Configuration entropy in the soft wall AdS/QCD model and the Wien law

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The soft wall AdS/QCD holographic model provides simple estimates for the spectra of light mesons and glueballs satisfying linear Regge trajectories. It is also an interesting tool to represent the confinement/deconfinement transition of a gauge theory, that is pictured as a Hawking-Page transition from a dual geometry with no horizon to a black hole space. A very interesting tool to analyze stability of general physical systems that are localized in space is the configuration (or complexity) entropy (CE). This quantity, inspired in Shannon information entropy, is defined in terms of the energy density of the system in momentum space. The purpose of this work is to use the CE to investigate the stability of the soft wall background as a function of the temperature. A nontrivial aspect is that the geometry is an anti-de Sitter black hole, that has a singular energy density. In order to make it possible to calculate the CE, we first propose a regularized form for the black hole energy density. Then, calculating the CE, it is observed that its behavior is consistently related to the black hole instability in anti-de Sitter space. Another interesting result that emerges from this analysis is that the regularized energy density shows a behavior similar to the Wien law, satisfied by black body radiation. That means: the momentum $k_{\text{max}}$ where the energy density is maximum, varies with the temperature $T$ obeying the relation: $T/k_{\text{max}} = \text{constant}$ in the deconfined phase.

1 Introduction

It was proposed in [1–3] that the configuration entropy (CE) works as an indicator of the stability of physical systems. The CE, also known as complexity entropy, is motivated by information theory, in particular by the Shannon information entropy [4]. Currently, one finds in the literature many interesting examples where the CE plays the role of representing stability, as for instance [5–29]. By comparing the CE for different states of the same system, it was observed, for diverse physical systems, that the smaller is the CE, the more stable is the state.

The Shannon information entropy [4] for a discrete variable with probabilities $p_n$ for the possible values that it can assume is defined as:

$$S = -\sum_n p_n \log p_n .$$

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It is a measure of the information contained in the variable. The configuration entropy is defined as a continuous version of (1.1), that for a one-dimensional system reads

$$ S_C[f] = - \int dk \, f(k) \ln f(k), $$

(1.2)

The quantity $f(k)$ is called modal fraction and is usually defined in terms of the energy density in momentum space, $\rho(k)$, namely,

$$ f(k) = \frac{|\rho(k)|^2}{|\rho(k)|^2_{\text{max}}} $$

(1.3)

where $|\rho(k)|^2_{\text{max}}$ is the maximum value assumed by $|\rho(k)|^2$. The modal fraction can alternatively be defined to be a normalized function, replacing in the denominator of eq. (1.3) the square of the maximum value of the energy density in momentum space by $\int |\rho(k)|^2 dk$. Such a definition would be more similar to the Shannon entropy, where the probabilities are normalized, but lead to negative values for the CE. This happens because, in contrast to (1.1) where the probabilities satisfy $p_n \leq 1$, the same rule does not apply to the continuous case, that involves densities.

There are many situations in physics where is important to understand what are the conditions that affect the stability of a system. In this work we are interested in the case of the holographic soft wall model [30]. This AdS/QCD holographic model provides simple estimates for meson and glueball masses and also for the confinement/deconfinement temperature [31,32]. The model assumes an approximate duality between fields in an anti-de Sitter (AdS) space with a scalar background and light mesons in a strongly coupled gauge theory. In the finite temperature case, the geometry changes. For temperatures $T$ above a critical value $T_c$ the geometry dual to the gauge theory is an AdS black hole, while for $T < T_c$ it is just a thermal AdS space. The black hole phase corresponds to the deconfined plasma phase of the gauge theory. Stability of the black hole space corresponds, in this case, to stability of the plasma. The confinement/deconfinement transition is represented in holography in terms of a Hawking-Page (HP) transition [33], as pointed out by Witten [34].

The main purpose of the present work is to analyse the stability of the soft wall model using the configuration entropy approach. As we will show, the energy density is singular, so it is necessary to develop a regularization procedure in order to find a density that can be used to define the modal fraction and then the CE. A remarkable property of this regularized density is that it obeys a relation similar to the Wien law for black body radiation. The maximum value of the energy density occurs in a momentum $k_{\text{max}}$ that is proportional to the temperature. The organization is the following. In section 2 we present a review of the soft wall model at finite temperature and explain how does the Hawking-Page transition shows up. Then in section 3 we develop a regularized form for the energy density. Section 4 is devoted to discuss the behaviour similar to the Wien law displayed by such density. Finally, in section 5 we present the result for the CE of the soft wall as a function of the temperature. A discussion about the results and some conclusions are presented in section 6.

2 Soft wall AdS/QCD model and Hawking-Page transition

In the soft wall AdS/QCD model [30], the gravitational part of the action is given by

$$ I = -\frac{1}{2\kappa^2} \int d^5x \sqrt{g} e^{-\Phi} \left( R + \frac{12}{L^2} \right), $$

(2.1)

where $\kappa$ is the gravitational coupling and $R$ the Ricci scalar. The dilaton field $\Phi = cz^2$ plays the role of introducing, in a phenomenological way, an infrared energy scale, namely $\sqrt{c}$, in the model. This action
corresponds to a negative cosmological constant $\Lambda = -\dfrac{12}{L^2}$ and the solution considered in the soft wall model has a constant negative curvature $R$. At zero temperature the geometry is an anti-de Sitter space with radius $L$. At finite temperature there are two solutions. One is the thermal AdS space and the other the AdS black hole space. One assumes that the dilaton field does not affect the gravitational dynamics of the theory. This can be interpreted as the assumption that the fluctuations of $\phi$ are very small as compared to the gravitational ones.

The thermal AdS space, in the Euclidean signature with compact time direction, is described by

$$ds^2 = \frac{L^2}{z^2} \left( dt^2 + d\vec{x}^2 + dz^2 \right),$$

(2.2)

and the AdS with a black hole, by

$$ds^2 = \frac{L^2}{z^2} \left( f(z) dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right),$$

(2.3)

with $f(z) = 1 - z^4/z_h^4$, being $z_h$ the position of the black hole horizon. For the black hole, the time has a period $\beta$ and the temperature is $T = 1/\beta = 1/(\pi z_h)$, to avoid a conical singularity of the metric on the horizon [33]. In the thermal AdS case, the periodicity of time is, in principle, not constrained. However, requiring that the asymptotic limit of the two geometries at $z = \epsilon$, with $\epsilon \to 0$, are the same, one finds it out that the period of the thermal AdS is $\beta' = \pi z_h \sqrt{f'(\epsilon)}$.

The AdS geometries (2.2) and (2.3) are both solutions of the Einstein action, with curvature $R = -20/L^2$. So that, in both cases, one obtains the on-shell action

$$I_{\text{on-shell}} = \frac{4}{L^2 R^2} \int d^5 x \sqrt{g} e^{-\Phi}.$$  

(2.4)

It is worth to define action densities $E(\epsilon) = I/V$, where $V$ is the trivial transverse tree-dimensional space. For the thermal AdS it reads

$$E_{\text{AdS}}(\epsilon) = \frac{4 L^3}{\kappa^2} \int_0^{\beta'} dt \int_\epsilon^\infty dz \, z^{-5} e^{-cz^2},$$  

(2.5)

and for the black hole it is:

$$E_{\text{BH}}(\epsilon) = \frac{4 L^3}{\kappa^2} \int_0^{\pi z_h} dt \int_\epsilon^\infty dz \, z^{-5} e^{-cz^2}.$$  

(2.6)

The regulator $\epsilon$ is necessary due to the presence of ultraviolet divergences in the integration over $z$. However, defining $\Delta E$ as the difference between the black hole AdS and the thermal AdS action densities

$$\Delta E = \lim_{\epsilon \to 0} [E_{\text{BH}}(\epsilon) - E_{\text{AdS}}(\epsilon)],$$  

(2.7)

the UV divergences are cancelled, and one obtains the following result

$$\Delta E = \frac{\pi L^3}{\kappa^2 z_h^3} \left[ e^{-cz_h^2} \left( -1 + cz_h^2 \right) + \frac{1}{2} + c^2 z_h^4 \text{Ei}(-cz_h^2) \right],$$

(2.8)

where $\text{Ei}(x) = -\int_x^\infty \dfrac{e^{-t}}{t} dt$. We show in Figure 1 a plot of $\Delta E$ as a function of the temperature. One can see in this figure that for temperatures above a critical value $T = T_c$, $\Delta E$ is negative while for smaller values it is positive. So that, there is a Hawking-Page transition [33]. When $T > T_c$ the black hole has a smaller action, so it is stable. For lower temperatures, the thermal AdS is the stable phase. The critical temperature corresponds to $cz_h^2 = 0.419035$, or, equivalently to:

$$T_c = 0.491720 \sqrt{c}.$$  

(2.9)
If one fixes the parameter $c$ by the spectrum of the $\rho$-meson, one gets \[31\] $\sqrt{c} = 318 \text{ MeV}$. The corresponding deconfinement temperature is $T_c = 191 \text{ MeV}$. For an interesting alternative study of deconfinement transition in holographic QCD using entanglement entropy see \[35\].

3 Black hole regularized energy density

The mass of a black hole can be determined from the expression \[33,34\]

$$M = \frac{\partial I}{\partial \beta}.$$  \hfill (3.1)

Considering the regularized difference between the black hole and AdS actions

$$I = I_{BH} - I_{AdS} ,$$  \hfill (3.2)

one finds an energy per unit of transverse volume $V$

$$E = M/V = \frac{\partial}{\partial \beta} \frac{\Delta \mathcal{E}}{\Delta \mathcal{E}} ,$$  \hfill (3.3)

with $\Delta \mathcal{E}$ defined by eq. (2.8). The natural definition of a black hole energy density $\rho_{BH}(z)$ in the relevant holographic $z$ coordinate would be one satisfying:

$$\int_0^\infty \rho_{BH}(z) \, dz = E .$$  \hfill (3.4)

In order to find such a density one can apply the derivative with respect to $\beta$ on $\Delta \mathcal{E}$ but before integrating over coordinate $z$. That means, considering $\Delta \mathcal{E}$ defined as the difference between the expression in eq. (2.6) and the one in eq. (2.5) , taking into account the fact that the limits of integrations depend on $\beta$.

It is convenient to introduce the coordinate $u = z/z_h$. Using equations (2.5) and (2.6), one finds

$$\Delta \mathcal{E}(u) = \frac{4L^3}{\kappa^2} \left[ \beta \int_{\epsilon \pi/\beta}^1 du \, u^{-5} \left( \frac{\beta}{\pi} \right)^{-4} e^{-c(\frac{u\beta}{\pi})^2} \left( \beta - \frac{\pi^4 e^4}{2\beta^2} \right) \int_0^\infty du \, u^{-5} \left( \frac{\beta}{\pi} \right)^{-4} e^{-c(\frac{u\beta}{\pi})^2} \right] .$$  \hfill (3.5)
The derivative of (3.5) w.r.t. $\beta$ yields, after taking the $\epsilon \to 0$ limit, \[
E = \lim_{\epsilon \to 0} \frac{\partial \triangle E}{\partial \beta} = \frac{4L^3}{\kappa^2} \left[ \frac{1}{2} \frac{\pi^2}{u^5} + \int_1^{\infty} du \left( \frac{3\pi^4}{u^7} + \frac{2\epsilon^2}{u^2} \right) e^{-c(u^3/\pi)} - \frac{7\epsilon^4}{2} \int_{\frac{u^3}{\pi}}^{\infty} du \frac{\pi^8}{u^3} e^{-c(u^3/\pi)} \right]. \tag{3.6}
\]

The first (non-integrated) term of eq. (3.6) comes from the differentiation of the integration limit with respect to $\beta$. It can be written in terms of the last term, since one can easily show that \[
\frac{1}{2} \frac{\pi^2}{u^5} \to 2 \epsilon^4 \int_{\frac{\epsilon \pi}{u^3/\beta}}^{\infty} du \frac{\pi^8}{u^3} e^{-c(u^3/\pi)} \tag{3.7}
\]
So, all the terms in the expression (3.6) for the energy $E$ can be written as integrals over the coordinate $z$ and one can associate the integrand with a density. Actually, the result of the integral (3.7) does not depend on the value of the upper limit of integration, as long as it is greater than the lower limit. So, we will take it to be at $u = 1$. Finally, returning to the $z$-variable, one finds \[
E = \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{z_h} \rho_{1BH}(z) \, dz + \int_{z_h}^{\infty} \rho_{2BH}(z) \, dz \right), \tag{3.8}
\]
where \[
\rho_{1BH}(z) = -\frac{4L^3}{\kappa^2} \frac{3\epsilon^4}{2} \frac{\pi^8}{u^3} \frac{e^{-cz^2}}{z_h^2}, \quad \text{if} \quad \epsilon \leq z \leq z_h, \nonumber \]
\[
\rho_{2BH}(z) = \frac{4L^3}{\kappa^2} \left( \frac{3}{z^5} + \frac{2\epsilon^2}{z^3} \right) e^{-c\epsilon^2}, \quad \text{if} \quad z > z_h. \tag{3.9}
\]

The expression (3.9) shows that the BH mass is concentrated in the ultraviolet and infrared regions, i.e., in the extremes of $z$-direction. Such a regularized result for the energy density makes it possible to calculate the corresponding momentum space density, that is necessary in order to find the CE.

4 Wien law in the soft wall model

The BH energy density, $\rho_{BH}(z)$, depends only on the Poincaré coordinate $z$. The corresponding Fourier transform takes the form \[
\tilde{\rho}(k) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{z_h} dz \rho_{1BH}(z) e^{ikz} + \int_{z_h}^{\infty} dz \rho_{2BH}(z) e^{ikz} \right), \tag{4.1}
\]
with $\rho_{1BH}(z)$ and $\rho_{2BH}(z)$ defined by eq. (3.9). The relevant quantity for the calculation of the modal fraction, and then the CE, is the squared absolute value of the black hole energy density in Fourier space, that can be written as: \[
|\tilde{\rho}(k)|^2 = \left[ \frac{1}{2\pi} \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{z_h} \rho_{1BH}(z) \cos(kz) \, dz + \int_{z_h}^{\infty} \rho_{2BH}(z) \cos(kz) \, dz \right) \right]^2 + \left[ \frac{1}{2\pi} \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{z_h} \rho_{1BH}(z) \sin(kz) \, dz + \int_{z_h}^{\infty} \rho_{2BH}(z) \sin(kz) \, dz \right) \right]^2. \tag{4.2}
\]

One can use numerical methods in order to compute these integrals, since they do not present analytical solutions. The results must be independent of the UV regulator $\epsilon$. As the numerical computation is
performed with a finite value of \( \epsilon \), the limit \( \epsilon \to 0 \) is effectively obtained by identifying the order of magnitude for which taking smaller values of \( \epsilon \) would not change the results, considering the degree of precision in which one is working. We found it out that from \( \epsilon \sim 10^{-16} \) to smaller values there is no change in any of the results of this work, so this value was used in our computations.

The functions \( |\tilde{\rho}(k)|^2 \) present a global maximum \( |\tilde{\rho}(k)|^2_{\text{max}} \), at a finite value of \( k \), that we will name as \( k_{\text{max}} \). Analysing the variation of the value of \( k_{\text{max}} \) with the temperature, it is found that

\[
\frac{k_{\text{max}}}{T} = K,
\]

where \( K \) is a constant. Using the normalization \( \frac{4L^3}{2\pi c^2} = 1 \) and taking \( c = 1 \), that corresponds to working with the dimensionless temperature \( T \equiv T/\sqrt{c} \), one finds that \( K = 8.790 \times 10^4 \), with an error of \( \pm 0.001 \times 10^4 \) or 0.01\%. In Figure 2-(A), we plot the absolute value of the BH energy density at \( T = 0.65 \), which shows a behaviour that is the same for all values of \( T \). Namely, \( |\tilde{\rho}(k)|^2 \) goes to zero as \( k \to 0 \), for \( k > 0 \) it increases until it reaches the global maximum, and then oscillates, tending to zero for large momentum. As \( T \) increases, \( |\tilde{\rho}(k)|^2_{\text{max}} \) and \( k_{\text{max}} \) increase, as shown in Figure 2-(B), where we plot \( |\tilde{\rho}(k)|^2 \) at three different temperatures, \( T_3 > T_2 > T_1 \).

This behaviour is represented in Figure 3, where we plot the points displayed in Table 1. For the confined phase, corresponding to \( T < T_c \), there is no black hole, so there is no energy density. For \( T > T_c \), the points represent the relation described by eq. (4.3).

The wavelength associated with the momentum \( k_{\text{max}} \) is \( \lambda_{\text{max}} = 2\pi/k_{\text{max}} \). Replacing this relation in (4.3), one gets the analogous of Wien’s displacement law

\[
\lambda_{\text{max}} T = C, \tag{4.4}
\]

wherein \( C = \frac{2\pi}{K} = 7.148 \times 10^3 \). This equation shows that the BH energy density will peak at different wavelength that are inversely proportional to the temperature, analogously to the black-body radiation spectrum.

5 Configuration entropy and stability

Now, let us study the stability of the soft wall model by computing the configuration entropy (CE) following the prescriptions presented in the previous sections. First we calculate the modal fraction (1.3),
Table 1: Maximum of BH density, $|\tilde{\rho}(k)|^2_{\text{max}}$, with the corresponding momentum $k_{\text{max}}$, at different temperatures.

| $T$     | $|\tilde{\rho}_{BH}(k)|^2_{\text{max}}$ | $k_{\text{max}}$ | $T$     | $|\tilde{\rho}_{BH}(k)|^2_{\text{max}}$ | $k_{\text{max}}$ |
|---------|----------------------------------------|------------------|---------|----------------------------------------|------------------|
| 0.50    | $1.2092 \times 10^{24}$                | 43949            | 1.00    | $7.7388 \times 10^{25}$                | 87900            |
| 0.55    | $2.1422 \times 10^{24}$                | 48345            | 1.05    | $1.0371 \times 10^{26}$                | 92295            |
| 0.60    | $3.6106 \times 10^{24}$                | 52740            | 1.10    | $1.3710 \times 10^{26}$                | 96683            |
| 0.65    | $5.8365 \times 10^{24}$                | 57135            | 1.15    | $1.7900 \times 10^{26}$                | 101088           |
| 0.70    | $9.1046 \times 10^{24}$                | 61530            | 1.20    | $2.3108 \times 10^{26}$                | 105474           |
| 0.75    | $1.3773 \times 10^{25}$                | 65924            | 1.25    | $2.9521 \times 10^{26}$                | 109873           |
| 0.80    | $2.0287 \times 10^{25}$                | 70321            | 1.30    | $3.7152 \times 10^{26}$                | 114275           |
| 0.85    | $2.9187 \times 10^{25}$                | 74712            | 1.35    | $4.6593 \times 10^{26}$                | 118665           |
| 0.90    | $4.1127 \times 10^{25}$                | 79117            | 1.40    | $5.7955 \times 10^{26}$                | 123066           |
| 0.95    | $5.6887 \times 10^{25}$                | 83505            | 1.45    | $7.1537 \times 10^{26}$                | 127457           |

Figure 3: $k_{\text{max}}$ versus $T$ for $T > T_c$.

as defined in the previous section, being $k_{\text{max}}$ the value of $k$ where $|\tilde{\rho}(k)|^2$ assumes the maximum value. The square of the Fourier transform of the energy density is obtained from equation (4.2) using the spatial density defined in eq. (3.9). Then the CE is calculated using equation (1.2). This way, one can determine the CE as a function of the temperature. Below $T_c = 0.491720$, the stable phase is the thermal AdS. So, there is no black hole and the CE is zero since, as expressed in (3.2), we are considering the AdS as the “empty space” background. In the deconfined phase, where the black holes are stable, the numerical computations of the CEs at the same temperatures of Table 1, lead to the results displayed in Table 2 and plotted in Figure 4.

One can see in Figure 4 that the CE varies linearly with the temperature. In particular, it is found that

$$S_C = K_c T, \quad (5.1)$$

where $K_c = (6.546 \pm 0.001) \times 10^5$, meaning that black holes become more unstable at higher temperatures. Such a result is consistent with the expectation that black holes are subject to a process of evaporation...
by Hawking radiation, and this effect should increase with the temperature.

Another outcome of this result for the CE is that there is also a linear relation between the CE and \( k_{\text{max}} \): \( S_C \sim k_{\text{max}} \). The jump from \( S_C = 0 \) in the confined hadronic phase to \( S_C \sim T \) in the plasma phase, represents in a consistent way the release of the color degrees of freedom that happens at \( T = T_c \). From the correlation between the Wien law and the CE, one concludes that states with smaller values of \( k_{\text{max}} \) are more stable.

### 6 Analysis of the results and conclusions

Studying the thermodynamics of the black hole geometry in the soft wall AdS/QCD model, we obtained, in eq. (3.9), a regularized form for the energy density in the holographic coordinate \( z \). This regularized expression made it possible the calculation of the square of the absolute value of energy density in momentum space \( |\tilde{\rho}(k)|^2 \), given by eq. (4.2). The results obtained for the black hole phase display a

| \( T \) | Black hole CE | \( T \) | Black hole CE |
|---|---|---|---|
| 0.50 | 327276 | 1.00 | 654774 |
| 0.55 | 360026 | 1.05 | 687282 |
| 0.60 | 392764 | 1.10 | 720018 |
| 0.65 | 425406 | 1.15 | 752790 |
| 0.70 | 458194 | 1.20 | 785476 |
| 0.75 | 490936 | 1.25 | 818200 |
| 0.80 | 523682 | 1.30 | 850874 |
| 0.85 | 556308 | 1.35 | 883674 |
| 0.90 | 589096 | 1.40 | 916450 |
| 0.95 | 621846 | 1.45 | 949192 |

Table 2: Black hole CE at different temperatures.

Figure 4: CE versus \( T \).
behaviour with the form of the Wien’s displacement law for black body radiation. Associating a wavelength \( \lambda \) with the inverse of \( k \) one finds it out that the the value \( k_{\text{max}} \) where the maximum of \( |\tilde{\rho}(k)|^2 \) occurs, corresponds to a wavelength \( \lambda_{\text{max}} \) that is inversely proportional to the temperature, as shown in eq. (4.4).

We also found that the CE is proportional to \( k_{\text{max}} \) and to the temperature. This is a nontrivial issue. Recently it was show in Ref. [29] that the BH configuration entropy in the hard wall holographic model is proportional to \( \log T \), showing a different behaviour with respect to the one found here for the soft wall. It is known that the soft wall presents a better holographic description of hadronic matter since the mass spectra obtained with this model present linear Regge trajectories, as opposed to the hard wall model. So, it is interesting to see that it also present a richer structure from the point of view of the configuration entropy. The gap between the thermal AdS and black hole phases is consistent with the transition from hadronic to the plasma phase, or confinement/deconfinement transition. This corresponds to the liberation of the color degrees of freedom at the Hawking-Page critical temperature.

The result that the CE increases with the temperature, indicating that the black hole becomes more unstable, is consistent with the expectation that black holes undergo a process of evaporation that increases in intensity with the temperature. The evaporation of anti-de Sitter black holes with planar horizon was discussed in Ref. [36]. There, it was shown that the rate of change in the mass is proportional to a positive power of the mass itself. So, the rate of change of the mass increases with the temperature. The conclusion that follows is that the CE captures consistently the dynamical instability of the black hole phase, associated with the evaporation. In the gauge theory side of the holographic duality, this instability could be associated with the instability of the plasma, that has a finite lifetime, undergoing a process of hadronization after this time. Another interesting point to be remarked is that the vanishing of the CE for \( T < T_c \) is a result consistent with the interpretation that the instability captured by the CE is associated with the black hole evaporation. In the confined phase there is no black hole, so there is no evaporation.

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