PERMANENCE PROPERTIES OF THE SECOND NILPOTENT PRODUCT OF GROUPS

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Abstract. We show that Amenability, the Haagerup Property, the Kazhdan’s Property (T) and Exactness are preserved under taking second nilpotent product of groups. We also define the restricted second nilpotent wreath product of groups, this is a semi-direct product akin to the restricted wreath product but constructed from the second nilpotent product. We then show that if two discrete groups have the Haagerup Property, the second nilpotent wreath product of them also has the Haagerup Property. We finally show that if a discrete group is abelian then the second nilpotent wreath product constructed from it is unitarizable if and only the acting group is amenable.

1. Introduction

Given a family of groups, the direct product and the free product provide ways of constructing new groups out of them. Even though both operations are quite different, they share the next common properties:

(1) Associativity.

(2) Commutativity.

(3) The product contains subgroups which generate the product.

(4) These subgroups are isomorphic to the original groups.

(5) The intersection of a given one of these subgroups with the normal subgroup generated by the rest of these subgroups is the identity.

In the first edition of his classic book “Theory of Groups” Kurosh asked if there were other operations on a family of groups that satisfy the above properties. This problem was solved in the affirmative by Golovin in [8] where he defined countably many such operations. Among these operations, the second nilpotent product stands out as the simplest to scrutinize. It is defined as follows:
1.1. **Definition.** For a family of groups \( \{H_i\}_{i \in I} \) indexed on a set \( I \), the second nilpotent product of the family is the group:

\[
\ast_2 H_i \overset{\text{def}}{=} \ast_{i \in I} H_i / \langle \prod_{i \in I} [H_i,H_j] \rangle_{j \neq i}
\]

The importance of having such operation resides in that it provides a “new” way of constructing groups. Unlike the direct product, the second nilpotent product of a family of abelian groups does not need to be abelian. On the other hand, the second nilpotent product of finitely many finite groups is necessarily a finite group.

Second nilpotent products of groups have an interesting application in Mathematical Logic. Indeed, they were used by Mekler in [12] to show that the isomorphism relation of contable groups are model complete for countable structures. This was later framed in the context of Borel Reducibility in the pioneering article of Friedman and Stanley [7]. Inspired by these results, a variant of a construction of Mekler involving certain semi-direct product related to the standard wreath product was developed by Törnquist and the author in [15, Section 5] to show non classification results for von Neumann algebras. The original motivation of the present article was to further analyze that construction. Here we study some permanence properties of the second nilpotent product of groups that come from representation theory and dynamics of group actions. Our first result is:

1.2. **Theorem (A).** Amenability, the Haagerup’s Approximation Property, the Kazhdan’s Property (T) and Exactness are preserved under taking second nilpotent product of two groups.

In geometric and measurable group theory, wreath products have been playing a significant role to produce examples of groups that verify or disprove long standing conjectures. Since the second nilpotent product allows a construction similar to the wreath product (that we shall call **restricted second nilpotent wreath product**) it is conceivable that it could be used in a similar manner, as it was manifested in [15]. To further illustrate this we present the following adaptation of a theorem of Cornulier, Stalder & Valette in [4, 5].

1.3. **Theorem (B).** Let \( G \) and \( H \) be countable groups with the Haagerup Property. Then the second nilpotent wreath product \( \left( \ast_{g \in G} H_g \right) \rtimes G \) has the Haagerup Property.

Wreath products have been used by Monod & Ozawa in [13] to give examples of non unitarizable groups that do not contain a copy of \( \mathbb{F}_2 \). Recall a group \( G \) is unitarizable if for every uniformly bounded representation \( \pi \) of \( G \) on a Hilbert space \( \mathcal{H} \), there exists \( T \in B(\mathcal{H}) \) such that \( T\pi T^{-1} \) is a unitary representation. Dixmier showed that
amenable groups are always unitarizable [6]. The Dixmier problem asks whether the converse holds (See for instance [14]). It will be an immediate corollary of our construction combined with in [13, Theorem 1] the following:

1.4. **Theorem (C).** Let $G$ and $A$ be countable groups with $A$ abelian. Then the second nilpotent wreath product $\left( \underset{g \in G}{\star} A_g \right) \rtimes G$ is unitarizable if and only if $G$ is amenable.

We will see that for an abelian group $A \neq \{1\}$, the group $\underset{g \in G}{\star} A_g$ is never abelian (unless $G = \{1\}$), and that $\underset{g \in G}{\star} \mathbb{Z}/p\mathbb{Z}$ is a nil-2 $p$-group. Thus our examples are apparently different from the ones that already appeared in the literature.

To the best of our knowledge, the papers of Golovin have not been studied much in recent years. A partial indication of this is the scarce number of citations they have. Thus another goal of this article is to bring back the second nilpotent product as a useful construction of groups.

2. **The second nilpotent product of two groups**

We record several properties of the second nilpotent product of two groups. Some of the results of this section can be found (at least implicitly) in the papers of Golovin [8, 9]. However, since that articles are not widely available and the proofs given there are somewhat obscured by the lack of modern terminology, we opted to provide short proofs of them.

2.1. **Convention.** In this article we adopt the convention $[a, b] = aba^{-1}b^{-1}$.

2.2. **Definition.** Given two groups $A$ and $B$, their second nilpotent product is the group

$$A \star B \overset{\text{def}}{=} A \ast B / [A \ast B, [A, B]]$$

In words, this means that we start from the free product and declare that all the commutators $[a, b], a \in A, b \in B$ are central. Historically, the construction of the second nilpotent product of (exactly) two groups first appeared in a paper of Levi [10]. Golovin, unaware of Levi’s work, treated the general case in [8, 9] to solve the problem of Kurosh mentioned in the introduction.

2.3. **Notation.** We denote with $[A, B]^{(2)}$ the subgroup of $A \star B$ generated by the commutators $[a, b], a \in A, b \in B$. 

2.4. **Proposition.** An element \( x \in A \hat{\ast} B \) admits a unique representation \( abc \) where \( a \in A, b \in B \) and \( c \in [A, B]^{(2)} \).

*Proof.* \( x \) can be represented as a word with letters in \( A \) and \( B \).

\[
x = a_1 b_1 a_2 b_2 ... a_n b_n
\]

The identity \( ba = [b, a]ab \) allows us to take all the \( a_i \) to the left. Having in mind that \([b_j, a_i]\) are central in \( A \hat{\ast} B \), we obtain \( x = a_1 ... a_n b_1 ... b_n c \) with \( c \in [A, B]^{(2)} \). Uniqueness follows easily applying the projections \( A \hat{\ast} B \xrightarrow{\pi_A} A \) and \( A \hat{\ast} B \xrightarrow{\pi_B} B \). \( \square \)

2.5. **Proposition.** \( B \xrightarrow{[a,-]} [A, B]^{(2)} \) is a group homomorphism for every element \( a \in A \).

*Proof.* Let \( b, c \in B \). We have the following equations in \( A \hat{\ast} B \).

\[
[a, bc] = [a, b][a, c]b^{-1} = [a, b][a, c]
\]

\( \square \)

2.6. **Corollary.** \( A \) commutes with \([B, B]\) inside \( A \hat{\ast} B \).

*Proof.* This follows from Proposition 2.5 together with the identity

\[
a[b, b'][a^{-1}] = [b, b'][[b, b']^{-1}, a].
\]

\( \square \)

In order to better understand the second nilpotent product it is convenient to use the tensor product of groups (not necessarily abelian). This tensor product and the corresponding results were already introduced by Whitney in \([17]\).

2.7. **Definition.** Let \( A \) and \( B \) be two groups. The tensor product \( A \tilde{\otimes} B \) is defined as

\[
\mathbb{F}_{A \times B} / \langle (a_1, a_2, b) \sim (a_1, b)(a_2, b), (a, b_1 b_2) \sim (a, b_1)(a, b_2) \rangle
\]

where \( \mathbb{F}_{A \times B} \) denotes the free group generated by the set \( A \times B \).

The natural application \( A \times B \rightarrow A \tilde{\otimes} B \) is a group homomorphism in each variable, and the following universal property holds:

\[
\begin{tikzcd}
A \times B \arrow{r}{\varphi} \arrow[swap]{d}{\sim} & \tilde{\otimes} B \arrow{r}{\exists!} & G
\end{tikzcd}
\]

for every group \( G \) and \( \varphi \) a morphism in each variable.
2.9. **Remark.** We use the provisory notation \( \hat{\otimes} \) because for abelian groups this definition differs *a priori* from the usual tensor product construction, where one takes the quotient from the free abelian group generated by \( A \times B \). However, we will see that in the abelian case it coincides with the usual tensor product of abelian groups.

The following proposition is due to Whitney [17, Theorem 11]. It was also reproved by MacHenry [11, Theorem 17] in the same context as ours. For the sake of completeness, we provide a simple proof of it.

2.10. **Proposition** (Whitney). \( A \hat{\otimes} B \) is an abelian group. Actually, it is equal to \( A/[A,A] \otimes B/[B,B] \). Thus the tensor product between nonabelian groups is just the usual tensor product between the abelianized groups.

**Proof.**

\[
(a_1a_2)(b_2b_1) = (a_1a_2)(a_1b_2)(a_2b_1) = (a_1b_2)(a_2b_1) = (a_1b_1)(a_2b_1) = (a_1b_1)(a_2b_1)
\]

Then \( (a_1b_1)(a_2b_1) = (a_2b_1)(a_1b_1) \), so it is abelian.

The natural arrow \( A \times B \rightarrow A/[A,A] \otimes B/[B,B] \) is a morphism in each variable and it is easy to check that it has universal property (2.8) but only for \( G \) abelian. As the abelian group \( A \hat{\otimes} B \) has the same universal property, they are isomorphic through the canonical morphism. □

We will need the next result about commutator subgroups of the free product of two groups. Its proof is elementary and can be found for instance in [16, Section 1.3, Proposition 4].

2.11. **Lemma.** \( [A, B] = \langle \{(a, b) \}_{a \in A, b \in B} \rangle \) is a free subgroup of \( A \ast B \) in the generators \( [a, b], a \in A, b \in B, a, b \neq 1 \). Moreover it is normal in \( A \ast B \).

The following Proposition is the main result of the article of MacHenry [11]. Arguably, the proof we exhibit here using universal properties is simpler than the one given in [11].

2.12. **Proposition.** \( [A, B]^{(2)} = A \otimes B \).

**Proof.** Observe first that since \( [A, B] \) is a normal subgroup of \( A \ast B \), then \( [A \ast B, [A, B]] \) is a subgroup of \( [A, B] \). The identity \( x[y, z]x^{-1} = [x^y, x^z] \) shows that it is normal. It follows that \( [A, B]^{(2)} = [A, B]/[A \ast B, [A, B]] \).

Let us give explicitly the isomorphisms \( [A, B]^{(2)} \xrightarrow{u} A \otimes B \). The application \( A \times B \rightarrow [A, B]^{(2)}, (a, b) \mapsto [a, b] \) is a morphism in each variable because of Proposition 2.5. This defines \( u \).

On the other hand the map \( [A, B] \xrightarrow{\tilde{u}} A \otimes B, [a, b] \mapsto a \otimes b \) extends to a group morphism thanks to the previous lemma. Thus in order to define \( u \) we just need to show that \( \tilde{u} \) vanishes on \( [A \ast B, [A, B]] \). For this
take an element \([p, q] \in [A \ast B, [A, B]]\) with \(p \in A \ast B\) and \(q \in [A, B]\). 
\[
\hat{u}(p, q) = \hat{u}(pq^{-1}q^{-1}) = \hat{u}(pq^{-1})\hat{u}(q^{-1}).
\]
Thus we must show that \(\hat{u}(pq^{-1}) = \hat{u}(q)\). If \(x \in A\):
\[
\hat{u}(x[a, b]x^{-1}) = \hat{u}((xa, b)[b, x]) = (xa \otimes b)(x \otimes b)^{-1} = 
= (x^{-1} \otimes b)(xa \otimes b) = a \otimes b = \hat{u}([a, b]).
\]
A similar argument works for \(x \in B\). Induction on the length of a word \(x \in A \ast B\) shows that \(\hat{u}(x[a, b]x^{-1}) = \hat{u}([a, b]).\) Since \([\{a, b : a \in A, b \in B\}]\) generates \([A, B]\) it follows that \([A \ast B, [A, B]] \subset \text{Ker}(\hat{u})\). In particular \([A, B]^{(2)} \to A \otimes B\) is well defined. It is clear now that \(uv = id\) and \(vu = id\). \(\square\)

This result is very useful. For instance, combined with Proposition 2.4, immediately implies that the second nilpotent product of two finite groups is finite. In fact it even allows to compute its order.

2.13. Corollary. \(|A \hat{\ast} B| = |A| |B| |A \otimes B|\). In particular if a group \(A\) is perfect, i.e. \(A = [A, A]\), then for any group \(B\) the second nilpotent product \(A \hat{\ast} B\) is equal to the direct product \(A \oplus B\).

Proof. By Proposition 2.12, if \(A\) is perfect then \(A \otimes B = \{e\}\). \(\square\)

The next Proposition identifies the derived group in the second nilpotent product of two groups.

2.14. Proposition. Let \(G = A \hat{\ast} B\). Its commutator subgroup \([G, G]\) is isomorphic to \([A, A] \oplus [B, B] \oplus [A, B]^{(2)}\). In particular, \(G\) is abelian only when both \(A\) and \(B\) are abelian and \(A \otimes B\) is trivial.

Proof. A commutator element in \(G\) is \([a_1b_1c_1, a_2b_2c_2]\) where \(a_i \in A, b_i \in B, c_i \in [A, B]^{(2)}\). It is equal to \([a_1, a_2][b_1, b_2]c\) for some \(c \in [A, B]^{(2)} \subset Z(G)\), (in fact \(c = [a_1, b_2][b_1, a_2]\)). The multiplication of two such elements:
\[
[a_1, a_2][b_1, b_2]c[a_3, a_4][b_3, b_4]d = [a_1, a_2][a_3, a_4][b_1, b_2][b_3, b_4]cd
\]
is exactly the multiplication in the direct product (have in mind that, by Corollary 2.6, \([B, B]\) commutes with \(A\) inside \(G\)). The subgroups \([A, A], [B, B], [A, B]^{(2)}\) are contained in \([G, G]\), so we have an explicit isomorphism \([A, A] \oplus [B, B] \oplus [A, B]^{(2)} \to [G, G]\). \(\square\)

2.15. Examples.

1. If \(p, q \in \mathbb{N}\) are coprime numbers, then \(\mathbb{Z}/p\mathbb{Z} \hat{\ast} \mathbb{Z}/q\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}\). This is just because whenever \((p, q) = 1\) the only \(\varphi\) that verifies the condition of the diagram (2.8) is \(\varphi = 0\). This means in particular that \(\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z} = 0\). Propositions 2.4 and 2.12 yields the desired result.
(2) More generally, if \( p, q \in \mathbb{N} \) and \( d = \gcd(p, q) \), it follows that 
\[ \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z} = \mathbb{Z}/d\mathbb{Z} \]. Thus the order of \( \mathbb{Z}/p\mathbb{Z} \ast \mathbb{Z}/q\mathbb{Z} \) is \( pqd \).

(3) \( \mathbb{Z}/n\mathbb{Z} \ast \mathbb{Z}/n\mathbb{Z} \) is isomorphic to the Heisenberg group:

\[
\text{Heis}(\mathbb{Z}/n\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}/n\mathbb{Z} \right\}
\]

A straightforward way to see this is to verify that the function

\[
\Psi : \mathbb{Z}/n\mathbb{Z} \ast \mathbb{Z}/n\mathbb{Z} \to \text{Heis}(\mathbb{Z}/n\mathbb{Z})
\]

\[
\Psi(abc) = \begin{pmatrix} 1 & a & -c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}
\]

is a group isomorphism (here of course we are using Proposition 2.4). Observe that in particular \( \mathbb{Z}/n\mathbb{Z} \ast \mathbb{Z}/n\mathbb{Z} \) is a non abelian group of order \( n^3 \) and when \( n = 2 \) this is also isomorphic to the dihedral group \( D_4 \). The same strategy shows that \( \mathbb{Z} \ast \mathbb{Z} \) is isomorphic to the Heisenberg group \( \text{Heis}(\mathbb{Z}) \).

(4) Denote with \( D_n \) the dihedral group of order \( 2n \), namely the group with presentation \( \langle a, b | a^n = 1, b^2 = 1, bab = a^{-1} \rangle \). It is plain that \( [D_n, D_n] = \langle a^2 \rangle \). If \( n \) is even \([D_n, D_n]\) has order \( n/2 \) in which case \( D_n/[D_n, D_n] = \langle \bar{a}, \bar{b} \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). Thus for \( n \) even \( D_n \ast D_n \) is a (non abelian) group of order \( 2^6n^2 \) and its derived subgroup is isomorphic to \( \oplus_1^2(\mathbb{Z}/4\mathbb{Z}) \oplus_1^4(\mathbb{Z}/2\mathbb{Z}) \). If \( n \) is odd a similar analysis shows that \( D_n \ast D_n \) is a (non abelian) group of order \( 2^3n^2 \) and its derived subgroup is isomorphic to \( \oplus_1^2\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

2.16. Remark. Example (2) and Proposition 2.5 can be used to prove that inside \( \mathbb{Z}/4\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \) the subgroup generated by the element of order 2 in \( \mathbb{Z}/4\mathbb{Z} \) together with \( \mathbb{Z}/2\mathbb{Z} \) is abelian and isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). This shows that if \( \tilde{A} \triangleleft A \) and \( \tilde{B} \triangleleft B \) the group \( \langle \tilde{A}, \tilde{B} \rangle \triangleleft A \ast B \) is not isomorphic to \( \tilde{A} \ast \tilde{B} \).

For some of the proofs in the following section it shall be useful to know that the second nilpotent product can also be viewed as certain central extension. What follows is not present in the articles of Golovin.

Recall first that if \( K \) and \( H \) are groups with \( K \) abelian, a 2-cocycle of \( H \) with coefficients in \( K \) is a function \( \mu : H \times H \to K \) such that

\[
\mu(g, h)\mu(gh, k) = \mu(g, hk)\mu(h, k) \text{ for all } g, h, k \in H.
\]

The central extension of \( H \) with cocycle \( \mu \) is the group \( H \times_\mu K \) where the underlying set is

\[
H \times K,
\]
the multiplication is given by
\[(g_1, a_1)(g_2, a_2) = (g_1g_2, \mu(g_1, g_2)a_1a_2),\]
and the identity is
\[(1, \mu(1, 1)^{-1}).\]

One then gets the short exact sequence \(0 \to K \to H \times_\mu K \to H \to 0\) and \(K\) is injected into the center of \(H \times_\mu K\). Notice also that an element \(k \in K\) is mapped to \((1, \mu(1, 1)^{-1}k)\).

In the case of the second nilpotent product of two groups \(A\) and \(B\), it is easy to check that the function
\[\mu : (A \oplus B) \times (A \oplus B) \to A \otimes B\]
\[\mu((a_1, b_1), (a_2, b_2)) = \mu(a_2, b_1)^{-1}\]
satisfies the 2-cocycle relation. Thus \(A \ast B\) is the central extension of \(A \oplus B\) induced by the 2-cocycle \(\mu\). In other words \(A \ast B = A \times B \times (A \otimes B)\) as a set with the product given by \((a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1a_2, b_1b_2, (a_2 \otimes b_1)^{-1}c_1c_2)\). And one gets the short exact sequence
\[(2.17) 0 \to A \otimes B \to A \ast B \to A \oplus B \to 0.\]

3. PERMANENCE PROPERTIES OF THE SECOND NILPOTENT PRODUCT OF TWO GROUPS

The purpose of this section is to prove Theorem (A) from the introduction. We rephrase it here.

3.1. Proposition. Let \(A\) and \(B\) be countable groups. \(G = A \ast B\) has one of the following properties:

1. Nilpotent,
2. Amenable,
3. Haagerup Approximation Property,
4. Exact (or boundary amenable, or satisfies property A of Yu),
5. Kazhdan Property (T),

if and only if \(A\) and \(B\) have the same property.

3.2. Remark.

(i) We refer to [1, 2, 3] for the definitions and thorough treatments of the properties of groups stated above.

(ii) Golovin showed that the second nilpotent product preserves nilpotency. For the sake of completeness we include a short proof here.

(iii) Except for nilpotency, Proposition 3.1 is obvious and not interesting when the groups at hand are finite.

Proof. Since \(A\) and \(B\) are subgroups of \(G\) and the properties (1), (2), (3) and (4) are inherited by subgroups, it follows that if \(G\) satisfies one of these three properties then both \(A\) and \(B\) must also satisfy it. The
fact that Property (T) is inherited by quotients (see [1, Theorem 1.3.4]) tells that if $G$ satisfies (5) then $A \simeq G/\overline{A}$ and $B \simeq G/\overline{B}$ must also satisfy (5), where $\overline{A}$ is the normal closure of $A$ in $G$. We are now left to show the reverse implications.

To prove (1), let $h \in [G,G]$ and $g \in G$, $g = abc$ as in Proposition 2.4. By Proposition 2.14, $h$ is of the form $h = a_1b_1c_1$, with $a_1 \in [A,A]$, $b_1 \in [B,B]$, $c_1 \in [A,B]^{(2)}$. Then $[g,h] = [abc,a_1b_1c_1] = [ab,a_1][b,b_1]$. This means that the third term in the lower central series of $G$, $G_3 = [G,[G,G]]$, equals $[A,[A,A]] \oplus [B,[B,B]] = A_3 \oplus B_3$. It follows by induction that for $n > 2$ the $n^{th}$ term of the lower central series of $G$ is equal to $A_n \oplus B_n$. In particular, if $A$ and $B$ are nilpotent groups of classes $n$ and $m$, then $G$ is either abelian or nilpotent of class $\max\{2, n, m\}$.

To prove (2) we consider the short exact sequence in (2.17). If $A$ and $B$ are amenable, then $A \oplus B$ is amenable. Since $A \varotimes B$ is abelian, by [1, Proposition G.2.2] $G$ is amenable.

To prove (3) recall that the Haagerup Property is preserved by taking subgroups and finite direct products, thus by Proposition 2.14 the subgroup $[G,G]$ has the Haagerup Property if both $A$ and $B$ have it. Since $G/[G,G]$ is abelian, and extensions with amenable quotients preserve the Haagerup Property ([3, Example 6.1.6]), then $G$ has the Haagerup Property.

To show (4), we consider the short exact sequence in (2.17). Then recall that abelian groups are exact, and that subgroups and extensions of exact groups are exact [2, Proposition 5.1.11].

Finally, to show (5), assume that both $A$ and $B$ have the Property (T). Their abelianizations are finite groups. Then by Proposition 2.10, $A \varotimes B$ is a finite group. In particular both ends of the short exact sequence (2.17) have the Property (T). We apply then of [1, Proposition 1.7.6] to obtain (5).

□

3.3. Remark. Proposition 2.4, Proposition 2.12 together with the proof of (1) shows that the 2-nilpotent product of finite abelian $p$-groups is a finite $p$-group of nilpotency class 2.

It might seem that (5) could be used to construct property (T) groups with large center. Unfortunately this is not the case since we proved that if $A$ and $B$ have the property (T) the group $[A,B]^{(2)}$ is finite.

4. Second nilpotent product indexed by a set

In this section we consider an index set $\mathcal{I}$ and for each $i \in \mathcal{I}$ a group $H_i$. Recall from the introduction the next:
4.1. **Definition.** For a family of groups \( \{H_i\}_{i \in I} \) indexed on a set \( I \) the second nilpotent product of the family is the group

\[
\bigstar_{i \in I} H_i = \bigstar_{i \in I} H_i / \langle \bigstar_{i \in I} [H_i, H_j] \rangle_{i \neq j}.
\]

The remainder of this section presents several facts about the 2-nilpotent product of arbitrary many groups that will be needed to prove Theorem B. This will also enable us to give a short proof of the associativity of the second nilpotent product, the only one among the five properties listed in the introduction and proved by Golovin that does not follow immediately from the definitions.

4.2. **Proposition.** The second nilpotent product is functorial. Explicitly, a family of morphisms \( H_i \to K_i \) induce a natural morphism \( \bigstar_{i \in I} H_i \to \bigstar_{i \in I} K_i \).

**Proof.** Functoriality is a straightforward consequence of the functoriality of the free product. \( \square \)

A universal property for the second nilpotent product is the following:

\[
\begin{array}{c}
H_i \\
\bigstar_{i \in I} H_i \\
\end{array} \rightarrow \begin{array}{c} G \\
\bigstar_{i \in I} H_i \end{array}
\]

where \( G \) is a group and the morphisms \( r_i \) verify \( [r_i(g), r_j(h)] \in Z(G) \) if \( i \neq j \).

The group generated by the commutators \( [h_i, h_j] \) with \( h_i \in H_i, h_j \in H_j, i \neq j \) is central in \( \bigstar_{i \in I} H_i \). Fixing a total order on \( I \) we have that

\[
\langle [h_i, h_j] | h_i \in H_i, h_j \in H_j, i \neq j \rangle = \langle [H_i, H_j]^{(2)} | i \neq j \rangle = \bigoplus_{i < j} [H_i, H_j]^{(2)}.
\]

This immediately implies the next generalization of Proposition 2.12.

4.4. **Proposition.**

\[
\langle [h_i, h_j] | h_i \in H_i, h_j \in H_j, i \neq j \rangle = \bigoplus_{i < j} [H_i, H_j]^{(2)} = \bigoplus_{i < j} H_i \otimes H_j \subset Z(\bigstar_{i \in I} H_i).
\]
Observe that because of functoriality, for any subset $S \subset I$ the natural projection $\bigast_{i \in I} H_i \xrightarrow{\pi_S} \bigast_{i \in S} H_i$ is a well defined group homomorphism. Moreover the projection $\pi_S(x)$ can be computed in any word $x$ by erasing all letters belonging to a group $H_i$ whose index $i \notin S$. Thus, for $S \subset J \subset I$ the composition of the two natural projections coincides with projecting directly to $S$.

For two elements $i \neq j \in I$ we denote with $\pi_{(i,j)}$ the projection $\bigast_{l \in I} H_l \xrightarrow{\pi_{(i,j)}} H_i \bigast H_j$.

4.5. Proposition. Fixing a total order in $I$, every element $x \in \bigast_{i \in I} H_i$ admits a unique representation

$$x = a_{i_1} a_{i_2} \ldots a_{i_l} \omega$$

where $a_{i_k} \in H_{i_k}$, $i_1 < i_2 < \ldots < i_l \in I$, and $\omega \in Z(\bigast_{i \in I} H_i)$ is of the form

$$\omega = \prod_{i_k < i_r \quad r \leq l} c_{i_k, i_r}$$

with $c_{i_k, i_r} \in [H_{i_k}, H_{i_r}]^{(2)}$.

Proof. Existence: Start from a word $P$ that represents $x$. By rearranging the elements (adding the corresponding commutators) we can obtain such a representation. Uniqueness: by Proposition 2.4, the projection $\pi_{(i_k, i_r)}(x)$ determines $a_{i_k}$, $a_{i_r}$ and $c_{i_k, i_r}$.

This result combined with Proposition 4.4 can be used to compute the order of the second nilpotent product of finitely many finite groups.

4.6. Example. $\bigast_{1 \leq i \leq n} \mathbb{Z}/p\mathbb{Z}$ is the universal nil-2 exponent $p$ group in $n$ generators. It has order $p^{\frac{n^2-n}{2}}$, and its derived subgroup has order $p^{\frac{n^2-n}{2}}$ and it is isomorphic to $\bigoplus_{1 \leq i < j \leq n} \mathbb{Z}/p\mathbb{Z}$.

4.7. Proposition. Let $J \subset I$. For $x \in \bigast_{j \in J} H_j$, the commutator $[x, -]$ defines a group homomorphism

$$\bigast_{i \in I \setminus J} H_i \xrightarrow{[x,-]} \bigoplus_{i \in I \setminus J, j \in J} [H_i, H_j]^{(2)} \subset Z(\bigast_{i \in I} H_i).$$

Proof. Since the identity $[x, yz] = [x, y]y[x, z]y^{-1}$ is always valid, it is enough to show that $[x, z] \in Z(\bigast_{i \in I} H_i)$ whenever $x \in \bigast_{j \in J} H_j$ and $z \in \bigast_{i \in I \setminus J} H_i$. Assume first that $x \in H_j$, $j \in J$. Any $z \in \bigast_{i \in I \setminus J} H_i$...
can be represented as a finite word \( z_{i_1}z_{i_2}\ldots z_{i_l} \), where for all \( 1 \leq k \leq l \), \( z_{i_k} \in H_{i_k} \) and \( i_k \notin J \). Since for all \( k \), \([x, z_{i_k}] \in Z(\bigodot_{i \in I} H_i)\), induction on \( l \) shows that \([x, z] = [x, z_{i_1}z_{i_2}\ldots z_{i_l}] = [x, z_{i_1}z_{i_2}\ldots z_{i_{l-1}}] [x, z_{i_l}] \in Z(\bigodot_{i \in I} H_i)\). Repeating the same induction argument but now for \( x \in \bigodot_{j \in J} H_j \) finishes the proof.

\[\square\]

We can now give a short proof of the associativity of the second nilpotent product.

4.8. Proposition. [8, Golovin] Let \( S = \bigsqcup_{i \in I} S_i \) be a disjoint union of index sets, and \( H_j \) a group for each \( j \in S \). Then
\[
\bigodot_{j \in S} H_j \simeq \bigodot_{i \in I} \left( \bigodot_{j \in S_i} H_j \right)
\]

Proof. The isomorphism will be given by the identity on each \( H_j \). As these generate both groups, the only nontrivial fact is that they are well defined. We will induce them with the help of the universal property of the second nilpotent product. Natural inclusions give the following diagram:

For \( u \) to be well defined, we must check that for every \( \alpha \in H_{k_1}, \beta \in H_{k_2} (k_1 \neq k_2) \), \([l_{i(k_1)}]_{k_1} t_{i(k_2)} l_{k_2}(\alpha), t_{i(k_2)} l_{k_2}(\beta)]\) belongs to the center of \( \bigodot_{j \in S_i} H_j \). In case \( i(k_1) \neq i(k_2) \), it is trivial. If \( i(k_1) = i(k_2) \), our element commutes trivially with every \( H_j \) for \( i(j) = i(k_1) \), but it also commutes with \( H_j \) for \( i(j) \neq i(k_1) \), due to proposition 4.7.

Now for \( v \), we must check \([s_{i_1}(\gamma), s_{i_2}(\delta)] \in Z(\bigodot_{j \in S} H_j)\), where \( \gamma \in \bigodot_{j \in S_i} H_j, \delta \in \bigodot_{j \in S_{i_2}} H_j, i_1 \neq i_2 \). Say \( \gamma = *\gamma_n, \delta = \delta_m \), with each \( \gamma_n, \delta_m \) a letter in a group \( H_j \) (* denotes a noncommutative finite product). By proposition 4.7, we have \([s_{i_1}(\gamma), s_{i_2}(\delta)] = \prod[\gamma_n, \delta_m] \), so it clearly belongs to the center. \[\square\]
4.9. **Corollary.** If \( \{H_i\}_{i \in I} \) is a countable family of discrete amenable (respectively Haagerup, resp. Exact) groups, the group \( \bigstar_{i \in I} H_i \) is amenable (resp. Haagerup, resp. exact).

**Proof.** If \( I \) is finite, the result follows from associativity together with Proposition 3.1. If \( I = \mathbb{N} \), then

\[
\bigstar_{i \in \mathbb{N}} H_i = \bigcup_{n \in \mathbb{N}} \bigstar_{i \in \{1, 2, \ldots, n\}} H_i.
\]

Amenability, the Haagerup Property and Exactness are preserved under countable increasing unions of discrete groups (see [1, Proposition G.2.2] [3, Proposition 6.1.1] and [2, Exercise 5.1.1]).

\[\square\]

5. **SECOND NILPOTENT WREATH PRODUCTS**

Fix \( H \) and \( G \) two countable groups. We consider the second nilpotent product of \( |G| \)-many copies of \( H \) indexed by \( G \), i.e. the group \( \bigstar_{g \in G} H_g \).

Since the shift action of \( G \) on the free product \( \bigstar_{g \in G} H_g \) leaves the set \( \{ [\bigstar_{g \in G} H_g, [H_h, H_k]] \}_{h \neq k} \) invariant, it follows that this action passes to the factor group \( \bigstar_{g \in G} H_g / \langle [\bigstar_{g \in G} H_g, [H_h, H_k]] \rangle \). In other words \( G \) acts on \( \bigstar_{g \in G} H_g \).

5.1. **Definition.** The semidirect product

\[ ( \bigstar_{g \in G} H_g ) \rtimes G \]

will be called the restricted second nilpotent wreath product of \( H \) and \( G \).

A variant of this construction that motivated this article appeared in [15, Section 5]. The goal of this section is to show Theorem (B) and Theorem (C) from the Introduction.

5.2. **Definition.** The support of an element \( x \in \bigstar_{i \in I} H_i \) is the subset of \( I \) whose elements are all the indices \( i \) such that one of the elements \( a_i, c_{ij}, j \neq i \) in the representation of \( x \) as in Proposition 4.5, is nontrivial. Equivalently,

\[
\text{supp}(x) = \{ i \in I : \exists j \neq i \text{ such that } \pi_{(i,j)}(x) \notin H_j \setminus \{ e_j \} \}.
\]

The support of the identity element is the empty set.

5.3. **Remark.** Some obvious properties of the support are:
(1) \( \text{supp}(x) \) is a finite set.
(2) \( \text{supp}(x) = \text{supp}(x^{-1}) \).
(3) \( \text{supp}(xy) \subset \text{supp}(x) \cup \text{supp}(y) \).
(4) When the index set is a group \( G \), we have that \( \forall g \in G \) and
\[ \forall x \in \bigotimes_{g \in G} H \ \text{supp}(g.x) = g.\text{supp}(x). \]

The proof of Theorem (B) that we exhibit here follows the general strategy developed by Cornulier et.al. in [5]. In fact the result will follow after setting up the premisses that allow us to apply [5, Theorem 5.1]. For the sake of completeness we will include its statement but first we recall the next definition.

5.4. Definition. [5, Definition 3.3] Let \( W \) be a group, and \( X \) be a set. \( \mathcal{A} = 2^{(X)} \) denotes the set of finite subsets of \( X \). A \( \mathcal{A} \)-invariant \( \mathcal{A} \)-gauge on \( W \) is a function \( \psi : W \rightarrow \mathcal{A} \) such that
\[
\psi(w) = \psi(w^{-1}) \quad \forall w \in W; \\
\psi(ww') \subset \psi(w) \cup \psi(w') \quad \forall w, w' \in W.
\]

5.5. Example. The first three items of Remark 5.3 say that the support function is a \( \bigotimes_{i \in I} H_i \)-invariant \( 2^{(I)} \)-gauge on \( \bigotimes_{i \in I} H_i \). While condition (4) of Remark 5.3 says that in under that hypothesis, the support function is \( G \)-equivariant.

5.6. Theorem. [5, Theorem 5.1]. Let \( W, G \) be groups, with \( G \) acting on \( W \) by automorphisms. Set \( \mathcal{A} = 2^{(G)} \). Let \( \psi \) be a left \( W \)-invariant, \( G \)-equivariant \( \mathcal{A} \)-gauge on \( W \). Assume that there exists a \( G \)-invariant conditionally negative definite function \( u \) on \( W \) such that, for every finite subset \( F \subset G \), the restriction of \( u \) to every subset of the form \( W_F := \{ w \in W : \psi(w) \subset F \} \) is proper. Then \( W \rtimes G \) is a Haagerup group if and only if \( G \) is Haagerup.

Thus to prove Theorem B from the introduction it is enough to show the next:

5.7. Proposition. Let \( G \) and \( H \) be discrete countable groups. Assume \( H \) Haagerup. Then there exists a \( G \)-invariant and conditionally negative definite function \( \bigotimes_{g \in G} H \rightarrow \mathbb{R} \) such that for every finite subset \( F \subset G \), the restriction of \( u \) to any subset of the form \( \{ x \in \bigotimes_{g \in G} H : \text{supp}(x) \subset F \} \) is proper.

Proof. Since \( H \) is Haagerup, by (3) of Proposition 3.1, \( H \bigotimes H \) is Haagerup. By definition, this means that there exists a proper
conditionally negative definite function $\varphi : H \ast H \to \mathbb{R}_{\geq 0}$. For $h, k \in G, h \neq k$, we have:

$$
\begin{align*}
\ast & : H \ast H \to H \ast H \\
(\pi(h,k)_g) & \mapsto \pi(h,k)_g(\ast) \mapsto H \ast H \to \mathbb{R}
\end{align*}
$$

Denote $v(h,k) = \varphi \circ \pi(h,k)$. Let $\Psi : G \to \mathbb{N}$ be an enumeration of $G$.

**Claim:** The function

$$
u = \sum_{h,k \in G, h \neq k} 1 \frac{1}{2\Psi(h^{-1}k)} v(h,k)
$$

satisfies the required conditions.

(i) $\nu$ is $G$-invariant: Since $\pi(h,k)(g.x) = \pi(g^{-1}h,g^{-1}k)(x)$ it follows that

$$
u(g.x) = \sum_{h,k \in G, h \neq k} 1 \frac{1}{2\Psi(h^{-1}k)} v(h,k)(g.x) = \sum_{h,k \in G, h \neq k} 1 \frac{1}{2\Psi((g^{-1}h^{-1})(g^{-1}k))} v(g^{-1}h,g^{-1}k)(x) = \nu(x)
$$

(ii) For every fixed $x$, $\nu(x)$ is finite: First notice that $\forall h, k \notin supp(x), \ v(h,k)(x) = \varphi(e) = 0$. Then

$$
\begin{align*}
u(x) &= \sum_{h,k \in supp(x), h \neq k} 1 \frac{1}{2\Psi(h^{-1}k)} v(h,k)(x) + \sum_{h \in supp(x)} \sum_{k \notin supp(x)} 1 \frac{1}{2\Psi(h^{-1}k)} v(h,k)(x) + \\
&= \sum_{h \in supp(x)} \sum_{k \notin supp(x)} 1 \frac{1}{2\Psi(h^{-1}k)} v(h,k)(x)
\end{align*}
$$

Since $supp(x)$ is finite, the first summand in (5.8) is finite. On the other hand if $h \in supp(x)$ is fixed, then $\forall k, k' \notin supp(x), \ v(h,k)(x) = v(h,k')(x)$. It follows that the sum

$$
\sum_{k \notin supp(x)} 1 \frac{1}{2\Psi(h^{-1}k)} v(h,k)(x)
$$

converges $\forall h \in supp(x)$. Thus the second summand in (5.8) is finite. The same method shows that the third summand in (5.8) is finite.

(iii) Restrictions are proper: Fix a finite subset $F \subset G$. Let $N = \max_{h,k \in F} \Psi(h^{-1}k)$.

Let $a, b \in F, a \neq b$, then for any $x \in \ast \ G H$ we have the inequalities:

$$
v_a(b)(x) \leq \sum_{h,k \in F, h \neq k} v(h,k)(x) \leq 2^N \nu(x)
$$
This means that the set
\[ \{ x \in \bigodot^*_{G} H : \text{supp}(x) \subset F, u(x) \leq M \} \]
is contained in
\[ \{ x \in \bigodot^*_{G} H : \text{supp}(x) \subset F, v_{(a,b)}(x) \leq 2^{N}M \text{ for all } a, b \in F \} \]

But this set is finite since \( \varphi \) is proper and for all \( x \neq x' \) whose supports are contained in \( F \) there exists \( a, b \in F \) such that \( \pi_{(a,b)}(x) \neq \pi_{(a,b)}(x') \).

(iv) \( u \) is a conditionally negative definite function (c.n.d.f.): This is obvious since the set of c.n.d.f. is a convex cone and pointwise limit of c.n.d.f. is a c.n.d.f. (see for instance [1, Proposition C.2.4]).

\[ \square \]

5.9. **Remark.** The case when \( H \) is finite and \( G = \mathbb{F}_2 \) could be shown by mimicking the proof given in [4] for the lamplighter group. This alternative approach has the advantage of being self contained since it only requires to transfer the space with walls structure from \( \mathbb{F}_2 \) to \( \bigodot_{g \in \mathbb{F}_2} H \times \mathbb{F}_2 \).

**Proof of Theorem C.** If \( A \) is abelian and \( G \) is amenable, Corollary 4.9 implies that the second nilpotent wreath product of \( A \) and \( G \) is amenable and thus unitarizable. To show the converse let \( \{ g_i : g_i \in G \}_{i \in \mathbb{N}} \) be an enumeration of \( G \). Since \( C = \bigoplus_{1 \leq j} [A_{g_i}, A_{g_j}]^{(2)} \) is a \( G \)-invariant central subgroup of \( \bigodot_{g \in G} A \), then \( \tilde{C} = (C, e_G) \) is a normal subgroup of \( \bigodot_{g \in G} A \rtimes G \). Their quotient is:
\[ \left( \bigodot_{g \in G} A \right) \rtimes G / \tilde{C} \simeq \left( \bigodot_{g \in G} A / C \right) \rtimes G \simeq \left( \bigoplus_{g \in G} A \right) \rtimes G. \]

When \( G \) is non amenable [13, Theorem 1] says that \( \left( \bigoplus_{g \in G} A \right) \rtimes G \) is non unitarizable. Since a quotient of a unitarizable group must be unitarizable, it follows that \( \left( \bigodot_{g \in G} A \right) \rtimes G \) is non unitarizable. \[ \square \]

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