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Phys. Rev. Lett. 116, 011601 — Published 6 January 2016
DOI: 10.1103/PhysRevLett.116.011601
Complexified path integrals, exact saddles and supersymmetry

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In the context of two illustrative examples from supersymmetric quantum mechanics we show that the semi-classical analysis of the path integral requires complexification of the configuration space and action, and the inclusion of complex saddle points, even when the parameters in the action are real. We find new exact complex saddles, and show that without their contribution the semi-classical expansion is in conflict with basic properties such as positive-semidefiniteness of the spectrum, and constraints of supersymmetry. Generic saddles are not only complex, but also possibly multi-valued, and even singular. This is in contrast to instanton solutions, which are real, smooth, and single-valued. The multi-valuedness of the action can be interpreted as a hidden topological angle, quantized in units of $\pi$ in supersymmetric theories. The general ideas also apply to non-supersymmetric theories.

Introduction: We address the question of how to properly define the semi-classical expansion of the path integral in quantum mechanics and quantum field theory. This question goes beyond the problem of studying the semi-classical approximation, because the theory of resurgence shows that the semi-classical expansion encodes perturbative as well as non-perturbative effects, and may provide a complete definition of the path integral \cite{1, 2}. We consider a set of examples for which we show that the path integral measure and action must be complexified, and that novel complex saddle points appear. The usefulness of complexification is not surprising from the point of view of the steepest descent method for ordinary integration, but important new effects appear in functional integrals. We show that in generic cases complexification is indeed essential. Our results go beyond proposals in the literature to complexify the path integral in cases where coupling constants are analytically continued away from their physical values, as described in the work of Witten on Chern-Simons theory \cite{3}, and Harlow, Maltz and Witten on Liouville theory \cite{4}, and is potentially related to the complexification of the phase space formulation of path integral \cite{5}. Complex saddles were previously studied as a computational tool in quantum mechanics, see e.g. \cite{6–9}. Complex path integrals were also studied in connection with the sign problem in the Euclidean path integral of QCD and related model systems at finite chemical potential \cite{10–14}. Here, we demonstrate the necessity of complexification even for the physical theory with real couplings. In \cite{15} we show that these complex saddles have a natural interpretation in terms of thimbles in Picard-Lefschetz theory.

There are several calculations in field theory that suggest the importance of complex saddle points. As an example consider $\mathcal{N} = 1$ supersymmetric gluodynamics on $\mathbb{R}^3 \times S_1$ with SUSY preserving boundary conditions. This theory is confining, and has a non-perturbatively generated bosonic potential for the Polyakov line. The potential for the Polyakov line can be computed using bions, molecules of monopole-instantons \cite{16, 17}. Bions also determine the vacuum energy, with the conclusion that supersymmetry is unbroken, e.g. for $SU(2)$ theory, $E_{gr} \propto -e^{-2\lambda_0} - e^{-2\lambda_0 \pm \pi} = 0$ where the first is from magnetic bion and the latter from neutral bion. This calculation agrees with a calculation based on supersymmetry and the monopole instanton induced superpotential \cite{18}. A puzzle concerning this result is that the sum over different bion types give zero vacuum energy, despite the fact that contribution of real saddles is universally negative-semidefinite \cite{33}.

The calculations in \cite{17, 19} are based on analytic continuation in the coupling constant. Ref. \cite{20} reinterprets the relative sign between the two different bion types as a hidden topological angle (HTA), a factor $\exp(i\pi)$ associated with the relative phase in the quasi-zero mode Lefschetz thimble, which is nothing but a direction in field space. This result suggests that the calculation can be done directly for real values of $g$, and that bions arise as exact (non-BPS) saddle point solutions of the complexified path integral, and furthermore that the HTA is related to the imaginary part of the complexified action.

SUSY gluodynamics on $\mathbb{R}^3 \times S_1$ is not an isolated case. Similar phenomena occur in $\mathcal{N} = 1 SU(2)$ SUSY QCD \cite{21}, in three-dimensional SUSY gauge theory \cite{22}, and in $\mathcal{N} = 2$ SUSY QM \cite{23}. In this paper we make the basic idea precise in the context of SUSY quantum mechanics.

Formalism and holomorphic Newton’s equation: Consider the Euclidean quantum mechanical path integral as a sum over real paths, $Z = \int \mathcal{D}x(t) \exp(-\frac{1}{\hbar}S_E)$, with $S_E = \int dt (\frac{1}{2} \dot{x}^2 + V(x))$. The critical points solve Newton’s equation in the inverted potential, $\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}$. This leads to the standard multi-instanton calculus in quantum mechanics. More general saddle points appear in the complexified path integral

$$Z = \int_{\Gamma} \mathcal{D}z \ e^{-\frac{1}{\hbar}S[z(t)]}, \quad S[z(t)] = \int dt \left( \frac{1}{2} \dot{z}^2 + V(z) \right); \quad (1)$$

where $\Gamma$ is an integration cycle that has the same dimensionality as the original real path integral. The critical points of the complexified path integral solve the \textit{holomorphic Newton’s equation} in the inverted potential $-V(z): \frac{d^2z}{dt^2} = -\frac{\partial V}{\partial \bar{z}}$. In terms of real and imaginary parts of the potential, $V(z) = V_r(x,y) + iV_i(x,y)$, we get

$$\frac{d^2x}{dt^2} = \frac{\partial V_r}{\partial x}, \quad \frac{d^2y}{dt^2} = -\frac{\partial V_i}{\partial y}, \quad (2)$$
where we have used the Cauchy-Riemann equations $\partial_x V_i = \partial_y V_i$, and $\partial_y V_i = -\partial_x V_i$. An important aspect of (2) is that it does not describe an ordinary two-dimensional classical mechanical system: the holomorphic classical mechanics is not the same as the motion of a particle in the two-dimensional inverted potential $-V_i(x,y)$. Instead of the usual Newton equations with force $\nabla V_i(x,y)$, the force in the $x$-direction is due to $\nabla x V_i(x,y)$ while the force in the $y$-direction is due to $-\nabla y V_i(x,y)$. This has interesting consequences.

Supersymmetric quantum mechanics: Consider supersymmetric quantum mechanics with the superpotential $W(y)$

$$S = \int dt \left( \frac{1}{2} \dot{\psi}^2 + \frac{1}{2} (W')^2 + [\overline{\psi}\psi + p W' \overline{\psi} \psi] \right), \quad (3)$$

corresponding to $p = 1$. The parameter $p$ will be used to deform the theory away from the supersymmetric point [9]. We choose $W(y)$ with more than one critical point, so that there will be real instantons. By projecting to fermion number eigenstates one obtains a pair of Hamiltonians $H_{\pm}$ [24]:

$$H_{\pm} = \frac{1}{2} \dot{\psi}^2 + V_{\pm}(\psi), \quad V_{\pm}(\psi) = \frac{1}{2} (W'(\psi))^2 \pm \frac{p}{2} W''(\psi). \quad (4)$$

In the following we consider superpotentials of the form $W(y) = \frac{1}{8} W(\sqrt{8} y)$, and rescale $x = \sqrt{8} y$. Then the Euclidean action takes the form $S_E = \frac{1}{8} \int dt (\frac{1}{2} \dot{\psi}^2 + V_{\pm}(\psi))$. We work with the bosonized description (4). Note that compared to the original bosonic potential $\frac{1}{2} (W')^2$ the bosonized theory contains an $O(g)$ term that arises from integrating out the fermions. The quantum modified holomorphic equations of motion in the inverted potential $-V_+(z)$ is

$$\frac{d^2 z}{dt^2} = W'(z) W''(z) + \frac{pg}{2} W'''(z). \quad (5)$$

Double well potential: Consider $W(x) = x^3/3 - x$, so that $V(x)$ is an asymmetric double well potential with an $O(g)$ “tilt”. The ground state energy of the system is zero to all orders in perturbation theory, but non-perturbatively supersymmetry is spontaneously broken and the ground state energy is non-zero and positive [24]. Note that the positivity of the ground state energy is a consequence of the SUSY algebra, $H = \frac{1}{8} \{ \mathcal{Q}, \bar{\mathcal{Q}} \}$, where $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ are the SUSY generators.

In the original formulation (3) this can be understood as the contribution from approximate instanton-anti-instanton solutions of the bosonic potential $\frac{1}{2} (W')^2$ [9]. In the bosonized version we seek classical solutions in the inverted potential $-V_+$. However, the real equations of motion in the inverted potential have no finite action configurations except for the trivial perturbative saddle, and an exact (real) bounce solution. But this bounce is related to the false vacuum and is not directly relevant for ground state properties, which are determined by saddles starting at the global maximum of the inverted potential. But the real motion of a classical particle starting at such a global maximum is unbounded, and has infinite action.

On the other hand, the holomorphic Newton’s equation (5) does support finite action solutions starting from the global maximum. There are exact finite action complex solutions that start at the global maximum of the inverted potential and bounce back from one of the two complex turning points, whose real part is located near the top of the local maximum, see Fig. 1. We refer to this as the “complex bion” solution:

$$z_{cb}(t) = z_{cr} \pm \frac{z_{cr} - z_T}{2} \coth \left( \frac{\omega_{cb} t + t_0}{2} \right) \left[ \tanh \left( \frac{\omega_{cb} (t + t_0)}{2} \right) - \tan \left( \frac{\omega_{cb} (t - t_0)}{2} \right) \right], \quad (6)$$

where $z_{cb}(\pm \infty) = z_{cr}$ is the global maximum of the inverted potential, and $z_T = -z_{cr} \pm i \sqrt{pg}/(z_{cr}^2)$ are the complex turn-
The inverted potential, Fig. 3, has global maxima at \( \frac{\pi}{2} = \pm \ell \pi \) and \( \frac{3\pi}{2} = \pm \ell \pi \), and \( \pi \), where \( \ell \) is an integer. At \( \ell = 1 \) we have a complex conjugate pair of turning points, and complex conjugate actions. This implies that we can deform the contour into a large circle in the \( \mathbb{C} \)-plane. If \( \gamma \) winds twice as \( n \) encircles the critical points. This proves the quantization of the HTA in the supersymmetric \( p = 1 \) limit.

For the \( p \neq 1 \) non-supersymmetric deformation of the theory, the perturbative ground state energy does not vanish anymore, but the energy spectrum must still be unambiguous. In that case, we show in [15] that the ambiguity inherent to the Borel resummation of perturbation theory cancels exactly the two-fold ambiguous complex bion amplitude, as an explicit illustration of resurgence.

**Periodic potential:** Now consider the superpotential \( W(x) = 4\cos(x/2) \). In this system supersymmetry is unbroken [24]. There are two degenerate ground states, one bosonic and one fermionic, both with vanishing ground state energy. After the fermion is integrated out we obtain the bosonic potential

\[
V_\pm(x) = 2\sin^2(x/2) \pm \frac{pg}{2} \cos(x/2).
\]

The inverted potential, Fig. 3, has global maxima at \( x = 4n\pi \), and local maxima at \( x = (4n + 2)\pi \). (The potential has period \( 4\pi \)). There is an exact real bounce solution starting at the local maximum, and bouncing from a real turning point, but again this is not directly relevant for ground state properties. Now we find two types of exact bion solutions, shown in Fig. 3. The first is a “real bion”, connecting neighboring
global maxima, say at \( x = 0 \) and \( x = 4\pi \). It has the form of an instanton-instanton solution, and as such has no analogue in the double-well case. There is also a complex bion solution that starts from a global maximum of the inverted potential, and is reflected from a complex turning point, with real part near the local maximum. This solution can be found directly or by analytic continuation from the real bounce, \( p \to pe^{i\theta} \), and leads to an exact finite action complex saddle:

\[
z_{\text{cb}}(t) = 2\pi \pm 4 \left( \arctan e^{-\omega_{\text{cb}}(t-t_0)} + \arctan e^{\omega_{\text{cb}}(t-t_0)} \right),
\]

where \( \omega_{\text{cb}} = \sqrt{V''(0)} = \sqrt{1 + \frac{pg}{S}} \). The complex parameter \( t_0 \simeq \frac{1}{2\omega_{\text{cb}}} \ln \left( \frac{-32}{pg} \right) \), where \( \text{Re}[2t_0] \) is the complex bion size. The action is

\[
S_{\text{cb}} \simeq \left( \frac{16}{g} + p \ln \frac{32}{pg} + \ldots \right) \pm i p \pi.
\]

The complex bion has the form of a complex instanton/anti-instanton molecule. An interesting new feature of this solution is that it is singular at \( t = \pm t_0 \), even though the action is finite. Physically this is because the real part of the holomorphic potential has ridges along the \( y \) direction, and the holomorphic equations of motion allow the particle to roll up (notice relative signs in (2)) along one ridge and then jump to the next ridge at infinity before rolling back again.

The analytic continuation in \( \theta \) smooths this singularity, and the solution is correspondingly multivalued as \( \theta \to \pi \pm \varepsilon \): see Fig. 4. As \( \theta \to \pi \) the real part of \( z_{\text{cb}}(t) \) has a discontinuity, and the imaginary part diverges. The action is finite, because the divergence in the action integral due to the singular behavior in \( \text{Re} z(t) \) and \( \text{Im} z(t) \) cancel. Fig. 5 shows the real and imaginary parts of the action as a function of the \( \theta \) parameter.

In the semi-classical limit the ground state can be described as a dilute gas of complex and real bions, with energy

\[
E_{gs} \sim -e^{-S_{\text{cb}}} - e^{-S_{\text{rb}}} = -e^{\pm i\pi} e^{-2S_{\text{rb}}} - e^{-2S_{\text{rb}}} = 0,
\]

consistent with the requirement of supersymmetry. The non-inclusion of the multi-valued saddle would result in a negative ground state energy and a conflict with the constraints of supersymmetry algebra. This proves that in order for the semi-
classical analysis to be consistent with the supersymmetry algebra, it is essential to include singular, multi-valued complex bion solution. This resolves a deep puzzle raised in [4].

Conclusions: We have presented two examples that demonstrate the need to include complex, and even singular and multi-valued, saddle point solutions of the path integral. We obtained exact finite action saddle points of the complexified path integral in supersymmetric quantum mechanics with a double well and Sine-Gordon potential. In both cases these new complex bion configurations are essential in order to obtain agreement with known results and the requirements of supersymmetry. This phenomenon is not restricted to quantum mechanics: analogous effects occur in several field theories, such as sigma models with fermions [2, 26–29] SUSY gluodynamics and Sine-Gordon potential, and [30, 31], SUSY QCD with one quark flavor [21, 32], and three dimensional SUSY N = 2 gauge theory [22]. Clearly, it is of interest to study these field theories, and ultimately QCD, using complexified path integrals.

Acknowledgments: We thank P. Argyres, D. Harlow, and E. Witten for useful comments and discussions. M. Ü was partially supported by the Center for Mathematical Sciences and Applications (CMSA) at Harvard University. We acknowledge support from DOE grants DE-FG02-03ER41260 and DE-SC0010339.

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