The solvability of a kind of generalized Riemann–Hilbert problems on function spaces $H_*$

Pingrun Li*

Abstract

In this paper, we study a kind of generalized Riemann–Hilbert problems (R-HPs) with several unknown functions in strip domains. We mainly discuss methods of solving R-HPs with two unknown functions and obtain general solutions and conditions of solvability on function spaces $H_*$. At the end of this paper, we consider in detail the behavior of the solution at $\infty$ and in different domains. Thus the results in this paper generalize and improve the theory of the classical Riemann–Hilbert problems.

MSC: 30E25; 30G35; 40G35

Keywords: Riemann–Hilbert problems; Singular integral equations; Sokhotski–Plemelj formula; Canonical function; Function spaces $H_*$

1 Introduction

It is well known that Riemann–Hilbert problems are a powerful mathematical tool widely applied in physics, fracture mechanics, engineering mechanics, engineering and technology, and many other fields [1–3]. Especially, the problem of finding solutions for some kinds of singular integral equations is often transformed to solving Riemann–Hilbert problems [4–15]. In [1, 2] the Riemann–Hilbert problems on an infinite straight line has systematically been studied, and the Riemann–Hilbert problems with unknown function on two parallel lines was further described. So far, the results of the boundary value problems for analytic functions have been mostly confined to the case of only one unknown function.

Motivated by the above researches, the main purpose of this paper is extending the theory to the R-HPs with $n \geq 2$ unknown functions on $n$ parallel straight lines, and we mainly discuss the case $n = 2$. Using the classical boundary value theory and principle of analytic continuation, we investigate the analytic solutions and the conditions of solvability on function spaces $H_*$ (the notation $H_*$ can be found in Sect. 2). At nodal points the asymptotic behavior of a solution of such a problem is discussed in detail. Our method of solving problems is innovate, different from those in classical cases. Meanwhile, this paper also improves some results of [2–4, 7].

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.
2 Definitions and lemmas

In this section, we introduce some definitions and lemmas.

**Definition 2.1** Let $F(x)$ be a continuous function in the real number field $\mathbb{R}$. A function $F(x)$ belongs to $\mathcal{H}$ if the following two conditions are fulfilled:

1. There exists $B \in \mathbb{R}^+$ such that
   \[
   |F(x_1) - F(x_2)| \leq B|x_1^{-1} - x_2^{-1}|
   \]  
   (2.1)
   on the neighborhood $N_{\infty}$ of $\infty$, that is, there exists a sufficiently large $M > 0$ such that (2.1) is satisfied for all $x_1, x_2 \in \mathbb{R}\setminus[-M, M]$.

2. $F \in H$ on $[-M, M]$, where $H$ is the class of Hölder continuous functions (for the notation $H$, see [1]).

**Definition 2.2** A function $F$ belongs to $H_\ast$ if it satisfies:

1. $F \in \mathcal{H}$,
2. $F \in L^2(\mathbb{R})$ (see [1, 6] for the definition of $L^2(\mathbb{R})$).

With respect to the function spaces $H_\ast$, one of its important properties is closedness under pointwise multiplication.

If a function $F$ satisfies the Hölder condition on a neighborhood $N_{\infty}$ of $\infty$, then we write $F \in H(N_{\infty})$.

**Definition 2.3** Let $f(t) \in H_\ast$. We define its Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ as follows:

\[
\mathcal{F}f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{itx} \, dt; \quad \mathcal{F}^{-1}F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(x)e^{-itx} \, dx.
\]  
(2.2)

**Lemma 2.1** Let functions $\Omega_1$ and $\Omega_2$ be analytic in the upper half-plane $\mathbb{C}^+$ and the lower half-plane $\mathbb{C}^-$ except their poles $z_0 = \infty$ and $z_k$ ($k = 1, 2, \ldots, n$). Suppose that the boundary values of $\Omega_1$ and $\Omega_2$ on $\text{Im} \, z = 0$ are equal. The main parts of the Laurent expansion of $\Omega_1$ and $\Omega_2$ at $z_0 = \infty$ are

\[
G_0(z) = c_1^0 z + c_2^0 z^2 + \cdots + c_m^0 z^m,
\]  
(2.3)

where $c_m^0 \neq 0$ and $c_k^0$ ($1 \leq k \leq m$) are constants. The main parts of the Laurent expansion for $\Omega_1$ and $\Omega_2$ at every pole $z_k$ ($k = 1, 2, \ldots, n$) are

\[
G_k \left( \frac{1}{z - z_k} \right) = \frac{c_1^k}{z - z_k} + \frac{c_2^k}{(z - z_k)^2} + \cdots + \frac{c_p^k}{(z - z_k)^p}, \quad \forall k \in \{1, 2, \ldots, n\},
\]  
(2.4)

where $c_l^k$ ($1 \leq l \leq p$) are constants, and $c_p^k \neq 0$, $p \geq 1$. Then $\Omega_1$ and $\Omega_2$ can be represented by the same function $\Omega$ in the complex plane $\mathbb{C}$, namely,

\[
\Omega(z) = c_0 + G_0(z) + \sum_{k=1}^{n} G_k \left( \frac{1}{z - z_k} \right),
\]  
(2.5)

where $c_0$ is a complex constant.
Proof. We can prove the lemma by using the generalized Liouville theorem [14, 16, 17] and the principle of analytic continuation [1, 2, 7].

The following lemmas are obvious facts, and we omit their proofs.

Lemma 2.2. If there exists a sufficiently large positive number $B$ such that $F(z) \in H$ for $|z| < B$ and $F(z)$ is analytic for $|z| \geq B$, then $F(z) \in \hat{H}$ on the complex plane $\mathbb{C}$.

Lemma 2.3 (see [1, 2, 4]). Let $f(t) \in [0]$ and $F(x) = \Phi(t)$. The Cauchy-type integral is defined as

$$\Theta(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - z} \, d\tau, \quad z \in \mathbb{C}^+ \cup \mathbb{C}^-.$$  

(2.6)

Then for $z \in \mathbb{C}^+$, we have

$$\Theta^+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - z} \, d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\tau z} \, dt,$$

(2.7)

and for $z \in \mathbb{C}^-$, we have

$$\Theta^-(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - z} \, d\tau = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\tau z} \, dt.$$  

(2.8)

It is easy to prove that $\Theta^+(z)$ and $\Theta^-(z)$ are analytical in $z \in \mathbb{C}^+$ and $z \in \mathbb{C}^-$, respectively.

3. Problem presentation and solution

We now propose boundary value problems for analytic functions with $n \geq 2$ unknown functions on $n$ parallel lines, and then we discuss the methods of solution of such problems.

Suppose that $n$ lines $\gamma_j$ ($1 \leq j \leq n$) are parallel to the $X$-axis and denote them by $L = \Sigma_{j=1}^{n} \gamma_j$, where $\gamma_j(1 \leq j \leq n)$ take the direction from left to right as the positive direction and can be expressed by $\xi_j = x + i R_j (R_n < \cdots < R_2 < R_1)$, where $x, R_j \in \mathbb{R}$. Our goal is to obtain functions $F_j(z)(1 \leq j \leq n)$ such that $F_j(z)$ are analytic in $R_j < \text{Im} z < R_{j-1}$ ($2 \leq j \leq n$) and $F_1(z)$ is analytic in $\text{Im} z > R_1$ and $\text{Im} z < R_n$, and the following boundary value conditions are fulfilled:

$$F_j^+(\xi) - A_j(\xi) F_{j-1}^-(\xi) = B_j(\xi), \quad \xi \in \gamma_j, j = 1, 2, \ldots, n,$$

(3.1)

when $j = n$, denote $F_{n+1}^- (\xi)$ as $F_n^-(\xi)$, where the given functions $A_j(\xi), B_j(\xi)(1 \leq j \leq n)$ belong to $H_+ \text{ on } \gamma_j$. Obviously, R-HP (3.1) can also be written in the following form:

$$\begin{align*}
F_1^+(\xi) - A_1(\xi) F_2^- (\xi) &= B_1(\xi), \quad \xi \in \gamma_1, \\
F_2^+(\xi) - A_2(\xi) F_3^- (\xi) &= B_2(\xi), \quad \xi \in \gamma_2, \\
&\vdots \\
F_{n-1}^+(\xi) - A_{n-1}(\xi) F_n^- (\xi) &= B_{n-1}(\xi), \quad \xi \in \gamma_{n-1}, \\
F_n^+(\xi) - A_n(\xi) F_1^- (\xi) &= B_n(\xi), \quad \xi \in \gamma_n.
\end{align*}$$

(3.2)
It follows from (3.1) that the orders of $F_j(z)(1 \leq j \leq n)$ are equal to each other at infinity. Thus, when the orders of $F_j(z)$ are $m$ at infinity, (3.1) can be denoted as $R_m$. In fact, the problem $R_0$ and problem $R_1$ are frequently discussed. On the problem $R_0$, $F_j(z)$ are supposed to be finite and nonzero at infinity. On problem $R_1$, $F_j(z)$ are assumed to be zero at infinity. When $A_j(\xi)(1 \leq j \leq n)$ are not zero on $L$, problem (3.1) is said to be of normal type; otherwise, it is called of nonnormal type or of exception type.

Note that since the positive direction of $\gamma_j(1 \leq j \leq n)$ is the direction from left to right, when the observer moves from left to right on $\gamma_j$, the boundary values of left domain of $\gamma_j$ are positive, that is, the positive boundary values of $F_j(z)(1 \leq j \leq n)$ are the boundary values above $\gamma_j$, and the negative boundary values of $F_j(z)$ are those below $\gamma_j$.

Without loss of generality, in this paper, we only discuss the case $n = 2$. As for R-HP with $n > 2$ unknown functions on $n$ parallel lines, there is no essential difference for the methods of solution with the case $n = 2$.

When $n = 2$, R-HP (3.1) can be stated as follows.

**Problem** Assume that $\gamma_1 : \xi = x + i\beta$ and $\gamma_2 : \xi = x + i\alpha$ are two oriented lines, where $\alpha$ and $\beta$ are real numbers with $\alpha < \beta$. Similarly to the above statement, we take the direction of $\gamma_1$ and $\gamma_2$ from left to right as the positive direction. We want to obtain functions $F_1(z)$ and $F_2(z)$ such that $F_1(z)$ is analytic in $\{\text{Im } z > \beta\} \cup \{\text{Im } z < \alpha\}$, and $F_2(z)$ is analytic in $\{z : \alpha < \text{Im } z < \beta\}$ and satisfies the following boundary value conditions on $\gamma_1$ and $\gamma_2$:

$$
\begin{align*}
F_1^+(\xi) - A_1(\xi)F_2^-(\xi) &= B_1(\xi), \quad \xi \in \gamma_1, \\
F_2^+(\xi) - A_2(\xi)F_1^-(\xi) &= B_2(\xi), \quad \xi \in \gamma_2.
\end{align*}
$$

(3.3)

In fact, (3.3) is the R-HP on two parallel straight lines $\text{Im } z = \beta$ and $\text{Im } z = \alpha$ with $z = \infty$ as a pole, and it is a generalization of the classical R-HP.

Here $F_1^+(\xi)$ is the boundary value of the analytic function $F_1(z)$, which is analytic in $\{z : \text{Im } z > \beta\}$, and belongs to $H_+$ on $\gamma_1$; $F_1^-(\xi)$ is the boundary value of the analytic function $F_1(z)$, which is analytic in $\{z : \text{Im } z < \alpha\}$, and belongs to $H_+$ on $\gamma_2$; $F_2^+(\xi)$ are the boundary values of the analytic function $F_2(z)$, which is analytic in $\{z : \alpha < \text{Im } z < \beta\}$, and belong to $H_+$ on $\gamma_1, \gamma_2$, respectively. Since $A_j(\xi), B_j(\xi)$ ($j = 1, 2$) belong to $H_+$ on $\gamma_j$. Thus, for the functions appearing in (3.3), their one-sided limits exist as $x \to \infty$ on $\gamma_1, \gamma_2$, and we have

$$
\lim_{x \to +\infty} A_1(i\beta + x) = \lim_{x \to -\infty} A_1(i\beta + x), \quad \lim_{x \to +\infty} A_2(i\alpha + x) = \lim_{x \to -\infty} A_2(i\alpha + x).
$$

(3.4)

Similarly to $B_j(\xi)$ ($j = 1, 2$), we also have

$$
B_1(i\beta + \infty) = B_1(i\beta - \infty), \quad B_2(i\alpha + \infty) = B_2(i\alpha - \infty).
$$

(3.5)

In this paper, we only solve problem (3.3) in problem $R_0$. For problem $R_m$, similar arguments can be done. Since $F_1(z) \in H_+$ on $\gamma_j$ ($j = 1, 2$), the $\lim_{x \to \infty} F_1(z)$ exists, that is, both $F_1^+(\infty)$ and $F_1^-(\infty)$ exist. Therefore, on $\gamma_1$ and $\gamma_2$, we have

$$
\lim_{x \to +\infty} F_1^+(i\beta + x) = \lim_{x \to +\infty} F_1^-(i\beta + x), \quad \lim_{x \to +\infty} F_1^+(i\alpha + x) = \lim_{x \to +\infty} F_1^-(i\alpha + x).
$$

(3.6)
Similarly, the \( \lim_{z \to \infty} F_2(z) \) exists, and so do \( F_1^\pm(\infty) \) and \( F_2^\pm(\infty) \). Then on \( \gamma_1 \) and \( \gamma_2 \), we also have

\[
F_2^+ (i \beta + \infty) = F_2^+ (i \beta - \infty), \quad F_2^- (i \alpha + \infty) = F_2^- (i \alpha - \infty). \tag{3.7}
\]

Hence, \( F_1(\infty) \) and \( F_2(\infty) \) are finite and nonzero. Here we only consider the case of normal type, that is, \( A_j(\xi), A_j^{-1}(\xi) \) have no zero point on \( \gamma_j \). For the case of nonnormal type, similar discussions also work (see [13–15]). To solve R-HP (3.3), we define \( \kappa_j \) as follows:

\[
k_j = \text{Ind}_{\gamma_j} A_j(\xi) = \frac{1}{2\pi i} \left[ \arg A_j(\xi) \right]_{\gamma_j}, \quad j = 1, 2,
\]

and

\[
\kappa = \kappa_1 + \kappa_2. \tag{3.8}
\]

Then we call \( \kappa \) as the index of R-HP (3.3). Set

\[
\begin{align*}
\lambda^{(1)}_\infty &= \mu^{(1)}_\infty + i \nu^{(1)}_\infty = \frac{1}{2\pi i} \ln \frac{A_1(+\infty + i\beta)}{A_1(-\infty + i\beta)}, \\
\lambda^{(2)}_\infty &= \mu^{(2)}_\infty + i \nu^{(2)}_\infty = \frac{1}{2\pi i} \ln \frac{A_2(+\infty + i\alpha)}{A_2(-\infty + i\alpha)}, \\
\gamma_\infty &= \lambda^{(1)}_\infty + \lambda^{(2)}_\infty, \quad \mu_\infty = \mu^{(1)}_\infty + \mu^{(2)}_\infty.
\end{align*} \tag{3.9}
\]

Since \( z = \infty \) is a branch point of \( \ln A_j(\xi) \) \( (j = 1, 2) \), on the neighborhood \( \mathcal{N}(\infty) \) of \( \infty \), \( \ln A_1(\xi) \) and \( \ln A_2(\xi) \) are taken to be continuous branches, respectively, such that \( 0 \leq \mu_\infty < 1 \). Without loss of generality, we take three points \( z_0, z_1, z_2 \) such that \( \alpha < \Im z_0 < \beta, \Im z_1 > \beta, \Im z_2 < \alpha \), and we define the functions \( Y_1(z) \) and \( Y_2(z) \) as follows:

\[
Y_1(z) = \begin{cases} 
\frac{e^{\Gamma_1(t)}}{(z-z_0)^{\kappa_1}}, & \text{Im } z > \beta, \\
\frac{e^{\Gamma_2(t)}}{(z-z_0)^{\kappa_2}}, & \text{Im } z < \beta,
\end{cases} \tag{3.10}
\]

and

\[
Y_2(z) = \begin{cases} 
\frac{e^{\Gamma_2(t)}}{(z-z_0)^{\kappa_2}}, & \text{Im } z > \alpha, \\
\frac{e^{\Gamma_1(t)}}{(z-z_0)^{\kappa_1}}, & \text{Im } z < \alpha,
\end{cases} \tag{3.11}
\]

where

\[
\begin{align*}
\Gamma_1(z) &= \frac{z - z_0}{2\pi i} \int_{-\infty + i\beta}^{+\infty + i\beta} \frac{\ln E_1(t)}{(t-z)(t-z_0)} \, dt, \\
\Gamma_2(z) &= \frac{z - z_0}{2\pi i} \int_{-\infty + i\alpha}^{+\infty + i\alpha} \frac{\ln E_2(t)}{(t-z)(t-z_0)} \, dt, \tag{3.12}
\end{align*}
\]

\[
E_j(t) = \left( \frac{t - z_0}{t - z_j} \right)^{\kappa_j} A_j(t), \quad j = 1, 2, \text{Im } z \neq \alpha, \beta,
\]
in which we have taken the analytic branches of \( \ln E_j(t) \) \((j = 1, 2)\), provided that we have chosen
\[
\lim_{t \to \infty} \frac{t - z_0}{t - z_j} = 0, \quad \forall j = 1, 2.
\]

Note that the integrands appeared in (3.12) belong to \( H_\alpha \) on \( \gamma_1, \gamma_2 \), respectively, and therefore their integrals exist. Due to (3.10) and (3.11), we know that \( Y_1^+(z) \) and \( Y_1^-(z) \) are analytic in \( \text{Im} \ z > \beta \) and \( \text{Im} \ z < \beta \), respectively; \( Y_2^+(z) \) and \( Y_2^-(z) \) are analytic in \( \text{Im} \ z > \alpha \) and \( \text{Im} \ z < \alpha \), respectively. Therefore, \( Y_1(z) \) and \( Y_2(z) \) are sectionally holomorphic functions, and
\[
Y_j(z) = O(|z|^{-\kappa}) (z \to \infty), \quad j = 1, 2.
\]

It is easy to see that \( Y_1(z) \), \( Y_2(z) \) are the canonical functions, and one has
\[
Y_j^+(t) = A_j(t)Y_j^-(t), \quad \forall j \in \{1, 2\}. \quad (3.13)
\]

Denote
\[
Y(z) = Y_1(z)Y_2(z),
\]

it is not difficult to verify that \( Y(z) \) satisfies the definition of a canonical function (see [1–3]), thus we call \( Y(z) \) as the canonical function of (3.3). Since \( A_j(\xi) \) \((j = 1, 2)\) belong to \( H_\alpha \), similar to the discussion in [4], one must have, near \( z = \infty \),
\[
Y(\xi) = \frac{Y_0(\xi)}{\xi}, \quad Y_0(\xi) \in H(N_\infty).
\]

Substituting (3.13) into (3.3), we obtain
\[
\begin{cases}
[Y_1^+(\xi)]^{-1}F_1^+(\xi) - [Y_1^-(\xi)]^{-1}F_2^-(\xi) = B_1(\xi)[Y_1^+(\xi)]^{-1}, & \xi \in \gamma_1; \\
[Y_2^+(\xi)]^{-1}F_1^+(\xi) - [Y_2^-(\xi)]^{-1}F_2^-(\xi) = B_2(\xi)[Y_2^+(\xi)]^{-1}, & \xi \in \gamma_2.
\end{cases} \quad (3.14)
\]

The first equality in (3.14) is multiplied by \( [Y_2^+(\xi)]^{-1} \), and the second one is multiplied by \( [Y_1^+(\xi)]^{-1} \), so we get
\[
\begin{cases}
[Y_1^+(\xi)Y_2^+(\xi)]^{-1}F_1^+(\xi) - [Y_1^+(\xi)]^{-1}F_2^+(\xi) = B_1(\xi)[Y_1^+(\xi)Y_2^+(\xi)]^{-1}, & \xi \in \gamma_1; \\
[Y_1^+(\xi)Y_2^+(\xi)]^{-1}F_1^+(\xi) - [Y_1^+(\xi)]^{-1}F_2^+(\xi) = B_2(\xi)[Y_1^+(\xi)Y_2^+(\xi)]^{-1}, & \xi \in \gamma_2.
\end{cases} \quad (3.15)
\]

Note that in (3.15), we cannot apply the Sokhotski–Plemelj formula [8, 16, 17] directly. In general, when \( \kappa = \kappa_1 + \kappa_2 > 0 \), \( B_1(\xi)[Y_1^+(\xi)Y_2^+(\xi)]^{-1} \notin H_\alpha \), and so does \( B_2(\xi)[Y_1^+(\xi)Y_2^+(\xi)]^{-1} \).

In order to unify, no matter how \( \kappa \) is chosen, let
\[
X_1(z) = \begin{cases}
e^{-\gamma_1(z)}, & \text{Im} \ z > \beta, \\
e^{\gamma_1(z)}(z - z_0)^{-\kappa_1}, & \text{Im} \ z < \beta,
\end{cases} \quad (3.16)
\]
Thus the first equality in (3.18) can be reduced to

\[
\begin{align*}
\{[X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - [X_1^-(\xi)X_2^-(\xi)]^{-1}F_2^-(\xi) &= B_1(\xi)[X_1^+(\xi)X_2^+(\xi)]^{-1}, \\
\xi &\in \gamma_1, \\
[X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - [X_1^- (\xi)X_2^- (\xi)]^{-1}F_2^- (\xi) &= B_2(\xi)[X_1^+(\xi)X_2^+(\xi)]^{-1}, \\
\xi &\in \gamma_2.
\end{align*}
\]  

(3.18)

The first and second equalities in (3.15) are multiplied by \((\xi - z_0)^{-x}\), respectively, and thus (3.15) can be transformed to

\[
\begin{align*}
\{[X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - [X_1^-(\xi)X_2^-(\xi)]^{-1}F_2^- (\xi) &= B_1(\xi)[X_1^+(\xi)X_2^+(\xi)]^{-1}, \\
\xi &\in \gamma_1, \\
[X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - [X_1^- (\xi)X_2^- (\xi)]^{-1}F_2^- (\xi) &= B_2(\xi)[X_1^+(\xi)X_2^+(\xi)]^{-1}, \\
\xi &\in \gamma_2.
\end{align*}
\]  

(3.18)

Since \(X_1(z), X_2(z)\) are bounded and nonzero on \(\gamma_1, \gamma_2\), respectively, and \(B_1(t) \in H_+\) by (3.16) and (3.17) we get \(\frac{B_1(t)}{X_1(t)X_2(t)} \in H_+\). Therefore we can define the sectionally analytic function

\[
\psi_1(z) = \frac{1}{2\pi i} \int_{\alpha - i\beta}^{\alpha + i\beta} \frac{B_1(t)[X_1^+(t)X_2^+(t)]^{-1}}{t - z} \, dt, \quad \text{Im} \, z \neq \beta.
\]  

(3.19)

According to Privalov’s theorem (see [1]), we easily see that \(\psi_1(z)\) is analytic in \(\text{Im} \, z > \beta\) and \(\text{Im} \, z < \beta\). Applying the Sokhotski–Plemelj formula to \(\psi_1(z)\) in (3.19), we have

\[
\psi_1^+(\xi) - \psi_1^-(\xi) = B_1(\xi)[X_1^+(\xi)X_2^+(\xi)]^{-1}, \quad \xi \in \gamma_1.
\]  

(3.20)

Thus the first equality in (3.18) can be reduced to

\[
\begin{align*}
[X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - [X_1^- (\xi)X_2^- (\xi)]^{-1}F_2^- (\xi) &= \psi_1^+(\xi) - \psi_1^-(\xi), \\
\xi &\in \gamma_1.
\end{align*}
\]  

(3.21)

Similarly, we also define the function

\[
\psi_2(z) = \frac{1}{2\pi i} \int_{\alpha - i\beta}^{\alpha + i\beta} \frac{B_2(t)[X_1^+(t)X_2^+(t)]^{-1}}{t - z} \, dt, \quad \text{Im} \, z \neq \alpha.
\]  

(3.22)

We again apply the Sokhotski–Plemelj formula to \(\psi_2(z)\) in (3.22), and thus the second equality in (3.18) can also be reduced to

\[
\begin{align*}
[X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - [X_1^- (\xi)X_2^- (\xi)]^{-1}F_2^- (\xi) &= \psi_2^+(\xi) - \psi_2^-(\xi), \\
\xi &\in \gamma_2.
\end{align*}
\]  

(3.23)

Combining (3.21) and (3.23), we obtain

\[
\begin{align*}
\{[X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - \psi_1^+(\xi) &= [X_1^+(\xi)X_2^+(\xi)]^{-1}F_2^- (\xi) - \psi_1^-(\xi), \quad \xi \in \gamma_1, \\
[X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - \psi_2^+(\xi) &= [X_1^+(\xi)X_2^+(\xi)]^{-1}F_2^- (\xi) - \psi_2^-(\xi), \quad \xi \in \gamma_2.
\end{align*}
\]  

(3.24)
Thus we need only to discuss problem (3.24) instead of (3.3). Adding $-\psi_2^-(\xi)$ in the first equality of (3.24) and subtracting $\psi_1^-(\xi)$ in the second one, we get

\[
\begin{cases}
X_1^-(\xi)X_2^-(\xi)F_1^-(\xi) - \psi_1^-(\xi) - \psi_2^-(\xi) \\
X_1^-(\xi)X_2^-(\xi)F_2^-(\xi) - \psi_1^-(\xi) - \psi_2^-(\xi)
\end{cases}, \quad \xi \in \gamma_1,
\]

\[
\begin{cases}
X_1^-(\xi)X_2^-(\xi)F_1^-(\xi) - \psi_1^-(\xi) - \psi_2^-(\xi) \\
X_1^-(\xi)X_2^-(\xi)F_2^-(\xi) - \psi_1^-(\xi) - \psi_2^-(\xi)
\end{cases}, \quad \xi \in \gamma_2.
\]

(3.25)

We denote the left side of the first equality in (3.25) by $\Phi_1^+(z)$, whereas the right side is denoted by $\Phi_1^-(z)$, that is,

\[
\Phi_1^+(z) = [X_1^-(z)X_2^-(z)]^{-1}F_1^-(z) - \psi_1^-(z) - \psi_2^-(z),
\]

\[
\Phi_1^-(z) = [X_1^-(z)X_2^-(z)]^{-1}F_2^-(z) - \psi_1^-(z) - \psi_2^-(z).
\]

(3.26)

We define the following function $\Phi_1(z)$:

\[
\Phi_1(z) = \Phi_1^+(z) \quad \text{when} \quad \text{Im} z > \beta; \quad \Phi_1(z) = \Phi_1^-(z) \quad \text{when} \quad \text{Im} z < \beta.
\]

(3.27)

By the boundary value conditions (3.25) we know that $\Phi_1^+(z) = \Phi_1^-(z)$ on $\text{Im} z = \beta$.

We first give explicit solutions of $\Phi_1^+(z)$ and $\Phi_1^-(z)$ in $\text{Im} z > \beta$ and $\text{Im} z < \beta$, respectively.

In $\text{Im} z > \beta$, since $[X_1^-(z)]^{-1}$ and $[X_2^-(z)]^{-1}$ are analytic, $[X_1^-(z)X_2^-(z)]^{-1}$ is also analytic. Since $\psi_j^-(z)$ ($j = 1, 2$) are analytic, it follows that $\Phi_1^+(z)$ is analytic, so the $\lim_{z \to \infty} \Phi_1^+(z)$ exists. For convenience, we assume that

\[
\Phi_1^+(z) = D_1(z),
\]

and thus

\[
[X_1^-(z)X_2^-(z)]^{-1}F_1^-(z) - \psi_1^-(z) - \psi_2^-(z) = D_1(z), \quad \text{Im} z > \beta,
\]

(3.28)

where $D_1(z)$ is analytic in $\text{Im} z > \beta$, and the $\lim_{z \to \infty} D_1(z)$ exists.

In $\text{Im} z < \beta$, when $\kappa \geq 0$, $[X_j^-(z)]^{-1}$ and $[X_1^-(z)X_2^-(z)]^{-1}$ have a pole $z_0$ of order $\kappa$; when $\kappa < 0$, $[X_1^-(z)X_2^-(z)]^{-1}$ has no singularity, but $X_1^-(z)X_2^-(z)$ has a pole $z_0$ of order $|\kappa|$. Hence, using Lemmas 2.1 and 2.2, the generalized Liouville theorem, and the principle of analytic continuation, it follows that

\[
\Phi_1^-(z) = \frac{q_k(z)}{(z - z_0)^\kappa},
\]

(3.29)

where we have set

\[
q_k(z) = c_0 + c_1z + c_2z^2 + \cdots + c_\kappa z^\kappa
\]

with arbitrary constants $c_j$ ($0 \leq j \leq \kappa$). When $\kappa > -1$, $q_k(z)$ is a polynomial of degree no more than $\kappa$; when $\kappa \leq -1$, $q_k(z) \equiv 0$. Due to (3.26), (3.28), and (3.29), we have

\[
[X_1^-(z)X_2^-(z)]^{-1}F_1^-(z) - \psi_1^-(z) - \psi_2^-(z) = \Phi_1(z), \quad \text{Im} z > \beta,
\]

\[
[X_1^-(z)X_2^-(z)]^{-1}F_2^-(z) - \psi_1^-(z) - \psi_2^-(z) = \Phi_1(z), \quad \text{Im} z < \beta.
\]

(3.30)
that is,
\[
\begin{align*}
F_1^+(z) &= X_1^+(z)X_2^+(z)\left[\psi_1^+(z) + \psi_2^+(z) + \Phi_1(z)\right], \quad \text{Im} \, z > \beta, \\
F_2^+(z) &= X_1^-(z)X_2^+(z)\left[\psi_1^-(z) + \psi_2^+(z) + \Phi_1(z)\right], \quad \text{Im} \, z < \beta,
\end{align*}
\]
(3.31)
where
\[
\Phi_1(z) = \begin{cases} 
D_1(z), & \text{Im} \, z > \beta, \\
\frac{q_k(z)}{|z|^\kappa}, & \text{Im} \, z < \beta.
\end{cases}
\]

Similarly, the left side of the second equality in (3.25) is denoted by \( \Phi_2^+(z) \), whereas the right side is denoted by \( \Phi_2^-(z) \), namely,
\[
\begin{align*}
\Phi_2^+(z) &= \left[X_1^-(z)X_2^+(z)\right]^{-1}F_1^+(z) - \psi_1^+(z) - \psi_1^-(z); \\
\Phi_2^-(z) &= \left[X_1^-(z)X_2^+(z)\right]^{-1}F_1^-(z) - \psi_2^+(z) - \psi_1^-(z).
\end{align*}
\]
(3.32)

We also have
\[
\Phi_2(z) = \begin{cases} 
\Phi_2^+(z), & \text{Im} \, z > \alpha, \\
\Phi_2^-(z), & \text{Im} \, z < \alpha.
\end{cases}
\]
(3.33)

In \( \text{Im} \, z < \alpha \), since \( [X_1^-(z)]^{-1} \) and \( \psi_j^-(z) \) \((j = 1, 2)\) are analytic, \( \Phi_2^-(z) \) is also analytic. Similarly to the above discussion, we may denote \( \Phi_2^+(z) = D_2(z) \), where \( D_2(z) \) is analytic in \( \text{Im} \, z < \alpha \), and the lim_{z \to \infty} D_2(z) exists.

In \( \text{Im} \, z > \alpha \), note that when \( \kappa \geq 0 \), \( [X_1^+(z)X_2^+(z)]^{-1} \) has a pole \( z = z_0 \) of order \( \kappa \); when \( \kappa < 0 \), \( [X_1^-(z)X_2^+(z)]^{-1} \) has no singularity, but \( X_1^-(z)X_2^+(z) \) has a pole \( z = z_0 \) of order \( |\kappa| \). Therefore we also obtain
\[
\Phi_2^+(z) = \frac{q_k(z)}{(z - z_0)^\kappa},
\]
(3.34)
where \( q_k(z) \) is as before. From the previous discussions we know that
\[
\begin{align*}
F_1^-(z) &= X_1^-(z)X_2^-(z)\left[\psi_1^-(z) + \psi_2^-(z) + \Phi_2(z)\right], \quad \text{Im} \, z < \alpha, \\
F_2^+(z) &= X_1^-(z)X_2^+(z)\left[\psi_1^+(z) + \psi_2^+(z) + \Phi_2(z)\right], \quad \text{Im} \, z > \alpha,
\end{align*}
\]
(3.35)
where
\[
\Phi_2(z) = \begin{cases} 
D_2(z), & \text{Im} \, z < \alpha, \\
\frac{q_k(z)}{|z-\alpha|}, & \text{Im} \, z > \alpha.
\end{cases}
\]

Combining (3.31) and (3.35), R-HP (3.3) has the general solutions given by the formulas
\[
\begin{align*}
F_1^+(z) &= X_1^+(z)X_2^+(z)\left[\psi_1^+(z) + \psi_2^+(z) + \Phi(z)\right], \quad \text{Im} \, z > \beta, \\
F_2^-(z) &= X_1^+(z)X_2^-(z)\left[\psi_1^-(z) + \psi_2^-(z) + \Phi(z)\right], \quad \text{Im} \, z < \alpha, \\
F_2(z) &= X_1^-z)X_2^+(z)\left[\psi_1^+(z) + \psi_2^+(z) + \Phi(z)\right], \quad \alpha < \text{Im} \, z < \beta,
\end{align*}
\]
(3.36)
in which we have put

$$\Phi(z) = \begin{cases} 
D_1(z), & \text{Im} z > \beta, \\
D_2(z), & \text{Im} z < \alpha, \\
q_k(z), & \alpha < \text{Im} z < \beta,
\end{cases}$$

(3.37)

where $q_k(z)$ is a polynomial of degree no more than $\kappa$ as before, and $D_1(z)$ and $D_2(z)$ are analytic in $\text{Im} z > \beta$ and $\text{Im} z < \alpha$, respectively.

Now we can formulate the main results with respect to the solutions of R-HP (3.3).

**Theorem 3.1** R-HP (3.3) with two unknown functions $F_1(z)$ and $F_2(z)$ on two parallel lines has solutions in $\{z: \alpha < \text{Im} z < \beta\}$ and $\{\text{Im} z > \beta\} \cup \{\text{Im} z < \alpha\}$, respectively. Its general solutions can be expressed by (3.36), where $X_j(z)$ ($j=1,2$) are defined by (3.16) and (3.17), and $\psi_1(z), \psi_2(z)$ are given by (3.19) and (3.22), respectively. When $\kappa > -1$, $q_k(z)$ is a polynomial of degree no more than $\kappa$; when $\kappa \leq -1$, $q_k(z) = 0$. In conclusion, the degree of freedom of the solution is equal to $\kappa + 1$.

If $F_1(\infty) = F_2(\infty) = 0$, then (3.3) gives a solution in problem $R_{-1}$. In this case, we have the following conclusion.

**Theorem 3.2** Under the conditions $F_1(\infty) = F_2(\infty) = 0$, R-HP (3.3) has a solution if and only if $B_1(\infty) = B_2(\infty) = 0$. In such a case, a solution of (3.3) is similar to (3.36), and the only difference lies in that $q_k(z)$ should be substituted by $q_{k-1}(z)$ in (3.36). However, the degree of freedom of the solution for (3.3) is equal to $\kappa$.

At the end of this section, we give an important example in practical application. In problem (3.3), we suppose that

$$A_1(z) = A_2(z) = 1, \quad B_1(z) = \frac{1}{2 + z^2}, \quad B_2(z) = \frac{1}{1 + z^2},$$

(3.38)

$$\gamma_1: z = 0, \quad \gamma_2: z = x + i, \quad -\infty < x < +\infty.$$

Then (3.3) can be transformed to the following form:

$$\begin{cases} 
F_1'(z) - F_2'(z) = \frac{1}{z + z}, & z \in \gamma_1, \\
F_2(z) - F_1(z) = \frac{1}{z + z}, & z \in \gamma_2.
\end{cases}$$

(3.39)

Without loss of generality, we assume that $z_0 = \frac{1}{2}i, z_1 = \frac{3}{2}i, z_2 = -\frac{3}{2}i$. Then we have $\kappa_1 = \kappa_2 = 0$ and $\kappa = 0$. Therefore we get

$$\Gamma_j(t) = 0, \quad X_j(z) = 1, \quad j = 1, 2.$$
Similarly to the discussion in [1–3], from (3.19), (3.22), and Lemma 2.3 we have

when \( \text{Im} z > 1 \),

\[
\psi_1^+(z) = \frac{1}{2\pi i} \int_{-\infty+i}^{\infty+i} \frac{dt}{(2 + t^2)(t - z)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i}^{\infty+i} e^{-iz} \frac{d\tau}{2 + \tau^2} d\tau = \frac{i}{2\sqrt{2}(z + \sqrt{2}i)},
\]

when \( \text{Im} z < 1 \),

\[
\psi_1^-(z) = \frac{1}{2\pi i} \int_{-\infty+i}^{\infty+i} \frac{dt}{(2 + t^2)(t - z)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i}^{\infty+i} e^{-iz} \frac{d\tau}{2 + \tau^2} d\tau = \frac{i}{2\sqrt{2}(z - \sqrt{2}i)},
\]

when \( \text{Im} z > 0 \),

\[
\psi_2^+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)(t - z)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iz} \frac{d\tau}{1 + \tau^2} d\tau = \frac{i}{2(z + i)},
\]

when \( \text{Im} z < 0 \),

\[
\psi_2^-(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)(t - z)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iz} \frac{d\tau}{1 + \tau^2} d\tau = \frac{i}{2(z - i)}.
\]

Then we obtain the following solutions of problem (3.39):

\[
F_1^+(z) = \frac{i}{2\sqrt{2}(z + \sqrt{2}i)} + \frac{i}{2(z + i)} + D(z), \quad \text{Im} z > 1,
\]

\[
F_1^-(z) = \frac{i}{2\sqrt{2}(z - \sqrt{2}i)} + \frac{i}{2(z - i)} + D(z), \quad \text{Im} z < 0,
\]

\[
F_2(z) = \frac{i}{2\sqrt{2}(z - \sqrt{2}i)} + \frac{i}{2(z + i)} + D(z), \quad 0 < \text{Im} z < 1,
\]

where \( D(z) \) is an analytic function in the whole complex plane \( \mathbb{C} \). We can easily verify that (3.42) are solutions of problem (3.39).

4 The conditions of solvability of R-HP (3.3)

Now we are concerned about solution (3.36) and the conditions of solvability of R-HP (3.3).

(1) It follows from the discussion in Sect. 3 that when \( \kappa < 0 \), \( z_0 \) is a \( |\kappa| \)th-order pole of \( X_1^-(z)X_2^+(z) \). To guarantee that (3.3) is solvable, the following \( -\kappa \) conditions are required:

\[
\int_{-\infty+i\beta}^{\infty+i\beta} \frac{B_1(t)(t - z)^j}{X_1^+(t)X_2^+(t)} dt = -\int_{-\infty+i\alpha}^{\infty+i\alpha} \frac{B_2(t)(t - z)^j}{X_1^-(t)X_2^+(t)} dt, \quad j \in \{-1, -2, \ldots, \kappa\}. \quad (4.1)
\]

(2) The cases of the solutions in \( \alpha < \text{Im} z < \beta \). For \( z \in \{z : \alpha < \text{Im} z < \beta\} \), if \( A_1(z) \) and \( A_2^+(z) \) have no zero points, then \( F_1(z) \) and \( F_2(z) \) can be obtained by (3.36). Otherwise, if \( \vartheta_1, \vartheta_2, \ldots, \vartheta_n \) are common zero points of \( A_1(z) \) and \( A_2^+(z) \) of orders \( s_1, s_2, \ldots, s_n \), respec-
tively, then we must have

\[ F_2^{(j)}(z)|_{z=\theta_k} = 0, \quad 1 \leq k \leq n, 1 \leq j \leq s_k. \] (4.2)

Therefore we obtain:

(a) when \( \kappa \geq 0 \), the following equations with the unknown elements \( c_0, c_1, \ldots, c_k \) have solutions:

\[
\left[ \frac{q_k(z)}{(z - z_0)^\mu} \right]_{z=\theta_k}^{(0)} = \frac{j^l}{2\pi i} \int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_2(\xi)[X_1^* (\xi)X_2^* (\xi)]^{-1} d\xi}{(\xi - \theta_k)^{\mu+1}(\xi - z_0)}
\]

\[
- \frac{j^l}{2\pi i} \int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_1(\xi)[X_1^* (\xi)X_2^* (\xi)]^{-1} d\xi}{(\xi - \theta_k)^{\mu+1}(\xi - z_0)}; \] (4.3)

(b) when \( \kappa < 0 \), the following equalities must be satisfied:

\[
\int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_2(\xi)[X_1^* (\xi)X_2^* (\xi)]^{-1} d\xi}{(\xi - \theta_k)^{\mu+1}(\xi - z_0)} = \int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_1(\xi)[X_1^* (\xi)X_2^* (\xi)]^{-1} d\xi}{(\xi - \theta_k)^{\mu+1}(\xi - z_0)}, \] (4.4)

where \( j = 0, 1, 2, \ldots, s_k \), \( k = 1, 2, \ldots, n \), and \( c_0, c_1, \ldots, c_k \) are the coefficients of \( q_k(z) \).

(3) The behavior of solution at \( z = \infty \). If \( z = \infty \) is an ordinary node, then \( 0 < \mu_\infty < 1 \). It follows from \( \psi_1(\infty) = \psi_2(\infty) = 0 \) that near \( z = \infty \),

\[
\psi_j(\xi) = \psi_j^*(\xi) \xi^{-\mu_j^*}, \quad 0 < \mu_j^* < \mu^{(j)}, \quad \psi_j^*(\xi) \in H, \quad j = 1, 2. \] (4.5)

In (3.36), since \( F_2(\xi) \) is bounded, \( F_2(\infty) \) is taken as a finite value. Note that \( 0 < \mu_\infty < 1 \). If \( \mu_\infty > \frac{1}{2} \), then we easily see that

\[
X_1^* (\xi)X_2^* (\xi)q_\kappa(\xi)(\xi - z_0)^{-\kappa} = O(\xi|^{-\mu_\infty}) \quad (\xi \to \infty) \] (4.6)

and

\[
X_1^* (\xi)X_2^* (\xi)(\psi_1(\xi) - \psi_2(\xi)) = O(\xi|^{-\mu_\infty + \epsilon}) \quad (\xi \to \infty), \] (4.7)

where \( \epsilon \) is a sufficiently small positive number such that \( \mu_\infty - \epsilon > \frac{1}{2} \).

If \( \mu_\infty \leq \frac{1}{2} \), when \( \kappa \geq 0 \), the coefficient \( e_\kappa \) of \( \kappa \)-th-power item in \( q_k(z) \) should be taken as

\[
e_\kappa = \frac{1}{2\pi i} \int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_1(\xi) d\xi}{X_1^* (\xi)X_2^* (\xi)(\xi - z_0)} + \frac{1}{2\pi i} \int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_2(\xi) d\xi}{X_1^* (\xi)X_2^* (\xi)(\xi - z_0)}; \] (4.8)

when \( \kappa < 0 \), (4.1) is required to fulfill.

If \( z = \infty \) is a special node, that is, \( \mu_\infty = 0 \), then we can transform it into the case \( \mu_\infty \leq \frac{1}{2} \) as an ordinary node, and similar arguments can be done (see [16–19]).

5 Conclusions

In this paper, we propose one kind of R-HP with several unknown functions in strip domains. By using the methods of the classical boundary value problems for analytic functions, we obtain the exact solution, defined by integrals, of problem (3.3) and the conditions of solvability on function spaces \( H_\kappa \). Our method is different from those of the classical R-HP and is novel and effective.
R-HP (3.1) has important applications in practical problems, such as elastic mechanics, heat conduction, and electrostatics. Many problems, such as piezoelectric material, voltage magnetic materials, and functional gradient materials, can often attribute the problem to find the solutions of (3.1). Therefore the solving method of (3.1) has important meaning not only in applications, but also in the theory of resolving the equation itself. This paper mainly deals with the solvability and explicit solutions of the R-HP with two unknown functions. Indeed, it is possible to solving the problem mentioned above in Clifford analysis, which is similar to that in [20–27]. We omit further discussion.

Acknowledgements
The author is thankful to the editor and anonymous referees for their valuable comments and suggestions.

Funding
Research is supported by the National Natural Science Foundation of China (Grant No. 11971015).

Availability of data and materials
Not applicable.

Competing interests
The author declares that no competing interests exist.

Author’s contributions
Author read and approved the final manuscript.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 17 April 2020  Accepted: 20 October 2020  Published online: 29 October 2020

References
1. Muskhelishvilli, N.I.: Singular Integral Equations. Nauka, Moscow (2002)
2. Lu, J.K.: Boundary Value Problems for Analytic Functions. World Scientific, Singapore (2004)
3. Li, P.R.: Generalized boundary value problems for analytic functions with convolutions and its applications. Math. Methods Appl. Sci. 42, 2631–2645 (2019)
4. Li, P.R., Ren, G.B.: Solvability of singular integro-differential equations via Riemann–Hilbert problem. J. Differ. Equ. 265, 5455–5471 (2018)
5. Du, H., Shen, J.H.: Reproducing kernel method of solving singular integral equation with cosecant kernel. J. Math. Anal. Appl. 348(1), 308–314 (2008)
6. Li, P.R., Ren, G.B.: Some classes of equations of discrete type with harmonic singular operator and convolution. Appl. Math. Comput. 284, 185–194 (2016)
7. Litvinchuc, G.S.: Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift. Kluwer Academic, London (2004)
8. Li, P.R.: Generalized convolution-type singular integral equations. Appl. Math. Comput. 311, 314–323 (2017)
9. Li, P.R.: Some classes of singular integral equations of convolution type in the class of exponentially increasing functions. J. Inequal. Appl. 2017, 307 (2017)
10. Jiang, Y., Xu, Y.: Fast Fourier–Galerkin methods for solving singular boundary integral equations: numerical integration and precondition. J. Comput. Appl. Math. 234, 2792–2807 (2010)
11. Li, P.R.: Singular integral equations of convolution type with reflection and translation shifts. Numer. Funct. Anal. Optim. 40(9), 1023–1038 (2019)
12. Li, P.R.: On solvability of singular integral-differential equations with convolution. J. Appl. Anal. Comput. 9(3), 1071–1082 (2019)
13. Li, P.R.: Non-normal type singular integral-differential equations by Riemann–Hilbert approach. J. Math. Anal. Appl. 483(2), 123643 (2020)
14. Li, P.R.: Solvability theory of convolution singular integral equations via Riemann–Hilbert approach. J. Comput. Appl. Math. 370(2), 112601 (2020)
15. Li, P.R.: The solvability and explicit solutions of singular integral-differential equations of non-normal type via Riemann–Hilbert problem. J. Comput. Appl. Math. 374(2), 112759 (2020)
16. Mau, N.V., Hoa, N.T.: On solvability of a boundary value problem for singular integral equations. Southeast Asian Bull. Math. 26, 235–244 (2002)
17. Li, P.R.: Singular integral equations of convolution type with Hilbert kernel and a discrete jump problem. Adv. Differ. Equ. 2017, 360 (2017)
18. Li, P.R.: Singular integral equations of convolution type with cosecant kernels and periodic coefficients. Math. Probl. Eng. 2017, Article ID 6148393 (2017)
19. Li, P.R.: Two classes of linear equations of discrete convolution type with harmonic singular operators. Complex Var. Elliptic Equ. 61(1), 67–75 (2016)
20. Li, P.R.: Solvability of some classes of singular integral equations of convolution type via Riemann–Hilbert problem. J. Inequal. Appl. 2019, 22 (2019)
21. Abreu Blaya, R., Bory Reyes, J., Brackx, F., De Schepper, H.: Boundary value problems for the quaternionic Hermitian in $R^4$. Bound. Value Probl. 2012, 74 (2012)
22. Li, P.R.: Linear BVPs and SIEs for generalized regular functions in Clifford analysis. J. Funct. Spaces 2018, Article ID 6967149 (2018)
23. Ren, G.B., Kaehler, U., Shi, J.H., Liu, C.W.: Hardy–Littlewood inequalities for fractional derivatives of invariant harmonic functions. Complex Anal. Oper. Theory 6(2), 373–396 (2012)
24. Wang, W.: On non-homogeneous Cauchy–Fueter equations and Hartogs phenomenon in several quaternionic variables. J. Geom. Phys. 58, 1203–1210 (2008)
25. Gentili, G., Stoppato, C., Struppa, D.C.: Regular Functions of a Quaternionic Variable. Springer, Berlin (2013)
26. Li, P.R.: One class of generalized boundary value problem for analytic functions. Bound. Value Probl. 2015, 40 (2015)
27. Li, P.R.: Singular integral equations of convolution type with Cauchy kernel in the class of exponentially increasing functions. Appl. Math. Comput. 344–345, 116–127 (2019)