Delta-operator based consensus analysis of multi-agent networks with link failures

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Abstract

In this paper, a discrete-time multi-agent system is presented which is formulated in terms of the delta operator. The proposed multi-agent system can unify discrete-time and continuous-time multi-agent systems. In a multi-agent network, in practice, the communication among agents is acted upon by various factors. The communication network among faulty agents may cause link failures, which is modeled by randomly switching graphs. First, we show that the delta representation of discrete-time multi-agent system reaches consensus in mean (in probability and almost surely) if the expected graph is strongly connected. The results induce that the continuous-time multi-agent system with random networks can also reach consensus in the same sense. Second, the influence of faulty agents on consensus value is quantified under original network. By using matrix perturbation theory, the error bound is also presented in this paper. Finally, a simulation example is provided to demonstrate the effectiveness of our theoretical results.

Index Terms

Consensus, multi-agent systems, delta operator, link failures, error bound.

I. INTRODUCTION

Distributed cooperative control problem of multi-agent systems has captured great attention. This interest is motivated by its diverse applications in various fields, from biology and sociology...
to control engineering and computer science. In order to finish different cooperative tasks, a variety of protocols have been established for multi-agent systems [1], [2], [3], [4]. Lots of criteria concerning multi-agent coordination have been provided [5], [6], [7], etc.

As a fundamental problem of multi-agent coordination, consensus characterizes a phenomenon that multiple agents achieve a common decision or agreement. For consensus problem, it has been studied for a long time in management science [8]. The rise of consensus problem in control filed is influenced by Vicsek model [9], which is a discrete-time model of multiple agents and each agent updates its state by using average of its own state as well as its neighbors’. The theoretical analysis of consensus for Vicsek model was finished in [10]. And then in [1], the authors proposed classical consensus protocols for multi-agent systems and provided several sufficient conditions to solve the consensus problem. Inspired by these results, many researchers devoted themselves to studying consensus problems [11], [12]. For a multi-agent system, it can be analyzed from two perspectives: one is dynamic model and the other is interaction network. From the viewpoint of dynamic model, the related researches include first-order dynamics [1], [11], second-order dynamics [13], hybrid dynamics [14], switched dynamics [15], heterogeneous dynamics [16], etc. From the viewpoint of interaction network, the related researches have fixed networks [16], switching networks [12], antagonistic networks [17], random networks [18], [19], and so on.

With the development of digital controller, in many cases, a continuous-time multi-agent system only obtains input signal at the discrete sampling instants. According to actual factor, researchers investigated the sampled control and the event-triggered control for multi-agent systems [3], [20], [21]. It is well known that some discrete-time multi-agent systems are obtained directly from continuous-time multi-agent systems based on sampled control, which are described by the shift operator. However, some applications may possess higher sampling rate, which will lead to ill-conditioning problems when the shift operator is applied to represent the discrete-time system [22]. And the shift operator can’t show the intuitive connection between the discrete-time system and the continuous-time system. To overcome these limitations, Goodwin et al. used the delta operator to represent the dynamics of sampled data system [22], [23]. Compared with shift operator approach, delta operator has several advantages [22], [23], [24], such as superior finite world length coefficient representation and convergence to its continuous-time counterpart as the sampling period tends to zero. It is worth pointing out that the delta operator makes
the smooth transition from the discrete-time representation to the underlying continuous-time system as sampled period tends to zero. Therefore, it can be used to unify discrete-time and continuous-time systems. Due to these advantages of the delta operator, there have existed many related research results [25], [26]. Inspired by these researches, we apply the delta operator to describe the multi-agent system with sampled data. A discrete-time representation is proposed for multi-agent systems.

It is well known that the communication may be destroyed in realistic multi-agent network due to link failures, node failures, etc. Thus, the consensus of multi-agent systems with random networks was also studied in [18], [19], [27], [28]. Based on the delta operator, we consider the consensus of multi-agent systems with random networks in this paper. We assume that there exist faulty agents that only receive information or send information, which lead to link failures of the network. The original network without faulty agents is an undirected connected graph. This phenomenon often occurs in practice. For instance, the receiver (emitter) of the agent is failure, which leads to the link failure of the communication network. Different from [19], we consider the consensus of discrete-time multi-agent system with directed random networks. Due to the variation of networks, however, the consensus value is changed. Therefore, we analyze the influence of faulty agents on the original network. The main contribution of this paper is twofold. First, we show that the delta representation of discrete-time multi-agent system reaches consensus in different sense (in mean, in probability and almost surely) if the expected graph is strongly connected. Based on the delta operator, we get that the consensus conditions are also appropriate for the continuous-time multi-agent system with random networks. Second, we analyze the influence of faulty agents on the consensus value under original network. By using matrix perturbation theory, the error bound between consensus values under network with link failures and original network is presented.

The structure of this paper is given as follows. In Section 2, based on the delta operator, a discrete-time multi-agent system is established. In Section 3, consensus in different sense is studied. In Section 4, we provide the error bound caused by faulty agents. In Section 5, a simulation example is presented. Finally, we give a short conclusion in Section 6.

**Notation:** Let $1$, $0$, $R$ and $R^{n \times n}$ denote the column vector of all ones, the column vector of all zeros, the set of real numbers and the $n \times n$ real matrices, respectively. The $i$th eigenvalue of matrix $A$ can be denoted as $\lambda_i(A)$. $\| \cdot \|_2$ denotes the standard Euclidean norm, i.e., $\| x(t) \|_2 =$

\[ \sqrt{x^T(t)x(t)}. \] We write \( \|x(t)\|_2 = x^T(t)x(t) \). For the vector 2-norm \( \|\cdot\|_2 \), the induced matrix norm is \( \|A\|_2 = \max_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\lambda_{\text{max}}(A^T A)} \). We write \( \|A\| = \lambda_{\text{max}}(A^T A) \). \( B = \begin{bmatrix} b_{ij} \end{bmatrix} \in \mathbb{R}^{n \times n}, B \geq 0 \) if all \( b_{ij} \geq 0 \). We say that \( B \) is a nonnegative matrix if \( B \geq 0 \). Moreover, if all its row sums are 1, \( B \) is said to be a row stochastic matrix. For a given vector or matrix \( A \), \( A^T \) denotes its transpose. Let \( \bar{d} = \max_{i \in I_n} \{d_{ii}\} \), \( H(t_k) = \max_{i \in I_n} \{x_i(t_k)\} \), \( h(t_k) = \min_{i \in I_n} \{x_i(t_k)\} \), \( \lambda (A) = \max \{\lambda_i^2(A)\} \) and \( \zeta = \max \{\zeta_k\} \). \( A^\dagger \) denotes the group inverse of matrix \( A \) [29].

II. Preliminaries

A. Graph theory

The communication relationship between agents is modeled as a graph \( G = (V, E, A) \) with vertex set \( V = \{\nu_1, \nu_2, \ldots, \nu_n\} \), edge set \( E = \{e_{ij}\} \subseteq V \times V \) and nonnegative matrix \( A = [a_{ij}]_{n \times n} \). If \( (\nu_j, \nu_i) \in E_i \), agents \( i \) and \( j \) are adjacent and \( a_{ij} = 1 \). The set of neighbors of agent \( i \) is denoted by \( N_i = \{\nu_j | (\nu_j, \nu_i) \in E\} \). The degree matrix \( D = [d_{ii}]_{n \times n} \) is a diagonal matrix with \( d_{ii} = \sum_{j \in N_i} a_{ij} \). The Laplacian matrix of the graph is defined as \( L = [l_{ij}]_{n \times n} = D - A \) with \( l_{ii} = -\sum_{j \in N_i} a_{ij} \) and \( l_{ij} = -a_{ij} \). The eigenvalues of \( L \) can be denoted as \( 0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_n(L) \). Graph \( G \) is said to be strongly connected if there exists a path between any two distinct vertices. A path that connects \( v_i \) and \( v_j \) in directed graph \( G \) is a sequence of distinct vertices \( v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_m} \), where \( v_{i_0} = v_i, v_{i_m} = v_j \) and \( (v_{i_r}, v_{i_{r+1}}) \in E, 0 \leq r \leq m - 1 \). When \( G \) is an undirected connected graph, then \( L \) is positive semi-definite and has a simple zero eigenvalue. Moreover, there exists \( \min_{\xi \neq 0, V \xi = 0} \frac{\xi^T L \xi}{\xi^T \xi} = \lambda_2(L) \) for any \( \xi \in \mathbb{R}^n \). Throughout this paper, we always assume that \( G \) is an undirected connected graph if there does not exist the faulty agent (agent not be able to receive or send information).

B. Problem statement

In this paper, we consider a multi-agent system which consists of \( n \) agents. The continuous-time dynamics of the \( i \)th agent is described by

\[ \dot{x}_i(t) = u_i(t), \quad i = 1, \ldots, n, \]  

where \( x_i(t) \in \mathbb{R} \) and \( u_i(t) \in \mathbb{R} \) are the state and control input of \( i \)th agent, respectively.
For continuous-time multi-agent system (1), a discrete-time representation can be obtained by using a traditional shift operator. It is given by

$$x_i(t_k + h) = x_i(t_k) + hu_i(t_k), \quad i = 1, \ldots, n,$$

(2)

where $h$ is the sampling period. It is worth noting that, as sampling period $h \to 0$, we lose all information about the underlying continuous-time multi-agent system (1) [30]. Moreover, it is difficult to describe the next value of $x_i(t_k)$. This difficulty can be avoided using the delta operator introduced in [22].

The delta operator is defined as follows:

$$\delta x(t) = \begin{cases} \dot{x}(t), & h = 0, \\ \frac{x(t + h) - x(t)}{h}, & h \neq 0. \end{cases}$$

Then, by using the delta operator, the discrete-time representation of system (1) is described by

$$\delta x_i(t_k) = u_i(t_k).$$

(3)

It can be seen that $\delta x_i(t_k) \to \dot{x}_i(t_k)$ as $h \to 0$. We know that $\delta x_i(t_k) = \dot{x}_i(t_k)$ when $h = 0$. Hence, there is a smooth transition from $\delta x_i(t_k)$ to $\dot{x}_i(t_k)$ as $h \to 0$, which ensures that discrete-time multi-agent system (5) converges to continuous-time multi-agent system (1) as $h \to 0$.

For system (1), we apply the classic consensus protocol $u_i(t) = \sum_{j \in N_i} a_{ij} (x_j(t) - x_i(t))$. By using zero-order hold, the protocol is given as:

$$u_i(t_k) = \sum_{j \in N_i} a_{ij} (x_j(t_k) - x_i(t_k)), \quad t \in [t_k, t_k + h).$$

(4)

Denote $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$. System (5) with protocol (4) can be represented by

$$\delta x(t_k) = -Lx(t_k).$$

(5)

Based on above discussion and analysis, we know that discrete-time multi-agent system (5) converges to the continuous-time multi-agent system

$$\dot{x}(t) = -Lx(t)$$

(6)

as $h \to 0$.

In this paper, the original multi-agent network is undirected connected. We know that each agent is influenced by the information of its neighbours. However, there may exist the agent that
is unable to receive or send information in the network. We call this type of agent as faulty agent in this paper. Without loss of generality, we assume that agents 1 and 2 are faulty agent, while other agents are normal. That is, they can always receive and send information in the network. Two scenarios are considered.

Scenario I: Four cases are considered: (1) only agent 1 can’t receive information; (2) only agent 2 can’t receive information; (3) agents 1 and 2 cannot receive information simultaneously; (4) all agents are normal. Networks $G_1$, $G_2$, $G_3$ and $G_4$ correspond to the four cases (1), (2), (3) and (4), respectively. We assume that $G_i$ randomly switches among distinct networks $G_i \in \{G_1, G_2, G_3, G_4\}$. Networks $G_1$, $G_2$, $G_3$ and $G_4$ correspond to the occurrence probabilities $1 > \alpha > 0$, $1 > \beta > 0$, $1 > \gamma > 0$ and $1 > \theta > 0$, respectively. Moreover, $\alpha + \beta + \gamma + \theta = 1$.

Scenario II: Four cases are considered: (1) only agent 1 can’t send information; (2) only agent 2 can’t send information; (3) agents 1 and 2 cannot send information simultaneously; (4) all agents are normal. Networks $G'_1$, $G'_2$, $G'_3$ and $G'_4$ correspond to the four cases (1), (2), (3) and (4), respectively. We assume that $G'_i$ randomly switches among distinct networks $G'_i \in \{G'_1, G'_2, G'_3, G'_4\}$. Networks $G'_1$, $G'_2$, $G'_3$ and $G'_4$ correspond to the occurrence probabilities $1 > \alpha > 0$, $1 > \beta > 0$, $1 > \gamma > 0$ and $1 > \theta > 0$, respectively. Moreover, $\alpha + \beta + \gamma + \theta = 1$.

System (5) under Scenario I or II can be written as:

$$\delta x(t_k) = -L_{t_k} x(t_k),$$

where $L_{t_k}$ is the Laplacian matrix at time point $t_k$. Note that the graph $G_{t_k}$ is invariant during the time interval $\Delta$. Corresponding adjacent matrix at time point $t_k$ is $A_{t_k}$. Throughout this paper, the sampling period satisfies $\Delta = \bar{k}h$ when $h \rightarrow 0$.

Two main objectives are considered in this paper. First, the consensus of system (7) is considered. Second, the error bound between consensus values of system (5) and system (7) is presented.

Remark 1: For simplicity, we focus on two faulty agents. However, the analytical methods concerning error bound in this paper can be extended to the Scenario of more than two faulty agents, which is left to the interested readers as an exercise.

Definition 1: System (7) reaches consensus

(a) in mean if for any $x_0 \in \mathbb{R}^n$ it holds that

$$\lim_{t_k \rightarrow \infty} E[x(t)] = v(x_0);$$

(b) in mean square if for any $x_0 \in \mathbb{R}^n$ it holds that

$$\lim_{t_k \rightarrow \infty} E[\|x(t)\|^2] = v(x_0);$$

(c) in probability if for any $x_0 \in \mathbb{R}^n$ it holds that

$$\lim_{t_k \rightarrow \infty} P[\|x(t)\| > \epsilon] = 0$$

for any $\epsilon > 0$.
(b) in probability if \( \forall \varepsilon > 0 \) and any \( x_0 \in R^n \) it holds that
\[
\lim_{t_k \to \infty} P \{ H(t_k) - h(t_k) \geq \varepsilon \} = 0; \tag{9}
\]
(c) almost surely if for any \( x_0 \in R^n \) it holds that
\[
P \left\{ \lim_{t_k \to \infty} (H(t_k) - h(t_k)) = 0 \right\} = 1. \tag{10}
\]

**Definition 2 ([31]):** Let \( W \) denote the transition matrix of Markov chain. Then the Markov chain is called the regular chain if there exist \( k > 0 \) such that \( W^k \) has only positive elements.

**Lemma 1 ([29]):** If \( T \) is the transition matrix of a regular chain, then \( A^\sharp = \sum_{k=0}^{\infty} (T^k - T^\infty) \) where \( A = I - T \).

**Lemma 2 ([32]):** If \( C \) and \( \tilde{C} \) are ergodic chains with transition matrices \( T \) and \( \tilde{T} = T - E \) and limiting probability vectors \( s \) and \( \tilde{s} \), respectively, then \( s - \tilde{s} = sE A^\sharp (I + EA^\sharp)^{-1} \) where \( E1 = 0 \) and \( A = I - T \).

**Lemma 3 ([23]):** The property of delta operator for any time function \( x(t_k) \) and \( y(t_k) \) can be represented as
\[
\delta(x(t_k)y(t_k)) = \delta(x(t_k))y(t_k) + x(t_k)\delta(y(t_k)) + h\delta(x(t_k))\delta(y(t_k)).
\]

**Lemma 4:** Assume that the sampling period \( 0 < h < \frac{1}{d_{max}} \). Then, system (5) can reach average consensus if the graph is undirected connected.

**Proof.** Let \( \nu(t_k) = x(t_k) - \frac{1}{n} x(0) \). Due to \( L1 = 0 \), one has \( \delta(\nu(t_k)) = -L\nu(t_k) \). Consider \( V(t_k) = \|\nu(t_k)\| \) as a Lyapunov function. By Lemma 3 it holds that
\[
\delta V(t_k) = \delta^T(\nu(t_k))\nu(t_k) + \nu^T(t_k)\delta(\nu(t_k)) + h\delta^T(\nu(t_k))\delta(\nu(t_k))
\]
\[
= \nu^T(t_k)(-2L + hLT)L\nu(t_k)
\]
\[
= \nu^T(t_k)\Xi\nu(t_k).
\]

Since the graph is undirected connected, which implies that the Laplacian matrix \( L \) is positive semi-definite. Hence, the eigenvalues of \( \Xi \) are represented by \(-2\lambda_i(L) + h\lambda^2_i(L) \). From Gersgorin Disk Theorem, we get \( \lambda_i(L) \leq 2d_{max} \). Then \(-2 + h\lambda_i(L) < -2 + \frac{1}{d_{max}}2d_{max} \leq 0 \). Owing to
\[
\min_{\xi \neq 0,1^T\xi = 0} \frac{\xi^T L \xi}{\xi^T \xi} = \lambda_2(L),
\]
then
\[
\delta V(t_k) \leq -(2\lambda(L) - h\lambda^2(L))\|\nu(t_k)\|
\]
where \(2\lambda(L) - h\lambda^2(L) = \min\{2\lambda_2(L) - h\lambda_2^2(L), 2\lambda_n(L) - h\lambda_n^2(L)\}\). Due to \(\lambda_2(L) > 0\) and \(\lambda_n(L) > 0\), this proves that \(\delta V(t_k) < 0\). Therefore, \(\nu(t_k)\) is converge to 0. That is, system (5) can achieve average consensus asymptotically.

**Remark 2:** From Lemma 4, there exists
\[
\delta V(t_k) = \frac{V(t_k + h) - V(t_k)}{h} < 0,
\]
which implies that
\[
\lim_{h \to 0} \delta V(t_k) = \lim_{h \to 0} \frac{V(t_k + h) - V(t_k)}{h} = \dot{V}(t) < 0.
\]
It can be seen that \(\delta V(t_k) < 0\) can be reduced to the \(\dot{V}(t) < 0\) as \(h \to 0\). Note that system (5) converges to system (6) as \(h \to 0\). Consequently, system (6) reaches average consensus under undirected connected graph.

### III. Consensus Analysis

In this section, it is shown that system (5) reaches consensus despite the existence of faulty agents. Supposed that Scenario I and Scenario II have the same expression pattern for the network. Hence, the following results can be viewed as the unified conclusions of system (5) under Scenarios I and II.

**Theorem 1:** Assume that the sampling period \(0 < h < \frac{1}{d_{max}}\). Then, system (7) reaches consensus in mean if the expected graph is strongly connected. Furthermore,
\[
\lim_{t_k \to \infty} E[x(t_k)] = \mathbf{1}\pi^T x(0),
\]
where
\[
\mathbf{1}\pi^T = \left\{\begin{array}{ll}
\mathbf{1}\pi_1^T = \lim_{k \to \infty} W_1^k : W_1 = E[(I - hL_{tk})^k], \\
\mathbf{1}\pi_2^T = \lim_{k \to \infty} W_2^k : W_2 = E[e^{-L_{tk}\Delta}], h \to 0,
\end{array}\right.
\]

vectors \(\pi_1 > 0\) and \(\pi_2 > 0\) are left eigenvectors of the matrices \(W_1\) and \(W_2\), respectively, such that \(\pi_1^T \mathbf{1} = 1\) and \(\pi_2^T \mathbf{1} = 1\).

**Proof.** As pointed out in [22], the solution to system (5) is \(x(t) = (I - hL)\hat{x}x(0)\). Due to the invariance of graph \(G_{tk}\) during the time interval \(\Delta\), it can be get that \(x(t_k + \Delta) = (I -
\(hL_{tk}) \overset{\Delta}{\to} x(t_k).\) Then
\[
\lim_{k \to \infty} E(x(t_k)) = \lim_{k \to \infty} [E((I - hL_{tk})^k)]^k x(0) = \lim_{k \to \infty} [(I - hL_1)^k \alpha + (I - hL_2)^k \beta + (I - hL_3)^k \gamma + (I - hL_4)^k \theta]^k x(0) = \lim_{k \to \infty} W_1^k x(0).
\]

According to \(0 < h < \frac{1}{d_{\max}},\) we have \(I - hL_i = I - hD_i + hA_i \geq 0\) with positive diagonal elements. Since \(\alpha > 0, \beta > 0, \gamma > 0\) and \(\theta > 0,\) it is immediate that \(E((I - hL_{tk})^k)\) is also nonnegative matrix with positive diagonal elements.

It follows that
\[
E((I - hL_{tk})^k) \geq \Pi(A_1 \alpha + A_2 \beta + A_3 \gamma + A_4 \theta),
\]
where \(\Pi\) is a positive diagonal matrix. Since the expected graph is strongly connected, matrix \(E((I - hL_{tk})^k)\) is a nonnegative irreducible with positive diagonal elements. Moreover, it is easy to verify that \(E((I - hL_{tk})^k) = 1.\) Then, by Geršgorin Disc theorem, one has \(|\lambda_i(E((I - hL_{tk})^k))| \leq 1.\) Hence, by Perron-Frobenius Theorem \([33],\) it can be deduced that \(\rho(E((I - hL_{tk})^k)) = 1\) is an algebraically simple eigenvalue. Consequently, matrix \(W_1\) is a primitive. By virtue of Theorem 8.5.1 in \([33],\) we obtain that \(\lim_{t_k \to \infty} E[x(t_k)] = 1_{\pi_1}^T x(0).\) Hence, system (7) reaches consensus in mean.

Next, we give the consensus value of system (7) as \(h \to 0.\) By Proposition 11.1.3 in \([34],\) it follows that \(x(t_k + \Delta) = (I - hL_{tk}) \overset{\Delta}{\to} x(t_k) = e^{-L_{tk} \Delta}\) as \(h \to 0.\) Hence
\[
\lim_{t_k \to \infty} E(x(t_k)) = \lim_{k \to \infty} [E((I - hL_{tk})^k)]^k x(0) = \lim_{k \to \infty} (e^{-L_{1} \Delta \alpha} + e^{-L_{2} \Delta \beta} + e^{-L_{3} \Delta \gamma} + e^{-L_{4} \Delta \theta})^k x(0) = \lim_{k \to \infty} W_2^k x(0).
\]

Let \(L_{tk} = d_{\max} I - \bar{A}_{tk},\) then \(e^{-L_{tk} \Delta} = e^{-d_{\max} t} e^{A_{tk}} \geq \zeta_{tk} \bar{A}_{tk}\) for \(\zeta_{tk} > 0\) where \(\bar{A}_{tk} \geq A_{tk}.\) Hence, \(E(e^{-L_{tk} \Delta}) \geq \zeta(\bar{A}_1 \alpha + \bar{A}_2 \beta + \bar{A}_3 \gamma + \bar{A}_4 \theta).\) That is, matrix \(E(e^{-L_{tk} \Delta})\) is a nonnegative irreducible with positive diagonal elements. Similar to the previous discussion, we have
\[
\lim_{t_k \to \infty} E[x(t_k)] = 1_{\pi_2}^T x(0)\) as \(h \to 0.\)

**Theorem 2:** Assume that the sampling period \(0 < h < \frac{1}{d_{\max}}.\) Then, system (7) reaches consensus in probability if the expected graph is strongly connected.
Proof. Since the expected graph is strongly connected, by Theorem 1, it follows that \( \lim_{t_k \to \infty} E(H(t_k) - h(t_k)) = 0 \). Let \((I - hL_{t_k})^k = [w_{ij}]_{n \times n}\), we have \( x_i(t_k + \Delta) = \sum_{j=1}^n w_{ij} x_j(t_k) \). Since matrix \((I - hL_{t_k})^k\) is a row stochastic matrix, we get \( H(t_k + \Delta) \leq H(t_k) \) and \( h(t_k + \Delta) \geq h(t_k) \). It can be verified that the \( H(t_k) - h(t_k) \) is nonincreasing.

Let \( t_{k+1} = t_k + \Delta \), hence \( 0 \leq H(t_{k+1}) - h(t_{k+1}) \leq H(t_k) - h(t_k) \), which yields
\[
E[(H(t_{k+1}) - h(t_{k+1}))^2] \leq E[H(t_k) - h(t_k)](H(0) - h(0)).
\] (14)

Hence
\[
\lim_{t_k \to \infty} E[(H(t_{k+1}) - h(t_{k+1}))^2] = 0.
\] (15)

As a result of Chebyshev’s inequality, for any \( \varepsilon > 0 \), it follows that
\[
P\left\{ H(t_k) - h(t_k) \geq \varepsilon \right\} \leq \frac{E[(H(t_k) - h(t_k))^2]}{\varepsilon^2}.
\] (16)

Therefore
\[
\lim_{t_k \to \infty} P\left\{ H(t_k) - h(t_k) \geq \varepsilon \right\} = 0.
\] (17)

It is shown from Theorem 1 that \( \lim_{h \to 0} (I - hL_{t_k})^{\frac{\Delta}{h}} = e^{-L_{t_k} \Delta} \). Matrix \( e^{-L_{t_k} \Delta} \) is also a row stochastic matrix. Similar to the above proof, it can be proved that (17) also holds as \( h \to 0 \).

**Theorem 3:** Assume that the sampling period \( 0 < h < \frac{1}{r_{\max}} \). Then, system (7) reaches consensus almost surely if the expected graph is strongly connected.

**Proof.** It follows from Theorem 2 that \( H(t_k) - h(t_k) \to 0 \) in probability. By Theorem 2.5.3 in [35], there exists a subsequence of \( \{H(t_k) - h(t_k)\} \) that converges almost surely to 0. Hence, for any \( \varepsilon > 0 \), there exists \( t_l \) such that for \( t_l \geq t_i \), \( H(t_l) - h(t_l) < \varepsilon \) almost surely. Since \( \{H(t_k) - h(t_k)\} \) is nonincreasing, it holds that \( 0 \leq H(t_{l+1}) - h(t_{l+1}) \leq H(t_l) - h(t_l) < \varepsilon \) almost surely. Therefore, for any \( t_k \geq t_{l+1} \), there holds \( 0 \leq \{H(t_k) - h(t_k)\} < \varepsilon \) almost surely. This implies that system (7) reaches consensus almost surely.

**Remark 3:** As pointed out in Theorem 1, one has \( x(t_k + \Delta) = e^{-L_{t_k} \Delta} x(t_k) \) as \( h \to 0 \). Since the network is invariant during time interval \( \Delta \), partial state of system (6) can be represented by \( x(t_k + \Delta) = e^{-L_{t_k} \Delta} x(t_k) \). It is shown from Theorem 1 that the sequence \( x(t_k) \) achieves consensus in mean. Then, using \( -L1 = 0 \), we can conclude that \( x(t) \) achieves consensus in mean. Therefore, system (6) with random networks reaches consensus in mean if the expected graph is strongly connected.
This indicates that the consensus result of system (7) with random networks reduces to the consensus result of system (6) under random networks as $h \to 0$. Moreover, Theorems 2 and 3 are also appropriate for the continuous-time multi-agent system as $h \to 0$.

IV. ERROR ANALYSIS

In this section, we consider the error bound on the consensus value $\lim_{t_k \to \infty} E(x(t_k))$ and the consensus value under original network $\frac{1}{n}x(0)$, i.e.,

\[
\lim_{t_k \to \infty} E(x(t_k)) - \frac{1}{n}x(0) = \pi^T x(0) - \frac{1}{n}x(0) = (\pi^T - \frac{1}{n})x(0).
\] (18)

To solve this problem, the matrix perturbation theory and the property of finite Markov chains are applied.

On the analysis of the consensus problem, we apply $E(x(t_k + \Delta)) = W_1 E(x(t_k))$. Suppose that the expected graph is strongly connected and $0 < h < \frac{1}{\max_{d}}$. Then, Theorem 1 shows that $W_1$ is a row stochastic matrix such that $W_1^k > 0$. By property of row stochastic matrix, each element $w_{ij}$ of matrix $W_1$ satisfies $0 \leq w_{ij} < 1$. Hence, by Definition 2, $W_1$ can be regarded as the transition matrix of a regular chain. Moreover, $W_1$ is a transition matrix of ergodic chain. It is noteworthy that the following analysis results are appropriate for Scenario I and Scenario II.

Theorem 4: Assume that the sampling period $0 < h < \frac{1}{\max_{d}}$ and the expected graph is strongly connected. Then

\[
\|\pi_1^T - \frac{1}{n}\| \leq \|D_1\| \frac{1}{1-\lambda(W_1 - \frac{1}{n})},
\] (19)

and

\[
\|\pi_2^T - \frac{1}{n}\| \leq \|D_2\| \frac{1}{1-\lambda(W_2 - \frac{1}{n})}, \quad h \to 0,
\] (20)

where $\bar{W}_1 = (I - hL_4)^k$, $D_1 = \sum_{i=1}^{3}((I - hL_1)^k - (I - hL_4)^k)p_i$, $\bar{W}_2 = e^{-L_4\Delta}$, $D_2 = \sum_{i=1}^{3}(e^{-L_i\Delta} - e^{-L_4\Delta})p_i$. $p_1$, $p_2$, $p_3$ correspond to $\alpha$, $\beta$, $\gamma$, respectively.

Proof. It is pointed out that $W_1 = (I - hL_1)^k\alpha + (I - hL_2)^k\beta + (I - hL_3)^k\gamma + (I - hL_4)^k\theta$ can be written as $W_1 = \bar{W}_1 + D_1$. From Theorem 1 we can derive that $\lim_{k \to \infty} \bar{W}_1^k = \frac{1}{n}\pi_1^T$ and $W_1$ is a row stochastic matrix. Hence, it proves that the row sums of $D_1$ are all equal to 0. Moreover, it follows from Theorem 1 that $\lim_{k \to \infty} W_1^k = \pi_1^T$. Let $e = \pi_1^T - \frac{1}{n}$, we analyze the error bound of $\|e\|$.
Using Lemmas 1 and 2 it holds that
\[ e = \frac{1}{n} D_1 F (I - D_1 F)^{-1}, \tag{21} \]
where \( F = \sum_{k=0}^{\infty} (\bar{W}_1^k - \frac{11^T}{n}). \) By some algebraic manipulations for (21), the following equation holds
\[ \frac{1}{n} D_1 F = e - \bar{\pi}_1^T D_1 F + \frac{11^T}{n} D_1 F, \tag{22} \]
i.e., \( e = \bar{\pi}_1^T D_1 F. \) This implies that \( \|e\| = \|\bar{\pi}_1^T D_1 F\| \leq \|\bar{\pi}_1^T\| \|D_1 F\|. \) Due to \( \bar{\pi}_1 > 0 \) and \( \bar{\pi}_1^T \mathbf{1} = 1, \) we get \( \|e\| \leq \|D_1 F\|. \)

It follows from \( F = \sum_{k=0}^{\infty} (\bar{W}_1^k - \frac{11^T}{n}) \) that \( \|e\| \leq \|D_1 F\| = \|D_1 \sum_{k=0}^{\infty} \bar{W}_1^k\|. \) Hence, we analyze \( \|D_1 \sum_{k=0}^{\infty} \bar{W}_1^k\|. \) To solve this problem, we introduce a vector \( y(0) \) such that \( \|y(0)\| \neq 0 \) and \( y(k + 1) = \bar{W}_1^k y(0). \) It is obvious that \( \mathbf{1}^T y(k) = \mathbf{1}^T y(0). \) Therefore,
\[
\|D_1 \bar{W}_1^k y(0)\| \leq \|D_1 \|\bar{W}_1^k y(0)\|
= \|D_1 \|\bar{W}_1 y(k)\|
= \|D_1 \|((\bar{W}_1 - \frac{11^T}{n})(y(k) - \frac{11^T}{n} y(0)))\|
= \|D_1 \|((\bar{W}_1 - \frac{11^T}{n})^k \|I - \frac{11^T}{n}\| \|y(0)\|). \tag{23} \]

Owing to \( \bar{W}_1 \frac{11^T}{n} = \frac{11^T}{n} \bar{W}_1, \) by Theorem 4.5.15 in [33], we have \( \lambda_i(\bar{W}_1 - \frac{11^T}{n}) = (1 - h \lambda_i(L_4))^k - \lambda_i(\frac{11^T}{n}). \) There exists the eigenvector \( \mathbf{1} \) corresponding to \( \lambda_1(\bar{W}_1) = 1 \) and \( \lambda_1(\frac{11^T}{n}) = 1, \) which implies \( \lambda_1(\bar{W}_1 - \frac{11^T}{n}) = 0 \) and \( \lambda_i(\bar{W}_1 - \frac{11^T}{n}) = (1 - h \lambda_i(L_4))^k, i = 2, \ldots, n. \) Moreover, by a similar analysis, we have \( \lambda_1(I - \frac{11^T}{n}) = 0 \) and \( \lambda_i(I - \frac{11^T}{n}) = 1, i = 2, \ldots, n. \) Due to \( 0 < h < \frac{1}{d_{\text{max}}}, \) it can be deduced that \( -1 < 1 - h \lambda_i(L_4) < 1. \) We know that matrix \( \bar{W}_1 - \frac{11^T}{n} \) is symmetric. Consequently, \( 0 \leq \bar{\lambda}(\bar{W}_1 - \frac{11^T}{n}) = \max\{\lambda_i((\bar{W}_1 - \frac{11^T}{n})^T(\bar{W}_1 - \frac{11^T}{n}))\} < 1. \) It follows that
\[
\|D_1 \bar{W}_1^k y(0)\| \leq \bar{\lambda}^k(\bar{W}_1 - \frac{11^T}{n}) \|D_1\| \|y(0)\|. \tag{24} \]

Then
\[
\|D_1 \sum_{k=0}^{\infty} \bar{W}_1^k y(0)\| \leq \sum_{k=0}^{\infty} \|D_1 \bar{W}_1^k y(0)\|
\leq \|D_1\|(1 + \bar{\lambda}(\bar{W}_1 - \frac{11^T}{n}) + \bar{\lambda}^2(\bar{W}_1 - \frac{11^T}{n}) + \cdots) \|y(0)\|. \tag{25} \]

On account of \( \|D_1 \bar{W}_1\|_2 = \max_{\|y(0)\|_2 \neq 0} \frac{\|D_1 \bar{W}_1^k y(0)\|_2}{\|y(0)\|_2} \) and \( \bar{\lambda}(\bar{W}_1 - \frac{11^T}{n}) < 1, \) we obtain \( \|e\| \leq \|D_1\| \frac{1}{1 - \bar{\lambda}(\bar{W}_1 - \frac{11^T}{n})}. \)
When $h \to 0$, from Theorem 11 we know that $\lim_{t_k \to \infty} E[x(t_k)] = 1, 1^T x(0)$. Then we analyze $\|e\| = \|\bar{\pi}_2 - \frac{1}{n}\|$. It is clear that $E(x(t_k + \Delta)) = W_2 E(x(t_k))$ for $h \to 0$, where

$$W_2 = \lim_{h \to 0} W_1 = e^{-L_1 \Delta} + e^{-L_2 \Delta} + e^{-L_3 \Delta} + e^{-L_4 \Delta} \pi_\Delta.$$

(26)

Similar to $W_1$, $W_2$ can be regarded as the transition matrix of a regular chain. Moreover $W_2 = \bar{W}_2 + D_2$ and $D_2 1 = 0$. When $h \to 0$, by using $\lim_{k \to \infty} \bar{W}_1^k = \frac{1}{n}$, we have $\lim_{k \to \infty} \bar{W}_2 = (e^{-L_4 \Delta})^k = \frac{1}{n}$. Hence

$$\|e\| \leq \sum_{k=0}^{\infty} \|D_2 \bar{W}_2^k\| \leq \|D_2\|(1 + \tilde{\lambda}(\bar{W}_2 - \frac{1}{n})) + \tilde{\lambda}^2(\bar{W}_2 - \frac{1}{n}) + \cdots).$$

(27)

Similar to the above analysis, we get that $0 \leq \tilde{\lambda}(\bar{W}_2 - \frac{1}{n}) < 1$. Therefore, we have $\|e\| \leq \frac{1}{1 - \tilde{\lambda}(\bar{W}_2 - \frac{1}{n})}$. $\blacksquare$

**Remark 4:** Agent that can’t receive information is considered in Scenario I, while agent that can’t send information is considered in Scenario II. We assume that there exist agents which can not receive or send information in Scenario III. For this scenario, similar to Theorem 4, error bound on consensus value can be calculated.

**Theorem 5:** Assume that the sampling period $\Delta = h < \frac{1}{\sigma_{max}}$ and the expected graph is strongly connected in Scenario I. Then

$$\|\bar{\pi}_1 - \frac{1}{n}\| \leq \frac{2c \max\{(\alpha + \beta)^2, (\beta + \gamma)^2\}}{1 - \tilde{\lambda}(\bar{W}_1 - \frac{1}{n})},$$

(28)

where $\bar{W}_1 = I - hL_4$ and $c = h^2 \max\{\sum_{j=1}^{n} l_{1j}^2, \sum_{j=1}^{n} l_{2j}^2\}$.

**Proof.** Due to $h = \Delta$, then we have $W_1 = (I - hL_1)\alpha + (I - hL_2)\beta + (I - hL_3)\gamma + (I - hL_4)\theta$. Matrix $W_1$ is expressed as $\bar{W}_1 + D_1$, where

$$D_1 = h \left[ \begin{array}{cccc} l_{11}(\alpha + \gamma) & l_{12}(\alpha + \gamma) & \cdots & l_{1n}(\alpha + \gamma) \\ l_{21}(\beta + \gamma) & l_{22}(\beta + \gamma) & \cdots & l_{2n}(\beta + \gamma) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} \right].$$

(29)

Similar to the analysis of Theorem 4, we have $\|e\| = \|\bar{\pi}_1^T D_1 F\| \leq \|D_1 F\| = \|D_1 \sum_{k=0}^{\infty} \bar{W}_1^k\|$. To calculate $\|D_1 \sum_{k=0}^{\infty} \bar{W}_1^k\|$, we introduce a vector $y(0)$ such that $\|y(0)\| \neq 0$ and $y(k+1) = \bar{W}_1^k y(0)$. 

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By utilizing (29), the following equation is obtained
\[
\|D_1 y(k)\| = h^2(\alpha + \gamma)^2(l_{11}y_1(k) + l_{12}y_2(k) + \cdots + l_{1n}y_n(k))^2 \\
+ h^2(\beta + \gamma)^2(l_{21}y_1(k) + l_{22}y_2(k) + \cdots + l_{2n}y_n(k))^2.
\] (30)

Due to \(l_{11} = -\sum_{j=2}^{n} l_{1j}\) and \(l_{22} = -\sum_{j=1,j\neq 2}^{n} l_{2j}\),
\[
\|D_1 y(k)\| = h^2(\alpha + \gamma)^2(-l_{12}(y_1(k) - y_2(k)) - \cdots - l_{1n}(y_1(k) - y_n(k)))^2 \\
+ h^2(\beta + \gamma)^2(-l_{21}(y_2(k) - y_1(k)) - \cdots - l_{2n}(y_2(k) - y_n(k)))^2.
\] (31)

Let \(\bar{y}(k) = \frac{1}{n}y(0)\), substituting \(\bar{y}(k)\) into equation (31) leads to that
\[
\|D_1 y(k)\| = h^2(\alpha + \gamma)^2(l_{11}(y_1(k) - \bar{y}(k)) - l_{12}(\bar{y}(k) - y_2(k)) - \cdots)^2 \\
+ h^2(\beta + \gamma)^2(l_{22}(y_2(k) - \bar{y}(k)) - l_{21}(\bar{y}(k) - y_1(k)) - \cdots)^2 \\
\leq h^2(\alpha + \gamma)^2\sum_{j=1}^{n} l_{1j}^2 \sum_{i=1}^{n} (y_i(k) - \bar{y}(k))^2 + h^2(\beta + \gamma)^2\sum_{j=1}^{n} l_{2j}^2 \sum_{i=1}^{n} (y_i(k) - \bar{y}(k))^2 \\
= h^2(\alpha + \gamma)^2\sum_{j=1}^{n} l_{1j}^2\|y(k) - \bar{1}\bar{y}(k)\| + h^2(\beta + \gamma)^2\sum_{j=1}^{n} l_{2j}^2\|y(k) - \bar{1}\bar{y}(k)\| \\
\leq 2c \max\{(\alpha + \gamma)^2, (\beta + \gamma)^2\}\|y(k) - \bar{1}\bar{y}(k)\|.
\] (32)

By using \(\|y(k+1) - \bar{1}\bar{y}(k)\| = \|(W_1 - \frac{1}{n}I)y(k)\|\), we get
\[
\|y(k+1) - \bar{1}\bar{y}(k)\| \leq \bar{\lambda}^k(W_1 - \frac{1}{n}I)\|y(0)\|,
\] (33)

where \(0 \leq \bar{\lambda}(W_1 - \frac{1}{n}I) < 1\). Hence
\[
\|D_1 y(k+1)\| \leq 2c \max\{(\alpha + \gamma)^2, (\beta + \gamma)^2\}\bar{\lambda}^k(W_1 - \frac{1}{n}I)\|y(0)\|.
\] (34)

Due to \(\|D_1 \sum_{k=0}^{\infty} \bar{W}_1^k y(0)\| \leq \sum_{k=0}^{\infty} \|D_1 \bar{W}_1^k y(0)\|\), it holds that
\[
\sum_{k=0}^{\infty} \|D_1 \bar{W}_1^k y(0)\| = \sum_{k=0}^{\infty} \|D_1 y(k)\| \\
\leq \frac{2c \max\{(\alpha + \beta)^2, (\beta + \gamma)^2\}}{1 - \bar{\lambda}(W_1 - \frac{1}{n}I)}\|y(0)\|.
\]

Therefore, \(\|e\| \leq \|D_1 \sum_{k=0}^{\infty} \bar{W}_1^k\| \leq \frac{2c \max\{(\alpha + \beta)^2, (\beta + \gamma)^2\}}{1 - \bar{\lambda}(W_1 - \frac{1}{n}I)}\). \(\blacksquare\)

**Corollary 1:** Assume that the sampling period \(\Delta = h < \frac{1}{\alpha_{max}}\) and the expected graph is strongly connected in Scenario II. Then
\[
\|\pi_1^T - \frac{1}{n}\| \leq \frac{4\bar{c} \max\{(\alpha + \beta)^2, (\beta + \gamma)^2\}}{1 - \bar{\lambda}(W_1 - \frac{1}{n}I)},
\] (35)
Due to interaction topology among agents randomly switches among Matrix \( \omega \). It can be seen that all the agents reach consensus. The interaction topology among agents randomly switches among Matrix \( \omega \). By utilizing (36), we have

\[
\| \tilde{D}_1 y(k) \| = h^2 l_{12}^2 (\alpha + \gamma)^2 (y_1 - y_2)^2 + h^2 l_{12}^2 (\beta + \gamma)^2 (y_1 - y_2)^2 + h l_{31} (\alpha + \gamma) (y_1 - y_3) + h l_{32} (\beta + \gamma) (y_2 - y_3) \]

\[+ \cdots + h l_{n1} (\alpha + \gamma) (y_1 - y_n) + h l_{n2} (\beta + \gamma) (y_2 - y_n)^2 \]

\[
= (h^2 l_{12}^2 (\alpha + \gamma)^2 + h^2 l_{12}^2 (\beta + \gamma)^2) (y_1 - \bar{y} + \bar{y} - y_2)^2 + \cdots + (h l_{n1} (\alpha + \gamma) (y_1 - \bar{y} + \bar{y} - y_n) + h l_{n2} (\beta + \gamma) (y_2 - \bar{y} + \bar{y} - y_n) \]

\[
\leq 4 h^2 (\alpha + \gamma)^2 \sum_{j=1}^{n} l_{j1}^2 \sum_{i=1}^{n} (y_i - \bar{y})^2 + 4 h^2 (\beta + \gamma)^2 \sum_{j=1}^{n} l_{j2}^2 \sum_{i=1}^{n} (y_i - \bar{y})^2 \]

\[
\leq 4 \tilde{c} \max \{(\alpha + \gamma)^2, (\beta + \gamma)^2\} \| y(k) - \bar{y}(k) \|. 
\]

Similar to the analysis of Theorem [5], we have

\[
\| e \| \leq \| \tilde{D}_1 \sum_{k=0}^{\infty} \tilde{W}_1^k \| \leq \frac{4 \tilde{c} \max \{(\alpha + \beta)^2, (\beta + \gamma)^2\}}{1 - \lambda(\tilde{W}_1 - \frac{1}{n})}. 
\]

V. SIMULATION

In this section, a simulation is presented to illustrate the effectiveness of our theoretical results.

**Example 1:** We consider that the communication network is chosen as in Figure [1]. The interaction topology among agents randomly switches among \( G_1, G_2, G_3 \) and \( G_4 \). Networks \( G_1, G_2, G_3 \) and \( G_4 \) correspond to the occurrence probabilities \( \alpha = 0.3, \beta = 0.3, \gamma = 0.2, \theta = 0.2 \), respectively. By calculation, we can get the sampling period \( 0 < h < 0.5 \). We choose \( h = 0.01, \Delta = 0.1 \) and initial value \( x(0) = [0.2, 0.8, 0.4, -1, -2]^T \). Figure [2] depicts the state trajectories of system [5] with random networks. It can be seen that all the agents reach consensus.
Fig. 1. Network topologies $G_1$, $G_2$, $G_3$ and $G_4$.

Fig. 2. State trajectories of system (5) with random networks.

The original network is denoted by graph $G_4$. The state trajectories of system (5) under network $G_4$ are shown in Figure 3. It is shown that all the agents reach consensus.

By calculation, we obtain $\|D_1\|\frac{1}{1-\lambda(W_1-n^{-1})} = 0.0716$. Therefore, based on Theorem 4, we can obtain $\|\pi^T - \frac{1^T}{n}\| \leq 0.0716$. It follows from (18) that, when $t_k \to \infty$, $\|E(x(t_k)) - \frac{1^T}{n} x(0)\| \leq 2.0918$. From Figures 2 and 3 it is easy to verify that error bound $\|E(x(t_k)) - \frac{1^T}{n} x(0)\|$ is less than 2.0918.
VI. CONCLUSION

In this paper, based on the delta operator, a discrete-time multi-agent system is proposed. It is pointed out that the proposed discrete-time multi-agent system can converge to the continuous-time multi-agent system as the sampling period tends to zero. We assume that there exist faulty agents that only send or receive information in the network. The communication network is described by randomly switching networks. Under the random networks, it is proved that the consensus in mean (in probability and almost surely) can be achieved when the expected graph is strong connected. Furthermore, the influence of faulty agents on the consensus value is analyzed. The error bound between consensus values under network with link failures and original network is presented. In the future, based on the delta operator, we will consider the formation control and containment control of multi-agent systems, etc.

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