Boundary time crystals engineered by long-range dissipation

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Dissipative time crystals can appear in spin systems, when the \( Z_2 \) symmetry of the Hamiltonian is broken by the environment, and the square of total spin operator \( S^2 \) is conserved. In this manuscript, we relax the latter condition and show that time-translation-symmetry breaking collective oscillations persist, in the thermodynamic limit, even in the absence of spin symmetry. We engineer an \( \text{ad hoc} \) Lindbladian using power-law decaying spin operators and show that time-translation symmetry breaking appears when the decay exponent obeys \( 0 < \eta \leq 1 \).

This model shows a surprisingly rich phase diagram, including the time-crystal phase as well as first-order, second-order, and continuous transitions of the fixed points. We study the phase diagram and the magnetization dynamics in the mean-field approximation, which we prove to be exact, in the thermodynamic limit, as the system does not develop sizable quantum fluctuations, up to the third order cumulant expansion.

I. INTRODUCTION

Spontaneous symmetry breaking is a cornerstone of physics occurring at the most diverse energy scales, from cosmology and high-energy physics to condensed matter, just to mention some relevant cases. Thermal or quantum fluctuations can drive a system into a state that breaks, in the thermodynamic limit, some of the symmetries present in its (thermo)dynamical potentials [1, 2]. Time-translational symmetry can also be spontaneously broken [3], as first conjectured by Wilczek, leading to the existence of time crystals (TCs). The mere definition of spontaneous breaking of time-translational invariance prompted an immediate and intense discussion [4–8]. Since then, the interest on the topic has grown enormously. A comprehensive review of this activity can be found in Refs. [9, 10].

Time-crystal ordering can occur only with nonlocal Hamiltonians [11] or under nonequilibrium conditions [12]. A key step in this direction has been achieved in Refs. [13, 14] where Floquet time crystals [13] were introduced. Here a unitary system, subject to an external periodic driving, has observables whose averages break the discrete time-translational symmetry imposed by the external drive. Floquet time crystals were intensively theoretically explored, see e.g. Refs. [15–21], and recently experimentally observed [22, 23].

The time-crystal phase can be also realized in many-body open quantum systems [24–39] where the competition between quantum driving and dissipation can give rise to persisting oscillations of a collective observable. For Markovian dynamics, the possibility of spontaneous time-translation symmetry breaking, in the steady state is embedded in the properties of the Lindbladian spectrum and its scaling with the system size [24, 25, 28, 32]. In the thermodynamic limit, the real part of the Lindbladian spectrum becomes gapless, giving rise to persistent oscillations related to the imaginary part of the corresponding eigenvalues. Dissipative time crystals can also be seen as cases in which only a macroscopic portion of a system (the boundary) undergoes symmetry-breaking when the remaining degrees of freedom (the bulk) acts as an effective nonequilibrium bath. We address these systems as \textit{boundary time crystals} (BTCs) as the boundary behaves effectively as a dissipative (or open) quantum system, possibly described by a Lindblad equation. The BTC phase has also been observed for \( Z_2 \)-symmetric generalized \((p, q)\)-spin models in the presence of collective dissipation [36], while the TC phase does not appear when the Hamiltonian is not parity invariant [40]. More recently, the origin of BTCs has been attributed to the Lindbladian steady-state being parity-time symmetric [41]. Most notably, in the case of open system dynamics it is possible to realize continuous time crystals, whose first experimental implementation has been reported in Ref. [42].

Despite the vibrant scientific activity surrounding this topic, many aspects of time crystals in open systems remain yet to be fully understood. The spontaneous generation of a collective periodic oscillation in classical dissipative systems has a long history. Examples in this sense are, for instance, the synchronization in the Kuramoto model [43, 44], the laser [45, 46], the salt oscillator [47] and, in some sense, also the Belousov-Zhabotinski reaction [48]. Moreover, already in the 90’s, Refs. [49, 50] provided an extensive analysis of when the generation of subharmonics can and cannot occur in the case of many-body classical dissipative driven systems (see also Ref. [51]). The great interest in time crystals has brought new examples of this sort [52–54], opening also the possibility for a deeper understanding of possible connection with these phenomena.

A very useful framework to understand time crystals in open systems is to link their existence to the emergence of decoherence-free subspaces [55, 56] in the thermodynamic limit. It is thus natural to expect that symmetries in the coupling to the external bath should play a major role. Indeed all spin models supporting dissipative TCs so far implicitly assume a collective (infinite-range) coupling to the external environment.

In this work we make one step further, and introduce a new class of long-range dissipators that spatially decay as a
power law and allow nevertheless a BTC phase. The resulting phase diagram is very rich as a function of the range of the dissipation and the coupling parameters, with both first-order, second-order, and continuous transitions in the fixed points, as well as a coexisting region.

BTC phases are believed to appear in spin models in the presence of two fundamental ingredients, that are (i) a $Z_2$ symmetry of the Hamiltonian that is explicitly broken by the environment [36], and (ii) “strong” rotational symmetry [57], which allows to decouple the eigenspaces of the total angular momentum operator. Here, “strong” means that both the Hamiltonian and the Lindblad operators are functions of collective spin operators [36], expressed as uniform sums of onsite spins. As a result, the square of the total angular momentum $\mathcal{S}^2 = \mathcal{S}_x^2 + \mathcal{S}_y^2 + \mathcal{S}_z^2$ is conserved at the operator level for any finite size.

In this paper we inquire the role of condition (ii). In particular, we break it and study if the boundary time crystal is still present. We focus on the Lindblad operators, and substitute the collective spins with power-law decaying operators. Our finding is that condition (ii) is not strictly necessary in order to have a boundary time crystal. In analogy with the Hamiltonian case [21, 58, 59] also here the time-crystal phase is supported up to a critical value of the exponent governing the power-law decay. Surprisingly, this is the first example of a phase transition engineered by tuning the range of dissipation.

The paper is organized as follows. In Sec. II, we introduce our model Lindbladian with power-law decaying Lindblad operators and discuss its general properties. In Sec. III, we resort to mean-field theory to derive the equations of motion of the magnetization components and we use them to study both the dynamics and the phase diagram of our model. Including quantum correlations in the dynamics, through a cumulant expansion, does not drastically affect the observables. We confirm this point in Sec. IV, where we discuss the dynamics of the magnetization at the third order in the cumulant expansion and show that correlations provide only a small correction to the mean-field dynamics for all dissipation ranges, in the thermodynamic limit. We finally summarize and comment on our results in Sec. V. Some technical aspects of our analysis are discussed in a number of dedicated Appendices.

II. MODEL AND PHASE DIAGRAM

Motivated by Refs. [60, 61] we consider a generalization of the model considered in Ref. [24] (see Appendix A). We consider a system of $N$ spin-1/2 particles described by Pauli matrices $\sigma_\alpha^\alpha$ with $\alpha \in \{x,y,z\}$. The dynamics is governed by a Lindblad equation

$$\dot{\rho} = -i[H, \rho] + \gamma \sum_{i=1}^{N} \left( L_i(\eta) \rho L_i^\dagger(\eta) - \frac{1}{2} \left[ L_i^\dagger(\eta) L_i(\eta), \rho \right] \right).$$

We take the same Hamiltonian as in Ref. [24], $H = 2J \mathcal{S}_z$ with

$$\mathcal{S}_\alpha = \frac{1}{2} \sum_{i=1}^{N} \sigma_\alpha^\alpha,$$

and the following Lindblad operators,

$$L_i(\eta) = \sum_{j=1}^{N} f_{ij}(\eta) \sigma_\alpha^\dagger \quad \text{and} \quad f_{ij}(\eta) = \frac{K^{(N)}(\eta)}{D(|i-j|)} \eta,$$

where $\sigma_\alpha^\pm = (\sigma_\alpha^x \pm i \sigma_\alpha^y)/2$, $\eta \in [0, \infty)$ is the power-law exponent, and $D(r)$ is a distance function between lattice sites defined as $D(r) = \min(r, N - r) + 1$ in order to provide periodic boundary conditions [62]. The Kac normalization factor $K^{(N)}(\eta)$ ensures that $\sum_{i=1}^{N} f_{ij}(\eta) = 1$ [63] and, for this choice of $D(r)$, it can be computed analytically for any $\eta$ (see Appendix B for details). Notice that $f_{ij}(\eta) = f_{i-j}(\eta)$.

These operators are sums of local spins which are not uniform, but have a coefficient decaying with distance as a power law, with exponent $\eta$. In this way, for any finite size and any $\eta > 0$, $S^2$ is not conserved, although the model is still symmetric under all the possible site permutations. In fact, not only is the Hamiltonian $H$ permutation invariant, but also the Lindblad operators preserve the permutation invariance of the model for any given $\eta$, despite breaking the conservation of $S^2$. The starting state has to be symmetric under all spin permutations. This symmetry is crucial for the forthcoming discussion.

We remark that, in order to have a permutation symmetric system, in this work we consider an all-to-all connected model where, due to the graph topology, the distance between any given spin pair is one bond. However, we label the spins using indices in $\{1,2,\ldots,N\}$ and use them to evaluate the distance function $D(r)$ similarly to what is done in unitary systems with Hamiltonians with power-law interactions, see Refs. [64–68] for some recent studies.

Let us briefly mention two limiting cases. For $\eta = 0$, the so-called infinite-range case, we have that

$$f_{ij}(0) = \frac{1}{N}, \quad L_i(0) = \frac{S_+}{N} \quad \forall i,j :$$

here, the Lindbladian of Eq. (1) coincides with the collective one of Eq. (A1). In the opposite, zero-range limit we have that

$$f_{ij}(\eta \to \infty) = \delta_{ij}, \quad L_i(\eta \to \infty) = \sigma_\alpha^+:$$

the corresponding Lindbladian acts independently on each spin, thus the zero-range Lindbladian cannot generate correlations during the open-system evolution. While, in most cases, collective and independent baths give rise only to quantitative changes in the physics [69, 70], in our model the two limiting cases are qualitatively different.

The Lindbladian of Eq. (1) interpolates between the infinite-range one of Eq. (A1) and a local Lindbladian where each spin is independently coupled to its own environment via $\sigma_\alpha^\dagger$. For every choice of $\eta$, the Lindblad operators act by orienting the spins in the $z$-direction, thus breaking the $Z_2$ symmetry as required. For the two limiting cases, $\eta = 0$ and $\eta = \infty$, correlations are negligible: here, the mean-field approximation is (for different reasons) exact. For intermediate values of $\eta$ less is known. In the following, we are going to show that quantum correlations are always negligible and that the
The mean-field equations of motion for the magnetization can be derived analytically by only exploiting the model’s permutation invariance, see Appendix C. In the thermodynamic limit, they read

\begin{align*}
\dot{m}_x &= -\frac{\eta}{2} m_x F_\eta - \frac{\gamma}{2} m_x m_z (1 - F_\eta); \\
\dot{m}_y &= 2 J m_z - \frac{\eta}{2} m_y F_\eta - \frac{\gamma}{2} m_y m_z (1 - F_\eta); \\
\dot{m}_z &= -2 J m_y + \gamma (1 - m_z) F_\eta + \frac{\gamma}{2} (m_x^2 + m_y^2) (1 - F_\eta),
\end{align*}

where, considering Eq. (2), we have defined the magnetization components as

\begin{align*}
m_x &= m \cos \theta, \\
m_y &= m \sin \theta, \\
m_z &= m.
\end{align*}

In this section we discuss the results obtained with the mean-field approximation, summarized in the phase diagram of Fig. 1, where the parameters are the exponent \(\eta\) and the normalized dissipation strength \(\gamma\).
components as the expectations of the total spin components

\[ m_\alpha(t) = 2 \lim_{N \to \infty} \frac{\text{Tr}[\rho(t)S_\alpha]}{N} \quad (\alpha \in \{x, y, z\}) \]  

(6)

and omitted time-dependences for shortness. The coefficient \( F_\eta \) is given by

\[ F_\eta = \lim_{N \to \infty} \frac{1}{N} \sum_{i,j=1}^{N} \zeta_i^2(\eta) = \begin{cases} \frac{2\zeta(2\eta) - 1}{[2\zeta(\eta) - 1]^2}, & \eta > 1, \\ 0, & 0 \leq \eta \leq 1. \end{cases} \]  

(7)

where \( \zeta(\cdot) \) is Riemann’s Zeta function. We plot \( F_\eta \) versus \( \eta \) in Fig. 2. We also plot the functions \( F^{(N)}(\eta) \) for finite values of \( N \).

The highlighted light-blue range \( 0 \leq \eta \leq 1 \) in Fig. 2 marks the parameter region where, in the thermodynamic limit, \( F_\eta = 0 \). This is the so-called long-range regime: here, the power-law Lindblad operators do not affect the thermodynamic limit behavior of the coefficients of local and collective pump. In this case, since \( F_\eta = 0 \), the equations of motion [Eqs. (5)] are exactly the same as the ones derived in Ref. [24] for the Lindbladian of Eq. (A1). Thus, the boundary time crystal phase is not affected by the power-law decay of the jump operators in the long-range regime and exists for all values of \( 0 \leq \eta \leq 1 \), as one can see in Fig. 1. Here, the two quantities

\[ N = m^2_x + m^2_y + m^2_z, \quad M = \frac{m_x}{m_y - 1/\chi} \]  

(8)

are constants of motion, even if \([L_i(\eta), S^2] \neq 0\).

In the next two subsections we are going to see how the properties of the system change by moving away from this range of parameters. In Sec. III.1 we consider the situation from the point of view of the fixed points of the dynamics, and in Sec. III.2 from the point of view of the dynamics.

### III.1. Fixed points

In order to find the fixed points, we impose \( \dot{m}_x = \dot{m}_y = \dot{m}_z = 0 \) in Eq. (5). Let us start by considering the situation in the long-range regime \( \eta < 1 \). In this case, the fixed-point solution is

\[ m_x = 0; \quad m_y = \frac{1}{\chi}; \quad m_z = \frac{\sqrt{\chi^2 - 1}}{\chi} \quad \text{if } \chi \geq 1, \]

\[ m_x = \sqrt{1 - \chi^2}; \quad m_y = \chi; \quad m_z = 0 \quad \text{if } \chi < 1. \]  

(9)

At \( \chi = 1 \), there is a second-order phase transition between the BTC phase (\( \chi < 1 \)) and the ordered ferromagnetic phase (\( \chi > 1 \)), because one can see a discontinuity of \( \partial_\eta m_z \) at \( \chi = 1 \). We notice also that at \( \chi = 1 \) the nature of the fixed point changes, from an elliptic fixed point around which the trajectories fall asymptotically, to a stable attractive fixed point onto which the trajectories fall asymptotically.

Let us now move to another simple limit, \( \eta \to \infty \) (zero-range), which has been thoroughly analyzed in Ref. [74]. In this limit, the Hamiltonian and the Lindblad operators act independently on single spins, thus correlations never build up in the system and the equations of motion for \( m_\alpha \) can be written down exactly without even resorting to the mean-field approximation. The resulting equations for the expectation values are given by Eq. (5) with \( F_\eta = 1 \) and the steady-state solution is

\[ m_x = 0; \quad m_y = \frac{2\chi}{2\chi^2 + 1}; \quad m_z = \frac{2\chi^2}{2\chi^2 + 1}. \]  

(10)

Here there are no critical points nor conserved quantities. We find that a qualitatively similar picture characterized by the absence of a phase transition holds whenever \( \eta \geq 1.625 \): in this short-range regime, the dynamics are mostly controlled by local pump processes.

When one crosses the \( \eta = 1 \) line from \( \eta < 1 \) to \( \eta > 1 \), the fixed point changes in a nonanalytic way and features therefore a phase transition. Crossing this line, there is a sudden change in the equations of motion for \( m_\alpha \), due to the fact that for \( \eta > 1 \) the two quantities in Eq. (8) are no longer conserved and the BTC phase is destroyed. For \( \eta > 1 \) the fixed-point solutions to Eqs. (5) are found from the following system of equations:

\[
\begin{align*}
\frac{m_x}{m_y - 1/\chi} &= m_z, \\
\frac{m_y}{\chi[F_\eta + m_z(1 - F_\eta)]} &= \lambda m_z, \\
\frac{m_z^3 - (1 - 2\lambda)m_z^2 - \lambda[2 - \lambda(1 + \frac{1}{2\chi^2 F_\eta^2})]}{m_z} &= 0, \\
\lambda^2 &= \prod_{k=1}^{3}(m_z - m_z^{(k)}) = 0,
\end{align*}
\]  

(11)

where \( \lambda = F_\eta/(1 - F_\eta) \). Since, in this range of \( \eta \), \( \lambda \neq 0 \), we find that \( m_z^{(k)} = 0 \) is never a solution to these equations.

A special regime occurs when \( 1 < \eta < 1.625 \). In this regime Eqs. (11) can either have one or three real solutions depending on the value of \( \chi \). Small values of \( \chi \) result in a unique gas-like steady state, while large values of \( \chi \) lead to a unique liquid-like steady state. In between the gas-like and liquid-like phases, there is a coexistence region, shown in Fig. 1 with a dashed filling pattern, where there are three real
solutions to Eqs. (11). The analysis of the stability of these fixed points shows that one of them is always unstable and repulsive, whereas the remaining two are stable and attractive. One of these solutions, \( m_z^{(1)} \), corresponds to a steady state with large total magnetization \( N \approx 1 \), while the other one, \( m_z^{(2)} \), has a small total magnetization \( N \approx 0 \). We show an example of the fixed-point \( m_z \) versus \( \chi \) for \( \eta = 1.2 \) in Fig. 3(a). The shaded area shows the coexistence of two stable fixed points, denoted by the solid lines. The dotted line represents the third, unstable fixed point. This situation corresponds to a first-order phase transition, because \( m_z \) changes discontinuously as a function of \( \chi \).

This coexistence situation disappears for \( \eta > 1.625 \), where the equation for the fixed-point \( m_z \) always has two complex roots and a real one, the latter corresponding to a stable fixed point. The magnetization curve is therefore continuous and regular without phase transition and coexistence regime [see Fig. 3(c) for \( \eta = 2 \)]. Exactly at \( \eta = \eta_C = 1.625 \), the magnetization curve is continuous and regular but at \( \chi = \chi_C = 1.225 \), where it has a vertical-tangent flex, as shown in Fig. 3(b). So, there is a discontinuity in \( \partial \chi m_z \), corresponding to the critical point marked as \( C \) in Fig. 1. So, decreasing \( \eta \), one has first one real stable and two complex solutions of Eq. (11) for all \( \chi \), then a value of \( \chi \) with three coinciding real solutions (the critical point), and then an interval of \( \chi \) where there are two real stable solution and a real unstable one (coexistence regime). This phenomenon is standard in nonlinear dynamics and is known as asymmetric pitchfork bifurcation [75].

Let us now comment on how the fixed points change when the \( \eta = 1 \) line is crossed. For \( \chi > 1 \) they change in a discontinuous way, featuring a first-order transition. For \( \chi < 1 \) the transition is between the BTC phase (\( \eta < 1 \)) and the gas-like phase (\( \eta > 1 \)) and is a continuous \( \chi \)-order transition for the fixed point. Indeed, crossing the \( \eta = 1 \) line, the fixed-point \( m_z \) is infinitely differentiable with continuous derivatives but is not analytic. This can be seen in Fig. 4, where we plot the fixed-point magnetization and its first and second derivative as a function of \( \eta \) for \( \chi = 0.5 \). We see that \( m_z \) is well fitted around \( \eta = 1 \) by a function of the form

\[
 m_z(\eta) = a e^{-\frac{\eta}{b-\eta C}}, \tag{12}
\]

where \( a, b \) and \( c \) are fitting parameters.

In the next subsection we discuss the dynamics around these fixed points, leading to persistent oscillations in the BTC phase, and to relaxation to an attractive fixed point otherwise, with interesting observations when two attractive fixed points coexist.

### III.2. Dynamics

**BTC phase.** Let us start by considering the BTC phase. Here the time crystal supported by the long-range power-law Lindblad operators with \( \eta \leq 1 \) is the same as the one studied in Ref. [24], i.e., a mean-field semiclassical time crystal. We can see some time traces for \( \eta = 0.5 \) in Figs. 5(a–c). Fig. 5(a) corresponds to the time-crystal phase, and one can

![Figure 3](image1.png)

**FIG. 3.** Steady-state magnetization as a function of \( \chi \). (a): \( \eta = 1.2 \); (b): \( \eta = \eta_C = 1.625 \); (c): \( \eta = 2 \). For \( \eta < \eta_C \), we observe the coexistence of two stable solutions in the shaded area (solid red lines) and a single unstable one (dotted red line). For \( \eta = \eta_C \), the derivative \( \partial \chi m_z \) is infinite at \( \chi = \chi_C = 1.225 \).

![Figure 4](image2.png)

**FIG. 4.** Magnetization \( m_z \) and its derivatives as a function of \( \eta \) for \( \chi = 0.5 \). The curve for \( m_z \) is well described by the function in Eq. (12), which is nonanalytic in \( \eta = 1 \) (best-fit parameters: \( a = 2.5 \), \( b = 4.4 \), \( c = 0.66 \)).
FIG. 5. Dynamics of the magnetization components, $m_{\alpha r}$, and of the two functions $M$ and $N$ described in the main text, for several choices of the exponent $\eta$ and the dissipation rate $\chi$. Top row: $\eta = 0.5$; center row: $\eta = 1.4$; bottom row: $\eta = 2.0$. Left column: $\chi = 0.7$; center column: $\chi = 1.0$; right column: $\chi = 1.3$. In panel (a), highlighted with a bold red frame, we observe a BTC phase. For $\eta = 0.5$, $M$ and $N$ are conserved quantities. Technical details: dynamics numerically performed with implicit backward differentiation formulas [76], initial condition $m_{\alpha r}(0) = 1/N^3\varphi$.

FIG. 6. Decay rate of the amplitude of $m_z(t)$ for $\eta = 1.1$ and $\chi = 0.7$.

see the persisting oscillations of the magnetization components. Figs. 5(b, c) correspond to the symmetry-broken phase for $\chi > 1$, and one can see the magnetization components reaching the asymptotic value given by Eq. (9). In all these cases, $M$ and $N$ are conserved.

Short-range regime. Some examples of dynamics in the short-range regime $\eta > 1$ are shown in Figs. 5(d–i). In Figs. 5(d, e) we see oscillations decaying to a value close to zero, in Fig. 5(f) the magnetization relaxes to an asymptotic value in the liquid phase. Both $M$ and $N$ [see Eq. (8)] are not conserved and decay. For $\eta = 2$, the oscillations of the magnetization are only seen in the gas phase [Fig. 5(g)] but they quickly decay in time, whereas in the liquid phase the magnetization immediately relaxes [Figs. 5(h, i)].

In summary, the BTC exists only in the region of the phase diagram given by $0 \leq \eta \leq 1$, $0 \leq \chi < 1$. The amplitude of oscillation of $m_z$ at infinite times has a finite value in this region due to the persistent BTC oscillations, and are zero everywhere else, with a finite discontinuity at the boundary. So, while the fixed point changes in a continuous way across the boundaries of the BTC region, the amplitude of the BTC oscillations changes discontinuously, featuring a first-order transition.

Focusing on the gas phase, for $\chi = 0.7$, we see that the magnetization immediately starts to oscillate and the oscillations decay in time [Fig. 5(d)]. We checked that the oscillations exponentially decay in time, with the amplitude of the envelope following the law $A(t) = A_0 \exp[-B(\eta)Jt]$. For $\eta \to 1^+$, the decay rate $B(\eta) = 0.7(\eta - 1)^2$ as shown in Fig. 6, hinting to long-lived oscillations near the BTC phase. For $\eta = 1$, one falls in the BTC phase, the oscillations are persistent, and $B(1) = 0$.

In order to better understand the short-range regime, let us consider some phase space portraits of the dynamics in the

FIG. 7. Phase-space flow portraits in the plane $m_x = 0$ for $\eta = 2$. Panel (a): $\chi = 0.7$; panel (b): $\chi = 1.3$. The fixed points are marked in blue. The shaded area corresponds to the region where $N \leq 1$. 
plane $m_\perp = 0$. For small values of $\chi$, the steady-state magnetization is close to zero and the system falls to the fixed point following rapidly decaying oscillations, signaled by spiraling orbits in the phase space portraits in the plane $m_\perp = 0$ [Fig. 7(a), compare with Fig. 5(g)]. This is the gas-like phase shown in the leftmost part of Fig. 1. Instead, for larger values of $\chi$ we are in the liquid-like phase (see Fig. 1), and the system converges to a fixed point with large $m_\perp$ with no oscillations [Fig. 7(b), compare with Fig. 5(i)].

Coexistence region. Some interesting dynamical behaviors occur when $\eta$ is close to one, inside the coexistence region of the phase diagram. Here, there is a regime where the magnetization starts oscillating around $m_\perp = 0$ after a delay. As an example, here we discuss the case of $\eta = 1.1$. For three choices of $\chi$, we numerically solve Eqs. (5) and plot the time evolution of $m_z(t)$ in Figs. 8(a–c).

For $1 < \chi \leq 1.4$, $m_z(t)$ starts to oscillate at a later time. The time at which oscillations start becomes longer with $\chi$ approaching 1.4. It also becomes longer for $\eta \to 1^+$ since, for $\eta = 1$, the system goes back to the infinite range case where, for $1 < \chi \leq 1.4$, the system is already in the magnetized phase and no oscillations are visible in $m_z(t)$. This behavior is shown in Figs. 8(b) for $\chi = 1.3$. Right after the critical point, the oscillations in $m_z(t)$ disappear completely, see Fig. 8(c) where we show the case of $\chi = 1.408$. In all cases, there is no trace of the persistent oscillations indicating a BTC phase, at least at the mean-field level of the analysis thus far. The endpoint of this region tends to $\chi = \sqrt{2}$ when $\eta \to 1^+$.

In order to explain the transient seen in Fig. 8, we have to consider that we are in the coexistence regime, where there are two stable fixed points (and an unstable one). Each of the stable fixed points has its own basin of attraction, and the dynamics eventually reaches one or another according to the chosen initial conditions. By studying the phase-space portraits in this regime, we see that, if the starting state has a total magnetization $N$ [Eq. (8)] above a certain threshold, the fixed point with $N \approx 0$ is never reached. We can see this in the phase-space portrait shown in Fig. 9 for $\eta = 1.1$ and $\chi = 2$ in the plane $m_\perp = 0$. The system converges to the fixed point $m_z^{(1)}$ (black line) if the starting state falls within the basin of attraction of this point, i.e., if the starting total magnetization is larger than $N = 0.5$. Instead, if $N < 0.5$, the system spirals towards the fixed point with $N \approx 0$.

The small-$N$ fixed point is continuously connected to the one in Fig. 7(a), and the spiraling dynamics of the trajectories are similar. The large-$N$ fixed point is continuously connected to the one in Fig. 7(b) and trajectories converge to it without spiraling. The coexistence of fixed points with these properties of the trajectories converging to them explains the results shown in Fig. 8. Here the system is initialized inside the basin of attraction of the small-$N$ fixed point, and near the boundary between the two attraction basins. So, there is a transient where the system behaves as it was converging to the large-$N$ fixed point, and then the trajectory spirals around the small-$N$ fixed point and converges to it.

IV. QUANTUM CORRELATIONS: THIRD-ORDER CUMULANT EXPANSION

Eqs. (5) are obtained performing the mean-field approximation, which is equivalent to neglect quantum fluctuations $\langle \sigma^\alpha_i \sigma^\beta_j \rangle \approx \langle \sigma^\alpha_i \rangle \langle \sigma^\beta_j \rangle$, $\forall j, l = 1, \ldots, N$ with $j \neq l$, and $\alpha, \beta = x, y, z$ [77]. Here we write $\langle \cdots \rangle_t \equiv \text{Tr}[\rho(t)\cdots]$ for brevity. This is a strong approximation, because from a physical point of view it is equivalent to state that the density matrix $\rho(t)$ gives rise to distributions of measurement outcomes of $\sigma^\beta_j$ with no fluctuations. This is a good approx-
imimation in the case of $\eta = 0$ (see Ref. [34] and Appendix A, as well as in the case $\eta \to \infty$ (see Ref. [74]). By contrast its validity is not known for intermediate values of $\eta$.

In order to clarify the relevance of quantum fluctuations, we apply a third order cumulant expansion [78]. In the mean field approximation one considers the distribution of the measurement outcomes of $\sigma_i^\alpha$ and imposes that the nontrivial second cumulants are vanishing, $C_2(\alpha, \beta, t) \equiv \langle \sigma_i^\alpha \sigma_i^\beta \rangle_t - \langle \sigma_i^\alpha \rangle_t \langle \sigma_i^\beta \rangle_t = 0$, with $j \neq l$ [77]. Notice that $C_2$ is independent of $j$ and $l$ due to invariance under permutations. The mean-field Eqs. (5) can be obtained by the exact equations of motion [see Eqs. (C5)] requiring that the second cumulant of the probability distributions of the magnetization components are zero. In the third-order cumulant expansion, instead, one imposes that the nontrivial third cumulants (and all the next ones) are vanishing, that is to say

$$C_3(\alpha, \beta, \gamma, t) = \langle \sigma_i^\alpha \sigma_j^\beta \sigma_k^\gamma \rangle_t - \langle \sigma_i^\alpha \sigma_j^\beta \rangle_t \langle \sigma_k^\gamma \rangle_t - \langle \sigma_i^\alpha \sigma_k^\gamma \rangle_t \langle \sigma_j^\beta \rangle_t + 2 \langle \sigma_i^\alpha \rangle_t \langle \sigma_j^\beta \rangle_t \langle \sigma_k^\gamma \rangle_t = 0. \quad (13)$$

This is the so-called Gaussian approximation, because one assumes that in the distribution of the measurement outcomes of $\sigma_i^\alpha$ only the first two cumulants are nonvanishing, as appropriate for Gaussian distributions.

Assuming the Gaussian approximation Eq. (13), the equations of motions are more involved and include also the correlations. Thanks to permutation invariance, this ODE system has nine equations, as shown in Appendix D [see Eqs. (D3)].

Here we prepare the system in the uncorrelated state with $m_z(0) = 1$, thus the second cumulant is vanishing. Letting the system evolve with the Gaussian-approximation equations, it has the freedom to develop nonvanishing second-order cumulants, and then get nontrivial quantum correlations. Whatever are the considered values of $\chi$ and $\eta$, we see that these second-order cumulants stay smaller than $10^{-3}$, as we show in Fig. 10, where we plot some examples of evolution of $\Delta_z(t) = C_2(z, z, t)$. In this figure we focus on the case of $\eta = 1.1$ as an example of short range, but we have checked that the results discussed here are representative of all other values of $\eta$.

The results obtained with the mean field are almost unchanged when quantum fluctuations are considered, because these fluctuations are vanishing small even if the system is allowed to develop them. We emphasize that this is a numerical observation, and further studies are needed to understand it from the analytical point of view.

Notice that the vanishing of the second-order cumulant occurs only in the thermodynamic limit. If we consider a finite-size system, we see that order-1 second-order cumulants (and the associated quantum fluctuations) develop. We show this in Fig. 11 for the case of $\eta = 1.1$ and $\chi = 0.7$. In the inset we use the Gaussian approximation at finite system size $N$ [see Eq. (D1)], where we keep terms proportional to $1/N$ to take into account finite-size effects, and evaluate the maximum over the dynamics of $\Delta_z(t)$. Plotting this quantity $\max_\Delta$ versus $N$ we see that it converges to the $N \to \infty$ value for large system sizes, but at small sizes it is order 1. So, at finite size, the Gaussian approximation is bad and further orders in the cluster expansion are needed. In the main panel of Fig. 11 we plot $\Delta_z(t)$ versus $t$ obtained by numerically integrating the Lindblad Eq. (1) for small $N$. We see that also in absence of any approximation $\Delta_z(t)$ reaches values of order 1 at finite size.

V. CONCLUSIONS

In conclusion, we have introduced a new class of long-range dissipative models that support a time-crystal phase, where the Lindblad operators decay as a power law with exponent $\eta$. The steady-state phase diagram of the model we considered appears quite rich with different transition lines and a coexistence phase. The form of long-range dissipation has been motivated by a recent proposal [60, 61] and can be implemented with Rydberg atoms in cavities.
In this model, as opposed to the one proposed in Ref. [24], the square of the collective spin operator is no longer conserved. Nevertheless, studying the dynamics of the total magnetization, we see that the time-translation symmetry breaking oscillations persist if \(0 < \eta \leq 1\) (long-range regime). In this regime there is a transition between a small-\(\chi\) time-translation symmetry breaking phase and a large-\(\chi\) phase with no time crystal, where the magnetization attains an asymptotic finite value. For \(0 < \eta \leq 1\) the transition point in \(\chi\) is independent of \(\eta\). So, in this regime, the thermodynamic-limit dynamics is independent of \(\eta\), similarly to what happens in Hamiltonian models with long-range interaction. Remarkably, although the square of the collective spin operator is not conserved for any finite size, its expectation in the thermodynamic limit (the square of the total magnetization) is conserved.

Outside this range of parameters, the system reaches an asymptotic steady state. Here, an interesting regime occurs for \(1 < \eta \leq 1.625\), where the system shows a first-order phase transition line (a discontinuity) of the \(z\)-magnetization in the \(\chi-\eta\) plane and there is a region where two stable steady states coexist. This phase transition line terminates with a critical point, after which the magnetization changes in an analytic way. The way the first-order phase transition develops corresponds to an asymmetric pitchfork bifurcation.

In order to study the thermodynamic-limit dynamics of the total magnetization, we use the mean-field approximation. That means that we impose zero quantum fluctuations in the Ehrenfest equations of the expectation of the magnetization. From a mathematical point of view, this is equivalent to impose that the distribution of the outcomes of the measurements of the spin components on the wave function has a vanishing second cumulant (that is to say no fluctuations). It was already known that this approximation is exact in the thermodynamic limit for \(\eta = 0 \) and \(\eta \to \infty\) [74]. Here we numerically prove that it is exact, up to the third cumulant, also for \(1 < \eta < \infty\).

In order to do that, we study the Ehrenfest equations beyond the mean-field approximation: imposing that the third-order cumulants are vanishing. So, we approximate that the distribution of the outcomes of the measurements of the spin components on the wave function has a vanishing second cumulant (with the associated fluctuations and correlations), but we numerically observe that it never does in the thermodynamic limit. This supports the correctness of the mean-field picture. We emphasize that this is true only in the thermodynamic limit. When we consider finite-size cases, the dynamics can build up nonvanishing second-order cumulants.

Perspectives of future work include to better understand analytically this absence of second-order cumulants, that we verify here only numerically. Moreover, it will be interesting to break the full permutation symmetry of the model (for instance applying disorder to the Lindbladians), in order to see if the time-crystal phase is stable to this further reduction of symmetry. Finally, we plan to apply the long-range dissipation to a model displaying quantum chaos with an infinite-range dissipator [79]. Our aim is to understand how the chaotic properties change, and if the mean-field approximation is still valid in the thermodynamic limit when there is chaos.

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**Appendix A: Collective spins model**

The simplest model providing a boundary time crystal phase is given by the following Lindbladian, first proposed and studied in Ref. [24]:

\[
\dot{\rho} = -i [2J S_x, \rho] + \frac{\gamma}{N} \left( S_+ \rho S_- - \frac{1}{2} \{ S_+ S_-, \rho \} \right),
\] (A1)

where \(S_\alpha = \sum_{i=1}^{N} \sigma_\alpha^i / 2\) with \(\alpha = \{x, y, z\}\) are collective spin operators with algebra \([S_\alpha, S_\beta] = i \epsilon_{\alpha \beta \gamma} S_\gamma\). \(S_\pm = S_x \pm i S_y\), and \(S\) is the total spin. The \(\sigma_\alpha^i\) are Pauli matrices and \(\sigma_\alpha^z = (\sigma_\alpha^z \pm i \sigma_\alpha^y) / 2\). The number of particles is \(N\), and the components of the magnetization are defined as \(m_\alpha = 2 \langle S_\alpha \rangle / N\), where the average is taken over the density operator \(\rho\).

The Hamiltonian part describes noninteracting (free) spins in a uniform magnetic field oriented in the \(x\) direction. The resulting Hamiltonian, as seen in the first term at the r.h.s. of Eq. (A1), is \(H = 2J S_x\). This Hamiltonian is time-independent: the bare system is invariant by continuous time translations.

The second term in the r.h.s. of Eq. (A1) is the environment, acting by orienting the spins in the \(z\) direction, towards the state with a positive magnetization. The same model but with \(S_x\) and \(S_-\) exchanged also features a BTC phase.

The Hamiltonian and the jump operators commute with \(S^2 = S_x^2 + S_y^2 + S_z^2\), and the conditions (i) and (ii), mentioned in Sec. 1, are satisfied, hence this model possesses a TC phase. This phase exists in the thermodynamic limit and can be analyzed using mean-field theory, since, for large \(N\), correlations between collective variables vanish as \(1/N\), \([S_+ S_-]/N^2 = O(1/N)\), and the magnetization behaves like a classical variable. The dissipative phase diagram of the model features a critical point \(\chi = \gamma / 4J \approx 1\) separating the boundary time crystal phase for weak dissipation \((\chi < 1)\) from an ordered magnetic phase where the spin state is magnetized \((\chi > 1)\) and the \(Z_2\) symmetry is manifestly broken.

**Appendix B: Kac normalization**

The power-law Lindblad operators are defined in Eq. (3), where

\[
f_{ij}(\eta) = \frac{K^{(N)}(\eta)}{[\min(|i-j|, N-|i-j|)+1]^{\eta}},
\] (B1)

The normalization factor \(K^{(N)}(\eta)\) is defined in such a way that \(\sum_{j=1}^{N} f_{ij}(\eta) = 1\). This relation must hold for all \(i\) due
Appendix C: Mean-field magnetization dynamics

When the time evolution is described by the Lindbladian of Eq. (1), then for any observable \( O \) we can write the dissipative Ehrenfest theorem as

\[
\langle \dot{O} \rangle = i\langle [H, O] \rangle + \frac{\gamma}{2} \sum_{i=1}^{N} \langle L_i^+ [O, L_i] + [L_i^+, O] L_i \rangle, \tag{C1}
\]

where we avoid to write the \( \eta \)- and time-dependence for convenience and the expectation values are taken over \( \rho(t) \). Evaluating this equation for the magnetization components \( m_{\alpha} = 2\langle \sigma_{\alpha} \rangle / N \), we obtain the following equations,

\[
\dot{m}_x = -\frac{\gamma}{2N} \sum_{i=1}^{N} \langle L_i^+ M_i + M_i L_i \rangle, \tag{C2}
\]

\[
\dot{m}_y = U_y^{(N)} + \frac{\gamma}{2N} \sum_{i=1}^{N} \langle L_i^+ M_i - M_i L_i \rangle, \tag{C3}
\]

\[
\dot{m}_z = U_z^{(N)} - \frac{2\gamma}{N} \sum_{i=1}^{N} \langle L_i^+ L_i \rangle, \tag{C4}
\]

where we defined \( M_i(\eta) = \sum_{j} f_{ij}(\eta) \sigma_i^z \) and \( U_{\alpha}^{(N)} = 2i \langle [H, S_{\alpha}] \rangle / N \). In terms of single-spin operators, these equations can be rewritten as follows:

\[
\dot{m}_x = -\frac{\gamma}{2N} \sum_{j=1}^{N} \mathcal{F}_{jj}(\eta) m_x - \frac{\gamma}{2N} \sum_{j=k \neq j}^{N} \mathcal{F}_{j*k}(\eta) C_{j*k}^{y,z},
\]

\[
\dot{m}_y = U_y^{(N)} - \frac{\gamma}{2N} \sum_{j=1}^{N} \mathcal{F}_{jj}(\eta) m_y - \frac{\gamma}{2N} \sum_{j=k \neq j}^{N} \mathcal{F}_{j*k}(\eta) C_{j*k}^{y,z},
\]

\[
\dot{m}_z = U_z^{(N)} + \frac{\gamma}{N} \sum_{j=1}^{N} \mathcal{F}_{j}^{(N)} (1 - m_z)
+ \frac{\gamma}{2N} \sum_{j=k \neq j}^{N} \mathcal{F}_{j*k}(\eta) (C_{j*k}^{x,y} + C_{j*k}^{y,x}).
\]

If we assume that the initial state is permutation invariant, then the correlation functions cannot depend on the spin lattice indices since the Lindbladian preserves the permutation symmetry, thus we can write \( C_{j*k}^{\alpha\beta} \equiv C_{\alpha\beta} \) and further simplify the equations of motion:

\[
\dot{m}_x = -\frac{\gamma}{2} F_{\eta} m_x - \frac{\gamma}{2} (1 - F_{\eta}^{(N)}) C_{x*z},
\]

\[
\dot{m}_y = U_y^{(N)} - \frac{\gamma}{2} F_{\eta}^{(N)} m_y - \frac{\gamma}{2} (1 - F_{\eta}^{(N)}) C_{y*z},
\]

\[
\dot{m}_z = U_z^{(N)} + \gamma F_{\eta}^{(N)} (1 - m_z) + \frac{\gamma}{2} (1 - F_{\eta}^{(N)}) (C_{x*x} + C_{y*y}).
\]

In the thermodynamic limit \( N \to \infty \), the coefficients of these equations can be evaluated analytically. In particular, we have that

\[
F_{\eta} = \lim_{N \to \infty} F_{\eta}^{(N)}
= \lim_{N \to \infty} \sum_{i=1}^{N} \langle \frac{[K^{(N)}(\eta)]^2}{(\min |i-j|, N - |i-j| + 1)^{2\eta}} \rangle.
\]

This relation must hold for all \( j \), thus we can fix \( j = N/2 \) and follow the same steps as those of Appendix B. In this way, we
easily obtain

\[
F_\eta = \begin{cases} 
2\zeta(2\eta) - 1 & \eta > 1, \\
(2\zeta(\eta) - 1)^2 & 0 \leq \eta \leq 1,
\end{cases}
\]

(C7)

where \( \zeta(z) = \lim_{n \to \infty} H_n^{(z)} \) is Riemann’s Zeta function, defined for \( \text{Re}(z) > 1 \). For finite \( N \), the sum in Eq. (C6) can be evaluated numerically.

In the mean field approximation, we suppose that the system state is uncorrelated and can be written as a tensor product of single-qubit density matrices (Gutzwiller ansatz):

\[
\rho = \bigotimes_{i=1}^{N} \rho_i.
\]

Thus, we assume \( C_{\alpha\beta} = m_{\alpha} m_{\beta} \) and obtain the mean-field equations of motion in the thermodynamic limit

\[
\dot{m}_x = -\frac{\gamma}{2} F_\eta m_x - \frac{\gamma}{2} (1 - F_\eta) m_x m_z,
\]

\[
\dot{m}_y = U^{(\alpha)}_\eta - \frac{\gamma}{2} F_\eta m_y - \frac{\gamma}{2} (1 - F_\eta) m_y m_z,
\]

\[
\dot{m}_z = U^{(\alpha)}_\eta + \gamma F_\eta (1 - m_z) + \frac{\gamma}{2} (1 - F_\eta) (m_x^2 + m_y^2),
\]

(C8)

also shown in the main text [Eqs. (5)].

**Appendix D: Correlations dynamics in the Gaussian approximation**

As mentioned in the main text, one way to keep some information regarding correlation is by requiring that the third cumulant is zero. This is known as the Gaussian approximation. The 3-body correlation functions appearing in the dynamical equations of the 2-body correlation functions can be decomposed as \( \langle x_1 x_2 x_3 \rangle = \langle x_1 x_2 \rangle \langle x_3 \rangle + \langle x_1 x_3 \rangle \langle x_2 \rangle + \langle x_2 x_3 \rangle \langle x_1 \rangle - 2 \langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle \), hence it is possible to write down a closed-form system of ordinary differential equations for the magnetization and 2-body correlations only. In the case of permutation invariant systems, this ODE system has nine equations.

When the system Hamiltonian is \( H = 2JS_z \), the time evolution of the correlation function \( C_{lm}^{\alpha\beta} = \langle r^\alpha r^\beta_m \rangle \) in the power-law dissipative model in the Gaussian approximation is given by

\[
C_{lm}^{\alpha\beta} = 2J (\epsilon_{x\alpha\theta} C_{lm}^{\alpha\theta} + \epsilon_{x\beta\theta} C_{lm}^{\beta\theta}) + \frac{\gamma}{2} \left[ \epsilon_{\beta x\theta} \left[ m_x (C_{lm}^{\alpha\theta} - 2m_{\alpha} m_{\theta}) + m_{\alpha} (F_{lm}(\delta_{\gamma\alpha} - \epsilon_{x\alpha\gamma} m_{\gamma}) + \sum_{j \neq l} \tilde{F}_{jm} C_{jm}^{\alpha\gamma}) + m_{\theta} (F_{nm}(\delta_{\gamma\theta} - \epsilon_{x\theta\gamma} m_{\gamma}) + \sum_{j \neq m} \tilde{F}_{jm} C_{jm}^{\gamma\theta}) \right] \\
- \epsilon_{\beta y\theta} \left[ m_y (C_{lm}^{\alpha\theta} - 2m_{\alpha} m_{\theta}) + m_{\alpha} (F_{lm}(\delta_{\gamma\alpha} + \epsilon_{y\alpha\gamma} m_{\gamma}) + \sum_{j \neq l} \tilde{F}_{jm} C_{jm}^{\alpha\gamma}) + m_{\theta} (F_{nm}(\delta_{\gamma\theta} + \epsilon_{y\theta\gamma} m_{\gamma}) + \sum_{j \neq m} \tilde{F}_{jm} C_{jm}^{\gamma\theta}) \right] \\
+ \epsilon_{ax\theta} \left[ m_x (C_{lm}^{\alpha\beta} - 2m_{\beta} m_{\theta}) + m_{\beta} (F_{lm}(\delta_{\gamma\beta} - \epsilon_{x\beta\gamma} m_{\gamma}) + \sum_{j \neq l} \tilde{F}_{jm} C_{jm}^{\beta\gamma}) + m_{\theta} (F_{nm}(\delta_{\gamma\theta} - \epsilon_{x\theta\gamma} m_{\gamma}) + \sum_{j \neq m} \tilde{F}_{jm} C_{jm}^{\gamma\theta}) \right] \\
- \epsilon_{ay\theta} \left[ m_y (C_{lm}^{\alpha\beta} - 2m_{\beta} m_{\theta}) + m_{\beta} (F_{lm}(\delta_{\gamma\beta} + \epsilon_{y\beta\gamma} m_{\gamma}) + \sum_{j \neq l} \tilde{F}_{jm} C_{jm}^{\beta\gamma}) + m_{\theta} (F_{nm}(\delta_{\gamma\theta} + \epsilon_{y\theta\gamma} m_{\gamma}) + \sum_{j \neq m} \tilde{F}_{jm} C_{jm}^{\gamma\theta}) \right] \right].
\]

(D1)

**Thermodynamic limit**

If we exploit the model’s permutation invariance \( (C_{lm}^{\alpha\beta} = C_{\alpha\beta}) \) and sum both sides of the equation over \( l \) and \( m \neq l \) (so that the spin operators in the expectation value always commute), we get the following equation in the thermodynamic limit:

\[
\dot{C}_{\alpha\beta} = 2J (\epsilon_{x\alpha\theta} C_{\alpha\theta} + \epsilon_{x\beta\theta} C_{\beta\theta}) + \frac{\gamma}{2} \left[ \epsilon_{\beta x\theta} \left[ m_x (C_{\alpha\theta} - 2m_{\alpha} m_{\theta}) + m_{\alpha} (F_{\alpha}(\delta_{\gamma\alpha} - \epsilon_{x\alpha\gamma} m_{\gamma}) + C_{\gamma\theta}(1 - F_{\gamma})) \right] \\
- \epsilon_{\beta y\theta} \left[ m_y (C_{\alpha\theta} - 2m_{\alpha} m_{\theta}) + m_{\alpha} (F_{\alpha}(\delta_{\gamma\alpha} + \epsilon_{y\alpha\gamma} m_{\gamma}) + C_{\gamma\theta}(1 - F_{\gamma})) \right] \\
+ \epsilon_{ax\theta} \left[ m_x (C_{\beta\theta} - 2m_{\beta} m_{\theta}) + m_{\beta} (F_{\beta}(\delta_{\gamma\beta} - \epsilon_{x\beta\gamma} m_{\gamma}) + C_{\gamma\theta}(1 - F_{\gamma})) \right] \\
- \epsilon_{ay\theta} \left[ m_y (C_{\beta\theta} - 2m_{\beta} m_{\theta}) + m_{\beta} (F_{\beta}(\delta_{\gamma\beta} + \epsilon_{y\beta\gamma} m_{\gamma}) + C_{\gamma\theta}(1 - F_{\gamma})) \right] \right].
\]

(D2)
Putting everything together, we obtain the following system of ordinary differential equations:

\[
\begin{align*}
\dot{m}_x &= \frac{\gamma}{2} F_H m_x - \frac{\gamma}{2} (1 - F_H) C_{xz}, \\
\dot{m}_y &= 2 J m_z - \frac{\gamma}{2} F_H m_y - \frac{\gamma}{2} (1 - F_H) C_{yz}, \\
\dot{m}_z &= -2 J m_y + \gamma F_H (1 - m_z) + \frac{\gamma}{2} (1 - F_H) (C_{xx} + C_{yy}), \\
\dot{C}_{xx} &= -\gamma \left\{ m_x \left[ F_H m_x + (2 - F_H) C_{xz} - 2 m_x m_z \right] + m_z C_{xx} \right\}, \\
\dot{C}_{xy} &= 2 J C_{xz} - \frac{\gamma}{2} \left\{ m_x \left[ F_H m_x + (2 - F_H) C_{yz} - 2 m_x m_z \right] + m_y \left[ F_H m_y + (2 - F_H) C_{xz} - 2 m_x m_z \right] + 2 m_y C_{xy} \right\}, \\
\dot{C}_{xz} &= -2 J C_{xy} + \frac{\gamma}{2} \left\{ m_x \left[ 2 F_H (1 - m_z) + (C_{xx} + C_{yy}) (1 - F_H) + 2 C_{xx} - 2 m_x^2 + 2 m_y^2 - C_{zz} \right] \\
&\quad + m_y \left[ 2 C_{xy} - 2 m_x m_y \right] - m_z \left[ F_H m_x + (2 - F_H) C_{xz} \right] \right\}, \\
\dot{C}_{yy} &= 4 J C_{yz} - \gamma \left\{ m_y \left[ F_H m_y + (2 - F_H) C_{yz} - 2 m_y m_z \right] + m_z C_{yy} \right\}, \\
\dot{C}_{yz} &= 2 J (C_{zz} - C_{yy}) + \frac{\gamma}{2} \left\{ m_x \left[ 2 C_{xy} - 2 m_x m_y \right] + m_y \left[ 2 C_{yy} - 2 m_y^2 + 2 F_H (1 - m_z) \right] \\
&\quad + (C_{xx} + C_{yy}) (1 - F_H) - C_{zz} + 2 m_z^2 \right\} - m_z \left[ F_H m_y + (2 - F_H) C_{yz} \right] \right\}, \\
\dot{C}_{zz} &= -4 J C_{yz} + \gamma \left\{ 2 m_x \left(C_{zz} - m_z m_y \right) + 2 m_y \left(C_{yz} - m_y m_z \right) + m_z \left[ 2 F_H (1 - m_z) + (C_{xx} + C_{yy}) (1 - F_H) \right] \right\}.
\end{align*}
\]

It is a stiff nonlinear system, thus its numerical solution is only stable when the evolution time is relatively short [76].

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