Theory of optical forces on small particles by multiple plane waves

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We theoretically investigate the optical force exerted on an isotropic particle illuminated by a superposition of plane waves. We derive explicit analytical expressions for the exerted force up to quadrupolar polarizabilities. Based on these analytical expressions, we demonstrate that an illumination consisting of two tilted plane waves can provide a full control on the optical force. In particular, optical pulling, pushing and lateral forces can be obtained by the proper tuning of illumination parameters. Our findings might unlock multiple applications based on a deterministic control of the spatial motion of small particles.

I. INTRODUCTION

Light as an electromagnetic radiation carries energy and momentum. The direction of energy flux of an electromagnetic wave at any point in space-time is given by the Poynting vector $\mathbf{S}(\mathbf{r},t)$ and the linear momentum density (i.e. linear momentum per unit volume) is given by $\mathbf{S}(\mathbf{r},t)/c^2$, where $c$ is the speed of light$^1$. The exchange of linear momentum of light with an electromagnetically interacting particle can lead to an exerted optical force, known as the radiation pressure. In general, the acceleration caused by the radiation pressure on heavy macroscopic objects is considerably small. However, this acceleration can be considerably large for small particles (compared to wavelength) when illuminated by a light beam with a moderate intensity. Therefore, light beams can be used to move, trap, or guide a particle. This became experimentally feasible after the invention of lasers and the challenge was overcome by Arthur Ashkin that used a single weakly focused laser beam/two counter-propagating beams to move/trap microparticles$^2$. From then on, the technique has been widely used to manipulate atoms, molecules$^3$, and biological cells$^4$, and it has opened a brand new field of research termed as optical manipulation$^5$. Besides, one can have further control on the direction of the exerted optical force and achieve counter-intuitive forces like the optical pulling or lateral force$^6$. The former being also called an optical tractor beam. These forces have been obtained by engineering the excitation and particle’s symmetry and material. In particular, optical pulling force on an isotropic particle can be obtained through the interference of multiple plane waves, solenoidal beams$^7$, or Bessel beams$^8$. Furthermore, employing chiral particles, gain media, or plasmonic interfaces can also allow achieving exerted lateral and pulling forces on the particle$^9$. Multipole expansion is a key tool to study several optical phenomena namely, light perfect absorption$^{10,11}$, directional light emission$^{12,13}$, manipulating and controlling spontaneous emission$^{14,15}$, electromagnetically-induced-transparency$^{16}$, Fano resonances$^{17}$, electromagnetic cloaking$^{18,19}$, and also optical force$^{20,21}$ and torque$^{22,23}$ among many others. For small particles compared to the wavelength, induced electric and magnetic dipole and quadrupole moments are usually sufficient to fully understand the underlying physics. In this paper, we derive analytical expressions for the exerted optical force based on multipole expansion$^{24,25}$, up to quadrupolar polarizabilities. Detailed derivations are given in the supplementary material. We theoretically and numerically study optical pulling, pushing, and lateral forces exerted on an isotropic particle for single and two tilted plane waves. The conditions for optical pushing, pulling, and lateral forces are discussed. In particular, we explore the effects of the illumination parameters i.e. the wavelength, the angle between the two plane waves, and the position of the particle on the exerted optical force. The analytical expressions (Theory) are verified with the numerical solution of Maxwell’s equations using COMSOL (Simulation). The electric and magnetic fields obtained through the simulations are employed to calculate the Maxwell stress tensor and finally the optical force can be calculated accordingly, see Eq. (1). In the following, we focus on the underlying theory and explain our results and its physical implications.

II. THEORY

The time averaged mechanical force exerted on an arbitrary particle by an optical wave can be calculated as$^{25,26}$

$$\langle \mathbf{F}\rangle = \left\langle \int_{\mathbf{x}} \mathbf{T}(\mathbf{r},t) \cdot \mathbf{n} dS \right\rangle,$$  

(1)

with $S$ being any closed surface surrounding the particle,
are the total (incoming plus scattered) electric and magnetic multipole moments. For a small polarizable particle (compared to wavelength), depending on the geometry, material, and the illumination, the decomposition of induced current leads to the definition of the electric and magnetic dipole and quadrupole polarizabilities (neglecting the higher order polarizabilities). The induced electric and magnetic dipoles in Cartesian coordinates for an isotropic particle are expressed in terms of electric ($\alpha^e$) and magnetic ($\alpha^m$) polarizabilities as $p = \varepsilon_0 \alpha^e E$ and $m = \alpha^m H$, respectively. The electric and magnetic quadrupole moments for an isotropic particle is defined as\cite{9, 10, 11}:

$$Q^e = \varepsilon_0 \alpha^{Qe} \nabla E + E \nabla,$$

$$Q^m = \alpha^{Qm} \nabla H + H \nabla,$$

where $\alpha^{Qe}$ and $\alpha^{Qm}$ are the Cartesian quadrupolar polarizabilities. Using the relation $(\nabla A + A \nabla)_{ij} = \partial_i A_j + \partial_j A_i$, any component of $Q^e$ and $Q^m$ is calculated as $Q^{m}_{ij} = \frac{\varepsilon_0 \alpha^{Qm}}{2} (\partial_i E_j + \partial_j E_i)$ and $Q^{m}_{ij} = \alpha^{Qm} (\partial_i H_j + \partial_j H_i)$, respectively.

The optical force exerted on a particle by an arbitrary illumination can be written as the following (truncated at quadrupole order)\cite{12, 13}:

$$F = [F_p + F_{Q^e} + ...] + [F_m + F_{Q^m} + ...]$$

$$+ [F_{pm} + F_{Q^e Q^m} + ...]$$

$$+ [F_{pQ^e} + ...] + [F_{mQ^m} + ...],$$

where $F_p$ ($F_m$) is the individual contribution of an electric (magnetic) dipole moment to the total optical force. $F_{Q^e}$ ($F_{Q^m}$) is the contribution of an individual electric (magnetic) quadrupole moment; and the other expressions are the contribution of the interference between two multipole moments.

Neglecting higher order terms, Eq. (3) can be rewritten in terms of Cartesian dipole and quadrupole moments as follows\cite{14, 15, 16}:

$$F_i = \frac{1}{2} \Re \left[ \sum_j p_j \nabla_i E^*_j \right] + \frac{1}{2} \Re \left[ \sum_j m_j \nabla_i B^*_j \right] - \frac{k^4}{12\pi \varepsilon_0 c} \Re \left[ \sum_{l,j,k} \varepsilon_{ijk} p_j m_k^* \right]$$

$$- \frac{k^5}{120\pi \varepsilon_0} \Im \left[ \sum_j (Q^e)_{ij} p_j^* \right] + \frac{1}{12} \Re \left[ (Q^e)_{jk} \nabla_i \nabla_j E^*_i \right] + \frac{k^5}{120\pi \varepsilon_0 c^2} \Im \left[ (Q^m)_{ij} m_j^* \right]$$

$$+ \frac{1}{12} \Re \left[ \sum_k (Q^m)_{jk} \nabla_i \nabla_k B^*_j \right] - \frac{k^6}{9 \times 240\pi \varepsilon_0 c} \Re \left[ \sum_{l,j,k} \varepsilon_{ijk} (Q^e)_{lj} (Q^m)_{ik}^* \right],$$

in which $E_i$ and $B_i$ are components of the incident electric and magnetic fields, respectively, and $\varepsilon_{ijk}$ is the Levi-Civita symbol.

As shown in Fig. 1, consider an illumination composed of two tilted plane waves with wave vectors $k_1 = k[\sin \psi, 0, \cos \psi]$ and $k_2 = k[-\sin \psi, 0, \cos \psi]$, illuminating an isotropic particle, where $\psi$ is the tilting angle. We define
the TE and TM illuminations as the following:

\[
\mathbf{E}^{\text{TE}} = \frac{E_0}{2} (e^{i \mathbf{k}_1 \cdot \mathbf{r}} + e^{i \mathbf{k}_2 \cdot \mathbf{r}}) \mathbf{e}_y, \\
\mathbf{E}^{\text{TM}} = \frac{E_0}{2} \left( \begin{bmatrix} \cos \psi \\ 0 \\ -\sin \psi \end{bmatrix} e^{i \mathbf{k}_1 \cdot \mathbf{r}} + \begin{bmatrix} \cos \psi \\ 0 \\ \sin \psi \end{bmatrix} e^{i \mathbf{k}_2 \cdot \mathbf{r}} \right). \tag{6}
\]

The time averaged optical forces exerted on an isotropic particle located at the position \( \mathbf{r}_0 = (x_0, y_0, z_0) \) by the TE and TM illuminations, i.e. Eqs. (5) and (6) read as (see supplementary material):

\[
\mathbf{F}^{\text{TE}} \approx \mathbf{F}^{\text{TE}}_p + \mathbf{F}^{\text{TE}}_m + \mathbf{F}^{\text{TE}}_{pm} + \mathbf{F}^{\text{TE}}_Q + \mathbf{F}^{\text{TE}}_{mQ} + \mathbf{F}^{\text{TE}}_{pQ} + \mathbf{F}^{\text{TE}}_{QQ}, \tag{7}
\]

\[
\mathbf{F}^{\text{TM}} \approx \mathbf{F}^{\text{TM}}_p + \mathbf{F}^{\text{TM}}_m + \mathbf{F}^{\text{TM}}_{pm} + \mathbf{F}^{\text{TM}}_Q + \mathbf{F}^{\text{TM}}_{mQ} + \mathbf{F}^{\text{TM}}_{pQ} + \mathbf{F}^{\text{TM}}_{QQ}, \tag{8}
\]

where we define \( \alpha_{e,m} = \alpha_{e,m} / \alpha_d, \alpha_{Q_{e,m}} = \alpha_{Q_{e,m}} / \alpha_Q \), and \( \delta = k \sin \psi x_0 \). \( \alpha_d = \frac{120 \pi}{k^3} \) and \( \alpha_d = 6 \pi / k^3 \) are the polarizability normalizations for dipoles and quadrupoles, respectively. Please note that a lateral change in \( x_0 \), being the spatial position of the particle, has the equivalent effect on the force as a phase shift \( \Delta \phi = -2 k x_0 \sin \psi \) in the two plane waves illuminating the particle. This makes sense as the spatial interference pattern that the two plane waves form only depends on the phase difference. The optical force exerted on the particle only depends on its position along the \( x \)-axis while it is independent on the position along the \( z \)- and \( y \)-axis (i.e. it depends on \( \delta = k \sin \psi x_0 \)). Throughout the paper, optical forces are normalized to \( F^{\text{norm}} = (I_0 / c) [\chi^2 / (2 \pi)] = I_0 k \alpha_d / (3c) \).
The normalization factor of $F_{\text{norm}}$ is of physical significance and $3F_{\text{norm}}$ corresponds to the upper bound for the exerted optical force on an isotropic electric/magnetic dipolar particle illuminated by a plane wave.

### III. THEORETICAL AND NUMERICAL RESULTS

In the following, we consider a dielectric sphere made from a material with a permittivity of $\varepsilon = 3.5^2$, and radius $a$. Figure 2 shows the calculated polarizabilities. They are calculated by using electric and magnetic Mie coefficients (i.e. $a_1, a_2$ and $b_1, b_2$):

$$
\alpha_e = \frac{6\pi}{k^3} a_1 = i\alpha_d a_1, \quad \alpha_m = \frac{i}{k^3} b_1 = i\alpha_q a_1, \quad \alpha_{Qe} = \frac{i}{k^3} a_2 = i\alpha_d a_2, \quad \alpha_{Qm} = \frac{i}{k^3} b_2 = i\alpha_q b_2.
$$

Alternatively, they can be extracted from exact multipole moments based on induced current. We restrict ourselves to the wavelength region where the lowest order polarizabilities have their lowest order resonances. Having the polarizabilities, through Eqs. (7) and (8), the exerted optical force due to the contribution of different multipole moments can be derived. Below, we consider several scenarios for the illumination.

#### A. Single plane wave illumination

Assuming a plane wave excitation, i.e. $\psi = 0$, using Eqs. (7) or (8) the normalized optical force exerted on an isotropic particle is calculated as:

$$
F \approx 3\text{Im}\left(\bar{\alpha}_e + \bar{\alpha}_m\right) + 5\text{Im}\left(\bar{\alpha}_{Qe} + \bar{\alpha}_{Qm}\right)
- 3\text{Re}\left(\bar{\alpha}_e\bar{\alpha}_{Qe}^* + \bar{\alpha}_m\bar{\alpha}_{Qm}^* + \bar{\alpha}_{Qe}\bar{\alpha}_{Qm}^* + \frac{5}{9}\bar{\alpha}_{Qe}\bar{\alpha}_{Qm}^*\right) e_z.
$$

The theoretical results based on the derived equation are shown in Fig. 3. As a verification, the results of the COMSOL simulation are also shown in Fig. 3(c).

As can be seen in Fig. 3(a) and (b), interference terms can have negative (pulling) contributions to the total optical force, while the individual contributions of the moments are positive across the entire spectrum (the imaginary part of the polarizabilities is always positive for passive particles). With a single plane wave illumination, the total optical force is always positive (pushing), Fig. 3(c).

Further, it can be observed that the quadrupolar terms have the dominant contributions to the highest values of the optical force at lower wavelengths. The optical force, as can be intuitively expected, is in the direction of the overall linear momentum, i.e. here, in the $z$-direction.

#### B. Two plane wave illumination: sphere at $r = 0$

Assuming the sphere to be located in the center of the coordinate system, i.e. $r = (0, 0, 0)$, and being illuminated with the wave expressed by Eqs. (5)-(6), the calculated optical force as a function of the tilting angle $\psi$ and the particle’s size parameter, i.e. $a/\lambda$, is shown in Fig. 4(a) and (b) for TE- and TM-polarization, respectively. An optical pulling force is achieved for certain tilting angles for both TE and TM illuminations. This can be expected already by inspecting Eqs. (7)-(8), since the contribution of negative terms can dominate for some large angles as
FIG. 4. Two plane wave illumination and the sphere located at r = 0: (a)-(b) Exerted optical forces by two tilted plane waves as a function of the tilting angle and the particle’s size parameter, i.e. a/λ for TE and TM illuminations, respectively.

FIG. 5. Two plane wave illumination and the sphere located at r = 0: (a)-(b) Exerted optical forces on the dielectric sphere, with permittivity ε = 3.5², by two tilted plane waves at a/λ = (1/7.52) and (1/5.26) for TE and TM illuminations, respectively. (c)-(d) Contribution of different orders of dipoles and their interference to the optical force.

The positive terms are attenuating faster as the angle increases.

To make a better analysis of the negative force, we choose a smaller sized sphere, where dipole moments are dominant, and neglect the quadrupolar moments. Therefore, the optical force simplifies to:

\[ \vec{F} \approx \vec{F}_p + \vec{F}_m + \vec{F}_{pm}, \]

\[ \vec{F}_{TE} \approx 3 \cos \psi \left[ \text{Im}(\bar{\alpha}_r) + \text{Im}(\bar{\alpha}_m) \cos^2 \psi - \text{Re}(\bar{\alpha}_r \bar{\alpha}_m^*) \right] \hat{e}_z, \]

\[ \vec{F}_{TM} \approx 3 \cos \psi \left[ \text{Im}(\bar{\alpha}_r) \cos^2 \psi + \text{Im}(\bar{\alpha}_m) - \text{Re}(\bar{\alpha}_r \bar{\alpha}_m^*) \right] \hat{e}_z. \]

The calculated optical force is shown in Fig. 5 for certain values of a/λ = (1/7.52), (1/5.26) for TE and TM illuminations, respectively.

Based on this figure, for small values of the tilting angle ψ, a pushing force is exerted on the particle. As the tilting angle increases, for the TE(TM) polarization the magnitude of the force reduces until it vanishes at ψ = 62° (76°) and beyond that the pulling force appears, showing a minimum at ψ = 75° (83°). For both TE and TM illuminations, the terms \( F_p \) and \( F_m \) are positive, however, the term \( F_{pm} \) in both cases is negative, canceling out the contributions of the positive terms at certain angles. Then, it becomes possible to reduce the positive contributions to the optical force and to achieve an overall negative force. In other words, according to Eq. (11) for TE (TM) polarization, the term \( F_{pm} \) \( \vec{F}_p \) vanishes due to the term \( \cos^3 \psi \) for large ψ [see Fig. 5 (a)-(d)]. Meanwhile, the overall force is decreased due to the term \( \cos \psi \), a factor which appears as a total pre-factor.

C. Two plane wave illumination: sphere at r = r₀

In order to investigate the effects of the particle position r₀ on the force exerted on it, which shows itself in the equations through the angle \( \delta = k \sin \psi r_0 \), the exerted optical forces as a function of the position and the particle’s size parameter, i.e. a/λ is shown in Fig. 6 (a)-(d). The tilting angles for the TE and TM polarizations are 76° and 73°, respectively. It can be seen that the particle experiences a periodic optical force. The existence of the
Two plane wave illumination using Eq. (12) and a sphere located at \( r = r_0 \): (a) The schematic of an isotropic particle illuminated by two tilted linearly polarized plane waves. (b)-(c) Optical forces versus \( \xi \) and \( x_0/a \) exerted on the dielectric sphere with permittivity \( \varepsilon = 3.5^2 \) and illuminated by two tilted linearly polarized plane waves at wavelength \( \lambda = 6a \) with angles \( (\psi = 75^\circ, x_0 = a/2) \). (c) for \( \psi = 75^\circ, \xi = 45^\circ \). The solid line is the simulation and the circles are the analytical results calculated by using Eq. (13).

A lateral force is due to the gradient of the field intensity along the x-axis. Moreover, according to these figures it can be realized that the quadrupolar terms (around \( a/\lambda = 0.2 \) and 0.25, see also Figure 2) cause major variations of the optical force in amplitude and sign for both TE and TM illuminations. Figure 6 (e)-(f) depict the theoretical and simulated exerted forces (i.e. both \( F_x \) and \( F_y \)) calculated for \( a/\lambda = 0.133 \) and 0.2 with the periodicity of \( \Lambda = \lambda/\sin \psi \) with respect to the x-axis (see the definition of \( \delta \)). Theoretical results using Eq. (13) are in perfect agreement with the simulation results.

In order to explore other possible influences of two-plane wave illumination in the \( x-z \)-plane on the exerted optical force, the following generalized excitation is defined:

\[
E = \frac{E_0}{2} \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{i k_z \cdot r} + \begin{bmatrix} \cos \sin \xi \\ \cos \xi \sin \xi \\ \sin \xi \cos \xi \end{bmatrix} e^{i k_2 \cdot r} \right), \quad (12)
\]

where \( \xi \) is the deviation angle between electric field polarization of the two plane waves. The time averaged optical force exerted on an isotropic dipolar particle by the above illumination can be derived as:

\[
\begin{align*}
F_x &= -\frac{3}{2} \left[ \Re(\tilde{\alpha}_e) + \Re(\tilde{\alpha}_m) \cos 2\psi - \Re(\tilde{\alpha}_e \tilde{\alpha}_m^*) \right] \\
& \quad \times \sin \psi \cos \xi \sin 2\delta, \\
F_y &= \frac{3}{2} \left[ \Re\left(e^{-2i\delta} \tilde{\alpha}_e \tilde{\alpha}_m^*\right) \right] \sin \psi \sin \xi \cos \psi, \\
F_z &= \frac{3}{2} \cos \psi \left\{ \left( 1 + \cos 2\psi \cos \xi \cos 2\delta \right) \Im(\tilde{\alpha}_m) \right. \\
& \quad + \left[ \Im(\tilde{\alpha}_e) - \Re(\tilde{\alpha}_e \tilde{\alpha}_m^*) \right] (1 + \cos \xi \cos 2\delta) \right\}. \quad (13)
\]

Intuitively, one might expect to have an exerted optical force only in the direction of the overall linear momentum, i.e. \( k_1 + k_2 = 2k \cos \psi \). However, according to Eq. (13), lateral forces (in both directions of \( x \) and \( y \)) can be experienced by a fully symmetric (isotropic) particle for certain angles of \( \psi \) and \( \xi \). These peculiar lateral forces can be elaborated on the basis of the symmetry breaking mediated by the illuminating wave rather than the particle. The variation of lateral optical forces exerted on the dielectric sphere for parameters \( \xi, \psi = 75^\circ, x_0 = a/2 \) and \( \psi = 75^\circ, \xi = 45^\circ \) are illustrated in Fig. 7 (a)-(b), respectively.

\section{IV. CONCLUSION}

In conclusion, we investigated the optical force exerted on an isotropic particle by two plane waves and demonstrated theoretically that pushing-pulling forces for either TE or TM illuminations is possible. Our method, based on the theoretical calculations of multipolar forces, revealed the contribution of each electric and magnetic moment up to quadrupole terms (including their interferences) to the optical force. Additionally, this approach elaborates the optical force in a closed form due to the electrodynamical formalism of all influential parameters i.e. the designated angles, polarizabilities and amplitudes for either TE or TM illuminations. According to this formalism, we also showed the existence of lateral forces, in both \( x \) and \( y \) directions, for certain angles of \( \psi, \xi \) and an interval of deviation of the object from the center \( x_0 \). Our approach and findings can be employed in the optical manipulations and sorting of micro/nano particles with different illuminations.

\section{V. ACKNOWLEDGMENTS}

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Appendix A: Optical force: TE illumination

1. Useful expressions

a. Optical force

The total optical force exerted on a small particle up to quadrupole moments (neglecting higher order terms) reads as:

\[
\langle F_i \rangle = \frac{1}{2} \text{Re} \left[ \sum_j p_j \nabla_i E_j^0 \right] + \frac{1}{2} \text{Re} \left[ \sum j m_j \nabla_i B_j^0 \right] - \frac{k^4}{12 \pi \varepsilon_0 c} \text{Re} \left[ \sum_{j,k} \epsilon_{ijk} p_j m_k \right]
\]

\[
- \frac{k^5}{120 \pi \varepsilon_0} \text{Im} \left[ \sum_j (Q^e)_{ij} p_j^* \right] + \frac{1}{12} \text{Re} \left[ (Q^e)_{jk} \nabla_i \nabla_k E_j^0 \right] + \frac{k^5}{120 \pi \varepsilon_0 c^2} \text{Im} \left[ (Q^m)_{ij} m_j^* \right]
\]

\[
+ \frac{1}{12} \text{Re} \left[ \sum_k (Q^m)_{jk} \nabla_i \nabla_k B_j^0 \right] - \frac{k^6}{9 \times 240 \pi \varepsilon_0 c} \text{Re} \left[ \sum_{l,j,k} \epsilon_{ijk} (Q^e)_{lj} (Q^m)_{lk}^* \right].
\]

(A1)

In the following sections, we will use Eq. (A1) to derive analytical expressions for the exerted optical force on a small particle up to quadrupolar moments.

b. Electric and magnetic fields and and their derivatives

Here, we derive the whole required electromagnetic fields and their derivatives which are necessary to derive the exerted optical forces on small particles by TE and TM illuminations. The TE illumination is defined as \( \mathbf{E} = E_0 (e^{ik_1 \cdot r} + e^{ik_2 \cdot r}) \mathbf{e}_y / 2 \), where \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) are wave vectors of the two plane waves and are defined as:

\[
\mathbf{k}_1 = k[\sin \psi, 0, \cos \psi]^T, \quad \mathbf{k}_2 = k[-\sin \psi, 0, \cos \psi]^T.
\]

(A2)

The field at any arbitrary point \( \mathbf{r}_0(x_0, y_0, z_0) \) can also be written as:

\[
\mathbf{E}_{|\mathbf{r}=\mathbf{r}_0} = E_0 \cos \delta e^{ik \cos \psi z_0} \mathbf{e}_y,
\]

(A3)

where \( \delta = k \sin \psi x_0 \). The corresponding magnetic field is calculated as:

\[
\mathbf{B} = B_x \mathbf{e}_x + B_z \mathbf{e}_z = \frac{1}{\omega} \left[ (\mathbf{k}_1 \times \mathbf{E}_1) + (\mathbf{k}_2 \times \mathbf{E}_2) \right],
\]

\[
= \frac{k}{\omega} \cos \psi \mathbf{e}_x \times (\mathbf{E}_1 + \mathbf{E}_2) + \frac{k}{\omega} \sin \psi \mathbf{e}_x \times (\mathbf{E}_1 - \mathbf{E}_2),
\]

(A4)

where \( \mathbf{E}_{i=1,2} = \frac{E_0}{2} e^{ik_i \cdot r} \mathbf{e}_y \). Therefore, the magnetic field at any arbitrary point \( \mathbf{r}_0(x_0, y_0, z_0) \) can be written as the following:

\[
B_x_{|\mathbf{r}=\mathbf{r}_0} = i E_0 \frac{k}{\omega} \sin \psi \sin \delta e^{ik \cos \psi z_0},
\]

\[
B_z_{|\mathbf{r}=\mathbf{r}_0} = -E_0 \frac{k}{\omega} \cos \psi \cos \delta e^{ik \cos \psi z_0}.
\]

(A5)

Some useful expressions for the derivatives of the electric and magnetic fields can be calculated as follows:

\[
\frac{\partial E_y}{\partial x}_{|\mathbf{r}=\mathbf{r}_0} = -E_0 k \sin \psi \sin \delta e^{ik \cos \psi z_0},
\]

\[
\frac{\partial E_y}{\partial x}_{|\mathbf{r}=\mathbf{r}_0} = 0,
\]

\[
\frac{\partial E_y}{\partial x}_{|\mathbf{r}=\mathbf{r}_0} = i E_0 k \cos \psi \cos \delta e^{ik \cos \psi z_0},
\]

(A6)
where the elements of the matrix are defined as components of the electric quadrupole moments: \( \mathbf{e} \) is a second rank tensor and a traceless matrix, i.e.

\[
\begin{align*}
\frac{\partial H_z}{\partial x} & |_{r=r_0} = iE_0 \frac{k^2}{\omega \mu} \sin^2 \psi \cos \delta e^{ik\cos \psi z_0}, \\
\frac{\partial H_z}{\partial z} & |_{r=r_0} = -iE_0 \frac{k^2}{\omega \mu} \cos^2 \psi \cos \delta e^{ik\cos \psi z_0}, \\
\frac{\partial H_x}{\partial x} & |_{r=r_0} = E_0 \frac{k^2}{\omega \mu} \sin \psi \cos \sin \delta e^{ik\cos \psi z_0}, \\
\frac{\partial H_x}{\partial z} & |_{r=r_0} = -E_0 \frac{k^2}{\omega \mu} \sin \psi \cos \sin \delta e^{ik\cos \psi z_0}, \\
\end{align*}
\] (A7)

\[
\begin{align*}
\nabla_x \nabla_x E_x^* & |_{r=r_0} = iE_0^* \frac{k^3}{\omega} \cos^3 \psi \cos \delta e^{-ik\cos \psi z_0}, \\
\nabla_x \nabla_x E_y^* & |_{r=r_0} = -E_0^* \frac{k^3}{\omega} \sin^2 \psi \cos \delta e^{-ik\cos \psi z_0}, \\
\nabla_x \nabla_x E_z^* & |_{r=r_0} = -E_0^* \frac{k^3}{\omega} \sin \psi \cos \sin \delta e^{-ik\cos \psi z_0}, \\
\nabla_x \nabla_x B_x^* & |_{r=r_0} = E_0 \frac{k^3}{\omega} \sin^2 \psi \cos \delta e^{-ik\cos \psi z_0}, \\
\nabla_x \nabla_x B_y^* & |_{r=r_0} = iE_0^* \frac{k^3}{\omega} \sin \psi \cos \sin \delta e^{-ik\cos \psi z_0}, \\
\nabla_x \nabla_x B_z^* & |_{r=r_0} = iE_0^* \frac{k^3}{\omega} \sin \psi \cos \sin \delta e^{-ik\cos \psi z_0}. \\
\end{align*}
\] (A8)

\[
\begin{align*}
\nabla_z \nabla_z B_z^* & |_{r=r_0} = E_0 \frac{k^3}{\omega} \cos^3 \psi \cos \delta e^{-ik\cos \psi z_0}, \\
\nabla_z \nabla_z B_x^* & |_{r=r_0} = -E_0^* \frac{k^3}{\omega} \sin^2 \psi \cos \delta e^{-ik\cos \psi z_0}, \\
\nabla_z \nabla_z B_y^* & |_{r=r_0} = -iE_0^* \frac{k^3}{\omega} \sin \psi \cos \sin \delta e^{-ik\cos \psi z_0}, \\
\nabla_z \nabla_z B_z^* & |_{r=r_0} = E_0 \frac{k^3}{\omega} \sin^2 \psi \cos \delta e^{-ik\cos \psi z_0}, \\
\nabla_z \nabla_z B_z^* & |_{r=r_0} = iE_0^* \frac{k^3}{\omega} \sin \psi \cos \sin \delta e^{-ik\cos \psi z_0}. \\
\end{align*}
\] (A9)

c. Induced multipole moments

Electric and magnetic dipole moments for an isotropic object are defined as \( \mathbf{p} = \varepsilon_0 \alpha_e \mathbf{E} \) and \( \mathbf{m} = \alpha_m \mathbf{H} \), where \( \mathbf{p} \) and \( \mathbf{m} \) are the electric and magnetic dipole moments, respectively. \( \alpha_e \) and \( \alpha_m \) denote the scalar electric and magnetic polarizabilities, respectively. Using the electric and magnetic fields in Eqs. (A3-A5), the induced moments for an isotropic particle reads as:

\[
\mathbf{p} = \varepsilon_0 \alpha_e \mathbf{E} = \varepsilon_0 \alpha_e E_0 \cos \delta e^{ik\cos \psi z_0} \mathbf{e}_y,
\] (A10)

\[
\mathbf{m} = \alpha_m \mathbf{H} = \alpha_m E_0 \frac{k}{\mu \omega} \left( \sin \psi \sin \delta e^{ik\cos \psi z_0} \mathbf{e}_x - \cos \psi \cos \delta e^{ik\cos \psi z_0} \mathbf{e}_z \right).
\] (A11)

The electric quadrupole moment (\( \mathbf{Q}^e \)) induced in an isotropic particle is defined as:

\[
\mathbf{Q}^e = \begin{bmatrix}
Q_{xx}^e & Q_{xy}^e & Q_{xz}^e \\
Q_{yx}^e & Q_{yy}^e & Q_{yz}^e \\
Q_{zx}^e & Q_{zy}^e & Q_{zz}^e
\end{bmatrix},
\] (A12)

where the elements of the matrix are defined as \( Q_{ij}^e = \varepsilon_0 \alpha_e \left( \partial_i E_j + \partial_j E_i \right) / 2 \) , \( i, j = 1, 2, 3 \) for \( x, y \) and \( z \). \( \mathbf{Q}^e \) is a second rank tensor and a traceless matrix, i.e. \( Q_{xx}^e + Q_{yy}^e + Q_{zz}^e = 0 \). Using Eq. (A10), we can calculate all the components of the electric quadrupole moments:
\[ Q_{xx}^e = \varepsilon_0 \alpha Q^e \frac{\partial_x E_x + \partial_y E_y}{2} = 0, \]
\[ Q_{yy}^e = \varepsilon_0 \alpha Q^e \frac{\partial_y E_y + \partial_y E_y}{2} = 0, \]
\[ Q_{zz}^e = \varepsilon_0 \alpha Q^e \frac{\partial_z E_x + \partial_z E_z}{2} = 0, \]
\[ Q_{xx}^e = \varepsilon_0 \alpha Q^e \frac{\partial_x E_x + \partial_x E_z}{2} = 0, \]
\[ Q_{yy}^e = \varepsilon_0 \alpha Q^e \frac{\partial_y E_x + \partial_y E_y}{2} = \varepsilon_0 \alpha Q^e \frac{\partial_z E_y}{2}, \]
\[ = \frac{i k E_0}{2} \varepsilon_0 \alpha Q^e \cos \psi / \cos \delta \varepsilon e^{ik \cos \psi z_0}, \]
\[ Q_{yy}^e = \varepsilon_0 \alpha Q^e \frac{\partial_y E_x + \partial_y E_y}{2} = \varepsilon_0 \alpha Q^e \frac{\partial_x E_y}{2}, \]
\[ = -\frac{k E_0}{2} \varepsilon_0 \alpha Q^e \sin \psi / \sin \delta \varepsilon e^{ik \cos \psi z_0}. \] (A13)

Similarly, the magnetic quadrupole moment \( (Q^m) \) for an isotropic particle is defined as:

\[ Q^m = \begin{bmatrix} Q_{xx}^m & Q_{xy}^m & Q_{xz}^m \\ Q_{yx}^m & Q_{yy}^m & Q_{yz}^m \\ Q_{zx}^m & Q_{zy}^m & Q_{zz}^m \end{bmatrix}, \] (A14)

where the elements of the matrix are defined as \( Q_{ij}^m = \alpha Q^m \frac{\partial_i H_j + \partial_j H_i}{2} \), and \( i, j = 1, 2, 3 \) for \( x, y \) and \( z \). \( Q^m \) is a second rank tensor and a traceless matrix, i.e. \( Q_{xx}^m + Q_{yy}^m + Q_{zz}^m = 0 \). Using the magnetic field in Eq. A5 and its derivative Eq. A7, we can calculate all components of the magnetic quadrupole moments:

\[ Q_{yy}^m = \alpha Q^m \frac{\partial_y H_y + \partial_y H_y}{2} = 0, \]
\[ Q_{yx}^m = \alpha Q^m \frac{\partial_x H_x + \partial_x H_y}{2} = 0, \]
\[ Q_{xx}^m = \alpha Q^m \frac{\partial_x H_x + \partial_x H_x}{2} = \alpha Q^m \partial_x H_x, \]
\[ = E_0 \frac{k^2}{\omega \mu} \alpha Q^m \sin \psi / \sin \delta \varepsilon e^{ik \cos \psi z_0}, \]
\[ Q_{zz}^m = -Q_{xx}^m, \]
\[ Q_{yz}^m = \alpha Q^m \frac{\partial_y H_z + \partial_z H_y}{2} = 0, \]
\[ Q_{zx}^m = \alpha Q^m \frac{\partial_z H_x + \partial_x H_z}{2}, \]
\[ = \frac{i}{2} \frac{E_0}{\omega \mu} \alpha Q^m \left( \sin^2 \psi - \cos^2 \psi \right) \cos \delta \varepsilon e^{ik \cos \psi z}. \] (A15)

In the following section, we calculate the components of the optical force by using the induced multipole moments, the electric and magnetic fields and their derivatives.

2. \( F_p \) contribution

According to Eq. A11, the electric dipole contribution reads as:

\[ F_{i(p)} = \frac{1}{2} \text{Re} \left( \sum_j p_j \nabla_i E_j^* \right), \] (A16)

\( i, j = 1, 2, 3 \) for \( x, y \) and \( z \). Using the electric field in Eq. A3 and the definition of the electric dipole moment \( p = \varepsilon_0 \alpha e \), it can be easily seen that the \( y \) component of \( F_p \) is zero.
Using Eq. [A16] the x component of $F_{p}$ reads as:

$$F_{x(p)} = \frac{1}{2} \text{Re} \left( p_x \frac{\partial}{\partial x} E^*_x + p_y \frac{\partial}{\partial y} E^*_y + p_z \frac{\partial}{\partial z} E^*_z \right) = \frac{1}{2} \text{Re} \left( p_y \frac{\partial}{\partial x} E^*_y \right). \tag{A17}$$

Now, by substituting Eq. [A10] and Eq. [A6] into Eq. [A17] we obtain:

$$F_{x(p)} = \frac{1}{2} \text{Re} \left[ \varepsilon_0 |E_0|^2 \alpha e \cos \sin \delta (-k \sin \psi) \right],$$

$$= -3 \left( \frac{k^3}{6\pi} \right) F_{\text{norm}} \sin \psi \cos \sin \delta \Re (\alpha_e),$$

$$= -3 \left( \frac{k^3}{12\pi} \right) F_{\text{norm}} \sin \psi \sin 2\Re (\alpha_e). \tag{A18}$$

Finally, by using the definition of the normalized force, i.e. $\bar{F} = F/F_{\text{norm}}$ and the normalized electric polarizability, i.e. $\bar{\alpha}_e = \alpha_e/\alpha_d$, we obtain

$$\bar{F}_{x(p)} = -\frac{3}{2} \sin \psi \sin 2\Re (\bar{\alpha}_e), \tag{A18}$$

where $\alpha_d = 6\pi/k^3$, and $F_{\text{norm}} = \frac{6\pi k^3}{2\pi}$. This expression is documented in Eq. 7 of the main manuscript.

Similarly, the z component of $F_{p}$ read as:

$$F_{z(p)} = \frac{1}{2} \text{Re} \left( p_x \frac{\partial}{\partial z} E^*_x + p_y \frac{\partial}{\partial z} E^*_y + p_z \frac{\partial}{\partial z} E^*_z \right) = \frac{1}{2} \text{Re} \left( p_y \frac{\partial}{\partial z} E^*_y \right). \tag{A19}$$

Now, by substituting Eq. [A10] and Eq. [A6] into Eq. [A19] we obtain:

$$F_{z(p)} = \frac{1}{2} \text{Re} \left[ \varepsilon_0 |E_0|^2 \alpha e \cos^2 (\delta) (-ik \cos \psi) \right],$$

$$= 3 \left( \frac{k^3}{6\pi} \right) F_{\text{norm}} \cos \psi \cos^2 \delta \Im (\alpha_e).$$

Then, the normalized contribution is derived as:

$$\bar{F}_{z(p)} = 3\cos \psi \cos^2 \delta \Im (\bar{\alpha}_e). \tag{A20}$$

This expression is documented in Eq. 7 of the main manuscript.

### 3. $F_{m}$ contribution

According to Eq. [A11] the magnetic dipole contribution reads as:

$$F_{i(m)} = \frac{1}{2} \text{Re} \left( \sum_j m_j \nabla_i B^*_j \right), \tag{A21}$$

$i, j = 1, 2, 3$ for $x, y$ and $z$. Using the magnetic field, i.e. Eq. [A5] and the definition of the magnetic dipole moment $m = \alpha_m H$, it can be easily seen that the y component of the $F_m$ is zero.

Using Eq. [A21] the x component of $F_m$ read as

$$F_{x(m)} = \frac{1}{2} \text{Re} \left( m_x \frac{\partial}{\partial x} B^*_x + m_y \frac{\partial}{\partial y} B^*_y + m_z \frac{\partial}{\partial z} B^*_z \right),$$

$$= \frac{1}{2} \text{Re} \left( m_z \frac{\partial}{\partial x} B^*_z + m_z \frac{\partial}{\partial x} B^*_z \right). \tag{A22}$$

Now, by substituting Eq. [A11] and Eq. [A7] into Eq. [A22] we obtain:
\[
F_{x(m)} = \frac{1}{2} \text{Re} \left[ \alpha_m (k \sin \psi) \left( \frac{k}{\mu} \right)^2 (\sin^2 \psi - \cos^2 \psi) \cos \delta \sin \delta |E_0|^2 \right],
\]
\[
= 3 \left( \frac{k^3}{6\pi} \right) F^{\text{norm}} \sin \psi \cos \delta \sin \delta (\sin^2 \psi - \cos^2 \psi) \text{Re} (\alpha_m),
\]
\[
= -3 \left( \frac{k^3}{6\pi} \right) F^{\text{norm}} \sin \psi \cos \sin \delta \cos \psi \text{Re} (\alpha_m),
\]
\[
= -\frac{3}{2} \left( \frac{k^3}{6\pi} \right) F^{\text{norm}} \sin \psi \sin 2 \cos \psi \text{Re} (\alpha_m).
\] \hspace{1cm} (A23)

Finally, by using the definition of the normalized force, i.e. \( \bar{F} = \frac{F}{F^{\text{norm}}} \) and the normalized magnetic polarizability, i.e. \( \bar{\alpha}_m = \frac{\alpha_m}{\alpha_d} \), we obtain:

\[
\bar{F}_{x(m)} = -\frac{3}{2} \sin \psi \sin 2 \cos \psi \text{Re} (\bar{\alpha}_m).
\] \hspace{1cm} (A24)

This expression is documented in Eq. 7 of the main manuscript.

Similarly, the \( z \) component of \( F_m \) reads as:

\[
F_{z(m)} = \frac{1}{2} \text{Re} \left( m_x \frac{\partial}{\partial z} B_x^* + m_y \frac{\partial}{\partial z} B_y^* + m_z \frac{\partial}{\partial z} B_z^* \right),
\]
\[
= \frac{1}{2} \text{Re} \left( m_x \frac{\partial}{\partial z} B_x^* + m_z \frac{\partial}{\partial z} B_z^* \right).
\] \hspace{1cm} (A25)

Now, by substituting Eq. A11 and Eq. A7 into Eq. A25 we obtain

\[
F_{z(m)} = \frac{1}{2} \text{Re} \left( -i \varepsilon_0 k \cos^3 \psi \cos^2 \delta \alpha_m |E_0|^2 \right)
\]
\[
+ \frac{1}{2} \text{Re} \left( -i \varepsilon_0 k \cos \psi \sin^2 \psi \sin^2 \delta \alpha_m |E_0|^2 \right),
\]
\[
= 3 \left( \frac{k^3}{6\pi} \right) F^{\text{norm}} \cos \psi (\cos^2 \psi \cos \sin \delta + \sin^2 \psi \sin^2 \delta) \text{Im} (\alpha_m),
\] \hspace{1cm} (A26)

Finally, the normalized contribution is derived as:

\[
\bar{F}_{z(m)} = 3 \cos \psi \left( \cos^2 \psi \cos \sin \delta + \sin^2 \psi \sin^2 \delta \right) \text{Im} (\bar{\alpha}_m).
\] \hspace{1cm} (A27)

this expression is documented in Eq. 7 of the main manuscript.

4. \( F_{pm} \) contribution

According to Eq. A11 the interference dipolar term reads as:

\[
F_{i(pm)} = -\frac{k^4}{12\pi \varepsilon_0 c} \text{Re} \left( \sum_{j,k} \epsilon_{ijk} p_j m_k^* \right),
\] \hspace{1cm} (A28)

where \( \epsilon_{ijk} \) is the Levi-Civita symbol and is defined as:

\[
\epsilon_{ijk} = \begin{cases} 
+1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), (3,1,2) \\
-1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), (2,1,3) \\
0 & \text{if } i = j \text{ or } j = k \text{ or } i = k
\end{cases}
\]

Using Eq. A28 the \( x \) component of \( F_{pm} \) read as

\[
F_{x(pm)} = -\frac{k^4}{12\pi \varepsilon_0 c} \text{Re} \left( \epsilon_{132} p_y m_z^* + \epsilon_{132} p_z m_y^* \right).
\] \hspace{1cm} (A29)
Now, by substituting Eq. A10 and Eq. A11 into Eq. A29, we obtain:

\[
F_{x(pm)} = -\frac{k^4}{12\pi \varepsilon_0 c} \text{Re} \left[ (\varepsilon_0 \alpha_e E_y) \left( \alpha_m \frac{B_z}{\mu} \right)^* \right],
\]

\[
= -\frac{k^4}{12\pi} \text{Re} \left[ -i \varepsilon_0 \alpha_e \alpha_m^* \sin \psi \cos \delta \sin \delta |E_0|^2 \right],
\]

\[
= -\frac{3}{2} F_{\text{norm}} \sin \psi \sin 2\delta \text{Im} \left[ \left( \frac{k^3}{6\pi} \right)^2 (\alpha_e \alpha_m^*) \right].
\]  

(A30)

Finally, by using the definition of the normalized force and the normalized polarizabilities, we obtain:

\[
\bar{F}_{x(pm)} = -\frac{3}{2} \sin \psi \sin 2\delta \text{Im} \left[ \left( \frac{k^3}{6\pi} \right)^2 (\bar{\alpha}_e \bar{\alpha}_m^*) \right].
\]  

(A31)

This expression is documented in Eq. 7 of the main manuscript.

Using Eq. A28 the \(z\) component of \(\mathbf{F}_{pm}\) read as

\[
F_{z(pm)} = -\frac{k^4}{12\pi \varepsilon_0 c} \text{Re} \left( \varepsilon_{312} p_x m_y^* + \varepsilon_{321} p_y m_x^* \right),
\]  

(A32)

Now, by substituting Eq. A10 and Eq. A11 into Eq. A29, we obtain

\[
F_{z(pm)} = -\frac{k^4}{12\pi \varepsilon_0 c} \text{Re} \left[ (-1) (\varepsilon_0 \alpha_e E_y) \left( \alpha_m \frac{B_z}{\mu} \right)^* \right]
\]

\[
= -\frac{k^4}{12\pi} \text{Re} \left[ \varepsilon_0 \alpha_e \alpha_m^* \cos \psi \cos^2 \delta \sin \delta |E_0|^2 \right]
\]

\[
= -3 F_{\text{norm}} \cos \psi \cos^2 \delta \text{Re} \left[ \left( \frac{k^3}{6\pi} \right)^2 (\alpha_e \alpha_m^*) \right]
\]

Finally, the normalized contribution derives as:

\[
\bar{F}_{z(pm)} = -3 \cos \psi \cos^2 \delta \text{Re} \left( \bar{\alpha}_e \bar{\alpha}_m^* \right),
\]  

(A33)

this expression is documented in Eq. 7 of the main manuscript.

5. \(F_{pQe}\) contribution

According to Eq. A11 the optical force caused by the interference of the electrical dipole and electric quadrupole reads as:

\[
F_{i(pQe)} = -\frac{k^5}{120\pi \varepsilon_0} \text{Im} \left[ \sum_j (Q^e)_{ij} p_j^* \right].
\]  

(A34)

Using Eq. A34 the \(x\) component of \(\mathbf{F}_{pQe}\) reads as:

\[
F_{x(pQe)} = -\frac{k^5}{120\pi \varepsilon_0} \text{Im} \left( Q_{xy} p_y^* \right).
\]  

(A35)

Now, by substituting Eq. A10 and Eq. A13 into Eq. A35, we obtain:

\[
F_{x(pQe)} = \frac{k^5}{120\pi} \frac{1}{2} \varepsilon_0 k |E_0|^2 \sin \psi \cos \delta \sin \delta \text{Im} \left( \alpha_{Q^-} \alpha_e^* \right),
\]

\[
= \frac{3}{2} F_{\text{norm}} \sin \psi \sin 2\delta \text{Im} \left[ \left( \frac{k^3}{6\pi} \right)^2 \frac{k^5}{120\pi} (\alpha_{Q^-} \alpha_e^*) \right].
\]
Finally, by using the definition of the normalized force, i.e. \( \bar{F} = F/F_{\text{norm}} \) and the normalized electric dipolar and quadrupolar polarizabilities, i.e. \( \bar{\alpha}_e = \alpha_e/\alpha_d, \bar{\alpha}_{Q_e} = \alpha_{Q_e}/\alpha_q \), we obtain:

\[
\bar{F}_x(pQ_e) = \frac{3}{2} \sin\psi / \sin(2\delta) \Im (\bar{\alpha}_{Q_e} \bar{\alpha}_e^*),
\]

where \( \alpha_q = 120\pi/k^5 \). This expression is documented in Eq. 7 of the main manuscript.

Similarly, using Eq. \[A34\] the \( z \) component of \( F_{pQ_e} \) reads as:

\[
F_z(pQ_e) = -\frac{k^5}{120\pi\varepsilon_0} \Im (Q_{zy} p_y^*).
\]

This expression is documented in Eq. 7 of the main manuscript.

6. \( F_{Q_e} \) contribution

According to Eq. \[A1\] the electric quadrupole \( (Q_e) \) contribution reads as:

\[
F_i(Q_e) = \frac{1}{12} \Re \left[ \sum_k (Q_e)_{jk} \nabla_j \nabla_k E_i^* \right].
\]

Using Eq. \[A39\] the \( x \) component of \( F_{Q_e} \) read as

\[
F_x(Q_e) = \frac{1}{12} \Re \left[ \sum_k (Q_e)_{yk} \nabla_x \nabla_k E_y^* \right],
\]

\[
= \frac{1}{12} \Re \left[ (Q_e)_{yx} \nabla_x \nabla_x E_y^* + (Q_e)_{yz} \nabla_x \nabla_z E_y^* \right].
\]

Now, by substituting Eq. \[A8\] and Eq. \[A13\] into Eq. \[A40\] we obtain

\[
F_x(Q_e) = \frac{1}{12} \Re \left[ -\frac{1}{2} \varepsilon_0 \alpha_{Q_e} E_0 \sin / \sin(\delta) \left( -E_0^* k^2 \cos \delta \sin^2 \psi \right) \right],
\]

\[
+ \frac{1}{12} \Re \left[ \frac{1}{2} \varepsilon_0 \alpha_{Q_e} E_0 \cos \delta \left( iE_0^* k^2 \cos \psi / \sin \psi / \sin(\delta) \right) \right],
\]

\[
= -\frac{5}{2} F_{\text{norm}} \sin \psi / \sin 2\delta \cos 2\psi \Re \left( \frac{k^5}{120\pi} \alpha_{Q_e} \right).
\]

Finally, by using the definition normalized force and the normalized polarizabilities, we obtain:

\[
\bar{F}_x(Q_e) = -\frac{5}{2} F_{pQ_e} \sin \psi / \sin 2\delta \cos 2\psi \Re (\bar{\alpha}_{Q_e}).
\]

This expression is documented in Eq. 7 of the main manuscript.

Using Eq. \[A39\] the \( z \) component of \( F_{Q_e} \) reads as:
Now, by substituting Eq. A9 and Eq. A15 into Eq. A44, we obtain:

\[
F_z(Q^r) = \frac{1}{12} \text{Re} \left[ \sum_k (Q_v)_y k \nabla_z \nabla_k E^*_y \right],
\]

\[
= \frac{1}{12} \text{Re} \left[ (Q_v)_{yz} \nabla_z \nabla_x E^*_y + (Q_v)_{yx} \nabla_z \nabla_x E^*_y \right].
\]  
(A41)

Finally, by using the definition of the normalized force and the normalized magnetic quadrupolar polarizabilities, this expression is documented in Eq. 7 of the main manuscript.

\[
F_z(Q^r) = 5F_{\text{norm}} \cos \psi (\cos^2 \psi \cos^2 \delta + \sin^2 \psi \sin^2 \delta) \text{Im} \left( \frac{k^5}{120\pi} \alpha_{Q^r} \right).
\]  
(A42)

7. \( F_{Q^m} \) contribution

According to Eq. A11, the magnetic quadrupole\((Q^m)\) contribution is given by:

\[
F_i(Q^m) = \frac{1}{12} \text{Re} \left[ \sum_k (Q^m)_{jk} \nabla_i \nabla_k B^*_j \right].
\]  
(A43)

Using Eq. A43, the \( x \) component of \( F_{Q^m} \) read as

\[
F_{x(Q^m)} = \frac{1}{12} \text{Re} \left[ \sum_k (Q^m)_{xz} \nabla_x \nabla_k B^*_z \right] + \frac{1}{12} \text{Re} \left[ \sum_k (Q^m)_{zx} \nabla_z \nabla_k B^*_x \right],
\]  
(A44)

Now, by substituting Eq. A9 and Eq. A15 into Eq. A44, we obtain:

\[
F_{x(Q^m)} = \frac{1}{12} \text{Re} \left\{ \text{i}Q^m E_0 \frac{k^2}{\omega \mu} \left( \sin^2 \psi - \cos^2 \psi \right) \cos \delta \right\} \left[ -\text{i}E_0^* \left( \frac{k^3}{\omega} \sin \psi \cos^2 \psi \sin \delta \right) \right]
\]

\[
+ \frac{1}{12} \text{Re} \left\{ \alpha_{Q^m} E_0 \frac{k^2}{\omega \mu} \sin \psi \cos \psi \sin \delta E_0^* \left( \frac{k^3}{\omega} \sin^2 \psi \cos \psi \cos \delta \right) \right\}
\]

\[
+ \frac{1}{12} \text{Re} \left\{ \text{i}Q^m E_0 \frac{k^2}{\omega \mu} \left( \sin^2 \psi - \cos^2 \psi \right) \cos \delta \right\} \left[ \text{i}E_0^* \left( \frac{k^3}{\omega} \sin^3 \psi \sin \delta \right) \right]
\]

\[
+ \frac{1}{12} \text{Re} \left[ -\alpha_{Q^m} E_0 \frac{k^2}{\omega \mu} \sin \psi \cos \psi \sin \delta \left( -E_0^* \right) \left( \frac{k^3}{\omega} \sin^2 \psi \cos \psi \cos \delta \right) \right]
\]

\[
= -\frac{5}{2} F_{\text{norm}} \sin \psi \sin^2 \delta \left( \cos^2 \psi - \sin^2 \psi \right) \text{Re} \left( \alpha_{Q^m} \right).
\]  
(A45)

Finally, by using the definition of the normalized force and the normalized magnetic quadrupolar polarizabilities, i.e. \( \tilde{\alpha}_{Q^m} = \alpha_{Q^m} / \alpha_q \), we obtain:

\[
\tilde{F}_{x(Q^m)} = -\frac{5}{2} \sin \psi \sin \delta \left( \cos^2 \psi - \sin^2 \psi \right) \text{Re} \left( \tilde{\alpha}_{Q^m} \right).
\]  
(A45)
This expression is documented in Eq. 7 of the main manuscript. Using Eq.  A43 the z component of $F_{Q^m}$ reads as:

$$F_{z(Q^m)} = \frac{1}{12}\text{Re}\left[\sum_k (Q^m)_{zk} \nabla_z \nabla_k B_z^*\right] + \frac{1}{12}\text{Re}\left[\sum_k (Q^m)_{zk} \nabla_z \nabla_k B_z^*\right],$$  \hfill (A46)

Now, by substituting Eq. A15 and Eq. A11 into Eq. A49, we obtain:

$$F_{z(Q^m)} = \frac{1}{12}\text{Re}\left[(Q^m)_{xz} \nabla_z \nabla_z B_x^* + (Q^m)_{xz} \nabla_z \nabla_z B_x^* + (Q^m)_{xy} \nabla_z \nabla_y B_x^*\right],$$

$$+ \frac{1}{12}\text{Re}\left[(Q^m)_{xz} \nabla_z \nabla_z B_x^* + (Q^m)_{yz} \nabla_z \nabla_y B_x^* + (Q^m)_{yy} \nabla_z \nabla_z B_x^*\right],$$

$$= \frac{1}{12}\text{Re}\left[(Q^m)_{xz} \nabla_z \nabla_z B_x^* + (Q^m)_{xz} \nabla_z \nabla_z B_x^*\right] + \frac{1}{12}\text{Re}\left[(Q^m)_{xz} \nabla_z \nabla_z B_x^* + (Q^m)_{yz} \nabla_z \nabla_y B_x^*\right] + \frac{1}{12}\text{Re}\left[(Q^m)_{xz} \nabla_z \nabla_z B_x^* + (Q^m)_{yy} \nabla_z \nabla_z B_x^*\right].$$

Finally, the normalized contribution derives as:

$$\tilde{F}_{z(Q^m)} = 5\cos(\psi) \cos(2\psi) \cos(2\delta) + \sin(2\psi) \sin(2\delta) \text{Im}\left(\frac{k^5}{120\pi} \alpha_{Q^m}\right).$$ \hfill (A47)

This expression is documented in Eq. 7 of the main manuscript.

8. $F_{m Q^m}$ contribution

According to Eq. A1 the term due to the interference of the magnetic dipole $(m)$ and quadrupole$(Q^m)$ is given by

$$F_{i(m Q^m)} = -\frac{k^5}{120\pi \varepsilon_0 c^2} \text{Im}\left[(Q^m)_{ij} m_i^*\right].$$ \hfill (A48)

Using Eq. A48 the $x$ component of $F_{m Q^m}$ reads as:

$$F_{x(m Q^m)} = -\frac{k^5}{120\pi \varepsilon_0 c^2} \text{Im}\left[(Q^m)_{xz} m_z^* + (Q^m)_{xx} m_x^*\right],$$ \hfill (A49)

Now, by substituting Eq. A15 and Eq. A11 into Eq. A49 we obtain:

$$F_{x(m Q^m)} = -\frac{k^5}{120\pi \varepsilon_0 c^2} \text{Im}\left(\alpha_{Q^m} E_0 \left[\frac{k^2}{\omega \mu} (\sin^2(\psi) - \cos^2(\psi)) \cos(\delta)\right] e^{i k \cos(\psi) z_0} m_z^*\right),$$

$$- \frac{k^5}{120\pi \varepsilon_0 c^2} \text{Im}\left(\alpha_{Q^m} E_0 \left[\frac{k^2}{\omega \mu} \sin(\psi) \cos(\psi) \sin(\delta)\right] e^{i k \cos(\psi) z_0} m_x^*\right),$$

$$= \frac{3}{2} F^{\text{norm}} \sin(\psi) \sin(2\delta) \cos(2\psi) + \cos(2\psi) \text{Im}\left(\frac{k^5}{120\pi \varepsilon_0 c^2} \alpha_{Q^m} \alpha_m^*\right).$$

Finally, by using the definition of the normalized force and the normalized polarizabilities, we obtain:

$$\tilde{F}_{x(m Q^m)} = \frac{3}{2} \sin(\psi) \sin(2\delta) \cos(2\psi) + \cos(2\psi) \text{Im}\left(\alpha_{Q^m} \alpha_m^*\right).$$ \hfill (A50)
This expression is documented in Eq. 7 of the main manuscript. Using Eq. [A48] the $z$ component of $F_{mQ^m}$ read as

$$F_{z(mQ^m)} = -\frac{k^5}{120\pi\varepsilon_0 c^2} \text{Im} \left[ (Q^m)_{zz} m^*_x + (Q^m)_{zz} m^*_z \right]. \tag{A51}$$

Now, by substituting Eq. [A15] and Eq. [A11] into Eq. [A49] we obtain:

$$F_{z(mQ^m)} = -\frac{k^5}{120\pi\varepsilon_0 c^2} \text{Im} \left\{ i\alpha Q^m - \frac{k^2}{\omega\mu} \left[ \frac{k^2}{\omega\mu} (\sin^2\psi - \cos^2\psi) \cos \delta \right] e^{ik\cos\psi z_0 m^*_z} \right\}$$

$$= -\frac{k^5}{120\pi\varepsilon_0 c^2} \text{Im} \left( -\alpha Q^m - \frac{k^2}{\omega\mu} \sin\psi/\sin\delta e^{ik\cos\psi z_0 m^*_z} \right),$$

$$= -3F_{\text{norm}} \cos\psi \left( \cos 2\psi/\cos^2\delta + 2\sin^2\psi/\sin^2\delta \right) \text{Re} \left( \frac{k^5}{120\pi} \frac{k^3}{6\pi} \alpha Q^m \bar{\alpha}^*_m \right).$$

Finally, the normalized contribution is derived as:

$$F_{z(mQ^m)} = -3\cos\psi \left( \cos 2\psi/\cos^2\delta + 2\sin^2\psi/\sin^2\delta \right) \text{Re} \left( \bar{\alpha} Q^m \alpha^*_m \right). \tag{A52}$$

This expression is documented in Eq. 7 of the main manuscript.

9. $F_{Q^c Q^m}$ contribution

According to Eq. [A11] the term due to the interference of the electric quadrupole ($Q^c$) and and magnetic quadrupole ($Q^m$) is given by:

$$F_{i(Q^c Q^m)} = -\frac{k^6}{9 \times 240\pi\varepsilon_0 c} \text{Re} \left[ \sum_{l,j,k} \varepsilon_{ijk} (Q^c)_{ij} (Q^m)_{lk}^* \right]. \tag{A53}$$

Using Eq. [A48] the $x$ component of $F_{Q^c Q^m}$ reads as:

$$F_{x(Q^c Q^m)} = -\frac{k^6}{9 \times 240\pi\varepsilon_0 c} \text{Re} \left[ \sum_{l,j,k} \varepsilon_{ijk} (Q^c)_{ij} (Q^m)_{lk}^* \right],$$

$$= -\frac{k^6}{9 \times 240\pi\varepsilon_0 c} \text{Re} \left\{ \sum_{l} \varepsilon_{123} (Q^c)_{12} (Q^m)_{13}^* \right\} + \left[ \sum_{l} \varepsilon_{132} (Q^c)_{13} (Q^m)_{12}^* \right],$$

$$= -\frac{k^6}{9 \times 240\pi\varepsilon_0 c} \text{Re} \left\{ (+1) [Q_{12}^c Q_{13}^m + Q_{22}^c Q_{23}^m + Q_{32}^c Q_{33}^m] + (-1) [Q_{13}^c Q_{12}^m + Q_{23}^c Q_{22}^m + Q_{33}^c Q_{32}^m] \right\},$$

$$= -\frac{k^6}{9 \times 240\pi\varepsilon_0 c} \text{Re} \left( Q_{xy}^c Q_{xz}^m + Q_{xy}^c Q_{xz}^m \right). \tag{A54}$$

Now, by substituting Eq. [A13] and Eq. [A15] into Eq. [A54] we obtain:

$$F_{x(Q^c Q^m)} = -\frac{5}{6} F_{\text{norm}} \sin\psi/\sin\delta \left( \cos 2\psi + 2\cos^2\psi \right) \text{Im} \left( \frac{k^5}{120\pi} \frac{k^3}{6\pi} \alpha Q^c \bar{\alpha}^*_m \right).$$

Finally, the normalized contribution is derived as:

$$F_{x(Q^c Q^m)} = -\frac{5}{6} \sin\psi/\sin\delta \left( \cos 2\psi + 2\cos^2\psi \right) \text{Im} \left( \bar{\alpha} Q^c \alpha^*_m \right).$$

This expression is documented in Eq. 7 of the main manuscript.
Using Eq. A15 the z component of $F_{Q',Q''}$ read as

$$F_z(Q',Q'') = -\frac{k^6}{9 \times 240 \pi \varepsilon_0 c} \text{Re} \left[ \sum_{i,j,k} \varepsilon_{ijk}(Q')_i (Q'')_{jk} \right]$$

$$= -\frac{k^6}{9 \times 240 \pi \varepsilon_0 c} \text{Re} \left[ \sum_{i} \varepsilon_{312}(Q')_{i1} (Q'')_{i1} + \sum_{i} \varepsilon_{321}(Q')_{i2} (Q'')_{i1} \right]$$

$$= -\frac{k^6}{9 \times 240 \pi \varepsilon_0 c} \text{Re} \{ (+1) [Q'_{11}Q''_{12} + Q'_{21}Q''_{22} + Q'_{31}Q''_{32}] 

+ (-1) [Q'_{12}Q''_{11} + Q'_{22}Q''_{21} + Q'_{32}Q''_{31}] \} .$$

$$= \frac{k^6}{9 \times 240 \pi \varepsilon_0 c} \text{Re} \left( Q'_{xy}Q''_{xz} + Q'_{xy}Q''_{xx} \right) .$$

(A55)

Now, by substituting Eq. A13 and Eq. A15 into Eq. A55 we obtain

$$F_z(Q',Q'') = \frac{k^6}{9 \times 240 \pi \varepsilon_0 c} \text{Re} \left( Q'_{xy}Q''_{xz} + Q'_{xy}Q''_{xx} \right)$$

$$= -\frac{5}{3} \text{norm cos} \psi \left( \cos^2 \psi \cos^2 \delta + 2 \sin^2 \psi \sin^2 \delta \right) \text{Re} \left[ \left( \frac{k^6}{120 \pi} \right)^2 \alpha_{Q'} \alpha_{Q''} \right] .$$

(A56)

Finally, the normalized contribution is derived as:

$$\tilde{F}_z(Q',Q'') = \frac{5}{3} \text{cos} \psi \left( \cos^2 \psi \cos^2 \delta + 2 \sin^2 \psi \sin^2 \delta \right) \text{Re} \left( \alpha_{Q'} \alpha_{Q''} \right) .$$

This expression is documented in Eq. 7 of the main manuscript.

**Appendix B: Optical force: TM illumination**

Using the duality in the Maxwell’s equations for the electric and magnetic fields/induced moments, similar expression for optical force can be obtained for a TM polarization. The results are as following:

$$F_p^{TM} \approx F_p^{TM} + F_m^{TM} + F_{pm}^{TM} + F_{Q}^{TM} + F_{Qm}^{TM} + F_{pQ}^{TM} + F_{mQm}^{TM} + F_{Q'Q''}^{TM} ,$$

(B1)

$$F_p^{TM} = \frac{3}{2} \text{sin} \psi \text{sin} 2 \text{cos} 2 \psi \text{Re} (\alpha_{Q'}) e_x + 3 \text{cos} \psi (\cos^2 \psi \cos^2 \delta + \sin^2 \psi \sin^2 \delta) \text{Im} (\alpha_{Q'}) e_z ,$$

$$F_m^{TM} = \frac{3}{2} \text{sin} \psi \text{sin} 2 \psi \text{Re} (\alpha_{m}) e_x + 3 \text{cos} \psi (\cos^2 \psi \cos^2 \delta + \sin^2 \psi \sin^2 \delta) \text{Im} (\alpha_{m}) e_z ,$$

$$F_{pm}^{TM} = \frac{3}{2} \text{sin} \psi \text{sin} 2 \psi \text{Im} (\alpha_{e}^* \alpha_{m}^*) e_x - 3 \text{cos} \psi (\cos^2 \psi \cos^2 \delta \text{Re} (\alpha_{e}^* \alpha_{m}^*) e_z ,$$

$$F_Q^{TM} = \frac{5}{2} \text{sin} \psi \text{sin} 2 \psi \text{cos} 4 \psi \text{Re} (\alpha_{Q'}) e_x + 5 \text{cos} \psi (\cos^2 \psi \cos^2 \delta + \sin^2 \psi \sin^2 \delta) \text{Im} (\alpha_{Q'}) e_z ,$$

$$F_{Qm}^{TM} = \frac{5}{2} \text{sin} \psi \text{sin} 2 \psi \cos 2 \psi \text{Re} (\alpha_{Qm}) e_x + 5 \text{cos} \psi (\cos^2 \psi \cos^2 \delta + \sin^2 \psi \sin^2 \delta) \text{Im} (\alpha_{Qm}) e_z ,$$

$$F_{pQ''}^{TM} = \frac{3}{2} \text{sin} \psi \text{sin} 2 \psi (\cos^2 \psi + 2 \cos^2 \psi) \text{Im} (\alpha_{Q''} \alpha_{*}) e_x - 3 \text{cos} \psi (\cos^2 \delta \text{cos} 2 \psi + 2 \sin^2 \psi \sin^2 \delta) \text{Re} (\alpha_{Q''} \alpha_{*}) e_z ,$$

$$F_{mQm}^{TM} = \frac{3}{2} \text{sin} \psi \text{sin} 2 \psi (\cos^2 \psi + 2 \cos^2 \psi) \text{Im} (\alpha_{Qm} \alpha_{*}) e_x - 3 \text{cos} \psi (\cos^2 \delta \text{cos} 2 \psi + 2 \sin^2 \psi \sin^2 \delta) \text{Re} (\alpha_{Qm} \alpha_{*}) e_z ,$$

$$F_{Q'Q''}^{TM} = \frac{5}{6} \text{sin} \psi \text{sin} 2 \psi (\cos^2 \psi + 2 \cos^2 \psi) \text{Im} (\alpha_{Q'} \alpha_{Q''}) e_x - \frac{5}{3} \text{cos} \psi (\cos^2 \delta \text{cos} 2 \psi + 2 \sin^2 \psi \sin^2 \delta) \text{Re} (\alpha_{Q'} \alpha_{Q''}) e_z .$$

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