A Map between Time-Dependent and Time-Independent Quantum Many-Body Hamiltonians

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Abstract—Given a time-independent Hamiltonian $\tilde{H}$, one can construct a time-dependent Hamiltonian $H_t$ by means of the gauge transformation $H_t = U_t \tilde{H} U_t^\dagger - i U_t \partial_t U_t^\dagger$. Here $U_t$ is the unitary transformation that relates the solutions of the corresponding Schrödinger equations. In the many-body case one is usually interested in Hamiltonians with few-body (often, at most two-body) interactions. We refer to such Hamiltonians as physical. We formulate sufficient conditions on $U_t$ ensuring that $H_t$ is physical as long as $\tilde{H}$ is physical (and vice versa). This way we obtain a general method for finding pairs of physical Hamiltonians $H_t$ and $\tilde{H}$ such that the driven many-body dynamics governed by $H_t$ can be reduced to the quench dynamics due to the time-independent $\tilde{H}$. We apply this method to a number of many-body systems. First we review the mapping of a spin system with isotropic Heisenberg interaction and arbitrary time-dependent magnetic field to a time-independent system without a magnetic field [F. Yan, L. Yang, and B. Li, Phys. Lett. A\textbf{251}, 289–293; \textbf{259}, 207–211 (1999)]. Then we demonstrate that essentially the same gauge transformation eliminates an arbitrary time-dependent magnetic field from a system of interacting fermions. Further, we apply the method to the quantum Ising spin system and a spin coupled to a bosonic environment. We also discuss a more general situation where $\tilde{H} = \tilde{H}_t$ is time-dependent but dynamically integrable.

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1. INTRODUCTION

Quantum many-body systems driven by external classical parameters varying in time are ubiquitous in nature and in labs. In particular, time-dependent electric and magnetic fields are routinely used to engineer and manipulate large quantum systems consisting of cold atoms in optical lattices, ions in ion traps, arrays of superconducting qubits and quantum dots, etc.

From the theoretical standpoint, the description of a driven quantum many-body system is, in general, a formidable task. The state of such a system is described by a multidimensional dynamical wave function $\Psi_t$ satisfying the Schrödinger equation

$$i \partial_t \Psi_t = H_t \Psi_t,$$

where $H_t$ is the time-dependent Hamiltonian of the system. A major difficulty of this equation is that it cannot, in general, be reduced to the stationary Schrödinger equation, in contrast to the case of a Hamiltonian independent of time. This difficulty adds to the complexity due to the many-body nature of the problem.

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In recent years it has been realized that certain quantum many-body models with driving are 
*dynamically integrable*, i.e., allow for explicit solutions. These are mostly time-dependent general-
izations of models solvable by the Bethe ansatz [1, 17, 20, 25, 44] and continuous-variable models 
with scaling invariance [4, 6, 7, 11, 24, 28, 35].

Here we elaborate upon a somewhat different approach to the driven many-body dynamics. 
Namely, we consider time-dependent many-body Hamiltonians that can be reduced either to time-
dependent Hamiltonians (integrable or not) or to dynamically integrable time-dependent Hamil-
tonians with the help of a suitable gauge transformation. This transformation implies a unitary 
map between the corresponding dynamical wave functions, and thus the dynamics induced by one 
Hamiltonian is straightforwardly restored from the dynamics induced by the other. This approach 
has been previously applied to a number of particular many-body systems [8, 21, 49, 50].

Importantly, we impose a physical requirement that both Hamiltonians contain only few-body 
interactions, and put forward a class of transformations that automatically satisfy this requirement. 
This general construction is described in the next section. In Sections 3–6 we apply this machinery 
to particular systems. A brief summary is given in Section 7. Some technical details are relegated 
to the appendices.

2. GAUGE TRANSFORMATION PRESERVING 
FEW-BODY NATURE OF INTERACTIONS

**Gauge transformation.** Consider two time-dependent Hamiltonians, $H_t$ and $\tilde{H}_t$. The 
corresponding wave functions, $\Psi_t$ and $\tilde{\Psi}_t$, satisfy the Schrödinger equations (1.1) and

$$i\partial_t \tilde{\Psi}_t = \tilde{H}_t \tilde{\Psi}_t,$$

respectively. Obviously, there always exists a unitary transformation $U_t$ with the property $U_0 = 1$ 
such that

$$\Psi_t = U_t \tilde{\Psi}_t$$

(2.2) 

for arbitrary common initial conditions $\Psi_0 = \tilde{\Psi}_0$. It is easy to verify that this unitary transformation 
of the wave function induces a gauge transformation of the Hamiltonian,

$$H_t = U_t \tilde{H}_t U_t^\dagger - W_t, \quad W_t \equiv iU_t \partial_t U_t^\dagger.$$

(2.3)

The above formula is well known and emerges in many contexts, including quantum electrodynamics [19], dynamical symmetries [45], adiabatic theorem [26, 29], counterdiabatic driving [2, 12, 13, 30, 43, 47], Berry phase [48] and Floquet engineering [15, 16, 27] (see also its analog for the 
Lindblad dynamics of open systems in [41, 42]).

Formula (2.3) generalizes the transformation to a uniformly rotating reference frame in the 
Hilbert space given by

$$H_t = e^{iWt} \tilde{H} e^{-iWt} - W = e^{iWt}(\tilde{H} - W)e^{-iWt}$$

(2.4)

with time-independent $W$ and $\tilde{H}$. The latter transformation has numerous applications, from Rabi 
oscillation [38] to the electron transport through a driven conformal quantum point contact [21]. An 
important difference between formulas (2.4) and (2.3) is that $H_t$ in (2.4) has a constant spectrum, 
while $H_t$ in (2.3) can have a spectrum varying in time in an arbitrary way.

**Preserving few-body nature of interactions.** Since the gauge transformation (2.3) can 
relate two arbitrary time-dependent Hamiltonians acting on the same Hilbert space, it is tempting 
to use this transformation along with (2.2) to reduce the dynamics governed by some complex time-
dependent $H_t$ to the dynamics governed by some simple (preferably time-independent) $\tilde{H}_t$. One
can also attempt to turn the tables and generate a new tractable Hamiltonian $H_t$ from a known tractable (for example, time-independent or dynamically integrable) Hamiltonian $\tilde{H}_t$.

However, when it comes to the many-body systems, the set of tractable and/or physically meaningful Hamiltonians is normally restricted by the Hamiltonians with few-body (often, at most two-body) interactions. We will refer to such Hamiltonians as physical throughout the paper. It is then natural to ask under what conditions both $H_t$ and $\tilde{H}_t$ in (2.3) are physical in this sense. Here we come to the central observation of the present paper: To ensure that both $H_t$ and $\tilde{H}_t$ are physical it is sufficient to require that

$$U_t = \prod_{j=1}^{K} U_j(t),$$

(2.5)

where each unitary $U_j(t)$ is a few-body operator, $K$ is at most polynomial in the system size $L$, and

$$[U_j(t), U_{j'}(t')] = 0 \quad \text{for} \quad j \neq j'.$$

(2.6)

Indeed, under these conditions, (2.3) implies that whenever $\tilde{H}_t$ is an extensive sum of few-body terms, so is $H_t$. The commutation relation (2.6) at $t = t'$ (at $t \neq t'$) ensures the few-body nature of $U_t \tilde{H}_t U_t^{\dagger}$ (of $W_t$). It should be emphasized that the commutation of $U_j(t)$ with $U_{j'}(t')$ is not required.

In general, there is no guarantee that there exists a unitary transformation relating two given physical Hamiltonians according to (2.3) and satisfying conditions (2.5) and (2.6). However, we will demonstrate that such a unitary transformation exists for a number of important models.

**Recipe summary.** Let us summarize our main idea. Assume one wishes to study quantum many-body dynamics governed by a physical time-dependent many-body Hamiltonian $H_t$ but is unable to solve the Schrödinger equation (1.1) directly. One can then attempt to find a unitary operator $U_t$ of the product form (2.5) satisfying condition (2.6), such that the gauge transformation (2.3) reduces $H_t$ to a more tractable $\tilde{H}_t$. Conditions (2.5) and (2.6) automatically ensure that $\tilde{H}_t$ is also physical. If one is then able to solve the Schrödinger equation (2.1) with $\tilde{H}_t$, then the solution to the original Schrödinger equation (1.1) is obtained by the unitary transformation (2.2).

While this recipe may seem to be quite abstract at this point, we will substantiate it by considering specific models in what follows. Before we turn to these models in the next sections, let us present some additional general considerations and introduce relevant notions.

**Special initial states.** Assume $\tilde{H}_t = \tilde{H}$ is independent of time and the system is initialized in an eigenstate of $\tilde{H}$ with eigenvalue $\tilde{E}$. Then

$$\Psi_t = e^{-i\tilde{E}t} U_t \Psi_0.$$  

(2.7)

This implies that the dynamics of any few-body observable $A$ can be found explicitly, $\langle \Psi_t | A | \Psi_t \rangle = \langle \Psi_0 | U_t^{\dagger} AU_t | \Psi_0 \rangle$, provided the few-body expectation values of the initial state are known. This follows from the fact that $U_t^{\dagger} AU_t$ is a few-body operator thanks to (2.5) and (2.6). We give an explicit example of the dynamics of a special initial state in Section 3.

**Floquet dynamics.** Assume that $\tilde{H}_t = \tilde{H}$ is independent of time and $U_t$ is periodic with period $T$. This implies that $H_t$ is also periodic, and we immediately obtain the stroboscopic Floquet dynamics

$$\Psi_{nT} = e^{-i\tilde{H}T} \Psi_0, \quad n \in \mathbb{Z},$$

(2.8)

with the Floquet Hamiltonian equal to $\tilde{H}$.

Remarkably, the exact Floquet Hamiltonian thus obtained contains only few-body interactions. This is in sharp contrast to typical Floquet Hamiltonians involving arbitrarily extended many-body interactions.
One particular consequence of this feature is that the periodic driving fails to heat the system to the infinite temperature, contrary to what is happening in generic systems [9, 31, 37]. It has been actually anticipated that if the time dependence can be eliminated by moving to a different reference frame, the indefinite heating hypothesis will fail [9]. Here we provide a receipt how to construct a broad range of local many-body models where this indeed happens.

Integrable driven dynamics. If $\tilde{H}_t = \tilde{H}$ is independent of time and integrable, the driven dynamics of $H_t$ is reduced to the integrable quench dynamics of $\tilde{H}$. Remarkably, the instantaneous integrability of $H_t$ can be absent in this case.

Integrals of motion $I_n$ of $\tilde{H}$ are mapped to the dynamical invariants

$$I_n(t) = U_t \tilde{I}_n U_t^\dagger.$$  

(2.9)

Dynamical invariants are operators that are, in general, explicitly time-dependent, satisfy the equation $i \partial_t I_n(t) = [H_t, I_n(t)]$, and thus possess conserved expectation values, $\partial_t \langle \Psi_t | I_n(t) | \Psi_t \rangle = 0$ (see [33]).

Further, if $\tilde{H}_t$ is time-dependent and dynamically integrable, so is $H_t$. This statement is illustrated by an explicit example in Section 5.

Variational approach and relation to counterdiabatic driving. While a gauge transformation is a standard tool in constructing counterdiabatic driving protocols [2, 12, 13, 30, 43], the meaning of $U_t$ there is very different from that in the present paper. Namely, here $U_t$ relates solutions of the dynamical Schrödinger equations with different Hamiltonians, while there $U_t$ relates instantaneous eigenstates of a time-dependent Hamiltonian at different times. As a consequence, $W_t$ in (2.3) is not the adiabatic gauge potential introduced in [30].

Nevertheless, formal similarities are strong enough to attempt to adapt tools developed for the counterdiabatic driving to the present problem. In particular, one can observe, following [43], that (2.3) with time-independent $\tilde{H}_t = \tilde{H}$ entails

$$\partial_t H_t = i[W_t, H_t] - \partial_t W_t.$$  

(2.10)

Note that this equation does not contain $\tilde{H}$. Given $H_t$, one can try to find $W_t$ from this equation. If successful, one can then restore $U_t$ from $W_t$ and find $\tilde{H}$ from (2.3). To summarize, solving (2.10) is equivalent to finding $U_t$ and $\tilde{H}$ for a given $H_t$. One can also address (2.10) variationally, as in [43]. We leave further research in this promising direction for future work.

3. HEISENBERG MODEL WITH TIME-DEPENDENT MAGNETIC FIELD

Model. Here we consider a system of quantum spins on an arbitrary lattice, with the isotropic Heisenberg interaction and a time-dependent spatially homogeneous magnetic field $B_t$:

$$H_t = H_H - B_t S_{\text{tot}},$$  

(3.1)

where $S_{\text{tot}} = \sum_j S_j$ is the total spin and

$$H_H = \sum_{i<j} J_{ij} S_i S_j$$  

(3.2)

describes the isotropic Heisenberg coupling between the spins with arbitrary coupling constants $J_{ij}$, the indices $i$ and $j$ labeling the sites of the lattice. We emphasize that our reasoning will apply to spins with arbitrary spin quantum number (not necessarily spins $1/2$).

The fact that the Hamiltonian (3.1) can be essentially reduced to a time-independent Hamiltonian was arguably first recognized in [49, 50]. We will first present this mapping in our own framework, and then comment on the presentation of [49, 50].

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1 We are grateful to A. Polkovnikov for pointing to this equation.
Eliminating the magnetic field. It turns out that for an arbitrary $B_t$ one can choose a $U_t$ of the form (2.5), (2.6) such that the time-dependent Hamiltonian (3.1) can be mapped onto the time-independent Hamiltonian $\tilde{H} = H_H$ by the gauge transformation (2.3). This $U_t$ reads

$$U_t = \exp(iK_tS_{\text{tot}}),$$

(3.3)

where $K_t$ is the solution of the system of differential equations

$$\dot{K}_t = B_t n_t, \quad \dot{n}_t = \frac{1}{2} n_t \times B_t + \frac{1}{2} \cot K_t \left(B_t - (B_t n_t) n_t\right),$$

(3.4)

with $n_t$ being a unit vector and $K_t = K_t n_t$. Here and in what follows the dot stands for the time derivative and the cross “×” stands for the vector product. The initial condition is $K_t = 0$, which ensures $U_0 = 1$. The ambiguity in the initial condition for $n_t$ is convenient to resolve as $n_0 = B_0 / |B_0|$. The derivation of the system of equations (3.4), as well as its alternative forms, is presented in Appendix A. The key point of the derivation is that $U_t H_H U_t^\dagger = H_H$ due to the rotational invariance of the isotropic Heisenberg interaction.

Interestingly, $U_t$ can be expressed in a quite different form tagged as “Gauss parametrization” [39, 46]. A remarkable feature of the Gauss parametrization is that it leads to a Riccati equation. We review this result in Appendix B.

Let us briefly comment on [49, 50] and related work. In [49] the Hamiltonian (3.1) was reduced to a Hamiltonian with Heisenberg interactions and magnetic field with a time-dependent magnitude but fixed direction along the $z$ axis. This was done by means of a unitary transformation similar to that in (B.1). This transformation had been earlier used to treat a time-dependent Hamiltonian constructed of a sum of SU(2) and SU(1, 1) generators [32] (in particular, a spin in a time-dependent magnetic field could be treated by the method of [32]). In [50] the Hamiltonian (3.1) was reduced to a time-independent one in a somewhat indirect way in the framework of the theory of dynamical invariants [33]. A dynamical invariant for a single spin in a time-dependent magnetic field was constructed in [22].

We also remark that since $H_t$ and $H_{t'}$ form a finite closed algebra with respect to commutation, the dynamics can be alternatively addressed by a technique described in [25].

We note that apart from the time-dependent dynamical invariant constructed in [22, 50], the Hamiltonian (3.1) has two independent conserved quantities, $H_I$ and $S_{\text{tot}}^2$. Further, if $H_I$ is one-dimensional with nearest-neighbor Heisenberg interactions (i.e., integrable), there is a large number of additional local, rotationally invariant, time-independent conserved quantities that can be found, for example, in [23].

**Special initial states.** It follows from (2.7) and the above considerations that any eigenstate of $\tilde{H} = H_H$ evolves as if there were no interactions between spins whatsoever, i.e., under the action of the Hamiltonian $-B_t S_{\text{tot}}$ (up to an irrelevant overall phase). The most simple eigenstates of $H_H$ are product states of spins pointing in the same direction, and this form of the state vector is preserved throughout the evolution, the direction of a spin being determined by the Schrödinger equation with a single-particle Hamiltonian $-B_t S_i$.

4. INTERACTING FERMIONS IN A TIME-DEPENDENT MAGNETIC FIELD

Here we consider a general Hamiltonian of interacting lattice fermions with spin $1/2$ in an external homogeneous time-dependent magnetic field,

$$H_t = H_e - \sum_{\alpha, \sigma, \bar{\sigma}} B_t^{\alpha \sigma} \bar{c}_{\alpha} \sum_i \gamma_i \sigma_i c_{\sigma_i},$$

(4.1)
where
\[
H_e = -\frac{1}{2} \sum_{\sigma,\tilde{\sigma}} \epsilon_{ij} \sigma_i^\dagger \sigma_j + \frac{1}{2} \sum_{\alpha} V_{ij} \sigma_i^\dagger \sigma_j^\dagger \sigma_j \sigma_i,
\] (4.2)
i and \( j \) label lattice sites, \( \sigma, \tilde{\sigma} \in \{ \uparrow, \downarrow \} \) are spin indices, \( \alpha \in \{ x, y, z \} \) labels \( 2 \times 2 \) spin matrices \( s_\alpha \) equal to the corresponding Pauli matrices divided by 2, \( c_i \) is a fermion annihilation operator at the lattice site \( i \), \( \epsilon_{ij} \) are tunneling matrix elements (for \( i \neq j \)) or local on-site potentials (for \( i = j \)), \( V_{ij} \) is a two-body interaction potential, and \( B_t \) is the time-dependent external magnetic field. A particular case of this Hamiltonian is the Hubbard model, which is integrable in one dimension [18].

In complete analogy with the argument in the previous section, we find that an arbitrary \( B_t \) can be eliminated by the gauge transformation (2.3) with \( H = H_e \) and
\[
U_t = \exp \left( \sum_{\alpha, \sigma, \tilde{\sigma}} K_t^\alpha s_\alpha^\dagger \sum_i c_i^\dagger \sigma_i \right),
\] (4.3)
where \( K_t \) satisfies the differential equations (3.4).

Alternatively, one can use \( U_t \) in the Gauss parametrization, analogously to (B.1).

Note that the results of the present section can be straightforwardly generalized to fermions or bosons with an arbitrary spin, to models with continuous variables, and to Kondo-type models with both itinerant and localized particle species.

5. ISING MODEL IN A TIME-DEPENDENT MAGNETIC FIELD

Ising model on a general lattice. Here we consider the quantum Ising model in a time-dependent magnetic field,
\[
H_t = H_t - \sum_i B_i(t) S_i,
\] (5.1)
where
\[
H_t = \sum_{i < j} J_{ij} S_i^x S_j^x
\] (5.2)
describes the Ising coupling. As in Section 3, the spins reside on an arbitrary lattice labeled by indices \( i \) and \( j \). In contrast to the previous examples, now the magnetic field \( B_i(t) \) need not be homogeneous, which is indicated by the index \( i \). For the latter reason we have to use a somewhat different notation for the magnetic field, \( B_i(t) \) instead of \( B_t \). In what follows the argument \( t \) in \( B_i(t) \) will be sometimes omitted for brevity.

We introduce \( U_t \) of the form (2.5), where each individual \( U_i(t) \) reads
\[
U_i(t) = \exp(i\phi_i(t)S_i^x)
\] (5.3)
and the phases \( \phi_i(t) \) are defined as
\[
\phi_i(t) = -\int_0^t B_i^x(t') dt'.
\] (5.4)
Then, provided that the magnetic field satisfies
\[
B_i^y(t) = B_i^y(0) \cos \phi_i(t) + B_i^z(0) \sin \phi_i(t),
\]
\[
B_i^z(t) = -B_i^y(0) \sin \phi_i(t) + B_i^z(0) \cos \phi_i(t),
\] (5.5)
the time-dependent Hamiltonian (5.1) can be mapped to the time-independent Hamiltonian
\[ \tilde{H} = H_I - \sum_i (B_y^i(0) S_y^i + B_z^i(0) S_z^i) \] (5.6)
by means of the gauge transformation (2.3).

Note that, in contrast to the previous cases, here we are able to eliminate the time dependence
of the magnetic field only for those \( B_i(t) \) that satisfy (5.5) and (5.4).

Note also that, in contrast to the previous cases, \( H_I \) and \( H_{I'} \) here do not form a finite closed
algebra with respect to commutation, and the technique of [25] cannot be utilized.

Remarkably, in the case of one-dimensional nearest-neighbor model, the Hamiltonian (5.6) is
integrable [36]. This motivates us to consider the latter case separately.

Nearest-neighbor one-dimensional Ising model. The Hamiltonian of this model reads
\[ H_I = \sum_i J_i S_i^x S_{i+1}^x - \sum_i B_i(t) S_i. \] (5.7)
Assume that the magnetic field satisfies
\[ B_y^i(t) = B^\perp_i(t) (B_y^i(0) \cos \phi_i(t) + B_z^i(0) \sin \phi_i(t)), \]
\[ B_z^i(t) = B^\perp_i(t) (-B_y^i(0) \sin \phi_i(t) + B_z^i(0) \cos \phi_i(t)), \] (5.8)
where
\[ B^\perp_i(t) = \sqrt{(B_y^i(t))^2 + (B_z^i(t))^2} \] (5.9)
and \( \phi_i(t) \) is given by (5.4). It should be emphasized that these conditions are less restrictive than
conditions (5.5). In particular, they allow for a time-dependent magnitude \( B^\perp_i(t) \) of the transverse
component of the magnetic field.

Under these conditions the gauge transformation (2.3) reduces the Hamiltonian (5.7) to the
Hamiltonian
\[ \tilde{H}_I = H_I - \sum_i B^\perp_i(t) (B_z^i(0) S_z^i + B_y^i(0) S_y^i), \] (5.10)
where the unitary transformation \( U_i \) is given by (2.5) with \( U_i \) defined in (5.3).

In contrast to the previous examples, here \( \tilde{H}_I \) is time-dependent. However, this Hamiltonian is
still tractable, since it can be mapped to a quadratic fermionic Hamiltonian with time-dependent
coefficients by means of the Jordan–Wigner transformation [10, 14, 34, 36]. This way we are able
to reduce the time-dependent many-body problem (5.7) to the one-body problem of a single fermion
in a time-dependent lattice.

Comment on instantaneous (non)integrability. Remarkably, the magnetic field in the
Hamiltonian (5.7) has, in general, a nonzero longitudinal component \( B_x^i(t) \). Consequently, this Hamiltonian is dynamically integrable despite being nonintegrable at a fixed \( t \). This contrasts to [1, 17, 20, 25, 44], where dynamical integrability is shown for Hamiltonians that are instantaneously
integrable at any moment of time.

Comment on localization. Assume that the magnetic field in \( \tilde{H} \) given by (5.6) is homogeneous,
\( B_y^i(0) = B_y^0, B_z^i(0) = B_z^0, \) and \( H_I \) does not contain any disorder (for example, it is
translation-invariant). Curiously, one still has a freedom to construct \( H_I \) that is many-body localized
at any fixed \( t \) by choosing appropriate disordered phases \( \phi_i(t) \) in the unitary transformation (5.3).
The lesson that we learn from this example is that even if some \( H_I \) is many-body localized at any
fixed \( t \), its dynamics may be reduced to the quench dynamics of a Hamiltonian without disorder
and localization.
Comment on driving through a critical point. In [14] the Ising model driven through a quantum critical point is studied and the Kibble–Zurek mechanism for the density of the created defects is verified. While a particular driving protocol is considered in [14], our method allows one to extend their analysis to a much wider range of paths in the parameter space crossing a phase transition. Note, however, that our method does not produce a counterdiabatic (or transitionless) driving, as in [5, 40, 43] (see the discussion in Section 2).

6. SPIN COUPLED TO BOSONIC ENVIRONMENT

Here we consider a spin in a time-dependent magnetic field coupled to a bosonic environment, the total Hamiltonian being

\[ H_t = -B_t \mathbf{S} + S^x \sum_k f_k (a_k^\dagger + a_k) + \sum_k \omega_k a_k^\dagger a_k. \tag{6.1} \]

Here \( a_k \) is the operator annihilating the bosonic mode with the quantum number \( k \), \( \omega_k \) is the energy of the corresponding mode, and \( f_k \) is the spin–boson coupling (it usually scales with the system size \( L \) as \( 1/\sqrt{L} \)). In the case of spin 1/2 this Hamiltonian can describe a qubit coupled to bosonic modes.

In complete analogy with the Ising model, we find that the Hamiltonian \( H_t \) given by (6.1) can be reduced to the time-independent

\[ \tilde{H} = -(B_0^y S_0^y + B_0^z S_0^z) + S^x \sum_k f_k (a_k^\dagger + a_k) + \sum_k \omega_k a_k^\dagger a_k \tag{6.2} \]

by means of the gauge transformation (2.3) with \( U_t \) given by (5.3), provided the magnetic field satisfies conditions (5.5) (where the indices \( i \) in the spin and magnetic field variables should be omitted).

7. SUMMARY

To summarize, we have presented a method to map certain time-dependent many-body Hamiltonians to time-independent Hamiltonians or to more simple (for example, dynamically integrable) time-dependent Hamiltonians while preserving the few-body nature of interactions. The method is based on a gauge transformation (2.3) with a unitary operator of a product form (2.5) satisfying condition (2.6). We have applied this general method to eliminate time-dependent magnetic fields from Heisenberg and Ising quantum spin systems, from a system of a spin coupled to a bosonic environment, as well as from a system of interacting fermions. The method opens new perspectives to study dynamical integrability, Floquet dynamics of periodically driven systems, and driving through quantum critical points.

APPENDIX A. ELIMINATING MAGNETIC FIELD: COVARIANT PARAMETRIZATION

Here we outline the derivation of equations (3.4). First we use a formula for the derivative of the exponential map [3] to obtain an integral representation of \( W_t \):

\[ W_t = i e^{ix \mathbf{K}_t \mathbf{S}_{\text{tot}}} \frac{d}{dt} e^{-ix \mathbf{K}_t \mathbf{S}_{\text{tot}}} = \frac{1}{ix} \int_0^1 dx e^{ix \mathbf{K}_t \mathbf{S}_{\text{tot}}} \mathbf{K}_t \mathbf{S}_{\text{tot}} e^{-ix \mathbf{K}_t \mathbf{S}_{\text{tot}}}. \tag{A.1} \]

To proceed further, we use

\[ e^{ia \mathbf{S}} e^{-ia \mathbf{S}} = \frac{1}{a^2} (ab)(a\mathbf{S}) - \frac{1}{a} (ab \mathbf{S}) \sin a + \left( (b \mathbf{S}) - \frac{1}{a^2} (ab)(a\mathbf{S}) \right) \cos a \tag{A.2} \]
valid for arbitrary vectors \( \mathbf{a} \) and \( \mathbf{b} \) and arbitrary spin \( \mathbf{S} \). Here \((\mathbf{ab})\) denotes the scalar product and \((\mathbf{ab}\mathbf{S})\) denotes the scalar triple product. With the help of this formula, (A.1) can be explicitly integrated. Taking into account that \( U_t H U_t^\dagger = H \) and, consequently, \( B_t S_{tot} = W_t \), according to (2.3) and (3.1)–(3.3) we obtain the equation

\[
B_t = \sin \frac{K}{K} \mathbf{K}_t - \frac{1 - \cos \frac{K}{K^2}}{K^2} \mathbf{K}_t \times \mathbf{K}_t + \frac{1}{K^2} \left( 1 - \frac{\sin \frac{K}{K}}{K} \right)(\mathbf{K}_t \mathbf{K}_t) \mathbf{K}_t. \tag{A.3}
\]

Introducing \( \mathbf{K}_t = K_t \mathbf{n}_t \), where \( \mathbf{n}_t \) is a unit vector, one reduces (A.3) to

\[
B_t = \dot{K}_t \mathbf{n}_t + \sin K_t \dot{\mathbf{n}}_t - (1 - \cos K_t)(\mathbf{n}_t \times \dot{\mathbf{n}}_t). \tag{A.4}
\]

By performing a scalar (vector) multiplication of this equation by \( \mathbf{n}_t \) (and doing some additional algebra in the second case), one obtains the first (second) equation in (3.4). Note that equations (A.3) and (A.4) can be more suitable for numerical integration than (3.4).

### Appendix B. Eliminating Magnetic Field: Gauss Parametrization

The Gauss parametrization of \( U_t \) reads [39]

\[
U_t = \exp(\xi^+_t S^+_t) \exp(\xi^-_t S^-_t) \exp(\xi^z_t S^z_t), \tag{B.1}
\]

where \( S^\pm_{tot} = S^x_{tot} \pm i S^y_{tot} \). This operator is unitary whenever

\[
\xi^+ = -(\xi^-)^* e^{i \text{Im} \xi^z} \quad \text{and} \quad |\xi^-|^2 + 1 = e^{\text{Re} \xi^z}. \tag{B.2}
\]

Note that the first condition above implies \( |\xi^+| = |\xi^-| \).

Equation (2.3) leads to the following differential equations for the functions \( \xi^+_t \) and \( \xi^-_t \):

\[
i \dot{\xi}^+_t = B^-_t (\xi^+_t)^2 - B^+_t \xi^+_t - B^+_t, \]

\[
i \dot{\xi}^-_t = 2B^-_t \xi^+_t - B^+_t, \tag{B.3}
\]

\[
i \dot{\xi}^z_t = -B^-_t \exp(\xi^z_t),
\]

where \( B^\pm_t \equiv (B^x_t \mp i B^y_t)/2 \) (this definition implies \( B_t S_{tot} = B^+_t S^+_t + B^-_t S^-_t + B^z_t S^z_t \)). The initial condition is \( \xi^+_0 = \xi^-_0 = 0 \). In a somewhat different context, the system of equations (B.3) was derived in [39] following the lines of the earlier work [46]. It can be verified that these equations are consistent with conditions (B.2). Note that the first equation in (B.3) is a Riccati equation with a single variable, and the other two are trivially integrated when the solution of the first one is plugged in. These equations are somewhat simpler than the equivalent system (3.4). In addition, the equations in (B.3) have nonsingular right-hand sides, which makes it clear that the solution exists for an arbitrary \( B_t \).

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