REPRESENTATIONS WHOSE MINIMAL REDUCTION HAS A TORIC IDENTITY COMPONENT

CLAUDIO GORODSKI AND ALEXANDER LYTCHEK

Abstract. We classify irreducible representations of connected compact Lie groups whose orbit space is isometric to the orbit space of a representation of a finite extension of (positive dimensional) toric group. They turn out to be exactly the non-polar irreducible representations preserving an isoparametric submanifold and acting with cohomogeneity one on it.

1. Introduction

A representation \( \rho : G \to O(V) \) of a compact Lie group on an Euclidean space \( V \) is called polar if there exists a subspace \( \Sigma \), called a section, that meets all \( G \)-orbits and meets them always orthogonally \cite{PTS87,BCO03}. In this case, the normalizer \( N(\Sigma) \) acts on \( \Sigma \) with kernel equal to the centralizer \( Z(\Sigma) \), the quotient group \( \mathcal{W} = N(\Sigma)/Z(\Sigma) \) is finite, and the inclusion \( \Sigma \to V \) induces an isometry between orbit spaces \( \Sigma/\mathcal{W} = V/G \) with the quotient metrics. Conversely, assume \( \rho : G \to O(V) \) is a representation whose orbit space \( X = V/G \) can also be given as the orbit space of a representation of a finite group. On one hand, the set of regular points \( X_{\text{reg}} \) is a flat Riemannian manifold. On the other hand, due to O’Neill’s formula, the horizontal distribution of the Riemannian submersion \( V_{\text{reg}} \to X_{\text{reg}} \) is integrable. It follows that \( \rho \) is polar \cite{Ale06,HLO06}. Thus we have the following characterization: a representation is polar if and only if its orbit space is isometric to the orbit space of a representation of a finite group.

This paper follows the program introduced in \cite{GL}, namely, to hierarchize the representations of compact Lie groups in terms of the complexity of their orbit spaces, viewed as metric spaces. Dadok \cite{Dad85} has classified polar representations of connected groups and showed that they are orbit-equivalent to the isotropy representations of symmetric spaces. Herein we enlarge the class of polar representations by replacing “finite group” by “finite extension of toric group” in the above characterization. Namely, we consider a class of representations defined by the condition that the orbit space is isometric to the orbit space of a representation of a finite extension of toric group, and we classify such representations in the irreducible case and for connected groups.

It turns out that such representations have a remarkable connection with isoparametric submanifolds and polar representations. And, rather surprisingly, only few families of non-polar representations can occur. Our result can be stated as follows.

\( \text{Date: May 6, 2014.} \)

The first author was partially supported by the CNPq grant 302472/2009-6 and the FAPESP project 2011/21362-2.

The second author was partially supported by a Heisenberg grant of the DFG and by the SFB 878 Groups, geometry and actions.
Theorem 1.1. Let $\rho : G \to O(V)$ be an effective irreducible representation of a connected compact Lie group $G$ on a Euclidean space $V$. Assume that $\rho$ is not polar. Then the following conditions are equivalent:

(a) There is a representation $\tau : H \to O(W)$ of a compact Lie group with toric identity component such that the orbit spaces $V/G$, $W/H$ are isometric.

(b) There is a connected subgroup $\hat{G}$ of $O(V)$ containing $\rho(G)$ which acts polarly on $V$ with cohomogeneity one less than the cohomogeneity of $\rho$.

(c) The action of $G$ leaves an isoparametric submanifold $S$ of $V$ invariant and acts with cohomogeneity one on $S$.

Moreover, the representations satisfying any of the above conditions consist of three disjoint families as follows:

(i) $\rho$ is one of the non-polar irreducible representations of cohomogeneity three:

| $G$                     | $\rho$                      | Conditions |
|-------------------------|-----------------------------|------------|
| $\text{SO}(2) \times \text{Spin}(9)$ | $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^{16}$ | $-$        |
| $U(2) \times \text{Sp}(n)$       | $\mathbb{C}^{2} \otimes_{\mathbb{C}} \mathbb{C}^{2n}$ | $n \geq 2$ |
| $\text{SU}(2) \times \text{Sp}(n)$ | $S^2(\mathbb{C}^{2}) \otimes_{\mathbb{H}} \mathbb{C}^{2n}$ | $n \geq 2$ |

(ii) The group $G$ is the semisimple factor of an irreducible polar representation of Hermitian type such that action of $G$ is not orbit-equivalent to the polar representation:

| $G$                     | $\rho$                      | Conditions |
|-------------------------|-----------------------------|------------|
| $\text{SU}(n)$         | $S^2\mathbb{C}^n$           | $n \geq 3$ |
| $\text{SU}(n)$         | $\Lambda^2\mathbb{C}^n$    | $n = 2p \geq 6$ |
| $\text{SU}(n) \times \text{SU}(n)$ | $\mathbb{C}^{n} \otimes_{\mathbb{C}} \mathbb{C}^{n}$ | $n \geq 3$ |
| $\text{E}_6$           | $\mathbb{C}^{27}$          | $-$        |

(iii) $\rho$ is one of the two exceptions: $\text{SO}(3) \otimes \text{G}_2$, $\text{SO}(4) \otimes \text{Spin}(7)$.

A further motivation for Theorem 1.1 comes from one of the main results of [GL], which gives another characterization of the above class. Namely, if $\rho : G \to O(V)$, $\rho' : G' \to O(V')$ are two non-polar representations of a compact Lie groups with isometric orbit spaces $V/G = V'/G'$ such that the restriction of $\rho$ to the identity component $G^\circ$ is irreducible but that of $\rho'$ to $(G')^\circ$ is reducible, then the restriction of $\rho$ to $G^\circ$ satisfies the condition (a), for some representation $\tau : H \to O(W)$.

We are grateful to Wolfgang Ziller for pointing out the connection with the work of Kollross that considerably simplifies our classification.

2. Non-classifying arguments

Since principal orbits of polar representations are isoparametric submanifolds of Euclidean space [PT87], (b) implies (c). Conversely, assume (c) holds. Then $S$ is full and irreducible, thus either it is homogeneous or it has codimension two in $V$, due to the theorem of Throbergsson [Tho91]. In the former case, the maximal connected subgroup of $O(V)$ preserving $S$ acts polarly on $V$ and has $S$ as a principal orbit [PT87]. In the latter case, the cohomogeneity of $G$ on $V$ is 3. Such actions are listed in (i) above, and we can directly check that (b) holds in each case by finding a larger connected group acting as the isotropy representation of a symmetric space of rank two, namely, $(\text{SO}(2) \times \text{SO}(16), \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^{16})$. 


$$(U(2) \times SU(2n), C^2 \otimes C^{2n}), (Sp(2) \times Sp(n), C^4 \otimes H C^{2n})$$, respectively. Hence (b) and (c) are equivalent.

Next we prove that (a) implies (b). If $\tau$ is as in (a), then the induced representation of $H^o$ is reducible, since $H^o$ is a torus. Using [GL, Theorem 1.7], we deduce that the action of $H^o$ on $W$ can be identified with that of the maximal torus $T^k$ of $SU(k+1)$ on $C^{k+1}$, for some $k \geq 1$ (possibly after replacing $H$ and $\tau$). Of course, $H$ is contained in the normalizer $N$ of $H^o$ in $O(W)$. Moreover, the connected component $N^o$ is the maximal torus (of rank $k+1$) of the unitary group of $W$, which acts polarly on $W$.

Let $X = V/G = W/H$, let $Y = X/(N/H) = W/N$ and denote by $\pi$ denote the composite map

$$V \to X \to Y.$$ 

This composite map $\pi$ is a submetry (see [Lyt02, GW11]), whose fibers are smooth equidistant submanifolds of $V$. Denote by $Y_{reg}$ the set of principal $N/H$-orbits in $X$, denote by $V_{reg}$ the set of $G$-regular points in $V$ and put $V' = \pi^{-1}(Y_{reg}) \cap V_{reg}$. Then $V'$ is an open dense subset of $V$ and $\pi : V' \to Y_{reg}$ is a Riemannian submersion. Since the action of $N$ on $W$ is polar, $Y_{reg}$ is flat, so O’Neill’s formula implies that the $\pi$-horizontal distribution in $V'$ is integrable. Hence the regular fibers of $\pi$ have flat normal bundles. Moreover, each regular fiber is equifocal (the focal points are given by the intersection of a horizontal geodesic with singular fibers). It follows that the fibers of $\pi$ in $V$ yield an isoparametric foliation $F$ by full irreducible submanifolds of $V$.

We have codim $F = \dim Y = \dim X - 1 = k + 1$. In case $k = 1$, the cohomogeneity of $\rho$ is 3 and the result follows as above, so we may assume $k \geq 2$. By the theorem of Thorbergsson [Tho91], $F$ is homogeneous, and the maximal connected subgroup $\hat{G} \subset O(V)$ which preserves the foliations acts transitively on the leaves. By definition, $\hat{G}$ is closed, acts polarly and contains $G$. By construction, $V/\hat{G} = Y$. Hence we have (b).

It was already remarked in [GL] that the representations in (i), (ii), (iii) all satisfy (a). More precisely, representations of cohomogeneity three have copolarity one ([GL, Example 1.9]; see also [Str94]). The representations listed in (iii) have copolarity 2 and 3 respectively [GL, Theorem 1.11]. Thus, due to Theorem 1.5 in the same paper, the representations listed in (i) and (iii) satisfy (a). That the representations coming from the Hermitian symmetric spaces and listed in (ii) satisfy (a) has been observed in [GL, Example 1.10]; see also [GOT04].

We shall finish the proof of Theorem 1.1 in the next section by proving that the representations satisfying (b) are listed in (i), (ii) and (iii) above.

3. The classification

Let $\hat{G}$ be as in (b). Consider the maximal connected subgroup $K$ of $O(V)$ with the same orbits as $\hat{G}$. It is known that the action of $K$ on $V$ is the isotropy representation of an irreducible symmetric space [BCO13, Prop. 4.3.9], say, of compact type. Let $(L, K)$ and $M = L/K$ be the corresponding symmetric pair and symmetric space, respectively. The cohomogeneity $c(K, V)$ is just the rank $r = \text{rk}(M)$ of the symmetric space $M$. We have $G \subset K$ and $c(G, V) = c(K, V) + 1 = r + 1$. We run over the cases of irreducible symmetric spaces of compact type [Hel78, Wol84].

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3.1. **Adjoint representations.** These are associated to symmetric spaces of type II. For the adjoint representation of $K$ on its Lie algebra $\mathfrak{g}$ we have that $\mathfrak{g}$, the Lie algebra of $G$, is a proper invariant subspace of $\mathfrak{k}$ for the $G$-action, hence the action of $G$ on $\mathfrak{k}$ is not irreducible.

3.2. **List of symmetric spaces of type I.** One says that the symmetric space $M = L/K$ has maximal rank if $\text{rk}(M) = \text{rk}(L)$. Such spaces are marked with $\ast$ below. The space marked $\ast\ast$ has maximal rank if and only if $|p - q| \leq 1$.

| $L/K$ | $\text{dimension}$ | $\text{rank}$ |
|-------|---------------------|---------------|
| $\text{SU}(n)/\text{SO}(n)\ast$ | $\frac{1}{2}(n - 1)(n + 2)$ | $n - 1$ |
| $\text{SU}(2n)/\text{Sp}(n)$ | $(n - 1)(2n + 1)$ | $n - 1$ |
| $\text{SU}(p + q)/\text{SU}(p) \times \text{U}(q)$ | $2pq$ | $\min\{p, q\}$ |
| $\text{SO}(p + q)/\text{SO}(p) \times \text{SO}(q)$ $\ast\ast$ | $pq$ | $\min\{p, q\}$ |
| $\text{SO}(2n)/\text{U}(n)$ | $n(n - 1)$ | $\left\lfloor \frac{n}{2} \right\rfloor$ |
| $\text{Sp}(n)/\text{U}(n)\ast$ | $n(n + 1)$ | $n$ |
| $\text{Sp}(p + q)/\text{Sp}(p) \times \text{Sp}(q)\ast$ | $4pq$ | $\min\{p, q\}$ |
| $E_6/\text{Sp}(4)\ast$ | $42$ | $6$ |
| $E_6/\text{SU}(6)\text{SU}(2)$ | $40$ | $4$ |
| $E_6/\text{Spin}(10)\text{U}(1)$ | $32$ | $2$ |
| $E_6/F_4$ | $26$ | $2$ |
| $E_7/\text{SU}(8)\ast$ | $70$ | $7$ |
| $E_7/\text{Spin}(12)\text{SU}(2)$ | $64$ | $4$ |
| $E_7/E_6\text{U}(1)$ | $54$ | $3$ |
| $E_8/\text{Spin}(16)\ast$ | $128$ | $8$ |
| $E_8/E_7\text{SU}(2)$ | $112$ | $4$ |
| $F_4/\text{Spin}(3)\text{Sp}(1)\ast$ | $28$ | $4$ |
| $F_4/\text{Spin}(9)$ | $16$ | $1$ |
| $G_2/\text{SO}(4)\ast$ | $8$ | $2$ |

3.3. **Spaces of low rank.** Here we deal with the case $r \leq 4$. We rely on the classification of irreducible representations of connected groups of cohomogeneity at most five ([Str96] and [GL] Theorem 1.11).

Recall that $c(G, V) = r + 1$. If $r = 1$, then $c(G, V) = 2$ and $\rho$ is polar.

If $r = 2$, then $c(G, V) = 3$ and $\rho$ is listed in (i) [Str96].

If $r = 3$, then $c(G, V) = 4$ and $\rho$ is given in [GL] Table 1. The only cases in the table not listed in (ii) or (iii) are the first two, for which $\dim M = \dim V \leq 8$ and $\text{rk}(M) = 3$. But there are no symmetric spaces under these conditions.

Finally, if $r = 4$ then $c(G, V) = 5$ and $\rho$ is given in [GL] Table 2. The only cases in the table not listed in (ii) or (iii) are the first two and the last. In those cases the dimension of $V$ is 8, 12 or 24. Going through the list of symmetric spaces of type I and rank 4, we see that $\dim(V) = \dim(M)$ cannot be 8 or 12 and we are left with the 24-dimensional case $M = \text{SO}(10)//(\text{SO}(6) \times \text{SO}(4))$. Thus we only have to exclude the case $K = \text{SO}(6) \times \text{SO}(4)$ and $G = \text{U}(3) \times \text{Sp}(2)$. However, $\text{U}(3) \times \text{Sp}(2)$ cannot be a subgroup of $\text{SO}(6) \times \text{SO}(4)$.

Henceforth we shall assume $r \geq 5$.

3.4. **Spaces of maximal rank.** In this case, the principal isotropy group $K_{\text{princ}}$ is finite, so also $G_{\text{princ}}$ is finite. Therefore $G$ has codimension one in $K$, so it is a normal subgroup of $K$. Since $K$ has a normal subgroup of codimension one, it has a normal (hence central)
subgroup of dimension 1. Thus $K$ has a circle factor and the corresponding symmetric space $M$ is Hermitian. Since the rank is maximal and at least 5, we deduce $M = \text{Sp}(n)/\text{U}(n)$. In this case, there is a unique codimension one subgroup of $K$ and we get one example $(\text{SU}(n), S^2\mathbb{C}^n)$ listed in (ii).

3.5. **Reformulation.** There remain five classical families of symmetric spaces that we are going to analyse in the following. For easy reference, we list their isotropy representations together with the identity components of their principal isotropy groups.

| $K$                          | $V$                              | $H := (K_{\text{princ}})^\circ$ | Conditions       |
|------------------------------|----------------------------------|----------------------------------|------------------|
| $\text{Sp}(n)$              | $[\Lambda^2\mathbb{C}^{2n} \oplus \mathbb{C}]_{\mathbb{R}}$ | $\text{Sp}(1)^n$               | $n \geq 6$      |
| $\text{SO}(p) \times \text{SO}(q)$ $(p \geq q)$ | $\mathbb{C}^p \otimes_{\mathbb{C}} \mathbb{C}^q^*$ | $\text{S}(\text{U}(p-q) \times \text{U}(1)^q)$ | $p \geq q \geq 5$ |
| $\text{SO}(p) \times \text{SO}(q)$ $(p \geq q)$ | $\mathbb{R}^p \otimes_{\mathbb{R}} \mathbb{R}^q$ | $\text{SO}(p-q)$               | $p \geq q + 2 \geq 7$ |
| $\text{U}(n)$               | $\Lambda^2\mathbb{C}^n$          | $\text{SU}(2)^{\frac{n}{2}}$ if $n$ is even  | $n \geq 10$     |
| $\text{Sp}(p) \times \text{Sp}(q)$ $(p \geq q)$ | $\mathbb{H}^p \otimes_{\mathbb{H}} \mathbb{H}^{q*}$ | $\text{SU}(2)^{\frac{n+1}{2}} \text{U}(1)$ if $n$ is odd | $p \geq q \geq 5$ |

3.6. **Relation with the work of Kollross.** Since $G$ acts on the Euclidean space $V$ with cohomogeneity one less than $K$, the action of $G$ on the principal orbits of $K$ has cohomogeneity one. Since those orbits are of type $K/K_{\text{princ}}$, this means that $G \times K_{\text{princ}}$, and therefore $G \times H$ where $H := K_{\text{princ}}^\circ$, acts on the Lie group $K$ with cohomogeneity one, where the first factor acts from the left and the second from the right. Such (and more general) cohomogeneity one actions have been extensively studied by Kollross. He has classified such actions in the case of simple groups $K$, unfortunately, only up to orbit equivalence. More precisely, he has shown that for any such action, one finds larger connected groups $G \subset \tilde{G} \subset K$ and $H \subset \tilde{H} \subset K$, such that the triple $(G, K, \tilde{H})$ is listed in [Kol02 Theorem B].

Using this theorem of Kollross it will be easy to finish the proof of our theorem. We only need to circumvent some minor difficulties related to the fact that our group $K$ needs not be simple and that our pair $(G, H)$ needs not be maximal. We will derive from [Kol02] the following:

**Lemma 3.1.** Let $F$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $K$ be respectively $\text{SO}(n)$, $\text{SU}(n)$, $\text{Sp}(n)$ acting on $V = F^n$, with $n \geq 5$. Let $H$ be a subgroup of $K$. Assume that $V$ decomposes into $H$-invariant subspaces as $V = V_1 \oplus V_2 \oplus V_3$, with $1 \leq \dim_F(V_1) \leq \dim_F(V_2) \leq \dim_F(V_3)$. Assume that for a proper subgroup $G \subset K$ the group $G \times H$ acts with cohomogeneity at most one on $K$. Then $F \neq H$. If $F = \mathbb{C}$ then $n$ is even and $G$ is contained in $\text{Sp}(\frac{n}{2})$. If $K = \text{SO}(7)$ then $G \subset \text{G}_2$. Finally, in all cases, $V_3$ has either $F$-dimension or $F$-codimension at most 3.

**Proof.** Assume first that $G$ acts reducibly on $V$, namely, leaves a proper subspace $W$ of $V$ invariant. Let $K_W = \{k \in K | k \cdot W = W\}$. Then $O = K/K_W$ is the Grassmannian of dim($W$)-dimensional subspaces of $V$ and, since $G \subset K_W$, the group $H$ acts with cohomogeneity at most one on $O$. However, this is impossible, since $H$ preserves the relative positions to the $V_i$ of any subspace $W'' \in O$ and this implies that $O/H$ is at least two-dimensional.

Thus $G$ acts irreducibly on $V$. Consider now a maximal connected subgroup $\tilde{G}$ of $K$ containing $G$. Set $\tilde{H} = (K_{V_3})^\circ$. Then $\tilde{H}$ is a maximal connected subgroup of $K$ containing
Hence \( G \times \tilde{H} \) acts on \( K \) with cohomogeneity at most one. Hence it must appear either in the list of Onishik [Oni62], or in the list of Kollross.

From the list of Onishik, we only get \( K = \text{SO}(7) \) and \( G = G_2 \). In the list of Kollross, we see the remaining statements.

3.7. General consequences. In case \( K \) is a direct product \( K = K_1 \times K_2 \), we denote by \( G_i \) and \( H_i \), respectively, the projections of \( G \) and of \( H \) to \( K_i \). Then \( 1 = \dim(K/G \times H) \geq \dim(K_1/G_1 \times H_1) + \dim(K_2/G_2 \times H_2) \).

In any case, observe now that for each one of the groups \( K \) appearing in Subsection 3.5 the assumption that the rank is at least 5 and the structure of the principal isotropy group \( K_{\text{princ}} \) implies the following: for any simple non-abelian factor \( K_i \) of \( K \), the triple \( (G_i, K_i, H_i) \) has the form as in Lemma 3.1 above. Moreover, the decomposition of the space \( F^p \) into \( H_i \)-irreducible subspaces has at least five summands. Next we proceed case by case.

3.8. Symplectic cases. Assume that \( K = \text{Sp}(n) \) or \( K = \text{Sp}(p) \times \text{Sp}(q) \). From Lemma 3.1 we see that the projection of \( G \) to each factor \( K_i \) coincides with \( K_i \). We deduce \( K = G \), unless possibly \( K = \text{Sp}(q) \times \text{Sp}(q) \) and \( G \) being the diagonal subgroup of \( K \) (up to conjugation), isomorphic to \( \text{Sp}(q) \). However, in the last case \( H = \text{Sp}(1)^q \) and \( G \times H \) cannot act on \( K \) with codimension 1 since \( \dim(K) - \dim(G) - \dim(H) > 1 \).

3.9. Real case. Assume that \( K = \text{SO}(p) \times \text{SO}(q) \), with \( p \geq q + 2 \geq 7 \). The projection of \( H \) to \( \text{SO}(q) \) is trivial, thus the projection of \( G \) to \( \text{SO}(q) \) has codimension at most 1 in \( \text{SO}(q) \). Hence this projection coincides with \( \text{SO}(q) \). The projection of \( H \) to \( \text{SO}(p) \) fixes pointwise a 5-dimensional subspace. If \( p \geq 8 \) we find an \( H \)-invariant decomposition \( R^p = V_1 \oplus V_2 \oplus V_3 \) with dimension and codimension of \( V_3 \) at least 4; and applying Lemma 3.1 we deduce that the projection \( G_1 \) of \( G \) to \( \text{SO}(p) \) is \( \text{SO}(p) \). We are left with the case \( p = 7, q = 5 \). Then the projection \( H_1 \) of \( H \) to \( \text{SO}(p) \) is \( \text{SO}(2) \) and, due to Lemma 3.1 the group \( G_1 \) must be a subgroup of the 14-dimensional group \( G_2 \). Then \( G_1 \times H_1 \) cannot act on \( \text{SO}(7) \) with cohomogeneity 1.

It follows that the projections of \( G \) to the simple factors of \( K \) coincide with the factors. Since by assumption \( p \neq q \), we get \( G = K \).

3.10. Complex case. Assume that \( K = \text{U}(n) \) with \( n \geq 10 \). Then, for the projection \( H_1 \) of \( H \) to the semisimple part \( \text{SU}(n) \), the space \( \mathbb{C}^n \) decomposes into a sum of at least five \( H_1 \)-invariant subspaces of complex dimension 2. From Lemma 3.1 we deduce that the projection \( G_1 \) of \( G \) to \( \text{SU}(n) \) coincides with \( \text{SU}(n) \). Therefore \( G \) has codimension 1 in \( K \). Hence \( G = \text{SU}(n) \) and we get one example \( (\text{SU}(n), \Lambda^2 \mathbb{C}^n) \), where \( n \) is even, listed in (ii) (note that in case \( n \) is odd, the action of \( \text{SU}(n) \) is orbit-equivalent to that of \( \text{U}(n) \), which is polar [EH99]).

3.11. Complex Grassmanian. Assume that \( K = \text{S}(\text{U}(p) \times \text{U}(q)) \) with \( p \geq q \geq 5 \). Applying Lemma 3.1 to the projection \( H_2 \) of \( H \) and \( G_2 \) of \( G \) to \( \text{SU}(q) \), we see that \( G_2 = \text{SU}(q) \), unless possibly \( q = 6 \) and \( G_2 \subset \text{Sp}(3) \). But in the latter case, \( H_2 = \text{S}(\text{U}(1)^6) \) and \( G_2 \times H_2 \) cannot act with cohomogeneity 1 on \( \text{SU}(6) \).

Similarly, for the projection of \( G_1 \) of \( G \) to \( \text{SU}(p) \) we deduce that \( G_1 = \text{SU}(p) \), unless possibly \( p = 6 \) and \( G_1 \subset \text{Sp}(3) \). In the latter case the projection \( H_1 \) of \( H \) to \( \text{SU}(p) \) is equal to \( \text{S}(\text{U}(1)^6) \), which is again impossible by dimensional reasons.
Thus the projection of $G$ to the simple factors of $K$ is surjective. Now either the projection $G$ of $G$ to $SU(p) \times SU(q)$ is surjective, or $p = q$ and the image $G$ is the twisted diagonal subgroup \( \{(g, \alpha(g)) : g \in SU(q)\} \) where $\alpha$ is an automorphism of $SU(q)$. The last case is again impossible by dimensional reasons. Thus $G = SU(p) \times SU(q)$.

Therefore, $G$ has codimension one in $K$. Hence $G = SU(p) \times SU(q)$ and we get one example $(SU(p) \times SU(p), \mathbb{C}^p \otimes \mathbb{C}^p)$ listed in (ii) (note that in case $p > q$ the action of $SU(p) \times SU(q)$ is orbit-equivalent to that of $S(U(p) \times U(q))$, which is polar [EH99]).

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Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, São Paulo, SP 05508-090, Brazil

E-mail address: gorodski@ime.usp.br

Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany

E-mail address: alytchak@math.uni-koeln.de