Hypergeometric solution of a certain polynomial Hamiltonian system of isomonodromy type

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Abstract

In [15] a unified description as polynomial Hamiltonian systems was established for a broad class of the Schlesinger systems including the sixth Painlevé equation and Garnier systems. The main purpose of this paper is to present particularsolutions of this Hamiltonian system in terms of a certain generalization of Gauß’s hypergeometric function. Key ingredients of the argument are the linear Pfaffian system derived from an integral representation of the hypergeometric function (with the aid of twisted de Rham theory) and Lax formalism of the Hamiltonian system.

1 Introduction

Fix integers $L \geq 2$ and $N \geq 1$. We consider the following completely integrable Hamiltonian system of partial differential equations:

\[
\frac{\partial q_n^{(i)}}{\partial x_j} = \frac{\partial H_j}{\partial p_n^{(i)}}, \quad \frac{\partial p_n^{(i)}}{\partial x_j} = -\frac{\partial H_j}{\partial q_n^{(i)}} \quad (i, j = 1, \ldots, N) \quad (n = 1, \ldots, L-1)
\]

with variables $x = (x_1, \ldots, x_N)$ and unknowns $q_n^{(i)}$ and $p_n^{(i)}$. Here the Hamiltonians function $H_i$ is given by

\[
x_i H_i = \sum_{n=0}^{L-1} e_n q_n^{(i)} p_n^{(i)} + \sum_{j=0}^{N} \sum_{0 \leq m < n \leq L-1} q_m^{(i)} p_m^{(j)} q_n^{(j)} p_n^{(i)} + \sum_{j=0}^{N} \frac{x_j}{x_i - x_j} \sum_{m, n=0}^{L-1} q_m^{(i)} p_m^{(j)} q_n^{(j)} p_n^{(i)}
\]

and

\[
x_0 = q_0^{(0)} = q_0^{(i)} = 1, \quad p_n^{(0)} = \kappa_n - \sum_{i=1}^{N} q_n^{(i)} p_n^{(i)}, \quad p_0^{(0)} = \theta_1 - \sum_{n=1}^{L-1} q_n^{(i)} p_n^{(i)}
\]
thereby, $H_i$ forms a polynomial in the canonical variables. This system was introduced by the author in [15] via a similarity reduction of the UC hierarchy [14] (an extension of the KP hierarchy); it is equivalent to a class of the Schlesinger systems [11] describing isomonodromic deformations of an $L \times L$ Fuchsian system with $N + 3$ poles on the Riemann sphere. The spectral type of this Fuchsian system is given by the $(N + 3)$-tuple

$$(1,1,\ldots,1), (1,1,\ldots,1), (L-1,1), \ldots, (L-1,1)$$

of partitions of $L$, which indicates how the characteristic exponents overlap at each of the $N + 3$ regular singularities. System (1.1) contains complex constants

$$(e, \kappa, \theta) = (e_0, e_1, \ldots, e_{L-1}, \kappa_0, \kappa_1, \ldots, \kappa_{L-1}, \theta_0, \theta_1, \ldots, \theta_N)$$

satisfying the linear constraints

$$\sum_{n=0}^{L-1} e_n = \frac{L-1}{2}$$

and

$$\sum_{n=0}^{L-1} \kappa_n = N$$

so the number of constant parameters is essentially $2L + N - 1$. Note that these parameters correspond to the characteristic exponents of the associated Fuchsian system. Henceforth, we denote by $\mathcal{H}_{L,N}$ the polynomial Hamiltonian system (1.1). For example, the case where $L = 2$ and any $N \geq 1$ ($\mathcal{H}_{2,N}$) coincides with the Garnier system in $N$ variables [5, 6] and, thus, the first nontrivial case ($\mathcal{H}_{2,1}$) with the sixth Painlevé equation $P_{VI}$ [8, 9]. For details refer to [15].

In this paper we present a family of particular solutions of the polynomial Hamiltonian system $\mathcal{H}_{L,N}$, which is parameterized by a point in the projective space $\mathbb{P}^{N(L-1)}$. These solutions are governed by a linear Pfaffian system of rank $N(L-1) + 1$ and, furthermore, expressed in terms of a generalization of Gauß’s hypergeometric function.

We begin by introducing the hypergeometric function crucial to this work. Fix the notation of multi-index; let $I = \{m = (m_1, \ldots, m_N) \mid m_i \in \mathbb{Z}_{\geq 0}\}$, and write $x^m = x_1^{m_1} \cdots x_N^{m_N}$ and $|m| = m_1 + \cdots + m_N$ for $m \in I$. We define a function $F_{L,N} = F_{L,N}(\alpha, \beta, \gamma; x)$ in $N$ variables $x = (x_1, \ldots, x_N)$ by means of the power series

$$F_{L,N}(\alpha, \beta, \gamma; x) \overset{\text{def}}{=} \sum_{m \in I} \frac{(\alpha_1 m_1) \cdots (\alpha_{L-1} m_{L-1}) (\beta_1 m_1) \cdots (\beta_N m_N)}{(\gamma_1 m_1) \cdots (\gamma_{L-1} m_{L-1}) (1 m_1) \cdots (1 m_N)} x^m$$

(1.2)

convergent in the polydisc $\{|x_1| < 1, \ldots, |x_N| < 1\} \subset \mathbb{C}^N$. Here

$$(\alpha, \beta, \gamma) = (\alpha_1, \ldots, \alpha_{L-1}, \beta_1, \ldots, \beta_N, \gamma_1, \ldots, \gamma_{L-1})$$

are complex constants such that $\gamma_i \notin \mathbb{Z}_{<0}$, and $(a)_n = \Gamma(a+n)/\Gamma(a)$. The series $F_{L,N}$ satisfies the system of linear differential equations

$$\left\{ x_i (\beta_i + \delta_i) \prod_{k=1}^{L-1} (\alpha_k + D) - \delta_i \prod_{k=1}^{L-1} (\gamma_k - 1 + D) \right\} y = 0, \quad i = 1, \ldots, N,$$

(1.3)

where

$$\delta_i = x_i \frac{\partial}{\partial x_i} \quad \text{and} \quad D = \sum_{i=1}^{N} \delta_i,$$

(1.4)
As is shown in the next section, system (1.3) is equivalent to a linear Pfaffian system of rank \( N(L-1) + 1 \). The holomorphic function \( F_{L,N}(\alpha, \beta, \gamma; x) \) at \( 0 \in \mathbb{C}^N \) can be analytically continued along any path on \( X = \{ x = (x_1, \ldots, x_N) \in \mathbb{C}^N \mid x_i \neq x_j (i \neq j), x_i \neq 0, 1 \} \), which is the complement of the singular locus of this Pfaffian system. Note that, if \((L, N) = (2, 1), (L, 1)\) and \((2, N)\), then the hypergeometric series \( F_{L,N} \) reduces to Gauß’s \(_2F_1\), Thomae’s \(_1F_{L-1}\) [13] and Appell–Lauricella’s \( F_D \) [2, 7], respectively.

We turn now to the particular solutions of the Hamiltonian system \( \mathcal{H}_{L,N} \). There is an equivalent formulation of \( \mathcal{H}_{L,N} \) (Lax formalism) as a compatibility condition of an auxiliary linear problem consisting of the foregoing Fuchsian system and its deformation equations; see [15]. Under a certain condition of parameters we find particular solutions of \( \mathcal{H}_{L,N} \) such that the associated Fuchsian system becomes reducible; furthermore, these solutions are governed by the same linear Pfaffian system of rank \( N(L-1) + 1 \) as \( F_{L,N} \). This fact leads us to the

**Theorem 1.1.** When \( \kappa_0 - \sum_{i=1}^{N} \theta_i = 0 \), the Hamiltonian system \( \mathcal{H}_{L,N} \) possesses an \( N(L-1) \)-parameter family of particular solutions, each of which is expressed in terms of a hypergeometric function, i.e., an arbitrary solution of (1.3).

(See Theorem 3.2.)

**Remark 1.2.** The hypergeometric solutions of \( P_{VI} (= \mathcal{H}_{2,1}) \) and the Garnier system (\( = \mathcal{H}_{2,N} \)) were first given by Fuchs [4] and Okamoto–Kimura [10], respectively. See also [5]. They linearized the Riccati-type equations for particular solutions by introducing new dependent variables, and then identified the resulting linear ones as the hypergeometric differential equations. Recently the case of \( \mathcal{H}_{L,1} \) was studied independently by Suzuki [12]; he obtained a power series solution through Frobenius’ method after a direct linearization. The present result covers all previous ones; however, it is based on a method quite different from theirs. We emphasize that key ingredients of the argument are a systematic derivation of the linear Pfaffian system from an integral representation of the hypergeometric function (with the aid of twisted de Rham theory) and investigation into its Lax formalism rather than the Hamiltonian system itself.

In Sect. 2 we present the integral representation of the hypergeometric function \( F_{L,N} \) (Proposition 2.1). Applying a twisted (co)homological technique we derive the linear Pfaffian system for \( F_{L,N} \) (Theorem 2.2). In Sect. 3 after a brief review of Lax formalism of \( \mathcal{H}_{L,N} \) (Theorem 3.1), we solve it with special values of parameters. Particular solutions thus obtained satisfy the same linear Pfaffian system as \( F_{L,N} \). This establishes a representation of the solutions in terms of the hypergeometric functions (Theorem 3.2). In the appendix we summarize the contiguity relations for \( F_{L,N} \).

### 2 Integral representation and Pfaffian system for hypergeometric function \( F_{L,N} \)

In this section we first introduce the integral representation of \( F_{L,N} \), then from which we derive the linear Pfaffian system by means of the viewpoint of twisted de Rham theory. Conversely, the hypergeometric function \( F_{L,N} \) can be characterized as the unique holomorphic solution of this Pfaffian system at \( 0 \in \mathbb{C}^N \).
2.1 Integral representations

The hypergeometric function $F_{L,N}$, (1.2), can be written as

$$F_{L,N}(\alpha, \beta, \gamma; x) = \prod_{k=1}^{L-1} \frac{\Gamma(\gamma_k)}{\Gamma(\alpha_k)\Gamma(\gamma_k - \alpha_k)} \times \int_{[0,1]^{L-1}} \prod_{k=1}^{L-1} z_k^{\alpha_k-1}(1-z_k)^{\gamma_k-\alpha_k-1} \prod_{i=1}^{N} (1 - x_i z_1 z_2 \cdots z_{L-1} - \beta) dz_1 \cdots dz_{L-1}, \quad (2.1)$$

provided $|x_i| < 1$ and $\operatorname{Re}(\gamma_k) > \operatorname{Re}(\alpha_k) > 0$ and the branch of the integrand is assigned as

$$\arg z_k = \arg(1-z_k) = 0 \quad \text{and} \quad |\arg(1-x_i z_1 z_2 \cdots z_{L-1})| < \frac{\pi}{2}. \quad (2.1)$$

Representation (2.1) can be verified in a standard manner, i.e., by means of the binomial theorem:

$$(1 - z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{(1)_n} z^n$$

and the relation between the beta function and gamma function:

$$B(a, b) = \int_0^1 z^{a-1}(1-z)^{b-1} dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \operatorname{Re}(a), \operatorname{Re}(b) > 0.$$

To transform the nonlinear form in the integrand of (2.1) into a linear one, we apply the change of integration variables

$$t_1 = z_1, \quad t_2 = z_1 z_2, \quad t_3 = z_1 z_2 z_3, \quad \ldots, \quad t_{L-1} = z_1 z_2 \cdots z_{L-1}.$$

The Jacobian of this transformation is calculated as

$$\left| \frac{\partial(z_1, \ldots, z_{L-1})}{\partial(t_1, \ldots, t_{L-1})} \right| = \frac{1}{t_1 t_2 \cdots t_{L-2}}.$$

Put $t_0 = 1$ for convenience. The integral representation of $F_{L,N}$ in which every factor of the integrand takes a linear form is, therefore, established.

**Proposition 2.1.** Assume $\operatorname{Re}(\gamma_k) > \operatorname{Re}(\alpha_k) > 0$. For $|x_i| < 1$ it holds that

$$F_{L,N}(\alpha, \beta, \gamma; x) = \prod_{k=1}^{L-1} \frac{\Gamma(\gamma_k)}{\Gamma(\alpha_k)\Gamma(\gamma_k - \alpha_k)} \times \int_{\Delta} t_1^{\alpha_1-1} \prod_{k=1}^{L-2} t_k^{\alpha_k-\gamma_k-1} \prod_{i=1}^{N} (1 - x_i t_{L-1})^{\gamma_k-\alpha_k-1} dt_1 \cdots dt_{L-1} \quad (2.2)$$

with the integration domain $\Delta$ being an $(L-1)$-simplex

$$\Delta = \left\{ 0 \leq t_{L-1} \leq \cdots \leq t_2 \leq t_1 \leq 1 \right\} \subset \mathbb{R}^{L-1}.$$

Here the branch of the integrand is assigned as

$$\arg t_k = \arg(t_{k-1} - t_k) = 0 \quad \text{and} \quad |\arg(1-x_i t_{L-1})| < \frac{\pi}{2}.$$

Based on this integral representation and twisted de Rham theory, we will discuss below the linear Pfaffian system characterizing $F_{L,N}$.
2.2 Pfaffian system

Consider a multivalued function

\[ U(t) = t_{L-1}^{-\alpha_{L-1}} \prod_{k=1}^{L-2} t_k^{\alpha_k - \gamma_k + 1} \prod_{k=1}^{L-1} (t_{k-1} - t_k)^{\gamma_k - \alpha_k} \prod_{i=1}^{N} (1 - x_i t_{L-1})^{-\beta_i} \]

with \( t_0 = 1 \) defined on

\[ T = \{ t = (t_1, \ldots, t_{L-1}) \in \mathbb{C}^{L-1} \mid t_k \neq 0, \ t_k \neq t_{k-1}, \ t_{L-1} \neq 1/x_i \}, \]

which is the complement of singular locus \( D = \bigcup_{k=1}^{L-1} (\{ t_k = 0 \} \cup \{ t_{k-1} - t_k = 0 \}) \cup \bigcup_{i=1}^{N} \{ 1 - x_i t_{L-1} = 0 \} \) of \( U(t) \) in \( \mathbb{C}^{L-1} \). Let \( \mathcal{L} \) be the local system of rank one determined by \( 1/U(t) \), i.e., a flat line bundle consisting of the local solutions of \( \nabla_{\omega} h = 0 \) on \( T \), where \( \nabla_{\omega} \) is the covariant differential operator given by

\[ \nabla_{\omega} = d + \omega \wedge, \quad \omega = d \log U(t). \quad (2.3) \]

Let \( \mathcal{L}' \) be the dual local system of \( \mathcal{L} \). Denote by \( H^p(T, \mathcal{L}) \) (resp. \( H^p(T, \mathcal{L}') \)) the \( p \)-th cohomology (resp. homology) group with coefficients in \( \mathcal{L} \) (resp. \( \mathcal{L}' \)). Under a certain genericity condition for the exponents \( \alpha_i, \beta_i, \gamma_i \in \mathbb{C} \setminus \mathbb{Z} \), it holds that

\[ \dim H^p(T, \mathcal{L}) = \dim H^p(T, \mathcal{L}') = \begin{cases} N(L - 1) + 1 & \text{if } p = L - 1 \\ 0 & \text{if } p \neq L - 1 \end{cases} \]

and, furthermore, bases of the top cohomology and homology groups are described as follows. First we notice the isomorphism

\[ H^p(T, \mathcal{L}) \cong H^p(\Omega^\ast \ast D, \nabla_{\omega}) = \{ \xi \in \Omega^p(\ast D) \mid \nabla_{\omega} \xi = 0 \} \big/ \nabla_{\omega} \Omega^{p-1}(\ast D), \]

where the right-hand side is the de Rham cohomology group determined by \( \nabla_{\omega} \) and \( \Omega^p(\ast D) \) stands for the space of rational \( p \)-forms holomorphic outside \( D \). It can be verified that the rational \((L - 1)\)-forms

\[ \varphi_0 = \frac{dt_1 \wedge \cdots \wedge dt_{L-1}}{t_{L-1} \prod_{k=1}^{L-1} (t_{k-1} - t_k)}, \]

\[ \varphi_n^{(i)} = \frac{dt_1 \wedge \cdots \wedge dt_{L-1}}{t_{L-1} (1 - x_i t_{L-1}) \prod_{k=1}^{L-1} (t_{k-1} - t_k)} \quad \left( 1 \leq i \leq N, \quad 1 \leq n \leq L - 1 \right) \]

are cocycles representing a basis of \( H^{L-1}(T, \mathcal{L}) \). On the other hand, a basis of \( H_{L-1}(T, \mathcal{L}') \) can be constructed from the set of bounded chambers in the real locus \( T \cap \mathbb{R}^{L-1} \) of \( T \). For simplicity, we fix the configuration \( x \in \mathbb{C}^N \) of \( N \) points to be real numbers such that \( 0 < x_N < \cdots < x_2 < x_1 < 1 \). Accordingly, the set of bounded chambers is given by

\[ \Delta_0' = \{ 0 < t_{L-1} < \cdots < t_1 < t_0 = 1 \}, \]

\[ \Delta_n^{(i)} = \left\{ \begin{array}{l} t_{L-1} > t_{L-2} > \cdots > t_{L-n-1} > 0 \\ 0 < t_{L-n-1} < \cdots < t_1 < t_0 = 1 \\ 1/x_i < t_{L-1} < 1/x_i \end{array} \right\} \quad \left( 1 \leq i \leq N, \quad 1 \leq n \leq L - 1 \right). \]
The regularizations [1, 16] of these cycles, denoted by $\Delta_0$ and $\Delta_n$, represent a basis of $H_{L-1}(T, \mathcal{L}^\nu)$. Now we introduce the integrals

$$y_0 = \int_\Delta U(t) \varphi_0 \quad \text{and} \quad y_n^{(i)} = \int_\Delta U(t) \varphi_n^{(i)}$$  \hspace{1cm} (2.4)

for any twisted cycle $\Delta \in H_{L-1}(T, \mathcal{L}^\nu)$. Then we have the

**Theorem 2.2.** The functions $y_0$ and $y_n^{(i)}$ satisfy the linear Pfaffian system

$$(x_i - 1) \frac{\partial y_0}{\partial x_i} = \beta_i \left( -y_0 + \sum_{m=1}^{L-1} y_m^{(i)} \right),$$  \hspace{1cm} (2.5a)

$$(x_i - x_j) \frac{\partial y_n^{(j)}}{\partial x_i} = \beta_i \left( y_n^{(i)} - y_n^{(j)} \right),$$  \hspace{1cm} (2.5b)

$$x_i \frac{\partial y_n^{(i)}}{\partial x_i} = -\alpha_n y_n^{(i)} + (\gamma_n - \alpha_n) \sum_{m=n}^{L-1} y_m^{(i)} + \frac{\gamma_n - \alpha_n}{x_i - 1} \left( -y_0 + \sum_{m=1}^{L-1} y_m^{(i)} \right) + \sum_{j=1}^{N} \frac{\beta_j x_j}{x_i - x_j} \left( y_n^{(j)} - y_n^{(i)} \right),$$  \hspace{1cm} (2.5c)

A proof of this theorem will be given in Sect. 2.3.

Let us consider the vector-valued function

$$\vec{y} = \vec{y}(x; \Delta) = (y_0, y_1^{(1)}, y_1^{(2)}, \ldots, y_{L-1}^{(1)}, y_{L-1}^{(2)}, \ldots, y_1^{(N)}, y_2^{(N)}, \ldots, y_{L-1}^{(N)})$$

defined by the integrals (2.4) for $\Delta \in H_{L-1}(T, \mathcal{L}^\nu)$. Then (2.5) takes the following expression:

$$d\vec{y} = \left( \sum_{i=1}^{N} \left( E_i d \log x_i + F_i d \log(x_i - 1) \right) + \sum_{1 \leq i < j \leq N} G_{ij} d \log(x_i - x_j) \right) \vec{y},$$  \hspace{1cm} (2.6)

where $E_i$, $F_i$ and $G_{ij}$ are the square matrices of size $N(L - 1) + 1$:

$$E_i = \begin{bmatrix} -a_1 & -\beta_1 I & -\beta_2 I & \cdots & b_{i,1} \\ -a_2 \\ -a_3 \\ \vdots \\ -a_{L-1} \end{bmatrix}$$

$$F_i = \begin{bmatrix} -\beta_i & \beta_i & \beta_i & \cdots & \beta_i \\ a_1 & -a_1 & -a_1 & \cdots & -a_1 \\ a_2 & -a_2 & -a_2 & \cdots & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{L-1} & -a_{L-1} & -a_{L-1} & \cdots & -a_{L-1} \end{bmatrix}.$$
and \(a_n = \alpha_n - \gamma_n\) and \(b_{i,n} = \sum_{j=i}^{n} \beta_j - \gamma_n\); the symbol \(I\) denotes the identity matrix of size \(L - 1\). We wrote a square matrix \(M\) of size \(N(L - 1) + 1\) as

\[
M = \begin{bmatrix}
M_{00} & M_{01} & \cdots & M_{0N} \\
M_{10} & M_{11} & \cdots & M_{1N} \\
\vdots & \vdots & \ddots & \vdots \\
M_{N0} & M_{N1} & \cdots & M_{NN}
\end{bmatrix}
\]

with dividing it into \((N + 1)^2\) blocks so that \(M_{00}\) becomes a scalar, \(M_{0j}\) \((j \neq 0)\) and \(M_{i0}\) \((i \neq 0)\) row and column \((L - 1)\)-vectors, respectively, and \(M_{ij}\) \((i, j \neq 0)\) a square matrix of size \(L - 1\).

The linear Pfaffian system \((2.5)\), or \((2.6)\), is of rank \(N(L - 1) + 1\) and the integrals

\[
\bar{y}(x; \Delta_0) \quad \text{and} \quad \bar{y}(x; \Delta_n^{(i)}) \quad \left(\begin{array}{c}
1 \\ 1 \\ \vdots \\ N
\end{array}\right) \quad \left(\begin{array}{c}
i \\ i \\ \vdots \\ N
\end{array}\right)
\]

provide a fundamental system of solutions. In particular, \(\bar{y}(x; \Delta_0)\) is the unique holomorphic solution at \(0 \in \mathbb{C}^N\) up to multiplication by constants; it is expressible in terms of the hypergeometric function \(F_{L,N}(\alpha, \beta, \gamma; x)\) according to the integral representation (see Proposition \((2.1)\)) as

\[
y_0 = cF_{L,N}, \quad y_1^{(i)} = \frac{\gamma_1 - \alpha_1}{\gamma_1} cF_{L,N}(\beta_i + 1, \gamma_1 + 1),
\]

\[
y_2^{(i)} = \frac{\alpha_1(\gamma_2 - \alpha_2)}{\gamma_1 \gamma_2} cF_{L,N}(\alpha_1 + 1, \beta_i + 1, \gamma_1 + 1, \gamma_2 + 1), \quad \ldots
\]

\[
y_n^{(i)} = \frac{\alpha_1 \cdots \alpha_{n-1}(\gamma_n - \alpha_n)}{\gamma_1 \cdots \gamma_n} cF_{L,N}(\alpha_1 + 1, \ldots, \alpha_{n-1} + 1, \beta_i + 1, \gamma_1 + 1, \ldots, \gamma_n + 1), \quad \ldots
\]

where \(c = \prod_{k=1}^{L-1} \Gamma(\alpha_k) \Gamma(\gamma_k - \alpha_k) / \Gamma(\gamma_k)\). For notational simplicity, we used the abbreviation \(F_{L,N}(\beta_i + 1, \gamma_1 + 1)\) to mean that among the parameters \((\alpha, \beta, \gamma)\) only the indicated ones \(\beta_i\) and \(\gamma_1\) are shifted by one, and so forth. We mention that the differential equations satisfied by the first element \(y = y_0\) of \(\bar{y}\) are indeed \((1.3)\).

### 2.3 Verification of Theorem \((2.2)\)

In general, it holds for an \((L - 1)\)-form \(\varphi\) that

\[
\frac{\partial}{\partial x_i} \int_{\Delta} U \varphi = \int_{\Delta} U \left( \frac{1}{U} \frac{\partial U}{\partial x_i} \varphi + \frac{\partial \varphi}{\partial x_i} \right).
\]

Hence Theorem \((2.2)\) is an immediate consequence of \((2.4)\) and the following lemma.
Lemma 2.3. Define a linear operator $\nabla_i$ \((i = 1, \ldots, N)\) acting on a differential form $\varphi$ by

$$
\nabla_i \varphi = \frac{1}{U} \frac{\partial U}{\partial x_i} \varphi + \frac{\partial \varphi}{\partial x_i}
$$

The rational \((L - 1)\)-forms $\varphi_0$ and $\varphi_n^{(i)}$ satisfy the relations

\begin{align}
(x_i - 1) \nabla_i \varphi_0 &= \beta_i \left( -\varphi_0 + \sum_{m=1}^{L-1} \varphi_m^{(i)} \right), \\
(x_i - x_j) \nabla_i \varphi_n^{(j)} &= \beta_i \left( \varphi_n^{(i)} - \varphi_n^{(j)} \right) \quad \text{(2.7a)} \\
x_i \nabla_i \varphi_n^{(i)} &= -\alpha_n \varphi_n^{(i)} + (\gamma_n - \alpha_n) \sum_{m=n+1}^{L-1} \varphi_m^{(i)} + \frac{\gamma_n - \alpha_n}{x_i - 1} \left( -\varphi_0 + \sum_{m=1}^{L-1} \varphi_m^{(i)} \right) \\
&\quad + \sum_{j=1}^{N} \frac{\beta_j x_j}{x_i - x_j} \left( \varphi_n^{(j)} - \varphi_n^{(i)} \right) \quad \text{(modulo $\nabla_\omega \Omega^{L-2}(\ast D)$).} \quad \text{(2.7b)}
\end{align}

Proof. We will use the notation

$$
\begin{align}
\text{d}t &= \text{d}t_1 \land \cdots \land \text{d}t_{L-1}, \\
\ast \text{d}t_j &= (-1)^{j-1} \text{d}t_1 \land \cdots \land \text{d}t_j \land \cdots \land \text{d}t_{L-1};
\end{align}
$$

therefore, $\text{d}t \land \ast \text{d}t_j = \frac{\text{d}t}{x_i - x_j}$. We abbreviate $\prod_{k=1}^{L-1}$ and $\prod_{k=n}^{L-1}$ respectively to $\prod_k$ and $\prod_{k \neq n}$, and so forth. From the definition

\begin{align}
\varphi_0 &= \frac{\text{d}t}{t_{L-1} \prod_k (t_{k-1} - t_k)}, \\
\varphi_n^{(i)} &= \frac{\text{d}t}{t_{L-1} (1 - x_i t_{L-1}) \prod_{k \neq n} (t_{k-1} - t_k)},
\end{align}

it is readily seen that

\begin{align}
\sum_{m=n+1}^{L-1} \varphi_m^{(i)} &= \frac{t_n - t_{L-1}}{t_{L-1} (1 - x_i t_{L-1}) \prod_k (t_{k-1} - t_k)} \text{d}t \quad (0 \leq n \leq L - 2), \\
-\varphi_0 + \sum_{m=1}^{L-1} \varphi_m^{(i)} &= \frac{x_i - 1}{(1 - x_i t_{L-1}) \prod_k (t_{k-1} - t_k)} \text{d}t. \quad \text{(2.10)}
\end{align}

Since $\varphi_0$ does not depend on $x_i$ it follows that

$$
\nabla_i \varphi_0 = \frac{1}{U} \frac{\partial U}{\partial x_i} \varphi_0 = \frac{\beta_i t_{L-1}}{1 - x_i t_{L-1}} \varphi_0 = \frac{\beta_i \text{d}t}{(1 - x_i t_{L-1}) \prod_k (t_{k-1} - t_k)},
$$

which coincides with \text{(2.7a)} according to \text{(2.10)}. Likewise \text{(2.7b)} can be verified as

\begin{align}
\nabla_i \varphi_n^{(j)} &= \frac{\text{d}t}{(1 - x_i t_{L-1}) \prod_{k \neq n} (t_{k-1} - t_k)} \left( \frac{1}{1 - x_i t_{L-1}} \right) \left( \frac{1}{1 - x_j t_{L-1}} \right) \\
&= \frac{\beta_i}{x_i - x_j} \left( \varphi_n^{(i)} - \varphi_n^{(j)} \right).
\end{align}
It is difficult to calculate directly $\nabla_i \varphi_n^{(i)}$ because $\varphi_n^{(i)}$ depends on $x_i$. So we first prepare an appropriate coboundary, by which we eliminate the $x_i$ dependence of $\varphi_n^{(i)}$ modulo $\nabla_n \Omega^{L-2}(\ast D)$. Consider the rational $(L-2)$-form

$$\Omega^{L-2}(\ast D) \ni \xi_n = \prod_{k \neq n} d \log(t_{k-1} - t_k) \wedge$$

$$= \frac{(-1)^{L+n-1}}{\prod_{k \neq n}(t_{k-1} - t_k)} \sum_{j=n}^{L-1} *dt_j.$$

Its covariant derivative reads (recall (2.3))

$$\nabla_\omega \xi_n = (-1)^{L+n} \left( \frac{\gamma_n - \alpha_n}{\prod_k (t_{k-1} - t_k)} - \frac{\alpha_{L-1}}{t_{L-1} \prod_{k \neq n}(t_{k-1} - t_k)} \right) \nabla_\omega \xi_n$$

$$- \sum_{j=1}^N \beta_j \frac{x_j}{(1 - x_j t_{L-1}) \prod_{k \neq n}(t_{k-1} - t_k)} + \sum_{j=1}^{L-2} \frac{\gamma_{j+1} - \alpha_j}{t_j \prod_{k \neq n}(t_{k-1} - t_k)} dt_j.$$

Hence

$$\sum_{j=1}^N \beta_j \varphi_n^{(j)} = \sum_{j=1}^N \beta_j \left( \frac{1}{t_{L-1} \prod_{k \neq n}(t_{k-1} - t_k)} + \frac{x_j}{(1 - x_j t_{L-1}) \prod_{k \neq n}(t_{k-1} - t_k)} \right) dt$$

$$\equiv \left( \frac{\gamma_n - \alpha_n}{\prod_k (t_{k-1} - t_k)} + \sum_{j=1}^N \beta_j - \frac{\alpha_{L-1}}{t_{L-1} \prod_{k \neq n}(t_{k-1} - t_k)} \right) \nabla_\omega \xi_n$$

modulo $\nabla_\omega \xi_n$. Now the derivative can be calculated as

$$\left( \frac{x_i}{\beta_i} \right) \left( \sum_{j=1}^N \beta_j \varphi_n^{(j)} \right) = \left( \frac{1}{1 - x_i t_{L-1}} + \frac{x_i}{\beta_i} \partial x_i \right) \sum_{j=1}^N \beta_j \varphi_n^{(j)}$$

$$\equiv \left( \frac{\gamma_n - \alpha_n}{(1 - x_i t_{L-1}) \prod_k (t_{k-1} - t_k)} + \frac{\sum_{j=1}^N \beta_j - \alpha_{L-1}}{t_{L-1} (1 - x_i t_{L-1}) \prod_{k \neq n}(t_{k-1} - t_k)} \right) \nabla_\omega \xi_n$$

Applying (2.8), (2.10) and Lemma 2.4 below, we thus arrive at

$$\left( \frac{x_i}{\beta_i} \right) \sum_{j=1}^N \beta_j \varphi_n^{(j)} \equiv \frac{\gamma_n - \alpha_n}{x_i - 1} \left( -\varphi_0 + \sum_{m=1}^{L-1} \varphi_m^{(i)} \right) + \left( \sum_{j=1}^N \beta_j - \alpha_n \right) \varphi_n^{(i)} + \left( \gamma_n - \alpha_n \right) \sum_{m=n+1}^{L-1} \varphi_m^{(i)}$$

which establishes (2.7c) by virtue of (2.7b). The proof of the lemma is complete. \hfill \Box

**Lemma 2.4.** One has

$$\sum_{j=n}^{L-2} \frac{\gamma_{j+1} - \alpha_j}{t_j (1 - x_j t_{L-1}) \prod_{k \neq n}(t_{k-1} - t_k)} dt \equiv \left( \alpha_{L-1} - \alpha_n \right) \varphi_n^{(i)} + \left( \gamma_n - \alpha_n \right) \sum_{m=n+1}^{L-1} \varphi_m^{(i)} \quad (2.11)$$

modulo $\nabla_\omega \Omega^{L-2}(\ast D)$. 

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Proof. Taking partial fraction decomposition yields

\[ \sum_{j=n}^{L-2} \frac{\gamma_{j+1} - \alpha_j}{t_j(1 - x_l t_{L-1}) \prod_{k \neq n}(t_{k-1} - t_k)} \, dt \]

\[ = \frac{1}{t_{L-1}(1 - x_l t_{L-1})} \sum_{j=n}^{L-2} (\gamma_{j+1} - \alpha_j) \left( \frac{1}{\prod_{k \neq n}(t_{k-1} - t_k)} - \frac{1}{t_j \prod_{m \neq n,m}(t_{k-1} - t_k)} \right) \, dt \]

\[ = \sum_{j=n}^{L-2} (\gamma_{j+1} - \alpha_j) \psi_n^{(i)} - \frac{1}{t_{L-1}(1 - x_l t_{L-1})} \sum_{j=n}^{L-2} \sum_{m=j+1}^{L-1} \gamma_{j+1} - \alpha_j \frac{\gamma_m - \alpha_m}{\prod_{k \neq n,m}(t_{k-1} - t_k)} \, dt. \quad (2.12) \]

On the other hand, we consider the \((L - 2)\)-form

\[ \Omega^{L-2}(D) \ni \psi_{n,m} = \left( \prod_{k \neq n,m} d \log(t_{k-1} - t_k) \right) dt_{L-1} \]

\[ = \frac{(-1)^{n+m-1}}{\prod_{k \neq n,m}(t_{k-1} - t_k)} \sum_{j=n}^{m-1} \star dt_j \]

for \(n < m\), of which the covariant derivative reads

\[ \nabla_\omega \psi_{n,m} = (-1)^{n+m} \left( \sum_{j=n}^{m-1} \frac{\gamma_{j+1} - \alpha_j}{t_j \prod_{k \neq n,m}(t_{k-1} - t_k)} + \frac{\gamma_n - \alpha_n}{\prod_{k \neq n}(t_{k-1} - t_k)} - \frac{\gamma_m - \alpha_m}{\prod_{k \neq n}(t_{k-1} - t_k)} \right) \, dt. \]

Observe that \(\nabla_\omega \psi_{n,n}\) still remains a coboundary if we multiply it by any rational function \(g \in \Omega^0(D)\) such that \(\partial g / \partial t_k = 0\) \((\forall k \neq L - 1)\). In particular, by choosing \(g = t_{L-1}^{-1}(1 - x_l t_{L-1})^{-1}\), we have

\[ \frac{1}{t_{L-1}(1 - x_l t_{L-1})} \sum_{j=n}^{m-1} \frac{\gamma_{j+1} - \alpha_j}{t_j \prod_{k \neq n,m}(t_{k-1} - t_k)} \, dt + (\gamma_n - \alpha_n) \psi_m^{(i)} - (\gamma_m - \alpha_m) \psi_n^{(i)} \equiv 0. \]

Summation over \(m = n + 1, \ldots, L - 1\) of this formula entails

\[ \frac{1}{t_{L-1}(1 - x_l t_{L-1})} \sum_{m=n+1}^{L-1} \sum_{j=n}^{m-1} \frac{\gamma_{j+1} - \alpha_j}{t_j \prod_{k \neq n,m}(t_{k-1} - t_k)} \, dt + (\gamma_n - \alpha_n) \sum_{m=n+1}^{L-1} \psi_m^{(i)} - (\gamma_m - \alpha_m) \psi_n^{(i)} \equiv 0. \]

Substituting the above into (2.12), we verify the desired result (2.11). \(\square\)

### 3 Hypergeometric solution of Hamiltonian system \(\mathcal{H}_{L,N}\)

In this section we first review Lax formalism of \(\mathcal{H}_{L,N}\) following [15]. Under a certain condition of parameters \(\mathcal{H}_{L,N}\) admits particular solutions such that the associated Fuchsian system becomes reducible. These solutions are governed by the Pfaffian system derived in the previous section and, thereby, expressible in terms of the hypergeometric function.
3.1 Lax formalism of $\mathcal{H}_{L,N}$

We begin with a brief review of Lax formalism of $\mathcal{H}_{L,N}$. Consider an $L \times L$ Fuchsian system

$$\frac{\partial \Phi}{\partial z} = A \Phi = \sum_{i=0}^{N+1} A_i \Phi$$

(3.1)

with $N + 3$ regular singularities $\{u_0 = 1, u_1, \ldots, u_N, u_{N+1} = 0, u_{N+2} = \infty\} \subset \mathbb{P}^1$, of which the characteristic exponents at each singularity $z = u_i$, i.e., the eigenvalues of each residue matrix $A_i$, are listed in the following table (Riemann scheme):

| Singularity | Exponents |
|-------------|-----------|
| $u_i$ ($0 \leq i \leq N$) | $(-\theta_i, 0, \ldots, 0)$ |
| $u_{N+1} = 0$ | $(e_0, e_1, \ldots, e_{L-1})$ |
| $u_{N+2} = \infty$ | $(\kappa_0 - e_0, \kappa_1 - e_1, \ldots, \kappa_{L-1} - e_{L-1})$ |

We can, and will, normalize the exponents as $\text{tr} A_{N+1} = \sum_{i=0}^{L-1} e_i = (L-1)/2$ without loss of generality. Assume $\sum_{i=0}^{L-1} \kappa_i = \sum_{i=0}^{N} \theta_i$ (Fuchs’ relation). Such Fuchsian systems as above then turn out to constitute a $2N(L-1)$-dimensional family and, actually, can be written in terms of the accessory parameters $b_n^{(i)}$ and $c_n^{(i)}$ in the following way:

$$A_i = \begin{pmatrix} e_0 & w_{0,1} & \cdots & w_{0,L-1} \\ e_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & w_{L-2,L-1} \\ e_{L-1} & & & \end{pmatrix},$$

$$A_{N+1} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix},$$

where $c_0 = 1$ and $w_{m,n} = -\sum_{i=0}^{N} b_n^{(i)} c_n^{(i)}$. We thus find the relations

$$(\text{tr} A_i = \sum_{n=0}^{L-1} b_n^{(i)} c_n^{(i)}) = -\theta_i \quad \text{and} \quad \sum_{i=0}^{N} b_n^{(i)} c_n^{(i)} = -\kappa_i,$$

(3.2)

the latter of which comes from the diagonal entries of the lower triangular matrix $A_{N+2} = -\sum_{i=0}^{N+1} A_i$. Since $A_{N+1}$ and $A_{N+2}$ are triangular, there still remains the degree of freedom of a similarity transformation by a diagonal matrix. Consequently, the essential number of the accessory parameters is confirmed to be $2N(L-1)$. In fact, they can be realized by the canonical variables $q_n^{(i)}$ and $p_n^{(i)}$ of $\mathcal{H}_{L,N}$; see (3.5) below.

The isomonodromic family of Fuchsian systems of the form (3.1) is described by the integrability condition of the extended linear system, i.e., (3.1) itself and its deformation equations

$$\frac{\partial \Phi}{\partial u_i} = B_i \Phi, \quad B_i = \frac{A_i}{u_i - z} + \frac{1}{u_i} \begin{pmatrix} -\frac{\theta_i}{L} & & \\ & \ddots & \vdots \\ & & -\frac{\theta_i}{L} \end{pmatrix} \quad (1 \leq i \leq N).$$

(3.3)

Here the lower triangular part ($\ast$) of the second term is exactly the same as $A_i$. 

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therefore, (3.1) is clearly reducible. On the other hand, we have
\[ \frac{\partial A}{\partial u_i} - \frac{\partial B_i}{\partial x} + [A, B_i] = 0 \tag{3.4} \]
of (3.1) and (3.3) is equivalent to the polynomial Hamiltonian system \( H_{L,N} \), (1.1), via the change of variables
\[ x_i = \frac{1}{u_i}, \quad q_n^{(i)} = \frac{c_n^{(i)}}{c_n^{(0)}}, \quad \text{and} \quad p_n^{(i)} = -b_n^{(i)}c_n^{(0)}. \tag{3.5} \]

### 3.2 Particular solution of \( H_{L,N} \)

Suppose that \( \kappa_0 - \sum_{i=1}^N \theta_i = 0 \). This condition enables us to restrict \( b_n^{(i)} \) and \( c_n^{(i)} \) to the subvariety
\[ b_0^{(0)} = 0, \quad b_0^{(i)} = -\theta_i \quad (i \neq 0), \]
\[ c_1^{(0)} = \cdots = c_{L-1}^{(0)} \quad \text{and} \quad c_n^{(i)} = 0 \quad (i \neq 0, n \neq 0) \]
while keeping consistency of the linear system (3.1) and (3.3). Notice here that this restriction amounts to \( q_n^{(0)} = 0 \). In view of (3.2) we see that the matrices \( A_i \) and \( B_i \) can be parameterized by the \( N(L-1) + 1 \) variables \( f := 1/c_1^{(0)} = \cdots = 1/c_{L-1}^{(0)} \) and \( \theta_i^{(i)} \) \((i \neq 0, n \neq 0)\). It actually follows that
\[ A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\kappa_0 f & -\kappa_1 & \cdots & -\kappa_1 \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa_{L-1} f & -\kappa_{L-1} & \cdots & -\kappa_{L-1} \end{pmatrix}, \quad A_i = \begin{pmatrix} -\theta_i & 0 & \cdots & 0 \\ b_1^{(i)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{L-1}^{(i)} & 0 & \cdots & 0 \end{pmatrix} \quad (1 \leq i \leq N), \]
\[ A_{N+1} = \begin{pmatrix} e_0 & 0 & 0 & \cdots & 0 \\ e_1 & \kappa_1 & \kappa_1 & \cdots & \kappa_1 \\ e_2 & \kappa_2 & \cdots & \kappa_2 \\ \vdots & \vdots & \ddots & \vdots \\ e_{L-1} & \kappa_{L-2} & \cdots & \cdots & \kappa_{L-1} \end{pmatrix}. \]

Therefore, (3.1) is clearly reducible. On the other hand, we have
\[ B_i = \frac{\theta_i}{Lu_i} \begin{pmatrix} 1 & L \\ 1 & \vdots \end{pmatrix} + \frac{z}{u_i(u_i - z)} \begin{pmatrix} -\theta_i & 0 & \cdots & 0 \\ b_1^{(i)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{L-1}^{(i)} & 0 & \cdots & 0 \end{pmatrix}. \]
Observe that in this situation only the \((n, 0)\)-entries \((1 \leq n \leq L - 1)\) of the matrix equation \((3.4)\) are nontrivial. In fact,

\[
\begin{align*}
\left( \frac{\partial A}{\partial u_{i}} - \frac{\partial B_{i}}{\partial z} \right)_{n, 0} &= -\kappa_{n} \frac{\partial f}{\partial u_{i}} + \sum_{j=1}^{N} \frac{1}{z - u_{j}} \frac{\partial b_{n}^{(j)}}{\partial u_{i}}, \\
[A, B_{i}]_{n, 0} &= \frac{\kappa_{n}}{u_{i}(z - 1)(z - u_{i})} \left( -\theta_{i} u_{i} f + z \sum_{m=1}^{L-1} b_{m}^{(i)} \right) \\
&\quad + \frac{1}{u_{i}(z - u_{i})} \left( (e_{0} - e_{n}) b_{n}^{(i)} - \kappa_{n} \sum_{m=n+1}^{L-1} b_{m}^{(i)} \right) \\
&\quad + \frac{1}{u_{i}(z - u_{i})} \sum_{j=1}^{N} \theta_{j} b_{n}^{(j)} \frac{u_{i} - \theta_{j} b_{n}^{(j)} z}{z - u_{j}}.
\end{align*}
\]

Residue calculus at \(z = 1, z = u_{j} (j \neq i)\) and \(z = u_{i}\) yields the system of linear differential equations for unknowns \(f\) and \(b_{n}^{(i)}\):

\[
\begin{align*}
u_{i}(1 - u_{i}) \frac{\partial f}{\partial u_{i}} &= -u_{i} \theta_{i} f + \sum_{m=1}^{L-1} b_{m}^{(i)}, \quad (3.6a) \\
(u_{i} - u_{j}) \frac{\partial b_{n}^{(i)}}{\partial u_{i}} &= \theta_{i} b_{n}^{(i)} - \frac{u_{i}}{u_{i}} \theta_{j} b_{n}^{(i)}, \quad (3.6b) \\
\frac{\partial b_{n}^{(i)}}{\partial u_{i}} &= \frac{1}{u_{i}} \left( (e_{n} - e_{0} + \theta_{i}) b_{n}^{(i)} + \kappa_{n} \sum_{m=n+1}^{L-1} b_{m}^{(i)} \right) + \frac{\kappa_{n}}{u_{i} - 1} \left( \theta_{i} f - \sum_{m=1}^{L-1} b_{m}^{(i)} \right) \\
&\quad - \sum_{j=1}^{N} \frac{\theta_{j} b_{n}^{(j)} - \theta_{j} b_{n}^{(i)}}{u_{i} - u_{j}}. \quad (3.6c)
\end{align*}
\]

If we apply the change of variables

\[
x_{i} = \frac{1}{u_{i}}, \quad y_{0} = \frac{f}{\prod_{j=1}^{N} u_{j}^{\theta_{j}}}, \quad \text{and} \quad y_{n}^{(i)} = \frac{b_{n}^{(i)}}{\theta_{i} \prod_{j=1}^{N} u_{j}^{\theta_{j}}},
\]

then \((3.6)\) is converted into the Pfaffian system for \(F_{L, N}\) (see \((2.5)\) in Theorem 2.2) with \(\alpha_{n} = e_{n} - e_{0}, \beta_{n} = -\theta_{n}\) and \(\gamma_{n} = e_{n} - e_{0} - \kappa_{n}\). Combining this fact with Theorem 3.1, we finally arrive at the

**Theorem 3.2.** When \(\kappa_{0} - \sum_{i=1}^{N} \theta_{i} = 0\), the Hamiltonian system \(H_{L, N}\) admits a particular solution of the form

\[
q_{n}^{(i)} = 0, \quad p_{n}^{(i)} = -\theta_{i} \frac{y_{n}^{(i)}}{y_{0}} \quad \left( \begin{array}{c} 1 \leq i \leq N \\ 1 \leq n \leq L - 1 \end{array} \right)
\]

where \(\{y_{0}, y_{n}^{(i)}\}\) is an arbitrary solution of the linear Pfaffian system, \((2.5)\) or \((2.6)\), with

\[
\alpha_{n} = e_{n} - e_{0}, \quad \beta_{n} = -\theta_{n}, \quad \gamma_{n} = e_{n} - e_{0} - \kappa_{n}.
\]
Remark 3.3. We already know that \( y_0 \) is a solution of the hypergeometric equation (1.3). Moreover, it is possible to write all the other elements \( y_n^{(i)} \) as linear combinations of derivatives of \( y_0 \). In fact, we can carry out it by the differential operators appearing in the contiguity relations for the hypergeometric series \( F_{L,N} \); cf. Sect. 2 and the appendix below.

Remark 3.4. Particular solutions of the Schlesinger system for the case of a general spectral type have been studied by Dubrovin–Mazzocco [5]. For instance they showed that if the monodromy group of the associated Fuchsian system is triangular, then the Schlesinger system can be solved in terms of solutions of linear differential equations (that are generally inhomogeneous).

A Contiguity relations for \( F_{L,N} \)

In this appendix we provide a table of the contiguity relations for \( F = F_{L,N}(\alpha, \beta, \gamma; x) \). We shall use again such an abbreviation as \( F(\alpha_n + 1) \) to represent the same function as \( F \) except increasing the indicated parameter \( \alpha_n \) by one. Recall the notation (1.4) of the Euler operators.

Theorem A.1. The hypergeometric function \( F = F_{L,N}(\alpha, \beta, \gamma; x) \) satisfies the contiguity relations

\[
F(\alpha_n + 1) = \frac{D + \alpha_n}{\alpha_n} F, \quad (A.1)
\]

\[
F(\beta_i + 1) = \frac{\delta_i + \beta_i}{\beta_i} F, \quad (A.2)
\]

\[
F(\gamma_n + 1) = \gamma_n \frac{\partial}{\partial x} \left[ \left( \frac{\partial}{\partial x} + \gamma_n \right) - \sum_{j=0}^{L-1} \epsilon_j \left( \frac{\partial}{\partial x} + \gamma_n \right)^{j-1} \right] F, \quad (A.3)
\]

\[
F(\alpha_n - 1) = \frac{\alpha_n - 1}{\epsilon' L} \left[ \sum_{j=1}^{N} x_j (\delta_i + \beta_i) \left( \frac{\partial}{\partial x} + \alpha_k \right) - \sum_{j=0}^{L-1} \epsilon'_j \left( \frac{\partial}{\partial x} + \alpha_n - 1 \right)^{j-1} \right] F, \quad (A.4)
\]

\[
F(\gamma_n - 1) = \frac{D + \gamma_n - 1}{\gamma_n - 1} F, \quad (A.5)
\]

\[
F(\beta_i + 1, \beta_j - 1) = \frac{(x_i - x_j) \frac{\partial}{\partial x} + \beta_i}{\beta_i} F, \quad (A.6)
\]

\[
F(\alpha_1 + 1, \ldots, \alpha_{L-1} + 1, \beta_i + 1, \gamma_1 + 1, \ldots, \gamma_{L-1} + 1) = \frac{\gamma_1 \cdots \gamma_{L-1}}{\alpha_1 \cdots \alpha_{L-1} \beta_i} \frac{\partial F}{\partial x_i}, \quad (A.7)
\]

Here each \( \epsilon_j \) denotes the \( j \)-th elementary symmetric polynomial in \( L \) variables \( \alpha_k - \gamma_n \) (\( k = 1, \ldots, L - 1 \)) and \( \sum_{i=1}^{N} \beta_i - \gamma_n \), and similarly \( \epsilon'_j \) does that in \( \gamma_k - \alpha_n \) (\( k = 1, \ldots, L - 1 \)) and \( 1 - \alpha_n \).

Proof. Notice the formula \( \delta_i x_i = x_i (\delta_i + 1) \) with \( x_i \) regarded as the operator multiplying \( x_i \). Then (A.1), (A.2), (A.5) and (A.7) are immediate from the definition (1.2) of the hypergeometric series. By an analogue of the classical factorization method, (A.3), (A.4) and (A.6) can be obtained. \( \square \)

Let \( S = S(\alpha, \beta, \gamma) \) be the linear space of solutions of the hypergeometric equation (1.3). In general, the linear operators appearing in Theorem A.1 induce isomorphisms of these spaces. For
instance, let $H$ and $B$ be the differential operators defined by

$$H = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \prod_{k=1, k \neq n}^{L-1} (\mathcal{D} + \gamma_k - 1) - \sum_{j=0}^{L-1} \varepsilon_j (\mathcal{D} + \gamma_n)^{L-1-j}, \quad B = \mathcal{D} + \gamma_n;$$

cf. (A.3) and (A.5). The linear homomorphisms

$$H : S \to S(\gamma_n + 1), \quad B : S(\gamma_n + 1) \to S$$

are isomorphisms if and only if $\gamma_n \varepsilon_L \neq 0$.

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