A concrete anti-de Sitter black hole with dynamical horizon having toroidal cross-sections and its characteristics

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We propose a special solution of Einstein equations in the general Vaidya form representing a dynamical black hole having horizon cross sections with toroidal topology. The concrete model enables us to study for the first time dynamical horizons with toroidal topology, its area law, and the question of matter flux inside the horizon, without using a cut-and-paste technology to construct the solution.

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I. INTRODUCTION

The topology of black hole horizons has been a matter of wide discussions in the past. Starting with the Hawking’s theorem, stating that each connected component of the event horizon of a stationary black hole in four dimensional space time has the topology of a 2-sphere [1], most of the authors have been interested in non-dynamical asymptotically flat space times, excluding any kind of cosmological black holes [10]. Gannon was the first who opened the possibility of a torus topology for a black hole horizon [2], generalizing Hawking’s theorem by replacing stationarity by some weaker assumptions.

On the other hand, Chrusciel and Wald [3] showed that each connected component of a cross-section of the event horizon of a stationary asymptotically flat black hole must have spherical topology. Jacobson and Venkataramani [4] proved that, under certain conditions, the topology of the event horizon of a four dimensional asymptotically flat black-hole space time must be 2-sphere. All these studies have been supported by the topological censorship theorem of Friedmann, Schleich and Witt, another statement indicating the impossibility of non spherical horizons [5]. The theorem states that in a globally hyperbolic, asymptotically flat spacetime, any two causal curves extending from past to future null infinity are homotopic, meaning that a black hole with toroidal surface topology is a possible violation of topological censorship theorem, as pointed out in [4]. In fact, as was shown by Shapiro, Teutolsky and Winicour [6], a temporarily toroidal horizon can be formed in a gravitational collapse, in a way consistent with the theorem.

Now, a concrete model of a black hole with toroidal horizon has been constructed by [7] for which the thermodynamics and area law is also considered. Vanzo’s model represents an isolated genus-one black hole in an asymptotically anti-de Sitter space time; its extension to d dimensions with a negative cosmological constant is given in [8]. Recently, toroidal and higher genus asymptotically AdS black holes have been put forward [9] through gluing of some special metrics such as Lemaitre-Tolman-Bondi-like and McVittie solution to toroidal or higher genus asymptotically AdS black holes. These pasted-manifolds, however, lack the dynamical features of the black hole we are interested in.

Our purpose is to construct a dynamical topological black hole in an asymptotically anti-de Sitter space time as a first step towards a better understanding of cosmological black holes [10] and their thermodynamics. For a more concrete definition of dynamical horizons and their difference to the isolated ones we refer to [11] and the references therein. A general formalism for the area law in the case of dynamical black holes is also formulated there for the
first time, including the possibility of the toroidal cross section of the horizon and the corresponding black hole area law. Using a 3+1 decomposition and the Gauss-Bonnet theorem, they also found a formalism which identifies the rate at which the radius of the cross-sections increases precisely according to the matter flux and gravitational wave on the horizon with cross-sections having a spherical topology and strictly positive cosmological constants. However, in the case of zero and negative cosmological constant, where the topology of horizon cross-sections may be toroidal, their formalism is not conclusive.

In this paper we construct a non-stationary space-time which is asymptotically anti-de Sitter and represents a dynamical horizon with toroidal cross-sections. An area law is then written for this dynamical black hole showing an increase of the horizon in accordance with the second law of black hole thermodynamics. The matter flux, however, is non-vanishing, although the total matter flux including the contribution from the cosmological constant vanishes. Our dynamical horizon model with toroidal cross-sections reduce to that of Vanzo [7] with an isolated horizon if the space-time is stationary.

We will follow in this paper definitions and notations used in [11]. They introduced a local definition of horizon as a three dimensional manifold $H$ in space time which can be foliated by closed 2-dimensional surfaces $S$, assuming special characteristic of the expansion on each leaf. The space-time metric $g_{ab}$, has signature $(-, +, +, +)$ and its covariant derivative operator will be denoted by $\nabla$. The Riemann tensor is defined by $R_{abcd} := 2\nabla_a \nabla_b g_{cd} - \nabla_{[a} g_{bc]} + \nabla_{[b} g_{ac]} - \nabla_{[a} g_{[b]}^c]$, the Ricci tensor by $R_{ab} := R_{abcd} n^d$, and the scalar curvature by $R := g^{ab} R_{ab}$. The unit normal to $H$ will be denoted by $\tilde{r}^a$; $g_{ab} \tilde{r}^a \tilde{r}^b = -1$. The intrinsic metric and the extrinsic curvature of $H$ are denoted by $g_{ab}$ and $K_{ab} := g_{ab}^{\prime} - \tilde{r}^a \tilde{r}^b$, respectively. $D$ is the covariant derivative operator on $H$ compatible with $g_{ab}$, $R_{ab}$ its Ricci tensor and $\nabla$ its scalar curvature. The unit space-like vector orthogonal to $S$ and tangent to $H$ is denoted by $\tilde{r}^a$. Quantities intrinsic to $S$ will be generally written with a tilde. Thus, the two-metric on $S$ is $\tilde{g}_{ab}$ and the extrinsic curvature of $S \subset H$ is $\tilde{K}_{ab} := \tilde{g}_{ab}^{\prime} - \tilde{r}^a \tilde{r}^b$. The derivative operator on $(S, \tilde{g}_{ab})$ is $\tilde{D}$ and its Ricci tensor is $\tilde{R}_{ab}$. Finally, we can fix the rescaling freedom in the choice of null normals $\ell^a := \tilde{\tau}^a + \tilde{r}^a$ and $n^a := \tilde{\tau}^a - \tilde{r}^a$ such that $\ell^a n_a = -2$. The shear tensor $\sigma_{ab}$ of the null vector $\ell^a$ and its trace $\sigma$ are defined as usual. A dynamical horizon is then defined as a three dimensional space-like sub-manifold of space time such that it can be foliated with closed orientable two dimensional surfaces on which the expansion $\Theta (\ell)$ vanishes, and the expansion $\Theta (n)$ of the other null normal is negative. Section 2 is devoted to a short review of the area law formalism for dynamical horizons in different cases of the cosmological constant. The area law for the dynamical horizon in a concrete metric with AdS back ground ($\Lambda < 0$) as an exact solution of the Einstein equations is discussed in section III. We then conclude in section IV.

II. AREA LAW FOR DYNAMICAL BLACK HOLES

Now, having the necessary definitions and notations, we may write the first consequence of the above definition of a dynamical horizon, using the relations $\Theta (\ell) = 0$ and $\Theta (n) < 0$, in the form

$$\tilde{K} = \tilde{q}^{ab} D_a \tilde{r}_b = \frac{1}{2} \tilde{q}^{ab} \tilde{\nabla}_a (\ell - n) = -\frac{1}{2} \Theta (n) > 0. \quad (1)$$

Hence, the area $a_S$ of $S$ will increase monotonically along $\tilde{r}^a$ by the change of the cross-sections, which is equivalent to the second law of black hole mechanics on $H$. To obtain an explicit expression for the area change the authors in [11] take first two fixed cross-sections $S_1$ and $S_2$ of $H$, and then integrate the result on a portion $\Delta H \subset H$ bounded by $S_1$ and $S_2$ with the corresponding radii $R_1$ and $R_2$. Note that $R$ is the area radius of $S$ defined by $a_S = 4\pi R^2$, $a_S$ being the surface area of $S$ independent of its topology. Following result is then obtained using Gauss-Bonnet theorem [11]:

$$\mathcal{I} (R_2 - R_1) = 16\pi G \int_{\Delta H} (T_{ab} - \frac{\Lambda}{8\pi G} g_{ab}) \hat{\zeta}^a (R) d^3V + \int_{\Delta H} N_R \left\{ |\sigma|^2 + 2|\zeta|^2 \right\} d^3V, \quad (2)$$

where $\mathcal{I}$ is the Euler characteristic of $S$. The scalar $|\zeta|$ is the length of the vector $\zeta^a = \tilde{q}^{ab} \tilde{\nabla}_b \ell_c$ and $\xi^a (R) = N_R l^a$ where the lapse function is given by $N_R = |\partial R|$. The first term on the right hand side is usually called the matter flux, $\mathcal{F}_{\text{matter}}^{(R)}$, and the second term is the gravitational wave flux, $\mathcal{F}_{\text{grav}}^{(R)}$. The discussion on the topology of $S$ is now divided in three cases depending on the cosmological constant.

Case 1: $\Lambda > 0$. Since the stress energy tensor $T_{ab}$ is assumed to satisfy the dominant energy condition, the right hand side is manifestly positive definite. Due to the fact that the area increases along $\tilde{r}^a$, we must have
Case 2 : $\Lambda = 0$. The right-hand side of eq.(2) is necessarily non-negative. Hence, the topology of hypersurface. We propose then a solution in the general Vaidya form:

$$\frac{R_2 - R_1}{2G} = \int_{\Delta H} (T_{ab} - \frac{\Lambda}{8\pi G} g_{ab}) \hat{\mathbf{r}}^a \xi^b_{(R)} \, d^3V$$

$$+ \frac{1}{16\pi G} \int_{\Delta H} N_R \{ |\sigma|^2 + 2|\zeta|^2 \} \, d^3V .$$  \hspace{1cm} (3)

**Case 2** : $\Lambda = 0$. The right-hand side of eq.(2) is necessarily non-negative. Hence, the topology of $S$ is either that of a 2-sphere (if the right hand side is positive) or that of a 2-torus (if the right hand side vanishes). The torus topology can occur if and only if $T_{ab}\xi^b$, $\sigma_{ab}$ and $\zeta^a$ all vanish everywhere on $H$. Therefore, it may be concluded that the scalar curvature $\mathcal{R}$ of $S$ must also vanish on every cross-section. The 2-manifold $S$ then has to be a flat torus. Using the fact that $H$ is space-like, we conclude that in this case $L_n \Theta(\ell) = 0$ everywhere on $H$. Thus, in this case the dynamical horizon cannot be a FOTH [11]. Furthermore, since $\Theta(\ell), \sigma_{ab}$, and $R_{ab}\xi^b$ all vanish on $H$, the Raychaudhuri equation now implies that $L_t \Theta(\ell)$ also vanishes.

Note the following cases:

- In the case of torus topology the transition to the stationary case and isolated horizon is not trivial [11]. Given that in the stationary case the topology cannot be a torus [1], a topology change is then unavoidable [12]. The procedure used in [11] to understand the transition to the isolated horizon is based on the relation

$$f L_n \Theta(\ell) = -\sigma^2 - R_{ab}\xi^a\xi^b ,$$  \hspace{1cm} (4)

where $f$ is the length of a space-like vector orthogonal to $S$ and tangent to $H$. Now, due to the fact that in the case of torus topology we have $L_n \Theta(\ell) = 0$, it cannot be concluded that $f = 0$ which is a necessary condition for the isolated horizon with the sphere topology.

- The familiar matter flux as defined in (2) does vanish in the case of the torus topology. However, by changing the definition of $\xi_{(R)}^b$ in (2) to $\xi_{(R)}^b = cn^b$, where $c$ is an appropriate coefficient related to $f$, we may arrive at a non-vanishing matter flux. That this is not always the case, we will see on hand a concrete example in the following section. We may also note that the positivity of the extrinsic curvature of $S$ along $r^a$ in the case of torus topology does not necessarily means an area increase. This is due to the vanishing of the left hand side of (2).

**Case 3**: $\Lambda < 0$. In this case there is no control on the sign of the right-hand side of eq.(2). Hence, any topology is permissible. Stationary solutions with quite general topologies are known for black holes which are locally asymptotically anti-de Sitter [7]. Event horizons of these solutions are potential asymptotic states of these dynamical horizons in the distant future.

### III. AREA LAW FOR A DYNAMICAL BLACK HOLE IN ADS BACK GROUND

We are interested in a solution of Einstein equations with negative cosmological constant representing a black hole with a dynamical horizon having torodial cross-sections. So far we have not found any exact solution of Einstein equations having these features and not constructed through a cut and paste technology. Solutions produced by cut and paste technology do not represent a genuine dynamical black hole due to the build-in freezing of the matching hypersurface. We propose then a solution in the general Vaidya form:

$$ds^2 = -f(v,r)dv^2 + 2dvdr + r^2(d\theta^2 + d\phi^2),$$  \hspace{1cm} (5)

with the arbitrary function $f$ of coordinates $v$ and $r$, where $v$ is the advanced time coordinate with $-\infty < v < \infty$, $r$ is the radial coordinate with $0 < r < \infty$, and $\theta, \phi$ are coordinates describing the two-dimensional zero-curvature space generated by the two-dimensional commutative Lie group $G_2$ of isometries [7]. The black-hole apparent horizon is space-like and located at

$$f(v,r) := 0.$$  \hspace{1cm} (6)
The expansions of the corresponding null normals are
\[ \Theta_t = \frac{f}{r}, \quad \text{and} \quad \Theta_n = -\frac{4}{r}. \]  
(7)

Note that \( \Theta_n \) is always negative and \( \Theta_t \) vanishes precisely at the horizon, as required by a dynamical horizon. The unit normal to the horizon is given by
\[ \hat{\tau}_a = \frac{1}{\sqrt{|2ff'|}}[-\hat{f}, \hat{f}'] \quad \text{and} \quad \hat{n}^a = \frac{1}{\sqrt{|2ff'|}}[\hat{f}', -\hat{f}]. \]  
(8)

The constant \( r \) surfaces are the preferred cross-sections of the horizon and the unit space-like normal \( \hat{\tau}^a \) to these cross sections is given by
\[ \hat{\tau}_a = \frac{1}{\sqrt{|2ff'|}}[-\hat{f}, \hat{f}'] \quad \text{and} \quad \hat{n}^a = \frac{1}{\sqrt{|2ff'|}}[\hat{f}', -\hat{f}]. \]  
(9)

The properly rescaled null normals are then given by
\[ \epsilon^a = \frac{2|f'|}{\sqrt{|2ff'|}}(0, 0, 0, 1) \quad \text{and} \quad n^a = \frac{2\hat{f}}{\sqrt{|2ff'|}}(1, 0, 0, 0). \]  
(10)

The lapse function corresponding to the radial coordinate \( r \), which in this case is identical to the area radius, is given by
\[ N_r = \left| \frac{f}{2\hat{f}} \right|^{1/2}. \]  
(11)

Thus the properly rescaled vector field corresponding to the radial coordinate \( r \) is \( \epsilon_{(r=R)} = N_r \epsilon^a = (\partial/\partial v)^a \).

Now, take the following special solution with toroidal horizon configurations suggested in [13]
\[ ds^2 = -\left( \alpha^2 r^2 - \frac{\beta m(v)}{r} \right) dv^2 + 2dvdr + r^2(d\theta^2 + d\phi^2), \]  
(12)

where \( 0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi \). The corresponding energy-momentum tensor is then given by
\[ T_{ab} = \frac{\beta}{8\pi r^2} \frac{dm(v)}{dv} k_a k_b, \]
\[ k_a = -\delta_a^i, \quad k_a k^a = 0, \]

where \( \alpha \equiv \sqrt{\frac{\Lambda}{3}}, \beta = q/\alpha, \) and \( m(v) \) is the Misner-Sharp mass. In these coordinates, lines with \( v = \)constant represent incoming radial null vectors having tangent vectors in the form \( k^a = (0, -1, 0, 0) \), or \( k_a = (-1, 0, 0, 0) \). The energy momentum tensor depends, in general, on \( \Lambda \) and diverges as \( \Lambda \to 0 \). To assume a toroidal cross-section and apply the eq.(2), we therefore ask for \( \beta \) to be independent of \( \Lambda \), leading to \( q = \frac{3\Lambda}{4\pi} \).

Now, noting that the energy-density of the radiation is \( \epsilon = \frac{q m}{4\pi r^3} \frac{dm}{dv} \), one sees that the weak energy condition for the radiation is satisfied whenever \( \frac{dm}{dv} \geq 0 \), i.e., the radiation is imploding.

The apparent horizon surface is now defined by
\[ qm|_{AH} = \alpha^3 r^3_{AH}. \]  
(14)

Having specified \( f \), and noting that \( f = -\frac{q m}{\alpha r^3} \) and \( f' = \frac{2\alpha^3 r^3 q m}{\alpha^3 r^3} \), we are able to calculate the gravitational flux leading to
\[ F_{grav}^{(R)} = \frac{1}{16\pi G} \int_{\Delta H} N_R \left\{ |\sigma|^2 + 2|\zeta|^2 \right\} d^3V = 0. \]  
(15)

Therefore, from the equation (2) we conclude that in our model of dynamical black hole with the toroidal topology the total matter flux term including the \( \Lambda \)-fluid across the horizon has to be zero, i.e. \( F_{total}^{(R)} = 0 \). This means
that the ordinary matter flux \( F_{\text{ordi}} = \int_{\Delta H} T_{ab} \hat{\tau}^a c_b \frac{d^3V}{\delta (R)} \), equals to the corresponding \( \Lambda \)-fluid term, i.e. \( F_{\Lambda} = \int_{\Delta H} \frac{\Lambda}{8\pi G} g_{ab} \hat{\tau}^a c_b \frac{d^3V}{\delta (R)} \).

Therefore, in the general case of non-stationary metric where \( m(v) \) is everywhere time dependent, the horizon is dynamical, the total matter flux is vanishing while the ordinary matter flux is non-zero, and the horizon is space-like. Now, let us differentiate two special cases:

- **\( m \) is constant everywhere**: In this case, the horizon is isolated. The following coordinate transformations will then give us the stationary metric suggested by Vanzo [7]:
  \[
  t = v - \int \frac{dr}{(\alpha^2 r^2 - \frac{qm(r)}{\alpha})}.
  \]
  Therefore, all results stated in [7] for this metric including the formula for the black hole area law are applicable here.

- **\( m \) is constant on the horizon \( (m|_{AH} = \text{constant}) \)**: In this limiting case we have \( \dot{m}|_{AH} = 0 \), i.e. the matter flux in addition to the total flux is zero, and \( \hat{\tau}^a \) will be null. Thus, the horizon, being null now, has a constant surface. It is therefore an isolated horizon although the metric is non-stationary.

### IV. CONCLUSION

We have constructed a dynamical black hole having toroidal topology in its cross-sections within an asymptotically anti-de Sitter space time as an exact solution of Einstein equations not produced by any cut-and-paste technology. The area law is written out and it has been shown that the total matter flux, including the \( \Lambda \)-matter, is zero while the ordinary matter flux is non-vanishing. This model for the first time, exemplifies the existence of dynamical horizons with toroidal topology. The vanishing of the total matter flux may be just a feature of the concrete model we have proposed. The model leads in a limiting case to an isolated horizon. Assuming our general metric to be stationary it reduces to the Vanzo metric [7].

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