A NOTE ON BIPARTITE GRAPHS
WHOSE $[1, k]$-DOMINATION NUMBER
EQUAL TO THEIR NUMBER OF VERTICES

Narges Ghareghani, Iztok Peterin, and Pouyeh Sharifani

Communicated by Dalibor Fronček

Abstract. A subset $D$ of the vertex set $V$ of a graph $G$ is called an $[1, k]$-dominating set if every vertex from $V - D$ is adjacent to at least one vertex and at most $k$ vertices of $D$. A $[1, k]$-dominating set with the minimum number of vertices is called a $\gamma_{[1,k]}$-set and the number of its vertices is the $[1, k]$-domination number $\gamma_{[1,k]}(G)$ of $G$. In this short note we show that the decision problem whether $\gamma_{[1,k]}(G) = n$ is an NP-hard problem, even for bipartite graphs. Also, a simple construction of a bipartite graph $G$ of order $n$ satisfying $\gamma_{[1,k]}(G) = n$ is given for every integer $n \geq (k + 1)(2k + 3)$.

Keywords: domination, $[1, k]$-domination number, $[1, k]$-total domination number, bipartite graphs.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset $D$ of $V(G)$ is called a dominating set, if every vertex from $V(G) - D$ has at least one neighbor in $D$. The minimum cardinality of a dominating set is called the domination number of $G$ and is denoted by $\gamma(G)$. A dominating set $D$ of $G$ is called a $[1, k]$-dominating set if every vertex of $V - D$ is adjacent to at most $k$ vertices of $D$. The minimum cardinality of a $[1, k]$-dominating set is the $[1, k]$-domination number of $G$ and denoted by $\gamma_{[1,k]}(G)$. We call a $[1, k]$-dominating set of cardinality $\gamma_{[1,k]}(G)$ a $\gamma_{[1,k]}(G)$-set. Clearly $\gamma(G) \leq \gamma_{[1,k]}(G) \leq |V(G)|$, which are the trivial bounds for $\gamma_{[1,k]}(G)$.

The invariant $\gamma_{[1,k]}(G)$ was introduced by Chellali et al. in [3] in the more general setting of the $[j,k]$-domination number of a graph. They proved that computing $\gamma_{[1,2]}(G)$ is an NP-complete problem. Among other results, it was shown that the trivial bounds are strict for some graphs in the case of $k = 2$. They also posed several questions; one of them was to characterize graphs for which the trivial lower bound is
strict for \( k = 2 \), that is \( \gamma_{[1,2]}(G) = \gamma(G) \). Recently, see [4], it was shown that there is no polynomial recognition algorithm for graphs with \( \gamma_{[1,k]}(G) = \gamma(G) \) unless \( P = NP \).

Some other problems from [3] have been considered in [1,2,5,9]. For instance, in [9] authors find planar graphs and bipartite graphs of order \( n \) with \( \gamma_{[1,k]}(G) = n \). More precisely, for integer \( n \) which is sufficiently large, they construct a bipartite graph \( G \) of order \( n \) with \( \gamma_{[1,2]}(G) = n \). The construction is complicated and work only for large \( n \).

In this note we present a simple construction of a bipartite graph \( G \) of order \( n \) with \( \gamma_{[1,k]}(G) = n \) for any integers \( k > 2 \) and \( n \geq (k + 1)(2k + 3) \). Hence, we generalize and simplify some results given in [9]. We also show that the decision problem \( \gamma_{[1,k]}(G) = n \) is NP-hard for a given bipartite graph \( G \) of order \( n \), \( n > k \geq 2 \).

2. PRELIMINARIES

Let \( G \) be a simple graph with vertex set \( V(G) \) and edge set \( E(G) \). An empty graph on \( n \) vertices \( K_n \) consists of \( n \) isolated vertices with no edges. A tree which has exactly one vertex of degree greater than two is said to be star-like. The vertex of maximum degree of such a tree is called the central vertex. The graph \( T - v \), where \( T \) is a star-like tree and \( v \) its central vertex, contains disjoint paths \( P_{n_1}, \ldots, P_{n_k} \) and is denoted by \( S(n_1, \ldots, n_k) \).

A subset \( D \) of \( V(G) \) is called a total dominating set if every \( v \in V(G) \) is adjacent to a vertex from \( D \). The minimum cardinality of a total dominating set in graph \( G \) is denoted by \( \gamma_t(G) \) and is called the total domination number. A total dominating set \( D \subseteq V(G) \) is a total \([1,k]-\text{dominating set} \), if for every vertex \( v \in V(G) \) is adjacent to at most \( k \) vertices from \( D \). While an \([1,k]-\text{dominating set}\) exists for every graph \( G \), there exist graphs which do not have any total \([1,k]-\text{dominating sets}\). By \( \gamma_{[1,k]}(G) \) we denote the minimum cardinality of a total \([1,k]-\text{dominating set}\) (if it exist), it is \( \infty \) if no total \([1,k]-\text{dominating set}\) exists. An example of a graph \( G_1 \cong S(2,2,2) \) with \( \gamma_{[1,2]}(G_1) = \infty \) is presented on Figure 2.

The lexicographic product of two graphs \( G \) and \( H \), denoted by \( G \circ H \), is a graph with the vertex set \( V(G \circ H) = V(G) \times V(H) \), where two vertices \((g,h)\) and \((g',h')\) are adjacent in \( G \circ H \) if \( g g' \in E(G) \) or \( g = g' \) and \( h h' \in E(H) \). It follows directly from the definition of the lexicographic product that \( G \circ H \) is bipartite if and only if one factor is the empty graph \( K_1 \) and the other is bipartite. Moreover, for a graph \( G \) on at least two vertices, the graph \( G \circ H \) is connected and bipartite if and only if \( G \) is connected and bipartite and \( H \cong K_1 \). See [7] for more informations about lexicographic and other products.

For any \( h_0 \in V(H) \), we call the set

\[
G^{h_0} = \{ (g,h_0) \in V(G \circ H) : g \in V(G) \}
\]

a \( G \)-layer of the graph \( G \circ H \). Similarly, for \( g_0 \in V(G) \), we call the set

\[
H^{g_0} = \{ (g_0,h) \in V(G \circ H) : h \in V(H) \}
\]

an \( H \)-layer of the graph \( G \circ H \).
Recently, see [8], $\gamma_{[1,k]}(G \circ H)$ was described as an optimization problem of some partitions of $V(G)$. For some special cases it is possible to present $\gamma_{[1,k]}(G \circ H)$ as an invariant of $G$. In particular, this is possible when $\gamma_{[1,k]}(H) > k$ and $H$ contains an isolated vertex.

**Theorem 2.1** ([8, Theorem 4.4]). Let $G$ be a connected graph, $H$ a graph and $k \geq 2$ an integer. If $\gamma_{[1,k]}(H) > k$ and $H$ contains an isolated vertex, then

$$\gamma_{[1,k]}(G \circ H) = \begin{cases} \gamma_{[1,k]}(G), & \text{if } \gamma_{[1,k]}(G) < \infty, \\ |V(G)| \cdot |V(H)|, & \text{otherwise.} \end{cases}$$

The following corollary is the direct consequence of Theorem 2.1 and will be useful later to construct bipartite graphs with $\gamma_{[1,k]}(G) = |V(G)|$.

**Corollary 2.2.** Let $G$ be a connected graph and $H \cong \overline{K}_{k+1}$. Then

$$\gamma_{[1,k]}(G \circ H) = |V(G \circ H)|$$

if and only if $G$ has no total $[1,k]$-dominating set.

3. COMPLEXITY

In this section we will show that it is $NP$-hard to check whether $\gamma_{[1,k]}(G) = |V(G)|$ for a bipartite graph $G$. For this aim, we first show that the related problem of checking whether $\gamma_{[1,k]}(G) = |V(G)|$ is $NP$-hard for a bipartite graph $G$. The problem is called a BipTotal $[1,k]$-set problem. To prove this we use reduction from a kind of a set cover problem, called $[1,k]$-triple set cover problem, which is known to be $NP$-hard as shown in [6]. Then, using Theorem 2.1, we prove that for a bipartite graph $G$, checking whether $\gamma_{[1,k]}(G) = |V(G)|$ is an $NP$-hard problem.

**Problem A:** $[1,k]$-triple set cover

**Input:** A finite set $X = \{x_1, \ldots, x_n\}$ and a collection $C = \{C_1, \ldots, C_t\}$ of 3-element subsets of $X$.

**Output:** Yes if there exists a $C' \subseteq C$ such that every element of $X$ appears in at least one and at most $k$ elements of $C'$, No otherwise.

**Problem B:** BipTotal $[1,k]$-set

**Input:** A bipartite graph $G$.

**Output:** Yes if there exists a $D \subseteq V(G)$ such that every element of $V(G)$ is adjacent to at least one and at most $k$ vertices of $D$, No otherwise.

We are going to prove that the BipTotal $[1,k]$-set problem is $NP$-hard by giving a polynomial time reduction from the $[1,k]$-triple set cover problem.
Definition 3.1. Let \( X = \{x_1, \ldots, x_n\} \) and \( C = \{C_1, \ldots, C_t\} \) be any given instance of Problem A. We construct a graph \( G_{X,C} \) as follows:

\[
V(G_{X,C}) = \bigcup_{i=1}^{t} (P_i \cup L_i) \cup X \cup \{c_1, \ldots, c_t\},
\]

where for each integer \( i, 1 \leq i \leq t \), we have \( P_i = \{p_{i,1}, \ldots, p_{i,k}\} \), \( L_i = \{l_{i,1}, \ldots, l_{i,k}\} \), and

\[
E(G_{X,C}) = \bigcup_{1 \leq j \leq t} \{c_j p_{j,1}, \ldots, c_j p_{j,k}, p_{j,1} l_{j,1}, \ldots, p_{j,k} l_{j,k}\} \cup \bigcup_{1 \leq i \leq t} \{x_i c_j : x_i \in C_j\}.
\]

Lemma 3.2. Let \( X = \{x_1, \ldots, x_n\} \) and \( C = \{C_1, \ldots, C_t\} \) be any collection of 3-element subsets of \( X \). Problem A for \( (X, C) \) is a YES instance if and only if \( G_{X,C} \) is a YES instance of Problem B.

Proof. Suppose that \( C' \) is a solution for the instance \( (X, C) \) of Problem A. We construct \( D \) as follows:

\[
D = \bigcup_{1 \leq j \leq t} P_j \cup \bigcup_{c_j \in C'} \{c_j\} \cup \bigcup_{c_j \notin C'} L_j.
\]

We can check easily that \( D \) is a \([1,k]\)-total set for \( G_{X,C} \). Conversely, suppose that \( G_{X,C} \) has a total \([1,k]\)-set \( D \). Clearly \( D \) must contain all vertices of \( P_i \) because every \( p_{j,j'} \) is adjacent to at least one leaf \( l_{j,j'} \). These vertices dominate every \( c_j \) exactly \( k \) times. Therefore, there is no vertex \( x_i \) in \( D \); in other words \( D \cap X = \emptyset \). So, every \( x_i \) must be dominated by a vertex of \( \{c_1, \ldots, c_t\} \). It is easy to see that there is a solution \( C' \subseteq C \) for \([1,k]\)-triple set cover problem if and only if the corresponding vertices \( C' \) of \( V(G) \) dominate all vertices of \( \{x_1, \ldots, x_n\} \) at least once and at most \( k \) times. These vertices dominate all vertices \( \{p_{j,1}, \ldots, p_{j,k}\} \) for \( c_j \in C' \). To dominate all other vertices we add \( \{l_{j,1}, \ldots, l_{j,k}\} \) to \( D \) for \( c_j \notin C' \).

The following example help us to understand the definition and the lemma.

Example 3.3. Let \( X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} \) and \( C = \{C_1, C_2, C_3, C_4, C_5\} \) such that \( C_1 = \{x_1, x_2, x_4\} \), \( C_2 = \{x_2, x_5, x_7\} \), \( C_3 = \{x_4, x_5, x_6\} \), \( C_4 = \{x_3, x_5, x_9\} \) and \( C_5 = \{x_3, x_8, x_9\} \). For \( k = 3 \), the corresponding graph \( G_{X,C} \) is shown in Figure 1. This is a YES-instance for the \([1,3]\)-set cover problem \((X, C)\), because \( C' = \{C_1, C_2, C_3, C_5\} \) has the desired property. The vertices of total \([1,3]\)-set are black vertices shown in Figure 1.

Theorem 3.4. The BipTotal \([1,k]\)-set problem is NP-hard

Proof. By Lemma 1 of [6] the \([1,k]\)-triple set cover problem is NP-hard. Hence, using Lemma 3.2 the BipTotal \([1,k]\)-set problem is also NP-hard.

The following theorem which is the main result of this section is a direct consequence of Theorem 3.4 and Corollary 2.2.

Theorem 3.5. For bipartite graphs it is NP-hard to decide whether we have \( \gamma_{[1,k]}(G) = |V(G)| \).
A note on bipartite graphs whose $[1,k]$-domination number.

Fig. 1. $G_{X,C}$ from Example 3.3

4. CONSTRUCTION

Here, for any integers $k \geq 2$ and $n \geq (k + 1)(2k + 3)$, we construct a bipartite graph $G$ of order $n$ with $\gamma_{[1,k]}(G) = n$. As already mentioned in [9], a bipartite graph $G$ of order $n$ was constructed for sufficiently large integer $n$ which satisfies $\gamma_{[1,2]}(G) = n$.

First, we give our construction in the case of $k = 2$, then we extend the result to the general case.

**Example 4.1.** If $G_1 \cong S(2,2,2)$, see Figure 2, and $H \cong K_3$, then $G = G_1 \circ H$, see Figure 3, is bipartite and $\gamma_{[1,2]}(G) = |V(G)|$ by Theorem 2.1.

Fig. 2. Bipartite graph $G_1 \cong S(2,2,2)$ with $\gamma_{(1,2)}(G_1) = \infty$

Fig. 3. Bipartite graph $G$ with $\gamma_{[1,2]}(G) = |V(G)|$
Theorem 4.2. For any integer $n \geq 21$, there exists a bipartite graph $\Gamma$ with $n$ vertices such that $\gamma_{[1,2]}(\Gamma) = n$.

Proof. Let $G_1 = S(2,2,2)$ and $G$ be graphs shown on Figures 2 and 3, respectively. By Example 4.1 $G$ is a bipartite graph with 21 vertices for which $\gamma_{[1,2]}(G) = 21$. Let $v_1, \ldots, v_7 \in V(G_1)$ be the vertices of $G_1$ as shown on Figure 2. For any integer $t \geq 1$, using the graph $G$ of Figure 3, we construct a new bipartite graph $\Gamma$ of order $n = 21 + t$ as follows:

$$\Gamma = (V(\Gamma), E(\Gamma)),$$

where

$$V(\Gamma) = V(G) \cup \{a_1, \ldots, a_t\} \quad \text{and} \quad E(\Gamma) = \bigcup_{h \in V(H)} \{a_1(v_2, h), \ldots, a_t(v_2, h)\} \cup E(G)$$

(see Figure 4).

Fig. 4. Bipartite graph $\Gamma$ with $\gamma_{[1,2]}(\Gamma) = |V(\Gamma)|$

Let $S$ be a $[1,2]$-set for $\Gamma$. First, we claim that there exists a vertex $h \in H$ with $(v_2, h) \in S$. To dominate the three vertices of the $H$-layer $H^{v_2}$, either there exists a vertex $h \in H$ with $(v_2, h) \in S$ or $H^{v_2} \subseteq S$. If there exists a vertex $h \in H$ with $(v_2, h) \in S$, then there is nothing to prove. If $H^{v_2} \subseteq S$, then every vertex of $H^{v_2}$ is dominated at least three times, hence $H^{v_2} \subseteq S$. Therefore the claim is true and $H^{v_2} \cap S \neq \emptyset$. By the same reasoning we have $H^{v_4} \cap S \neq \emptyset$ and $H^{v_6} \cap S \neq \emptyset$. Hence, by the definition of lexicographic product of graphs, every vertex of $H^{v_7}$ is dominated at least three times. Therefore, we have

$$H^{v_7} \subseteq S. \quad (4.1)$$

Now, by (4.1) every vertex in $H^{v_2} \cup H^{v_4} \cup H^{v_6}$ is dominated at least three times and so we have

$$H^{v_2} \cup H^{v_4} \cup H^{v_6} \subseteq S. \quad (4.2)$$
A note on bipartite graphs whose $[1,k]$-domination number... 381

And, then by (4.2) we conclude that

$$H^{v_1} \cup H^{v_3} \cup H^{v_5} \cup \{a_1, \ldots, a_t\} \subseteq S.$$ 

Therefore, $S = V(\Gamma)$, as desired. □

We end with a generalization of the above result from $k = 2$ to $k \geq 2$.

**Theorem 4.3.** For integers $k \geq 2$ and $n \geq (k + 1)(2k + 3)$, there exists a bipartite graph $\Gamma$ with $n$ vertices such that $\gamma_{[1,k]}(\Gamma) = n$.

**Proof.** Let $G_1 = S(2, 2, \ldots, 2)$ be a star-like tree with $2k+3$ vertices and let $H \cong K_{k+1}$. Clearly $G = G_1 \circ H$ is a bipartite graph. Let $v_1 \in V(G_1)$ be a vertex of degree one and $v_2 \in V(G_1)$ be its only neighbor. For any integer $t \geq 1$, using the graph $G$, we construct a new bipartite graph $\Gamma$ of order $n = (k + 1)(2k + 3) + t$ as follows:

$$\Gamma = (V(\Gamma), E(\Gamma)),$$

where

$$V(\Gamma) = V(G) \cup \{a_1, \ldots, a_t\} \quad \text{and} \quad E(\Gamma) = \bigcup_{h \in V(H)} \{a_1(v_2, h), \ldots, a_t(v_2, h)\} \cup E(G).$$

Let $S$ be a $\gamma_{[1,k]}(\Gamma)$-set. By the same reasoning as in the proof of Theorem 4.2 one can show that $H^{v_i} \cap S \neq \emptyset$ for every vertex $v_i \in V(G_1)$ with $\deg_{G_1}(v_i) = 2$. Since there are $k + 1$ such vertices in $G_1$, all vertices of $H^v$ must be in $S$ for a central vertex $v$ of $G_1$. This clearly leads to $\gamma_{[1,k]}(\Gamma) = |V(\Gamma)|$ because $|H^v| = k + 1$. □

**Acknowledgements**

The work of the second author was partially supported by Slovenian research agency under the grants P1-0297 and J1-9109.

**REFERENCES**

[1] A. Bishnu, K. Dutta, A. Gosh, S. Pual, $[1,j]$-set problem in graphs, Discrete Math. 339 (2016), 2215–2525.

[2] M. Chellali, O. Favaron, T.W. Haynes, S.T. Hedetniemi, A. McRae, Independent $[1,k]$-sets in graphs, Australasian J. Combin. 59 (2014), 144–156.

[3] M. Chellali, T.W. Haynes, S.T. Hedetniemi, A. McRae, $[1,2]$-sets in graphs, Discrete Appl. Math. 161 (2013), 2885–2893.

[4] O. Etesami, N. Ghareghani, M. Habib, M. Hooshmandasl, R. Naserasr, P. Sharifani, When an optimal dominating set with given constraints exists, Theoret. Comput. Sci. 780 (2019), 54–65.

[5] A.K. Goharshady, M.R. Hooshmandasl, M.A. Meybodi, $[1,2]$-sets and $[1,2]$-total sets in trees with algorithms, Discrete Appl. Math. 198 (2016), 136–146.
[6] P. Golovach, J. Kratochvíl, *Computational complexity of generalized domination: a complete dichotomy for chordal graphs*, [in:] *International Workshop on Graph-Theoretic Concepts in Computer Science*, Springer, 2007, pp. 1–11.

[7] R. Hammack, W. Imrich, S. Klavžar, *Handbook of Product Graphs*, CRC Press, 2011.

[8] N. Ghareghani, I. Peterin, P. Sharifani, *[1, k]-domination number of lexicographic products of graphs*, Manuscript (2019).

[9] X. Yang, B. Wu, *[1, 2]-domination in graphs*, *Discrete Appl. Math.* 175 (2014), 79–86.

Narges Ghareghani
ghareghani@ut.ac.ir

University of Tehran
Department of Industrial Design
College of Fine Arts
Tehran, Iran

Iztok Peterin (corresponding author)
iztok.peterin@um.si

University of Maribor
Faculty of Electrical Engineering and Computer Science
Koroška 46, 2000 Maribor, Slovenia

Institute of Mathematics, Physics, and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia

Pouyeh Sharifani
pouyeh.sharifani@gmail.com

Institute for Research in Fundamental Sciences (IPM)
School of Mathematics
Tehran, Iran

Received: October 6, 2019.
Revised: February 25, 2020.
Accepted: February 26, 2020.