Work distribution in a time-dependent logarithmic–harmonic potential: exact results and asymptotic analysis

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Abstract

We investigate the distribution of work performed on a Brownian particle in a time-dependent asymmetric potential well. The potential has a harmonic component with a time-dependent force constant and a time-independent logarithmic barrier at the origin. For an arbitrary driving protocol, the problem of solving the Fokker–Planck equation for the joint probability density of work and particle position is reduced to the solution of the Riccati differential equation. For a particular choice of the driving protocol, an exact solution of the Riccati equation is presented. An asymptotic analysis of the resulting expression yields the tail behavior of the work distribution for small and large work values. In the limit of a vanishing logarithmic barrier, the work distribution for the breathing parabola model is obtained.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Stochastic thermodynamics is an advancing field with many applications to small systems of current interest [1–3]. Work performed on a small system by an external driving becomes a stochastic variable because of the strong influence of fluctuations mediated by the environment. In a standard setting, a particle moves in a thermal environment and experiences a time-dependent external force. The particle position by itself is a stochastic process, say \( X(t) \). Any single trajectory of the particle in a time interval \([0, t]\) yields a single value of the work \( W(t) \) performed on the particle. The work \( W(t) \) thus is a functional of the position process \( X(t') \), \( 0 \leq t' \leq t \), and is distributed with a probability density function (PDF) \( p(w; t) \). The
probability $p(w; t) dw$ that the work $W(t)$ falls into an infinitesimal interval $(w, w + dw)$ equals the probabilistic weight of all trajectories giving work values in that interval. An important aspect of stochastic thermodynamics is that relations between work distributions for forward and reverse driving protocols and averages over functions of the work allow one to determine thermodynamic potential differences between equilibrium states of small systems. Perhaps the most widely known is the Jarzynski equality [4], which relates the free-energy difference between two equilibrium configurations of a system to the average

$$
\langle e^{-\beta W(t)} \rangle = \int_{-\infty}^{+\infty} dw \, e^{-\beta w} p(w; t),
$$

where $\beta^{-1} = k_B T$ is the thermal energy. In computer simulations and experiments, sampled values of $w$ lie typically within one or two standard deviations of the maximum of $p(w; t)$, while the most important values in equation (1) are those near the maximum of $\exp(-\beta w) p(w; t)$. The corresponding regimes overlap significantly only if the function $\exp(-\beta w)$ does not change much over one standard deviation. If this is not the case, the relevant contributions to the integral come from those rare trajectories with a work value belonging to the tails of $p(w; t)$. In experiments, these rare trajectories are almost never observed and even in simulations, it is difficult to generate them with the sufficient statistical weight. Hence, to evaluate averages as in equation (1), information on the tails of the PDF is essential. An important part of this study is to gain insight into the asymptotic PDF behavior of the work performed on a Brownian particle in an asymmetric time-dependent potential well.

We will calculate the characteristic function for the work in a simple setting which, however, may be realized in experiments [5–7]. In this setting, an overdamped motion of a Brownian particle is considered in the logarithmic–harmonic potential

$$
U(x, t) = -g \ln(x) + \frac{1}{2} k(t)x^2, \quad g > 0, \quad x > 0,
$$

where the parameter $g > 0$ specifies the strength of the logarithmic repulsive wall near the origin, and $k(t)$ is a time-dependent force constant. In the deterministic limit, i.e. in the absence of thermal noise, the particle moves along the positive $x$-axis as driven by the time-dependent force $-\partial U(x, t)/\partial x$ (without inertia). Taking into account the thermal noise, the combined process $(X(t), W(t))$ is described by the system of Langevin equations:

$$
dX(t) = \left[ \frac{g}{X(t)} - k(t)X(t) \right] dt + \sqrt{2D} dB(t),
$$

$$
dW(t) = \frac{1}{2} k(t)X^2(t) dt,
$$

where $D$ quantifies the strength of the noise and $B(t)$ is the standard Wiener process. Specifically, in the case of a thermal environment, the noise strength is proportional to the temperature, i.e. $D = k_B T$ (when the particle mobility is set to 1).

While the work process has not yet been studied for the logarithmic–harmonic potential, an exact solution of the Fokker–Planck equation for the position process was obtained [8]. In the following, we recover this solution from our Lie algebraic approach. The solvability of this problem stems from the fact that the operators entering the Fokker–Planck equation form a Lie algebra [9, 10]. If one considers the Fokker–Planck equation for the joint PDF of position and work, the corresponding differential operators no longer form a Lie algebra. However, if one starts with the Fokker–Planck equation for the joint PDF and performs a Laplace transformation with respect to the work variable $w$, a Lie algebra is obtained. The solution of the Fokker–Planck equation then provides the characteristic function for the work process and the tails of the work PDF can be extracted using an asymptotic analysis of Laplace transforms.
The work PDF for the problem of the ‘breathing parabola’ \((g = 0)\) has been studied analytically in \([11–14]\). In \([11, 12]\), the authors considered an expansion around a single trajectory attributed to a prescribed rare value of the work and derived asymptotic results for the tails of the work PDF in the small temperature limit. The solution reported in \([13]\) is formally exact for the arbitrary protocol \(k(t)\). Explicit results are given in the limit of slow driving, where the process is close to a quasi-static equilibrium and the work PDF can be approximated by a Gaussian. In \([14]\), the work-weighted propagator was derived by the path integral method. Another closely related setting, where the work PDF can be calculated analytically, is for a parabolic potential with a time-dependent position of the minimum (sliding parabola) \([14–18]\). This work broadens the list of few exact results in this field.

2. Solution of Fokker–Planck equation for an arbitrary protocol

2.1. Green’s function for a logarithmic potential

An important auxiliary quantity in deriving all subsequent results is Green’s function

\[
q(x; t|x_0) = \exp \left[ t \left( \frac{\partial^2}{\partial x^2} - \frac{g \partial}{D \partial x} \right) \right] \delta(x - x_0), \quad x_0 > 0, \tag{5}
\]

which represents the solution of the Fokker–Planck equation:

\[
\frac{\partial}{\partial t} q(x; t|x_0) = \left( \frac{\partial^2}{\partial x^2} - \frac{g \partial}{D \partial x} \right) q(x; t|x_0) \tag{6}
\]

for the diffusion in the time-independent logarithmic potential with the initial condition \(q(x; 0|x_0) = \delta(x - x_0)\). The explicit form of the solution is \([19, 20]\)

\[
q(x; t|x_0) = \frac{x_0}{2r} \left( \frac{x}{x_0} \right)^{v+1} \exp \left( -\frac{x^2 + x_0^2}{4r} \right) I_v \left( \frac{x x_0}{2r} \right), \tag{7}
\]

where \(I_v(.)\) is the modified Bessel function of order \(v\), and

\[
v = \frac{1}{2} \left( \frac{g}{D} - 1 \right) \tag{8}
\]

measures the strength of the logarithmic potential in relation to the intensity of the thermal noise. In all subsequent results, the parameter \(g\) enters solely through \(v\) defined in equation (8).

Equation (7) is the unique norm-preserving solution of the diffusion problem in the domain \(x > 0\), i.e. the probability current at \(x = 0\) vanishes. Therefore, performing the limit \(g \to 0\) and using \(I_{v+1/2}(z) = \sqrt{\frac{2}{\pi}} \cosh(z)/\sqrt{z}\), we obtain the standard solution for free diffusion with a reflecting boundary at \(x = 0\) \([21]\).

2.2. Joint Green’s function for work and position

Let us denote by \(p(x, w; t|x_0, 0)\) the joint PDF for the process \(\{X(t), W(t)\}\) given that at time \(t = 0\), the particle is at position \(x_0, x_0 > 0\), and no work has been performed yet

\[
p(x, w; 0|x_0, 0) = \delta(x - x_0)\delta(w), \quad x_0 > 0. \tag{9}
\]

The time evolution of the joint PDF is given by the Fokker–Planck equation:

\[
\frac{\partial}{\partial t} p(x, w; t|x_0, 0) = \left[ D \frac{\partial^2}{\partial x^2} - \frac{g}{x} \frac{\partial}{\partial x} - k(t) x^2 \frac{\partial^2}{\partial w^2} \right] p(x, w; t|x_0, 0). \tag{10}
\]

The differential operators on the right-hand side do not exhibit closed commutation relations. However, after performing the two-sided Laplace transformation \([22]\)

\[
\tilde{p}(x, \xi; t|x_0) = \int_{-\infty}^{+\infty} dw \ e^{-\xi w} p(x, w; t|x_0, 0), \tag{11}
\]
the Fokker–Planck equation (10) assumes the form
\[
\frac{\partial}{\partial t} \tilde{p}(x, \xi; t|x_0, 0) = \left[D \tilde{J}_0 + 2k(t) \tilde{J}_1 - 4\xi k(t) \tilde{J}_2 + (v + 1) k(t)\right] \tilde{p}(x, \xi; t|x_0, 0),
\]
where the differential operators
\[
\tilde{J}_0 = \frac{\partial^2}{\partial x^2} - \frac{g}{D} \frac{\partial}{\partial x}, \quad \tilde{J}_1 = \frac{1}{2} \left(x \frac{\partial}{\partial x} - v\right), \quad \tilde{J}_2 = \frac{1}{8} x^2,
\]
satisfy the closed commutation relations
\[
[\tilde{J}_0, \tilde{J}_1] = \tilde{J}_0, \quad [\tilde{J}_0, \tilde{J}_2] = \tilde{J}_1, \quad [\tilde{J}_1, \tilde{J}_2] = \tilde{J}_2.
\]
This allows us to apply the Lie algebraic method, as discussed, e.g., in [9, 23]. First, we write the solution of equation (12) in the factorized form
\[
\tilde{p}(x, \xi; t|x_0) = \exp \left[ (v + 1) \int_0^t \mathrm{d} t' k(t') \right] \exp(b_2(t) \tilde{J}_2) \exp(b_1(t) \tilde{J}_1) \exp(b_0(t) \tilde{J}_0) \delta(x - x_0),
\]
where the time-dependent coefficient \(b_2(t)\) is obtained by solving the Riccati differential equation:
\[
\dot{b}_2(t) = \frac{D}{2} b_2(t) + 2k(t) b_2(t) - 4\xi k(t), \quad b_2(0) = 0.
\]
Knowing \(b_2(t)\), the other coefficients are given by
\[
b_1(t) = 2 \int_0^t \mathrm{d} t' k(t') + D \int_0^t \mathrm{d} t' b_2(t'), \quad b_0(t) = D \int_0^t \mathrm{d} t' \exp[b_1(t')].
\]
In the last step, we evaluate, using equations (5) and (7), the action of the operator \(\exp(b_0(t) \tilde{J}_0)\) on the delta function in equation (15), and subsequently apply to the corresponding result the two remaining exponential operators in equation (15). This yields
\[
\tilde{p}(x, \xi; t|x_0) = \exp \left( \int_0^t \mathrm{d} t' k(t') - \frac{1}{2} vD \int_0^t \mathrm{d} t' b_2(t') + \frac{1}{8} b_2(t) x^2 \right) q(xe^{2 b_2(t)}; b_0(t) |x_0).
\]
In the derivation, we have utilized the operator identity
\[
\exp \left( \eta x \frac{\partial}{\partial x} \right) f(x) = f(x \exp(\eta)).
\]
The exact solution (18) is the central result of the present section and constitutes the starting point of all subsequent analyses.

3. PDF of particle position and its long-time asymptotics

After integrating the joint PDF \(p(x, w; t|x_0, 0)\) over the work variable, the transition PDF \(p(x; t|x_0)\) for the particle coordinate is obtained. Equivalently, the \(w\)-integration is accomplished by evaluating the result (18) at \(\xi = 0\) (cf equation (11)). Note that the variable \(\xi\) enters the solution (18) only through the Riccati equation (16). When taking \(\xi = 0\), this equation reduces to the Bernoulli differential equation, where the unique solution satisfying \(b_2(0) = 0\) is the trivial one, i.e. \(b_2(t) = 0\). The remaining coefficients in equation (18) are then given by
\[
b_1(t) = 2 \int_0^t \mathrm{d} t' k(t'), \quad b_0(t) = D \int_0^t \mathrm{d} t' \exp \left[ 2 \int_0^t \mathrm{d} t'' k(t'') \right].
\]
Hence, the PDF for the particle position reads
\[
p(x; t|x_0) = \frac{x_0 e^{-\frac{1}{2} b_1(t)}}{2b_0(t)} \left( \frac{x}{x_0} \right)^{v+1} \exp \left( - \frac{x^2}{4b_0(t)} \right) I_v \left( \frac{x_0 e^{\frac{1}{2} b_1(t)}}{2b_0(t)} \right).
\] (21)

This finding is valid for an arbitrary driving protocol \( k(t) \). If \( k(t) \) is a positive constant, say \( k(t) = k_0 > 0 \), then the system approaches the Gibbs canonical equilibrium at long times. If the constant force constant is superimposed with a periodically oscillating component, a gradual formation of a nontrivial steady state occurs. In this steady state, the PDF does not depend on the initial condition \( x_0 \) and, for any given \( x > 0 \), is a periodic function of time with the fundamental period given by that of \( k(t) \).

To exemplify the PDF in the steady state, let us take
\[
\lim_{t \to \infty} \exp \left( - \frac{x^2}{4b_0(t)} \right) = 1, \quad k_0 > 0.
\] (22)

The asymptotic analysis of equation (21) for long times requires the evaluation of the limit
\[
\lim_{t \to \infty} \exp \left( - \frac{x^2}{4b_0(t)} \right), \quad \alpha \leq 1.
\] (23)

If \( \alpha < 1 \), the limit exists, and using L'Hôpital’s rule, it equals zero. Hence for any finite \( x \) and \( x_0 \), the argument \( z = \frac{x_0^2}{2b_0(t)} \) in the Bessel function appearing in equation (21) becomes small for large \( t \) and we can write \( I_v(z) \sim \left( \frac{1}{2} \right)^v / v! \). If \( \alpha = 1 \), the limit does not exist and \( \exp(-x^2/(4b_0(t))) \sim 1/f(t) \) for \( t \to \infty \), where
\[
f(t) = D \exp \left( \frac{2k_1}{\omega} \cos(\omega t) \right) \sum_{n=-\infty}^{+\infty} I_v \left( \frac{2k_1}{\omega} \right) \frac{\sin(\omega n t)}{2k_0 + i\omega}, \quad k_0 > 0.
\] (24)

Accordingly, for \( t \to \infty \)
\[
p(x; t|x_0) \sim p_{\text{as}}(x; t) = \frac{1}{\Gamma(v+1)} \left( \frac{1}{f(t)} \right)^{v+1} \left( \frac{x}{2} \right)^{2v+1} \exp \left( - \frac{x^2}{2f(t)} \right).
\] (25)

In the limit \( k_1 \to 0 \) or \( \omega \to 0 \), \( f(t) \to D/(2k_0) \), and \( p(x; t|x_0) \) approaches the Gibbs equilibrium distribution.

Finally, the limit \( g \to 0 \) in equation (21) (equation (25)) yields the exact transition PDF (exact time-asymptotic PDF) for the breathing parabola model with the reflecting boundary at the origin. Since the parameter \( g \) enters only via \( v \) in equation (8), \( g \to 0 \) corresponds to \( v \to -\frac{1}{2} \) in equations (21) and (25).

4. Work fluctuations

4.1. Characteristic functions

By the integration of the joint PDF in equation (18) over the spatial variable \( x \), we obtain the characteristic function for the work performed on the particle during the time interval \([0, t]\).

Let us first consider the particle dynamics conditioned on the initial position \( x_0 \). In this case, the characteristic function for the work reads
\[
\Phi(\xi; t|x_0) = \int_0^{+\infty} dx \, \phi(x; \xi; t|x_0).
\] (26)

3 This result agrees with equation (19) in [24], where it has been derived in connection with a diffusion problem with logarithmic factors in drift and diffusion coefficients.
Carrying out the integration, we find
\[ \Phi(\xi; t | x_0) = \left( \frac{2^b(t) - \frac{1}{2} D \int_0^t dt' b_2(t')}{2e^{b(t)} - b_0(t)b_2(t)} \right)^{v+1} \exp \left[ \left( \frac{x_0}{2} \right)^2 \frac{b_2(t)}{2e^{b(t)} - b_0(t)b_2(t)} \right]. \tag{27} \]

A physically more important situation is when the particle coordinate is initially equilibrated with respect to the initial value \( k(0) = k_0 > 0 \) of the force constant. In order to obtain the characteristic function for this situation, we have to integrate over \( x_0 \) the product of the characteristic function \( \Phi(\xi; t | x_0) \) and the equilibrium PDF \( p_{eq} \) for \( f(t) = D/(2k_0) \) given in equation (25). The result is
\[ \Phi(\xi; t) = \langle e^{-\xi W(t)} \rangle = \left( \frac{4k_0e^{b(t)} - \frac{1}{2} D \int_0^t dt' b_2(t')}{4k_0e^{b(t)} - (D + 2k_0b_0(t))b_2(t)} \right)^{v+1}. \tag{28} \]

Note that for \( \xi = \beta \), the average in equation (1) equals \( \Phi(\beta; t) \). Equation (28) is valid for an arbitrary driving protocol \( k(t) \). The Laplace variable \( \xi \) enters \( \Phi(\xi; t) \) implicitly through the function \( b_2(t) \) (cf the Riccati equation (16)) and through the functions \( b_1(t) \) and \( b_0(t) \) (cf equation (17)).

In the limit \( g \to 0 \) (\( \nu \to -\frac{1}{2} \)), equations (27) and (28) give the corresponding characteristic functions for the breathing parabola model with the reflecting boundary at \( x = 0 \). These characteristic functions are also valid for the breathing parabola model without the reflecting boundary, if rather obvious changes are made of the meaning of the initial coordinate \( x_0 \) in equation (27), and of the initial Gibbs equilibrium state underlying equation (28). In the breathing parabola model without the reflecting boundary, equation (27) is valid for \( x_0 \in (-\infty, +\infty) \) and equation (28) corresponds to the initial Gibbs equilibrium in the parabolic potential \( U(x_0) = k_0x_0^2/2 \). The equivalence of the characteristic functions for the problems with and without the reflecting boundary is due to the symmetry of the parabolic potential, which implies that the work performed on the particle that crosses the origin is the same as the work performed on the particle reflected at the origin. This reasoning can be supported by an independent calculation if one notes that both models, the present model with the logarithmic–harmonic potential and the breathing parabola one, have the same operator algebra.

4.2. Simple example

A driving protocol, where an explicit solution of the Riccati equation (16) in terms of elementary functions can be given, is
\[ k(t) = \frac{k_0}{1 + \gamma t}, \quad k_0 > 0, \quad \gamma > 0. \tag{29} \]

Note that the same protocol was considered before in [11, 12]. According to equation (29), the potential well widens with time and hence the work performed on the particle is negative for any \( x > 0 \). The solution of the Riccati equation (16) reads
\[ b_2(t) = -\frac{2}{D} \frac{d}{dt} \ln \left( 1 + \gamma t \right)^{\frac{1}{2}(2k_0 \gamma - A(\xi))} \left[ 1 + \gamma t (A(\xi) - \frac{(2k_0 + \gamma)}{(2k_0 + \gamma) - A(\xi)}) \right], \tag{30} \]
where
\[ A(\xi) = \sqrt{(2k_0 + \gamma)^2 - 8k_0 \gamma D\xi}. \tag{31} \]

For simplicity, we take \( \gamma = 1 \) and \( k_0 = 1 \) in the following. After calculating \( b_0(t) \) and \( b_1(t) \) from equations (17), the characteristic function in equation (28) becomes
\[ \Phi(\xi; t) = \left( \frac{2A(\xi)(1 + t)^{\frac{1}{2}(1 + A(\xi))}}{A(\xi)[(1 + t)^{A(\xi)} + 1] + (3 - 2D\xi)[(1 + t)^{A(\xi)} - 1]} \right)^{v+1}. \tag{32} \]
For $\xi = 1/D$ in particular, we obtain $\Phi(1/D; t) = (1 + t)^{v+1}$ which exemplifies the Jarzynski equality for the driving protocol (29).

Successive derivatives of the characteristic function with respect to $\xi$ evaluated at $\xi = 0$ yield the cumulants of the work distribution. The mean work performed on the particle during the time interval $[0, t]$ is

$$\langle W(t) \rangle = -(v + 1) \frac{D}{9} \left[ 6 \ln(1 + t) + \frac{t^3 + 3t^2 + 3t}{(1 + t)^3} \right].$$

(33)

It is a monotonically decreasing function of $t$, where for small times, the decrease is linear, while in the long-time limit, it is logarithmic. For the variance, we find

$$\langle W^2(t) \rangle - \langle W(t) \rangle^2 = (v + 1) \left( \frac{D}{9} \right)^2 \frac{t^3 + 3t^2 + 3t}{(1 + t)^6} [t^3 + 3t^2 + 3t + 24(1 + t)^3 \ln(1 + t)].$$

(34)

This increases monotonically, where the increase is quadratic for small times and logarithmic for long times. The strength $g$ of the logarithmic potential barrier enters the above formulas only through the multiplicative prefactor $(v + 1)$. This holds true for all cumulants. For stronger repulsion, the particle predominantly diffuses in a region further away from the origin. The decrease in its typical potential energy results in a larger absolute value of the mean work. At the same time, the width of the work PDF increases since the initial particle position is sampled from a broader Gibbs distribution.

Equation (32) entails the complete information about the work distribution $p(w; t)$. In particular, it allows one to derive the tails of the work PDF for both $w \to 0^-$ and $w \to -\infty$ without carrying out the inverse Laplace transformation of $\Phi(\xi; t)$.

The asymptotics of $p(w; t)$ for $w \to 0^-$ is related to the asymptotics of $\Phi(\xi; t)$ for $\xi \to -\infty$, which follows from (32):

$$\Phi(\xi; t) \sim \left( \sqrt{8(1 + t)} \exp[-\sqrt{2 \ln(1 + t) \sqrt{-DE}}] \right)^{v+1}, \quad \xi \to -\infty.$$

(35)

By taking the inverse Laplace transform of this asymptotic form (cf [25]), we obtain the parabolic cylinder function (cf [26]) with an argument proportional to $\sqrt{D/|w|}$. Considering the limit of large arguments of this function [26], we find

$$p(w; t) \sim c_1(t) \left( \frac{|w|}{D} \right)^{v-\frac{1}{2}} e^{-c_2(t) \frac{D}{2|w|}}, \quad w \to 0^-,$$

(36)

where

$$c_2(t) = \left( \frac{v + 1}{\sqrt{2}} \ln(1 + t) \right)^2, \quad c_1(t) = \left( \frac{\sqrt{8}}{D} \pi \right)^{\frac{1}{4}} (1 + t)^{\frac{v}{2}} \left( \frac{2 (1 + t)^{\frac{3}{2}}}{(v + 1) \ln(1 + t)} \right)^{v}.$$

(37)

For any $g$ and any $t$, the PDF almost vanishes in an interval $(w_0(t), 0)$, where its width $|w_0(t)|$ is controlled by the ‘damping constant’ $c_2(t)$. The width increases both with time and strength of the logarithmic potential. This can be understood from the fact that any trajectory yielding a small (absolute) value of the work must necessarily depart from a position close to the origin and remain in its vicinity during the whole time interval $[0, t]$. The probabilistic weight of such trajectories decreases with both $t$ and $g$.

The asymptotics of the work PDF $p(w; t)$ for $w \to -\infty$ (at fixed $t$) is determined by the expansion of the characteristic function $\Phi(\xi; t)$ at $\xi_0(t)$, which gives the singularity of $\Phi(\xi; t)$ lying closest to its domain of analyticity [22]. To find $\xi_0(t)$, we numerically solved the transcendental equation $1/\Phi(\xi_0(t); t) = 0$. In the vicinity of the singularity,

$$\Phi(\xi; t) \sim r(t) \left( \frac{1}{D(\xi - \xi_0(t))} \right)^{v+1}, \quad \xi \to \xi_0(t).$$

(38)
Figure 1. Positions of the singularity in expression (38) (left panel) and prefactor (39) (right panel) as functions of time. These functions control, through equation (40), the large $|w|$ asymptotics of the work PDF. In the right panel, we have taken $D = 1$ and $g = 1$.

with

$$r(t) = (-D)^{v+1} \lim_{\xi \to \xi_0(t)} (\xi - \xi_0(t))^{v+1} \Phi(\xi; t).$$

(39)

From this result, we obtain [22]

$$p(w; t) \sim \frac{1}{D} \frac{r(t)}{\Gamma(v + 1)} \left( \frac{|w|}{D} \right)^v e^{-D\xi_0(t)\frac{|w|}{D}}, \quad w \to -\infty,$$

(40)

where $D\xi_0(t)$ is a real, positive, decreasing function of $t$; cf figure 1. For small $t$, the work PDF is very narrow ($D\xi_0(t)$ large). With the increasing $t$, the weight of the trajectories yielding large (absolute) values of the work increases. This is reflected by the decrease of $D\xi_0(t)^\frac{|w|}{D}$. Contrary to the function $c_2(t)$ in (36), $D\xi_0(t)$ does not depend on the strength of the logarithmic potential. The parameter $g$ enters only the pre-exponential factor in equation (40).

In order to verify the exact asymptotic expansions of the work PDF, we have performed extensive Langevin dynamics simulations using the Heun algorithm [27] for several sets of parameters and different time intervals. A typical PDF together with the predictions for its asymptotic behavior according to equations (36) and (40) is shown in figure 2. In order to avoid nonphysical negative values of the particle position in the numerics, which can originate from a fixed time discretization, we have implemented a time-adapted Heun scheme. If a negative (attempted) coordinate along a trajectory is generated, the time step $\Delta t$ is reduced until the attempted particle position is positive. To allow for a better comparison of the analytical findings with the simulated data in figure 2 for small $|w|$, we have also derived the second leading term in the asymptotic expansion for $|w| \to 0$. After somewhat lengthy but straightforward calculation, we obtain

$$p(w; t) \sim c_1(t) \left[ \left( \frac{|w|}{D} \right)^{v+1} - \frac{4}{(v + 1)\ln(1 + t)} \left( \frac{|w|}{D} \right)^{v+1} \right] e^{-c_2(t)\frac{|w|}{D}}, \quad w \to 0^-, \quad (41)$$

where $c_1(t)$ and $c_2(t)$ are given in equation (37).

4 For $t = 2$ and $g = 0$, we obtain $D\xi_0(2) \approx 1.827$, $r(2)/\Gamma(1/2) \approx 1.021$, which is in perfect agreement with [12, equation (5.30)].
Figure 2. (a) Simulated work PDF in comparison with the asymptotic behavior predicted by equation (40) (|w| large, solid line) and equation (41) (|w| small, dashed line) for parameters g = 1.5, D = 1 and t = 1. In the simulations, $10^6$ trajectories were generated with a time step $\Delta t = 0.001$ (adapted when the particle is near the origin; see the text). (b) A semi-logarithmic plot of simulated $p(w; t)$ versus w (circles), demonstrating the agreement with equation (40) (solid line) for large |w|. (c) A semi-logarithmic plot of simulated $p(w; t)$ versus w (circles) in comparison with the first leading term of the asymptotic expansion for small |w| (equation (36), solid line), and when including the second leading term according to equation (41) (dashed line).

5. Concluding remarks

Based on a Lie algebraic approach, we succeeded in deriving equation (18) for the joint PDF of work and position for a Brownian particle in a time-dependent logarithmic–harmonic potential. In order to derive explicit results from equation (18) for a given protocol, the Riccati equation (16) needs to be solved. This nonlinear differential equation is equivalent to the linear second-order differential equation

$$\ddot{y}(t) - 2k(t)\dot{y}(t) - 2D\xi\dot{k}(t)y(t) = 0, \quad \dot{y}(0) = 0. \quad (42)$$

Specifically, if $y(t)$ solves (42), then the logarithmic derivative

$$b_2(t) = -\frac{2}{D}\frac{\dot{y}(t)}{y(t)} \quad (43)$$

is the solution of equation (16). Hence, the characteristic function (28) can be expressed in terms of the function $y(t)$.

The solution of (42) for several reasonable driving protocols, e.g., for $k(t) = k_0 \exp(\pm \gamma t)$, $k(t) = k_0 + k_1t$, or $k(t) = k_1t^n$, can be written in terms of higher transcendental functions. Corresponding results are quite involved and will be published elsewhere. Here, we have focused on the simple protocol (29) which should exemplify typical asymptotic features of the work PDF for monotonic driving. Note that, if $\dot{k}(t) > 0$ and $\xi \to \infty$ along the real axis, one can use the WKB approximation and derive a generic expression for $y(t)$ valid for any protocol, i.e. also a generic approximative expression for the work characteristic function (28).

For non-monotonic driving protocols, the work can assume any real value. Then, the work PDF has the support $(-\infty, +\infty)$ and its two-sided Laplace transform will be analytic within
a stripe parallel to the imaginary axis. The $w \to +\infty$ ($w \to -\infty$) tail of the work PDF is determined by the singularity, which is closest to the stripe on its left (right) side. We hence expect an asymptotics

$$p(w; t) \sim \frac{1}{D} \frac{r_\pm(t)}{\Gamma(v + 1)} \left( \frac{|w|}{D} \right)^v e^{-D\xi_\pm(t) |w|^2}, \quad w \to \pm \infty,$$

(44)

where the coefficients $\xi_\pm(t)$, $r_\pm(t)$ depend on the driving protocol $k(t)$. Periodic driving protocols play an important role in the analysis of Brownian motors. A deeper analysis of the work PDF for this class of protocols seems to be worthy of further study.

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