3D Topological Quantum Computing

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July 20, 2021

Abstract

In this paper we will present some ideas to use 3D topology for quantum computing extending ideas from a previous paper. Topological quantum computing used “knotted” quantum states of topological phases of matter, called anyons. But anyons are connected with surface topology. But surfaces have (usually) abelian fundamental groups and therefore one needs non-abelian anyons to use it for quantum computing. But usual materials are 3D objects which can admit more complicated topologies. Here, complements of knots do play a prominent role and are in principle the main parts to understand 3-manifold topology. For that purpose, we will construct a quantum system on the complements of a knot in the 3-sphere (see arXiv:2102.04452 for previous work). The whole system is designed as knotted superconductor where every crossing is a Josephson junction and the qubit is realized as flux qubit. We discuss the properties of this systems in particular the fluxion quantization by using the A-polynomial of the knot. Furthermore we showed that 2-qubit operations can be realized by linked (knotted) superconductors again coupled via a Josephson junction.

1 Introduction

Quantum computing exploits quantum-mechanical phenomena such as superposition and entanglement to perform operations on data, which in many cases, are infeasible to do efficiently on classical computers. Topological quantum computing seeks to implement a more resilient qubit by utilizing non-Abelian forms of matter like non-abelian anyons to store quantum information. Then, operations (what we may think of as quantum gates) are performed upon these qubits through “braiding” the worldlines of the anyons. We refer to the book [1] for an introduction of these ideas.

In a previous paper [2] we described first ideas to use 3-manifold topology for topological quantum computing. There, we discussed the knot complement as the main part of a 3-manifold. Main topological invariant of a knot complement
is the fundamental group $\pi_1$. In the previous paper we discussed the representation of the fundamental group into $SU(2)$ to produce the 1-qubit operations. Furthermore we argued that linking is able to produce 2-qubit operations showing the universality of the approach. The operations are realized by the consideration of the Berry phase. In contrast, in this paper we will realize 1-qubit operations of the knot group via knotted superconductors. Here, the crossings are given by Josephson junctions to mimic the knot. Furthermore, the 2-qubit operations is again realized by the linking of two knotted superconductors via a Josephson element, in agreement with [2].

In the next section we will introduce the concept of a fundamental group and manifold. Furthermore we explain the importance of knot complements. In section 3 we explain the representation of knot groups partly from [2] to make the paper self-contained. Then in section 4 we introduce the model of a knotted superconductor with flux qubit. Here we will discuss the fluxion quantization (using the A-polynomial) and the influence of the Josephson junction. Main result is the realization of the 2-qubit operation by linking.

2 Some preliminaries and motivation: 3-manifolds and knot complements

In this paper we need two concepts, the fundamental group and the manifold, which will be introduced now. We assume that the reader is familiar with the definition of a topological space. Let $X, Y$ be topological spaces. First consider the definition of homotopy as applied to a pair of maps, $f, g : X \to Y$:

- Let $f, g : X \to Y$ be continuous functions; $f$ and $g$ are homotopic to each other, denoted by $f \simeq g$ if there is a continuous function $F : X \times [0, 1] \to Y$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$.

- The function $F$ provides a deformation of one map into the other. Clearly, this relation is an equivalence relation. The equivalence class of homotopic maps between $X$ and $Y$ will be denoted by

$$[X, Y] = \{ f : X \to Y \text{ continuous } \} / \simeq .$$

This relation leads to the notion of homotopy-equivalence of spaces.

- Two topological spaces $X$ and $Y$ are homotopy-equivalent, if there are two smooth maps $f : X \to Y$ and $g : Y \to X$ so that

$$f \circ g \simeq Id_Y \quad g \circ f \simeq Id_X$$

where $Id_X$ and $Id_Y$ are the identity maps on $X$ and $Y$, respectively.

In general define

$$\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)] .$$
the homotopy equivalence class of maps of the pointed sphere into the pointed space. Since we have used the word “group” to refer to them we must define a combining operation.

Here we need the $n = 1$ case, $\pi_1$, the fundamental group. This is the loop space, modulo smooth deformations or contractions. There are several ways to define the group combining operation. We choose one which is easily extendable from $n = 1$ to the general case. Let $S^1 \vee S^1$ be the one-point union defined by

$$S^1 \vee S^1 = S^1 \times \{s_1\} \cup_{s_0 = s_1} \{s_0\} \times S^1 \subset S^1 \times S^1 \ni (s_0, s_1).$$

Now define the product $\gamma_1 \star \gamma_2 : S^1 \to X$ of the maps $\gamma_1, \gamma_2 : S^1 \to X$ to $\gamma_{12} : S^1 \to X$ as defined geometrically by the process to identify two opposite points on the circle $S^1$, with naturally defined map. We then combine this map with the maps $\gamma_1, \gamma_2$ to define the product, $\gamma_1 \star \gamma_2$. This provides a group product structure which will in general not be abelian, since there is no map homotopic to the identity which switches the upper and lower circles in this diagram. The proof of the associativity of this product can be found for instance in [4] Proposition 14.16. These formal definitions have a simple interpretation:

- elements of the fundamental group $\pi_1(X)$ are closed non-contractible curves (the unit element $e$ is the contractible curve);
- the group operation is the concatenation of closed curves up to homotopy (to guarantee associativity);
- the inverse group element is a closed curve with opposite orientation;
- the fundamental group $\pi_1(X)$ is a topological invariant, i.e. homeomorphic spaces $X, Y$ have isomorphic fundamental groups;
- the space $X$ is (usually) path-connected so that the choice of the point $x_0$ is arbitrary.

Clearly, every closed curve in $\mathbb{R}^2, \mathbb{R}^3$ is contractible, therefore $\pi_1(\mathbb{R}^2) = 0 = \pi_1(\mathbb{R}^3)$. The group is nontrivial for the circle $\pi_1(S^1)$. Obviously, a curve $a$ going around the circle is a closed curve which cannot be contract (to a point). The same is true for the concatenation $a^2$ of two closed curves, three curves $a^3$ etc. Therefore a closed curve in $S^1$ is characterized by the winding number of the closed curve or $\pi_1(S^1) = \mathbb{Z}$, see [4] for more details. In general the fundamental group consists of sequences of generators (the alphabet) restricted by relations (the grammar). For $\pi_1(S^1)$, we have one generator $a$ but no relations, i.e.

$$\pi_1(S^1) = \langle a | \emptyset \rangle = \mathbb{Z}$$

A similar argumentation can be used for $S^1 \vee S^1$, the one-point union of two circles. The closed curve in the first circle is generator $a$ and in the second circle it is $b$. There is no relation, i.e.

$$\pi_1(S^1 \vee S^1) = \langle a, b | \emptyset \rangle = \mathbb{Z} \ast \mathbb{Z}$$

the group is non-abelian (the sequences $ab$ and $ba$ are different). The second ingredient of our work is the concept of a manifold defined by:
Let $M$ be a Hausdorff topological space covered by a (countable) family of open sets, $\mathcal{U}$, together with homeomorphisms, $\phi_U : U \ni U \rightarrow U_R$, where $U_R$ is an open set of $\mathbb{R}^n$. This defines $M$ as a topological manifold. For smoothness we require that, where defined, $\phi_U \cdot \phi_V^{-1}$ is smooth in $\mathbb{R}^n$, in the standard multi-variable calculus sense. The family $\mathcal{A} = \{\mathcal{U}, \phi_U\}$ is called an atlas or a differentiable structure. Obviously, $\mathcal{A}$ is not unique. Two atlases are said to be compatible if their union is also an atlas. From this comes the notion of a maximal atlas. Finally, the pair $(M, \mathcal{A})$, with $\mathcal{A}$ maximal, defines a smooth manifold of dimension $n$.

Now we will concentrate on two- and three-dimensional manifolds, 2-manifold and 3-manifold for short. For 2-manifolds, the basic elements are the 2-sphere $S^2$, the torus $T^2$ or the Klein bottle $\mathbb{R}P^2$. Then one gets for the classification of 2-manifolds:

- Every compact, closed, oriented 2-manifold $S_g$ is homeomorphic to either $S^2$ ($\pi_1(S^2) = 0$) or to the connected sum

$$S_g = T^2 \# T^2 \# \ldots \# T^2,$$

$$\pi_1(S_g) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$$

of $T^2$ for a fixed genus $g$. Every compact, closed, non-oriented 2-manifold is homeomorphic to the connected sum

$$\underbrace{\mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_g,$$

$$\pi_1(\tilde{S}_g) = \langle a_1, \ldots, a_g \mid a_1^2 \cdots a_g^2 = e \rangle$$

of $\mathbb{R}P^2$ for a fixed genus $g$.

- Every compact 2-manifold with boundary can be obtained from one of these cases by cutting out the specific number of disks $D^2$ from one of the connected sums.

The connected sum operation $\#$ is defined by: Let $M, N$ be two $n$-manifolds with boundaries $\partial M, \partial N$. The connected sum $M \# N$ is the procedure of cutting out a disk $D^n$ from the interior $\text{int}(M) \setminus D^n$ and $\text{int}(N) \setminus D^n$ with the boundaries $S^{n-1} \cup \partial M$ and $S^{n-1} \cup \partial N$, respectively, and gluing them together along the common boundary component $S^{n-1}$. This operation is important for 3-manifolds too. But the classification of 3-manifolds is more complex. Following Thurston’s idea, one needs eight pieces which are arranged by using two sums (sum along a torus and connected sum). We don’t want to go into the details and refer to [5, 6] for a description of 3-manifolds (the conjecture of Thurston was proved by Perelman [7, 8, 9] in 2003). The following facts are a consequence of this classification:

- 3-manifolds are mainly classified by the fundamental group (and the Reidemeister torsion for lens spaces),
• the fundamental groups of 3-manifolds can be non-abelian which is impossible for oriented 2-manifolds

• the simplest pieces of 3-manifolds are mainly given by the complement of a knot or link.

Even the last point is the main motivation of this paper. In contrast to topological quantum computing with anyons, we cannot directly use 3-manifolds (as submanifolds) like surfaces in the fractional Quantum Hall effect. Surfaces (or 2-manifolds) embed into a 3-dimensional space like $\mathbb{R}^3$ but 3-manifolds require a 5-dimensional space like $\mathbb{R}^5$ as an embedding space. Therefore, we cannot directly use 3-manifolds. However, as we argued above, there is a group-theoretical substitute for a 3-manifolds, the fundamental group of a knot complement also known as knot group which will be introduced now.

A knot in mathematics is the embedding $K : S^1 \to S^3$ of a circle into the 3-sphere $S^3$ (or in $\mathbb{R}^3$), i.e. a closed knotted curve $K(S^1)$ (or $K$ for short). To form the knot complement, we have to consider a thick knot $K \times D^2$ (knotted solid torus). Then the knot complement is defined by

$$C(K) = S^3 \setminus (K \times D^2)$$

and the knot group $\pi_1(C(K))$ is the fundamental group of the knot complement. The knot complement $C(K)$ is a 3-manifold with boundary $\partial C(K) = T^2$. It was shown that prime knots are divided into two classes: hyperbolic knots ($C(K)$ admits a hyperbolic structure) and non-hyperbolic knots ($C(K)$ admits one of the other seven geometric structures). An embedding of disjoints circles into $S^3$ is called a link $L$. Then, $C(L)$ is the link complement. If we speak about 3-manifolds then we have to consider $C(K)$ as one of the basic pieces. Furthermore, there is the Gordon–Luecke theorem: if two knot complements are homeomorphic, then the knots are equivalent (see in [3] for the statement of the exact theorem). Interestingly, knot complements of prime knots are determined by its fundamental group.

### 3 Knot groups and quantum computing representations

Any knot can be represented by a projection on the plane with no multiple points which are more than double. As an example let us consider the simplest knot, the trefoil knot $3_1$ (knot with three crossings). The plane projection of the trefoil is shown in Fig. [1]. This projection can be divided into three arcs, around each arc we have a closed curve as generator of $\pi_1(C(3_1))$ denoted by $a, b, c$ (see Fig. [2]). Now each crossing gives a relation between the corresponding generators: $c = a^{-1} ba, b = c^{-1} ac, a = b^{-1} cb$, i.e. we obtain the knot group

$$\pi_1(C(3_1)) = \langle a, b, c | c = a^{-1} ba, b = c^{-1} ac, a = b^{-1} cb \rangle$$
Then we substitute the expression $c = a^{-1}ba$ into the other relations to get a representation of the knot with two generators and one relation. From relation $a = b^{-1}cb$ we will obtain $a = b^{-1}(a^{-1}ba)b$ or $bab = aba$ and the other relation $b = c^{-1}ac$ gives nothing new. Finally we will get the result

$$\pi_1(C(3_1)) = \langle a, b \mid bab = aba \rangle$$

But this group is also well-known, it is the braid group $B_3$ of three strands. Now we will get in touch with quantum computing. The main idea is the interpretation of the braid group $B_3$ as operations (gates) on qubits. From the mathematical point of view, we have to consider the representation of $B_3$ into $SU(2)$, i.e. a homomorphism

$$\phi : B_3 \to SU(2)$$

mapping sequences of generators (called words) into matrices as elements of $SU(2)$. At first we note that a matrix in $SU(2)$ has the form

$$M = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \quad |z|^2 + |w|^2 = 1$$

where $z$ and $w$ are complex numbers. Now we choose a well-known basis of $SU(2)$:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad j = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad k = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$
so that

\[ M = a1 + bi + cj + dk \]

with \( a^2 + b^2 + c^2 + d^2 = 1 \) (and \( z = a + bi, w = c + di \)). The algebra of 1, i, j, k are known as quaternions. Among all representations, there is the simplest example

\[ g = e^{i\pi i/10}, f = i\tau + k\sqrt{\tau}, h = fgf^{-1} \]

where \( \tau^2 + \tau = 1 \) and we have the matrix representation

\[
\begin{pmatrix}
    e^{i\pi/10} & 0 \\
    0 & e^{-i\pi/10}
\end{pmatrix}
\quad f =
\begin{pmatrix}
    i\tau & i\sqrt{\tau} \\
    i\sqrt{\tau} & -i\tau
\end{pmatrix}
\]

Then \( g, h \) satisfy \( ghg = hgh \) the relation of \( B_3 \). This representation is known as the Fibonacci representation of \( B_3 \) to \( SU(2) \). The Fibonacci representation is dense in \( SU(2) \), see [12, 1]. The Fibonacci representation is usually used in anyonic quantum computing. It denotes a special representation of the braid group into \( SU(2) \). In case of the trefoil knot, the knot group is the braid group \( B_3 \) so that the representation of the knot group agrees with the representation in anyonic quantum computing. The 3-manifold associated to the trefoil knot is the Poincare sphere.

However, there are more complicated knots. The complexity of knots is measured by the number of crossings. There is only one knot with three crossings (trefoil) and with four crossings (figure-8). For the figure-8 knot \( 4_1 \) (see Figure 3), the knot group is given by

\[
\pi_1 (C(4_1)) = \langle a, b \mid bab^{-1}ab = aba^{-1}ba \rangle
\]

admitting a representation \( \phi \) into \( SU(2) \), see [13]. Here, we remark that the figure-8 knot is part of a large class, the so-called hyperbolic knots. Hyperbolic knots are characterized by the property that the knot complement admits a hyperbolic geometry. Hyperbolic knot complements have special properties, in particular topology and geometry are connected in a special way. Central property is the so-called Mostow-Prasad rigidity [14, 15]: every deformation of the space is an isometry, or geometric properties like volume or curvature are
topological invariants. In particular, the hyperbolic structure can be used to get new invariants (but only for hyperbolic knots). A well-known invariant is the A-polynomial [16, 17] which will be used in the next section.

4 Knotted superconductors and knot groups

In this section we will present first ideas to realize the knot complement by using a superconducting ring which is knotted.

The usage of superconductors for quantum computing is divided into three possible realizations: charge qubit, flux qubit and phase qubit but also many hybridizations exist like Fluxonium [18], Transmon [19], Xmon [20] and Quantromunium [21]. The qubit implementation as the logical quantum states \(|0\rangle, |1\rangle\) is realized by the mapping to the different states of the system. Therefore we have to deal with the states in the knotted superconductor. For a superconductor, there is a single wave function of the condensate. For the qubit realization, we have to consider a superposition of two wave functions in different energy states. The knotted superconductor has Josephson junctions at the over and under crossings. Let \(\psi_1\) and \(\psi_2\) be the wave functions at the over or under crossing. By using the Ginsburg-Landau theory we will get the current

\[ j = \frac{eHV}{m} |\psi_1|^2 \sin(\Phi_{12}) \]

where \(V\) is the potential difference and \(\Phi_{12}\) is the phase difference between the two wave functions. If the knotted superconductor consists of a single state, say \(|0\rangle\), then the Josephson junctions at the over or under crossings have no effect because there is no potential difference. In case of the two state \(|0\rangle\) and \(|1\rangle\), there is an energy difference and one gets a coupling between the two states \(|0\rangle, |1\rangle\) which is given by

\[ i\hbar \frac{d}{dt} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = \begin{pmatrix} E_1 & Ve^{i\phi} \\ Ve^{-i\phi} & E_2 \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = H \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} \]

where \(V\) depends on the energy difference \(E_1 - E_2\) and the coupling between the states. Then we get the Hamiltonian

\[ H_{\text{loop}} = E_1 \frac{1 + \sigma_z}{2} + E_2 \frac{1 - \sigma_z}{2} + V \cdot \cos \phi \cdot \sigma_y + V \cdot \sin \phi \cdot \sigma_x \]

acting on the single state in terms of Pauli matrices (as generators of the Lie algebra of \(SU(2)\)). The same Hamiltonian also works for Josephson junctions which are placed at the generators \(a, b, c\) of the knot group (see Fig. 2). In the previous section we described the representation of the generators abstractly. Here, the energy and coupling determines the representation. Now let us choose the Josephson junction for the generator \(a\). Furthermore we trim the energy levels via the junction so that \(|0\rangle\) has energy \(E\) and \(|1\rangle\) has energy \(-E\). For simplicity we neglect the phase shift \(\phi = 0\). Then we obtain the Hamiltonian

\[ H_1 = E \cdot \sigma_z + V \cdot \sigma_z \]
and for small couplings one gets the generator \( g \) of the Fibonacci representation (see the previous section). If the coupling is stronger, then we get in principle the other generator \( f \) of the Fibonacci representation (by a suitable choice of the energy).

In a previous paper \([2]\) we constructed the 1-qubit operations from the knot complement. Furthermore we argued that the linking of two knots is needed to generate the 2-qubit operations. The simplest link is the Hopf link (denoted as \( L_{2a1} \), see Figure 4), the linking of two unknotted curves. The knot group of the Hopf link \( L_{2a1} \), i.e. the fundamental group \( \pi_1(C(L_{2a1})) \) of the Hopf link complement \( C(L_{2a1}) \), can be calculated to be

\[
\pi_1(C(L_{2a1})) = \langle a, b | aba^{-1}b^{-1} = [a, b] = e \rangle = \mathbb{Z} \oplus \mathbb{Z}
\]

In \([13]\), the representation of knot and link groups is discussed. As shown in \([22]\), there is a relation of this representations to the so-called skein modules. Skein modules can be seen as the deformation quantization of these representations. Then at the view of skein modules, the representation of the Hopf link can be interpreted as to put a \( SU(2) \) representing on each component. That means that every component is related to a \( SU(2) \) representation, i.e., the knot group \( \pi_1(C(L_{2a1})) \) is represented by \( SU(2) \otimes SU(2) \). From the superconducting point of view, one puts one qubit \( \psi_1 \) on one ring and the second qubit \( \psi_2 \) on the other ring. Again, the coupling is realized by a Josephson junction between \( \psi_1, \psi_2 \) described by the equation

\[
\frac{i\hbar}{\hbar} \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} E & V \\ V & -E \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

where we tuned the energy levels so that \( \psi_1 \) has energy \( E \) and \( \psi_2 \) has energy \( -E \). Then the Hamiltonian is given by

\[
H_2 = E \cdot \sigma_z + V \cdot \sigma_x
\]

acting on the 2-qubit state. But both states \( \psi_1, \psi_2 \) are decomposed into \((|0\rangle, |1\rangle)\) vectors so that at the level of these vectors we have the Hamiltonian \( H_{1\otimes2} = \)
This Hamiltonian has the right structure, i.e. couplings like $\sigma_z \otimes \sigma_x$, to realize 2-qubit operations. For example, the CNOT gate given by the Hamiltonian 

$$(1 - \sigma_z) \otimes (1 - \sigma_x).$$

Finally via the scheme above, one can realize a universal quantum computer by linked superconductors coupled by Josephson junctions. As noted above, there are three types of qubit, the phase, charge and flux qubit. Now we start with the realization of the flux qubit and we made the following assumptions:

- every over-crossing or under-crossing is made into a Josephson junction, i.e. the two superconductor parts are coupled;
- the qubit is realized by a Flux qubit, abstractly described by the Hamiltonian

$$H = \frac{q^2}{2C_J} + \left( \frac{\Phi_0}{2\pi} \right)^2 \frac{\phi^2}{2L} - E_J \cos \left[ \phi - \Phi_\frac{2\pi}{\Phi_0} \right],$$

with $\Phi_0 = \hbar c/2e$, $\Phi$ the flux quant and flux, $L$ inductance of the ring, $C_J$ the junction capacity, $E_J$ junction energy and $\phi$ phase shift.

The states correspond to a symmetrical $|0\rangle$ and an anti-symmetrical superposition $|1\rangle$ of zero or single trapped flux quanta, sometimes denoted as clockwise and counterclockwise loop current states. The different energy levels are given by different (integer) numbers of magnetic flux quanta trapped in the (knotted) superconducting ring. Therefore to realize the flux qubit, we have to understand the relation between the shape of the knotted superconductor and the flux (or better the fluxion quantization). At first we state that in the case of the trefoil knot, the flux of the knotted superconductor can be controlled by the two loops $a, b$ (two generators of the knot group), the third loop $c$ is determined by the relation $c = a^{-1}ba$.

The flux properties of the knotted superconductor was described in [23]. Main result of this work is the description of these properties by considering the space around the knotted superconductor (by using the Meissner effect). Then (from the formal point of view) we have to consider a function over the knot complement $C(K) \to U(1)$ representing the vector potential which generates the flux. This map induces a map of the fundamental groups $\pi_1(C(K)) \to \pi_1(U(1)) = \mathbb{Z}$. Here the mapping into the integers is given by the integer part of the ratio $\Phi/\Phi_0$ (number of flux quanta). This map seem to imply that the flux or better the fluxion quantization does not depend on knot. In [23], the fluxion quantization was described for knotted superconductors, which are knotted like the trefoil knot. Usually for a ring $C$, one has the well-known relation (Stokes theorem)

$$N = \frac{1}{\Phi_0} \oint_{C=\partial S} A = \frac{1}{\Phi_0} \int_S F$$

between the electromagnetic potential (seen as 1-form) and the flux through a surface $S$ (a disk $D^2$). This theorem can be generalized to the knot $K$: for
every knot there is a surface $S_K$ with minimal genus $g$ so that $K = \partial S_K$. It is known that only the unknot has a Seifert surface of genus 0. The two knots $3_1$ (trefoil) and $4_1$ (figure-8) have Seifert surfaces of genus 1. In case of the trefoil knot, a simple picture was found to express the flux quantization, see [23]. There, the flux $\Phi_{SS}$ through the Seifert surface is decomposed into a linear combination of two (approximately) conserved fluxes $\Phi_R, \Phi_Q$ so that $\Phi_{SS} = 3\Phi_R + 2\Phi_Q$. The coefficients are given by another representation of the knot group $\pi_1(C(3_1)) = \langle \alpha, \beta | \alpha^3 = \beta^2 \rangle$ (with different generators, see Fig. 2 in [23]). The trefoil knot belongs to the class of torus knots, i.e. a closed curve winding around the torus.

In case of non-torus knots like the figure-8 knot, we cannot use these ideas. Instead we will follow another path. The knot complement has a boundary, which is a torus. For the torus, we know the fluxion quantization. Fora general knot, we have to know how the knot lies inside of the knot complement. In case of the flux qubit, we have to understand how the boundary torus (where we know the flux) lies inside of the knot group representations. Here, one has to use the so-called $A$-polynomial to get these information and to express the fluxion quantization. Let $C(4_1)$ be the knot complement of the figure-8 knot $4_1$. As we remarked above this knot complement carries a hyperbolic structure which will be used to define the $A$-polynomial. The hyperbolic structure is given by the choice of a homomorphism $\pi_1(C(4_1)) \to SL(2,\mathbb{C})$ (up to conjugation). Every knot complement $C(K)$ has the boundary $\partial C(K) = T^2$. The problem is now how the torus boundary lies inside of the space of all hyperbolic structures (character variety) $ChV(K) = Hom(\pi_1(K), SL(2,\mathbb{C}))/SL(2,\mathbb{C})$. Then the torus inside of $ChV(4_1)$ is defined by the zero set of a polynomial, the $A$-polynomial (for the details consult [16, 17]). For the figure-8 knot $4_1$, the $A$-polynomial is given by

$$A(M, L) = -2 + M^4 + M^{-4} - M^2 - M^{-2} - L - L^{-1}$$

and the decomposition of $A(\pm 1, L) = (L - 1)^2L^{-1}$ gives the first possible values $(2, -1)$ how the torus (via the slopes) lies in $ChV(4_1)$ (interpreted as eigenvalues of the torus slopes). Now we can use these eigenvalues to get the decomposition of the flux into the two (approximately) conserved fluxes $\Phi_R, \Phi_Q$ so that $\Phi_{SS} = 2\Phi_R - \Phi_Q$ for the first possible values. In contrast to the trefoil knot, there are more than one possible values for the combination $\Phi_{SS}$. In particular, this example showed that the fluxion quantization is more complex for hyperbolic knots (in contrast to torus knots). Then for the flux qubit one gets different combinations of the flux $\Phi_{SS}$ in dependence on the generator of the knot group.

The discussion above for the flux qubit showed that the relation between the flux, the operation and the qubit is complicated. Now we will discuss another possibility to realize the operations. Above we discussed the Hamiltonian operators $H_1$ and $H_{1\otimes 2}$ to describe the Josephson junctions. There, we found the interesting result that the Josephson junction at the over or under crossing has the same effect then a gate (as element of the knot group representation) acting on the qubit. The reason for this unexpected behavior is rooted in the
energy/potential difference between the two states $|0\rangle$ and $|1\rangle$, respectively. For a controlled behavior of the operations, one needs extra Josephson junctions. Every junction is located at the parts which are represented by the generators $a, b, c$ of the knot group (see Fig. 2). As discussed above, we can trim the energy levels so that we get the Hamiltonian

$$H_1 = E \cdot \sigma_z + Ve^{i\phi} \sigma_x$$

with a phase shift $\phi$ as induced by the coupling in the junctions. Every junction gives rise to an operation $\exp(iH_1t)$ of one qubit which is related to a representation of the knot group. The 2-qubit operations are induced by the linking where one has to add an Josephson junction near the linking (which is also a Josephson junction). This Josephson junction is described by the Hamiltonian $H_1$ and together with the linking we get the Hamiltonian $H_1 \otimes 2$ above. Then we obtain the 2-qubit operation $\exp(iH_1\otimes 2t)$.

We will close this section with a remark about the decoherence. The knotting of the superconductor (via the Josephson junctions at the over and under crossings) gives a self-coupling which has the ability to stabilize the state. We conjecture that a qubit (flux or phase qubit) on knotted superconductor has an increased decoherence time. We will discuss it in a forthcoming work.

## 5 Conclusion

In this paper we presented some ideas to use 3-manifolds for quantum computing. As explained above, the best representative is the fundamental group of a manifold. The fundamental group is the set of closed curves up to deformation with concatenation as group operation up to homotopy. It is known that every 3-manifold can be decomposed into simple pieces so that every piece carries a geometric structure (out of 8 classes). In principle, the pieces consist of complements of knots and links. Then the fundamental group of the knot complement, known as knot group, is an important invariant of the knot or link. Why not use this knot group for quantum computing? In [24, 25, 26, 27] M. Planat et.al. studied the representation of knot groups and the usage for quantum computing. Here we discussed a realization of knot complements by knotted superconductors where the crossings are Josephson junctions. The qubit is given by the flux qubit but we also discuss the phase qubit. As shown in [2], the knot group determines the operations and we got all 1-qubit operations for a knot. Then we discussed the construction of 2-qubit operations by linking the two knotted superconductors.

### Acknowledgments

I want to thank the anonymous referee for the helpful remarks which increases the readability of this paper.
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