Interacting Individuals Leading to Zipf’s Law

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We present a general approach to explain the Zipf’s law of city distribution. If the simplest interaction (pairwise) is assumed, individuals tend to form cities in agreement with the well-known statistics

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Zipf [1], half a century ago, has found that the city sizes obey an astonishingly simple distribution law, which is attributed to the more generic least effort principle of human behavior. Let us denote by \( q_m \) the number of cities having the population size \( m \), then \( R(m) = \int_m^\infty q_{m'} \, dm' \) defines the rank, i.e. the largest city has \( R = 1 \), the second largest \( R = 2 \), etc. Zipf found that empirically, \( R(m) \sim 1/m^\gamma \), with \( \gamma \approx 1 \) (see figure [1]). Remarkably Zipf showed that the scaling exponent \( \gamma = 1 \) is very close to reality for many different societies and during various time periods. More recent data [2] shows some variations from the pure \( \gamma = 1 \) result as also shown in figure [1]. Countries which have a peculiar social structure, such as the former USSR or China, do not follow Zipf law. For other developed countries Zipf law remains a rather good approximation. Such a generic law calls for generic explanation, since different countries (e.g. Germany and USA) have different cultural, economic structures, and their people have innumerable reasons to choose whether to live in a big or small city. Yet collectively the society self-organises, without express wishes of authorities, to obey the Zipf’s law.

Human settlements on the earth’s surface appear to be clustered, hence cities. This is because individuals interact with each other through social, economic and cultural ties. In very primitive times an individual (or a family) should perform all basic activities to survive; there was no need to form a large cluster beyond the size of a tribe. In modern times, ever refined mutual cooperation/competition brings people to live together. Yet this tendency does not seem to lead to a single megacity in the world: many of us prefer to live in a big city, equally many may hate and escape it for all its negative impacts. Somehow the ensuing compromise results in a robust statistical distribution, the Zipf’s law.

All these very general features call for a equally general approach to model city distribution. Below we propose a general framework using master equations. Let there be \( Q \) cities and \( m_i \) citizens in the \( i \)th city. The model is defined in terms of a master equation, assigning transition rates for the growth \( w_u(m_i) \) or decrease \( w_d(m_i) \) of the population of a city of size \( m_i \). In other words we assume that with a probability \( w_u(m_i)dt \) a new citizen arrives in city \( i \) in the time interval \( (t,t+dt) \), so that \( m_i \to m_i + 1 \). With a probability \( w_d(m_i)dt \) one of the \( m_i \) citizens departs so that \( m_i \to m_i - 1 \). We also assume that there is a small probability \( p \, dt \) that a new city is created, with a single citizen. Assigning the transition rates \( w_u(m) \) and \( w_d(m) \) specifies the model. Note that birth and death are not explicitly considered, these can be included in the transition probabilities. Note also that once an individual leaves a city, not necessarily he/she will settle in another city right away, thus the total number of city dwellers is not conserved. Thus the whole system is composed of the city dwellers and a reservoir of unsettled travellers whose number is unregistered.

One can study this problem introducing the average number \( q_{m,t} \) of cities of size \( m \) at time \( t \), which satisfies the master equation:

\[
\frac{\partial q_{m,t}}{\partial t} = w_u(m+1)q_{m+1,t} - w_d(m)q_{m,t} + p\delta_{m,1}
\]

\[
+ w_u(m-1)q_{m-1,t} - w_a(m)q_{m,t}.
\]  

(1)

The parameters of the equation are the transition rates \( w_u(m) \), \( p \), \( w_a(m) \). The total number \( n(t) = \sum m q_{m,t} \) of persons and the total number \( Q \) of cities are generally not constant:

\[
\frac{\partial Q}{\partial t} = p - w_d(1)q_{1,t}
\]  

(2)

\[
\frac{\partial n}{\partial t} = p - \sum_{m=1}^{\infty} [w_d(m) - w_a(m)]q_{m,t}.
\]  

(3)

\[
\text{FIG. 1. Zipf plot for cities of population larger than } 10^5 \text{ in the US and in India, and for a simulation of the master equation with } p = 10^{-3} \text{ and } W = 10^6, \alpha = 2.
\]
Let us focus on the stationary state solutions $\partial_t q_{m,t} = 0$, for which $q_{m,t}$ is simply denoted $q_m$, independent of time, and $n$ and $Q$ are constant on average. This leads us to the equation

$$w_d(m+1)q_{m+1} - w_d(m)q_m + p\delta_{m,1} + w_a(m-1)q_{m-1} - w_a(m)q_m = 0,$$

where the rate $w$ are also independent of time. The problem can be readily handled with the aid of the generating function $g(s) = \sum_m s^m q_m$.

Let us first consider the linear case $w_d(m) = Dm$ and $w_a(m) = A m$. This corresponds to independent decisions by the individuals. It is convenient to choose the constants to be $A = (1-p)/n, D = 1/n$, where $n$ counts the average total number of citizens (constant). In general there is a deficit for each existing city $w_d > w_a$, since a departing individual have finite chance to create a new city. The above equation is readily solved with the result:

$$q_m = \frac{np}{1-p} \frac{(1-p)^m}{m} \sim \frac{1}{m} e^{-pm}. \quad (5)$$

One can verify that the average number of citizens is indeed $\sum mq_m = n$ while the average number of cities is:

$$Q = \frac{np |\ln p|}{1-p}. \quad (6)$$

Several remarks are in order: due to the above deficit, the odds are always against the existing cities. If the deficit is small (or $p \ll 1$), fluctuations can still make a large city arise. The parameter $p$ sets the cutoff size $m^* \simeq 1/p$ of the power law behavior $q_m \sim 1/m$. The condition $p \ll 1$ is equivalent to the statement that the above deficit is small, or that the creation of a new city is a rare event, which seems realistic. The distribution $q_m \sim 1/m$ is very broad. Inverting the rank function $R(m)$, we find $m(R) = m^* \exp(-R+1)$, very different from the observed Zipf’s law.

The linear model implies no interaction among citizens. From an individual’s viewpoint, the chance to leave (or arrive at) a city is independent of the city’s size $m$, everybody is free to move around. The simplest interaction we may consider is pair-wise type, this leads us to $w \sim m^2$. In this case we choose $w_a(m) = (1-p)m^2/W$ and $w_d(m) = m^2/W$. The calculation is, again, standard and the result is that

$$n = \frac{pW|\ln p|}{1-p}, \quad Q = \frac{n}{|\ln p|} \sum_{m=1}^{\infty} \frac{(1-p)^m}{m^2} \sim \frac{\pi^2n}{6|\ln p|},$$

and

$$q_m = \frac{Wp}{1-p} \frac{(1-p)^m}{m^2}. \quad (7)$$

Note that $q_m \sim 1/m^2$, for $m \ll m^* \simeq 1/p$. Again using the rank relation, we find $R(m) \sim 1/m$, or in the more familiar form of Zipf $m(R) \sim 1/R$. Therefore we may draw the conclusion that it is the pairwise interaction that is behind the Zipf law for city distribution, a general explanation indeed. Recently Zanette and Manrubia have proposed a city formation model [3]. They use a multiplication and diffusion process and find that their results also reproduce the Zipf’s law. A possible motivation for such a multiplicative process is that citizens of the same city are subject to the same aggregate shocks, which tend to increase or decrease the population size. The implicit assumption is that the strength of such random shocks does not depend on the size $m_i$ of city $i$.

Our present approach is complementary to theirs: since there are more or less $m^2$ departures and arrivals, the net increase (or decrease) $\delta m$ of the population of a city in a unit time interval, is linearly proportional to $m$. This leads to a system with multiplicative noise. This argument shows, on one side that multiplicative noise results from a pairwise interaction, on the other that aggregate shocks, i.e. random events which affect equally each citizen of a given city, also would lead to the $m^2$ transition rates in our model.

On the other hand, the linear case $w_i \sim m$ is characterized by fluctuations $\delta m$ which are proportional to $\sqrt{m}$. As it has been discussed in another reproduction-diffusion system [4], this is typical of a system of non-interacting individuals.

These results are confirmed by numerical simulations. We show in fig. 2 the Zipf plot of a population of cities obtained for $w_d(m), w_d(m) \sim m^2$. This compares well with actual data [4].

One may argue that, in reality, both linear and square terms should be present, since an individual’s decision must have both his independent as well as interactive parts. Therefore it is natural to consider the mixed case. For the ease of presentation we assume the transition probabilities to be:

$$w_a(m) = \frac{m^2}{m_0} + m, \quad w_d(m) = e^{1/m^*} (\frac{m^2}{m_0} + am). \quad (8)$$

The simplest way to derive the distribution of city sizes, in this case, is to use detailed balance: The number of cities of size $m$ becoming of size $m-1$ is $q_{m-1}w_a(m)$. In the steady state, this has to balance the number of cities of size $m-1$ becoming of size $m$, i.e. $q_{m-1}w_a(m-1)$. This readily gives

$$q_m = C e^{-m/m^*} \frac{\Gamma(m + m_0)}{\Gamma(m + 1 + am_0)},$$

where the constant $C$ depends on the rate $p$ at which new cities are created.

For sizes $m < m_0$ the distribution is practically the same as that for the linear case, $q_m \sim 1/m$. If the cut-off size $m^* < m_0$, we conclude that the quadratic term is not relevant. If, on the other hand $m^* \gg m_0$, the linear
behavior \( q_m \sim 1/m \) holds for \( m \ll m_0 \) and then it crosses over to a power law behavior

\[
q_m \sim m^{-2-(a-1)m_0} \text{ for } m_0 \ll m \ll m^*.
\]

The Zipf exponent is therefore \( \gamma = 1/[1 + (a-1)m_0] \). Zipf’s law \( \gamma \simeq 1 \), obtains only if the pairwise interaction is dominant, i.e. more precisely if \( |m_0(a-1)| \ll 1 \). In the original Zipf’s work [8], the scaling law is valid only for large cities. The above results with \( |a-1| \ll 1 \), allow for a scenario where the distribution of city sizes crosses over from Zipf’s law with \( \gamma \simeq 1 \) for large cities \( m \gg m_0 \), to a non-interacting situation \( q_m \sim 1/m \) for small towns \( m \ll m_0 \). The population dynamics in small towns is dominated by the linear term in the transition rates, which describes non-interacting individuals. In large cities, on the other hand, interaction dominates and it leads to Zipf’s law. Note that Zipf’s law, with a general value of \( \gamma \) can also occur if the condition \( |m_0(a-1)| \ll 1 \) is not met.

It is also interesting to consider a general model with \( w_0(m) = (1-p)m^0/W \) and \( w_0(m) = m^\alpha/W \). This indeed allows us to investigate the effects of multi-person interactions. For example \( \alpha = 3 \) would represent a three person interaction. Having found that for \( \alpha = 1,2 \) the solution is:

\[
q_m = \frac{pW}{1-p} \frac{(1-p)^m}{m^\alpha}, \tag{9}
\]

we can try this solution in the equation. Treating separately the cases \( m = 1 \) and \( m > 1 \) we see indeed that this is the solution. Such a solution is also readily found using detailed balance. The numbers \( n \) and \( Q \) are given by

\[
Q = \sum_{m=1}^{\infty} q_m = \frac{pW}{1-p} \int_0^\infty \frac{a^{a-1}dt}{e^{t/a} - 1 + p}. \tag{11}
\]

For \( \alpha > 1 \) the integral is finite for \( p = 0 \), so that \( Q \sim pW \).

The interesting point here is that the average size of cities is finite for \( \alpha > 2 \), \( n/Q \simeq \zeta(\alpha-1)/\zeta(\alpha) \), where \( \zeta(x) \) is Riemann’s Zeta function. This observation raises the question of what happens when the population density \( \rho = n/Q \) grows beyond the value \( \rho_c = \zeta(\alpha-1)/\zeta(\alpha) \).

The answer is that the excess population concentrates in just one city, which therefore has a finite, large fraction of the total population. This effect has been studied recently [3] in an equilibrium model where an Hamiltonian

\[
H = \sum_i \ln m_i \] was considered. The equilibrium distribution at inverse temperature \( \beta = \alpha \) is clearly given by a power law distribution. However for \( \alpha > 2 \), there are two phases: a fluid phase for \( p < \rho_c(\alpha) \) which is well described by a power law distribution with an exponential cutoff, and a condensed phase where a “mega-city” nucleates, containing a finite fraction of the total population. It is easy to understand this transition from equilibrium considerations: Let us consider the state in which we assign to each city \( i \) a random number of citizens \( m_i \) drawn from a power law distribution with exponent \( \alpha > 2 \). This state minimizes the entropy and it has a density which is given by \( \rho = n/Q = \langle m_i \rangle = \rho_c(\alpha) \). If we want to find a state with a lower density, one can introduce a chemical potential, which is equivalent to an exponential cutoff on the distribution of \( m_i \). On the other hand if one wants to build a state of higher density \( \rho > \rho_c \), one gets into troubles. There are two ways out: The first is to put some \( \delta m_i = \rho - \rho_c \) extra citizens in each city. This results in a state which has a free energy cost \( \delta F \simeq Q \ln(\rho/\rho_c) \). The second way out is to put all the \( Q(\rho - \rho_c) \) excess citizens in only one city. This leads to a free energy cost \( \delta F \simeq \ln(Q(\rho/\rho_c - 1)) \). This only grows logarithmically with the system size whereas the first variant leads to an extensive increase of the free energy. It is then clear that in the equilibrium state the system prefers to create a “mega-city” to accommodate the excess population.

This discussion clearly refers to an equilibrium model. Metropolis or Montecarlo dynamics of this equilibrium system [3] is different from the dynamics of our master equation. Furthermore, and more importantly, we deal with a system in which nor the number of cities \( Q \) nor the population size \( n \) are fixed. In other words the density \( \rho \) is not fixed. For \( p \ll 1 \) the average density is very close to \( \rho_c(\alpha) \). The density, no matter how it fluctuates, will sweep across \( \rho_c \) in time. This suggests that, in our model, dynamic nucleation of a “mega-city” should occur for \( \alpha > 2 \). This agrees indeed with the results of numerical simulations. We show in fig.3 the distribution of the fraction \( m_{\max}/n \) of citizens which live in the biggest city. For \( \alpha = 3 \), the distribution is peaked at values very close to 1 (for \( \alpha = 2 \) the critical density diverges \( \rho_c = \infty \). Some precursor effects of the transition are however visible).

This leads to our second observation, that processes with \( \alpha > 2 \) must result in the nucleation of a “mega-city”, which is not very realistic. We can conclude that interactions of higher orders (than pairwise) are not relevant in the dynamics of city formation.

We see that the interaction leading to the Zipf’s law is, on one hand, the simplest possible (pairwise interaction). On the other it is a rather special one, since it is the “lowest order” of interaction which does not lead to the formation of a mega-city, which draws a good portion of the whole population. The absence of a mega-city suggests that in an expansion of the transition rates in powers of \( m \), we should neglect terms of order higher
than the second. This leads us to the mixed interaction case, which gives very realistic results.

In many disparate societies, it is not unnatural to assume that individuals make their city-dwelling decision based on their own opinions as well as on its interaction with other citizens. If this indeed is the case, we show that the larger cities obey approximately the Zipf’s law.

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