Solvability of abstract semilinear equations by a global
diffeomorphism theorem

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Abstract

In this work we provided a new simpler proof of the global diffeomorphism theorem from [9] which we
further apply to consider unique solvability of some abstract semilinear equations. Applications to the second
order Dirichlet problem driven by the Laplace operator are given.

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1 Introduction

The idea of applying global invertibility results to boundary value problems and integral, integro-differential
equations has been known in the literature for some time now. There is a variational tool concerning global
invertibility which we are going to use.

Theorem 1. [9] Theorem 3.1]Let $X$ be a real Banach space and let $H$ be a real Hilbert space. Suppose that $F : X \to H$
is a $C^1$ mapping such that:

D1 for every $y \in H$ functional $\varphi_y : X \to \mathbb{R}$ given by $\varphi_y(x) := \frac{1}{2} \|F(x) - y\|^2$ satisfies Palais-Smale condition,
i.e. every sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that $(\varphi_y(x_n))_{n \in \mathbb{N}}$ is bounded and $\varphi'_y(x_n) \to 0_{X^*}$ admits convergent
subsequence;

D2 for every $x \in X$ an operator $F'(x)$ is bijective.

Then $F$ is diffeomorphism.

The proof of Theorem 1 relies on the application of the celebrated Mountain Pass Theorem due to Am-
brosetti and Rabinowitz, see [11] and relies in checking that the functional $\varphi$ satisfies the mountain geometry.
Precisely speaking the fact that $f$ is onto is reached through the classical Ekeland’s Variational Principle. The
injectivity part is obtained by contradiction assuming to the contrary and arguing by the application of the
Mountain Pass Theorem. The most difficult part of the proof is the estimation of $\varphi$ on some sphere around 0.
However, we will show using some ideas from [18] that the proof can be performed in a different and more
readable manner thus simplifying the arguments from [9]. Theorem 1 proposes some approach towards the
existence of solutions to nonlinear equations which is variational in spirit, i.e. concerns the usage of certain
functional which is at the same different from the classical energy (Euler type) action functional. Moreover,
it allows for obtaining uniqueness of a solutions without any notion of convexity, again contrary to what
is known in the application of a direct method, see for example [15], Corollary 1.3]. However up to now
Theorem 1 and related global implicit function theorem from [10] have been applied to various first order
integro-differential problems which cover also the so called fractional case (with the fractional derivative) and correspond to Urysohn and Volterra type equations, see [4][11][12]. Some comments on the global invertibility results from [9], relation with other approaches and possible applications are contained in [8]. There was also an attempt to examine second order Dirichlet problem for O.D.E. in [2], but for some specific problem and without any abstract scheme allowing for considering boundary value problems in some unified manner. Results for continuous problem in [2] are related to the existence result obtained in [17], although the methods are different, both yield the existence with similar assumptions. This suggests that possibly the abstract framework here is to be obtained with some different global invertibility result. Our applications are meant for partial differential equations and thus do not have their counterparts in [19].

In this work we aim at proposing some abstract approach in order to examine solvability of some semilinear equations pertaining to second order Dirichlet problems for both ordinary and partial differential equations using the approach suggested by Theorem [11]. Our results towards abstract approach were inspired by some recent abstract approaches developed in [6][17] which were based on the variational framework due to [17] and which utilized relations between critical points to actions functional and fixed points to certain mappings. Nevertheless, our approach towards solvability is different and relies on different abstract tools. Moreover, the setting is now somehow different since for the sake of global invertibility densely defined operators are insufficient. In fact one need to consider the domain of the operator with its natural topology induced by a suitable norm.

2 Problem formulation and main results

Let \((H, \langle \cdot | \cdot \rangle)\) be a real Hilbert space with a norm denoted by \(\| \cdot \|\) and let \(A\) be a self-adjoint operator on \(H\) with the domain \(D(A)\). Recall that \((D(A), \langle \cdot | \cdot \rangle_A)\) is a real Hilbert space, where \(\langle \cdot | \cdot \rangle_A = \langle \cdot | \cdot \rangle + \langle A \cdot | A \cdot \rangle\). By \(\| \cdot \|_A\) we denote its norm, i.e. the graph norm of \(A\). Let \((B, \| \cdot \|_B)\) be a real Banach space and let \(N : (B, \| \cdot \|_B) \to (H, \langle \cdot | \cdot \rangle)\) be an operator which is not necessarily linear. In this framework we shall study in \(D(A)\) the following equation

\[
Ax = N(x)
\]  

(1)

In order to consider [11] we will make the following assumptions:

(A1) \(D(A) \subset B \subset H\) and the embedding \((D(A), \langle \cdot | \cdot \rangle_A) \hookrightarrow (B, \| \cdot \|_B)\) is compact;

(A2) \(\langle Au | u \rangle \geq \alpha \|u\|^2\) for some \(\alpha > 0\) and all \(u \in D(A)\);

(N1) \(N\) is of class \(C^1\) with \(N'(u)\) symmetric for all \(u \in B\);

(N2) there exist constants \(0 < \beta < 1\) and \(0 < \gamma < \alpha\) such that:

i) \(\|N(u)\| < \beta \|Au\| + \delta\) for all \(u \in D(A)\);

ii) \(\langle N'(u)h | h \rangle < \gamma \|h\|^2\) for all \(u, h \in D(A)\).

Our main result reads as follows.

Theorem 2. Assume that (A1)-(A2) and (N1)-(N2) are satisfied. Then equation (1) has a unique solution in \(D(A)\).

In this Theorem we may replace assumption (A1) by the following one:

(A1') \((B, \| \cdot \|_B) = (H, \langle \cdot | \cdot \rangle)\) and \(A : D(A) \to H\) is a self-adjoint operator with purely discrete spectrum,

because then the embedding \((D(A), \langle \cdot | \cdot \rangle_A) \hookrightarrow (H, \langle \cdot | \cdot \rangle)\) is compact, by [20], Proposition 5.12.

Remark. While all spaces which we consider are real, the theory developed in [20] works for complex spaces. Nevertheless, results which we use (namely: Proposition 3.10, Proposition 5.12, Proposition 10.19) can be clearly taken to the setting of a real space using the spirit of a book by Brezis [5]. Moreover Kato-Rellich Theorem for the setting of a real space is contained in [7].
3 Proofs

For the proof of Theorem 1 we need the following Theorem.

**Theorem 3.** Let \( X \) be a Banach space and let \( f : X \to \mathbb{R} \) be a \( C^1 \) functional satisfying Palais-Smale condition with \( 0_X \) its strict local minimum. If there exists \( e \neq 0_X \) such that \( f(e) \leq f(0_X) \), then there is a critical point \( \bar{x} \) of \( f \), with \( f(\bar{x}) > f(0_X) \), which is not a local minimum.

**Proof of Theorem 1** Firstly, we show that operator \( F \) is "onto". Fix \( y \in H \). As \( F \) is of class \( C^1 \), \( \varphi_y(x) = \frac{1}{2}\|F(x) - y\|^2 \) is of the same type and its differential \( \varphi'_y(x) \) at \( x \in X \) is given by

\[
\varphi'_y(x)h = \langle F(x) - y, F'(x)h \rangle.
\]

for all \( x \in X \). Clearly, \( \varphi_y \) is bounded from below and it satisfies Palais-Smale condition, by [13]. Hence, \( \varphi_y \) has a critical point (see [13], Chapter 3, Corollary 3.3]). In other words, there exists \( x_0 \in X \) such that \( \langle F(x_0) - y, F'(x_0)h \rangle = 0 \) for all \( h \in X \). Since \( F'(x_0) \) is surjective, \( F(x_0) - y = 0 \) and so \( F(x_0) = y \).

Now we show that \( F \) is "one-to-one". Aiming for a contradiction, suppose that there exist \( x_1, x_2 \in X \) such that \( x_1 \neq x_2 \) and \( F(x_1) = F(x_2) \). Define \( e := x_1 - x_2 \) and put \( \psi : X \to \mathbb{R} \) by formula

\[
\psi(x) := \frac{1}{2}\|F(x + x_2) - F(x_1)\|^2 = \varphi_{F(x_1)}(x + x_2).
\]

Then \( \psi \) is of class \( C^1 \) and \( \varphi(0_X) = \psi(e) = 0 \). Moreover, \( 0_X \) is a strict local minimum of \( \psi \), since otherwise, in any neighbourhood of \( 0_X \) we would have a nonzero \( x \) with \( F(x + x_2) - F(x_1) = 0 \) and this would contradict the fact that \( F \) defines a local diffeomorphism. Therefore we can apply Theorem 3 and, in consequence, there exists \( \bar{x} \in X \) such that \( \varphi(\bar{x}) > 0 \) and \( \psi'('bar{x}) = 0 \). Hence

\[
\psi'(\bar{x})h = \langle F(\bar{x} + x_2) - F(x_1), F'(\bar{x} + x_2)h \rangle = 0
\]

for all \( h \in X \). Again, by surjectivity of \( F'(\bar{x} + x_2) \), we have \( F(\bar{x} + x_2) - F(x_1) = 0 \) and so \( \psi(\bar{x}) = 0 \), which contradicts \( \psi(\bar{x}) > 0 \). Obtained contradiction ends the proof.

Now we can present the proof of the main Theorem.

**Proof of Theorem 2** By (A2) we have \( \|Au\| \geq \alpha\|u\| \) for \( u \in D(A) \) and so

\[
\|Au\| \geq \frac{\alpha}{1+\beta}\|Au\| \geq \frac{\alpha}{1+\beta}\|u\|.
\]

Let \( X := (D(A), \|\cdot\|_A) \) and let the operator \( \tilde{N} : X \to H \) be defined by \( \tilde{N} = N \circ i \), where \( i : X \to (B, \|\cdot\|_B) \) is a compact embedding given by (A1). Then \( \tilde{N} \in C^1(X, H) \) and operator \( \tilde{N}'(u) \) is symmetric compact and linear for all \( u \in X \), by (N1). Since \( i(u) = u \), any solution of equation

\[
Au = \tilde{N}(u)
\]

is also a solution of equation (1).

Let us define \( F : X \to H \) by

\[
F(u) := Au - \tilde{N}(u).
\]

Fix \( y \in H \) and consider the mapping \( \varphi_y : X \to \mathbb{R} \) given by

\[
\varphi_y(u) := \frac{1}{2}\|F(u) - y\|^2.
\]

Then \( \varphi_y \in C^1(X, \mathbb{R}) \), \( F \in C^1(X, H) \) and its derivatives are given, respectively, by the following formulas

\[
\varphi'_y(u)h = \langle Au - \tilde{N}(u) - y, Ah - \tilde{N}'(u)h \rangle
\]
and
\[ F'(u)h = Ah - \tilde{N}'(u)h \]
for every \( u, h \in X \).

In order to be able to use Theorem [3] we must show that \( \varphi_y \) satisfies Palais - Smale condition and \( F'(u) \) is bijective for all \( u \in X \).

By applying (N2) we see that
\[ \|F(u) - y\| = \|Au - N(u) - y\| \leq \|Au\| - \beta \|Au\| - \|y\| \geq (1 - \beta) \|u\| - \delta - \|y\| \]
for every \( u \in X \). This implies that \( \varphi_y \) is coercive. Thus any (PS) sequence can be assumed to be weakly convergent.

Now we show that the functional \( \varphi_y \) satisfies (PS) condition on \( X \). Assume that \( (u_n)_{n \in \mathbb{N}} \subset X \) is such that:

(PS1) \( (\varphi_y(u_n))_{n \in \mathbb{N}} \) is bounded;

(PS2) \( \varphi'_y(u_n) \to 0 \) if \( n \to \infty \).

Since \( \varphi_y \) is coercive, (PS1) shows that \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( X \), and then after a subsequence, it is weakly convergent to some \( u_0 \in X \). From (A1) there exists another subsequence, denote it once again by \( (u_n)_{n \in \mathbb{N}} \), which is convergent in \( (B, \| \cdot \|_B) \). So, by our assumptions we have

- \( u_n \to u_0 \) in \( B \);
- \( \tilde{N}(u_n) \to \tilde{N}(u_0) \) in \( H \);
- \( \tilde{N}'(u_n) \to \tilde{N}'(u_0) \) in \( \mathcal{L}(B, H) \);
- \( (A(u_n))_{n \in \mathbb{N}} \) is bounded in \( H \).

Now, a direct calculation yields
\[ \varphi'_y(u_n)(u_n - u_0) - \varphi'_y(u_0)(u_n - u_0) = \|Au_n - Au\|^2 + \sum_{k=1}^{4} \psi_k(u_n), \]  
\[ \psi_1(u_n) = \left\langle Au_0 - \tilde{N}(u_0), \tilde{N}'(u_0)(u_n - u_0) \right\rangle, \]
\[ \psi_2(u_n) = \left\langle \tilde{N}(u_0) - \tilde{N}(u_n), Au_n - Au_0 \right\rangle, \]
\[ \psi_3(u_n) = \left\langle \tilde{N}(u_n) - Au_n, \tilde{N}'(u_n)(u_n - u_0) \right\rangle, \]
\[ \psi_4(u_n) = \left\langle y, (\tilde{N}'(u_0) - \tilde{N}'(u_n))(u_n - u_0) \right\rangle. \]

Then, using observations made above, we obtain
\[ |\psi_1(u_n)| \leq \|Au_0 - \tilde{N}(u_0)\| \|\tilde{N}'(u_0)(u_n - u_0)\| \to 0, \]
\[ |\psi_2(u_n)| \leq \|Au_n - Au_0\| \|\tilde{N}(u_n) - \tilde{N}(u_0)\| \to 0, \]
\[ |\psi_3(u_n)| \leq \|\tilde{N}(u_n) - Au_n\| \|\tilde{N}'(u_n)(u_n - u_0)\| \to 0, \]
\[ |\psi_4(u_n)| \leq \|y\| \|(\tilde{N}'(u_0) - \tilde{N}'(u_n))(u_n - u_0)\| \to 0. \]
as \( n \to \infty \). On the other hand, by (PS2) and be the weak convergence of \((u_n)_{n \in \mathbb{N}}\) to \(u_0\) in \(X\), we have
\[
|\varphi_y'(u_n)(u_n - u_0)| \leq ||\varphi_y'(u_n)||_{X^*}||u_n - u_0||_A \to 0
\]
and
\[
|\varphi_y'(u_0)(u_n - u_0)| \to 0
\]
as \( n \to \infty \). Coining the above observations together, we can now show that equality (3) implies
\[
||Au_n - Au_0|| \to 0
\]
as \( n \to \infty \) which means, by (2), that \((u_n)_{n \in \mathbb{N}}\) converges strongly to \(u_0\) in \(X\). This shows that \(\varphi_y\) satisfies (PS) condition.

Now, we show that \(F'(u)\) is bijective for any \(u \in X\). Fix \(u \in X\). Since \(A\) is self-adjoint operator and since \(\tilde{N}'(u)\) is a symmetric compact linear operator, it follows that \(F'(u)\) is self-adjoint operator, by [7] RKNG Theorem in real Hilbert space. Using (A2) and (N2) we get
\[
||Ah - \tilde{N}'(u)h||_H \geq \langle Ah - \tilde{N}'(u)h | h \rangle = \langle Ah | h \rangle - \langle N'(u)h | h \rangle \geq \alpha ||h||^2 - \gamma ||h||^2.
\]
Hence, equivalently
\[
||F'(u)h|| = ||Ah - \tilde{N}'(u)h|| \geq (\alpha - \gamma) ||h||
\]
for all \(h \in H\). Then, as \(F'(u)\) is linear, it is injective. Applying [20] Proposition 3.10, \(F'(u)\) is also surjective, and so bijective.

Now we can apply Theorem [1] and obtain a unique \(u^* \in X\) such that \(0 = F(u^*) = Au^* - \tilde{N}(u^*)\).

4 Applications

As an application of Theorem [2] we study the following nonlinear Dirichlet problem
\[
\begin{cases}
-\Delta u(x) = f(x,u(x)), \\
u|_{\partial \Omega} = 0.
\end{cases}
\]
Here \(\Omega \subset \mathbb{R}^m\) is an open and bounded set of class \(C^2\) and \(f: \Omega \times \mathbb{R} \to \mathbb{R}\) is a \(C^1\)-Caratheodory function, i.e. for a.e. \(x \in \Omega\), \(f(x,\cdot)\) is of class \(C^1\) and for all \(u \in \mathbb{R}\), \(f(\cdot,u), f'_u(\cdot,u)\) are measurable.

An unbounded linear operator \(A\) on \(H = L^2(\Omega)\) defined by \(Au = -\Delta u\) is self-adjoint if \(D(A) = H^1_0(\Omega) \cap H^2(\Omega)\), see [20] Proposition 10.19. By the Poincaré inequality
\[
c_\Omega \int_\Omega |u(x)|^2 dx \leq \sum_{k=1}^m \int_\Omega |\partial_k u(x)|^2 dx
\]
and Green’s formula we have
\[
\langle Au | u \rangle \geq c_\Omega^2 ||u||^2, u \in D(A),
\]
where \(c_\Omega\) is a constant in Poincaré inequality and \(\langle \cdot | \cdot \rangle\) and \(|| \cdot ||\) denote the scalar product and the norm in \(H\), respectively. On \(D(A)\) the graph norm of \(A\) and norm \(|| \cdot ||_{H^2(\Omega)}\) are equivalent, see [20] p. 240. Therefore if we put
\[
B_m(\Omega) := \begin{cases}
C(\overline{\Omega}) & \text{if } m \leq 3, \\
L^{p_m}(\Omega) & \text{if } m \geq 4,
\end{cases}
\]

where \( p_m > 2 \) for \( m = 4 \) and \( p_m \in (2, \frac{2m}{m-1}) \) for \( m > 4 \), we obtain the compact embedding \((D(A), \langle \cdot , \cdot \rangle_A) \hookrightarrow (B_m(\Omega), \| \cdot \|_{B_m})\), see [15] Theorem 1.51. For \( m \leq 3 \) let \( c_m > 0 \) be such that \( \| u \|_{\infty} \leq c_m \| Au \| \) for all \( u \in D(A) \), where \( \| \cdot \|_{\infty} \) denotes the supremum norm.

We will need the following assumptions on \( f \):

(P1m)

(if \( m \leq 3 \)) there exist \( a_1, b_1 \in L^2(\Omega) \), \( \| b_1 \| < c_m^{-1} \) such that \( |f(x,u)| \leq a_1(x) + b_1(x)|u| \) for a.e. \( x \in \Omega \) and every \( u \in \mathbb{R} \);

(if \( m > 4 \)) there exist \( a_1 \in L^2(\Omega) \) and \( b_1 \in (0,c_\Omega) \) such that \( |f(x,u)| \leq a_1(x) + b_1|u| \) for a.e. \( x \in \Omega \) and every \( u \in \mathbb{R} \);

(P2m)

(if \( m \leq 3 \)) there exist \( a_2 \in L^2(\Omega) \) and \( g \in C(\mathbb{R}, \mathbb{R}) \) such that \( |f'_u(x,u)| \leq a_2(x)g(u) \) for a.e. \( x \in \Omega \) and every \( u \in \mathbb{R} \);

(if \( m > 4 \)) there exist \( a_2 \in L^q(\Omega) \) and \( b_2 > 0 \) such that \( |f'_u(x,u)| \leq a_2(x) + b_2|u|^r \) for a.e. \( x \in \Omega \) and every \( u \in \mathbb{R} \), where \( r = \frac{p_m-2}{2} \) and \( q = \frac{2p_m}{p_m-2} \);

(P3) there exists \( b_3 \in (0,c_\Omega^2) \) such that \( f'_u(x,u) < b_3 \) for a.e. \( x \in \Omega \) and every \( u \in \mathbb{R} \).

Under the above assumptions the operator \( N_f : B_m(\Omega) \rightarrow L^2(\Omega) \) given by formula \( N_f(u)(x) = f(x,u(x)) \) for \( x \in \Omega \) is of class \( C^1 \) with \( N'_f(u)(h) = N_f(u)(h) \) for all \( u,h \in B_m(\Omega) \). For case \( m \geq 4 \) see [15] Proposition 2.78 and for \( m \leq 3 \) see Appendix. Clearly, \( N'_f(u) \) is symmetric operator for all \( u \in B_m(\Omega) \).

In order to check (N2), take some \( u \in B_m(\Omega) \). Using (P1m) we have for \( m \leq 3 \)

\[
\|N_f(u)\| = \left( \int_\Omega |f(x,u(x))|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_\Omega |a_1(x)|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_\Omega b_1(x)|u(x)|^2 \, dx \right)^{\frac{1}{2}} = \|a_1\| + \|b_1\||u|_{\infty} \leq \|a_1\| + c_m\|b_1\|\|Au\|.
\]

and for \( m \geq 4 \)

\[
\|N_f(u)\| = \left( \int_\Omega |f(x,u(x))|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_\Omega |a_1(x)|^2 \, dx \right)^{\frac{1}{2}} + b_1 \left( \int_\Omega |u(x)|^2 \, dx \right)^{\frac{1}{2}} = \|a_1\| + b_1\|u\| \leq \frac{b_1}{c_\Omega}\|Au\| + \|a_1\|.
\]

Assumption (P3) provides that for every \( u,h \in B_m(\Omega) \) there is

\[
\langle N'_f(u)h | h \rangle = \int_\Omega f'_u(x,u(x))h(x)h(x) \, dx \leq b_3 \int_\Omega |h(x)|^2 \, dx = b_3\|h\|^2.
\]

As a conclusion, we obtained

**Theorem 4.** Assume that \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a \( C^1 \)-Caratheodory function such that (P1m), (P2m) and (P3) hold. Then problem \( \text(7) \) has an unique solution in \( H^1_0(\Omega) \cap H^2(\Omega) \).
As an example the following problem
\[
\begin{aligned}
-\Delta u(x) &= \left(1 - \frac{1}{|x|^s}\right) (cu(x) - 1), \\
\text{for all } x \in \Omega, \\
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^3 \) is any open and bounded set of class \( C^2 \), \( c > 0 \) is a suitable constant and \( | \cdot | \) denotes the Euclidean norm, has a unique solution in \( H^1_0(\Omega) \cap H^2(\Omega) \).

\section{Appendix}

In this appendix we show that for \( m \leq 3 \) if \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a \( C^1 \)-Caratheodory function such that (P1m), (P2m) hold, then an operator \( N_f : C(\overline{\Omega}) \to L^2(\Omega) \) given by formula \( N_f(u)(x) = f(x, u(x)) \) for \( x \in \Omega \) is of class \( C^1 \) with \( N'_f(u)(h) = N_{f_h}(u)(h) \) for all \( u, h \in C(\overline{\Omega}) \). By Theorem B in [1], if \( |f(x, u)| \leq a(x)g(u) \) for all \( x \in \Omega \) and \( u \in \mathbb{R} \) with \( a \in L^2(\Omega) \) and \( g \in C(\mathbb{R}, \mathbb{R}) \), then \( N_f \) is continuous from \( C(\overline{\Omega}) \) into \( L^2(\Omega) \).

First, we show that for all \( u, h \in C(\overline{\Omega}) \), \( N_{f_h}(u)(h) \in L^2(\Omega) \). Indeed, we have
\[
\|N_{f_h}(u)(h)\| = \left( \int_{\Omega} |f'_u(x, u(x))h(x)|^2 dx \right)^{\frac{1}{2}} \leq \|h\| \sup_{x \in \Omega} |a(x)g(u(x))| < \infty.
\]

Now, fix \( u \in C(\overline{\Omega}) \) and let
\[
w(h) = N_f(u + h) - N_f(u) - N_{f_h}(u)(h)
\]
for all \( h \in C(\overline{\Omega}) \). Let \( h \in C(\overline{\Omega}) \). We have
\[
f(x, u(x) + h(x)) - f(x, u(x)) = \int_0^1 f'_u(x, u(x) + \tau h(x)) h(x) \, d\tau
\]
Hence, using Fubini’s theorem, we obtain
\[
\|w(h)\| = \left( \int_{\Omega} \left| f(x, u(x) + h(x)) - f(x, u(x)) - f'_u(x, u(x))h(x) \right|^2 dx \right)^{\frac{1}{2}} \leq \|h\| \sup_{x \in \Omega} |a(x)g(u(x))| < \infty.
\]

Since \( N_{f_h} : C(\overline{\Omega}) \to L^2(\Omega) \) is continuous, the above implies that
\[
\frac{\|w(h)\|}{\|h\|} \to 0
\]
as \( \|h\|_\infty \to 0 \). The continuity of the map \( N_f^j : C(\Omega) \to \mathcal{L}(C(\Omega), L^2(\Omega)) \) follows now from the continuity of \( N_{f_h} : C(\Omega) \to L^2(\Omega) \). Indeed, suppose that \( u_n \to u_0 \) in \( C(\Omega) \). Then

\[
\|N_f^j(u_n) - N_f^j(u_0)\|_{\mathcal{L}(C(\Omega), L^2(\Omega))} = \sup_{\|h\|_\infty = 1} \left( \int_\Omega |f'_u(x, u_n(x))h(x) - f'_u(x, u_0(x))h(x)|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_\Omega |f'_u(x, u_n(x)) - f'_u(x, u_0(x))|^2 \, dx \right)^{\frac{1}{2}} \to 0
\]
as \( n \to 0 \).

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