Generalized triangle inequalities in thick Euclidean buildings of rank 2

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Abstract

We give the generalized triangle inequalities which determine the possible $\Delta$-valued side lengths of $n$-gons in thick Euclidean buildings of rank 2.

1 Introduction

Let $X$ be a symmetric space of noncompact type or a thick Euclidean building. We are interested in the following geometric question:

Which are the possible side lengths of polygons in $X$?

In this context the appropriate notion of length of an oriented geodesic segment is given by a vector in the Euclidean Weyl chamber $\Delta_{\text{euc}}$ associated to $X$. If $X = G/K$ is a symmetric space, the full invariant of a segment modulo the action of $G$ is precisely this vector-valued length since we can identify $X \times X/G \cong \Delta_{\text{euc}}$ (cf. [KLM09a]). For $X$ a Euclidean building the same notion of vector-valued length can be defined (cf. [KLM09b]). We denote by $\mathcal{P}_n(X) \subset \Delta_{\text{euc}}^n$ the set of all possible $\Delta_{\text{euc}}$-valued side lengths of $n$-gons in $X$.

An algebraic question (the so-called Eigenvalue Problem), which goes back to 1912 when it was already studied by H. Weyl, is closely related to a special case of the geometric question above, namely, for the symmetric space $X = SL(m, \mathbb{C})/SU(m)$. It is one of the motivations for considering this geometric problem. The Eigenvalue Problem asks:

How are the eigenvalues of two Hermitian matrices related to the eigenvalues of their sum?

We refer to [KLM09a] for more information on the relation between these two questions and [Fu00] for more history on this problem.

In [KLM09a] and [KLM09b] it is shown that the set $\mathcal{P}_n(X)$ depends only on the spherical Coxeter complex associated to $X$ (i.e. on the spherical Weyl chamber $\Delta_{\text{sph}}$). We will therefore sometimes refer to $\mathcal{P}_n(\Delta_{\text{sph}})$ as the set of side lengths of $n$-gons in $X$ a symmetric space or a Euclidean building with $\Delta_{\text{sph}}$ as spherical Weyl chamber.

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For a symmetric space \( X = G/K \) the set of possible side lengths has been completely determined in [KLM09a]: \( P_n(X) \) is a finite sided convex polyhedral cone and it can be described as the solution set of a finite set of homogeneous linear inequalities in terms of the Schubert calculus in the homology of the generalized Grassmannian manifolds associated to the symmetric space \( G/K \). It follows, that for a Euclidean building \( X' \) with the same associated spherical Weyl chamber \( \Delta_{\text{sph}} \) as \( X \), the set \( P_n(X') \) is also a finite sided convex polyhedral cone determined by the same inequalities as \( P_n(X) = P_n(\Delta_{\text{sph}}) \).

As already pointed out in [KLM09a] for the case of exotic spherical Coxeter complexes (i.e. when it is the Coxeter complex of a Euclidean building but it does not occur for a symmetric space) the structure of the set \( P_n(\Delta_{\text{sph}}) \) cannot be described with this method, since we do not have a Schubert calculus for these Coxeter complexes. Thus, the structure of \( P_n(\Delta_{\text{sph}}) \) for these Coxeter complexes and even its convexity were unknown. It is clear that we can restrict our attention to irreducible Coxeter complexes. By a result of Tits [Ti77], exotic irreducible Coxeter complexes occur only in rank 2. Our main result is the description of \( P_n(X) \) in this case (compare with Theorem 6.14).

**Theorem 1.1.** For a Euclidean building \( X \) of rank 2, the space \( P_n(X) \) is a finite sided convex polyhedral cone. The set of inequalities defining \( P_n(X) \) can be given in terms of the combinatorics of the spherical Coxeter complex associated to \( X \).

The inequalities given in our main theorem coincide with the so-called weak triangle inequalities (cf. [KLM09a, Sec. 3.8]). Moreover, our arguments also work (see Remark 6.12) to prove the weak triangle inequalities for buildings of arbitrary rank (cf. [KLM09a, Thm. 3.34]). For symmetric spaces, these inequalities correspond to specially simple intersections of Schubert cells in the description of \( P_n(X) \) given in [KLM09a]. Their description depend only in the Weyl group of \( X \) and therefore, they can be defined for arbitrary Coxeter complexes.

Consider the side length map \( \sigma : Pol_n(X) = X^n \rightarrow \Delta_{\text{vac}}^n \). The set \( P_n(X) \) which we are interested in is nothing else than the image of \( \sigma \). We use a direct geometric approach to describe this image. Our main idea is to study the singular values of \( \sigma \) by deforming the sides of a given polygon in \( X \). This strategy was already used for the case of symmetric spaces by B. Leeb in [Le] to give a simple proof of the Thompson Conjecture (cf. [KLM09a, Theorem 1.1]). In this paper we adapt this variational method to the case of Euclidean buildings and use it to describe the space \( P_n(X) \).

Throughout this paper we state the results, whenever possible, in such a way that they apply to Euclidean buildings of arbitrary rank. In particular, Sections 4, 5 and 6.1 (except Lemma 6.6 and Proposition 6.7) do not use the assumption on the rank of the building. And when we do use the assumption, we indicate it explicitly in the statement of the corresponding result.

The set of inequalities obtained in Theorem 1.1 constitute an irredundant system defining the polyhedral cone \( P_n(X) \). The inequalities given by Schubert calculus in [KLM09a] are known to be irredundant for the cases of type \( A_n \) (see [KTW04]), however, these seem to be the only cases. A smaller set of inequalities is given in [BK06] by defining a new product in the cohomology of flag varieties. The irredundancy of this set has been recently shown in [Re10].

After a first version of this paper was written, the author learned about a recent related
paper of Berenstein and Kapovich \cite{BKa10}, where the generalized triangle inequalities for rank 2 are also determined by a different approach.

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2 Preliminaries

A very good introduction to the concepts used in this paper is the work \cite[ch. 2-4]{KL98}. We refer also to \cite{BH99} for more information on metric spaces with upper curvature bounds and to \cite[ch. 2-3]{KLM09b} for the different concepts of length in Euclidean buildings.

2.1 CAT(0) spaces

Recall that a complete geodesic metric space $X$ is said to be $CAT(0)$ if the geodesic triangles in $X$ are not thicker than the corresponding triangles in the Euclidean space.

For two points $x, y \in X$ we denote with $xy$ the geodesic segment between them. The link $\Sigma_x X$ is the completion of the space of directions at $x$ with the angle metric. $\vec{xy} \in \Sigma_x X$ denotes the direction of the segment $xy$ at $x$. 
Two complete geodesic lines $\gamma_1, \gamma_2$ are said to be parallel if they have finite Hausdorff distance, or equivalently, if the functions $d(\cdot, \gamma_i)|_{\gamma_3}$ are constant. The parallel set $P_\gamma$ is defined as the union of all geodesic lines parallel to $\gamma$. It is a closed convex set that splits as a metric product $P_\gamma \cong \mathbb{R} \times Y$, where $Y$ is also a CAT(0) space.

For a polygon $p$, or more precisely, an $n$-gon in $X$ we mean the union of $n$ oriented geodesic segments $x_0x_1, \ldots, x_{n-1}x_n$ with $x_n = x_0$. Since geodesic segments in CAT(0) spaces between two given points are unique, we can also describe $p$ by its vertices. We write $p = (x_0, \ldots, x_n)$.

**2.2 Coxeter complex**

A spherical Coxeter complex is a pair $(S, W)$ consisting of a unit sphere $S$ with its usual metric and a finite group $W$ of isometries, the Weyl group, generated by reflections on total geodesic spheres of codimension one. A Weyl chamber in $S$ is a fundamental domain of the action $W \curvearrowright S$. The model Weyl chamber is defined as $\Delta_{sph} := S/W$. We say that two points in $S$ have the same $W$-type (or just type) if they belong to the same $W$-orbit.

A Euclidean Coxeter complex is a pair $(E, W_{aff})$ consisting of a Euclidean space $E$ and a group of isometries $W_{aff}$, the affine Weyl group, generated by reflections on hyperplanes and such that its rotational part $W := rot(W_{aff})$ is finite. The set of fixed points of reflections in $W_{aff}$ are called walls of $(E, W_{aff})$. We define the $W_{aff}$-type of a point in $E$ as above. To $(E, W_{aff})$, we can associate the spherical Coxeter complex $(S, W)$, where $S := \partial_\infty E$ is the Tits boundary of $E$. The Euclidean model Weyl chamber $\Delta_{euc}$ is the complete Euclidean cone over $\Delta_{sph}$.

The link $\Sigma_x E$ of a point $x \in E$ is naturally a spherical Coxeter complex with Weyl group $\text{Stab}_{W_{aff}}(x)$. We will also use another structure on $\Sigma_x E$ as a Coxeter complex with Weyl group $W$. This will be given by the natural identification $\Sigma_x E \cong \partial_\infty E$.

The refined length of the oriented geodesic segment $xy \subset E$ is defined as the image of $(x, y)$ under the projection $E \times E \to (E \times E)/W_{aff}$. The $\Delta$-valued length, or just length, is the image of the refined length under the natural forgetful map $(E \times E)/W_{aff} \to \Delta_{euc}$. We denote with $\sigma$ the length map assigning to a segment its $\Delta$-valued length.

We can also define the refined length of an oriented segment $xy$ in the spherical Coxeter complex $(S, W)$ analogously as the image of $(x, y)$ under the projection $S \times S \to (S \times S)/W$.

**2.3 Buildings**

For an introduction to spherical and Euclidean buildings from the point of view of metric geometry, we refer to [KL98].

Let $X$ be a thick Euclidean building modelled in the Euclidean Coxeter complex $(E, W_{aff})$. The concepts of refined length and $\Delta$-valued length of an oriented geodesic segment $xy \subset X$ can be also defined naturally by identifying an apartment containing $xy$ with the Coxeter complex $(E, W_{aff})$. 


For a polygon $p = (x_0, \ldots, x_{n-1})$ in $X$, we write $\sigma(p) = (\sigma(x_0x_1), \ldots, \sigma(x_{n-1}x_0)) \in \Delta^n_{euc}$ and call $\sigma : X \to \Delta^n_{euc}$ the side length map. The space $P_n(X) := \sigma(X^n)$ is the set of possible $\Delta$-valued side lengths of $n$-gons in $X$. We say that a polygon in $X$ is regular if all its sides are regular, that is if their $\Delta$-valued lengths lie in the interior of $\Delta$. The space of regular polygons is an open dense subset of $X^n$.

We will use following result from [KLM09b] concerning the refined side lengths of polygons in $X$. We reproduce here its statement for the convenience of the reader.

**Theorem 2.1 (Transfer theorem).** Let $X$ and $X'$ be thick Euclidean buildings modelled on the same Euclidean Coxeter complex $(E, W_{aff})$. Let $p = (x_0, \ldots, x_{n-1})$ be a polygon in $X$ and let $x_0x_1$ be a segment in $X'$ with the same refined length as $x_0x_1$. Then there exists a polygon $p' = (x'_0, x'_1, \ldots, x'_{n-1})$ in $X'$ with the same refined side lengths as of $p$.

### 3 The set of functionals $\mathcal{L}_n$

We fix a vertex $v_0$ of $(E, W_{aff})$ with $\text{Stab}_{W_{aff}}(v_0) \cong W$. We obtain in this way an identification $E \cong \mathbb{R}^{\dim E}$. By fixing $v_0$ we get an embedding $W \hookrightarrow W_{aff}$ and also the (coarser) structure $(E, W)$ as Euclidean Coxeter complex. We will think of the Euclidean Weyl chamber $\Delta_{euc} \cong E/W$ as embedded in $E$, such that $\Delta_{euc}$ is a fundamental domain of the action $W \curvearrowright E$. Hence, the cone point of $\Delta_{euc}$ corresponds to $v_0$.

Let $\eta \in E$ be a maximal singular unit vector, i.e. $\overrightarrow{v_0\eta}$ is a vertex of $(\Sigma_{v_0} E, W)$. We define the following linear functional:

$$l_\eta : \Delta_{euc} \to \mathbb{R}$$

$$v \mapsto \langle v, \eta \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^{\dim E}$. We denote with $\mathcal{L}_n$ the finite set of functionals on $\Delta_{euc}^n$ of the form $L(e_1, \ldots, e_n) = l_{\eta_1}(e_1) + \cdots + l_{\eta_n}(e_n)$ where all the $\eta_i$ have the same $W$-type. We write $L = (l_{\eta_1}, \ldots, l_{\eta_n})$ for such a functional.

Let $H_L$ denote the hyperplane $L^{-1}(0) \cap \Delta_{euc}^n$ for $L \in \mathcal{L}_n$. We call $H_L$ a wall in $\Delta_{euc}^n$. The set of walls $H_L$ divide $\Delta_{euc}^n$ in finitely many convex polyhedral cones. We denote with $\mathcal{C}_n$ the family of the interiors of these cones, i.e. $\mathcal{C}_n$ is the set of the connected components of $\text{int}(\Delta_{euc}^n) \setminus \bigcup_{L \in \mathcal{L}_n} H_L$.

### 4 Polygons

#### 4.1 Holonomy map

Let $p = (x_0, \ldots, x_{n-1})$ be an $n$-gon in $X$. We say that a $n$-tuple $F = (F_1, \ldots, F_n)$ of apartments in $X$ supports the polygon $p$ if $e_i := x_{i-1}x_i \subset F_i$ and the convex set $F_i \cap F_{i+1}$ is top dimensional and contains $x_i$ in its interior.

**Remark 4.1.** If $p$ is a regular polygon then there always exists an $n$-tuple $F$ supporting $p$. $F$ can be constructed as follows: Let $A \in \Sigma_{x_0} X$ be an apartment containing $\overrightarrow{x_0x_1}$ and $\overrightarrow{x_0x_{n-1}}$ and
take \( v \in A \) antipodal to \( \vec{x}_0 \vec{x}_1 \). Extend the segment \( x_0x_1 \) a little further than \( x_0 \) in direction of \( v \) to a segment \( x_0'x_1 \). Inductively for \( i = 1, \ldots, n - 1 \) choose \( F_i \in X \) to be an apartment containing \( x_{i-1}'x_i \) and an initial part of \( x_ix_{i+1} \) and extend \( x_ix_{i+1} \) in \( F_i \) a little further than \( x_i \) to a segment \( x_i'x_{i+1} \). Finally choose \( F_n \) to contain \( x_{n-1}'x_0 \) and an initial part of \( x_0x_1 \) and \( x_0x_0' \). This last step is possible because of our first choice of \( x_0' \). The polyhedron \( F_i \cap F_{i+1} \) contains a regular segment with \( x_i \) in its interior. In particular \( F_i \cap F_{i+1} \) is top dimensional. 

Let now \( p \) be a polygon and \( F \) an \( n \)-tuple supporting it. Notice that the convex set \( F_i \cap F_{i+1} \) is a neighborhood of \( x_i \) in \( F_i \) and \( F_{i+1} \). Therefore we have:

\[
S_i := \Sigma x_i F_i = \Sigma x_i F_{i+1} = \Sigma x_i (F_i \cap F_{i+1}).
\]

So we have a natural map \( \phi_i : S_i \to S_{i+1} \) (just take parallel transport in \( F_{i+1} \) along the side \( e_{i+1} \)) and an associated holonomy map \( \phi_p : S_i \to S_i \) defined as the composition \( \phi_p = \phi_{i+n-1} \circ \cdots \circ \phi_{i+1} \circ \phi_i \). We introduce also the following notation:

\[
\phi_i^k := \phi_{i+k-1} \circ \cdots \circ \phi_i : S_i \to S_{i+k}
\]

If we identify \( S_i \) with \( \partial \infty F_i \) in the natural way, we obtain a structure of spherical Coxeter complex on \( S_i \) with Weyl group \( W \). With this structure the maps \( \phi_i \) are isomorphisms of Coxeter complexes and the holonomy map \( \phi_p \) is an element of the Weyl group \( W \). In particular the set of fixed points of \( \phi_p \) is a singular sphere in \( (S_i, W) \). Notice that the holonomy map (and therefore also its fixed points set) depends on the choice of the \( n \)-tuple \( F \) supporting \( p \). We will make use of this flexibility later.

### 4.2 Opening a polygon in an apartment

Let \( p = (x_0, \ldots, x_{n-1}) \) be a \( n \)-gon in \( X \) and let \( F \) be an \( n \)-tuple supporting it. We construct points \( x_i' \in F_1, i = 1, \ldots, n \) inductively as follows: for \( i = 0, 1 \) just set \( x_0' = x_0 \) and \( x_1' = x_1 \) and suppose we have already constructed \( x_i' \). For each \( x \in F_1 \) we can identify naturally \( \Sigma x F_1 \) with \( \partial \infty F_1 \) thus giving it a structure of spherical Coxeter complex with Weyl group \( W \). Let \( \psi_i : S_i \to \Sigma x_i' F \) be an isomorphism of spherical Coxeter complexes such that \( \psi_i (\vec{x}_i \vec{x}_{i+1}^{-1}) = \vec{x}_i' \vec{x}_{i+1}'^{-1} \). Notice that if \( p \) is regular such an isomorphism is unique. Let now \( x'_{i+1} \) be the point in \( F_i \) such that \( d(x_i', x'_{i+1}) = d(x_i, x_{i+1}) \) and \( \vec{x}_i' \vec{x}_{i+1}' = \psi_i (\vec{x}_i \vec{x}_{i+1}^{-1}) \) (see Fig. [ ]). We remark that in general \( x_n' \neq x_0' \) and \( (x_0', \ldots, x_n') \) is a polygonal path hence the expression “opening a polygon”. We can continue this process and define \( x_j' \in F_1 \) for \( j > n \).

The isomorphisms \( \psi_i \) can be chosen (and we do so) so that the induced automorphisms of \((S_1, W)\)

\[
\begin{array}{ccc}
S_i & \xrightarrow{\phi_i} & S_{i+1} \\
\downarrow{\psi_i} & & \downarrow{\psi_{i+1}} \\
\Sigma x_i' F & \cong S_1 & \longrightarrow \Sigma x_{i+1}' F \cong S_1
\end{array}
\]

are just the identity map.
4.3 Folding a polygon into an apartment

This construction was first considered in [KLM08, Sec. 6.1].

For simplicity on the notation, suppose \( p = (x_0, x_1, x_2) \) is a triangle in \( X \). There is a partition \( y_1 = x_1, y_1, \ldots, y_k = x_2 \) of the segment \( x_1x_2 \) such that the triangles \( (x_0, y_i, y_{i+1}) \) for \( i = 1, \ldots, k-1 \) are contained in an apartment \( A_i \). We define points \( \hat{y}_i \) in the apartment \( A_1 \) inductively as follows: for \( i = 1 \) set \( \hat{y}_1 = y_1 = x_1 \) and suppose we have already defined \( \hat{y}_i \). Let \( \beta_i : A_i \to A_1 \) be an isomorphism of Euclidean Coxeter complexes, such that \( \beta(x_0y_i) = x_0\hat{y}_i \). We define \( \hat{y}_{i+1} := \beta(y_{i+1}) \). We say that the polygon \( \hat{p} = (x_0, \hat{y}_1, \ldots, \hat{y}_k) \) is the result of folding the triangle \( p \) into \( A_1 \). We say that the points \( \hat{y}_i \) for \( i = 2, \ldots, k-1 \) are the break points of the folded polygon \( \hat{p} \). Notice that the segments \( x_0x_1 \) and \( x_0x_2 \) have the same refined side lengths as the segments \( x_0\hat{y}_i \) and \( x_0\hat{y}_k \) respectively. Write \( y_0 = x_0 \) and define \( \zeta_i := \overrightarrow{y_ii-1} \) and \( \xi_i := \overrightarrow{y_iy_{i+1}} \), analogously \( \hat{\zeta}_i := \overrightarrow{\hat{y}_i\hat{y}_{i-1}} \) and \( \hat{\xi}_i := \overrightarrow{\hat{y}_i\hat{y}_{i+1}} \).

A billiard triangle is a polygon \( \hat{p} = (x_0, \hat{y}_1, \ldots, \hat{y}_k) \) in an apartment \( A_1 \) such that for \( i = 2, \ldots, k-1 \) the directions \( \hat{\zeta}_i \) and \( \hat{\xi}_i \) are antipodal in the spherical Coxeter complex \( (\Sigma_{\hat{y}}, A_1, Stab_{W_{aff}}(\hat{y})) \) modulo the action of the Weyl group \( Stab_{W_{aff}}(\hat{y}) \). Clearly, a folded triangle is a billiard triangle. Conversely, the next condition is necessary and sufficient for a billiard triangle to be a folded triangle.

For \( i = 2, \ldots, k-1 \) there is a triangle \((\zeta_i', \xi_i', \zeta_i')\) in the spherical building \( \Sigma_{\hat{y}}X \) such that \( d(\zeta_i', \xi_i') = \pi \) and the refined lengths of \( \zeta_i', \xi_i' \) and \( \zeta_i, \xi_i \) are the same as of \( \hat{\zeta}_i, \hat{\xi}_i, x_0 \) and \( \hat{\xi}_i, \hat{\xi}_i, x_0 \) respectively.

We investigate now the relation between the constructions of opening and folding a polygon. Let \( p = (x_0, x_1, x_2) \) be a triangle in \( X \) and let \( F \) be a triple supporting \( p \). Observe that we can choose \( A_1 = F_1 \). Let \( \hat{p} = (x_0, \hat{y}_1, \ldots, \hat{y}_k) \) be the folded triangle. Again we identify naturally \( \Sigma_xF_j \cong S_j \) with \( \partial_xF_j \) for each \( x \in F_j \) and give the structure of spherical Coxeter complex with Weyl group \( W \).

For \( i = 1, \ldots, k-1 \). Let \( \alpha_i : S_2 \cong \Sigma_{y_i}F_2 \to S_1 \cong \Sigma_{\hat{y}_i}F_1 \) be an isomorphism of spherical Coxeter complexes so that \( \alpha_i(\zeta_i) = \hat{\zeta}_i \). Notice that for \( i = 1 \) we just have \( \alpha_1 = \phi_1^{-1} \). Analogously, let \( \alpha_k : S_3 \cong \Sigma_{x_2}F_3 \to S_1 \) be an isomorphism so that \( \alpha_k(\zeta_k = \overrightarrow{x_2x_1}) = \hat{\zeta}_k \) and let \( \alpha_0 : S_1 \cong \Sigma_{x_0}F_1 \to S_1 \) be so that \( \alpha_0(\overrightarrow{x_0x_2}) = \overrightarrow{\hat{y}_0\hat{y}_k} = \hat{\zeta}_0 \). Observe that if \( p \) is regular, then the \( \alpha_i \) are
unique.

Since \( \hat{p} = (x_0, \hat{y}_1, \ldots, \hat{y}_k) \) is a billiard triangle, there are isometries \( \mu_i \) of \( F_1 \) in the affine Weyl group \( W_{\text{aff}} \) for \( i = 0, \ldots, k \) such that \( \hat{\zeta}_i \hat{y}_i \mu_i(\hat{y}_{i+1}) \) has the same refined length as \( \zeta_i \xi_i \). In particular for \( i = 2, \ldots, k-1 \) the points \( \hat{y}_{i-1}, \hat{y}_i, \mu_i(\hat{y}_{i+1}) \) lie on a geodesic segment. Hence, we call the \( \mu_i \) the *straightening* isometries. It holds:

\[
\mu_1 \circ \cdots \circ \mu_k \circ \mu_0(\hat{y}_1) = x'_{n+1}
\]

where \( x'_{n+1} \) is constructed as in Section 4.2. Consider the natural action of \( \mu_i \) on \( S_1 \). The straightening isometries can be chosen (if \( p \) is regular then they are unique) such that

\[
\alpha_i = \mu_i \circ \alpha_{i+1} \quad \text{for } i = 1, \ldots, k - 2
\]

\[
\alpha_i = \mu_i \circ \alpha_{i+1} \circ \phi_{i-1} \quad \text{for } i = k - 1, k, 0.
\]

It follows that

\[
\mu_0^{-1} \circ \mu_k^{-1} \circ \cdots \circ \mu_1^{-1} = \phi_0 \circ \phi_2 \circ \phi_1 = \phi_p : S_1 \to S_1
\]

is the holonomy map at \( x_1 \).

The constructions for \( n \)-gons \((n > 3)\) are analogous.

### 5 Critical values of the side length map \( \sigma \)

For a *regular* value of the side length map \( \sigma \) we mean a value \( s \in P_n(X) \) for which there is a polygon \( p \) with \( \sigma(p) = s \) and such that \( \sigma \) is an open map at \( p \). First we give a sufficient condition in terms of the holonomy map for \( \sigma(p) \) being a regular value of \( \sigma \).

**Proposition 5.1.** Let \( p \) be an \( n \)-gon in \( X \) and \( \mathcal{F} \) an \( n \)-tuple supporting \( p \). Suppose that the holonomy map \( \phi_p \) has no fixed points, then the space \( P_n(X) \) is a neighborhood of \( \sigma(p) \) in \( \Delta_{\text{euc}}^n \).

The constructions for \( n \)-gons \((n > 3)\) are analogous.

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**Figure 2:** Folding and opening a triangle
Proof. Choose $\epsilon > 0$ so that $B_{x_i}(n\epsilon) \subset F_i \cap F_{i+1}$ for all $i$. For $v \in S_i$ and for $0 < t < n\epsilon$ we write $\exp(tv)$ to denote the point $x \in F_i \cap F_{i+1}$ with $d(x, x_i) = t$ and $\overline{x_i x} = v$.

We want to vary the polygon $p$ along $v \in S_i$ to a polygon $p_v = (x_0^v, \ldots, x_1^v)$ with side lengths $\sigma(e_i^v) = \sigma(e_j)$ for $j \neq i$. For this, let $t < \epsilon$ and define $x_k^v := \exp(t \phi_k^v(v))$ for $k = 0, \ldots, n-1$ where the subindices are considered modulo $n$. Notice that for $j \neq i$ the segment $e_j^v = x_{j-1}^v x_j^v$ is just a translation in the apartment $F_j$ of the segment $e_j$. Hence the condition on the side lengths above is clearly fulfilled. But since $\phi_i^v(v) = \phi_p(v) \neq v$ we get (see Fig. 3)

$$\sigma(e_i^v) = \sigma(\exp(d(x_{i-1}, x_i) \overline{x_{i-1} x_i}) - t (\phi_p(v) - v)).$$

Figure 3: Variation of the side $e_i$

Since $\phi_p$ has no fixed points the set $\{\sigma(e_i^v) \mid v \in S_i, 0 \leq t < \epsilon\}$ is a neighborhood of $\sigma(e_i)$ in $\Delta_{euc}$. This means that we can deform every side length of $p$ independently, thus $P_n(X)$ is a neighborhood of $\sigma(p)$ in $\Delta^n_{euc}$.

The next proposition says that for a building with only one vertex the critical values of $\sigma$ must lie in the walls $H_L$.

**Proposition 5.2.** Let $p$ be an $n$-gon in a thick Euclidean building $X$ which has only one vertex. Let $F$ be an $n$-tuple supporting $p$. Suppose that the holonomy map $\phi_p$ fixes a maximal singular direction. Then there exists a functional $L \in \mathcal{L}_n$, such that $L(\sigma(p)) = 0$.

Proof. First observe that we have a natural identification of any apartment with $\mathbb{R}^{\dim X}$ since we assumed that $X$ has only one vertex. This gives us also an identification $W_{eff} = W$. Let $\eta \in S_1$ be a maximal singular direction fixed by $\phi_p : S_1 \to S_1$. Let $v \in F_1$ be a unit vector with direction $\eta \in S_1$. Now open the polygon $p = (x_0, \ldots, x_n)$ in the apartment $F_1$ to the polygonal path $p' = (x_1', \ldots, x_{n+1}')$. We can also fold $p$ into $F_1$ and obtain the straightening isometry $\mu := \mu_0^{-1} \circ \mu_k^{-1} \circ \cdots \circ \mu_1^{-1}$. Recall that $\mu(x_{n+1}') = x_1'$ and $\mu(v) = v$ since $\mu$ induces the holonomy map. It then follows that $\langle x'_1, v \rangle = \langle x_{n+1}', v \rangle$.

Now let $\eta_i \in E$ be a maximal singular unit vector of the same $W$-type as $\eta$, such that $l_n(\sigma(x_{i-1} x_i)) = \langle x'_i - x'_{i-1}, v \rangle$. Set $L = (l_1, \ldots, l_n)$, then

$$L(\sigma(p)) = \int_{p'} \langle \cdot, v \rangle = \langle x'_{n+1}, v \rangle - \langle x'_1, v \rangle = 0.$$
We use next the result in [KLM09b] that \( \mathcal{P}_n(X) \) depends only on the spherical Coxeter complex to transfer the result above to arbitrary buildings.

**Corollary 5.3.** Let \( s \in \mathcal{P}_n(X) \cap \text{int} \Delta_{\text{euc}}^n \) and suppose that \( L(s) \neq 0 \) for all functionals \( L \in \mathcal{L}_n \). Then \( \mathcal{P}_n(X) \) is a neighborhood of \( s \) in \( \Delta_{\text{euc}}^n \).

**Proof.** By [KLM09b] we may assume that \( X \) has only one vertex. Let \( p \) be a regular polygon with \( \sigma(p) = s \) and let \( \mathcal{F} \) be an \( n \)-tuple supporting \( p \). By Proposition 5.2 the holonomy map has no fixed points. The result now follows from Proposition 5.1.

**Lemma 5.4.** Let \( p_k \) be a sequence of regular \( n \)-gons in \( X \) such that \( \sigma(p_k) \rightarrow s \) in \( \Delta_{\text{euc}}^n \), then there exists an \( n \)-gon \( p \) in \( X \) such that \( \sigma(p) = s \).

**Proof.** We assume again that \( X \) has only one vertex. Let \( p_k = (x_k^0, \ldots, x_k^n) \) and let \( \mathcal{F}_k = (F_k^1, \ldots, F_k^n) \) be \( n \)-tuples supporting \( p_k \). After transferring the polygons \( p_k \) (cf. Theorem 2.1) we may assume that the sides \( x_k^0 x_k^1 \) lie in the same apartment \( F \) and that \( x_k^0 \) lie in the same Euclidean Weyl chamber \( \Delta_{\text{euc}} \subseteq F \). After a small perturbation of the polygons we may also suppose that \( x_k^0 \) lie in the interior of \( \Delta_{\text{euc}} \). We open now the polygons \( p_k \) in the apartment \( F \) to polygonal paths \( p'_k = (x_k^0', \ldots, x_k^{n'}) \).

If \( x_0^k \rightarrow \infty \) in \( F \), then for \( k \) big enough \( p'_k \) must be completely contained in the interior of \( \Delta_{\text{euc}} \). In particular, folding the polygon \( p_k \) into \( F \) cannot have break points. This implies that \( p_k \) is contained in the apartment \( F \) for \( k \) big enough. Since \( \sigma(p_k) \rightarrow s \), then it is clear that the polygons \( p_k \) subconverge in \( F \) modulo translations in \( F \) to a polygon \( p \) with \( \sigma(p) = s \).

Suppose now that \( x_k^0 \) stay in a bounded region. Then after taking a subsequence we can assume that the polygonal paths \( p'_k \) converge to a polygonal path \( p' = (x_0', \ldots, x'_n) \) with \( \Delta \)-valued side lengths \( s \). We want now to lift this polygonal path near the polygons \( p_k \). Let \( \rho^k_i : F_i^k \rightarrow F \) be the isomorphisms of Euclidean Coxeter complexes that send \( x_{i-1}^k x_i^k \) to \( x_{i-1}^{k'} x_i^{k'} \). So we have \( x_i^k \in \rho^k_i(F_k^i \cap F_{i+1}^k) = \rho^k_{i+1}(F_k^i \cap F_{i+1}^k) \). Hence, for \( k \) big enough we have \( x_i^k \in \rho^k_i(F_i^k \cap F_{i+1}^k) = \rho^k_{i+1}(F_i^k \cap F_{i+1}^k) \) and we can define \( z_i^k := (\rho^k_i)^{-1}(x_i') = (\rho^k_{i+1})^{-1}(x_i') \in F_i^k \cap F_{i+1}^k \). Then \( q_k := (z_0^k, \ldots, z_n^k) \) is a polygonal path with the same side lengths as \( p' \), i.e. \( \sigma(q_k) = s \).

However \( q_k \) may still not be a closed polygon.

Notice that \( d(z_0^k, x_0^k) = d(x'_0, x_0^k) \) and \( d(z_n^k, x_n^k) = d(x'_n, x_n^k) \), thus \( d(z_0^k, z_n^k) \leq d(x'_0, x_n^k) + d(x'_n, x_0^k) \rightarrow 0 \). On the other hand, observe that \( x_0^k \) and \( x_n^k \) have the same \( W_{\text{aff}} \)-type and therefore also \( z_0^k \) and \( z_n^k \) have the same type. But \( W_{\text{aff}} \) is finite, so \( d(z_0^k, z_n^k) \) can only take finitely many values. It follows that for \( k \) big enough \( z_0^k = z_n^k \) and \( q_k \) is a closed polygon with \( \Delta \)-valued side lengths \( s \).

**Corollary 5.5.** For any open cone \( C \in \mathcal{C}_n \) the intersection \( \mathcal{P}_n(X) \cap C \) is empty or \( C \). Moreover, if \( C \subset \mathcal{P}_n(X) \), then \( \bar{C} \subset \mathcal{P}_n(X) \).

**Proof.** The intersection \( \mathcal{P}_n(X) \cap C \) is open by Corollary 5.3 and closed by Lemma 5.4.
6 The generalized triangle inequalities

6.1 Crossing the walls $H_L$

Suppose $p$ is a polygon in $X$ with $\sigma(p) = s \in H_L$ for some functional $L \in \mathcal{L}_n$. Considering Corollary 5.5, the natural question is if there is a cone $C \in \mathcal{C}_n$ such that $s \in \tilde{C} \subset \mathcal{P}_n(X)$. We would also like to describe all cones in $\mathcal{C}_n$ with this property. With this in mind we investigate in this section following question. When can we find polygons $p'$ with $\Delta$-valued side lengths near $s$ and such that $L \circ \sigma(p) > 0$ (or $< 0$)? For this we might try to study the side lengths of small perturbations of $p$. However since a Euclidean building has dimension equal to his rank, we do not have much flexibility to perturbate the polygon. Thus we must be more compliant with the variations of $p$ that we want to admit. Therefore we will often have to translate the polygon to other place in $X$ where we can perform the perturbations.

Let $L = (l_{\eta_1}, \ldots, l_{\eta_n})$ be a functional in $\mathcal{L}_n$. For the rest of this section $p = (x_0, \ldots, x_{n-1})$ will be always a regular $n$-gon such that $\sigma(p) \in H_L$.

Let $F$ be a $n$-tuple of apartments supporting $p$. Let $v_i, w_i \in S_i$ be maximal singular directions (in the structure coming from $S_i \simeq \partial_\infty F_i$ with Weyl group $W$) such that if $y_i \in F_i$, $z_i \in F_{i+1}$ are unit vectors with base point $x_i$ and directions $v_i$ and $w_i$ respectively, then $l_{\eta_i}(\sigma(e_i)) = \langle e_i, y_i \rangle$ and $l_{\eta_{i+1}}(\sigma(e_{i+1})) = \langle e_{i+1}, z_i \rangle$. Observe that $v_i, w_i$ are of the same $W$-type as $\eta_i, \ldots, \eta_n$. We will therefore sometimes write $l_{\eta_i} \circ \sigma = \langle \cdot, v_i \rangle$ and $l_{\eta_{i+1}} \circ \sigma = \langle \cdot, w_i \rangle$. Notice that $y_i$ is just the parallel translation along $e_i$ in $F_i$ of $z_i$, that is $v_i = \varphi_i(w_{i-1})$.

Lemma 6.1. If in the notation above $v_i \neq w_i$ for some $i$, then for any neighborhood $U$ of $\sigma(p)$ in $\Delta^e_{\text{euc}}$ there exist $n$-gons $p_1, p_2$ in $X$ with $\sigma(p_i) \in U$ and $L \circ \sigma(p_1) > 0 > L \circ \sigma(p_2)$.

Proof. The proof is similar to the one of Proposition 5.1. For $\epsilon > 0$ small, let $x'_i := \exp(\epsilon v_i)$. Consider the polygon $p_1 := (x_0, \ldots, x'_i, \ldots, x_{n-1})$, then

$$
L(\sigma(p_1)) = l_{\eta_i}(\sigma(x_0x_i)) + \cdots + \langle x_{i-1}x_i + \epsilon y_i, y_i \rangle + \langle x_i, x_{i+1} - \epsilon y_i, z_i \rangle + \cdots + l_{\eta_i}(\sigma(x_{n-1}x_0)) = L(\sigma(p)) + \epsilon(\langle y_i, y_i \rangle - \langle y_i, z_i \rangle) = 0.
$$

Analogously for $p_2 := (x_1, \ldots, \exp(\epsilon w_i), \ldots, x_{n-1})$ we have $L(\sigma(p_2)) < L(\sigma(p)) = 0$. $\square$

Assume now that $v_i = w_i \in S_i$ for all $i$. In particular, the holonomy map $\phi_p : S_i \rightarrow S_i$ has the fixed point $v_i$. Let $\gamma_i$ (resp. $\lambda_i$) be the line (i.e. complete geodesic) in $F_i$ (resp. $F_{i+1}$) with $x_i = \gamma_i(0) = \lambda_i(0)$ and $v_i = \dot{\gamma}_i(0) = \dot{\lambda}_i(0)$. If $\gamma_i = \lambda_i$ for all $i$, then the polygon $p$ is contained in a parallel set, namely the set $P_{\gamma_0}$ of all lines parallel to $\gamma_0$.

Lemma 6.2. Suppose $p$ is not contained in any parallel set $P_{\gamma}$, where $\gamma$ is a geodesic line with $\eta = \gamma(\infty)$ such that $v_i = \overrightarrow{x_i\gamma}$ for all $i$. Then for any neighborhood $U$ of $\sigma(p)$ in $\Delta^e_{\text{euc}}$ there exist $n$-gons $p_1, p_2$ in $X$ with $\sigma(p_i) \in U$ and $L \circ \sigma(p_1) > 0 > L \circ \sigma(p_2)$.

Proof. Let $P = (\nu_0, \ldots, \nu_{n-1})$ be an $n$-tuple of geodesic segments $\nu_i : [s^-, s^+] \rightarrow X$ with $\nu_i(0) = x_i$, $\nu_i = v_i$, and such that the convex hull $CH(\nu_i, \nu_{i+1})$ is a (2-dimensional) flat quadrilateral. Such a $P$ exists, just take the initial parts of the geodesics $\gamma_i \cap \lambda_i$. Suppose now that $P$ is maximal, i.e. the segments $\nu_i$ cannot be extended. If $|s^\pm| = \infty$, then the $\nu_i$ are
parallel geodesic lines and \( p \subset P_{\gamma} \). Hence at least one of \( s^+ \) or \( -s^- \) must be \(< \infty \). Suppose \( s = s^+ < \infty \) (the other case is analogous).

Now we want to displace \( p \) along \( \nu_i \) to the region, where it does not look locally like a parallel set anymore: set \( p' = (x'_0, \ldots, x'_{n-1}) = (\nu_0(s), \ldots, \nu_{n-1}(s)) \). Then \( p' \) is an \( n \)-gon with \( \sigma(p') = \sigma(p) \). Choose apartments \( A_i \) containing the convex sets \( CH(\nu_{i-1}, \nu_i) \). Let \( u_i := -\nu_i(s) \in \Sigma_{x'_i}(A_i \cap A_{i+1}) \) and let \( v'_i \in \Sigma_{x'_i}A_i \), \( w'_i \in \Sigma_{x'_i}A_{i+1} \) be the antipodes of \( u_i \) in \( \Sigma_{x'_i}A_i \) and \( \Sigma_{x'_i}A_{i+1} \) respectively.

If \( v'_i = w'_i \) for all \( i \), then we can extend the \( \nu_i \) inside \( A_i \cap A_{i+1} \) contradicting the maximality of \( P \). Hence, there is a \( j \) such that \( v'_j \neq w'_j \).

Moreover, if it holds for all \( i \) that \( d(x'_ix'_{i+1}, v'_i) = d(x'_ix'_{i+1}, w'_i) \), then \( u_ix'_ix'_{i+1}v'_i \) is a geodesic segment in \( \Sigma_{x'_i}X \) of length \( \pi \). Let \( z_i+1 \in A_{i+1} \) be a point near \( x'_{i+1} \) with \( x'_{i+1}z_i+1 = v'_i \). We can choose \( z_i+1 \) close enough to \( x'_{i+1} \), so that \( x'_{i}z_i+1 \) is a regular point in the same Weyl chamber as \( \overrightarrow{x'_ix'_{i+1}} \). It follows that \( x'_iz_i+1 \) lies in the intersection of the segments \( u_ix'_ix'_{i+1}v'_i \) and \( u_ix'_ix'_{i+1}w'_i \). Thus \( u_ix'_iz_i+1v'_i \) is a geodesic segment of length \( \pi \). Let now \( z_i \in A_i \) be a point with \( x'_iz_i = v'_i \) and so that \( CH(x'_i, z_i, z_i+1) \) is a flat triangle. It follows that the union of the \( (2\text{-dimensional}) \) flat convex sets \( CH(x_i, x_{i+1}, x'_i, x'_1), \) \( CH(x'_i, z_{i+1}, z_i) \) and \( CH(x'_i, z_i, z_{i+1}) \) is a flat convex quadrilateral. (See Figure 4.) Notice also that \( \nu_i(s^-)z_i \) are extensions of the geodesic segments \( \nu_i(s^-)u_i(s^+) \). Thus this contradicts as well the maximality of \( P \). Hence, there is a \( j \) such that \( d(x'_jx'_{j+1}, v'_j) > d(x'_jx'_{j+1}, w'_j) \).

![Figure 4: Extending the geodesics \( \nu_i \)](image)

Let \( \tilde{x}_j := \exp(\epsilon v'_j) \) in \( A_j \) for some small \( \epsilon > 0 \). Then \( \sigma(\tilde{x}_jx'_{j+1}) = \sigma(x'_jx'_{j+1}) - \epsilon \tilde{\eta} = \sigma(x_jx_{j+1}) - \epsilon \tilde{\eta} \) for some unit vector \( \tilde{\eta} \in E \) of the same type as \( \eta_{j+1} \). By the above consideration we must have \( \tilde{\eta} \neq \eta_{j+1} \), otherwise \( d(x'_jx'_{j+1}, v'_j) = d(x'_jx'_{j+1}, w'_j) \). In particular \( l_{\eta_{j+1}}(\sigma(\tilde{x}_jx'_{j+1})) = (\sigma(x_jx_{j+1}) - \epsilon \tilde{\eta}, \eta_{j+1}) > l_{\eta_{j+1}}(\sigma(x_jx_{j+1})) - \epsilon \). On the other hand, \( l_{\eta_j}(\sigma(x'_{j-1}\tilde{x}_j)) = (x'_{j-1}x'_j + \epsilon v'_j, v'_j) = l_{\eta_j}(\sigma(x'_{j-1}x'_j)) + \epsilon \). Thus, for \( p_1 := (x'_1, \ldots, \tilde{x}_j, \ldots, x'_{n-1}) \) we have \( L(\sigma(p_1)) > L(\sigma(p)) = 0 \).

Analogously for \( p_2 := (x'_1, \ldots, \exp(\epsilon w'_j), \ldots, x'_{n-1}) \) we get \( L(\sigma(p_2)) < L(\sigma(p)) = 0 \).

The next question is what happens when \( p \) is contained in such a parallel set \( P_{\gamma} \). In this last situation we cannot always get the same conclusion as in Lemmata 6.1 and 6.2. For instance, if the wall \( H_L \) lies in the boundary of \( \mathcal{P}_{\alpha}(X) \), then we can cross \( H_L \) in one direction but not in the opposite one.
Remark 6.3. Suppose that $p$ is contained in $P_\gamma$. Let $b_{\eta_-} : X \to \mathbb{R}$ be a Busemann function associated to $\eta_- = \gamma(-\infty)$ (see e.g. [KLM09b, Sec. 2.2] for a definition). Then by considering an apartment parallel to $\gamma$ containing the side $x_{i-1}x_i$, we see that $l_{\eta_-}(\sigma(x_{i-1}x_i)) = b_{\eta_-}(x_i) - b_{\eta_-}(x_{i-1})$. In particular

$$L(\sigma(p)) = l_{\eta_-}(\sigma(x_0x_1)) + \cdots + l_{\eta_-}(\sigma(x_{n-1}x_n)) = \sum_{i=1}^{n}(b_{\eta_-}(x_i) - b_{\eta_-}(x_{i-1})) = 0.$$  

Thus, if $p'$ is the result of a small variation of the polygon $p$ within the parallel set $P_\gamma$, it still holds $L(\sigma(p')) = 0$.

The next lemma gives a condition that let us cross the wall $H_L$ in the positive direction.

Suppose $p$ is contained in $P_\gamma$. Assume also that there are vertices $x_i, x_j, x_{j+1}$ of $p$ with the following property. Let $A_0, A_1$ be apartments in $P_\gamma$ containing the segment $x_jx_{j+1}$ and an initial part of the segment $x_jx_i$ and $x_{j+1}x_i$ respectively. Let $y_k \in A_k$ for $k = 0, 1$ be points in the initial parts of the segments $x_jx_i$ and $x_{j+1}x_i$ respectively. Thus $x_jx_{j+1}y_k$ are flat triangles in $A_k$. Suppose that for some $k = 0, 1$ there is a singular hyperplane $w_k \subset A_k$ such that the directions $\eta = \gamma(\infty), \overrightarrow{x_jx_{j+1}}$ and $(-1)^k\overrightarrow{y_kx_{j+k}}$ lie in the same open half space determined by $w_k$ (after the natural identification of $\partial_+ A_i$ and $\Sigma_xA_i$ for $x \in A_i$). (See Figure 5)

![Figure 5: Setting of Lemma 6.4](image)

Lemma 6.4. Under the assumptions above, for any neighborhood $U$ of $\sigma(p)$ in $\Delta^n_{\text{cuc}}$ there is an $n$-gon $\bar{p}$ in $X$ with $\sigma(\bar{p}) \in U$ and $L \circ \sigma(\bar{p}) > 0$.

Proof. We show the lemma when the singular hyperplane $w_k$ exists for $k = 0$. The other case $k = 1$ is analogous.

Denote $h^\pm$ the open half space of $A_0$ determined by $w_0$ containing the direction $\gamma(\pm\infty)$. Let $\epsilon > 0$ be small. First we displace the polygon $p$ along $\gamma$ such that $x_j$ lies in $h^+$ and $d(x_j, w_0) < \epsilon$. Let $A'_0$ be an apartment in $X$ such that $A_0 \cap A'_0 = \overline{h^-}$. Let $x'_j \in A'_0$ be the point such that $d(y_0, x'_j) = d(y_0, x_j)$ and $\overrightarrow{y_0x'_j} = \overrightarrow{y_0x_j}$. Let $z \in A_0$ be the reflection of $x_j$ in the hyperplane $w_0$.

Observe that $x'_j \notin A_0$ and $x_{j+1} \notin A'_0$. It follows that $\sigma(x'_jx_{j+1}) = \sigma(zx_{j+1})$. In particular

$$l_{\eta_{j+1}}(\sigma(x'_jx_{j+1})) = l_{\eta_{j+1}}(\sigma(zx_{j+1})) = \langle zx_{j+1}, \eta \rangle = l_{\eta_{j+1}}(\sigma(x_jx_{j+1})) + \langle zx_j, \eta \rangle > l_{\eta_{j+1}}(\sigma(x_jx_{j+1})).$$

Notice that the refined length of $x'_jx_i$ is the same as of $x_jx_i$. Hence, by Theorem 2.1 we can transfer the polygon $(x_i, x_{i+1}, \ldots, x_j)$ to a polygon $(x'_i, x'_{i+1}, \ldots, x'_j)$ with the same $\Delta$-valued side lengths. The $n$-gon $\bar{p} = (x'_i, x'_{i+1}, \ldots, x'_j, x_{j+1}, \ldots, x_{i-1})$ satisfy the conclusion of the lemma. □
If the polygon \( p \) is completely contained in an apartment in \( P_{\gamma} \), then the condition for the lemma above can be stated more easily.

**Corollary 6.5.** Suppose \( p \) is contained in an apartment \( A \subset P_{\gamma} \). Suppose there are two sides \( x_i x_{i+1}, x_j x_{j+1} \) of \( p \) and a singular hyperplane \( w \subset A \), such that the directions \( \eta = \gamma(\infty), \overrightarrow{x_i x_{i+1}} \) and \( \overrightarrow{x_j x_{j+1}} \) lie in the same open half space determined by \( w \). Then for any neighborhood \( U \) of \( \sigma(p) \) in \( \Delta_{\text{euc}}^n \) there is an \( n \)-gon \( \bar{p} \) in \( X \) with \( \sigma(\bar{p}) \in U \) and \( L \circ \sigma(\bar{p}) > 0 \).

**Proof.** Consider the segments \( d_1 = x_i x_j \) and \( d_2 = x_j x_i \). After a small variation of the polygon \( p \) inside of the apartment \( A \), we may assume that \( d_1 \) (and therefore also \( d_2 \)) is regular. Then for one \( k = 1, 2 \), it must hold, that \( d_k \) and \( \eta \) lie in the same open half space determined by \( w \). Suppose w.l.o.g. \( k = 1 \). Then Lemma 6.4 applies for the vertices \( x_i, x_j, x_{j+1} \). \( \square \)

Let us assume now that the building \( X \) has rank 2. We explain another method special for this case to cross the wall \( H_L \).

Let \( p = (x_0, x_1, x_2) \) be a regular triangle contained in \( P_{\gamma} \) but not contained in any apartment. It is easy to see, that when we fold \( p \) into an apartment \( A \), it has exactly one break point. After relabeling the vertices we can assume that the break point \( y \) lies in the side \( x_1x_2 \) and that the sides of the folded triangle \( \bar{p} = (\hat{x}_0 = x_0, \hat{x}_1 = x_1, y, \hat{x}_2) \) do not intersect in their interiors (see Figure 6). After displacing \( \bar{p} \) along \( \gamma \) we can assume that \( y \) is a vertex of \( X \). We can take \( \gamma \) to be contained in \( A \) and go through \( y \).

**Lemma 6.6.** We use the setting above (in particular, rank(\( X \)) = 2). Suppose that the Weyl chamber containing \( yx_1^2 \) is not adjacent to \( \Sigma_y \gamma \). Then for any neighborhood \( U \) of \( \sigma(p) \) in \( \Delta_{\text{euc}}^3 \) there are triangles \( p_1, p_2 \) in \( X \) with \( \sigma(p_1) \in U \) and \( L \circ \sigma(p_1) > 0 \).

**Proof.** We identify \( A \) with \( \mathbb{R}^2 \) by taking \( y \) to the origin. For a unit vector \( a \in A \) we write \( h_a^+ := \{ \pm \langle \cdot, a \rangle > 0 \} \). Let \( \ell \subset A \) be the singular line through \( y \) such that \( \Sigma_y \ell \) is adjacent to the simplicial convex hull of \( yx_1^2x_2 \) and the directions \( \eta = \gamma(\infty), \overrightarrow{yx_1^2} \) and \( \overrightarrow{yx_2} \) are in the same open half plane determined by \( \ell \). It exists by the assumptions of the lemma. Let \( \ell' \subset A \) be the reflection of \( \ell \) in \( \gamma \). Let \( u, v, v' \) be unit vectors orthogonal to \( \gamma, \ell \) and \( \ell' \) respectively and such that \( x_0 \in h_u^+ \) and \( \eta = \gamma(\infty) \in h_v^+ \cap h_{v'}^+ \). Then the simplicial convex hull of \( \overrightarrow{yx_1^2x_2} \) is \( \Sigma_y (h_v^+ \cap h_{v'}^+) \). (See Figure 6)

Let \( A_3 \) be an apartment in \( X \) such that \( A \cap A_3 = \overrightarrow{h_u^- \cap h_v^-} \). Let \( x_2' \in A_3 \) be the point so that \( d(x_0, x_2') = d(x_0, \hat{x}_2) \) and \( \overrightarrow{x_0x_2'} = \overrightarrow{x_0\hat{x}_2} \). Notice that \( \hat{x}_2 \notin A_3 \), thus, \( x_2' \neq \hat{x}_2 \). Observe also that \( x_2' \notin P_{\gamma} \), hence, \( x_2' \neq x_2 \).

Let \( \zeta := \gamma(-\infty) \). The concatenation of the segments \( \overrightarrow{yx_1^2z} \in \Sigma_y A \) and \( \overrightarrow{yz_2x_2} \in \Sigma_y A_3 \) gives a segment in \( \Sigma_y X \) of length \( \pi \) (see Figure 7). Therefore \( x_1x_2 \) is a geodesic segment and the triangle \( p' = (x_0, x_1, x_2') = (z_0, z_1, z_2) \) has the same side lengths as \( p \). Set \( A_1 := A \) and let \( A_2 \) be an apartment in \( X \) containing the segment \( z_1z_2 \).

Let \( \nu_i \) be the geodesic rays with \( \nu_i(0) = z_i \) and \( \nu_i(-\infty) = \zeta \). Then \( CH(\nu_i, \nu_{i+1}) \) are (2-dimensional) flat stripes. We want to see that the \( \nu_i \) cannot be extended to parallel geodesic lines. Suppose then the contrary: there are parallel geodesic lines \( \nu'_i \) containing \( \nu_i \). Set \( \eta' := \nu'_i(\infty) \). Then \( p' \subset Y := P_{\nu'_i} \) and in particular, \( \overrightarrow{y\zeta}, \overrightarrow{y\zeta_2} \in \Sigma_y Y \). Since \( \overrightarrow{y\zeta_1}, \overrightarrow{y\zeta_2} \in \Sigma_y A_2 \) are antipodal regular points, the apartment containing them is unique. Therefore \( \Sigma_y A_2 \subset \Sigma_y Y \).
Let $k \in \{1, 2\}$ so that the Weyl chamber containing $\overrightarrow{yy_k}$ is adjacent to $\Sigma_y\ell'$. Let $\sigma_k \subset \Sigma_yA_{2k-1}$ be the Weyl chamber containing $\overrightarrow{yz_k}$ and let $\tilde{\sigma}_k \in \Sigma_y(A_1 \cap A_3)$ be the antipodal chamber to $\sigma_k$. (See Figure 7 for $k = 2$.) Notice that $\overrightarrow{yz_0}$ intersects $\tilde{\sigma}_k$ in its interior. In particular $\tilde{\sigma}_k \subset \Sigma_yY$. It follows that the unique apartment containing $\sigma_k$ and $\tilde{\sigma}_k$ is contained in $\Sigma_yY$, i.e. $\Sigma_yA_{2k-1} \subset \Sigma_yY$.

![Figure 6: The folded triangle $\hat{p}$](image)

![Figure 7: $\Sigma_yX$](image)

Let $\sigma \subset \Sigma_y(A_1 \cap A_3) \subset \Sigma_yY$ be the Weyl chamber adjacent to $\ell$. The Weyl chamber containing $\overrightarrow{yz_{3-k}}$ is antipodal to $\sigma$. Hence, the unique apartment containing $\sigma$ and $\overrightarrow{yz_{3-k}}$ is contained in $\Sigma_yY$, i.e. $\Sigma_yA_{5-2k} \subset \Sigma_yY$.

We have conclude that $A_1, A_3 \subset \Sigma_yY = \Sigma_yP_{y_0'}$, but this is not possible because of the construction of $A_3$. Therefore the geodesic rays $\nu_i$ cannot be extended to complete parallel geodesic lines. The lemma now follows from Lemma 6.2 and its proof.

We can show now that for rank 2 the space $\mathcal{P}_n(X)$ is a polyhedral cone. Its convexity will be shown in the next section.

**Proposition 6.7.** If $X$ has rank 2, then $\mathcal{P}_n(X)$ is a union of the closures of polyhedral cones in $\mathcal{C}_n$. 


Proof. We have already seen in Corollary 5.5 that if for \( C \in C_n \) holds \( \mathcal{P}_n(X) \cap C \neq \emptyset \), then \( \overline{C} \subset \mathcal{P}_n(X) \). Now let \( p = (x_0, \ldots, x_{n-1}) \) be a polygon in \( X \). We want to show that \( \sigma(p) \) is contained in \( \overline{C} \) for some \( C \in C_n \). Since any polygon can be approximated by regular polygons, we may assume that \( p \) is regular. Suppose now \( s := \sigma(p) \in H_L \). If for any neighborhood \( U \) of \( s \) we can find polygons with side lengths in \( U \setminus H_L \), then we are done. Indeed, in this case, there is an open cone \( C \in C_n \) such that \( \mathcal{P}_n(X) \cap C \neq \emptyset \) and \( s \in \overline{C} \).

Suppose then that for some neighborhood \( U \) of \( \sigma(p) \) we cannot find polygons \( p' \) with side lengths in \( U \) and \( L \circ \sigma(p') \neq 0 \). Lemmata 6.1 and 6.2 implies that \( p \) lies in a parallel set \( P_\gamma \) and the functional \( L \) is given in \( p \) by taking scalar product with the direction of \( \eta = \gamma(\infty) \). Suppose first that the triangle \( t = (x_0, x_1, x_2) \) lies in an apartment parallel to \( \gamma \). Then it is easy to see that Lemma 6.4 must apply for one of the functionals \( L' = (l_{\eta_1}, l_{\eta_2}, l_{\eta'}) \) or \( -L' \), where \( \eta' \) is so that \( l_{\eta'}(\sigma(x_2x_0, \eta)) = (x_2x_0, \eta) \). If \( t \) is not contained in an apartment, then we fold it into an apartment as in the setting of Lemma 6.6. Then, either Lemma 6.6 applies or the Weyl chamber containing the direction \( \vec{x_i}x_{i+1} \) of the side of \( t \) with the break point must be adjacent to \( \gamma \). If the last occurs, it is again easy to see, that Lemma 6.4 must apply for \( L' \) or \( -L' \). In either case, we find a triangle \( t'' = (x''_0, x''_1, x''_2) \) with \( L' \circ \sigma(t) \neq 0 \) and such that (modulo displacement along \( \gamma \)) the refined side lengths of \( t'' \) are as near as we want to the ones of \( t \). After a small variation of the polygon \( (x_0, x_2, \ldots, x_{n-1}) \) inside the parallel set \( P_\gamma \) and displacing it along \( \gamma \), we obtain a polygon \( q = (x''_0, x''_2, \ldots, x''_{n-1}) \) so that the refined side length of \( x''_0x''_2 \) is the same as of \( x''_0x''_2 \). Then by the Transfer Theorem 2.1 we can glue \( t \) and \( q \) along \( x''_0x''_2 \) and \( x''_0x''_2 \) to a polygon \( p' \) with \( \Delta \)-valued side lengths near \( s \) and \( L(\sigma(p)) \neq 0 \).

Remark 6.8. Proposition 6.7 is also true in rank > 2 by the results of [KLM09a] and [KLM09b]. However our proof here uses Lemma 6.6 which we only showed in rank 2.

6.2 The boundary of \( \mathcal{P}_n(X) \)

We have seen in the previous section different methods which allows to cross certain walls \( H_L \) within the space \( \mathcal{P}_n(X) \). We will show in this section that for the case of buildings of rank 2 the walls where this method cannot be applied are precisely the walls that determine the boundary of \( \mathcal{P}_n(X) \). That is, if a wall cannot be crossed with the methods of Section 6.1, it is because that wall cannot be crossed at all.

First we characterize the walls \( H_L \) that cannot be crossed with the methods above in terms of the combinatorics of the associated spherical Coxeter complex \( (S, W) \). Let \( \eta \in \Delta_{euc} \subset E \) be a maximal singular unit vector (we use the same notation as in Section 3). We define the following set of singular hyperplanes of \( E \) through \( v_0 \) (i.e. walls of \( (E, W) \)):

\[
T_\eta := \{ w \subset E \mid w \text{ is a wall of } (E, W) \text{ not containing } \eta \}.
\]

For each element \( \omega \in W \cong \text{Stab}_{W_{aff}}(v_0) \) we define the subset of \( T_\eta \)

\[
T_\eta^\omega := \{ w \in T \mid \eta \text{ and } \omega \Delta_{euc} \text{ lie in the same half space determined by } w \}.
\]

Finally define \( B_\eta \) as the set of \( n \)-tuples \((\eta_1, \ldots, \eta_n) \in (W\eta)^n \) such that for \( i = 1, \ldots, n \) there are \( \omega_i \in W \) with \( \omega_i \eta_i = \eta \) and with the following properties:
\((*)\) \(T^\omega_i \cap T^\omega_j = \emptyset\) for all \(i \neq j\),

\((**)\) \(\bigcup_{i=1}^n T^\omega_i = T^\eta\).

For \(\bar{\eta} = (\eta_1, \ldots, \eta_n) \in (W\eta)^n\) write \(L_{\bar{\eta}} = (l_{\eta_1}, \ldots, l_{\eta_n})\). Let \(B_n \subseteq \mathcal{L}_n\) be the union of the sets \(\{L_{\bar{\eta}} \mid \bar{\eta} \in B_\eta\}\) for all maximal singular unit vectors \(\eta \in \Delta_{\text{euc}}\).

We will see in Lemma 6.13 below that the walls \(H_L\) that cannot be crossed with our previous methods are precisely the ones of the form \(L_{\bar{\eta}}\) with \(\bar{\eta} = (\eta_1, \ldots, \eta_n)\) satisfying the property \((*)\). A motivation for this property \((*)\) can already be seen in Corollary 6.5. The property \((**)\) is introduced to avoid later obvious redundancies in the set of generalized triangle inequalities. This can be seen in the Proposition 6.11.

**Lemma 6.9.** If \((E, W)\) has rank 2, then \(\bar{\eta} \in B_\eta\) if and only if for \(i = 1, \ldots, n\) we can find \(\omega_i \in W\) with \(\omega_i \eta_i = \eta\) such that there exist \(j, j' \in \{1, \ldots, n\}\), \(j \neq j'\) with \(\omega_j \Delta_{\text{euc}}\) antipodal to \(\omega_{j'} \Delta_{\text{euc}}\) and \(\omega_i \Delta_{\text{euc}}\) adjacent to \(-\eta\) for \(i \neq j, j'\).

**Proof.** \((\Leftarrow)\). \(\omega_j \Delta_{\text{euc}}\) is antipodal to \(\omega_{j'} \Delta_{\text{euc}}\) if and only if \(T^\omega_j = T^{\omega_{j'}} = T^\eta \setminus T_{\eta_j}^{\omega_j} = T_{\eta_j}^{\omega_j} = T_{\eta_{j'}}^{\omega_{j'}}\). On the other hand, \(\omega_i \Delta_{\text{euc}}\) is adjacent to \(-\eta\) if and only if \(T_{\eta_i}^{\omega_i} = \emptyset\).

\((\Rightarrow)\). By property \((**)\), there is a \(j\) with \(T_{\eta_j}^{\omega_j} \neq \emptyset\). If \(T_{\eta_j}^{\omega_j} = T_{\eta_j}^{\omega_{j'}} = T^\eta\), then \(\omega_j \Delta_{\text{euc}} = \Delta_{\text{euc}}\) and the assertion is clear. Otherwise let \(\ell_1 \in T_{\eta_j}^{\omega_j}\) be the singular line adjacent to \(\omega_j \Delta_{\text{euc}}\). Let \(\ell_2\) be the other singular line adjacent to \(\omega_{j'} \Delta_{\text{euc}}\). Then \(\ell_2 \notin T_{\eta_j}^{\omega_{j'}}\). Let \(j'\) be such that \(\ell_2 \in T_{\eta_{j'}}^{\omega_{j'}}\), it follows that \(\ell_1 \notin T_{\eta_{j'}}^{\omega_{j'}}\) and \(\omega_{j'} \Delta_{\text{euc}}\) must be antipodal to \(\omega_j \Delta_{\text{euc}}\). The rest follows as in the first part. \(\square\)

**Remark 6.10.** The \(\Leftarrow\) direction in Lemma 6.9 holds for arbitrary rank. Let \(B^w_n \subseteq \mathcal{L}_n\) be the set of functionals \(L_{\bar{\eta}}\) for \(\bar{\eta}\) with this property (the assumption in the \(\Leftarrow\) direction). The inequalities \(L \leq 0\) for \(L \in B^w_n\) are the so-called \textit{weak triangle inequalities} (cf. [KLM09a, Section 3.8]). Thus, Lemma 6.9 states that \(B^w_n \subseteq B_n\) and for rank 2 also holds \(B_n = B^w_n\).

![Figure 8: \(B^w_n\): weak triangle inequalities](image)

**Proposition 6.11.** Suppose \(X\) has rank 2. For any \(n\)-gon \(p\) in \(X\) and any functional \(L \in B_n\) holds \(L \circ \sigma(p) \leq 0\). That is, \(\mathcal{P}_n(X) \subseteq \bigcap_{L \in B_n} \{L \leq 0\}\).
Moreover, if $\bar{\eta} \in (W\eta)^n$ satisfies the property (*) but not the property (**), then there is a $\bar{\eta}' \in B_{\eta}$ so that $L_{\bar{\eta}} \circ \sigma(p) \leq L_{\bar{\eta}'} \circ \sigma(p)$ for all n-gons $p$ in $X$. If $p$ is regular, then the strict inequality holds.

Proof. Let $p = (x_0, \ldots, x_{n-1})$ be an n-gon in $X$. For the functional $L = (l_{\eta_1}, \ldots, l_{\eta_n}) \in B_n$, let $\omega_i \in W$ and $j, j'$ be as in Lemma 6.9. Notice that since for $i \neq j, j'$, $\omega_i \Delta_{\text{euc}}$ is adjacent to $-\eta$ and $\omega_i \eta_i = \eta$ then we have $l_{\eta_i} \leq l_{\eta'}$ in $\Delta_{\text{euc}}$ for all $\eta'$ of the same type as $\eta$. That is, $l_{\eta_i}$ is the smallest functional of the same type as $\eta$. After shifting the subindices of the polygon and the functional we can assume that $j = 1$.

Suppose first that $j' = j - 1$, that is $j' = n$. Fold the polygon $p$ into an apartment $A$, so that the broken sides are $x_1x_2, \ldots, x_{n-2}x_{n-1}$. Let $\rho : A \to E$ be an isometry that sends $x_0$ to the vertex of $\Delta_{\text{euc}} \subset E$, induces an isomorphism of the Coxeter complexes $(\partial_\infty A, W)$ and $(E, W)$ and so that $\rho(x_0x_1) \subset \omega_1 \Delta_{\text{euc}}$. Notice that $\rho$ is not necessarily an isomorphism of Coxeter complexes with the Weyl group $W_{aff}$. Denote with $q$ the image under $\rho$ of the folded polygon. By folding $E$ onto the Euclidean Weyl chamber $\omega_1 \Delta_{\text{euc}}$ with the natural “accordion” map, we obtain a further folded polygon $q' = (y_0, \ldots, y_k)$ where $y_0$ is the vertex of $\Delta_{\text{euc}}$ and the $\Delta$-valued side lengths of $y_0y_1, y_ky_0 \subset \omega_1 \Delta_{\text{euc}}$ are the same as for $x_0x_1$ and $x_{n-1}x_0$ respectively. Observe that $q'$ is not necessarily a billiard polygon in $(E, W_{aff})$, but if the side $x_rx_{r+1}$ of $p$ is broken in $q'$ to the sides $y_ry_{r+1}, y_{r+1}y_{r+2}, \ldots, y_{t-1}y_t$, then the vectors $\sigma(y_ry_{r+1}), \ldots, \sigma(y_{t-1}y_t)$ are just multiples of $\sigma(x_rx_{r+1})$. This means, that if $W_{aff}$ is the group generated by $W_{aff}$ and the whole translation group of $E$, then $q'$ is a billiard polygon in $(E, W_{aff})$. Notice also that for $r \neq 1, n$ holds $l_\eta_i (\sigma(y_r y_{r+1})) \leq \langle y_{r+1}, \eta \rangle - \langle y_r, \eta \rangle$ because of the observation at the beginning of the proof. It follows that

$$l_{\eta_2}(\sigma(x_1x_2)) + \cdots + l_{\eta_{n-1}}(\sigma(x_{n-2}x_{n-1})) \leq \langle y_k, \eta \rangle - \langle y_1, \eta \rangle.$$ 

On the other hand, since $y_0y_1, y_ky_0 \subset \omega_1 \Delta_{\text{euc}}$ and $\omega_n \Delta_{\text{euc}}$ is antipodal to $\omega_1 \Delta_{\text{euc}}$, it follows that

$$l_{\eta_1}(\sigma(x_0x_1)) = l_{\eta_1}(\sigma(y_0y_1)) = \langle y_1, \eta \rangle - \langle y_0, \eta \rangle$$

and

$$l_{\eta_n}(\sigma(x_{n-1}x_0)) = l_{\eta_n}(\sigma(y_ky_0)) = \langle y_0, \eta \rangle - \langle y_k, \eta \rangle.$$

Hence, $L(\sigma(p)) \leq \langle y_k, \eta \rangle - \langle y_1, \eta \rangle + \langle y_1, \eta \rangle - \langle y_0, \eta \rangle + \langle y_0, \eta \rangle - \langle y_k, \eta \rangle = 0$.

The general case now follows from the special case above by considering the polygons $p_1 = (x_{j'-1}, x_{j-1}, x_j, \ldots, x_{j'-2})$, i.e., $p_1$ is the polygon $p$ with the vertices $x_{j'}, x_{j'+1}, \ldots, x_{j-2}$ deleted, and $p_2 = (x_{j-1}, x_{j'-1}, x_j', \ldots, x_{j-2})$ with the functionals $(l_{\eta_j}, l_{\eta_{j}}, l_{\eta_{j+1}}, \ldots, l_{\eta_{j'-1}})$ respectively $(l_{\eta_j}^-, l_{\eta_{j'}}, l_{\eta_{j'+1}}, \ldots, l_{\eta_{j-1}}^-)$. Indeed, notice that since $\omega_j \Delta_{\text{euc}}$ and $\omega_{j'} \Delta_{\text{euc}}$ are antipodal, it follows

$$l_{\eta_j}(\sigma(x_{j'-1}x_{j-1})) = -l_{\eta_j}^-(\sigma(x_{j-1}x_{j'-1})).$$

For the second assertion, let $\tilde{\omega}_j \eta_i = \eta$ satisfy the property (*). It is easy to see that in rank 2 at most for two indices $i$ can hold $T_\eta^{\tilde{\omega}_j} \neq \emptyset$. Let $j \neq j'$ be so that $T_\eta^{\tilde{\omega}_j} = \emptyset$ for all $i \neq j, j'$. If $\tilde{\eta}$ does not satisfy the property (**), then $\tilde{\omega}_j \Delta_{\text{euc}}$ is not antipodal to $\omega_j \Delta_{\text{euc}}$. Let $\hat{\omega}_j \in W$ be so that $\hat{\omega}_j \Delta_{\text{euc}}$ is antipodal to $\omega_j \Delta_{\text{euc}}$. From the property (*) follows that for $\hat{\eta}_j := \hat{\omega}_j^{-1} \eta$ holds $l_{\eta_j^=} \leq l_{\hat{\eta}_j}$ and since $\hat{\eta}_j \neq \eta_j$ the strict inequality holds for regular segments.

Remark 6.12. The same proof as for the first assertion of Proposition 6.11 works for buildings of arbitrary rank to prove the weak triangle inequalities (see Remark 6.10). That is,

$$\mathcal{P}_n(X) \subset \bigcap_{L \in B_n^\infty} \{ L \leq 0 \}.$$
Lemma 6.13. Suppose $X$ has rank 2 and let $p$ be a regular $n$-gon in $X$. Suppose that $\sigma(p) \in H_L$ for some functional $L$ with $L, -L \in \mathcal{L}_n \setminus \mathcal{B}_n$. Then for any neighborhood $U$ of $\sigma(p)$ in $\Delta_{euc}^n$ there exist $n$-gons $p_1, p_2$ in $X$ with $\sigma(p_i) \in U$ and $L \circ \sigma(p_1) > L \circ \sigma(p_2)$.

Proof. Suppose that for a neighborhood $U$ of $\sigma(p)$ in $\Delta_{euc}^n$, we cannot find a polygon $p_1$ in $X$ with $\sigma(p_1) \in U$ and $L \circ \sigma(p_1) > 0$. (The other inequality follows considering the functional $-L$.) It follows from Lemmata 6.1 and 6.2 that $p$ lies in a parallel set $P_\gamma$ and the functional $L$ in $p$ is just given by taking scalar product with the direction of $\gamma(\infty)$. Fold the polygon in an apartment $A \subset P_\gamma$ so that the broken sides are $x_1 x_2, \ldots, x_{n-2} x_{n-1}$. Let $\rho : A \to E$ be an isomorphism that sends $\gamma = \gamma(\infty)$ to the singular direction in $\Delta_{euc}$ of the same type. By the notation, we write also $\eta$ to denote the unit vector in $\Delta_{euc}$ with direction $\rho(\eta)$.

Suppose $X$ has only one vertex and $\gamma$ goes through it. Then the break points of the folded polygon all lie on $\gamma$. We may assume that the folded polygon has only one break point because any two consecutive break points can be simultaneously unfolded. Let $k$ be so that the break point $y$ lies on the side $x_k x_{k+1}$ (if there is no break point we take $k = n - 1$). Then the folded polygon has the form $p' = (x_0, x_1, \ldots, x_k, y, \hat{x}_{k+1}, \ldots, \hat{x}_{n-1})$. Let $\omega_i \in W$ be so that $\omega_i \Delta_{euc}$ contains the direction $\rho(\hat{x}_{i-1} x_i)$ for $1 \leq i \leq k$, $\rho(\hat{x}_k \hat{y})$ for $i = k + 1$, $\rho(\hat{x}_{k-1} x_i)$ for $k + 2 \leq i \leq n - 1$, and $\rho(\hat{x}_{n-1} x_0)$ for $i = n$, respectively. Then the functional $L$ is just given by $(l_{\eta_1}, \ldots, l_{\eta_n})$ for $\eta_i = \omega_i^{-1} \eta$. After a small variation inside the parallel set $P_\gamma$ we may assume that the segments $x_0 x_k$ and $x_0 \hat{x}_{k+1}$ are regular. Let $\alpha, \beta \in W$ be so that $\alpha \Delta_{euc}$ contains the direction $\rho(\hat{x}_0 \hat{x}_k)$ and $\beta \Delta_{euc}$ contains $\rho(\hat{x}_{k+1} x_0)$. Let $\delta \in W$ be such that $\Delta_{euc}$ and $\delta \Delta_{euc}$ are antipodal. For $\omega \in W$ set $\tilde{\omega} := \delta \omega$.

Consider the regular polygon $q = (x_0, \ldots, x_k) \subset A$ and the functional $L' = (l_{\eta_1}, \ldots, l_{\eta_k}, l_\omega)$ for $\omega' := \tilde{\omega}^{-1} \eta$. That is, $L'$ is the functional given in $q$ by taking scalar product with the direction $\eta$. Hence $L'(\sigma(q)) = 0$. Set $(\tau_1, \ldots, \tau_k, \tau_{k+1}) := (\omega_1, \ldots, \omega_k, \tilde{\alpha})$. Suppose that there are $1 \leq i < j \leq k + 1$ such that $T_{\eta_i} \cap T_{\eta_j} \neq \emptyset$. Corollary 6.5 and its proof imply that there is a polygon $q' = (z_0, \ldots, z_k)$ with $L'(\sigma(q')) > 0$ and with refined side lengths as near as we want to those of $q$ modulo displacement along $\gamma$. We can then choose $x_k' \in P_\gamma$ near $x_k$ such that $x_0 x_k'$ has the same refined side length (again modulo displacement along $\gamma$) as $z_0 z_k$. The functional $(-l_{\eta'}, l_{\eta_{k+1}}, \ldots, l_{\eta_n})$ applied to the polygon $(x_0, x_k', x_{k+1}, \ldots, x_{n-1})$ is 0 because it is contained in the parallel set $P_\gamma$. After displacing the polygon $(x_0, x_k', x_{k+1}, \ldots, x_{n-1})$ along $\gamma$ we can glue it together to $q'$ and obtain a polygon $p_1$ with $\Delta$-valued side lengths as near as we want to those of $p$ and with $L(\sigma(p_1)) > 0$ (compare with the proof of Proposition 6.7). This contradicts the assumption at the beginning of the proof. Thus, $T_{\eta_i} \cap T_{\eta_j}$ is $\emptyset$ for all $1 \leq i < j \leq k + 1$. Since $q$ is a regular polygon with $L(\sigma(q)) = 0$, then by the second claim in Proposition 6.11 we must also have $T_{\eta_i} = T_{\eta_i} \setminus \bigcup_{i=1}^{k} T_{\omega_i}$, or equivalently, $T_{\eta_i} = \bigcup_{i=1}^{n} T_{\omega_i}$.

Analogously, considering the polygon $(x_0, \hat{x}_{k+1}, \ldots, \hat{x}_{n-1})$ we obtain $T_{\eta_i} \cap T_{\omega_i} = \emptyset$ for all $k + 2 \leq i < j \leq n$ and $T_{\eta_i}^\beta = \bigcup_{i=k+2}^{n} T_{\omega_i}$.

Consider now the triangle $t = (x_0, x_k, x_{k+1})$ with the functional $L'' = (l_{\alpha^{-1} \eta}, l_{\eta_{k+1}}, l_{\beta^{-1} \eta})$. Let $\omega_{k+1}' \in W$ be so that $\omega_{k+1}' \Delta_{euc}$ contains the direction $\rho(y x_{k+1})$. Then $\omega_{k+1}' = \omega_{k+1} \eta_{k+1} = \eta$. We want to show that $(\alpha, \omega_{k+1}, \beta)$ or $(\alpha, \omega_{k+1}', \beta)$ have the property $(\ast)$. By Lemma 6.4, applied to the side $x_0 x_k$ we get $T_{\eta_i}^\alpha \cap T_{\eta_i}^\beta = \emptyset = T_{\eta_i}^\alpha \cap T_{\eta_i}^{\omega_{k+1}}$. Again by Lemma 6.4, now applied to the
side \(x_{k+1}x_0\) we obtain \(T^\beta_\eta \cap T^{0\kappa}_{k+1} = \emptyset\). Therefore if \(T^\alpha_\eta\) or \(T^\beta_\eta = \emptyset\), then we are done, so suppose both are nonempty. Now by Lemma 6.6 one of \(\alpha\Delta_{\text{euc}}, \beta\Delta_{\text{euc}}\) or \(\omega_{k+1}\Delta_{\text{euc}}\) must be adjacent to \(\rho(\gamma)\). Notice that for \(\omega \in W\), \(\omega\Delta_{\text{euc}}\) is adjacent to \(\rho(\gamma)\) if and only if \(T^\omega_\eta \in \{\emptyset, T_\eta\}\). This and \(T^\alpha_\eta \cap T^\beta_\eta = \emptyset\) imply that \(\omega_{k+1}\Delta_{\text{euc}}\) must be adjacent to \(\rho(\gamma)\). \(T^0_\eta \cap T^{0\kappa}_{k+1} = \emptyset\) implies that \(T^{0\kappa}_{k+1}\) must be empty and we are also done in this case.

So we have conclude that \(\bar{\eta} = (\eta_1, \ldots, \eta_n)\) has the property (*) and since \(p\) is a regular polygon with \(L(\sigma(p)) = 0\), it follows from Proposition 6.11 that \(L \in B_n\).

Now we are ready to prove our main theorem.

**Theorem 6.14.** Let \(X\) be a building of rank 2. \(\mathcal{P}_n(X)\) is a convex polyhedral cone determined by the inequalities \(\{L \leq 0\}\) for \(L \in B_n\). That is,

\[
\mathcal{P}_n(X) = \bigcap_{L \in B_n} \{L \leq 0\}
\]

This inequalities constitute an irredundant set of inequalities.

**Proof.** Let \(Q \subset \mathcal{C}_n\) be the subset of open cones such that \(\bigcap_{L \in B_n} \{L \leq 0\} = \bigcup C \subset Q\). Analogously, let \(Q' \subset \mathcal{C}_n\) be the subset of open cones such that \(\mathcal{P}_n(X) = \bigcup C \subset Q'\) (this can be done by Proposition 6.7). We have shown in Proposition 6.11 that \(Q' \subset Q\). Let \(C_0 \in Q'\) and \(C \in Q\). Take a chain \(C_0, C_1, \ldots, C_k = C \in Q\) such that \(C_i \cap C_{i+1}\) is a face of codimension one. We prove now inductively that \(C_i \in Q'\). Suppose then that \(C_i \in Q'\) and take a regular polygon \(p\) with \(\sigma(p)\) in the interior of the face \(C_i \cap C_{i+1}\). Since \(C_i \cap C_{i+1}\) is not in the boundary of \(\bigcap_{L \in B_n} \{L \leq 0\}\), it lies in a wall \(H_L\) with neither \(L\), \(-L\) in \(B_n\). It follows from Lemma 6.13 that \(\mathcal{P}_n(X) \cap C_{i+1}\) is not empty and therefore \(C_{i+1} \subset \mathcal{P}_n(X)\). Thus \(C \in Q'\), and \(Q = Q'\).

For \(L \in B_n\) it is clear that we can find a regular polygon \(p\) in an apartment \(A\) and \(\gamma \subset A\) a maximal singular line, such that the functional \(L\) in \(p\) is given by taking scalar product with the direction of \(\eta = \gamma(\infty)\). In particular, \(L(\sigma(p)) = 0\). It is also clear that we can find a regular polygon \(p'\) in \(P\), but not contained in any apartment and such that the functional \(L\) in \(p'\) is also given by taking scalar product with the direction of \(\eta\). It follows from Lemmata 6.1 and 6.2 that \(L\) is the only functional in \(B_n\) for which it can hold \(L(\sigma(p')) = 0\). Thus the inequalities \(\{L \leq 0\}\) with \(L \in B_n\) are irredundant.

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