Reversible coagulation-fragmentation processes and random combinatorial structures: asymptotics for the number of groups

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Abstract

The equilibrium distribution of a reversible coagulation-fragmentation process (CFP) and the joint distribution of components of a random combinatorial structure (RCS) are given by the same probability measure on the set of partitions. We establish a central limit theorem for the number of groups (=components) in the case 

\[ a(k) = qk^{p-1}, \quad k \geq 1, \quad q, p > 0, \]

where \( a(k), \quad k \geq 1 \) is the parameter function that induces the invariant measure. The result obtained is compared with the ones for logarithmic RCS’s and for RCS’s, corresponding to the case \( p < 0 \).

1 Summary.

Our main result is a central limit theorem (Theorem 4.6) for the number of groups at steady state for a class of reversible CFP’s and for the corresponding class of RCS’s.

In Section 2, we provide a definition of a reversible k-CFP admitting interactions of up to \( k \) groups, as a generalization of the standard 2-CFP. The steady state of the processes considered is fully defined by a parameter function \( a \geq 0 \) on the set of integers. It was observed by F. Kelly ([11], p. 183) that for all \( 2 \leq k \leq N \) the k-CFP’s have the same invariant measure on the set of partitions of a given integer \( N (= \) the number of particles).

Section 3 explains the idea of A. Khintchine’s probabilistic method for derivation of asymptotic formulae. In the spirit of the method, we construct a representation of the probability function of the number of groups via the probability function of the sum of i.i.d. random variables.

In section 4 we study the case when the parameter function \( a \) is of the form:

\[ a(k) = qk^{p-1}, \quad k \geq 1, \quad q, p > 0. \]

We prove a local and a central normal limit theorems for the number of groups at equilibrium, as \( N \to \infty \). To achieve this, we employ a new (for this field) tool: the Poisson summation formula. We conclude the section by providing some intuition for the main result of the paper.

In Section 5 we recall that the invariant measure of a reversible CFP can be viewed as a joint distribution of components of a RCS, known as an assembly. Comparing our main result with the known ones for RCS’s, we identify \( p = 0 \) as a point of phase transition of the invariant measure, as \( N \to \infty \). We also provide a few examples of RCS’s that conform to the case \( p > 0 \).

2 CFP’s with multiple interactions: Definition.

Following [11], [7], we treat a CFP as a continuous-time Markov chain on the finite set \( \Omega_N = \{\eta\} \) of all partitions \( \eta = (n_1, \ldots, n_N) \) of a given integer \( N \):

\[ \sum_{j=1}^{N} jn_j = N, \quad n_j \geq 0, \quad j = 1, \ldots, N. \tag{2.1} \]

Here \( N \) codes the total population of indistinguishable particles partitioned into groups (=clusters) of different sizes. A group of size \( j \geq 2 \) may split into a number, say \( s, \quad 2 \leq s \leq j \), of groups of sizes
\(j_1, \ldots, j_s : j_1 + \ldots + j_s = j\) and, conversely, the above \(s\) groups may coagulate into one large group of size \(j\). We will call these \(s\)-interactions (=\(s\)-transitions), \(J_s\) -fragmentation and \(J_s\) -coagulation respectively, where \(J_s = (j_1, \ldots, j_s)\). A stochastic process that admits interactions of up to \(k \leq N\) groups will be denoted \(k\)-CFP. Note that both types of the interactions conserve the total number of particles.

We now provide a formal definition of a \(k\)-CFP that naturally extends the definition of the standard 2-CFP (see [7], [11]). A \(k\)-CFP is given by the rates of infinitesimal transitions that are assumed to depend only on the sizes of interacting groups. For given \(N\) and \(s\), \(2 \leq s \leq k\), let \(\psi_s, \phi_s \geq 0\) be a pair of functions defined on the same set of \(s\)-tuples of positive integers, depicting the sizes of interacting groups. The functions \(\psi_s, \phi_s\) are the rates of \(J_s\)-coagulation and \(J_s\)-fragmentation respectively.

We also assume that \(\psi_s, \phi_s\) are invariant w.r.t. all \(s!\) permutations of \(j_1, \ldots, j_s\). To complete the definition of a \(k\)-CFP, it is left to determine the total rates of all possible \(s\)-transitions from one partition \(\eta \in \Omega_N\) to another. Assume that the given \(J_s = (j_1, \ldots, j_s) \in J_s\) and \(\eta = (n_1, \ldots, n_N) \in \Omega_N\) are such that \(n_l \geq m_l \geq 1\), \(l \in J_s\), where \(m_l\) counts the number of components in \(J_s\) that are equal to \(l\). In other words, we assume, that a \(J_s\)-coagulation of some groups in the partition \(\eta\) is possible. Clearly, a given \(J_s\)-coagulation of any groups in \(\eta\) transforms \(\eta\) into the same partition that will be denoted \(\eta^{(J_s)} \in \Omega_N\). Similarly, the result of \(J_s\)-fragmentation of any groups in \(\eta \in \Omega_N\) will be denoted \(\eta_{(J_s)} \in \Omega_N\). By a simple combinatorial calculation, the total rate \(\Psi_s(J_s; \eta)\) of all possible \(J_s\)-coagulations at the partition \(\eta\) is

\[
\Psi_s(J_s; \eta) = \left( \prod_{j_l} \frac{n_{j_l}!}{(n_{j_l} - m_l)!} \right) \psi_s(J_s), \quad J_s \in J_s,
\]

(2.2)

where the product is taken over all distinct components \(j_l\) of \(J_s\). By the same logic we define the total rate \(\Phi(J_s; \eta)\) of all possible \(J_s\)-fragmentations at the partition \(\eta\) to be equal to

\[
\Phi_s(J_s; \eta) = n_{|J_s|} \phi_s(J_s), \quad J_s \in J_s,
\]

(2.3)

where \(|J_s| = j_1 + \ldots + j_s\). Now we see that a \(k\)-CFP is fully defined by the \((k - 1)\) pairs of functions \(\psi_s, \phi_s\), \(s = 2, \ldots, k\).

In this paper we will be concerned only with reversible CFP’s. Define the ratio of \(s\)-interactions

\[
g_s(J_s) = \begin{cases} 
\frac{\psi_s(J_s)}{\phi_s(J_s)}, & \text{if } J_s : \psi_s(J_s)\phi_s(J_s) > 0, \quad J_s \in J_s, \\
0, & \text{otherwise}.
\end{cases}
\]

(2.4)

A natural extension of Theorem 1 in [7] gives the following characterization of reversible \(k\)-CFP’s with positive transition rates.
Proposition 2.1 Let
\[ \psi_s(J_s) \phi_s(J_s) > 0, \quad J_s \in J_s, \quad s = 2, \ldots, k. \] (2.5)

Then the corresponding k-CFP is reversible iff the ratios \( q_s, s = 2, \ldots, k \) are of the form:
\[ q_s(J_s) = \frac{a(|J_s|)}{\prod_{j=1}^{s} a(j)}, \quad J_s = (j_1, \ldots, j_s) \in J_s, \quad s = 2, \ldots, k, \] (2.6)

where \( a > 0 \) is a function on the set of positive integers.

The proof is deferred until after Proposition 2.2.

A historical remark: It was already noted in [7], that the characterization of reversible 2-CFP’s was motivated by the following two completely independent lines of research: the seminal paper of F. Spitzer [17] on nearest-particle systems and F. Kelly’s and P. Whittle’s works in the 1970-s on networks and clustering process in polymerization (see [11], [19]).

We will call \( a \) the parameter function of a reversible CFP and we will write \( a_j = a(j), \quad j \geq 1 \). The following result gives the explicit form of the steady state of the processes.

Proposition 2.2 ([7], [17], Ch.8)
For a given \( N \), all k-CFP’s, \( k = 2, \ldots, N \) satisfying (2.6) for some parameter function \( a \) have the same invariant measure \( \mu_N \) on \( \Omega_N \):

\[ \mu_N(\eta) = (c_N)^{-1} \frac{a_1^{n_1} a_2^{n_2} \cdots a_N^{n_N}}{n_1! n_2! \cdots n_N!}, \quad \eta = (n_1, \ldots, n_N) \in \Omega_N, \] (2.7)

where \( a_j > 0, \quad j = 1, 2, \ldots, N \) and \( c_N = c_N(a_1, \ldots, a_N) \) is the partition function of the measure \( \mu_N \):

\[ c_0 = 1, \quad c_N = \sum_{\eta \in \Omega_N} \frac{a_1^{n_1} a_2^{n_2} \cdots a_N^{n_N}}{n_1! n_2! \cdots n_N!}, \quad N \geq 1. \] (2.8)

Proof: We have to show that the measure \( \mu_N \) given by (2.7) satisfies the detailed balance condition:

\[ \mu_N(\eta)V(\eta, \xi) = \mu_N(\xi)V(\xi, \eta), \quad \eta, \xi \in \Omega_N, \] (2.9)

where \( V(\eta, \xi) \) is the total rate of the infinitesimal (in time) transition from \( \eta \) to \( \xi \). If \( V(\eta, \xi) = 0 \), then (2.9) is trivially true. If now \( \xi = \eta(J_s) \) for some \( J_s \in \mathbb{J}_s, \quad 2 \leq s \leq k \), then we see from (2.7) and (2.10) that

\[ \frac{\mu_N(\eta(J_s))}{\mu_N(\eta)} = \frac{\Psi_s(J_s; \eta)}{\Phi_s(J_s; \eta(J_s))}. \] (2.10)

In a similar manner one can verify (2.9) in the case \( \xi = \eta(J_s) \) for some \( J_s \in \mathbb{J}_s \).
Proof of Proposition 2.1: If a k-CFP, \( k \geq 2 \) is reversible, then it follows from [7], Theorem 1, that (2.6) should hold for \( s=2 \). The latter fact implies that the unique invariant measure of all k-CFP-s, \( k = 2, \ldots, N \) is given by (2.7). Consequently, (2.6) should hold for \( 3 \leq s \leq k \), by Proposition 2.2. The same reasoning proves the converse part of the claim. ■

The preceding discussion shows that the steady state of a reversible k-CFP is uniquely determined by a parameter function \( a \).

3 Khintchine’s type representation for the probability function of the number of groups.

Our objective will be the study of the asymptotic behavior, as \( N \to \infty \), of the number of groups \( \nu_N \) at equilibrium given by the measure \( \mu_N \). It follows from (2.7) and (2.8) that

\[
\mathbb{P}(\nu_N = n) = (c_N)^{-1} \left( \sum_{\eta \in \Omega_N} a_{\eta 1}^{n_1} a_{\eta 2}^{n_2} \ldots a_{\eta N}^{n_N} \frac{1}{n_1! n_2! \ldots n_N!} (\sum_{k=1}^{N} n_k = n) \right), \quad n \leq N.
\]

(3.11)

As in [8], [9], our tool will be the probabilistic method by A. Khintchine introduced in the 1950’s in his seminal book [12]. The idea of the method is to construct the representation of the quantity of interest via the probability function of a sum of independent integer valued random variables depending on a free parameter, and subsequent implementation of a local limit theorem. This allows for the derivation of the desired asymptotic formula.

In number theory, the implementation of Khintchine’s method was developed by G. Freiman et al (for references see [15], [9]). In particular, a general scheme of the method for asymptotic problems related to partitions was outlined by G. Freiman and J. Pitman in [10]. The method was applied to CFP’s for the first time in [8], for derivation of the asymptotic formula for the partition function of the measure \( \mu_N \) in the case \( a_k \sim k^{p-1}, \ k \to \infty, \ p > 0 \). In [9] the method was used for the study of the asymptotic behavior of some quantities related to clustering of groups at the steady state, when \( a_k \sim k^{p-1} L(k), \ k \to \infty, \ p > 0 \), where \( L \) is a slowly varying function.

Though the implementation of Khintchine’s method goes along the standard scheme, the related asymptotic analysis varies from problem to problem. In contrast to the aforementioned research, the problem treated in the present paper requires knowledge of the second term in the asymptotic expansions considered. In light of this, we employ the Poisson summation formula, a new tool for this field.

We will always assume that the parameter functions \( a \) considered are positive and s.t. the power series

\[
\sum_{k=1}^{\infty} a_k x^k, \quad x \in \mathbb{C}
\]

(3.12)

has a finite radius of convergence \( R > 0 \). Since the transformation \( a_k \Rightarrow h^k a_k, \ h > 0, \ k = 1, 2, \ldots, N \) does not change the measure \( \mu_N \) given by (2.7), we assume w.l.g. that \( R = 1 \).
Now let \( \xi_1, \ldots, \xi_n \) be i.i.d. integer valued nonzero r.v.'s defined by

\[
P(\xi_1 = l) = \frac{a_le^{-\delta l}}{S(\delta)}, \quad a_l > 0, \quad l \geq 1,
\]

where \( \delta > 0 \) is a free parameter and

\[
S(\delta) = \sum_{k=1}^{\infty} a_k e^{-\delta k}.
\]

Note that the r.v. \( \xi_1 \) has finite moments of all orders for all \( \delta > 0 \), since \( R = 1 \). We start with the following representation of the probability \( P(\nu_N = n) \).

**Lemma 3.1** Define

\[
T_n = \sum_{k=1}^{n} \xi_k, \quad n \geq 1,
\]

where \( \xi_k, \ k = 1, 2, \ldots \) are i.i.d. r.v.'s given by (3.13), (3.14). Then

\[
P(\nu_N = n) = (c_N n!)^{-1} S^n(\delta) e^{\delta N} P(T_n = N), \quad \delta > 0.
\]

**Proof:** It follows from (3.13), (3.14) that

\[
P(T_n = N) = \frac{\left( \sum_{\eta \in \Omega_N} \frac{a_1^{n_1} a_2^{n_2} \ldots a_N^{n_N}}{n_1! n_2! \ldots n_N!} \1_{(\sum_{k=1}^{n} n_k = n)} \right) n!}{S^n(\delta) e^{\delta N}}. \quad (3.16)
\]

By (2.7) and (3.11) this implies the claim (3.15). \( \blacksquare \)

**Remark.** (3.15) has a form of a typical representation in Khintchine’s method. It can be also viewed as a version of the representation formula for the total number of components in the generalized scheme of allocation (see [13], Lemma 1.3.3).

Our study will be heavily based on the assumption

\[
\sum_{k=1}^{\infty} ka_k = \infty.
\]

Denote

\[
M_1 = M_1(n; \delta) = ET_n = nE\xi_1 = nS^{-1}(\delta) \sum_{k=1}^{\infty} ka_k e^{-\delta k}, \quad \delta > 0
\]

and choose the free parameter \( \delta \) as a solution of the equation

\[
M_1 = N, \quad n \leq N.
\]

Such a choice of the free parameter is typical for Khintchine’s method ([12], p.110) and is designed to make the probability \( P(T_n = N) \) in (3.15) large, as \( n, N \to \infty, \ n \leq N \).
Lemma 3.2 Under condition (3.17), the equation (3.19) has a unique solution \( \delta = \delta_{n,N} \) for all \( n \leq N \).

Proof: We first show that \( M_1 \) is decreasing in \( \delta > 0 \):

\[
M'_1(n; \delta) = n \frac{-S(\delta) \sum_{k=1}^{\infty} k^2 a_k e^{-\delta k} + \left( \sum_{k=1}^{\infty} k a_k e^{-\delta k} \right)^2}{S(\delta)} < 0, \quad \delta > 0,
\]

(3.20)

by the Cauchy-Schwarz inequality. Consequently, \( M_1(n; 0) = \sup_{\delta > 0} M_1(n; \delta) := n M^*_1 \) where \( M^*_1 \) does not depend on \( n \). If now \( S(0) < \infty \), then (3.17) immediately implies \( M^*_1 = \infty \). In the case \( S(0) = \infty \), supposing \( M^*_1 < \infty \) leads to the contradiction:

\[
0 > M_1(n; \delta) - n M^*_1 = n \frac{\sum_{k=1}^{\infty} a_k e^{-\delta k} (k - M^*_1)}{S(\delta)}
\]

(3.21)

where \( A(\delta) \) is bounded for any \( \delta \geq 0 \), while the sum in the brackets tends to \( +\infty \) as \( \delta \to 0^+ \), by (3.17). Hence, \( M_1(n; 0) = +\infty \). Finally,

\[
M_1(n; \infty) = n \lim_{\delta \to \infty} \frac{\sum_{k=1}^{\infty} k a_k e^{-\delta (k-1)}}{\sum_{k=1}^{\infty} a_k e^{-\delta (k-1)}} = n.
\]

(3.22)

4 A central limit theorem for the number of groups.

Our paper is devoted exclusively to the case when the parameter function \( a \) has a polynomial rate of growth, namely:

\[
a_k = qk^{p-1}, \quad q, p > 0, \quad k \geq 1.
\]

(4.23)

We first consider the case \( q = 1 \). The following lemma which is basic for our subsequent asymptotic analysis is a particular case of the Poisson summation formula (see [3],[2]).

Lemma 4.1 ([2], p.82). Let \( p > 1 \) and \( \text{Re}(z) > 0 \). Then we have

\[
\sum_{k=1}^{\infty} e^{-zk} k^{p-1} = \Gamma(p) \left( \sum_{l=-\infty}^{\infty} (z + 2\pi il)^{-p} \right).
\]

(4.24)

With the help of this remarkable identity we derive the following asymptotic formula that holds for \( p > 0 \).
Lemma 4.2 If \( p > 0 \), \( |z| \to 0 \), \( \text{Re}(z) \to 0^+ \), then
\[
\sum_{k=1}^{\infty} e^{-zk}k^{p-1} = \Gamma(p) \left( z^{-p} + A(p) \right) + O(z),
\]
(4.25)
where \( A(p) \) is a constant which in the case \( p > 1 \) is given explicitly by
\[
A(p) = 2(2\pi)^{-p} \zeta(p) \cos \frac{\pi p}{2}.
\]
(4.26)
(Here \( \zeta(p) \) is the Riemann zeta function).

**Proof:** First consider the case \( p > 1 \). Let
\[
G(z) = \sum_{l=-\infty}^{-1} (z + 2\pi il)^{-p} + \sum_{l=1}^{\infty} (z + 2\pi il)^{-p}, \quad \text{Re}(z) \geq 0, \quad p > 1.
\]
(4.27)
The two series in the RHS of (4.27) converge absolutely when \( \text{Re}(z) > 0 \), while, by straightforward calculations, \( G(0) = A(p) \), \( G'(0) = -pA(p+1) \) with \( A(p) \) given by (4.26). Consequently, by (4.24) we have for \( p > 1 \)
\[
\sum_{k=1}^{\infty} e^{-zk}k^{p-1} - \Gamma(p) z^{-p} = \Gamma(p) G(z) = \Gamma(p) \left( G(0) + G'(0) z + o(z) \right),
\]
\[
|z| \to 0, \quad \text{Re}(z) \to 0^+.
\]
(4.28)
This proves (4.25) for \( p > 1 \). So, we write for \( p > 0 \)
\[
\sum_{k=1}^{\infty} e^{-zk}k^{p} = \Gamma(p+1) \left( z^{-p-1} + A(p+1) \right) + O(z), \quad |z| \to 0, \quad \text{Re}(z) \to 0^+.
\]
(4.29)
Next, integrating (4.29) w.r.t. \( z \) gives (4.25) with a constant \( A(p) \) that is not known explicitly.

Now we are in a position to derive the asymptotic formula for the free parameter \( \delta \).

**Proposition 4.3** Assume that \( n \) is s.t. \( \alpha := N/n \to \infty \), as \( N \to \infty \). Then
\[
\delta = p\alpha^{-1} \left( 1 - A(p)p^{-p} \right) + o(\alpha^{-p-1}), \quad \alpha \to \infty.
\]
(4.30)
**Proof:** First observe that in the case of the function \( a \) considered, it follows from (3.18), (3.19) and the proof of Lemma 3.2, that \( \alpha = N/n \to \infty \) implies \( \delta_{n,N} \to 0^+ \). Implementing (4.25) gives
\[
\mathbb{E} \xi_1 = \frac{\sum_{k=1}^{\infty} k^pe^{-\delta k}}{\sum_{k=1}^{\infty} k^{p-1}e^{-\delta k}} = \frac{\delta^{-p-1} + A(p+1) + O(\delta)}{\delta^{-p} + A(p) + O(\delta)} = \frac{p\delta^{-1} \left( 1 - A(p)p^{-p} + o(p) \right)}{\delta - p + o(\delta)}, \quad \delta \to 0^+, \quad p > 0.
\]
(4.31)
Now (3.19) leads to
\[ \delta = p_0 \alpha^{-1} \left( 1 - A(p) \delta^p + o(\delta^p) \right), \quad \delta = \delta_{n,N} \to 0^+, \quad \alpha \to \infty. \quad (4.32) \]

Iterating this equation w.r.t. \( \delta \) gives (4.30).

We will focus now on asymptotics, as \( \alpha \to \infty \), of the probability \( \mathbb{P}(T_n = N) \) under \( \delta \) given by (4.30). We have
\[ \mathbb{P}(T_n = N) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \varphi(t) e^{-itN} dt, \quad (4.33) \]
where \( \varphi \) is the characteristic function of \( T_n \).

The basic idea of the Khintchine’s method is that for a wide class of models, choosing the free parameter from the condition (3.19) guarantees that the main contribution to the integral in the RHS of (4.33) comes from a set which is some neighborhood of zero.

We will demonstrate that this is in effect true in the case considered and will prove the normal local limit theorem for the sum \( T_n \), as \( \alpha \to \infty \). As a preliminary step, we verify the validity of the Lyapunov’s condition
\[ \lim_{n \to \infty} \frac{M_3}{M_2^3} = 0, \quad (4.34) \]
where \( M_2 = \text{Var}_{\delta} T_n = n \text{Var}_{\delta} \xi_1 = n \left( S^{-1}(\delta) \sum_{k=1}^{\infty} k^2 a_k e^{-\delta k} - \alpha^2 \right) \), \( \delta > 0 \)
and
\[ M_3 = E(T_n - N)^3 = nE(\xi_1 - \alpha)^3 = n(E\xi_1^3 - 3\alpha E\xi_1^2 + 2\alpha^3). \quad (4.35) \]

Applying now (4.25) and (4.30) gives the following asymptotic expressions for the moments considered:
\[ M_2 \sim n \left( p(p+1)\delta^{-2} - p^2\delta^{-2} \right) = np\delta^{-2}, \quad (4.36) \]
\[ M_3 \sim n \left( p(p+1)(p+2)\delta^{-3} - 3p^2(p+1)\delta^{-3} + 2p^3\delta^{-3} \right) = 2p\delta^{-3}n, \quad n, \alpha \to \infty. \quad (4.37) \]
This proves (4.31). The condition (4.34) provides the existence of \( \beta = \beta(n, \delta) > 0 \) s.t.
\[ \beta^2 M_2 \to \infty, \quad \beta^3 M_3 \to 0, \quad n, \alpha \to \infty. \quad (4.38) \]
Explicitly, in view of (4.37),
\[ \beta = \delta n^{-\frac{1-\epsilon}{2}}, \quad 0 < \epsilon < 1/3 \quad (4.39) \]
satisfies (4.38). As it will be shown below, \([-\beta, \beta]\) is just the required neighborhood of zero.

**Lemma 4.4 (The local limit theorem for \(T_n\)).**

Set

\[ n = Q_p N_p^{\frac{p}{p+1}} + sN^{\frac{n}{2p+2}}, \quad Q_p = p^{-1} (\Gamma(p+1))^{\frac{1}{p+1}}, \quad s \in R, \quad p > 0. \]  

(4.40)

Then

\[ P(T_n = N) \sim \frac{1}{\sqrt{2\pi Var T_n}}, \quad N \to \infty. \]  

(4.41)

**Proof:** We write

\[ I = I_1 + I_2, \]  

(4.42)

where \( I = \int_{\pi}^{\pi} \varphi(t)e^{-itN} dt \) and \( I_1, I_2 \) are integrals of the integrand of \( I \) taken over the sets \([-\beta, \beta]\) and \([-\pi, -\beta] \cup [\beta, \pi]\) respectively.

**Step 1.** We find the asymptotics of the integral \( I_1 \), when \( \beta \) is as given by (4.39).

By the definition of \( \alpha \),

\[ \varphi(t)e^{-itN} = \varphi_1^n(t), \quad t \in R, \]  

(4.43)

where \( \varphi_1 \) is the characteristic function of the r.v. \( \xi_1 - \alpha \). Next, denoting by \( \varphi_2 \) the characteristic function of the r.v. \( (p^{-1/2}\delta) (\xi_1 - \alpha) \), we get from (4.35)–(4.37)

\[ \varphi_2(t) = 1 - \frac{1}{2} t^2 + O(t^3), \quad t \to 0. \]  

(4.44)

Combining this with the relationship

\[ \varphi_2(t) = \varphi_1(p^{-1/2}\delta t), \quad t \in R, \]  

(4.45)

(4.43) becomes

\[ \varphi(t)e^{-itN} = \left(1 - \frac{1}{2} (\sqrt{p}\delta^{-1}t)^2 + O(\delta^{-3}t^3)\right)^n \sim \right. \]

\[ \exp \left(-\frac{1}{2} (\sqrt{p}\delta^{-1}t)^2 + nO(\delta^{-3}t^3)\right), \quad t\delta^{-1} \to 0, \quad n \to \infty. \]  

(4.46)

Consequently, by virtue of (4.37) and (4.38),

\[ I_1 \sim \sqrt{\frac{2\pi}{Var T_n}}, \quad n, \alpha \to \infty, \quad p > 0. \]  

(4.47)

**Step 2.** We are to show that

\[ I_2 = o(I_1), \quad N \to \infty, \quad p > 0. \]  

(4.48)
We apply (4.24) with $z = \delta - it$, $t \in [\beta, \pi]$ to obtain from (4.27) the following analog of (4.28):

$$
\sum_{k=1}^{\infty} e^{-zk}k^{p-1} = \Gamma(p) \left( z^{-p} + G(-it) + \delta G'(-it) \right) + o(\delta), \quad p > 1, \quad t \in [\beta, \pi], \quad \delta \to 0^+,
$$

(4.49)

where $|G(it)|, |G'(it)| \leq \text{const} := B(p), \quad t \in [\beta, \pi], \quad p > 1$. By the same argument as for (4.29) the latter yields

$$
|\varphi_1(t)| \leq \frac{(\delta^2 + \beta^2)^{-\frac{1}{2}} + B(p) + O(\delta)}{\delta^{p} + A(p) + O(\delta)}, \quad t \in [\beta, \pi], \quad p > 0, \quad \delta \to 0^+.
$$

(4.50)

In the rest of this section it is always assumed that $n$ is specified as in (4.40). In view of (4.40) we have

$$
n\alpha^{-p} = \frac{n^{p+1}}{N^p} = p^{-p}\Gamma(p) + \epsilon_N, \quad p > 0, \quad N \to \infty,
$$

(4.51)

where

$$
\epsilon_N = (p + 1)Q_0 p^s N^{-p\frac{1}{p+1}} + \frac{p(p+1)}{2}Q_0^{-1}s^2 N^{-\frac{1}{p+1}} + O(N^{-\frac{3p}{p+1}}), \quad p > 0, \quad N \to \infty.
$$

(4.52)

This fact will be repeatedly used in our subsequent asymptotic analysis. Consequently,

$$
n\delta^p = \Gamma(p) + p^p\epsilon_N + O(\alpha^{-p}), \quad p > 0, \quad N \to \infty.
$$

(4.53)

Employing (4.53) we further obtain from (4.50) and (4.40)

$$
|\varphi_1(t)|^n \leq O \left( \exp \left( -\frac{p}{2}n^s \right) \right), \quad t \in [\beta, \pi], \quad p > 0, \quad n \to \infty,
$$

(4.54)

for any $0 < \epsilon < \frac{1}{3}$. This together with (4.47) proves (4.48).

To establish our main result, it is left to find the asymptotic formulae for the rest of the factors in (3.15). First we make use of the following result of [8]:

$$
c_N \sim \frac{1}{\sqrt{2\pi B_N^2}} \exp \left( N\sigma + \sum_{k=1}^{N} k^{p-1}e^{-k\sigma} \right), \quad N \to \infty, \quad p > 0,
$$

(4.55)

where

$$
\sigma = \sigma_N \sim \left( \frac{N}{\Gamma(p+1)} \right)^{-\frac{1}{p+1}}, \quad p > 0, \quad N \to \infty
$$

(4.56)

is the unique solution of the equation

$$
\sum_{k=1}^{N} e^{-k\sigma}k^{p} = N, \quad p > 0,
$$

(4.57)

while

$$
B_N^{2} = \sum_{k=1}^{N} e^{-k\sigma}k^{p+1}, \quad p > 0.
$$

(4.58)
For our purpose, we need to know the second term in the asymptotic expansion (4.56) of \( \sigma \). In what follows we denote by \( \epsilon_N \) different quantities tending to zero, as \( N \to \infty \). By our asymptotic formula (4.25),

\[
\sum_{k=1}^{\infty} k^p e^{-k\sigma} = \Gamma(p+1) \left( \sigma^{-(p+1)} + A(p+1) \right) + O(\sigma), \quad p > 0, \quad N \to \infty,
\]

(4.59)

where \( A(p) \) is as in Lemma 4.2, while, by the Euler summation formula, we have for any \( \gamma > 0 \), and \( \sigma = \sigma_N \) given by (4.56),

\[
\sum_{k=N+1}^{\infty} k^p e^{-k\sigma} \sim \int_{N+1}^{\infty} x^p e^{-x \sigma} \, dx = \sigma^{-(p+1)} \int_{\sigma(N+1)}^{\infty} x^p e^{-x} \, dx = o(N^{-\gamma}), \quad p > 0, \quad N \to \infty.
\]

(4.60)

In view of (4.59), (4.60), the equation (4.57) can be rewritten now as

\[
\Gamma(p+1) \left( \sigma^{-(p+1)} + A(p+1) \right) + O(\sigma) = N.
\]

(4.61)

Consequently, we get

\[
\sigma = \left( \frac{N}{\Gamma(p+1)} - A(p+1) + \epsilon_N \right)^{-\frac{1}{p+1}} = \left( \frac{N}{\Gamma(p+1)} + A(p+1) \right)^{-\frac{1}{p+1}} N^{-\frac{1}{p+1}} + \frac{A(p+1)}{p+1} \left( \frac{N}{\Gamma(p+1)} \right)^{\frac{p+2}{p+1}} + o(N^{-\frac{p+2}{p+1}}), \quad p > 0, \quad N \to \infty.
\]

(4.62)

This yields

\[
N\sigma = N^{-\frac{1}{p+1}} \left( \Gamma(p+1) \right)^{\frac{1}{p+1}} + \epsilon_N, \quad p > 0, \quad N \to \infty.
\]

(4.63)

Next, we obtain from (4.25), (4.58), (4.60) and (4.62)

\[
B_N^2 \sim \left( \frac{N}{\Gamma(p+1)} \right)^{\frac{p+2}{p+1}} \Gamma(p+2), \quad p > 0, \quad N \to \infty
\]

(4.64)

and

\[
\sum_{k=1}^{N} k^{p-1} e^{-k\sigma} = \sum_{k=1}^{\infty} k^{p-1} e^{-k\sigma} + \epsilon_N = \Gamma(p) \left( \frac{N}{\Gamma(p+1)} \right)^{\frac{p}{p+1}} + \Gamma(p) A(p) + \epsilon_N, \quad p > 0, \quad N \to \infty.
\]

(4.65)

Substituting the above expressions in (4.55) gives the desired asymptotic formula for \( c_N \):

\[
c_N \sim h_1 N^{-\frac{p+2}{p+1}} \exp(h_2 N^{\frac{p}{p+1}} + h_3), \quad p > 0, \quad N \to \infty,
\]

(4.66)

where the constants \( h_i, \ i = 1, 2, 3 \) are given by

\[
h_1 = \left( \frac{\Gamma(p+1)}{2\pi(p+1)} \right)^{\frac{1}{2(p+1)}},
\]

and

\[
h_2 = \frac{1}{2(p+1)} \sqrt{2\pi(p+1)},
\]

(4.67)
\[ h_2 = (p + 1)Q_p, \]

\[ h_3 = \Gamma(p)A(p), \quad p > 0. \]  

Next, (4.51), (4.40) and (4.30) give

\[ \exp(\delta N) \sim \exp \left( pm - A(p)\Gamma(p + 1) \right), \quad p > 0, \quad N \to \infty, \]  

while (4.53) gives

\[ S^n(\delta) = \left( \Gamma(p)\delta^{-p} + A(p)\Gamma(p) + O(\delta) \right)^n \sim \left( \Gamma(p) \right)^n \delta^{-pn} \exp \left( A(p)\Gamma(p) \right), \]

\[ p > 0, \quad N \to \infty. \]  

We again apply (4.51) to get

\[ \delta^{-pn} \sim p^{-pn} \alpha^{pn} \exp \left( A(p)\Gamma(p + 1) \right), \]

\[ \alpha^{pn} \sim \frac{n^n p^n \exp \left( -n \left( \frac{p^2}{2\Gamma(p)} \epsilon_N - \frac{1}{2} \left( \frac{p^2}{2\Gamma(p)} \right)^2 \epsilon_N^2 + O(\epsilon_N^3) \right) \right)}{\left( \Gamma(p) \right)^n}, \quad p > 0, \quad N \to \infty, \]  

where \( \epsilon_N \) is given by (4.52). Observing that \( n\epsilon_N^3 \to 0, \quad N \to \infty \), we write out now the asymptotic expressions for \( n\epsilon_N \) and \( n\epsilon_N^2 \) to obtain

\[ S^n(\delta) \sim n^n \exp \left( -\frac{s^2}{2d_p} - s(p + 1)N \frac{p^2}{2(p + 1)} + A(p)(\Gamma(p) + \Gamma(p + 1)) \right), \]

\[ d_p = \frac{Q_p}{p + 1}, \quad p > 0, \quad N \to \infty. \]  

Finally, substituting in (3.15) the preceding asymptotic expansions and employing Stirling’s asymptotic formula gives the following result.

**Theorem 4.5 (The local limit theorem for \( \nu_N \)).**

Let \( n \) be given as in (4.40). Then

\[ P(\nu_N = n) \sim \frac{1}{\sqrt{2\pi d_p}} N^{-\frac{p}{2(p + 2)}} \exp \left( -\frac{s^2}{2d_p} \right) := f(N; s), \quad s \in R, \quad p > 0, \quad N \to \infty. \]  

This leads to our main result that says that almost all the mass of the probability distribution of the r.v. \( \nu_N \) is concentrated, as \( N \to \infty \), in a neighborhood of size \( O(N^{-\frac{p}{p + 2}}) \) of the point \( Q_p N^{-\frac{p}{p + 2}} \).
Theorem 4.6 (The central limit theorem for $\nu_N$).

$$
\frac{\nu_N - Q_p N^{p+1}}{\sqrt{d_p N^{2p+2}}} \Rightarrow N(0,1), \quad p > 0, \quad N \to \infty,
$$

(4.73)

where $\Rightarrow$ denotes weak convergence and $d_p, Q_p$ are as in (4.71), (4.40) respectively.

**Proof:** We provide a sketch of the proof that is done by the implementation of the standard technique of passing from the local theorem to the integral theorem (for more details see ([16], p.59, [6], p.81).

It follows from (4.72) that for any $a \leq b, \ a, b \in \mathbb{R},$

$$
P\left(\frac{\nu_N - Q_p N^{p+1}}{N^{2p+2}} \in [a,b]\right) = \sum_{s \in R_N} f(N; s), \quad p > 0, \quad N \to \infty,
$$

(4.74)

where $f(N; s)$ is given by (4.72) and $R_N = \{s \in [a,b] : n = Q_p N^{p+1} + s N^{2p+2} \in \mathbb{N}\}$.

Since $|R_N| = O(N^{2p+2})$, as $N \to \infty$, we have

$$
\sum_{s \in R_N} \exp \left(-\frac{s^2}{2d_p}\right) N^{-2p} \to \int_a^b \exp \left(-\frac{x^2}{2d_p}\right) dx, \quad p > 0, \quad N \to \infty. \quad \blacksquare
$$

(4.75)

**Remark:** Since the transformation $a_k \Rightarrow h^k a_k, \ h > 0, \ k \geq 1$ of the parameter function $a$ does not change the measure $\mu_N$, the results of our paper are true for $a_k = h^k k^{p-1}, \ k \geq 1, \ h > 0, \ p > 0. \quad \blacksquare$

One more extension of Theorem 4.6 is provided by the following result.

**Theorem 4.7:** If $\tilde{a}_k = q k^{p-1}, \ p > 0, \ k \geq 1, \ where \ q > 0 \ is \ a \ constant, \ then$

$$
\frac{\nu_N - \tilde{Q}_p N^{p+1}}{\sqrt{\tilde{d}_p N^{2p+2}}} \Rightarrow N(0,1), \quad p > 0, \quad N \to \infty,
$$

(4.76)

where $\tilde{Q}_p = q^{p+1} Q_p$, $\tilde{d}_p = q^{p+1} d_p$.

**Proof:** Denote by $\bullet$ the quantities related to the parameter function $\tilde{a}$. We see from (3.13), (4.57) and (4.62) that the r.v. $\tilde{\xi}_1$ has the same distribution as $\xi_1$ and that

$$
\tilde{\sigma}_N \sim q^{p+1} \sigma_N, \quad N \to \infty.
$$

(4.77)

Repeating the preceding asymptotic analysis gives the claimed change in the scaling induced by $q > 0. \quad \blacksquare$

We conclude this section by providing some intuition for the scaling in the central limit Theorem 4.6.
For this purpose we employ the following result established in [9] for the model in question. Denote by $\bar{q}(\eta), \underline{q}(\eta)$ the size of the largest (resp. smallest) group in a random partition $\eta \in \Omega_N$. Then

$$\lim_{N \to \infty} P \left( N^{\frac{1}{p+1}-\epsilon} < \bar{q}(\eta) < N^{\frac{1}{p+1}+\epsilon} \right) = 1, \quad p > 0,$$

(4.78)

for all $\epsilon > 0$, while

$$\lim_{N \to \infty} P \left( q(\eta) \geq l \right) =
\begin{cases} 
0, & \text{if } l = N^\beta, \quad 0 < \beta \leq 1, \\
\exp \left( - \sum_{j=1}^{l-1} a_j \right), & \text{if } l \geq 2 \text{ is a fixed number.}
\end{cases}$$

(4.79)

From (4.78) and (4.79) one may conjecture, that for large $N$, the main "mass", of size $O(N)$, is partitioned into groups (=clusters) of sizes $O\left(N^{\frac{1}{p+1}}\right)$, while the rest of the mass, of size $o(N)$, is partitioned into groups of small sizes. Adopting this conjecture gives the expectation of the number of groups $\nu_N$ as

$$\mathbb{E} \left[ \nu_N \right] = O\left( N^{\frac{p}{p+1}} \right).$$

5 RCS’s: examples and comparison with known results.

It was already observed in [8] and [9] that particular cases of the expression (2.7) for the equilibrium measure $\mu_N$ conform to joint distributions of components of a variety of RCS’s, known as assemblies. By a combinatorial structure (CS) of a size $N$ one means a union of nondecomposable components (=components) of different sizes. Formally, such a structure is given by the two sets of integers $\{p_N, \ N \geq 1\}$ and $\{m_N, \ N \geq 1\}$ that count respectively the total number of instances of size $N$ and the number of components of size $N$. An example of a CS is a graph on $N$ vertices treated as a union of its connected components. Therefore, an instance of a CS of size $N$ is given by a partition $\eta = (n_1, \ldots, n_N) \in \Omega_N$, where $n_k$ is the number of components of size $k$ in the instance. By assuming that for a given $N$ an instance is chosen randomly from all $p_N$ instances, one induces a RCS that is completely determined by the random component size counting process, the latter being a random vector with values in $\Omega_N$. With an obvious abuse of notation we denote the random vector by $\eta$. A remarkable fact in the theory of RCS’s is that a great variety of them obey the conditioning relation

$$\mathcal{L}(\eta) = \mathcal{L}(Z_1, \ldots, Z_N) \left| \sum_{j=1}^{N} jZ_j = N \right.,
\quad \text{where} \ Z_1, \ldots, Z_N \ \text{are independent integer valued r.v.’s.}
\quad \text{In particular, a class of CS’s known as assemblies is characterized by the fact that} \ Z_k \ \text{are Poisson r.v.’s s.t.} \ Z_k \sim P\left(m_k/k!\right), \ k = 1, 2, \ldots, N.
\quad \text{Consequently, by a straightforward calculation we find from (5.1) that for assemblies}
\quad \mathbb{P}(\eta) = \mu_N(\eta), \quad \eta \in \Omega_N,$n_2) \quad \text{is a fixed number.}$n_3) \quad \text{is a fixed number.}$
with \( a_k = \frac{m_k}{k!} \) and \( c_k = \frac{p_k}{k!} \), \( k = 1, 2, \ldots, N \). In particular, the case \( m_k \sim \theta(k-1)! \), \( \theta > 0 \) conforms to logarithmic RCS’s that encompass permutations \( (m_k = (k-1)!) \) and the Evens sampling formula \( (m_k = \theta(k-1)! \), \( \theta > 0 \)). The novel theory of general logarithmic RCS’s is presented in [11]. From the analytical point of view, the common feature of logarithmic RCS’s is that they do not obey the condition \( p > 0 \) adopted in the present paper. As a result, our asymptotic analysis based on the Poisson summation formula is not applicable for logarithmic RCS’s.

The case of permutations has a long history. For this case, the seminal result by V.L. Goncharov (1944) and L. Shepp and S. Lloyd (1966) states the following central limit theorem for \( \nu_n \):

\[
\frac{\nu_n - \log N}{\sqrt{\log N}} \Rightarrow N(0,1).
\] (5.3)

A version of (5.3) for general logarithmic RCS’s is also known.

L. Mutafchiev ([14]) proved a local limit theorem for \( \nu_n \) under some assumptions on the asymptotic behavior as \( x \to 1 \) of the generating function of the sequence \( \{c_n\}_1^\infty \). It can be verified that in the case \( p > 0 \) the assumptions (2.7) and (2.8) in [14] do not hold. However, Mutafchiev ([14],p.425) conjectured, that a result similar to his Theorem 2.4 holds for a wider class of RCS’s. Our Theorem 4.6 confirms this conjecture. In fact, in the case \( p > 0 \), we have \( \mathbb{E}\nu_n \sim Q_p N^{\frac{p}{p+1}} \) and it is not hard to see that, in agreement with the claim in [14], \( Q_p N^{\frac{p}{p+1}} \sim S(e^{-\sigma N}), N \to \infty \), where \( \sigma_N \) is as in (4.5).

A. Barbour and B. Granovsky ([4]) explored the case when \( Z_k \) in (5.1) are quite general r.v.’s obeying \( \mathbb{E}Z_k \sim k^{p-1}, k \to \infty, \ p < 0 \). (This includes assemblies with \( m_k \sim k^{p-1}k!, k \to \infty, \ p < 0 \).) It was shown in ([4]) that such RCS’s exhibit a completely different asymptotic behavior. In particular, in this case \( \nu_n \) is finite with probability 1.

Hence, comparing the asymptotic behavior of \( \nu_n \) (e.g. \( \mathbb{E}\nu_n \)) in the cases \( p < 0 \) and \( p \geq 0 \), one sees that \( p = 0 \) can be viewed as a point of phase transition for the measure \( \mu_N \) as \( N \to \infty \).

In conclusion, we provide a few examples of assemblies that conform to the setting of the present paper:

**Example 1. Forests of labelled and colored linear trees.** A linear tree (see [5]) is a graph with no cycles, where each vertex has no more than two neighbors. Assuming that a vertex is labelled and is colored into one of \( q \) (\( q \geq 1 \)) colors gives \( m_k = q^k k!, a_k = q^k, k \geq 1 \). So, by the remark following Theorem 4.6, this RCS corresponds to the case \( p = 1 \).

**Example 2. Forests of labelled rooted linear trees.** In this case we have \( m_k = k^k, k \geq 1 \), which gives \( a_k = k, k \geq 1 \), that corresponds to the case \( p = 2 \).

**Example 3. Compositions.** (see [18]) Consider a space \( \Upsilon_N \) of ordered m-tuples \( \mathbf{x} = (x_1, \ldots, x_m), m = 1, \ldots, N \), where \( x_i \) are positive integers, summing to \( N \). In other words, \( \Upsilon_N \)
is a space of all ordered partitions (=compositions) of \( N \). We define the probability measure \( \lambda_N \) on \( \Upsilon_N \):

\[
\lambda_N(\mathbf{x}) = \frac{1}{(m(\mathbf{x}))!}(r_N)^{-1}, \quad \mathbf{x} = (x_1, \ldots, x_m) \in \Upsilon_N,
\]  

(5.4)

where \((r_N)^{-1}\) is the normalizing constant and \( m = m(\mathbf{x}) \) is the number of components of \( \mathbf{x} \). This means that \( \lambda_N \) prescribes to \( \mathbf{x} \in \Upsilon_N \) the weight \( \frac{1}{(m(\mathbf{x}))!} \). Denote \( n_i = n_i(\mathbf{x}), \ i = 1, \ldots, N \) the number of components of \( \mathbf{x} \in \Upsilon_N \) that are equal to \( i \). Then we have

\[
r_N = \sum_{m=1}^{N} \frac{1}{m!} \sum_{\mathbf{x} \in \Upsilon_N : m(\mathbf{x})=m} \frac{1}{m!} \sum_{\eta \in \Omega_N, \ |\eta| = m} \frac{m!}{n_1! \ldots n_m!} = \sum_{\eta \in \Omega_N} \frac{1}{n_1! \ldots n_N!},
\]  

(5.5)

where, as before, \( \Omega_N \) is the set of all (unordered) partitions of \( N \), while \( |\eta| = n_1 + \ldots + n_N \). (5.5) says that \( r_N = c_N \), where \( c_N \) is the partition function of the measure \( \mu_N \) in the case \( a_k = 1, \ k = 1, \ldots, N \). Therefore, we can apply our results for \( p = 1 \) to the number of summands in the random composition drawn according to \( \lambda_N \).
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