A CONSTRUCTION OF GENERATORS OF \( Z(\mathfrak{so}_n) \)

KENJI TANIGUCHI

Abstract. We construct generators of the center of the universal enveloping algebra of the complex orthogonal Lie algebra realized as the alternative matrices of size \( n \). These elements are constructed in accordance with the Iwasawa decomposition of the real rank one indefinite orthogonal Lie algebra. We also discuss the Iwasawa decomposition of the Pfaffian.

1. Introduction and Main results

Let \( \mathfrak{g} = \mathfrak{so}_n \) be the complex orthogonal Lie algebra realized as the alternative matrices of size \( n \). Denote by \( U(\mathfrak{g}) \) the universal enveloping algebra of \( \mathfrak{g} \) and by \( Z(\mathfrak{g}) \) the center of \( U(\mathfrak{g}) \). It is well known for experts that a set of generators of \( Z(\mathfrak{g}) \) is given with determinant and Pfaffian \([3], [7], [4]\).

The author is now interested in the structure of the space of Whittaker functions on real reductive Lie groups. In order to determine the composition series of the standard Whittaker \((\mathfrak{g}, K)\)-modules \([9]\), he needed to write the action of central elements on the space of Whittaker functions. For the case of indefinite unitary group \( U(n-1,1) \), he succeeded in this task by using the determinant type generators of \( Z(\mathfrak{gl}_n) \). He also tried the case of indefinite orthogonal group \( \mathit{SO}_0(n-1,1) \), but, in his narrow idea, it seems that it is very difficult to write the differential equations characterizing Whittaker functions on \( \mathit{SO}_0(n-1,1) \) by using the above determinant type generators of \( Z(\mathfrak{so}_n) \). Under such backgrounds, he tried to write the action of \( Z(\mathfrak{so}_n) \) in a different way. As a result, a new construction of the generators of \( Z(\mathfrak{so}_n) \) is obtained. This is the main object of this paper.

In order to express the main result, we introduce some notation. In general, for a real Lie group \( L \), the Lie algebra of it is denoted by \( l_0 \) and its complexification by \( l \). This notation will be applied to groups denoted by other Roman letters in the same way without comment. The Kronecker delta is denoted by \( \delta_{i,j} \). Let \( E_{i,j} := (\delta_{i,k} \delta_{j,l})_{k,l=1}^n \) be the matrix units and define \( A_{j,i} = E_{j,i} - E_{i,j} \). These are the standard generators of the space of alternative matrices. The diagonal \( n \times n \) matrix \( \sum_{i=1}^{n-1} E_{i,i} - E_{n,n} \) is denoted by \( I_{n-1,1} \). The field of real (resp. complex) numbers is denoted by \( \mathbb{R} \) (resp. \( \mathbb{C} \)). For a complex matrix \( Z = (z_{i,j})_{i,j} \), define \( \overline{Z} = (\overline{z_{i,j}})_{i,j} \), where \( \overline{z} \) is the complex conjugate of a complex number \( z \).

We realize the group \( \mathit{SO}(n,\mathbb{C}) \) as the subgroup of \( SL(n,\mathbb{C}) \) consisting of those elements which satisfy \( \overline{g} = g^{-1} \). Its Lie algebra \( \mathfrak{g} = \mathfrak{so}_n \) is spanned by \( A_{j,i} \), \( 1 \leq i < j \leq n \). The Lie group \( G = \mathit{SO}_0(n-1,1) \) is the identity component of the real form of \( \mathit{SO}(n,\mathbb{C}) \) defined by the complex conjugation \( g \rightarrow I_{n-1,1} \overline{g} I_{n-1,1} \). Let \( \theta(g) = I_{n-1,1} g I_{n-1,1} \) be a Cartan involution on \( \mathit{SO}_0(n-1,1) \). Denote by \( K \) the maximal compact subgroup of \( G \) consisting of the fixed points of \( \theta \). Let \( \mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0 \).
be the corresponding Cartan decomposition of $\mathfrak{g}_0 = \mathfrak{so}(n - 1, 1)$. More explicitly,

$$K \simeq SO(n - 1),$$

$$\mathfrak{t}_0 = \mathbb{R}\text{-span}\{A_{j, i} \mid 1 \leq i < j \leq n - 1\}, \quad \mathfrak{p}_0 = \mathbb{R}\text{-span}\{\sqrt{-1}A_{n, i} \mid 1 \leq i \leq n - 1\}.$$

As a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$, we choose

$$\mathfrak{a}_0 = \mathbb{R}H, \quad H := \sqrt{-1}A_{n, n-1}.$$

The subgroup $\exp \mathfrak{a}_0$ is denoted by $A$. Define a basis $\alpha$ of the complex dual space $\mathfrak{a}^*$ by $\alpha(H) = 1$. Then $\Sigma^+ = \{\alpha\}$ is a positive system of the root system $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$. Dente by $\mathfrak{n}_0$ the nilpotent subalgebra corresponding to $\Sigma^+$. We choose

$$X_i := A_{n-1, i} + \sqrt{-1}A_{n, i}, \quad i = 1, \ldots, n - 2$$

as a basis of $\mathfrak{n}_0$. Define $\mathfrak{n} := \exp \mathfrak{n}_0$. Then we get Iwasawa decompositions $G = NAK$ and $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{t}_0$. As a consequence of Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{n}) \otimes U(\mathfrak{a}) \otimes U(\mathfrak{t})$.

As usual, denote by $M$ the centralizer of $A$ in $K$. This group is isomorphic to $SO(n - 2)$. Define a Cartan subalgebra $\mathfrak{t}_m$ of $\mathfrak{m}$ by

$$\mathfrak{t}_m := \sum_{i=1}^{[n-2]/2} C T_i, \quad T_i := \sqrt{-1}A_{n-2i, n-2i}.$$

We set $\mathfrak{t} = \mathfrak{t}_m$ if $n$ is even, and $\mathfrak{t} = \mathfrak{t}_m + C T_i^{(n-1)/2}$ if $n$ is odd. Here $T_i^{(n-1)/2} := \sqrt{-1}A_{n-1, 1}$. This $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Define a basis $\{e_1, \ldots, e_{(n-1)/2}\}$ on $\mathfrak{t}^*$ by $e_i(T_j) = \delta_{i,j}$. We regard $\{e_1, \ldots, e_{(n-2)/2}\}$ as a basis of $(\mathfrak{t}_m)^*$.

Choose a Borel subalgebra $\mathfrak{b}_m = \mathfrak{t}_m \oplus \mathfrak{u}$ of $\mathfrak{m}$. Set $\mathfrak{h} := \mathfrak{t}_m \oplus \mathfrak{a}$ and $\mathfrak{R} := \mathfrak{n} \oplus \mathfrak{u}$. The nilpotent subalgebras opposite to $\mathfrak{n}$, $\mathfrak{u}$ and $\mathfrak{R}$ are denoted by $\mathfrak{N}$, $\mathfrak{U}$ and $\mathfrak{R}$, respectively. Denote by $\gamma$ the Harish-Chandra map defined by the projection $U(\mathfrak{g}) \simeq U(\mathfrak{h}) \oplus (\mathfrak{R}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{R}) \to U(\mathfrak{h})$ composed by rho shift.

**Theorem 1.1.** Suppose $\mathfrak{g} = \mathfrak{so}_n$. Let

$$\Omega_{n-2} = \sum_{1 \leq i < j \leq n-2} (A_{j, i})^2$$

be a multiple of the Casimir element of $\mathfrak{so}_{n-2}$. For a parameter $u \in \mathbb{C}$, define elements $C_n(u) \in U(\mathfrak{g})$ inductively by the following formulas:

$$C_0(u) = C_1(u) = 1,$$

(1.1)

$$C_n(u) = - \left\{ \left( H - \frac{n - 2}{2} \right)^2 - u^2 + \sum_{i=1}^{n-2} X_i^2 \right\} C_{n-2}(u)$$

$$+ \sum_{i=1}^{n-2} X_i \left( H - \frac{n - 5}{2} \right) [A_{n-1, i}, C_{n-2}(u)] + 2 \sum_{i=1}^{n-2} X_i C_{n-2}(u) A_{n-1, i}$$

$$- \frac{1}{2} \sum_{i=1}^{n-2} X_i \left[ \Omega_{n-2}, [A_{n-1, i}, C_{n-2}(u)] \right]$$

$$- \frac{1}{2} \sum_{i,j=1}^{n-2} X_i X_j [A_{n-1, i}, [A_{n-1, j}, C_{n-2}(u)]] \quad \text{for} \quad n = 2, 3, \ldots$$

Then $C_n(u)$ is an element of $Z(\mathfrak{g})$ for any $u \in \mathbb{C}$.
Moreover, the image $\gamma(C_n(u))$ of the Harish-Chandra map $\gamma$ is

$$\gamma(C_n(u)) = (u^2 - H^2)(u^2 - T_1^2) \cdots (u^2 - T_{(n-2)/2}^2).$$

This paper is organized as follows. In [2] we explain the $K$-type shift operators and write it explicitly. [3] is the main part of this paper, in which Theorem 1.1 is proved. For the proof, we relate the element $C_n(u)$ and some composition of $K$-type shift operators. This relationship is explained in Lemma 3.2. The proof of this lemma is done in [4]. For completeness, we discuss the Iwasawa decomposition of the Pfaffian in [5] and we relate it to a $K$-type shift operator.

2. Shift operators

In order to show that $C_n(u)$ is invariant under the adjoint action of $K$, we use the $K$-type shift operators on the space of smooth functions on $G$.

Let us review the definition of shift operators briefly. Denote by $g = t \oplus p$ the complexified Cartan decomposition. For a finite dimensional representation $(\tau, V)$ of $K$, define

$$C_r^\infty(K\backslash G) := \{ f : G \to C_r^\infty V \mid f(kg) = \tau(k)f(g), \, k \in K, g \in G \}.$$

This space is isomorphic to the intertwining space $\text{Hom}_K(V^*, C^\infty(G)_{K\text{-finite}})$, where $V^*$ is the contragredient representation of $(\tau, V)$.

Choose an orthonormal basis $\{W_i\}$ of $p_0$ with respect to an invariant bilinear form $(\ , \ )$ on $g$ which is negative (resp. positive) definite on $t_0$ (resp. $p_0$). Define a differential-difference operator $\nabla$ by

$$\nabla \phi_\tau := \sum_i L(W_i)\phi_\tau \otimes W_i, \quad \phi_\tau \in C_r^\infty(K\backslash G).$$

Here, $L(*)$ is the left regular representation. It is easy to see that $\nabla$ does not depend on the choice of an orthonormal basis $\{W_i\}$ of $p_0$. As a consequence, the image of $\nabla$ is an element of $C_r^\infty_{\tau \otimes \text{Ad}}(K\backslash G)$:

$$\nabla \phi_\tau(kg) = (\tau \otimes \text{Ad})(k)\nabla \phi_\tau(g), \quad k \in K, g \in G,$$

Here “Ad” is the adjoint representation of $K$ on $p$.

Let $\lambda \in t^*$ be a dominant integral weight of $K$ and let $(\tau_\lambda, V_\lambda)$ be the irreducible representation of $K$ with highest weight $\lambda$. For notational convenience, set $e_{-\ell} = -e_\ell$ for $\ell = 1, 2, \ldots$ and $e_0 = 0$. In the case when $G = SO_0(n - 1, 1)$ and $\lambda$ is sufficiently regular, the irreducible decomposition of $V_\lambda \otimes p$ is

$$V_\lambda \otimes p \simeq \bigoplus_{1 \leq |\ell| \leq [(n-1)/2]} V_{\lambda + e_\ell} \oplus V_{\lambda + e_0} \text{ if } n \text{ is even}.$$

The projection operator from $V_\lambda \otimes p$ to $V_{\lambda + e_\ell}$ is denoted by $pr_\ell$. Define a $K$-type shift operator $P_\ell$ by

$$P_\ell = pr_\ell \circ \nabla : C_r^\infty(K\backslash G) \to C_r^\infty_{\tau_\lambda + e_\ell}(K\backslash G).$$

The basis $\{\sqrt{-1}A_{n,i} \mid 1 \leq i \leq n - 1\}$ of $p_0$ is orthonormal with respect to an appropriately normalized invariant bilinear form. Therefore, the operator $\nabla$ is

$$\nabla \phi_\tau(g) = \sum_{i=1}^{n-1} L(\sqrt{-1}A_{n,i}) \phi_\tau(g) \otimes \sqrt{-1}A_{n,i}.$$
For an irreducible representation $\tau$ of $K \simeq SO(n - 1)$, the shift operators are explicitly calculated in [8]. To state the results, we introduce the Gelfand-Tsetlin basis of irreducible representations of $SO(n - 1)$.

**Definition 2.1.** Let $\lambda = (\lambda_1, \ldots, \lambda_{(n-1)/2})$ be a dominant integral weight of $SO(n - 1)$. A $(\lambda, \cdot)$Gelfand-Tsetlin pattern is a set of vectors $Q = (q_1, \ldots, q_{n-2})$ such that

1. $q_i = (q_{i,1}, q_{i,2}, \ldots, q_{i,\lfloor (i+1)/2 \rfloor})$.
2. The numbers $q_{i,j}$ are all integers.
3. $q_{2i+1,j} \geq q_{2i,j} \geq q_{2i+1,j+1}$, for any $j = 1, \ldots, i - 1$.
4. $q_{2i+1,i} \geq q_{2i,i} \geq |q_{2i+1,i+1}|$.
5. $q_{2i,j} \geq q_{2i-1,j} \geq q_{2i,j+1}$, for any $j = 1, \ldots, i - 1$.
6. $q_{2i,i} \geq q_{2i-1,i} \geq -q_{2i,i}$.
7. $q_{n-2,j} = \lambda_j$.

The set of all $\lambda$-Gelfand-Tsetlin patterns is denoted by $GT(\lambda)$.

**Notation 2.2.** For any set or number $*$ depending on $Q \in GT(\lambda)$, we denote it by $*Q$, if we need to specify $Q$. For example, $q_{i,j}(Q)$ is the $q_{i,j}$ part of $Q \in GT(\lambda)$.

**Theorem 2.3** ([2]). For a dominant integral weight $\lambda$ of $SO(n - 1)$, the set $GT(\lambda)$ of Gelfand-Tsetlin patterns is identified with a basis of $(\tau_\lambda, V_\lambda)$.

The action of the elements in $\mathfrak{so}(n - 1)$ is expressed as follows. For $j > 0$, let

$$
\begin{align*}
    l_{2i-1,j} &:= q_{2i-1,j} + i - j, & l_{2i-1,-j} &:= -l_{2i-1,j}, \\
    l_{2i,j} &:= q_{2i,j} + i + 1 - j, & l_{2i,-j} &:= -l_{2i,j} + 1,
\end{align*}
$$

and let $l_{2i,0} = 0$. Define $a_{p,q}(Q)$ by

$$a_{2i-1,j}(Q) = \text{sgn} \left( \prod_{1 \leq |k| \leq i-1} (l_{2i-1,j} + l_{2i-2,k}) \prod_{1 \leq |k| \leq i} (l_{2i-1,j} + l_{2i,k}) \right) \frac{4}{\prod_{k \neq j \neq i} (l_{2i-1,j} + l_{2i-1,k}) (l_{2i-1,j} + l_{2i-1,k} + 1)},$$

for $j = \pm 1, \ldots, \pm i$, and

$$a_{2i,j}(Q) = \text{sgn} \left( \prod_{1 \leq |k| \leq i-1} (l_{2i,j} + l_{2i-1,k}) \prod_{1 \leq |k| \leq i} (l_{2i,j} + l_{2i+1,k}) \right) \frac{4(l_{2i,j}^2 - 1)}{\prod_{k \neq j} (l_{2i,j} + l_{2i,k}) (l_{2i,j} - l_{2i,k})},$$

for $j = 0, \pm 1, \ldots, \pm i$, where $\text{sgn}(Q)$ is $\text{sgn}(Q)$ if $j \neq 0$, and $\text{sgn}(q_{2i-1,i}, q_{2i+1,i+1})$ if $j = 0$.

Let $\sigma_{a,b}$ be the shift operator, sending $q_a$ to $(0, \ldots, \text{sgn}(b), 0, \ldots, 0)$. Under the above notation, the action of the Lie algebra is expressed as

$$
\begin{align*}
    \tau_\lambda(A_{2i+1,2i})Q &= \sum_{1 \leq |j| \leq i} a_{2i-1,j}(Q) \sigma_{2i-1,j}Q, \\
    \tau_\lambda(A_{2i+2,2i+1})Q &= \sum_{0 \leq |j| \leq i} a_{2i,j}(Q) \sigma_{2i,j}Q.
\end{align*}
$$

**Remark 2.4.** This basis is compatible with the restriction to smaller orthogonal groups. More precisely, the restriction of $\tau_\lambda$ to $SO(n - 2)$ is multiplicity free, and the vector $Q = (q_1, \ldots, q_{n-2})$ is contained in the irreducible representation of $SO(n - 2)$ whose highest weight is $q_{n-2}$. 

In order to write the projection operator $pr_{\ell}$ explicitly, we embed $V_\lambda$ and $V_{\lambda+e_\ell}$ into an appropriately chosen irreducible representation of $SO(n)$. For example, when we consider the projection $pr_{\tau}$, we embed $V_\lambda$ and $V_{\lambda+e_\ell}$ into the irreducible representation of $SO(n)$ whose highest weight is $\lambda = (\lambda_1 + 1, \lambda_2, \ldots)$. If we do so, then \"$a_{n-2,\ell}(Q)\sigma_{n-2,\ell}Q$\" in the following (for example in (2.3)) makes sense.

Just in the way as the proof of [5, Proposition 4.3], we get the following formulas.

**Lemma 2.5.** For $Q \in GT(\lambda)$ and $\ell = 0, \pm 1, \ldots, \pm \lfloor (n-1)/2 \rfloor$,

\[(2.3) \quad pr_{\ell}(Q \otimes \sqrt{-1}A_{n,n-1}) = a_{n-2,\ell}(Q)\sigma_{n-2,\ell}Q.\]

**Remark 2.6.** This lemma says that, if we embed $V_\lambda$ and $V_{\lambda+e_\ell}$ into an irreducible representation $V_\chi$ of $SO(n)$, then we may identify $pr_{\ell}(Q \otimes \sqrt{-1}A_{n,n-1})$ with the $V_\chi|SO(n-1)$ component of $\tau_{\chi}(A_{n,n-1})Q \in V_\chi$.

Let us write the operator $P_{\ell}$ explicitly. The action of $\ell$ on $\phi_{\tau_\lambda}$ is given by

\[(2.4) \quad L(W)\phi_{\tau_\lambda}(a) = -\tau_\lambda(W)\phi_{\tau_\lambda}(a) \quad \text{for} \quad W \in \ell.\]

Let $\varpi_{\ell}$ be operators from $GT(\lambda)$ to $GT(\lambda+e_\ell)$ defined by

\[(2.5) \quad \varpi_{\ell}Q := a_{n-2,\ell}(Q)\sigma_{n-2,\ell}Q, \quad \ell = 0, \pm 1, \ldots, \lfloor (n-1)/2 \rfloor.\]

Here, $\varpi_0$ is defined only when $n$ is even. Then $\varpi_{\ell}Q := pr_{\ell}(Q \otimes \sqrt{-1}A_{n,n-1}) = \varpi_{\ell}Q$, and

\[
pr_{\ell}(Q \otimes \sqrt{-1}A_{n,i}) = pr_{\ell}(\tau_\lambda(A_{n-1,i})Q \otimes \sqrt{-1}A_{n,n-1}) - pr_{\ell}(\{\tau_\lambda \otimes \text{ad})(A_{n-1,i})(Q \otimes \sqrt{-1}A_{n,n-1}))
\]

\[
= \varpi_{\ell}\tau_\lambda(A_{n-1,i})Q - \tau_{\lambda+e_\ell}(A_{n-1,i})\varpi_{\ell}Q.
\]

For simplicity, we omit the symbols $\tau_\lambda$ and $\tau_{\lambda+e_\ell}$ hereafter. For example, we write $[\varpi_{\ell}, A_{n-1,i}]$ instead of $\varpi_{\ell}\tau_\lambda(A_{n-1,i}) - \tau_{\lambda+e_\ell}(A_{n-1,i})\varpi_{\ell}$, so the projection above is

\[pr_{\ell}(Q \otimes \sqrt{-1}A_{n,i}) = [\varpi_{\ell}, A_{n-1,i}]Q.\]

In order to express $\phi_{\tau_\lambda}(g) \in C^\infty(K\setminus G)$ explicitly, we use the Gelfand-Tsetlin basis. The coefficient function of $Q$ is denoted by $c(Q; g)$. Namely, we write

\[\phi_{\tau_\lambda}(g) = \sum_{Q \in GT(\lambda)} c(Q; g)Q.\]

**Lemma 2.7 ([5]).** For $\ell = 0, \pm 1, \ldots, \pm \lfloor (n-1)/2 \rfloor$, the following formulas hold:

\[(2.6) \quad P_{\ell}\phi_{\tau_\lambda}(g) = \sum_{Q \in GT(\lambda)} \left\{ (L(H) + l_{n-2,\ell} - \left\lfloor \frac{n-2}{2} \right\rfloor)c(Q; g) \varpi_{\ell}Q \right. \]

\[+ \left. \sum_{i=1}^{n-2} L(X_i)c(Q; g)[\varpi_{\ell}, A_{n-1,i}]Q \right\}.\]
Proof. When $n$ is odd, (2.6) is obtained in [8] Proposition 5.1.4. The calculation there is also valid if $n$ is even. (2.7) can be obtained by composing two operators $P_{-\ell}$ and $P_{\ell}$. For the proof of (2.7), we use the following identity: If $\ell \neq 0$,
\[
\ell_{n-2,\ell}(\sigma_{n-2,\ell}Q) - \lfloor \frac{n-2}{2} \rfloor = \ell_{n-2,\ell}(Q) - 1 - \lfloor \frac{n-2}{2} \rfloor
\]
(2.7)
\[
= \begin{cases} 
-\ell_{n-2,\ell}(Q) - \frac{n-2}{2} & (n \text{ is even}) \\
-\ell_{n-2,\ell}(Q) - 1 - \frac{n-3}{2} & (n \text{ is odd}) 
\end{cases}
\]
The conclusion of this equality is valid when $n$ is even and $\ell = 0$. \qed

3. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 1.1. For the proof, we introduce notation.

Definition 3.1. Define
\[
u_{\ell} = \begin{cases} 
\ell_{\lambda} + n/2 - \ell, & \text{if } \ell > 0, \\
1 - (\lambda|\ell) + n/2 - |\ell|, & \text{if } \ell < 0, \\
0, & \text{when } n \text{ is even and } \ell = 0.
\end{cases}
\]
In other words, $\nu_{\ell} = \ell_{n-2,\ell} + 1/2$ when $n$ is odd and $\nu_{\ell} = \ell_{n-2,\ell}$ when $n$ is even.

Next lemma is a key to show our main theorem.

Lemma 3.2. Assume that $C_{n-2}(u) \in Z(\mathfrak{so}_{n-2})$ and that it satisfies (1.2). For $\ell = 0, \pm 1, \ldots, \pm [(n-1)/2]$ and for every irreducible representation $(\tau_{\lambda}, V_{\lambda})$ of $K \simeq SO(n-1)$, there exists a non-zero constant $d_{\lambda,\ell}$ determined by $\ell$ and the highest weight $\lambda$ such that
\[
P_{-\ell}P_{\ell}\phi_{\tau_{\lambda}}(g) = P_{-\ell}P_{\ell}\phi_{\tau_{\lambda}}(g) = d_{\lambda,\ell} L(C_{n}(u_{\ell}))\phi_{\tau_{\lambda}}(g), \\
\phi_{\tau_{\lambda}} \in C_{\tau_{\lambda}}^{\infty}(K\backslash G).
\]

This lemma is proved by direct calculation. Since it is elementary but messy, we prove it in the next section. Here we complete the proof of Theorem 1.1.

Proposition 3.3. Assume that $C_{n-2}(u) \in Z(\mathfrak{so}_{n-2})$ and that it satisfies (1.2). For any $k \in K \simeq SO(n-1)$, $\text{Ad}(k)C_{n}(u) = C_{n}(u)$.

Proof. Let $X(\mu, \nu)$, $\mu \in \widetilde{M}$, $\nu \in \mathfrak{a}^{*}$ be the Harish-Chandra module of the principal series representation induced from $\mu \boxtimes e^{\nu+\rho}$. If $G = SO_{0}(n-1, 1)$, then $X(\mu, \nu)$ is $K$-multiplicity free because of the Frobenius reciprocity $\text{Hom}_{K}(V_{\lambda}^{*}, X(\mu, \nu)) \simeq \text{Hom}_{M}(V_{\lambda}^{*}, \mu)$ and the multiplicity freeness of $V_{\lambda}|SO(n-2)$. 

Suppose that a function \( \phi_{\tau_\lambda} \in C_c^\infty(K\backslash G) \) corresponds to an intertwining operator in \( \text{Hom}_K(V_{\lambda}^\tau, X(\mu, \nu)) \). Since \( X(\mu, \nu) \) is \( K \)-multiplicity free, the function \( P_{-\ell}P_{\ell}\phi_{\tau_\lambda} \) is a constant multiple of \( \phi_{\tau_\lambda} \). It follows that
\[
L(k)P_{-\ell}P_{\ell}L(k^{-1})\phi_{\tau_\lambda} = P_{-\ell}P_{\ell}\phi_{\tau_\lambda}.
\]
By Lemma 3.2, \( P_{-\ell}P_{\ell}\phi_{\tau_\lambda} = d_{\lambda, \ell} L(C_n(u_\ell))\phi_{\tau_\lambda} \). Therefore,
\[
L(\text{Ad}(k)C_n(u_\ell))\phi_{\tau_\lambda} = L(k) L(C_n(u_\ell)) L(k^{-1})\phi_{\tau_\lambda} = (d_{\lambda, \ell})^{-1} L(k) P_{-\ell}P_{\ell} L(k^{-1})\phi_{\tau_\lambda} = (d_{\lambda, \ell})^{-1} P_{-\ell}P_{\ell}\phi_{\tau_\lambda}.
\]
By the definition of \( l_{n-2, \ell} \) and \( u_\ell \), the \( u_\ell \)'s satisfy \( u_1 > u_2 > \cdots > u_{(n-1)/2} > 0 \) if \( n \) is odd and \( u_1 > u_2 > \cdots > u_{([n-1]/2)} > u_0 = 0 \) if \( n \) is even. It follows that \( L(\text{Ad}(k)C_n(u))\phi_{\tau_\lambda} \) and \( L(C_n(u))\phi_{\tau_\lambda} \) are identical for \( [n/2] \) points \( u^2 = u_1^2, \ldots, u_{n/2}^2 \). By the definition \( \boxed{[1.1]} \) of \( C_n(u) \), it is a monic polynomial in \( u^2 \) of degree \( [n/2] \). Therefore \( L(\text{Ad}(k)C_n(u))\phi_{\tau_\lambda} = L(C_n(u))\phi_{\tau_\lambda} \) for any \( u \in C \) and for every \( K \)-type \( (\tau_*^\lambda, V_*^\lambda) \) in \( X(\mu, \nu) \). It follows that \( \text{Ad}(k)C_n(u) - C_n(u) \) annihilates every principal series. By the subrepresentation theorem, it annihilates every irreducible Harish-Chandra module. According to the Plancherel formula for \( G \), the element \( \text{Ad}(k)C_n(u) - C_n(u) \) acts trivially on the space \( C_c^\infty(G)(\subset L^2(G)) \) of smooth functions of compact support, so it is the zero element in \( U(g) \).

**Proof of Theorem 1.1.** We shall show \( C_n(u) \in Z(so_n) \) for any \( u \in C \) and it satisfies \( \boxed{[1.2]} \) by induction on \( n \). If \( n = 0, 1 \), then this is trivial since \( C_0(u) = C_1(u) = 1 \) by definition.

Assume that \( C_{n-2}(u) \in Z(so_{n-2}) \) and that it satisfies \( \boxed{[1.2]} \). Consider the (generalized) Harish-Chandra maps
\[
\gamma_n : U(g) \simeq U(m \oplus a) \oplus (nU(g) + U(g)\overline{\mathfrak{m}}) \rightarrow U(m \oplus a) \xrightarrow{\rho \text{ shift}} U(m \oplus a),
\]
\[
\gamma_n : U(m \oplus a) \simeq U(h) \oplus (U(m \oplus a) + U(m \oplus a)\overline{\mathfrak{m}}) \rightarrow U(h) \xrightarrow{\rho \text{ shift}} U(h).
\]
For notation, see \( \boxed{[1]} \). Then by \( \boxed{[1.1]} \).
\[
\gamma_n(C_n(u)) = (u^2 - h^2)C_{n-2}(u).
\]
By the hypothesis of induction, \( C_{n-2}(u) \) is a central element of \( U(so_{n-2}) \) and \( \gamma_n(C_{n-2}(u)) \) is \( (u^2 - T_1^2) \cdots (u^2 - T_{(n-2)/2}^2) \). Since the Harish-Chandra map \( \gamma \) given in \( \boxed{[1]} \) is the composition \( \gamma = \gamma_n \circ \gamma_n \), \( \boxed{[1.2]} \) is shown.

The image \( \gamma(C_n(u)) \) given in \( \boxed{[1.2]} \) is invariant under the action of the Weyl group of \( g \). Therefore, there exists an element \( z \in Z(g) \) such that \( \gamma(z) = \gamma(C_n(u)) \). By \( \boxed{[1.1]} \) and the hypothesis of induction, \( \gamma_n(C_n(u) - z) \) is an element of \( Z(m \oplus a) \).

Since the restriction of \( \gamma_n \) to \( Z(m \oplus a) \) is injective, \( \gamma_n(C_n(u) - z) = 0 \).

Consider the projection
\[
p : U(g) \simeq U(n) \otimes U(a) \otimes U(\mathfrak{t}) \simeq (U(a) \otimes U(\mathfrak{t})) \oplus nU(g) \rightarrow U(a) \otimes U(\mathfrak{t}).
\]
Recall the definition \( \boxed{[1.1]} \) of \( C_n(u) \). The right hand of \( \boxed{[3.1]} \) is the same as \( C_n(u) \ mod nU(g) \) composed by rho shift. It follows that \( \gamma_n(C_n(u)) = (\rho \text{ shift}) \circ p(C_n(u)) \). By a result of Lepowsky’s \( \boxed{[6]} \), the restriction of \( p \) to the subalgebra
Lemma 4.1. Let \( C \) be an element of \( U(\mathfrak{g}) \). It follows from \( \gamma_u(C_n(u) - z) = 0 \) that \( C_n(u) - z \in U(\mathfrak{g})^\ell \cap \text{Ker} \). This shows \( C_n(u) = z \in Z(\mathfrak{g}) \). \( \square \)

4. A Proof of Lemma 3.2

We shall prove Lemma 3.2, so we assume that \( C_{n-2}(u) \in Z(\mathfrak{so}_{n-2}) \) and that it satisfies (1.2) in this section. We first write \( L(C_n(u)) \phi_{\tau_k} \) explicitly.

For \( Y_1, \ldots, Y_p \in \mathfrak{k} \), the action of \( Y_1 \cdots Y_p \in U(\mathfrak{k}) \) on \( \phi_{\tau_k} \) is given by

\[
L(Y_1 \cdots Y_p) \phi_{\tau_k}(g) = (Y_1 \cdots Y_p)^{opp} \phi_{\tau_k}(g), \quad (Y_1 \cdots Y_p)^{opp} := (-Y_p) \cdots (-Y_1).
\]

Note that, as we have noticed, the symbol \( \tau_k \) is omitted. The map “opp” satisfies

\[
[A, B]^{opp} = -[A^{opp}, B^{opp}].
\]

By the assumption, \( C_{n-2}(u) \in Z(\mathfrak{so}_{n-2}) \). Since the shifted Harish-Chandra map \( \gamma_u \) does not depend on the choice of \( u \),

\[
\gamma_u(C_{n-2}(u)) = \prod_{p=1}^{[n-2]/2} (u^2 - T_p^2) = \gamma_u(C_{n-2}(u))^{opp}
\]

\[
= \gamma_\Pi(C_{n-2}(u)^{opp}) = \gamma_u(C_{n-2}(u)^{opp}).
\]

Therefore, \( C_{n-2}(u)^{opp} = C_{n-2}(u) \). Analogously, we have \( \Omega_{n-2}^{opp} = \Omega_{n-2} \). By the definition (1.1) of \( C_n(u) \), we have the following lemma:

Lemma 4.1. Let \( \phi_{\tau_k}(g) = \sum_{Q \in GT(\lambda)} c(Q; g) Q \) be an element of \( C_{\tau_k}^\infty(K \backslash G) \). The action of \( C_n(u) \) on it is given by

\[
L(C_n(u)) \phi_{\tau_k}(g) = \sum_{Q \in GT(\lambda)} \left[ -\left( L(H) - \frac{n-2}{2} \right)^2 + u^2 - \sum_{i=1}^{n-2} L(X_i)^2 \right] c(Q; g) C_{n-2}(u) Q
\]

\[
+ \sum_{i=1}^{n-2} L(X_i) \left( L(H) - \frac{n-5}{2} \right) c(Q; g) [A_{n-1,i}, C_{n-2}(u)] Q
\]

\[
- 2 \sum_{i=1}^{n-2} L(X_i) c(Q; g) A_{n-1,i} C_{n-2}(u) Q
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n-2} L(X_i) c(Q; g) [\Omega_{n-2}, [A_{n-1,i}, C_{n-2}(u)]] Q
\]

\[
- \frac{1}{2} \sum_{i,j=1}^{n-2} L(X_i) L(X_j) c(Q; g) [A_{n-1,i}, A_{n-1,j}, C_{n-2}(u)] Q
\]

(4.1)

In the proof of lemmas below, we treat the case when \( \ell \neq 0 \). The difference between these cases and the case when \( \ell = 0 \) is only one point: \( a_{n-2,-\ell}(\sigma_{n-2,-\ell} Q) = -a_{n-2,-\ell}(Q) \) if \( \ell \neq 0 \), but \( a_{n-2,-0}(\sigma_{n-2,-0} Q) = a_{n-2,-0}(Q) \) if \( n \) is even and \( \ell = 0 \). If you modify this point, you get the proof of the latter case.

Next, we see the relationship between \( \varpi_{-\ell, -\ell} \) and \( C(u_\ell) \).
Lemma 4.2. There exists a non-zero constant $d_{\lambda,\ell}$ which does not depend on the $q_1, \ldots, q_{n-3}$ parts of $Q = (q_1, \ldots, q_{n-3}, q_{n-2}) \in GT(\lambda)$ such that
\begin{equation}
\varpi_{-\ell} \varpi_{\ell} Q = -d_{\lambda,\ell} C_{n-2}(u_\ell) Q, \tag{4.2}
\end{equation}

Proof. By the definition of the Gelfand-Tsetlin basis and $\varpi_{\ell}$, the action of $\varpi_{-\ell} \varpi_{\ell}$ on $Q \in GT(\lambda)$ is given by
\begin{equation}
\varpi_{-\ell} \varpi_{\ell} Q = a_{n-2,-\ell}(\sigma_{n-2,\ell}(Q)) \sigma_{n-2,-\ell} a_{n-2,\ell}(Q) \sigma_{n-2,\ell} Q = -a_{n-2,\ell}(Q)^2 Q \tag{4.3}
\end{equation}
where $d_{\lambda,\ell}$ is a constant which does not depend on the $q_1, \ldots, q_{n-3}$ parts of $Q = (q_1, \ldots, q_{n-3}, q_{n-2}) \in GT(\lambda)$.

On the other hand, $C_{n-2}(u)$ acts on $Q$ by a scalar, since $Q = (q_1, \ldots, q_{n-3}, q_{n-2})$ is contained in the irreducible representation of $SO(n-2)$ with highest weight $q_{n-3}$ (cf. Remark 2.4) and $C_{n-2}(u)$ is an element of $Z(\mathfrak{so}_{n-2})$.

Let us calculate this scalar. By the assumption,
\begin{equation}
\gamma_{\alpha}(C_{n-2}(u)) = \prod_{i=1}^{\lfloor(n-2)/2\rfloor} (u^2 - T_i^2). \tag{4.4}
\end{equation}
The “rho” of $\mathfrak{so}_{n-2}$ is $\rho_{\mathfrak{so}_{n-2}} := \frac{1}{2} \sum_{i=1}^{\lfloor(n-2)/2\rfloor} (n-2-2i)e_i$. It follows that

\begin{equation*}
\prod_{1 \leq |i| \leq \lfloor(n-2)/2\rfloor} (l_{n-2,\ell} + l_{n-3,i}) = \prod_{i=1}^{m-1} (l_{n-2,\ell} + l_{n-3,i})(l_{n-2,\ell} - l_{n-3,i} + 1)
\end{equation*}
(4.5)
\begin{equation*}
= \prod_{i=1}^{m-1} \left\{ \left(l_{n-2,\ell} + \frac{1}{2}\right)^2 - \left(q_{n-3,i} + \frac{n-2-2i}{2}\right)^2 \right\}.
\end{equation*}

When $n = 2m + 1$ is odd and $i > 0$, then $q_{n-3,i} + (n-2-2i)/2 = l_{n-3,i} - 1/2$ and $l_{n-3,-i} = 1 - l_{n-3,i}$. Therefore
\begin{equation*}
\prod_{1 \leq |i| \leq \lfloor(n-2)/2\rfloor} (l_{n-2,\ell} + l_{n-3,i}) = \prod_{i=1}^{m-1} (l_{n-2,\ell} + l_{n-3,i})(l_{n-2,\ell} - l_{n-3,i} + 1)
\end{equation*}
(4.6)
\begin{equation*}
= \prod_{i=1}^{m-1} \left\{ (l_{n-2,\ell})^2 - \left(q_{n-3,i} + \frac{n-2-2i}{2}\right)^2 \right\}.
\end{equation*}

Then (4.2) follows from (4.3), (4.4), (4.5), (4.6) and Definition 3.1. If $d_{\lambda,\ell}$ is not zero, then the lemma is proved.

Consider the case when $d_{\lambda,\ell}$ in (4.3) is zero. By the definition of $a_{n-2,\ell}(Q)$ and (4.3), $d_{\lambda,\ell}$ is zero if and only if one of the following conditions is satisfied:
1. $1 < \ell \leq \lfloor n/2 \rfloor$ and $\lambda_\ell = \lambda_{\ell-1}$.
2. $\lfloor n/2 \rfloor + 1 \leq \ell \leq -1$ and $\lambda_\ell = \lambda_{\ell+1}$.
3. $n = 2m + 1$ is odd, $\lambda_{m-1} = -\lambda_m$ and $\ell = -m + 1, -m$. 

CENTRAL ELEMENT OF $U(\mathfrak{so}_n)$
In the case (1), \( q_{n-3,t-1} = \lambda_t \) since \( \lambda_{t-1} \geq q_{n-3,t-1} \geq \lambda_t \). By Definition 3.1, \( q_{n-3,t-1} + \{ n - 2 - 2(\ell - 1) \}/2 = \lambda_t + n/2 - \ell = u_\ell \).

In the case (2), \( q_{n-3,|\ell|} = \lambda_{|\ell|} \) since \( \lambda_{|\ell|} \geq q_{n-3,|\ell|} \geq \lambda_{|\ell|+1} \). By Definition 3.1, \( q_{n-3,|\ell|} + \{ n - 2 - 2|\ell| \}/2 = (\lambda_{|\ell|} + n/2 - |\ell|) - 1 = -u_\ell \).

In the case (3), \( \lambda_{m-1} = q_{2m-2,m-1} = -\lambda_m \) since \( \lambda_{m-1} \geq q_{2m-2,m-1} \geq -\lambda_m \). The numbers \( u_{-m+1}, u_{-m} \) and \( q_{n-3,m-1} + \{ n - 2 - 2(m-1) \}/2 \) are

\[
\begin{align*}
u_{-m+1} & = 1 - \{ \lambda_{m-1} + (2m+1)/2 - m + 1 \} = - (\lambda_{m-1} + 1/2), \\
u_{-m} & = 1 - \{ \lambda_m + (2m+1)/2 - m \} = - \lambda_m + 1/2 \\
q_{n-3,m-1} + \frac{2m + 1 - 2 - 2(m-1)}{2} & = q_{2m-2,m-1} + 1/2 = -u_{-m+1} = u_{-m}.
\end{align*}
\]

In every case, we get \( C_{n-2}(u_\ell) Q = 0 \) by (4.4).

On the other hand, if \( d_{\lambda,\ell} \times \varpi_{-\ell} \varpi_{\ell} Q = 0 \). Therefore, if we replace \( d_{\lambda,\ell} \) by a non-zero constant, then (4.2) holds.

In the reminder of this section, we show that (2.7) and (4.3) are identical when \( u = u_\ell \). We first show that the terms which do not contain \( L(X_i) \) in these are identical.

**Lemma 4.3.** For \( Q \in GT(\lambda), \)

\[
(4.7) \quad \left( L(H) - l_{n-2,\ell} - \left\lfloor \frac{n-1}{2} \right\rfloor \right) \left( L(H) + l_{n-2,\ell} - \left\lfloor \frac{n-2}{2} \right\rfloor \right) \varpi_{-\ell} \varpi_{\ell} Q
\]

\[
= -d_{\lambda,\ell} \left\{ \left( L(H) - \frac{n-2}{2} \right)^2 - u_\ell^2 \right\} C_{n-2}(u_\ell) Q.
\]

**Proof.** By Definition 3.1 and

\[
\left\lfloor \frac{n-1}{2} \right\rfloor = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd}, \\
\frac{n-2}{2} & \text{if } n \text{ is even}, \end{cases} \quad \left\lfloor \frac{n-2}{2} \right\rfloor = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ is odd}, \\
\frac{n-3}{2} & \text{if } n \text{ is even}, \end{cases}
\]

we have

\[
(4.8) \quad -l_{n-2,\ell} - \left\lfloor \frac{n-1}{2} \right\rfloor = - \frac{n-2}{2} - u_\ell, \quad l_{n-2,\ell} - \left\lfloor \frac{n-2}{2} \right\rfloor = - \frac{n-2}{2} + u_\ell.
\]

Therefore, this lemma follows from (4.2). \( \square \)

Next, we check the terms containing \( L(X_i) L(X_j), 1 \leq i \leq j \leq n - 2 \). We know \( X_i \) and \( X_j \) commute. Moreover,

\[
[A_{n-1,i}, [A_{n-1,j}, C_{n-2}(u)]] = [A_{j,i}, C_{n-2}(u)] + [A_{n-1,i}, [A_{n-1,j}, C_{n-2}(u)]]
\]

\[
= [A_{n-1,j}, [A_{n-1,i}, C_{n-2}(u)]],
\]

since \( A_{j,i} \in \mathfrak{so}_{n-2} \) and \( C_{n-2}(u) \in Z(\mathfrak{so}_{n-2}) \) by the assumption. Therefore, what we should show is the following lemma:

**Lemma 4.4.** For \( Q \in GT(\lambda), \)

\[
[\varpi_{-\ell} A_{n-1,i}] [\varpi_{\ell}, A_{n-1,j}] Q + [\varpi_{-\ell} A_{n-1,j}] [\varpi_{\ell}, A_{n-1,i}] Q
\]

\[
= -d_{\lambda,\ell} \{ 2\delta_{i,j} C_{n-2}(u) + [A_{n-1,i}, [A_{n-1,j}, C_{n-2}(u)]] \} Q.
\]

(4.9)
Proof. By \((4.2)\), we have
\[
\begin{align*}
[A_{n-1,i}, [A_{n-1,j}, -d_{\lambda,\ell} C_{n-2}(u_\ell)]]Q &= [A_{n-1,i}, A_{n-1,j}, -d_{\lambda,\ell} \varpi_{\ell}]Q \\
&= [A_{n-1,i}, [A_{n-1,j}, \varpi_{-\ell}]]Q + [A_{n-1,j}, \varpi_{-\ell}][A_{n-1,i}, \varpi_{\ell}]]Q \\
&
+ [A_{n-1,i}, \varpi_{-\ell}][A_{n-1,j}, \varpi_{\ell}]]Q + \varpi_{-\ell}[A_{n-1,i}, [A_{n-1,j}, \varpi_{\ell}]]Q.
\end{align*}
\]
As we remarked in Remark 2.6, \(\varpi_{\ell} Q\) is identified with the \(V_{\lambda+\epsilon+\ell}\) component of \(\tau_\chi(A_{n,n-1})Q\). Therefore, \([A_{n-1,i}, [A_{n-1,j}, \varpi_{-\ell}]]Q\) is identified with the \(V_{\lambda+\epsilon+\ell}\) component of
\[
\begin{align*}
[A_{n-1,i}, [A_{n-1,j}, A_{n,n-1}]]Q &= -[A_{n-1,i}, A_{n,j}]Q = -\delta_{i,j} A_{n,n-1}Q \\
&= -\delta_{i,j} \sum_k a_{n-2,k}(Q)\varpi_{n-2,k}Q = -\delta_{i,j} \sum_k \varpi_k Q,
\end{align*}
\]
namely, identified with \(-\delta_{i,j} \varpi_{\pm\ell}\). Then we get
\[
\begin{align*}
[A_{n-1,i}, [A_{n-1,j}, \varpi_{-\ell}]]Q &= \varpi_{-\ell}[A_{n-1,i}, [A_{n-1,j}, \varpi_{\ell}]]Q \\
&= -\delta_{i,j} \varpi_{-\ell}\varpi_{\ell} = \delta_{i,j} d_{\lambda,\ell} C_{n-2}(u_\ell)Q.
\end{align*}
\]
It follows that
\[
\begin{align*}
[A_{n-1,i}, \varpi_{-\ell}][A_{n-1,j}, \varpi_{\ell}]Q &= [A_{n-1,j}, \varpi_{-\ell}][A_{n-1,i}, \varpi_{\ell}]Q \\
&= -2\delta_{i,j} d_{\lambda,\ell} C_{n-2}(u_\ell)Q - d_{\lambda,\ell} [A_{n-1,n-2}, [A_{n-1,n-2}, C_{n-2}(u_\ell)]]Q,
\end{align*}
\]
so \((4.9)\) holds.

Finally, we check the terms containing \(L(X_i), i = 1, \ldots, n-2\).

Consider the actions of \(M \simeq SO(n-2)\) on \(n_0 = \mathbb{R}\)-span \(\{X_i \mid i = 1, 2, \ldots, n-2\}\) and on \(\mathbb{R}\)-span \(\{A_{n-1,i} \mid i = 1, 2, \ldots, n-2\}\). We can find elements \(m_i \in M (1 \leq i \leq n-2)\) such that
\[
\text{Ad}(m_i) X_{n-2} = X_i \quad \text{and} \quad \text{Ad}(m_i) A_{n-1,n-2} = A_{n-1,i}.
\]
If \(Y_1, Y_2\) are \(M\)-invariant elements in \(U(\mathfrak{g})\), then
\[
\text{Ad}(m_i)(X_{n-2} Y_1 A_{n-1,n-2} Y_2) = X_i Y_1 A_{n-1,i} Y_2.
\]

By the definition \((2.23)\) of \(\varpi_{\ell}\), its action commutes with \(m \in M\). Moreover, \(\Omega_{n-2}\) is \(M\)-invariant; so is \(C_{n-2}(u)\) by the assumption. It follows that, if we can show the terms containing \(L(X_{n-2})\) in \((2.24)\) and \(d_{\lambda,\ell} \times (4.11)\) are identical for \(u = u_\ell\), then the terms containing \(L(X_i), i = 1, \ldots, n-3\), are also identical. Therefore, the next lemma will complete the proof of Lemma 3.2.

Lemma 4.5. For \(Q \in GT(\lambda)\),
\[
\begin{align*}
\left( L(H) - l_{n-2,\ell} - \frac{n-3}{2} \right) \varpi_{\ell} [\varpi_{\ell}, A_{n-1,n-2}]Q &
+ \left( L(H) + l_{n-2,\ell} - \frac{n-2}{2} \right) [\varpi_{-\ell}, A_{n-1,n-2}]\varpi_{\ell}Q \\
\tag{4.10} &
= d_{\lambda,\ell} \left\{ \left( L(H) - \frac{n-5}{2} \right) [A_{n-1,n-2}, C_{n-2}(u_\ell)] \\
&
- 2A_{n-1,n-2} C_{n-2}(u_\ell) + \frac{1}{2}[\Omega_{n-2}, [A_{n-1,n-2}, C_{n-2}(u_\ell)]] \right\} Q.
\end{align*}
\]
Proof. By (1.2) and (1.3), the difference of both sides of (1.10) is
\[
\left( L(H) - u_\ell - \frac{n - 4}{2} \right) \varpi_\ell [\varpi_\ell, A_{n-1,n-2}] Q \\
+ \left( L(H) + u_\ell - \frac{n - 2}{2} \right) [\varpi_\ell, A_{n-1,n-2}] \varpi_\ell Q \\
+ \left( L(H) - \frac{n - 5}{2} \right) [A_{n-1,n-2}, \varpi_\ell \varpi_\ell] Q \\
- 2A_{n-1,n-2} \varpi_\ell \varpi_\ell Q + \frac{1}{2} [\Omega_{n-2}, [A_{n-1,n-2}, \varpi_\ell \varpi_\ell]] Q
\]
(4.11) \[= \left( u_\ell + \frac{1}{2} \right) \varpi_\ell [A_{n-1,n-2}, \varpi_\ell] Q + \left( -u_\ell + \frac{3}{2} \right) [A_{n-1,n-2}, \varpi_\ell] \varpi_\ell Q \\
- 2A_{n-1,n-2} \varpi_\ell \varpi_\ell Q + \frac{1}{2} [\Omega_{n-2}, [A_{n-1,n-2}, \varpi_\ell \varpi_\ell]] Q.\]

We shall show that this is zero. For simplicity, we denote
\[
A := A_{n-1,n-2}, \quad \Omega_{n-2} := \Omega, \quad a_\ell(Q) := a_{n-2,\ell}(Q),
\]
\[
a_j(Q) := a_{n-3,j}(Q), \quad \sigma_\ell := \sigma_{n-2,\ell} \quad \sigma_j := \sigma_{n-3,j},
\]
\[
l_\ell := l_{n-2,\ell} \quad \text{and} \quad l_j := l_{n-3,j}.
\]
By the definitions of \( \varpi_\ell \) and the Gelfand-Tsetlin basis,
\[
\varpi_\ell Q = a_\ell(Q) \sigma_\ell Q, \quad AQ = \sum_j a_j(Q) \sigma_j Q
\]
for \( Q \in GT(\lambda) \). By the definition of \( a_\ell(Q) \) and \( a_j(Q) \), we have
\[
\frac{a_\ell(Q) a_j(Q)}{a_j(Q) a_\ell(Q)} = \frac{l_\ell + l_j}{l_\ell + l_j - 1} \quad \text{and} \quad a_{-\ell}(\sigma_j Q) = -a_\ell(Q).
\]
It follows that
\[
\varpi_\ell [A, \varpi_\ell] Q = \sum_j a_{-\ell}(\sigma_j Q) \{ a_\ell(Q) a_j(Q) - a_j(Q) a_\ell(Q) \} \sigma_j Q
\]
(4.12) \[= - \sum_j a_\ell(Q) a_j(Q) \sigma_j Q.\]

By analogous calculations, we obtain
\[
[A, \varpi_\ell] \varpi_\ell Q = \sum_j \frac{a_\ell(Q)^2 a_j(Q)}{l_\ell + l_j} \sigma_j Q,
\]
(4.13) \[A \varpi_\ell \varpi_\ell Q = - \sum_j a_\ell(Q)^2 a_j(Q) \sigma_j Q,
\]
(4.14) \[A, \varpi_\ell \varpi_\ell] Q = \sum_j \{ a_\ell(Q)^2 - a_\ell(Q)^2 \} a_j(Q) \sigma_j Q.
\]
(4.15)

Since the “rho” of \( \mathfrak{so}_{n-2} \) is \( \rho_{\mathfrak{so}_{n-2}} = \frac{1}{2} \sum_{i=1}^{(n-2)/2} (n - 2 - 2i) e_i \), \( \Omega \) acts on \( Q \) by the scalar
\[
-|q_{n-3} + \rho_{\mathfrak{so}_{n-2}}|^2 + |\rho_{\mathfrak{so}_{n-2}}|^2 = - \sum_{i=1}^{(n-2)/2} \left\{ \left( \frac{l_i - l_{i-1}}{2} \right)^2 - |\rho_{\mathfrak{so}_{n-2}}|^2 \right\}.
\]
It follows that

\[ [\Omega, [A, \varpi^{e} \varpi^{f}]] Q \]

\[ = \sum_{j} \{ a_{\ell}(\sigma_{j}) Q^{2} - a_{\ell}(Q^{2}) a_{j}(Q) \left\{ \left( \frac{l_{j} - l_{-j}}{2} \right)^{2} - \left( \frac{l_{j} - l_{-j}}{2} + 1 \right)^{2} \right\} \sigma_{j} Q \]

\[ = \sum_{j} \{ a_{\ell}(Q^{2}) a_{\ell}(\sigma_{j}) Q^{2} a_{j}(Q) (l_{j} - l_{-j} + 1) \sigma_{j} Q. \]

By (4.12), (4.13), (4.14), (4.15) and (4.16) and

\[ \frac{a_{\ell}(\sigma_{j}) Q^{2}}{a_{\ell}(Q^{2})} = \frac{(l_{\ell} + l_{j} + 1)(l_{\ell} + l_{-j} - 1)}{(l_{\ell} + l_{j})(l_{\ell} + l_{-j})}, \]

the coefficient of \( \sigma_{j} Q \) in (4.11) is

\[- \left( u_{\ell} + \frac{1}{2} \right) \frac{a_{\ell}(\sigma_{j}) Q^{2} a_{j}(Q)}{l_{\ell} + l_{-j} - 1} + \left( -u_{\ell} + \frac{3}{2} \right) \frac{a_{\ell}(Q^{2}) a_{j}(Q)}{l_{\ell} + l_{j}} + 2a_{\ell}(Q^{2}) a_{j}(Q) + \frac{1}{2} \{ a_{\ell}(Q^{2}) - a_{\ell}(\sigma_{j}) Q^{2} \} a_{j}(Q) (l_{j} - l_{-j} + 1) \]

\[ = \frac{a_{\ell}(Q^{2}) a_{j}(Q)}{(l_{\ell} + l_{j})(l_{\ell} + l_{-j})} \times \left\{ -\left( u_{\ell} + \frac{1}{2} \right) (l_{\ell} + l_{j} + 1) + \left( -u_{\ell} + \frac{3}{2} \right) (l_{\ell} + l_{-j}) + 2(l_{\ell} + l_{j})(l_{\ell} + l_{-j}) \right\} \]

\[ + \frac{1}{2} \left( (l_{\ell} + l_{j})(l_{\ell} + l_{-j}) - (l_{\ell} + l_{j} + 1)(l_{\ell} + l_{-j} - 1) \right) (l_{j} - l_{-j} + 1) \}

If \( n \) is odd, then \( u_{\ell} = l_{\ell} + 1/2 \) and \( l_{-j} = 1 - l_{j} \). In this case, the term in the braces is

\[ -(l_{\ell} + 1)(l_{\ell} + l_{j} + 1) + (-l_{\ell} + 1)(l_{\ell} - l_{j} + 1) + 2(l_{\ell} + l_{j})(l_{\ell} - l_{j} + 1) + 2(l_{j})^{2} = 0. \]

If \( n \) is even, then \( u_{\ell} = l_{\ell} \) and \( l_{-j} = -l_{j} \). In this case, the term in the braces is

\[ -(l_{\ell} + \frac{1}{2})(l_{\ell} + l_{j} + 1) + (-l_{\ell} + \frac{3}{2})(l_{\ell} - l_{j}) + 2(l_{\ell} + l_{j})(l_{\ell} - l_{j}) + \frac{1}{2}(2l_{j} + 1)^{2} = 0. \]

Therefore, (4.11) is zero. \( \square \)

5. Pfaffian

When \( n = 2m \) is even, there is another generator of \( Z(\mathfrak{so}_{2m}) \), which is called Pfaffian. In this section, we obtain the Iwasawa decomposition of this element and relate it to the K-type shift operator \( P_{0} \).

For a set \( \{ i_{1}, i_{2}, \ldots, i_{2k} \} \) of \( 2k \) different positive integers, define the Pfaffian \( \text{Pf}_{2k}(i_{2k}, i_{2k-1}, \ldots, i_{1}) \) inductively by

\[ \text{Pf}_{2}(i_{2}, i_{1}) = A_{i_{2}, i_{1}}, \]

\[ \text{Pf}_{2k}(i_{2k}, i_{2k-1}, \ldots, i_{1}) = \sum_{j=1}^{2k-1} (-)^{j+1} A_{i_{2k}, i_{j}} \text{Pf}_{2k-2}(i_{2k-1}, \ldots, i_{j}, \ldots, i_{1}), \]

and define

\[ \mathbb{Pf}_{2k} = \text{Pf}_{2k}(2k, 2k - 1, \ldots, 1). \]
Lemma 5.1.  

(1) For every permutation $\sigma \in S_{2k}$,

\[
Pf_{2k}(i_{\sigma(2k)}, i_{\sigma(2k-1)}, \ldots, i_{\sigma(1)}) = \text{sgn}\sigma Pf_{2k}(i_{2k}, i_{2k-1}, \ldots, i_{1}).
\]

(2) $\mathbb{P}F_{2m}$ is an element of $Z(so_{2m})$.

Proof. (1) It is enough to show that $\mathbb{P}F_{2k}$ satisfies (5.1). We will show it by induction on $k$.

If $k = 1$, then $\mathbb{P}F_2 = A_{2,1}$ is alternative under the action of $S_2$. Assume that (5.1) holds for adjacent transpositions $\sigma = (j, j + 1), j = 1, \ldots, 2k - 2$.

Next, consider the case when $\sigma$ is the transposition $(2k - 1, 2k)$. By definition,

\[
\mathbb{P}F_{2m} = A_{2m,2m-1} Pf_{2m-2}(2m - 2, \ldots, 1)
\]

\[
+ \sum_{1 \leq i < j \leq 2m-1} (-)^{i+j} [A_{2m,i} A_{2m-1,j}] Pf_{2m-4}(2m - 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 1)
\]

\[
= A_{2m,2m-1} Pf_{2m-2}(2m - 2, \ldots, 1)
\]

(2) By (1), it is enough to show that $\mathbb{P}F_{2m}$ and $A_{2m,2m-1}$ commute, but this is clear from (5.2).

\[
\mathbb{P}F_{2m} = H Pf_{2m-2} + \sum_{1 \leq i < j \leq 2m-1} (-)^{i+j} [A_{2m-1,i} A_{2m-1,j}] Pf_{2m-4}(2m - 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 1)
\]

\[
= H Pf_{2m-2} + \sum_{1 \leq i < j \leq 2m-1} (-)^{i+j} X_{i,j} Pf_{2m-4}(2m - 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 1)
\]

Let us consider the Iwasawa decomposition of $\mathbb{P}F_{2m}$. Substitute

\[
H = \sqrt{-1} A_{2m,2m-1}, \quad X_i = A_{2m-1,i} + \sqrt{-1} A_{2m,i} \quad (i = 1, 2, \ldots, 2m - 2)
\]

into (5.2). Then we get

\[
\mathbb{P}F_{2m} = H Pf_{2m-2} + \sum_{1 \leq i < j \leq 2m-1} (-)^{i+j} X_{i,j} Pf_{2m-4}(2m - 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 1)
\]
On the other hand,
\[ (2m - 2)^{PF_{2m-2}} \]
\[ = \sum_{i=1}^{2m-2} (-)^i \text{Pf}_{2m-2}(i, 2m - 2, \ldots, \hat{i}, \ldots, 1) \]
\[ = \sum_{i=1}^{2m-2} (-)^i \sum_{j=i+1}^{2m-2} (-)^j A_{i,j} \text{Pf}_{2m-4}(2m - 2, \ldots, \hat{j}, \ldots, \hat{i}, \ldots, 1) \]
\[ + \sum_{i=1}^{2m-2} (-)^i \sum_{j=1}^{i-1} (-)^{j-1} A_{i,j} \text{Pf}_{2m-4}(2m - 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 1) \]
\[ = -2 \sum_{1 \leq i < j \leq 2m-2} (-)^{i+j} A_{j,i} \text{Pf}_{2m-2}(2m - 2, \ldots, \hat{j}, \ldots, \hat{i}, \ldots, 1). \]
Moreover,
\[ [A_{2m-1,i}, PF_{2m-2}] \]
\[ = [A_{2m-1,i}, (-)^i \text{Pf}_{2m-2}(i, 2m - 2, \ldots, \hat{i}, \ldots, 1)] \]
\[ = [A_{2m-1,i}, (-)^i \sum_{j=i+1}^{2m-2} (-)^j A_{i,j} \text{Pf}_{2m-4}(2m - 2, \ldots, \hat{j}, \ldots, \hat{i}, \ldots, 1)] \]
\[ + [A_{2m-1,i}, (-)^i \sum_{j=1}^{i-1} (-)^{j+1} A_{i,j} \text{Pf}_{2m-4}(2m - 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 1) \]
\[ = (-)^i \sum_{j=i+1}^{2m-2} (-)^j A_{2m-1,i,j} \text{Pf}_{2m-4}(2m - 2, \ldots, \hat{j}, \ldots, \hat{i}, \ldots, 1) \]
\[ + (-)^i \sum_{j=1}^{i-1} (-)^{j+1} A_{2m-1,i,j} \text{Pf}_{2m-4}(2m - 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 1) \]
\[ = (-)^i \text{Pf}_{2m-2}(2m - 1, \ldots, \hat{i}, \ldots, 1). \]
Therefore, we get the following Iwasawa decomposition of \( PF_{2m} \):

**Proposition 5.2.**

\[ (5.3) \quad \sqrt{-1}PF_{2m} = (H - m + 1)PF_{2m-2} - \sum_{i=1}^{2m-2} X_i [A_{2m-1,i}, PF_{2m-2}], \]

Let us calculate the action of \( PF_{2m} \) on \( C_{\gamma}^\infty(K\backslash G) \). We use the Cartan subalgebra \( \mathfrak{h} = \mathfrak{t}_{\mathfrak{n}} \oplus \mathfrak{a} \), its basis \( H, T_1, \ldots, T_{m-1} \) and the dual basis \( \{ \alpha, e_1, \ldots, e_{m-1} \} \) defined in 312.

By \( (5.3) \) and \( 32 \),
\[ \sqrt{-1} \gamma(\mathfrak{h}PF_{2m}) = H \mathfrak{PF}_{2m-2}. \]

By induction on \( m \), the image of the (shifted) Harish-Chandra map is
\[ \sqrt{-1} \gamma(\mathfrak{PF}_{2m}) = HT_1 \cdots T_{m-2}T_{m-1}. \]

Suppose that \( Q = (q_1, \ldots, q_{2m-2}) \) is a Gelfand-Tsetlin base of the representation \( V_\lambda \) of \( SO(2m-1) \). Then \( Q \) is contained in the irreducible representation of \( SO(2m-1) \).
2) whose highest weight is $q_{2m-3}$. By [5.4], the image of Harish-Chandra map of $(\mathbb{PF}_{2m-2})^{opp}$ is

$$(\sqrt{-1})^{m-1}\gamma_0((\mathbb{PF}_{2m-2})^{opp}) = (-)^{m-1}T_{m-1} \cdots T_1.$$ 

It follows that $(\mathbb{PF}_{2m-2})^{opp}$ acts on $Q$ by the scalar

$$(\sqrt{-1})^{m-1}(q_{2m-3,1} + m - 2) \cdots (q_{2m-3,m-2} + 1)q_{2m-3,m-1} = (\sqrt{-1})^{m-1}\left(\prod_{i=1}^{m-1} l_{2m-3,i}\right).$$

Since

$$a_{2m-2,0}(Q) = \sqrt{-1}\prod_{i=1}^{m-1} l_{2m-3,i} \prod_{k=1}^{m} l_{2m-2,k}(l_{2m-2,k} - 1),$$

there exists a constant $d_\lambda$, which depends on $\lambda$ but not on $Q \in GT(\lambda)$, such that

$$-d_\lambda(\sqrt{-1})^m \left(\prod_{i=1}^{m-1} l_{2m-3,i}\right) Q = a_{2m-2,0}(Q)Q = \varpi_0 Q.$$ 

We have proved the following proposition:

**Proposition 5.3.** For $\phi_{\tau_\lambda}(g) = \sum_{Q \in GT(\lambda)} c(Q; g)Q \in C^\infty_c(K \backslash G)$, the action of $\mathbb{PF}_{2m}$ is given by

$$d_\lambda L(\mathbb{PF}_{2m})\phi_{\tau_\lambda}(g)$$

$$= \sum_{Q \in GT(\lambda)} \left\{(L(H) - m + 1)c(Q; g)\varpi_0 Q + \sum_{i=1}^{2m-2} L(X_i)c(Q; g)[\varpi_0, A_{2m-2,i}]Q\right\}$$

$$= P_0 \phi_{\tau_\lambda}(g).$$

**References**

[1] Capelli, A.: Sur les opérations dans la théorie des formes algébriques. Math. Ann. 37 (1890), 1–37.

[2] Gelfand, I. M.; Tsetlin, M. L.: Finite-dimensional representations of the group of orthogonal matrices. Doklady Akad. Nauk SSSR 71 (1950), 1017–1020 (Russian). English transl. in: I. M. Gelfand, Collected Papers, vol. II, Springer Verlag, Berlin, 1988.

[3] Howe, R.; Umeda, T.: The Capelli identity, the double commutant theorem, and multiplicity-free actions. Math. Ann. 290 (1991), no. 3, 565–619.

[4] Itoh, M.; Umeda, T.: On Central Elements in the Universal Enveloping Algebras of the Orthogonal Lie Algebras. Compositio Math. 127 (2001), 333–359.

[5] Kraljević, H.: Representations of the universal covering group of the group $SU(n, 1)$. Glas. Mat. Ser. III 8(28) No. 1 (1973), 23–72.

[6] Lepowsky, J.: Algebraic results on representations of semisimple Lie groups. Trans. Amer. Math. Soc., 176 (1973), 1–44.

[7] Molev, A.; Nazarov, M.: Capelli identities for classical Lie algebras. Math. Ann. 313 (1999), no. 2, 315–357.

[8] Taniguchi, K.: Discrete Series Whittaker Functions of $SU(n, 1)$ and $Spin(2n, 1)$. J. Math. Sci. Univ. Tokyo 3 (1996), 331–377.

[9] Taniguchi, K.: On the composition series of the standard Whittaker $(\mathfrak{g}, K)$-modules. to appear in Trans. Amer. Math. Soc., [arXiv:1102.4966]