On the connection problem for Painlevé I

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Abstract
We study the dependence of the tau function of the Painlevé I equation on the
generalized monodromy of the associated linear problem. In particular, we
compute the connection constants relating the tau function asymptotics on five
canonical rays at infinity. The result is expressed in terms of the dilogarithms
of cluster-type coordinates in the space of the Stokes data.

Keywords: Painlevé equations, tau function, connection problem

(Some figures may appear in colour only in the online journal)

1. Introduction
The present note is concerned with the first Painlevé equation, whose standard form reads

\[ q'' = 6q^2 + t. \]  (1.1)

This equation represents the shortest entry of the Painlevé–Gambier list of 2nd order ODEs,
with the property that their solutions have no movable branch points. As is well known, it
appears as a similarity reduction of integrable PDEs such as KdV and Boussinesq equations,
and also in the context of matrix models and two-dimensional quantum gravity, see [FIKN, Kap3] for details and further references. Among more recent applications, let us mention that specific Painlevé I transcendents arise in the description of the universal behavior of solutions of the nonlinear Schrödinger equation [DGK], analysis of the cubic anharmonic oscillator [Mas], and as a model for topological recursion [IS]. The general solution of Painlevé I has been conjecturally related to the partition function of the superconformal Argyres–Douglas theory of type \( H_0 \) [BLMST].

Painlevé I can be rewritten as a non-autonomous Hamiltonian system \( q_t = H_p, \ p_t = -H_q \),
where the time-dependent Hamiltonian is given by

\[ H = \frac{p^2}{2} - 2q^3 - tq. \]  (1.2)
The Hamiltonian itself satisfies the so-called $\sigma$-form of the Painlevé I equation

$$H_{tt}^2 = 2 (H - tH_t) - 4H_t^3,$$

which is easily deduced by taking into account that $H_t = -q$. The Painlevé I tau function is defined up to a factor independent of $t$ by

$$\partial_t \ln \tau = H.$$  \hspace{1cm} (1.4)

In the case of equation (1.1), the Painlevé property means that every solution $q(t)$ is a meromorphic function on the whole complex $t$-plane with only double poles, see e.g. [Del, chapter 2] for a detailed proof. Moreover, the tau function $\tau(t)$ is holomorphic, with only simple zeros. The asymptotic behavior of the Painlevé I transcendents as $t \to \infty$ is rather intricate. The asymptotics is trigonometric along the canonical rays $R_k = \{ \text{arg } t = \pi - \frac{2\pi k}{5} \}$, \hspace{1cm} $k \in \mathbb{Z}/5\mathbb{Z}$, whereas its generic form inside the sectors shown in figure 1 is described by the modulated elliptic functions [Bou, JK].

The relation of Painlevé I to the theory of monodromy-preserving deformations [JMU, FIKN] provides an avenue for solving the connection problem between different asymptotic directions at infinity. For that one needs to express the parameters of the asymptotic behavior in terms of the Stokes data of the associated linear problem. In the case of $q(t)$, the latter task was accomplished by Kapaev [Kap1, Kap2] on canonical rays and by Kapaev and Kitaev [KK] for the elliptic asymptotics. As far as the tau function is concerned, there remains the problem of evaluating certain constant factors. To explain what we have in mind, let us now formulate a sharp question concerning such connection constants in the asymptotics of $\tau(t)$.

As will be discussed below, the results of [Kap1, Kap2] imply that for generic monodromy, the asymptotic behavior of $\tau(t)$ on five canonical rays is given by

$$\tau (t \to \infty) \simeq C_k x^{\frac{3}{5}} e^{\frac{2}{5} t^{\frac{1}{2}} + \frac{i}{45} x^{\frac{5}{4}} [1 + o(1)]}, \hspace{1cm} t \in R_k,$$

where $x = 24^{\frac{1}{5}} |t|^\frac{4}{5}$. The parameters $\nu_1, \ldots, \nu_5$ may be expressed in terms of Stokes multipliers. Any pair of them can be taken as Painlevé I integrals of motion. This knowledge fixes the other three $\nu_k$ as well as all subleading corrections to the asymptotics. The factors $C_k$ in (1.5) are individually undefined since the equation (1.4) only fixes $\tau(t)$ up to a multiplicative constant. The ratios $C_k/C_{k'}$ describing the relative tau function normalization on different rays are, on the other hand, unambiguously fixed by the appropriate Painlevé I function $q(t)$. Our main result, formulated in theorem 3.6, provides an explicit evaluation of these ratios in terms of the asymptotic parameters $\nu_1, \ldots, \nu_5$. 

\hspace{1cm} Figure 1. Painlevé I canonical rays in the $t$-plane.
The computation of the connection constants in the asymptotics of particular Painlevé tau functions was initially motivated by applications in random matrix theory and integrable systems, see e.g. [BT, Tra, Ehr, DIK, DKV], and other references therein, as well as in [ILP]. Their systematic study was initiated in [ILT13, ILT14], where evaluations of these constants were conjectured in terms of monodromy for general solutions of Painlevé VI and Painlevé III\(_D_8\). The latter result has been proved by Its and Prokhorov [IP] using the idea, first suggested in [Ber], of extending the Jimbo–Miwa–Ueno differential [JMU], which defines the isomonodromic tau function, to the space of the monodromy data. General construction of the localized formulas for this extended differential was developed in [ILP] and used there to derive connection constants for generic Painlevé VI and zero-parameter Painlevé II tau functions. The present work implements the approach outlined in [ILP] in the case of Painlevé I.

2. Monodromy and quasiperiodicity

2.1. Associated linear problem

Consider the system of linear differential equations

\[
\begin{align*}
\partial_t \Phi &= A(z,t) \Phi, \\
\partial_z \Phi &= B(z,t) \Phi,
\end{align*}
\]

with \(A\) and \(B\) given by

\[
A(z,t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2 + \begin{pmatrix} 0 & q \\ 4 & 0 \end{pmatrix} z + \begin{pmatrix} -p & q^2 + t/2 \\ -4q & p \end{pmatrix}, \quad (2.2a)
\]

\[
B(z,t) = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & q \\ 2 & 0 \end{pmatrix}. \quad (2.2b)
\]

Equations (2.1a) and (2.2a) provide a canonical form for \(2 \times 2\) linear systems with a single irregular singular point of Poincaré rank three on the Riemann sphere \(\bar{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})\) with the non-diagonalizable highest polar part; here the singularity is located at \(\infty\). The parameters \(p,q,t\) may be thought of as coordinates on the moduli space of the appropriate irregular connections. The global asymptotic behavior of the fundamental matrix solution \(\Phi(z,t)\) as \(z \to \infty\) is characterized by a set of Stokes matrices which will be defined below. They constitute the generalized monodromy data for the linear system (2.1a). Requiring their invariance under simultaneous variation of \(t\), \(p\) and \(q\) leads to the second equation in the Lax pair (2.1). The flatness condition \(\partial_t A - \partial_z B + [A,B] = 0\) is equivalent to the Painlevé I Hamiltonian system described above.

Another Lax pair frequently used in the literature is obtained from (2.2) by a combination of a gauge transformation and a quadratic change of variable. Introducing

\[
\Psi(\xi,t) = K(\xi)^{-1} \Phi(\xi^2,t), \quad K(\xi) = \begin{pmatrix} \sqrt{\xi}/2 & \sqrt{\xi}/2 \\ 1/\sqrt{\xi} & -1/\sqrt{\xi} \end{pmatrix},
\]

the new fundamental solution \(\Psi(\xi,t)\) satisfies

\[
\begin{align*}
\partial_{\xi} \Psi &= \tilde{A}(\xi,t) \Psi, \\
\partial_t \Psi &= \tilde{B}(\xi,t) \Psi,
\end{align*}
\]

where the transformed matrices

\[
\tilde{A}(\xi,t) = \begin{pmatrix} \sqrt{\xi}/2 & \sqrt{\xi}/2 \\ 1/\sqrt{\xi} & -1/\sqrt{\xi} \end{pmatrix}, \quad \tilde{B}(\xi,t) = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} \frac{\xi + \sqrt{\xi^2 + 1}}{\xi - \sqrt{\xi^2 + 1}}.
\]
\[ \tilde{A}(\xi, t) = 2\xi K(\xi)^{-1} A(\xi^2, t) K(\xi) - K(\xi)^{-1} K'(\xi), \quad \tilde{B}(\xi, t) = K(\xi)^{-1} B(\xi^2, t) K(\xi) \]

are explicitly given by

\[ \tilde{A}(\xi, t) = (4\xi^4 + 2q^2 + t) \sigma_z - \left(2p\xi + \frac{1}{2\xi}\right) \sigma_x - (4q\xi^2 + 2q^2 + t) i\sigma_y, \quad (2.4a) \]

\[ \tilde{B}(\xi, t) = \left(\xi + \frac{q}{\xi}\right) \sigma_z - \frac{i\sigma_y}{\xi}. \quad (2.4b) \]

and \( \sigma_{x,y,z} \) denote the Pauli matrices,

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The coefficient of the highest polar part of \( \tilde{A}(\xi, t) \) at \( \xi = \infty \) is diagonalizable (in fact, diagonal), which means that the corresponding irregular singular point is unramified. This is the main advantage of the Lax pair (2.4) in comparison with (2.2).

2.2. Stokes data

The monodromy data of the linear system (2.1a) or its transformed version (2.3a) are integrals of motion of the Painlevé I equation, which uniquely determine the solution \( q(t) [KK] \). Let us now describe them in more detail. In the neighborhood of \( \infty \), the equation (2.3a) possesses a unique formal solution of the form

\[ \Psi_{\text{form}}(\xi) = G(\xi) e^{\Theta(\xi)}, \quad (2.5a) \]

\[ \Theta(\xi) = \left(\frac{4}{5} \xi^3 + t\xi\right) \sigma_z, \quad G(\xi) = \mathbb{1} + \sum_{k=1}^{\infty} g_k \xi^{-k}. \quad (2.5b) \]

The coefficients \( g_k \) can in principle be iteratively determined from (2.3a). Below we will need the first five of these coefficients, explicitly given by

\[ g_1 = -H\sigma_z, \quad g_2 = \frac{H^2}{2} + \frac{q}{2}\sigma_y, \]

\[ g_3 = \left(-\frac{H^3}{6} + \frac{2p - t^2}{24}\right) \sigma_z + \left(\frac{qH}{2} + \frac{p}{3}\right) i\sigma_y, \]

\[ g_4 = \left(-\frac{H^4}{24} + \frac{2p - t^2}{24}H + \frac{q^2}{8}\right) \sigma_z + \left(\frac{qH^2}{4} + \frac{pH}{8} + \frac{2q^2 + t^2}{8}\right) i\sigma_y, \]

\[ g_5 = \left(-\frac{H^5}{120} - \frac{2p - t^2}{48}H^2 - \frac{5q^2 - 2q^2 t - 4pq + 1}{40}\right) \sigma_z + \left(\frac{qH^3}{12} + \frac{pH^2}{8} + \frac{2q^2 + t^2}{16}\right) \sigma_y, \]

\[ (2.6) \]

where \( H \) is the Hamiltonian (1.2).

Ten genuine canonical solutions \( \Psi_k(\xi) \) are uniquely specified by their asymptotic behavior \( \Psi_k(\xi) \sim \Psi_{\text{form}}(\xi) \) as \( \xi \to \infty \) inside the Stokes sectors

\[ \Omega_k = \left\{ \xi \in \mathbb{C}: \frac{(2k - 3)\pi}{10} < \arg\xi < \frac{(2k + 1)\pi}{10} \right\}, \quad k \in \mathbb{Z}/10\mathbb{Z}, \]
of the Stokes data is therefore generically two-dimensional, and this data can be expressed in terms of a pair of complex monodromy parameters which provide Painlevé I transcendents can therefore be labeled as \( \Psi_\kappa (\xi e^{i\theta}) \) for \( \kappa \). Indeed, from (2.7) it follows that both sides of (2.8) satisfy the same equation (2.1a), hence to show their equality it suffices to compare their asymptotics inside the sector \( \mathcal{S}_k \). Furthermore, combining (2.7) and (2.8), one obtains a cyclic identity

\[
S_1 S_2 S_3 S_4 S_5 = i \sigma_x.
\]

This is equivalent to the equations

\[
s_k + 3 = i (1 + s_k s_k + 1), \quad s_k + 5 = s_k,
\]

which imply that there are at most two independent Stokes parameters. For example, for \( s_2 s_3 \neq -1 \) one may express \( s_1, s_4 \) and \( s_5 \) in terms of \( s_2 \) and \( s_3 \):

\[
s_1 = \frac{i - s_3}{1 + s_2 s_3}, \quad s_4 = \frac{i - s_2}{1 + s_2 s_3}, \quad s_5 = i (1 + s_2 s_3).
\]

The space \( \mathcal{S} \) of the Stokes data is therefore generically two-dimensional, and this data can be expressed in terms of a pair of complex monodromy parameters which provide Painlevé I conserved quantities. In what follows, we assume the genericity condition \( s_k \neq 0 \) for \( k = 1, \ldots, 5 \), which excludes the so-called *tronquée* solutions from our consideration.

Define \( v_k = -i v_{2k} \) and further rewrite the relations (2.10) as

\[
v_{k-1} v_{k+1} = 1 - v_k.
\]

It is easy to check that the sequence defined by the latter equation is 5-periodic. In fact, the recurrence (2.12) describes mutations in the simplest rank 2 cluster algebra of finite type, associated with the Dynkin diagram \( A_2 \). Also, introduce new monodromy parameters \( v_k \) by

\[
v_k = e^{2\pi i v_k}, \quad k \in \mathbb{Z}/5\mathbb{Z}.
\]

It will turn out later that the most convenient sets of local coordinates on \( \mathcal{S} \) are provided by the pairs \((v_k, v_{k+1})\) associated with different clusters.

Painlevé I transcendents can therefore be labeled as \( q(t | \nu) \), where \( \nu \in \mathcal{S} \) denotes the appropriate point in the space of the Stokes data. The reader with no prior knowledge of Painlevé theory should think of \( \nu \) as a nonlinear analog of the parameter \( \alpha \) in the Bessel function \( J_\alpha (t) \), or the parameters \( a, b, c \) in the hypergeometric function \( _2F_1 (a, b; c; t) \). It is important to recall that the tau function \( \tau (t | \nu) \) has so far only been defined up to a multiplicative factor \( \mathcal{C} (\nu) \), independent of time.

1 If the condition \( s_2 s_3 \neq -1 \) does not hold, then \( s_2 = s_3 = i, s_5 = 0 \) and \( s_1 + s_4 = i \), which corresponds to a one-dimensional stratum of \( \mathcal{S} \).
In contrast to all other Painlevé equations, Painlevé I does not contain parameters and does not possess affine Bäcklund transformations. However, it does have a finite \( \mathbb{Z}_5 \)-symmetry. If \( q(t) \) and \( H(t) \) are solutions of (1.1) and (1.3), then clearly so are \( \tilde{q}(t) = \zeta^2 q(\zeta t) \) and \( \tilde{H}(t) = \zeta H(\zeta t) \), where \( \zeta = \exp\left( -\frac{2\pi i}{5} \right) \) is a fifth root of unity. This in turn implies that if \( \tau(t) \) is a Painlevé I tau function, then so is \( \tilde{\tau}(t) = \tau(\zeta t) \).

This symmetry can be lifted to solutions of the linear system (2.3a) as \( \Psi(\zeta \xi, t; \tilde{q}(t)) = \Psi(\xi, \zeta t; q(\zeta t)) \). As a consequence, the Stokes parameters \( \tilde{s}_k, \tilde{v}_k \) and \( \tilde{\nu}_k \) corresponding to the transformed solution \( \tilde{q}(t) \) are expressed as \( \tilde{s}_k = s_{k+2}, \tilde{v}_k = v_{k+1} \) and \( \tilde{\nu}_k = v_{k+1} \).

Introducing the operator \( T \) of cyclic permutation, which acts on the monodromy parameters as \( T \nu_k = \nu_{k+1} \) with \( k = 1, \ldots, 5 \), we may then write

\[
q(\zeta t \mid \nu) = \zeta^3 q(t \mid T\nu).
\]

(2.14)

The analog of this relation for the tau function is

\[
\tau(\zeta t \mid \nu) = \Upsilon(\nu) \cdot \tau(t \mid T\nu).
\]

(2.15)

The appearance of the prefactor \( \Upsilon(\nu) \) is related to the ambiguity in the definition of the tau function by a solution of the \( \sigma \)-Painlevé I equation (1.3). However, once the normalization of the tau function \( \tau(t \mid \nu) \) is fixed, the connection coefficient \( \Upsilon(\nu) \) becomes a well-defined function of monodromy.

One way of choosing the normalization is to require \( \tau(0 \mid \nu) = 1 \), so that we trivially have \( \Upsilon(\nu) = 1 \). Although this is legitimate in the generic situation where \( t = 0 \) is not a pole of \( q(t \mid \nu) \), it is more natural, both conceptually and from the point of view of applications, to normalize the asymptotic behavior of the tau function at the only genuine Painlevé I singular point \( t = \infty \). Our main goal in the next sections is the determination of the explicit form of \( \Upsilon(\nu) \) in this setting.
3. Extended Painlevé I tau function

3.1. General setup

A normalization of the Painlevé I tau function can be introduced by constructing a closed 1-form \( \hat{\omega} \in \Lambda^1(\mathbb{C} \times \mathcal{M}) \), whose restriction to the first factor coincides with \( Hdr \). Then the tau function is defined up to a constant independently of the monodromy data by

\[
\tau(t|\nu) = \exp \int \hat{\omega}.
\]

A general approach for constructing \( \hat{\omega} \) was developed in [ILP], using the earlier results of the works [Ber, IP]. It can be summarized as follows:

- For a linear system \( \partial_t \Phi = A(\xi) \Phi \) with rational \( A(\xi) \), one should write formal solutions at each singular point \( a_i \) of \( A(\xi) \) as \( \Phi(\xi, t) = G_i(\xi, t) e^{\Theta_i(t)} \), where

\[
G_i(\xi, t) = \left[ 1 + \sum_{k=1}^{\infty} g_{i,k}(\xi - a_i)^k \right], \quad (3.1a)
\]

\[
\Theta_i(t) = -\sum_{k=-r_i}^{1} \Theta_{i,k}(\xi - a_i)^k + \Theta_{i,0} \ln (\xi - a_i). \quad (3.1b)
\]

In the last formula, \( r_i \) denotes the Poincaré rank of \( a_i \) and the matrices \( \Theta_{i,k} \) are all diagonal. Their elements (with the exception of \( \Theta_{i,0} \)) and the positions \( a_i \) play the role of isomonodromic times. For a singular point at \( \infty \), the expressions (3.1) should be appropriately modified.

- Define a 1-form \( \omega \) by

\[
\omega = \sum_i \text{res}_{\xi=a_i} \text{Tr} \left( A(\xi) \, dG(i)(\xi) \, G(i)(\xi)^{-1} \right), \quad (3.2)
\]

where the differential \( d = d_T + d_M \) is taken both with respect to times \( T \) and the monodromy parameters \( M \). The differential \( \Omega = d\omega \) was shown in [ILP] to be a closed 2-form on \( \mathcal{M} \) only, which is furthermore independent of the isomonodromic times.

- The 2-form \( \Omega \) can in principle be calculated explicitly using the asymptotics of the solutions of the deformation equations expressed in terms of monodromy. Once the expression for \( \Omega \) is found, one may look for a 1-form \( \omega_0 \) on \( \mathcal{M} \) such that \( d\omega_0 = \Omega \) and define \( \hat{\omega} = \omega - \omega_0 \). Of course, \( \omega_0 \) is determined by \( \Omega \) up to the addition of an exact differential on \( \mathcal{M} \), which is the origin of the tau function normalization ambiguity.

The results of [ILP] were obtained under the assumption that the highest polar contribution to \( A(\xi) \) at each singular point is diagonalizable; for Fuchsian singularities, it is required in addition to be non-resonant. The matrix \( A(z, t) \) given by (2.2a) violates the diagonalizability condition at \( z = \infty \), whereas its transformed version \( \tilde{A}(\xi, t) \) from (2.4a) has a resonant (though trivial) Fuchsian singularity at \( \xi = 0 \).

We are not going to develop a general theory for the ramified irregular singularities in this paper. Instead, consider the formula (3.2) as an ansatz for \( \omega \) for the transformed system (2.3a), simply ignoring the undefined contribution of the resonant singular point \( \xi = 0 \). We thus introduce

\[
\omega := \text{res}_{\xi=\infty} \text{Tr} \left( \tilde{A}(\xi, t) \, dG(\xi) \, G(\xi)^{-1} \right),
\]
with $G(\xi)$ defined by (2.5). Since $\tilde{A}(\xi,i)$ is a Laurent polynomial of degree 4 in $\xi$, the residue is given by

$$
\omega = \text{Tr} \left\{ -4\sigma_z (dg_1 h_4 + dg_2 h_3 + dg_3 h_2 + dq \sigma_y (dg_1 h_2 + dg_2 h_1 + dg_3) \\
+ 2p\sigma_x (dg_1 h_1 + dg_2) - (2q^2 + t) (\sigma_z - i\sigma_y) dg_1 \right\},
$$

(3.3)

where $h_k$ denotes the expansion coefficients of the inverse matrix $G(\xi)^{-1} = 1 + \sum_{k=1}^{\infty} h_k \xi^{-k}$. Let us express them in terms of $g_k$,

$$
h_1 = -g_1, \quad h_2 = -g_2 + g_1^2, \quad h_3 = -g_3 + g_2 g_1 + g_1 g_2 - g_1^3, \quad h_4 = -g_4 + g_3 g_1 + g_1 g_3 + g_2^2 - g_2 g_1^2 - g_1 g_2 - g_1^4,
$$

and substitute into (3.3). Using the explicit expressions (2.6) for $g_1, \ldots, g_4$, after a lengthy but straightforward simplification we find that

$$
\omega = 2 \left( H dt + Q_a dm_a + Q_b dm_b \right),
$$

(3.4)

where $m_{a,b}$ are two arbitrary independent local coordinates on the space $\mathcal{S}$ of the Painlevé I Stokes data, and the coefficients $Q_{a,b}$ are given by

$$
Q_k = \frac{1}{2} \left( 4t q_k q_m a + 3 q_k q_m a - 2 q_k q_m - 24 t q_k q_m - 4 t^2 q_k q_m \right)
= \frac{1}{2} \left( 4t H_m a + 3 q_k q_m a - 2 q_k q_m \right), \quad k = a, b.
$$

(3.5)

It can now be checked directly that $\omega$ indeed has all the necessary properties for realization of the scheme outlined above. Note, however, the appearance of an extra factor of 2 in (3.4).

**Proposition 3.1.** The differential $\Omega = d\omega$ of the form $\omega$ defined by (3.4)–(3.5) is a closed constant 2-form on $\mathcal{S}$.

**Proof.** Differentiating $H$ with respect to the monodromy, one obtains

$$
\partial_m H = q_m q_m a - (6 q_t^2 + t) q_m a.
$$

Compute the time derivative of (3.5) and use the Painlevé I equation (1.1) to eliminate all second order time derivatives in the resulting expressions. Simplifying the result, we observe that $\partial_t Q_k = \partial_m H$, which shows that $\Omega$ is a 2-form on $\mathcal{S}$ only.

One also has

$$
\partial_m Q_b = -\frac{1}{2} \left( 4 t q_m a q_m a + 4 t q_m a q_m a + 3 q_m q_m a + 3 q_m a q_m a \\
- 2 q_m q_m a - 2 q_m a q_m a - 24 t q_m a q_m a - 48 t q_m a q_m a - 4 t^2 q_m a q_m a \right).
$$

Almost all the terms in this formula are symmetric with respect to the exchange $a \leftrightarrow b$, and therefore do not contribute to $\Omega = 2 (\partial_m Q_b - \partial_m Q_a) dm_a \wedge dm_b$. The non-symmetric part yields

$$
\Omega = 2 (q_m a q_m a - q_m a q_m a) dm_a \wedge dm_b.
$$

(3.6)
It remains to show that $\partial_i (q_{m_b} q_{m_b} - q_{m_b} q_{m_b}) = 0$, which is an easy consequence of (1.1).

### 3.2. Asymptotics on five canonical rays

Our task in this section is to construct a 1-form $\omega_0 \in \Lambda^1(S)$ such that $d\omega_0 = \Omega$. In order to achieve this goal, we will first compute the explicit form of $\Omega$ in terms of Stokes parameters using the results of Kapaev [Kap1, Kap2] (see also [Tak]) for the asymptotics of the Painlevé I transcendent on the rays

$$\mathcal{R}_k = \left\{ t \in e^{\pi i - \frac{2\pi i}{5}} | \Re t \geq 0 \right\}, \quad k \in \mathbb{Z}/5\mathbb{Z},$$

as $|t| \to \infty$. Recall that the Stokes multipliers are parameterized as

$$s_{2k} = i e^{2\pi i \nu}, \quad k \in \mathbb{Z}/5\mathbb{Z}.$$

The pairs $(\nu_k, \nu_{k+1})$ correspond to five different choices of local coordinates on $S$, adapted for description of the asymptotics on different rays $\mathcal{R}_k$.

**Theorem 3.2 ([Kap2]).** The asymptotic behavior of the Painlevé I function $q(t | \nu)$ as $|t| \to \infty$, $\arg t = \pi - \frac{2\pi k}{5}$ with $k \in \mathbb{Z}/5\mathbb{Z}$ is described by the following formulas:

- For $|\Re \nu_k| < \frac{1}{5}$, one has

$$q(t | \nu) = e^{i \frac{2\pi}{5}} \sqrt{e^{\frac{2\pi i}{5} - \nu i t}} \left[ \frac{1}{\sqrt{6}} + \sum_{\nu = \pm} \alpha_{\nu} (\nu_k) x^{\nu} e^{\frac{2\pi i \nu}{5} + 2\pi i \nu_{k+1}} + O \left( x^{1+|\Re \nu_k|} \right) \right],$$

where $x = 24^{\frac{1}{2}} \left( e^{\frac{2\pi i}{5} - \nu i} \right)^{\frac{1}{2}}$ and the coefficients $\alpha_{\nu} (\nu)$ are given by

$$\alpha_{+} (\nu) = 48 e^{-\nu \pi i} \sqrt{2\pi} \Gamma (1 + \nu) / \Gamma (\nu), \quad \alpha_{-} (\nu) = 48 e^{-\nu \pi i} \sqrt{\pi} \Gamma (\nu).$$

- For $0 < \Re \nu_k < 1$, one has

$$q(t | \nu) = e^{i \frac{2\pi}{5}} \sqrt{e^{\frac{2\pi i}{5} - \nu i t}} \left[ \frac{1}{\sqrt{6}} + \sum_{\nu = \pm} \beta_{\nu} (\nu_k) x^{\nu} e^{\frac{2\pi i \nu}{5} + 2\pi i \nu_{k+1}} + O \left( x^{2|\Re \nu_k|} \right) \right],$$

where $\beta_{+} (\nu) = \frac{\sqrt{6}}{\alpha_{+} (\nu)}$ and $\beta_{-} (\nu) = \frac{\alpha_{-} (\nu)}{4\sqrt{6}}$.

Of course, the two behaviors (3.8) and (3.10) are compatible on the overlap of the corresponding domains. The latter covers all possible values of the Stokes parameters except the pathological one-dimensional strata described above. It turns out that both formulas produce the same asymptotic form of the tau function. Moreover, iteratively computing the next terms in the expansion of $\tau(t | \nu)$, one may observe the following periodic pattern:

\footnote{Let us mention the paper [HRZ], which is concerned with another type of Painlevé I tau function expansions; namely, around movable poles of $q(t | \nu)$, i.e. zeros of $\tau(t | \nu)$. This study highlights the fact that Painlevé I can be considered as a deautonomization of the differential equation for the Weierstrass $\wp$-function.}
Conjecture 3.3 ([BLMST], section 3.1). The asymptotic expansion of the Painlevé I tau function \( \tau(t | \nu) \) as \( t \to \infty \) along the ray \( R_k \) has the structure of a Fourier transform,

\[
\tau(t | \nu) \simeq C_1(\nu) x^{-\nu} e^{\frac{\nu \pi}{2}} \sum_{n \in \mathbb{Z}} e^{2\pi i n y_{k-1}} B(n, x), \tag{3.11a}
\]

\[
B(n, x) \simeq C(\nu) x^{-\nu} e^{\frac{\nu \pi}{2}} \left[ 1 + \sum_{k=1}^{\infty} \frac{B_k(\nu)}{x^k} \right], \tag{3.11b}
\]

\[
C(\nu) = 48^{-\frac{\nu}{2}} (2\pi)^{-\frac{\nu}{2}} e^{-\frac{\nu \pi^2}{2}} G(1 + \nu), \tag{3.11c}
\]

where as before \( x = 24^{\frac{1}{4}} \left( e^{\frac{i\pi}{2} - \pi t} \right) \), and \( G(z) \) denotes the Barnes \( G \)-function.

This proposal has been verified by calculating explicitly over 50 first terms in the asymptotic expansion of \( \tau(t | \nu) \). Setting for definiteness \( |\Re \nu| \leq \frac{1}{2}, \) this corresponds to taking the values \( |n| \leq 4 \) in \( (3.11a) \) and going up to \( k = 7 \) in \( (3.11b) \). The coefficients \( B_k(\nu) \) of \( B(n, x) \) are polynomials of degree \( 3k \) in \( \nu \) with rational coefficients; the first few of them are given by

\[
B_1(\nu) = -\frac{i\nu (94\nu^2 + 17)}{96},
\]

\[
B_2(\nu) = \frac{44180\nu^6 + 170320\nu^4 + 74985\nu^2 + 1344}{92160},
\]

\[
B_3(\nu) = \frac{i\nu (4 \times 152920\nu^8 + 45777060\nu^6 + 156847302\nu^4 + 124622833\nu^2 + 13059000)}{26542080}.
\]

\[ \ldots \ldots \ldots \ldots \]

We are now in a position to determine the explicit form of \( \Omega \) defined by \((3.6)\). It is very useful to rewrite the latter formula as

\[
\Omega = 2d_{\mathcal{S}}q_{\nu} \wedge d_{\mathcal{S}}q_{\nu}, \tag{3.12}
\]

where \( d_{\mathcal{S}} \) denotes the differential taken with respect to the Stokes data. The formal expansion of \( q(t | \nu) \) on the ray \( R_k \) can be written as

\[
q(t | \nu) = e^{\frac{\nu t}{2}} e^{\frac{\nu \pi}{2} - \nu t} \left[ \frac{1}{\sqrt{6}} + \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} a_{l-2j} p^{l-2j}(x) \right],
\]

where

\[
p(x) = x^{-\nu} e^{\frac{\nu x}{2}}, \quad x = 24^{\frac{1}{4}} \left( e^{\frac{i\pi}{2} - \pi t} \right) ^{\frac{1}{2}}.
\]

Although the structure of this expansion is not as transparent as for the tau function \( \tau(t | \nu) \), we only need the few first terms of it. It suffices to compute \( \Omega \) under the assumption of \( |\Re \nu| < \frac{1}{6} \), in which case the terms present in \((3.8)\), i.e. with \( l = 1 \), are already sufficient. Their straightforward substitution into \((3.12)\) gives

\[
\Omega = -4i x d_{\mathcal{S}} \left( \alpha_+(\nu_k) x^{-\nu_k} e^{\frac{\nu_k}{2} + 2\pi i n_{k+1}} \right) \wedge d_{\mathcal{S}} \left( \alpha_-(\nu_k) x^{-\nu_k} e^{-\frac{\nu_k}{2} - 2\pi i n_{k+1}} \right) + O \left( x^{-\frac{1}{4} + \frac{1}{6} |\Re \nu|} \right). \tag{3.13}
\]
Using the fact that the coefficients $\alpha_{\pm}(\nu)$ are independent of $\nu_{k+1}$, the first term in (3.13) can be simplified to $-8\pi d (\alpha_+ (\nu) - \alpha_- (\nu)) \wedge d\nu_{k+1}$, and further reduced to $4\pi i d\nu_k \wedge d\nu_{k+1}$ using (3.9). The error term is actually absent, as it has been shown in proposition 3.1 that $\Omega$ is independent of $t$. We thus obtain:

**Proposition 3.4.** The 2-form $\Omega$ can be expressed as

$$\Omega = 4\pi i d\nu_k \wedge d\nu_{k+1}, \quad k \in \mathbb{Z}/5\mathbb{Z},$$

(3.14)

where $(\nu_k, \nu_{k+1})$ is any of the five pairs of local coordinates on $\mathcal{S}$ defined by (3.7).

In other words, each of the five pairs $(\nu_k, \nu_{k+1})$ provides canonical coordinates on the space $\mathcal{S}$ of the Stokes data of the linear system associated with Painlevé I. One may also rewrite the formula (3.14) directly in terms of the Stokes parameters,

$$\Omega = \frac{i}{\pi} \frac{ds_2 + 2k \wedge ds_3 + 2k}{1 + s_2 + s_3 + 2k}, \quad k \in \mathbb{Z}/5\mathbb{Z}.$$  

(3.15)

The tau function normalization is determined up to a factor that is independent of monodromy by choice of the form $\omega_0 \in A^1(\mathcal{S})$ such that $d\omega_0 = \Omega$; recall that the freedom exists to add an exact differential to $\omega_0$ on $\mathcal{S}$. In principle, we are already able to set, e.g., $\omega_0 = 4\pi i \nu_k d\nu_{k+1}$, and define the extended tau function on $\mathbb{C} \times \mathcal{S}$ as $d\ln \tau = (\omega - \omega_0)/2$. This choice of $\omega_0$ turns out to be compatible with setting $\mathcal{C}_k(\nu) = 1$ in the asymptotics (3.11a) on the ray $\mathcal{R}_k = e^{\pi i - 2\pi k} \mathbb{R}_{>0}$.

**Proposition 3.5.** Given $k \in \mathbb{Z}/5\mathbb{Z}$, let us introduce

$$\omega_{0,k} := 4\pi i \nu_k d\nu_{k+1}.$$  

(3.16)

Let $|\Re \nu_k| < \frac{1}{2}$. The extended tau function $\tau_k(t|\nu)$ defined by

$$d\ln \tau_k(t|\nu) = \frac{\omega - \omega_{0,k}}{2},$$

(3.17)

with $\omega$ given by (3.4)–(3.5), is characterized by the following asymptotic behavior as $t \to \infty$ along $\mathcal{R}_k$:

$$\tau_k(t|\nu) = \mathcal{C}_k(\nu_k) x^{-\frac{1}{2} - \frac{\nu_2}{4} + 2i\nu_k} \left[ 1 + o(1) \right],$$

(3.18)

where $x = 24^{\frac{3}{4}} \left( e^{\pi i - \pi t} \right)^{\frac{1}{6}}$ and $\mathcal{C}_k$ is independent of the Stokes data.

**Proof.** Consider the leading terms in (3.11a)–(3.11b) (note that the conjectural part of the statement concerns only the full expansion), one finds that as $t \to e^{\pi i - 2\pi k} \infty$,

$$d\ln \tau_k(t|\nu) = d \left( \frac{x^2}{45} + \frac{4\nu_2 x}{5} - \frac{\nu_2^2 \ln x}{2} + \ln \mathcal{C}(\nu_k) \right) + d\ln \mathcal{C}_k(\nu) + o(1).$$

On the other hand, from (3.4)–(3.5) and the asymptotics (3.8), one may deduce the corresponding asymptotics of $\omega$. From the expressions (3.9) for $\alpha_{\pm}(\nu)$ combined with the classical formula

$$d\ln (1 + z) = zd\ln \Gamma (1 + z) - d \left( \frac{z^2}{2} \right) + \frac{\ln 2\pi - 1}{2} dz,$$

(3.19)
after a somewhat lengthy simplification, it follows that
\[
\omega = 2d \left( \frac{x^2}{45} + \frac{4i\nu_kx}{5} - \frac{\nu_k^2 \ln x}{2} + \ln C(\nu_k) \right) + 4\pi i\nu_k \nu_{k+1} + o(1),
\]
which yields the statement of the proposition. \(\square\)

3.3. Connection constant

Let us now set \(\nu_k = 1\) in (3.18). This defines five distinct tau function normalizations \(\tau_k(t|\nu)\) corresponding to normalized asymptotic behaviors on different rays. The connection coefficients that we are after can be alternatively defined as

\[
\Upsilon_{k\ell'}(\nu) = \frac{\tau_{k'}(t|\nu)}{\tau_k(t|\nu)}.
\]

Proposition 3.5 implies that
\[
d \ln \Upsilon_{k\ell'}(\nu) = \frac{\omega_{0,k} - \omega_{0,k'}}{2} = 2\pi i (\nu_k \nu_{k+1} - \nu_{k'} \nu_{k'+1}).
\]
Thus \(\ln \Upsilon_{k\ell'}(\nu)\) coincides with the generating function of the canonical transformation between the pairs \((\nu_k, \nu_{k+1})\) and \((\nu_{k'}, \nu_{k'+1})\) of the local Darboux coordinates on the Painlevé I monodromy manifold. To obtain its explicit form (up to an additive constant independent of monodromy), it clearly suffices to compute the antiderivative
\[
2\pi i \int (\nu_{k-1} \nu_k - \nu_k \nu_{k+1}),
\]
which enters into the expression for the connection constant \(\Upsilon_{k-1,k}(\nu)\) between adjacent rays.

We are now finally ready to state our main result:

**Theorem 3.6.** The connection coefficient \(\Upsilon_{k-1,k}(\nu)\) is expressed in terms of the Stokes data as
\[
\Upsilon_{k-1,k}(\nu) = e^{\chi} \left( 2\pi \right)^{-\nu_k} e^{2\pi i\nu_{k-1} - \frac{\nu_k^2}{4\nu_k}} \hat{G}(\nu_k),
\]
where \(\hat{G}(z) = \frac{G(z + 1)}{G(z - 1)}\) and \(k \in \mathbb{Z}/5\mathbb{Z}.

**Proof.** Let us rewrite the recurrence relation (2.12) in terms of \(\nu_k\) with the help of (2.13):
\[
e^{2\pi i\nu_{k+1}} = (1 - e^{2\pi i\nu_k}) e^{-2\pi i\nu_{k-1}}.
\]
This transforms the antiderivative (3.21) into
\[
\ln \Upsilon_{k-1,k}(\nu) = 2\pi i\nu_{k-1} - \nu_k - \nu_k d \ln (1 - e^{2\pi i\nu_k}).
\]
The identity (3.19) implies the differentiation formula \(d \ln \hat{G}(z) = 2\pi dz - zd \ln \sin \pi z\). Using this to compute the last integral, we obtain
\[
\Upsilon_{k-1,k}(\nu) = \chi \cdot \left( 2\pi \right)^{-\nu_k} e^{-\frac{\nu_k^2}{4\nu_k} + 2\pi i\nu_{k-1} \nu_k} \hat{G}(\nu_k),
\]
where $\chi$ is a numerical constant that is independent of monodromy. Analogous constants for the Painlevé VI and Painlevé II have been fixed in [ILP] with the help of special solutions (respectively, the algebraic/Picard and Hastings–McLeod) of the corresponding equations. Even though such solutions are not available for Painlevé I, we will be able to find an explicit evaluation for $\chi$ by exploiting the $\mathbb{Z}_5$-symmetry.

Denote by $\tau^{(0)}(t | \nu)$ the Painlevé I tau function associated with the Stokes data $\nu$ and normalized as $\tau^{(0)}(0 | \nu) = 1$. Recall that $\tau^{(0)}(\zeta t | \nu) = \tau^{(0)}(t | T\nu)$, where $\zeta = \exp\left(-\frac{2\pi i}{5}\right)$ and $T$ cyclically permutes the Stokes parameters, see section 2.3. We can then write

$$\tau_k(t | \nu) = \tilde{\Upsilon}(T^k \nu) \tau^{(0)}(\zeta^{-kt} | T^k \nu) = \tilde{\Upsilon}(T^k \nu) \tau^{(0)}(t | \nu).$$

where $\tilde{\Upsilon}(\nu)$ stands for the coefficient of relative normalization of the tau functions $\tau^{(0)}$ and $\tau^{(0)}_0$ (i.e. the connection constant between $-\infty$ and $0$). As a consequence, the coefficients (3.20) have the structure

$$\Upsilon_{kk'}(\nu) = \tilde{\Upsilon}(T^k \nu) \tilde{\Upsilon}(T^{k'} \nu).$$

Although the explicit form of $\tilde{\Upsilon}(\nu)$ is unknown, this structure implies that for a point $\nu_f \in \mathcal{S}$ fixed by $T$ we should have $\Upsilon_{kk'}(\nu_f) = 1$. One, however, has to check that the tau function $\tau(t | \nu_f)$ associated with this specific monodromy does not vanish at $t = 0$ (or, equivalently, that $q(t | \nu_f)$ does not have a pole there). This condition ensures the existence of $\tau^{(0)}_0$, which appears in the above argument, and it is verified for at least one of the two fixed points of $T$, namely, for $[Kit]$

$$\nu_1 = \ldots = \nu_5 = e^{2\pi i \nu_f} = \sqrt{5} - \frac{1}{2}.$$

It follows that the numerical constant we are looking for is given by

$$\chi = (2\pi)^{\nu_f} e^{\sum_{j=1}^{5} \frac{\nu_{j}}{\nu_f}} \tilde{G}(-\nu_f).$$

In order to further simplify this representation, let us note that the function $\hat{G}(z)$ is closely related to the classical dilogarithm $\text{Li}_2(e^{2\pi i z})$:

$$\hat{G}(z) = \left(\frac{\sin \pi z}{\pi}\right)^{-2} \exp \left\{ \frac{\pi iz (1 - z)}{2} - \frac{\pi i}{12} - \frac{\text{Li}_2(e^{2\pi i z})}{2\pi i} \right\}.\quad (3.25)$$

The relevant quantity

$$\text{Li}_2\left(\frac{\sqrt{5} - 1}{2}\right) = \frac{\pi^2}{10} - \ln^2 \frac{\sqrt{5} + 1}{2}$$

is one of the few dilogarithm values with explicit elementary evaluations. Putting (3.24)–(3.26) together and simplifying the result, we end up with $\chi = e^{\frac{3\pi i}{2}}$. □

**Remark 3.7.** Using the recurrence relation $\hat{G}(z + 1) = -\frac{\pi}{\sin \pi z} \hat{G}(z)$ and the formula (3.23), it is straightforward to check that the answer (3.22) satisfies the quasiperiodicity relations
\[ \Upsilon_{k-1,k}(\nu_k - 1, \nu_k) = e^{2\pi i \nu_k} \Upsilon_{k-1,k}(\nu_k - 1, \nu_k), \]
\[ \Upsilon_{k-1,k}(\nu_k - 1, \nu_k + 1) = e^{-2\pi i \nu_k} \Upsilon_{k-1,k}(\nu_k - 1, \nu_k), \]
which may be considered as further evidence for conjecture 3.3. From the definition of \( \Upsilon_{k-1,k}(\nu) \) it is also clear that this quantity should satisfy the cyclic identity \( \prod_{k=1}^{5} \Upsilon_{k-1,k}(\nu) = 1 \). It indeed holds for the above result (3.22). However, the verification is not completely trivial; the relation
\[ \prod_{k=1}^{5} \hat{G}(\nu_k) = e^{-\pi i/6} \prod_{k=1}^{5} (2\pi)^{\nu_k} e^{\frac{\nu_k^2}{2} - 2\pi i \nu_k + \nu_{k-1}} \]
turns out to be equivalent to Abel’s five-term identity
\[ \sum_{k=1}^{5} L(\nu_k) = \frac{\pi^2}{2}, \]
satisfied by the Rogers \( L \)-function \( L(z) = \text{Li}_2(z) + \frac{1}{2} \ln z \ln (1 - z) \).

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