The response of self-gravitating protostellar discs to slow reduction in cooling timescale: the fragmentation boundary revisited

C.J. Clarke 1, E. Harper-Clark2, G. Lodato 3
1 Institute of Astronomy, Madingley Road, Cambridge, CB3 0HA
2 Department of Astronomy and Astrophysics, 50 St. George Street, Toronto, Ontario, Canada, M5S 3H4
3 Department of Physics and Astronomy, University of Leicester, University Road, Leicester, LE1 7RH

ABSTRACT

A number of previous studies of the fragmentation of self-gravitating protostellar discs have involved suites of simulations in which radiative cooling is modeled in terms of a cooling timescale (t_{cool}) which is parameterised as a simple multiple (\beta_{cool}) of the local dynamical timescale. Such studies have delineated the 'fragmentation boundary' in terms of a critical value of \beta_{cool} (\beta_{crit}) such that the disc fragments if \beta_{cool} < \beta_{crit}. Such an approach however begs the question of how in reality a disc could ever be assembled in a state with \beta_{cool} < \beta_{crit}.

Here we adopt the more realistic approach of effecting a gradual reduction in \beta_{cool}, as might correspond to changes in thermal regime due to secular changes in the disc density profile. We find that the effect of gradually reducing \beta_{cool} (on a timescale longer than t_{cool}) is to stabilise the disc against fragmentation, compared with models in which \beta_{cool} is reduced rapidly (over less than t_{cool}). We therefore conclude that the ability of a disc to remain in a self-regulated, self-gravitating state (without fragmentation) is partly dependent on the disc's thermal history, as well as its current cooling rate. Nevertheless, the effect of a slow reduction in t_{cool} appears only to lower the fragmentation boundary by about a factor two in t_{cool} and thus only permits maximum ‘\alpha’ values (which parameterise the efficiency of angular momentum transfer in the disc) that are about a factor two higher than determined hitherto. Our results therefore do not undermine the notion that there is a fundamental upper limit to the heating rate that can be delivered by gravitational instabilities before the disc is subject to fragmentation. An important implication of this work, therefore, is that self-gravitating discs can enter into the regime of fragmentation via secular evolution and it is not necessary to invoke rapid (impulsive) events to trigger fragmentation.

Key words: accretion, accretion discs – star: formation – gravitation– instabilities – stars: formation

1 INTRODUCTION

Following the seminal work of Gammie (2001), there has been considerable progress in recent years in understanding the behaviour of self-gravitating accretion discs (see Durisen et al. (2007) and references therein). A number of simulations (Gammie 2001; Rice et al. 2003; Lodato & Rice 2004, 2005) have demonstrated that if the thermodynamic properties of the disc are evolved according to a thermal equation (involving a cooling term parameterised in terms of a cooling timescale, t_{cool}), then the disc may be able to establish a self-gravitating, self-regulated state. In this state, the Toomre Q parameter:

\[ Q = \frac{c_s \kappa}{\pi G \Sigma}, \] (1)

(where \( c_s \) is the sound speed, \( \kappa \) is the epicyclic frequency (equal to the angular velocity \( \Omega \) in a Keplerian disc) and \( \Sigma \) is the disc surface density) hovers at a value somewhat greater than unity over an extended region of the disc. Whereas the state \( Q = 1 \), corresponds to a situation of marginal stability against axisymmetric perturbations, in the self-regulated state the disc is instead subject to a variety of non-axisymmetric self-gravitating modes whose effect, through the action of weak shocks, is to dissipate mechanical energy (i.e. kinetic and potential energy of the accretion flow) as heat. Thermal equilibrium is then attained through the balancing of such heating by the prescribed radiative cooling: in essence, self-regulation results when the amplitude of these modes is able to self-adjust so as to maintain thermodynamic equilibrium against the relevant energy loss processes.
The above studies have all found, however, that such self-regulation is only possible in the case that the cooling timescale is not too short: stability demands that \( \beta_{\text{cool}} = t_{\text{cool}}/\Omega \) exceeds a critical value which, for discs with adiabatic index of 5/3, is \( \sim 7 \) \cite{Rice2003}. In the case of more rapid cooling, the disc instead fragments.

Such simulations however approach the ‘fragmentation boundary’ in a manner that is unlikely ever to apply to discs in reality. In the simulations, the discs are set up without additional heating mechanisms and are subject to cooling at some prescribed value of \( \beta_{\text{cool}} \). Discs with \( \beta_{\text{cool}} < \beta_{\text{crit}} \) then fragment on the local cooling timescale (i.e., a few times the local dynamical timescale), thus begging the question of how such unstable initial conditions could ever have been set up in the first place.

A more likely scenario for disc fragmentation is that the disc is instead set up in self-regulated, self-gravitating state and then conditions gradually change so that \( \beta_{\text{cool}} \) is lowered. (For example, continued infall of material onto a disc or secular re-arrangement of material in the disc due to the action of gravitational torques could alter the surface density profile of the disc and allow it to enter a new cooling regime with lower \( \beta_{\text{cool}} \)). It is not however clear that the fragmentation boundary would be the same in the case that \( \beta_{\text{cool}} \) is gradually reduced.

In this paper we conduct a suite of idealised simulations in which we explore whether the fragmentation boundary just depends on the instantaneous value of \( \beta_{\text{cool}} \) (as has been assumed hitherto) of whether the system ‘remembers’ the history of how it evolved to a point of given \( \beta_{\text{cool}} \). Such a (“toy model”) approach, is complementary to studies \cite{Boley2006, Mayer2007, Stamatellos2007}, see also the analytical estimates by \cite{Rafikov2005a, Rafikov2005b} which attempt to achieve ever-increasing verisimilitude via the incorporation of more realistic treatments of radiative transfer. Here, instead, we make no claims that the simplified cooling law (for example, the assumption that \( \beta_{\text{cool}} \) is spatially uniform) actually corresponds to a situation encountered in a real disc, because our aim is to isolate a particular physical effect (i.e., the timescale on which the fragmentation boundary is approached). The computational expense of ‘realistic’ simulations however prevents their use to study secular effects: even in the case of the present ‘toy’ simulations, it is impracticable to run simulations over the long timescales on which the \( \Sigma \) profile changes due to gravitational torques or infall. We can nevertheless assess the effect of relatively slow changes in \( \beta_{\text{cool}} \) on the fragmentation boundary through imposing an ad hoc reduction in the value of \( \beta_{\text{cool}} \) and can apply this insight to the secular evolution of real discs.

In particular, we want to examine the cause of the fragmentation for \( \beta_{\text{cool}} < \beta_{\text{crit}} \), that has been found in previous simulations. Is this (i) due to the disc’s inability to maintain - under any circumstances - a gravitational heating rate that can match the imposed high cooling rate? This is the hypothesis of \cite{Lodato2005}, who identify the minimum value of \( \beta_{\text{cool}} \) with a maximum value of the gravitationally induced angular momentum transfer that can be delivered by a disc without its fragmenting. They parameterise this state of maximal angular momentum transfer in terms of the ratio of the \( r, \phi \) component of the stress tensor to the thermal pressure, i.e., by analogy with the equivalent expression for a viscous disc, in terms of a maximum in the well known viscous ‘\( \alpha \)’ parameter \cite{Shakura1973}. A critical value of \( \beta_{\text{cool}} \) of \( \sim 7 \) corresponds to a maximum \( \alpha \) of \( \sim 0.06 \).

Alternatively, (ii) does fragmentation instead reflect the disc’s inability to set up the required high heating rate on the short timescale (\( t_{\text{cool}} \)) on which the disc is cooling? If this were the case, then with sufficiently gradual approach to the regime of low \( \beta_{\text{cool}} \), the disc could in principle deliver a value of \( \alpha \) that exceeded the above limit by a generous margin.

We can obviously distinguish between these alternatives by investigating the case in which \( \beta_{\text{cool}} \) is reduced on a timescale \( \tau \) that is longer than \( t_{\text{cool}} \); since in this case the disc temperature will fall via a sequence of thermal equilibrium states (on timescale \( \tau \)), rather than dropping on timescale \( t_{\text{cool}} \). The aim of this investigation is thus to see whether the disc is more resistant to fragmentation in the regime that \( \tau > t_{\text{cool}} \). If it is not, then the manner in which the disc approaches the fragmentation boundary is unimportant. If, on the other hand, it is found that rapid changes in cooling regime are required, then it may be necessary to invoke impulsive events (such as an external dynamical interaction) to trigger fragmentation.

In Section 2 we describe the numerical setup, discuss our results in Section 3 and in Section 4 we present some conclusions.

## 2 NUMERICAL SETUP

### 2.1 The SPH code

Our three-dimensional numerical simulations are carried out using SPH, a Lagrangian hydrodynamic scheme \cite{Benz1990, Monsaert1992}. The general implementation is very similar to \cite{Lodato2004, Lodato2005} and \cite{Rice2005}. The gas disc is modeled with 250,000 SPH particles (500,000 in a run used as a convergence test) and the local fluid properties are computed by suitably averaging over the neighbouring particles. The disc is set in almost Keplerian rotation (allowing from slight departures from it to account for the effect of pressure forces and of the disc gravitational force) around a central point mass onto which gas particles can accrete if they get closer than the accretion radius, taken to be equal to 0.5 code units.

The gas disc can heat up due to \( p dV \) work and artificial viscosity. The ratio of specific heats is \( \gamma = 5/3 \). Cooling is here implemented in a simplified way, i.e. by parameterizing the cooling rate in terms of a cooling timescale:

\[
\frac{du}{dt}_{\text{cool}} = -\frac{n_e}{t_{\text{cool}}},
\]

where \( n_e \) is the internal energy of a particle and the cooling timescale \( t_{\text{cool}} \) is assumed to be proportional to the dynamical timescale, \( t_{\text{cool}} = \beta_{\text{cool}} \Omega^{-1} \), where \( \beta_{\text{cool}} \) is varied according to a time-dependent prescription (see Section 2.3 below).

Artificial viscosity is introduced using the standard SPH formalism. The actual implementation is very similar to the one used in \cite{Rice2005}, that is we set the two relevant numerical parameters to \( \alpha_{\text{SPH}} = 0.1 \) and \( \beta_{\text{SPH}} = 0.2 \) and we have not included here (consistent with \cite{Rice2005}) the so-called Balsara switch \cite{Balsara1995} to reduce shear viscosity.

### 2.2 Disc setup

The main physical properties of the disc at the beginning of the simulation are again similar to those of \cite{Lodato2004, Lodato2005}. The disc surface density \( \Sigma \) is initially proportional to \( R^{-1} \) (where \( R \) is the cylindrical radius), while the temperature is initially proportional to \( R^{-1/2} \). Given our simplified form of the cooling function, the computations described here are essentially scale free and can be rescaled to different disc sizes and masses. For reference, we will assume that the unit mass (which is the mass of the central star) is...
1M⊙ and that the unit radius is 1AU. In this units the disc extends from Rin = 0.25AU to Rout = 25AU. The normalization of the surface density is generally chosen such as to have a total disc mass of Mdisc = 0.1M⊙, while the temperature normalization is chosen so as to have a minimum value of Q = 2, which is attained at the outer edge of the disc.

Initially, the disc is evolved with constant βcool = 7.5, this value of βcool being in the regime where previous work [Gammie (2001), Rice et al. (2005)] has shown that the disc does not fragment. The general features of this initial evolution is described in detail in Lodato & Rice (2004). The disc starts cooling down until the vertical scale-length H is reduced such that H/R ≈ Mdisc/M* = 0.1. At this point the disc becomes Toomre unstable and develops a spiral structure that heats up the disc and maintains it close to marginal stability. We have evolved the disc with this value of βcool for 7.8 outer disc orbits. At this stage it is close to Q = 1 over most of the disc (i.e. over the radial range R ≈ 3 – 23 A.U.).

| Simulation | x     | βhold | N     | fragmentation |
|-----------|-------|-------|-------|---------------|
| F1        | 10.5  | —     | 250K  | yes           |
| F2        | 10.5  | 3     | 250K  | yes           |
| V         | 105-10.5 | 3     | 250K  | yes           |
| S1        | 105   | 3     | 250K  | no            |
| Sh        | 105   | 3     | 500K  | yes           |
| S2        | 105   | 2.75  | 250K  | no            |
| S3        | 105   | 2.62  | 250K  | yes           |
| VS1       | 314   | 3     | 250K  | no            |
| VS2       | 314   | 2.75  | 250K  | yes           |

Table 1. Details of the various simulations discussed in this paper. The different columns indicate: the name of the run, the value of the parameter x determining the speed of the reduction of the cooling time, the value of βhold (if any) at which the cooling time was held fixed after reduction, the number of particles used in the run N and whether fragmentation did occur or not. Simulation V was performed with an initially slow reduction of β (with x = 105), followed by a fast reduction (with x = 10.5), so that it would reach β = 3 with a fast reduction at the same time as simulation S1.

2.3 Evolution of βcool

After evolution of the disc with βcool = βcool(0) = 7.5 for a cooling timescale, we effect a linear reduction of βcool on a timescale T, i.e.

βcool(t) = βcool(0) \left(1 - \frac{t}{T}\right) \quad (3)

where we set T = xΩ−1(Rout).

Such a prescription implies that the timescale τ(t) on which the local instantaneous value of tcool (i.e. tcool(R, t)) drops to zero is

τ(t) = \frac{βcool(t)}{βcool(0)} = \frac{x}{(R/Rout)^{1.5}} tcool(R, t). \quad (4)

We adopt three values of x: x = 10.5 (fast), x = 105 (slow) and x = 314 (very slow). In the fast case, τ(R, t) ~ tcool(R, t) in the outer disc so tcool is changing faster than the disc can come into thermal equilibrium at that value of tcool. In the slow case, τ(r, t) > tcool(r, t) so that the disc is everywhere able to come into thermal equilibrium at that tcool. This situation is even more amply satisfied in the very slow case.

For each value of x, we run the simulation until a fragmentation occurs. The dotted line is the result of simulation F1 and the thick solid line is the highest resolution run (Sh). Finally, the dashed line is the simulation with variable rate of change of βcool (S3). An asterisk at the end of the line indicates fragmentation at this time, whereas runs without an asterisk are unfragmented at the end of the simulation.

The reduction in βcool when it attains a value equal to βhold (> βcool).

We then experiment with values of βhold in order to find the minimum value of βcool at which the disc does not fragment over the duration of the numerical experiment. Table 1 and Fig. 1 summarize the main details of the various runs we have performed, where the ‘F’ simulations are the ‘fast’ ones, the ‘S’ are the ‘slow’ ones and the ‘VS’ are the ‘very slow’ ones (see discussion in Section 3 below).

2.4 Resolution issues

One important aspect that needs to be taken into account is whether the resolution of our simulation is high enough to reproduce fragmentation, when it occurs. Resolution criteria for fragmentation with SPH codes have been discussed by Bate & Burkert (1997). They obtained that SPH correctly reproduces fragmentation if the relevant Jeans mass contains at least 100 SPH particles, that is twice the typical number of neighbors (Nneigh = 50) within one smoothing region. More recently, Nelson (2006) has revisited this issue focussing on fragmentation in self-gravitating discs and has found a slightly more stringent criterion, requiring that the Jeans mass is resolved with three times as many particles as required by Bate & Burkert (1997). In a gravitationally unstable disc, the most unstable wavelength is given by λ = 2c2/GΣ. The Jeans mass (or, as Nelson 2006 calls it, the “Toomre mass”) is then given by:
M_\text{tot} = \pi \Sigma \right)^{2} = \frac{4 \pi \Sigma^{2}}{G \Sigma} = 4 \pi \Omega^{2} \left( \frac{H}{R} \right)^{2} \Sigma R^{2}. \quad (5)

The cumulative disc mass at radius R is given by:

M_{\text{disc}}(R) = 2 \pi \Sigma R^{2} = M_{\text{disc}} \frac{R}{R_{\text{out}}}, \quad (6)

since in our setup \Sigma \approx R^{-1} (see above). We can then rewrite the Jeans mass using eq. (6), as:

M_{J} = 2 \pi^{2} \Omega^{2} \left( \frac{H}{R} \right)^{2} \left( \frac{R}{R_{\text{out}}} \right) m_{p} N_{\text{out}}, \quad (7)

where we have also used \( M_{\text{disc}} = m_{p} N_{\text{out}} \), where \( N_{\text{out}} \) is the total number of particles used and \( m_{p} \) is the mass of an individual SPH particle. In order to properly resolve fragmentation, we require that \( M_{J} > m_{p} N_{\text{out}} \), where \( N_{\text{res}} = 2N_{\text{neigh}} = 100 \), according to Nelson’s more restrictive criterion, or \( N_{\text{res}} \approx \beta_{\text{cool}} \), which controls fragmentation.

We have tested whether this resistance to fragmentation in the slow case is simply because the disc takes longer to reach \( \beta_{\text{cool}} = 3 \) and is therefore of lower mass, due to accretion onto the central star. However, in the control run \( V \) (in which the disc attains \( \beta_{\text{cool}} = 3 \) at the same time, but with rapid \( (x = 10.5) \) reduction in \( \beta_{\text{cool}} \) between \( \beta_{\text{cool}} = 6 \) and \( \beta_{\text{cool}} = 3 \), the disc fragments promptly. Thus we are satisfied that it is indeed the value of \( x \) which controls fragmentation.

3 RESULTS

We find that in the fast case (\( x = 10.5 \)), a fragment forms when \( \beta_{\text{cool}} = 0.75 \), i.e. about 8.9 outer disc dynamical times after the rapid reduction in \( \beta_{\text{cool}} \) commenced. Since fragmentation always takes about a dynamical timescale to get under way, it follows that, as expected, the ‘fast’ case behaves like the usual case where a fixed \( \beta_{\text{cool}} \) is imposed.

We however see different behaviour in the slow case: here we find that when \( \beta_{\text{cool}} \) is reduced to, and then held at, \( \beta_{\text{hold}} = 2.75 \) (the evolution of this simulation is not shown in Fig. 1), the disc does not fragment even when the disc is then integrated for a further 44 outer dynamical timescales. Likewise, for the slow case, the disc does not fragment when held at \( \beta_{\text{hold}} = 3 \), even after integration for 63 outer dynamical timescales at this \( \beta_{\text{cool}} \) value. On the other hand, when \( \beta_{\text{cool}} \) was instead held at 2.62, it fragmented after a further ~ 18 outer dynamical timescales, so it would appear that the fragmentation boundary is at around 2.7. This is in strong contrast with the value of ~ 7 derived in previous work where a fixed \( \beta_{\text{cool}} \) is imposed. We hesitate to say that we have proved that a disc will never fragment when brought to such a low value of \( \beta_{\text{cool}} \) value at this slow rate, since our experience shows that where one is close to the limit of marginally stable \( \beta_{\text{cool}} \) fragmentation may ensue after long timescales, and that its timing may depend on numerical noise that can be affected by resolution. Indeed, we found that when we re-ran the \( x = 105 \) simulation at higher resolution (\( N = 500,000 \)), and held it at \( \beta_{\text{hold}} = 3 \), it eventually did form a fragment at large radius. We however show the disc structure in this simulation at the point of fragmentation and contrast it with the corresponding situation when the cooling time is rapidly

\[ H = \frac{Q M_{\text{disc}}(R)}{2 M_{*}}. \quad (9) \]

that, even at the lower resolution of 250,000 particles, the average smoothing length is a fraction \( \approx 0.5 \) of the disc thickness.
The fragmentation boundary revisited

4 CONCLUSIONS

We have found that the rate at which the cooling timescale is changed indeed affects the minimum value of $\beta_{cool}$ at which the disc can exist in a stable, self-regulated state. As expected, this effect is only manifest when the the cooling timescale is varied on a timescale ($\tau$) that is longer than the cooling timescale, since for $\tau < t_{cool}$, the temperature always falls on a timescale $t_{cool}$, irrespective of $\tau$. We find that when $\tau > t_{cool}$, the self-regulated state is sustainable at cooling times that are about a factor two less than those that are possible when a fixed cooling timescale is imposed at the outset of the simulation. This implies that (in the slow cooling case) the gravitational instabilities are able to deliver about twice the heating rate without the disc fragmenting. In terms of the ‘viscous alpha’ description of such instabilities (Shakura and Sunyaev 1973, Gammie 2001, Lodato & Rice 2005), the maximum $\alpha$ deliverable by such a disc is then increased from $\sim 0.06$ to $\sim 0.12$. It should be noted that such ‘local’ description of the transport induced by gravitational instabilities is only possible in the limit in which global, wave-like transport does not play an important role. (see in particular their Fig. 13), which employ more realistic cooling properties, confirm that in the limit of small disc mass, the transport induced by gravitational instabilities is essentially local.

We have thus found that thermal history can affect the ability of the disc to exist in a self-regulated state without fragmentation but that this affects the location of the stability boundary at only the factor two level. The fact that there was negligible change in the fragmentation boundary when even slower changes in $t_{cool}$ were employed, demonstrates that thermal history is only part of the story. Our results suggest that, however slowly the disc is cooled through a sequence of thermal equilibria, there is still a fundamental upper limit to the heating that can be provided by gravitational instabilities in a non-fragmenting disc. Thus it would appear that the initiation of fragmentation in a self-gravitating disc does not require that the disc enter the regime of rapid cooling on a short timescale. It is thus unnecessary to invoke sudden events (e.g. impulsive interactions with passing stars, Lodato et al 2007) to tip a previously self-regulated disc into the fragmenting regime. Instead our results suggest that fragmentation can in principle be approached via the secular evolution of a self-gravitating disc.

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Mejía et al. (2005), using a constant cooling time, argue that global mass is small (see in particular their Fig. 13), which employ more realistic cooling properties, confirm that in the limit of small disc mass, the transport induced by gravitational instabilities is essentially local.

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Mejía et al. (2005), using a cooling prescription similar to ours, have shown that this is the case, as long as the total disc mass is small (see in particular their Fig. 13), which employ more realistic cooling properties, confirm that in the limit of small disc mass, the transport induced by gravitational instabilities is essentially local.