Stability of Closed Timelike Curves in the Gödel Universe

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We study, in some detail, the linear stability of closed timelike curves in the Gödel universe. We show that these curves are stable. We present a simple extension (deformation) of the Gödel metric that contains a class of closed timelike curves similar to the ones associated to the original metric. This extension correspond to the addition of matter whose energy-momentum tensor is analyzed. We find the conditions to have matter that satisfies the usual energy conditions. We study the stability of closed timelike curves in the presence of usual matter as well as in the presence of exotic matter (matter that does satisfy the above mentioned conditions). We find that the closed timelike curves in the Gödel universe with or without the inclusion of regular or exotic matter are stable under linear perturbations. We also find a sort of structural stability.

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I. INTRODUCTION

The Gödel universe, a spacetime in which the matter takes the form of a pressure-free perfect fluid with a negative cosmological constant, is the most celebrated solution of Einstein field equations that contains closed timelike curves (CTCs). This spacetime has a five-dimensional group of isometries which is transitive. The matter everywhere rotates relative to the compass of inertia with the angular velocity proportional to the square root of the matter density [1] [2]. Dynamical conditions for time traveling in this spacetime are not sufficient to exclude the existence of CTCs [3].

The Gödel metric has some qualitative features like the projections of geodesics onto the 2-surface \((r, \phi)\) being simple closed curves. This property can be extended to a set of metrics called Gödel-type. It is possible to show explicitly that when the characteristic vector field that defines a Gödel-type metric is also a Killing vector we have closed timelike or null curves [4]. The study of the geodesic motion of free test particles in these Gödel-type spacetimes can be extended to a family of homogeneous Gödel-type spacetimes [5]. Güres et al. [6] introduce and used Gödel-types metrics to find charged dust solutions to the Einstein field equations in D dimensions. In the Gödel spacetime timelike geodesics are not closed as was independently proved by Kunt [7] and Chandrashekhar and Wrigth [8]. However there exist timelike curves subject to an external force that are closed.

Exact solutions of Einstein-Maxwell equations that contain CTCs at least for some values of the parameters are studied in [9]. It turns out that magnetic fields can give rise to non-trivial chronology violations. A sufficiently large magnetic field can always ensure chronology violation. The spacetime of an infinite rotating cylindrical shell of charged perfect fluid contains CTCs [10]. For the conditions for the existence of CTCs in the spacetime associated to a rigidly rotating cylinder of charged dust see Ivanov [11]. In spacetimes with conic singularities that represent cosmic strings we can have CTCs for one spinning string [12] and for two parallel moving strings [13].

The stability of Gödel’s cosmological model with respect to scalar, vector, and tensor perturbation modes using a gauge covariant formalism is considered in [14]. It is found that the background vortical energy contributes to the gravitational pull of matter, while gradients add to the pressure support. The balance between these two agents effectively determines the stability of Gödel universe against matter aggregations.

The possibility to have no violation of causality for geodesics that in finite intervals of time goes back in time \((dt/ds < 0)\) was considered by [15]. As it was already mentioned, the existence of CTCs contradicts the usual notion of causality. Beyond the usual paradoxes, it seems to induce physical impossibilities, like the necessity to work with negative energy densities. One could speculate that these impossibilities will be eliminated by quantum-gravitational
effects. All our experience seems to indicate that the physical laws do not allow the appearance of CTCs. This is that, essentially, says the Chronology Protection Conjecture (CPC) proposed by Hawking in 1992 [16].

Other possible explanation for CTCs is that the metrics studied until now are unrealistic [17]. But, there exists an exact, asymptotically flat solution of vacuum Einstein equations which contains CTCs that can represent the exterior of a spinning rod of finite length [18].

The existence of CTCs, in principle, should be a matter of experimentation. If the General Relativity predicts them in a physically reasonable situation and they are not found, we will have that this theory is in trouble. If they are found we will have a bigger problem, our usual notion of causality will need a deep revision.

The possibility that a spacetime associated to a realistic model of matter may contain CTCs leads us to ask how permanent is the existence of these curves. Perhaps, one may rule out the CTCs by simple considerations about their linear stability. Otherwise, if these curves are stable under linear perturbations the conceptual problem associated to their existence is enhanced.

In this first paper about linear stability of CTCs we consider the Gödel case due to its historical relevance (first paradigm for CTCs) as well as its mathematical simplicity. We intend to study, in subsequent works, the stability of CTCs in Gödel-type metrics [4, 6], Bonnor metric [18] and other relevant spacetimes. All these metrics present their own peculiarities that deserve a special treatment.

The paper is divided as follows, in Section 2, we review in some detail the CTCs in the Gödel metric. In particular we present the CTCs and its corresponding forces in the usual Gödel coordinates as well as in Cartesian coordinates. In Section 3 we study the stability of CTCs when they suffer a small perturbation. We solve the linear system of equations for the evolution of the perturbation in the original Gödel coordinates in an exact form. For future reference the same system in Cartesian coordinates is solved numerically for a representative range of initial conditions. In Section 4 we present a simple extension (deformation) of the Gödel metric that corresponds to the addition of matter. We look for the conditions for geodesics to be timelike and to be closed. We prove that these two conditions can not be satisfied at the same time. We present the CTCs and its corresponding forces in the usual Gödel coordinates as well as in Cartesian coordinates.

II. GÖDEL METRIC, GEODESICS AND CTCs

In this section we review the CTCs in Gödel metric following mainly references [1] [3] [7]. In standard coordinates, \( X^\mu = (\tilde{t}, r, \varphi, z) \), the Gödel metric is [3],

\[
ds^2 = \frac{4}{\beta^2} \left[ d\tilde{t}^2 - dr^2 + \left( \sinh^4 r - \sinh^2 r \right) d\varphi^2 + 2\sqrt{\beta} \sinh^2 r \, dt \, d\varphi \right] - dz^2, \tag{1}
\]

where \( \beta \) is a constant that describes the vorticity of the four-vector \( u^\mu = (1, 0, 0, 0) \), \(-\infty < \tilde{t} < \infty, -\infty < z < \infty, r > 0 \) and \( \varphi \in [-\pi, \pi] \). The limits \( \varphi = -\pi \) and \( \varphi = \pi \) are topologically identified.

The metric (1) satisfies the Einstein field equations,

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -T_{\mu\nu}, \tag{2}
\]

with \( T^{\mu\nu} = \rho u^\mu u^\nu \), \( \Lambda = -\frac{\beta^2}{2} \) and \( \rho = \beta^2 \). We use units such that \( c = 8\pi G = 1 \).

Let us denote by \( \gamma \) the particular closed curve given in its parametric form by

\[
\tilde{t} = \tilde{t}_0, \quad r = r_0, \quad \varphi \in [-\pi, \pi], \quad z = 0, \tag{3}
\]

where \( t_0 \) and \( r_0 \) are constants. When \( \gamma \) is parametrized with an arbitrary variable \( u \) the condition to be timelike is \( \frac{dX^\mu}{du} \frac{dX^\nu}{du} > 0 \). The curve \( \gamma \) is timelike when \( r_0 > \log(\sqrt{2} + 1) \). The four-acceleration of this CTC is

\[
\bar{a}^\mu = \delta_0^\mu \sinh r_0 \cosh r_0 \left( 2 \cosh^2 r_0 - 3 \right) \varphi^2. \tag{4}
\]

In the “Cartesian” coordinates \( X^\mu = (t, x, y, z) \) defined by

\[
y = (\cosh(2r) + \sinh(2r) \cos \varphi)^{-1}, \tag{5}
\]

\[
\beta x = (\sinh(2r) \sin \varphi) \, y, \tag{6}
\]

\[
\frac{\varphi}{2} + \left( \frac{\beta t - 2\tilde{t}}{2\sqrt{2}} \right) = \arctan \left( e^{-2r} \tan \frac{\varphi}{2} \right), \tag{7}
\]

\[
z = z, \tag{8}
\]
with $-\infty < t < \infty$, $-\infty < x < \infty$, $-\infty < z < \infty$, and $\infty > y > 0$ we find that the line element (1) reduces to

$$ds^2 = \left[ dt + \frac{\sqrt{2} \, dx}{\beta y} \right]^2 - \frac{dx^2 + dy^2}{(\beta y)^2} - dz^2.$$  \hfill (9)

In these coordinates the geodesic equation gives us,

\begin{align}
\ddot{t} &= \frac{2}{y} \dot{y} + \frac{\sqrt{2}}{\beta y^2} \dot{x} \dot{y}, \\
\ddot{x} &= -\beta \sqrt{2} \, \dot{t} \dot{x}, \\
\ddot{y} &= \beta \sqrt{2} \, \dot{t} \dot{x} + \frac{x^2}{y} + \frac{y^2}{y}, \\
\ddot{z} &= 0,
\end{align}

where the overdot indicates derivation with respect to the proper time $s$. This system of equations admits the first integrals,

\begin{align}
\dot{t} &= \frac{2y - y_0}{\sqrt{2C}}, \\
\dot{x} &= \frac{\beta y}{C} (y_0 - y), \\
\dot{y} &= \frac{\beta y}{C} (x - x_0), \\
\dot{z} &= \frac{d}{C},
\end{align}

where $x_0$, $y_0$, $d$ and $C$ are integration constants. The solution of this system is

\begin{align}
t &= \frac{\sqrt{2}}{\beta} \left[ 2 \arctan \left( \sqrt{N} \tan \sigma \right) - \frac{N + 1}{2 \sqrt{N}} \sigma \right] + t_0, \\
x &= \frac{2\sqrt{N} \, \eta \, \sin \sigma \cos \sigma}{\cos^2 \sigma + N \sin^2 \sigma} + x_0, \\
y &= \frac{y_0 - \eta}{\cos^2 \sigma + N \sin^2 \sigma}, \\
z &= \frac{2\sigma \, d}{\beta \sqrt{y_0^2 - \eta^2}} + z_0,
\end{align}

with

\begin{align}
\sigma &= \frac{\beta \sqrt{y_0^2 - \eta^2}}{2C} (s - s_0), \\
N &= \frac{y_0 - \eta}{y_0 + \eta},
\end{align}

where $t_0$ is an integration constant, and $\eta$ is given by $\eta^2 := (x - x_0)^2 + (y - y_0)^2$.

The relation $\dot{X}^a \dot{X}_a = 1$ (with $z = 0$) reduces to

$$\left( \frac{t + \sqrt{2}}{\beta y} \frac{\dot{t}}{\dot{x}} \right)^2 - \frac{\dot{x}^2 + \dot{y}^2}{(\beta y)^2} = 1.$$  \hfill (24)

From (14)-(16) we get

$$y_0^2 = 2(C^2 + \eta^2).$$  \hfill (25)

Therefore

$$y_0^2 > 2\eta^2.$$  \hfill (26)
The condition for a geodesic to be closed, i.e., \( t(-\pi/2) = t(\pi/2), x(-\pi/2) = x(\pi/2) \) \( \text{e} \) \( y(-\pi/2) = y(\pi/2) \), is

\[
4\sqrt{N} = N + 1 \Rightarrow \eta_0^2 = \frac{4}{3}\eta^2
\]  

(27)

that contradicts (26). Hence we have no timelike closed geodesics in the Gödel universe.

In order to obtain a CTC in Cartesian coordinates we use another differential equation for \( t \) obtained from (24) and equations (19) and (20). Then

\[
\left[ \dot{t} + \frac{\sqrt{2}}{C}(y_0 - y) \right]^2 = \frac{C^2 + \eta^2}{C^2},
\]

(28)

whose solution is

\[
t = \frac{2\sqrt{2}}{\beta} \left[ \arctan \left( \sqrt{N} \tan \sigma \right) - \frac{y_0 \sqrt{2} - \sqrt{C^2 + \eta^2}}{\sqrt{2}(y_0^2 - \eta^2)} \right] + t_0.
\]

(29)

These timelike curves are closed when

\[
C^2 = 2(y_0 - \sqrt{y_0^2 - \eta^2})^2 - \eta^2.
\]

(30)

By replacing these parametric equations for the CTC into the geodesic equations we find that the four-acceleration satisfies the relations

\[
a^0 = -(x - x_0)\frac{\lambda \sqrt{2}}{\beta}, \quad a^1 = \lambda y(x - x_0), \quad a^2 = -\lambda y(y_0 - y), \quad a^3 = 0,
\]

(31)

where

\[
\lambda = \frac{\beta^2}{C^2} \left[ \sqrt{2(C^2 + \eta^2)} - y_0 \right].
\]

(32)

The right hand side of (31) can be interpreted as the components of a specific external force \( F^\mu \) associated to \( \gamma \).

It is instructive to compare the force given in these two systems of coordinates. In Gödel standard coordinates we have \( g_{\varphi \varphi} \varphi^2 = 1 \). Therefore along the CTC \( \gamma \) we have

\[
\varphi^2 = \frac{\beta^2}{4(\sinh^4 r_0 - \sinh^2 r_0)}.
\]

(33)

From equation (11) we can write the non zero component of the force in following way,

\[
\hat{F}^1 = \frac{\beta^2 \sinh r_0 \cosh (2 \cosh^2 r_0 - 3)}{4(\sinh^4 r_0 - \sinh^2 r_0)} = \frac{\beta^2 \sinh 2r_0(\cosh 2r_0 - 2)}{2(\cosh 2r_0 - 2)^2 - 1}.
\]

(34)

The non zero components of force in Cartesian coordinates are,

\[
F^0 = -\frac{2\sqrt{2}}{\eta \beta} (x - x_0) \hat{F}^1,
\]

(35)

\[
F^1 = \frac{2y}{\eta} (x - x_0) \hat{F}^1,
\]

(36)

\[
F^2 = -\frac{2y}{\eta} (y_0 - y) \hat{F}^1,
\]

(37)

\[
F^3 = 0.
\]

(38)

On \( \gamma \) the constants in (22) and (23) are now \( N = e^{-2r_0}, y_0 = \cosh 2r_0 \) and \( \eta = \sinh 2r_0 \). Therefore

\[
\hat{F}^1 = \frac{\beta^2 y(y_0 - 2)}{2((y_0 - 2)^2 - 1)}.
\]

(39)
By using the condition (33) we can cast \( \lambda \) as

\[
\lambda = \frac{\beta^2(y_0 - 2)}{(y_0 - 2)^2 - 1}.
\]

Hence

\[
\tilde{F}^1 = \frac{\eta}{2} \lambda,
\]

that is force calculated in (31).

### III. LINEAR PERTURBATION OF CTCS IN THE GöDEL UNIVERSE

A generic CTC \( \gamma \) satisfies the system of equations given by

\[
\frac{D}{ds} \dot{X}^\mu = F^\mu(X),
\]

where \( \frac{D}{ds} \) is the covariant derivative of the vector field \((\cdot)^\mu\) along \(\gamma(s)\) and \(F^\mu\) is a given external force.

Let \( \tilde{\gamma} \) be the curve obtained from \( \gamma \) after a perturbation \( \xi, \lambda \), i.e., \( \tilde{X}^\mu = X^\mu + \xi^\mu \). Let \( e_\alpha \) be a given basis. In this basis \( F^\mu \) is represented by the equation \( \frac{Du}{ds} = F \), where \( \frac{Du}{ds} = (\frac{d_\alpha}{ds} + \Gamma^\alpha_{\beta \mu} u^\beta u^\mu)e_\alpha \), \( F = F^\alpha e_\alpha \), \( u = u^\alpha e_\alpha \) and \( u^\alpha = X^\alpha \). In order to find the behavior of \( \xi \) we calculate the variation, in first approximation, of both sides of \( \frac{Du}{ds} = F \). We find,

\[
\frac{\delta D u}{ds} = \left( \frac{d^2 \xi^\alpha}{ds^2} + 2 \Gamma^\alpha_{\beta \mu} \delta u^\beta u^\mu + \Gamma^\alpha_{\beta \mu, \lambda} \xi^\lambda u^\beta u^\mu \right) e_\alpha + F^\alpha \delta e_\alpha
\]

\[
\delta F = F^\alpha_{, \beta} \delta e_\alpha + F^\alpha \delta e_\alpha.
\]

where \( (),_\lambda = \frac{\partial (\cdot)}{\partial x^\lambda} \). Comparing the last two equations we get the system of differential equation satisfied by the perturbation \( \xi \)

\[
\frac{d^2 \xi^\alpha}{ds^2} + 2 \Gamma^\alpha_{\beta \mu} \delta u^\beta u^\mu + \Gamma^\alpha_{\beta \mu, \lambda} \xi^\lambda u^\beta u^\mu = F^\alpha \xi^\beta.
\]

This last equation can be cast in a manifestly covariant form by noticing that its left-hand-side is the well known geodesic deviation equation as pointed out in [19] and its right-hand-side is the Lie derivative of the force along \( \xi^\mu \).

In standard coordinates \( \tilde{X}^\mu = [t, \rho, \varphi, z] \), the system (45) reduces to

\[
\begin{align*}
\dot{\xi}^0 + a \xi^1 &= 0, \\
\dot{\xi}^1 + b \xi^0 + c \xi^2 + d \xi^1 &= 0, \\
\dot{\xi}^2 + e \xi^1 &= 0, \\
\xi^3 &= 0,
\end{align*}
\]

where

\[
\begin{align*}
a &= (\beta \sinh^2 r_0 \sqrt{2}) / (\cosh r_0 \sqrt{\sinh^2 r_0 - 1}), \\
b &= \beta \sqrt{2} \cosh r_0 / \sqrt{\sinh^2 r_0 - 1}, \\
c &= \beta \cosh r_0 (2 \cosh^2 r_0 - 3) / \sqrt{\sinh^2 r_0 - 1}, \\
d &= \beta^2 \cosh^2 r_0 (2 \cosh^2 r_0 - 3)^2 / (2 \sinh^2 r_0 (\sinh^2 r_0 - 1)^2), \\
e &= \beta / (\sinh r_0 \cosh r_0 \sqrt{\sinh^2 r_0 - 1})
\end{align*}
\]

The solution of system of equations (46) is,

\[
\begin{align*}
\xi^0 &= -a(c_3 \sin(\omega s + c_4) / \omega + \tau s) + c_1 s + c_5, \\
\xi^1 &= c_3 \cos(\omega s + c_4) + \tau, \\
\xi^2 &= -e(c_3 \sin(\omega s + c_4) / \omega + \tau s) + c_2 s + c_6, \\
\xi^3 &= c_7 s + c_8,
\end{align*}
\]
where $c_i$, $i = 1, \ldots, 8$ are integration constants, $\omega = \sqrt{d - ab - ce}$, and $\tau = -(bc_1 + cc_2)/\omega^2$. In order that the perturbed curve, $\tilde{\gamma}$, remains on the plane $z = 0$ we take initial conditions such that $c_7 = c_8 = 0$, i.e., $\xi^3 = 0$. The solution (48) shows the typical behavior for stability, we have vibrational modes untangled with translational ones that can be eliminated by a suitable choice of the initial conditions.

For future reference, we study the linear stability of the CTCs in Cartesian coordinates. In these coordinates the system (45) can be written as

\[
\dot{\xi}^0 = a_0\xi^0 + a_0^1\xi^1 + a_0^2\xi^2 + b_0^1\xi^1 + b_0^2\xi^2 \\
\dot{\xi}^1 = a_1^0\xi^0 + a_1^1\xi^1 + a_1^2\xi^2 + b_1^1\xi^1 + b_1^2\xi^2 \\
\dot{\xi}^2 = a_2^0\xi^0 + a_2^1\xi^1 + a_2^2\xi^2 + b_2^1\xi^1 + b_2^2\xi^2 \\
\dot{\xi}^3 = 0,
\]

(49)

where $(a_{ij})$ and $(b_{ij})$ are given by

\[
(a_{ij}) = \frac{1}{C} \begin{bmatrix}
\frac{2\beta Se^{-2r_0}}{D} & \sqrt{2}S & \frac{\sqrt{2}}{2}(1 + (y_0 - 2)De^{2r_0}) \\
-\sqrt{2}\beta^2 e^{-4r_0}D^2 & 0 & -\beta \left( \frac{e^{-2r_0}}{D} - 1 \right) \\
\frac{\sqrt{2}\beta e^{-2r_0}}{2D} \left( y_0 - \frac{e^{-2r_0}}{D} \right) & \beta(y_0 - 1) & \frac{2\beta Se^{-2r_0}}{D}
\end{bmatrix},
\]

(50)

\[
(b_{ij}) = \begin{bmatrix}
-\lambda\sqrt{2} \beta & \frac{y_0 - 1}{C^2} 4\sqrt{2}S \\
\frac{e^{-2r_0}}{\cos^2 \sigma + e^{-4r_0} \sin^2 \sigma} & \frac{2e^{-2r_0}S}{\cos^2 \sigma + e^{-4r_0} \sin^2 \sigma} \\
0 & \lambda \left[ \frac{2e^{-2r_0}}{\cos^2 \sigma + e^{-4r_0} \sin^2 \sigma} - y_0 \right] - \eta^2 \beta^2 \frac{C^2}{2}
\end{bmatrix},
\]

(51)

and

\[
D = \cos^2 \sigma + e^{-4r_0} \sin^2 \sigma, \quad S = \eta \sin \sigma \cos \sigma.
\]

(52)

As before, we take $\xi^3 = 0$ in order that $\dot{\gamma}$ does not leave the plane $z = 0$.

To describe the behavior of perturbed curve $\tilde{\gamma}$ we introduce two distance and one angle functions,

\[
R_2^2 = (x - x_0)^2 + (y - y_0)^2, \quad (53) \\
R_3^2 = (t - t_0)^2 + (x - x_0)^2 + (y - y_0)^2, \quad (54) \\
\phi = \arctan \frac{y - y_0}{x - x_0}, \quad (55)
\]

The first function is a constant equal to $\eta$ when $x$ and $y$ are on the CTC $\gamma$. The second represent a “radius” in spacetime and the third one is an angle on the usual space. The variation of these functions along $\gamma(s)$ are,

\[
\delta R_2 = [(x - x_0)\xi^1 + (y - y_0)\xi^2]/\eta, \quad (56) \\
\delta R_3 = [(t - t_0)\xi^0 + (x - x_0)\xi^1 + (y - y_0)\xi^2]/R_3, \quad (57) \\
\delta \phi = [(x - x_0)\xi^2 - (y - y_0)\xi^1]/\eta. \quad (58)
\]

To study these functions we solve the system (49) by running the independent variable $s$, first from $-\pi/2$ to 0 and second from $\pi/2$ to 0, we recall that the points $-\pi/2$ and $\pi/2$ on the curve are identified. We keep the initial position of the perturbation equal to zero and analyze the behavior of $\delta R_2$, $\delta \phi$ and $\delta R_3$. The perturbation initial velocity is taken each time with only one component different from zero.

In Fig. 1 we show graphics that represent the variation of radius $R_2$ of perturbed curve, with $r_0 = 1.5(\eta \sim 10)$ and $\beta = 1$. For the first graph the initial conditions are: a) $\xi^0(-\frac{\pi}{2}) = [0, 0, 0, 0], \xi^0(-\frac{\pi}{2}) = [10^{-3}, 0, 0, 0]$ (solid line), b)
Variation of the radius $R$ and stability of the CTCs when we add matter with pressure or tension.

In Figs. 2 and 3 we show graphics that represent $\delta \phi$ and $\delta R$, respectively, computed with the same initial conditions used for $\delta R_2$.

We consider a generic perturbation $\delta u$ small when $\frac{(\delta u)^2}{u_0^2} \leq 0.01$, i.e., $(\delta u)^2 << \delta u$. All the variations presented in the graphics satisfy by large this smallness conditions. We have typical vibrational modes for $\delta R_2$ and $\delta R_3$ and mainly translational modes for $\delta \phi$. We tested this smallness condition for a significant variety of initial conditions and values of $r_0$, we find that this condition is always satisfied even for curves that are near the bound given by $r_0 > \log(\sqrt{2} + 1)$.

### IV. CTCS AND MATTER

The material source of the Gödel metric is a pressure-free perfect fluid. In this section we study the persistence and stability of the CTCs when we add matter with pressure or tension.

Let us consider the metric,

$$ds^2 = (dt + \frac{\sqrt{2}h_1}{\beta y}dx)^2 - \frac{h_2^2}{(\beta y)^2}(dx^2 + dy^2) - dz^2$$

that for $h_1 = h_2 = 1$ reduces to Gödel metric. This metric can be considered as a deformation of the original Gödel metric (See Appendix A).
First, to understand the physical meaning of the changes introduced in \[59\] we compute the associated energy-momentum tensor from the Einstein field equations with the same cosmological constant as in the Gödel metric \((\Lambda = -\beta^2/2)\). We get,

\[
(T^\mu_\nu) = \begin{bmatrix}
\frac{\beta^2(3h_2^2 - 2h_1^2 + h_4^2)}{2h_2^2} & \frac{\sqrt{2}h_1(2h_2^2 - h_1^2)}{2h_2^2} & 0 & 0 \\
0 & \frac{\beta^2(h_2^2 - h_1^2)}{2h_2^2} & 0 & 0 \\
0 & 0 & \frac{\beta^2(h_1^2 + h_4^2)}{2h_2^2} & 0 \\
0 & 0 & 0 & \frac{\beta^2(h_1^2 - 2h_2^2 + h_4^2)}{2h_2^2}
\end{bmatrix}.
\]  

(60)

By solving the eigenvector equation,

\[
T^\mu_\nu \xi^\nu = \lambda \xi^\mu,
\]

we find the eigenvalues: \(\lambda_0 = \frac{AB^2}{B}\), \(\lambda_1 = \lambda_2 = \frac{C\beta^2}{B}\) and \(\lambda_3 = \frac{D\beta^2}{B}\), where \(\Lambda = 3h_1^2 - 2h_2^2 + h_4^2\), \(B = 2h_2^4\), \(C = h_2^4 - h_1^2\) and \(D = h_1^2 - 2h_2^2 + h_4^2\). The timelike eigenvector \(u^\mu = [1, 0, 0, 0]\) is associated to \(\lambda_0\), and the spacelike eigenvectors \(X^\mu = [-\frac{\sqrt{2}h_1}{h_2}, \frac{\beta y}{h_2}, 0, 0]\), \(Y^\mu = [0, 0, \frac{\beta y}{h_2}, 0]\), and \(Z^\mu = [0, 0, 0, 1]\) are associated to \(\lambda_1\), \(\lambda_2\), and \(\lambda_3\), respectively. We can write the energy-tensor in its canonical form as,

\[
T^{\mu\nu} = \lambda_0 u^\mu u^\nu + \lambda_1 (X^\mu X^\nu + Y^\mu Y^\nu) + \lambda_3 (Z^\mu Z^\nu).
\]  

(62)

In order to have realistic matter, the eigenvalue \(\lambda_0\) that represents energy density, and the eigenvalues \(\lambda_1 = \lambda_2\) and \(\lambda_3\) that describe pressures (or tensions) are restricted. \(\lambda_0\) must be non-negative (weak energy condition) and \(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3 > 0\) (strong energy condition) \[2\]. The weak energy condition is satisfied when \(h_1^2 > \frac{2h_2^2 - h_4^2}{3}\). If, furthermore, we have \(h_1^2 > \frac{h_4^2}{3}\) the strong energy condition is also satisfied.

Simple modifications as the one presented in \[59\] of the Gödel metric written in standard coordinates are not associated with an energy-momentum tensor with a simple physical interpretations nor to geodesic equations that can exactly be solved. This justify our use of Cartesian coordinates.

Returning to the analysis of the curves in this new geometry, we have that the geodesic equations are,

\[
\ddot{t} - \frac{2h_1^2}{y h_2^2} \dot{t} \dot{y} - \sqrt{2}h_1(2h_1^2 - h_2^2) \dot{x} \dot{y} = 0
\]  

(63)

\[
\ddot{x} + \frac{\beta \sqrt{2} h_1}{h_2^2} \dot{t} \dot{y} + \frac{2 \dot{x} \dot{y}}{y h_2^2} (h_1^2 - h_2^2) = 0
\]  

(64)

\[
\ddot{y} - \frac{\beta \sqrt{2} h_1}{h_2^2} \dot{t} \dot{x} - \frac{2h_1^2}{y h_2^2} (\dot{x})^2 - \frac{1}{y} \dot{y}^2 = 0
\]  

(65)

\[
\ddot{z} = 0.
\]  

(66)

FIG. 3: Variation of \(R_3\) with same initial conditions of Fig. 1.
We have timelike geodesics for

\[ i = \frac{1}{\sqrt{2Ch_1}}[2y h_1^2 - y_0(2h_1^2 - h_2)], \]

(67)

Therefore

\[ t = \frac{\sqrt{2}}{\beta}[2\arctan(\sqrt{N} \tan \sigma) h_1 - \frac{(2h_1^2 - h_2)(N + 1)}{h_1} \frac{1}{2\sqrt{N}} \sigma] + x_0^0. \]

(68)

We have timelike geodesics for

\[ y_0 = \cosh 2r_0 < \frac{\sqrt{2}h_1}{\sqrt{2h_1^2 - h_2^2}}, \]

(69)

that are closed whenever

\[ y_0 = \cosh 2r_0 = \frac{2h_1^2}{2h_1^2 - h_2^2}. \]

(70)

Hence, as before, we have no closed timelike geodesics in the spacetime whose metric is given by (69). In this metric we can find equations for CTCs in the same way as before. For \( z = 0 \), we have

\[ t = \frac{2\sqrt{2}}{\beta} \left[ \arctan \left( \sqrt{N} \tan \sigma \right) h_1 - \frac{\sqrt{2y_0 h_1 - \sqrt{C^2 + h_2^2 \eta^2}}}{\sqrt{2y_0^2 - \eta^2}} \sigma \right] + t_0. \]

(71)

The condition for closed curves is

\[ C^2 = 2h_1^2(y_0 - \sqrt{y_0^2 - \eta^2})^2 - h_2^2 \eta^2. \]

(72)

The components of force are,

\[ F_p^0 = -(x - x_0) \frac{\lambda(h_1, h_2) \sqrt{2}}{\beta}, \]

(73)

\[ F_p^1 = (x - x_0) y \lambda(h_1, h_2), \]

(74)

\[ F_p^2 = -y \lambda(h_1, h_2)(y_0 - y), \]

(75)

\[ F_p^3 = 0, \]

(76)

where

\[ \lambda(h_1, h_2) = \frac{\beta^2(\sqrt{2} \sqrt{C^2 + h_2^2 \eta^2} h_1 - y_0 h_1^2)}{C^2 h_2}. \]

(77)

We have CTCs when \( C^2 > 0 \). From (72) we find the relation between \( h_1 \) and \( h_2 \) to have CTCs,

\[ h_1^2 > \frac{\eta^2 h_2^2}{2(y_0 - 1)^2}. \]

(78)

From the equations for the CTCs it is possible to analyze its stability under linear perturbation. As before, we write the coordinates of the perturbed curve as \( X^\mu = X^\mu + \xi^\mu \) and find a system like (15), where \((a_{ij}) \), \((b_{ij}) \) are now given by

\[ (a_{ij}) = \frac{1}{Ch_2^2} \begin{bmatrix} \frac{4\beta Se^{-2\eta h_1^2}}{D} & 2\sqrt{2}Sh_1 N_1 & \frac{2\sqrt{2}h_1 (2h_1 N_1+ (y_0 N_1 - 2h_1) D e^{2\eta})}{D} \\ -2\sqrt{2} Se^{-4\eta h_1^2} & \frac{4\beta Se^{-2\eta N_2}}{D} & \frac{-2\beta(e^{-2\eta}(N_2 + h_1) D (h_1 + N_2 y_0))}{D} \\ \sqrt{2} Se^{-2\eta h_1 (y_0 D - e^{-2\eta})} & 2\beta(e^{-2\eta}(h_1 - N_2) D (h_1^2 - N_1 y_0)) & \frac{2\beta Se^{-2\eta}}{D} \end{bmatrix}, \]

(79)
$$\begin{align*}
(b_{ij}) = & \begin{bmatrix}
-\frac{\lambda \sqrt{2}}{\beta} & -\frac{e^{-2r_0}(N_1 - h_1) + D(h_1 - N_1y_0)}{Dh_2^2C^2} + \sqrt{2} \beta S \\
\frac{e^{-2r_0} \lambda}{D} & \frac{e^{-2r_0} S}{D} \left[ \frac{4\beta^2 N_2}{h_2^2C^2} \frac{e^{-2r_0}}{D} - y_0 + 2\lambda \right] \\
0 & \lambda \left[ \frac{2e^{-2r_0}}{\cos^2 \sigma + e^{-4r_0} \sin^2 \sigma} - y_0 \right] - \beta^2 \frac{N_1}{h_2^2C^2} \left[ \frac{e^{-2r_0}}{D} - y_0 \right]^2 + \frac{4\beta^2 S^2}{D^2} e^{-4r_0}
\end{bmatrix},
\end{align*}$$

$$N_1 = 2h_1^2 - h_2^2, \quad N_2 = h_2^2 - h_1^2, \quad \text{and} \quad \lambda \text{ is like in (77). As before}, \quad \xi^3 = 0 \quad \text{and we do} \quad \xi^3 = 0.$$
Therefore, when \((h_2^2, h_1^2) \in I\) the spacetime contains CTCs and ordinary matter. For \((h_2^2, h_1^2) \in II\) the spacetime contains ordinary matter, but not CTCs. If \((h_2^2, h_1^2) \in III\) the spacetime contains CTCs and exotic matter. And when \((h_2^2, h_1^2) \in IV\) the spacetime contains exotic matter, but not CTCs. The isolated point represent the Gödel universe \((h_1, h_2) = (1, 1)\). As we can see in this figure we have an open neighborhood of the point \((1, 1)\) (Gödel spacetime) where the matter is ordinary, the CTCs are present and are stable as depicted in Fig. 6.

Loosely speaking, we have structural stability of a vector field when the equations that it satisfies are slightly changed we also have a small change in the trajectories represented by the vector field (for a simple introduction to this subject see reference [21]). We see that the addition of matter with constants \(h_1 \sim 1\) and \(h_2 \sim 1\) changes “slightly” the original geodesic equations for the Gödel metric. We still have closed curved that are similar to the original Gödel CTCs. Therefore we can say that the Gödel CTCs are structural stable under the inclusion of the special matter represented by the energy-momentum tensor (60).

V. CONCLUSIONS

In this work we verify that the closed timelike curves in Gödel spacetime are stable under linear perturbations. We also show that the Gödel spacetime has a stable structure under a special class of deformations. We found explicit CTCs in the deformed spacetime and proved that closed timelike geodesics do not exist. The energy-momentum tensor of the deformed spacetime was studied in some detail, specially we examined the conditions to have exotic and usual matter. We studied the stability of the new CTCs under linear perturbation and found that these curves are also stable. We tested these curves in a spacetime with exotic matter and find the same properties of stability as in the case of ordinary matter. We also find a kind of structural stability of the CTCs.

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Appendix A
Gödel in his seminal article [1] mentions that his metric has five isometries [1]; Kundt [7] shows explicitly four out of the five Killing vectors of the above mentioned metric in Cartesian coordinates,

\[ \zeta^{(0)} = \frac{\partial}{\partial t}, \quad \zeta^{(1)} = \frac{\partial}{\partial x}, \quad \zeta^{(3)} = \frac{\partial}{\partial z}, \]

\[ \zeta^{(2)} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \]

The fifth Killing vector is not so trivial we find,

\[ \zeta^{(4)} = \sqrt{\beta} (y - 1) \frac{\partial}{\partial t} + \frac{1}{2} (1 - y^2 + x^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \]

For a discussion of the topology of Gödel metric and the five Killing vectors in a different system of coordinates, see [20].

The five Gödel Killing vectors plus the discrete symmetry of reflection on the \( z = 0 \) plane ( \( z \rightarrow -z \) ) give us a family of metrics, that we will named the Gödel family,

\[ ds^2 = k_1 dt^2 + 2 \frac{k_2}{y} dtdx + \frac{k_3}{y^2} dx^2 + \frac{k_4}{y^2} dy^2 + k_5 dz^2; \]

where \( k_i, i = 1, \ldots, 4 \) satisfy the following relations:

\[ k_2 - \frac{\sqrt{2}}{\beta} k_1 = 0, \quad \frac{\sqrt{2}}{\beta} k_2 - k_3 + k_4 = 0. \]

The corresponding constants that appear in the metric (59) do not satisfy the above relation. In fact, only four of the above mentioned five Killings vectors (\( \zeta^{(0)}, ..., \zeta^{(3)} \)) are symmetries of (59), therefore this last metric can be considered as a deformation of the Gödel metric.

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