Physics-Enhanced Bifurcation Optimisers: All You Need is a Canonical Complex Network

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Abstract—Many physical systems with dynamical evolution that at their steady state give a solution to optimization problems were proposed and realized as promising alternatives to conventional computing. Systems of oscillators such as coherent Ising and XY machines based on lasers, optical parametric oscillators, memristors, polariton and photon condensates are particularly promising due to their scalability, low power consumption and room temperature operation. They achieve a solution via the bifurcation of the fundamental supermode that globally minimizes either the power dissipation of the system or the system Hamiltonian. We show that the canonical Andronov-Hopf networks can capture the bifurcation behaviour of the physical optimizer. Furthermore, a continuous change of variables transforms any physical optimizer into the canonical network so that the success of the physical XY-Ising machine depends primarily on how well the parameters of the networks can be controlled. Our work, therefore, places different physical optimizers in the same mathematical framework that allows for the hybridization of ideas across disparate physical platforms.

Index Terms—Artificial neural networks, bifurcation, Hopfield neural networks, injection-locked oscillators, memristors, microelectromechanical systems, minimization, minimization methods, nonlinear dynamical systems, optimization, optimization methods, optoelectronic devices, photonics, spin systems.

I. INTRODUCTION

Optimization problems are ubiquitous in technology and applications, from machine learning and artificial intelligence to industrial designs of vehicles, new materials, and drugs [1], [2], [3]. The complexity of such problems grows fast, often with an exponential increase in the number of candidate solutions with the number of unknowns. Given the importance of finding a reasonable solution quickly while searching a large hyperspace of an ever-increasing number of variables, there have been intense efforts to design analogue physical hardware capable of performing this task. The feasibility of this task has been assured by the complexity theory stating that most if not all optimization problems can be mapped into universal spin Hamiltonians with a polynomial overhead on the number of additional variables (spins) [4], [5], [6].

A physical system can be arranged and controlled, so that the Hamiltonian is globally minimised. In that case, this physical system can potentially be used as an unconventional physics-enhanced computing device for solving these types of tasks. XY, $H_{XY}$, Ising, $H_I$ and k-local, $H_K$ classical Hamiltonians formulated as

$$H_{XY} = \min_{\{\theta_i \in [0,2\pi]\}} - \sum_{i,j} J_{ij} \cos(\theta_i - \theta_j),$$

$$(1)$$

$$H_I = \min_{\{s_i = \pm 1\}} - \sum_{i,j} J_{ij} s_i \cdot s_j,$$

$$(2)$$

$$H_K = \min_{\{s_i = \pm 1\}} - \sum_{i,j,,...,k} J_{ij,...,k} s_i \cdot s_j \cdot s_k ..., s_k,$$

$$(3)$$

are all universal, meaning that for the most general coupling matrix $J$, finding the global minimum of the corresponding Hamiltonian is an NP-hard problem that requires exponentially fast-growing resources.

Various physics-based Ising and XY spin minimisers – XY-Ising machines – have been created using lasers [7], [8], [9], spatial light modulators [10], [11], [12] optical parametric oscillators [13], [14], [15], superconducting qubits [16], [17], [18], memristors [19], trapped ions [20], polariton condensates [21], [22], and photon condensates [23] among many others. Some of these solvers achieve minimisation using the underlying physical principle of minimum power dissipation subject to constraints on voltage, amplitude, gain, etc. [24]. Others use quantum or classical annealing. For instance, in the vicinity of a power-dissipation minimum, the system evolves in time following the gradient of the power-dissipation function. However, a successful XY-Ising machine cannot be based on gradient descent alone to offer a computational advantage. The success of the XY-Ising devices depends on their ability to search the low-energy part of the spin Hamiltonian without being trapped by the local minima. This is through the bifurcation of an additional degree of freedom – the amplitude of the laser field or condensate wavefunction – that the system selects the fundamental mode, and, therefore, the minimum of the power-dissipation function.
The amplitude bifurcation often is a key to other physical minimisation principles, for instance, when the system evolves adiabatically or anneals to the minimum of the system Hamiltonian. Annealing here concerns the changing Hamiltonian during the system evolution that can occur adiabatically [25] or not [26]. A simulated bifurcation machine is an example of such bifurcation on the route to minimise the Ising Hamiltonian [27], [28], [29].

Many lasers, photonic, polaritonic and biological systems exhibit the so-called Andronov-Hopf bifurcation at the threshold that leads to the birth of the limit cycle out of an equilibrium point. The canonical model describing this bifurcation is the network of the Andronov-Hopf oscillators (AHO) that can be written in the most general form as

$$\dot{\psi}_i = (\gamma_i + i\omega_i)\psi_i - (\sigma_i + iU_i)|\psi_i|^2\psi_i + \sum_{j \neq i} Q_{ij}(\psi_j).$$

(4)

Here $\psi_i(t)$ is a complex function of time that characterises the state of the $i$th oscillator, $Q_{ij}$ describes the coupling between the $i$th and $j$th oscillators and $\gamma_i$, $\omega_i$, $\sigma_i$, $U_i$ represent the effective gain, self-frequency, nonlinear dissipation and self-interactions, respectively. In particular, polaritonic or lasers networks [8] show a potential of controlling all these terms independently [22], [30]. In what follows, we show that (4) is a general framework that describes different classes of physical optimisers considered in the literature and formulate when this model achieves the minimum of a universal spin Hamiltonian.

II. ANDRONOV-HOPF OSCILLATORS AS COHERENT XY-ISING MACHINES

Coherent Ising Machines: The Hopfield models are perhaps the best-known networks used to minimise the Ising Hamiltonians. They are also used to describe the dynamics of the coherent Ising machines (CIMs) [14], [31]. They are trivially reduced to the AHO networks under parametric pumping that projects the phases to 0 or $\pi$. In CIMs, the state $x_i$ corresponds to the in-phase amplitude of the $i$-th oscillator pulse, whose dynamics are described by

$$\frac{dx_i}{dt} = (p - 1 - x_i^2)x_i + \sum_{j \neq i} G_{ij}x_j,$$

where $p$ is the photon injection rate and $\xi$ is the suitable scaling factor. Equation (5) coincides with (4) if $\theta_i \in \{0, \pi\}$. In this case, $\psi_i = r_i \exp[i\theta_i + i\omega_i]$. To see this we let $x_i = r_i \exp[i\theta_i]$, $\theta_i \in \{0, \pi\}$, $\omega_i = \omega$, $\gamma_i = p - 1$, $\sigma_i = 1$, $U = 0$, $Q_{ij}(\psi_j) = \xi G_{ij} \psi_j$. The projection of phases onto 0 or $\pi$ will be automatically achieved if only the real parts of the fields are coupled so the coupling in (4) takes the form $Q_{ij}(\psi_j) = \xi G_{ij}(\psi_j + \psi_j^*)$.

Various modifications of the CIM and/or the Hopfield networks can be accomplished by (4). For instance, the success of the Hopfield network minimisers (CIMs) is improved with the introduction of the chaotic amplitude method [32] that anneals the coupling terms as $\xi = \epsilon t$ where $\epsilon_t$ depends on how far away each oscillator is from its saturation amplitude. These annealing schedules can be introduced into the canonical form of (4) and, therefore, in principle, realised by any optical network described by (4) that offers that kind of control.

Higher order binary optimisation: The minimisers of the higher order binary optimisation problem can be obtained by AHO networks [33] if the coupling term in (4) is replaced by the higher order coupling leading to

$$\dot{\psi}_i = (\gamma_i + i\omega_i)\psi_i - (\sigma_i + iU_i)|\psi_i|^2\psi_i + \sum_{j, k \ldots l} J_{ijkl}\psi_j\psi_k\psi_l\psi_l^*.$$

(6)

No projection of the phases to the discrete values 0 and $\pi$ are needed in this case, as such projection is automatically achieved by mixing $\psi_j$ and $\psi_j^*$ in the coupling terms as was argued in [33].

Weakly Interacting Networks: The development of the XY-Ising machines postdated extensive research on networks of neural oscillators near multiple Andronov-Hopf bifurcation points. In particular, weakly interacting networks were proposed as oscillatory neurocomputers capable of emulating an associative memory network. Networks consist of $N$ neural oscillators comprised of two populations of neurons excitatory, described by a scalar function of time $x_i(t)$ and inhibitory, $y_i(t)$, that evolve according to the dynamical equations [34], [35]

$$\dot{x}_i = f(x_i, y_i, \lambda_i) + \varepsilon q_i(x_i, \gamma_i, q_i),$$

(7)

$$\dot{y}_i = g(x_i, y_i, \lambda_i) + \varepsilon q_i(x_i, \gamma_i, q_i),$$

(8)

where $\lambda_i$ is a bifurcation parameter and $\varepsilon$ is a small parameter describing the strength of interactions between neurons. Functions $f, g, q_i, q_i$ describe the self-evolution and couplings among the neurons.

The dynamical system described by (7), (8) with $\varepsilon = 0$ is near an Andronov-Hopf bifurcation if the Jacobian matrix

$$\frac{Df(x, g)}{D(x, y)} = \left(\begin{array}{cc} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{array}\right)$$

(9)

has a pair of purely imaginary eigenvalues that we denote $\pm i\Omega$. The corresponding column (row) eigenvectors we denote as $x$ and $\bar{x}$ ($w$ and $\bar{w}$) and form a matrix $V = (x, \bar{x})$. Changing the variables to

$$\begin{aligned}
\left(\begin{array}{c} x_i(t) \\ y_i(t) \end{array}\right) &= \sqrt{\varepsilon} V^{-1} \left(\begin{array}{c} \exp[i\Omega] \psi_i(\tau) \\ \exp[-i\Omega] \psi_i^*(\tau) \end{array}\right) + \mathcal{O}(\varepsilon),
\end{aligned}$$

(10)

introducing slow time $t = \tau$ and considering the dynamics near multiple Andronov-Hopf bifurcation points with $\lambda \to \epsilon \lambda$ reduces (7), (8) to the canonical form of (4) in the order $O(\sqrt{\varepsilon})$ [36]. The parameters $\gamma_i, \sigma_i, U_i$ and $\omega_i$ in (4) depend on $\lambda$ and $\Omega$ (so on the structure of $f$ and $g$) and the couplings become $Q_{ij}(\psi_j) = w \cdot D(p_i, q_i)/D(x_j, y_j) \cdot v \psi_j$.

There are two further popular examples of XY-Ising machines that operate near multiple Andronov-Hopf bifurcation points.

MEMs and Conformal Hamiltonian Systems: Coupled micro-electromechanical systems (MEMs) [37] are governed by

$$\dot{x}_i = i(F(x_i, \lambda_i)\dot{x}_i + G(x_i) = \sum_{j \neq i} (e_{ij}x_j + k_{ij}x_j),$$

(11)

where $x_i$ is the displacement from the rest position, $F$ and $G$ are damping and stiffness parameters, $e_{ij}$ and $k_{ij}$ are electric conductances [mechanical spring] constants coupling the i-th and the j-th oscillators. Clearly, (11) can be written as (7), (8) by letting $y_i = \dot{x}_i$. The reduction to the canonical AHO is achieved
by writing $\psi = \phi + i \sqrt{\lambda} x_i$ [37]. The relationship between the coefficients of (11) and the coefficients of AHO in (4) are given in [37].

The dynamics of (11) can be viewed as a particular case of gradient descent accelerated by momentum, also known as Nesterov’s accelerated gradient method [38], [39]:

$$\dot{x}_i + \lambda(t) \dot{x}_i + \xi \nabla E(x) = 0,$$

(12)

where $\lambda(t) > 0$ is the friction coefficient and $E(x) = -\frac{1}{2} \sum_{i,j} J_{ij} x_i x_j$. $E(x)$ becomes the Ising Hamiltonian if we replace $x_i$ with $\psi(x_i)$. More generally, (12) is a special case of a conformal Hamiltonian system of the form

$$\dot{x} = \nabla_y H(x,y),$$

(13)

$$\dot{y} = - \nabla_x H(x,y) - \lambda y.$$  

(14)

If $H$ is a separable Hamiltonian, $H(x,y) = \frac{1}{2} \lambda \|y\|_2^2 + \xi E(x) + \lambda \|x\|_2^2$, we get back to (12) [40]. The reduction to AHO is obtained similar to [37] using $\psi_i = \dot{x}_i + i \sqrt{\lambda} x_i$ giving $Q_{ij} = i \frac{1}{2} \xi J_{ij} (\dot{x}_j - \dot{x}_i)$, $\omega_i = \sqrt{\lambda}$, $\gamma_i = -\frac{1}{2} (\lambda - \lambda_H)$, where $\lambda_H$ is the threshold for the bifurcation. The nonlinear terms controlled by the coefficients $\sigma_i$ and $U_i$ do not appear after the transformation at this order, but their introduction into the equations helps to saturate the gain faster.

**Simulated Bifurcation Machine:** Simulated bifurcation machine (SBM) [27], [28], [29] has demonstrated an improvement over the CIM by employing adiabatic evolutions of energy-conservative systems motivated by purely adiabatic quantum annealing. Its dynamics are governed by

$$\dot{x}_i = a_0 y_i,$$

(15)

$$\dot{y}_i = -(a_0 - a(t)) x_i + \xi \sum_{j \neq i} J_{ij} x_j,$$

(16)

where the state of each oscillator is described by two real variables $x_i$ and $y_i$ and the annealing is performed by letting $a = a(t)$ approach $a_0$ as $t \rightarrow t_{\text{max}}$, where $t_{\text{max}}$ is the terminal time of the dynamics. As in CIM, the spins are associated with the $s_i = \text{sign}(x_i)$ at the fixed point of the dynamics.

To get the canonical AHO equations at the onset of bifurcation we assume that $a_0 - a$ is constant and write

$$x_i = \frac{1}{2} (\psi_i + \psi_i^*) \sqrt{a_0}, \quad y_i = \frac{1}{2} (\psi_i - \psi_i^*) \sqrt{a_0 - a},$$

(17)

so that $\psi_i = x_i/\sqrt{a_0 + iy_i/\sqrt{a_0 - a}}$. The AHO network of (4) becomes

$$\dot{\psi}_i = -i \sqrt{a_0} \sqrt{a_0 - a} \psi_i + \frac{i \xi}{\sqrt{a_0 - a}} \sum_{j \neq i} J_{ij} (\psi_j + \psi_j^*).$$

(18)

The canonical form given by (4), (18) is capable, therefore, of capturing the dynamics of the system close to the bifurcation point.

**Minimisation of the XY Hamiltonian:** Next, we will clarify the relationship between (4) and the XY Hamiltonian minimisation (while the correspondence with other classical spin Hamiltonians follows when one takes into account the structure of the spins and the coupling terms). Let the coupling term $Q_{ij}(\psi_j) = s_{ij} \exp[i \phi_{ij}] \psi_j$ and $\psi_i = r_i \exp[i \theta_i]$ so that (4) can be written in polar coordinates

$$\dot{r}_i = \gamma_i r_i - \sigma_i r_i^3 + \sum_{j \neq i} s_{ij} r_j \cos(\theta_i - \theta_j - \phi_{ij}),$$

(19)

$$\dot{\theta}_i = \omega - U r_i^2 - \frac{1}{r_i} \sum_{j \neq i} s_{ij} r_j \sin(\theta_i - \theta_j - \phi_{ij}),$$

(20)
where we assumed that $\omega_i = \omega$, $\sigma_i = \sigma$, $U_i = U$. The relationship with the XY model can be obtained by either assuming that $\sigma \gg \max s_{ij}$ [37] or by using feedback on the gain coefficients, [41], [42]. We will discuss both approaches below.

If $\sigma \gg \max s_{ij}$ and all oscillators are pumped with the same intensity $\gamma_i = \gamma$, the last term on the right-hand side of (19) is negligible in comparison with other terms, so that the amplitudes of the oscillators take on stationary values $r_i = r = \sqrt{\gamma}/\sigma$. Equation (20) reduces to the Kuramoto-Sakaguchi model of oscillators with identical natural frequency $\tilde{\omega} = \omega - U\gamma/\sigma$ and natural phase lag $\phi_{ij}$

$$\dot{\theta}_i = \tilde{\omega} - \sum_{j \neq i} s_{ij} \sin(\theta_i - \theta_j - \phi_{ij}).$$

(21)

If the couplings are real, so that $\phi_{ij} \in \{0, \pi\}$ then (21) reduces to the Kuramoto model

$$\dot{\theta}_i = \tilde{\omega} - \sum_{j \neq i} J_{ij} \sin(\theta_i - \theta_j),$$

(22)

where $J_{ij} = \pm s_{ij}$. Starting from any initial condition (22) follows the gradient descent to the minimum of the classical XY Hamiltonian $H_{XY}$.

It was shown that the network of oscillators reproducing (4) and using the Hebbian learning rule has associative memory similar to that of Hopfield–Grossberg networks but a greater memory capacity [43]. While gradient descent is sufficient for pattern recognition and other associative memory applications, optimization tasks require the system to escape the local minima in its search for the global one. This search benefits from unequal and dynamically changing amplitudes that bifurcate from zero as the gain increases. However, these amplitudes must all reach the same value at the steady state to minimise the Hamiltonian with the given coupling matrix [41]. This is achieved by complementing (4) with time-evolving gains

$$\gamma_i = \tilde{\epsilon}(1 - r_i^2),$$

(23)

where the parameter $\tilde{\epsilon}$ controls the rate of change of $\gamma_i$ with respect to the amplitudes of the oscillators.

As the network of oscillators approaches the steady state, the amplitudes approach one, while the phases start evolving according to (22) and the total occupancy (mass) of the system of $N$ oscillators becomes

$$N\sigma = \sum_{i=1}^{N} \gamma_i + H_{XY}.$$  

(24)

It follows that if the total effective gain $\sum_{i=1}^{N} \gamma_i$ is globally minimised, then $H_{XY}$ is also globally minimised.
III. XY-Ising Machines for Global Minimisation

In the previous section, we argued that the physical optimisers could be reduced to the canonical complex AHO networks in the vicinity of the bifurcation. However, all considered optimisers involve time-varying (annealed) parameters, so they all will have different dynamics before and after the bifurcation. We argue, however, that as follows from the fundamental theorem of weakly connected neural network theory [36] only the region close to the bifurcation is essential for the global minimisation; therefore, we can always choose the annealing schedule to bring different systems to the same behaviour at the bifurcation point, and, therefore, to the same solution. In this section, we illustrate this by using numerical simulations of the canonical complex AHO networks for XY and Ising Hamiltonian minimisation and demonstrate that AHO behaviour corresponds to the operation of vastly different machines considered in the previous section if annealing schedules are suitably chosen.

**XY Machine:** For XY minimisation we use (4) with additive noise and (23) with \( \omega_i = 0, \sigma_i = 1, U_i = 0 \) and \( Q_{ij}(\psi_j) = J_{ij} \psi_j \). Fig. 1(a)–(d) illustrate the typical numerical evolution of the system. Fig. 1(e) and (f) show the statistics of finding the global minimum compared to a brute-force Monte Carlo method. In most cases, the AHO finds the global minimum with a very high probability. In contrast, the system still seeks out a local minimum close to the ground state for the coupling matrices where the success probability of finding the true ground state is low. Comparison to a quasi-Newton method, on the other hand, shows that the actual distribution of local minima is far more spread out.

**Comparison of Ising Machines:** To illustrate that AHO captures the behaviours of the CIMs and the SBMs, we numerically simulate (4) using the mapping presented and compare the results with the dynamical behaviour of (5) and (16) on two different graphs. The results are displayed in Fig. 2, where we compare the time evolution of the CIM and SBM described by (5) and (15)–(16) respectively with that of the canonical AHO described by (4). For both linearly annealed gains, \( \gamma_i \) and gains controlled by (23), the canonical AHO manages to replicate the behaviour of the CIM and SBM close to the bifurcation. When minimising the Ising energy on the random graph used in Fig. 2(d) and (e), we observe that some spins exhibit a delayed bifurcation due to frustration effects. The evolution of AHO captures a delayed bifurcation as well.

IV. Conclusion

In the work of any physical XY-Ising machine, the dynamics near the bifurcation are the most important for optimisation as well as for the associative memory task because each oscillator must be near a bifurcation to make a nontrivial contribution to the entire network dynamics as follows from the fundamental theorem of weakly connected neural network theory [36]. This theorem is a consequence of the centre manifold reduction that lies at the heart of the dynamical behaviour of the dynamical systems and bifurcation theories. In particular, it assures that those neurons (e.g., oscillators, spins) that do not participate in the local bifurcation dynamics can be removed from the network without changing its local dynamics. However, in XY-Ising machines, all oscillators undergo bifurcation from vacuum to well-defined states, so all oscillators should participate in that bifurcation. In general, even small external perturbations profoundly affect the network dynamics following the bifurcation. The local events at the bifurcation can determine the future behaviour of such a network. However, when the bifurcation captures the transition to the fixed point of the dynamical system that corresponds to the global minimum of the spin Hamiltonian, any network that achieves this bifurcation near the same fixed point can be used as an XY-Ising machine. The difference in the demonstrated behaviour of various XY-Ising devices comes, therefore, not from the particularities of the dynamical network that lies at the core of their operation but the annealing schedule of the parameters that allows achieving the same bifurcation.

High parallelism, processing speed, shared memory, energy efficiency and other advantages of analogue physical simulators led to the development of many competing platforms and physics-inspired optimisation methods. The analogue mode of operation of such platforms typically emulates interacting dynamical systems, and their behaviour near critical regimes, such as bifurcations, determines their optimisation properties. From the mathematics of dynamical systems, we know that many diverse systems behave similarly close to the bifurcation points and share similar universal descriptions by canonical models. Such canonical models can describe the systems’ operation near criticality even when the exact mathematical description of that system is not known or too complex. Here, we show how the popular physical platforms used as optimisers can all be described as canonical AHO networks.

When the physical platforms are presented with vastly different mathematical formulations, it is hard to directly compare the existing methods and the performance of such platforms. Such comparison requires optimal parameters for each platform that can be different for different problem structures. However, as we argue in our paper, as long as the primary mechanism for optimisation is based on the behaviour at the bifurcation point, the canonical complex AHO networks can represent all such models. The performance of the method and the physical platform depends only on the annealing schedule of the coefficients and the feasibility to realise such controls in practice.

The consequences of such canonical representation are rather significant. On the one hand, it removes the need to compare the structure of the dynamical model description of various spin Hamiltonian optimisers that use the behaviour at bifurcation to define the spins. On the other hand, it formulates a universal problem of determining the annealing schedule of the parameters of the canonical complex AHO model as the route to global minimisation. The comparison between different XY-Ising bifurcation machines reduces to the comparison of actual physical parameters of their operation, such as speed, scalability, accuracy, energy efficiency, quantum speed-up, etc. As such, one of the main questions we hope will be addressed in the future is, therefore, as follows. Given the coupling matrix \( J_{ij} \), what is the optimal time-dependence of the parameters in the canonical complex network that bifurcates the network elements to the global minimum of universal spin Hamiltonians?
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