Online Spanners in Metric Spaces

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Abstract

Given a metric space \( \mathcal{M} = (X, \delta) \), a weighted graph \( G \) over \( X \) is a metric \( t \)-spanner of \( \mathcal{M} \) if for every \( u, v \in X \), \( \delta(u, v) \leq \delta_G(u, v) \leq t \cdot \delta(u, v) \), where \( \delta_G \) is the shortest path metric in \( G \). In this paper, we construct spanners for finite sets in metric spaces in the online setting. Here, we are given a sequence of points \( (s_1, \ldots, s_n) \), where the points are presented one at a time (i.e., after \( i \) steps, we saw \( S_i = \{s_1, \ldots, s_i\} \)). The algorithm is allowed to add edges to the spanner when a new point arrives, however, it is not allowed to remove any edge from the spanner. The goal is to maintain a \( t \)-spanner \( G_i \) for \( S_i \) for all \( i \), while minimizing the number of edges, and their total weight.

We construct online \( (1+\varepsilon) \)-spanners in Euclidean \( d \)-space, \( (2k-1)(1+\varepsilon) \)-spanners for general metrics, and \( (2+\varepsilon) \)-spanners for ultrametrics. Most notably, in Euclidean plane, we construct a \( (1+\varepsilon) \)-spanner with competitive ratio \( O(\varepsilon^{-3/2} \log \varepsilon^{-1} \log n) \), bypassing the classic lower bound \( \Omega(\varepsilon^{-2}) \) for lightness, which compares the weight of the spanner, to that of the MST.

1 Introduction

Let \( \mathcal{M} = (P, \delta) \) be a finite metric space. Let \( G = (P, E) \) be a graph on the points of \( P \) in \( \mathcal{M} \), where the edges are weighted with the distances between their endpoints. The graph \( G \) is a \( t \)-spanner, for \( t \geq 1 \), if \( \delta_G(u, v) \leq t \cdot \delta(u, v) \) for all \( u, v \in P \), where \( \delta_G(u, v) \) is the length of the shortest path between \( u \) and \( v \) in \( G \), and \( \delta(u, v) \) is the distance between \( u \) and \( v \) in \( \mathcal{M} \). The stretch factor \( t \) of \( G \) is the maximum distortion between the metrics \( \delta \) and \( \delta_G \). Spanners were first introduced by Peleg and Schäffer [PS89], and since then they have turned out to be one of the fundamental graph structures with numerous applications in the area of distributed systems and communication, distributed queuing protocol, compact routing schemes, etc. [DH98, HTW01, PU89a, PU89b].

The study of Euclidean spanners, where \( P \subset \mathbb{R}^d \) with \( L_2 \)-norm, was initiated by Chew [Che89]. Since then a large body of research has been devoted to Euclidean spanners due to its vast range of applications across domains, such as topology control in wireless networks, efficient regression in metric spaces, approximate distance oracles, data structures, and many more [GKK17, GLNS08, SVZ07, Yao82]. Some of the results generalize to metric spaces with constant doubling dimensions [BLW19] (the doubling dimension of \( \mathbb{R}^d \) is \( d \).

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1Often in the literature, the input metric is the shortest path metric of a graph \( G = (V, E, w) \), and a spanner is required to be a subgraph of the input graph (see e.g. [ADD+93]). Here we study metric spanners where there is no such requirement.
Lightness and sparsity are two fundamental parameters for spanners. The lightness of a spanner $G = (P, E)$ is the ratio $w(G)/w(MST)$ between the total weight of $G$ and the weight of a minimum spanning tree (MST) on $P$. The sparsity of $G$ is the ratio $|E(G)|/|E(MST)| \approx |E(G)|/|P|$ between the number of edges of $H$ and an MST. Since every spanner is connected and thus contain a spanning tree, the lightness and sparsity of a spanner $G$, resp., are trivial lower bounds for the ratio of $w(G)$ and $|E(G)|$ to the optimum weight and the number of edges.

**Online Spanners.** We are given a sequence of points $(s_1, \ldots, s_n)$ in a metric space, where the points are presented one-by-one, i.e., point $s_i$ is revealed at the step $i$, and $S_i = \{s_1, \ldots, s_i\}$ for $i \in \{1, \ldots, n\}$. The objective of an online algorithm is to maintain a $t$-spanner $G_i$ for $S_i$ for all $i$. The algorithm is allowed to add edges to the spanner when a new point arrives, however it is not allowed to remove any edge from the spanner. Moreover, the algorithm does not know the value of the total number points in advance.

The performance of an online algorithm ALG is measured by comparing it to the offline optimum OPT using the standard notion of competitive ratio [BE98, Ch. 1]. The competitive ratio of an online $t$-spanner algorithm ALG is defined as $\sup_\sigma \frac{\sigma_{ALG}(\sigma)}{\sigma_{OPT}(\sigma)}$, where the supremum is taken over all input sequences $\sigma$, OPT$(\sigma)$ is the minimum weight of a $t$-spanner for the (unordered) set of points in $\sigma$, and ALG$(\sigma)$ denotes the weight of the $t$-spanner produced by ALG for this input sequence. Note that, in order to measure the competitive ratio it is important that $\sigma$ is a finite sequence of points.

In the online minimum spanning tree problem, points of a finite metric space arrive one-by-one, and we need to connect each new point to a previous point to maintain a spanning tree. Imase and Waxman [IW91a] proved $\Theta(\log n)$-competitiveness, which is the best possible bound. Later, Alon and Azar [AA93] studied this problem for points in Euclidean plane, and proved a lower bound $\Omega(\log n/\log \log n)$ for the competitive ratio. Their result was the first to analyze the impact of auxiliary points (Steiner points) on a geometric network problem in the online setting. Several algorithms were proposed over the years for the online minimum Steiner tree and Steiner forest problems, on graphs in both weighted and unweighted settings; see [AAA*06, AAB04, BC97, HLP17, NPS11]. However, these algorithms do not provide any guarantee on the stretch factor. This leads to the following open problem.

**Problem.** Determine bounds on the competitive ratios for the weight and the number of edges of online $t$-spanners, for $t \geq 1$.

Previously, Gupta et al. [GRTU17, Theorem 1.5] constructed online spanners for terminal pairs in the same model we consider here. The analysis of [GRTU17] implicitly implies that, given a sequence of $n$ points in an online fashion in a general metric space, one can maintain a $O(\log n)$-spanner with $O(n)$ edges and $O(\log n)$ lightness, as pointed out by one of the authors [Umb21]. Recent work on online directed spanners [GLQ21] is not comparable to our results.

In the geometric setting, $(1 + \varepsilon)$-spanners are possible in any constant dimension $d \in \mathbb{N}$. Tight worst-case bounds $\Theta_d(\varepsilon^{-d})$ and $\Theta_d(\varepsilon^{1-d})$ on the lightness and sparsity of offline $(1+\varepsilon)$-spanners have recently been established by Le and Solomon [LS19]. Online Euclidean spanners in $\mathbb{R}^d$ have been introduced by Bhaore and Tóth [BT21c]. In the real line (1D), they have established a tight bound of $O(\varepsilon^{-1}/\log \varepsilon^{-1}) \log n$ for the competitive ratio of any online $(1+\varepsilon)$-spanner algorithm for $n$ points. In dimensions $d \geq 2$, the dynamic algorithm DEFSPANNER of Gao et al. [GGN06] maintains a $(1+\varepsilon)$-spanner with $O_d(\varepsilon^{-(d+1)}n)$ edges and $O_d(\varepsilon^{-(d+1)}\log n)$ lightness, and works under the
online model (as it never deletes edges when new points arrive). However, no lower bound better
than the 1-dimensional $\Omega((\varepsilon^{-1}/\log\varepsilon^{-1})\log n)$ is currently known in higher dimensions.

### 1.1 Our Contribution

See Table 1 for an overview of our results.

| Family            | Stretch     | # of edges | Lightness                     | Ref/comments       |
|-------------------|-------------|------------|-------------------------------|--------------------|
| General metrics   | $(2k-1)(1+\varepsilon)$ | $O(\varepsilon^{-1}\log(\frac{1}{\varepsilon}))n^{1+\frac{\varepsilon}{k}}$ | $O(n^{\frac{1}{k}}\varepsilon^{-1}\log^2 n)$ | Theorem 4           |
|                   | $O(\log n)$ | $O(n)$     | $O(\log n)$                  | [GRTU17, Umb21]    |
| $\alpha$-HST      | $2^{\frac{\alpha}{n-1}}$ | $n-1$      | $1$                           | Lemmas 8 and 9     |
| Ultrametric       | $O(\varepsilon^{-1})$ | $n-1$      | $1+\varepsilon$              | Theorem 6           |
| Doubling $d$-space| $1+\varepsilon$ | $\varepsilon^{-O(d)}n$ | $\varepsilon^{-O(d)}\log n$ | DefSpanner [GGN06] |
| Euclidean $d$-space| $1+\varepsilon$ | $O_d(\varepsilon^{-d})n$ | $O_d(\varepsilon^{-(d+1)}\log n)$ | DefSpanner [GGN06] |
|                  | $1+\varepsilon$ | $O_d(\varepsilon^{1-d})n$ | $\Omega(\varepsilon^{-1}n)$ | ordered Θ-graph [RS91] |
|                  | $1+\varepsilon$ | $O_d(\varepsilon^{1-d})n$ | $O_d(\varepsilon^{-d}\log n)$ | Theorem 1           |
| Real line         | $1+\varepsilon$ | $O(n)$     | $\Theta(\varepsilon^{-1}\log n)$ | ordered greedy [BT21c] |

Table 1: Overview of online spanners algorithms. In the last three rows, we compare the spanner weight directly with the optimum weight (rather than the MST) to bound the competitive ratio.

**Upper Bounds for Points in $\mathbb{R}^d$.** Under the $L_2$-norm in $\mathbb{R}^d$, for arbitrary constant $d \in \mathbb{N}$, we present an online algorithm for $(1+\varepsilon)$-spanner with lightness $O_d(\varepsilon^{-d}\log n)$ and sparsity $O(\varepsilon^{1-d}\log \varepsilon^{-1})$ (Theorem 1 in Section 2.1). This improves upon the previous lightness bound of $O_d(\varepsilon^{-(d+1)}\log n)$ by Gao et al. [GGN06, Lemma 3.8]. In the plane, we give a tighter analysis of the same algorithm and achieve an almost quadratic improvement of the competitive ratio to $O(\varepsilon^{-3/2}\log \varepsilon^{-1}\log n)$ (Theorem 2 in Section 2.2). Recall that in the offline setting, $\Theta(\varepsilon^{-2})$ is a tight worst-case bound for the lightness of a $(1+\varepsilon)$-spanner in the plane [LS19]. We obtain a better dependence on $\varepsilon$ by comparing the online spanner with an instance-optimal spanner directly, bypassing the comparison to an MST (i.e., lightness). The logarithmic dependence on $n$ cannot be eliminated in the online setting, based on the lower bound in $\mathbb{R}^1$ [BT21c].

**Lower Bounds for Points in $\mathbb{R}^d$.** As a counterpart, we design a sequence of points that yields a $\Omega_d(\varepsilon^{-d})$ lower bound for the competitive ratio for online $(1+\varepsilon)$-spanner algorithms in $\mathbb{R}^d$ under the $L_1$-norm (Theorem 3 in Section 3). This improves the previous bound of $\Omega(\varepsilon^{-2}/\log \varepsilon^{-1})$ in $\mathbb{R}^2$ under the $L_1$-norm. It remains open whether a similar lower bound holds in $\mathbb{R}^d$ under the $L_2$-norm;
the current best lower bound is $\Omega((\varepsilon^{-1}/\log \varepsilon^{-1}) \log n)$, established in [BT21c], holds already for the real line ($d = 1$).

**Points in General Metrics.** In Section 4, we study online spanners in general metrics. Note that it is not possible to obtain a spanner with stretch less than 3 with a subquadratic number of edges, even in the offline settings, for general metrics. We analyze an online version of the celebrated greedy spanner algorithm, dubbed ordered greedy. With stretch factor $t = (2k - 1)(1 + \varepsilon)$ for $k \geq 2$ and $\varepsilon \in (0, 1)$, we show that it maintains a spanner with $O(\varepsilon^{-1} \log \frac{1}{k}) \cdot n^{1 + \frac{1}{k}}$ edges and $O(\varepsilon^{-1} n^{\frac{1}{k}} \log^{2} n)$ lightness for a sequence of $n$ points in a metric space (Theorem 4). We show (in Theorem 5) that these bounds cannot be significantly improved, by introducing an instance where every online algorithm will have $\Omega(\frac{1}{k} \cdot n^{1/k})$ competitive ratio on both sparsity and lightness. Next, we establish the trade-off among stretch, number of edges and lightness for points in ultrametrics. Specifically, we show that it is possible to maintain a $(2 + \varepsilon)$-spanner with $O(\varepsilon^{-1} \log \varepsilon^{-1}) \cdot n$ edges and $O(\varepsilon^{-2})$ lightness in ultrametrics (Theorem 7). Note that as the uniform metric (shortest path on a clique) is an ultrametric, any subquadratic spanner must have stretch at least 2.

### 1.2 Related Work

#### 1.2.1 Dynamic & Streaming Algorithms for Graph Spanners

A $t$-spanner in a graph $G = (V,E)$ is subgraph $H = (V,E')$ such that $\delta_H(u,v) \leq t \cdot \delta_G(u,v)$ for all pairs of vertices $u,v \in V$. That is, the stretch $t$ is the maximum distortion between the graph distances $\delta_G$ and $\delta_H$. Importantly, when $G$ changes (under edge/vertex insertions or deletions), the underlying metric $\delta_G$ changes, as well. The distance $\delta_G(u,v)$ may dramatically decrease upon the insertion of the edge $uv$. In contrast, our model assumes that the distances in the underlying metric space $M = (P,\delta)$ remain fixed, but the algorithm can only see the distances between the points that have been presented. For this reason, our results are not directly comparable to models where the underlying graph changes dynamically.

For unweighted graphs with $n$ vertices, the current best fully dynamic and single-pass streaming algorithms can maintain spanners that achieve almost the same stretch-sparsity trade-off available for the static case: $2k - 1$ stretch and $O(n^{1 + \frac{1}{k}})$ edges, for $k \geq 1$, which is attained by the greedy algorithm [ADD+93], and conjectured to be optimal due to the Erdős girth conjecture [Erd64]. In the dynamic model, the objective is design algorithms and data structures that minimize the worst-case update time needed to maintain a $t$-spanner for $S$ over all steps, regardless of its weight, sparsity, or lightness. See [BKS12, BHG+21, BFH19, BK16] for some excellent work on dynamic spanners. In the streaming model the input is a sequence (or stream) of edges representing the edge set $E$ of the graph $G$. A (single-pass) streaming algorithm decides, for each newly arriving edge, whether to include it in the spanner. The graph $G$ is too large to fit in memory, and the objective is to optimize work space and update time [Bas08, BFKL21, Elk11, FKM+08, FKN21, McG14].

#### 1.2.2 Incremental Algorithms for Geometric Spanners

We briefly review three previously known incremental $(1 + \varepsilon)$-spanner algorithms in Euclidean $d$-space from the perspective of competitive analysis.
Deformable Spanners. Gao et al. [GGN06] designed a dynamic DeF\textsc{Spanner} algorithm that maintains a $(1 + \varepsilon)$-spanner for a dynamic set $S$ in Euclidean $d$-space. For point insertions, it only adds new edges, so it is an online algorithm, as well. It maintains a $(1 + \varepsilon)$-spanner with $O_d(\varepsilon^{-d}) \cdot n$ edges and $O_d(\varepsilon^{-(d+1)} \log n)$ lightness. Since the $\|\text{MST}(S)\|$ is a lower bound for the optimal spanner weight, its competitive ratio is also $O_d(\varepsilon^{-(d+1)} \log n)$. The key ingredient of DeF\textsc{Spanner} is hierarchical nets [HM06, KL04, Rod12], a form of hierarchical clustering, which can be maintained dynamically. Hierarchical nets naturally generalize to doubling spaces, and so DeF\textsc{Spanner} also maintains a $(1 + \varepsilon)$-spanner with $\varepsilon^{-O(d)} \cdot n$ edges and lightness $\varepsilon^{-O(d)}$ in doubling dimension $d$ [GR08, Rod12].

Well-Separated Pair Decomposition (WSPD). Well-separated pair decomposition was introduced by Callahan and Kosaraju [CK93] (see also [GK18, Har11, NS07, Smi18]). For a set $S$ in a metric space, a WSPD is a collection of unordered pairs $W = \{\{A_i, B_i\} : i \in I\}$ such that (1) $A_i, B_i \subset S$ for all $i \in I$; (2) $\min\{\|ab\| : a \in A_i, b \in B_i\} \leq g \cdot \max\{\text{diam}(A_i), \text{diam}(B_i)\}$ for all $i \in I$, where $g$ is the separation ratio; (3) for each point pair $\{a, b\} \subset S$ there exists a pair $\{A_i, B_i\}$ such that $A_i$ and $B_i$ each contain one of $a$ and $b$. Given a WSPD with separation ratio $g > 4$, any graph that contains at least one edge between $A_i$ and $B_i$, for all $i \in I$, is a spanner with stretch $t = 1 + 8/g(\varepsilon - 1)$. Setting $g \geq 12\varepsilon^{-1}$ for $0 < \varepsilon < 1$, we obtain $t \leq 1 + \varepsilon$.

Hierarchical clustering provides a WSPD [Har11, Ch. 3]. Perhaps the simplest hierarchical subdivisions in $\mathbb{R}^d$ are quadtrees. Let $T$ be a quadtree for a finite set $S \subset \mathbb{R}^d$. The root of $T$ is an axis-aligned cube of side length $a_0$, which contains $S$; it is recursively subdivided into $2^d$ congruent cubes until each leaf cube contains at most one point in $S$. For all pairs of cubes $\{Q_1, Q_2\}$ at level $\ell$ of $T$, create a pair $\{A_i, B_i\}$ with $A_i = Q_1 \cap S$ and $B_i = Q_2 \cap S$ whenever $D_{\ell} \leq \text{dist}(Q_1, Q_2) < 2D_{\ell}$ for $D_{\ell} = g \cdot \text{diam}(Q_1) = 12\varepsilon^{-1} \cdot \sqrt{d} \cdot a_0 / 2^\ell$; and repeat for all levels $\ell \geq 0$. Properties (1)–(3) of a WSPD are easily verified [Har11, Ch. 3]. The resulting $(1 + \varepsilon)$-spanner has $O_d(\varepsilon^{-d}) \cdot n$ edges [Har11, HM06] and lightness $O_d(\varepsilon^{-(d+1)} \log n)$ [BT21c].

For point insertions in $\mathbb{R}^d$, a dynamic quadtree only adds nodes, which in turn creates new pairs in the WSPD, and new edges in the spanner. This is an online algorithm with the same guarantees as DeF\textsc{Spanner} [BT21c, HM06] (see also [FH05] for an efficient implementation).

Ordered Yao-Graphs and Θ-Graphs. One of the first constructions for (offline) sparse $(1 + \varepsilon)$-spanner in Euclidean $d$-space were the Yao- and Θ-graphs [Cla87, Kei88, RS91]. Incremental versions of Yao-graphs and Θ-graphs were introduced by Bose et al. [BGM04]. Let $S = \{s_1, \ldots, s_n\}$ be an ordered set of points in $\mathbb{R}^2$. For each $s_i \in S$, partition the plane into $k$ cones with apex $s$ and aperture $2\pi/k$. The ordered Yao-graph $Y_k(S)$ contains an edge between $s_i$ and a closest previous point in $\{s_j : j < i\}$ in each cone. The graph $\Theta_k(S)$ is defined similarly, but in each cone the distance to the apex is measured by the orthogonal projection to a ray within the cone. Bose et al. [BGM04] showed that the ordered Yao- and Θ-graphs have spanning ratio at most $1/(1 - 2\sin(\pi/k))$ for $k > 8$; tighter bounds were later obtained in [BCM+16]. In particular, the ordered Yao- and Θ-graphs are $(1 + \varepsilon)$-spanners for $k \geq \Omega(\varepsilon^{-1})$.

The construction generalizes to $\mathbb{R}^d$ for all $d \in \mathbb{N}$ [RS91]. For an angle $\alpha \in (0, \pi)$, let $A \subset \mathbb{S}^{d-1}$ be a maximal set of points in the $(d - 1)$-sphere such that $\min_{a,b \in A} \text{dist}(a, b) \leq \alpha$ (in radians). A standard volume argument shows that $|A| \leq O_d(\alpha^{-d})$. For each $a_i \in A$, create a cone $C_i$ with apex at the origin $o$, aperture $\alpha$, and symmetry axis $oa_i$. Note that $\mathbb{R}^d \subseteq \bigcup_i C_i$. Given a finite set $P \subset \mathbb{R}^d$, we translate each cone $C_i$ to a cone $C_i(p)$ with apex $p \in P$. For every cone $C_i(p)$, the Yao-graph

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contains an edge between \( p \) and a closest point in \( P \cap C_i(p) \). For every \( \varepsilon > 0 \) and \( d \in \mathbb{N} \), there exists an angle \( \alpha = \alpha(d, \varepsilon) = \Theta_d(\varepsilon) \) for which the Yao-graph is a \((1 + \varepsilon)\)-spanner for every finite set \( P \subset \mathbb{R}^d \).

Ordered Yao- and \( \Theta \)-graphs give online algorithms for maintaining a \((1 + \varepsilon)\)-spanner for a sequence of points in \( \mathbb{R}^d \). The sparsity of these spanners is bounded by the number of cones per vertex, \( O_d(\varepsilon^{-1-d}) \), which matches the (offline) lower bound of \( \Omega_d(\varepsilon^{1-d}) \) [LS19]. However, their weight may be significantly higher than optimal: For \( n \) equally spaced points in a unit circle, in any order, Yao- and \( \Theta \)-graphs yield \((1 + \varepsilon)\)-spanners of weight \( \Omega(\varepsilon^{-1}n) \), hence lightness \( \Omega(\varepsilon^{-1}n) \), while the optimum weight is \( O(\varepsilon^{-2}) \) [LS19].

**Online Steiner Spanners.** An important variant of online spanners is when it is allowed to use auxiliary points (Steiner points) which are not part of the input sequence of points, but are present in the metric space. An online algorithm is allowed to add Steiner points, however, the spanner must achieve the given stretch factor only for the input point pairs. It has been observed through a series of work in recent years, that Steiner points allow for substantial improvements over the bounds on the sparsity and lightness of Euclidean spanners in the offline settings and highly nontrivial insights are required to argue the bounds for Steiner spanners, and often they tend to be even more intricate than their non-Steiner counterpart; see [BT21a, BT21b, LS19, LS20]. Bhore and Tóth [BT21c] showed that if an algorithm can use Steiner points, then the competitive ratio for weight improves to \( O(\varepsilon^{1-d/2}\log n) \) in the Euclidean \( d \)-space.

### 2 Upper Bounds in Euclidean Spaces

We present an online algorithm for a sequence of \( n \) points in Euclidean \( d \)-space (Section 2.1). It combines features from several previous approaches, and maintains a \((1 + \varepsilon)\)-spanner of lightness \( O_d(\varepsilon^{-d}\log n) \) and sparsity \( O_d(\varepsilon^{1-d}\log\varepsilon^{-1}) \) for \( d \geq 1 \). Lightness is an upper bound for the competitive ratio for weight; the sparsity almost matching the optimal bound \( O_d(\varepsilon^{1-d}) \) attained by ordered Yao-graphs. In the plane \((d = 2)\), we show that the same algorithm achieves competitive ratio \( O(\varepsilon^{-3/2}\log\varepsilon^{-1}\log n) \) using a tighter analysis: A charging scheme that charges the weight of the online spanner to a minimum weight spanner (Section 2.2).

#### 2.1 An Improvement in All Dimensions

We combine features from two incremental algorithms for geometric spanners, and obtain an online \((1 + \varepsilon)\)-spanner algorithm for a sequence of \( n \) points in \( \mathbb{R}^d \). We maintain a dynamic quadtree for hierarchical clustering, and use a modified ordered Yao-graph in each level of the hierarchy. In particular, we limit the weight of the edges in the Yao-graph in each level of the hierarchy (thereby avoiding heavy edges). We start with an easy observation.

**Lemma 1.** Let \( G = (S, E) \) be a \( t \)-spanner and let \( w > 0 \). Let \( G' = (S, E') \), where \( E' = \{ e \in E : \|e\| \leq w \} \) is the set of edges of weight at most \( w \). Then for every \( a, b \in S \) with \( \|ab\| < w/t \), graph \( G' \) contains an \( ab \)-path of weights at most \( t\|ab\| \).

**Proof.** Since \( G \) is a \( t \)-spanner, it contains an \( ab \)-path \( P_{ab} \) of weight at most \( t\|ab\| \leq w \). By the triangle inequality, every edge in this path has weight at most \( w \), hence present in \( G' \). Consequently \( G' \) contains \( P_{ab} \).
**Online Algorithm ALG.** The input is a sequence of points \((s_1, s_2, \ldots)\) in \(\mathbb{R}^d, d \geq 1\). The set of the first \(n\) points is denoted by \(S_n = \{s_i : 1 \leq i \leq n\}\). For every \(n\), we dynamically maintain a quadtree \(T_n\) for \(S_n\). Every node of \(T_n\) corresponds to a cube. The root of \(T_n\), at level 0, corresponds to a cube \(Q_0\) of side length \(a_0 = \Theta(\text{diam}(S_n))\). At every level \(\ell \geq 0\), there are at most \(2^{d\ell}\) interior-disjoint cubes, each of side length \(a_\ell = a_0 2^{-\ell}\). A cube \(Q \in T_n\) is nonempty if \(Q \cap S_n \neq \emptyset\). For every nonempty cube \(Q\), we maintain a representative \(s(Q) \in Q \cap S_n\), selected at the time when \(Q\) becomes nonempty. At each level \(\ell\), let \(P_\ell\) be the sequence of representatives, in the order in which they are created.

For each level \(\ell\), we maintain a modified ordered Yao-graph \(G_\ell = (P_\ell, E_\ell)\) as follows. When a new point \(p\) is inserted into \(P_\ell\), cover \(\mathbb{R}^d\) with \(\Theta_d(\varepsilon^{1-d})\) cones of aperture \(\alpha(d, \varepsilon)\) as in the construction of Yao-graphs. In each cone \(C_i\), find a point \(q_i \in C_i \cap P_\ell\) closest to \(p\); and add \(pq_i\) to \(E_\ell\) if \(\|pq_i\| < 24a_\ell \sqrt{d} \cdot \varepsilon^{-1}\). The algorithm maintains the spanner \(G = \bigcup_{\ell=0}^{n} G_\ell\).

**Theorem 1.** Let \(d \geq 1\) and \(\varepsilon \in (0, 1)\). The online algorithm ALG maintains, for a sequence of \(n\) points in Euclidean \(d\)-space, an \((1 + O(\varepsilon))\)-spanner with weight \(O_d(\varepsilon^{-d}\log n) \cdot \|\text{MST}\|\) and \(O_d(\varepsilon^{1-d}\log \varepsilon^{-1}) \cdot n\) edges.

Note that Theorem 1 implies that the competitive ratio of this algorithm is also \(O_d(\varepsilon^{-d}\log n)\).

**Proof.** **Stretch Analysis.** We give a bound on the stretch factor in two steps: First, we define an auxiliary graph \(H = (S, E')\) which is a \((1 + \varepsilon)\)-spanner for \(S\) by the analysis of WSPDs. Then we show that \(G\) contains an \(ab\)-path of weight at most \((1 + \varepsilon)\|ab\|\) for each edge of \(H\). Overall, the stretch of \(G\) is at most \((1 + \varepsilon)^2 = (1 + O(\varepsilon))\) for all \(a, b \in S\).

*First Layer: WSPD.** For each level \(\ell \geq 0\), let \(H_\ell = (P_\ell, E'_\ell)\) be the graph that contains an edge between two representatives \(a, b \in P_\ell\) whenever \(\|ab\| \leq 12a_\ell \sqrt{d} \cdot \varepsilon^{-1}\). Let \(H = \bigcup_{\ell=0}^{n} H_\ell\). The auxiliary graph \(H_\ell\) contains an edge between the representatives of any such pair of cubes at level \(\ell\). As noted Section 1.2.2, \(H = \bigcup_{\ell=0}^{n} H_\ell\) is a \((1 + \varepsilon)\)-spanner (cf. [Har11, HM06]).

*Second Layer: Near-Sighted Yao-graphs.* As \(H\) is a \((1 + \varepsilon)\)-spanner, for every \(a, b \in S_n\), it contains an \(ab\)-path of weight at most \((1 + \varepsilon)\|ab\|\). Consider such a path \(P_{ab} = (a = p_0, \ldots, p_m = b)\). Each edge \(p_{i-1}p_i\) is in \(H_\ell\) for some \(\ell \geq 0\). By construction, every edge in \(H_\ell\) has weight at most \(12a_\ell \sqrt{d} \cdot \varepsilon^{-1}\). For every level \(\ell\), theordered Yao-graph \(Y(P_\ell)\) with angle \(\alpha(d, \varepsilon)\) is a \((1 + \varepsilon)\)-spanner. The graph \(G_\ell = (P_\ell, E_\ell)\) constructed by ALG at level \(\ell\) is a subgraph of \(Y(P_\ell)\). By Lemma 1, for every \(p, q \in P_\ell\) with \(\|pq\| \leq 12a_\ell \sqrt{d} \cdot \varepsilon^{-1}\), graph \(G_\ell\) contains a \(pq\)-path of weight at most \((1 + \varepsilon)\|pq\|\).

Overall, \(H\) contains an \(ab\)-path \(P_{ab} = (p_0, \ldots, p_m)\) of weight at most \((1 + \varepsilon)\|ab\|\). For each edge \(p_{i-1}p_i\) of \(P_{ab}\), graph \(G\) contains a \(p_{i-1}p_i\)-path of weight \((1 + \varepsilon)\|p_{i-1}p_i\|\). The concatenation of these paths is an \(ab\)-path of weight \((1 + \varepsilon)^2 \|ab\| \leq (1 + O(\varepsilon)) \|ab\|\).

**Weight Analysis.** We may assume w.l.o.g. that the root of the quadtree \(T_n\) is the unit cube \([0, 1]^d \subset \mathbb{R}^d\), which has diameter \(\sqrt{d}\). This implies \(\text{diam}(S_n) \leq \sqrt{d} = O_d(1)\). Assume further that \(n > 1\), and \(\frac{1}{4} \leq \text{diam}(S_n) \leq \|\text{MST}(S_n)\|\).

Every edge in \(E_\ell\) at level \(\ell\) has weight \(O_d(\varepsilon^{-1} 2^{-\ell})\). In particular, every edge at level \(\ell \geq 2\log n\) has weight \(O_d(\varepsilon^{-1} / n^2)\); and the total weight of these edges is \(O_d(\varepsilon^{-1} \|\text{MST}(S_n)\|)\).

It remains to bound the weight of the edges on levels \(\ell = 1, \ldots, \lfloor 2\log n \rfloor\). At level \(\ell\) of the quadtree \(T_n\), there are at most \(2^{d\ell}\) nodes, hence \(\|P_\ell\| \leq 2^{d\ell}\). If \(\|P_\ell\| < 3^d\), then \(G_\ell\) has at most \(O(3^d) = O_d(1)\) edges, each of weight at most \(\text{diam}(P_\ell) \leq \text{diam}(S_n) \leq \|\text{MST}(S_n)\|\), and so \(\|E_\ell\| \leq O_d(\|\text{MST}(S_n)\|)\).
Assume now that \(|G_t| \geq 3^d\). By the definition of ordered Yao-graphs, each vertex inserted into \(P_t\) adds \(\Theta(\varepsilon^{1-d})\) new edges, each of weight \(O(\varepsilon^{-1}2^{-\ell})\). The total weight of the edges in \(G_t\) is at most

\[
\|E_t\| \leq |P_t| \cdot \varepsilon^{1-d} \cdot \max_{e \in E_t} \|e\| \leq O_d(|P_t| \varepsilon^{-d}2^{-\ell}). \tag{1}
\]

We next derive a lower bound for \(\|MST(S_n)\|\) in terms of \(|P_t|\), when \(|P_t| > 1\) and \(\ell > 2\), using a standard volume argument. Define a graph on the vertex set \(\text{MST}\) all \(\ell\) adjacent iff \(p\) and \(q\) lie in neighboring quadtree cells of level \(\ell\). Since every quadtree cell has \(3^d - 1\) neighbors, this graph is \((3^d - 1)\)-degenerate, and contains an independent set \(I_\ell\) of size at least \((3^d - 1)^{-1}|P_t| = \Omega_d(|P_t|)\). The distance between any two disjoint quadtree cells at level \(\ell\) is at least \(2^{-\ell}\). Consequently, the open balls of radius \(2^{-\ell+1}\) centered at the points in \(I_\ell\) are pairwise disjoint. None of the balls contains \(S_n\) for \(\ell > 2\), as the diameter of each of ball is \(2^{-\ell}\) while \(\text{diam}(S_n) \geq \frac{1}{2}\). For all \(\ell > 2\), \(\text{MST}(S_n)\) contains the center of each ball and a point in its interior; hence the intersection of \(\text{MST}(S_n)\) and each ball contains a path from the center to a boundary point, which has weight at least \(2^{-\ell+1}\). Summation over \(|I_\ell|\) disjoint balls yields

\[
\|\text{MST}(S_n)\| \geq |I_\ell| \cdot 2^{-(\ell+1)} \geq \Omega_d(|P_t|2^{-\ell}). \tag{2}
\]

Comparing inequalities (1) and (2), we obtain \(\|E_t\| \leq O_d(\varepsilon^{-d})\cdot\|\text{MST}(S_n)\|\). Summation over all levels \(\ell \in \mathbb{N}\) yields \(\|E\| \leq O_d(\varepsilon^{-d}\log n)\cdot\|\text{MST}(S_n)\|\), as claimed.

**Sparsity Analysis.** We show that \(G\) has \(O_d(\varepsilon^{1-d}\log \varepsilon^{-1})\cdot n\) edges. Har-Peled proved that the auxiliary graph \(H\) is \(O(\varepsilon^{-d})\)-degenerate, and so it has \(O_d(\varepsilon^{-d})\cdot n\) edges [Har11, HM06, Lemma 3.9]. As \(G\) is a subgraph of \(H\), hence has \(O_d(\varepsilon^{-d})\cdot n\) edges as well. We improve this bound using a charging scheme.

For the quadtree \(T_n\) maintained by algorithm \(\text{ALG}_1\), let \(T'_n\) denote the **compressed quadtree**, which is obtained from \(T_n\) by removing all leaves that correspond to empty cubes, and suppressing nodes with a single child [BCKO08, Har11]. For \(n\) points in \(\mathbb{R}^d\), the compressed quadtree has \(O_d(n)\) nodes (which are nodes of the original quadtree, as well). For each node \(Q\) of \(T'_n\), algorithm \(\text{ALG}_1\) adds \(O_d(\varepsilon^{-d})\) edges between the representative \(s(Q)\) and the closest points in each cone \(C_i\) (in \(P_t\), where \(\ell \geq 0\) is the level of \(Q\) in \(T_n\)). The total number of these edges for all nodes of \(T'_n\) is \(O(\varepsilon^{1-d})\cdot n\).

It remains to consider the nodes of the quadtree \(T_n\) that are compressed in \(T'_n\). Every compressed node is part of a descending chain of single-child nodes in \(\text{universal}T_n\). The number of such chains is \(O_d(n)\), as each chain has a unique direct descendant in \(T'_n\). Let \(Q_k, \ldots, Q_\ell\) be a maximal chain of single-child nodes in \(T_n\), where \(Q_j\) is on level \(j\) of \(T_n\) for \(j = k, \ldots, \ell\). These are nested cubes \(Q_k \subset Q_{k-1} \subset \ldots \subset Q_1\) with a common representative, \(s = q(Q_k) = \ldots = s(Q_\ell)\); see Figure 1. Let \(C_i\) be a cones with apex \(s\) and aperture \(\alpha(d, \varepsilon)\) in algorithm \(\text{ALG}_1\); and let \(q_{i,j}\) denote the closest point to \(s\) in \(C_i \cap P_j\) for \(j = k, \ldots, \ell\). If a point \(q_{i,j} \in P_j\) represents some compressed cube \(Q'\) (in another compressed chain), then \(q_{i,j}\) represents the parent of \(Q'\), as well. In this case, \(q_{i,j} \in P_{\ell-1}\), which implies \(q_{i,j} = q_{i,j-1}\). Consequently, \(q_{i,j} = q_{i,j-1} = \ldots = q_{i,k}\). We may assume that only \(q_{i,k}\) represents a compressed node.

The first \(\leq \lceil \log \varepsilon^{-1} \rceil\) nodes (i.e., \(Q_j\) for \(k \leq \lceil \log \varepsilon^{-1} \rceil\)) jointly contribute \(O(\varepsilon^{1-d} \log \varepsilon^{-1})\) edges to \(G\). Summation over all compressed chains yields \(O(\varepsilon^{1-d} \log \varepsilon^{-1})\cdot n\) edges. For the remaining nodes in the chain (that is, nodes \(Q_j\) for \(k < j \leq \ell - \log \varepsilon^{-1}\)), we use the following charging scheme: Charge the edge \(sq_{i,j}\) to \(q_{i,j}\). Since \(j + k\), then \(q_{i,j}\) represents a noncompressed node at level \(j\) of \(T_n\). Next we bound the charges received by \(q_{i,j}\).
Figure 1: A point \( s \) is the representative of five nested squares in the quadtree. The closest point to \( s \) is \( q_{i,\ell} \in C_i \cap P_\ell \) in the cone \( C_i \) at level \( \ell = 3, \ldots, 7 \).

We claim that for every noncompressed node \( Q \), the representative \( q = s(Q) \) receives at most \( O_d(1) \) units of charges. Indeed, suppose that \( q \in P_\ell \) and an edge \( sq \) has been charged to \( q \). Then \( \|sq\| \leq 24a_\ell \sqrt{d} \cdot \varepsilon^{-1} \). However, \( s \) is the only point in the cube \( Q_s' := Q_j \cap \{ \log \varepsilon^{-1} \} \) of side length \( a_\ell \cdot 2^{\log \varepsilon^{-1}} \geq a_\ell \cdot \varepsilon^{-1} \) and \( \text{diam}(Q_s') \geq a_\ell \sqrt{d} \cdot \varepsilon^{-1} \). Consequently, \( Q_s' \) lies in the ball \( B_q \) of radius \( 25a_\ell \sqrt{d} \cdot \varepsilon^{-1} \) centered at \( q \). However, comparing the volumes of \( B_q \) and \( Q_s' \) shows that \( B_q \) contains \( O_d(1) \) interior-disjoint cubes \( \epsilon \), and so \( q \) is charged at most \( O_d(1) \) times. Summation over all \( O_d(n) \) noncompressed nodes over all levels \( \ell > 0 \) shows that the total number of edges that participate in the charging scheme is \( O_d(n) \).

Overall, we have shown that \( G \) has at most \( O(\varepsilon^{1-d} \log \varepsilon^{-1}) \cdot n \) edges. \( \square \)

### 2.2 Further Improvements in the Plane

We present a tighter analysis of algorithm ALG\(_1\) for \( d = 2 \) that compares the spanner weight to the offline optimum weight, and bypasses the comparison with the MST (i.e., lightness).

**Minimum-Weight Euclidean \((1+\varepsilon)\)-Spanner.** For any \( a, b \in \mathbb{R}^d \), an \( ab \)-path \( P_{ab} \) of Euclidean weight at most \( (1+\varepsilon)\|ab\| \) lies in the ellipsoid \( E_{ab} \) with foci \( a \) and \( b \) and great axes of weight \( (1+\varepsilon)\|ab\| \); see Figure 2. A key observation is that the minor axis of \( E_{ab} \) is \( ((1+\varepsilon)^2 - 1/2) \|ab\| \approx \sqrt{2\varepsilon} \|ab\| \).

Furthermore, Bhore and Tóth [BT21b] recently observed that the directions of “most” edges of the path \( P_{ab} \) are “close” to the direction of \( ab \). Specifically, if we denote by \( E(\alpha) \) the set of edges \( e \) in \( P_{ab} \) with \( \angle(ab,e) \leq \alpha \), then the following holds.

**Lemma 2** (Bhore and Tóth [BT21b]). Let \( a, b \in \mathbb{R}^d \) and let \( P_{ab} \) be an \( ab \)-path of weight \( \|P_{ab}\| \leq (1+\varepsilon)\|ab\| \). Then for every \( i \in \{1, \ldots, \lfloor 1/\sqrt{\varepsilon} \rfloor \} \), we have \( \|E(i \cdot \sqrt{\varepsilon})\| \geq (1 - 2/i^2) \|ab\| \).

Let \( R(a,b) = E_{ab} \cap N(a,b) \), where \( N(a,b) \) is the annulus bounded by two concentric spheres centered at \( a \), of radii \( \frac{1+\varepsilon}{2} \|ab\| \) and \( \|ab\| \); see Figure 2 for an example.

**Lemma 3.** If \( 0 < \varepsilon < \frac{1}{3} \), then every \( ab \)-path \( P_{ab} \) of weight at most \( \|P_{ab}\| \leq (1+\varepsilon)\|ab\| \) contains interior-disjoint line segments \( s \subset R(a,b) \) of total weight at least \( \frac{1}{2} \|ab\| \) such that \( \angle(ab,s) \leq 3 \cdot \sqrt{\varepsilon} \).

**Proof.** Since the distance between the two concentric circles is \( \frac{1-\varepsilon}{2} \|ab\| \), every \( ab \)-path contains a subpath of weight at least \( \frac{1-\varepsilon}{2} \|ab\| \) in the annulus \( N(a,b) \).

Let \( P_{ab} \) be an \( ab \)-path of weight at most \( (1+\varepsilon)\|ab\| \). As noted above \( P_{ab} \subset E_{ab} \). Hence, \( \|P_{ab}\cap N(a,b)\| = \|P_{ab}\cap R(a,b)\| \geq \frac{1-\varepsilon}{2} \|ab\| \) in \( R(ab) \); and so \( \|P_{ab}\cap R(a,b)\| = \|P_{ab}\| - \|P_{ab}\cap R(a,b)\| \leq \frac{1+3\varepsilon}{2} \|ab\| \).
Applying Lemma 2 with \( i = 3 \), the total weight of the edges \( e \) of \( P_{ab} \) with \( \text{dir}(ab, e) \leq 3 \cdot \sqrt{\varepsilon} \) is at least \( \frac{7}{9} \| ab \| \). The parts of these edges lying outside of \( R(a, b) \) have weight at most \( \| P_{ab} \setminus R(a, b) \| \leq \frac{1+3\varepsilon}{2} \| ab \| \). Consequently, the remaining part of these edges are in \( R(a, b) \), and their weight is at least \( \left( \frac{7}{9} - \frac{1+3\varepsilon}{2} \right) \| ab \| \leq \frac{7-27\varepsilon}{18} \| ab \| \leq \frac{2}{9} \| ab \| \) if \( \varepsilon < \frac{1}{9} \), as claimed.

We also need an observation from elementary geometry; see Figure 2.

**Lemma 4.** For \( a, b \in \mathbb{R}^4 \), let \( cd \) be the minor axis of the ellipsoid \( E_{ab} \). Then \( \angle cad \leq 2\varepsilon^{1/2} \).

**Proof.** We may assume w.l.o.g. that \( \| ab \| = 1 \). Let \( o \) be the center of the ellipsoid \( E_{ab} \). Then \( \sec \angle cao = (\cos \angle cao)^{-1} = \left| \frac{ac}{ao} \right| = 1 + \varepsilon \). From the Taylor estimate \( \sec(x) = 1 + \frac{1}{2} x^2 + \frac{5}{24} x^4 + \ldots \leq 1 + x^2 \) for \( 0 < x < 1 \), we have \( \angle cao \geq \varepsilon^{1/2} \). Consequently, \( \angle cad = 2 \angle cao \geq 2\varepsilon^{1/2} \).

**Theorem 2.** Let \( d = 2 \) and \( \varepsilon \in (0, 1) \). The online algorithm \( \text{ALG}_1 \) maintains, for a sequence of \( n \) points in Euclidean plane, an \((1 + \varepsilon)\)-spanner of weight \( O(\varepsilon^{-3/2} \log \varepsilon^{-1} \log n) \cdot \text{OPT} \), where \( \text{OPT} \) denotes the minimum weight of an \((1 + \varepsilon)\)-spanner for the same point set.

**Proof.** Theorem 1 has established that algorithm \( \text{ALG}_1 \) maintains a \((1 + \varepsilon)\)-spanner. The tighter competitive analysis uses Lemmas 3 and 4.

**Competitive Analysis.** Assume w.l.o.g. that \( \text{diam}(S_n) = \Theta(1) \), hence the side length of every quadtree square at level \( \ell \) is \( \Theta(2^{-\ell}) \). For a set \( S_n = \{s_1, \ldots, s_n\} \subset \mathbb{R}^2 \), let \( G^* = (S_n, E^*) \) be a \((1 + \varepsilon)\)-spanner of minimum weight, and let \( \text{OPT} = \| G^* \| \). Let \( G = (S_n, E) \) be the spanner returned by the online algorithm \( \text{ALG}_1 \). Recall that \( G = \bigcup_{\ell \geq 0} G_\ell \), where the total weight of all edges at levels \( \ell > 2 \log n \) is less than \( \text{diam}(S_n) \), so it is enough to consider \( \ell = 0, \ldots, \lfloor 2 \log n \rfloor \).

**Claim 1.** \( \| G_\ell \| \leq O(\varepsilon^{-3/2} \log \varepsilon^{-1}) \cdot \text{OPT} \) for all \( \ell \geq 0 \).

Claim 1 immediately implies \( \| G \| \leq O(\varepsilon^{-3/2} \log \varepsilon^{-1} \log n) \cdot \text{OPT} \). For every level \( \ell \geq 0 \), \( G_\ell = (P_\ell, E_\ell) \) is a graph on the representatives \( P_\ell \). Note that \( G^* \) is a Steiner spanner with respect to the point set \( P_\ell \), as \( G^* \) is a spanner on all \( n \) points of the input.

We prove Claim 1 using a charging scheme: We charge the weight of every edge in \( G_\ell \) to \( G^* \) (more precisely, to line segments along the edges of \( G^* \)), and then show that each line segment of weight \( w \) in \( G^* \) receives \( O(\varepsilon^{-3/2} \log \varepsilon^{-1}) \cdot w \) charge.

For every point \( p \in P_\ell \), algorithm \( \text{ALG}_1 \) greedily covers \( \mathbb{R}^2 \) by \( \Theta(\varepsilon^{-1}) \) cones of aperture \( \pi/k = \Theta(\varepsilon^{-1}) \) and apex \( p \), and adds an edge \( pq_i \) in each nonempty cone \( C_i \). For the competitive analysis, we greedily cover \( \mathbb{R}^2 \) by \( \Theta(\varepsilon^{-1/2}) \) cones of aperture \( \sqrt{\varepsilon} \) and apex \( p \). We use translates of the same cone cover for all \( p \in P_\ell \). Standard volume argument implies that a cone of aperture \( \sqrt{\varepsilon} \) intersects \( O(\varepsilon^{-1/2}) \) cones of aperture \( \Theta(\varepsilon^{-1}) \). We describe the charging scheme for each such cone \( \hat{C} \).
**Charging Scheme.** Consider a cone $\hat{C}$ with apex $p$ and aperture $\sqrt{\varepsilon}$. Let $E(\hat{C})$ be the set of edges $pq$, $q \in \hat{C}$ that algorithm ALG$_1$ adds to $G_\ell$ when $p$ is inserted into $P_\ell$. Since $\hat{C}$ intersects $O(\varepsilon^{-1/2})$ cones of the ordered Yao-graph, then $|E(\hat{C})| \leq O(\varepsilon^{-1/2})$. By construction, every edge in $G_\ell$ has weight at most $O(\varepsilon^{-1/2}2^{-\ell})$.

$$
\|E(\hat{C})\| = \sum_{pq \in E(\hat{C})} \|pq\| \leq \|E(\hat{C})\| \cdot O(\varepsilon^{-1/2}2^{-\ell}) \leq O(\varepsilon^{-3/2}2^{-\ell}). \quad (3)
$$

Let $q_0 = q_0(\hat{C})$ be a closest point in $P_\ell \cap C$ to $p$. (Possibly, $q_0$ arrived after $p$.) We distinguish between two cases:

**Case 1:** $\|pq_0\| < 2 \cdot 2^{-\ell}$. Since $q_0 \in P_\ell$, $\ell$ is bounded by $O(\varepsilon^{-3/2}2^{-\ell})$. On the one hand, the sum of weights over all $p \in P_\ell$ and all cones $\hat{C}$ with $\|pq_0\| < 2 \cdot 2^{-\ell}$ is bounded by $O(|P_\ell| \cdot \varepsilon^{-3/2}2^{-\ell})$. On the other hand, $\text{OPT} \geq \Omega(\|\text{MST}(P_\ell)\|) \geq \Omega(|P_\ell| \cdot 2^{-\ell})$. Consequently, the total weight of all edges handled in Case 1 is $O(\varepsilon^{-3/2}) \text{OPT}$.

**Case 2:** $\|pq_0\| \geq 2 \cdot 2^{-\ell}$. The optimal spanner $G^*$ contains a $pq_0$-path $P_0$ of weight at most $(1 + \varepsilon)\|pq_0\|$. Recall $P_0$ lies in the ellipse $E_0$ with foci $p$ and $q_0$, and $R(p,q_0)$ is the half of $E_0$ that contains $q_0$ (cf. Figure 2). Let $E^*(\hat{C})$ be the set of maximal line segments $e$ along edges in $E^*$ such that $e \in P_0 \cap R(p,q_0)$ and $\varepsilon(e,pq_0) \leq 3 \cdot \sqrt{\varepsilon}$. By Lemma 3, we have $\|E^*(\hat{C})\| \geq \frac{1}{3}\|pq_0\|$. We distribute the weight of all edges in $E(\hat{C})$ uniformly among the line segments in $E^*(\hat{C})$. That is, each segment of weight $w$ in $E^*(\hat{C})$ receives a charge of

$$
\frac{\|E(\hat{C})\|}{\|E^*(\hat{C})\|} \cdot w \leq \frac{O(\varepsilon^{-3/2}2^{-\ell})}{\Omega(2^{-\ell})} \cdot w \leq O(\varepsilon^{-3/2}) \cdot w. \quad (4)
$$

This completes the description of the charging scheme in Case 2.

Figure 3: Left: There consecutive cones, $\hat{C}_0$, $\hat{C}_1$, and $\hat{C}_2$, with apex $p$ and aperture $\sqrt{\varepsilon}$. Point $q_0$ is the closest to $p$ in $P_\ell \cap C_1$; and $R(p,q_0) \in \hat{K}_1 = C_0 \cup C_1 \cup C_2$. Right: No point in $P_\ell$ is in the blue sector $\hat{K}$, but there may be points in the pink sectors.

**Charges Received.** A point along an edge of the optimal spanner $G^*$ may receive charges from several cones $\hat{C}$, possibly with different apices $p \in P_\ell$. Let $L$ be a maximal line segment along an edge of $G^*$ such that every point in $L$ receives the same charges.
For a cone $\hat{C}$ of aperture $\sqrt{\varepsilon}$, let $\hat{K}$ denote a cone with the same apex and axis as $\hat{C}$, but aperture $3\sqrt{\varepsilon}$; refer to Figure 3.

Claim 2. If $L$ receives charges from $\hat{C}$, then $L \subset \hat{K}$.

Indeed, if $L$ receive charges from $\hat{C}$, then $L \subset R(p, q_0) \subset E_0$, where $E_0$ is the ellipse with foci $p$ and the closest point $q_0 \in \hat{C} \cap P_j$. By Lemma 4, $R(p, q_0)$ lies in a cone with apex $p$, aperture $2\sqrt{\varepsilon}$, and axis $pq_0$. Consequently $L \subset R(p, q_0) \subset \hat{K}$, which proves Claim 2.

Note that if $L$ receives positive charge from a cone $\hat{C}$ with apex $p$ and closest point $q_0$, then $\angle Lpq_0 \leq 3\cdot\sqrt{\varepsilon}$. Since the aperture of the cones $\hat{C}$ is $\sqrt{\varepsilon}$, then $L$ receives charges from cones $\hat{C}$ with at most $O(1)$ different orientations. We may restrict ourselves to cones $\hat{C}$ that are translates of each other (but have different apices in $P_\ell$).

Let $A$ be the set of all translates of a cone $\hat{C}$ with aperture $\sqrt{\varepsilon}$ and apices in $P_\ell$, and $L$ receives positive charge from $\hat{C}$. We partition $A$ into $O(\log\varepsilon^{-1})$ classes as follows. For $j = 1, \ldots, |\log(2\varepsilon^{-1})|$, let $A_j$ be the set of cones $\hat{C} \in A$ such that $2^{1-j} \leq \|pq_0\| < 2^{1-j-\ell}$, where $p \in P_\ell$ is the apex of $\hat{C}$ and $q_0$ is the closest point in $P_\ell \cap \hat{C}$ to $p$.

Claim 3. For each $j$, segment $L$ receives $O(\varepsilon^{-3/2}) \cdot \|L\|$ total charges from all cones in $A_j$.

By refining (4) for a cone in $\hat{C} \in A_j$, we see that $L$ receives a charge

$$\frac{\|E(\hat{C})\|}{\|E^*(\hat{C})\|} \cdot \|L\| \leq O(\varepsilon^{-3/2}2^{j-\ell}) \cdot \|L\| \leq O(\varepsilon^{-3/2}2^{-j}) \cdot \|L\|$$

(5)

from each cone in $A_j$. To prove Claim 3, it is enough to show that $|A_j| \leq O(2^j)$.

![Figure 4: The union $U$ of triangles $\hat{C} \cap h^-$, where $L$ receives charges from the cones $\hat{C}$.](image)

We may assume w.l.o.g. that the symmetry axis of every cone in $A_j$ is parallel to the $x$-axis, and their apex is their leftmost point. Let $h$ be a vertical line that contains the left endpoint of $L$, and let $h^-$ be the left halfplane bounded by $h$; see Figure 4. The intersections $\bar{C} \cap h$ and $\bar{K} \cap h$ are vertical line segment of length $O(2^{j-\ell} \tan \sqrt{\varepsilon})$. We have $L \cap h \subset \bar{K} \cap h$ by Claim 2; and obviously $\bar{C} \cap h \subset \bar{K} \cap h$. Consequently, a vertical line segment of length $O(2^{j-\ell} \tan \sqrt{\varepsilon})$ contains $h \cap \hat{C}$ for all $\hat{C} \in A_j$.

Let $U$ be the union of the triangles $\hat{C} \cap h^-$ for all $\hat{C} \in A_j$. The interior of the $\bar{C} \cap h^-$ does not contain any point in $P_\ell$. Consequently, the apices of all cones lie on the boundary $\partial U$ of $U$. The part of $\partial U$ in $h^-$ is a $y$-monotone curve with slopes $\pm \sqrt{\varepsilon}$. It follows that the length of $\partial U$ is $O(2^{j-\ell} \tan \sqrt{\varepsilon} / \sin \sqrt{\varepsilon}) = O(2^{j-\ell} \csc \sqrt{\varepsilon}) = O(2^{j-\ell})$. This, in turn, implies that $\partial U$ intersects $O(2^j)$ cubes of side length $a_02^{-\ell}$ at level $\ell$ of the quadtree, and so $|A_j| \leq O(2^j)$, as required. This completes the proof Claim 3, and hence the proof of Theorem 2.

$\blacksquare$
3 Lower Bounds in $\mathbb{R}^d$ Under the $L_1$ Norm

In this section we introduce a strategy based on the points on the integer lattice $\mathbb{Z}^d$, that achieves a new lower bound for the competitive ratio of an online $(1+\varepsilon)$-spanner algorithm in $\mathbb{R}^d$ under the $L_1$ norm.

**Construction.** We describe an adversary strategy with $\Omega(d(\varepsilon^{-d})^d$ points and show that any online algorithm returns a $(1+\varepsilon)$-spanner whose weight is $\Omega(d(\varepsilon^{-d})^d$ times the optimum weight. One can extend this result for arbitrary number of points, but that does not necessarily improve the lower bound. The final point set $X$ consists of the points of the integer lattice $\mathbb{Z}^d$ in the hypercube $[0,1/\varepsilon d]^d$, where $\varepsilon < 1$. The points are presented in stages in order to deceive the online algorithm to add more edges than needed. In step $2i$, where $0 \leq i < \frac{1}{\varepsilon d}$, points $x \in X$ such that $\|x\|_1 = i$ will be given to the algorithm. In step $2i + 1$, where $0 \leq i < \frac{1}{\varepsilon d}$, the adversary presents points $x \in X$ such that $\|x\|_1 = \lceil 1/\varepsilon \rceil - i$ (Figure 5). In other words, points are presented in batches according to their $L_1$ norms.

**Competitive Ratio.** Denote by $X_i$ the set of points presented in step $i$. The idea is to show that there has to exist many edges between $X_i$ and $X_{i+1}$ in order to guarantee the $1+\varepsilon$ stretch-factor. Specifically, we define an **ordered-pair** as follows.

**Definition 1** (ordered-pair). A pair of points $(x,y)$ in $\mathbb{R}^d$ is an ordered-pair if $x \in X_{2i}$ and $y \in X_{2i+1}$ for some $i$, and $x_k \leq y_k$ for all $k$, where $x_k$ and $y_k$ are the $k$-th coordinates of $x$ and $y$ respectively.

Now we show that any ordered-pair $(x,y) \in X_{2i} \times X_{2i+1}$ requires an edge in the spanner immediately after $x$ and $y$ are presented. To prove this, we show that previously presented points cannot serve as via points in a $(1+\varepsilon)$-path between $x$ and $y$.
Lemma 5. Let \((x, y)\) be an ordered-pair. Then there is no \((1 + \epsilon)\)-path between \(x\) and \(y\) that goes through any other point \(z \in X_j\) with \(j \leq i + 1\).

Proof. Let \(x_k, y_k, z_k\) be the \(k\)-th coordinate of \(x, y, z\), respectively. Then the equality \(\|x - z\|_1 + \|y - z\|_1 = \|x - y\|_1\) holds if and only if \(x_k \leq z_k \leq y_k\), for all \(k\). Since \(z \neq x\) and \(z \neq y\), we can conclude that \(\|x\|_1 < \|z\|_1 < \|y\|_1\), which means that \(z\) is not added in the previous steps, which is a contradiction. So the equality does not hold and \(\|x - z\|_1 + \|y - z\|_1\) is strictly larger than \(\|x - y\|_1\). As both expressions are integers, we have

\[
\|x - z\|_1 + \|y - z\|_1 \geq 1 + \|x - y\|_1 \\
> \epsilon \|x - y\|_1 + \|x - y\|_1 \\
= (1 + \epsilon) \|x - y\|_1.
\]

The second inequality follows from the fact that \(\|x - y\|_1 < \epsilon^{-1}\) which holds for any two points in \(X\). The above inequality shows that a \((1 + \epsilon)\)-path between \(x\) and \(y\) cannot go through \(z\) and completes the proof of the lemma.

We next show that the total weight of the edges between ordered pairs is \(\Omega_d(\epsilon^{-2d})\).

Lemma 6. The total weight of the edges between the ordered-pairs is \(\Omega_d(\epsilon^{-2d})\).

Proof. Let \(x = (x_1, \ldots, x_d)\) and \(y = (y_1, \ldots, y_d)\) be two points in \(X\). We show that if \(x_k \in \left[\frac{1}{4\epsilon(d + 0.25)} - \frac{1}{4\epsilon d}\right]\) for all \(1 \leq k \leq d\), and \(y_k \in \left[\frac{3}{4\epsilon(d + 0.25)} - \frac{3}{4\epsilon d}\right]\) for all \(1 \leq k \leq d - 1\), then there is choice of \(y_d\) that makes \((x, y)\) an ordered-pair. This would imply that there are \(\Omega_d(\epsilon^{-2d+1})\) ordered-pairs and by Lemma 5, each pair requires an edge of weight \(\Omega_d(\epsilon^{-1})\), thus the total weight of required edges would be \(\Omega_d(\epsilon^{-2d})\).

In order to find such a \(y_d\), recall that \(\|x\|_1 + \|y\|_1 = [\epsilon^{-1}]\) holds because \((x, y)\) is an ordered-pair. This equality uniquely determines the value of \(y_d\),

\[
y_d = \left[\epsilon^{-1}\right] - \sum_{k=1}^{d} x_k - \sum_{k=1}^{d-1} y_k.
\]

We just need to prove the inequalities \(y_k \geq x_k\) and \(y_k \leq 1/(\epsilon d)\) for this unique \(y_k\). This can simply be done by plugging the maximum (and minimum) values of \(x_k\)’s and other \(y_k\)’s and calculating the result,

\[
y_d \geq \frac{1}{\epsilon} - \frac{d}{4\epsilon d} - \frac{3(d-1)}{4\epsilon d} = \frac{3}{4\epsilon d} > x_d.
\]

Also,

\[
y_d \leq \frac{1}{\epsilon} + 1 - \frac{d}{4\epsilon(d + 0.25)} - \frac{3(d-1)}{4\epsilon(d + 0.25)} = 1 + \frac{1}{\epsilon(d + 0.25)} < \frac{1}{\epsilon d}.
\]

Now we can prove the main theorem of this section.

Theorem 3. The competitive ratio of any online \((1 + \epsilon)\)-spanner algorithm in \(\mathbb{R}^d\) under the \(L_1\)-norm is \(\Omega_d(\epsilon^{-d})\).

Proof. For the point set \(X \in \mathbb{R}^d\), the unit-distance graph is a Manhattan network: It contains a path of weight \(\|xy\|_1\) for all \(x, y \in X\). Its weight is \(\Theta_d(\epsilon^{-d})\) which is an upper bound for the weight of a \((1 + \epsilon)\)-spanner for any \(\epsilon \geq 1\). By Lemma 6, any online algorithm returns a spanner of weight \(\Omega_d(\epsilon^{-2d})\). Thus its competitive ratio is \(\Omega_d(\epsilon^{-d})\).
4 General Metrics: The Ordered Greedy Spanner

In this section we study the online spanners problem on general metric spaces. The points arrive one by one, where for each new point we also receive its distances to all previously introduced points.

In the offline setting, the celebrated greedy spanner algorithm [ADD+93] sorts the edges by increasing weight, and then processes them one by one, adding each edge if by the time of examination, the distance between its endpoints is too large. This algorithm achieves the existentially optimal\(^2\) sparsity and lightness as a function of the stretch factor [FS20]. However, in the online model, we do not receive the edges in a sorted order, and therefore cannot execute the greedy algorithm. As an alternative, we propose here the ordered greedy algorithm. This is a deterministic algorithm working against an adaptive adversary. The algorithm receives a stretch factor \(t\), and works naturally as follows: We maintain a spanner \(H\). When a point \(v_i\) arrives, we order its edges\(^3\) in the original metric by weight. Each edge \(\{v_i, v_i\}\) is added to the spanner \(H\) if currently \(d_H(v_i, v_i) > t \cdot d_X(v_i, v_i)\). Note that this algorithm can be easily executed in an online fashion.

**Theorem 4.** Given an \(n\)-point metric space \((X, d_X)\) in an (adaptive) adversarial order, with stretch factor \(t = (2k - 1)(1 + \varepsilon)\) for \(k \geq 2\) and \(\varepsilon \in (0, 1)\), the ordered greedy algorithm returns a spanner with \(O(\varepsilon^{-1} \log \frac{1}{t}) \cdot n^{1 + \frac{1}{t}}\) edges and weight \(O(\varepsilon^{-1} n^{1 + \frac{1}{t}} \log^2 n) \cdot w(MST)\).

**Proof.** The bounded stretch of our spanner is straightforward by construction, as every pair was examined at some point, and taken care of. Next we analyze the lightness.

In the online spanning tree problem, points of a finite metric space arrive one-by-one, and we need to connect each new point to a previous point to maintain a spanning tree. The ordered greedy algorithm connects each vertex \(v_i\) to the closest vertex in \(\{v_1, \ldots, v_{i-1}\}\). As was shown by Imase and Waxman [IW91b], the tree created by the ordered greedy algorithm has lightness \(O(\log n)\), which is the best possible [IW91b]. Denote the online spanning tree by \(T_G\). Note that the ordered greedy spanner \(H\) will contain \(T_G\), as a shortest edge between a new vertex to a previously introduced vertex is always added to the spanner \(H\). The following clustering lemma is frequently used for spanner constructions (see e.g. [ADF+22, CW18, ES16]). We provide a proof for the sake of completeness.

**Claim 4.** For every \(i \in \mathbb{N}\), the point set \(X\) can be partitioned into clusters \(C_i\) of diameter at most \(D_i = \varepsilon \cdot (1 + \varepsilon)^i\) w.r.t. the metric \(d_{T_G}\) such that \(|C_i| = O\left(\frac{w(T_G)}{\varepsilon \cdot (1 + \varepsilon)^i}\right)\).

**Proof.** Let \(N_i\) be a maximal set of vertices such that for every \(x, y \in N_i\), \(d_{T_G}(x, y) > \frac{1}{2} \cdot D_i\). For every vertex \(x \in N_i\) let \(C_x = \{z : z = \text{argmin}_{y \in N_i} d(x, y)\}\) be the Voronoi cell of \(x\). Clearly, \(\text{diam}(C_x) \leq D_i\) for all \(x\). Further, consider a continuous version of \(T_G\) (where each edge is an interval). Then as the graph \(T_G\) is connected, each cluster \(C_x\) contains at least \(\frac{1}{4} D_i\) length of edges (as the balls \(\{B_{T_G}(x, 1/4 D_i)\}_{x \in N_i}\) are pairwise disjoint). It follows that

\[
|C_i| = |N_i| \leq \frac{w(T_G)}{\frac{1}{4} D_i} = O\left(\frac{w(T_G)}{\varepsilon \cdot (1 + \varepsilon)^i}\right),
\]

as claimed. \(\square\)

\(^2\)Specifically, if a \(t\)-spanner construction achieves an upper bound \(m(n, t)\) and \(l(n, t)\), resp., on the size and lightness of an \(n\)-vertex graph then this bound also holds for the greedy \(t\)-spanner [FS20].

\(^3\)By edges we mean point pairs in the metric space, we will often use notation from graph theory.
For every \( i \), consider the scale \( E_i = \{ e = \{ u, v \} \in H : (1 + \epsilon)^{i-1} \leq d_X(u, v) < (1 + \epsilon)^i \} \). We now ready to bound the lightness.

**Claim 5.** The weight of the ordered greedy spanner is \( O(n^{1/2} \cdot \epsilon^{-2} \log^2 n) \cdot w(MST) \).

**Proof.** For scale \( i \), consider the clusters \( C_i \) from Claim 4. We create an (unweighted) cluster graph \( G_i \) by contacting all the edges in each cluster and adding the edges \( E_i \) (i.e., for every \( u, v \in E_i \) such that \( u \in C_u \) and \( v \in C_v \), we add the edge \( \{ c_u, c_v \} \to G_i \). Consider a cluster \( C \in C_i \) where \( C = (u_1, u_2, \ldots, u_{|C|}) \) are the vertices ordered w.r.t. arrival times. We argue that for every \( j = 1, \ldots, |C| \), the induced subgraph \( T_G[\{u_1, \ldots, u_j\}] \) is connected. Assume for contradiction otherwise, and let \( j \) be the first index violating this rule. Let \( T^j_G \) be the tree \( T_G \) right after the arrival of \( u_j \). On the one hand, \( T^j_G \) is connected, and so it contains a path \( P \) from \( u_j \) to \( \{ u_1, \ldots, u_{j-1} \} \). By the assumption that \( T_G[\{u_1, \ldots, u_j\}] \) is disconnected, the a path \( P \) has interior vertices that are not \( \{ u_1, \ldots, u_j \} \). On the other hand, there is a path \( P' \) from \( u_j \) to \( \{ u_1, \ldots, u_{j-1} \} \) in \( T_G[C] \). We conclude that \( T_G \) contains two different paths from \( u_1 \) to \( u_j \), a contradiction to the fact that \( T_G \) is a tree.

Furthermore, note that as \( T_G \) is a tree, the diameter of \( T_G[\{u_1, \ldots, u_j\}] \) is bounded as well by \( D_i \).

We next argue that \( C_i \) is a simple graph. Suppose for contradiction that there is a cluster \( C \in C_i \) with a self loop. This implies that there are \( v_a, v_b \in C \) such that \( \{ v_a, v_b \} \in E_i \). But this is impossible as \( d_X(v_a, v_b) \leq d_{T_G}(v_a, v_b) < D_i = \epsilon \cdot (1 + \epsilon)^i \).

Next, suppose for contradiction that there is an edge \( \{ C, C' \} \) in \( G_i \) of multiplicity two or higher. Then there are vertices \( x_1, x_2 \in C \) and \( y_1, y_2 \in C' \) such that \( \{ x_1, y_1 \}, \{ x_2, y_2 \} \in E_i \). Assume w.l.o.g. that \( y_2 \) is the last arriving vertex among \( \{ x_1, x_2, y_1, y_2 \} \). At the time \( x_2, y_2 \) is examined by the ordered greedy algorithm, there are paths from \( x_1 \) to \( x_2 \) and from \( y_1 \) to \( y_2 \) of weight at most \( D_i \). As \( \{ x_1, y_1 \} \) were already added to \( H \), the spanner contains a \( x_2 y_2 \)-path of weight at most \( 2D_i + d_X(x_1, y_1) \leq 2 \cdot \epsilon \cdot (1 + \epsilon)^i + (1 + \epsilon)^i < t \cdot (1 + \epsilon)^i \leq t \cdot d_X(x_2, y_2) \), which contradicts to the fact that the algorithm chose to add \( \{ x_2, y_2 \} \). We conclude that \( G_i \) is indeed a simple graph.

Next, we argue that \( G_i \) has girth at least \( 2k + 1 \). Suppose for contradiction that there is a cycle \( C_0 C_1 C_2 \ldots C_{2k} C_0 \) in \( G_i \), with \( \beta \leq 2k - 1 \), where the edge \( C_j C_{j+1} \) corresponds to the edge \( \{ x_j, y_{j+1} \} \in E_i \), modulo \( \beta \). Assume w.l.o.g. that the edge \( \{ x_j, y_0 \} \) was added last. Note that at the time the algorithm examines \( \{ x_j, y_0 \} \), for every \( j \), there is a path in \( H \) from \( y_j \) to \( x_j \) of weight at most \( D_i \). Denote by \( \tilde{H} \) the spanner \( H \) at this time. We conclude that

\[
d_{\tilde{H}}(y_0, x_\beta) \leq \sum_{j=0}^{\beta} d_{\tilde{H}}(y_j, x_j) + \sum_{j=0}^{\beta-1} d_{\tilde{H}}(x_j, y_j) \\
\leq (\beta + 1) \cdot D_i + \beta \cdot (1 + \epsilon)^i \\
\leq (2k - 1) (1 + 3\epsilon)^i (1 + \epsilon)^{i-1} \leq (2k - 1) (1 + 3\epsilon) \cdot d_X(y_0, x_{2k-1}),
\]

which contradicts the fact that the edge \( \{ x_\beta, y_0 \} \) was added to the algorithm.

A graph with girth \( 2k + 1 \) contains at most \( O(n^{1+\frac{1}{k}}) \) edges (see e.g. [Bol78]). Hence the total weight of all the edges in \( E_i \) is bounded by

\[
(1 + \epsilon)^i \cdot |E_i| = O(|C_i|^{1+\frac{1}{k}}) \cdot (1 + \epsilon)^i = O(n^{\frac{k}{2}}) \cdot \frac{w(T_G)}{\epsilon} \cdot (1 + \epsilon)^i \cdot (1 + \epsilon)^i = O(\epsilon^{-1} n^{\frac{k}{2}}) \cdot w(T_G).
\]

Let \( \epsilon_{\text{max}} \) be the heaviest edge in \( H \), and let \( i_{\text{max}} \) be the index such that \( \{ x, y \} \in E_{i_{\text{max}}} \). Note that for every scale \( i \leq i_{\text{max}} - \alpha \) have weight at most

\[
w(E_i) \leq \left( \frac{n}{2} \right)^i \cdot (1 + \epsilon)^i \leq n^2 \cdot w(\epsilon_{\text{max}}) \cdot (1 + \epsilon)^{-\alpha} \leq n^2 \cdot w(T_G) \cdot (1 + \epsilon)^{-\alpha}.
\]

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We conclude that the weight of the spanner is bounded by

\[
w(H) = \sum_{i \leq \text{max}} w(E_i) = \sum_{i = \text{max} - \log s}^{\text{max}} \sum_{n^2}^{n^2} w(E_i) + \sum_{i = \text{max} - \log s}^{\text{max}} w(E_i) \\
\leq \log_{1+\varepsilon} n^2 \cdot O\left(\varepsilon^{-1} n^{1/k}\right) \cdot w(T_G) + \sum_{j \geq 1} \log_{1+\varepsilon} n^2 \cdot (1 + \varepsilon)^{-j} \\
\leq O\left(\frac{\log n}{\log(1+\varepsilon)} \cdot \frac{n^{\frac{1}{k}}}{\varepsilon} + \frac{1}{\varepsilon}\right) \cdot w(T_G) \\
= O\left(n^{\frac{1}{k}} \cdot \frac{\log n}{\varepsilon^2}\right) \cdot w(T_G) = O\left(n^{\frac{1}{k}} \cdot \frac{\log^2 n}{\varepsilon^2}\right) \cdot w(MST).
\]

We next bound the sparsity of the ordered greedy spanner.

**Claim 6.** The ordered greedy spanner has \(O(\varepsilon^{-1} \log \frac{1}{\varepsilon}) \cdot n^{1+\frac{1}{k}}\) edges.

**Proof.** We will assume for simplicity that the algorithm was executed with parameter \(t = (2k - 1)(1 + 2\varepsilon)\), later one can scale the results accordingly. Let \(\{v_1, \ldots, v_n\}\) be the order in which the vertices arrived. Let \(H_i\) be the state of the spanner just after the arrival of \(v_i\). We greedily construct a laminar set system \(N_0 \subseteq N_1 \subseteq \ldots\), where every pair of point in \(N_i\) will be at distance at least \((1 + \varepsilon)^i\) w.r.t. the spanner \(H\). Specifically, given a newly arrived vertex \(v_j\) which already joined \(N_i\), \(v_j\) will join \(N_{i+1}\) if there is no vertex \(v_j'\) (where \(j' < j\)) at distance \(d_{H_j}(v_j, v_j') \leq (1 + \varepsilon)^{i+1}\) in the current spanner. Let \(\Delta_i = \frac{(1+\varepsilon)^{i+1}-1}{\varepsilon}\). We will call each set \(N_i\) a net, and every point \(v_j \in N_i\) a net point. We argue that the set \(N_i\) is \(\Delta_i\) dominating, that is every vertex \(v_j\) has a net point \(v_j' \in N_i\), such that at the time \(v_j\) arrived, \(d_{H_j}(v_j, v_j') \leq \Delta_i\).

Indeed, by induction there is a net point \(v_q \in N_{i-1}\) such that \(d_{H_j}(v_j, v_q) \leq \Delta_{i-1}\), and \(q < j\). If \(v_q \in N_i\) then we are done. Otherwise, there is a point \(v_p \in N_i\) such that \(d_{H_j}(v_q, v_p) \leq (1 + \varepsilon)^i\) and \(s < q\). Implying \(d_{H_j}(v_j, v_p) \leq d_{H_j}(v_j, v_q) + d_{H_j}(v_q, v_p) \leq \frac{(1+\varepsilon)^{i+1}}{\varepsilon} + (1 + \varepsilon)^i = \frac{1}{\varepsilon}\). For \(i\) too small, let \(N_i = X\), and \(\Delta_i = 0\).

For every \(i\), consider the *scale* \(E_i = \{e = \{u, v\} \in H : (1 + \varepsilon)^{i-1} \leq d_X(u, v) < (1 + \varepsilon)^i\}\). Set \(s = \left[\log_{1+\varepsilon}(\frac{4k}{2k-1} \cdot \frac{1+\varepsilon}{\varepsilon^2})\right]\).

We argue that for every \(i, |E_i| \leq O(|N_{i-s} \setminus N_{i+s}|)^{1+\frac{1}{2}}\). For this goal, we construct an auxiliary graph \(G_i\), which contains vertices and \(E_i\) as edges. Specifically, for every \(\{x, y\} \in E_i\), let \(v_x v_y \in N_i\) be the closest vertices to \(x, y\) in \(N_i\) at the time they were added. Then we will add the edge \(\{v_x, v_y\}\) to \(G_i\).

Clearly \(G_i\) does not contain self loops, as the distance between two vertices \(x, y\) who has the same closest vertex in \(N_i\) is bounded by \(2\Delta_{i-s} < (1 + \varepsilon)^{i-s}\). Suppose for contradiction that there is an edge \(\{v, u\}\) in \(G_i\) of multiplicity two or higher. Then there are vertices \(x_1, x_2, y_1, y_2\) such that \(v\) was the closest vertex to \(x_1, x_2\), \(u\) was the closest vertex to \(y_1, y_2\), and \(\{x_1, y_1\}, \{x_2, y_2\} \in E_i\). Assume w.l.o.g. that \(y_2\) is the last arriving vertex among \(\{x_1, x_2, y_1, y_2\}\). At the time \(\{x_2, y_2\}\) is examined by the ordered greedy algorithm, the pairs \((x_1, x_2)\) and \((y_1, y_2)\) already were examined, and hence \(H\) contain path from \(x_1\) to \(x_2\) and from \(y_1\) to \(y_2\) of weight at most \(2 \cdot \Delta_{i-s}\). By our
corresponding to \( G \) in \( \mathbb{N} \) vertices in \( G \) modulo \( a \). A contradiction to the fact that both \( u \) and \( v \) are in \( E \) is thus reached. Assume w.l.o.g. that the edge \( \{x, y\} \) was added last. Note that at the time the algorithm examines \( x, y \), for every \( j \), there is a path in \( H \) from \( y \) to \( x \) of weight at most \( 2 \cdot \Delta_{i-s} \). Denote by \( \bar{H} \) the spanner \( H \) at this time. We conclude that

\[
d_{\bar{H}}(y_0, x_\beta) \leq \sum_{j=0}^{\beta} d_{\bar{H}}(y_j, x_j) + \sum_{j=0}^{\beta-1} d_{\bar{H}}(x_j, y_j) \\
\leq (\beta + 1) \cdot 2\Delta_{i-s} + \beta \cdot (1 + \varepsilon)^i \\
\leq 2k \cdot \frac{2(1 + \varepsilon)^{i-s}}{\varepsilon} + (2k - 1) \cdot (1 + \varepsilon)^i \\
= (2k - 1)(1 + \varepsilon + \frac{4k}{2k-1} \cdot \frac{1}{\varepsilon(1 + \varepsilon)^{s-1}}) \cdot (1 + \varepsilon)^i \\
\leq (2k - 1)(1 + 2\varepsilon) \cdot d_X(x_0, x) \leq (1 + \varepsilon)^i 
\]

which contradicts the fact that the edge \( \{x_\beta, y_0\} \) was added to the algorithm.

Consider a pair of net points \( u, v \in N_{i+s} \). Then the distance between \( u, v \) in \( \mathcal{G}_i \) has to be at least 3. Otherwise, if \( d_{\mathcal{G}_i}(u, v) \leq 2 \), there is a net point \( z \in N_{i-s} \) and two edges \( \{x_0, y_1\}, \{x_1, y_2\} \in E_i \) corresponding to \( \{u, z\}, \{z, v\} \) in \( \mathcal{G}_i \). Then following the logic above,

\[
d_{\bar{H}}(u, v) \leq d_{\bar{H}}(u, x_0) + d_{\bar{H}}(x_0, y_1) + d_{\bar{H}}(y_1, x_1) + d_{\bar{H}}(x_1, y_2) + d_{\bar{H}}(y_2, v) \\
\leq 4\Delta_{i-s} + 2 \cdot (1 + \varepsilon)^i \\
\leq \left( \frac{8}{\varepsilon(1 + \varepsilon)^s} + 2 \right) \cdot (1 + \varepsilon)^i < (1 + \varepsilon)^{i+s},
\]

a contradiction to the fact that both \( u, v \) joined \( N_i \). It follows that there are no edges between vertices in \( N_{i+s} \), and furthermore, each vertex in \( N_{i-s} \setminus N_{i+s} \) is connected to at most a single vertex in \( N_{i+s} \). We conclude that the number of edges incident on \( N_{i+s} \) vertices is bounded by \( |N_{i-s} \setminus N_{i+s}| \). As the induced graph \( \mathcal{G}_i[N_{i-s} \setminus N_{i+s}] \) has girth \( 2k+1 \), it contains at most \( O\left(|N_{i-s} \setminus N_{i+s}|^{1+\frac{1}{2}}\right) \) edges (see e.g. [Bol78]). We conclude

\[
|E_i| = E(\mathcal{G}_i) = E(G[N_{i-s} \setminus N_{i+s}]) + |N_{i-s} \setminus N_{i+s}| = O\left(|N_{i-s} \setminus N_{i+s}|^{1+\frac{1}{2}}\right).
\]
We conclude a bound on the number of edges:

\[ |E(H)| = \sum_{i \geq 0} |E_i| \leq \sum_{i \geq 0} O\left(\left|N_{i-s} \setminus N_{i+s}\right|^{1+\frac{1}{k}}\right) \]

\[ \leq O(n^{\frac{1}{k}}) \cdot \sum_{i \geq 0} |N_{i-s} \setminus N_{i+s}| = O(s \cdot n^{1+\frac{1}{k}}) = O\left(\frac{\log \frac{1}{\varepsilon}}{\varepsilon} \cdot n^{1+\frac{1}{k}}\right), \]

where the second to last equality follows since each vertex can participate in at least 2s different addends in the sum.

The theorem now follows.

\[ \square \]

5 Lower Bound for General metrics

In this section we prove an \( \Omega(\frac{1}{k} \cdot n^{\frac{1}{k}}) \) lower bound on the competitive ratio of an online \((2k - 1)\)-spanner of \( n \)-vertex graphs. Our lower bound holds in both cases where the quality is measured by number of edges or the weight. It follows that our upper bound in Theorem 4 cannot be substantially improved, even if we consider competitive ratio instead of lightness/sparsity.

Recall that the Erdős Girth Conjecture \([Erd64]\) states that for every \( n, k \geq 1 \), there exists an \( n \)-vertex graph with \( \Omega(n^{1+\frac{1}{k}}) \) edges and girth \( 2k + 2 \). The proof of the following lemma is based on a counting argument form the recent lower bound proof for (static) vertex fault tolerant emulators by Bodwin, Dinitz, and Nazari \([BDN22]\).

**Lemma 7.** Assuming the Erdős girth conjecture, for every \( n, k \geq 1 \), there exists an \( n \)-point metric space \( (X, d_X) \) with diameter \( 2k - 1 \), such that every \((2k - 1)\)-spanner has \( \Omega(\frac{1}{k} \cdot n^{1+\frac{1}{k}}) \) edges and weight \( \Omega(n^{1+\frac{1}{k}}) \).

**Proof.** Let \( G = (V, E_G) \) be the graph fulfilling the Erdős girth conjecture. That is, \( G \) is an unweighted \( n \)-vertex graph with girth \( 2k + 2 \) and \( |E_G| = \Omega(n^{1+\frac{1}{k}}) \) edges. Set a metric \( d_X \) over \( V \) as follows, \(^4\)

\[ \forall u, v \in V \quad d_X(u, v) = \min \{d_G(u, v), 2k - 1\}. \]

Suppose that \( H = (V, E_H) \) is a \((2k - 1)\)-spanner for \((V, d_X)\) with weight function \( w_H \), where the weight of an edge \( e' \in \{u, v\} \in E_H \) is \( w_H(e') = d_X(u, v) \). Let \( E' = E_H \setminus E_G \) be the edges of \( H \) which are not in \( G \). We say that an edge \( e' \in E' \) covers an edge \( e \in E_G \), if there is a shortest path in \( G \) between the endpoints of \( e' \) going through \( e \) of weight at most \( k \). Note that as \( e' \) has weight at most \( k \), there is a unique shortest path in \( G \) between its endpoints. In particular, each edge \( e \in E' \) can cover at most \( k \) edges in \( E_G \).

Consider an edge \( e = \{v_0, v_s\} \in E_G \setminus E_H \). We argue that some edge \( e' \in E' \) must cover \( e \). Suppose for contradiction otherwise, and let \( P = (v_0, v_1, \ldots, v_s) \) be the shortest path in \( H \) between the endpoints \( v_0, v_s \) of \( e \). Suppose first that \( P \) contains an edge \( v_i, v_{i+1} \) of weight at least \( w_H(\{v_i, v_{i+1}\}) \geq k+1 \). In particular, \( d_G(\{v_i, v_{i+1}\}) \geq k+1 \). Then by the triangle inequality, \( d_G(v_0, v_i) + d_G(v_{i+1}, v_s) \geq d_G(v_i, v_{i+1}) \geq k \). It follows that \( P \) has weight at least \( 2k + 1 \), a contradiction to the fact that \( H \) is a \((2k - 1)\)-spanner. We conclude that for every \( i \in \{0, \ldots, s - 1\} \), \( d_X(v_i, v_{i+1}) = \)

\(^4\)Note that \( \forall x, y, z \in V \quad d_X(x, z) = \min \{d_G(x, z), 2k - 1\} \leq \min \{d_G(x, y) + d_G(y, z), 2k - 1\} \leq \min \{d_G(x, y), 2k - 1\} + \min \{d_G(y, z), 2k - 1\} = d_X(x, y) + d_X(y, z) \). Thus \( d_X \) is a metric space.
In particular, in $G$ there is a unique path $P_i = (u_0^i, \ldots, u_n^i)$ between $v_i$ to $v_{i+1}$ of weight $d_G(v_i, v_{i+1}) \leq k$. As no edge covers $e$, $e$ does not belong to any of these paths. The concatenation of this path $P_0 \circ P_1 \circ \cdots \circ P_{s-1}$ is a path in $G$ of at most $2k-1$ edges between the endpoints of $e$. It follows that $G$ contains a 2$k$-cycle, a contradiction.

For conclusion, as every edge in $E_G \setminus E_H$ is covered, and every edge in $E' = E_H \setminus E_G$ can cover at most $k$ edges, it follows that $|E_H \setminus E_G| \geq \frac{1}{k} \cdot |E_G \setminus E_H|$. In particular,

$$|E_H| = |E_H \cap E_G| + |E_H \setminus E_G| \geq |E_H \cap E_G| + \frac{1}{k} \cdot |E_G \setminus E_H| \geq \frac{1}{k} \cdot |E_G|.$$ 

To bound the weight, for each edge $e' = \{s, t\} \in E'$, let $A_{e'}$ be the set of edges in $E_G$ covered by $e'$. Note that $w_H(e') = d_G(s, t) = |A_{e'}|$. As all the edges in $E_G \setminus E_H$ are covered, we conclude

$$w_H(E_H) = w_H(E_H \cap E_G) + w_H(E_H \setminus E_G) = |E_H \cap E_G| + \sum_{e' \in E'} |A_{e'}| \geq |E_H \cap E_G| + |E_G \setminus E_H| = |E_G| = \Omega(n^{1+\frac{1}{k}}),$$

the lemma now follows.

**Theorem 5.** Assuming Erdős girth conjecture, the competitive ratio of any online (2$k$-1)-spanner algorithm for $n$-point metrics is $\Omega\left(\frac{1}{k} \cdot n^{\frac{1}{k}}\right)$, for both weight and edges.

In more details, there is an $n$-point metric space $(X, d_X)$ with a (2$k$-1)-spanner $H_{OPT} = (X, E_{OPT})$, and order over $X$ for which every (2$k$-1)-spanner produced by an online algorithm will have

$$\Omega\left(\frac{1}{k} \cdot n^{\frac{1}{k}}\right) \cdot |E_{OPT}| \text{ edges, and } \Omega\left(\frac{1}{k} \cdot n^{\frac{1}{k}}\right) \cdot w(H_{OPT}) \text{ weight.}$$

**Proof.** Consider the metric space $(X, d_X)$ from Lemma 7 with parameters $n-1$ and $k$. Let $X'$ be the metric space $X$ with an additional point $r$ at distance $\frac{k-1}{2}$ from all the points in $X$. Note that no pairwise distance is changed due to the introduction of $r$. The adversary provides the online algorithm the points in $X$ first (in some arbitrary order), and the point $r$ last. After the algorithm received all the points in $X'$, it has a 2$k$-1-spanner $H_{n-1}$. According to Lemma 7, $H_{n-1}$ has

$$\Omega\left(\frac{1}{k} \cdot (n-1)^{1+\frac{1}{k}}\right) = \Omega\left(\frac{1}{k} \cdot n^{1+\frac{1}{k}}\right) \text{ edges, and } \Omega(n^{1+\frac{1}{k}}) \text{ weight.}$$

Next the algorithm introduces $r$. Consider the spanner $S = (X', E_S)$ consisting of $n-1$ edges with $r$ as a center. Note that the maximum distance in $S$ is $2k-1$, and hence $S$ is a 2$k$-1 spanner as required. Note that $S$ contains $n-1$ edges of weight $\frac{2k-1}{2}$ each, and thus have total weight of $O(nk)$. We conclude

$$|E_{H_{n-1}}| \geq |E_{H_{n-1}}| = \Omega\left(\frac{1}{k} \cdot n^{1+\frac{1}{k}}\right) = \Omega\left(\frac{1}{k} \cdot n^{1+\frac{1}{k}}\right) \cdot |E_S|. $$

$$w(E_{H_{n-1}}) \geq w(E_{H_{n-1}}) = \Omega(n^{1+\frac{1}{k}}) = \Omega\left(\frac{1}{k} \cdot n^{1+\frac{1}{k}}\right) \cdot w(S).$$

\end{proof}

### 6 Ultrametrics

An ultrametric $(X, d)$ is a metric space satisfying a strong form of the triangle inequality, that is, for all $x, y, z \in X$, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. A related notion is a $k$-hierarchical well-separated tree ($k$-HST).
**Definition 2** ($\alpha$-HST). A metric $(X, d_X)$ is an $\alpha$-hierarchical well-separated tree ($\alpha$-HST) if there exists a bijection $\varphi$ from $X$ to leaves of a rooted tree $T$ in which:

- Each node $v \in T$ is associated with a label $\ell(v)$ such that $\ell(v) = 0$ if $v$ is a leaf and $\ell(v) \geq \alpha \ell(u)$ if $v$ is an internal node and $u$ is any child of $v$.

- $d_X(x, y) = \ell(lca(\varphi(x), \varphi(y)))$ where $lca(u, v)$ is the least common ancestor of any two given nodes $u, v$ in $T$.

It is well known that any ultrametric is a 1-HST, and any $k$-HST is an ultrametric [BLMN05].

Suppose that we are given an HST in the online model. Construct a spanner $H$ using the following algorithm: for every arriving vertex $v$, let $u$ be the first vertex in the order of arrival among all the nearest neighbors of $v$. We add the edge $\{u, v\}$ to the spanner $H$. Note that $H$ is a spanning tree at all times (we will later argue that it is actually an MST).

We show that for general ultrametrics, the online algorithm can maintain a spanner of lightness arbitrarily close to 1 (with constant stretch).

**Lemma 8.** If $U$ is an $\alpha$-HST, then the spanner $H$ has distortion $2 \cdot \frac{\alpha}{\alpha - 1}$.

**Proof.** Think of the representation of the HST as a tree with labeled internal nodes. For every internal node $\chi$, we call the first descendent in the order of arrival the center of $\chi$. Consider a vertex $v$ at the time of its arrival, let $\chi$ be an internal node which is an ancestor of $v$, and let $u$ be the center of $\chi$. We argue that $d_H(v, u) \leq t \cdot d_U(v, u)$ for $t = \frac{\alpha}{\alpha - 1}$. The proof is by induction. The induction step is immediate if the edge $\{u, v\}$ was added to $H$. Otherwise, let $\chi'$ be the highest internal node which is an ancestor of $v$ but has a center other than $u$. Let $x$ be the center of $\chi'$. At the time when $x$ arrives, it was the only descendent of $\chi'$. In particular, the closest neighbors of $x$ at this time is $u$ (as otherwise, there must be an internal vertex $\chi''$ between $\chi'$ and $\chi$ with center other than $u$). As $u$ is the center of $\chi$, it is the first arriving descendent of $\chi$. In other words, $u$ is the first vertex in the order of arrival among all the nearest neighbors of $u$. We conclude that $\{x, u\} \in H$. As $U$ is an $\alpha$-HST $\ell(\chi') \leq \frac{1}{\alpha} \ell(\chi)$. By the induction hypothesis, $d_H(v, x) \leq t \cdot d_U(v, x)$. We conclude

$$d_H(v, u) \leq d_H(v, x) + d_H(x, u) \leq t \cdot d_U(v, x) + d_U(x, u)$$

$$= t \cdot \ell(\chi') + \ell(\chi) \leq \left(\frac{t}{\alpha} + 1\right) \cdot \ell(\chi) = t \cdot \ell(\chi) = t \cdot d_U(v, u).$$

For two arbitrary vertices $u, v$, let $\chi = lca(u, v)$, and let $x$ be the center of $\chi$. By the definition of HST, $d_U(v, x), d_U(x, u) \leq d_U(v, u)$. Using the previous argument,

$$d_H(v, u) \leq d_H(v, x) + d_H(x, u) \leq t \cdot (d_U(v, x) + d_U(x, u)) = 2t \cdot d_U(v, u).$$

**Lemma 9.** The spanner $H$ is an MST of $U$.

**Proof.** Assume for contradiction otherwise. Then $w(MST(U)) < w(H)$. Let $T$ be an MST of $U$ containing the maximum number edges of $H$. Let $\{u, v\} = e \in H \setminus T$ be some edge. Assume w.l.o.g. that $u$ arrived before $v$, and let $\chi = lca(u, v)$. As the algorithm added edge $\{u, v\}$ to $H$, necessarily $u$ is the center of $\chi$. Further, there is a child node $\chi_v$ of $\chi$, where $v$ is a unique descendent of $\chi_v$. Therefore, there is a path $P = \chi_v \rightarrow \chi \rightarrow u \rightarrow \chi'$ in $H$. Let $x$ be the first arriving descendent of $\chi'$ and $x' \in U$ be the center of $\chi'$. Note that $d_U(x', x) = d_U(x', x')$. Assume $x'$ is not an ancestor of $\chi'$ and let $\chi''$ be any child of $\chi'$. Then $d_U(x', \chi'') > d_U(x', x')$, which contradicts the definition of $x'$. Therefore, $x'$ is the center of $\chi'$, implying $d_U(x', x) = d_U(x', x')$. Thus, $x'$ is an ancestor of $\chi'$, contradicting the assumption. Therefore, $H$ is an MST of $U$. 

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Proof. Let \( S_v \) be the set of all descendants of \( \chi_v \) in \( U \). Then \( T \) contains at least one edge from the vertices of \( S_v \) to a vertex outside of \( S_v \). Let \( e' \in T \) be such an edge that is on the unique \( w \)-path in \( T \). Then \( w(e') \geq \ell(\chi) = w(e) \), and \( T \cup \{e\} \setminus \{e'\} \) is a spanning tree of \( U \), of weight at most \( w(T) \). A contradiction to the maximality of \( T \). \( \square \)

**Theorem 6.** Given an ultrametric \( U \), for every \( \alpha \geq 1 \), an online algorithm can maintain a \( \frac{2\alpha^2}{\alpha-1} \)-spanner of weight \( \alpha \cdot w(\text{MST}) \). Alternatively, for every \( \varepsilon > 0 \), it can maintain a spanner of weight \( (1 + \varepsilon) \cdot w(\text{MST}) \) and stretch \( \frac{2(1+\varepsilon)^2}{\varepsilon} = O(\varepsilon^{-1}) \).

**Proof.** Let \( U_\alpha \) be the \( \alpha \)-HST for \( U \) where we round every distance up to the next integer power of \( \alpha \). That is, \( d_{U_\alpha}(u, v) = \alpha^{\left\lfloor \log_\alpha d_U(u, v) \right\rfloor} \). Note that \( d_{U_\alpha}(u, v) \leq d_{U_\alpha}(u, v) < \alpha \cdot d_U(u, v) \). In particular, the weight of the MST in \( U_\alpha \) is larger than the MST of \( U \) by at most a factor \( \alpha \). We run the online algorithm above on \( U_\alpha \) instead of \( U \). As a result, we get a spanner \( H_\alpha \) of \( U_\alpha \) with stretch \( 2 \cdot \frac{\alpha}{\alpha-1} \) and lightness 1 (w.r.t. \( U_\alpha \)). Let \( H \) be the same spanner with the original weights. Then for every pair of vertices \( u, v \)

\[
d_H(u, v) \leq d_{H_\alpha}(u, v) \leq \frac{2\alpha}{\alpha-1} \cdot d_{U_\alpha}(u, v) \leq \frac{2\alpha^2}{\alpha-1} \cdot d_U(u, v).
\]

The weight of \( H \) is bounded by \( w(H) \leq w(H_\alpha) = w(\text{MST}(U_\alpha)) \leq \alpha \cdot w(\text{MST}(U)) \). \( \square \)

**Remark 1.** The minimal possible stretch in the Theorem 6 above is 8, which is obtained for lightness \( \alpha = 2 \). This stretch is the best possible stretch obtained by a spanning tree. Indeed, consider the metric induced on the leaves of the full binary tree. One can observe that this is an ultrametric. Chan et al. [CXKR06] showed that for every metric induced on the leaves of the full binary tree, it holds that \( \kappa + 1 = O(\log_\alpha \varepsilon^{-1}) = O(\varepsilon^{-1} \log \varepsilon^{-1}) \) trees. For the lightness, let \( T \) be an MST of the ultrametric \( X \). Denote by \( T_i \) the MST of \( U_i \). For every edge \( e \in T_i \), it holds that

\[
\sum_{i=0}^{\kappa} w_{U_i}(e) \leq w_U(e) \sum_{i=0}^{\kappa} (1 + \varepsilon)^i = \frac{(1 + \varepsilon)^{\kappa+1} - 1}{\varepsilon} \cdot w_U(e) = O(\varepsilon^{-2}) \cdot w_U(e).
\]
We now can bound the weight of $H$ as follows:

$$w_U(H) \leq \sum_{i=0}^{K} w_U(H_i) \leq \sum_{i=0}^{K} w_U(T_i) = \sum_{i=0}^{K} w_U(T)$$

$$= \sum_{i=0}^{K} \sum_{e \in T_i} w_U(e) = \sum_{e \in T} O(\varepsilon^{-2}) w_U(e) = O(\varepsilon^{-2}) w_U(T).$$

It remains to analyze the stretch of $H$. For every pair of vertices $u, v \in U$, there are unique indices $i, j$ such that $(1 + \varepsilon)^{i-1} \cdot \varepsilon^{-j} < d_U(u, v) \leq (1 + \varepsilon)^{j} \cdot \varepsilon^{-j}$. Hence in $U_i$ it holds that $d_{U_i}(u, v) \leq (1 + \varepsilon) \cdot d_U(u, v)$. As $U_i$ is an $\varepsilon^{-1}$-HST, it holds that

$$d_H(u, v) \leq d_{U_i}(u, v) \leq \frac{2\varepsilon^{-1}}{\varepsilon^{-1} - 1} \cdot d_U(u, v) \leq \frac{2}{1 - \varepsilon} \cdot (1 + \varepsilon) \cdot d_U(u, v) \leq 2(1 + 3\varepsilon) \cdot d_U(u, v).$$

One can obtain the stretch factor $2 + \varepsilon$, stated in the theorem, by scaling $\varepsilon$ accordingly.

\[ \square \]

7 Conclusion

We studied online spanners for points in metric spaces. In the Euclidean $d$-space, we presented an online $(1 + \varepsilon)$-spanner algorithm with competitive ratio $O(\varepsilon^{1-d} \log n)$, improving the previous bound of $O_d(\varepsilon^{-(d+1)} \log n)$ from [BT21c]. In fact, the spanner maintained by the algorithm has $O_d(\varepsilon^{1-d} \log \varepsilon^{-1}) \cdot n$ edges, almost matching the (offline) optimal bound of $O_d(\varepsilon^{1-d}) \cdot n$. Moreover, in the plane, a tighter analysis of the same algorithm provides an almost quadratic improvement of the competitive ratio to $O(\varepsilon^{-3/2} \log \varepsilon^{-1} \log n)$, by comparing the online spanner with an instance-optimal spanner directly, circumventing the comparison to an MST (i.e., lightness). Note that, the logarithmic dependence on $n$ is unavoidable due to a $\Omega((\varepsilon^{-1}/\log \varepsilon^{-1}) \log n)$ lower bound in the real line [BT21c]. However, our lower bound $\Omega(\varepsilon^{-d})$ under $L_1$-norm in $\mathbb{R}^d$ shows a dependence on the dimension. This leads to the following question.

Question. Does the competitive ratio of an online $(1 + \varepsilon)$-spanning algorithm for $n$ points in $\mathbb{R}^d$ necessarily grow proportionally with $\varepsilon^{-f(d)} \cdot \log n$, where $\lim_{d \to \infty} f(d) = \infty$?

Interestingly, for $t \in [(1 + \varepsilon)\sqrt{2}, (1 - \varepsilon)2]$, we can show that every online $t$-spanner algorithm in $\mathbb{R}^d$ must have competitive ratio $2^{\Omega(\varepsilon^2 d)}$ (see Theorem 8 in Appendix A).

Next, we studied online spanners in general metrics. We showed that the ordered greedy algorithm maintains a spanner with $O(\varepsilon^{-1} \log \frac{k}{\varepsilon}) \cdot n^{1 + \frac{1}{k}}$ edges and $O(\varepsilon^{-1} n^{\frac{1}{2}} \log^2 n)$ lightness, with stretch factor $\varepsilon = (2k - 1)(1 + \varepsilon)$ for $k \geq 2$ and $\varepsilon \in (0, 1)$, for a sequence of $n$ points in a metric space. Moreover, we show that these bounds cannot be significantly improved, by introducing an instance that achieves an $\Omega(k^{\frac{1}{k}} \cdot n^{1/k})$ competitive ratio on both sparsity and lightness. Finally, we established the trade-off among stretch, number of edges and lightness for points in ultrametrics, showing that one can maintain a $(2 + \varepsilon)$-spanner for ultrametrics with $O(n \cdot \varepsilon^{-1} \log \varepsilon^{-1})$ edges and $O(\varepsilon^{-2})$ lightness.
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A High Dimensional Euclidean Lower Bound

**Theorem 8.** For \( t \in \left[ (1 + \varepsilon)\sqrt{2}, (1 - \varepsilon)^2 \right] \), the competitive ratio of any online \( t \)-spanner algorithm in \( \mathbb{R}^d \) under the Euclidean norm is \( 2^{\Omega(\varepsilon^2 d)} \).

**Proof sketch.** Let \( A \subseteq \{\pm 1\}^d \) be a set of \( 2^{\Omega(\varepsilon^2 d)} \) points such that every \( u, v \in A \) differ in \( (1 \pm \varepsilon)^2 \) coordinates (such a set can be constructed randomly using Chernoff). In particular, \( \|u - v\|_2 \) is in \( \sqrt{(1 \pm \varepsilon)^2 d} \). Every \( t \)-spanner for \( A \) must contain all \( \frac{|A|}{2} \) edges, this is as the weight of any two edges is at least \( 2\sqrt{(1 - \varepsilon)^2 d} > t \cdot \sqrt{(1 + \varepsilon)^2 d} \).

Next the adversary introduces the point \( \tilde{0} \) with all zeros, which is at distance \( \sqrt{d} \) from all other points. Let \( H \) be the star with \( \tilde{0} \) as a center. Then for every \( u, v \in A \), there is a path in \( H \) of weight \( 2\sqrt{d} \leq t \cdot \sqrt{(1 - \varepsilon)^2 d} \leq t \cdot \|v - u\|_2 \). The competitive ratio is \( \Omega\left(\frac{d^3}{|A|}\right) = 2^{\Omega(\varepsilon^2 d)} \). \( \square \)