On the Capacity of the Multiuser Vector Adder Channel

Alexey Frolov, Pavel Rybin and Victor Zyablov
Inst. for Information Transmission Problems
Russian Academy of Sciences
Moscow, Russia
Email: {alexey.frolov, prybin, zyablov}@iitp.ru

Abstract—We investigate the capacity of the \(Q\)-frequency \(S\)-user vector adder channel (channel with intensity information) introduced by Chang and Wolf. Both coordinated and uncoordinated types of transmission are considered. Asymptotic (under the conditions \(Q \to \infty\), \(S = \gamma Q\) and \(0 < \gamma < \infty\)) upper and lower bounds on the relative (per subchannel) capacity are derived. The lower bound for the coordinated case is shown to increase when \(\gamma\) grows. At the same time the relative capacity for the uncoordinated case is upper bounded by a constant.

I. INTRODUCTION

In [1] two multiuser channel models were introduced: the A-channel (or the channel without intensity information) and the B-channel (or the channel with intensity information). The capacity of the A-channel was investigated in [1], [2] for the case of coordinated transmission and in [3], [4], [6], [5], [7] for the case of uncoordinated transmission (the terminology is from [8], [9]). Note that the A-channel is in fact a vector disjunctive channel (OR channel) [10], [11].

In this paper we investigate the capacity of the B-channel. The B-channel is a noiseless multiuser vector adder channel. Let us denote the number of active users by \(S\), \(S \geq 2\). For a certain time instant \(\tau\) the channel inputs are binary vectors \(x_i^{(\tau)}\), \(i = 1, 2, \ldots, S\), of length \(Q\) (the number of frequencies or subchannels) and of weight 1 and the channel output at time instant \(\tau\) is given by an elementwise sum of vectors at input

\[
y^{(\tau)} = \sum_{i=1}^{S} x_i^{(\tau)}.\]

Note that the elements are added as real numbers.

The capacity of the B-channel for the coordinated case was investigated in [1] when \(Q\) is fixed and \(S \to \infty\). In this paper we are interested in the following asymptotics: \(Q \to \infty\), \(S = \gamma Q\) \((0 < \gamma < \infty)\). If we take the limit as \(Q \to \infty\), then the result of [1] corresponds to the case \(\gamma \to \infty\). We also investigate the asymptotic capacity of the B-channel for the uncoordinated transmission, i.e. the type of transmission in which a user transmits the information independently of other users. This fact allows us to consider another users as noise. An uncoordinated transmission is preferable for high-rate applications where a joint decoding is not possible for the complexity reasons.

Our contribution is as follows. Asymptotic (under the conditions \(Q \to \infty\), \(S = \gamma Q\) and \(0 < \gamma < \infty\)) upper and lower bounds on the relative (per subchannel) capacity are derived. The lower bound on the relative capacity for the coordinated case is shown to increase when \(\gamma\) grows. At the same time the relative capacity for the uncoordinated case is upper bounded by a constant. The comparison with the result for the A-channel is done.

II. COORDINATED TRANSMISSION

Let us consider the case of coordinated transmission first. An example of a multiple-access system with coordinated transmission for a binary adder channel is given in [12]. Uniquely decodable codes are the major element of the system. Note that the system requires symbol and block synchronizations.

Let us denote by \(X_i\) a vector sent by the \(i\)-th user \((i = 1, \ldots, S)\) at a certain time instant, by \(Y\) we denote the output of the channel at the time instant. The capacity (sum capacity) of the channel \(C_c\) for the coordinated transmission is defined as follows

\[
C_c(Q, S) = \max \{I(X_1, X_2, \ldots, X_S) ; Y)\} = \max \{H(Y) - H(Y|X_1, X_2, \ldots, X_S)\} = \max \{H(Y)\},
\]

where \(H(X)\) is the binary entropy of a random variable, the maximum is taken over all possible independent distributions of random variables \(X_1, X_2, \ldots, X_S\).

Since only the vectors of length \(Q\) with the sum of elements equal to \(S\) may be the channel outputs, then

\[
C_c(Q, S) \leq C_c^{(U)}(Q, S) = \log_2 \left(\frac{S + Q - 1}{S}\right), \tag{1}
\]

as the number of such vectors is equal to \((S+Q-1)/S\).

In what follows we are interested in such an asymptotics: \(Q \to \infty\), \(S = \gamma Q\) \((0 < \gamma < \infty)\). Let us introduce the notation of the asymptotic relative capacity

\[
c_c(\gamma) = \lim_{Q \to \infty} \{C_c(Q, \gamma Q)/Q\}.
\]

The existence of the limit and the convexity of the function \(c_c(\gamma)\) can be easily proved by corresponding frequency division (see [2]).
From (10) we obtain
\[ c_c(\gamma) \leq c_c^{(U)}(\gamma) = (\gamma + 1) \log_2(\gamma + 1) - \gamma \log_2(\gamma). \]

Now we derive a lower bound \( c_c^{(L)}(\gamma) \). In \( \Pi \) a formula for the entropy of the distribution at output \( H(Y) \) is obtained when all variables \( X_1, X_2, \ldots, X_S \) are distributed uniformly, i.e.
\[
P(X_i = j) = \frac{1}{Q}, \quad i = 1, \ldots, S, \ j = 1, \ldots, Q,
\]
and an asymptotics of the quantity if found when \( Q \) is fixed and \( S \to \infty \). If we take the limit as \( Q \to \infty \), we obtain that when \( \gamma \to \infty \)
\[
c_c^{(L)}(\gamma) \sim \frac{1}{2} \log_2(2\pi e\gamma).
\]

**Remark 1.** Here and in what follows by \( P(X_i = j) \) we mean \( P(X_i = x_j) \), where \( x_j \) is a binary vector of length \( Q \) with a single unit in the \( j \)-th position (the positions are enumerated from 1 to \( Q \)).

Let us consider the case when \( \gamma \) is finite.

**Theorem 1.** Let \( 0 < \gamma < \infty \), then
\[
c_c(\gamma) \geq c_c^{(L)}(\gamma) = \sum_{i=0}^{\infty} \frac{\gamma^i}{i!} e^{-\gamma} \log_2(i!) - \gamma \log_2\left(\frac{2}{e}\right).
\]

**Proof:** Let all the users use uniform distributions at input, then the probability to obtain the vector \( y = (y_1, y_2, \ldots, y_Q) \) at output of the channel can be calculated as follows
\[
p(y) = \left(\frac{1}{Q}\right)^S \cdot \frac{S!}{y_1! y_2! \cdots y_Q!} \left(\frac{1}{Q}\right)^S.
\]

Thus,
\[
C_c^{(L)}(Q,S) = H(Y) = -\sum_y [p(y) \log_2 p(y)]
\]
\[
= -\sum_y \left[p(y) \log_2 \left(\frac{S}{y_1, y_2, \ldots, y_Q} \left(\frac{1}{Q}\right)^S\right)\right]
\]
\[
= \sum_y \left[p(y) \sum_{j=1}^Q \{\log_2 (y_j!)}\right] - \log_2 \left(\frac{S!}{Q^S}\right)
\]
\[
= Q \sum_{j=1}^Q \left[p(y) \log_2 (y_j!}\right] - \log_2 \left(\frac{S!}{Q^S}\right)
\]
\[
= Q \sum_{i=0}^{S} \left[\left(\frac{S}{i}\right)^i \left(\frac{1}{Q}\right)^i (1 - \frac{1}{Q})^{S-i} \log_2 (i!}\right] - \log_2 \left(\frac{S!}{Q^S}\right),
\]
the last transition is done in accordance to Lemma 11 (see the appendix).

After dividing by \( Q \) and taking the limit as \( Q \to \infty \) we obtain the needed result.

In Fig. 1 the derived bounds \( c_c^{(L)}(\gamma) \) and \( c_c^{(U)}(\gamma) \) are shown. For the comparison we also added the lower bound on the relative capacity \( c_c^{(disj)}(\gamma) \) for the disjunctive channel (A-channel from \( \Pi \)). The last bound was derived in [2].

### III. UNCOORDINATED TRANSMISSION

Let us consider an uncoordinated transmission, i.e. the type of transmission where another users are considered as noise. The use of an uncoordinated transmission is preferable in the multiple-access systems with large number of active users with strict requirements to the transmission rate. An example of a multiple-access system with uncoordinated transmission for a disjunctive channel (OR channel) is given in [8] and for a vector disjunctive channel in [10], [11]. Note that block synchronization is no more required.

In what follows we only consider the case when all the users use the same distributions of input symbols, i.e.
\[
P(X_i = j) = p_j, \quad i = 1, \ldots, S, \ j = 1, \ldots, Q.
\]

Note that this constraint is very natural for the uncoordinated transmission.

The single-user capacity \( C_i \) for the \( i \)-th user can be calculated as follows
\[
C_i(Q,S) = \max \{ I(X_i;Y) \},
\]
where the maximum is taken over all the distributions \( X_i \).

Since all the users are "equal", then the capacity (sum capacity) \( C_{\text{uc}} \) for the uncoordinated case can be calculated as a sum of single-user capacities
\[
C_{\text{uc}}(Q,S) = \sum_{i=1}^S C_i(Q,S) = S \max_{p_1,p_2,\ldots,p_Q} \{ I(X;Y) \}.
\]

In the last equality we used the fact that all the users use the same input distributions. For the same reason we dropped out the index \( i \) in the notation of input \( X \).
Analogously to the case of coordinated transmission we introduce the notation $(Q \to \infty, S = \gamma Q)$

$$c_{uc}(\gamma) = \lim_{Q \to \infty} \{C_{uc}(Q, \gamma Q)/Q\}.$$ 

The proofs of the existence of the limit and of the convexity of the function $c_{uc}(\gamma)$ are little bit different here as all the users use the same distributions. We omit the proofs here.

A. Upper bound

It is clear that $C_{uc}(Q, S) \leq C_c(Q, S)$, then

$$C_{uc}(Q, S) \leq \log_2 \left(\frac{S + Q - 1}{S}\right). \tag{2}$$

Now we derive a stronger bound for large number of users.

**Theorem 2.** The inequality holds

$$C_{uc}(Q, S) \leq (Q - 1) \log_2 e = (Q - 1)1.4427...$$

**Proof:** Note that

$$p(y) = \sum_{y_1, y_2, \ldots, y_Q} \frac{p(y_1, y_2, \ldots, y_Q)}{p(y)}.$$ 

Thus,

$$I(X; Y) = \sum_x \sum_y \left[ p(x, y) \log_2 \left( \frac{p(y|x)}{p(y)} \right) \right]$$

$$= \sum_{j=1}^Q \sum_y \left[ p_j \sum_{y_1, \ldots, y_{j-1}, y_Q} p(y_1, \ldots, y_{j-1}, y_Q) \times p(y_1, \ldots, y_{j-1}, y_Q) \log_2 \left( \frac{y_j}{Sp_j} \right) \right]$$

$$= \sum_{j=1}^Q \sum_{i=0}^{S-1} \left[ p_j \sum_{y_1, \ldots, y_{j-1}, y_Q} \frac{S - 1}{i} p_j^i (1 - p_j)^{S-1-i} \times \log_2 \left( \frac{i + 1}{Sp_j} \right) \right], \tag{3}$$

the last transition is done in accordance to Lemma II (see the appendix).

Applying the inequality

$$\ln(1 + x) \leq x,$$

we obtain

$$SI(X; Y) \leq \log_2 e \sum_{j=1}^Q \sum_{i=0}^{S-1} \left( \frac{S - 1}{i} \right) p_j^i (1 - p_j)^{S-1-i} (i + 1)$$

$$= \log_2 e \sum_{j=1}^Q [(S - 1)p_j + 1 - Sp_j] = (Q - 1) \log_2 e,$$

this completes the proof. \hfill \square

From (2) and Theorem II we obtain such an upper bound

$$c_{uc}(\gamma) \leq c_{uc}^{unif}(\gamma) = \min \{ (\gamma + 1) \log_2 (Q + 1) - \gamma \log_2 \gamma, \log_2 e \}. \tag{4}$$

**Remark 2.** Sure the derived upper bound is not tight and can be improved. But already this rough bound shows that the quantity $c_{uc}(\gamma)$ is upper bounded by a constant.

B. Lower bound

In this section using several input distributions we obtain a lower bound on $c_{uc}$.

1) Uniform distribution: Let $p_1 = p_2 = \ldots = p_Q = 1/Q$.

**Statement 1.** Let $0 < \gamma < \infty$, then

$$c_{uc}(\gamma) \geq c_{uc}^{unif}(\gamma) = \gamma \sum_{i=0}^{\infty} \frac{\gamma^i}{i!} e^{-\gamma} \log_2 \left( \frac{i + 1}{\gamma} \right).$$

**Proof:** After substituting of the uniform distribution for (3) we obtain

$$C_{uc}^{unif}(Q, S)$$

$$= S \sum_{i=0}^{S-1} \left[ \frac{S - 1}{i} \frac{1}{Q^i} (1 - \frac{1}{Q})^{S-1-i} \log_2 \left( \frac{i + 1}{\gamma} \right) \right].$$

After dividing on $Q$ and taking the limit as $Q \to \infty$ we obtain the needed result. \hfill \square

The dependency $c_{uc}^{unif}(\gamma)$ is shown in Fig. 2. Let us introduce some notations

$$\gamma^* = \arg \max_\gamma \{ c_{uc}^{unif}(\gamma) \} = 1.3382 \ldots$$

$$c_{uc}^* = \max_\gamma \{ c_{uc}^{unif}(\gamma) \} = 0.8371 \ldots$$

Let us consider the case $\gamma \to \infty$.

**Statement 2.** The equality follows

$$\lim_{\gamma \to \infty} \{ c_{uc}^{unif}(\gamma) \} = \log_2 e \frac{2}{\gamma} = 0.7213 \ldots$$
Proof: We need to use Lemma 2 (see appendix).

2) Distorted distribution: Let $S \geq \gamma^*(Q-1)$, we introduce the distorted distribution as follows

$$
\begin{cases}
    p_1 = p_2 = \ldots = p_{Q-1} = \frac{\gamma^*}{S} \\
    p_Q = 1 - (Q-1) \frac{\gamma^*}{S}
\end{cases}
$$

Statement 3. Let $\gamma \geq \gamma^*$, then

$$
c_{uc}(\gamma) \geq c_{uc}^s.
$$

Proof: After substituting of the distorted distribution for $\gamma^*$ we obtain

$$
c_{uc}^{\text{distort}}(Q, S) = \gamma^*(Q-1) \sum_{i=0}^{S-1} \left( \frac{S-1}{i} \right) \left( \frac{\gamma^* S}{S} \right)^i \left( 1 - \frac{\gamma^* S}{S} \right)^{S-1-i} \log_2 \left( 1 + \frac{i+1}{\gamma^*} \right)
$$

After dividing on $Q$ and taking the limit as $Q \to \infty$, we obtain

$$
c_{uc}^{\text{distort}}(\gamma) = c_{uc}^* + f(\gamma),
$$

where

$$
f(\gamma) = \lim_{Q \to \infty} \left( \frac{\gamma - \gamma^* S}{S} \sum_{i=0}^{S-1} \left( \frac{S-1}{i} \right) \left( 1 - \frac{\gamma^* S}{S} \right)^{S-1-i} \log_2 \left( 1 + \frac{i+1}{\gamma^* S} \right) \right).
$$

In accordance to Lemma 2 (see appendix)

$$
f(\gamma) = \lim_{Q \to \infty} \left( \frac{\gamma - \gamma^*}{S} \frac{1}{2 \left( 1 - \frac{\gamma^* S}{S} \right) + \frac{1}{2}} \log_2 e \right) = 0.
$$

Thus we proved the following

Theorem 3. The inequality follows

$$
c_{uc}(\gamma) \geq c_{uc}^{(L)}(\gamma) = \left\{ \begin{array}{ll}
    c_{uc}^*(\gamma), & \gamma < \gamma^* \\
    c_{uc}^* = 0.8371, & \gamma \geq \gamma^*.
\end{array} \right.
$$

In Fig. 3 the derived bounds $c_{uc}^{(L)}(\gamma)$ and $c_{uc}^{(U)}(\gamma)$ are shown, as for the coordinated case we add a lower bound $c_{uc}^{(\text{dis})}(\gamma)$ on the capacity of the vector disjunctive channel (A-channel from [1]). The last bound is from [3], [4], [5], [6]. One can see that the relative capacity for the uncoordinated case is upper bounded by a constant. We also note that the gain in comparison to the A-channel is not big.

IV. CONCLUSION

Asymptotic (under the conditions $Q \to \infty$, $S = \gamma Q$ and $0 < \gamma < \infty$) upper and lower bounds on the relative (per subchannel) capacity are derived. The lower bound on the relative capacity for the coordinated case is shown to increase when $\gamma$ grows. At the same time the relative capacity for the uncoordinated case is upper bounded by a constant.

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Lemma 2. Let \( p_i \geq 0, \ i = 1, \ldots, Q, \sum_{i=1}^{Q} p_i = 1 \) and \( f(\cdot) \) be any function, then
\[
\sum_{m_1+\ldots+m_Q=S} \left[ \binom{S}{m_1, m_2, \ldots, m_Q} \right] p_{m_1}^{1} p_{m_2}^{2} \ldots p_{m_Q}^{m_Q} f(m_1) = \sum_{i=0}^{S} \left\{ \binom{S}{i} p_i (1 - p_1)^{S-i} f(i) \right\}.
\]

Proof:
\[
\sum_{m_1+\ldots+m_Q=S} \left[ \binom{S}{m_1, m_2, \ldots, m_Q} \right] p_{m_1}^{1} p_{m_2}^{2} \ldots p_{m_Q}^{m_Q} f(m_1)
\]
\[
= \sum_{i=0}^{S} \left\{ \binom{S}{i} p_i (1 - p_1)^{S-i} f(i) \right\}.
\]

Lemma 1. Let \( p_i \geq 0, \ i = 1, \ldots, Q, \sum_{i=1}^{Q} p_i = 1 \) and \( f(\cdot) \) be any function, then
\[
\sum_{m_1+\ldots+m_Q=S} \left[ \binom{S}{m_1, m_2, \ldots, m_Q} \right] p_{m_1}^{1} p_{m_2}^{2} \ldots p_{m_Q}^{m_Q} f(m_1) = \sum_{i=0}^{S} \left\{ \binom{S}{i} p_i (1 - p_1)^{S-i} f(i) \right\}.
\]

Proof: Let us consider the function
\[
G(p, N) = N \sum_{i=0}^{N} \left[ \binom{N}{i} p^i (1-p)^{N-i} \ln \left( \frac{i+1}{pN} \right) \right].
\]

Let \( \mu = pN, \varepsilon \) is an arbitrarily small positive value, let us divide the sum into three parts:
\[
G(p, N) = S_1 + S_2 + S_3,
\]
where
\[
S_1 = N \sum_{i=0}^{(1-\varepsilon)\mu} \left[ \binom{N}{i} p^i (1-p)^{N-i} \ln \left( \frac{i+1}{pN} \right) \right],
\]
\[
S_2 = N \sum_{i=(1-\varepsilon)\mu}^{(1+\varepsilon)\mu} \left[ \binom{N}{i} p^i (1-p)^{N-i} \ln \left( \frac{i+1}{pN} \right) \right],
\]
\[
S_3 = N \sum_{i=(1+\varepsilon)\mu}^{N} \left[ \binom{N}{i} p^i (1-p)^{N-i} \ln \left( \frac{i+1}{pN} \right) \right].
\]

In accordance to the Chernoff bound
\[
\lim_{N \to \infty} S_1 = \lim_{N \to \infty} S_3 = 0,
\]
and we only need to work with \( S_2 \).

Let us apply the following inequalities for logarithm \( -\varepsilon \leq x \leq \varepsilon \)
\[
L(x) = x - \frac{x^2}{2} - \frac{\varepsilon^3}{3(1-\varepsilon)} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{\varepsilon^3}{3} = \tilde{L}(x).
\]

Let us consider two functions
\[
\bar{S}_2(p, N) = N \sum_{i=(1-\varepsilon)\mu}^{(1+\varepsilon)\mu} \left[ \binom{N}{i} p^i (1-p)^{N-i} \tilde{L} \left( \frac{i+1}{pN} \right) \right]
\]
and
\[
\underline{S}_2(p, N) = N \sum_{i=(1-\varepsilon)\mu}^{(1+\varepsilon)\mu} \left[ \binom{N}{i} p^i (1-p)^{N-i} \tilde{L} \left( \frac{i+1}{pN} \right) \right],
\]
it is clear, that \( \bar{S}_2(p, N) \leq S_2(p, N) \leq \underline{S}_2(p, N) \).

Consider \( \lim_{N \to \infty} \underline{S}_2(p, N) \). It is easy to check, that
\[
\lim_{N \to \infty} \left\{ \left| \frac{1}{p} \right| \right\} = \left| \frac{1}{p} \right|,
\]
\[
\lim_{N \to \infty} \left\{ \left| \frac{1}{2} \right| \right\} = \left| \frac{1}{2} \right|,
\]
\[
\lim_{N \to \infty} \left\{ \left| \frac{1}{3} \right| \right\} = \left| \frac{1}{3} \right|
\]
Thus,
\[
\lim_{N \to \infty} \underline{S}_2(p, N) = \frac{1}{2p} + \frac{1}{2}.
\]

Similarly
\[
\lim_{N \to \infty} \bar{S}_2(p, N) = \frac{1}{2p} + \frac{1}{2} - \frac{\varepsilon^3}{3(1-\varepsilon)}.
\]
As \( \varepsilon \) can be chosen arbitrarily small, then
\[
\lim_{N \to \infty} S_2(p, N) = \frac{1}{2p} + \frac{1}{2}.
\]