Spin-spin interaction and spin-squeezing in an optical lattice

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We show that by displacing two optical lattices with respect to each other, we may produce interactions similar to the ones describing ferro-magnetism in condensed matter physics. We also show that particularly simple choices of the interaction lead to spin-squeezing, which may be used to improve the sensitivity of atomic clocks. Spin-squeezing is generated even with partially, and randomly, filled lattices, and our proposal may be implemented with current technology.

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Simulation of quantum many-body problems on a classical computer is difficult because the size of the Hilbert space grows exponentially with the number of particles. As suggested by Feynman (1) the growth in computational requirements is only linear on a quantum computer (2), which is itself a quantum many-body system, and such a computer containing only a few tens of quantum bits may outperform a classical computer. A quantum computer aimed at the solution of a quantum problem is a general purpose quantum computer in an optical lattice.

In this Letter we describe how atoms in an optical lattice may be manipulated to simulate spin-spin interactions which are used to describe ferro-magnetism in condensed matter physics. We also show that with a specific choice of interaction we may generate spin squeezed states (3) which may be used to enhance spectroscopic resolution (4), e.g., in atomic clocks.

In Refs. (3, 4) two different methods to perform a coherent evolution of the joint state of pairs of atoms in an optical lattice were proposed. Both methods involve displacement of two identical optical lattices with respect to each other. Each lattice traps one of the two internal states |0⟩ and |1⟩ of the atoms. Initially, the atoms are in the same internal state |0⟩, the two lattices are on top of each other and the atoms are assumed to be cooled to the vibrational ground state in the lattice. Using a resonant pulse the atoms may be prepared in any superposition of the two internal states. The lattice containing the |1⟩ component of the wavefunction is now displaced so that if an atom (at the lattice site k) is in |1⟩, it is transferred to the vicinity of the neighbouring atom (at the lattice site k + 1) if this is in |0⟩, causing an interaction between the two atoms. See Fig. 1. The procedures described in this Letter follow the proposal in Ref. (3) where, the atoms interact through controlled collisions. Also the optically induced dipole-dipole interactions proposed in (3) may be adjusted to fit into this framework. After the interaction, the lattices are returned to their initial position and the internal states of each atom may again be subject to single particle unitary evolution. The total effect of the displacement and the interaction with the neighbour can be tailored to yield a certain phaseshift φ on the |1⟩k|0⟩k+1 component of the wavefunction, i.e.,

|0⟩k|0⟩k+1 → |0⟩k|0⟩k+1 + |0⟩k|1⟩k+1
|1⟩k|0⟩k+1 → e^{iφ}|1⟩k|0⟩k+1 + |1⟩k|1⟩k+1 → |1⟩k|1⟩k+1, (1)

where |a⟩k (a = 0 or 1) refers to the state of the atom at the k'th lattice site. In (3) it is suggested to build a general purpose quantum computer in an optical lattice. Such a general computer requires two-atom gates, which may be accomplished through the dynamics in (3) and single atom control, which is possible by directing a laser beam on each atom. We shall show that even without allowing access to the individual atoms, the lattice may be used to perform a highly non-trivial computational task: Simulation of a ferro-magnet.

Our two level quantum systems conveniently describe spin 1/2 particles with the two states |0⟩k and |1⟩k representing |jm⟩k = |1/2, − 1/2⟩k and |1/2, 1/2⟩k, where states |jm⟩k are eigenstates of the jz,k-operator jz,k|jm⟩k = mj|jm⟩k (ℏ = 1). The phase-shifted component of the wavefunction in Eq. (1) may be isolated by applying the operator (jz,k + 1/2)(jz,k+1 − 1/2), and the total evolution composed of the lattice translations and the interaction induced phaseshift may be described by the unitary operator e^{tHt} with Hamiltonian

\[ H = \chi (j_{z,k} + 1/2)(j_{z,k+1} - 1/2) \]

and time t = φ/χ. In a filled lattice the evolution is described by the Hamiltonian

\[ H = \chi \sum_{k,l} j_{z,k} + 1/2(j_{z,k+1} - 1/2), \]

and if we are only interested in the bulk behaviour of the atoms we may apply periodic boundary conditions, so that the Hamiltonian reduces to

\[ H_{zz} = \chi \sum_{<k,l>} j_{z,k}j_{z,l}, \]

where the sum is over nearest neighbours. By appropriately displacing the lattice we may extend the sum to nearest neighbours in two and three dimensions. \( H_{zz} \)
coincides with the celebrated Ising-model Hamiltonian introduced to describe ferro-magnetism. Hence, by elementary lattice displacements we perform a quantum simulation of a ferro-magnet.

A more general Hamiltonian of the type

$$H_f = \sum_{<k,l>} \chi jz,k,jz,l + \eta jx,k,jx,l + \chi jy,k,jy,l$$

may be engineered using multiple resonant pulses and displacements of the lattices: A resonant $\pi/2$-pulse acting simultaneously on all atoms rotates the $j_x$-operators into $j_x$-operators, $\exp(i\eta_{k,l} \pi/2 j_z,k e^{-i\eta_{k,l} \pi/2} = j_z,k$. Hence, by applying $\pi/2$-pulses, in conjunction with the displacement sequence, we turn $H_{zz}$ into $H_{xx}$, the second term in Eq. (3). Similarly we may produce $H_{yy}$, the third term in Eq. (3), and by adjusting the duration of the interaction with the neighbours we may adjust the coefficients $\chi$, $\eta$ and $\lambda$ to any values. We cannot, however, produce $H_f$ by simply applying $H_{xz}$ for the desired time $t$, followed by $H_{xx}$ and $H_{yy}$, because the different Hamiltonians do not commute. Instead we apply a physical implementation of a well-known numerical scheme: The split operator technique. If we choose short time steps, i.e., small phaseshifts $\phi$ in Eq. (1), the error will only be of order $\phi^2$, and by repeated application of $H_{xz}$, $H_{xx}$ and $H_{yy}$ we may stroboscopically approximate $H_f$.

For a few atoms the system may be simulated numerically on a classical computer. In Fig. 2 we show the propagation of a spin wave in a one-dimensional string of 15 atoms which are initially in the $|1/2, -1/2\rangle$ state. The central atom is flipped at $t = 0$ and a spin wave propagates to the left and right. The figure shows the evolution of $\langle j_z,k \rangle$ for all atoms, obtained by repeatedly applying the Hamiltonians $H_{xz}$, $H_{xx}$ and $H_{yy}$ with $\chi = \eta = \lambda$ and periodic boundary conditions. Small time steps $dt \approx 0.1 \chi^{-1}$ result in a stroboscopic approximation almost indistinguishable from the results of a direct numerical integration of $H_f$.

A host of magnetic phenomena may be simulated on our optical lattice: Solitons, topological excitations, two magnon bound states, etc. By pumping a fraction of the atoms into the $|1/2, 1/2\rangle$ state, we may also perform micro-canonical ensemble calculations for non-vanishing temperature. Other procedures for introducing a non-vanishing temperature are described in Ref. [3].

We now show how to generate spin squeezed states using the same techniques as discussed above. Signals obtained in spectroscopic investigations of a sample of two level atoms are expressed by the collective spin operators $J_i = \sum_k j_{k,i}$, and their quantum mechanical uncertainty limits the measurement accuracy, and e.g., the performance of atomic clocks. In standard spectroscopy with $N$ uncorrelated atoms starting in the $|1/2, -1/2\rangle$ state, the uncertainties $\Delta J_x = \sqrt{\langle J_x^2 \rangle - \langle J_x \rangle^2}$ and $\Delta J_y$ are identical, and the standard quantum limit resulting from the uncertainty relation of angular momentum operators

$$(\Delta J_x)^2 (\Delta J_y)^2 \geq |\langle J_z/2 \rangle|^2$$

predicts a spectroscopic sensitivity proportional to $1/\sqrt{N}$. Polarization rotation spectroscopy and high precision atomic fountain clocks are now limited by this sensitivity. In [4] it is suggested to produce spin squeezed states which redistribute the uncertainty unevenly between components like $J_x$ and $J_y$, so that measurements, sensitive to the component with reduced uncertainty, become more precise. Spin squeezing resulting from absorption of non-classical light has been suggested [2] and demonstrated experimentally [3]. Ref. [9] presents an analysis of squeezing obtained from the non-linear couplings $H = \chi J_x^2$ and $H = \chi (J_x^2 - J_y^2)$. For neutral atoms, such a coupling has been suggested in the spatial overlap of two components of a Bose-Einstein condensate [4]. Spin squeezing in an optical lattice has two main advantages compared to the condensates: The interaction can be turned on and off easily, and the localization at lattice sites increases the density and thus the interaction strength. The product of two collective spin operators involves terms $j_x,k,j_x,l$ for all atoms $k$ and $l$, and this coupling may be produced by displacing the lattices several times so that the $|1/2, 1/2\rangle$ component of each atom visits every lattice site and interacts with all other atoms. In a large lattice such multiple displacements are not desirable. We shall show, however, that substantial spin-squeezing occurs through interaction with only a few nearby atoms, i.e., for Hamiltonians

$$H = \sum_{k,l} \chi_{k,l} j_x,k,j_x,l$$

and

$$H = \sum_{k,l} \chi_{k,l} (j_x,k,j_x,l - j_y,k,j_y,l),$$

where the coupling constants $\chi_{k,l}$ between atoms $k$ and $l$ vanishes except for a small selection of displacements of the lattices.

Expectation values of relevant angular momentum operators and the variance of the spin operator $J_\theta = \cos(\theta) J_x + \sin(\theta) J_y$ may be calculated for an initially uncorrelated state with all atoms in $|1/2, -1/2\rangle$, propagated by the simple coupling (5). If each atom visits one neighbour $\chi_{k,l} = \chi_{k+1,l}$, we get the time dependent variance of the spin component $J_{-\pi/4} = \frac{1}{\sqrt{2}}(J_x - J_y)$

$$(\Delta J_{-\pi/4})^2 = \frac{N}{4} \left[ 1 + \frac{1}{4} \sin^2(\chi t) - \sin(\chi t) \right].$$

The mean spin vector is in the negative $z$ direction and has the expectation value
For small values of $\chi t$, $\Delta J_{-\pi/4}$ decreases linearly with $\chi t$ whereas $| < J_z > |$ decreases proportional to $(\chi t)^2$, hence $\Delta J_{-\pi/4}$ falls below $| < J_z > / 2 |$, and the spin is squeezed.

In Fig. 3 we show numerical results for 15 atoms in a one-dimensional lattice with periodic boundary conditions. Fig. 3 (a) shows the evolution of $(\Delta J_\theta)^2$ when we apply the coupling (5) and visit 1, 2, and 3 neighbours. The squeezing angle $\theta = -\pi/4$ is optimal for short times $\chi t << 1$. For longer times the optimal angle deviates from $-\pi/4$, and we plot the variance $(\Delta J_\theta)^2$ minimized with respect to the angle $\theta$. We assume the same phaseshift for all collisions, i.e., all non-vanishing $\chi_{k,l}$ are identical.

For spectroscopic investigations not only the variance of a spin component is relevant. In [8] it is shown that if spectroscopy is performed with $N$ particles, the reduction in the frequency variance due to squeezing is given by the quantity

$$
\xi^2 = \frac{N(\Delta J_\theta)^2}{(J_z)^2}.
$$

In Fig. 3 (b) we show the minimum value of $\xi^2$ obtained with the couplings (3) and (6) as functions of the number of neighbours visited. Fig. 3 (b) shows that the coupling (3) produces better squeezing than (6). The coupling (3), however, is more attractive from an experimental viewpoint. Firstly, all $j_{x,k}$ operators commute and we do not have to apply several displacements with infinitesimal durations to produce the desired Hamiltonian. We may simply displace the atoms so that they interact with one neighbour to produce the desired phaseshift $\phi$, and then go on to interact with another neighbour. Secondly, if the $j_{x,k}j_{x,l}$ coupling involves a phaseshift $\phi$, the operator $-j_y,kj_y,l$ requires the opposite phaseshift $-\phi$. This requires a long interaction producing $2\pi - \phi$, or a change of the interaction among the atoms, i.e., a change of the sign of the scattering length in the implementation of (8).

Like the analytic expression for $\chi^2$ obtained from (5), the results shown in Fig. 3 (b) are independent of the total number of atoms as long as this exceeds the “number of neighbours visited”. When all lattice sites are visited we approach the results obtained in Ref. [8], i.e., a variance scaling as $N^{1/3}$ and a constant for the couplings (3) and (6).

So far we have assumed that the lattice contains one atom at each lattice site and that all atoms are cooled to the vibrational ground state. The present experimental status is that atoms can be cooled to the vibrational ground-state, but with a filling factor below unity [13]. A mean filling factor of unity is reported in [17], but when at most a single atom is permitted at each lattice site a mean occupation of 0.44 is achieved. It has been suggested that a single atom per lattice site may be achieved by filling the lattice from a Bose-Einstein condensate [17].

To describe a partially filled lattice it is convenient to introduce stochastic variables $h_k$, describing whether the $k$'th lattice site is filled $h_k = 1$ or empty $h_k = 0$. The interaction may be described by the Hamiltonian

$$
H = \sum_{k,l} \chi_{k,l} h_k j_{x,k} h_l j_{x,l} + \frac{1}{2} h_1 (j_{x,1} - 1/2),
$$

where the sum is over all lattice sites $k$ and $l$. If we, rather than just displacing the atoms in one direction, also displace the lattice in the opposite direction, so that $\chi_{k,l}$ is symmetric in $k$ and $l$, we may produce the Hamiltonian

$$
H = \sum_{k,l} \chi_{k,l} h_k j_{x,k} h_l j_{x,l}.
$$

This Hamiltonian models ferromagnetism in random structures, and it might shed light on morphology properties, and, e.g., percolation [13]. Here we shall restrict our analysis to spin- squeezing aspects, since these are both of practical interest, and they represent an ideal experimental signature of the microscopic interaction.

In Fig. 4 we show the result of a simulation of squeezing in a partially filled one dimensional lattice. Each lattice site contains an atom with a probability $p$, and the size of the lattice is adjusted so that it contains 15 atoms. In Fig. 4 (a) we show the decrease in the variance of $J_\theta$, averaged over 20 realizations and minimized with respect to $\theta$. Lines indicate the predictions from the time derivatives at $t = 0$

$$
\frac{d}{dt} (\Delta J_{-\pi/4})^2 = -\frac{1}{2} \sum_{k,l} \chi_{k,l} < h_k h_l >
$$

where $< h_k h_l >$ denotes the ensemble average over the distribution of atoms in the lattice, i.e., the two atom correlation function. In Fig. 4 (b) we show the minimum value of $\xi^2$ for different filling factors $p$ as a function of the number of neighbours visited. The calculations confirm that even in dilute lattices, considerable squeezing may be achieved by visiting a few neighbours.

In conclusion we have suggested a method to simulate condensed matter physics in an optical lattice, and we have shown how the dynamics may be employed to produce spin-squeezing. We emphasize the moderate experimental requirement for our scheme. With the two internal states represented as hyperfine structure states in alkaline atoms, all spin rotations may be performed by Raman or RF-pulses acting on all atoms simultaneously, and lattice displacements may be performed by simply rotating the polarisation of the lasers [7]. With the parameters in [13], the duration of the sequence in Fig. 4 can be as low as a few micro-seconds. Following our suggestion spin-squeezing may be produced in dilute optical lattices, and implementation is possible with current...
technology. The resulting macroscopic decrease in projection noise has several promising applications in technology and quantum physics, and it provides an experimental signature of the microscopic interaction between the atoms.

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FIG. 1. (a) Two overlapping lattices trapping the two internal states |0⟩ (black circle) and |1⟩ (white circle). By resonant laser pulses the atoms can be prepared in any superposition of the two internal states. (b) The lattices are displaced so that if an atom is in the |1⟩ state, it is moved close to the neighboring atom if this is in |0⟩ causing an interaction between the two atoms. (c) The lattices are returned to their initial position and the atoms may be prepared in new superpositions by resonant pulses.

FIG. 2. Propagation of a spin wave in a one dimensional string. The central atom is flipped at t = 0, and repeated application of Hzz, Hxx and Hyy results in a wave propagating to the left and right. The figure shows the evolution of <j_{z,k}> for all atoms (k).

FIG. 3. Squeezing in a one-dimensional lattice with 15 atoms. (a) Evolution of (ΔJθ)2 with the coupling (5) and interaction with 1, 2, and 3 neighbours (full, dashed, and short dashed line, respectively). (b) The optimal value of the squeezing parameter ξ2 obtained with the coupling (5) and (6). Lines are shown to guide the eye.

FIG. 4. Spin squeezing in a partially filled one dimensional lattice containing 15 atoms. (a) Evolution of (ΔJθ)2 in a lattice with a filling factor p = 50% and displacements to 1, 2, and 3 neighbouring sites (full, dashed, and short dashed curve respectively). Dotted lines represent the predictions from Eq. (11). (b) Minimum attainable squeezing parameter ξ2 for filling factors p=100% (∗), 50% (+), 25% (□), and 10% (×) as functions of the number of sites visited.