\(\infty\)-JETS OF DIFFEOMORPHISMS PRESERVING ORBITS OF VECTOR FIELDS

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Abstract. Let \(F\) be a \(C^\infty\) vector field defined near the origin \(O \in \mathbb{R}^n\), \(F(O) = 0\), and \((F_t)\) be its local flow. Denote by \(\hat{\mathcal{E}}(F)\) the set of germs of orbit preserving diffeomorphisms \(h : \mathbb{R}^n \to \mathbb{R}^n\) at \(O\), and let \(\hat{\mathcal{E}}_{id}(F)^r\), \((r \geq 0)\), be the identity component of \(\hat{\mathcal{E}}(F)\) with respect to \(C^r\) topology. Then \(\hat{\mathcal{E}}_{id}(F)^\infty\) contains a subset \(\hat{\mathcal{S}}h(F)\) consisting of maps of the form \(F_\alpha(x)\), where \(\alpha : \mathbb{R}^n \to \mathbb{R}\) runs over the space of all smooth germs at \(O\). It was proved earlier by the author that if \(F\) is a linear vector field, then \(\hat{\mathcal{S}}h(F) = \hat{\mathcal{E}}_{id}(F)^0\).

In this paper we present a class of examples of vector fields with degenerate singularities at \(O\) for which \(\hat{\mathcal{S}}h(F)\) formally coincides with \(\hat{\mathcal{E}}_{id}(F)^1\), i.e. on the level of \(\infty\)-jets at \(O\).

We also establish parameter rigidity of linear vector fields and “reduced” Hamiltonian vector fields of real homogeneous polynomials in two variables.

Keywords: orbit preserving diffeomorphism, parameter rigidity, Borel’s theorem.

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1. Introduction

Let \(F\) be a smooth \((C^\infty)\) vector field on a smooth manifold \(M\), \((F_t)\) be the local flow generated by \(F\), and \(\Sigma_F\) be the set of singular points of \(F\). In this paper we consider smooth maps \(h : M \to M\) preserving the (singular) foliation on \(M\) by orbits of \(F\), i.e. \(h(M \cap \gamma) \subset \gamma\) for every orbit \(\gamma\) of \(F\).

The groups of leaf preserving diffeomorphisms and homeomorphisms of foliations are intensively studied. Most of the results concern with regular foliations, see e.g. [B77, Ryb1, Ryb2, AF03] and references in these papers. For singular foliations the situation is much more difficult. Therefore usually foliations by orbits of actions of finite-dimensional Lie groups are considered, e.g. [Sch75, Ma77, AF01, Ryb3]. Homeomorphisms preserving foliations of vector fields are studied e.g. in [CN77, GM77].

The approach used in this paper is specific for the case of flows. By definition for every \(x \in M\) its image \(h(x)\) belongs to the orbit \(\gamma_x\) of \(x\). Therefore we want to associate to \(x\) the time \(\alpha_h(x)\) between \(x\) and \(h(x)\) along \(\gamma_x\), so that

\[
h(x) = F^{\alpha_h(x)}(x).
\]
We will call $\alpha_h$ a *shift function* for $h$, which in turn will be called the *shift* along orbits of $F$ via $\alpha_h$.

Such ideas were used e.g. in [Hopf37, Ch66, To66, Ko72, Pa72, Koc73] and others for reparametrizations of measure preserving flows and study their mixing properties. In these papers $\alpha_h$ is required to be measurable, so it can even be discontinuous and its values on subsets of measure 0 can be ignored. In [OS78] continuity of shift functions was investigated. In contrast, we work in $C^\infty$ category and require that $\alpha_h$ *is $C^\infty$ whenever so is $h$*. One of the main problems here is to define $\alpha_h$ near a singular point of a vector field, see 2.1-2.5.

Smooth shift functions were used in authors papers [M1, M2, M3] for calculations of homotopy types of certain infinite-dimensional spaces. The present paper brings another application of shift functions to smooth reparametrization of flows and in particular to parameter rigidity.

In [M1] the problem of finding representation (1.1) was solved for linear flows. It was shown that if $F$ is a linear vector field on $\mathbb{R}^n$, then for every diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ preserving orbits of $F$ and being isotopic to the identity map $\text{id}_{\mathbb{R}^n}$ via an orbit preserving isotopy there exists a $C^\infty$ shift function $\alpha_h$. Moreover, if a family $h_s$ of orbit preserving diffeomorphisms smoothly depends on some $k$-dimensional parameter $s$, then so does the family $\alpha_{h_s}$ of their shift functions\(^1\).

Our first result claims that the last two properties easily imply parameter rigidity of a vector field, see Theorem 4.4. In particular, as a consequence of [M1], we obtain parameter rigidity of linear vector fields and their regular extensions. Notice that this statement together with the result of S. Sternberg [St57] implies parameter rigidity of a large class of “hyperbolic” flows, which agrees with discovered about thirty years ago rigidity of locally free hyperbolic actions of certain Lie groups [KS94]. Though we consider actions of the one-dimensional group $\mathbb{R}$ only, Theorem 4.4 is nevertheless stronger in the part that we admit fixed points, i.e. non-locally free actions.

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\(^1\) I must warn the reader that my paper [M1] contains mistakes in the estimations of continuity of the correspondence $h \mapsto \alpha_h$ regarded as a map between certain functional spaces. In particular in [M1, Defn. 15] it should be additionally required that the image $\varphi_V(M)$ is at least $C^\infty_W$-open in the image of the map $\varphi_V$. Without this assumption [M1, Th. 17] is not true. Moreover in [M1, Lm. 31] the mapping $Z^{-1}$ is $C_{W,W}^{r+1,r}$-continuous in the real case and only $C_W^{r,\infty}$-continuous in the complex case. As a result the formulations of [M1, Th 1, Th.27 (part concerning (S)-points) & Lm. 28] should be changed.

Unfortunately [M1, Th. 1] was essentially used in [M2] for the calculations of the homotopy types of stabilizers and orbits of Morse functions on surfaces. We will repair the mistakes of [M1] in another paper and show that the part of results [M1] used in [M2] remains true.

Also notice that the formula [M1, Eq. (10)] for the shift functions at regular points is misprinted. It must be read as follows: $\alpha(x) = p_1 \circ f(x) - p_1 \circ \Phi(x, a) + a.$
Further we deal with the situation when $F$ is a vector field defined on some neighbourhood $V$ of the origin $O \in \mathbb{R}^n$ being its singular point. Denote by $\hat{Sh}(F)$ the group of germs at $O$ of smooth shifts, i.e. maps of the form (1.1). In §5 we describe the structure of $\infty$-jets of elements of $\hat{Sh}(F)$ and establish a necessary and sufficient condition for a certain group $\mathcal{G}$ of germs of diffeomorphisms of $(\mathbb{R}^n, O)$ to coincide with $\hat{Sh}(F)$ on the level of $\infty$-jets at $O$, see Theorem 5.1.

Let $\mathcal{J}(F, V)$ be the space of all smooth maps $h : V \to \mathbb{R}^n$ whose $\infty$-jet $j^{\infty}(h)$ at $O$ coincides with the $\infty$-jet of some shift $F_{\alpha_h}$, where $\alpha_h \in C^\infty(V, \mathbb{R})$. Though in general such a function $\alpha_h$ is not unique, we show in §§6,7 that it can be chosen so that the correspondence $h \mapsto \alpha_h$ becomes a continuous (and in a certain sense smooth) map $\Lambda : \mathcal{J}(F, V) \supset \mathcal{X} \to C^\infty(V, \mathbb{R})$ defined on some subset $\mathcal{X}$ of $\mathcal{J}(F, V)$ (Theorem 7.6). Actually §6 contains a variant of a well-known theorem of E. Borel about smooth functions with given Taylor series (Theorem 6.1). This theorem is then used in §7 for the construction of $\Lambda$. In §7.3 we also present application of $\Lambda$ to the problem of resolving (1.1).

Denote by $\hat{D}(F)$ the group of germs of orbit preserving diffeomorphisms for $F$, and let $\hat{D}_{id}(F)$ be its path component with respect to weak $C^1_W$ topology. In §8 we introduce a certain condition ($\ast$) on $F$ guaranteeing that $\hat{Sh}(F)$ coincides with $\hat{D}_{id}(F)$ on the (formal) level of $\infty$-jets, see Theorem 8.5. The proof of this theorem is given in §§9,10.

Finally, in §11 we present a class of vector fields on $\mathbb{R}^2$ satisfying condition ($\ast$) and explain that for these vector fields $\hat{Sh}(F) = \hat{D}_{id}(F)$. This improves results of [M4], which were based on a previous (unpublished) version of this paper (Theorem 11.1).

In another paper the last theorem will be used to extend calculations of [M2] to a large class of functions with degenerate singularities on surfaces.

1.1. Preliminaries. Let $A$ and $B$ be smooth manifolds. Then for every $r = 0, 1, \ldots, \infty$ we can define the weak $C^r_W$ topology on $C^\infty(A, B)$, see e.g. [GG, Hi]. We will assume that the reader is familiar with these topologies. It easily follows from definition that topology $C^0_W$ coincides with the compact open one. Moreover, let $J^r(A, B)$ be the manifold of $r$-jets of $C^r$ maps $A \to B$. Associating to every $h \in C^\infty(A, B)$ its $r$-jet extension being an element of $J^r(A, B)$, we obtain a natural inclusion $C^\infty(A, B) \subset C^r(A, J^r(A, B))$. Then $C^r_W$ topology on $C^\infty(A, B)$ can be defined as the topology induced by $C^0_W$ topology of $C^r(A, J^r(A, B))$.

We say that a subset $\mathcal{X} \subset C^\infty(A, B)$ is $C^r_W$-open if it is open with respect to the induced $C^r_W$-topology of $C^\infty(A, B)$.

1.2. Definition. Let $H : A \times I \to B$ be a homotopy such that for every $t \in I$ the mapping $H_t : A \to B$ is $C^r$. We will call $H$ an $r$-homotopy
if the following map
\[ j^r H : A \times I \to J^r(A, B), \quad (a, t) \mapsto j^r(H_t)(a), \]
associating to every \((a, t) \in A \times I\) the \(r\)-th jet of \(H_t\) at \(a\), is continuous. In local coordinates this means that \(H_t\) and all its partial derivatives “along \(A\)” are continuous in \((a, t)\). In particular, every \(C^r\)-homotopy is an \(r\)-homotopy as well.

Equivalently, regarding a homotopy \(H\) as a path \(\hat{H} : I \to C^\infty(A, B)\) defined by \(\hat{H}(t)(a) = H(a, t)\), we see that \(H\) is an \(r\)-homotopy if and only if \(\hat{H}\) is a continuous path into \(C^r\)-topology of \(C^\infty(A, B)\).

If \(H\) is an \(r\)-homotopy consisting of embeddings, it will be called an \(r\)-isotopy.

Let \(C\) and \(D\) be some other smooth manifolds and \(Y \subset C^\infty(C, D)\) be a subset. A map \(u : \mathcal{X} \to Y\) will be called \(C^s_{W,W}\)-continuous if it is continuous from \(C^s_{W}\)-topology of \(X\) to \(C^r_{W}\)-topology of \(Y\), \((r, s = 0, 1, \ldots, \infty)\).

1.3. Definition. We will say that \(u : \mathcal{X} \to Y\) preserves smoothness if for any \(C^\infty\) map \(H : A \times \mathbb{R}^n \to B\) such that \(H_t = H(\cdot, t) \in \mathcal{X}\) for all \(t \in \mathbb{R}^n\) the following mapping
\[ u(H) : C \times \mathbb{R}^n \to D, \quad u(H)(c, t) = u(H_t)(c) \]
is \(C^\infty\) as well.

2. Obstructions for shift functions

In this section we briefly discuss obstructions for smooth resolvability of (1.1).

2.1. Evidently, the value \(\alpha_h(z)\) is uniquely defined only if \(z\) is regular and non-periodic for \(F\). If \(z\) is a periodic point of period \(\theta\), then \(\alpha_h(z)\) is defined only up to a constant summand \(n\theta\), \((n \in \mathbb{Z})\), while if \(z\) is fixed, we can set \(\alpha_h(z)\) to arbitrary number.

In applications to ergodic flows this problem usually does not appear: the union of periodic and fixed points is an invariant subset, therefore it can be assumed to have measure 0. Hence the values of \(\alpha_h\) on this set may be ignored. Sometimes it is also assumed that the set of periodic points is empty, e.g. [Ko72, p.357].

2.2. If \(z\) is a regular (even periodic) point of \(F\), then \(\alpha_h\) can be smoothly defined on some neighbourhood of \(z\), see [M1, §3.1] and footnote on page 2 for correct reading of [M1, Eq. (10)].

2.3. Representation (1.1) with smooth \(\alpha_h\) implies that \(h\) is homotopic to the identity \(id\) via a smooth orbit preserving homotopy. For instance we can take the following one: \(h_t(x) = F(x, t\alpha_h(x))\).
2.4. Conversely, if $h$ is homotopic to id via some smooth orbit preserving homotopy, then (using this homotopy and 2.2) we can smoothly define $\alpha_h$ on the set of regular points of $F$, see [M1, Th. 25], though $\alpha_h$ can even be discontinuous at singular points of $F$. In general, $\alpha_h$ depends on a particular homotopy.

2.5. If $\alpha_h$ can be defined at some singular point $z$ of $F$ so that it becomes smooth near $z$, then $h$ must be an embedding at $z$, [M1, Cor. 21].

3. Shift map

Observations of the previous section lead to the following construction of shift map used in [M1].

Let $M$ be a smooth manifold and $F$ be a vector field on $M$ tangent to $\partial M$. Then for every $x \in M$ its orbit with respect to $F$ is a unique mapping $\gamma_x : \mathbb{R} \ni (a_x, b_x) \to M$ such that $\gamma_x(0) = x$ and $\dot{\gamma}_x = F(\gamma_x)$, where $(a_x, b_x) \subset \mathbb{R}$ is the maximal interval on which a map with the previous two properties can be defined. Then

$$\text{dom}(F) = \bigcup_{x \in M} x \times (a_x, b_x),$$

is an open neighbourhood of $M \times 0$ in $M \times \mathbb{R}$, and by definition the local flow of $F$ is the following map

$$F : M \times \mathbb{R} \ni \text{dom}(F) \longrightarrow M, \quad F(x, t) = \gamma_x(t).$$

If $M$ is compact, or more generally if $F$ has compact support, then $\text{dom}(F) = M \times \mathbb{R}$ and thus $F$ is global, i.e. is defined on all of $M \times \mathbb{R}$, see e.g. [PM].

For every open $V \subset M$ denote by $\text{func}(F, V)$ the subset of $C^{\infty}(V, \mathbb{R})$ consisting of functions $\alpha$ whose graph $\Gamma_\alpha = \{(x, \alpha(x)) : x \in V\}$ is contained in $\text{dom}(F)$. Then we can define the following map

$$\varphi_V : C^{\infty}(V, \mathbb{R}) \supset \text{func}(F, V) \longrightarrow C^{\infty}(V, M),$$

(3.1)

$$\varphi_V(\alpha)(x) = F(x, \alpha(x)),$$

which will be called the shift map of $F$ on $V$. Its image in $C^{\infty}(V, M)$ will be denoted by $\text{Sh}(F, V)$. If $F$ is global, then $\text{func}(F, V) = C^{\infty}(V, \mathbb{R})$.

It is easy to see that $\varphi_V$ is $C^{r}_{W,W}$-continuous for all $r = 0, 1, \ldots, \infty$, [M1, Lemma 2]. Moreover, if the set $\Sigma_F$ of singular points of $F$ is nowhere dense in $V$, then $\varphi_V$ is locally injective with respect to any $C^{r}_{W}$ topology of $\text{func}(F, V)$, [M1, Prop. 14].

Denote by $\mathcal{E}(F, V) \subset C^{\infty}(V, M)$ the subset consisting of all smooth maps $h : V \to M$ such that

- $h(\omega \cap V) \subset \omega$ for every orbit $\omega$ of $F$, in particular $h$ is fixed on $\Sigma_F \cap V$, and
- $h$ is a local diffeomorphism at every $z \in \Sigma_F \cap V$. 

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Then the following statements hold true.

**Proof.**

(a) Since the independence of solutions of ODE on initial values that for arbitrary large $A \geq 0$, there exists a neighbourhood $W_A$ of $z$ such that $W_A \times [-A, A] \subset \text{dom}(\mathbf{F})$. Hence if $A > |z|$ then $\mathbf{F}_\alpha$ is defined on some neighbourhood of $z$ contained in $W_A$. The proof for $\mathbf{F}_{h,\beta}$ is similar.

(b) is proved in [M1, Eqs.(8),(9) and Corollary 21], see also 2.5.

(c) Eq. (3.4) just means that $\mathbf{F}_{h,\beta}(x) = \mathbf{F}(h(x), \beta \circ h^{-1} \circ h(x))$.

(d) Finally, the verification of (3.5) is direct. 

\qed
Let \( F(z) = 0 \). Then we have a well-defined local shift map at \( z \)
\[
\hat{\varphi} : \hat{F}_z(M) \to \hat{D}(M, z), \quad \hat{\varphi}(\alpha) = F_\alpha.
\]
Denote its image \( \hat{\varphi}(\hat{F}_z(M)) \) in \( \hat{D}(M, z) \) by \( \hat{Sh}(F, z) \).

Let also \( \hat{D}(F, z) \) be the subset of \( \hat{D}(M, z) \) consisting of orbit preserving germs, i.e. \( h \in \hat{D}(M, z) \) provided there exists a neighbourhood \( V \) of \( z \) such that \( h(\gamma \cap V) \subset \gamma \) for every orbit \( \gamma \) of \( F \).

For every \( r = 0, 1, \ldots, \infty \) let \( \hat{D}_{\text{id}}(F, z)^r \) be the “identity component” of \( \hat{D}(F, z) \) with respect to \( C^r \)-topology, i.e. \( \hat{D}_{\text{id}}(F, z)^r \) consists of all \( h \in \hat{D}(F, z) \) for which there exists a neighbourhood \( V \subset M \) of \( z \) and an \( r \)-isotopy \( H : V \times I \to M \) such that \( H_0 = i_V : V \subset M, H_t \in \hat{D}(F, z) \) for all \( t \in I \), and \( H_1 = h \).

Then similarly to (3.2) we have the following inclusions:
\[
\hat{Sh}(F, z) \subset \hat{D}_{\text{id}}(F, z)_{\infty} \subset \cdots \subset \hat{D}_{\text{id}}(F, z)^1 \subset \hat{D}_{\text{id}}(F, z)^0.
\]

4. Parameter rigidity

In recent years there were obtained many results concerning rigidity of hyperbolic and locally free actions of certain Lie groups and their lattices, see e.g. [KS94, Hur94, Ka96, KS97, MM03, D07, EF07] and references there. Roughly speaking a rigidity of an action \( T \) means that every action \( T' \) which is sufficiently close in a proper sense to \( T \) is conjugate to \( T \).

For instance, in a recent paper [Sa07] by N. dos Santos parameter rigidity of locally free actions of contractible Lie groups on closed manifolds are considered. In the case of vector fields, i.e. actions of \( \mathbb{R} \), local freeness means regularity of orbits. In contrast we will consider certain classes of vector fields with singular points, i.e. not locally free \( \mathbb{R} \)-actions, and prove their parameter rigidity, see 4.4 and 4.6.

4.1. Definition. (c.f. [Sa07]) We say that a vector field \( F \) on a manifold \( M \) is parameter rigid if for any vector field \( G \) on \( M \) such that every orbit of \( G \) is contained in some orbit of \( F \) there exists a \( C^\infty \)-function \( \alpha \) such that \( G = \alpha F \).

Let \( \Sigma_F \) and \( \Sigma_G \) be the sets of singular points of \( F \) and \( G \) respectively. The assumption that orbits of \( G \) are contained in orbits of \( F \) implies that \( \Sigma_F \subset \Sigma_G \) and that \( F \) and \( G \) are parallel on \( M \setminus \Sigma_G \). Therefore there exists a \( C^\infty \) function \( \mu : M \setminus \Sigma_F \to \mathbb{R} \setminus \{0\} \) such that \( G = \mu F \) on \( M \setminus \Sigma_F \). Then Definition 4.1 requires that \( \mu \) smoothly extends to all of \( M \) for any such \( G \).

4.2. Lemma (Extensions of shift functions under homotopies). Let \( V \subset M \) be an open subset, \( \alpha_0 \in \text{func}(F, V) \), and \( H : V \times I \to M \) be a \( C^\infty \)-homotopy such that \( H_0 = \varphi(\alpha_0) \) and \( H_t \in Sh(F, V) \) for every
Then there exists a unique $C^\infty$ function $\Lambda : (V \setminus \Sigma_F) \times I \to \mathbb{R}$ such that
\begin{equation}
\Lambda(x, 0) = \alpha_0(x), \quad H(x, t) = F(x, \Lambda(x, t)),
\end{equation}
for all $(x, t) \in (V \setminus \Sigma_F) \times I$. Thus $\Lambda$ is a shift function for $H$ on $(V \setminus \Sigma_F) \times I$ which extends $\alpha_0$.

\textbf{Proof.} This statement was actually established during the proof of [M1, Th. 25] for the case $\alpha_0 \equiv 0$. But the same arguments show that the proof holds for any $\alpha_0 \in \text{func}(F, V)$. We leave the details for the reader. \hfill \Box

\begin{table}
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4.3. Definition. Let $V \subset M$ be an open subset such that $\Sigma_F$ is nowhere dense in $V$. We will say that the shift map $\varphi_V$ of $F$ satisfies smooth path-lifting condition if for every $C^\infty$-homotopy $H : V \times I \to M$ and $\alpha_0 \in \text{func}(F, V)$ such that $H_0 = \varphi(\alpha_0)$ and $H_t \in \text{Sh}(F, V)$, $(t \in I)$ the shift function $\Lambda : (V \setminus \Sigma_F) \times I \to \mathbb{R}$ of $H$ satisfying (4.1) smoothly extends to all of $V \times I$.

See also [Sch80], where the problem of lifting smooth homotopies of orbit spaces of Lie groups is considered.

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4.4. Theorem. Let $F$ be a vector field on a manifold $M$. Suppose that for every singular point $z \in \Sigma_F$ there exists an open neighbourhood $V$ such that $\text{Sh}(F, V) = \mathcal{E}_{id}(F, V)^\infty$ and the corresponding shift map $\varphi_V$ satisfies smooth path-lifting condition. Then $F$ is parameter rigid.

\textbf{Proof.} Let $G$ be a vector field on $M$ such that every orbit of $G$ is contained in some orbit of $F$. Then there exists a smooth function $\mu : M \setminus \Sigma_F \to \mathbb{R}$ such that $G = \mu F$. We have to show that $\mu$ smoothly extends to all of $M$.

We can assume that $G$ generates a global flow $G : M \times \mathbb{R} \to M$. Otherwise, there exists a smooth function $\beta : M \to (0, \infty)$ such that the vector field $G' = \beta G$ generates a global flow, see e.g. [Hr83, Corollary 2]. Then $G$ and $G'$ have the same orbit foliation. Moreover, if $G' = \gamma F$ for some smooth function $\gamma : M \to \mathbb{R}$, then $G = \frac{\gamma}{\beta} F$, where $\frac{\gamma}{\beta}$ is smooth on all of $M$ as well.

Let $z \in \Sigma_F$ and $V$ be a neighbourhood of $z$ such that $\varphi_V$ satisfies smooth path-lifting condition. Then for each $t \in \mathbb{R}$ we have a well-defined embedding $G_t|_V : V \to M$ belonging to $\mathcal{E}(F, V)$. Moreover, since $G_0 = \text{id}_M = \varphi(0)$ and $G$ is $C^\infty$, it follows that $G_t|_V \in \mathcal{E}_{id}(F, V)^\infty = \text{Sh}(F, V)$.

Then by smooth path-lifting condition for $\varphi_V$ there exists a smooth function $\bar{\mu} : V \times \mathbb{R}$ such that
\begin{equation}
G(x, t) = F(x, \bar{\mu}(x, t))
\end{equation}
and $\bar{\mu}(x, 0) = 0$ for all $x \in V$. Let us differentiate both parts of (4.2) in $t$ and set $t = 0$. Then we will get

$$G(x) = \frac{\partial G}{\partial t}(x, 0) = \frac{\partial F}{\partial t}(x, \bar{\mu}(x, 0)) \cdot \bar{\mu}'(x, 0) = F(x) \cdot \bar{\mu}'(x, 0).$$

Hence $\mu \equiv \bar{\mu}'(x, 0)$. Since $\Sigma_F$ is nowhere dense in $V$, we obtain that $\mu$ smoothly extends to all of $V$. Applying these arguments to all $z \in \Sigma_F$ we will get that $\mu$ is smooth on all of $M$. \hfill \Box

As an application of this theorem and results of [M1] we will now obtain parameter rigidity of linear vector fields and their regular extensions.

4.5. Definition. Let $M, N$ be two manifolds, $G : M \to TM$ be a vector field of $M$ and $F : M \times N \to T(M \times N) = TM \times TN$ be a vector field of $M \times N$ regarded as sections of the corresponding tangent bundles. Say that $F$ is a regular extension of $G$ provided that

$$F(x, y) = (G(x), H(x, y)), \quad (x, y) \in M \times N,$$

for some smooth map $H : M \times N \to TN$ such that $H(x, y) \in T_y N$. In other words the “first” coordinate function of $F$ “coincides with $G$” and does not depend on $y \in N$.

For instance if $G_i$ is a vector field on a manifold $M_i$, $(i = 1, 2)$, then their product $F(x, y) = (G_1(x), G_2(y))$ on $M_1 \times M_2$ is a regular extension of either of $G_i$. Every linear vector field $F(x) = Ax$ on $\mathbb{R}^n$ is a product of linear vector fields generated by Jordan cells of real Jordan form of $A$. Moreover, every Jordan cell vector field is a regular extension of a linear vector fields defined by the one of the following matrices: $(\lambda)$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where either $\lambda \neq 0$ or $b \neq 0$, see also [V66].

4.6. Corollary. Let $F$ be a vector field on a manifold $M$ and $V \subset M$ be an open subset. Suppose that the restriction of $F$ to $V$ is a regular extension of some non-zero linear vector field. This means that there exist a non-zero $(m \times m)$-matrix $A$, a smooth manifold $N$, and a diffeomorphism $\eta : V \to \mathbb{R}^m \times N$ such that the induced vector field $\eta^* F$ on $\mathbb{R}^k \times N$ is a regular extension of the linear vector field $G(y) = Ay$ on $\mathbb{R}^m$. Then $Sh(F, V) = E_{id}(F, V)^0$ and the shift map $\varphi_V$ satisfies smooth path-lifting condition.

Hence if every $z \in \Sigma_F$ has a neighbourhood $V$ with the above property, then $F$ is parameter rigid.

Proof. The proof follows from [M1, Theorem 25 & Theorem 27, statement about (E)-point]. \hfill \Box

In [Si52, St57, V66] and others there were obtained sufficient conditions for a vector field $F$ defined in a neighbourhood of $O \in \mathbb{R}^n$ to
be linear in some local coordinates at $O$. These results together with
Corollary 4.6 imply parameter rigidity for a large class of “hyperbolic”
flows which is in the spirit of mentioned above results of [KS94, Hur94,
Ka96, KS97, MM03, D07, EF07, Sa07] concerning rigidity of hyperbolic
locally free actions of Lie groups. A new feature of Corollary 4.6
is that $F$ has singularity, and therefore the corresponding $\mathbb{R}$-action (i.e.
the flow of $F$) is not locally free.

The following lemma gives sufficient condition for a vector field to
satisfy smooth path-lifting condition.

4.7. Lemma. Suppose that $\Sigma_F$ is nowhere dense in $V$ and for every
$h \in Sh(F, V)$ there exists a $C^\infty_W$-neighbourhood $N$ in $Sh(F, V)$ and a
preserving smoothness map (not necessarily continuous in any sense)

$$\sigma : Sh(F, V) \supset N \to \text{func}(F, V) \subset C^\infty(V, \mathbb{R}),$$

such that $\tau(x) = F(x, \sigma(\tau)(x))$ for all $\tau \in N$. In other words, $\sigma$
is a section of $\varphi_V$, i.e. $\varphi_V \circ \sigma = \text{id}(N)$. Then $\varphi_V$
satisfies smooth path-lifting condition.

Proof. Before proving this lemma let us make two remarks.

R1. Let $h \in Sh(F, V)$ and $\sigma_1, \sigma_2 : N \to \text{func}(F, V)$ be two
preserving smoothness sections of $\varphi_V$ defined on some $C^\infty_W$-neighbourhood $N$
of $h$, and $H : V \times I \to M$ be a $C^\infty$ map such that $H_0 = h$ and $H_t \in N$
for all $t \in I$. Since $\sigma_i$ preserves smoothness we have that the following
function $\Lambda_i : V \times I \to \mathbb{R}$ given by

$$\Lambda_i(x, t) = \sigma_i(H_t)(x)$$
is $C^\infty$ as well as $H$. If $\sigma_1(h) = \sigma_2(h)$, i.e. $\Lambda_1(\cdot, 0) = \Lambda_2(\cdot, 0)$, then
$\Lambda_1 \equiv \Lambda_2$ on all of $V \times I$.

Indeed, since $\Lambda_1(\cdot, 0) = \Lambda_2(\cdot, 0)$, it follows from Lemma 4.2 that
$\Lambda_1(x, t) = \Lambda_2(x, t)$ for all $(x, t) \in \left(V \setminus \Sigma_F\right) \times I$. But $\Sigma_F$ is nowhere
dense in $V$ and each $\Lambda_i$ is continuous. Therefore $\Lambda_1 = \Lambda_2$ on all of $V \times I$.

R2. Let $\sigma : N \to \text{func}(F, V)$ be a preserving smoothness section of $\varphi_V$
defined on some neighbourhood $N$ of $h$ and $\alpha \in \varphi_V^{-1}(h)$ be any shift
function for $h$. Notice that $\sigma(h) \in \varphi_V^{-1}(h)$ as well. Then there exists
(possibly) another preserving smoothness section $\sigma' : N \to \text{func}(F, V)$
such that $\sigma'(h) = \alpha$.

Suppose that $\sigma(h) \neq \alpha$. Then $\varphi_V$ is not injective map. Since $\Sigma_F$
is nowhere dense, it follows from [M1, Lm. 5 & Th. 12(2)] that there exists a smooth function $\nu : V \to (0, +\infty)$ such that $F(x, \nu(x)) \equiv x$
for all $x \in V$, and $\alpha = \sigma(h) + n\nu$ for some $n \in \mathbb{Z}$. Define the following
map $\sigma' : N \to \text{func}(F, V)$ by $\sigma'(\tau) = \sigma(\tau) + n\nu$ for $\tau \in N$. Then $\sigma'$ is
also a preserving smoothness section of $\varphi_V$ and $\sigma'(h) = \sigma(h) + n\nu = \alpha$.

Now we are ready to complete Lemma 4.7. Let $\alpha_0 \in \text{func}(F, V)$ and $H : V \times I \to M$ be a $C^\infty$ map such that $H_0 = \varphi(\alpha_0)$ and $H_t \in Sh(F, V)$
for all $t \in I$, i.e. $H_t = \varphi_V(\alpha_t)$ for some (not necessarily unique) $\alpha_t \in \func(F, V)$.

We will show that under assumptions of lemma it is possible to choose $\alpha_t$ so that the correspondence $(x, t) \mapsto \alpha_t(x)$ becomes a $C^\infty$ shift function $\Lambda' : V \times I \to \mathbb{R}$ for $H$ such that $\Lambda'(x, 0) = \alpha_0(x)$. Let also $\Lambda : (V \setminus \Sigma_F) \times I \to \mathbb{R}$ be a unique $C^\infty$ shift function for $H$ such that $\Lambda(x, 0) = \alpha_0(x) = \Lambda'(x, 0)$, see Lemma 4.2. Then it will follow from uniqueness of such shift function that $\Lambda = \Lambda'$ on $(V \setminus \Sigma_F) \times I$.

Since $\Sigma_F$ is nowhere dense in $V$, we will get $\Lambda \equiv \Lambda'$ on all of $V \times I$. This will imply smooth path-lifting condition for $\varphi_V$.

Notice that $H$ can be regarded as a continuous path into $C^\infty_W$ topology of $\sh(F, V)$. Since the image of this path is compact, there exist finitely many points $0 = t_0 < t_1 < \cdots < t_n = 1$ and for each $k = 0, 1, \ldots, n - 1$ a $C^\infty$ neighbourhood $N_k$ of $H_{t_k}$ in $\sh(F, V)$ such that

- $H_{t_k} \in N_k$ for all $t \in [t_k, t_{k+1}]$,
- and for every $\alpha \in \func(F, V)$ such that $\varphi_V(\alpha) = H_{t_k}$ there exists a preserving smoothness section $\sigma_\alpha : N_k \to \func(F, V)$ of $\varphi_V$ such that $\sigma_\alpha(H_{t_k}) = \alpha$, (this follows from $\mathbf{R2}$).

Since $H_0 = \varphi_V(\alpha_0)$, the map $\sigma_{\alpha_0}$ is well defined and we put

$$\Lambda(x, t) = \sigma_{\alpha_0}(H_t)(x), \quad (x, t) \in V \times [0, t_1].$$

Then $\Lambda$ is smooth on $V \times [0, t_1]$.

Denote $\alpha_{t_1} = \sigma_{\alpha_0}(H_{t_1})$. Then $H_{t_1} = \varphi_V(\alpha_{t_1})$ and $\sigma_{\alpha_1}$ is well defined. Therefore we also put

$$\Lambda(x, t) = \sigma_{\alpha_1}(H_t)(x), \quad (x, t) \in V \times [t_1, t_2].$$

Then $\Lambda$ is smooth on $V \times [t_1, t_2]$. Moreover $\sigma_{\alpha_0}$ and $\sigma_{\alpha_1}$ are two sections of $\varphi_V$ such that $\sigma_{\alpha_0}(H_{t_1}) = \sigma_{\alpha_1}(H_{t_1}) = \alpha_{t_1}$. Therefore by $\mathbf{R1}$ $\sigma_{\alpha_0}(H_t) = \sigma_{\alpha_1}(H_t) = \Lambda(\cdot, t)$ for all $t$ sufficiently close to $t_1$. Since $\sigma_{\alpha_0}$ and $\sigma_{\alpha_1}$ preserve smoothness, it follows that $\Lambda$ is smooth on all of $[0, t_2]$.

Using induction on $n$ we can smoothly extend $\Lambda$ on all of $V \times I$. ~

5. $\infty$-JETS OF SHIFTS

Let $F$ be a smooth vector field near the origin $O \in \mathbb{R}^n$ such that $F(O) = 0$ and $(F_t)$ be the local flow of $F$. Define the following map

$$j^\infty : \mathcal{D}(\mathbb{R}^n) \to (\mathbb{R}[[x_1, \ldots, x_n]])^n$$

associating to every $h \in \mathcal{D}(\mathbb{R}^n)$ its $\infty$-jet at $O$. Let also

$$j^\infty : \mathcal{D}(\mathbb{R}^n) \to (\mathbb{R}[[x_1, \ldots, x_n]])^n$$

Thus $\mathcal{J}(F)$ is the subgroup of $\mathcal{D}(\mathbb{R}^n)$ consisting of germs $h$ for which there exists a smooth function $\alpha_h \in \mathcal{F}(\mathbb{R}^n)$ such that $j^\infty(h) = j^\infty(F_{\alpha_h})$. 

Evidently, $\check{S}h(F) \subset \check{J}(F)$. Our first result gives necessary and sufficient conditions for a subgroup $G \subset D(\mathbb{R}^n)$ containing $\check{S}h(F)$ to be included into $\check{J}(F)$.

5.1. Theorem. Suppose that $F$ is not flat at $O$, i.e. there exists $p \geq 1$ such that $j^{p-1}(F) = 0$ and $P = j^p(F) : \mathbb{R}^n \to \mathbb{R}^n$ is a non-zero homogeneous map of degree $p$. For $p = 1$ we will write $P(x) = L \cdot x$, where $L$ is a certain non-zero $(n \times n)$-matrix.

Let $G$ be a subgroup of $D(\mathbb{R}^n)$ having the following properties:

(A1) $\check{S}h(F) \subset G$.

(A2) For every $h \in G$ there exists $\omega_0 \in \mathbb{R}$ such that

$$j^p(h)(x) = j^p(F_{\omega_0})(x) = \begin{cases} e^{L \omega_0} \cdot x, & p = 1, \\ x + P(x) \cdot \omega_0, & p \geq 2. \end{cases}$$

(A3) Moreover, if $j^{k-1}(h) = j^{k-1}(\text{id})$ for some $k \geq p$, then there exists a unique homogeneous polynomial $\omega_l$ of degree $l = k - p$ such that

$$j^k(h)(x) = j^k(F_{\omega_l})(x) = x + P(x) \cdot \omega_l(x).$$

Then $j^\infty(\check{S}h(F)) = j^\infty(G)$. In other words, $\check{S}h(F) \subset G \subset \check{J}(F)$.

The rest of this section is devoted to the proof of Theorem 5.1 which will be completed in §5.9. Our aim is to establish Lemma 5.7 and statement (2) of Corollary 5.8 below. They will be used in the proof of Theorem 5.1.

5.2. Spaces of jets. Let $\hat{E}(\mathbb{R}^n, O; \mathbb{R}^m)$ be the space of germs at the origin $O \in \mathbb{R}^n$ of smooth maps $h : \mathbb{R}^n \to \mathbb{R}^m$ and $\hat{E}(\mathbb{R}^n, O; \mathbb{R}^m, O)$ be its subset consisting of all germs such that $h(O) = O$. For $n = m$, we will write $\hat{E}(\mathbb{R}^n)$ instead of $\hat{E}(\mathbb{R}^n, O; \mathbb{R}^n, O)$. Let also $\hat{D}(\mathbb{R}^n) \subset \hat{E}(\mathbb{R}^n)$ be the subset consisting of germs of diffeomorphisms at $O$. The space $\hat{E}(\mathbb{R}^n, O; \mathbb{R}^1)$ of germs of smooth functions will be denoted by $\hat{F}(\mathbb{R}^n)$.

For $h \in \hat{E}(\mathbb{R}^n, O; \mathbb{R}^m)$ denote by $j^k(h)$ its $k$-jet at $O \in \mathbb{R}^n$. It will be convenient to formally assume that $(-1)$-jet of $h$ is identically zero:

$$j^{(-1)}(h) \equiv 0. \tag{5.2}$$

We will say that $h \in \hat{E}(\mathbb{R}^n, O; \mathbb{R}^m)$ is $k$-small, $(k \geq 0)$, at $O$ provided $j^{k-1}(h) = 0$. In particular, by assumption (5.2) every $h \in \hat{E}(\mathbb{R}^n, O; \mathbb{R}^m)$ is 0-small and $h$ is 1-small iff $h \in \hat{E}(\mathbb{R}^n, O; \mathbb{R}^m, O)$, i.e. $j^1(h) = h(O) = O$. If $j^\infty(h) = 0$ then $h$ is called flat.

Let $\alpha, \beta \in \hat{F}(\mathbb{R}^n)$ be such that $\alpha$ is $a$-small, and $\beta$ is $b$-small for some $a, b \geq 0$. Then their product $\alpha \beta$ is $(a + b)$-small. In other words

$$j^{a-1}(\alpha) = j^{b-1}(\beta) = 0 \Rightarrow j^{a+b-1}(\alpha \beta) = 0. \tag{5.3}$$
We also say that \( h \in \hat{\mathcal{E}}(\mathbb{R}^n, O; \mathbb{R}^m, O) \) is \textit{homogeneous of degree} \( k \) if its coordinate functions are homogeneous polynomials of degree \( k \). Let \( \hat{j}^k(\mathbb{R}^n) \), \( 0 \leq k < \infty \), be the space of all polynomial maps \( h : \mathbb{R}^n \to \mathbb{R}^n \) of degree \( \leq k \) such that \( h(O) = O \). Similarly, put
\[
\hat{j}^\infty(\mathbb{R}^n) = \{ \tau \in (\mathbb{R}[[x_1, \ldots, x_n]])^n : \tau(O) = 0 \}.
\]

Define the following linear map \( j^k : \hat{\mathcal{E}}(\mathbb{R}^n) \to \hat{j}^k(\mathbb{R}^n) \) associating to every \( h \in \hat{\mathcal{E}}(\mathbb{R}^n) \) its \( k \)-jet \( j^k(h) \) at \( O \). Then it is easy to verify that:
\[
j^k(f \circ g) = j^k(f) \circ j^k(g), \quad f, g \in \hat{\mathcal{E}}(\mathbb{R}^n),
\]
see e.g. [St57, §4]. The following lemma is a direct corollary of (5.4).

5.3. Lemma. Let \( f, g \in \hat{\mathcal{D}}(\mathbb{R}^n) \). Then for every \( k = 0, \ldots, \infty \) the following conditions are equivalent:
\[
j^k(f) = j^k(g) \iff j^k(f^{-1} \circ g) = j^k(id) \iff j^k(f^{-1}) = j^k(g^{-1}).
\]
Moreover if \( \alpha, \beta \in \hat{\mathcal{F}}(\mathbb{R}^n) \), then
\[
j^k(\alpha) = j^k(\beta) \iff j^k(\alpha \circ f) = j^k(\beta \circ f).
\]
\[\square\]

5.4. Jets of a flow. First we introduce the following notation. For a smooth mapping \( G = (G^1, \ldots, G^n) : \mathbb{R}^n \to \mathbb{R}^n \) denote
\[
\nabla G = \begin{bmatrix}
\frac{\partial G^1}{\partial x_1} & \frac{\partial G^1}{\partial x_2} & \cdots & \frac{\partial G^1}{\partial x_n} \\
\frac{\partial G^2}{\partial x_1} & \frac{\partial G^2}{\partial x_2} & \cdots & \frac{\partial G^2}{\partial x_n} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial G^n}{\partial x_1} & \frac{\partial G^n}{\partial x_2} & \cdots & \frac{\partial G^n}{\partial x_n}
\end{bmatrix}.
\]

Thus \( \nabla G \) is an \((n \times n)\)-matrix whose rows are the gradients of the corresponding coordinate functions of \( G \).

Now let \( F = (F^1, \ldots, F^n) \) be a smooth vector field defined on some neighbourhood \( V \) of \( O \in \mathbb{R}^n \) and
\[
F : V \times \mathbb{R} \supset \text{dom}(F) \to \mathbb{R}^n
\]
be the local flow of \( F \), so
\[
\frac{\partial F}{\partial t}(x, t) = F(F(x, t)) \quad \text{and} \quad F(x, 0) = x.
\]

Hence the Taylor expansion of \( F \) in \( t \) at \( x = O \) is given by
\[
F(x, t) = x + v_1(x) t + v_2(x) \frac{t^2}{2} + \cdots + v_n(x) \frac{t^n}{n!} + \cdots,
\]
where \( v_i(x) = \frac{\partial^i F}{\partial t^i}(x, t) \big|_{t=0} \). It follows from (5.5) that \( v_1 = F \).
5.5. **Lemma.** For every $i \geq 1$ we have that

$$v_{i+1} = \nabla v_i \cdot F = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \cdots & \frac{\partial v_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial x_2} & \cdots & \frac{\partial v_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} F^1 \\ F^2 \\ \vdots \\ F^n \end{bmatrix} = \begin{bmatrix} \nabla v_1 \cdot F \\ \nabla v_2 \cdot F \\ \vdots \\ \nabla v_n \cdot F \end{bmatrix},$$

where $v_i^j$ is the $j$-th coordinate function of $v_i$. Moreover, $\frac{\partial F}{\partial t}(x, t) = v_1(F(x, t))$. If $j^{p-1}(F) = 0$, then

$$(5.7) \quad j^{i(p-1)}(v_i) = 0, \quad \forall i \geq 1.$$ 

**Proof.** We will now calculate $v_2$. Let $F^j$ be the $j$-th coordinate function of $F$. Then

$$\frac{\partial^2 F^j}{\partial t^2}(x, t) = \frac{\partial}{\partial t} \left( \frac{\partial F^j}{\partial t}(x, t) \right) \overset{(5.5)}{=} \frac{\partial}{\partial t} F^j(F(x, t)) \overset{(5.5)}{=} \sum_{k=1}^n \frac{\partial F^j}{\partial x_k}(F(x, t)) \cdot F^k(F(x, t)) = \left( \sum_{j=1}^n \frac{\partial F^j}{\partial x_k} \cdot F^k \right) \circ F(x, t) = \langle \nabla F^j, F \rangle \circ F(x, t).$$

Therefore

$$\frac{\partial^2 F}{\partial t^2}(x, t) = (\nabla F \cdot F) \circ F(x, t) = v_2 \circ F(x, t).$$

Calculations for other $v_i$ are similar and are left to the reader.

**Proof of (5.7).** We have $j^{1(p-1)}(v_1) = j^{p-1}(F) = 0$. Suppose by induction that $j^{i(p-1)}(v_i) = 0$ for some $i$.

Then $j^{i(p-1)-1}(\nabla v_i) = 0$. Since $j^{p-1}(F) = 0$, it follows from (5.3) that $j^{i(p-1)+p-1}(\nabla v_i \cdot F) = j^{(i+1)(p-1)}(v_{i+1}) = 0$. \hfill \square

5.6. **Initial non-zero jets of smooth shifts for non-flat vector fields.** Suppose now that there exists $p \geq 1$ such that $j^{p-1}(F) = 0$ and

$$P = j^p(F) : \mathbb{R}^n \to \mathbb{R}^n$$

is a non-zero homogeneous map of degree $p$. For $p = 1$ we will write

$$P(x) = L \cdot x,$$

where $L$ is a certain non-zero $n \times n$-matrix.

5.7. **Lemma.** Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be an $l$-small germ at $O$ for some $l \geq 0$, so $j^l(\alpha) = \omega$ is a homogeneous polynomial of degree $l$. Put $F_\alpha(x) = F(x, \alpha(x))$. Then

$$(5.8) \quad j^{p+l}(F_\alpha)(x) = \begin{cases} e^{L \omega} \cdot x, & \text{if } p = 1 \text{ and } l = 0, \\ x + P(x) \cdot \omega(x), & \text{otherwise.} \end{cases}$$
Thus the \((p + l)\)-jet of \(F_\alpha\) depends only on the \(l\)-jet of \(\alpha\). Moreover,
\[
j^{p+l}(F_\alpha^{-1}) = j^{p+l}(F_{-\alpha}).
\]

**Proof.** (5.8). Substituting \(\alpha\) into (5.6) instead of \(t\) we get
\[
F_\alpha(x) = F(x, \alpha(x)) = x + F(x) \alpha(x) + \cdots + v_i(x) \frac{\alpha(x)^i}{i!} + \cdots
\]
Though this series converges near \(z\), the infinitesimal orders of its summands at \(O\) do not necessarily increase when \(n \to \infty\). Therefore in order to find the initial non-zero jet of \(F_\alpha\) we should investigate each of \(v_i\alpha^i\). In fact we will get only one exceptional case \(p = 1\) and \(l = 0\).

First we calculate \(j^1(F_\alpha)\). Suppose that \(j^1(F) = L \cdot x\) for some possibly zero matrix \(L\). Then \(j^0(\nabla F) = \nabla F(O) = L,\)
\[
j^1(v_2) = j^1(\nabla F \cdot F) = j^0(\nabla F) \cdot j^1(F) + j^1(\nabla F) \cdot j^0(F) = L \cdot L \cdot x + j^1(\nabla F) \cdot 0 = L^2 \cdot x,
\]
and by induction on \(i\) we obtain
\[
j^1(v_i) = j^1(\nabla v_{i-1} \cdot F) = L^i \cdot x.
\]

Therefore
\[
j^1(v_i \alpha^i) = j^1(v_i) \cdot j^0(\alpha^i) + j^0(v_i) \cdot j^1(\alpha^i) = L^i \cdot x \cdot \alpha(O) + 0 \cdot j^1(\alpha^i) = \alpha(O) \cdot L^i \cdot x,
\]
whence
\[
j^1(F_\alpha)(x) = x + \sum_{i=1}^{\infty} \frac{\alpha(O)^i}{i!} \cdot L^i \cdot x = e^{L \cdot \alpha(O)} \cdot x.
\]

Thus if \(p = 1\) and \(l = 0\), i.e \(L \neq 0\) and \(j^0(\alpha) = \alpha(O) = \omega \neq 0\), we obtain that \(j^{p+l}(F_\alpha)(x) = j^1(F_\alpha)(x) = e^{L \cdot \omega \cdot x} \cdot x\).

Otherwise \(p + l \geq 2\). We claim in this case \(j^{p+l}(v_i \cdot \alpha^i) = 0\) for \(i \geq 2\). This will imply that
\[
j^{p+l}(F_\alpha)(x) = j^{p+l}(x + F(x) \alpha(x)) = x + P(x) \cdot \omega(x).
\]
To calculate \(j^{p+l}(v_i \cdot \alpha^i)\) notice that by (5.7) \(j^{(p-1)}(v_i) = 0\) and by assumption \(j^{i-1}(\alpha) = 0\) as well. Then it follows from (5.3) that \(j^{i-1}(\alpha^i) = 0\) and
\[
j^{i(p-1)+i}(v_i \cdot \alpha^i) = j^{i(p+l-1)}(v_i \cdot \alpha^i) = 0.
\]

It remains to note that \(p + l < i(p + l - 1)\) if \(i \geq 2\) and \(p + l \geq 2\). Hence \(j^{p+l}(v_i \alpha^i) = 0\) for \(i \geq 2\).

**Proof of (5.9).** Recall that by (3.3) \(F_\alpha^{-1} = F_{-\alpha} \circ F_\alpha^{-1}\). We will show that \(j^1(\alpha \circ F_\alpha^{-1}) = j^1(\alpha)\). Then it will follow from (5.8) that
\[
j^{p+l}(F_\alpha^{-1}) = j^{p+l}(F_{-\alpha} \circ F_\alpha^{-1}) \overset{(5.8)}{=} j^{p+l}(F_{-\alpha}).
\]

Suppose that \(l = 0\). Since \(F_t(O) = O\) for all \(t \in R\), we obtain that \(j^0(\alpha \circ F_\alpha^{-1}) = j^0(\alpha) = \alpha(O)\).
If \( l \geq 1 \), then it follows from (5.8) that \( j^l(F_\alpha)(x) = x \), whence \( j^l(\alpha \circ F_\alpha^{-1}) = j^l(\alpha) \). \( \square \)

5.8. **Corollary.** Let \( \alpha, \beta \in \mathcal{F}(\mathbb{R}^n) \), \( h \in \mathcal{D}(\mathbb{R}^n) \), \( l, k = 0, 1, \ldots, \infty \).

1. The following conditions (A)-(C) are equivalent:
   - (A) \( j^l(\alpha) = j^l(\beta) \),
   - (B) \( j^{p+l}(F_\alpha) = j^{p+l}(F_\beta) \),
   - (C) \( j^{p+l}(F_{h,\alpha}) = j^{p+l}(F_{h,\beta}) \).

2. The following conditions (D) and (E) are equivalent:
   - (D) \( j^k(h) = j^k(F_\alpha) \),
   - (E) \( j^k(F_{h,-\alpha}) = j^k(\text{id}) \).

**Proof.** (1) (A) \( \iff \) (B). Notice that

\[
F_\alpha \circ F_\beta^{-1} = F_\alpha \circ F_{-\beta \circ F_\beta^{-1}} = F_{\alpha \circ F_\beta^{-1}} = F_{(\alpha - \beta) \circ F_\beta^{-1}}.
\]

Then the following statements are equivalent:

- (A) \( j^l(\alpha) = j^l(\beta) \),
- (b) \( j^l(\alpha - \beta) = 0 \),
- (d) \( j^{p+l}(F_\alpha \circ F_\beta^{-1}) = j^{p+l}(\text{id}) \),
- (E) \( j^k(F_{h,-\alpha}) = j^k(\text{id}) \).

The equivalence of (A), (b), and (d) is trivial, (b) \( \iff \) (c) holds by (5.8), (c) \( \iff \) (d) by (5.10), and (d) \( \iff \) (B) by Lemma 5.3.

(A) \( \iff \) (C) Recall that by (3.4) \( F_{h,\alpha} = F_{\alpha \circ h^{-1}} \circ h \). Then the following conditions are equivalent:

- (C) \( j^{p+l}(F_{h,\alpha}) = j^{p+l}(F_{h,\beta}) \),
- (e) \( j^{p+l}(F_{\alpha \circ h^{-1}}) = j^{p+l}(F_{\beta \circ h^{-1}}) \),
- (A) \( j^l(\alpha) = j^l(\beta) \).

The equivalence (C) \( \iff \) (e) holds by (3.4), (e) \( \iff \) (f) by the equivalence (B) \( \iff \) (A) which is already proved, and (f) \( \iff \) (A) by Lemma 5.3.

(2) (D) \( \iff \) (E). Suppose that \( j^k(h) = j^k(F_\alpha) \). Then

\[
\begin{align*}
\text{(5.11)} & \quad j^k(F^{-1}_\alpha \circ h) = j^k(\text{id}), \\
\text{(5.12)} & \quad j^k(\alpha) = j^k(\alpha \circ F^{-1}_\alpha \circ h).
\end{align*}
\]

Notice also that

\[
\text{(5.13)} \quad F^{-1}_\alpha \circ h \overset{(3.3)}{=} F_{-\alpha \circ F^{-1}_\alpha} \circ h = F_{h,-\alpha \circ F^{-1}_\alpha \circ h}.
\]

Hence

\[
\begin{align*}
\text{(5.14)} & \quad F_{\alpha \circ F_{h,-\alpha}} \circ F_{h,-\alpha} = h.
\end{align*}
\]

This identity simply means that \( F(F(h(x), -\alpha(x)), \alpha(x)) = h(x) \). Suppose that \( j^k(F_{h,-\alpha}) = j^k(\text{id}) \). Then

\[
\begin{align*}
\text{(5.15)} & \quad j^k(\alpha \circ F_{h,-\alpha}) = j^k(\alpha),
\end{align*}
\]
whence
\[ j^k(h) \xrightarrow{(5.14) \& (E)} j^k(F_{\alpha \circ F_{h^{-1}}} \circ \hat{\alpha}) \xrightarrow{(5.15) \& (5.8)} j^k(F_{\alpha}). \]

Corollary 5.8 is proved. \[ \square \]

5.9. Proof of Theorem 5.1. Let \( F \) be a vector field defined on some neighbourhood of the origin in \( \mathbb{R}^n \) and \( \mathcal{G} \) be a subgroup of \( \mathcal{D}(\mathbb{R}^n) \) satisfying (A1)-(A3). We have to show that \( j^\infty(\hat{S}h(F)) = j^\infty(\mathcal{G}) \). Due to (A1) it remains to verify that \( j^\infty(\hat{S}h(F)) \supset j^\infty(\mathcal{G}) \).

Let \( h \in \mathcal{G} \). We will find a germ of a smooth function \( \alpha \in \hat{\mathcal{F}}(\mathbb{R}^n) \) such that \( j^\infty(\mathbf{F}_{h^{-1}} \circ \alpha) = j^\infty(h) \), i.e. \( j^\infty(h) \in j^\infty(\hat{S}h(F)) \).

Since \( \mathcal{G} \) is a group, it follows from (A1) and (3.4) that
\[
(5.16) \quad \mathbf{F}_{h,\alpha} = \mathbf{F}_{\alpha \circ h^{-1}} \circ h \in \mathcal{G}, \quad \forall \alpha \in \hat{\mathcal{F}}(\mathbb{R}^n).
\]

Put \( h_0 = h \). Then by (A2) there exists \( \omega_0 \in \mathbb{R} \) such that \( j^p(h_0) = j^p(\mathbf{F}_{\omega_0}) \). Denote
\[
h_1(x) = \mathbf{F}_{h^{-1},\omega_0}(x) = \mathbf{F}(h(x), -\omega_0) = \mathbf{F}_{-\omega_0} \circ h(x).
\]

Then \( h_1 \in \mathcal{G} \) and by (2) of Corollary 5.8 \( j^p(h_1) = j^p(\text{id}_{\mathbb{R}^n}) \).

Therefore by (A3) there exists a homogeneous polynomial \( \omega_1 \) of degree 1 such that \( j^{p+1}(h_1) = j^{p+1}(\mathbf{F}_{\omega_1}) \), where \( \mathbf{F}_{\omega_1}(x) = \mathbf{F}(x, \omega_1(x)) \). Denote
\[
h_2(x) = \mathbf{F}_{h_1,\omega_1}(x) = \mathbf{F}(h_1(x), -\omega_1(x)) = \mathbf{F}(h_0(x), -\omega_0, -\omega_1(x)) = \mathbf{F}(h(x), -\omega_0 - \omega_1(x)) = \mathbf{F}_{h^{-1},\omega_0-\omega_1}(x).
\]

Then by (5.16) \( h_2 \in \mathcal{G} \) and by (2) of Corollary 5.8 \( j^{p+1}(h_2) = j^{p+1}(\text{id}) \).

Therefore we can again apply (A3) to \( h_2 \) and so on. Using induction we will construct a sequence of homogeneous polynomials \( \{\omega_i\}_{i=0}^\infty \), \( (\deg \omega_i = l) \), and a sequence \( \{h_l\} \) in \( \mathcal{G} \) such that for every \( l \geq 0 \), we have that
\[
(5.17) \quad j^{p+i-1}(h_l) = j^{p+i-1}(\text{id}), \quad j^{p+i}(h_l) = j^{p+i}(\mathbf{F}_{\omega_i}),
\]
\[
(5.18) \quad h_{l+1}(x) = \mathbf{F}_{h_l,\omega_l}(x) = \mathbf{F}(h(x), -\sum_{i=0}^l \omega_i(x)).
\]

Put \( \tau = \sum_{l=0}^\infty \omega_l(x) \). Then by a well-known theorem of E. Borel, see also §6, there exists a smooth function \( \alpha : \mathbb{R}^n \to \mathbb{R} \) whose \( \infty \)-jet at \( O \) coincides with \( \tau \).

We claim that \( j^\infty(\mathbf{F}_{h^{-1}} \circ \alpha) = j^\infty(\text{id}) \). Evidently, it suffices to show that \( j^{p+i}(\mathbf{F}_{h^{-1}} \circ \alpha) = j^{p+i}(\text{id}) \) for arbitrary large \( l \).
For \( l \geq 0 \) put \( \alpha_l = \sum_{i=0}^{l} \omega_i \) and \( \alpha_{>l} = \alpha - \alpha_l \). Then
\[
F_{h,-\alpha}(x) = F(h(x), -\alpha_l(x) - \alpha_{>l}(x)) = F(F(h(x), -\alpha_l(x)), -\alpha_{>l}(x)) = F(h_{l+1}(x), -\alpha_{>l}(x)) = F_{h_{l+1},-\alpha_{>l}}.
\]

Notice that by (5.17) \( j^{p+l}(h_{l+1}) = j^{p+l}(id) \). Moreover, since \( j^l(\alpha_{>l}) = 0 \), it follows from Lemma 5.7 that \( j^{p+l}(F_{\alpha_{>l}}) = j^{p+l}(id) \) for \( l \geq 1 \). Thus \( j^{p+l}(h_{l+1}) = j^{p+l}(F_{\alpha_{>l}}) = j^{p+l}(id) \). Then by (2) of Corollary 5.8 \( j^{p+l}(F_{h,-\alpha}) = j^{p+l}(F_{h_{l+1},-\alpha_{>l}}) = j^{p+l}(id) \).

6. BOREL’S THEOREM

In this section we present a variant of a well-known theorem of E. Borel. It will be used in the next section for the construction of \( j^\infty \)-sections of a shift map.

Let \( V \) be an open subset of \( \mathbb{R}^n \) and
\[
f = (f_1, \ldots, f_m) : V \to \mathbb{R}^m
\]
be a smooth mapping. For every compact \( K \subset V \) and \( r \geq 0 \) define the \( r \)-norm of \( f \) on \( K \) by \( \|f\|_K = \sum_{j=1}^{m} \sum_{|i| \leq r} \sup_{x \in K} |D^i f_j(x)| \), where \( i = (i_1, \ldots, i_n) \), \( |i| = i_1 + \cdots + i_n \), and \( D^i = \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} \). For a fixed \( r \) the norms \( \|f\|_K \), where \( K \) runs over all compact subsets of \( V \), define the weak Whitney \( C^r_W \) topology on \( C^\infty(V, \mathbb{R}^m) \). A \( C^\infty_W \)-topology on \( C^\infty(V, \mathbb{R}^m) \) is generated by \( C^r_W \)-topologies for all finite \( r \geq 0 \).

For every \( i \geq 0 \) denote by \( \mathcal{P}_i \) the space of real homogeneous polynomials in \( n \) variables \( x_1, \ldots, x_n \) of degree \( i \). Associating to every \( \omega \in \mathcal{P}_i \) its coefficients we can identify \( \mathcal{P}_i \) with \( \mathbb{R}^{C^{i-1}_{n+i-1}} \), where \( C^{i-1}_{n+i-1} = \frac{(n+i-1)!}{i! (i-1)!} \).

On the other hand \( \mathcal{P}_i \subset C^\infty(\mathbb{R}^n, \mathbb{R}) \). It is easy to show that every \( C^r_W \)-topologies on \( \mathcal{P}_i \) induced from \( C^\infty(\mathbb{R}^n, \mathbb{R}) \) coincides with the Euclidean one.

Let \( A, B \) be smooth manifolds and \( \mathcal{X} \subset C^\infty(A, B) \) a subset. We will say that a map \( \lambda : \mathcal{X} \to \mathcal{P}_i \) is \( C^\infty_W \)-continuous provided it is continuous from \( C^\infty_W \)-topology of \( \mathcal{X} \) to the Euclidean topology of \( \mathcal{P}_i \).

6.1. Theorem. Let \( A, B \) be smooth manifolds, \( \mathcal{X} \subset C^\infty(A, B) \) a subset, and \( V \subset \mathbb{R}^n \) an open neighbourhood of \( O \subset \mathbb{R}^n \). Suppose that for every \( i \geq 0 \) we are given a number \( s_i \geq 0 \) and a \( C^\infty_W \)-continuous map
\[
\lambda_i : \mathcal{X} \to \mathcal{P}_i.
\]
Thus we can define the following mapping
\[
\lambda : \mathcal{X} \to \mathbb{R}[[x_1, \ldots, x_n]], \quad \lambda(\alpha) = \sum_{i=0}^{\infty} \lambda_i(\alpha).
\]
Then there exists a $C_{W,W}^{\infty,\infty}$-continuous map $\Lambda : \mathcal{X} \to C^{\infty}(V, \mathbb{R})$ such that $\infty$-jet of $\Lambda(\alpha)$ at $O$ coincides with $\lambda(\alpha)$ for every $\alpha \in \mathcal{X}$.

Moreover if every $\lambda_i$ preserves smoothness (in the sense of Definition 1.3), then so does $\Lambda$.

Proof. The proof uses a theorem of E. Borel claiming that there exists a map $B : \mathbb{R}[[x_1, \ldots, x_n]] \to C^{\infty}(V, \mathbb{R})$ such that for every formal series $\tau \in \mathbb{R}[[x_1, \ldots, x_n]]$ the Taylor series of the function $B(\tau)$ at $O$ coincides with $\tau$, see e.g. [GG].

Such a map $B$ can be assumed to have the following properties (a)-(c) below. Let $\tau = \sum_{i=0}^{\infty} \omega_i$ be a formal series, where $\omega_i \in \mathcal{P}_i$. Then

(a) for every $i \geq 0$ the restriction $B|_{\mathcal{P}_i} : \mathcal{P}_i \to C^{\infty}(V, \mathbb{R})$ is continuous from the Euclidean topology of $\mathcal{P}_i$ to $C^{\infty}_W$-topology of $C^{\infty}(V, \mathbb{R})$;

(b) $B(\sum_{i=0}^{\infty} \omega_i) = \sum_{i=0}^{\infty} B(\omega_i)$;

(c) for every compact $K \subset V$, $\varepsilon > 0$, and an integer number $r \geq 0$ there exists $s = s(K, \varepsilon, r) \geq 0$ which does not depend on $\tau$ such that

$$\left\| B \left( \sum_{i=s+1}^{\infty} \omega_i \right) \right\|_K^r < \varepsilon.$$

Indeed, for every $i \geq 0$ let $\rho_i : \mathbb{R}^n \to \mathbb{R}$ be a smooth function supported in a sufficiently small neighbourhood $V_i \subset V$ of $O$ and equal to 1 in a smaller neighbourhood of 0. Put $B(\omega) = \omega \rho_i$ for all $\omega \in \mathcal{P}_i$. Then (a) holds true.

Property (b) is just the definition of $B$ on all of $\mathbb{R}[[x_1, \ldots, x_n]].$

Finally, (c) can be reached if the supports of $\rho_i$ decrease sufficiently fast when $i$ tends to $\infty$, see for details e.g. [GG].

Assuming that $B$ has properties (a)-(c) we will now show that the following map

$$\Lambda = B \circ \lambda : \mathcal{X} \xrightarrow{\lambda} \mathbb{R}[[x_1, \ldots, x_n]] \xrightarrow{B} C^{\infty}(V, \mathbb{R}),$$

(6.1)$$\Lambda(q)(x) = \sum_{i=0}^{\infty} \rho_i(x) \lambda_i(q)(x), \quad q \in \mathcal{X}, \ x \in V,$$

satisfies the statement of our theorem.

Indeed, for each $\alpha \in \mathcal{X}$ the Taylor series of $\Lambda(\alpha) = B \circ \lambda(\alpha)$ at $O$ coincides with $\lambda(\alpha)$.

Let us verify $C_{W,W}^{\infty,\infty}$-continuity of $\Lambda$ at $\alpha$. It suffices to prove that for every compact subset $K \subset V$, $\varepsilon > 0$, and $r \geq 0$ there exist $c = c(K, \varepsilon, r) \geq 0$ which does not depend on $\alpha$, and a $C_{W}^{\infty}$-neighbourhood $\mathcal{U}_\alpha$ of $\alpha$ in $\mathcal{X}$ such that $\|\Lambda(\alpha) - \Lambda(\beta)\|_K^r < \varepsilon$ for all $\beta \in \mathcal{U}_\alpha$.

For every $s \geq 0$ define the following two maps:

$$\Lambda_s, \Lambda_{>s} : \mathcal{X} \to C^{\infty}(V, \mathbb{R}),$$
\[ \Lambda_s(\alpha) = B \circ \bigoplus_{i=0}^{s} \lambda_i(\alpha), \quad \Lambda_{>s} = \Lambda - \Lambda_s. \]

Then

\[ (6.2) \quad \|\Lambda(\alpha) - \Lambda(\beta)\|_K^r \leq \|\Lambda_s(\alpha) - \Lambda_s(\beta)\|_K^r + \|\Lambda_{>s}(\alpha)\|_K^r + \|\Lambda_{>s}(\beta)\|_K^r. \]

For every \( s \geq 0 \) let \( c(s) = \max\{s_i\}_{i=0}^{s} \). Then it follows from (a) and assumptions about continuity of \( \lambda_i \) that \( \Lambda_s \) is \( C^{c(s), \infty}_{W,W} \)-continuous. Hence \( \Lambda_s \) is \( C^{c(s), r}_{W,W} \)-continuous for every \( r \geq 0 \).

Fix \( \varepsilon > 0, r \geq 0, \) and a compact subset \( K \subset V \). Then by (c) there exists \( s > 0 \) which does not depend on \( \alpha \) and \( \beta \) such that each of the last two terms in (6.2) is less than \( \frac{\varepsilon}{3} \). Moreover it follows from \( C^{c(s), r}_{W,W} \)-continuity of \( \Lambda_s \) that there exists a \( C^{c(s)}_{W} \)-neighbourhood \( \mathcal{U}_\alpha \) of \( \alpha \) in \( X \) such that for every \( \beta \in \mathcal{U}_\alpha \) the first term on the right side of (6.2) is less than \( \frac{\varepsilon}{3} \) as well. Then \( \|\Lambda(\alpha) - \Lambda(\beta)\|_K^r < \varepsilon \).

Suppose now that every \( \lambda_i \) preserves smoothness. Let \( q : A \times \mathbb{R}^k \to B \) be a \( C^\infty \) map such that \( q_t \in \mathcal{X} \) for every \( t \in \mathbb{R}^k \). Then for every \( i \geq 0 \) the following map

\[ \lambda_i(q) : V \times \mathbb{R}^k \to \mathbb{R}, \quad \lambda_i(q)(x,t) = \lambda_i(q_t)(x) \]

is also \( C^\infty \). Therefore it follows from (6.1) that the following map \( \Lambda(q) : V \times \mathbb{R}^k \to \mathbb{R} \) defined by

\[ \Lambda(q)(x,t) = \Lambda(q_t)(x) = \sum_{i=0}^{\infty} \rho_i(x) \lambda_i(q_t)(x) \]

is \( C^\infty \) as well. Hence \( \Lambda \) preserves smoothness.

\[ \square \]

7. \( j^\infty \)-Sections of the Shift Map

7.1. Notation. Let \( V \) be an open neighbourhood of \( O \) in \( \mathbb{R}^n \), \( F \) be a vector field on \( V \) such that \( F(O) = 0 \), \( \mathbf{F} : V \times \mathbb{R} \supset \text{dom}(\mathbf{F}) \to \mathbb{R}^n \) be the local flow of \( F \), \( \text{func}(\mathbf{F}, V) \) be the subset of \( C^\infty(V, \mathbb{R}) \) consisting of functions \( \alpha \) whose graph is contained in \( \text{dom}(\mathbf{F}) \), and

\[ \varphi : \text{func}(\mathbf{F}, V) \to C^\infty(V, \mathbb{R}^n), \quad \varphi(\alpha)(z) = \mathbf{F}_\alpha(x) = \mathbf{F}(z, \alpha(z)) \]

be the corresponding shift map of \( F \). Denote its image by \( Sh(\mathbf{F}, V) \).

For every \( k \geq 0 \) and \( h \in C^\infty(V, \mathbb{R}^n) \) let \( j^k(h) \) be the \( k \)-jet of \( h \) at \( O \).

Let also \( \mathcal{J}(F, V) \subset C^\infty(V, \mathbb{R}^n) \) be the subset consisting of maps \( h \) for which there exists a smooth function \( \alpha_h \in C^\infty(V, \mathbb{R}) \) such that \( j^\infty(h) = j^\infty(\mathbf{F}_{\alpha_h}), \) c.f. (5.1).

In this section we will show how to choose \( \alpha_h \) so that the correspondence \( h \mapsto \alpha_h \) becomes a continuous and preserving smoothness map. Such a map will be called a \( j^\infty \)-section of \( \varphi_V \).
7.2. Definition. Let $\mathcal{X} \subset J(F, V)$ be a subset. We will say that a mapping $\Lambda : \mathcal{X} \to C^\infty(V, \mathbb{R})$ is a $j^\infty$-section of $\varphi_V$ on $\mathcal{X}$ provided $\Lambda$ is $C^\infty_{W,W}$-continuous, preserves smoothness, and

$$j^\infty(h) = j^\infty(\varphi_V \circ \Lambda(h)), \quad \forall h \in \mathcal{X}.$$  

If $\mathcal{X} = J(F, V)$ then $\Lambda$ will be called a global $j^\infty$-section of $\varphi_V$.

Recall that we use the following notation

$$\varphi_V \circ \Lambda(h)(x) = F_{\Lambda(h)}(x) = F(x, \Lambda(h)(x)),$$

$$F_{h,-\Lambda(h)}(x) = F(h(x), -\Lambda(h)(x)).$$

Then by (2) of Corollary 5.8 relation (7.1) is equivalent to each of the following conditions

$$j^\infty(h) = j^\infty(F_{\Lambda(h)}) \iff j^\infty(F_{h,-\Lambda(h)}) = j^\infty(id).$$

Our main result proves existence of $j^\infty$-sections on certain subspaces of $J(F, V)$, see Theorem 7.6. Before formulating this theorem, let us show how $j^\infty$-sections can be used for constructing real sections of $\varphi_V$.

7.3. Applications of $j^\infty$-sections. Denote by $E^\infty_\infty(F, V) \subset E(F, V)$ the subset consisting of maps $h$ such that $j^\infty(h) = j^\infty(id)$. Since $j^\infty(h) = j^\infty(F_0)$, we obtain that

$$E^\infty_\infty(F, V) \subset E(F, V) \cap J(F, V).$$

Let $\mathcal{X} \subset E(F, V) \cap J(F, V)$ be a subset. Thus every $h \in \mathcal{X}$ is an orbit preserving map $V \to \mathbb{R}^n$ being a diffeomorphism at every singular point $z \in \Sigma_F$ and such that $j^\infty(h) = j^\infty(F_{\alpha_h})$ for some $\alpha_h \in C^\infty(V, \mathbb{R})$.

Suppose that there exists a $j^\infty$-section $\Lambda : \mathcal{X} \to C^\infty(V, \mathbb{R})$ of $\varphi_V$. Then by (7.2) $F_{h,-\Lambda(h)} \in E^\infty_\infty(F, V)$ for every $h \in \mathcal{X}$, whence the following map is well defined:

$$\mathcal{H} : \mathcal{X} \to E^\infty_\infty(F, V), \quad \mathcal{H}(h) = F_{h,-\Lambda(h)}.$$  

7.4. Lemma. c.f. [M4, Pr. 3.4] Suppose also that there exists a $C^\infty_{W,W}$-continuous and preserving smoothness section

$$\Psi : E^\infty_\infty(F, V) \to C^\infty(V, \mathbb{R})$$

of $\varphi_V$, i.e. $h(x) = F(x, \Psi(h)(x))$ for all $h \in E^\infty_\infty(F, V)$. Then the following map $\sigma : \mathcal{X} \to C^\infty(V, \mathbb{R})$ defined by

$$\sigma(h) = \Lambda(h) + \Psi \circ \mathcal{H}(h) = \Lambda(h) + \Psi(F_{h,-\Lambda(h)}).$$

is a section of $\varphi_V$ defined on all of $\mathcal{X}$.

Proof. This statement was actually established in [M4, Pr. 3.4]. For the convenience of the reader we recall the proof. It suffices to show
that $F(h(x), -\sigma(h)(x)) \equiv x$ for all $x \in V$ and $h \in \mathcal{X}$ but this easily follows from definitions:

$$F(h(x), -\sigma(h)(x)) = F(h(x), -\Lambda(h)(x) - \Psi \circ \mathcal{H}(h)(x))$$
$$= F(F(h(x), -\Lambda(h)(x)), -\Psi \circ \mathcal{H}(h)(x))$$
$$= F(\mathcal{H}(h)(x), -\Psi \circ \mathcal{H}(h)(x)) = x. \quad \square$$

Thus existence of $j^\infty$-sections reduces the problem of resolving (1.1) to the case when $j^\infty(h) = j^\infty(id)$.

7.5. Main result. Suppose that $F$ is not flat at $O$, i.e. there exists $p \geq 1$ such that $j^{p-1}(F) = 0$ and $P = j^p(F)$ is a non-zero homogeneous vector field. For $p = 1$ we will write $P(x) = L \cdot x$, where $L$ is a certain non-zero $(n \times n)$-matrix. In this case we have the exponential map

$$\exp_L : \mathbb{R} \to GL(\mathbb{R}, n), \quad \exp_L(t) = e^{Lt}.$$

Denote its image by $E_L = \{e^{Lt}\}_{t \in \mathbb{R}}$. Then the following three cases of $E_L$ will be separated:

(G1) $E_L \approx \mathbb{R}$ and is a closed subgroup of $GL(\mathbb{R}, n)$;

(G2) $E_L \approx SO(2)$;

(G3) $E_L \approx \mathbb{R}$ and is a non-closed subset of $GL(\mathbb{R}, n)$.

For every $k \geq 1$ denote

$$\mathcal{J}_k(F, V) = \{h \in \mathcal{J}(F, V) \mid j^{k-1}(h) = j^{k-1}(id)\}, \quad k \geq 1.$$

Since $h(O) = O$, i.e. $j^0(h) = j^0(id)$, for all $h \in \mathcal{J}(F, V)$, we see that $\mathcal{J}(F, V) = \mathcal{J}_1(F, V)$. Moreover, it follows from Lemma 5.7 that $j^{p-1}(h) = j^{p-1}(id)$ for $p \geq 2$. Hence for arbitrary $p \geq 1$ we have the following inclusions:

$$\mathcal{J}(F, V) = \mathcal{J}_1(F, V) = \cdots = \mathcal{J}_p(F, V) \supset \mathcal{J}_{p+1}(F, V) \supset \cdots$$

It will also be convenient to define local variants of the above constructions. Recall that $\hat{\mathcal{J}}(F)$ is the subgroup of $\hat{D}(\mathbb{R}^n)$ consisting of space of germs at $O$ of maps from $\hat{\mathcal{J}}(F)$, see (5.1). Therefore we define $\hat{\mathcal{J}}_k(F)$ to be the subspace of $\hat{\mathcal{J}}(F)$ consisting of germs at $O$ of maps from $\mathcal{J}_k(F, V)$. Then again

$$\hat{\mathcal{J}}(F) = \hat{\mathcal{J}}_1(F) = \cdots = \hat{\mathcal{J}}_p(F) \supset \hat{\mathcal{J}}_{p+1}(F) \supset \cdots$$

7.6. Theorem. (1) There exists a $j^\infty$-section $\Lambda$ of $\varphi_V$ on $\mathcal{J}_2(F, V)$. Hence if $p \geq 2$, then $\mathcal{J}(F, V) = \mathcal{J}_2(F, V)$, and thus $\Lambda$ is defined on all of $\mathcal{J}(F, V)$.

(2) If $p = 1$ and $E_L$ satisfies (G1), then there exists a global $j^\infty$-section $\Lambda$ of $\varphi_V$ as well.

(3) If $p = 1$ and $E_L$ satisfies (G2), then for every $g \in \mathcal{J}(F, V)$ the shift map $\varphi_V$ has a $j^\infty$-section defined on some $C^1_W$-neighbourhood of $g$ in $\mathcal{J}(F, V)$.
Remark. If $p = 1$ but $E_L$ satisfies (G3) then it seems that $\varphi_V$ has no even local $(C^\infty_W \text{-continuous})$ $j^\infty$-sections on $\mathcal{J}_1(F, V)$, though by (1) it has a $j^\infty$-section on $\mathcal{J}_2(F, V)$.

The proof will be given in §7.10. It is similar to the proof Theorem 5.1 but we have to estimate continuity of certain correspondences. We need the following two statements.

7.8. Proposition. Let $p = 1$. In the case (G1) of $E_L$ there exists a $C^1_W$-continuous map $\Delta_0 : \mathcal{J}(F, V) \rightarrow \mathbb{R}$ such that for every $h \in \mathcal{J}(F, V)$ the germ at $O$ of the mapping

$$F_{h, -\Delta_0(h)}(x) = F(h(x), -\Delta_0(h))$$

belongs to $\hat{\mathcal{J}}_2(F)$.

Suppose that $E_L$ satisfies (G2). Then for every $g \in \mathcal{J}(F, V)$ there exists a $C^1_W$-neighbourhood $\mathcal{N}_g$ in $\mathcal{J}(F, V)$ and a $C^1_W$-continuous mapping $\Delta_0 : \mathcal{N}_g \rightarrow \mathbb{R}$ such that for every $h \in \mathcal{N}_g$ the germ at $O$ of the mapping $F_{h, -\Delta_0(h)}$ belongs to $\hat{\mathcal{J}}_2(F)$.

In both cases $\Delta_0$ preserves smoothness.

Proof. Let $j^1 : \mathcal{J}(F, V) \rightarrow GL(\mathbb{R}, n)$ be the map associating to every $h \in \mathcal{J}(F, V)$ its Jacobi matrix $J(h, O)$ at $O$. Then it follows from Lemma 5.7 that the image of $j^1$ coincides with $E_L$. Thus we have two maps:

$$\mathcal{J}(F, V) \xrightarrow{j^1} E_L \xrightarrow{\exp_L} \mathbb{R}.$$ 

Notice that in the cases (G1) and (G2) $E_L$ is a closed subgroup of $GL(\mathbb{R}, n)$. Moreover, in the case (G1) $\exp_L$ is an embedding, so we can put

$$\Delta_0 : \mathcal{J}(F, V) \rightarrow \mathbb{R}, \quad \Delta_0 = \exp_L^{-1} \circ j^1.$$ 

In the case (G2) $\exp_L$ is a smooth $\mathbb{Z}$-covering map, whence for every $g \in \mathcal{J}(F, V)$ there exists only a $C^1_W$-neighbourhood $\mathcal{N}_g$ in $\mathcal{J}(F, V)$ such that the map $\Delta_0 = \exp_L^{-1} \circ j^1 : \mathcal{N}_g \rightarrow \mathbb{R}$ is well-defined.

It is easy to see that in both cases $\Delta_0$ has the desired properties. $\square$

7.9. Proposition. Let $p + l \geq 2$. Then there exists a $C^p_W$-continuous and preserving smoothness map $\Delta_l : \mathcal{J}_{p+l}(F, V) \rightarrow \mathcal{P}_l$ such that for every $h \in \mathcal{J}_{p+l}(F, V)$ the germ at $O$ of the map

$$F_{h, -\Delta_l(h)}(x) = F(h(x), -\Delta_l(h)(x))$$

belongs to $\hat{\mathcal{J}}_{p+l+1}(F)$.

Proof. Let $h \in \mathcal{J}_{p+l}(F, V)$. Since $p + l \geq 2$ or $p = 1$ but $l \geq 1$. It follows from Lemma 5.7 that in both cases there exists a unique homogeneous polynomial $\omega_l$ of degree $l$ such that $j^{p+l}(h)(x) = x + P(x) \cdot \omega_l(x)$. The correspondence $h \mapsto \omega_l$ is a well-defined map $\Delta_l : \mathcal{J}_{p+l}(F, V) \rightarrow \mathcal{P}_l$ and by (2) of Corollary 5.8 the germ at $O$ of the mapping $F_{h, -\Delta_l(h)}$ belongs to $\hat{\mathcal{J}}_{p+l+1}(F)$.
Let us verify the continuity of $\Delta_l$. Consider the following map

$$j^{p+l} : C^\infty(V, \mathbb{R}^n) \to J^{p+l}(V, \mathbb{R}^n), \quad h \mapsto j^{p+l}(h),$$

associating to each $h \in C^\infty(V, \mathbb{R}^n)$ its $(p+l)$-jet $j^{p+l}(h)$ at $O$. Evidently, $j^{p+l}$ is a $C^p_l$-continuous and preserving smoothness map. Moreover, the image $j^{p+l}(\mathcal{E}_{p+l}(F,V,O))$ is contained in the following set $\mathcal{A}_l = \{ x + P(x) \cdot \omega(x) \mid \omega \in \mathcal{P}_l \} \subset J^{p+l}(V, \mathbb{R}^n)$. Further, it follows from smoothness of the Euclid algorithm of division of polynomials that the correspondence $x + P(x) \cdot \omega(x) \mapsto \omega$ is a well-defined smooth map $D : \mathcal{A}_l \to \mathcal{P}_l$. Therefore $\Delta_l = D \circ j^{p+l}$ is $C^p_l$-continuous and preserves smoothness.

**7.10. Proof of Theorem 7.6.** Let $\mathcal{X}$ be one of the spaces $\mathcal{J}_2(F,V)$, $\mathcal{J}(F,V)$, or $\mathcal{N}_g$ with respect to the cases (1), (2), or (3) of our theorem, where $\mathcal{N}_g$ is a $C^1_W$-neighbourhood of $g \in \mathcal{J}(F,V)$ constructed in Propositions 7.8.

Then similarly to the proof of Theorem 5.1 for every $h \in \mathcal{X}$ we can construct a sequence of homogeneous polynomials $\{\omega_i\}_{i=0}^\infty, (\deg \omega_i = i)$ via the following rule:

$$\omega_0 = \Delta_0(h), \quad \omega_1 = \Delta_1(F_{h,-\omega_0}), \quad \ldots, \quad \omega_l = \Delta_l(F_{h,-\sum_{i=0}^{l-1} \omega_i}).$$

It follows from Propositions 7.8, 7.9, and formulae for $\omega_i$ that for every $i \geq 0$ the correspondence $h \mapsto \omega_i$ is a $C^p_l$-continuous and preserving smoothness map $\lambda_i : \mathcal{X} \to \mathcal{P}_l, \quad \lambda_i(h) = \omega_i$.

Put $\lambda(h) = \sum_{i=0}^\infty \lambda_i(h)$. Then $j^\infty(F_{h,-\lambda(h)}) = j^\infty(id)$ as well as in Theorem 5.1.

Now it follows from Borel’s Theorem 6.1 applied to $\mathcal{X}$ that there exists a $C^\infty_{W,W}$-continuous and preserving smoothness map $\Lambda : \mathcal{X} \to C^\infty(V, \mathbb{R})$

such that $j^\infty(\Lambda(h)) = \lambda(h)$ for $h \in \mathcal{X}$. Hence $j^\infty(F_{h,-\Lambda(h)}) = j^\infty(id)$. Then by (2) of Corollary 5.8 $j^\infty(h) = j^\infty(F_{\Lambda(h)})$ for all $h \in \mathcal{X}$. □

**8. Property (\textasteriskcentered)**

In this section we describe a class of vector fields $F$ on $\mathbb{R}^n$ for which $j^\infty S h(F) = j^\infty \mathcal{D}_{id}(F)^1$, see Theorem 8.5. This class is rather special since it consists of completely integrable (i.e. having $n-1$ almost everywhere independent integrals) vector fields satisfying certain non-degeneracy conditions.
8.1. **Cross product.** Let \((x_1, \ldots, x_n)\) be coordinates in \(\mathbb{R}^n\). For every smooth function \(f : \mathbb{R}^n \to \mathbb{R}\) define the following “gradient” vector field with respect to these coordinates:

\[
\nabla_x f = (f'_x, \ldots, f'_{x_n}).
\]

If \(f_1, \ldots, f_{n-1} : \mathbb{R}^n \to \mathbb{R}\) is an \((n-1)\)-tuple of smooth functions, then we can define the following *cross-product* vector field:

\[
(8.1) \quad H = [\nabla_x f_1, \ldots, \nabla_x f_{n-1}] = \begin{vmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial f_{n-1}}{\partial x_1} & \frac{\partial f_{n-1}}{\partial x_2} & \cdots & \frac{\partial f_{n-1}}{\partial x_n} \\
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n}
\end{vmatrix}
\]

being an analogue of the cross-product \([a, b]\) of two vectors \(a, b\) in \(\mathbb{R}^3\). Notice that the first \(n-1\) rows of this \((n \times n)\)-matrix consist of smooth functions, while the \(n\)-th row is the standard basis \(\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle\) of the space of vector fields on \(\mathbb{R}^n\). Therefore the corresponding determinant is a well-defined vector field.

Equivalently, let us fix the standard Euclidean metric on \(\mathbb{R}^n\). Then we have a Hodge isomorphism \(* : \Lambda^{n-1}(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\) between the spaces of differential forms and the isomorphism \(\phi : \Lambda^1(\mathbb{R}^n) \to \Gamma(\mathbb{R}^n)\) between the space of 1-forms and the space of vector fields on \(\mathbb{R}^n\). Then it is easy to see that

\[
[\nabla_x f_1, \ldots, \nabla_x f_{n-1}] = \phi \circ * (df_1 \wedge \cdots \wedge df_{n-1}).
\]

It is easy to see that every \(f_i\) is constant along orbits of \(H\), i.e. \(f_i\) is an integral for \(H\). Indeed, substituting \(\nabla_x f_i\) in (8.1) instead of last row, we will get \(df_i(H) = 0\). Thus \(H\) is completely integrable in the sense that it has \(n-1\) integrals and its singular set coincides with the set of points where the gradients \(\nabla_x f_1, \ldots, \nabla_x f_{n-1}\) are linearly dependent.

8.2. **Example.** Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be a smooth function. Then

\[
H = [\nabla_x f] = \begin{vmatrix}
\frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \\
\end{vmatrix} = -f'_x \frac{\partial}{\partial x} + f'_x \frac{\partial}{\partial y}
\]

is the corresponding Hamiltonian vector field of \(f\).

8.3. **Lemma.** Let \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) be two local coordinate systems at \(O\) related by a germ of diffeomorphism \(h = (h_1, \ldots, h_n)\) of \((\mathbb{R}^n, O)\), i.e. \(x = h(y)\). Let also \(H_x\) and \(H_y\) be vector fields defined by (8.1) in the coordinates \((x_i)\) and \((y_i)\) respectively, and \(h^*H_x\) be the vector field induced by \(h\), i.e. this is \(H_x\) in the coordinates \((y_i)\). Then

\[
H_y = |J(h)| \cdot h^*H_x.
\]
Proof. Notice that if \( H_x(x) = \sum_{i=1}^{n} T_i(x) \frac{\partial}{\partial x_i} \), then in the coordinates \((y_i)\) we can also write

\[
H_x(y) = \sum_{i=1}^{n} T_i(y) \frac{\partial}{\partial x_i},
\]

where \( \frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j} \). Hence

\[
(8.2) \quad H_x(y) = \left| \begin{array}{ccc}
\frac{\partial f_1}{\partial y_1}(y) & \cdots & \frac{\partial f_n}{\partial y_1}(y) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_1}{\partial y_n}(y) & \cdots & \frac{\partial f_n}{\partial y_n}(y)
\end{array} \right|
\]

On the other hand,

\[
H_y(y) = \left| \begin{array}{ccc}
\frac{\partial f_1}{\partial y_1}(y) & \cdots & \frac{\partial f_n}{\partial y_1}(y) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_1}{\partial y_n}(y) & \cdots & \frac{\partial f_n}{\partial y_n}(y)
\end{array} \right|
\]

\[
= \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i}(y) \cdot \frac{\partial x_i}{\partial y_1} \cdots \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i}(y) \cdot \frac{\partial x_i}{\partial y_n}
\]

\[
= \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i}(y) \cdot \frac{\partial x_i}{\partial y_1} \cdots \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i}(y) \cdot \frac{\partial x_i}{\partial y_n}
\]

\[
= \left| \begin{array}{ccc}
\frac{\partial f_1}{\partial x_1}(y) & \cdots & \frac{\partial f_n}{\partial x_1}(y) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_1}{\partial x_n}(y) & \cdots & \frac{\partial f_n}{\partial x_n}(y)
\end{array} \right| \cdot \left| \begin{array}{ccc}
\frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n}
\end{array} \right|
\]

\[
(8.2) \quad H_x(y) \cdot |J(h)|.
\]

Lemma is proved. \( \square \)

8.4. Definition. Let \( F \) be a vector field defined on some neighbourhood \( V \) of \( O \in \mathbb{R}^n \). Say that \( F \) has property (*) at \( O \) if there exist \( p \in \mathbb{N} \) and \( n \) smooth non-flat at \( O \) functions

\[
\eta, f_1, \ldots, f_{n-1} : V \to \mathbb{R}
\]

such that

(a) \( j^{p-1}(F) = 0 \),
(b) $P = j^p(F)$ is a non-zero homogeneous vector field being **non-divisible by homogeneous polynomials**, i.e. $P$ can not be represented as a product $P = \omega Q$, where $\omega$ is a homogeneous polynomial of degree $\deg \omega \geq 1$ and $Q$ is a homogeneous vector field of degree $\deg Q \geq 1$.

(c) vector fields $\nabla_x f_1, \ldots, \nabla_x f_{n-1}$ are linearly independent on an everywhere dense subset of $V$ and

\[(8.3) \quad \eta \cdot F = [\nabla_x f_1, \ldots, \nabla_x f_{n-1}].\]

Let us explain this definition.

1) Since $1 \leq p < \infty$, we have that $F(O) = 0$ and $F$ is not flat at $O$.

2) We allow $\eta$ and therefore $\eta \cdot F$ vanish at some points of $V$ which can be even non-singular for $F$. But due to (c) the set of zeros of $\eta \cdot F$ and therefore $\eta^{-1}(0)$ are nowhere dense in $V$.

3) Let $k = \text{ord}(\eta, O)$, $p_i = \text{ord}(f_i, O) < \infty$, $(i = 1, \ldots, n - 1)$. Thus

\[(8.4) \quad \gamma = j^k(\eta), \quad \Gamma_i = j^{p_i}(f_i), \quad (i = 1, \ldots, n - 1)\]

are non-zero homogeneous polynomials of degrees $k, p_1, \ldots, p_{n-1}$ respectively. Then $R = [\nabla_x \Gamma_1, \ldots, \nabla_x \Gamma_{n-1}]$ is a homogeneous vector field of degree $\sum_{i=1}^{n-1} (p_i - 1)$ and $\gamma \cdot P = R$. Since $P$ is non-divisible by homogeneous polynomials, it follows that $\gamma$ is the greatest common divisor of coordinate functions of $R$ in the ring $\mathbb{R}[x_1, \ldots, x_n]$.

**8.5. Theorem.** If $F$ has property $(\ast)$ then $j^\infty \tilde{Sh}(F) = j^\infty \tilde{D}_{id}(F)^1$. Moreover, for every neighbourhood $V$ of $O$ and every $g \in \mathcal{E}_{id}(F, V)^1$ there exist a $C^1_W$-neighbourhood $\mathcal{N}_g$ in $\mathcal{E}_{id}(F, V)^1$ and a $j^\infty$-section of the shift map $\varphi_V$ on $\mathcal{N}_g$.

Thus due to Lemma 7.4 in order to completely resolve (1.1) we have to construct a section of $\varphi_V$ on $\mathcal{E}_\infty(F, V)$. This was done in [M4] for the case of homogeneous polynomial vector fields on $\mathbb{R}^2$ satisfying $(\ast)$, see also §11 for more general result.

The proof of Theorem 8.5 will be given in §§9, 10. The following lemma presents a class of examples of vector fields with property $(\ast)$.

**8.6. Lemma.** Let $f_1, \ldots, f_{n-1} : \mathbb{R}^n \to \mathbb{R}$ be homogeneous polynomials such that $\nabla_x f_1, \ldots, \nabla_x f_{n-1}$ are linearly independent on everywhere dense subset of $\mathbb{R}^n$, and let $\eta$ be the greatest common divisor of the coordinate functions of $H = [\nabla_x f_1, \ldots, \nabla_x f_{n-1}]$. Then the homogeneous vector field $F = H/\eta$ has property $(\ast)$. \(\Box\)

**8.7. Example.** Let $n = 2$, $f(x, y) = x^3y^4$, and

$$H = [\nabla f] = (-f_y', f_x') = (-4x^3y^3, 3x^2y^4) = x^2y^3(-4x, 3y) = \eta F.$$

Then $F$ is non-divisible. Notice also that the singular set of $H$ consists of $x$- and $y$-axes while the singular set of $F$ is the origin only.
8.8. Lemma. Property (⋆) does not depend on a particular choice of local coordinates at O.

Proof. Suppose that in coordinates $x = (x_1, \ldots, x_n)$ at O conditions (a)-(c) of Definition 8.4 are satisfied. Let $y = (y_1, \ldots, y_n)$ be another coordinates at O related to $(x_1, \ldots, x_n)$ by a germ of a diffeomorphism $h = (h_1, \ldots, h_n)$ of $(\mathbb{R}^n, O)$, i.e. $x = h(y)$. We have to show that conditions (a)-(c) of Definition 8.4 hold in the coordinates $(y_1, \ldots, y_n)$ for the induced vector field $h^*F = Th^{-1} \circ F \circ h$.

Let $A = J(h, O)$ be the Jacobi matrix of $h$ at O. Then it easily follows from condition (a) for $F$ that

$$j^{p-1}(h^*F) = 0, \quad j^p(h^*F)(y) = A^{-1}P(Ay).$$

The latter identity implies that the initial non-zero jet of $h^*F$ is non-divisible by homogeneous polynomials iff so is $P$. This proves (a) and (b) for $h^*F$.

To establish (c) apply $h$ to both parts of (8.3). Then

$$h^*(\eta \cdot F) = \eta \circ h \cdot h^*F,$$

$$h^* \left[ \nabla_x f_1, \ldots, \nabla_x f_{n-1} \right] \frac{1}{|J(h)|} \cdot \left[ \nabla_y f_1, \ldots, \nabla_y f_{n-1} \right].$$

Denote $\eta' = \eta \circ h \cdot |J(h)|$. Then $\eta'$ is smooth and

$$\eta' \cdot h^*F = \left[ \nabla_y f_1, \ldots, \nabla_y f_{n-1} \right].$$

This proves (c). \qed

9. Stabilizers of functions and polynomials

In this section we present some statements which will be used in the proof of Theorem 8.5.

9.1. Lemma. Let $f \in \hat{F}(\mathbb{R}^n)$, $h \in \hat{E}(\mathbb{R}^n)$, and $\delta = f \circ h - f$. Suppose that $j^{p-1}(f) = 0$ and $j^{k-1}(h) = j^{k-1}(\text{id})$ for some $p, k \geq 1$. In particular $\Gamma = j^p(f)$ is a homogeneous polynomial of degree $p$.

If $k = 1$ and $j^1(h)(x) = Ax$ for some $(n \times n)$-matrix, then

$$j^p(\delta)(x) = \Gamma(A \cdot x) - \Gamma(x).$$

If $k \geq 2$ and $j^k(h)(x) = x + v(x)$ for some homogeneous map $v$ of degree $k$, then

$$j^{p-1+k}(\delta) = \langle \nabla \Gamma, v \rangle.$$

The proof of this lemma is direct and we leave it for the reader.

9.2. Corollary. Suppose that $h$ preserves $f$, i.e. $f \circ h = f$, and thus $\delta \equiv 0$. If $k = 1$, then $\Gamma(A \cdot x) = \Gamma(x)$. If $k \geq 2$, then $\langle \nabla \Gamma, v \rangle = 0$. 

9.3. Stabilizers of polynomials. Consider the right action of the group \( \text{GL}(\mathbb{R}, n) \) on the space of polynomials \( \mathbb{R}[x_1, \ldots, x_n] \) by:

\[
\Phi : \mathbb{R}[x_1, \ldots, x_n] \times \text{GL}(\mathbb{R}, n) \to \mathbb{R}[x_1, \ldots, x_n]
\]

(9.1)

\[\Phi(\Gamma, A) = \Gamma \circ A, \quad \text{i.e.} \quad \Phi(\Gamma, A)(x) = \Gamma(A \cdot x),\]

where \((\Gamma, A) \in \mathbb{R}[x_1, \ldots, x_n] \times \text{GL}(\mathbb{R}, n)\). For \( \Gamma \in \mathbb{R}[x_1, \ldots, x_n] \) let

\[S(\Gamma) = \{ A \in \text{GL}(\mathbb{R}, n) \mid \Gamma(A \cdot x) = \Gamma(x) \}\]

be its stabilizer with respect to \( \Phi \). Then \( S(\Gamma) \) is a closed (and therefore a Lie) subgroup of \( \text{GL}(\mathbb{R}, n) \).

9.4. Lemma. For every \( \Gamma \in \mathbb{R}[x_1, \ldots, x_n] \) the tangent space \( T_E S(\Gamma) \) to the stabilizer \( S(\Gamma) \) of \( \Gamma \) at \( E \) consists of matrices \( V \in M(\mathbb{R}, n) \) such that \( \langle \nabla \Gamma(x), V x \rangle = 0 \) for all \( x \in \mathbb{R}^n \).

Proof. Let \( V \in M(\mathbb{R}, n) \) and \( A : \mathbb{R} \to \text{GL}(\mathbb{R}, n) \) be the following homomorphism \( A(t) = e^{Vt} \). Evidently, \( A(0) = E \) and \( A_t'(0) = V \).

Notice that

\[
\frac{\partial}{\partial t} \Gamma(A(t) x) = \langle \nabla \Gamma(A(t) x), A'_t(t) x \rangle = \langle \nabla \Gamma(e^{Vt} x), V e^{Vt} x \rangle.
\]

Then the following statements are equivalent:

(i) \( V \in T_E S(\Gamma) \);

(ii) \( A(t) \in S(\Gamma) \), i.e. \( \Gamma(A(t) x) = \Gamma(x) \), for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \);

(iii) \( \frac{\partial}{\partial t} \Gamma(A(t) x) = \langle \nabla \Gamma(e^{Vt} x), V e^{Vt} x \rangle = 0 \), for all \( x \in \mathbb{R}^n \);

(iv) \( \frac{\partial}{\partial t} \Gamma(A(t) x)|_{t=0} = \langle \nabla \Gamma(x), V x \rangle = 0 \) for all \( x \in \mathbb{R}^n \).

The implications \( (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \) are evident and \( (iv) \Rightarrow (iii) \) can be obtained by substituting \( e^{Vt} x \) instead of \( x \) in \( (iv) \). It remains to note that our lemma claims that \( (i) \Leftrightarrow (iv) \).

\[\square\]

Let \( \Gamma_1, \ldots, \Gamma_{n-1} \in \mathbb{R}[x_1, \ldots, x_n] \) and

(9.2)

\[S = \bigcap_{i=1}^{n-1} S(\Gamma_i)\]

be the intersection of their stabilizers. Then \( S \) is a closed Lie subgroup of \( \text{GL}(\mathbb{R}, n) \). Denote by \( T_E S \) the tangent space of \( S \) at the unity matrix \( E \), and let \( S_E \) be the unity component of \( S \).

9.5. Lemma. Let \( H = [\nabla \Gamma_1, \ldots, \nabla \Gamma_{n-1}] \) be the vector field on \( \mathbb{R}^n \) defined by (8.4), \( \eta \) be the greatest common divisor of coordinate functions of \( H \), and \( P = H/\eta \). Suppose that \( P \neq 0 \). Then \( P \) is non-divisible by polynomials, i.e. if \( P(x) = \omega(x) U(x) \), where \( \omega \) is a polynomial and \( U \) is a polynomial vector field, then either \( \deg \omega = 0 \), or \( \deg U = 0 \).

(i) If \( \deg P = 1 \), i.e. \( P(x) = L x \) for some non-zero matrix \( L \in M(\mathbb{R}, n) \), then \( T_E S = \{ L t \}_{t \in \mathbb{R}} \) and \( S_E = \{ e^{Lt} \}_{t \in \mathbb{R}} \).

(ii) If \( \deg P \geq 2 \) then \( S_E = \{ E \} \).
Proof. Notice that $T_E \mathcal{S} = \bigcap_{i=1}^{n-1} T_E \mathcal{S}(\Gamma_i)$. Let $U \in T_E \mathcal{S}$. Then by Lemma 9.4 $\langle \nabla \Gamma_i(x), Ux \rangle = 0$ for every $i = 1, \ldots, n - 1$. Therefore $Ux$ is parallel to the cross product $H(x)$ of gradients $\nabla \Gamma_i$ and therefore to $P(x)$ at every $x \in \mathbb{R}^n$. If $U \neq 0$, then there exists a non-zero polynomial $\omega$ such that

$$P(x) = \omega(x) \cdot Ux$$

Since $P$ is non-divisible, this identity is possible only if $\omega$ is a constant and in this case deg $P = 1$.

Hence if deg $P \geq 2$, then $U$ is always zero, whence $T_E \mathcal{S} = \{0\}$, and $\mathcal{S}_E = \{E\}$. This proves (ii).

(i) Suppose that deg $P = 1$. Then it follows from Lemma 9.4 that $\{Lt\}_{t \in \mathbb{R}} \subset T_E \mathcal{S}$. On the other hand, as noted above for every $U \in T_E \mathcal{S}$ there exists $\omega \in \mathbb{R}$ such that (9.3) holds true, whence $L = \omega U$. Therefore $U \in \{Lt\}_{t \in \mathbb{R}}$, and thus $\{Lt\}_{t \in \mathbb{R}} = T_E \mathcal{S}$. □

10. PROOF OF THEOREM 8.5

Suppose that $F$ has property $(\ast)$ at $O$. Thus

$$\eta \cdot F = [\nabla_x f_1, \ldots, \nabla_x f_{n-1}],$$

where $\eta, f_1, \ldots, f_{n-1} : V \to \mathbb{R}$ are germs of smooth functions satisfying assumptions of Definition 8.4. We have to show that

$$(10.1) \quad j^\infty \hat{S} h(F) = j^\infty \hat{D}_{id}(F)^1$$

and for every open neighbourhood $V$ of $O$ and $g \in \mathcal{E}_{id}(F, V)^1$ construct a local $j^\infty$-section of $\varphi_V$ defined on some $C^1_{W}$-neighbourhood $N_g$ of $g$ in $\mathcal{E}_{id}(F, V)^1$.

Proof of (10.1). Notice that $\mathcal{G} = \hat{D}_{id}(F)^1$ is a group which contains $\hat{S} h(F)$. Therefore it suffices to verify conditions (A2) and (A3) of Theorem 5.1.

Similarly to (8.4) set $k = \text{ord}(\eta, O)$, $p_i = \text{ord}(f_i, O)$, $\gamma = j^k(\eta)$, and $\Gamma_i = j^p(f_i)$, $(i = 1, \ldots, n - 1)$. Denote $p = \sum_{i=1}^{n-1} (p_i - 1) - k$. Then

$$(10.2) \quad \gamma \cdot P = [\nabla_x \Gamma_1, \ldots, \nabla_x \Gamma_{n-1}],$$

where $P = j^p(F)$ is a homogeneous vector field of degree $p$. By assumption $P$ is non-divisible by homogeneous polynomials. For $p = 1$ we assume that $P(x) = Lx$ for some non-zero matrix $L \in M(\mathbb{R}, n)$.

Let $h \in \hat{D}(F)$. Then $h$ leaves invariant every orbit of $F$ and therefore preserves every function $f_i$, i.e. $f_i \circ h = f_i$ for all $i = 1, \ldots, n$.

(A2) Let $A$ be the Jacobi matrix of $h$ at $O$, thus $j^1(h)(x) = A \cdot x$. We have to show that $A = e^{\omega_0 L}$ for some $\omega_0 \in \mathbb{R}$ if $p = 1$, and $A = E$ for $p \geq 2$. This is implied by the following lemma and Lemma 9.5.
10.1. Lemma. Let \( S = \bigcap_{i=1}^{n-1} S(f_i) \) be the intersection of the stabilizers of \( f_i \) with respect to the action of \( \text{GL}(\mathbb{R}, n) \), see (9.2), and \( S_E \) be the unity component of \( S \). Then for every \( h \in \mathcal{D}_{id}(F)^1 \) its Jacobi matrix \( A \) at \( O \) belongs to \( S_E \).

Proof. Since \( f_i \circ h = f_i, (i = 1, \ldots, n - 1) \), we get from Corollary 9.2 that \( \Gamma_i(Ax) = \Gamma_i(x) \). In other words \( A \) belongs to the intersection of the stabilizers \( S = \bigcap_{i=1}^{n-1} S(\Gamma_i) \). On the other hand the assumption \( h \in \mathcal{D}_{id}(F)^1 \) means that there exists a 1-isotopy \( (h_t) \) in \( \mathcal{D}(F) \) between \( h_0 = \text{id}_{\mathbb{R}^n} \) and \( h_1 = h \). Let \( A_t \) be the Jacobi matrix of \( h_t \) at \( O \). Since \( (h_t) \) is 1-isotopy, we have that \( (A_t) \) continuously depend on \( t \). Moreover, \( A_0 = E \), whence \( A = A_1 \) belongs to the unity component \( S_E \) of \( S \). \( \square \)

(A3) Suppose that \( j^{p+l}(h)(x) = x + v(x) \), where \( v \) is a non-zero homogeneous map of degree \( p + l \geq 2 \). Since \( f_i \circ h = f_i, (i = 1, \ldots, n - 1) \), we obtain from Corollary 9.2 that \( \langle \nabla \Gamma_i, v \rangle = 0 \), whence \( v \) is parallel to the cross-product of gradients \( \nabla \Gamma_i \) and therefore to \( P \). Since \( P \) is non-divisible, it follows that \( k \geq p \) and there exists a unique non-zero homogeneous polynomial \( \omega_l \in \mathcal{P}_l \) such that \( v = P \cdot \omega_l \).

This completes the proof of (10.1).

It remains to construct \( j^\infty \)-sections of \( \varphi_V \). Let \( V \) be a neighbourhood of \( O \). Then (10.1) implies that \( \mathcal{E}_{id}(F,V)^1 \subset \mathcal{J}(F,V) \). If \( p \geq 2 \), then by (1) of Theorem 7.6 there exists a \( j^\infty \)-section of \( \varphi_V \) on all of \( \mathcal{J}(F,V) \) and therefore on \( \mathcal{E}_{id}(F,V)^1 \).

Suppose that \( p = 1 \), so \( P(x) = Lx \) is a linear vector field. Notice that the corresponding one-parametric subgroup \( E_L = \{e^{Lt}\}_{t \in \mathbb{R}} \) is closed in \( \text{GL}(\mathbb{R}, n) \) as the unity component of the intersection of closed subgroups of \( \text{GL}(\mathbb{R}, n) \) (stabilizers of \( f_i \)). Then by (2) and (3) of Theorem 7.6 for every \( g \in \mathcal{J}(F,V) \) there exists a local \( j^\infty \)-section of \( \varphi_V \). In particular, this holds for all \( g \in \mathcal{E}_{id}(F,V)^1 \subset \mathcal{J}(F,V) \). \( \square \)

11. Reduced Hamiltonian vector fields

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a real homogeneous polynomial in two variables, so we can write

\[
(11.1) \quad f(x, y) = \prod_{i=1}^{l} L_i^l(x, y) \cdot \prod_{j=1}^{q} Q_j^q(x, y),
\]

where \( l, q \geq 1, every \ L_i \) is a linear function, every \( Q_j \) is a definite (i.e. irreducible over \( \mathbb{R} \)) quadratic form, \( L_i/L' \neq \text{const} \) for \( i \neq i' \), and \( Q_j/Q_{j'} \neq \text{const} \) for \( j \neq j' \). Then it can easily be shown that the polynomial

\[
D = \prod_{i=1}^{l} L_i^{l-1} \cdot \prod_{j=1}^{q} Q_j^{q-1}
\]
is the greatest common divisor of its partial derivatives $f'_x$ and $f'_y$.

Hence the following homogeneous vector field of degree $p = l + 2q - 1$ on $\mathbb{R}^2$

$$F = [\nabla f]/D = -(f'_y/D) \frac{\partial}{\partial x} + (f'_x/D) \frac{\partial}{\partial y}$$

is non-divisible by homogeneous polynomials. Thus $F$ has property $(\ast)$. We will call $F$ the reduced Hamiltonian vector field of $F$. Notice that $O$ is a unique singular point of $F$.

The following theorem improves [M4, Theorem 3.2] which was based on the previous version of this paper.

11.1. **Theorem.** Let $f$ be a real homogeneous polynomial in two variables, $F$ be its reduced Hamiltonian vector field, and $V$ be an open neighbourhood of $O$. Then

$$\mathcal{S}h(F, V) = \mathcal{E}_{id}(F, V)^1$$

and for every $g \in \mathcal{E}_{id}(F, V)^1$ there exist a $C^1_W$-neighbourhood $\mathcal{N}_g$ in $\mathcal{E}_{id}(F, V)^1$ and a $C^\infty,W,W$-continuous and preserving smoothness section $\Lambda : \mathcal{N}_g \to C^\infty(V, \mathbb{R})$ of $\varphi_V$, i.e. for every $h \in \mathcal{N}_g$

$$h(x) = \varphi_V(\Lambda(h))(x) = F(x, \Lambda(h)(x)).$$

In particular, it follows from Theorem 4.4 and Lemma 4.7 that $F$ is parameter rigid.

**Proof.** Since $F$ has property $(\ast)$, it follows from Theorem 8.5 that for every $g \in \mathcal{E}_{id}(F, V)^1$ there exists a $j^\infty$-section defined on some $C^1_W$-neighbourhood of $g$ in $\mathcal{E}_{id}(F, V)^1 \subset \mathcal{E}(F, V) \cap \mathcal{J}(F, V)$. Therefore by Lemma 7.4 it suffices to construct a $C^\infty,W,W$-continuous and preserving smoothness section $\Psi : \mathcal{E}_{id}(F, V)^1 \to C^\infty(V, \mathbb{R})$ of $\varphi_V$.

For the case $D \equiv 1$, i.e., when $f$ has no multiple factors, such a section was constructed in [M4, Theorem 3.2]. The detailed analysis of the proof shows that [M4, Theorem 3.2] uses only the assumption that coordinate functions of $F$ are relatively prime in $\mathbb{R}[x, y]$, i.e. that $F$ has property $(\ast)$, but not the assumption that $f$ has no multiple factors. This implies that the same arguments prove an existence of $\Psi$ for arbitrary $f$. The details are left for the reader. \[ \square \]

12. **Acknowledgments**

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