Abstract. Let \( \{q_n^{(\alpha,\beta,m)}(x)\}_{n \geq 0} \) be the orthonormal polynomials respect to the Sobolev-type inner product
\[
\langle f, g \rangle_{\alpha,\beta,m} = \sum_{k=0}^{m} \int_{-1}^{1} f^{(k)}(x)g^{(k)}(x) \, dw_{\alpha+k,\beta+k}(x), \quad \alpha, \beta > -1, \quad m \geq 1,
\]
where \( dw_{a,b}(x) = (1-x)^a(1+x)^b \, dx \). We obtain necessary and sufficient conditions for the uniform boundedness of the partial sum operators related to this sequence of polynomials in the Sobolev space \( W^{p,m}_{\alpha,\beta} \). As a consequence we deduce the convergence of such partial sums in the norm of \( W^{p,m}_{\alpha,\beta} \).

1. Introduction

The study of orthogonal polynomials with respect to the Sobolev-type inner product
\[
\langle f, g \rangle = \sum_{k=0}^{m} \int_{-1}^{1} f^{(k)}(x)g^{(k)}(x) \, d\mu_k,
\]
has attracted the interest of many researchers in the last years (see, for example, the survey [9] and the references therein). In this paper we contribute to that study with the analysis of the Fourier series in terms of orthonormal polynomials associated with a particular Sobolev-type inner product. Specifically, for each \( m \in \mathbb{N} \setminus \{0\} \), we consider the inner product
\[
\langle f, g \rangle_{\alpha,\beta,m} = \sum_{k=0}^{m} \int_{-1}^{1} f^{(k)}(x)g^{(k)}(x) \, d\mu_{\alpha+k,\beta+k}(x), \quad \alpha, \beta > -1,
\]
where \( d\mu_{a,b}(x) = (1-x)^a(1+x)^b \, dx \). We exclude the case \( m = 0 \) of our analysis because it corresponds with the classical inner product related to Jacobi polynomials.

By using the Rodrigues formula, the Jacobi polynomials \( \{P_n^{(\alpha,\beta)}(x)\}_{n \geq 0} \) are defined as
\[
P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha}(1+x)^{-\beta} \, \frac{d^n}{dx^n} \left( (1-x)^{n+\alpha} (1+x)^{\beta+n} \right).
\]

2010 Mathematics Subject Classification. Primary: 42A20. Secondary: 33C47.

Key words and phrases. Sobolev-type inner product, Sobolev polynomials, Jacobi polynomials, partial sum operator.

The authors were supported by grant MTM2015-65888-C04-4-P from Spanish Government.
They are orthogonal in the interval \([-1,1]\) with the measure \(d\mu_{\alpha,\beta}\) and then the sequence \(\{p_n^{(\alpha,\beta)}(x)\}_{n \geq 0}\), given by

\[
p_n^{(\alpha,\beta)}(x) = \frac{1}{\|P_n^{(\alpha,\beta)}\|_{L^2([-1,1],d\mu_{\alpha,\beta})}} \sqrt{(2n + \alpha + \beta + 1) n! \Gamma(n + \alpha + \beta + 1)}
\]

is orthonormal and complete in \(L^2([-1,1],d\mu_{\alpha,\beta})\). Moreover, the Jacobi polynomials are eigenfunctions of the second order differential operator

\[
L_{\alpha,\beta} f(x) = (1 - x^2) f''(x) + ((\beta + 1)(1 - x) - (\alpha + 1)(1 + x)) f'(x).
\]

In fact,

\[
L_{\alpha,\beta} p_n^{(\alpha,\beta)}(x) = -\lambda_n^{(\alpha,\beta)} p_n^{(\alpha,\beta)}(x),
\]

with

\[
\lambda_n^{(\alpha,\beta)} = n(n + \alpha + \beta + 1).
\]

The identity (see [16], p. 63, eq. (4.21.7))

\[
\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{n + \alpha + \beta + 1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x),
\]

taking into account that \(u_n^{(\alpha,\beta)} = 2\sqrt{\frac{n}{n+\alpha+\beta+1}} u_{n-1}^{(\alpha+1,\beta+1)}\), implies

\[
\frac{d}{dx} p_n^{(\alpha,\beta)}(x) = \sqrt{\lambda_n^{(\alpha,\beta)}} p_{n-1}^{(\alpha+1,\beta+1)}(x)
\]

and, more generally,

\[
\frac{d^k}{dx^k} p_n^{(\alpha,\beta)}(x) = \sqrt{r_n^{(\alpha,\beta)}} p_{n-k}^{(\alpha+k,\beta+k)}(x)
\]

where

\[
r_n^{(\alpha,\beta)} = \prod_{j=0}^{k-1} \lambda_n^{(\alpha+j,\beta+j)}, \quad k \geq 1,
\]

and \(r_{n,0} = 1\). In this way, the polynomials

\[
q_n^{(\alpha,\beta,m)}(x) = \frac{2_{n}^{(\alpha,\beta)}}{s_{n,m}^{(\alpha,\beta)}},
\]

with

\[
s_{n,m}^{(\alpha,\beta)} = \sum_{k=0}^{m} r_{n,k}^{(\alpha,\beta)},
\]

are orthonormal with respect the Sobolev-type inner product \(\langle \cdot, \cdot \rangle_{\alpha,\beta,m}\); i.e., they satisfy

\[
\langle q_n^{(\alpha,\beta,m)}(x), q_j^{(\alpha,\beta,m)}(x) \rangle_{\alpha,\beta,m} = \delta_{n,j}.
\]

Given \(1 \leq p < \infty\), we will write \(L^p_{\alpha,\beta}\) to denote \(L^p([-1,1],d\mu_{\alpha,\beta})\), the space of all measurable functions on \([-1,1]\) for which

\[
\|f\|_{L^p_{\alpha,\beta}} := \left( \int_{-1}^{1} |f(x)|^p \, d\mu_{\alpha,\beta}(x) \right)^{1/p} < \infty.
\]
For $p = \infty$, we consider the standard definition in terms of essential supremum. We define the space $W_{p,m}^{\alpha,\beta}$, for $1 \leq p < \infty$, as the space of measurable functions $f$ defined on $[-1,1]$ such that there exist $f', f'', \ldots, f^{(m)}$ almost everywhere and

$$
\|f\|_{W_{p,m}^{\alpha,\beta}} := \left( \sum_{k=0}^{m} \|f^{(k)}\|_{L_{p,\alpha+k,\beta+k}^{p}} \right)^{1/p} < \infty.
$$

We denote by $S_{n}^{(\alpha,\beta,m)} f$ the $n$-th partial sum operator as

$$
S_{n}^{(\alpha,\beta,m)} f = \sum_{j=0}^{n} c_{j}^{(\alpha,\beta,m)}(f) q_{j}^{(\alpha,\beta,m)}(x),
$$

where $c_{j}^{(\alpha,\beta,m)}(f) = \langle f, q_{j}^{(\alpha,\beta,m)} \rangle_{\alpha,\beta,m}$ are the Fourier-Jacobi-Sobolev coefficients. Our main result characterizes the uniform boundedness of the operators $S_{n}^{(\alpha,\beta,m)}$ in the spaces $W_{p,m}^{\alpha,\beta}$. In fact, we will prove the following theorem.

**Theorem 1.** Let $f \in W_{p,m}^{\alpha,\beta}$ with $\alpha \geq \beta > -1$, $m \in \mathbb{N} \setminus \{0\}$ and $1 < p < \infty$. Then

$$
\|S_{n}^{(\alpha,\beta,m)} f\|_{W_{p,m}^{\alpha,\beta}} \leq C \|f\|_{W_{p,m}^{\alpha,\beta}}
$$

with a constant $C$ independent of $n$ and $f$, if and only if

$$
\frac{4(\alpha + m + 1)}{2(\alpha + m) + 3} < p < \frac{4(\alpha + m + 1)}{2(\alpha + m) + 1}.
$$

The restriction $\alpha \geq \beta$ is imposed to simplify the proof of the result but we do not lose generality with it. In fact, when $\beta \geq \alpha$ the uniform boundedness (5) holds if and only if

$$
\frac{4(\beta + m + 1)}{2(\beta + m) + 3} < p < \frac{4(\beta + m + 1)}{2(\beta + m) + 1},
$$

and, in general, for $\alpha, \beta > -1$ (5) is verified if and only if

$$
\max \left\{ \frac{4(\alpha + m + 1)}{2(\alpha + m) + 3}, \frac{4(\beta + m + 1)}{2(\beta + m) + 3} \right\} < p < \min \left\{ \frac{4(\alpha + m + 1)}{2(\alpha + m) + 1}, \frac{4(\beta + m + 1)}{2(\beta + m) + 1} \right\}.
$$

The analysis of the Fourier series of Jacobi polynomials has a long history. Pollard in [12] and [13] studied the uniform boundedness of the partial sums for the Fourier series of Gegenbauer and Jacobi polynomials, respectively. A general result including weights for Jacobi expansions can be seen in [10]. In [4], by applying the boundedness with weights of the Hilbert transform, the authors did a complete study of the boundedness of the partial sum operators related to generalized Jacobi weights. The same authors studied the generalized Jacobi weights with mass points on the interval $[-1,1]$ (see [5]).

In [3], the authors gave a complete characterization of the uniform boundedness of the partial sum operators for the Fourier series related to orthonormal polynomials with respect to $\mu_{0}$ where $d\mu_{0} = d\mu_{\alpha} + M(\delta_{1} + \delta_{-1})$, with $d\mu_{\alpha}$ the probability measure corresponding to the Gegenbauer polynomials, $d\mu_{1} = N(\delta_{1} + \delta_{-1})$, and $d\mu_{k} = 0$, $k \geq 2$. That was the first result of this type in the literature. In fact, as it is observed in [9], the main obstacle to analyze this kind of problems is the lack of a Christoffel-Darboux formula for Sobolev orthogonal polynomials. So it is necessary to look for alternative ways to deal with the problem.
In [8], the authors considered the Fourier series for polynomials associated to the Sobolev-type inner product

$$\langle f, g \rangle_S = \int_{-1}^{1} f(x)g(x)w_\alpha(x) \, dx + \int_{-1}^{1} f'(x)g'(x)w_{\alpha+1}(x) \, dx,$$

where $w_\alpha(x) = (1 - x^2)^\alpha$, $x \in [-1, 1]$ and $\alpha > -1$. They analyzed the uniform boundedness of the partial sum operators for the orthogonal polynomials with respect (7) using the Pollard decomposition but, unfortunately, the given results are not completely satisfactory. Theorem 1 gives, as a particular case, necessary and sufficient conditions for this case.

Due to the denseness of the polynomials in the spaces $W^{p,m}_{\alpha,\beta}$ [14], applying the uniform boundedness theorem in the complete space $W^{p,m}_{\alpha,\beta}$, it is verified that (5) is equivalent to the convergence of the partial sums $S_n^{(\alpha,\beta,m)}$ in the spaces $W^{p,m}_{\alpha,\beta}$.

Then, we have the following result.

**Corollary 2.** Let $f \in W^{p,m}_{\alpha,\beta}$ with $\alpha \geq \beta > -1$, $m \in \mathbb{N} \setminus \{0\}$, and $1 < p < \infty$. Then

$$\lim_{n \to \infty} \| S_n^{(\alpha,\beta,m)}f - f \|_{W^{p,m}_{\alpha,\beta}} = 0$$

if and only if

$$\frac{4(\alpha + m + 1)}{2(\alpha + m) + 3} < \frac{4(\alpha + m + 1)}{2(\alpha + m) + 1}.$$

The next section contains the proof of Theorem 1 and it is divided into two subsections, one for the sufficient conditions and the other for the necessary ones. The last section is devoted to the proof of a technical result involved in the proof of Theorem 1.

### 2. Proof of Theorem 1

#### 2.1. Sufficient conditions.

For $n$ big enough, it is clear that

$$S_n^{(\alpha,\beta,m)} f(x) = \sum_{k=0}^{m} S_n^{(\alpha,\beta,k)} f(x),$$

where

$$S_n^{(\alpha,\beta,k)} f(x) = \sum_{j=k}^{n} \sqrt{r_j^{(\alpha,\beta)}} \frac{s_j^{(\alpha,\beta)}}{s_j^{(\alpha,\beta)}} b_j^{(\alpha,\beta,k)}(f(k)) q_j^{(\alpha,\beta,m)}(x)$$

and

$$b_j^{(\alpha,\beta,k)}(f(k)) = \int_{-1}^{1} f(k)(y) p_{j-k}^{(\alpha+k,\beta+k)}(y) \, d\mu_{\alpha+k,\beta+k}(y).$$

To obtain the uniform boundedness (8), it is enough to prove that

$$\left\| \left( S_n^{(\alpha,\beta,k)} f \right)^{(\ell)} \right\|_{L^p_{\alpha+k,\beta+k}} \leq C \| f(k) \|_{L^p_{\alpha+k,\beta+k}}, \quad 0 \leq k, \ell \leq m,$$

under the conditions (9). First, note that

$$\left( S_n^{(\alpha,\beta,k)} f \right)^{(\ell)} = \sum_{j=\max\{k,\ell\}}^{n} \sqrt{r_j^{(\alpha,\beta)}} \frac{s_j^{(\alpha,\beta)}}{s_j^{(\alpha,\beta)}} b_j^{(\alpha,\beta,k)}(f(k)) p_j^{(\alpha+k,\beta+k)}.$$
To obtain (8), we distinguish three cases $2m - 2 \geq k + \ell$, $2m - 1 = k + \ell$, and $2m = k + \ell$.

**Case 2m – 2 ≥ k + ℓ.** For $\alpha \geq \beta > -1$, it is well known (see [5]) the equivalence

$$
\|P_n^{(\alpha,\beta)}\|_{L^p_{\alpha,\beta}} \simeq \begin{cases} 
1, & 1 < p \leq \frac{4(\alpha+1)}{2\alpha+1}, \\
(\log n)^{1/p}, & p = \frac{4(\alpha+1)}{2\alpha+1}, \\
\eta(2\alpha+4(\alpha+1)p)/2, & p > \frac{4}{2\alpha+1}.
\end{cases}
$$

(9)

Then, using that

$$
\sqrt{r_{j,k}^{(\alpha,\beta)}} = \frac{1}{(j+1)^{2m-k-\ell}} \left( A \frac{B}{j+1} + O \left( \frac{1}{(j+1)^2} \right) \right),
$$

for some constants $A$ and $B$, we have

$$
\left\| \left( S^{(\alpha,\beta)}_n f \right)^{(\ell)} \right\|_{L^p_{\alpha+\ell,\beta+\ell}} \leq \sum_{j=\max\{k,\ell\}}^{n} \left\| b_j^{(\alpha,\beta)}(f(k)) \right\|_{L^p_{\alpha+k,\beta+k}} \left\| P_j^{(\alpha+\ell,\beta+\ell)} \right\|_{L^p_{\alpha+\ell,\beta+\ell}}.
$$

By Hölder inequality,

$$
\left\| b_j^{(\alpha,\beta)}(f(k)) \right\| \leq \left\| P_j^{(\alpha+k,\beta+k)} \right\|_{L^p_{\alpha+k,\beta+k}} \left\| f(k) \right\|_{L^p_{\alpha+k,\beta+k}}.
$$

where $p'$ is the conjugate value of $p$ and it satisfies $1/p + 1/p' = 1$, and

$$
\left\| \left( S^{(\alpha,\beta)}_n f \right)^{(\ell)} \right\|_{L^p_{\alpha+\ell,\beta+\ell}} \leq \left\| f(k) \right\|_{L^p_{\alpha+k,\beta+k}} \sum_{j=\max\{k,\ell\}}^{n} \left\| P_j^{(\alpha+k,\beta+k)} \right\|_{L^p_{\alpha+k,\beta+k}} \left\| P_j^{(\alpha+\ell,\beta+\ell)} \right\|_{L^p_{\alpha+\ell,\beta+\ell}} \left\| f(k) \right\|_{L^p_{\alpha+k,\beta+k}}
$$

where in the last step we have used the inequality

$$
\left\| P_j^{(\alpha+k,\beta+k)} \right\|_{L^p_{\alpha+k,\beta+k}} \left\| P_j^{(\alpha+\ell,\beta+\ell)} \right\|_{L^p_{\alpha+\ell,\beta+\ell}} \leq \left\| P_j^{(\alpha+m,\beta+m)} \right\|_{L^p_{\alpha+m,\beta+m}} \left\| P_j^{(\alpha+m,\beta+m)} \right\|_{L^p_{\alpha+m,\beta+m}}.
$$

(10), the restriction (6), and the condition $2m - 2 \geq k + \ell$.

**Case 2m – 1 = k + ℓ.** In this case we have to prove (8) for the pairs $(k, \ell) = (m, m - 1)$ and $(k, \ell) = (m - 1, m)$. We will focus in the last pair because the estimate for the other pair can be obtained by using similar arguments and duality. By (10), we have

$$
\left( S_n^{(\alpha,\beta,m-1)} f \right)^{(m)}(x) = A T_1 f^{(m-1)}(x) + T_2 f^{(m-1)}(x),
$$

for some constant $A$, where

$$
T_1 f^{(m-1)}(x) = \sum_{j=1}^{n-m+1} \frac{b_j^{(\alpha,\beta,m-1)}(f^{(m-1)})}{j+m} P_{j-1}^{(\alpha+m,\beta+m)}(x)
$$
and
\[ |T_2 f^{(m-1)}(x)| \leq C \sum_{j=m}^{n} \frac{|b_j^{(\alpha,\beta,m-1)}(f^{(m-1)})|}{(j+1)^2} |p_j^{(\alpha+m,\beta+m)}(x)|. \]

The estimate
\[ \|T_2 f^{(m-1)}\|_{L^p_{\alpha+m,\beta+m}} \leq C \|f^{(m-1)}\|_{L^p_{\alpha+m-1,\beta+m-1}}, \]
under the restrictions (6), can be obtained by using Hölder inequality as in the previous case. To prove the boundedness of \( T \) under the restrictions (6), can be obtained by using Hölder inequality as in the previous case. To prove the boundedness of \( T_1 \) we write it as the composition of two operators. For \( \alpha, \beta > 0 \), we define
\[ T_{\alpha,\beta} g(x) = \sum_{j=1}^{\infty} \frac{e_j^{(\alpha-1,\beta-1)}(g)}{j+m} p_j^{(\alpha,\beta)}(x) \]
with
\[ e_j^{(\alpha,\beta)}(g) = \int_{-1}^{1} g(y) p_j^{(\alpha,\beta)}(y) \, d\mu_{\alpha,\beta}. \]
Moreover, for \( \alpha, \beta > -1 \), we consider the partial sum operator for the Jacobi expansions
\[ S_n^{(\alpha,\beta)} h(x) = \sum_{j=0}^{n} e_j^{(\alpha,\beta)}(h) p_j^{(\alpha,\beta)}(x). \]
It is known [13] that, for \( \alpha, \beta \geq -1/2 \),
\[ \|S_n^{(\alpha,\beta)} h\|_{L^p_{\alpha,\beta}} \leq C \|h\|_{L^p_{\alpha,\beta}}, \]
with a constant \( C \) independent of \( n \) and \( f \), if and only if
\[ \max \left\{ \frac{4(\alpha+1)}{2\alpha+3}, \frac{4(\beta+1)}{2\beta+3} \right\} < p < \min \left\{ \frac{4(\alpha+1)}{2\alpha+1}, \frac{4(\beta+1)}{2\beta+1} \right\}. \]

About the boundedness properties of the operator \( T_{\alpha,\beta} \) we have the following result.

**Proposition 3.** For \( \alpha, \beta > 0 \) and \( 1 < p < \infty \), it is verified that
\[ \|T_{\alpha,\beta} g\|_{L^p_{\alpha,\beta}} \leq C \|g\|_{L^p_{\alpha-1,\beta-1}}, \]
for each \( g \in L^p_{\alpha-1,\beta-1} \).

The proof of this proposition is highly technical and it is postponed to the last section.

Now, it is easy to check that
\[ T_1 f^{(m-1)}(x) = T_{\alpha+m,\beta+m} \left( S_n^{(\alpha+m-1,\beta+m-1)} f^{(m-1)}(x) \right). \]

Then, by Proposition 3 and (11), it is clear that
\[ \|T_1 f^{(m-1)}\|_{L^p_{\alpha+m,\beta+m}} \leq C \|f^{(m-1)}\|_{L^p_{\alpha+m-1,\beta+m-1}}, \]
when the conditions (6) hold.

**Case 2m = k + \ell.** In this last case \( k = \ell = m \) and, by (10), we have the decomposition
\[ (S_n^{(\alpha,\beta,m)} f^{(m)}(x) = A_n^{(\alpha+m,\beta+m)} f^{(m)}(x) + B_{\ell} f^{(m)}(x) + P_2 f^{(m)}(x), \]
where
\[ P_1 f^{(m)}(x) = \sum_{j=0}^{n-m} \frac{e_j^{(\alpha+m,\beta+m)}(f^{(m)})}{j+m+1} p_j^{(\alpha+m,\beta+m)}(x). \]
\[ |P_2 f^{(m)}(x)| \leq C \sum_{j=m}^{n} \frac{|b_j^{(\alpha,\beta,m)}(f^{(m)})|}{(j+1)^2} |p_{j-m}^{(\alpha+m,\beta+m)}(x)| \]

The estimate
\[ \|P_2 f^{(m)}\|_{L^p_{\alpha+m,\beta+m}} \leq C \|f^{(m)}\|_{L^p_{\alpha+m,\beta+m}} \]

when the conditions (6) hold, it is obtained by applying Hölder as in the two previous cases. When we consider (11) with \( \alpha + m \) and \( \beta + m \) instead of \( \alpha \) and \( \beta \), with \( \alpha \geq \beta \), we obtain (6), so we deduce that
\[ \|S_{n-m}^{(\alpha+m,\beta+m)} f^{(m)}\|_{L^p_{\alpha+m,\beta+m}} \leq C \|f^{(m)}\|_{L^p_{\alpha+m,\beta+m}}. \]

Finally to analyze the operator \( P_1 \) we need an auxiliary operator and its boundedness properties. We define
\[ R^{(\alpha,\beta)} f(x) = \sum_{j=0}^{\infty} \frac{c_j^{(\alpha,\beta)}(f)}{j + m + 1} p_j^{(\alpha,\beta)}(x). \]

**Lemma 4.** Let \( \alpha \geq \beta > -1 \), \( j \in \mathbb{N} \setminus \{0\} \), and
\[ \frac{4(\alpha + j + 1)}{2(\alpha + j) + 3} < p < \frac{4(\alpha + j + 1)}{2(\alpha + j) + 1}. \]

Then,
\[ \|R^{(\alpha+j,\beta+j)} f\|_{L^p_{\alpha+j,\beta+j}} \leq C\|f\|_{L^p_{\alpha+j,\beta+j}}. \]

This lemma is a particular case of [11] Theorem 1.10] because the multiplier \( 1/(j + m + 1) \) belongs to the class \( M(1,1) \) there defined.

Now, it is clear that
\[ P_1 f^{(m)}(x) = R^{(\alpha+m,\beta+m)}(S_{n-m}^{(\alpha+m,\beta+m)} f^{(m)})(x) \]
and the estimate
\[ \|P_1 f^{(m)}\|_{L^p_{\alpha+m,\beta+m}} \leq C \|f^{(m)}\|_{L^p_{\alpha+m,\beta+m}} \]
is an immediate consequence of the previous lemma and the boundedness of the partial sum operator for the Jacobi expansions.

### 2.2. Necessary conditions.
If (5) holds, it is clear that
\[ c_n^{(\alpha,\beta,m)}(f) q_n^{(\alpha,\beta,m)} W_{\alpha,\beta}^p = \|S_n^{(\alpha,\beta,m)} f - S_{n-1}^{(\alpha,\beta,m)} f\|_{W_{\alpha,\beta}^p} \leq C\|f\|_{W_{\alpha,\beta}^p} \]

By [7] Theorem 4.3], each functional in \( T \in (W_{\alpha,\beta}^p)^' \), with \( 1 \leq p < \infty \), can be written as
\[ T(f) = \sum_{k=0}^{m} \int_{-1}^{1} f^{(k)}(x)v_k(x) d\mu_{\alpha+k,\beta+k}, \]
where \( v = (v_0, \ldots, v_m) \) belongs to the space \( \prod_{k=0}^{m} L_{\alpha+k,\beta+k}^q \) equipped with the norm
\[ \|v\|_{\prod_{k=0}^{m} L_{\alpha+k,\beta+k}^q} = \sum_{k=0}^{m} \|v_k\|_{L_{\alpha+k,\beta+k}^q}. \]
Moreover, \(\|T\| = \|v\|\prod_{k=0}^{m} L_{\alpha+k,\beta+k}^{\alpha+k,\beta+k}\) and the function \(v\) is unique for \(1 < p < \infty\).

From this fact, it is clear that the norm as operator of \(c_{n}^{(\alpha,\beta,m)}(f)\) is given by \(\|q_{n}^{(\alpha,\beta,m)}\|_{W_{\alpha,\beta}^{p,m}}\) and, by (13), the inequality
\[
(14) \quad \|q_{n}^{(\alpha,\beta,m)}\|_{W_{\alpha,\beta}^{p,m}} \|q_{n}^{(\alpha,\beta,m)}\|_{W_{\alpha,\beta}^{q,m}} \leq C
\]
holds with a constant independent of \(n\) when (3) is verified.

Now, taking into account that
\[
\|q_{n}^{(\alpha,\beta,m)}\|_{W_{\alpha,\beta}^{p,m}} = \sum_{k=0}^{m} \sqrt{\frac{p_{n}^{(\alpha,\beta)}(\alpha+k,\beta+k)}{s_{n,m}}} \|p_{n-k}^{(\alpha+k,\beta+k)}\|_{L_{\alpha+k,\beta+k}^{p}}
\]
and the inequality \(\|p_{n-k}^{(\alpha+k,\beta+k)}\|_{L_{\alpha+k,\beta+k}^{p}} \leq C\|p_{n-k}^{(\alpha+m,\beta+m)}\|_{L_{\alpha+m,\beta+m}^{p}}\), for \(0 \leq k \leq m\), we can deduce that
\[
\|q_{n}^{(\alpha,\beta,m)}\|_{W_{\alpha,\beta}^{p,m}} \approx \|p_{n-k}^{(\alpha+m,\beta+m)}\|_{L_{\alpha+m,\beta+m}^{p}}.
\]
Then, (14) and (9) imply (6).

3. PROOF OF THE PROPOSITION (3)

It is easy to check that
\[
T_{\alpha,\beta}g(x) = \int_{-1}^{1} g(y)L(x, y) \, d\mu_{\alpha-1,\beta-1}(y),
\]
with
\[
L(x, y) = \sum_{j=1}^{\infty} \frac{p_{j-1}^{(\alpha,\beta)}(x)p_{j}^{(\alpha-1,\beta-1)}(y)}{j + m}
\]
Then, with the change of variable \(x = \cos \theta\) and \(y = \cos \omega\), the inequality (12) is equivalent to
\[
(15) \quad \int_{0}^{\pi} |T_{\alpha,\beta}G(\theta)|^{p} W_{\alpha,\beta}(\theta) \, d\theta \leq C \int_{0}^{\pi} |G(\theta)|^{p} W_{\alpha-1,\beta-1}(\theta) \, d\theta,
\]
where \(W_{\alpha,\beta}(\theta) = (\sin \theta/2)^{(\alpha+1)/2}(\cos \theta/2)^{(\beta+1)/2}\).

\[
T_{\alpha,\beta}G(\theta) = \int_{-1}^{1} G(\omega)L(\theta, \omega) \, d\omega,
\]
with
\[
L(\theta, \omega) = \sum_{j=1}^{\infty} \frac{G_{j}^{(\alpha,\beta)}(\theta)G_{j}^{(\alpha-1,\beta-1)}(\omega)}{j + m}
\]
and
\[
G_{k}^{(\alpha,\beta)}(\theta) = 2^{(\alpha+\beta+1)/2}(\sin \theta/2)^{\alpha+1/2}(\cos \theta/2)^{\beta+1/2}p_{k}^{(\alpha,\beta)}(\cos \theta).
\]
Now, for any integer \(d\) and \(0 < r < 1\), we consider the auxiliary kernel
\[
L_{r,d,m}^{(\alpha,\beta),(\alpha-1,\beta-1)}(\theta, \omega) = \sum_{j=\max\{0, -d\}}^{\infty} r^{j} \frac{\theta_{j}^{(\alpha,\beta)}(\theta)\phi_{j}^{(\alpha-1,\beta-1)}(\omega)}{j + m + 1},
\]
Lemma 5. For \( \alpha, \beta > 0 \), \( 0 < r < 1 \), and \( 0 < \theta, \omega < \pi \), it is verified that

\[
|\mathcal{L}_{r-1, m}^{(\alpha, \beta), (\alpha-1, \beta-1)}(\theta, \omega)| \leq C \begin{cases} 
\omega^{\alpha/2} (\pi - \theta)^{\beta+1/2} / \theta^{\alpha-1/2} (\pi - \omega)^{\beta+1/2}, & 0 < \omega \leq M(\theta), \\
\log \left( \frac{2\theta}{|\theta - \omega|} \right), & M(\theta) < \omega < m(\theta), \\
\theta^{\alpha+1/2} (\pi - \omega)^{\beta-1/2} / \omega^{\alpha+1/2} (\pi - \theta)^{\beta-1/2}, & m(\theta) \leq \omega < \pi,
\end{cases}
\]

with

\[
M(\theta) = \max \left\{ \frac{\theta}{2} - \frac{3\theta - \pi}{2} \right\} \quad \text{and} \quad m(\theta) = \min \left\{ \frac{3\theta}{2} - \frac{\theta + \pi}{2} \right\}
\]

Proof. To prove the estimate in the first line of (16) note that when \( 0 < \theta, \omega < 3\pi/4 \), the right-hand side is equivalent to \( \omega^{\alpha-1/2} \theta^{1/2-\beta} \) and this can be deduced from [11, Theorem 7.1] (in fact, we have to consider in that theorem \( d = 1 \), \( s = 1 \), and \( g(j) = 1/(j+m) \)). When \( \pi/4 < \theta, \omega < \pi \), the required estimate is comparable with \( (\pi - \theta)^{\beta+1/2} (\pi - \omega)^{-\beta-1/2} \), and the required bound is obtained by using the identities (remember that \( P_n^{(b,a)}(-x) = (-1)^n P_n^{(b,a)}(x) \))

\[
\mathcal{L}_{r-1, m}^{(\alpha, \beta), (\alpha-1, \beta-1)}(\theta, \omega) = -\mathcal{L}_{r-1, m}^{(\beta, \alpha), (\beta-1, \alpha-1)}(\pi - \theta, \pi - \omega) = r \mathcal{L}_{r-1, m+1}^{(\alpha-1, \beta-1), (\alpha, \beta)}(\omega, \theta)
\]

and again [11, Theorem 7.1] (in this case with \( d = 1 \), \( s = 1 \), and \( g(j) = 1/(j + m + 1) \)). Finally, for \( 3\pi/4 \leq \theta < \pi \) and \( 0 < \omega \leq \pi/4 \) the bound is equivalent to \( \omega^{\alpha-1/2} (\pi - \theta)^{\beta+1/2} \) and this one is contained in [11, Theorem 5.1].

The estimate in the third line of (16) is obtained in similar manner because is the dual of the bound in the first line.

To obtain the bound in the second line of (16) we proceed as in the analysis of the kernel \( L_{r-1}^{1, -1} \) in [2, Proposition 3.2, pp. 365–366].

Following the ideas in the proof of [2, Proposition 3.3], it is possible to prove that

\[
\mathcal{L}(\theta, \omega) = \lim_{r \to 1^-} \mathcal{L}_{r-1, m}^{(\alpha, \beta), (\alpha-1, \beta-1)}(\theta, \omega).
\]

Moreover, \( |\mathcal{L}(\theta, \omega)| \) is bounded by the right-hand side of (16). Then the operator \( T_{\alpha, \beta} G(\theta) \) can be controlled by the sum of

\[
R_1 G(\theta) = \frac{(\pi - \theta)^{1/2}}{\theta^{1/2}} \int_0^\theta \frac{\omega^{\alpha-1/2}}{(\pi - \omega)^{\beta+1/2}} |G(\omega)| d\omega,
\]

\[
R_2 G(\theta) = \int_{M(\theta)}^{m(\theta)} \log \left( \frac{2\theta}{|\theta - \omega|} \right) |G(\omega)| d\omega,
\]

and

\[
R_3 G(\theta) = \frac{\theta^{\alpha+1/2}}{(\pi - \theta)^{\beta-1/2}} \int_0^\pi \frac{(\pi - \omega)^{\beta-1/2}}{\omega^{\alpha+1/2}} |G(\omega)| d\omega.
\]

It is known, it is a consequence of [1, Theorem A], that the inequality

\[
\int_0^\pi \left| U(\theta) \int_0^\theta h(\omega) d\omega \right|^p d\theta \leq C \int_0^\pi |V(\theta) h(\theta)|^p d\theta
\]
holds if and only if

\[(17) \sup_{0 < r < \pi} \left( \int_0^\pi U^p(\theta) \, d\theta \right)^{1/p} \left( \int_0^\pi V^{-p'}(\theta) \, d\omega \right)^{1/p'} < \infty.\]

Moreover, from [1, Theorem B], we have

\[
\int_0^\pi \left| U(\theta) \int_\theta^\pi h(\omega) \, d\omega \right|^p \, d\theta \leq C \int_0^\pi |V(\theta)h(\theta)|^p \, d\theta
\]

if and only if

\[(18) \sup_{0 < r < \pi} \left( \int_0^\pi U^p(\theta) \, d\theta \right)^{1/p} \left( \int_0^\pi V^{-p'}(\theta) \, d\omega \right)^{1/p'} < \infty.\]

In this way the boundedness

\[
\int_0^\pi |R_1 G(\theta)|^p W_{\alpha,\beta}(\theta) \leq C \int_0^\pi |G(\theta)|^p W_{\alpha-1,\beta-1}(\theta) \, d\theta
\]

will follow checking the condition \[(17)\] for the weights

\[
U(\theta) = W^{1/p}_{\alpha,\beta}(\theta) \frac{\varphi_p - \varphi_0}{\varphi_0^{\alpha-1/2}} \quad \text{and} \quad V(\theta) = W^{1/p}_{\alpha-1,\beta-1}(\theta) \frac{\varphi_p - \varphi_0}{\varphi_0^{\alpha-1/2}}.
\]

For these weights the supremum in \[(17)\] is equivalent to

\[
\sup_{0 < r < \pi} \left( \int_0^\pi \theta^2(1-p) \left( \frac{\varphi_p - \varphi_0}{\varphi_0^{\alpha-1/2}} \right)^{1/p} \left( \int_0^\pi \theta^2(1-p) \left( \frac{\varphi_p - \varphi_0}{\varphi_0^{\alpha-1/2}} \right)^{1/p'} d\theta \right)^{1/p'} d\theta \right)^{1/p'}
\]

and this quantity is finite for $1 < p < \infty$ and $\alpha, \beta > 0$. To obtain the inequality

\[
\int_0^\pi |R_1 G(\theta)|^p W_{\alpha,\beta}(\theta) \leq C \int_0^\pi |G(\theta)|^p W_{\alpha-1,\beta-1}(\theta) \, d\theta
\]

we proceed in the same way but checking the condition \[(18)\] for the appropriate weights.

To complete the proof of the inequality \[(15)\] we have to check that

\[
\int_0^\pi |R_2 G(\theta)|^p W_{\alpha,\beta}(\theta) \leq C \int_0^\pi |G(\theta)|^p W_{\alpha-1,\beta-1}(\theta) \, d\theta.
\]

With some elementary manipulations, the previous inequality follows from

\[
\int_0^{\pi/2} \int_{\beta/2}^{\beta/2} \log \left( \frac{2\theta}{\theta - \omega} \right) h_1(\omega) \, d\omega \left| \theta^{(2-p)} \right| d\theta \leq \int_0^{\pi/2} |h_1(\theta)|^p d\theta
\]

and

\[
\int_{\pi/2}^{\pi/2} \int_{(3\theta-\pi)/2}^{(3\theta-\pi)/2} \log \left( \frac{2\theta}{\theta - \omega} \right) h_2(\omega) \, d\omega \left| (\pi - \theta)^{(2-p)} \right| d\theta \leq \int_{\pi/2}^{\pi/2} |h_2(\theta)|^p d\theta
\]

and both of them can be deduced applying Hölder inequality. Now, the proof of the proposition is finished.
FOURIER SERIES OF JACOBI-SOBOLEV POLYNOMIALS

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