STRONGLY DISTRIBUTIONAL CHAOS IN THE SETS OF TWELVE DIFFERENT TYPES OF NON-RECURRENT POINTS

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Abstract. In present paper we mainly focus on non-recurrent dynamical orbits with empty syndetic center and show that twelve different statistical structures over mixing expanding maps or transitive Anosov diffeomorphisms all have dynamical complexity in the sense of strongly distributional chaos.

1. Introduction

The study of the thermodynamic formalism and multifractal analysis for maps with some hyperbolicity has drawn the attention of many researchers from the theoretical physics and mathematics communities in the last decades. The general concept of multifractal analysis (or dimension theory), that can be traced back to Besicovitch, is to decompose the phase space in (invariant) subsets of points which have a similar dynamical behavior and to describe the size of each of such subsets from the geometrical or topological viewpoint by using the concepts of Hausdorff dimension, topological entropy or pressure and Lebesgue measure etc.

Birkhoff ergodic theorem is one classical and basic way to study dynamical orbits by describing asymptotic behavior from the probabilistic viewpoint of a given observable function. Some concepts in the theory of multifractal analysis, for example, irregular set and level set, derive from the Birkhoff ergodic theorem. As for characterizing the complexity of these sets, there are some prevalent indexes. For example, Pesin and Pitskel [44] are the first to notice the phenomenon of the irregular set carrying full topological entropy in the case of the full shift on two symbols. Since then, there are lots of advanced results to show that the irregular points can carry full entropy (and topological pressure) in symbolic systems, hyperbolic systems and systems with specification-like or shadowing-like properties [7, 6, 39, 40, 41, 43, 19, 10, 64, 15, 63, 62]. In addition, Lebesgue measure [60, 33], Hausdorff dimension [44, 7, 43, 10, 64, 15, 5, 61, 19] and chaos [9] are all used afterwards to describe how complex the irregular set and level set can be. We refer to [18, 43, 4] for more contents of fractals and dimension theory.

Another way to differ asymptotic behavior of dynamical orbits is from the perspective of periodic-like recurrence. There are many such concepts, for example, periodic points, almost periodic points, weakly almost periodic points, transitive points etc [13, 28, 29, 23, 24, 22, 21, 51, 70, 66]. For periodic points, it is well-known that the exponential growth of the periodic points equals to topological entropy for hyperbolic systems but Kaloshin showed that in general periodic points can grow much faster than entropy [28]. Moreover, it is well known that for $C^1$ generic diffeomorphisms, all periodic points are hyperbolic so that countable and they form a dense subset of the non-wandering set (by classical Kupka-Smale theorem, Pugh’s or Mañé’s ergodic closing lemma from smooth ergodic theory, for example, see [32, 57, 49, 48, 37]). However, periodic point does not exist naturally. For example there is no periodic points in any irrational rotation. Almost periodic point is a good generalization which exists naturally since it is equivalent that it belongs to a minimal set (see [8, 22, 21, 23, 38]) and by Zorn’s lemma any dynamical system contains at least one minimal invariant subset. There are many examples of subshifts which are strictly ergodic (so that every point in the subshift is almost periodic) and has positive entropy, for example, see [24].

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other words, almost periodic points have strong dynamical complexity. For weakly almost periodic points, quasi-weakly almost periodic points and Banach-recurrent points, it is shown recently that all their gap-sets with same asymptotic behavior carry high dynamical complexity in the sense of full topological entropy over dynamical system with specification-like property and expansiveness [26, 66]. In addition to studying the entropy of periodic-like recurrent point sets, authors in [9] consider their complexity from the perspective of chaos. The results of [26, 66, 9] are all restricted on studying transitive points (or dense orbits).

In contract to transitive points, non-dense points (i.e., points that are not transitive) have been studied a lot. For example, non-dense points (i.e., points that are not transitive) form a set with full Hausdorff dimension for hyperbolic systems, see [67]. An effective tool to show full Hausdorff dimension is Schmidt’s game, which was first introduced by Schmidt in [54]. A winning set for such games is large in the following sense: it is dense in the metric space, and its intersection with any nonempty open subset has full Hausdorff dimension when the metric space is $\mathbb{R}^n$ or a manifold. In [54] Schmidt showed that the set of badly approximable numbers is winning for Schmidt’s game and hence has full Hausdorff dimension 1. There are many applications of Schmidt’s game in homogeneous dynamics due to the well known connection between Diophantine approximation and bounded orbits of flows (for example, see [11, 12, 14, 31, 69] and references therein). Recently it was showed in [16] that for hyperbolic or expanding systems the set of non-dense points carries full topological entropy by using a way different with Schmidt’s game. Moreover, this set may contain eighteen different types for which twelve fractals correspond to nonrecurrent points and other six fractals correspond to recurrent (but not transitive) points, and every one of eighteen fractals has full topological entropy by using a way different with Schmidt’s game. Moreover, this set may contain eighteen different types for which twelve fractals correspond to nonrecurrent points and other six fractals correspond to recurrent (but not transitive) points, and every one of eighteen fractals has full topological entropy even though the twelve different fractals corresponding to non-recurrent points always have totally zero measure (since the set of recurrent points has totally full measure).

In present paper, we adopt this new pattern of classification of non-recurrent points and show that twelve different fractals of non-recurrent points all are strongly distributional chaotic.

1.1. Non-recurrent Points and Statistical $\omega$-limit Sets. Let $(X, d)$ be a nondegenerate (i.e, with at least two points) compact metric space, and $f : X \to X$ be a continuous map. Such $(X, f)$ is called a dynamical system. Given a dynamical system $(X, f)$, let $\mathcal{C} = \{x_n\}_{n=1}^{\infty}$, we define $\omega(\mathcal{C}) := \bigcap_{n \geq 1} \bigcup_{k \geq n} \{x_k\}$. For any $x \in X$, the orbit of $x$ is the sequence $(x, fx, f^2x, \cdots)$, which we denote by $\text{orb}(x, f)$. We call $\omega(\text{orb}(x, f))$ the $\omega$-limit set of $x$, written as $\omega_f(x)$ briefly. A point $x \in X$ is recurrent, if $x \in \omega_f(x)$. We denote all recurrent points of $X$ by $\text{Rec}$ or $\text{Rec}(f)$. We call $X \setminus \text{Rec}$ the non-recurrent point set, which we denote by $\text{NR}$ or $\text{NR}(f)$. A point $x$ is transitive if $\omega_f(x) = X$. It is equivalent to $\text{orb}(x, f) = X$ if the system is surjective. Let $\text{Tran}, \text{ND}$ denote the set of transitive points and non-transitive points. Note that

$$ND = NR \sqcup (\text{Rec} \setminus \text{Tran}).$$

If for every pair of nonempty open sets $U, V$ there is an positive integer $n$ such that $f^n(U) \cap V \neq \emptyset$ then we call $(X, f)$ topologically transitive. Furthermore, if for every pair of nonempty open sets $U, V$ there exists an positive integer $N$ such that $f^n(U) \cap V \neq \emptyset$ for every $n > N$, then we call $(X, f)$ topologically mixing.

**Theorem A.** Suppose $(X, f)$ is a mixing expanding map or a transitive Anosov diffeomorphism on a compact manifold. Then the set of non-recurrent set $\text{NR}(f)$ is strongly distributional chaotic.

Strongly distributional chaos is a kind of chaos which is stronger than usual distributional chaos and Li-Yorke chaos. See their definitions in section 2.3.

Since $\text{NR}(f)$ is contained in the set of non-dense points $\text{ND}(f)$, then one also has that $\text{ND}$ is strongly distributional chaotic. We will also show that $\text{Rec}(f) \setminus \text{Tran}(f)$ is distributional chaotic of type 1 in section 4.4. Since $\text{NR}(f)$ has zero measure for any invariant measure so that we have a following corollary.
For convenience, it is called \( \xi \omega \) (1.1)

**Corollary 1.1.** Suppose \((X, f)\) is a mixing expanding map or a transitive Anosov diffeomorphism on a compact manifold. Then there is set \( S \subseteq X \) which is strongly distributional chaotic such that for any \( \mu \in \mathcal{M}_f(X), \mu(S) = 0. \)

The set of non-recurrent points \( NR \) also can be decomposed into many different levels by using different asymptotic behavior. One natural question is that

 Which layer does strongly distributional chaos occur in?

Here we use some concepts from [16] to describe different statistical structure of dynamical orbits.

**Definition 1.2.** Let \( S \subseteq \mathbb{N} \), define

\[
\bar{d}(S) := \limsup_{n \to \infty} \frac{|S \cap \{0, 1, \ldots, n-1\}|}{n}, \quad \underline{d}(S) := \liminf_{n \to \infty} \frac{|S \cap \{0, 1, \ldots, n-1\}|}{n},
\]

where \(|Y|\) denotes the cardinality of the set \( Y \). These two concepts are called **upper density** and **lower density** of \( S \), respectively. If \( \bar{d}(S) = \underline{d}(S) = d \), we call \( S \) to have density of \( d \). Define

\[
B^+(S) := \limsup_{|I| \to \infty} \frac{|S \cap I|}{|I|}, \quad B_+(S) := \liminf_{|I| \to \infty} \frac{|S \cap I|}{|I|},
\]

here \( I \subseteq \mathbb{N} \) is taken from finite continuous integer intervals. These two concepts are called **Banach upper density** and **Banach lower density** of \( S \), respectively. A set \( S \subseteq \mathbb{N} \) is called **syndetic**, if there is \( N \in \mathbb{N} \) such that for any \( n \in \mathbb{N}, S \cap \{n, n+1, \ldots, n+N\} \neq \emptyset \).

These concepts of density are basic and have played important roles in the field of dynamical systems, ergodic theory and number theory, etc. Let \( U, V \subseteq X \) be two nonempty open subsets and \( x \in X \). Define sets of visiting time

\[
N(U, V) := \{n \geq 1 | U \cap f^{-n}(V) \neq \emptyset \} \quad \text{and} \quad N(x, U) := \{n \geq 1 | f^n(x) \in U \}.
\]

**Definition 1.3.** (Statistical \( \omega \)-limit sets) For \( x \in X \) and \( \xi = d, d_*, B^+, B_* \), a point \( y \in X \) is called \( x-\xi \)-accessible, if for any \( \varepsilon > 0 \), \( N(x, V_\varepsilon(y)) \) has positive density w. r. t. \( \xi \), where \( V_\varepsilon \) denotes the ball centered at \( x \) with radius \( \varepsilon \). Let

\[
\omega_\xi(x) := \{y \in X | y \text{ is } x-\xi \text{- accessible}\}.
\]

For convenience, it is called \( \xi-\omega \)-**limit set** of \( x \). We also call \( \omega_{B_*}(x) \) to be **syndetic center** of \( x \).

It is obvious that

\[
(1.1) \quad \omega_{B_*}(x) \subseteq \omega_d(x) \subseteq \omega_{\pi}(x) \subseteq \omega_{B^+}(x) \subseteq \omega_f(x).
\]

In [2] for maps and [3] for flows that \( \omega_{\pi}(x) \) is called essential \( \omega \)-limit set of \( x \). In [70, 16], authors show that these concept have a close relationship with measure space. From [16], we know that for any \( x \in X \), if \( \omega_{B_*}(x) = \emptyset \), then \( x \) satisfies only one of following twelve cases:

**Case (1)**: \( \emptyset = \omega_{B_*}(x) \subseteq \omega_d(x) = \omega_{\pi}(x) \subseteq \omega_{B^+}(x) = \omega_f(x) \);

**Case (1’)**: \( \emptyset = \omega_{B_*}(x) \subseteq \omega_d(x) = \omega_{\pi}(x) \subseteq \omega_{B^+}(x) \subseteq \omega_f(x) \);

**Case (2)**: \( \emptyset = \omega_{B_*}(x) = \omega_d(x) \subseteq \omega_{\pi}(x) = \omega_{B^+}(x) = \omega_f(x) \);

**Case (2’)**: \( \emptyset = \omega_{B_*}(x) = \omega_d(x) \subseteq \omega_{\pi}(x) \subseteq \omega_{B^+}(x) \subseteq \omega_f(x) \);

**Case (3)**: \( \emptyset = \omega_{B_*}(x) \subseteq \omega_d(x) \subseteq \omega_{\pi}(x) = \omega_{B^+}(x) = \omega_f(x) \);

**Case (3’)**: \( \emptyset = \omega_{B_*}(x) \subseteq \omega_d(x) \subseteq \omega_{\pi}(x) \subseteq \omega_{B^+}(x) \subseteq \omega_f(x) \);

**Case (4)**: \( \emptyset = \omega_{B_*}(x) = \omega_d(x) \subseteq \omega_{\pi}(x) \subseteq \omega_{B^+}(x) = \omega_f(x) \);

**Case (4’)**: \( \emptyset = \omega_{B_*}(x) = \omega_d(x) \subseteq \omega_{\pi}(x) \subseteq \omega_{B^+}(x) \subseteq \omega_f(x) \);

**Case (5)**: \( \emptyset = \omega_{B_*}(x) \subseteq \omega_d(x) \subseteq \omega_{\pi}(x) \subseteq \omega_{B^+}(x) = \omega_f(x) \);

**Case (5’)**: \( \emptyset = \omega_{B_*}(x) \subseteq \omega_d(x) \subseteq \omega_{\pi}(x) \subseteq \omega_{B^+}(x) \subseteq \omega_f(x) \);

**Case (6)**: \( \emptyset = \omega_{B_*}(x) \subseteq \omega_d(x) = \omega_{\pi}(x) = \omega_{B^+}(x) = \omega_f(x) \);

**Case (6’)**: \( \emptyset = \omega_{B_*}(x) \subseteq \omega_d(x) = \omega_{\pi}(x) \subseteq \omega_{B^+}(x) \subseteq \omega_f(x) \).
If \( \omega_{B_i}(x) \neq \emptyset \), then the relation between \( \omega_{B_i}(x) \), \( \omega_B(x) \), \( \omega_{\mathcal{B}}(x) \), \( \omega_f(x) \) has sixteen possible cases, which are unknown. In this paper, we just consider the systems with \( \omega_{B_i}(x) = \emptyset \). A surprising discovery is that strongly distributional chaos does not just occur in one fixed layer. Here we find there are twelve different layers for which every layer is strongly distributional chaotic, as a refined version of Theorem A.

**Theorem B.** Suppose \((X, f)\) is a mixing expanding map or a transitive Anosov diffeomorphism on a compact manifold. Then the set \( \{x \in X \mid x \text{ satisfies Case (i)}\} \cap NR(f) \) is strongly distributional chaotic for any fixed \( i \in \{1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'\} \).

### 1.2. Combination with Irregular Set and Level Set.

Let us recall irregular set and level sets which were studied a lot in the dimension theory (or multifractal analysis), for example, see [7, 6, 64, 15, 63, 62]. Let \( \mathcal{M}(X) \), \( \mathcal{M}_f(X) \), \( \mathcal{M}_f^c(X) \) denote the space of probability measures, \( f \)-invariant, \( f \)-ergodic probability measures respectively.

For a continuous function \( \varphi \) on \( X \), denote the \( \varphi \)-irregular set by

\[
I_{\varphi}(f) := \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \text{ diverges} \right\},
\]

Denote:

\[
L_{\varphi} := \left[ \inf_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu, \sup_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu \right], \quad \text{Int}(L_{\varphi}) := \left[ \inf_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu, \sup_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu \right].
\]

For any \( a \in L_{\varphi} \), denote the level set by

\[
R_{\varphi}(a) := \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) = a \right\}.
\]

The following two theorems are to say that there are no effect on strongly distributional chaos in the twelve layers of Theorem B when they intersect irregular set and level sets.

**Theorem C.** Suppose \((X, f)\) is a mixing expanding map or a transitive Anosov diffeomorphism on a compact manifold. Let \( \varphi \) be a continuous function on \( X \). If \( I_{\varphi}(f) \neq \emptyset \), then the set \( \{x \in X \mid x \text{ satisfies case (i)}\} \cap NR(f) \cap I_{\varphi}(f) \) is strongly distributional chaotic for any \( i \in \{1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'\} \).

**Theorem D.** Suppose \((X, f)\) is a mixing expanding map or a transitive Anosov diffeomorphism on a compact manifold. Let \( \varphi \) be a continuous function on \( X \). If \( \text{Int}(L_{\varphi}) \neq \emptyset \), then for any \( a \in \text{Int}(L_{\varphi}) \), the set \( \{x \in X \mid x \text{ satisfies case (i)}\} \cap NR(f) \cap R_{\varphi}(a) \) is strongly distributional chaotic for any \( i \in \{1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'\} \).

In this paper, we mainly consider continuous maps on a compact metric space. Everything can also be formulated for homeomorphisms on a compact metric space. The difference is that we must consider expansivity in place of positive expansivity, and shadowing property of homeomorphisms is defined for bilateral pseudo-orbit rather than unilateral pseudo-orbit. We leave the details to the readers.

**Organization of the paper.** Section 2 is a review of definitions to make precise the statements of the theorems and some basic facts useful for the proof of main results. In Section 3 we give some key technique lemmas for which Lemma A and B are the crucial lemmas to prove the main theorems and in Section 4 we extend the proof of main theorems. In Section 5, we apply the results in the previous sections to more systems, including mixing subshifts of finite type, \( \beta \)-shifts, homoclinic classes and hyperbolic ergodic measures.
2. Preliminaries

2.1. Internally Chain Mixing and Shadowing Property. Given a dynamical system \((X, f)\).
For \(l \in \mathbb{N}\), a sequence \(\mathcal{C}^l = \langle x_1, \ldots, x_l \rangle\) is called a chain with length \(l\). Define
\[
\delta_{\mathcal{C}^l} := \frac{1}{l} \sum_{i=1}^{l} \delta_{x_i}.
\]
For \(A \subseteq X\), we say a chain \(\mathcal{C}^l = \langle x_1, \ldots, x_l \rangle\) is in \(A\) if \(\{x_i\}_{i=1}^{l} \subseteq A\). Furthermore, if \(d(f(x_i), x_{i+1}) < \varepsilon, 1 \leq i \leq l - 1\), we call \(\mathcal{C}^l\) an \(\varepsilon\)-chain with length \(l\). We say an \(\varepsilon\)-chain \(\mathcal{C}_{ab}^l = \langle x_1, \ldots, x_l \rangle\) connects \(a\) and \(b\) if \(x_1 = a\) and \(d(f(x_i), b) < \varepsilon\). For two \(\varepsilon\)-chain \(\mathcal{C}_{ab}^{l_1} = \langle x_1, \ldots, x_{l_1} \rangle\) and \(\mathcal{C}_{bc}^{l_2} = \langle y_1, \ldots, y_{l_2} \rangle\), define
\[
\mathcal{C}_{ab}^{l_1} \mathcal{C}_{bc}^{l_2} = \langle x_1, \ldots, x_{l_1}, y_1, \ldots, y_{l_2} \rangle.
\]
Obviously, \(\mathcal{C}_{ab}^{l_1} \mathcal{C}_{bc}^{l_2}\) is \(\varepsilon\)-chain with length \(l_1 + l_2\) connecting \(a\) and \(c\). For \(x \in X\), we define
\[
\text{orb}(x, n) := \langle x, f^n x, \ldots, f^{n-1}x \rangle.
\]

2.1.1. Internally Chain Mixing.

Definition 2.1. Let \(A \subseteq X\) be a nonempty closed invariant set. We call \(A\) internally chain transitive if for any \(a, b \in A\) and any \(\varepsilon > 0\), there is an \(n \in \mathbb{N}\) and an \(\varepsilon\)-chain \(\mathcal{C}_{ab}^n\) in \(A\) connecting \(a\) and \(b\). We call \(A\) internally chain mixing if for any \(a, b \in A\) and any \(\varepsilon > 0\), there is an \(N \in \mathbb{N}\), such that for any \(n \geq N\), there is an \(\varepsilon\)-chain \(\mathcal{C}_{ab}^n\) in \(A\) connecting \(a\) and \(b\). We denote the collection of internally chain transitive(mixing) sets by \(ICT(ICC)\). Obviously \(ICC \subseteq ICT\).

Lemma 2.2. [25] For any \(x \in X\), \(\omega_f(x) \in ICT\).

Lemma 2.3. Suppose \(\Lambda \in ICT\), \(A \in ICT\) and \(A \subseteq \Lambda\). Then \(\Lambda \in ICT\).

Proof. Fix a point \(x \in A\). Then for any \(\varepsilon > 0\), there is an \(L \in \mathbb{N}\) such that for any \(l \geq L\), there is an \(\varepsilon\)-chain \(\mathcal{C}_{xx}^l\). Note that \(\Lambda \in ICT\). Then for any \(a, b \in \Lambda\), there are two \(\varepsilon\)-chain \(\mathcal{C}_{ax}^{l_1}\) and \(\mathcal{C}_{xb}^{l_2}\) for some integers \(l_1\) and \(l_2\). Then for any \(l \geq L + l_1 + l_2\), there is an \(\varepsilon\)-chain \(\mathcal{C}_{ab}^l = \mathcal{C}_{ax}^{l_1} \mathcal{C}_{xx}^L \mathcal{C}_{xb}^{l_2}\) connecting \(a\) and \(b\) where \(l = l_1 + l_2\).

Corollary 2.4. Suppose \(\Lambda \in ICT\) and \(\Lambda\) contains a fixed point. Then \(\Lambda \in ICT\).

2.1.2. Shadowing Property and Exponential Shadowing Property.

Definition 2.5. An infinite sequence \((x_n)_{n=1}^{+\infty}\) of points in \(X\) is a \(\delta\)-pseudo-orbit for a dynamical system \((X, f)\) if \(d(x_{n+1}, f(x_n)) < \delta\) for each \(n \geq 1\). We say that a dynamical system \((X, f)\) has the shadowing property if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that any \(\delta\)-pseudo-orbit \((x_n)_{n=1}^{+\infty}\) can be \(\varepsilon\)-shadowed by a point \(y \in X\), that is, \(d(f^n(y), x_n) < \varepsilon\) for all \(n \geq 1\).

Given \(x \in X\) and \(i \geq 1\), let
\[
\{x, i\} := \{f^j(x) : j = 0, 1, \ldots, i - 1\}.
\]
For a sequence of points \((x_n)_{n=1}^{+\infty}\) in \(X\) and a sequence of positive integers \((i_n)_{n=1}^{+\infty}\) we call \(\{x_n, i_n\}_{n=1}^{+\infty}\) a \(\delta\)-pseudo-orbit, if \(d(f^i_n(x_n), x_{n+1}) < \delta\) for all \(n \geq 1\). Given \(\varepsilon > 0\) and \(\lambda > 0\), we call a point \(x \in X\) an (exponentially) \((\varepsilon, \lambda)\)-shadowing point for a pseudo-orbit \(\{x_n, i_n\}_{n=1}^{+\infty}\), if
\[
d(f^{i_n+j}(x), f^j(x_n)) < \varepsilon \cdot e^{-\min\{j, i_n-1-j\}\lambda}
\]
for all \(0 \leq j \leq i_n - 1\) and \(n \geq 1\), where \(c_i\) is defined as
\[
c_n = \begin{cases} 0, & \text{for } n = 1 \\ \sum_{m=1}^{n-1} i_m, & \text{for } n > 1. \end{cases}
\]

Definition 2.6. Let \(\lambda > 0\). We say that a dynamical system \((X, f)\) has the exponential shadowing property with exponent \(\lambda\) if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that any \(\delta\)-pseudo-orbit \(\{x_n, i_n\}_{n=1}^{+\infty}\) can be \((\varepsilon, \lambda)\)-shadowed by a point \(x \in X\).
**Remark 2.7.** From the definitions, if a dynamical system \((X, f)\) has exponential shadowing property, then it has shadowing property.

**Definition 2.8.** A sequence \(\langle x_i \rangle_{i=1}^{+\infty}\) is called a limit-pseudo-orbit if
\[
\lim_{i \to \infty} d(f(x_i), x_{i+1}) = 0.
\]
Moreover, \(\langle x_i \rangle_{i=1}^{+\infty}\) are limit-shadowed by \(y \in X\) if
\[
\lim_{i \to \infty} d(f^{i-1}y, x_i) = 0.
\]
Then we say \((X, f)\) has the limit-shadowing property if any limit-pseudo-orbit can be limit-shadowed.

**Definition 2.9.** For any \(\delta > 0\), a sequence \(\langle x_i \rangle_{i=1}^{+\infty}\) is called a \(\delta\)-limit-pseudo-orbit if \(\langle x_n \rangle_{n=1}^{+\infty}\) is both a \(\delta\)-pseudo-orbit and a limit-pseudo-orbit. Furthermore, \(\langle x_n \rangle_{n=1}^{+\infty}\) is \(\epsilon\)-limit-shadowed by some \(y \in X\) if \(\langle x_n \rangle_{n=1}^{+\infty}\) is both \(\epsilon\)-shadowed and limit-shadowed by \(y\). Finally, we say that \((X, f)\) has the \(\delta\)-limit-shadowing property if for any \(\epsilon > 0\), there exists \(\delta > 0\) such that any \(\delta\)-limit-pseudo-orbit can be \(\epsilon\)-limit-shadowed.

**Remark 2.10.** When \(f : X \to X\) is a homeomorphism, on a compact metric space, various shadowing properties are defined for \(\langle x_n \rangle_{n=-\infty}^{+\infty}\) or \(\{x_n, i_n\}_{n=-\infty}^{+\infty}\). For example, see [34] for the definitions of shadowing property, limit-shadowing property, see [65] for the definition of exponential shadowing property.

We say that \((X, f)\) is positively expansive if there exists a constant \(e > 0\) such that for any \(x \neq y \in X\), \(d(f^i(x), f^i(y)) > e\) for some integer \(i \geq 0\). We call \(e\) the expansive constant.

**Lemma 2.11.** [52] If \((X, f)\) is positively expansive and has the shadowing property, then \((X, f)\) has the \(s\)-limit shadowing property.

**Lemma 2.12.** If \((X, f)\) is positively expansive and has exponential shadowing property with exponent \(\lambda\), then for any \(\epsilon > 0\), there exists \(\delta > 0\) such that for any \(\delta\)-limit-pseudo-orbit \(\{x_n, i_n\}_{n=1}^{+\infty}\), there exists \(y \in X\) such that \(\{x_n, i_n\}_{n=1}^{+\infty}\) is both \((\epsilon, \lambda)\)-shadowed and limit-shadowed by \(y\).

**Proof.** Suppose that the expansive constant is \(e\). WLOG, we assume that \(0 < \epsilon < \frac{\lambda}{2}\). By Remark 2.7 and Lemma 2.11, \((X, f)\) has the \(s\)-limit shadowing property, there exists \(\delta_1\), such that for any \(\delta_1\)-limit-pseudo-orbit \(\{x_n, i_n\}_{n=1}^{+\infty}\), there exists \(y_1\) such that \(\{x_n, i_n\}_{n=1}^{+\infty}\) is both \(\epsilon\)-shadowed and limit-shadowed by \(y_1\). Since \((X, f)\) has exponential shadowing property with exponent \(\lambda\), there exists \(\delta_2 > 0\) such that for any \(\delta_2\)-limit-pseudo-orbit \(\{x_n, i_n\}_{n=1}^{+\infty}\), there exists \(y_2 \in X\) such that \(\{x_n, i_n\}_{n=1}^{+\infty}\) is \((\epsilon, \lambda)\)-shadowed by \(y_2\). Denote \(\delta = \min\{\delta_1, \delta_2\}\), by expansiveness, we finish the proof. \(\square\)

**2.1.3. Specification Property.**

**Definition 2.13.** Suppose that \(f\) is a continuous map on a nondegenerate (i.e., with at least two points) compact metric space \((X, d)\). We say \(X\) has strong specification property, if for any \(\epsilon > 0\), there is a positive integer \(K_\epsilon\) such that for any integer \(s \geq 2\), any set \(\{y_1, y_2, \ldots, y_s\}\) of \(s\) points of \(X\), and any sequence
\[
0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s
\]

of \(2s\) integers with
\[
a_{m+1} - b_m \geq K_\epsilon
\]
for \(m = 1, 2, \cdots, s - 1\), there is a point \(x \in X\) such that the following two conditions hold:

(a): \(d(f^n(x), f^n(y_m)) < \epsilon\) for all positive integers \(m \leq s\) and all integers \(i\) with \(a_m \leq i \leq b_m\);

(b): \(f^n(x) = x\), where \(n = b_s + K_\epsilon\).

If the periodicity condition (b) is omitted, we say that \(X\) has specification property. Obviously, if \(X\) has specification property, \(X \in ICM\).
Definition. If \((X, f)\) is positively expansive, mixing and has the shadowing property, then it has strong specification property.

Lemma 2.15. [13, Proposition 21.12] Suppose that \((X, f)\) is a dynamical system satisfying strong specification property, then the set of invariant measures with full support is residual in \(\mathcal{M}_f(X)\).

Lemma 2.16. [13, Proposition 21.9] Suppose that \((X, f)\) is a dynamical system satisfying strong specification property, then \(\mathcal{M}_f^s(X)\) is residual in \(\mathcal{M}_f(X)\).

2.2. Anosov diffeomorphisms and expanding maps. Let \(M\) be a compact smooth Riemann manifold without boundary. \(f : M \rightarrow M\) is a diffeomorphism. A \(f\)-invariant set \(\Lambda \subset M\) is said to be uniformly hyperbolic, if for any \(x \in \Lambda\) there is a splitting of the tangent space \(T_xM = E^s(x) \oplus E^u(x)\) which is preserved by the differential \(Df\) of \(f\):

\[
Df(E^s(x)) = E^s(f(x)), \quad Df(E^u(x)) = E^u(f(x)),
\]

and there are constants \(C > 0\) and \(0 < \lambda < 1\) such that for all \(n \geq 0\)
\[
|Df^n(v)| \leq C\lambda^n|v|, \quad \forall x \in \Lambda, \quad v \in E^s(x),
\]
\[
|Df^{-n}(v)| \leq C\lambda^n|v|, \quad \forall x \in \Lambda, \quad v \in E^u(x).
\]

If \(M\) is a uniformly hyperbolic set, then \(f\) is called an Anosov diffeomorphism. A hyperbolic set \(\Lambda\) is said to be locally maximal for \(f\) if there exists a neighborhood \(U\) of \(\Lambda\) in \(M\) such that \(\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(U)\).

When \(f : X \rightarrow X\) is a homeomorphism on a compact metric space, we say that \((X, f)\) is expansive if there exists a constant \(e > 0\) such that for any \(x, y \in X\), \(d(f^i(x), f^i(y)) > e\) for some integer \(i\).

Theorem 2.17. Every transitive Anosov diffeomorphism on a compact connected manifold or a system restricted on a mixing locally maximal hyperbolic set is expansive, mixing and has the exponential shadowing property.

Proof. A system restricted on a locally maximal hyperbolic set has exponential shadowing property by Proposition [65, Proposition 2.7] and is expansive by [30, Corollary 6.4.10]. By spectral decomposition, every transitive Anosov diffeomorphism on a compact connected manifold is mixing [30, Corollary 18.3.5]. Finally every Anosov diffeomorphism is locally maximal, so it has the exponential shadowing property. \(\square\)

A \(C^1\) map \(f : M \rightarrow M\) is said to be expanding if there are constants \(C > 0\) and \(0 < \lambda < 1\) such that for all \(n \geq 0\)
\[
|Df^n(v)| \geq C\lambda^{-n}|v|, \quad \forall x \in M, \quad v \in T_xM.
\]

Theorem 2.18. Every expanding map on a compact connected manifold is positively expansive and has the exponential shadowing property.

Proof. Every expanding map is positively expansive by [1, Theorem 1.2.1] and has the exponential shadowing property by [27, Proposition 6.1]. \(\square\)

2.3. Distributional Chaos and Strongly Distributional Chaos. The notion of chaos, as well as Li-Yorke pair and scrambled set, was invented in 1975 by Li and Yorke in the context of continuous transformations of the interval [35]. Since then various extensions definition of chaos have been developed. One style bases on topological perspective, which specifies how the scrambled set is placed in the space, such as dense chaos [47] and generic chaos [47, 59]. Another one derives from statistical perspective by adding some statistical restriction to the definition of Li-Yorke pair, which results in distributional chaos.

Definition 2.19. A pair \(x, y \in X\) is DC1-scrambled if the following two conditions hold:

\[
\forall t > 0, \quad \limsup_{n \to \infty} \frac{1}{n} \left| \{i \in [0, n - 1] : d(f^i(x), f^i(y)) < t \} \right| = 1.
\]
\[
\exists t_0 > 0, \quad \liminf_{n \to \infty} \frac{1}{n} \left| \{i \in [0, n - 1] : d(f^i(x), f^i(y)) < t_0 \} \right| = 0.
\]
In other words, the orbits of $x$ and $y$ are arbitrarily close with upper density one, but for some distance, with lower density zero. A set $S$ is called a DC1-scrambled set if any pair of its distinct points is DC1-scrambled. A map $f$ is called distributional chaotic of type 1 (DC1 chaotic for brevity), if there is an uncountable DC1-scrambled set $S \subseteq X$. In this paper, we focus on DC1 chaotic. Readers can refer to [17, 55, 58] for the definition of DC2 and DC3 if necessary.

Now we recall from [27] a kind of chaos, strongly distributional chaos, which is stronger than usual distributional chaos and Li-Yorke chaos. For any positive integer $n$, points $x, y \in M$ and $t \in \mathbb{R}$ let
\[
\Phi(n)_{xy}(t, f) = \frac{1}{n} \{0 \leq i \leq n - 1 : d(f^i x, f^i y) < t\},
\]
where $|A|$ denotes the cardinality of the set $A$. Let us denote by $\Phi_{xy}$ the following function:
\[
\Phi_{xy}(t, f) = \liminf_{n \to \infty} \Phi(n)_{xy}(t, f).
\]
Define $\mathcal{A} = \{\alpha(\cdot) : \alpha$ is a nondecreasing map form $\mathbb{N}$ to $[0, +\infty)$, $\lim_{n \to \infty} \alpha(n) = +\infty$ and $\lim_{n \to \infty} \frac{\alpha(n)}{n} = 0\}$. For any positive integer $n$, points $x, y \in M$, $t \in \mathbb{R}$ and $\alpha \in \mathcal{A}$, let
\[
\Phi^\alpha_{xy}(t, f, \alpha) = \frac{1}{n} \{1 \leq i \leq n : \sum_{j=0}^{i-1} d(f^j x, f^j y) < \alpha(i)t\}.
\]
Let us denote by $\Phi^\alpha_{xy}(t, f, \alpha)$ the following functions:
\[
\Phi^\alpha_{xy}(t, f, \alpha) = \limsup_{n \to \infty} \Phi(n)_{xy}(t, f, \alpha).
\]

**Definition 2.20.** A pair $x, y \in X$ is $\alpha$-DC1-scrambled if the following two conditions hold:
\[
\Phi_{xy}(t_0, f) = 0 \text{ for some } t_0 > 0 \text{ and } \Phi^\alpha_{xy}(t, f, \alpha) = 1 \text{ for all } t > 0.
\]

A set $S$ is called a $\alpha$-DC1-scrambled set if any pair of distinct points in $S$ is $\alpha$-DC1-scrambled. A subset $Y \subseteq M$ is said to be strongly distributional chaotic if it has an uncountable $\alpha$-DC1-scrambled set for any $\alpha \in \mathcal{A}$.

**Remark 2.21.** Strongly distributional chaos is stronger than distributional chaos of type 1 (see [27, Proposition 2.5]).

### 2.4. Basic facts for statistical $\omega$-sets.

For any $\mu \in \mathcal{M}_f(X)$, we denote $S_\mu = \text{supp}(\mu) = \{x \in X | \mu(U) > 0 \text{ for any neighborhood } U \text{ of } x\}$ the support of $\mu$. Given $x \in X$, denote $V_f(x) \subseteq \mathcal{M}_f(X)$ the set of all accumulation points of the empirical measures
\[
\mathcal{E}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)},
\]
where $\delta_x$ is the Dirac measure concentrate on $x$. As is known, $V_f(x)$ is a non-empty compact connected subset of $\mathcal{M}_f(X)$ [13]. For any two positive integers $a_k < b_k$, denote $[a_k, b_k] = \{a_k, a_k + 1, \cdots, b_k\}$ and $[a_k, b_k] = [a_k, b_k - 1], (a_k, b_k) = [a_k + 1, b_k - 1], (a_k, b_k) = [a_k + 1, b_k]$. A point $x$ is called quasi-generic for some measure $\mu$, if there is a sequence of positive integer intervals $I_k = (a_k, b_k)$ with $b_k - a_k \to \infty$ such that
\[
\lim_{k \to \infty} \frac{1}{b_k - a_k} \sum_{j=a_k}^{b_k-1} \delta_{f^j(x)} = \mu
\]
in weak* topology. Let $V^*_f(x) = \{\mu \in M(f, X) : x \text{ is quasi-generic for } \mu\}$. This concept is from [20] and from there it is known $V^*_f(x)$ is always nonempty, compact and connected. Note that $V_f(x) \subseteq V^*_f(x)$.

**Proposition 2.22.** [16, Theorem 1.4] Suppose $(X, f)$ is a topological dynamical system.
(1): For any \( x \in X \), \( \omega_d(x) = \bigcap_{\mu \in V_f(x)} S_{\mu} \).

(2): For any \( x \in X \), \( \omega_d^c(x) = \bigcup_{\mu \in V_f(x)} S_{\mu} \neq \emptyset \).

(3): For any \( x \in X \), \( \omega_{B_\epsilon}(x) = \bigcap_{\mu \in V_f(x)} S_{\mu} = \bigcap_{\mu \in \mathcal{M}_f(\omega_f(x))} S_{\mu} = \bigcap_{\mu \in \mathcal{M}_f(\omega_f(x))} S_{\mu} \). If \( \omega_{B_\epsilon}(x) \neq \emptyset \), then \( \omega_{B_\epsilon}(x) \) is minimal.

(4): For any \( x \in X \), \( \omega_{B}(x) = \bigcup_{\mu \in V_f(x)} S_{\mu} = \bigcup_{\mu \in \mathcal{M}_f(\omega_f(x))} S_{\mu} = \bigcup_{\mu \in \mathcal{M}_f(\omega_f(x))} S_{\mu} \neq \emptyset \);

(5): For any invariant measure \( \mu \) and \( \mu \) a.e. \( x \in X \),

\[
\omega_d(x) = \omega_d^c(x) = \omega_B(x) = \omega_f(x) = S_{\mu}.
\]

3. Technique Lemmas

3.1. Distributional Chaos in Saturated Sets. Consider a dynamical system \((X, f)\). Given \( x \in X \), it is known that \( V_f(x) \) is a non-empty compact connected subset of \( \mathcal{M}_f(X) \) [13]. So for any non-empty compact connected subset \( K \) of \( \mathcal{M}_f(X) \), it is logical to define the following set

\[
G_K := \{ x \in X \mid V_f(X) = K \}.
\]

\( G_K \) is known as the saturated set of \( K \). Particularly, if \( K = \{ \mu \} \) for some ergodic measure \( \mu \), then \( G_\mu \) is just the generic points of \( \mu \). The existence of saturated sets are studied by Sigmund in [56]. The Bowen entropy of saturated sets are studied by Pfi ster and Sullivan in [46]. Here we consider distributional chaos in saturated sets. Let \( \Lambda \subseteq X \) be a closed invariant subset and \( K \) is a non-empty compact connected subset of \( \mathcal{M}_f(\Lambda) \). Define

\[
G^\Lambda_K := G_K \cap \{ x \in X \mid \omega_f(x) = \Lambda \}.
\]

We say a pair \( p, q \in X \) is distal if

\[
\liminf_{i \to \infty} d(f^i(p), f^i(q)) > 0.
\]

Obviously, \( \inf\{d(f^i(p), f^i(q)) \mid i \in \mathbb{N}\} > 0 \) if \( p, q \) is distal. We say a subset \( M \subseteq X \) has distal pair if there are distinct \( p, q \in M \) such that \( p, q \) is distal.

Lemma A. Suppose that \((X, f)\) is positively expansive and transitive, \( \Lambda \in ICM \). Let \( K \subseteq \mathcal{M}_f(\Lambda) \) be non-empty compact connected. If there is a \( \mu \in K \) such that \( \mu = \theta \mu_1 + (1 - \theta) \mu_2 \) (\( \mu_1 = \mu_2 \) are legal), where \( \theta \in [0, 1] \), and \( G_{\mu_1}, G_{\mu_2} \) both have distal pair. Then

(a): if \((X, f)\) has the shadowing property, then for any non-empty open set \( U \subseteq X \), there is an uncountable DC1-scrambled set \( S \subseteq G^\Lambda_K \cap U \). Particularly, \( S \subseteq G^\Lambda_K \cap NR \) when \( U \subseteq X \setminus \Lambda \); (b): if further \((X, f)\) has exponential shadowing property, then for any \( \alpha \in \Lambda \) and for any non-empty open set \( U \subseteq X \) there is an uncountable \( \alpha \)-DC1-scrambled set \( S^\alpha \subseteq G^\Lambda_K \cap U \).

3.1.1. Some Lemmas. We write \( \mathbb{N} = \{0, 1, 2, \cdots\} \) and \( \mathbb{N}^+ = \{1, 2, \cdots\} \). The cardinality of a finite set \( \Lambda \) is denoted by \( |\Lambda| \). We set

\[
\langle f, \mu \rangle := \int_X f d\mu.
\]

There exists a countable and separating set of continuous function \( \{f_1, f_2, \cdots\} \) with \( 0 \leq f_k(x) \leq 1 \), and such that

\[
d(\mu, \nu) := \| \mu - \nu \| := \sum_{k \geq 1} 2^{-k} | \langle f_k, \mu - \nu \rangle |
\]

defines a metric for the weak*-topology on \( \mathcal{M}_f(x) \). We refer to [46] and use the metric on \( X \) as following defined by Pfister and Sullivan.

\[
d(x, y) := d(\delta_x, \delta_y),
\]

which is equivalent to the original metric on \( X \). Readers will find the benefits of using this metric in our proof later.
Lemma 3.1. For any $\varepsilon > 0$, $\delta > 0$ and two sequences $\{x_i\}_{i=0}^{n-1}$, $\{y_i\}_{i=0}^{n-1}$ of $X$ such that $d(x_i, y_i) < \varepsilon$ holds for any $i \in [0, n-1]$, then for any $J \subseteq \{0, 1, \ldots, n-1\}$, $\frac{n-|J|}{n} < \delta$, one has:

(a): $d\left(\frac{1}{n} \sum_{i=0}^{n-1} \delta_{x_i}, \frac{1}{|J|} \sum_{i \in J} \delta_{y_i}\right) < \varepsilon$.

(b): $d\left(\frac{1}{n} \sum_{i=0}^{n-1} \delta_{x_i}, \frac{1}{|J|} \sum_{i \in J} \delta_{y_i}\right) < \varepsilon + 2\delta$.

Lemma 3.1 is easy to be verified and shows us that if any two orbit of $x$ and $y$ in finite steps are close in the most of time, then the two empirical measures induced by $x, y$ are also close.

Given a metric space $(X, d)$. We denote the Hausdorff distance between two nonempty subsets of $X$, $A$ and $B$, by

$$d_H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(y, x) \right\}.$$ 

A point $x \in X$ is periodic if there is a $n \in \mathbb{N}$ such that $f^n(x) = x$. We denote $\text{Per}$ the set of periodic points. We denote $\text{Per}_n := \{x \in X \mid f^n(x) = x\}$.

Lemma 3.2. Suppose that $(X, f)$ is positively expansive and has the shadowing property, $\Lambda \in \text{ICM}$. Then for any $\varepsilon > 0$, any $x \in \Lambda$, any $\mu \in \mathcal{M}_f(\Lambda)$ and its neighborhood $F_\mu$, there exists an $M \in \mathbb{N}$ such that for any $n > M$, there exists $p \in B(x, \varepsilon)$ satisfying

(a): $p \in \text{Per}_n$, $\mathcal{E}_n(p) \in F_\mu$;

(b): $d_H(\text{orb}(p), \Lambda) < \varepsilon$.

Proof. Since $F_\mu$ is a neighborhood of $\mu$, there is an $a > 0$ such that $B(\mu, a) \subseteq F_\mu$. By the ergodic decomposition Theorem, there exists a finite convex combination of ergodic measure $\sum_{i=1}^m c_i \nu_i \in B(\mu, a/5)$. Moreover, by the denseness of rational numbers, we can choose each $c_i = \frac{b_i}{a}$ with $b_i \in \mathbb{N}$ and $\sum_{i=1}^m b_i = a$. It is known that $\nu_i(G_{\nu_i}) = 1$. So we can choose $\{p_i\}_{i=1}^m \subseteq \Lambda$ with $p_i \in G_{\nu_i}$, $i = 1, 2, \ldots, m$. Then there is a common $N_c \in \mathbb{N}$ such that for any $n > N_c$ and any $i \in \{1, 2, \ldots, m\}$, $\mathcal{E}_n(p_i) \in B(\nu_i, a/5)$.

Let $\varepsilon^* = \min\{\frac{a}{2}, \frac{b}{a}, \frac{a}{5}\}$, where $e$ is the expansive constant. By shadowing property, there is a $\delta \in (0, \varepsilon^*)$ such that any $\delta$-pseudo-orbit can be $\varepsilon^*$-shadowed by some point in $X$. Since $\Lambda$ is compact we can choose finite ball open balls $\{B(x_i, \delta)\}_{i=1}^n$ which covers $\Lambda$ with $\{x_i\}_{i=1}^n \subseteq \Lambda$. Note that $\Lambda \in \text{ICM}$, then there exists an $L \in \mathbb{N}$ such that for any $l \geq L$, there is a $\delta$-chain $\mathcal{E}_{yz}$ connecting $y$ and $z$ for any $y, z \in \{x_i\}_{i=1}^n \cup \{x\} \cup \{p_i\}_{i=1}^m$. For any $p \in \Lambda$, we define

$$x(p, n) := \min\{x \in \mathbb{R} | f^n(x) \in B(x, \delta)\}.$$

(Taking the minimum index here has no explicit meanings. We just want to fix a point in $\{x_i\}_{i=1}^n$.)

Now, let $k \in \mathbb{N}$ large enough such that

$$kb_i \geq N_c, \quad 1 \leq i \leq m; \quad \text{and}$$

$$\frac{(s + m + 1)L + b}{(s + m + 1)L + kb + b} < \frac{a}{5}.$$ 

Let $M = (s + m + 1)L + b$; then for any $n > M$, $n = M + tb + c$, where $c < b, t \in \mathbb{N}$. Here we construct $\mathcal{C} = \mathcal{C}_x \mathcal{C}_y^{n} \cdots$, where

$$\mathcal{C}_x = \mathcal{C}_{x_1}^{L^*} \mathcal{C}_{x_2}^{L^*} \cdots \mathcal{C}_{x_{s-1}}^{L^*} \mathcal{C}_{x_s}^{L^*} \mathcal{E}_{x_1}^{L} \cdots \mathcal{E}_{x_{s-1}}^{L} \mathcal{E}_{x_s}^{L} \mathcal{E}_{x_1}^{L} \cdots \mathcal{E}_{x_{s-1}}^{L} \mathcal{E}_{x_s}^{L} \mathcal{E}_{p_1}^{L} \cdots \mathcal{E}_{p_m}^{L} \mathcal{E}_{y}^{L} \mathcal{E}_{z}^{L} \mathcal{C}_y.$$ 

It is easy to check $\mathcal{C}$ is a $\delta$-pseudo-orbit. By shadowing property, $\mathcal{C}$ can be $\varepsilon^*$-shadowed by a point $p \in X$. Note that $\mathcal{C}$ is $n$ periodic. So $d(f^i(p), f^{i+n}(p)) < \varepsilon^* + \varepsilon^* = 2\varepsilon^* < \varepsilon$ for any $i \in \mathbb{N}$, which
implies $f^n(p) = p$ by expansiveness. By (3.1), (3.2) and Lemma 3.1, we have
\[
\begin{align*}
d(\mathcal{E}_n(p), \sum_{i=1}^m c_i \nu_i) & \leq d(\mathcal{E}_n(p), \delta \varepsilon \sum_{i=1}^m c_i \nu_i) \\
& \leq \varepsilon^* + d(\mathcal{E}_n(p), \sum_{i=1}^m c_i \nu_i) \\
& \leq \varepsilon^* + d(\mathcal{E}_n(p), \sum_{i=1}^m \frac{b_i}{b} \delta \text{orb}(p_i, (k+\varepsilon)b_i)) + d(\sum_{i=1}^m \frac{b_i}{b} \delta \text{orb}(p_i, (k+\varepsilon)b_i), \sum_{i=1}^m c_i \nu_i) \\
& < \varepsilon^* + 2 \cdot \frac{a}{5} + \frac{a}{5} \\
& < \frac{4a}{5}.
\end{align*}
\]
Thus $\mathcal{E}_n(p) \subseteq B(\mu, a) \subseteq F_\mu$.

Note that $\mathcal{C}_{xx}$ contains $\{x_i \}_{i=1}^4$ and $\{B(x_i, \delta)\}_{i=1}^4$ covers $\Lambda$. So $\{B(f^i(p), \delta + \varepsilon^*)\}_{i=0}^{n-1}$ covers $\Lambda$. On the other side, $\{f^i(p)\}_{i=0}^{n-1} \subseteq B(\Lambda, \varepsilon^*)$ and $\mu \in \mathcal{C}_n$. So $d_H(\mathcal{C}_n, \Lambda) < \varepsilon^* + \delta < 2\varepsilon^* \leq \varepsilon$. □

Lemma 3.3. Suppose that $(X, f)$ is positively expanding and has the shadowing property, $\Lambda \in ICM$. Suppose there are $\mu_1, \mu_2 \in \mathcal{M}_f(\Lambda)$ such that $\mu_1, \mu_2$ has distal pair $(g_1, e_1), (g_2, e_2)$ respectively. Let
\[
\zeta = \min\{\inf\{d(f^i(g_1), f^i(e_1))| i \in \mathbb{N}\}, \inf\{d(f^i(g_2), f^i(e_2))| i \in \mathbb{N}\}\}.
\]
Then for any $\tau > 0$, any $\varepsilon \in (0, \zeta/2)$, any $\theta \in [0, 1]$ and any $x \in \Lambda$, there exists an $M \in \mathbb{N}$ such that for any $n > M$, there exists $p^1, p^2 \in B(x, \varepsilon)$ satisfying
(a): $p^1 \in \mathcal{C}_n, \mathcal{E}_n(p^1) = B(\theta \mu_1 + (1 - \theta)\mu_2, \varepsilon)$, $i = 1, 2$;
(b): $d_H(\text{orb}(p^1), \Lambda) < \varepsilon$, $i = 1, 2$;
(c): $\frac{1}{n} \sum_{i=1}^n d(f^{i}(p^1), f^{i}(p^2)) < \varepsilon - \zeta < \tau$.

Proof. By the denseness of rational numbers, we can assume $r_1, r_2 \in \mathbb{N}, r = r_1 + r_2$ and $\theta = \frac{r_1}{r_2}$. Since $g_1, e_1 \in G_{\mu_1}, i = 1, 2$, there is a common $N_\varepsilon \in \mathbb{N}$ such that for any $n \geq N_\varepsilon$, $\mathcal{E}_n(g_1) \in B(\mu_1, \varepsilon/4)$ and $\mathcal{E}_n(e_1) \in B(\mu_2, \varepsilon/4), i = 1, 2$. Let $\varepsilon^* = \min\{\frac{a}{2}, \frac{a}{2}\}$. By shadowing property, there is a $\delta \in (0, \varepsilon^*)$ such that any $\delta$-pseudo-orbit can be $\varepsilon^*$-shadowed by some point in $X$. Since $\Lambda$ is compact we can choose finite ball open balls $\{B(x_i, \delta)\}_{i=1}^4$ which covers $\Lambda$ with $\{x_i\}_{i=1}^4 \subseteq \Lambda$. Note that $\Lambda \in ICM$, then there exists an $L \in \mathbb{N}$ such that for any $l \geq L$, there is a $\delta$-chain $\mathcal{C}_y$ connecting $y$ and $z$ for any $y, z \in \{x_i\}_{i=1}^4 \cup \{x\} \cup \{g_1\}_{i=1}^4 \cup \{e_1\}_{i=1}^4$.

Now, let $k \in \mathbb{N}$ large enough such that
\[
kr_i \geq L, i = 1, 2; \quad \text{and}
\]
\[
\frac{(s + 3)L + r}{(s + 3)L + kr + r} < \min\{\tau, \varepsilon^*\}.
\]
Let $M = (s + 3)L + kr$, then for any $n > M, n = M + tr + c, c < r, t \in \mathbb{N}$. Here we construct $\mathcal{C}_1 = \mathcal{C}_{xx, 1}^n \mathcal{C}_{xx, 2} \cdots$ and $\mathcal{C}_2 = \mathcal{C}_{xx, 2} \mathcal{C}_{xx, 2} \cdots$, where
\[
\begin{align*}
\mathcal{C}_{xx, 1}^n &= \mathcal{C}_{xx, 1}^{L_1} \mathcal{C}_{xx, 2}^{L_2} \cdots \mathcal{C}_{xx, 1}^{L_k} \mathcal{C}_{xx, 2}^{L_k+1} \mathcal{C}_{xx, 1}^{L_k+1} \mathcal{C}_{xx, 2}^{L_k+1} \mathcal{C}_{xx, 1}^{L_k+1} \mathcal{C}_{xx, 2}^{L_k+1} \cdots,
\mathcal{C}_{xx, 2}^n &= \mathcal{C}_{xx, 2}^{L_1} \mathcal{C}_{xx, 1}^{L_2} \cdots \mathcal{C}_{xx, 2}^{L_k} \mathcal{C}_{xx, 1}^{L_k+1} \mathcal{C}_{xx, 2}^{L_k+1} \mathcal{C}_{xx, 1}^{L_k+1} \mathcal{C}_{xx, 2}^{L_k+1} \cdots,
\end{align*}
\]
It is easy to check $\mathcal{C}_1, \mathcal{C}_2$ are both $\delta$-pseudo-orbit. By shadowing property, $\mathcal{C}_i$ can be $\varepsilon^*$-shadowed by a point $p^i \in X, i = 1, 2$. With the similar analysis in the proof of Lemma 3.2, item (a) and (b) are satisfied. Observing $\mathcal{C}_{xx, 1}^n$ and $\mathcal{C}_{xx, 2}^n$, one can find that
\[
d(f^{i+1}(s+1)L+c(p^1), f^i(g_1)) < \varepsilon^*, j \in [0, (k+\varepsilon)t_r - 1],
\]
by Lemma 3.3. Choosing a strictly increasing integer sequence \( X, f \), since (3.13)

Now, fix a point \( \{ \}

Note that \( \zeta = \min\{ \inf\{ d(f^i(g_1), f^i(e_1)) \mid i \in \mathbb{N} \}, \inf\{ d(f^i(g_2), f^i(e_2)) \mid i \in \mathbb{N} \} \} \). So, with the combination of (3.5), (3.6), (3.7) and (3.8), one has

Combining (3.9), (3.10) and (3.11), one has

By (3.4), we have

Combining (3.9), (3.10) and (3.11), one has

\[
\frac{(k + t) \tau}{n} > 1 - \tau. 
\]

3.1.2. Proof of Lemma A. (a): By the proof of [46, Theorem 5.1], there is a sequence \( \{ \alpha_i \}_{i=1}^\infty \) in \( K \) such that for any \( n \in \mathbb{N} \), \( \{ \alpha_i : i \in \mathbb{N}, i > n \} = K \) and

For any \( \varepsilon > 0 \) and any \( t \in \mathbb{N}^+ \), there exists a sequence \( \{ \beta_i \}_{i=1}^t \subseteq K \) such that \( \beta_1 = \mu, \beta_t = \alpha_t \) and \( d(\beta_i, \beta_{i+1}) < \varepsilon, i = 1, 2, \cdots, t-1 \) since \( K \) is connected. So we can assume that such sequence \( \{ \alpha_i \}_{i=1}^t \) has a subsequence \( \{ \alpha_{i_k} \}_{k=1}^\infty \) satisfying

\[
i_{k+1} - i_k \geq 2, \text{ and } \alpha_{i_k} = \alpha_{i_{k+1}} = \mu \text{ for any } k \in \mathbb{N}^+.\]

(If not, add the sequence \( \{ \beta_i \}_{i=1}^t \) to the original sequence.) Define the index set

Let \( (g_1, e_1), (g_2, e_2) \) be the distal pair of \( G_{\mu_1}, G_{\mu_2} \) respectively and

\[
\zeta = \min\{ \inf\{ d(f^i(g_1), f^i(e_1)) \mid i \in \mathbb{N} \}, \inf\{ d(f^i(g_2), f^i(e_2)) \mid i \in \mathbb{N} \} \}.
\]

Fix a point \( z \in U \), there is a \( \rho > 0 \) such that \( B(z, \rho) \subseteq U \). Let \( \varepsilon_1 = \{ \rho/2, \zeta \} \) and \( \varepsilon_{i+1} = \varepsilon_i/2 \) for \( i \geq 1 \). By Lemma 2.11, \( (X, f) \) has the s-limit shadowing property. Then there is a \( \delta_i \in (0, \varepsilon_i) \) such that any \( \delta_i \)-limit-pseudo-orbit can be \( \varepsilon_i \)-limit-shadowed by some point in \( X \), and \( \lim_{i \to \infty} \delta_i = 0 \). Now, fix a point \( z_\Lambda \in \Lambda \). There is an \( m \in \mathbb{N} \) and a \( p_0 \in B(z, \varepsilon_1) \) such that \( f^m(p_0) \in B(z_\Lambda, \delta_1/2) \) since \( (X, f) \) is transitive. Let \( \{ \tau_i \}_{i=1}^\infty \) be a strictly decreasing sequence with \( \lim_{i \to \infty} \tau_i = 0 \).

For any \( i \in S_2 \), using Lemma 3.2 on \( \delta_i/2, z_\Lambda, \alpha_i, B(\alpha_i, \delta_i/2) \), we get a positive integer sequence \( \{ M_i \}_{i \in S_2} \). For any \( i \in S_1 \), using Lemma 3.3 on \( \tau_i, \delta_i/2, \theta, z_\Lambda \), we get a positive integer sequence \( \{ M_i \}_{i \in S_1} \). Let \( \{ n_i \}_{i=1}^\infty \) be an integer sequence with

Then for \( i \in S_2 \), we get \( p_i \in \text{Per}_{n_i} \) by Lemma 3.2. For \( i \in S_1 \), we get \( p_i^1 \in \text{Per}_{n_i} \) and \( p_i^2 \in \text{Per}_{n_i} \) by Lemma 3.3. Choosing a strictly increasing integer sequence \( \{ N_i \}_{i=1}^\infty \) with

\[
n_{i+1} \leq \tau_i \sum_{j=1}^i n_j N_j, \quad \text{and} 
\]

\[
x_{i+1} \leq \tau_i \sum_{j=1}^i n_j N_j, \quad \text{and} 
\]
For any $\xi = \{\xi_1, \xi_2, \cdots\} \in \{1, 2\}^\infty$, we denote
$$C(\xi) = \text{orb}(p_0, m_1) \xi_1 \cdots \xi_3 \xi_3 \cdots \xi_3 \cdots,$$
where
$$C_i = \begin{cases} \text{orb}(p_i, n_i) & \text{if } i \in S_2; \\ \text{orb}(p_i, n_i) & \text{if } i = i_k \text{ for some } k. \end{cases}$$
where $[k] :=$ the minimum positive integer of $\{k - \sum_{i=0}^{t}i\}_{t \in \mathbb{N}}$. One can check that $C(\xi)$ is a $\delta_1$-limit-pseudo-orbit. So there is a point $S_{C(\xi)} \in X$ which $\varepsilon_1$-limit-shadows $C(\xi)$. Denote
$$S := \bigcup_{\xi \in \{1, 2\}^\infty} S_{C(\xi)}.$$
We complete this proof by proving the following four facts:

1. $S \subseteq G_K$;
2. For any $y \in S$, $\omega_f(y) = \Lambda$;
3. For any distinct $x, y \in S$, $x, y$ is a DC1-scrambled pair;
4. $S_{C(\xi)} \neq S_{C(\eta)}$ if $\xi \neq \eta$, which implies $S$ is uncountable.

(1) The method in the proof of item (1) is mainly referring to [46]. Firstly, we define two stretched sequences $\{n'_l\}_{l=1}^\infty$ by
$$n'_l = n_k \text{ if } \sum_{j=1}^{k-1} N_j + 1 \leq l \leq \sum_{j=1}^{k} N_j,$$
and $\{\alpha'_l\}_{l=1}^\infty$ by
$$\alpha'_l := \alpha_k \text{ if } l \leq m; \alpha'_k := \alpha_k \text{ if } \sum_{j=1}^{k-1} n_jN_j + 1 \leq l - m \leq \sum_{j=1}^{k} n_jN_j.$$
The sequence $\{\alpha'_l\}$ has the same limit point set as the sequence $\{\alpha_k\}$. If for any $y \in S$,
$$\lim_{n \to \infty} d(C_n(y), \alpha'_n) = 0,$$
then the two sequence $\{C_n(y)\}$ and $\{\alpha'_n\}$ have the same limit point set and thus $S \subseteq G_K$. Let
$$M_k := m + \sum_{j=1}^{k} n'_j.$$
Because of (3.13) and the definition of $\{\alpha'_m\}$, it is sufficient to show that
$$\lim_{k \to \infty} d(C_{M_k}(y), \alpha'_M) = 0.$$
Suppose that $M_k = m + \sum_{i=1}^{j} n_iN_i + n_{j+1}\mathcal{R}$ with $i \in \mathbb{N}$ and $1 \leq \mathcal{R} \leq N_{j+1}$, hence $\alpha'_{M_k} = \alpha_{j+1}$. By (3.14), we have
$$d(C_{M_k}(y), \alpha'_{M_k}) \leq \frac{m + \sum_{i=1}^{j-1} n_iN_i}{M_k} d(C_{m + \sum_{i=1}^{j-1} n_iN_i}(y), \alpha'_{M_k})$$
$$+ \frac{n_jN_j}{M_k} d(C_{n_jN_j}(f^{m + \sum_{i=1}^{j-1} n_iN_i}(y), \alpha'_{M_k})$$
$$+ \frac{n_{j+1}\mathcal{R}}{M_k} d(C_{n_{j+1}\mathcal{R}}(f^{m + \sum_{i=1}^{j} n_iN_i}(y), \alpha'_{M_k})$$
$$\leq \tau_j + d(C_{n_jN_j}(f^{m + \sum_{i=1}^{j} n_iN_i}(y), \alpha_{j+1}) + d(C_{n_{j+1}\mathcal{R}}(f^{m + \sum_{i=1}^{j} n_iN_i}(y), \alpha_{j+1}).$$
By Lemma 3.2 and Lemma 3.3, we have

\begin{equation}
(3.17)\quad d(\mathcal{E}_{n_jN_j}(f^{m+\sum_{i=1}^{j-1}n_iN_i}, \alpha_{j+1}) \leq d(\mathcal{E}_{n_jN_j}(f^{m+\sum_{i=1}^{j-1}n_iN_i}, \sum_{i=1}^{N_j} \frac{1}{N_j} \mathcal{E}_{n_j}(p_j)))
\end{equation}

\begin{equation}
+ d\left(\sum_{i=1}^{N_j} \frac{1}{N_j} \mathcal{E}_{n_j}(p_j), \alpha_j\right) + d(\alpha_j, \alpha_{j+1})
\end{equation}

\begin{equation}
\leq d(\mathcal{E}_{n_jN_j}(f^{m+\sum_{i=1}^{j-1}n_iN_i}, \sum_{i=1}^{N_j} \frac{1}{N_j} \mathcal{E}_{n_j}(p_j))) + \delta_j/2 + d(\alpha_j, \alpha_{j+1}).
\end{equation}

and

\begin{equation}
(3.18)\quad d(\mathcal{E}_{n_j+1N}(f^{m+\sum_{i=1}^{j-1}n_iN_i}(y), \alpha_{j+1}) \leq d(\mathcal{E}_{n_j+1N}(f^{m+\sum_{i=1}^{j-1}n_iN_i}(y), \sum_{i=1}^{N_j+1} \frac{1}{N_j} \mathcal{E}_{n_j+1}(p_{j+1})))
\end{equation}

\begin{equation}
+ d\left(\sum_{i=1}^{N_j+1} \frac{1}{N_j} \mathcal{E}_{n_j+1}(p_{j+1}), \alpha_{j+1}\right)
\end{equation}

\begin{equation}
\leq d(\mathcal{E}_{n_j+1N}(f^{m+\sum_{i=1}^{j-1}n_iN_i}(y), \sum_{i=1}^{N_j+1} \frac{1}{N_j} \mathcal{E}_{n_j+1}(p_{j+1})) + \delta_{j+1}/2.
\end{equation}

Note that \( y \) limit-shadows some \( \mathcal{C}(\xi) \), so by Lemma 3.1

\begin{equation}
(3.19)\quad \lim_{j \to \infty} d(\mathcal{E}_{n_jN_j}(f^{m+\sum_{i=1}^{j-1}n_iN_i}(y), \sum_{i=1}^{N_j} \frac{1}{N_j} \mathcal{E}_{n_j}(p_j))) = 0,
\end{equation}

and

\begin{equation}
(3.20)\quad \lim_{j \to \infty} d(\mathcal{E}_{n_j+1N}(f^{m+\sum_{i=1}^{j-1}n_iN_i}(y), \sum_{i=1}^{N_j+1} \frac{1}{N_j} \mathcal{E}_{n_j+1}(p_{j+1}))) = 0.
\end{equation}

(3.16)-(3.20) and (3.12) result in

\begin{equation}
\lim_{k \to \infty} d(\mathcal{E}_{M_k}(y), \alpha'_{M_k}) = 0.
\end{equation}

Thus we have proved \( S \subseteq G_K \).

(2): For any \( y = S_\xi(\xi) \), \( y \) limit-shadows some \( \mathcal{C}(\xi) \). Thus \( \omega_f(y) = \omega(\mathcal{C}(\xi)) \). Note that \( \lim_{j \to \infty} d_{H}(\mathcal{C}_j, \Lambda) = 0 \), so \( \omega_f(y) = \Lambda \).

(3): For distinct \( x, y \in S \), we can assume that \( x = S_\xi(\xi) \), and \( y = S_\eta(\eta) \), where \( \mathcal{C}(\xi) = \langle u_1, u_2, \ldots \rangle \), \( \mathcal{C}(\eta) = \langle z_1, z_2, \ldots \rangle \) and \( \xi \neq \eta \). Note that \( x, y \) limit-shadows \( \mathcal{C}(\xi), \mathcal{C}(\eta) \) respectively. So for any \( t > 0 \), there exists a \( k_0 \in \mathbb{N} \) such that for any \( n \geq m + \sum_{j=1}^{k_0} n_j N_j \)

\begin{equation}
(3.21)\quad d(f^{n-1}(x), u_n) < t/2, \quad d(f^{n-1}(y), z_n) < t/2.
\end{equation}

Note that (3.15), then for any \( k \in \mathbb{N} \) and any \( s \in [\sum_{j=1}^{k} n_j N_j + 1, \sum_{j=1}^{k+1} n_j N_j] \),

\begin{equation}
(3.22)\quad u_{m+s} = z_{m+s}.
\end{equation}

By (3.21) and (3.22), we have for any \( k \geq k_0 \) and any \( s \in [\sum_{j=1}^{k} n_j N_j + 1, \sum_{j=1}^{k+1} n_j N_j] \),

\begin{equation}
(3.23)\quad d(f^{s-1}(x), f^{s-1}(y)) < t.
\end{equation}
Combining (3.23) and (3.14),
\[
\limsup_{n \to \infty} \frac{1}{n} \left| \left\{ j \in [0, n-1] : d(f^j(x), f^j(y)) < t \right\} \right|
\geq \limsup_{k \to \infty} \frac{1}{m + \sum_{j=1}^{n_k} n_j N_j} \left| \left\{ j \in [0, m + \sum_{j=1}^{n_k} n_j N_j - 1] : d(f^j x, f^j y) < t \right\} \right|
\geq \limsup_{k \to \infty} \frac{n_{i_k} + 1}{m + \sum_{j=1}^{n_{i_k} + 1} n_j N_j}
\]
\[
= 1.
\]

On the other hand, there exists a \( h \in \mathbb{N} \) such that \( \xi_h \neq \eta_h \) since \( \xi \neq \eta \). By the definition of \([k]\), there is a strictly increasing integer sequence \( \{k_j\}_{j=1}^\infty \) such that \( [k_j] = h \) for any \( j \in \mathbb{N} \), which implies
\[
\xi_{k_j} \neq \eta_{k_j}.
\]

By the item (c) of Lemma 3.3 and (3.24), there is a \( j_0^* \in \mathbb{N} \) such that for any \( j \geq j_0^* ,
\[
| \left\{ i \in [\sum_{j=1}^{i_{k_j}} n_i N_i + 1, \sum_{j=1}^{i_{k_j}} n_i N_i] \left| d(u_{m+i}, z_{m+i}) < \zeta - \delta_{i_{k_j}}/2 \right\} \right| < \tau_{i_{k_j}}.
\]

Note that \( x, y \) limit-shadows \( \mathcal{C}(\xi), \mathcal{C}(\eta) \) respectively. So there exists a \( j_0 \in \mathbb{N} \) such that for any \( n \geq m + \sum_{j=1}^{i_{k_j}} n_i N_i \)
\[
d(f^{n-1}(x), u_n) < \zeta/8, \quad d(f^{n-1}(y), z_n) < \zeta/8.
\]
Let \( \hat{j} \) large enough such that \( \delta_{i_{k_j}} \leq \zeta/2 \) and \( \hat{j} \geq \max\{j_0, j_0^*\} \). Combining (3.25) and (3.26),
\[
\liminf_{n \to \infty} \frac{1}{n} \left| \left\{ j \in [0, n-1] : d(f^j(x), f^j(y)) < \zeta/2 \right\} \right|
\leq \liminf_{j \to \infty} \frac{1}{m + \sum_{j=1}^{i_k} n_i N_i} \left| \left\{ i \in [0, m + \sum_{j=1}^{i_k} n_i N_i - 1] : d(f^i(x), f^i(y)) < \zeta/2 \right\} \right|
\leq \liminf_{j \to \infty} \frac{1}{m + \sum_{j=1}^{i_k} n_i N_i} \left\{ (m \sum_{j=1}^{i_k} n_i N_i + \left\{ i \in [0, m + \sum_{j=1}^{i_k} n_i N_i] \left| d(f^i(x), f^i(y)) < \zeta/2 \right\} \right) \right\}
\leq \liminf_{j \to \infty} \frac{m + \sum_{j=1}^{i_k} n_i N_i}{m + \sum_{j=1}^{i_k} n_i N_i + \tau_{k_j}}
\]
\[
= 0.
\]

So \( x, y \) is a DC1-scrambled pair.

(4): Implied by (3).

(b): Suppose that \((X, f)\) has exponential shadowing property with exponent \( \lambda \). Given \( \alpha \in \mathcal{A} \), we will construct an uncountable \( \alpha \)-DC1-scrambled set \( S^\alpha \). The construction is similar to the construction in the proof of item (a). We put the differences in the following

(1) By Lemma 2.12 there is a \( \delta_i \in (0, \varepsilon_i) \) such that any \( \delta_i \)-limit-pseudo-orbit can be both \((\varepsilon_i, \lambda)\)-shadowed and limit-shadowed by some point in \( X \), and \( \lim_{i \to \infty} \delta_i = 0 \).
(2) \( N_i \) chose here should be even number, satisfies (3.13), (3.14) and

\[
\frac{n_{k+1}N_{k+1} - c_k}{m + \sum_{i=1}^{k+1} n_iN_i} > 1 - \tau_k \quad \text{for any } k \geq 1,
\]

where

\[
c_k := \min \{c \in \mathbb{N}^+ \mid \alpha(m + \sum_{j=1}^{k} n_jN_j + c)\tau_k > m + \sum_{j=1}^{k} n_jN_j + \frac{4\varepsilon_1}{1 - e^{-\lambda}} \} \quad \text{for any } k \geq 1.
\]

(3.28) can be satisfied since \( \lim_{n \to \infty} \alpha(n) = \infty \).

(3) \( C(\xi) \) also can be seen as

\[
C(\xi) = \text{orb}(p_0, m) \cap \{ C_1 \cdots C_1 \cap C_2 \cap C_3 \cdots C_3 \cdots \},
\]

it is a \( \delta_1 \)-limit-pseudo-orbit. Then by Lemma 2.12, \( C(\xi) \) is both \( (\varepsilon_1, \lambda) \)-shadowed and limit-shadowed by some point \( S_{C(\xi)}^\alpha \). Denote

\[
S^\alpha := \bigcup_{\xi \in \{1,2,\ldots\}} S_{C(\xi)}^\alpha.
\]

We complete this proof by proving the following four facts:

(i): \( S^\alpha \subseteq G_K \);

(ii): For any \( y \in S^\alpha \), \( \omega_f(y) = \Lambda \);

(iii): For any distinct \( x, y \in S^\alpha \), \( x, y \) is a \( \alpha \)-DC1-scrambled pair;

(iv): \( S_{C(\xi)}^\alpha \neq S_{C(\eta)}^\alpha \) if \( \xi \neq \eta \), which implies \( S^\alpha \) is uncountable.

Item (iv) is directly from item (iii), the proofs of item (i) and item (ii) are the same as item (1) and item (2) in the proof of item (a). We only need to prove the different part in item (iii).

For distinct \( x, y \in S^\alpha \), we can assume that \( x = S_{C(\xi)}^\alpha \), \( y = S_{C(\eta)}^\alpha \), where \( C(\xi) = \langle u_1, u_2, \cdots \rangle \), \( C(\eta) = \langle z_1, z_2, \cdots \rangle \) and \( \xi \neq \eta \). Recall that \( \{ \tau_i \}_{i=1}^\infty \) is a strictly decreasing sequence with \( \lim_{t \to \infty} \tau_t = 0 \).

So for any \( t > 0 \), there exists a \( k_0 \in \mathbb{N}^+ \) such that for any \( n \geq k_0 \), we have \( \tau_n < t \). Denote \( I_k := [\sum_{j=1}^{i_k} n_jN_j + 1, \sum_{j=1}^{i_k+1} n_jN_j] \), note that (3.15), then for any \( k \in \mathbb{N} \) and any \( s \in I_k \),

\[
u_{m+s} = z_{m+s}.
\]

By (3.29) and the definitions of \( x \) and \( y \), we have

\[
ds_{s \in I_k} d(f^{m+s-1}(x), f^{m+s-1}(y)) = 2 \cdot 2\varepsilon_1 \sum_{r=0}^{n_{i_k+1}N_{i_k+1} - 1} e^{-r\lambda} < \frac{4\varepsilon_1}{1 - e^{-\lambda}}.
\]

Combining (3.30) and (3.28), for any \( k \geq k_0 \) and \( c \in I_k \), we have

\[
m+c-1 \sum_{j=0}^{m+c-1} d(f^j(x), f^j(y)) \leq (m + \sum_{i=1}^{i_k} n_iN_i) \text{diam } X + \frac{4\varepsilon_1}{1 - e^{-\lambda}} < \alpha(m + \sum_{j=1}^{i_k} n_jN_j + c_k)t.
\]
As a result, we have
\[
\limsup_{n \to \infty} \frac{1}{n} \{1 \leq i \leq n : \sum_{j=0}^{i-1} d(f^j(x), f^j(y)) < \alpha(i)t\} \\
\geq \limsup_{k \to \infty} \frac{1}{m + \sum_{j=1}^{i_k} n_j N_j} |\{i \in [m + \sum_{j=1}^{i_k} n_j N_j + c_{i_k}, m + \sum_{j=1}^{i_k+1} n_j N_j] : \sum_{j=0}^{i-1} d(f^j(x), f^j(y)) < \alpha(i)t}\} \\
\geq \limsup_{k \to \infty} \frac{n_{i_k+1} N_{i_k+1} - c_{i_k}}{m + \sum_{j=1}^{i_k+1} n_j N_j} = 1.
\]

The rest of the proof of item (iii) is the same as item (3) in the proof of item (a).

3.2. Strong-elementary-dense Property. For any \(m \in \mathbb{N}\) and \(\{\nu_i\}_{i=1}^m \subseteq M(X)\), we write \(\text{cov}\{\nu_i\}_{i=1}^m\) for the convex combination of \(\{\nu_i\}_{i=1}^m\), namely,
\[
\text{cov}\{\nu_i\}_{i=1}^m = \text{cov}(\nu_1, \cdots, \nu_m) := \left\{\sum_{i=1}^m t_i \nu_i : t_i \in [0, 1], 1 \leq i \leq m \text{ and } \sum_{i=1}^m t_i = 1 \right\}.
\]

\textbf{Definition 3.4.} We say that \((X, f)\) satisfies the strong-elementary-dense property if for any \(K = \text{cov}\{\mu_i\}_{i=1}^m \subseteq M_f(X)\) and any \(\varepsilon > 0\), there exist compact invariant subset \(\Lambda_i \subseteq \Lambda \subseteq X\), \(1 \leq i \leq m\) such that
\begin{enumerate}[(a)]
\item \((\Lambda, f)\) has the strong specification property, more generally \(\Lambda \in ICM\).
\item \(d_H(K, M_f(\Lambda)) < \varepsilon\), \(d_H(\mu_i, M_f(\Lambda_i)) < \varepsilon\).
\item There is no fixed point in \(\Lambda\).
\end{enumerate}

\textbf{Lemma B.} Suppose \((X, f)\) is positively expansive, mixing and has the shadowing property. Then \((X, f)\) satisfies the strong-elementary-dense property.

\textbf{Proof.} Actually, it is enough to finish the proof of our main theorems if this lemma holds for any \(K = \text{cov}\{\mu_1, \mu_2\} \subseteq M_f(X)\). So, for brevity, we just prove this lemma for such \(K\). Let \(e\) be the expansive constant and \(\varepsilon^* = \min\{\varepsilon/3, e/3\}\). By shadowing property, there is a \(\delta \in (0, \varepsilon^*)\) such that any \(\delta\)-pseudo-orbit is \(\varepsilon^*\)-shadowed by some point in \(X\). Note \(X \in ICM\) since \((X, f)\) is mixing. So, by Lemma 3.2 for a fix \(x \in X\), there is an \(n \in \mathbb{N}\) and \(p_1, p_2 \in B(x, \delta/2)\) such that \(p_1 \in \text{Per}_n, p_2 \in \text{Per}_{n+1}\) with \(E_n(p_1) \in B(\mu_1, \varepsilon^*), E_{n+1}(p_2) \in B(\mu_2, \varepsilon^*)\) Denote \(I = \{p_1, f(p_1), \cdots, f^{n-1}(p_1), p_2, f(p_2), \cdots, f^n(p_2)\}\). Let \(\Sigma \subseteq \Sigma_I\) be a shubf of finite type with transition matrix \(L\):

\[
\begin{pmatrix}
p_1 & f(p_1) & \cdots & p_2 & \cdots & f^n(p_2) \\
0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\end{pmatrix}\]

where \(\Sigma_I := \{\langle \theta_1, \theta_2, \cdots \rangle : \theta_i \in I, i \in \mathbb{N}^+\}\) is a full shift with alphabet \(I\). Actually we define a subshift whose points is the infinite connection of the orbits of \(p_1\) and \(p_2\) such like \((f^{n-2}(p_1), f^{n-1}(p_1), p_1, f(p_1), \cdots, f^{n-1}(p_1), p_2, f(p_2), \cdots, f^n(p_2))\) repeating \(\cdots\). And this is what the transition matrix exactly implies. The directed graph deduced by \(L\) is as follows:
Obviously, it is a strongly connected graph, which implies \( L \) is irreducible. By [13, Proposition 17.9], \((\Sigma, \sigma)\) is transitive. For any \( k \geq n^2\), we can write \( k \) as \( k = sn + t \) with \( s \geq n \) and \( t \leq n \). Thus \( k = (s - t)n + t(n + 1) \). In other words, for any \( k \geq n^2\), there is a block with length \( k \) as
\[ \langle \text{orb}(p_1, n) \cdots \text{orb}(p_1, n) \text{orb}(p_2, n + 1) \cdots \text{orb}(p_2, n + 1) \rangle. \]

Then for any \( a, b \in I \), there exists an \( m_0(\geq n^2) \) such that for any \( m \geq m_0 \), \( \{x \in \Sigma \mid x_1 = a, x_m = b\} \neq \emptyset \). So \((\Sigma, \sigma)\) is mixing by [13, Proposition 17.8, 17.10 (3)]. Further, \((\Sigma, \sigma)\) has strong specification property by Lemma 2.14. For any \( \theta \in \Sigma \), one can observe that it is a \( \delta \)-pseudo-orbit. Denote
\[ Y_\theta := \{x \in X \mid d(f^{i-1}(x), \theta_i) \leq \varepsilon/3, \ i \geq 1\}. \]
Then \( Y_\theta \) is nonempty by shadowing property and singleton by expansiveness. So \( f(Y_\theta) = Y_{\sigma(\theta)} \).
Define \( \Lambda = \bigcup_{\theta \in \Sigma} Y_\theta \), and
\[ \pi : \Sigma \rightarrow \Lambda \]
\[ \theta \mapsto Y_\theta. \]

One can check that \( \pi \) is continuous. Thus \( \Lambda \) is closed and \((\Lambda, f)\) is a factor of \((\Sigma, \sigma)\). By [13, Proposition 21.4 (c)], \((\Lambda, f)\) has strong specification property.

Let \( \Lambda_i = \text{orb}(p_i), i = 1, 2 \). Then \( \Lambda_i \subseteq \Lambda \) and \( d_H(\mu_i, \mathcal{M}_f(\Lambda_i)) < \varepsilon^* \). For any \( \tilde{\mu} \in K \), let \( \mu \in B(\tilde{\mu}, \varepsilon^*) \) with \( \mu = \frac{c_1}{c} \mu_1 + \frac{c_2}{c} \mu_2 \), where \( c_1, c_2 \in \mathbb{N}^+, c = c_1 + c_2 \). Let \( \mathcal{C}_1 = \text{orb}(p_1, n), \mathcal{C}_2 = \text{orb}(p_2, n + 1), \)
\[ \mathcal{C}_\mu = \mathcal{C}_1 \cdots \mathcal{C}_i \cdots \mathcal{C}_2, \]
and \( \theta = \mathcal{C}_\mu \mathcal{C}_\mu \cdots \). Then \( \theta \in \Sigma \) and \( Y_\theta \in \text{Per}_{n(n+1)c} \), which implies \( V_f(Y_\theta) = \mathcal{E}_{n(n+1)c}(Y_\theta) \).
\[ d(\mathcal{E}_{n(n+1)c}(Y_\theta), \mu) \leq d(\mathcal{E}_{n(n+1)c}(Y_\theta), \delta_{\mu_1}) + d(\delta_{\mu_1}, \frac{c_1}{c} \mu_1 + \frac{c_2}{c} \mu_2) \]
\[ \leq \varepsilon^* + d(\frac{n(n+1)c_1}{n(n+1)c} \delta_{\varepsilon_1} + \frac{n(n+1)c_2}{n(n+1)c} \delta_{\varepsilon_2}, \frac{c_1}{c} \mu_1 + \frac{c_2}{c} \mu_2) \]
\[ \leq \varepsilon^* + \frac{c_1}{c} \varepsilon^* + \frac{c_2}{c} \varepsilon^* \]
\[ < 2\varepsilon^*. \]
Thus \( d(V_f(Y_\theta), \tilde{\mu}) < 3\varepsilon^* < \varepsilon \). Further, \( K \subseteq B(\mathcal{M}_f(\Lambda), \varepsilon) \). One the other hand, \( \mathcal{M}_f(\Lambda) \subseteq B(K, \varepsilon) \) by its definition. So \( d_H(K, \mathcal{M}_f(\Lambda)) < \varepsilon \).

Finally, from the proof of Lemma 3.2, we can assume that \( p_1 \) and \( p_2 \) are not fixed points, so there is no fixed point in \( \Lambda \). \qed

4. Proof of Main Theorems

4.1. Some Lemmas. For any \( x \in X \), we define the measure center of \( x \) as
\[ C^*_x := \bigcup_{\mu \in \mathcal{M}_f(\omega_f(x))} S_{\mu}. \]
Furthermore, we define the measure center of a closed invariant set $\Lambda \subseteq X$ as

$$C^*_\Lambda := \bigcup_{\mu \in M_f(\Lambda)} S_\mu.$$ 

**Lemma 4.1.** For $(X, f)$, let $\Lambda \subseteq X$ be closed $f$-invariant and $K \subseteq M_f(\Lambda)$ be a nonempty compact connected set.

1. Suppose $\bigcap_{\mu \in K} S_\mu = C^*_\Lambda$. Then

$$G^\Lambda_K \subseteq \{x \in X : \bigcap_{\mu \in K} S_\mu = \omega_d(x) = \omega_f(x) = C^*_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$ 

2. Suppose $\bigcap_{\mu \in K} S_\mu = \bigcup_{\nu \in K} S_\nu \subseteq C^*_\Lambda$. Then

$$G^\Lambda_K \subseteq \{x \in X : \bigcap_{\mu \in K} S_\mu = \omega_d(x) = \omega_f(x) \subseteq \omega_f(x) = C^*_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$ 

3. Suppose $\bigcap_{\mu \in K} S_\mu \subseteq \bigcup_{\nu \in K} S_\nu = C^*_\Lambda$. Then

$$G^\Lambda_K \subseteq \{x \in X : \bigcap_{\mu \in K} S_\mu = \omega_d(x) \subseteq \omega_f(x) \subseteq \omega_f(x) = C^*_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$ 

4. Suppose $\bigcap_{\mu \in K} S_\mu \subseteq \bigcup_{\nu \in K} S_\nu \subseteq C^*_\Lambda$. Then

$$G^\Lambda_K \subseteq \{x \in X : \bigcap_{\mu \in K} S_\mu = \omega_d(x) \subseteq \omega_f(x) \subseteq \omega_f(x) = C^*_\Lambda \subseteq \omega_f(x) = \Lambda\}.$$ 

**Proof.** For any $x \in G^\Lambda_K$, $\omega_f(x) = \Lambda$ by definition. So by item (4) of Proposition 2.22,

$$\omega_f(x) = C^*_{\omega_f} = \bigcup_{\mu \in M_f(\omega_f(x))} S_\mu = \bigcup_{\mu \in M_f(\Lambda)} S_\mu = C^*_\Lambda.$$ 

Consequently, one uses (1.1) and obtains that

$$\omega_d(x) \subseteq \omega_f(x) \subseteq \omega_f(x) = C^*_\Lambda \subseteq \omega_f(x) = \Lambda.$$ 

Note that $V_f(x) = K$, then

$$\omega_d(x) = \bigcap_{\mu \in V_f(x)} S_\mu = \bigcap_{\mu \in K} S_\mu \text{ and } \omega_f(x) = \bigcup_{\mu \in V_f(x)} S_\mu = \bigcup_{\nu \in K} S_\nu$$

by Proposition 2.22. Therefore, a convenient use of (1.1) and (4.2) yields (1)-(4). \hfill \Box

Given an invariant compact subset $\Lambda \subseteq X$ and a continuous function $\varphi : X \to \mathbb{R}$, we denote

$$L^\Lambda_{\varphi} = \left[ \inf_{\mu \in M_f(\Lambda)} \int \varphi d\mu, \sup_{\mu \in M_f(\Lambda)} \int \varphi d\mu \right].$$

**Lemma 4.2.** Suppose $(X, f)$ is positively expansive, mixing and has the shadowing property. Let $\varphi : X \to \mathbb{R}$ be a continuous function and assume that $\text{Int}(L^\Lambda_{\varphi}) \neq \emptyset$. Then for any $a \in \text{Int}(L^\Lambda_{\varphi})$, there are two $f$-invariant compact subsets $\Lambda \subseteq \Theta \subseteq X$ such that

1. $\Lambda$ has the strong specification property, more generally $\Lambda \in ICM$ with

$$C^*_\Lambda = \Lambda, \quad a \in \text{Int}(L^\Lambda_{\varphi}).$$

2. $\Theta$ is internally chain mixing with

$$\Lambda = C^*_\Theta \subseteq \Theta, \quad a \in \text{Int}(L^\Lambda_{\varphi}).$$

3. There is no fixed point in $\Lambda$ and $\Theta$.

In particular, $\text{Int}(L^\Lambda_{\varphi}) = \text{Int}(L^\Theta_{\varphi}) \neq \emptyset$. 

Proof. (1): For any $a \in \text{Int}(L_\varphi)$, we can choose $\mu, \nu, \nu' \in \mathcal{M}_f(X)$ with $\int \varphi d\mu < a < \int \varphi d\nu$. Denote

$$\zeta = \min\{a - \int \varphi d\mu, \int \varphi d\nu - a\}.$$ 

By Lemma B, we obtain a $\Lambda \subseteq X$ and $\Lambda$ has strong specification property with two measure $\tilde{\mu}, \tilde{\nu} \in \mathcal{M}_f(\Lambda)$ such that

$$\left| \int \varphi d\tilde{\mu} - \int \varphi d\mu \right| < \zeta/2, \quad \left| \int \varphi d\tilde{\nu} - \int \varphi d\nu \right| < \zeta/2.$$ 

Then $\int \varphi d\tilde{\mu} < a < \int \varphi d\tilde{\nu}$, which implies $a \in \text{Int}(L_\varphi^\Lambda)$. By Lemma 2.15, there is a $\rho \in \mathcal{M}_f(\Lambda)$ with full support. Thus $C^*_\Lambda = \Lambda$.

(2): By [16, Corollary 4.21], we can find a $z \notin \Lambda$ such that $\omega_f(z) = \Lambda$. Let $A = \text{orb}(z, f) \cup \Lambda$. Note that $A$ is closed and $f$-invariant. And approximate product property is weaker than specification property from their definitions. So we have the following.

**Proposition 4.4.** [45, Proposition 2.3(1)] Suppose that $(X, f)$ is a dynamical system satisfying specification property. Then $(X, f)$ has entropy-dense property.

**Proposition 4.5.** Suppose that $(X, f)$ is a dynamical system satisfying specification property. Then for any $n \in \mathbb{N}$, there exist $f$-invariant ergodic measures $\{\omega_k\}_{k=1}^n$ such that $\{S_{\omega_k}\}_{k=1}^n$ are pairwise disjoint and $\cup_{k=1}^n S_{\omega_k} \neq X$.

**Proof.** Since $(X, f)$ has specification property and $(X, d)$ be a nondegenerate (i.e, with at least two points) compact metric space, then $(X, f)$ is not uniquely ergodic. Thus there exists $f$-invariant measures $\mu \neq \nu$ on $X$ such that $\nu \neq \omega$. Choose $0 < \theta_1 < \theta_2 < \cdots < \theta_{n+1} < 1$. Let $\nu_k := \theta_k \mu + (1 - \theta_k) \nu$. Then $\nu_i \neq \nu_j$ for any $1 \leq i < j \leq n+1$. Denote $\rho_0 = \min\{\rho(\nu_i, \nu_j) : 1 \leq i < j \leq n+1\}$. By Proposition 4.4, for each $k = 1, \cdots, n+1$, there exists $\omega_k \in \mathcal{M}_f(X)$ with $\mathcal{M}_f(S_{\omega_k}) \subseteq B(\nu_k, \frac{1}{2}\rho_0)$. Note that $\{B(\nu_k, \frac{1}{2}\rho_0)\}_{k=1}^{n+1}$ are pairwise disjoint. So $\{\mathcal{M}_f(S_{\omega_k})\}_{k=1}^{n+1}$ are pairwise disjoint. As a result, $\{S_{\omega_k}\}_{k=1}^{n+1}$ are pairwise disjoint and $\cup_{k=1}^n S_{\omega_k} \neq X$. □

**Proposition 4.6.** Suppose that $(X, f)$ is a dynamical system. If there is no fixed point in $X$, then $G_\mu$ has distal pair for any $\mu \in \mathcal{M}_f(X)$.
**Proof.** $G_\mu \neq \emptyset$ since $\mu \in M^f_\nu(X)$. Let $x \in G_\mu$, then $f(x) \in G_\mu$. Assume that $x, f(x)$ is not distal, then $\omega_f(x)$ contains a fixed point.

It is easy to check Theorem A and B can be deduced from Theorem C or D so we only need to show Theorem C or D.

**4.2. Proof of Theorem C.** Now we state a abstract result on strongly distributional chaos of $\{x \in X \mid x \text{ satisfies case (i)}\} \cap NR(f) \cap I_\varphi(f)$. Combining with Theorem 2.17 and Theorem 2.18, we have Theorem C.

**Theorem 4.7.** Suppose $(X, f)$ is positively expansive, mixing and has the exponential shadowing property. Let $\varphi$ be a continuous function on $X$. If $I_\varphi(f) \neq \emptyset$, then $\{x \in X \mid x \text{ satisfies case (i)}\} \cap NR(f) \cap I_\varphi(f)$ is strongly distributional chaotic for any $i \in \{1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'\}$.

**Proof.** $I_\varphi(f) \neq \emptyset$ implies $Int(L_\varphi) \neq \emptyset$. Then there are $\Lambda \subseteq \Theta \subseteq X$ satisfying Lemma 4.2. By Proposition 4.5, Lemma 2.15 and Lemma 16, we can take $\mu_1, \mu_2 \in M^f_\nu(X)$ with disjoint support, and $\mu, \nu \in M^f_\nu(\Lambda)$ with full support satisfying

$$
\int \varphi d\mu_1, \int \varphi d\mu_2, \int \varphi d\nu, \text{ and } \int \varphi d\mu \text{ are mutually unequal.}
$$

Also, $S_{\mu_1} \cup S_{\mu_2} \neq \Lambda$ by Proposition 4.5. Let $0 < \theta_1 < \theta_2 < 1$ and

$$
K_1 := \text{cov}\{\theta_1 \mu_1 + (1 - \theta_1) \mu_2, \theta_2 \mu_1 + (1 - \theta_2) \mu_2\},
$$

$$
K_2 := \text{cov}\{\mu_1, \mu\} \cup \text{cov}\{\mu_2, \mu\},
$$

$$
K_3 := \text{cov}\{\mu_1, \theta_1 \mu_1 + (1 - \theta_1) \mu\},
$$

$$
K_4 := \text{cov}\{\mu_1, \mu_2\},
$$

$$
K_5 := \text{cov}\{\mu_1, \theta_1 \mu_1 + (1 - \theta_1) \mu_2\},
$$

$$
K_6 := \text{cov}\{\mu, \nu\}.
$$

Now, using Lemma A on $K_i$, $i = 1, 2, 3, 4, 5, 6$ and $\Lambda$, we get $\{x \in X \mid x \text{ satisfies case (i)}\} \cap NR(f) \cap I_\varphi(f)$ is strongly distributional chaotic for any $i \in \{1, 2, 3, 4, 5, 6\}$ by Lemma 4.1 and Proposition 4.6. Using Lemma A on $K_i$, $i = 1, 2, 3, 4, 5, 6$ and $\Theta$, we get $\{x \in X \mid x \text{ satisfies case (i)}\} \cap NR(f) \cap I_\varphi(f)$ is strongly distributional chaotic for any $i \in \{1', 2', 3', 4', 5', 6'\}$ by Lemma 4.1 and Proposition 4.6.

**4.3. Proof of Theorem D.** Now we state a abstract result on strongly distributional chaos of $\{x \in X \mid x \text{ satisfies case (i)}\} \cap NR(f) \cap R_\varphi(a)$. Combining with Theorem 2.17 and Theorem 2.18, we have Theorem D.

**Theorem 4.8.** Suppose $(X, f)$ is positively expansive, mixing and has the exponential shadowing property. Let $\varphi$ be a continuous function on $X$. If $Int(L_\varphi) \neq \emptyset$, then for any $a \in Int(L_\varphi)$, $\{x \in X \mid x \text{ satisfies case (i)}\} \cap NR(f) \cap R_\varphi(a)$ is strongly distributional chaotic for any $i \in \{1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'\}$.

**Proof.** Take $\Lambda \subseteq \Theta \subseteq X$ satisfying Lemma 4.2. By Proposition 4.5, we can take $\mu_1, \mu_2, \mu_3, \mu_4 \in M^f_\nu(\Lambda)$ with mutually disjoint support satisfying

$$
\int \varphi d\mu_1 < \int \varphi d\mu_2 < a < \int \varphi d\mu_3 < \int \varphi d\mu_4.
$$

Similarly, $\bigcup_{i=1}^4 S_{\mu_i} \neq \Lambda$ by Proposition 4.5. Now, we can choose proper $\{\theta_i\}_{i=1}^3 \subseteq (0, 1)$ such that

$$
\theta_1 \int \varphi d\mu_1 + (1 - \theta_1) \int \varphi d\mu_3 = a;
$$

$$
\theta_2 \int \varphi d\mu_2 + (1 - \theta_2) \int \varphi d\mu_4 = a;
$$

and
Denote
\[ \nu_1 = \theta_1 \mu_1 + (1 - \theta_1) \mu_3; \]
\[ \nu_2 = \theta_2 \mu_2 + (1 - \theta_2) \mu_4; \]
\[ \nu_3 = \theta_3 \mu_1 + (1 - \theta_3) \mu_4. \]
By Lemma 2.15 and Lemma 2.16, there are \( \omega_1, \omega_2 \in M^f_\mu(X) \) with full support and
\[ \int \varphi d\omega_1 < a < \int \varphi d\omega_2. \]
Now, we can choose proper \( \theta \subseteq (0, 1) \) such that
\[ \theta \int \varphi d\omega_1 + (1 - \theta) \int \varphi d\omega_2 = a. \]
Denote \( \mu = \theta \omega_1 + (1 - \theta) \omega_2. \) Let
\[ K_1 := \{ \nu_1 \}, \]
\[ K_2 := \text{cov}\{\nu_1, \mu\} \cup \text{cov}\{\nu_2, \mu\}, \]
\[ K_3 := \text{cov}\{\nu_1, \mu\}, \]
\[ K_4 := \text{cov}\{\nu_1, \nu_2\}, \]
\[ K_5 := \text{cov}\{\nu_1, \nu_3\}, \]
\[ K_6 := \{ \mu \}. \]
Now, using Lemma A on \( K_i, i = 1, 2, 3, 4, 5, 6 \) and \( \Lambda \), we get \( \{ x \in X \mid x \text{ satisfies case } (i) \} \cap NR(f) \cap R_\phi(a) \) is strongly distributional chaotic for any \( i \in \{1, 2, 3, 4, 5, 6 \} \) by Lemma 4.1 and Proposition 4.6. Using Lemma A on \( K_i, i = 1, 2, 3, 4, 5, 6 \) and \( \Theta \), we get \( \{ x \in X \mid x \text{ satisfies case } (i) \} \cap NR(f) \cap R_\phi(a) \) is strongly distributional chaotic for any \( i \in \{1', 2', 3', 4', 5', 6' \} \) by Lemma 4.1 and Proposition 4.6. \( \square \)

4.4. Recurrent Points That Are Not Transitive. Recall that
\[ ND = NR \sqcup (\text{Rec} \setminus \text{Tran}). \]
In previous sections, we have obtained that for mixing expanding maps or transitive Anosov diffeomorphisms, the set of non-recurrent set \( NR(f) \) is strongly distributional chaotic. In this subsection we show that \( \text{Rec} \setminus \text{Tran} \) is distributional chaotic of type 1.

**Theorem 4.9.** Suppose \( (X, f) \) is positively expansive, mixing and has the shadowing property. Let \( \varphi \) be a continuous function on \( X \). If \( I_\varphi(f) \neq \emptyset \), then for any \( i \in \{1, 2, 3, 4, 5, 6 \} \)

(1): \( \{ x \in X \mid x \text{ satisfies case } (i) \} \cap (\text{Rec}(f) \setminus \text{Tran}(f)) \cap I_\varphi(f) \) contains an uncountable DC1-scrambled set,

(II): \( \{ x \in X \mid x \text{ satisfies case } (i) \} \cap (\text{Rec}(f) \setminus \text{Tran}(f)) \cap R_\varphi(a) \) contains an uncountable DC1-scrambled set for any \( a \in \text{Int}(L_\varphi) \).

**Proof.** We first recall the following result.

**Lemma 4.10.** [9, Theorem F] Suppose that \( (X, f) \) is a dynamical system with the specification property and let \( K \) be a connected non-empty compact subset of \( M_f(X) \). If there is a \( \mu \in K \) such that \( \mu = \theta \mu_1 + (1 - \theta) \mu_2 \) (\( \mu_1 = \mu_2 \) could happen) where \( \theta \in [0, 1] \), and \( G_{\mu_1}, G_{\mu_2} \) both have distal a pair, then there exists an uncountable DC1-scrambled set \( S_K \subseteq G_K \cap \text{Tran}(f) \).
Note that for any invariant compact subset $\Lambda \subseteq X$, we have $\text{Tran}(f|_\Lambda) \subseteq \text{Rec}(f) \setminus \text{Tran}(f)$. Now, using Lemma 4.10 on $K_i$ and $\Lambda$ of Theorem 4.7, we get $\{x \in X \mid x \text{ satisfies case (i)}\} \cap (\text{Rec}(f) \setminus \text{Tran}(f)) \cap I_{\varphi}(f)$ contains an uncountable DC1-scrambled set for any $i \in \{1, 2, 3, 4, 5, 6\}$ by Lemma 4.1 and Proposition 4.6. Using Lemma 4.10 on $K_i$ and $\Lambda$ of Theorem 4.8, we get $\{x \in X \mid x \text{ satisfies case (i)}\} \cap (\text{Rec}(f) \setminus \text{Tran}(f)) \cap R_{\varphi}(a)$ contains an uncountable DC1-scrambled set for any $i \in \{1, 2, 3, 4, 5, 6\}$ by Lemma 4.1 and Proposition 4.6.

5. Other Dynamical Systems

In this section, we apply the results in the previous sections to more systems, including mixing subshifts of finite type, $\beta$-shifts and nonuniformly hyperbolic systems. Before that, we give a abstract result as a generalization of Theorem 4.7, 4.8 and 4.9.

**Theorem 5.1.** Suppose that $(X, f)$ is a dynamical system, and $Y$ is an invariant compact sub-
set of $X$. Assume that $(Y, f|_Y)$ is positively expansive, mixing and has the exponential shading-
ning property. Let $\varphi$ be a continuous function on $X$. If $I_{\varphi}(f) \cap Y \neq \emptyset$, then for any $i \in \{1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'\}$

(I): $\{x \in X \mid x \text{ satisfies case (i)}\} \cap \text{NR}(f) \cap I_{\varphi}(f)$ is strongly distributional chaotic,

(II): $\{x \in X \mid x \text{ satisfies case (i)}\} \cap \text{NR}(f) \cap I_{\varphi}(f) \cap R_{\varphi}(a)$ is strongly distributional chaotic for any $a \in \text{Int}(I_{\varphi}(f))$.

And for any $i \in \{1, 2, 3, 4, 5, 6\}$

(III): $\{x \in X \mid x \text{ satisfies case (i)}\} \cap (\text{Rec}(f) \setminus \text{Tran}(f)) \cap I_{\varphi}(f)$ contains an uncountable DC1-scrambled set,

(IV): $\{x \in X \mid x \text{ satisfies case (i)}\} \cap (\text{Rec}(f) \setminus \text{Tran}(f)) \cap R_{\varphi}(a)$ contains an uncountable DC1-scrambled set for any $a \in \text{Int}(L_{\varphi}(f))$.

**Proof.** Note that $\text{NR}(f|_Y) \subseteq \text{NR}(f)$, $I_{\varphi}(f|_Y) \subseteq I_{\varphi}(f)$, and $R_{\varphi}(a) \cap Y \subseteq R_{\varphi}(a)$. Then by Theorem 4.7, 4.8 and 4.9 we finish the proof.

5.1. Symbolic Dynamics

5.1.1. Subshifts of Finite Type. For any finite alphabet $A$, the full symbolic space is the set $A^\mathbb{Z} = \{\cdots x_{-1}x_0x_1 \cdots : x_i \in A\}$, which is viewed as a compact topological space with the discrete product topology. The set $A^{\mathbb{N}^+} = \{x_1x_2 \cdots : x_i \in A\}$ is called one side full symbolic space. The shift action on one side full symbolic space is defined by

$$\sigma : A^{\mathbb{N}^+} \to A^{\mathbb{N}^+}, \quad x_1x_2 \cdots \mapsto x_2x_3 \cdots.$$  

$(A^{\mathbb{N}^+}, \sigma)$ forms a dynamical system under the discrete product topology which we called a shift. We equip $A^{\mathbb{N}^+}$ with a compatible metric $d$ given by

$$d(\omega, \gamma) = \begin{cases} n^{-\min\{k \in \mathbb{N}^+: \omega_k \neq \gamma_k\}}, & \omega \neq \gamma, \\
0, & \omega = \gamma. \end{cases}$$

where $n = \#A$. A closed subset $X \subseteq A^{\mathbb{N}^+}$ or $A^\mathbb{Z}$ is called subshift if it is invariant under the shift action $\sigma$. $w \in A^n \triangleq \{x_1x_2 \cdots : x_i \in A\}$ is a word of subshift $X$ if there is an $x \in X$ and $k \in \mathbb{N}^+$ such that $w = x_kx_{k+1} \cdots x_{k+n-1}$. Here we call $n$ the length of $w$, denoted by $|w|$. The language of a subshift $X$, denoted by $L(X)$, is the set of all words of $X$. Denote $L_n(X) \triangleq L(X) \cap A^n$ all the words of $X$ with length $n$.

Let $B$ be a set of words occurring in $A^{\mathbb{N}^+}$. Then

$$A_B := \{x \in A^{\mathbb{N}^+} \mid \text{no block } w \in B \text{ occurs in } x\}$$

is a compact $\sigma$-invariant subset. $(A_B, \sigma)$ is a subshift defined by $B$. A subshift $(\Lambda, \sigma)$ is called a (one-side) subshift of finite type if there exists a finite word set $B$ such that $\Lambda_B = \Lambda$. By [68], a subshift satisfies shadowing property if and only if it is a subshift of finite type. By [27,
5.2. Homoclinic Classes. The results of Theorem 4.7 and 4.8 hold for every mixing (one-side) subshift of finite type.

Remark 5.3. Theorem 5.2 is also true for every mixing (two-side) subshift of finite type.

5.1.2. β-Shifts. Next we present one type of subshift, (one-side) β-shift, basically referring to [50, 53, 45]. Let β > 1 be a real number. We denote by \([x]\) and \(\{x\}\) the integer and fractional part of the real number \(x\). Considering the β-transformation \(f_\beta : [0,1) \to [0,1)\) given by
\[
f_\beta(x) = \beta x \pmod{1}.
\]

For \(\beta \notin \mathbb{N}\), let \(b = [\beta]\) and for \(\beta \in \mathbb{N}\), let \(b = \beta - 1\). Then we split the interval \([0,1)\) into \(b + 1\) partition as below
\[
J_0 = \left[0, \frac{1}{\beta}\right), \ J_1 = \left[rac{1}{\beta}, \frac{2}{\beta}\right), \ldots, \ J_b = \left[rac{b}{\beta}, 1\right).
\]

For \(x \in [0,1)\), let \(i(x, \beta) = (i_n(x, \beta))_1^\infty\) be the sequence given by \(i_n(x, \beta) = j\) when \(f_\beta^{n-1}(x) \in J_j\). We call \(i(x, \beta)\) the greedy β-expansion of \(x\) and we have \(x = \sum_{n=1}^\infty i_n(x, \beta)\beta^{-n}\). We call \((\Sigma_\beta, \sigma)\) (one-side) β-shift, where \(\sigma\) is the shift map, \(\Sigma_\beta\) is the closure of \(\{i(x, \beta)\}_{x \in [0,1)}\) in \(\prod_{i=1}^\infty \{0, 1, \ldots, b\}\).

From the discussion above, we can define the greedy β-expansion of 1, denoted by \(i(1, \beta)\). Parry showed that the set of sequence with belong to \(\Sigma_\beta\) can be characterised as
\[
\omega \in \Sigma_\beta \iff f^k(\omega) \leq i(1, \beta) \text{ for all } k \geq 1,
\]
where \(\leq\) is taken in the lexicographic ordering [42]. By the definition of \(\Sigma_\beta\) above, \(\Sigma_{\beta_1} \subseteq \Sigma_{\beta_2}\) for \(\beta_1 < \beta_2\) [42].

From [53], \(\{\beta \in (1, +\infty) : (\Sigma_\beta, \sigma)\) has the shadowing property\} is dense in \((1, +\infty)\). For every \(\beta_1 > 1\), there exists \(1 < \beta_2 < \beta_1\) such that and \((\Sigma_{\beta_2}, \sigma)\) satisfies the shadowing property. By [27, Proposition 6.4] every subshift with shadowing property satisfies exponential shadowing property. So \((\Sigma_{\beta_2}, \sigma)\) has exponential shadowing property. By definition every β-shift is positively expansive and mixing. So we have the following.

Theorem 5.4. The results of Theorem 5.1 hold for every (one-side) β-shift.

Remark 5.5. Theorem 5.4 is also true for every (two-side) β-shift.

5.2. Homoclinic Classes and Hyperbolic Ergodic Measures.

5.2.1. Homoclinic Classes. Let \(M\) be a compact Riemannian manifold \(M\), and \(\text{Diff}^1(M)\) denote the space of \(C^1\) diffeomorphisms on \(M\). Given \(f \in \text{Diff}^1(M)\), we recall that the homoclinic class of a hyperbolic saddle \(p\), denoted by \(H(p)\), is the closure of the set of hyperbolic saddles \(q\) homoclinically related to \(p\) (the stable manifold of the orbit of \(q\) transversely meets the unstable one of the orbit of \(p\) and vice versa). In this subsection we consider homoclinic class \(H(p)\) in which there is a hyperbolic periodic point \(q\) such that \(q\) is homoclinic related to \(p\) and the periods of \(p\) and \(q\) are coprime. Since \(p_1\) is homoclinic related to \(p_2\), there is \(\Lambda \subseteq H(p)\) such that \((\Lambda, f)\) a transitive locally maximal hyperbolic set that contains \(p\) and \(q\). By Theorem 2.17, \((\Lambda, f)\) is expansive and has the exponential shadowing property. Next we show that it is mixing.

Proposition 5.6. Suppose that a dynamical system \((X, f)\) is transitive and has the shadowing property. If there exist two periodic points \(p_1\) and \(p_2\) such that \(n_1\) and \(n_2\) are coprime where \(n_i\) is the period of \(p_i\) for \(i = 1, 2\), then \((X, f)\) is mixing.
Proof. Since $n_1$ and $n_2$ are coprime, there exist two integers $m_1$ and $m_2$ such that $m_1n_1 + m_2n_2 = 1$. We assume that $m_2 > 0$, then for any $n ≥ |m_1|n_1^2$, $n$ is expressed as $n = mn_1 + l$ where $m ≥ |m_1|n_1$ and $0 ≤ l < n_1$ are integers. Then one has

$$n = mn_1 + l = mn_1 + (m_1n_1 + m_2n_2) = (m + lm_1)n_1 + lm_2n_2$$

and $m + l n_1 ≥ |m_1|n_1 + l m_1 > 0$. So for any $n ≥ |m_1|^2$ there exist nonnegative $l_1$ and $l_2$ such that $n = l_1 n_1 + l_2 n_2$.

For any nonempty open sets $U$ and $V$ in $X$, there exist $x_1$, $x_2$ and $ε > 0$ such that $B(x_1, 2ε) ⊂ U$ and $B(x_2, 2ε) ⊂ V$. Let $δ > 0$ be provided for the $ε$ by shadowing property. Let $0 < δ_0 < \min\{δ, ε\}$. Since $(X, f)$ is transitive, there exist $y_1 ∈ B(x_1, δ_0)$, $y_2 ∈ B(p_1, δ_0)$, $y_3 ∈ B(p_2, δ_0)$ and $s_1$, $s_2$, $s_3 ∈ \mathbb{N}^+$ such that $f^{s_1}(y_1) ∈ B(p_1, δ_0)$, $f^{s_2}(y_2) ∈ B(p_2, δ_0)$ and $f^{s_3}(y_3) ∈ B(x_2, δ_0)$. For any $n ≥ |m_1|^2$ there exist nonnegative $l_1$ and $l_2$ such that $n = l_1 n_1 + l_2 n_2$. For the $n$ we define $C$ by

$$C^n = orb(y_1, s_1)orb(p_1, l_1 n_1)orb(y_2, s_2)orb(p_2, l_2 n_2)orb(y_3, s_3 + 1).$$

Then $C^n$ is a $δ$-chain connecting $x_1$ and $x_2$. By shadowing property there exists $z ∈ X$ such that $C^n$ is $ε$-traced by $z$. Then we have $z ∈ B(x_1, 2ε) ⊂ U$ and $f^{s_1 + l_1 n_1 + s_2 + l_2 n_2 + s_3}(z) ∈ B(x_2, 2ε) ⊂ V$. So $U ∩ f^{-t}V ≠ \emptyset$ for any $t ≥ s_1 + s_2 + s_3 + |m_1|^2$, i.e., $(X, f)$ is mixing. □

Thus $(Λ, f)$ is expansive, mixing and has the exponential shadowing property. So we have the following.

**Theorem 5.7.** For every homoclinic class $H(p)$ in which there is a hyperbolic periodic point $q$ such that $q$ is homoclinic related to $p$ and the periods of $p$ and $q$ are coprime, the results of Theorem 5.1 hold.

5.2.2. **Hyperbolic Ergodic Measures.** Given $f ∈ Diff^1(M)$, we recall that an ergodic $f$-invariant Borel probability measure is said to be hyperbolic if it has positive and negative but no zero Lyapunov exponents, In [29], for any ergodic and non-atomic hyperbolic measure $μ$ of a $C^{1+α}$ diffeomorphism the author finds a hyperbolic periodic point $p$ such that $S_μ$ is contained in the closure of the transverse homoclinic points of $p$, i.e., $H(p)$. Following Katok’s idea, in [36] for any weakly mixing hyperbolic measure $μ$ of a $C^{1+α}$ diffeomorphism G. Liao et al find a hyperbolic periodic point $p$ and a mixing locally maximal hyperbolic set $Λ$ such that $S_μ ⊂ H(p)$ and $p ∈ Λ ⊂ H(p)$. Thus we can state a similar result as a corollary of Theorem 5.7 for hyperbolic weakly mixing measures.

**Corollary 5.8.** Let $f : M → M$ be a diffeomorphism on a compact Riemannian manifold. Assume that $f$ is $C^{1+α}$ and there is a weakly mixing hyperbolic measure. Then the results of Theorem 5.1 hold.

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