sequence of norms
\[
\{\|\bar{A}_n B_n \cdots \bar{A}_1B_1\|, n = 1, 2, \ldots\}
\]
is not bounded. Theorem 1 is proved.

\section{Nonnegative Matrices With Nonzero Rows}

The purpose of this section is to relax the restrictions imposed on the matrix sets \( \mathcal{A} \) and \( \mathcal{B} \) in Section 4. As in Section 4, the matrix sets \( \mathcal{A} \) and \( \mathcal{B} \) are assumed to be finite, but instead of nondegeneracy (9) of the matrix products \( AB \), where \( A \in \mathcal{A} \), \( B \in \mathcal{B} \), nonnegativity of the corresponding matrices is assumed:
\[
a_{ij} \geq 0, \quad b_{ij} \geq 0, \quad \forall A = (a_{ij}) \in \mathcal{A}, \quad B = (b_{ij}) \in \mathcal{B}, \quad (18)
\]
and it is also assumed that no matrix of \( \mathcal{A} \) and \( \mathcal{B} \) can have ‘zero’ rows, i.e. rows consisting entirely of zero elements:

\[
\begin{align*}
\min_i \sum_j a_{ij} > 0, & \quad \forall A = (a_{ij}) \in \mathcal{A}, \\
\min_i \sum_j b_{ij} > 0, & \quad \forall B = (b_{ij}) \in \mathcal{B}.
\end{align*}
\]

Let us show that under these conditions the answer to Question 1 is positive:

**Theorem 2.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be finite sets of matrices satisfying (18) and (19), and let any sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) be \( \mathcal{B} \)-right-bounded. Then there is a constant \( C > 0 \) such that for any sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) inequalities (1) hold.

Before proceeding with the proof of Theorem 2, we make a few remarks and note a number of auxiliary facts.

Note that conditions (18) and (19) are satisfied by the sets of matrices \( \mathcal{A} \) and \( \mathcal{B} \), each of which has strictly positive elements, i.e.,

\[
\begin{align*}
\min_{i,j} a_{ij} > 0, & \quad \forall A = (a_{ij}) \in \mathcal{A}, \\
\min_{i,j} b_{ji} > 0, & \quad \forall B = (b_{ij}) \in \mathcal{B}.
\end{align*}
\]

Note also that under condition (19), since the sets \( \mathcal{A} \) and \( \mathcal{B} \) are finite, there exists \( \gamma > 0 \) that

\[
\begin{align*}
\min_i \sum_j a_{ij} \geq \gamma, & \quad \forall A = (a_{ij}) \in \mathcal{A}, \\
\min_i \sum_j b_{ij} \geq \gamma, & \quad \forall B = (b_{ij}) \in \mathcal{B}.
\end{align*}
\]

If we introduce a finite set of matrices

\[
\mathcal{F} = \{AB: A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{M}(N,N),
\]

then we can assume without loss of generality that under conditions (18) and (19) the following inequality holds:

\[
\min_i \sum_j f_{ij} \geq \gamma, \quad \forall F = (f_{ij}) \in \mathcal{F}.
\]

Finally, note that if condition (18) is satisfied, condition (21) follows from condition (19) but is not equivalent to it.

By \( e \) we denote the vector in \( \mathbb{R}^N \) (or \( \mathbb{R}^M \)) consisting of the unit elements:

\[
e = \{1, 1, \ldots, 1\}.
\]

Then the following equivalent description of conditions (19) and (21) is true.

**Lemma 3.** Condition (19) is true if and only if there exists \( \gamma > 0 \) such that

\[
Ae \geq \gamma e, \quad Be \geq \gamma e, \quad \forall A \in \mathcal{A}, \ B \in \mathcal{B},
\]

where the inequalities between vectors are understood coordinate-wise.

If condition (18) is satisfied, then condition (21) is satisfied if and only if there is \( \gamma > 0 \) such that

\[
Fe \geq \gamma e, \quad \forall F \in \mathcal{F},
\]

where the inequalities between vectors are understood coordinate-wise.

**Proof.** Evident.

Note a general fact in connection with nonnegative matrices. By the end of this section, let \( \| \cdot \| \) be the so-called max-norm in \( \mathbb{R}^N \), that is, the norm given by the equality

\[
\|x\| = \max_i |x_i|, \quad x = \{x_1, x_2, \ldots, x_N\}.
\]

The fact that the same notation applies to elements from different spaces will hopefully not cause any confusion!
Lemma 4. If $A \in \mathcal{M}(N, N)$ is a matrix with nonnegative elements, then $\|A\| = \|Ae\|$, where $e = \{1, 1, \ldots, 1\}$.

Proof. Evident. \hfill \Box

According to Lemma 4, for any matrix $A$ with nonnegative entries, the equality $\|A\| = \|Ae\|$ holds. Therefore, in the following, when estimating the norm of the matrix products under consideration, it will suffice to estimate the norms of the matrices on element $e$.

Now we can proceed with the proof of Theorem 2.

Proof of Theorem 2. As in the case of Theorem 1, we perform the proof by contradiction: We shall show that the absence of the required constant $C$ implies the existence of a sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$, for which the sequence of norms $\{\|A_nB_n \cdots A_1B_1\|\}_{n=1}^\infty$ is unbounded for any choice of the sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$.

So, suppose that the required constant $C$ does not exist, then inequalities (9) are not true for any $C$, that is,

$$
\limsup_{n \to \infty} \mu_n(\mathcal{A}, \mathcal{B}) = \infty. \tag{22}
$$

Let us take an arbitrary positive number $\varepsilon_1$, for example $\varepsilon_1 = 1$. Then, due to (22), there is an integer $n_1 \geq 1$ such that

$$
\mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_1 = 1. \tag{23}
$$

Let $\bar{A}_1, \ldots, \bar{A}_{n_1}$ denote the set of matrices from $\mathcal{A}$ for which the right-hand side in (3) for $n = n_1 \geq 1$ reaches the maximum over $\{A_i\} \in \mathcal{A}^{n_1}$. Then, due to (23) and the definition (3) of the quantities $\mu_n(\mathcal{A}, \mathcal{B})$,

$$
\min_{\{B_i\} \in \mathcal{B}^{n_1}} \nu_{n_1}(\{\bar{A}_i\}_{i=1}^{n_1}, \{B_i\}) = \mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_1 = 1
$$

and thus by the definition of $\nu_n(\cdot, \cdot)$ the relations

$$
\max_{1 \leq k \leq n_1} \|\bar{A}_kB_k \cdots \bar{A}_1B_1\| = \nu_{n_1}(\{\bar{A}_i\}_{i=1}^{n_1}, \{B_i\}) = \mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_1 = 1
$$

hold for any sequence of matrices $\{B_i\} \in \mathcal{B}^{n_1}$.

Now note that because of the assumptions (19) or (21) for any sequences of matrices $\{B_i\} \in \mathcal{B}^{n_1}$ each vector $\bar{A}_1B_{n_1} \cdots \bar{A}_1B_1e$ does not vanish and, moreover, is coordinate-bounded from below by a positive factor of the vector $e$. Since the set of matrices $\mathcal{B}$ is finite, it follows that there exists a constant $\omega_1 > 0$ for which

$$
\bar{A}_1B_{n_1} \cdots \bar{A}_1B_1e \geq \omega_1 e, \quad \forall \{B_i\} \in \mathcal{B}^{n_1}. \tag{24}
$$

Let us now choose an arbitrary number $\varepsilon_2 \geq 2\omega_1^{-1}$. Then, again due to (22), there is an integer $n_2 \geq 1$ such that

$$
\mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_2. \tag{25}
$$

Let $\bar{A}_{n_1+1}, \ldots, \bar{A}_{n_1+n_2}$ denote the set of matrices from $\mathcal{A}$ for which the right-hand side in (3) for $n = n_2 \geq 1$ reaches its maximum over $\{A_i\} \in \mathcal{A}^{n_2}$. Then, due to (25) and the definition (3) of the quantities $\mu_n(\mathcal{A}, \mathcal{B})$,

$$
\min_{\{B_i\} \in \mathcal{B}^{n_2}} \nu_{n_2}(\{\bar{A}_i\}_{i=n_1+1}^{n_1+n_2}, \{B_i\}) = \mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_2
$$

and therefore by the definition of $\nu_n(\cdot, \cdot)$ for any sequence of matrices $\{B_i\} \in \mathcal{B}^{n_2}$ the following relations hold:

$$
\max_{1 \leq k \leq n_2} \|\bar{A}_{n_1+k}B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1}\| = \nu_{n_2}(\{\bar{A}_i\}_{i=n_1+1}^{n_1+n_2}, \{B_i\}) = \mu_n(\mathcal{A}, \mathcal{B}) \geq \varepsilon_2,
$$

which by Lemma 4 are equivalent to the inequalities

$$
\max_{1 \leq k \leq n_2} \|\bar{A}_{n_1+k}B_{n_1+k} \cdots \bar{A}_{n_1+1}B_{n_1+1}e\| \geq \varepsilon_2. \tag{26}
$$
Now consider for $k = 1, 2, \ldots, n_2$ the product of the matrices

$$W_k = \tilde{A}_{n_1+k}B_{n_1+k} \cdots \tilde{A}_{n_1+1}B_{n_1+1} \cdot \tilde{A}_{n_1}B_{n_1} \cdots \tilde{A}_1B_1$$

and estimate its norm $\|W_k\|$ from below. For this, due to Lemma 4, it suffices to estimate the norm of the vector $\|W_ke\| = \|W_k\|$ from below.

We set

$$P_k = \tilde{A}_{n_1+k}B_{n_1+k} \cdots \tilde{A}_{n_1+1}B_{n_1+1}, \quad Q = \tilde{A}_{n_1}B_{n_1} \cdots \tilde{A}_1B_1,$$

Now note that in the representation $W_k = P_k Q e$, the lower bound holds for the vector $Q e$ due to (24):

$$Q e \geq \omega_1 e,$$

which, since the matrices $P_k$ are positive, for each $k = 1, 2, \ldots, n_2$ the relations

$$\tilde{A}_{n_1+k}B_{n_1+k} \cdots \tilde{A}_{n_1+1}B_{n_1+1} \cdot \tilde{A}_{n_1}B_{n_1} \cdots \tilde{A}_1B_1e = W_k e = P_k Q e \geq P_k(\omega_1 e) = \omega_1 P_k e$$

From this and from Lemma 4

$$\|\tilde{A}_{n_1+k}B_{n_1+k} \cdots \tilde{A}_{n_1+1}B_{n_1+1} \cdot \tilde{A}_{n_1}B_{n_1} \cdots \tilde{A}_1B_1\| \geq \omega_1 \|P_k e\|,$$

and then, due to inequalities (26),

$$\max_{1 \leq k \leq n_2} \|\tilde{A}_{n_1+k}B_{n_1+k} \cdots \tilde{A}_{n_1+1}B_{n_1+1} \cdot \tilde{A}_{n_1}B_{n_1} \cdots \tilde{A}_1B_1\| \geq \omega_1 \omega_2 \geq 2, \quad \forall \{B_i\} \in \mathscr{B}^{n_1+n_2},$$

and hence, more, for any sequence of matrices $\{B_i\} \in \mathscr{B}^{n_1+n_2}$ we have the estimates

$$\max_{1 \leq k \leq n_2+n_2} \|\tilde{A}_kB_k \cdots \tilde{A}_1B_1\| \geq \max_{1 \leq k \leq n_2} \|\tilde{A}_{n_1+k}B_{n_1+k} \cdots \tilde{A}_{n_1+1}B_{n_1+1} \cdot \tilde{A}_{n_1}B_{n_1} \cdots \tilde{A}_1B_1\| \geq 2.$$

Similarly, we define the constant $\omega_2 > 0$ to satisfy the inequality

$$\tilde{A}_{n_1+n_2}B_{n_1+n_2} \cdots \tilde{A}_1B_1 e \geq \omega_2 e, \quad \forall \{B_i\} \in \mathscr{B}^{n_1+n_2},$$

and now choose an arbitrary number $\omega_2 \geq 3\omega_2^{-1}$. 

Then, as in the previous case, one can specify such an integer $n_3 \geq 1$ and a set of matrices

$$\tilde{A}_{n_1+n_2+1}, \ldots, \tilde{A}_{n_1+n_2+n_3}$$

for which the inequality

$$\max_{1 \leq k \leq n_3} \|\tilde{A}_{n_1+n_2+k}B_{n_1+n_2+k} \cdots \tilde{A}_{n_1+n_2+1}B_{n_1+n_2+1}e\| \geq \omega_2, \quad \forall \{B_{n_1+n_2+i}\} \in \mathscr{B}^{n_3},$$

holds, and so does the inequality

$$\max_{1 \leq k \leq n_1+n_2+n_3} \|\tilde{A}_kB_k \cdots \tilde{A}_1B_1\| \geq 3, \quad \forall \{B_i\} \in \mathscr{B}^{n_1+n_2+n_3}.$$

Acting similarly, for each $m = 1, 2, \ldots$ we can specify an integer $n_m \geq 1$ and have a set of matrices

$$\tilde{A}_{n_1+\cdots+n_m+1}, \ldots, \tilde{A}_{n_1+\cdots+n_m+n_m},$$

for which

$$\max_{1 \leq k \leq n_1+\cdots+n_m} \|\tilde{A}_kB_k \cdots \tilde{A}_1B_1\| \geq m, \quad \forall \{B_i\} \in \mathscr{B}^{n_1+\cdots+n_m}. \quad (27)$$

The resulting estimate contradicts the assumption from Question 1 that for every sequence of matrices $\{A_n\} \in \mathscr{A}^\infty$ there is a sequence of matrices $\{B_n\} \in \mathscr{B}^\infty$ for which the sequence of norms $\{\|A_nB_n \cdots A_1B_1\|, n = 1, 2, \ldots\}$ is bounded. Indeed, due to inequalities (27) for the constructed sequence of matrices $\{A_i\} \in \mathscr{A}^\infty$ and any sequence of matrices $\{B_i\} \in \mathscr{B}^\infty$ the sequence of norms $\{\|\tilde{A}_kB_k \cdots \tilde{A}_1B_1\|\}_{n=1}^\infty$ is unbounded. Theorem 2 is proved. \qed

In [9, Thm. 4] the following statement was made:
Proposition 3. Let the sets of matrices $\mathcal{A}$ and $\mathcal{B}$ be such that for every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ there exists a natural $k$ such that for a set of matrices $B_1, \ldots, B_k \in \mathcal{B}$ the inequality $\|A_k B_k \cdots A_1 B_1\| < 1$ holds. Then there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ there is a sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$, for which $\|A_n B_n \cdots A_1 B_1\| \leq C \lambda^n$ for all $n = 1, 2, \ldots$.

It follows from this statement that if the matrix sets $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Proposition 3, the answer to Question 1 is also positive. Therefore, when analyzing the boundedness of matrix products with factors from two sets of matrices, it suffices to restrict to the case where the conditions of Proposition 3 are not satisfied for the sets of matrices $\mathcal{A}$ and $\mathcal{B}$, i.e., in the set of matrices $\mathcal{A}$ there is a sequence of matrices $\{A_n\}$ such that the inequality $\|A_k B_k \cdots A_1 B_1\| \geq 1$ holds for any set of matrices $B_1, \ldots, B_k \in \mathcal{B}$.

6 Pointwise Setting

It is known [1, 2, 5, 6, 8] that Proposition 1 on boundedness of matrix products remains valid if one replaces in it the assumption that every sequence of norms $\{\|A_n \cdots A_1 x\|\}_{n=1}^\infty$ is bounded, by the assumption that the sequence of norms $\{\|A_n \cdots A_1 x\|\}_{n=1}^\infty$ is ‘pointwise bounded’, that is, it is bounded for every vector $x \in \mathbb{R}^N$ and every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$.

In this context, the question arises of the validity of a similar extension for matrix products with alternating factors. We say that a sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is pointwise $\mathcal{B}$-right-bounded, if for every vector $x \in \mathbb{R}^N$ and every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ there are such a constant $C = C(x, \{A_n\})$ and a sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$ for which the following inequality holds:

$$\|A_n B_n \cdots A_1 B_1 x\| \leq C \|x\|, \quad n = 1, 2, \ldots.$$  

Question 2. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets of matrices such that every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is pointwise $\mathcal{B}$-right-bounded. In this case, is there a universal constant $C > 0$ such that for every vector $x \in \mathbb{R}^N$ and every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ there is a sequence of matrices $\{B_n\} \in \mathcal{B}^\infty$, for which for all $n = 1, 2, \ldots$ the inequalities $\|A_n B_n \cdots A_1 B_1 x\| \leq C \|x\|$ hold? In other words, is it true that if every $\{A_n\} \in \mathcal{A}^\infty$ is pointwise $\mathcal{B}$-right-bounded, then every sequence of matrices $\{A_n\} \in \mathcal{A}^\infty$ is $\mathcal{B}$-right-bounded?

Let us show that in the general case the answer to Question 2 is negative.

Example 1. In [11, Example 2] D.P. Stanford constructed an example of a set $\mathcal{H}$ consisting of two matrices

$$H_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}, \quad H_2 = \begin{bmatrix} \sqrt{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

with the property that for any sequence of matrices $\{A_n\} \in \mathcal{H}^\infty$ the following inequalities hold:

$$\|A_n \cdots A_1\| \geq 1, \quad n = 1, 2, \ldots, \tag{28}$$

and for each vector $x \in \mathbb{R}^2$ there is a sequence of matrices $\{\bar{A}_n\} \in \mathcal{H}^\infty$ such that

$$\bar{A}_n \cdots \bar{A}_1 x \to 0 \quad \text{for} \quad n \to \infty. \tag{29}$$

Relation (28) follows from the inequality

$$\|A\| \geq \rho(A) \geq |\det A|^{1/N}, \tag{30}$$

valid for any $(N \times N)$-matrix $A$, where $\rho(\cdot)$ is the spectral radius of a matrix.

For matrices $H_1$ and $H_2$, the equalities $\det H_1 = \det H_2 = 1$ are obvious, and hence $\det(A_n \cdots A_1) = 1$. 

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From this and from (30) it follows that
\[ \rho(A_n \cdots A_1) \geq \sqrt{\det(A_n \cdots A_1)} = 1, \quad \forall \{A_n\} \in \mathcal{H}_\infty. \]
And then, due to (30),
\[ \|A_n \cdots A_1\| \geq \rho(A_n \cdots A_1) \geq 1, \quad n = 1, 2, \ldots. \]

The construction (for the chosen vector \( x \in \mathbb{R}^2 \)) of a sequence of matrices \( \{\bar{A}_n\} \in \mathcal{H}_\infty \) for which (29) holds is based on the following observation. Since \( H_2 \) is the 30° rotation matrix, for the vector \( x \) there are always at most 6 matrices \( \bar{A}_1, \ldots, \bar{A}_i = H_2 \) (i.e., \( i \leq 6 \)) whose sequential application to the vector \( x \), under the assumption \( \|x\| \leq 1 \), where \( \|\cdot\| \) is the Euclidean norm in \( \mathbb{R}^2 \), leads to the hitting of the vector \( \bar{A}_i \cdots \bar{A}_1 x \) into one of the sectors \( S \) or \( \bar{S} \), each of which has a gap 30° and is symmetric about the abscissa axis, see Fig. 1. In this case, applying the matrix \( \bar{A}_{i+1} = H_1 \) to the corresponding product \( \bar{A}_i \cdots \bar{A}_1 x \) leads, as can be easily calculated, to the hitting of the vector \( x_1 = \bar{A}_{i+1} \cdot \bar{A}_i \cdots \bar{A}_1 x \) into one of the sets \( H_1 S \) or \( H_1 \bar{S} \), and for this vector the following estimation holds:
\[ \|x_1\| \leq q\|x\|, \quad (31) \]
where \( q < 1 \) is a constant, see Fig. 1.

Figure 1: Illustration for the example of Stanford

Continuing this procedure, we obtain a sequence of vectors \( x_n \to 0 \) satisfying the relations \( \|x_n\| \leq q\|x_{n-1}\| \), from which follows (29).

Remark 1. The procedure described remains in force if instead of the matrices \( H_1, H_2 \) the matrices \( \alpha H_1, \alpha H_2 \) are taken, with the constant \( \alpha > 1 \) chosen such that the validity of the estimate (31) with a constant \( q = q_{\alpha} < 1 \) is still guaranteed. In this case, as before, for each vector \( x \in \mathbb{R}^2 \) there is a sequence of matrices \( \{A_n\} \in \mathcal{H}_\infty \) for which the convergence (29) is valid. But since now \( \det A_i = \alpha^2 \), instead of (28) a stronger statement holds:
\[ \|A_n \cdots A_1\| \geq \rho(A_n \cdots A_1) \geq \sqrt{\det(A_n \cdots A_1)} = \alpha^n \to \infty \quad \text{for} \quad n \to \infty. \]

Example 2. In addition to Example 2 from [11] cited above, we note without proof another example of the set \( \mathcal{H} \) for which the relations (28) and (29) hold, see [4, p. 50]. Denote by \( \mathcal{H} \) the set consisting of two matrices
\[
H_1 = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}, \quad H_2 = \begin{bmatrix} \cos \beta & -\gamma \sin \beta \\ \gamma^{-1} \sin \beta & \cos \beta \end{bmatrix},
\]
where \( \gamma > 0, \gamma \neq 1 \) and the quantities \( \alpha \) and \( \beta \) are incommensurable with \( \pi \).

In this case, for any sequence of matrices \( \{A_n\} \in \mathcal{H}_\infty \) relations (28) hold, i.e., \( \|A_n \cdots A_1\| \geq 1 \) for \( n = 1, 2, \ldots, \) and furthermore, for each nonzero vector \( x \in \mathbb{R}^2 \), one can specify a sequence of
matrices \( \{\tilde{A}_n\} \in \mathcal{H}^\infty \) for which convergence \((29)\) takes place: \( \tilde{A}_n \cdots \tilde{A}_1 x \to 0 \) for \( n \to \infty \). Moreover, for any nonzero vector \( x \in \mathbb{R}^2 \), one can also specify a sequence of matrices \( \{\tilde{A}_i \in \mathcal{H}\} \) for which the divergence takes place: \( \|\tilde{A}_n \cdots \tilde{A}_1 x\| \to \infty \) for \( n \to \infty \).

Now we can return to Question \((2)\) and answer it negatively. To do this, it suffices to consider the following sets of \((2 \times 2)\)-matrices:

\[
\mathcal{A} := \{I\}, \quad \mathcal{B} := \{H_1, H_2\},
\]

where \(H_1, H_2\) are matrices of Remark \([1]\). As follows from Remark \([1]\) in this case any sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) (and such a sequence is unique: \( A_n \equiv I \)) is pointwise \( \mathcal{B} \)-right-bounded, but not \( \mathcal{B} \)-right-bounded, since due to Remark \([1]\)

\[
\|A_n B_n \cdots A_1 B_1\| = \|B_n \cdots B_1\| \geq \alpha^n \to \infty \quad \text{for} \quad n \to \infty,
\]

for any sequence of matrices \(B_n \in \mathcal{B} = \{H_1, H_2\}\).

The example constructed permits the following generalization.

**Theorem 3.** Let \( \mathcal{A} \) be a finite set of \((2 \times 2)\)-matrices whose determinants are equal to 1. Then there exists a finite set \( \mathcal{B} \) of nondegenerate \((2 \times 2)\)-matrices such that every sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) is pointwise \( \mathcal{B} \)-right-bounded, but not \( \mathcal{B} \)-right-bounded.

**Proof.** Let us choose arbitrary matrices \(H_1, H_2\), defined in Remark \([1]\). Furthermore, for each matrix \(A_n \in \mathcal{A}\) we construct two matrices

\[
B_{1,n} = A_n^{-1} H_1, \quad B_{2,n} = A_n^{-1} H_2
\]

and define \( \mathcal{B} \) to be the set of all matrices of the form \( B_{1,n}, B_{2,n} \).

Now note that due to the definition of the matrices \(H_1, H_2\) in Remark \([1]\),

\[
\det H_1 = \det H_2 = \alpha^2 > 1,
\]

which means that

\[
\det A_n = 1, \quad \det B_n = \alpha^2 > 1 \quad \forall \ A_n \in \mathcal{A}, B_n \in \mathcal{B}.
\]

Therefore,

\[
\|A_n B_n \cdots A_1 B_1\| \geq \sqrt{\det(A_n B_n \cdots A_1 B_1)} = \alpha^n \to \infty \quad \text{for} \quad n \to \infty,
\]

for any sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) and any sequence of matrices \( \{B_n\} \in \mathcal{B}^\infty \), from which it follows that no sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) is \( \mathcal{B} \)-right bounded.

It remains to show that nevertheless any sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) is pointwise \( \mathcal{B} \)-right-bounded. To do this, we specify any sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \), and a nonzero vector \( x \in \mathbb{R}^2 \). Next, we choose a sequence of matrices \( H_{i_n} \) that satisfy the relation

\[
H_{i_n} \cdots H_{i_1} x \to 0. \tag{32}
\]

This is possible because of the remark \([1]\). Finally, we construct a sequence of matrices \( \{B_n\} \in \mathcal{B}^\infty \) by taking

\[
B_n = B_{i_n,n} = A_n^{-1} H_{i_n}.
\]

With the chosen sequence of matrices \( \{B_n\} \in \mathcal{B}^\infty \), the corresponding sequence \( A_n B_n \cdots A_1 B_1 x \) takes the following form

\[
A_n B_n \cdots A_1 B_1 x = H_{i_n} \cdots H_{i_1} x,
\]

whence, due to \((32)\) \( A_n B_n \cdots A_1 B_1 x \to 0 \). The pointwise \( \mathcal{B} \)-right-boundedness of the sequence of matrices \( \{A_n\} \in \mathcal{A}^\infty \) is proved. \( \square \)
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