WELL-LOCALIZED OPERATORS ON MATRIX WEIGHTED $L^2$ SPACES

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Abstract. Nazarov-Treil-Volberg recently proved an elegant two-weight T1 theorem for “almost diagonal” operators that played a key role in the proof of the $A_2$ conjecture for dyadic shifts and related operators. In this paper, we obtain a generalization of their T1 theorem to the setting of matrix weights. Our theorem does differ slightly from the scalar results, a fact attributable almost completely to differences between the scalar and matrix Carleson Embedding Theorems. The main tools include a reduction to the study of well-localized operators, a new system of Haar functions adapted to matrix weights, and a matrix Carleson Embedding Theorem.

1. Introduction

In this paper, the dimension $d$ is fixed and $L^2$ will denote $L^2(\mathbb{R}, \mathbb{C}^d)$, namely the set of vector-valued functions satisfying

$$\|f\|^2_{L^2} \equiv \int_\mathbb{R} \|f(x)\|^2_{\mathbb{C}^d} dx < \infty.$$ 

We will be primarily interested in matrix weights, $d \times d$ positive definite matrix-valued functions with locally integrable entries. Given such a weight $W$, let $L^2(W)$ be the set of functions satisfying

$$\|f\|^2_{L^2(W)} \equiv \int_\mathbb{R} \|W^{1/2}(x)f(x)\|^2_{\mathbb{C}^d} dx = \int_\mathbb{R} \langle W(x)f(x), f(x) \rangle_{\mathbb{C}^d} dx < \infty.$$ 

Given matrix weights $V$ and $W$, a natural question is: when does a bounded operator $T$ mapping $L^2$ to itself extend to a bounded operator mapping $L^2(W)$ to $L^2(V)$ and what is the norm of $T$ as a map from $L^2(W)$ to $L^2(V)$?

If we consider the special one-dimensional case when $V = W = w$, this question has a classical answer. Indeed, a Calderón-Zygmund operator $T$ extends to a bounded operator on $L^2(w)$ if and only if $w$ is an $A_2$ Muckenhoupt weight, namely:

$$[w]_{A_2} \equiv \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < \infty,$$

where the supremum is taken over all intervals $I$ and $\langle w \rangle_I \equiv \frac{1}{|I|} \int_I w(x) dx$. In contrast, the question of the operator norm of $T$ on $L^2(w)$, and its sharp dependence on $[w]_{A_2}$, called the $A_2$ conjecture, remained open for decades. Lacey-Petermichl-Reguera made substantial progress on this question in [7] by establishing the sharp bound for dyadic shifts and as a corollary, obtained new proofs of the bound for simple Calderón-Zygmund operators including the Hilbert transform, Riesz transforms, and Beurling transform. Their proof rested on an

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elegant two-weight T1 theorem due to Nazarov-Treil-Volberg [10] coupled with technical testing estimates.

Using a refined method of decomposing Calderón-Zygmund operators as sums of dyadic shifts and an improvement of the Lacey-Petermichl-Reguera estimates, Hytönen resolved the $A_2$ conjecture in 2012 in [4] and showed

$$\|T\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_2}$$

for all Calderón-Zygmund operators $T$.

We are interested in the analogue of the $A_2$ conjecture in the setting of matrix weights. However, due to complications arising in the matrix case, the current literature is less developed. Still, the boundedness of Calderón-Zygmund operators is known. In 1997, Treil-Volberg showed in [13] that the Hilbert transform $H$ extends to a bounded operator on $L^2(W)$ if and only if $W$ is an $A_2$ matrix weight, i.e. if and only if

$$[W]_{A_2} \equiv \sup_I \left\| \frac{1}{|I|^2} \langle W \rangle_I \langle W^{-1} \rangle_I \right\|^2 < \infty,$$

where $\| \cdot \|$ denotes the norm of the matrix acting on $\mathbb{C}^d$. Soon after, Nazarov-Treil [11] extended this result to general (classical) Calderón-Zygmund operators and in the interim, the study of operators on matrix-weighted spaces has received a great deal of attention. See [2, 3, 5, 8, 9, 14]. However, the question of the sharp dependence on $[W]_{A_2}$ is still open and this seems to be a very difficult problem. In [1], the two authors with S. Petermichl showed that

$$\|H\|_{L^2(W) \to L^2(W)} \lesssim [W]_{A_2}^3 \log [W]_{A_2},$$

for all $A_2$ weights $W$, but this bound is unlikely to be sharp.

Rather, a proof yielding a sharp estimate would likely follow, as in the scalar case, from the combination of (1) a sharp T1 theorem and (2) appropriate testing estimates. The goal of this paper is to establish the T1 theorem and specifically, obtain matrix generalizations of the two-weight T1 theorems of Nazarov-Treil-Volberg from [10] about “almost diagonal” operators including Haar multipliers and dyadic shifts. These generalizations are interesting in their own right because they give two-weight results for all pairs of matrix $A_2$ weights, which is a new development. It seems likely that, as in the scalar case, these T1 theorems will prove a robust tool for studying the dependence of operator norms on the $A_2$ characteristic. Before discussing the main results in more detail, we require several definitions.

### 1.1. The Main Results.

Throughout the paper, $\mathcal{D}$ denotes the standard dyadic grid on $\mathbb{R}$ and $A \lesssim B$ means $A \leq C(d)B$, where $C(d)$ is a (absolute) dimensional constant. For $I \in \mathcal{D}$, let $h_I$ be the standard Haar function defined by

$$h_I \equiv |I|^{-\frac{1}{2}} \left( 1_{I_+} - 1_{I_0} \right),$$

where $I_+$ is the right half of $I$ and $I_0$ is the left half of $I$. To the dyadic grid $\mathcal{D}$, associate the unique binary tree where each $I$ is connected to its two children $I_-$ and $I_+$. Given that dyadic tree, let $d_{\text{tree}}(I, J)$ denote the “tree distance” between $I$ and $J$, namely, the number of edges on the shortest path connecting $I$ and $J$. The “almost diagonal” operators of interest possess a band structure defined as follows:

**Definition 1.1.** A bounded operator $T$ on $L^2$ is a called a band operator with radius $r$ if $T$ satisfies

$$\langle Th_I e, h_J v \rangle_{L^2} = 0$$
for all intervals $I, J \in \mathcal{D}$ with $d_{tree}(I, J) > r$ and vectors $e, v \in \mathbb{C}^d$.

Given a matrix weight $W$ and interval $I$ in $\mathcal{D}$, define the matrices:

$$W(I) \equiv \int_I W(x) \, dx$$

and

$$\langle W \rangle_I \equiv \frac{1}{|I|} \int_I W(x) \, dx = \frac{W(I)}{|I|}.$$  

In this paper, we will only consider weights $W$ with the property of being an $A_2$ weight, and without loss of generality, we can focus on the question of when a band operator $T$ extends to a bounded operator from $L^2(W^{-1})$ to $L^2(V)$ with norm $C$ for matrix weights $V, W$. It is not hard to show that this occurs precisely when

$$\left\| M_{V} \frac{1}{2} T M_{W} \frac{1}{2} \right\|_{L^2 \rightarrow L^2} = C.$$  

The main results of this paper are then the following theorems.

**Theorem 1.2.** Let $W, V$ be matrix $A_2$ weights and let $T$ be a band operator with radius $r$. Then $M_{V} \frac{1}{2} T M_{W} \frac{1}{2}$ extends to a bounded operator on $L^2$ if and only if

(1) $$\|T W 1_I e\|_{L^2(V)} \leq A_1 \langle W(I) e, e \rangle_{\mathbb{C}^d}^{\frac{1}{2}}$$

(2) $$\|T^* V 1_I e\|_{L^2(W)} \leq A_2 \langle V(I) e, e \rangle_{\mathbb{C}^d}^{\frac{1}{2}}$$

for all intervals $I \in \mathcal{D}$ and vectors $e \in \mathbb{C}^d$. Furthermore,

$$\left\| M_{V} \frac{1}{2} T M_{W} \frac{1}{2} \right\|_{L^2 \rightarrow L^2} \leq 2^{2r} C(d) \left( A_1 B(W) + A_2 B(V) \right) ,$$

where $C(d)$ is a dimensional constant and $B(W)$ and $B(V)$ are constants depending on $W$ and $V$ from an application of the matrix Carleson Embedding Theorem.

The definitions of the constants $B(W)$ and $B(V)$ are given in Theorem 3.5, the matrix Carleson Embedding Theorem used in this paper, and discussed further in Remark 3.6. As in [10], the conditions of Theorem 1.2 can be relaxed slightly to yield the following result:

**Theorem 1.3.** Let $W, V$ be matrix $A_2$ weights and let $T$ be a band operator with radius $r$. Then $M_{V} \frac{1}{2} T M_{W} \frac{1}{2}$ extends to a bounded operator on $L^2$ if and only if the following two conditions hold:

(i) For all intervals $I \in \mathcal{D}$ and vectors $e \in \mathbb{C}^d$,

$$\|1_I T W 1_I e\|_{L^2(V)} \leq A_1 \langle W(I) e, e \rangle_{\mathbb{C}^d}^{\frac{1}{2}}$$

$$\|1_I T^* V 1_I e\|_{L^2(W)} \leq A_2 \langle V(I) e, e \rangle_{\mathbb{C}^d}^{\frac{1}{2}} .$$

(ii) For all intervals $I, J \in \mathcal{D}$ satisfying $2^{-r} |I| \leq |J| \leq 2^r |I|$ and vectors $e, v \in \mathbb{C}^d$,

$$\left| \langle T W 1_I e, 1_J v \rangle_{L^2(V)} \right| \leq A_3 \langle W(I) e, e \rangle_{\mathbb{C}^d}^{\frac{1}{2}} \langle V(J) v, v \rangle_{\mathbb{C}^d}^{\frac{1}{2}} .$$

Furthermore,

$$\left\| M_{V} \frac{1}{2} T M_{W} \frac{1}{2} \right\|_{L^2 \rightarrow L^2} \leq 2^{2r} C(d) \left( A_1 B(W) + A_2 B(V) + A_3 \right) ,$$

where $C(d)$ is a dimensional constant and $B(W)$ and $B(V)$ are constants depending on $W$ and $V$ from an application of the matrix Carleson Embedding Theorem.
Remark 1.4. An observant reader, and expert in the area, will notice that Theorems 1.2 and 1.3 are strictly weaker than the results of Nazarov-Treil-Volberg [10] in two respects. First, our results are only proved for pairs $V, W$ of matrix $A_2$ weights and second, they introduce additional constants $B(V)$ and $B(W)$ in the norm estimates, which do not come from the testing conditions.

However, it is worth pointing out that both of these shortcomings are the direct result of differences between the scalar Carleson Embedding Theorem and the current matrix Carleson Embedding Theorem. In the scalar case, the Carleson Embedding Theorem holds for all weights and the embedding constant is an absolute multiple of the constant obtained from the testing condition. In the matrix case, the current Carleson Embedding Theorem, Theorem 3.5, is only known for matrix $A_2$ weights and the embedding constant is the testing constant times an additional constant $B(W)$, depending upon the weight $W$.

A careful reading of our paper reveals that, if one can improve the underlying matrix Carleson Embedding Theorem in these two respects, then our arguments will give T1 theorems with sharp constants that hold for all pairs of matrix weights. It then seems likely that these results could be used as a tool to approach the matrix $A_2$ conjecture, at least in the setting of dyadic shifts and related operators.

It is also worth observing that related but weaker results are obtained by R. Kerr in [6]. He shows that band operators on $L^2$ will be bounded from $L^2(W)$ to $L^2(V)$ if the matrix weights $V$ and $W$ are both in the matrix analogue of $A_\infty$ (denoted $A_{2,0}$) and satisfy a joint $A_2$ condition.

Remark 1.5. If the entries of $W, V$ are not locally square-integrable, i.e. not in $L^2_{loc}(\mathbb{R})$, one needs to be a little careful about interpreting the expressions on the left-hand sides of (1) and (2) and the analogous expressions in Theorem 1.3. This technicality can be handled in a way similar to that found in [10]. Indeed, observe that if $W, W'$ are matrix weights satisfying $W' \leq W$, then

$$\left\| M_{W' \frac{1}{2}} T^* M_{V \frac{1}{2}} \right\|_{L^2 \to L^2} \leq \left\| M_{W \frac{1}{2}} T^* M_{V \frac{1}{2}} \right\|_{L^2 \to L^2}$$

and taking adjoints gives

$$\left\| M_{V \frac{1}{2}} T M_{W' \frac{1}{2}} \right\|_{L^2 \to L^2} \leq \left\| M_{V \frac{1}{2}} T M_{W \frac{1}{2}} \right\|_{L^2 \to L^2}.$$ 

Now, to interpret the first necessary condition appropriately, let $\{W_n\}$ be a sequence of matrix weights with entries in $L^2_{loc}(\mathbb{R})$ increasing to $W$. Then, the boundedness of $M_{V \frac{1}{2}} T M_{W \frac{1}{2}}$ implies that

$$\|TW_n 1_{I e}\|_{L^2(V)} \leq C < \infty$$

for some constant $C$ uniformly in $n$. It is not difficult to show that this implies $\left\{M_{V \frac{1}{2}} TW_n 1_{I e}\right\}$ has a limit in $L^2$, which is independent of the sequence $\{W_n\}$ chosen. So, there is no ambiguity in calling this limit function $V^{-\frac{1}{2}} TW 1_{I e}$ and interpreting the lefthand side of (1) as its $L^2$ norm. The dual expressions are interpreted analogously. We can similarly interpret the term in (ii) from Theorem 1.3 as the inner product between $V^{-\frac{1}{2}} TW 1_{I e}$ and $V^{-\frac{1}{2}} 1_{J \nu}$ in $L^2$.

To interpret the sufficient condition, fix any sequences $\{W_n\}$ and $\{V_n\}$ in $L^2_{loc}(\mathbb{R})$ increasing to $W$ and $V$ respectively. Conditions (1) and (2) can be interpreted as the estimates

$$\|TW_n 1_{I e}\|_{L^2(W_n)} \leq A_1 \langle W_n(I)e, e \rangle_{\mathcal{C}^d}^{\frac{1}{2}}$$

and

$$\|T^* V_n 1_{I e}\|_{L^2(W_n)} \leq A_2 \langle V_n(I)e, e \rangle_{\mathcal{C}^d}^{\frac{1}{2}}.$$
which are uniform in $n$, $e$, and $I$. Then Theorem 1.2 gives the bound for $\left\| M_{V}^{\frac{1}{2}}T_{W}^{\frac{1}{2}} \right\|_{L^{2} \rightarrow L^{2}}$ which implies the desired bound for $\left\| M_{V}^{\frac{1}{2}}T_{W}^{\frac{1}{2}} \right\|_{L^{2} \rightarrow L^{2}}$. The analogous interpretations of the expressions in Theorem 1.3 should also be clear.

1.2. Summary and Outline of the Paper. The remainder of the paper consists of the proofs of Theorems 1.2 and 1.3. To outline the proof technique, assume that $W$, $V$ are matrix $A_2$ weights. It is not hard to show that $M_{V}^{\frac{1}{2}}T_{W}^{\frac{1}{2}} : L^2 \rightarrow L^2$ is bounded with operator norm $C$ if and only if the operator

$$T_{W} \equiv T_{W} : L^2(W) \rightarrow L^2(V) \text{ satisfies } ||T_{W}||_{L^2(W) \rightarrow L^2(V)} = C.$$  

Because $T$ is a band operator, $T_{W}$ will have a particularly nice structure. Following the language and proof strategy of Nazarov-Treil-Volberg [10], we will show $T_{W}$ is well-localized. Section 4 contains the details of well-localized operators, their connections to band operators, and the analogues of Theorems 1.2 and 1.3 for well-localized operators. We call these results Theorems 4.3 and 4.4. These theorems will immediately imply our main results: Theorems 1.2 and 1.3.

In Sections 2 and 3, the paper develops the tools need to prove Theorems 4.3 and 4.4. In Section 2, we define and outline the properties of a system of Haar functions adapted to a general matrix weight $W$. This system appears to be new in the context of matrix weights. We also require a matrix Carleson Embedding Theorem. We use the ideas of Treil-Volberg [13] and Isralowitz-Kwon-Pott [5] to obtain such a theorem with the best known constant. Details are given in Section 3.

Section 5 contains the proofs of Theorems 4.3 and 4.4. The well-localized structure of $T_{W}$ makes $T_{W}$ amenable to separate analyses of its diagonal part and upper and lower triangular parts, which behave like nice paraproducts. We compute the norm by duality and as part of the argument, decompose the functions in question relative to weighted Haar bases adapted to $W$ and $V$ respectively. To control the upper and lower triangular pieces, we define associated paraproducts and show they are bounded using the testing hypothesis and matrix Carleson Embedding Theorem. We bound the diagonal pieces using the well-localized structure of $T_{W}$ coupled with properties of the system of Haar functions and the given testing conditions.

2. Weighted Haar Basis

Let $W$ be a matrix weight, and let $\| \cdot \|$ denote the operator norm of a matrix on $\mathbb{C}^d$. In this section, we construct a set of disbalanced Haar functions adapted to $W$, which we denote $H_{W}$. First, fix $J \in \mathcal{D}$ and let $v_{j}^{1}, \ldots, v_{j}^{d}$ be a set of orthonormal eigenvectors of the positive matrix:

$$W(J_{-})W(J_{+})^{-1}W(J_{-}) + W(J_{-}) = W(J_{-})W(J_{+})^{-1}W(J_{-}) + W(J_{+})W(J_{+})^{-1}W(J_{-})$$

$$= W(J)W(J_{+})^{-1}W(J_{-}).$$

(3)

Furthermore, for $1 \leq j \leq d$, define the constant

$$w_{j}^{2} \equiv \left\| (W(J)W(J_{+})^{-1}W(J_{-}))^{\frac{1}{2}} v_{j}^{2} \right\|. $$

Since the matrix (3) is positive and $v_{j}^{2}$ is a normalized eigenvector, it follows that:

$$(w_{j}^{2})^{-1}v_{j}^{2} = (W(J)W(J_{+})^{-1}W(J_{-}))^{-\frac{1}{2}} v_{j}^{2} \quad \forall 1 \leq j \leq d.$$
Definition 2.1. For each \( J \in \mathcal{D} \), define the vector-valued Haar functions on \( J \) adapted to \( W \) as follows:

\[
\begin{array}{l}
\forall 1 \leq j \leq d.
\end{array}
\]

If the constant function \( 1_{[0,\infty)} \) is in \( L^2(W) \) for any nonzero \( e \in \mathbb{C}^d \), let \( \{e_1, \ldots, e_p\} \) be an orthonormal basis of the subspace of \( \mathbb{C}^d \) satisfying \( 1_{[0,\infty)}e \in L^2(W) \). Define

\[
h_{1,W}^i \equiv c_1^i 1_{[0,\infty)}e_i \quad \text{for} \ i = 1, \ldots, p_1,
\]

where \( c_1^i \) is chosen so that \( \|h_{1,W}^i\|_{L^2(W)} = 1 \). Define the functions

\[
h_{2,W}^i \equiv c_2^i 1_{(-\infty,0]}\nu_i \quad \text{for} \ i = 1, \ldots, p_2,
\]

where \( \{\nu_1, \ldots, \nu_{p_2}\} \) is an orthonormal basis of the subspace of \( \mathbb{C}^d \) satisfying \( 1_{(-\infty,0]}\nu \in L^2(W) \), in an analogous way. Define \( H_W \), the system of Haar functions adapted to \( W \), by:

\[
H_W \equiv \{h_{1,W}^j, h_{2,W}^j\} \cup \{h_{k,W}^i\}_{k=1}^d.
\]

We now show that \( H_W \) is an orthonormal basis of \( L^2(W) \).

Lemma 2.2. The system \( H_W \) is an orthonormal system in \( L^2(W) \).

Proof. We first prove that the system \( \{h_{1,W}^j\} \) is orthogonal. Fix \( h_{1,W}^j \) and \( h_{1,W}^i \). First, assume \( I \neq J \). Then, one interval must be strictly contained in the other because otherwise, the inner product trivially vanishes by support conditions. Without loss of generality, assume \( I \subset J \). This implies that \( h_{1,W}^j \) equals a constant vector on \( I \), which we will denote by \( e \). Then

\[
\langle h_{1,W}^i, h_{1,W}^j \rangle_{L^2(W)} = \int_I \langle W(x)h_{1,W}^i, e \rangle_{\mathbb{C}^d} dx
\]

\[
= \int_I (w_j^i)^{-1} \langle W(I_+)W(I_+)^{-1}W(I_-)v_j^i 1_{I_+} - v_j^i 1_{I_-}, e \rangle_{\mathbb{C}^d} dx
\]

\[
= (w_j^i)^{-1} \langle W(I_+)W(I_+)^{-1}W(I_-)v_j^i e, \mathbb{C}^d \rangle - (w_j^i)^{-1} \langle W(I_-)v_j^i e, \mathbb{C}^d \rangle = 0.
\]

One should notice that the definition of \( e \) played no role; in fact, the above arguments show that each \( h_{1,W}^j \) has mean zero with respect to \( W \). Now assume \( I = J \) and \( i \neq j \). Observe that:

\[
\langle h_{1,W}^i, h_{1,W}^j \rangle_{L^2(W)} = \int_I \langle W(x)h_{1,W}^i, h_{1,W}^j \rangle_{\mathbb{C}^d} dx
\]

\[
= (w^j_i)^{-1}(w^i_j)^{-1} \int_I \langle W(x)\left(W(J_+)^{-1}W(J_-)v_j^i 1_{J_+} - v_j^i 1_{J_-}\right), W(J_+)^{-1}W(J_-)v_j^j 1_{J_+} - v_j^j 1_{J_-}\rangle_{\mathbb{C}^d} dx
\]

\[
= (w^j_i)^{-1}(w^i_j)^{-1} \left( \langle W(J_+)W(J_+)^{-1}W(J_-)v_j^i v_j^i, W(J_+)^{-1}W(J_-)v_j^j v_j^j \rangle_{\mathbb{C}^d} + \langle W(J_-)v_j^i, v_j^j \rangle_{\mathbb{C}^d} \right)
\]

\[
= (w^j_i)^{-1}(w^i_j)^{-1} \left( \langle W(J_-)W(J_+)^{-1}W(J_-) + W(J_-) \rangle v_j^i v_j^j \rangle_{\mathbb{C}^d} \right)
\]

\[
= 0,
\]

since \( v_j^i \) and \( v_j^j \) are orthonormal eigenvectors of \( W(J_-)W(J_+)^{-1}W(J_-) + W(J_-) \). Since each \( h_{1,W}^j \) has mean zero with respect to \( W \) and since each \( h_{1,W}^i \) is either supported in \((-\infty,0]\) or \([0,\infty)\), it is clear that

\[
\langle h_{1,W}^j, h_{k,W}^i \rangle_{L^2(W)} = 0 \quad \forall J \in \mathcal{D}
\]
and for all indices \( i, j, k \). By construction, it is also clear that \( \{ h_{W,j}^{W,j} \} \) is an orthonormal set in \( L^2(W) \). Finally, to see that \( \{ h_{W,j}^{W,j} \} \) is normalized, fix \( h_{W,j}^{W,j} \) and observe that

\[
\langle h_{W,j}^{W,j}, h_{W,j}^{W,j} \rangle_{L^2(W)} = (w_j^*)^{-2} \langle (W(J_--)W(J_+)^{-1}W(J_-) + W(J_-)) v_j^*, v_j^* \rangle_{C^d} \\
= \langle (W(J_--)W(J_+)^{-1}W(J_-) + W(J_-)) (W(J_-)W(J_+)^{-1}W(J_-) + W(J_-))^{-1} v_j^*, v_j^* \rangle_{C^d} \\
= 1,
\]

using the properties of \( v_j^* \) and the definition of \( w_j^* \). This completes the proof. \( \square \)

**Lemma 2.3.** The orthonormal system \( H_w \) is complete in \( L^2(W) \).

**Proof.** Fix \( f \) in \( L^2(W) \), and assume \( f \) is orthogonal to every function in \( H_w \). Specifically, \( f \) is orthogonal to the set \( \{ h_{W,j}^{W,j} \} \). Then, for each \( J \in \mathcal{D} \) and \( j = 1, \ldots, d \),

\[
0 = \langle f, h_{W,j}^{W,j} \rangle_{L^2(W)}.
\]

Multiplying by a constant gives:

\[
0 = |J_-|^{-1} \langle W(J_--)^{-1}W(J_-) v_j^* 1_{J_+} - v_j^* 1_{J_-}, f \rangle_{L^2(W)} \\
= |J_-|^{-1} \int_J \langle W(J_--)^{-1}W(J_-) v_j^* 1_{J_+} - v_j^* 1_{J_-}, W(x)f(x) \rangle_{C^d} dx \\
= \langle W(J_--)^{-1}W(J_-) v_j^*, \langle W f \rangle_{J_+} \rangle_{C^d} - \langle v_j^*, \langle W f \rangle_{J_-} \rangle_{C^d} \\
= \langle v_j^*, W(J_-)W(J_+)^{-1}(W f)_{J_+} - \langle W f \rangle_{J_-} \rangle_{C^d}.
\]

Since this holds for each \( j \) and \( v_1^*, \ldots, v_d^* \) is an orthonormal basis of \( C^d \), we can conclude that

\[
\langle W f \rangle_{J_+} = W(J--)W(J_+)^{-1}(W f)_{J_+}.
\]

Adding \( \langle W f \rangle_{J_+} \) to both sides gives

\[
2 \langle W f \rangle_{J_+} = W(J--)W(J_+)^{-1}(W f)_{J_+} + \langle W f \rangle_{J_+} = (W(J--)W(J_+)^{-1} + W(J_+))W(J_+)^{-1} \langle W f \rangle_{J_+}.
\]

Rearranging by factoring out \( W(J_+)^{-1} \) on the right from the term in parentheses and using the definitions gives

\[
\langle W \rangle_{J_+}^{-1} \langle W f \rangle_{J_+} = \langle W \rangle_{J_+}^{-1} \langle W f \rangle_{J_+}.
\]

Solving (5) for \( \langle W f \rangle_{J_+} \) and using analogous arguments, one can show:

\[
\langle W \rangle_{J_+}^{-1} \langle W f \rangle_{J_+} = \langle W \rangle_{J_-}^{-1} \langle W f \rangle_{J_-}.
\]

Now fix any \( x, y \in (0, \infty) \) and choose some dyadic interval \( J_0 \) so that \( x, y \in J_0 \). Define two sequence of dyadic intervals:

\[
J_0 = I_0 \supset I_1 \supseteq I_2 \cdots \supseteq I_i \supseteq I_{i+1} \cdots \\
J_0 = K_0 \supset K_1 \supset K_2 \cdots \supset K_k \supset K_{k+1} \cdots
\]

such that each \( I_i \) is a parent of \( I_{i+1} \) and \( x \in I_i \) for all \( i \) and similarly, each \( K_k \) is a parent of \( K_{k+1} \) and \( y \) is in each \( K_k \). Our previous arguments imply that

\[
\langle W \rangle_{I_i}^{-1} \langle W f \rangle_{I_i} = \langle W \rangle_{I_0}^{-1} \langle W f \rangle_{I_0} = \langle W \rangle_{K_k}^{-1} \langle W f \rangle_{K_k} \quad \forall i, k \in \mathbb{N}.
\]
Now we can use the Lebesgue Differentiation Theorem to conclude that

\[ W(x)^{-1}W(x)f(x) = W(y)^{-1}W(y)f(y) \]

for almost every \( x, y \) in \((0, \infty)\) and so \( f(x) = f(y) \) for almost every \( x, y \) in \([0, \infty)\). Analogous arguments imply \( f \) must be constant on \((-\infty, 0]\). But, by assumption, \( f \) is also orthogonal to the set \( \{h^W_k\} \), which implies \( f \) is orthogonal to all of the nonzero constant functions supported on \([0, \infty)\) or \((-\infty, 0]\) in \(L^2(W)\). Thus, we can conclude \( f \equiv 0 \).  

We require one additional fact about the weighted Haar system:

**Lemma 2.4.** The orthonormal system \( H_W \) satisfies

\[
\| W(J_\pm)^{\frac{1}{2}} h^W_j (J_\pm) \|_{C^d} \leq C(d) \\
\| W(J_\pm)^{\frac{1}{2}} h^W_j (J_\pm) \|_{C^d} \leq C(d)
\]

for all \( J \in \mathcal{D} \) and \( 1 \leq j \leq d \), where \( h^W_j (J_\pm) \) is the constant value \( h^W_j \) takes on \( J_\pm \).

**Proof.** We only prove the first inequality as the second is proved similarly. Observe that

\[
\begin{align*}
\| W(J_-)^{\frac{1}{2}} h^W_j (J_-) \|_{C^d}^2 &\leq \| W(J_-)^{\frac{1}{2}} (W(J)W(J_-)^{-1}W(J_-))^{-\frac{1}{2}} \| \leq \| W(J_-)^{\frac{1}{2}} W(J_-)^{-1}W(J_+)W(J)^{-1}W(J_-)^{\frac{1}{2}} \|
\leq C(d) \text{Tr} \left( W(J_-)^{\frac{1}{2}} W(J_-)^{-1}W(J_+)W(J)^{-1}W(J_-)^{\frac{1}{2}} \right)
= C(d) \text{Tr} \left( W(J)^{-\frac{1}{2}} W(J_+)W(J)^{-\frac{1}{2}} \right)
\leq C(d) \| W(J)^{-\frac{1}{2}} W(J_+)W(J)^{-\frac{1}{2}} \|
\leq C(d) \| W(J)^{-\frac{1}{2}} W(J)W(J)^{-\frac{1}{2}} \|
= C(d),
\end{align*}
\]

where we used the fact that trace and operator norm are equivalent (up to a dimensional constant) for positive, self-adjoint matrices. This completes the proof.  

**Remark 2.5.** In the proofs of Theorems 4.3 and 4.4, we will expand functions in \(L^2(W)\) with respect to the basis \( H_W \). Specifically, if \( f \in L^2(W) \), we can expand \( f \) as

\[
f = \sum_{J \in \mathcal{D}} \sum_{1 \leq j \leq d} \left\langle f, h^W_j \right\rangle_{L^2(W)} h^W_j + \sum_{1 \leq k \leq 2} \sum_{1 \leq j \leq p_k} \left\langle f, h^W_k \right\rangle_{L^2(W)} h^W_k.
\]
This means that for $K \in \mathcal{D}$, we can express the weighted average of $f$ on $K$ as
\[
\langle W \rangle_K^{-1} \langle W f \rangle_K = \sum_{J \in \mathcal{D}} \left\langle f, h^{W,j}_J \right\rangle_{L^2(W)} \langle W \rangle_K^{-1} \left\langle W h^{W,j}_J \right\rangle_K + \sum_{1 \leq k \leq 2} \sum_{1 \leq j \leq p_k} \left\langle f, h^{W,j}_k \right\rangle_{L^2(W)} \langle W \rangle_K^{-1} \left\langle W h^{W,j}_k \right\rangle_K
\]
\[
= \sum_{J \in \mathcal{D}} \left\langle f, h^{W,j}_J \right\rangle_{L^2(W)} h^{W,j}_J(K) + \sum_{1 \leq k \leq 2} \sum_{1 \leq j \leq p_k} \left\langle f, h^{W,j}_k \right\rangle_{L^2(W)} h^{W,j}_k(K),
\]
where $h^{W,j}_J(K)$ is the constant value that $h^{W,j}_J$ takes on $K$ and $h^{W,j}_k(K)$ is the constant value that $h^{W,j}_k$ takes on $K$. Now assume $f$ is compactly supported, and so supp$(f)$ is contained in at most two dyadic intervals. Call them $I_1$ and $I_2$, where $I_1 \subset [0, \infty)$ and $I_2 \subset (-\infty, 0]$. Then we can write $f$ as
\[
f = \sum_{J \in \mathcal{D}} \left\langle f, h^{W,j}_J \right\rangle_{L^2(W)} h^{W,j}_J + \sum_{1 \leq k \leq 2} \sum_{1 \leq j \leq p_k} \left\langle f, h^{W,j}_k \right\rangle_{L^2(W)} h^{W,j}_k
\]
(6)\[
= \sum_{J \in \mathcal{D}} \left\langle f, h^{W,j}_J \right\rangle_{L^2(W)} h^{W,j}_J + \sum_{1 \leq \ell \leq 2} E^W_{I_\ell} f,
\]
where for each $I \in \mathcal{D}$, the expectation $E^W_{I} f$ is defined to be $\langle W \rangle_I^{-1} \langle W f \rangle_I, 1_I$.

3. Matrix Carleson Embedding Theorem

Let $W$ be a matrix weight such that for all positive semi-definite matrices $A$ and intervals $J \in \mathcal{D}$, there is a uniform constant $C$ satisfying
\[
\frac{1}{|J|} \int_J \|AW(x)A\|dx \leq C \left( \frac{1}{|J|} \int_J \|AW(x)A\|^{\frac{3}{2}}dx \right)^2.
\]
(7)
Define $[W]_{R_2}$ to be the smallest such constant $C$. Treil-Volberg’s arguments in Lemma 3.5 and Lemma 3.6 in [13] show that, if $W$ is an $A_2$ matrix weight, then
\[
[W]_{R_2} \leq C(d)[W]_{A_2}.
\]
In Theorem 6.1 in [13], Treil-Volberg prove an embedding theorem for a specific sequence of positive semi-definite matrices. Their arguments generalize easily to arbitrary sequences of matrices, yielding the following matrix Carleson Embedding Theorem:

**Theorem 3.1.** Let $W$ be a matrix weight satisfying (7) and let $\{A_I\}_{I \in \mathcal{D}}$ be a sequence of positive semi-definite $d \times d$ matrices. Then
\[
\sum_{I \in \mathcal{D}} \langle A_I (f)_I, (f)_I \rangle_{C^d} \leq C_1 \|f\|^2_{L^2(W^{-1})} \text{ if } \frac{1}{|J|} \sum_{I : I \subseteq J} \left\| \langle W \rangle_I^\frac{1}{2} A_I \langle W \rangle_I^\frac{1}{2} \right\| \leq C_2 \quad \forall J \in \mathcal{D},
\]
where $C_1 = C_2C(d)[W]_{R_2}$ and $C(d)$ is a dimensional constant.
Remark 3.2. Treil-Volberg’s arguments in [13] actually establish a seemingly stronger result. Namely, they show that if \( \{B_I\}_{I \in \mathcal{D}} \) is a sequence of positive semi-definite matrices, then
\[
\sum_{I \in \mathcal{D}} \left\| \langle W \rangle_I^{-\frac{1}{2}} B_I \langle W \rangle_I^{-\frac{1}{2}} \right\|^2_{C^d} \leq C_1 \|g\|_{L^2}^2 \quad \text{if} \quad \frac{1}{|J|} \sum_{I : I \subseteq J} \left\| \langle W \rangle_I^{-\frac{1}{2}} B_I \langle W \rangle_I^{-\frac{1}{2}} \right\| \leq C_2,
\]
for all \( J \in \mathcal{D} \). To recover Theorem 3.1 from (9), note that
\[
\sum_{I \in \mathcal{D}} \left\| \langle W \rangle_I^{-1} B_I \langle W \rangle_I^{-1} \langle W \frac{1}{2} g \rangle_I \right\|_{C^d}^2 \leq \sum_{I \in \mathcal{D}} \left\| \langle W \rangle_I^{-\frac{1}{2}} B_I \langle W \rangle_I^{-\frac{1}{2}} \right\|^2_{C^d} \left\| \langle W \rangle_I^{-\frac{1}{2}} \langle W \frac{1}{2} g \rangle_I \right\|^2_{C^d}.
\]
If one is given \( \{A_I\}_{I \in \mathcal{D}} \) and \( f \in L^2(W^{-1}) \), then pairing the above inequality with (9) using \( B_I \equiv \langle W \rangle_I A_I \langle W \rangle_I \) and \( g \equiv W^{-\frac{1}{2}} f \) gives the inequalities in Theorem 3.1.

Equation (9) is proved via arguments similar to those used in [12] to establish the standard Carleson Embedding Theorem. Specifically, Treil-Volberg defines an associated embedding operator and show it is bounded using the Senichkin-Vinogradov Test:

**Theorem 3.3** (Senichkin-Vinogradov Test). Let \( \mathcal{Z} \) be a measure space, and let \( k \) be a locally summable, nonnegative, measurable function on \( \mathcal{Z} \times \mathcal{Z} \). If
\[
\int_{\mathcal{Z}} k(s,t) k(s,x) \, ds \leq C [k(x,t) + k(t,x)] \quad \text{a.e. on } \mathcal{Z},
\]
then for all nonnegative \( g \in L^2(\mathcal{Z}) \),
\[
\int_{\mathcal{Z}} \int_{\mathcal{Z}} k(s,t) g(s) g(t) \, ds dt \leq 2C \|g\|_{L^2(\mathcal{Z})}^2.
\]

For the ease of the reader, we sketch the proof of (9). We focus on the first half of the proof, as the second half is given in detail in [13].

**Proof.** First define \( \mu_I = \left\| \langle W \rangle_I^{-\frac{1}{2}} B_I \langle W \rangle_I^{-\frac{1}{2}} \right\| \). Then, by assumption, \( \{\mu_I\}_{I \in \mathcal{D}} \) is a scalar Carleson sequence with testing constant \( C_2 \). Define the embedding operator \( \mathcal{J} : L^2 \to \ell^2(\{\mu_I\}, \mathbb{C}^d) \) by
\[
\mathcal{J} f = \left\{ \langle W \rangle_I^{-\frac{1}{2}} \langle W \frac{1}{2} f \rangle_I \right\}_{I \in \mathcal{D}}
\]
and observe that (9) is equivalent to \( \mathcal{J} \) having operator norm bounded by \( \sqrt{C_1} \). To prove the norm bound, one shows that the formal adjoint \( \mathcal{J}^* : \ell^2(\{\mu_I\}, \mathbb{C}^d) \to L^2 \) defined by
\[
\mathcal{J}^* \{\alpha_I\} = \sum_{I \in \mathcal{D}} \frac{\mu_I}{|I|} 1_I W^{-\frac{1}{2}} \langle W \rangle_I^{-\frac{1}{2}} \alpha_I \quad \forall \{\alpha_I\} \in \ell^2(\{\mu_I\}, \mathbb{C}^d)
\]
has the desired norm bound. First observe that
\[
\mathcal{J} \mathcal{J}^* \{\alpha_I\} = \left\{ \langle W \rangle_J^{-\frac{1}{2}} \sum_{I \in \mathcal{D}} \frac{\mu_I}{|I|} \langle W 1_I \rangle_J \langle W \rangle_I^{-\frac{1}{2}} \alpha_I \right\}_{J \in \mathcal{D}}.
\]
One can use this to immediately show that for any \( \{\alpha_I\} \) in \( \ell^2(\{\mu_I\}, \mathbb{C}^d) \),
\[
\|\mathcal{J}^* \{\alpha_I\}\|_{L^2}^2 = \left\langle \mathcal{J} \mathcal{J}^* \{\alpha_I\}, \{\alpha_I\} \right\rangle_{\ell^2(\{\mu_I\}, \mathbb{C}^d)}
\]
\[
= \sum_{J \in \mathcal{D}} \sum_{I : I \subseteq J} \frac{\mu_I \mu_J}{|J|} \left\langle \langle W \rangle_J^{-\frac{1}{2}} \langle W \rangle_I^{-\frac{1}{2}} \alpha_I, \alpha_J \right\rangle_{\mathbb{C}^d} + \sum_{I \in \mathcal{D}} \sum_{J : J \subseteq I} \frac{\mu_I \mu_J}{|I|} \left\langle \langle W \rangle_J^{-\frac{1}{2}} \langle W \rangle_I^{-\frac{1}{2}} \alpha_I, \alpha_J \right\rangle_{\mathbb{C}^d}.
\]
Now, for $K, L \in \mathcal{D}$, define $T_{LK}$ by
\[
T_{LK} \equiv \frac{1}{|L|} \left\| \langle W \rangle^{\frac{3}{2}}_K \langle W \rangle^{-\frac{3}{2}}_L \right\| = \frac{1}{|L|} \left\| \langle W \rangle^{-\frac{3}{2}}_L \langle W \rangle^{\frac{3}{2}}_K \right\|
\]
if $K \subseteq L$ and $T_{KL} = 0$ otherwise. By symmetry in the sums, it is easy to show that
\[
\|J^*\{\alpha_t\}\|_{L^2}^2 \leq 2 \sum_{J \in \mathcal{D}} \sum_{I : I \subseteq J} \mu_I \mu_J T_{JI} \|\alpha_I\|_{C^d} \|\alpha_J\|_{C^d}.
\]
Thus, the result will be proved if one can show that the righthand side of (10) is bounded by $C_1 \|\{\alpha_t\}\|_{l^2(\{\mu_t\}, C^d)}$. This is where one uses the Senichkin-Vinogradov Test. Let $Z$ be $\mathcal{D}$, the set of dyadic intervals, with point mass $\mu_I$ on each interval $I$. Then, $L^2(Z)$ is equivalent to $l^2(\{\mu_I\}, \mathbb{C})$. Indeed, $\{\beta_t\} \in l^2(\{\mu_t\}, \mathbb{C})$ if and only if the function $\beta$ defined by $\beta(I) = \beta_I$ is in $L^2(Z)$. Moreover,
\[
\|\{\beta_t\}\|_{l^2(\{\mu_t\}, \mathbb{C})} = \|\beta\|_{L^2(Z)},
\]
so we can treat these as the same objects. Now, define the nonnegative function $k : Z \times Z \to \mathbb{R}^+$ by
\[
k(K, L) \equiv \sum_{J \in \mathcal{D}} \sum_{I : I \subseteq J} T_{JI} \delta_I(K) \delta_J(L),
\]
where $\delta_I(K) = 1$ if $K = I$ and zero otherwise. Fix a sequence $\{\alpha_t\} \in l^2(\{\mu_t\}, \mathbb{C}^d)$. Then the sequence $\{a_t\}$ defined by $a_I \equiv \|\alpha_t\|_{C^d}$ is a nonnegative sequence in $l^2(\{\mu_t\}, \mathbb{C})$ or equivalently, $a$ (defined by $a(I) = a_I$) is a nonnegative function in $L^2(Z)$, and the norms of the two sequences are equal. It is easy to show that
\[
\int_Z \int_Z k(K, L)a(K)(a(L)\, dKdL = \sum_{J \in \mathcal{D}} \sum_{I : I \subseteq J} \mu_I \mu_J T_{JI} a_I a_J = \sum_{J \in \mathcal{D}} \sum_{I : I \subseteq J} \mu_I \mu_J T_{JI} \|\alpha_I\|_{C^d} \|\alpha_J\|_{C^d},
\]
which is exactly the object we need to control. Indeed, if we can establish the conditions of the Senichkin-Vinogradov test with constant $C_1$, then the result will be proved. Let us first rewrite the desired conditions. The definition of $k$ implies that
\[
\int_Z k(K, J)k(K, J') \, dK = \sum_{I : I \subseteq J,J'} T_{JI} T_{J'I} \mu_I \quad \forall J, J' \in \mathcal{D}.
\]
Again using the definition of $k$, we have
\[
k(J, J') + k(J', J) = T_{JI'} + T_{J'I} \quad \forall J, J' \in \mathcal{D}.
\]
Since we only sum over dyadic $I \subseteq J \cap J'$, to have a nonzero sum, we must have $J \subseteq J'$ or $J' \subseteq J$. Without loss of generality, assume $J' \subseteq J$. Then, to establish the conditions of the Senichkin-Vinogradov test, one must simple show:
\[
\sum_{I : I \subseteq J'} T_{JI} T_{J'I} \mu_I = \sum_{I : I \subseteq J'} \mu_I \frac{1}{|J'|} \left\| \langle W \rangle^{-\frac{3}{2}}_J \langle W \rangle^{\frac{3}{2}}_I \right\| \frac{1}{|J'|} \left\| \langle W \rangle^{\frac{3}{2}}_J \langle W \rangle^{-\frac{3}{2}}_I \right\| \leq C_1 \frac{1}{|J'|} \left\| \langle W \rangle^{\frac{3}{2}}_J \langle W \rangle^{\frac{3}{2}}_J \right\|.
\]
This inequality is proven in detail in [13]. The proof uses simple results about matrix weights including the fact that all matrix $A_2$ weights satisfy a reverse Hölder estimate as in (7). The reverse Hölder estimate is used to turn the sum of interest into a sum of averages of a function weighted by the constants $\mu_I$. Since $\{\mu_I\}_{I \in \mathcal{D}}$ is a scalar Carleson sequence, one can use the scalar Carleson Embedding Theorem to complete the proof. \qed
Remark 3.4. A more general Carleson Embedding Theorem, which holds for all $A_p$ matrix weights, is proven by Isralowitz-Kwon-Pott in [5]. Using arguments from Isralowitz-Kwon-Pott [5], we obtain the following Carleson Embedding Theorem; its testing conditions are particularly well-suited to the objects appearing in the proofs of Theorems 4.3 and 4.4, the well-localized analogues of Theorems 1.2 and 1.3.

It should be noted that the existence of this result, albeit with a different constant, is mentioned in the final remarks of [5]. Indeed, according to these remarks, if one modifies their previous arguments and tracks all constants closely, one could obtain this Carleson Embedding Theorem with constant $C(d)[W]_{A_2}$. However, in light of Equation (8), our constant is very likely smaller than the one appearing in [5].

**Theorem 3.5.** Let $W$ be an $A_2$ weight and let $\{A_I\}_{I \in \mathcal{D}}$ be a sequence of positive semi-definite $d \times d$ matrices. Then

$$\sum_{I \in \mathcal{D}} \langle A_I \langle f \rangle_I, \langle f \rangle_I \rangle_{C^d} \leq C_1 \|f\|_{L^2(W^{-1})}^2 \quad \text{if} \quad \frac{1}{|J|} \sum_{I: J \subseteq I} \langle W \rangle_{I} A_I \langle W \rangle_{I} \leq C_2 \langle W \rangle_{J} \quad \forall J \in \mathcal{D},$$

where $C_1 = C_2 C(d)[W]_{R_2}[W]_{A_2}$.

**Remark 3.6.** In Theorems 1.2, 1.3 and Theorems 4.3, 4.4, the constants $B(W)$ and $B(V)$ appear. Since dimensional constants are already included in the statement of those theorems, it should be clear from Theorem 3.5 that

$$B(W) = [W]_{R_2}^{\frac{1}{2}}[W]_{A_2}^{\frac{1}{2}} \quad \text{and} \quad B(V) = [V]_{R_2}^{\frac{1}{2}}[V]_{A_2}^{\frac{1}{2}}.$$

The existence of Theorem 3.5 with a different constant is mentioned at the end of [5]. Since the details of the proof are not given and we obtain a different constant, we include the proof. We first need the decaying stopping tree from Isralowitz-Kwon-Pott. Specifically, fix $I \in \mathcal{D}$ and let $\mathcal{J}(I)$ be the collection of maximal dyadic $J \subseteq I$ such that

$$\left\| \langle W \rangle_{J}^{-\frac{1}{2}} \langle W \rangle_{J}^{\frac{1}{2}} \right\|^2 > \lambda \quad \text{or} \quad \left\| \langle W \rangle_{J}^{\frac{1}{2}} \langle W \rangle_{J}^{-\frac{1}{2}} \right\|^2 > \lambda,$$

for $\lambda > 1$ to be determined later. Set $\mathcal{F}(I)$ to be the collection of $J \subseteq I$ such that $J$ is not contained in any interval in $\mathcal{J}(I)$. It is clear that $I$ is always in $\mathcal{F}(I)$. Set $\mathcal{F}^0(I) \equiv \{I\}$.

Inductively define $\mathcal{J}^j(I)$ and $\mathcal{F}^j(I)$ by

$$\mathcal{J}^j(I) = \bigcup_{J \in \mathcal{J}^{j-1}(I)} \mathcal{J}(J) \quad \text{and} \quad \mathcal{F}^j(I) = \bigcup_{J \in \mathcal{J}^{j-1}(I)} \mathcal{F}(J).$$

One can then prove the following lemma.

**Lemma 3.7** (Lemma 2.1, [5]). Given the stopping-tree set-up, if $\lambda = 4C(d)[W]_{A_2}$, then

$$\left| \bigcup_{J \in \mathcal{J}^j(I)} \mathcal{J}(J) \right| \leq 2^{-j}|I| \quad \forall I \in \mathcal{D}.$$

We can now provide the proof of Theorem 3.5:

**Proof of Theorem 3.5.** Using the equivalence, up to a dimensional constant, of norm and trace for positive semi-definite matrices, our hypothesis implies

$$\sum_{I: J \subseteq K} \left\| \langle W \rangle_{K}^{-\frac{1}{2}} \langle W \rangle_{I} A_I \langle W \rangle_{I} \langle W \rangle_{K}^{\frac{1}{2}} \right\| \lesssim C_2 |K| \quad \forall K \in \mathcal{D}.$$
We will use this to obtain the testing condition from Theorem 3.1. Specifically, fix $J \in \mathcal{D}$. Then

$$
\frac{1}{|J|} \sum_{I : \mathcal{J} \subseteq J} \left\| \langle W \rangle_{T}^{\frac{1}{2}} A_{I} \langle W \rangle_{K}^{\frac{1}{2}} \right\| = \frac{1}{|J|} \sum_{j = 1}^{\infty} \sum_{K \in (\mathcal{J}^{-1}(J))} \sum_{I \in \mathcal{F}(K)} \left\| \langle W \rangle_{T}^{\frac{1}{2}} A_{I} \langle W \rangle_{K}^{\frac{1}{2}} \right\|
$$

$$
\leq \frac{1}{|J|} \sum_{j = 1}^{\infty} \sum_{K \in (\mathcal{J}^{-1}(J))} \sum_{I \in \mathcal{F}(K)} \left\| \langle W \rangle_{I}^{\frac{1}{2}} A_{I} \langle W \rangle_{K}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{T}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{K}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{K}^{\frac{1}{2}} \right\|
$$

$$
= \frac{1}{|J|} \sum_{j = 1}^{\infty} \sum_{K \in (\mathcal{J}^{-1}(J))} \sum_{I \in \mathcal{F}(K)} \left\| \langle W \rangle_{I}^{\frac{1}{2}} A_{I} \langle W \rangle_{K}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{T}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{K}^{\frac{1}{2}} \right\|
$$

$$
\lesssim \frac{[W]_{A_{2}}}{|J|} \sum_{j = 1}^{\infty} \sum_{K \in (\mathcal{J}^{-1}(J))} \sum_{I \in \mathcal{F}(K)} \left\| \langle W \rangle_{I}^{\frac{1}{2}} A_{I} \langle W \rangle_{K}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{T}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{K}^{\frac{1}{2}} \right\|
$$

$$
\leq \frac{[W]_{A_{2}}}{|J|} \sum_{j = 1}^{\infty} \sum_{K \in (\mathcal{J}^{-1}(J))} \sum_{I \in \mathcal{F}(K)} \left\| \langle W \rangle_{I}^{\frac{1}{2}} A_{I} \langle W \rangle_{K}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{T}^{\frac{1}{2}} \right\| \left\| \langle W \rangle_{K}^{\frac{1}{2}} \right\|\right\|
$$

$$
\lesssim C_{2}[W]_{A_{2}} \sum_{j = 1}^{\infty} \sum_{K \in (\mathcal{J}^{-1}(J))} |K|\right\|
$$

$$
\leq C_{2}[W]_{A_{2}} \sum_{j = 1}^{\infty} 2^{-j}
$$

$$
= C_{2}[W]_{A_{2}}
$$

In the fourth line from the top we use the stopping criteria, which introduces the value $[W]_{A_{2}}$. Pairing this estimate with Theorem 3.1 gives the desired result. □

4. Well-Localized Operators

We say an operator $T_{W}$ acts formally from $L^{2}(W)$ to $L^{2}(V)$ if the bilinear form

$$
\langle T_{W} 1_{I} e, 1_{J} v \rangle_{L^{2}(V)}
$$

is given for all $I, J \in \mathcal{D}$ and $e, v \in \mathbb{C}^{d}$ is well-defined. Then, the formal adjoint $T_{V}^{*}$ is defined by

$$
\langle T_{V}^{*} 1_{I} e, 1_{J} v \rangle_{L^{2}(W)} = \langle 1_{I} e, T_{W} 1_{J} v \rangle_{L^{2}(V)}
$$

Given this, we can define:

**Definition 4.1.** An operator $T_{W}$ acting (formally) from $L^{2}(W)$ to $L^{2}(V)$ is called r-lower triangular if for all $1 \leq j \leq d$ and $I, J \in \mathcal{D}$ with $|J| \leq 2|I|$ and all $e \in \mathbb{C}^{d}$, $T_{W}$ satisfies

$$
\langle T_{W} 1_{I} e, h_{j}^{V} \rangle_{L^{2}(V)} = 0
$$

whenever $J \not\subset I^{(r+1)}$ or $|J| \leq 2^{-r}|I|$ and $J \not\subset I$. Here, $\{h_{j}^{V}\}$ is the set of $V$-weighted Haar functions on $J$ as defined in (4) and $I^{(r+1)}$ is the $(r+1)^{th}$ ancestor of $I$. We say $T_{W}$ is well-localized with radius $r$ if both $T_{W}$ and its formal adjoint $T_{V}^{*}$ are r-lower triangular.
Remark 4.2. This definition of well-localized is slightly different than the one appearing in [10]. Indeed, to define lower triangular, Nazarov-Treil-Volberg only impose conditions on $T_W$ when $|J| \leq |I|$, rather than $|J| \leq 2|I|$. Nevertheless, their ideas are clearly the correct ones and their definition is essentially correct; the difference is likely attributable to a typographical error.

However, to see why imposing conditions on only $|J| \leq |I|$ is not quite sufficient, let us consider the role of the well-localized property in the proofs of the $T_1$ theorems for well-localized operators, our Theorems 4.3 and 4.4. It is used to show that for each fixed $I$, there is at most a finite number of $J$ with $2^{-r}|I| \leq |J| \leq 2^r|I|$ such that

$$\left| \langle T_W h_I^{W,i}, h_J^{V,j} \rangle_{L^2(V)} \right| \neq 0.$$ 

This allows one to control related sums given in (14). However, the definition of well-localized given by Nazarov-Treil-Volberg is not quite enough for this, as it does not handle the case where $|I| = |J|$. In this case, one would need control over terms such as

$$\left| \langle T_W h_I^{W,i}(I_+), h_J^{V,j} \rangle_{L^2(V)} \right| \text{ or } \left| \langle h_I^{W,i}, T_V^* h_J^{V,j}(J_+) \rangle_{L^2(W)} \right|,$$

which are not addressed in their definition of well-localized since $|I_+| < |J|$ and $|J_+| < |I|$. This case is no longer a problem if we impose conditions on all $I, J$ with $|J| \leq 2|I|$ as in Definition 4.1. For an example of what can go wrong, fix $K_0 \in \mathcal{D}$. Fix a sequence $\{c_K\}$ in $\ell^2(\mathcal{D})$ with no nonzero terms, and define the operator $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$Th_{K_0} \equiv \sum_{K: |K| = |K_0|} c_K h_K \text{ and } Th_L \equiv 0 \text{ for } L \neq K_0.$$ 

It is not difficult to show $T$ is well-localized (with radius 0) from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ according to the definition in [10]. Indeed, if $|J| \leq |I|$, then

$$\langle T \mathbf{1}_I, h_J \rangle_{L^2} = 0 = \langle T^* \mathbf{1}_I, h_J \rangle_{L^2}.$$ 

To see these equalities, first write

$$\mathbf{1}_I = \sum_{K: |I| \subseteq K} \langle \mathbf{1}_I, h_K \rangle_{L^2} h_K.$$ 

Thus, if $I$ is not strictly contained in $K_0$, then $T \mathbf{1}_I = 0$. So, we can assume $I \subseteq K_0$. Then $|J| \leq |I| < |K_0|$ so

$$\langle T \mathbf{1}_I, h_J \rangle_{L^2} = \sum_{K: |K| = |K_0|} \langle \mathbf{1}_I, h_{K_0} \rangle_{L^2} c_K \langle h_K, h_J \rangle_{L^2} = 0.$$ 

Now consider $T^*$. If $|J| \leq |I|$ and $J \neq K_0$, then

$$\langle T^* \mathbf{1}_I, h_J \rangle_{L^2} = \langle \mathbf{1}_I, T h_J \rangle_{L^2} = \langle \mathbf{1}_I, 0 \rangle_{L^2} = 0$$

immediately. If $J = K_0$, then

$$\langle T^* \mathbf{1}_I, h_J \rangle_{L^2} = \sum_{K: |K| = |K_0|} c_K \langle \mathbf{1}_I, h_K \rangle_{L^2} = 0,$$

since $|K_0| = |J| \leq |I|$ implies $K \subseteq I$ or $K \cap I = 0$. However, for this operator $T$,

$$\langle Th_{K_0}, h_J \rangle_{L^2} = c_J \neq 0,$$
for all $J$ with $|J| = |K_0|$. Since there are is infinite number of such $J$, this means we could not use the well-localized property to control the sums from (14) for this operator.

The main results about well-localized operators are the following two theorems, which are the well-localized analogues of Theorems 1.2 and 1.3:

**Theorem 4.3.** Let $V, W$ be matrix $A_2$ weights, and assume $T_W$ is a well-localized operator of radius $r$ acting formally from $L^2(W)$ to $L^2(V)$. Then $T_W$ extends to a bounded operator from $L^2(W)$ to $L^2(V)$ if and only if

$$
\|T_W 1_I e\|_{L^2(V)} \leq A_1 \langle W(I) e, e \rangle_{C^d}^{\frac{1}{2}}
$$

$$
\|T_W^* 1_I e\|_{L^2(W)} \leq A_2 \langle V(I) e, e \rangle_{C^d}^{\frac{1}{2}}
$$

for all $I \in D$ and $e \in C^d$. Furthermore,

$$
\|T_W\|_{L^2(W) \to L^2(V)} \leq 2^{2r} C(d) \left( A_1 B(W) + A_2 B(V) \right),
$$

where $C(d)$ is a dimensional constant and $B(W)$ and $B(V)$ are constants depending on $W$ and $V$ from an application of the matrix Carleson Embedding Theorem.

**Theorem 4.4.** Let $V, W$ be matrix $A_2$ weights, and assume $T_W$ is a well-localized operator of radius $r$ acting formally from $L^2(W)$ to $L^2(V)$. Then $T_W$ extends to a bounded operator from $L^2(W)$ to $L^2(V)$ if and only if the following two conditions hold:

(i) For all intervals $I \in D$ and $e \in C^d$,

$$
\|1_I T_W 1_I e\|_{L^2(V)} \leq A_1 \langle W(I) e, e \rangle_{C^d}^{\frac{1}{2}}
$$

$$
\|1_I T_W^* 1_I e\|_{L^2(W)} \leq A_2 \langle V(I) e, e \rangle_{C^d}^{\frac{1}{2}}.
$$

(ii) For all intervals $I, J$ in $D$ satisfying $2^{-r} |I| \leq |J| \leq 2^r |I|$ and vectors $e, \nu$ in $C^d$,

$$
\left| \langle T_W 1_I e, 1_J \nu \rangle_{L^2(V)} \right| \leq A_3 \langle W(I) e, e \rangle_{C^d}^{\frac{1}{2}} \langle V(J) \nu, \nu \rangle_{C^d}^{\frac{1}{2}}.
$$

Furthermore,

$$
\|T_W\|_{L^2(W) \to L^2(V)} \leq 2^{2r} C(d) \left( A_1 B(W) + A_2 B(V) + A_3 \right),
$$

where $C(d)$ is a dimensional constant and $B(W)$ and $B(V)$ are constants depending on $W$ and $V$ from an application of the matrix Carleson Embedding Theorem.

Theorems 1.2 and 1.3 will follow immediately from these theorems once we establish the following lemma:

**Lemma 4.5.** If $V, W$ are matrix weights whose entries are in $L^2_{loc} (\mathbb{R})$ and if $T$ is a band operator of radius $r$, then $T_W$ is a well-localized operator of radius $r$ acting formally from $L^2(W)$ to $L^2(V)$.

**Proof.** Assume $T: L^2 \to L^2$ is a band operator with radius $r$, and $W, V$ are matrix weights whose entries are in $L^2_{loc}$. Then the operators

$$
T_W \equiv TM_W \quad \text{and} \quad T_V^* \equiv T^* M_V
$$

act formally from $L^2(W)$ to $L^2(V)$ and $L^2(V)$ to $L^2(W)$ respectively since

$$
\langle TW 1_I e, V 1_J \nu \rangle_{L^2} = \langle TW 1_I e, 1_J \nu \rangle_{L^2(V)} \quad \text{and} \quad \langle W 1_I e, T^* V 1_J \nu \rangle_{L^2} = \langle 1_I e, T_V^* 1_J \nu \rangle_{L^2(W)}
$$
are well-defined. To show $T_W$ is a well-localized operator with radius $r$, by symmetry, it suffices to show that $T_W$ is $r$-lower triangular. First, fix an orthonormal basis $\{ e_i \}_{i=1}^d$ of $\mathbb{C}^d$ and for $I \in \mathcal{D}$, define $H_I \equiv \{ h_I e_i \}_{1 \leq i \leq d}$. Then we can write

$$T = \sum_{I,J \in \mathcal{D}} T_{IJ} \text{ where } T_{IJ} : H_I \to H_J,$$

and each $T_{IJ}$ is given by

$$T_{IJ} = \sum_{1 \leq i,j \leq d} \langle Th_I e_i, h_J e_j \rangle_{L^2} \langle \cdot, h_I e_i \rangle_{L^2} h_J e_j.$$

Since the entries of $W$ are in $L^2_{\text{loc}}(\mathbb{R})$, then $W_I e$ is in $L^2$ and so, $T_W I e \equiv TW I e$ makes sense for each $I \in \mathcal{D}$ and $e \in \mathbb{C}^d$. Given $h_{VJ}$, a vector-valued Haar function on $J$ adapted to $V$, one can write:

$$\left\langle T_W I e, h_{VJ} \right\rangle_{L^2(V)} = \sum_{K,L \in \mathcal{D}} \left\langle T_K L W I e, h_{VJ} \right\rangle_{L^2(V)}$$

$$= \sum_{K,L \in \mathcal{D}} \sum_{1 \leq k,l \leq d} \langle Th_K e_k, h_L e_l \rangle_{L^2} \langle W_I e, h_K e_k \rangle_{L^2} \langle h_L e_l, h_{VJ} \rangle_{L^2(V)}.$$

Observe that $\left\langle T_K L W I e, h_{VJ} \right\rangle_{L^2(V)}$ is zero if $d_{\text{tree}}(K, L) > r$, if $I \cap K = \emptyset$, or if $L \not\subseteq J$. So, we only need consider terms where $d_{\text{tree}}(K, L) \leq r$, $I \cap K \neq \emptyset$, and $L \subseteq J$.

To show $T_W$ is $r$-lower triangular let $|J| \leq 2|I|$. First, assume that $J \not\subseteq I^{(r+1)}$ and by contradiction, assume there is a nonzero term $\left\langle T_K L W I e, h_{VJ} \right\rangle_{L^2(V)}$ in the above sum for some $K, L \in \mathcal{D}$. By our previous assertions, we must have

$$|K| \leq 2^r |L| \leq 2^r |J| \leq 2^{r+1} |I|.$$

Since $I \cap K \neq \emptyset$, this implies that $K \subseteq I^{(r+1)}$. Since $L \subseteq J$, $|L| \leq 2|I|$ and $L \not\subseteq I^{(r+1)}$. But, this immediately implies that $d_{\text{tree}}(K, L) \geq r + 1$, a contradiction.

Similarly, assume $|J| \leq 2^{-r}|I|$ and $J \not\subseteq I$ and by contradiction, assume there is a nonzero term $\left\langle T_K L W I e, h_{VJ} \right\rangle_{L^2(V)}$ for some $K, L$. Then $|L| \leq 2^{-r}|I|$ and $L \not\subseteq I$. Furthermore, since $d_{\text{tree}}(K, L) \leq r$, this implies $|K| \leq |I|$, so $K \subseteq I$. But $|L| \leq 2^{-r}|I|$, $L \not\subseteq I$, and $K \subseteq I$ implies that $d_{\text{tree}}(K, L) \geq r + 1$, a contradiction.

Thus, $T_W$ is $r$-lower triangular and symmetric arguments give the result for $T_W^*$. This implies $T_W$ is well-localized with radius $r$.

Remark 4.6. In Theorems 4.3 and 4.4, one must interpret the testing conditions correctly when the matrix weights’ entries are not in $L^2_{\text{loc}}(\mathbb{R})$. We already outlined the remedy for this problem in Remark 1.5. Similarly, one should notice that Lemma 4.5 only handles the case where the matrix weights have entries in $L^2_{\text{loc}}(\mathbb{R})$. Nevertheless, this result is sufficient to allow us to pass from Theorems 4.3 and 4.4 to Theorems 1.2 and 1.3. This is easy to see since, as detailed in Remark 1.5, we interpret all statements about weights with locally integrable (but not necessary square-integrable) entries in Theorems 1.2 and 1.3 using limits of weights with entries in $L^2_{\text{loc}}(\mathbb{R})$. 


5. Proofs of Theorems 4.3 and 4.4

5.1. Paraproducts. To prove Theorems 4.3 and 4.4, we require several results about related paraproducts. As before, let $T_W$ be a well-localized operator of radius $r$ acting formally from $L^2(W)$ to $L^2(V)$ with formal adjoint $T_V^*$. Using these operators, define the following paraproducts:

$$\Pi_W f \equiv \sum_{I \in \mathcal{D}} \sum_{1 \leq j \leq d} \left\langle T_W E_I^W f, h^V_j \right\rangle_{L^2(V)} h^V_j$$

$$\Pi^V g \equiv \sum_{I \in \mathcal{D}} \sum_{1 \leq j \leq d} \left\langle T_V^* E_I^V g, h^W_j \right\rangle_{L^2(W)} h^W_j$$

for $f \in L^2(W)$ and $g \in L^2(V)$. Recall that the $W$-weighted expectation of $f$ on $I$ is defined by $E_I^W f \equiv \langle W \rangle_I^{-1} \langle W f \rangle_I 1_I$. Now, observe that, as demonstrated by the following lemma, these paraproducts mimic the behavior of $T_W$ and $T_V^*$ respectively.

**Lemma 5.1.** Let $I, J \in \mathcal{D}$ and let $\Pi_W$ be the paraproduct defined above using the well-localized operator $T_W$ with radius $r$ acting (formally) from $L^2(W)$ to $L^2(V)$.

1. If $|J| \geq 2^{-r}|I|$, then

$$\left\langle \Pi_W h^W_i, h^V_j \right\rangle_{L^2(V)} = 0 \quad \forall \ 1 \leq i, j \leq d.$$

2. If $|J| < 2^{-r}|I|$, then

$$\left\langle \Pi_W h^W_i, h^V_j \right\rangle_{L^2(V)} = \left\langle T_W h^W_i, h^V_j \right\rangle_{L^2(V)} \quad \forall \ 1 \leq i, j \leq d.$$

If $J \varsubsetneq I$, then both sides of the equality are zero.

Furthermore, analogous statements hold for the paraproduct $\Pi^V$ and formal adjoint $T_V^*$.

**Proof.** First, observe that

$$\left\langle \Pi_W h^W_i, h^V_j \right\rangle_{L^2(V)} = \sum_{K \in \mathcal{D}} \sum_{1 \leq \ell \leq d} \left\langle T_W E_K^W h^W_i, h^V_{\ell} \right\rangle_{L^2(V)} \left\langle h^V_{\ell}, h^V_j \right\rangle_{L^2(V)}$$

$$= \left\langle T_W E_J^W h^W_i, h^V_j \right\rangle_{L^2(V)} ,$$

where $J^{(r)}$ is the $r^{th}$ ancestor of $J$. Now assume $|J| \geq 2^{-r}|I|$ or $J \varsubsetneq I$. Then, either $I \subseteq J^{(r)}$ or $I \cap J^{(r)} = \emptyset$. In either case,

$$E_J^W h^W_i = 0 ,$$

so the corresponding inner product is zero. Now assume $|J| < 2^{-r}|I|$, so that $|J| \leq 2^{-r}|I_+| = 2^{-r}|I_+|$. If $J \varsubsetneq I$, then $J \varsubsetneq I_-, I_+$ and since $T_W$ is well-localized with radius $r$,

$$\left\langle T_W h^W_i, h^V_j \right\rangle_{L^2(V)} = \left\langle T_W h^W_i (I_-) 1_{I_-}, h^V_j \right\rangle_{L^2(V)} + \left\langle T_W h^W_i (I_+) 1_{I_+}, h^V_j \right\rangle_{L^2(V)} = 0.$$
This gives equality if $J \not\subset I$. Now assume $|J| < 2^{-r}|I|$ and $J \subseteq I$. Then
\[
\left\langle \Pi^W h_I^W, h_J^V \right\rangle_{L^2(V)} = \left\langle T_W E_{J(r)}^W h_I^W, h_J^V \right\rangle_{L^2(V)}
= \left\langle T_W h_I^W (J(r)) 1_{J(r)}, h_J^V \right\rangle_{L^2(V)}
= \left\langle T_W h_I^W, h_J^V \right\rangle_{L^2(V)},
\]
since for all $I' \subset I \setminus J(r)$, the tree distance $d_{\text{tree}}(I', J) > r$ and so
\[
\left\langle T_W h_I^W (I') 1_{I'}, h_J^V \right\rangle_{L^2(V)} = 0.
\]
Analogous statements hold for $\Pi^V$, since it is defined using the operator $T_V^*$, which is also well-localized with radius $r$.

Now, we show that the testing condition (i) from Theorem 4.4 and hence, the stronger testing condition from Theorem 4.3, implies the boundedness of the paraproducts $\Pi^W$ and $\Pi^V$. We state the result for $\Pi^W$, but analogous arguments give the result for $\Pi^V$.

**Lemma 5.2.** Let $\Pi^W$ be the paraproduct defined above and assume that the well-localized operator $T_W$ satisfies:
\[
\| 1_I T_W 1_e \|_{L^2(V)} \leq C \langle W(I) e, e \rangle_{C^d}^{\frac{1}{4}} \quad \forall I \in D, \ e \in C^d.
\]
Then $\Pi^W$ is bounded from $L^2(W)$ to $L^2(V)$ and
\[
\| \Pi^W \|_{L^2(W) \to L^2(V)} \leq CB(W),
\]
where $B(W)$ is the constant obtained from applying the matrix Carleson Embedding Theorem.

**Proof.** Fix $f \in L^2(W)$, which implies $Wf \in L^2(W^{-1})$, and observe that
\[
\| \Pi^W f \|_{L^2(V)}^2 = \sum_{K \in D} \sum_{1 \leq \ell \leq d} \left| \left\langle T_W E_K^W f, h_L^{V, \ell} \right\rangle_{L^2(V)} \right|^2
= \sum_{K \in D} \sum_{1 \leq \ell \leq d} \left| \left\langle E_K^W f, T_{V, L}^{V, \ell} h_L^{V, \ell} \right\rangle_{L^2(W)} \right|^2
= \sum_{K \in D} \sum_{1 \leq \ell \leq d} \left| \left\langle (W)_{K}^{-1} \langle Wf \rangle_K, \alpha_{L, \ell} \right\rangle_{C^d} \right|^2,
\]
where we have set $\alpha_{L, \ell}$ to be the vector
\[
\alpha_{L, \ell} \equiv \int_{L(r)} W(x) T_{V, L}^{V, \ell}(x) dx.
\]
And so,
\[
\| \Pi^W f \|_{L^2(V)}^2 = \sum_{K \in D} \sum_{1 \leq \ell \leq d} \left\langle \alpha_{L, \ell} \right\rangle_{C^d} \langle W \rangle_{K}^{-1} \langle Wf \rangle_K, \langle W \rangle_{K}^{-1} \langle Wf \rangle_K \rangle_{C^d}
= \sum_{K \in D} \langle A_K \langle Wf \rangle_K, \langle Wf \rangle_K \rangle_{C^d},
\]
where we have set
\[ A_K \equiv \sum_{1 \leq \ell, \ell' \leq d} \langle W \rangle_K^{-1} \alpha_{L, \ell} (\alpha_{L, \ell'})^* \langle W \rangle_K^{-1}. \]

This is exactly the setup where we can apply Theorem 3.5. Specifically, we need to show that for all \( J \in \mathcal{D} \),
\[ \sum_{K \subseteq J} \langle W \rangle_K A_K \langle W \rangle_K \leq C^2 W(J). \]

To prove this matrix inequality, fix \( e \in \mathbb{C}^d \) and observe that
\[
\sum_{K \subseteq J} \langle W \rangle_K A_K \langle W \rangle_K e, e \rangle_{\mathbb{C}^d} = \sum_{K \subseteq J} \sum_{1 \leq \ell, \ell' \leq d} \langle \alpha_{L, \ell} (\alpha_{L, \ell'})^* e, e \rangle_{\mathbb{C}^d}
\]
\[ = \sum_{K \subseteq J} \sum_{1 \leq \ell, \ell' \leq d} |\langle \alpha_{L, \ell} e, e \rangle_{\mathbb{C}^d}|^2
\]
\[ = \sum_{K \subseteq J} \sum_{1 \leq \ell, \ell' \leq d} \left| \left\langle h_{L, \ell}^{V, \ell}, T_W e \mathbf{1}_K \right\rangle_{L^2(V)} \right|^2. \]

Notice that if \( I \subseteq J \setminus K \), then \( d_{\text{tree}}(L, I) > r \), and so
\[ \left\langle h_{L, \ell}^{V, \ell}, T_W e \mathbf{1}_{J \setminus K} \right\rangle_{L^2(V)} = 0. \]

This means that
\[ \sum_{K \subseteq J} \langle W \rangle_K A_K \langle W \rangle_K e, e \rangle_{\mathbb{C}^d} = \sum_{K \subseteq J} \sum_{1 \leq \ell, \ell' \leq d} \left| \left\langle h_{L, \ell}^{V, \ell}, T_W e \mathbf{1}_J \right\rangle_{L^2(V)} \right|^2
\]
\[ \leq \| \mathbf{1}_J T_W e \|_{L^2(V)}^2
\]
\[ \leq C^2 \langle W(J)e, e \rangle_{\mathbb{C}^d}. \]

Since \( e \in \mathbb{C}^d \) was arbitrary, the matrix inequality follows, so we can apply Theorem 3.5 to obtain:
\[ \| \Pi^W f \|^2_{L^2(V)} = \sum_{K \in \mathcal{D}} \langle A_K \langle W \rangle_K, \langle W \rangle_K \rangle_{\mathbb{C}^d} \leq C^2 B(W)^2 \| W f \|^2_{L^2(W^{-1})} = C^2 B(W)^2 \| f \|^2_{L^2(W)}, \]

as desired. \( \square \)

5.2. **Small Lemmas.** In this subsection, we verify several small lemmas that are trivial in the scalar situation. As before, \( T_W \) is a well-localized operator with radius \( r \) that satisfies the testing conditions from Theorem 4.3 or 4.4.

**Lemma 5.3.** Let \( T_W \) be a well-localized operator with radius \( r \) acting (formally) from \( L^2(W) \) to \( L^2(V) \) that satisfies the testing condition from Theorem 4.3 with constant \( A_1 \). Then
\[ \left| \left\langle T_W h_{L, i}^{W, i}, h_{L, j}^{V, j} \right\rangle_{L^2(V)} \right| \leq C(d) A_1 \quad \forall I, J \in \mathcal{D}, 1 \leq i, j \leq d. \]
Similarly, if $T_W$ satisfies the testing condition (ii) from Theorem 4.4 with constant $A_3$, then

$$\left| \left\langle T_W h_{I_i}^{W,i}, h_{J_j}^{V,j} \right\rangle \right|_{L^2(V)} \leq C(d) A_3 \quad \forall I, J \in \mathcal{D}, 1 \leq i, j \leq d.$$ 

Proof. For the first part of the lemma, we can use Cauchy-Schwarz to obtain:

$$\left| \left\langle T_W h_{I_i}^{W,i}, h_{J_j}^{V,j} \right\rangle \right|_{L^2(V)} \leq \left\| T_W h_{I_i}^{W,i} \right\|_{L^2(V)} \leq \left\| T_W h_{I_i}^{W,i} (I_-) 1_{I_-} \right\|_{L^2(V)} + \left\| T_W h_{I_i}^{W,i} (I_+) 1_{I_+} \right\|_{L^2(V)}.$$

It suffices to prove the desired bound for one term in the sum, since the arguments are symmetric. Using the testing condition and Lemma 2.4, we have:

$$\left\| T_W h_{I_i}^{W,i} (I_-) 1_{I_-} \right\|_{L^2(V)} \leq A_1 \left\langle W(I_-) h_{I_i}^{W,i} (I_-), h_{I_i}^{W,i} (I_-) \right\rangle_{\mathbb{C}^d}^{\frac{1}{2}}$$

$$= A_1 \left\| W(I_-)^{\frac{1}{2}} h_{I_i}^{W,i} (I_-) \right\|_{\mathbb{C}^d} \leq C(d) A_1,$$

which completes the first part of the lemma. For the second part, we can write:

$$\left| \left\langle T_W h_{I_i}^{W,i}, h_{J_j}^{V,j} \right\rangle \right|_{L^2(V)} \leq \left| \left\langle T_W h_{I_i}^{W,i} (I_-), h_{J_j}^{V,j} (J_-) 1_{J_-} \right\rangle \right|_{L^2(V)} + \left| \left\langle T_W h_{I_i}^{W,i} (I_+), h_{J_j}^{V,j} (J_+) 1_{J_+} \right\rangle \right|_{L^2(V)} + \left| \left\langle T_W h_{I_i}^{W,i} (I_-) 1_{I_-}, h_{J_j}^{V,j} (J_-) 1_{J_-} \right\rangle \right|_{L^2(V)} + \left| \left\langle T_W h_{I_i}^{W,i} (I_+) 1_{I_+}, h_{J_j}^{V,j} (J_+) 1_{J_+} \right\rangle \right|_{L^2(V)}.$$

By Lemma 2.4 and testing hypothesis (ii), we can conclude:

$$\left| \left\langle T_W h_{I_i}^{W,i} (I_-), h_{J_j}^{V,j} (J_-) 1_{J_-} \right\rangle \right|_{L^2(V)} \leq A_3 \left\langle W(I_-) h_{I_i}^{W,i} (I_-), h_{I_i}^{W,i} (I_-) \right\rangle_{\mathbb{C}^d}^{\frac{1}{2}} \left\langle V(I_-) h_{J_j}^{V,j} (J_-), h_{J_j}^{V,j} (J_-) \right\rangle_{\mathbb{C}^d}^{\frac{1}{2}}$$

$$= A_3 \left\| W(I_-)^{\frac{1}{2}} h_{I_i}^{W,i} (I_-) \right\|_{\mathbb{C}^d} \left\| V(I_-)^{\frac{1}{2}} h_{J_j}^{V,j} (I_-) \right\|_{\mathbb{C}^d} \leq C(d) A_3.$$

The other three terms in the sum can be handled similarly.

\[ \Box \]

Lemma 5.4. Let $f \in L^2(W)$. Then for all $I \in \mathcal{D}$,

$$|I|^{\frac{1}{2}} \left\| \left\langle W \right\rangle_I^{\frac{1}{2}} (W f) I \right\|_{\mathbb{C}^d} \leq C(d) \| f \|_{L^2(W)}.$$
Proof. Using Hölder’s inequality, we can compute

\[ |I| \left\| \langle W \rangle_I^{-\frac{1}{2}} \langle W f \rangle_I \right\|_{\mathcal{C}^d}^2 = |I|^{-1} \left\| \int_I \langle W \rangle_I^{-\frac{1}{2}} W(x) f(x) \, dx \right\|_{\mathcal{C}^d}^2 \]

\[ \leq |I|^{-1} \left( \int_I \left\| \langle W \rangle_I^{-\frac{1}{2}} W(x) f(x) \right\|_{\mathcal{C}^d} \, dx \right)^2 \]

\[ \leq |I|^{-1} \left( \int_I \left\| \langle W \rangle_I^{-\frac{1}{2}} W(x) \right\|^2 \, dx \right) \left( \int_I \left\| \langle W \rangle_I^{\frac{1}{2}} f(x) \right\|^2 \, dx \right) \]

\[ = \left( |I|^{-1} \int_I \left\| \langle W \rangle_I^{-\frac{1}{2}} W(x) \right\|^2 \, dx \right) \left\| f 1_I \right\|_{L^2(W)}^2 \]

\[ \leq C(d) \left\| f 1_I \right\|_{L^2(W)}^2 \left\| |I|^{-1} \int_I \langle W \rangle_I^{-\frac{1}{2}} W(x) \, dx \right\| \]

\[ = C(d) \left\| f 1_I \right\|_{L^2(W)}^2, \]

which gives the needed inequality.

5.3. Proofs of Theorems 4.3 and 4.4. We first prove Theorem 4.3:

Proof. We prove \( T_W \) extends to a bounded operator from \( L^2(W) \) to \( L^2(V) \) using duality. Specifically we show

\[ \left| \langle T_W f, g \rangle_{L^2(V)} \right| \leq C \| f \|_{L^2(W)} \| g \|_{L^2(V)}, \]

for a fixed constant \( C \) and all \( f \) and \( g \) in dense sets of \( L^2(W) \) and \( L^2(V) \) respectively. Without loss of generality, we can assume \( f \) and \( g \) are compactly supported and so, we can choose disjoint \( I_1, I_2 \in \mathcal{D} \) such that \( \text{supp}(f), \text{supp}(g) \subseteq I_1 \cup I_2 \) and \( |I_1| = |I_2| = 2^m \), for some \( m \in \mathbb{N} \). Using (6), we can write

\[ f = f_1 + f_2 = \sum_{I: |I| \leq 2^m, 1 \leq i \leq d} \left\langle f, h^W_{i, j} \right\rangle_{L^2(W)} h^W_{i,j} + \sum_{k=1}^2 E^W_k f \]

\[ g = g_1 + g_2 = \sum_{J: |J| \leq 2^m, 1 \leq j \leq d} \left\langle g, h^V_{j, j} \right\rangle_{L^2(V)} h^V_{j,j} + \sum_{\ell=1}^2 E^V_{\ell} g. \]

Using these decompositions, it suffices to show

\[ \left| \langle T_W f_i, g_j \rangle_{L^2(V)} \right| \leq C \| f \|_{L^2(W)} \| g \|_{L^2(V)} \quad \forall 1 \leq i, j \leq 2. \]
First, consider \( f_1 \) and \( g_1 \). Using Lemma 5.1, we can write

\[
\langle Tw f_1, g_1 \rangle_{L^2(V)} = \sum_{|I| \leq 2^m} \sum_{1 \leq i \leq d} \left\langle f, h^W_{I,i} \right\rangle_{L^2(W)} \left\langle g, h^V_{j} \right\rangle_{L^2(V)} \left\langle Tw h^W_{I,i}, h^V_{j} \right\rangle_{L^2(V)}
\]

\[
= \sum_{|I| \leq 2^m} \sum_{1 \leq i \leq d} \left\langle f, h^W_{I,i} \right\rangle_{L^2(W)} \left\langle g, h^V_{j} \right\rangle_{L^2(V)} \left\langle Tw h^W_{I,i}, h^V_{j} \right\rangle_{L^2(V)}
\]

\[
+ \sum_{|J| \leq 2^m} \sum_{1 \leq j \leq d} \left\langle f, h^W_{I,i} \right\rangle_{L^2(W)} \left\langle g, h^V_{j} \right\rangle_{L^2(V)} \left\langle Tw h^W_{I,i}, h^V_{j} \right\rangle_{L^2(V)}
\]

\[
+ \sum_{|I| \leq 2^m} \sum_{1 \leq i \leq d} \left\langle f, h^W_{I,i} \right\rangle_{L^2(W)} \left\langle g, h^V_{j} \right\rangle_{L^2(V)} \left\langle Tw h^W_{I,i}, h^V_{j} \right\rangle_{L^2(V)}
\]

\[
= \langle \Pi^W f_1, g_1 \rangle_{L^2(V)} + \langle f_1, \Pi^V g_1 \rangle_{L^2(W)}
\]

\[
+ \sum_{|I| \leq 2^m} \sum_{1 \leq i \leq d} \left\langle f, h^W_{I,i} \right\rangle_{L^2(W)} \left\langle g, h^V_{j} \right\rangle_{L^2(V)} \left\langle Tw h^W_{I,i}, h^V_{j} \right\rangle_{L^2(V)}
\]

Lemma 5.2 implies that

\[
\left| \langle \Pi^W f_1, g_1 \rangle_{L^2(V)} \right| + \left| \langle f_1, \Pi^V g_1 \rangle_{L^2(W)} \right| \leq (A_1 B(W) + A_2 B(V)) \| f \|_{L^2(W)} \| g \|_{L^2(V)}.
\]

So, we just need to bound the last sum. We first apply Cauchy-Schwarz and exploit symmetry in the sums to obtain:

\[
\sum_{|I| \leq 2^m} \sum_{1 \leq i \leq d} \left| \left\langle f, h^W_{I,i} \right\rangle_{L^2(W)} \left\langle g, h^V_{j} \right\rangle_{L^2(V)} \left\langle Tw h^W_{I,i}, h^V_{j} \right\rangle_{L^2(V)} \right|
\]

\[
\leq \left( \sum_{|I| \leq 2^m} \sum_{1 \leq i \leq d} \left| \left\langle f, h^W_{I,i} \right\rangle_{L^2(W)} \right|^2 \left| \left\langle Tw h^W_{I,i}, h^V_{j} \right\rangle_{L^2(V)} \right|^2 \right)^{1/2}
\]

\[
\times \left( \sum_{|J| \leq 2^m} \sum_{1 \leq j \leq d} \left| \left\langle g, h^V_{j} \right\rangle_{L^2(V)} \right|^2 \left| \left\langle Tw h^W_{I,i}, h^V_{j} \right\rangle_{L^2(V)} \right|^2 \right)^{1/2}
\]

\[
(14)
\]
Now, fix $I \in \mathcal{D}$. Since $T_W$ is well-localized, it is not hard to show that there are only finitely many $J$ satisfying $2^{-r}|I| \leq |J| \leq 2^r|I|$ such that

$$\left\langle T_W h_I^{W,i}, h_J^{V,j} \right\rangle_{L^2(V)} \neq 0.$$ 

Specifically, the number of such $J$ will always be bounded by a fixed constant times $2^{2r}$. Similarly, if we fix $J$, there are only finitely many $I$ satisfying $2^{-r}|J| \leq |I| \leq 2^r|J|$ such that

$$\left\langle T_W h_I^{W,i}, h_J^{V,j} \right\rangle_{L^2(W)} = \left\langle h_I^{W,i}, T^*_V h_J^{V,j} \right\rangle_{L^2(V)} \neq 0.$$ 

The number of such $I$ will also be bounded by a fixed constant times $2^{2r}$. Thus, we can use the testing conditions and Lemma 5.3 to estimate

$$\leq A_1 2^{2r} C(d) \|f\|_{L^2(W)} \|g\|_{L^2(V)}.$$ 

The other terms are much simpler. First observe that for each $k, \ell$:

$$\left| \left\langle T_W E_{k}^W f, E_{\ell}^V g \right\rangle_{L^2(V)} \right| \leq \left\| T_W E_{k}^W f \right\|_{L^2(V)} \left\langle \langle V \rangle_{I_k}^{-1} \langle V g \rangle_{I_{\ell}} 1_{I_{\ell}} \right\|_{L^2(V)}$$

$$\leq A_1 \left\| W(I_k)^{1/2} \langle V \rangle_{I_k}^{-1} \langle W f \rangle_{I_k} \right\|_{C^d} \left\| V(I_{\ell})^{1/2} \langle V g \rangle_{I_{\ell}} \right\|_{C^d}$$

$$= A_1 |I_k|^{1/2} \left\| (W(I_k))^{1/2} \langle W f \rangle_{I_k} \right\|_{C^d} |I_{\ell}|^{1/2} \left\| (V(I_{\ell}))^{1/2} \langle V g \rangle_{I_{\ell}} \right\|_{C^d}$$

$$\leq A_1 C(d) \|f\|_{L^2(W)} \|g\|_{L^2(V)},$$

by Lemma 5.4. This immediately implies the desired bound for $\langle T_W f_2, g_2 \rangle_{L^2(V)}$. The mixed terms are similarly straightforward. Specifically, observe that

$$\left| \left\langle T_W f_2, g_1 \right\rangle_{L^2(V)} \right| \leq \|g_1\|_{L^2(V)} \sum_{k=1}^2 \left\| T_W E_{k}^W f \right\|_{L^2(V)} \leq A_1 C(d) \|f\|_{L^2(W)} \|g\|_{L^2(V)},$$

using the arguments that appeared in the previous bound. Similarly,

$$\left| \left\langle T_W f_1, g_2 \right\rangle_{L^2(V)} \right| = \left| \left\langle f_1, T^*_V g_2 \right\rangle_{L^2(W)} \right| \leq \|f_1\|_{L^2(W)} \sum_{\ell=1}^2 \left\| T_{\ell}^* E_{\ell}^V g \right\|_{L^2(W)}$$

$$\leq A_2 C(d) \|f\|_{L^2(W)} \|g\|_{L^2(V)},$$

using Lemma 5.4 and the testing condition on $T^*_V$. This completes the proof.

We now turn to the proof of Theorem 4.4.

**Proof.** This theorem is established in basically the same manner as Theorem 4.3. We simply need to check that the weaker conditions (i) and (ii) in Theorem 4.4 allow us to deduce the same estimates. As before, we establish boundedness by duality as in (11), fix $f, g$ compactly supported in $I_1 \cup I_2$ with $|I_1| = |I_2| = 2^m$, and decompose

$$f = f_1 + f_2 \quad \text{and} \quad g = g_1 + g_2$$

as in (12) and (13). As before,

$$\langle T_W f_1, g_1 \rangle_{L^2(V)} = \left\langle \Pi^W f_1, g_1 \right\rangle_{L^2(V)} + \left\langle f_1, \Pi^V g_1 \right\rangle_{L^2(W)}$$

$$+ \sum_{I:|I| \leq 2^m} \sum_{1 \leq i \leq d} \left\langle f, h_I^{W,i} \right\rangle_{L^2(W)} \left\langle g, h_J^{V,j} \right\rangle_{L^2(V)} + \left\langle T_W h_I^{W,i}, h_J^{V,j} \right\rangle_{L^2(V)}.$$
The first two terms can be controlled by testing hypothesis (i) and Lemma 5.2. For the sum, we can use Lemma 5.3 and testing hypothesis (ii) to conclude
\[ \left| \left< T_W h_{i}^{W_i}, h_{j}^{V_j} \right>_{L^2(V)} \right| \leq C(d) A_3. \]
Since \( T_W \) is still well-localized with radius \( r \), we can use the strategy from the proof of Theorem 4.3 to immediately conclude:
\[ \left| \left< T_W f_1, g_1 \right>_{L^2(V)} \right| \leq 2^{2r} C(d) (A_1 B(W) + A_2 B(V) + A_3) \| f \|_{L^2(W)} \| g \|_{L^2(V)}. \]
The other terms are also straightforward. First observe that since \( |I_k| = |I_\ell| \), assumption (ii) paired with Lemma 5.4 implies that for each \( k, \ell \):
\[
\left| \left< T_W E_{I_k}^{W} f, E_{I_\ell}^{V} g \right>_{L^2(V)} \right| \leq A_3 \left\| W(I_k) \frac{1}{2} \left( W(I_\ell) \frac{1}{2} \langle W f \rangle_{I_k} \right) \left( W(I_\ell) \frac{1}{2} \langle W g \rangle_{I_\ell} \right) \right\|_{C^d} \\
= A_3 |I_k| \frac{1}{2} \left\| \langle W \rangle_{I_k} \frac{1}{2} \langle W f \rangle_{I_k} \right\|_{C^d} \left\| \langle W \rangle_{I_\ell} \frac{1}{2} \langle W g \rangle_{I_\ell} \right\|_{C^d} \\
\leq A_3 C(d) \| f \|_{L^2(W)} \| g \|_{L^2(V)}. 
\]
This immediately gives the desired bound for \( \left< T_W f_2, g_2 \right>_{L^2(V)} \). The mixed terms require a bit more work. We consider \( \left< T_W f_2, g_1 \right>_{L^2(V)} \). The other term can be handled analogously. Observe that
\[
\left| \left< T_W f_2, g_1 \right>_{L^2(V)} \right| \leq \sum_{k=1}^{2} \sum_{\substack{J: |J| \leq 2^m \\substack{1 \leq j \leq d}}} \left| \left< g, h_{j}^{V_j} \right>_{L^2(V)} \left< T_W E_{I_k}^{W} f, h_{j}^{V_j} \right>_{L^2(V)} \right| \\
= \sum_{k=1}^{2} \sum_{J: |J| \leq 2^m, J \subseteq I_k} \left| \left< g, h_{j}^{V_j} \right>_{L^2(V)} \left< T_W E_{I_k}^{W} f, h_{j}^{V_j} \right>_{L^2(V)} \right| \\
+ \sum_{k=1}^{2} \sum_{J: |J| \leq 2^m, J \not\subseteq I_k} \left| \left< g, h_{j}^{V_j} \right>_{L^2(V)} \left< T_W E_{I_k}^{W} f, h_{j}^{V_j} \right>_{L^2(V)} \right|. 
\]
We have to handle (16) and (17) separately. To handle (16), simply use Cauchy-Schwarz, Lemma 5.4, and assumption (i) to conclude
\[
\sum_{k=1}^{2} \sum_{\substack{J: |J| \leq 2^m \\substack{1 \leq j \leq d}}} \left| \left< g, h_{j}^{V_j} \right>_{L^2(V)} \left< T_W E_{I_k}^{W} f, h_{j}^{V_j} \right>_{L^2(V)} \right| \leq \sum_{k=1}^{2} \left\| I_k T_W E_{I_k}^{W} f \right\|_{L^2(V)} \left\| I_k g \right\|_{L^2(V)} \\
\leq A_1 \left\| g \right\|_{L^2(V)} \sum_{k=1}^{2} \left\| I_k \right\|_{L^2(W)} \left\| \langle W \rangle_{I_k} \frac{1}{2} \langle W f \rangle_{I_k} \right\|_{C^d} \\
\leq A_1 C(d) \| f \|_{L^2(W)} \| g \|_{L^2(V)}. 
\]
Now, consider (17). Since \( T_W \) is well-localized with radius \( r \), one can easily that show that for each \( I_k \), there are at most a fixed constant times \( 2^{2r} \) intervals \( J \) that satisfy
\[ \left< T_W E_{I_k}^{W} f, h_{j}^{V_j} \right>_{L^2(V)} \neq 0, \]
$|J| \leq 2^m$, and $J \not\subset I_k$. Indeed, for the inner product to be nonzero, $J$ must satisfy $J \subset I_k^{(r+1)}$ and $|J| > 2^{-r}|I_k|$. Now, using assumption $(ii)$, Lemma 5.4, and Lemma 2.4, we can establish the following sequence:

$$\left(17\right) = \sum_{k=1}^{2} \sum_{J : 2^{-r}|I_k| < |J| \leq |I_k|, J \subset I_k^{(r+1)}, J \not\subset I_k} \left| \langle g, h_j^V \rangle_{L^2(V)} \langle T_W E_{I_k}^W f, h_j^V \rangle_{L^2(V)} \right|$$

$$\leq \|g\|_{L^2(V)} \sum_{k=1}^{2} \sum_{J : 2^{-r}|I_k| < |J| \leq |I_k|, J \subset I_k^{(r+1)}, J \not\subset I_k} \left| \langle T_W E_{I_k}^W f, h_j^V \rangle_{L^2(V)} \right|$$

$$\leq A_3 \|g\|_{L^2(V)} \sum_{k=1}^{2} \sum_{J : 2^{-r}|I_k| < |J| \leq |I_k|, J \subset I_k^{(r+1)}, J \not\subset I_k} |I_k|^{1/2} \left| \langle W \rangle_{I_k}^{-1/2} \langle W f \rangle_{I_k} \right|_{C^d} \left| \langle V (I_+) \rangle_{I_k}^{1/2} h_j^V (J) \right|_{C^d}$$

$$\leq A_3 \|g\|_{L^2(V)} \sum_{k=1}^{2} \sum_{J : 2^{-r}|I_k| < |J| \leq |I_k|, J \subset I_k^{(r+1)}, J \not\subset I_k} |I_k|^{1/2} \left| \langle W \rangle_{I_k}^{-1/2} \langle W f \rangle_{I_k} \right|_{C^d} \left| \langle V (J_+) \rangle_{I_k}^{1/2} h_j^V (J) \right|_{C^d}$$

$$\leq 2^{2r} C(d) A_3 \|g\|_{L^2(V)} \|f\|_{L^2(W)},$$

which completes the proof. □

**Remark 5.5.** In this paper, we only considered band operators defined on $L^2(\mathbb{R}, \mathbb{C}^d)$. However, we anticipate that these T1 theorems will generalize without substantial difficulty to band operators on $L^2(\mathbb{R}^n, \mathbb{C}^d)$. One must define a slightly more complicated Haar system, but in general, the tools and proof strategy seem to work without issue.
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