A few remarks on the Generalized Vanishing Conjecture

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Abstract. We show that the Generalized Vanishing Conjecture
\[ \forall m \geq 1 [\Lambda^m f^m = 0] \implies \forall m \gg 0 [\Lambda^m (gf^m) = 0] \]
for a fixed differential operator \( \Lambda \in k[\partial] \) follows from a special case of it, namely that the additional factor \( g \) is a power of the radical polynomial \( f \). Next we show that in order to prove the Generalized Vanishing Conjecture (up to some bound on the degree of \( \Lambda \)), we may assume that \( \Lambda \) is a linear combination of powers of distinct partial derivatives. At last, we show that the Generalized Vanishing Conjecture holds for products of linear forms in \( \partial \), in particular homogeneous differential operators \( \Lambda \in k[\partial_1, \partial_2] \).

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Introduction. The Jacobian Conjecture has been the topic of many papers (see [1] and [4] and its references). Until recently, there was no framework available in which this notorious conjecture could be studied. Based on work in [2], Wenhua Zhao published several papers ([7], [8], [9], [10]) which have changed this situation dramatically.

In these papers, he introduced various conjectures which imply the Jacobian Conjecture. One of these conjectures is the so-called Generalized Vanishing Conjecture. To describe it, we fix the following notations. Let \( k[x] = k[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over a field \( k \). By \( D = k[\partial_1, \ldots, \partial_n] \) we denote the ring of differential operators with constant coefficients. By \( k \)-linearity of taking partial derivative, the following defines \( \Lambda f \in \)

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k[x] with \( \Lambda \in D \) and \( f \in k[x] \) uniquely:

\[
(\Lambda_1 + \Lambda_2)f = \Lambda_1 f + \Lambda_2 f, \quad (\Lambda_1 \Lambda_2)f = \Lambda_1 (\Lambda_2 f), \quad \partial_i f = \frac{\partial}{\partial x_i} f,
\]

where \( \Lambda_1, \Lambda_2 \in D \) and \( f \in k[x] \).

**Generalized Vanishing Conjecture (GVC).** Let \( \Lambda \in D \) and \( f \in k[x] \) be such that

\[
\Lambda^m f^m = 0 \quad \text{for all } m \geq 1.
\]

Then for all \( g \in k[x] \), we have

\[
\Lambda^m (gf^m) = 0 \quad \text{for all } m \gg 0.
\]

It was shown in [8, Theorem 7.2] that for a field \( k \) of characteristic zero, a positive answer to this conjecture (in all dimensions), with \( \Lambda \) being the Laplace operator \( \Delta \) (and \( g = f \)), implies the Jacobian Conjecture. For a field of positive characteristic \( p \), the GVC can easily be proved because \( \Lambda^p g = 0 \) for all \( \Lambda \in k[\partial] \) with trivial constant part and all \( g \in k[x] \).

The main results of this paper can be described as follows. First we show that the \( g \)'s in the formulation of the GVC can be replaced by powers of \( f \). We will do that in a corollary of the following theorem.

**Theorem 1.** Let \( \tilde{f}, g \in k[x] \) and \( m \geq d \). Suppose that

\[
\Lambda^{m-d} \tilde{f} = 0
\]

for some \( \Lambda \in D \). If \( \deg g \leq d \), then

\[
\Lambda^m (g \tilde{f}) = 0
\]

as well.

**Corollary 2.** The GVC (for some \( \Lambda \in D \)) is equivalent to the following statement: if \( f \in k[x] \) is such that

\[
\Lambda^m f^m = 0 \quad \text{for all } m \geq 1,
\]

then for each \( d \geq 1 \), we have

\[
\Lambda^m f^{m+d} = 0 \quad \text{for all } m \gg 0.
\]

**Proof.** The statement of Corollary 2 follows from the GVC (for \( \Lambda \in D \)) by taking \( g = f^d \), so it remains to prove the converse. For that purpose, let \( g \in k[x] \) and choose \( d \geq \deg g \). Combining the condition \( \Lambda^m f^m = 0 \) for all \( m \geq 1 \) of the GVC (for \( \Lambda \in D \)) and the statement of corollary 2, we get \( \Lambda^m f^{m+d} = 0 \) for all \( m \gg 0 \), which is equivalent to \( \Lambda^{m-d} f^m = 0 \) for all \( m \gg 0 \). By taking \( \tilde{f} = f^m \) in Theorem 1, we subsequently obtain \( \Lambda^m (g f^m) = 0 \) for all \( m \gg 0 \). □

In the proof of [6, Theorem 1.5], Corollary 2 is proved for \( \Lambda = \Delta \), the Laplace operator. The claim of [6, Theorem 1.5] is that one can even take \( d = 1 \) in Corollary 2 when \( \Lambda = \Delta \), which subsequently follows from (3) \( \Rightarrow \) (2) of [8, Theorem 6.2]. Hence we can take \( g = f \) in the GVC when we restrict ourselves to \( \Lambda = \Delta \).