Optimal Attitude Estimation and Filtering Without Using Local Coordinates
Part I: Uncontrolled and Deterministic Attitude Dynamics

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Abstract—There are several attitude estimation algorithms in existence, all of which use local coordinate representations for the group of rigid body orientations. All local coordinate representations of the group of orientations have associated problems. While minimal coordinate representations exhibit kinematic singularities for large rotations, the quaternion representation requires satisfaction of an extra constraint. This paper treats the attitude estimation and filtering problem as an optimization problem, without using any local coordinates for the group of rotations. An attitude determination algorithm and attitude estimation filters are developed, that minimize the attitude and angular velocity estimation errors. For filter propagation, the attitude kinematics and deterministic dynamics equations (Euler’s equations) for a rigid body in an attitude dependent potential are used. Vector attitude measurements are used for attitude and angular velocity estimation, with or without angular velocity measurements.

I. INTRODUCTION

Attitude estimation is often a prerequisite for controlling aerospace and underwater vehicles, mobile robots, and other mechanical systems moving in space. Hence, attitude estimation of a rigid body has applications in spacecraft and aircraft dynamics, unmanned vehicle dynamics, and robot dynamics, including walking robots. While attitude sensors and the control tasks for which attitude feedback are required may be different in these different applications, the fundamental importance of obtaining accurate attitude data remains common to all these applications. In this paper, a new look at the attitude estimation problem is provided, which has two essentially new features: (1) the attitude is globally represented without using any local coordinates and the nonlinear attitude dynamics equation (Euler’s equation) for rigid bodies is used, and (2) the filter obtained is not a Kalman or extended Kalman filter. A global attitude representation has been recently used for partial attitude estimation with a linear dynamics model (see [16]). However, to the author’s knowledge, total attitude estimation using a global attitude representation and a full nonlinear attitude kinematics and dynamics model (without linearization) has not been done before.

Spacecraft attitude determination and filtering is perhaps the oldest application for attitude estimation algorithms, and the attitude determination problem for a spacecraft from vector measurements was first posed in [21]. A sample of the literature in this area can be found in [1], [6], [13], [18], [19], [21]. Applications of attitude estimation to unmanned vehicles and robots can be found in [2], [16], [17], [20]. Algorithms that are typically used for attitude estimation in such applications are based upon local coordinate representations of the group of rotations, like quaternions, Rodrigues parameters, or Euler angles. As is well known, minimal coordinate representations of the rotation group, like Euler angles, Rodrigues parameters, and modified Rodrigues parameters (see [7]), usually lead to geometric or kinematic singularities. Quaternion representation of the attitude matrix is commonly used, particularly in spacecraft applications, where the quaternion estimation (QUEST) algorithm and its several variants have been in use for quite some time ([1], [19], [15]). Besides the extra constraint (of unit norm) that one needs to impose on the quaternion, the quaternion representation for a given rotation depends on the sense of rotation used to define the principal angle, and hence can be defined in one of two ways. Local coordinate representations of the attitude usually lead to use of the extended Kalman filter (EKF) as an estimator for attitude and angular velocity. It is well known that the EKF has problems with convergence and stability in the case of large initial condition errors [5]. The attitude determination algorithm presented here does not use any local coordinate representation of the attitude, and is hence free of the drawbacks associated with such local representations. Nonlinear attitude estimation filters for a rigid body in an attitude-dependent potential field are also developed using this attitude determination algorithm. These are optimal nonlinear filters that minimize the attitude and angular velocity estimation errors at each measurement instant, and are hence free of the stability issues confronting extended Kalman filters.

A brief outline of this paper is given here. In Section II the attitude determination problem for vector measurements with measurement noise is introduced, and a global attitude determination algorithm which gives a global minimum of the attitude estimation error is presented. This section also presents some simulation results that demonstrate the applicability of this attitude determination algorithm. Section III introduces an attitude dynamics model for a free rigid body in a potential field, where the inertia properties of the body are assumed to be perfectly known. This deterministic dynamics model is used to create a filter that estimates both the attitude and the angular velocity. Two cases are considered here: the presence of angular velocity measurements,
and their absence; and the filter algorithms for both cases are presented. The paper is concluded in Section \[\text{IV}\] with a summary of results presented, and a discussion on future enhancements to the filter algorithm developed here.

II. ATTITUDE DETERMINATION PROBLEM

The attitude of a rigid body in space is a representation of the orientation of a body-fixed coordinate frame to an inertial frame. The principal axes of the body and inertial frames are related by a linear transformation given by a proper orthogonal matrix, which is usually referred to as the rotation or orientation matrix. The rotation matrix may be represented by various sets of coordinates, like the Euler angles, quaternions, or Rodrigues parameters (see [8], [9]).

The rotation matrices are proper (determinant=+1) orthogonal matrices that form a group under matrix multiplication; this abstract group is denoted SO (3). Hence, the group SO (3) is the compact Lie group of orientation-preserving isometries on \(\mathbb{R}^3\), and we represent it using the set of 3 \(3\) proper orthogonal matrices,

\[
C \in \mathbb{R}^{3 \times 3} ; \quad \text{s. t.} \quad C^T C = I_3 = C C^T ; \quad \det(C) = 1 ;
\]

A. Attitude Determination from Vector Measurements

We now formulate the attitude determination problem from vector measurements. Let the direction vectors of a few known points in an inertial frame \(I\) for \(\mathbb{R}^3\) be given by

\[
e_i ; \quad i = 1; 2; \ldots ; n ;
\]

and their corresponding direction vectors in a body-fixed frame \(B\) (fixed to a rigid body of interest) for \(\mathbb{R}^3\) be

\[
b_i ; \quad i = 1; \ldots ; n ;
\]

The inertial and body-fixed direction vectors are related by the rotation matrix \(C\) which rotates the body frame into the inertial frame, such that

\[
e_i = C b_i ; \quad i = 1; 2; \ldots ; n ;
\]

Note that the convention followed for the rotation matrix \(C\) in \[1\] is that in [3], [12], while the reverse convention of a rotation matrix taking the inertial frame to the body frame is used in most of the other literature cited here. The convention used here makes it easier to represent the body kinematics and dynamics in the body frame and the equations of motion are left-invariant, i.e., invariant to left multiplication of \(C\) by a non-singular matrix. The direction vectors \(b_i\) when measured from the body (e.g., a spacecraft), usually contain additive measurement errors and the measured direction vectors may not coincide with the actual \(b_i = C^T e_i\). Let the measured direction vectors be given by

\[
\hat{b}_i = b_i + e_i ;
\]

where \(e_i\) are measurement errors that are usually assumed to be Gaussian with zero mean.

The attitude determination problem consists of finding an estimate \(\hat{C}\) of the rotation matrix \(C\) such that the errors \(e_i = C \hat{b}_i\) are minimized. The least squares attitude estimation problem would be to

\[
\text{Minimize } \frac{1}{2} \sum_{i=1}^{n} w_i (e_i \cdot \hat{b}_i) (e_i \cdot \hat{b}_i) \text{ with respect to } \hat{C} \text{ subject to } \hat{C} \in \text{SO}(3),
\]

where \(w_i\) is a known weight factor (positive) usually taken to correspond to the statistical standard deviation of the \(i\th\) measured vector. This problem is also known as Wahba’s problem [21]. In this work, the weight factors are considered as design parameters.

We define the \(3 \times n\) matrices

\[
E = [e_1 \quad e_2 \quad \ldots \quad e_n] ; \quad \bar{E} = [b_1 \quad b_2 \quad \ldots \quad b_n] ;
\]

The assumption here and throughout the rest of this paper is that both \(E\) and \(\bar{E}\) are of rank 3; otherwise the attitude determination problem is ill-posed. We introduce the trace inner product on the space of real \(n\times n\) matrices, i.e., if \(A_1 \neq A_2 \in \mathbb{R}^{n \times n}\), then

\[
\text{trace}(A_1 A_2) ;
\]

The above attitude determination problem can then be restated as follows:

Minimize \(J_0 = \frac{1}{2} \text{trace}(E \hat{C} \bar{E}^T)\) \(\text{subject to } \hat{C} \in \text{SO}(3)\);

(2)

where the estimate of the rotation matrix (attitude) \(\hat{C}\) is the only unknown and \(\bar{W} = \text{diag}(w_i)\) is the positive diagonal weight matrix. We can extremize the cost function \(J_0\) by taking the first variation with respect to \(\hat{C}\) and setting it to zero since \(\hat{C}\) is the only unknown to be determined in this problem. The extremal solution to this problem is given by

\[
\hat{C} = \hat{U} ; \quad \bar{U} \in \text{SO}(3) ;
\]

Hence, from (3) and (4), we get

\[
\text{trace}(\bar{W} E (\hat{C} \bar{E})^T \hat{U}) = 0 ; \quad \bar{W} E (\hat{C} \bar{E})^T \bar{U}^T = \text{symmetric} ; \quad \bar{W} E (\hat{C} \bar{E})^T \bar{U}^T \quad \text{is symmetric} ;
\]

(5)

since \(U\) is skew-symmetric. The above result can be recast into the following form:

\[
L \hat{C} = \hat{C} L ; \quad L = E W \bar{U}^T ;
\]

(6)

and \(L\) is known since \(E\) is known, \(\bar{W}\) is known from measurements, and \(\bar{U}\) is known as a design parameter.

The following result gives a necessary condition for the attitude matrix \(C\) that satisfies (3), and is equivalent to equation (6).

Lemma 1. Define the linear map \(M_L : \text{SO}(3) \rightarrow \mathbb{R}^3\) by

\[
M_L(C) = C L^T C ; \quad C \in \text{SO}(3) ;
\]

(7)
where $L$ is as defined by (6). If $C \in 2 \mathbb{SO}(3)$ is in the kernel of this map, then $C$ is of the form

$$C = SL; \quad S = ST; \quad \text{i.e., } C = SL\text{ where } S\text{ is a } 3 \times 3\text{ symmetric matrix.}$$

**Proof:** If $C$ is in the kernel of $M_L$, then

$$C^T L = L^T C,$$

Hence, $D = LC^T$ is symmetric and we could express $L = DC$, where $D = C^T$ is symmetric. Now from our earlier assumptions, $L = EWB^T$ is non-singular, since $E$ and $B$ are of rank 3, and $W R_n$ is positive definite. Thus $D = LC^T$ is also non-singular. Hence, we get

$$D = L \quad \Rightarrow \quad C = D L = SL;$$

where $S = D^{-1}$ is symmetric. This proves the result. 

This result is a special case of Proposition 1 in [4], in which $C$ is replaced by a matrix whose row vectors form an orthonormal set. However, the above result does not give the unique solution to the attitude determination problem (6) since it does not give an expression for $S$, from which the estimate $\hat{C}$ of the unknown attitude can be determined.

To obtain the $\hat{C}$ that minimizes the cost function $J_0$, we apply the sufficient condition for a minimum by taking its first variation with respect to $\hat{C}$. The first variation of $J_0$ in (3) can be written as

$$\hat{C} J_0 = \text{trace} L^T C U^2 \quad \hat{C} = \text{trace} L^T C U^2 > 0; \quad \text{(9)}$$

This condition, along with Lemma 1, leads to the following result.

**Proposition 1.** The cost function $J_0$ in (3) is minimized by $\hat{C} = SL$ such that the symmetric matrix $S$ is positive definite.

**Proof:** From Lemma 1, we know that a necessary condition for the minimizing $\hat{C}$ would be $\hat{C} = SL$, where $S$ is a symmetric matrix. Hence

$$D = L^T \hat{C} = L^T SL$$

is symmetric. From condition (9), we have $\text{trace} D U^2 < 0$. Since $U 2 \mathbb{SO}(3)$, $U^2$ is symmetric and has negative definite trace. Let $Q_1, Q_2 \in 2 \mathbb{SO}(3)$ be such that

$$D = Q_1 T Q_1; \quad U^2 = Q_2 T Q_2;$$

are the spectral decompositions of $D$ and $U^2$ respectively, and $1, 2$ are diagonal. Then

$$c = \text{trace} (D U^2) = \text{trace} Q_1 T Q_2 > 0 \quad \text{and} \quad \text{trace} Q_2 T Q_1 > 0.$$
since \( Q \in SO(3) \). This proves that \( \Phi \in SO(3) \), and is hence the unique minimal solution to the attitude determination problem \( \Phi \).

Although we have used the QR decomposition for the matrix \( L \) here, one can use the singular value decomposition or any other decomposition using orthogonal matrices, to show this result.

We next show that the attitude estimate given by this algorithm is unbiased, i.e., in the absence of measurement errors, this estimate gives the actual attitude.

**Proposition 2.** The attitude determination algorithm given by equations (10)–(11) gives an unbiased estimate of the attitude.

**Proof:** Let us assume that there is no error in the measurement of body vectors, i.e., \( \Phi = B = [b_0 \ b_1 \ \cdots \ b_7] \). In that case, we have \( E = CB \), and

\[
L = L_0 = EWE^T = EW(E^T)^{-1}; \quad C = EWE^T; \quad L_0 = Q_0R_0.
\]

which, by Theorem 11 implies that

\[
S_0 = Q_0(R_0R_0^T)^{-1}Q_0^T = (EWE^T)^{-1};
\]

where \( L_0 = Q_0R_0 \). This is equivalent to

\[
EWE^T = Q_0(R_0R_0^T)^{-1}Q_0^T; \quad (EWE^T)^{-1} = Q_0R_0R_0^TQ_0^T;
\]

But the right-hand side above is \( L_0L_0^T = (EWE^T)(EWE^T)^{-1} \), which is equal to the left-hand side. Thus, we have \( C = S_0L_0 \) where \( S_0 = (EWE^T)^{-1} \) as required. This proves that this algorithm is unbiased.

These results are used as the basis for attitude estimation filters obtained in Section 111.

**B. Simulation Results for Attitude Determination Algorithm**

We end this section with a simulated example of an attitude determination problem, where an attitude matrix is obtained from ‘measurements’ of seven (unit) vectors, representing seven different directions in Euclidean 3-space. The “measured” vectors are given as normalized (unit) vectors. The simulation is carried out using a MATLAB program, which implements the attitude determination algorithm given in Theorem 1. The simulated vectors “measured” in the body frame have added Gaussian noise with a standard deviation of \( \sigma = 0.002 \) radians \( \pm 1.15 \) (which is relatively large compared to the capabilities of most modern attitude sensors).

The data used in this simulation, in terms of the inertial unit vectors, and their “measured” counterparts in the body frame, are as follows:

**Simulation Data:**

1. \( E = 4 \):
   - \( E = 4 \):
     - \( 0.3817 \)
     - \( 0.3077 \)
     - \( 0.2324 \)
     - \( 0.3374 \)
   - \( E = 4 \):
     - \( 0.5450 \)
     - \( 0.6045 \)
     - \( 0.5824 \)
     - \( 0.5675 \)
   - \( E = 4 \):
     - \( 0.7465 \)
     - \( 0.7347 \)
     - \( 0.7789 \)
     - \( 0.7511 \)
   - \( E = 4 \):
     - \( 0.6582 \)
     - \( 0.6046 \)
     - \( 0.5912 \)
     - \( 0.5912 \)
   - \( E = 4 \):
     - \( 0.6832 \)
     - \( 0.7389 \)
     - \( 0.7561 \)

2. \( E = 4 \):
   - \( 0.1287 \)
   - \( 0.0975 \)
   - \( 0.1580 \)
   - \( 0.1264 \)
   - \( 0.9628 \)
   - \( 0.9843 \)
   - \( 0.9833 \)
   - \( 0.9750 \)
   - \( 0.2394 \)
   - \( 0.4517 \)
   - \( 0.0862 \)
   - \( 0.4094 \)
   - \( 0.0210 \)
   - \( 0.1020 \)
   - \( 0.0249 \)
   - \( 0.9904 \)
   - \( 0.9829 \)
   - \( 0.9836 \)
   - \( 0.0141 \)
   - \( 0.1404 \)
   - \( 0.0279 \)

Note that the vectors in the inertial frame are clustered together, and hence, so are the vectors in the body frame. This simulates direction vectors as would be measured by an optical instrument with a finite field of view, e.g., a star tracker. The “actual” attitude matrix which takes the “actual” body directions to the inertial directions, is assumed to be known for this simulation, and is given by

\[
C = 4 \begin{pmatrix} 0.6385 & 0.7191 & 0.2743 \\ 0.7424 & 0.6694 & 0.0269 \end{pmatrix}
\]

As mentioned before, these simulated measurements of body directions correspond to added Gaussian noise of 0.002 radian to the “actual” body directions.

The results of this simulation, in the form of the attitude matrix determined by this algorithm \( \Phi \), the error between the known “actual” attitude matrix \( C \) and the attitude \( \Phi \), and the error \( e = E \Phi \Phi \), are given below.

**Simulation Results:**

1. \( \Phi = 4 \):
   - \( 0.2042 \)
   - \( 0.1856 \)
   - \( 0.9612 \)
   - \( 0.9612 \)
   - \( 0.2475 \)
   - \( 0.5 ; \)
   - \( 0.7420 \)
   - \( 0.6698 \)
   - \( 0.0283 \)
   - \( 0.0000 \)
   - \( 0.0006 \)
   - \( 0.0012 \)
   - \( 0.0006 \)
   - \( 0.0008 \)
   - \( 0.0000 \)
   - \( 0.0012 \)
   - \( 0.0008 \)
   - \( 0.0006 \)
   - \( 0.0000 \)
   - \( 0.0007 \)

2. \( \Phi = 4 \):
   - \( 0.0007 \)
   - \( 0.0008 \)
   - \( 0.0000 \)
   - \( 0.0005 \)
   - \( 0.0007 \)
   - \( 0.0012 \)
   - \( 0.0006 \)
   - \( 0.0012 \)
   - \( 0.0007 \)
   - \( 0.0009 \)
   - \( 0.0008 \)
   - \( 0.0003 \)
   - \( 0.0009 \)
   - \( 0.0009 \)
   - \( 0.0016 \)
   - \( 0.0016 \)
   - \( 0.0011 \)

Note that the error in the attitude matrix is here specified as \( \Phi \Phi \) minus the identity matrix; one can also specify this error as \( C \Phi \). As defined here, the matrix \( I_3 + \varepsilon_C = \Phi \Phi \), is the measure (in the group of rigid body rotations) of the attitude error. The maximum errors in these results are of the order of the measurement errors in the body vectors, which demonstrates the applicability of this attitude determination algorithm.
III. ATTITUDE ESTIMATION FILTERS FOR A FREE RIGID BODY IN A POTENTIAL FIELD

In this section, we develop an attitude estimation filter based on the attitude determination algorithm of Theorem II developed in the last section. We assume that the attitude dynamics is perfectly known, and that of a free rigid body in a potential field, i.e., there are no applied (control) forces on the body. We leave the potential field to be general (could be uniform or central gravity, for example). Two cases are dealt with here: (1) the case without, and (2) the case with angular velocity measurements. Since we use the actual (continuous) nonlinear dynamics equations for filter propagation, the filters developed here are not extended Kalman filters; they are nonlinear filters. We also assume that the vector attitude measurements (and angular velocity measurements, if any), are made at discrete time instants. Hence, the filter equations obtained are of the continuous-discrete type.

Let $f$, $g$, $k$ be non-negative, denote an increasing sequence of non-negative real numbers that coincide with the actual (continuous) nonlinear dynamics equations for the measured body vectors. If $E_k \in \mathbb{R}^{3 \times n_k}$ denotes the vectors in the inertial frame, and $C_k$ is the actual attitude matrix from the body to the inertial frame, then

$$\mathcal{E}_k = C_k^T E_k + N_k;$$

where the columns of $N_k \in \mathbb{R}^{3 \times n_k}$ are the measurement errors in the body vectors. The measured body vectors are usually expressed as unit vectors. The angular velocity measurement at time $t_k$, $k = 1$, the attitude and angular velocity estimates obtained by propagating using the attitude kinematics and dynamics equations from time $t_{k-1}$ are denoted $\hat{\theta}_k$ and $\hat{b}_k$, respectively, and the updated attitude at this time instant obtained from the attitude determination part of the filter (which is based on the algorithm of Section III) is denoted $\hat{C}_k$. The angular velocity is also updated at the measurement time $t_k$, and the updated angular velocity estimate is denoted $\hat{b}_k$.

A. Dynamics of Free Rigid Body in a Potential Field

We first obtain the dynamics (equations of motion) of a free rigid body in a potential field in a compact geometric form, that is free of any particular coordinate description. The attitude kinematics is given by

$$\dot{C} = \Omega (C);$$

where $\Omega (\cdot)$ denotes the angular velocity in the body frame. Let $\Omega$ denote the symmetric positive definite inertia matrix of the rigid body. The Lagrangian for the rigid body in a potential field is given by

$$L (C) = \frac{1}{2} \Omega u^T C;$$

where the first term is the kinetic energy, and $V (C)$ denotes the potential energy that is dependent on the attitude of the body.

The equations of motion are obtained by applying Hamilton’s principle to the action quantity

$$S = \int_0^T L (C) \, dt;$$

and taking reduced variations on the group $SO (3)$ (see [3], [12]). The reduced variations at the point $C$ are given by ([3], [12])

$$C = C; \quad \theta = - \omega_\tau C; \quad \hat{b} = 0;$$

where $\omega_\tau = \Omega C$ and $J : SO (3)$ is a positive definite operator on $SO (3)$ that is defined by

$$J (\cdot) = T_{\omega_\tau} \Omega_\tau; \quad (16)$$

The second term arising from the potential in the first variation above, can be rendered as

$$\int C \cdot \Omega C \, dt = \frac{1}{2} \int \Omega C \, dt;$$

Using the reduced variations as given by (16) and the relations (see [14])

$$\Omega C \cdot \Omega C = \frac{\Omega C \cdot \Omega C}{\Omega C \cdot \Omega C};$$

where $\Omega C = \Omega C \cdot \Omega C$, and $J : SO (3)$ is a positive definite operator on $SO (3)$ that is defined by

$$J (\omega) = T_{\omega} \Omega \tau; \quad (17)$$

The reduced variations at the point $C$ are given by $\Delta \Omega C$ and $\omega = \Omega C$ and $J : SO (3)$ is a positive definite operator on $SO (3)$ that is defined by

$$J (\omega) = T_{\omega} \Omega \tau; \quad (18)$$
Proof: Since $X$ is skew, there exists a unitary matrix $L$, i.e.,

$$LL^T - L^TL = I_n,$$

such that $LX = X = L^TL = 1$, where $X$ is a real diagonal matrix ($X$ is unitarily diagonalizable). Thus if $KX + XK = 0$, then

$$LK L^T + LK^T L^T = 0;$$

where $LK L^T$ is a positive definite Hermitian matrix. If $e$ is an eigenvector of $\bar{P} = LK L^T$, then

$$\bar{P} e = e; > 0;$$

and

$$\bar{P} + e \bar{P} = 0 \Rightarrow \bar{P} (e) + (e) = 0;$$

Hence $e$ is also an eigenvector of $\bar{P}$ with eigenvalue $< 0$. But $\bar{P}$ is positive definite and so all its eigenvalues are strictly positive. Thus, we have a contradiction, unless $e = 0$ and hence $X = 0$.

This lemma and its proof are also given in [4]. Note that we only need this result for $so(3)$ here, although we have stated and proved it for all $so(n)$. Thus, if the momentum $M = \mathbf{J} (\dot{\theta})$ is given, then one can obtain the unique angular velocity corresponding to it, $\omega = \mathbf{J}^{-1} (M) \in so(3)$. From equation (18) and Lemma 2, this uniquely determines $\dot{\omega}$ given the values of $\omega$ and $C$ at any instant. We use this dynamics equation, along with the attitude kinematics equation (14), to propagate the attitude and angular velocity between discrete sets of measurements.

### B. Attitude Estimation Filter without Angular Velocity Measurements

We use the attitude determination algorithm given in Section II to form an attitude estimation filter, by augmenting angular velocity data. The algorithm presented here works when there are body vector measurements at discrete time instants, but no available angular velocity measurements. However, we assume that we know the initial angular velocity. We use equations (14) and (18) to integrate the attitude and angular velocity in time between the sets of body vector measurements; this corresponds to the propagation phase of the filter. The attitude and angular velocity are then updated based on the body vector measurements, in a manner similar to that used in a Kalman filter.

We obtain the filter as an optimal filter from a suitable cost function that minimizes errors between the estimated and measured attitude, as well as the difference between the propagated and updated estimates. The update of the attitude estimate $\bar{E}_k^+$ at measurement instant $t_k$ is obtained by minimizing the following cost function with respect to $\bar{E}_k$:

$$J_a = \frac{1}{2} \sum_{k=0}^{N-1} \mathbf{h} \mathbf{E}_k + \mathbf{E}_k^+ \mathbf{F}_k; \mathbf{E}_k \mathbf{E}_k^+ \mathbf{W}_k \mathbf{E}_k^+ + \mathbf{h} \mathbf{E}_k^+ \mathbf{T} \mathbf{E}_k^+ + \mathbf{I}; (\mathbf{E}_k^+ \mathbf{T} \mathbf{E}_k^+) \mathbf{I},$$

where $\mathbf{E}_k^+$ is a symmetric positive definite matrix that can be chosen as a design parameter for the filter. In the case that we are considering now, there are no angular velocity measurements; therefore we update the angular velocity by minimizing the difference between the rates of change of attitude at a measurement instant:

$$J_r = \frac{1}{2} \sum_{k=0}^{N-1} \left( \mathbf{E}_k \mathbf{E}_k^+ \mathbf{C}_k \mathbf{C}_k^+ \right) \mathbf{I};$$

where $\mathbf{C}_k$ is a symmetric positive definite matrix that can be chosen as a design parameter for the filter. Here $\mathbf{E}_k$ and $\mathbf{C}_k$ are obtained by integrating equations (14) and (18) respectively, from time $t_{k-1}$ to time $t_k$, with initial conditions $\bar{E}_{k-1}^+$ and $\bar{I}_{k-1}^+$ respectively.

The first variation of $\mathbf{E}_k^+$ is given by

$$\mathbf{E}_k^+ = \mathbf{E}_k^+ \mathbf{U}_k^+ + \mathbf{U}_k^+ 2 \mathbf{so}(3);$$

Setting the first variation of $J_a$ in (19) to zero, we get:

$$J_a = \frac{1}{2} \sum_{k=0}^{N-1} \mathbf{h} \mathbf{E}_k + \mathbf{E}_k^+ \mathbf{F}_k; \mathbf{E}_k \mathbf{E}_k^+ \mathbf{W}_k \mathbf{E}_k^+ + \mathbf{h} \mathbf{E}_k^+ \mathbf{T} \mathbf{E}_k^+ + \mathbf{I},$$

where $\mathbf{W}_k = \mathbf{E}_k^+ \mathbf{T} \mathbf{E}_k^+ + \mathbf{I}$. Taking into account the initial condition $\mathbf{E}_0^+ = \mathbf{E}_0$, which is either assumed to be known from a given initial attitude, or obtained from an initial set of measurements $\bar{E}_0$ using the algorithm of Theorem 1. From the expression (21), we get the result

$$\mathbf{E}_k^+ T L_k$$

is symmetric, where $L_k = \mathbf{E}_k + \mathbf{E}_k^+ \mathbf{W}_k \mathbf{E}_k^+ T$.

Now the result of Theorem 1 can be applied to obtain the update of the attitude estimate $\mathbf{E}_k^+$ in terms of the QR decomposition of $L_k$. Given this update of the attitude estimate, one can obtain an update of the angular velocity estimate by minimizing $J_r$ in (20) with respect to $\mathbf{C}_k$. This gives us:

$$J_r = \frac{1}{2} \sum_{k=0}^{N-1} \left( \mathbf{E}_k \mathbf{E}_k^+ \mathbf{C}_k \mathbf{C}_k^+ \right) \mathbf{I}; (\mathbf{E}_k^+ \mathbf{T} \mathbf{E}_k^+) \mathbf{I},$$

where $\mathbf{C}_k^+$ is a symmetric positive definite matrix that can be chosen as a design parameter for the filter. The initial angular velocity $\mathbf{C}_0$ is assumed to be known, and from (20), $\mathbf{C}_0 = \mathbf{C}_0^+$. The above analysis can be formalized into the following result, which is one of the main results of this paper.

**Theorem 2.** The attitude estimation filter obtained from minimizing the cost functions $J_a$ and $J_r$ is given by the attitude and angular velocity updates:

$$\mathbf{E}_k^+ = \mathbf{S}_k L_k; \mathbf{C}_k^+ + \mathbf{C}_k^+ = (\mathbf{E}_k^+ \mathbf{T} \mathbf{E}_k^+) \mathbf{C}_k$$

$$\mathbf{C}_k + \mathbf{E}_k^+ \mathbf{W}_k \mathbf{E}_k^+ T$$

where

$$Q_k R_k = L_k = \mathbf{E}_k + \mathbf{E}_k^+ \mathbf{W}_k \mathbf{E}_k^+ T$$

(24)
is the QR decomposition of $L_k$, and
\[ S_k = Q_k \quad (R_k R_k^T)^{-1} Q_k^T \quad (25) \]
is symmetric. The initial conditions for the filter are
\[ E_0^+ = E_0; \quad C_0 = C_0^+ \quad (26) \]
where $E_0$ is either given or obtained from an initial set of measurements $\bar{\theta}_0$, and $C_0$ is given. The propagation equations for the filter are given by:
\[ \dot{C}_{k+1} = \mathcal{C}_k \quad \dot{E}_{k+1} = \mathcal{C}_k \quad (27) \]
\[ \{k+1 = J k \quad Z \quad \} \quad \dot{C}_k \quad \mathcal{C}_k \quad \mathcal{C}(C), \quad (28) \]
where $\mathcal{C} = \mathcal{C}_k^+$ and $(t_k) = C_k^+$. 

Note that according to Lemma 2, equations (25) and (28) uniquely determine $C_k^+$ and $E_{k+1}$ respectively. The result of Proposition 2 also holds, i.e., the attitude estimate given by this result is unbiased. This can be shown in a similar manner to the proof of Proposition 2. If $I$ is the identity matrix, then equation (24) for the update of the angular velocity estimate simplifies to
\[ C_k^+ = \frac{1}{2} (E_k^+)^T \mathcal{E}_k \quad C_k \quad + \quad C_k \quad (E_k^+)^T \mathcal{E}_k^+ \quad (29) \]

This equation can be readily used for updating angular velocity estimates without angular velocity measurements in the filter implementation. For the propagation equations (27) and (28), may be implemented by numerical integration software, including variational integrators (see [10], [11]) that preserve the group structure of $S O(3)$.

C. Attitude Estimation Filter using Angular Velocity Measurements

The creation of an attitude estimation filter from the basic attitude determination algorithm in Section B is made easier when angular velocity measurements are available. In this case, we assume that the sampling instants for attitude and angular velocity measurements are the same. The body vector (attitude) measurements are given by (13), while the angular velocity measurements are given by
\[ \bar{\omega}_k = \kappa_k + \mathcal{P}_k \quad (30) \]
where $\kappa_k = (t_k)$ is the actual angular velocity and $\mathcal{P}_k$ is a zero mean measurement error from a stochastic process with known statistics. Extensions can also be made to deal with the case when the sampling instants for attitude and angular velocity measurements are different.

The optimal filter when attitude and angular measurements are available is obtained by minimizing the following cost function with respect to $E_k^+$ and $C_k^+$:
\[ J_b = J_a + \sum_{k=0}^{\infty} h_C^k \quad \mathcal{F}_k \quad (C_k^+)^T \quad X_k \quad \mathcal{I}_a \quad (31) \]

where $J_a$ is as defined in (19), and $C_k^+$ and $E_k^+$ are symmetric positive definite matrices that can be chosen as design parameters for the filter. The matrix $X_k$ is also a symmetric positive definite matrix, which can be assigned as the error covariance matrix for the angular velocity measurement error $\mathcal{P}_k$ in (30).

We take reduced variations on $S O(3)$ with the first variations of $E_k^+$ and $C_k^+$ given by:
\[ E_k^+ = E_k^+ \quad U_k^+; \quad C_k^+ = U_k^+ + \{C_k^+ \quad U_k^+ \} \quad (32) \]
where $U_k^+ \geq 0$. The necessary condition for optimality is given by equating the first variation of $J_b$ with respect to $E_k^+$ and $C_k^+$ to zero. This gives us:
\[ \sum_{k=0}^{\infty} h[C_k^+ \quad E_k^+ \quad J T L_k; U_k^+ = 0; \quad (32) \]

where
\[ G_k = (C_k^+)^T \quad X_k \quad (C_k^+ \quad C_k^+) \quad L_k = E_k W_k \quad E_k^+ \quad (33) \]

Since $U_k^+$ and $U_k^+$ are independent of each other, the second term in equation (32) implies that $G_k$ is symmetric. This in turn implies that $[E_k^+ \quad C_k^+]$ is also symmetric, and hence $h[C_k^+ \quad E_k^+] \quad U_k^+ = 0$. Thus, the first term in equation (32) implies that $(E_k^+)^T \quad L_k$ is symmetric. Now we can apply Theorem 1 to obtain the update of the attitude estimate. The attitude and angular velocity updates are given in the following result.

Theorem 3. The attitude estimation filter obtained from minimizing the cost function $J_a$ in equation (31) is given by the attitude and angular velocity updates:
\[ E_k^+ = S_k L_k \quad J X_k \quad (C_k^+) = J X_k \quad (F_k) \quad + \quad J (C_k^+) \quad (34) \]

where $J_a : so(n) \quad so(n)$ for a symmetric positive definite matrix $K$ is as defined in Lemma 2.

\[ Q_k R_k = L_k = E_k \quad + \quad E_k W_k \quad E_k^+ \quad (35) \]

is the QR decomposition of $L_k$, and
\[ S_k = Q_k \quad (R_k R_k^T)^{-1} Q_k^T \quad (36) \]
is symmetric. The initial conditions for the filter are
\[ E_0^+ = E_0; \quad C_0 = C_0^+ \quad (37) \]

where $E_0$ and $C_0$ are either given or obtained from initial measurements $\bar{\theta}_0$ and $\mathcal{P}_0$.

The propagation equations for the filter are equations (27)-(28) where $\mathcal{C} = \mathcal{E}_k^+$ and $(t_k) = C_k^+$. Thus, the propagation phases for the filters given by theorems 2 and 3 are identical. Note that, by Lemma 2, equation (34) determines $C_k^+$ uniquely since $X_k$ is positive definite. Also, the result of Proposition 2 holds for the attitude estimate given by this filter. The angular velocity estimate
is also unbiased, since in the absence of angular velocity measurement errors, \( f_k = c_k = k \), and equation (34) gives \( c_k^* = k \). The filters developed in this section can also be extended to estimate a constant bias in measurements, if sensor bias is present.

IV. Conclusions

This paper presents an attitude determination algorithm and attitude estimation filters that can be used for attitude estimation of robots, spacecraft, and other vehicles. The attitude determination algorithm is obtained from an optimization process with the cost function equal to a weighted attitude estimation error on the group of rigid-body orientations. This algorithm is global, and does not use any local coordinate representation (like Euler angles or quaternions) for the group of orientations. The optimization is carried out with variations on the smooth manifold (Lie group) of rigid body orientations, and the estimate obtained is shown to (globally) minimize the attitude estimation error. It is also shown to provide an unbiased estimate of the attitude, i.e., in the absence of measurement errors, the estimate of the attitude obtained is the actual attitude. A numerical simulation of this attitude determination algorithm, with a set of seven simulated body unit vector measurements with noise for seven given inertial unit vectors, is carried out. The order of error in the attitude estimate obtained using this algorithm is found to be no more than the order of the error in measurements.

The attitude estimation filters are of the continuous-discrete type, which work with a continuous deterministic dynamics model supplemented by discrete sets of noisy measurements. These filters are obtained for two cases: when there are body vector measurements but no angular velocity measurements, and when there are attitude and angular velocity measurements at the same measurement instant. It is assumed that the attitude dynamics is deterministic, and accurately known. The (nonlinear) attitude kinematics and dynamics equations are used for propagation of the attitude and angular velocity between successive sets of measurements. The attitude and angular velocity estimates are obtained from an optimization process that minimizes the weighted sum of errors between the estimates and measurements, and between the estimates and propagated values for these quantities at each measurement instant. These filters are shown to be unbiased, i.e., in the absence of measurement errors, the estimates they give are equal to the actual attitude and angular velocity.

This work is a preliminary exploration into attitude estimation techniques without using any local coordinate representation for the attitude. The results obtained thus far are encouraging, and they show that one can obtain unbiased filters that minimize the errors in attitude and angular velocity estimation without using local coordinate representations and without using an extended Kalman filter. While local coordinate representations of attitude have problems associated with convergence of estimates for large initialization errors. These drawbacks are not present in the filters developed here, since they do not use local coordinates, and since they give optimal nonlinear filters that minimize the attitude and angular velocity estimation errors at each measurement instant. Thus, the attitude estimation filter algorithms developed here fill a gap in the existing research in this direction, besides improving upon the filters currently in use for attitude estimation of mechanical systems.

Future work would include extension of the attitude and angular velocity estimation filters developed here to the case when the dynamics has modeling errors or noise. Numerical and/or experimental studies in implementation of these filters could also be explored. Numerical simulation results for the filters developed here, and numerical comparisons with estimation algorithms using local coordinates and extended Kalman filters for the deterministic dynamics case, have not been obtained yet. Such results are very likely to be reported in the near future.

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