Topological Transitions with an Imaginary Aubry-André-Harper Potential

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We study one-dimensional lattices with imaginary-valued Aubry-André-Harper (AAH) potentials. Such lattices can host edge states with purely imaginary eigenenergies, which differ from the edge states of the Hermitian AAH model and are stabilized by a non-Hermitian particle-hole symmetry. The edge states arise when the period of the imaginary potential is a multiple of four lattice constants. They are topological in origin, and can manifest on domain walls between lattices with different modulation periods and phases, as predicted by a bulk polarization invariant. Interestingly, the edge states persist and remain localized even if the gap in the real spectrum closes. These features can be used in laser arrays to select topological lasing modes under spatially extended pumping.

I. INTRODUCTION

The Aubry-André-Harper (AAH) model is a foundational theoretical model that illustrates the deep connections between quasicrystals, localization, and band topology [1-3]. It consists of a one dimensional (1D) periodic discrete lattice, on which is applied a sinusoidal potential with a mismatched period. Varying the potential’s period and phase produces an assortment of spectral gaps, which map to the band gaps of a two dimensional (2D) quantum Hall lattice [4-7]. The boundary states in certain gaps of the 1D AAH model likewise map to topological edge states of the 2D lattice, which are linked to bulk topological invariants (Chern numbers). These interesting features have inspired numerous investigations into variants of the AAH model. For example, an AAH-type model with commensurate hopping modulations was found to have a separate class of topological boundary states [8]; zero modes whose energies are pinned to zero by particle-hole symmetry [9] and are linked to the topological properties of the Majorana chain [10].

Over the past decade, there has been increasing interest in non-Hermitian extensions of the AAH model [11-29], as part of a broader program to explore the properties and uses of non-Hermitian systems [21-23]. These models have included AAH-type lattices with parity/time-reversal (PT) symmetric gain/loss [11-14, 18], and lattices with asymmetric hoppings violating both Hermiticity and reciprocity [15, 19, 20]. For example, PT symmetric AAH models have been found to exhibit fractal spectra, similar to the Hermitian AAH model, in the real part of their eigenenergies [12]. Their PT symmetry breaking transition points also have interesting properties [13, 14, 18], such as governing the formation of boundary states [13] and mobility edges [18].

The boundary states in these non-Hermitian AAH models are directly related to the boundary states of the original AAH model. Similar persistence of topological boundary states into the non-Hermitian regime has been observed in other models; for example, in PT symmetric Su-Schrieffer-Heeger (SSH) models [24, 28], topological zero modes can be stabilized by particle-hole symmetry (as in the original Hermitian SSH model) or a non-Hermitian particle-hole symmetry [29, 30]. Very recently, researchers have also found lattice models that host intrinsically non-Hermitian boundary states with no direct link to the Hermitian case [31-33]. For instance, Takata and Notomi discovered a periodic 1D lattice, with four atoms per unit cell, that hosts zero modes induced purely by gain and loss [33]. In view of these advances, it is worthwhile to examine zero modes in non-Hermitian AAH models. Can such modes be induced by gain/loss? What topological properties govern them, and how are they influenced by the AAH-style potential?

Here, we investigate a non-Hermitian AAH model with imaginary commensurate potentials. We find that when the modulation has period \( \lambda = p/q \), where \( p \) and \( q \) are coprime integers and \( p \) is a multiple of 4, there arise topological boundary states whose energies have zero real part, which we refer to as “zero modes”. The case of \( \lambda = 4 \) corresponds to the Takata-Notomi lattice [33]. The zero modes are stabilized by a non-Hermitian particle-hole symmetry [24, 30], and are linked to a non-Hermitian topological invariant based on the electric polarization [34, 37], which depends on the modulation parameters. We derive the topological phase diagrams, and show that they predict the existence of zero modes at domain walls between different modulation functions (including those with different periods). Interestingly, the zero modes can
survive and retain their localized character even if the gap in the real spectrum closes.

As the zero modes are governed by an imaginary sinusoidal potential, it may be possible to use them for mode selection in laser arrays. In existing implementations and proposals for topological lasers [27, 28, 38–47], including those based on the 1D SSH lattice [38, 39, 44] or its PT symmetric variant [27, 28], it is typically necessary to selectively pump the spatial regions where the desired topological modes are localized [27, 28, 38, 46]. This induce the topological modes, rather than the numerous other non-topological modes, to lase. Using our non-Hermitian AAH model and its zero modes, a topological lasing mode can be selected via a spatially extended pump, such as the interference pattern formed by two optical pumping beams. The topological lasing mode can even be enabled or disabled by tuning the phase and period of the pumping pattern.

II. MODEL

We consider a one-dimensional chain with coupling $t$ between nearest neighbors and a purely imaginary on-site potential described by a sinusoidal modulation, as depicted in Fig. 1(a). The Schrödinger equation is

$$t(\psi_{n+1} + \psi_{n-1}) + i V_n \psi_n = E \psi_n,$$

where $\psi_n$ is the wavefunction at site $n$, $E$ is the eigenenergy, and $V$, $\alpha$ and $\delta$ are the amplitude, inverse period, and phase of the potential modulation function. We will set the unit of energy so that $t = 1$. We consider rational values of $\alpha = q/p$, where $p$ and $q$ are coprime positive integers; hence, the modulation function is commensurate with the underlying lattice, and the model is periodic with $p$ sites per unit cell [3].

If $p$ is even, the bulk Hamiltonian $\hat{H}_k$ (a $p \times p$ matrix) satisfies the non-Hermitian particle-hole symmetry [50]

$$-\hat{H}_k = \hat{C} \hat{H}^*_{-k} \hat{C} = \hat{C} \hat{H}^*_k \hat{C},$$

where $\hat{C} = I_{p/2} \otimes \sigma_z$, with $I_{p/2}$ denoting the $p/2 \times p/2$ identity matrix.
identity matrix and $\sigma_z$ denoting the third Pauli matrix, and $\hat{T}$ is the complex conjugation (time-reversal) operator. Eq. (2) implies that the bulk eigenstates either occur in pairs with energies $\{E_1, E_2\}$ satisfying $\hat{E}_1(k) = -\hat{E}_2(k)$, or form a flat band with purely imaginary energy [30]. Moreover, for a finite lattice with $N$ sites (with $N$ even), the Hamiltonian $\hat{H}$ obeys the non-Hermitian particle-hole symmetry

$$\{\hat{H}, \hat{C}\hat{T}\} = 0,$$

where $\hat{C} = \mathbb{I}_{N/2} \otimes \sigma_z$.

We will focus on the case of $p = 4N$, where $N \in \mathbb{Z}^+$. In this case, the bulk bandstructure can host a real line gap, meaning a gap in the real part of the spectrum [22, 48], around $\text{Re}(E) = 0$. Such a gap does not appear for other choices of $\alpha$ (see Supplemental Materials [52]). As an example, Fig. 1(b) plots the complex spectrum for $\alpha = 3/8$, using a lattice of $N = 200$ sites with open boundary conditions (OBC). In the bulk spectrum, calculated using PBC with the same lattice parameters, the real line gap closes at $m\pi/4$ where $m \in \mathbb{Z}$. In Fig. 1(b), it appears that the gap does not fully close at certain of these points (e.g., at $\delta = \pi/2$), but this is a finite-size effect; for larger $N$, the OBC spectrum has gap-closings at the same points as the PBC spectrum (for details, see the Supplemental Materials [59]).

Within half of the gaps, the lattice with OBC exhibits eigenenergies with $\text{Re}(E) = 0$, plotted as red curves in Fig. 1(b). The wavefunctions of these “zero modes” are exponentially localized to the boundary lattice, as shown in Fig. 1(c)–(d). The zero modes preserve the non-Hermitian particle-hole symmetry: each eigenvector $|\psi\rangle$ obeys $|\psi\rangle = e^{i\theta}\hat{C}\hat{T}|\psi\rangle$, where $\theta$ is some global phase factor [29]. Note also that the zero modes need not have $\text{Im}(E) = 0$; in fact, we see from the lower panel of Fig. 1(b) that they can have larger $\text{Im}(E)$ than the bulk states. We will explore the possibility of using this feature for lasing in Section IV.

In Fig. 1(b), we can also see some in-gap states in the other bandgaps, away from $\text{Re}(E) = 0$. These are similar to the topological boundary states of the original AAH model [17], and are not the focus of the present work.

Takata and Notomi [35] have studied the case of $p = 4, q = 1$, which corresponds to the repeating gain/loss sequence $\{g_1, -g_2, -g_1, g_2\}$. In particular, they noted the existence of zero modes induced by the imaginary potential. The present work extends these results to a wider range of gain/loss modulations based on non-Hermitian AAH models.

### III. TOPOLOGICAL PHASES

The zero modes introduced in the previous section are linked to topological features of the non-Hermitian bandstructure. These are expressible using the non-Hermitian Berry connection, calculated via a biorthogonal product instead of the Hermitian inner product [49, 51].

The non-Hermitian band topology can be characterized in two complementary ways [59]. The first approach involves the non-Hermitian generalization [31, 37] of the electric polarization [52, 53]. When there is a real line gap, we can calculate the non-Abelian, non-Hermitian Berry connection for all bands with $\text{Re}(E) < 0$, and use the nested Wilson loop method [53, 55] to integrate it around the Brillouin zone. This procedure has previously been shown to yield quantized polarizations in other non-Hermitian systems with real line gaps, e.g. non-Hermitian higher-order topological insulators [34]. The second approach to characterizing the band topology is the global Berry phase [27, 33, 50], which involves integrating the non-Hermitian Berry connections for all bands (with care taken to fix the gauge and sort the bands [57, 59]). Both methods are based on the bulk band structure, derived under PBC.

When there is a real line gap at $\text{Re}(E) = 0$, the polarization and global Berry phase calculations are in agreement, and yield the topological phase diagrams shown in Fig. 1(c). These phase diagrams are plotted using the modulation parameters $(V, \delta)$ as polar coordinates, for various $\alpha = q/p$ with $p, q$ coprime and $p$ a multiple of 4. In the orange regions, the bandstructure gives quantized polarization $p_x = 1/2$ and a global Berry phase of $2\pi$. In the blue regions, the polarization and global Berry phase vanish. In the black regions, there is no real line gap at $\text{Re}(E) = 0$ and the polarization calculation is inapplicable; we will discuss the lattice’s behavior in this regime later in this section. Evidently, the real line gap phases form $p$ spokes in the phase diagram, extending outward from the origin $V = 0$, and alternating between trivial and nontrivial phases. The phase diagrams for other $\alpha = q/p$ are consistent with the pattern shown in Fig. 1(c).

To test whether the phase diagrams correctly predict the existence of zero modes, we examine the behavior at domain walls between different modulation functions [25, 33, 59, 61, 62]. The lattice shown in Fig. 2(a) consists of two adjacent domains with different gain/loss distributions. The two modulation functions have different $\alpha$ (3/8 and 1/4), as well as different $\delta$, as indicated by the phase diagrams in the lower panels of Fig. 2(a). With the two domains chosen to be topologically inequivalent, we see that the complex spectrum, plotted Fig. 2(b), contains a zero mode (highlighted in green). Its wavefunction is exponentially localized to the domain wall, as shown in Fig. 2(c). (Note that this zero mode has the largest $\text{Im}(E)$ among all the eigenstates; we will discuss the significance of this in Sec. IV.) The other zero mode that can be seen in Fig. 2(b) is localized to the opposite end of the topologically nontrivial domain, rather than the domain wall.) In the Supplemental Materials [59], we show other combinations of modulation parameters, which all behave as expected. In particular, if the domains are both trivial or both nontrivial, there is no zero
FIG. 2. (a) Lattice formed by joining two chains with different gain/loss modulations. In the upper panel, the left (right) domain, marked in orange (blue), is topologically nontrivial (trivial). In the middle panel, the black bars indicate $V_n$, the gain/loss on site $n$, and the solid curves plot the modulation functions, which notably have different periods in the two domains. For the left domain, \( \alpha = \frac{3}{8}, \ V = 1.5, \) and \( \delta = \frac{0.4}{\pi} \); for the right domain, \( \alpha = \frac{1}{4}, \ V = 1.5, \) and \( \delta = -0.4\pi \). In the lower panels, the phase diagrams for the two domains are shown, with the choice of modulation parameters marked by yellow stars. (b) Complex eigenenergy spectrum for the lattice, with a total of \( N = 48 \) sites (24 in each domain). The mirror symmetry around \( \text{Re}(E) = 0 \) is due to the non-Hermitian particle-hole symmetry in Eq. (2). (c) Spatial distribution of the zero mode highlighted in green in (b). Vertical dashes indicate the domain wall.

An interesting feature of the non-Hermitian zero modes is that they can persist for a short but nonzero interval after the closing of the real line gap in bulk spectrum, pinpointed to \( \text{Re}(E) = 0 \). This contrasts with the Hermitian case, where the closing of the band gap causes zero modes and other localized boundary states to hybridize with bulk states and lose their localized character. In Fig. 3(a), we plot a parametric trajectories in the \( \alpha = 3/8 \) phase diagram, extending into the gapless (i.e., no real line gap in bulk spectrum) phases to each side of the gapped phase. The complex band energies along these trajectories are plotted in Fig. 3(b)–(c). When the real line gap in bulk spectrum closes, the complex-valued zero mode energies and bulk energy bands (specifically, their imaginary parts) do not overlap. Hence, the zero modes remain spatially localized, as shown in Fig. 3(d)–(e). In the Supplemental Materials [59], we show that zero modes vanish by coalescing with each other at exceptional points \[21, 23\], rather than hybridizing with bulk states; moreover, within the gapless phase, they are robust against disorder that preserves particle-hole symmetry. Related behavior has recently been pointed out in the context of topological crystalline insulators, where higher-order topological modes can persist despite having \( \text{Re}(E) \) degenerate with the bulk bands \[63, 64\].

IV. MODE SELECTION IN LASER ARRAYS

The non-Hermitian AAH model can be used as the basis for a topological laser distinct from the other topo-
logical lasers studied to date [27, 28, 38, 40, 57]. It has previously been noted that lasers are a natural setting for realizing and exploiting non-Hermitian topological phenomena, since they necessarily contain gain (stimulated emission) and loss (outcoupling and material dissipation). Thus, for instance, researchers have implemented laser arrays based on the PT symmetric SSH model, with lasing modes based on the non-Hermitian zero modes of that model [27, 28].

In this context, the non-Hermitian AAH model’s most striking feature is that its properties are governed directly by the gain/loss modulation function, whose period differs from (and can be significantly larger than) that of the photonic lattice. One interesting possibility is to excite the lattice using a sinusoidally varying pump profile, such as an interference pattern of two optical pumping beams. The period and phase of the pump profile could be easily varied to access different parts of the non-Hermitian AAH model’s phase diagram.

To investigate this, we consider a laser model consisting of a non-Hermitian AAH chain with a nonlinear imaginary potential $iV_n$, where

$$V_n = \frac{\Gamma_n}{1 + |\psi_n|^2} - \gamma,$$

$$\lambda_n = \frac{1}{2} [1 + \sin(2\pi \alpha n + \delta)].$$

Here, $\Gamma$ is an overall pump strength, $\lambda_n \in [0, 1]$ is the spatial modulation of the pump strength, $|\psi_n|^2$ is the local intensity on site $n$, and $\gamma$ is a passive loss rate (which can include outcoupling loss). The $1 + |\psi_n|^2$ denominator represents the effects of gain saturation [29, 40].

Taking $\alpha = 3/8$, $\delta = 0.4\pi$, and $\gamma = 3$, we performed time-domain simulations by numerically integrating the nonlinear equation $i\partial_t \psi = H(|\psi|)\psi$ [29, 40, 63].

The wavefunction is initialized to the random values $\psi_n(t=0) = (\alpha_n + i\beta_n)\psi_0$, where $\alpha_n, \beta_n$ are drawn independently from the standard normal distribution, $n$ is the site index and $\psi_0 = 0.01$ is a scale factor. The simulated time interval is $t \in [0, 5000]$, long enough for transient oscillations to cease. The laser output $I_{\text{out}}$ is obtained by averaging the on-site intensities $|\psi_n|^2$ over the evolution interval of $t \in [2000, 5000]$ (we assume equal outcoupling from each site, with normalized power units).

The resulting plot of $I_{\text{out}}$ versus pump strength $\Gamma$ is shown in Fig. 4(a). Because the zero mode of the linear lattice has (in this case) the highest relative gain, it lases first. As shown as the inset of Fig. 4(a), the frequency spectrum at $\Gamma = 3.6$ consists of a single peak at $\omega = 0$. (Note that in an actual laser, $\omega$ is a frequency detuning, relative to the natural frequency of the decoupled resonators.) The intensity is localized to one boundary of the lattice, as shown in the upper panel of inset in Fig. 4(a). These two features of the lasing mode—the pinning of the frequency to $\omega = 0$ and the spatial localization—are inherited from the linear non-Hermitian AAH model, and are selected by the choice of the pump’s spatial modulation $\lambda_n$, which can be easily adjusted (e.g., by changing the interference pattern of an optical pump). When $\Gamma$ is further increased, the additional modes of the lattice also start to lase [59], and the system enters the multi-mode lasing regime, as shown for the case of $\Gamma = 4$ in Fig. 4(b) and the bottom panel of inset in Fig. 4(a).

An alternative way to access the non-Hermitian AAH model with a laser array is to use loss engineering. We can modulate the (linear) loss on individual sites, and then pump the entire lattice, as described by the imaginary potential $iV_n$, where

$$V_n = \frac{\Gamma}{1 + |\psi_n|^2} + \gamma_n - \gamma,$$
For this case, we suppose $\gamma_n - \gamma$ is formed by two modulation functions with a domain wall at the center, as shown in Fig. 1(c). Within each domain, $\gamma_n = V \sin(2\pi \alpha n + \delta)$, and we pick the same values of $V, \alpha$ and $\delta$ as in Fig. 2. We also include an additional constant loss $\gamma = \text{Im}(E_0) + 0.5$, where $E_0$ is the eigenenergy of the desired zero mode (square marker in Fig. 2(b)). As shown in Fig. 4(d), this sets the laser threshold to $\Gamma = 0.5$. For some range of pump strengths above threshold, the lasing frequency is pinned to $\omega = 0$ and localized at the domain wall, as shown in Fig. 4(e) and the upper panel of Fig. 4(d) for pump strength $\Gamma = 0.6$. Multi-mode lasing is observed at higher pump strengths (e.g., $\Gamma = 0.8$). Although the spatial modulation in lattice recalls the conventional distributed-feedback (DFB) laser, the basic mechanism is quite different. As the spatial modulation in DFB lasers only provide reflection (optical feedback), while the imaginary potential so that the zero modes can have a wider family of AAH-type imaginary potentials. The complex bandstructure has two distinct phases with real line gaps at $\text{Re}(E) = 0$, characterized by non-Hermitian topological invariants. The invariants correctly predict the existence of zero modes, even for domain walls between modulations with different periods.

Previously, Hermitian zero modes have been observed in a variant of AAH model that has commensurate modulations in the hoppings (rather than the on-site potential) [8]. However, that model was based on a Hermitian particle-hole symmetry different from Eq. (2), and we have not found any deeper relationship between these sets of results.

In the non-Hermitian AAH model, it is possible to tune the imaginary potential so that the zero modes can have the highest relative gain of all the eigenstates. This property can be exploited for mode selection in laser arrays, as we showed using simulations. One interesting possibility is to use optical pumping beams in an interference pattern (corresponding to spatially modulated gain) to control the lasing of the zero modes; alternatively, one can modulate the loss in the laser array and pump uniformly. In both cases, our simulations results show that a non-Hermitian zero mode can be the first lasing mode, and retain its key characteristics (frequency pinning and spatial localization) from the lasing threshold up to the onset of multi-mode lasing.

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V. DISCUSSION

We have demonstrated that a non-Hermitian variant of the AAH model, consisting of a sinusoidally-modulated potential that is not real but rather purely imaginary, can exhibit topological boundary modes with purely imaginary energy. These zero modes are found when the modulation period is a multiple of four lattice constants, and are pinned to $\text{Re}(E) = 0$ by an unbroken non-Hermitian particle-hole symmetry [29]. Our results generalize the period-four lattice found by Takata and Notomi [33] to a wider family of AAH-type imaginary potentials. The complex bandstructure has two distinct phases with real line gaps at $\text{Re}(E) = 0$, characterized by non-Hermitian topological invariants. The invariants correctly predict the existence of zero modes, even for domain walls between modulations with different periods.

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[1] P. G. Harper, Single Band Motion of Conduction Electrons in a Uniform Magnetic Field, Proc. Phys. Soc. Sec. A 68, 874 (1955).
[2] S. Aubry and G. André, Analyticity breaking and Anderson localization in incommensurate lattices, Ann. Isr. Phys. Soc. 3, 133 (1980).
[3] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential, Phys. Rev. Lett. 49, 405 (1982).
[4] D. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential, Phys. Rev. Lett. 49, 405 (1982).
[5] J. Faist, Quasicrystals, 1st ed. (Oxford University Press, Oxford, 1994).
[6] Y. E. Kraus and O. Zilberberg, Quasiperiodicity and topology transcend dimensions, Nat. Phys. 9, 981 (2018).
[7] O. Zilberberg, Topology in quasicrystals, Opt. Mat. Express 11, 1143 (2021).
[8] S. Ganeshan, K. Sun, and S. D. Sarma, Topological Zero-Energy Modes in Gapless Commensurate Aubry-André-Harper Models, Phys. Rev. Lett. 110, 180403 (2013).
[9] S. Ryu and Y. Hatsugai, Topological Origin of Zero-Energy Edge States in Particle-Hole Symmetric Systems, Phys. Rev. Lett. 89, 077002 (2002).
[10] A. Y. Kitaev, Unpaired Majorana fermions in quantum wires, Phys.-Usp. 44 (2001).
[11] S. Longhi, PT-symmetric optical superlattices, J. Phys. A 47, 165302 (2014).
[12] C. Yuce, PT-symmetric Aubry-André model, Phys. Lett. A 378, 2024 (2014).
[13] A. K. Harter, T. E. Lee, and Y. N. Joglekar, PT-breaking threshold in spatially asymmetric Aubry-André and Harper models: Hidden symmetry and topological states, Phys. Rev. A 93, 062101 (2016).
[14] S. Longhi, Topological Phase Transition in non-
Hermitian Quasicrystals, Phys. Rev. Lett. 122, 237601 (2019).
[15] H. Jiang, L. Lang, C. Yang, S. Zhu, and S. Chen, Interplay of non-Hermitian skin effects and Anderson localization in nonreciprocal quasiperiodic lattices, Phys. Rev. B 100, 054301 (2019).
[16] Q. Zeng, Y. Yang, and R. Lu, Topological phases in one-dimensional nonreciprocal superlattices, Phys. Rev. B 101, 125418 (2020).
[17] T. Liu, H. Guo, Y. Pu, and S. Longhi, Generalized Aubry-André self-duality and mobility edges in non-Hermitian quasiperiodic lattices, Phys. Rev. B 102, 024205 (2020).
[18] Y. Liu, X. P. Jiang, J. Cao, and S. Chen, Non-Hermitian mobility edges in one-dimensional quasicrystals with parity-time symmetry, Phys. Rev. B 101, 174205 (2020).
[19] S. Longhi, Phase transitions in a non-Hermitian Aubry-André-Harper model, Phys. Rev. B 103, 054203 (2021).
[20] S. Weidemann, M. Kremer, S. Longhi, and A. Szameit, Topological triple phase transition in non-Hermitian Floquet quasicrystals, Nature 601, 354 (2022).
[21] S. K. Özdemir, S. Rotter, F. Nort, and L. Yang, Parity-time symmetry and exceptional points in photonics, Nat. Mater. 18, 783 (2019).
[22] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Symmetry and Topology in Non-Hermitian Physics, Phys. Rev. X 9, 041015 (2019).
[23] Y. Ashida, Z. Gong, and M. Ueda, Non-Hermitian Physics, Adv. Phys. 69, 249 (2020).
[24] H. Schomerus, Topologically protected midgap states in complex photonic lattices, Opt. Lett. 38, 1912 (2013).
[25] L. Lang, Y. Wang, H. Wang and Y. D. Chong, Effects of non-Hermiticity on Su-Schrieffer-Heeger defect states, Phys. Rev. B 98, 094307 (2018).
[26] M. Pan, H. Zhao, P. Miao, S. Longhi and L. Feng, Photonic zero mode in a non-Hermitian photonic lattice, Nat. Commun. 9, 1308 (2018).
[27] M. Parto, S. Wittek, H. Hodaei, G. Harari, M. A. Bandres, J. Ren, M. C. Rechtsman, M. Segev, D. N. Christodoulides and M. Khajavikhan, Edge-Mode Lasing in 1D Topological Active Arrays, Phys. Rev. Lett. 120, 113901 (2018).
[28] H. Zhao, P. Miao, M. H. Teimourpour, S. Malzard, R. E. Ganainy, H. Schomerus and L. Feng, Topological hybrid silicon microlasers, Nat. Commun. 9, 981 (2018).
[29] L. Ge, Symmetry-protected zero-mode laser with a tunable spatial profile, Phys. Rev. A 95, 023812 (2017).
[30] B. Qi, L. Zhang, and L. Ge, Defect States Emerging from a Non-Hermitian Flatband of Photonic Zero Modes, Phys. Rev. Lett. 120, 093901 (2018).
[31] T. E. Lee, Anomalous Edge State in a Non-Hermitian Lattice, Phys. Rev. Lett. 116, 133903 (2016).
[32] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, Edge Modes, Degeneracies, and Topological Numbers in Non-Hermitian Systems, Phys. Rev. Lett. 118, 040401 (2017).
[33] K. Takata and M. Notomi, Photonic Topological Insulating Phase Induced Solely by Gain and Loss, Phys. Rev. Lett. 121, 213902 (2018).
[34] X. Luo and C. Zhang, Higher-Order Topological Corner States Induced by Gain and Loss, Phys. Rev. Lett. 123, 073601 (2019).
[35] E. Lee, H. Lee, and B.-J. Yang, Many-body approach to non-hermitian physics in fermionic systems, Phys. Rev. B 101, 121109 (2020).
[36] C. Ortega-Taberner, L. Rodland, and M. Hermanns, Polarization and entanglement spectrum in non-hermitian systems, Phys. Rev. B 105, 075103 (2022).
[37] J. Hu, C. A. Perroni, G. D. Filippis, S. Zhuang, L. Marrucci, and F. Cardano, Electric polarization and its quantization in one-dimensional non-Hermitian chains, Phys. Rev. B 107, L121101 (2023).
[38] P. St-Jean, V. Goblot, E. Galopin, A. Lemaître, T. Ozawa, L. Le Gratiet, I. Sagnes, J. Bloch, and A. Amo, Lasing in topological edge states of a one-dimensional lattice, Nat. Photon. 11, 651 (2017).
[39] Y. Ota, R. Katsumi, K. Watanabe, S. Iwamoto and Y. Arakawa, Topological photonic crystal nanocavity laser, Commun. Phys. 1, 1 (2018).
[40] G. Harari, Miguel. A. Bandres, Y. Lumer, M. C. Rechtsman, Y. D. Chong, M. Khajavikhan, D. N. Christodoulides, and M. Segev, Topological insulator laser: Theory, Science 359, eaar4003 (2018).
[41] M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. N. Christodoulides, and M. Khajavikhan, Topological insulator laser: Experiments, Science 359, eaar4005 (2018).
[42] Y. Zeng, U. Chattopadhyay, B. Zhu, B. Qiang, J. Li, Y. Jin, L. Li, A. G. Davies, E. H. Linfield, B. Zhang, Y. D. Chong, and Q. J. Wang, Electrically pumped topological laser with valley edge modes, Nature 578, 246 (2020).
[43] Z. Yang, Z. Shao, H. Chen, X. Mao, and R. Ma, Spin-momentum-locked edge mode for topological vortex lasing, Phys. Rev. Lett. 125, 013903 (2020).
[44] T. H. Harder, M. Sun, O. A. Egorov, I. Vakulchyk, J. Beierlein, P. Gagel, M. Emmerling, C. Schneider, U. Peschel, S. Klembt, and S. Hofling, Coherent topological polariton laser, ACS Photonics 8, 1377 (2021).
[45] A. Dikopoltsev, T. H. Harder, E. Lustig, O. A. Egorov, J. Beierlein, A. Wolf, Y. Lumer, M. Emmerling, C. Schneider, S. Hofling, M. Segev, and S. Klembt, Topological insulator vertical-cavity laser array, Science 373, 1514 (2021).
[46] J. H. Choi, W. E. Hayenga, Y. G. Liu, M. Parto, B. Bharati, D. N. Christodoulides and M. Khajavikhan, Room temperature electrically pumped topological insulator lasers, Nat. Commun. 12, 1 (2021).
[47] B. Qi, H. Chen, L. Ge, P. Berini, and R. Ma, Parity-Time Symmetry Synthetic Lasers: Physics and Devices, Adv. Opt. Mater. 7, 1900694 (2019).
[48] E. J. Bergholtz, J. C. Budich, F. K. Kunst, Exceptional topology of non-Hermitian systems, Rev. Mod. Phys. 93, 015005 (2021).
[49] J. C. Garrison and E. M. Wright, Complex geometrical phases for dissipative systems, Phys. Lett. A 128, 177 (1988).
[50] S. D. Liang and G. Y. Huang, Topological invariance and global Berry phase in non-Hermitian systems, Phys. Rev. A 87, 012118 (2013).
[51] S. Lieu, Topological phases in the non-Hermitian Su-Schrieffer-Heeger model, Phys. Rev. B 97, 045106 (2018).
[52] R. King-Smith and D. Vanderbilt, Theory of polarization of crystalline solids, Phys. Rev. B 47, 1651 (1993).
[53] R. Resta, Quantum-mechanical position operator in extended systems, Phys. Rev. Lett. 80, 1800 (1998).
[54] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes,
Electric multipole moments, topological multipole moment pumping, and chiral hinge states in crystalline insulators, Phys. Rev. B 96, 245115 (2017).

[55] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Quantized electric multipole moments, Science 357, 61 (2017).

[56] M. Blanco de Paz, C. Devescovi, G. Giedke, J. J. Saenz, M. G. Vergniory, B. Bradlyn, D. Bercioux and A. Garcia Etxarri, Tutorial: Computing topological invariants in 2D photonic crystals, Adv. Quantum Technol. 3, 1900117 (2020).

[57] P. Comaron, V. Shahnazaryan, W. Brzezicki, T. Hyart, and M. Matuszewski, Non-Hermitian topological end-mode lasing in polariton systems, Phys. Rev. Res. 2, 022051(R) (2020).

[58] M. Wagner, F. Dangel, H. Cartarius, J. Main, and G. Wunner, Numerical calculation of the complex berry phase in non-Hermitian systems, Acta. Polytechnica 57, 470 (2017).

[59] See online Supplemental Materials [...]

[60] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, and O. Zilberberg, Topological States and Adiabatic Pumping in Quasicrystals, Phys. Rev. Lett. 109, 106402 (2012).

[61] M. Verbin, O. Zilberberg, Y. E. Kraus, Y. Lahini, and Y. Silberberg, Observation of Topological Phase Transitions in Photonic Quasicrystals, Phys. Rev. Lett. 110, 076403 (2013).

[62] M. Verbin, O. Zilberberg, Y. Lahini, Y. E. Kraus, and Y. Silberberg, Topological pumping over a photonic Fibonacci quasicrystal, Phys. Rev. B 91, 064201 (2015).

[63] W. A. Benalcazar and A. Cerjan, Bound states in the continuum of higher-order topological insulators, Phys. Rev. B 101, 161116(R) (2020).

[64] A. Cerjan, M. Jürgensen, W. A. Benalcazar, S. Mukherjee, and Mikael C. Rechtsman, Observation of a Higher-Order Topological Bound State in the Continuum, Phys. Rev. Lett. 125, 213901 (2020).

[65] Z. Yang, E. Lustig, G. Harari, Y. Plotnik, Y. Lumer, M. A. Bandres and M. Segev, Mode-Locked Topological Insulator Laser Utilizing Synthetic Dimensions, Phys. Rev. X 10, 011059 (2020)