TOEPLITZ CORONA THEOREMS FOR THE POLYDISK AND THE UNIT BALL

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The main purpose of this paper is to extend and refine some work of Agler-McCarthy [1] and Amar [2] concerning the Corona problem for the polydisk and the unit ball in $\mathbb{C}^n$. In 1962, Carleson [4] proved his famous Corona theorem with bounds:

**Corona Theorem.** Let $\{f_j\}_{j=1}^m \subseteq H^\infty(D)$. Assume that

$$0 < \epsilon^2 \leq \sum_{j=1}^m |f_j(z)|^2 \leq 1 \text{ for all } z \in D.$$

There exists a constant $C(\epsilon, m) < \infty$ and $\{g_j\}_{j=1}^m \subseteq H^\infty(D)$, so that

$$\sup_{z \in D} \sum_{j=1}^m |g_j(z)|^2 \leq C(\epsilon, m)^2 \text{ and } \sum_{j=1}^m f_j(z)g_j(z) = 1 \text{ for all } z \in D.$$

The Corona theorem and especially the techniques utilized in its proof have been very influential. See, for example, Garnett [7]. Among many questions raised by this theorem, we wish to consider the analogous Corona problem for the polydisk and the unit ball in $\mathbb{C}^n$.

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We will need some notation:

\[ D^n = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1 \text{ for } j = 1, \ldots, n \} \]

\[ B^n = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1 \} \]

\( \Omega \) denotes either \( D^n \) or \( B^n \)

\( \partial \Omega \) denotes \( \partial B^n \) if \( \Omega = B^n \) or else, if \( \Omega = D^n \), \( \partial \Omega \) denotes \( T^n \), the distinguished boundary of \( D^n \)

\( \sigma \) denotes normalized Lebesgue measure on \( \partial \Omega \)

\( \mu \) denotes a probability measure on \( \partial \Omega \)

\[ P^2(\mu) = \{ \text{analytic polynomials on } \mathbb{C}^n \} - L^2(\mu) \]

\( F(z) = (f_1(z), f_2(z), \ldots) \) for \( f_j \in H^\infty(\Omega) \)

\( \mathcal{F}(z) = [f_{ij}(z)]_{i,j=1}^\infty \) for \( f_{ij} \in H^\infty(\Omega) \)

\( T^\mu_F \) the multiplication operator from \( \bigoplus_1^\infty P^2(\mu) \) into \( P^2(\mu) \) by \( F \)

\( T^\mu_F \) the multiplication operator from \( \bigoplus_1^\infty P^2(\mu) \) into \( \bigoplus_1^\infty P^2(\mu) \) by \( \mathcal{F} \)

\( I_\mu \) the identity operator on \( P^2(\mu) \)

\( \mathcal{H} = \{ H \in H^\infty(\Omega) : H \text{ nonvanishing in } \Omega, \frac{1}{H} \in L^\infty(\partial \Omega, d\sigma), \text{ and } \|H\|_2 = 1 \} \)

\( T^H_F \) defined as above, when \( d\mu = |H|^2 d\sigma \) and \( H \in \mathcal{H} \)

For the case of the bidisk, \( D^2 \), Agler and McCarthy proved the following:

**Theorem (Agler and McCarthy).** Let \( \{f_j\}_{j=1}^m \subseteq H^\infty(D^2) \). Then there exist \( \{g_j\}_{j=1}^m \subseteq H^\infty(D^2) \) with

\[ \sum_{j=1}^m f_j g_j \equiv 1 \quad \text{and} \quad \sup_{z \in D^2} \sum_{j=1}^m |g_j(z)|^2 \leq \frac{1}{\delta^2} \]

if and only if

\[ T^\mu_F(T^\mu_F)^* \geq \delta^2 I_\mu \]

for all probability measures \( \mu \) on \( T^2 \).

Although the Agler-McCarthy theorem and its proof seemed to be restricted to \( n = 2 \) by the classical and beautiful counterexample of
Parrot [10]; nevertheless, Amar managed to extend it to $D^n$ (and to $B^n$).

**Theorem (Amar).** Let $\{f_j\}_{j=1}^m \subseteq H^\infty(\Omega)$. Then there exist $\{g_j\}_{j=1}^m \subseteq H^\infty(\Omega)$ with

$$\sum_{j=1}^m f_j g_j \equiv 1 \quad \text{and} \quad \sup_{z \in \Omega} \sum_{j=1}^m |g_j(z)|^2 \leq \frac{1}{\delta^2}$$

if and only if

$$T^\mu_F(T^\mu_F)^* \geq \delta^2 I_\mu$$

for all probability measures $\mu$ on $\partial\Omega$.

In other words, Amar shows that for $\{f_j\}_{j=1}^m \subseteq H^\infty(\Omega)$ and $\delta > 0$ the following are equivalent:

(i) There exist $\{g_j\}_{j=1}^m \subseteq H^\infty(\Omega)$ with

$$\sum_{j=1}^m f_j g_j = 1 \quad \text{on } \Omega \quad \text{and} \quad \sup_{z \in \Omega} \sum_{j=1}^m |g_j(z)|^2 \leq \frac{1}{\delta^2}.$$

(ii) For all probability measures $\mu$ on $\partial\Omega$ and all $h \in P^2(\mu)$ there exists $\{k_j\}_{j=1}^m \subseteq H^\infty(\Omega)$

$$\sum_{j=1}^m f_j k_j = h \quad \text{and} \quad \sum_{j=1}^m \|k_j\|_\mu^2 \leq \frac{1}{\delta^2} \|h\|_\mu^2.$$

By results of Andersson-Carlsson [3] for the unit ball and Varopoulos [17], Li [8], Lin [9], Trent [16], and Treil-Wick [15] for the polydisk case, we know that if the input functions are bounded away from 0 on $\Omega$, we have an $H^p(\Omega)$ Corona theorem for $1 \leq p < \infty$. That is, if

$$\{f_j\}_{j=1}^\infty \subseteq H^\infty(\Omega) \quad \text{and} \quad 0 < \epsilon^2 \leq \inf_{z \in \Omega} \sum_{j=1}^\infty |f_j(z)|^2 \leq 1,$$

then for $1 \leq p < \infty$ there exists a $\delta_p > 0$ so that

$$T_F T_F^* \geq \delta_p^2 I_{H^p(\Omega)},$$

where $F = (f_1, f_2, \ldots)$. Unfortunately, the best of these estimates have $\delta_p \downarrow 0$ as $p \uparrow \infty$.

Thus Amar’s theorem tells us that a solution to the Corona problem for $H^\infty(\Omega)$ follows from the following statement:

$$T_F T_F^* \geq \delta^2 I, \quad \text{for } \delta > 0 \Rightarrow \exists \epsilon > 0 \quad \text{such that} \quad T^\mu_F(T^\mu_F)^* \geq \epsilon^2 I_\mu$$
for all probability measures $\mu$ on $\partial \Omega$.

Of course, necessity in Amar’s theorem is trivial; so we will concentrate on weakening the sufficient conditions to get the same Corona output.

We will extend Amar’s theorem to an infinite number of input functions and refine his theorem, so that we need only consider probability measures, $\mu$, of the form $|H|^2 \, d\sigma$, where $H \in \mathcal{H}$. In addition, we weaken the hypotheses to just have our operators dominate a certain rank one operator. We begin with a series of lemmas.

**Lemma 1.** Let $\mathcal{F}(z) = [f_{ij}(z)]_{i,j=1}^\infty$, $f_{ij} \in H^\infty(\Omega)$. Then

$$\|T_\mathcal{F}\|_{B(\bigoplus_1 H^2(\Omega))} = \sup_{z \in \Omega} \|\mathcal{F}(z)\|_{B(l^2)}.$$ 

**Proof.** Let $h \in \bigoplus_1 H^2(\Omega)$. Then

$$\|T_\mathcal{F}h\|_{\bigoplus_1 H^2(\Omega)}^2 = \sup_{0 \leq r < 1} \left( \int_{\partial \Omega} \|\mathcal{F}(re^{it})h(re^{it})\|_{l^2}^2 \, d\sigma \right) \leq \sup_{z \in \Omega} \|\mathcal{F}(z)\|_{B(l^2)}^2 \sup_{0 \leq r < 1} \left( \int_{\partial \Omega} \|h(re^{it})\|_{l^2}^2 \, d\sigma \right) \leq \sup_{z \in \Omega} \|\mathcal{F}(z)\|_{B(l^2)}^2 \|H\|_{\bigoplus_1 H^2(\Omega)}^2.$$ 

For $x \in \text{Ball}_1(l^2)$ and $z \in \Omega$

$$\left\|T_\mathcal{F}^* \left( \frac{k_z}{\|k_z\|_{H^2(\Omega)}} x \right) \right\|_{\bigoplus_1 H^2(\Omega)}^2 = \left\|\mathcal{F}(z)^* x \frac{k_z}{\|k_z\|_{H^2(\Omega)}} \right\|_{\bigoplus_1 H^2(\Omega)}^2 = \left\|\mathcal{F}(z)^* x \right\|_{l^2}^2.$$

Thus,

$$\|T_\mathcal{F}\| = \|T_\mathcal{F}^*\| \geq \sup_{z \in \Omega} \sup_{x \in \text{Ball}_1(l^2)} \|\mathcal{F}(z)^* x\|_{l^2}^2 \geq \sup_{z \in \Omega} \|\mathcal{F}(z)^*\|_{B(l^2)} = \sup_{z \in \Omega} \|\mathcal{F}(z)\|. \qedhere$$

For a Hilbert space, $K$, and vectors $x, y, h \in K$, we let $x \otimes y$ denote the rank one operator defined on $K$ by

$$(x \otimes y)(h) = \langle h, y \rangle x.$$
The next lemma will be used repeatedly with $A = T^H_F$ and $k = H$, for $H \in \mathcal{H}$.

**Lemma 2.** Assume that for $A \in B(K)$ and $k \in K$ with $\|k\|_K = 1$, $AA^* \geq \delta^2 k \otimes k$. Then there exists $u_k \in (\text{Ker} A)^\perp$, so that $Au_k = k$ and $\|u_k\|_K \leq \frac{1}{\delta}$.

**Proof.** By the Douglas Range Inclusion Theorem, see [5], there exists $s \in B(H, \text{Ker} A^\perp)$ such that $A = k \otimes k$ and $\|s\| \leq \frac{1}{\delta}$. Let $u_k = Ck$. □

**Lemma 3.** For $f$ a positive, bounded, lower semi-continuous function on $\partial \Omega$, there exists a nonvanishing $H \in H^\infty(\Omega)$, so that $f = |H|^2$ $\sigma$-a.e. on $\partial \Omega$.

**Proof.** For $\Omega = D^n$ and $\partial \Omega = T^n$, this is a result of Rudin [12]. For $\Omega = B^n$ and $\partial \Omega = \partial B^n$, this is a theorem of Alexandrov (see Rudin [13], p. 32). □

Recall that $\mathcal{H} \triangleq \{H \in H^\infty(\Omega) : H \text{ nonvanishing in } \Omega, \frac{1}{H} \in L^\infty(\partial \Omega, d\sigma), \text{ and } \|H\|_2 = 1\}$.

For $\{a_j\}_{j=1}^\infty$ a fixed countable dense set in $\Omega$ with $a_1 = 0$, define for each $N = 1, 2, \ldots$

$$C_N \triangleq \text{co} \{\frac{|k_{a_j}|^2}{\|k_{a_j}\|^2_2} : j = 1, \ldots, N\}$$

Here $k_a(\cdot)$ is the reproducing kernel for $H^2(\Omega)$. It is clear that $C_N$ is compact and convex in $L^1(\partial \Omega, d\sigma)$.

Calculating, we see that for $\Omega = D^n$ and $g \in C_N$, we have

$$0 < \left(\frac{1 - \|a\|}{1 + \|a\|}\right)^n \leq g(z) \leq \left(\frac{1 + \|a\|}{1 - \|a\|}\right)^n < \infty \text{ for all } z \in \Omega,$$

where $\|a\| = \max \{\|a_j\| : j = 1, \ldots, N\}$.

For $\Omega = B^n$ and $g \in C_N$, we have

$$0 < \left(\frac{1 - \|a\|}{1 + \|a\|}\right)^n \leq g(z) \leq \left(\frac{1 + \|a\|}{1 - \|a\|}\right)^n < \infty \text{ for all } z \in \Omega,$$

where $\|a\| = (\sum_{j=1}^N \|a_j\|^2_2)^{\frac{1}{2}}$.

Note that for $g \in C_N$, the above calculation shows that, as sets, $P^2(g \, d\sigma)$ equals $H^2(\Omega)$.

Assume that $T_FT_F^* \geq \delta^2 I_1 \otimes I_1$ and choose $x \in \bigoplus_1^\infty H^2(\Omega)$
so that \( T_F x = 1 \) and \( \|x\|_2 \leq \frac{1}{\delta} \).

For \( N = 1, 2, \ldots \) define
\[
\mathcal{F}_N : C_N \times \bigoplus_1 H^2(\Omega) \to [0, \infty)
\]
by
\[
\mathcal{F}_N(g, \mathbf{a}) \triangleq \int_{\partial \Omega} \|\mathbf{x} - P_{\text{Ker}(T_F)} \mathbf{a}\|_2^2 g \, d\sigma
\]
for \( g \in C_N \) and \( \mathbf{a} \in \bigoplus_1 H^2(\Omega) \).

Since \( g \in C_N \),
\[
\mathbf{x} - P_{\text{Ker}(T_F)} \mathbf{a} \in \bigoplus_1 H^2(\Omega) \text{ and } \mathcal{F}_N(g, \mathbf{a}) \text{ is finite and positive.}
\]

For fixed \( \mathbf{a} \in \bigoplus_1 L^2(d\sigma) \), \( g \mapsto \mathcal{F}_N(g, \mathbf{a}) \) is linear and thus concave on the compact convex set \( C_N \). For fixed \( g \in C_N \), \( \mathbf{a} \mapsto \mathcal{F}(g, \mathbf{a}) \) is convex and continuous on \( \bigoplus_1 H^2(\Omega) \).

**Lemma 4.** Assume that \( T_F T_F^* \geq \delta^2 1 \otimes 1 \). For each \( N = 1, 2, \ldots \),
\[
\inf_{\mathbf{a} \in \bigoplus_1 H^2(\Omega)} \sup_{g \in C_N} \mathcal{F}_N(g, \mathbf{a}) = \sup_{g \in C_N} \inf_{\mathbf{a} \in \bigoplus_1 H^2(\Omega)} \mathcal{F}_N(g, \mathbf{a}).
\]

**Proof.** By our remarks above, we may apply von Neumann’s minimax theorem. See, for example, Gamelin [6]. \( \square \)

We are now ready to present our extension of Amar’s theorem.

**Theorem 1.** Assume that for some \( \delta > 0 \), \( T_F^H (T_F^H)^* \geq \delta^2 H \otimes H \) for all \( H \in \mathcal{H} \). Then there exists a \( G \in \bigoplus_1 H^2(\Omega) \) with
\[
F G \equiv 1 \text{ in } \Omega \text{ and } \sup_{z \in \Omega} \|G(z)\|_H \leq \frac{1}{\delta}.
\]
That is, \( T_F T_G^* \equiv I \text{ in } H^2(\Omega) \).

**Proof.** Since \( T_F T_F^* \geq \delta^2 1 \otimes 1 \), we may choose \( x \in \bigoplus_1 H^2(\Omega) \) so that \( T_F x = 1 \) and \( \|x\|_2 \leq \frac{1}{\delta} \).

Fix any positive integer, \( N \), and any \( g \in C_N \). By Lemma 3, we may find an \( H \in \mathcal{H} \), so that \( |H| = g \) \( \sigma\)-a.e. on \( \partial \Omega \).
By our assumption
\[ T^H_F(T^H_F)^* \geq \delta^2 H \otimes H, \]
so there exists an \( \overline{x}_H \in \bigoplus^\infty_1 H^2(\Omega) \) with
\[ T^H_F(\overline{x}_H) = 1 \text{ and } \|\overline{x}_H\|_{2, g \, d\sigma} \leq \frac{1}{\delta}. \] (1)

Since \( x - \overline{x}_H \in \text{Ker}(T_F) \), we have \( \overline{x}_H - x = P_{\text{Ker}(T_F)} \alpha \) for \( \alpha = x - \overline{x}_H \).
Thus (1) says that
\[ \int_{\partial \Omega} \|x - P_{\text{Ker}(T_F)} \alpha\|^2 g \, d\sigma = F_N(g, \alpha) \leq \frac{1}{\delta^2}. \]
Since this is true for every \( g \in C_N \), we may apply the minimax theorem, Lemma 4, and deduce that
\[ \inf_{\alpha \in \bigoplus^\infty_1 H^2(\Omega)} \sup_{g \in C_N} F_N(g, \alpha) \leq \frac{1}{\delta^2}. \] (2)

Then using (2), choose \( \alpha \in \bigoplus^\infty_1 H^2(\Omega) \) so that
\[ \int_{\partial \Omega} \|x - P_{\text{Ker}(T_F)} \alpha\|^2 g \, d\sigma \leq \left( \frac{1}{\delta^2} + \frac{1}{N} \right) \text{ for all } g \in C_N. \] (3)

Since \( \frac{|k_{a_j}|^2}{\|k_{a_j}\|^2} \in C_N \) for \( j = 1, 2, \ldots, N \), we see that if
\[ G^{(N)} \triangleq x - P_{\text{Ker}(T_F)} \alpha, \]
then
(a) \( \|G^{(N)}(a_j)\|^2 \leq \int_{\partial \Omega} \|G^{(N)}\|^2 \frac{|k_{a_j}|^2}{\|k_{a_j}\|^2} d\sigma \leq \frac{1}{\delta^2} + \frac{1}{N} \), for \( j = 1, 2, \ldots, N \)
(b) \( \|G^{(N)}\|^2 \leq \frac{1}{\delta^2} + \frac{1}{N} \), and
(c) \( F_{G^{(N)}} \equiv 1 \text{ in } \Omega. \)

Repeating this argument for each \( N = 1, 2, \ldots, \), we get a sequence of elements, \( G^{(N)} \in \bigoplus^\infty_1 H^2(\Omega) \), satisfying (a), (b), and (c).

By relabeling the sequence of elements, \( \{G^{(N)}\} \), if necessary, let \( G \) be a weak limit of \( \{G^{(N)}\}_{N=1}^\infty \) in \( \bigoplus^\infty_1 H^2(\Omega) \). Fix any \( a_p \in \{a_j\}_{j=1}^\infty \). Then
\[ \|G(a_p)\|_{l^2} = \lim_{N \to \infty} \|G^{(N)}(a_j)\|_{l^2} \leq \frac{1}{\delta} \text{ by (b)}. \]
Since $G$ is continuous in $\Omega$ and $\{a_j\}_{j=1}^\infty$ is dense in $\Omega$, we have shown that
\[ \sup_{z \in \Omega} \|G(z)\|_{l^2} \leq \frac{1}{\delta}. \]

By (c),
\[ I = \text{weak limit } \lim_{N \to \infty} \int T_F(G^{(N)}_N) = T_F(G). \]

Thus, by Lemma 1, $T_F T_G = I$. This completes the proof of Theorem 1. \(\square\)

For the next theorem, we need the fact that $\text{Ker}(T_F) = \text{Ran}(T_F)$, for an appropriate analytic $F$. For $\Omega = B^n$, the unit ball in $\mathbb{C}^n$, the fact that $\text{Ker}(T_F) = \text{Ran}(T_F)$ follows from results of Andersson and Carlsson [3]. For $\Omega = D^2$, $\text{Ker}(T_F) = \text{Ran}(T_F)$ follows from Taylor spectrum results of Putinar [11]. That $\text{Ker}(T_F) = \text{Ran}(T_F)$ in the general case, $\Omega = D^n$, follows from an extension of the techniques of Trent [16] and will appear in a forthcoming paper concerning the Taylor spectrum of $T_F$.

The following shows that the Corona theorem for the polydisk or unit ball, reduces to an estimation of a lower bound for $T^*_F(T_F^*)^*H_{\infty} \otimes H_{\infty}$ where $H \in H$, but $H$ is not cyclic for $H^2(\Omega)$. (Note that we always have $\frac{1}{\Pi} \in L^\infty(\partial \Omega, d\sigma)$.)

**Theorem 2.** For $H \in H$ and $H$ cyclic in $H^2(\Omega)$, then
\[ T_F T_F^* \geq \delta^2 1 \otimes 1 \Rightarrow T^*_F(T^*_F)^* \geq \delta^2 H \otimes H. \]

**Proof.** To show that, when $H$ is cyclic, $T^*_F(T^*_F)^* \geq \delta^2 H \otimes H$, it suffices to find a $u_H \in \bigoplus_1^\infty H^2(\Omega)$, satisfying
\[ F u_H = 1 \quad \text{(so } F(H u_H) = H) \]
and
\[ \|H u_H\|_{H^2(\Omega)} \leq \frac{1}{\delta} \|H\|_{H^2(\Omega)} = \frac{1}{\delta}. \]  

Let $x = T_F(T_F^*)^{-1} 1$. Then such a $u_H$ must have the form $u_H = x - P_{\text{Ker}(T_F)} u_H$ for some $u_H \in \bigoplus_1^\infty H^2(\Omega)$. To see that such an $u_H$ exists, satisfying (4), we compute
\[ \inf_{\alpha \in \bigoplus_1^\infty H^2(\Omega)} \int_{\partial \Omega} \|x - P_{\text{Ker}(T_F)} \alpha\|^2_{l^2} |H|^2 d\sigma \]
\[ = \inf_{\alpha \in \bigoplus_1^\infty H^2(\Omega)} \int_{\partial \Omega} \|x H - T_F(H \alpha)\|^2_{l^2} d\sigma \quad \text{(since }\text{Ker}(T_F) = \text{Ran}(T_F)) \]
\begin{align*}
&= \inf_{\beta \in H^2(\Omega)} \int_{\partial \Omega} \|xH - T_F(\beta)\|_2^2 \, d\sigma \quad \text{(since } H \text{ is cyclic)} \\
&= \|P_{\text{ran}(T_F)}(xH)\|_2^2 \quad \text{(since } \text{ ran}(T_F) = \text{ Ker}(T_F)) \\
&= \|P_{\text{ker}(T_F)}(xH)\|_2^2 \quad \text{(since } \text{ ran}(T_F) = \text{ Ker}(T_F)) \\
&= \|T_F(T_F^*T_F)^{-1}T_FHT_F(T_F^*T_F)^{-1}1\|_2^2 \quad \text{in } H^2(\Omega) \\
&\leq \frac{1}{\delta^2} \|H\|_{H^2(\Omega)} = \frac{1}{\delta^2}.
\end{align*}

□

In the case that \( n = 1 \), we may choose \( H \) in Lemma 3 to be outer and thus cyclic for \( H^2(D) \). So Carleson’s corona theorem for \( H^\infty(D) \) follows from Theorems 1 and 2.

A very natural and interesting question arises from our work. Thanks to Treil’s remarkable example \[14\], we know that for an analytic \( \mathcal{F} = [f_{ij}]_{i,j=1}^\infty \) with

\[ e^2 I_{l^2} \leq \mathcal{F}(z)\mathcal{F}(z)^* \leq I_{l^2} \quad \text{for all } z \in \Omega, \]

there does not necessarily exist an analytic \( \mathcal{G} = [g_{ij}]_{i,j=1}^\infty \) with

\[ \mathcal{F}(z)\mathcal{G}(z) = I_{l^2} \quad \text{for all } z \in \Omega \]

and

\[ \sup_{z \in \Omega} \|\mathcal{G}(z)\|_{B(l^2)} < \infty. \]

How do we know when such a \( \mathcal{G} \) must exist? For the case of the unit disk, \( D \), it is necessary and sufficient that there exist a \( \delta > 0 \) with

\[ \delta^2 I \leq T_F T_F^*. \]

For the polydisk and ball in \( \mathbb{C}^n \), a natural question is: Does \( T_F^H T_F^{H^*} \geq \delta^2 I_H \) for some \( \delta > 0 \) and for all \( H \in \mathcal{H} \) imply the existence of a bounded analytic Toeplitz operator \( T_G \) with

\[ T_F T_G = I_{l^2}? \]
For $T(z)$, a $q \times \infty$ matrix with $q < \infty$, a modification of our techniques works, but we only get an estimate

$$\|T_\psi\| \leq \frac{q}{\delta}.$$

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