Configurations related to combinatorial Veronesians representing a skew perspective

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August 22, 2019

Abstract

A combinatorial object representing schemas of, possibly skew, perspectives, called a configuration of skew perspective has been defined in [12], [4]. Here we develop the theory of configurations generalizing perspectives defined in combinatorial Veronesians. The complete classification of thus obtained \((15 \times 20)_3\)-configurations is presented.

key words: Veblen (Pasch) configuration, combinatorial Veronesian, binomial configuration, complete (free sub)subgraph, perspective.

MSC(2000): 05B30, 51E30.

Introduction

A project to characterize and classify so called binomial partial Steiner triple systems via the arrangement of their free complete subgraphs was started in [12]. In particular, we know that if a configuration \(K\) contains the maximal number (with respect to its parameters, i.e. \(= m + 2\), where \(m\) is the rank of a point in \(K\)) of free \(K_{m+1}\)-subgraphs then \(K\) is a so called combinatorial Grassmannian (cf. [7]) and if \(K\) contains \(m\) free complete subgraphs then it is a multi veblen configuration (cf. [9]). One of the most fruitful observation used to obtain a required classification is quoted in 1.2 after [12]:

\[
\text{a configuration } K \text{ with two free subgraphs } K_{m+1} \text{ can be considered as a schema of an abstract perspective between these graphs.}
\]

Let us stress on the words schema and abstract: ‘ordinary’ projections, as used and investigated e.g. in [1], [2], or [3] can be considered as examples (realizations) of our perspectives, but configurations considered in this paper do not necessarily have any realization in a desarguesian projective space.

The above observation enables us to reduce the problem to a classification of line perspectives (maps between edges of graphs, we call them also ‘skews’) and a classification of axial configurations (defined on intersection points of lines containing perspective edges); these axial configurations have vertices with on 2 smaller point rank. If \(K\) has three free \(K_{m+1}\), a similar technique involving a triple perspective can be used; for \(m = 4\) the complete classification was given in [6]. If the line perspective preserves intersection of edges a simple theory presenting the case can
be developed (see [1]). In result, the complete classification of such ‘cousins’ of the Desargues configuration for \( m = 4 \) could be obtained – and presented in [13]. Even in this small case \( m = 4 \) there are, generally, \( 10! \geq 3 \cdot 10^6 \) admissible perspectives. One has to look for some ways to distinguish among them some more regular and interesting.

On a second side, there is a family of known and investigated configurations other than combinatorial Grassmannians: combinatorial Veronesians. This family contains binomial partial Steiner triple systems with exactly three maximal free subgraphs. Computing formulas which define in this case the line perspectives we obtain a class of functions that can determine an (interesting) family of configurations: keeping invariant skew taken from the theory of combinatorial Veronesians we vary axial configurations.

Rudiments of the theory of so obtained Veronese-like perspectivities are presented in this note. We close the paper with the complete classification of \( (15_4 \cdot 20_3) \)-configurations which can be presented as a such Veronese-like perspective: there are 18 such configurations, and 14 of them have not been found before.

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1 Underlying ideas and basic definitions

Let us begin with introducing some, standard, notation. Let \( X \) be an arbitrary set. The symbol \( S_X \) stands for the family of permutations of \( X \). Let \( k \) be a positive integer; we write \( \wp_k(X) \) for the family of \( k \)-element subsets of \( X \). Then \( K_X = \langle X, \wp_2(X) \rangle \) is the complete graph on \( X \); \( K_n \) is \( K_X \) for any \( X \) with \( |X| = n \). Analogously, \( S_n = S_X \).

A \((\nu, b, \kappa)\)-configuration is a configuration (a partial linear space i.e. an incidence structure with blocks (lines) pair wise intersecting in at most a point) with \( \nu \) points, each of rank \( r \), and \( b \) lines, each of rank (size) \( \kappa \). A partial Steiner triple system (in short: a PSTS) is a partial linear space with all the lines of size 3. A \( \left( \binom{n}{2} \cdot n-2 \binom{n}{3} \cdot 3 \right) \)-configuration is a partial Steiner triple system, it is called a binomial partial Steiner triple system.

We say that a graph \( G \) is freely contained in a configuration \( \mathcal{B} \) iff the vertices of \( G \) are points of \( \mathcal{B} \), each edge \( e \) of \( G \) is contained in a line \( \tau \) of \( \mathcal{B} \), the above map \( e \mapsto \tau \) is an injection, and lines of \( \mathcal{B} \) which contain disjoint edges of \( G \) do not intersect in \( \mathcal{B} \). If \( \mathcal{B} \) is a \( \left( \binom{n}{2} \cdot n-2 \binom{n}{3} \cdot 3 \right) \)-configuration and \( G = K_X \) then \( |X| + 1 \leq n \). Consequently, \( K_{n-1} \) is a maximal complete graph freely contained in a binomial \( \left( \binom{n}{2} \cdot n-2 \binom{n}{3} \cdot 3 \right) \)-configuration. Further details of this theory are presented in [12], relevant results will be quoted in the text, when needed.
Underlying ideas and basic definitions

Construction 1.1. ([4] Constr. 1.1)] Let $I$ be a nonempty finite set, $n := |I| \geq 2$. In most parts, without loss of generality, we assume that $I = I_n = \{1, \ldots, n\}$. Let $A = \{a_i: i \in I\}$ and $B = \{b_i: i \in I\}$ be two disjoint $n$-element sets, let $p \not\in A \cup B$. Then we take a $\binom{n}{3}$-element set $C = \{c_u: u \in \wp_2(I)\}$ disjoint with $A \cup B \cup \{p\}$. Set
\[ \mathcal{P} = A \cup B \cup \{p\} \cup C. \]

Let us fix a permutation $\sigma$ of $\wp_2(I)$ and write
\[
\begin{align*}
\mathcal{L}_p & := \{\{p, a_i, b_i\}: i \in I\}, \\
\mathcal{L}_A & := \{\{a_i, a_j, c_{i,j}\}: \{i, j\} \in \wp_2(I)\}, \\
\mathcal{L}_B & := \{\{b_i, b_j, c_{\sigma^{-1}(i,j)}\}: \{i, j\} \in \wp_2(I)\}.
\end{align*}
\]

Finally, let $\mathcal{L}_C$ be a family of 3-subsets of $C$ such that $\mathfrak{N} = \langle C, \mathcal{L}_C \rangle$ is a $\binom{n}{3}$-configuration. Set
\[ \mathcal{L} = \mathcal{L}_p \cup \mathcal{L}_A \cup \mathcal{L}_B \cup \mathcal{L}_C \]
and $\Pi(n, \sigma, \mathfrak{N}) := \langle \mathcal{P}, \mathcal{L} \rangle$.

The structure $\Pi(n, \sigma, \mathfrak{N})$ will be referred to as a *skew perspective* with the skew $\sigma$.

We frequently shorten $c_{i,j}$ to $c_{i,j}$. Sometimes the parameter $\mathfrak{N}$ will not be essential and then it will be omitted, we shall write simply $\Pi(n, \sigma)$. In essence, the names “$a_i$”, “$c_{i,j}$” are – from the point of view of mathematics – arbitrary, and could be replaced by any other labelling (cf. analogous problem of labelling in [10], Constr. 3, Repr. 3] or in [6] Rem 2.11, Rem 2.13, [10] Exmpl. 2]). Formally, one can define $J = I \cup \{a, b\}$, $x_i = \{x, i\}$ for $x \in \{a, b\} =: p$ and $i \in I$, and $c_u = u$ for $u \in \wp_2(I)$. After this identification $\Pi(n, \sigma)$ becomes a structure defined on $\wp_2(J)$.

Then, it is easily seen that
\[ \Pi(n, \sigma, \mathfrak{N}) \text{ is a } \binom{n+2}{2} \binom{n+2}{3} - \text{configuration. } \quad (1) \]

In particular, it is a partial Steiner triple system, so we can use standard notation: $x, y$ stands for the line which joins two collinear points $x, y \in \mathcal{P}$, and then we define on $\mathcal{P}$ the partial operation $\oplus$ with the following requirements: $x \oplus x = x$, $\{x, y, x \oplus y\} \in \mathcal{L}$ whenever $x, y$ exists. Observe that (cf. [7] Eq. (1), the definition of combinatorial Grassmannian $G_2(n)$)
\[ G_2(n+2) = G_2(J) = (\wp_2(J), \wp_3(J), C) \cong B(n, \text{id}_{I_n}, G_2(I_n)). \quad (2) \]

It is clear that $A^* = A \cup \{p\}$ and $B^* = B \cup \{p\}$ are two $K_{n+1}$-graphs freely contained in $\Pi(n, \sigma, \mathfrak{N})$. Applying the results [12] Prop. 2.6 and Thm. 2.12 we immediately obtain the following fact.

Fact 1.2. Let $N = n + 2$. The following conditions are equivalent.

(i) $\mathfrak{N}$ is a binomial $\binom{N}{2} - \binom{N}{3}$-configuration which freely contains two $K_{N-1}$-graphs.
(ii) $\mathcal{M} \cong \Pi(n, \sigma, \mathcal{N})$ for a $\sigma \in S_{\mathcal{P}_2(I_n)}$ and a $\begin{pmatrix} n \\ 0 \end{pmatrix} \begin{pmatrix} n-2 \ \ 0 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix}$-configuration $\mathcal{R}$ defined on $\mathcal{P}_2(I_n)$.

The map
$$\pi = (a_i \mapsto b_i, \ i \in I)$$

is a point-perspective of $K_A$ onto $K_B$ with centre $p$. Moreover, the map
$$\xi = (a_i, a_j \mapsto b_i', b_j', \ \sigma(\{i, j\}) = \{i', j'\} \in \mathcal{P}_2(I))$$

is a line-perspective, where $\mathcal{R}$ is the axial configuration of our perspective. With each permutation $\sigma_0 \in S_I$ we associate the permutation $\sigma_0$ defined by
$$\sigma_0(\{i, j\}) = \{\sigma_0(i), \sigma_0(j)\} \quad (3)$$

for every $\{i, j\} \in \mathcal{P}_2(I)$.

**Note 1.3.** If $\sigma_0 \in S_I$ we frequently identify $\sigma_0$, $\sigma_0$, and the corresponding map $\xi$. Consequently, if $\sigma \in S_I$ we write $\Pi(n, \sigma, \mathcal{N})$ in place of $\Pi(n, \sigma, \mathcal{N})$.

**Proposition 1.4** (comp. [4 Prop. 2.2]). Let $f \in S_p$, $f(p) = p$, $\sigma_1, \sigma_2 \in S_{\mathcal{P}_2(I)}$, and $\mathcal{N}_1, \mathcal{N}_2$ be two $\begin{pmatrix} n \ \ 0 \end{pmatrix} \begin{pmatrix} n-2 \ \ 0 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix}$-configurations defined on $\mathcal{P}_2(I)$. The following conditions are equivalent.

(i) $f$ is an isomorphism of $\Pi(n, \sigma_1, \mathcal{N}_1)$ onto $\Pi(n, \sigma_2, \mathcal{N}_2)$.

(ii) There is $\phi \in S_I$ such that one of the following holds
$$\phi \ (\text{comp. } (3)) \ \text{is an isomorphism of } \mathcal{N}_1 \text{ onto } \mathcal{N}_2, \quad (4)$$
$$f(x_i) = x_{\phi(i)}, \ x = a, b, \quad f(c_{\{i,j\}}) = c_{\{\phi(i),\phi(j)\}}, \ i, j \in I, i \neq j, \quad (5)$$

or
$$\sigma_2^{-1} \phi \ \text{is an isomorphism of } \mathcal{N}_1 \text{ onto } \mathcal{N}_2, \quad (7)$$
$$f(a_i) = b_{\phi(i)}, \ f(b_i) = a_{\phi(i)}, \quad f(c_{\{i,j\}}) = c_{\sigma_2^{-1}\{\phi(i),\phi(j)\}}, \ i, j \in I, i \neq j, \quad (8)$$

$$\phi \circ \sigma_1 \ \text{=} \ \sigma_2^{-1} \circ \phi. \quad (9)$$

**Lemma 1.5** (Comp. [4 Lem. 2.1]). Assume that $\Pi(n, \sigma, \mathcal{N})$ freely contains a complete $K_{n+1}$-graph $G \neq K_{A^*}, K_{B^*}$, $\sigma \in S_{\mathcal{P}_2(I)}$. Then there is $i_0 \in I$ such that $S(i_0) = \{c_a \colon i_0 \in u \in \mathcal{P}_2(I)\}$ is a collinearity clique in $\mathcal{R}$ freely contained in it and $\sigma$ satisfies
$$i_0 \in u \implies i_0 \in \sigma(u) \ \text{for every } u \in \mathcal{P}_2(I). \quad (10)$$

Moreover,
$$G = G(i_0) := \{a_{i_0}, b_{i_0}\} \cup S(i_0). \quad (11)$$
2 Vergras-like skew

We start this Section with a presentation of the combinatorial Veronesian $V_k(X)$ of \[1\], as, in essence, it will be generalized in the paper. Besides, this example shows that not every "sensibly roughly presented" perspective $\Pi(n, \sigma, \mathcal{M})$ between complete graphs has necessarily a 'Desarguesian axis' nor its skew preserves the adjacency of edges of the graphs in question.

**Example 2.1.** Let $|X| = 3$, $X = \{a, b, c\}$. Then the combinatorial Veronesian $V_k(X) =: \mathcal{M}$ is a $\binom{k+2}{2} \binom{k+2}{3}$-configuration; its point set is the set $\mathbf{\eta}_k(X)$ of the $k$-element multisets with elements in $X$ and the lines have form $eX^s$, $e \in \mathbf{\eta}_{k-s}(X)$. $V_1(X)$ is a single line, $V_2(X)$ is the Veblen configuration, and $V_3(X)$ is the known Kantor configuration (comp. [11, Prop's. 2.2, 2.3], [5, Repr. 2.7]). Consequently, we assume $k > 3$. The following was noted in [12, Fct. 4.1]:

The $K_{k+1}$ graphs freely contained in $V_k(X)$ are the sets $X_{a,b} := \mathbf{\eta}_b(\{a, b\})$, $X_{b,c} := \mathbf{\eta}_c(\{b, c\})$, and $X_{c,a} := \mathbf{\eta}_a(\{c, a\})$.

In particular, $\mathcal{M}$ freely contains two complete subgraphs $X_{a,b}$, $X_{b,c}$, which cross each other in $p = a^k$. We shall present $\mathcal{M}$ as a perspective between these two graphs.

Let us re-label the points of $V_k(X)$:

$$c_i = b^i a^{k-i}, \ b_i = c^i a^{k-i}, \ i \in \{1, \ldots, k\} =: I, \ e_{i,j} = c_i \oplus c_j, \ \{i, j\} \in \varphi_2(I).$$

Assume that $i < j$, then $c_i, c_j = b^{k-j}a^i X^{j-i}$, so $e_{i,j} = a^{k-j}b^i c^{j-i}$. Clearly, $p \oplus c_i = b_i$ so, the map $(c_i \mapsto b_i, \ i \in I)$ is a point-perspective. Let us define the permutation $\zeta$ of $\varphi_2(I)$ by the formula

$$\zeta((i, j)) = \{j - i, j\} \text{ when } 1 \leq i < j \leq k.$$

It is seen that $\zeta = \zeta^{-1}$. After routine computation we obtain $b_i \oplus b_j = e_{\zeta((i,j))}$ whenever $i < j$; moreover, in this representation the axial configuration consists of the points in $b \varphi_{k-2}(X)$ so, it is isomorphic to $V_{k-2}(X)$. Consequently,

**Fact 2.2.** $V_k(X) \cong \Pi(k, \zeta, V_{k-2}(X)).$

Recall (cf. [11] Thm. 5.9) that for $k > 3$ the structure $V_k(X)$ cannot be embedded into any desarguesian projective space.

Let us fix an integer $n$ and define the map $\zeta = \zeta_n : \varphi_2(I_n) \rightarrow \varphi_2(I_n)$ by the formula

$$\zeta((i, j)) := \{j - i, j\} \text{ for } 1 \leq i < j \leq n. \tag{12}$$

Note: $i < j, \ u = \{i, j\} \in \varphi_2(I_n)$, and $\zeta(u) = \{i', j'\}$, $i' < j'$ yields $j = j'$.

Clearly, $\zeta = \zeta^{-1}$.

**Lemma 2.3.** $\zeta_n = \sigma$ for $a \sigma \in S_{I_n}$ iff $n \leq 3$.

**Proof.** It is seen that $\zeta_2 = \mathrm{id}_{\varphi_2(I_2)} = \overline{\mathrm{id}_{I_2}}$. Let $n = 3$; define $\sigma = (1, 2)(3)$; it is evident that $\zeta_3 = \overline{\tau}$. Now assume that $n > 3$ and $\zeta = \zeta_n = \overline{\tau}$ for $a \sigma \in S_{I_n}$. Take $u_i = \{i, n\}$ for $i = 1, 2, 3$, so $\zeta(u_i) = \{n - i, n\} = \{\sigma(i), \sigma(n)\}$. This yields, in particular, $n = \sigma(n)$ and, next, $\sigma(i) = n - i$ for all $i < n$. Since $i < j < n$ gives $n - j < n - i < n$ and $\zeta((i, j)) = \{j - i, j\} = \{n - j, n - i\}$ we infer: $i < j < n$ yields $n - i = j$, which is impossible.

\[\square\]
Let \( G \) be a complete \( K_{n+1} \)-graph freely contained in \( \mathcal{M} \). From \( \mathcal{M} = \Pi(n, \zeta, \mathcal{M}) \), \( G \) is a complete clique freely contained in \( \mathcal{M} \). If \( S(n) \) is not a clique in \( \mathcal{M} \) then \( \mathcal{M} \) does not contain any third complete free \( K_{n+1} \)-graph.

Proof. Let \( G \) be a complete \( K_{n+1} \)-graph freely contained in \( \mathcal{M} \). From \( \mathcal{M} = \Pi(n, \zeta, \mathcal{M}) \), \( G \) is a suitable clique. Moreover, \( b_{j0} \) is collinear with all the elements of \( S(j0) \), which means that for every \( i \in I_n \), \( i \neq j0 \) there is \( i' \) such that \( c_{j0,i} = b_{j0} \oplus b_{i'} = c_{i'j0} \), which gives

\[
\text{for every } i \in I_n, i \neq j0 \text{ there exist } i' \neq j0 \text{ s.t. } \zeta(\{j0, i\}) = \{j0, i'\}. \tag{13}
\]

It is seen that \( \zeta(\{i, n\}) = \{n - i, n\} \) whenever \( i < n \).

Now suppose \( j0 < n \). Let \( j0 < i \); then \( \zeta(\{j0, i\}) = \{i - j0, i\} = \{j0, i'\} \) for some \( i' \). Since \( i = j0 \), \( i' = i - j0 \) is impossible, we conclude with \( i = i' \), \( i = 2j0 \) for all \( i > j0 \). In particular, \( j0 + 1 = 2j0 \) gives, inconsistently, \( j0 = 1, n = 2 \).

It needs only a routine computation to justify that, conversely, when \( \zeta(\{i, n\}) = \{n - i, n\} \) is valid and \( S(n) \) is a free clique in \( \mathcal{M} \) then \( G(n) \) is a free \( K_{n+1} \)-graph in \( \mathcal{M} \).

As an immediate consequence we obtain

**Corollary 2.5.** The structure \( \Pi(n, \zeta, G_2(I_n)) \) freely contains exactly three \( K_{n+1} \)-graphs.

Let us make the following immediate observation

**Lemma 2.6.** Let \( \mathcal{M} \) be a \( \left( \begin{array}{c} n \\ 2 \end{array} \right) \)-configuration defined on \( \varphi_2(I_n) \). Clearly, the \( \zeta \)-image \( \zeta(\mathcal{M}) \) of \( \mathcal{M} \) is a \( \left( \begin{array}{c} n \\ 2 \end{array} \right) \)-configuration. Then the (involuntary) map

\[
\varphi: \quad \begin{array}{ccc}
  a:i & b:j & c:u \\
  ↓ & ↓ & ↓ \\
  b:i & a:j & c:u \\
\end{array}
\]

for all \( i, j \in I_n \), \( u \in \varphi_2(I_n) \) \( \tag{14} \)

is an isomorphism of \( \Pi(n, \zeta, \mathcal{M}) \) onto \( \Pi(n, \zeta, \zeta(\mathcal{M})) \). \( \varphi \) maps \( S(n) \) onto \( S(n) \).

**Note 2.7.** Using 2.6 we can reformulate condition \( 7 \) in 1.4 characterizing isomorphisms between skew perspectives to the following, more similar to 4.7

\[
\varphi \quad \text{is an isomorphism of } \mathcal{M}_1 \text{ onto } \sigma_2(\mathcal{M}_2) \tag{7}
\]

In essence, in most parts, \( S \) is the unique automorphims of \( \Pi(n, \zeta, \mathcal{M}) \) (when \( \mathcal{M} = \zeta(\mathcal{M}) \). First, note a technical

**Lemma 2.8.** Let \( \mathcal{M} = \Pi(n, \zeta, \mathcal{M}) \) for a \( \left( \begin{array}{c} n \\ 2 \end{array} \right) \)-configuration \( \mathcal{M} \). Next, let \( n > 3 \) and \( k \in I_n, k > 3 \). The following conditions are equivalent:
Let us determine the skew of this perspective and its axis. A particular example let us say a few general words more on the case when our relatively small figures. Before we shall study in more detail these three free $K_{n+1}$-graphs. This rough classification can be made more exact when considering that $\gamma$ holds in $\mathfrak{M}$; $n = k$.

Proof. It is evident that (ii) implies (i): we take $j = n - i$ in $\text{Cross}(n)$. Suppose that $\text{Cross}(k)$ holds for $3 < k < n$. Take $i = k + 1$ in $\text{Cross}(k)$; then we obtain $\zeta(\{k, k + 1\}) = \{k, j\}$ for some $j$. This gives $k = 1, j = 2$. If $i = n$ then $n = 2$ and if $i + 1 = n$ then $n = 3$: this contradicts assumptions. So, $k + 2 < n, k = 1$. Considering $\text{Cross}(k)$ again we obtain $\{1, \text{something}\} = \zeta(1, 3) = \{2, 3\}$ and we arrive to a contradiction.

As a corollary to 2.8 we obtain the following rigidity property:

**Proposition 2.9.** Let $\mathfrak{M}$ be as in 2.8 with $n > 3$. Assume that $f \in \text{Aut}(\mathfrak{M})$ with $f(p) = p$. Then either $f = \text{id}$ or $f = \mathfrak{S}$ and $\mathfrak{M} = \zeta \mathfrak{M}$.

Proof. Evidently, either $f(A) = A$ or $f(A) = B$ (in the notation of 1.1). From 2.8 we obtain $f(a_n) = a_n$. Then, let us restrict $\mathfrak{M}$ to points with indices in $I_{n-1}$; in this structure $\text{Cross}(n - 1)$ holds and therefore $f(a_{n-1}) = a_{n-1}$ as well. Step by step we get $f(a_i) = a_i$ for $3 < i \leq n$. Next, we look at $c_{4,2} = a_4 \oplus a_2$, it goes under $f$ onto $a_4 \oplus a_{\alpha(2)} = c_{4, \alpha(2)}$ for a permutation $\alpha \in \mathcal{S}_4$. Simultaneously, $c_{4,2} = b_4 \oplus b_2$ and thus $c_{4, \alpha(2)} = c_{4, 4-\alpha(2)}$ which gives $\alpha(2) = 2$. Similarly we compute $\alpha(3) = 3$ and $\alpha(1) = 1$.

If $f(A) = B$ the reasoning is provided analogously; we obtain $f(a_i) = b_i$ for $3 < i \leq n$ and then $f(a_i) = b_i$ for all $i \in I_n$.

In view of 1.4 this yields, in particular,

**Lemma 2.10.** Let $\varphi \in \mathcal{S}_{I_n}, n > 3$. If $\zeta \varphi = \zeta$ then $\varphi = \text{id}_{I_n}$.

In view of 2.4 any structure $\Pi(n, \zeta, \mathfrak{M})$ contains either exactly two or exactly three free $K_{n+1}$-graphs. This rough classification can be made more exact when we consider relatively small figures. Before we shall study in more detail these particular examples let us say a few general words more on the case when our structures contain three $K_{n+1}$-graphs.

Now, let us suppose that $S(n)$ is a free clique in $\mathfrak{M}$. In this case $\mathfrak{M}$ can be presented as a perspective between two other simplices contained in $\mathfrak{M}$: between $A^* \setminus \{a_n\} := A$ and $G(n) \setminus \{a_n\} := D$, with $q = a_n$ as the centre of the perspective. Let us determine the skew of this perspective and its axis.

First, we ‘renumber’ the points in $A = \{a_1', \ldots, a_n'\}$; next we number the points in $D = \{d_1, \ldots, d_n\}$ so as

$$d_i \in \overline{p, a_i'}, \quad i \in I_n.$$ (15)
This is done as follows:

\[
\begin{array}{cccccc}
  a_1 & \ldots & a_{n-1} & p \\
  \parallel & \ldots & \parallel & \parallel \\
  d_1' & \ldots & d_{n-1}' & d_n' \\
  \downarrow & \ldots & \downarrow & \downarrow \\
  c_1,n & \ldots & c_{n-1,n} & b_n \\
\end{array}
\]  

(16)

Then we set

\[ e_{i,j} := d_i' \oplus d_j'. \]

From the definitions we get

\[ e_{i,j} = c_{i,j}, \quad e_{i,n} = b_i \text{ for all } i, j < n, \; i \neq j. \]  

(17)

Finally, we compute for \( i, j < n \):

\[ d_i' \oplus d_j' = c_{i,n} \oplus c_{j,n} = c_{\rho^{-1}(\langle i, j \rangle)} = e_{\rho^{-1}(\langle i, j \rangle)} \]  

\text{for a map } \rho_0: \rho_2(I_{n-1}) \to \rho_2(I_{n-1}).

The map \( \rho_0 \) is entirely determined by the configuration \( \mathfrak{R} \).

To complete determining \( \mathfrak{g} \) we must compute \( d_i' \oplus d_n \) and compare it with suitable \( e_{i',n} \): Recall: \( c_{n-i,n} = c_{\xi(i,n)} = b_i \oplus b_n \). Thus \( e_{i,n} = b_i = b_{n-(n-i)} = b_n \oplus e_{n-i,n} = d_n \oplus d_{n-i} \).

This can be noted as \( d_i' \oplus d_n = e_{n-i,n} = e_{\rho^{-1}(\langle i, n \rangle)} \). Summarizing, we see that the following defines \( \rho \):

\[
\rho^{-1}(\{i, j\}) = \begin{cases} 
\{i', j'\} & \text{iff } c_{i,n} \oplus c_{j,n} = c_{i',j'} \text{ for } i, j < n \\
\{n-i, n\} & \text{for } i < n, \; j = n .
\end{cases}
\]

(18)

At the very end we characterize the axis \( \mathfrak{R} \) of our perspective: the subconfiguration of \( \mathfrak{M} \) with the points in \( E := \{e_{i,j} : 1 \leq i < j \leq n\} \). To do so it suffices to make use the following consequence of (17): \( E = (E \cap C) \cup (B \setminus \{b_n\}) = (C \setminus S(n)) \cup (B \setminus \{b_n\}) \). So, \( \mathfrak{R} \) contains all the lines of \( \mathfrak{R} \) which miss \( S(n) \):

\[ \text{if } i, j, k, l < n \text{ then } e_{i,j} \oplus e_{k,l} = c_{i,j} \oplus c_{k,l}(= c_{s,t} = e_{s,t} \text{ for some } s, t < n) . \]

(19)

And for \( i < j < n \) we have

\[ e_{i,n} \oplus e_{j,n} = b_i \oplus b_j = c_{j-i,j} = e_{j-i,j} . \]

(20)

Conditions (19) and (20) fully characterize the structure \( \mathfrak{R} \), so we obtain

**Proposition 2.11.** Let \( M = \Pi(n, \zeta, \mathfrak{R}) \) and \( S(n) \) be a free clique in \( \mathfrak{R} \). Then

\[ \mathfrak{M} = \Pi(n, \varrho, \mathfrak{R}) , \]

where \( \mathfrak{R} \) is characterized by (19) and (20), while \( \varrho \) is defined by (18).

As a particular instance of the investigations above let us substitute \( \mathfrak{R} = \mathfrak{G}_2(I_n) \); then \( \rho_0 = \text{id}_{I_{n-1}} \). To make an impression how much “non-Veblenian” figures may \( \mathfrak{R} \) contain we present in Figure 1 the schema of a fragment of \( \mathfrak{R} \), when \( \mathfrak{M} = \mathfrak{G}_2(I_n) \).

Besides, with the help of [1.4] we get that there is no automorphism of \( \mathfrak{M} \) which maps \( p \) onto \( q \). Moreover, \( \mathfrak{G}_2(I_n) \) contains \( L := \varrho_2(\{1, 2, n\}) \) as a line, while for \( n > 3 \) the set \( \zeta(L) \) is not any line of \( \mathfrak{G}_2(I_n) \) so, \( \mathfrak{G}_2(I_n) \neq \zeta(\mathfrak{G}_2(I_n)) \). And therefore, from [2.10] we conclude with the following
3 n = 4: the axis is the Veblen Configuration

![Veblen Configuration Diagram]

Figure 1: Let $i < j < n$; then $n - j < n - i$. Moreover, let $i < n - j$ (then $j < n - i$) and $j < n - j$ (then $i < n - i$). Note that we need $n > 4$ to draw such a figure! 

**Corollary 2.12.** Let $f \in \text{Aut}(\Pi(n, \zeta, G_2(I_n)))$ and $n > 3$. Then $f = \text{id}$.

At the end of this section let us try to show how to decide whether our $\zeta$-perspective has three ‘geometrically equivalent’ perspective centres i.e. whether it has an automorphism which interchanges its three free complete subgraphs. In view of [1.4] and [2.11] we need to find a permutation $\alpha \in S_n$ such that (notation of the reasoning which leads to [2.11]) $f(p) = q$ i.e. $f(p) = a_n$ and

(a) $f(a_i) = a'_{\alpha(i)}$, $f(b_i) = d_{\alpha(i)}$, $f(c_{i,j}) = e_{\alpha(i),\alpha(j)}$, or

(b) $f(a_i) = a'_{\alpha(i)}$, $f(b_i) = d_{\alpha(i)}$, $f(c_{i,j}) = e_{\zeta(\alpha(i),\alpha(j))}$.

In particular, $\pi$ must be an automorphism of $\mathcal{R}$ and $\mathcal{R}$ characterized in [2.11]. Substituting values of $a'_j$, $d_j$ we obtain more explicit requirements.

3 n = 4: the axis is the Veblen Configuration

In this section we present a classification of configurations $\Pi(4, \zeta_4, \mathcal{R})$: then $\mathcal{R}$ is a $((6_24_3))$-configuration i.e. $\mathcal{R}$ is the Veblen (Pasch) configuration suitably labelled. Let us quote after [2] definitions of the labellings of the Veblen configuration defined on $\varphi_2(I_4)$ together with the star-triangles $S(i)$ contained in them: $(Y \in \varphi_2(I_4): T(Y) := \varphi_2(Y); i_0 \in I_4: T(i_0) := T(I_4 \setminus \{i_0\})$)

- **veblen type (i):** $G_2(I_4)$ – all four $S(i)$ with $i \in I_4$.

- **veblen type (ii):** $G_2^*(I_4)$: its lines are the $\gamma$-images of the lines of $G_2(I_4)$ and we briefly write $G_2^*(I_4) = \gamma(G_2(I_4))$ – no star-triangle.

- **veblen type (iii):** $\mathcal{B}(2) = \left< \varphi_2(I_4), \{T(3), T(4), \{1, 4\}, \{3, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 3\}\} \right> - S(4)$ and $S(3)$ are its unique star-triangles.

- **veblen type (iv):** $\mathcal{V}_5 = \left< \varphi_2(I_4), \{T(4), \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2\}, \{1, 4\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\} \right> - S(4)$ is its unique star-triangle;

- **veblen type (v):** $\gamma(\mathcal{B}(2)) =: \mathcal{V}_4$ – no star-triangle.
veblen type (vi): \( \varphi(V_5) =: V_6 \) — no star-triangle.

For any Veblen configuration \( \mathfrak{V} \) defined on \( \varphi_2(I_4) \) there is an isomorphism \( \varpi \) with \( \alpha \in S_{I_4} \) of \( \mathfrak{V} \) and a one (exactly one!) of the above six. To avoid technical troubles we have slightly changed our labelling in comparison with the definitions of [5]; we have applied permutation \((1,3)(2,4)\) to the ‘original’ labeling in case \([iii]\), and \((1)(2)(3,4)\) in case \([iv]\).

Evidently, for every \( \varphi \in S_{I_4} \) the \( \varphi \)-image of any structure \( \mathfrak{V} \) of the above list is again a Veblen configuration isomorphic to \( \mathfrak{V} \); but \( \Pi(4, \zeta, \mathfrak{V}) \) and \( \Pi(4, \zeta, \varphi(\mathfrak{V})) \) may stay non-isomorphic.

**Proposition 3.1.** Let \( \varphi \in S_{I_4} \), \( \mathfrak{V} \) be a one among the above labellings of the Veblen configuration, and \( \mathfrak{M} = \Pi(4, \zeta, \varphi(\mathfrak{V})) \). Then one of the following holds.

\( \mathfrak{M} = G_2(I_4) \): \( \mathfrak{M} \) contains exactly three free \( K_5 \)-graphs.

\( \mathfrak{M} = G_2^*(I_4), \mathfrak{V}_4, \mathfrak{V}_6 \): \( \mathfrak{M} \) contains two free \( K_5 \).

\( \mathfrak{M} = \mathfrak{P}(2) \): \( 4 \in \{ \varphi(4), \varphi(3) \} \): \( \mathfrak{M} \) contains three free \( K_5 \);

\( 4 \not\in \{ \varphi(4), \varphi(3) \} \): \( \mathfrak{M} \) contains two free \( K_5 \);

\( \varphi(4) = 4 \): \( \mathfrak{M} \) contains three free \( K_5 \);

\( \varphi(4) \neq 4 \): \( \mathfrak{M} \) contains two free \( K_5 \).

In essence, in view of [2,6] the situation is more complex, as, formally, for every \( \varphi \) as above we need to determine \( \zeta(\varphi(\mathfrak{V})) \). Actually, we need to determine all labellings of the Veblen configuration by the elements of \( \varphi_2(I_4) \). By a way of an example on Figure 2 we show the drawing presenting the schema of \( \Pi(4, \zeta, G_2^*(I_4)) \); but remember that this is merely a one among 18 others (cf. [3,4]!)

Suppose that \( \mathfrak{V} \) contains \( S(i_0) \) as a triangle, and then it contains \( T(i_0) \) as a line. Let us introduce a numbering of the sides of \( S(i_0) \) and of points of \( T(i_0) \), invariant under permutations of \( S_{I_4\setminus\{i_0\}} \):

\[ c_{i_4, j_4, k_4} \sim k \text{ and } c_{i, j} \sim k \iff \{i, j, k\} = I_4 \setminus \{i_0\}. \]

Then the definition of \( \mathfrak{V} \) corresponds to a \( \mu \in S_{I_4\setminus\{i_0\}} \) with the following rule

\[ k \sim c_{i, j, k} \Rightarrow c_{i, i_0} \oplus c_{j, i_0} = c_{i', j', k} \sim \mu(k). \]  

(21)

Next, suppose that \( \mathfrak{V} \) contains \( T(i_0) \) as a triangle, and then it contains \( S(i_0) \) as a line. Analogously to the above we introduce a numbering of the sides of \( T(i_0) \) and of points of \( S(i_0) \):

\[ c_{i, k, j, k} \sim k \text{ and } c_{k, i_0} \sim k \iff \{i, j, k\} = I_4 \setminus \{i_0\}. \]

Then the definition of \( \mathfrak{V} \) corresponds to a \( \mu \in S_{I_4\setminus\{i_0\}} \) with the following rule

\[ k \sim c_{i, k, j, k} \Rightarrow c_{i, k} \oplus c_{j, k} = c_{i', j', k} \sim \mu(k). \]  

(22)

Let \( \mu \in S_{I_4\setminus\{i_0\}} \); we write \( V_5(\mu) \) for the Veblen configuration defined by (21): it has \( T(i_0) \) as a line, and \( V_6(\mu) \) for the Veblen configuration defined by (22): it has \( S(i_0) \) as a line.
3 \ n = 4: the axis is the Veblen Configuration

Next, note that in accordance with the rules above, $\mathcal{W} = V_s(\mu)$ ($\mu \in \mathcal{S}_{I_4 \setminus \{i_0\}}$) has another star-triangle $S(i_0')$ ($s = 5$) or another top-triangle $T(i_0')$ ($s = 6$) iff $\mu(i_0') = i_0'$. In other words, $\mu = (i_0')(j_1, j_2)$. It is easy to compute that then $\mathcal{W} = V_s((i_0)(j_1, j_2))$.

Since under every labelling by the elements of $\wp(\mathcal{I}_4)$ the Veblen configuration contains either at least one top-line or at least one star-line, each Veblen configuration has either the form $V_5(\mu)$ or $V_6(\mu)$ for some $\mu \in \mathcal{S}_{I_4 \setminus \{i_0\}}$ and $i_0 \in \mathcal{I}_4$. So, each Veblen configuration $\mathcal{W}$ can be uniquely associated with a permutation $\mu \in \mathcal{S}_4$ with at least one fixed point (not a derangement of $I_4$) and a ‘switch’ $s \in \{5, 6\}$ so as $\mathcal{W} = V_s(\mu)$.

Let us note the following observation, justified on Figure 3, that will be used in the sequel

**Fact 3.2.** For every $s \in \{5, 6\}$ and $\mu \in \mathcal{S}_4$ with $\text{Fix}(\mu) \neq \emptyset$ the following holds

$\kappa(V_s(\mu)) = V_{11-s}(\mu)$.

Finally, note that $\zeta(S(4)) = S(4)$ and, consequently, $\zeta(T(4)) = T(4)$ and therefore if $\mathcal{W}$ contains $T(4)$ as a line (as a triangle) then $\zeta(\mathcal{W})$ contains $T(4)$ as a line (as a triangle, resp). The following is evident

$V_5(\text{id}) = G_2(I_4), \quad V_6(\text{id}) = G^*_2(I_4),$

$V_5((3)(4)(1, 2)) = \mathcal{B}(2), \quad V_6((3)(4)(1, 2)) = \mathcal{V}_4,$

$V_5((4)(1, 2, 3)) = \mathcal{V}_5$ and $V_6((4)(1, 2, 3)) = \mathcal{V}_6.$

Moreover, if $\mathcal{M} = \Pi(4, \zeta, \mathcal{W})$ freely contains three $K_5$ and we represent $\mathcal{M}$ as a perspective as above, then $\mathcal{M}$ defined in 2.11 is again the Veblen configuration

![Figure 2: The structure $\Pi(4, \zeta_4, G^*_2(I_4)) = \Pi(4, \zeta_4, V_6(\text{id}))$ (cf. definition of $V_s(\mu)$ below).](image)
suitably labelled, so it is in the list above. And $\mathcal{M}$ must be a one among those defined in [6]! what are they?, are they all distinct?

Let us start with a slight reminder of the representation technique of [6]. We arrange the vertices of three triangles of $\mathcal{M}$: $\Delta_1 = \{a_1, a_2, a_3\}$, $\Delta_2 = \{b_1, b_2, b_3\}$, and $\Delta_3 = \{c_{i,4}, c_{j,4}, c_{k,4}\}$ in three rows of a $3 \times 3$-matrix so as when we join in pairs points in the same two columns, the obtained lines of $\mathcal{M}$ have a common point. So obtained three common points form the line $T(4)$. After that we join points in distinct rows when there is a line in $\mathcal{M}$ which joins them: these lines for every pair of rows should meet in a common point. On Figure 4 we visualize a schema of this procedure.

It is known that after such a representation the obtained structures are (with a few exceptions) isomorphic when the associated diagrams are isomorphic (can be mapped one onto the other by a permutation of rows and columns). From Figure 4 we read that the diagram is determined by the permutation $\{i, j\} \mapsto \{i', j'\}$: $c_{i',j'} = c_{i,4} \oplus c_{j,4}$ with $1 \leq i, j \leq 3$.

So, any structure $\Pi(4, \zeta, \mathcal{M})$ with three free $K_3$ inside is uniquely determined by a permutation $\mu \in S_{K_3}$ ($\mu \in S_{I_4}$, $\mu(4) = 4$) such that $\mathcal{M} = V_6(\mu)$. It remains to determine their isomorphism types. Analogous method is used to classify all the $\Pi(4, \zeta, V_s(\mu))$. In view of 2.6 and 1.4 the following fact is essential. Its proof is
quite elementary, but needs a quite pouring computation.

**Fact 3.3.** In the following table we enumerate all the pairs \((s|\mu) \leftrightarrow (s'|\mu')\) such that \(\zeta(V_s(\mu)) = V_{s'}(\mu')\) with \(\mu \in S_{I_4}\) and \(s \in \{5, 6\}\).

| \(i\) | \((5;4)(3,2,1)(4)\) | \(5|(1,2)(3)(4) \leftrightarrow 5|(1,3)(2)(4)\) |
| --- | --- | --- |
| \(3\) | \(6|(1,2,3)(4)\) | \(6|(4)(1,3,2) \leftrightarrow 6|(4)(1,3,2)\) |
| \(4\) | \(6|(4)(1,2,3)\) | \(6|(4)(1,2,3) \leftrightarrow 6|(4)(3)(1,2)\) |
| \(5\) | \(6|(1,2,3)(4) \leftrightarrow 5|(2)(3)(1,4)\) | \(6|(1)(2,4,3) \leftrightarrow 6|(1)(2,4,3)\) |
| \(6\) | \(6|(1)(2,4,3) \leftrightarrow 5|(2)(1,3,4)\) | \(6|(1)(3)(2,4) \leftrightarrow 5|(3)(1,4,2)\) |
| \(7\) | \(6|(2)(1,3,4) \leftrightarrow 6|(3)(1,4,2)\) | \(6|(2)(1,4,3) \leftrightarrow 5|(1)(2)(3,4)\) |
| \(8\) | \(6|(2)(1,3,4) \leftrightarrow 5|(1)(2,4,3)\) | \(6|(3)(1,2,4) \leftrightarrow 5|(1)(3)(2,4)\) |
| \(9\) | \(5|(1)(2,3,4) \leftrightarrow 5|(1)(2,3,4)\) | \(5|(2)(1,4,3) \leftrightarrow 5|(3)(1,2,4)\) |

This technique allows us to formulate a complete characterization of the structures \(\Pi(4, \zeta, \mathcal{V})\).

**Theorem 3.4.** Let \(\mathcal{M} = \Pi(4, \zeta, \mathcal{V})\), where \(\mathcal{V}\) is a Veblen configuration defined on \(\mathcal{V}_2(I_4)\). Then one of the following holds

**\(\mathcal{V}\) contains \(S(4)\) as a triangle:** Then \(\mathcal{V} = \mathcal{V}_5(\mu)\), \(\mu \in S_{I_4}\) is the following one, and \(\mathcal{M}\) can be found among those listed in [6]:

(i) \(\mu = \text{id}\) \(\mathcal{M}\) has the type (viii), \((\sigma_x, \rho, \text{id})\) in [6, Classification 2.8]

(ii) \(\mu = (1)(2,3)\) \(\mathcal{M}\) has the type (vi), \((\sigma_x, \sigma_y, \sigma_z)\) in [6, Classification 2.8]

(iii) \(\mu = (2)(1,3)\) \(\mathcal{M}\) has the type (xii), \((\sigma_x, \sigma_y, \sigma_z)\) in [6, Classification 2.8]

(iv) \(\mu = (1,2,3)\) \(\mathcal{M}\) has the type (xiii), \((\sigma_x, \rho^{-1}, \rho) \cong (\rho, \rho, \sigma_x)\) in [6, Classification 2.8]

**\(S(4)\) is not a triangle in \(\mathcal{V}\):** There are 14 such \(\mathcal{M}\) and they are of the form \(\mathcal{V}_s(\mu)\), where \(s|\mu\) are enumerated in lines 3–9 of the table in 3.3

The above structures are pair wise non isomorphic.

**Proof.** If \(S(4)\) is not a triangle in \(\mathcal{M}\) then, in accordance with [6, Classification 2.8] \(\mathcal{M}\) has form \(\mathcal{V}_s(\mu)\), where \(\mu \in S_{I_4}\) and \(\text{Fix}(\mu) \neq \emptyset\), \(s = 3\) and \(4 \notin \text{Fix}(\mu)\), or \(s = 6\) and \(\mu \neq \text{id}\) is arbitrary. By [2.10] and [1.4] table in 3.3 enumerates all the possible types of perspectives \(\mathcal{M}\).

Now, suppose that \(s(4)\) is a triangle in \(\mathcal{V}\). Then the diagram-representation of \(\mathcal{M}\) depends on a permutation \(\mu \in S_{I_4}\) and \(\mathcal{V} = \mathcal{V}_5(\mu)\).

If \(\text{Fix}(\mu) \neq \emptyset\) and \(\mu \neq \text{id}\), \(\mathcal{V}\) contains exactly two top-lines: \(T(4)\) and \(T(\text{Fix}(\mu))\). In the corresponding cases it suffices to draw a respective diagram in accordance with the rules on Figure 3 and observe that it is isomorphic to the structure defined in [6, Classification 2.8] as claimed in the statement. An example is presented in Figure 5. The same technique works when \(\mu = (1,2,3)\).

To close the proof let us observe again the lines 1–2 of the Table in 3.3 and note that the perspectives associated with \(\mu = \text{id}\) and \(\mu = (3,2,1)\) are isomorphic under the map \(S\) defined in (14); analogously, the perspectives associated with \(\mu = (3)(1,2)\) and with \(\mu = (2)(1,3)\) are isomorphic under \(S\). \(\square\)
\[ n = 4: \text{the axis is the Veblen Configuration} \]

\[ \Delta_1 : a_3 \quad \Delta_2 : b_3 \quad \Delta_3 : c_{3,4} \]

Figure 5: The diagram of the line \( \{c_{1,2}, c_{2,3}, c_{1,3}\} = T(4) \) in \( \Pi(4, \zeta, \mathfrak{P}(2)) = \Pi(4, \zeta_4, \mathfrak{V}_5((3)(1,2))) \): comp. rules on Figure 4.

Lemma 3.5. In every case of 3.1 in which \( M \) freely contains three \( K_5 \) the axis \( \mathfrak{R} \) is isomorphic to \( \mathfrak{P}(2) \). Moreover, in all these cases the permutation \( \varrho \) coincides with \( \zeta \) on \( S(4) \).

Proof. It suffices to note that the formula (20), defining \( e_{n,i} \oplus e_{n,j} \), does not depend on \( \mathfrak{V} \), while in case \( n = 4 \) the formula (19) for every admissible \( \mathfrak{V} \) yields \( e_{1,2} \oplus e_{1,3} = e_{2,3} \).

The last statement of the Lemma is immediate.

The automorphisms of the structures \( \Pi(4, \zeta, \mathfrak{V}) \) which freely contain three \( K_5 \) are determined in [6] so, there is no need to write them down explicitly here.

Theorem 3.6. Let \( \mathfrak{V} = \mathfrak{V}_s(\mu) \) be a Veblen configuration defined on \( \mathfrak{V}_2(I_4) \) and \( \mathfrak{M} = \Pi(4, \zeta_4, \mathfrak{V}), s \in \{5, 6\}, \mu \in S_{I_4} \). Assume that \( \mathfrak{M} \) contains exactly two \( K_5 \).

Then \( \text{Aut}(\mathfrak{M}) \) is nontrivial only when

- \( s = 6, \mu = (4)(1, 2, 3) \)
- \( s = 6, \mu = (4)(1)(2, 3) \)
- \( s = 6, \mu = (1)(2, 4, 3) \)
- \( s = 5, \mu = (1)(2, 3, 4) \)

If \( \text{Aut}(\mathfrak{M}) \) is not trivial then

- \( \text{Aut}(\mathfrak{M}) = \{\text{id}, \mathfrak{S}\} \cong C_2 \).

Proof. The claim is an immediate consequence of [3.3] and [2.9].

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