On the weak Roman domination number of lexicographic product graphs

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Abstract

A vertex v of a graph G = (V, E) is said to be undefended with respect to a function f : V → {0, 1, 2} if f(v) = 0 and f(u) = 0 for every vertex u adjacent to v. We call the function f a weak Roman dominating function if for every v such that f(v) = 0 there exists a vertex u adjacent to v such that f(u) ∈ {1, 2} and the function f′ : V → {0, 1, 2} defined by f′(v) = 1, f′(u) = f(u) − 1 and f′(z) = f(z) for every z ∈ V \ {u, v}, has no undefended vertices. The weight of f is w(f) = ∑v∈V(G)f(v). The weak Roman domination number of a graph G, denoted by γ_r(G), is the minimum weight among all weak Roman dominating functions on G. Henning and Hedetniemi [Discrete Math. 266 (2003) 239-251] showed that the problem of computing γ_r(G) is NP-Hard, even when restricted to bipartite or chordal graphs. This suggests finding γ_r(G) for special classes of graphs or obtaining good bounds on this invariant. In this article, we obtain closed formulae and tight bounds for the weak Roman domination number of lexicographic product graphs in terms of invariants of the factor graphs involved in the product.

Keywords: Domination number; weak Roman domination number; domination in graphs; lexicographic product.

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1 Introduction

Cockaine et al. [6] defined a Roman dominating function (RDF) on a graph G to be a function f : V(G) → {0, 1, 2} satisfying the condition that every vertex u for which f(u) = 0 is adjacent to at least one vertex v for which f(v) = 2. The weight of f is w(f) = ∑v∈V(G)f(v), and for X ⊆ V(G) we define the weight of X
to be \( f(X) = \sum_{v \in X} f(v) \). The Roman domination number, denoted by \( \gamma_R(G) \), is the minimum weight among all Roman dominating functions on \( G \), i.e.,

\[
\gamma_R(G) = \min \{ w(f) : f \text{ is a RDF on } G \}.
\]

The Roman domination theory was motivated by an article by Ian Stewart entitled “Defend the Roman Empire!” [19]. Each vertex in our graph represents a location in the Roman Empire and the value \( f(v) \) corresponds to the number of legions stationed in \( v \). A location \( v \) is unsecured if no legions are stationed there (i.e., \( f(v) = 0 \)) and secured otherwise (i.e., \( f(v) \in \{1, 2\} \)). An unsecured location \( u \) can be secured by sending a legion to \( u \) from an adjacent location \( v \). But a legion cannot be sent from a location \( v \) if doing so leaves that location unsecured (i.e., if \( f(v) = 1 \)). Thus, two legions must be stationed at a location \( (f(v) = 2) \) before one of the legions can be sent to an adjacent location. A RDF of weight \( \gamma_R(G) \) corresponds to such an optimal assignment of legions to locations.

Henning and Hedetniemi [13] explored the potential of saving substantial costs of maintaining legions, while still defending the Roman Empire (from a single attack). They proposed the use of weak Roman dominating functions (WRDF) as follows. Using the terminology introduced earlier, they defined a location to be undefended if the location and every location adjacent to it are unsecured (i.e., have no legion stationed there). Since an undefended location is vulnerable to an attack, we require that every unsecured location be adjacent to a secure location in such a way that the movement of a legion from the secure location to the unsecured location does not create an undefended location. Hence every unsecured location can be defended without creating an undefended location. Such a placement of legions corresponds to a WRDF and a minimum such placement of legions corresponds to a minimum WRDF. More formally, a vertex \( v \in V(G) \) is undefended with respect to a function \( f : V(G) \rightarrow \{0, 1, 2\} \) if \( f(v) = 0 \) and \( f(u) = 0 \) for every vertex \( u \) adjacent to \( v \). We call the function \( f \) a WRDF if for every \( v \) such that \( f(v) = 0 \) there exists a vertex \( u \) adjacent to \( v \) such that \( f(u) \in \{1, 2\} \) and the function \( f' : V(G) \rightarrow \{0, 1, 2\} \) defined by \( f'(v) = 1, f'(u) = f(u) - 1 \) and \( f'(z) = f(z) \) for every \( z \in V(G) \setminus \{u, v\} \), has no undefended vertices.

Let \( f \) be a WRDF on \( G \) and let \( V_0, V_1 \) and \( V_2 \) be the subsets of vertices assigned the values 0, 1, and 2, respectively, under \( f \). Notice that there is a one-to-one correspondence between the set of weak Roman dominating functions \( f \) and the set of ordered partitions \( (V_0, V_1, V_2) \) of \( V(G) \). Thus, in order to specify the partition of \( V(G) \) associated to \( f \), the function \( f \) will be denoted by \( f(V_0, V_1, V_2) \).

The weak Roman domination number, denoted by \( \gamma_r(G) \), is the minimum weight among all weak Roman dominating functions on \( G \), i.e.,

\[
\gamma_r(G) = \min \{ w(f) : f \text{ is a WRDF on } G \}.
\]

A WRDF of weight \( \gamma_r(G) \) is called a \( \gamma_r(G) \)-function. For instance, for the tree shown in Figure 1 a \( \gamma_r(G) \)-function can place 2 legions in the vertex of degree three and one legion in the other black-coloured vertex. Notice that \( \gamma_r(G) = 3 < 4 = \gamma_R(G) \).

It was shown in [13] that the problem of computing \( \gamma_r(G) \) is NP-hard, even when restricted to bipartite or chordal graphs. This suggests finding the weak Roman
Figure 1: Two placements of legions which correspond to two different weak Roman dominating functions on the same tree.

domination number for special classes of graphs or obtaining good bounds on this invariant. In this paper we develop the theory of weak Roman domination in lexicographic product graphs.

The remainder of the paper is structured as follows. Section 2 covers basic results on the weak Roman domination number of a graph. The case of lexicographic product graphs is studied in Sections 3-5. Specifically, Section 3 covers general bounds, Section 4 covers closed formulae, while Section 5 contains the proof of Theorem 36, which is very long. Finally, in Section 6 we collect some open problems derived from this work.

Throughout the paper, we will use the notation $K_n$, $K_{1,n-1}$, $C_n$, $N_n$ and $P_n$ for complete graphs, star graphs, cycle graphs, empty graphs and path graphs of order $n$, respectively. We use the notation $u \sim v$ if $u$ and $v$ are adjacent vertices, and $G \cong H$ if $G$ and $H$ are isomorphic graphs. For a vertex $v$ of a graph $G$, $N(v)$ will denote the set of neighbours or open neighbourhood of $v$ in $G$, i.e. $N(v) = \{u \in V(G) : u \sim v\}$. The closed neighbourhood, denoted by $N[v]$, equals $N(v) \cup \{v\}$. We denote by $\delta(v) = |N(v)|$ the degree of vertex $v$, as well as $\delta = \min_{v \in V(G)} \{\delta(v)\}$ and $\Delta = \max_{v \in V(G)} \{\delta(v)\}$. The subgraph of $G$ induced by a set $S$ of vertices is denoted by $\langle S \rangle$.

For the remainder of the paper, definitions will be introduced whenever a concept is needed.

## 2 Some remarks on $\gamma_r(G)$

In this section we will discuss some basic but useful remarks on the weak Roman domination number of a graph. For nonconnected graphs we have the following remark.

**Remark 1.** For any graph $G$ of $k$ components, $G_1, G_2, \ldots, G_k$,

$$\gamma_r(G) = \sum_{i=1}^{k} \gamma_r(G_k).$$

According to the remark above, we can restrict ourselves to the case of connected graphs.

Recall that a set $D \subseteq V(G)$ is dominating in $G$ if every vertex in $V(G) \setminus D$ has at least one neighbour in $D$, i.e., $N(u) \cap D \neq \emptyset$ for every $u \in V(G) \setminus D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in $G$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The reader is referred to the books [11, 12] for details on domination in graphs.
Remark 2. [13] For any graph $G$,

$$\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G).$$

Graphs with $\gamma_R(G) = 2\gamma(G)$ are called Roman graphs [6]. We say that $G$ is a weak Roman graph if $\gamma_r(G) = 2\gamma(G)$. Notice that any weak Roman graph is a Roman graph. In general, the converse does not hold. For instance, the graph shown in Figure 1 is a Roman graph, as $\gamma_R(G) = 2\gamma(G) = 4$, while $\gamma_r(G) = 3$.

A support vertex of a tree is a vertex adjacent to a leaf, while a strong support vertex is a support vertex that is adjacent to more than one leaf.

**Lemma 3.** [13] If $T$ is a tree with a unique $\gamma(T)$-set $S$, and if every vertex in $S$ is a strong support vertex, then $T$ is a weak Roman tree.

The reader is refereed to [13] for a complete characterization of all weak Roman forests.

**Remark 4.** Let $G$ be a graph of order $n$. Then $\gamma_r(G) = 1$ if and only if $G \cong K_n$.

According to this remark, for any noncomplete graph $G$ we have that $\gamma_r(G) \geq 2$. Before discussing the limit case of this trivial bound, we need to introduce some additional notation and terminology. A set $S \subseteq V(G)$ is a secure dominating set of $G$ if $S$ is a dominating set and for every $v \in V(G) \setminus S$ there exists $u \in S \cap N(v)$ such that $(S \setminus \{u\}) \cup \{v\}$ is a dominating set [4]. The secure domination number, denoted by $\gamma_s(G)$, is the minimum cardinality among all secure dominating sets. As observed in [4], since for any secure dominating set $S$ we can construct a weak Roman dominating function $f(W_0, W_1, W_2)$, where $W_0 = V(G) \setminus S$, $W_1 = S$ and $W_2 = \emptyset$, we have that $\gamma_s(G) \geq \gamma_r(G)$.

**Remark 5.** Let $G$ be a noncomplete graph. The following statements are equivalent.

(i) $\gamma_r(G) = 2$.

(ii) $\gamma(G) = 1$ or $\gamma_s(G) = 2$.

**Proof.** If $\gamma_r(G) = 2$, then any $\gamma_r(G)$-function $f(X_0, X_1, X_2)$ has weight $w(f) = |X_1| + 2|X_2| = 2$, which implies that either $X_1 = \emptyset$ and $|X_2| = 1$ or $|X_1| = 2$ and $X_2 = \emptyset$. In the first case we have $\gamma(G) = 1$, and in the second one $X_1$ is a secure dominating set, which implies that $2 = \gamma_r(G) \leq \gamma_s(G) \leq |X_1| = 2$. Therefore, from (i) we deduce (ii).

Now, since $G$ is not a complete graph, $\gamma_r(G) \geq 2$. Obviously, if $\gamma(G) = 1$, then $\gamma_r(G) = 2$. On the other hand, if there exists a secure dominating set of cardinality two, then $2 \leq \gamma_r(G) \leq \gamma_s(G) \leq 2$. Therefore, from (ii) we deduce (i).

Given a graph $G$ and an edge $e \in E(G)$, the graph obtained from $G$ by removing the edge $e$ will be denoted by $G - e$, i.e., $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. Since any $\gamma_r(G - e)$-function is a WRDF for $G$, we deduce the following basic result.
Remark 6. [13] For any spanning subgraph $H$ of a graph $G$, 
$$\gamma_r(G) \leq \gamma_r(H).$$

Proposition 7. [13] For any $n \geq 4$, 
$$\gamma_r(C_n) = \gamma_r(P_n) = \left\lceil \frac{3n}{7} \right\rceil.$$

By Remark 6 and Proposition 7 we deduce the following result.

Theorem 8. For any Hamiltonian graph $G$ of order $n \geq 4$, 
$$\gamma_r(G) \leq \left\lceil \frac{3n}{7} \right\rceil.$$

Obviously, the bound above is tight, as it is achieved for $G \cong C_n$.

3 Preliminary results on lexicographic product graphs

Let $G$ and $H$ be two graphs. The lexicographic product of $G$ and $H$ is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \circ H)$ if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$. Notice that for any $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to $H$. For simplicity, we will denote this subgraph by $H_u$, and if a vertex of $G$ is denoted by $u_i$, then the referred subgraph will be denoted by $H_i$. For any $u \in V(G)$ and any WRDF $f$ on $G \circ H$ we define 
$$f(H_u) = \sum_{v \in V(H)} f(u, v)$$
and 
$$f[H_u] = \sum_{x \in N[u]} f(H_x).$$

Remark 9. Let $G$ and $H$ be two graphs. The following assertions hold.

- $G \circ H$ is connected if and only if $G$ is connected.
- If $G = G_1 \cup \ldots \cup G_t$, then $G \circ H = (G_1 \circ H) \cup \ldots \cup (G_t \circ H)$.

From Remarks 1 and 9 we deduce the following result.

Remark 10. For any graph $G$ of components $G_1, G_2, \ldots, G_k$ and any graph $H$, 
$$\gamma_r(G \circ H) = \sum_{i=1}^{k} \gamma_r(G_i \circ H).$$

For basic properties of the lexicographic product of two graphs we suggest the books [10, 15]. A main problem in the study of product of graphs consists of finding exact values or sharp bounds for specific parameters of the product of two graphs and express these in terms of invariants of the factor graphs. In particular, we cite the following works on domination theory of lexicographic product graphs. For instance,
the domination number was studied in [16, 18], the Roman domination number was studied in [20], the rainbow domination number was studied in [21], the super domination number was studied in [7], while the doubly connected domination number was studied in [1].

To begin our analysis we would point out the following result, which is a direct consequence of Remark 6.

**Remark 11.** Let $G$ be a connected graph of order $n$ and let $H$ be a nonempty graph. For any spanning subgraph $G_1$ of $G$,

$$
\gamma_r(K_n \circ H) \leq \gamma_r(G \circ H) \leq \gamma_r(G_1 \circ H).
$$

In particular, if $G$ is a Hamiltonian graph, then

$$
\gamma_r(G \circ H) \leq \gamma_t(C_n \circ H).
$$

### 3.1 Upper bounds on $\gamma_r(G \circ H)$

A total dominating set of a graph $G$ with no isolated vertex is a set $S$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the cardinality of a smallest total dominating set, and we refer to such a set as a $\gamma_t(G)$-set. Notice that for any graph $G$ with no isolated vertex,

$$
\gamma_r(G) \leq 2\gamma_t(G) \leq 2\gamma_t(G). \tag{1}
$$

The reader is referred to the book [14] for details on total domination in graphs. This book provides and explores the fundamentals of total domination in graphs.

Notice that if $D$ is a total dominating set of $G$ and $h \in V(H)$, then $D \times \{h\}$ is a dominating set of $G \circ H$, so that $\gamma_t(G \circ H) \leq \gamma_t(G)$. Hence, from the first inequality in chain (1) we deduce the following theorem, which can also be derived from the inequality $\gamma_r(G \circ H) \leq 2\gamma_t(G)$ observed in [20] and the second inequality in Remark 2.

**Theorem 12.** If $G$ is a graph with no isolated vertex, then for any graph $H$,

$$
\gamma_r(G \circ H) \leq 2\gamma_t(G).
$$

The total domination number of a path is known and is easy to compute. For every integer $n \geq 3$ we have $\gamma_t(P_n) = \lfloor n/2 \rfloor + \lfloor n/4 \rfloor - \lfloor n/4 \rfloor$. We will show in Corollary 38 that if $\gamma(H) \geq 4$, then $\gamma_r(P_n \circ H) = 2\gamma_t(P_n)$. Thus, the bound above is tight. Furthermore, as we will show in Proposition 33, if $n \geq 3$ and $\gamma(H) \geq 4$, then $\gamma_r(K_{1,n} \circ H) = 4 = 2\gamma_t(K_{1,n})$. Notice that $K_{1,n}$ is a graph of diameter two and minimum degree $\delta = 1$. In general, for any graph $G$ of diameter two and minimum degree $\delta$, the total domination number of $G$ is bounded above by $\delta + 1$. Moreover, if $G$ is a graph of order $n$ with no isolated vertex and maximum degree $\Delta \geq n - 2$, then $\gamma_t(G) = 2$. Therefore, the following result is a direct consequence of Theorem 12.
Corollary 13. The following assertions hold for any graph $H$.

- If $G$ is a graph of order $n$ with no isolated vertex and maximum degree $\Delta \geq n-2$, then $\gamma_t(G \circ H) \leq 4$.
- If $G$ has diameter two and minimum degree $\delta$, then $\gamma_t(G \circ H) \leq 2(\delta + 1)$.

It was shown in [5] that for any connected graph of order $n \geq 3$, $\gamma_t(G) \leq \frac{2}{3}n$. Hence, Theorem 12 leads to the following result.

Corollary 14. For any connected graph $G$ of order $n \geq 3$ and any graph $H$,

$$\gamma_t(G \circ H) \leq 2 \left\lfloor \frac{2n}{3} \right\rfloor.$$ 

In Proposition 35 we will show that the bound above is tight.

Chellali and Haynes [3] established that the total domination number of a tree $T$ of order $n \geq 3$ is bounded above by $(n + s)/2$, where $s$ is the number of support vertices of $T$. Therefore, Theorem 12 leads to the following corollary.

Corollary 15. For any graph $H$ and any tree $T$ of order $n \geq 3$ having $s$ support vertices,

$$\gamma_t(T \circ H) \leq n + s.$$ 

The bound above is tight. For instance, Proposition 35 shows that for any $n = 3k$ and any graph $H$ with $\gamma(H) \geq 4$, $\gamma_t(T_n \circ H) = n + s = 4k$, where $T_n$ is a comb defined prior to Proposition 35.

As stated by Goddard and Henning [9], if $G$ is a planar graph with diameter two, then $\gamma_t(G) \leq 3$. Hence, as an immediate consequence of Theorem 12, we have the following result.

![Figure 2: A planar graph of diameter two.](image)

Corollary 16. If $G$ is a planar graph of diameter two, then for any graph $H$,

$$\gamma_t(G \circ H) \leq 6.$$ 

The bound above is achieved, for instance, for the planar graph $G$ shown in Figure 2 and any graph $H$ with $\gamma(H) \geq 4$. An optimum placement of legions in $G \circ H$ can be done by assigning two legions to the copies of $H$ corresponding to the gray-coloured vertices of $G$. 

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Corollary 17. For any graph $G$ with no isolated vertex and any noncomplete graph $H$,
\[ \gamma_r(G \circ H) \leq 4\gamma(G). \]

Proof. It is well known that for every graph $G$ with no isolated vertex, $\gamma(G) \leq 2\gamma(G)$ (see, for instance, [2]). Hence, by Theorem 12 we have $\gamma_r(G \circ H) \leq 4\gamma(G)$. Therefore, the result follows. \hfill \Box

The bound $\gamma_r(G \circ H) \leq 4\gamma(G)$ is achieved, for instance, for graphs $G$ and $H$ satisfying the assumptions of Theorem 29.

Theorem 18. For any graph $G$ and any noncomplete graph $H$,
\[ \gamma_r(G \circ H) \leq \gamma(G)\gamma_r(H). \]

Proof. Let $f_1(V_0, V_1, V_2)$ be a $\gamma_r(H)$-function and $X$ a $\gamma(G)$-set. Notice that $\gamma_r(H) \geq 2$, as $H$ is not complete. It is readily seen that $f(W_0, W_1, W_2)$ defined by $W_1 = X \times V_1$ and $W_2 = X \times V_2$ is a WRDF of $G \circ H$. Hence,
\[ \gamma_r(G \circ H) \leq |X \times V_1| + 2|X \times V_2| = |X||V_1| + 2|V_2| = \gamma(G)\gamma_r(H). \]

Therefore, the result follows. \hfill \Box

The bound $\gamma_r(G \circ H) \leq \gamma(G)\gamma_r(H)$ is achieved, for instance, for any comb graph $T_{3k}$ defined prior to Proposition 35 and any graph $H$ with $\gamma_r(H) = 4$. Besides, the bound is attained for any $G$ and $H$ satisfying the assumptions of Theorem 27.

A double total dominating set of a graph $G$ with minimum degree greater than or equal to two is a set $S$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least two vertices in $S$, [14]. The double total domination number of $G$, denoted by $\gamma_{2,t}(G)$, is the cardinality of a smallest double total dominating set, and we refer to such a set as a $\gamma_{2,t}(G)$-set.

Theorem 19. Let $G$ be a graph of minimum degree greater than or equal to two. The following assertions hold.

(i) $\gamma_r(G) \leq \gamma_{2,t}(G)$.

(ii) For any graph $H$, $\gamma_{2,t}(G \circ H) \leq \gamma_{2,t}(G)$.

(iii) For any graph $H$, $\gamma_r(G \circ H) \leq \gamma_{2,t}(G)$.

Proof. For every $\gamma_{2,t}(G)$-set $S$ we can define a WRDF $f(X_0, X_1, X_2)$ on $G$ by $X_0 = V(G) \setminus S$, $X_1 = S$ and $X_2 = \emptyset$. Hence, (i) follows.

Now, let $D$ be a $\gamma_{2,t}(G)$-set and $y \in V(H)$. Thus, for every $(x, y) \in V(G) \times V(H)$, there exist $a, b \in D \cap N(x)$, which implies that $(a, y), (b, y) \in (D \times \{y\}) \cap N((x, y))$, and so $D \times \{y\}$ is a double total dominating set of $G \circ H$. Hence, (ii) follows.

Finally, from (i) and (ii) we deduce (iii), as $\gamma_r(G \circ H) \leq \gamma_{2,t}(G \circ H) \leq \gamma_{2,t}(G)$. \hfill \Box
In order to show an example of graphs where $\gamma_r(G \circ H) = \gamma_{2,t}(G)$, we define the family $\mathcal{G}$ as follows. A graph $G_{r,s} = (V, E)$ belongs to $\mathcal{G}$ if and only if there exist two positive integers $r, s$ such that $V = \{x_1, x_2, x_3, y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s\}$ and $E = \{x_1 y_i : 1 \leq i \leq r\} \cup \{x_1 z_i : 1 \leq i \leq s\} \cup \{x_2 y_i : 1 \leq i \leq r\} \cup \{x_2 z_i : 1 \leq i \leq s\} \cup \{x_2x_3\}$. Figure 3 shows the graph $G_{4,4}$.

It is not difficult to check that for any graph $G_{r,s} \in \mathcal{G}$ and any graph $H$ with $\gamma(H) \geq 3$ we have $\gamma_r(G_{r,s} \circ H) = 5 = \gamma_{2,t}(G_{r,s})$.

**Corollary 20.** For any graph $H$ and any graph $G$ of order $n$ and minimum degree greater than or equal to two, 

$$\gamma_r(G \circ H) \leq n.$$ 

As we will show in Corollary 37, the bound above is tight.

### 3.2 Lower bounds on $\gamma_r(G \circ H)$

In order to deduce our next result we need to state the following basic lemma.

**Lemma 21.** Let $G$ be a graph and $H$ a noncomplete graph. For any $u \in V(G)$ and any $\gamma_r(G \circ H)$-function $f$,

$$f[H_u] = \sum_{x \in N[u]} f(H_x) \geq 2.$$ 

**Proof.** Suppose that $f$ is a $\gamma_r(G \circ H)$-function and there exists $u \in V(G)$ such that $f[H_u] \leq 1$. If $f[H_u] = 1$, then the placement of a legion in a non-universal vertex of $H_u$ produces undefended vertices, which is a contradiction. Now, if $f[H_u] = 0$, then there are undefended vertices in $H_u$, which is a contradiction again. Therefore, the result follows.

A set $X \subseteq V(G)$ is called a 2-packing if $N[u] \cap N[v] = \emptyset$ for every pair of different vertices $u, v \in X$. The 2-packing number $\rho(G)$ is the cardinality of any largest 2-packing of $G$. A 2-packing of cardinality $\rho(G)$ is called a $\rho(G)$-set.

**Theorem 22.** For any graph $G$ of minimum degree $\delta \geq 1$ and any noncomplete graph $H$,

$$\gamma_r(G \circ H) \geq \max\{\gamma_r(G), \gamma_t(G), 2\rho(G)\}.$$ 

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Proof. Let \( f(W_0, W_1, W_2) \) be a \( \gamma_r(G \circ H) \)-function. In order to show that \( \gamma_r(G \circ H) \geq \gamma_r(G) \), we will show that there exists a WRDF \( f_1(X_0, X_1, X_2) \) of \( G \) where \( X_0 = \{x : (x, y) \in W_0\}, X_2 = \{x : (x, y) \in W_2 \text{ or } |\{x\} \times W_1| \geq 2\} \) and \( X_1 = V(G) \setminus (X_0 \cup X_2) \).

Notice that, since \( W_1 \cup W_2 \) is a dominating set of \( G \circ H \), \( X_1 \cup X_2 \) is a dominating set of \( G \). Now, for every \((x, y) \in W_0\) there exists \((x', y') \in N((x, y)) \cap (W_1 \cup W_2)\) and a function \( f' : V(G \circ H) \rightarrow \{0, 1, 2\} \) defined by \( f'(x', y') = f(x', y') - 1 \), \( f'(x, y) = 1 \) and \( f'(a, b) = f(a, b) \) for every \((a, b) \notin \{(x, y), (x', y')\}\), which has no undominated vertex. Hence, the function \( f'_1 : V(G) \rightarrow \{0, 1, 2\} \) defined by \( f'_1(x') = f_1(x') - 1 \), \( f'_1(x) = 1 \) and \( f'_1(a) = f_1(a) \) for every \( a \notin \{x, x'\} \) has no undominated vertex. Thus, \( \gamma_r(G \circ H) \geq \gamma_r(G) \).

Now, let \( X \subset V(G) \) be a \( \rho(G) \)-set. By Lemma 21 we have

\[
\gamma_r(G \circ H) = w(f) = \sum_{u \in V(G)} f(H_u) \geq \sum_{u \in X} f[H_u] \geq 2|X| = 2\rho(G),
\]

as required.

In order to prove that \( \gamma_r(G \circ H) \geq \gamma_l(G) \), we define \( U_i = \{x \in V(G) : f(H_x) = i\} \), where \( i \in \{0, 1\} \), and \( U_2 = \{x \in V(G) : f(H_x) \geq 2\} \). By Lemma 21 we have that if \( x \in U_1 \), then there exists \( x' \in N(x) \cap (U_1 \cup U_2) \). Now, let \( U_2^* = \{x \in U_2 : \sum_{x' \in N(x)} f(H_{x'}) = 0\} \). Since \( \delta \geq 1 \), there exists \( U_0^* \subseteq U_0 \) such that \( |U_0^*| \leq |U_2^*| \) with the property that for every \( x \in U_0^* \) there exists \( x_x \in U_0^* \cap N(x) \). Notice that \( U_1 \cup U_2 \cup U_0^* \) is a total dominating set. Therefore,

\[
\gamma_r(G \circ H) = w(f) = \sum_{u \in V(G)} f(H_u) \geq |U_1 \cup U_2 \cup U_0^*| \geq \gamma_l(G),
\]

as required. \( \square \)

In the next section we give closed formulae for \( \gamma_r(G \circ H) \). In particular, we discuss several cases in which \( \gamma_r(G \circ H) = \max\{\gamma_r(G), \gamma_l(G), 2\rho(G)\} \).

An example of a graph with \( \gamma_r(G) > \max\{\gamma_l(G), 2\rho(G)\} \) is the complete bipartite graph \( G \cong K_{3,3} \), where \( \gamma_r(K_{3,3}) = 3 \) and \( \gamma_l(K_{3,3}) = 2 \rho(K_{3,3}) = 2 \). An example of a graph with \( 2 \rho(G) > \max\{\gamma_r(G), \gamma_l(G)\} \) is the path graph \( P_n, n \geq 4 \), as \( \gamma_r(P_n) = \lceil 3n/4 \rceil \), \( \gamma_l(P_n) = \lfloor n/2 \rfloor + \lfloor n/4 \rfloor - \lceil n/4 \rceil \) and \( 2 \rho(P_n) = 2 \gamma(P_n) = 2 \lceil n/4 \rceil \). Finally, for the graph shown in Figure 4 we have \( \gamma_l(G) = 5 > 4 = \max\{\gamma_r(G), 2\rho(G)\} \).

![Figure 4: \( \gamma_r(G) = 4 \), the labels correspond to an optimum placement of legions.](image)

It is well known that for any graph \( G \), \( \gamma(G) \geq \rho(G) \). Meir and Moon [17] showed in 1975 that \( \gamma(T) = \rho(T) \) for any tree \( T \). We remark that in general, these \( \gamma(T) \)-sets and \( \rho(T) \)-sets are not identical. Notice that for any weak Roman tree \( T \) we have \( \gamma_r(T) = 2\rho(T) \), while if \( T \) is not a weak Roman tree, then \( \gamma_r(T) < 2\gamma(G) = 2\rho(T) \).
Corollary 23. For any tree $T$ and any noncomplete graph $H$,
\[ \gamma_r(T \circ H) \geq 2\gamma(T). \]

The bound above is achieved for any tree $T$ and any graph $H$ satisfying the assumptions of Theorem 28.

4 Closed formulae for $\gamma_r(G \circ H)$

To begin this section we consider the case of lexicographic product graphs in which the second factor is a complete graph.

Proposition 24. For any graph $G$ and any integer $n \geq 1$,
\[ \gamma_r(G \circ K_n) = \gamma_r(G). \]

Proof. The result is straightforward. We leave the details to the reader. \qed

From Theorems 12 and 22 we have the following result.

Theorem 25. For any graph $G$ with $\gamma_t(G) = \frac{1}{2} \max\{\gamma_r(G), 2\rho(G)\}$ and any noncomplete graph $H$,
\[ \gamma_r(G \circ H) = 2\gamma_t(G). \]

To show some families of graphs for which $\gamma_r(G) = 2\gamma_t(G) = 2\rho(G)$, we introduce the corona product of two graphs. Let $G_1$ be a graph of order $n$ and let $G_2$ be a graph. The corona product of $G_1$ and $G_2$, denoted by $G_1 \circ G_2$, was defined in [8] as the graph obtained from $G_1$ and $G_2$ by taking one copy of $G_1$ and $n$ copies of $G_2$ and joining by an edge each vertex from the $i$-th copy of $G_2$ with the $i$-th vertex of $G_1$.

Theorem 26. For any graph $G_1$ with no isolated vertex and any noncomplete graph $G_2$,
\[ \gamma_r(G_1 \circ G_2) = 2\gamma_t(G_1 \circ G_2) = 2\rho(G_1 \circ G_2). \]

Proof. Since $\gamma_r(G_1 \circ G_2) = |V(G_1)|$, we have that $\gamma_r(G_1 \circ G_2) \leq 2|V(G_1)|$. Now, we denote by $\langle g_i \rangle + G_2$ the subgraph of $G_1 \circ G_2$ induced by $g_i \in V(G)$ and the vertex set of the $i$-th copy of $G_2$. Since $G_2$ has two nonadjacent vertices and $g_i$ is the only vertex of $\langle g_i \rangle + G_2$ which is adjacent to some vertex outside $\langle g_i \rangle + G_2$, we deduce that every $\gamma_r(G_1 \circ G_2)$-function assigns at least two legions to the vertex set of $\langle g_i \rangle + G_2$, which implies that $\gamma_r(G_1 \circ G_2) \geq 2|V(G_1)|$. Now, since $G_1$ is a graph with no isolated vertex, $V(G_1)$ is a total dominating set. Hence, $\gamma_r(G_1 \circ G_2) = 2|V(G_1)| = 2\gamma_t(G_1 \circ G_2)$.

The proof of the equality $\gamma_t(G_1 \circ G_2) = \rho(G_1 \circ G_2)$ is straightforward. \qed

If $\gamma_r(G) = 2\gamma(G)$, then for the Cocktail-party graph $K_{2k} - F$, where $F$ is a perfect matching of $K_{2k}$, we have $\gamma_r(G \circ (K_{2k} - F)) = \gamma_r(G)$. This example is a particular case of the next result which is derived from Theorems 18 and 22.
Theorem 27. For any weak Roman graph $G$ and any graph $H$ such that $\gamma_r(H) = 2$, 

$$\gamma_r(G \circ H) = 2\gamma(G).$$

The study of weak Roman graphs was initiated in [13] by Henning and Hedetniemi, where they characterized forests for which the equality holds. The general problem of characterizing all weak Roman graphs remains open.

From Lemma 3 and Theorem 27 we derive the following result.

Theorem 28. If $T$ is a tree with a unique $\gamma(T)$-set $S$, and if every vertex in $S$ is a strong support vertex, then for any graph $H$ with $\gamma_r(H) = 2$,

$$\gamma_r(T \circ H) = 2\gamma(T).$$

Our next result shows that the inequality $\gamma_r(G \circ H) \leq 4\gamma(G)$ stated in Corollary 17 is tight.

Theorem 29. If $G$ is a graph with $\gamma_l(G) = 2\gamma(G)$ and there exists a $\gamma(G)$-set $D$ such that every vertex in $D$ is adjacent to a vertex of degree one, then for any graph $H$ with $\gamma(H) \geq 4$,

$$\gamma_r(G \circ H) = 4\gamma(G).$$

Proof. Assume that $\gamma_l(G) = 2\gamma(G)$, $\gamma(H) \geq 4$ and let $D$ be a $\gamma(G)$-set such that every vertex in $D$ is adjacent to a vertex of degree one. We will show that $\gamma_r(G \circ H) \geq 4\gamma(G)$. Since $\gamma_l(G) = 2\gamma(G)$, the vertex set of $G$ can be partitioned by the closed neighbourhoods of vertices in $D$, i.e., $V(G) = \cup_{x \in D} N[x]$ and $N[x] \cap N[y] = \emptyset$, for every $x, y \in D$, $x \neq y$. Now, let $f(W_0, W_1, W_2)$ be a $\gamma_r(G \circ H)$-function and let $x' \in N(x)$ be a vertex of degree one, for $x \in D$. Suppose that $f$ assigns at most three legions to $N[x] \times V(H)$. We differentiate the following cases for the set $W = W_1 \cup W_2$.

Case 1. $|W \cap (\{x\} \times V(H))| = 3$. Since $\gamma(H) \geq 4$, there exists at least one vertex in $\{x\} \times V(H)$ which is not dominated by the elements in $W$, which is a contradiction.

Case 2. $|W_2 \cap (\{x\} \times V(H))| = 1$ or $|W_1 \cap (\{x\} \times V(H))| = 2$. In both cases there exists $y \in N(x)$ such that $|W_1 \cap (\{y\} \times V(H))| = 1$. Since $\gamma(H) \geq 4$, the movement of a legion from the vertex in $W_1 \cap (\{y\} \times V(H))$ to any vertex in $W_0 \cap (\{x\} \times V(H))$ produces undefended vertices in $\{x\} \times V(H)$, which is a contradiction.

Case 3. $|W_1 \cap (\{x\} \times V(H))| = 1$. Since $\gamma(H) \geq 4$, the movement of a legion from the vertex in $W_1 \cap (\{x\} \times V(H))$ to any vertex in $W_0 \cap (\{x\} \times V(H))$ produces undefended vertices in $\{x'\} \times V(H)$, which is a contradiction.

Case 4. $|W \cap (\{x\} \times V(H))| = 0$. Since $\gamma(H) \geq 4$, there exists at least one vertex $(x', h) \in W_0$ which is not dominated by the elements in $W$, which is a contradiction.

According to the four cases above, for every $x \in D$ we have that $f$ assigns at least four legions to $N[x] \times V(H)$, which implies that $\gamma_r(G \circ H) \geq 4\gamma(G)$.

Furthermore, by Corollary 17, $\gamma_r(G \circ H) \leq 4\gamma(G)$. Therefore, the result follows. \qed

For the tree shown in Figure 5 we have $\gamma(T) = \rho(T) = 3$. Notice that in this case the set of support vertices of $T$ is the only $\gamma(T)$-set and a $\rho(T)$-set.
Figure 5: The set of gray-coloured vertices is the only dominating set of $G$. In this case $\gamma_t(G) = 2\gamma(G) = 6$. By Theorem 29, for any graph $H$ with $\gamma(H) \geq 4$ we have $\gamma_r(G \circ H) = 12 = 4\gamma(G)$.

**Corollary 30.** If the set of support vertices of a tree $T$ is a $\rho(T)$-set, then for any graph $H$ with $\gamma(H) \geq 4$, 
\[ \gamma_r(T \circ H) = 4\gamma(T). \]

From Theorems 19 and 22 we have the following result.

**Theorem 31.** If $G$ is a graph such that $\gamma_{2,t}(G) = \max\{\gamma_r(G), 2\rho(G)\}$, then for any noncomplete graph $H$,
\[ \gamma_r(G \circ H) = \gamma_{2,t}(G). \]

According to Theorem 31, the problem of characterizing the graphs for which $\gamma_{2,t}(G) = \gamma_r(G)$ or $\gamma_{2,t}(G) = 2\rho(G)$ deserves being considered in future works. We will construct a family $\mathcal{H}_k$ of graphs such that $\gamma_r(G) = \gamma_{2,t}(G)$, for every $G \in \mathcal{H}_k$. A graph $G = (V, E)$ belongs to $\mathcal{H}_k$ if and only if it is constructed from a cycle $C_k$ and $k$ empty graphs $N_{s_1}, \ldots, N_{s_k}$ of order $s_1, \ldots, s_k$, respectively, and joining by an edge each vertex from $N_{s_i}$ with the vertices $v_i$ and $v_{i+1}$ of $C_k$. Here we are assuming that $v_i$ is adjacent to $v_{i+1}$ in $C_k$, where the subscripts are taken modulo $k$. Figure 6 shows a graph belonging to $\mathcal{H}_k$, where $k = 4$, $s_1 = s_3 = 3$ and $s_2 = s_4 = 2$.

Figure 6: The set of gray-coloured vertices is a double dominating set.

For any graph $G \in \mathcal{H}_k$ we have $\gamma_r(G) = \gamma_{2,t}(G) = k$. Therefore, by Theorem 31, for any $G \in \mathcal{H}_k$ and any graph $H$, $\gamma_r(G \circ H) = k$.

From now on we say that a vertex $a \in V(H)$ satisfies Property $\mathcal{P}$ if $\{a, b\}$ is a dominating set of $H$, for every $b \in V(H) \setminus N[a]$. In other words, $a \in V(H)$ satisfies Property $\mathcal{P}$ if the subgraph induced by $V(H) \setminus N[a]$ is a clique.

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Proposition 32. For any integer $n \geq 3$ and any noncomplete graph $H$, 

$$2 \leq \gamma_r(K_n \circ H) \leq 3.$$ 

Furthermore, $\gamma_r(K_n \circ H) = 2$ if and only if $\gamma_r(H) = 2$ or there exists a vertex of $H$ which satisfies $P$.

Proof. By Remark 4 we have $\gamma_r(K_n \circ H) \geq 2$ and by Theorem 19 we have that $\gamma_r(K_n \circ H) \leq 3$.

To characterize the graphs with $\gamma_r(K_n \circ H) = 2$ we first assume that $\gamma_r(H) = 2$, and we will apply Remark 5 to the graph $H$. Let $u \in V(G)$ and let $\{a, b\} \subseteq V(H)$ a secure dominating set. We claim that the function $f$ which satisfies $P$ is tight. To this end, we have that $f$ satisfies $\gamma_r(K_n \circ H) = 2$ if and only if $\gamma_r(H) = 2$ or there exists a vertex of $H$ which satisfies $P$.

Proposition 33. Let $H$ be a graph and let $n \geq 3$ be an integer. Then the following statements hold.

(i) If $\gamma_r(H) \in \{2, 3\}$, then $\gamma_r(K_{1,n} \circ H) = \gamma_r(H)$.

(ii) If $\gamma_r(H) \geq 4$, then $3 \leq \gamma_r(K_{1,n} \circ H) \leq 4$.

(iii) If $\gamma_r(H) \geq 4$, then $\gamma_r(K_{1,n} \circ H) = 4$.

Proof. By Remark 4 we have $\gamma_r(K_{1,n} \circ H) \geq 2$ and by Theorem 12, $\gamma_r(K_{1,n} \circ H) \leq 2 \gamma_r(K_{1,n}) = 4$.

Let $g$ be a $\gamma_r(H)$-function. Assume that $\gamma_r(H) \in \{2, 3\}$. The function $f : V(K_{1,n} \circ H) \rightarrow \{0, 1, 2\}$ defined by $f(u_0, v) = g(v)$, for every $v \in V(H)$, and $f(u, v) = 0$, for every $u \in V(K_{1,n}) \setminus \{u_0\}$ and $v \in V(H)$, is a WRDF of $K_{1,n} \circ H$, which implies that $\gamma_r(K_{1,n} \circ H) \leq \gamma_r(H)$. Hence, if $\gamma_r(H) = 2$, then we are done. Since $n \geq 3$, for any $\gamma_r(K_{1,n} \circ H)$-function we have $f(H_{u_0}) \geq 2$ and, if $\gamma_r(H) \geq 3$, then $w(f) \geq 3$. Thus, (i) and (ii) follow.

Finally, if $\gamma_r(H) \geq 4$, then Theorem 29 leads to $\gamma_r(K_{1,n} \circ H) = 4$.

We will now show that the bound given in Corollary 14 is tight. To this end, we need to introduce some additional notation. Given a graph $G$, let $P_3(G)$ be the family of ordered sets $S = \{x_1, x_2, x_3\} \subseteq V(G)$ such that $\langle S \rangle \cong P_3$, $\delta(x_1) \geq 2$, $\delta(x_2) = 2$ and $\delta(x_3) = 1$. 

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Lemma 34. Let $G$ and $H$ be two graphs, and $\{x_1, x_2, x_3\} \in P_3(G)$. If $\gamma(H) \geq 4$, then for any $\gamma_r(G \circ H)$-function $f$,

$$\sum_{i=1}^{3} f(H_i) = 4.$$  

Furthermore, there exists a $\gamma_r(G \circ H)$-function $f$, such that $f(H_2) = 2$ and $f(H_3) = 0$.

Proof. Suppose that there exists a $\gamma_r(G \circ H)$-function $f$ with

$$\sum_{i=1}^{3} f(H_i) \leq 3.$$  

We differentiate the following cases according to the value of $f(H_1)$.

1. $f(H_1) = 0$. If $f(H_2) = 0$ (resp. $f(H_3) = 0$), then there is an undefended vertex in $H_3$ (resp. $H_2$). If $f(H_2) = 1$ (resp. $f(H_3) = 1$), then the movement of the legion from $H_2$ to $H_3$ (resp. from $H_3$ to $H_2$) produces an undefended vertex in $H_3$ (resp. from $H_2$).

2. $f(H_1) = 1$. If $f(H_2) = 0$, then there is an undefended vertex in $H_3$. If $f(H_2) = 1$, then the movement of the legion from $H_2$ to $H_3$ produces an undefended vertex in $H_3$. Finally, If $f(H_2) = 2$, then the movement of the legion from $H_1$ to $H_2$ produces an undefended vertex in $H_2$.

3. $f(H_1) = 2$. If $f(H_2) = 0$, then there is an undefended vertex in $H_3$. If $f(H_2) = 1$, then the movement of the legion from $H_2$ to $H_3$ produces an undefended vertex in $H_3$.

4. $f(H_1) = 3$. In this case the vertices in $H_3$ are undefended.

In all cases above we obtain a contradiction, which implies that $f(H_1) + f(H_2) + f(H_3) \geq 4$. To conclude the proof we only need to observe that we can construct a $\gamma_r(G \circ H)$-function $f$ with $f(H_1) + f(H_2) + f(H_3) = 4$, as we can take $f(H_1) = f(H_2) = 2$ and $f(H_3) = 0$.  

We will now prove that there exists a family of trees $T_n$, which we will call combs, such that for any graph $H$ with $\gamma(H) \geq 4$, $\gamma_r(T_n \circ H) = 2 \left\lceil \frac{2n}{3} \right\rceil$. With this end we will now describe this family of combs. Take a path $P_k$ of length $k = \left\lceil \frac{n}{3} \right\rceil$, with vertices $v_1, \ldots, v_k$, and attach a path $P_3$ to each vertex $v_1, \ldots, v_{k-1}$, by identifying each $v_i$ with a leaf of its corresponding copy of $P_3$. Finally, we attach a path of length $r = n - 3 \left\lceil \frac{n}{3} \right\rceil + 2$ to $v_k$. Notice that

$$n - 3 \left\lceil \frac{n}{3} \right\rceil + 2 = \begin{cases} 0 & \text{if } n \equiv 1 \pmod{3}; \\ 1 & \text{if } n \equiv 2 \pmod{3}; \\ 2 & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

Figure 7 shows the construction of $T_n$ for different values of $n$. Notice that the comb of order six is simply $T_6 \cong P_6$.  

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Proposition 35. For any \( n \geq 4 \) and any graph \( H \) with \( \gamma(H) \geq 4 \),

\[
\gamma_r(T_n \circ H) = 2 \left\lceil \frac{2n}{3} \right\rceil.
\]

Proof. By Corollary 14 we have \( \gamma_r(T_n \circ H) \leq 2 \left\lfloor \frac{2n}{3} \right\rfloor \). In order to show that \( \gamma_r(T_n \circ H) \geq 2 \left\lceil \frac{2n}{3} \right\rceil \) we differentiate three cases.

If \( n = 3k \), then Lemma 34 leads to \( \gamma_r(T_n \circ H) = 4k = 2 \left\lfloor \frac{2n}{3} \right\rfloor \). Now, if \( n = 3(k-1) + 1 \), then Lemma 34 leads to \( \gamma_r(T_n \circ H) \geq 4(k-1) = 2 \left\lfloor \frac{2n}{3} \right\rfloor \). Finally, if \( n = 3(k-1) + 2 \), then Lemma 34 leads to \( \gamma_r(T_n \circ H) \geq 4(k-1) + 2 = 2 \left\lceil \frac{2n}{3} \right\rceil \). \( \square \)

Given a graph \( G \), let \( \mathcal{P}_4(G) \) be the family of ordered sets \( S = \{x_1, x_2, x_3, x_4\} \subset V(G) \) such that \( \langle S \rangle \cong P_4 \), \( \delta(x_1) \geq 2 \), \( \delta(x_2) = \delta(x_3) = 2 \) and \( \delta(x_4) \geq 2 \). For any \( G \) such that \( \mathcal{P}_4(G) \neq \emptyset \) we define the family \( \mathcal{O}_4(G) \) of graphs \( G^* \) constructed from \( G \) as follows. Let \( S \in \mathcal{P}_4(G) \) such that \( \langle S \rangle = P_4 = (x_1, x_2, x_3, x_4) \), \( X = \{x_1x_2, x_2x_3, x_3x_4\} \) and \( Y = \{ab : a \in N(x_1) \setminus \{x_2\} \text{ and } b \in N(x_4) \setminus \{x_3\}\} \). The vertex set of \( G^* \) is \( V(G^*) = V(G) \setminus S \) and the edge set is \( E(G^*) = (E(G) \setminus X) \cup Y \).

Theorem 36. Let \( G \) be a graph such that \( \mathcal{P}_4(G) \neq \emptyset \) and let \( H \) be a graph. If \( \gamma(H) \geq 4 \), then for any \( G^* \in \mathcal{O}_4(G) \),

\[
\gamma_r(G \circ H) = \gamma_r(G^* \circ H) + 4.
\]

Proof. We will prove this result in Section 5. \( \square \)

A simple case analysis shows that for \( n \in \{3, 4, 5, 6\} \) and any graph \( H \) such that \( \gamma(H) \geq 4 \) we have \( \gamma_r(C_n \circ H) = n \). Hence, Theorem 36 immediately leads to the following corollary.

Corollary 37. Let \( n \geq 3 \) be an integer and let \( H \) be a graph. If \( \gamma(H) \geq 4 \), then

\[
\gamma_r(C_n \circ H) = n.
\]

It is readily seen that if \( \gamma(H) \geq 4 \), then \( \gamma_r(P_2 \circ H) = \gamma_r(P_3 \circ H) = \gamma_r(P_4 \circ H) = 4 \) and \( \gamma_r(P_5 \circ H) = 6 \). Therefore, Theorem 36 leads to the following result.

Corollary 38. Let \( n \geq 2 \) be an integer and let \( H \) be a graph. If \( \gamma(H) \geq 4 \), then

\[
\gamma_r(P_n \circ H) = \begin{cases} 
n, & n \equiv 0 \pmod{4}; 
n + 2, & n \equiv 2 \pmod{4}; 
n + 1, & \text{otherwise}. 
\end{cases}
\]
5 Proof of Theorem 36

To prove Theorem 36 we need the following lemma.

Lemma 39. Let $G$ and $H$ be nontrivial connected graphs. If $\gamma(H) \geq 4$, then there exists a $\gamma_r(G \circ H)$-function $f$ such that $\sum_{u' \in N(u)} f(H_{u'}) \geq 2$, for every $u \in V(G)$.

Proof. Let $u, u' \in V(G)$ such that $u' \in N(u)$ and $v' \in V(H)$. First, suppose that $\sum_{z \in N(u)} f(H_z) = f(u', v') = 1$. If $f(H_u) < \gamma(H) - 1$, then there exists $v \in V(H)$ such that $\sum_{h \in N[v]} f(u, h) = 0$, so that the movement of the legion from $(u', v')$ to $(u, v)$ produces undefended vertices, which is a contradiction. Hence, $\gamma(H) - 1 \geq 3$ and we can construct a $\gamma_r(G \circ H)$-function $f_1$ from $f$ as follows. For some $(u, v)$ such that $f(u, v) \geq 1$ we set $f_1(u, v) = f(u, v) - 1$, for some $v'' \neq v''$ we set $f_1(u', v''') = 1$ and $f_1(x, y) = f(x, y)$ for every $(x, y) \in V(G \circ H) \setminus \{(u, v), (u', v'')\}$.

Hence, $\sum_{z \in N(u)} f_1(H_z) = f_1(u', v') + f_1(u, v''') = 2$.

Now, if $\sum_{z \in N(u)} f_1(H_z) = 0$, then we proceed as above to construct a $\gamma_r(G \circ H)$-function $f_1$ from $f$ by the movement of two legions from $H_u$ to $(u', v')$. In this case, $\sum_{z \in N(u)} f_1(H_z) = f_1(u', v') = 2$.

For each $u \in V(G)$ such that $\sum_{u' \in N(u)} f(H_{u'}) \leq 1$ we can repeat the procedure above until finally obtaining a $\gamma_r(G \circ H)$-function satisfying the result. \hfill \Box

The proof of Theorem 36. Let $S \in \mathcal{P}_4(G)$ such that $\langle S \rangle \cong P_4 = (x_1, x_2, x_3, x_4)$. We will first show that $\gamma_r(G \circ H) \leq \gamma_r(G^* \circ H) + 4$. Let $f$ be a $\gamma_r(G^* \circ H)$-function and define $\alpha_1$ and $\alpha_4$ as follows:

$$\alpha_1 = \sum_{x \in N(x_1) \setminus \{x_2\}} f(H_x) \quad \text{and} \quad \alpha_4 = \sum_{x \in N(x_4) \setminus \{x_3\}} f(H_x).$$

We will construct a WRDF $f_1$ on $G \circ H$ from $f$ such that $w(f_1) \leq w(f) + 4$. For each vertex $(u, v) \in V(G^* \circ H)$ we set $f_1(u, v) = f(u, v)$ and now we will describe the following six cases for the vertices $(u, v) \in S \times V(H)$, where symmetric cases are omitted. In all these cases we fix $y \in V(H)$.

1. $\alpha_1 \geq 2$ and $\alpha_4 \geq 2$. We set $f_1(x_1, y) = f_1(x_4, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_1, y), (x_4, y)\}$.

2. $\alpha_1 \geq 2$ and $\alpha_4 = 1$. We set $f_1(x_1, y) = f_1(x_3, y) = 1$, $f_1(x_4, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_1, y), (x_3, y), (x_4, y)\}$.

3. $\alpha_1 \geq 2$ and $\alpha_4 = 0$. We set $f_1(x_3, y) = f_1(x_4, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_3, y), (x_4, y)\}$.

4. $\alpha_1 = 1$ and $\alpha_4 = 1$. We set $f_1(x_1, y) = f_1(x_2, y) = f_1(x_3, y) = f_1(x_4, y) = 1$ and $f_1(u, v) = 0$ for every $v \neq y$ and $u \notin \{x_1, x_2, x_3, x_4\}$.

5. $\alpha_1 = 1$ and $\alpha_4 = 0$. We set $f_1(x_2, y) = f_1(x_4, y) = 1$, $f_1(x_3, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_2, y), (x_3, y), (x_4, y)\}$.

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6. \( \alpha_1 = 0 \) and \( \alpha_4 = 0 \). We set \( f_1(x_2, y) = f_1(x_3, y) = 2 \) and \( f_1(u, v) = 0 \) for every \((u, v) \notin \{(x_2, y), (x_3, y)\}\).

\[
\begin{align*}
\alpha_1 &= 0 \\
\alpha_4 &= 0 \\
f_1(x_2, y) &= f_1(x_3, y) = 2 \\
f_1(u, v) &= 0
\end{align*}
\]

Figure 8: The six cases above are described in this scheme.

A simple case analysis shows that the vertices of \( G \odot H \) are defended by the assignment of legions produced by \( f_1 \). Therefore,

\[
\gamma_r(G \odot H) \leq w(f_1) \leq w(f) + 4 = \gamma_r(G^* \odot H) + 4.
\]

(2)

Now we will show that the equality holds. Let \( g \) be a \( \gamma_r(G \odot H) \)-function satisfying Lemma 39. We will construct a WRDF \( g_1 \) on \( G^* \odot H \) from the function \( g \) such that \( w(g_1) \leq w(g) - 4 \). We also need to define \( B_1 \) and \( B_4 \) as follows.

\[
B_1 = \bigcup_{x \in N(x_1) \setminus \{x_2\}} V(H_x) \quad \text{and} \quad B_4 = \bigcup_{x \in N(x_4) \setminus \{x_3\}} V(H_x).
\]

We define \( g_1 \) according to the following six cases:

1'. \( g(B_1) \geq 2 \) and \( g(B_4) \geq 2 \). In this case, we set \( g_1(x, y) = g(x, y) \) for every \((x, y) \in V(G^* \odot H)\).

2'. \( g(B_1) \geq 2 \) and \( g(B_4) = 1 \). Depending on \( g(H_1) \) we will consider the following two cases:

2'.1 \( g(H_1) \leq 1 \). Since \( g(H_1) \leq g(B_4) \), we can set \( g_1(x, y) = g(x, y) \) for every \((x, y) \in V(G^* \odot H)\).

2'.2 \( g(H_1) \geq 2 \). We will show that \( g(S \times V(H)) \geq 5 \). To see this, we will try to place four legions in \( S \times V(H) \) as shown in Figure 9, where \( a+b = 2 \) (in Figures 9-13 black vertices represent a contradiction with Lemma 39). Since in all these cases we have a contradiction, we can conclude that \( g(S \times V(H)) \geq 5 \). Hence, we place the legions in the following way: for some \((x_0, y_0) \in B_4\) we set \( g_1(x_0, y_0) = g(x_0, y_0) + 1 \) and \( g_1(x, y) = g(x, y) \) for every \((x, y) \in V(G^* \odot H) \setminus \{(x_0, y_0)\}\).
3'. $g(B_1) \geq 2$ and $g(B_4) = 0$. In this case, we consider the following three cases depending on the value of $g(H_1)$:

3'.1 $g(H_1) = 0$. In this case $g(H_1) = g(B_4)$ so we set $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^{*} \circ H)$.

3'.2 $g(H_1) = 1$. We will show that $g(S \times V(H)) \geq 5$. To see this, we will try to place four legions in $S \times V(H)$ as shown in Figure 10, where $2 \leq a + b \leq 3$ and $c + d = 1$. In both cases we have a contradiction with Lemma 39. Hence, $g(S \times V(H)) \geq 5$ and so we place the legions in the following way: for some $(x_0, y_0) \in B_4$ we set $g_1(x_0, y_0) = 1$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^{*} \circ H) \setminus \{(x_0, y_0)\}$.

3'.3 $g(H_1) \geq 2$. We will show that $g(S \times V(H)) \geq 6$. To see this, we will try to place five legions in $S \times V(H)$ as shown in Figure 11, where $2 \leq a + b \leq 3$ and $c + d = 1$. In both cases we have a contradiction with Lemma 39. Hence, $g(S \times V(H)) \geq 6$ and so we place the legions in the following way: for some $(x_0, y_0) \in B_4$ we set $g_1(x_0, y_0) = 2$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^{*} \circ H) \setminus \{(x_0, y_0)\}$.

4'. $g(B_1) = g(B_4) = 1$. In this case, we consider the following three cases depending on the value of $g(H_1)$ and $g(H_4)$:

4'.1 $g(H_1) \leq 1$ and $g(H_4) \leq 1$. In this case $g(H_1) \leq g(B_4)$ and $g(H_4) \leq g(B_1)$, so we set $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^{*} \circ H)$. 

Figure 9: Scheme corresponding to Case 2'.2.

Figure 10: Scheme corresponding to Case 3'.2.

Figure 11: Scheme corresponding to Case 3'.3.
4.2 $g(H_1) \geq 2$ and $g(H_4) \leq 1$ (this case is symmetric to $g(H_1) \leq 1$ and $g(H_4) \geq 2$). We will show that $g(S \times V(H)) \geq 5$. To see this, we will try to place four legions in $S \times V(H)$ as shown in Figure 12, where $a+b=1$. In all cases we have a contradiction with Lemma 39. Hence, $g(S \times V(H)) \geq 5$ and so we define $g_1$ as follows: for some $(x_0, y_0) \in B_4$ we set $g_1(x_0, y_0) = g(x_0, y_0) + 1$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0)\}$.

![Figure 12: Scheme corresponding to Case 4.2.](image)

4.3 $g(H_1) \geq 2$ and $g(H_4) \geq 2$.

We will show that $g(S \times V(H)) \geq 6$. To see this, we will try to place five legions in $S \times V(H)$ as shown in Figure 13, where $a+b=1$. In this case we have a contradiction with Lemma 39. Hence, $g(S \times V(H)) \geq 6$ and so we place the legions in the following way: for some $(x_0, y_0) \in B_4$ we set $g_1(x_0, y_0) = g(x_0, y_0) + 1$, for some $(x_0', y_0') \in B_4$ we set $g_1(x_0', y_0') = g(x_0, y_0') + 1$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0), (x_0', y_0')\}$.

![Figure 13: Scheme corresponding to Case 4.3.](image)

5. $g(B_1) = 1$ and $g(B_4) = 0$. Notice that if $g(H_2) = 0$ or $g(H_3) \leq 1$, then we have a contradiction with Lemma 39, so that $g(H_2) \geq 1$ and $g(H_3) \geq 2$. We differentiate two cases according to the value of $g(H_2)$:

5.1 $g(H_2) = 1$. By Lemma 39, we have that $g(H_4) \geq 1$. Thus, we place the legions in the following way: for some $(x_0, y_0) \in B_1$ we set $g_1(x_0, y_0) = \min\{2, g(H_4) - 1\}$, for some $(x_0', y_0') \in B_4$ we set $g_1(x_0', y_0') = \min\{2, g(H_1)\}$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0), (x_0', y_0')\}$.

5.2 $g(H_2) \geq 2$. In this case, we place the legions in the following way: for some $(x_0, y_0) \in B_1$ we set $g_1(x_0, y_0) = \min\{2, g(H_4)\}$, for some $(x_0', y_0') \in B_4$ we set $g_1(x_0', y_0') = \min\{2, g(H_1)\}$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0), (x_0', y_0')\}$.

6. $g(B_1) = g(B_4) = 0$. Notice that if $g(H_2) \leq 1$ or $g(H_3) \leq 1$, then we have a contradiction with Lemma 39, so that $g(H_2) \geq 2$ and $g(H_3) \geq 2$. We place the legions
in the following way: for some \((x_0, y_0) \in B_1\) we set \(g_1(x_0, y_0) = \min\{2, g(H_4)\}\), for some \((x_0', y_0') \in B_4\) we set \(g_1(x_0', y_0') = \min\{2, g(H_1)\}\) and \(g_1(x, y) = g(x, y)\) for every \((x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0), (x_0', y_0')\}\).

A simple case analysis shows that the vertices of \(G \circ H\) are defended by the assignment of legions produced by \(g_1\). Therefore,

\[
\gamma_r(G^* \circ H) \leq w(g_1) \leq w(g) - 4 \leq \gamma_r(G \circ H) - 4.
\]  

Finally, by (2) and (3) we can conclude that \(\gamma_r(G \circ H) = \gamma_r(G^* \circ H) + 4\), as claimed. \(\square\)

6 Open problems

Some closed formulae for \(\gamma_r(G \circ H)\), obtained in Section 4, were derived under the assumption that \(\gamma_l(G) = \frac{1}{2} \max\{\gamma_r(G), 2\rho(G)\}\) or \(\gamma_{2,t}(G) = \max\{\gamma_r(G), 2\rho(G)\}\) or \(\gamma_r(G) = 2\gamma(G)\). This suggests the following open problems.

Problem 1. Characterize the graphs with \(\gamma_r(G) = 2\gamma(G)\).

Problem 2. Characterize the graphs with \(\gamma_r(G) = 2\gamma_l(G)\).

Problem 3. Characterize the graphs with \(\gamma_l(G) = \rho(G)\).

Problem 4. Characterize the graphs with \(\gamma_r(G) = \gamma_{2,t}(G)\).

Problem 5. Characterize the graphs with \(\gamma_{2,t}(G) = 2\rho(G)\).

Notice that \(\gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G) \leq 2\gamma_l(G)\). Hence, \(\gamma_r(G) = 2\gamma(G)\) if and only if \(\gamma_r(G) = \gamma_R(G)\) and \(G\) is a Roman graph. Furthermore, \(\gamma_r(G) = 2\gamma_l(G)\) if and only if all equalities hold true in the previous domination chain. Therefore, the starting point to solve Problems 1 and 2 is a deep investigation of Roman graphs.

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