Lower bounds for the scalar curvatures of Ricci flow singularity models

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Abstract

In a series of papers, Bamler [Bam20a, Bam20b, Bam20c] further developed the high-dimensional theory of Hamilton’s Ricci flow to include new monotonicity formulas, a completely general compactness theorem, and a long-sought partial regularity theory analogous to Cheeger–Colding theory. In this paper we give an application of his theory to lower bounds for the scalar curvatures of singularity models for Ricci flow. In the case of 4-dimensional non-Ricci-flat steady soliton singularity models, we obtain as a consequence a quadratic decay lower bound for the scalar curvature.

1 Introduction and background

To help formulate Ricci flow with surgery, one would like to better understand singularity models of the Ricci flow. This has been highly successful in dimension 3 by the Hamilton–Perelman theory leading to the proof of the Poincaré and Thurston geometrization conjectures by Hamilton’s Ricci flow [Per02, Per03a, Per03b]. A deeper understanding of Ricci flow and singularity formation in dimension 3 has led to solutions to a pair of fundamental conjectures of Perelman; see Brendle [Bre13, Bre20] and Bamler [Bam18] and the references therein. Bamler and Kleiner [BK22, BK21a] proved the generalized Smale conjecture using 3-dimensional Ricci flow with surgery. Furthermore, the understanding of 3-dimensional ancient solutions essentially complete. See Brendle, Daskalopoulos, and Sesum [BDS21], Angenent, Brendle, Daskalopoulos, and Sesum [ABDS22], Bamler and Kleiner [BK21b], and Lai [Lai20, Lai22].

Among the higher dimensions, dimension 4 is the most hopeful. In this case, Bamler’s theory [Bam20a, Bam20b, Bam20c] yields the strongest results regarding Ricci flow singularity formation and compactness. In general, regarding singularity models, one is interested in Ricci solitons since they are prototypical [Ham88, Ham93] and [Per02, Per03a]. Among a vast literature on Ricci solitons, important progress has been made by Munteanu and Wang (see [MW19] and the references therein), especially on the geometric understanding of shrinkers.

A general question is: what curvature estimates hold for singularity models, and in particular, Ricci solitons? In this paper we apply Bamler’s theory to make some further progress on lower bounds for the scalar curvature.

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If a singularity model is Ricci flat, then by Perelman’s no local collapsing theorem [Per02], it has Euclidean volume growth. If, in addition, the dimension is 4, then by a theorem of Cheeger and Naber [CN15, Corollary 8.86], it is an asymptotically locally Euclidean (ALE) space. The Eguchi–Hanson metric, which is such a space, occurs as a singularity model of 4-dimensional Ricci flow by Appleton [App19]. In this paper we consider the complementary case of non-Ricci-flat singularity models.

Let \((M^n, g_t)_{t \leq 0}\) be an ancient Ricci flow, i.e., \(\partial_t g_t = -2 \text{Ric}_g\), and \(t \leq 0\) means that \(t \in (-\infty, 0]\), whose time slices are complete. The conjugate heat operator is defined as \(\Box^{\ast} = -\partial_t - \Delta_{g_t} + R_{g_t}\). We write the conjugate heat kernel based at the space-time point \((o, 0)\) as

\[
K(o, 0 \mid x, t) = (4\pi|t|)^{-\frac{n}{2}} e^{-f_t(x)}.
\]

So, \(\Box^{\ast} K = 0\) and \(\lim_{t \to o} K(o, 0 \mid \cdot, t) = \delta_o\). The pointed Nash entropy based at \((o, 0)\) is

\[
\mathcal{N}_{o,0}(\tau) := \int_M f_t(x) K(o, 0 \mid x, t) \, dg_t(x) - \frac{n}{2}, \tag{1.1}
\]

where \(\tau = |t|\) (we will henceforth use this notation).

Henceforth, we assume that \((M^n, g_t)_{t \leq 0}\) either is a finite-time singularity model or has bounded curvature over compact time intervals with a pointed Nash entropy lower bound, namely,

\[
\inf_{\tau > 0} \mathcal{N}_{o,0}(\tau) =: \mu_\infty > -\infty, \tag{NELB}
\]

for some point \(o \in M\) (and thus for any point since \(\inf \mathcal{N}\) is independent of the base point by [MZ21, Proposition 4.6]). By a finite-time singularity model we mean a complete and non-flat ancient solution \((M^n, g_t)_{t \leq 0}\) which arises as a blow-up limit of a Ricci flow \((\overline{M^n}, \overline{g}(t))_{t \in [0,T]}\) on a closed manifold \(\overline{M}\), with a finite-time singularity at \(T < \infty\). More precisely, one can find sequences of \((x_i, t_i) \in \overline{M} \times [0, T)\) and \(\lambda_i \to \infty\) such that the rescaled solutions \((\overline{M}, \lambda_i \overline{g}(t_i + \frac{1}{\lambda_i^2})), t \in (-\lambda_i t_i, 0]\), \(\mathcal{F}\)-converge on compact time intervals to \((M, g(t)), t \in (-\infty, 0]\), in the sense of Bamler [Bam20b] (see also [CFSZ20, BCDMZ21]).

For each \(t < 0\), let \((z_t, t)\) be an \(H_n\)-center of \((o, 0)\), as defined in [Bam20a]. That is, \(\text{Var}_t(\delta_{z_t}, K(o, 0 \mid \cdot, t) \, dg_t) \leq H_n |t|\), where \(H_n := \frac{(n-1)\pi^2}{2} + 4\). Such space-time points effectively represent the past of a point \((o, 0)\) of a Ricci flow. We shall consider the following \(H_n\)-center scalar curvature lower bound assumption

\[
\inf \left\{(1 + \tau) R(x, t) \mid x \in B_t(z_t, D\sqrt{1 + \tau}), t < 0\right\} := a_0 > 0, \tag{SCLB}
\]

for some constant \(D < \infty\). As we will see below, this assumption is suitable for applications to noncollapsed steady solitons.

## 2 Statements of the results

The main results of this paper stem from the following global scalar curvature decay estimate for the ancient solutions in which we are interested.
Theorem 2.1. Assume that a complete ancient solution \((M^n, g_t)\) is a finite-time singularity model or that \((M, g_t)\) has bounded curvature over compact time intervals with (NELB). Further assume (SCLB). If the constant \(D\) in (SCLB) satisfies \(D \geq D(n, \mu_\infty)\), then
\[
(1 + \tau) R(x, t) \geq \frac{c(n, \mu_\infty, a_0)}{f_t(x) + C(n) - \mu_\infty},
\]
for \((x, t) \in M \times (-\infty, 0)\), where \(a_0\) is the infimum in (SCLB) and \(\mu_\infty\) is as in (NELB).

The following says that we can replace the assumption of an \(H_n\)-center scalar curvature lower bound by the smoothness of the tangent flow at infinity.

Corollary 2.2. Assume that \((M, g_t)\) is a finite-time singularity model or that \((M, g_t)\) has bounded curvature over compact time intervals with (NELB). Suppose further that each tangent flow at infinity of \((M, g_t)\) (as defined in [Bam20b]) is smooth. Then
\[
(1 + \tau) R(x, t) \geq \frac{a}{f_t(x) + C(n) - \mu_\infty},
\]
where \(a > 0\) is a constant depending on the geometry of \((M, g_t)\) at \(t \leq 0\).

Alternative versions of this result include that it holds under the type-I assumption or under another more technical assumption stated below. In particular, we have the following.

Corollary 2.3. Let \((M^n, g_t)_{t \leq 0}\) be a complete and non-flat ancient flow with (NELB). Suppose that either:

1. \((M, g_t)\) is of type-I, i.e., \(\tau |\text{Rm}| \leq C\) uniformly over \(M \times (-\infty, 0]\) for some constant \(C < \infty\); or

2. \((M, g_t)\) satisfies Hamilton’s trace Harnack estimate, i.e., for any smooth vector field \(X\) on \(M\),
\[
\frac{\partial R}{\partial t} - 2\langle X, \nabla R \rangle + 2 \text{Ric}(X, X) \geq 0,
\]
and \(|\text{Rm}| \leq CR\) for some constant \(C < \infty\).

Then
\[
(1 + \tau) R(x, t) \geq \frac{a}{f_t(x) + C(n) - \mu_\infty},
\]
where \(a > 0\) is a constant depending on the geometry of \((M, g_t)_{t \leq 0}\), and \(\mu_\infty\) is as in (NELB).

We then apply the results above to steady gradient Ricci solitons. Recall that a triple \((M^n, g, f)\) is called a steady gradient Ricci soliton, if \((M^n, g)\) is a Riemannian manifold and \(f\) is a smooth function on \(M\) satisfying
\[
\text{Ric} = \nabla^2 f.
\]

If we denote by \((\Phi_t)_{t \in \mathbb{R}}\) the 1-parameter group of diffeomorphisms generated by \(-\nabla f\) with \(\Phi_0 = \text{id}\), then \(g_t := \Phi_t^* g\) solves the Ricci flow, and we call \((M^n, g_t)_{t \in \mathbb{R}}\) the canonical form of \((M^n, g, f)\). It was proved by Hamilton [Ham93] that on any steady gradient Ricci soliton, \(\nabla (|\nabla f|^2 + R) \equiv 0\) holds, and hence
\[
|\nabla f|^2 + R = C_1,
\]
for some nonnegative constant \(C_1\). Let \(|xy| := d(x, y)|\) denote the distance between \(x\) and \(y\); if we add a subscript \(t\), then this means that the distance is with respect to \(g_t\).
Definition 2.4. A steady gradient Ricci soliton \((M^n, g, f)\) is called noncollapsed if its canonical form \((M^n, g_t)\) for some \(o \in M\).

It is to be remarked that our definition of noncollapsing is different from that of Perelman [Per02]. Bamler [Bam20a, Theorem 6.1] and [MZ21, Proposition 4.6] show that Definition 2.4 implies Perelman’s \(\kappa\)-noncollapsing on all scales. Nevertheless, Definition 2.4 is natural for applications, since due to the entropy monotonicity, any finite-time singularity model is necessarily noncollapsed in the sense of Definition 2.4.

In the case of noncollapsed steady solitons with nonnegative Ricci curvature, we can use the tangent flows at infinity to obtain curvature bounds near the \(H_n\)-centers, and then our estimate provides a global scalar curvature lower bound. Thus, we have the following.

Corollary 2.5. Let \((M^n, g, f)\) be a complete and noncollapsed steady gradient Ricci soliton. Suppose that \(\text{Ric} \geq 0\) and \(|\text{Rm}| \leq CR\) for some constant \(C < \infty\). If \((M^n, g)\) is not Ricci flat, then for any \(x \in M, \tau \geq \tau(n)\), we have that

\[
R(x) \geq \frac{a}{(C - \mu_\infty)\tau + |xx_\tau|^2},
\]

where \(x_\tau \in \partial B_\tau(o)\) satisfies \(\lambda(x_\tau, \tau_0) \leq C\), for some \(\tau_0 \in [\tau/C, C\tau]\). (See the definition of \(\lambda\) in Section 3.3 or [BCMZ21, Section 2.1].) Here, \(C = C(n)\), and \(a > 0\) depends on the geometry of \(g\).

We remark that this lower bound is not sharp for the hypothesized class of steady solitons, e.g., Lai’s noncollapsed examples in dimension 4 [Lai20]. However, this estimate is sharp for 3-cylindrical steady solitons. In particular, if \(f\) grows linearly and the level sets \(\Sigma_\tau = \{f = \tau\}\) have diameters bounded by \(C\sqrt{\tau}\) (this is the case when \(R(x) \leq \frac{C}{\tau + |xo|}\); see Deng and Zhu [DZ20b, Proposition 3.3]), then Corollary 2.5 gives the improved estimate

\[
R(x) \geq c/\tau \geq c' \frac{1}{|xo|},
\]

for any \(x \in \Sigma_\tau\), if \(\tau \geq \tau(n)\). This can be viewed as an alternative proof of a result by Deng and Zhu [DZ20a, Theorem 2.5] in this case. Note that we do not need to assume the existence of critical points of \(f\).

In fact, the arguments in [DZ20a, Theorem 2.5] or [DZ18, Proposition 4.3] prove the following.

Theorem 2.6 (Deng and Zhu). Let \((M^n, g, f)\) be a complete noncollapsed steady gradient soliton. Suppose that \(\text{Ric} > 0\) and \(|\text{Rm}| \leq CR\) for some constant \(C < \infty\). If \(f\) admits a critical point, then

\[
R(x) \geq c/|xo|,
\]

for any \(x \notin B_{r_0}(o)\), for some constant \(c > 0\) and \(r_0 < \infty\).

Sketch of the proof. [DZ20a, Theorem 2.5] uses the same arguments as [DZ18, Proposition 4.3], where they assumed nonnegative curvature operator or sectional curvature only to obtain Hamilton’s trace Harnack estimate (2.1) which implies a Harnack estimate for the scalar curvature [DZ18, (3.3)]. As observed in [MZ21, Lemma 5.1], nonnegative Ricci curvature on steady solitons implies Hamilton’s trace Harnack estimate. The rest of the arguments are the same as in the proof of [DZ18, Proposition 4.3].
The estimate in the Corollary 2.5 implies that
\[ \limsup_{x \to \infty} R(x)|xo| > 0, \]
which is a statement stronger than infinite ASCR. Indeed, by taking \( x = x_\tau \in \partial B_\tau(o) \), we see that for all \( \tau \gg 1 \),
\[ R(x_\tau) \geq \frac{a}{(C - \mu_\infty)\tau} = \frac{a}{(C - \mu_\infty)|x_\tau o|}. \]
Hence, \( \limsup_{x \to \infty} R(x)|xo| > 0 \).

A main consequence of Theorem 2.1, applicable to 4-dimensional steady soliton singularity models, is:

**Corollary 2.7.** Let \( (M^4, g, f) \) be a 4-dimensional noncollapsed and non-Ricci-flat steady gradient Ricci soliton with bounded curvature. Let \( o \in M \) be a fixed point. Then there exists a positive constant \( c \) depending on the soliton and \( o \) such that
\[ R(x) \geq \frac{c}{1 + |xo|^2}. \]

Hence, any 4-dimensional steady soliton singularity model either is \( \mathbb{R}^4/\Gamma \), or is a Ricci flat ALE, or has scalar curvature decaying at most quadratically.

By Munteanu, Sung, and Wang [MSW19] (generalizing [CLY11]), the general exponential lower bound for steady solitons \( (M^n, g, f) \) with potential functions bounded from below is:
\[ R \geq ce^{-f}. \]

Another asymptotic curvature estimate for steady solitons satisfying the conditions in Corollary 2.5 is due to Ma and Zhang [MZ21, Corollary 5.2]:
\[ \text{ASCR}(g) := \limsup_{x \to \infty} R(x)|xo|^2 = \infty. \]

We mention a related result by Han in [Han20], where he proves the following asymptotic curvature estimates: Let \( (M^n, g, f) \) be a complete steady gradient Ricci soliton with \( \sec \geq 0 \), \( \text{Ric} > 0 \), and where the scalar curvature decays uniformly. Then
\[ \liminf_{x \to \infty} R(x)|xo|^\alpha = 0, \]
for any \( \alpha \in (0,1) \).

We remark that one may conjecture that \( R \) has an inverse linear lower bound in distance for general steady gradient Ricci solitons without nonnegative curvature conditions.

3 Proofs of the results

3.1 Proof of Theorem 2.1

**Proof of Theorem 2.1.** By [Bam20a, Theorem 7.2], for any \( x \in M, t < 0 \), we have the Gaussian upper bound for the conjugate heat kernel
\[ |t|^{n/2}K(o,0|x,t) \leq C\exp\left(-\mu_\infty - \frac{|x_\tau|^2}{9|t|}\right), \]
where \((z_t, t)\) is an \(H_n\)-center of \((o, 0)\) and \(\mu_\infty\) is the constant in (NELB). Thus,

\[
 f_t(x) \geq -C_0 + \mu_\infty + 1 + \frac{|xz_t|^2}{9\tau},
\]

for some \(C_0 = C_0(n) > 0\). In the following, we write

\[
 \rho_t(x) := \sqrt{1 + \tau} \left( f_t(x) + C_0 - \mu_\infty \right) \geq \sqrt{1 + \tau} \left( 1 + \frac{|xz_t|^2}{9\tau} \right).
\]

(3.1)

Since \(\partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{\omega_0}{2\tau}\), we have that \(\tau \Box f = f - n\), where \(\Box = \partial_t - \Delta g_t\) is the heat operator. By Perelman’s differential Harnack estimate [Per02, §9], we have

\[
 w := \tau(2\Delta f - |\nabla f|^2 + R) + f - n \leq 0.
\]

So

\[
 \tau \Box \rho \geq -\frac{\tau}{2(1 + \tau)} \rho + \sqrt{1 + \tau} \left( f - \frac{n}{2} \right) \geq \frac{1}{2} \rho - A\sqrt{1 + \tau},
\]

where

\[
 A = A(n, \mu_\infty) := C_0 - \mu_\infty + \frac{n}{2}.
\]

Then

\[
 \tau \Box \rho^{-1} = -\rho^{-2} \Box \rho - 2\rho^{-3} |\nabla f|^2 \\
 \leq -\frac{1}{2} \rho^{-1} + A\sqrt{1 + \tau} \rho^{-3}.
\]

Similarly,

\[
 \tau \Box \rho^{-2} \leq -\rho^{-2} + 2A\sqrt{1 + \tau} \rho^{-3},
\]

and

\[
 \tau \Box (\sqrt{1 + \tau} \rho^{-2}) \leq -\sqrt{1 + \tau} \rho^{-2} + 2A(1 + \tau) \rho^{-3}.
\]

So

\[
 \tau \Box (\rho^{-1} + 4A\sqrt{1 + \tau} \rho^{-2}) \leq -\frac{1}{2} \left( \rho^{-1} + 4A\sqrt{1 + \tau} \rho^{-2} \right) - A\sqrt{1 + \tau} \rho^{-3} \left( \rho - 8A\sqrt{1 + \tau} \right).
\]

Recall that

\[
 \Box (\sqrt{1 + \tau} R) = -\frac{R}{2\sqrt{1 + \tau}} + 2\sqrt{1 + \tau} |\text{Ric}| \geq -\frac{R}{2\sqrt{1 + \tau}}.
\]

Let us define

\[
 F := \sqrt{1 + \tau} R - c\rho^{-1} - 4cA\sqrt{1 + \tau} \rho^{-2},
\]

(3.2)

where \(c = c(n, \mu_\infty, a_0)\) will be determined in the course of the proof, and where \(\mu_\infty\) and \(a_0\) are the constants in (NELB) and (SCLB), respectively. Then

\[
 \tau \Box F \geq -\frac{1}{2} F + cA\sqrt{1 + \tau} \rho^{-3} \left( \rho - 8A\sqrt{1 + \tau} \right). \tag{3.3}
\]
3 PROOFS OF THE RESULTS

Next, we aim to show that $\inf_{M \times (-\infty, 0)} F \geq 0$. To apply the maximum principle to (3.3), we need to verify the boundary conditions. Note that since the operator $\tau \Box$ and the function $\rho$ become singular as $t \to 0^-$, we also check the boundary condition on $M \times \{0\}$ to avoid invoking advanced maximum principles.

Claim. By taking $c \leq \overline{c}(n, \mu_{\infty}, a_0) := \frac{a_0}{1 + 4A}$, we have

$$\liminf_{x \to \infty} F(x, t) \geq 0,$$

(3.4)

uniformly for $t \in I$, where $I \subset (-\infty, 0)$ is any compact interval, and

$$\liminf_{t \to 0^-} \left( \inf_{x \in M} F(x, t) \right) \geq 0,$$

(3.5)

and

$$\liminf_{t \to -\infty} \left( \inf_{x \in M} F(x, t) \right) \geq 0.$$

(3.6)

Proof of the claim. Inequality (3.4) is a direct consequence of Chen’s result that $R \geq 0$ (see [Che09, Corollary 2.5]), the quadratic growth of $\rho$ from (3.1), the equivalence of the metrics on a compact time interval, and the fact that the $H_n$-centers of a fixed point do not drift with infinite velocity (a consequence of the estimate of Perelman’s $\ell$-distance).

To prove (3.5), we fix an arbitrary time $t \in (\infty, 0)$. Let $x \in M$ be an arbitrary point. If $x \in B_t(z_t, D\sqrt{1 + \tau})$, then by (SCLB) and (3.1), we have

$$F(x, t) \geq \frac{a_0}{\sqrt{1 + \tau}} - \frac{c(1 + 4A)}{\sqrt{1 + \tau}} \geq 0,$$

provided we take $c \leq \frac{a_0}{1 + 4A}$. If $x \notin B_t(z_t, D\sqrt{1 + \tau})$, then we have

$$\rho(x, t) \geq \sqrt{1 + \tau} \left( 1 + \frac{D^2(1 + \tau)}{\tau} \right) \geq \frac{D^2}{\tau},$$

and thus

$$F(x, t) \geq -\frac{c\tau}{D^2} \frac{4cA\tau^2\sqrt{1 + \tau}}{D^4}.$$

Therefore, we have

$$\inf_{x \in M} F(x, t) \geq -C(A, D)\tau \to 0 \quad \text{as} \quad t \to 0^-,$$

and (3.5) follows immediately.

Finally, for (3.6), recall that $R \geq 0$ and $\rho \geq \sqrt{1 + \tau}$. We have

$$\inf_{x \in M} F(x, t) \geq -\frac{c(1 + 4A)}{\sqrt{1 + \tau}} \quad \text{for all} \quad t < 0,$$

and (3.6) is proved.
Assume, for a contradiction, that $\inf_{M \times (-\infty, 0)} F < 0$. By (3.4), (3.5), and (3.6), we have that the space-time infimum of $F$ must be attained at some point $(x_0, t_0) \in M \times (-\infty, 0)$. Applying (3.3) at $(x_0, t_0)$, we have

$$0 \geq \tau \Box F \geq -\frac{1}{2} F + cA\sqrt{1 + \tau} \rho^{-3} (\rho - 8A\sqrt{1 + \tau}) \quad \text{at} \quad (x_0, t_0).$$

Since $F(x_0, t_0) < 0$, we have

$$8A\sqrt{1 + |t_0|} \geq \rho(x_0, t_0) \geq \sqrt{1 + |t_0|} \left(1 + \frac{|x_0z_{t_0}|^2}{9|t_0|}\right),$$

where we have applied (3.1). Thus, we have

$$|x_0z_{t_0}|_{t_0} \leq \sqrt{72A(1 + |t_0|)}.$$  \hfill (3.7)

If we require the constant $D$ in (SCLB) to satisfy $D \geq \sqrt{100A}$, then by (SCLB), we have

$$R(x_0, t_0) \geq \frac{a_0}{1 + |t_0|}.$$  

By the definitions of $F$ in (3.2) and of $\rho$ in (3.1), we have

$$F(x_0, t_0) \geq \frac{a_0}{\sqrt{1 + |t_0|}} - \frac{c(1 + 4A)}{\sqrt{1 + |t_0|}} \geq 0,$$

provided we take $c \leq \frac{a_0}{1 + 4A}$; this is a contradiction, and the proof of Theorem 2.1 is complete. \hfill $\Box$

### 3.2 Proofs of Corollaries 2.2 and 2.3

**Proof of Corollary 2.2.** First of all, the ancient solution in question cannot be Ricci-flat. Since the tangent flow at infinity of a Ricci-flat static Ricci flow is its own blow-down limit, which necessarily contains a singularity unless the manifold is flat.

We show by contradiction that (SCLB) holds for any $D > 0$. Suppose that (SCLB) is false for some $D > 0$. Since non-Ricci-flat ancient Ricci flows have positive scalar curvature [Che09], there must be a sequence $(x_i, t_i)$ with $t_i \to -\infty$ and $x_i \in B_{t_i}(z_{t_i}, D\sqrt{1 + |t_i|})$, such that $(1 + |t_i|)R(x_i, t_i) \to 0$. Take $|t_i|$ as the scaling factors when obtaining the tangent flow and let $z_{t_i} \to z_\infty \in M_\infty$. Then there is a point $x_\infty \in B(z_\infty, D)$ such that $R_\infty(x_\infty) = 0$. By the strong maximum principle, we immediately have that $(M_\infty, g_\infty, f_\infty)$ is the Euclidean space, and $\mu_\infty = 0$. The ancient solution in question must also be Euclidean space by Perelman’s monotonicity formula; this is a contradiction. \hfill $\Box$

**Proof of Corollary 2.3.** By [MZ21, Proposition 2.4], under the assumptions of Hamilton’s trace Harnack estimate (2.1) and the curvature pinching condition $|Rm| \leq CR$, Perelman’s asymptotic shrinkers exist and are smooth. Thus, by [CMZ21a], each tangent flow at infinity is smooth. We may now apply Corollary 2.2.

For the type-I case, Cao and Zhang proved the existence of Perelman’s asymptotic shrinkers in [CZ11, Theorem 4.1] and thus we can apply Corollary 2.2. \hfill $\Box$
3.3 Proofs of Corollaries 2.5 and 2.7

Corollary 2.5 bounds the scalar curvature from below according to the distance to certain anchor points on the level sets which play the same role as $H_n$-centers or $\ell$-centers (points where the $\ell$-distance is bounded by a fixed constant).

For steady solitons, it is often more convenient to work with the static metric as compared to the induced Ricci flow. Let us recall some notions from [BCMZ21]. Let $(M^n, g, f)$ be a steady gradient Ricci soliton and let $\Phi_t$ be the 1-parameter group of diffeomorphisms generated by $-\nabla f$. Fix a point $o \in M$. We define

$$\Lambda(x, \tau) := \inf \int_0^\tau \sqrt{s} \left( R_g + |\dot{\gamma} - \nabla f|^2_g \right)(\gamma(s)) ds,$$

where the infimum is taken over all $\gamma : [0, \tau] \to M$ with $\gamma(0) = o$ and $\gamma(\tau) = x$. Accordingly, define

$$\lambda(x, \tau) := \ell(\Phi_{-\tau}(x), \tau) =: \frac{1}{2\sqrt{\tau}} \Lambda(x, \tau),$$

where $\ell$ is Perelman’s $\ell$-function. Arguing as Perelman in [Per02, Section 7.1], we have that, for any $\tau > 0$, there is a point $p_\tau \in M$ such that

$$\lambda(p_\tau, \tau) = \ell(\Phi_{-\tau}(p_\tau), \tau) \leq \frac{n}{2}.$$  

Any such point $p_\tau$ is called a $\lambda$-center at time $-\tau$. See [BCMZ21, Section 2.1] for more details.

**Proof of Corollary 2.5.** Recall that by Perelman’s space-time comparison geometry, the trace Harnack estimate (2.1) implies that

$$\tau |\nabla \ell| \leq C \ell.$$  

By [BCMZ21, Lemma 2.4], for $\tau \geq \bar{\tau}(n)$, we can find a point $x_\tau \in \partial B(0)$ satisfying

$$\lambda(x_\tau, \tau_0) \leq C$$

for some $\tau_0 \in [\tau/C, C\tau]$, where $C = C(n)$. We write $\bar{x}_\tau = \Phi_{\tau_0}(x_\tau)$. Then

$$\ell(\bar{x}_\tau, \tau_0) = \lambda(x_\tau, \tau_0) \leq C.$$

By [Per02, Corollary 9.5] and (3.8), for any $\bar{x} = \Phi_{\tau_0}(x) \in M,$

$$f_{-\tau_0}(\bar{x}) \leq \ell(\bar{x}, \tau_0) \leq C + C \frac{|\bar{x}\bar{x}_\tau|}{\tau_0} \leq C + C \frac{|xx_\tau|}{\tau},$$

where $(4\pi|t|)^{-\frac{1}{2}}e^{-\frac{f}{4}}(x) = K(o, 0 | x, t)$.

By [MZ21, Theorem 1.13], Perelman’s asymptotic shrinkers exist and are smooth. We may now apply Corollary 2.2 to conclude that, for any $\bar{x} = \Phi_{\tau_0}(x) \in M, \tau \geq \bar{\tau}(n),$

$$R(x) = R(\Phi_{-\tau_0}(\bar{x})) = R(\bar{x}, -\tau_0) \geq \frac{a}{\tau_0(f_{-\tau_0}(\bar{x}) + C(n) - \mu_\infty)} \geq \frac{a'}{(C(n) - \mu_\infty)^2 + |xx_\tau|^2},$$

$\square$
Proof of Corollary 2.7. Let \((M, g_t)_{t \in (-\infty, 0]}\) be the canonical form of the steady soliton in question. Since the curvature and the pointed Nash entropy are both uniformly bounded, Bamler’s theorems [Bam20a, Bam20b, Bam20c] can be applied, due to Bamler’s explanation in [Bam21]. Thus, we may apply [BCDMZ21, Proposition 3.1] to the ancient solution \((M, g_t)_{t \in (-\infty, 0]}\). By the classification of tangent flows at infinity therein and the non-Ricci-flat assumption, we have that the tangent flow at infinity is unique and smooth. Thus, we may apply Corollary 2.2. Finally, fixing, say, \(\tau = 1\), and applying a standard Gaussian lower bound to the conjugate heat kernel (see e.g. [CCGHIKLLN10, Theorem 26.31]), we obtain the quadratic decay lower bound for the scalar curvature. More precisely, by the Bishop–Gromov volume comparison theorem, the volumes of unit balls are uniformly bounded from above: for any \(x \in M\),

\[ |B_1(x)|_g \leq C, \]

where \(|B_1(x)|_g\) is the volume of the unit geodesic ball taken with respect to the soliton metric \(g\) and \(C\) depends on the curvature bound of \(g\). It can be seen from [CCGHIKLLN10, Theorem 26.31] that

\[ (4\pi)^{-\frac{n}{2}} e^{-f_{-1}(x)} = K(o, 0 | x, -1) \geq \frac{c_1}{\sqrt{|B_1(o)|_g \sqrt{|B_1(x)|_g}}} e^{-\frac{|ax|^2}{2}} \geq \frac{c_1}{C} e^{-\frac{|ax|^2}{2}}, \]

where \(c_1\) and \(c_2\) are positive constants depending only on the curvature bound of the soliton. Hence, by \(|\nabla f| \leq \sqrt{C_1(2.2)}\), for any \(x \in M\),

\[ f_{-1}(\Phi_1(x)) \leq C(|o\Phi_1(x)|^2 + 1) \leq 2C(|x\Phi_1(x)|^2 + |ax|^2 + 1) \leq 2C(|ax|^2 + C_1 + 1). \]

The quadratic decay bound of the scalar curvature now follows by applying Corollary 2.2 to the canonical form \((M, g_t)_{t \in (-\infty, 0]}\). \(\square\)

### 3.4 Recovering the shrinker estimate for singularity models

In this section we show that we can consider Theorem 2.1 as a generalization of the lower scalar curvature estimate for shrinkers of [CLY11], at least in the case of singularity models.

Let \((M^n, g, f)\) be a complete shrinking gradient Ricci soliton satisfying

\[ \text{Ric} + \nabla^2 f = \frac{1}{2} g, \quad R + |\nabla f|^2 = \tilde{f}. \]

Let \((\Phi_t)_{t \in \mathbb{R}}\) be the 1-parameter group of diffeomorphisms generated by \(\nabla \tilde{f}\). Then

\[ g_t := |t|\Phi_{-\ln|t|}^* g \]

solves the Ricci flow and is called the canonical form generated by the shrinking soliton. Recall that if we set

\[ \tilde{f}_t := \Phi_{-\ln|t|}^* \tilde{f}, \]

then

\[ \text{Ric}_{g_t} + \nabla^2_{g_t} \tilde{f}_t = \frac{1}{2} g_t, \quad \tau(R_{g_t} + |\nabla \tilde{f}_t|_{g_t}^2) = \tilde{f}_t. \]

We have the following simple lemma.

**Lemma 3.1.** Let \((M^n, g_t)_{t < 0}\) be the canonical form induced by a complete shrinking gradient Ricci soliton \((M^n, g, f)\). Then for any \(x \in M, t < -1\),

\[ \int_M |t| \tilde{f}_t \, d\nu_{x,-1|t} = \tilde{f}(x) + \frac{\tau}{2}(-1 - t). \]
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Proof. For any $t < -1$, writing $\nu_t = \nu_{x,-1|t}$,
\[
\partial_t \int_M \tilde{f}_t \, d\nu_t = \int_M \Box \tilde{f}_t \, d\nu_t = \frac{1}{\tau} \int_M \tilde{f}_t \, d\nu_t - \frac{n}{2\tau}.
\]
See, e.g., [LW20] for the justification of integration by parts at infinity. Integrating the equation above from $t$ to $-1$, we have
\[
\tilde{f}(x) - \tau \int_M \tilde{f}_t \, d\nu_t = -\frac{\tau}{2}(-1 - t),
\]
and the conclusion follows.

As a corollary, the $H_n$-centers of any minimum point $o$ of the potential function $\tilde{f}$ on shrinkers are not far away from $o$.

**Proposition 3.2.** Let $(M^n, g_t)_{t < 0}$ be the canonical form induced by a complete shrinking gradient Ricci soliton $(M^n, g, \tilde{f})$. Let $o$ be a minimum point of $\tilde{f}$. Then for any $t < -1$, if $(z, t)$ is an $H_n$-center of $(o, -1)$, then
\[
|oz_t|_t \leq C(n)\sqrt{\tau}.
\]

**Proof.** By the fundamental growth estimate of the potential function $\tilde{f}$ by Cao and Zhou [CZ10], for any $y \in M, t < -1$,
\[
\tau \tilde{f}_t(y) \geq \frac{1}{4} (|oy_t|_t - C\sqrt{\tau})^2,
\]
where $C = C(n)$. Write $\nu_t = \nu_{o,-1|t}$. Then for any $t < -1$,
\[
\int_M |oy|^2 \, d\nu_t(y) \leq \int_M (|oy_t|_t - C\sqrt{\tau})^2 \, d\nu_t(y) + C\tau
\]
\[
\leq C\tau + 4\tau \int_M \tilde{f}_t \, d\nu_t
\]
\[
\leq C\tau,
\]
where we have used the previous lemma and the fact that $\tilde{f}(o) \leq n/2$ in the last inequality. Thus, by [Bam20a, Lemma 3.2],
\[
|oz_t|_t \leq \text{dist}^\text{W}^o_{1}(\delta_o, \nu_t) + \text{dist}^\text{W}^o_{1}(\delta_z, \nu_t) \leq C\sqrt{\tau}.
\]

**Corollary 3.3.** Let $(M^n, g_t)_{t < 0}$ be the canonical form induced by a complete non-flat shrinking gradient Ricci soliton $(M^n, g, \tilde{f})$. Suppose also that it is a singularity model or $(M^n, g)$ has bounded curvature. Then for any $D < \infty$, the condition (SCLB) holds for some $a_0 > 0$, after shifting the time by $-1$. As a consequence of Theorem 2.1, we can recover the main theorem of [CLY11] in this case.

**Proof.** Let $o$ be a minimum point of $\tilde{f}$. By Proposition 3.2, for any $t < -1$, and any $H_n$-center $(z_t, t)$ of $(o, -1)$,
\[
|oz_t|_t \leq C\sqrt{\tau}.
\]
Since \((M^n, g)\) is not Ricci-flat, we can choose \(a_0 > 0\) such that

\[
\inf_{B(o, 2(D+C))} R \geq a_0.
\]

Note that

\[
B_t(z, D\sqrt{1+\tau}) \subset B_t(o, (D+C)\sqrt{1+\tau}) = \Phi_{\ln |t|} \left( B(o, (D+C)\sqrt{(1+\tau)/\tau}) \right).
\]

Recall that \(R(x, t) = R(\Phi_{-\ln |t|}(x))/\tau\). Thus, (SCLB) holds. 

References

[ABDS22] Angenent, Sigurd; Brendle, Simon; Daskalopoulos, Panagiota; Sesum, Natasa. *Unique asymptotics of compact ancient solutions to three-dimensional Ricci flow.* Comm. Pure Appl. Math. 75 (2022), no. 5, 1032–1073.

[App19] Appleton, Alexander. *Eguchi-Hanson singularities in \(U(2)\)-invariant Ricci flow.* arXiv preprint arXiv:1903.09936 (2019).

[Bam18] Bamler, Richard H. *Long-time behavior of 3-dimensional Ricci flow – Introduction.* Geometry & Topology 22-2 (2018), 757–774.

[Bam20a] Bamler, Richard H. *Entropy and heat kernel bounds on a Ricci flow background.* arXiv preprint arXiv:2008.07093 (2020).

[Bam20b] , *Compactness theory of the space of super Ricci flows.* arXiv preprint arXiv:2008.09298 (2020).

[Bam20c] , *Structure theory of non-collapsed limits of Ricci flows.* arXiv preprint arXiv:2009.03243 (2020).

[Bam21] , *On the fundamental group of non-collapsed ancient Ricci flows,* arXiv preprint arXiv:2110.02254 (2021).

[BCMZ21] Bamler, Richard H.; Chan, Pak-Yeung; Ma, Zilu; Zhang, Yongjia. *An optimal volume growth estimate for noncollapsed steady gradient Ricci solitons.* arXiv preprint arXiv:2110.04661 (2021).

[BCDM21] Bamler, Richard H.; Chow, Bennett; Deng, Yuxing; Ma, Zilu; Zhang, Yongjia, *Four-dimensional steady gradient Ricci solitons with 3-cylindrical tangent flows at infinity.* Adv. Math. 401 (2022), Paper No. 108285, 21 pp.

[BK22] Bamler, Richard H.; Kleiner, Bruce. *Ricci flow and diffeomorphism groups of 3-manifolds.* J. Amer. Math. Soc. DOI: https://doi.org/10.1090/jams/1003 Published electronically: August 12, 2022.

[BK21a] Bamler, Richard H.; Kleiner, Bruce. *Diffeomorphism groups of prime 3-manifolds.* arXiv preprint arXiv:2108.03302 (2021).
REFERENCES

[BK21b] Bamler, Richard H.; Kleiner, Bruce. On the rotational symmetry of 3-dimensional \(\kappa\)-solutions. J. Reine Angew. Math. 779 (2021), 37–55.

[Bre13] Brendle, Simon. Rotational symmetry of self-similar solutions to the Ricci flow. Inventiones Mathematicae 194, 731–764 (2013).

[Bre20] Brendle, Simon. Ancient solutions to the Ricci flow in dimension 3. Acta Mathematica 225, 1–102 (2020).

[BDS21] Brendle, Simon; Daskalopoulos, Panagiota; Sesum, Natasa. Uniqueness of compact ancient solutions to three-dimensional Ricci flow. Invent. Math. 226 (2021), no. 2, 579–651.

[CZ10] Cao, Huai-Dong; Zhou, De-Tang. On complete gradient shrinking Ricci solitons. J. Differential Geom. 85 (2010), 175–186.

[CZ11] Cao, Xiaodong; Zhang, Qi S. The conjugate heat equation and ancient solutions of the Ricci flow. Adv. Math. 228 (2011), no. 5, 2891–2919.

[CMZ21a] Chan, Pak-Yeung; Ma, Zilu; Zhang, Yongjia. Ancient Ricci flows with asymptotic solitons. arXiv preprint arXiv:2106.06904 (2021).

[CN15] Cheeger, Jeff; Naber, Aaron. Regularity of Einstein manifolds and the codimension 4 conjecture. Ann. of Math. (2) 182 (2015), no. 3, 1093–1165.

[Che09] Chen, Bing-Long, Strong uniqueness of the Ricci flow. J. Differential Geom. 82 (2009), 363–382.

[CCGGIIKLLN10] Chow, B.; Chu, S.; Glickenstein, D.; Guenther, C.; Isenberg, J.; Ivey, T.; Knopf, D.; Lu, P.; Luo, F.; Ni, L. The Ricci flow: techniques and applications. Part III. Geometric-Analytic Aspects, Mathematical Surveys and Monographs, vol. 163, AMS, Providence, RI, 2010.

[CFSZ20] Chow, Bennett; Freedman, Michael; Shin, Henry; Zhang, Yongjia Curvature growth of some 4-dimensional gradient Ricci soliton singularity models. Adv. Math. 372 (2020), 107303, 17 pp.

[CLY11] Chow, Bennett; Lu, Peng; Yang, Bo Lower bounds for the scalar curvatures of noncompact gradient Ricci solitons. C. R. Math. Acad. Sci. Paris 349 (2011), no. 23-24, 1265–1267.

[DZ18] Deng, Yuxing; Zhu, Xiaohua. Asymptotic behavior of positively curved steady Ricci solitons, Trans. Amer. Math. Soc. 370 (2018), 2855-2877.

[DZ20a] Deng, Yuxing; Zhu, Xiaohua. Higher dimensional steady Ricci solitons with linear curvature decay. J. Eur. Math. Soc. (JEMS) 22 (2020), no. 12, 4097–4120.

[DZ20b] Deng, Yuxing; Zhu, Xiaohua. Classification of gradient steady Ricci solitons with linear curvature decay, Sci. China Math. 63 (2020), no. 1, 135–154.
[Ham88] Hamilton, Richard S. *The Ricci flow on surfaces*. Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

[Ham93] Hamilton, Richard S. *The formation of singularities in the Ricci flow*. Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995.

[Han20] Han, Daoyuan. *Asymptotic curvature estimate for steady solitons*. arXiv preprint arXiv:2009.04665 (2020).

[Lai20] Lai, Yi. *A family of 3d steady gradient solitons that are flying wings*. arXiv preprint arXiv:2010.07272 (2020), J. Differential Geom., to appear.

[Lai22] Lai, Yi. *3D flying wings for any asymptotic cones*. arXiv preprint arXiv:2207.02714 (2022).

[LW20] Li, Yu; Wang, Bing. *Heat kernel on Ricci shrinkers*. Calc. Var. Partial Differential Equations 59 (2020), no. 6, Paper No. 194, 84 pp.

[MZ21] Ma, Zilu; Zhang, Yongjia. *Perelman’s entropy on ancient Ricci flows*. J. Funct. Anal. 281 (2021), no. 9, Paper No. 109195, 31 pp.

[MSW19] Munteanu, Ovidiu; Sung, Chiung-Jue Anna; Wang, Jiaping. *Poisson equation on complete manifolds*. Adv. Math. 348 (2019), 81–145.

[MW19] Munteanu, Ovidiu; Wang, Jiaping. *Structure at infinity for shrinking Ricci solitons*. Annales Scientifiques de l’Ecole Normale Superieure 52 (2019), 891–925.

[Per02] Perelman, Grisha. *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159 (2002).

[Per03a] Perelman, Grisha. *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109 (2003).

[Per03b] Perelman, Grisha. *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv preprint math/0307245 (2003).