Non-crossing geometric spanning trees with bounded degree and monochromatic leaves on bicolored point sets

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Abstract

Let $R$ and $B$ be a set of red points and a set of blue points in the plane, respectively, such that $R \cup B$ is in general position, and let $f : R \rightarrow \{2, 3, 4, \ldots\}$ be a function. We show that if $2 \leq |B| \leq \sum_{x \in R} (f(x) - 2) + 2$, then there exists a non-crossing geometric spanning tree $T$ on $R \cup B$ such that $2 \leq \deg_T(x) \leq f(x)$ for every $x \in R$ and the set of leaves of $T$ is $B$, where every edge of $T$ is a straight-line segment.

1 Introduction and related work

Let $R$ be a set of red points and $B$ be a set of blue points in the plane. We always assume that $R$ and $B$ are disjoint and $R \cup B$ is in general position (i.e,
no three points of $R \cup B$ are collinear). Several works \cite{1,8} have considered problems on non-crossing geometric spanning trees and geometric graphs (where edges are straight-line segments) on $R \cup B$. See also the survey \cite{7}.

### 1.1 Contributions of this work

For a tree $T$ and a vertex $v$ of $T$, let us denote by $\deg_T(v)$ the degree of $v$ in $T$. A vertex of $T$ with degree one is called a leaf of $T$, and the set of leaves of $T$ will be denoted by $\text{Leaf}(T)$. Further, let us denote by $|T|$ the order of a tree $T$ (i.e., its number of vertices) and by $|X|$ the cardinality of a set $X$.

In this paper, given $R$ and $B$ in the plane as above, we aim for non-crossing geometric spanning trees $T$ on $R \cup B$ such that $\text{Leaf}(T) = B$. We prove the following theorem:

**Theorem 1** Assume that $R$ and $B$ are given in the plane and a function $f : R \to \{2, 3, 4, \ldots\}$ is given. If $2 \leq |B| \leq \sum_{x \in R} (f(x) - 2) + 2$, then there exists a non-crossing geometric spanning tree $T$ on $R \cup B$ such that $\text{Leaf}(T) = B$ and $2 \leq \deg_T(x) \leq f(x)$ for every $x \in R$. Moreover, if $|B| = \sum_{x \in R} (f(x) - 2) + 2$, then $T$ satisfies that $\deg_T(x) = f(x)$ for every $x \in R$ (see (2) and (3) of Figure 1).

![Figure 1](image_url)

Figure 1: (1) A non-crossing geometric alternating spanning tree on 7 red points and 7 blue points with maximum degree 3; (2) A red point set $R$ with labels $f(x)$ and a blue point set $B$; (3) A non-crossing geometric spanning tree $T$ on $R \cup B$ such that $\text{Leaf}(T) = B$ and $\deg_T(x) = f(x)$ for every $x \in R$.

By setting $f(x) = k$ for every $x \in R$ in the above theorem, we obtain the following corollary:

**Corollary 2** Let $k \geq 2$ be an integer. Assume that $R$ and $B$ are given in the plane. If $2 \leq |B| \leq (k-2)|R| + 2$, then there exists a non-crossing geometric spanning tree $T$ on $R \cup B$ such that $\text{Leaf}(T) = B$ and the maximum degree
of $T$ is at most $k$. Moreover, if $|B| = (k - 2)|R| + 2$, then $T$ satisfies that $\deg_T(x) = k$ for every $x \in R$.

**Observation 3** For $k = 2$, the above corollary says that if $|B| = 2$ and $|R| \geq 1$, then there exists a non-crossing geometric path which passes through all the points of $R \cup B$ and whose endvertices are the two blue points.

### 1.2 Related work

Ikebe et al. [4] proved that, given one red point $R = \{r\}$ and a set $B$ of blue points in the plane, any rooted tree $T$ with root $w$ of order $|B| + 1$ can be straight-line embedded on $\{r\} \cup B$ in such a way that $w$ is mapped to $r$ and no crossings arise. Kaneko and Kano [6] proved that, given two red points $R = \{r_1, r_2\}$ and a set $B$ of blue points in the plane, together with two rooted trees $T_1$ with root $w_1$ and $T_2$ with root $w_2$, if $|T_1| + |T_2| = |B| + 2$, then $T_1 \cup T_2$ can be straight-line embedded on $\{r_1, r_2\} \cup B$, without crossings, in such a way that $w_1$ and $w_2$ are mapped to $r_1$ and $r_2$, respectively.

Kaneko [5] considered sets $R$ and $B$ in the plane with $|R| = |B|$ and proved that, then, there exists a non-crossing geometric spanning tree $T$ on $R \cup B$ such that every edge of $T$ joins a red point to a blue point and the maximum degree of $T$ is at most 3 (see (1) of Figure 1). Finally, Biniaz et al. [2] considered sets $R$ and $B$ in the plane with $|B| \leq |R|$ and proved that, then, there exists a non-crossing geometric spanning tree $T$ on $R \cup B$ such that every edge of $T$ joins a red point to a blue point and the maximum degree of $T$ is at most $\max\{3, \lceil(|R| - 1)/|B|\rceil + 1\}$.

### 2 Proof of Theorem 1

In this section we prove Theorem 1. We first state the following proposition, which is a special case of Theorem 1.

**Proposition 4** Assume that $R$ and $B$ are given in the plane and a function $f : R \to \{2, 3, 4, \ldots\}$ is given. If $|B| = \sum_{x \in R}(f(x) - 2) + 2$, then there exists a non-crossing geometric spanning tree $T$ on $R \cup B$ such that Leaf$(T) = B$ and $\deg_T(x) = f(x)$ for every $x \in R$ (see (2) and (3) of Figure 1).

In order to prove Proposition 4 we will use the following lemma:

**Lemma 5** (Theorem 3.6 of [3], and Exercises 2.1.12 of [9]) Let $n \geq 2$ be an integer, and let $d_1, d_2, \ldots, d_n$ be positive integers. If $d_1 + d_2 + \cdots + d_n = 2n - 2$, then there exists a tree $T$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ that satisfies $\deg_T(v_i) = d_i$ for every $1 \leq i \leq n$. 


Proof of Proposition 4. We first show that there exists a geometric spanning tree $Q$ on $R \cup B$ that might have crossings but satisfies
\[
\deg_Q(x) = f(x) \quad \text{for all } x \in R, \quad \text{and} \quad \deg_Q(y) = 1 \quad \text{for all } y \in B. \tag{1}
\]
It follows from the condition of Proposition 4 that
\[
\sum_{x \in R} f(x) + \sum_{y \in B} 1 = \sum_{x \in R} (f(x) - 2) + 2|R| + |B|
\]
\[
= |B| - 2 + 2|R| + |B| = 2(|R \cup B| - 2).
\]
Hence by Lemma 5 there exists a geometric spanning tree $Q$ that satisfies the condition (1) but might have some crossings. Among all geometric spanning trees $Q$ satisfying (1), choose a geometric spanning tree $T$ on $R \cup B$ such that the sum $\sum_{xy \in E(T)} |xy|$ is minimum, where $|xy|$ denotes the length of the straight-line edge $x$ to $y$. We shall show that $T$ has no crossings.

The following three possible types of crossings could arise. First, that two edges $st$ and $uv$ of $T$ intersect, where $s, t, u, v$ are red points (see (1) of Figure 2). Since $T - st - uv$ consists of three components, by symmetry, we may assume that $u$ and $t$ are contained in the same component of $T - st - uv$, that is, $u$ and $t$ are connected by a path in $T - st - uv$. Then $T - st - uv + su + vt$ is another geometric spanning tree on $R \cup B$ satisfying the degree condition (1) and its total sum of edge lengths is smaller than that of $T$. This contradicts the choice of $T$. Hence this case does not occur.

Second, that two edges $st$ and $ux$ of $T$ intersect, where $s, t, u$ are red points and $x$ is a blue point (see (2) of Figure 2). Since $T - st - ux$ consists of three components and $\{x\}$ forms one component, $u$ and $t$ are connected by a path in $T - st - ux$ or $u$ and $s$ are connected by a path in $T - st - ux$. By symmetry, we may assume that $u$ and $t$ are connected by a path in $T - st - ux$. Then $T - st - ux + su + tx$ is another geometric spanning tree on $R \cup B$ satisfying the degree condition (1) and its total sum of edge lengths is smaller than that of $T$. This is a contradiction.

Third, that two edges $sy$ and $ux$ of $T$ intersect, where $s, u$ are red points and $x, y$ are blue points (see (3) of Figure 2). Since $T - sy - ux$ consists of three components and $\{x\}$ and $\{y\}$ form two components, $s$ and $u$ are connected by a path in $T - sy - ux$. Then $T - sy - ux + sx + uy$ is another geometric spanning tree on $R \cup B$ satisfying the degree condition (1) and its total sum of edge lengths is smaller than that of $T$. This is a contradiction.

Note that blue points being leaves implies that these three were the only possible cases for crossings and, therefore, $T$ has no crossings. Consequently, $T$ is the desired non-crossing geometric spanning tree on $R \cup B$, and Proposition 4 is proved. \(\square\)
Figure 2: (1) Two intersecting edges $st$ and $uv$ and two new edges $su$ and $vt$, which satisfy $|st| + |uv| > |su| + |vt|$; (2) Two crossing edges $st$ and $ux$ and two new edges $su$ and $tx$, where $x$ is a blue point and a leaf of $T$; (3) Two crossing edges $sy$ and $ux$ and two new edges $sx$ and $uy$, where $x$ and $y$ are blue points and leaves of $T$.

We next prove Theorem 1 by making use of Proposition 3.

Proof of Theorem 1. We may assume that $2 \leq |B| < \sum_{x \in R} (f(x) - 2) + 2$ since if $|B| = \sum_{x \in R} (f(x) - 2) + 2$, then the theorem holds by Proposition 3. It is easy to see that there exists a mapping $f' : R \to \{2, 3, 4, \ldots\}$ that satisfies $f'(x) \leq f(x)$ for all $x \in R$ and $|B| = \sum_{x \in R} (f'(x) - 2) + 2$. By Proposition 3, there exists a non-crossing geometric spanning tree $T$ such that $Leaf(T) = B$ and $\deg_T(x) = f'(x)$ for all $x \in R$. Hence, $T$ is the desired geometric spanning tree on $R \cup B$. Consequently Theorem 1 is proved. 

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