Nonexistence of global solutions of wave equations with weak time-dependent damping and combined nonlinearity

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Abstract
In our previous two works, we studied the blow-up and lifespan estimates for damped wave equations with a power nonlinearity of the solution or its derivative, with scattering damping independently. In this work, we are devoted to establishing a similar result for a combined nonlinearity. Comparing to the result of wave equation without damping, one can say that the scattering damping has no influence.

1 Introduction
Recently, the small data Cauchy problem of damped semilinear wave equations with time dependent variable coefficients attracts more and more attention. The works of Wirth [18, 19, 20] showed that the behavior of the solution of the following linear problem

$$\begin{cases}
  u_{tt}^0 - \Delta u^0 + \frac{\mu}{(1+t)^\beta} u_t^0 = 0, & \text{in } \mathbb{R}^n \times [0, \infty), \\
  u^0(x,0) = u_1(x), \quad u_t^0(x,0) = u_2(x), & x \in \mathbb{R}^n,
\end{cases}$$

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heavily relies on the decay rate $\beta$ and the size of the positive constant $\mu$. Then people get interested in the corresponding nonlinear problem, i.e., the following small data Cauchy problem

$$\begin{cases}
  u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p \quad \text{in} \quad \mathbb{R}^n \times [0, \infty), \\
  u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), \quad x \in \mathbb{R}^n,
\end{cases} \tag{1.1}$$

where $\mu > 0$, $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$, and $\varepsilon$ measures the smallness of the data. Before going on, it is necessary to mention two corresponding nonlinear problems without damping

$$\begin{cases}
  u_t - \Delta u = |u|^p \quad \text{in} \quad \mathbb{R}^n \times [0, \infty), \\
  u(x,0) = \varepsilon f(x), \quad x \in \mathbb{R}^n,
\end{cases} \tag{1.2}$$

and

$$\begin{cases}
  u_{tt} - \Delta u = |u|^p \quad \text{in} \quad \mathbb{R}^n \times [0, \infty), \\
  u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), \quad x \in \mathbb{R}^n.
\end{cases} \tag{1.3}$$

For Cauchy problem (1.2) we know that it admits the critical value of $p$ by

$$p_F(n) := 1 + \frac{2}{n},$$

which is so-called Fujita exponent, while the one for problem (1.3) is so-called Strauss exponent $p_S(n)$, which is the positive root of the quadratic equation,

$$\gamma(p,n) := 2 + (n + 1)p - (n - 1)p^2 = 0.$$

**Remark 1.1** “critical” here means the borderline which divides the domain of $p$ into the blow-up part and the global existence part of the solution.

**Remark 1.2** It is easy to prove that

$$p_F(n) < p_S(n) \quad \text{for} \quad n \geq 2.$$
main influence on the behavior of the solution, which means that this case has the same critical exponent as that of problem (1.2). See the works [1, 2]. But, if \( \mu \) is relatively small, we may conjecture that the influence of \( u_{tt} \) will dominate over \( \{\mu/(1 + t)\}u_t \), which means that the critical exponent is related to \( p_S(n) \). See the work [10] by the authors and Wakasa for \( 0 < \mu < (n^2 + n + 2)/(2(n + 2)) \), which was extended to \( 0 < \mu < (n^2 + n + 2)/(n + 2) \) by Ikeda and Sobajima [9] and Tu and Lin [15, 16]. Unfortunately, till now we are not clear of the boardline of \( \mu \), which determines that the critical power of Cauchy problem (1.1) with \( \beta = 1 \) will be Fujita or Strauss. We refer the reader to a very recent work by Palmieri and Reissig [14].

In a recent work [12] by the authors, we study the blow-up for the small data Cauchy problem

\[
\begin{cases}
  u_{tt} - \Delta u + \frac{\mu}{(1 + t)^\beta} u_t = |u_t|^p & \text{in } \mathbb{R}^n \times [0, \infty), \\
  u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbb{R}^n.
\end{cases}
\]  

(1.4)

If \( \beta > 1 \), then we showed that the problem has no global solution for \( 1 < p \leq p_G(n) \), where

\[ p_G(n) := \frac{n + 1}{n - 1}, \]

which denotes the critical exponent for Glassey conjecture. In this work, we are devoted to studying the small data Cauchy problem with combined nonlinear terms, that is:

\[
\begin{cases}
  u_{tt} - \Delta u + \frac{\mu}{(1 + t)^\beta} u_t = |u_t|^p + |u|^q & \text{in } \mathbb{R}^n \times [0, \infty), \\
  u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbb{R}^n.
\end{cases}
\]  

(1.5)

where \( \beta > 1 \). Inspired by the work [5], in which Han and Zhou studied the Cauchy problem (1.5) without damping and obtained the blow-up result for

\[ \max \left(1, \frac{2}{n - 1}\right) < p \leq \frac{2n}{n - 1} \]  

(1.6)

and

\[ 1 < q < \min \left\{1 + \frac{4}{(n - 1)p - 2}, \frac{2n}{n - 2}\right\}, \]

we want to show that whether we have the same blow-up result for Cauchy problem (1.5). The difficulty comes from the damping term, which prevents us from getting the lower bound of some functional by using the test function method, and we overcome this by using a multiplier which was first introduced in the authors [11]. Also, due to the damping term, we can’t get the blow-up result and lifespan estimate by using Kato’s Lemma, and we do it by using an iteration argument similar to that in [11].
Remark 1.3 Hidano, Wang and Yokoyama [6] established global existence result for Cauchy problem (1.5) without damping for \( n = 2, 3 \) and
\[ p > p_G(n), q > q_S(n) \text{ and } (q - 1)((n - 1)p - 2) \geq 4. \]

In the following we are going to find out that whether the global existence result holds for Cauchy problem (1.5).

2 Main Result

First we introduce the definition of the solution as follows.

**Definition 2.1** As in [11], we say that \( u \) is an energy solution of (1.5) on \([0, T)\) if
\[ u \in \bigcap_{i=0}^{1} C^i([0, T), H^{1-i}(\mathbb{R}^n)) \cap C^1((0, T), L^p(\mathbb{R}^n)) \cap L^q_{\text{loc}}(\mathbb{R}^n \times (0, T)) \]
satisfies \( u(x, 0) = \varepsilon f(x) \) in \( H^1(\mathbb{R}^n) \) and
\[
\int_{\mathbb{R}^n} u_t(x, t) \phi(x, t) dx - \int_{\mathbb{R}^n} \varepsilon g(x, \omega(x, 0)) dx \\
+ \int_0^t ds \int_{\mathbb{R}^n} \{ -u_t(x, s) \phi_t(x, s) + \nabla u(x, s) \cdot \nabla \phi(x, s) \} dx \\
+ \int_0^t ds \int_{\mathbb{R}^n} \sum_{i=0}^{1} \mu u_i(x, s) (1 + s)^{\beta} \phi(x, s) dx \\
= \int_0^t ds \int_{\mathbb{R}^n} |u_t(x, s)|^p \phi(x, s) dx + \int_0^t ds \int_{\mathbb{R}^n} |u(x, s)|^q \phi(x, s) dx
\] (2.1)
with any \( \phi \in C_0^{\infty}(\mathbb{R}^n \times [0, T)) \) and any \( t \in [0, T) \).

Employing the integration by parts in (2.1) and letting \( t \to T \), we get the weak solution of (1.5)
\[
\int_{\mathbb{R}^n \times [0, T]} u(x, s) \left\{ \phi_t(x, s) - \Delta \phi(x, s) - \left( \frac{\mu \phi(x, s)}{(1 + s)^{\beta}} \right) \right\} dx ds \\
= \int_{\mathbb{R}^n} \mu \varepsilon f(x, \omega(x, 0)) dx - \int_{\mathbb{R}^n} \varepsilon f(x) \phi_t(x, 0) dx \\
+ \int_{\mathbb{R}^n} \varepsilon g(x, \omega(x, 0)) dx + \int_{\mathbb{R}^n \times [0, T]} |u_t(x, s)|^p \phi(x, s) dx ds \\
+ \int_{\mathbb{R}^n \times [0, T]} |u(x, s)|^q \phi(x, s) dx ds.
\]

Our main theorem is the following.
Theorem 2.1 Let $\mu > 0$, $\beta > 1$ and $n \geq 1$. Assume that both $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ are non-negative, compactly supported, and $g$ does not vanish identically. Suppose that an energy solution $u$ of (1.5) on $[0,T)$ satisfies
\[ \text{supp } u \subset \{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x| \leq t + R\} \quad (2.2) \]
with some $R \geq 1$. If
\[ p > 1 \quad (2.3) \]
and
\[ \begin{cases} 1 < q < \min \left\{ 1 + \frac{4}{(n-1)p-2}, \frac{2n}{n-2} \right\} & \text{for } n \geq 2, \\ 1 < q & \text{for } n = 1, \end{cases} \quad (2.4) \]
then there exists a constant $\varepsilon_0 = \varepsilon_0(f,g,n,p,\mu,\beta,R) > 0$ such that $T$ has to satisfy
\[ T \leq C\varepsilon^{-2p(q-1)/(2q+2-(n-1)p(q-1))} \quad (2.5) \]
for $0 < \varepsilon \leq \varepsilon_0$, where $C$ is a positive constant independent of $\varepsilon$.

Remark 2.1 We have less restriction for $p$, by comparing the conditions (2.3) and (1.6), since we use an iteration argument instead of Kato’s type lemma. Which means that we may get blow-up result even for large $p$ but small $q$. What is more, for relatively large $p$ and small $q$, we can establish an improved lifespan estimate. See Theorem 2.2 below.

Remark 2.2 The restriction $q < 2n/(n-2)$ for $n \geq 2$ is necessary to guarantee the integrability of the nonlinear term $|u|^q$.

Remark 2.3 As in [5], we should point out that there exist pairs of $(p,q)$ satisfying
\[ p > p_G(n), \quad q > q_S(n), \]
but still blow-up will occur. For example, since
\[ \gamma \left( n, 1 + \frac{4}{n-1} \right) = -\frac{8}{n-1} < 0, \]
we may choose such an appropriate pair $(p_0,q_0)$ by setting small constants $\delta_1$ and $\delta_2$, such that
\[ p_0 := \frac{n+1}{n-1} + \delta_1 > p_G(n) \]
and
\[ q_S(n) < q_0 := 1 + \frac{4}{n-1+(n+1)\delta_1} < 1 - \delta_2 + \frac{4}{n-1} < 1 + \frac{4}{n-1+(n-1)\delta_1} = 1 + \frac{4}{(n-1)p_0-2}. \]
We also have an improvement on the estimate of the lifespan for relatively large $p$ and small $q$ as follows.

**Theorem 2.2** Let $\mu > 0$, $\beta > 1$ and $n \geq 2$. Assume that both $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ are non-negative, compactly supported, and $g$ does not vanish identically. Suppose that an energy solution $u$ of (1.5) on $[0,T)$ satisfies

$$\text{supp } u \subset \{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x| \leq t + R\}$$

with some $R \geq 1$. If

$$p > \frac{2n}{n-1} \quad \text{and} \quad 1 < q < \frac{n+1}{n-1},$$

then there exists a constant $\varepsilon_0 = \varepsilon_0(f,g,n,p,\mu,\beta,R) > 0$ such that $T$ has to satisfy

$$T(\varepsilon) \leq C\varepsilon^{-(q-1)/(q+1-n(q-1))}$$

for $0 < \varepsilon \leq \varepsilon_0$, where $C$ is a positive constant independent of $\varepsilon$.

**Remark 2.4** Under the assumption (2.7), the lifespan estimate (2.8) is better than (2.5). For this, we should have

$$\frac{q-1}{q+1-n(q-1)} < \frac{2p(q-1)}{2q+2-(n-1)p(q-1)}$$

which is equivalent to

$$p > \frac{2(q+1)}{2(q+1)-(n+1)(q-1)}.\quad (2.10)$$

On the other hand, $q < (n+1)/(n-1)$ is equivalent to

$$\frac{2(q+1)}{2(q+1)-(n+1)(q-1)} < \frac{2n}{n-1},$$

which means that assumption (2.7) guarantees the inequality (2.9). In section 6 we will give the reason why we have to pose the restriction on $p$ in the form

$$p > \frac{2n}{n-1}$$

instead of (2.10).
3 Lower bound of the first functional

One of the key ingredients to the blow-up result is to get the lower bound of

\[ F_1(t) := \int_{\mathbb{R}^n} u(x,t)\psi(x,t)dx, \]

where

\[ \psi(x,t) := e^{-t}\phi_1(x), \quad \phi_1(x) := \begin{cases} \int_{S^{n-1}} e^{x\omega}dS_\omega & \text{for } n \geq 2, \\ e^x + e^{-x} & \text{for } n = 1, \end{cases} \quad (3.1) \]

which was first introduced in Yordanov and Zhang [21]. Another key point is a multiplier,

\[ m(t) := \exp\left(\frac{\mu(1+t)^{1-\beta}}{1-\beta}\right), \quad (3.2) \]

which is crucial for our proof and was first introduced in [11]. We note that \( m(t) \) is bounded as

\[ 0 < m(0) \leq m(t) \leq 1. \]

Then we have the following lemma.

**Lemma 3.1** Let \( u \) be an energy solution of (1.5) on \([0,T)\). Under the same assumption of Theorem 2.1, it holds that

\[ F_1(t) \geq \frac{m(0)e}{2} \int_{\mathbb{R}^n} f(x)\phi_1(x)dx \geq 0 \quad \text{for } t \geq 0. \quad (3.3) \]

**Proof.** The proof of Lemma 3.1 is almost the same as that of Lemma 3.1 in [12], which is established by neglecting the spatial integral of the nonlinear term

\[ \int_{\mathbb{R}^n} |u(x,t)|^pdx \]

due to its positivity. Replacing this quantity by

\[ \int_{\mathbb{R}^n} \{|u_t(x,t)|^p + |u(x,t)|\}^qdx, \]

we get the desired proof immediately.
4 Lower bound of the second functional

With Lemma 3.1 in hand, we may prove a key inequality for

\[ F_2(t) := \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx. \]

**Lemma 4.1** Let \( u(x, t) \) and \( \psi(x, t) \) be as in section 3. Under the same assumption of Theorem 2.1, it holds that

\[ F_2(t) \geq \frac{m(0)\varepsilon}{2} \int_{\mathbb{R}^n} g(x)\phi_1(x) dx \geq 0 \quad \text{for } t \geq 0. \quad (4.1) \]

**Proof.** Actually Lemma 4.1 is a partial result of the proof of Theorem 2.1 in [12]. For convenience we rewrite the detail. By direction calculation we have

\[
\frac{d}{dt} \left[ m(t) \int_{\mathbb{R}^n} \{u_t(x, t) + u(x, t)\} \psi(x, t) dx \right] = \frac{\mu}{(1 + t)^\beta} m(t) \int_{\mathbb{R}^n} \{u_t(x, t) + u(x, t)\} \psi(x, t) dx + m(t) \frac{d}{dt} \int_{\mathbb{R}^n} \{u_t(x, t) + u(x, t)\} \psi(x, t) dx. \quad (4.2)
\]

Replacing the test function \( \phi \) in the definition (2.1) with \( \psi \) and taking derivative to both sides with respect to \( t \), we have that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \{u_t(x, t) + u(x, t)\} \psi(x, t) dx - \int_{\mathbb{R}^n} u_t(x, t) \psi_t(x, t) dx + \int_{\mathbb{R}^n} \nabla u(x, t) \cdot \nabla \psi(x, t) dx + \frac{\mu}{(1 + t)^\beta} \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \]

\[ = \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx + \int_{\mathbb{R}^n} |u(x, t)|^q \psi(x, t) dx. \quad (4.3) \]

Since for \( \psi(x, t) \) we have

\[ \psi_t = -\psi, \quad \psi_{tt} = \Delta \psi = \psi, \]

then by integration by parts in the first term in the second line of the last equality yields that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \{u_t(x, t) + u(x, t)\} \psi(x, t) dx - \int_{\mathbb{R}^n} u_t(x, t) \psi_t(x, t) dx + \frac{\mu}{(1 + t)^\beta} \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \]

\[ = \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx + \int_{\mathbb{R}^n} |u(x, t)|^q \psi(x, t) dx. \quad (4.4) \]
By combining (4.2) and (4.4) we have

\[
\frac{d}{dt} \left[ m(t) \int_{\mathbb{R}^n} \{ u_t(x, t) + u(x, t) \} \psi(x, t) dx \right] \\
= m(t) \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx + m(t) \int_{\mathbb{R}^n} |u(x, t)|^q \psi(x, t) dx \\
+ \frac{\mu}{(1 + t)\beta} m(t) F_1(t)
\]

for \( t \geq 0 \). Then (4.5) and the positivity of \( F_1 \) by Lemma 3.1 yield

\[
m(t) \int_{\mathbb{R}^n} \{ u_t(x, t) + u(x, t) \} \psi(x, t) dx \\
\geq m(0) \varepsilon \int_{\mathbb{R}^n} \{ f(x) + g(x) \} \phi_1(x) dx \\
+ \int_0^t ds \int_{\mathbb{R}^n} m(s)|u_t(x, s)|^p \psi(x, s) dx \\
+ \int_0^t ds \int_{\mathbb{R}^n} m(s)|u(x, s)|^q \psi(x, s) dx.
\]

On the other hand, noting that

\[
\frac{m'(t)}{m(t)} = \frac{\mu}{(1 + t)^{\beta}}
\]

then (4.3) implies that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx + \frac{m'(t)}{m(t)} \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \\
+ \int_{\mathbb{R}^n} \{ u_t(x, t) - u(x, t) \} \psi(x, t) dx \\
= \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx + \int_{\mathbb{R}^n} |u(x, t)|^q \psi(x, t) dx.
\]

Multiplying the above equality by \( m(t) \), we get

\[
\frac{d}{dt} \left[ m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \right] \\
+ m(t) \int_{\mathbb{R}^n} \{ u_t(x, t) - u(x, t) \} \psi(x, t) dx \\
= m(t) \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx + m(t) \int_{\mathbb{R}^n} |u(x, t)|^q \psi(x, t) dx.
\]
Adding (4.6) and (4.7) together, we obtain that
\[
\frac{d}{dt} \left[ m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \right] + 2m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \\
\geq m(0) \varepsilon \int_{\mathbb{R}^n} \{ f(x) + g(x) \} \phi_1(x) dx + m(t) \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx \\
+ m(t) \int_{\mathbb{R}^n} |u(x, t)|^q \psi(x, t) dx \quad (4.8)
\]

Setting
\[
G(t) := m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx - m(0) \varepsilon \int_{\mathbb{R}^n} g(x) \phi_1(x) dx \\
- \frac{1}{2} \int_0^t m(s) ds \int_{\mathbb{R}^n} |u_t(x, s)|^p \psi(x, s) dx,
\]
then we have
\[
G(0) = \frac{m(0) \varepsilon}{2} \int_{\mathbb{R}^n} g(x) \phi_1(x) dx > 0.
\]

It is easy to get from (4.8) that
\[
G'(t) + 2G(t) \\
\geq \frac{m(t)}{2} \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx + m(0) \varepsilon \int_{\mathbb{R}^n} \phi_1(x) f(x) dx \\
\geq 0
\]
which implies
\[
G(t) \geq e^{-2t} G(0) > 0 \quad \text{for } t \geq 0.
\]

Hence, by the definition (4.9), it holds that
\[
m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \\
\geq \frac{1}{2} \int_0^t m(s) ds \int_{\mathbb{R}^n} |u_t(x, s)|^p \psi(x, s) dx \\
+ \frac{m(0) \varepsilon}{2} \int_{\mathbb{R}^n} g(x) \phi_1(x) dx,
\]

(4.10)
which implies that
\[
\int_{\mathbb{R}^n} u_t(x,t)\psi(x,t)\,dx \geq \frac{m(0)\varepsilon}{2m(t)} \int_{\mathbb{R}^n} g(x)\phi_1(x)\,dx \\
\geq \frac{m(0)\varepsilon}{2} \int_{\mathbb{R}^n} g(x)\phi_1(x)\,dx,
\]
which is exactly the desired inequality in Lemma 4.1.

5 Iteration argument

As mentioned in the introduction, we can’t establish the blow-up result and lifespan estimate by using Kato’s lemma, instead of which we will use an iteration argument, following the idea in [11]. Set
\[
F_0(t) := \int_{\mathbb{R}^n} u(x,t)\,dx.
\]
Choosing the test function \( \phi = \phi(x,s) \) in (2.1) to satisfy \( \phi \equiv 1 \) in \( \{(x,s) \in \mathbb{R}^n \times [0,t] : |x| \leq s + R\} \), we get
\[
\int_{\mathbb{R}^n} u_t(x,t)\,dx - \int_{\mathbb{R}^n} u_t(x,0)\,dx + \int_0^t ds \int_{\mathbb{R}^n} \frac{\mu u_t(x,s)}{(1+s)^{\beta}}\,dx \\
= \int_0^t ds \int_{\mathbb{R}^n} |u_t(x,s)|^p\,dx + \int_0^t ds \int_{\mathbb{R}^n} |u(x,s)|^q\,dx,
\]
which implies that by taking derivative with respect to \( t \) on the both sides
\[
F_0'(t) + \frac{\mu}{(1+t)^{\beta}} F_0'(t) = \int_{\mathbb{R}^n} |u_t(x,t)|^p\,dx + \int_{\mathbb{R}^n} |u(x,s)|^q\,dx.
\]
Multiplying with \( m(t) \) on the both sides yields
\[
\{m(t) F_0'(t)\}' = m(t) \int_{\mathbb{R}^n} |u_t(x,t)|^p\,dx + m(t) \int_{\mathbb{R}^n} |u(x,t)|^q\,dx,
\]
which means that
\[
F_0'(t) \geq m(0) \int_0^t ds \int_{\mathbb{R}^n} |u_t(x,s)|^p\,dx + m(0) \int_0^t ds \int_{\mathbb{R}^n} |u(x,s)|^q\,dx.
\]

Lemma 5.1 (Inequality (2.5) of Yordanov and Zhang [21]) There exists a constant \( C_1 = C_1(n,p,R) > 0 \) such that
\[
\int_{|x| \leq t+R} [\psi(x,t)]^{\beta/(p-1)}\,dx \leq C_1(1+t)^{(n-1)(1-p/(2(p-1)))} \quad \text{for } t \geq 0.
\]
By Hölder’s inequality, (5.3) and (4.1), we may estimate the nonlinear term

\[ \int_{\mathbb{R}^n} |u_t(x,t)|^p \, dx \geq F_2^p(t) \left( \int_{|x| \leq t+R} [\psi(x,t)]^{p/(p-1)} \, dx \right)^{-(p-1)} \]

\[ \geq C_2 \varepsilon^p (1 + t)^{-(n-1)(p-2)/2}, \]

where

\[ C_2 := C_1^{1-p} \left( \frac{m(0)}{2} \int_{\mathbb{R}^n} g(x) \phi_1(x) \, dx \right)^{\frac{1}{p}}. \]

Plugging which into (5.2) we have

\[ F_0(t) \geq m(0) C_2 \varepsilon^p \int_0^t \int_s^t (1 + r)^{n-1-(n-1)p/2} \, dr \, ds \]

\[ \geq m(0) C_2 \varepsilon^p (1 + t)^{-(n-1)p/2} \int_0^t \int_0^t r^{n-1} \, dr \, ds \]

\[ \geq C_3 \varepsilon^p (1 + t)^{-(n-1)p/2} t^{n+1}, \]

where

\[ C_3 := \frac{m(0) C_2}{n(n+1)}. \]

By Hölder’s inequality again, it follows from (5.2) that

\[ F_0(t) \geq C_4 m(0) \int_0^t \int_0^t (1 + r)^{(q-1)n} F_0^q(r) \, dr \, ds \]

(5.5)

with some positive constant $C_4$ independent of $\varepsilon$. In this way, we find two key ingredients for our iteration argument.

Assuming that

\[ F_0(t) \geq A_j (1 + t)^{-a_j t^{b_j}} \quad \text{for} \quad t \geq 0 \quad (j = 1, 2, 3 \cdots) \]

(5.6)

with

\[ A_1 = C_3 \varepsilon^p, \quad a_1 = \frac{(n-1)p}{2}, \quad b_1 = n + 1. \]

(5.7)

Plugging (5.6) into (5.5) we have

\[ F_0(t) \geq A_{j+1} (1 + t)^{-qa_j n(q-1)p^{b_j} + 2}, \]

where

\[ A_{j+1} \geq \frac{C_4 m(0) A_j^q}{(qb_j + 2)^2}, \quad a_{j+1} = qa_j + n(q - 1), \quad b_{j+1} = qb_j + 2. \]

(5.8)
By combining (5.7) and (5.8) we come to
\[
a_j = q^{j-1} \left( \frac{(n-1)p}{2} + n \right) - n, \\
b_j = q^{j-1} \left( n + 1 + \frac{2}{q-1} \right) - \frac{2}{q-1}, \\
A_j \geq \frac{C_5 A_{j-1}^q}{q^{2(j-1)}}
\]
with
\[
C_5 := \frac{C_4 m(0)}{\left( n + 1 + \frac{2}{q-1} \right)^2}.
\]
Hence we have
\[
\log A_j \\
\geq q \log A_{j-1} - 2(j-1) \log q + \log C_5 \\
\geq q^2 \log A_{j-2} - 2(q(j-2) + (j-1)) \log q + (q+1) \log C_5.
\]
Repeating this procedure, we have
\[
\log A_j \geq q^{j-1} \log A_1 - \sum_{k=1}^{j-1} \frac{2k \log q - \log C_5}{q^k},
\]
which yields that
\[
A_j \geq \exp \left\{ q^{j-1} \left( \log A_1 - S_q(j) \right) \right\},
\]
where
\[
S_q(j) := \sum_{k=1}^{j-1} \frac{2k \log q - \log C_5}{q^k}.
\]
By d’Alembert’s criterion we know that \( S_q(j) \) converges for \( q > 1 \) as \( j \rightarrow \infty \). And therefore we obtain that
\[
A_j \geq \exp \left\{ q^{j-1} \left( \log A_1 - S_q(\infty) \right) \right\}.
\]
So if we come back to (5.6) we have
\[
F_0(t) \geq A_j(1 + t)^{-a_j} t^{b_j} \\
\geq (1 + t)^{a_j} t^{-2/(q-1)} \exp \left( q^{j-1} J(t) \right), \quad t > 0,
\]
(5.9)
where

\[ J(t) = -\left( (n - 1) \frac{p}{2} + n \right) \log(1 + t) + \left( n + 1 + \frac{2}{q - 1} \right) \log t \\
+ \log A_1 - S_q(\infty). \]

Then for \( t \geq 1 \), \( J(t) \) can be estimated as

\[ J(t) \geq -\left( (n - 1) \frac{p}{2} + n \right) \log(2t) + \left( n + 1 + \frac{2}{q - 1} \right) \log t \\
+ \log A_1 - S_q(\infty) \\
= \left( n + 1 + \frac{2}{q - 1} - (n - 1) \frac{p}{2} - n \right) \log t + \log A_1 \\
\quad - \left( (n - 1) \frac{p}{2} + n \right) \log 2 - S_q(\infty) \\
= \log \left( t^{1 + 2/(q - 1) - (n - 1)p/2} A_1 \right) - C_6, \tag{5.10} \]

where

\[ C_6 := \left( (n - 1) \frac{p}{2} + n \right) \log 2 + S_q(\infty). \]

Recall the definition of \( A_1 \) in (5.7), we have that \( J(t) > 1 \) if

\[ t \geq C_7 \varepsilon^{-2p(q - 1)/(2q + 2 - (n - 1)p(q - 1))} \]

with

\[ C_7 := \left( C_3^{-1} e^{1 + C_6} \right)^{2(q - 1)/(2q + 2 - (n - 1)p(q - 1))}. \]

By (5.9), it is easy to get

\[ F_0(t) \to \infty \text{ as } j \to \infty. \]

Hence we get the lifespan estimate in Theorem 2.1.

**Remark 5.1** *In the last line of (5.10), we should require that

\[ 1 + \frac{2}{q - 1} - \frac{(n - 1)p}{2} > 0, \]

which leads to the restriction (2.4) for \( q \) in the case \( n \geq 2. \)***
6 Proof of Theorem 2.2

Due to (5.4), we roughly get an estimate of the form,
\[ F_0(t) \geq C\varepsilon p \frac{t}{n} + 1 - (n-1)p/2 \]
for large \( t \) with some positive constant \( C \) independent of \( \varepsilon \). So if
\[ p > \frac{2n}{n-1}, \]
then we have
\[ n + 1 - (n-1)p/2 < 1, \]
which means that (5.4) is weaker than the linear growth. And hence it is natural to get a better result if we have linear growth in the first step in the iteration argument. Actually, due to the assumption of the initial data, we get from (5.1) that
\[ F_0'(t) \geq \frac{m(0)}{m(t)} F_0(0) \geq \left( m(0) \int_{\mathbb{R}^n} g(x)dx \right) \varepsilon, \]
which implies that
\[ F_0(t) \geq C_8 \varepsilon t, \quad t \geq 0, \quad (6.1) \]
where
\[ C_8 := m(0) \int_{\mathbb{R}^n} g(x)dx. \]
Plugging (6.1) into (5.5) we obtain
\[ F_0(t) \geq C_9 \varepsilon q \int_0^t \int_0^r (1 + s)^{-n(q-1)} t^q dr ds \]
\[ \geq C_{10} \varepsilon q (1 + t)^{-n(q-1)} t^q + 2, \quad (6.2) \]
where
\[ C_9 := C_4 m(0) C_8^q \quad \text{and} \quad C_{10} := \frac{C_9}{(q + 1)(q + 2)}. \]
Then as in section 5, we may assume that
\[ F_0(t) \geq \tilde{A}_j (1 + t)^{-\tilde{a}_j} t^{\tilde{b}_j} \quad \text{for } t \geq 0 \quad (j = 1, 2, 3 \cdots) \quad (6.3) \]
with
\[ \tilde{A}_1 = C_{10} \varepsilon q, \quad \tilde{a}_1 = n(q - 1), \quad \tilde{b}_1 = q + 2. \quad (6.4) \]
Plugging (6.3) into (5.5) we have
\[ F_0(t) \geq \tilde{A}_{j+1}(1 + t)^{-q\tilde{a}_j - q \tilde{b}_j - q}, \]
where
\[ \tilde{A}_{j+1} \geq \frac{C_4 m(0) \tilde{A}_j^q}{(q \tilde{b}_j + 2)^2}, \quad \tilde{a}_{j+1} = q\tilde{a}_j + n(q - 1), \quad \tilde{b}_{j+1} = q\tilde{b}_j + 2, \]
from which we get that
\[
\begin{align*}
\tilde{a}_j &= nq^j - n, \\
\tilde{b}_j &= q^{j-1}\{q + 2 + 2/(q - 1)\} - 2/(q - 1), \\
\tilde{A}_j &\geq C_{11} \tilde{A}_{j-1}/q^{2(j-1)}
\end{align*}
\]
with
\[ C_{11} := \frac{C_4 m(0)}{\{q + 2 + 2/(q - 1)\}^2}. \]
In the same way as in section 5, we conclude that
\[ \tilde{A}_j \geq \exp \left\{ q^{j-1}\left( \log \tilde{A}_1 - \tilde{S}_q(\infty) \right) \right\} \]
with
\[ \tilde{S}_q(\infty) := \lim_{j \to \infty} \tilde{S}_q(j) := \lim_{j \to \infty} \sum_{k=1}^{j-1} \frac{2k \log q - \log C_{11}}{q^k}, \]
and
\[ F_0(t) \geq (1 + t)^n t^{-2/(q - 1)} \exp \left( q^{j-1} \tilde{J}(t) \right) \]
with
\[ \tilde{J}(t) = -nq \log(1 + t) + \left( q + 2 + \frac{2}{q - 1} \right) \log t + \log \tilde{A}_1 - \tilde{S}_q(\infty). \]
Therefore, if \( t \geq 1 \), we come to
\[ \tilde{J}(t) \geq -nq \log(2t) + \left( q + 2 + \frac{2}{q - 1} \right) \log t + \log \tilde{A}_1 - \tilde{S}_q(\infty) \\
= \left( q + 2 + \frac{2}{q - 1} - nq \right) \log t + \log \tilde{A}_1 - \tilde{S}_q(\infty) - nq \log 2 \\
= \log \left( t^{q+2+2/(q-1)-nq} \tilde{A}_1 \right) - C_{12} \]
with
\[ C_{12} := \tilde{S}_q(\infty) + nq \log 2. \]
If
\[ t \geq C_{13} \varepsilon^{-(q-1)/(q+1-n(q-1))}, \]
where
\[ C_{13} := \left( \frac{e^{C_{12}+1}}{C_{10}} \right)^{1/(q+2+2/(q-1)-nq)}, \]
then we have \( \tilde{J}(t) \geq 1 \), which will lead to by (6.6)
\[ F_0(t) \to \infty \quad \text{as} \quad j \to \infty, \]
and we finish the proof of Theorem 2.2.

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