A NOTE ON THE REPRESENTABILITY OF A CERTAIN
HAMILTONIAN CAPACITY

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ABSTRACT. In this note we establish a representation property for a certain
Hamiltonian capacity on $\mathbb{R}^{2n}$ with the standard symplectic structure. We
demonstrate that the value of this capacity on an open set with a contact type
boundary is an element of the action spectrum of the boundary.

1. Introduction

Consider $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$.
Recall the definition of a symplectic capacity on $(\mathbb{R}^{2n} \cong \mathbb{C}^n, \omega_0)$.

Definition 1. A (normalized) symplectic capacity is a map which associates to a
given set $U \subset \mathbb{C}^n$ a number $c(U)$ with the following properties,

1) Monotonicity: If $U \subset V$ then $c(U) \leq c(V)$,
2) Symplectic invariance: $c(\phi(U)) = c(U)$, for any symplectomorphism $\phi$ of
$\mathbb{C}^n$, $\phi$,$n$
3) Homogeneity: $c(aU) = a^2 c(U)$ for any real number $a$.
4) Normalization: $c(B^{2n}(1)) = c(Z(1)) = \pi$, where $B^{2n}(1)$ is the unit ball in
$\mathbb{C}^n$, centered at the origin and $Z(1) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| < 1\}$

Notice that it is sufficient to define such a map $c$ with the above properties on
open and bounded subsets of $\mathbb{C}^n$, afterwards we can extend it to any open set as
follows,

$c(U) = \sup \{c(V) \mid V \text{ is bounded and connected and } V \subset U \}$
and to any subset by:

$c(E) = \inf \{c(U) \mid U \text{ is open and } E \subset U \}$

The notion “symplectic capacity” was introduced by Ekeland and Hofer, [6, 7],
although the first symplectic invariant, satisfying the axioms of Definition 1, was
presented by Gromov in [13]. Now there are many capacity functions derived from
various constructions in symplectic topology and for these we refer to the survey
paper, [4].

Consider a bounded domain $U \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ with smooth boundary $S = \partial U$.
The restriction of the symplectic form $\omega_0$ on $S$ gives rise to a line bundle over $S$,
$L_S = \ker \omega_0|_S \subset TS$ that is,

$L_S = \{(x, \xi) | \omega_0(\xi, \eta) = 0 \text{ for all } \eta \in T_x S\}$
The integral curves of the line bundle $L_S$, are called characteristics of $S$. Let $\alpha$ be
any primitive of $\omega_0$ on $\mathbb{C}^n$. If $\gamma$ is a closed characteristic on $S$ we define its action
\( A(\gamma) = \int_\gamma \alpha \) and we define the \textit{action spectrum} of \( S \) to be the set:
\[
\Sigma(S) = \{ k | A(\gamma) | k \in \mathbb{N}, \gamma \text{ is a closed characteristic of } S \}
\]

We point out that if \( S \) is a regular level surface of autonomous Hamiltonian function \( H = \text{const} \), then the Hamiltonian vector field \( X_H \) is a section of \( L \) and so in this case the closed characteristics of \( S \) are the periodic orbits of \( X_H \) on \( S \). The problem of the existence of closed characteristics on a hypersurface of \( \mathbb{R}^{2n} \) has been studied extensively and it depends on the symplectic properties of \( S \). Indeed, the results of M. Hermann, [18], and V. Ginzburg, [12], show that one should not expect the existence of closed characteristics on an arbitrary hypersurface. On the other hand, the validity of Weinstein’s conjecture in \( \mathbb{R}^{2n} \), proved by C. Viterbo, [25], guarantees existence of closed characteristics provided that the hypersurface \( S \) is of contact type. We recall that a hypersurface \( S \) in \( \mathbb{C}^n \) is called \textit{contact}, if there is a Liouville vector field \( X \), i.e. \( L_X \omega_0 = \omega_0 \), which is defined in a neighborhood of \( S \) and is transversal to \( S \). The hypersurface \( S \) is called of \textit{restricted contact type} if the vector field \( X \) is globally defined on \( \mathbb{C}^n \). We remark that S. Bates, [2], found examples of hypersurfaces which are of contact type, but not of restricted contact type.

There are several interesting questions to be answered regarding the capacities and one of them concerns their \textit{representability} that is when \( c(U) \in \Sigma(\partial U) \)? There are several known results in this direction. Most of them concern open sets with restricted contact type (RCT) boundary. It is known that Ekeland-Hofer capacities, [6, 7], Hofer-Zehnder capacity, [19], Viterbo capacity, [27], the Floer-Hofer capacity and etc. enjoy this property for RCT open sets. In general the question of representability of the capacities is a delicate one. D. Hermann showed in [17], that not all symplectic capacities are representable even if \( U \) is a RCT open set. In a recent paper, [1], the main result implies the existence of dynamically convex star-shaped levels in \( \mathbb{R}^4 \) where the Gromov width is not representable as the action of a closed characteristic. On the other hand all of the above mentioned representable capacities fall in the class of Hamiltonian capacities in D. Hermann’s terminology, [17]. In the same reference, [17], Question 1.2.2, he poses the question if all Hamiltonian capacities are representable on open sets with contact type boundary?

In this note we give positive answer to this question for the Floer-Hofer capacity in the terminology [15, 16, 5]. It falls in the class of Hamiltonian capacities. Our main theorem is:

**Theorem 1.** Let \( c \) be the Floer-Hofer capacity and \( U \) be a bounded domain in \( \mathbb{R}^{2n} \) with contact type boundary. Then \( c(U) \in \Sigma(\partial U) \).

We mention that there are similar representation theorems concerning the Viterbo capacity, and Ekeland-Hofer capacity, for particular open sets with contact type boundary, e.g. \( U \) is a tubular neighborhood of Lagrangian torus or a tubular neighborhood of hyperbolic Lagrangian submanifold, cf. [26, 2, 23].

As a consequence of the above theorem, we can strengthen a little bit Theorem 1.3 in view of Remark 1.4 in [5]. We have,

**Corollary 1.** Let \( S \) be a compact hypersurface in \( (\mathbb{R}^{2n}, \omega_0) \) of contact type. Let \( \phi \) be the time-1 map of a compactly supported Hamiltonian \( H \) on \( [0, 1] \times \mathbb{R}^{2n} \) such that \( E(\phi) \leq c(S) \). Then there exists \( x \in S \) such that \( \phi(x) \in L_x S \).
Next, we provide a brief description of the strategy of the proof of Theorem 1. It exploits ideas similar to the ones used in [25, 20, 6]. We thicken the boundary $S = \partial U$, i.e., using the contact definition we foliate a neighborhood of $S$, into diffeomorphic images of $S$, $\{S_\delta\}$, $\delta \in (0, \epsilon)$ for small $\epsilon > 0$ and $S = S_0$. Then we find a sequence of Hamiltonians $H_{\delta_k} \in \mathcal{H}_{ad}(U)$, (defined in the next section), and a sequence of critical points $\{x_{\delta_k}\}$ of $A_{H_{\delta_k}}$, with the property that each $x_{\delta_k}$ is a closed characteristic of $S_{\delta_k}$ and

\[ \sigma(H_{\delta_k}) = A_{H_{\delta_k}}(x_{\delta_k}) \]

where $\sigma$ is an action selector coming from certain type of Morse homology for the Hamiltonian action functional $A_H$, (see Section 2). Then under certain assumptions we would have that

\[ c(U) = \lim_{k \to \infty} \sigma(H_{\delta_k}) \]

Now, if the lengths of the closed characteristics $x_{\delta_k}$ are uniformly bounded, we could find a subsequence which of $\{x_{\delta_k}\}$ converging to $x_0$ - a closed characteristic of $S = S_0$. Assuming that $\lim_{k \to \infty} H_{\delta_k}(x_{\delta_k}) = 0$, we would get using, (1), that

\[ c(U) = A(x_0) \in \Sigma(S) \]

This way, the crucial point is to establish the uniform boundedness of the lengths of the closed characteristics $\{x_{\delta_k}\}$. In case $U$ is RCT open this is a consequence of the global definition of the Liouville vector field or equivalently, cf. [19], the existence of a 1-form $\alpha_0$ on $\mathbb{C}^n$ so that $\omega_0 = \alpha_0$ on $\mathbb{C}^n$ and $\alpha_0 \wedge (d\alpha_0)^{n-1}$ is a volume form on $S$. In case $U$ is with contact type boundary we establish the boundedness by utilizing an elegant idea of Ph. Bolle, [3], which was successfully applied by the author in [5]. Namely using the fact that each $x_{\delta_k}$ is a deformation of a constant solution for $H_{\delta_k}$, an application of Stokes’ theorem implies the desired boundedness.

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2. The Floer-Hofer capacity.

The Hamiltonian capacity we are concerned here may be obtained by the so-called action selector method, cf. [19, 11]. We describe it briefly. The idea is to consider the set of admissible Hamiltonians for a given bounded domain $U$. In general these are functions which are bounded from below, (or above depending on the convention), on $U$, (usually by 0), which are allowed to grow rapidly near $\partial U$ and have certain growth rate at infinity. We denote this set by $\mathcal{H}_{ad}(U)$. For $H \in \mathcal{H}_{ad}(U)$, we consider the Hamiltonian action functional $A_H(x)$, $x \in C^\infty(S^1, \mathbb{R}^{2n})$,

\[ A_H(x) = \int_{S^1} x^* \lambda_0 - \int_0^1 H(t, x(t))dt \]

where $\lambda_0$ is some primitive of $\omega_0$. Then one “selects”, $\sigma(H)$ - a critical value of $A_H(x_0)$ where $x_0$ is some “topologically visible” periodic orbit of the Hamiltonian vector field $X_H$ and the capacity of $U$ is defined by,

\[ c(U) = \inf\{\sigma(H)|H \in \mathcal{H}_{ad}(U)\} \]

We would like to outline how the action selector method we just described, may be used to construct the Floer-Hofer capacity. For the sake of brevity we will just
present the important steps in the construction, to fix the notation. We omit some of the details for which, however, we present references. The Floer-Hofer capacity is based on the computations of the symplectic homology, [9] for balls in [10].

Define the set of admissible Hamiltonians, \( \mathcal{H}_{\text{ad}}(U) \) for an open bounded set \( U \subset \mathbb{C}^n \),

\[
\mathcal{H}_{\text{ad}}(U) = \{ H : S^1 \times \mathbb{C}^n \to \mathbb{R} | H(t, z) \leq 0 \text{ on } U \text{ and } H(t, z) = \mu|z|^2 \text{ outside of a compact set} \}
\]

The Floer homology, [8] [21], can be thought as an infinite-dimensional version of Morse theory for the Hamiltonian action functional, (4). Choose a compatible almost complex structure \( J \) on \( \mathbb{C}^n \), that is an almost complex structure, \( J^2 = -1 \), so that \( \omega_0(\cdot, J \cdot) = g_J(\cdot, \cdot) \) is a Riemannian metric on \( \mathbb{C}^n \). Then the gradient lines of (4), satisfy Cauchy-Riemann type PDE,

\[
\frac{\partial u}{\partial s} + J(t, u) \frac{\partial u}{\partial t} + \nabla H(t, u) = 0 \quad \text{for } u \in C^\infty(\mathbb{R} \times S^1, \mathbb{C}^n)
\]

Given two critical points \( x_+, x_- \) of \( A_H \), or equivalently periodic orbits for the Hamilton equations \( \dot{x} = X_H(x) \); \( x(0) = x(1) \), consider the set of solutions, \( \mathcal{M}(x_-, x_+, J, H) \), of (7), such that, \( \lim_{s \to \pm\infty} u(s, t) = x_\pm(t) \). An element of \( \mathcal{M}(x_-, x_+, J, H) \) is called a Floer trajectory. The difference of the actions between the ends of a Floer trajectory, \( u \), is given by the energy, \( E_J(u) \).

\[
A_H(x_-) - A_H(x_+) = \int_{\mathbb{R} \times S^1} g_J(J_\nu, \frac{\partial u}{\partial s}) ds dt \equiv E_J(u) \geq 0
\]

One can assign to each critical point, \( x \) of \( A_H \), under some non-degeneracy assumptions, an index, \( \mu_{CZ}(x) \), called the Conley-Zehnder index, cf. [22]. Studying the combinatorics of the solutions of (6), yields the Floer homology groups, \( HF_{k}^{(a, b)}(H, J) \), which are independent of \( J \) for generic choice of the almost complex structure. These groups consist of linear combinations with \( \mathbb{Z}_2 \) coefficients of critical points, \( x \) of \( A_H \), with \( \mu_{CZ}(x) = k, a \leq A_H(x) < b \). Now using the assumptions on \( \mathcal{H}_{\text{ad}}(U) \) and the functoriality of the Floer homology, one defines, following [9], the Floer homology groups \( HF_k^{(a, b)}(H, J) \) for \( H \in \mathcal{H}_{\text{ad}}(U) \). Then the symplectic homology groups \( S_k^{(a, b)}(U) \) are defined as direct limits of the Floer homology groups. In other words for an exhausting (cofinal) family, \( (H_\lambda, J_\lambda) \), for \( \mathcal{H}_{\text{ad}}(U) \), which is 1-parameter family such that if \( K \in \mathcal{H}_{\text{ad}}(U) \), there exists \( \lambda' \) so that \( H_\lambda \geq K \), whenever \( \lambda > \lambda' \). Then,

\[
S_k^{(a, b)}(U) = \lim_{\lambda \to \infty} HF_k^{(a, b)}(H_\lambda, J_\lambda)
\]

Recall from [10],

**Lemma 1.** The symplectic homology groups of an open ball of radius \( R \), \( B_R = B^{2n}(R) \subset \mathbb{C}^n \), satisfy

\[
S_k^{(a, b)}(B^{2n}(R)) = \mathbb{Z}_2 \text{ for } 0 < a < b \leq \pi R^2, \text{ and } 0 \text{ otherwise.}
\]

\[
S_k^{(a, b)}(B^{2n}(R)) = \mathbb{Z}_2 \text{ for } 0 < a \leq \pi R^2 < b, \text{ and } 0 \text{ otherwise.}
\]

\[
S_k^{(a, b)}(B^{2n}(R)) = 0 \text{ for } k < n \text{ or } n < k < 3n
\]
Let $U$ be an open and bounded subset of $\mathbb{C}^n$. Without loss of generality we may assume that the origin, $0 \in U$. Let $r > 0$ be a number such that $B^{2n}(r) \subset U$. Pick numbers $\varepsilon > 0$ such that $\varepsilon < \pi r^2$ and a number $b > \pi r^2$. Observe that for large $b$, the natural map,

$$Z_2 = S_n^{(0,\varepsilon)}(B^{2n}(\rho)) \to S_n^{(0,b)}(B^{2n}(\rho))$$

vanishes, (see [24]). Let $R$ be sufficiently large so that $B_r = B^{2n}(r) \subset U \subset B^{2n}(R) = B_R$, then we have

$$Z_2 = S_n^{(0,\varepsilon)}(B_R) \xrightarrow{i_R} S_n^{(0,\varepsilon)}(U) \xrightarrow{i_r} S_n^{(0,\varepsilon)}(B_r) = Z_2$$

Where $i_r, i_R$ are the inclusion morphisms, cf. [15,16]. Since the composition $i_R \circ i_r$ is an isomorphism, it follows that $0 \neq \alpha_U = i_R(1) \in S_n^{(0,\varepsilon)}(U)$. One then considers the natural map

$$i_U^b : S_n^{(0,\varepsilon)}(U) \to S_n^{(0,b)}(U)$$

and the Floer-Hofer (homological) capacity is defined as

$$c(U) = \inf \{ b | i_U^b(\alpha_U) = 0 \}$$

Next we present an alternative definition of the Floer-Hofer capacity utilizing the action selector method. We follow [16], Section 3.2. Let $x_0 \in U$ and $B_r(x_0) \subset U$, where $B_r(x_0)$ is the open ball centered at $x_0$ with radius $r$. Denote by $\mathcal{H}_{ad}^r(U)$ the subset of $\mathcal{H}_{ad}(U)$, consisting of functions $H$ with unique minimum at $x_0$ so that $0 > \min H = H(x_0) > -\pi r^2$. Now given function $H \in \mathcal{H}_{ad}^r(U)$, we consider a function $K \in \mathcal{H}_{ad}(B_r(x_0))$, so that $\min K = K(x_0) = \delta < \pi r^2$ and $H \geq K$ on $S^1 \times \mathbb{C}^n$. For $\delta < \varepsilon < \pi r^2$, we have that

$$HF_n^{(0,\varepsilon)}(K) \simeq \mathbb{Z}_2$$

and the generator is $x_0$. Consider the monotonicity morphism, [9],

$$m : HF_n^{(0,\varepsilon)}(K) \to HF_n^{(0,\varepsilon)}(H)$$

and denote by $\alpha_H = m(x_0)$. Consider the natural inclusion map,

$$i^b : HF_n^{(0,\varepsilon)}(H) \to HF_n^{(0,b)}(H)$$

and define the number

$$\sigma(H) = \inf \{ b | i^b(\alpha_H) = 0 \}$$

It is proven in [16], Proposition 3.3 that $\sigma(H)$ is a positive critical value of $A_H$.

**Remark 1.** We remark that if $H \in \mathcal{H}_{ad}^{x_0,r}(U)$ as above, it follows from the definition of Floer homology groups that for some compatible almost complex structure $J$ there exists a Floer trajectory for $(H,J)$, connecting $x_0$ and $x(t)$ where $x_t$ is a 1-periodic orbit of $X_H$, so that $HF_{n+1}^z(x) = n+1$ and $\sigma(H) = A_H(x)$.

The action selector $\sigma$, defined above can be extended for all $H \in \mathcal{H}_{ad}(U)$ and the Floer-Hofer capacity is defined as

$$c(U) = \inf \{ \sigma(H) | H \in \mathcal{H}_{ad}(U) \}$$

We refer to [16], for the details.
3. Proof of Theorem 1

We recall the following lemma from [3], Lemma 1. Here we will use the version for $\mathbb{R}^{2n}$ and $p = 1$.

**Lemma 2.** Let $U$ be a bounded domain in $\mathbb{R}^{2n}$ with contact type boundary $S = \partial U$. There exists $\epsilon > 0$, an open neighborhood $V$ of $S$ in $\mathbb{R}^{2n}$ and a diffeomorphism $\psi : S \times (-\epsilon, \epsilon) \to V$ so that:

- For all $x \in S$ we have $\psi(x, 0) = x$;
- $\psi^* \omega = (1 + t)q^*(\omega_0|_S) + dt \wedge q^*(\alpha_0)$;

where $\alpha_0$ is a 1-form on $S$, such that, $d\alpha_0 = \omega_0$ and $\alpha_0 \wedge \omega_0^{n-1} \neq 0$ on $S$, $q : S \times (-\epsilon, \epsilon) \to S$ is the projection and $t$ is a coordinate on $(-\epsilon, \epsilon)$.

Set

$$r = q \circ \psi^{-1} : V \to S$$

$$\beta_0 = r^*\alpha_0$$

$$\Omega = r^*(\omega_0|_S)$$

$$z = t \circ \psi^{-1}$$

then

$$\omega_0 = (1 + z)\Omega + dz \wedge \beta_0$$

Denote the Hamiltonian vector field corresponding to $z$, by $X_z$, then we get

$$\beta_0(X_z) = 1$$

$$\Omega(X_z, \cdot) = 0$$

Let $\epsilon > 0$ be the number given by Lemma 2 we may assume in addition that $1 > \epsilon > 0$. Fix $\epsilon'$ such that $0 < \tau \leq \epsilon$, denote by

$$V_\tau = \psi(N \times (-\tau, \tau)) = \{x \in V|\psi^2(x) < \tau^2\}$$

Consider the 1-form $B_0$ defined on $\mathbb{C}^n$ by $B_0 = f \beta_0$, where $f$ is a smooth function on $\mathbb{C}^n$ such that $f = 1$ on $V'$ and $f = 0$ on $\mathbb{R}^{2n} \setminus V'$. This way we get one-form defined on $\mathbb{R}^{2n}$ such that

$$B_0 = \beta_0 \text{ on } V'$$

(11)

and

$$B_0 = 0 \text{ on } \mathbb{C}^n \setminus V$$

(12)

Our next task is to build exhausting family for the set $U$. Before we proceed we need, the following proposition.

**Proposition 1.** The action spectrum $\Sigma(S)$ of a hypersurface $S$ of contact type is a nowhere dense set.

This proposition appears in many papers as a statement without a proof. B. Gürel has carried out the details of the proof and I thank her and V. Ginzburg for emailing me detailed outline of the proof, [14].

Consider, for sufficiently large $\lambda \notin \Sigma(S)$, smooth functions $g$ and $h$ on $\mathbb{R}^+$ so that.

- $h'(t) = \lambda$ for $t \in [3\lambda^{-1}, 2\lambda^{-1} + \lambda^{-1/2}]$,
- $h(t) = -\lambda^{-1}$ for $t \in [0, \lambda^{-1}]$,
- $h(t) = -\lambda^{-1} + \lambda^{1/2}$ for $t \geq 3\lambda^{-1} + \lambda^{-1/2}$,
- $h$ is convex on $[\lambda^{-1}, 3\lambda^{-1}]$ and concave on $[2\lambda^{-1} + \lambda^{-1/2}, 3\lambda^{-1} + \lambda^{-1/2}]$,
The perturbation in such a way, so that each $x$ derivatives by $J$ slightly, if necessary, to assure that $h'(t) \notin \Sigma(S_t)$, for $t \in [3\lambda^{-1}, 2\lambda^{-1} + \lambda^{-1/2}]$. Now define $H_\lambda$ as follows.

- $H_\lambda(x) = -\lambda^{-1}$ for $x \in U$,
- $H_\lambda(x) = h(z(x))$ for $x \in \bigcup_{\nu < 3\lambda^{-1} + \lambda^{-1/2}} S_\nu$,
- $H_\lambda(x) = g(|x|^2)$ for $|x| > \lambda^{1/6}$,
- $H_\lambda(x) = -\lambda^{-1} + \lambda^{1/2}$ for $x \in B^{2n}(\lambda^{1/6}) \setminus \{U \bigcup_{\nu < 3\lambda^{-1} + \lambda^{-1/2}} S_\nu\}$.

It is clear that $\{H_\lambda\}$ is an exhausting family for $U$. Next perturb each $H_\lambda(x)$, for $x \in U \bigcup_{\tau \in [0, \lambda^{-1}]} S_\tau$, as in [13, 15], to get non-degenerate family with unique prescribed global minimum at the origin $0 \in \mathbb{C} \setminus \{\lambda \}$, which we assumed to be in $U$. Abusing the notation we call the new family again $H_\lambda$. We can arrange the perturbation in such a way, so that each $H_\lambda$, is bounded together with its derivatives by $\lambda^{-1}$ on $\bar{U}$. In particular we choose a constant $C_3 > 0$ so that for $x \in \bar{U}$ we have,

$$|X_{H_\lambda}(x)| \leq C_3 \lambda^{-1}$$

Next couple each $H_\lambda$ with a compatible non-degenerate almost complex structure $J_\lambda$.

Standard arguments as in [20, 15], show that for sufficiently large, but fixed $\lambda$, $0 < \sigma(H_\lambda) = A_{H_\lambda}(x_{\tau(\lambda)})$, where $x_{\tau(\lambda)} \in S_{\tau(\lambda)}$ and $0 \leq \tau(\lambda) < 3\lambda^{-1} + \lambda^{-1/2}$. Moreover $x_{\tau(\lambda)}$ solves an equation of the form, $\dot{x} = \rho X_x(x)$, where $\rho = h'(\tau(\lambda))$. Our goal is to show that the length of $x_{\tau(\lambda)}$ is bounded independently of $\lambda$.

We know from Remark 1 that there is a Floer trajectory for $H_\lambda, J_\lambda$, $u$ s.t. $\lim_{s \to -\infty} u = 0$ and $\lim_{s \to \infty} u = x_{\tau(\lambda)}(t)$. Then,

$$(13) \quad A_{H_\lambda}(x_{\tau(\lambda)}) - A_{H_\lambda}(0) = E_{J_\lambda}(u) = \int_{\mathbb{R} \times S^1} \omega_0(u_s, u_t - X_{H_\lambda}(u)) ds dt$$

Consider now holomorphic change of the variables $\sigma : \dot{D} \to \mathbb{R} \times S^1$, where $\dot{D} = D \setminus \{0\}$, and $D$ is the unit disk in $\mathbb{C}$. Set $v(s, t) = u(\sigma(s, t))$, here, by abusing the notation we consider $s + it$ to be the holomorphic coordinate on $D$. Then we have that $v$, satisfies,

$$(14) \quad \frac{\partial v}{\partial s} + J_\lambda(t, v) \frac{\partial v}{\partial t} + \nabla H_\lambda(v) = 0$$

with $\lim_{s \to 0} v(s, t) = 0$ and $\lim_{s \to 1} v(s, t) = x_{\tau(\lambda)}(t)$. Moreover we have that,

$$E_{J_\lambda}(u) = E_{J_\lambda}(v) = \frac{1}{2} \int_D (|v_s|^2_{g_{J_\lambda}} + |v_t - X_{H_\lambda}(v)|^2_{g_{J_\lambda}}) ds dt$$

Write $\int_D = \int_{\overline{D}}^{I} + \int_{\overline{D}}^{I I} + \int_{\overline{D}}^{I I I}$, where $\int_{\overline{D}}^{I}$, means that the integration is taken over $(s, t)$, for which $v(s, t) \in U$, in $\int_{\overline{D}}^{I I}$, integration is over $(s, t)$ for which $v(s, t) \in \overline{U \bigcup V_s \setminus \bar{U}}$, and in $\int_{\overline{D}}^{I I I}$, the integration is over the remaining values of $(s, t) \in \bar{D}$.

Consider the space of all almost complex structures $J$ on $\mathbb{C}^n$, compatible with $\omega_0$. Denote, as before, by $g_J$ the corresponding metric, i.e., $g_J(\cdot, \cdot) = \omega_0(\cdot, J \cdot)$.
Since the set $\overline{U \cup V_\varepsilon}$ is a compact subset of $\mathbb{C}^n$, there is a constant $C_2 > 0$ so that on $\overline{U \cup V_\varepsilon}$ we have that,

$$|\xi|_{g_{JJ}} \geq \sqrt{C_2} |\xi|_{g_{J0}} = \sqrt{C_2} |\xi|$$

for any $\xi \in \mathbb{C}^n$. Here $J_0 = i$ is the standard complex structure on $\mathbb{C}^n$. Let $C_1 > 0$ be a constant such that,

$$C_1 |\xi||\eta| \geq |dB_0(\xi, \eta)|$$

for all $\xi, \eta \in \mathbb{C}^n$. Observe that we can extend $X_{H_{\lambda}}(v)$ smoothly over the puncture $s = 0$ by setting it equal to 0. We have then,

$$\frac{1}{2} \int_D (|v_s|^2 + |v_t - X_{H_{\lambda}}(v)|^2)dsdt \geq \frac{C_2}{2} \int_D (|v_s|^2 + |v_t - X_{H_{\lambda}}(v)|^2)dsdt$$

$$\geq \frac{C_2}{\sqrt{2}} \int_D |v_s|^2 + |v_t|^2/2 - |X_{H_{\lambda}}(v)|^2)dsdt$$

$$\geq C_2/\sqrt{2} \int_D |v_s| |v_t| dsdt - C_2/2 \int_D |X_{H_{\lambda}}(v)|^2 dsdt$$

$$\geq C_2/(\sqrt{2}C_1) \int_D |dB_0(v_s, v_t)| dsdt - C_2/2 \int_D C_2 \lambda^2 dsdt$$

$$\geq C_2 \int_D |dB_0(v_s, v_t)| dsdt - \pi C_2 C_3 \lambda^2/2$$

On the other hand,

$$\frac{1}{2} \int_D (|v_s|^2 + |v_t - X_{H_{\lambda}}(v)|^2)dsdt \geq \frac{C_2}{2} \int_D (|v_s|^2 + |v_t - X_{H_{\lambda}}(v)|^2)dsdt$$

$$\geq \frac{C_2}{C_1} \int_D |dB_0(v_s, v_t - X_{H_{\lambda}}(v)| dsdt = \frac{C_2}{C_1} \int_D |dB_0(v_s, v_t)| dsdt$$

The last inequality follows from the fact that $dB_0(X_{H_{\lambda}}(v), \cdot) = 0$, when $v \in U \cup V_\varepsilon$. Observe that $\int_D |dB_0(v_s, v_t)| dsdt = 0$. In view of the inequalities above we conclude that,

$$E_{J_\lambda}(v) \geq \frac{C_2}{\sqrt{2}C_1} \int_D |dB_0(v_s, v_t)| dsdt - \pi C_2 C_3 \lambda^2/2$$

Further, applying Stokes’ theorem, we get,

$$\int_D |dB_0(v_s, v_t)| dsdt \geq |\int_D dB_0(v_s, v_t)| dsdt = |\int_{S^1} x^*(\tau(\lambda)) \beta_0| = |h'(\tau(\lambda))|$$

And we obtain that

$$\sigma(H_{\lambda}) - H_{\lambda}(0) = A_{H_{\lambda}}(x^*(\lambda)) - A_{H_{\lambda}}(0) = E_{J_\lambda}(u) \geq \frac{C_2}{\sqrt{2}C_1} |h'(\tau(\lambda))| - \pi C_2 C_3 \lambda^2/2$$

and this shows,

$$|\rho| = |h'(\tau(\lambda))| \leq \frac{\sqrt{2}C_1}{C_2} (\sigma(H_{\lambda}) - H_{\lambda}(0) + \pi C_2 C_3 \lambda^2/2)$$

Since $\sigma(H_{\lambda})$ is bounded independently of $\lambda$, (it is less than the capacity of some large ball containing $U \cup V_\varepsilon$), we conclude that $|\rho|$ is bounded independently of $\lambda$, and therefore so is the length of $x^*(\lambda)$. Applying Arzela - Ascoli theorem, we can
find a sequence \( \{\lambda_k \to \infty\} \), s.t., \( \lim_{k \to \infty} \tau(\lambda_k) = 0 \), \( c(U) = \lim_{k \to \infty} \sigma(H_{\lambda_k}) \), and \( x_0 = \lim_{k \to \infty} x_{\tau(\lambda_k)} \) is a closed characteristic on \( S_0 = S \). Moreover

\[
c(U) = \lim_{k \to \infty} \left( \int_{S^1} x_{\tau(\lambda_k)}^* \lambda_0 - \int_0^1 H_{\lambda_k}(x_{\tau(\lambda_k)}) \, dt \right) = \int_{S^1} x_0^* \lambda_0 \in \Sigma(S)
\]

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