ORBITS IN REAL $\mathbb{Z}_m$-GRADED SEMISIMPLE LIE ALGEBRAS

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Abstract. In this note we propose a method to classify homogeneous nilpotent elements in a real $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}$. Using this we describe the set of orbits of homogeneous elements in a real $\mathbb{Z}_2$-graded semisimple Lie algebra. A classification of 4-vectors (resp. 4-forms) on $\mathbb{R}^8$ can be given using this method.

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1. Introduction

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ be a real $\mathbb{Z}_m$-graded semisimple Lie algebra. If $m \neq 2$ we cannot associate to this $\mathbb{Z}_m$-gradation a compatible finite order automorphism of $\mathfrak{g}$ as in the case of complex $\mathbb{Z}_m$-graded Lie algebras, unless $m$ is even and the only nonzero components of $\mathfrak{g}$ have degree 0 or $m/2$. To get around this problem we extend the $\mathbb{Z}_m$-gradation on $\mathfrak{g}$ linearly to a $\mathbb{Z}_m$-gradation on the complexification $\mathfrak{g}^C$. Denote by $\theta^C$ the automorphism of $\mathfrak{g}^C$ associated with this $\mathbb{Z}_m$-gradation, i.e. $\theta^C_k = \exp \frac{2\pi i k}{m} \cdot \text{Id}.$

Let $G^C$ be the connected simply-connected Lie group whose Lie algebra is $\mathfrak{g}^C$. Clearly, $\theta^C$ can be lifted to an automorphism $\Theta^C$ of $G^C$. Denote by $G_0^C$ the connected Lie subgroup in $G^C$ whose Lie algebra is $\mathfrak{g}_0^C$. A result by Steinberg in [30], Theorem 8.1, implies that $G_0^C$ is the Lie subgroup consisting of fixed points of $\Theta^C$. Note that the adjoint action of
group $G_C^0$ on $\mathfrak{g}^C$ preserves the induced $\mathbb{Z}_m$-gradation on $\mathfrak{g}^C$. Let $G$ be the connected Lie subgroup in $G_C$ whose Lie algebra is $\mathfrak{g}$. Denote by $G_0$ the connected Lie subgroup in $G$ whose Lie algebra is $\mathfrak{g}_0$. The adjoint action of $G_0$ on $\mathfrak{g}$ preserves the $\mathbb{Z}_m$-gradation. We note that the adjoint action of $G_0$ on $\mathfrak{g}$ coincides with the adjoint action of any connected Lie subgroup $\tilde{G}_0$ of a connected Lie group $\tilde{G}$ having Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}$ correspondingly. In [33] Vinberg observed that by considering a new $\mathbb{Z}_{\tilde{m}}$-graded Lie algebra $\tilde{\mathfrak{g}}$, $\tilde{m} = \frac{m}{(m,k)}$ and $\tilde{\mathfrak{g}}_p = \mathfrak{g}_{pk}$ for $p \in \mathbb{Z}_{\tilde{m}}$ we can regard the adjoint action of $G_0$ on $\mathfrak{g}_k$ as the action of $G_0$ on $\tilde{\mathfrak{g}}_1$. Thus in this note we will consider only the adjoint action of $G_0$ on $\mathfrak{g}_1$. We also write “the adjoint action/orbit(s)”, or simply “orbits”, if no misunderstanding can occur.

The problem of classification of the adjoint orbits in real or complex graded semisimple Lie algebras $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ is related to many important algebraic and geometric questions. In [32] Vinberg proposed a method to classify the adjoint orbits in complex $\mathbb{Z}_m$-graded semisimple Lie algebras. His work developed further the theory of $\mathbb{Z}_2$-graded complex semisimple Lie algebras by Kostant and Rallis [15], and the theory of automorphisms of finite order on complex simple Lie algebras by Kac [19]. It is known that all Cartan subspaces in $\mathfrak{g}_1$ are conjugate [33]. Thus the classification of semisimple elements in $\mathfrak{g}_1$ is reduced to the classification of the orbits of the associated Weyl group on the Cartan subspace in $\mathfrak{g}_1^C$ [33]. To classify nilpotent elements in $\mathfrak{g}_1^C$, Vinberg proposed a method of support, which associates to each nilpotent element $e$ in $\mathfrak{g}_1$ a $\mathbb{Z}$-graded semisimple Lie algebra defined by a characteristic $h(e)$ of $e$, see section 4 for more details. In a complex $\mathbb{Z}_m$-graded semisimple Lie algebra a nilpotent element $e$ in $\mathfrak{g}_1$ is defined uniquely up to conjugacy by its characteristic $h(e)$ [32]. If $m = 1$, we can also classify nilpotent orbits in a simple Lie algebra $\mathfrak{g}$ over an algebraic closed field of characteristic 0, or of primitive characteristic $p$, provided $p$ is sufficient large. We refer the reader to the book by Collingwood and McGovern [4] and the book by Humphreys [31] for surveys.

In a real $\mathbb{Z}_m$-graded semisimple Lie algebras $\mathfrak{g}$ there are many conjugacy classes of Cartan subspaces. Furthermore, a given characteristic element in real $\mathbb{Z}_m$-graded Lie algebra can be associated with many conjugacy classes of nilpotent elements in $\mathfrak{g}_1$. These phenomena are main difficulties when we want to classify the adjoint orbits in a real $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}$. If $m = 1$, i.e. $\mathfrak{g}$ is without gradation, a classification of the adjoint orbits of nilpotent elements in $\mathfrak{g}$ can be obtained, using the Cayley transformation [9], [28] and a classification of nilpotent elements in an associated $\mathbb{Z}_2$-graded complex semisimple Lie algebra see e.g. [4], [10]. Furthermore, a classification of the adjoint orbits of semisimple elements in $\mathfrak{g}$ can be obtained from the classification of Cartan subalgebras in $\mathfrak{g}$ by Kostant [16] and Sugiura [29]. We also like to mention here the work by Rothschild on the adjoint orbit space in a real reductive algebra [27], as well as the work by Djokovic on the adjoint orbits of nilpotent elements in $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}_{8(8)}$ [8]. An essential part of our method of classification of nilpotent orbits in real $\mathbb{Z}_m$-graded semisimple Lie algebras is a combination of certain ideas in their works.

In this note we propose a method to classify the adjoint orbits of homogeneous nilpotent elements in a real $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}$. Roughly speaking, our method
of classification of homogeneous nilpotent elements in \( g \) consists of two steps. In the first step we classify the conjugacy classes of characteristics in a given real \( \mathbb{Z}_m \)-graded semisimple Lie algebra. In the second step we classify the conjugacy classes of nilpotent elements associated with a given conjugacy class of a characteristic. The first step uses the Vinberg classification of characteristics in the complexification \( g^C_1 \) [35] combining with our observation that there is an injective map from the set of \( \text{Ad}_{G_0} \)-conjugacy classes of characteristics in \( g_1 \) to the set of \( \text{Ad}_{G_0^C} \)-conjugacy classes of characteristics in \( g^C_1 \), see Lemma 4.1 and Remark 4.2. To perform the second step we analyze the set of singular elements in a real \( \mathbb{Z} \)-graded semisimple Lie algebra defined by a given characteristic, see section 4 for more details. It turns out that we can apply algorithms in real algebraic geometry to distinguish the conjugacy classes of nilpotent elements associated with given characteristic. Our recipe to classify nilpotent elements is summarized in Remark 4.10. We note that the related algorithm in real algebraic geometry is highly complicated. To implement our algorithm we will need a powerful computer system together with a suitable software, see Remark 4.8.

For \( m = 2 \) a classification of Cartan subspaces in \( g_1 \) has been obtained by Oshima and Matsuki [23]. Using their classification and our results in previous section, we describe the set of orbits of homogeneous elements of degree 1 in a \( \mathbb{Z}_2 \)-graded semisimple Lie algebra, following the same scheme proposed by Elashvili and Vinberg in [12], see Remark 5.6. The plan of our note is as follows. In section 2 we recall main notions and prove a version of Jacobi-Morozov-Vinberg theorem for real \( \mathbb{Z}_m \)-graded semisimple Lie algebras, see Theorem 2.1. In section 3 we prove the existence of a \( \mathbb{R} \)-compatible Cartan involution on \( g = \bigoplus_{i \in \mathbb{Z}_m} g_i \), which provides us an isomorphism between the \( \text{Ad}_{G_0} \)-orbit spaces on \( g_i \) and \( g_{-i} \), see Corollary 3.5. We also give many important examples of real \( \mathbb{Z}_m \)-graded semisimple Lie algebras in this section. In section 4 we propose a method to classify homogeneous nilpotent elements in a real \( \mathbb{Z}_m \)-graded semisimple Lie algebra. In section 5 we describe the set of homogeneous elements in a real \( \mathbb{Z}_2 \)-graded semisimple Lie algebra. In this section we also explain the relation between a classification of homogeneous elements in real \( \mathbb{Z}_m \)-graded semisimple Lie algebras and a classification of \( k \)-vectors (resp. \( k \)-forms) on \( \mathbb{R}^8 \). We briefly recount the history of this classification problem which motivated the author to write this note.

2. Semisimple elements and nilpotent elements of a real \( \mathbb{Z}_m \)-graded semisimple Lie algebra

Let \( g = \bigoplus_{i \in \mathbb{Z}_m} g_i \) be a real \( \mathbb{Z}_m \)-graded semisimple Lie algebra. An element \( x \in g_i, i = 0, m - 1 \), is called semisimple (resp. nilpotent), if \( x \) is semisimple (resp. nilpotent) in \( g \). In this section we explain the Jordan decomposition for an element \( x \in g_i \). We also prove an analog of the Jacobson-Morozov-Vinberg theorem for \( g_i \), and we introduce the notion of a Cartan subspace in \( g_1 \).
Jordan decomposition in a real $\mathbb{Z}_m$-graded semisimple Lie algebra. Any $x \in g_i$ has a unique decomposition $x = x_s + x_n$, where $x_s$ is semisimple, $x_n$ is nilpotent, $x_s, x_n \in g_i$, $[x_s, x_n] = 0$.

For a real form $g$ of $g^C$ let us denote by $\tau_g$ the complex conjugation of $g^C$ w.r.t. $g$. It is easy to see that the existence and the uniqueness of the Jordan decomposition for $x \in g_i$ follows from the existence and the uniqueness of the Jordan decomposition for $x$ in $g^C_i$ \[33\], since this decomposition is invariant under the complex conjugation $\tau_g$, which preserves the $\mathbb{Z}_m$-gradation on $g^C$.

The case $m = 1$ has been treated before, see e.g. [13], chapter IX, exercise A.6, and the references therein.

The following Theorem 2.1 is an analogue of the Jacobson-Morozov-Vinberg theorem in [35], Theorem 2(1). Some partial cases of Theorem 2.1 has been proved in [8], Lemma 6.1, and in [4], Theorem 9.2.3.

For any element $e \in g$ let us denote by $Z_{G_0}(e)$ the centralizer of $e$ in $G_0$.

**Theorem 2.1** (Jacobson-Morozov-Vinberg (JMV) theorem for a real $\mathbb{Z}_m$-graded semisimple Lie algebra). Let $e \in g_1$ be a nonzero nilpotent element.

i) There is a semisimple element $h \in g_0$ and a nilpotent element $f \in g_{-1}$ such that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$ 

ii) Element $h$ is defined uniquely up to conjugacy via an element in $Z_{G_0}(e)$.

iii) Given $e$ and $h$, element $f$ is defined uniquely.

Remark 2.2. - The JMV Theorem plays a key role in the study of nilpotent elements. This Theorem associate to each nilpotent element $e$ a semisimple element $h \in g_0$, which is defined by $e$ uniquely up to a conjugation. Element $h$ in Theorem 2.1 is called characteristic of $e$. We also denote a characteristic of $e$ by $h(e)$. We call an element $h \in g_0$ characteristic, if it is characteristic of some nilpotent element $e \in g_1$.

- Each assertion in Theorem 2.1 has its counterpart in the complex case [35], Theorem 1. The converse is not true. We do not have an analogue of Theorem 1(4) in [35], since $e$ is not defined uniquely by $h$ up to $Z_{G_0}(e)$. This makes the classification of nilpotent elements over reals more complicated than over complexes.

Proof of Theorem 2.1. i) Theorem 2.1i is obtained by combining the JMV theorem in [35] for complex algebras with a Jacobson’s trick, see [4], Lemma 9.2.2. Using the JMV theorem in [35], we choose a triple $(h_R + \sqrt{-1}h'_R, e, f_R + \sqrt{-1}f'_R) \in g^C_0, e, f_R + \sqrt{-1}f'_R \in g^C_{-1})$ such that $h_R, h'_R, f_R, f'_R \in g$ and

$$[h_R, e] = 2e, \quad [e, f_R] = h_R.$$
A Jacobson’s trick [4], Lemma 9.2.2, provides us with an element $z$ in the centralizer $Z_g(e)$ of $e$ in $g$ such that

$$ (ad_{h_R} + 2)z = -[h_R, f_R] - 2f_R. \tag{2.1} $$

It is easy to see that we can assume that $z \in g_{-1}$. Then $(h_R, e, f_R + z)$ satisfies our condition in Theorem 2.1.i. Any $h$ satisfying the relation in Theorem 2.1.i is semisimple, since it is a semisimple element in the Lie algebra $sl(2, \mathbb{R}) = \langle e, f, h \rangle_{\mathbb{R}}$. This proves Theorem 2.1.i.

ii) There are two proofs of this assertion. In the first proof we adapt the argument in [4], proof of Theorem 3.4.10 (Theorem of Kostant), which is Theorem 1(2) in [35] for non-graded Lie algebras. Their proof, based on the $sl_2$-theory, works also for field $\mathbb{R}$. Let us explain their argument adapted to our case. Denote by $Z_{g_0}(e)$ the centralizer of $e$ in $g_0$. If $h'$ is another element satisfying the condition in Theorem 2.1.i, then $h - h' \in Z_{g_0}(e)$. The relations in Theorem 2.1.i imply that $h - h' \in [g_{-1}, e]$. Set $u_{g_0}(e) := Z_{g_0}(e) \cap [g_{-1}, e]$. Then $h' - h \in u_{g_0}(e)$.

Next, we note that $u_{g_0}(e)$ is an $ad_h$-invariant nilpotent ideal of $Z_{g_0}(e)$ (see Lemma 3.4.5 in [4] for the ungraded case and observing that, if a $Z_m$-graded ideal is nilpotent then its 0-graded component is a nilpotent ideal in the corresponding 0-graded subalgebra.)

Set $U_0(e) := \exp u_{g_0}(e) \subset Z_{G_0}(e)$. We will show that

$$ Ad_{U_0(e)}(h) = h + u_{g_0}(e), \tag{2.2} $$

which is a version of Lemma 3.4.7 in [4] for our graded Lie algebra. The proof of Lemma 3.4.7 in [4] carries to our case easily, since $u_{g_0}(e)$ is $ad_h$-invariant. In particular, we also have the following decomposition

$$ u_{g_0}(e) = \oplus_{i=1}^m u(e)_k $$

for some finite positive integer $m$, where

$$ u(e)_k := \{ x \in u_{g_0}(e) | [h, x] = kx \}. $$

For a given $v \in u_{g_0}(e)$ we will find $z \in u_{g_0}(e)$ such that $Ad_{exp_z}(h) = v$, which would imply (2.2). We approximate $z$ by $z_j$ inductively such that

$$ z_j \in \oplus_{1 \leq i \leq j} u(e)_i, \text{ and} \tag{2.3} $$

$$ Ad_{exp_{z_j}}(h - (h + v)) \in \oplus_{j+1 \leq i \leq m} u(e)_i. \tag{2.4} $$

Set

$$ z_{j+1} := \text{ the component of } (Ad_{exp_{z_j}}(h - (h + v))) \text{ in } u(e)_{j+1}. $$

Let

$$ z_{j+1} = z_j + \frac{1}{j+1} z'_{j+1} \in \oplus_{1 \leq i \leq j+1} u(e)_i. $$

Then we check immediately that properties (2.3) and (2.4) carry over to $z_{j+1}$. Thus if we set $z_1 := -v_1$, where $v_1$ is the component of $v$ is $u(e)_1$, and we put $z = z_m$, we get $Ad_{exp_z}(h) = v$, as desired. This proves (2.2), hence, Theorem 2.1.ii.
The second proof adapts the Vinberg argument in [35]. We do not produce this argument here, but we remark that the only place we need to take care when working over $\mathbb{R}$ instead over $\mathbb{C}$ is the closedness of the orbit $Ad_{U_0(e)}h$. This closedness holds, since this orbit is a component of the intersection of the complexified orbit $Ad_{U_0^C(e)}h$, which is closed, with $g_1$.

iii) Theorem 2.1iii is a direct consequence of Theorem 1(3) in [35], which is an analog of Theorem 2.1ii for complex algebras. \hfill \Box

We call a triple $(h, e, f)$ satisfying the condition in 2.1.i a $sl_2$-triple, and we denote by $sl_2(e)$ the Lie subalgebra of $g$ generated by $e, f, h$.

Thanks to the JMV theorem we can characterize semisimple elements and nilpotent elements in $g_1$ using the geometry of their $Ad_{G_0}$-orbits.

**Lemma 2.3.** Element $x \in g_1$ is nilpotent if and only if the closure of its orbit under the $Ad_{G_0}$-action contains zero. Element $x \in g_1$ is semisimple if and only if its orbit under the action of $Ad_{G_0}$ is closed.

**Proof.** Suppose that $x \in g_1$ is nilpotent. By Theorem 2.1 there is an element $h \in g_0$ such that $[h, x] = x$. Clearly $\lim_{t \to -\infty} Ad_{\exp(t h)}(x) = 0$. This proves the “only if” part of the first assertion of Lemma 2.3.

Now we suppose that the closure of the orbit $Ad_{G_0}(x)$ contains zero. Then the orbit $Ad_{\rho(G_0)}(x)$ contains zero, in particular $Ad_{G_0^C}(x)$ contains zero. By Proposition 1 in [33], $x$ is a nilpotent element in $g_1^C$. Hence $x$ is a nilpotent element in $g_1$. This proves the “if” part of the first assertion.

Let us prove the second assertion of Lemma 2.3. If $x$ is not semisimple, let us consider its Jordan decomposition $x = x_s + x_n$. As in [33], proof of Proposition 3, by Morozov’s theorem we find an element $t$ in the centralizer $Z_{g_0^C}(x_s)$ such that $[t, x_n] = x_n$. Clearly, we can choose $t$ as an element in $g_0$. Then $\lim_{t \to -\infty} Ad_{\exp(t x_n)}(x) = x_s$. Hence the orbit $Ad_{G_0}(x)$ is not closed. This proves the “if” part of the second assertion.

Now assume that $x$ is semisimple. Then the orbit $Ad_{G_0^C}(x)$ in $g_1^C$ is closed. Hence the intersection of this orbit with $g_1 \subset g_1^C$ is closed. Let $y \in Ad_{G_0^C}(x) \cap g_1$. Denote by $T_y(Ad_{G_0^C}(x) \cap g_1)$ the tangent cone of $Ad_{G_0^C}(x) \cap g_1$ at $y$, i.e.

$$T_y(Ad_{G_0^C}(x) \cap g_1) := T_y(Ad_{G_0^C}(x)) \cap g_1.$$ 

Then $T_y(Ad_{G_0^C}(x) \cap g_1) = [g_0^C, y] \cap g_0 = [g_0, y] = T_y(Ad_{G_0}(y))$. Hence this intersection is a disjoint union of $Ad_{G_0}$-orbits of elements in $g_1$. Since each orbit $Ad_{G_0}(y)$ is a submanifold in $g_1$, it follows that each $Ad_{G_0}$-orbit in this intersection is also closed. This proves the “only if” part of the second assertion. \hfill \Box
We adopt the following definition in [33]. Let \( g = \bigoplus_{i=1}^{m} g_i \) be a \( \mathbb{Z}_m \)-graded semisimple Lie algebra. A Cartan subspace in \( g_1 \) (resp. \( g_1^C \)) is a maximal subspace in \( g_1 \) (resp. in \( g_1^C \)) consisting of commuting semisimple elements. The classification of Cartan subspaces in \( g_1 \) is well-known for \( m \leq 2 \), see [16], [29], [23], and unknown for \( m \geq 3 \).

### 3. R-compatible Cartan involutions

In this section we show the existence of a Cartan involution of a real \( \mathbb{Z}_m \)-graded semisimple Lie algebra \( g \) which reverses the \( \mathbb{Z}_m \)-gradation on \( g \), see Theorem 3.4. As a consequence, there is a 1-1 correspondence between \( Ad_{G_0^C} \)-orbits (resp. \( Ad_{G_0} \)-orbits) on \( g_1^C \) and \( g_{-1}^C \), (resp. on \( g_i \) and \( g_{-i} \)), see Corollary 3.5. We also give important examples of real \( \mathbb{Z}_m \)-graded semisimple Lie algebras.

Let \( g = \bigoplus_{i=0}^{m-1} g_i \) be a \( \mathbb{Z}_m \)-graded semisimple Lie algebra and \( \theta^C \) the automorphism of \( g^C \) associated with this induced gradation. It is easy to check that

\[
\tau_g \theta^C = (\theta^C)^{-1} \tau_g.
\]

Since \( \tau_g^2 = Id \), (3.1) holds if and only if

\[
\tau_g (\theta^C)^{-1} = (\theta^C) \tau_g.
\]

Now let \( g \) be a real form in \( g^C \) with a \( \mathbb{Z}_m \)-gradation generated by \( \theta^C \). If \( g \) satisfies the relation (3.1), then for any \( x \in g_k^C \)

\[
\theta^C(\tau_g(x)) = \tau_g(\theta^C)^{-1}(x) = \tau_g(\exp^{\frac{-2\pi\sqrt{-1} k}{m}} x) = \exp^{\frac{2\pi\sqrt{-1} k}{m} \tau_g(x)}.
\]

Hence \( \tau_g(g_k^C) = g_k^C \), and therefore

\[
\tau_g = \bigoplus_i (g \cap g_i^C).
\]

Thus we say that a real form \( g \) of \( g^C \) is compatible with \( \theta^C \), if (3.1) holds. Equivalently (3.2) holds, and equivalently (3.3) holds.

**Remark 3.1.** If \( m \neq 2 \), any real form \( g \) compatible with \( \theta^C \) is not invariant under \( \theta^C \) unless \( m \) is even and the only nonzero components of \( g \) have degree 0 or \( m/2 \). A real form \( g \) is invariant under \( \theta^C \), if and only if \( \tau_g \) commutes with \( \theta^C \).

Let \( u \) be a compact real form of \( g^C \) which is compatible with \( g \), i.e. \( \tau_u \) is R-compatible with the \( \mathbb{Z}_m \)-gradation, if \( u \) is invariant under the automorphism \( \theta^C \) associated with this gradation: \( \tau_u \theta^C = (\theta^C) \tau_u \).

Clearly, \( \tau_u \) is R-compatible with the \( \mathbb{Z}_m \)-gradation, if and only if \( \tau_u \) reverses the gradation on \( g \) : \( \tau_u(g_k) = g_{-k} \).
Example 3.3. i) Any real $\mathbb{Z}_2$-graded semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ has a $\mathbb{R}$-compatible Cartan involution, see [3], Lemma 10.2. The classification of all $\mathbb{Z}_2$-graded simple Lie algebras has been given in [3].

ii) Let us consider the split algebra $\mathfrak{g} = \mathfrak{e}_7(7)$ - a normal real form of the complex Lie algebra $\mathfrak{e}_7$. The complex algebra $\mathfrak{g}^C$ has the following root system

$\Sigma = \{\varepsilon_i - \varepsilon_j, \varepsilon_p + \varepsilon_q + \varepsilon_r + \varepsilon_s, |i \neq j, (p, q, r, s \text{ distinct})\}, \sum_{i=1}^{8} \varepsilon_i = 0\}.$

Let us choose a Cartan subalgebra $\mathfrak{h}_0^C$ of $\mathfrak{g}^C$. Denote by $E_\alpha$, $\alpha \in \Sigma$, the corresponding root vectors such that $[E_\alpha, E_{-\alpha}] = \frac{2H_\alpha}{\alpha(H_\alpha)} \in \mathfrak{h}_0^C$, see e.g. [13], p.258. We decompose $\mathfrak{g}$ as

\begin{equation}
\mathfrak{g} = \oplus_{\alpha \in \Sigma} \langle H_\alpha \rangle_{\mathbb{R}} \oplus_{\alpha \in \Sigma} \langle E_\alpha \rangle_{\mathbb{R}} \oplus_{\alpha \in \Sigma} \langle E_{-\alpha} \rangle_{\mathbb{R}}.
\end{equation}

$\mathfrak{g}^C$ has the following compact form $\mathfrak{u}$, which is compatible with $\mathfrak{g}$:

\begin{equation}
\mathfrak{u} = \oplus_{\alpha \in \Sigma} \langle iH_\alpha \rangle_{\mathbb{R}} \oplus_{\alpha \in \Sigma} \langle i(E_\alpha + E_{-\alpha}) \rangle_{\mathbb{R}} \oplus_{\alpha \in \Sigma} \langle (E_\alpha - E_{-\alpha}) \rangle_{\mathbb{R}}.
\end{equation}

Let $\theta^C$ be the involution of $\mathfrak{e}_7$ defined in [1] as follows

\begin{equation}
\theta^C|_{\mathfrak{h}_0} = Id,
\end{equation}

\begin{equation}
\theta^C(E_\alpha) = E_\alpha, \text{ if } \alpha = \varepsilon_i - \varepsilon_j,
\end{equation}

\begin{equation}
\theta^C(E_\alpha) = -E_{-\alpha}, \text{ if } \alpha = \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l.
\end{equation}

Then $\theta^C(\mathfrak{g}) = \mathfrak{g}$, and $\theta^C(\mathfrak{u}) = \mathfrak{u}$. Hence $\theta^C$ commutes with $\tau_\mathfrak{g}$ as well as with $\tau_\mathfrak{u}$. Denote by $\theta$ the restriction of $\theta^C$ to $\mathfrak{g}$. Automorphism $\theta$ defines a $\mathbb{Z}_2$-gradation: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{sl}(8, \mathbb{R})$. Clearly $\tau_\mathfrak{u}$ is a $\mathbb{R}$-compatible with this $\mathbb{Z}_2$-gradation.

iii) Let $x \in \mathfrak{g}_1$. Let $\mathcal{Z}_\mathfrak{g}(x)$ be the centralizer of $x$ in $\mathfrak{g}$. Clearly, its complexification $\mathcal{Z}_{\mathfrak{g}^C}(x)$ is invariant under the action of $\theta^C$. Hence $\mathcal{Z}_\mathfrak{g}(x)$ inherits the $\mathbb{Z}_m$-grading, and the commutant $\mathcal{Z}_\mathfrak{g}(x)'$ of $\mathcal{Z}_\mathfrak{g}(x)$ is also a real $\mathbb{Z}_m$-graded semisimple Lie algebra. If $m = 2$ and $x \in \mathfrak{g}_1 \cap \mathfrak{p}$ or $x \in \mathfrak{g}_1 \cap \mathfrak{t}$, the Cartan compatible involution $\tau_\mathfrak{u}$ also preserves $\mathcal{Z}_\mathfrak{g}(x)$.

iv) If $(\mathfrak{g}, \tau_\mathfrak{u})$ and $(\mathfrak{g}', \tau_{\mathfrak{u}'})$ are real $\mathbb{Z}_m$-graded semisimple Lie algebras with R-compatible Cartan involutions $\tau_\mathfrak{u}$ and $\tau_{\mathfrak{u}'}$, then their direct sum $\mathfrak{g} \oplus \mathfrak{g}'$ is also a real $\mathbb{Z}_m$-graded semisimple Lie algebra equipped with the R-compatible Cartan involution $\tau_{\mathfrak{u} \oplus \mathfrak{u}'}$. Conversely, if $m$ is prime any real $\mathbb{Z}_m$-graded semisimple Lie algebra is a direct sum of real $\mathbb{Z}_m$-graded simple Lie algebras (see [33] for a similar assertion over $\mathbb{C}$, which implies our assertion).

v) Let us consider a real $\mathbb{Z}_3$-graded simple Lie algebra $\mathfrak{e}_8(8)$ which is a normal form of the complex algebra $\mathfrak{e}_8$. The root system $\Sigma$ of $\mathfrak{e}_8$ is

$\Sigma = \{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)\}, (i, j, k \text{ distinct}), \sum_{i=1}^{9} \varepsilon_i = 0\}.$

In [12] Vinberg and Elashvili proved that there is an automorphism $\theta^C$ of order 3 on $\mathfrak{e}_8$ defined by the following formulas

\begin{equation}
\theta^C_{|\langle H_\alpha, E_\alpha, \alpha = \varepsilon_i - \varepsilon_j \rangle_C} = Id,
\end{equation}

\begin{equation}
\theta^C_{|\langle H_\alpha, E_\alpha, \alpha = \varepsilon_i + \varepsilon_j + \varepsilon_k \rangle_C} = Id,
\end{equation}

\begin{equation}
\theta^C_{|\langle H_\alpha, E_\alpha, \alpha = \varepsilon_i + \varepsilon_j + \varepsilon_k \rangle_C} = Id.
\end{equation}
\[ \theta^C_\ell (E_{\alpha}, \alpha = (\epsilon_i + \epsilon_j + \epsilon_k)) = \exp (i2\pi/3) \cdot \Id, \]
\[ \theta^C_\ell (E_{\alpha}, \alpha = (\epsilon_i + \epsilon_j + \epsilon_k)) = \exp (-i2\pi/3) \cdot \Id. \]

It is easy to see that \( \theta^C \) defines a \( \mathbb{Z}_3 \)-grading on \( \mathfrak{e}_8 \) as well as on \( \mathfrak{e}_{8(8)} \). Namely, we have \( \mathfrak{e}_{8(8)} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{-1} \) where

\[ \mathfrak{g}_0 = \langle H_{\alpha}, E_{\alpha}, \alpha = \epsilon_i - \epsilon_j \rangle, \]
\[ \mathfrak{g}_1 = \langle E_{\alpha}, \alpha = (\epsilon_i + \epsilon_j + \epsilon_k) \rangle, \]
\[ \mathfrak{g}_{-1} = \langle E_{\alpha}, \alpha = - (\epsilon_i + \epsilon_j + \epsilon_k) \rangle. \]

The compact form \( u \) of \( \mathfrak{e}_8 \) defined as in \( 3 \) is \( \mathfrak{R} \)-compatible with this \( \mathbb{Z}_3 \)-grading of \( \mathfrak{e}_{8(8)} \).

In [12] Vinberg and Elashvili proved that the space \( \mathfrak{g}_1^C \) is linearly isomorphic to the space of 3-vectors on \( \mathbb{C}^9 \) and the space \( \mathfrak{g}_{-1}^C \) is linearly isomorphic to the space of 3-forms on \( \mathbb{C}^9 \). Let \( C_0^C \subset E_8^C \) be the connected Lie subgroup with the Lie subalgebra \( \mathfrak{g}_0 \). Vinberg and Elashvili showed that the adjoint action of the connected Lie subgroup on \( g_0^C \) (resp. \( g_{-1}^C \)) is exactly the canonical action of \( SL(9, \mathbb{C}) \) on the space \( \Lambda^3(\mathbb{C}^9) \) (resp. \( \Lambda^3(\mathbb{C}^9)^* \)) of 3-vectors (resp. 3-forms) in \( \mathbb{C}^9 \). It is easy to see that the adjoint action of \( G_0 \) on \( g_1 \) is the canonical action of \( SL(9, \mathbb{R}) \) on the space of 3-vectors on \( \mathbb{R}^9 \).

The following Theorem is an analogue of Theorem 7.1 in [13] for real \( \mathbb{Z}_m \)-graded Lie semisimple Lie algebras. The case \( m = 2 \) is well-known, see [3].

**Theorem 3.4.** Let \( \mathfrak{u}' \) be a real compact form of \( \mathfrak{g}^C \), which is invariant under \( \theta^C \).

1) There exists an automorphism \( \phi \) of \( \mathfrak{g}^C \), which commutes with \( \theta^C \), such that \( u = \phi(u') \) is invariant under \( \tau_\theta \) and under \( \theta^C \).

2) Any real \( \mathbb{Z}_m \)-graded semisimple Lie algebra has a Cartan involution, which reserves the gradation.

**Proof.** 1) We use the idea of the proof in [13], p. 183. Let \( B \) denote the Killing form on \( \mathfrak{g}_c \times \mathfrak{g}_C \). The Hermitian form \( B_{u'} \) defined on \( \mathfrak{g}_c \times \mathfrak{g}_c \) by

\[ B_{u'}(X, Y) = -B(X, \tau_{u'}(Y)) \]

is strictly positive definite, since \( u' \) is compact. The composition \( \tau_\theta \tau_{u'} \) is an automorphism of \( \mathfrak{g}_c \), so it leaves the Killing form invariant. The argument in [13] shows that \( \tau_\theta \tau_{u'} \) is self-adjoint w.r.t. \( B_{u'} \). Hence \( (\tau_\theta \tau_{u'})^2 \) is positive self-adjoint w.r.t. \( B_{u'} \), moreover it commutes with \( \theta^C \), because \( \tau_\theta \theta^C = (\theta^C)^{-1} \tau_\theta \) and \( \tau_{u'} \) commutes with \( \theta^C \). It follows that the automorphism \( \phi := [(\tau_\theta \tau_{u'})^2]^{1/4} \) commutes with \( \theta^C \). (To see it, we choose an orthogonal basis \( (e_j) \) of \( \mathfrak{g}_c \) w.r.t. \( B_{u'} \) which are also eigenvectors with eigenvalues \( a_i > 0 \) of \( (\tau_\theta \tau_{u'})^2 \) for all \( i \). We note that \( \theta^C \) commutes with \( (\tau_\theta \tau_{u'})^2 \) if and only if \( \theta(e_i) \) is also eigenvector of \( (\tau_\theta \tau_{u'})^2 \) with value \( a_i \) for all \( i \). Clearly, \( (e_i) \) and \( \theta^C(e_i) \) are also eigenvectors of \( [(\tau_\theta \tau_{u'})^2]^{1/4} \) with eigenvalue \( (a_i)^{1/4} \). Therefore \( \theta^C \) commutes also with \( [(\tau_\theta \tau_{u'})^2]^{1/4} \). Hence \( \phi(u') \) is invariant under \( \theta^C \). The proof of Theorem 7.1 in [13] shows that \( \phi(u') \) is invariant under \( \tau_\theta \). This proves the first assertion of Theorem 3.4.
2) By Lemma 5.2, chapter X in [13], p. 491, there is a real compact form \( u' \) of \( \mathfrak{g}^C \) which is invariant under \( \theta^C \). Taking into account the first assertion of Theorem 3.4, we prove the second assertion.

Here is another short proof of the second assertion due to Vinberg [36]. Let us consider the group \( G(\theta^C, \tau_0) \) generated by \( \theta^C \) and \( \tau_0^C \) acting on the space \( G^C/U \) of all compact real forms of \( \mathfrak{g}^C \). This group is finite, since \( \tau_0 \theta^C = (\theta^C)^{-1} \tau_0 \). As E. Cartan proved [6], see also Theorem 13.5, chapter I in [13] for a modern treatment, any compact group of motions of a simply connected symmetric space of non-positive curvature has a fixed point. It is known that \( G^C/U \) is a symmetric space of noncompact type, hence it has nonpositive curvature, [13], chapter VI. The fixed point of \( G(\theta^C, \tau_0) \) is the required compact form. □

**Corollary 3.5.** A \( R \)-compatible involution \( \tau_u \) gives an isomorphism between \( \text{Ad}_{G_0} \)-orbits in \( \mathfrak{g}_i \) and \( \mathfrak{g}_{-i} \). The \( C \)-linear extension \( \tau_u^C (= \tau_u \circ \tau_0) \) of \( \tau_u \) gives an isomorphism between \( \text{Ad}_{G_0^C} \)-orbits in \( \mathfrak{g}_i^C \) and \( \mathfrak{g}_{-i}^C \).

**Proof.** Denote by \( \tau_u^C \) the involutive automorphism on \( G^C \) whose differential is \( \tau_u \). Since \( \tau_u^C(\mathfrak{g}_0) = \mathfrak{g}_0 \) and \( \tau_u^C(\mathfrak{g}_0^C) = \mathfrak{g}_0^C \), we get

\[
\tau_u^C(G_0) = G_0, \quad \tau_u^C(G_0^C) = G_0^C.
\]

For any \( v \in \mathfrak{g}_0^C \) and \( e \in \mathfrak{g}_i^C \) we have \( \tau_u^C(\exp v) = \exp(\tau_u^C(v)) \) and

\[
\tau_u^C(\text{Ad}_{\exp v} e) = \text{Ad}_{\exp(\tau_u^C(v))}(\tau_u^C(e)).
\]

Consequently

\[
\tau_u(\text{Ad}_{G_0} e) = \text{Ad}_{G_0}(\tau_u e), \quad \tau_u^C(\text{Ad}_{G_0^C} e) = \text{Ad}_{G_0^C}(\tau_u^C e).
\]

This proves our corollary. □

### 4. Classification of homogeneous nilpotent elements

The set of orbits of homogeneous nilpotent elements in a real \( \mathbb{Z}_m \)-graded semisimple Lie algebra \( \mathfrak{g} \) is more complicated than the set of orbits of nilpotent elements in the complex case, since the \( \text{Ad}_{G_0} \)-conjugacy class of a nilpotent element \( e \) in the real case is not defined uniquely by its characteristic. If \( m = 1 \), i.e. \( \mathfrak{g} \) is regarded without gradation, a complete classification of nilpotent elements in \( \mathfrak{g} \) can be obtained using the Cayley transformation and the Vinberg method of classification of nilpotent elements in an associated complex \( \mathbb{Z}_2 \)-graded semisimple Lie algebra, see e.g. [10]. We do not know how to generalize this method for \( m \geq 2 \). Our study of the set of orbits of homogeneous nilpotent elements in a real \( \mathbb{Z}_m \)-graded Lie algebra \( \mathfrak{g} \) is divided in the following steps. In Lemma 4.1 we prove that there is an injective map from the set of the \( \text{Ad}_{G_0} \)-conjugacy classes of characteristics in \( \mathfrak{g} \) to the set of \( \text{Ad}_{G_0^C} \)-conjugacy classes of characteristics in \( \mathfrak{g}^C \). Recall that a classification of characteristics in \( \mathfrak{g}^C \) can be obtained by the Vinberg method of support [35]. In Remark 4.2 we summarize these results in an algorithm to classify characteristics in \( \mathfrak{g} \). Then we
show in Theorem 4.3 that there is a 1-1 correspondence between $\text{Ad}_{G_0}$-orbits of nilpotent elements $e \in \mathfrak{g}_1$ with a given characteristic $h$ and the set of open $\mathbb{Z}_{G_0}(h)$-orbits in $\mathfrak{g}_1(\frac{h}{2})$. This set is closely related to the set of connected components of a semialgebraic set in $\mathfrak{g}_1(\frac{h}{2})$. In Remark 4.10 we explain our algorithm to count the number of conjugacy classes of nilpotent elements in $\mathfrak{g}_1$ as well as to choose a sample representative for each conjugacy class. We note that this algorithm is highly complicated, and it can be implemented with a sufficient computer power and a suitable software package in future, see Remark 4.8.

Let $e$ be a nilpotent element in $\mathfrak{g}_1$ and $h \in \mathfrak{g}_0$ its characteristic. Then $h$ is also a characteristic of $e$ in $\mathfrak{g}^C$. A classification of $\text{Ad}_{G_0}^C$-conjugacy classes of characteristics in $\mathfrak{g}_0^C$ can be obtained by using the support method of Vinberg in [35]. Recall that a complex support $\mathfrak{s}^C(h)$ of $e$ in a complex $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}^C$ is a locally flat (i.e. $\dim \mathfrak{g}_0(h) = \dim \mathfrak{g}_1(h)$) $\mathbb{Z}$-graded semisimple Lie algebra in $\mathfrak{g}^C$ whose defining element is a characteristic $h$ of $e$, namely $\mathfrak{s}^C(h) := \mathfrak{g}'(\mathfrak{h}, \phi)$ - the commutant of $\mathfrak{g}(\mathfrak{h}, \phi)$. Here $\mathfrak{h}$ is a Cartan subspace in the normalizer $\mathcal{N}_{\mathfrak{g}_0}(e)$ such that $\mathfrak{h} \ni h$, and $\phi$ is the character of $\mathfrak{h}$ defined by

$$[u, e] = \phi(u)(e) \text{ for all } u \in \mathfrak{h}$$

is a $\mathbb{Z}$-graded reductive Lie algebra in $\mathfrak{g}$, see [35]. Clearly $\mathfrak{s}^C(h)$ is defined by $h$ uniquely up to conjugacy by element in $\mathcal{N}_{\mathfrak{g}_0}(e)$.

We define a real support $\mathfrak{s}(h)$ of a nilpotent element $e$ in a real $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}$ in the same way. Here we choose $\mathfrak{h}$ to be a maximal $\mathbb{R}$-diagonalizable Cartan subspace in $\mathcal{N}_{\mathfrak{g}_0}(e)$ containing $h$. Such a choice is unique up to a conjugacy by elements in $\mathcal{N}_{G_0}(e)$. Clearly, the complexification of a real support of $e$ is a complex support of $e$ in $\mathfrak{g}^C$.

It is known that the $\text{Ad}_{G_0}^C$-conjugacy classes of characteristic elements $h \in \mathfrak{g}_0^C$ are in a 1-1 correspondence with the $\text{Ad}_{G_0}^C$-conjugacy classes of locally flat $\mathbb{Z}$-graded semisimple Lie subalgebras $\mathfrak{s}(h)$ in $\mathfrak{g}^C$ [35]. We refer the reader to [35] and [7] for more details on $\mathbb{Z}$-graded semisimple Lie algebras and $\mathbb{Z}$-graded locally flat semisimple Lie algebras over $\mathbb{C}$ or over $\mathbb{R}$.

**Lemma 4.1.** i) There exists an injective map from the set of $\text{Ad}_{G_0}$-orbits of characteristics in $\mathfrak{g}$ to the set of $G_0^C$-orbits of characteristics in $\mathfrak{g}^C$.

ii) Let $h \in \mathfrak{g}_0$ be a characteristic of a nilpotent element in $\mathfrak{g}_1$. Then $\text{Ad}_{G_0}^C(h) \cap \mathfrak{g}_0 = \text{Ad}_{G_0}(h)$.

**Proof.** i) First we note that if $h \in \mathfrak{g}$ is a characteristic element then it is also a characteristic element in $\mathfrak{g}^C$. Thus we have a map from the conjugacy classes of characteristics in $\mathfrak{g}$ to the conjugacy classes of characteristics in $\mathfrak{g}^C$. We will show that this map is injective. Suppose
that $h_1, h_2 \in \mathfrak{g}_0$ are characteristics in $\mathfrak{g}$ such that $\text{Ad}_X h_1 = h_2$ for $X \in G_0^C$. Let $\tau_u$ be a $\mathbb{R}$-compatible Cartan involution in Theorem 3.3.1. Note that the restriction of $\tau_u$ to $\mathfrak{g}_0$ leaves the center of $\mathfrak{g}_0$ as well as the commutant $\mathfrak{g}_0' \cap \mathfrak{g}_0$ fixed. Moreover the restriction of $\tau_u$ to $\mathfrak{g}_0'$ is also a Cartan involution of $\mathfrak{g}_0'$. By the theory of Cartan subalgebras in real reductive Lie algebras, see e.g. [13], chapter IX, Corollary 4.2, we can assume that $h_1, h_2 \in \mathcal{Z}(\mathfrak{g}_0) \oplus \mathfrak{p}_0'$, where $\mathfrak{p}_0' = \mathfrak{t}_0' \oplus \mathfrak{p}_0'$ is the Cartan decomposition of $\mathfrak{g}_0'$ w.r.t. $\tau_u$. By Theorem 2.1 in [27], there exists $Y \in G_0$ such that $\text{Ad}_Y h_1 = h_2$.

ii) Clearly Lemma 4.1.ii is a consequence of Lemma 4.1.i.

**Remark 4.2.** Using Lemma 4.1 we obtain a classification of conjugacy classes of characteristics in $\mathfrak{g}$ as follows. First we find all complex supports in $\mathfrak{g}_0^C$ by Vinberg method in [35]. There are only a finite number of them. Next, we find the real forms of these complex supports using the Djokovic classification of real forms of complex $\mathbb{Z}$-graded semisimple Lie algebras in [7]. In the third step we decide which real form of a given complex support admits an embedding to $\mathfrak{g}$ whose complexification is the given complex support. This step can be done using the theory of representations of real semisimple Lie algebras, see e.g. [13], [34]. Lemma 4.1 shows that in the third step there exists not more than one real form for each given complex support. The defining element of the corresponding real support is our desired characteristic.

Now let us fix a characteristic $h \in \mathfrak{g}_0$ corresponding to a nilpotent element $e \in \mathfrak{g}_1$. Let us consider the following $\mathbb{Z}$-graded algebra

$$\mathfrak{g}^h_0 := \bigoplus_k \mathfrak{g}_k^h = \{ x \in \mathfrak{g}_k \mod m : [h, x] = kx \}.$$

Clearly the centralizer $\mathcal{Z}_{G_0}(h)$ of $h$ in $G_0$ acts on $\mathfrak{g}_0^h$ preserving the $\mathbb{Z}$-gradation. The Lie algebra of $\mathcal{Z}_{G_0}(h)$ is $\mathfrak{g}_0^h$. It is known [35], proof of Theorem 1 (4), that $e \in \mathfrak{g}_1^h$, moreover $[\mathfrak{g}_0^h, e] = \mathfrak{g}_1^h$. Equivalently, $e$ belongs to an open $\text{Ad}_{\mathcal{Z}_{G_0}(h)}$-orbit in $\mathfrak{g}_1^h$. An element $e \in \mathfrak{g}_1$ (resp. $\mathfrak{g}_1^C$) is called generic, if orbit $\text{Ad}_{\mathcal{Z}_{G_0}(h)}(e)$ is open in $\mathfrak{g}_1$ (resp. $\text{Ad}_{\mathcal{Z}_{G_0}(h)}(e)$ in $\mathfrak{g}_1^C$). Otherwise $e$ is called singular. By the definition the genericity of an element $e \in \mathfrak{g}_1$ implies the genericity of any element in the orbit $\text{Ad}_{\mathcal{Z}_{G_0}(h)}(e)$. The following Theorem 4.3 generalizes Djokovic’s theorem in [8], Theorem 6.1.

**Theorem 4.3.** Let $(h, e, f)$ be a sl$_2$-triple. The inclusion $\mathfrak{g}_1^h \to \mathfrak{g}_1$ induces a bijection between the open $\text{Ad}_{\mathcal{Z}_{G_0}(h)}$-orbits in $\mathfrak{g}_1^h$ and the $\text{Ad}_{G_0}$-orbits contained in $\text{Ad}_{G_0}(\mathfrak{g}_1^C)$.

**Proof.** Suppose that $\text{Ad}_{\mathcal{Z}_{G_0}(h)}(e')$ is an open orbit in $\mathfrak{g}_1^h$. By Vinberg’s theorem, [35], proof of Theorem 1(4), $e'$ belongs to the orbit $\text{Ad}_{\mathcal{Z}_{G_0}(h)}(e)$ in $\mathfrak{g}_1^C$. This defines a map from the set of open $\text{Ad}_{\mathcal{Z}_{G_0}(h)}$-orbits in $\mathfrak{g}_1^h$ to the set of $\text{Ad}_{G_0}$-orbits containing in $\text{Ad}_{G_0}(\mathfrak{g}_1^C)$.\[\Box\]
We will show that this map is surjective. Let $e' \in Ad_{G_0}(e) \cap g_1$. Let $h' \in g_0$ be a characteristic of $e$. By JMV theorem for the complex case, $h$ and $h'$ belong to the same $Ad_{G_0}$-orbit. Lemma 4.4 implies that there exists $X \in G_0$ such that $Ad_X(h') = h$. Clearly $Ad_X(e') \in g_1(h)$, since $[Ad_X(h'), Ad_X(e')] = Ad_X(e')$. Element $Ad_X(e')$ is generic in $g_1(h)$, since it lies in the orbit $Ad_{Z_{G_0}}(h)(e)$. This proves the surjectivity of the considered map.

It remains to show that this map is injective. First we will prove the following

**Lemma 4.4.** (cf. Lemma 6.4 in [8]) Let $e'$ be a generic element in $g_1(h)$. Then there exists $f' \in g_{-1}(h)$ such that $(h, e', f')$ is a $sl_2$-triple.

**Proof.** Let $e$ be a nilpotent element in a $sl_2$-triple $(h, e, f)$. By a Vinberg result [35], proof of Theorem 1.4, there is an element $Y \in Z_{G_0}(h)$ such that $Ad_Y(e) = e'$. Clearly $(h, e', Ad_Y(f))$ is a $sl_2^C$-triple in $g_1^C$, moreover $Ad_Y(f) \in g_{-1}(h)$, since $f \in g_{-1}(h)$. Since $h$ and $e'$ define their $sl_2$-triple uniquely (see Theorem 2.1 and its version in the complex case, [35], Theorem 1.3), we get $Ad_Y(f) \in g_{-1}(h)$. □

Let us complete the proof of Theorem 4.3. Suppose that $e$ and $e'$ are generic elements of $g_1(h)$ such that $e' = Ad_X e$ for some $X \in G_0$. We will show that $e$ and $e'$ are in the same open orbit of $Z_{G_0}(h)$. By Lemma 4.4 there are elements $f$ and $f'$ in $g_{-1}(h)$ such that $(h, e, f)$ and $(h, e', f')$ are $sl_2$-triples in $g$. Note that $(Ad_X h, e', Ad_X f)$ is a $sl_2$-triple in $g$. By Theorem 2.1 there exists an element $Y \in G_0$ such that $Ad_Y(e') = e'$, $Ad_Y(Ad_X h) = h$ and $Ad_Y(Ad_X f) = f'$. Thus $e' = Ad_Y e$, where $Y \cdot X \in Z_{G_0}(h)$. This proves the injectivity of our map. □

Now we proceed to classify the open $Z_{G_0}(h)$-orbits in $g_1(h)$.

Denote by $g_i(h)$ the $i$-th component of the commutant of $g(h)$ which has the induced $Z$-gradation from $g(h)$. Since $g_1(h) = [g_0(h), g_1(h)]$, we get

$$g_1(h) = g_0(h).$$  \hspace{1cm} (4.1)

Since $Z(g(h)) \subset g_0(h)$, we have $g_0(h) = Z(g(h)) \oplus g_0(h)'. \hspace{1cm}$ Hence

$$[g_0(h'), g_1(h)] = g_1(h).$$  \hspace{1cm} (4.2)

Denote by $Z_{G_0}(h)'$ the connected subgroup in $G_0$ whose Lie algebra is $g_0(h)'$. An element $e_i \in g_i(h)'$ is called generic, if the orbit $Ad_{Z_{G_0}}(e_i)$ is open in $g_i(h)$. Equivalently, $[g_0(h)', e_i] = g_i(h)'. \hspace{1cm}$

Let $Z_{G_0}(h)^0$ be the connected component of $Z_{G_0}(h)$. From (4.1) and (4.2) we get immediately
Lemma 4.5. There exists a 1-1 correspondence between the set of open \( Ad_{Z_{G_0}(h)} \)-orbits in \( g_1(\frac{h}{2}) \) and the set of open \( Ad_{Z_{G_0}(h)} \)-orbits in \( g_1(\frac{h}{2})' = g_1(\frac{h}{2}) \).

Remark 4.6. Clearly, all elements in \( g_1^C(\frac{h}{2})' \) are nilpotent, if \( i \neq 0 \). Proposition 2 in [33] asserts that there is only a finite number of \( Z_{C_0}^e(h)' \)-conjugacy classes of nilpotent elements in \( g_1^C(\frac{h}{2})' \). Hence it follows that the set of generic nilpotent elements in \( g_1^C(\frac{h}{2})' \) is open and dense in \( g_1^C(\frac{h}{2})' \). Since the number of \( Ad'_{Z_{G_0}(h)} \)-orbits in a \( Ad_{Z_{C_0}(h)} \)-orbit is finite [31], Proposition 2.3, it follows that for any \( i \neq 0 \) the set of generic elements in \( g_1(\frac{h}{2})' \) is open and dense.

Let us analyze the set of open \( Ad_{Z_{G_0}(h)} \)-orbits in \( g_1 \). An element \( e \) in \( g_1(\frac{h}{2})' \) (resp. in \( g_1^C(\frac{h}{2})' \)) is called singular, if it is not generic. Equivalently

\[
\dim[g_0(\frac{h}{2})', e] \leq \dim g_1(\frac{h}{2}) - 1.
\]

Let \( f_1, \ldots, f_m \) be a basis in \( g_0(\frac{h}{2})' \). Let us choose an basis \( e_1, \ldots, e_n \) in \( g_1 \). We write \( e = \sum_j a_j(e)e_j, a_j \in \mathbb{R} \). Then \( [e, f_i] = \sum a_j(e)|e_j, f_i] = \sum_j a_j(e)c_{ij}f_k \). Set \( b_{ik} = \sum_j a_j(e)c_{ij}^k \). Note that \( e \) is singular, if and only if the matrix \((b_{ij})_{i=1}^m \) has rank less than or equal to \( n - 1 \). Note that \( m \geq n \). Denote by \( P_l, l = 1, (\binom{n}{m}) \), the sub-determinants of \((b_{ij})\). Clearly \( e \) is singular, if and only if \( P_l(e) = 0 \) for all \( l \).

Lemma 4.7. There is a 1-1 correspondence between the set of open \( Ad_{Z_{G_0}(h)} \)-orbits in \( g_1(\frac{h}{2}) \) and the set of connected components of the semialgebraic set \( \{ x \in g_1(\frac{h}{2}) | \sum_{l=1}^m P_l^2(x) > 0 \} \). The number of open \( Ad_{Z_{G_0}(h)} \)-orbits in \( g_1(\frac{h}{2}) \) is finite.

Proof. The first assertion follows from Lemma 4.5 and our consideration above. The second assertion follows from the first one.

Remark 4.8. In [2], chapter 16, the authors offer an algorithm to compute the number of the connected components of a semialgebraic set and produce sample representative for each connected component. Their algorithm also allows to recognize, whether given two points in a semialgebraic set belong to the same connected component of this set. This algorithm is highly complicated and we hope to implement it in future using an appropriate software package.

It remains to consider whether two given open connected \( Ad_{Z_{G_0}(h)} \)-orbits in \( g_1(\frac{h}{2})' \) belong to the same \( Ad_{Z_{G_0}(h)} \)-orbit in \( g_1(\frac{h}{2}) \). Let \( e_i, i = 1, M, \) be representatives of the connected open \( Ad_{Z_{G_0}(h)} \)-orbits in \( g_1(\frac{h}{2}) \) obtained by the algorithm in [2], see Remark 4.8. Since the group \( Z_{Ad_{G_0}(h)} \) is connected [17], Lemma 5, the group \( Z_{G_0}(h) \) is generated by \( Z_{G_0}(h)^0 \) and the center \( Z(G_0) \) of \( G_0 \). Denote by \( F(e_k) \) the set of all elements \( X \in Z(G_0) \) such that \( Ad_X(e_k) \) belongs to the orbit \( Ad_{Z_{G_0}(h)^0}(e_k) \). Clearly \( F(e_k) \) is a subgroup of \( Z(G_0) \).
Lemma 4.9. The quotient $Z(G_0)/F'(e_k)$ is a finite abelian group. There exists an algorithm to find representatives $Y_{k,i}, i = 1, N,$ of the coset $Z(G_0)/N(e_k)$ in $Z(G_0)$. The orbit $\text{Ad}_{ZG_0}(h)(e_k)$ is a disjoint union of $N$ open connected orbits $\text{Ad}_{ZG_0(h)}(Y_{k,i}(e_k))$.

Proof. We know that $Z(G_0)$ is a finitely generated abelian group, see e.g. [34], which can be find explicitly [34]. Let $X_1, \cdots, X_l$ be generators of $Z(G_0)$. Since there is only finite number of connected open $\text{Ad}_{ZG_0(h)}(h)$-orbits in $g_1(\frac{h}{2})$, for each $j \in \overline{1,l}$ there exists a finite number $p(j)$ such that $\text{Ad}_{X_i^p(j)}(e_k)$ belongs to the orbit $\text{Ad}_{ZG_0(h)}(e_k)$. This proves the first assertion of Lemma 4.9. The second assertion follows from the proof of the first assertion using the algorithm in [2], see Remark 4.8. The last assertion follows from the second assertion. □

Remark 4.10. We summarize our result in the following algorithm to find conjugacy classes of nilpotent elements of degree 1 in a real $\mathbb{Z}_m$-graded semisimple Lie algebra $g$.

5. Orbits in a real $\mathbb{Z}_2$-graded semisimple Lie algebra

In this section, using results in the previous sections, we describe the set of homogeneous elements in a real $\mathbb{Z}_2$-graded Lie algebra $g$, see Remark 5.6 for a summarization.

The restriction to real $\mathbb{Z}_2$-graded semisimple Lie algebras is motivated by the fact that we do not have a classification of Cartan subspaces in $g_1$, if $m \geq 3$. A classification of Cartan subspaces in $g_1$ in a $\mathbb{Z}_2$-graded real semisimple Lie algebra has been given by Matsuki and Oshima [23], based on an earlier work by Matsuki [21].

Let us first consider the class of semisimple elements in $g_1$. Any semisimple element in $g_1$ belongs to a Cartan subspace in $g_1$.

Lemma 5.1 ([23]). Let $\tau_u$ be a $R$-compatible Cartan involution of a real $\mathbb{Z}_2$-graded semisimple Lie algebra $g$. Every Cartan subspace $\mathfrak{h} \subset g_1$ is $\text{Ad}_{G_0}$-conjugate to a Cartan subspace $\mathfrak{h}_{st}$ in $g_1$ which is invariant under the action of $\tau_u$.

A Cartan subspace $\mathfrak{h}_{st}$ in $g_1$ which is invariant under the action of $\tau_u$ is called a standard Cartan subspace. It is known that there are only finite number of standard Cartan subspaces, moreover there is algorithm to find them [23]. Let $g = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $g$ w.r.t. $\tau_u$. Then $\mathfrak{h}_{st} = (\mathfrak{h}_{st} \cap \mathfrak{k}) \oplus (\mathfrak{h}_{st} \cap \mathfrak{p})$. Denote by $K_0$ the connected Lie subgroup in $G_0$ with Lie algebra $\mathfrak{k}$. 
Proposition 5.2. Suppose that \( h, h' \in \mathfrak{h}_{st} \) are \( \text{Ad}_{G_0} \)-conjugate. Then they are \( \text{Ad}_{K_0} \)-conjugate.

Proof. We employ idea in [27] for our proof. Let \( h = h_t + h_p \) and \( h' = h_t' + h_p' \) be the decomposition of \( h \) and \( h' \) into elliptic and vector parts. Suppose that \( h = \text{Ad}_X(h') \), where \( X \in G_0 \). Since \( \text{Ad}_X \) does not change the eigenvalues, \( h_p = \text{Ad}_X(h'_p) \). Suppose that \( h_p \neq 0 \). We note that \( G_0 = \exp(\mathfrak{g}_0 \cap \mathfrak{p}) \cdot K_0 \), and \( \exp(\mathfrak{g}_0 \cap \mathfrak{p}) \subset \exp \mathfrak{iu}_0 \). Now suppose that \( X = A \cdot Y \) where \( Y \in K_0 \) and \( A \in \exp \mathfrak{iu}_0 \). Let \( y = \text{Ad}_Y h_p \in \sqrt{-1} \mathfrak{u}_1 \). Then \((\text{Ad}_A)\sqrt{-1} y = \sqrt{-1} h'_p = \tau_0(\text{Ad}_A \sqrt{-1} y) = \text{Ad}_A^{-1} \sqrt{-1} y \), so \( \text{Ad}_A^2 y = y \). If \( A \neq \text{Id} \) this implies that \( \text{Ad}_A \) has at least one eigenvalue \((-1)\), which contradicts the fact that \( \text{Ad}_A \) is a positive definite transformation. Hence \( A = \text{Id} \) and \( X = Y \in K_0 \subset G_0 \). This proves the first assertion, if \( h_p = 0 \) then \( h_t \neq 0 \) and we can apply the same argument to conclude that \( X \in K_0 \). \( \square \)

Since any semisimple element in \( \mathfrak{g}_1 \) is \( \text{Ad}_{G_0} \)-conjugate to an element in some standard Cartan subspace in \( \mathfrak{g}_1 \), using the Cartan theory of symmetric spaces, see e.g. [13], we get

Corollary 5.3. The set of \( \text{Ad}_{G_0} \)-conjugacy classes of semisimple elements in \( \mathfrak{g}_1 \) with pure imaginary or zero eigenvalues (elliptic semisimple elements) coincides with the quotient set of a Cartan subspace (maximal abelian subspace) \( \mathfrak{h}_{st} \subset (\mathfrak{g}_1 \cap \mathfrak{k}) \) under the action of the Weyl group of the \( \mathbb{Z}_2 \)-graded symmetric Lie algebra \( \mathfrak{e}_0 \oplus \mathfrak{k} \cap \mathfrak{g}_1 \). The set of \( \text{Ad}_{G_0} \)-conjugacy classes of real semisimple elements in \( \mathfrak{g}_1 \) coincides with the quotient set of a Cartan subspace (maximal abelian subspace) \( \mathfrak{h}_{st} \subset (\mathfrak{g}_1 \cap \mathfrak{p}) \) under the action of the Weyl group of the \( \mathbb{Z}_2 \)-graded symmetric Lie algebra \( \mathfrak{e}_0 \oplus \mathfrak{g}_1 \cap \mathfrak{p} \).

Now we want to define the conjugacy class of general semisimple elements \( h = h_t + h_p \in \mathfrak{h}_{st} \), where \( h_t \) is elliptic semisimple element and \( h_p \) is a real semisimple element in \( \mathfrak{h}_p \), moreover \([h_t, h_p] = 0\). Thus any semisimple element in \( \mathfrak{g}_1 \) admits a decomposition into sum of two commuting elliptic and real semisimple elements. Clearly this decomposition is unique.

By Corollary 5.3 \( h_t \) is conjugate to some element in a Cartan subspace \( \mathfrak{h}_{st} \subset \mathfrak{g}_1 \cap \mathfrak{p} \). Thus to classify all semisimple elements in \( \mathfrak{g}_1 \) it suffices to classify all semisimple elements in \( \mathfrak{g}_1 \) whose elliptic part is an element in \( \mathfrak{h}_{st} \).

Corollary 5.4. The set of \( \text{Ad}_{G_0} \)-equivalent elements \( h \) with given elliptic part \( h_t \in \mathfrak{h}_{st} \) coincides with the quotient set of a Cartan subspace in \( \mathbb{Z}_{\mathfrak{g}_1 \cap \mathfrak{p}}(h_t) \) under the action of the Weyl group of the \( \mathbb{Z}_2 \)-graded symmetric Lie algebra \( \mathbb{Z}_{\mathfrak{h}_0}(h_t) \oplus (\mathbb{Z}_{\mathfrak{g}_1 \cap \mathfrak{p}}(h_t)) \).

The following theorem describes the set of orbits a general mixed element in \( \mathfrak{g}_1 \). Recall that for an element \( e \in \mathfrak{g}_1 \) we denote by \( e_s + e_n \) its Jordan decomposition.

Theorem 5.5. Two elements \( e_s + e_n, e'_s + e'_n \in \mathfrak{g}_1 \) are in the same \( \text{Ad}_{G_0} \)-orbit, if and only if \( e_s \) and \( e'_s \) are in the same \( \text{Ad}_{G_0} \)-orbit and \( e_n \) and \( e'_n \) are in the same \( \mathbb{Z}_{\mathfrak{g}_0}(e_s) \)-orbit of the \( \mathbb{Z}_m \)-graded reductive Lie algebra \( \mathbb{Z}_{\mathfrak{g}}(e_s) \).
We note that $Ad_{ZG}(e_s)$ may disconnected, but it is a subgroup in the connected group $Ad_{ZG}(e_s)$ (by the Kostant theorem in [17]), so it seems possible to determine this subgroup.

**Remark 5.6.** We summarize our results in the following description of the set of the adjoint orbits in $g_1$. Any element in $g_1$ is $Ad_{G_0}$-conjugate to an element of the form $h_t + h_p + e_n$ such that

i) $h_t$ is an elliptic semisimple element in $h_{1n}$,

ii) $h_p$ is a real semisimple element, commuting with $h_t$,

iii) $e_n$ is a nilpotent element, commuting with $h_t + h_p$.

Furthermore, two elements $h_t + h_p + e_n$ and $h_t' + h_p' + e_n'$ are conjugate, only if $h_t$ is conjugate to $h_t'$ under the action of the associated Weyl group, see Corollary 5.3. Thus we can assume that $h_t = h_t'$. Two elements $h_t + h_p + e_n$ and $h_t + h_p' + e_n'$ are conjugate, only if $h_p$ and $h_p'$ are conjugate under the action of the associated Weyl group, see Corollary 5.4. Thus we can assume that $h_p = h_p'$. Finally, two elements $h_t + h_p + e_n$ and $h_t + h_p + e_n'$ are conjugate, if and only $e_n$ and $e_n'$ are in the same orbit of nilpotent elements of the associated $Z_{m}$-graded reductive Lie algebra, see Theorem 5.5. The classification of these nilpotent orbits can be obtained using the method in section 4.

We finish this section by showing the relation between the set of orbits on real (resp. complex) $Z_{m}$-graded Lie algebras and the $GL(8, \mathbb{R})$-orbit spaces (resp. the $GL(8, \mathbb{C})$-orbit space) of $k$-vectors and $k$-forms on $\mathbb{R}^8$ (resp. on $\mathbb{C}^8$). To find a classification of $k$-forms on $\mathbb{R}^8$ is an important problem in classical invariant theory. Many interesting applications in geometry, [11], [14], [20], are related to this classification problem. This problem motivates the author to write this note.

Kac observed that the orbit space of homogeneous elements of degree 1 in the $Z_3$-graded complex algebra $\mathfrak{e}_8$ (see example 3.3 vi) can be identified with the $SL(9, \mathbb{C})$-orbit space of 3-vectors on $\mathbb{C}^9$, and the orbit space of homogeneous elements of degree 1 in the $Z_2$-graded complex algebra $\mathfrak{e}_7$ (see example 3.3 ii) can be identified with the orbit space of 4-vectors in $\mathbb{C}^8$ [19]. In [12] Elashvili and Vinberg classified all homogeneous elements of degree 1 in the $Z_3$-graded Lie algebra $\mathfrak{e}_8$. They also observed that, all 3-vectors in $\mathbb{C}^k$, $k \leq 8$, can be considered as nilpotent elements of degree 1 in this $Z_3$-graded Lie algebra $\mathfrak{e}_8$, furthermore a classification of $GL(k, \mathbb{C})$-orbits on $\Lambda^3(\mathbb{C}^k)$ is equivalent to a classification of these homogeneous nilpotent elements. In [8], based on this remark, Djokovic classified all 3-vectors in $\mathbb{C}^8$ and $\mathbb{R}^8$. His classification is reduced to a classification of homogeneous nilpotent elements of degree 1 in a $Z$-graded Lie algebra $\mathfrak{e}_8$ (resp. $\mathfrak{e}_8(8)$). His method is close to our one (more precisely, our method is a generalization of his method), but he used a method of the Galois cohomology theory, first used by Revoy in [25], to compute the number of the open orbits in $Z$-graded $\mathfrak{e}_8(8)$. Djokovic used the Vinberg method of support to find a representative for each open orbit in $Z$-graded $\mathfrak{e}_8$.
A classification of $4$-vectors in $\mathbb{C}^8$ has been given by Antonyan in [1]. Using his classification and our method in this note it is possible to classify all $4$-vectors in $\mathbb{R}^8$, which is reduced to the classification of homogeneous elements of degree 1 in the $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{e}_7(7)$, (see example 3.3ii).

A classification of $SL(9, \mathbb{C})$-orbits of $3$-forms on $\mathbb{C}^9$ (resp. $SL(9, \mathbb{R})$-orbits on $\Lambda^3(\mathbb{R}^9)^*$) is equivalent to a classification of homogeneous elements of degree (-1) in the $\mathbb{Z}_3$-graded Lie algebra $\mathfrak{e}_8$ (resp. $\mathfrak{e}_{8(8)}$) [12]. By Corollary 3.5 this classification can be obtained from a classification of $3$-vectors on $\mathbb{C}^9$ (resp. on $\mathbb{R}^9$). In particular, a classification of $3$-forms on $\mathbb{R}^8$ can be obtained from the classification of $3$-vectors in $\mathbb{R}^8$ in [8].

We note that a classification of $GL(8, \mathbb{R})$-orbits on the space $\Lambda^k(\mathbb{R}^8)$ can be obtained easily from a classification of $SL(8, \mathbb{R})$-orbits on the same space.

Given a volume element $vol^* \in \Lambda^8(\mathbb{R}^8)^*$, there is a unique element $vol_* \in \Lambda^8(\mathbb{R}^8)$ such that $\langle vol^*, vol_* \rangle = 1$. Further there is a natural Poincare isomorphism $P_* : \Lambda^k(\mathbb{R}^8)^* \to \Lambda^{8-k}(\mathbb{R}^8)$, $\langle P_*(x), y \rangle = \langle x \wedge y, vol_* \rangle$, which commutes with the $SL(8, \mathbb{R})$-action.

Thus we can get a classification of all $k$-vectors and $k$-forms on $\mathbb{R}^8$ (resp. on $\mathbb{C}^8$) using the theory of real (resp. complex) $\mathbb{Z}_m$-graded semisimple Lie algebras.

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