A NOTE ON CLASSICAL AND P-ADIC FRÉCHET FUNCTIONAL EQUATIONS WITH RESTRICTIONS

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ABSTRACT. Given $X, Y$ two $\mathbb{Q}$-vector spaces, and $f : X \to Y$, we study under which conditions on the sets $B_k \subseteq X$, $k = 1, \ldots, s$, if $\Delta_{h_1 h_2 \cdots h_s} f(x) = 0$ for all $x \in X$ and $h_k \in B_k$, $k = 1, 2, \ldots, s$, then $\Delta_{h_1 h_2 \cdots h_s} f(x) = 0$ for all $(x, h_1, \ldots, h_s) \in X^{s+1}$.

1. Introduction

Let $X, Y$ be two $\mathbb{Q}$-vector spaces and let $f : X \to Y$. We say that $f$ satisfies Fréchet’s functional equation of order $s - 1$ if

\[ \Delta_{h_1 h_2 \cdots h_s} f(x) = 0 \quad (x, h_1, h_2, \ldots, h_s \in X), \]

where $\Delta_h f(x) = f(x + h) - f(x)$ and $\Delta_{h_1 h_2 \cdots h_s} f(x) = \Delta_{h_1} (\Delta_{h_2 \cdots h_s} f)(x)$, $s = 2, 3, \ldots$. In particular, after Fréchet’s seminal paper \cite{Fr}, the solutions of this equation are named “polynomials” by the Functional Equations community, since it is known that, under very mild regularity conditions on $f$, if $f : \mathbb{R} \to \mathbb{R}$ satisfies (1), then $f(x) = a_0 + a_1 x + \cdots + a_{s-1} x^{s-1}$ for all $x \in \mathbb{R}$ and certain constants $a_i \in \mathbb{R}$. For example, in order to have this property, it is enough for $f$ being locally bounded \cite{Fr, Almira}, but there are stronger results \cite{Gar, Ger, Ger2}. The general solutions of (1) are characterized as functions of the form $f(x) = A_0 + A_1(x) + \cdots + A_n(x)$, where $A_0$ is a constant and $A_k(x) = A^k(x, x, \ldots, x)$ for a certain $k$-additive symmetric function $A^k : X \to Y$ (we say that $A_k$ is a diagonalization of $A^k$). Furthermore, it is known that $f : X \to Y$ satisfies (1) if and only if it satisfies the (apparently less restrictive) equation

\[ \Delta^s_h f(x) := \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} f(x + kh) = 0 \quad (x, h \in X). \]

A proof of this fact follows directly from \cite[Theorem 9.2, p. 66]{Sch}, where it is proved that the operators $\Delta_{h_1 h_2 \cdots h_s}$ satisfy the equation

\[ \Delta_{h_1 h_2 \cdots h_s} f(x) = \sum_{\varepsilon_1, \ldots, \varepsilon_s = 0}^{1} (-1)^{\varepsilon_1 + \cdots + \varepsilon_s} \Delta^s_{\alpha_{\varepsilon_1, \ldots, \varepsilon_s}(h_1, \ldots, h_s)} f(x + \beta_{\varepsilon_1, \ldots, \varepsilon_s}(h_1, \ldots, h_s)), \]

where $\alpha_{\varepsilon_1, \ldots, \varepsilon_s}(h_1, \ldots, h_s) = (-1) \sum_{r=1}^{s} \varepsilon_r h_r$ and $\beta_{\varepsilon_1, \ldots, \varepsilon_s}(h_1, \ldots, h_s) = \sum_{r=1}^{s} \varepsilon_r h_r$. (Note that $\Delta^s_h f(x)$ results from $\Delta_{h_1 h_2 \cdots h_s} f(x)$ by imposing $h_1 = \cdots = h_s = h$.)

Given $D \subseteq X$, the function $f : D \to Y$ is named a “polynomial on $D$” if $f$ satisfies (2) for a certain $s \in \mathbb{N}$ and all $x, h \in X$ such that $\{x, x + h, \ldots, x + sh\} \subseteq D$. A natural problem, that has been solved by R. Ger \cite{Ger}, is to study the conditions under which a

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polynomial $f$ on $D$ can be extended (and how to make this) to a polynomial on $X$. In this paper we study the analogous problem when we try to extend not the function $f$, which is assumed to be defined on all the space $X$, but the domain of validity of the equation (1) for the steps $h_1, h_2, \cdots, h_s$. We prove our results for the special cases $X = \mathbb{R}$ and $X = \mathbb{Q}_p$, where $\mathbb{Q}_p$ denotes the field of $p$-adic numbers (which is, of course, a very special $\mathbb{Q}$-vector space). We also consider the case of $X$ being a topological vector space in a more general setting.

2. The case $X = \mathbb{R}$

In this section we assume that $f : \mathbb{R} \to Y$ for a certain $\mathbb{Q}$-vector space $Y$.

**Lemma 2.1.** Let us assume that $I = (a, b)$ is a nonempty open interval of the real line. If $\Delta_h f(x) = 0$ for all $x \in \mathbb{R}$ and $h \in I$ then $\Delta_h f(x) = 0$ for all $x, h \in \mathbb{R}$.

**Proof.** We assume, with no loss of generality, that $0 \leq a < b$. Obviously, $\Delta_h f(x) = 0$ for all $x, h \in \mathbb{R}$ if and only if $f$ is a constant function. As a first step, we prove that $f$ is constant on an interval of the form $(\alpha, \infty)$ for a certain real number $\alpha$.

Given $x_0 \in \mathbb{R}$, we know that $\Delta_h f(x_0) = f(x_0 + h) - f(x_0) = 0$ for all $h \in (a, b)$. Hence $f_{x_0+I} = f(x_0)$. What is more, if $k \in \mathbb{N}$ and $h \in (a, b)$, then

$$0 = \Delta_h f(x_0 + (k - 1)h) = f(x_0 + kh) - f(x_0 + (k - 1)h),$$

$$0 = \Delta_h f(x_0 + (k - 2)h) = f(x_0 + (k - 1)h) - f(x_0 + (k - 2)h),$$

$$\vdots$$

$$0 = \Delta_h f(x_0 + (k - 1)h) = f(x_0 + h) - f(x_0),$$

so that $f(x_0) = f(x_0 + kh)$. This implies that $f(x)$ is constant on the set $M = x_0 + \bigcup_{k=1}^{\infty} (ka, kb)$. Now, the intervals $(ka, kb)$ and $((k+1)a, (k+1)b)$ overlap for all $k \geq k_0$ for a certain $k_0 \in \mathbb{N}$, since $0 \leq a/b < 1$ and $k/(k+1)$ converges monotonically to 1. Hence, there exists a real number $\alpha$ such that $f_{(\alpha, \infty)} = f(x_0)$.

Let us prove that $f$ is constant on all the real line. We know that $f(x) = C$ for a certain constant $C$ and all $x \in (\alpha, \infty)$. On the other hand, the same argument we used for the first part of the proof shows that, for all $x_1 \leq \alpha$, if $h \in (a, b)$ and $k \in \mathbb{N}$ is big enough, then $x_1 + kh > \alpha$ and $f(x_1) = f(x_1 + kh) = C$.

**Theorem 2.2.** Let us assume that $I_k = (a_k, b_k)$ are nonempty open intervals of the real line, $k = 1, 2, \cdots, s$. If $\Delta_{h_1h_2\cdots h_s} f(x) = 0$ for all $x \in \mathbb{R}$ and $h_k \in I_k$, $k = 1, 2, \cdots, s$, then $\Delta_{h_1h_2\cdots h_s} f(x) = 0$ for all $(x, h_1, \cdots, h_s) \in \mathbb{R}^{s+1}$.

**Proof.** By hypothesis, given $(h_2, h_3, \cdots, h_s) \in I_2 \times I_3 \times \cdots \times I_s$, the function $F(x) = \Delta_{h_2h_3\cdots h_s} f(x)$ satisfies the hypothesis of Lemma 2.1 with $a = a_1, b = b_1$. It follows that $\Delta_{h_1h_2\cdots h_s} f(x) = \Delta_{h_1} (\Delta_{h_2h_3\cdots h_s} f(x)) = 0$ for all $x \in \mathbb{R}$ and $(h_1, h_2, \cdots, h_s) \in \mathbb{R} \times I_2 \times \cdots \times I_s$. On the other hand, it is well known [2, Corollary 9.1, p. 66] that, for any permutation $\sigma$ of the indices $\{1, 2, \cdots, s\}$, we have that $\Delta_{h_1h_2\cdots h_s} = \Delta_{h_{\sigma(1)}h_{\sigma(2)}\cdots h_{\sigma(s)}}$. Thus, if we take $\sigma_2 = (12)$ (i.e., the transposition of the indices $\{1, 2\}$, see [7, p. 49] for the definition) and we apply the argument above to $\Delta_{h_{\sigma_2(1)}h_{\sigma_2(2)}\cdots h_{\sigma_2(s)}}$, we will get $\Delta_{h_1h_2\cdots h_s} f(x) = 0$ for all $x \in \mathbb{R}$ and $(h_1, h_2, \cdots, h_s) \in \mathbb{R} \times \mathbb{R} \times I_3 \times \cdots \times I_s$. The proof follows by repetition of the argument, taking into account the transpositions $\sigma_k = (1k), k = 3, 4, \cdots, s$. □
Theorem 2.3. Let $I = (−δ, 0)$ for a certain $δ > 0$. If $\Delta_k^n f(x) = 0$ for all $x \in \mathbb{R}$ and $h \in I$, then $\Delta_{h, h, \ldots, h} f(x) = 0$ for all $(x, h, \ldots, h) \in \mathbb{R}^{s+1}$. An analogous result holds for $I = (0, δ)$.

Proof. Let us assume that $−\delta/s ≤ h_k ≤ 0$ for $k = 1, 2, \ldots, s$. Take $(\epsilon_1, \ldots, \epsilon_s) \in \{0, 1\}^s$ and set $\alpha(\epsilon_1, \ldots, \epsilon_s)(h_1, \ldots, h_s) = (−1)^s \sum_r \epsilon_r h_r$. Then

$$0 ≤ \alpha(\epsilon_1, \ldots, \epsilon_s)(h_1, \ldots, h_s) = (−1)^s \sum_r \epsilon_r h_r ≤ \frac{1}{s} \left( \sum_r \epsilon_r \right) \delta ≤ \delta.$$ 

and, taking into account the equation (3) above, it follows that we can use Theorem 2.2 with $I_k = (−\delta/s, 0)$ for $k = 1, \ldots, s$.

The last claim of the theorem follows from the relation that exists between the operators $\Delta^n_{−h}$ and $\Delta^n_{h}$:

$$\Delta^n_{−h} f(x) = (−1)^n \Delta^n_{h} f(x − sh).$$ \hfill \blacksquare

Remark 1. Take $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = x^2$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $\Delta_{h, h, h_3} f(x) = 0$ for all $x \in \mathbb{R}$ and all $(h_1, h_2, h_3) \in \mathbb{Q}^3$. On the other hand, $f$ can not be a solution of Fréchet’s equation $\Delta^n_{h} f(x) = 0$ for all $x \in \mathbb{R}$ and all $h \in \mathbb{R}$ for any $s \in \mathbb{N}$, since $f$ is not a polynomial function (in the ordinary sense) and $f$ is locally bounded. This shows that, in order to extend the set of validity of the parameters $h_i$ (as in Theorems 2.2, 2.3 above), it is quite natural to assume that the equation holds true for $h_i$ in a certain open set.

Remark 2. Let $f : X → Y$, where $X, Y$ are $\mathbb{Q}$-vector spaces and $X$ admits a topology $\tau_X$ with the property that all neighborhoods of the origin are “naturally absorbent” sets (we say that $B$ is naturally absorbent if for each $x \in X$ there exists $k \in \mathbb{N}$ such that $x \in kB = \{kz : z \in B\}$). If we assume that $\Delta_{h} f(x) = 0$ for all $x \in X$ and $h \in B$ for a certain neighborhood $B$ of the origin, then the arguments of the first part of the proof of Lemma 2.1 lead to the conclusion that $\Delta_{h} f(x) = 0$ for all $x, h \in X$, since $X = \bigcup_{k \geq 1} kB$. It follows that Theorems 2.2, 2.3 admit the following natural generalizations:

Theorem 2.4. Let $f : X → Y$, where $X, Y$ are $\mathbb{Q}$-vector spaces and $X$ admits a topology $\tau_X$ with the property that all neighborhoods of the origin are naturally absorbent sets. Let $B$ be a neighborhood of the origin. If $\Delta_{h, h, \ldots, h} f(x) = 0$ for all $x \in X$ and $h_k \in B$, $k = 1, 2, \ldots, s$, then $\Delta_{h, h, \ldots, h} f(x) = 0$ for all $(x, h_1, \ldots, h_s) \in X^{s+1}$.

Theorem 2.5. Let $f : X → Y$, where $X, Y$ are $\mathbb{Q}$-vector spaces and $X$ is a real normed vector space. Let $B = B(0, \varepsilon) = \{x : \|x\|_X < \varepsilon\}$ be an open ball centered in the origin of $X$. If $\Delta_{h} f(x) = 0$ for all $x \in X$ and $h \in B$, then $\Delta_{h, h_2, \ldots, h_s} f(x) = 0$ for all $(x, h_1, \ldots, h_s) \in X^{s+1}$.

3. The $p$–adic case

Let $\mathbb{Q}_p$ denote the field of $p$-adic numbers, which are expressions of the form

$$x = a_m p^m + a_{m+1} p^{m+1} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots = \sum_{n \geq m} a_n p^n,$$

where $m \in \mathbb{Z}$, $1 ≤ a_n ≤ p − 1$ and $0 ≤ a_k ≤ p − 1$ for all $k > m$ (see, for example, [6], [10] for the definition and basic properties of these numbers and their field extensions.
Given $x$ as in (4), its $p$-adic absolute value is given by $|x| = p^{-m}$. The set of $p$-adic numbers with $|x|_p ≤ 1$ is denoted by $\mathbb{Z}_p$. Obviously, $|x|_p ≤ p^n$ if and only if $x ∈ p^{-n}\mathbb{Z}_p := \{p^{-n}h : h ∈ \mathbb{Z}_p\}$. An important property of the absolute value $|\cdot|_p$ we will use is the following one:

$$ (|x|_p > |y|_p) \Rightarrow (|x + y|_p = |x|_p) $$

In this section we assume that $f : \mathbb{Q}_p → Y$, where $Y$ is a $\mathbb{Q}$-vector space. Previous to any work about the extension of Fréchet functional equation in this context, it would be appropriate to say something about the equation in the context of $p$-adic analysis. In particular, we show that Fréchet’s original result has a natural extension to this new setting:

**Theorem 3.1** (p-adic version of Fréchet’s theorem). Let $(\mathbb{K}, | \cdot |_\mathbb{K})$ be a valued field such that $\mathbb{Q}_p ⊆ \mathbb{K}$ and the inclusion $\mathbb{Q}_p → \mathbb{K}$ is continuous. Let us assume that $f : \mathbb{Q}_p → \mathbb{K}$ is continuous. Then $f$ satisfies $Δ^{n+1}_f(x) = 0$ for all $x, h ∈ \mathbb{Q}_p$ if and only if $f(x) = a_0 + \cdots + a_nx^n$ for certain constants $a_k ∈ \mathbb{K}$.

**Proof.** Assume $Δ^{n+1}_f(x) = 0$ for all $x, h ∈ \mathbb{Q}_p$. Let $x_0, h_0 ∈ \mathbb{Q}_p$ and let $p_0(t) ∈ \mathbb{K}[t]$ be the polynomial of degree $≤ n$ such that $f(x_0 + kh_0) = p_0(x_0 + kh_0)$ for all $k ∈ \{0, 1, \cdots, n\}$ (this polynomial exists and it is unique, thanks to Lagrange’s interpolation formula). Then

$$ 0 = Δ^{n+1}_f(x_0) = \sum_{k=0}^{n} \binom{n+1}{k} (-1)^{n+1-k}f(x_0 + kh_0) + f(x_0 + (n+1)h_0) $$

$$ = p_0(x_0 + (n+1)h_0) + f(x_0 + (n+1)h_0) $$

since $0 = Δ^{n+1}_f(x_0) = \sum_{k=0}^{n} \binom{n+1}{k} (-1)^{n+1-k}p_0(x_0 + kh_0)$. This means that $f(x_0 + (n+1)h_0) = p_0(x_0 + (n+1)h_0)$. In particular, $p_0 = q$, where $q$ denotes the polynomial of degree $≤ n$ which interpolates $f$ at the nodes $\{x_0 + kh_0\}_{k=1}^{n+1}$. This argument can be repeated (forward and backward) to prove that $p_0$ interpolates $f$ at all the nodes $x_0 + h_0\mathbb{Z}$.

Let $m ∈ \mathbb{Z}$ and let us use the same kind of argument, taking $h_0^* = h_0/p^m$ instead of $h_0$. Then we get a polynomial $p_0^*$ of degree $≤ n$ such that $p_0^*$ interpolates $f$ at the nodes $x_0 + h_0^*\mathbb{Z}$. Now, $p_0 = p_0^*$ since the set

$$ \frac{h_0}{p^m}\mathbb{Z} ∩ h_0\mathbb{Z} = \begin{cases} h_0\mathbb{Z} & \text{if } m ≥ 0 \\ h_0^*\mathbb{Z} & \text{if } m < 0 \end{cases} $$

is infinite. Thus, we have proved that $p_0$ interpolates $f$ at all the points of

$$ Γ_{x_0, h_0} := x_0 + \bigcup_{m=-∞}^{∞} \frac{h_0}{p^m}\mathbb{Z} $$

Now, $Γ_{0,1}$ is a dense subset of $\mathbb{Q}_p$, so that the continuity of $f$ implies that $f = p_0$ everywhere. □

**Corollary 3.2** (Local p-adic version of Fréchet’s theorem). Let $(\mathbb{K}, | \cdot |_\mathbb{K})$ be a valued field such that $\mathbb{Q}_p ⊆ \mathbb{K}$ and the inclusion $\mathbb{Q}_p → \mathbb{K}$ is continuous. Let us assume that
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3.1 $f : \mathbb{Q}_p \to \mathbb{K}$ is continuous and let $N \in \mathbb{N}$. If $f$ satisfies $\Delta_{p^N}^{n+1} f(x) = 0$ for all $x \in \mathbb{Q}_p$, then for each $a \in \mathbb{Q}_p$ there exists constants $a_k \in \mathbb{K}$ such that $f(x) = a_0 + \cdots + a_n x^n$ for all $x \in a + p^N \mathbb{Z}_p$.

**Proof.** Just repeat the argument of the first part of the proof of Theorem 3.1 with $x_0 = a$ and $h_0 = p^N$. This will show that there exists a polynomial of degree $\leq n$, $p_0 \in \mathbb{K}[t]$ such that $(p_0)(u+p^N Z) = f|_{u+p^N Z}$. Now, $f$ is continuous and $a + p^N \mathbb{Z}_p$ is a dense subset of $a + p^N \mathbb{Z}_p$. □

It is well known that there are non constant locally constant functions $f : \mathbb{Q}_p \to Y$. In particular, the characteristic function associated to $\mathbb{Z}_p$, given by $\phi(x) = 1$ for $x \in \mathbb{Z}_p$ and $\phi(x) = 0$ otherwise, is continuous, non-constant, and $\Delta_k \phi(x) = 0$ for all $x \in \mathbb{Q}_p$ and $h$ such that $|h|_p \leq 1$. Furthermore, $\Delta_p \phi(0) = \phi(p^{-1}) - \phi(0) = -1 \neq 0$. This is in contrast with the results we got for $X = \mathbb{R}$.

**Lemma 3.3.** Let $f : \mathbb{Q}_p \to Y$ and let $N \in \mathbb{Z}$, $a \in \mathbb{Q}_p$ be such that $\Delta_a f(x) = 0$ for all $x \in \mathbb{Q}_p$ and all $h \in \mathbb{Q}_p \setminus (a + p^{-N} \mathbb{Z}_p)$ then $\Delta_h f(x) = 0$ for all $x, h \in \mathbb{Q}_p$.

**Proof.** We divide the proof in two cases:

**Case 1:** $a = 0$. Given $x_0 \in \mathbb{Q}_p$, we know that $\Delta_{h} f(x_0) = f(x_0 + h) - f(x_0) = 0$ for all $h \in \mathbb{Q}_p \setminus p^{-N} \mathbb{Z}_p$. Let us take $h \in p^{-N} \mathbb{Z}_p$. Then there exists $m \in \mathbb{N}$ such that $\frac{h}{p^m} \notin p^{-N} \mathbb{Z}_p$ and

$$0 = \Delta_{\frac{h}{p^m}} f(x_0) = f(x_0 + \frac{(p^m - 1)h}{p^m}) - f(x_0 + \frac{(p^m - 1)h}{p^m})$$

$$0 = \Delta_{\frac{h}{p^m}} f(x_0) = f(x_0 + \frac{(p^m - 2)h}{p^m}) - f(x_0 + \frac{(p^m - 2)h}{p^m})$$

$$\vdots$$

$$0 = \Delta_{\frac{h}{p^m}} f(x_0) = f(x_0 + \frac{h}{p^m}) - f(x_0)$$

so that $f(x_0) = f(x_0 + h)$.

**Case 2:** $a \neq 0$. Let $k_0 \in \mathbb{Z}$ be such that $|a|_p = p^{k_0}$ and let $h \in \mathbb{Q}_p$ be such that $|h|_p = p^n$. If $m \neq k_0$ then $|h - a|_p = \max\{p^{k_0}, p^n\}$, so that the imposition of $m \geq M = \max\{N, k_0\} + 1$ implies that $|h - a|_p = p^n > p^N$. Hence $\mathbb{Q}_p \setminus p^{-M} \mathbb{Z}_p \subseteq \mathbb{Q}_p \setminus (a + p^{-N} \mathbb{Z}_p)$ and we are again in Case 1. □

**Remark 3.** Another proof of Case 1 above reads as follows: Take $x \in \mathbb{Q}_p$ and $h \in p^{-N} \mathbb{Z}_p$. We want to show that $\Delta_h f(x) = 0$ or, in other words, that $\Delta_h f(x + h) = f(x)$. Let $u \in \mathbb{Q}_p \setminus p^{-N} \mathbb{Z}_p$. Then $f(x + h + u) = f(x + h)$ because $\Delta_u f(x + h) = 0$. On the other hand, $u + h \in \mathbb{Q}_p \setminus p^{-N} \mathbb{Z}_p$, so that $f(x) = f(x + u + h)$ because $\Delta_{u+h} f(x) = 0$. This ends the proof.

**Theorem 3.4.** Let $f : \mathbb{Q}_p \to Y$ and let $(N_1, \cdots, N_s) \in \mathbb{Z}^s$, $(a_1, \cdots, a_s) \in \mathbb{Q}_p^s$ be such that $\Delta_{h_1 h_2 \cdots h_s} f(x) = 0$ for all $x \in \mathbb{Q}_p$ and all $(h_1, \cdots, h_s) \in (\mathbb{Q}_p \setminus (a_1 + p^{-N_1} \mathbb{Z}_p)) \times \cdots \times (\mathbb{Q}_p \setminus (a_s + p^{-N_s} \mathbb{Z}_p))$. Then $\Delta_{h_1 h_2 \cdots h_s} f(x) = 0$ for all $(x, h_1, \cdots, h_s) \in \mathbb{Q}_p^{s+1}$.

**Proof.** It is enough to apply the same arguments we used for the proof of Theorem 2.2, just replacing Lemma 2.1 by Lemma 3.3. □
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