Reinforcement learning for linear-convex models with jumps via stability analysis of feedback controls

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Abstract. We study finite-time horizon continuous-time linear-convex reinforcement learning problems in an episodic setting. In this problem, the unknown linear jump-diffusion process is controlled subject to nonsmooth convex costs. We show that the associated linear-convex control problems admit Lipschitz continuous optimal feedback controls and further prove the Lipschitz stability of the feedback controls, i.e., the performance gap between applying feedback controls for an incorrect model and for the true model depends Lipschitz-continuously on the magnitude of perturbations in the model coefficients; the proof relies on a stability analysis of the associated forward-backward stochastic differential equation. We then propose a least-squares algorithm which achieves a regret of the order $O(\sqrt{N \ln N})$ on linear-convex learning problems with jumps, where $N$ is the number of learning episodes; the analysis leverages the Lipschitz stability of feedback controls and concentration properties of sub-Weibull random variables. Numerical experiment confirms the convergence and the robustness of the proposed algorithm.

Key words. Continuous-time reinforcement learning, linear-convex, jump-diffusion, Lipschitz stability, least-squares estimation, sub-Weibull random variable

AMS subject classifications. 93E35, 62G35, 93E24, 68Q32

1 Introduction

Reinforcement learning (RL) seeks optimal strategies to control an unknown dynamical system by interacting with the random environment through exploration and exploitation [38]. This paper studies a reinforcement learning problem for controlled linear-convex models with unknown drift parameters. The controlled dynamics are with possible jumps, the objectives are extended real-valued nonsmooth convex functions, and the learning is in an episodic setting for a finite-time horizon.

Regret analysis of RL algorithm and stability of controls. RL algorithms are in general characterized by iterations of exploitation and exploration (see e.g. [1, 30, 5]). In the model-based approach, for instance, the agent interacts with the environment via policies based on the present estimation of the unknown model parameters, and then incorporates the responses of these interactions to improve their knowledge of the system. One of the main performance criteria for RL algorithm, called regret, is to measure its deviation from the optimality over the learning process.

One key component in regret analysis is the Lipschitz stability of feedback controls which quantifies the mismatch between the assumed and actual models, or the stability of controls with respect to model perturbations. It is to analyze the precise derivation of a pre-computed feedback control from the optimal one, and is also known as the robustness of control policies in the learning community [30, 5, 23]).

Despite the long history of stability of controls in the control literature, its main focus in classical control theory has been restricted to the continuity of value functions and optimal open-loop controls (see e.g. [2, 44, 4, 6, 23]). Studies of high-order stability of controls such as the Lipschitz stability, has only attracted attention very recently, largely due to its crucial importance in characterizing the precise regret.

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order of learning algorithms (see [30, 5, 34]). Analyzing Lipschitz stability of feedback control is technically more challenging. It requires analyzing the derivatives of the value function in a suitable function space, as optimal feedback controls are usually characterized via the derivatives of the value function.

Due to this technical difficulty, most existing works on regret analysis of RL algorithms concentrate on the linear-quadratic (LQ) control framework. In this special setting, the optimal feedback control is an affine function of state variables, whose coefficients satisfy an associated algebraic or ordinary Riccati equation. Consequently, the Lipschitz stability of feedback controls is simplified by analyzing the robustness of the Riccati equation (see e.g. [1, 30, 5]). Unfortunately, these techniques developed specifically for Riccati equations in LQ-RL problems are clearly not applicable for general RL problems (see e.g. [7, 11, 17, 27]). In particular, optimal policies are typically nonlinear in the state variable, especially with the inclusion of entropy regularization for the exploration strategy in the optimization objective (see e.g. [43, 19, 37, 34]).

Our work. This paper consists of three parts.

• It first establishes the Lipschitz stability for finite-time horizon linear-convex control problems, whose dynamics are linear jump-diffusion processes with controlled drifts and possibly degenerate additive noises, and objectives are extended real-valued lower semicontinuous convex functions. Such control problems include as special cases LQ problems with convex control constraints, sparse and switching control of linear systems, and entropy-regularized relaxed control problems (see Examples 2.1 and 2.2). It shows that these control problems admit Lipschitz continuous optimal feedback controls with linear growth in the spatial variables (Theorem 2.5). It further proves that the performance gap between applying feedback controls for an incorrect model and for the true model depends Lipschitz-continuously on the magnitude of perturbations in the model coefficients, even with lower semicontinuous cost functions (Theorem 2.7). The Lipschitz stability of feedback controls is extended to entropy-regularized control problems with controlled diffusion in Proposition 4.1.

• It then proposes a greedy least-squares (GLS) algorithm for a class of continuous-time linear-convex RL problems in an episodic setting. At each iteration, the GLS algorithm estimates the unknown drift parameters by a regularized least-squares estimator based on observed trajectories, and then designs a feedback control for the estimated model. It establishes that the regret of this GLS algorithm is sublinear, i.e., of the magnitude $O(\sqrt{N \ln N})$ with $N$ being the number of learning episodes, provided that the least-squares estimator satisfies a general concentration inequality (Theorem 3.2). It further characterizes the explicit concentration behaviour of the least-squares estimator (and hence the precise regret bound of the GLS algorithm), depending on tail behaviours of the random jumps in the state dynamics (Theorem 3.3). In the pure diffusion case, a sharper regret bound has been obtained (Theorem 3.4).

• It finally verifies the theoretical properties of the proposed GLS algorithm through numerical experiment on a three-dimensional LQ RL problem. It shows the convergence of the least-squares estimations to the true parameters as the number of episodes increases, as well as a sublinear regret as indicated in theoretical results. It also demonstrates the GLS algorithm is robust with respect to initializations.

Our approaches and related works. Optimal control of stochastic systems with parametric uncertainty has been studied in the classical adaptive control literature (see [16, 36, 22, 3]), where stationary policy is constructed to minimize the long term average cost and where the asymptotic stability and convergence of an adaptive control law is analyzed when the time horizon goes to infinity. However, research on rate of convergence is virtually non-existent. The problem studied here is different. The main objective is to construct optimal (and time-dependent) policies for finite-horizon problems, with the finite-sample regret analysis for the learning algorithm. Compared with the classical adaptive control literature, the regret analysis in this work, also known as the non-asymptotic performance analysis, requires novel techniques, consisting of a precise performance estimate of a greedy policy (namely the Lipschitz stability of feedback controls) and a finite-sample analysis of the parameter estimation scheme.

Analyzing the Lipschitz stability of feedback controls in a continuous-time setting requires quantifying the impact of parameter uncertainty on the derivatives of the value functions. [34] studies the so-called exit time problem and the Lipschitz stability of regularized relaxed controls of diffusion processes via a partial differential equation (PDE) approach, which assumes that the diffusion coefficients are non-degenerate.
and the state process takes values in a compact set. In contrast, we consider (see Section 2) unconstrained jump-diffusion process with unbounded drift and (uncontrolled) degenerate noise, and the cost functions are nonsmooth and unbounded. Consequently, the PDE approach requires to deal with a degenerate nonlocal PDE with non-Lipschitz nonlinearity, whose solution (i.e., the value function) is unbounded and may be nonsmooth due to the lack of regularization from the Laplacian operator. Here the Lipschitz stability of feedback controls is established by analyzing the stability of the associated coupled forward-backward stochastic differential equations (FBSDEs). This is possible by (a) first exploiting the linear-convex structure of the control problem, which enables constructing a Lipschitz continuous feedback control via solutions of coupled FBSDEs, and then (b) by extending the stochastic maximum principle in [40] to feedback controls with nonsmooth costs. To the best of our knowledge, this is the first time FBSDE has been used to study stability of feedback controls.

Analyzing the (finite-sample) accuracy of the least-squares estimator for jump-diffusion models involves integrations of the state and control processes with respect to Brownian motions and Poisson random measures. Now, the nonlinearity of feedback controls renders it impossible to analyze the tail behaviour of these stochastic integrals as [19] does for LQ problems with analytical solutions; Additionally, the presence of random jumps implies that the state process is no longer sub-Gaussian, and hence the stochastic integrals in the least-squares estimator no longer sub-exponential. To overcome these difficulties, a convex concentration inequality is employed for SDEs with jumps [29], along with Burkholder’s inequality and the Girsanov theorem to characterize precisely the sub-Weibull behaviour of the required stochastic integrals in terms of their Orlicz norms (Lemmas 3.6 and 3.7). Leveraging recent developments in the theory of sub-Weibull random variables, the precise parameter estimation error of the least-squares estimator is quantified in terms of the sample size.

It is worth pointing out that the stability analysis of feedback controls can be extended (see Section 4) to entropy-regularized control problems with controlled diffusion and without the linear-convex structure. Instead of the maximum principle for the linear-convex setting, regularity analysis of the associated fully-nonlinear parabolic PDEs may be needed for nondegenerate noise with regular (such as bounded and high-order differentiable) coefficients. (See the discussion after Proposition 4.1 for more details).

**Notation.** For each $T > 0$, filtered probability space $(\Omega, \mathcal{F}, \mathbb{P} = \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ satisfying the usual condition and Euclidean space $(E, |\cdot|)$, we introduce the following spaces:
- $L^p(0,T; E)$, $p \in [2, \infty]$, is the space of (Borel) measurable functions $\phi : [0,T] \rightarrow E$ satisfying $\|\phi\|_{L^p} = (\int_0^T |\phi(t)|^p \, dt)^{1/p} < \infty$ if $p \in [2, \infty)$ and $\|\phi\|_{L^\infty} = \text{ess sup}_{t \in [0,T]} |\phi(t)| < \infty$ if $p = \infty$;
- $L^2(\Omega; E)$ is the space of $E$-valued $\mathcal{F}$-measurable random variables $X$ satisfying $\|X\|_{L^2} = \mathbb{E}[|X|^2]^{1/2} < \infty$;
- $S^2(t, T; E)$, $t \in [0,T]$, is the space of $E$-valued $\mathcal{F}$-progressively measurable càdlàg processes $Y : \Omega \times [t,T] \rightarrow E$ satisfying $\|Y\|_{S^2} = \mathbb{E}[\sup_{t \in [0,T]} |Y_s|^2]^{1/2} < \infty$;
- $\mathcal{H}^2(t, T; E)$, $t \in [0,T]$, is the space of $E$-valued $\mathcal{F}$-progressively measurable processes $Z : \Omega \times [t,T] \rightarrow E$ satisfying $\|Z\|_{\mathcal{H}^2} = \mathbb{E}[\int_t^T |Z_s|^2 \, ds]^{1/2} < \infty$;
- $\mathcal{H}^2_0(t, T; E)$, $t \in [0,T]$, is the space of $E$-valued $\mathcal{F}$-progressively measurable processes $M : \Omega \times [t,T] \times \mathbb{R}^p_0 \rightarrow E$ satisfying $\|M\|_{\mathcal{H}^2_0} = \mathbb{E}[\int_t^T \int_{\mathbb{R}^p} |M_s(u)|^2 \nu(du) \, ds]^{1/2} < \infty$, where $\mathbb{R}^p_0 := \mathbb{R}^p \setminus \{0\}$ and $\nu$ is a $\sigma$-finite measure on $\mathbb{R}^p_0$.

For notational simplicity, we denote $S^2(E) = S^2(0, T; E)$, $\mathcal{H}^2(E) = \mathcal{H}^2(0, T; E)$ and $\mathcal{H}^2_0(E) = \mathcal{H}^2_0(0, T; E)$. We shall also denote by $\langle \cdot, \cdot \rangle$ the usual inner product in a given Euclidean space, by $|\cdot|$ the norm induced by $\langle \cdot, \cdot \rangle$, by $A^T$ the transpose of a matrix $A$, and by $C \in [0, \infty)$ a generic constant, which depends only on the constants appearing in the assumptions and may take a different value at each occurrence.

## 2 Lipschitz stability of linear-convex control problems

### 2.1 Problem formulation with nonsmooth costs

In this section, we introduce the linear-convex control problems with nonsmooth costs.

Let $T > 0$ be a given terminal time and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, in which two mutually independent processes, a $d$-dimensional Brownian motion $W$ and a Poisson random measure $N(dt, du)$ with compensator $\nu(du)dt$, are defined. We assume that $\nu$ is a $\sigma$-finite measure on $\mathbb{R}^p$ equipped with its Borel
where for each process of compensated process of $N$, we denote by $\hat{N}(dt, du) = N(dt, du) - \nu(du) dt$ the compensated process of $N$ and by $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ the filtration generated by $W$ and $N$ and augmented by the $\mathbb{P}$-null sets.

For any given initial state $x_0 \in \mathbb{R}^n$, we consider the following minimization problem

$$V(x_0) = \inf_{\alpha \in \mathcal{H}^2(\mathbb{R}^k)} J(\alpha; x_0), \quad \text{with} \quad J(\alpha; x_0) = \mathbb{E} \left[ \int_0^T f(t, X_t^{x_0, \alpha}, \alpha_t) dt + g(X_T^{x_0, \alpha}) \right], \quad (2.1)$$

where for each $\alpha \in \mathcal{H}^2(\mathbb{R}^k)$, the process $X^{x_0, \alpha}$ satisfies the following controlled dynamics:

$$dX_t = b(t, X_t, \alpha_t) dt + \sigma(t) dW_t + \int_{\mathbb{R}_+} \gamma(t, u) \hat{N}(dt, du), \quad t \in [0, T], \quad X_0 = x_0, \quad (2.2)$$

where $b, \sigma, \gamma, f$ and $g$ are given functions satisfying the following conditions:

**H.1.** $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n, \sigma : [0, T] \to \mathbb{R}^{n \times n}, \gamma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \cup \{ \infty \},$ $g : \mathbb{R}^n \to \mathbb{R}$ are measurable functions such that for some $L \geq 0$ and $\lambda > 0,$

1. there exist measurable functions $(b_0, b_1, b_2) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times k}$ such that $b(t, x, a) = b_0(t) + b_1(t) x + b_2(t)a$ for all $(t, x, a) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k$, with $\|b_0\|_{L^2} + \|b_1\|_{L^\infty} + \|b_2\|_{L^\infty} + \|\sigma\|_{L^2} + (\int_{\mathbb{R}_+} \|\gamma(u)\|_{L^2} du) / 2 \leq L$.

2. $g$ is convex and differentiable with an $L$-Lipschitz derivative such that $|\nabla g(0)| \leq L$.

3. there exist functions $f_0 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ and $R : \mathbb{R}^k \to \mathbb{R} \cup \{ \infty \}$ such that

$$f(t, x, a) = f_0(t, x, a) + R(a), \quad \forall (t, x, a) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k.$$ For all $(t, x) \in [0, T] \times \mathbb{R}^n$, $f_0(t, \cdot, \cdot)$ is convex in $\mathbb{R}^k$, $f_0(t, \cdot, \cdot)$ is differentiable in $\mathbb{R}^n \times \mathbb{R}^k$ with an $L$-Lipschitz derivative, and $|f_0(t, 0, 0)| + |\partial_{(x, a)} f_0(t, 0, 0)| \leq L$. Moreover, $R$ is proper, lower semicontinuous, and convex.

4. for all $t \in [0, T], (x, a), (x', a') \in \mathbb{R}^n \times \mathbb{R}^k$, and $\eta \in [0, 1]$,

$$\eta f(t, x, a) + (1 - \eta) f(t, x', a') \geq f(t, \eta x + (1 - \eta) x', \eta a + (1 - \eta) a') + \eta(1 - \eta) \frac{1}{2} |a - a'|^2. \quad (2.3)$$

**Remark 2.1.** Throughout this paper, let $\text{dom} \mathcal{R} = \{ a \in \mathbb{R}^k \mid R(a) < \infty \}$ be the effective domain of $\mathcal{R}$ (or equivalently the effective domain of $f$). Under (H.1), we can show that both the function $f$ and its conjugate function

$$[0, T] \times \mathbb{R}^n \times \mathbb{R}^k \ni (t, x, z) \mapsto f^*(t, x, z) := \sup\{ \langle a, z \rangle - f(t, x, a) \mid a \in \mathbb{R}^k \} \in \mathbb{R} \cup \{ \infty \} \quad (2.4)$$

are normal convex integrands in the sense of [35, Section 14] and hence measurable, which are crucial for the well-definedness of the control problem (2.1) and the characterization of optimal controls. Furthermore, the strong convexity condition (H.1(4)) enables us to establish the Lipschitz stability of feedback controls to (2.1), which is essential for the analysis of learning algorithms.

Our analysis and results can be extended to control problems with time-space dependent nonsmooth cost function $\mathcal{R} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \cup \{ \infty \}$ by assuming $\mathcal{R}$ is a normal convex integrand and satisfies suitable subdifferentiability conditions. For notational simplicity and clarity, we choose to refrain from further generalization.

Note that (H.1) allows the diffusion coefficient $\sigma$ to be degenerate, hence the stability results in Section 2.3 apply to deterministic control problems. Moreover, (H.1) requires neither the effective domain $\text{dom} \mathcal{R}$ to be closed nor the function $\mathcal{R}$ to be bounded or continuous on $\text{dom} \mathcal{R}$, which is important for problems in engineering and machine learning, as shown in the following examples.

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1. We say a function $\mathcal{R} : \mathbb{R}^k \to \mathbb{R} \cup \{ \infty \}$ is proper if it has a nonempty effective domain $\text{dom} \mathcal{R} := \{ a \in \mathbb{R}^k \mid \mathcal{R}(a) < \infty \}$. 

Example 2.1 (Sparse and switching controls). Let $A \subset \mathbb{R}^k$ be a nonempty closed convex set, $\delta_A$ be the indicator of $A$ satisfying $\delta_A(x) = 0$ for $x \in A$ and $\delta_A(x) = \infty$ for $x \in \mathbb{R}^k \setminus A$, and $\ell : \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous and convex function. Then $R := \ell + \delta_A$ satisfies (H.1(3)). In particular, by setting $\ell \equiv 0$, we can consider the linear-convex control problems with smooth running costs and control constraints (see e.g. [7] and [44, Theorem 5.2 on p. 137]), which include the most commonly used linear-quadratic models as special cases.

More importantly, it is well-known in optimal control literature (see e.g. [11, 17, 27] and references therein) that, one can employ a nonsmooth function $\ell$ involving $L^1$-norm of controls to enhance the sparsity and switching property of optimal controls, which are practically important for minimum fuel problems and optimal device placement problems. Here by sparsity we refer to the situation where the whole vector $\alpha_t$ is zero, while by switching control we refer to the phenomena where at most one coordinate of $\alpha_t$ is non-zero at each $t$.

Example 2.2 (Regularized relaxed controls). Consider a regularized control problem arising from reinforcement learning (see e.g. [43, 19, 37, 34]), whose cost function $f$ is of the following form:

$$f(t, x, a) = f_0(t, x) + \langle f_1(t, x), a \rangle + \rho D_f(a||\mu) \quad \forall (t, x, a) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k,$$

(2.5)

where $f_0 : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $f_1 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^k$ are given functions, $\rho > 0$ is a regularization parameter, and $D_f(\cdot||\mu) : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$ is an $f$-divergence defined as follows. Let $\Delta_k := \{a \in [0, 1]^k \mid \sum_{i=1}^k a_i = 1\}$, $\mu = (\mu_i)_{i=1}^k \in \Delta_k \cap (0, 1)^k$, and $f : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function which satisfies $f(0) = \lim_{x \to 0} f(x)$, $f(1) = 0$ and $f$ is $\kappa_\mu$-strongly convex on $[0, \min \mu_i]$ with a constant $\kappa_\mu > 0$. Then, the $f$-divergence $D_f(\cdot||\mu) : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$ satisfies $D_f(a||\mu) = \infty$ for $a \notin \Delta_k$ and

$$D_f(a||\mu) := \sum_{i=1}^k \mu_i f\left(\frac{a_i}{\mu_i}\right) \in \mathbb{R} \cup \{\infty\} \quad \forall a \in \Delta_k.$$

One can easily see from $f(1) = 0$ and the lower semicontinuity of $f$ that $D_f(\cdot||\mu)$ is a proper, lower semicontinuous function with effective domain $\text{dom} D_f(\cdot||\mu) \subset \Delta_k$. Moreover, by the $\kappa_\mu$-strong convexity of $f$, we have for all $a, \tilde{a} \in \Delta_k$, $\eta \in [0, 1]$ that

$$\eta D_f(a||\mu) + (1 - \eta) D_f(\tilde{a}||\mu) \geq \sum_{i=1}^k \mu_i \left(\eta f\left(\frac{a_i}{\mu_i}\right) + (1 - \eta) f\left(\frac{\tilde{a}_i}{\mu_i}\right)\right) \geq \sum_{i=1}^k \mu_i \left(f\left(\frac{\eta a_i + (1 - \eta) \tilde{a}_i}{\mu_i}\right) + \eta (1 - \eta) \kappa_\mu |a_i - \tilde{a}_i|^2\right)$$

$$\geq D_f(\eta a + (1 - \eta) \tilde{a}||\mu) + (1 - \eta) \frac{\kappa_\mu}{\max \mu_i} |a - \tilde{a}|^2,$$

which implies the $\frac{\kappa_\mu}{\max \mu_i}$-strong convexity of $D_f(\cdot||\mu)$ in $\mathbb{R}^k$. It is clear that for suitable choices of $f_0, f_1$, the function $f$ in (2.5) satisfies (H.1(3)).

It is important to notice that an $f$-divergence $D_f(\cdot||\mu)$ is in general non-differentiable and unbounded on its effective domain. For example, one may consider the reverse relative entropy (with $f(a) = -\log a$) and the squared Hellinger divergence (with $f(a) = 2(1 - \sqrt{a})$), which are not subdifferentiable at the boundary of $\Delta_k$. Moreover, the reverse relative entropy (with $f(a) = -\log a$) and the Neyman’s $\chi^2$ divergence (with $f(a) = \frac{a - 1}{a} - 1$) are unbounded near the boundary of $\Delta_k$.

2.2 Construction of optimal feedback controls

In this section, we apply the maximum principle to (2.1) and explicitly construct optimal feedback controls of (2.1) based on the associated coupled FBSDE.

The following proposition shows that under (H.1), the control problem (2.1) admits a unique optimal open-loop control.

Proposition 2.1. Suppose (H.1) holds and let $x_0 \in \mathbb{R}^n$. Then the cost functional $J(\alpha; x_0) : \mathcal{H}^2(\mathbb{R}^k) \to \mathbb{R} \cup \{\infty\}$ is proper, lower semicontinuous, and $\lambda$-strongly convex. Consequently, $J(\cdot; x_0)$ admits a unique minimizer $\alpha^{x_0}$ in $\mathcal{H}^2(\mathbb{R}^k)$. 

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Proof. The desired properties of $J$ follow directly from the corresponding properties of $f$, $g$ in (H.1) and the fact that (2.2) has affine coefficients. The well-posedness of minimizers then follows from the standard theory of strongly convex minimization problems on Hilbert spaces (see e.g. [9, Lemma 2.33 (ii)]).

We then proceed to study optimal feedback controls of (2.1). The classical control theory shows that under suitable coercivity and convexity conditions, the optimal open-loop control of (2.1) can be expressed in a feedback form, i.e., there exists a measurable function $\psi : [0, T] \times \mathbb{R}^n \to \mathbb{R}^k$ such that $\alpha^{x_0} = \psi(t, X_t^{x_0, \alpha^{x_0}})$ for $d\mathbb{P} \otimes dt$ a.e. (see [31] for the case with controlled jump-diffusions and smooth costs and [20] for the case with controlled diffusions and nonsmooth costs). However, since these non-constructive proofs are based on a measurable selection theorem, the resulting feedback policy $\psi$ may not be unique, and may be unstable with respect to perturbations of the state dynamics.

In the subsequent analysis, we give a constructive proof of the existence of Lipschitz continuous feedback controls by exploiting the linear-convex structure of the control problem (2.1)-(2.2). Such a feedback control can be explicitly represented as solutions of a suitable FBSDE, and hence is Lipschitz stable with respect to perturbations of underlying models (see Theorem 2.6).

We first present the precise definitions of feedback controls and the associated state processes.

**Definition 2.1.** Let $\mathcal{V}$ be the following space of feedback controls:

$$
\mathcal{V} := \left\{ \psi : [0, T] \times \mathbb{R}^n \to \mathbb{R}^k \left| \psi \text{ is measurable and there exists } C \geq 0 \text{ such that} \right. \right. \\
\left. \left. \text{for all } (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \| \psi(t, 0) \| \leq C \right. \right. \\
\left. \left. \text{and } |\psi(t, x) - \psi(t, y)| \leq C|x - y| \right. \right. \right\} 
$$

(2.6)

For any given $x_0 \in \mathbb{R}^n$ and $\psi \in \mathcal{V}$, we say $X^{x_0, \psi} \in \mathcal{S}^2(\mathbb{R}^n)$ is the state process associated with $\psi$ if it satisfies the following dynamics:

$$
dX_t = b(t, X_t, \psi(t, X_t)) \, dt + \sigma(t) \, dW_t + \int_{\mathbb{R}^k} \gamma(t, u) \tilde{N}(dt, du), \quad t \in [0, T], \quad X_0 = x_0.
$$

(2.7)

We say $\psi \in \mathcal{V}$ is an optimal feedback control of (2.1) if it holds for $d\mathbb{P} \otimes dt$ a.e. that $\alpha^{x_0} = \psi(t, X_t^{x_0, \psi})$, where $\alpha^{x_0} \in H^2(\mathbb{R}^n)$ is the optimal control of (2.1).

We then proceed to establish a maximum principle for feedback controls of the control problem (2.1) with non-smooth costs. Let $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and $\phi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ such that for all $(t, x, a, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$,

$$
H(t, x, a, y) := \langle b(t, x, a), y \rangle + f(t, x, a), \quad \phi(t, x, y) := \arg \min_{a \in \mathbb{R}^k} H(t, x, a, y) \in \text{dom } \mathcal{R}.
$$

(2.8)

The following lemma shows that the function $\phi$ is well-defined and measurable.

**Lemma 2.2.** Suppose (H.1) holds. Then the function $\phi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ defined in (2.8) is measurable and satisfies for all $t \in [0, T], \ x, y \in \mathbb{R}^n$ that

$$
\phi(t, x, y) = \partial_z f^*(t, x, -b_2(t)^T y),
$$

(2.9)

where the function $f^*$ is defined in (2.4).

Proof. Let $f^* : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$ be the function defined in (2.4). Recall that for each $(t, x) \in [0, T] \times \mathbb{R}^n$, $f^*(t, x, \cdot)$ is $\lambda$-strongly convex and lower semicontinuous. Hence by [35, Theorems 11.3 and 11.8], $f^*(t, x, \cdot)$ is finite and differentiable on $\mathbb{R}^k$, and $\partial_z f^*(t, x, z) = \arg \max_{a \in \text{dom } \mathcal{R}} \langle (a, z) - f(t, x, a) \rangle$ for all $z \in \mathbb{R}^k$. Moreover, by [21, Theorem E4.2.1], $\mathbb{R}^k \ni z \mapsto \partial_z f^*(t, x, z) \in \mathbb{R}^k$ is $1/\lambda$-Lipschitz continuous. Hence, from the definition of $\phi$ and (H.1(1)), for all $t \in [0, T], \ x, y \in \mathbb{R}^n$,

$$
\phi(t, x, y) = \arg \min_{a \in \mathbb{R}^k} \left( \langle b(t, x, a), y \rangle + f(t, x, a) \right) = \arg \max_{a \in \mathbb{R}^k} \left( \langle a, -b_2(t)^T y \rangle - f(t, x, a) \right) = \partial_z f^*(t, x, -b_2(t)^T y).
$$

(2.10)

Note that the measurability of $f^*$ (see Remark 2.1) implies that the derivative $\partial_z f^*$ is measurable, which along with the continuity of $z \mapsto \partial_z f^*(t, x, z)$ leads to the measurability of $\phi$. \qed
Then there exists a constant $\lambda$ in (2.8), which are essential for the well-posedness and stability of (2.11).

With the measurable function $\phi$ in hand, for each $(t, x) \in [0, T] \times \mathbb{R}^n$, let us consider the following coupled FBSDE on $[t, T]$: for all $s \in [t, T]$,

$$
\begin{align*}
\text{d}X_s &= b(s, X_s, \phi(s, X_s, Y_s)) \text{d}s + \sigma(s) \text{d}W_s + \int_{\mathbb{R}^d} \gamma(s, u) \tilde{N}(\text{d}s, \text{d}u), \quad X_t = x, \quad (2.11a) \\
\text{d}Y_s &= -\partial_x H(s, X_s, \phi(s, X_s, Y_s), Y_s) \text{d}s + Z_s \text{d}W_s + \int_{\mathbb{R}^d} M_s \tilde{N}(\text{d}s, \text{d}u), \quad Y_T = \nabla g(X_T). \quad (2.11b)
\end{align*}
$$

We say a tuple of processes $(X^{t,x}, Y^{t,x}, Z^{t,x}, M^{t,x}) \in \mathcal{S}(t, T) := \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{H}^2(t, T; \mathbb{R}^n)$ is a solution to (2.11) (on $[t, T]$ with initial condition $X^t_t = x$) if it satisfies (2.11) $\mathbb{P}$-almost surely.

The next lemma presents several important properties of the Hamiltonian $H$ and the function $\phi$ defined in (2.8), which are essential for the well-posedness and stability of (2.11).

**Lemma 2.3.** Suppose (H.1) holds. Let $\phi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ be the function defined in (2.8). Then there exists a constant $C$ such that for all $t \in [0, T]$ and $(x, y), (x', y') \in \mathbb{R}^n \times \mathbb{R}^n$, $|\phi(t, x, y) - \phi(t, x', y')| \leq C(|x - x'| + |y - y'|)$ and

$$
\begin{align*}
(b(t, x, \phi(t, x, y)) - b(t, x', \phi(t, x', y')) - y - y') \\
&+ (\langle -\partial_x H(t, x, \phi(t, x, y), y), y \rangle - \langle -\partial_x H(t, x', \phi(t, x', y'), y'), y - x' \rangle)
\end{align*}
$$

$$
\leq -\lambda|\phi(t, x, y) - \phi(t, x', y')|^2,
$$

with the constant $\lambda$ in (H.1).

**Proof.** We start by showing the boundedness of $\phi(\cdot, 0, 0)$ by considering $a(t) := (\partial_x f^*)(t, 0, 0)$ for each $t \in [0, T]$, where $f^* : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ is defined as in (2.4). The fact that $f(t, 0, \cdot)$ is proper, lower semicontinuous and convex implies that $0 \notin \partial_a f(t, 0, a(t))$ for all $t \in [0, T]$, where $\partial_a f(t, 0, a(t))$ is the subdifferential of $f(t, 0, \cdot)$ at $a(t)$. Note that $f_0(t, 0, \cdot)$ and $R$ are proper, lower semicontinuous, and convex, and $\text{dom } R \subset \text{dom } f_0(t, 0, \cdot) = \mathbb{R}^k$. Hence by [35, Corollary 10.9], $\partial_a f(t, 0, a) = \partial_a f_0(t, 0, a) + \partial R(a)$ for all $(t, a) \in [0, T] \times \mathbb{R}^k$, where $\partial R(a)$ is the subdifferential of $R$ at $a$. Now fix an arbitrary $t_0 \in [0, T]$ and set $a_0 = a(t_0)$. The fact that $0 \notin \partial_a f(t_0, 0, a_0)$ implies that $-\partial_a f_0(t_0, 0, a_0) \in \partial R(a)$ and hence $\partial_a f_0(t_0, 0, a_0) - \partial_a f_0(t_0, 0, a_0) \in \partial_a f(t_0, 0, a_0)$ for all $t \in [0, T]$. By the strong convexity condition (2.3), for all $t \in [0, T]$, $\xi_1 \in \partial_a f(t, 0, a_0)$ and $\xi_2 \in \partial_a f(t, 0, a(t))$,

$$
\lambda|a_0 - a(t)|^2 \leq (\xi_1 - \xi_2, a_0 - a(t)) \leq |\xi_1 - \xi_2||a_0 - a(t)|.
$$

Taking $\xi_1 = \partial_a f_0(t_0, 0, a_0) - \partial_a f_0(t_0, 0, a_0)$ and $\xi_2 = 0$ in the above inequality yields

$$
|a_0 - a(t)| \leq |\partial_a f_0(t_0, 0, a_0) - \partial_a f_0(t_0, 0, a_0)|/\lambda \leq C,
$$

by the linear growth of $\partial_a f_0(t_0, 0, \cdot)$. This implies that $|(\partial_a f^*)(t, 0, 0)| \leq C$ for all $t \in [0, T]$, which along with (2.10) leads to the desired uniform boundedness of $\phi(\cdot, 0, 0)$.

We proceed to establish the Lipschitz continuity of $\phi$ with respect to $(x, y)$. The $1/\lambda$-Lipschitz continuity of $\partial_x f^*(t, x, \cdot)$ and the boundedness of $b_2$ imply that $\phi$ is Lipschitz continuous in $y$, uniformly with respect to $(t, x)$. It remains to show the Lipschitz continuity of $\partial_x f^*$ with respect to $x$, which along with (2.10) leads to the desired Lipschitz continuity of $\phi$. For any given $(t, z) \in [0, T] \times \mathbb{R}^n$ and $x', z' \in \mathbb{R}^n$, let $a = \partial_x f^*(t, x, z)$ and $a' = \partial_x f^*(t, x', z)$. Then we have $z \in \partial_a f(t, x, a)$ and $z \in \partial_a f(t, x', a')$. Moreover, by the convexity of $f(t, x, \cdot)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and similar arguments as above, we can show that $z - \partial_a f_0(t, x', a') + \partial_a f_0(t, x, a') \in \partial_a f(t, x, a')$, which together with the convexity condition (2.3) and $z \in \partial_a f(t, x, a)$ leads to

$$
\lambda|a' - a| \leq |z - \partial_a f_0(t, x', a') + \partial_a f_0(t, x, a') - z| \leq L|x - x'|,
$$

where we have used the $L$-Lipschitz continuity of $\partial_a f_0(t, \cdot, \cdot)$ and $\phi(t, \cdot, \cdot)$. This finishes the proof of the Lipschitz continuity of $\partial_x f^*(t, \cdot, \cdot)$ and $\phi(t, \cdot, \cdot)$.
Finally, we establish the monotonicity condition \((2.12)\). By \((H.1(4))\), for all \(t \in [0, T]\), \(x, x' \in \mathbb{R}^n, a, a' \in \mathbb{R}^k, y \in \mathbb{R}^n\), the function \(\mathbb{R}^n \times \mathbb{R}^k \ni (x, a) \mapsto H(t, x, a, y) \in \mathbb{R}^n \cup \{\infty\}\) satisfies the same convexity condition \((2.3)\) as the function \(f\), and hence
\[
H(t, x', a', y) - H(t, x, a, y) \geq \langle \xi, x' - x, a' - a \rangle + \frac{1}{2} \|a' - a\|^2 \quad \forall \xi \in \widehat{\partial}_{(x,a)} H(t, x, a, y),
\] for all \(t \in [0, T]\), \(x, x' \in \mathbb{R}^n, a, a' \in \mathbb{R}^k, y \in \mathbb{R}^n\), the function \(H(t, x, a, y)\) denotes the subdifferential of the function \(H(t, \cdot, y)\) at \((x, a)\). Moreover, for any given \(t \in [0, T]\) and \(x, y \in \mathbb{R}^n\), the definition of \(\phi\) in \((2.8)\) implies that \(0 \in \widehat{\partial}_{x} H(t, x, \phi(t, x, y), y)\), where \(\widehat{\partial}_{x} H(t, x, \phi(t, x, y), y)\) denotes the subdifferential of the function \(H(t, \cdot, y)\) at \(\phi(t, x, y)\). Now recall that for any Euclidean space \(E\), convex function \(F : E \to \mathbb{R} \cup \{\infty\}\) and \(x \in \text{dom} F, v \in \partial F(x)\) if and only if \(\liminf_{\tau \to 0, \omega \to w} \frac{F(x + \tau w) - F(x)}{\tau} \geq \langle v, w \rangle\) for all \(w \in E\) (see e.g., Exercise 8.4 and Proposition 8.12 in \([35]\)). Thus, for any \(t \in [0, T]\) and \(x, y \in \mathbb{R}^n, 0 \in \widehat{\partial}_{x} H(t, x, \phi(t, x, y), y)\) yields for all \(z \in \mathbb{R}^k\),
\[
\liminf_{\tau \to 0, \tilde{z} \to z} \frac{H(t, x, \phi(t, x, y) + \tau \tilde{z}, y) - H(t, x, \phi(t, x, y), y)}{\tau} \geq \langle 0, z \rangle = 0.
\] Moreover, by the convexity of \(H\) and the continuity of \(\partial_x H\) in \((x, a)\), for any \(t \in [0, T]\) and \(x, y, w \in \mathbb{R}^n\) and \(z \in \mathbb{R}^k\),
\[
\liminf_{\tau \to 0, \tilde{w} \to w} \frac{H(t, x + \tau \tilde{w}, \phi(t, x, y) + \tau \tilde{z}, y) - H(t, x, \phi(t, x, y) + \tau \tilde{z}, y)}{\tau} \geq \langle \partial^2_{w} H(t, x, \phi(t, x, y), y), \tilde{w} \rangle,
\] provided that \(\phi(t, x, y) + \tilde{z} \in \text{dom} R\) (cf. \((2.8)\)). Then for any \(t \in [0, T]\) and \(x, y \in \mathbb{R}^n\), adding up \((2.14)\) and \((2.15)\) and using the fact that \(\phi(t, x, y) \in \text{dom} R\) give for all \((w, z) \in \mathbb{R}^n \times \mathbb{R}^k\),
\[
\liminf_{\tau \to 0, (\tilde{w}, \tilde{z}) \to (w, z)} \frac{H(t, x + \tau \tilde{w}, \phi(t, x, y) + \tau \tilde{z}, y) - H(t, x, \phi(t, x, y), y)}{\tau} \geq \langle \partial^2_{w} H(t, x, \phi(t, x, y), y), w \rangle + \langle 0, z \rangle,
\] which implies
\[
\langle \partial^2_{x} H(t, x, \phi(t, x, y), y), y \rangle \subset \widehat{\partial}_{(x,a)} H(t, x, \phi(t, x, y), y).
\] Hence for all \(t \in [0, T]\), \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^n \times \mathbb{R}^n\), we can define \(a_1 = \phi(t, x_1, y_1), a_2 = \phi(t, x_2, y_2)\) and deduce that
\[
\begin{align*}
&\langle b(t, x_1, a_1) - b(t, x_2, a_2), y_1 - y_2 \rangle + \langle -\partial^2_{x} H(t, x_1, a_1, y_1) + \partial^2_{x} H(t, x_2, a_2, y_2), x_1 - x_2 \rangle
&= H(t, x_1, a_1, y_1) - H(t, x_2, a_2, y_1) - \langle \partial^2_{x} H(t, x_1, a_1, y_1), x_1 - x_2 \rangle
&- H(t, x_2, a_2, y_2 - \langle \partial^2_{x} H(t, x_2, a_2, y_2), x_1 - x_2 \rangle
&\leq -\lambda \|a_1 - a_2\|^2,
\end{align*}
\] which finishes the proof of the desired monotonicity condition. \(\square\)

The following proposition shows that \((2.11)\) admits a unique solution, which is Lipschitz continuous with respect to the initial state. The proof is based on the stability of \((2.11)\) under the generalized monotonicity condition \((2.12)\) (see Lemma A.1), and follows \([33, \text{Corollary 2.4}\) for the case without jumps.

**Proposition 2.4.** Suppose \((H.1)\) holds. Then for any given \((t, x) \in [0, T] \times \mathbb{R}^n\), the FBSDE \((2.11)\) admits a unique solution \((X^{t,x}, Y^{t,x}, Z^{t,x}, M^{t,x}) \in \mathcal{S}(t,T)\). Moreover, there exists a constant \(C\) such that for all \(t \in [0, T]\) and \(x, x' \in \mathbb{R}^n\), \(\|X^{t,x} - X^{t,x'}\|_{\mathcal{S}(t,T)} \leq C(1 + |x|)\) and \(\|X^{t,x} - X^{t,x'}, Y^{t,x} - Y^{t,x'}, Z^{t,x - Z^{t,x'}}, M^{t,x} - M^{t,x'}\|_{\mathcal{S}(t,T)} \leq C|x - x'|\).

Now we are ready to present the main result of this section, which constructs an optimal feedback control of \((2.1)\) based on the Hamiltonian \((2.8)\) and the solutions to the FBSDE \((2.11)\).
Theorem 2.5. Suppose (H.1) holds. Let \( \psi : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^k \) be the function defined as
\[
\psi(t, x) := \phi(t, x, Y_t^{x_0}), \quad (t, x) \in [0, T] \times \mathbb{R}^n,
\]
where the function \( \phi \) is defined in (2.8). Then there exists a constant \( C \) such that \( |\psi(t, 0)| \leq C \) and \( |\psi(t, x) - \psi(t, x')| \leq C|x - x'| \) for all \( t \in [0, T], x, x' \in \mathbb{R}^n \). Moreover, for all \( x_0 \in \mathbb{R}^n \), \( \psi \) is an optimal feedback control of (2.1).

Proof. We first analyze the mapping \([0, T] \times \mathbb{R}^n \ni (t, x) \mapsto v(t, x) := Y_t^{x_0} \in \mathbb{R}^n\). Note by Proposition 2.4, for any given \((t, x) \in [0, T] \times \mathbb{R}^n\), the solution to (2.11) (with initial time \( t \) and initial state \( x \)) is pathwise unique and Lipschitz continuous with respect to the initial state \( x \in \mathbb{R}^n \). Hence, it is well-known that (see e.g., Theorem 3.1 and Remarks 3.2-3.3 in [28]) that the map \( v \) can be identified with a deterministic function in the space \( \mathcal{V} \) and it holds for all \((t, x) \in [0, T] \times \mathbb{R}^n\) that \( \mathbb{P}(\forall s \in [t, T], Y_s^{x_0} = v(s, X_s^{x_0}) = 1\). Thus, from the regularity of \( \phi \) and \( v \), \(|\psi(t, 0)| \leq C\) and \( |\psi(t, x) - \psi(t, x')| \leq C|x - x'|\) for all \( x, x' \in \mathbb{R}^n \), i.e., \( \psi \) is in the space \( \mathcal{V} \).

Now let \( x_0 \in \mathbb{R}^n \) be a given initial state and \( \tilde{\alpha} \in \mathcal{A} \) satisfy for \( \mathbb{P} \otimes dt \) a.e. that \( \tilde{\alpha}_t = \phi(t, X_t^{0, x_0}, Y_t^{0, x_0}) \). Then for \( \mathbb{P} \otimes dt \) a.e., \( \tilde{\alpha}_t = \phi(t, X_t^{0, x_0}, v(t, 0)) = \psi(t, X_t^{0, x_0}) \), and \( X_0^{0, x_0} \) is the solution to (2.2) controlled by \( \tilde{\alpha} \), because \((X_t^{0, x_0}, Y_t^{0, x_0})\) satisfy (2.11a). Since the control problem (2.1) admits an unique optimal control in \( \mathcal{H}^2(\mathbb{R}^k) \), it suffices to show that \( \tilde{\alpha} \) is optimal. By (2.16), for \( \mathbb{P} \otimes dt \) a.e.,
\[
(\partial_x H(t, X_t^{0, x_0}, \phi(t, X_t^{0, x_0}, Y_t^{0, x_0}), Y_t^{0, x_0}), 0) \subset \partial_{\tilde{\alpha}(x, 0)} H(t, X_t^{0, x_0}, \phi(t, X_t^{0, x_0}, Y_t^{0, x_0}), Y_t^{0, x_0}).
\]
Then for any given \( \alpha \in \mathcal{H}^2(\mathbb{R}^n) \) with the state process \( X_t^{0, x_0} \) satisfying the controlled dynamics (2.2), by the definition of \( H \) in (2.8), (H.1(2)) and (2.13),
\[
J(\alpha; x_0) - J(\tilde{\alpha}; x_0) = \mathbb{E} \left[ g(X_T^{x_0, \alpha}) - g(X_T^{0, x_0}) + \int_0^T \left( H(t, X_t^{0, x_0}, \alpha_t, Y_t^{0, x_0}) - H(t, X_t^{0, x_0}, \tilde{\alpha}_t, Y_t^{0, x_0}) \right) dt \right]
\]
\[
- \int_0^T \left( b(t, X_t^{\alpha, x_0}, \alpha_t) - b(t, X_t^{0, x_0}, \tilde{\alpha}_t), Y_t^{0, x_0} \right) dt \right]
\]
\[
\geq \mathbb{E} \left[ \nabla g(X_T^{0, x_0}), X_T^{\alpha, x_0} - X_T^{0, x_0} \right] + \int_0^T \left( \partial_x H(t, X_t^{0, x_0}, \phi(t, X_t^{0, x_0}, Y_t^{0, x_0}), Y_t^{0, x_0}), X_t^{0, x_0} - X_t^{0, x_0} \right) dt
\]
\[
- \int_0^T \left( b(t, X_t^{\alpha, x_0}, \alpha_t) - b(t, X_t^{0, x_0}, \tilde{\alpha}_t), Y_t^{0, x_0} \right) dt \right) = 0,
\]
where the last equality is by applying Itô’s formula to the process \((X_t^{0, x_0} - X_t^{0, x_0}, Y_t^{0, x_0})\) \(t \geq 0\) and by the FBSDE (2.11). That is, \( \psi \in \mathcal{V} \) is an optimal feedback control of (2.1).

\[\square\]

2.3 Lipschitz stability of optimal feedback controls and associated costs

In this section, we establish the Lipschitz stability of the optimal feedback controls constructed in Theorem 2.5 and their associated costs: that is, they are are Lipschitz continuous with respect to the perturbation in the coefficients of (2.2). Such a Lipschitz stability property is crucial for the subsequent analysis of learning algorithms.

More precisely, for any given \( x_0 \in \mathbb{R}^n \), we consider a perturbed control problem where the cost functions \( f, g \) are the same as those in (2.1), and for each \( \alpha \in \mathcal{H}^2(\mathbb{R}^n) \), the corresponding state dynamics satisfies the following perturbed dynamics:
\[
dX_t = \tilde{b}(t, X_t, \alpha_t) dt + \tilde{\sigma}(t) dW_t + \int_{\mathbb{R}^d_0} \tilde{\gamma}(t, u) \tilde{N}(dt, du), \quad t \in [0, T], \quad X_0 = x_0,
\]
whose coefficients satisfy the following assumption:

**H.2.** \( \tilde{b} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n \), \( \tilde{\sigma} : [0, T] \rightarrow \mathbb{R}^{n \times d} \) and \( \tilde{\gamma} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfy (H.1(1)) with the same constant \( L \), i.e., there exist measurable functions \((\tilde{b}_0, \tilde{b}_1, \tilde{b}_2) : [0, T] \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) such that \( \tilde{b}(t, x, a) = \tilde{b}_0(t) + \tilde{b}_1(t)x + \tilde{b}_2(t)a \) for all \((t, x, a) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \) and \( \|\tilde{b}_0\|_{L^2} + \|\tilde{b}_1\|_{L^\infty} + \|\tilde{b}_2\|_{L^2} + \left( \int_0^T \int_{\mathbb{R}^d_0} |\tilde{\gamma}(t, u)|^2 \nu(du) dt \right)^{1/2} \leq L \).
Under (H.1) and (H.2), Theorem 2.5 ensures that an optimal feedback control of the perturbed control problem can be obtained by

\[ [0, T] \times \mathbb{R}^n \ni (t, x) \mapsto \tilde{\psi}(t, x) := \tilde{\phi}(t, x, \tilde{Y}^{t,x}) \in \mathbb{R}^k, \]

(2.19)

where \( \tilde{\phi} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k \) satisfies for all \( (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \) that

\[ \tilde{\phi}(t, x, y) := \arg\min_{\hat{a} \in \mathbb{R}^k} \hat{H}(t, x, a, y), \quad \hat{H}(t, x, a, y) := (\hat{b}(t, x, a), y) + f(t, x), \]

(2.20)

and for each \( (t, x) \in [0, T] \times \mathbb{R}^n \), \((\tilde{X}^{t,x}, \tilde{Y}^{t,x}, \tilde{Z}^{t,x}, \tilde{M}^{t,x}) \in \mathcal{S}(t, T)\) is the solution to the following perturbed FBSDE: for all \( s \in [t, T] \),

\[
\begin{align*}
\text{d}X_s &= \hat{b}(s, X_s, \tilde{\phi}(s, X_s, Y_s)) \, ds + \hat{\sigma}(s) \, dW_s + \int_{\mathbb{R}_0^+} \tilde{\gamma}(s, u) \, \tilde{N}(ds, du), \quad X_t = x, \\
\text{d}Y_s &= -\partial_x \hat{H}(s, X_s, \tilde{\phi}(s, X_s, Y_s), Y_s) \, ds + Z_s \, dW_s + \int_{\mathbb{R}_0^+} M_s \, \tilde{N}(ds, du), \quad Y_T = \nabla g(X_T).
\end{align*}
\]

(2.21)

The following theorem quantifies the difference of optimal feedback controls in terms of the magnitude of perturbations in the coefficients.

**Theorem 2.6.** Suppose (H.1) and (H.2) hold. Let \( \psi, \tilde{\psi} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k \) be the functions defined in (2.17) and (2.19), respectively. Then there exists a constant \( C \) such that \( |\psi(t, x) - \tilde{\psi}(t, x)| \leq C(1 + |x|) \mathcal{E}_{\text{per}} \) for all \( (t, x) \in [0, T] \times \mathbb{R}^n \), with the constant \( \mathcal{E}_{\text{per}} \) defined by

\[
\mathcal{E}_{\text{per}} := \|b_0 - \tilde{b}_0\|_{L^2} + \|b_1 - \tilde{b}_1\|_{L^\infty} + \|b_2 - \tilde{b}_2\|_{L^\infty} + \|\sigma - \tilde{\sigma}\|_{L^2} + \left( \int_0^T \int_{\mathbb{R}_0^+} |\gamma(t, u) - \tilde{\gamma}(t, u)|^2 \nu(du)dt \right)^{1/2}.
\]

(2.22)

**Proof.** Throughout this proof, for each \( (t, x) \in [0, T] \times \mathbb{R}^n \), let \((X^{t,x}, Y^{t,x}, Z^{t,x}, M^{t,x}) \in \mathcal{S}(t, T)\) and \((\tilde{X}^{t,x}, \tilde{Y}^{t,x}, \tilde{Z}^{t,x}, \tilde{M}^{t,x}) \in \mathcal{S}(t, T)\) be the solutions to (2.11) and (2.21), respectively, and let \( C \) be a generic constant which is independent of \( (t, x) \in [0, T] \times \mathbb{R}^n \). Then by Proposition 2.4, there exists \( C \geq 0 \) such that for all \( (t, x) \in [0, T] \times \mathbb{R}^n \), \( \|(X^{t,x}, Y^{t,x}, Z^{t,x}, M^{t,x})\|_{\mathcal{S}(t, T)} \leq C(1 + |x|) \) and \( \|(\tilde{X}^{t,x}, \tilde{Y}^{t,x}, \tilde{Z}^{t,x}, \tilde{M}^{t,x})\|_{\mathcal{S}(t, T)} \leq C(1 + |x|) \).

We first estimate the difference between \((X^{t,x}, Y^{t,x}, Z^{t,x}, M^{t,x})\) and \((\tilde{X}^{t,x}, \tilde{Y}^{t,x}, \tilde{Z}^{t,x}, \tilde{M}^{t,x})\) for a given \((t, x) \in [0, T] \times \mathbb{R}^n \). By Lemmas 2.3 and A.1,

\[
\begin{align*}
\|(X^{t,x} - \tilde{X}^{t,x}, Y^{t,x} - \tilde{Y}^{t,x}, Z^{t,x} - \tilde{Z}^{t,x}, M^{t,x} - \tilde{M}^{t,x})\|_{\mathcal{S}(t, T)} &\leq C \left\{ \|b(\cdot, \tilde{X}^{t,x}, \phi(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})) - \tilde{b}(\cdot, \tilde{X}^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}))\|_{\mathcal{H}^2} \\
&\quad + \|\partial_x \hat{H}(\cdot, \tilde{X}^{t,x}, \phi(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}), \tilde{Y}^{t,x}) - \partial_x \hat{H}(\cdot, \tilde{X}^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}), \tilde{Y}^{t,x})\|_{\mathcal{H}^2} \\
&\quad + \|\sigma - \tilde{\sigma}\|_{L^2} + \left( \int_0^T \int_{\mathbb{R}_0^+} |\gamma(t, u) - \tilde{\gamma}(t, u)|^2 \nu(du)dt \right)^{1/2} \right\}.
\end{align*}
\]

(2.23)

It remains to estimate the first two terms on the right-hand side of the above inequality. By (H.1(1)),

\[
\begin{align*}
&\|b(\cdot, \tilde{X}^{t,x}, \phi(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})) - \tilde{b}(\cdot, \tilde{X}^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}))\|_{\mathcal{H}^2} \\
&\leq \|b(\cdot, \tilde{X}^{t,x}, \phi(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})) - b(\cdot, \tilde{X}^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}))\|_{\mathcal{H}^2} \\
&\quad + \|b(\cdot, \tilde{X}^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})) - \tilde{b}(\cdot, \tilde{X}^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}))\|_{\mathcal{H}^2} \\
&\leq \|b_0\|_{L^\infty}\|\phi(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}) - \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})\|_{\mathcal{H}^2} + \|b_0 - \tilde{b}_0\|_{L^\infty}\|X^{t,x}\|_{\mathcal{H}^2} \\
&\quad + \|b_2 - \tilde{b}_2\|_{L^\infty}\|\phi(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})\|_{\mathcal{H}^2}.
\end{align*}
\]

(2.24)
Note that by (2.10), for all $t \in [0, T]$ and $x, y \in \mathbb{R}^n$, $\phi(t, x, y) = (\partial_z f^*)(t, x, -b_2(t)^T y)$ and $\tilde{\phi}(t, x, y) = (\partial_z f^*)(t, x, -\tilde{b}_2(t)^T y)$, where $f^*$ is the function defined in (2.4). Hence, from the $1/\lambda$-Lipschitz continuity of $\partial_z f^*(t, x, \cdot)$ (see the proof of Lemma 2.2),

$$\|\phi(\cdot, X^{t,x}, \tilde{Y}^{t,x}) - \phi(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})\|_{\mathcal{H}^2} \leq C\|b_2 - \tilde{b}_2\|_{L^\infty} \|	ilde{Y}^{t,x}\|_{\mathcal{H}^2} \leq C(1 + |x|)\mathcal{E}_{\text{per}},$$

(2.23)

where the last inequality follows from the moment estimate of $\tilde{Y}^{t,x}$. Moreover, the regularity of $\tilde{\phi}$ (see Lemma 2.3) and the moment estimate of $(\tilde{X}^{t,x}, \tilde{Y}^{t,x})$ imply that $\|\tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})\|_{\mathcal{H}^2} \leq C(1 + |x|)$, which shows that $\|b(t, \tilde{X}^{t,x}, \phi(\tilde{X}^{t,x}, -\tilde{Y}^{t,x})) - b(t, X^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})))\|_{\mathcal{H}^2} \leq C(1 + |x|)\mathcal{E}_{\text{per}}$. By the definitions of $H$ and $\tilde{H}$, the Lipschitz continuity of $\partial_x f_0$ in (H.1(3)) and (2.23),

$$\|\partial_z H(\cdot, X^{t,x}, \phi(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}), \tilde{Y}^{t,x}) - \partial_z \tilde{H}(\cdot, \tilde{X}^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x}), \tilde{Y}^{t,x})\|_{\mathcal{H}^2} \leq (\|b_1 - \tilde{b}_1\|_{L^\infty} \|\tilde{Y}^{t,x}\|_{\mathcal{H}^2} + \|\partial_x f_0(\cdot, \tilde{X}^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})) - \partial_x f_0(\cdot, X^{t,x}, \tilde{\phi}(\cdot, \tilde{X}^{t,x}, \tilde{Y}^{t,x})))\|_{\mathcal{H}^2} \leq C(1 + |x|)\mathcal{E}_{\text{per}}.$$

Thus, we have proved the stability estimate that $\|(X^{t,x} - \tilde{X}^{t,x}, Y^{t,x} - \tilde{Y}^{t,x}, Z^{t,x} - \tilde{Z}^{t,x}, M^{t,x} - \tilde{M}^{t,x})\|_{\mathcal{B}(t,T)} \leq C(1 + |x|)\mathcal{E}_{\text{per}}$.

We now establish the stability of feedback controls. By (2.10) and the $1/\lambda$-Lipschitz continuity of $\partial_z f^*(t, x, \cdot)$, for all $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$|\psi(t, x) - \tilde{\psi}(t, x)| = |(\partial_z f^*)(t, x, -b_2(t)^T Y^{t,x}_t) - (\partial_z f^*)(t, x, -\tilde{b}_2(t)^T \tilde{Y}^{t,x}_t)| \leq |b_2(t)^T Y^{t,x}_t - \tilde{b}_2(t)^T \tilde{Y}^{t,x}_t| \leq C(\|b_2 - \tilde{b}_2\|_{L^\infty} \|Y^{t,x}_t\| + \|Y^{t,x}_t - \tilde{Y}^{t,x}_t\|) \leq C(1 + |x|)\mathcal{E}_{\text{per}}.$$

An important application of the Lipschitz stability of feedback controls (Theorem 2.6) is the analysis of model misspecification error of a given learning algorithm. One essential component is to examine the performance of the feedback control $\tilde{\psi}$, computed based on the control problem (2.1) with the perturbed coefficients $(\tilde{b}, \tilde{\sigma}, \tilde{\gamma}, \tilde{f}, \tilde{g})$, on the true model with coefficients $(b, \sigma, \gamma, f, g)$. For any given $x_0 \in \mathbb{R}^n$, implementing the feedback control $\psi$ on the original system (2.2) will lead to the sub-optimal cost:

$$J(\psi; x_0) := \mathbb{E} \left[ \int_0^T f(t, X^{t,x}_t, \psi(t, X^{t,x}_t)) \, dt + g(X^{t,x}_T) \right],$$

(2.24)

where $X^{x_0, \psi} \in \mathcal{S}^2(\mathbb{R}^n)$ is the state process (with coefficients $b$, $\sigma$ and $\gamma$) associated with $\psi$ (see Definition 2.1). The following theorem shows that the difference between this sub-optimal cost $J(\psi; x_0)$ and the optimal cost $V$ in (2.1) depends Lipschitz-continuously on the magnitude of perturbations in the coefficients.

**Theorem 2.7.** Suppose (H.1) and (H.2) hold. Let $\psi \in \mathcal{V}$ (resp. $\tilde{\psi} \in \mathcal{V}$) be defined in (2.17) (resp. (2.19)), and for each $x_0 \in \mathbb{R}^n$, let $X^{x_0, \psi} \in \mathcal{S}^2(\mathbb{R}^n)$ (resp. $X^{x_0, \tilde{\psi}} \in \mathcal{S}^2(\mathbb{R}^n)$) be the state process (2.2) associated with $\psi$ (resp. $\tilde{\psi}$), and let $V(x_0)$ (resp. $\tilde{V}(x_0)$) be defined in (2.1) (resp. (2.24)). Then there exists a constant $C$ such that for all $x_0 \in \mathbb{R}^n$, $\|X^{x_0, \psi} - X^{x_0, \tilde{\psi}}\|_{\mathcal{S}^2} \leq C(1 + |x_0|)\mathcal{E}_{\text{per}}$ and $|V(x_0) - J(\psi; x_0)| \leq C(1 + |x_0|^2)\mathcal{E}_{\text{per}}$, with the constant $\mathcal{E}_{\text{per}}$ defined in (2.22).

To prove Theorem 2.7, we first establish that the composition of $f$ and the optimal feedback control is Lipschitz continuous, even though the cost function $f$ is merely lower semicontinuous in the control variable (cf. (H.1(3))). The proof is based on the Fenchel-Young identity:

$$f(t, x, \partial_z f^*(t, x, z)) = \langle z, \partial_z f^*(t, x, z) \rangle - f^*(t, x, z) \in \mathbb{R}, \quad \forall (t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k,$$

the regularity of $f^*$ and Theorem 2.6, and has been given in Appendix B.

**Lemma 2.8.** Suppose (H.1) and (H.2) hold. Let $\psi, \tilde{\psi} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^k$ be the functions defined in (2.17) and (2.19), respectively. Then there exists a constant $C$ such that for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$,

$$|f(t, x, \psi(t, x)) - f(t, x', \tilde{\psi}(t, x'))| \leq C \left( (1 + |x| + |x'|)|x - x'| + (1 + |x|^2 + |x'|^2)\mathcal{E}_{\text{per}} \right),$$

where the constant $\mathcal{E}_{\text{per}}$ is defined in (2.22).
Moreover, from Lemma 2.8 and the Cauchy-Schwarz inequality, the parameter \( \alpha \) where for each of (2.7) yield
\[
\|X^{x_0,\psi} - X^{x_0,\tilde{\psi}}\|_{S^2} \leq C\|b(\cdot, X^{x_0,\psi}, \psi(\cdot, X^{x_0,\psi})), - b(\cdot, X^{x_0,\tilde{\psi}}, \tilde{\psi}(\cdot, X^{x_0,\tilde{\psi}}))\|_{H^2}
\]
\[
\leq C\|\psi(\cdot, X^{x_0,\psi} - \tilde{\psi}(\cdot, X^{x_0,\tilde{\psi}}))\|_{H^2} \leq C(1 + \|X^{x_0,\psi}\|_{H^2})\mathcal{E}_{\text{per}} \leq C(1 + |x_0|)\mathcal{E}_{\text{per}}.
\]

We now proceed to estimate \(|V(x_0) - \tilde{V}(x_0)|\) for any given \( x_0 \in \mathbb{R}^n \). By the mean value theorem, (H.1(2)) and the Cauchy-Schwarz inequality,
\[
\mathbb{E}[\|g(X_T^{x_0,\psi}) - g(X_T^{x_0,\tilde{\psi}})\|] \leq C\mathbb{E}[\|(1 + |X_T^{x_0,\psi}| + |X_T^{x_0,\tilde{\psi}}|)|X_T^{x_0,\psi} - X_T^{x_0,\tilde{\psi}}|]
\]
\[
\leq C(1 + \|X_T^{x_0,\psi}\|_{L^2} + \|X_T^{x_0,\tilde{\psi}}\|_{L^2})\|X_T^{x_0,\psi} - X_T^{x_0,\tilde{\psi}}\|_{L^2}
\]
\[
\leq C(1 + |x_0|^2)\mathcal{E}_{\text{per}}.
\]

Moreover, from Lemma 2.8 and the Cauchy-Schwarz inequality,
\[
\mathbb{E}\left[\left|\int_0^T f(t, X_t^{x_0,\psi}, \psi(t, X_t^{x_0,\psi})) - f(t, X_t^{x_0,\tilde{\psi}}, \tilde{\psi}(t, X_t^{x_0,\tilde{\psi}}))\right| dt\right]
\]
\[
\leq C\mathbb{E}\left[\int_0^T \left((1 + |X_t^{x_0,\psi}| + |X_t^{x_0,\tilde{\psi}}|)|X_t^{x_0,\psi} - X_t^{x_0,\tilde{\psi}}| + (1 + |X_t^{x_0,\psi}|^2 + |X_t^{x_0,\tilde{\psi}}|^2)\mathcal{E}_{\text{per}}\right) dt\right]
\]
\[
\leq C\left((1 + \|X_T^{x_0,\psi}\|_{H^2}^2 + \|X_T^{x_0,\tilde{\psi}}\|_{H^2}^2)\|X_T^{x_0,\psi} - X_T^{x_0,\tilde{\psi}}\|_{H^2} + (1 + \|X_T^{x_0,\psi}\|_{H^2}^2 + \|X_T^{x_0,\tilde{\psi}}\|_{H^2}^2)\mathcal{E}_{\text{per}}\right)
\]
\[
\leq C(1 + |x_0|^2)\mathcal{E}_{\text{per}}.
\]

Since \( \psi \) is an optimal feedback control of (2.1) with the initial state \( x_0 \in \mathbb{R}^n \), the desired estimate \(|V(x_0) - J(\psi; x_0)| \leq C(1 + |x_0|^2)\mathcal{E}_{\text{per}}\) follows. \( \square \)

### 3 Regret analysis for linear-convex reinforcement learning

The focus of this section is the linear-convex reinforcement learning (RL) problem, where the drift coefficient of the state dynamics (2.2) is unknown to the controller, and the objective is to control the system optimally while simultaneously learning the dynamics. We shall propose a greedy least-squares algorithm to solve such problems, and show that the algorithm provides a sublinear regret with high probability guarantees. The analysis of the regret bounds for the algorithm relies on the Lipschitz stability of feedback controls established in Section 2.1.

#### 3.1 Reinforcement learning problem and least-squares algorithm

The RL problem goes as follows. Let \( x_0 \in \mathbb{R}^n \) be a given initial state and \( \theta^* = (A^*, B^*) \in \mathbb{R}^{n \times (n + k)} \) be fixed but unknown constants, consider the following problem:

\[
V(x_0; \theta^*) = \inf_{\alpha \in \mathcal{H}^2(\mathbb{R}^k)} \theta^*(\alpha; x_0), \quad \text{with} \quad J^\theta(\alpha; x_0) = \mathbb{E}\left[\int_0^T f(t, X_t^{x_0,\theta^*,\alpha}, \alpha_t) dt + g(X_T^{x_0,\theta^*,\alpha})\right], \quad (3.1)
\]

where for each \( \alpha \in \mathcal{H}^2(\mathbb{R}^k) \), the process \( X_{x_0,\theta^*,\alpha} \) satisfies the following controlled dynamics associated with the parameter \( \theta^* \):

\[
dX_t = (A^* X_t + B^* \alpha_t) dt + \sigma dW_t + \int_{\mathbb{R}_0^d} \gamma(u) \tilde{N}(dt, du), \quad t \in [0,T], \quad X_0 = x_0,
\]

with a given constant \( \sigma \in \mathbb{R}^{n \times d} \) and given functions \( \gamma : \mathbb{R}_0^p \to \mathbb{R}^n \), \( f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \cup \{\infty\} \) and \( g : \mathbb{R}^n \to \mathbb{R} \). If \( \theta^* = (A^*, B^*) \) were known, then (3.1) is a control problem.
It is clear that (3.1)-(3.2) is a special case of (2.1)-(2.2) with $b(t, x, a) = A^*x + B^*a$, $\sigma(t) = \sigma$ and $\gamma(t, u) = \gamma(u)$ for all $(t, x, a, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}_0^p$. Hence, if $f$ and $g$ satisfy (H.1) with for some $L \geq 0$ and $\lambda > 0$, then (3.1)-(3.2) admits an optimal feedback control $\psi^{\theta^*} \in \mathcal{V}$ as shown in Theorem 2.5. Note that to simplify the presentation, we assume that (3.2) has time homogenous coefficients as in [1, 30, 5], but similar analysis can be performed if the drift is a linear combination of given time-and-space-dependent basis functions with unknown weights or the diffusion/jump coefficients are also unknown.

To solve (3.1)-(3.2) with unknown $\theta^*$, in an episodic reinforcement learning framework, the controller improves their knowledge of the parameter $\theta^*$ through successive learning episodes. In particular, for each episode $i \in \mathbb{N}$, based on her observations in the past episodes, the controller executes a suitable control policy in $\psi_i \in \mathcal{V}$, whose associated state dynamics (3.2) leads to an expected cost $J^{\theta^*}(\psi_i; x_0)$. To measure the performance of an learning algorithm in this setting, one widely adopted criteria is the (expected) regret of the algorithm defined as follows (see e.g. [12, 5]):

$$R(N) = \sum_{i=1}^{N} \left( J^{\theta^*}(\psi_i; x_0) - V(x_0; \theta^*) \right), \quad \forall N \in \mathbb{N},$$

(3.3)

where $N$ denotes the total number of learning episodes. Intuitively, this regret characterizes the cumulative loss from taking sub-optimal policies in all episodes.

To start, let us consider a greedy algorithm, which chooses the optimal feedback control based on the current estimation of the parameter, and provides a sublinear regret with respect to the number of episodes $N$. More precisely, let $\theta = (A, B) \in \mathbb{R}^n \times (n+k)$ be the current estimate of $\theta^*$, then the controller would exercise the optimal feedback control $\psi^\theta \in \mathcal{V}$ defined in Theorem 2.5 for the control problem (3.1)-(3.2) with $\theta^*$ replaced by $\theta$, which leads to the state process $X^{\theta^*, \theta} \in \mathbb{S}^2(\mathbb{R}^n)$ satisfying:

$$dX_t = (A^*X_t + B^*\psi^\theta(t, X_t)) \, dt + \sigma \, dW_t + \int_{\mathbb{R}_0^p} \gamma(u) \, \tilde{N}(dt, du), \quad t \in [0, T], \quad X_0 = x_0.$$

(3.4)

By the martingale properties of stochastic integrals, we can then estimate $\theta^*$ based on the process $Z_t^{\theta^*, \theta} := \left( X_t^{\theta^*, \theta}, \psi^\theta(t, X_t^{\theta^*, \theta}) \right)$, $t \in [0, T]$, as follows:

$$\langle \theta^* \rangle^T = \left( \mathbb{E} \left[ \int_0^T Z_t^{\theta^*, \theta}(Z_t^{\theta^*, \theta})^T \, dt \right] \right)^{-1} \mathbb{E} \left[ \int_0^T Z_t^{\theta^*, \theta}(dX_t^{\theta^*, \theta})^T \right],$$

(3.5)

provided that $\mathbb{E} \left[ \int_0^T Z_t^{\theta^*, \theta}(Z_t^{\theta^*, \theta})^T \, dt \right] \in \mathbb{R}^{(n+k) \times (n+k)}$ is invertible. This motivates us to introduce an iterative procedure to estimate $\theta^*$, where the expectations in (3.5) are replaced by empirical averages over independent realizations. More precisely, let $m \in \mathbb{N}$ and $(X_{t}^{\theta^*, \theta, i}, \psi^\theta(t, X_{t}^{\theta^*, \theta, i}))_{t \in [0,T]}, \ i = 1, \ldots, m$, be trajectories of $m$ independent realizations of the state and control processes, we shall update the estimate $\theta$, denoted by $\hat{\theta}$, according to (3.5):

$$\langle \hat{\theta} \rangle^T := \left( \frac{1}{m} \sum_{i=1}^{m} \int_0^T Z_t^{\theta^*, \theta, i}(Z_t^{\theta^*, \theta, i})^T \, dt + \frac{1}{m} I_{n+k} \right)^{-1} \left( \frac{1}{m} \sum_{i=1}^{m} \int_0^T Z_t^{\theta^*, \theta, i}(dX_t^{\theta^*, \theta, i})^T \right),$$

(3.6)

where $Z_t^{\theta^*, \theta, i} := \left( X_t^{\theta^*, \theta, i}, \psi^\theta(t, X_t^{\theta^*, \theta, i}) \right)$ for all $t \in [0, T]$ and $i = 1, \ldots, m$, and $I$ is the $(n+k) \times (n+k)$ identity matrix used to ensure the existence of the required matrix inverse. This leads to the following greedy least-squares (GLS) algorithm:

**Algorithm 1 Greedy least-squares (GLS) algorithm**

1. **Input:** Choose an initial estimation $\theta_0$ of $\theta^*$ and numbers of learning episodes $\{m_t\}_{t \in \mathbb{N} \cup \{0\}}$.
2. for $\ell = 0, 1, \ldots$ do
3. Obtain the optimal feedback control $\psi^\theta_\ell$ for (3.1)-(3.2) with $\theta^* = \theta_\ell$ as in Theorem 2.5.
4. Execute the feedback control $\psi^\theta_\ell$ for $m_\ell$ independent episodes, and collect the trajectory data $(X_{t}^{\theta^*, \theta, i}, \psi^\theta_\ell(t, X_{t}^{\theta^*, \theta, i}))_{t \in [0, T]}, \ i = 1, \ldots, m_\ell$.
5. Obtain an updated estimation $\hat{\theta}_{\ell+1}$ by using (3.6) and the $m_\ell$ trajectories collected above.
6. end for.
3.2 Structural assumptions for learning problems

In this section, we analyze the regret of Algorithm 1 based on the following assumptions of the learning problem (3.1)-(3.2).

**H.3.**
1. Let \( x_0 \in \mathbb{R}^n, \theta^* = (A^*, B^*) \in \mathbb{R}^{n \times (n+k)}, \sigma \in \mathbb{R}^{n \times d}, \gamma : \mathbb{R}^n_0 \to \mathbb{R}^n, f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \cup \{ \infty \} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) satisfy (H.1) with some constants \( L \geq 0 \) and \( \lambda > 0 \).

2. There exist \( \gamma_{\max} \geq 0 \) and \( \vartheta \in [0, 1] \) such that \( \sup_{q \geq 2} q^{-\vartheta} \left( \int_{\mathbb{R}^n} |\gamma(u)|^q \nu(du) \right)^{1/q} \leq \gamma_{\max} \).

**Remark 3.1.** Condition (H.3(1)) implies that for each \( \theta = (A, B) \), the control problem of (3.1)-(3.2) with \( \theta^* \) replaced by \( \theta \) is a non-smooth linear-convex control problem studied in Section 2.

Condition (H.3(2)) describes the large jumps of the pure jump process \( L_t := \int_0^t \int_{\mathbb{R}^n_0} \gamma(u) \tilde{N}(ds, du), t \in [0, T] \), which enables estimating the tail behaviour of the state process \( X^\theta \), and subsequently quantifying the parameter estimation error of the least-squares estimator (3.6) (see Section 3.4.2). If the jump coefficient \( \gamma \) is bounded, then one can easily see from \( \int_{\mathbb{R}^n_0} |\gamma(u)|^2 \nu(du) < \infty \) that (H.3(2)) holds with \( \vartheta = 0 \).

Another important case is when \( \gamma(u) = u \) for all \( u \in \mathbb{R}^n_0 \), under which the process \( (L_t)_{t \in [0, T]} \) is a Lévy process of pure jumps with Lévy measure \( \nu(du) \). In this case, (H.3(2)) holds with \( \vartheta \in (0, 1] \) if and only if \( (\int_{\mathbb{R}^n_0} u^q \nu(du))^{1/q} \leq O(q) \) as \( q \to \infty \).

**H.4.** \( \theta^* \) is identifiable, i.e., the optimal control \( \alpha^{x_0, \theta^*} \in \mathcal{H}^2(\mathbb{R}^k) \) and the optimal state process \( X^{x_0, \theta^*, \alpha^*} \in \mathcal{S}^2(\mathbb{R}^n) \) of (3.1)-(3.2) with initial state \( x_0 \) and parameter \( \theta^* \) satisfy the following linear independence condition: if \( u_1 \in \mathbb{R}^n \) and \( u_2 \in \mathbb{R}^k \) satisfy \( u_1^T X_t^{x_0, \theta^*, \alpha^*} + u_2^T \alpha_t = 0 \) for \( d\mathbb{P} \otimes dt \) a.e., then \( u_1 \) and \( u_2 \) are zero vectors.

Condition (H.4) implies that the true parameter \( \theta^* \) can be uniquely identified if we observe sufficiently many trajectories of the optimal state and control processes of (3.1)-(3.2). Such a self-exploration property allows us to design exploration-free learning algorithms for (3.1)-(3.2).

The following proposition shows that if the laws of the state processes are supported on the whole space, then (H.4) is equivalent to a self-exploration property of the optimal feedback control. The proof essentially follows the argument of [39, Lemma 6.1], and hence is omitted.

**Proposition 3.1.** Assume (H.3(1)). Let \( \psi \in \mathcal{V} \). Assume that for all \( t \in (0, T] \), and any open set \( O \subset \mathbb{R}^n \) with positive Lebesgue measure, the state process \( X^{\theta, \psi} \) (defined by (3.4) with \( \psi^\theta = \psi \)) satisfies that \( \mathbb{P}(\{ \omega \in O \mid X^{\theta, \psi}_t(\omega) \in O \}) > 0 \). Then the following two statements are equivalent:

1. if \( u_1 \in \mathbb{R}^n \) and \( u_2 \in \mathbb{R}^k \) satisfy \( u_1^T X_t^{\theta, \psi} + u_2^T \psi(t, X_t^{\theta, \psi}) = 0 \) for \( d\mathbb{P} \otimes dt \) a.e., then \( u_1 \) and \( u_2 \) are zero vectors;
2. if \( u_1 \in \mathbb{R}^n \) and \( u_2 \in \mathbb{R}^k \) satisfy \( u_1^T x + u_2^T \psi(t, x) = 0 \) for almost every \( (t, x) \in [0, T] \times \mathbb{R}^n \), then \( u_1 \) and \( u_2 \) are zero vectors.

Consequently, suppose that (H.3(1)) holds and \( \sigma \theta^T \) is positive definite, then (H.4) holds if and only if the optimal feedback control \( \psi^{\theta^*} \) of (3.1) satisfies Item (b).

Proposition 3.1 allows for more explicit expressions of (H.4). For instance, as shown in [5, Proposition 3.9], for quadratic cost functions \( g = 0 \) and \( f(t, x, a) = x^T Q x + a^T R a \) with positive definite matrices \( Q \) and \( R \), (H.4) holds if and only if \( B^* \) in (3.2) is full column rank. Alternatively, by [39, Proposition 3.3], if (3.1)-(3.2) has a bounded action set, i.e., \( \mathcal{R} \) in (H.1(3)) has a bounded domain \( \mathcal{R} \) (cf. Example 2.1), then (H.4) holds if and only if the range of \( \psi^{\theta^*} \) contains \( k \) linearly independent vectors.

We remark that for general linear-convex learning problems without (H.4), an explicit exploration is necessary for learning [39]. Instead of merely employing greedy policies as in Algorithm 1, they dedicate certain episodes to actively explore the environment with some exploration policy \( \psi^{\theta^*} \) satisfying Proposition 3.1 Item (b). The numbers of exploration and exploitation episodes are then balanced based on the performance gap in Theorem 2.7 and the finite-sample accuracy of the parameter estimator. Note, however, this explicit exploration may yield larger regrets for algorithm in [39] than that in Theorem 3.2.
3.3 Main results on sublinear regret bounds

We now state the main result which shows that the regret of Algorithm 1 grows at most sublinearly with respect to the number of episodes, provided that the hyper-parameters $\theta_0$ and $\{m_i\}_{i \in \mathbb{N} \cup \{0\}}$ are chosen properly. In particular, we shall choose an initial guess $\theta_0$ of $\theta^*$ which satisfies the identifiability condition in (H.4) and we shall also double the number of learning episodes between two successive updates of the estimation of $\theta^*$, which is a commonly used strategy (the so-called doubling trick) in the design of online learning algorithms (see e.g. [5]). The proof of this theorem is given in Section 3.4.3.

To simplify the notation, we introduce the following quantities for each $x_0 \in \mathbb{R}^n$, $\theta = (A,B) \in \mathbb{R}^{n \times (n+k)}$ and $m \in \mathbb{N}$:

\[
U_{x_0,\theta} := \mathbb{E} \left[ \int_0^T Z_{x_0,\theta}(Z_{x_0,\theta}^\top)^T \, dt \right], \quad V_{x_0,\theta} := \mathbb{E} \left[ \int_0^T Z_{x_0,\theta}(dX_{x_0,\theta})^\top \, dt \right],
\]

\[
U_{x_0,\theta,m} := \frac{1}{m} \sum_{i=1}^m \int_0^T Z_{x_0,\theta,i}(Z_{x_0,\theta,i}^\top)^T \, dt, \quad V_{x_0,\theta,m} := \frac{1}{m} \sum_{i=1}^m \int_0^T Z_{x_0,\theta,i}(dX_{x_0,\theta,i})^\top, \tag{3.7}
\]

where $X_{x_0,\theta} \in \mathcal{S}^2(\mathbb{R}^n)$ is the solution of (3.4), $(X_{x_0,\theta,i})_{i=1}^m$ are independent copies of $X_{x_0,\theta}$, and $Z_{x_0,\theta}$ and $(Z_{x_0,\theta,i})_{i=1}^m$ are defined as in (3.5) and (3.6), respectively. For any given symmetric matrix $A$, we denote by $\lambda_{\text{min}}(A)$ the smallest eigenvalue of $A$.

**Theorem 3.3.** Suppose (H.3(1)) and (H.4) hold. Assume further that $\lambda_{\text{min}}(U_{x_0,\theta_0}) > 0$, and for any given bounded set $K \subset \mathbb{R}^{n \times (n+k)}$, there exist constants $C_1, C_2 > 0$ and $\beta \geq 1$, such that the following concentration inequality holds for all $\varepsilon > 0$, $m \in \mathbb{N}$ and $\theta \in K$,

\[
\max \left\{ \mathbb{P}(U_{x_0,\theta,m} - U_{x_0,\theta} \geq \varepsilon), \mathbb{P}(V_{x_0,\theta,m} - V_{x_0,\theta} \geq \varepsilon) \right\} \leq C_2 \exp \left( -C_1 \min \left\{ \frac{m\varepsilon^2}{C_2^2}, \frac{m\varepsilon}{C_2} \right\} \right). \tag{3.8}
\]

Then there exists a constant $C_0 > 0$, such that for all $C \geq C_0$ and $\delta \in (0,1/4)$, if we set $m_0 = C(-\ln \delta)^\beta$ and $m_{\ell} = 2^\ell m_0$ for all $\ell \in \mathbb{N}$, then the regret of Algorithm 1 (cf. (3.3)) satisfies the following properties:

1. It holds with probability at least $1 - 4\delta$ that $R(N) \leq C' \left( \sqrt{N \ln N} + \sqrt{-\ln \delta} \sqrt{N} + (-\ln \delta)^\beta \ln N \right)$ for all $N \in \mathbb{N}$, where $C'$ is a constant independent of $N$ and $\delta$.

2. It holds with probability 1 that $R(N) = \mathcal{O}(\sqrt{N \ln N})$ as $N \to \infty$.

The following theorem presents a precise sublinear regret bound of Algorithm 1 for the jump-diffusion model (3.2), depending on the jump sizes of the Poisson random measure. The proof follows from Theorem 3.2 and Proposition 3.9.

**Theorem 3.3.** Suppose (H.3) and (H.4) hold, and $\lambda_{\text{min}}(U_{x_0,\theta_0}) > 0$. Then there exists a constant $C_0 > 0$, such that for all $C \geq C_0$ and $\delta \in (0,1/4)$, if we set $m_0 = C(-\ln \delta)^{3+\beta}$ and $m_{\ell} = 2^\ell m_0$ for all $\ell \in \mathbb{N}$, then the regret of Algorithm 1 (cf. (3.3)) satisfies the following properties:

1. It holds with probability at least $1 - 4\delta$ that $R(N) \leq C' \left( \sqrt{N \ln N} + \sqrt{-\ln \delta} \sqrt{N} + (-\ln \delta)^{3+\beta} \ln N \right)$ for all $N \in \mathbb{N}$, where $C'$ is the constant in (H.3(2)) and $C'$ is a constant independent of $\theta$, $N$ and $\delta$.

2. It holds with probability 1 that $R(N) = \mathcal{O}(\sqrt{N \ln N})$ as $N \to \infty$.

In the case where (3.2) is only driven by the Brownian motion, we can exploit the sub-Gaussianity of the state process and obtain a sharper regret bound based on Theorem 3.2 and Proposition 3.10.

**Theorem 3.4.** Suppose (H.3) and (H.4) hold with $\gamma_{\text{max}} = 0$, and $\lambda_{\text{min}}(U_{x_0,\theta_0}) > 0$. Then there exists a constant $C_0 > 0$, such that for all $C \geq C_0$ and $\delta \in (0,1/4)$, if we set $m_0 = C(-\ln \delta)$ and $m_{\ell} = 2^\ell m_0$ for all $\ell \in \mathbb{N}$, then the regret of Algorithm 1 (cf. (3.3)) satisfies the following properties:

1. It holds with probability at least $1 - 4\delta$ that $R(N) \leq C' \left( \sqrt{N \ln N} + \sqrt{-\ln \delta} \sqrt{N} + (-\ln \delta) \ln N \right)$ for all $N \in \mathbb{N}$, where $C'$ is a constant independent of $N$ and $\delta$.  

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For every $\beta$, random variables and (3.8) holds with some $\beta$ in Section 3.4.1. We then show in Section 3.4.2 that (3.6) behaves like sub-Weibull random variables as in [26] and establishing that both deterministic and stochastic integrals preserve sub-

$X$ variable $\beta$ state dynamics creates a crucial difficulty in quantifying the precise value of $\beta$ in (3.8) since the state

equality (3.8) for the least-squares estimator (3.6). Compared to the classical

Section 3.4.3). Note that to analyze algorithm regre
ts, it is common to assume some a-priori information on the true parameter and the algorithm being initialized with sufficiently many learning episodes (see e.g., [13]). Obtaining an explicit dependence of $C_0$ on model parameters, however, could be challenging. A practical strategy for validating (H.4) and for choosing the initial episode $m_0$ is to ensure that the obtained estimations $(\theta_t)_{t \in \mathbb{N}}$ remain bounded and that the resulting greedy policies $(\psi^{\theta_t})_{t \in \mathbb{N}}$ satisfy Proposition 3.1 Item (b). Our numerical experiments in Section 5 demonstrate that the performance of Algorithm 1 is stable with respect to $m_0$, and that a small $m_0$ in general suffices to guarantee a sublinear regret.

3.4 Proofs of sublinear regret bounds

This section is devoted to the proofs of Theorem 3.2, 3.3 and 3.4.

As we have seen in Theorems 3.3-3.4, an essential step for estimating the regret of Algorithm 1 is to establish the concentration inequality (3.8) for the least-squares estimator (3.6). Compared to the classical

learning problems with Brownian-motion-driven state dynamics (see e.g. [5]), the presence of jumps in the state dynamics creates a crucial difficulty in quantifying the precise value of $\beta$ in (3.8), since the state variable $X^\theta$ is in general not sub-Gaussian, and hence (3.8) does not hold with $\beta = 1$.

In the subsequent analysis, we overcome the above difficulty by introducing a notation of sub-Weibull random variables as in [26] and establishing that both deterministic and stochastic integrals preserve sub-Weibull random variables in Section 3.4.1. We then show in Section 3.4.2 that (3.6) behaves like sub-Weibull random variables and (3.8) holds with some $\beta \geq 1$, provided that the jumps of the state dynamics are sub-exponential. Finally, we prove the general regret result Theorem 3.2 for Algorithm 1 in Section 3.4.3.

3.4.1 Step 1: Analysis of sub-Weibull random variables

The first step is to analyze integrals of sub-Weibull random variables. We start by recalling the precise definition of sub-Weibull random variables in terms of their Orlicz norms (see [26]).

Definition 3.1. For every $\alpha > 0$, let $\Psi_\alpha : [0, \infty) \to \mathbb{R}$ such that $\Psi_\alpha(x) = e^{x^\alpha} - 1$ for all $x \geq 0$, and let $\| \cdot \|_{\Psi_\alpha}$ be the corresponding $\Psi_\alpha$-Orlicz (quasi-)norm such that for any given random variable $X$,

$$
\| X \|_{\Psi_\alpha} := \inf \left\{ t > 0 : \mathbb{E} \left[ \Psi_\alpha \left( \frac{|X|}{t} \right) \right] \leq 1 \right\}.
$$

Then a random variable $X$ is said to be sub-Weibull of order $\alpha > 0$, denoted by $X \in \text{subW}(\alpha)$, if $\| X \|_{\Psi_\alpha} < \infty$.

Note that $\| \cdot \|_{\Psi_\alpha}$ is a norm if and only if $\alpha \geq 1$, as otherwise the triangle inequality does not hold. Examples of sub-Weibull random variables include sub-Gaussian and sub-exponential random variables, which correspond to subW(2) and subW(1), respectively. We point out that the class of sub-Weibull random variables is closed under multiplication and addition, and for all $\alpha > 0$, there exists a constant $C_\alpha$, depending only on $\alpha$, such that

$$
C_\alpha^{-1} \sup_{q \geq 1} q^{-1/\alpha} \| X \|_{L_q} \leq \| X \|_{\Psi_\alpha} \leq C_\alpha \sup_{q \geq 1} q^{-1/\alpha} \| X \|_{L_q}
$$

(3.9)

for all random variables $X$ (see [18, Appendix A] for a proof of these properties).

We now present several important lemmas regarding the behavior of integrals of sub-Weibull random variables. The first lemma shows that deterministic integral of a product of sub-Weibull random variables is still sub-Weibull. The proof is based on Definition 3.1 and Hölder’s inequality, and is given in Appendix B.
Lemma 3.5. For all $\alpha > 0$ and every stochastic process $X, Y : \Omega \times [0, T] \to \mathbb{R}$,
\[
\left\| \int_0^T XY \, dt \right\|_{\Psi_{\alpha/2}} \leq \left\| \left( \int_0^T |X|^2 \, dt \right)^{\frac{1}{2}} \right\|_{\Psi_{\alpha}} \left\| \left( \int_0^T |Y|^2 \, dt \right)^{\frac{1}{2}} \right\|_{\Psi_{\alpha}}.
\]

The second lemma shows that stochastic integrals preserve the property of being sub-Weibull random variables. The proof is based on the equivalent characterization (3.9) of sub-Weibull random variables and Burkholder’s inequality, whose details are given in Appendix B.

Lemma 3.6. There exists $C \geq 0$ such that for all $\sigma \in \mathbb{R}^d$, $X \in \mathcal{S}^2(\mathbb{R})$ and every measurable function $\gamma : \mathbb{R}^d \to \mathbb{R}$ satisfying (H.3(2)), $\| \int_0^T X_t \sigma^T \, dW_t \|_{\Psi_{1/2}} \leq C|\sigma|\left( \int_0^T |X|^2 \, dt \right)^{\frac{1}{2}} \|\gamma\|_{L_1}$ and
\[
\left\| \int_0^T \int_{\mathbb{R}^d} X_t \gamma(u) \tilde{N}(dt, du) \right\|_{\Psi_{1/(3+\vartheta)}} \leq C \gamma_{\max} \left( \sup_{p \geq 2} \left( \int_0^T |X|^q \, dt \right)^{\frac{1}{q}} \right),
\]
with the constants $\gamma_{\max}$ and $\vartheta$ in (H.3(2)).

Lemma 3.6 focuses on the case where $(\int_0^T |X|^2 \, dt)^{1/2} \in \text{subW}(1) \setminus \text{subW}(2)$, which is important for control problems whose state dynamics is driven by a Poisson random measure. Hence we establish the sub-Weibull properties of the stochastic integrals by applying the Burkholder’s inequality to estimate the growth of their $L^q$-norms, precise order of which depends on the constants $C_q$ and $\bar{C}_q$ in the inequalities (A.7) and (A.8).

In the case where $(\int_0^T |X|^2 \, dt)^{1/2} \in \text{subW}(2)$, we can establish the optimal sub-Weibull order $\int_0^T X_t \sigma^T \, dW_t$ in subW(1). Such a characterization is essential for obtaining a sharper regret bound of Algorithm 1 when the state dynamics is only driven by the Brownian motion. The proof is based on the Girsanov theorem and is given in Appendix B.

Lemma 3.7. There exists $C \geq 0$ such that for all $\sigma \in \mathbb{R}^d$ and $X \in \mathcal{S}^2(\mathbb{R})$, $\| \int_0^T X_t \sigma^T \, dW_t \|_{\Psi_1} \leq C|\sigma|\left( \int_0^T |X|^2 \, dt \right)^{\frac{1}{2}} \|\gamma\|_{L_1}$.

3.4.2 Step 2: Concentration inequalities for the least-squares estimator

Based on the fact that sub-Weibull properties are preserved under algebraic and integral operations as shown in Section 3.4.1, we now quantify the precise tail behavior of the least-squares estimator (3.6), namely the constant $\beta$ in (3.8), for the jump-diffusion model (3.2).

We start by establishing the sub-exponential properties of Lipschitz functionals of the state process $X^\theta$ driven by both Brownian motions and Poisson random measures as in (3.4). The proof follows as a special case of [29] and is given in Appendix B.

Lemma 3.8. Suppose (H.3) holds. Let $K \subset \mathbb{R}^n$ and $\theta = (A, B) \in \mathbb{R}^{n \times (n+k)}$ satisfy $|\theta| \leq K$. Then there exists $C \geq 0$, depending only on $K$, $T$ and the constants in (H.3), such that for all $x_0 \in \mathbb{R}^n$ and for every Lipschitz continuous function $f : (\mathcal{D}(\mathbb{R}^n), d_\infty) \to \mathbb{R}$, the solution $X^{x_0, \theta}$ of (3.4) satisfies $\|f(X^{x_0, \theta})\|_{\Psi_1} \leq C(||f||_{\text{Lip}} + |E[f(X^{x_0, \theta})]|)$, where $\mathcal{D}(\mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued càdlàg functions on $[0, T]$ endowed with the uniform metric $d_\infty$, and $||f||_{\text{Lip}}$ is the Lipschitz constant of $f$. (cf. Lemma A.3).

We now characterize the parameter $\beta$ in the concentration inequality (3.8) based on Lemmas 3.5, 3.6 and 3.8.

Proposition 3.9. Suppose (H.3) holds and let $\mathcal{K} \subset \mathbb{R}^{n \times (n+k)}$ be a bounded set. Then there exist constants $C_1, C_2 \geq 0$ such that (3.8) holds for all $\varepsilon \geq 0$, $m \in \mathbb{N}$ and $\theta \in K$ with $\beta = 3 + \vartheta$, where $\vartheta$ is the constant in (H.3(2)).

Proof. Throughout this proof, let $\theta$ be a given constant satisfying $|\theta| \leq K$ for some $K \geq 0$. For notational simplicity, we shall omit the dependence on $(x_0, \theta)$ in the subscripts of all random variables, and denote by $C_2$ a generic constant, which is independent of $m$ and the precise value of $\theta$, and depends possibly on $K$, $x_0$, the constants in (H.3) and the dimensions.
Note that for each \( i = 1, \ldots, m \), the entries of \( \int_0^T Z_i^T \frac{dX_i}{dt} \) are one of the three cases:

\[
\int_0^T X_{i,t} X_{i,t}^\varepsilon dt, \quad \int_0^T X_{i,t} \psi^\varepsilon(t, X_{i,t}^\varepsilon) dt, \quad \int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) \psi^\varepsilon(t, X_{i,t}^\varepsilon) dt
\]

where \( X_{i,t}^\varepsilon \) and \( \psi^\varepsilon(t, X_{i,t}^\varepsilon) \) are the \( \ell \)-th entry of \( X_{i,t}^\varepsilon \) and \( \psi^\varepsilon(t, X_{i,t}^\varepsilon) \), respectively. Similarly, the entries of \( \int_0^T Z_i^T \frac{dX_i^\varepsilon}{dt} \) are one of the two cases:

\[
\int_0^T X_{i,t}^\varepsilon (A^* X_{i,t}^\varepsilon) dt + \int_0^T X_{i,t}^\varepsilon (B^* \psi^\varepsilon(t, X_{i,t}^\varepsilon))dt + \int_0^T X_{i,t}^\varepsilon \sigma_j dt + \int_0^T \int_{\mathbb{R}^d} X_{i,t}^\varepsilon (u) \tilde{N}^i(dt, du),
\]

\[
\int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) (A^* X_{i,t}^\varepsilon) dt + \int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) (B^* \psi^\varepsilon(t, X_{i,t}^\varepsilon))dt + \int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) \sigma_j dt + \int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) \sigma_j dt \]

(3.11)

where \( \sigma_j \) is the \( j \)-th row of \( \sigma \in \mathbb{R}^{n \times d} \), \( \gamma_j \) is the \( j \)-th entry of the function \( \gamma : \mathbb{R}^p \to \mathbb{R}^n \), \( (W_i^m)_{i=1}^\inf \) are \( m \)-independent \( d \)-dimensional Brownian motion, and \( (\tilde{N}^i)_{i=1}^\inf \) are \( m \)-independent compensated Poisson random measures. By the definitions of \( U^{x_0,\theta,m}, V^{x_0,\theta,m} \) in (3.7), and the inequality that \( P(\{ \sum_{i=1}^\ell X_i \geq \varepsilon \}) \leq \sum_{i=1}^\ell P(\{ |X_i| \geq \varepsilon / \ell \}) \) for all \( \varepsilon \in \mathbb{N} \) and random variables \( (X_i)_{i=1}^\inf \), it suffices to obtain a concentration inequality for each term in (3.10) and (3.11).

Since \( |\theta| \leq K \), by Theorem 2.5, there exists \( C_2 \geq 0 \) such that \( |\psi^\varepsilon(t, 0)| \leq C_2 \) and \( |\psi^\varepsilon(t, x) - \psi^\varepsilon(t, x')| \leq C_2 |x - x'| \) for all \( t \in [0, T] \), \( x, x' \in \mathbb{R}^n \). Then standard moment estimates of (3.4) (with the initial condition \( x_0 \)) shows that \( \|X_i\|_{Lip} \leq C_2 \) for all \( i = 1, \ldots, m \), with a constant \( C_2 \) depending on \( x_0 \). Then, for each \( q \geq 2, \ell = 1, \ldots, n \), and \( j = 1, \ldots, k \), we consider the functions \( f_j(q), \tilde{f}_j(q) : (\mathbb{D}([0, T]; \mathbb{R}^n), d_\infty) \to \mathbb{R} \) satisfying for all \( \rho \in \mathbb{D}([0, T]; \mathbb{R}^n) \) that \( f_j(q)(\rho) = \int_0^T |\rho(t_j)|^q dt \frac{1}{q} \) and \( \tilde{f}_j(q)(\rho) = \int_0^T |\psi^\varepsilon(t, \rho(t_j))|^q dt \) for all \( \psi^\varepsilon \). One can easily show that \( f_j(q)(0) = 0 \) and \( \|f_j(q)(0)\|_{Lip}, ||f_j(q)\|_{Lip} \leq C \), which along with Lemma 3.8 implies that \( \|f_j(q)(\psi^\varepsilon)\|_{Lip} \leq C \), and \( \|f_j(q)(\psi^\varepsilon)\|_{Lip} \leq C \), uniformly with respect to \( i, \ell, j, q, \theta \). Hence, we can obtain from Lemmas 3.5 and 3.6 a uniform bound for the \( \|\cdot\|_{1/(3+\theta)} \)-norms of all the terms in (3.10) and (3.11).

Consequently, we can deduce the desired concentration inequality by applying Lemma A.4 (with \( \alpha = 1/(3+\theta), \beta = m \)) to each component of the zero-mean random variables \( \{ \int_0^T Z_i^T (dX_i^\varepsilon)^T - \overline{U} \}_{i=1}^\inf \) and \( \{ \int_0^T Z_i^T (dX_i)^T - \overline{U} \}_{i=1}^\inf \). 

The following proposition improves the concentration inequality in Proposition 3.9 for the case without jumps.

**Proposition 3.10.** Suppose (H.3) holds with \( \gamma_{\text{max}} = 0 \) and let \( K \subset \mathbb{R}^{n \times (n+k)} \) be a bounded set. Then there exist constants \( C_1, C_2 \geq 0 \) such that (3.8) holds for all \( \varepsilon \geq 0, m \in \mathbb{N} \) and \( \theta \in K \) with \( \beta = 1 \).

**Proof.** We first refine the result of Lemma 3.8 and prove Lipschitz functionals of the state process \( X^{x_0, \theta} \) is sub-Gaussian. By [15, Theorem 1.1 and Corollary 4.1], there exists \( C_0 \geq 0 \) such that for all \( x_0 \in \mathbb{R}^n \) and for every Lipschitz continuous function \( f : (\mathbb{D}([0, T]; \mathbb{R}^n), d_\infty) \to \mathbb{R} \) with \( \|f\|_{Lip} \leq 1 \), \( \mathbb{E}[\exp(\lambda (||f(X^{x_0, \theta})|| - ||f(X^{x_0, \theta})||))] \leq \exp(C^2 \lambda^2) \) for all \( \lambda > 0 \), which along with [42, Proposition 2.5.2 (v)] implies that \( \|f(X^{x_0, \theta}) - ||f(X^{x_0, \theta})||\|_{Lip} \leq C \) for some constant \( C \), uniformly with respect to \( x_0 \in \mathbb{R}^n \), \( \theta \in K \) and \( f : (\mathbb{D}([0, T]; \mathbb{R}^n), d_\infty) \to \mathbb{R} \). Then, we can deduce from the fact that \( \|f\|_{Lip} \) is a norm that \( \|f(X^{x_0, \theta})\|_{Lip} \leq C(\|f\|_{Lip} + \mathbb{E}[f(X^{x_0, \theta})])] \) for all \( x_0 \in \mathbb{R}^n \), \( \theta \in K \) and Lipschitz continuous functions \( f \).

We then proceed along the proof of Proposition 3.9. For each \( i = 1, \ldots, m \), all entries of \( \int_0^T Z_i^T (dX_i^\varepsilon)^T dt \) are given in (3.10), and all entries of \( \int_0^T Z_i^T (dX_i)^T dt \) are given by (cf. (3.11)):

\[
\int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) (A^* X_{i,t}^\varepsilon) dt + \int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) (B^* \psi^\varepsilon(t, X_{i,t}^\varepsilon))dt + \int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) \psi^\varepsilon(t, X_{i,t}^\varepsilon) dt + \int_0^T \psi^\varepsilon(t, X_{i,t}^\varepsilon) \psi^\varepsilon(t, X_{i,t}^\varepsilon) \sigma_j dt \]

(3.12)
for all \( \ell = 1, \ldots, n \) and \( j = 1, \ldots, k \), where we have omitted the dependence on \((x_0, \theta)\) in the subscripts for notational simplicity. Hence, by following the same argument as in Proposition 3.9, we can show that there exists a constant \( C \), such that for all \( i = 1, \ldots, m, \ell = 1, \ldots, n \) and \( j = 1, \ldots, k \) and \( \theta \in K \), we have
\[
\| (\int_0^T |X_{i,\ell}^j|^2 \, dt)^{\frac{1}{2}} \|_{\mathcal{F}_2} \leq C \quad \text{and} \quad \| (\int_0^T |\psi^j(t, X_i^j)|^2 \, dt)^{\frac{1}{2}} \|_{\mathcal{F}_2} \leq C.
\]
Then, we can obtain from Lemmas 3.5 and 3.7 a uniform bound for the \( \| \cdot \|_{\mathcal{F}_2} \)-norms of all entires of \( (\int_0^T Z_i^j(Z_i^T)^T \, dt \) and \( (\int_0^T Z_i^j(dX_i^j)^T)^T \). Consequently, we can apply Lemma A.4 (with \( \alpha = 1, N = m \) and \( \epsilon' = mc \)) to each entry of \( (\int_0^T Z_i^j(Z_i^T) \, dt - \overline{U})^m_{i=1} \) and \( (\int_0^T Z_i^j(dX_i^j)^T - \overline{V})^m_{i=1} \), and deduce the desired concentration inequality.

3.4.3 Step 3: Proof of general regret bounds

After demonstrating how to verify (3.8) based on the precise jump sizes in the state dynamics, it remains to establish the general regret result in Theorem 3.2 under the assumption that (3.8) holds for some \( \beta \geq 1 \).

We start by showing that under (H.3(1)) and (H.4), the expression (3.5) is well-defined if \( \theta \) is a sufficiently accurate estimation of the true parameter \( \theta^* \).

**Lemma 3.11.** Suppose (H.3(1)) and (H.4) hold. Then there exist constants \( \varepsilon_0 > 0 \) and \( \tau_0 > 0 \), such that for all \( \theta \in K_0 := \{ \theta \in \mathbb{R}^{n \times (n+k)} \mid |\theta - \theta^*| \leq \varepsilon_0 \} \), we have \( \lambda_{\text{min}}(U^{x_0, \theta}) \geq \tau_0 \), where \( U^{x_0, \theta} \) is defined as in (3.7) and \( \lambda_{\text{min}}(A) \) is the smallest eigenvalue of a symmetric matrix \( A \).

**Proof.** Since \( U^{x_0, \theta} \) is positive semidefinite, we shall prove \( \lambda_{\text{min}}(U^{x_0, \theta}) > 0 \) by assuming that \( \lambda_{\text{min}}(U^{x_0, \theta}) = 0 \). Then we see there exists a non-zero vector \( u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^{n+k} \) with \( u_1 \in \mathbb{R}^{n} \) and \( u_2 \in \mathbb{R}^{k} \), such that \( u^TU^{x_0, \theta}u = 0 \). By the definition of \( U^{x_0, \theta} \) in (3.7), we can deduce that \( \mathbb{E}\left[\int_0^T |u^T z_0^{x_0, \theta}|^2 \, dt\right] = 0 \), which along with the definition of \( z_0^{x_0, \theta} \) in (3.5) implies for \( d\mathbb{P} \otimes dt \) a.e. that \( u_1^T z_t^{x_0, \theta} + u_2^T z_t^{x_0, \theta} = 0 \). This contradicts to (H.4), which leads to the desired inequality that \( \lambda_{\text{min}}(U^{x_0, \theta}) > 0 \).

We then show that the map \( \mathbb{R}^{n \times (n+k)} \ni \theta \mapsto U^{x_0, \theta} \in \mathbb{R} \) is continuous. Theorem 2.7 shows that the map \( \mathbb{R}^{n \times (n+k)} \ni \theta \mapsto X^{x_0, \theta} \in \mathcal{H}^2(\mathbb{R}^n) \) is continuous. Moreover, Theorems 2.5 and 2.6 imply that there exists a constant \( C \geq 0 \), such that for all \( \theta \in \mathbb{R}^{n \times (n+k)} \) satisfying \( |\theta - \theta^*| \leq 1 \), \( t \in [0, T] \) and \( x, x' \in \mathbb{R}^n \), we have that
\[
|\psi(t, 0)| \leq C, \quad |\psi^j(t, x) - \psi^j(t, x')| \leq C|x - x'| \quad \text{and} \quad |\psi(t, x) - \psi^* \psi(t, x)| \leq C(1 + |x|)|\theta - \theta^*|,
\]
from which we can deduce that
\[
|\psi^j(t, x) - \psi^* \psi(t, x)| \leq |\psi^j(t, x) - \psi^* \psi(t, x)| + |\psi^* \psi(t, x) - \psi^* \psi(t, x')| \\
\leq C(1 + |x|)|\theta - \theta^*| + C|x - x'|.
\]

Hence, for all \( \theta \in \mathbb{R}^{n \times (n+k)} \) with \( |\theta - \theta^*| \leq 1 \),
\[
||\psi^j(\cdot, X^{x_0, \theta}) - \psi^j(\cdot, X^{x_0, \theta^*})||_{\mathcal{H}^2} \leq C(1 + \| X^{x_0, \theta} - X^{x_0, \theta^*} \|_{\mathcal{H}^2})|\theta - \theta^*| + C\| X^{x_0, \theta} - X^{x_0, \theta^*} \|_{\mathcal{H}^2},
\]
which along with the continuity of the map \( \mathbb{R}^{n \times (n+k)} \ni \theta \mapsto X^{x_0, \theta} \in \mathcal{H}^2(\mathbb{R}^n) \) implies that the map \( \mathbb{R}^{n \times (n+k)} \ni \theta \mapsto \psi^j(\cdot, X^{x_0, \theta}) \in \mathcal{H}^2(\mathbb{R}^k) \) is continuous. Since the entires of \( U^{x_0, \theta} \) involve only the expectations of products of \( X^{x_0, \theta} \) and \( \psi^j(\cdot, X^{x_0, \theta}) \), the desired continuity of the map \( \mathbb{R}^{n \times (n+k)} \ni \theta \mapsto U^{x_0, \theta} \in \mathbb{R} \) follows.

Finally, by the continuity of the minimum eigenvalue function, clearly \( \mathbb{R}^{n \times (n+k)} \ni \theta \mapsto \lambda_{\text{min}}(U^{x_0, \theta}) \in \mathbb{R} \) is continuous, which along with the fact that \( \lambda_{\text{min}}(U^{x_0, \theta}) > 0 \) leads to the desired result.

We then quantify the estimation error of the least-squares estimator (3.6) by assuming the concentration inequality (3.8) holds for the compact set \( K_0 \) in Lemma 3.11.

**Proposition 3.12.** Suppose (H.3(1)) and (H.4) hold. Let \( K_0 \) be the set in Lemma 3.11. Assume further that there exist constants \( C_1, C_2 > 0 \) and \( \beta \geq 1 \) such that (3.8) holds for all \( \varepsilon \geq 0 \), \( m \in \mathbb{N} \) and \( \theta \in K_0 \). Then there exist constants \( C_1, C_2 \geq 0 \), such that for all \( \theta \in K_0 \) and \( \delta \in (0, 1/2) \), if \( m \geq C_1(-\ln \delta)^{\beta} \), then we have with probability at least \( 1 - 2\delta \) that
\[
|\hat{\theta} - \theta^*| \leq \tilde{C}_2 \left( \sqrt{\frac{-\ln \delta}{m}} + \frac{(-\ln \delta)^{\beta}}{m} + \frac{(-\ln \delta)^{2\beta}}{m^2} \right). \tag{3.13}
\]
where \( \hat{\theta} \) denotes the transpose of the left-hand side of (3.6) associated with \( \theta \).

**Proof.** Throughout the proof, let \( \delta \in (0, 1/2) \) and \( \theta \in \mathcal{K}_0 \) be fixed and let \( \| \cdot \|_2 \) be the matrix norm induced by Euclidean norms. The invertibility of \( U_{x, \theta} \) (see Lemma 3.11) implies that (3.5) is well-defined, which along with (3.6) leads to

\[
\| \hat{\theta} - \theta^* \|_2 = \| (U_{x, \theta, m} + \frac{1}{m} I)^{-1} V_{x, \theta, m} - (U_{x, \theta})^{-1} V_{x, \theta} \|_2 
\leq \| (U_{x, \theta, m} + \frac{1}{m} I)^{-1} - (U_{x, \theta})^{-1} \|_2 \| V_{x, \theta, m} \|_2 + \| (U_{x, \theta})^{-1} \|_2 \| V_{x, \theta, m} - V_{x, \theta} \|_2. \tag{3.14}
\]

We now estimate each term in the right-hand side of (3.14). By Lemma 3.11, \( \lambda_{\min}(U_{x, \theta}) \geq \tau_0 \) for some \( \tau_0 > 0 \), which implies that \( \| (U_{x, \theta})^{-1} \|_2 \leq 1/\tau_0 \). Moreover, by setting the right-hand side of (3.8) to be \( \delta \), we can deduce with probability at least \( 1 - 2\delta \) that \( |U_{x, \theta, m} - U_{x, \theta}| \leq \delta_m \) and \( |V_{x, \theta, m} - V_{x, \theta}| \leq \delta_m \) with the constant \( \delta_m \) given by

\[
\delta_m := \max \left\{ \left( \frac{C_2^2}{C_1 m} \ln \left( \frac{C_2}{\delta} \right) \right) \frac{C_1}{C_2} \left( \frac{1}{C_2} \ln \left( \frac{C_2}{\delta} \right) \right)^\beta \right\}, \tag{3.15}
\]

where we have assumed without loss of generality that \( C_2 \geq 1 \).

Let \( m \) be a sufficiently large constant satisfying \( \delta_m + 1/m \leq \tau_0/2 \). The fact that \( \| \cdot \|_2 \leq \| \cdot \| \) indicates with probability at least \( 1 - 2\delta \) that \( \| U_{x, \theta, m} + \frac{1}{m} I - U_{x, \theta} \|_2 \leq \frac{1}{m} + \delta_m \leq \frac{\tau_0}{2} \), which in turn yields

\[
\lambda_{\min}(U_{x, \theta, m} + \frac{1}{m} I) \geq \lambda_{\min}(U_{x, \theta}) - \| U_{x, \theta, m} + \frac{1}{m} I - U_{x, \theta} \|_2 \geq \frac{\tau_0}{2},
\]

or equivalently \( \| (U_{x, \theta, m} + \frac{1}{m} I)^{-1} \| \leq 2/\tau_0 \). Then, since \( A^{-1} - (A + B)^{-1} = (A + B)^{-1}BA^{-1} \) for all nonsingular matrices \( A \) and \( A + B \), we have with probability at least \( 1 - 2\delta \) that,

\[
\| (U_{x, \theta, m} + \frac{1}{m} I)^{-1} - (U_{x, \theta})^{-1} \|_2 = \| (U_{x, \theta} + U_{x, \theta, m} + \frac{1}{m} I - U_{x, \theta}) - (U_{x, \theta})^{-1} \|_2 
\leq \| (U_{x, \theta} + U_{x, \theta, m} + \frac{1}{m} I - U_{x, \theta})^{-1} \|_2 \| (U_{x, \theta, m} + \frac{1}{m} I - U_{x, \theta}) \|_2 
\leq \frac{2\delta}{\tau_0} (\frac{1}{m} + \delta_m),
\]

which along with the inequality that \( \| V_{x, \theta, m} \|_2 \leq \| V_{x, \theta} \|_2 + |V_{x, \theta, m} - V_{x, \theta}| \) allows us to derive the following estimate from (3.14):

\[
\| \hat{\theta} - \theta^* \|_2 \leq \frac{2\delta}{\tau_0} (\frac{1}{m} + \delta_m) (\| V_{x, \theta} \|_2 + \delta_m) + \frac{\delta_m}{\tau_0}.
\]

Note that \( \| V_{x, \theta} \|_2 \) is uniformly bounded for all \( \theta \in \mathcal{K}_0 \) by the compactness of \( \mathcal{K}_0 \) and the continuity of the map \( \theta \mapsto V_{x, \theta} \) (cf. Lemma 3.11). Thus, by the condition that \( \beta \geq 1 \) and the definition of \( \delta_m \) in (3.15), we see that there exists a constant \( C_2 \), depending only on \( C_1, C_2, \beta, \tau_0 \), and the constants in (H.3(1)), such that the desired estimate (3.13) holds with probability at least \( 1 - 2\delta \), provided that \( m \) satisfies \( \delta_m + 1/m \leq \tau_0/2 \). Since \( \beta \geq 1 \) and \( \delta \leq 1/2 \), we see there exists \( C_1 \geq 0 \), independent of \( m, \delta \) and \( \theta \), such that the inequality (3.13) holds for all \( m \) satisfying \( m \geq C_1 (\ln \delta) \).

Now we are ready to present the proof of Theorem 3.2.

**Proof of Theorem 3.2.** We start by proving Item (1). Then, by the assumptions that \( \lambda_{\min}(U_{x, \theta}) > 0 \) and (3.8) holds for \( K = \mathcal{K}_0 = \mathcal{K}_0 \cup \{ \theta_0 \} \) with \( \mathcal{K}_0 \) from Lemma 3.11, we can extend Proposition 3.12 to show that (3.13) holds for all \( \theta \in \mathcal{K}_\delta \), \( \delta \in (0, 1/2) \) and \( m \geq C_1 (\ln \delta) \), with some constants \( C_1, C_2 \geq 1 \) depending on \( \mathcal{K}_0 \). In the subsequent analysis, we fix \( \delta \in (0, 1/4) \) and for all \( \ell \in \mathbb{N} \cup \{ 0 \} \), we define \( \delta_{\ell} = 2^{-\ell} \delta \), and let \( \theta_{\ell+1} \) be generated by using (3.6) with \( m = m_\ell \) and \( \theta = \theta_\ell \). We shall specify the precise choice of \( m_0 \) later.
In the sequel, we assume without loss of generality that \( \varepsilon_0/(3\bar{C}_2) \leq 1 \) and \( \bar{C}_2/\varepsilon_0 \geq \bar{C}_1 \), where \( \varepsilon_0 > 0 \) is the constant in the definition of \( \mathcal{K}_0 \) (see Lemma 3.11). We first show that there exists \( \bar{C}_0 > 0 \), independent of \( \delta \), such that if \( m_0 \geq \bar{C}_0(-\ln \delta)^\beta \), then for all \( \ell \in \mathbb{N} \cup \{0\}, \)

\[
\bar{C}_2 \left( \sqrt{\frac{-\ln \delta_\ell}{m_\ell}} + \frac{(-\ln \delta_\ell)^\beta}{m_\ell} + \frac{(-\ln \delta_\ell)^{2\beta}}{m_\ell^2} \right) \leq \varepsilon_0. \tag{3.16}
\]

By the assumption that \( \varepsilon_0/(3\bar{C}_2) \leq 1 \), it suffices to show that for all \( \ell \in \mathbb{N} \cup \{0\}, -\ln \delta_\ell/m_\ell \leq (\varepsilon_0/(3\bar{C}_2))^2 \) and \((\ln \delta_\ell)^\beta/m_\ell \leq \varepsilon_0/(3\bar{C}_2)\). Given \( \beta \geq 1 \) and \( \delta_\ell < 1/4 \), it suffices to ensure \( m_\ell \geq C(-\ln \delta_\ell)^\beta \) for all \( \ell \in \mathbb{N} \cup \{0\} \), where \( C \) is a sufficiently large constant independent of \( \delta \) and \( \ell \). By the definitions of \((\delta_\ell)_{\ell \in \mathbb{N}}\) and \((m_\ell)_{\ell \in \mathbb{N}}\) and the fact that \( \delta < 1/4 \), the desired condition can be achieved by choosing \( m_0 \geq \bar{C}_0(-\ln \delta)^\beta \), for a sufficiently large constant \( \bar{C}_0 \) satisfying

\[
\sup_{\ell \in \mathbb{N} \cup \{0\}, \delta \in (0, \frac{1}{4})} \frac{(-\ln (2-\ell/\delta))^\beta}{2^\ell (-\ln \delta)^\beta} = \sup_{\ell \in \mathbb{N} \cup \{0\}, \delta \in (0, \frac{1}{4})} 2^{-\ell} \left( \frac{\ell \ln 2}{-\ln \delta} + 1 \right)^\beta \leq \sup_{\ell \in \mathbb{N} \cup \{0\}} 2^{-\ell} \left( \frac{\ell}{2} + 1 \right)^\beta \leq \bar{C}_0 < \infty.
\]

Now we choose \( m_0 \geq \max(\bar{C}_0, \bar{C}_1)(-\ln \delta)^\beta \), and show by induction that for all \( k \in \mathbb{N} \cup \{0\} \), it holds with probability at least \( 1 - 2^{\sum_{\ell=0}^{k-1} \delta_\ell} \) that \( \theta_\ell \in \bar{K}_0 \) for all \( \ell = 0, \ldots, k \) and

\[
|\theta_k - \theta^*|^2 \leq \left\{ \frac{\theta_0 - \theta^*|^2_0}{2}, \sqrt{\bar{C}_2 \left( \frac{-\ln \delta_k}{m_k} + \frac{(-\ln \delta_k)^\beta}{m_k} + \frac{(-\ln \delta_k)^{2\beta}}{m_k^2} \right)} \right\}, \quad k = 0, \ldots, n. \tag{3.17}
\]

The statement clearly holds for \( k = 0 \). Now suppose that the induction statement holds for some \( k \in \mathbb{N} \cup \{0\} \). Conditioning on \( \theta_k \in \bar{K}_0 \), we can apply (3.13) with \( \theta = \theta_k, \delta = \delta_k < 1/2 \) and \( m = m_k \geq \bar{C}_1(-\ln \delta_k)^\beta \) (see (3.16) and \( \bar{C}_2/\varepsilon_0 \geq \bar{C}_1 \)), and deduce with probability at least \( 1 - 2\delta_k \) that (3.17) holds for the index \( k + 1 \), which along with (3.16) shows that \( \theta_{k+1} \in \bar{K}_0 \subseteq \bar{K}_0 \). Since the induction hypothesis implies that \( \theta_k \in \bar{K}_0 \) holds with probability at least \( 1 - 2^{\sum_{\ell=0}^{k} \delta_\ell} \), one can deduce that the induction statement also holds \( k + 1 \).

The above induction argument shows that if \( m_0 = C(-\ln \delta)^\beta \) for any constant \( C \geq C_0 := \max(\bar{C}_0, \bar{C}_1) \), then with probability at least \( 1 - 2^{\sum_{\ell=0}^{\infty} \delta_\ell} = 1 - \delta_0 \), \( \theta_k \in \bar{K}_0 \) and (3.17) holds for all \( k \in \mathbb{N} \cup \{0\} \). Now let us assume such a setting, and observe that the i-th trajectory is generated with control \( \psi_{\ell} \) if \( i = \sum_{j=0}^{\ell-1} m_j, j_{\ell} = \sum_{j=0}^{\ell} m_j \) for \( \ell \in \mathbb{N} \cup \{0\} \) (cf. Algorithm 1). Then we can apply Theorem 2.7 and deduce for all \( N \in \mathbb{N} \) that

\[
R(N) \leq \sum_{\ell=0}^{\log_2\left( \frac{N^{2(\ell+1)}}{m_0} + 1 \right)-1} \sum_{\ell=0}^{\log_2\left( \frac{N}{m_0} + 1 \right)-1} m_\ell \left( J^\theta(\psi_{\ell}; x_0) - V(x_0; \theta^*) \right) \leq C' \sum_{\ell=0}^{\log_2\left( \frac{N}{m_0} + 1 \right)-1} m_\ell |\theta_\ell - \theta^*| \leq C' m_0 + C' \sum_{\ell=1}^{\log_2\left( \frac{N}{m_0} + 1 \right)-1} \left( \sqrt{\frac{(-\ln \delta_\ell)m_\ell}{m_\ell}} + (-\ln \delta_\ell)^\beta \left( 1 + \frac{(-\ln \delta_\ell)^2}{m_\ell} \right) \right) \leq C'(-\ln \delta)^\beta + C' \sum_{\ell=1}^{\log_2\left( \frac{N}{m_0} + 1 \right)-1} \left( \sqrt{\frac{(-\ln \delta_\ell)m_\ell}{m_\ell}} + (-\ln \delta_\ell)^\beta \right), \tag{3.18}
\]

where we have denoted by \( C' \) a generic constant independent of \( \ell, N, \delta \), and used the fact that \( (-\ln \delta_\ell)^\beta/m_\ell \leq C' \) for the last inequality (cf. the choice of \( C_0 \)). We then derive an upper bound of (3.18). By virtue of the inequality that \( \sqrt{(-\ln \delta_\ell)m_\ell} = \sqrt{(-\ln 2 - \ln \delta)^2 m_\ell} \leq C' \sqrt{(-\ln \delta)^2 m_\ell} \leq \sqrt{C'} \sqrt{(\ln N - \ln \delta)m_\ell 2^{\log_2\left( \frac{N}{m_0} + 1 \right)}} \leq \sqrt{C'} \sqrt{(\ln N - \ln \delta)(N + (-\ln \delta)^2)} \), for all \( \ell \in \mathbb{N} \), we have

\[
\sum_{\ell=1}^{\log_2\left( \frac{N}{m_0} + 1 \right)-1} \sqrt{(-\ln \delta_\ell)m_\ell} \leq C' \sqrt{(\ln N - \ln \delta)m_\ell 2^{\log_2\left( \frac{N}{m_0} + 1 \right)}} \leq C' \sqrt{(\ln N - \ln \delta)(N + (-\ln \delta)^2)}.
\]

Moreover, by \( \ln \delta_\ell = -\ell \ln 2 + \ln \delta \) and Hölder’s inequality,

\[
\sum_{\ell=1}^{\log_2\left( \frac{N}{m_0} + 1 \right)-1} (-\ln \delta_\ell)^\beta \leq C' \sum_{\ell=1}^{\log_2\left( \frac{N}{m_0} + 1 \right)-1} \left( \frac{\ell \ln 2)^{\beta}}{m_\ell} + (-\ln \delta_\ell)^\beta \right) \leq C'((\ln N)^{\beta+1} + \ln N(-\ln \delta)^\beta). \]
Consequently, from (3.18), $\beta \geq 1$ and the inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for all $x, y \geq 0$, it is clear for all $N \in \mathbb{N}$, $R(N) \leq C'\sqrt{N\ln N + \ln \delta \sqrt{N} + (\ln \delta)^{\beta} \ln N}$ for some constant $C'$ independent of $\beta$ and $N$, which finishes the proof of Item (1).

We are ready to show Item (2). For each $N \in \mathbb{N} \cap [3, \infty)$, we define $\delta_N = 1/N^2$ and the event $A_N = \{R(N) > C'\sqrt{N\ln N + \ln \delta \sqrt{N} + (\ln \delta)^{\beta} \ln N}\}$, Item (1) shows that $\sum_{N=3}^{\infty} P(A_N) \leq 4\sum_{N=3}^{\infty} \delta_N < \infty$. Hence, from the Borel-Cantelli lemma, $P(\limsup_{N \to \infty} A_N) = 0$, which along with the definition of $\delta_N$ implies the desired conclusion.

4 Extension: RL problems with controlled diffusion

In this section, we extend our framework to analyze the regret order of learning algorithms for general continuous-time RL problems, whose state dynamics involves controlled diffusion. To simplify the presentation, we focus on entropy-regularized problems studied in [43, 37, 34, 41] and outline the essential steps of the argument.

For each $\theta = (A, B) \in \mathbb{R}^{n \times (n+k)}$, define $V(\cdot; \theta) : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ by

$$V(t, x; \theta) := \inf_{\alpha \in H^2(\mathbb{R}^k)} \mathbb{E} \left[ \int_t^T f(s, X_s^{t,x,\alpha}, \alpha_s) \, ds + g(X_T^{t,x,\alpha}) \right], \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, \quad (4.1)$$

where for each $\alpha \in H^2(\mathbb{R}^k)$, $X^{t,x,\alpha} \in S^2(\mathbb{R}^n)$ satisfies the controlled dynamics:

$$dX_s = (AX_s + B\alpha_s) \, ds + \sigma(s, X_s, \alpha_s) \, dW_s, \quad s \in [t, T], \quad X_t = x. \quad (4.2)$$

The functions $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{n \times d}$ are such that for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $a = (a_i)_{i=1}^k \in \mathbb{R}^k$,

$$f(t, x, a) = \sum_{i=1}^k \mathcal{I}_i(t, x) a_i + \mathcal{R}_{\text{en}}(a), \quad \sigma(t, x, a) \, \sigma(t, x, a)^T = \sum_{i=1}^k \sigma_i(t, x) \sigma_i(t, x)^T a_i, \quad (4.3)$$

where for each $i = 1, \ldots, k$, $\mathcal{I}_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $\sigma_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are some given functions and $\mathcal{R}_{\text{en}} : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$ is Shannon’s entropy function (cf. Example 2.2) such that

$$\mathcal{R}_{\text{en}}(a) = \begin{cases} \sum_{i=1}^k a_i \ln(a_i), & a \in \Delta_k := \{a \in [0, 1]^k \ | \ \sum_{i=1}^k a_i = 1\}, \\ \infty, & a \in \mathbb{R}^k \setminus \Delta_k. \end{cases} \quad (4.4)$$

To avoid needless technicalities, we assume $(\mathcal{I}_i)_{i=1}^k$, $(\sigma_i)_{i=1}^k$ and $g$ to be bounded and sufficiently regular as in Proposition 4.1.

Note that (4.4) restricts control processes to those taking values in $\Delta_k$. Hence, if $\sigma_i(t, x) \equiv \sigma$ for some $\sigma \in \mathbb{R}^{n \times d}$, then (4.1)-(4.2) is a special case of the linear-convex model studied in Sections 2-3. Consequently, Theorem 3.2 can be applied to study the regret order of GLS algorithms for (4.1)-(4.2) with given initial time and state $(t, x) \in [0, T] \times \mathbb{R}^n$ but with unknown parameter $\theta$.

To analyze the regret order of learning algorithms with general $\sigma$, a crucial step is to extend Theorem 2.7 and quantify the performance of a greedy policy from an incorrect model. The fact that control affects the diffusion coefficients complicates the stability analysis of optimal feedback controls (i.e., Theorem 2.6) for (4.1)-(4.2). The following proposition proves a linear performance gap under the condition that the value function $V(t, x; \theta)$ in (4.1) is sufficiently regular in $t, x$ and $\theta$. Recall that the first and second-order derivatives of a sufficiently regular value function can be represented by solutions to the associated FBSDE (2.11) (see e.g., [44, Theorem 4.1, p. 250]). Hence the linear performance gap can also be established by assuming sufficient regularity of the solution process $(Y^{t,\sigma}, Z^{t,\sigma})$ to (2.11), whose details are omitted here.

**Proposition 4.1.** For each $\theta \in \mathbb{R}^{n \times (n+k)}$, let $V(\cdot; \theta) : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be defined by (4.1). Suppose that $(\mathcal{I}_i)_{i=1}^k$, $(\sigma_i)_{i=1}^k \subset C^{0,1}([0, T] \times \mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n)$, and there exists $m : [0, \infty) \to [0, \infty)$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\theta, \theta' \in \mathbb{R}^{n \times (n+k)}$, $\frac{\partial}{\partial \theta} V(\cdot; \theta)$ is continuous, $\|V(\cdot; \theta)\|_{C^{0,1}([0, T] \times \mathbb{R}^n)} \leq m(\|\theta\|)$,

$$|\nabla_x V(t, x; \theta) - \nabla_x V(t, x; \theta'_i)| + \|\text{Hess}_x V(t, x; \theta) - \text{Hess}_x V(t, x; \theta'_i)\| \leq (m(\|\theta\|) + m(\|\theta'_i\|))|\theta - \theta'_i| \left(1 + |x|\right). \quad (4.5)$$

Then for all $\theta \in \mathbb{R}^{n \times (n+k)}$, there exists $\psi(\theta) \in \mathcal{V}$ such that
(1) $\psi^\theta$ is an optimal feedback control of (4.1)-(4.2) satisfying for all $x_0 \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^{n \times (n+k)}$, $V(0, x_0; \theta) = J(\psi^\theta; x_0, \theta)$, where for each $\psi \in \mathcal{V}$,

$$J(\psi; x_0, \theta) := \mathbb{E} \left[ \int_0^T f(t, X_t^{x_0, \psi}, \psi(t, X_t^{x_0, \psi})) dt + g(X_T^{x_0, \psi}) \right],$$

and $X^{x_0, \psi} \in \mathcal{S}^2(\mathbb{R}^n)$ satisfies the following dynamics:

$$dX_t = \theta \left( \psi(t, X_t) \right) dt + \sigma(t, X_t, \psi(t, X_t)) dW_t, \quad t \in [0, T], \quad X_0 = x_0,$$

(2) for all $x_0 \in \mathbb{R}^n$ and $R \geq 0$ there exists a constant $C$ such that for all $\theta, \theta' \in \mathbb{R}^{n \times (n+k)}$ with $|\theta|, |\theta'| \leq R$,

$$|J(\psi^{\theta'}; x_0, \theta) - J(\psi^\theta; x_0, \theta)| \leq C|\theta' - \theta|.$$

Proposition 4.1 relies on the regularity and Lipschitz stability of the value function $V$. For instance, if all coefficients are bounded and sufficiently smooth, and $\sigma$ satisfies the uniform parabolic condition, then $C^{2+\alpha}$ regularity results for fully nonlinear parabolic PDEs (see e.g., the Evan-Krylov theorem in [24, Theorems 6.4.3 and 6.4.4, p. 301]) and a bootstrap argument would ensure that for any given $\theta$, the function $V(\cdot, \theta)$ is continuously differentiable in $t$ and three-time continuously differentiable in $x$. Due to the unbounded drift coefficient of (4.2), the boundedness in the $C^{0,3}([0, T] \times \mathbb{R}^n)$-norm and the locally Lipschitz continuity of $V$ in $\theta$ follow from an extension of the Schauder estimate (see e.g., [25]) to nonlinear parabolic equations with unbounded coefficients in the whole space.

With Proposition 4.1, we can then quantify the regrets of GLS algorithms (see Algorithm 1) for (4.1)-(4.2) with unknown drift parameter $\theta$ and known diffusion coefficient $\sigma$. By the boundedness of $\sigma$ and the regularity of $\psi^\theta$, one can prove Proposition 3.10 in the present setting. Hence, Theorem 3.2 (with $\beta = 1$ in (3.8)) shows that Algorithm 1 enjoys a sublinear regret as shown in Theorem 3.4.

Proof of Proposition 4.1. For any given $\theta = (A, B) \in \mathbb{R}^{n \times (n+k)}$, the regularity of $V(\cdot; \theta)$ and [44, Proposition 3.5, p. 182] imply that $V(\cdot; \theta)$ is the unique classical solution to the associated HJB equation. That is, for all $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\frac{\partial}{\partial t} V(t, x) + \inf_{a \in \Delta_k} \left( \frac{1}{2} \text{tr}(\sigma(t, x, a) \sigma(t, x, a)^T \text{Hess}_x V(t, x)) + (Ax + Ba, \nabla_x V(t, x)) + f(t, x, a) \right) = 0,$$

and $V(T, x) = g(x)$ for all $x \in \mathbb{R}^d$. By (4.3), for all $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$\psi^\theta(t, x) := \arg\min_{a \in \Delta_k} \left( \frac{1}{2} \text{tr}(\sigma(t, x, a) \sigma(t, x, a)^T \text{Hess}_x V(t, x; \theta)) + (Ba, \nabla_x V(t, x; \theta)) + f(t, x, a) \right)$$

$= \nabla R^*_{en}(\cdot),

\quad - \frac{1}{2} \text{tr}(\sigma(t, x) \sigma(t, x)^T \text{Hess}_x V(t, x; \theta)) - B^T \nabla_x V(t, x; \theta) - f(t, x),

where for all $z \in \mathbb{R}^k$, $R^*_{en}(z) = \sup_{a \in \Delta_k} \langle (a, z) - R_{en}(a) \rangle = \ln \sum_{i=1}^k \exp(z_i),$ and

$$\text{tr}(\sigma(t, x) \sigma(t, x)^T \text{Hess}_x V(t, x; \theta)) = \begin{pmatrix} \text{tr}(\sigma(t, x) \sigma(t, x)^T \text{Hess}_x V(t, x; \theta)) \\ \vdots \\ \text{tr}(\sigma_k(t, x) \sigma_k(t, x)^T \text{Hess}_x V(t, x; \theta)) \end{pmatrix}, \quad f(t, x) = \begin{pmatrix} f_1(t, x) \\ \vdots \\ f_k(t, x) \end{pmatrix}.$$

The Lipschitz continuity of $\nabla R^*_{en}$ and the regularity assumptions imply that $\psi^\theta \in \mathcal{V}$ and the corresponding state process $X^{x_0, \psi^\theta}$ is well defined. Then a standard verification argument (see e.g., [44, Theorem 6.6, p. 278]) shows $\psi^\theta$ is an optimal feedback control and finishes the proof of Item (1).

To prove Item (2), Fix $x_0 \in \mathbb{R}^n$ and $R \geq 0$ and let $C$ be a generic constant independent of $\theta$. Note that the Fenchel-Young identity gives that $R^*_{en}(\nabla R^*_{en}(z)) = \langle z, \nabla R^*_{en}(z) \rangle - R^*_{en}(z)$ for all $z \in \mathbb{R}^k,$ which along with (4.3) implies that for all $(t, x, \theta) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times (n+k)},$

$$f(t, x, \psi^\theta(t, x)) = -\frac{1}{2} \text{tr}(\sigma(t, x) \sigma(t, x)^T \text{Hess}_x V(t, x; \theta)) + B^T \nabla_x V(t, x; \theta, \psi^\theta(t, x))$$

$$- R^*_{en}(\cdot) \frac{1}{2} \text{tr}(\sigma(t, x) \sigma(t, x)^T \text{Hess}_x V(t, x; \theta)) - B^T \nabla_x V(t, x; \theta) - f(t, x).$$

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By the regularity assumptions of the coefficients and the function $V$, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$ and $\theta, \theta' \in \mathbb{R}^{n \times (n+k)}$ with $|\theta|, |\theta'| \leq R$, there exists $C > 0$ such that
\[
|\psi^{\theta}(t, x') - \psi^{\theta}(t, x)| + |f(t, x', \psi^{\theta}(t, x')) - f(t, x, \psi^{\theta}(t, x))| \leq C(|x - x'| + (1 + |x'| + |x|)|\theta - \theta'|).
\]

Proceeding along the lines of the proof of Theorem 2.7 leads to the desired estimate in Item (2).

\section{Numerical experiments}

In this section, we test the theoretical findings and Algorithm 1 through numerical experiment on a three-dimensional LQ RL problem considered in [13, 14]. Our experiments show the convergence of the least-squares estimations to the true parameters as the number of episodes increases, as well as the sublinear cumulative regret as indicated in Theorem 3.4. Moreover, it confirms that the state coefficient $A^*$ is easier to learn than the control coefficient $B^*$, consistent with the observations in [14].

Our numerical results show that a rough estimation of the control parameter $B^*$ is often sufficient to design a nearly optimal feedback control, and that the Algorithm 1 is robust with respect to the initial batch size $m_0$.

\textbf{Problem setup.} We consider a three-dimensional LQ RL problems over the time horizon $[0, T]$ with $T = 1.5$, where the linear state dynamics (3.2) has the initial state $x_0$ and unknown coefficients $\theta^* = (A^*, B^*) \in \mathbb{R}^{3 \times (3+3)}$, chosen as in [13, 14]:
\[
A^* = \begin{bmatrix} 1.01 & 0.01 & 0 \\ 0.01 & 1.01 & 0.01 \\ 0 & 0.01 & 1.01 \end{bmatrix}, \quad B^* = I_3, \quad \sigma = I_3, \quad \gamma \equiv 0, \quad x_0 = 0,
\]
with $I_3$ being the $3 \times 3$ identity matrix, and the cost functional (3.1) involves quadratic functions $g \equiv 0$ and $f(t, x, a) = (x^TQx + a^TRa)/2$, with $Q = 0.1I_3$ and $R = I_3$. As mentioned in [13, 14], this state dynamics corresponds to a marginally unstable graph Laplacian system where adjacent nodes are weakly connected, which arises naturally from consensus and distributed averaging problems. Since the cost penalizes the control inputs more than the states, it is essential to learn the unstable components of $A^*$ and perform control on these components in order to achieve an optimal cost. Note that this LQ RL problem satisfies (H.3); see the last paragraph of Remark 3.1.

The numerical experiments are coded using Python. Algorithm 1 is initialized with $m_0 = 4$ and the initial guess $A_0 = \begin{bmatrix} -1.0730 & 0.8654 & -2.3015 \\ 0.6241 & -0.6118 & -0.5283 \\ 1.7448 & -0.7612 & 0.3190 \end{bmatrix}$ and $B_0 = \begin{bmatrix} -0.3224 & -0.3841 & 1.338 \\ 0.4621 & -2.0601 & -0.1224 \\ -0.10999 & -0.1724 & -0.8779 \end{bmatrix}$, whose entries are sampled independently from the standard normal distribution. For each $\ell \in \mathbb{N} \cup \{0\}$, given the current estimate $\theta_\ell = (A_\ell, B_\ell)$ of $\theta^*$, classical LQ control theory (see e.g., [44]) shows that solutions to (2.11) can be found analytically via Riccati equations, and the greedy policy $\psi^{\theta_\ell}$ is given by $\psi^{\theta_\ell}(t, x) = -R^{-1}B^TP^{\theta_\ell}x$, where $P^{\theta_\ell}$ is the unique positive semidefinite solution to
\[
\frac{d}{dt}P_t + A_t^TP_t + P_tA_t - P_t(B_tR^{-1}B_t^T)P_t + Q = 0, \quad t \in (0, T); \quad P_T = 0.
\]

We solve (5.1) numerically via a high-order Runge-Kutta method on a uniform time grid with stepsizes $T/100$, and then simulate $m_\ell = 2^\ell m_0$ independent trajectories of the state dynamics (3.4) (controlled by $\psi^{\theta_\ell}$) using the Euler-Maruyama method on the same time grid. To estimate statistical properties of the algorithm regret (3.3), we execute Algorithm 1 for 100 independent runs, where among different executions, the observed state trajectories are simulated based on independent Brownian motion increments.

\textbf{Performance with $m_0 = 4$.} Figure 1 exhibits the performance of Algorithm 1 for this LQ-RL problem, where the solid lines and the shallow areas indicate the sample mean and the 95% confidence interval over 100 repeated experiments. The numerical results indicate that algorithm 1 manages to learn the parameters over time while incurring a desirable sublinear regret, which is consistent with our theoretical result in Theorem 3.4. More precisely,

- Figure 1(a) presents the logarithmic relative error of the estimate $(A_\ell, B_\ell)$ (in the Frobenius norm) after the $\ell$-th update for $\ell \in \{0, \ldots, 10\}$. One can observe that the estimate $(A_\ell, B_\ell)_\ell$ converge to the
true parameter \((A^*, B^*)\) as the number of episodes increases. Our experiment shows that it is much easier to learn the state coefficient \(A^*\) than the control coefficient \(B^*\), which is consistent with the observation in [14] for other adaptive control schemes.

- Figure 1(b) presents the relative error between the expected cost \(J^π(\psi^{θ_ℓ}; x_0)\) and the optimal expected cost \(J^π(\psi^{θ_∗}; x_0)\). One can see that a rough estimate of the control parameter \(B^*\) is often sufficient to design a nearly optimal feedback control. In particular, after the 10-th update \((ℓ = 10)\), although the relative approximation errors of \(A_ℓ\) and \(B_ℓ\) are 2.7% and 24.9%, respectively, the cost of \(\psi^{θ_ℓ}\) approximates the optimal cost accurately with a relative error 0.6%.

- Figure 1(c) presents the cumulative regret over episodes. One can see that the small performance gap results in a slowly growing algorithm regret. In fact, performing a linear regression for logarithms of expected regret and episode shows that the regret after the \(N\)-th episode is of the magnitude \(O(N^{0.34})\), which is slightly better than the theoretical upper bound in Theorem 3.4.

![Figure 1](image1.png)

(a) Parameter estimation errors.  
(b) Cost suboptimality gap.  
(c) Algorithm regret.

Figure 1: Performance of Algorithm 1 for the LQ-RL problem \((m_0 = 4)\).

Robustness with respect to the initial batch size \(m_0\). We next demonstrate the robustness of Algorithm 1 by performing computations with \(m_0 = 1\) and fixing other settings as above. The results are shown in Figure 2. Note that the smaller initial batch size \(m_0\) makes the learning more challenging. By comparing the results against those with \(m_0 = 4\), one can see that our algorithm is robust and performs well with the small \(m_0\). In particular, we see that

- Estimating parameters with fewer sample trajectories leads to larger parameter estimation errors with suboptimality gaps, especially for the first few iterations. It also leads to a wider range of \((A_ℓ, B_ℓ)\) among different algorithm executions and hence a larger variance of the algorithm regret.

- As the number of episodes increases, the estimate \((A_ℓ, B_ℓ)\) converge to the true parameter \((A^*, B^*)\) and the suboptimality gap quickly converges to 0, see Figures 2(a) and 2(b). The algorithm regret grows sublinearly (see Figure 2(c)), and the regret after the \(N\)-th episode is of the magnitude \(O(N^{0.51})\). This confirms the theoretical results in Theorem 3.4 even for a small \(m_0\).

![Figure 2](image2.png)

(a) Parameter estimation errors.  
(b) Cost suboptimality gap.  
(c) Algorithm regret.

Figure 2: Performance of Algorithm 1 for the LQ-RL problem \((m_0 = 1)\).
Appendix A Preliminaries

Here, we collect some fundamental results which are used for our analysis.

We start with a stability result for coupled FBSDEs under a generalized monotonicity condition, which is crucial for our stability analysis of feedback controls. For any given $t \in [0,T]$ and $\lambda \in [0,1]$, we consider the following FBSDE defined on $[t,T]$; for $s \in [t,T]$,

\begin{align}
\text{d}X_s &= (\lambda \bar{b}(s,X_s,Y_s) + \mathcal{I}^b_s) \text{d}s + \sigma(s) \text{d}W_s + \int_{R^d} \gamma(s,u) \tilde{N}(ds,du), \quad X_t = \xi, \\
\text{d}Y_s &= -\lambda \bar{f}(s,X_s,Y_s) + \mathcal{I}^f_s) \text{d}t + Z_s \text{d}W_s + \int_{R^d} M_s \tilde{N}(ds,du), \quad Y_T = \lambda \bar{g}(X_T) + \mathcal{I}^g,
\end{align}

with given $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^n)$, $(\mathcal{I}^b, \mathcal{I}^f) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^n)$, $\mathcal{I}^g \in L^2(\mathcal{F}_T;\mathbb{R}^n)$ and measurable functions $\sigma : [0,T] \to \mathbb{R}^{n \times d}$, $\gamma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^n$, $\bar{b}, \bar{f} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{g} : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma A.1. Let $K \geq 0$, for each $i \in \{1,2\}$, let $\bar{b}_i, \bar{f}_i : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\bar{g}_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $\int_0^T \mathbb{E} [\bar{b}_i(t,0,0)^2 + \bar{f}_i(t,0,0)^2] \text{d}t < \infty$ and for all $t \in [0,T]$ that $\bar{f}_i(t, \cdot), \bar{g}_i$ are K-Lipschitz continuous, let $\bar{\sigma}_i : [0,T] \to \mathbb{R}^{n \times d}$ satisfy $\int_0^T \mathbb{E} [\bar{\sigma}_i(t)^2] \text{d}t < \infty$ and let $\bar{\gamma}_i : [0,T] \times \mathbb{R}^d \to \mathbb{R}^n$ satisfy $\int_0^T \mathbb{E} [\bar{\gamma}_i(t, u)^2] \nu(du)dt < \infty$. Assume further that there exists $\tau > 0$ and a measurable function $\eta : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ such that for all $t \in [0,T], (x,y), (x',y') \in \mathbb{R}^n$,

$$
\begin{align}
\langle \bar{b}_1(t,x,y) - \bar{b}_1(t,x',y'), y - y' \rangle + \langle -\bar{f}_1(t,x,y) + \bar{f}_1(t,x',y'), x - x' \rangle &\leq -\tau \eta(t,x,y,x',y'), \\
\langle \bar{b}_1(t,x,y) - \bar{b}_1(t,x',y'), y - y' \rangle &\leq K \langle x - x', x - x' \rangle,
\end{align}
$$

$$
\begin{align}
\langle \bar{g}(x) - \bar{g}(x'), x - x' \rangle &\geq 0.
\end{align}
$$

Then there exists $C > 0$, depending only on $T, K, \lambda$ and the dimensions, such that for all $t \in [0,T], \lambda \in [0,1], i \in \{1,2\}$, for every $(X_1, Y_1, Z_1, M_1) \in \mathcal{S}^2(t,T;\mathbb{R}^n)$ and $\mathcal{S}^2(t,T;\mathbb{R}^n) \times \mathcal{H}^2(t,T;\mathbb{R}^{n \times d}) \times \mathcal{H}^2(t,T;\mathbb{R}^n)$ satisfying (A.1) with $\lambda = \lambda_0, \bar{b}, \bar{f}, \bar{g} = (\bar{b}_1, \bar{f}_1, \bar{g}_1, \xi), \xi \in L^2(\mathcal{F}_t;\mathbb{R}^n)$, $\mathcal{I}^b = \mathcal{I}^f = \mathcal{I}^g$ in $\mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^n)$, we have that

$$
\begin{align}
\|X_1 - X_2\|_{\mathcal{S}^2}^2 &+ \|Y_1 - Y_2\|_{\mathcal{S}^2}^2 + \|Z_1 - Z_2\|_{\mathcal{S}^2}^2 + \|M_1 - M_2\|_{\mathcal{S}^2}^2 \\
&\leq C \left\{ \|\xi_1 - \xi_2\|_{\mathcal{S}^2}^2 + \|\lambda_0 (\bar{g}_1(X_1,T) - \bar{g}_2(X_2,T)) + \mathcal{I}^b_T - \mathcal{I}^f_T \|_{\mathcal{S}^2}^2 + \|\bar{\sigma}_1 - \bar{\sigma}_2\|_{\mathcal{H}^2}^2 + \|\bar{\gamma}_1 - \bar{\gamma}_2\|_{\mathcal{H}^2}^2 \\
&+ \|\lambda_0 (\bar{f}_1(X_1, Y_1, Z_1, M_1) - \bar{f}_2(X_2, Y_2, Z_2, M_2)) + \mathcal{I}^f_T - \mathcal{I}^f_T \|_{\mathcal{S}^2}^2 \right\}.
\end{align}
$$

Proof. Throughout this proof, let $C$ be a generic constant depending only on $T, K, \lambda$ and the dimensions, let $t \in [0,T], \lambda \in [0,1]$, let $\delta \sigma = \sigma_1 - \sigma_2, \delta \gamma = \gamma_1 - \gamma_2, \delta \mathcal{I}^b = T_1^b - T_2^b$, and for each $s \in [t,T]$ let $\delta \mathcal{I}^b_s = T_1^b - T_2^b, \delta \mathcal{I}^f_s = T_1^f - T_2^f, \delta \mathcal{I}^g_s = (\bar{b}_1(Y_1, s) - \bar{b}_2(Y_2, s)), \delta \mathcal{I}^f_s = (\bar{f}_1(X_1, s) - \bar{f}_2(X_2, s))$. Similarly, we introduce the notation $\delta \mathcal{I}^g_s$ for $s \in [t,T]$.

By applying Itô’s formula to $(Y_{1,s} - Y_{2,s}, X_{1,s} - X_{2,s})_{s \in [t,T]}$, we can obtain from (A.1) that

$$
\begin{align}
\mathbb{E}[(\mathcal{I}^b_T + \mathcal{I}^f_T + \mathcal{I}^g_T) - \langle \mathcal{I}^b_s, \mathcal{I}^f_s, \mathcal{I}^g_s \rangle] \\
= \mathbb{E} \left[ \int_t^T \left( \lambda_0 (\bar{g}_1(Y_1, s) - \bar{g}_2(Y_2, s)) + \mathcal{I}^b_s, \mathcal{I}^f_s, \mathcal{I}^g_s \right) ds \right] \\
\leq \mathbb{E} \left[ \int_t^T \left( -\lambda_0 \eta(s, X_1, Y_1, Z_1, M_1) + \lambda_0 (\bar{g}_1(Y_1, s) - \bar{g}_2(Y_2, s)) + \mathcal{I}^b_s, \mathcal{I}^f_s, \mathcal{I}^g_s \right) ds \right]
\end{align}
$$

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where for the last inequality, we have added and subtracted the terms \( \lambda_0 \tilde{b}_1(\Theta_{2,s}) \) and \(-\lambda_0 \tilde{f}_1(\Theta_{2,s}) \), and applied (A.2). Then, we can further deduce from (A.4) that

\[
\lambda_0 \tau \mathbb{E} \left[ \int_t^T \eta(s, X_{1,s}, Y_{1,s}, X_{2,s}, Y_{2,s}) \, ds \right] 
\leq -\mathbb{E} \left[ \lambda_0 (\tilde{g}_1(X_{2,T}) - \tilde{g}_2(X_{2,T})) + \delta \tilde{I}^q, \delta X_T - \langle \delta Y_t, \delta \xi \rangle \right] 
+ \mathbb{E} \left[ \int_t^T \left( \langle \lambda_0 (\tilde{b}_1(\Theta_{2,s}) - \tilde{b}_2(\Theta_{2,s})) + \delta \tilde{I}_s^b, \delta Y_s \rangle 
- \langle \lambda_0 (\tilde{f}_1(\Theta_{2,s}) - \tilde{f}_2(\Theta_{2,s})) + \delta \tilde{I}_s^f, \delta X_s \rangle \right) 
+ \langle \delta \sigma(s), \delta Z_s \rangle + \int_{\mathbb{R}_+^2} (\delta \gamma(s, u), \delta M_s) \nu(du) \right] \, ds,
\]

from which we can apply Young’s inequality and obtain for all \( \varepsilon > 0 \) that

\[
\lambda_0 \mathbb{E} \left[ \int_t^T \eta(s, X_{1,s}, Y_{1,s}, X_{2,s}, Y_{2,s}) \, ds \right] 
\leq \varepsilon \left( \| \delta X_T \|_{L^2}^2 + \| \delta Y_t \|_{L^2}^2 + \| \delta X \|_{H^2}^2 + \| \delta Y \|_{H^2}^2 + \| \delta Z \|_{H^2}^2 + \| \delta M \|_{H^2}^2 \right) + \text{RHS}/\varepsilon
\]

where RHS denotes the terms at the right-hand side of (A.5).

By (A.3) and a standard stability estimate of (A.1a), we can deduce that

\[
\| \delta X \|_{H^2}^2 \leq C \left( \lambda_0 \mathbb{E} \left[ \int_t^T \eta(s, X_{1,s}, Y_{1,s}, X_{2,s}, Y_{2,s}) \, ds \right] + \text{RHS} \right)
\leq \varepsilon C \left( \| \delta Y_t \|_{L^2}^2 + \| \delta Y \|_{H^2}^2 + \| \delta Z \|_{H^2}^2 + \| \delta M \|_{H^2}^2 \right) + \text{RHS}/\varepsilon
\]

for all small enough \( \varepsilon > 0 \). Moreover, by the Lipschitz continuity of \( \tilde{f}_1, \tilde{g}_1 \) and the stability estimate of (A.1b) (see e.g. [32, Proposition A4]), we have that

\[
\| \delta Y \|_{L^2}^2 + \| \delta Z \|_{H^2}^2 + \| \delta M \|_{H^2}^2 
\leq C \left( \| \lambda_0 (\tilde{g}_1(X_{1,T}) - \tilde{g}_2(X_{2,T})) + \delta \tilde{I}^q \|_{L^2}^2 + \| \lambda_0 (\tilde{f}_1(\cdot, X_1, Y_2) - \tilde{f}_2(\cdot, X_2, Y_2)) + \delta \tilde{I}^f \|_{H^2}^2 \right)
\leq C \left( \| \delta X \|_{L^2}^2 + \| \lambda_0 (\tilde{g}_1(X_{2,T}) - \tilde{g}_2(X_{2,T})) + \delta \tilde{I}^q \|_{L^2}^2 + \| \lambda_0 (\tilde{f}_1(\cdot, X_1, Y_2) - \tilde{f}_2(\cdot, X_2, Y_2)) + \delta \tilde{I}^f \|_{H^2}^2 \right)
\leq \text{CRHS},
\]

where we have applied (A.6) with a sufficiently small \( \varepsilon \) for the last inequality. This completes the desired stability estimate. \( \square \)

We then present a version of Burkholder’s inequality for the \( \| \cdot \|_{L^q} \)-norm of stochastic integrals, which not only extends [10, Corollary 2.2] to stochastic integrals with respect to general Poisson random measures on \([0, T] \times \mathbb{R}_+^p \), but also improves the bounding constants there with a sharper dependence on the index \( q \).

**Lemma A.2.** For all \( v \in \mathcal{H}^2(0, T; \mathbb{R}^d) \), \( w \in \mathcal{H}^2(0, T; \mathbb{R}) \) and \( q \geq 2 \), we have

\[
\mathbb{E} \left[ \left( \int_0^T v_t^T \, dW_t \right)^q \right] \leq C_q \mathbb{E} \left[ \left( \int_0^T |v_t|^2 \, dt \right)^{q/2} \right], 
\]

\[
\mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{R}_+^p} w(t, u) \hat{N}(dt, du) \right)^q \right] \leq \tilde{C}_q \left( \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}_+^p} |w(t, u)|^q \, \nu(du) \, dt \right] + \mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{R}_+^p} |w(t, u)|^2 \, \nu(du) \, dt \right)^{q/2} \right] \right),
\]

where \( C_q = (\sqrt{e}/2)^q \) and \( \tilde{C}_q = 21e^q q^{2q} \).
Hence, recursively applying the above estimate yields for all $q \geq 2$ and $n \in \mathbb{N}$ with $q/2^{n-1} \geq 2$ that

$$
\mathbb{E}\left[\left\|\int_0^T \int_{\mathbb{R}_0^p} w(t, u) \tilde{N}(dt, du)\right\|^q\right] \leq \left(\prod_{j=1}^{n} 2^{n-j} C_{\frac{q}{2^{n-j}}}\right) \mathbb{E}\left[\|(K^{(2^n)})_{T}|q/2^n\|\right] \\
+ \sum_{k=1}^{n} \left(\prod_{j=k}^{n} 2^{n-j} C_{\frac{q}{2^{n-j}}}\right) \mathbb{E}\left[\left\|\int_0^T \int_{\mathbb{R}_0^p} |w(t, u)|^{2^k} \nu(du)dt\right\|^{q/2^k}\right].
$$

(A.9)

Now let $q \geq 2$ be fixed and set $n = \lceil \log_2 q \rceil$ such that $q \in \{2^n, 2^{n+1}\}$. Since $q/2^n \in [1, 2)$, the constant $C_{q/2^n}$ in Burkholder’s inequality satisfies $C_{q/2^n} \leq 20$, from which we can show that (see [10, Lemma 2.1]):

$$
\mathbb{E}\left[\|(K^{(2^n)})_{T}|q/2^n\|\right] \leq 20 \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0^p} |w(t, u)|^q \nu(du)dt\right].
$$

Moreover, by proceeding along the lines of [10, Corollary 2.2], we obtain for all $k = 1, \ldots, n$ that

$$
\mathbb{E}\left[\left\|\int_0^T \int_{\mathbb{R}_0^p} |w(t, u)|^{2^k} \nu(du)dt\right\|^{q/2^k}\right] \\
\leq \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0^p} |w(t, u)|^q \nu(du)dt\right] + \mathbb{E}\left[\left\|\int_0^T \int_{\mathbb{R}_0^p} |w(t, u)|^{2^k} \nu(du)dt\right\|^{q/2^k}\right].
$$

Hence, we can deduce from (A.9) that

$$
\mathbb{E}\left[\left\|\int_0^T \int_{\mathbb{R}_0^p} w(t, u) \tilde{N}(dt, du)\right\|^q\right] \leq 21 \sum_{k=1}^{\lceil \log_2 q \rceil} \left(\prod_{j=1}^{k} 2^{\frac{q}{2^{j}}} C_{\frac{q}{2^{j}}}\right) \left(\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0^p} |w(t, u)|^q \nu(du)dt\right] + \mathbb{E}\left[\left\|\int_0^T \int_{\mathbb{R}_0^p} |w(t, u)|^{2^k} \nu(du)dt\right\|^{q/2^k}\right]\right).
$$

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We now obtain an upper bound of the constant $21 \sum_{k=1}^{[\log_2 q]} \left( \prod_{j=1}^{k} 2^{\frac{q}{2^{j-1}}} \frac{1}{C_{2^{j-1}}} \right)$ as follows:

$$
21 \sum_{k=1}^{[\log_2 q]} \left( \prod_{j=1}^{k} 2^{\frac{q}{2^{j-1}}} \frac{1}{C_{2^{j-1}}} \right) = 21 \sum_{k=1}^{[\log_2 q]} \prod_{j=1}^{k} 2^{\frac{q}{2^{j-1}}} \frac{1}{C_{2^{j-1}}} 
\leq 21 \left( \sum_{k=1}^{[\log_2 q]} 2^{-k} \right)^{\frac{q}{2}} \prod_{j=1}^{[\log_2 q]} \left( \frac{q}{2^{j-1}} \right)^{\frac{q}{2^{j-1}}} 
\leq 21 e^{2^{\sum_{j=1}^{[\log_2 q]} \frac{q}{2^{j-1}} \log_2 q}} \leq 21 e^{2^{\sum_{j=1}^{[\log_2 q]} \frac{q}{2^{j-1}} \log_2 q}} = 21 e^{2^{q^2 \log_2 q}} = 21 e^{q^2 q} := \tilde{C}_q,
$$

which leads us to the desired conclusion.

The following lemma estimates the tail behaviors of solutions to SDEs with jumps. The result has been established in Lemma 2.1 and Theorem 2.8 of [29] for SDEs with time homogenous coefficients and bounded Lipschitz continuous functions $f$ via Malliavin Calculus, which can be extended to SDEs with time inhomogeneous coefficients and unbounded $f$ (via Fatou’s lemma) in a straightforward manner.

**Lemma A.3.** Let $T \geq 0$ and $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^p \to \mathbb{R}^n$ be measurable functions such that there exist $K, \sigma_{\text{max}} \geq 0$ and a measurable function $\tilde{\gamma} : \mathbb{R}^n_+ \to \mathbb{R}$ satisfying for all $(t, u) \in [0, T] \times \mathbb{R}^n_+$, $x, x' \in \mathbb{R}^n$ that $b(t, 0) \leq K$, $|b(t, x) - b(t, x')| \leq K|x - x'|$, $|\sigma(t)| \leq \sigma_{\text{max}}$ and $\gamma(t, u) \leq \tilde{\gamma}(u)$, $\nu$-a.e.. Let $\beta : [0, \infty) \to [0, \infty]$ be defined by $\beta(\lambda) := \int_{\mathbb{R}^n_+} (e^{\lambda \tilde{\gamma}(u)} - \lambda \tilde{\gamma}(u) - 1) \nu(du)$ for any $\lambda \geq 0$. Assume that $\beta(\lambda) < \infty$ for some $\lambda > 0$.

Then there exists a constant $C > 0$, depending only on $K$ and $T$, such that for all $x \in \mathbb{R}^n$, the unique solution $X^x \in \mathcal{S}^2(\mathbb{R}^n)$ to the following SDE

$$
dX_t = b(t, X_t) \, dt + \sigma(t) \, dW_t + \int_{\mathbb{R}^n_+} \gamma(t, u) \, \tilde{N}(dt, du), \quad t \in [0, T], \quad X_0 = x,
$$
satisfies for every Lipschitz continuous function $f : (\mathbb{D}([0, T]; \mathbb{R}^n), d_\infty) \to \mathbb{R}$ that

$$
\mathbb{E} \left[ e^{\lambda(f(X^x) - \mathbb{E}(f(X^x)))} \right] \leq e^{C \beta(\lambda) \|f\|_{\text{Lip}}}, \quad \forall \lambda > 0, \quad (A.10)
$$

where $\mathbb{D}([0, T]; \mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued càdlàg functions on $[0, T]$, $d_\infty$ is the uniform metric defined by $d_\infty(p_1, p_2) := \sup_{t \in [0, T]} |p_1(t) - p_2(t)|$ for any $p_1, p_2 \in \mathbb{D}([0, T]; \mathbb{R}^n)$, $\|f\|_{\text{Lip}}$ is the constant defined by $\|f\|_{\text{Lip}} := \sup_{\lambda \neq \lambda'} \frac{\|f(p_1) - f(p_2)\|_\infty}{d_\infty(p_1, p_2)}$, and $\eta : [0, \infty) \to [0, \infty]$ is the function defined by $\eta(\lambda) := \beta(\lambda) + \sigma_{\text{max}}^2 \lambda^2/2$ for any $\lambda \geq 0$.

The next lemma presents a concentration inequality for the sum of independent sub-Weibull random variables, which follows directly from Theorem 3.1 and Proposition A3 in [26].

**Lemma A.4.** Let $\alpha \in (0, 1]$, $N \in \mathbb{N}$ and $X_1, \ldots, X_N \in \text{subW}(\alpha)$ be independent random variables satisfying $\mathbb{E}[X_i] = 0$ for all $i = 1, \ldots, N$. Then there exists a constant $C \geq 0$, depending only on $\alpha$, such that

$$
\mathbb{P} \left( \left| \sum_{i=1}^{N} X_i \right| \geq \varepsilon' \right) \leq 2 \exp \left( - C \min \left\{ \frac{(\varepsilon')^2}{\sum_{i=1}^{N} \|X_i\|^2_{\Psi_\alpha}}, \left( \frac{\varepsilon'}{\max_i \|X_i\|_{\Psi_\alpha}} \right)^\alpha \right\} \right), \quad \forall \varepsilon' \geq 0.
$$

**Appendix B** Proofs of Lemmas 2.8, 3.5, 3.6, 3.7, 3.8

**Proof of Lemma 2.8.** We start by establishing the regularity of $\mathbb{R}^n \times \mathbb{R}^k \ni (x, z) \mapsto f(t, x, \partial_z f^*(t, x, z)) \in \mathbb{R} \cup \{\infty\}$ for a given $t \in [0, T]$. Observe that for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $f(t, x, \cdot)$ is proper, convex, and lower semicontinuous, which along with the Fenchel-Young identity implies

$$
f(t, x, \partial_z f^*(t, x, z)) = \langle z, \partial_z f^*(t, x, z) \rangle - f^*(t, x, z) \in \mathbb{R}, \quad \forall (t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k. \quad (B.1)
$$
Given \( t [0, T] \) and \((x_1, z_1), (x_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^k \), by (B.1),
\[
|f(t,x_1, \partial_z f^*(t,x_1,z_1)) - f(t,x_2, \partial_z f^*(t,x_2,z_2))|
\leq |\langle z_1, \partial_z f^*(t,x_1,z_1) \rangle - \langle z_2, \partial_z f^*(t,x_2,z_2) \rangle| + |f^*(t,x_1,z_1) - f^*(t,x_2,z_1)|
+ |f^*(t,x_2,z_1) - f^*(t,x_2,z_2)|.
\]

We now estimate all the terms on the right hand side of the above inequality. By the Lipschitz continuity and local boundedness of \( \partial_z f^*(t, \cdot) \) (see the proof of Lemma 2.3), we can obtain the following upper bound for the first and third terms:
\[
|\langle z_1, \partial_z f^*(t,x_1,z_1) \rangle - \langle z_2, \partial_z f^*(t,x_2,z_2) \rangle| + |f^*(t,x_2,z_1) - f^*(t,x_2,z_2)|
\leq C(1 + |x_1| + |x_2| + |z_1| + |z_2|)(|x_1 - x_2| + |z_1 - z_2|),
\]

where the last inequality is by the mean value theorem. Moreover, by applying (B.1) to \( f^*(t,x_1,z_1) \) and by the definition of \( f^*(t,x_2,z_1) \) in (2.4), (H.1(3)), the linear growth of \( \partial_z f^*(t, \cdot) \),
\[
f^*(t,x_1,z_1) - f^*(t,x_2,z_1) \leq \langle z_1, \partial_z f^*(t,x_1,z_1) \rangle - \langle z_2, \partial_z f^*(t,x_2,z_1) \rangle
- \langle z_1, \partial_z f^*(t,x_1,z_1) \rangle - f(t,x_2, \partial_z f^*(t,x_1,z_1))
\leq f_0(t,x_1, \partial_z f^*(t,x_1,z_1)) + f_0(t,x_2, \partial_z f^*(t,x_1,z_1))
\leq C(1 + |x_1| + |x_2| + |z_1| + |z_2|)(|x_1 - x_2| + |z_1 - z_2|).
\]

Then, by interchanging the roles of \( x_1, x_2 \) in (B.3) and taking account of (B.2), we can obtain the following estimate for all \( t [0, T] \) and \( (x_1, z_1), (x_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^k \):
\[
|f(t,x_1, \partial_z f^*(t,x_1,z_1)) - f(t,x_2, \partial_z f^*(t,x_2,z_2))|
\leq C(1 + |x_1| + |x_2| + |z_1| + |z_2|)(|x_1 - x_2| + |z_1 - z_2|).
\]

Therefore, by (2.10), (2.17) and (2.19), for all \( t [0, T] \) and \( x, x' \in \mathbb{R}^n \),
\[
|f(t, x, \psi(t,x)) - f(t, x', \psi(t,x'))| = |f(t, x, \psi(t,x, Y^t_x)) - f(t, x', \psi(t,x', Y^t_{x'}))|
\leq C(1 + |x| + |x'| + \|b_2(t)\|^2 + \|\tilde{b}_2(t)\|^2)(|x - x'| + \|Y^t_x - Y^t_{x'}\|^2)
\leq C(1 + |x| + |x'| + \|Y^t_x\|^2 + \|Y^t_{x'}\|^2)
\leq C(1 + |x| + |x'|)(|x - x'| + \mathcal{E}_{\psi}(1 + |x|)),
\]

which along with Young’s inequality leads to the desired conclusion.

---

**Proof of Lemma 3.5.** It suffices to show the statement for processes \( X, Y \) such that \( \|X\|_{L^2(0,T)}, \|Y\|_{L^2(0,T)} \in \text{subW} \) with \( \|X\|_{L^2(0,T)} := (\int_0^T |X|^2 \, dt)^{1/2} \) and \( \|Y\|_{L^2(0,T)} := (\int_0^T |Y|^2 \, dt)^{1/2} \), as otherwise the right-hand side of the inequality would be infinity. Since \( \cdot \|_{\Psi_\alpha} \) is a quasi-norm for any \( \alpha > 0 \), we shall assume without loss of generality that \( \|X\|_{L^2(0,T)} \|_{\Psi_\alpha} = \|Y\|_{L^2(0,T)} \|_{\Psi_\alpha} = 1 \). Then, we can deduce from Hölder’s inequality and Young’s inequality that
\[
E\left[ \exp\left( \int_0^T XY \, dt \right)^{\frac{\alpha}{2}} \right] \leq E\left[ \exp\left( \|X\|_{L^2(0,T)} \|Y\|_{L^2(0,T)} \right)^{\frac{\alpha}{2}} \right]
\leq E\left[ \exp\left( \frac{1}{2} \|X\|_{L^2(0,T)}^2 + \frac{1}{2} \|Y\|_{L^2(0,T)}^2 \right) \exp\left( \frac{1}{2} \|X\|_{L^2(0,T)}^2 \right) \right]
\leq \left( E\left[ \exp\left( \|X\|_{L^2(0,T)}^2 \right) \right] \right)^{\frac{1}{2}} \left( E\left[ \exp\left( \|Y\|_{L^2(0,T)}^2 \right) \right] \right)^{\frac{1}{2}} \leq 2,
\]
which implies that \( \|\int_0^T XY \, dt\|_{\Psi_{\alpha/2}} \leq 1 \) and finishes the proof.
Proof of Lemma 3.6. Note that (3.9) and Hölder’s inequality suggest that it suffices to estimate the growth of \(|| \cdot ||_{L^q}\)-norms of the stochastic integrals for \(q \geq 2\). Hence, by (A.7), there exists a constant \(C\) such that for all \(q \geq 2\),

\[
q^{-2}\left\| \int_0^T X_t \sigma T \, dW_t \right\|_{L^q} \leq q^{-2} C q \left( \left\| \int_0^T |X_t|\, dt \right\|_{L^q} \right)^\frac{1}{2} \leq C|\sigma| \sup_{q \geq 1} \left( q^{-1} \left\| \int_0^T |X_t|\, dt \right\|_{L^q} \right)^\frac{1}{2},
\]

which along with (3.9) leads to the desired estimate for \(\| \int_0^T X_t \sigma T \, dW_t \|_{\Psi_1/2}\).

Similarly, by (A.8), there exists a constant \(C\) satisfying for all \(q \geq 2\) that

\[
\left\| \int_0^T \int_{\mathbb{R}^p} X_t \gamma(u) \, \tilde{N}(dt, du) \right\|_{L^q} \leq C q^2 \left\{ \left( \mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{R}^p} |X_t \gamma(u)|^q \, \nu(du) \, dt \right)^\frac{1}{q} \right] \right)^{\frac{1}{q}} + \left( \mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{R}^p} |X_t \gamma(u)|^q \, \nu(du) \right)^\frac{2}{q} \right] \right)^{\frac{1}{2}} \right\} \leq C q^2 \left\{ \left( \int_0^T \int_{\mathbb{R}^p} |\gamma|\, \nu(du) \left( \int_0^T |X_t|^q \, dt \right)^\frac{1}{q} \right) \, d\nu(du) \right\} + \left( \int_0^T \int_{\mathbb{R}^p} |\gamma|\, \nu(du) \left( \int_0^T |X_t|^2 \, dt \right)^\frac{1}{2} \right) \, d\nu(du) \right\}.
\]

Hence by (H.3(2)) and (3.9), for all \(q \geq 2\),

\[
q^{-3(3+\delta)}\left\| \int_0^T \int_{\mathbb{R}^p} X_t \gamma(u) \, \tilde{N}(dt, du) \right\|_{L^q} \leq C \left( \sup_{q \geq 2} q^{-\delta} \left( \int_{\mathbb{R}^p} |\gamma(u)|^q \, \nu(du) \right) \right) \left\{ q^{-1} \left\| \int_0^T |X_t|^q \, dt \right\|_{L^q} + q^{-1} \left\| \int_0^T |X_t|^2 \, dt \right\|_{L^q} \right\} \leq C \gamma_{\text{max}} \left\{ \left( \int_0^T |X_t|^q \, dt \right)^\frac{1}{q} \right\} + \left( \int_0^T |X_t|^2 \, dt \right)^\frac{1}{2} \right\}.
\]

Therefore, taking the supremum over \(q \geq 2\) in the above inequality leads to the desired estimate of \(\| \int_0^T \int_{\mathbb{R}^p} X_t \gamma(u) \, \tilde{N}(dt, du) \|_{\Psi_1/(3+\delta)}\) from (3.9).

\[\square\]

Proof of Lemma 3.7. Let us assume without loss of generality that \(|\sigma| > 0\) and \(\tau := \|(\int_0^T |X_t|^2 \, dt)^{1/2}\|_{\Psi_2} < \infty\), which implies that \(\|(\int_0^T 2|X_t|^2 \, dt)^{1/2}\|_{\Psi_2} \leq \sqrt{2}|\sigma|\tau\). Then, by the characterization of sub-Gaussian random variable in [42, Proposition 2.5.2(iii)], there exists \(C \geq 0\) such that

\[
\mathbb{E} \left[ \exp \left( 2\lambda^2 \int_0^T |X_t|^2 \, dt \right) \right] \leq \exp (2C^2 \lambda^2 |\sigma|^2 \tau^2) < \infty \quad \forall |\lambda| \leq \frac{1}{\sqrt{2C}|\sigma|\tau}.
\]

Hence, it holds for all \(|\lambda| \leq 1/(\sqrt{2C}|\sigma|\tau)\) that the process \((M_{\lambda, t})_{t \in [0, T]}\) defined by:

\[
M_{\lambda, t} := \exp \left( \int_0^t \left( 2\lambda X_s \sigma T - \frac{1}{2} \int_0^t 2\lambda^2 |X_s|^2 \, ds \right) \right) \quad \forall t \in [0, T]
\]

is a martingale, since Novikov’s condition is satisfied, which implies that \(\mathbb{E}[M_{\lambda, T}] = 1\). Thus, for any given
\[ |\lambda| \leq 1/(\sqrt{2C}|\sigma|\tau), \] by the Cauchy-Schwarz inequality,

\[
\mathbb{E} \left[ \exp \left( \lambda \int_0^T X_t \sigma^T \, dW_t \right) \right] \\
= \mathbb{E} \left[ \exp \left( \int_0^T \lambda X_t \sigma^T \, dW_t - \frac{(2\lambda)^2}{4} \int_0^T |X_t\sigma|^2 \, dt \right) \exp \left( \frac{(2\lambda)^2}{4} \int_0^T |X_t\sigma|^2 \, dt \right) \right] \\
\leq \mathbb{E}[M_{\lambda,T}]^{1/2} \mathbb{E} \left[ \exp \left( 2\lambda^2 \int_0^T |X_t\sigma|^2 \, dt \right) \right]^{1/2} \leq \exp(C^2\lambda^2|\sigma|^2\tau^2),
\]

which along with the fact that \( \mathbb{E}[\int_0^T X_t \sigma^T \, dW_t] = 0 \) and the characterization of sub-exponential random variable \cite[Proposition 2.7.1(v)]{42} yields that \( \| \int_0^T X_t \sigma^T \, dW_t \|_{\Psi_1} \leq C|\sigma|\tau. \)

**Proof of Lemma 3.8.** Throughout this proof, let \( x_0 \in \mathbb{R}^n \) and \( \theta \in \mathbb{R}^{n \times (n+k)} \) be given constants satisfying \( |\theta| \leq K \), and let \( C \) be a generic constant depend on \( K, T \) and the constants in (H.3), but independent of \( x_0 \) and \( \theta \).

By (3.4), we see that the process \( X^{x_0,\theta} \) satisfies the SDE:

\[
dX_t = b^\theta(t, X_t) \, dt + \sigma \, dW_t + \int_{\mathbb{R}_0^p} \gamma(u) \, \tilde{N}(dt, du), \quad t \in [0, T], \quad X_0 = x_0,
\]

where \( b^\theta(t, x) = A^*x + B^*\psi^\theta(t, x) \) for all \((t, x) \in [0, T] \times \mathbb{R}^n\). The definition of the feedback control \( \psi^\theta \), (H.3(1)) and Theorem 2.5 show that there exists \( C \geq 0 \) such that \( |\psi^\theta(t, 0)| \leq C \) and \( |\psi^\theta(t, x) - \psi^\theta(t, x')| \leq C|x - x'| \) for all \( t \in [0, T], \ x, x' \in \mathbb{R}^n \), which implies the same properties for the function \( b^\theta \). Then, by Lemma A.3, for every Lipschitz continuous function \( f : (\mathbb{D}([0, T]; \mathbb{R}^n), d_\infty) \to \mathbb{R} \), \( \mathbb{E} \left[ \exp \left( \lambda(\mathbb{E}[f(X^{x_0,\theta})] - \mathbb{E}[f(X^{x_0,\theta})]) \right) \right] \leq \exp(C\lambda|f||\mathbb{E}[f]|_{\text{Lip}}) \) for all \( \lambda > 0 \), with the function \( \eta : [0, \infty) \to [0, \infty] \) defined by:

\[
\eta(\lambda) := \int_{\mathbb{R}_0^p} \left( e^{\lambda \gamma(u)} - \lambda \gamma(u) - 1 \right) \nu(du) + \frac{\sigma^2}{2} \lambda^2 \forall \lambda > 0.
\]

By (H.3(2)) and Stirling’s approximation \( q! \geq (q/e)^q \) for all \( q \geq 2 \), we have for each \( \lambda \in [0, 1/(2\gamma_{\text{max}}e)) \),

\[
\int_{\mathbb{R}_0^p} \left( e^{\lambda \gamma(u)} - \lambda \gamma(u) - 1 \right) \nu(du) \\
= \int_{\mathbb{R}_0^p} \sum_{q=2}^\infty \frac{|\gamma(u)|^q}{q!} \nu(du) = \sum_{q=2}^\infty \frac{\lambda^q}{q!} \int_{\mathbb{R}_0^p} |\gamma(u)|^q \nu(du) \leq \sum_{q=2}^\infty \frac{\lambda^q}{q!} \gamma_{\text{max}}^q q^{\theta q} \\
\leq \sum_{q=2}^\infty \frac{(\lambda \gamma_{\text{max}})^q}{q^{1-\theta q}} \leq \frac{(\lambda \gamma_{\text{max}})^2}{1 - \lambda \gamma_{\text{max}}^2} \leq 2(\lambda \gamma_{\text{max}})^2,
\]

which implies for all \( 0 \leq \lambda \leq 1/C \) and \( f : (\mathbb{D}([0, T]; \mathbb{R}^n), d_\infty) \to \mathbb{R} \) satisfying \( ||f||_{\text{Lip}} \leq 1 \) that \( \mathbb{E} \left[ \exp \left( \lambda(\mathbb{E}[f(X^{x_0,\theta})] - \mathbb{E}[f(X^{x_0,\theta})]) \right) \right] \leq \exp(C^2\lambda^2) \). Replacing \( f \) with \( -f \) shows that the same estimate holds for all \( \lambda \leq 1/C \), which, along with the characterization of sub-exponential random variable in \cite[Proposition 2.7.1(v)]{42}, leads to \( ||f(X^{x_0,\theta}) - \mathbb{E}[f(X^{x_0,\theta})]||_{\Psi_1} \leq C \) for some constant \( C \), uniformly with respect to \( x_0 \in \mathbb{R}^n, \ |\theta| \leq K \) and \( f : (\mathbb{D}([0, T]; \mathbb{R}^n), d_\infty) \to \mathbb{R} \) satisfying \( ||f||_{\text{Lip}} \leq 1 \).

Since \( || \cdot ||_{\Psi_1} \) is a norm and \( ||\mathbb{E}[f(X)]||_{\Psi_1} \leq ||\mathbb{E}[f(X)]||/\ln 2, ||f(X^{x_0,\theta})||_{\Psi_1} \leq C(1 + ||\mathbb{E}[f(X^{x_0,\theta})]||) \) for all \( f \) with ||f||_{\text{Lip}} \leq 1. The estimate for a general Lipschitz continuous function \( f \) follows by considering \( ||f||_{\text{Lip}} \) and by using the fact that \( || \cdot ||_{\Psi_1} \) is a norm. \( \square \)

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