The discrete flavor symmetry $D_5$

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Abstract

We consider the standard model (SM) extended by the flavor symmetry $D_5$ and search for a minimal model leading to viable phenomenology. We find that it contains four Higgs fields apart from the three generations of fermions whose left- and left-handed conjugate parts do not transform in the same way under $D_5$. We provide two numerical fits for the case of Dirac and Majorana neutrinos to show the viability of our low energy model. The fits can accommodate all data with the neutrinos being normally ordered. For Majorana neutrinos two of the right-handed neutrinos are degenerate. Concerning the Higgs sector we find that all potentials constructed with three SM-like Higgs doublets transforming as $\mathbf{1}+\mathbf{2}$ under $D_5$ have a further unwanted global $U(1)$ symmetry. Therefore we consider the case of four Higgs fields forming two $D_5$ doublets and show that this potential leads to viable solutions in general, however it does not allow spontaneous CP-violation (SCPV) for an arbitrary vacuum expectation value (VEV) configuration. Finally, we discuss extensions of our model to grand unified theories (GUTs) as well as embeddings of $D_5$ into the continuous flavor symmetries $SO(3)_f$ and $SU(3)_f$.

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1 Introduction

Gauge interactions and charge quantization of quarks and leptons can successfully be described by the mathematical concept of Lie groups, e.g. in the framework of GUTs. Albeit the number of fermion generations, the diverse masses and mixing parameters of quarks and leptons remain free parameters. It is tempting to assume that these properties can also be explained by some (flavor) symmetry \( G_f \). For several reasons \( G_f \) is chosen to be discrete and non-abelian in many models. In the literature the permutation symmetries \( S_3 \) \cite{1}, \( A_4 \) \cite{2} and \( S_4 \) \cite{3}, the single- and double-valued dihedral groups such as \( D_4 \) \cite{4} and \( D_2' \) \cite{5} and groups \( D_n, D'_n \) with larger index \( n \) \cite{6-8} have been discussed. Furthermore the two-valued group \( T' \) \cite{9} and subgroups of \( SU(3) \), \( \Delta(48) \) and \( \Delta(75) \) \cite{10}, belonging to the series of \( \Delta(3n^2) \) and \( \Delta(6n^2) \) with \( n \in \mathbb{N} \) have been studied. Most of these groups have been used to maintain a certain fermion mass texture. However, proceeding in this way does not answer the question which fundamental group structure of a discrete symmetry is favorable for describing nature and which is not.

In order to investigate the generic features of a certain group structure it is enough to discuss the smallest group which reveals this structure. Therefore we choose the flavor symmetry to be \( D_5 \) which is the smallest group with two irreducible (faithful) inequivalent two-dimensional representations. This group is used in \cite{6} to produce certain mass textures for the lepton sector, but mass matrices for the quarks as well as the Higgs sector are not discussed. Apart from \( D_5 \) only the discussed groups \( D_n \) for \( n \geq 6 \), \( D'_n \) for \( n > 2 \) and \( T' \) have more than one irreducible two-dimensional representation. However, in general the groups differ in the product structure.

Our starting point is thus the SM gauge group extended by the flavor group \( D_5 \). Both groups are broken only spontaneously at the electroweak scale. We require a partial unification for left- and left-handed conjugate fields, i.e. both should transform as \( 1 + 2 \) under \( D_5 \) where \( 1 \) and \( 2 \) do not need to be the same for both. Since we do not want to give up the idea of unified gauge groups we further require that our model is embeddable into the Pati-Salam group \( SU(4)_C \times SU(2)_L \times SU(2)_R, SU(5), SO(10) \) or \( E(6) \). The resulting mass matrices should allow a viable fit of all data which will be demonstrated by numerical examples. For this and for the spontaneous breaking of \( D_5 \), we have to take at least three \( SU(2)_L \) doublet Higgs fields which transform non-trivially under \( D_5 \). Since there exist strong bounds on flavor changing neutral currents (FCNCs), the number of Higgs fields should be as small as possible and they should be sufficiently heavy. Furthermore we discard the possible existence of \( SU(2)_L \) triplet and SM gauge singlet (scalar) fields. Taking all these constraints and the requirement that there are no left-over massless Goldstone bosons coming from accidental symmetries of the Higgs potential we will show that we need at least four Higgs fields. With these it turns out to be favorable to have different transformation properties of left- and left-handed conjugate fermions under \( D_5 \). The neutrinos can be either Dirac or Majorana particles. In the second case two of the right-handed neutrinos are degenerate, since there are no SM gauge singlets in the theory. We contrast this minimal \( D_5 \) invariant model with the corresponding one invariant under the flavor symmetry \( D_3 \) which is isomorphic to \( S_3 \) and considered very often in the literature \cite{1}.

We also discuss the three Higgs potential in detail and show the existence of an accidental global \( U(1) \) symmetry in the potential. Furthermore we study the phenomenology of the
four Higgs sector analytically and numerically and demonstrate that the VEV configurations chosen in the numerical examples of the fermion mass matrices cannot be minima of the potential, if CP is only spontaneously violated. In the case of explicit CP-violation a numerical analysis indicates the possibility that the chosen VEV configurations can be minima of the general $D_5$ invariant potential. The $D_5$ invariant three Higgs sector as well as the four Higgs sector are compared with the corresponding Higgs sectors invariant under the dihedral groups $D_3$, $D_4$ and $D_6$. Thereby we show the importance to classify the symmetries according to their product structure rather than to pick one freely. Finally, we briefly mention the possible embeddings of our minimal model into GUT groups and continuous flavor groups.

The paper is organized as follows: Section 2 contains the group theory of the dihedral symmetries. Our minimal model is presented in Section 3 and the numerical analysis in Section 4. Section 5 is dedicated to the Higgs sectors of $D_5$ and the differences to $D_3$, $D_4$ and $D_6$. Section 6 contains possible extensions of our model from a low to a high energy theory. Finally, we conclude in Section 7 and comment on non-trivial subgroups of $D_5$. Clebsch Gordan coefficients and embeddings of $D_5$ are delegated to Appendix A. Appendix B lists the numerical solutions for the Yukawa couplings and Higgs VEVs and Appendix C contains the used experimental data.

2 Group Structure of Dihedral Groups

2.1 General Properties of Dihedral Groups $D_n$

The groups $D_n$ are well-known in solid state and molecular physics. Their double-valued counterparts are the groups $D'_n$. Since $n \in \mathbb{N}$, there are infinitely many of them. Apart from the two trivial groups with $n = 1, 2$ all groups $D_n$ are non-abelian. They only contain real one- and two-dimensional irreducible representations. If its index $n$ is even, the group $D_n$ has four one- and $\frac{n}{2} - 1$ two-dimensional representations and for $n$ being odd $D_n$ has two one- and $\frac{n-1}{2}$ two-dimensional representations. The order of the group $D_n$ is $2n$. The four smallest non-abelian discrete groups can be found among the family of the dihedral symmetries: $D_3$, $D_4$, $D'_2$ and $D_5$. Generators of the two-dimensional representations can be given for all $n$ [11]:

$$A = \begin{pmatrix} e^{\frac{2\pi i}{n}}j & 0 \\ 0 & e^{-\frac{2\pi i}{n}}j \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(1)

with $j = 1, \ldots, \frac{n}{2} - 1$ for $n$ even and $j = 1, \ldots, \frac{n-1}{2}$ for $n$ odd. They fulfill the relations:

$$A^n = \mathbb{1}, \quad B^2 = \mathbb{1}, \quad ABA = B.$$ (2)

The corresponding character tables can also be found in [11]. Note that we have chosen complex generators for the two-dimensional representations. Since these are real, there exists a unitary matrix $U$ which links their generators to its complex conjugates: $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For any $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim 2$ the combination $U \begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix} = \begin{pmatrix} a_2^* \\ a_1^* \end{pmatrix}$ transforms as $\mathbf{2}$ instead of $\begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix}$, as it would be the case for real generators A and B.
Table 1: Character table of the group $D_5$. \( \alpha \) and \( \beta \) are given as \( \alpha = \frac{1}{2} \left( -1 + \sqrt{5} \right) = 2 \cos \left( \frac{2\pi}{5} \right) \) and \( \beta = \frac{1}{2} \left( -1 - \sqrt{5} \right) = 2 \cos \left( \frac{4\pi}{5} \right) \) and therefore \( \alpha + \beta = -1 \). For further explanations see text.

### 2.2 The Group $D_5$

$D_5$ is of order ten and has two one- and two two-dimensional irreducible representations, since its index is odd. They are denoted as \( \mathbf{1}_1, \mathbf{1}_2, \mathbf{2}_1 \) and \( \mathbf{2}_2 \). Both two-dimensional representations are faithful. Their characters \( \chi \), i.e. the traces of their representation matrices, are given in the character table, shown in Table 1. We use the following notations: \( C_i \) with \( i = 1, \ldots, 4 \) are the four classes of the group, \( ^oC_i \) is the order of the \( i \)th class, i.e. the number of distinct elements contained in this class, \( ^o h_{C_i} \) is the order of the elements \( R \) in the class \( C_i \), i.e. the smallest integer (\( \geq 0 \)) for which the equation \( R ^o h_{C_i} = 1 \) holds. Furthermore the table contains one representative for each class \( C_i \) given as product of the generators \( A \) and \( B \) of the group. The elements belonging to the classes \( C_i \) are: \( C_1 = \{ 1 \} \), \( C_2 = \{ B, BA, BA^2, BA^3, BA^4 \} \), \( C_3 = \{ A, A^4 \} \) and \( C_4 = \{ A^2, A^3 \} \). With the help of the character table the Kronecker products can be calculated. They are \( \mathbf{1}_i \times \mathbf{1}_j = \mathbf{1}_{\text{mod}2+1} \), \( \mathbf{1}_i \times \mathbf{2}_j = \mathbf{2}_j \) for \( \{i,j\} \in \{1,2\} \) and \( \mathbf{2}_1 \times \mathbf{2}_2 = \mathbf{2}_1 + \mathbf{2}_2 \) and \( \mathbf{2}_i \times \mathbf{2}_j = \mathbf{1}_1 + \mathbf{2}_j \), \( \mathbf{2}_i \times \mathbf{2}_j = \mathbf{1}_2 \) for \( i \neq j \) where \( [\mu \times \mu] \) is the symmetric part of the product \( \mu \times \mu \) and \( \{\mu \times \mu\} \) the anti-symmetric one. Note further that \( \mu \times \nu = \nu \times \mu \) for all representations \( \mu \) and \( \nu \). Taking \( n = 5 \) in Eq. (1) and Eq. (2) gives the generators \( A \) and \( B \) and their relations for \( D_5 \). They are required for the calculation of the Clebsch Gordan coefficients being shown in Appendix A. They actually coincide with the matrices chosen in [6]. The embedding of \( D_5 \) into continuous groups is very interesting with respect to grand unified model building. Therefore we show how \( D_5 \) can be embedded into \( SO(3) \) and \( SU(3) \) in Appendix A. We will discuss this in more detail in Section 6.

### 3 Minimal Model

Here we present a minimal model which leads to viable mass spectra and mixing parameters for quarks as well as leptons with the Higgs potential being free from accidental symmetries (see Section 5). We assign the left-handed quarks \( Q_i = (u_i, d_i)^T \) and their conjugates \( u^c_{L_i} \),
The resulting Dirac mass matrices are:

\[
Q_1 \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \left( \begin{array}{c} Q_2 \\ Q_3 \end{array} \right) \sim \begin{pmatrix} 2 \\ 2 \end{pmatrix}; \quad u_{L1}^c, d_{L1}^c \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \left( \begin{array}{c} u_{L2}^c \\ u_{L3}^c \end{array} \right), \left( \begin{array}{c} d_{L2}^c \\ d_{L3}^c \end{array} \right) \sim \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

The left-handed lepton doublets \( L_i = (\nu_i, e_i)^T_L \) and its conjugates \( e_{L_i}^c \) and \( \nu_{L_i}^c \) transform in a similar way, i.e.:

\[
L_1 \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \left( \begin{array}{c} L_2 \\ L_3 \end{array} \right) \sim \begin{pmatrix} 2 \\ 2 \end{pmatrix}; \quad e_{L1}^c, \nu_{L1}^c \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \left( \begin{array}{c} e_{L2}^c \\ e_{L3}^c \end{array} \right), \left( \begin{array}{c} \nu_{L2}^c \\ \nu_{L3}^c \end{array} \right) \sim \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

The four Higgs fields \( \chi_i \) and \( \psi_i \) which are \( SU(2)_L \) doublets with hypercharge \( Y = -1 \) (like the Higgs field in the SM) transform as \( \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) under \( D_5 \).

The fermion mass matrices arise from the coupling \( y_{ij} L_i^T \epsilon \xi L_j^c \) for down-type quarks \( (L_i = Q_i, L_i = d_{L_i}^c) \) and charged leptons \( (L_i = L_i, L_i^c = e_{L_i}^c) \) and up-type quarks \( (L_i = Q_i, L_i^c = u_{L_i}^c) \) and neutrinos \( (L_i = L_i, L_i^c = \nu_{L_i}^c) \). Thereby, the Higgs field \( \xi \) is \( \xi^T = (\xi^0, \xi^-)^T \) and its complex conjugate \( \bar{\xi} \) is \( \xi = \epsilon \xi^* \) with \( \epsilon \) being the anti-symmetric 2-by-2 matrix in \( SU(2)_L \) space and the star \( * \) denotes the complex conjugation.

The resulting Dirac mass matrices are:

\[
\mathcal{M}_{u,\nu} = \begin{pmatrix}
0 & \alpha_{22}^{u,\nu} \langle \chi_1 \rangle^* & \alpha_{32}^{u,\nu} \langle \chi_2 \rangle^* \\
\alpha_{21}^{u,\nu} \langle \psi_1 \rangle^* & 0 & \alpha_{31}^{u,\nu} \langle \chi_1 \rangle^* \\
\alpha_{31}^{u,\nu} \langle \psi_2 \rangle^* & \alpha_{32}^{u,\nu} \langle \chi_2 \rangle^* & 0
\end{pmatrix}, \quad
\mathcal{M}_{d,l} = \begin{pmatrix}
0 & \alpha_{22}^{d,l} \langle \chi_2 \rangle & \alpha_{32}^{d,l} \langle \chi_1 \rangle \\
\alpha_{21}^{d,l} \langle \psi_2 \rangle & 0 & \alpha_{31}^{d,l} \langle \chi_2 \rangle \\
\alpha_{31}^{d,l} \langle \psi_1 \rangle & \alpha_{32}^{d,l} \langle \chi_1 \rangle & 0
\end{pmatrix}, \quad (3)
\]

where \( \langle \xi \rangle \) denotes the VEV of the field \( \xi = \psi_i, \chi_i \). The VEVs and the Yukawa couplings \( \alpha_{j}^{u,d,l,\nu} \) are in general complex. The (1,1) element of the mass matrices is zero, since there is no Higgs field transforming trivially under \( D_5 \). Even though there are more parameters in our model than observables to fit, this is a rather non-trivial task, since apart from the number of free parameters also the structure of the mass matrices plays an important role in fitting the observables.

The number of parameters could obviously be reduced, if some of the Yukawa couplings were assumed to be equal. Since our flavor symmetry \( D_5 \) cannot explain this, we do not use such assumptions. Another way to reduce the number of parameters could be to set some of the VEVs to be equal or zero. For two VEVs being zero we either have two massless quarks or cannot generate CP-violation, since \( J_{CP} \propto \det \left( \begin{pmatrix} \mathcal{M}_u \mathcal{M}_d^\dagger, \mathcal{M}_d \mathcal{M}_d^\dagger \end{pmatrix} \right) \) vanishes. Furthermore some of these configurations lead to the appearance of accidental symmetries in the Higgs potential (see Section 5.2). For one VEV being zero or two VEVs being equal we cannot find an obvious reason to exclude these assumptions, but one does not gain much in doing so, since most of the free parameters in our model come from the (in total) 16 Yukawa couplings which have to be compared with the 20 (22) observable masses and mixing parameters in the quark and lepton sector for Dirac (Majorana) neutrinos. Therefore we do not make such assumptions in the following numerical study.

We have chosen a structure which is similar to a mass texture which has already been discussed in the literature [13]. It actually arises from our mass matrix for real parameters and in the limit that all VEVs are equal in Eq.(3) together with \( \alpha_2^i = \alpha_3^i \) for \( i = u, d, l, \nu \) or
for $\langle \chi_1 \rangle = \langle \chi_2 \rangle$, $\langle \psi_1 \rangle = \langle \psi_2 \rangle$ and $\alpha_3^{u,\nu} = \langle \chi_2 \rangle^\dagger \alpha_2^{u,\nu}$, $\alpha_3^{d,\nu} = \langle \chi_2 \rangle^\dagger \alpha_2^{d,\nu}$. Then all mass matrices are invariant under the interchange of the second and third generation which always leads to mixing angles $\theta_{13} = 0$ and $\theta_{23} = \frac{\pi}{4}$ with unconfined $\theta_{12}$. In the leptonic sector this is called $\mu - \tau$ interchange symmetry [14].

Here we assume that the first generation transforms as 1_1 and the second and third one as 2_1 under $D_5$. This choice is inspired by the observation that the masses of the particles which belong to the first generation are much smaller than the masses of the ones of the second and third one and by the fact that the mixing in the 2-3 sector of the leptons is large, possibly maximal. In general there are six possibilities to assign the three generations {1, 2, 3} to 1_1 + 2_1: {{[1], [2, 3]}, {[1], [3, 2]}, {[2], [1, 3]}, {[2], [3, 1]}, {[3], [1, 2]} and {[3], [2, 1]} where [...] the two-dimensional one under $D_5$. If the left-handed fields are permuted by $P$: $P_1 \{[1], [2, 3]\} = \left\{\left[P_1 \cdot (1, 2, 3)^T\right], \left[P_2 \cdot (1, 2, 3)^T\right], \left[P_3 \cdot (1, 2, 3)^T\right]\right\}$ with $P_i$ being the $i$th row of the matrix $P$ and the left-handed conjugate fields by $Q$, the mass matrix $M$ changes to $\tilde{M} = P M Q^T$, since all permutations are orthogonal. As one can see these permutations do neither change the eigenvalues of the mass matrix, i.e. $\det(M M^T - \Lambda \text{diag}(1, 1, 1)) = 0$ remains invariant, nor the mixing matrices $V_{CKM}$ and $U_{MNS}$. If the mass matrix of the up-type quarks $M_u$ is diagonalized by $U_u$ fulfilling $U_u^T M_u U_u = \text{diag}(m_1^2, m_2^2, m_3^2)$ and the same holds for $M_d$ and $\bar{U}_u$, then $U_u$ and $\bar{U}_u$ are connected by $\bar{U}_u = P U_u$. Similarly one gets for the down-type quarks $\tilde{U}_d = P U_d$ and therefore, for example, $V_{CKM} = \bar{U}_u U_d^* = U_d^T P^T P^* U_d^* = U_d^T U_d$, such that $V_{CKM}$ is not affected by this permutation. For the mass matrix texture it seems to be most convenient to have a vanishing (1, 1) element instead of, for example, a (2, 3) or (3, 3) one.

Apart from permuting the three generations among each other one can interchange the transformation properties of the left-handed and left-handed conjugate fields. This leads to matrices which are transposed to the ones shown in Eq. (3). Furthermore one can ask whether there is a considerable change, if the first generation is not assigned to 1_1, but to 1_2. The answer is no, since it only introduces a relative sign between the (1, 2) and (1, 3) and (2, 1) and (3, 1) elements of the mass matrix.

Our choice for the assignment of fermion generations allows an embedding into the Pati-Salam gauge group, where all left-handed fields are unified into one representation as well as all left-handed conjugate fields into the conjugated one. One can also attempt to embed the model into $SO(10)$, but then all fermions have to transform in the same way under $D_5$. In doing so one arrives at mass matrices which have two additional texture zeros in the (2, 3) and (3, 2) element (and one Yukawa coupling less than the matrices shown above). In case of hermitian matrices such a texture is excluded for quarks [15]. This does not strictly apply in our case, because our matrices are in general not hermitian, but we believe that this does not change the result of [15]. For an embedding into $SU(5)$, the generations $Q_i$ and $u_{Li}$ would have to transform in the same way under $D_5$, since these fields are unified into the 10-plet of $SU(5)$. Then again the mass matrix for the up-type quarks has to have three texture zeros in the positions (1, 1), (2, 3) and (3, 2) combined with a mass matrix for the down-type quarks with one zero in the (1, 1) element, since $Q_i$ and $d_{Li}$ do not belong to the same $SU(5)$ representation and therefore can transform differently under $D_5$.\footnote{The permutations have to be both $P$, since $u_L$ and $d_L$ transform in the same representation of the SM.}
Generally, such a structure is not excluded, but taking into account the various relations among the non-vanishing matrix elements, it seems to be unfavorable. Therefore we do not discuss this possibility here. Concerning the number of possible different assignments for quarks and leptons the Pati-Salam group has an advantage over $SU(5)$, since the sixteen fermions of one generation (i.e. the right-handed neutrino is always included in our considerations) are unified into two and not into three representations of the gauge group. Its disadvantage is the fact that the three SM gauge factors are not unified into a single group, but rather in a product one.

If neutrinos are Majorana particles, the Majorana mass matrix for the right-handed neutrinos looks very simple, since our model does not include SM gauge singlets transforming non-trivially under $D_5$:

$$M_{RR} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & M_2 \\ 0 & M_2 & 0 \end{pmatrix}.$$  

(4)

The resulting mass matrix for the light neutrinos is then given through the type I seesaw \[ 16 \] formula

$$M_\nu = (-)M_\nu M_{RR}^{-1} M_\nu^T.$$  

(5)

As one can see, two of the right-handed neutrinos are degenerate at tree-level. This can be used for resonant leptogenesis \[ 17 \].

An important aspect of our symmetry driven discussion is that different from the usual assumption in papers treating a certain texture of the mass matrices (like \[ 13 \]) the Majorana mass matrix for the right-handed neutrinos strongly differs from the structure of the Dirac masses. Therefore also the effective mass matrix for the light neutrinos is in general distinct from the (Dirac) mass matrices of the other fermions. The reason for this simply lies in the fact that Majorana and Dirac masses do arise from completely different mechanisms with different symmetry aspects: first the Dirac masses connect different fields whereas Majorana masses connect the same field with itself and second Dirac masses arise through the coupling of $SU(2)_L$ doublet Higgs fields with hypercharge $Y = \pm 1$ unlike Majorana masses which are direct mass terms for right-handed neutrinos and are mediated by $SU(2)_L$ Higgs triplets for left-handed ones.

$D_5$ has two distinct two-dimensional representations instead of only one like $D_3$. The differences in the mass matrices which follow from this fact will be studied next. In \[ 18 \] the authors assigned the fermion generations and three Higgs fields to $\mathbf{1} + \mathbf{2}$ under $D_3$. We observe that we cannot use the same representation structure in the Higgs sector in our $D_5$ model for a realistic theory due to an accidental $U(1)$ symmetry in the potential (see Section 5.1). If we do so anyway, we can distinguish two cases in $D_5$: both fermion generations and Higgs fields transform as $\mathbf{1} + \mathbf{2}$ under $D_5$ or the fermions are in $\mathbf{1} + \mathbf{2}$ and the Higgs fields are in $\mathbf{1} + \mathbf{2}$ with $i \neq j$. In the first case the $D_5$ invariance leads to a mass matrix with two zeros on its diagonal, i.e. the $(2,2)$ and $(3,3)$ element vanish, since $\mathbf{2} \times \mathbf{2} = \mathbf{2}$ for $i = 1,2$ in contrast to $\mathbf{2} \times \mathbf{2} \not\equiv \mathbf{2}$ in $D_3$. In the latter case the first generation transforming trivially under $D_5$ is decoupled from the two others forming a two-dimensional representation, since $\mathbf{1} \times \mathbf{2} = \mathbf{2}$ for $i = 1,2$.

Thus the existence of two two-dimensional representations in the flavor group has two main consequences on the structure of the mass matrices: on the one hand it tends to reduce
the number of allowed Yukawa couplings and so maintaining texture zeros becomes easier, on the other hand it leaves the freedom of assigning the three generations of fermions to different two-dimensional representations (as done here).

4 Phenomenological Analysis

One appropriate example for a starting point of our numerical analysis is given by

\[ M_{\text{start}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}. \] (6)

With this matrix one can already fit the masses of the second and third generation fermions by fixing \( a \) and \( b \). The eigenvalues of \( M_{\text{start}} \) are \((0, a - b, a + b)\). The mass of the third generation can be taken to be \( a - b \) and the one of the second one \( a + b \). It is clear then that sign\((a) = -\text{sign}(b)\). The mass of the third generation determines the absolute values of \( a \) and \( b \) and the second generation the difference of \(|a|\) and \(|b|\). The vanishing eigenvalue of \( M_{\text{start}} \) also explains the smallness of the first generation compared to the two other ones. Such a matrix is closely connected to the mass matrix of the light neutrinos for \( b \neq 0 \) [14, 19] where it leads to maximal atmospheric mixing. Although it contains this large mixing angle we can use it for the description of quarks, because taking this form for up-type as well as down-type quark mass matrices makes the two large mixing angles cancel such that the angle \( \theta_{23} \) can be arbitrarily small in this sector.

The matrix in Eq. (6) arises from Eq. (3) for \( \langle \chi_1 \rangle = \langle \chi_2 \rangle, \langle \psi_1 \rangle = \langle \psi_2 \rangle \) and \( \alpha_{i,2,3}^i = 0 \) for \( i = u, d, l, \nu \). As argued in Section 5.2 one can arrange the Higgs potential to have an extremum for VEVs being pairwise equal. Since the difference of \(|a|\) and \(|b|\) is determined by the mass of the second generation, \(|a| \approx |b|\) holds. This can be maintained if all VEVs are nearly equal and \(|\alpha_0^i| \approx |\alpha_1^i|\). As shown in Section 5.2 also this is allowed by the minimization conditions. Note that \( D_5 \) does not restrict the Yukawa couplings \( \alpha_j^i \). Therefore \(|\alpha_0^i| \approx |\alpha_1^i|\) is not favored by the flavor symmetry. Also our assumption \(|\alpha_{2,3}^i| \ll |\alpha_{0,1}^i|\) is not guaranteed by any symmetry of the model. In order to achieve this, one could for example introduce a \( U(1)_{FN} \) factor acting non-trivially in flavor space to implement the Froggatt Nielsen (FN) mechanism [20]. We could assign a non-vanishing charge +q to the first generation and let the second and third generation be neutral under this \( U(1)_{FN} \). We then gain a suppression factor of \( \epsilon^{q} \) with \( \epsilon \equiv \frac{\langle \theta \rangle}{M} \) for the matrix elements of the first row and column compared to the others. \( \langle \theta \rangle \) is the VEV of the scalar SM gauge singlet \( \theta \) having charge −1 under \( U(1)_{FN} \) and \( M \) is the mass of some vector-like fermions. These fields are assumed to be very heavy and therefore actually decouple from our low energy theory. Note that the second and third generation of fermions have to transform in the same way under \( U(1)_{FN} \), since otherwise the \( U(1)_{FN} \) would not commute with our flavor symmetry \( D_5 \). Note further that the zero in the \((1,1)\) element is independent of the FN mechanism, since it comes from our assignment of fermions and Higgs fields under \( D_5 \).

Next we present our numerical examples for Dirac and Majorana neutrinos. As already stated above, the mass matrices contain in general too many parameters to make predictions. In order to reduce the number of free parameters we restrict ourselves to real
Yukawa couplings and allow the VEVs \(\langle \chi_{1,2} \rangle\) and \(\langle \psi_1 \rangle\) to have non-vanishing (complex) phases. We show the numerical values of the Yukawa couplings and VEVs in Appendix B. With these the best fit values of the measured quantities shown in Appendix C can be accommodated within the given error bars. Interestingly, all phases of the VEVs turn out to be small. Although the Yukawa couplings are chosen to be real, SCPV is excluded, since the parameters in the Higgs sector have to be complex in order to allow the shown VEV configurations to be minima of the potential. This fact will be explained in detail in Section 5.2. The mass ordering of the (light) neutrinos is normal in both examples. This is not a general feature of our model, but rather chosen by us for simplicity. As we can fit all measured quantities, we only discuss the results for the unmeasured ones.

In case of Dirac neutrinos the sum of the neutrino masses is 0.2255 eV. This is below the current bound obtained from cosmology, even if the Lyman \(\alpha\) data are included [21]. However, it will be measurable in the next five to ten years [22]. \(s_{13}^2\) is about 0.012 and hence a factor of three below the current CHOOZ bound, but detectable quite soon in the next generation of reactor experiments [23]. The Dirac phase \(\delta\) is \(\sim 3.6\) radian. The quantity \(m_\beta\) measured in beta decay experiments is 0.07 eV. This is below the current limit of 2.2 eV [24] and also a factor of three below the one of the planned KATRIN experiment [25].

Before presenting the corresponding results in case of Majorana neutrinos, we comment on the generic problem of Dirac neutrinos. As one can see in Appendix B the Yukawa couplings of the neutrinos \(\alpha'_\nu\) have to be suppressed by nine to twelve orders of magnitude compared to the other fermions to ensure that the neutrinos have masses of the order 1 eV. Clearly, our flavor symmetry \(D_5\) does not explain this, but an additional \(U(1)_{FN}\) family symmetry can do so. If the right-handed neutrinos have a charge \(q_f + 10\) under \(U(1)_{FN}\) where \(q_f\) is the charge of any other fermion under \(U(1)_{FN}\), the neutrino couplings can be suppressed by an additional factor \(\epsilon^{10}\). For \(\epsilon \sim 0.1\) this gives the right order of magnitude for the neutrino masses. However, then the model cannot be embedded into the Pati-Salam group.

Next we consider the neutrinos to be Majorana particles. In this case the type I seesaw [16] explains the smallness of the neutrino masses without an extra suppression of their Yukawa couplings \(\alpha'_\nu\). The masses for the light neutrinos are \((0.1146, 0.1149, 0.1242)\) eV. The sum of their masses is therefore still below the current bound, but could be measured by the planned experiments. The scale of the right-handed neutrino masses is about \(10^{14}\) GeV, but it can be rescaled by proper redefinition of the neutrino Yukawa couplings \(\alpha'_\nu\). Interestingly, \(s_{13}^2\) is around the \(2\sigma\) limit of the CHOOZ experiment. The CP phases which are not constrained by experiments are \((\delta, \varphi_1, \varphi_2) \sim (3.9, 0.74, 0.33)\) radian. \(m_\beta\) is - similar to the Dirac case - a factor of two smaller than the bound which can be obtained by the KATRIN experiment. \(|m_{ee}|\) which is measured in neutrinoless double beta decay is about 0.1 eV. This is an order of magnitude below the upper bound [26], but can be measured in the next five to ten years [27].

The smallness of \(m_\beta\) and \(|m_{ee}|\) is due to the normal ordering of the (light) neutrinos. Finally, we summarize all mentioned quantities in Table 2.
Dirac neutrinos & Majorana neutrinos \\
\hline
\(m_1\) [eV] & 0.0701 & 0.1146 \\
\(m_2\) [eV] & 0.0706 & 0.1149 \\
\(m_3\) [eV] & 0.0848 & 0.1242 \\
\(\sum_i m_i\) [eV] & 0.2255 & 0.3537 \\
\hline
\(M_{R1}\) [GeV] & - & 1.878 \times 10^{14} \\
\(M_{R2,3}\) [GeV] & - & 2.011 \times 10^{14} \\
\(s_{13}^2\) & 0.0119 & 0.0303 \\
\(\delta\) [rad.] & 3.5775 & 3.8619 \\
\(\varphi_1\) [rad.] & - & 0.7396 \\
\(\varphi_2\) [rad.] & - & 0.3312 \\
\(m_\beta\) [eV] & 0.0704 & 0.1150 \\
\(|m_{ee}|\) [eV] & - & 0.1002 \\
\hline
\end{tabular}

Table 2: Numerical values for the unmeasured quantities of the leptonic sector.

The Majorana phases \(\varphi_{1,2}\) are given by the convention: \(U_{MNS} = \tilde{V}_{CKM} \cdot \text{diag}(e^{i\varphi_1}, e^{i\varphi_2}, 1)\) with \(0 \leq \varphi_{1,2} \leq \pi\).

5 Minimal Higgs Potentials in \(D_5\)

5.1 Three Higgs Potential

In this Subsection we discuss the potential arising from the three Higgs fields \(\phi, \psi_1\) and \(\psi_2\) where \(\phi\) transforms as any one-dimensional representation and \(\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\) forms any doublet under \(D_5\). The potential reads:

\[
V_3(\phi, \psi_i) = -\mu_1^2 \phi^\dagger \phi - \mu_2^2 \sum_{i=1}^{2} \psi_i^\dagger \psi_i + \lambda_s \left( \phi^\dagger \phi \right)^2 + \lambda_1 \left( \sum_{i=1}^{2} \psi_i^\dagger \psi_i \right)^2 \\
+ \lambda_2 \left( \psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2 \right)^2 + \lambda_3 |\psi_1^\dagger \psi_2|^2 \\
+ \sigma_1 \left( \phi^\dagger \phi \right) \left( \sum_{i=1}^{2} \psi_i^\dagger \psi_i \right) + \left\{ \sigma_2 \left( \phi^\dagger \psi_1 \right) \left( \phi^\dagger \psi_2 \right) + \text{h.c.} \right\} + \sigma_3 \sum_{i=1}^{2} |\phi^\dagger \psi_i|^2
\] (7)

where only \(\sigma_2\) is complex. It can be made real by appropriate redefinition of the field \(\phi\), for example. We want to show that there exists an accidental \(U(1)\) symmetry in this potential apart from the gauge symmetry \(U(1)_Y\). In order to see this let the Higgs fields \(\phi\) and \(\psi_i\) transform as

\[
\phi \to e^{i\alpha} \phi, \quad \psi_1 \to e^{i\beta} \psi_1, \quad \psi_2 \to e^{i\gamma} \psi_2.
\] (8)

The only non-trivial condition for the phases \(\alpha, \beta\) and \(\gamma\) arises from the term \(\sigma_2\):

\[
2\alpha - \beta - \gamma = 0,
\] (9)

i.e. \(\alpha\) can be expressed as \(\frac{1}{2}(\beta + \gamma)\) while \(\beta\) and \(\gamma\) can have any value. Consequently, there exist two \(U(1)\) symmetries, called \(U(1)_\beta\) and \(U(1)_\gamma\), under which the three fields

9
have the charges: $Q(\phi; \beta) = Q(\phi; \gamma) = \frac{1}{2}$, $Q(\psi_1; \beta) = 1$, $Q(\psi_1; \gamma) = 0$ and vice versa for $\psi_2$: $Q(\psi_2; \beta) = 0$, $Q(\psi_2; \gamma) = 1$. Taking the two linear independent combinations of the charges $Q(\chi; Y) = -[Q(\chi; \beta) + Q(\chi; \gamma)]$ and $Q(\chi; X) = Q(\chi; \beta) - Q(\chi; \gamma)$ for $\chi = \phi, \psi_1, \psi_2$ one recovers the $U(1)_Y$ and a further $U(1)_X$ under which the two fields $\psi_i$ transform with opposite charges and $\phi$ remains invariant. Alternatively, the $U(1)_X$ could be defined such that $Q(\phi; X) = -Q(\psi_i; X)$ and $Q(\psi_j; X) = 0$ with $i \neq j$. Taking the first definition of the $U(1)_X$ charges one sees that any non-vanishing VEV for a field $\psi_i$ leads to the spontaneous breaking of the $U(1)_X$ and therefore to the appearance of a massless Goldstone boson which is phenomenologically unacceptable. There are two ways to circumvent this: first introduce terms in the potential which explicitly break $U(1)_X$, but also the $D_5$ symmetry \footnote{This is similar to the soft breaking terms invoked in the MSSM.} or second leave $U(1)_X$ unbroken. The first possibility increases the number of parameters by at least four and is not explained in terms of any (further) symmetry while the second one cannot be realized, if our model should accommodate the fermion masses at tree-level without further fields. Hence we abandon this three Higgs potential which actually contains the minimal set of Higgs fields needed for the construction of viable mass matrices.

The accidental $U(1)$ symmetry found here becomes obvious in the basis where the generators $A$ and $B$ of $D_5$ are taken to be the ones shown in Eq.(1). If one chooses for example real representation matrices (found in [11]), the resulting potential still contains the extra $U(1)$, but it is rather non-trivial to show this.

If one sets $\sigma_2 = 0$, the symmetry of the potential is further increased to $U(1)^3$, since then the condition Eq.(1) is no longer valid. The $U(1)^2$ which then exists in the $(\psi_1, \psi_2)$ space can be enhanced to an $SU(2)$ by setting $\lambda_3 = 4 \lambda_2$. Then the terms $\lambda_2$ and $\lambda_3$ can be written as $\sum_a (\Psi^T \tau_a \Psi)^2$ with $\Psi = (\psi_1, \psi_2)^T$ and $\tau_a$ are the Pauli matrices, i.e. it equals the invariant arising from $(2 \times 2)^2 \equiv 3 \times 3 \equiv 1$ in $SU(2)$. $\sigma_2 = 0$ and/or $\lambda_3 = 4 \lambda_2$ can be enforced by the VEV conditions. One example for this is given by the configuration where the VEVs of $\phi$ and $\psi_1$ are unequal zero and $\langle \psi_2 \rangle = 0$.

Let us comment on the origin of this accidental $U(1)$. For this we compare our $D_5$ invariant Higgs potential to one being invariant under $D_5$ and $D_4$, respectively. The $D_5$ invariant version of our potential has already been discussed in the literature [28]. Apart from the terms contained in the $D_5$ invariant potential it allows a further term, namely:

\[
\left\{ \tau \left[ (\phi^\dagger \psi_1) \left( \psi_2^\dagger \psi_1 \right) \pm (\phi^\dagger \psi_2) \left( \psi_1^\dagger \psi_2 \right) \right] + \text{h.c.} \right\}
\]

with $+ \phi \sim 1_1$ and $- \phi \sim 1_2$ (under $D_5$). This term is $D_3$ invariant, since the product $2 \times 2$ contains the representation $2$ itself and therefore $2 \times 2 \times 2 = 3 \equiv 1_1$ for $i = 1, 2$. In $D_5$ the corresponding coupling is of the form $2_i \times 2_i \neq 2_i$ for both $i = 1, 2$. Clearly, the $\tau$ term does not allow for a further $U(1)$ symmetry, since it enforces the relations $2 \beta - \alpha - \gamma = 0$ and $2 \gamma - \alpha - \beta = 0$ for the phases $\alpha, \beta$ and $\gamma$. This term has to vanish, if the potential should be invariant under the reflection symmetry $\phi \rightarrow -\phi$ and $\psi_{1,2} \rightarrow \psi_{1,2}$ as mentioned in [29]. Then there exists an accidental $U(1)$ which was already realized in [30].

To compare our potential to the one being invariant under $D_4$ one has to notice that the product $2 \times 2$ decomposes into $\sum_{i=1}^4 1_i$ there. Hence the quartic coupling $\lambda_3$ has to be replaced by

\[
\lambda_3 \left( \psi_1^\dagger \psi_2 - \psi_1^\dagger \psi_1 \right)^2 + \tilde{\lambda}_3 \left( \psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1 \right)^2.
\]
The rest of the potential remains the same. Thereby the field $\phi$ can transform as any one-dimensional representation of $D_4$. $\lambda_3$ and $\tilde{\lambda}_3$ lead to $\beta = \gamma$ such that $\alpha = \beta = \gamma$ is enforced. The accidental $U(1)$ can be restored, if $\tilde{\lambda}_3 = -\lambda_3$ is chosen, since then Eq. (10) simplifies to $-4 \lambda_3 |\psi_1^+ \psi_2|^2$.

Since $D_6$ has also been mentioned as flavor symmetry in the literature and is the next smallest $D_n$ symmetry after $D_5$, we briefly comment on $D_6$ invariant three Higgs potentials. If the three fields transform as faithful two-dimensional and as trivial representation, their potential incorporates an accidental $U(1)$ symmetry. However, using instead one of the two further one-dimensional representations of $D_6$ which are not present in $D_5$ one can get rid of this $U(1)$. Products of the faithful representation with these have the structure $1 \times 2 = 2'$ and therefore lead together with $2 \times 2 = 1 + 1' + 2'$ to a potential which coincides with the one obtained from $D_3$.

This demonstrates that a thorough discussion of the Higgs potential is always necessary to ensure the validity of the model as a whole. A more complete discussion about the possible potentials arising from $D_n$ flavor symmetries and also $D_n'$ symmetries will be given elsewhere [31].

5.2 Four Higgs Potential

In this Subsection we consider a potential containing four Higgs fields. There exist two possible choices. First we can augment our three Higgs potential with a further Higgs field $\chi$ transforming as one-dimensional representation. If $\phi \sim 1_i$ then $\chi$ should transform as $1_j$ with $i \neq j$. Writing down all possible $D_5$ invariant couplings shows that they cannot break the $U(1)_\chi$ symmetry. Therefore we will consider a four Higgs potential with fields $\chi_i$ and $\psi_i$, $i = 1, 2$. Each pair forms a doublet under $D_5$, without loss of generality: $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \sim 2_1$ and $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim 2_2$. The potential then has the following form:

$$V_4(\chi_i, \psi_i) = -\mu_1^2 \sum_{i=1}^2 \chi_i^\dagger \chi_i - \mu_2^2 \sum_{i=1}^2 \psi_i^\dagger \psi_i + \lambda_1 \left( \sum_{i=1}^2 \chi_i^\dagger \chi_i \right)^2 + \tilde{\lambda}_1 \left( \sum_{i=1}^2 \psi_i^\dagger \psi_i \right)^2 + \lambda_2 \left( \chi_1^\dagger \chi_1 - \chi_2^\dagger \chi_2 \right)^2 + \lambda_3 |\chi_1^\dagger \chi_2|^2 + \tilde{\lambda}_2 \left( \psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2 \right)^2 + \tilde{\lambda}_3 |\psi_1^\dagger \psi_2|^2$$

$$+ \sigma_1 \left( \sum_{i=1}^2 \chi_i^\dagger \chi_i \right) \left( \sum_{j=1}^2 \psi_j^\dagger \psi_j \right) + \sigma_2 \left( \chi_1^\dagger \chi_1 - \chi_2^\dagger \chi_2 \right) \left( \psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2 \right)$$

$$+ \left\{ \tau_1 \left( \chi_1^\dagger \psi_1 \right) \left( \chi_2^\dagger \psi_2 \right) + \text{h.c.} \right\} + \left\{ \tau_2 \left( \chi_1^\dagger \psi_2 \right) \left( \chi_2^\dagger \psi_1 \right) + \text{h.c.} \right\}$$

$$+ \left\{ \kappa_1 \left[ \left( \chi_1^\dagger \chi_2 \right) \left( \chi_1^\dagger \psi_2 \right) + \left( \chi_2^\dagger \chi_1 \right) \left( \chi_2^\dagger \psi_1 \right) \right] + \text{h.c.} \right\}$$

$$+ \left\{ \kappa_2 \left[ \left( \psi_1^\dagger \psi_2 \right) \left( \chi_2^\dagger \psi_2 \right) + \left( \psi_2^\dagger \psi_1 \right) \left( \chi_1^\dagger \psi_1 \right) \right] + \text{h.c.} \right\}$$

$$+ \kappa_3 \left[ |\chi_1^\dagger \psi_1|^2 + |\chi_2^\dagger \psi_2|^2 \right] + \kappa_4 \left[ |\chi_1^\dagger \psi_2|^2 + |\chi_2^\dagger \psi_1|^2 \right]$$

(11)

where the couplings $\tau_{1,2}$ and $\kappa_{1,2}$ are in general complex. We checked that this potential does not have any accidental (global) symmetries. Assuming that the fields $\chi_{1,2}, \psi_{1,2}$
transform in the following way:

\[ \chi_1 \rightarrow \chi_1 e^{i\alpha}, \quad \chi_2 \rightarrow \chi_2 e^{i\beta}, \quad \psi_1 \rightarrow \psi_1 e^{i\gamma}, \quad \psi_2 \rightarrow \psi_2 e^{i\delta}, \]

one finds that the couplings \( \mu_{1,2}, \lambda_{1,2,3}, \tilde{\lambda}_{1,2,3}, \sigma_{1,2}, \kappa_{3,4} \) leave the full \( U(1)^4 \) invariant, \( \tau_{1,2} \) breaks it down to \( U(1)^3 \) and \( \kappa_{1,2} \) down to \( U(1)^2 \), i.e. none of the couplings itself is only invariant under \( U(1)^Y \). \( \tau_{1,2} \) leave the same \( U(1)^3 \) invariant with the condition \( \alpha = \gamma + \delta - \beta \).

The \( U(1)^2 \) symmetries which are preserved by \( \kappa_{1,2} \) are constrained by the conditions \( 2\alpha = \beta + \delta, \quad 2\beta = \alpha + \gamma \) and \( 2\delta = \beta + \gamma, \quad 2\gamma = \alpha + \delta \), respectively. As one can see only \( \kappa_1 \neq 0 \) and \( \kappa_2 \neq 0 \) can reduce \( U(1)^4 \) to \( U(1)^Y \), i.e. taking the \( \tau_{1,2} \) terms with only the \( \kappa_1 \) term still leaves the potential invariant under \( U(1)^2 \). Consequently, none of the VEV conditions should enforce \( \kappa_1 \) or \( \kappa_2 \) to vanish. A simple example for this is the configuration \( \langle \chi_1 \rangle \neq 0, \langle \psi_2 \rangle \neq 0 \) and \( \langle \chi_2 \rangle = \langle \psi_1 \rangle = 0 \) with all VEVs being real. It leads to \( \kappa_1 = 0 \). However, it cannot produce phenomenological viable mass matrices anyway as discussed above.

In the following we show that the VEV configuration which is used in the zeroth order approximation in our numerical study represents one possible minimum of the Higgs potential \( V_4 \). As one can see, the equivalence of all four VEVs is not obligatory, since for example \( \mu_1 \) and \( \mu_2 \) and \( \lambda_1 \) and \( \tilde{\lambda}_1 \) are not restricted to have the same value, respectively. Therefore we search for a symmetry which can maintain these restrictions such that the equivalence of all four VEVs becomes more natural. The simplest choice is to first interchange the fields \( \chi_i \) with \( \psi_i \) in order to enforce for example the equivalence of \( \mu_1 \) and \( \mu_2 \) and to further exchange the fields \( \chi_1 \) and \( \chi_2 \) preventing the couplings \( \kappa_{1,2} \) from being set to zero \(^3\). This symmetry will be called \( T \) in the following. It restricts the parameters as follows:

\[
\begin{align*}
\mu_1 &= \mu_2, \quad \lambda_i = \tilde{\lambda}_i, \quad \sigma_2 = 0, \quad \tau_1 = \tau_2, \quad \kappa_1 = \kappa_2, \quad \kappa_3 = \kappa_4.
\end{align*}
\]

Note that setting \( \sigma_2 \) to zero does not lead to an accidental continuous symmetry. Especially, we do not enforce \( \kappa_{1,2} \) to vanish. Note also that changing the order of the actions \( \chi_i \leftrightarrow \psi_i \) and \( \chi_1 \leftrightarrow \chi_2 \) does not change the result.

Next we analyze the potential invariant under \( D_5 \times T \) for real VEVs \( \langle \chi_1 \rangle = \frac{v}{\sqrt{2}} \cos(\alpha), \langle \chi_2 \rangle = \frac{v}{\sqrt{2}} \sin(\alpha), \langle \psi_1 \rangle = \frac{v}{\sqrt{2}} \cos(\beta) \) and \( \langle \psi_2 \rangle = \frac{v}{\sqrt{2}} \sin(\beta) \). The form of the potential at the extremum is:

\[
V_{4T_{min}} = -\frac{1}{2} \mu_1^2 \left( u^2 + v^2 \right) + \frac{1}{32} \left( u^4 + v^4 \right) \left( 8 \lambda_1 + 4 \lambda_2 + \lambda_3 \right) + \frac{1}{4} u^2 v^2 \left( \sigma_1 + \kappa_3 \right) + \frac{1}{32} \left( u^4 \cos(4\alpha) + v^4 \cos(4\beta) \right) \left( 4 \lambda_2 - \lambda_3 \right) + \frac{1}{4} u v \left[ u^2 \cos(\alpha - \beta) \sin(2\beta) \right.
\]

\[
+ v^2 \sin(2\alpha) \sin(\alpha + \beta) \] \( \Re(\kappa_1) \] \( + \frac{1}{4} u^2 v^2 \sin(2\alpha) \sin(2\beta) \Re(\tau_1) \)

The minimization conditions which can be deduced from \( V_{4T_{min}} \) are:

\[
\frac{\partial V_{4T_{min}}}{\partial \alpha} = -\frac{1}{8} u^4 \sin(4\alpha) y + \frac{1}{2} u^2 v^2 \cos(2\alpha) \sin(2\beta) \Re(\tau_1) 
\]

\[
+ \frac{1}{4} u v \left[ v^2 \left( \cos(2\alpha) \sin(\alpha + \beta) + \sin(3\alpha + \beta) \right) - u^2 \sin(\alpha - \beta) \sin(2\beta) \right] \Re(\kappa_1) \]

\(^3\)The exchange of the fields \( \psi_1 \) and \( \psi_2 \) gives the same result.
\[
\frac{\partial V_{4,T_{\min}}}{\partial \beta} = -\frac{1}{8} u^4 \sin(4\beta) y + \frac{1}{2} u^2 v^2 \sin(2\alpha) \cos(2\beta) \Re(\tau_1) + \frac{1}{4} u v [u^2 (\cos(\alpha - \beta) \cos(2\beta) + \cos(\alpha - 3\beta)) + v^2 \sin(2\alpha) \cos(\alpha + \beta)] \Re(\kappa_1)
\]

where \( y = 4\lambda_2 - \lambda_3 \). Eq.(14a) and Eq.(14b) are fulfilled for \( \alpha = \frac{\pi}{4} \) and \( \beta = \frac{\pi}{4} \). Then each of the terms vanishes separately, especially there is no constraint on \( \Re(\tau_1) \), \( \Re(\kappa_1) \) or \( 4\lambda_2 - \lambda_3 \). This is important, since constraining these parameters to be zero could lead to accidental symmetries. For \( \alpha = \frac{\pi}{4} \) and \( \beta = \frac{\pi}{4} \) there is also one solution with \( u = v \). Therefore the equivalence of all (real) VEVs is a natural result of the potential.

Apart from this zeroth order solution it is important to check whether phenomenological viable VEV configurations can be a minimum of the potential for an appropriate choice of parameters. As we tried to restrict ourselves above to SCPV, it is especially necessary to find out whether this is possible for the chosen VEVs in our numerical examples. Unfortunately, it turns out to be impossible for the potential invariant under \( D_5 \times T \). For VEVs parameterized as \( \langle \chi_1 \rangle = \frac{v_1}{\sqrt{2}} e^{i\alpha}, \langle \chi_2 \rangle = \frac{v_2}{\sqrt{2}} e^{i\beta}, \langle \psi_1 \rangle = \frac{v_3}{\sqrt{2}} e^{i\gamma} \) and \( \langle \psi_2 \rangle = \frac{v_4}{\sqrt{2}} \) one can deduce, for example, the following equations from the minimization conditions for \( v_i \neq 0 \), \( \alpha \neq 0 \), \( \beta \neq 0 \), \( \gamma \neq 0 \):

\[
\begin{align*}
5 v_3 v_4 \Re(\kappa_1) (v_1 v_3 \sin(\alpha - 2\gamma) - v_2 v_4 \sin(\beta + \gamma)) &= 0 \quad (15a) \\
5 v_1 v_2 \Re(\kappa_1) (v_1 v_4 \sin(2\alpha - \beta) + v_2 v_3 \sin(\alpha - 2\beta + \gamma)) &= 0 \quad (15b)
\end{align*}
\]

These directly lead to the conclusion that \( \Re(\kappa_1) = 0 \). As we consider SCPV, also \( \Im(\kappa_1) = 0 \) and therefore the coupling \( \kappa_1 \) vanishes \(^4\). This increases the symmetry of the potential, as explained above. With \( v_i \neq 0 \), \( \alpha \neq 0 \), \( \beta \neq 0 \), \( \gamma \neq 0 \) it is then clear that this additional symmetry will be broken and hence further massless Goldstone bosons will appear which are phenomenologically unacceptable. In this case we have not gained anything by discussing the four Higgs potential compared to the three Higgs one. Abandoning the \( T \) symmetry and only requiring that the potential is invariant under \( D_5 \) does not change the situation, since then one can deduce the equations:

\[
\begin{align*}
5 v_3 v_4 \Re(\kappa_2) (v_1 v_3 \sin(\alpha - 2\gamma) - v_2 v_4 \sin(\beta + \gamma)) &= 0 \quad (16a) \\
5 v_1 v_2 \Re(\kappa_1) (v_1 v_4 \sin(2\alpha - \beta) + v_2 v_3 \sin(\alpha - 2\beta + \gamma)) &= 0 \quad (16b)
\end{align*}
\]

These enforce the vanishing of \( \Re(\kappa_1) \) and \( \Re(\kappa_2) \) for general VEV configurations. Again \( \Im(\kappa_{1,2}) \) are already set to zero, since we want to study the case of SCPV. In the end, the constraints \( \kappa_1 = 0 \) and \( \kappa_2 = 0 \) lead to an increase of the symmetry of the potential. Similar to the case above this further symmetry is broken by arbitrary VEV configurations resulting in extra Goldstone bosons. This proves that SCPV can only exist for special VEV configurations, but not in general.

For a general \( D_5 \) invariant four Higgs potential with complex parameters one can successfully solve all minimization conditions without the necessity to set parameters to zero. Furthermore one is able to maintain that all masses of the Higgs fields at this extremum are positive, i.e. this extremum can be a minimum of the potential. As all relevant equations are invariant under \( v_i \rightarrow -v_i \), the VEV configurations \( \langle \chi_1 \rangle = \frac{v_1}{\sqrt{2}} e^{i\alpha}, \langle \chi_2 \rangle = \frac{v_2}{\sqrt{2}} e^{i\beta}, \)

\(^4\)Actually in general even more parameters of the potential are constrained to be zero or have to fulfill certain relations.
⟨ψ₁⟩ = \frac{v_3}{\sqrt{2}} e^{i\gamma}, \langle ψ₂⟩ = \frac{v_4}{\sqrt{2}} \text{ and } \langle χ₁⟩ = -\frac{v_1}{\sqrt{2}} e^{i\alpha}, \langle χ₂⟩ = -\frac{v_2}{\sqrt{2}} e^{i\beta}, \langle ψ₁⟩ = -\frac{v_3}{\sqrt{2}} e^{i\gamma}, \langle ψ₂⟩ = -\frac{v_4}{\sqrt{2}} \text{ are degenerate. Finally, one can check numerically whether the potential is stable as a whole. This clearly is not a proof of the stability of the potential, but is enough for our considerations.}

All this has been done for the two VEV configurations used in the numerical examples. The parameters of the potential can be chosen in such a way that all constraints are fulfilled. The mass of the lightest Higgs field is usually smaller (\sim 40 \text{ GeV}) than the experimental bounds (\leq 114.4 \text{ GeV}) [32], if the mass parameters \(\mu_i\) are of the order of the electroweak scale (100 – 200 \text{ GeV}) and the quartic couplings are in the perturbative range. This problem can be cured by simply assuming that the \(\mu_i\)'s are larger than \(O(100 \text{ GeV})\) or adding some other mass dimension two terms which break \(D_5\). In order to pass not only the direct Higgs mass bounds, but also the stringent bounds on FCNCs, the Higgs masses should be even larger than a few \text{ TeV}. The mechanism of adding \(D_5\) breaking terms is unmotivated from the theoretical point of view, but seems to be necessary for a phenomenological viable model in this context.

One could ask whether it is also possible to achieve that arbitrary VEV configurations can be minima of the potential, if this is invariant under \(D_5 \times T\). The answer is no, since one can deduce three linear independent equations containing \(\text{Re}(\kappa_1), \text{Im}(\kappa_1)\) and \(\text{Re}(\tau_1)\) which are in general only solved, if \(\text{Re}(\kappa_1) = 0, \text{Im}(\kappa_1) = 0\) and \(\text{Re}(\tau_1) = 0\). Again, the minimization conditions enforce a parameter setup which leads to an additional global symmetry in the Higgs potential.

Finally, we compare the \(D_5\) invariant potential of four Higgs fields to the equivalent one in \(D_6\). Similar to \(D_5\) also \(D_6\) has two inequivalent two-dimensional representations (one faithful and one unfaithful one). However, in contrast to \(D_5\) the \(D_6\) invariant four Higgs potential contains a further \(U(1)\) symmetry. The reason for this is the \(D_6\) product structure \(2_i \times 2_i = 1_1 + 1_4 + 2_2\) for \(i = 1, 2\) and \(2_1 \times 2_2 = 1_2 + 1_3 + 2_1\) which does not allow for invariant couplings of the form \(2_i 2_j\) with \(i \neq j\). Precisely, these couplings, \(\kappa_1\) and \(\kappa_2\), exist in \(D_5\) and therefore prevent the potential from having an accidental \(U(1)\).

### 6 Extensions of the Model

Finally, we would like to comment on how the model has to be changed in order to be embedded into an \(SO(10)\) GUT and - maybe simultaneously - into a continuous flavor symmetry, like \(SO(3)_f\) or \(SU(3)_f\). This is desirable, since GUTs turned out to be very successful in unifying the SM gauge interactions and fermions of one generation and in explaining, for example, charge quantization. These features should not be given up when flavored models are considered. Second, the embedding of a discrete flavor symmetry into a continuous group \(G_f\) allows one to unify it with the GUT group being also continuous into one group containing gauge and flavor symmetries. Attempts to find such a group can be found in the literature [33]. Albeit these have not been very successful, the idea is still appealing. Furthermore gauged symmetries are the only ones which remain unbroken in the presence of quantum gravitational corrections [34] which suggests that any flavor symmetry should also be gauged. However, gauging a discrete symmetry can be performed in the easiest way, if it is embedded into a continuous one which is then gauged. Nevertheless, in the context of string theory discrete flavor symmetries could also arise without such an
embedding.
Since all fermions of one generation reside in the \(16\) of \(SO(10)\) they need to transform in the same way under \(D_5\), for example as \(1_1 + 2_1\). In our minimal model with just the four Higgs fields \(\chi_i\) and \(\psi_i\) the resulting mass matrices do hardly lead to phenomenological viable masses at tree-level and at low energies. Therefore we have to extend the Higgs sector by at least one Higgs field \(\phi\) transforming trivially under \(D_5\). The mass matrices are then of the form

\[
\begin{align*}
M_{u,\nu} &= \begin{pmatrix}
\alpha_0 \langle \phi \rangle & \alpha_1 \langle \chi_1 \rangle & \alpha_1 \langle \chi_2 \rangle \\
\alpha_2 \langle \chi_1 \rangle & \alpha_3 \langle \phi \rangle & \alpha_4 \langle \psi_2 \rangle \\
\alpha_2 \langle \chi_2 \rangle & \alpha_3 \langle \phi \rangle & \alpha_4 \langle \psi_2 \rangle
\end{pmatrix}, & M_{d,l} &= \begin{pmatrix}
\alpha_0 d \langle \phi \rangle & \alpha_1 d \langle \chi_2 \rangle & \alpha_1 d \langle \chi_1 \rangle \\
\alpha_2 d \langle \chi_2 \rangle & \alpha_4 d \langle \psi_2 \rangle & \alpha_3 d \langle \phi \rangle \\
\alpha_2 d \langle \chi_1 \rangle & \alpha_4 d \langle \phi \rangle & \alpha_4 d \langle \psi_1 \rangle
\end{pmatrix},
\end{align*}
\]

i.e. the Higgs field \(\phi\) fills the zeros in the \((1,1),(2,3)\) and \((3,2)\) elements. Note that the form of the right-handed Majorana mass terms does not change. In a complete \(SO(10)\) model the Higgs doublet fields have to be embedded into the representations \(10, 120\) and \(126\), since these do couple to \(16 \times 16\). Still this setup has to be embedded into the continuous flavor group \(G_f\). For \(G_f\) being \(SO(3)_f\) this is not possible, since we cannot identify \(1_1 + 2_1\) with the fundamental representation of \(SO(3)_f\). The same holds for \(SU(3)_f\). In order to do so, the first generation has to transform as \(1_2\) rather than \(1_1\). This leads to a sign in the \((1,3)\) and \((3,1)\) elements of the mass matrices in Eq.\((17)\), but does not alter the discussion. The five Higgs fields \(\chi_i, \psi_i\) and \(\phi \sim 1_1 + 2_1 + 2_2\) can be identified with the \(5\) of \(SO(3)_f\) and together with an additional field \(\phi' \sim 1_1\) also with the six-dimensional representation of \(SU(3)_f\).

A more minimal choice for an embedding into \(SO(10) \times G_f\) would be given by the three generations transforming as \(1_2 + 2_1\) and three Higgs fields doing the same. Unfortunately, this leads to traceless mass matrices for the fermions which seem to be highly disfavored by the observed mass hierarchies among the generations. This problem can be cured by adding another Higgs field transforming trivially under \(D_5\). Furthermore this increases the number of allowed Yukawa couplings by two. Since the added Higgs field transforms as \(1_1\), the model can still be embedded into the continuous flavor symmetries \(SO(3)_f\) and \(SU(3)_f\) with this field being identified with the singlet of \(SO(3)_f\) or \(SU(3)_f\). Although we showed that the Higgs sector is not phenomenologically viable in this case (see Section \(5\)), we cannot exclude it as a GUT model, because the Higgs couplings might change through the embedding of the \(SU(2)_L\) Higgs doublet fields into \(SO(10)\) representations.

### 7 Conclusions and Outlook

In this paper, we constructed a minimal model with the SM gauge group enlarged by the flavor symmetry \(D_5\). Both are broken only spontaneously at the electroweak scale. We chose \(D_5\), since it is the smallest discrete group with two inequivalent irreducible two-dimensional representations. We demanded the left- and left-handed conjugate fields of the three generations to unify partially, i.e. transform as \(1 + 2\) under \(D_5\), combined with the requirement that our model should be embeddable at least into the Pati-Salam gauge group. Furthermore we have chosen the minimal possible number of Higgs doublets with a potential free of accidental symmetries and did not include scalar fields transforming
as $SU(2)_L$ triplets or gauge singlets. We showed that under these constraints a minimal model can be built in which the left-handed fields transform as $\mathbf{1}_1 + \mathbf{2}_2$ under $D_5$, the left-handed conjugate ones as $\mathbf{1}_1 + \mathbf{2}_1$ and the four Higgses $\chi_i$ and $\psi_i$ $(i = 1, 2)$ as $\mathbf{2}_1 + \mathbf{2}_2$. By a numerical study we showed that all fermion masses and mixing parameters can be accommodated at tree-level. We considered the case of Majorana as well as Dirac neutrinos and we discussed the results of the unmeasured leptonic quantities. By our choice the spectrum of the light neutrinos is always normally ordered. The structure of the right-handed neutrino mass matrix is (almost) trivial, since we did not include SM gauge singlets. As a consequence two of the right-handed neutrinos are degenerate at tree-level. We compared the structure of the $D_5$ invariant mass matrices with those of $D_3$ invariant ones which are often discussed in the literature. The main difference is the tendency to get more texture zeros for a similar assignment of fermions and Higgs fields arising from the existence of the two inequivalent two-dimensional representations in $D_5$. We then turned to a discussion of the Higgs sector and found that all potentials with three Higgs fields transforming as $\mathbf{1} + \mathbf{2}$ are not only $D_5$ invariant, but also incorporate an accidental $U(1)$ symmetry which is broken by any VEV configuration leading to phenomenological viable mass matrices for the fermions at tree-level. To find the group theoretical reason for this accidental $U(1)$ we considered similar potentials invariant under $D_3$ and $D_4$, respectively, and found that they do not have an accidental $U(1)$ symmetry. The difference lies in the $D_5$ product structure $\mathbf{1}_{1,2} \times \mathbf{2} = \mathbf{2}$ and $\mathbf{2} \times \mathbf{2} = \mathbf{1}_1 + \mathbf{1}_2 + \mathbf{2}'$ such that the coupling $\mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{1}_{1,2}$ is not invariant under $D_5$. Therefore we had to extend the Higgs sector to four fields $\chi_i$ and $\psi_i$ transforming as the doublets of $D_5$. We explicitly showed that this potential is free of accidental symmetries and analyzed its VEV configurations. For a zeroth order solution we imposed a further discrete symmetry - called $T$ - on the potential in order to maintain the configuration that all (real) VEVs are equal as natural outcome of the minimization conditions. In a second step we proved that SCPV is not possible for the VEV configurations used in our numerical examples of the fermion mass matrices. Nevertheless these configurations can be minima of the $D_5$ invariant four Higgs potential, if its parameters are complex. Furthermore we calculated the masses for the Higgs fields. We found that they are naturally of the order $\mathcal{O}(100 \text{ GeV})$ up to $\mathcal{O}(1 \text{ TeV})$ with the smallest mass below the LEP bound of $114.4 \text{ GeV}$ [32], if the mass parameters of the potential are of the order of the electroweak scale and the quartic couplings are in the perturbative regime. Therefore FCNCs might be a problem which can probably be cured by adding large mass dimension two terms which break $D_5$. In our numerical examples the FCNCs involving the first generation get additionally suppressed, since the relevant Yukawa couplings are at most $10^{-4}$. Finally, we considered extensions of our low energy model and showed the necessary changes in the particle assignment and content to achieve the embedding into $SO(10) \times G_f$ where $G_f$ can be either $SO(3)_f$ or $SU(3)_f$. In our numerical examples the VEVs of the fields $\chi_i$ and $\psi_i$ break $D_5$ completely. However, $D_5$ has two non-trivial abelian subgroups $Z_2$ and $Z_5$ which can be generated by the generator $B$ and the generator $A$ alone, respectively. As one can see, $Z_5$ is always broken by a non-vanishing VEV of $\chi_i$ and $\psi_i$. In contrast to this a residual $Z_2$ is preserved in the Lagrangian, if $\langle \chi_1 \rangle = \langle \chi_2 \rangle$ and $\langle \psi_1 \rangle = \langle \psi_2 \rangle$. Interestingly, the resulting mass matrices $M_{u,\nu}$ and $M_{d,\ell}$ are then invariant under the interchange of the second and third generation and therefore produce a maximal mixing in the 2-3 sector, a vanishing mixing in the 1-3 sector and leave the mixing angle $\theta_{12}$ undetermined. For an exact $Z_2$ thus the mixing
matrices $V_{CKM}$ and $U_{MNS}$ have two vanishing mixing angles $\theta_{13}$ and $\theta_{23}$, since the maximal mixing angles in the 2-3 sectors of the up-type quarks (neutrinos) and the down-type quarks (charged leptons) cancel each other. This leads to the conclusion that this residual $Z_2$ is only weakly broken in the quark sector, but strongly broken in the lepton/neutrino sector. Actually the equalities $\langle \chi_1 \rangle = \langle \chi_2 \rangle$ and $\langle \psi_1 \rangle = \langle \psi_2 \rangle$ have also been employed when we searched for an appropriate zeroth order structure of the fermion mass matrices in our phenomenological analysis (see Section III). By considering the non-trivial subgroups of $D_5$ this choice gains further significance.

This discussion can be compared with the studies of the non-trivial subgroups of $A_4$ [35]. $A_4$ can be broken to either $Z_2$ or $Z_3$ by different VEV configurations of Higgs fields forming a triplet under $A_4$. It turns out that preserving the subgroup $Z_3$ for charged fermions leads to $V_{CKM} = 1$ whereas $Z_2$ is preserved in the neutrino sector leading to tri-bi-maximal mixing. Concerning the quark sector our flavor symmetry $D_5$ has the advantage that its breaking to $Z_2$ leads to vanishing 1-3 and 2-3 mixing, but does not constrain the Cabibbo angle. In this way we can explain why the Cabibbo angle is about one order of magnitude larger than the two other mixing angles whereas models using $A_4$ might have problems to generate a 1-2 mixing angle being large enough to accommodate the data. On the other hand in our minimal model shown here the residual subgroups of $D_5$ can hardly give reason for the tri-bi-maximal or bi-maximal mixing pattern observed in the leptonic sector which can nicely be explained by the residual $Z_2$ symmetry of $A_4$ in the neutrino sector.

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A Details of Group Theory

Here, we show the Clebsch Gordan coefficients for all Kronecker products in case that none of the representations is complex conjugated. This choice corresponds to the Yukawa couplings for the down-type quarks and charged leptons (see Section III). All other Clebsch Gordan coefficients needed for example for the quartic couplings in the Higgs sector and the Yukawas for the up-type quarks which involve at least one complex conjugated representation can be generated from the given Clebsch Gordan coefficients taking into account the similarity transformation between the representation matrices and its complex conjugates as shown in Section 2.4.

For $A \sim \mathbf{1}_i$ and $B \sim \mathbf{1}_j$ the product is $AB \sim \mathbf{1}_{(i+j) \mod 2 + 1}$. Combining the two-dimensional representation $\left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) \sim 2_i$ with the trivial singlet $A \sim \mathbf{1}_1$ leads to $\left( \begin{array}{c} Aa_1 \\ Aa_2 \end{array} \right) \sim 2_i$. Similarly for the non-trivial singlet $B \sim \mathbf{1}_2$ one finds $\left( \begin{array}{c} B a_1 \\ -B a_2 \end{array} \right) \sim 2_i$.

The $D_5$ covariant combinations of $2_1 \times 2_1$ for $\left( \begin{array}{c} a_1 \\ a_2 \end{array} \right)$, $\left( \begin{array}{c} a_1' \\ a_2' \end{array} \right) \sim 2_1$ are $a_1 a_2' + a_2 a_1' \sim 1_1$, $a_1 a_2' - a_2 a_1' \sim 1_2$ and $\left( \begin{array}{c} a_1 a_2' \\ a_2 a_1' \end{array} \right) \sim 2_2$ and for the product $2_2 \times 2_2$ they read $b_1 b_2' + b_2 b_1' \sim 1_1$, $b_1 b_2' - b_2 b_1' \sim 1_2$ and $\left( \begin{array}{c} b_1 b_2' \\ b_1 b_1' \\ b_2 b_1' \end{array} \right) \sim 2_1$ with $\left( \begin{array}{c} b_1 \\ b_2 \\ b_2' \end{array} \right) \sim 2_2$. For the mixed product
we find \( \begin{pmatrix} a_2 b_1 \\ a_1 b_2 \end{pmatrix} \sim 2_1 \) and \( \begin{pmatrix} a_2 b_1 \\ a_1 b_2 \end{pmatrix} \sim 2_2 \) with \( a \) being the upper and lower components of \( 2_1 \) and \( b \) of \( 2_2 \), respectively.

The Clebsch Gordan coefficients for the product \( \nu \times \mu \) can be constructed from the ones given for \( \mu \times \nu \) by simply taking the transpose of these. Therefore the shown Clebsch Gordan coefficients are sufficient for the calculation of all Yukawa and Higgs couplings.

Finally, we display the resolution of the smallest representations of \( SO(3) \) \( (SU(3)) \) into irreducible ones of \( D_5 \).

| \( SO(3) \) | \( \rightarrow \) \( D_5 \) | \( SU(3) \) | \( \rightarrow \) \( D_5 \) |
|---|---|---|---|
| 1 | \( 1 \) | 1 | \( 1 \) |
| 3 | \( 1 \) \( + \) \( 2 \) \( 1 \) | 3 | \( 1 \) \( + \) \( 2 \) \( 1 \) |
| 5 | \( 1 \) \( + \) \( 2 \) \( 1 \) \( + \) \( 2 \) \( 2 \) | 6 | \( 2 \) \( 1 \) \( + \) \( 2 \) \( 1 \) \( + \) \( 2 \) \( 2 \) |
| 7 | \( 1 \) \( + \) \( 2 \) \( 1 \) \( + \) \( 2 \) \( 2 \) | 8 | \( 1 \) \( + \) \( 2 \) \( 1 \) \( + 2 \) \( 2 \) \( 1 \) \( + \) \( 2 \) \( 2 \) |
| 9 | \( 1 \) \( + \) \( 2 \) \( 2 \) \( + \) \( 2 \) \( 2 \) | 10 | \( 2 \) \( 1 \) \( + \) \( 2 \) \( 2 \) \| 1 \( + \) \( 2 \) \( 2 \) \| 2 \( 2 \) |

One can interchange \( 2_1 \) with \( 2_2 \) to get an alternative possible embedding. These breaking sequences can be calculated with the methods shown in [36].

### B Tables of Numerical Examples

| Yukawas | \( \alpha_0^i \) | \( \alpha_1^i \) | \( \alpha_2^i \) | \( \alpha_3^i \) |
|---|---|---|---|---|
| \( i = u \) | -0.993443 | 0.994834 | 0.00047857 | 0.000151179 |
| \( i = d \) | -0.0169925 | 0.0163996 | 0.0000874833 | 0.000127194 |
| \( i = l \) | -0.0107912 | 0.00954078 | -0.00060939 | 0.00056597 |
| \( i = \nu \) | -1.00616 | 0.979207 | 1.3039 | 1.45344 |

| VEVs | \( \langle \chi_1 \rangle \) | \( \langle \chi_2 \rangle \) | \( \langle \psi_1 \rangle \) | \( \langle \psi_2 \rangle \) |
|---|---|---|---|---|
| abs. \( [\text{GeV}] \) | 97.3856 | 71.386 | 101.872 | 68.0594 |
| phase \( [\text{rad}] \) | -0.0076515 | 0.0071711 | 0.014899 | - |

| Yukawas | \( \alpha_0^i \) | \( \alpha_1^i \) | \( \alpha_2^i \) | \( \alpha_3^i \) |
|---|---|---|---|---|
| \( i = u \) | -0.972907 | 0.98535 | 0.00044789 | 0.000161089 |
| \( i = d \) | -0.0166799 | 0.016192 | 0.0000831123 | 0.00013383 |
| \( i = l \) | -0.0106418 | 0.00937877 | -0.00060939 | 0.0000211836 |
| \( i = \nu \) | 0.83783 | -0.983826 | 1.22405 | 1.22214 |

| VEVs | \( \langle \chi_1 \rangle \) | \( \langle \chi_2 \rangle \) | \( \langle \psi_1 \rangle \) | \( \langle \psi_2 \rangle \) |
|---|---|---|---|---|
| abs. \( [\text{GeV}] \) | 60.385 | 106.489 | 56.6084 | 110.964 |
| phase \( [\text{rad}] \) | -0.0313569 | -0.0358665 | -0.0500026 | - |

| \( M_{RR} \) | \( M_1 = 1.878 \times 10^{14} \text{ GeV} \) | \( M_2 = 2.011 \times 10^{14} \text{ GeV} \) |

Table 3: Numerical solution for Dirac neutrinos. The Yukawa couplings of the neutrinos have to be multiplied by \( 10^{-12.35} \) and \( \sum_{i=1}^{2} (|\langle \chi_i \rangle|^2 + |\langle \psi_i \rangle|^2) = (172.02 \text{ GeV})^2 \).

Table 4: Numerical solution for Majorana neutrinos. The sum of the squares of the absolute values of the VEVs is \( \approx (174.65 \text{ GeV})^2 \).
C  Experimental Data

The masses for the quarks and charged leptons at $\mu = M_Z$ are [37, 38]:

$$m_u(M_Z) = (1.7 \pm 0.4) \text{ MeV}, \quad m_c(M_Z) = (0.62 \pm 0.03) \text{ GeV}, \quad m_t(M_Z) = (171 \pm 3) \text{ GeV};$$

$$m_d(M_Z) = (3.0 \pm 0.6) \text{ MeV}, \quad m_s(M_Z) = (54 \pm 11) \text{ MeV}, \quad m_b(M_Z) = (2.87 \pm 0.03) \text{ GeV},$$

$$m_e(M_Z) = (0.48684727 \pm 0.0000014) \text{ MeV}, \quad m_{\mu}(M_Z) = (102.75138 \pm 0.00033) \text{ MeV},$$

$$m_{\tau}(M_Z) = 1.74669^{+0.00040}_{-0.00027} \text{ GeV}.$$

The CKM mixing angles hardly depend on the scale $\mu$ at low energies. Therefore we take the values found in [39] which are measured in tree-level processes only:

$$\sin(\theta_{12}) \equiv s_{12} = 0.2243 \pm 0.0016, \quad \sin(\theta_{23}) \equiv s_{23} = 0.0413 \pm 0.0015, \quad \sin(\theta_{13}) \equiv s_{13} = 0.0037 \pm 0.0005, \quad \delta = 1.05 \pm 0.24$$

and $\mathcal{J}_{CP} = (2.88 \pm 0.33) \times 10^{-5}$. In the neutrino sector only the two mass squared differences measured in atmospheric and solar neutrino experiments are known [40]:

$$\Delta m^2_{21} = m_2^2 - m_1^2 = (7.9^{+0.6}_{-0.6}) \times 10^{-5} \text{ eV}^2, \quad |\Delta m^2_{31}| = |m_3^2 - m_1^2| = (2.2^{+0.7}_{-0.5}) \times 10^{-3} \text{ eV}^2.$$ 

The leptonic mixing angles are constrained: $s^2_{13} \leq 0.031, \quad s^2_{12} = 0.3^{+0.04}_{-0.05} \text{ and } s^2_{23} = 0.5^{+0.14}_{-0.12}$. All values observed in neutrino oscillations are given at $2\sigma$ level. Three further quantities connected to the neutrinos are measurable: the sum of the neutrino masses from cosmology, $m_\beta$ in beta decay experiments and $|m_{ee}|$ in neutrinoless double beta decay. The experimental bounds on these quantities are:

$$\sum_{i=1}^{3} m_i \leq (0.42 \ldots 1.8) \text{ eV} \quad [21], \quad m_\beta = \left( \sum_{i=1}^{3} |U^e_{MNS}^i|^2 m_i^2 \right)^{1/2} \leq 2.2 \text{ eV} \quad [24] \text{ and} \quad |m_{ee}| = |\sum_{i=1}^{3} (U^e_{MNS}^i)^2 m_i| \leq 0.9 \text{ eV} \quad [26].$$

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