Non-equilibrium Floquet steady states of time-periodic driven Luttinger liquids

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Time-periodic driving facilitates a wealth of novel quantum states and quantum engineering. The interplay of Floquet states and strong interactions is particularly intriguing, which we study using time-periodic fields in a one-dimensional quantum gas, modeled by a Luttinger liquid with periodically changing interactions. By developing a time-periodic operator algebra, we are able to solve and analyze the complete set of non-equilibrium steady states in terms of a Floquet-Bogoliubov ansatz and known analytic functions. Complex valued Floquet eigenenergies occur when integer multiples of the driving frequency approximately match twice the dispersion energy, which correspond to resonant states. In experimental systems of Lieb-Liniger bosons we predict a change from powerlaw correlations to dominant collective density wave excitations at the corresponding wave numbers as the frequency is lowered below a characteristic cutoff.

Introduction. Controlled time-periodic driving of quantum systems has recently pushed the development of fascinating quantum phenomena such as topological phases [1, 2], many body localization [3], cavity optomechanics [4–12], Floquet time crystals [13, 14], artificial gauge fields [15–20], transmission resonances [21–23], dynamic localization [24–29], pairing [30, 31], driven Bose-Einstein condensates [32–39], and anyons [40–44]. However, when complications from strong correlations could give much needed insight in Floquet-induced strong correlations near resonances.

In this letter we now provide the many-body eigenstate solution and report resonance phenomena in one-dimensional (1D) interacting quantum systems with time-periodically driven parameters. Our analysis applies to time-periodic driving of generic Tomonaga-Luttinger liquids (TLL), which describe a large class of effectively 1D many-body systems [48] and can also be realized using ultra-cold gases [49–52]. The time-evolution of an initially prepared state in a TLL has been calculated before [53–60], but much less is known about the nature of possible non-equilibrium steady states under periodic driving. It is therefore desirable to obtain the full eigenbasis of the Floquet eigenvalue problem, which gives systematic information about all stable steady states and their corresponding dominant correlations. We now obtain the explicit steady state solutions of the time-dependent Schrödinger equation of a quantum many-body system in terms of a time-periodic operator algebra, which not only allow a full analysis using closed analytic functions, but also show regions of instabilities in frequency and momentum space. We therefore predict large-amplitude density waves at the characteristic wave vectors in trapped ultracold boson systems.

Model. We will develop a Floquet-Bogoliubov ansatz for general driven TLL models. To make concrete predictions for decoupled 1D tubes of interacting bosons in optical lattices [49–52] we choose the Lieb-Liniger Hamiltonian [61] as a starting point

\[ H_0 = -\frac{1}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i<j}^{N} \delta(x_i - x_j) \]

where \( \hbar = 1 \) and \( g = \frac{2\alpha_0}{m a_\perp (a_\perp - 1.93 a_0)} \) is the 1D onsite interaction strength, which is tunable via the 3D scattering length \( a_0 \) and the perpendicular confinement length \( a_\perp \) [62–64]. The static system is integrable and correlations are known to be well described by a TLL model in the long-wavelength limit \( q < q_c \) [48, 65, 66].

\[ H_{TLL} = \frac{v}{2\pi} \int dx \left[ K (\partial_x \theta)^2 + K^{-1} (\partial_x \phi)^2 \right] \sum_{q > 0} H_q \]

\[ H_q = v_F q [(1 + g_2) 2 J_{0,q} + g_2 (J_{+q} + J_{-q})] \]

where \( 2 J_{0,q} = b_{L,q}^\dagger b_{L,q} + b_{R,q}^\dagger b_{R,q} \) and \( J_{\pm q} = J_{\pm q}^\dagger = b_{L,q}^\dagger b_{R,q}^\dagger \) in terms of bosonic operator \( b_{L/R,q}^\dagger \) which create left- and right-moving density waves at wave-vector \( q \) [48]. For the Lieb-Lininger model the Luttinger parameter \( K \) is known exactly [65]. It depends only on the ratio \( mg/n \), where \( n \) is the 1D particle density. The cutoff \( q_c \) above which the TLL description fails has also been determined [64]. The scattering amplitudes \( g_2 \) and \( q_c \) are rescaled from the traditional “g-ology” scheme [65] and related to \( v \) and \( K \) by [48]

\[ v K = v_F (1 + g_4 - g_2), \quad \frac{v}{K} = v_F (1 + g_4 + g_2) \]
where $v_F = \pi n/m$. Values of $g_2 = g_4 = (1/K^2 - 1)/2$ for the Lieb-Liniger model are shown in Fig. 1.

We now turn to systems with time-periodically changing control parameters $a_0$ and $a_\perp$, which will result in time-periodic couplings $g_2$ and $g_4$, all of which can be determined exactly. Any desired time-periodic couplings can be created by suitable fields given by the inverted relations, including a pure sinusoidal behavior \cite{71}.

$$2g_2(t) = 2g_4(t) = \tilde{\rho} + \rho \cos \omega t$$ \hspace{1cm} (4)

with constant parameters $\tilde{\rho}$ and $\rho$. We will later consider more general behavior. For the Lieb-Liniger model it is known that $vK = v_F$ and $K > 1$ \cite{63, 64}, so that from Eq. (3) $-1/2 < g_2 = g_4 < 0$ as shown in Fig. 1.

**Floquet ansatz.** We now seek to solve the time-dependent Schrödinger equation $i\hbar \partial_t |\Psi(t)\rangle = H_0(t) |\Psi(t)\rangle$ for each momentum $q$ separately, which remains a good quantum number and can be omitted in the following. According to Floquet theory \cite{27, 46, 47, 71}, there exists a complete set of quantum numbers $n$ for steady state solutions $|\Psi_n(t)\rangle = e^{-i\epsilon_n t} |u_n(t)\rangle$. Here $|u_n(t)\rangle = |u_n(t + T)\rangle$ with $T = 2\pi/\omega$ obey the Floquet equation

$$(H(t) - i\partial_t)|u_n(t)\rangle = \epsilon_n |u_n(t)\rangle$$ \hspace{1cm} (5)

where $\epsilon_n$ are the Floquet quasienergies. We now wish to map this problem onto a static eigenvalue problem \cite{27}.

$$\hat{H}|n\rangle = (HQ\dagger - iQ\partial_t Q)|n\rangle = \epsilon_n |n\rangle.$$ \hspace{1cm} (6)

Floquet theory has been reviewed extensively \cite{27, 46, 47, 71}, but the ansatz goes beyond the usual time-evolution approach since it makes the problem static, diagonalizes it in the original Hilbert space, and determines all steady states for all times in one single unitary transformation $Q(t)$, which is an ambitious goal. The relation of $Q = \sum_n |n\rangle\langle u_n(t)|$ to Floquet concepts is discussed in the Appendix: While the time-evolution operator $W(t)$ is not the topic here, it can be simply obtained $W(t) = Q(t)e^{-i\hat{H}t}Q(0)$. However, it is not possible to construct $Q$ using $W$. Likewise, the so-called Floquet Hamiltonian $\hat{H}_F = Q\dagger(0)\hat{H}Q(0)$ can be found using $Q$. We now proceed to find an explicit expression for $Q(t)$ for the model in Eq. (2).

**Floquet Bogoliubov solution.** The goal is to find a static eigenbasis in the rotating frame, which can be achieved if $\hat{H}$ becomes diagonal and time-independent. The interacting model $H_0(t)$ in Eq. (2) is defined in left and right oscillator Hilbert spaces $\chi = L, R$, so a static solution must be of the form $\hat{H} = \Delta \sum b_k^\dagger b_k$. The characteristic commutation relation $[\hat{H}, b_{k, R}] = -\Delta b_{L, R}$ transforms to

$$[(H(t) - i\partial_t), \beta_{L, R}(t)] = -\Delta \beta_{L, R}(t)$$ \hspace{1cm} (7)

$$\beta_{L, R}(t) = Q^\dagger(t) b_{L, R}Q(t) = \gamma_1(t) b_{L, R} + \gamma_2(t) \beta_{L, R}$$ \hspace{1cm} (8)

where we have used a general Floquet-Bogoliubov ansatz for $Q$ in Eq. (8) with the canonical constraint $|\gamma_1(t)|^2 - |\gamma_2(t)|^2 = 1$. The defining relation in Eq. (7) provides differential equations for the time-periodic coefficients $\gamma_{1,2}$

$$i\dot{\gamma}_{1,2} = (\Delta \mp \lambda_1)\gamma_{1,2} \mp \lambda_2 \gamma_{2,1}$$ \hspace{1cm} (9)

where $\lambda_1 = v_F q(1 + g_4)$ and $\lambda_2 = v_F qg_2$. The relation \cite{9} applies to general TLL, but for the Lieb-Liniger model it simplifies since $\lambda_1 - \lambda_2 = vF$ is constant due to Galilean invariance. Using $f_{\pm}(t) = e^{i\Delta t}(\gamma_{1,2}(t) \pm \gamma_{2,2}(t))$ and Eq. (4), we obtain a Mathieu equation

$$\dot{f}_{-}(t) + q^2 v_F^2 (1 + \tilde{\rho} + \rho \cos \omega t) f_{-}(t) = 0$$ \hspace{1cm} (10)

and $f_+ = -i\dot{f}_-/q\omega$. The solution can be expressed as

$$f_{-}(t) = c_1 C(a, p, \tau) + c_2 S(a, p, \tau)$$ \hspace{1cm} (11)

where $a = 4q^2/v_F^2(1 + \tilde{\rho})$, $p = -2q^2/v_F^2 \rho$, $\tau = \omega t/2$, and $C(a, p, \tau)$, $S(a, p, \tau)$ are even and odd Mathieu functions normalized with $C(a, p, 0) = S(a, p, \pi) = 1$. The coefficients $c_{1,2}$ are determined by the time-periodicity of steady states $|u_n(t)\rangle$ and operators $\beta(t)$, which also fixes the quantization condition for $\Delta$: We use Floquet’s theorem to write the solution of Eq. (10) $f_{-}(t) = e^{i\nu t} P_{\nu}(\tau)$ with $P_{\nu}(\tau) = P_{\nu}(\tau + \pi)$ \cite{72}. Since $\gamma_{1/2}$ are periodic, we find that the Mathieu characteristic exponent is $\nu = 2\Delta/\omega$, which must be real for stable steady states, just like for the Mathieu stability chart \cite{73} of Paul traps \cite{74}. From the normalization above follows $\cos(\pi \nu) = C(a, p, \pi)$, which gives (see Appendix)

$$\Delta = \arccos[C(a, p, \pi)/T], \quad c_2 = ic_1 \sin T\Delta,$$ \hspace{1cm} (12)

and $c_1$ is fixed by $|\gamma_1|^2 - |\gamma_2|^2 = 1$. Last but not least, we can use the solutions of $\gamma_{1,2}$ to uniquely define three real time periodic functions $\theta, \phi, r$, which parametrize an explicit expression of $Q(t)$ in Eq. (8) in terms of the SU(1,1) generators $J_0$, $J_-$, and $J_+$ in Eq. (2),

$$Q(t) = e^{i\theta J_0} e^{i(\gamma_1 J_- - \gamma_2 J_+)} e^{-i\phi J_0}$$ \hspace{1cm} (13)

with

$$\gamma_1 = e^{i(\theta - \phi)/2} \cosh r, \quad \gamma_2 = e^{i(\theta + \phi)/2} \sinh r.$$ \hspace{1cm} (14)
Before calculating physical observables, we need to analyse the stability of the differential equations, which may not always have a solution due to the periodicity constraint. In Fig. 2 (top) we plot the value of $\nu = 2\Delta/\omega$ as a function of $q v_F/\omega$ using $\rho = -0.6$ and amplitude $\rho = 0.25$. We observe that for certain regions of momenta there are no real solutions for $\Delta$. These “instability regions” will have interesting physical implications as discussed below. The stable regions are shown as a function of amplitude $\rho$ in Fig. 2 (bottom) for $\rho = -0.6$. For small $\rho$ the instability regions are equally spaced at integer values $\ell \in \mathbb{N}$ corresponding to $a = \ell^2$ or $\ell = 2\sqrt{1 + \rho q v_F/\omega}$. Defining an average velocity $\bar{v} = v_F/K = v_F/\sqrt{1 + \rho}$ the instability regions therefore correspond to integer multiples of frequency which match twice the interacting dispersion relation $\ell \omega = 2q v_F$, so the physical cause can be traced to resonant excitations on the linear branches of left movers from $-q v_F$ to right movers at $q v_F$ and vice versa.

As shown in Fig. 3 the region of instabilities also occur for more general TLL models where the restriction $g_3(t) = g_4(t)$ in Eq. (4) is lifted and/or contain higher harmonics. A general analytic solution remains elusive, but the corresponding differential equation (5) is still valid, which we have solved numerically by Fourier decomposition for several parameters. Instability regions are always expected since the problem is analogous to forbidden energy regions in a band structure of a periodic potential \[71\], which is of course generic. In Fig. 3 we show the behavior of $\nu$ as a function of $q v_F/\omega$ for the case that only the $g_2$ scattering process is periodically modulated in time with $\rho = -0.6$ and $\rho = 0.25$ in Fig. (4) while $g_4 = -0.4$. While quantitative changes compared to Fig. 2 (top) can be identified, the regions of instabilities are again found at resonant wave vectors. In Figs. 2 (top) and 3 we see that $\nu \rightarrow 1$ near the unstable regions and the ratio $c_2/c_1$ in Eq. (12) becomes singular.

To understand the physical significance of the instability regions, it is essential to consider damping. Intrinsic damping is always present in the TLL description due to higher order boson-boson interaction terms \[48\], which lead to a finite quasiparticle life-time. A corresponding broadening of spectral peaks is seen numerically for finite energies and in finite systems \[76, 77\]. The size of damping is not universal since it depends on microscopic details including the system size, but it can be assumed to be smaller than all other energy scales. In Ref. \[74\] it was shown that solutions of damped Mathieu equations become always stable for amplitudes below a given threshold. We also find that a finite life-time $\tau_0$ in form of an imaginary energy correction $\text{Im} \lambda_1 = -1/\tau_0$ leads to convergence of instabilities as discussed below.

**Results.** We are now in the position to calculate physical observables. The main effect of the time periodic driving is the excitation of density waves in the steady...
For large driving amplitudes $\rho = -0.6$ using different amplitudes $\rho$. For $q \rightarrow 0$ the static limit $|\bar{\gamma}_2|^2 = (1/\bar{K} + \bar{K} - 2)/4$ is recovered (red dot). Inset: With finite life-time $\tau_0 = 10^4/\nu_F q$ the divergent regions are turned into overwhelmingly large maxima.

state. The number of density excitations $b_\chi^\dagger b_{\chi'}^\dagger |u_0(t)\rangle$ in the transformed ground state $|u_0(t)\rangle$ is given by

$$\eta_q = \langle u_0(t)|b_\chi^\dagger b_{\chi'}^\dagger |u_0(t)\rangle = \langle 0|\beta_\chi^\dagger \beta_{\chi'}^\dagger |0\rangle = |\gamma_2(t)|^2. \quad (15)$$

In Fig. 4 we plot the time average $\bar{\eta}_q$. For small $q$ we find that $\bar{\eta}_q$ approaches the static limit, but a strong divergence is observable as the instability region around $q_c$ is approached. In the inset of Fig. 4 we exemplarily show that a finite life-time $\tau_0 = 10^4/\nu_F q$ turns the divergences of $\bar{\eta}_q$ into large maxima around $q_c$. The height of the maxima can be tuned by the product $\rho \tau_0$.

A universal physical picture emerges analogous to a resonance catastrophe: A finite life-time has little effect away from resonance, but the resonance response is overwhelmingly large and proportional to $|\gamma_2(t)|^2$. Everywhere $n$ agrees with the local density approximation (LDA) of TLL correlations for the local trap potential. The trapping potential is therefore turning into an advantage: Instead of changing the frequency $\omega$, different regimes can be reached using the changing density $n$. As a function of $n$ we know $\bar{\nu} = \sqrt{ng/m}$, which in turn determines the resonant wave-vectors $q_0 = \omega/2\bar{\nu} < q_c$, in the TLL regime, such maxima will therefore dominate the correlations. We find that $q_c \sim \bar{\nu} m/2$ is a good estimate for the cutoff.

It is well known how TLL correlation are calculated, which is reviewed in the Appendix for the example of density-density correlations. An overwhelming maximum of $\bar{\eta}_q$ will dominate the correlations and lead to long-range density order (see Appendix)

$$\langle u_0|n(x)n(y)|u_0\rangle \propto \cos q_c (x - y). \quad (16)$$

For large driving amplitudes $\rho$ the magnitude of the induced density waves can become larger than the background density, which may lead to fragmentation into irregular density grains.

Discussion. The three energy scales $\omega$, $\bar{\nu} q_c$, and $\nu_F q$ determine the behavior of the system, which undergoes three different regimes as the frequency is changed:

1.) High frequencies: For $\omega \gg \bar{\nu} q_c$, the instability regions are outside the TLL regime, so the physical relevant region is free of resonances. The transformation $Q$ results in a systematic change of $\tilde{\eta}_q$ shown in Fig. 4 which approaches the static limit as $q \rightarrow 0$. The famous power-law correlations are corrected for intermediate distances, but the asymptotic static limit is recovered.

2.) Intermediate frequencies: As the frequency is lowered, the resonant wave-numbers $q_0 = \omega/2\bar{\nu} \approx q_c$ into the TLL regime. The number of density waves $\tilde{\eta}_q$, becomes very large, dominating the correlations in Eq. (11). Instead of powerlaw correlations, standing density waves at wave-numbers $q_c$ become stable throughout the system.

3.) Very low frequencies: For $\omega \ll \bar{\nu} q_c$, extended regions of instability will lead to a large number of excitations and heating, destroying the correlations.

Using typical experimental parameters for a 1D $^{87}$Rb gas from Ref. [51] of $n = 6.2 \times 10^6/m$ and $mg/n = 0.6$, we arrive at $K \approx 4$ and a cutoff of $\omega_c = \bar{\nu} = 2\pi \times 1.4$KHz in the middle of the trap. Driving the perpendicular confinement with a frequency of $\omega = 2\pi \times 500$Hz results in a resonance at $q_1 = \omega/2\bar{\nu} = 444 \times 10^3/m$. We therefore predict a standing density wave with wavelength $\lambda = 2\pi/q_1$ in the $\mu$m range, which is observable in real space with optical methods or an electron beam [51, 52].

The confining trapping potential leads to lower local densities $n$ near edges [51, 52] and reduced velocities $v_F = \pi n/m$. Everywhere $n$ agrees with the local density approximation (LDA) of TLL correlations for the local trap potential. The trapping potential is therefore turning into an advantage: Instead of changing the frequency $\omega$, different regimes can be reached using the changing density $n$. As a function of $n$ we know $\bar{\nu} = \sqrt{ng/m}$, which in turn determines the resonant wave-vectors $q_0 = \omega/2\bar{\nu} \approx q_c$, dominating the correlations. Note, that the density wavenumbers $q_c$ increase near edges in contrast to Fermionic Friedel density wavenumbers, which decrease with lower densities in a trap. In the proposed experiment, we therefore predict standing density waves at $\lambda \sim 14 \mu$m in the middle of the trap, which become shorter $\lambda \propto \sqrt{n}$ weaker near the edge. It is an interesting open problem if significant corrections would be observed when going beyond the present LDA analysis for a typical trap size of $120 \mu$m in Ref. [51].

Interesting many-body density excitations have been experimentally observed in driven 1D and 2D systems. For 1D elongated bosonic $^7$Li gases $\mu$m-size density grains emerge at $2\pi \times 80$Hz driving, which were identified as stable many-body effects. Experimental images show grains that appear smaller and weaker near edges which resemble features predicted above, but in a random pattern. All correlations disappear for very low frequencies $\omega$. A future grain size analysis as a function of $\omega$ and $n$ may clarify if there is a relation to TLL density waves in Eq. (16).

Conclusion. We have considered time-periodically
driven interacting systems in the steady state, corresponding to generic TLL models in general or the Lieb-Liniger model in particular, which e.g. applies to 1D confined atoms in ultra-cold gas experiments with tunable parameters. As we have shown, this setup is one of the very rare cases where the combination of non-equilibrium steady states with many-body physics can be analyzed in great detail. In particular, we have developed a Floquet-Bogoliubov approach by constructing time-periodic creation and annihilation operators, which solve the eigenvalue equation for the steady state by acting on the entire Floquet space. We also identify regions in frequency-momentum space where damped resonant behavior leads to a large number of density excitations. The known static powerlaw correlations \[ \frac{\omega}{\omega_0} \] are recovered for large distances \( \gg \omega / \omega_0 \), but for frequencies below the cutoff \( \omega_c \), characteristic density waves at integer-spaced resonant wave numbers \( \omega_c = \omega_0 / 2\pi \) will become dominant.

We emphasize that the proposed Floquet-Bogoliubov algebra is completely general and can be used to solve any time-periodically driven model with Bogoliubov-type interactions exactly. The explicitly known transformation \( Q \) maps all steady states onto a diagonal static oscillator basis for all times, which paves the way for a complete analysis of time-dependent effects in strongly interacting systems using a combination of powerful experimental, analytic, and numerical techniques.

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APPENDIX

Here we give details on the Floquet-Bogoliubov transformation, its relation to Floquet theory, the explicit form of the transformed ground state density-density correlations, and the application of Floquet’s theorem to Mathieu functions.

Relation of the time-dependent transformation to Floquet theory

The goal is to find all possible steady state solutions \( |u_n(t)\rangle = |u_n(t + T)\rangle \) under time-periodic driving at each time \( t \), which are defined by the Floquet eigenvalue equation

\[
(H - i\partial_t)|u_n(t)\rangle = \epsilon_n|u_n(t)\rangle,
\]

where \( \epsilon_n \) are real quasi-energies. It should be noted that it is not always possible to find steady state solutions, but if they exist they form a complete basis in the original Hilbert space. The underlying Floquet theory has been discussed in a number of review articles \cite{27,16,47,71}, where different approaches are presented: By Fourier transforming into frequency space, the eigenvalue problem becomes static in an extended Hilbert space. Different frequency components can be perturbatively decoupled using a Magnus expansion, which is helpful in defining a so-called Floquet Hamiltonian \( H_F \). The Floquet Hamiltonian is useful since it determines the quasi-energies and the stroboscopic time evolution. The eigenstates of \( H_F \) are the steady states \( |u_n(0)\rangle \) at one instant in time only, so for the full time evolution it is necessary to additionally know the micromotion operator \( U(t) = \sum_n |u_n(t)\rangle\langle u_n(0)\rangle \), which is in general more difficult.

Our novel approach is now to solve the Floquet eigenvalue problem in one single step by mapping it to a static problem in the original Hilbert space

\[
\tilde{H}|n\rangle = (QHQ^\dagger - iQ\partial_t Q^\dagger)|n\rangle = \epsilon_n|n\rangle.
\]

If solutions to the original problem in Eq. (17) exist the unitary transformation \( Q \) can formally always be written as

\[
Q(t) = \sum_n |n\rangle\langle u_n(t)|,
\]

which transforms the entire basis of steady state solutions at each time into a diagonal static basis. This new transformation \( Q \) therefore does three things at once: It maps the system to a static problem in the original Hilbert space, it diagonalizes the eigenvalue problem, and it provides the time-dependent steady states for all times. All this is done without using a Fourier transform into an extended Hilbert space. Needless to say, each of the above steps is normally highly non-trivial, so finding such a transformation \( Q \) into a diagonal rotating frame is very ambitious indeed. Note, that \( Q(t) = Q(t + T) \) is time periodic, but we need not assume that \( Q(t) \) becomes the identity at the initial time or any other time.

The operator \( Q \) must therefore not be confused with the time-evolution operator \( W \)

\[
W(t) = \sum_n |u_n(t)\rangle\langle u_n(0)|e^{-i\epsilon_n t} = U(t)e^{-iH_F t},
\]

which can be used to study the time-dependence of a given initial state. In particular, knowing the time evolution cannot be used to construct \( Q \), but the time evolution can always be expressed as

\[
W(t) = Q^\dagger(t)e^{-i\tilde{H}t}Q(0).
\]

Moreover, the Floquet Hamiltonian can be obtained by \( H_F = Q^\dagger(0)\tilde{H}Q(0) \), but again just knowing \( H_F \) cannot be used to extract the steady states for all times unless \( Q \) is known. Finally, also the micromotion operator \( U(t) = Q^\dagger(t)Q(0) \) and all steady states \( |u_n(t)\rangle = Q^\dagger(t)|n\rangle \) can be obtained with \( Q \), so such a transformation truly contains a complete solution of the many-body driven system.
Explicit form of the Floquet Bogoliubov transformation

The model of interest can conveniently be expressed in terms of SU(1,1) generators

\[ H(t) = \lambda_1 J_0 + \lambda_2 (J_+ + J_-) , \]

(22)

where

\[ 2J_0 = b_L^\dagger b_L + b_R^\dagger b_R , \quad J_+ = J_- = b_L^\dagger b_R^\dagger , \]

(23)

and \( \lambda_1 = \nu F q(1 + g_4) \) and \( \lambda_2 = \nu F^2 g_2 \) are the time-periodic coupling parameters. For the static case it is known that the transformation \( U_1 = e^{r(J_+ - J_-)} \) can be used for diagonalization, using the following relations for transformed operators \( \tilde{\Lambda} = U_1 \Lambda U_1^\dagger \) [64, 68]:

\[ \tilde{b}_R = b_R \cos r - b_L^\dagger \sinh r \]

(24)

\[ \tilde{b}_L = b_L \cos r - b_R^\dagger \sinh r \]

(25)

\[ \tilde{J}_0 = J_0 \cosh 2r - J_+ + J_- \sinh 2r \]

(26)

\[ \tilde{J}_+ = J_0 \sinh 2r + J_+ + J_- \cosh 2r \]

(27)

\[ \tilde{J}_- = -2J_0 \sinh 2r + (J_+ + J_-) \cosh 2r \]

(28)

For the time-dependent transformation, we need a more general ansatz parametrized in terms of three real time-periodic parameters \( \theta, \phi, r \):

\[ Q(t) = e^{i\theta J_0} e^{r(J_+ - J_-)} e^{-i\phi J_0} \]

(29)

\[ Q^\dagger = Q^\dagger = e^{i\phi J_0} e^{-r(J_+ - J_-)} e^{-i\theta J_0} . \]

(30)

Using relations Eqs. (24)-(28) together with gauge transformations, we find that the general time-dependent Bogoliubov transformation can be written as

\[ \beta_\chi = Q^\dagger b_\chi Q = \gamma_1 b_\chi + \gamma_2 b_\chi^\dagger \]

(31)

\[ Q b_\chi Q^\dagger = \gamma_1^\dagger b_\chi - \gamma_2^\dagger b_\chi \]

(32)

with \( \chi = L, R \) and

\[ \gamma_1 = e^{i(\theta - \phi)/2} \cosh r \]

(33)

\[ \gamma_2 = e^{i(\theta + \phi)/2} \sinh r \]

(34)

With this parametrization the transformed operators \( \tilde{\Lambda} = QAQ^\dagger \) can again be straightforwardly derived from Eqs. (31)-(33).

Note, that the three real parameters \( \theta, \phi, r \) give a general one-to-one parametrization of the complex functions \( \gamma_1 \) and \( \gamma_2 \) which obey \( |\gamma_1|^2 - |\gamma_2|^2 = 1 \). The functions \( \gamma_1 \) and \( \gamma_2 \) have been extensively discussed in the paper so the transformation \( Q \) is already explicitly known, but what is left to show in the following is that the Hamiltonian in Eq. (22) indeed becomes static and diagonal when using those functions.

The defining differential equation is given in Eq. (9) of the paper in terms of \( \gamma_1 \) and \( \gamma_2 \):

\[ i\dot{\gamma}_1 = (\Delta - \lambda_1)\gamma_1 + \lambda_2 \gamma_2 \]

(40)

\[ i\dot{\gamma}_2 = (\Delta + \lambda_1)\gamma_2 - \lambda_2 \gamma_1 \]

(41)

where \( \Delta \) is a real constant which is fixed by the constraint that both \( \gamma_1 \) and \( \gamma_2 \) are periodic as discussed in the paper. In terms of the parametrization \( \theta, \phi, r \), the differential equations become after multiplying by \( \exp(-i\theta) \) respectively:

\[ i\dot{r} \sinh r - \frac{\dot{\theta}}{2} \cosh r = (\Delta - \lambda_1) \cosh r + \lambda_2 e^{i\phi} \sinh r \]

\[ i\dot{r} \cosh r - \frac{\dot{\theta}}{2} \sinh r = (\Delta + \lambda_1) \sinh r - \lambda_2 e^{-i\phi} \cosh r \]

The imaginary parts of both equations give the same relation

\[ \dot{r} = \lambda_2 \sin \phi \]

(42)

The real parts give

\[ 0 = (\Delta + \dot{\theta}/2 - \lambda_1 - \dot{\phi}/2) \cosh r + \lambda_2 \cos \phi \sinh r \]

(43)

\[ 0 = (\Delta + \dot{\theta}/2 + \lambda_1 + \dot{\phi}/2) \sinh r - \lambda_2 \cos \phi \cosh r \]

(44)

For later use we take \( \sinh r - \theta/2 \cosh r \), which gives

\[ 0 = - (\lambda_1 + \dot{\phi}/2) \sinh 2r + \lambda_2 \cos \phi \cosh 2r \]

(45)

Likewise \( \sinh r - \theta/2 \cosh r \) gives

\[ \Delta = - \dot{\theta}/2 + (\lambda_1 + \dot{\phi}/2) \cosh 2r - \lambda_2 \cos \phi \sinh 2r \]

(46)

We now turn to identify the different parts in the transformed Hamiltonian

\[ \tilde{H} = QHQA^\dagger - iQ\partial_t Q^\dagger \]

(47)

Collecting all the terms of \( \tilde{H} \) from Eqs. (85)-(89) we find that the prefactor of the diagonal part \( 2J_0 \) reads

\[ (\lambda_1 + \dot{\phi}/2) \cosh 2r - \lambda_2 \cos \phi \sinh 2r - \dot{\theta}/2 \]

(48)

which is exactly \( \Delta \) according to Eq. (46) and therefore time-independent. The prefactor of the off-diagonal part \( e^{i\theta}J_+ \) is given by

\[ - \lambda_1 \sinh 2r + \lambda_2 (\cos \phi \cosh 2r - i \sin \phi) + i\dot{r} - \frac{\dot{\phi}}{2} \sinh 2r. \]

(49)
Using Eq. (12) for the imaginary part and Eq. (15) for the real part, we see that this expression is indeed zero, so that we have shown that the model in Eq. (22) transforms to
\[
\hat{H} = QHQ^\dagger - iQ\partial_t Q^\dagger = 2\Delta J_0 = \Delta(b_L^\dagger b_L + b_R^\dagger b_R) \tag{50}
\]
where the constant \(\Delta\) is determined by the constraint of periodicity and Floquet’s theorem as described in the text.

The transformed ground state

We give an explicit expression of the transformed ground state \(|u_0(t)\rangle = Q^\dagger|0\rangle\) and show that it indeed satisfies the condition
\[
\beta_L,R(t)|u_0(t)\rangle = 0 \quad \forall t. \tag{51}
\]
With Eq. (50), the calculation of \(Q^\dagger|0\rangle\) is split into three steps, one for each operator exponential. As \(|0\rangle\) is an eigenstate of \(J_0\), the first step yields \(e^{-iQ}\omega J_0|0\rangle = e^{-i\omega/2}|0\rangle\). Using the relation \[57\]
\[
e^{-r(J_+ - J_-)} = e^{-\tanh(r)J_+} e^{-2\ln(cosh(r))J_0} e^{\tanh(r)J_-} \tag{52}
\]
and \(J_-(0) = 0\), we find as an intermediate result
\[
Q^\dagger|0\rangle = e^{-i\omega/2} e^{i\omega J_0} e^{-\tanh(r)J_+} e^{-\ln(cosh(r))|0\rangle}, \tag{53}\]
With the definition of \(\gamma_1\) and \(\gamma_2\) in Eqs. (33) and (64) we further simplify \(e^{-\ln(cosh(r))} = 1/|\gamma_1|\) and \(\tanh(r) = |\gamma_2|/|\gamma_1|\). The action of the last part of the transformation is found to be
\[
e^{i\omega J_0} e^{-|\gamma_2|/|\gamma_1| J_+}|0\rangle = e^{i\omega/2} \sum_{n=0}^{\infty} (-|\gamma_2|/|\gamma_1| e^{i\omega})^n |n\rangle_L |n\rangle_R. \tag{54}\]
With \(e^{i\omega}|\gamma_2|/|\gamma_1| = |\gamma_2|/|\gamma_1|\) we finally find an explicit expression for the transformed ground state
\[
|u_0(t)\rangle = \frac{1}{\gamma_1} e^{-\frac{2\gamma_2}{\gamma_1} b_L^\dagger b_R^\dagger}|0\rangle. \tag{55}\]
As the first bracket in (56) vanishes trivially, the state \[56\] is indeed the ground state of the \(\beta_L(t)\) operator obeying Eq. (51) and analogously also for \(\beta_R(t)\). This is an important result, as \(|u_0(t)\rangle\) serves as base case for generating the entire set of steady states \(|u_n(t)\rangle\) by application of \((\beta_L(t))^{n+1}(\beta_R(t))^{nR}\) using Eq. (7) in the main article.

Correlation functions

It is well known how to calculate correlation functions of physical operators in terms of the diagonal boson model \(H^{48, 63, 64}\). Of particular interest for ultra-cold gases is the density-density correlation, which we will consider here to exemplify the calculation. The fluctuating density is given in terms of the bosonic field \(n(x) = \partial_x \phi(x) / \pi\), which has the mode expansion \[48, 63, 64\]
\[
\partial_x \phi = \sum_{q > 0} \left[ \frac{\pi q}{2L} e^{iqx} \left( b_{L,q} \right) + \text{h.c.} \right]. \tag{57}\]
For the density-density correlation function we find in the transformed ground state \(|u_0(t)\rangle = Q^\dagger|0\rangle\)
\[
\langle u_0|n(x)n(y)|u_0\rangle = \sum_{q > 0} \frac{q}{L^2} |\gamma_1 + \gamma_2|^2 \cos(q(x - y)). \tag{58}\]
where we have used Eq. (31). If the parameters \(\gamma_1,2\) are constant we recover the known asymptotic powerlaw behavior \[48, 63, 64\] \(\frac{\pi}{2\nu^2} |\gamma_1 + \gamma_2|^2 / |x - y|^2\). However, if a resonance \(q_r = \ell \omega / 2\nu\) is part of the linear TLL regime, the parameters \(\gamma_1,2\) will become very large as discussed in the main article. Therefore, the sum in Eq. (58) will be dominated by the corresponding instability region, leading to a long-range density order of the form
\[
\langle u_0|n(x)n(y)|u_0\rangle \propto \cos(q_r(x - y)). \tag{59}\]

Floquet solution in terms of Mathieu functions

The solution of the Mathieu equation
\[
\ddot{y}(\tau) + (a - 2p \cos 2\tau) y(\tau) = 0 \tag{60}\]
is usually discussed in terms of even and odd solutions, known respectively as Mathieu cosine \(C\) and Mathieu sine \(S\) functions. A general solution can be therefore written as
\[
y(\tau) = c_1 C(a, p, \tau) + c_2 S(a, p, \tau), \tag{61}\]
with \(\tau = \omega t / 2\). Floquet’s theorem states that the solutions of a time-periodic differential equation can always be written in the form
\[
y(\tau) = e^{\nu \tau} P_\nu(\tau) \tag{62}\]
with \( P_e(\tau) = P_e(\tau \pm \pi) \). We want to use the quantum number \( \nu \), which is commonly referred to as Mathieu characteristic exponent. Therefore, in this section we clarify the relation between the latter and the Mathieu functions. Comparing Eqs. \([61]\) and \([62]\) and employing the periodicity of \( P_e(\tau) \), we get the following relation

\[
c_1 C(a, p, \tau) + c_2 S(a, p, \tau) = e^{\mp i \nu \tau} (c_1 C(a, p, \tau \pm \pi) + c_2 S(a, p, \tau \pm \pi)) .
\]

(63)

Evaluating this expression in \( \tau = 0 \) and normalizing the Mathieu functions such that \( C(a, p, 0) = S(a, p, \pi) = 1 \), we obtain

\[
c_1 (e^{\pm i \nu \pi} - C(a, p, \pi)) = \pm c_2 S(a, p, \pi) = \pm c_2 ,
\]

(64)

from which we finally get

\[
\cos \pi \nu = C(a, p, \pi) , \quad \text{and} \quad c_2 = i c_1 \sin \pi \nu .
\]

(65)

**Static Bogoliubov transformation**

For the time-independent case, Hamiltonian \((2)\) in the main article can be expressed as

\[
H_q = v_F q [(1 + g_4) (2 J_{0,q} - 1) + g_2 (J_{+,q} + J_{-,q})] \tag{66}
\]

where \( 2 J_{0,q} = b_R^\dagger b_R + b_L^\dagger b_L, J_{+,q} = b_R^\dagger b_L^\dagger, J_{-,q} = J_{+,q}^\dagger \). For the sake of simplicity, in the following we will drop the index \( q \). We observe that \( J_i(i = 0, \pm) \) form a \( su(1,1) \) algebra; therefore the Tomonaga-Luttinger Hamiltonian can be diagonalized by the Schrieffer-Wolff transformation obtained through the unitary operator

\[
U = e^z \quad z = \theta (J_+ - J_-) \tag{67}
\]

\[
H \rightarrow \tilde{H} = U H U^\dagger \tag{68}
\]

\[
|GS\rangle \rightarrow |\tilde{GS}\rangle = U |GS\rangle .
\]

The transformed Hamiltonian \( \tilde{H} \) is diagonal in the old bosonic basis

\[
\tilde{H} = \Delta \left( b_R^\dagger b_R + b_L^\dagger b_L \right) + \Delta - v_F q (1 + g_4)
\]

\[
= 2 \Delta J_0 - v_F q (1 + g_4)
\]

(69)

with \( \Delta = v_F q \sqrt{(1 + g_4)^2 - g_2^2} \) if \( \tanh(2\theta) = \tanh(2\tilde{\theta}) \equiv g_2/(1 + g_4) \), implying \( \cosh(2\tilde{\theta}) = v_F q (1 + g_4)/\Delta \), \( \sinh(2\tilde{\theta}) = v_F q g_2/\Delta \).

Notice that we have used a passive transformation, where the operators are rotated, while the states defined by the original creation and annihilation operators stay the same.

**Classical model**

The Hamiltonian

\[
H(t) = (A + B(t)) (a^\dagger a + b^\dagger b) + C(t) (ab + a^\dagger b^\dagger) \tag{70}
\]

with real coefficients \( A, B(t), C(t) \) can be mapped to the classical model

\[
H(t) = \frac{1}{2} (A+B(t)) (x^2 + y^2 + p_x^2 + p_y^2) + C(t) (xy - p_x p_y) \tag{71}
\]

with the substitution

\[
a = \frac{1}{\sqrt{2}} (x + iy), \quad b = \frac{1}{\sqrt{2}} (y - ip_y) . \tag{72}
\]

The Hamilton’s equations for Hamiltonian \((71)\) are

\[
\dot{x} = \frac{\partial H}{\partial p_x} = (A + B(t)) p_x - C(t) p_y \tag{73}
\]

\[
\dot{y} = \frac{\partial H}{\partial p_y} = (A + B(t)) p_y - C(t) p_x \tag{74}
\]

\[
\dot{p}_x = -\frac{\partial H}{\partial y} = -(A + B(t)) x - C(t) y \tag{75}
\]

\[
\dot{p}_y = -\frac{\partial H}{\partial x} = -(A + B(t)) y - C(t) x . \tag{76}
\]

By summing Eqs. \((73)\) and \((74)\), and Eqs. \((75)\) and \((76)\), we get the system

\[
\dot{v} = (A + B(t) - C(t)) p_v \tag{77}
\]

\[
\dot{p}_v = -(A + B(t) + C(t)) v \tag{78}
\]

for \( v = x + y \) and \( p_v = p_x + p_y \), which yields

\[
\dot{v} - \frac{\dot{B}(t) - \dot{C}(t)}{A + B(t) - C(t)} \dot{v} + \left[ (A + B(t))^2 - C^2(t) \right] v = 0 . \tag{79}
\]

In the special case \( C(t) = B(t) = \tilde{\rho} + \rho \cos(\omega t) \), one recovers the Mathieu equation

\[
\ddot{v} + [A^2 + 2A\tilde{\rho} + 2A\rho \cos(\omega t)] v = 0 . \tag{80}
\]

The classical equations of motions also give insight to the role of damping, which generally reduces the regions of instability of the Mathieu equation. In fact, in \([74]\) it is shown that the presence of linear damping pushes the instability zone upwards in Fig. 2, so that below a critical value of amplitude \( \rho \) stable solution are always possible and no instabilities occur. In \([74]\) it is also shown that nonlinear effects can generate subharmonic stable motions. These cubic terms correspond to band curvature in the original band structure, so in a real system the non-linearity of the band further stabilizes the system. While a large number of density waves is still expected to occur at the critical \( q \)-values, the sum over all momenta becomes well defined leading to the predicted density order of scenario 2) in the main paper.
Floquet theory

Analogously to the Bloch theorem, the Floquet theorem asserts that the Schrödinger equation for a time-periodic Hamiltonian admits steady-state solutions of the form

$$\Psi(t) = e^{-i\epsilon t} |u(t)\rangle,$$

where the modes $|u(t)\rangle = |u(t + T)\rangle$ inherit the periodicity from the Hamiltonian, and the quantity $\epsilon$ is the so-called Floquet quasienergy. Indeed the steady-state Schrödinger equation can be recast in the form of an eigenvalue equation for the quasienergy operator $\mathcal{H} = \hat{H}(t) - i\partial_t$ in the extended Hilbert space generated by the product of the state space of the quantum system and the space of square-integrable $T$-periodic functions:

$$\mathcal{H}|u(t)\rangle = \epsilon|u(t)\rangle.$$  

(82)

By expanding both the Hamiltonian and the Floquet mode in Fourier series

$$H(t) = \sum_m e^{im\omega t} H^{(m)},$$

$$|u(t)\rangle = \sum_m e^{im\omega t} |u_m\rangle,$$

(83)  
(84)

Eq. (82) yields

$$\left(\hat{H}^{(0)} + m\omega\right)|u_m\rangle + \hat{H}^{(1)}(|u_{m-1}\rangle + |u_{m+1}\rangle) = \epsilon|u_m\rangle,$$

which turns out to be an eigenvalue equation for the infinite tridiagonal matrix

$$M_F = \begin{pmatrix}
    \vdots & \vdots & \vdots & \vdots \\
    \ddots & H^{(0)} - \omega & H^{(1)} & 0 & \cdots \\
    \ddots & H^{(1)} & H^{(0)} & H^{(1)} & \cdots \\
    \ddots & 0 & H^{(1)} & H^{(0)} + \omega & \ddots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$  

(86)

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The applied fields are not necessarily purely sinosoidal, but \( a_0(g_2(t)) \) or \( a_\perp(g_2(t)) \) can be obtained exactly for a given situation and therefore also the required fields. For small amplitudes a linear expansion will generally also result in sinooidal fields.

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While the Floquet problem always becomes static in the extended Floquet Hilbert space using frequency decomposition \[ V(t) \] we seek to obtain a static diagonal solution in the original Hilbert space and a transformation for all times.

Here, \( \nu \) is also analogous to the quasimomentum in the well-known bandstructure of a cosine-potential \[ H(t) = H_0 + \delta H_0 \cos(q \cdot r - \omega t) \].

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