1 Introduction

The theory of gravitational radiation from isolated sources, in the context of general relativity, takes on importance because of the possibility of comparing the theory with contemporary astrophysical observations. We have in mind the large-scale optical interferometers LIGO, VIRGO, GEO and TAMA, which should routinely observe the gravitational waves produced by massive and rapidly evolving systems such as inspiralling compact binaries. To prepare these experiments the required theoretical work consists of (i) carrying out a sufficiently general solution of the Einstein field equations, valid for a large class of matter systems, (ii) applying the latter solution to specific systems like binaries.

For general isolated sources the basic problem is to relate the asymptotic gravitational-wave form, at the location of a detector in the source’s wave zone, to the stress-energy tensor $T^{\alpha\beta}$ of the matter fields. In this article we shall explore this problem using the post-Newtonian expansion, when the speed of light $c \to +\infty$, as the basic approximation. This approximation has proved in the past to constitute an invaluable tool for understanding many aspects of the problems of radiation and motion in general relativity (for reviews see Refs. 1, 2, 3, 4, 5, 6).

Concerning the application to compact binaries, the point is that a model made of two structureless point-particles, characterized by two mass parameters $m_1$ and $m_2$ (and possibly two spins), should be sufficient to describe the “inspiral” phase, driven by gravitational radiation reaction, which pre-
cedes the final binary coalescence. Most of the non-gravitational effects, usually plaguing the dynamics of binary star systems, are dominated by (post-Newtonian) gravitational effects. However the real justification of the model of point particles is that the effects due to the finite size of the compact bodies are small\(^a\). Inspiralling compact binaries are ideally suited for application of a post-Newtonian wave generation formalism, because it has been shown\(^b\) that the post-Newtonian precision required to implement successively the optimal filtering technique in the LIGO/VIRGO detectors corresponds grossly, in the case of neutron-star binaries, to the 3PN approximation, namely \(1/c^6\) beyond the quadrupole moment formalism.

In this article we review (Part A) and apply (Part B) a wave-generation formalism based on systematic multipole decompositions of Einstein’s theory, and initiated by Blanchet and Damour\(^9\),\(^10\), following preceding work by Bonnor\(^11\) and Thorne\(^12\). The basic multipole moments in this approach are some appropriately defined source moments, which are some functions of time, supposed to describe the actual physical source. They are iterated by means of a post-Minkowskian expansion of the vacuum field equations (valid in the source’s exterior). Technically, the post-Minkowskian approximation scheme is greatly simplified by the assumption of multipolar expansion, because one can consider separately the iteration of the different multipole pieces composing the exterior field. In our formalism, the radiation field at future null infinity is described by the standard radiative multipole moments\(^12\), which are obtained as some explicit non-linear functionals of the more basic source moments\(^13\),\(^14\),\(^15\). In the current situation, the closed-form expressions of the source multipole moments have been established in the case where the source is post-Newtonian\(^16\),\(^17\), (existence of a small parameter which can be regarded as a slow motion estimate \(\sim v/c\)). For post-Newtonian sources the domain of validity of the post-Newtonian expansion (\(viz\) the near zone) overlaps the exterior weak-field region, so that there exists an intermediate zone in which the post-Newtonian and multipolar expansions can be matched together\(^1\). Furthermore, the complete expression of the gravitational field inside the source, valid formally up to any post-Newtonian order, has also been determined by the latter matching\(^2\).

The post-Newtonian wave-generation formalism permitted to compute the total energy flux, \(\mathcal{L}\), emitted by inspiralling compact binaries. \(\mathcal{L}\) was

\(^a\)For instance the influence of the Newtonian quadrupole moments, \(Q_1\) and \(Q_2\), induced by tidal interaction between two compact objects (for which \(a_1 \approx Gm_1/c^2\) and \(a_2 \approx Gm_2/c^2\)), is, by a simple Newtonian argument, comparable (formally) with gravitational effects at the 5PN order.

\(^b\)This is an application of the method of matched asymptotic expansions\(^4\),\(^8\).
completed first to the 2PN order by Blanchet, Damour and Iyer \cite{blanchet1990nonlinear}, and, independently, by Will and Wiseman using their own formalism \cite{will1991electromagnetic,will1991quadrupole}. Higher-order tail effects at the 2.5PN and 3.5PN orders, as well as a crucial contribution of tails generated by the tails themselves (the so-called “tails of tails”) arising at the 3PN order, were subsequently obtained \cite{jaranowski1998note}. Notice that unlike the 1.5PN, 2.5PN and 3.5PN orders which are entirely composed of tail terms, the 3PN order involves also, besides the tails of tails, many non-tail contributions coming from the relativistic corrections in the binary’s multipole moments. These have been recently computed by Blanchet, Iyer and Joguet \cite{blanchet2018higher}.

The dynamics of compact binaries, i.e. their equations of motion, is also needed in this theoretical work. The 1PN approximation in the equations of motion is standard \cite{damour1992post}. The 2PN terms have been derived by Damour and Deruelle \cite{damour1993post}, Kopeikin and Grishchuk \cite{kopeikin1993post}, Blanchet, Faye and Ponsot \cite{blanchet2018higher}, Itoh et al \cite{itohe1991post}, Pati and Will \cite{pati1992post}. The 3PN order has been obtained independently by two groups: Jaranowski and Schäfer \cite{jaranowski1998note, jaranowski1998note2}, Damour, Jaranowski and Schäfer \cite{damour1998post} employ the ADM-Hamiltonian formalism of general relativity; Blanchet and Faye \cite{blanchet2018higher, blanchet2018higher2, blanchet2018higher3}, de Andrade, Blanchet and Faye \cite{deandrade2018post} compute directly the equations of motion (instead of a Hamiltonian) in harmonic coordinates. To the 3PN order the end results have been shown to be physically equivalent in the sense that there exists a unique “contact” transformation of the binary’s dynamical variables, that changes the harmonic-coordinates Lagrangian of Ref. \cite{blanchet2018higher} into a new Lagrangian whose Legendre transform coincides exactly with the Hamiltonian given in Ref. \cite{damour1998post}. This equivalence is of course quite satisfying.

**PART A : POST-NEWTONIAN SOURCES**

We describe the general solution of the field equations, valid for isolated post-Newtonian sources, in a series of mathematical statements, presented here without proofs. This presentation has the advantage of emphasizing some key results and outlining the basic “skeleton” of the formalism. We define $h^{\alpha \beta} = \sqrt{-g} g^{\alpha \beta} - \eta^{\alpha \beta}$, where $g^{\alpha \beta}$ is the metric, $\eta^{\alpha \beta}$ is the Minkowski metric, and $g = \det(g_{\alpha \beta})$. The field equations in harmonic coordinates, $\partial_\beta h^{\alpha \beta} = 0$, read

\begin{align}
\Box h^{\alpha \beta} &= \frac{16 \pi G}{c^4} \tau^{\alpha \beta}, \\
\tau^{\alpha \beta} &= |g| T^{\alpha \beta} + \frac{c^4}{16 \pi G} \Lambda^{\alpha \beta}(h, \partial h, \partial^2 h),
\end{align}
where $\square$ is the flat d’Alembertian operator, $T^{\alpha\beta}$ is the matter tensor and $\Lambda^{\alpha\beta}$ denotes the non-linear gravitational source term.

### 2 Linearized vacuum equations: general solution

The first result concerns the solution of the linearized field equations in the vacuum region exterior to the isolated source.

**Theorem 1** The most general solution of the linearized vacuum field equations, $\square h^\alpha_1 = 0 = \partial_\beta h^\alpha_1$, outside some time-like world tube enclosing the source, and stationary in the past (when $t \leq -T$), reads

$$h^\alpha_1 = k^\alpha_1 + \partial^\alpha \varphi^\beta_1 + \partial^\beta \varphi^\alpha_1 - \eta^{\alpha\beta} \partial_\gamma \varphi^\gamma_1.$$  

(3)

The vector $\varphi^\alpha_1$ parametrizes a linearized gauge transformation. The first term is given by

$$k^\alpha_1 = 12c^2 \sum_{i \geq 0} \frac{(-1)^i}{i!} \partial_L \left( \frac{1}{r} I_L(u) \right),$$  

(4)

$$k^0_i = 4c^3 \sum_{i \geq 1} \frac{(-1)^l}{l!} \left\{ \partial_{L-1} \left( \frac{1}{r} I^{(1)}_{iL-1}(u) \right) + \frac{l}{l+1} \varepsilon_{iab} \partial_{aL-1} \left( \frac{1}{r} J_{bL-1}(u) \right) \right\},$$  

(5)

$$k^{ij} = -4c^4 \sum_{i \geq 2} \frac{(-1)^l}{l!} \left\{ \partial_{L-2} \left( \frac{1}{r} I^{(2)}_{iL}(u) \right) + \frac{2l}{l+1} \partial_{aL-2} \left( \frac{1}{r} \varepsilon_{ij}(r) \partial_{bL-2}(u) \right) \right\},$$  

(6)

where $I_L$ and $J_L$ are two sets of STF-tensorial multipole moments, functions of the retarded time $u = t - r/c$, and satisfying the conservation laws: $I = \text{const}$, $I_i = \text{const}$, $J_i = \text{const}$.

### 3 Exterior vacuum solution: post-Minkowskian iteration

Consider the so-called post-Minkowskian expansion, or formal series in powers of Newton’s constant $G$.

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\[^c\text{Multi-indices are denoted by } L = \underbrace{i_1 i_2 \cdots i_l}; \text{ the derivative operator } \partial_L \text{ is a short-hand for } \partial_{i_1} \cdots \partial_{i_l}; \text{ a tensor } I_L \text{ is STF if and only if for any pair of indices } i_p, i_q \in L, \text{ we have } I_{i_p i_q} = I_{i_q i_p}, \text{ and } \delta_{i_p i_q} I_{i_p i_q} = 0. \text{ The superscript } (p) \text{ refers to the time derivation.} \]
\[ h_{\text{ext}}^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n h_n^{\alpha\beta}, \tag{7} \]

that is based on the general linearized exterior solution \( h_t^{\alpha\beta} [I_L, J_L] \) of Theorem 1. We obtain a particular solution, at any post-Minkowskian order \( n \), in the form

\[ h_n^{\alpha\beta} = \mathcal{FP} \Box_{\text{ret}}^{-1} [\Lambda_n^{\alpha\beta}] + v_n^{\alpha\beta}, \tag{8} \]

where \( \Box_{\text{ret}}^{-1} \) is the usual retarded integral and \( \mathcal{FP} \) a specific operation (defined by complex analytic continuation) of taking the finite part. The finite part is necessary because we are looking for the exterior solution, in the form of a multipole expansion which is singular at the origin \( r = 0 \), supposed to be located inside the compact-support source. The term \( v_n^{\alpha\beta} \) is a particular source-free retarded solution constructed in such a way that the harmonic coordinate condition \( (\partial_\beta h_n^{\alpha\beta} = 0) \) is satisfied.

**Theorem 2**

The most general solution of the Einstein field equations, in the vacuum region outside the source, admitting post-Minkowskian and multipolar expansions (and stationary in the past), is given by the construction (7)-(8). It depends on the source multipole moments \( I_L(u) \) and \( J_L(u) \), and on four supplementary functions parametrizing the gauge vector in Eq. (3).

4 General structure of the exterior solution

The solution is valid in all the (weak-field) exterior region of the source. We investigate its expansion when \( r \to 0 \) and \( r \to +\infty \).

**Theorem 3**

The structure of the expansion of the post-Minkowskian exterior metric in the near-zone (when \( r \to 0 \)) is of the type

\[ h_n^{\alpha\beta}(x, t) = \sum \hat{n}_L F^m (\ln r)^p F_L^{p, m, p, n}(t) + o(r^N), \tag{9} \]

where \( m \in \mathbb{Z} \), with \( m_0 \leq m \leq N \) (and \( m_0 \) becoming more and more negative when \( n \) grows), \( p \in \mathbb{N} \) with \( p \leq n - 1 \). The functions \( F_L^{p, m, p, n} \) are multilinear functionals of the source multipole moments.

\[ d \text{We denote by } \hat{n}_L \text{ (or sometimes } n_{<L}> \text{) the STF projection of the product of unit vectors } n_L = n_{i_1} \cdots n_{i_l}; \text{ an expansion into STF tensors } \hat{n}_L = \hat{n}_L(\theta, \phi) \text{ is equivalent to the usual expansion in spherical harmonics } Y_{m}(\theta, \phi). \text{ Similarly we pose } x_L = x_{i_1} \cdots x_{i_l} = r^n n_L \text{ and } \hat{x}_L = r^n \hat{n}_L. \]
This theorem shows in particular that the general form of the post-Newtonian expansion (i.e. when $c \to +\infty$) is necessarily of the type $\sum (\ln c)^p / c^m$.

**Theorem 4** The most general multipolar and post-Minkowskian solution, stationary in the past, (i) admits radiative coordinates $(T, X)$, for which the expansion at infinity, $R \to +\infty$ with $U \equiv T - R/c = \text{const}$, takes the form:

$$H_n^{\alpha\beta}(X, T) = \sum \frac{\hat{N}_L}{R^k} K_{L,k,n}^{\alpha\beta}(U) + O\left(\frac{1}{R^N}\right),$$

(10)

where the functions $K_{L,k,n}^{\alpha\beta}$ are computable functionals of the source multipole moments, (ii) is asymptotically simple in the sense of Penrose [42], perturbatively to any post-Minkowskian order.

In radiative coordinates the retarded time $U$ is a null coordinate at future null infinity. Theorem 4 permits to connect our work with the study by Bondi et al [43] of the asymptotic structure of radiative gravitational fields. Most importantly, it permits to calculate the relations between the source multipole moments $I_L$ and $J_L$ and the radiative ones, say $U_L$ and $V_L$, which parametrize the asymptotic wave form. These relations involve many non-linear multipole interactions associated with tails and related effects [13], [14], [15].

5 Post-Newtonian wave generation

Suppose that there exists a solution $h^{\alpha\beta}$, valid everywhere inside and outside a post-Newtonian source, whose multipole expansion coincides with the previously constructed exterior field: $\mathcal{M}(h^{\alpha\beta}) \equiv h_{\text{ext}}^{\alpha\beta}$. Our basic assumption is that of the matching between the near-zone expansion, $r \to 0$, of the exterior multipole expansion: i.e. $\mathcal{M}(h^{\alpha\beta})$, and the far-zone expansion, $r \to +\infty$, of the inner post-Newtonian expansion: i.e. $\mathcal{M}(\overline{h^{\alpha\beta}})$. The notation is $\mathcal{M}$ for the operation of taking the multipole expansion, and an overline for the post-Newtonian (re-)expansion. As shown by the matching, the near-zone expansion of the outer field is actually a post-Newtonian expansion, and the far-zone expansion of the inner field is identical to the multipole decomposition. The matching equation reads

$$\mathcal{M}(\overline{h^{\alpha\beta}}) = \sum \hat{n}_L r^m (\ln r)^p F_{L,m,p}^{\alpha\beta}(t) = \mathcal{M}(\overline{h^{\alpha\beta}}),$$

(11)

where we have given the common structure of both types of asymptotic expansions, which is given, after formal post-Minkowskian re-summation, by Theorem 3.
Theorem 5 Under the hypothesis of matching, Eq. (11), the multipole expansion of the field exterior to a post-Newtonian source reads

\[ \mathcal{M}(h^{\alpha\beta}) = \mathcal{F}\mathcal{P} \partial_{r}^{-1} [\mathcal{M}(\Lambda^{\alpha\beta})] - \frac{4Gc^{4}}{c^{4}} \sum_{l=0}^{+\infty} \frac{(-1)^{l}}{l!} \partial_{l} \left\{ \frac{1}{r} \mathcal{F}_{L}^{\alpha\beta}(t - r/c) \right\} , \]  

(12)

where the “multipole moments” are given by

\[ \mathcal{F}_{L}^{\alpha\beta}(u) = \mathcal{F}\mathcal{P} \int d^{3}y \hat{y}_{L} \int_{-1}^{1} dz \delta_{l}(z) \mathcal{T}^{\alpha\beta}(y, u - z|y|/c) . \]  

(13)

Here, \( \mathcal{T}^{\alpha\beta} \) denotes the post-Newtonian expansion of the stress-energy pseudo-tensor defined by Eq. (3).

The function \( \delta_{l}(z) \) entering the moments is

\[ \delta_{l}(z) = \frac{2l + 1)!!}{2^{l+1}l!} (1 - z^{2})^{l} . \]  

(14)

A different formalism, alternative to Theorem 5, has been developed by Will and Wiseman, and proved to be strictly equivalent to ours.

With Theorem 5 in hands, one can obtain the expressions of the source multipole moments \( I_{L} \) and \( J_{L} \) parametrizing the exterior field of Sections 2-3. These are the ones to be inserted into the linearized metric (3)-(6) and iterated by means of the post-Minkowskian algorithm (7)-(8).

\[ I_{L}(u) = \mathcal{F}\mathcal{P} \int d^{3}y \int_{-1}^{1} dz \left\{ \delta_{l}\hat{y}_{L}\Sigma - \frac{4(2l + 1)}{c^{4}(l + 1)(2l + 3)} \delta_{l+1}\hat{y}_{iL}\Sigma_{i}^{(1)} ight\} 
\left\{ \hat{y}_{j} - z|y|/c \right\} , \]

(15)

\[ J_{L}(u) = \mathcal{F}\mathcal{P} \int d^{3}y \int_{-1}^{1} dz \varepsilon_{abc} \delta_{l}\hat{y}_{L-1>ac}\Sigma_{b} 
\left\{ \hat{y}_{b} - z|y|/c \right\} , \]

(16)

where we have posed

\[ \varepsilon \text{This function approaches the Dirac delta-function in the limit of large } l : \lim_{l\to+\infty} \delta_{l}(z) = \delta(z), \text{ and its integral is normalized to one} : \int_{-1}^{1} dz \delta_{l}(z) = 1. \]

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\[ \Sigma = \frac{\tau^{00} + \tau^{ii}}{c^2}, \quad \Sigma_i = \frac{\tau^{0i}}{c}, \quad \Sigma_{ij} = \tau^{ij}. \]  

(17)

6 Post-Newtonian field in the near zone

The post-Newtonian metric can be systematically computed by means of a specific variant of the operator of the “symmetric” potentials,

\[ \mathcal{I}^{-1} = \frac{1}{\Delta - \frac{1}{c^2} \partial_t^2} = \sum_{k=0}^{+\infty} \frac{1}{c^{2k}} \partial_t^{2k} \Delta^{-k-1}. \]  

(18)

Theorem 6 The expression of the post-Newtonian field inside the source, satisfying correct boundary conditions at infinity (no incoming radiation), follows from the matching equation (11) as

\[ h_{\alpha\beta} = \frac{16\pi G}{c^4} \mathcal{F} \mathcal{P} \mathcal{I}^{-1} \left[ \tau_{\alpha\beta} \right] - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \left( -\right)^l \frac{l!}{l!} \partial_L \left\{ \frac{A^\alpha_{L\beta}(t - r/c) - A^\alpha_{L\beta}(t + r/c)}{2r} \right\}. \]  

(19)

The “anti-symmetric” waves are parametrized by

\[ A^\alpha_{L\beta}(u) = \mathcal{F}_{L\alpha\beta}(u) + \mathcal{R}_{L\alpha\beta}(u), \]

where \( \mathcal{F}_{L\alpha\beta} \) is given by Eq. (13), and where

\[ \mathcal{R}_{L\alpha\beta}(u) = \mathcal{F} \mathcal{P} \int d^3 y \, \hat{y}_L \int_1^{+\infty} dz \, \gamma_l(z) \, M(\tau_{\alpha\beta})(y, u - z|y|/c). \]  

(20)

The anti-symmetric waves are associated with radiation-reaction effects. For instance the \( \mathcal{F}_{L\alpha\beta} \)-contribution yields the dominant radiation reaction force at the 2.5PN order, while the \( \mathcal{R}_{L\alpha\beta} \)-contribution contains the gravitational-wave tails in the radiation reaction, which arise at the 4PN order.

PART B : COMPACT BINARIES

We recall that the forthcoming detection and analysis of the gravitational waves emitted by inspiralling compact binaries should necessitate the prior knowledge of the equations of motion and gravitational radiation up to the high 3PN order.

\(^\dagger\)We denote \( \gamma_l(z) = -2\delta_l(z) \). The normalization of \( \gamma_l(z) : \int_1^{+\infty} dz \, \gamma_l(z) = 1 \) (where the integral is computed by analytic continuation in \( l \in \mathbb{C} \)), is consistent with that of \( \delta_l(z) \).
The problem of self-field regularization

It makes sense in post-Newtonian approximations to model the compact objects by point-like particles, but we then face the problem of the regularization of the infinite self-field of the particles. The Hadamard regularization has been extensively used in this field, but it turned out, as we now review, to yield an unknown coefficient at the 3PN order (and not before that order).

Jaranowski and Schäfer noticed that, in their computation of the equations of motion within the ADM-Hamiltonian formulation of general relativity, the “standard” Hadamard regularization leads to some ambiguous results for the computation of certain integrals at the 3PN order. They showed that there are two and only two types of ambiguous terms in the 3PN Hamiltonian, which they parametrized by two unknown numerical coefficients $\omega_{\text{static}}$ and $\omega_{\text{kinetic}}$.

Motivated by this result, Blanchet and Faye introduced an “improved” Hadamard regularization, based on a theory of pseudo-functions and generalized distributional derivatives, which is free of ambiguities, and leads to unique results for the computation of all the integrals occurring at the 3PN order. Unfortunately, this regularization turned out to be incomplete, because it was found that the 3PN equations of motion involve one and only one unknown numerical constant, called $\lambda$, which cannot be determined within the method. The comparison of this result with the work of Jaranowski and Schäfer on the basis of the computation of the invariant energy of binaries moving on circular orbits, revealed that

$$\omega_{\text{kinetic}} = \frac{41}{24} ,$$

$$\omega_{\text{static}} = -\frac{11}{3} \lambda - \frac{1987}{840} .$$

The ambiguity $\omega_{\text{kinetic}}$ is therefore fixed, while $\lambda$ is equivalent to the other ambiguity $\omega_{\text{static}}$.

Damour, Jaranowski and Schäfer recovered the value of $\omega_{\text{kinetic}}$ given in Eq. (21) by proving that this value is the unique one for which the global Poincaré invariance of their formalism is verified. Since the coordinate conditions associated with the ADM approach do not manifestly respect the Poincaré symmetry, they had to prove that the Hamiltonian is compatible with the existence of generators for the Poincaré algebra. By contrast, the harmonic-coordinates condition preserves the Poincaré invariance, and therefore the associated equations of motion should be Lorentz-invariant, as was indeed found to be the case by Blanchet and Faye, thanks to their use of
a Lorentz-invariant regularization \[^{40}\] (hence their determination of \(\omega_{\text{kinetic}}\)).

More recently, the other parameter \(\omega_{\text{static}}\) was computed by Damour, Jaranowski and Schäfer \[^{36}\] by means of a dimensional regularization, instead of an Hadamard-type one, within the ADM-Hamiltonian formalism. The result, which in principle fixes \(\lambda\) according to Eq. (22), is

\[
\omega_{\text{static}} = 0 \iff \lambda = \frac{-1987}{3080}.
\] (23)

As Damour et al \[^{36}\] argue, clearing up the ambiguity is made possible by the fact that the dimensional regularization, contrary to the Hadamard one, respects all the basic properties of the algebraic and differential calculus of ordinary functions \[^{3}\].

Blanchet, Iyer and Joguet \[^{24}\], in their computation of the 3PN radiation field of two point particles, used the (standard) Hadamard regularization and found necessary to introduce three additional regularization constants \(\xi, \kappa, \zeta\), playing a role analogous to the equation-of-motion constant \(\lambda\). Such unknowns come from the computation of the 3PN binary’s quadrupole moment \(I_{ij}\). It was found \[^{24}\] that the total gravitational-wave flux, in the case of circular orbits, depends in fact only on a single combination of these three constants,

\[
\theta = \xi + 2\kappa + \zeta.
\] (24)

### 8 The equations of binary motion

The equations are expressed in Newtonian-like form, by means of the coordinate positions \(y^i_1, y^i_2\), and velocities \(v^i_1 = dy^i_1/ dt, v^i_2 = dy^i_2/ dt\), of the bodies in the harmonic coordinate system. The 3PN acceleration of one of the bodies, say 1, reads

\[
\frac{dv^i_1}{dt} = A^i_N + \frac{1}{c^2} A^i_{1\text{PN}} + \frac{1}{c^4} A^i_{2\text{PN}} + \frac{1}{c^6} A^i_{2.5\text{PN}} + \frac{1}{c^8} A^i_{3\text{PN}} + \mathcal{O}\left(\frac{1}{c^{10}}\right),
\] (25)

The first term is the famous Newtonian law,

\[
A^i_N = -\frac{Gm_2}{r_{12}^2} n^i_{12},
\] (26)

\[^{9}\text{For instance, the Hadamard regularization unavoidably violates at some stage the Leibniz rule for the differentiation of a product.}\]
(where \( r_{12} = |\mathbf{y}_1 - \mathbf{y}_2| \) and \( \mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12} \)). All the other terms represent the successive post-Newtonian corrections. The 3PN equations of motion, that we show below, possess the correct perturbative limit given by the geodesics of the Schwarzschild metric, stay invariant when we perform a global Lorentz transformation (when the radiation-reaction term is neglected). The 1PN term is the Einstein-Infeld-Hoffmann acceleration:

\[
A_{1PN}^i = \left[ \frac{5G^2m_1m_2}{r_{12}^3} + 4 \frac{G^2m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^3} \left( \frac{3}{2} (n_{12}v_2)^2 - v_1^2 + 4(v_1v_2) - 2v_2^2 \right) \right] n_{12}^i \\
+ \frac{Gm_2}{r_{12}^3} \left[ 4(n_{12}v_1) - 3(n_{12}v_2) \right] v_{12}^i.
\]

The parenthesis indicate the usual Euclidean scalar product, e.g. \((n_{12}v_1)v_1 = \mathbf{n}_{12} \cdot \mathbf{v}_1\), and \(v_{12}^i \equiv v_1^i - v_2^i\). At the 2PN order, the result is:

\[
A_{2PN}^i = \left[ -\frac{57}{4} \frac{G^3m_1^2m_2}{r_{12}^5} - \frac{69}{2} \frac{G^3m_1m_2^2}{r_{12}^5} - \frac{9}{2} \frac{G^3m_2^3}{r_{12}^5} \\
+ \frac{Gm_2}{r_{12}^3} \left( -\frac{15}{8} (n_{12}v_2)^2 + \frac{3}{2} (n_{12}v_2)^2 v_2^i - 6(n_{12}v_2)^2(v_1v_2) - 2(v_1v_2)^2 + \frac{9}{2} (n_{12}v_2)^2 v_2^2 + 4(v_1v_2)v_2^2 - 2v_2^i \right) \\
+ \frac{G^2m_1m_2}{r_{12}^3} \left( \frac{39}{2} (n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) \\
+ \frac{17}{2} (n_{12}v_2)^2 - \frac{15}{4} v_1^2 - \frac{5}{2} (v_1v_2) + \frac{5}{4} v_2^2 \right) \\
+ \frac{G^2m_2^2}{r_{12}^3} \left( 2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2 - 8(v_1v_2) + 4v_2^2 \right) \right] n_{12}^i \\
+ \frac{G^2m_1m_2}{r_{12}^3} \left( -2(n_{12}v_1) - 2(n_{12}v_2) \right) \\
+ \frac{G^2m_1m_2}{r_{12}^3} \left( -\frac{63}{4} (n_{12}v_1) + \frac{55}{4} (n_{12}v_2) \right) \\
+ \frac{Gm_2}{r_{12}^3} \left( -6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2} (n_{12}v_2)^3 + (n_{12}v_2)v_1^2 - 4(n_{12}v_1)(v_1v_2) \\
+ 4(n_{12}v_2)(v_1v_2) + 4(n_{12}v_1)v_2^2 - 5(n_{12}v_2)v_2^2 \right) \right] v_{12}^i.
\]
The 2.5PN term – an “odd” term $\sim 1/c^3$ – corresponds to the damping effect of radiation reaction:

$$A_{2.5PN}^i = \frac{4}{5} \frac{G^2 m_1 m_2}{r_{12}^3} \left[ -6 \frac{G m_1}{r_{12}} + \frac{52}{3} \frac{G m_2}{r_{12}} + 3 v_{12}^2 \right] (n_{12} v_{12}) n_{12}^i$$

$$+ \frac{4}{5} \frac{G^2 m_1 m_2}{r_{12}^3} \left[ 2 \frac{G m_1}{r_{12}} - 8 \frac{G m_2}{r_{12}} - v_{12}^2 \right] v_{12}^i .$$

(29)

Finally, the very lengthy 3PN coefficient is given by

$$A_{3PN}^i = \left[ \frac{G m_2}{r_{12}^5} \left( \frac{35}{16} (n_{12} v_{12})^6 - \frac{15}{8} (n_{12} v_{12})^4 v_1^2 + \frac{15}{2} (n_{12} v_{12})^4 (v_1 v_2) 
+ 3 (n_{12} v_{12})^2 (v_1 v_2)^2 - \frac{15}{2} (n_{12} v_{12})^4 v_2^2 + \frac{3}{2} (n_{12} v_{12})^2 v_1^2 v_2^2 
- 12 (n_{12} v_{12})^2 v_1 v_2 v_2^2 - 2 (v_1 v_2)^2 v_2^2 
+ \frac{15}{2} (n_{12} v_{12})^2 v_2^4 + 4 (v_1 v_2) v_2^4 - 2 v_2^6 \right) 
+ \frac{G^2 m_1 m_2}{r_{12}^3} \left( - \frac{171}{8} (n_{12} v_1)^4 
+ \frac{171}{2} (n_{12} v_1)^3 (n_{12} v_2) - \frac{723}{4} (n_{12} v_1)^2 (n_{12} v_2)^2 + \frac{383}{2} (n_{12} v_1) (n_{12} v_2)^3 
- \frac{455}{8} (n_{12} v_2)^4 + \frac{229}{4} (n_{12} v_1)^2 v_1^2 - \frac{205}{2} (n_{12} v_1) (n_{12} v_2) v_1^2 
+ \frac{191}{4} (n_{12} v_2)^2 v_1^2 - \frac{91}{8} v_1^4 - \frac{229}{2} (n_{12} v_1)^2 (v_1 v_2) + 244 (n_{12} v_1) (n_{12} v_2) (v_1 v_2) 
- \frac{225}{2} (n_{12} v_2)^2 (v_1 v_2) + \frac{91}{2} v_1^2 (v_1 v_2) - \frac{177}{4} (v_1 v_2)^2 + \frac{229}{4} (n_{12} v_1)^2 v_2^2 
- \frac{283}{2} (n_{12} v_1) (n_{12} v_2) v_2^2 + \frac{259}{4} (n_{12} v_2)^2 v_2^2 - \frac{91}{4} v_1^2 v_2^2 + 43 (v_1 v_2) v_2^2 - \frac{81}{8} v_2^4 \right) 
+ \frac{G^2 m_2^2}{r_{12}^3} \left( - 6 (n_{12} v_1)^3 (n_{12} v_2)^2 + 12 (n_{12} v_1) (n_{12} v_2)^3 + 6 (n_{12} v_2)^4 
+ 4 (n_{12} v_1) (n_{12} v_2) (v_1 v_2) + 12 (n_{12} v_2)^2 (v_1 v_2) + 4 (v_1 v_2)^2 
- 4 (n_{12} v_1) (n_{12} v_2) v_2^2 - 12 (n_{12} v_2)^2 v_2^2 - 8 (v_1 v_2) v_2^2 + 4 v_2^4 \right) 
+ \frac{G^2 m_1 m_2^2}{r_{12}^3} \left( - (n_{12} v_1)^2 + 2 (n_{12} v_1) (n_{12} v_2) + \frac{43}{2} (n_{12} v_2)^2 + 18 (v_1 v_2) - 9 v_2^2 \right) 
+ \frac{G^2 m_1 m_2^2}{r_{12}^3} \left( \frac{415}{8} (n_{12} v_1)^2 - \frac{375}{4} (n_{12} v_1) (n_{12} v_2) + \frac{1113}{8} (n_{12} v_2)^2 \right)$$
\[- \frac{615}{64} (n_{12} v_{12})^2 \pi^2 + \frac{123}{64} v_{12}^2 \pi^2 + 18 v_1^2 + 33(v_1 v_2) - \frac{33}{2} v_2^2 \]
\[+ \frac{G^4 m_1^4 m_2}{r_{12}^3} \left( - \frac{45887}{168} (n_{12} v_1)^2 + \frac{24025}{42} (n_{12} v_1)(n_{12} v_2) - \frac{10469}{42} (n_{12} v_2)^2 + \frac{48197}{840} v_1^2 - \frac{36227}{420} (v_1 v_2) + \frac{36227}{840} v_2^2 \right)\]
\[+ 110 (n_{12} v_{12})^2 \ln \left( \frac{r_{12}}{v_1} \right) - 22 \ln \left( \frac{r_{12}}{v_1} \right) \]
\[+ \frac{G^4 m_1^2 m_2}{r_{12}^5} \left( - \frac{3187}{1260} + \frac{44}{3} \ln \left( \frac{r_{12}}{v_1} \right) \right) \]
\[+ \frac{G^4 m_1^2 m_2}{r_{12}^5} \left( \frac{34763}{210} - \frac{44}{3} \lambda - \frac{41}{16} \pi^2 \right) + 16 \frac{G^4 m_1^4}{r_{12}^5} \]
\[+ \frac{G^4 m_1^3 m_3}{r_{12}^7} \left( \frac{10478}{63} - \frac{44}{3} \lambda - \frac{41}{16} \pi^2 - \frac{44}{3} \ln \left( \frac{r_{12}}{r_2} \right) \right) \]
\[n_{12} \]
\[+ \frac{G m_2}{r_{12}^3} \left( \frac{15}{2} (n_{12} v_1)(n_{12} v_2)^4 - \frac{45}{8} (n_{12} v_2)^5 - \frac{3}{2} (n_{12} v_2)^3 v_1^2 \right) \]
\[+ 6 (n_{12} v_1)(n_{12} v_2)(v_1 v_2) - 6 (n_{12} v_2)^3 (v_1 v_2) - 2 (n_{12} v_2)(v_1 v_2)^2 \]
\[- 12 (n_{12} v_1)(n_{12} v_2)^2 v_1^2 + 12 (n_{12} v_2)^3 v_1^2 + (n_{12} v_2) v_1^2 v_2^2 \]
\[- 4 (n_{12} v_1)(v_1 v_2) v_2^2 + 8 (n_{12} v_2)(v_1 v_2) v_2^2 \]
\[+ 4 (n_{12} v_1) v_2^2 - 7 (n_{12} v_2) v_2^2 \]
\[+ \frac{G^2 m_2^2}{r_{12}^4} \left( - 2 (n_{12} v_1)^2 (n_{12} v_2) + 8 (n_{12} v_1)(n_{12} v_2)^2 + 2 (n_{12} v_2)^3 \right) \]
\[+ 2 (n_{12} v_1)(v_1 v_2) + 4 (n_{12} v_2)(v_1 v_2) - 2 (n_{12} v_1) v_2^2 - 4 (n_{12} v_2) v_2^2 \]
\[+ \frac{G^2 m_1 m_2}{r_{12}^4} \left( - \frac{243}{4} (n_{12} v_1)^3 + \frac{565}{4} (n_{12} v_1)^2 (n_{12} v_2) \right) \]
\[- \frac{269}{4} (n_{12} v_1)(n_{12} v_2)^2 - \frac{95}{12} (n_{12} v_2)^3 + \frac{207}{8} (n_{12} v_1) v_1^2 \]
\[- \frac{137}{8} (n_{12} v_2) v_1^2 - 36 (n_{12} v_1)(v_1 v_2) \]
\[+ \frac{27}{4} (n_{12} v_2)(v_1 v_2) + \frac{81}{8} (n_{12} v_1) v_2^2 + \frac{83}{8} (n_{12} v_2) v_2^2 \]
\[+ \frac{G^3 m_3^3}{r_{12}^4} \left( 4 (n_{12} v_1) + 5 (n_{12} v_2) \right) \]
\[
\begin{align*}
&+ \frac{G^3 m_1 m_2}{r_{12}^3} \left( - \frac{307}{8} (n_{12} v_1) + \frac{479}{8} (n_{12} v_2) + \frac{123}{32} (n_{12} v_{12})^2 \pi^2 \right) \\
&+ \frac{G^3 m_1^2 m_2}{r_{12}^3} \left( \frac{31397}{420} (n_{12} v_1) - \frac{36227}{420} (n_{12} v_2) \\
&- 44 (n_{12} v_{12}) \ln \left( \frac{r_{12}}{r_1'} \right) \right) v_{12}.
\end{align*}
\]

The latter harmonic-coordinates equations depend at the 3PN order on two arbitrary constants \(r_1'\) and \(r_2'\) parametrizing some logarithmic terms, however these constants are not physical in the sense that they can be removed by a coordinate transformation. By contrast, the equations of motion in ADM-coordinates\[^3\,^4\,^5\] do not involve logarithms and are free of such type of constants.

9 Energy of circular-orbit binaries

Most inspiralling compact binaries will have been circularized at the time when they become visible by the LIGO and VIRGO detectors. In the case of circular orbits the equations (25)-(30) simplify drastically. Also we can translate the origin of coordinates to the binary’s center of mass by imposing that the mass-dipole be \(I_i = 0\) (notation of Part A). Mass parameters are the total mass \(m = m_1 + m_2\), the reduced mass \(\mu = m_1 m_2/m\), and the useful symmetric mass ratio

\[
\nu \equiv \frac{\mu}{m} \equiv \frac{m_1 m_2}{(m_1 + m_2)^2},
\]

such that \(0 < \nu \leq \frac{1}{4}\), with \(\nu = \frac{1}{4}\) in the case of equal masses, and \(\nu \to 0\) in the “test-mass” limit for one of the bodies. The energy \(E\) of the binary in the center-of-mass frame is deduced from the 3PN harmonic-coordinates Lagrangian\[^4\,^5\] and expressed in invariant form (the same in different coordinate systems). This is achieved with the help of the angular orbital frequency \(\omega\) of the circular orbit, or, more conveniently, the parameter defined by

\[
x \equiv \left( \frac{G m \omega}{c^3} \right)^{2/3}.
\]

After replacement we discover that the two gauge constants \(r_1'\) and \(r_2'\) cancel out. The end result is
\[ E = -\frac{\mu c^2 x^2}{2} \left\{ 1 + \left( -\frac{3}{4} - \frac{1}{12} \nu \right) x + \left( -\frac{27}{8} + \frac{19}{8} \nu - \frac{1}{24} \nu^2 \right) x^2 \right. \\
+ \left. \left( \frac{675}{64} + \frac{209323}{4032} - \frac{205}{96} \pi^2 - \frac{110}{9} \lambda \nu - \frac{155}{96} \nu^2 - \frac{35}{5184} \nu^3 \right)x^3 \right\}. \quad (33) \]

As an application of Eq. (33) let us determine the location of the minimum of the energy \( E \) as a function of the orbital frequency for circular orbits \[^4\], say \( E_{\text{min}} = E(\omega_{\text{min}}) \).

In Fig. 1 we display the results given by the successive 1PN, 2PN and 3PN orders [the 1PN and 2PN orders are defined by the obvious truncation of Eq. (33)], and compare them with the prediction recently calculated numerically by Gourgoulhon, Grandclément and Bonazzola \[^{47,48}\] under the assumption of exactly circular orbits (helicoidal symmetry). As we see the post-Newtonian
and numerical predictions agree reasonably well. The post-Newtonian approximation seems to converge well toward the “exact” solution, even at this very relativistic stage where the orbital velocity reaches $v/c \sim (Gm\omega/c^3)^{1/3} \sim 0.5$. However, the 1PN order is clearly not precise enough at this stage.

10 Flux of circular-orbit binaries

The calculation of the flux necessitates, following the formalism described in Part A, the expressions of the source multipole moments $I_L$ and $J_L$. The crucial moment to be calculated is the mass-quadrupole, which is to be determined with the maximal post-Newtonian precision. It has been obtained by Blanchet, Iyer and Joguet\cite{24} at the 3PN order in the case of circular orbits\cite{3}. We compute the time-derivatives of the moments with the help of the 3PN equations of motion (25)-(30), replace them into the total energy flux developed at the 3PN order, and add all the non-linear tail and related effects found up to the 3.5PN order in Ref.\cite{15}. We must also express the result in terms of the frequency-dependent parameter (32). Finally we get

$$\mathcal{L} = \frac{32c^5}{5G}\nu^2 x^5 \left\{ 1 + \left( \frac{1247}{336} - \frac{35}{12} \nu \right) x + 4 \pi x^{3/2} \right. $$

$$+ \left( \frac{-4471}{9072} + \frac{9271}{504} \nu + \frac{65}{18} \nu^2 \right) x^2 + \left( \frac{-8191}{672} - \frac{535}{24} \nu \right) \pi x^{5/2}$$

$$+ \left( \frac{6643739519}{69854400} - \frac{16}{3} \nu^2 - \frac{1712}{105} C - \frac{856}{105} \ln(16 x) \right) x^3$$

$$+ \left[ \frac{-11497453}{272160} + \frac{41}{48} \nu^2 + \frac{176}{9} \lambda - \frac{88}{3} \theta \right] \nu - \frac{94403}{3024} \nu^2 - \frac{775}{324} \nu^3 \right) x^3$$

$$+ \left( \frac{-16285}{504} + \frac{176419}{1512} \nu + \frac{19897}{378} \nu^2 \right) \pi x^{7/2} \right\}.$$  \hspace{1cm} (34)

We have $\theta = \xi + 2\kappa + \zeta$ (and $C \approx 0.577$ is the Euler constant). The flux\cite{24} agrees perfectly, in the test-mass limit $\nu \to 0$, with the result of linear black-holes perturbations calculated by Tagoshi and Sasaki\cite{49}.

As an application of Eqs. (34) and (33) one can obtain the orbital phase $\phi = \int \omega dt$ of an inspiralling compact binary at the 3.5PN order simply by

\hspace{1cm} (34)\hfill

The 3PN quadrupole is found to involve three ambiguity constants $\xi$, $\kappa$ and $\zeta$ [see Eq. (24)], as well as some $r_0$ coming from the definition of the finite part process $FP$ employed in Part A, and some $r'_0$ which is in fact the “logarithmic barycenter” of the constants $r'_1$ and $r'_2$ present in the equations of motion (30). Both $r_0$ and $r'_0$ disappear from the result at the end, in agreement with general expectations.
using the energy balance equation

\[
\frac{dE}{dt} = -L \implies \phi = -\int \frac{\omega dE}{L}.
\] (35)

The orbital phase is the crucial observable to be implemented in the construction of accurate templates for the detection and analysis of inspiralling-binary signals in the LIGO and VIRGO experiments.

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