CLASS OF OPERATORS WITH SUPERIORITYLY CLOSED NUMERICAL RANGES AND GENERALIZED NUMERICAL RANGES

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Abstract. The aim of this paper is to propose a class of operators acting on a complex Hilbert space. This class will contain, among others, non-zero compact operators. We will give a characterization of this class in term of generalized numerical ranges. We will deduce that if \( A \) is a compact operator, then \( w(A) = |\lambda| \) with \( \lambda \in W(A) \), where \( W(A) \) and \( w(A) \) are the numerical range and the numerical radius of \( A \) respectively. We will give some new necessary conditions for an operator to be compact. We will also shed some light on the generalized numerical ranges of the elementary operators \( \delta_{2,A,B} \) and \( M_{2,A,B} \).

1. Introduction

Let \( \mathcal{A} \) be a complex unital Banach algebra with unit \( e \) and let \( \mathcal{A}' \) be its dual space. Define the state space of \( \mathcal{A} \) by
\[
S(\mathcal{A}) = \{ f \in \mathcal{A}' : f(e) = \|f\| = 1 \}.
\]
For \( a \in \mathcal{A} \), the algebraic numerical range of \( a \) is given by
\[
V(a) = \{ f(a) : f \in S(\mathcal{A}) \}.
\]

Let \( \mathcal{B}(\mathcal{H}) \) be the Banach algebra of all bounded linear operators acting on an infinite dimensional complex Hilbert space \( \mathcal{H} \). For \( A \in \mathcal{B}(\mathcal{H}) \), the usual numerical range of \( A \) is defined by
\[
W(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}.
\]

It is well known that \( V(a) \) (\( a \in \mathcal{A} \)) is convex, compact and contains the convex hull of the spectrum of \( a \); this result follows at once from the corresponding properties of the set \( S(\mathcal{A}) \). For more details, see [17, Theorem 1]. If the last inclusion becomes an equality, we say that \( a \) is convexoid.

One of the most fundamental properties of \( W(A) \) (\( A \in \mathcal{B}(\mathcal{H}) \)) is its convexity, stated by the famous Toeplitz-Hausdorff Theorem [15, 18]. Other important properties of \( W(A) \) is that its closure \( \overline{W(A)} = V(A) \), see [17, Theorem 6].

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Also, $W(A)$ is a connected set and it is compact in the finite dimensional case.

For $A \in \mathcal{B}(\mathcal{H})$, let $\sigma(A)$, $\sigma_p(A)$, $r(A)$ and $w(A)$ denote the spectrum, the point spectrum (the set of eigenvalues), the spectral radius and the numerical radius of $A$ respectively and defined as follows

$\sigma(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible} \}$,

$\sigma_p(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not injective} \}$,

$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ and $w(A) = \sup\{|\lambda| : \lambda \in W(A)\}$.

For basic facts about these concepts, we refer to [3, 10] and their references. Then the operator $A$ is called convexoid, as cited above, if $\text{co}\{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\} = W(A)$ (co denotes the convex hull), and we say that $A$ is normaloid if $\|A\| = w(A)$ (equivalent condition $r(A) = \|A\|$, see [11]). Recall that an operator $N \in \mathcal{B}(\mathcal{H})$ is called normal if $N^*N = NN^*$, where $N^*$ denotes the adjoint of $N$. Recall also that an operator $S \in \mathcal{B}(\mathcal{H})$ is called subnormal if there is a normal operator $N$ on $K \supset \mathcal{H}$ such that $\mathcal{H}$ is invariant for $N$ and $N|_{\mathcal{H}} = S$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called hyponormal operator if $A^*A - AA^* \geq 0$ (i.e., $(A^*A - AA^*)x, x \geq 0$ for all $x \in \mathcal{H}$) and it is known that any subnormal operator is an hyponormal operator.

For $A, B \in \mathcal{B}(\mathcal{H})$, the generalized derivation $\delta_{A,B}$ and the bimultiplication $\mathcal{M}_{A,B}$ associated with $(A, B)$ are defined on $\mathcal{B}(\mathcal{H})$ by

$\delta_{A,B}(X) = AX - XB$ and $\mathcal{M}_{A,B}(X) = AXB$,

respectively.

The properties of $\delta_{A,B}$ have been much studied, see for instance [11] [8] [13] [14] [15] [16]. For example, Lumer and Rosenblum [14] proved the identity

$\sigma(\delta_{A,B}) = \sigma(A) - \sigma(B)$, \hspace{1cm} (1.1)

and Kyle [13] proved that

$V(\delta_{A,B}) = V(A) - V(B)$, \hspace{1cm} (1.2)

Let $\mathcal{C}_2(\mathcal{H})$ denote the class of Hilbert-Schmidt operators on $\mathcal{H}$. Recall that $\mathcal{C}_2(\mathcal{H}) = \{ X \in \mathcal{B}(\mathcal{H}) : \|X\|_2 < \infty \}$, where $\|X\|_2^2 = tr(X^*X)$, $(X \in \mathcal{C}_2(\mathcal{H}))$, and $tr$ stands for the usual trace functional. Recall also that $\mathcal{C}_2(\mathcal{H})$ is a Hilbert space with respect to the inner product associated with the norm $\|\cdot\|_2$.

For $A, B \in \mathcal{B}(\mathcal{H})$, define on $\mathcal{C}_2(\mathcal{H})$ the following elementary operators

(i) the generalized derivation $\delta_{2,A,B}$ by

$\delta_{2,A,B}(X) = AX - BX \ (X \in \mathcal{C}_2(\mathcal{H}))$;

(ii) the bimultiplication $\mathcal{M}_{2,A,B}$ by

$\mathcal{M}_{2,A,B}(X) = AXB \ (X \in \mathcal{C}_2(\mathcal{H}))$. 
The numerical range of the generalized derivation $\delta_{2,A,B}$ was studied by several authors, including Seddik [15] and Chraibi [6]. In [15, Corollary 4.2], it is proved that
\[ V(\delta_{2,A,B}) = V(A) - V(B), \quad (1.3) \]
and in [6, Theorem 13], it is proved that
\[ W(\delta_{2,A,B}) = W(A) - W(B). \quad (1.4) \]
Note that from the identities (1.3) and (1.4), we obtain
\[ V(\delta_{2,A,B}) = V(\delta_{A,B}). \quad (1.5) \]

The determination of the numerical range of the bimultiplication $M_{2,A,B}$ was proposed in 1991 by Fialkow [9]. Several mathematicians worked on this question including Gustafson [10], Seddik [15], Chraibi [6] and Boumazgour-Nabwey [2]. In [15], Seddik gives an example where the identity (1.5) doesn’t hold for $M_{2,A,B}$ and $M_{A,B}$. In the study of this question, three important answers deserve to be cited.

(1) The first answer is due to Gustafson-Rao [10, Theorem 5.4-3], who proved that in the finite dimensional case and $A$ is normal, we have
\[ W(A \otimes B) = \text{co}(W(A)W(B)). \]
Recall that $M_{2,A,B}$ and $A \otimes B$ are unitary equivalent and have the same numerical range; see [5].

(2) The second answer is due to Chraibi [7, Theorem 11], who proved that
\[ W(M_{2,A,B}) \subseteq W(A)W(B) + S(A)S(B), \quad (1.6) \]
where $S(A) = \{\langle Ax, y \rangle : x \in \mathcal{H}, y \in \mathcal{H}, \|x\| = \|y\| = 1 \text{ and } \langle x, y \rangle = 0\}$ which is exactly the disk centered at the origin and of radius $d(A) = \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|$, see [6, Lemma 7].

(3) The third answer is also due to Chraibi [7, Theorem 10], who proved that if either $A$ or $B$ is a subnormal operator, then
\[ V(M_{2,A,B}) = \text{co}(W(A)W(B)), \quad (1.7) \]
and if either $A$ or $B$ is a diagonal operator, then
\[ W(M_{2,A,B}) = \text{co}(W(A)W(B)). \quad (1.8) \]
Recently, Boumazgour-Nabwey [2, Corollary 3.3] generalized the identity (1.7) and showed that if either $A$ or $B$ is an hyponormal operator, then
\[ V(M_{2,A,B}) = \text{co}(W(A)W(B)). \quad (1.9) \]
However, the determination of $W(M_{2,A,B})$ in the general case is still open.
We end this historical by considering the following sets, proposed by Halmos [11], called generalized algebraic numerical range and generalized numerical range of $A$, defined by

$$V_B(A) = \{ f(A) : f \in (\mathcal{B}(\mathcal{H}))' \}, \quad f(I) = \|f\| \leq 1,$$

where $I$ is the identity operator, and

$$W_B(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \|x\| \leq 1 \},$$

respectively. Note that we always have $V(A) \subseteq V_B(A)$, $W(A) \subseteq W_B(A)$, $0 \in W_B(A)$ and $W_B(A) \subseteq V_B(A)$. We will need these sets for the sequel.

This paper is organized as follows. The second section is devoted to generalized numerical ranges. More precisely, we show that $V_B(A)$ and $W_B(A)$ are convex. We give a necessary and sufficient condition for each of the equalities $V_B(A) = V(A)$ and $W_B(A) = W(A)$. We also prove that $V_B(A)$ is compact and $W_B(A) = W(A)$. We also prove that $\text{co}(\sigma(A) \cup \{0\}) \subseteq W_B(A)$ and this inclusion becomes an equality whenever $A$ is convexoid.

In the third section, we define the class of operators with superiorly closed numerical ranges. We give a characterization of this class in terms of generalized numerical ranges. We prove that if $A$ is in this class, then $w(A) = |\lambda|$ with $\lambda \in W(A)$. We show that this class contains strictly the class of non zero compact operators, then we deduce the previous property for compact operators. We conclude this section with some new necessary conditions for an operator to be compact.

In the last section, we study the generalized numerical ranges of the elementary operators $\delta_{2,A,B}$ and $\mathcal{M}_{2,A,B}$. We give a necessary and sufficient condition to have $V_B(\delta_{2,A,B}) = W(A) - W(B)$. We prove that the inclusion (1.6) holds when we replace the numerical range by the generalized numerical range, we even show that

$$W_B(\mathcal{M}_{2,A,B}) \subseteq W(A)W_B(B) + S(A)S(B) = W_B(A)W_B(B) + S(A)S(B).$$

We also prove, under the same assumptions as in [2, Corollary 3.3] and [7, Theorem 10], that

$$V_B(\mathcal{M}_{2,A,B}) = \text{co}(W(A)W_B(B)),$$

and

$$W_B(\mathcal{M}_{2,A,B}) = \text{co}(W(A)W_B(B)),$$

respectively. We give a sufficient but not necessary condition to have $V_B(\mathcal{M}_{2,A,B}) = \text{co}(W(A)W_B(B))$.

Throughout this paper, $\mathcal{A}$ will denote a complex unital Banach algebra with unit $e$, $\mathcal{A}'$ will denote its dual space and $\mathcal{B}(\mathcal{H})$ will denote the Banach algebra of all bounded linear operators acting on an infinite dimensional complex Hilbert space $\mathcal{H}$. 

2. Generalized numerical ranges

**Definition 2.1.** Let $A \in \mathcal{B}(\mathcal{H})$. The generalized algebraic numerical range and the generalized numerical range of $A$ are defined by

$$\mathcal{V}_\mathcal{B}(A) = \{ f(A) : f \in \langle \mathcal{B}(\mathcal{H}) \rangle^+ \}, \quad f(I) = \| f \| \leq 1,$$

and

$$\mathcal{W}_\mathcal{B}(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \| x \| \leq 1 \},$$

respectively, where $I$ is the identity operator.

**Remark 2.2.** Let $A \in \mathcal{B}(\mathcal{H})$. Then

$$\mathcal{V}_\mathcal{B}(A) = [0, 1] V(A) \quad \text{and} \quad \mathcal{W}_\mathcal{B}(A) = [0, 1] W(A).$$

Therefore, to study the properties of $\mathcal{V}_\mathcal{B}(A)$ and $\mathcal{W}_\mathcal{B}(A)$, we study those of the product $[0, 1] L$ where $L$ is a subset of the complex plane $\mathbb{C}$. The following two lemmas are designed for this purpose and will be used throughout the sections of this paper.

**Lemma 2.3.** Let $L, S$ are two subsets of $\mathbb{C}$. Then, $\text{co}(\text{co}(L) \cup \text{co}(S)) = \text{co}(L \cup S)$.

*Proof.* The inclusion $\text{co}(L \cup S) \subset \text{co}(\text{co}(L) \cup \text{co}(S))$ is obvious. For the reverse inclusion, we have $\text{co}(L) \subset \text{co}(L \cup S)$ and $\text{co}(S) \subset \text{co}(L \cup S)$, then $\text{co}(L \cup S)$ is convex, it follows that $\text{co}(\text{co}(L) \cup \text{co}(S)) \subset \text{co}(L \cup S)$. \hfill $\square$

**Lemma 2.4.** Let $L$ be a subset of $\mathbb{C}$ and $a, b$ are two real numbers such that $b > a \geq 0$. Then,

$$[a, b] \text{co}(L) = \text{co}(aL \cup bL).$$

In particular, $[0, 1] \text{co}(L) = \text{co}(L \cup \{0\})$. Furthermore, if $L$ is convex and $0 \in L$, then $[0, 1] L = L$.

*Proof.* Let $z \in [a, b] \text{co}(L)$, then $z = co$, with $c \in [a, b]$ and $\alpha \in \text{co}(L)$. So, there exists $t \in [0, 1]$ such that $z = (ta + (1-t)b)\alpha = t(\alpha a) + (1-t)(\alpha b)$. Then, $z \in \text{co}(a \text{co}(L) \cup b \text{co}(L)) = \text{co}(a \text{co}(L) \cup b \text{co}(L))$. By Lemma 2.3, $z \in \text{co}(aL \cup bL)$ and $[a, b] \text{co}(L) \subset \text{co}(aL \cup bL)$.

Now, let $z \in \text{co}(aL \cup bL)$. By the Caratheodory’s theorem, $z = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3$, where $z_i \in aL \cup bL$, $\alpha_i \geq 0$ for $i = 1, 2, 3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. By symmetry, we have to study two cases.

**First case:** $z_1, z_2, z_3 \in aL$. In this case, $z \in \text{co}(aL) = \text{aco}(L) \subset [a, b] \text{co}(L)$.

**Second case:** $z_1 \in aL$ and $z_2, z_3 \in bL$. There exists $l_1, l_2, l_3 \in L$ such that $z = \alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3$. If $\alpha_2 + \alpha_3 = 0$, then $\alpha_2 = \alpha_3 = 0$, $z = al_1$ and we conclude as in the first case. Therefore, we assume that $\alpha_2 + \alpha_3 \neq 0$. Writing $z = (\alpha_1 + \alpha_2 + \alpha_3) \frac{\alpha_2}{\alpha_2 + \alpha_3} l_2 + \frac{\alpha_3}{\alpha_2 + \alpha_3} l_3$, we see that $z = \alpha_1 l_1 + (\alpha_2 + \alpha_3) l_4$, where $l_4 \in \text{co}(L)$. We repeat the same argument, we obtain $z = (\alpha_1 + \alpha_2 + \alpha_3) \frac{\alpha_1}{\alpha_1 a + (\alpha_2 + \alpha_3) b} l_1 + (\alpha_2 + \alpha_3) \frac{\alpha_2 + \alpha_3}{\alpha_1 a + (\alpha_2 + \alpha_3) b} l_4$.
\((\alpha_1a + (\alpha_2 + \alpha_3)b)l_5\), with \(l_5 \in co(L)\). Since \(\alpha_1a + (\alpha_2 + \alpha_3)b \in [a, b]\), then \(z \in [a, b] co(L)\). Consequently, \(co(aL \cup bL) \subset [a, b] co(L)\). This completes the proof. \(\square\)

**Remark 2.5.** If \(L\) is convex, the set \([a, b] L\) is exactly the convex hull of \(aL \cup bL\), therefore it is convex. If in addition, \(L\) is compact, then \(aL\) and \(bL\) are compacts, so \([a, b] L = co(aL \cup bL)\) is compact. Lemma 2.4 is then an improvement of Bouldin’s Lemma [3] Lemma 3.

Now we give some properties of generalized numerical ranges of an operator which will be used in the next section.

**Proposition 2.6.** Let \(A \in \mathcal{B}(\mathcal{H})\). Then, \(V_\mathcal{B}(A)\) and \(W_\mathcal{B}(A)\) are convex, furthermore,

(i) \(V_\mathcal{B}(A) = V(A)\) if and only if \(0 \in V(A)\).

(ii) \(W_\mathcal{B}(A) = W(A)\) if and only if \(0 \in W(A)\).

**Proof.** We have \(V_\mathcal{B}(A) = [0, 1] V(A)\) and \(W_\mathcal{B}(A) = [0, 1] W(A)\). Take \(L = V(A)\) which is convex, then by Lemma 2.4 we obtain \(V_\mathcal{B}(A) = co(V(A) \cup \{0\})\) which is convex. A similar argument shows that \(W_\mathcal{B}(A) = co(W(A) \cup \{0\})\), so \(W_\mathcal{B}(A)\) is convex.

Using the convexity of both \(V(A)\) and \(W(A)\), the equalities \(V_\mathcal{B}(A) = co(V(A) \cup \{0\})\) and \(W_\mathcal{B}(A) = co(W(A) \cup \{0\})\) give immediately the equivalences (i) and (ii). \(\square\)

**Remark 2.7.** Let \(\Delta\) be a bounded subset of \(\mathbb{C}\). Define the modulus of \(\Delta\) by

\[|\Delta| = \sup \{|z| : z \in \Delta\}\]

It is clear that for any \(A \in \mathcal{B}(\mathcal{H})\), \(w(A) = |W(A)| \leq |W_\mathcal{B}(A)|\). On the other hand \(W_\mathcal{B}(A) = [0, 1] W(A)\), then \(|W_\mathcal{B}(A)| \leq |[0, 1] W(A)| = |W(A)|\). So \(|W(A)| = |W_\mathcal{B}(A)|\), that is

\[w(A) = \sup \{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\} = \sup \{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| \leq 1\}\]

We estimate it is better to make calculations of \(\langle Ax, x \rangle\) for \(\|x\| \leq 1\); we then obtain more informations about \(w(A)\).

**Proposition 2.8.** Let \(A \in \mathcal{B}(\mathcal{H})\). Then,

(i) \(V_\mathcal{B}(A)\) is closed and \(V_\mathcal{B}(A) = \overline{V_\mathcal{B}(A)}\);

(ii) \(co(\sigma(A) \cup \{0\}) \subseteq W_\mathcal{B}(A)\).

**Proof.** (i) We have \(V_\mathcal{B}(A) = [0, 1] V(A) = [0, 1] W(A) = [0, 1] W_\mathcal{B}(A)\). So \(V_\mathcal{B}(A)\) is closed.

(ii) By Lemma 2.4 we get \(co(\sigma(A) \cup \{0\}) = [0, 1] co(\sigma(A)) \subseteq [0, 1] W(A) = W_\mathcal{B}(A)\). \(\square\)

It is natural to ask when the inclusion (ii) in Proposition 2.8 is an equality. The following proposition gives a sufficient condition for this.
Proposition 2.9. Let $A \in \mathcal{B}(\mathcal{H})$. If $A$ is convexoid, then $W_B(A) = \text{co}(\sigma(A) \cup \{0\})$.

Proof. Since $A$ is convexoid, then $W(A) = \text{co}(\sigma(A))$, so $W_B(A) = [0,1]W(A) = [0,1] W(A) = [0,1] \text{co}(\sigma(A))$. By Lemma 2.3, $W_B(A) = \text{co}(\sigma(A) \cup \{0\})$. □

3. Operator with superiorly closed numerical range

We start this section with some definitions and notations that will be used after. Let $A$ be a non-zero operator in $\mathcal{B}(\mathcal{H})$. Let $D_{W(A)}$ denote the set of all half-lines of the complex plane of origin $O$ and intersect $W(A)$ at a non-zero point.

Consider the following set $S_{W(A)} = \{\lambda \in W(A) : \exists D \in D_{W(A)}, |\lambda| = \sup_{z \in D \cap W(A)} |z|\}$. The set $S_{W(A)}$ is formed by the points of $W(A)$ that are further away from the origin relatively to some element $D \in D_{W(A)}$. Note that we always have $S_{W(A)} \subseteq W(A)$ and $0 \not\in S_{W(A)}$. Note also that $S_{W(A)} \subseteq \partial W(A)$, where $\partial W(A)$ is the boundary of $W(A)$. The last inclusion can be strict, indeed let $\mathcal{H} = \mathbb{C}^2$ and let $A$ be the operator defined on $\mathcal{H}$ by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$ 

We have $W(A) = [1,2], \partial W(A) = \{1,2\}$ and $S_{W(A)} = \{2\}$. Note also that the set $S_{W(A)}$ does not need to be contained in $W(A)$ in general; the operator $B$ defined in Remark 3.10(ii) below produces a counter-example.

Definition 3.1. Let $0 \neq A \in \mathcal{B}(\mathcal{H})$. We say that $A$ is with superiorly closed numerical range if for any $0 \neq z \in W(A)$, there exists $\lambda \in W(A)$ such that $z$ is in the semi-open line segment $(0, \lambda]$.

We denote the class of operators with superiorly closed numerical ranges by $\mathcal{SC}(\mathcal{H})$.

Remark 3.2.

(i) If $A \neq 0$ and $W(A)$ is closed, then $A \in \mathcal{SC}(\mathcal{H})$. In other word, the class $\mathcal{SC}(\mathcal{H})$ contains non-zero operators with closed numerical ranges.

(ii) If $A \in \mathcal{SC}(\mathcal{H})$ and $0 \in W(A)$, then by convexity, $W(A)$ is closed.

We give the first characterizations of the class $\mathcal{SC}(\mathcal{H})$.

Theorem 3.3. Let $A \in \mathcal{B}(\mathcal{H}), A \neq 0$. Then the following statements are equivalent

(i) $A \in \mathcal{SC}(\mathcal{H})$;

(ii) $S_{W(A)} \subseteq W(A)$. 

that loss of generality, we can assume that nothing to show. Therefore, suppose that 

\[
\{ \lambda \in W(A) \mid |\lambda| \geq |\mu| \}
\]

It results that \( \lambda = \mu \in W(A) \) which gives the inclusion \( S_{W(A)} \subseteq W(A) \).

Proof. (i) \( \Rightarrow \) (ii). Assume that \( A \in SC(\mathcal{H}) \). Let \( \lambda \in S_{W(A)} \), then \( 0 \neq \lambda \in W(A) \). There exists \( \mu \in W(A) \) such that \( \lambda \in (0, \mu] \). But \( |\lambda| = \sup_{z \in D \cap W(A)} |z| \) for some \( D \in D_{W(A)} \), so we get \( \mu \in D \cap W(A) \) and \( |\lambda| \geq |\mu| \). Therefore, suppose that \( \lambda = \mu \in W(A) \) which gives the inclusion \( S_{W(A)} \subseteq W(A) \).

(ii) \( \Rightarrow \) (i). Assume that \( S_{W(A)} \subseteq W(A) \). Let \( 0 \neq z \in W(A) \) and let \( \lambda \in S_{W(A)} \) such that \( z \in (0, \lambda] \) (it may be that \( \lambda = z \)), since \( S_{W(A)} \subseteq W(A) \), we have \( \lambda \in W(A) \), hence \( A \in SC(\mathcal{H}) \).

\[\Box\]

We now give a second characterization of the class \( SC(\mathcal{H}) \) in term of generalized numerical ranges. For that, we need the following lemma.

Lemma 3.4. Let \( A \in SC(\mathcal{H}) \). Then \( S_{W(A)} \cup \{0\} \) is closed.

Proof. Let \( (z_n) \subset S_{W(A)} \cup \{0\} \) be a convergent sequence to \( z \). If \( z = 0 \), there is nothing to show. Therefore, suppose that \( z \neq 0 \) and that \( z \notin S_{W(A)} \). Without loss of generality, we can assume that \( z_n \neq 0 \) for all \( n \). Since \( 0 \neq z \in W(A) \) and \( A \in SC(\mathcal{H}) \), there exists \( \lambda \in S_{W(A)} \) such that \( z \in (0, \lambda] \). Let \( z_k \) be any element of \( (z_n) \). Take a disc \( D \) centred at \( z \) and of nonzero radius such that there is no intersection of \( D \) and both the lines \( (O z_k) \) and \( (z_k \lambda) \). So we can find in \( D \) an element \( z_1 \) of argument which is between the arguments of \( z_k \) and \( \lambda \), otherwise we repeat the same for \( z_1 \). The scalars \( z_k, z_1 \) and \( \lambda \) are all in \( W(A) \), so by convexity, the triangle \( T \) with vertices \( z_k, z_1 \) and \( \lambda \) is in \( W(A) \). The half-line belonging to \( D_{W(A)} \) and containing \( z_1 \) meets the triangle \( T \) at some point \( \mu \) such that \( |\mu| > |z_1| \). This contradicts the fact that \( z_1 \in S_{W(A)} \), so \( z \in S_{W(A)} \).

\[\Box\]

Remark 3.5. It is easy to see that \( S_{W(A)} \cup \{0\} \) is not closed in general.

Theorem 3.6. Let \( A \in B(\mathcal{H}) \), \( A \neq 0 \). Then the following statements are equivalent

1. \( A \in SC(\mathcal{H}) \);
2. \( W_\beta(A) = [0,1] S_{W(A)} \);
3. \( W_\beta(A) \) is closed (i.e., \( W_\beta(A) = V_\beta(A) \)).

Proof. (i) \( \Rightarrow \) (ii). Suppose that \( A \in SC(\mathcal{H}) \). By Theorem 3.3, \( S_{W(A)} \subseteq W(A) \), then \( [0,1] S_{W(A)} \subseteq [0,1] W(A) = W_\beta(A) \). We have to show the reverse inclusion. Note that \( 0 \in [0,1] S_{W(A)} \), therefore let \( 0 \neq z \in W_\beta(A) = [0,1] W(A) \). There exists \( t \in (0,1] \) and \( \alpha \in W(A) \) (\( \alpha \neq 0 \)) such that \( z = t \alpha \). Now \( z \in (0, \alpha] \subseteq (0, \lambda] \), where \( \lambda \in S_{W(A)} \). Then, there exists \( s \in (0,1] \) such that \( z = s \lambda \in [0,1] S_{W(A)} \). So, \( W_\beta(A) \subseteq [0,1] S_{W(A)} \).

(ii) \( \Rightarrow \) (iii). First, note that for any subset \( L \subset \mathbb{C} \) we have \( [0,1] L = [0,1] (L \cup \{0\}) \). Now suppose that \( W_\beta(A) = [0,1] S_{W(A)} \). Then \( W_\beta(A) = [0,1] S_{W(A)} = [0,1] (S_{W(A)} \cup \{0\}) = [0,1] (S_{W(A)} \cup \{0\}) \). By Lemma 3.4, \( S_{W(A)} \cup \{0\} \) is closed, then \( W_\beta(A) = [0,1] (S_{W(A)} \cup \{0\}) = [0,1] S_{W(A)} \). So, \( W_\beta(A) = W_\beta(A) \), consequently, \( W_\beta(A) \) is closed.
(iii) $\Rightarrow$ (i). Suppose that $W_B(A)$ is closed; that is $[0,1] W(A) = [0,1] W(A)$. Let $0 \neq z \in W(A) \subseteq [0,1] W(A) = [0,1] W(A)$, then $z = t\lambda$ with $t \in (0,1]$ and $\lambda \in W(A)$. Thus $z \in (0,\lambda]$ with $\lambda \in W(A)$. This is exactly to say that $A \in SC(H)$. □

Remark 3.7. If $A \in SC(H)$, then $V_\delta(A) = W_B(A)$, that is $[0,1] V(A) = [0,1] W(A)$. But we do not have $V(A) = W(A)$ in general; indeed the operator $A$ in Remark 3.10(iii) below confirms this fact.

Theorem 3.8. Let $A \in B(H)$, $A \neq 0$. Then the following statements are equivalent

(i) $A \in SC(H)$ and $0 \in W(A)$;
(ii) $W(A) = W_B(A)$ (i.e., $W(A) = \{(Ax, x) : x \in H, \|x\| \leq 1\}$).

Proof. (i) $\Rightarrow$ (ii). Assume that $A \in SC(H)$ and $0 \in W(A)$. By Theorem 3.6 $W_\delta(A) = V_\delta(A) = [0,1] W(A)$, so by Lemma 2.4, $W_\delta(A) = W(A)$.

(ii) $\Rightarrow$ (i). Assume that $W_\delta(A) = W(A)$. Then $W_\delta(A)$ is closed, therefore $A \in SC(H)$ (by Theorem 3.6) and it is clear that $0 \in W(A)$. □

Theorem 3.9. Let $A \in SC(H)$. Then $w(A) = |\lambda|$ where $\lambda \in W(A)$.

Proof. Let $0 \neq z \in W(A)$ such that $w(A) = |z|$. Since $A \in SC(H)$, then $z \in (0,\lambda]$ where $\lambda \in W(A)$. But $|\lambda| \leq |z|$, so $z = \lambda$. □

Remark 3.10.

(i) If $A \in SC(H)$, then for any $\mu \in W(A)$ such that $w(A) = |\mu|$, we have $\mu \in W(A)$. Note that we can particularly deduce this from the fact that $S_{W(A)} \subseteq W(A)$ (see Theorem 3.8).

(ii) The converse of Theorem 3.9 is false. Indeed, let $A$ be the operator on $\ell_2$ the Hilbert space of square summable sequences, defined by

$$A(x_1, x_2, ..., x_n, ...) = (x_1, \frac{1}{2}x_2, ..., \frac{1}{n}x_n, ...).$$

By a simple calculation, we obtain $W(A) = (0,1]$. Let $B = A - \frac{I}{2}$, we have $W(B) = (-\frac{1}{2}, \frac{1}{2})$ and $w(B) = \frac{1}{2}$, however $B \notin SC(H)$ (since $\frac{1}{2} \in S_{W(B)}$ and $-\frac{1}{2} \notin W(B)$).

The following corollary gives a necessary and sufficient condition for $A \in SC(H)$ to be normaloid.

Corollary 3.11. Let $A \in SC(H)$. $A$ is normaloid if and only if there exists $\lambda \in \sigma_p(A)$ such that $|\lambda| = \|A\|$.

Proof. Since we always have $\sigma_p(A) \subseteq \sigma(A)$, the sufficiency is evident. Passing to the necessity: suppose that $A \in SC(H)$, then by Theorem 3.9 $w(A) = |\lambda|$ where $\lambda \in W(A)$. If $A$ is normaloid, then $|\lambda| = \|A\|$ and by [10] Theorem 1.3-3, $\lambda \in \sigma_p(A)$. □
Now we show that the class $\mathcal{SC}(\mathcal{H})$ contains strictly the class of non zero compact operators, then we deduce some necessary conditions for an operator to be compact.

**Theorem 3.12.** Let $A \in \mathcal{B}(\mathcal{H})$, $A \neq 0$. If $A$ is compact, then $A \in \mathcal{SC}(\mathcal{H})$.

**Proof.** Let $0 \neq z \in \overline{W(A)}$, there exists a sequence $\{\langle Ax_n, x_n \rangle\}$, where $\|x_n\| = 1$ for all $n$, converging to $z$. Since the unit ball is weakly compact, there exists a subsequence $\{x_{n_k}\}$ which is weakly convergent to an $x$ with $\|x\| \leq 1$. But $A$ is a compact operator, then $\{Ax_{n_k}\}$ is strongly convergent to $Ax$. We have

\[ |\langle Ax_{n_k}, x_{n_k} \rangle - \langle Ax, x \rangle| \leq |\langle Ax_{n_k}, x_{n_k} \rangle - \langle Ax, x_{n_k} \rangle| + |\langle Ax, x_{n_k} \rangle - \langle Ax, x \rangle| \]
\[ \leq \|x_{n_k}\| \|Ax_{n_k} - Ax\| + |\langle x_{n_k}, Ax \rangle - \langle x, Ax \rangle| \]
\[ \leq \|x_{n_k}\| \|Ax_{n_k} - Ax\| + |\langle Ax, x_{n_k} - x \rangle| . \]

Therefore $\{\langle Ax_{n_k}, x_{n_k} \rangle\}$ converges to $\langle Ax, x \rangle$, so $\langle Ax, x \rangle = z$. Since $z \neq 0$, then $x \neq 0$. Put $\lambda = \frac{2}{\|x\|^2} = \frac{A x}{\|x\|^2}$, we see that $\lambda \in W(A)$ and $z = \lambda \|x\|^2 \in (0, \lambda]$ since $\|x\|^2 \leq 1$, thus $A \in \mathcal{SC}(\mathcal{H})$, as desired. $\square$

**Remark 3.13.** Theorem 3.12 shows that the class $\mathcal{SC}(\mathcal{H})$ contains the class of non zero compact operators. This inclusion is strict, indeed let $A$ be the operator on $\ell_2$ defined by

\[ A(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, \ldots). \]

It is easy to show that $W(A) = [0, 1]$, then by Remark 3.12(i), $A \in \mathcal{SC}(\mathcal{H})$. However, $A$ is the orthogonal projection of $\ell_2$ onto the closed infinite dimensional subspace of sequences with zero in the first coordinate, so it is not compact.

**Corollary 3.14.** Let $A \in \mathcal{B}(\mathcal{H})$, $A \neq 0$. If $A$ is compact, then the following statements hold

(i) $S_{W(A)} \subseteq W(A)$.
(ii) $W_B(A) = [0, 1] S_{W(A)}$.
(iii) $W_B(A)$ is closed.
(iv) $w(A) = |\lambda|$ where $\lambda \in W(A)$.
(v) $A$ is normaloid if and only if there exists $\lambda \in \sigma_p(A)$ such that $|\lambda| = \|A\|$.
(vi) $W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| \leq 1\}$.

**Proof.** If $A$ is a non zero compact operator, by Theorem 3.12 $A \in \mathcal{SC}(\mathcal{H})$. The first four statements derive from Theorem 3.12, Theorem 3.10 and Theorem 3.9 respectively.

The statement (v) is known, but we give here another proof: it is an immediate consequence of Theorem 3.12 and Corollary 3.11.

Since $A$ is compact, $0 \in \sigma(A) \subseteq W(A)$. By Theorem 3.12 $W(A) = W_B(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| \leq 1\}$. This gives (vi) as desired. $\square$
Remark 3.15.

(i) Corollary 3.14 can be used to deduce that an operator is not compact. Let us see some examples.

1. Let \( S \) be the unilateral shift on \( \ell^2 \) defined by

\[
S(x_1, x_2, ...) = (x_2, x_3, ...).
\]

It is known that \( W(S) = \{ z \in \mathbb{C} : |z| < 1 \} \) (for more details, see \cite[Example 2, p.2]{10}), then \( w(S) = 1 \). Since \( W(S) \) contains no scalar of modulus 1, \( S \) is not compact. Note that \( S \) is not even in \( SC(\mathcal{H}) \) and, since \( S \neq 0 \), we obtain the same result by Theorem 3.12.

2. Let \( A \) be a self-adjoint operator such that \( W(A) \) is the one of the following half-open intervals:

- \((\alpha, \beta) \) or \((\alpha, \beta]\) with \( \alpha < 0 \) and \( \alpha < \beta \);
- \([\alpha, \beta) \) or \([\alpha, \beta]\) with \( \alpha < \beta \) and \( 0 < \beta \).

Then \( A \) is not compact: for example in the first cases, \( \alpha \in S_{W(A)} \) and \( \alpha \notin W(A) \).

3. More generally, we can use the same argument for a normal operator which has numerical range as a line segment.

(ii) Conditions (i), (ii), (iii), (iv) and (vi) of Corollary 3.14 are all necessary but not sufficient. Indeed, let \((d_n)\) be a sequence of positive numbers that decreases monotonically to \( d > 0 \) and let \( D \) be the diagonal operator on \( \ell^2 \) defined by

\[
D = \begin{pmatrix}
d_1 & 0 & \cdots \\
0 & -d_1 & \\
& \ddots & \ddots \\
& & d_2 & -d_2 \\
& & & \ddots
\end{pmatrix}.
\]

Since the diagonal sequence does not converge to 0, then \( D \) is not compact. However \( W(D) = [-d_1, d_1] \) and

(a) \( S_{W(D)} = \{-d_1, d_1\} \subset W(D) \);

(b) \( W_B(D) = W(D) \), then \( W_B(D) \) is closed;

(c) \( W_B(D) = [0, 1] S_{W(D)} \);

(d) \( w(D) = d_1 \in W(D) \).

4. Generalized numerical range of \( \delta_{2,A,B} \) and \( M_{2,A,B} \)

Let \( a \in \mathcal{A} \), by the same way we define the generalized algebraic numerical range of \( a \) as follows

\[
V_{\mathcal{B}}(a) = \{ f(a) : f \in \mathcal{A}', \ f(e) = \| f \| \leq 1 \},
\]
Let $A, B \in \mathcal{B}(\mathcal{H})$. From the identity (4.3), we derive that $V_B(\delta_{2,A,B}) = V_B(\delta_{A,B})$. On the other hand, we always have the inclusions

$$W(A) - W(B) \subseteq V(A) - V(B) \subseteq V_B(\delta_{A,B}).$$

The last inclusion can be strict. Indeed, let $A = I$ and $B = 0$. We have $W(A) - W(B) = V(A) - V(B) = \{1\}$ and $V_B(\delta_{A,B}) = [0, 1]$. The following theorem gives a sufficient and necessary condition for $V_B(\delta_{A,B})$ to be minimal in the sense of inclusion.

**Theorem 4.1.** Let $A, B \in \mathcal{B}(\mathcal{H})$. The following are equivalent:

1. $W(A), W(B)$ are closed and $W(A) \cap W(B) \neq \emptyset$;
2. $V_B(\delta_{A,B}) = W(A) - W(B)$.

Before proving Theorem 4.1, we first establish two lemmas.

**Lemma 4.2.** Let $K, L$ are two convex compact subsets of $\mathbb{R}^2$, where $\mathbb{R}$ denotes the real line, and let $b$ be an extrem point of $L$. There exists $a \in K$ such that if $a + b = x + y$, then $y = b$ for any $(x, y) \in K \times L$.

**Proof.** Let $a_2$ and $b_2$ be the maximum of the second coordinates of the elements of $K$ and $L$ respectively. By rotating, we can assume that $b = (b, b_2)$. Set $K' = \{x' \in \mathbb{R} : (x', a_2) \in K\}$ and $L' = \{y' \in \mathbb{R} : (y', b_2) \in L\}$. It takes a little work to show that $K'$ and $L'$ are two nonempty convex compact subsets of $\mathbb{R}$ and $b'$ is an extrem point of $L'$. Therefore $K'$ and $L'$ are two closed line segments of $\mathbb{R}$, say $K' = [x_1', x_2']$ and $L' = [b', y_2']$. Take $a = (x_1', a_2)$. If $x = (x_1, x_2) \in K$ and $y = (y_1, y_2) \in L$ such that $a + b = x + y$, then we obtain $x_1 + b' = x_1 + y_1$ and $a_2 + b_2 = x_2 + y_2$. Since $x_1' \leq x_1$ and $b' \leq y_1$, then $x_1' = x_1$ and $b' = y_1$. The same argument shows that $a_2 = x_2$ and $y_2 = b_2$, so $y = b$. \hfill \square

**Lemma 4.3.** Let $K, L$ are two bounded convex subsets of $\mathbb{C}$. If $K + L$ is compact, then $K$ and $L$ are compact.

**Proof.** Since $K + L = \overline{K} + \overline{L}$, we have to show that if $\overline{K} + \overline{L} = K + L$, then $\overline{K} = K$ and $\overline{L} = L$. Suppose that $\overline{K} + \overline{L} = K + L$, then $\overline{K} + \overline{L} = \overline{K} + L$. Let $b$ be an extrem point of $\overline{L}$ and take $a \in \overline{K}$ as given by Lemma 4.2. We have $a + b \in \overline{K} + L$, then there exists $x \in \overline{K}$ and $y \in L$ such that $a + b = x + y$. Since $y \in \overline{L}$, we conclude by Lemma 4.2 that $y = b$, and hence $b \in L$. Since $L$ is bounded and convex, then by Krein-Milman’s theorem $\overline{L}$ is the convex hull of its extreme points, it follows that $\overline{L} = L$. We obtain by the same argument $\overline{K} = K$. \hfill \square

**Remark 4.4.** Not that Lemma 4.3 is false if one of $K$ and $L$ is not convex. Indeed, let $K = [0, 1]$ and $L = \mathbb{Q} \cap K$, the set of rational numbers in $K$, then $L$ is not convex. However, $K + L = [0, 2]$ is compact and $L$ is not compact.
We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** (i) ⇒ (ii). Suppose that \( W(A), W(B) \) are closed and \( W(A) \cap W(B) \neq \emptyset \). Then \( W(A) = V(A), W(B) = V(B) \) and \( 0 \in V(A)-V(B) \). Consequently, \( W(A) - W(B) = V(t)-V(B) = [0, 1](V(A)-V(B)) = [0, 1]V(\delta_{A,B}) = V_2(\delta_{A,B}) \).

(ii) ⇒ (i). Suppose that \( V_2(\delta_{A,B}) = W(A) - W(B) \). Then \( 0 \in W(A) - W(B) \), so \( W(A) \cap W(B) \neq \emptyset \). On the other hand, since \( V(A) - V(B) \) is convex and contains \( 0 \), then by Lemma 2.4, we have \( V(A) - V(B) = [0, 1](V(A) - V(B)) = [0, 1]V(\delta_{A,B}) = V_2(\delta_{A,B}) = W(A) - W(B) \). Thus \( V(A) - V(B) = W(A) - W(B) \). It follows by Lemma 4.3 that \( V(A) = W(A) \) and \( V(B) = W(B) \). So \( W(A) \) and \( W(B) \) are closed. This completes the proof.

We end this section by giving some properties of the generalized numerical range of \( \mathcal{M}_{2,A,B} \). Let us start with a lemma that we use for the sequel.

**Lemma 4.5.** Let \( L, S \) be two subsets of \( C \). Then

(i) \( L_2S_2 = L_2S = LS_2 \),
(ii) \( \co(LS)_2 = \co(LS_2) \), where \( L_2 = [0, 1] L \).

**Proof.** (i) \( L_2S_2 = [0, 1] L [0, 1] S = L [0, 1] S = LS_2 \).

(ii) It is clear that \( \co(LS) \subseteq \co(LS_2) \), then \( \co(LS)_2 \subseteq \co(LS_2)_2 \). Since \( 0 \in \co(LS_2)_2 \) and \( \co(LS_2)_2 \) is convex, by Lemma 2.4, \( \co(LS_2)_2 = \co(LS_2) \). So, \( \co(LS)_2 \subseteq \co(LS_2) \). Now, let \( z \in LS_2 \), so \( z = l(tS) \) where \( l \in L, s \in S \) and \( t \in [0, 1] \). Writing \( z = tS + (1-t)S \), we see that \( z \in \co(LS) \cup \{0\} = \co(LS)_2 \). So, \( LS_2 \subseteq \co(LS)_2 \). Since \( \co(LS)_2 \) is convex, we obtain the inclusion \( \co(LS)_2 \subseteq \co(LS)_2 \).

**Proposition 4.6.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then

\[ W_2(\mathcal{M}_{2,A,B}) \subseteq W(A)W(B) + S(A)S(B) \]

**Proof.** We have \( W_2(\mathcal{M}_{2,A,B}) = [0,1] W(\mathcal{M}_{2,A,B}) \), using the inclusion \( \bullet \), we obtain

\[ W_2(\mathcal{M}_{2,A,B}) \subseteq [0,1] W(A)W(B) + S(A)S(B) \]
\[ \subseteq [0,1] W(A)W(B) + [0,1] S(A)S(B) \]
\[ = [0,1] W(A)W(B) + [0,1] S(A)S(B) \]
\[ = W(A)W_2(B) + S(A)S(B) \]

since \( 0 \in S(A) \) and \( S(A) \) is convex (Lemma 2.4). \( \square \)

**Theorem 4.7.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \),

(i) if either \( A \) or \( B \) is hyponormal, then

\[ V_2(\mathcal{M}_{2,A,B}) = \overline{\co(W(A)W_2(B))} \]
(ii) if either $A$ or $B$ is diagonal, then

\[ W_\beta(M_{2,A,B}) = \text{co}(W(A)W_\beta(B)). \]

**Proof.** (i) Using the identity (1.9),

\[ V_\beta(M_{2,A,B}) = [0,1]co(W(A)W(B)) = [0,1]co(W(A)W(B)) = \text{co}(W(A)W(B))_B \]

and by Lemma 4.5(ii),

\[ V_\beta(M_{2,A,B}) = \text{co}(W(A)W_\beta(B)). \]

(iii) We use the identity (1.8) and the same argument as in (i). \qed

**Corollary 4.8.** Let $A, B \in \mathcal{B}(\mathcal{H})$ such that either $A$ or $B$ is hyponormal. If $0 \in W(A) \cup W(B)$, then

\[ V_\beta(M_{2,A,B}) = \text{co}(W(A)W(B)). \]

**Proof.** Assume, for example, that $0 \in W(A)$, then par Proposition 2.3(ii),

\[ W_\beta(A) = W(A) \text{ and by Lemma 4.5(i), } W(A)W_\beta(B) = W_\beta(A)W(B) = W(A)W(B). \]

Since either $A$ or $B$ is hyponormal, the derived result derives by Theorem 4.7. \qed

**Remark 4.9.** The converse of Corollary 4.8 is false. Indeed, let $A$ and $B$ are the operators on $\mathcal{H} = \mathbb{C}^2$ defined by

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2i \end{pmatrix}. \]

The numerical ranges $W(A)$ and $W(B)$ are the closed segments $[1, i]$ and $[1, 2i]$ respectively. Since $1 \in W(A)W(B)$ and $-2 \in W(A)W(B)$, then $0 \in \text{co}(W(A)W(B))$ and by Lemma 4.5(ii),

\[ \text{co}(W(A)W(B)) = \text{co}(W(A)W_\beta(B)). \]

Since $A$ is an hyponormal operator, then by Theorem 4.7, $V_\beta(M_{2,A,A}) = \text{co}(W(A)W_\beta(A)) = \text{co}(W(A)W(A))$. However, $0 \notin W(A) \cup W(B)$.

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