Study of Diffusion Normalized Least Mean M-estimate Algorithms

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Abstract—This work proposes diffusion normalized least mean M-estimate algorithm based on the modified Huber function, which can equip distributed networks with robust learning capability in the presence of impulsive interference. In order to exploit the system’s underlying sparsity to further improve the learning performance, a sparse-aware variant is also developed by incorporating the $\ell_0$-norm of the estimates into the update process. We then analyze the transient, steady-state and stability behaviors of the algorithms in a unified framework. In particular, we present an analytical method that is simpler than conventional approaches to deal with the score function since it removes the requirements of integrals and Price’s theorem. Simulations in various impulsive noise scenarios show that the proposed algorithms are superior to some existing diffusion algorithms and the theoretical results are verifiable.

Index Terms—Diffusion networks, Impulsive interference, M-estimate, Performance analysis, Sparse regularization.

I. INTRODUCTION

In the last decade, distributed adaptive estimation over networks with numerous sensing agents/nodes has been studied frequently [1], [2], and applied to many areas such as spectrum estimation in cognitive radios [3], [4], [5] and beamforming [6]. In distributed adaptive estimation, the nodes cooperate with each other through the network links, to estimate parameters of interest using the streaming measurements. In the light of different cooperation strategies among interconnected nodes, distributed adaptive algorithms can be categorized into incremental [7], consensus [8], [9], [10], [11], and diffusion [12], [13], [14], [15], [16], [17] strategies.

The incremental strategy requires a cyclic communication path that covers all nodes, which is prone to link and node failures [7]. For both consensus and diffusion strategies, the nodes exchange local information with their neighboring nodes without the need for such a cycle. The consensus strategy ensures an agreement constraint among the nodes. To solve the global mean-square-error (MSE) minimization problem under this constraint, the alternating-direction method of multipliers (ADMM) [8] and the stochastic gradient descent (SGD) [9] Section 5.13.4] methods were used and different distributed consensus algorithms were presented. The work in [10] has proved that the stability of the SGD-based consensus algorithm is dependent of the network topology, while the diffusion algorithm does not have this limitation and has better performance. It has been shown in [15] that the diffusion algorithm slightly outperforms the ADMM-based consensus algorithm in terms of the steady-state performance for ideal communication links. So, the focus of this work is on diffusion-based distributed algorithms, among which the adapt-then-combine (ATC) implementation is of particular interest, since it usually outperform the combine-then-adapt (CTA) implementation [15]. Moreover, the CTA implementation can also be obtained straightforwardly from the ATC one. For brevity, we would leave out ‘ATC’ in the following texts.

In signal processing, the measurement noise is usually assumed to be Gaussian. In this case, the MSE criterion is widely used for developing many algorithms such as diffusion least square (DLS) [12] and normalized DLMS (NDLMS) [13]. In real-world applications, however, the measurements may be corrupted by the non-Gaussian noise including Gaussian and impulsive components. Its probability density function (pdf) has heavier tail than that of Gaussian noise. Such noise may be natural or from man-made, e.g., biological noise, pulse electromagnetic interference, and keyboard clicking or pen dropping in teleconferences [19], [20], [21], [22], [23], [24]. Although the realization of impulsive noise in the time domain is sparse, its amplitude is much higher than that of the nominal measurement; thus, it severely leads to the performance degeneration of the above algorithms, or even divergence. Especially due to the cooperation between nodes, impulsive noise that occurs even at one of the nodes could be propagated over the entire network.

In order to obtain good estimation performance over networks disturbed by impulsive noise, several robust diffusion algorithms have been proposed [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54]. Specifically, some examples based on different robust minimization criteria are the diffusion least mean $p$-th error (DLMp) where $1 \leq p < 2$ [49], diffusion sign error LMS (DSE-LMS) [50], diffusion error-nonlinearity LMS (DENLMS) [51], and diffusion least logarithmic absolute difference (D-LLAD) [52] algorithms. However, the robustness of DLMp against impulsive noise depends on the value of $p$ and the parameters in the $\alpha$-stable impulsive noise. The DSE-LMS is a particular case of the DLMp when $p = 1$, which is
usually a good performance benchmark as compared to other algorithms in impulsive noise. As shown in [51], the DNLMS converges more slowly than the DSE-LMS in Laplacian noise environments. The D-LLAD achieves faster convergence than the DSE-LMS [52]. Owing to the capability of Huber M-estimate function for removing outliers, references [53] and [54] proposed the diffusion Huber LMS and NLMS algorithms, respectively, and their estimation performance in impulsive noise relies on an empirical threshold. Also, the theoretical behaviors of both Huber-based algorithms have not been studied.

On the other hand, the parameter vector of interest may be sparse, which means only a fraction of elements in the parameter vector are relatively large and the remaining coefficients are small enough to be negligible or zero. Such sparsity is often encountered in many situations; to name just a few, spectrum estimation [3, 4], compressed sensing [55], and Digital TV transmission channels [56]. Thus, exploiting the underlying sparsity is able to improve the estimation performance. A cost-effective approach is to add a sparse regularization constraint (e.g., the\( l_1 \)-norm or the\( l_0 \)-norm) on the estimated vector into the cost function, known as the sparsity-aware technique [57], [58], [59], [60], [61]. Recently, the merit of sparsity has also been extended to distributed estimation, and several sparsity-aware diffusion algorithms were developed [62], [63], [64], [65], [66]. Nevertheless, it should be remarked that distributed estimation with sparsity in impulsive noise has not yet drawn much attention.

This work focuses on developing and analyzing the M-estimate based diffusion algorithms over distributed networks in the presence of impulsive noise. The main contributions are:

1) We propose a novel diffusion normalized least mean M-estimate (D-NLMM) algorithm by applying the modified Huber (MH) function with adaptive thresholds, which is robust against impulsive noise. For scenarios with sparse parameter vectors, we further develop the sparsity-aware D-NLMM (D-SNLMM) algorithm by incorporating the \( l_0 \)-norm based regularization technique.

2) Based on the contaminated-Gaussian (CG) noise model, the mean and mean-square behaviors of the proposed algorithms are analyzed and then supported by simulations. These analyses are addressed in a unified manner from the D-SNLMM update. In particular, we provide a closed-form expression for predicting the steady-state performance of the D-NLMM algorithm. The analysis results show that the stability conditions for both D-NLMM and D-SNLMM algorithms are independent of the powers of input regressors.

3) We also present proximal variants of the D-SNLMM algorithm which exploits the sparsity by virtue of the forward-backward splitting method.

4) Simulation results in various noise environments demonstrate the superiority of the proposed algorithms.

The paper is organized as follows. Section II introduces the signal model and the DNLMS algorithm. In Section III, we present the derivations of the D-NLMM and the D-SNLMM algorithms. In Section IV, we carry out a stochastic analysis of the proposed algorithms. Simulation results are given in Section V. Section VI draws some conclusions.

\textbf{Notations:} We use the subscript on the time index \( i \) to denote matrices and vectors, and the parentheses on the time index \( i \) to denote scalars. Operators \((\cdot)^T\), \(\|\cdot\|_2\), \(E\{\cdot\}\), \(\text{col}\{\cdot\}\), \(\text{diag}\{\cdot\}\), \(\text{Tr}\{\cdot\}\), \(\lambda_{\text{max}}\{\cdot\}\), and \(\otimes\) represent the transpose, \(l_2\)-norm of a vector, mathematical expectation, deployment of a vector by successively staking its arguments, diagonal or block diagonal operation over its arguments, trace of a matrix, maximum eigenvalue of a matrix, and Kronecker product of two matrices, respectively. \(\text{vec}(\cdot)\) stacks the columns of an \(L \times L\) matrix to form an \(L^2 \times 1\) vector, and \(\text{vec}^{-1}(\cdot)\) is its inverse operator. Also, \(I_L\) is an \(L \times L\) identity matrix, and \(1_N\) is a \(N \times 1\) vector with 1’s value.

\section{Signal Model and DNLMS Algorithm}

Consider a connected network consisting of \(N\) sensor nodes geographically distributed. Every node \(k\) communicates only with its single-hop neighbors and the communication between interconnected nodes is bidirectional. The set of single-hop neighbors to node \(k\) (including itself) is denoted by \(\mathcal{N}_k\). At every time instant \(i\), every node \(k\) acquires a desired output scalar \(d_k(i)\) and an \(L \times 1\) input regressor \(u_{k,i}\), in which \(d_k(i)\) and \(u_{k,i}\) are related by the linear model:

\[ d_k(i) = u_{k,i}^T w^o + v_k(i), \]

where \(w^o\) is an \(L \times 1\) sparse parameter vector and \(v_k(i)\) is the additive noise at node \(k\) independent of \(u_{m,j}\) for any \(m\) and \(j\). The model \([1]\) can be found in many applications \([11], [67]\).

The objective of the network nodes is to use the streaming data \(\{d_k(i), u_{k,i}\}_{k=1}^N\) to perform the estimation of the vector \(w^o\) in a recursive way. To this end, the commonly used global MSE minimization problem \([1]\) is stated as

\[ \min_{w_o} \sum_{k=1}^N E \{ (d_k(i) - u_{k,i}^T w_o)^2 \}. \]

Based on the diffusion strategy that every node fuses linearly its own information and the received information from its neighbors, \([2]\) is equivalent to minimizing the local MSE cost functions for all the nodes \(k = 1, \ldots, N\) \([1], [15]\):

\[ \min_{w_k} J_{k}^{\text{loc}}(i), \]

\[ J_{k}^{\text{loc}}(i) = \sum_{m \in \mathcal{N}_k} c_{m,k} E \{ (d_m(i) - u_{m,i}^T w_k)^2 \}, \]

where \(c_{m,k}\) represents a weight that node \(k\) assigns to the information coming from node \(m\), also called the combination coefficients. Note that \(\{c_{m,k}\}\) requires \(c_{m,k} \geq 0\), \(c_{m,k} = 0\) if \(m \notin \mathcal{N}_k\), and \(\sum_{m \in \mathcal{N}_k} c_{m,k} = 1\) \([69]\).

By using the SGD rule to solve \((3)\), the DLMS algorithm \([12]\) is obtained as

\[ \psi_{k,i+1} = \psi_{k,i} + \mu_k u_{k,i} e_k(i), \]

\[ w_{k,i+1} = \sum_{m \in \mathcal{N}_k} c_{m,k} \psi_{m,i+1}, \]

\[ \min_{w} \left\{ \lambda^{i+1} ||w||^2 + \frac{1}{N} \sum_{j=1}^N \sum_{k=1}^N (d_k(j) - u_{k,j}^T w)^2 \right\} \]

\([68]\).
where at node $k$, 
\[ e_k(i) = d_k(i) - u_{k,i}^T w_{k,i} \]  
(5)
denotes the output error and $\mu_k > 0$ is a constant step size. Specifically, in the adaptation step (4a), each node $k$ updates from the current estimate $w_{k,i}$ to the intermediate estimate $\psi_{k,i+1}$. Then, in the combination step (4b), each node $k$ fuses all the intermediate estimates of nodes $m \in N_k$ to yield an innovative estimate $w_{k,i+1}$. To make the step size range independent of the covariance matrices of input regressors, the DNLMS algorithm [18] modifies (4a) with the property 
\[ \phi \]
Moreover, the impulsive noise appears randomly with a small probability or appears with a short duration of times. In order to prevent from outliers, there is a positive number $|w_{k,i}|$ to yield (9). Specifically, in the adaptation step (4a), each node $k$, after $k$, where at node $k$, 
\[ u_{k,i}^T w_{k,i} \]  
(6)
where $0 < \mu_k < 2$ is to guarantee the algorithm convergence.

For the scenario that $\psi_{k,i}$ contains impulsive noise, the measurements of $d_k(i)$ have many outliers with large amplitudes. However, the MSE criterion cannot distinguish these outliers. Moreover, the impulsive noise appears randomly with a small probability or appears with a short duration of times. In this case, the DLMS and DNLMS algorithms will experience poor convergence or even divergence.

III. PROPOSED DIFFUSION M-ESTIMATE ALGORITHMS

In this section, we present the derivations of the D-NLMM and D-SNLMM algorithms.

A. Derivation of D-NLMM

To estimate $\omega$ in impulsive noise, we define the robust minimization problem for nodes $k = 1, ..., N$:
\[ \min_{w_k} J_{k}^{loc}(i), \]
\[ J_{k}^{loc}(i) = \sum_{m \in N_k} c_{m,k} g_m^{-1} E \{ \phi(d_m(i) - u_{m,i}^T w_k) \}, \]  
(7)
where $c_m > 0$ is a free-specified parameter, and $\phi(x)$ is an $M$-estimate function on variable $x$. To ensure that $J_{k}^{loc}(i)$ converges to the minimum, $\phi(x)$ is a continuous even function with the property $\phi(x_1) > \phi(x_2) > 0$ for $|x_1| > |x_2| > 0$ and is sub-differentiable at least [20]. Moreover, to prevent from outliers, there is a positive number $\xi^*$, and after $|x| > \xi^*$, the score function $\phi(x) \triangleq \frac{\partial \phi(x)}{\partial x}$ holds that $|\phi'(x)| \leq |\phi(\xi^*)|$, e.g., the below MH function for such $\phi(x)$. In other words, $\phi(x)$ is convex but may not be strictly convex at the points $|x| = \xi^*$. Additionally, $\phi'(x)$ may also equal a positive number multiplied by $\text{sgn}(x)$, where $\text{sgn}(\cdot)$ is the signum operator [4] as an example, the well-known sign strategy is $\phi(x) = |x|$ so that $\phi'(x) = \text{sgn}(x)$, which leads to the DSE-LMS algorithm [50].

At time instant $i$, the instantaneous sub-gradient of (7) with respect to $w_k$ is formulated as
\[ \nabla w J_{k}^{loc}(i) \triangleq - \sum_{m \in N_k} c_{m,k} g_m^{-1} u_{m,i} \phi'(d_m(i) - u_{m,i}^T w_{k,i}). \]  
(8)

Based on the SGD rule, the update equation for estimating $\omega$ is established:
\[ w_{k,i+1} = w_{k,i} + \mu_k \sum_{m \in N_k} c_{m,k} g_m^{-1} u_{m,i} \phi'(d_m(i) - u_{m,i}^T w_{k,i}). \]  
(9)
Following the diffusion cooperation [15, 3], at iteration $i$, the current estimate $w_{k,i}$ and new estimate $w_{k,i+1}$ are given by
\[ w_{k,i} = \sum_{m \in N_k} c_{m,k} \psi_{m,i} \]  
(10)
and [45], respectively. By plugging them into (9), we obtain
\[ \psi_{m,i+1} = \psi_{m,i} + \mu_k g_m^{-1} u_{m,i} \phi'(d_m(i) - u_{m,i}^T w_{k,i}). \]  
(11)
In (11), although $w_{k,i}$ is unavailable for node $m$, we can approximate it with $w_{m,i}$ because both are estimates of $\omega$. Also, we replace $\psi_{m,i}$ with $w_{m,i}$, since the latter contains more information through (10). Under these considerations, we arrive at the recursion for the D-NLMM algorithm [4]:
\[ \psi_{k,i+1} = w_{k,i} + \mu_k \frac{u_{k,i} \phi'(e_k(i))}{\|u_{k,i}\|^2}, \]  
(12a)
\[ w_{k,i+1} = \sum_{m \in N_k} c_{m,k} \psi_{m,i+1}. \]  
(12b)
where we also choose $\psi_{k,i} = \|u_{k,i}\|^2$ to yield (12b).

In adaptive filters, several M-estimate strategies have been studied for $\phi(x)$ such as the Huber function [53, 54], the MH function [71], and the Hampel’s three-part re-scaling function [72], to develop robust adaptive algorithms in impulsive noise. Due to the MH’s simplicity, we focus on it for presenting the D-NLMM algorithm in distributed estimation [4].

Interestingly, the algorithm using the MH function has also a comparable performance to that using either the Huber function or the Hampel’s three-part re-scaling function, as can be seen in Fig. 3.

The MH is a piecewise continuous function:
\[ \phi(e_k) = \begin{cases} e_k^2/2, & \text{if } |e_k| < \xi_k, \\ \xi_k^2/2, & \text{if } |e_k| \geq \xi_k, \end{cases} \]  
(13)
and its score function is
\[ \phi'(e_k) = \begin{cases} e_k, & \text{if } |e_k| < \xi_k, \\ 0, & \text{if } |e_k| \geq \xi_k, \end{cases} \]  
(14)
where $\xi_k$ is a threshold. By combining (12) and (14), it turns out that, at time instant $i$, when the magnitude of $e_k(i)$ is smaller than $\xi_k$, $\phi'(e_k(i))$ is equal to $e_k(i)$, and the proposed algorithm performs the DNLMS update. When $|e_k(i)| \geq \xi_k$ (which means the appearance of impulsive noise), $\phi'(e_k(i))$ will become zero, thereby stopping the adaptation of the algorithm. Towards this goal, the threshold $\xi_k$ is adaptively adjusted by
\[ \xi_k = \kappa \sigma_{e_k(i)}, \]  
(15)
\[ \text{sign}(x) = 1, 0, \text{and } -1 \text{ for } x > 0, x = 0, \text{and } x < 0. \]  
\[ \text{sign}(x) = 1, 0, \text{and } -1 \text{ for } x > 0, x = 0, \text{and } x < 0. \]
where $\sigma^2_{e,k}(i)$ is the variance of $e_k(i)$ excluding impulsive noise. Typically, $\kappa = 2.576$ for the suppression of impulsive noise, which means, under the assumption that $e_k(i)$ is Gaussian distributed except when being polluted accordingly by impulsive noise, the confidence level of preventing $e_k(i)$ from contributing to the update is 99% when $|e_k| \geq \xi_k$ [22]. And, $\sigma^2_{e,k}(i)$ can be estimated by the following recursion:

$$\sigma^2_{e,k}(i) = \zeta \sigma^2_{e,k}(i-1) + (1-\zeta)\text{med}(A_{e,k,i}),$$

where $0 < \zeta \leq 1$ is a forgetting factor except $\zeta = 0$ at the starting time $i = 0$, $\text{med}(\cdot)$ is the median operator of the error data sliding window $A_{e,k,i} = [e^2_k(i), e^2_k(i-1), \ldots, e^2_k(i-N_w+1)]$ which helps to avoid the effect of impulsive noise on $\sigma^2_{e,k}(i)$. The window length $N_w$ is usually chosen between 5 and 9; also, it should be increased appropriately when the occurrence probability of impulsive noise is high.

### B. Derivation of D-SNLMM

In order to enforce the sparsity of $\mathbf{w}^o$, we propose to incorporate a real-valued sparse regularization $F(\mathbf{w}_k)$ on $\mathbf{w}_k$ into $J^{\text{loc}}(\mathbf{w}_k)$ and obtain:

$$J^{\text{loc}}(\mathbf{w}_k) = \sum_{m \in N_k} c_{m,k} f_{\text{norm}}^{-1} \{ \varphi(d_m(i) - \mathbf{u}^T_{m,i}\mathbf{w}_k) \} + \beta F(\mathbf{w}_k),$$

for nodes $k = 1, 2, \ldots, N$, where the regularization parameter $\beta > 0$ controls the intensity given to $F(\mathbf{w}_k)$.

By following the derivation procedure in the above subsection to (17), the D-SNLMM algorithm for estimating $\mathbf{w}^o$ is formulated as

$$\psi_{k,\tau+1} = \mathbf{w}_{k,\tau} + \mu_k \frac{\mathbf{u}_{k,\tau} \varphi'(e_k(i))}{\|\mathbf{u}_{k,\tau}\|_2} - \mu_k \beta f(\mathbf{w}_{k,\tau}),$$

(18a)

$$\mathbf{w}_{k,\tau+1} = \sum_{m \in N_k} c_{m,k} \psi_{m,\tau+1},$$

(18b)

where $f(\mathbf{w}_{k,\tau})$ is referred to as the zero attractor:

$$f(\mathbf{w}_{k,\tau}) = \frac{\partial F(\mathbf{w}_{k,\tau})}{\partial \mathbf{w}_{k,\tau}} = [f([\mathbf{w}_{k,\tau}]_1), \ldots, f([\mathbf{w}_{k,\tau}]_L)]^T$$

(19)

Note that, (20) is strictly equivalent to the $l_0$-norm when $\nu \to \infty$. Thus, the zero attractor $f(\mathbf{w})$ is given by

$$f([\mathbf{w}]_l) = \nu \text{sign}([\mathbf{w}]_l) \exp^{-\nu}||[\mathbf{w}]_l||, \ l = 1, \ldots, L.$$  

(21)

Furthermore, by taking advantage of the first-order Taylor expansions of the exponential function, the low complexity version of (21) is obtained:

$$f([\mathbf{w}]_l) = \begin{cases} 
-\nu^2 [\mathbf{w}]_l - \nu, & \text{if } -\frac{1}{\nu} \leq [\mathbf{w}]_l < 0 \\
-\nu^2 [\mathbf{w}]_l + \nu, & \text{if } 0 < [\mathbf{w}]_l \leq \frac{1}{\nu} \\
0, & \text{otherwise.}
\end{cases}$$  

(22)

**Remark 1:** The zero attractor $f(\mathbf{w}_{k,\tau})$ imposes an attraction towards zero on small elements of the vector $\mathbf{w}_{k,\tau}$ and those elements are in the majority, thereby bringing about a performance improvement of the D-SNLMM algorithm when estimating a sparse vector $\mathbf{w}^o$. As can be seen from (22) that the elements attracted are within a range of $[a^2/v, 1/v]$, and the attraction intensity will be greater if the element is closer to zero. It is worth noting that as $\nu$ increases, the attraction intensity will become strong but the attraction range will become narrow. Also, the proper choice of $\beta$ will be explained later on in the analysis.

**Remark 2:** The proposed D-NLMM and many existing robust diffusion algorithms can be described in a unified recursion (12), but the main difference is the score function $\varphi'(e_k(i))$ to resist impulsive noise, which depends on the specific robust strategy shown in Table I. In the DEN-LMS algorithm, the score function is a weighted combination of preselected sign-preserving basis functions and the weights need to be optimized. In comparison, the D-NLMM algorithm is simpler in implementation, since $\varphi'(e_k(i))$ only originates from the MH function. Moreover, the D-NLMM algorithm considers the normalization of input regressors in the adaptation. Based on the Huber function, when $|e_k(i)| \geq b$, the DNHuber algorithm performs the adaptation $\psi_{k,\tau+1} = \mathbf{w}_{k,\tau} + \mu_k \frac{\mathbf{u}_{k,\tau} \varphi'(e_k(i))}{\|\mathbf{u}_{k,\tau}\|_2}$ rather than the freezing like the D-NLMM algorithm. However, the D-NLMM algorithm uses an adaptive threshold instead of the constant one in the DNHuber algorithm so that the former outperforms the latter in practice. Importantly, the D-SNLMM algorithm further improves the D-NLMM performance in sparse parameter vector scenarios. Particularly, if we set $g_k = 1$, the non-normalized versions, i.e., the D-LMM and D-SLMM algorithms are obtained; they can be considered as extensions of the MH function to the context of detection problems [28].
### TABLE I

Existing Functions for Approximating the $l_0$-norm.

| regularization function: $F(w)$ | zero attractor: $f(|w_j|)$, $j = 1, \ldots, L$ |
|----------------------------------|----------------------------------|
| (a) $\|w\|_1$ | $\text{sign}(|w_j|)$ |
| (b) $\|w\|_1^p$ with a variable $p$ in $0 < p < 1$ | $\text{sign}(|w_j|)\exp(-|w_j|)$ |
| (c) $\sum_{i=1}^{L} (1 - \exp(-v|w_i|))$ | $v\text{sign}(|w_i|)\exp(-v|w_i|)$ |
| (d) $\sum_{i=1}^{L} |w_i|^p$ with a variable $p$ in $0 < p < 1$ | $\text{sign}(|w_i|)\exp(-v|w_i|)$ |
| (e) $\sum_{i=1}^{L} (1 - \exp(-\frac{1}{2}v^2|w_i|^2))$ | $v^2\text{sign}(|w_i|)\exp(-\frac{1}{2}v^2|w_i|^2)$ |
| (f) $\frac{L}{v} \left( 1 + \frac{|w_i|^p}{|w_j|^p} \right)^\frac{1}{p - 1}$, if $|w_i| > 1$ $\frac{|w_i|(1 - (v - 1)|w_j|)}{v - 1}$, if $|w_i| \leq 1$ 0, elsewhere, |

### TABLE II

$\varphi(e_k(i))$ and $\varphi'(e_k(i))$ for Existing Robust Diffusion Algorithms.

| robust cost function: $\varphi(e_k(i))$ | the score function: $\varphi'(e_k(i))$ |
|------------------------------------------|------------------------------------------|
| DSNLMM [59] | $|e_k(i)|^{p-1}\text{sign}(e_k(i))$ |
| DLMP [49] | $|e_k(i)|^{p-1}\text{sign}(e_k(i))$ |
| D-LAD [53] | $|e_k(i)| - \ln(1 + |e_k(i)|)$, where $k > 0$ |
| DEN-LMS [51] | $\sum_{i} |e_k(i)|$, where $k > 0$ |
| DNHuber [51] | $|e_k(i)|^2$, if $|e_k(i)| < b$ |

Note: 1) $h_k(i)$ is a linear combination of preselected sign-preserving basis functions \{$\phi_{b,k}(e_k(i))$, $b = 1, \ldots, B_k$\}, where $B_k \geq 1$. 2) the parameters’ notations (i.e., $p$, $\alpha$, $b$, and $B_k$) are the same as the ones in references.

### TABLE III

Proposed D-SNLMM Algorithm and Its Special Versions.

| Initialization: $w_{k,0} = 0$, $\sigma_{k,0} = 0$ | Parameters: \(0 < \zeta < 1, 0 < \mu_k < 2, \text{and } \beta > 0\) |
|-----------------------------------------------|--------------------------------------------------|
| D-SNLMM algorithm: $g_{k,i} = \|u_{k,i}||e_k(i)|| > 0$ | D-LMM algorithm: $g_{k,i} = 1, \beta > 0$ |
| D-SLMM algorithm: $g_{k,i} = \|u_{k,i}||e_k(i)|| > 0$ | D-LMM algorithm: $g_{k,i} = 1, \beta > 0$ |
| for iteration $i > 0$ do | for iteration $i > 0$ do |
| for each node $k$ do | for each node $k$ do |
| $e_{k,i} = d_{k,i} - u_{k,i}$, $A_{k,i} = [e_{k,i}, e_{k,i} - 1, \ldots, e_{k,i} - N_{w_k} + 1]$ | $e_{k,i} = d_{k,i} - u_{k,i}$, $A_{k,i} = [e_{k,i}, e_{k,i} - 1, \ldots, e_{k,i} - N_{w_k} + 1]$ |
| $\sigma_{k,i} = \zeta \sigma_{k,i}^{i+1} (i) + (1 - \zeta) \text{med} (A_{k,i})$ | $\sigma_{k,i} = \zeta \sigma_{k,i}^{i+1} (i) + (1 - \zeta) \text{med} (A_{k,i})$ |
| $\xi_k = 2.576 \sigma_{k,i} (i)$ | $\xi_k = 2.576 \sigma_{k,i} (i)$ |
| $f(|w_{k,i}|) = \left\{ \begin{array}{ll} -v^2 |w_{k,i}| - v, & 0 \leq |w_{k,i}| < v \\ -v^2 |w_{k,i}| - v + \frac{1}{v}, & v \leq |w_{k,i}| \leq v + \frac{1}{v} \end{array} \right.$ | $f(|w_{k,i}|) = \left\{ \begin{array}{ll} -v^2 |w_{k,i}| - v, & 0 \leq |w_{k,i}| < v \\ -v^2 |w_{k,i}| - v + \frac{1}{v}, & v \leq |w_{k,i}| \leq v + \frac{1}{v} \end{array} \right.$ |
| $\psi_{k,i+1} = w_{k,i} + \mu_k g_{i}^{-1} w_{k,i} e_{k,i} - \mu_k \beta g_{i} f_{k,i}$ | $\psi_{k,i+1} = w_{k,i} - \mu_k \beta \frac{f_{k,i}}{|u_{k,i}|^2}$ |
| end for each node $k$ do | end for each node $k$ do |
| $w_{k,i+1} = \sum_{m \in N_k} c_{m,k} \psi_{m,i+1}$ | $w_{k,i+1} = \sum_{m \in N_k} c_{m,k} \psi_{m,i+1}$ |

IV. PERFORMANCE ANALYSIS

In this section, the mean and mean-square behaviors of the proposed algorithms in impulsive noise are studied. As stated in Remark 2, we will start the analysis from the D-SNLMM algorithm.

From (19), we know that the difficulty of the performance analysis is how to evaluate the score term $\varphi'(e_k(i))$. Although the literature has studied the performance of the MH-based algorithms for adaptive filters [71], [119], the extension to diffusion algorithms is not straightforward owing to the cooperation of interconnected nodes. More importantly, the existing analysis method for dealing with $\varphi'(e_k(i))$ is complicated, which involves the conditioned expectation, Price’s theorem, and three complicated integrals. Therefore, we propose to move the score function $\varphi'(\cdot)$ out of (18), which makes the analysis simpler. Specifically, since the D-SNLMM algorithm performs the update of the estimate at node $k$ when $|e_k(i)| < \xi_k$, we denote the probability of the update as

$$P_{u,k}(i) = P\{ |e_k(i)| < \xi_k \},$$

whose calculation will be discussed in Section IV.C. Accordingly, we can equivalently express (18) in the mean as

$$\psi_{k,i+1} = w_{k,i} + \mu_k g_{i}^{-1} w_{k,i} e_{k,i} - \mu_k \beta f_{k,i},$$

(24a)

$$w_{k,i+1} = \sum_{m \in N_k} c_{m,k} \psi_{m,i+1}.$$  

(24b)

It is noticed that (24) does not contain the score function $\varphi'(\cdot)$, which is absorbed into the evaluation of $P_{u,k}(i)$. This approach can also be extended to simplify the analyses of adaptive filtering algorithms in [71], [119].
Subtracting $w^o$ from both (24a) and (24b), respectively, which yields

\[
\tilde{\psi}_{k,i} = \tilde{w}_{k,i} - \mu_k P_{u,k}(i) \frac{u_{k,i} e_k(i)}{\|u_{k,i}\|^2} + \mu_k \beta f(w_{k,i}), \quad (25a)
\]

\[
\tilde{w}_{k,i+1} = \sum_{m \in N_k} c_{m,k} \tilde{\psi}_{m,i+1}, \quad (25b)
\]

where $\tilde{w}_{k,i} \triangleq w^o - w_{k,i}$ and $\tilde{\psi}_{k,i+1} \triangleq \tilde{w}^o - \tilde{\psi}_{k,i+1}$ indicate the error vector and the intermediate error vector, respectively. The relation (25) will be the starting point of performance analysis. For convenience of analysis, we make the following assumptions.

**Assumption 1:** The regressors $u_{k,i}$ are zero-mean with correlation matrices $R_k = E\{u_{k,i} u_{k,i}^T\}$ in spatial independence.

**Assumption 2:** The regressors $u_{k,i}$ are independent of the estimation deviation $\tilde{w}_{m,i}$ for $j \leq i$ and all $k,m$. This is the well-known independence assumption in the performance analyses of adaptive filtering algorithms [67, 120, 121] and distributed algorithms [1].

**Assumption 3:** At every node $k$, the additive noise $v_k(i)$ includes the background noise $\theta_k(i)$ and the impulsive noise $\eta_k(i)$, namely, $v_k(i) = \theta_k(i) + \eta_k(i)$. The background noise $\theta_k(i)$ is drawn from a zero-mean white Gaussian process with variance $\sigma_\theta^2_k$.

**Assumption 4:** The impulsive noise $\eta_k(i)$ is modelled by the Bernoulli-Gaussian (BG) process: $\eta_k(i) = b_k(i) \cdot g_k(i)$, where $b_k(i)$ is a Bernoulli process whose pdf is expressed as $P[b_k(i) = 1] = p_k$ and $P[b_k(i) = 0] = 1 - p_k$, and

$g_k(i)$ is drawn from a zero-mean white Gaussian process with variance $\sigma_g^2_k$, with $\sigma_g^2_k \gg \sigma_\theta^2_k$. Note that, $p_k$ also stands for the probability of occurrence of impulsive noise.

According to assumptions 3 and 4, it is seen that the additive noise $v_k(i)$ is a CG process with zero-mean and variance $\sigma_v^2 = p_k \sigma_\theta^2 + (1 - p_k) \sigma_g^2$, where $\sigma_o^2 = \sigma_\theta^2 + \sigma_g^2$. The CG model is used frequently for analyzing the algorithms in impulsive noise [71, 50, 119].

Under the condition of $P_{u,k}(i)$, we can use assumption 3 to get the relation $e_k(i) = u_{k,i}^T \tilde{w}_{k,i} + \theta_k(i)$. Hence, (25a) becomes

\[
\tilde{\psi}_{k,i+1} = (I_L - \mu_k P_{u,k}(i) A_{k,i}) \tilde{w}_{k,i} - \mu_k P_{u,k}(i) b_{k,i} + \mu_k \beta f(w_{k,i}), \quad (26)
\]

where $A_{k,i} = \frac{u_{k,i} u_{k,i}^T}{\|u_{k,i}\|^2}$ and $b_{k,i} = \frac{u_{k,i} \theta_k(i)}{\|u_{k,i}\|^2}$.

Some global quantities on all the nodes are defined as follows:

\[
\tilde{w}_i \triangleq \text{col}\{\tilde{w}_{1,i}, ..., \tilde{w}_{N,i}\}
\]

\[
\tilde{\psi}_i \triangleq \text{col}\{\tilde{\psi}_{1,i}, ..., \tilde{\psi}_{N,i}\},
\]

\[
b_i \triangleq \text{col}\{b_{1,i}, ..., b_{N,i}\},
\]

\[
A_i \triangleq \text{diag}\{A_{1,i}, ..., A_{N,i}\},
\]

\[
\mathcal{M} \triangleq \text{diag}\{\mu_1 I_L, ..., \mu_N I_L\}
\]

\[
\mathcal{P}_i \triangleq \text{diag}\{P_{u,1}(i) I_L, ..., P_{u,N}(i) I_L\}
\]

\[
f(w_i) \triangleq \text{col}\{f(w_{1,i}), ..., f(w_{N,i})\}
\]

\[
C \triangleq C \otimes I_L,
\]

where the matrix $C$ collects all the combination coefficients $\{\epsilon_{m,k}\}$, thus each column of $C$ sums up to one (i.e., $C^T 1_N = 1_N$). Using the above quantities, we rearrange (26) and (25b) in a compact form:

\[
\tilde{w}_{i+1} = C^T \tilde{\psi}_{i+1}
\]

\[
= C^T (I_{NL} - \mathcal{M} P_i A_i) \tilde{w}_i - C^T \mathcal{M} P_i b_i + \beta C^T \mathcal{M} f(w_i)
\]

which shows how the network error vector evolves over time.

**A. Mean Behavior**

By taking the expectation of both sides of (28) under assumptions 2 and 3, we obtain that the mean of $w_{k,i}$ evolves according to the recursion on the time instant $i$:

\[
E\{\tilde{w}_{i+1}\} = \Gamma_i E\{\tilde{w}_i\} + \beta C^T \mathcal{M} E\{f(w_i)\}, \quad (29)
\]

where

\[
\Gamma_i = C^T (I_{NL} - \mathcal{M} P_i E\{A_i\}). \quad (30)
\]

From (29), we have the following statement.

**Theorem 1:** The D-SNLMM algorithm converges in the mean if the step sizes are chosen to satisfy

\[
0 < \mu_k < \frac{2}{P_{u,k}(i) \lambda_{\max}(E\{A_{k,i}\})}, \quad k = 1, ..., N. \quad (31)
\]

Furthermore, in the steady-state, the estimates across all nodes for this algorithm are biased with respect to $w^o$, i.e.,

\[
E\{w_{k,\infty}\} = w^o - \beta (I_{NL} - \Gamma_i)^{-1} C^T \mathcal{M} E\{f(w_{i-1})\} \quad \text{bias}
\]

for $k = 1, ..., N$.

**Proof:** Repeatedly iterating (29), we have

\[
E\{\tilde{w}_{i+1}\} = \prod_{j=0}^i \Gamma_j E\{\tilde{w}_0\} + y_i, \quad (33)
\]

where

\[
y_i = \beta \sum_{j=0}^i \left( \prod_{s=i-j+1}^i \Gamma_s \right) C^T \mathcal{M} E\{f(w_{i-j})\}. \quad (34)
\]

Let us introduce the block-maximum-norm of the $MN \times MN$ matrix $\Psi$ with block entries of size $M \times M$ each, which is defined as [1]:

\[
\|\Psi\|_{b,\infty} \triangleq \max \left\{ \|\Psi x\|_{b,\infty} : x \neq 0 \right\}, \quad (35)
\]

\[
\|x\|_{b,\infty} \triangleq \max_{1 \leq k \leq N_N} \|x_k\|_2,
\]

where $x_{b,\infty} \triangleq \sum_{i=1}^N \|x_i\|_2$.
where $x = \text{col}\{x_1, \ldots, x_N\}$ is an $MN \times 1$ vector with block entries $\{x_k\}$ of size $M \times 1$ each. Thus, $\|C^T\|_{b,\infty} = 1$ holds. Then, enforcing the block-maximum-norm on both sides of (33) yields:

$$\|E\{\tilde{w}_{i+1}\}\|_{b,\infty} \leq \left\| \prod_{j=0}^{i} \Gamma_j E\{\tilde{w}_0\} \right\|_{b,\infty} + \|y_i\|_{b,\infty}, \quad (36)$$

where

$$\left\| \prod_{j=0}^{i} \Gamma_j E\{\tilde{w}_0\} \right\|_{b,\infty} \leq \prod_{j=0}^{i} \left\| \Gamma_j \right\|_{b,\infty} \cdot \|E\{\tilde{w}_0\}\|_{b,\infty}, \quad (37)$$

and

$$\|y_i\|_{b,\infty} \leq \beta \sum_{j=0}^{i} \left( \max_{-1 \leq s \leq i} \left\| \Gamma_s \right\|_{b,\infty} \right)^j \times \|C^T\|_{b,\infty} \cdot \left\| \left| ME\{f(w_{i-j})\}\right|_{b,\infty} \right\|_{b,\infty}, \quad (38)$$

If we can ensure $\|\Gamma_i\|_{b,\infty} < 1$ for any $i$, it is easy to check that as $i \to \infty$, (37) will approach to zero so that (36) will converge to

$$\|E\{\tilde{w}_{i+1}\}\|_{b,\infty} \leq \|y_i\|_{b,\infty}, \quad (38)$$

where the term $\max ||E\{f(w_{i})\}|_{b,\infty}$ is finite, because $f(w_{k,i})$ given by (22) has bounded elements. Equation (39) means that the algorithm is mean stable under the condition that

$$\|\Gamma_i\|_{b,\infty} = \|C^T(I_{NL} - MP_i E(A_i))\|_{b,\infty} \leq \|C^T\|_{b,\infty} \cdot \|I_{NL} - MP_i E(A_i)\|_{b,\infty} = \|I_{NL} - MP_i E(A_i)\|_{b,\infty} \leq \frac{\mu_k}{1 - \max_{1 \leq k \leq N} \|\Gamma_i\|_{b,\infty}}, \quad (40)$$

where the rationale behind (a) is the block diagonal property of the matrices $I_{NL}, M, P_i,$ and $E\{A_i\}$. It follows from (40) that the step sizes are bounded by (31). Also, when $i \to \infty$, we have $E\{\tilde{w}_{i+1}\} = E\{\tilde{w}_i\}$ so that (36) can be deduced from (29), thereby completing the proof of Theorem 1.

From Theorem 1 and by setting $\beta = 0$, we know that the mean convergence condition of the D-NLMM algorithm is also shown in (31), and this algorithm is unbiased for estimating $w^o$ across all nodes in impulsive noise, i.e., $E\{\tilde{w}_{k,\infty}\} = w^o$, $k = 1, \ldots, N$.

**B. Mean-Square Behavior**

Let us define the covariance matrix of the network error vector $\tilde{w}_i$ as

$$W_i \triangleq E\{\tilde{w}_i \tilde{w}_i^T\} \quad (41)$$

where the $k$-th $L \times L$ diagonal block, denoted as $W_{k,i} \triangleq E\{\tilde{w}_{k,i} \tilde{w}_{k,i}^T\}$, is the covariance matrix of the error vector $\tilde{w}_{k,i}$ at node $k$. Then, post-multiplying (29) by its transpose, and then taking the expectation on both sides of the equation under assumptions 2 and 3, we find the recursive relation:

$$W_{i+1} = C^T(I_{NL} - MP_i E(A_i)) W_i (I_{NL} - MP_i E(A_i))^T C + C^T M P_i B P_i E \{\tilde{w}_i \tilde{w}_i^T\} (C^T M)^T + \beta C^T M E\{f(w_i)^T (w_i)\} (C^T M)^T,$$

where

$$\mathcal{B} \triangleq E\{b_i b_i^T\} = \text{diag} \left\{ \sigma_{0,1}^2 \left( \frac{u_{1,i}}{||u_{1,i}||_2} \right)^2, \ldots, \sigma_{0,N}^2 \left( \frac{u_{N,i}}{||u_{N,i}||_2} \right)^2 \right\}.$$

Enforcing the vectorization operation on both sides of (42) and applying the Kronecker product property vec$(X\Sigma Y) = (Y^T \otimes X) \text{vec}(\Sigma)$ for any matrices $(X, \Sigma, Y)$ of compatible dimensions [67], we can establish that

$$\text{vec}(W_{i+1}) = \mathcal{F}_i \text{vec}(W_i) + (C^T \otimes C^T) (I_{NL} - MP_i E\{A_i\}) - (MP_i E\{A_i\}) \otimes I_{NL} + \beta \text{vec}(E\{\tilde{w}_i \tilde{w}_i^T\}) + \beta C^T M \text{vec}(E\{f(w_i)^T (w_i)\}) + \beta^2 C^T MC \text{vec}(E\{f(w_i)^T (w_i)\} (C^T M)^T) \quad (43)$$

where

$$\mathcal{F}_i := (C^T \otimes C^T) [I_{NL} - I_{NL} \otimes (MP_i E\{A_i\}) - (MP_i E\{A_i\}) \otimes I_{NL} + (MP_i \otimes MP_i) E\{A_i \otimes A_i\}] \quad (45)$$

The mean-square-deviation (MSD) at node $k$ is defined as $\text{MSD}_k(i) \triangleq \text{Tr}\{W_{k,i}\}$, and the network MSD over all the nodes is defined as $\text{MSD}_n(i) = \frac{1}{N} \sum_{k=1}^{N} \text{MSD}_k(i) = \text{Tr}\{W_i\}/N$. Naturally, by giving $\beta = 0$ and performing the inverse operator vec$^{-1}()$, (44) will model the MSD evolution behavior of the D-NLMM algorithm. However, when this model is used for describing the MSD evolution behavior of the D-SNLMM algorithm, it still requires knowing the moments $E\{f^T(w_i)\}$, $E\{w_i f^T(w_i)\}$, and $E\{f^T(w_i) f^T(w_i)\}$ beforehand. To calculate them, we resort to two assumptions:

**Assumption 5:** All entries in $\tilde{w}_i$ are Gaussian. The assumption is widely used and can be verified by the central limit theorem [57, 58, 59]. Interested readers for its studies can refer to [123, 124]. For the $l$-th entry of $\tilde{w}_i$, its mean and variance can be calculated by $x_l(i) = E\{\tilde{w}_{l,i}\}$, and $\sigma^2_{z,l}(i) = [W_{l,i}]_{l,l} - [E\{\tilde{w}_{l,i}\}]^2$, respectively, where $[\cdot]_{l,l}$ denotes the $l$-th diagonal entry of a matrix. Thus, the mean and variance of $[\tilde{w}_{l,i}]$ are obtained as $x_l(i) = [1_N \otimes w^o]_{l,l} - x_l(i)$ and $\sigma^2_{l,l}(i)$, respectively.

**Assumption 6:** When $k \neq m$ and $l \neq j$, the approximations $E\{[w_{k,i}]_l [f(w_{m,j})]_j\} \approx E\{w_{k,i}\} E\{f(w_{m,j})\}_j$ and $E\{[f(w_{k,i})]_l [f(w_{m,j})]_j\} \approx E\{f(w_{k,i})\} E\{f(w_{m,j})\}_j$ are valid.

6The excess MSE (EMSE) evolution behavior of the algorithm can also be described by (24) according to the definition EMSE$_n(i) \triangleq \text{Tr}\{W_{k,i} R_k\}$. 


made \([125, 59, 57]\). Although this is a strong “separable assumption”, it leads to the simplification of the analysis.

Note that, the above two assumptions hold in simulations, see Figs. 12 and 13.

Rewriting \(E\{\hat{w}_i f^T(w_i)\}\) yields

\[E\{\hat{w}_i f^T(w_i)\} = \omega^o E\{f^T(w_i)\} - E\{w_i f^T(w_i)\}. \quad (46)\]

Based on (22) and assumption 5, \(E\{f(w_i)|l\}\) can be computed as

\[
E\{f(x)\} = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} f(x) \exp\left(-\frac{(x - \bar{x})^2}{2\sigma_x^2}\right) dx = \frac{\nu^2 \sigma_x^2}{\sqrt{2\pi}} (a_1 - a_2) - \frac{\nu^2 \bar{x}}{2} (b_1 + b_2)
\]

\[
+ \frac{\nu}{2} (b_1 - b_2 + 2b_3),
\]

where \(\text{erf}(x) \triangleq 2 \int_{-\infty}^{x} \exp(-t^2) dt\), \(a_1 = \exp\left(-\frac{(1/\nu - \bar{x})^2}{2\sigma_x^2}\right)\), \(a_2 = \exp\left(-\frac{(1/\nu + \bar{x})^2}{2\sigma_x^2}\right)\), \(b_1 = \text{erf}(\frac{1/\nu - \bar{x}}{\sqrt{2\sigma_x}})\), \(b_2 = \text{erf}(\frac{1/\nu + \bar{x}}{\sqrt{2\sigma_x}})\), and \(b_3 = \text{erf}(\frac{\bar{x}}{\sqrt{2\sigma_x}})\).

We define the block matrices \(\Xi_k \triangleq E\{w_k f^T(w_k)\}\) and \(\Pi_i \triangleq E\{f(w_i)|f^T(w_i)\}\) which have \(N^2\) blocks with every block size of \(L \times L\), where \(\Xi_{k,m,i} \triangleq E\{w_k f^T(w_m,i)\}\) and \(\Pi_{k,m,i} \triangleq E\{f(w_k)|f^T(w_m,i)\}\) are the \((k,m)\)-th block of the \(\Xi_k\) and \(\Pi_i\), respectively.

For the \(k\)-th diagonal block matrices \(\Xi_{k,k,i}\) and \(\Pi_{k,k,i}\) with \(k = 1, ..., N\), their off-diagonal entries can be directly given using assumption 6; based on assumption 5, their the \(l\)-th diagonal entries are computed respectively as follows:

\[
E\{[w_{k,i}][f^T(w_{k,i})]\} \triangleq E\{xf(x)\} = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} f(x) \exp\left(-\frac{(x - \bar{x})^2}{2\sigma_x^2}\right) dx = \frac{\nu^2 \sigma_x^2}{\sqrt{2\pi}} (a_1 - a_2) - \frac{\nu^2 \bar{x}}{2} (b_1 + b_2)
\]

\[
+ \frac{\nu}{2} (b_1 - b_2 + 2b_3),
\]

\[
E\{f(w_{k,i})\} = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} f^2(x) \exp\left(-\frac{(x - \bar{x})^2}{2\sigma_x^2}\right) dx = \frac{\nu^2 \sigma_x^2}{\sqrt{2\pi}} (a_1 + a_2 - 2a_3) - \frac{\nu^2 \bar{x}^2}{2} (b_1 + b_2) - \nu^2 \bar{x}(b_1 - b_2 + 2b_3),
\]

where \(a_3 = \exp\left(-\frac{\bar{x}^2}{2\sigma_x^2}\right)\).

For the \((k,m)\)-th off-diagonal block matrices \(\Xi_{k,m,i}\) and \(\Pi_{k,m,i}\) where \(k \neq m \in \{1, ..., N\}\), we can also use assumption 6 to compute their off-diagonal entries. Nevertheless, the “separable assumption” cannot be used for computing the \(l\)-th diagonal entries \(E\{[w_{k,i}][f^T(w_{m,i})]\}\) and \(E\{[f(w_{k,i})][f^T(w_{m,i})]\}\) since \(w_{k,i}\) and \(w_{m,i}\) are of significant similarity for estimating \(w^o\). In view of the difficulty for obtaining the joint probability density function of \([w_{k,i}]\) and \([w_{m,i}]\), we propose the following symmetric approximations:

\[
E\{[w_{k,i}][f^T(w_{m,i})]\} = E\{[w_{m,i}][f^T(w_{k,i})]\} \approx E\{[w_{k,i}][f^T(w_{k,i})]\} + E\{[w_{m,i}][f^T(w_{m,i})]\},
\]

\[
E\{[f(w_{k,i})][f^T(w_{m,i})]\} \approx E\{[f(w_{k,i})][f^T(w_{k,i})]\} + E\{[f(w_{m,i})][f^T(w_{m,i})]\},
\]

\[
(46)\]

\[
(47)\]

\[
(49)\]

\[
(50)\]

\[
(51)\]

\[
(53)\]

\[
(54)\]

\[
(55)\]

\[
(56)\]

\[
\]
where

\[
\beta_u = -\text{vec}(I_{LN})^T(I_{LN^2} - F)^{-1}x \\
\left[(C^TM \otimes \Gamma_i)\text{vec}(E\{\tilde{w}_i f^T(w_i)\}) + (\Gamma_i \otimes C^TM)\text{vec}(E\{\tilde{w}_i f^T(w_i)\}^T)\right],
\]

(57)

\[
\beta_b = \text{vec}(I_{LN})^T(I_{LN^2} - F)^{-1}x \\
(C^TM \otimes C^TM)\text{vec}(E\{f(w_i) f^T(w_i)\}).
\]

(58)

\[\text{Remark 3: As shown in (56), the accuracy range of } \beta \text{ depends on the true } w^0, \text{ but it can reveal the feasibility of the D-SNLMM algorithm outperforming the D-NLMM algorithm in sparse estimation problem. This phenomenon can also be observed in Figs. 6 and 7. For the D-SNLMM algorithm, } \binom{54}{54} \text{ is an implicit equation on the steady-state MSD, but we may obtain its numerical solution by running the transient model } \binom{49}{49} \text{ to the steady-state.}\]

\[\text{C. Calculation of } P_{u,k}(i)\]

To implement the above mean and mean-square models, now we will show how to compute the probability \( P_{u,k}(i) \) defined in (23). Recalling the CG noise model and applying the law of total probability, we obtain

\[
P_{u,k}(i) = p_k P\{e_{s,k}(i) < \xi_k\} + (1 - p_k) P\{e_{\theta,k}(i) < \xi_k\},
\]

(59)

where \( e_{s,k}(i) \equiv u_{k,i}^T \tilde{w}_{k,i} + \theta_k(i) + g_k(i) \) and \( e_{\theta,k}(i) \equiv u_{k,i}^T \tilde{w}_{k,i} + \theta_k(i) \). It has been pointed out in [12] that based on the central limit theorem, \( u_{k,i}^T \tilde{w}_{k,i} \) can be assumed to be Gaussian for \( L \gg 1 \). Therefore, we are able to assume that \( e_{s,k}(i) \) and \( e_{\theta,k}(i) \) are zero mean Gaussian variables so that

\[
P\{e(i) < \xi\} \equiv \frac{1}{\sqrt{2\pi} \sigma e} \int_{-\infty}^{\xi} \exp\left(-\frac{e^2}{2\sigma^2 e}\right) de
\]

\[
= \text{erf}(\xi/\sqrt{2}\sigma e),
\]

(60)

where the subscripts \( k, s \) and \( \theta \) are omitted for concision. Subsequently, (59) can be rewritten as

\[
P_{u,k}(i) = p_k \text{erf} \left( \frac{\xi_k}{\sqrt{2\sigma_{e,s,k}(i)}} \right) + (1 - p_k) \text{erf} \left( \frac{\xi_k}{\sqrt{2\sigma_{e,\theta,k}(i)}} \right),
\]

(61)

where \( \sigma_{e,s,k}^2(i) \equiv \text{Tr}\{W_{k,i}R_k\} + \sigma^2_{e,k} \) and \( \sigma_{e,\theta,k}^2(i) \equiv \text{Tr}\{W_{k,i}R_k\} + \sigma^2_{\theta,k} \). Similar to (15), \( \xi_k \) is computed according to

\[
\xi_k = \kappa \sigma_{e,s,k}(i).
\]

(62)

In the steady-state, we introduce two approximations:

\[
\sigma_{e,s,k}(\infty) \approx \sigma^2_{e,k} \text{ and } \sigma_{e,\theta,k}(\infty) \approx \sigma^2_{\theta,k}.
\]

due to \( \text{Tr}\{W_{k,i}R_k\} < \sigma^2_{e,k} \ll \sigma^2_{e,k} \) for small step sizes. Consequently, \( \Sigma_k \infty \) in (53) can be computed:

\[
P_{u,k}(\infty) = p_k \text{erf} \left( \frac{\kappa \sigma_{e,k}}{\sqrt{2\sigma_{e,k}}} \right) + (1 - p_k) \text{erf} \left( \frac{\kappa}{\sqrt{2}} \right).
\]

(63)

\[\text{D. Some Insights}\]

\[\text{Remark 4: From (51) and (52), the step size range that guarantees the convergence of both the D-NLMM and D-SNLMM algorithms in both mean and mean-square senses is formulated as}\]

\[
0 < \mu_k < \frac{1}{P_{u,k}(i)} \min \left\{ \frac{2}{\lambda_{\max}(E\{A_k\})}, \frac{2}{\lambda_{\max}(R_k(\text{DNLMS}))} \right\}
\]

(64)

\[
\Rightarrow 0 < \mu_k < \frac{2}{P_{u,k}(i)}
\]

for all nodes \( k = 1, ..., N \). It should be stressed that the term \( b \) stems from \( \lambda_{\max}(E\{A_{k,i}\}) \leq 1 \). This is also because \( E\{A_{k,i}\} \) can also be considered as the normalized covariance matrix of input regressors, i.e., \( E\{A_{k,i}\} \approx R_k(\text{DNLMS}) \) when \( L \gg 1 \) [53]. If \( P_{u,k}(i) = 1 \), then (64) reduces to the convergence condition of the DLMS algorithm in the absence of impulsive noise. Because of \( 0 < P_{u,k}(i) \leq 1 \), one can infer from (64) that the step size range of the D-NLMM algorithm is slightly wider than that of the DLMS algorithm. More importantly, the convergence condition (64) for the proposed algorithms is derived in the presence of impulsive noise.

\[\text{Remark 5: The previous analysis focuses on both the D-NLMM and D-SNLMM algorithms. However, following the previous analysis procedures, their non-normalized forms: the D-LMM and D-SSLMM algorithms can also be easily analyzed, except the following differences:}\]

1) In terms of the mean convergence condition, transient MSD, and steady-state MSD, we only need to replace \( E\{A_i\} \) with \( R, E\{A_i \otimes A_i\} \) with \( R \otimes R \), and \( B \) with

\[
B = \text{diag}\{\sigma^2_{e,k}R_1, ..., \sigma^2_{e,N}R_N\},
\]

(65)

where \( R = \text{diag}\{R_1, ..., R_N\} \).

2) On the mean-square convergence condition, \( \Sigma_k \) becomes

\[
\Sigma_k \approx (I_L - \mu_k P_{u,k}(i)R_k)^T(I_L - \mu_k P_{u,k}(i)R_k).
\]

(66)

From (66), then we derive the bounds on step sizes that

\[
0 < \mu_k < \frac{1}{P_{u,k}(i)} \frac{2}{\lambda_{\max}(R_k(\text{DLMS}))}, k = 1, ..., N.
\]

(67)

Assuming \( P_{u,k}(i) = 1 \), (67) degrades into the step size range of the DLMS algorithm [10]. From (64) and (67), the convergence conditions of the D-NLMM and D-SNLMM algorithms do not depend on the maximum eigenvalues of the covariance matrices of input regressors, as opposed to the D-LMM and D-SSLMM algorithms.

\[\text{V. SIMULATION RESULTS}\]

Computer simulations are conducted over a distributed network with \( N = 20 \) nodes (shown in Fig. [11]a), unless otherwise specified. The network MSD is used as a performance metric. All diffusion algorithms only consider the cooperation of
the estimates in the combination step, and the combination coefficients \( \{c_{m,k}\} \) are computed by the Metropolis rule [69]:

\[
c_{m,k} = \begin{cases} 
1 / \max(n_m, n_k), & \text{if } m \in \mathcal{N}_k, \ m \neq k \\
1 - \sum_{m \neq k} c_{m,k}, & \text{if } m = k \\
0, & \text{otherwise,}
\end{cases}
\]

where \( n_k \) is the number of neighbors of node \( k \) including itself. All results are the average of 100 independent trials.

As shown in (67), the stability of the D-LMM algorithm is controlled by \( \lambda_{max}(\mathbf{R}_k) = \sigma_{z,k}^2 \).

\[\lambda_{max}(\mathbf{R}_k) = \sigma_{z,k}^2 \]

Thus, in this algorithm, we set the same step size in the range \( 0 < \mu_k < 2 \) at all the nodes owing to \( \mu_k = \lambda_{max}(\mathbf{R}_k) \). The results for estimating non-sparse \( \mathbf{w}^o \) are shown in Fig. 2. It is seen that unlike D-LMM, the stability range of D-NLMM on step sizes is not affected by the maximum eigenvalue of \( \mathbf{R}_k \) because it normalizes the adaptation process by the energies of input regressors. The D-NLMM with 'non-coop' is that every node runs independently an NLMM algorithm [119], i.e., no cooperation among nodes. As we know, NLMM diverges when \( \mu = 2 \). However, D-NLMM still converges when \( \mu = 2 \), since \( \mu = 2 \) is derived based on the relaxed inequalities (A.7) and (A.12). Thanks to the cooperation of interconnected nodes, D-NLMM has much better performance than NLMM.

**A. Performance of Algorithms**

Let \( Q \) denote the number of non-zero entries in the vector \( \mathbf{w}^o \) of length \( L = 32 \), and a smaller \( Q \) means sparser \( \mathbf{w}^o \). We set randomly values of non-zero entries from a Gaussian distribution, and then \( \mathbf{w}^o \) is normalized by \( \| \mathbf{w}^o \|_2 = 1 \). The input regressor of node \( k \) is given by \( \mathbf{u}_k = [u_k(0), u_k(1), \ldots, u_k(i-L+1)]^T \) [7], [59], with \( u_k(i) \) being drawn from a first-order autoregressive model, \( u_k(i) = \tau_k u_k(i-1) + \varepsilon_k(i) \), where \( \varepsilon_k(i) \) is a zero mean white Gaussian process with variance \( \sigma_{z,k}^2 \) and \( \tau_k \) controls the correlation of \( u_k(i) \) over time. Fig. 1(b) and (c) illustrate values of \( \sigma_{z,k}^2 \) and \( \tau_k \) at the nodes. The additive noise \( \nu_k(i) \) interfering the desired output \( d_k(i) \) at node \( k \) is drawn from a CG process described in assumptions 3 and 4, where Fig. 1(d) illustrates values of \( \sigma_{\nu,k}^2 \) for the Gaussian background noise.

To begin with, we compare the stability of the D-LMM algorithm with that of the D-NLMM algorithm on step sizes. For conveniently calculating the maximum eigenvalue of the correlation matrix \( \mathbf{R}_k \), we set Gaussian white input regressors for all \( k \), i.e., \( u_k(i) = \varepsilon_k(i) \), which makes \( \lambda_{max}(\mathbf{R}_k) = \sigma_{z,k}^2 \).

As shown in (67), the stability of the D-LMM algorithm is controlled by \( \lambda_{max}(\mathbf{R}_k) \) for all \( k \). Thus, in this algorithm, we set the same step size at all the nodes to \( \mu = t \cdot (2 / \max(\sigma_{z,k}^2 | k = 1, \ldots, N) \) ), where \( 0 < t < 1 \). For the D-NLMM algorithm, we choose the same step size in the range \( 0 < \mu_k < 2 \) at all the nodes owing to \( \mu_k = \lambda_{max}(\mathbf{R}_k) \leq 1 \) in (64). The results for estimating non-sparse \( \mathbf{w}^o \) are shown in Fig. 2. It is seen that unlike D-LMM, the stability range of D-NLMM on step sizes is not affected by the maximum eigenvalue of \( \mathbf{R}_k \) because it normalizes the adaptation process by the energies of input regressors. The D-NLMM with 'non-coop' is that every node runs independently an NLMM algorithm [119], i.e., no cooperation among nodes. As we know, NLMM diverges when \( \mu = 2 \). However, D-NLMM still converges when \( \mu = 2 \), since \( \mu = 2 \) is derived based on the relaxed inequalities (A.7) and (A.12). Thanks to the cooperation of interconnected nodes, D-NLMM has much better performance than NLMM.

\[\lambda_{max}(\mathbf{R}_k) = \sigma_{z,k}^2 \]

\[\mu_k = \lambda_{max}(\mathbf{R}_k) \leq 1 \]

Then, Fig. 4 examines the effect of the regularization parameter \( \beta \) on the steady-state performance of the D-SNLMM algorithm, where the D-NLMM algorithm (\( \beta = 0 \)) is used as a comparison benchmark when estimating the sparse vector \( \mathbf{w}^o \). The results are obtained by averaging over the last 100 MSD values after convergence to steady-state. As one can see,
there is a region describing the choices of $\beta$ so that D-SNLMM has better steady-state performance than D-NLMM in the estimation of parameters with sparsity, as also indicated in [56]. According to Fig. 4(a), a proper range of $\upsilon$ can be determined as $\upsilon \in [15, 40]$. Moreover, in Fig. 4(b), as the sparsity degrees of $\omega^o$ decrease (i.e., values of $Q$ increase), the superiority region on $\beta$ for the D-SNLMM algorithm becomes narrow as compared to the D-NLMM algorithm, until it becomes null when $\omega^o$ is non-sparse. That is to say, the D-SNLMM and D-NLMM algorithms are suitable for sparse and non-sparse scenarios, respectively.

From Figs. 5 and 6, one can also see that when identifying a sparse parameter vector $\omega^o$, the D-SNLMM reduces about 7 dB in the steady-state MSD as compared to that of D-NLMM, as the former takes advantage of the sparsity of $\omega^o$.

In Fig. 6 we investigate the performance of the proposed algorithms in the presence of impulse noise, where the occurrence probability of impulse noise is set to $p_k = 0.01$ and $0.05$. The DLMP, D-LLAD, and DNHuber [54] algorithms are also required to a performance comparison. In this situation, the DNLSMs and $l_0$-DNLSMs algorithms have degraded performance, or even are of divergence in Fig. 6(b), yet other algorithms are insensitive to impulse noise. For the DLMP algorithm, its performance is better when setting a smaller parameter $p$ in a strong impulse noise case, as it becomes the DLMS and DSE-LMS algorithms when $p = 2$ and $p = 1$, respectively. The D-LLAD performs better than DSE-LMS but worse than DNHuber. Since D-NLMM employs the adaptive rule (15) to select the threshold, it exhibits faster convergence rate than DNHuber using the fixed threshold. From Figs. 5 and 6 one can also see that when identifying a sparse parameter vector $\omega^o$, the D-SNLMM reduces about 7 dB in the steady-state MSD as compared to that of D-NLMM, as the former takes advantage of the sparsity of $\omega^o$. 

Fig. 5 compares the performance of the DNLMS, $l_0$-DNLMS, DSE-LMS, and proposed D-NLMM and D-SNLMM algorithms in the case of no impulsive noise, where the $l_0$-DNLMS is the normalized version of the ATC $l_0$-LMS in [42]. For a fair comparison, the parameters of algorithms are chosen based on the rule that the algorithms hold the same steady-state or convergence performance. As we can see from Fig. 5 the DSE-LMS algorithm is the slowest in the convergence. The D-NLMM and D-SNLMM algorithms keep almost the same performance as the DNLMS and $l_0$-DNLMS algorithms, respectively.

Fig. 6. Network MSD curves of diffusion algorithms under CG noise with $\sigma^2_{g,k} = 10^2\sigma^2_{\theta,k}$ and (a) $p_k = 0.01$, (b) $p_k = 0.05$. [Sparse $\omega^o$ with $Q = 2$]. Parameters of some algorithms are chosen as follows: $\mu_k = 0.0058$ (DSE-LMS); $\mu_k = 0.01$ (b) 0.005, $p = 1.4$ (DLMP); $\mu_k = 0.042$, $\alpha = 0.5$ (b) $\mu_k = 0.18$, $\alpha = 1.4$ (D-LLAD); $\mu_k = 0.7$, (a) $b = 0.4$ (b) $b = 0.3$ (DNHuber); (a) $N_w = 9$ (b) $N_w = 16$ (D-NLMM, D-SNLMM), and the remaining parameters are chosen as in Fig. 5.

In Fig. 7 we reset the CG noise parameters when the impulsive probability is increased to $p_k = 0.1$ and the impulsive component $g_k(i)$ follows from the zero mean Laplacian distribution with variance $\sigma^2_{g,k} = 10^2\sigma^2_{\theta,k}$. As in [51], the DEN-LMS algorithm uses the basis functions $\varphi_k^{anh} = \{\phi_k,1 = x, \phi_k,2 = \tanh(x)\}$ and other parameters are chosen as $\mu_k = 0.018$, $\nu_k = 0.99$, and $\epsilon = 10^{-6}$. The parameters of other algorithms are re-tuned to reach the same convergence

\[ \text{Steady-state network MSD curves of diffusion algorithms under Gaussian noise.} \]

\[ \text{Fig. 5. Network MSD curves of diffusion algorithms under Gaussian noise.} \]

\[ \text{Fig. 6. Network MSD curves of diffusion algorithms under CG noise with } \sigma^2_{g,k} = 10^2\sigma^2_{\theta,k} \text{ and (a) } p_k = 0.01, \text{ (b) } p_k = 0.05. \text{ [Sparse } \omega^o \text{ with } Q = 2\}. \text{ Parameters of some algorithms are chosen as follows: } \mu_k = 0.0058 \text{ (DSE-LMS); } \mu_k = 0.01 \text{ (b) 0.005, } p = 1.4 \text{ (DLMP); } \mu_k = 0.042, \alpha = 0.5 \text{ (b) } \mu_k = 0.18, \alpha = 1.4 \text{ (D-LLAD); } \mu_k = 0.7, \text{ (a) } b = 0.4 \text{ (b) } b = 0.3 \text{ (DNHuber); (a) } N_w = 9 \text{ (b) } N_w = 16 \text{ (D-NLMM, D-SNLMM), and the remaining parameters are chosen as in Fig. 5.} \]

In Fig. 7 we reset the CG noise parameters when the impulsive probability is increased to $p_k = 0.1$ and the impulsive component $g_k(i)$ follows from the zero mean Laplacian distribution with variance $\sigma^2_{g,k} = 10^2\sigma^2_{\theta,k}$. As in [51], the DEN-LMS algorithm uses the basis functions $\varphi_k^{anh} = \{\phi_k,1 = x, \phi_k,2 = \tanh(x)\}$ and other parameters are chosen as $\mu_k = 0.018$, $\nu_k = 0.99$, and $\epsilon = 10^{-6}$. The parameters of other algorithms are re-tuned to reach the same convergence

\[ \text{Fig. 7. Network MSD curves of diffusion algorithms under CG noise with } \sigma^2_{g,k} = 10^2\sigma^2_{\theta,k} \text{ and (a) } p_k = 0.01, \text{ (b) } p_k = 0.05. \text{ [Sparse } \omega^o \text{ with } Q = 2\}. \text{ Parameters of some algorithms are chosen as follows: } \mu_k = 0.0058 \text{ (DSE-LMS); } \mu_k = 0.01 \text{ (b) 0.005, } p = 1.4 \text{ (DLMP); } \mu_k = 0.042, \alpha = 0.5 \text{ (b) } \mu_k = 0.18, \alpha = 1.4 \text{ (D-LLAD); } \mu_k = 0.7, \text{ (a) } b = 0.4 \text{ (b) } b = 0.3 \text{ (DNHuber); (a) } N_w = 9 \text{ (b) } N_w = 16 \text{ (D-NLMM, D-SNLMM), and the remaining parameters are chosen as in Fig. 5.} \]
or steady-state MSD as the DEN-LMS algorithm, except that both DNLMS and $l_0$-DNLMS algorithms have poor convergence in impulsive noise. As can be seen from Fig. 7 the convergence performance of DEN-LMS is superior to that of the DLMP and D-LLAD algorithms and approaches that of the DSE-LMS algorithm in the presence of Laplacian noise. However, the D-NLMM algorithm has faster convergence than than the D-LMS algorithm. Due to the use of sparsity, the D-SNLMM further improves the D-NLMM’s steady-state performance.

Fig. 7. Network MSD curves of diffusion algorithms under CG noise. [Sparse $\mathbf{w}_o$ with $Q = 2$]. Parameters of algorithms are chosen as follows: $\mu_k = 0.008$ (DNLMS); $\mu_k = 0.007$, $\rho = 1.5$ (DLMP); $\mu_k = 0.024$, $\alpha = 0.8$ (D-LLAD); $N_\sigma = 16$, $\zeta = 0.95$ (M-estimator). The other parameters of the DNLMS, $l_0$-DNLMS, D-NLMM, and D-SNLMM algorithms. The sparsity parameters of $l_0$-DNLMS and D-SNLMM are the same $\rho = \beta = 1 \times 10^{-4}$.

On the other hand, the $\alpha$-stable process is used to model the additive noise $v_n$ with impulsive behavior, also called the $\alpha$-stable noise, whose characteristic function is expressed as [20]

$$\Phi(t) = \exp(-\gamma |t|^{\alpha}). \quad (68)$$

The characteristic exponent $\alpha$ describes the impulsiveness of the noise (smaller $\alpha$ leads to more outliers) and $\gamma > 0$ represents the dispersion degree of the noise. Specifically, when $\alpha = 1$ or 2, it becomes the Cauchy noise or the Gaussian noise, respectively. In this example, we set $\alpha = 1.3$ and $\gamma = 2/15$. Fig. 8 compares the performance of the previous algorithms. As we have known, in $\alpha$-stable noise environments, the least $p$-th ($L_p$) moment (where $p < \alpha$) is a proper criterion to devise the DLMP algorithm, and here we set $p = 1.25$. Accordingly, we should reselect the basis functions for the DEN-LMS algorithm as $\varphi_{L_p}^k = \{\phi_{k,1} = x, \phi_{k,2} = |x|^{p-1}\text{sign}(x)\}$ and set the step size to $\mu_k = 0.009$. It can be seen that the DEN-LMS version with the tanh basis exhibits poor convergence, like the DNLMS and $l_0$-DNLMS algorithms. By using the $L_p$ basis, the DEN-LMS version can converge as fast as the DLMP algorithm and outperforms the DSE-LMS and D-LLAD algorithms. However, among these robust algorithms, the proposed D-NLMM and D-SNLMM still achieve better estimation performance.

Fig. 8. Network MSD curves of diffusion algorithms in $\alpha$-stable noise. [Sparse $\mathbf{w}_o$ with $Q = 2$]. Parameters of some algorithms are as follows: $\mu_k = 0.006$ (DSE-LMS); $\mu_k = 0.009$ (DLMP); $\mu_k = 0.03$, $\alpha = 0.6$ (D-LLAD); $N_\sigma = 16$, $\zeta = 0.95$ (M-estimator). The other parameters of the DNLS, $l_0$-DNLMS, D-NLMM, and D-SNLMM algorithms are the same as Fig. 5.

B. Verification of Analysis

In this subsection, the simulation setup is the same as in the above Fig. 6 unless otherwise specified. The distributed network has $N = 10$ nodes and the length of $\mathbf{w}_o$ is $L = 5$.

In Figs. 9 and 10 we check the analysis results of the DNLMM algorithm for estimating the non-sparse vector $\mathbf{w}_o$ with $Q = 5$. The theoretical transient results are computed by (44), and the theoretical steady-state results are computed by (53) and (63), where $\beta = 0$. Some expectations associated only with input regressors (e.g., $R_k$) in the analyses are obtained by the ensemble average. Fig. 9 shows the network MSD performance, and Fig. 10 shows the steady-state MSDs at every node. As one can see, the theoretical results have a good agreement with the simulated results. For the node-wise steady-state results in Fig. 10b), there is a small discrepancy when $\mu_k = 0.5$ and 1, thanks mainly to the fact that (63) is derived under small step sizes.
To test the theoretical model in (44), for the D-SNLMM algorithm, the sparse vector is set as \( w^0 = [0 \ 0 \ 0 \ 1 \ 0]^T \). The comparison results in impulsive noise are shown in Fig. 11. As can be seen, the difference between the theoretical results and the simulated results in the convergence stage of curves is relatively large, and it is negligible in the steady-state. The mechanism behind this difference is that assumptions 5 and 6 hold well in the steady-state. Furthermore, based on 500 independent trials, Fig. 12 takes the simulated pdfs of the 1st and 4-th entries of the estimation error vector \( \tilde{w}_{k,i} \) for this algorithm, at nodes \( k = 2, 6 \) and at iteration \( i = 100 \) (transient). It is clear to see that for both zero and large coefficients in the sparse vector \( w^0 \), the corresponding entries’ estimation error approximately follow the Gaussian distribution. In the steady-state, the corresponding distribution is more like Gaussian, but here the figure is omitted. Fig. 13 gives a simulated verification for assumption 6. From Figs. 12 and 13, it is concluded that assumptions 5 and 6 can be applied to simplify the analysis of sparsity-aware algorithms.

C. Comparison with Proximal Variants

Finally, Fig. 14 compares the performance of the D-SNLMM algorithm with that of the proximal (prox.) variants with \( l_1 \)-norm and \( l_0 \)-norm (presented in Appendix B) in -stable noise. These algorithms are employed to solve the distributed sparsity-aware minimization problem (17). The simulation setting is the same as Fig. 8 except \( \beta = 2 \times 10^{-4} \) for the ‘prox.’ with \( l_1 \) variant and \( \beta = 6 \times 10^{-5} \) for the ‘prox.’ with \( l_0 \) variant. As illustrated in Fig. 14, due to the \( l_0 \)-norm based sparsity-regularization, both D-SNLMM and ‘prox. with \( l_0 \)’ algorithms works better than the ‘prox. with \( l_1 \)’ algorithm in terms of convergence and steady-state performance. Although the ‘prox. with \( l_0 \)’ algorithm could slightly outperform the D-SNLMM algorithm, its complexity is slightly high. In particular, the additional complexity of the proximal algorithm originates from (B.11), i.e., \( \text{prox}_{\mu_k\beta} \|\|_0 (\Psi_{k,i+1}) = \max(\|\Psi_{k,i+1} - z\|_0) \odot \text{sign}(\Psi_{k,i+1}) \) which costs \( 4L \) additions per each iteration at every node, where the comparisons required are counted as additions. Also, the ‘prox. with \( l_0 \)’ algorithm requires an extra memory of size \( L \) for storing \( \Psi_{k,i+1} \) at node \( k \).
bounded and usually is more simple than the conventional approach for dealing behaviors of the proposed algorithms including the transient and into the D-NLMM recursion. The mean and mean-square be-

integrals on the score function and the Price’s theorem, which at:

of (26), setting algorithm for simplicity.

the stability condition of the D-SNLMM algorithm (see [63],

\[ \delta_{k,\text{min}}(i) \leq \delta_{k,\text{max}}(i) \leq \delta_{k,\text{max}}(i) \]

where

\[ \delta_{k,\text{min}}(i) = 1 - \mu_k P_{u,k}(i) (2 - \mu_k P_{u,k}(i)) \lambda_{\text{max}}(E\{A_{k,i}\}) \]

\[ \delta_{k,\text{max}}(i) = 1 - \mu_k P_{u,k}(i) (2 - \mu_k P_{u,k}(i)) \lambda_{\text{min}}(E\{A_{k,i}\}) \]

(A.6)

In view of (25b) being a linear convex combination of \( \{ \psi_{m,i+1} \} \), by taking advantage of Jensen’s inequality [16, 17], the following inequality holds

\[ E \{ \| \tilde{w}_{k,i+1} \|^2 \} \leq \sum_{m \in \Lambda_k} c_{m,k} E \{ \| \tilde{\psi}_{k,i+1} \|^2 \}. \]

(A.7)

Introducing the following vectors:

\[ X_i \triangleq \text{col} \left\{ E \{ || \tilde{\psi}_{1,i} ||^2 \}, \ldots, E \{ || \tilde{\psi}_{N,i} ||^2 \} \right\} \]

\[ Y_i \triangleq \text{col} \left\{ E \{ || \tilde{w}_{1,i} ||^2 \}, \ldots, E \{ || \tilde{w}_{N,i} ||^2 \} \right\} \]

\[ Z_i \triangleq \text{col} \left\{ \mu_1^2 P_{u,1}(i) \sigma_{\theta,1}^2 E \left\{ \frac{1}{||u_{1,i}||^2} \right\}, \ldots, \right. \]

\[ \left. \mu_N^2 P_{u,N}(i) \sigma_{\theta,N}^2 E \left\{ \frac{1}{||u_{N,i}||^2} \right\} \right\}, \]

(A.8)

and the diagonal matrix

\[ \delta_i \triangleq \text{diag} \left\{ \delta_{1,\text{max}}(i), \ldots, \delta_{N,\text{max}}(i) \right\}, \]

(A.9)

then gathering (A.3), (A.5), and (A.7) at all nodes implies that

\[ X_{i+1} \preceq \delta_i Y_i + Z_i, \]

(A.10a)

\[ Y_{i+1} \preceq C^T X_{i+1}, \]

(A.10b)

where \( \preceq \) denotes the entry-wise comparison. Since the entries of the matrix \( C \) are nonnegative, (A.10a) and (A.10b) further lead to

\[ Y_{i+1} \preceq C^T \delta_i Y_i + C^T Z_i, \]

(A.11)

Taking the \( \infty \)-norm of both sides of (A.11) and again using \( ||C^T||_\infty = 1 \), we obtain

\[ ||Y_{i+1}||_\infty \leq ||C^T||_\infty \cdot ||\delta_i||_\infty \cdot ||Y_i||_\infty + ||C^T||_\infty \cdot ||Z_i||_\infty \]

\[ = ||\delta_i||_\infty \cdot ||Y_i||_\infty + ||Z_i||_\infty. \]

(A.12)

Recalling that \( E\{A_{k,i}\} \) is the normalized covariance matrix of input regressor at node \( k \), so it is symmetric and non-negative definite. Thus, using the eigenvalue decomposition of \( E\{A_{k,i}\} \), namely, \( E\{A_{k,i}\} = U_k \Lambda_k U_k^T \) with \( \Lambda_k = \text{diag} \{ \lambda_{k,1}, \ldots, \lambda_{k,L} \} \) collecting the eigenvalues and \( U_k \) being the corresponding orthogonal matrix, then we rewrite (A.1) and (A.2) as follows:

\[ E\{ ||\tilde{\psi}_{k,i+1} ||^2 \} = E\{ ||\tilde{w}_{k,i} ||^2 \} + \mu_k^2 P_{u,k}(i) \sigma_{\theta,k}^2 E \left\{ \frac{1}{||u_{k,i}||^2} \right\}, \]

\[ \Sigma_k = I_L - \mu_k P_{u,k}(i) (2 - \mu_k P_{u,k}(i)) E\{A_{k,i}\}. \]

(A.3)

VI. Conclusion

We have derived the D-NLMM algorithm over distributed networks by applying the MH function, which is robust against impulsive noise. Furthermore, considering the underlying sparsity for parameters of interest, the D-NLMM algorithm was proposed by incorporating the \( l_0 \)-norm based zero-attractor into the D-NLMM recursion. The mean and mean-square behaviors of the proposed algorithms including the transient and steady-state were analyzed. This analysis does not require the integrals on the score function and the Price’s theorem, which is more simple than the conventional approach for dealing with the score function. The theoretical expression has shown that D-SNLMM outperforms D-NLMM in sparse systems. Simulations in various noise environments have shown the advantages of our proposed algorithms over some existing diffusion algorithms. The theoretical models are also supported by simulations.

Appendix A

Proof of Theorem 2

Recalling that the elements in \( f(w_{k,i}) \) given by (22) are bounded and usually \( \beta \) is a small positive number, we may neglect the effect of some terms associated with \( \beta f(w_{k,i}) \) on the stability condition of the D-NLMM algorithm (see [63], [62], [65], [64] for similar paradigms). In other words, the bounds on step sizes of the D-SNLMM are the same as those of the D-NLMM algorithm. Therefore, we will deduce the convergence condition in the mean-square from the D-NLMM algorithm for simplicity.

By resorting to the squared Euclidean norms of both sides of (26), setting \( \beta = 0 \), and taking the expectation, we arrive at:

\[ E\{ ||\tilde{\psi}_{k,i+1} ||^2 \} = E\{ ||\tilde{w}_{k,i} ||^2 \} + \mu_k^2 P_{u,k}(i) \sigma_{\theta,k}^2 E \left\{ \frac{1}{||u_{k,i}||^2} \right\}, \]

(A.1)

where

\[ E\{ ||\tilde{w}_{k,i} ||^2 \} \triangleq E\{ \tilde{w}_{k,i}^T \Sigma_k \tilde{w}_{k,i} \}, \]

\[ \Sigma_k = I_L - \mu_k P_{u,k}(i) (2 - \mu_k P_{u,k}(i)) E\{A_{k,i}\}. \]

(A.2)
Given that \(|Z_i|_\infty = \max_{1 \leq k \leq N} \mu_k^2 P_{u,k}(i) \sigma_{\theta,k}^2 \mathbb{E}\left\{ \frac{1}{|u_{k,i}|_2} \right\}\) is bounded, (A.12) converges if and only if \(|\delta_i|_\infty < 1\), which is equivalent to requiring
\[
|1 - \mu_k P_{u,k}(i) (2 - \mu_k P_{u,k}(i)) \lambda_{\min}(\mathcal{E}\{A_{k,i}\})| < 1
\]
for \(k = 1, \ldots, N\). From (A.13), the inequality (52) is acquired.

**APPENDIX B**

**Proximal Variants of the D-SNLMM Algorithm**

In Section III. B, we have derived the D-SNLMM algorithm for solving the distributed estimation problem (17) by virtue of the known SGD rule. In this appendix, we will show briefly how to deduce the distributed proximal solution. Accordingly, we recast (17) as
\[
\min_{w_k} J_{k}^{loc}(i),
\]
\[
J_{k}^{loc}(i) = G_k(i) + \beta F(w_k),
\]
where
\[
G_k(i) = \sum_{m \in N_k} c_{m,k} g_m^{-1} \mathbb{E}\{ \varphi(d_m(i) - u_{m,i}^T w_k) \}. \tag{B.2}
\]

The forward-backward splitting method [128, 129] to get the minimizer of (B.1) consists of two steps. It firstly performs the forward step to minimize \(G_k(i)\) according to the SGD rule (also see the derivation in Section III. A):
\[
\begin{align*}
\Psi_{k,i+1} &= w_{k,i} + \mu_k g_k^{-1} u_{k,i} \varphi'(\epsilon_k(i)), \tag{B.3a} \\
\Psi_{k,i+1} &= \sum_{m \in N_k} c_{m,k} \Psi_{m,i+1}. \tag{B.3b}
\end{align*}
\]
where \(g_k = 1\) and \(g_k = \|u_{k,i}\|_2^2\) correspond to the non-normalized and normalized types, respectively. Subsequently, it performs the proximal step:
\[
w_{k,i+1} = \operatorname{prox}_{\mu_k \beta}[w_{k,i+1}], \tag{B.4}
\]
where \(\operatorname{prox}(\cdot)\) is referred to as the proximal operator of index \(\mu_k \beta \in (0, +\infty)\), defined by
\[
\operatorname{prox}_{\mu_k \beta}[w_{k,i+1}] = \min_{w_k} \beta F(w_k) + \frac{1}{2 \mu_k} \|w_{k,i+1} - w_k\|_2^2. \tag{B.5}
\]
For ease of implementation, a closed-form solution of (B.5) is necessary. Benefited from the subdifferential feature of the function [130], we can solve (B.5) to get
\[
0 \in \beta \partial F(w_k) - \frac{1}{2 \mu_k} (\Psi_{k,i+1} - w_k). \tag{B.6}
\]
If the sparse regularization \(F(w_k) = \sum_{l=1}^L \|w_l\|_1\) (the \(l_1\)-norm in Table I) is used, then (B.6) will become
\[
0 \in \beta \operatorname{sign}(w_k) - \frac{1}{2 \mu_k} (\Psi_{k,i+1} - w_k), \tag{B.7}
\]
which leads to a popular proximal operator, called the soft-threshold [131]:
\[
\operatorname{prox}_{\mu_k \beta, \|\cdot\|_1}(\Psi_{k,i+1}) = \max(\|\Psi_{k,i+1} - \mu_k \beta, 0\) \odot \operatorname{sign}(\Psi_{k,i+1}), \tag{B.8}
\]
where \(\odot\) denotes the element-wise product of two vectors. To significantly utilize the sparsity, we still consider the \(l_0\)-norm approximation in (20) for \(F(w_k)\) so that
\[
0 \in \beta f_w(w_k) \odot \operatorname{sign}(w_k) - \frac{1}{2 \mu_k} (\Psi_{k,i+1} - w_k), \tag{B.9}
\]
where the column vector \(f_w(w_k)\) consists of \(L\) entries
\[
f_w(|w_l|) = \exp^{-v|w_l|}, l = 1, \ldots, L. \tag{B.10}
\]
It is clear that (B.9) can be considered as a weighted variant of (B.7), and then the associated proximal step with the \(l_0\)-norm is represented by the weighted soft-threshold below:
\[
\operatorname{prox}_{\mu_k \beta, \|\cdot\|_0}(\Psi_{k,i+1}) = \max(\Psi_{k,i+1} - \mu_k \beta f_w(\Psi_{k,i+1}), 0) \odot \operatorname{sign}(\Psi_{k,i+1}). \tag{B.11}
\]

In the algorithm’s implementation, we also employ the first order Taylor series of the exponential function which helps to reduce the complexity of (B.11), namely,
\[
f_w(|\Psi_l|) = \begin{cases} 
\upsilon^2 |\Psi_l| + \upsilon, \text{ if } -\frac{1}{\upsilon} \leq \|\Psi_l\| < 0 \\
-\upsilon^2 |\Psi_l| + \upsilon, \text{ if } 0 < \|\Psi_l\| \leq \frac{1}{\upsilon} \\
0, \text{ elsewhere.}
\end{cases} \tag{B.12}
\]
Note that other sparse regularization criteria in Table I can also be extended under this weighted proximal formalism. Moreover, these distributed proximal algorithms have also the same stability condition (64) as the D-NLMM algorithm, with the aid of references [63, 131]. However, the detailed proof is omitted due to lack of space. In fact, because of involving the proximal step (B.11), the theoretical analysis of the proximal algorithm will also be more complicated than that of the D-SNLMM algorithm.

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