Escaping Points of the Cosine Family

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Abstract

We study the dynamics of iterated cosine maps $E: z \mapsto ae^z + be^{-z}$, with $a, b \in \mathbb{C} \setminus \{0\}$. We show that the points which converge to $\infty$ under iteration are organized in the form of rays and, as in the exponential family, every escaping point is either on one of these rays or the landing point of a unique ray. Thus we get a complete classification of the escaping points of the cosine family, confirming a conjecture of Eremenko in this case. We also get a particularly strong version of the “dimension paradox”: the set of rays has Hausdorff dimension 1, while the set of points these rays land at has not only Hausdorff dimension 2 but infinite planar Lebesgue measure.

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1 Introduction

The dynamics of iterated polynomials has been investigated quite successfully, particularly in the past two decades. The study begins with a description of the escaping points: those points which converge to \( \infty \) under iteration. It is well known that the set of escaping points is an open neighborhood of \( \infty \) which can be parametrized by dynamic rays. The Julia set can then be studied in terms of landing properties of dynamic rays.

For entire transcendental functions, the point \( \infty \) is an essential singularity (rather than a superattracting fixed point as for polynomials). This makes the investigation of the dynamics much more difficult. In particular, there is no obvious structure of the set of escaping points. Eremenko [E] showed that for every entire transcendental function, the set of escaping points is always non-empty, and he asked whether each connected component (or even every path component) was unbounded. In [SZ], this question was answered in the affirmative for the special case of exponential functions \( z \mapsto \lambda e^z \): every escaping point can be connected to \( \infty \) along a unique curve running entirely along escaping points. In this paper, we extend this result to the family of cosine maps \( E_{a,b}: z \mapsto a e^z + b e^{-z} \) with \( a, b \in \mathbb{C}^* \).

In many ways, the escaping points of the cosine family behave quite similarly to those of the exponential family; therefore, the present paper is very similar to [SZ]. It is based on the Diploma thesis [Ro]. Based on experience with these and other families of maps, we believe that similar results should hold for much larger classes of entire transcendental functions, possibly of bounded type (which means that the set of asymptotic or critical values is bounded). This paper is thus a contribution to the program to make polynomials tools, in particular dynamic rays, available for the study of iterated entire transcendental functions.

Our main result is a classification of escaping points for every map \( E_{a,b} \); this classification is the same for all such maps (with natural exceptions if one or both critical orbits escape). A byproduct is an affirmative answer to Eremenko’s question as mentioned above: every path connected component of the set of escaping points is a curve starting at \( \infty \). Similarly as for exponential functions, but quite unlike the polynomial case, certain of these curves land at points in \( \mathbb{C} \) which are escaping points themselves. A dynamic ray is a connected component of the escaping set, removing the landing points (for those curves which land at escaping points). It turns out that the union of all the uncountably many dynamic rays has Hausdorff dimension 1 (in analogy to results [K, SZ] for the exponential family). However, by a result of McMullen [McM], the set of escaping points in the cosine family has infinite planar Lebesgue measure (this is one main difference to the exponential...
case, where the set of escaping points has zero measure). Therefore, the entire measure of the escaping set sits in the landing points of those rays which land at escaping points.

In [S], these results are extended even further: if both critical orbits of $E_{a,b}$ are strictly preperiodic, then the set $R$ of dynamic rays still has Hausdorff dimension 1 and every dynamic ray lands somewhere in $\mathbb{C}$. Therefore, $\mathbb{C} \setminus R$ is “most of $\mathbb{C}$” (the complement of a one-dimensional set). It turns out that each $z \in \mathbb{C} \setminus R$ is the landing point of one or several of the rays in $R$: the rays in the one-dimensional set $R$ manage to connect all the remaining points to $\infty$ by curves in $R$! This highlights another difference between the dynamics in the cosine family and in the exponential family: in the exponential family, there is no case known where the Julia set is the entire complex plane and every dynamic ray lands [RG, Theorem 1.6]; this is due to the enormous contraction in the asymptotic tract of the asymptotic value 0, while the singular values in the cosine family are simply two critical values with much better-behaved properties.

We start this paper in Section 2 by setting up a partition for symbolic dynamics. In Section 3 we construct ray tails which are curves of escaping points terminating at $\infty$. These ray tails are extended to entire dynamic rays in Section 4. In Section 5 we prove that points on the same dynamic ray move away from each other very quickly, and this leads to a complete classification of escaping points in Section 6. Finally, in Section 7 we discuss the implications in terms of Hausdorff dimension.

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Notation. We consider the maps

$$E_{a,b}(z) := ae^z + be^{-z} \quad a, b \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$$

and their iterates $E_{a,b}^n(z)$. Usually we will omit the parameters $a$ and $b$ and write $E(z)$ for $E_{a,b}(z)$. Set $c := \frac{1}{2} \ln \left( \frac{a}{b} \right)$, where the branch of the logarithm is chosen such that $|\text{Im}(c)| \leq \pi/2$. The critical points of $E$ are

$$C_{\text{crit}} = \{c + in\pi, n \in \mathbb{Z} \}$$

and the critical values $v_{1/2} = \pm 2\sqrt{ab}$, choosing signs so that $v_1$ is the image of $c + 2\pi i \mathbb{Z}$, while $v_2$ is the image of $c + i \pi + 2\pi i \mathbb{Z}$. There are no asymptotic
values in \( \mathbb{C} \). We will use the notation \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) and \( \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \). Also, we will need \( F: \mathbb{R}^*_+ \to \mathbb{R}^*_+ \), \( F(t) := \exp(t) - 1 \). Now let \( \alpha := \ln a \) and \( \beta := \ln b \), choosing branches so that \(|\text{Im}(\alpha)| \leq \pi \), \(|\text{Im}(\beta)| \leq \pi \). Let furthermore \( K := \min \{|a|, |b|\} \), \( K_{\max} := \max \{|a|, |b|\} \) and \( M := \max\{|\alpha|, |\beta|\} \). Given \( a, b \in \mathbb{C}^* \), let

\[
T_{a,b} \geq \max \left\{ \sqrt{\frac{2b}{a}}(|a| + |b|), \sqrt{\frac{2a}{b}}(|a| + |b|), 8|ab|, 1, \frac{1}{2} \ln \left| \frac{b}{a} \right|, \frac{1}{2} \ln \left| \frac{a}{b} \right|, \ln \frac{4}{|a|}, \ln \frac{4}{|b|} \right\} + M + 2 \quad (1)
\]

be the least value for which \( F(T_{a,b}) \geq T_{a,b} + M + 4 \). Note that

\[
2\sqrt{|ab|} \leq \max\{8|ab|, 1\} \quad (2)
\]

if \(|ab| < 1/16\), then \(2\sqrt{|ab|} < 1/2 < 1\), while if \(|ab| \geq 1/16\), then \(2\sqrt{|ab|} \leq 8|ab|\). Thus \( T_{a,b} \geq 2\sqrt{|ab|} + M + 2 \).

We will use the following sets (their significance will be explained in Section 2 and Figure 2): if \( \text{Im}(v_1) \geq \text{Im}(v_2) \), then set

\[
\mathcal{A} := \{z \in \mathbb{C}: z = \lambda v_1 + (1 - \lambda)v_2; \lambda \in [0, 1]\} \cup \{z \in \mathbb{C}: \text{Re}(z) = \text{Re}(v_1), \text{Im}(z) \geq \text{Im}(v_1)\} \quad (3)
\]

this is the segment between \( v_1 \) and \( v_2 \), together with the vertical ray starting at \( v_1 \) in upwards direction. If \( \text{Im}(v_1) < \text{Im}(v_2) \), we reverse the last inequality in the definition of \( \mathcal{A} \): the last ray is replaced by the downwards vertical ray at \( v_1 \). In both cases, set \( \mathbb{C}' := \mathbb{C} \setminus \mathcal{A} \).

2 Escaping Points and Symbolic Dynamics

In this section, we set up a partition of the complex plane and define symbolic dynamics of escaping points.

**Definition 2.1 (Escaping Points)**

A point \( z \in \mathbb{C} \) with \(|E^\circ_k(z)| \to \infty\) for \( k \to \infty \) is called an escaping point and its orbit is called escaping orbit.

**Lemma 2.2 (Real Parts of Escaping Orbits)**

If \((z_k)\) is an orbit with \(|z_k| \to \infty\) for \( k \to \infty \), then \(|\text{Re}(z_k)| \to \infty\).
Proof. This follows from the standard estimate
\[ |z_{k+1}| \leq |a| \exp(\text{Re}(z_k)) + |b| \exp(\text{Re}(-z_k)) \]  \hspace{1cm} (4)

In order to introduce symbolic dynamics we need a useful partition of the complex plane. First define
\[ Z_S := \mathbb{Z} \times \{L, R\}, \quad Z_R := \mathbb{Z} \times \{R\} \quad \text{and} \quad Z_L := \mathbb{Z} \times \{L\}, \]
and we write \( n_R := (n, R) \) and \( n_L := (n, L) \) for \( n \in \mathbb{Z} \). Let us first consider the “straight partition” (Figure 1): for \( j \in Z_R \) define the strip \( R'_j \) as follows
\[ R'_j := \{ z \in \mathbb{C} : \text{Im}(c) + 2\pi j < \text{Im}(z) < \text{Im}(c) + 2\pi(j + 1); \text{Re}(c) < \text{Re}(z) \} \]
and for \( j \in Z_L \)
\[ R'_j := \{ z \in \mathbb{C} : \text{Im}(c) + 2\pi j < \text{Im}(z) < \text{Im}(c) + 2\pi(j + 1); \text{Re}(z) < \text{Re}(c) \} . \]

Then the \( R'_j \) form a rather simple partition of \( \mathbb{C} \). The image of the boundary of each strip is the part of the straight line through the critical values ending in \( v_1 \). It will be convenient to modify the partition so that the real part of the image of the boundary is bounded. To introduce our partition (see Figure 2) now define \( A \) as in (3) and set \( \mathbb{C}' := \mathbb{C} \setminus A \). Then define the strips \( R_j \) as connected components of \( E^{-1}(\mathbb{C} \setminus A) \), so that
\[ E: R_j \to \mathbb{C}' \]
is a conformal isomorphism for all \( j \in Z_S \). For \( s \in Z_S \) we denote the inverse mapping of \( E: R_s \to \mathbb{C}' \) by \( L_s: \mathbb{C}' \to R_s \). Note that the strips are open and that the union of their closures is \( \mathbb{C} \). Label the strips such that \( \overline{R}_0 R \) contains some right end of \( \mathbb{R}^+ \), \( \overline{R}_0 L \) contains some left end of \( \mathbb{R}^- \), and that \( z \in R_{n,L} \) iff \( z + 2\pi i \in R_{(n+1,L)} \) and \( z \in R_{n,R} \) iff \( z + 2\pi i \in R_{(n+1,R)} \). The following property of the strips will be used throughout:

Lemma 2.3 (Height of Strips Bounded by \( 3\pi \))
If \( z, w \in R_{n,L} \cup R_{n,R} \), then \( |\text{Im}(z) - \text{Im}(w)| < 3\pi \) and \( |\text{Im}(z) - 2\pi i n| < 3\pi \).

Proof. The straight line through \( v_1 \) and \( v_2 \) divides \( \mathbb{C} \) into two halfplanes, and the preimages under \( E_{a,b} \) of each of them are horizontal half-strips of height \( \pi \). Therefore, each \( R_s \) is contained in three of those half-strips. Since \( \overline{R}_0 L \) and \( \overline{R}_0 R \) intersect the real axis, the second part follows immediately. \( \square \)
Figure 1: The partition formed by the $R_j$: the strips have simple shapes, but the real part of $A_1$ is unbounded.

Figure 2: The partition we use: the set $A$ has bounded real parts.

Figure 3: A partition with bounded real parts of $A_2$ which could be used.
Lemma 2.4 (Sign of Real Part)

If $|E(z)| > |a| + |b|$ for some $z \in \mathbb{C}$, then $\text{Re}(z) > 0$ iff $z \in R_{n,R}$ for some $n \in \mathbb{Z}$, and $\text{Re}(z) < 0$ iff $z \in R_{n,L}$ for some $n \in \mathbb{Z}$.

Proof. Suppose that $\text{Re}(c) < 0$. Then $\{z \in R_{n,L} : \text{Re}(z) > 0\}$ is empty for all $n$, while every $z \in R_{n,R}$ with $\text{Re}(z) < 0$ has real parts between $\text{Re}(c)$ and 0. The vertical line through $c$ (containing all the critical points) is mapped under $E$ to the segment connecting the two critical values $\pm 2\sqrt{ab}$, while the imaginary axis maps to an ellipse with major axis $|a| + |b|$. Therefore if $|E(z)| > |a| + |b|$, then $|\text{Re}(z)|$ has the same sign as the unbounded part of the strip $R_{n,L}$ or $R_{n,R}$ containing $z$.

The case $\text{Re}(c) \geq 0$ is analogous. □

There is a number of further conceivable partitions, such as the one in Figure 3, all with their particular advantages but with a different syntax of symbolic sequences.

Definition 2.5 (External Address)

Let $S := \mathbb{Z}_S^\infty = \{(s_1s_2s_3\ldots) : s_k \in \mathbb{Z}_S\}$ be the sequence space over $\mathbb{Z}_S$ and let $\sigma : S \to S, (s_1s_2s_3s_4\ldots) \mapsto (s_2s_3s_4\ldots)$ be the shift on $S$. We will often use the notation $\underline{s} = (s_1s_2s_3\ldots)$. For all $z \in \mathbb{C}$ with $E^\alpha(z) \in \mathbb{C}'$ for all $n \in \mathbb{N}$ the external address $S(z) \in S$ is the sequence of the symbols of the strips containing $z, E(z), E^{\circ 2}(z), \ldots$

Set $|(n, L)| = |(n, R)| := |n|$ for $n \in \mathbb{Z}$.

Definition 2.6 (Minimal Potential of External Addresses)

For sequence $\underline{s} = s_1s_2s_3\ldots \in S$, define its minimal potential $t_{\underline{s}} \in \mathbb{R}^+_0 \cup \{\infty\}$ via

$$t_{\underline{s}} = \inf \left\{ t > 0 : \lim_{k \geq 1} \frac{|s_k|}{F^\circ(k-1)(t)} = 0 \right\}.$$ 

Notice that $t_{\sigma(\underline{s})} = F(t_{\underline{s}})$.

Definition 2.7 (Exponentially Bounded)

A sequence $\underline{s} \in S$ is exponentially bounded if there is an $x > 0$ such that $|s_k| \leq F^\alpha(k-1)(x)$ for all $k \geq 1$.

This condition is preserved under the shift, but the constant changes:

$$\underline{s}' = \sigma(\underline{s}) \implies |s'_k| \leq F^\alpha(k-1)(F(x)).$$

An equivalent definition of exponential boundedness (as used in [SZ]) is the existence of $x, A > 0$ with $|s_k| \leq AF^\alpha(k-1)(x)$ for all $k \geq 1$. It was shown in [SZ, Theorem 4.2 (1)] that a sequence $\underline{s}$ is exponentially bounded iff $t_{\underline{s}} < \infty$. 

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Lemma 2.8 (External Addresses are Exponentially Bounded)
For $E = E_{a,b}$ choose $\delta > 0$ with $|a| + |b| \leq e^\delta - (\delta + 1)$. Then every orbit $(z_k)$ satisfies the bound
\[
\max\{|\text{Re}(z_k)|, |\text{Im}(z_k)|\} \leq |z_k| < F^{o(k-1)}(|z_1| + \delta).
\]
In particular, every orbit in $\mathbb{C}'$ has exponentially bounded external address.

Proof. For all $k$ we can estimate
\[
|z_{k+1}| + \delta = |ae^{z_k} + be^{-z_k}| + \delta \leq |a| \exp(\text{Re}(z_k)) + |b| \exp(\text{Re}(-z_k)) + \delta \\
\leq (|a| + |b|) \exp(|z_k|) + \delta \leq (e^\delta - (\delta + 1)) \exp(|z_k|) + \delta \\
= e^{\delta |z_k|} + (\delta + 1)e^{\delta |z_k|} + \delta \leq e^{\delta |z_k|} + (\delta + 1) + \delta = F(|z_k| + \delta).
\]
Induction yields
\[
|z_k| + \delta < F^{o(k-1)}(|z_1| + \delta)
\]
for all $k \geq 1$. If the orbit avoids $\mathcal{A}$, then the external address $s_1 s_2 s_3 \ldots$ is defined and we have (using Lemma 2.3 in the first inequality):
\[
2\pi |s_k| \leq |\text{Im}(z_k)| + 3\pi \leq |z_k| + \delta + 3\pi - \delta < F^{o(k-1)}(|z_1| + \delta) + 3\pi - \delta.
\]

3 Tails of Dynamic Rays

In this section, we show that the set of escaping points of $E$ contains uncountably many curves starting at $\infty$; in Section 4 these curves will be extended further, and in Section 6 we will show that all escaping points are associated to these curves.

Definition 3.1 (Tail of Ray)
A ray tail with external address $\underline{s} \in S$ is an injective curve $g_{\underline{s}}: [\tau, \infty) \to \mathbb{C}$ ($\tau > 0$) with the following properties:

- each point on the curve escapes within $\mathbb{C}'$
- each point on the curve has external address $\underline{s}$
- either $\lim_{t \to \infty} \text{Re}(g_{\underline{s}}(t)) = +\infty$ or $\lim_{t \to \infty} \text{Re}(g_{\underline{s}}(t)) = -\infty$
- all $t \geq \tau$ satisfy $E^{o(k)}(g_{\underline{s}}(t)) = \pm F^{o(k-1)}(t) + 2\pi i s_k + O(1)$ as $k \to \infty$. 
The real value $t$ is called the potential of $z = g_{s}(t)$. 

Given an exponentially bounded external address $s$, let $T_{s} ≥ T_{a,b}$ be the least value for which $4π|s_{n+1}| < F^{n}(T_{s})$ for all $n \in \mathbb{N}$ (such a finite $T_{s}$ exists since $s$ is exponentially bounded). Note that $T_{σ(ζ)} ≤ F(T_{s})$.

Lemma 3.2 (Minimal Potentials)
For every exponentially bounded $s$ and every $t > T_{s}$, there is an $N \in \mathbb{N}$ such that for all $n ≥ N$, $F^{n}(t) > T_{σ^{n}(ζ)}$.

Proof. Since $t > T_{s}$, there is an $N > 0$ such that $F^{n}(t) > 4π|s_{n+1}|$ for all $n ≥ N$; enlarge $N$ if necessary so that also $F^{n}(t) > T_{a,b}$. Then $F^{n}(t) = F^{n}(F^{n}(t)) > F^{n}(T_{σ^{n}(ζ)}) ≥ T_{σ^{n}(ζ)}$. $\square$

Proposition 3.3 (Existence of Tails of Rays)
For all $a, b \in \mathbb{C}^{*}$ and for every exponentially bounded sequence $s ∈ S$ there exists a ray tail $g_{s}([T_{s}, ∞])$ with external address $s$. Furthermore

$$g_{s}(t) = \begin{cases} t - α + 2πis_{1} + r_{s}(t) & \text{if } s_{1} ∉ \mathbb{Z}_{R} \\ -t + β + 2πis_{1} + r_{s}(t) & \text{if } s_{1} ∉ \mathbb{Z}_{L} \end{cases}$$

with

$$|r_{s}(t)| ≤ (C_{1} + 8π|s_{2}|)e^{-t}$$

with a constant $C_{1}$ depending only on $a$ and $b$. Moreover, for $t ≥ T_{s}$,

$$E(g_{s}(t)) = g_{σ(ζ)}(F(t)) .$$

Overview: We construct a family of maps $g_{s}^{n}: \mathbb{R}^{+} → \mathbb{C}$ for $n ∈ \mathbb{N}$ as follows:

$$g_{s}^{n}(t) := L_{s_{1}} \circ \ldots \circ L_{s_{n}} \circ (±F^{n}(t) + 2πis_{n+1}) . \tag{5}$$

The $±$ depends on $s_{n+1}$; it is $+$ for $s_{n+1} ∈ \mathbb{Z}_{R}$ and $-$ for $s_{n+1} ∈ \mathbb{Z}_{L}$ (then $±F^{n}(t) + 2πis_{n+1} ∈ R_{s_{n+1}}$ for large $t$). We will show that the $g_{s}^{n}$ are defined for all $t ≥ T_{s}$ independently of $n$ and converge uniformly to the desired function $g_{s}(t)$. For this proof we need several lemmas. The first of them gives control over the inverse branches of $E$.

Lemma 3.4 (Control on $L_{s}$)
Let $w ∈ \mathbb{C}$ and $z = L_{s}(w)$ with $|w| > \max \left\{ \sqrt{\frac{2a}{α}}(|a| + |b|), \sqrt{\frac{2a}{β}}(|a| + |b|), 8|ab|, 1 \right\}$ and $s ∈ \mathbb{Z}_{R}$. Then

$$z = \ln w - α + 2πip + r^{*}$$
for some \( p \in \mathbb{Z} \) with \(|r^*| < 2\frac{b|e^{-2z}|}{a} < 1 \) and \(|r^*| < 8|ab| \cdot |w|^{-2} < \frac{1}{|w|} < 1\).

Similarly, for \( s \in \mathbb{Z}_L \) we get
\[
z = -\ln w + \beta + 2\pi ip + r^*
\]
for some \( p \in \mathbb{Z} \) with \(|r^*| < 2\frac{b|e^{-2z}|}{a} < 1 \) and \(|r^*| < 8|ab| \cdot |w|^{-2} < \frac{1}{|w|} < 1\).

Note that the branch of \( \ln w \) in this lemma is immaterial because of the ambiguity in \( p \).

**Proof.** Heuristically, \( E(z) \approx ae^z \) if \( \text{Re}(z) \gg 0 \) and \( E(z) \approx be^{-z} \) if \( \text{Re}(z) \ll 0 \). We discuss the case \( s \in \mathbb{Z}_R \); the other case is similar. We have
\[
w = E(z) = ae^z + be^{-z} = ae^z \left( 1 + \frac{b}{a} e^{-2z} \right)
\]
\[
\Rightarrow z \in \ln \frac{w}{a} - \ln \left( 1 + \frac{b}{a} e^{-2z} \right) + 2\pi i \mathbb{Z}
\]

(6)

Lemma 2.4 implies \( \text{Re}(z) > 0 \), so we have
\[
\sqrt{2\frac{b}{a}} (|a| + |b|) < |w| \leq |a|e^{\text{Re}(z)} + |b|e^{-\text{Re}(z)} \leq (|a| + |b|)e^{\text{Re}(z)},
\]
hence \( e^{\text{Re}(z)} > \sqrt{2\frac{b}{a}} \) and thus
\[
\Rightarrow \left| \frac{b}{a} e^{-2z} \right| = \left| \frac{b}{a} \right| e^{-2\text{Re}(z)} < \frac{1}{2}.
\]

Since \( \ln(1 + u) < (2 \ln 2)|u| \) for \(|u| < \frac{1}{2} \), it follows that
\[
\left| \ln \left( 1 + \frac{b}{a} e^{-2z} \right) \right| < 2 \ln 2 \left| \frac{b}{a} e^{-2z} \right| < 1
\]
(here we use the principal branch of \( \ln \)).

Together with (3) the first claim follows. For the estimate of \(|r^*| \) in terms of \(|w|\) it follows from
\[
2 \ln 2 \left| \frac{b}{a} e^{-2z} \right| < 1 \Rightarrow |be^{-z}| < \frac{1}{2\ln 2} |ae^z| \text{ and hence by (4) we get } |w| < 2|ae^z|.
\]

We thus obtain
\[
|ab| \frac{2}{|ae^z|^2} < 8|ab| \cdot |w|^{-2} < |w|^{-1} < 1.
\]

The next lemma gives us control on \( g_1^L = L_{s_1} \circ (\pm F + 2\pi i s_2) \).
Lemma 3.5 (Control on $L_s \circ F$)
Choose $a, b \in \mathbb{C}^*$ and $n \in \mathbb{Z}$. Then for $t > 0$ such that
\[ F(t) > \max \left\{ \sqrt{2b} \left| |a| + |b| \right|, \sqrt{2a} \left| |a| + |b| \right|, 8|ab|, 1 \right\} \]
and $F(t) > 4\pi|n|$, 
\[ L_s(\pm F(t) + 2\pi in) \in t - \alpha + \pi i\mathbb{Z} + r \quad \text{with} \quad |r| < (4 + 8\pi|n|)e^{-t} < 4 \]
for $s \in \mathbb{Z}_R$ and 
\[ L_s(\pm F(t) + 2\pi in) \in -t + \beta + \pi i\mathbb{Z} + r \quad \text{with} \quad |r| < (4 + 8\pi|n|)e^{-t} < 4 \]
for $s \in \mathbb{Z}_L$.

PROOF. For $s \in \mathbb{Z}_R$, Lemma 3.4 gives
\[ L_s(\pm F(t) + 2\pi in) \in \ln \left( \pm F(t) \left( 1 \pm \frac{2\pi in}{F(t)} \right) \right) - \alpha + 2\pi i\mathbb{Z} + r^* \]
\[ = t + \ln(1 - e^{-t}) + \ln \left( 1 + \frac{2\pi in}{F(t)} \right) - \alpha + 2\pi i\mathbb{Z} + r^* + (i\pi) \]
with $|r^*| < |F(t) + 2\pi in|^{-1} \leq F(t)^{-1} = e^{-t} \frac{e^t}{e^t - 1} < 2e^{-t} < 1$. Here, the last term $(+i\pi)$ only occurs in the case $L_s(-F(t)\ldots)$.

Since $2\pi|n|/F(t) < 1/2$, we get
\[ \left| \ln \left( 1 + \frac{2\pi in}{F(t)} \right) \right| \leq (2\ln 2) \frac{2\pi|n|}{F(t)} \leq 8\pi|n|e^{-t} < 2 \]
and
\[ |\ln(1 - e^{-t})| < 2e^{-t} < 1. \]
Thus we get
\[ L_s(\pm F(t) + 2\pi in) \in t - \alpha + \pi i\mathbb{Z} + r \]
with $|r| < (4 + 8\pi|n|)e^{-t} < 4$.

The proof for $s \in \mathbb{Z}_L$ is analogous. 

Lemma 3.6 (Bound on Real Parts)
For all $s \in S$, $n \in \mathbb{N}$ and $t > 0$ with $t \geq T_{a,b}$
\[ \left| \text{Re} \left( g^n_s(t) \right) \right| > t - (M + 2). \]
Moreover, if $|\text{Re}(z)| > \max \left\{ \frac{1}{2} \ln \left| \frac{1}{\alpha} \right|, \frac{1}{2} \ln \left| \frac{1}{b} \right|, \ln \left| \frac{1}{|a|} \right|, \ln \left| \frac{1}{|b|} \right| \right\}$, then
\[ |E'(z)| > \frac{1}{2} K e^{\text{Re}(z)} > 2. \]
Proof. We will prove the first part via induction simultaneously for all external addresses $s$ and all $t \geq T_{a,b}$. For $n = 0$ clearly

$$|\text{Re}(g_s^0(t))| = t > t - (M + 2).$$

Assume that $|\text{Re}(g_s^{n-1}(t))| > t - (M + 2)$. Then by Lemma 3.3 and inductive hypothesis we get

$$|\text{Re}(g_s^n(t))| = |\text{Re} \left( \pm \ln \left( g_{\alpha(\#)}^{n-1}(F(t)) \right) \pm \{\alpha, \beta\} + 2\pi i \rho + r^* \right)\right|$$

$$\geq |\text{Re} \left( \ln \left( g_{\alpha(\#)}^{n-1}(F(t)) \right) \right) - M - 1 = |\text{Re} \left( g_{\alpha(\#)}^{n-1}(F(t)) \right) - M - 1$$

$$\geq |\ln |F(t) - (M + 2)|| - M - 1 = \ln(e^t - M - 3) - M - 1$$

$$> t - M - 2.$$

The last step needs an elementary calculation based on $F(t) \geq F(T_{a,b}) \geq T_{a,b} + M + 4$.

For the second part, we write:

$$|E'(z)| = \begin{cases} |a|e^{\text{Re}(z)} \cdot |1 - \frac{b}{a}e^{-2\text{Re}(z)}| & \text{if } \text{Re}(z) > 0 \\ |b|e^{\text{Re}(z)} \cdot |1 - \frac{a}{b}e^{-2\text{Re}(z)}| & \text{if } \text{Re}(z) < 0 \end{cases}.$$

By hypothesis, $|\frac{b}{a}e^{-2\text{Re}(z)}| < \frac{1}{2}$, resp. $|\frac{a}{b}e^{-2\text{Re}(z)}| < \frac{1}{2}$, so we get

$$|E'(z)| > \frac{1}{2} \min\{|a|, |b|\} e^{\text{Re}(z)}$$

and hence $|E'(z)| > 2$. \qed

We can now finish the construction of dynamic ray tails.

Proof of 3.3. We show first that $g_s^n(t)$ converges uniformly in $t$ to a limit function $g_s([T_s, \infty]) \rightarrow \mathbb{C}$. For $t \geq T_s$, we write

$$|g_s^{n+1}(t) - g_s^n(t)| = L_{s_1} \circ \ldots \circ L_{s_n} \circ L_{s_{n+1}} (\pm F(F^{\text{un}}(t)) + 2\pi is_{n+2})$$

$$- L_{s_1} \circ \ldots \circ L_{s_n} (\pm F^{\text{un}}(t) + 2\pi is_{n+1})$$

$$=: |\mu_n - \nu_n|.$$ 

Since $\mu_n, \nu_n \in \overline{R}_{s_{n+1}}$, we get $|\text{Im}(\mu_n - \nu_n)| \leq 3\pi$ by Lemma 2.3, by Lemma 3.3 we get $|\text{Re}(\mu_n - \nu_n)| < 4 + M$, hence $|\mu_n - \nu_n| < 4 + 3\pi + M$. 

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By construction, all \( w \in A \) have \( |\text{Re}(w)| \leq 2\sqrt{|ab|} \). However, by Lemma 3.6 and (2),
\[
|\text{Re}(L_{s_k} \circ \ldots \circ L_{s_n}(\nu_n))| > F^{(k-1)}(t) - (M + 2) \geq T_s - (M + 2)
\]
and similarly for \( \mu_n \), so the same branch of \( L_{s_k-1} \) applies to \( L_{s_k} \circ \ldots \circ L_{s_n}(\nu_n) \) and \( L_{s_k} \circ \ldots \circ L_{s_n}(\mu_n) \).

By Lemma 3.6, \( L_{s_k} \circ \ldots \circ L_{s_n}(\nu_n) \) and \( L_{s_k} \circ \ldots \circ L_{s_n}(\mu_n) \) are both in the domain \( \{ z \in \mathbb{C}: \text{Re}(z) > T_s - (M + 2) \} \) or both in \( \{ z \in \mathbb{C}: \text{Re}(z) < -T_s + (M + 2) \} \) on which \( |E'(z)| > \frac{1}{2} Ke^{\text{Re}(z)} | \geq 2 \). In particular,
\[
|\text{Re}(\nu_n)| = F^{\circ n}(t) > t + (M + 4)
\]
and so \( |\text{Re}(\mu_n)| > t \).

Therefore,
\[
|L_{s_n}(\mu_n) - L_{s_n}(\nu_n)| < \frac{|\mu_n - \nu_n|}{\frac{1}{2}Ke^t} < \frac{8 + 6\pi + 2M}{K} e^{-t}.
\]

After repeated application, we get
\[
|g_{s_n}^{n+1}(t) - g_{s_n}^n(t)| < \frac{8 + 6\pi + 2M}{2^{n-1}K} e^{-t}.
\]
Therefore, the \( g_{s_n}^n \) converge uniformly to a continuous limit function \( g_{s_n}: [T_s, \infty) \to \mathbb{C} \).

By construction, \( E(g_{s_n}^n(t)) = g_{s_{(\nu_n)}}^{n-1}(F(t)) \), so in the limit we obtain the desired relation
\[
E(g_{s_n}(t)) = g_{s_{(\nu_n)}}(F(t))
\]
for \( t \geq T_s \). In order to estimate \( r \), we get
\[
|g_{s_n}(t) - g_{s_n}(1)| \leq \sum_{n=1}^{\infty} |g_{s_n}^{n+1}(t) - g_{s_n}^n(t)|
\]
\[
< \sum_{n=1}^{\infty} \frac{8 + 6\pi + 2M}{2^{n-1}K} e^{-t} = \frac{2}{K} \frac{8 + 6\pi + 2M}{K} e^{-t}.
\]

By Lemma 3.5, we have \( g_2(t) = t - \alpha + 2\pi is_1 + r_2(t) \), resp. \( g_2(t) = -t + \beta + 2\pi is_1 + r_2(t) \) with
\[
r_2(t) \leq \left( \frac{8 + 6\pi + 2M}{K} + 4 + 8\pi|s_2| \right) e^{-t}.
\]
Finally we prove injectivity. If \( g_\sigma(t_1) = g_\sigma(t_2) \) for \( t_2 \geq t_1 \geq T_\sigma \), then we get
\[
g_{\sigma^n(\underline{s})}(F^{\circ n}(t_1)) = g_{\sigma^n(\underline{s})}(F^{\circ n}(t_2))
\]
for all \( n \geq 0 \) with the bounds
\[
|g_{\sigma^n(\underline{s})}(F^{\circ n}(t_j)) - (\pm F^{\circ n}(t_j) + \{ -\alpha \text{ or } + \beta \} - 2\pi i s_{n+1})| = |r_{\sigma^n(\underline{s})}(F^{\circ n}(t_j))| < (C_1 + 8\pi |s_{n+2}|) e^{-F^{\circ n}(t_j)}
\]
\[
= 8\pi |s_{n+2}| / F^{\circ (n+1)}(t_j) + o(1).
\]

Since \( t_1, t_2 > T_\sigma \), the right hand side is bounded; this implies that \( |F^{\circ n}(t_1) - F^{\circ n}(t_2)| \) must be bounded as well as \( n \to \infty \). This implies that \( t_1 = t_2 \).

4 Dynamic Rays

In this section we construct dynamic rays by extending ray tails to as low potentials as possible. The idea is to use the relation \( E(g_\sigma(t)) = g_{\sigma(\underline{s})}(F(t)) \) to pull ray tails back by the dynamics: \( g_\sigma \) is a branch of \( E^{-1} \circ g_{\sigma(\underline{s})} \circ F \).

**Theorem 4.1 (Existence of Dynamic Rays)**

(1) If neither of the two critical orbits escapes then for every exponentially bounded \( \underline{s} \) there exists an injective curve \( g_\sigma: t_\sigma, \infty [\to C \) consisting of escaping points such that
\[
E(g_\sigma(t)) = g_{\sigma(\underline{s})}(F(t)) \text{ for all } t > t_\sigma
\]
which extends the ray tail with external address \( \underline{s} \) as constructed in Proposition 3.3. In particular, it inherits its asymptotics for large \( t \).

(2) If at least one of the critical orbits escapes, then (1) is still true for every \( \underline{s} \), unless \( \underline{s} \) is such that there is an \( n \geq 1 \) and a \( t_0 > F^{\circ n}(t_\sigma) \) with \( g_{\sigma^n(\underline{s})}(t_0) \in \{v_1, v_2\} \). For those exceptional \( \underline{s} \), there is an injective curve \( g_\sigma: t_\sigma^*, \infty [\to C \) with the same properties as before, where \( t_\sigma^* \) is the largest potential which has an \( n \geq 1 \) such that \( g_{\sigma^n(\underline{s})}(F^{\circ n}(t_\sigma^*)) \in \{v_1, v_2\} \).

The curve \( g_\sigma: t_\sigma, \infty [\to C \) is called the **dynamic ray at external address** \( \underline{s} \).

**Remark.** Notice that it is no longer required that all points on the ray share the external address \( \underline{s} \). Since the partition \( \{R_s\} \) of \( C \) is unnatural from the dynamical point of view, there is no reason why the rays should respect it. Only the points with large potential (those which are on ray tails) have external address \( \underline{s} \). Note that the definition of \( t_\sigma \) is the same as for the case of exponential functions \( \lambda \exp \) in [SZ]: since \( E_{a,b}(z) \approx ae^z \text{ resp. } E_{a,b}(z) \approx be^{-z} \)
in far right resp. left half planes, this coincidence of the values of \( t_2 \) reflects the fact that the value of \( t_2 \) for exponential functions does not depend on the complex parameter \( \lambda \).

**Proof.** We want to show that \( g_2(t) \) exists for \( t > t_2 \). Choose \( \varepsilon > 0 \). By Lemma 3.2 there is an \( N \in \mathbb{N} \) such that \( F^N(t_2 + \varepsilon) > T_{\sigma N}(s) \).

By Proposition 3.3 the ray tail \( g_{\sigma N}(s)(t) \) exists for \( t \geq T_{\sigma N}(s) \). Since \( E^N \) maps the ray tail \( g_2 \) to a tail of \( g_{\sigma N} \), it follows that there is a branch of \( E^- \) which sends (a subset of) the ray tail \( g_{\sigma N} : [T_{\sigma N}(s), \infty) \to \mathbb{C} \) to a curve \( g_2 : [t_2 + \varepsilon, \infty) \to \mathbb{C} \). In other words, the ray tail \( g_2 \) can be extended to potentials \( t_2 + \varepsilon \). Since \( \varepsilon \) was arbitrary, we have shown the existence of the dynamic ray \( g_2 : (t_2, \infty) \to \mathbb{C} \). Injectivity follows from Proposition 3.3. This proves the second claim.

Note that the pull-back of a ray tail is possible if and only if it contains no critical value. Therefore, this construction can be carried out for all ray tails except those mentioned in the exceptions of the third claim. This proves the theorem.

**Remark.** Note that the pull-back in the proof of the previous theorem need not respect the partition we had initially used for constructing the ray tails: the branch of every \( E^{-1} \) is determined using the ray tails and is then continued analytically. It may well happen that a dynamic ray crosses the partition boundary, but only at potentials below \( T_2 \).

Next we investigate under which conditions an escaping point is on a ray.

**Theorem 4.2 (Fast Escaping Points are on Ray)**

Let \((z_k)\) be an escaping orbit and let \( s \) be such that \( z_k \in \overline{R}_{s_k} \) for all \( k \); suppose also that there exists a \( t' > t_2 \) with \(|\text{Re}(z_k)| \geq F^{(k-1)}(t')\) for infinitely many \( k \). If in addition

\[
|\text{Re}(z_k)| > 2\sqrt{|ab|} + 1 \quad \text{for all} \; k, \text{as well as} \; t' \geq T_2 \tag{8}
\]

then there is a \( t \geq t' \) such that \( z_1 = g_2(t) \).

If (8) is not satisfied for all \( k \), then there is an \( N \in \mathbb{N} \) such that \( z_{N+1} = g_{\sigma N}(s)(F^{\sigma N}(t)) \) for some \( t \geq t' \); moreover, at least if the two critical orbits do not escape, there is an external address \( s' \) which differs from \( s \) only in finitely many entries such that \( z_1 = g_2(t) \) for some \( t \geq t' > t_2 \).

**Proof.** For \( k \in \mathbb{N} \) pick \( t_k > 0 \) such that \( F^{(k-1)}(t_k) = |\text{Re}(z_k)| \). By assumption, \( t_k \geq t' \) for infinitely many \( k \), and by Lemma 2.3 the sequence \((t_k)\) is bounded above. Moreover, by Lemma 2.3 we have \(|\text{Im}(z_k)| \leq 2\pi(|s_k| +
Suppose first that $|\text{Re}(z_k)| > 2\sqrt{|ab|} + 1$ for all $k$. Then all $z_k \notin A$, hence $z_k \in R_s$, and thus

$$z_l = L_{s_l} \circ \ldots \circ L_{s_{k-1}}(z_k)$$

for all $l \in \{k-1, \ldots, 2, 1\}$.

Consider the points $w_k := \pm F^{(k-1)}(t_k) + 2\pi is_k$, where the signs are chosen such that for all $k$, $z_k$ and $w_k$ are in the same (right or left) half plane; then, using Lemma 2.3 again,

$$|z_k - w_k| \leq 3\pi. \quad (9)$$

Therefore, given $\varepsilon \in [0, 1]$, the derivative bound in Lemma 3.6 implies $|z_k - L_{s_{k-1}}(w_k)| < \varepsilon$ for sufficiently large $k$. But this implies that the same branch of $L_{s_{k-2}}$ applies to both points, and repeated application of this argument shows that if $k$ is sufficiently large, then for all $l \in \{k-1, \ldots, 1\}$ we have

$$|z_l - L_{s_l} \circ \ldots \circ L_{s_{k-1}}(w_k)| < \varepsilon.$$ 

If in addition $t_k \geq T_\sigma$, then $g_\sigma^{k-1}(t_k) = L_{s_\sigma} \circ \ldots \circ L_{s_{k-1}}(w_k)$, and we have $|z_l - g_\sigma^{k-1}(t_k)| < \varepsilon$. Now suppose $t' \geq T_\sigma$. Let $t$ be a limit point of the sequence $(t_k)$ (restricted to such $k$ as mentioned above). Obviously $t \geq t' \geq T_\sigma$. By uniform convergence of $g_\sigma^{k-1}$ to $g_\sigma$ for potentials at least $T_\sigma$, we get $|g_\sigma^{k-1}(t_k) - g_\sigma(t_k)| < \varepsilon$ (possibly by enlarging $k$). Finally, for $t_k$ close enough to $t$ ($t$ is a limit point of $(t_k)$) $|g_\sigma(t_k) - g_\sigma(t)| < \varepsilon$. Combining this, it follows

$$|z_l - g_\sigma(t)| \leq |z_l - g_\sigma^{k-1}(t_k)| + |g_\sigma^{k-1}(t_k) - g_\sigma(t_k)| + |g_\sigma(t_k) - g_\sigma(t)| < 3\varepsilon$$

for certain sufficiently large $k$. Hence $g_\sigma(t) = z_1$ because $\varepsilon > 0$ was arbitrary.

If the condition $|\text{Re}(z_k)| > 2\sqrt{|ab|} + 1$ does not hold for all $k$, then there is an $N \in \mathbb{N}$ such that it holds for all $k \geq N$; similarly, by Lemma 3.2 if $t' \geq T_\sigma$ is not satisfied, then for sufficiently large $N$, we have $F^{oN}(t') \geq T_\sigma^{oN}(A)$. Therefore, there is an $N \in \mathbb{N}$ with $z_{N+1} = g_{oN}(F^{oN}(t))$ for some $t \geq t'$. Pulling back $N$ times along the orbit from $z_1$ to $z_{N+1}$, it follows that $z_l = g_{oN}(t)$ for some external address $s'$ which can differ from $s$ only in the first $N$ entries; thus $t_{s'} = t_s$. However, if the pull-back runs through a dynamic ray which contains an escaping critical value, then this pull-back is impossible — and only then.

Proposition 4.3 (Controlled Escape for Points on Rays)

For every exponentially bounded external address $s$ and for every $t > t_s$, the orbit of $g_s(t)$ satisfies the asymptotic bound

$$E^{\alpha k}(g_s(t)) = \begin{cases} F^{\alpha k}(t) - \alpha + 2\pi is_{k+1} + o(1) & \text{if } s_{k+1} \in \mathbb{Z}_R \\ -F^{\alpha k}(t) + \beta + 2\pi is_{k+1} + o(1) & \text{if } s_{k+1} \in \mathbb{Z}_L \end{cases}$$
as $k \to \infty$. In particular, for every real $p > 0$ it satisfies
\[
\frac{|\text{Im}(E^{\circ k}(g_s(t)))|^p}{\text{Re}(E^{\circ k}(g_s(t)))} \to 0.
\]

**Proof.** By Proposition 3.3 we have good error bounds for the dynamic rays $g_{\sigma^k(s)}$ for potentials greater than $T_{\sigma^k(s)}$. Since $t > t_s$, there exists a $k_0$ such that $F^{\circ k}(t) > T_{\sigma^k(s)}$ for all $k \geq k_0$ (Lemma 3.2). Then
\[
E^{\circ k}(g_s(t)) = g_{\sigma^k(s)}(F^{\circ k}(t)) = F^{\circ k}(t) - \alpha + 2\pi is_{k+1} + r_{\sigma^k(s)}(F^{\circ k}(t))
\]

with
\[
|r_{\sigma^k(s)}(F^{\circ k}(t))| < (C_1 + 8\pi|s_{k+2}|)e^{-F^{\circ k}(t)},
\]
where $C_1$ only depends on $a$ and $b$. This tends to 0 as $k \to \infty$ (extremely fast). Along the orbit of $g_s(t)$, the absolute values of real parts thus grow like $F^{\circ k}(t)$, while the imaginary parts are bounded in absolute value by the asymptotically much smaller quantity $AF^{\circ k}(t_s + \varepsilon)$ for any $\varepsilon \in [0, t - t_s]$. In particular we have
\[
\ln \left( \frac{|\text{Im}(E^{\circ k}(g_s(t)))|^p}{|\text{Re}(E^{\circ k}(g_s(t)))|} \right) < pF^{\circ(k-1)}(t_s + \varepsilon) - F^{\circ(k-1)}(t) + O(1) \to -\infty.
\]
This proves the last claim.

5 Eventually Horizontal Escape

The main result in this section shows essentially that, for any given external address, there is at most one point with this external address which is not on a dynamic ray.

Define
\[
R_h := \max \left\{ \ln \left( \frac{2h + 8\pi}{|a|\pi} \right), \ln \left( \frac{2h + 8\pi}{|b|\pi} \right), \frac{1}{2} \ln \left| \frac{2h}{a} \right|, \frac{1}{2} \ln \left| \frac{2h}{b} \right|, |a| + |b| \right\}
\]
(depending on the parameters $a, b \in \mathbb{C}^*$).

**Lemma 5.1 (Exponential Separation of Orbits)**

Let $R > R_h$ and let $(z_k)$ and $(w_k)$ be two escaping orbits with $|\text{Re}(z_k)| > R$, $|\text{Re}(w_k)| > R$, $\text{Re}(z_k)/\text{Re}(w_k) > 0$ and $|\text{Im}(z_k - w_k)| < h$ for some $h > 0$ and all $k$. Define $d_k := \text{Re}(z_k - w_k)$ for all $k$. If $|\text{Re}(z_1)| - |\text{Re}(w_1)| \geq 3$ and none of the critical values escapes, then the following holds
1. \(|d_{k+1}| \geq \exp(|d_k|)\) and \(|\text{Re}(z_k)| = |\text{Re}(w_k)| + |d_k|\) for all \(k \geq 1\);

2. if \(s \in S\) is such that \(z_k \in R_s\) for all \(k\), then \(z_1 = g_s(t')\) for some \(t' > t_s\) and some \(s'\) which differs from \(s\) in only finitely many entries; if \(|d_1|\) is sufficiently large, then \(s' = s\);

3. if \(w_1 = g_s(t'')\) for some external address \(s''\) and some \(t'' > t_s''\), then \(t' > t''\).

If one or both the critical values escape then the first and third statements are still true, and there is an \(N \in \mathbb{N}\) such that \(z_{N+1} = g_{s^N}(\omega)(F^N(t))\) for some \(t > t_s\).

**Proof.** 1. Let \(w_k = t_k + iu_k\) with real \(t_k, u_k\). We write

\[
|w_k| = |E(w_{k-1})| = \begin{cases} 
|a|e^{t_{k-1}} \cdot |1 + \frac{b}{a} e^{-2w_{k-1}}| & \text{if } t_{k-1} > 0 \\
|b|e^{-t_{k-1}} \cdot |1 + \frac{a}{b} e^{2w_{k-1}}| & \text{if } t_{k-1} < 0 
\end{cases}
\]

By assumption, \(\text{Re}(z_k)\) and \(\text{Re}(w_k)\) always have the same sign. By definition of \(d_k\), the property \(|\text{Re}(z_k)| = |\text{Re}(w_k)| + |d_k|\) is equivalent to the fact that \(d_k\) has the same sign as \(\text{Re}(z_k)\) and \(\text{Re}(w_k)\), i.e. \(\frac{t_k}{d_k} > 0\). Since \(|\text{Re}(w_{k-1})| > R\), we have \(e^{-2|\text{Re}(w_{k-1})|} < \min \{|\frac{a}{2b}|, |\frac{b}{2a}|\}\) and thus

\[
\frac{3}{2}|a|e^{t_{k-1}} \geq |w_k| \geq \frac{1}{2}|a|e^{t_{k-1}} \quad \text{if } t_{k-1} > 0 ,
\]

and

\[
\frac{3}{2}|b|e^{t_{k-1}} \geq |w_k| \geq \frac{1}{2}|b|e^{t_{k-1}} \quad \text{if } t_{k-1} < 0 .
\]

Since \(\text{Re}(w_k)\) and \(\text{Re}(z_k)\) always have the same signs, there exists a \(K' \in \{|a|, |b|\}\) (depending on \(s_k\)) such that

\[
\frac{3K'}{2}e^{\text{Re}(z_k-1)} \geq |z_k| \geq \frac{K'}{2}e^{\text{Re}(z_k-1)}
\]

and

\[
\frac{3K'}{2}e^{\text{Re}(w_k-1)} \geq |w_k| \geq \frac{K'}{2}e^{\text{Re}(w_k-1)}.
\]

Now let \(|\text{Re}(z_{k-1})| - |t_{k-1}| \geq 3\) be the inductive hypothesis; then \(|d_{k-1}| \geq 3\) and \(\frac{t_{k-1}}{d_{k-1}} > 0\). Since \(|\text{Im}(z_k - w_k)| < h\), the Pythagorean Theorem implies

\[
(\text{Re}(z_k))^2 = |z_k|^2 - (\text{Im}(z_k))^2 \geq |z_k|^2 - (|w_k| + h)^2
\]

\[
\geq \left(\frac{K'}{2} \exp |t_{k-1} + d_{k-1}|\right)^2 - \left(\frac{3}{2}K' \exp |t_{k-1}| + h\right)^2 .
\]
This difference is positive: from $|t_{k-1}| > R_h > \max \left\{ \ln \left( \frac{2h}{|\theta|} \right), \ln \left( \frac{2h}{|\theta|} \right) \right\}$, it follows $\frac{2h}{K'e^{\left|t_{k-1}\right|}} < \pi$; since $|d_{k-1}| \geq 3$, we get $e^{d_{k-1}} \geq e^3 > 3 + \pi > 3 + \frac{2h}{K'e^{\left|t_{k-1}\right|}}$ and therefore

$$\frac{K'}{2} e^{\left|t_{k-1}\right|} e^{d_{k-1}} \geq \frac{3K'}{2} e^{\left|t_{k-1}\right|} + h.$$ 

Hence we can extract the root in (14) and obtain

$$|t_k + d_k| \geq \frac{K'}{2} \exp |t_{k-1} + d_{k-1}| \sqrt{1 - \left( \frac{3 \exp |t_{k-1}| + 2h/K'}{\exp |t_{k-1} + d_{k-1}|} \right)^2}. \quad (15)$$

Since $|d_{k-1}| \geq 3$ and $\frac{t_{k-1}}{d_{k-1}} > 0$, we get as above

$$0 < \frac{3 \exp |t_{k-1}| + 2h/K'}{\exp |t_{k-1} + d_{k-1}|} = \frac{3 + (2h/K') \exp(-|t_{k-1}|)}{\exp |d_{k-1}|}$$

$$< \frac{3 + \pi}{\exp |d_{k-1}|} \leq \frac{3 + \pi}{e^3} < \frac{1}{2}. \quad (16)$$

The radicand in (15) is thus in $(\frac{1}{2}, 1)$. Since $|t_k| = |\Re(w_k)| \leq |w_k|$, we get

$$|t_k + d_k| - |t_k| \geq \frac{K'}{4} \exp |t_{k-1} + d_{k-1}| - \frac{3}{2} K' \exp |t_{k-1}|$$

$$= \frac{K'}{4} \exp |t_{k-1}| \left( \exp |d_{k-1}| - 6 \right) > 0.$$ 

Thus $|\Re(z_k)| > |\Re(w_k)|$ and $\frac{h}{d_k} > 0$. Since $|t_{k-1}| \geq R$, we get $\frac{K'}{4} \exp |t_{k-1}| > \frac{K'}{4} \exp (\ln (8/K')) = 2$. Thus

$$|d_k| = |t_k + d_k| - |t_k| > 2(\exp |d_{k-1}| - 6)$$

$$= \exp |d_{k-1}| + (\exp |d_{k-1}| - 12) \geq \exp |d_{k-1}|.$$ 

This shows the first claim.

2. The second claim will be done in four steps:

(a). $|t_k + d_k| \geq F^{\circ(k-N)}(t_N + d_N + \gamma') - \gamma'$ for some $\gamma' \in \mathbb{R}$ and $N \in \mathbb{N}$ for all $k \geq N$ (using the estimates from the first step)

(b). $|t_k| + \gamma \leq F^{\circ(k-N)}(t_N + \gamma)$ for some $\gamma \in \mathbb{R}$ and all $k > N$ (from the maximal growth rate along orbits)

(c). this will imply an upper bound on $|w_k|$, hence on $|\Im(w_k)|$ and on $|s_k|$; we will get $F^{\circ(N-1)}(t_k) \leq |t_N| + \gamma$

(d). the conclusion will then follow.
Choose $\gamma' < \ln \left( \frac{K}{4} \right)$ (possibly negative). By (15), (16) and since $K' \geq K$, we have

$$|t_k + d_k| = |\text{Re}(z_k)| \geq \frac{K}{4} e^{\left|t_{k-1} + d_{k-1}\right|} = e^{\left|t_{k-1} + d_{k-1}\right| + \ln \frac{K}{4}} \geq e^{\left|t_{k-1} + d_{k-1}\right| + \gamma'} - \gamma' - 1 = F\left(\left|t_{k-1} + d_{k-1}\right| + \gamma'\right) - \gamma'$$

provided $|t_{k-1} + d_{k-1}|$ is sufficiently large. There is thus an $N > 0$ such that inductively for all $k \geq N$

$$|t_k + d_k| + \gamma' \geq F^{\circ(k-N)}(|t_N + d_N| + \gamma') \quad (17)$$

Choose $\gamma > \ln \left( \frac{3K_{\max}}{2} \right)$. We get

$$|t_k| = |\text{Re}(w_k)| \leq \left|w_k\right| \leq \frac{3K_{\max}}{2} e^{\left|t_{k-1}\right|} = F\left(\left|t_{k-1}\right| + \ln \frac{3K_{\max}}{2}\right) + 1 \leq F\left(\left|t_{k-1}\right| + \gamma\right) - \gamma,$$

again provided $t_{k-1}$ is sufficiently large. Possibly by enlarging $N$, we have for all $k \geq N$

$$|t_k| + \gamma \leq F^{\circ(k-N)}(|t_N| + \gamma).$$

For every $\varepsilon > 0$ the definition of $\underline{\sigma}$ shows that there are infinitely many $k \geq N$ with

$$F^{\circ(k-N)}(F^{\circ(N-1)}(\underline{\sigma} - \varepsilon)) = F^{\circ(k-1)}(\underline{\sigma} - \varepsilon) < 2\pi |s_k| \leq |\text{Im}(w_k)| + 3\pi \leq |w_k| + 3\pi \leq \frac{3K_{\max}}{2} e^{\left|t_{k-1}\right|} + 3\pi < e^{\left|t_{k-1}\right| + \gamma} + 3\pi \leq F^{\circ(k-N)}(|t_N| + \gamma) + 1 + 3\pi.$$

Since this holds for arbitrarily large $k$, we get

$$F^{\circ(N-1)}(\underline{\sigma} - \varepsilon) \leq |t_N| + \gamma.$$

Since $\varepsilon > 0$ was arbitrary, it follows

$$t_{\sigma^{N-1}(\underline{\sigma})} = F^{\circ(N-1)}(\underline{\sigma}) \leq |t_N| + \gamma.$$

(Enlarge $N$ if necessary so that $|t_N + d_N| + \gamma' > |t_N| + \gamma + 2$ and $F^{\circ(N-1)}(\underline{\sigma} + 2) > T_{\sigma^{N-1}(\underline{\sigma})} + 1$ (such $N$ exists by Lemma (3.2)). Then

$$|\text{Re}(z_k)| = |t_k + d_k| \geq F^{\circ(k-N)}(|t_N + d_N| + \gamma') - \gamma' \geq F^{\circ(k-N)}(|t_N| + \gamma + 2) - \gamma' \geq F^{\circ(k-N)}(t_{\sigma^{N-1}(\underline{\sigma})} + 2) - \gamma' \gg F^{\circ(k-N)}(t_{\sigma^{N-1}(\underline{\sigma})} + 1).$$

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for infinitely many \(k > N\). Now Theorem 4.2 implies that there is a \(k \geq N\) and a \(t^* > t_{\sigma^{k-1}(z)}\) such that

\[ z_{k+1} = g_{\sigma^k(z)}(t^*) . \]

If no critical value escapes, then we can pull back \(k\) times, and it follows that there is an \(s'\) which differs from \(s\) only in finitely many entries, and a \(t' > t_z = t'_z\) with

\[ z_1 = g_z(t') . \]

If \(|d_1|\) is large enough so that \(|t_k + d_k| \geq \max\{ F^{o(k-1)}(T_z), 2\sqrt{|ab|} + 1\} \) for all \(k\), then the conditions of Theorem 4.2 are satisfied immediately so that \(z_1 = g_z(t)\) for some \(t \geq T_z\).

3. If \(w_1 = g_{z'}(t'')\) for \(t'' > t_z\), then \(t > t''\) by Proposition 4.3.

\section{Classification of Escaping Points}

In this section we show that all escaping points are organized in the form of dynamic rays which are associated to exponentially bounded external addresses, and we complete the classification of escaping points.

\textbf{Definition 6.1 (Limit Set, Landing Point, Uniform Escape)}

The limit set of the ray \(g_z\) is defined as the set of all possible limit points of \(g_z(t_k)\) for \(t_k \searrow t_z\). We say that the ray \(g_z\) lands at a point \(w\) if \(\lim_{t'} g_z(t')\) exists and is equal to \(w\) (the limit set consists of only one point). If \(g_z\) lands at an escaping point \(w = g_z(t_z)\), we say that ray and landing point escape uniformly if for every \(R \in \mathbb{R}\) there exists an \(N \geq 0\) such that for every \(n \geq N\), we have \(|\text{Re}(g_z^n([t_z, \infty]))| > R\).

For \(R > 0\) let \(Y_R\) be defined as

\[ Y_R := \{ z \in \mathbb{C} : |\text{Re}(z)| < R \} . \]

\textbf{Lemma 6.2 (Escaping Set Connected)}

Let \(R \geq R_3\) with

\[ \frac{K e^R}{2} \geq \frac{2 + 3\pi}{1 - 3/e^2} \]

and let \((z_k)\) be an escaping orbit which is completely contained in \(\mathbb{C} \setminus Y_{R+2}\). Then there exists a closed connected set \(C \subset \overline{C} \setminus Y_R\) with \(\{z_1, \infty\} \subset C\) such that the orbit of every \(z \in C\) is completely contained in \(\mathbb{C} \setminus Y_R\) and escapes such that

\[ |\text{Re}(F^{o(k-1)}(z))| \geq |\text{Re}(z_k)| - 2. \]
Let $s$ be the external address of $z_1$. All points in $C \setminus \{\infty\}$ have external address $s$ and

- either $C \setminus \{\infty\} = g_s([t, \infty])$ for some $t > t_s$,
- or the ray $g_s$ lands at $z_1$, and $C = \{z_1, \infty\} \cup g_s([t_2, \infty])$.

In particular, all points in $C \setminus \{\infty\}$, except possibly $z_1$, lie on the ray $g_s$.

**Remark.** The external address of $z_1$ is defined uniquely because the orbit of $z_1$ never enters the vertical strip $Y_R \supset A$, so it can never hit the boundary of our partition.

**Proof.** By (12), for every $k$ there is a $K' \in \{|a|, |b|\}$ (depending on $k$) such that

$$\frac{3}{2} K' \exp |\text{Re}(z_k)| \geq |E(z_k)| = |z_{k+1}| \geq \frac{K'}{2} \exp |\text{Re}(z_k)| \geq \frac{K'}{2} e^R \geq \frac{2 + 3\pi}{1 - 3/e^2} > 0$$

and thus

$$|z_{k+1}| - 2 - 3\pi \geq \frac{3}{e^2} |z_{k+1}|.$$

For $k \geq 1$ define

$$S_k := \left\{ z \in R_{s_k} : \begin{cases} \text{Re}(z) > \text{Re}(z_k) - 2, & \text{if Re}(z_k) > 0 \\ \text{Re}(z) < \text{Re}(z_k) + 2, & \text{if Re}(z_k) < 0 \end{cases} \right\}.$$

Our first claim is $E(S_k) \supset S_{k+1}$ for all $k$. Every $z \in R_{s_k}$ with $|\text{Re}(z)| \leq |\text{Re}(z_k)| - 2$ satisfies as in (18)

$$|E(z)| \leq \frac{3}{2} K' \exp |\text{Re}(z)| \leq \frac{3}{2} K' \exp(|\text{Re}(z_k)| - 2) \leq \frac{3}{e^2} |z_{k+1}|.$$

Since $E : \overline{R_{s_k}} \to \mathbb{C}$ is surjective, it follows that $E(S_k)$ contains every point $w \in S_{k+1}$ with

$$|w| > \frac{3}{e^2} |z_{k+1}|.$$

But $w \in S_{k+1}$ implies

$$|\text{Re}(w)| \geq |\text{Re}(z_{k+1})| - 2 \quad \text{and} \quad |\text{Im}(w) - \text{Im}(z_{k+1})| < 3\pi$$

and thus

$$|w| \geq |z_{k+1}| - 2 - 3\pi \geq \frac{3}{e^2} |z_{k+1}|.$$
Hence we get \( w \in E(S_k) \) and the first claim is proved.

Since \( \mathcal{A} \cap S_{k+1} = \emptyset \) we obtain a connected set \( C'_k \subset S_k \) such that \( E: C'_k \to S_{k+1} \) is a conformal isomorphism. For \( k \geq 1 \) consider the sets

\[
C_k := \left\{ z \in S_1 : E^{oi}(z) \in \overline{S}_{i+1}, \text{ for } i = 0, \ldots, k - 1 \right\} \cup \{ \infty \}.
\]

The sets \( C_k \) are non-empty (because \( \{ z_1, \infty \} \subset C_k \)), compact and nested: \( C_{k+1} \subset C_k \). We just proved that \( E^{o(k-1)} : C_k \to S_k \cup \{ \infty \} \) is a homeomorphism for all \( k \). Therefore, all \( C_k \) are connected. The nested intersection of non-empty connected compact sets is non-empty connected and compact; therefore

\[
C := \bigcap_{k \geq 1} C_k
\]

is a closed connected and compact subset of \( \overline{C} \) with \( \{ z_1, \infty \} \subset C \). Set \( C' := C \setminus \{ \infty \} \). For \( z \in C' \) we have \( E^{o(k-1)}(z) \in \overline{S}_k \) for all \( k \) and thus \( |\text{Re}(E^{o(k-1)}(z))| > |\text{Re}(z_1)| - 2 \to \infty \). Hence \( C' \) only consists of escaping points with orbits in \( \mathbb{C} \setminus Y_R \) which have the same external address as \( z_1 \).

It remains to show that there is at most one point in \( C' \) which is not on the dynamic ray \( g_{\bar{2}} \). By Lemma 5.1 there is a \( \xi > 0 \) such that every \( z \in C' \) with \( |\text{Re}(z)| > \xi \) is on \( g_{\bar{2}} \). In particular, \( C' \) contains an unbounded connected part of the tail of \( g_{\bar{2}} \).

Suppose that there are two points \( z_1, w_1 \in C' \) with orbits \( (z_k) \) and \( (w_k) \) such that \( |\text{Re}(z_k - w_k)| < 3 \) for all \( k \). By Lemma 2.3 we also have \( |\text{Im}(z_k - w_k)| < 3\pi \), hence \( |z_k - w_k| < 3 + 3\pi \). The derivative \( E' \) is bounded below along the orbits of \( z_1 \) resp. \( w_1 \) by \( |ae^R - be^{-R}| \geq \frac{K_e}{2} e^R > 4 \). By pulling back choosing the branch \( E^{-1} : S_{k+1} \to C_k \), we get

\[
|z_{k-1} - w_{k-1}| < (3\pi + 3)/4
\]

and thus inductively \( |z_{k-j} - w_{k-j}| < (3\pi + 3)/4^j \). Hence \( |z_1 - w_1| < (3\pi + 3)/4^k \) for all \( k \) and thus \( z_1 = w_1 \). Therefore, if \( z_1 \neq w_1 \), then by Lemma 5.1 at least one of these points (say \( z_1 = g_{\bar{2}}(t) \)) for an external address \( \bar{s}' \) which differs from \( \bar{s} \) in only finitely many positions, and \( t > t_{\bar{s}'} = t_{\bar{s}} \); say \( \sigma^N(\bar{s}') = \sigma^N(\bar{s}) \). But since \( E^{oN}(C') \cup \{ \infty \} \) is connected, a single branch of \( E^{-N} \) maps \( E^{oN}(C') \to C' \), avoiding \( \mathcal{A} \) in the pull-back, so it follows even that \( \bar{s}' = \bar{s} \).

Therefore, every point in \( C' \) with at most one exception is on \( g_{\bar{2}} \). Since \( g_{\bar{2}} : [t_{\bar{s}}, \infty] \to \overline{\mathbb{C}} \) is continuous and \( C' \) is closed, the set \( \{ t \in [t_{\bar{s}}, \infty] : g_{\bar{2}}(t) \in C' \} \) is closed in \( [t_{\bar{s}}, \infty] \). We use this to show that if \( g_{\bar{2}}(t) \in C' \), then \( g_{\bar{2}}(t') \in C' \) for all \( t' > t \); otherwise, there would be \( t_2 > t_1 > t_s \) with \( g_{\bar{2}}(t_1), g_{\bar{2}}(t_2) \in C' \), but \( g_{\bar{2}}([t_1, t_2]) \cap C' = \emptyset \). But then at least for large \( N \), there would be a large
gap between \(g_{\sigma}^{N}(F^{N}(t_2))\) and \(g_{\sigma}^{N}(F^{N}(t_1))\) which could not be filled by points in \(g_{\sigma}^{N}(\{t_2, t_1\})\) or \(g_{\sigma}^{N}(\{t_2, \infty\})\) (Lemma 5.1(3)) or by the single exceptional point, so \(E^{\infty}(C')\) could not be connected.

Therefore, except for possibly a single point, \(C'\) equals either \(g_{\sigma}(\{t, \infty\})\) for some \(t > t_{\sigma}\) or \(g_{\sigma}(\{t_{\sigma}, \infty\})\). But since \(C'\) is closed and connected, an extra point can (and must) occur only in the second case, and this is what we claimed. □

For an external address \(s\) let \(S(s)\) be the space of external addresses \(s'\) which differ from \(s\) at only finitely many entries. Clearly, all \(s' \in S(s)\) have \(t_{s'} = t_{s}\).

**Lemma 6.3 (Limit Set does Not Intersect Ray)**

Let \(g_{\sigma}\) be a ray with the property that the entire orbits of all of its points avoid \(Y_R\) for an \(R > 2\sqrt{|ab|} + M + 2\) and \(R > R_h\) for \(h = 2\pi\). Let \(L_{s'}\) be the limit set of \(g_{\sigma}\). Then \(L_{s'}\) is disjoint from \(g_{\sigma}'\) for all \(s' \in S(s)\).

**Proof.** Suppose there exists a \(s' \in S(s)\) and a \(t > t_{s'}\) with \(g_{\sigma}(t) \in L_{s'}\). Since all \(|\text{Re}(E^{\infty}(g_{\sigma}(t)))| > R\) for all \(t > t_{s}\) and all \(k \in \mathbb{N}\), all points \(g_{\sigma}(t)\) have external address \(s\), and so has the limit \(g_{\sigma}'(t)\) (Theorem 4.2).

Now choose a potential \(t' \in \{t_{s}, t\}\). By Proposition 4.3 (and possibly after finitely many iterations) we get \(|\text{Re}(g_{\sigma}(t))| - |\text{Re}(g_{\sigma}(t'))| \geq 4\). Then there is a potential \(t'' > t_{s}\) arbitrarily close to \(t_{s}\) such that \(g_{\sigma}(t'')\) is arbitrarily close to \(g_{\sigma}'(t)\). More precisely we assume that \(t'' < t'\) and \(|g_{\sigma}(t'') - g_{\sigma}'(t)| < 1\), hence \(|\text{Re}(g_{\sigma}(t''))| - |\text{Re}(g_{\sigma}'(t'))| > 3\). Since \(g_{\sigma}(t'')\) and \(g_{\sigma}'(t')\) both have external address \(s\), we have \(|\text{Im}(E^{\infty}(g_{\sigma}(t'')) - E^{\infty}(g_{\sigma}'(t')))| \leq 2\pi\) for all \(k\). By Lemma 5.1 this means \(t'' > t'\) and we get a contradiction. Hence \(L_{s'} \cap g_{\sigma}'(\{t_{s}, \infty\}) = \emptyset\) for all \(s' \in S(s)\). □

**Theorem 6.4 (Escaping Points are Organized in Rays)**

For every escaping point \(w\) there exists a unique exponentially bounded external address \(s\) and a unique potential \(t \geq t_{s}\) such that exactly one of the following holds:

- either \(t > t_{s}\) and \(w = g_{s}(t)\),
- or \(t = t_{s}\) and the dynamic ray \(g_{s}\) lands at \(w\) such that \(w\) and the ray \(g_{s}\) escape uniformly,
- or, if one of the singular values escapes: \(v_i = g_{s}(t)\) for some \(s\) and \(t \geq t_{s}\), and the point \(w\) maps to \(g_{s}(t')\) with \(t_{s} \leq t' < t\) after finitely many iterations.
Proof. Let $R$ be as defined in Lemma 6.2. If the entire orbit of $w$ lies in $\mathbb{C} \setminus Y_{R+2}$, then let $s$ be the external address of $w$. By Lemma 6.2, either $w = g_s(t)$ for some $t > t_s$, or the ray $g_s$ lands at $w$ with uniform escape. In both cases, the external address $s$ is uniquely determined by the orbit of $w$, and so is $t$ along dynamic rays because each ray is an injective curve. To finish the uniqueness claim, we have to show that $g_s(t) \neq g_s(t_s)$ for all $t > t_s$; this follows from Lemma 6.3.

If not the entire orbit of $w$ is in $\mathbb{C} \setminus Y_R$, there exists a finite iterate of $w$ whose orbit has that property. This iterate is either on a dynamic ray or landing point of a ray. By pulling back along the orbit of $w$ the claim is proved for all $w$; if a singular value lies on a dynamic ray along this pull-back, then an exception may occur as stated in the third case of the claim.

Now we want to show under which conditions a landing point escapes. We need slow and fast external addresses, just like in [SZ].

**Definition 6.5 (Slow and Fast External Addresses)**

We say that an external address $s$ is slow if there are $A, x > 0$ and infinitely many $n$ for which $|s_{n+k}| \leq AF^{(k-1)}(x)$ for all $k \geq 1$. Otherwise we call $s$ fast.

Note that every external address $s$ with $t_s > 0$ is fast, but the converse is not true: the two external addresses $1 2 1 3 1 4 1 5 \ldots$ and $1 2 1 2 3 1 2 3 4 1 2 3 4 5 \ldots$ (with arbitrary entries $L$ or $R$) are both unbounded with $t_s = 0$, but the first one is fast while the second one is slow.

Now the following result holds in complete analogy to [SZ, Proposition 6.8], so we omit the proof.

**Proposition 6.6 (Uniform Escape for Fast Addresses)**

An external address $s$ is fast if and only if the ray $g_s$ lands at an escaping point so that ray and landing point escape uniformly.

**Corollary 6.7 (Uniform Escape of Ray and Landing Point)**

If a dynamic ray lands at an escaping point, then the ray and its landing point escape uniformly.

Proof. Let $g_s$ be a dynamic ray and $w$ be its escaping landing point. If $s$ is fast (in particular, if $t_s > 0$), then $g_s$ lands at $w$ with uniform escape by Proposition 6.6. Therefore, $s$ is slow and $t_s = 0$.

The point $w$ cannot be on any other dynamic ray $g_{s'}$ by the same argument as in [SZ, Corollary 6.9]: every point $g_{s'}(t)$ with $t > t_{s'}$ can be approximated arbitrarily closely by pieces of other rays almost parallel to $g_{s'}$ so that $g_{s'}(t)$ is
not accessible by any ray. Therefore, by our classification, \( w \) is the escaping landing point of some other dynamic rays \( g_s' \), and thus \( s' \) is necessarily fast. This means that the dynamic rays \( g_s' \) and \( g_s \) land together at \( w \), so for every \( N \in \mathbb{N} \) there is a \( k \in \mathbb{N} \) such that the first entries of \( \sigma^k(s') \) and \( \sigma^k(s) \) differ by at least \( N \), while \( g_{\sigma^k(s')} \) and \( g_{\sigma^k(s)} \) land together at \( E^{\circ k}(w) \). But the \( 2\pi \)in-translates of the same rays must land at the \( 2\pi \)in-translates of \( E^{\circ k}(w) \), and this is a topological impossibility if \( N > 1 \).

As a result, we obtain the following classification of escaping points:

**Theorem 6.8 (Classification of Escaping Points)**
If no critical value escapes, then the set of escaping points is classified by external addresses \( s \in S \) so that

- for \( s \) with \( t_s < \infty \) so that \( s \) is slow, the associated set of escaping points is the ray \( g_s([t_s, \infty[) \);
- for \( s \) with \( t_s < \infty \) so that \( s \) is fast, the associated set of escaping points is \( g_s([t_s, \infty[) \), i.e. the ray including its escaping landing point with uniform escape;
- for \( s \) with \( t_s = \infty \), there is no associated set of escaping points.

The sets of escaping points to all \( s \in S \) are disjoint, and their union is the entire set of escaping points of \( E \).

Moreover, each path component of the set of escaping points is exactly the set of escaping points associated to any particular exponentially bounded external address.

If one or both critical values escape, the necessary modifications are straightforward.

### 7 Epilogue: Hausdorff Dimension

**Theorem 7.1 (Dimension Paradox)**
The union of all dynamic rays has Hausdorff dimension 1, while the set of the escaping landing points has Hausdorff dimension 2 and even infinite planar Lebesgue-measure.

**Sketch of proof.** McMullen \[McM\] has shown that the set of escaping points for every \( E_{a,b} \) has infinite planar Lebesgue measure. By our classification, every escaping point is either on a dynamic ray or the landing point.
of a dynamic ray. It thus suffices to prove that the union of all dynamic rays has Hausdorff dimension 1. We sketch a lemma of Karpinska [K] which generalizes to show that the set of rays has dimension less than $1 + 1/p$ for every $p > 0$.

Given $p > 0$ and $\xi > 0$, let

$$P_{p, \xi} := \{ (x + iy) \in \mathbb{C} : |x| > \xi, |y| < |x|^{1/p} \}.$$ 

We will show that $S_{p, \xi} := \{ z \in \mathbb{C} : E^{\circ k}(z) \in P_{p, \xi} \text{ for all } k \}$ has Hausdorff dimension at most $1 + 1/p$ for large $\xi$.

Let $Q \subset P_{p, \xi}$ be a square of side length $2\pi$ with boundaries parallel to the coordinate axes, and with real parts in $[x, x + 2\pi]$. If $|x| > \xi$ is large, then $E(Q)$ is almost an annulus between radii $e^x$ and $e^{2\pi}e^x$ (up to a factor $|a|$ or $|b|$ which will not matter). Then $E(Q) \cap P_{p, \xi}$ is contained in two approximate rectangles of width $e^x(e^{2\pi} - 1)$ and height $2e^{x/p}e^{2\pi/p}$. Filling it with squares of side length $2\pi$, we need approximately $Ae^{x(1+1/p)}$ squares (where $A > 0$ is some bounded factor). Pulling those back into the original square $Q$, we obtain the same number of squares, with diameters approximately $e^{-x}/2\pi$.

Now

$$\frac{\log(Ae^{x(1+1/p)})}{-\log(e^{-x}/2\pi)} = \frac{x(1 + 1/p) + \log(A)}{x + \log(2\pi)} = 1 + \frac{1}{p} + O(1/\xi).$$

We can refine the tiling of $S_{p, \xi}$ by iterating this procedure: the squares with side length $2\pi$ in $E(Q) \cap P_{p, \xi}$ can be replaced themselves by smaller squares. In the limit, we obtain a covering of $S_{p, \xi}$ with small tiles, and we obtain an upper bound of $1 + 1/p$ for the Hausdorff dimension of $S_{p, \xi}$.

In Proposition 4.3 we have shown that every point $z$ on a ray satisfies the parabola condition $|\text{Im}(E^{\circ n}(z))|^p < |\text{Re}(E^{\circ n}(z))|$ for any $p > 0$ for all but finitely many $n$. Therefore, the union of all rays is contained in the countable union

$$\bigcup_{n \geq 0} E^{-N}(S_{p, \xi})$$

and still has Hausdorff dimension at most $1 + 1/p$ for every $p > 0$. Therefore, the union of all rays has Hausdorff dimension 1. \qed

As mentioned in the introduction, the dynamics is particularly easy to understand when both critical orbits are strictly preperiodic: see [S]. In this case, every dynamic ray lands somewhere in $\mathbb{C}$, and it can be read off from the external address whether the landing point is periodic, recurrent, escaping, whether its orbits is dense in $\mathbb{C}$ etc.. Conversely, every point in $\mathbb{C}$ which is not on a dynamic ray is the landing point of a ray. The set of dynamic rays still
has Hausdorff dimension 1, so the set of landing points of this 1-dimensional set of rays is the entire complex plane minus this 1-dimensional set of rays. In that case, almost every $z \in \mathbb{C}$ (with respect to planar Lebesgue measure) is an escaping point.

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