Boundedness of some sublinear operators on Herz-Morrey spaces with variable exponent

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Abstract: In this paper, the boundedness of some sublinear operators is proved on homogeneous Herz-Morrey spaces with variable exponent.

Keywords: sublinear operator; Lebesgue space with variable exponent; Herz-Morrey space with variable exponent

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1 Introduction

Function spaces with variable exponent are being watched with keen interest not in real analysis but also in partial differential equations and in applied mathematics because they are applicable to the modeling for electrorheological fluids and image restoration. The theory of function spaces with variable exponent has rapidly made progress in the past twenty years since some elementary properties were established by Kováčik-Rákosník [18]. One of the main problems on the theory is the boundedness of the Hardy-Littlewood maximal operator on variable Lebesgue spaces. By virtue of the fine works [4–6, 8–10, 15, 19, 23, 24], some important conditions on variable exponent, for example, the log-Hölder conditions and the Muckenhoupt type condition, have been obtained.

The class of the Herz spaces is arising from the study on characterization of multipliers on the classical Hardy spaces. The well-known Morrey spaces is used to show that certain systems of partial differential equations (PDEs) had Hölder continuous solutions. And the homogeneous Herz-Morrey spaces \( M_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) \) coordinate with the homogeneous Herz space \( K_{q}^{\alpha,p}(\mathbb{R}^n) \) when \( \lambda = 0 \). One of the important problems on Herz spaces and Herz-Morrey spaces is the boundedness of sublinear operators. Hernández, Li, and Yang [11, 20, 22] have proved that if a sublinear operator \( T \) is bounded on \( L^p(\mathbb{R}^n) \) and satisfies the size condition

\[
|Tf(x)| \leq C \int_{\mathbb{R}^n} |x-y|^{-n}|f(y)|dy, \quad \text{a.e.} \quad x \notin \text{supp} f
\]

for all \( f \in L^1(\mathbb{R}^n) \) with compact support, then \( T \) is bounded on the homogeneous Herz space \( K_{q}^{\alpha,p}(\mathbb{R}^n) \). In 2005, Lu and Xu [21] established the boundedness for some sublinear operators on homogeneous Herz-Morrey spaces.

In 2010, Izuki [14] proves the boundedness of some sublinear operators on Herz spaces with variable exponent; and recently Izuki [12, 13] also considers the boundedness of some operators on Herz-Morrey spaces with variable exponent.

Motivated by the study on the Herz spaces and Lebesgue spaces with variable exponent, when \( (1.1) \) is replaced by some more general size conditions, the main purpose of this paper is to establish some boundedness results of sublinear operators on Herz-Morrey spaces with variable exponent. This size condition is satisfied by many important operators in harmonic analysis.
Let us explain the outline of this article. In Section 2 we state some important properties of $L^{q(\cdot)}(\mathbb{R}^n)$ based on \cite{[8,12,18,23]}, and give some lemmas which will be needed for proving our main theorem. In Section 3 we prove the boundedness of sublinear operators on Herz-Morrey spaces with variable exponent $M^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$, and obtain the corresponding corollaries.

Throughout this paper, we will denote by $|S|$ the Lebesgue measure and by $\chi_S$ the characteristic function for a measurable set $S \subset \mathbb{R}^n$. $C$ denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $1 < q(x) < \infty$, we denote by $q'(x)$ its conjugate index, namely, $q'(x) = \frac{q(x)}{q(x)-1}$. For $A \sim D$, we mean that there is a constant $C > 0$ such that $C^{-1}D \leq A \leq CD$.

## 2 Preliminaries and Lemmas

In this section, we give the definition of Lebesgue and Herz-Morrey spaces with variable exponent, and state their properties. Let $E$ be a measurable set in $\mathbb{R}^n$ with $|E| > 0$. We first define Lebesgue spaces with variable exponent.

**Definition 2.1** Let $q(\cdot) : E \rightarrow [1, \infty)$ be a measurable function.

1) The Lebesgue spaces with variable exponent $L^{q(\cdot)}(E)$ is defined by

$$L^{q(\cdot)}(E) = \{ f \text{ is measurable function : } \int_E \left( \frac{|f(x)|}{\eta} \right)^{q(x)} \, dx < \infty \text{ for some constant } \eta > 0 \}.$$  

2) The space $L^{q(\cdot)}_{loc}(E)$ is defined by

$$L^{q(\cdot)}_{loc}(E) = \{ f \text{ is measurable function : } f \in L^{q(\cdot)}(K) \text{ for all compact subsets } K \subset E \}.$$  

$L^{q(\cdot)}(E)$ is a Banach space with the norm defined by

$$\|f\|_{L^{q(\cdot)}(E)} = \inf \left\{ q > 0 : \int_E \left( \frac{|f(x)|}{\eta} \right)^{q(x)} \, dx \leq 1 \right\}.$$  

Now, we define two classes of exponent functions. Given a function $f \in L^{1}_{loc}(E)$, the Hardy-Littlewood maximal operator $M$ is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r) \cap E} |f(y)| \, dy \quad (x \in E),$$  

where $B(x,r) = \{ y \in \mathbb{R}^n : |x-y| < r \}$.

**Definition 2.2** 1) The set $\mathcal{P}(E)$ consists of all measurable functions $q(\cdot)$ satisfying

$$1 < \operatorname{ess \ inf}_{x \in E} q(x) = q_- , \quad q_+ = \operatorname{ess \ sup}_{x \in E} q(x) < \infty.$$

2) The set $\mathcal{P}(E)$ consists of all measurable functions $q(\cdot) \in \mathcal{P}(E)$ such that the Hardy-Littlewood maximal operator $M$ is bounded on $L^{q(\cdot)}(E)$.

Next we define the Herz-Morrey spaces with variable exponent. Let $B_k = B(0,2^k) = \{ x \in \mathbb{R}^n : |x| \leq 2^k \}$, $A_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

**Definition 2.3** Let $\alpha \in \mathbb{R}$, $0 \leq \lambda < \infty$, $0 < p < \infty$, and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The Herz-Morrey space with variable exponent $M^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ is defined by

$$M^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = \{ f \in L^{q(\cdot)}_{loc}(\mathbb{R}^n \setminus \{ 0 \}) : \|f\|_{M^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty \},$$  

where $\|f\|_{M^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_0 p} \|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.$
Compare the Herz-Morrey space with variable exponent \( M K_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) \) with the Herz space with variable exponent \( K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \) \cite{13}, where

\[
K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{q(\cdot)}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \| f \chi_k \|_{L_{q(\cdot)}^{q(\cdot)}(\mathbb{R}^n)} < \infty \right\}.
\]

Obviously, \( M K_{p,q}^{\alpha,0}(\mathbb{R}^n) = K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \).

In 2012, Almeida and Drihem \cite{1} discussed the boundedness of a wide class of sublinear operators, including maximal, potential and Calderón-Zygmund operators, on variable Herz spaces \( K_{q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n) \) and \( K_{q(\cdot)}^{\alpha,\lambda,p}(\mathbb{R}^n) \). Meanwhile, they also established Hardy-Littlewood-Sobolev theorems for fractional integrals on variable Herz spaces. In other papers \cite{2,16,17} the boundedness of operators of harmonic analysis are established in variable exponent Morrey spaces. In this paper, the author considers Herz-Morrey space \( M K_{p,q(\cdot)}^{\alpha}(\mathbb{R}^n) \) with variable exponent \( q(\cdot) \) for fixed \( \alpha \in \mathbb{R} \) and \( p \in (0, \infty) \). However, for the case of the exponent \( \alpha(\cdot) \) is variable as well, we can refer to the furthermore work of the author of this paper.

Next we state some properties of variable exponent. Cruz-Uribe et al \cite{6} and Nekvinda \cite{23} proved the following sufficient conditions for the boundedness of \( M \) in variable exponent space independently. We note that Nekvinda \cite{23} gave a more general condition in place of \cite{2}.

**Proposition 2.1** \cite{23} Suppose that \( E \) is an open set. If \( q(\cdot) \in \mathcal{P}(E) \) satisfies the inequality

\[
|q(x) - q(y)| \leq \frac{-C}{\ln(|x - y|)} \quad \text{if } |x - y| \leq 1/2, \tag{2.1}
\]

\[
|q(x) - q(y)| \leq \frac{C}{\ln(e + |x|)} \quad \text{if } |y| \geq |x|, \tag{2.2}
\]

where \( C > 0 \) is a constant independent of \( x \) and \( y \), then we have \( q(\cdot) \in \mathcal{B}(E) \).

The next proposition is due to Diening \cite{35} (see Theorem 8.1). We remark that Diening has also proved general results on Musielak-Orlicz spaces. We only describe partial results we need in this paper.

**Proposition 2.2** \cite{8} Suppose that \( q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), then \( q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \) iff \( q'(\cdot) \in \mathcal{B}(\mathbb{R}^n) \).

In order to prove our main theorems, we also need the following result which is the Hardy-Littlewood-Sobolev theorem on Lebesgue spaces with variable exponent due to Capone, Cruz-Uribe and Fiorenza \cite{3} (see Theorem 1.8). We remark that this result was proved by Diening \cite{7} provided that \( q_{i}(\cdot) \) is constant outside of a large ball.

**Proposition 2.3** \cite{3} Suppose that \( q_{i}(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) satisfies conditions \cite{2,1} and \cite{2} in Proposition \cite{2,1} 0 < \( \beta < n/(q_{i}^{\cdot}+) \) and define \( q_{2}(\cdot) \) by

\[
\frac{1}{q_{i}(x)} - \frac{1}{q_{2}(x)} = \frac{\beta}{n}.
\]

Then we have

\[
\| I_{\beta} f \|_{L_{q_{2}(\cdot)}^{q_{2}}(\mathbb{R}^n)} \leq C \| f \|_{L_{q_{1}(\cdot)}^{q_{1}}(\mathbb{R}^n)},
\]

where fractional integral \( I_{\beta} \) is defined by \( I_{\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\beta}} dy \).

In addition, by Proposition \cite{2,3} we can obtain

**Proposition 2.4** Let \( q_{1}(\cdot), q_{2}(\cdot) \) and \( \beta \) be the same as in Proposition \cite{2,3} Then we have

\[
\| \chi_{B_{k}} \|_{L_{q_{2}(\cdot)}^{q_{2}}(\mathbb{R}^n)} \leq C 2^{-k\beta} \| \chi_{B_{k}} \|_{L_{q_{1}(\cdot)}^{q_{1}}(\mathbb{R}^n)}
\]

for all balls \( B_{k} = \{ x \in \mathbb{R}^n : |x| \leq 2^{k} \} \) with \( k \in \mathbb{Z} \).
**Proof** It is easy to see that

\[ I_\beta(\chi_{B_k})(x) \geq I_\beta(\chi_{B_k})(x) \cdot \chi_{B_k}(x) = \int_{B_k} \frac{dy}{|x - y|^{n - \beta}} \cdot \chi_{B_k}(x) \geq C 2^{k \beta} \chi_{B_k}(x). \]

Thus, applying Proposition 2.3, we get

\[ \|\chi_{B_k}\|_{L^{q'_2}(\mathbb{R}^n)} \leq C 2^{-k \beta} \|I_\beta(\chi_{B_k})\|_{L^{q}(\mathbb{R}^n)} \leq C 2^{-k \beta} \|\chi_{B_k}\|_{L^{q'_1}(\mathbb{R}^n)}. \]

This completes the proof of Proposition 2.4.

The next lemma describes the generalized Hölder’s inequality and the duality of \( L^{q}(E) \). The proof can be found in [13].

**Lemma 2.1** [13] Suppose that \( q(\cdot) \in \mathcal{P}(E) \). Then the following statements hold.

1) (generalized Hölder’s inequality) For all \( f \in L^{q}(E) \) and all \( g \in L^{q'}(E) \), we have

\[ \int_{E} |f(x)g(x)|dx \leq r_q \|f\|_{L^{q}(E)} \|g\|_{L^{q'}(E)}, \tag{2.3} \]

where \( r_q = 1 + 1/q - 1/q' + 1 \).

2) For all \( f \in L^{q}(E) \), we have

\[ \|f\|_{L^{q}(E)} \leq \sup \left\{ \int_{E} |f(x)g(x)|dx : \|g\|_{L^{q'}(E)} \leq 1 \right\}. \]

**Lemma 2.2** [12] If \( q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), then there exist positive constants \( \delta \in (0, 1) \) and \( C > 0 \) such that

\[ \frac{\|\chi_{S}\|_{L^{q}(\mathbb{R}^n)}}{\|\chi_{B}\|_{L^{q}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta}, \tag{2.4} \]

holds for all balls \( B \) in \( \mathbb{R}^n \) and all measurable subsets \( S \subset B \).

**Lemma 2.3** [12] If \( q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), then there exists a positive constant \( C > 0 \) such that

\[ C^{-1} \leq \frac{1}{|B|}\|\chi_{B}\|_{L^{q'}(E)} \|\chi_{B}\|_{L^{q}(E)} \leq C. \tag{2.5} \]

holds for all balls \( B \) in \( \mathbb{R}^n \).

### 3 Main theorems and their proofs

Let \( q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) satisfy conditions (2.1) and (2.2) in Proposition 2.1. Then so does \( q'(\cdot) \). In particular, we can see that \( q(\cdot), q'(\cdot) \in \mathcal{B}(\mathbb{R}^n) \) from Proposition 2.1. Therefore applying Lemma 2.2, we can take constants \( \delta_1 \in (0, 1/(q')_+) \), \( \delta_2 \in (0, 1/(q)_+) \) such that

\[ \frac{\|\chi_{S}\|_{L^{q'}(\mathbb{R}^n)}}{\|\chi_{B}\|_{L^{q'}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_1}, \quad \frac{\|\chi_{S}\|_{L^{q}(\mathbb{R}^n)}}{\|\chi_{B}\|_{L^{q}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_2} \tag{3.1} \]

holds for all balls \( B \) in \( \mathbb{R}^n \) and all measurable subsets \( S \subset B \). And when \( q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), applying Lemma 2.2 we can take constants \( \delta_3 \in (0, 1/(q'_2)_+) \), \( \delta_4 \in (0, 1/(q_1)_+) \) such that

\[ \frac{\|\chi_{S}\|_{L^{q'_2}(\mathbb{R}^n)}}{\|\chi_{B}\|_{L^{q'_2}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_3}, \quad \frac{\|\chi_{S}\|_{L^{q}(\mathbb{R}^n)}}{\|\chi_{B}\|_{L^{q}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_4} \tag{3.2} \]
holds for all balls $B$ in $\mathbb{R}^n$ and all measurable subsets $S \subset B$.

In this section, we will give some size condition which are more general than (1.1), and prove the boundedness of some sublinear operators, satisfying this size condition on Herz-Morrey spaces with variable exponent. This size condition is satisfied by many important operators in harmonic analysis. Our main result can be stated as follows.

**Theorem 3.1** Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (2.1) and (2.2) in Proposition 2.1, and $0 < p < \infty$, $\lambda > 0$, $\lambda - n\delta_2 < \alpha < \lambda + n\delta_1$, where $\delta_1 \in (0, 1/(q')_+)$ and $\delta_2 \in (0, 1/(q)_+)$ are the constants satisfying (2.2). Suppose that a sublinear operator $T$ also satisfies the assumptions of Theorem 3.1, where the supremum is taken over all balls $B \ni x$.

Let $q(\cdot)$ be the same as in Theorem 3.1. If a sublinear operator $T$ satisfies

(i) $T$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$;

(ii) for suitable function $f$ with supp $f \subset A_k$ and $|x| \geq 2^{k+1}$ with $k \in \mathbb{Z}$,

\[ |Tf(x)| \leq C|x|^{-n} \|f\|_{L^1(\mathbb{R}^n)}; \tag{3.3} \]

(iii) for suitable function $f$ with supp $f \subset A_k$ and $|x| \leq 2^{k-2}$ with $k \in \mathbb{Z}$,

\[ |Tf(x)| \leq C2^{-kn} \|f\|_{L^1(\mathbb{R}^n)}. \tag{3.4} \]

Then $T$ is also bounded on $M^{\mathcal{K}_{p,q(\cdot)}}(\mathbb{R}^n)$.

Note that if $T$ satisfies the size condition (1.1), then $T$ satisfies (3.3) and (3.4). Thus, by Theorem 3.1, we have

**Corollary 3.1** Let $q(\cdot)$, $p$, $\lambda$, $\alpha$, $\delta$ be the same as in Theorem 3.1. If a sublinear operator $T$ satisfies the size condition (1.1) and is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, then $T$ is also bounded on $M^{\mathcal{K}_{p,q(\cdot)}}(\mathbb{R}^n)$.

**Remark 1** For any $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (2.1) and (2.2) in Proposition 2.1 the Hardy-Littlewood maximal operator $M$, defined by

\[ M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap \Omega} |f(y)|dy, \]

also satisfies the assumptions of Theorem 3.1 where the supremum is taken over all balls $B$ containing $x$, and $\Omega \subset \mathbb{R}^n$ is an open set.

**Proof (Proof of Theorem 3.1)** For all $f \in M^{\mathcal{K}_{p,q(\cdot)}}(\mathbb{R}^n)$. If we denote $f_j := f \cdot \chi_{A_j}$ for each $j \in \mathbb{Z}$, then we can write

\[ f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_{A_j}(x) = \sum_{j=-\infty}^{\infty} f_j(x). \]

We have

\[ \|Tf\|^p_{M^{\mathcal{K}_{p,q(\cdot)}}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} \sum_{k=-\infty}^{k_0} 2^{k0\lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k0\alpha} \|T(f) \cdot \chi_k\|_{L^q(\mathbb{R}^n)}^p \right) \]

\[ \leq C \sup_{k_0 \in \mathbb{Z}} \sum_{k=-\infty}^{k_0} 2^{k0\lambda} \left( \sum_{j=-\infty}^{k-2} \|T(f_j) \cdot \chi_j\|_{L^q(\mathbb{R}^n)}^p \right) \]

\[ + C \sup_{k_0 \in \mathbb{Z}} \sum_{k=-\infty}^{k_0} 2^{k0\lambda} \|T\left( \sum_{j=k-1}^{k+1} f_j \right) \cdot \chi_k\|_{L^q(\mathbb{R}^n)}^p \]
Using Proposition 2.1, Proposition 2.2, Lemma 2.2, Lemma 2.3 and (3.1), we obtain

\[ C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k = -\infty}^{k_0} 2^{k\alpha} \left( \sum_{j=k+2}^{\infty} \| T(f_j) \cdot \chi_k \|_{L^q(\mathbb{R}^n)}^p \right) \right) \]

\[ = C(E_1 + E_2 + E_3). \]

First we estimate \( E_2 \). Applying the \( L^q()(\mathbb{R}^n) \)-boundedness of \( T \), we have

\[ E_2 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k = -\infty}^{k_0} 2^{k\alpha} \left\| \chi_k \right\|_{L^q(\mathbb{R}^n)}^p \right) \]

\[ \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k = -\infty}^{k_0} 2^{k\alpha} \left( \sum_{j=k+1}^{k+1} \| f_j \|_{L^q(\mathbb{R}^n)}^p \right) \right) \]

\[ \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k = -\infty}^{k_0} 2^{k\alpha} \| f \cdot \chi_k \|_{L^q(\mathbb{R}^n)}^p \right) \]

\[ = C \| f \|_{M^{(\alpha,\lambda)}_{p,q}(\mathbb{R}^n)}. \]

For \( E_1 \), we notice the facts that \( j \leq k - 2 \) and a.e. \( x \in A_k \) with \( k \in \mathbb{Z} \), then using the size condition (3.3) and the generalized Hölder’s inequality (see (2.3) in Lemma 2.1), we have

\[ |T(f_j)(x)| \leq C|x|^{-n} \| f_j \|_{L^1(\mathbb{R}^n)} \leq C2^{-kn} \| f_j \|_{L^1(\mathbb{R}^n)} \]

\[ \leq C2^{-kn} \| f_j \|_{L^q(\mathbb{R}^n)} \| \chi_j \|_{L^p(\mathbb{R}^n)}. \]  \hspace{1cm} (3.5)

Using Proposition 2.1, Proposition 2.2, Lemma 2.2, Lemma 2.3 and (3.1), we obtain

\[ 2^{-kn} \| \chi_k \|_{L^p(\mathbb{R}^n)} \| \chi_j \|_{L^q(\mathbb{R}^n)} \leq 2^{-kn} \| \chi_{B_k} \|_{L^p(\mathbb{R}^n)} \| \chi_j \|_{L^q(\mathbb{R}^n)} \]

\[ \leq C \| \chi_{B_k} \|_{L^p(\mathbb{R}^n)}^{1-\delta} \| \chi_j \|_{L^q(\mathbb{R}^n)} \]

\[ = C \| \chi_{B_k} \|_{L^p(\mathbb{R}^n)} \leq C2(j-k)^n \delta. \]  \hspace{1cm} (3.6)

On the other hand, note the following fact

\[ \| f_j \|_{L^q(\mathbb{R}^n)} = 2^{-ja} \left( 2^{ja} \| f_j \|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \]

\[ \leq 2^{-ja} \left( \sum_{i = -\infty}^{j} 2^{ja} \| f_i \|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \]

\[ = 2^{j(\lambda - \alpha)} \left( 2^{-ja} \left( \sum_{i = -\infty}^{j} 2^{ja} \| f_i \|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \right) \]

\[ \leq C2^{j(\lambda - \alpha)} \| f \|_{M^{(\alpha,\lambda)}_{p,q}(\mathbb{R}^n)}. \]  \hspace{1cm} (3.7)
Therefore, combining (3.5), (3.6), (3.7), and (3.8), and using $\alpha < \lambda + n\delta_2$, it follows that

$$E_1 \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} 2^{(j-k)\alpha \delta_2} \left\| f_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \right)^p \right)$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} 2^{(j-k)\alpha \delta_2} \left\| f_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \right)^p \right)$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} 2^{(j-k)\alpha \delta_2} \left\| f_j \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)} \right)^p \right)$$

$$\leq C \left\| f \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} 2^{(j-k)\alpha \delta_2} \left\| f_j \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)} \right)^p \right) \leq C \left\| f \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)}.$$

Now, let us estimate $E_3$. For every $j \geq k+2$ and a.e. $x \in A_k$ with $k \in \mathbb{Z}$, applying the size condition (3.4) and the generalized H"{o}lder’s inequality (see (2.3) in Lemma 2.1), we have

$$\left| T(f_j)(x) \right| \leq C 2^{-j\alpha \lambda} \left\| f_j \right\|_{L^1(\mathbb{R}^n)} \leq C 2^{-j\alpha \lambda} \left\| f_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)}.$$ (3.8)

Using Proposition 2.2, Lemma 2.2, Lemma 2.3 and (3.4), we obtain

$$2^{-j\alpha \lambda} \left\| x_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \leq C \left\| x_{B_j} \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_{B_j} \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_{B_j} \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)}$$

$$\leq C \left\| x_{B_j} \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_{B_j} \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \leq C 2^{(j-k)\alpha \delta_2}.$$ (3.9)

Thus, combining (3.7), (3.8) and (3.9), and using $\alpha > \lambda - n\delta_2$, it follows that

$$E_3 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} \left\| T(f_j) \cdot x_k \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \right)^p \right)$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} 2^{-j\alpha \lambda} \left\| f_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_k \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \right)^p \right)$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} 2^{(j-k)\alpha \delta_2} \left\| f_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_j \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \left\| x_k \right\|_{L^{\alpha \lambda}(\mathbb{R}^n)} \right)^p \right)$$

$$\leq C \left\| f \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} 2^{(j-k)\alpha \delta_2} \left\| f_j \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)} \right)^p \right)$$

$$\leq C \left\| f \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \alpha \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_2} \left( \sum_{j=-\infty}^{k_2} 2^{(j-k)\alpha \delta_2} \left\| f_j \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)} \right)^p \right) \leq C \left\| f \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)}.$$

Combining the estimates for $E_1$, $E_2$ and $E_3$ yields

$$\left\| T f \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)} \leq C \left\| f \right\|_{M^{\alpha \lambda}_{\infty \infty}(\mathbb{R}^n)},$$

and then completes the proof of Theorem 3.1.
Now, let us turn to consider the fractional singular integrals. We first have the following theorem similar to 3.1.

**Theorem 3.2** Let \( q_i(\cdot) \in \mathscr{P}(\mathbb{R}^n) \) satisfies conditions (2.14) and (2.22) in Proposition 2.1. Define the variable exponent \( q_i(\cdot) \) by

\[
\frac{1}{q_i(x)} - \frac{1}{q_2(x)} = \frac{\beta}{n}.
\]

And let \( 0 < p_1 \leq p_2 < \infty, \lambda > 0, \) \( 0 < \beta < n/(q_1)_+, \lambda - n\delta_4 + \alpha < \lambda + n\delta_3, \) where \( \delta_3 \in (0, 1/(q_1)_+) \) and \( \delta_4 \in (0, 1/(q_2)_+) \) are the constants appearing in (3.2). Suppose that a sublinear operator \( T_\beta \) satisfies

(i) \( T_\beta \) maps from \( L^{q_1}(\mathbb{R}^n) \) to \( L^{q_2}(\mathbb{R}^n) \);

(ii) for any function \( f \) with \( \text{supp} \ f \subset A_k \) and any \( |x| \geq 2^{k+1} \) with \( k \in \mathbb{Z} \),

\[
|T_\beta(f)(x)| \leq C|x|^{\beta-n}\|f\|_{L^1(\mathbb{R}^n)};
\]

(iii) for any function \( f \) with \( \text{supp} \ f \subset A_k \) and any \( |x| \leq 2^{k-2} \) with \( k \in \mathbb{Z} \),

\[
|T_\beta(f)(x)| \leq C2^{k(\beta-n)}\|f\|_{L^1(\mathbb{R}^n)}.
\]

Then \( T_\beta(f) \) is bounded from \( M\hat{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n) \) to \( M\hat{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n) \).

Notice that if \( T_\beta(f) \) satisfies the size condition

\[
|T_\beta(f)(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^\beta} dy, \quad x \notin \text{supp} \ f
\]

for any integrable function \( f \) with compact support, then \( T_\beta(f) \) obviously satisfies the assumptions of Theorem 3.2. Therefore, by Theorem 3.2 we have

**Corollary 3.2** Let \( q_i(\cdot), q_2(\cdot), p_1, p_2, \lambda, \beta, \alpha, \delta_3 \) and \( \delta_4 \) be the same as in Theorem 3.2. If a sublinear operator \( T_\beta \) satisfies the size condition (3.12) and is bounded from \( L^{q_1}(\mathbb{R}^n) \) to \( L^{q_2}(\mathbb{R}^n) \), then \( T_\beta \) is also bounded from \( M\hat{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n) \) to \( M\hat{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n) \).

**Remark 2** We can see that when \( \beta = 0 \), Theorem 3.2 is just Theorem 3.1. In particular, if \( T_\beta(f) \) is a (standard) fractional integral, then \( T_\beta(f) \) obviously satisfies (3.12). The fractional maximal function \( M_\beta(f) \), defined by

\[
M_\beta(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\beta/n}} \int_{B \cap \Omega} |f(y)| dy,
\]

also satisfies the conditions of Theorem 3.2 where the supremum is taken over all balls \( B \) which contain \( x \), and \( \Omega \subset \mathbb{R}^n \) is an open set.

**Proof** (Proof of Theorem 3.2) For all \( f \in M\hat{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n) \). If we denote \( f_j := f \cdot \chi_j = f \cdot \chi_{A_j} \) for each \( j \in \mathbb{Z} \), then we can write

\[
f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).
\]

Because of \( 0 < p_1/p_2 \leq 1 \), we apply inequality

\[
\left( \sum_{i=-\infty}^{\infty} |a_i| \right)^{p_1/p_2} \leq \sum_{i=-\infty}^{\infty} |a_i|^{p_1/p_2},
\]

(3.13)
and obtain

\[ \|T_\beta(f)\|_{M^{k,\lambda}_{p_1,p_2}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_2} \|T_\beta(f) \cdot \chi_k\|_{L^{p_2}(\mathbb{R}^n)}^{p_1/p_2} \right) \]

\[ \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_2} \|T_\beta(f) \cdot \chi_k\|_{L^{p_2}(\mathbb{R}^n)}^{p_1} \right) \]

\[ \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_2} \left( \sum_{j=-\infty}^{k-2} \|T_\beta(f_j) \cdot \chi_k\|_{L^{p_2}(\mathbb{R}^n)} \right)^{p_1} \right) \]

\[ + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_2} \left( \sum_{j=k+2}^{\infty} \|T_\beta(f_j) \cdot \chi_k\|_{L^{p_2}(\mathbb{R}^n)} \right)^{p_1} \right) \]

\[ = C(E_1 + E_2 + E_3). \]

For \( E_2 \), using the boundedness of \( T_\beta \) from \( L^{p_1}(\mathbb{R}^n) \) to \( L^{q_2}(\mathbb{R}^n) \), we have

\[ E_2 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_2} \|T_\beta\|_{L^{q_2}(\mathbb{R}^n)} \right) \]

\[ \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_2} \left( \sum_{j=-\infty}^{k-1} \|f_j\|_{L^q(\mathbb{R}^n)} \right)^{p_1} \right) \]

\[ \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_2} \left( \sum_{j=k}^{\infty} \|f_j\|_{L^q(\mathbb{R}^n)} \right)^{p_1} \right) \]

\[ = C\|f\|_{M^{k,\lambda}_{p_1,q_2}(\mathbb{R}^n)}. \]

For \( E_1 \), note that \( j \leq k - 2 \) and a.e. \( x \in A_k \) with \( k \in \mathbb{Z} \), then using the size condition (3.10) and the generalized Hölder’s inequality (see (2.3) in Lemma 2.1), we have

\[ |T_\beta(f_j)(x)| \leq C|x|^{\beta-n} \|f_j\|_{L^1(\mathbb{R}^n)} \leq C2^{k(\beta-n)} \|f_j\|_{L^1(\mathbb{R}^n)} \]

\[ \leq C2^{k(\beta-n)} \|f_j\|_{L^{q_1}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'}(\mathbb{R}^n)}. \] (3.14)

Using Proposition 2.1, Proposition 2.2, Proposition 2.4, Lemma 2.2, Lemma 2.3 and (3.2), we obtain

\[ 2^{k(\beta-n)} \|\chi_k\|_{L^{q_2}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'}(\mathbb{R}^n)} \leq C2^{-kn} 2^{k\beta} \|\chi_{B_k}\|_{L^{q_2}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'}(\mathbb{R}^n)} \]

\[ \leq C2^{-kn} \|\chi_{B_k}\|_{L^{q_1'}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'}(\mathbb{R}^n)} \]

\[ \leq C\|\chi_{B_k}\|_{L^{q_1'}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'}(\mathbb{R}^n)} \]

\[ = C\|\chi_{B_k}\|_{L^{q_1'}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'}(\mathbb{R}^n)} \leq C2^{(j-k)n\delta_3}. \] (3.15)
On the other hand, note the following fact

\[
\left\| f_j \right\|_{L^{q_1}(\mathbb{R}^n)} = 2^{-j\alpha} \left( 2^{j\alpha p_1} \left\| f_j \right\|_{L^{q_1}(\mathbb{R}^n)} \right)^{1/p_1} \\
\leq 2^{-j\alpha} \left( \sum_{i=-\infty}^{j} 2^{i\alpha p_1} \left\| f_i \right\|_{L^{q_1}(\mathbb{R}^n)} \right)^{1/p_1} \\
= 2^{j(\lambda-\alpha)} \left( \sum_{i=-\infty}^{j} 2^{i\alpha p_1} \left\| f_i \right\|_{L^{q_1}(\mathbb{R}^n)} \right)^{1/p_1} \\
\leq C 2^{j(\lambda-\alpha)} \left\| f \right\|_{M\ln K_{p_1,q_1}(\mathbb{R}^n)}.
\]  

(3.16)

Hence, combining (3.14), (3.15) and (3.16), and using \( \alpha < \lambda + n\delta \), it follows that

\[
E_1 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_1} \left( \sum_{j=-\infty}^{k-2} \left\| T_{\beta}(f_j) \cdot \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \right)^{p_1} \right) \\
\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_1} \left( \sum_{j=-\infty}^{k-2} 2^{j(\beta-\alpha)} \left\| f_j \right\|_{L^{q_1}(\mathbb{R}^n)} \left\| \chi_j \right\|_{L^{q_2}(\mathbb{R}^n)} \left\| \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \right)^{p_1} \right) \\
\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_1} \left( \sum_{j=-\infty}^{k-2} 2^{j(\alpha + n\delta)} \right)^{p_1} \right) \\
\leq C \left\| f \right\|_{M\ln K_{p_1,q_1}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \lambda p_1} \left( \sum_{j=-\infty}^{k-2} 2^{j(\lambda-\alpha)} \right)^{p_1} \right) \\
\leq C \left\| f \right\|_{M\ln K_{p_1,q_1}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \lambda p_1} \right) \leq C \left\| f \right\|_{M\ln K_{p_1,q_1}(\mathbb{R}^n)}.
\]

Now, let us estimate \( E_3 \). For every \( j \geq k+2 \) and a.e. \( x \in A_k \) with \( k \in \mathbb{Z} \), applying the size condition (3.11) and the generalized Hölder’s inequality (see (2.3) in Lemma 2.1), we have

\[
\left| T_{\beta}(f_j)(x) \right| \leq C 2^{j(\beta-n)} \left\| f_j \right\|_{L^1(\mathbb{R}^n)} \leq C 2^{j(\beta-n)} \left\| f_j \right\|_{L^{q_1}(\mathbb{R}^n)} \left\| \chi_j \right\|_{L^{q_1}(\mathbb{R}^n)}.
\]  

(3.17)

Using Proposition 2.1 Proposition 2.2 Proposition 2.4 Lemma 2.2 Lemma 2.3 and (3.2), we obtain

\[
2^{j(\beta-n)} \left\| \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \left\| \chi_j \right\|_{L^{q_1}(\mathbb{R}^n)} \leq 2^{j(\beta-n)} \left\| \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \left\| \chi_j \right\|_{L^{q_1}(\mathbb{R}^n)} \leq C \left\| \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \cdot 2^{j(\beta-n)} \left\| \chi_j \right\|_{L^{q_1}(\mathbb{R}^n)} \\
\leq C \left\| \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \cdot 2^{j(\beta-n)} \left\| \chi_j \right\|_{L^{q_1}(\mathbb{R}^n)} \\
\leq C \left\| \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \cdot \left\| \chi_j \right\|_{L^{q_1}(\mathbb{R}^n)} \\
\leq C \left\| \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \cdot \left\| \chi_j \right\|_{L^{q_1}(\mathbb{R}^n)} \\
= C \left\| \chi_k \right\|_{L^{q_2}(\mathbb{R}^n)} \left\| \chi_j \right\|_{L^{q_1}(\mathbb{R}^n)} \leq C 2^{(k-j)\alpha p_1}.
\]  

(3.18)
Therefore, combining (3.16), (3.17) and (3.18), and using \(\alpha > \lambda\), it follows that

\[
E_3 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k = -\infty}^{k_0} 2^{k \lambda p_1} \left( \sum_{j = k + 2}^{\infty} \|T_\beta(f_j) \cdot \chi_k\|_{L^{q_2}(\mathbb{R}^n)} \right)^{p_1} \right)
\]

\[
\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k = -\infty}^{k_0} 2^{k \lambda p_1} \left( \sum_{j = k + 2}^{\infty} 2^{j(\beta - n)} \|f_j\|_{L^{q_1}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'_1}(\mathbb{R}^n)} \|\chi_k\|_{L^{q_2}(\mathbb{R}^n)} \right)^{p_1} \right)
\]

\[
\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k = -\infty}^{k_0} 2^{k \lambda p_1} \left( \sum_{j = k + 2}^{\infty} 2^{(k-j)n\delta_4} \|f_j\|_{L^{q_1}(\mathbb{R}^n)} \right)^{p_1} \right)
\]

\[
\leq C \||f|^{p_1}_{M^{K^{\alpha,\lambda}_{p_1,q_1}()}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k = -\infty}^{k_0} 2^{k \lambda p_1} \left( \sum_{j = k + 2}^{\infty} 2^{(k-j)n\delta_4} 2^{j(\lambda - \alpha)} \right)^{p_1} \right)
\]

\[
\leq C \||f|^{p_1}_{M^{K^{\alpha,\lambda}_{p_1,q_1}()}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k = -\infty}^{k_0} 2^{k \lambda p_1} \left( \sum_{j = k + 2}^{\infty} 2^{(k-j)(\alpha - \lambda + n\delta_4)} \right)^{p_1} \right)
\]

\[
\leq C \||f|^{p_1}_{M^{K^{\alpha,\lambda}_{p_1,q_1}()}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k = -\infty}^{k_0} 2^{k \lambda p_1} \right) \leq C \||f|^{p_1}_{M^{K^{\alpha,\lambda}_{p_1,q_1}()}(\mathbb{R}^n)}
\]

Combining the estimates for \(E_1\), \(E_2\) and \(E_3\), we conclude that

\[
\|T_\beta(f)\|_{M^{K^{\alpha,\lambda}_{p_2,q_2}()}(\mathbb{R}^n)} \leq C \||f|^{p_1}_{M^{K^{\alpha,\lambda}_{p_1,q_1}()}(\mathbb{R}^n)}
\]

and then completes the proof of Theorem 3.2.

**Remark 3** It is easy to see that when \(\lambda = 0\), the above results are also true on the Herz space with variable exponent, and containing some main results for [14].

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