GLOBAL MARTINGALE SOLUTIONS TO A STOCHASTIC SUPERQUADRATIC CROSS-DIFFUSION POPULATION SYSTEM

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ABSTRACT. The existence of global nonnegative martingale solutions to a stochastic cross-diffusion population system with superquadratic transition rate is shown. The entropy method adapting to the deterministic cross-diffusion system may not be able to provide strong enough uniform estimates for the tightness proof in the stochastic environment. Under a strengthened coefficients of the diffusion matrix condition, an application of the Itô formula to a linear transformation between variables is suffice to provide us with strong enough uniform estimates. After the tightness property be proved based on the estimation, a space changing result be used to confirm the limit is a weak solution to the cross-diffusion system. Nonnegative property is proved before uniform estimation estimated. We apply the existence and uniqueness theorem for the stochastic differential equation to establish the existence of a unique strong solution result, then we fix the random factor, and an entropy with a “variable-nonnegative” factor be applied to a sequence of deterministic equations provides us with the nonnegativeness of the strong solution, \( \mathbb{P} \)-a.s.

1. INTRODUCTION

1.1. Description of the model. The dynamics and motions of interacting population species can largely be described by cross-diffusion equations. A well known example is the deterministic Shigesada-Kawasaki-Teramoto population system [25]. Generalized cross-diffusion models have also been derived when the dependence of the transition rates on the densities is nonlinear.

The existence of global weak solutions to these deterministic models has been proved for an arbitrary number of species in several papers, e.g. [4, 5, 6, 7, 8, 16, 19, 20, 21] and [28]. Those papers mainly adopt a so-called entropy method with an application of Aubin-Lions Lemma [10, 14]. If we allow for a random influence of the environment, then a different method comparing to deterministic ones be adopted to prove the existence of a global martingale solution to a cross-diffusion population system with stochastic factors involved.

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Denote $u = (u_1, ..., u_n)$, we consider the below stochastic partial differential equations

(1) \[ du_i - \text{div} \left( \sum_{j=1}^{n} A_{ij}(u) \nabla u_j \right) dt = \sum_{j=1}^{n} \sigma_{ij}(u) dW_j(t) \quad \text{in } \mathcal{O}, \ t > 0, \ 1 \leq i \leq n, \]

with no-flux boundary and initial conditions

(2) \[ \sum_{j=1}^{n} A_{ij}(u) \nabla u_j \cdot \nu = 0 \quad \text{on } \partial \mathcal{O}, \ t > 0, \ u_i(0) = u_i^0 \quad \text{in } \mathcal{O}, \ 1 \leq i \leq n, \]

where $\mathcal{O} \subset \mathbb{R}^d$, $d \leq 3$ is a bounded domain with Lipschitz boundary, $\nu$ is the exterior unit normal vector to $\partial \mathcal{O}$ and $u_i^0$ is a possibly random initial datum. The concentrations $u_i(\omega, x, t)$ are defined on $\Omega \times \mathcal{O} \times [0, T]$, where $\omega \in \Omega$ represents the stochastic variable, $x \in \mathcal{O}$ the spatial variable, and $t \in [0, T]$ the time parameter. $A(u) = (A_{ij}(u))$ is the diffusion matrix, and $\sigma(u) = (\sigma_{ij}(u))$ is the multiplicative noise term, and $W(t) = (W_1(t), W_2(t), ..., W_n(t))$ is an $n$-dimensional Wiener process.

The diffusion coefficients are given by

(3) \[ A_{ii}(u) = a_{i0} + (s + 1)a_{ii}u_i^s + \sum_{k=1, k \neq i}^{n} a_{ik}u_k^s, \quad \text{and } A_{ij}(u) = sa_{ij}u_iu_j^{s-1}, \quad \text{if } i \neq j, \]

where $a_{i0} > 0$, $a_{ik} > 0$ and $s > 0$.

In [13], if $s = 2$, the existence of global martingale solutions to (1)-(3) for an arbitrary number of species has been shown. We extend the result of [13] to the case if $s > 2$.

1.2. Key ideas. Uniform estimates in the stochastic environment are derived through an application of the Itô formula to the stochastic process $\sum_{i=1}^{n} \int_{\mathcal{O}} \pi_i(u_i^{(N)})^2 dx$, where $u_i^{(N)}$ is the strong solution to each Galerkin approximated equation indexed by $N \in \mathbb{N}$, with $\pi_i > 0$. Coefficients conditions applied in [13], which are originated from the entropy method, may not be able to provide strong enough uniform estimates for a tightness proof, when $s > 2$. Our coefficients scheme, with its character reflected in the matrix analysis section, provides us with strong enough estimation.

During the estimation process, we notice that in order to derive strong enough estimation for a tightness proof, we in addition have to estimate a replacement variable. In the Lemma [10] the replacement variable $v_i^{(N)}$ is given by $v_i^{(N)} = (u_i^{(N)})^\frac{2}{s}$, if $u_i^{(N)} \geq 0$, $\mathbb{P}$-a.s. If $s = 2$, $v_i^{(N)} = u_i^{(N)}$, with the estimation of this replacement variable coincides with the estimation of the original $u^{(N)}$.

When the case $s = 2$ being analyzed in [13], the nonnegativeness of the weak solution is an auxiliary result presented in the end of the discussion. We prove the nonnegativeness of the strong solution $u^{(N)}$ before its uniform estimation estimated, for the reason the uniform estimates hold only if $u_i^{(N)} \geq 0$. The above change of variables technique in the Lemma [10] has also to be guaranteed by the nonnegativeness of $u^{(N)}$.

Another reason we try to confirm the nonnegativeness of the strong solution $u^{(N)}$ at the beginning of our discussion is, if $s$ is an irrational number, or of the form $s = \frac{p}{q}$, $p, q \in \mathbb{N}$, $p, q > 0$, $(p, q) = 1$ (we denote $(p, q)$ as the greatest common denominator of $p$ and $q$), with
p an odd number, q an even number. In this situation, no existence result for \( u^{(N)} \) can be derived without nonnegativeness being guaranteed. Combining these factors, we prove the existence of \( u^{(N)} \) and \( u_i^{(N)} \geq 0, \mathbb{P}\text{-a.s.} \) together.

We adopt the existence and uniqueness theorem for a stochastic differential equation to show a unique strong solution exists. Actually in this step we have only shown that a unique strong solution exists when the Galerkin approximation been implemented to another system given by

\[
du_i - \text{div} \left( \sum_{j=1}^{n} M_{ij}(u) \nabla u_j \right) dt = \sum_{j=1}^{n} \sigma_{ij}(u) dW_j(t) \quad \text{in } \mathcal{O}, \quad t > 0, \quad 1 \leq i \leq n,
\]

with no-flux boundary and initial conditions

\[
\sum_{j=1}^{n} M_{ij}(u) \nabla u_j \cdot \nu = 0 \quad \text{on } \partial \mathcal{O}, \quad t > 0, \quad u_i(t) = u_i^0 \quad \text{in } \mathcal{O}, \quad 1 \leq i \leq n,
\]

and the diffusion coefficients are

\[
M_{ii}(u) = a_{i0} + (s + 1)a_{ii}|u_i|^s + \sum_{k=1,k\neq i}^{n} a_{ik}|u_k|^s, \quad \text{and} \quad M_{ij}(u) = sa_{ij}|u_i| \cdot |u_j|^{s-1}, \quad \text{if } i \neq j.
\]

After the entropy method adapting to deterministic equations derived through the Wong-Zakai approximation been applied to show this strong solution is nonnegative, \( \mathbb{P}\text{-a.s.} \) then we are able to conclude that the system when Galerkin method been applied to (4)-(6), which has been proved with a unique strong solution, coincides with the system when Galerkin method been applied to (1)-(3).

In [11, 12], the fixed point theorem has been applied to show the existence of a unique strong solution derived by the Galerkin approximation. In [13], the existence and uniqueness result for a stochastic differential equation has been adopted. The fixed point theorem has only provided us with a local solution, and we choose the existence and uniqueness theorem to show a unique strong solution obtained through the Galerkin approximation exists globally.

We rely on the entropy method, which is the main tool in the deterministic cross-diffusion system, to conclude the nonnegativeness of the solution to the stochastic model. In our coefficients scheme, we are able to prove a weak solution exists to the deterministic cross-diffusion equation indexed by \( \eta \), which is derived through the Wong-Zakai approximation. We implement the entropy density

\[
h_s(u) = \sum_{i=1}^{n} \pi_i^{s} \left( \frac{u_i^{s}}{s} + u_i (\log u_i - 1) + 1 \right),
\]

and do not have to consider adding a “regularizer” with the term \( u_i (\log u_i - 1) + 1 \) multiplied by \( \varepsilon \), and this \( \varepsilon \) vanishes through the regularization process.

We observe that uniform estimates based on the matrix analysis can be extended to \( 1 \leq s < 2 \), if we are able to first of all establish the existence of a strong solution obtained
through the Galerkin approximation. Actually, we are allowed to extend the existence result to the case when $s = 1$.

For $1 < s < 2$, so far to our best efforts, we are not able to locate a constant $C > 0$ independent of $y, z$, $y = (y_1, \ldots, y_n) \in \mathbb{R}_n^+$, $z = (z_1, \ldots, z_n) \in \mathbb{R}_n^+$, such that for every $y, z$, $\left| \sum_{i,j=1}^{n} (A_{ij}(y) - A_{ij}(z)) \right| \leq C \sum_{i=1}^{n} |y_i - z_i|$. Without this Lipschitz property can be verified, no existence result of a strong solution $u^{(N)}$ can be derived. If $s = 1$, such phenomenon disappears and we can proceed with a nonnegative strong solution for an estimation.

1.3. Main steps. We mainly derive proving strategies from [2, 3, 6, 9, 11, 12, 13]. Firstly, we prove the existence of a unique strong solution to a sequence of approximated equations obtained through the Galerkin approximation, by applying the existence and uniqueness result for a stochastic differential equation ([24], Theorem 3.1.1, original edition from [23]).

The second step is by the Wong-Zakai approximation method ([26, 27, 29]), we show that this strong solution is nonnegative, $\mathbb{P}$-a.s. The system proved with a unique strong solution in the first step coincides with our original Galerkin approximated equations system.

The third step is to provide uniform estimates of those approximated solutions, in the purpose of proving the tightness of this approximated sequence in a certain topological space.

The last step is, by applying some fundamental tools in stochastic analysis ([11, 12, 13, 18]), we find another sequence possessing same laws to the existing one, and show that this sequence indeed converges $\mathbb{P}$-a.s. to a weak solution of (1)-(3).

In the Wong-Zakai approximation after the Galerkin approximation step, the entropy method with Aubin-Lions Lemma been applied to each deterministic equation, with the entropy itself guarantees the nonnegativeness of solutions to those approximating deterministic equations.

Concerning the existence of weak solutions proof to these deterministic equations, a regularized entropy density is the key factor in the existence proof of [6, 20]. We have actually cited this similar idea, which is also reflected in [11], Section 3.2.

As the assumption for coefficients has been strengthened, we are allowed to derive the existence result by directly applying an entropy density with its derivative’s range all real numbers, but without regularization. The entropy $h_s(u)$ is not the same entropy applied in other references, e.g. [6, 8, 9], and we do not make much omittance in the proof of the Lemma 8 when the existence of weak solutions to a sequence of deterministic equations being discussed.

For the $s = 1$ case, we choose the entropy as $h_1(u) = \sum_{i=1}^{n} \pi_i (u_i (\log u_i - 1) + 1)$, with this case been intensively investigated in [5, 6]. As we focus our attention on the superquadratic case, we do not include the $s = 1$ case in our main theorem.

1.4. Stochastic background, assumptions and main results. First of all, we declare in the following discussion, notations $C, C_i, \alpha_i, \beta_i, \gamma_i, i \in \mathbb{N}$, are constants that may change their values from line to line.
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a complete right continuous filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\). The space \(L^2(\mathcal{O})\) is the vector space of all square integrable functions \(u : \mathcal{O} \to \mathbb{R}\) with the inner product \((\cdot, \cdot)_{L^2(\mathcal{O})}\). The space \(H^1(\mathcal{O})\) contains every \(u \in L^2(\mathcal{O})\) such that the distributional derivatives \(\partial u / \partial x_1, ..., \partial u / \partial x_n\) belong to \(L^2(\mathcal{O})\). Let \(H\) be a Hilbert space, \(L^2(\Omega; H)\) represents the space consisting of all \(H\)-valued random variables \(u\) with \(\mathbb{E}\|u\|^2_H = \int_{\Omega} \|u(\omega)\|^2_H d\mathbb{P}(\omega) < \infty\).

In subsequent sections, \(H\) often refers to a space with variables time and space involved. The \(L^2(\Omega)\) norm of a vector-valued random variable \(u = (u_1, u_2, ..., u_n)\) is understood as \(\|u\|^2_{L^2(\Omega)} = \sum_{i=1}^n \|u_i\|^2_{L^2(\mathcal{O})}\). A Hilbert basis denoted as \(\{\eta_k\}_{k=1}^\infty\) is given to the space \(L^2(\mathcal{O})\). Choose an orthonormal basis of \(\mathbb{R}^n\) as \(\{\eta_k\}_{k=1}^\infty\), and

\[
\mathcal{L}(\mathbb{R}^n; L^2(\mathcal{O})) = \left\{ L : \mathbb{R}^n \to L^2(\mathcal{O}) \text{ linear continuous: } \sum_{k=1}^n \|L \eta_k\|^2_{L^2(\mathcal{O})} < \infty \right\}
\]

as the space of Hilbert-Schmidt operators from \(\mathbb{R}^n\) to \(L^2(\mathcal{O})\) endowed with the norm \(\|L\|^2_{\mathcal{L}(\mathbb{R}^n; L^2(\mathcal{O}))} = \sum_{k=1}^n \|L \eta_k\|^2_{L^2(\mathcal{O})}\).

Let \((\beta_{jk})_{j=1, ..., n, k \in \mathbb{N}}\) be a sequence of independent one-dimensional Brownian motions, and for \(j = 1, ..., n\), let \(W_j(x, t, \omega) = \sum_{k \in \mathbb{N}} \eta_k(x) \beta_{jk}(t, \omega)\) be a cylindrical Brownian motion. The expression \(\sigma_{ij}(u) dW_j(t)\) formally means that \(\sigma_{ij}(u) dW_j(t) = \sum_{k, l \in \mathbb{N}} \sigma_{ijkl}(u) e_l d\beta_{jk}(t)\), where \(\sigma_{ijkl}(u) = (\sigma_{ij}(u))_{k, l \in \mathbb{N}}\).

Let us give assumptions on multiplicative noise terms \(\sigma = \sigma_{ij}(u, t, \omega) : L^2(\mathcal{O}) \times [0, T] \times \Omega \to \mathcal{L}(\mathbb{R}^n; L^2(\mathcal{O}))\). Noise terms \(\sigma = \sigma_{ij}(u, t, \omega)\) are assumed to be \(\mathcal{B}(L^2(\mathcal{O}) \otimes [0, T] \otimes \mathcal{F}; \mathcal{B}(\mathbb{R}^n; L^2(\mathcal{O})))\)-measurable and \(\mathcal{F}\)-adapted with the property that there exists a positive constant \(C\), such that for every \(u, v \in L^2(\mathcal{O})\) and \(1 \leq i, j, l \leq n\),

\[
\|\sigma_{ij}(u)\|^2_{\mathcal{L}(\mathbb{R}^n; L^2(\mathcal{O}))} + \|u^\top \sigma_{ij}(u)\|^2_{\mathcal{L}(\mathbb{R}^n; L^2(\mathcal{O}))} \leq C(1 + \|u\|^2_{L^2(\mathcal{O})}),
\]

and

\[
\|\partial_{u^l} \sigma_{ij}(u)\|_{\mathcal{L}(\mathbb{R}^n; L^2(\mathcal{O}))} \leq C, \quad \|\partial_{u^l} \sigma_{ij}(u) - \partial_{u^l} \sigma_{ij}(v)\|_{\mathcal{L}(\mathbb{R}^n; L^2(\mathcal{O}))} \leq C\|u - v\|_{L^2(\mathcal{O})},
\]

and

\[
\|\partial_{u^l} (u^l \sigma_{ij}(u))\|^2_{\mathcal{L}(\mathbb{R}^n; L^2(\mathcal{O}))} \leq C(1 + \|u\|^2_{L^2(\mathcal{O})}).
\]

Let us give the definition of the solutions to (1)-(3). After this definition been given, we devote the rest part showing such kind of solutions exist for (1)-(3).

**Definition 1.** Let \(T > 0\) be an arbitrary positive number, the system \((\tilde{U}, \tilde{W}, \tilde{u})\) is a global martingale solution to (1)-(3) if \(\tilde{U} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}})\) is a stochastic basis with filtration \(\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \in (0, T)}\), \(\tilde{W}\) is a cylindrical Wiener process, and \(\tilde{u}(t) = (\tilde{u}_1(t), \tilde{u}_2(t), ..., \tilde{u}_n(t))\) is an \(\tilde{\mathcal{F}}_t\)-adapted stochastic process for every \(t \in (0, T)\) such that for \(1 \leq i \leq n\),

\[
\tilde{u}_i \in L^2(\tilde{\Omega}; C^0([0, T]; L^2(\mathcal{O}))) \cap L^2(\tilde{\Omega}; L^2(0, T; H^1(\mathcal{O}))),
\]
the law of $\tilde{u}_i(0)$ is the same as for $u_i^0$, and $\tilde{u}$ satisfies for every $\phi \in H^1(\mathcal{O})$ and $1 \leq i \leq n$,
\[
(\tilde{u}_i(t), \phi)_{L^2(\mathcal{O})} = (\tilde{u}_i(0), \phi)_{L^2(\mathcal{O})} - \sum_{j=1}^{n} \int_{0}^{t} \langle (A_{ij}(\tilde{u}(r))) \nabla \tilde{u}_j(r), \nabla \phi \rangle \, dr \\
+ \left( \sum_{j=1}^{n} \int_{0}^{t} \sigma_{ij}(\tilde{u}(r)) d\tilde{W}_j(r), \phi \right)_{L^2(\mathcal{O})}.
\]

$C^0([0,T]; L^2(\mathcal{O}))$ represents all weakly continuous functions $u : [0,T] \rightarrow L^2(\mathcal{O})$ having the property $\sup_{0 < t < T} ||u(t)||_{L^2(\mathcal{O})} < \infty$. This notation will reappear in Section 2.3, along with several notations of spaces and respective topologies. They have detailed explanation in [2] and [13], so we will present them directly without much explanation.

In the existence of weak solutions analysis of this stochastic cross-diffusion population model, a key factor is coefficients for those transition rates need to satisfy a detailed-balance condition, plus an assumption indicating self-diffusion “dominates” cross-diffusion. The condition is given below as (9). There exists a sequence of positive numbers $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ such that

\[
\pi_i a_{ij} = \pi_j a_{ji}, \quad (s + 1)a_{ii} > \frac{s^2}{4} \sum_{j=1, j \neq i}^{n} a_{ij}, \quad \text{for every } 1 \leq i, j \leq n.
\]

We also present a condition originated from [6], which is a weaker condition than (9), given as

\[
\pi_i a_{ij} = \pi_j a_{ji}, \quad (s + 1)a_{ii} > (s - 1) \sum_{j=1, j \neq i}^{n} a_{ij}, \quad \text{for every } 1 \leq i, j \leq n.
\]

For deterministic cross-diffusion equations, readers may be more familiar with (10) when applying the entropy method. Coefficients $(a_{ij})_{1 \leq i, j \leq n}$ satisfying (9) automatically satisfy (10). When $s = 2$, (9) and (10) coincide.

For the main theorem, we also need a further assumption describing the interaction between the entropy and noise terms, which is the below (11). We notice that $\frac{\partial h_{s}}{\partial u_i} = \pi_i(u_i^{s-1} + \log u_i)$ and $h_s''(u) = PQ(u)$, with $P = diag(\pi_1, \ldots, \pi_n)$, $Q(u) = diag(\frac{1}{u_1} + (s - 1)u_1^{s-2}, \ldots, \frac{1}{u_n} + (s - 1)u_n^{s-2})$. The assumption (11) is required when applying the entropy $h_s(u)$, which is, for every $u \in (0, \infty)^n$,

\[
\frac{\max_{j=1, \ldots, n} \left| \sum_{i=1}^{n} \pi_i \sigma_{ij}(u)(u_i^{s-1} + \log u_i) \right| + \frac{1}{2} \left| \sum_{i,j,l=1}^{n} \pi_i \frac{\partial \sigma_{ij}}{\partial u_l}(u_\sigma_{ij}(u)) (u_i^{s-1} + \log u_i) \right|}{\sum_{i=1}^{n} \left( \frac{u_i^s}{s} + u_i(\log u_i - 1) + 1 \right)} \leq C \sum_{i=1}^{n} \left( \frac{u_i^s}{s} + u_i(\log u_i - 1) + 1 \right).
\]

For the initial data and the dimension $d$, we require that for every $1 \leq i \leq n$,

\[
u_i^0 \geq 0 \text{ a.e. in } \mathcal{O}, \quad E\|u_i^0\|_{L^2(\mathcal{O})}^p < \infty, \quad p = \frac{24}{4 - d}, \quad d \leq 3.
\]
In below proof sections, we always assume that \( s \geq 2, (7), (8), (9), (11) \) and (12) hold, with (10) be automatically satisfied.

**Theorem 1.** (Existence of a global martingale solution) Let \( T > 0 \) be an arbitrary positive number, and let \( \sigma = (\sigma_{ij})_{i,j=1}^{n} \) with \( \sigma_{ij} : L^{2}(\mathcal{O}) \times [0, T] \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^{n}, L^{2}(\mathcal{O})) \). If (7), (8), (9), (11) and (12) hold, \( s \geq 2 \), then there exists a global nonnegative martingale solution to (1)–(3), \( \mathbb{P} \)-a.s.

2. **Matrix analysis**

We again mention that \( s \geq 2 \), and for every \( 1 \leq i, j \leq n \), the relation \( \pi_{i}a_{ij} = \pi_{j}a_{ji} \) holds for below all four matrix analysis lemmas, with the application of this condition been implicitly contained in computations.

**Lemma 1.** For every \( z = (z_{1}, z_{2}, ..., z_{n}) \in \mathbb{R}^{n} \) and \( u = (u_{1}, u_{2}, ..., u_{n}) \), \( u_{i} \geq 0 \), \( 1 \leq i \leq n \), there exist constants \( \alpha_{1} > 0, \alpha_{2} > 0 \) such that

\[
\sum_{i,j=1}^{n} \pi_{i}A_{ij}(u)z_{i}z_{j} \geq \alpha_{1} \sum_{i=1}^{n} z_{i}^{2} + \alpha_{2} \sum_{i=1}^{n} u_{i}^{s}z_{i}^{2}.
\]

**Proof.** Let us define a new matrix \( \bar{A}(u) = (\bar{A}_{ij}(u)) \), with \( \bar{A}_{ii}(u) = \frac{s^{2}}{4} \sum_{k=1, k \neq i}^{n} a_{ik}u_{i}^{s} + \sum_{k=1, k \neq i}^{n} a_{ik}u_{k}^{s} = \sum_{k=1, k \neq i}^{n} \left( \frac{s^{2}}{4} a_{ik}u_{i}^{s} + a_{ik}u_{k}^{s} \right) \), and \( \bar{A}_{ij}(u) = A_{ij}(u) \) if \( i \neq j \).

By the assumption (9), \((s + 1)a_{ii} > \frac{s^{2}}{4} \sum_{k=1, k \neq i}^{n} a_{ik} \), there exist positive constants \( \{\beta_{i}\}_{i=1,...,n} \), such that for every \( \pi_{i} > 0 \), \( \pi_{i}A_{ii}(u) - \pi_{i}A_{ii}(u) \geq \pi_{i}a_{i0} + \beta_{i}u_{i}^{s} \), and

\[
\sum_{i,j=1}^{n} \pi_{i}A_{ij}(u)z_{i}z_{j} \geq \sum_{i,j=1}^{n} \pi_{i}\bar{A}_{ij}(u)z_{i}z_{j} + \sum_{i=1}^{n} \pi_{i}a_{i0}z_{i}^{2} + \sum_{i=1}^{n} \beta_{i}u_{i}^{s}z_{i}^{2}.
\]

Provided that \( \sum_{i,j=1}^{n} \pi_{i}\bar{A}_{ij}(u)z_{i}z_{j} \geq 0 \), we have from (13) that \( \sum_{i=1}^{n} \pi_{i}A_{ij}(u)z_{i}z_{j} \geq \sum_{i=1}^{n} \pi_{i}a_{i0}z_{i}^{2} + \sum_{i=1}^{n} \beta_{i}u_{i}^{s}z_{i}^{2} \), choose \( \alpha_{1} = \min\{\pi_{i}a_{i0} : 1 \leq i \leq n\} \), \( \alpha_{2} = \min\{\beta_{i} : 1 \leq i \leq n\} \), we can show this lemma.
We are left to show that $\sum_{i,j=1}^{n} \pi_i \bar{A}_{ij}(u) z_i z_j \geq 0$. Denote $\bar{A}_i^k(u) = \frac{s^2}{4} a_{ik} u_i^s + a_{ik} u^2_i$, $k \neq i$, then $\bar{A}_i(u) = \sum_{k=1,k \neq i}^{n} \bar{A}_i^k(u)$, and therefore,

$$\sum_{i,j=1}^{n} \pi_i \bar{A}_{ij}(u) z_i z_j = \sum_{i=1}^{n} \pi_i \bar{A}_i^2(u) z_i^2 + \sum_{i=1}^{n} \sum_{j=1,j \neq i}^{n} \pi_i \bar{A}_{ij}(u) z_i z_j$$

$$= \sum_{i=1}^{n} \sum_{j=1,j \neq i}^{n} \left( \pi_i \bar{A}_i^2(u) z_i^2 + \pi_i \bar{A}_{ij}(u) z_i z_j \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1,j < i}^{n} \left( \pi_i \bar{A}_i^2(u) z_i^2 + \pi_i \bar{A}_{ij}(u) z_i z_j \right) + \sum_{i=1}^{n} \sum_{j=1,j > i}^{n} \left( \pi_i \bar{A}_i^2(u) z_i^2 + \pi_i \bar{A}_{ij}(u) z_i z_j \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1,j < i}^{n} \left( \pi_i \bar{A}_i^2(u) z_i^2 + \pi_i \bar{A}_{ij}(u) z_i z_j \right) + \sum_{i=1}^{n} \sum_{j=1,j < i}^{n} \left( \pi_j \bar{A}_{ij}(u) z_j^2 + \pi_j \bar{A}_{ji}(u) z_j z_i \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1,j < i}^{n} \left[ \pi_i \bar{A}_i^2(u) z_i^2 + \pi_i \bar{A}_{ij}(u) z_i z_j + \pi_j \bar{A}_{ji}(u) z_j z_i + \pi_j \bar{A}_{jj}(u) z_j^2 \right].$$

Let us show that if $i \neq j$, either $u_i > 0$ or $u_j > 0$, then $\pi_i \bar{A}_{i}(u) z_i^2 + \pi_j \bar{A}_{j}(u) z_j^2 \geq 0$, which is equivalent to show $(\pi_i \bar{A}_{i}(u) + \pi_j \bar{A}_{j}(u))^2 \leq 4\pi_i \bar{A}_{i}(u) \pi_j \bar{A}_{j}(u)$.

We check that $\pi_i \bar{A}_{i}(u) + \pi_j \bar{A}_{j}(u) = s \pi_i a_{i}(u, u_j^{-1} + u_j u_i^{-1})$, so $(\pi_i \bar{A}_{i}(u) + \pi_j \bar{A}_{j}(u))^2 = \pi_i^2 a_{i}^2 (s^2 u_i^{2s-2}u_j^2 + 2s^2 u_i^{-s}u_j^s + s^2 u_i^{2s}u_j^{-2s})$. Also, $4\pi_i \bar{A}_{i}(u) \pi_j \bar{A}_{j}(u) = \pi_i^2 a_{i}^2 (s^2 u_i^{2s} + (\frac{s^4}{4} + 4) u_i^{2s}u_j^2 + s^2 u_i^{2s}u_j^{-2s}).$

So long as $u_i^{2s} + u_j^{2s} - u_i^{2s-2}u_j^2 - u_j^{2s-2} = (u_i^{2s-2} + u_j^{2s-2})(u_i^2 - u_j^2)$, $s \geq 2$, thus $u_i^{2s} + u_j^{2s} - u_i^{2s-2}u_j^2 - u_j^{2s-2}u_i^2 \geq 0$, so $s^2 u_i^{2s} + s^2 u_j^{2s} \geq s^2 u_i^{2s-2}u_j^2 + s^2 u_j^{2s-2}u_i^2$.

The fact that $\frac{s^4}{4} + 4 \geq 2s^2$ indicates that $(\frac{s^4}{4} + 4) u_i^{2s}u_j^2 \geq 2s^2 u_i^{2s}u_j^2$, and we conclude that $(\pi_i \bar{A}_{i}(u) + \pi_j \bar{A}_{j}(u))^2 \leq 4\pi_i \bar{A}_{i}(u) \pi_j \bar{A}_{j}(u)$.

If $i \neq j$, $u_i = u_j = 0$, then $\pi_i \bar{A}_{i}(u) z_i^2 + \pi_j \bar{A}_{j}(u) z_j^2 = 0$, and we finish the proof of this lemma.

\[\square\]

**Lemma 2.** For every $z = (z_1, z_2, ..., z_n) \in \mathbb{R}^n$ and $u = (u_1, u_2, ..., u_n) \in \mathbb{R}_+^n$, there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\sum_{i,j=1}^{n} \frac{\pi_i}{u_i} A_{ij}(u) z_i z_j \geq \alpha_1 \sum_{i=1}^{n} \frac{z_i^2}{u_i} + \alpha_2 \sum_{i=1}^{n} u_i^{s-1} z_i^2.$$

**Proof.** More precisely, $\frac{\pi_i}{u_i} A_{ij}(u) = \frac{\pi_i a_{ij}}{u_i} + (s+1) \pi_i a_{i}(u) u_i^{-s-1} + \frac{\pi_i}{u_i} \sum_{k=1,k \neq i}^{n} a_{ik} u_k^{s-1}$, and $\frac{\pi_i}{u_i} A_{ij}(u) = s \pi_i a_{i}(u) u_i^{s-1}$, if $i \neq j$. Still adopting the notation $\bar{A}(u) = (\bar{A}_{ij}(u))$, and with the same
definition to the above Lemma 1. Since $(s+1)\alpha_i > \frac{s^2}{4} \sum_{k=1, k \neq i}^{n} a_{ik}$, there exist positive constants \( \{\beta_i\}_{i=1, \ldots, n} \), such that for every \( \pi_i > 0 \),

\[
\sum_{i, j=1}^{n} \pi_i \bar{A}_{ij}(u)z_i z_j \geq \sum_{i, j=1}^{n} \pi_i \bar{A}_{ii}(u)z_i^2 + \sum_{i=1}^{n} \frac{n}{n} \sum_{j=1, j \neq i}^{n} \frac{\pi_i}{\pi_j} \bar{A}_{ij}(u)z_i z_j + \sum_{i=1}^{n} \beta_i u_i^{s-1} z_i^2.
\]

Provided that \( \sum_{i, j=1}^{n} \pi_i \bar{A}_{ij}(u)z_i z_j \geq 0 \), we have from (14) that \( \sum_{i, j=1}^{n} \pi_i \bar{A}_{ij}(u)z_i z_j \geq \sum_{i, j=1}^{n} \pi_i u_i^{s-1} z_i^2 + \sum_{i=1}^{n} \beta_i u_i^{s-1} z_i^2 \), choose \( \alpha_1 = \min\{\pi_i a_{ii} : 1 \leq i \leq n\} \), \( \alpha_2 = \min\{\beta_i : 1 \leq i \leq n\} \), we can show this lemma.

We are left to prove that \( \sum_{i, j=1}^{n} \frac{\pi_i}{\pi_j} \bar{A}_{ij}(u)z_i z_j \geq 0 \), and

\[
\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u)z_i^2 + \frac{\pi_i}{\pi_j} \bar{A}_{ij}(u)z_i z_j \right)
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u)z_i^2 + \frac{\pi_i}{\pi_j} \bar{A}_{ij}(u)z_i z_j \right)
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u)z_i^2 + \frac{\pi_i}{\pi_j} \bar{A}_{ij}(u)z_i z_j \right) + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left( \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u)z_j^2 + \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u)z_j z_i \right)
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u)z_i^2 + \frac{\pi_j}{\pi_i} \bar{A}_{ij}(u)z_i z_j \right) + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left( \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u)z_j^2 + \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u)z_j z_i \right)
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left[ \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u)z_i^2 + \frac{\pi_j}{\pi_i} \bar{A}_{ij}(u)z_i z_j \right] + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left[ \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u)z_j^2 + \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u)z_j z_i \right].
\]

Let us show that if \( i \neq j \), \( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u)z_i^2 + \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u)z_j^2 \geq 0 \), which is equivalent to show \( \left( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u) + \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u) \right)^2 \leq 4 \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u) \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u) \).

Since \( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u) + \frac{\pi_j}{\pi_i} \bar{A}_{ii}(u) = s \pi_i a_{ii} (u)_{j=1}^{s-1} + u_{i}^{s-1} \), so \( \left( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u) + \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u) \right)^2 = \pi_i a_{ii}^2 (u) + 2 s u_{i}^{s-1} u_{j}^{s-1} + s^2 u_{j}^{s-2} \). Also, \( 4 \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u) \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u) = a_{ii}^2 (u) + 4 u_{i}^{s-1} u_{j}^{s-1} + s^2 u_{j}^{s-2} \).

So long as \( u_{i}^{s-1} + u_{j}^{s-1} - u_{i}^{s-1} u_{j}^{s-1} = (u_{i}^{s-1} - u_{j}^{s-1})(u_{i} - u_{j}) \), \( s \geq 2 \), \( u_{i}^{s-1} + u_{j}^{s-1} - u_{i}^{s-1} u_{j}^{s-1} \geq 0 \), so \( \frac{s^2}{u_{i}^{s-1} u_{j}^{s-1}} \geq s^2 u_{i}^{s-2} + s^2 u_{j}^{s-2} \).

Since \( (\frac{s^2}{u_{i}^{s-1}} + 4) u_{i}^{s-1} u_{j}^{s-1} \geq 2 s^2 u_{i}^{s-1} u_{j}^{s-1} \), we conclude that \( \left( \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u) + \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u) \right)^2 \leq 4 \frac{\pi_i}{\pi_j} \bar{A}_{ii}(u) \frac{\pi_j}{\pi_i} \bar{A}_{jj}(u) \), then we finish the proof of this lemma.
Lemma 3. For every \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \) and \( u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}_+^n \), there exist constants \( \alpha_1 > 0 \), \( \alpha_2 > 0 \) such that

\[
\sum_{i,j=1}^{n} \pi_i u_i^{s-2} A_{ij}(u) z_i z_j \geq \alpha_1 \sum_{i=1}^{n} u_i^{s-2} z_i^2 + \alpha_2 \sum_{i=1}^{n} u_i^{2s-2} z_i^2.
\]

Proof. More precisely, \( \pi_i u_i^{s-2} A_{ii}(u) = \pi_i a_{i0} u_i^{s-2} + (s+1) \pi_i a_{ii} u_i^{2s-2} + \pi_i u_i^{s-2} \sum_{k=1, k \neq i}^{n} a_{ik} u_k^s \), and \( \pi_i u_i^{s-2} A_{ij}(u) = s \pi_i a_{ij} u_i^{s-1} u_j^{s-1} \), if \( i \neq j \). In this lemma, the matrix \( \bar{A}(u) = (\bar{A}_{ij}(u)) \) is given by \( \bar{A}_{ii}(u) = (s-1) \sum_{k=1, k \neq i}^{n} a_{ik} u_i^s + \sum_{k=1, k \neq i}^{n} a_{ik} u_k^s = \sum_{k=1, k \neq i}^{n} ((s-1) a_{ik} u_i^s + a_{ik} u_k^s) \), and \( \bar{A}_{ij}(u) = A_{ij}(u) \) if \( i \neq j \).

Based on the assumption \( (s+1)a_{ii} > (s-1) \sum_{k=1, k \neq i}^{n} a_{ik} \), there exist positive constants \( \{ \beta_i \}_{i=1, \ldots, n} \), such that for every \( \pi_i > 0 \), \( \pi_i A_{ii}(u) - \pi_i \bar{A}_{ii}(u) \geq \pi_i a_{i0} + \beta_i u_i^s \), and

\[
\sum_{i,j=1}^{n} \pi_i u_i^{s-2} A_{ij}(u) z_i z_j \geq \sum_{i=1}^{n} \pi_i u_i^{s-2} A_{ii}(u) z_i^2 + \sum_{i=1}^{n} \pi_i a_{i0} u_i^{s-2} z_i^2 + \sum_{i=1}^{n} \beta_i u_i^{2s-2} z_i^2.
\]

Provided that \( \sum_{i,j=1}^{n} \pi_i u_i^{s-2} \bar{A}_{ij}(u) z_i z_j \geq 0 \), from (15) we have \( \sum_{i,j=1}^{n} \pi_i u_i^{s-2} A_{ij}(u) z_i z_j \geq \sum_{i=1}^{n} \pi_i a_{i0} u_i^{s-2} z_i^2 + \sum_{i=1}^{n} \beta_i u_i^{2s-2} z_i^2 \), choose \( \alpha_1 = \min \{ \pi_i a_{i0} : 1 \leq i \leq n \} \), \( \alpha_2 = \min \{ \beta_i : 1 \leq i \leq n \} \), we can show this lemma.

We are left to prove that \( \sum_{i,j=1}^{n} \pi_i u_i^{s-2} \bar{A}_{ij}(u) z_i z_j \geq 0 \). Denote \( \bar{A}_{ii}(u) = (s-1) a_{ik} u_i^s + a_{ik} u_k^s \), \( k \neq i \), then \( \bar{A}_{ii}(u) = \sum_{k=1, k \neq i}^{n} \bar{A}_{ii}(u) \), and therefore,

\[
\sum_{i,j=1}^{n} \pi_i u_i^{s-2} \bar{A}_{ij}(u) z_i z_j = \sum_{i=1}^{n} \pi_i u_i^{s-2} \bar{A}_{ii}(u) z_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \pi_i u_i^{s-2} \bar{A}_{ij}(u) z_i z_j
\]

\[
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \left( \pi_i u_i^{s-2} \bar{A}_{ii}(u) z_i^2 + \pi_i u_i^{s-2} \bar{A}_{ij}(u) z_i z_j \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1, j < i}^{n} \pi_i u_i^{s-2} \left( \bar{A}_{ii}(u) z_i^2 + \bar{A}_{ij}(u) z_i z_j \right) + \sum_{i=1}^{n} \sum_{j=i}^{n} \pi_i u_i^{s-2} \left( \bar{A}_{ii}(u) z_i^2 + \bar{A}_{ij}(u) z_i z_j \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1, j < i}^{n} \pi_i u_i^{s-2} \left( \bar{A}_{ij}(u) z_1^2 + \bar{A}_{ij}(u) z_i z_j \right) + \sum_{i=1}^{n} \sum_{j=1, j < i}^{n} \pi_j u_j^{s-2} \left( \bar{A}_{jj}(u) z_j^2 + \bar{A}_{jj}(u) z_j z_i \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1, j < i}^{n} \left[ \pi_i u_i^{s-2} \bar{A}_{ii}(u) z_i^2 + \left( \pi_i u_i^{s-2} \bar{A}_{ij}(u) + \pi_j u_j^{s-2} \bar{A}_{ji}(u) \right) z_i z_j + \pi_j u_j^{s-2} \bar{A}_{jj}(u) z_j^2 \right].
\]

Let us show \( \pi_i u_i^{s-2} \bar{A}_{ii}(u) z_i^2 + \left( \pi_i u_i^{s-2} \bar{A}_{ij}(u) + \pi_j u_j^{s-2} \bar{A}_{ji}(u) \right) z_i z_j + \pi_j u_j^{s-2} \bar{A}_{jj}(u) z_j^2 \geq 0 \), if \( i \neq j \), which is equivalent to \( \left( \pi_i u_i^{s-2} \bar{A}_{ij}(u) + \pi_j u_j^{s-2} \bar{A}_{ji}(u) \right)^2 \leq 4 \pi_i u_i^{s-2} \bar{A}_{ii}(u) \pi_j u_j^{s-2} \bar{A}_{jj}(u) \).

Since \( \pi_i u_i^{s-2} \bar{A}_{ij}(u) + \pi_j u_j^{s-2} \bar{A}_{ji}(u) = 2 s \pi_{a_{ij}} u_i^{s-1} u_j^{s-1} \), and \( s \geq 2 \), \( (s-1)^2 + 1 = 2(s-1)^2 \), \( u_i^2 u_j^2 + (s-1)^2 + 1 \), thus \( (s-1) u_i^2 + (s-1)^2 + 1 \), \( u_i^2 u_j^2 + (s-1) u_i^2 \geq s^2 u_i^2 u_j^2 \), which
is equivalent to $((s-1)u_i^s + u_j^s)(u_i^s + (s-1)u_j^s) \geq s^2u_i^su_j^s$. Then

\[
\left(\pi_iu_i^{s-2}\bar{A}_{ij}(u) + \pi_ju_j^{s-2}\bar{A}_{ji}(u)\right)^2 = 4s^2\pi_i^2a_{ij}^2u_i^{2s-2}u_j^{2s-2} = 4\pi_i^2a_{ij}^2u_i^{s-2}u_j^{s-2}s^2u_i^s u_j^s
\]

\[
\leq 4\pi_i^2a_{ij}^2((s-1)u_i^{2s-2} + u_i^{s-2}u_j^s)(u_i^{s-2}u_j^s + (s-1)u_j^{2s-2}) \leq 4\pi_iu_i^{s-2}\bar{A}_{ij}(u)\pi_ju_j^{s-2}\bar{A}_{jj}(u),
\]

and we finish the proof of this lemma.

\[
\square
\]

We define a new matrix $A^H(v)$, with $A^H(v) = a_{i0} + (s+1)a_{ii}v_i^2 + \sum_{k \neq i}^n a_{ik}v_k^2$, and $A^H_{ij} = sa_{ij}v_iv_j$, if $i \neq j$.

**Lemma 4.** For every $z = (z_1, z_2, ..., z_n) \in \mathbb{R}^n$ and $v = (v_1, v_2, ..., v_n)$, $v_i \geq 0$, $1 \leq i \leq n$, there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ such that

\[
\sum_{i,j=1}^n \pi_iA^H_{ij}(v)zi_zj \geq \alpha_1\sum_{i=1}^n z_i^2 + \alpha_2\sum_{i=1}^n v_i^2z_i^2.
\]

**Proof.** The idea of the proof basically follows the proof of the Lemma 11 and Lemma 12. We define a new matrix $A^H(v) = (A^H_{ij}(v))$, with $A^H_{ii}(v) = (s-1)\sum_{k=1, k \neq i}^n a_{ik}v_k^2 + \sum_{k=1, k \neq i}^n a_{ik}v_k^2 = \sum_{k=1, k \neq i}^n ((s-1)a_{ik}v_i^2 + a_{ik}v_k^2)$, and $A^H_{ij}(v) = A^H_{ji}(v)$ if $i \neq j$.

Based on the assumption $(s+1)a_{ii} > (s-1)\sum_{k=1, k \neq i}^n a_{ik}$, there exist positive constants $\{\beta_i\}_{i=1,...,n}$, such that for every $\pi_i > 0$, $\pi_iA^H_{ii}(v) - \pi_i\bar{A}^H_{ii}(v) \geq \pi_ia_{i0} + \beta_iv_i^2$, and

\[
\tag{16}
\sum_{i,j=1}^n \pi_iA^H_{ij}(v)zi_zj \geq \sum_{i,j=1}^n \pi_i\bar{A}^H_{ij}(v)zi_zj + \sum_{i=1}^n \pi_ia_{i0}z_i^2 + \sum_{i=1}^n \beta_iv_i^2z_i^2.
\]

Provided that $\sum_{i,j=1}^n \pi_i\bar{A}^H_{ij}(v)zi_zj \geq 0$, we have from (16) that $\sum_{i,j=1}^n \pi_iA^H_{ij}(v)zi_zj \geq \sum_{i=1}^n \pi_ia_{i0}z_i^2 + \sum_{i=1}^n \beta_iv_i^2z_i^2$, choose $\alpha_1 = \min\{\pi_ia_{i0} : 1 \leq i \leq n\}$, $\alpha_2 = \min\{\beta_i : 1 \leq i \leq n\}$, we can show this lemma.
We are left to prove that \( \sum_{i,j=1}^n \pi_i \bar{A}_{ij}^H(v) z_i z_j \geq 0 \). Denote \( (\bar{A}_{ii}^H)^k(v) = (s-1)a_{ik}v_i^2 + a_{ik}v_k^2, \) \( k \neq i \), then \( \bar{A}_{ii}^H(v) = \sum_{k=1,k\neq i}^n (\bar{A}_{ii}^H)^k(v) \), and therefore,

\[
\begin{align*}
\sum_{i,j=1}^n \pi_i \bar{A}_{ij}^H(v) z_i z_j &= \sum_{i=1}^n \pi_i \bar{A}_{ii}^H(v) z_i^2 + \sum_{i=1}^n \sum_{j=1,j\neq i}^n \pi_i \bar{A}_{ij}^H(v) z_i z_j \\
&= \sum_{i=1}^n \sum_{j=1,j\neq i}^n (\pi_i (\bar{A}_{ii}^H)^i(v) z_i^2 + \pi_i \bar{A}_{ij}^H(v) z_i z_j)
\end{align*}
\]

\[
= \sum_{i=1}^n \sum_{j=1,j<i}^n (\pi_i (\bar{A}_{ii}^H)^i(v) z_i^2 + \pi_i \bar{A}_{ij}^H(v) z_i z_j) + \sum_{i=1}^n \sum_{j=1,j>i}^n (\pi_i (\bar{A}_{ii}^H)^i(v) z_i^2 + \pi_i \bar{A}_{ij}^H(v) z_i z_j)
\]

\[
= \sum_{i=1}^n \sum_{j=1,j<i}^n (\pi_i (\bar{A}_{ii}^H)^i(v) z_i^2 + \pi_i \bar{A}_{ij}^H(v) z_i z_j) + \sum_{i=1}^n \sum_{j=1,j<i}^n (\pi_j (\bar{A}_{jj}^H)^j(v) z_j^2 + \pi_j \bar{A}_{jj}^H(v) z_j z_i)
\]

\[
= \sum_{i=1}^n \sum_{j=1,j<i}^n \left[ \pi_i (\bar{A}_{ii}^H)^i(v) z_i^2 + (\pi_i \bar{A}_{ij}^H(v) + \pi_j \bar{A}_{jj}^H(v))^i(v) z_i z_j + \pi_j (\bar{A}_{jj}^H)^j(v) z_j^2 \right].
\]

Let us show that if \( i \neq j \), either \( v_i > 0 \) or \( v_j > 0 \), then \( \pi_i (\bar{A}_{ii}^H)^i(v) z_i^2 + (\pi_i \bar{A}_{ij}^H(v) + \pi_j \bar{A}_{jj}^H(v)) z_i z_j + \pi_j (\bar{A}_{jj}^H)^j(v) z_j^2 \geq 0 \), which is equivalent to show \( (\pi_i \bar{A}_{ij}^H(v) + \pi_j \bar{A}_{jj}^H(v))^2 \leq 4\pi_i (\bar{A}_{ii}^H)^i(v)\pi_j (\bar{A}_{jj}^H)^j(v) \).

Since \( (s-1)v_i^4 - 2(s-1)v_i^2v_j^2 + (s-1)v_j^4 \geq 0 \), then \( (s-1)v_i^4 + (s-2s+2)v_i^2v_j^2 + (s-1)v_j^4 \geq s^2v_i^2v_j^2 \), which is equivalent to \( ((s-1)v_i^2 + v_j^2)(v_i^2 + (s-1)v_j^2) \geq s^2v_i^2v_j^2 \). We check that

\[
(\pi_i \bar{A}_{ij}^H(v) + \pi_j \bar{A}_{jj}^H(v))^2 \leq 4\pi_i a_{ij}^2 ((s-1)v_i^2 + v_j^2) v_i^2 + (s-1)v_j^2 \]

\[
\leq 4\pi_i (\bar{A}_{ii}^H)^i(v)\pi_j (\bar{A}_{jj}^H)^j(v).
\]

If \( i \neq j, v_i = v_j = 0 \), then \( \pi_i (\bar{A}_{ii}^H)^i(v) z_i^2 + (\pi_i \bar{A}_{ij}^H(v) + \pi_j \bar{A}_{jj}^H(v)) z_i z_j + \pi_j (\bar{A}_{jj}^H)^j(v) z_j^2 = 0 \), and we finish the proof of this lemma.

\[
\square
\]

3. Stochastic Galerkin Approximation and the Existence of a Strong Solution \( u^{(N)} \), with \( u^{(N)} \) Nonnegative, Probably Almost Sure

We fix an orthonormal basis \((e_k)_{k \geq 1}\) of \( L^2(\mathcal{O}) \) and a number \( N \in \mathbb{N} \) such that the Hilbert space \( H_N = \text{span}\{e_1, ..., e_N\} \) satisfies \( H_N \subset H^1(\mathcal{O}) \cap L^\infty(\mathcal{O}) \). We introduce the projection operator \( \Pi_N : L^2(\mathcal{O}) \to H_N \), with

\[
\Pi_N(v) = \sum_{i=1}^N (v, e_i)_{L^2(\mathcal{O})} e_i, \quad v \in L^2(\mathcal{O}).
\]
We implement the Wong-Zakai approximation process in below, by discretizing the Wiener process as
\[ W^{(n)}_j(t) = W_j(t_k) + \frac{t - t_k}{\eta} (W_j(t_{k+1}) - W_j(t_k)), \quad t \in [t_k, t_{k+1}], \quad k = 0, ..., M - 1, \]
with the above discretization process indexed by \( \eta = \frac{T}{M}, M \in \mathbb{N}, \) and \( t_k = k\eta. \) Please refer to [26, 27, 29] for a further understanding of this Wong-Zakai approximation technique. This technique has also been carried out in [11] and [12].

For every \( 1 \leq i \leq n, \) the Galerkin approximation equation for the system (1)-(3) is
\[ du^{(N)}_i - \Pi_N \div \left( \sum_{j=1}^{n} A_{ij}(u^{(N)}) \nabla u^{(N)}_j \right) dt = \sum_{j=1}^{n} \Pi_N (\sigma_{ij}(u^{(N)}) \nabla W_j(t)), \quad u^{(N)}_i(0) = \Pi_N (u^0_i), \]
and the ultimate goal for this section is to show a strong solution exists for (17), which is nonnegative, \( \mathbb{P} \)-a.s. In order to achieve this, we first of all consider below equations, with the random factor \( \omega \) be fixed, thus deterministic. For every \( \phi_i \in H_N, 1 \leq i \leq n, \)
\[ \langle u^{(N,\eta)}_i, \phi_i \rangle = \langle u^{(N,\eta)}_i(0), \phi_i \rangle - \int_0^t \sum_{j=1}^{n} \langle A_{ij}(u^{(N,\eta)}(r)) \nabla u^{(N,\eta)}_j(r), \nabla \phi_i \rangle dr + \int_0^t \langle f_i(u^{(N,\eta)}(r)), \phi_i \rangle dr, \]
with
\[ f_i(u^{(N,\eta)}) = \sum_{j=1}^{n} \Pi_N (\sigma_{ij}(u^{(N,\eta)})) \frac{dW^{(n)}_j}{dt}(t) - \frac{1}{2} \Pi_N \left( \sum_{j=1}^{n} \frac{\partial \sigma_{ij}}{\partial u_l}(u^{(N,\eta)}) \sigma_{lj}(u^{(N,\eta)}) \right), \]
and \( u^{(N,\eta)}_i(0) = \Pi_N (u^0_i). \)

For convenience, we short write \( u^{(N,\eta)} \) as \( u \) in the proof and statement of the below lemma. We also divide the below proof into three steps, for a better understanding.

**Lemma 5.** There exists a \( u = (u_1, ..., u_n) \), such that for every \( 1 \leq i \leq n, \phi_i \in H_N, \) \( u_i \) satisfies (18)-(19), and \( u_i \geq 0. \)

**Proof.** Step 1: solution of an approximated equation. Set \( \tau = \frac{T}{T}, L \in \mathbb{N}, \) \( \epsilon > 0 \) and \( u^{k-1} \in L^\infty(\mathcal{O}; \mathbb{R}^n) \) be given, and our purpose is to show the existence of such \( w^k \in H^1(\mathcal{O}; \mathbb{R}^n), \)
\[ \frac{1}{\tau} \int_\mathcal{O} (u(w^k) - u(w^{k-1})) \cdot \phi dx + \int_\mathcal{O} \sum_{i,j=1}^{n} B_{ij}(w^k) \nabla \phi_i \cdot \nabla w^k_j dx \]
\[ + \epsilon \int_\mathcal{O} w^k \cdot \phi dx = \int_\mathcal{O} f(u(w^k), t_k) \cdot \phi dx, \quad u^{(N,\eta)}(0) = \Pi_N (u^0), \]
for every \( \phi \in H^1(\mathcal{O}; \mathbb{R}^n), \) where \( u(w^k) = (h^k_\epsilon)^{-1}(w^k), B(w^k) = A(u(w^k))h^k_\epsilon(u(w^k))^{-1}. \)
Let $y \in L^\infty(O)$ and $\vartheta \in [0, 1]$ be given, and we claim that there exists a unique solution $w = (w_1, \cdots, w_n) \in H_N$ to the linear problem
\begin{equation}
(21) 
\quad a(w, \phi) = F(\phi), \quad \text{for all } \phi \in H_N,
\end{equation}
where
\begin{align*}
\quad a(w, \phi) &= \int_{O} \sum_{i,j=1}^{n} B_{ij}(y) \nabla \phi_i \cdot \nabla w_j \, dx + \varepsilon \int_{O} w \cdot \phi \, dx, \\
\quad F(\phi) &= -\frac{\vartheta}{\tau} \int_{O} (u(y) - u(w^{k-1})) \cdot \phi \, dx + \vartheta \int_{O} f(u(y), t_k) \cdot \phi \, dx.
\end{align*}

By the Cauchy-Schwartz inequality, we have
\begin{align*}
\frac{1}{2} \left( \int_{O} (u(y) - u(w^{k-1})) \cdot \phi \, dx \right)^2 &\leq \left( \int_{O} u(y) \cdot \phi \, dx \right)^2 + \left( \int_{O} u(w^{k-1}) \cdot \phi \, dx \right)^2 \\
&\leq \|u(y)\|_{L^2(O)}^2 \|\phi\|_{L^2(O)}^2 + \|u(w^{k-1})\|_{L^2(O)}^2 \|\phi\|_{L^2(O)}^2 \\
&\leq C \|\phi\|_{H^N}^2,
\end{align*}
and by the assumption $(\mathcal{A})$,
\begin{align*}
\left( \int_{O} f(u(y), t_k) \cdot \phi \, dx \right)^2 &\leq \|f(u(y), t_k)\|_{L^2(O)}^2 \|\phi\|_{L^2(O)}^2 \\
&\leq C_\eta \|\phi\|_{L^2(O)}^2 \cdot \left( \sum_{i,j=1}^{n} \|\sigma_{ij}(u)\|_{L^2(\mathbb{R}^n ; L^2(O))}^2 + \sum_{i,j,l=1}^{n} \| \frac{\partial \sigma_{ij}}{\partial u_l} (u) \sigma_{ij}(u) \|_{L^2(\mathbb{R}^n ; L^2(O))}^2 \right) \\
&\leq C_\eta \|\phi\|_{L^2(O)}^2 \cdot (1 + \|u\|_{L^2(O)}^2) \leq C_\eta \|\phi\|_{H^N}^2,
\end{align*}
which implies the boundedness of $F$ on $H_N$. Also, we check that
\begin{align*}
B_{ii}(y) &= \frac{(a_{ii} + (s + 1) a_{ii} y_i^s + \sum_{k=1, k \neq i}^{n} a_{ik} y_k^s y_i)}{\pi_i ((s - 1) y_i^{s-1} + 1)}, \\
B_{ij}(y) &= \frac{s a_{ij} y_i y_j^s}{\pi_j ((s - 1) y_j^{s-1} + 1)}, \quad i \neq j,
\end{align*}
and by the boundedness of $y$, for every $1 \leq i, j \leq n$, $B_{ij}(y)$ is bounded. Thus
\begin{align*}
\left( \int_{O} B_{ij}(y) \nabla \phi_i \cdot \nabla w_j \, dx \right)^2 &\leq \|B_{ij}(y)\|_{L^\infty(O)}^2 \|\nabla w_j\|_{L^2(O)}^2 \|\nabla \phi_i\|_{L^2(O)}^2 \leq C \|w\|_{H^N}^2 \|\phi\|_{H^N}^2,
\end{align*}
and by the fact that
\begin{align*}
\left( \int_{O} w \cdot \phi \, dx \right)^2 &\leq \|w\|_{L^2(O)}^2 \|\phi\|_{L^2(O)}^2 \leq C \|w\|_{H^1(O)}^2 \|\phi\|_{H^1(O)}^2 \leq C \|w\|_{H^N}^2 \|\phi\|_{H^N}^2,
\end{align*}
we obtain the boundedness of $a$ on $H_N$.

By the Lemma $2$ and the Lemma $3$ for every $y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$, $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$, we denote $(h^a_n(y))^{-1} z =: \bar{z} = (\bar{z}_1, \cdots, \bar{z}_n)$, and $A : B = \sum_{i,j=1}^{n} a_{ij} b_{ij}$ for matrices $A = (a_{ij})$. 

and $B = (b_{ij})$, then
\[
    z^\top B(y)z = (h_s''(y))^{-1}z : h_s''(y)A(y)(h_s''(y))^{-1}z = \bar{z} : h_s''(y)A(y)\bar{z}
\]
\[
    = (s - 1) \sum_{i,j=1}^n \pi_{ij} y_i^{s-2} A_{ij}(y) \bar{z}_i \bar{z}_j + \sum_{i,j=1}^n \pi_{ij} y_i^{-1} A_{ij}(y) \bar{z}_i \bar{z}_j \geq 0,
\]
and $B(w)$ is positive semi-definite.

Since $B(y), y \in \mathbb{R}_+^n$ is positive semi-definite, and all norms are equivalent in finite dimensions, $a(w, w) \geq \varepsilon \|w\|_{L^2(O)}^2 \geq \varepsilon C(N)\|w\|_{H_N}^2$, $a$ is coercive on $H_N$. By the Lax-Milgram lemma, there exists a unique solution $w \in H_N$ to \((21)\) and it holds that $w \in L^\infty(O; \mathbb{R}^n)$. This defines the fixed point operator $S : H_N \times [0, 1] \to H_N, S(y, \vartheta) = w$, where $w$ solves \((21)\).

Let us verify the assumptions of the Leray-Schauder theorem. The only solution to \((21)\) with $\vartheta = 0$ is $w = 0$, thus $S(y, 0) = 0$. For the continuous property of $S$, please refer to Lemma 5, \[20\] for the proof.

Since $H_N$ is finite dimensional, $S$ is compact. It remains to prove a uniform bound for all fixed points of $S(\cdot, \vartheta)$. Let $w \in H_N$ be such a fixed point, then $w$ solves \((21)\) with $y$ replaced by $w$. Choose the test function $\phi = w$, $\mathbb{P}$-a.s.

\[
\begin{aligned}
    \frac{\partial}{\partial \tau} \int_O (u(w) - u(w^{k-1})) \cdot wdx + \int_O \sum_{i,j=1}^n B_{ij}(w) \nabla w_i \nabla w_j dx \\
    + \varepsilon \int_O |w|^2 dx = 0 \int O f(u(w), t_k) \cdot wdx.
\end{aligned}
\]

\[(22)\]

The convexity of $h_s$ indicates that $\frac{\partial}{\partial \tau} \int_O (u(w) - u(w^{k-1})) \cdot wdx \geq \frac{\partial}{\partial \tau} \int_O (h_s(u(w)) - h_s(u(w^{k-1}))) dx$. $B(w)$ is positive semi-definite, we have $\sum_{i,j=1}^n B_{ij}(w) \nabla w_i \nabla w_j \geq 0$. For the reaction term, by the assumption \((11)\),

\[
    f(u(w), t_k) \cdot h_s'(u(w)) = \frac{1}{\eta} \sum_{i,j=1}^n \sigma_{ij}(u(w))(W_j(t_{k+1}) - W_j(t_k)) \cdot \frac{\partial h_s}{\partial w_i}(u(w))
\]

\[
    - \frac{1}{2} \sum_{i,j,l=1}^n \frac{\partial \sigma_{ij}}{\partial w_l}(u(w)) \sigma_{ij}(u(w)) \cdot \frac{\partial h_s}{\partial w_i}(u(w))
\]

\[
    \leq \frac{1}{\eta} \sum_{j=1}^n |W_j(t_{k+1}) - W_j(t_k)| \max_{j=1, \ldots, n} \left| \sum_{i=1}^n \sigma_{ij}(u(w)) \cdot \frac{\partial h_s}{\partial w_i}(u(w)) \right|
\]

\[
    + \frac{1}{2} \left| \sum_{i,j,l=1}^n \frac{\partial \sigma_{ij}}{\partial w_l}(u) \sigma_{ij}(u) \cdot \frac{\partial h_s}{\partial w_i}(u(w)) \right| \leq C_\eta (1 + h_s(u(w))).
\]

The right-hand-side of \((22)\) is bounded uniformly in $\vartheta$ and $w$, and $\varepsilon \int_O |w|^2 dx \leq C_\eta$, which follows for each fixed $\eta > 0, \|w\|_{H^1(O)} \leq C, \mathbb{P}$-a.s. Then we are able to apply the
Leray-Schauder fixed point theorem to show the existence of a weak solution \( w^k := w \in H_N \) to (20).

Step 2: uniform estimates. If we denote \( w^{(\tau)}(\omega, x, t) = w^k(\omega, x) \) and \( u^{(\tau)}(\omega, x, t) = u(w^k(\omega, x)) \) for \( \omega \in \Omega, x \in \mathcal{O} \), and \( t \in ((k-1)\tau, k\tau] \), \( k = 1, \ldots, L \). At time \( t = 0 \), we set \( w^{(\tau)}(\cdot, 0) = h'_s(u^0) \) and \( u^{(\tau)}(\cdot, 0) = u^0 \). Also we define a shift operator as \( (\Gamma u^{(\tau)})(\omega, x, t) = u(w^{k-1}(\omega, x)) \) for \( \omega \in \Omega, x \in \mathcal{O} \), and \( t \in ((k-1)\tau, k\tau] \). Then \( u^{(\tau)} \) solves

\[
\frac{1}{\tau} \int_0^T \int_\mathcal{O} (u^{(\tau)} - \Gamma u^{(\tau)}) \cdot \phi dx dt + \int_0^T \int_\mathcal{O} \sum_{i,j=1}^n A_{ij}(u^{(\tau)}) \nabla \phi_i \cdot \nabla u^{(\tau)}_j dx dt
\]

\[
+ \varepsilon \int_0^T \int_\mathcal{O} w^{(\tau)} \cdot \phi dx dt = \int_0^T \int_\mathcal{O} f(u^{(\tau)}) \cdot \phi dx dt,
\]

for piecewise constant functions \( \phi : (0, T) \to H_N \).

By the Lemma 2 and the Lemma 3, \( (26) \)

\[
\sum_{i,j=1}^n \left( h''_s(u(w^k))A(u(w^k)) \right)_{ij} \nabla u_i(w^k) \nabla u_j(w^k) \geq \gamma_1 \sum_{i=1}^n (u_i(w^k))^{s-2} |\nabla u_i(w^k)|^2
\]

\[
+ \gamma_2 \sum_{i=1}^n (u_i(w^k))^{2s-2} |\nabla u_i(w^k)|^2 + \gamma_3 \sum_{i=1}^n \frac{|\nabla u_i(w^k)|^2}{u_i(w^k)} + \gamma_4 \sum_{i=1}^n (u_i(w^k))^{s-1} |\nabla u_i(w^k)|^2
\]

\[
\geq \sum_{i=1}^n \left( \gamma_1 |\nabla (u_i(w^k))^{\frac{1}{2}}|^2 + \gamma_2 |\nabla (u_i(w^k))^{s-2}|^2 + \gamma_3 |\nabla \sqrt{u_i(w^k)}|^2 + \gamma_4 |\nabla (u_i(w^k))^{\frac{1}{2}}|^2 \right).
\]

Choose the test function \( \phi = w^k \) in (20), based on gradient estimates above in (25) and assumptions listed as (11), we conclude that \( (26) \)

\[
(1 - C_\eta \tau) \int_\mathcal{O} h_s(u(w^k)) dx + C_\tau \sum_{i=1}^n \int_\mathcal{O} (|\nabla (u_i(w^k))^{s-2}|^2 + |\nabla (u_i(w^k))^{s-1}|^2 + |\nabla (u_i(w^k))^{\frac{1}{2}}|^2) dx
\]

\[
+ \varepsilon \tau \|w^k\|_{L^2(\mathcal{O})}^2 \leq \int_\mathcal{O} h_s(u(w^{k-1})) dx + C_\eta \tau \text{meas}(\mathcal{O}).
\]

Summing (26) over \( k = 1, \ldots, m \) with \( m \leq L \), we derive that for every \( 1 \leq m \leq L, \mathbb{P}\text{-a.s.} \)

\[
(1 - C_\eta \tau) \int_\mathcal{O} h_s(u(w^m)) dx + \varepsilon \tau \sum_{k=1}^m \|w^k\|_{L^2(\mathcal{O})}^2
\]

\[
+ C_\tau \sum_{k=1}^m \sum_{i=1}^n \int_\mathcal{O} (|\nabla (u_i(w^k))^{s-2}|^2 + |\nabla (u_i(w^k))^{s-1}|^2 + |\nabla (u_i(w^k))^{\frac{1}{2}}|^2) dx
\]

\[
\leq \int_\mathcal{O} h_s(u^0) dx + C_\eta \tau \sum_{k=1}^{m-1} \int_\mathcal{O} h_s(u^k) dx + C_\eta \tau \text{meas}(\mathcal{O}),
\]
and applying the discrete Gronwall inequality, choose \( \tau < \frac{1}{C\eta} \), then
\[
\int_{\mathcal{O}} h_s(u(w^m)) dx + \tau \sum_{k=1}^{m} \sum_{i=1}^{n} \int_{\mathcal{O}} \left( |\nabla (u_i(w^k))^s|^2 + |\nabla (u_i(w^k))^\frac{s}{2}|^2 + |\nabla (u_i(w^k))^{\frac{s}{2}}|^2 \right) dx \\
+ \varepsilon \tau \sum_{k=1}^{m} \|w^k\|_{L^2(\mathcal{O})}^2 \leq C,
\]
where \( C > 0 \) depends on \( h_s(u^0), \eta, \) but not on \( \varepsilon \) or \( \tau \). For every \( t > 0 \),
\[
\int_{\mathcal{O}} |\nabla u_i(w^k)|^2 dx = \int_{\{x \in \mathcal{O} : |u(x, t)| < 1\}} |\nabla u_i(w^k)|^2 dx + \int_{\{x \in \mathcal{O} : |u(x, t)| \geq 1\}} |\nabla u_i(w^k)|^2 dx \\
\leq C \int_{\mathcal{O}} \left( |\nabla (u_i(w^k))^\frac{s}{2}|^2 + |\nabla u_i(w^k)|^2 \right) dx,
\]
thus
\[
\int_{\mathcal{O}} h_s(u(w^m)) dx + \tau \sum_{k=1}^{m} \sum_{i=1}^{n} \int_{\mathcal{O}} \left( |\nabla (u_i(w^k))^s|^2 + |\nabla u_i(w^k)|^2 \right) dx + \varepsilon \tau \sum_{k=1}^{m} \|w^k\|_{L^2(\mathcal{O})}^2 \leq C,
\]
one step further,
\[
\int_{\mathcal{O}} h_s(u(w^m)) dx + \tau \sum_{k=1}^{m} \sum_{i=1}^{n} \int_{\mathcal{O}} \left( |\nabla (u_i(w^k))^s|^2 + |\nabla u_i(w^k)|^2 \right) dx + \varepsilon \tau \int_{0}^{T} \|w^k\|_{L^2(\mathcal{O})}^2 dt \leq C.
\]
(27)

Together with the dominance of the entropy density \( h_s(u(w^m)) \) on the \( L^1 \) norm, which is for every \( 1 \leq i \leq n, \)
\[
\sup_{t \in (0, T)} \int_{\mathcal{O}} (u_i^{(\tau)})^s dx + \sup_{t \in (0, T)} \int_{\mathcal{O}} u_i^{(\tau)} dx \\
\leq C \sup_{t \in (0, T)} \int_{\mathcal{O}} \left( \frac{(u_i^{(\tau)})^s}{s} + u_i^{(\tau)} (\log u_i^{(\tau)} - 1) + 1 \right) dx \leq C,
\]
and applying the Gagliardo-Nirenberg inequality and Hölder inequality,
\[
\int_{0}^{T} \|(u_i^{(\tau)})^s\|_{L^2(\mathcal{O})}^2 dt \leq C \int_{0}^{T} \|\nabla (u_i^{(\tau)})^s\|_{L^4(\mathcal{O})} \|\nabla (u_i^{(\tau)})^s\|_{L^2(\mathcal{O})} dt \\
\leq C \sup_{0 \leq t \leq T} \|(u_i^{(\tau)})^s\|_{L^2(\mathcal{O})} \|\nabla (u_i^{(\tau)})^s\|_{L^2(\mathcal{O})} \int_{0}^{T} \|\nabla (u_i^{(\tau)})^s\|_{L^2(\mathcal{O})}^2 dt \\
\leq C T \|\nabla (u_i^{(\tau)})^s\|_{L^2(\mathcal{O})} \|\nabla (u_i^{(\tau)})^s\|_{L^2(\mathcal{O})} \leq C.
\]
Using the same argument to the above, we obtain \( \int_0^T \| u_i^{(r)} \|_{L^2(O)}^2 dt \leq C. \) (27) yields the below
\[
\| (u^{(r)})^s \|_{L^\infty(0,T;L^1(O))} + \| u^{(r)} \|_{L^\infty(0,T;L^1(O))} + \| u^{(r)} \|_{L^2(0,T;H^1(O))} \\
+ \| (u^{(r)})^s \|_{L^2(0,T;H^1(O))} + \sqrt{\varepsilon} \| w^{(r)} \|_{L^2(0,T;L^2(O))} \leq C.
\]
Applying the Gagliardo-Nirenberg inequality, \( \mathbb{P} \)-a.s. we have
\[
\int_0^T \| (u_i^{(r)})^s \|_{L^\frac{2}{s}(O)}^{2+\frac{2}{s}} dt \leq C \int_0^T \| (u_i^{(r)})^s \|_{L^\frac{2}{s+1}(O)}^{2(d+1)} \| (u_i^{(r)})^s \|_{L^\frac{2}{s+1}(O)}^{2(d+1)} dt \\
\leq C \| (u_i^{(r)})^s \|_{L^\infty(0,T;L^1(O))} \| (u_i^{(r)})^s \|_{L^2(0,T;L^2(O))} \leq C.
\]
Let us consider each term in (24) in the purpose of showing that \( \frac{1}{\tau} (u^{(r)} - \Gamma_\tau u^{(r)}) \) is uniformly bounded in \( L^2(0,T;H^1(O))' \). By what we have shown in the above, we deduce that \( \int_0^T \int_O w^{(r)} \cdot \phi dx dt \leq C \), and \( \int_0^T \int_O f(u^{(r)}) \cdot \phi dx dt \leq C \). For the second term on the left-hand-side of (24), if \( i = j \), is
\[
\int_0^T \int_O A_{ij}(u^{(r)}) \nabla u_j^{(r)} \cdot \nabla \phi_i dx dt \leq \int_0^T \| A_{ij}(u^{(r)}) \|_{L^{\frac{2}{d+1}}(O)} \| \nabla u_j^{(r)} \|_{L^2(O)} \| \nabla \phi_i \|_{L^{2(d+1)}(O)} dt \\
\leq C \| A_{ij}(u^{(r)}) \|_{L^{\frac{2}{d+1}}(0,T;L^{\frac{2}{d+1}}(O))} \| \nabla u_j^{(r)} \|_{L^2(0,T;L^2(O))} \| \nabla \phi_i \|_{L^{2(d+1)}(0,T;L^{2(d+1)}(O))} \leq C,
\]
if \( i \neq j \), it is
\[
\int_0^T \int_O A_{ij}(u^{(r)}) \nabla u_j^{(r)} \cdot \nabla \phi_i dx dt = \int_0^T \int_O a_{ij} u_i^{(r)} (u_j^{(r)})^{s-1} \nabla u_j^{(r)} \cdot \nabla \phi_i dx dt \\
= \int_0^T \int_O a_{ij} u_i^{(r)} \nabla (u_j^{(r)})^s \nabla \phi_i dx dt \\
\leq C \| u_i^{(r)} \|_{L^2(0,T;L^2(O))} \| \nabla (u_j^{(r)})^s \|_{L^2(0,T;L^2(O))} \| \nabla \phi_i \|_{L^\infty(O)} \\
\leq C \| u_i^{(r)} \|_{L^2(0,T;L^2(O))} \| \nabla (u_j^{(r)})^s \|_{L^2(0,T;L^2(O))} \| \phi_i \|_{H^{2(d+1)}(O)} \leq C,
\]
for every \( \phi \in H^{2(d+1)}(O) \). By the density argument and (24), we conclude that \( \frac{1}{\tau} (u^{(r)} - \Gamma_\tau u^{(r)}) \) is uniformly bounded in \( L^2(0,T;H^1(O))' \).

**Step 3: the limit** \( (\varepsilon, \tau) \to 0 \). The uniform estimates allow us to apply the Aubin-Lions Lemma (10, 14), which provides the existence of a subsequence of \( (u^{(r)}) \) not relabeled, such that, as \( (\varepsilon, \tau) \to 0 \), \( u^{(r)} \to u \) strongly in \( L^2(O \times (0,T)) \), \( \mathbb{P} \)-a.s. Moreover by weak compactness, \( \mathbb{P} \)-a.s.
\[
\nabla u^{(r)} \to \nabla u \quad \text{weakly in} \quad L^2(0,T;L^2(O)), \\
\varepsilon w^{(r)} \to 0 \quad \text{strongly in} \quad L^2(0,T;L^2(O)), \\
\frac{1}{\tau} (u^{(r)} - \Gamma_\tau u^{(r)}) \to \partial_t u \quad \text{weakly in} \quad L^2(0,T;H^1(O))'.
\]
Performing the limit \( (\varepsilon, \tau) \to 0 \) in (24) shows that for every test function \( \phi \in H^{2(d+1)}(O) \), \( u \) solves (18), with \( f_i(u) \) given by (19). Thus \( u \) solves (18)-(19), for every test function.
\( \phi \in H^{2(d+1)}(O) \). By the density argument, \( u \) solves (18)-(19), for every test function \( \phi \in H^1(O) \).

It remains to show that \( u(0) \) satisfies the initial datum. Please refer to the Lemma 5, [20], page 1980 for the proof of this fact. Nonnegativeness of the entropy itself.

Let us consider another equations system

\[
\pi du = a(u)dt + b(u)dW(t), \quad t > 0, \quad u(0) = \Pi_N(u_0),
\]

where \( a = (a_1, \ldots, a_n) : H_N \to \mathbb{R}^n \), \( b_{ij} : H_N \to \mathcal{L}(\mathbb{R}^n; H_N) \), \( 1 \leq i, j \leq n \), and

\[
a_i(u) = \Pi_N \text{div} \left( \sum_{j=1}^{n} \pi_i M_{ij}(u) \nabla u_j \right), \quad b_{ij}(u) = \pi_i \Pi_N \sigma_{ij}(u).
\]

We still implement the Wong-Zakai approximation technique to the above (28)-(29). For every \( 1 \leq i \leq n \), we have

\[
\frac{du_i^{(N,\eta)}}{dt} = \Pi_N \text{div} \left( \sum_{j=1}^{n} M_{ij}(u^{(N,\eta)}(r)) \nabla u_j^{(N,\eta)}(r) \right) + f_i(u^{(N,\eta)}(r)), \quad u_i^{(N,\eta)}(0) = \Pi_N(u_i^0),
\]

and the weak form for the above (30) is, for every \( \phi_i \in H_N \),

\[
\langle u_i^{(N,\eta)}, \phi_i \rangle = \langle u_i^0, \phi_i \rangle - \int_0^t \sum_{j=1}^{n} \langle M_{ij}(u^{(N,\eta)}(r)) \nabla u_j^{(N,\eta)}(r), \nabla \phi_i \rangle dr + \int_0^t \langle f_i(u^{(N,\eta)}(r)), \phi_i \rangle dr,
\]

with \( f = (f_1, \ldots, f_n) \).

\[
f_i(u^{(N,\eta)}(t)) = \sum_{j=1}^{n} \Pi_N(\sigma_{ij}(u^{(N,\eta)}(t))) \frac{dW_j^{(n)}}{dt}(t) - \frac{1}{2} \Pi_N \left( \sum_{j,l=1}^{n} \frac{\partial \sigma_{ij}}{\partial u_l}(u^{(N,\eta)}) \sigma_{ij}(u^{(N,\eta)}) \right),
\]

and \( u_i^{(N,\eta)}(0) = \Pi_N(u_i^0) \).

Concerning the above (28)-(29) and (30) indexed by \( \eta \), we have the below Lemma 6.

**Lemma 6.** (i). There exists a unique strong solution \( u^{(N)} \) to (28)-(29), \( \mathbb{P} \)-a.s. (ii). For every \( \eta > 0 \), there exists a unique strong solution \( u^{(N,\eta)} \) to (30), \( \mathbb{P} \)-a.s.

**Proof.** Let us first of all prove (i). Let \( R > 0 \), \( T > 0 \), \( \omega \in \Omega \) and let \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), \( z = (z_1, \ldots, z_n) \in \mathbb{R}^n \), and \( y, z \in H_N \) with \( \|y\|_{H_N}, \|z\|_{H_N} \leq R \). Since

\[
M_{ij}(y) - M_{ij}(z) = (s+1)a_{ii}(|y_i|^s - |z_i|^s) + \sum_{k=1, k \neq i}^{n} a_{ik}(|y_k|^s - |z_k|^s), \quad i = j.
\]

\[
M_{ij}(y) - M_{ij}(z) = sa_{ij}(|y_i|^s - |z_i|^s - |z_j|^s - |z_j|^s) - sa_{ij}(|y_j|^s - |z_j|^s - |z_j|^s) + sa_{ij}(|y_j|^s - |z_j|^s - |z_j|^s) |z_i|, \quad i \neq j.
\]
plus in finite dimensional space $H_N$, the assumption that $\|y\|_{H_N}, \|z\|_{H_N} \leq R$ is equivalent to the fact that for every $1 \leq i \leq n$, a.e. $x \in \mathcal{O}$, $|y_i|, |z_i|$ are bounded uniformly by a positive constant.

There exists $\zeta_i(x), \lambda_i(x) \geq 0$, such that for $x \in \mathcal{O}$, $\min\{|y_i(x)|, |z_i(x)|\} \leq \zeta_i(x), \lambda_i(x) \leq \max\{|y_i(x)|, |z_i(x)|\}$, with

$$\|\|y_i\|^s - |z_i|^s\|_{L^2(\mathcal{O})} = \|s(|y_i| - |z_i|)^s \zeta_i^{s-1}\|_{L^2(\mathcal{O})} \leq \|s \zeta_i^{s-1}\|_{L^\infty(\mathcal{O})} \|y_i| - |z_i||_{L^2(\mathcal{O})} \leq C\|y_i - z_i\|_{L^2(\mathcal{O})},$$

and

$$\|\|y_i\|^{s-1} - |z_i|^{s-1}\|_{L^2(\mathcal{O})} = \|(s-1)(|y_i| - |z_i|)^s \lambda_i^{s-2}\|_{L^2(\mathcal{O})} \leq \|(s-1)\lambda_i^{s-2}\|_{L^\infty(\mathcal{O})} \|y_i| - |z_i||_{L^2(\mathcal{O})} \leq C\|y_i - z_i\|_{L^2(\mathcal{O})},$$

which follows if $i = j$,

$$\|M_{ij}(y) - M_{ij}(z)\|^2_{L^2(\mathcal{O})} = \|(s+1)a_{ii}(|y_i|^s - |z_i|^s) + \sum_{k=1,k \neq i}^n a_{ik}(|y_k|^s - |z_k|^s)\|^2_{L^2(\mathcal{O})} \leq (s+1)^2 a_{ii}^2 \|\|y_i\|^{s-1} - |z_i|^{s-1}\|_{L^2(\mathcal{O})} + \sum_{k=1,k \neq i}^n a_{ik}^2 \|\|y_k\|^{s-1} - |z_k|^{s-1}\|_{L^2(\mathcal{O})} \leq C\sum_{i=1}^n \|y_i - z_i\|_{L^2(\mathcal{O})}^2 = C\|y - z\|_{L^2(\mathcal{O})}^2,$$

and if $i \neq j$, then

$$\|M_{ij}(y) - M_{ij}(z)\|^2_{L^2(\mathcal{O})} = \|sa_{ij}(|y_i| - |z_i|)|y_j|^{s-1} + sa_{ij}(|y_j|^{s-1} - |z_j|^{s-1})|z_i|\|^2_{L^2(\mathcal{O})} \leq s^2 a_{ij}^2 \|\|(|y_i| - |z_i|)|y_j|^{s-1}\|_{L^2(\mathcal{O})}^2 + \|(|y_j|^{s-1} - |z_j|^{s-1})|z_i|\|_{L^2(\mathcal{O})}^2 \leq C\|\|y_j|^{s-1}\|_{L^\infty(\mathcal{O})} \|y_i| - |z_i||_{L^2(\mathcal{O})}^2 + C\|\|z_i||_{L^\infty(\mathcal{O})} \|y_j|^{s-1} - |z_j|^{s-1}\|_{L^2(\mathcal{O})}^2 \leq C\sum_{i=1}^n \|y_i - z_i\|_{L^2(\mathcal{O})}^2 = C\|y - z\|_{L^2(\mathcal{O})}^2.$$

By the semi positive-definiteness of $PA(u)$, if $u_i \geq 0, 1 \leq i \leq n$ (lemma 1), we conclude the semi positive-definiteness of $PM(y), y \in \mathbb{R}^n$. Also by the fact that norms are equivalent in finite dimensional spaces, such like $H_N$ in this case, it follows that $\|\nabla(y_i - z_i)\|_{L^2(\mathcal{O})} \leq$
\[ \|y_i - z_i\|_{H^1(\mathcal{O})} \leq C \|y_i - z_i\|_{H_N} \leq C \|y - z\|_{H_N}, \]

Thus

\[ (a(y) - a(z), y - z)_{H_N} = -\sum_{i,j=1}^n \int_\mathcal{O} \pi_i M_{ij}(y) \nabla(y_i - z_i) \cdot \nabla(y_j - z_j) \, dx \]

\[ + \sum_{i,j=1}^n \int_\mathcal{O} \pi_i (M_{ij}(z) - M_{ij}(y)) \nabla(y_i - z_i) \cdot \nabla z_j \, dx \]

\[ \leq C \sum_{i,j=1}^n \|M_{ij}(y) - M_{ij}(z)\|_{L^2(\mathcal{O})} \|\nabla(y_i - z_i)\|_{L^2(\mathcal{O})} \|\nabla z_j\|_{L^\infty(\mathcal{O})} \]

\[ \leq C \sum_{i=1}^n \|y_i - z_i\|_{H_N}^2 \leq C \|y - z\|_{H_N}^2, \]

and \[ \|b(y) - b(z)\|_{L^2(\mathbb{R}^n; H_N)}^2 \leq C \|\sigma(y) - \sigma(z)\|_{L^2(\mathbb{R}^n; H_N)}^2 \leq C \|y - z\|_{H_N}^2. \]

To verify the weak coercivity condition, we take \( y \in H_N \) with \( \|y\|_{H_N} \leq R, 1 \leq i \leq n, \)

\[ (a(y), y)_{H_N} + \|b(y)\|_{L^2(\mathbb{R}^n; H_N)}^2 = -\sum_{i,j=1}^n \int_\mathcal{O} \pi_i M_{ij}(y) \nabla y_i \cdot \nabla y_j \, dx + \|P \sigma(y)\|_{L^2(\mathbb{R}^n; H_N)}^2 \]

\[ \leq C(1 + \|y\|_{H_N}^2). \]

The existence and uniqueness result for strong solutions to stochastic differential equations in \[ 23 \quad 24 \] indicates that for every \( N \in \mathbb{N} \), a unique pathwise strong solution \( u^{(N)} \) to \[ 28 \quad 29 \] exists.

For (ii), the method in showing a strong solution \( u^{(N, \eta)} \) exists for \[ 30 \] be completely referred to the existence proof of a strong solution \( u^{(N)} \) to \[ 28 \quad 29 \]. We omit the proof, by mention that the Lipschitz and the coercivity property of \( f = (f_1, \ldots, f_n) \) in \[ 30 \] is guaranteed by our assumptions \[ 7 \quad 8 \].

After the above preparation, we are allowed to give a positive answer whether a nonnegative \( \mathbb{P} \)-a.s. strong solution exists for \[ 17 \]. Before the formal proof of the Lemma \[ 8 \] we need a preliminary result, which has already been given as Proposition 6 in \[ 12 \]. Please be aware a slight difference exists as no such stopping time technique has been introduced. We state this lemma without proof, and for a further understanding of the below lemma, refer to \[ 17 \], Chapter 6, Theorem 7.2.

**Lemma 7.** Let \( u^{(N, \eta)} \) be the solution to \[ 30 \], let \( u^{(N)} \) be the unique strong (in the probabilistic sense) solution to \[ 28 \quad 29 \]. Then \( u^{(N, \eta)} \rightarrow u^{(N)} \) in probability, as \( \eta \rightarrow 0 \).

**Lemma 8.** The strong solution \( u^{(N)} \) to \[ 28 \quad 29 \] is a strong solution to \[ 17 \], and \( u^{(N)} \) is nonnegative, \( \mathbb{P} \)-a.s.

**Proof.** By the Lemma \[ 6 \] a strong solution \( u^{(N, \eta)} \) exists for \[ 30 \]. By the Lemma \[ 5 \] this strong solution \( u^{(N, \eta)} \) is nonnegative, \( \mathbb{P} \)-a.s.
We mention that we have only shown that for each fixed \( \eta > 0 \), the strong solution denoted as \( u^{(N, \eta)} \) exists. Guaranteed by the Lemma 7, then we are allowed to pass the limit as \( \eta \to 0 \), which will be reflected in below.

Let us show that \( u^{(N)} \) is nonnegative, \( \mathbb{P} \)-a.s. If not, for some \( 1 \leq i \leq n \) in (30), there exist \( \varepsilon_0, \varepsilon_1 > 0 \), such that \( \mathbb{P}(\omega : u_i^{(N)} \leq -\varepsilon_1) \geq \varepsilon_0 > 0 \). By the Lemma 7, \( u^{(N, \eta)} \to u^{(N)} \) in probability as \( \eta \to 0 \), i.e. \( \lim_{\eta \to 0} \mathbb{P}(\omega : |u_i^{(N)} - u_i^{(N, \eta)}| \geq \varepsilon_1) = 0 \). There exists \( \eta_0 > 0 \), such that \( \mathbb{P}(\omega : u_i^{(N)} \geq u_i^{(N, \eta_0)} + \varepsilon_1 \text{ or } u_i^{(N)} \leq u_i^{(N, \eta_0)} - \varepsilon_1) \leq \frac{\varepsilon_0}{2} \).

So long as \( \{\omega : u_i^{(N)} \leq -\varepsilon_1\} \subset \{\omega : u_i^{(N)} \geq u_i^{(N, \eta_0)} + \varepsilon_1 \text{ or } u_i^{(N)} \leq u_i^{(N, \eta_0)} - \varepsilon_1\} \), then

\[ \varepsilon_0 \leq \mathbb{P}(\omega : u_i^{(N)} \leq -\varepsilon_1) \leq \mathbb{P}(\omega : u_i^{(N)} \geq u_i^{(N, \eta_0)} + \varepsilon_1 \text{ or } u_i^{(N)} \leq u_i^{(N, \eta_0)} - \varepsilon_1) \leq \frac{\varepsilon_0}{2}, \]

which is a contradiction. Thus (28)-(29) coincide with (17), up to a multiplication by the constant \( \pi_i \). There exists a strong solution \( u^{(N)} \) to (17), and is nonnegative, \( \mathbb{P} \)-a.s.

\[ \square \]

4. Energy estimates of approximated strong solutions \( u^{(N)} \)

We divide the estimation into two lemmas, with the second estimation lemma relies on a change of variables technique and an application of the Gagliardo-Nirenberg inequality.

**Lemma 9.** For every \( T > 0 \), there exists a constant \( C > 0 \), which does not depend on \( N \), such that

\begin{equation}
(33) \quad \sup_{N \in \mathbb{N}} \mathbb{E} \left( \sup_{t \in (0, T)} \| u^{(N)} \|^2_{L^2(\mathcal{O})} \right) \leq C,
\end{equation}

\begin{equation}
(34) \quad \sup_{N \in \mathbb{N}} \mathbb{E} \left( \int_0^T \| \nabla u^{(N)} \|^2_{L^2(\mathcal{O})} dr \right) \leq C.
\end{equation}

**Proof.** Let us apply the Itô formula (please refer to [17] [22] for a understanding of the kind of Itô formula we use in this proof) to the process \( X(t) = u^{(N)}(t) \), where \( u^{(N)}(t) \) is a strong solution to (17), \( P^\frac{1}{2} = \text{diag}(\pi_1^{\frac{1}{2}}, \ldots, \pi_n^{\frac{1}{2}}) \), \( t \in [0, T] \), and

\[
\frac{1}{2} \| P^\frac{1}{2} u^{(N)}(t) \|^2_{L^2(\mathcal{O})} - \frac{1}{2} \| \Pi_N(P^\frac{1}{2} u^0) \|^2_{L^2(\mathcal{O})} = \frac{1}{2} \int_0^t \| \Pi_N(P^\frac{1}{2} \sigma(u^{(N)}(r))) \|^2_{L^2(\mathcal{O})} dr
\]

\[
+ \sum_{i,j=1}^n \int_0^t (\Pi_N(\sigma_{ij}(u^{(N)}(r))) \nabla u_j^{(N)}(r), u_i^{(N)}(r))_{L^2(\mathcal{O})} dr
\]

\[
+ \sum_{i,j=1}^n \int_0^t (\Pi_N(\pi_{ij} u^{(N)}(r))) dW_j(r), u_i^{(N)}(r))_{L^2(\mathcal{O})}.
\]
which is equivalent to
\[
\frac{1}{2} \| P^\frac{1}{2} u^{(N)}(t) \|^2_{L^2(\mathcal{O})} - \frac{1}{2} \| \Pi_N(P^\frac{1}{2} u^0) \|_{L^2(\Omega)}^2 = \frac{1}{2} \int_0^t \| \Pi_N(P^\frac{1}{2} \sigma(u^{(N)}(r))) \|_{L(\mathbb{R}^n;L^2(\mathcal{O}))}^2 \, dr
\]
(35) \[
- \sum_{i,j=1}^n \int_0^t (\pi_i A_{ij}(u^{(N)}(r)) \nabla u_j^{(N)}(r), \nabla u_i^{(N)}(r))_{L^2(\mathcal{O})} \, dr
\]
\[
+ \sum_{i,j=1}^n \int_0^t (\pi_i \sigma_{ij}(u^{(N)}(r)) dW_j(r), u_i^{(N)}(r))_{L^2(\mathcal{O})},
\]
and the second term on the right-hand-side of (35) can be estimated by the Lemma 1, that for \( z_i, z_j \in \mathbb{R} \), there exist constants \( \alpha_1 > 0, \alpha_2 > 0 \), with \( \alpha_1, \alpha_2 \) changing from line to line, such that if each component of \( u \) is nonnegative, \( \mathbb{P} \)-a.s. then \( \sum_{i,j=1}^n \pi_i A_{ij}(u) z_i z_j \geq \alpha_1 \sum_{i=1}^n z_i^2 + \alpha_2 \sum_{i=1}^n u_i^2. \) Since we have already shown \( u^{(N)}(t) \) are nonnegative \( \mathbb{P} \)-a.s.
\[
\sum_{i,j=1}^n (\pi_i A_{ij}(u^{(N)}(r)) \nabla u_j^{(N)}(r), \nabla u_i^{(N)}(r))_{L^2(\mathcal{O})}
\]
\[
\geq \alpha_1 \sum_{i=1}^n \int_{\mathcal{O}} |\nabla u_i^{(N)}|^2 \, dx + \alpha_2 \sum_{i=1}^n \int_{\mathcal{O}} |u_i^{(N)}|^2 |\nabla u_i^{(N)}|^2 \, dx
\]
\[
\geq \alpha_1 \| \nabla u^{(N)} \|^2_{L^2(\mathcal{O})} + \alpha_2 \| (u^{(N)})^{\frac{n+2}{2}} \|^2_{L^2(\mathcal{O})}.
\]
Therefore, we have
\[
\frac{1}{2} \| P^\frac{1}{2} u^{(N)}(t) \|^2_{L^2(\mathcal{O})} + \alpha_1 \int_0^t \| \nabla u^{(N)}(r) \|^2_{L^2(\mathcal{O})} \, dr + \alpha_2 \int_0^t \| (u^{(N)}(r))^{\frac{n+2}{2}} \|^2_{L^2(\mathcal{O})} \, dr
\]
(36) \[
\leq \frac{1}{2} \| P^\frac{1}{2} u^0 \|^2_{L^2(\mathcal{O})} + \frac{1}{2} \int_0^t \| P^\frac{1}{2} \sigma(u^{(N)}(r)) \|^2_{L(\mathbb{R}^n;L^2(\mathcal{O}))} \, dr
\]
\[
+ \sum_{i,j=1}^n \int_0^t (\pi_i \sigma_{ij}(u^{(N)}(r)) dW_j(r), u_i^{(N)}(r))_{L^2(\mathcal{O})},
\]
and for the second term on the right-hand-side of (36), by the assumption (7),
\[
\frac{1}{2} \int_0^t \| P^\frac{1}{2} \sigma(u^{(N)}(r)) \|^2_{L(\mathbb{R}^n;L^2(\mathcal{O}))} \, dr \leq C \int_0^T (1 + \| u^{(N)} \|^2_{L^2(\mathcal{O})}) \, dr.
\]
For the third term on the right-hand-side of (36), the process given by the below
\[
\mu^{(N)}(t) = \sum_{i,j=1}^n \int_0^t (\pi_i \sigma_{ij}(u^{(N)}(r)) dW_j(r), u_i^{(N)}(r))_{L^2(\mathcal{O})}, \quad t \in [0,T],
\]
is an \( \mathcal{F}_t \)-martingale. By the Burkholder-Davis-Gundy inequality, which states that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |\mu^{(N)}(t)| \right) \leq C \mathbb{E} \left( (\sum_{i,j=1}^n (\pi_i \sigma_{ij}(u^{(N)}(T)))^2 \right),
\]
with the notation $\langle \mu^{(N)}(T) \rangle$ represents the quadratic variation of $\mu^{(N)}(T)$. Thus

$$
\mathbb{E}\left( \sup_{t \in (0,T)} \left| \sum_{i,j=1}^{n} \int_{0}^{t} (\pi_{i}\sigma_{ij}(u^{(N)}(r))dW_{j}(r), u^{(N)}_{i}(r)\right|_{L^{2}(\Omega)} \right)
\leq C\mathbb{E}\left( \sum_{i,j=1}^{n} \int_{0}^{T} (\int_{\Omega} \pi_{i}\sigma_{ij}(u^{(N)}(r))u^{(N)}_{i}(r)dx)^{2}dr \right)^{\frac{1}{2}}
\leq C\mathbb{E}\left( \int_{0}^{T} \left( \int_{\Omega} \sigma_{ij}(u^{(N)}(r))u^{(N)}_{i}(r)dx \right)^{2}dr \right)^{\frac{1}{2}}
\leq C\mathbb{E}\left( \int_{0}^{T} \left( \int_{\Omega} \sigma_{ij}(u^{(N)}(r))dx \right) \cdot \left( \int_{\Omega} \sum_{i=1}^{n} (u^{(N)}_{i}(r))^{2}dx \right)dr \right)^{\frac{1}{2}}
= C\mathbb{E}\left( \int_{0}^{T} \|u^{(N)}(r)\|_{L^{2}(\Omega)}^{2} \|\sigma(u^{(N)}(r))\|_{L^{2}(\Omega)}^{2}dr \right)^{\frac{1}{2}}.
$$

One step further, by the assumption on $\sigma$, the H"older inequality and the Young inequality, for every positive constant $\varepsilon_{0},$

$$
\mathbb{E}\left( \sup_{t \in (0,T)} \left| \sum_{i,j=1}^{n} \int_{0}^{t} (\pi_{i}\sigma_{ij}(u^{(N)}(r))dW_{j}(r), u^{(N)}_{i}(r)\right|_{L^{2}(\Omega)} \right)
\leq C\mathbb{E}\left( \int_{0}^{T} \|u^{(N)}(r)\|_{L^{2}(\Omega)}^{2} \|\sigma(u^{(N)}(r))\|_{L^{2}(\Omega)}^{2} \right)^{\frac{1}{2}}
\leq C\mathbb{E}\left( \sup_{t \in (0,T)} \|u^{(N)}(t)\|_{L^{2}(\Omega)}^{2} \right)^{\frac{1}{2}} \left( \int_{0}^{T} (1 + \|u^{(N)}(r)\|_{L^{2}(\Omega)}^{2})dr \right)^{\frac{1}{2}}
\leq C\varepsilon_{0}\mathbb{E}\left( \sup_{t \in (0,T)} \|u^{(N)}(t)\|_{L^{2}(\Omega)}^{2} \right) + \frac{C}{4\varepsilon_{0}} \cdot (T + \mathbb{E} \int_{0}^{T} \|u^{(N)}(r)\|_{L^{2}(\Omega)}^{2}dr).
$$

Combining (36) and (37), for every positive constant $\varepsilon_{0}$, and $t \in [0, T],$

$$
\frac{1}{2}\mathbb{E}\left( \sup_{t \in (0,T)} \|P_{\frac{1}{2}}u^{(N)}(T)\|_{L^{2}(\Omega)}^{2} \right) + \alpha_{1}\mathbb{E} \int_{0}^{t} \left\| \nabla u^{(N)}(r) \right\|_{L^{2}(\Omega)}^{2}dr
+ \alpha_{2}\mathbb{E} \int_{0}^{t} \left\| \nabla u^{(N)}(r) \right\|_{L^{2}(\Omega)}^{2}dr \leq \frac{1}{2}\mathbb{E}\left( \sup_{t \in (0,T)} \|P_{\frac{1}{2}}u^{0}\|_{L^{2}(\Omega)}^{2} \right) + C\mathbb{E} \int_{0}^{T} (1 + \|u^{(N)}(r)\|_{L^{2}(\Omega)}^{2})dr
+ C\varepsilon_{0}\mathbb{E}\left( \sup_{t \in (0,T)} \|u^{(N)}(t)\|_{L^{2}(\Omega)}^{2} \right) + \frac{C}{4\varepsilon_{0}} \cdot (T + \mathbb{E} \int_{0}^{T} \|u^{(N)}(r)\|_{L^{2}(\Omega)}^{2}dr)
\leq C + C (1 + \frac{1}{4\varepsilon_{0}}) \mathbb{E} \int_{0}^{T} \|u^{(N)}(r)\|_{L^{2}(\Omega)}^{2}dr + C\varepsilon_{0}\mathbb{E}\left( \sup_{t \in (0,T)} \|u^{(N)}(t)\|_{L^{2}(\Omega)}^{2} \right),
$$
and we conclude that there exist positive constants \( \alpha_i, i = 1, 2, 3 \), with values subject to change in below, such that

\[
\begin{align*}
\alpha_3 & \mathbb{E} \left( \sup_{t \in (0,T)} \|u^{(N)}(t)\|_{L^2(\Omega)}^2 \right) + \alpha_1 \mathbb{E} \int_0^T \|\nabla u^{(N)}(r)\|_{L^2(\Omega)}^2 dr \\
+ \alpha_2 & \mathbb{E} \int_0^T \|\nabla (u^{(N)}(r)) \|^2_{L^2(\Omega)} dr \leq C + C \left( 1 + \frac{1}{4\varepsilon_0} \right) \mathbb{E} \int_0^T \left( \sup_{\tau \in (0,r)} \|u^{(N)}(\tau)\|_{L^2(\Omega)}^2 \right) dr \\
+ C & \varepsilon_0 \mathbb{E} \left( \sup_{t \in (0,T)} \|u^{(N)}(t)\|_{L^2(\Omega)}^2 \right).
\end{align*}
\]

(38)

Let us choose this \( \varepsilon_0 \) such that \( C \varepsilon_0 < \alpha_3 \), then for some positive constants \( C_0, C_1 \),

\[
\begin{align*}
\alpha_3 & \mathbb{E} \left( \sup_{t \in (0,T)} \|u^{(N)}(t)\|_{L^2(\Omega)}^2 \right) + \alpha_1 \mathbb{E} \int_0^T \|\nabla u^{(N)}(r)\|_{L^2(\Omega)}^2 dr \\
+ \alpha_2 & \mathbb{E} \int_0^T \|\nabla (u^{(N)}(r)) \|^2_{L^2(\Omega)} dr \leq C_0 + C_1 \mathbb{E} \int_0^T \left( \sup_{\tau \in (0,r)} \|u^{(N)}(\tau)\|_{L^2(\Omega)}^2 \right) dr,
\end{align*}
\]

(39)

and by the Gronwall lemma,

\[
\mathbb{E} \left( \sup_{t \in (0,T)} \|u^{(N)}(t)\|_{L^2(\Omega)}^2 \right) \leq \frac{C_0}{\alpha_3} \left( 1 + \frac{C_1 T}{\alpha_3} e^{\frac{C_1 T}{\alpha_3}} \right),
\]

with all constants \( C, C_0, C_1, \alpha_1, \alpha_2, \alpha_3 \) independent of \( N \), though subject to change line by line. This proves (33), and (34) follows immediately.

\[ \Box \]

**Lemma 10.** There exists a constant \( C > 0 \), which does not depend on \( N \), such that

\[
\sup_{N \in \mathbb{N}} \mathbb{E} \left( \int_0^T \|\nabla (u^{(N)})^s\|_{L^2(\Omega)}^2 dr \right) \leq C,
\]

(40)

\[
\sup_{N \in \mathbb{N}} \mathbb{E} \left( \int_0^T \| (u^{(N)})^s \|_{L^2(\Omega)}^3 dr \right) \leq C.
\]

(41)

**Proof.** We choose \( v = (v_1, ..., v_n) \) such that for every \( 1 \leq i \leq n \), \( v_i = u_i^\top, \ u_i \geq 0 \). Let us denote two matrices \( H(v) = (H_{ij}(v)) \) and \( G(v) = (G_{ij}(v)) \), with \( H_{ij}(v) = \frac{2}{v_i} v_i^{\frac{2}{v_i} - 1} \) if \( i = j \), and \( H_{ij}(v) = 0 \) if \( i \neq j \), \( G_{ij}(v) = (1 - \frac{2}{v_i} \sum_{j=1}^n \frac{v_j}{v_i}) \) if \( i = j \), and \( G_{ij}(v) = 0 \) if \( i \neq j \). We can check that \( \partial_i u = H(v) \partial_i v, H^{-1}(v) A(u) H(v) = A^H(v) \) and \( \nabla H^{-1}(v) \cdot A(u) H(v) = G(v) A^H(v) \).

In fact,

\[
\begin{align*}
\left( H^{-1}(v) A(u) H(v) \right)_{ij} &= \sum_{k,l=1}^n H^{-1}_{ik}(v) A_{kl}(u) H_{lj}(v) \\
&= H_{ii}^{-1}(v) A_{ij}(u) H_{jj}(v) = v_i^{1 - \frac{2}{v_i}} v_j^{\frac{2}{v_j} - 1} A_{ij}(u) = A_{ij}^H(v),
\end{align*}
\]

and we conclude that there exist positive constants \( \alpha_i, i = 1, 2, 3 \), with values subject to change in below, such that
\[
\left( \nabla H^{-1}(v) \cdot A(u)H(v) \right)_{ij} = \sum_{k,l=1}^{n} \nabla H^{-1}_{ik}(v)A_{kl}(u)H_{lj}(v) = \nabla H^{-1}_{ii}(v)A_{ij}(u)H_{jj}(v)
\]
\[
= (1 - \frac{2}{s}) \nabla v_i \cdot v_i - \frac{2}{s} v_j - A_{ij}(u) = (1 - \frac{2}{s}) \nabla v_i \cdot v_i - \frac{2}{s} v_j - A_{ij}(u) = (G(v)A^H(v))_{ij},
\]
where \((\cdot)_{ij}\) represents the \(i\)-th row and \(j\)-th column of the matrix.

Denote \(q(v) = (q_1(v), \ldots, q_n(v))\) with \(q_i(v)dt = \sum_{j=1}^{n} \sigma_{ij}(v)dW_j(t)\), and since we have shown \(u^{(N)}(t)\) are nonnegative, \(\mathbb{P}\)-a.s. we are allowed to transform (17) into equations with the variable \(v^{(N)}(t) = (u^{(N)}(t))^\frac{1}{2}\), which is

\[
\partial_t v^{(N)} = \Pi_N \left( \text{div} \left( A^H(v^{(N)}) \nabla v^{(N)} \right) - G(v^{(N)})A^H(v^{(N)}) \nabla v^{(N)} + H^{-1}(v^{(N)})q(v^{(N)}) \right),
\]

with \(v_i^{(N)}(0) = \Pi_N(v_i^0)\), \(1 \leq i \leq n\). The existence and uniqueness of \(v^{(N)}\) is guaranteed by the existence and uniqueness of \(u^{(N)}\) and how we define this approximated solution \(v^{(N)}\). \(v^{(N)}\) is nonnegative, \(\mathbb{P}\)-a.s.

Still applying the Itô formula to \(v^{(N)}(t)\), which are strong solutions to (42),

\[
\frac{1}{2} \|P_i^\frac{1}{2} v^{(N)}(t)\|^2_{L^2(O)} = \frac{1}{2} \|P_i^\frac{1}{2} v^{(N)}(0)\|^2_{L^2(O)} + \sum_{i,j=1}^{n} \int_{0}^{t} \left\langle \Pi_N \text{div} \left( \pi_i A^H_{ij}(v^{(N)}(r)) \nabla v_j^{(N)} \right), v_i^{(N)} \right\rangle dr
\]
\[
+ \sum_{i,j=1}^{n} \int_{0}^{t} \left\langle \pi_i \Pi_N \left( A^H_{ij}(v^{(N)}(r)) G_{ii}(v^{(N)}(r)) \nabla v_j^{(N)} \right), v_i^{(N)} \right\rangle dr
\]
\[
+ \sum_{i,j=1}^{n} \int_{0}^{t} \left( \Pi_N(\pi_i H_{ii}^{-1}(v^{(N)}(r)) \sigma_{ij}(v^{(N)}(r)))dW_j(r), v_i^{(N)} \right)_{L^2(O)} dr
\]
\[
+ \frac{1}{2} \int_{0}^{t} \left\| \Pi_N(\pi_i H_{ii}^{-1}(v^{(N)}(r)) \sigma(v^{(N)}(r))) \right\|^2_{L^2(O)} dr,
\]
which is equivalent to the below

\[
\frac{1}{2} \|P_i^\frac{1}{2} v^{(N)}(t)\|^2_{L^2(O)} = \frac{1}{2} \|P_i^\frac{1}{2} v^{(N)}(0)\|^2_{L^2(O)} - \sum_{i,j=1}^{n} \int_{0}^{t} \int_{O} \pi_i A^H_{ij}(v^{(N)}(r)) \nabla v_i^{(N)} \nabla v_j^{(N)} dx dr
\]
\[
+ (1 - \frac{2}{s}) \sum_{i,j=1}^{n} \int_{0}^{t} \int_{O} \pi_i A^H_{ij}(v^{(N)}(r)) \nabla v_i^{(N)} \nabla v_j^{(N)} \cdot v_i^{(N)} dx dr
\]
\[
+ \frac{s}{2} \sum_{i,j=1}^{n} \int_{0}^{t} \left( \pi_i \sigma_{ij}(v^{(N)}(r))dW_j(r), (v_i^{(N)})^{2-\frac{2}{s}} \right)_{L^2(O)} dr
\]
\[
+ \frac{1}{2} \int_{0}^{t} \left\| \Pi_N(\pi_i H_{ii}^{-1}(v^{(N)}(r)) \sigma(v^{(N)}(r))) \right\|^2_{L^2(O)} dr,
\]
then
\begin{align*}
\frac{1}{2}\|P^{\frac{1}{2}}u^{(N)}(t)\|_{L^2(\Omega)}^2 &= \frac{1}{2}\|P^{\frac{1}{2}}u^{(N)}(0)\|_{L^2(\Omega)}^2 - \frac{2}{s} \sum_{i,j=1}^n \int_0^t \int_\Omega \pi_i A_{ij}(v^{(N)}(r)) \nabla v_i^{(N)} \nabla v_j^{(N)} \, dx \, dr \\
&\quad + \frac{1}{2} \int_0^t \|\Pi_N(P^{\frac{1}{2}}H^{-1}(v^{(N)}(r))\sigma(v^{(N)}(r)))\|_{L^2(\mathbb{R}^n;L^2(\Omega))}^2 \, dr \\
&\quad + \frac{s}{2} \sum_{i,j=1}^n \int_0^t \left( (v_i^{(N)}(r))^{2-\frac{2}{s}} \Pi_N(\pi_i \sigma_{ij}(v^{(N)}(r))) dW_j(r) \right)_{L^2(\Omega)}.
\end{align*}

For the third term on the right-hand-side of (43), by the assumption (7),
\begin{align*}
\frac{1}{2} \int_0^t \|P^{\frac{1}{2}}H^{-1}(v^{(N)}(r))\sigma(v^{(N)}(r))\|_{L^2(\mathbb{R}^n;L^2(\Omega))}^2 \, dr \\
= \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\sqrt{s}}{2} (v_i^{(N)}(r))^{\frac{2-2}{s}} \sigma_{ij}(v^{(N)}(r)) \|_{L^2(\mathbb{R}^n;L^2(\Omega))}^2 \, dr &\leq C \int_0^T (1 + \|v^{(N)}(r)\|_{L^2(\Omega)}) \, dr.
\end{align*}

For the fourth term on the right-hand-side of (43), the process given by the below
\begin{align*}
\mu^{(N)}(t) = \sum_{i,j=1}^n \int_0^t (\pi_i \sigma_{ij}(v^{(N)}(r)) dW_j(r), (v_i^{(N)}(r))^{2-\frac{2}{s}})_{L^2(\Omega)} = \mu^{(N)}(t), \quad t \in (0, T),
\end{align*}
is an $\mathcal{F}_t$-martingale. Still by the Burkholder-Davis-Gundy inequality, plus the assumption on $\sigma$, the Hölder inequality and the Young inequality, for every positive constant $\varepsilon_0$, we have
\begin{align*}
\mathbb{E}\left( \sup_{t \in (0,T)} \left| \sum_{i,j=1}^n \int_0^t \pi_i (v_i^{(N)}(r), (v_i^{(N)}(r))^{1-\frac{2}{s}} \sigma_{ij}(v^{(N)}(r)) dW_j(r) \right)_{L^2(\Omega)} \right| \right) \\
\leq C \mathbb{E}\left( \int_0^T \|v^{(N)}(r)\|_{L^2(\Omega)}^2 \|v_i^{(N)}(r))^{1-\frac{2}{s}} \sigma(v^{(N)}(r))\|_{L^2(\mathbb{R}^n;L^2(\Omega))}^2 \, dr \right)^{1/2} \\
\leq C \mathbb{E}\left( \left( \sup_{t \in (0,T)} \|v^{(N)}(t)\|^2_{L^2(\Omega)} \right) \left( \int_0^T (1 + \|v^{(N)}(r)\|^2_{L^2(\Omega)}) \, dr \right)^{\frac{1}{2}} \right) \\
\leq C \varepsilon_0 \mathbb{E}\left( \sup_{t \in (0,T)} \|v^{(N)}(t)\|^2_{L^2(\Omega)} \right) + C \cdot \frac{1}{4\varepsilon_0} \left( T + \mathbb{E} \int_0^T \|v^{(N)}(r)\|^2_{L^2(\Omega)} \, dr \right).
\end{align*}
Combining those above deduction, we conclude the below
\[ \frac{1}{2} \mathbb{E} \left( \sup_{t \in (0, T)} \| P_{t}^{2} v^{(N)}(t) \|_{L^{2}(\mathcal{O})}^{2} \right) + \frac{2}{s} \sum_{i,j=1}^{n} \mathbb{E} \int_{0}^{T} \int_{\mathcal{O}} \pi^{H}_{i,j}(v^{(N)}(r)) \nabla v_{i}^{(N)} \nabla v_{j}^{(N)} \, dx \, dr \]
\[ \leq \frac{1}{2} \mathbb{E} \| P_{1}^{2} v^{(N)}(0) \|_{L^{2}(\mathcal{O})}^{2} + C \mathbb{E} \int_{0}^{T} \left( 1 + \| v^{(N)}(r) \|_{L^{2}(\mathcal{O})} \right) \, dr + C \cdot \frac{\varepsilon_{0} s}{2} \mathbb{E} \left( \sup_{t \in (0, T)} \| v^{(N)}(t) \|_{L^{2}(\mathcal{O})}^{2} \right) + C \cdot \frac{\varepsilon_{0} s}{8 \varepsilon_{0}} \left( T + \mathbb{E} \int_{0}^{T} \| v^{(N)}(r) \|_{L^{2}(\mathcal{O})}^{2} \, dr \right). \]

The Lemma 4 indicates that for \( z_{i}, z_{j} \in \mathbb{R} \), there exist constants \( \alpha_{1} > 0, \alpha_{2} > 0 \), with \( \alpha_{1}, \alpha_{2} \) changing from line to line, such that if each component of \( v \) is nonnegative, \( \mathbb{P} \)-a.s. then \( \sum_{i,j=1}^{n} \pi_{i,j}^{H}(v) z_{i} z_{j} \geq \alpha_{1} \sum_{i=1}^{n} z_{i}^{2} + \alpha_{2} \sum_{i=1}^{n} v_{i}^{2} z_{i}^{2} \). \( v^{(N)} \) is nonnegative, \( \mathbb{P} \)-a.s. thus
\[ \alpha_{3} \mathbb{E} \left( \sup_{t \in (0, T)} \| v^{(N)}(t) \|_{L^{2}(\mathcal{O})}^{2} \right) + \alpha_{1} \mathbb{E} \int_{0}^{T} \| \nabla v^{(N)}(r) \|_{L^{2}(\mathcal{O})}^{2} \, dr + \alpha_{2} \mathbb{E} \int_{0}^{T} \| \nabla (v^{(N)}(r)) \|_{L^{2}(\mathcal{O})}^{2} \, dr \]
\[ \leq C + C \left( 1 + \frac{s}{8 \varepsilon_{0}} \right) \int_{0}^{T} \mathbb{E} \left( \sup_{\tau \in (0, r)} \| v^{(N)}(\tau) \|_{L^{2}(\mathcal{O})}^{2} \right) \, dr + C \cdot \frac{\varepsilon_{0} s}{2} \mathbb{E} \left( \sup_{t \in (0, T)} \| v^{(N)}(t) \|_{L^{2}(\mathcal{O})}^{2} \right). \]

Choose this \( \varepsilon_{0} \) such that \( C \varepsilon_{0} s < 2 \alpha_{3} \), then for some positive constants \( C_{0}, C_{1}, \)
\[ \alpha_{3} \mathbb{E} \left( \sup_{t \in (0, T)} \| v^{(N)}(t) \|_{L^{2}(\mathcal{O})}^{2} \right) + \alpha_{1} \mathbb{E} \int_{0}^{T} \| \nabla v^{(N)}(r) \|_{L^{2}(\mathcal{O})}^{2} \, dr \]
\[ + \alpha_{2} \mathbb{E} \int_{0}^{T} \| \nabla (v^{(N)}(r)) \|_{L^{2}(\mathcal{O})}^{2} \, dr \leq C_{0} + C_{1} \mathbb{E} \int_{0}^{T} \left( \sup_{\tau \in (0, r)} \| v^{(N)}(\tau) \|_{L^{2}(\mathcal{O})}^{2} \right) \, dr, \]
and by the Gronwall lemma,
\[ \mathbb{E} \left( \sup_{t \in (0, T)} \| v^{(N)}(t) \|_{L^{2}(\mathcal{O})}^{2} \right) \leq \frac{C_{0}}{\alpha_{3}} \left( 1 + \frac{C_{1} T}{\alpha_{3}} e^{-\frac{\alpha_{3} s}{T}} \right), \]
which follows that
\[ \sup_{N \in \mathbb{N}} \mathbb{E} \left( \sup_{t \in (0, T)} \| v^{(N)}(t) \|_{L^{2}(\mathcal{O})}^{2} \right) \leq C, \]
where \( C \) depends on \( \mathbb{E} \| v^{0} \|_{L^{2}(\mathcal{O})}^{2}, T \) and constants in the assumption (7). By the definition that \( v^{(N)}(t) = (u^{(N)}(t))^{2} \), we insert (45) into (44), (40) immediately follows.

In order to show (40), we need to list a higher-order moment estimate, and we need this estimation as a direct result, which is the below (46). The proof of (40) rely on (45), with (45) already been proved. Detailed proof of this result be referred to [13], Lemma 6. Let \( p = \frac{24}{\varepsilon_{0}} \) and by the assumption (12), \( \mathbb{E} \| v^{0} \|_{L^{2}(\mathcal{O})}^{p} < \infty \) (please refer to Remark 2, Remark 18 of [13] for a further understanding of this initial condition), then we have
\[ \sup_{N \in \mathbb{N}} \mathbb{E} \left( \sup_{t \in (0, T)} \| v^{(N)}(t) \|_{L^{2}(\mathcal{O})}^{p} \right) \leq C. \]
By the Gagliardo-Nirenberg inequality with \( \theta = \frac{d}{d+2} \), and the Hölder inequality with \( \frac{d}{d+2} + \frac{2}{d+2} = 1 \), we have

\[
\mathbb{E} \int_0^T \| (v^{(N)}(r))^{2} \|_{L^2(\Omega)}^2 \, dr \leq C \mathbb{E} \int_0^T \| \nabla (v^{(N)}(r))^{2} \|_{L^2(\Omega)}^2 \| (v^{(N)}(r))^{2} \|_{L^1(\Omega)}^2 \, dr
\]

\[
= C \mathbb{E} \int_0^T \| \nabla (v^{(N)}(r))^{2} \|_{L^2(\Omega)}^2 \| (v^{(N)}(r))^{2} \|_{L^1(\Omega)}^2 \, dr
\]

\[
\leq C \mathbb{E} \left\{ \left( \int_0^T \| \nabla (v^{(N)}(r))^{2} \|_{L^2(\Omega)}^2 \, dr \right)^{\frac{4}{d+2}} \right\} \left( \int_0^T \| (v^{(N)}(r))^{2} \|_{L^2(\Omega)}^2 \, dr \right)^{\frac{d}{d+2}}
\]

and

\[
\int_0^T \| (v^{(N)}(r))^{4} \|_{L^2(\Omega)}^2 \, dr \leq T \sup_{t \in (0,T)} \| v^{(N)}(t) \|_{L^2(\Omega)}^4,
\]

thus

\[
\mathbb{E} \int_0^T \| (v^{(N)}(r))^{2} \|_{L^2(\Omega)}^2 \, dr
\]

\[
\leq CT \frac{2}{d+2} \mathbb{E} \left\{ \left( \sup_{t \in (0,T)} \| v^{(N)}(t) \|_{L^2(\Omega)} \right)^{\frac{4}{d+2}} \right\} \left( \int_0^T \| \nabla (v^{(N)}(r))^{2} \|_{L^2(\Omega)}^2 \, dr \right)^{\frac{d}{d+2}}
\]

\[
\leq CT \frac{2}{d+2} \left( \mathbb{E} \sup_{t \in (0,T)} \| v^{(N)}(t) \|_{L^2(\Omega)}^4 \right)^{\frac{4}{d+2}} \left( \mathbb{E} \int_0^T \| \nabla (v^{(N)}(r))^{2} \|_{L^2(\Omega)}^2 \, dr \right)^{\frac{d}{d+2}} \leq C.
\]

The low dimensional assumption plays a major role in the proof so long as \( d \) can not be chosen a large number, which is satisfied by the assumption that \( d \leq 3 \). Since

\[
\mathbb{E} \int_0^T \| (v^{(N)})^{3} \|_{L^2(\Omega)}^3 \, dr \leq C \mathbb{E} \int_0^T \| (v^{(N)})^{2} \|_{H^1(\Omega)}^2 \| (v^{(N)})^{2} \|_{L^1(\Omega)}^2 \, dr
\]

\[
\leq C \mathbb{E} \left( \sup_{t \in (0,T)} \| v^{(N)}(t) \|_{L^2(\Omega)} \right)^{\frac{12}{d+2}} \left( \int_0^T \| (v^{(N)})^{2} \|_{H^1(\Omega)}^2 \, dr \right)^{\frac{6}{d+2}}
\]

\[
\leq C \left( \mathbb{E} \left( \sup_{t \in (0,T)} \| v^{(N)}(t) \|_{L^2(\Omega)} \right)^{\frac{24}{d+6}} \right)^{\frac{6}{d+2}} \left( \mathbb{E} \int_0^T \| (v^{(N)})^{2} \|_{H^1(\Omega)}^2 \, dr \right)^{\frac{6}{d+2}}
\]

by (40), (46) and (47), we have

\[
\mathbb{E} \int_0^T \| (v^{(N)})^{2} \|_{L^2(\Omega)}^3 \, dr \leq C.
\]

Since \( v^{(N)}(t) = (u^{(N)}(t))^\frac{2}{3} \), (41) holds, and we finish the proof of this lemma.

\[\square\]

5. Tightness of the approximated sequence and the converging sequence

A weak solution of the cross-diffusion system

We introduce those topological spaces established in [13]:

\[ Z_T = C^0([0,T]; H^3(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap L^2(0,T; L^2(\Omega)) \cap C^0([0,T]; L^2(\Omega)) \]
endowed with the topology $\mathcal{T}$, with $\mathcal{T}$ the maximum one of above topological spaces. On this space, we can formulate a compactness criterion. The origins and proving details for this criterion are from \cite{[1], [2], [3]}. For a detailed explanation of the Aldous condition, refer to \cite{[2]}. Please also see \cite{[1]} for some preparation materials to have a better knowledge of those tightness criteria we use in the following proof.

**Lemma 11.** The set of measures $\{\mathcal{L}(u^{(N)}): N \in \mathbb{N}\}$ is tight on $(Z_T, \mathcal{T})$.

**Proof.** Theorem 10 in \cite{[3]} provides criterions for the tightness of the approximated sequence $(u^{(N)})_{N \in \mathbb{N}}$ in $Z_T$, and we have verified two of those criterions in the Lemma \cite{[9]} which are

$$
\sup_{N \in \mathbb{N}} \mathbb{E}\left( \sup_{t \in (0,T)} \|u^{(N)}\|_{L^2(\Omega)}^2 \right) \leq C,
\sup_{N \in \mathbb{N}} \mathbb{E}\left( \int_0^T \|u^{(N)}\|_{H^1(\Omega)}^2 \, dt \right) \leq C.
$$

By the fact that embeddings $H^3(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)$ are dense and continuous and the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it remains to show that $(u^{(N)})_{N \in \mathbb{N}}$ satisfies the Aldous condition in $H^3(\Omega)'$. It is sufficient to show that for $(\tau_N)_{N \in \mathbb{N}}$ a sequence of $\mathbb{F}$-stopping times with $0 \leq \tau_N \leq T$, for every $\varepsilon > 0$, $\kappa > 0$, there exists $\delta > 0$ such that

$$
\sup_{N \in \mathbb{N}} \sup_{0 < \theta < \delta} \mathbb{P}\left( \|u^{(N)}(\tau_N + \theta) - u^{(N)}(\tau_N)\|_{H^3(\Omega)} \geq \kappa \right) \leq \varepsilon.
$$

Let $t \in (0, T)$ and $\phi \in H^3(\Omega)$, then

$$
\langle u^{(N)}_i(t), \phi \rangle = \langle \Pi_N(u^0_i), \phi \rangle - \sum_{j=1}^n \int_0^t \langle A_{ij}(u^{(N)}), \nabla u^{(N)}_j, \nabla \Pi_N \phi \rangle \, dr
$$

$$
+ \sum_{j=1}^n \left( \int_0^t \Pi_N(\sigma_{ij}(u^{(N)}(r))) \, dW_j(r), \phi \right) = J^{(N)}_0 + J^{(N)}_1(t) + J^{(N)}_2(t),
$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $H^3(\Omega)'$ and $H^3(\Omega)$. Denote $I_1 = \{ \omega \in \Omega : 0 \leq \int_0^T \|u^{(N)}(r)\|_{L^2(\Omega)}^3 \, dr \leq 1 \}$, with the complement of $I_1$ given by $I_1^c = \{ \omega \in \Omega : \int_0^T \|u^{(N)}(r)\|_{L^2(\Omega)}^3 \, dr > 1 \}$. Thus $\mathbb{E}\left( \|u^{(N)}(\cdot, T)\|_{L^3(\Omega)}^3 \right) = \mathbb{E}\left( \int_0^T \|u^{(N)}(r)\|_{L^2(\Omega)}^3 \, dr \right)^{\frac{1}{3}} = \int_{I_1 \cup I_1^c} \left( \int_0^T \|u^{(N)}(r)\|_{L^3(\Omega)}^3 \, dr \right)^{\frac{1}{3}} \mathbb{P}(d\omega) \leq 1 + \mathbb{E} \int_0^T \|u^{(N)}(r)\|_{L^3(\Omega)}^3 \, dr \leq C.$

By the Hölder inequality, we have

$$
\left( \mathbb{E}\left( \|u^{(N)}(\cdot, T)\|_{L^3(\Omega)}^3 \right) \right)^2 \leq C \mathbb{E}\left( \|u^{(N)}(\cdot, T)\|_{L^2(\Omega)}^2 \right) \mathbb{E}\left( \|\nabla u^{(N)}(\cdot, T)\|_{L^2(\Omega)}^2 \right) \leq C \mathbb{E}\left( \|\nabla u^{(N)}(\cdot, T)\|_{L^2(\Omega)}^2 \right)^2.
$$

Using the continuous embedding of $H^3(\Omega) \hookrightarrow W^{1, \infty}(\Omega)$, when $d \leq 3$, consider below terms presuming $0 < \theta < 1$ beforehand. Firstly, we denote $\chi_{(\tau_N, \tau_N + \theta)}(t) = 1$, if $\tau_N \leq t \leq \tau_N + \theta$,
\( \tau_N + \theta \), and otherwise, \( \chi(\tau_N, \tau_N + \theta) = 0 \). We observe that
\[
\int_{\tau_N}^{\tau_N + \theta} 1 \cdot \|u^{(N)}\|^s L^2(\Omega) \|\nabla u^{(N)}\| L^2(\Omega) dr = \int_0^T \chi(\tau_N, \tau_N + \theta)(r) \|u^{(N)}\|^s L^2(\Omega) \|\nabla u^{(N)}\| L^2(\Omega) dr
\]
\[
\leq \|\chi(\tau_N, \tau_N + \theta)(r)\| L^2((0, T)) \|u^{(N)}\|^s L^2(0, T; L^2(\Omega)) \|\nabla u^{(N)}\| L^2(0, T; L^2(\Omega))
\]
\[
= \|1\| L^2((\tau_N, \tau_N + \theta)) \|u^{(N)}\|^s L^2(0, T; L^2(\Omega)) \|\nabla u^{(N)}\| L^2(0, T; L^2(\Omega))
\]
\[
\leq \theta^\frac{1}{2} \|u^{(N)}\|^s L^2(0, T; L^2(\Omega)) \|\nabla u^{(N)}\| L^2(0, T; L^2(\Omega)),
\]
and for \( J_1^{(N)}(t) \), if \( i = j \),
\[
\mathbb{E}\left[ \int_{\tau_N}^{\tau_N + \theta} \langle A_{ij}(u^{(N)}) \nabla u_j^{(N)}, \nabla \Pi \phi \rangle dr \right] \leq \mathbb{E}\left[ \int_{\tau_N}^{\tau_N + \theta} \|A_{ij}(u^{(N)})\| L^2(\Omega) \|\nabla u_j^{(N)}\| L^2(\Omega) \|\nabla \phi\| L^\infty(\Omega) dr \right]
\]
\[
\leq C \mathbb{E}\left[ \int_{\tau_N}^{\tau_N + \theta} \|A_{ij}(u^{(N)})\| L^2(\Omega) \|\nabla u_j^{(N)}\| L^2(\Omega) \|\nabla \phi\| H^2(\Omega) dr \right]
\]
\[
\leq C \mathbb{E}\left( \theta^\frac{1}{2} + \theta^\frac{1}{2} \|u^{(N)}\|^s L^2(0, T; L^2(\Omega)) \|\nabla u^{(N)}\| L^2(0, T; L^2(\Omega)) \|\nabla \phi\| H^2(\Omega) \right)
\]
\[
\leq C \left\{ \theta^\frac{1}{2} \left( \mathbb{E}\left( \int_0^T \|u^{(N)}\|^s L^2(\Omega) dr \right)^\frac{1}{2} \right)^\frac{1}{2} \cdot \left( \mathbb{E}\left( \int_0^T \|\nabla u^{(N)}\|^2 L^2(\Omega) dr \right)^\frac{1}{2} \right) \right\} \|\nabla \phi\| H^2(\Omega)
\]
if \( i \neq j \),
\[
\mathbb{E}\left[ \int_{\tau_N}^{\tau_N + \theta} \langle A_{ij}(u^{(N)}) \nabla u_j^{(N)}, \nabla \Pi \phi \rangle dr \right]
\]
\[
\leq \mathbb{E}\left[ \int_{\tau_N}^{\tau_N + \theta} \|\nabla u_j^{(N)}\| L^2(\Omega) \|u^{(N)}\|^s L^2(\Omega) \nabla \phi \| dx dr \right]
\]
\[
\leq \mathbb{E}\left[ \int_{\tau_N}^{\tau_N + \theta} \|u^{(N)}\|^s L^2(\Omega) \|\nabla \phi\| L^\infty(\Omega) dr \right]
\]
\[
\leq C \mathbb{E}\left\{ \left( \int_{\tau_N}^{\tau_N + \theta} \|u^{(N)}\|^2 L^2(\Omega) dr \right)^\frac{1}{2} \left( \int_{\tau_N}^{\tau_N + \theta} \|\nabla u^{(N)}\|^2 L^2(\Omega) dr \right)^\frac{1}{2} \right\} \|\nabla \phi\| H^2(\Omega)
\]
\[
\leq C \theta^\frac{1}{2} \left( \mathbb{E}\left( \sup_{t \in (0, T)} \|u^{(N)}\|^2 L^2(\Omega) \right)^\frac{1}{2} \right) \cdot \left( \mathbb{E}\left( \int_0^T \|\nabla u^{(N)}\|^2 L^2(\Omega) dr \right)^\frac{1}{2} \right) \|\nabla \phi\| H^2(\Omega) \leq C \theta^\frac{1}{2} \|\nabla \phi\| H^2(\Omega).
\]

Denote \( I_2 = \{ \omega \in \Omega : 0 \leq J_1^{(N)}(t) \leq 1 \} \), \( I_2^c = \{ \omega \in \Omega : J_1^{(N)}(t) > 1 \} \), and
\[
\mathbb{E}\left( \int_0^T \|u^{(N)}\|^3 L^2(\Omega) dr \right)^\frac{2}{3} = \int_{I_2 \cup I_2^c} \left( \int_0^T \|u^{(N)}\|^3 L^2(\Omega) dr \right)^\frac{2}{3} \mathbb{P}(d\omega) \leq 1 + \mathbb{E}\int_0^T \|u^{(N)}\|^3 L^2(\Omega) dr \leq
\]
$1 + T \cdot \sup_{N \in \mathbb{N}} \mathbb{E}\left( \sup_{t \in (0, T)} \|u^{(N)}(t)\|^3_{L^2(\mathcal{O})} \right) \leq C$. For $J_2^{(N)}(t)$,

\[
\mathbb{E}\left( \int_{\tau_N}^{\tau_N + \theta} \Pi_N(\sigma_{ij}(u^{(N)}(r))dW_j(r), \phi) \right)^2 \leq \mathbb{E}\left( \int_{\tau_N}^{\tau_N + \theta} \|\sigma(u^{(N)})\|^2_{L^2(\mathbb{R}^n; L^2(\mathcal{O}))} dr \right) \|\phi\|^2_{L^2(\mathcal{O})}
\]

\[
\leq C\mathbb{E}\left( \int_{\tau_N}^{\tau_N + \theta} (1 + \|u^{(N)}\|^2_{L^2(\mathcal{O}))} dr \right) \|\phi\|^2_{L^2(\mathcal{O})} = C\left( \theta + \mathbb{E} \int_0^{T} \chi(\tau_N, \tau_N + \theta) \|u^{(N)}\|^2_{L^2(\mathcal{O}))} dr \right) \|\phi\|^2_{L^2(\mathcal{O})}
\]

\[
\leq C\left( \theta + \mathbb{E}\right) \left( \int_0^{T} \|u^{(N)}\|^3_{L^2(\mathcal{O}))} dr \right)^{3/2} \|\phi\|^2_{L^2(\mathcal{O})} \leq C\theta^{3/2} \|\phi\|^2_{H^3(\mathcal{O})}.
\]

Let $\kappa > 0$, $\varepsilon > 0$, by the definition of the $H^3(\mathcal{O})'$ norm and the Chebyshev inequality, for every $\phi$, with $\|\phi\|_{H^3(\mathcal{O})} = 1$,

\[
\mathbb{P}\left\{ |J_1^{(N)}(\tau_N + \theta) - J_1^{(N)}(\tau_N)| \geq \kappa \right\} \leq \frac{1}{\kappa^2} \mathbb{E}|J_1^{(N)}(\tau_N + \theta) - J_1^{(N)}(\tau_N)| \leq \frac{C\theta^{3/2}}{\kappa^2},
\]

choose $\delta_1 = (\kappa \varepsilon / C)^3$, we infer that

\[
\sup_{N \in \mathbb{N}} \sup_{0 < \theta < \delta_1} \mathbb{P}\left\{ |J_1^{(N)}(\tau_N + \theta) - J_1^{(N)}(\tau_N)| \geq \kappa \right\} \leq \varepsilon.
\]

Still applying the Chebyshev inequality,

\[
\mathbb{P}\left\{ |J_2^{(N)}(\tau_N + \theta) - J_2^{(N)}(\tau_N)| \geq \kappa \right\} \leq \frac{1}{\kappa^2} \mathbb{E}|J_2^{(N)}(\tau_N + \theta) - J_2^{(N)}(\tau_N)|^2 \leq \frac{C\theta^{3/2}}{\kappa^2},
\]

choose $\delta_2 = (\kappa^2 \varepsilon / C)^3$, we infer that

\[
\sup_{N \in \mathbb{N}} \sup_{0 < \theta < \delta_2} \mathbb{P}\left\{ |J_2^{(N)}(\tau_N + \theta) - J_2^{(N)}(\tau_N)| \geq \kappa \right\} \leq \varepsilon.
\]

This verifies the Aldous condition for $J_1^{(N)}(t)$ and $J_2^{(N)}(t)$, and $J_0^{(N)}$ is automatically satisfied since no parameter $t$ is involved for $J_0^{(N)}$. The set of measures $\{\mathcal{L}(u^{(N)}): N \in \mathbb{N}\}$ is tight on $(Z_T, \mathcal{T})$. 

\[\square\]

After we have shown the tightness property of the approximated sequence, it remains to show that this sequence converges to a weak solution of (1)-(3). The rest part largely rely on before work, i.e. [11, 12, 13, 18].

Based on results from [18], we can find a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and on this space $Z_T \times C^0([0, T]; Y_0)$-valued random variables $(\tilde{u}, \tilde{W})$, $(\tilde{u}^{(N)}, \tilde{W}^{(N)})$ with $N \in \mathbb{N}$ such that $(\tilde{u}^{(N)}, \tilde{W}^{(N)})$ has the same law as $(u^{(N)}, W)$ on $\mathcal{B}(Z_T \times C^0([0, T]; Y_0))$ and

\[
(\tilde{u}^{(N)}, \tilde{W}^{(N)}) \to (\tilde{u}, \tilde{W}) \quad \text{in} \quad Z_T \times C^0([0, T]; Y_0), \quad \tilde{\mathbb{P}}\text{-a.s., as } N \to \infty.
\]
We have shown that $u_i^{(N)}$ is nonnegative for every $i = 1, \ldots, n$. By the Lemma 13 of [11], we can show that for a.e. $(x, t) \in \mathcal{O} \times (0, T)$, $\tilde{P}$-a.s. $\tilde{u}_i^{(N)}$ is nonnegative, and its limit $\tilde{u}_i(x, t) \geq 0$, for every $i = 1, \ldots, n$.

One step further, we state another important result without proof. A similar result has been proved twice in the Lemma 16 in [12], Lemma 10 in [13]. In Lemma 14 of [11], the proof is omitted, for the reason the method is basically the same one. Readers are referred to those details for a detailed explanation.

**Lemma 12.** For every $r, t \in (0, T)$ with $r \leq t$, $\phi_1 \in L^2(\mathcal{O})$ and $\phi_2 \in H^3(\mathcal{O})$ satisfying $\nabla \phi_2 \cdot \nu = 0$ on $\partial \mathcal{O}$, we have

$$\lim_{N \to \infty} \tilde{E} \int_0^T (\tilde{u}^{(N)}(t) - \tilde{u}(t), \phi_1)^2_{L^2(\mathcal{O})} dt = 0,$$

$$\lim_{N \to \infty} \tilde{E}(\tilde{u}^{(N)}(0) - \tilde{u}(0), \phi_1)^2_{L^2(\mathcal{O})} = 0,$$

$$\lim_{N \to \infty} \tilde{E} \int_0^T \left| \sum_{j=1}^n \int_0^t \left( A_{ij}(\tilde{u}^{(N)}(r))\nabla \tilde{u}_j^{(N)}(r) - A_{ij}(\tilde{u}(r))\nabla \tilde{u}_j(r), \nabla \phi_2 \right) dr \right| dt = 0,$$

$$\lim_{N \to \infty} \tilde{E} \int_0^T \left| \sum_{j=1}^n \int_0^t \left( \sigma_{ij}(\tilde{u}^{(N)}(r))d\tilde{W}_j^{(N)}(r) - \sigma_{ij}(\tilde{u}(r))d\tilde{W}_j(r), \phi_1 \right) \right|_{L^2(\mathcal{O})}^2 dt = 0.$$

The Lemma [12] is a key tool in the proof of the main theorem.

**Proof.** (Proof of the main Theorem 1) Let us define

$$\Lambda_i^{(N)}(\tilde{u}^{(N)}, \tilde{W}^{(N)}, \phi)(t) = (\Pi_N(\tilde{u}_i^{(N)}), \phi)_{L^2(\mathcal{O})} + \sum_{j=1}^n \int_0^t \langle \Pi_N \text{div}(A_{ij}(\tilde{u}^{(N)}(r))\nabla \tilde{u}_j^{(N)}(r)), \phi \rangle dr$$

$$+ \left( \sum_{j=1}^n \int_0^t \Pi_N \sigma_{ij}(\tilde{u}^{(N)}(r))d\tilde{W}_j^{(N)}(r), \phi \right)_{L^2(\mathcal{O})},$$

and

$$\Lambda_i(\tilde{u}, \tilde{W}, \phi)(t) = (\tilde{u}_i(0), \phi)_{L^2(\mathcal{O})} + \sum_{j=1}^n \int_0^t \langle \text{div}(A_{ij}(\tilde{u}(r))\nabla \tilde{u}_j(r)), \phi \rangle dr$$

$$+ \left( \sum_{j=1}^n \int_0^t \sigma_{ij}(\tilde{u}(r))d\tilde{W}_j(r), \phi \right)_{L^2(\mathcal{O})},$$

for $t \in (0, T)$ and $i = 1, \ldots, n$.

$u^{(N)}$ is the strong solution, it satisfies the identity $(u_i^{(N)}(t), \phi)_{L^2(\mathcal{O})} = \Lambda_i^{(N)}(u^{(N)}, W, \phi)(t)$, $\tilde{P}$-a.s., for all $t \in (0, T)$, $i = 1, \ldots, n$, $\phi \in H^1(\mathcal{O})$. In particular, we have

$$\int_0^T \mathbb{E}|(u_i^{(N)}(t), \phi)_{L^2(\mathcal{O})} - \Lambda_i^{(N)}(u^{(N)}, W, \phi)(t)|dt = 0.$$
Since the law of $L(u^{(N)}, W)$ coincides with the law of $L(\tilde{u}^{(N)}, \tilde{W}^{(N)})$, we have

$$\int_0^T \tilde{E}(\tilde{u}_i^{(N)}(t), \phi)_{L^2(O)} - \Lambda_i^{(N)}(\tilde{u}^{(N)}, \tilde{W}^{(N)}, \phi)(t) \, dt = 0, \quad i = 1, \ldots, n.$$ 

Let $N \to \infty$, then $\int_0^T \tilde{E}(\tilde{u}_i(t), \phi)_{L^2(O)} - \Lambda_i(\tilde{u}, \tilde{W}, \phi)(t) \, dt = 0, \quad i = 1, \ldots, n$. This identity holds for every $\phi \in H^3(O)$ satisfying $\nabla \phi \cdot \nu = 0$ on $\partial O$, and by the density argument, it holds for every $\phi \in H^1(O)$. Hence, for Lebesgue-a.e. $t \in [0, T]$ and $\tilde{P}$-a.e. $\omega \in \tilde{\Omega}$, we deduce that $(\tilde{u}_i(t), \phi)_{L^2(O)} - \Lambda_i(\tilde{u}, \tilde{W}, \phi)(t) = 0, \quad i = 1, \ldots, n$. By the definition of $\Lambda_i$, this means that for Lebesgue-a.e. $t \in [0, T]$ and $\tilde{P}$-a.e. $\omega \in \tilde{\Omega}$,

$$(\tilde{u}_i(t), \phi)_{L^2(O)} = (\tilde{u}_i(0), \phi)_{L^2(O)} + \sum_{j=1}^n \int_0^t \langle \text{div} (A_{ij}(\tilde{u}(r)) \nabla \tilde{u}_j(r)), \phi \rangle \, dr$$

$$+ \left( \sum_{j=1}^n \int_0^t \sigma_{ij}(\tilde{u}(r)) d\tilde{W}_j(r), \phi \right)_{L^2(O)}.$$

We set $\tilde{U} = (\tilde{\Omega}, \tilde{F}, \tilde{P}, \tilde{E})$, and the system $(\tilde{U}, \tilde{W}, \tilde{u})$ is a martingale solution to (1)-(3), and we complete the proof.

\[\square\]

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