CPT symmetry and The Equality of Mass and Lifetime

J.C. Yoon

University of Washington

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CPT theorem has been known to imply the equality of mass and lifetime between particle and antiparticle even if C(charge conjugation) symmetry is violated. However, its mathematical verification is insufficient and limited as it considers only one type of C violation. Here C, not CPT, symmetry will be shown to imply the equality of mass and coupling constants and the lifetime could be the same regardless of C and CPT violation. And we conclude that the equality of mass and lifetime is prerequisite for the CPT theorem to be valid and it is not implied by CPT symmetry.

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I. INTRODUCTION

It is commonly believed that the CPT theorem implies that a particle and its antiparticle must have the same mass and lifetime[1]. However, no rigorous theoretical investigation has been provided especially when C symmetry is violated. Let us first review and define C symmetry and particle and antiparticle symmetry with care and investigate the proofs of the mass and lifetime equality.

II. THE DEFINITION OF SYMMETRY

A particle and its antiparticle can be described by two independent Dirac equations

\[(i\partial - m)\psi = +eA_\mu \gamma^\mu \psi \quad \text{for particle}\]
\[(i\partial - m)\psi^c = -\tau A^c_\mu \gamma^\mu \psi^c \quad \text{for antiparticle}\]

and they are expected to obey the same equation with the same mass \((m = \bar{m})\) and opposite charges \((\tau = \bar{\tau})\). Under the transformation of particle and antiparticle on \(\psi\) and the potential \(A\),

\[\psi \rightarrow \psi^c = \eta_c C\psi^T\]
\[A_\mu \rightarrow A^c_\mu = -A_\mu.\]

we have

\[(i\partial - m)\psi^c = -eA^c_\mu \gamma^\mu \psi^c\]
\[(i\partial - m)\psi^c = -\tau A^c_\mu \gamma^\mu \psi^c \quad \text{for antiparticle}\]

However, these charge conjugation transformations of \(\psi \rightarrow \psi^c\), \(A_\mu \rightarrow A^c_\mu\) is insufficient and it requires the same mass and coupling constant \((\bar{m} = m\) and \(\bar{\tau} = \tau)\) to obtain the equation for antiparticle.

Let us define charge conjugation \((C)\) as a symmetry that we can find between the equations for particle and antiparticle and particle and antiparticle symmetry by the equality of mass and lifetime. C symmetry could be violated when an antiparticle, for example, simply have different mass or coupling constants in strength, i.e. \(m \neq \bar{m}\) or \(e \neq \bar{\tau}\),

\[\psi \rightarrow \psi^c = \eta_c C\psi^T\]
\[A_\mu \rightarrow A^c_\mu = -A_\mu.\]

and particle and antiparticle symmetry is also violated as they have different masses and lifetimes. Another type of C violation could be found when more than two interactions involve. Let us introduce the interactions with coupling constants \(g_1, g_2\) and a potential that violates C symmetry as they have different signs from \(eA_\mu \gamma^\mu \psi\) for antiparticle,

\[(i\partial - m)\psi = +eA_\mu \gamma^\mu \psi + g_1 A_\mu \Gamma^\mu \psi + g_2 B_\mu \Gamma^\mu \psi\]
\[(i\partial - m)\psi^c = -eA^c_\mu \gamma^\mu \psi^c + g_1 A^c_\mu \Gamma^\mu \psi^c + g_2 B^c_\mu \Gamma^\mu \psi^c\]

where \(\Gamma_\mu\) represents the general structure of interaction. The lifetime \(\tau\) of the particle is given by the reciprocal of the total decay rate \(\Gamma = 1/\tau\)

\[\Gamma \propto |M_e|^2 + |M_{g1}|^2 + |M_{g2}|^2\]

if the final particles are different and

\[\Gamma \propto |M_e \pm M_{g1}|^2 + |M_{g2}|^2\]

if we have the interference of \(M_e\) and \(M_{g1}\) interactions decaying into the same final particles. In general, we have

\[\Gamma \propto |M_e e^{i\phi_1} \pm M_{g1} e^{i\phi_2}|^2 + |M_{g2}|^2\]

where \(\phi_{e\pm}\), \(\phi_{g1\pm}\) four independent phases for particle and antiparticle. We may have different lifetimes for particle and antiparticle due to the possible interferences, but, the relative signs of interference and interaction terms in the Dirac equation may not be explicitly related due to phase factors. Therefore, C symmetry could be violated even if a particle and its antiparticle have the same mass and lifetime and the lifetimes of particle and antiparticle could be different even when C symmetry is conserved as the interference of interactions occurs. It is because
the equations based on the definition of C symmetry are not fundamental to the practical calculation of interactions, lifetime. When we define C symmetry, the interactions are considered as a part of the equations,

\[(i\overrightarrow{\partial} - m)\psi = \pm eA_\mu\gamma^\mu\psi\] for defining C

but the practical calculation of lifetime is based on the free Dirac equation and interactions in separate, which neglects their relative signs

\[(i\overrightarrow{\partial} - m)\psi = 0\] and \(\pm eA_\mu\gamma^\mu\psi\) for lifetime

and the opposite signs are known from other independent physical observations indicating two opposite charges.

### III. PROOF OF MASS EQUALITY

Let us review how the mass and lifetime equality between particle and antiparticle are proved and investigate these proofs\[3, 4\].

**Theorem.** Mass equality between particles and antiparticles

**Proof.** Let \(|p\rangle_m\) be the state of an elementary particle at rest with its z component angular momentum \(m\). Since the mass of the particle is given by the expectation value,

\[mass_p = \langle p|H|p\rangle_m\]

where \(H\) is the total Hamiltonian, real and independent of \(m\). Hence it equals its complex conjugation. We have

\[mass_p = \langle p|H|p\rangle_m = \langle p|\Theta^{-1}\Theta\Theta^{-1}\Theta|p\rangle_m\]

where the operator \(\Theta\) is defined to be

\[\Theta \equiv CPT\]

Apart from a multiplicative phase factor, under \(C\) the state becomes \(|\overline{p}\rangle_m\), under \(P\) it remains itself, but under \(T\), \(m\) is changed into \(-m\), as given by

\[T|j, m\rangle = UT|j, m\rangle^* = U|j, m\rangle = e^{i\pi J_z}|j, m\rangle = (-1)^{j+m}|j, -m\rangle\]

Therefore,

\[\Theta|p\rangle_m = e^{i\theta}|\overline{p}\rangle_m\]

Since \(\Theta\Theta^{-1} = H\) by the CPT theorem, the above expression can also be written as

\[mass_p = \langle \overline{p}|H|\overline{p}\rangle_m\]

\[mass_\overline{p} \equiv \langle \overline{p}|\overline{H}|\overline{p}\rangle_m\]

Therefore,

\[mass_p = mass_\overline{p}\]

when \(H = \overline{H}\).

In this proof, not only do we need the CPT theorem \((\Theta\Theta^{-1} = H)\), but also it requires that \(H = \overline{H} \equiv CHC^{-1}\). The operation of CPT on a particle already implies \(C\) symmetry \((H = \overline{H})\). If the particle remains itself under \(P\), the Hamiltonian should be also be the same

\[H = PHP^{-1}\]

and since the mass of a particle is the same regardless of \(z\) component of angular momentum, \(m\) can be ignored and thus

\[H_{mass} = TH_{mass}T^{-1}\]

for the mass term

Now we have

\[\Theta H_{mass}\Theta^{-1} = CPTH_{mass}T^{-1}P^{-1}C^{-1}\]

\[= CPH_{mass}P^{-1}C^{-1}\]

\[= CH_{mass}C^{-1}\]

\[= H_{mass}\]

by definition \(\overline{H} \equiv CHC^{-1}\) and thus \(H = \overline{H}\).

However, this proof would fail with \(C\) violation even if \(CPT\) symmetry is conserved. For example, if \(CPT\) theorem holds \((\Theta\Theta^{-1} = H)\) but \(C\) is violated in a way \(\overline{H} \equiv CHC^{-1} = -H\), then the mass of particle and antiparticle is different.

\[mass_p = \langle p|H|p\rangle_m\]

\[= -\langle p|C^{-1}CHC^{-1}C|p\rangle_m\]

\[= -\langle \overline{p}|\overline{H}|\overline{p}\rangle_m\]

\[\equiv -mass_\overline{p}\]

And in order to investigate whether the masses of particle and antiparticle, the equation of particle and antiparticle is of interest, not that of CPT transformed particle is unless it is identified as antiparticle \((\Theta\Theta^{-1} = CHC^{-1})\).

\[(i\overrightarrow{\partial} - m)\psi = 0\]

for particle

\[(i\overrightarrow{\partial} - m)\psi^{CPT} = 0\]

for CPT transformed particle

\[(i\overrightarrow{\partial} - m)\psi^{CPT} = 0\]

for antiparticle

Therefore, the mass of particle and antiparticle should be proved by the conservation of \(C(H = \overline{H} \equiv CHC^{-1})\), not by the \(CPT\) theorem, as in

\[mass_p = \langle p|H|p\rangle_m\]

\[= \langle p|C^{-1}CHC^{-1}C|p\rangle_m\]

\[= \langle \overline{p}|\overline{H}|\overline{p}\rangle_m\]

\[\equiv mass_\overline{p}\]

To be more accurate, the equality of mass is prerequisite for \(C\) symmetry since the transformation of equation \((CHC^{-1})\) cannot be equated to \(\overline{H}\) unless its mass and coupling constants consisting of the equation are the same. The free Dirac equation(or the Dirac
Hence, for example, cannot be transform from one to another \((\mathcal{L}_0 \rightarrow \overline{\mathcal{L}_0})\) unless \(m = \overline{m}\).

\[
\mathcal{L}_0 = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\
\overline{\mathcal{L}_0} = \overline{\psi}(i\gamma^\mu \partial_\mu - \overline{m})\psi
\]

Therefore, the mass of particle and antiparticle is required to be the same for the conservation of \(C\) symmetry \((H = \overline{H} \equiv CHC^{-1})\) and the proper definition of mass representation should show that \(C\) symmetry, not the \(CPT\) theorem, implies the mass equality.

**IV. PROOF OF LIFETIME EQUALITY**

Consider a Hamiltonian

\[
H = H_{\text{strong}} + H_{\text{weak}}
\]

where both terms are invariant under a proper Lorentz transformation and \(H_{\text{strong}}\) is assumed to be invariant under \(C, P\) (Parity), and \(T\) (Time reversal), for example,

\[
CH_{\text{strong}}C^{-1} = H_{\text{strong}}
\]

**Theorem.** If a particle \(A\) decays through the interaction \(H_{\text{weak}}\), and if the particle and its antiparticle \(\overline{A}\) do not decay into the same final products (as e.g. when \(A\) is charged), then to the lowest order of \(H_{\text{weak}}\) the lifetimes of \(A\) and \(\overline{A}\) are the same, even if \(H_{\text{weak}}\) is not invariant under charge conjugation.

**Proof.** Consider particle \(A\) with spin zero and the final states \(B\) and \(\overline{B}\) in the decays would also have spin zero.

\[
A \rightarrow B, \quad \overline{A} \rightarrow \overline{B},
\]

Using the identity

\[
\langle \psi_1 | \psi_2 \rangle^* = \langle T\psi_1 | T\psi_2 \rangle,
\]

one obtains

\[
\langle B | H_{\text{weak}} | A \rangle^* = \langle TB | TH_{\text{weak}}T^{-1} | TA \rangle
\]

\[
= \langle TB | C^{-1}P^{-1}H_{\text{weak}}PC | TA \rangle,
\]

by the CPT theorem. If \(H_{\text{weak}}\) commutes (or anticommutes) with \(P\), then

\[
\langle B | H_{\text{weak}} | A \rangle^* = \pm \langle TB | C^{-1}H_{\text{weak}}C | TA \rangle.
\]

For a spinless system,

\[
| TA \rangle = | A \rangle, | TB \rangle = | B \rangle.
\]

Hence

\[
\langle B | H_{\text{weak}} | A \rangle^* = \pm \langle B | C^{-1}H_{\text{weak}}C | A \rangle
\]

\[
= \pm \langle CB | H_{\text{weak}} | CA \rangle = \pm \langle \overline{B} | H_{\text{weak}} | \overline{A} \rangle.
\]

This shows that the lifetimes of \(A\) and \(\overline{A}\) are the same.

Following this proof, one can also show the lifetime equality of particle and antiparticle under \(CPT\) violation.

**Theorem.** The lifetimes of \(A\) and \(\overline{A}\) are the same, even if \(H_{\text{weak}}\) is not invariant under \(CPT\).

**Proof.** The Hamiltonian \(H\) commutes with \(\Theta \equiv CPT\) by the CPT theorem

\[
\Theta H\Theta^{-1} = +H
\]

and if CPT is violated, then we have

\[
\Theta H\Theta^{-1} = -H
\]

Following the same steps as before, one obtains

\[
\langle B | H_{\text{weak}} | A \rangle^* = -\langle TB | C^{-1}P^{-1}H_{\text{weak}}PC | TA \rangle,
\]

if CPT is violated \((C^{-1}P^{-1}T^{-1}H_{\text{weak}}TPC = -H_{\text{weak}})\). If \(H_{\text{weak}}\) commutes (or anticommutes) with \(P\), then

\[
\langle B | H_{\text{weak}} | A \rangle^* = \mp \langle TB | C^{-1}H_{\text{weak}}C | TA \rangle.
\]

For a spinless system,

\[
\langle B | H_{\text{weak}} | A \rangle^* = \mp \langle \overline{B} | H_{\text{weak}} | \overline{A} \rangle.
\]

This shows that the lifetimes of \(A\) and \(\overline{A}\) are the same.

These theorems are limited as they exclude \(C\) violations of different mass and coupling constants \((m \neq \overline{m}\) or \(e \neq \overline{e}\) where we cannot find a proper transformation of equations \((\overline{H} \neq CHC^{-1}\) and thus \(\overline{|A\rangle \neq |C|A}\)). If \(C\) symmetry is violated with different coupling constants, then the lifetimes of particle and antiparticle would be different. However, if particle and antiparticle have the same mass and coupling constants, the lifetimes of particle and antiparticle are the same regardless of \(C\) and \(CPT\) symmetry if no interference is assumed

\[
\Gamma \propto |M_w|^2 + |M_g|^2
\]

where the signs of \(S - matrix M_w, M_g\) are irrelevant to the physical observation of lifetime.

**V. CONCLUSION**

The definitions and differences of \(C\) symmetry and particle and antiparticle symmetry are discussed and the proofs of mass and lifetime equality are reviewed and criticized for their ambiguity and exclusiveness. The conservation of \(C\), not \(CPT\), symmetry requires the mass and coupling constants to be the same between particle and antiparticle and the lifetime could be the same regardless of \(C\) and \(CPT\) violation. Therefore, we can conclude that the \(CPT\) theorem does not imply the equality of mass and lifetime and particle and antiparticle symmetry, not \(CPT\), is appropriate when the equality of mass and lifetime is implied \([1, 5, 6, 7, 8, 9, 10]\).
[1] H. Murayama, Phys. Lett. B 597 (2004) 73.
[2] C. Itzykson and J.B. Zuber, Quantum Field Theory, McGraw-Hill, Inc., (1980)
[3] T.D. Lee, Particle Physics and Introduction to Field Theory, Harwood Academic Publishers, (1981).
[4] T.D. Lee and C.N. Yang, Phys. Rev. 106, (1957) 340.
[5] R.D. McKeown and P. Vogel, Phys. Rept. 394 (2004) 315.
[6] John N. Bahcall, M.C. Gonzalez-Garcia and Carlos Pena-Garay, JHEP 0408 (2004) 016.
[7] V. Barger, D. Marfatia and K. Whisnant, Int. J. Mod. Phys. E 12 (2003) 569.
[8] KamLAND Collaboration, K. Eguchi et al., Phys. Rev. Lett. 90 (2003) 021802.
[9] KTeV Collaboration, A. Alavi-Harati et al., Phys. Rev. D 67 (2003) 012005.
[10] G. Barenboim, L. Borissov, J. Lykken and A. Yu. Smirnov, JHEP 0210 (2002) 001.