Limit shape for infinite rank limit of tensor power decomposition for Lie algebras of series $\mathfrak{so}_{2n+1}$

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Abstract

We consider the Plancherel measure on irreducible components of tensor powers of the spinor representation of $\mathfrak{so}_{2n+1}$. The irreducible representations correspond to the generalized Young diagrams. With respect to this measure the probability of an irreducible representation is the product of its multiplicity and dimension, divided by the total dimension of the tensor product. We study the limit shape of the generalized Young diagram when the tensor power $N$ and the rank $n$ of the algebra tend to infinity with $N/n$ fixed. We derive an explicit formula for the limit shape and prove convergence to it in probability. We prove central limit theorem for global fluctuations around the limit shape.

Keywords: limit shapes, Lie algebras, special orthogonal group, central limit theorem, determinantal point process, Young diagram, Berele insertion

(Some figures may appear in colour only in the online journal)
1. Introduction

The emergence of the limit shapes of the random Young diagrams goes back to Ulam’s problem on the length of the maximal increasing subsequence in a uniform random sequence [61]. Through the use of Robinson–Schensted–Knuth correspondence [37, 57, 59] a pair of Young tableaux is associated to the random sequence. Then the length of the maximal increasing subsequence is equal to the length of the first row of Young diagram. Uniform distribution on number sequences after Robinson–Schensted–Knuth mapping gives rise to Plancherel measure on Young diagrams [45, 63].

The limit shape problem for the Young diagram could also be stated for the tensor product decomposition of irreducible representations of semisimple Lie algebras. Due to the Schur–Weyl duality the multiplicities of the irreducible components in the $N$th tensor power of the vector fundamental representation of $sl_{n+1}$ are the dimensions of the irreducible representations of $S_N$. The Plancherel-type measure (8) associated with this decomposition was first considered by Kerov [36]. The asymptotic behavior of this measure was studied in three regimes: $N \to \infty$ with $n$ fixed, $N \to \infty$, $n \to \infty$ with $N/n$ fixed and $N,n \to \infty$ with $N/n^2$ fixed. The first case was studied [36] and later generalized to all simple Lie algebras in [49, 55, 60]. For the second case Kerov discovered that Vershik–Kerov–Logan–Shepp limit shape of Young diagrams with respect to the Plancherel measure on $S_N$ as $N \to \infty$ also appears as the limit shape with respect to this measure. Later, Biane [9, 10] described the limit shapes for the third case. But the asymptotical behavior of the Plancherel measure for $N$th tensor powers of representations of Lie algebras of types $so_{2n+1}, sp_{2n}, so_{2n}$ has not been studied yet in the limit $N,n \to \infty$.

Limit shape problem for Young diagrams is closely related to lozenge tilings and dimer models in statistical physics, and is also connected to random matrices, as explained in [22]. Consider tilings of a hexagon with sides $K,M,N,K,M,N$ by three types of lozenges and take a line parallel to a side of a hexagon that splits the hexagon in two parts. To specify the tiling it is enough to specify the positions of one specific type of lozenges. The positions of these lozenges on the chosen line can be encoded by Young diagram. Thus uniform measure on lozenge tilings of a hexagon induces some non-trivial measure on Young diagrams. In particular, Plancherel measure appears in the limit, when $M,N,K$ go to infinity. We can see the lozenge tiling as a 3d image of ‘cubes in the corner’ and the heights of the piles of cubes are encoded by plane partitions. Plane partitions can be viewed as a model for crystal melting and through dualities related to topological strings [54]. Such problems are usually studied by the use of free-fermionic representation [52, 53]. The use of free fermions for Lie algebras of series $so_{2n+1}, sp_{2n}, so_{2n}$ is under an active investigation [3, 6, 7], but there is no free-fermionic construction for our problem yet. An alternative visualization of the lozenge tiling can be obtained by drawing the lines along the main diagonals of lozenges. Then this model becomes dimer model on a hexagonal lattice. Since the works of Kasteleyn [34, 35] dimer models are very popular in physics, in particular, the solution of the two-dimensional Ising model by the use of dimers is very elegant [23]. The probability measure on Young diagrams that appears from the tensor power decompositions can be also seen as induced from dimer configurations on the Aztec diamond [32]. Dimer or domino tilings of the Aztec diamond attained great importance in last decades due to the limit shape phenomenon [18, 19].

In the present paper we consider the statistics of irreducible components in the $N$th tensor power of the spinor representation $V^{\omega_4}$ of the algebra $so_{2n+1}$ in the limit $N,n \to \infty$. We derive the limit shape in the limit when $N \to \infty$ and $N/n$ is finite. It is convenient to use the description
of irreducible components in terms of generalized Young diagrams \([33, 47]\), that we call below ‘diagrams’. The probability measure on the diagrams can be seen as induced from the uniform almost symmetric lozenge tilings of the skew hexagon or almost symmetric domino tilings for glued Aztec diamond (see figure 5). Our main result is most conveniently stated in coordinates that correspond to the diagrams (see figure 4):

**Theorem 1.** As \(n \to \infty, N \to \infty\), \(c = \lim_{n,N \to \infty} \frac{N+2n-1}{n} = \text{const}\), the upper boundary \(f_\alpha\) of a rotated and scaled generalized Young diagram for a highest weight in the decomposition of tensor power of the spinor representation \((V^{\otimes \alpha})^g\) of simple Lie algebra \(\mathfrak{so}_{2n+1}\) into irreducible representations converges in probability in the supremum norm \(\| \cdot \|_\infty\) to the limiting shape given by the formula \(f(x) = 1 + \int_0^1 (1 - 4\rho(t)) dt\), where the limit density \(\rho(x)\) is written explicitly as:

\[
\rho(x) = \theta \left( \frac{x}{2} - |x| \right) \frac{1}{2\pi} \arccos \frac{c - 4}{\sqrt{c^2 - 4x^2}},
\]

where \(\theta \left( \frac{x}{2} - |x| \right)\) is the Heaviside step function.

That is, for all \(\varepsilon > 0\) we have:

\[
P \left( \sup_{x \in \mathbb{R}} |f_\alpha(x) - f(x)| > \varepsilon \right) \xrightarrow{n \to \infty} 0.
\]

This result was previously announced in \([48]\), where the proof was outlined. Here we present the full proof of this theorem.

We also demonstrate, that it is easy to prove the central limit theorem for the global fluctuations of diagrams that is presented below. The proof is based upon the use of the biorthogonal ensembles techniques \([12]\).

**Theorem 2 (Central limit theorem).** For a linear statistics \(X_h = \sum_{i=1}^n h(x_i^2)\), where \(h \in C^1([0, 2c - 4])\) and \(x_i = \frac{d}{2n}\) are the midpoints of the (scaled) intervals, where the upper boundary of a random Young diagram, rotated \(45^\circ\) decreases, we have:

\[
X_h - E X_h \to \mathcal{N} \left( 0, \sum_{k \geq 1} k |\hat{h}_k|^2 \right),
\]

in distribution as \(n, N \to \infty\) with \(c = \lim_{n \to \infty} \frac{N+2n-1}{n}\), where the Fourier coefficients \(\hat{h}_k\) are defined as:

\[
\hat{h}_k = \frac{1}{2\pi} \int_0^{2\pi} h \left( (c - 2)(\cos \theta + 1) \right) e^{-ik\theta} d\theta.
\]

The paper is organized as follows. In section 2 we describe tensor power decomposition for spinor representation of \(\mathfrak{so}_{2n+1}\), that is used to introduce the probability measure. We fix the required notations, introduce the coordinates \(x\) and describe the generalized Young diagrams and their boundaries \(f_n\) that are used to state theorem 1. We then explain the multiplicity formula by skew Howe duality and discuss the insertion algorithm, that can be used to sample random Young diagrams. We also discuss the relation of the probability measure to almost symmetric tilings of a skew hexagon and glued tilings of the Aztec diamond and thus establish the connection of the present work to the dimer models of statistical physics.

The proof of the theorem 1 is contained in the sections 3 and 4. In the section 3 the variational problem for the limit shape is stated and solved and function (1) is obtained. In section 4 we prove the convergence of diagrams in probability, thus completing the proof of the theorem 1. The proof of the theorem 2 is presented in section 5.

In Conclusion we state open problems related to the presented results.
2. Notations and probability measure

First we recall the definition of the Plancherel measure for tensor products. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of rank $n$ and $V^\nu$ be its irreducible finite-dimensional highest-weight representation. Denote simple roots of $\mathfrak{g}$ by $\alpha_1, \ldots, \alpha_n$ and fundamental weights by $\omega_1, \ldots, \omega_n$, $(\alpha_i, \omega_j) = \delta_{ij}$. The Weyl group is denoted by $W$, the main Weyl chamber by $C_0$, the root system by $\Delta$ and the set of positive roots by $\Delta^+$. The root lattice $\mathbb{Z}\Delta$ is denoted by $Q$ and for a weight $\nu$ we denote by $Q(\nu)$ the set of weights that are obtained by repeated subtractions of positive roots from $\nu$: $Q(\nu) = \nu - \mathbb{Z}_+\Delta^+$.

Tensor power of $V^\nu$ is a completely reducible representation and can be decomposed as:

$$(V^\nu)^{\otimes N} \cong \bigoplus_{\lambda \in Q(\nu) \cap C_0} W_\lambda(V^\nu, N) \otimes V^\lambda. \quad (5)$$

The sum is taken over irreducible components of the tensor product and $W_\lambda(V^\nu, N)$ is the space of multiplicities:

$$W_\lambda(V^\nu, N) \simeq \text{Hom}_\mathfrak{g}((V^\nu)^{\otimes N}, V^\lambda). \quad (6)$$

Its dimension $M^N_{\lambda} \equiv M^N_\lambda(V^\nu)$ is the multiplicity of $V^\lambda$ in the tensor product. We can write tensor product decomposition as $(V^\nu)^{\otimes N} \cong \bigoplus_{\lambda \in Q(\nu) \cap C_0} M^N_{\lambda} V^\lambda$, when we want to indicate what irreducible representations appear in the decomposition and how many times, and are not interested in their embeddings.

This decomposition gives the identity:

$$(\dim V^\nu)^N = \sum_{\lambda \in Q(\nu) \cap C_0} M^N_{\lambda} \dim V^\lambda. \quad (7)$$

This formula can be used to introduce the probability measure on the set of dominant integral weights $Q(\nu) \cap C_0$. By the analogy with the representation theory of the permutation group we call it Plancherel measure:

$$p^N(\lambda) = \frac{M^N_{\lambda} \dim V^\lambda}{(\dim V^\nu)^N}. \quad (8)$$

From now on we focus on $\mathfrak{g} = \mathfrak{so}_{2n+1}$. In the standard orthogonal basis the simple roots are $\{\alpha_i = e_i - e_{i+1} | i = 1, \ldots, n-1\} \cup \{\alpha_n = e_n\}$. The root system $B_n$ consists of the roots $\Delta = \{\pm e_i \pm e_j | i \neq j\} \cup \{\pm e_i\}$, positive roots are $\Delta^+ = \{e_i + e_j | i < j\} \cup \{e_i\} \cup \{e_j - e_i | j < i\}$. The fundamental weights of $B_n$ in the same basis are given by formulae:

$$\omega_1 = e_1$$
$$\omega_2 = e_1 + e_2$$
$$\ldots$$
$$\omega_{n-1} = e_1 + \cdots + e_{n-1}$$
$$\omega_n = \frac{1}{2}(e_1 + \cdots + e_n). \quad (9)$$

We consider tensor powers $(V^{\omega_n})^{\otimes N}$ of the last fundamental representation $\nu = \omega_n$, that is also known as spinor representation.

Dominant integral weights $\lambda$ are linear combinations of fundamental weights with non-negative integer coefficients $l_i$, that are called Dynkin labels:

$$\lambda = \sum_{i=1}^{n} l_i \omega_i. \quad (10)$$
Figure 1. The generalized Young diagram for $B_5$ weight $\lambda$ with coordinates $[6, 4, 2, 2, 1]$ in orthogonal basis and Dynkin labels $(2, 2, 0, 1, 2)$.

In orthogonal coordinates such a weight is written as:

$$\lambda = \sum_{i=1}^{n} \left( l_i + l_{i+1} + \cdots + \frac{l_n}{2} \right) e_i.$$  \hspace{1cm} (11)

Dominant integral weights can be depicted by the generalized Young diagrams. For algebras of series $\mathfrak{so}_{2n+1}$ it is convenient to use the diagrams with boxes of two different widths, one being twice the other [33, 47] (see also [29]). Below we will the generalized Young diagrams for the series $\mathfrak{so}_{2n+1}$ 'the diagrams'. In the present case the analogue of Littlewood–Richardson rule for tensor product decomposition is more difficult than for ordinary Young diagrams for $\mathfrak{sl}_n$ and number of boxes in the diagram is not equal to the tensor power $N$. Since there are boxes of two different widths it is important to distinguish between the number of boxes in a row and the length of the row. The length of the diagram’s row $\lambda_i$ is equal to the corresponding orthogonal coordinate. The number of boxes is equal to $\sum_{j=i}^{n} l_j$. In such diagrams first $l_n$ boxes are of width one half. See an example in figure 1.

For convenience we use the coordinates $\{a_i\}$ given by the formula:

$$a_i = 2 \sum_{j=i}^{n-1} l_j + l_n + 2(n-i) + 1.$$  \hspace{1cm} (12)

Such coordinates are positive integer numbers for integral dominant weights and $a_i \geq a_j$ for $i < j$. The coordinates $\{a_i\}_{i=1}^{n}$ has natural interpretation if we scale the diagram by the factor $2\sqrt{2}$, rotate it $45^\circ$ counterclockwise and shift it in such a way that lowest point has coordinate $(2n)$. Then the upper border of the diagram is a graph of piecewise linear function and $a_i$ is an
Figure 2. Rotated and scaled generalized Young diagram and the geometrical meaning of the coordinates $\{a_i\}_{i=1}^n$.

The $x$-coordinate of the middle of decreasing piece number $i$, if we count decreasing pieces from the right. See figure 2.

The probability measure on the integral dominant weights is introduced by the formula (8) as:

$$
\mu_{n,N}(\lambda) = \frac{M^n_\lambda \dim V^\lambda}{2^{nN}}.
$$

We consider the limit $N,n \to \infty$ such that ratio of $n$ and $N$ tends to a finite constant:

$$
c = \lim_{N,n \to \infty} \frac{N + 2n - 1}{n}, \quad c = \text{const}.
$$

We are interested in the limiting probability distribution on the irreducible components of the tensor power decomposition. Since dominant integral weights are depicted by the diagrams, the measure (13) can be seen as a probability measure on the diagrams. Therefore we are interested in the limit shape of generalized Young diagrams with respect to the measure $\mu_{n,N}$.

Random diagrams with respect to the measure (13) can be efficiently sampled with the following algorithm, introduced by Benkart and Stroomer [4]. The algorithm uses Sundaram tableaux for $so_{2n+1}$ and modified Berele insertion [5], which works as follows. The tableau that corresponds to some basis element in the irreducible representation with the diagram $\lambda$ has the shape $\lambda$ that is filled with the numbers $1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}, \infty$ with the order $1 < \bar{1} < 2 < \bar{2} < \ldots < \infty$, and can have first column of boxes with width $\frac{1}{2}$ that can not contain $\infty$. The numbers strictly increase along the columns and weakly increase along the rows of the tableau, moreover, there are no numbers smaller than $\bar{i}$ below the $i$th row. The tableau for the spinor
representation $V^\omega_n$ is the full column with the width $\frac{1}{2}$. Due to the condition we have two choices $i, \bar{i}$ in the row number $i$, so that total number of such tableaux is $2^n$ and they are in the bijection with the basis elements of the spinor representation.

Tensor product decomposition $V^\lambda \otimes V^\omega_n = \bigoplus_{\mu} V^\mu$ is represented by the insertion of the column, corresponding to $V^\omega_n$. Insertion of this column to the tableau that does not contain first half-width column is done by erasing all the boxes with $\infty$ and adjoining the column from the left. Insertion to the diagram with the first half-width column is done by taking the inserted column and the first column, reading them from the bottom up row-by-row and producing a sequence of numbers from the two half-columns by the following rule:

$$(k, \bar{k}) \rightarrow k, (\bar{k}, k) \rightarrow \bar{k}, (\bar{k}, k) \rightarrow \emptyset, (k, k) \rightarrow \bar{k}, k.$$ We then insert these numbers one-by-one with a modified Berele insertion, that inserts the number by the Schensted insertion \[37\], but if the insertion of some number $i$ into some box should push the number $\bar{i}$ from this box to some row below the row number $i$, then the box is erased. The empty box is then shifted by jeu de taquin slides to the edge of the diagram and is filled by $\infty$.

Therefore to sample the diagrams with the measure \eqref{13} we produce $N$ uniform random vectors of zeros and ones with length $n$, that represent signs of the numbers in the half-width one-column tableaux, and then insert these half-width one-column tableaux one-by-one. In figure 3 we present a random diagram for $so_{101}$ and tensor power 300 sampled using this algorithm.

We scale the diagram by the factor $\frac{\sqrt{2}}{n}$, rotate it $45^\circ$ counterclockwise and shift along $x$ axis in such a way that the lowest point has coordinates $(1, 0)$. This corresponds to a rescaling of the coordinates $\{a_i\}$ as $x_i = n a_i$. See figure 4 for an example of the most probable diagram for $n = 20, N = 200$ and limit shape for $c = 12$. The upper border of the diagram is a graph of piecewise linear function $f_n(x)$, which is almost everywhere differentiable and $f'_n(x) = \pm 1$ if $x \neq \frac{20}{21}$. We will prove that piece-wise linear functions $f_n(x)$ converge in probability w.r.t. to the probability measure \eqref{13} to a continuous smooth function $f(x)$ when $n \rightarrow \infty$. 

\begin{figure}[h]
\centering
\includegraphics[width=\linewidth]{figure3}
\caption{Rotated and scaled random diagram for $so_{101}$ and $N = 300$ and limit shape $f(x)$ of the generalized Young diagrams for $c = \frac{2}{n} + 2 = 8$.}
\end{figure}
Figure 4. Rotated and scaled diagram for $B_{20}$ and $N = 200$ and limit shape $f(x)$ of the generalized Young diagrams for $c = \frac{N}{n} + 2 = 12$.

To derive the limit shape it is convenient to consider the diagrams as the particle point processes with coordinates $\{x_i\}_{i=1}^n$. Introduce the piecewise constant function $\rho_n(x) = \frac{1}{4}(1 - f_n'(x))$. It is equal to zero on an interval of the length $\frac{1}{n}$ if there is no particle in the middle of the interval and is equal to $\frac{1}{2}$ if there is a particle, which means that there is one particle on two intervals of length $\frac{1}{2n}$. So the function $\rho_n(x)$ can be called particle density. The convergence of the diagrams to the limit shape leads to the convergence of particle density functions $\rho_n$ to a limit particle density $\rho(x)$.

Due to our choice of normalization, limit density $\rho(x)$ is connected to a derivative of limit function $f(x)$ of the diagrams by the formula:

$$f'(x) = 1 - 4\rho(x),$$

and limit shape can be recovered from the explicit expression for $\rho(x)$ by the formula:

$$f(x) = 1 + \int_0^x (1 - 4\rho(t))\,dt.$$  \hspace{1cm} (16)

It is more convenient to solve the variational problem for the limit density $\rho(x)$.

In order to state the variational problem we need to write the explicit formula for the probability measure $\mu_{n,N}$. To do so we recall that Kulish et al derived the explicit formulae for tensor
product decomposition multiplicities using Weyl group symmetry and recurrence relations [39–42].

For the case of $\mathfrak{so}_{2n+1}$ and $(V^\lambda)^{\otimes N}$, for $\lambda$ written in the coordinates $\{a_i\}$ that are described above, the formula for the multiplicity of irreducible representation is:

$$
\tilde{M}_{\lambda(a_1,\ldots, a_n)}^{\omega_1, N} = \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^k \left(\binom{N+a_i+2n-1}{2} \cdots \binom{-a_i+2n-1}{2}\right)!} \prod_{i=1}^n a_i! \prod_{i<j} (a_i^2 - a_j^2). \quad (17)
$$

Note that the factors in the numerator vanish at the boundaries of the Weyl chambers, shifted by Weyl vector $-\rho = -\sum_{i=1}^n \omega_i$ and the denominator provides that $\tilde{M}_{\lambda}^{\omega_1, N}$ satisfies the boundary conditions and also ensures that the whole expression is anti invariant w.r.t. Weyl group transformations. Because of this anti-invariance we add tilde to $M$, since it is equal to multiplicity only for dominant integral weights $\lambda$.

Note that there are two congruence classes of weights, one is parametrized by even values of $a_i$, while another by odd. The class is determined by the parity of $i$.

This result can be understood as a consequence of the skew Howe duality between $\mathfrak{so}_{2n+1}$ and $\mathfrak{osp}_{2n+1}$. By theorem, presented, for example, in [40], the exterior algebra $\bigwedge (\mathbb{C}^{2n+1} \otimes \mathbb{C}^k)$ admits a multiplicity-free decomposition $\bigwedge (\mathbb{C}^{2n+1} \otimes \mathbb{C}^k) \simeq \bigoplus_{\lambda} V_{\lambda}^{\mathfrak{osp}_{2n+1}} \otimes V_{\lambda}'^{\mathfrak{osp}_{2n+1}}$, where $\lambda'$ is a generalized Young diagram that is complement conjugate to the diagram $\lambda$ inside of the $n \times k$-rectangle. From this decomposition the multiplicity formula $(V_{\lambda^{\mathfrak{osp}_{2n+1}}}^{\mathfrak{osp}_{2n+1}})^{(2k)} \simeq \bigoplus_{\lambda} \left(2^{1-k} \dim V_{\lambda^{\mathfrak{osp}_{2n+1}}}^{\mathfrak{osp}_{2n+1}}\right) V_{\lambda^{\mathfrak{osp}_{2n+1}}}^{\mathfrak{osp}_{2n+1}}$ can be deduced [50]. Setting $N = 2k$ and expressing the dimension of the $\mathfrak{osp}_N$-representation in terms of the coordinates $\{a_i\}$ we arrive at the formula (17).

We use the Weyl dimension formula:

$$
\dim V^\lambda = \prod_{\alpha \in \Delta^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)} = \frac{2^{-n^2+2n} n!}{(2n)! (2n-2)! \ldots 2!} \prod_{i<j} (a_i^2 - a_j^2) \prod_{i=1}^n a_i, \quad (18)
$$

thus we obtain the discrete probability measure with the explicit density function (or the probability mass function):

$$
\mu_{n,N}(\lambda) = \mu_{n,N}(\{a_i\}) = \frac{\tilde{M}_{\lambda(a_1,\ldots, a_n)}^{\omega_1, N} \dim V^\lambda}{(2n)^N} = \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^k \left(\binom{N+a_i+2n-1}{2} \cdots \binom{-a_i+2n-1}{2}\right)!} \prod_{i<j} (a_i^2 - a_j^2) \prod_{i=1}^n a_i^2 \frac{2^{-n^2+2n} n!}{(2n)! (2n-2)! \ldots 2!}. \quad (19)
$$

This measure can be related to the lozenge tilings of the hexagon or domino tilings of the Aztec diamond and therefore to the dimer models of statistical mechanics as follows. The dimension of an irreducible representation with a diagram $\lambda$ is equal to the number of (generalized) Young tableaux of the shape $\lambda$. For example, for the algebra $\mathfrak{so}_{2k}$ we should consider so-called ‘orthogonal tableaux’ [56], that are in one-to-one correspondence with the Gelfand-Tsetlin half patterns [65]. Each row of Gelfand-Tsetlin pattern encodes the positions of horizontal lozenges along the vertical line. If the number of horizontal lozenges is even, the positions should be symmetric with respect to the horizontal symmetry axis, but if it is odd the middle horizontal lozenge is free and can be above or below the symmetry axis [16]. Such tilings are called almost symmetric. Similarly, $\mathfrak{so}_{2n+1}$-patterns encode almost symmetric
Figure 5. Almost symmetric skew tiling of a hexagon, left half corresponds to $so_7$-tableau, right half to $so_6$-tableau with a complementary shape. Corresponding glued tiling of the Aztec diamond is presented on the right. Positions of the white lozenges in the left half correspond to positions of white dominoes on the lower left part of the Aztec diamond, positions of the white lozenges in the right half correspond to positions of blue dominoes on the upper right part of the Aztec diamond.

3. Variational problem for the limit shape

To prove the convergence of generalized Young diagrams to the limit shape, we need to consider the upper boundaries of the rotated diagrams as functions $f_n$ with bounded derivative. Then we can prove convergence in a space of such functions with respect to certain distance. Our approach to the proof of convergence is very similar to the proof of Vershik–Kerov–Logan–Shepp theorem in the book by Romik [58].
We first rewrite the probability of configuration (13) as an exponent of a quadratic functional on rotated diagram boundaries. Looking for a minimum of this functional, we obtain a variational problem that we solve in lemmas 3–5.

### 3.1. Probability of configuration as an exponent of a quadratic functional

We need to rewrite the formula for the probability measure in the form that is more convenient for analysis. We do so in the following Lemma.

**Lemma 1.** Denote by \(a_{-i}, i > 0\) a ‘mirror image’ of \(a_i\):

\[
a_{-i} \equiv -a_i,
\]

Then we can rewrite the measure (19) in the form:

\[
\mu_n(\{a_i\}_{i=-n,i\neq 0}^n) = \frac{1}{Z} \prod_{i<j, i,j \neq 0} |a_i - a_j| \cdot \prod_{i=-n,i\neq 0} \exp \left[ -(2n) V_0 \left( \frac{a_i}{2n} \right) - e_n (a_i) \right],
\]

where:

\[
V_0(u) = \frac{1}{4} \left[ \left( \frac{e}{2} + u \right) \ln \left( \frac{e}{2} + u \right) + \left( \frac{e}{2} - u \right) \ln \left( \frac{e}{2} - u \right) \right],
\]

\[
e_n(u) = \frac{1}{2} \ln \left( (cn)^2 - u^2 \right) + \frac{1}{2} \ln |u| + O \left( \frac{1}{n} \right) \text{ and } Z_n \text{ does not depend on } a_i \text{ and an additional condition (20) is satisfied.}
\]

The function \(V_0(u)\) is twice continuously differentiable on \((-\frac{e}{2}, \frac{e}{4})\). \(|V_0''(u)| \leq C(1 + |u - c/2|^{-1} + |u - c/2|^{-1})\) and \(e_n\) is uniformly bounded by \(C \ln 2n\).

**Proof.** We first extract the contribution that does not depend on \(a_i\) from the expression (19):

\[
C_n = \frac{2^{-n^2+2n-nN} n!}{(2n)! (2n-2)! \ldots 2!} \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k}!}.
\]

The exponential term can be written for \(l = 1, \ldots, n\) as:

\[
\exp \left[ \ln \left( \frac{N + a_l + 2n - 1}{2} \right)! \right] \cdot \exp \left[ \ln \left( \frac{N - a_l + 2n - 1}{2} \right)! \right] \cdot \exp |a_l|,
\]

note that we keep another product \(\prod_{k} |a_k|\) outside of the exponent for future use. Use the notation (13). First we use Stirling approximation formula for factorials to write the exponential term as:

\[
2\pi \exp \left[ - \frac{cn + a_l}{2} \ln \frac{cn + a_l}{2} + \frac{1}{2} \ln \frac{cn + a_l}{2} \right] + \frac{1}{2} \ln \frac{cn - a_l}{2} + |a_l| + O \left( \frac{1}{n} \right).
\]

Collecting the terms and combining in the leading contributions \(a_l\) with \(2n\), we get:

\[
2\pi \exp \left[ cn \ln n + (2n) \cdot \left( \frac{c}{2} + \frac{a_l}{2n} \right) \ln \left( \frac{c}{2} + \frac{a_l}{2n} \right) + \left( \frac{c}{2} - \frac{a_l}{2n} \right) \ln \left( \frac{c}{2} - \frac{a_l}{2n} \right) - cn \right.
\]
\[
\left. + \frac{1}{2} \ln \left( (cn)^2 - a_l^2 \right) - \ln 2 + |a_l| + O \left( \frac{1}{n} \right) \right].
\]
Now we denote by \( V_0(u) \) (one half of) the main contribution with \( u = \frac{c}{2n} \):

\[
V_0(u) = \frac{1}{4} \left[ \left( \frac{c}{2} + u \right) \ln \left( \frac{c}{2} + u \right) + \left( \frac{c}{2} - u \right) \ln \left( \frac{c}{2} - u \right) \right],
\]

and by \( e_n \) (one half of) the remainder:

\[
e_n(u) = \frac{1}{4} \ln \left( (cn)^2 - u^2 \right) + \frac{1}{2} \ln |u| + O\left( \frac{1}{n} \right).
\]

We also get \( a_i \)-independent contribution for each \( l = 1, \ldots, n \):

\[
\tilde{C}_n = \pi \exp \left[ cn \ln n - cn + O\left( \frac{1}{n} \right) \right].
\]

Combining the exponential terms (26) with the notations (27), (28) and \( a_i \)-independent contributions (29), (23), we arrive at the expression of the form:

\[
\mu_n(\{a_i\}) = \frac{1}{Z_n} \prod_{i \neq j, j=-n}^n (a_i - a_j)^2 \cdot \prod_{i < j, j=-n}^n (a_i + a_j)^2 \cdot \prod_{i \neq j, j=-n}^n \exp \left\{ 2 \left( -2n \cdot V_0 \left( \frac{a_i}{2n} \right) - e_n(a_i) \right) \right\},
\]

where \( \frac{1}{Z_n} = \frac{C_n}{\tilde{C}_n} \).

Let us now introduce the mirror images \( a_{-i} \equiv -a_i \) and use the equality:

\[
\prod_{i < j, j=-n}^n \prod_{i = -n}^{-1} |a_i - a_j| = \prod_{i < j, j=-n}^n \prod_{i = -n}^{-1} |a_i - a_j| \cdot |a_i - a_j| = \prod_{i < j, j=-n}^n \prod_{i = -n}^{-1} |a_i - a_j|.
\]

The second factor can be expanded as:

\[
\prod_{i = -n}^{-1} \prod_{j = 1}^{n} \prod_{i \neq j, j=-n}^n |a_i - a_j| = 2^n \prod_{j=1}^{n} |a_j| \cdot \prod_{i < j} \prod_{i \neq j, j=-n}^n |a_i + a_j| = 2^n \prod_{j=1}^{n} |a_j| \prod_{i < j} \prod_{i \neq j, j=-n}^n |a_i + a_j|^2,
\]

while the first and the third terms coincide. Thus we see that:

\[
\prod_{i < j, j=-n}^n \prod_{i \neq j, j=-n}^n |a_i - a_j| = 2^n \prod_{i < j, j=-n}^n (a_i^2 - a_j^2)^2 \cdot \prod_{k=1}^n |a_k|.
\]

So we can write the measure for \( l = -n, \ldots, -1, 1, \ldots, n \) in the form (21) with:

\[
\frac{1}{Z_n} = \frac{C_n}{\tilde{C}_n}^{-n} 2^{-n}.
\]

Note that for \( V_0''(u) \leq A \left( 1 + \frac{1}{|u-c/2|} + \frac{1}{|u+c/2|} \right) \) for \( u \in [-c/2, c/2] \) and \( e_n \) is uniformly bounded by \( B \ln n \) for some constants \( A, B \), since \( |a_i| \leq cn \) for \( l = -n, \ldots, -1, 1, \ldots, n \).

Using lemma 1 we can rewrite the probability of a highest weight \( \lambda \) in the limit \( N, n \rightarrow \infty, N \sim n \) and of the corresponding diagram as the exponent of the functional of the rotated diagram’s boundary \( f_n(x) \):

\[
\mu_n(\lambda) = \mu_n, N (\{a_i\}_{i=1}^n) = e^{-2n^2 f_n(x)} + O(n \ln n).
\]
In order to write down the functional $J[f]$ explicitly, we need to recall that the density $\rho_n(x)$ is closely related to the derivative $f'_n(x)$. We can interpret $\rho_n$ as a density of middle points of small intervals of length $\frac{1}{n}$ (or $\frac{1}{2n}$ for $i = n$) where $f'_n(x) = -1$. Note also, that in order to obtain the expression (21) we had to continue the function $\rho_n(x)$ to negative values of $x$ so that it becomes an even function. This corresponds to the continuation of the boundary $f_n$ such that $f'_n$ is even and $f_n$ is continuous at $x = 0$. The continuation is shown in the figure 6.

**Lemma 2.** Consider a dominant integral weight $\lambda$ with the coordinates $\{a_i\}$, defined in section 2. Then the probability measure $\mu_n(\lambda)$ can be approximated by the exponent of the quadratic functional $J$ of $f_n$:

$$
\mu_n(\lambda) = e^{-(2n)^2J[f_n]+O(n\ln n)}
= \exp \left( -(2n)^2 \left[ \frac{1}{3} \int_{-c/2}^{c/2} \int_{-c/2}^{c/2} f'_n(x) f'_n(y) \ln |x - y|^{-1} \, dx \, dy + C \right] + O(n\ln n) \right),
$$

where $f_n$ is an upper boundary of the diagram for $\lambda$, rotated and scaled as described in section 2, and the constant $C$ is given by the formula:

$$
C = -\frac{1}{32}c^2 \ln c + \frac{(c - 2)^2}{16} \ln (c - 2) + \frac{c - 1}{4} \ln 2 - \frac{3}{64}(c - 4)^2.
$$

**Proof.** We start with the functional and demonstrate that integrals are the approximations to the sums in the probability measure, written in the form (21).

First, we need to show that an integral in the functional can be written as a sum of two integrals, one of which contains the potential $V_0(x)$:

$$
\int_{-c/2}^{c/2} \int_{-c/2}^{c/2} f'_n(x) f'_n(y) \ln |x - y|^{-1} \, dx \, dy + \frac{c^2}{16} \ln c - \frac{3c^2}{32} + \frac{c}{2}
= \int_{-c/2}^{c/2} \int_{-c/2}^{c/2} \frac{1}{16} (1 - f'_n(x)) \cdot \frac{1}{4} (1 - f'_n(y)) \ln |x - y|^{-1} \, dx \, dy
$$

\[13\]
\[ + 2 \int_{-c/2}^{c/2} \frac{1}{4} (1 - f_n'(x)) V_0(x) \, dx. \] (38)

It is easily done by expanding the brackets in the double integral on the right hand side and using the equalities:

\[ \int_{-c/2}^{c/2} \ln|x-y|^{-1} \, dy = c - \left[ \left( \frac{c}{2} + x \right) \ln \left( \frac{c}{2} + x \right) + \left( \frac{c}{2} - x \right) \ln \left( \frac{c}{2} - x \right) \right] = c - 4V_0(x), \] (39)

\[ \int_{-c/2}^{c/2} \frac{\ln|x-y|^{-1}}{x} \, dx \, dy = \frac{3}{2} c^2 - c^2 \ln c, \] (40)

and also the equation:

\[ \int_{-c/2}^{c/2} f_n'(x) \, dx = \int_{-c/2}^{c/2} (1 - 4\rho_n(x)) \, dx = c - 4. \] (41)

Then we approximate the integrals on right-hand side of the equation (38) by the sums and demonstrate that these sums are the same as in (21). To do so we split the intervals \((0, \frac{c}{2})\) and \((-\frac{c}{2}, 0)\) into small intervals of length \(\frac{1}{2n}\) (for odd \(a_0\) the first interval should be of length \(\frac{1}{2n}\)):

\[ \frac{1}{16} \sum_{k=-[cn/2]}^{[cn/2]} \sum_{l=-[cn/2]}^{[cn/2]} \int_{k/n}^{(k+1)/n} \int_{l/n}^{(l+1)/n} (1 - f_n'(x))(1 - f_n'(y)) \ln|x-y|^{-1} \, dx \, dy \]
\[ + \frac{1}{2} \sum_{k=-[cn/2]}^{[cn/2]} \int_{k/n}^{(k+1)/n} (1 - f_n'(x)) V_0(x) \, dx. \] (42)

On each interval \((\frac{k}{n}, \frac{k+1}{n})\) we have \(f_n'(x) = \pm 1\). Recall that the coordinates of midpoints of intervals, where \(f_n'(x) = -1\), are \(\frac{a}{2n}\).

Approximating the double integral using the first order of Taylor expansion on each small interval with \(k \neq l\), we obtain the contribution:

\[ \frac{1}{4n^2} \sum_{i=-[n/2]}^{n} \sum_{j=-[n/2]}^{n} \ln \left| \frac{a_i}{2n} - \frac{a_j}{2n} \right|^{-1} = \frac{1}{4n^2} \sum_{i=-[n/2]}^{n} \sum_{j=-[n/2]}^{n} \ln |a_i - a_j|^{-1} + \ln |2n|. \] (43)

To estimate the correction to the sum we need to consider the difference of the integral over one of the squares, where \(f_n'(x) = -1\) and the corresponding term in the sum:

\[ E_{ij} = \frac{1}{16} \int_{k/n}^{(k+1)/n} \int_{l/n}^{(l+1)/n} (1 - f_n'(x))(1 - f_n'(y)) \]
\[ \times \ln|x-y|^{-1} \, dx \, dy = -\frac{1}{4n^2} \ln \left| \frac{a_i}{2n} - \frac{a_j}{2n} \right|^{-1}. \] (44)

Denote \(\frac{a}{2n}, \frac{a}{2n}\) by \(x_i, y_i\) correspondingly, then the difference is rewritten as:

\[ E_{ij} = \frac{1}{4} \int_{x_i-1/2n}^{x_i+1/2n} \int_{y_j-1/2n}^{y_j+1/2n} \left( \ln|x-y|^{-1} - \ln|x_i-y_j|^{-1} \right) \, dx \, dy. \] (45)

Changing the variables to \(\bar{x} = x - x_i, \bar{y} = y - y_j\), we arrive at the integral:

\[ E_{ij} = \frac{1}{4} \int_{-1/2n}^{1/2n} \int_{-1/2n}^{1/2n} \ln \left( 1 + \frac{\bar{x} - \bar{y}}{x_i - y_j} \right)^{-1} \, d\bar{x} \, d\bar{y}. \] (46)
Since $|\ln(1 + t)| < 2|t|$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$, this integral is estimated as $|E_{ij}| < \frac{1}{4\pi^2} \frac{1}{2n(3n - 2)}$. Thus in the sum $\sum_{i} \sum_{j} E_{ij}$ we have at most $2n$ terms with $|i - j| = 1$ that are estimated by $\frac{1}{4\pi^2} \frac{1}{4}$, at most $2n$ terms with $|i - j| = 2$ that are estimated by $\frac{1}{4\pi^2} \frac{1}{8}$, then we have $2n$ terms that are estimated by $\frac{1}{4\pi^2} \frac{1}{16}$ and so on. The total sum is of the order $(2n)^2 \sum_{i,j} E_{ij} = \mathcal{O}(n \ln n)$.

For contributions with $k = l$ we use the integral:

$$\frac{1}{4} \int_{k/n}^{(k+1)/n} \int_{k/n}^{(k+1)/n} \ln|x - y|^{-1} \, dx \, dy = \frac{1}{n^2} \ln n + \frac{3}{2n^2},$$

(47)

so the total contribution of $n$ such diagonal terms with a factor $(2n)^2$ in the exponent of the expression (36) is of order $\mathcal{O}(n \ln n)$.

We approximate the integral with $V_0(x)$ in a similar way and obtain:

$$\frac{1}{2} \sum_{k = [cn/2]}^{\lfloor cn/2 \rfloor} \int_{k/n}^{(k+1)/n} (1 - f'_n(x)) V_0(x) \, dx = \sum_{i = -2n, j \neq 0}^{2n} \frac{1}{n} V_0 \left( \frac{a_2}{2n} \right) + \mathcal{O} \left( \frac{1}{n^2} \right).$$

(48)

We substitute expressions (48), (47), (43) into (36) and see that we get a leading $a_i$-dependent part of the expression (21) correctly. But we need also to compute the $a_i$-independent part and compare it to the asymptotic behavior of $Z_n$ in $n$. We use the formulas (29), (23), (34) to derive the asymptotic behavior of $Z_n$:

$$\frac{1}{Z_n} = \exp \left( -n \ln 2 - n \ln C_n + \ln C_n \right) = \exp \left( -n \ln 2 - n \ln \pi - cn^2 \ln n + cn^2 + \ln C_n \right).$$

(49)

For $\ln C_n$ we have:

$$\ln C_n = \left[ -cn^2 + \mathcal{O}(n) \right] \ln 2 + \ln(a) - \sum_{k=1}^{n} \ln((2k)!) + \sum_{k=1}^{n} \ln[(N + 2k - 2)!!].$$

(50)

To estimate factorials it is convenient to combine them in the following way:

$$\ln \left[ \prod_{k=0}^{n-1} \frac{(N + 2k)!}{(2n - 2k)!} \right] = \sum_{k=0}^{n-1} \sum_{j=1}^{N-2n+4k-1} \ln(2n - 2k + j)$$

$$= \sum_{k=0}^{n-1} \sum_{j=1}^{N-2n+4k-1} \ln 2n + \sum_{k=0}^{n-1} \sum_{j=1}^{N-2n+4k-1} \ln \left( 1 - \frac{k}{n} + \frac{j}{2n} \right).$$

(51)

The first double sum gives us $\sum_{k=0}^{n-1} \sum_{j=1}^{N-2n+4k-1} \ln 2n = (c - 2)n^2 \ln 2n - 3n \ln 2 + \mathcal{O}(n)$, the second sum is a Riemann sum for an integral:

$$n^2 \int_0^1 dx \int_0^{(c-4)n+4x} dy \ln \left( 1 - x + \frac{y}{2} \right)$$

$$= -n^2 \left[ \frac{1}{4} e^2 \ln(c - 2) - \frac{1}{4} e^2 \ln c + \frac{3}{2} c - c \ln(c - 2) + c \ln 2 + \ln(c - 2) - 3 - \ln 2 \right].$$

(52)

Combining these results we can write down leading contributions to $\ln Z_n$ in $n$. We preserve the terms of the order $n^2 \ln n$ from expressions (51), (49) and of order $n^2$ from formulas (49), (50), (52):

$$\ln Z_n = -cn^2 \ln n + cn^2 + (c - 2)n^2 \ln n + (c - 2)n^2 \ln 2 - cn^2 \ln 2$$
Let us now combine the contributions of the order \( n^2 \ln n \) from \( \ln Z_n \) and from the equation (43). Substituting the equation (43) into (36), we get \( \frac{1}{2} (2n)^2 \ln |2n| = 2n^2 \ln n + 2n^2 \ln 2 \). Thus we see that terms of the order \( n^2 \ln n \) are cancelled in the equation (36).

Now combining the contributions of order \( n^2 \) from (38), (53), (43), we obtain the expression for a constant \( C \):

\[
C = \frac{1}{32} c^2 \ln c - \frac{3}{64} c^2 + \frac{1}{4} c - \frac{1}{2} \ln 2 + \frac{1}{2} \ln 2 - \frac{c}{4} - \frac{1}{16} c^2 \ln c + \frac{1}{16} c^2 \ln (c - 2) - \frac{1}{4} (c - 1) \ln (c - 2) + \frac{c}{4} \ln 2 - \frac{3}{4} + \frac{3}{8} c \\
= - \frac{1}{32} c^2 \ln c + \frac{(c - 2)^2}{16} \ln (c - 2) + \frac{c - 1}{4} \ln 2 - \frac{3}{64} (c - 4)^2.
\] (54)

Substituting the equations (48), (43) into (38), and combining it with the expression for \( C \), we see that equation (36) indeed reproduces the probability measure (21) in the leading order in \( n \).

3.2. Minimizer of the functional

The functional \( J[f] \) is clearly quadratic. Rewriting the functional \( J[f] \) in terms of densities and searching for its minimum, we arrive at the following variational problem. The particle density \( \rho(x) \) that is related to the limit shape \( f(x) \) by \( f'(x) = 1 - 4 \rho(x) \) (15), is the minimizer of the functional:

\[
\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho(x) \rho(y) \ln |x - y|^{-1} \, dx \, dy + \frac{1}{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho(x) \left[ \left( \frac{c}{2} + x \right) \ln \left( \frac{c}{2} + x \right) + \left( \frac{c}{2} - x \right) \ln \left( \frac{c}{2} - x \right) \right] \, dx.
\] (55)

We first recall necessary and sufficient conditions for the minimizer (proposition 1). Our functional is strictly convex (see [21], theorem 6.27), therefore the minimizer is unique.

**Proposition 1 ([21], theorem 6.132).** Suppose \( \rho(x) \) is a continuous function on \([-c/2, c/2]\). Then \( \rho(x) \) is the minimizer of (55) if and only if there exists a constant \( \ell \in \mathbb{R} \) such that:

(I) \[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln |x - y|^{-1} \rho(y) \, dy + V_0(x) \geq \ell \quad \text{for any} \quad x \in \left[ -\frac{c}{2}, \frac{c}{2} \right],
\]

(II) \[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln |x - y|^{-1} \rho(y) \, dy + V_0(x) = \ell \quad \text{for} \quad x \in \text{supp} \rho.
\]

We construct the minimizer by an explicit integral formula, which is obtained as a solution of Riemann–Hilbert problem, as described in the book by Deift [21]. This solution is presented in lemma 3 below, where we also check the necessary and sufficient conditions.
Lemma 3. For $c \geq 4$

$$\rho(x) = \frac{1}{\pi^2} \Re \left[ \sqrt{2c - 4 - x^2} \int_{-\sqrt{2c - 4}}^{\sqrt{2c - 4}} \frac{1}{\sqrt{2c - 4 - x^2}} \left( \ln \left( \frac{x + s}{\sqrt{2c - 4 - x^2}} \right) - \ln \left( \frac{x - s}{\sqrt{2c - 4 - x^2}} \right) \right) ds \right], \quad (56)$$

is the minimizer of the functional (55).

Proof. We want to find a minimizer of (55). We assume that the minimizer is a continuous density $\rho(x)$ and $\rho(x) = \rho(-x)$ due to condition (20). Thus we can assume that $\text{supp } \rho = [-a, a]$ with $a < \frac{1}{2}$. Additionally, since total number of particles is $2n$, we have $\int_{-c}^c \rho(x) = 1$, so the function $\rho$ should satisfy the condition:

$$\int_{-a}^a \rho(x) \, dx = 1. \quad (57)$$

Then we have a variational problem with the convex functional $\mathcal{E}[\rho]$:

$$\mathcal{E}[\rho] = \int_{-a}^a \int_{-a}^a \left( -\frac{1}{2} \ln |x - y| + \frac{1}{2} V_0(x) + \frac{1}{2} V_0(y) \right) \rho(x) \rho(y) \, dx \, dy. \quad (58)$$

Taking variation by $\rho$, we obtain Euler–Lagrange equation for $x \in \text{supp } \rho$:

$$\int_{-a}^a \ln |x - y|^{-1} \rho(y) \, dy + V_0(x) = \text{const.} \quad (59)$$

This equation has a natural electrostatic interpretation, since $\ln |x - y|^{-1}$ is a two-dimensional electrostatic potential at the point $x$ of a unit charge at $y$. Taking the derivative with respect to $x$ of the equation (59), we get the equilibrium condition for the charge distribution with density $\rho$ in an external field with the potential $V_0$:

$$-\int_{-a}^a \frac{\rho(y)}{y - x} \, dy + V_0'(x) = 0. \quad (60)$$

The charge distribution $\rho(x)$ creates a potential $\varphi(z)$ on a complex plane with a cut from $-a$ to $a$ along the real axis $\mathbb{C} \setminus [-a,a]$. The potential $\varphi(z)$ satisfies Laplace equation and is a harmonic function. Its gradient is called field strength and thus its components along the real and the imaginary axes are analytic functions on $\mathbb{C} \setminus [-a,a]$. Denote by $G(z)$ the field strength component along the imaginary axis. Then:

$$G(z) = -i \int_{-a}^a \frac{\rho(y)}{y - z} \, dy, \quad (61)$$

is a Hilbert transform of $\rho(x)$. We know that $G(z)$ is analytic on $\mathbb{C} \setminus [-a,a]$ and its limit values are:

$$G_\pm(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi(x + i \varepsilon)} \int \frac{\rho(y) \, dy}{y - (x + i \varepsilon)} = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int \frac{y - x \pm i \varepsilon}{(y - x)^2 + \varepsilon^2} \rho(y) \, dy$$

$$= -i \text{p.v.} \int \frac{\rho(y) \, dy}{y - x} \pm \pi \rho(x), \quad (62)$$

where we have used $\frac{\varepsilon}{\pi(x + i \varepsilon)} \to \delta(x)$. Thus we arrive at:

$$G_\pm(x) = \pm \pi \rho(x) + iV_0'(x),$$

so on the support of $\rho(x)$ we have:

$$G_+(x) + G_-(x) = 2iV_0'(x), \quad x \in [-a,a], \quad (64)$$
and outside of \([-a,a]\) the following conditions appear:
\[
G_+(x) - G_-(x) = 0, \quad x \not\in [-a,a], \\
G(z) \to 0, \; z \to \infty.
\] (65)

This is Riemann–Hilbert problem for \(G(z)\), but the equation (64) is in a non-standard form with the sum instead of a difference. We need to redefine \(G\) in such a way as to obtain a standard problem that can be solved by the Plemelj formula [21]:
\[
\tilde{G}(z) = \frac{G(z)}{\sqrt{z^2 - a^2}}.
\] (66)

Then we get:
\[
\tilde{G}_+(x) - \tilde{G}_-(x) = \frac{G_+(z)}{\sqrt{x^2 - a^2}}_+ - \frac{G_-(z)}{\sqrt{x^2 - a^2}}_- = \frac{G_+(z) + G_-(z)}{\sqrt{x^2 - a^2}}_+ = \frac{2V_0'(s)}{\sqrt{x^2 - a^2}}_+, \quad \text{where the branch of the square root changes the sign crossing the real line:}
\]
\[
\left(\sqrt{x^2 - a^2}\right)_+ = -\left(\sqrt{x^2 - a^2}\right)_-, \quad x \in [-a,a].
\] (68)

The condition (65) is preserved for \(\tilde{G}\):
\[
\tilde{G}_+(z) - \tilde{G}_-(z) = 0, \quad z \not\in [-a,a], \\
\tilde{G}(z) \to 0, \; z \to \infty.
\] (69)

Then \(\tilde{G}(z)\) is a solution of the standard Riemann-Hilbert problem and is given by the Plemelj formula:
\[
\tilde{G}(z) = \frac{1}{2\pi i} \int_{-a}^{a} \frac{2iV_0'(s)ds}{\left(\sqrt{s^2 - a^2}\right)_+(s-z)},
\] (70)

\[
G(z) = \frac{\sqrt{z^2 - a^2}}{\pi} \int_{-a}^{a} \frac{V_0'(s)ds}{\left(\sqrt{s^2 - a^2}\right)_+(s-z)}. \quad \text{(71)}
\]

To find the support of \(\rho\) we need to consider the asymptotics of \(G(z)\) as \(z \to \infty\). We expand the above expression into series:
\[
G(z) = \frac{z + \ldots}{\pi} \left(1 + \int_{-a}^{a} \frac{V_0'(s)}{\left(\sqrt{s^2 - a^2}\right)_+} \left(1 + \frac{s}{z} + \ldots\right) ds\right). \quad \text{(72)}
\]

Consider the first term in the series, for \(G(z) \to 0, z \to \infty\) we need to have:
\[
\int_{-a}^{a} \frac{V_0'(s)}{\left(\sqrt{s^2 - a^2}\right)_+} ds = 0. \quad \text{(73)}
\]
This condition is automatically satisfied, since \( V_0(x) \) is an even function and \( V'_0(s) \) is an odd function. At the same time:

\[
G(z) = -i \int \frac{\rho(y)dy}{y-z} \simeq \frac{i}{z} \int \rho(y)dy + \mathcal{O}\left(\frac{1}{z^3}\right),
\]

and comparing it to the second term in the series (72) we arrive at:

\[
-\frac{1}{\pi} \int \frac{V'_0(s)s}{(\sqrt{s^2-a^2})^+} \frac{ds}{z^2} = \frac{i}{z}.
\]

Taking the derivative of the equation (27) and substituting to the above equation, we get:

\[
\frac{1}{4} \int_{-a}^{a} \frac{s}{\sqrt{s^2-a^2}} \ln \left| \frac{s + c/2}{s - c/2} \right| ds = i.
\]

By taking a derivative we can check that:

\[
\int_{-a}^{a} \frac{s}{\sqrt{s^2-a^2}} \ln \left| \frac{s + c/2}{s - c/2} \right| ds = \frac{1}{2} \left( 2\sqrt{c^2-a^2} - \sqrt{c^3-4a^2} \right) \log(c-2s)
+ \left( \sqrt{c^2-4a^2} - 2\sqrt{s^2-a^2} \right) \log(c+2s)
- \sqrt{c^2-4a^2} \log \left( \sqrt{c^2-4a^2}\sqrt{s^2-a^2} - 2a^2 - cs \right)
+ \sqrt{c^2-4a^2} \log \left( \sqrt{c^2-4a^2}\sqrt{s^2-a^2} - 2a^2 + cs \right)
- 2c \log \left( \sqrt{s^2-a^2} + s \right) + \text{const.}
\]

Substituting the integration limits we obtain the equation:

\[
\frac{c}{4} \left[ 1 - \sqrt{1 - \left( \frac{2a}{c} \right)^2} \right] = 1,
\]

which can be solved for \( c \geq 4 \) and we get:

\[
a = \sqrt{2c} - 4.
\]

We see that indeed \( a < \frac{c}{2} \) and the solution \( \rho \) of the variational problem (55) is given by the formula:

\[
\rho(x) = \frac{1}{\pi} \Re[G_\rho(x)] = \frac{1}{\pi^2} \Re \left[ \sqrt{x^2-2c+4} \int_{-\sqrt{x^2-2c+4}}^{\sqrt{x^2-2c+4}} \frac{1}{2} (\ln(z+x) - \ln(z-x)) \right].
\]

By choosing the proper branch of the square roots we obtain formula (56).

We constructed \( \rho(x) \) in such a way that that minimization condition (II) of the proposition 1 is satisfied. Using the condition (II) for we rewrite the condition (I) for \( x > a \) as:

\[
\int_{a}^{x} (\rho(y) \ln |x-y| - V_0(y))dy \leq 0.
\]
Suppose to describe that:

an obvious restriction basically the same proof). Then for

\[ x < |ρ| \]

no longer the case and we restate the criterion for the minimizer in the following way (with the minimizer only because the condition

\[ \begin{aligned}
\int_{c} \frac{V'_{0}(s)}{\sqrt{s^{2} - a^{2}}} (s - z) \, ds = \\
\int_{c} \frac{s^{2m-1}}{\sqrt{s^{2} - a^{2}}} (s - z) \, ds = \\
i^{2m-1} - i \sqrt{z^{2} - a^{2}} \times \\
\left( z^{2m-2} + \sum_{j=1}^{m-1} z^{2m-2-2j} a^{2j} \prod_{l=1}^{j} \frac{2l - 1}{2l} \right). \\
\end{aligned} \]

Combining all the terms in the series, we get:

\[ G(z) = iV'_{0}(z) - i \sqrt{z^{2} - a^{2}} h(z), \]

where \( h(z) \) is an analytic function, which has the series with strictly positive coefficients.

Then for \( |x| < a \) we have \( \rho(x) = \Re \{G_{\pm}(x)\} = -i \left( \sqrt{x^{2} - a^{2}} \right) h(x) \). For \( x > a \) we substitute the equations (61), (85) to the formula (81) and arrive at the inequality:

\[ - \int_{a}^{x} \sqrt{y^{2} - a^{2}} h(y) \, dy \leq 0, \]

which holds, since the function under the integral is positive.

For \( c < 4 \) we can no longer solve the variational problem with a smooth function \( \rho \) such that \( \text{supp} \rho \subset [-\xi, \xi] \). The potential \( V_{0}(x) \) becomes weaker when \( c \) tends to 4, and when \( c = 4 \) we have a phase transition. In the latter case particles are not confined strictly inside the interval \([-2, 2]\) anymore, instead we have a constant density \( \rho(x) \equiv 1/4 \) on the whole interval. We have an obvious restriction \( \rho(x) \leq 1/2 \), therefore for \( c < 4 \) it is reasonable to expect:

\[ \rho(x) = \frac{1}{2} - \rho_{1}(x), \]

where \( \text{supp} \rho_{1} \subset [-\xi, \xi] \) (see figure 8). For \( c \geq 4 \) it is possible to use proposition 1 to describe the minimizer only because the condition \( \rho(x) < 1/2 \) is satisfied in this case. For \( c < 4 \) it is no longer the case and we restate the criterion for the minimizer in the following way (with basically the same proof).

**Proposition 2.** Suppose \( \rho(x) = \frac{1}{2} - \rho_{1}(x) \) is a continuous function on \([-\xi, \xi] \), and \( \frac{1}{2} > \rho_{1}(x) \geq 0 \). Then \( \rho(x) \) is the minimizer of (55) if and only if there exists a constant \( \ell \in \mathbb{R} \) such that:

\[ \]
and solve (91) The density $3$ for $3$ is the same as in formula (92) for lemma $\left[\frac{\rho}{\text{for}}\right]$ the normalization condition (93) to the variational problem (94).

In the next lemma we show that $\rho_1(x)$ can be computed the same way as in lemma 3 and solve the variational problem for $c \in [2, 4]$.

**Lemma 4.** For $2 \leq c \leq 4$

$$\rho(x) = \frac{1}{2} - \frac{1}{\pi^2} \int \left[ \sqrt{x^2 - 2c} + 4 \right] \frac{1}{\sqrt{x^2 - 2c + 4}} \left[ \frac{1}{2} \left( \ln \left( \frac{x + s}{s - x} \right) - \ln \left( \frac{x - s}{s - x} \right) \right) - dx \right].$$

(88)

**Proof.** Denote by $\rho_0(x) \equiv \frac{1}{4}$ a constant solution to the equation (69). If we look for a solution to the variational problem (55) with the normalization condition (57) in the form (87) so that:

$$\int_{-c/2}^{c/2} \rho(y)dy = \int_{-c/2}^{c/2} (2\rho_0(y) - \rho_1(y))dy = -2V'_0(x) + \int_{-c/2}^{c/2} \rho_1(y)dy = -V'_0(x),$$

(89)

then the function $\rho_1(x)$ should also be a solution of (69), but with a different normalization condition:

$$\int_{-c/2}^{c/2} \rho_1(x)dx = -\int_{-c/2}^{c/2} \rho_0(x)dx + 2 \int_{-c/2}^{c/2} \rho_0(x)dx = \frac{c - 2}{2}.$$  

(90)

The integral representation of $\rho_1(x)$ is obtained in the same way as in lemma 3, but equation (78) becomes:

$$\frac{c}{4} \left[ 1 - \sqrt{1 - \left( \frac{2a}{c} \right)^2} \right] = \frac{c - 2}{2},$$

(91)

and we again get $a = \sqrt{2c - 4}. \] We see that $\rho_1(x)$ is given by the formula (56) and satisfies the normalization condition (90). Thus we have derived the expression (88) for $\rho(x)$ for $c \in [2, 4]$. The check of the minimization conditions (I) and (II) of proposition 2 is the same as in lemma 3 $\square$.

The integral in the formula (56) can be calculated and we can write down the expression for $\rho(x)$ in terms of inverse trigonometric functions.

**Lemma 5.** The density $\rho(x)$ can be written explicitly as:

$$\rho(x) = \left\{ \begin{array}{ll} \theta(\sqrt{2c - 4} - |x|) & \frac{\arctan \left( \frac{c(x+4)-8}{(c-4)\sqrt{2c-4-x^2}} \right) + \arctan \left( \frac{c(x-4)-8}{(c-4)\sqrt{2c-4-x^2}} \right)}{4\pi} \bigg| c \geq 4, \end{array} \right.$$  

$$\left\{ \begin{array}{ll} \frac{1}{2} \theta(\sqrt{2c - 4} - |x|) & \frac{\arctan \left( \frac{-c(x-4)-8}{(4-c)\sqrt{2c-4-x^2}} \right) + \arctan \left( \frac{c(x+4)-8}{(4-c)\sqrt{2c-4-x^2}} \right)}{4\pi} \bigg| c \in [2, 4], \end{array} \right.$$  

(92)

where $\theta(\sqrt{2c - 4} - |x|)$ is the Heaviside step function. By the use of trigonometric identities formula (92) can be rewritten as (1).
Proof. We need to compute the integral in (56), we can combine logarithms the same way as we did in equation (76):

\[ \frac{1}{\pi} \int_{-\sqrt{2c-4}}^{\sqrt{2c-4}} \frac{\ln \left( \frac{x}{s} + 1 \right) - \ln \left( \frac{x}{s} - 1 \right)}{\sqrt{2c-4 - s^2(s-x)}} \, ds = \frac{1}{\pi} \int_{-\sqrt{2c-4}}^{\sqrt{2c-4}} \frac{1}{\sqrt{2c-4 - s^2(s-x)}} \cdot \frac{1}{\pi} \ln \left| s - \frac{x}{s} \right| \, ds. \]

(93)

To calculate this integral notice that the function \( \frac{1}{\pi} \ln \left| \frac{x}{s} \right| \) is the Hilbert transform of the indicator function \( 1_{[-c/2, c/2]} \). Then we can use the following well-known relation (see, for example, [25]):

\[ \int_{-\infty}^{\infty} f(s) g(s) \, ds = -\int_{-\infty}^{\infty} \hat{f}(s) g(s) \, ds, \]

(94)

where \( \hat{f} \) is a Hilbert transform of \( f \) and \( f \in L^p(\mathbb{R}) \), \( g \in L^q(\mathbb{R}) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). In particular, substituting the indicator function \( g = 1_{[-c/2, c/2]} \) we obtain:

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} f(s) \ln \left| \frac{s - c/2}{s + c/2} \right| \, ds = -\int_{-c/2}^{c/2} \hat{f}(s) \, ds. \]

(95)

Thus, we need to compute the Hilbert transform for the function \( f(y) = \frac{1}{\pi} \frac{1}{\sqrt{\sqrt{2c-4 + (y-a)}}} \) for \( y \in [-\sqrt{2c-4}, \sqrt{2c-4}] = [-a, a] \), and \( f(y) = 0 \) for \( y \notin [-a, a] \) and then integrate it from \(-c/2\) to \( c/2\). Integral in the Hilbert transform is computed explicitly by the change of variables:

\[ y = a^2 - t^2, \quad \frac{dy}{\sqrt{a^2 - t^2}} = \frac{2a \, dr}{a^2 + t^2}, \]

(96)

and we obtain:

\[ \tilde{f}(z) = \frac{1}{\pi} \int_{-a}^{a} \frac{dx}{\sqrt{a^2 - s^2(s-x)(s-z)}} = \frac{1}{\pi} \left( \frac{1}{\sqrt{a^2 - a^2}} - \frac{1}{\sqrt{a^2 - z^2}} \right). \]

(97)

Now we need to compute the integral:

\[ \rho(x) = \frac{1}{\pi} \Re \left[ \sqrt{a^2 - x^2} \int_{-c/2}^{c/2} \frac{1}{\pi} \left( \frac{1}{(x-z)\sqrt{z^2 - a^2}} - \frac{1}{(x-z)\sqrt{z^2 - a^2}} \right) \, dz \right]. \]

(98)

Here again we can use the substitution (96) or find the indefinite integral in the [26] and obtain:

\[ \rho(x) = -\frac{1}{4\pi} \left[ 3 \left( \log \left( \sqrt{(c-4)^2\sqrt{x^2 - 2c + 4} - c(x-4)} - 8 \right) + \log \left( \sqrt{(c-4)^2\sqrt{x^2 - 2c + 4} + c(x+4)} - 8 \right) \right) - \pi \right]. \]

(99)

This answer is easily rewritten in terms of the inverse trigonometric functions for \( c \geq 4, |x| \leq \sqrt{2c-4} \) as:

\[ \rho(x) = \frac{1}{4\pi} \left[ \arctan \left( \frac{-c(x-4) - 8}{(c-4)\sqrt{2c-4 - x^2}} \right) + \arctan \left( \frac{c(x+4) - 8}{(c-4)\sqrt{2c-4 - x^2}} \right) \right]. \]

(100)
Using lemma 4, choosing the positive square root of $\sqrt{(c-4)^2}$ and taking into account the values of the imaginary part of logarithms in formula (99) for $x^2 > 2c-4$, we obtain the desired formula (92).

Using the identities $\arctan x + \arctan y = \arctan \frac{x+y}{1+xy}$ and $\cos \frac{\theta}{2} = \pm \sqrt{\frac{1}{2} + \frac{1}{\sqrt{1 + \tan^2 \theta}}}$ and taking care of the signs we arrive at the final formula (1).

Thus we have solved the variational problem for the limit shape. The graphs of the density are presented in figure 7. In the next section we prove the convergence of the generalized Young diagrams to the limit shape.

4. Convergence of the probability measure

In this section we use the functional and the limit shape to introduce a pseudo-distance on the space of functions with bounded derivative. Then we estimate the probability of weights with a given deviation according to this distance and use this estimate to show that probability of deviation goes to zero as $n$ goes to infinity. We use the fact that quadratic part of the functional is the same as in the case of Vershik–Kerov–Logan–Shepp limit shape to conclude that convergence with respect to the pseudo-distance entails the convergence in the supremum norm.

We’ve proven that the probability of a weight is given by a quadratic functional of a rotated diagram boundary $f_n$ (see lemma 2). We use the notation that is very similar to Dan Romik’s book [58], the functional is denoted by $J[f_n]$, its quadratic in the derivative of $f_n$ part is denoted by $Q[f_n]$:

$$J[f_n] = Q[f_n] + C, \quad Q[f_n] = \frac{1}{2} \int_{-c/2}^{c/2} \int_{-c/2}^{c/2} \frac{1}{16} f_n'(x)f_n'(y) \ln |x-y|^{-1} \, dx \, dy. \quad (101)$$

Since our definition of $Q$ differs from a definition in the book [58] only by a factor $\frac{1}{16}$, we can use the proposition 1.15 there and see that $Q$ is positive-definite on compactly-supported Lipschitz functions.
Then for a compactly supported Lipschitz function \( f: \mathbb{R} \to [0, \infty) \) the quadratic part \( Q \) of the functional \( J \) is used to introduce a norm:
\[
\|f\|_Q = Q[f]^{1/2}.
\]
(102)

Consider a space of 1-Lipschitz functions \( f_1, f_2 \), such that the derivative \( f'_1(x) = 1 \) for \(|x| > \frac{c}{2} \). Then the difference \( f_1 - f_2 \) is a compactly supported Lipschitz function and we can use its norm to introduce a metric \( d_Q \):
\[
d_Q(f_1, f_2) = \|f_1 - f_2\|_Q.
\]
(103)

We can use lemma 1.21 in [58] to obtain an estimate on the supremum norm for a Lipschitz function \( f \) with a compact support:
\[
\|f\|_\infty = \sup_x |f(x)| \leq C_1 Q[f]^{1/4},
\]
(104)

where \( C_1 \) is some constant.

**Lemma 6.** The value of the functional \( J[f] \) on the limit shape \( f \) is non-negative:
\[
J[f] = \frac{1}{2} \int_{-c/2}^{c/2} \int_{-c/2}^{c/2} \frac{1}{16} f'(x)f'(y) \ln|x-y|^{-1} \, dx \, dy + C \geq 0,
\]
(105)

where \( C \) is given by the equation (37).

**Proof.** We prove this lemma by constructing a sequence of weights \( \{\lambda_n\} \) with the corresponding diagram boundaries \( f_n \) such that \( \{J[f_n]\} \) converges to \( J[f] \) and \( \mu_n(\lambda_n) \) converges to \( e^{-(2n)^2 J[f]} \). Since \( \mu_n(\lambda_n) \leq 1 \) we will see that \( J[f] \geq 0 \).

For \( n > 0 \) take a weight \( \lambda_n \) such that its diagram has \( n \) rows and the diagram boundary \( f_n(x) \leq f(x) \) and \( f(x) - f_n(x) \leq \frac{2c}{n} \), i.e. take the diagram with the longest possible rows that do not cross the limit shape.

Denote by \( \delta f_n \) a difference of \( f_n \) and \( f \):
\[
\delta f_n(x) = f_n(x) - f(x).
\]
(106)

The function \( \delta f_n(x) \) is a Lipschitz with a compact support. Then from lemma 2 we can write the probability \( \mu_n(\lambda_n) = \exp \left( -(2n)^2 J[f_n] + \mathcal{O}(n \ln n) \right) \) as:
\[
\mu_n(\delta f_n) = \exp \left( -(2n)^2 \left( J[f] + Q[\delta f_n] + \frac{1}{16} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} f'(x) \ln|x-y|^{-1} \delta f_n'(y) \, dx \, dy \right) \right) \cdot \exp \left( \mathcal{O}(n \ln n) \right). \]
(107)

Denote the last term in the exponent by \( L[\delta f_n] \):
\[
L[\delta f_n] := \frac{1}{16} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} f'(x) \ln|x-y|^{-1} \delta f_n'(y) \, dx \, dy.
\]
(108)

We need to estimate the terms \( Q[\delta f_n] \) and \( L[\delta f_n] \). First consider \( L[\delta f_n] \). Note, that both \( f(x) \) and \( f_n(x) \) satisfy the condition \( \int_{-c/2}^{c/2} f'(x) \, dx = c - 4 \) (41). Thus for \( \delta f_n(x) \) we have:
\[
\int_{-c/2}^{c/2} \delta f_n'(x) \, dx = 0.
\]
(109)
Also for \(|y| < \sqrt{2c - 4}\) we can substitute \(f'(x) = 1 - 4p(x)\) and due to proposition 1 and equations (59), (39) we see that the integral over \(x\) is a constant \(c - 4\ell\):

\[
\int_{-\frac{c}{2}}^{\frac{c}{2}} f'(x) \ln |x - y|^{-1} \, dx = c - 4\ell. \tag{110}
\]

If we subtract \(\frac{1}{16} \int_{-\frac{c}{2}}^{\frac{c}{2}} (c - 4\ell) \delta f'_n(y) \, dy = 0\) from \(L[\delta f_n]\), we get:

\[
L[\delta f_n] = \frac{1}{16} \int_{-\frac{c}{2}}^{\frac{c}{2}} \left[ \int_{-\frac{c}{2}}^{\frac{c}{2}} f'(x) \ln |x - y|^{-1} \, dx - (c - 4\ell) \right] \delta f'_n(y) \, dy. \tag{111}
\]

The expression inside the brackets is zero if \(|y| < \sqrt{2c - 4}\) and less than zero if \(y > \sqrt{2c - 4}\). Since \(f'(x) = 1\) and \(f'_n(x) = 1\) for \(|x| > \sqrt{2c - 4}\), we see that \(\delta f'_n(y) \leq 0\) for \(y > \sqrt{2c - 4}\) and greater than zero for \(y < -\sqrt{2c - 4}\) and conclude that value of \(L[\delta f_n]\) is non-negative.

Since \(\text{supp} \delta f_n \subset [-\sqrt{2c - 4}, \sqrt{2c - 4}]\), i.e. \(\text{supp} f_n \subset \text{supp} f\), the term \(L[\delta f_n]\) is exactly zero. Now we need to estimate the contribution \(Q[\delta f_n]\). To do so we need the inequality:

\[
-\int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \delta f'_n(x) \delta f'_n(y) \ln |x - y| \, dx \, dy \leq \frac{C}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\delta f_n(x) - \delta f_n(y)}{x - y} \right)^2 \, dx \, dy,
\]

for some constant \(C\), that is valid for all piecewise continuously differentiable functions with a compact support. This inequality can be obtained using the proposition 1.15 in [58], where it is shown that \(Q[f] = \frac{1}{16} \int_{-\infty}^{\infty} |x| \cdot |\hat{f}(x)|^2 \, dx\), where \(\hat{f}\) is a Fourier transform of \(f\). Then the left-hand side of the inequality is a definition of a norm on a Sobolev space \(H^2\). Right-hand side is the Slobodeckij norm on the same space and it is an equivalent one [28, 43, 44], so the inequality follows. It is also possible to prove that it is actually the equality with \(C = 1\), using the integration by parts (see e.g. [64] lemma 4).

Since \(\left|\delta f_n(x)\right| \leq \frac{2c}{n}\) for all \(x\), we can estimate the right hand side of the equation (112) by dividing the domain of integration into two parts: \(|x - y| > \frac{2c}{n}\), where we us this estimate and obtain contribution of the order \(\frac{c}{n}\) and \(|x - y| \leq \frac{2c}{n}\), where we use Lipschitz property of \(\delta f_n\) and obtain the contribution of the order \(\frac{c}{n}\). Thus we see that \(Q[\delta f_n] = O\left(\frac{\ln n}{n}\right)\) and we conclude that \(\left\{ -\frac{1}{(2\pi)^d} \ln \mu_n(\lambda) \right\}\) converges to \(J[f]\), so \(J[f] \geq 0\).

\[\square\]

**Lemma 7.** For a highest weight \(\lambda\) with the boundary of rotated Young diagram given by a function \(f_n(x)\) such that \(d_Q(f_n,f) = \varepsilon\), the probability is bounded by:

\[
\mu_n(\lambda) \leq C_2 e^{-n^2 \varepsilon^2 + O(n \ln n)}.
\]

**Proof.** Denote by \(\delta f_n\) a difference of \(f_n\) and \(f\): \(\delta f_n(x) = f_n(x) - f(x)\). The function \(\delta f_n(x)\) is a Lipschitz with a compact support. Then from lemma 2 we can write the probability \(\mu_n(\lambda) = \exp\left(\frac{-2n^2 J[f_n]}{O(n \ln n)}\right)\) in the same way as in lemma 6 as

\[
\mu_n(\delta f_n) = \exp\left(\frac{-2n^2 J[f_n]}{O(n \ln n)}\right) \cdot \exp(O(n \ln n)).
\]

By the condition of the lemma we have \(Q[\delta f_n] = \varepsilon^2\).
Similarly to lemma 6 we demonstrate that the term:

\[
L[\delta f_n] := \frac{1}{16} \int \int f'(x) \ln |x - y|^{-1} \delta f_n(y) dx dy,
\]

can be written as \( \frac{1}{16} \int \int f'(x) \ln |x - y|^{-1} dx (c - 4 \ell) \delta f_n(y) dy. \) The expression inside the brackets is zero if \( |y| < \sqrt{2c - 4} \) and less than zero if \( y > \sqrt{2c - 4} \) (greater than zero for \( y < -\sqrt{2c - 4} \)), since logarithm is a monotonic function. Since \( f'(x) = 1 \) and \( |f'_n(x)| = 1 \) for \( |x| > \sqrt{2c - 4} \), we see that \( \delta f_n(y) \leq 0 \) for \( y > \sqrt{2c - 4} \) and greater than zero for \( y < -\sqrt{2c - 4} \) and conclude that value of \( L[\delta f_n] \) is non-negative. Also if \( \text{supp} \delta f_n \subset [-\sqrt{2c - 4}, \sqrt{2c - 4}] \) this contribution is exactly zero.

From lemma 6 we have \( f[f] \geq 0 \), so we see that \( \exp \left( -2n \right) (J[f] + L[\delta f_n]) \) is non-negative.

**Lemma 8.** As \( n \to \infty \) rotated Young diagrams for highest weights in the decomposition of tensor power of the spinor representation of simple Lie algebra \( \mathfrak{so}_{2n+1} \) into irreducible representations converge in probability in the metric \( d_Q \) to the limiting shape given by the formulas (16), (15), (1). That is, for all \( \varepsilon > 0 \) we have:

\[
P( ||f_n - f||_Q > \varepsilon ) \xrightarrow{n \to \infty} 0.
\]

**Proof.** By lemma 7 the probability of each highest weight \( \lambda \) with a rotated Young diagram with boundary \( f_n \) such that \( ||f_n - f||_Q > \varepsilon \) is bounded by \( \exp \left( -n^2 \varepsilon^2 + O(n \ln n) \right) \). We also know that number of such highest weights is not greater than the total number of dominant integral weights in the reducible representation \( (V^{\otimes n})^\otimes N \). Let us estimate this number. Weight diagram of \( (V^{\otimes n})^\otimes N \) is an \( n \)-dimensional hypercube with a side \( 2N \) which contains at most \( (2N)^n \) integral weights. But this hypercube is then divided into Weyl chambers. Total number of Weyl chambers is equal to the order of Weyl group of \( B_n \), which is \( 2^n n! \). Thus we can estimate the number of integral dominant weights as being not greater than \( \frac{2^n n!}{\sqrt{2\pi n}} c^{\ln n} \). Combining the bound on number of highest weights and the bound on probability we come to the conclusion that:

\[
P( ||f_n - f||_Q > \varepsilon ) \xrightarrow{n \to \infty} 0.
\]

Now from this lemma and from (104) follows theorem 1.

We have obtained the limit shape for Young diagrams for tensor product decomposition of tensor powers of last fundamental representation of Lie algebras of series \( \mathfrak{so}_{2n+1} \). In the figure 8 we present limit shapes for \( c = 3, c = 4 \) and \( c = 6 \) as well as the most probable diagram for \( n = 20 \) and \( N = 40 \) (\( c \approx 4 \)).

5. **Proof of the Central limit theorem**

In general, Central Limit Theorems for the determinantal point processes can be proven by the use of discrete orthogonal polynomials and deducing their continuous asymptotic, as described in the book [2] and many papers, for example [12]. In our case the ensemble is a little bit different, since the measure (19) contains a Vandermonde determinant written in terms of squares of variable \( a_i \). Therefore some care is required, below we describe how to adapt the results of [12] to our case.
Introduce the variables $y_i = \frac{a_{2i}^2}{(2n)^2}$, the measure (19) takes the determinantal form:

$$\mu_{n,N}(y_1,\ldots,y_n) = \frac{1}{Z_{N,n}} \prod_{i<j} (y_i - y_j)^2 \prod_{i=1}^n \frac{(N + 2n - 1)!}{\left(\frac{N+2n-1}{2} - \frac{2n\sqrt{y_i}}{2}\right)\left(\frac{N+2n-1}{2} + \frac{2n\sqrt{y_i}}{2}\right)!},$$

(117)

where we have rewritten the weight in the binomial form and collected the terms that do not depend on $y_i$ into the normalization constant $Z_{N,n}$. Denote the weight by $W^{(n)}(y)$:

$$W^{(n)}(y) = \frac{(N + 2n - 1)!}{\left(\frac{N+2n-1}{2} - \frac{2n\sqrt{y}}{2}\right)\left(\frac{N+2n-1}{2} + \frac{2n\sqrt{y}}{2}\right)!},$$

(118)

and introduce the discrete orthonormal polynomials $p^{(n)}_m(y)$ on a quadratic lattice:

$$\sum_{l=1}^{N+2n-1} p^{(n)}_m\left(\frac{l^2}{(2n)^2}\right) p^{(n)}_l\left(\frac{l^2}{(2n)^2}\right) W^{(n)}\left(\frac{l^2}{(2n)^2}\right) = \delta_{m,l}.$$  

(119)

The orthogonal polynomials satisfy the three-term recurrence relation:

$$x p^{(n)}_m(x) = a^{(n)}_{m+1} p^{(n)}_{m+1}(x) + b^{(n)}_m p^{(n)}_m(x) + a^{(n)}_m p^{(n)}_{m-1}(x).$$

(120)

The measure is then rewritten as a determinantal ensemble $\mathcal{P}^{(n)}$:

$$\mu_{n,N}(y_1,\ldots,y_n) = \det \left(\sqrt{W^{(n)}(y_i)W^{(n)}(y_j)} \sum_{l=0}^{n-1} p^{(n)}_l(y_i) p^{(n)}_l(y_j)\right)_{i,j=1}^n.$$  

(121)
Let \(c_j\) correspond to the orthonormal polynomials \(a_j\). From now on it is more convenient to work with monic orthogonal polynomials. Then the recursion coefficients \(a_j\) are:

\[
\begin{align*}
\hat{a}_j = a_{j+1} &\rightarrow a, \quad \hat{b}_j \rightarrow b, \\
\hat{a}_{j+k} &\rightarrow a, \quad \hat{b}_{j+k} \rightarrow b,
\end{align*}
\]

as \(j \rightarrow \infty\). Then for any real-valued \(h \in C^4(\mathbb{R})\) we have:

\[
X_h^{(j)} \rightarrow \mathbb{E}X_h^{(j)} \rightarrow N \left(0, \sum_{l \geq 0} l|\hat{h}_l|^2 \right), \quad \text{as } j \rightarrow \infty,
\]

in distribution, where the coefficients \(\hat{h}_l\) are defined as:

\[
\hat{h}_l = \frac{1}{2\pi} \int_0^{2\pi} h(2a\cos \theta + b)e^{-il\theta} d\theta,
\]

for \(l \geq 1\). When \(n_j = j\), that is the subsequence is the whole sequence, \((123)\) is equivalent to:

\[
a_n^{(j)} \rightarrow a, \quad b_n^{(j)} \rightarrow b.
\]

To use this theorem we need to establish the convergence of the recursion coefficients. We do this by relating the polynomials \(p_m^{(n)}\) to the Krawtchouk polynomials using the ‘lifting’ procedure from [17] which is a variant of the QR-algorithm for the Christoffel transformation of orthogonal polynomials [13, 24].

First note, that the polynomials \(p_m^{(n)}(y)\) with \(y = x^2\) are just even polynomials from the set of polynomials \(\tilde{p}_m^{(n)}(x)\) that are orthonormal with respect to the weight \(W^{(n)}(x^2)\):

\[
\sum_{i=0}^{N+2n-1} \tilde{p}_m^{(i)}(x) \tilde{p}_m^{(i)}(x) W^{(i)}(x) \left(\frac{i^2}{(2n)^2}\right) = \delta_{m,i}.
\]

Since weight \(W^{(n)}(x^2)\) is even as a function of \(x\) and since each value of \(y\) appears in \((127)\) twice, we have:

\[
p_m^{(n)}(x^2) = \sqrt{2} \tilde{p}_m^{(n)}(x).
\]

Then the recursion coefficients \(\tilde{b}_k^{(n)} = 0\) for all \(k\) for the polynomials \(\tilde{p}_m^{(n)}\), and the recursion coefficients \(\tilde{a}_m^{(n)}, \tilde{b}_m^{(n)}\) are expressed in terms of \(a_k^{(n)}\) as:

\[
\tilde{a}_m^{(n)} = a_{2m}^{(n)}, \quad \tilde{b}_m^{(n)} = \left(2\tilde{a}_{2m}^{(n)}\right)^2 + \left(a_{2m+1}^{(n)}\right)^2.
\]

From now on it is more convenient to work with monic orthogonal polynomials \(\tilde{P}_m^{(n)}(x)\), that correspond to the orthonormal polynomials \(\tilde{p}_m^{(n)}(x)\) and satisfy the recurrence relation:

\[
x^2 \tilde{P}_m^{(n)}(x) = \tilde{P}_{m+1}^{(n)}(x) + \left(\tilde{a}_m^{(n)}\right)^2 \tilde{P}_{m-1}^{(n)}(x).
\]
Now consider monic Krawtchouk polynomials $\tilde{K}_i(z; q, M)$ that satisfy the orthogonality relation [38]:

$$
\sum_{i=0}^{M} \binom{M}{i} q^i (1-q)^{M-i} \tilde{K}_i(z; q, M) \tilde{K}_j(z; q, M) = \delta_{ij} \cdot \frac{q^j}{j!} \prod_{i=0}^{j-1} (M-i).
$$

(131)

and recurrence relation:

$$
z \tilde{K}_m(z; q, M) = \tilde{K}_{m+1}(z; q, M) + [q(M-m) + m(1-q)] \tilde{K}_m(z; q, M) + mq(1-q)(M+1-m) \tilde{K}_{m-1}(z; q, M),
$$

(132)

for $z = 0, \ldots, M$. Take $q = \frac{1}{2}$, $M = N + 2n - 1$, shift and rescale the argument $\bar{z} = \frac{1}{2}(z - \frac{N+2n-1}{2})$, then monic orthogonal polynomials $\tilde{K}_m^e(\bar{z}) = n^{-m} \tilde{K}_m\left(\frac{N+2n-1+2m}{2}; \frac{1}{2}, N + 2n - 1\right)$ have weight $W^e(\bar{z}) = \frac{(1+\bar{m}+\bar{n})}{(1+\bar{m}+\bar{n})!}$ and satisfy the recursion relation:

$$
\bar{z} \tilde{K}_m^e(\bar{z}) = \tilde{K}_{m+1}^e(\bar{z}) + \frac{m}{4n^2} (N+2n-m) \tilde{K}_{m-1}^e(\bar{z}).
$$

(133)

We see that the weight $W^e(x^2)$ of the polynomials $\tilde{P}_m^{(n)}$ differs from the weight $W^e(x)$ of $\tilde{K}_m^e$ by a factor $x^2$. Therefore the polynomials $P_m$ are expressed in terms of the polynomials $K_m^e$ and their derivatives $\left(\tilde{K}_m^e\right)$ by the Christoffel transformation [31, section 2.5]:

$$
\tilde{P}_m^{(n)}(x) = \frac{1}{C_{m,2}x^2}\begin{bmatrix}
\tilde{K}_m^{e}(0) & \tilde{K}_{m+1}^{e}(0) & \tilde{K}_{m+2}^{e}(0) \\
\tilde{K}_m^e(x) & \tilde{K}_{m+1}^e(x) & \tilde{K}_{m+2}^e(x)
\end{bmatrix},
$$

(134)

where:

$$
C_{m,2} = \begin{bmatrix}
\tilde{K}_m^e(0) & \tilde{K}_{m+1}^e(0) \\
\tilde{K}_m^e(x) & \tilde{K}_{m+1}^e(x)
\end{bmatrix}.
$$

(135)

Such transformation of polynomials leads to the following transformation of recursion coefficients: take three-diagonal matrix of the recursion coefficients, compute its QR-factorization and multiply the factors in reversed order [24].

In the case of monic polynomials with recurrent relation of the form $\bar{z} \tilde{K}_m^e(\bar{z}) = \tilde{K}_{m+1}^e(\bar{z}) + \beta_m \tilde{K}_{m-1}^e(\bar{z})$ this transformation is given by the relations [17][formula (2.10)]:

$$
\hat{\beta}_m = \frac{\beta_m \beta_{2m+1}}{\beta_{2m-1}},
$$

$$
\hat{\beta}_{m+1} = \beta_{2m+1} + \beta_{2m+2} - \beta_m,
$$

(136)

where $\beta_m = \frac{2m}{4n^2}(N + 2n - m)$. We were not able to solve these relations explicitly since the coefficients $\beta_m$ are given by the continuous fractions of $\beta_l$ and are cumbersome, but we can write asymptotic solutions in the form:

$$
\hat{\beta}_m = \beta_m - \frac{m}{3n^2} + O\left(\frac{1}{n^2}\right),
$$

$$
\hat{\beta}_{m+1} = \beta_{m+1} + \frac{N}{2n^2} - \frac{5m}{3n^2} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right).
$$

(137)
Substituting these asymptotics to the equation (136) for \( \tilde{\beta}_{2m+2} \) we check that:

\[
\tilde{\beta}_{2m+2} = \frac{\beta_{2m+2} \beta_{2m+3}}{\beta_{2m+1}} + \frac{N}{2n^2} - \frac{5m}{3n^2} - \frac{1}{2n^2} + O\left(\frac{1}{n^2}\right)
\]

\[
= \beta_{2m+3} + \frac{m+1}{3n^2} + O\left(\frac{1}{n^2}\right)
\]

Therefore for \( m = 2n \) we have the asymptotics \( \tilde{\beta}_{2n} = \frac{N}{2n} + O\left(\frac{1}{n}\right) = \frac{c-2}{2} + O\left(\frac{1}{n}\right) \) and thus using (129), (130) we conclude that \( a_n^{(n)} \to \frac{c-2}{2} \) and \( b_n^{(n)} \to c - 2 \). Then we can apply theorem 3 to conclude the proof of the Central limit theorem.

6. Conclusion

We have proven the convergence of irreducible components in tensor powers of the spinor representation of \( so_{2n+1} \) to the limit shape. It is possible to write an alternative proof based on the general results on discrete beta-ensembles, presented in the book by Guionnet [27].

Similar limit shapes can be obtained for the tensor powers of the certain reducible representations of \( gl_n, so_{2n}, sp_{2n} \). This result is presented in the paper by Nazarov et al [50].

It would be interesting to obtain similar result for the tensor powers of other irreducible representations and for Lie algebras of other classical series. Unfortunately, there are no known explicit formulas for the tensor product decomposition coefficients for the tensor powers of the irreducible representations in the cases except \( V^{\omega_i} \) for \( A_{n-1} \) and \( V^{\omega_i} \) for \( so_{2n+1} \). So this generalization remains an unsolved problem for the future.

Another possible generalization is to consider the character measure:

\[
p^N(\lambda, t) = \frac{M^N(\chi_{\lambda}(e^t))}{\chi_{\nu}(e^t)^N}.
\]

In the paper [60] an asymptotic formula for tensor product decomposition coefficients was obtained. The character measure was studied in the case of Lie algebras of fixed rank \( n \) in the papers [55]. In a separate publication we will consider the limit \( n, N \to \infty \) for the character measure.

It would be also very interesting to establish the connection of the present limit shape with random matrices. Such a connection is well known for Vershik–Kerov–Logan–Shepp limit shape [11, 15].

We also plan to study the local fluctuations around the limit shape presented in this paper. We expect to obtain an analogue of Baik–Deift-Johansson theorem [1].

The entropy of the Plancherel measure for the \( S_n \)-representations and \( sl_n \)-representations was considered in the papers [14, 46, 62]. Our preliminary numerical calculations demonstrate that similar result holds for the series \( so_{2n+1} \).

In the papers [8, 50, 51] we discuss similar measures on Young diagrams that are related to skew Howe dualities for classical Lie algebras. In particular, in [50] we explain the relation between tilings of skew hexagon, non-intersecting lattice paths and probability measures for series \( so_{2n+1}, sp_{2n}, so_{2n} \). As far as we know, the limit shapes for such skew tilings of a hexagon and their fluctuations or, alternatively, for domino tilings of glued Aztec diamond, are not studied yet. It would be interesting to derive the shapes of the Arctic curves, that separate frozen and free regions by the use of tangent method [20].
Data availability statement

No new data were created or analysed in this study.

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