FRACTIONAL LAPLACIANS ON DOMAINS,
A DEVELOPMENT OF HÖRMANDER’S THEORY OF
MU-TRANSMISSION PSEUDODIFFERENTIAL OPERATORS

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To the memory of Lars Hörmander 1931–2012

Abstract. Let $P$ be a classical pseudodifferential operator of order $m \in \mathbb{C}$ on an $n$-
dimensional $C^\infty$ manifold $\Omega$. For the truncation $P_{\Omega}$ to a smooth subset $\Omega$ there is a well-
known theory of boundary value problems when $P_{\Omega}$ has the transmission property (preserves $C^\infty(\Omega)$) and is of integer order; the calculus of Boutet de Monvel. Many interesting op-
erators, such as for example complex powers of the Laplacian $(-\Delta)^{\mu}$ with $\mu \notin \mathbb{Z}$, are not
covered. They have instead the $\mu$-transmission property defined in Hörmander’s books, mapping $x_n^\mu C^\infty(\Omega)$ into $C^\infty(\Omega)$. In an unpublished lecture note from 1965, Hörmander described
an $L_2$-solvability theory for $\mu$-transmission operators, departing from Vishik and Eskin’s re-
sults. We here develop the theory in $L_p$ Sobolev spaces ($1 < p < \infty$) in a modern setting.
It leads to not only Fredholm solvability statements but also regularity results in full scales
of Sobolev spaces ($s \to \infty$). The solution spaces have a singularity at the boundary that
we describe in detail. We moreover obtain results in Hölder spaces, which radically improve
recent regularity results for fractional Laplacians.

Introduction. Pseudodifferential operators (ψdo’s) of integer order with the transmission
property (preserving $C^\infty$ up to the boundary in a domain) and their boundary problems
have been studied since the basic theory was developed by Boutet de Monvel in [B71]. The
theory includes differential operators and the parametrices of elliptic such ones, and also
operators whose symbols are rational functions of $\xi$.

This was preceded by works of Vishik and Eskin ([VE65], [VE67] etc., included for the
major part in Eskin’s book [E81]), which treated operators of a more general type, having
a factorization of the principal symbol at the boundary of a smooth open set $\Omega$, in two
factors extending analytically to $\{\text{Im } \xi_n > 0\}$ resp. $\{\text{Im } \xi_n < 0\}$ as functions of the conormal
variable $\xi_n$, with each their degree of homogeneity $m - \kappa(x')$ resp. $\kappa(x')$, $x' \in \partial \Omega$. When
$\Omega$ is compact, such operators will under mild restrictions on the factorization index $\kappa(x')$ define Fredholm operators on Sobolev spaces with exponent $s$ in a certain open interval
$]s_- , s_+[ \leq 1$. For larger $s$ one has to add suitable boundary conditions, and for
smaller $s$ potential terms, in order to get Fredholmness. The results have been extended
to $L_p$-based Sobolev spaces by Shargorodsky [S94] and Chkadua and Duduchava [CD01].
In an unpublished (photocopy distributed) lecture note at Princeton 1965 [H65], Hörmander introduced, with Vishik and Eskin's work as a starting point, a generalized transmission condition of type $\mu \in \mathbb{C}$ (where the condition in [B71] is the case $\mu = 0$), reflecting the properties of the general operators studied by Vishik and Eskin in the case $\kappa(x') = \mu_0$ constant. Here he showed not only the Fredholm property in Sobolev spaces for $s$ in an interval, but he moreover determined the $L_2$ Sobolev regularity of solutions with data given for all larger $s$, or given in $C^\infty(\overline{\Omega})$, finding the domain spaces for Fredholm solvability and describing the associated boundary conditions.

The transmission condition of type $\mu$ was briefly characterized in [H85], Sect. 18.2. An application to propagation of singularities was given by Hirschowitz and Pirion [HP79].

Fractional powers of the Laplacian $(-\Delta)^a$ are of type $\mu = a$; they have recently received increased attention both in probability theory, cf. e.g. Bogdan, Grzywny and Ryznar [BGR10], Ros-Oton and Serra [RS14], in differential geometry, cf. e.g. Gonzalez, Mazzeo and Sire [GMS12], and in Schrödinger theory, cf. e.g. Frank and Geisinger [FG14], and the references in these papers. Only a little seems to be known about the regularity of solutions on domains. Inspired by this, we have in the present paper worked out an extension of Hörmander's theory to $L_p$-Sobolev spaces, $1 < p < \infty$, with additional results, moreover leading to solvability results in Hölder spaces. Applications include fractional powers of strongly elliptic differential operators.

In this process, the presentation could benefit from the theories developed since 1965, namely the theory of boundary value problems of type 0, as introduced by Boutet de Monvel for integer-order cases in [B71], and further developed by the present author, e.g. in [G96]. The work [G90] is particularly useful, extending the Boutet de Monvel calculus to the $L_p$-setting and introducing refined order-reduction techniques. A joint work with Hörmander [GH90] treated operators of type 0 and arbitrary real order $m$ (including $S^m_{0,\delta}$ symbols).

Here are some of the main results. We consider a smooth subset $\Omega$ of an $n$-dimensional Riemannian $C^\infty$ manifold $\Omega_1$, and denote by $d(x)$ a $C^\infty(\overline{\Omega})$-function equal to $\text{dist}(x, \partial \Omega)$ near $\partial \Omega$ and positive on $\Omega$. Restriction to $\Omega$ is denoted $r_\Omega$ (or $r^\pm$), extension by zero on $\Omega_1 \setminus \Omega$ is denoted $e_\Omega$ (or $e^+\Omega$). For $\mu \in \mathbb{C}$ with $\text{Re} \mu > -1$, $E_\mu(\overline{\Omega})$ denotes the space of functions $u$ such that $u = e_\Omega d(x)^\mu v$ with $v \in C^\infty(\overline{\Omega})$. The definition is generalized in a distribution sense to lower values of $\mu$. On $\Omega_1$ we consider a classical $\psi$do $P$ of order $m \in \mathbb{C}$, with symbol in local coordinates $p(x, \xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi)$ where $p_j(x, t\xi) = t^{m-j}p_j(x, \xi)$. The $\mu$-transmission property was described in [H85], Th. 18.2.18:

**Proposition 1.** A necessary and sufficient condition in order that $r_\Omega Pu \in C^\infty(\overline{\Omega})$ for all $u \in E_\mu(\overline{\Omega})$ is that $P$ satisfies the $\mu$-transmission condition (in short: is of type $\mu$), namely that

\[
\partial_x^\alpha \partial_{\xi}^\beta p_j(x, -N) = e^{\pi i (m-2\mu+j-|\alpha|)} \partial_x^\alpha \partial_{\xi}^\beta p_j(x, N), \quad x \in \partial \Omega,
\]

for all $j, \alpha, \beta$, where $N$ denotes the interior normal to $\partial \Omega$ at $x$.

In the following theorems we take $\overline{\Omega}$ compact.

Define the special spaces $H_{p}^{\mu(s)}(\overline{\mathbb{R}_+^n})$ (Hörmander's $\mu$-spaces), for $s > \text{Re} \mu - 1/p'$:

\[
H_{p}^{\mu(s)}(\overline{\mathbb{R}_+^n}) = \{ u \in \dot{H}_p^{\text{Re} \mu - 1/p' + 0}(\mathbb{R}_+^n) | r^+ \text{OP}((\xi') + i\xi_n)^\mu u \in \overline{\mathbb{R}_+^n}^{\text{Re} \mu}(\overline{\mathbb{R}_+^n}) \}.
\]
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(The notation used for \( L_p \) Sobolev spaces is listed below in Section 1.) The definition extends to define \( H^{\mu(s)}_p(\Omega) \) by use of local coordinates. This is the solution space for \( Pu = f \) on \( \Omega \):

**Theorem 2.** Assume that \( P \) is elliptic of order \( m \in \mathbb{C} \) and type \( \mu_0 \in \mathbb{C} \) (mod 1), and has factorization index \( \mu_0 \), and let \( s > \text{Re} \mu_0 - 1/p' \). When \( u \in \dot{H}^{\text{Re} \mu_0 - 1/p' + 0}_p(\Omega) \), then \( r_\Omega Pu \in \overline{H}^{s - \text{Re} m}_p(\Omega) \) implies \( u \in H^{\mu_0(s)}_p(\Omega) \). The mapping

\[
(3) \quad r_\Omega P: H^{\mu_0(s)}_p(\Omega) \to \overline{H}^{s - \text{Re} m}_p(\Omega)
\]

is Fredholm. Moreover, \( r_\Omega Pu \in C^\infty(\Omega) \) implies \( u \in E^{\mu_0}(\Omega) \), and the mapping \( r_\Omega P \) from \( E^{\mu_0}(\Omega) \) to \( C^\infty(\Omega) \) is Fredholm.

The spaces \( H^{\mu(s)}_p(\Omega) \) allow a definition of boundary values \( \gamma^{\mu,j} u \), that generalize the mapping \( u \mapsto \partial_j x_n(x_n^\mu u)|_{x_n=0} \), defined for \( u \in E^{\mu}(\mathbb{R}^n_+) \) when \( \text{Re} \mu > -1 \).

**Theorem 3.** When \( P \) and \( s \) are as in Theorem 2, and \( \mu = \mu_0 - M \) for a positive integer \( M \), then the following operator is Fredholm:

\[
(4) \quad \{r_\Omega P, \gamma^{\mu,0}, \ldots, \gamma^{\mu,M-1}\}: H^{\mu(s)}_p(\Omega) \to \overline{H}^{s - \text{Re} m}_p(\Omega) \times \prod_{0 \leq j < M} B^{s - \text{Re} \mu - j - 1/p}(\partial \Omega).
\]

Now follow some applications to fractional powers. Let \( a > 0 \) and let \( P_a \) equal the power \( A^a \) of a strongly elliptic second-order differential operator \( A \) with \( C^\infty \)-coefficients on \( \Omega_1 \) (a special case is \( P_a = (-\Delta)^a \)). Then \( P_a \) is of order \( 2a \), of type \( a \), and has factorization index \( a \). Theorems 2 and 3 give e.g. the following results in Hölder spaces (where \( \dot{C}^t(\Omega) \) stands for \( \{u \in C^t(\Omega_1) \mid \text{supp} u \subset \overline{\Omega}\} \)):

**Theorem 4.** Let \( u \in \dot{H}^{a-1/p' + 0}_p(\Omega) \) for some \( 1 < p < \infty \) (this holds if \( u \in e^+ L_\infty(\Omega) \) when \( a < 1 \), \( u \in C^{a-1+0}(\Omega) \) when \( a \geq 1 \)). The solutions of

\[
(5) \quad r_\Omega P_a u = f
\]

satisfy for \( t \geq 0 \):

\[
(6) \quad f \in C^{t+0}(\Omega) \implies u \in e^+ d(x)^a C^{t+a-0}(\Omega) \cap C^{t+2a-0}(\Omega).
\]

(For \( t = 0 \), \( f \in e^+ L_\infty(\Omega) \) suffices.) A solution exists under a finite dimensional linear condition on \( f \). Moreover,

\[
(7) \quad f \in C^{\infty}(\Omega) \iff u \in e^+ d(x)^a C^{\infty}(\Omega),
\]

with Fredholm solvability.

This theorem is concerned with the homogeneous Dirichlet problem for \( P_a \). We can moreover treat a nonhomogeneous Dirichlet problem (8):
Theorem 5. Let \( u \in H_p^{(a-1/2)}(\Omega) \) with \( s > a - 1/p' \). The solutions of

\[
\begin{align*}
\gamma_0 d(x)^{1-a} u &= \varphi,
\end{align*}
\]

satisfy

\[
\begin{align*}
\gamma_0 d(x)^{1-a} u &= \varphi,
\end{align*}
\]

\((\text{For } t = 0, \ f \in e^t L_{\infty}(\Omega) \text{ suffices.})\) A solution exists under a finite dimensional linear condition on \( \{f, \varphi\} \). Moreover,

\[
\begin{align*}
\gamma_0 d(x)^{1-a} u &= \varphi,
\end{align*}
\]

with Fredholm solvability.

Ros-Oton and Serra have recently shown in [RS14] for (5) with \( P_a = (-\Delta)^a \), \( 0 < a < 1 \), that \( f \in L_{\infty} \) implies \( u \in d(x)^{\alpha} C^\alpha \) for an \( \alpha < \min\{a, 1-a\} \) when \( \Omega \) is \( C^{1,1} \), by potential theoretic methods. Theorem 4 sharpens this result, allows more general operators, and extends it to higher regularity, when \( \Omega \) is smooth. We are not aware of any published precedents to the other theorems given above. One can also replace the condition in (8) by a Neumann condition \( \gamma_{a-1,1} u = \psi \) or more general conditions.

The theory of \( \mu \)-transmission \psi
do’s presented here provides a missing link between, on one hand, Boutet de Monvel’s theory of boundary value problems for integer-order 0-transmission \psi
do’s, and on the other hand the very general boundary value theories of other authors. There is a rich literature; let us for example point to the works of Schulze and coauthors, see e.g. Rempel-Schulze [RS84], Harutyunyan-Schulze [HS08] and their references, and the works of Melrose and coauthors, e.g. Melrose [M93], Albin and Melrose [AM09] and their references.

Outline. In Section 1, the relevant function spaces are introduced, including Hörmander’s \( \mu \)-spaces, along with important order-reducing operators. Section 2 defines the \( \mu \)-transmission property and the corresponding boundary behavior for smooth functions. Section 3 recalls the result of Vishik and Eskin. In Section 4 we show the Sobolev mapping properties of \( \mu \)-transmission operators and deduce the regularity results for solutions of elliptic homogeneous boundary problems. Section 5 defines the appropriate boundary operators, and analyzes the structure of the solution spaces. In Section 6, solvability of nonhomogeneous elliptic boundary problems is established, with a description of parametrices. Finally in Section 7, consequences are drawn for fractional powers of strongly elliptic differential operators, and their solvability properties in Hölder spaces.

1. Function spaces

1.1 \( L_p \)-Sobolev spaces. The function spaces used in [H65] are \( L_2 \)-Sobolev spaces and their anisotropic variants as introduced in [H63], together with a hitherto unpublished interesting case describing a special boundary behavior adapted to symbols with the \( \mu \)-transmission property.
In the present paper we generalize this to $L_p$-Sobolev spaces, mainly of Bessel-potential type, $1 < p < \infty$, to which the results of Eskin’s book [E81] were extended in [S94] and [CD01]. The notation will be a compromise between the nowadays common style where the regularity exponent $s$ is an upper index without parentheses, giving room for $p$ as a lower index (in [H63, H65, H85], a lower index ($s$) is used), and on the other hand Hörmander’s notation of indicating by $\mathcal{H}(\mathbb{R}^n_+)$ resp. $\mathcal{H}(\mathbb{R}^n_{++})$ the distributions restricted from $\mathbb{R}^n$ resp. supported in $\mathbb{R}^n_+$. The spaces are all Banach spaces with the indicated norms.

In the Euclidean space $\mathbb{R}^n$, the points are written $x = \{x_1, \ldots, x_n\} = \{x', x_n\}$, $\mathbb{R}^n_+ = \{x \mid x_n \geq 0\}$, $\langle x \rangle = (1 + |x|^2)^{\frac{j}{2}}$, and we denote by $[\xi]$ a smoothed version of $|\xi|$: 

$$[\xi] \in C^\infty(\mathbb{R}^n, \mathbb{R}_+), \ [\xi] = |\xi| \text{ for } |\xi| \geq 1, \ [\xi] \geq \frac{1}{2} \text{ for all } \xi.$$ 

Restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ is denoted $r^\pm$, extension by zero from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ is denoted $e^\pm$.

$\mathcal{F}$ denotes the Fourier transformation

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot \xi} f(x) \, dx,$$

defined on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing $C^\infty$-functions, and extended to distribution in $\mathcal{S}'(\mathbb{R}^n)$ and in function spaces in a well-known way. Note the minus-sign, standard in the Western literature, whereas there is usually a plus-sign in the definition used in the literature originating from Russian and other East-european authors.

We shall consider classical pseudodifferential operators ($\psi$do’s) $P$ of order $m \in \mathbb{C}$; this means that the symbol has an expansion in homogeneous terms $p(x, \xi) \sim \sum_0^\infty p_j(x, \xi)$, where $p_j$ is homogeneous of degree $m - j$ in $\xi$:

$$p_j(x, t\xi) = t^{m-j} p_j(x, \xi) = t^{\text{Re} \, m-j} e^{i \text{Im} \, m \log t} p_j(x, \xi), \text{ for } t > 0.$$ 

(We just take one-step polyhomogeneous symbols here, although [H65] allows general order sequences $m_j$ with $\text{Re} \, m_j \to -\infty$.) The operator is defined by

$$Pu = p(x, D)u = \text{OP}(p(x, \xi))u = (2\pi)^{-n} \int e^{ix\cdot \xi} p(x, \xi) \hat{u} \, d\xi,$$

suitably interpreted. Some boundary problems are treated e.g. in [B71, G90, G96, G09]. By truncation to $\mathbb{R}^n_{++}$, $P$ defines $P_{++} = r^\pm P e^\pm$.

For $s, t \in \mathbb{R}$ and $1 < p < \infty$, the Bessel-potential spaces over $\mathbb{R}^n$ are defined by

$$H^s_p(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}((\xi)^s \hat{u}) \in L_p(\mathbb{R}^n)\},$$

with norm $\|u\|_{H^s_p(\mathbb{R}^n)} = \|u\|_s = \|\mathcal{F}^{-1}((\xi)^s \hat{u})\|_{L_p(\mathbb{R}^n)}$,

$$H^{s,t}_p(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}((\xi)^s (\xi')^t \hat{u}) \in L_p(\mathbb{R}^n)\},$$

with norm $\|u\|_{H^{s,t}_p(\mathbb{R}^n)} = \|u\|_{s,t} = \|\mathcal{F}^{-1}((\xi)^s (\xi')^t \hat{u})\|_{L_p(\mathbb{R}^n)}$.

The latter anisotropic spaces are used in [H63, G96, G09, CD01]; [S94] includes other anisotropic cases. Note that $H^s_p = H^{s,0}_p$, and that $H^0_p = L_p$. 

The pseudodifferential symbols $p(x, \xi)$ of order $m \in \mathbb{C}$ are in $S^r_{1, 0}^{\operatorname{rem}}(\mathbb{R}^n \times \mathbb{R}^n)$, hence the operators are continuous from $H^s_p(\mathbb{R}^n)$ to $H^{s-\operatorname{rem}}_p(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, as accounted for e.g. in [G90]. The continuity extends to the map from $H^s_{p, t}(\mathbb{R}^n)$ to $H^{s-\operatorname{rem}, t}_p(\mathbb{R}^n)$ for all $t \in \mathbb{R}$, cf. e.g. [CD01]. The operators we consider in this paper are scalar.

From the spaces in (1.3) we define with a notation extended from [H63, H65, H85]:

\begin{align}
\hat{H}^{s, t}_p(\mathbb{R}^n) &= \{ u \in H^{s, t}_p(\mathbb{R}^n) \mid \text{supp } u \subseteq \mathbb{R}^n_+ \}, \\
\overline{\mathcal{H}}^{s, t}_p(\mathbb{R}^n) &= \{ u \in \mathcal{D}'(\mathbb{R}^n_+) \mid u = r^+ U \text{ for some } U \in H^{s, t}_p(\mathbb{R}^n) \},
\end{align}

the first space is a closed subspace of $H^{s, t}_p(\mathbb{R}^n)$, and in the second space, homeomorphic to $H^{s, t}_p(\mathbb{R}^n)/\hat{H}^{s, t}_p(\mathbb{R}^n)$, the norm

$$
\|u\|_{\overline{\mathcal{H}}^{s, t}_p(\mathbb{R}^n)} = \inf \{ \|U\|_{H^{s, t}_p(\mathbb{R}^n)} \mid u = r^+ U \}, \text{ also denoted } \|u\|_{s, t},
$$
is used. $\hat{H}$ was denoted $\hat{H}$ in the book [H63] and in [H65]. In some other texts it is marked as $H_0$ (e.g. in [G90]), or $\widetilde{H}$ (e.g. in [E81, T95, S94, CD01]). When $s - 1/p$ is integer, Triebel’s use of $\hat{H}$ in [T95] (first edition 1978) differs from Hörmander’s original 1963 definition.

The use of both $\overline{\mathcal{H}}$ and $\hat{H}$ is practical, since it allows leaving out the indication of the domain $\mathbb{R}^n_+$. We recall that $\hat{H}^{s, t}_p(\mathbb{R}^n)$ and $\overline{\mathcal{H}}^{s, t}_p(\mathbb{R}^n)$ $(1/p' = 1 - 1/p)$ are dual spaces to one another with respect to an extension of the sesquilinear form $(u,v) = \int_{\mathbb{R}^n_+} u(x) \overline{v}(x) \, dx$.

We shall denote

\begin{align}
\bigcup_{\varepsilon > 0} \hat{H}^{s+\varepsilon}_p &= \hat{H}^{s+0}_p, \\
\bigcap_{\varepsilon > 0} \hat{H}^{s-\varepsilon}_p &= \hat{H}^{s-0}_p, \\
\bigcup_{\varepsilon > 0} \overline{\mathcal{H}}^{s+\varepsilon}_p &= \overline{\mathcal{H}}^{s+0}_p, \\
\bigcap_{\varepsilon > 0} \overline{\mathcal{H}}^{s-\varepsilon}_p &= \overline{\mathcal{H}}^{s-0}_p.
\end{align}

The notation $\hat{S}(\mathbb{R}^n_+)$, $\hat{S}'(\mathbb{R}^n_+)$, will be used for Schwartz functions resp. distributions supported in $\mathbb{R}^n_+$, and $\overline{S}(\mathbb{R}^n_+)$, $\overline{S}'(\mathbb{R}^n_+)$, will be used for Schwartz functions resp. distributions restricted to $\mathbb{R}^n_+$. Here $\hat{S}(\mathbb{R}^n_+)$ (and $C_0^\infty(\mathbb{R}^n_+)$) is dense in the spaces $\hat{H}^{s, t}_p(\mathbb{R}^n)$, and $\overline{S}(\mathbb{R}^n_+)$ is dense in $\overline{\mathcal{H}}^{s, t}_p(\mathbb{R}^n)$.

We shall also need the Besov spaces $B^s_p(\mathbb{R}^n)$, which enter as range spaces for trace maps, recalling that for $0 < s < 2$,

$$
f \in B^s_p(\mathbb{R}^n) \iff \|f\|_{L^p} + \int_{\mathbb{R}^{2n}} \frac{|f(x) + f(y) - 2f((x+y)/2)|^p}{|x+y|^{n+ps}} \, dx \, dy < \infty;
$$

and $B^{s-t}_p(\mathbb{R}^n) = (1 - \Delta)^{t/2} B^s_p(\mathbb{R}^n)$ for all $t \in \mathbb{R}$.

Embedding, interpolation and other properties are found e.g. in Triebel [T95].

Let $\gamma_j$ denote the trace operator $\gamma_j : u(x', x_n) \mapsto D_{n} u(x', 0)$, defined to begin with on smooth functions: it extends to a continuous linear map $\gamma_j : \overline{\mathcal{H}}^{s, t}_p(\mathbb{R}^n) \to B^{s-1/p}_p(\mathbb{R}^{n-1})$, for $s > 1/p$. It is surjective with a continuous right inverse. In fact, defining the column vector $\varrho_M = \{ \gamma_0, \ldots, \gamma_{M-1} \}$ for a positive integer $M$, we have that

\begin{align}
\varrho_M : \overline{\mathcal{H}}^{s, t}_p(\mathbb{R}^n) \to \prod_{0 \leq j < M} B^{s-j-1/p}_p(\mathbb{R}^{n-1}) \text{ for } s > M - 1/p,
\end{align}
continuous and surjective, having a right inverse (row vector) \( \mathcal{K}_M = \{K_0, \ldots, K_{M-1}\} \) (a Poisson operator, cf. [G90]), that in addition is continuous from \( \prod_{0 \leq j < M} B_{p}^{1-j-1/p}(\mathbb{R}^{n-1}) \) to \( \overline{H}_p(C_{n}) \) for all \( t \in \mathbb{R} \). As \( \mathcal{K}_M \) one can for example take the Poisson operator \( \varphi \mapsto u \) solving the Dirichlet problem for \((1 - \Delta)^M\),

\[
(1 - \Delta)^M u = 0 \text{ in } \mathbb{R}^n, \quad \varphi_M u = \varphi \text{ on } \mathbb{R}^{n-1}
\]

(an elementary treatment of the case \( M = 1 \) is found in [G90], Ch. 9). We shall here use the closely related choice, cf. (1.1) \((e^+ \text{ is sometimes left out}):
\[
\mathcal{K}_M = \{K_0, \ldots, K_{M-1}\},
\]

with

\[
K_j : \varphi_j \mapsto (-1)^j \mathcal{F}^{-1}_{\xi \to x}(\varphi_j(\xi') \partial_{\xi_n}^j ([\xi'] + i\xi_n) - 1) = \frac{i^j}{\pi} x_n^{j} \mathcal{F}^{-1}_{\xi' \to x'}(e^{+r+e^{-|\xi'|x_n}}\varphi_j(\xi')).
\]

It can also be convenient to use (1.7) with \([\xi']\) replaced by \([\xi']\), more closely related to \(1 - \Delta\). Still another choice is given in [H63], Th. 2.5.7 (also recalled in [G96, G99]).

It is known that there are natural identifications

\[
\begin{align*}
\hat{H}_p^s(\mathbb{R}^n_+) &= \{u \in \overline{H}_p^s(\mathbb{R}^n_+) \mid \varphi_M u = 0\}, \text{ for } M + 1/p > s > M + 1/p - 1; \\
\hat{H}_p^s(\mathbb{R}^n_+) &= \overline{H}_p^s(\mathbb{R}^n_+), \text{ for } 1/p > s > 1/p - 1 = -1/p'.
\end{align*}
\]

In the borderline case \( s = 1/p, \overline{H}_p^{1/p}(\mathbb{R}^n_+) \) is strictly larger than \( \hat{H}_p^{1/p}(\mathbb{R}^n_+) \); the latter carries the norm \( \|u\|_{\overline{H}_p^s} + \|x_n^{-1/p}u\|_{L_p} \). However, \( C_0^\infty(\mathbb{R}^n_+) \) is dense in both of these spaces. (Cf. [G90] (2.15)ff. and its references.)

The definitions carry over to the manifold situation by use of local coordinates.

**1.2 Order-reducing operators.** Homeomorphisms between the various spaces play an important role in the theory. The operator \( \text{OP}((\xi)^\mu) \) defines homeomorphisms from \( H^s_p(\mathbb{R}^n) \) to \( H^{s-\text{Re} \mu}_p(\mathbb{R}^n) \) for all \( s \in \mathbb{R} \). Likewise for any \( \mu \in \mathbb{C} \), cf. (1.1),

\[
\Xi^\mu = \text{OP}(\chi^\mu), \quad \text{where } \chi^\mu = [\xi]^\mu, \text{ defines homeomorphisms}
\]

\[
\Xi^\mu : H^s_p(\mathbb{R}^n) \rightleftharpoons H^{s-\text{Re} \mu}_p(\mathbb{R}^n), \text{ all } s \in \mathbb{R}, \text{ with inverse } \Xi^{-\mu}.
\]

In the following, we can either use \( \langle \xi \rangle \), \( \langle \xi' \rangle \) as in [H65], or replace them by \([\xi], [\xi']\) to profit from the homogeneity. The operators defined by the two choices have the same mapping properties. The explicit formulas in the following will be written with \([\xi']\), since this is useful in the definition of \( \Lambda_\pm^\mu \) further below.

For the spaces defined relative to \( \mathbb{R}^n_\pm \), there are several interesting choices. One is the simple family

\[
\chi^\mu_\pm = ([\xi'] + i\xi_n)^\mu, \quad \text{resp. } \chi^\mu_\pm = ([\xi'] - i\xi_n)^\mu, \quad \text{OP}((|\xi'| \pm i\xi_n)^\mu) = \Xi^\mu_{\pm},
\]

(or, if needed, the corresponding formulas with \( \langle \xi' \rangle \)). Here \( \chi^\mu_\pm \) (resp. \( \chi^\mu_\pm \)) extends analytically as a function of \( \xi_n \) into \( \mathbb{C}_- = \{\text{Im } \xi_n < 0\} \) resp. \( \mathbb{C}_+ = \{\text{Im } \xi_n > 0\} \). (The imaginary halfspaces play the opposite roles in the works [E81, S94, CD01] because of the opposite sign in the definition of \( \mathcal{F} \).) Since \( \chi^\mu_+ \) extends analytically to \( \text{Im } \xi_n < 0 \), the operator \( \Xi^\mu_+ \) preserves support in \( \mathbb{R}^n_+ \); hence we have for all \( s \in \mathbb{R} \) that

\[
\Xi^\mu_+ : \hat{H}_p^s(\mathbb{R}^n_+) \rightleftharpoons \hat{H}_p^{s-\text{Re} \mu}(\mathbb{R}^n_+), \text{ with inverse } \Xi^{-\mu}_+.
\]

The adjoint mapping is \( \Xi^\mu_{-+} : \overline{H}_p^{s-\text{Re} \mu}(\mathbb{R}^n_+) \rightleftharpoons \overline{H}_p^s(\mathbb{R}^n_+) \); this shows for general \( s, p, \mu,
\]

\[
\Xi^\mu_{-+} : \overline{H}_p^s(\mathbb{R}^n_+) \rightleftharpoons \overline{H}_p^{s-\text{Re} \mu}(\mathbb{R}^n_+), \text{ with inverse } \Xi^{-\mu}_{-+}.
\]
Remark 1.1. For \( s > -1/p' \), \( \Xi^{\mu}_{-,+} \) in (1.12) identifies with \( r^+ \Xi^\mu_+ e^+ \) (\( e^+ \) is only defined then). For lower \( s \), the mapping in (1.12) can be understood, besides being a specific adjoint, as the extension by continuity from the operator defined on the dense subspace \( \mathcal{S}(\mathbb{R}^n_+) \) (as noted in [G93], p. 174). There is also a third formulation worth mentioning, used in [G90], namely that for any extension operator \( \ell : \dot{H}^s_p(\mathbb{R}^n_+) \to H^s_p(\mathbb{R}^n) \) with \( r^+ \ell = \text{Id} \),

\[
\Xi^{\mu}_{-,+} f = r^+ \Xi^\mu_- \ell f.
\]

This holds since \( r^+ \Xi^\mu_- g = 0 \) for any distribution \( g \) supported in \( \mathbb{R}^n_- \), using that since \( \chi^\mu_- \) extends analytically to \( \text{Im} \xi_n > 0 \), the operator \( \Xi^\mu_- \) preserves support in \( \mathbb{R}^n_- \). The formula (1.13) is independent of the choice of \( \ell \).

The symbols \( \chi^\mu_\pm \) are not truly pseudodifferential (although the \( \text{OP}(\chi^\mu_\pm) \) have a good meaning by Lizorkin’s criterion, cf. e.g. [G90]), since the higher \( \xi^\ell \)-derivatives do not have the correct fall-off for \( |\xi| \to \infty \). But there exists another choice with true \( \psi \)-do symbols given in [G90] (inspired from the unpublished [F86]), that also has the above mapping properties. Define

\[
\lambda^\mu_\pm = (\lambda^1_\pm)^\mu, \quad \lambda^1_- = [\xi^\ell] \psi \left( \frac{\xi_n}{a|\xi|} \right) - i\xi_n, \quad \lambda^1_+ = \overline{\lambda^1_-},
\]

with \( \psi \in \mathcal{S}(\mathbb{R}) \) having \( \psi(0) = 1 \) and \( \text{supp} \mathcal{F}^{-1} \psi \subset \mathbb{R}_- \). We set \( \psi(\pm \infty) = 0 \), then \( \psi \) is \( C^\infty \) on the extended real axis. Here the constant \( a > 0 \) is chosen so large that the negative powers are well-defined, cf. [G90] pp. 317-322. The functions \( \lambda^\mu_+ \) (resp. \( \lambda^\mu_- \)) extends analytically into \( \{ \text{Im} \xi_n < 0 \} \) resp. \( \{ \text{Im} \xi_n > 0 \} \). Denoting \( \text{OP}(\lambda^\mu_\pm) = \Lambda^\mu_\pm \), we have for all \( s \in \mathbb{R} \) that

\[
\Lambda^\mu_+: \dot{H}^s_p(\mathbb{R}^n_+) \sim \dot{H}^s_{p'}(\mathbb{R}^n_+)^{\text{Re} \mu}, \quad \text{with inverse} \quad \Lambda^{-\mu}_+;
\]

\[
\Lambda^{-\mu}_{-,+}: \dot{H}^s_p(\mathbb{R}^n_-) \sim \dot{H}^s_{p'}(\mathbb{R}^n_-)^{\text{Re} \mu}, \quad \text{with inverse} \quad \Lambda^{\mu}_{-,+};
\]

here \( \Lambda^{\mu}_{-,+} \) is the adjoint of \( \Lambda^{\mu}_{+,+}: \dot{H}^{s-Re\mu}_{p'}(\mathbb{R}^n_+) \sim \dot{H}^s_{p'}(\mathbb{R}^n_+)^{\text{Re} \mu} \), and again there are interpretations as in Remark 1.1. The proofs are given in [G90], (cf. (4.11), (4.24) there) using that for \( a \) taken sufficiently large in (1.14) (as we assume),

\[
\eta^\mu_\pm(\xi) = (\lambda^1_\pm(\xi)/\lambda^1_\pm(\xi))^\mu = 1 + q^\mu_\pm(\xi) \quad \text{with} \quad |q^\mu_\pm(\xi)| \leq \frac{1}{2},
\]

analytic for \( \text{Im} \xi_n \leq 0 \); they define \( \psi \)-do’s \( \eta^\mu_\pm(\xi', D_n) = \text{OP}_n(\eta^\mu_\pm(\xi', \xi_n)) \) of order 0 that are homeomorphisms in \( L_2(\mathbb{R}) \), uniformly in \( \xi' \). Since they preserve support in \( \mathbb{R}^n_- \) respectively (and the inverses do so too), \( r^\pm \eta^\mu_\pm(\xi', D_n) e^\pm \) are homeomorphism in \( L_2(\mathbb{R}^n_\pm) \), respectively. This allows transferring the mapping properties of the \( \Xi^\mu_\pm \) to the \( \Lambda^\mu_\pm \), cf. [G90]. The operators \( \Xi^\mu_\pm, \Lambda^\mu_\pm \) and \( \eta^\mu_\pm(\xi', D_n) \) belong to the so-called “plus-operators” of Eskin [E81], and the operators \( \Xi^\mu_-, \Lambda^\mu_- \) and \( \eta^\mu_-(\xi', D_n) \) belong to the “minus-operators”. The symbols are said to be “plus-symbols” resp. “minus-symbols”. (The sub-indices \( \pm \) here should not be confounded with the \( \pm \) used to indicate truncation — added on as an extra index.)

In addition to what was shown in [G90], we observe:
Lemma 1.2. Let \( Y^\mu_+ = \text{OP}(\eta^\mu_+(\xi)) \), then \( Y^\mu_{+,+} = r^+ Y^\mu_+ e^+ \) is a homeomorphism of \( \overline{H}^{s,t}_p(\mathbb{R}^n_+) \) onto itself for all \( s, t \in \mathbb{R} \). For any \( s, t \in \mathbb{R} \),

\[
||r^+ \Xi^\mu_+ u||_{H^{s,t}_p(\mathbb{R}^n_+)} \simeq ||r^+ \Lambda^\mu_+ u||_{H^{s,t}_p(\mathbb{R}^n_+)}.
\]

The equivalence also holds if \( [\xi'] \) is replaced by \( \langle \xi' \rangle \) in the definition of \( \Xi^\mu_+ \).

**Proof.** The proof needs some care, because \( Y^\mu_+ \) is not a standard \( \psi \)-do on \( \mathbb{R}^n \); however it is so at the one-dimensional level where we just use the definition with respect to \( \xi_n \). Here the Boutet de Monvel calculus on \( \mathbb{R} \) shows that \( r^+ \eta^\mu_+(\xi', D_n) e^+ \) is a homeomorphism in \( \overline{H}^{s,t}_2(\mathbb{R}^n_+) \) with inverse \( r^+ \text{OP}_n((\eta^\mu_+(\xi))^{-1}) e^+ \) for all \( m \in \mathbb{Z} \), since the left-over operators such as \( G^+(\text{OP}_n(\eta^\mu_+^n)) G^- (\text{OP}_n(\eta^\mu_-)) \) arising in the composition have the \( G^- \)-factor equal to 0, hence vanish. The norms are bounded in \( \xi' \). Interpolation extends the homeomorphism property to all real \( s \).

Estimating the norms simply by Fourier transformation, we find for \( p = 2 \) that the full operator \( r^+ Y^\mu_+ e^+ \) is a homeomorphism in \( \overline{H}^{s,t}_2(\mathbb{R}^n_+) \) with inverse \( r^+ (Y^\mu_+)^{-1} e^+ \). Both \( Y^\mu_+ \) and \( (Y^\mu_+)^{-1} = Y^{-\mu}_- \) are continuous in \( H^{s,t}_p(\mathbb{R}^n) \) by Lizorkin’s criterion. The \( L_2 \)-calculations apply in particular to functions \( u \in \mathcal{S}(\mathbb{R}^n_+) \), showing that \( r^+ Y^\mu_+ e^+ u = r^+ e^+ Y^\mu_+ u, r^+ Y^{-\mu}_- e^+ u = r^+ e^+ Y^{-\mu}_- u \) for such \( u \); this extends to \( u \in \overline{H}^{s,t}_p(\mathbb{R}^n_+) \) by closure, and completes the proof of the homeomorphism property.

Now

\[
r^+ \Lambda^\mu_+ u = r^+ Y^\mu_+ \Xi^\mu_+ u = r^+ Y^\mu_+ e^+ r^+ \Xi^\mu_+ u,
\]

where the corresponding term with \( e^- r^- \) in the middle vanishes since \( r^- \Xi^\mu_+ u \) does so. Then in view of the homeomorphism property of \( r^+ Y^\mu_+ e^+ \),

\[
||\Lambda^\mu_+ u||_{s,t} \leq C ||\Xi^\mu_+ u||_{s,t},
\]

Similarly, an inequality the other way follows by use of \( Y^{-\mu}_- \).

For the last statement, the operators \( \text{OP}((\langle \xi' \rangle + i \xi_n)^\mu) \) and \( \text{OP}((\langle \xi' \rangle + i \xi_n)^\mu) \) can be compared in a similar way, since \( ((\langle \xi' \rangle + i \xi_n)/(\langle \xi' \rangle + i \xi_n))^\mu = (1 + (\langle \xi' \rangle - \langle \xi' \rangle)/(\langle \xi' \rangle + i \xi_n))^\mu \) is an invertible plus-symbol of order 0. \( \square \)

It is important to observe that the operators \( \Lambda^m_+, m \in \mathbb{Z}, \) that act homeomorphically in the scale \( H^s_p(\mathbb{R}^n_+) \), can also be applied to the scale \( \overline{H}^{s,t}_p(\mathbb{R}^n_+) \) for \( s > -1/p' \) after truncation, \( \Lambda^m_{+,+} = r^+ \Lambda^m_+ e^+ \), since they belong to the Boutet de Monvel calculus. But here they must in general be supplied with trace or Poisson operators to define homeomorphisms. E.g. for integer \( m > 0 \),

\[
(\Lambda^m_{+,+}, q_m) : \overline{H}^s_p(\mathbb{R}^n_+) \Rightarrow \overline{H}^{s,m-m}_p(\mathbb{R}^n_+) \times \prod_{0 \leq j < m} B^{s+1/p,j-1/p}(\mathbb{R}^{n-1})
\]

(shown in [G90], Th. 4.3); it is an elliptic boundary value problem. A similar mapping property holds with \( \Xi^m_+ \) instead of \( \Lambda^m_+ \).

The construction of these operators extends to the manifold situation, by the method described in [G90]. Let \( \overline{\Omega} \) be a compact \( n \)-dimensional \( C^\infty \) manifold with interior \( \Omega \) and
boundary $\partial \Omega = \Sigma$, and let $E$ be a Hermitean $C^\infty$ vector bundle over $\overline{\Omega}$ of dimension $N$, its restriction to $\Sigma$ denoted $E'$. We can assume that $\overline{\Omega}$ is smoothly embedded in a compact boundaryless $n$-dimensional manifold $\Omega_1$ (e.g. the double of $\overline{\Omega}$) such that $\Sigma$ is the boundary of $\Omega$ there, and we assume that $E$ is the restriction to $\overline{\Omega}$ of a smooth vectorbundle $E_1$ given over $\Omega_1$. Then there is a standard way to generalize the definitions of Sobolev spaces over $\mathbb{R}^n$, $\mathbb{R}^n_+$, to spaces of distributions over $\overline{\Omega}$, $\Sigma$, $\Omega_1$, valued in the bundles, by use of local trivializations. The definition of $\psi$do's likewise generalizes to the manifold and vector bundle situation. In the present paper, our application deals with scalar $\psi$do's, so we shall drop the vector bundle aspect to simplify notations, but declare at this point that the constructions of order-reducing operators generalize to bundles as in [G90], easily taken up when needed. We denote by $r_{\Omega}$, or for brevity $r^+$, the restriction from $\Omega_1$ to $\Omega$, and by $e_\Omega$ or $e^+$ the extension from $\Omega$ by zero on $\Omega_1 \setminus \overline{\Omega}$. For an operator $P$ over $\Omega_1$, we denote $r_{\Omega}P e_{\Omega}$ (also called $e^+ P r^+$) by $P_{\Omega}$ or $P_+$.

**Theorem 1.3.** There exists a family of elliptic $\psi$do's $\Lambda_{+}^{(\mu)}$ on $\Omega_1$, classical of order $\mu$ and with principal symbol $\lambda_{+}^{\mu}$ at the boundary of $\Omega$, preserving support in $\overline{\Omega}$ and defining homeomorphisms

\begin{equation}
\Lambda_{+}^{(\mu)} : H^s_{p}(\overline{\Omega}) \xrightarrow{\sim} H^{s-\text{Re}\mu}_{p}(\Omega),
\end{equation}

for all $s \in \mathbb{R}$, with inverses $(\Lambda_{+}^{(\mu)})^{-1}$ likewise preserving support in $\overline{\Omega}$. The family of adjoints are classical elliptic operators $\Lambda_{-}^{(\mu)}$, with principal symbol $\lambda_{-}^{\mu}$ at the boundary of $\overline{\Omega}$, such that $\Lambda_{-}^{(\mu)} = r^+ \Lambda_{+}^{(\mu)} e^+$ are homeomorphisms

\begin{equation}
\Lambda_{-}^{(\mu)} : \overline{H}^s_{p}(\Omega) \xrightarrow{\sim} \overline{H}^{s-\text{Re}\mu}_{p}(\Omega),
\end{equation}

for all $s \in \mathbb{R}$, with inverses $((\Lambda_{-}^{(\mu)})^{-1})_+$. 

*Proof.* The construction is explained in detail in [G90], Sections 4 and 5, which we use with minor adaptations that we shall explain here. We provide $\Omega_1$ and $\Sigma$ with Riemannian metrics, such that a tubular neighborhood $\Omega_2$ of $\Sigma$ in $\Omega_1$ is isometric with $\Sigma \times ]-2,2[;\text{ the coordinates in }\Sigma \text{ resp. }] -2,2[ \text{ will be denoted } x' \text{ and } x_n, \text{ and we write } \Sigma_c = \Sigma \times ]-c,c[\text{ for } c \leq 2. \text{ Fix } \mu.}$ In the definition of $\lambda_{+}^{\mu}$ (1.14) we can insert an extra parameter $\zeta \geq 0$ (called $\mu$ in [G90]), defining

\begin{equation}
\lambda_{+,-}^{\mu} = (\lambda_{+,-}^{1})^{\mu}, \quad \lambda_{-,+}^{\mu} = [(\xi', \zeta)]\psi(\xi_n/a([\xi', \zeta])) - i \xi_n, \quad \lambda_{+,+}^{\mu} = \lambda_{-,+}^{1}.
\end{equation}

Now the construction of the $\psi$do $\Lambda_{+,-}^{(\mu)}$ defined on $\Omega_1$ is carried out similarly to the description in [G90] around (5.1), using $\lambda_{+,-}^{\mu}$ near the boundary and $[(\xi', \zeta)]^{\mu}$ at a distance from the boundary:

$\lambda_{+,+}^{(\mu)} = (\lambda_{+,+}^{1})^{\mu}(x_n)[(\xi', \zeta)][1-\alpha(x_n)]$

on $\Sigma_2$, extended by $[(\xi', \zeta)]^{\mu}$ on the rest of $\Omega_1$; here $\alpha(x_n) \in C^\infty(\mathbb{R},[0,1])$ equal to 1 on $[-1,1]$ and 0 on the complement of $[-\frac{3}{2}, \frac{3}{2}].$ The symbol extends analytically to $\text{Im} \xi_n < 0$. The operator $\Lambda_{+,-}^{(\mu)}$ is pieced together from this by use of a finite partition of unity.
subordinate to a covering of $\Omega_1$ by open sets in $\Sigma_\frac{n}{2}$ and open sets in $\Omega_1 \setminus \Sigma_\frac{n}{2}$, whereby $\Lambda^{(\mu)}_{+,\zeta}$ preserves support in $\overline{\Omega}$.

The construction with $\mu$ replaced by $-\mu$ gives the operator $\Lambda^{(-\mu)}_{+,\zeta}$, likewise elliptic on $\Omega_1$ and preserving support in $\overline{\Omega}$. Now

$$\tag{1.22} \Lambda^{(\mu)}_{+,\zeta} \Lambda^{(-\mu)}_{+,\zeta} = I + U_1(\zeta), \quad \Lambda^{(-\mu)}_{+,\zeta} \Lambda^{(\mu)}_{+,\zeta} = I + U_2(\zeta),$$

with $U_1, U_2$ of order $-1$, hence compact operators in $H^t_p(\Omega_1)$ for all $t, p$; they also preserve support in $\overline{\Omega}$. Standard elliptic theory shows that $\Lambda^{(\mu)}_{+,\zeta}$ is a Fredholm operator from $H^s_p(\Omega_1)$ to $H^{s-Re\mu}_p(\Omega_1)$ for all $s, p$, with a finite dimensional $C^\infty$ kernel and range complement independent of $s, p$. We have in particular that $\Lambda^{(\mu)}_{+,\zeta}$ maps $\dot{H}^s_p(\overline{\Omega})$ into $\dot{H}^{s-Re\mu}_p(\overline{\Omega})$, and $\Lambda^{(-\mu)}_{+,\zeta}$ maps the other way, with (1.22) valid there, so $\Lambda^{(\mu)}_{+,\zeta}$ is Fredholm between those spaces, with a finite dimensional $C^\infty$ kernel $K_1$ and range complement $K_2$ independent of $s, p$. The idea with the parameter $\zeta$ is that we can apply the calculus of [G96] (just for $\psi$do symbols), where our symbols are of regularity $\nu = +\infty$ as functions of $(\xi, \zeta)$; then the norms of $U_1$ and $U_2$ are $\leq \frac{1}{2}$ for $\zeta$ sufficiently large, so that $I + U_1$ and $I + U_2$ are invertible, and it follows that $\Lambda^{(\mu)}_{+,\zeta}$ over $\overline{\Omega}$ is invertible for large $\zeta$. Since it depends continuously on $\zeta$, it follows that $\Lambda^{(\mu)}_{+,0}$ has index 0. For $p = 2$, the kernel and range complement are spanned by orthonormal systems of smooth functions $\{\varphi_1, \ldots, \varphi_N\}$ and $\{\psi_1, \ldots, \psi_N\}$ supported in $\overline{\Omega}$, and when we define the order $-\infty$ operator $\Psi$ by $\Psi u = \sum_{j,k=1}^N \psi_j(u, \varphi_k)$,

$$\Lambda^{(\mu)}_+ = \Lambda^{(\mu)}_{+,0} + \Psi,$$

has the desired bijectiveness property.

An operator $\Lambda^{(\mu)}_{-,+}$ with the desired properties is now found as the adjoint of $\Lambda^{(\mu)}_{+,\zeta}$ in (1.19), in the same way as for $\mathbb{R}^n_+$. \(\square\)

For negative $s$ in (1.20) the operator is understood as in Remark 1.1. The assertion (1.18) generalizes to these operators. More properties are shown in Example 2.8 later.

It is the introduction of these $\psi$do’s that allows a relatively elegant deduction of solvability properties for the equations we consider in this paper. They had not been found when [H65] was written (and there is a remark there that such operators would be helpful).

Occasionally we shall refer to the spaces $C^t(\overline{\Omega})$ and $C^t(\Omega)$ for $t \geq 0$; in integer cases they are the usual spaces of functions with continuous derivatives up to order $t$ on $\overline{\Omega}$ resp. $\Omega$, and when $t = k + s$, $k \in \mathbb{N}_0$, $s \in [0, 1]$, they are the Hölder spaces also denoted $C^{k,s}(\overline{\Omega})$ resp. $C^{k,s}(\Omega)$. We denote $\bigcup_{\varepsilon > 0} C^{t+\varepsilon} = C^{t+0}$, and $\bigcap_{\varepsilon > 0} C^{t-\varepsilon} = C^{t-0}$ if $t > 0$. There are embeddings

$$\tag{1.23} \overline{H}^t_p(\Omega) \subset C^{t-n/p-0}(\overline{\Omega}) \text{ when } t > n/p, \quad C^{t+0}(\overline{\Omega}) \subset \overline{H}^t_p(\Omega) \text{ when } t \geq 0;$$

in the first embedding, “$-0$” can be left out if $t - n/p$ is not integer, in the second we assume $\overline{\Omega}$ compact. We shall denote \{\(u \in C^t(\Omega_1) \mid \supp u \subset \overline{\Omega}\)\} = \dot{C}^t(\overline{\Omega}).
1.3 Hörmander’s $\mu$-spaces. In the notes [H65] there are introduced (for $p = 2$) the following spaces that mix the features of the supported and the restricted Sobolev spaces in a particular way by use of the mappings $\Xi_+^\mu$. (Actually, [H65] uses $(\langle D' \rangle + \partial_n)^\mu$ instead of $\Xi_+^\mu = ([D'] + \partial_n)^\mu$; they are equivalent.)

**Definition 1.4.** Let $\mu \in \mathbb{C}$, and let $s > \Re \mu - 1/p'$. An element $u \in \dot{S}'(\mathbb{R}^n_+)$ is in $H_p^{\mu(s)}(\mathbb{R}^n_+)$ if and only if $\Xi_+^\mu u \in \dot{H}^{-1/p'+0}(\mathbb{R}^n_+)$ and

\[(1.24) \quad \|r^+\Xi_+^\mu u\|_{\mathcal{H}_p^{-Re \mu}(\mathbb{R}^n_+)} < \infty;\]

the topology is defined by the norm (1.24), also denoted $\|u\|_{\mu(s)}$.

In this definition, $\Xi_+^\mu$ can be replaced by $\Lambda_+^\mu$.

The last statement is justified by the properties shown in Section 1.2, in particular Lemma 1.2.

The condition $\Xi_+^\mu u \in \dot{H}^{-1/p'+0}(\mathbb{R}^n_+)$ can also be expressed as

\[u \in \dot{H}^{Re \mu - 1/p'+0}(\mathbb{R}^n_+),\]

in view of the homeomorphism properties (1.11). Note that the inequality in (1.24) implies, since $s - \Re \mu > -1/p'$, that the elements satisfy for $0 < \varepsilon < \min\{1, s - \Re \mu + 1/p'\}$:

\[(1.25) \quad \Xi_+^\mu u \in \mathcal{T}^{-1/p'}(\mathbb{R}^n_+),\]

using the identification of $r^+ v$ and $e^+ r^+ v$ in spaces with $-1/p' < s < 1/p$, cf. (1.8). So the norm (1.24) is stronger than the norm on the spaces in (1.25), which need not be mentioned in the definition of the topology.

If $s < \Re \mu + 1/p$, the condition in (1.24) reduces to $\Xi_+^\mu u \in \dot{H}^{s-\Re \mu}(\mathbb{R}^n_+)$; therefore

\[(1.26) \quad H_p^{\mu(s)}(\mathbb{R}^n_+) = \dot{H}^{s}(\mathbb{R}^n_+) \text{ when } -1/p' < s - \Re \mu < 1/p,\]

and $C_0^\infty(\mathbb{R}^n_+)$ is dense in the space. When $s$ is larger, (1.24) gives a nontrivial restriction on $u$.

We can then extend the definition to all $s$, consistently with the above:

**Definition 1.5.** Let $\mu \in \mathbb{C}$, and let $s < \Re \mu + 1/p$. Then we define

\[(1.27) \quad \dot{H}^{\mu(s)}_p(\mathbb{R}^n_+) = \dot{H}^{s}_{p}(\mathbb{R}^n_+).\]

Note that $\dot{H}^{s}_{p}(\mathbb{R}^n_+) \subset H^{\mu(s)}_p(\mathbb{R}^n_+)$ holds for all $s$ and $\mu$.

**Example 1.6.** Let $\mu = m \in \mathbb{N}$ and $s > m - 1/p'$. Then $u \in H^{m(s)}_p$ if and only if $u \in \dot{H}^{-1/p'+0}$ and $r^+(\partial_n)^m u \in \mathcal{T}_{p}^{-s-m}$. The first condition implies that $\varrho_m u = 0$, and the second condition holds if $u \in \mathcal{T}_{p}^{-s}$. The second condition can also be written $\Lambda_+^{m(s)} u \in \mathcal{T}_{p}^{-s-m}$, and in view of the ellipticity of the system $\{\Lambda_+^{m+\varrho_m}\}$ in the Boutet de Monvel calculus, cf. (1.18), we see that $u$ must lie in $\mathcal{T}_{p}^{-s}$.

This shows that $H^{m(s)}_p = \{u \in \mathcal{T}_{p}^{-s} \mid \varrho_m u = 0\}$. Note that for $s > m + 1/p$, the space is a proper subspace of $\mathcal{T}_{p}^{-s}$, different from $\dot{H}^{s}_p$.

This example is still within the Boutet de Monvel calculus; the novelty of the spaces $H^{\mu(s)}_p$ lies more in what happens for noninteger $\mu$.

The following observation will be very useful:
**Proposition 1.7.** Let $s > \text{Re } \mu - 1/p'$. The mapping $r^+ \Xi_+^\mu$ is a homeomorphism of $H_p^{\mu(s)}(\mathbb{R}_+^n)$ onto $H_p^{s-\text{Re } \mu}(\mathbb{R}_+^n)$ with inverse $\Xi_+^{-\mu} e^+$. In particular, $H_p^{\mu(s)}(\mathbb{R}_+^n)$ is a Banach space.

The analogous result holds with $\Lambda_+^\mu$-operators, and with $\Xi_+^\mu$-operators where $[\xi']$ is replaced by $\langle \xi' \rangle$.

**Proof.** By definition, $r^+ \Xi_+^\mu$ is continuous.

Surjectiveness is seen as follows: Let $v \in H_p^{s-\text{Re } \mu}$, and set $w = \Xi_+^{-\mu} e^+ v$. Then $\Xi_+^\mu w = \Xi_+^\mu \Xi_+^{-\mu} e^+ v = e^+ v$. Since $s - \text{Re } \mu > -1/p'$, $e^+ v \in H_p^{-1/p'+0}$, so $\Xi_+^\mu w \in H_p^{-1/p'+0}$ as required in Definition 1.4. Moreover,

$$r^+ \Xi_+^\mu w = r^+ \Xi_+^\mu \Xi_+^{-\mu} e^+ v = r^+ e^+ v = v$$

is in $H_p^{\mu(s)}$ by hypothesis, so $v$ is the image of $w \in H_p^{\mu(s)}$.

The injectiveness. When $u$ satisfies the hypotheses of Definition 1.2, then $u$ is reconstructed from $v = r^+ \Xi_+^\mu u$ as follows: Since $\Xi_+^\mu u \in H_p^{-1/p'+0}(\mathbb{R}_+^n)$, we can write

$$\Xi_+^\mu u = e^+ r^+ \Xi_+^\mu u + e^+ r^+ \Xi_+^\mu u.$$  

Here $r^+ \Xi_+^\mu u = 0$, since $\Xi_+^\mu$ preserves support in $\mathbb{R}_+^n$. Hence

$$u = \Xi_+^{-\mu} \Xi_+^\mu u = \Xi_+^{-\mu} e^+ r^+ \Xi_+^\mu u = \Xi_+^{-\mu} e^+ v.$$  

Thus $r^+ \Xi_+^\mu$ is an isometry of $H_p^{\mu(s)}$ onto $H_p^{s-\text{Re } \mu}$, with inverse $\Xi_+^{-\mu} e^+$. In particular, $H_p^{\mu(s)}$ is a Banach space.

The proof for $\Lambda_+^\mu$ and for the other version of $\Xi_+^\mu$ goes in the same way. □

The spaces can also be defined in the manifold situation. By use of the operators $\Lambda_+^{(\mu)}$ introduced in Theorem 1.3, we can formulate the definition as follows:

**Definition 1.8.** Let $\mu \in \mathbb{C}$. When $s > \text{Re } \mu - 1/p'$, then $H_p^{\mu(s)}(\Omega)$ consists of the elements $u \in \mathcal{E}'(\Omega)$ such that $\Lambda_+^{(\mu)} u \in H_p^{-1/p'+0}(\Omega)$ and

$$||r_\Omega \Lambda_+^{(\mu)} u||_{H_p^{-\text{Re } \mu}(\Omega)} < \infty;$$

it is a Banach space with the norm (1.29), also denoted $||u||_{\mu(s)}$.

When $s < \text{Re } \mu + 1/p$, we define

$$H_p^{\mu(s)}(\Omega) = \hat{H}_{ps}^s(\Omega).$$

Here the space $\mathcal{E}'(\Omega)$ denotes the distributions supported in $\Omega$ (compactly supported in $\Omega$, if $\Omega$ is allowed to be merely paracomplete). Again we observe that the norm in (1.29) is stronger than the norm in $H_p^{s-1/p'}(\Omega)$ for small $\varepsilon$, and that the space equals $\hat{H}_{ps}^s(\Omega)$ when $-1/p' < s - \text{Re } \mu < 1/p$, so that the last part of the definition allowing lower values of $s$ is consistent with the first part. Also Proposition 1.7 extends.

There are of course embeddings

$$H_p^{\mu(s)} \subset H_p^{\mu(s')}$$  

for $s' < s$.

On the other hand, embeddings between spaces with different $\mu, \mu'$ do not hold in general. An exception is when $\mu - \mu'$ is integer, see Proposition 4.3 later.

The structure of the spaces will be further described below, particularly their importance for $\psi$do’s with the transmission property of type $\mu$. 
Remark 1.9. In [H65], \( H^s_2(\Omega) \) is defined as the completion of \( \mathcal{E}_\mu(\overline{\Omega}) \) in the topology defined by the seminorms \( u \mapsto \| r^m Pu \|_{\mathcal{F}^{s-2m}_\infty(\Omega)} \), where \( P \) runs through the operators of type \( \mu \) and any order \( m \in \mathbb{C} \). The proof that this is equivalent with Definition 1.4 (when localized) fills a large section. It is here covered by Proposition 4.1 and Theorem 4.2 below.

2. The \( \mu \)-transmission condition

The \( \mu \)-transmission condition is defined and characterized in [H85] at the end of Section 18.2. Since the explanation is quite compressed there, we have incorporated some of the original detailed deductions from [H65] here, slightly modified if necessary. (We remark that the conventions in [H85] are a little different from here: The space called \( \mathcal{C}^\infty_\mu \) there on pp. 110–111 is the same as \( \mathcal{E}_{-\mu-1} \) here, and \( \mu \) in Th. 18.2.18 there corresponds to \( -\mu \) in Definition 2.5 below.)

Let \( \Omega \) be a fixed paracompact \( C^\infty \) manifold, and let \( \Omega \) be an open subset of \( \Omega_1 \) with a \( C^\infty \) boundary \( \partial \Omega \). Our purpose is to study boundary problems for the pseudodifferential operator \( P \) in \( \Omega \). This means that we shall look for distributions \( u \) with support in \( \overline{\Omega} \) such that \( Pu = f \) is given in \( \Omega \) and \( u \) satisfies some conditions on \( \partial \Omega \) in addition. In particular we shall make a detailed study of the regularity of \( u \) at the boundary when \( f \) and the boundary data are smooth. Examples involving \( \alpha \)-potentials due to M. Riesz and extended in part by Wallin show that one should not expect \( u \) to be smooth up to the boundary but that one has to expect \( u \) to behave as the distance to the boundary raised to some power. This leads us to define a family of spaces of distributions \( \mathcal{E}_\mu \) as follows.

Definition 2.1. If \( \text{Re} \mu > -1 \) and if \( d \) is a real valued function in \( C^\infty(\Omega_1) \) such that

\[
\Omega = \{ x \mid d(x) > 0 \}
\]

and \( d \) vanishes only to the first order on \( \partial \Omega \), then \( \mathcal{E}_\mu(\overline{\Omega}) \) consists of all functions \( u \) such that \( u = 0 \) in \( \Omega_1 \) and \( u = d^\mu v \) in \( \overline{\Omega} \) for some \( v \in C^\infty(\Omega) \).

For lower values of \( \text{Re} \mu \), \( \mathcal{E}_\mu \) is defined successively so that \( \mathcal{E}_{\mu-1} \) is always the linear hull of the spaces \( D\mathcal{E}_\mu \) when \( D \) varies over the first order differential operators with \( C^\infty \) coefficients.

This definition is independent of the choice of \( d \), for if \( d_1, d_2 \) are two functions with the required properties, the quotient \( d_1/d_2 \) is positive and infinitely differentiable.

To justify the second part of the definition we note that if \( D \) is a first order differential operator with \( C^\infty \) coefficients, and if \( \text{Re} \mu > 0 \), then \( D\mathcal{E}_\mu \subset \mathcal{E}_{\mu-1} \), for \( D(d^\mu v) = d^{\mu-1} V \) for some \( V \in C^\infty \). The linear hull of the spaces \( D\mathcal{E}_\mu \) when \( D \) varies is in fact equal to \( \mathcal{E}_{\mu-1} \). It is sufficient to prove that it contains any element in \( \mathcal{E}_{\mu-1} \) with support in a coordinate patch where \( \Omega \) is defined by \( x_n > 0 \). Then we can take \( D = \partial/\partial x_n \), noting that if \( v \in C^\infty \) then

\[
\int_0^{x_n} t^{\mu-1} v(x', t) \, dt = x_n^\mu V(x),
\]

where

\[
V(x) = \int_0^1 t^{\mu-1} v(x', x_n t) \, dt
\]

is a \( C^\infty \) function. If \( u = x_n^{\mu-1} v \) and \( U = x_n^\mu V \chi \), both functions being defined as 0 when \( x_n < 0 \), and \( \chi \in C_0^\infty \) is 1 in a neighborhood of \( \text{supp} \, u \), then \( u = \partial U/\partial x_n \) is a \( C^\infty \) function.
Conversely, given such $u \in \partial \mathcal{E}_\mu / \partial x_n + \mathcal{E}_\mu$. It is thus legitimate to define $\mathcal{E}_\mu$ successively for decreasing $\text{Re} \mu$ as indicated.

The spaces $\mathcal{E}_\mu$ so obtained have the local property that $u \in \mathcal{E}_\mu(\overline{\Omega})$ and $\varphi \in C^\infty(\Omega_1)$ implies that $\varphi u \in \mathcal{E}_\mu(\overline{\Omega})$. In fact, if $D$ again denotes a first order differential operator we have

$$\varphi D \mathcal{E}_{\mu+1} \subset D \varphi \mathcal{E}_{\mu+1} + \mathcal{E}_{\mu+1} \subset D \mathcal{E}_{\mu+1} + \mathcal{E}_\mu \subset \mathcal{E}_\mu,$$

where we have assumed that the assertion is already proved with $\mu$ replaced by $\mu + 1$. The spaces $\mathcal{E}_\mu$ are thus determined by local properties. Inside the set, the condition $u \in \mathcal{E}_\mu$ only means that $u$ is a $C^\infty$ function.

To determine the meaning of the condition $u \in \mathcal{E}_\mu$ at a boundary point we consider the case when $u$ has compact support in a coordinate patch where $\Omega$ is defined by the condition $x_n > 0$.

**Remark 2.2.** It will be useful to recall some formulas for power functions in one variable $t$ and their Fourier transforms. Denote as in [H65]

$$I^\mu(t) = \begin{cases} t^\mu / \Gamma(\mu + 1) & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

when $\text{Re} \mu > -1$; it is called $\chi_+^\mu(t)$ in [H83], Section 3.2. It is shown there that the distribution $I^\mu$ extends analytically from $\text{Re} \mu > -1$ to $\mu \in \mathbb{C}$. (For negative integers, $I^{-k} = \delta_0^{k-1}$.) Moreover, [H65] uses the notation $(z^\pm)^a$ for the boundary values of $z^a$ from the half-planes $\mathbb{C}_\pm = \{ z \in \mathbb{C} \mid \text{Im } z \geq 0 \}$, defined to be real and positive on the positive real axis (they are denoted $(z \pm i0)^a$ in [H83]). Explicitly,

$$(z^+)^a = \begin{cases} z^a & \text{for } z > 0, \\ |z|^a e^{i\pi a} & \text{for } z < 0; \end{cases} \quad (z^-)^a = \begin{cases} z^a & \text{for } z > 0, \\ |z|^a e^{-i\pi a} & \text{for } z < 0. \end{cases}$$

Then, cf. [H83], Ex. 7.1.17, $I^\mu(t)$ has the Fourier transform

$$\mathcal{F}_{t \to \tau} I^\mu = e^{-i\pi(\mu+1)/2} \tau^{-\mu-1}.$$

We also note that when $\sigma > 0$, translation by $-i\sigma$ gives

$$e^{-i\pi(\mu+1)/2} \mathcal{F}^{-1}(\tau - i\sigma)^{-\mu-1} = \mathcal{F}^{-1}(\tau + i\sigma)^{-\mu-1} = I^\mu e^{-t\sigma}.$$

**Lemma 2.3.** An element $u \in \mathcal{E}'(\mathbb{R}^n)$ belongs to $\mathcal{E}_\mu(\overline{\mathbb{R}^n_+})$, if and only if $u$ vanishes when $x_n < 0$ and one can find $u_0, u_1, \ldots \in C^\infty(\mathbb{R}^n)$ such that for every $N$

$$\hat{u}(\xi) - \sum_{0}^{N-1} (\xi_n - i)^{-\mu-j-1} \hat{u}_j(\xi') = O(|\xi|^{-\text{Re} \mu-N-1}), \xi \to \infty.$$

Conversely, given such $u_0, u_1, \ldots$ one can find $u \in \mathcal{E}_\mu(\overline{\mathbb{R}^n_+})$ satisfying this condition.

Here the argument of $\xi_n - i$ is chosen so that it tends to zero when $\xi_n \to +\infty$.

**Proof.** Any element $u \in \mathcal{E}_\mu$ can be written $u = v + \partial w / \partial x_n$ where $v$ and $w$ belong to $\mathcal{E}_{\mu+1}$. If the necessity of (2.6) has been proved when $\mu$ is replaced by $\mu + 1$ it follows therefore for
\( \mu \). Hence we may assume that \( \text{Re} \mu > 0 \), thus \( u = vx_n^\mu \) when \( x_n > 0 \), where \( v \in C_0^\infty(\mathbb{R}^n) \).

By forming a Taylor expansion of \( ve^{x_n} \) we can write for every \( N \)
\[
v = e^{-x_n} \sum_{j=0}^{N} v_j(x')x_n^j + R_N(x)
\]
where \( v_j \in C_0^\infty(\mathbb{R}^{n-1}) \) and \( R_N(x) = O(x_n^N) \) when \( x_n \to 0 \), \( R_N(x) = O(e^{-x_n}/2) \) when \( x_n \to \infty \). Set \( R_0^N(x) = e^{x_n}R_N(x) \). Then \( R_N^j(x)x_n^j \) has integrable derivatives of order \( N \), so the Fourier transform is \( O(|\xi|^{-N}) \). Now
\[
\hat{u} = \sum_{j=0}^{\infty} \hat{v}_j(\xi')F_{x_n}^{-\xi_n}(e^{x_n}e^{-x_n}x_n^{j+\mu}) + F_{x_n}^{-\xi_n}(R_0^N(x)x_n^\mu).
\]
By (2.5), \( F_{x_n}^{-\xi_n}(e^{x_n}e^{-x_n}x_n^{j+\mu}) = \Gamma(\mu + j + 1)e^{-\pi i(\mu+j+1)/2}(\xi_n - i)^{-\mu-j-1} \), so if we set
\[
u_j = v_j \Gamma(\mu + j + 1)e^{-\pi i(\mu+j+1)/2},
\]
it follows that (2.6) holds with the error term \( O(|\xi|^{-N}) \). Taking a few additional terms in the left hand side of (2.6) and noting that they can all be estimated in terms of the quantity on the right, we thus conclude that (2.6) is valid.

On the other hand, if \( u \) satisfies (2.6) we obtain with \( v_j \) defined by (2.7) that \( u - e^{-x_n} \sum_{j=0}^{N-1} v_j x_n^{j+\mu} \) will be arbitrarily smooth if \( N \) is large. This proves the sufficiency of (2.6). To prove the last statement we again assume that \( \text{Re} \mu > 0 \), take \( \chi \in C_0^\infty(\mathbb{R}) \), equal to 1 when \( |x_n| < 1 \) and define
\[
u(x) = 0, x_n \leq 0, \quad \nu(x) = \sum_{j=0}^{\infty} e^{-x_n} v_j(x')x_n^{j+\mu} \chi(x_n a_j), x_n > 0,
\]
where \( a_j \) is chosen so large that the derivatives of the \( j \)th term of order \( \leq j \) are all \( \leq 2^{-j} \). This is possible since \( (x_n a_j)^{\nu} \chi^{(k)}(x_n a_j) \) is bounded uniformly in \( x_n \) and \( a_j \) if \( \text{Re} \nu \geq 0 \). This completes the proof. \( \Box \)

The particular case where \( \mu \) is an integer is of special importance. When \( \mu \geq 0 \) the space \( E_{\mu} \) then consists of all functions in \( C_0^\infty(\overline{\Omega}) \) which vanish to the order \( \mu \) at the boundary (that is, the derivatives of order \( < \mu \) vanish there), extrapolated by 0 outside. When \( \mu < 0 \) we have the sum of a function in \( C_0^\infty(\overline{\Omega}) \) extrapolated as 0 in the complement of \( \overline{\Omega} \), and multiple layers with \( C_0^\infty \) densities and of order \( < -\mu \) on \( \partial \Omega \). This is the only case when \( E_{\mu} \) contains elements supported by \( \partial \Omega \); in other words, the restriction of an element in \( E_{\mu} \) to \( \Omega \) determines it uniquely except when \( \mu \) is a negative integer.

**Remark 2.4.** It was convenient in the proof of Lemma 2.1 to work with powers of \( \xi_n - i \) instead of powers of \( \xi_n \), and one could also work with powers of \( \xi_n - i\sigma \) with a \( \sigma > 0 \), e.g. \( \sigma = [\xi'] \); however \( (\xi_n^{-a}) \) are more convenient in some applications. In terms of these functions we can rewrite (2.6) in the form
\[
\hat{u}(\xi) - \sum_{j=0}^{N-1} (\xi_n)^{-\mu-j-1} \hat{u}_j(\xi') = O(|\xi|^{-\text{Re} \mu-N-1}), \xi \to \infty, |\xi_n| > 1,
\]
where \( \mu > 0 \).
where \( u_j' \) is a linear combination of \( u_0, \ldots, u_j \) with coefficient 1 for \( u_j \). Namely, insert Taylor expansions \((z - i)^a = (z^-)^a + (-i)a(z^-)^{a-1} + (-i)^2 \frac{1}{2!} a(a - 1)(z^-)^{a-2} + \ldots \) of the terms \((\xi_n - i)^{-\mu-j-1}\), and regroup the resulting sums. Thus the \( u_j' \) occurring in (2.8) are in one to one correspondence with the \( u_j \) in (2.6) and can be chosen arbitrarily.

In particular, when \( \mu = 0 \), so that \( \mathcal{E}_\mu(\overline{\Omega}) = \epsilon_\Omega C^\infty(\overline{\Omega}) \),

\[
(2.9) \quad u_0 = u_0' = -i\gamma_0 u,
\]

where \( \gamma_0 u \) is the boundary value from \( \Omega \).

Consider a classical pseudodifferential operator \( P \) in \( \Omega_1 \) of order \( m \in \mathbb{C} \). Recall the notation for derivatives of the symbol in local coordinates:

\[
(2.10) \quad p^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha \partial^\beta p(x, \xi).
\]

The first question to investigate is when \( P \) maps \( \mathcal{E}_\mu \) into \( C^\infty(\overline{\Omega}) \) (more precisely, the restrictions to \( \Omega \) belong to \( C^\infty(\overline{\Omega}) \)). By the pseudo-local property of \( \psi \text{do}'s \) we know that \( Pu \in C^\infty(\Omega) \) for all \( u \in \mathcal{E}_\mu \). We shall therefore only expect a restriction on \( P \) at points on \( \partial \Omega \). Of course it is no restriction to assume \( P \) compactly supported when studying a regularity problem.

**Definition 2.5.** A classical pseudodifferential operator of order \( m \) in \( \Omega_1 \) is said to satisfy the \( \mu \)-transmission condition relative to \( \Omega \) (in short: be of type \( \mu \)), when the symbol in any local coordinate system satisfies

\[
(2.11) \quad p^{(\alpha)}_{j(j)}(x, -N) = e^{\pi i(m-2\mu-j-|\alpha|)} p^{(\alpha)}_{j(j)}(x, N), \quad x \in \partial \Omega,
\]

for all \( j, \alpha, \beta \), where \( N \) denotes the interior normal of \( \partial \Omega \) at \( x \).

**Theorem 2.6.** Let \( P \) be a classical compactly supported pseudodifferential operator of order \( m \) in \( \Omega_1 \). In order that \( \tau_\Omega Pu \in C^\infty(\overline{\Omega}) \) for all \( u \in \mathcal{E}_\mu(\overline{\Omega}) \), it is necessary and sufficient that \( P \) satisfies the \( \mu \)-transmission condition.

Since every polynomial satisfies this hypothesis with \( \mu = 0 \) it follows from the rules for coordinate changes that (2.11) is invariant under any change of variables. In the proof of the theorem we may therefore use local coordinates such that \( \Omega \) is defined by the inequality \( x_n > 0 \). The statement is local, so it is enough to consider \( Pu \) for \( u \in \mathcal{E}_\mu(\mathbb{R}^n_+) \) with compact support in the coordinate patch \( U \subset \mathbb{R}^n \). After modifying \( P \) by an operator with symbol 0 we may assume that \( P \) is a compactly supported operator in \( U \).

A key observation is the following elementary lemma.

**Lemma 2.7.** Let \( q \) be a positively homogeneous function on \( \mathbb{R} \) of degree \( \sigma \), \( \Re \sigma < -1 \). For \( t > 0 \) we set \( \varphi_{\sigma}(t) = t^{-\sigma-1} \) if \( \sigma \) is not an integer and \( \varphi_{\sigma}(t) = t^{-\sigma-1} \log t \) if \( \sigma \) is an integer. Then

\[
\int_{|\tau| > 1} e^{it\tau} q(\tau) \, d\tau, \quad t > 0,
\]

is on \( \mathbb{R}_+ \) equal to the sum of a function in \( C^\infty(\mathbb{R}_+) \) and \( C \varphi_{\sigma}(t) \). Here \( C = 0 \) if and only if \( q(-1) = e^{i\pi\sigma} q(1) \), that is, if \( q(\tau) = q(1)(\tau^+)^\sigma \).
We shall introduce a Taylor expansion of \( p \sigma \) where
\[
p(2.13)
\]
for positively homogeneous functions of degree \( \sigma \).

Then the two functions
\[
p(2.12)
\]
are integrals of \( e^{it\tau} \) over semi-circles, hence obviously entire analytic functions of \( t \). By Cauchy’s integral formula one concludes that the integral over \( \gamma_+ (\gamma_-) \) vanishes for \( t > 0 \) \((t < 0)\), and that it is homogeneous of degree \(-\sigma - 1\) when \( t < 0 \) \((t > 0)\). When \( \sigma \) is not an integer, the two functions \((\tau^+)^\sigma\) and \((\tau^-)^\sigma\) are linearly independent, hence form a basis for positively homogeneous functions of degree \( \sigma \). This proves the lemma for non-integral \( \sigma \).

To complete the proof it only remains to study
\[
\int_{|\tau|>1} (\tau^\pm)^{\sigma-1}|\tau|e^{it\tau} d\tau
\]
when \( \sigma \) is an integer \( \leq -2 \). When \( \sigma = -2 \) the last integral is equal to
\[
2 \int_{1}^{\infty} \tau^{-2} t\tau d\tau = 2t \int_{1/t}^{\infty} \tau^{-2} \sin \tau d\tau.
\]

A Taylor expansion of \( \sin \tau \) shows that the integral is equal to \( \log 1/t \) plus a function in \( C^\infty(\mathbb{R}^n) \). This proves the statement when \( \sigma = -2 \), and by successive integration it follows for all integers \( \sigma < -2 \).

**Proof of Theorem 2.6.** Suppose that the theorem were already proved with \( \mu \) replaced by \( \mu + 1 \). The necessity of (2.11) is then obvious for it holds with \( \mu \) replaced by \( \mu + 1 \) and \( e^{-2\pi i t} = 1 \). To prove its sufficiency we have to show that \( PDu \in C^\infty(\Omega) \) if \( u \in \mathcal{E}_{\mu+1} \) and \( D \) is a first order differential operator. Since \( PDu = DPu + [P,D]u \) and \([P,D] \) satisfies (2.11) if \( P \) does, the assertion follows. Hence we may assume in what follows that \( \text{Re} \mu > \text{Re} m \). Then the product of \( p(x,\xi) \) by the Fourier transform of any compactly supported \( u \in \mathcal{E}_\mu(\mathbb{R}_+^n) \) is integrable, so by an obvious regularization we obtain
\[
(2.12) \quad p(x,D)u = (2\pi)^{-n} \int p(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi} d\xi.
\]

We shall introduce a Taylor expansion of \( p \) in (2.12),
\[
(2.13) \quad p(x,\xi) = \sum_{|\alpha|<\nu} (\partial^{\alpha}\rho(x',0,0,\xi_n)/\partial \xi^{\alpha'} \partial x_n^{\alpha_n})x_n^{\alpha_n}\xi^{\alpha'}/\alpha! + \sum_{|\alpha|=\nu} r^{\alpha}(x,\xi)x_n^{\alpha_n}\xi^{\alpha'},
\]
where
\[
r^{\alpha}(x,\xi) = |\alpha|/\alpha! \int_0^1 (1-t)^{|\alpha|-1} p^{(\alpha')}(x',tx_n,t\xi',\xi_n) dt,
\]
where somewhat incorrectly we have used the notation \( \alpha' \) for \((\alpha',0)\) and \( \alpha_n \) for \((0,\alpha_n)\). When \( |\alpha'| \geq \text{Re} m \) we can estimate \( r^{\alpha} \) by \((1+|\xi_n|)^{\text{Re} m-|\alpha'|} \), and when \( |\alpha'| \leq \text{Re} m \) we can estimate by \((1+|\xi|)^{\text{Re} m-|\alpha'|} \) instead. Now we have
\[
\int r^{\alpha}(x,\xi)x_n^{\alpha_n}\xi^{\alpha'}\hat{u}(\xi)e^{ix\cdot\xi} d\xi = \int (i\partial_{\xi_n})^{\alpha_n}(r^{\alpha}(x,\xi)\xi^{\alpha'}\hat{u}(\xi))e^{ix\cdot\xi} d\xi.
\]
Here the factor $x_n^\alpha$ was removed by an integration by parts with respect to $\xi_n$ (using that $x_n^\alpha e^{ix_n \xi_n} = (-i \partial_{\xi_n})^\alpha e^{ix_n \xi_n}$). In view of (2.6) we conclude that the integral and its derivatives of order $\leq k$ are absolutely convergent, thus the integral defines a $C^l$ function, provided that

$$l + \Re m - |\alpha'| - \alpha_n - \Re \mu < 0.$$  

If we choose $\nu > k + \Re (m - \mu)$, the error term in (2.13) will therefore only contribute a $C^l$ term to $p(x, D)u$. The remaining problem is only to study the regularity of the partial sums of the series obtained by replacing $p(x, \xi)$ by its Taylor expansion in (2.12). Since $\hat{u}$ is rapidly decreasing when $\xi \to \infty$ with $|\xi_n| < 1$, this part of the integral in (2.12) is infinitely differentiable. In view of (2.8) — where we drop the prime on $u'_j$ — it only remains to examine when the partial sums of the series

$$\sum_{\alpha, j, k} (2\pi)^{-n} \int_{|\xi_n| > 1} p_j^{(\alpha')}(x', 0, 0, \xi_n) x_n^\alpha \xi_k (\xi_n^-)\hat{u}_k(\xi') (\xi_n^-)^{-\mu - k - 1} e^{ix_n \xi} d\xi / \alpha!$$

become arbitrarily smooth when the order of the sum goes to infinity. Here we can remove $x_n^\alpha$ by an integration by parts with respect to $\xi_n$ as above. The boundary terms which then occur will give rise to only $C^\infty$ terms. Thus we are reduced to examining the differentiability of the partial sums of the series

$$\sum_{\alpha, j, k} D^{\alpha'} u_k(x')(2\pi)^{-1} \int_{|\xi_n| > 1} (i \partial_{\xi_n})^\alpha p_j^{(\alpha')}(x', 0, 0, \xi_n)(\xi_n^-)^{-\mu - k - 1} e^{ix_n \xi} d\xi_n / \alpha!.$$  

Since the functions $D^{\alpha'} u_k$ can be chosen arbitrarily in the neighborhood of any point, or rather, linear combinations of them are arbitrary, we conclude that for $P$ to have the required property it is necessary and sufficient that for any $\alpha'$ and $k = 0, 1, \ldots$ the partial sums of higher order of the series

$$(2.14) \quad \sum_{\alpha, n, j} (2\pi)^{-1} \int_{|\xi_n| > 1} (i \partial_{\xi_n})^\alpha p_j^{(\alpha')}(x', 0, 0, \xi_n)(\xi_n^-)^{-\mu - k - 1} e^{ix_n \xi} d\xi_n / \alpha!$$

are in $C^\nu(\mathbb{R}^+) = r^+ C^\nu(\mathbb{R})$ for any given $\nu$. Here $(i \partial_{\xi_n})^\alpha p_j^{(\alpha')}(x', 0, 0, \xi_n)(\xi_n^-)^{-\mu - k - 1}$ is homogeneous of degree $m - j - |\alpha| - \mu - k - 1$, so if $m - j - |\alpha| - \mu - 1 = \sigma$, the degree is $\sigma - k$.

Now we shall apply Lemma 2.7. Noting that a finite sum $\sum c_j \varphi_{\sigma_j}(t)$ with different $\sigma_j$ is in $C^\nu(\mathbb{R}^+)$ if and only if $c_j = 0$ when $-\sigma_j - 1 \leq \nu$, we conclude that (2.14) has the desired differentiability properties if and only if for each complex number $\sigma$, each $\alpha'$ and $k = 0, 1, \ldots$, each $x'$, the sum

$$(2.15) \quad q(\xi_n) \equiv \sum_{m - j - |\alpha| - \mu - 1 = \sigma} (i \partial_{\xi_n})^\alpha p_j^{(\alpha')}(x', 0, 0, \xi_n)(\xi_n^-)^{-\mu - k - 1} / \alpha!$$

is proportional to $(\xi_n^+)^{\sigma - k}$. (The sum of course contains only finitely many terms.)
In view of the homogeneity of \( p_j^{(\alpha')}(x', 0, 0, \xi_n) \) of degree \( m - j - |\alpha'| \), we have for each term in the sum:

\[
(i\partial_{\xi_n})^{\alpha_n}(p_j^{(\alpha')}(x', 0, 0, \xi_n)(\xi_n^-)^{-\mu - k - 1}) \quad \text{for } \xi_n > 0 \text{ equals}
\]

\[
= i^{\alpha_n} \partial_{\xi_n}^{\alpha_n}(p_j^{(\alpha')}(x', 0, 0, 1)\xi_n^{m - j - |\alpha'| - \mu - k - 1})
\]

\[
(2.16)
\]

\[
= i^{\alpha_n}(m - j - |\alpha'| - \mu - k - 1)\cdots(m - j - |\alpha'| - \mu - k) p_j^{(\alpha')}(x', 0, 0, 1)\xi_n^{m - j - |\alpha'| - \mu - k - 1}
\]

\[
= i^{\alpha_n}(m - j - |\alpha'| - \mu - k - 1)\cdots(m - j - |\alpha'| - \mu - k) p_j^{(\alpha')}(x', 0, 0, 1)\xi_n^{m - j - |\alpha'| - \mu - k - 1}
\]

whereas (cf. also (2.3))

\[
(i\partial_{\xi_n})^{\alpha_n}(p_j^{(\alpha')}(x', 0, 0, \xi_n)(\xi_n^-)^{-\mu - k - 1}) \quad \text{for } \xi_n < 0 \text{ equals}
\]

\[
= i^{\alpha_n} \partial_{\xi_n}^{\alpha_n}(p_j^{(\alpha')}(x', 0, 0, -1)|\xi_n|^{m - j - |\alpha'| - \mu - k - 1}e^{-\pi i(-\mu - k - 1)}
\]

\[
= (-i)^{\alpha_n}(m - j - |\alpha'| - \mu - k - 1)\cdots(m - j - |\alpha'| - \mu - k) p_j^{(\alpha')}(x', 0, 0, -1)|\xi_n|^{\mu - k}e^{\pi i(\mu + k + 1)}.
\]

A function equal to (2.16) on \( \mathbb{R}_+ \) will be proportional to \((\xi_n^+)^{\sigma - k}\) exactly when it on \( \mathbb{R}_- \) has the value

\[
i^{\alpha_n}(m - j - |\alpha'| - \mu - k - 1)\cdots(m - j - |\alpha'| - \mu - k) p_j^{(\alpha')}(x', 0, 0, 1)|\xi_n|^{\sigma - k}e^{\pi i(\sigma - k)}.
\]

Thus \( q(\xi_n) \), where we for fixed \( \alpha', k, \sigma \), take the sum over \( m - j - |\alpha'| - \mu - 1 = \sigma \), is proportional to \((\xi_n^+)^{\sigma - k}\) if and only if

\[
\sum_{m - j - |\alpha'| - \mu - 1 = \sigma} (m - j - |\alpha'| - \mu - k - 1)\cdots(m - j - |\alpha'| - \mu - k)(-1)^{\alpha_n} p_j^{(\alpha')}(x', 0, 0, -1)e^{\pi i(\mu + k + 1)}/\alpha_n! = \sum (m - j - |\alpha'| - \mu - k - 1)\cdots(m - j - |\alpha'| - \mu - k)(-1)^{\alpha_n} p_j^{(\alpha')}(x', 0, 0, 1)e^{\pi i(\mu + k + 1)}/\alpha_n!.
\]

After the exponential factors have been moved to the same side and integer powers of \( e^{2\pi i} \) have been eliminated, we find that \( k \) occurs only in the polynomial factors, which are of degree \( \alpha_n \), all different. It follows that the coefficients have to agree, that is

\[
p_j^{(\alpha')}(x', 0, 0, 1)e^{\pi i(m - j - |\alpha'| - 2\mu)} = p_j^{(\alpha')}(x', 0, 0, -1).
\]

This gives is a necessary and sufficient condition for \( r^+ P \) to map \( \mathcal{E}_\mu(\mathbb{R}_+^m) \) into \( C^\infty(\mathbb{R}_+^m) \). But (2.17) is a consequence of (2.11), and conversely, by differentiating (2.17) with respect to \( x' \) and using the homogeneity with respect to \( \xi_n \) we obtain (2.11). This completes the proof of Theorem 2.6. □

Note that it suffices that the conditions in (2.11) hold for the subset of derivatives \( p_j^{(\alpha')}(\alpha_n) \) indicated in (2.17). A similar sharpening is proved in [GH90] for more general, not necessarily polyhomogeneous symbols, in the case \( \mu = 0 \).

In [B69], Bouet de Monvel with reference to the notes [H65] showed that (2.11) for \( \psi \)-do’s with analytic symbols implies a mapping property as in Theorem 2.6 for functions analytic up to \( \partial \Omega \).

The product of two symbols of type \( \mu_1 \) resp. \( \mu_2 \) is clearly of type \( \mu_1 + \mu_2 \).
Example 2.8. As simple examples, let us mention \((-\Delta)^{\nu}\) and \(\Lambda^{\nu}_{\pm}\) on \(\mathbb{R}^n_{+}\) \((\nu \in \mathbb{C})\). For \((-\Delta)^{\nu}\), of order \(m = 2\nu\), the symbol \(|\xi|^{2\nu}\) equals 1 for \(\xi' = 0\), \(\xi_n = \pm 1\), so (2.11) is satisfied with \(\mu = \nu\); it is of type \(\nu\).

For \(\lambda^{\nu}_{\pm}\), the principal symbol \((\lambda^{\nu}_{\pm})_0\) is \((|\xi'|^{\nu}|(\xi_n/(\alpha|\xi'|)) + i\xi_n)^{\nu}\) (recall that \(\psi(\pm \infty) = 0\)), so \((\lambda^{\nu}_{\pm})_0(0, \pm 1) = (\pm i)^{\nu}\), satisfying (2.11) with \(m = \nu\), \(\mu = \nu\). The difference between \(\lambda^{\nu}_{\pm}\) and \((\lambda^{\nu}_{\pm})_0\) is of order \(-\infty\), since it has compact support in \(\xi'\) and is rapidly decreasing in \(\xi_n\). This shows that \(\lambda^{\nu}_{\pm}\) is of type \(\nu\).

A similar study of \(\lambda^{\nu}_{-}\) gives that it satisfies (2.11) with \(m = \nu\), \(\mu = 0\), since the principal part clearly does so, and the remainder is of order \(-\infty\). Hence it is of type 0.

Moreover, the modified symbols \(\lambda^{(\mu, 0)}\), used in the construction of order-reducing operators on a manifold (Theorem 1.3), are of type \(\mu\) resp. 0, since the exact symbols \(\lambda^{(\mu)}_{\pm}\) are used near \(\partial \Omega\), modulo smoothing terms.

We also have, when \(\Omega_1\) is compact:

Lemma 2.9. Let \(A\) be a strongly elliptic second-order differential operator with \(C^\infty\)-coefficients, and let \(\nu \in \mathbb{C}\). Then the pseudodifferential operator \(A^{\nu}\) is of order \(2\nu\), and of type \(\nu\) for any smooth set \(\Omega\).

Proof. \(A^{\nu}\) is constructed by the method of Seeley \([S67]\) (we recall that if 0 is an eigenvalue of \(A\), \(A^{\nu}\) is taken zero on the generalized eigenspace). First it is found that the resolvent \(Q = (A - \lambda)^{-1}\) has the symbol in local coordinates

\[
q(x, \xi, \lambda) \sim \sum_{l \geq 0} q_{-l}(x, \xi, \lambda), \text{ where } q_0 = (a_0(x, \xi) - \lambda)^{-1},
\]

\[
q_{-1} = b_{1,1}(x, \xi)q_0^2, \ldots, q_{-l} = \sum_{k=l/2}^{2l} b_{l,k}(x, \xi)q_0^{k+1}, \ldots;
\]

with symbols \(b_{l,k}\) independent of \(\lambda\) and polynomial of degree \(2k - l\) in \(\xi\). (References are given e.g. in \([G96]\), Remark 3.3.7.) The symbol of the \(\nu\)-th power of \(A\) is essentially constructed from this by a Cauchy integral together with \(\lambda^{\nu}\) around the spectrum. The principal term gives \((a_0(x, \xi))^{\nu}\), where, at boundary points, \(a_0 = s_0(x')\xi_n^2 + O(|\xi_n||\xi'|) + O(|\xi'|^2), \quad s_0(x') \neq 0,\)

with similar properties as the Laplacian symbol above; the \(\nu\)-th power satisfies (2.11) with \(m = 2\nu\) and \(\mu = \nu\). In the next terms, when \(q_0^{k+1} = c\partial^{k}a_0\) is inserted in the integral and the \(\lambda\)-derivative is carried over to \(\lambda^{\nu}\), we get powers \((a_0(x, \xi))^{\nu-k}\), that likewise satisfy (2.11) with \(\mu = \nu\), since the factors \(a_0^{-k}\) are of type 0. It follows that \(A^{\nu}\) is of type \(\nu\). \(\square\)

Remark 2.10. Consider \(A\) as above and assume moreover that it has product structure near the boundary \(\partial \Omega\), i.e., coordinates can be chosen near \(\partial \Omega\) such that \(A = D^2 + A'(x', D')\) there with \(A'\) strongly elliptic on \(\partial \Omega\). Then the associated Dirichlet-to-Neumann operator \(P_{\partial \Omega}\) (sending \(\gamma_0u\) to \(\gamma_1u\) when \(Au = 0\)) is essentially a constant times \((A')^{\frac{1}{2}}\), which is of order 1 and type \(\frac{1}{2}\) with respect to smooth subsets of \(\partial \Omega\).
Remark 2.11. When the equations (2.11) are satisfied with $\mu = 0$ and $m$ integer, they hold also if the normal vectors $N$ and $-N$ exchange roles. Then $P$ is of type 0 also for the exterior domain $\Omega_1 \setminus \Omega$; the so-called two-sided transmission property. This is the case treated in the Boutet de Monvel calculus.

Noninteger transmission properties have been used in another context by Hirschowitz and Pirio [HP79] to investigate lacunas by application of Fourier integral operators; see also the survey by Boutet de Monvel [B79].

3. The Vishik-Eskin estimates

Consider a $C^\infty$ manifold $\Omega_1$, a relatively compact subset $\Omega$ with $C^\infty$ boundary $\partial\Omega$, and a classical pseudodifferential operator $P$ in $\Omega_1$. The operator $P$ we assume to be elliptic in $\Omega_1$, that is, in a local coordinate system where the symbol is $\sum p_j(x, \xi)$, the terms being homogeneous of degree $m - j$, we have

\[ p_0(x, \xi) \neq 0 \text{ for } 0 \neq \xi \in \mathbb{R}^n. \]  

Further we assume that the $\mu$-transmission condition is fulfilled at least for $j = \alpha = \beta = 0$, that is, we assume that there is a number $\mu$ such that

\[ p_0(x, -N) = e^{\pi i (m - 2\mu)}p_0(x, N), \quad x \in \partial\Omega, \]  

where $N$ denotes the interior normal of $\partial\Omega$ at $x$. If $n > 2$ the set $\{\xi \mid \xi \in \mathbb{R}^n, \xi \neq 0\}$ is simply connected, so for fixed $x$ we can define $\log p(x, \xi)$ uniquely by fixing the value at one point. When $n = 2$, we impose this as a condition on $p$, called the root condition in analogy with the corresponding condition in the case of differential equations. Then we have

\[ \log p_0(x, \xi + \tau N) - \log p_0(x, \tau N) = \log \left( \frac{p_0(x, \xi + \tau N)}{p_0(x, \tau N)} \right) \to 0, \quad \tau \to \infty. \]

Hence

\[ \log p_0(x, \xi + \tau N) - m \log |\xi| \to a_\pm(x), \quad \tau \to \pm\infty, \]

where $\exp a_\pm = p_0(x, \pm N)$. It follows from (3.2) that $e^{a_-} = e^{\pi i (m - 2\mu) + a_+}$, that is, $\mu \equiv m/2 + (a_+ - a_-)/2\pi i \pmod{1}$. We define the factorization index $\mu_0$ by

\[ \mu_0 = m/2 + (a_+ - a_-)/2\pi i, \]  

noting that for reasons of continuity this number, which is always congruent to $\mu$, must be a constant on connected components of $\partial \Omega$. (There is a remark in Hörmander [H65] that much of the theory goes through with light modifications when $m$ and $\mu_0$ are allowed to be variable, referring to the 1964 Doklady notes preceding [VE65, VE67].) Note that we may replace $\mu$ by $\mu_0$ in (3.2).

We can now state the basic existence theorem for the Dirichlet problem, due to Vishik and Eskin in the case $p = 2$, cf. [VE65, E81], and extended to $1 < p < \infty$ by Shargorodsky [S94].
**Theorem 3.1.** Let $P$ be elliptic of order $m$ satisfying (3.2) (and the root condition if $n = 2$), and assume the factorization index $\mu_0$ introduced above to be constant on $\partial \Omega$. Then the mapping

$$
(3.5) \quad \hat{H}_p^s(\Omega) \ni u \mapsto r_\Omega Pu \in \overline{H}_p^{s-\text{Re}^m}(\Omega)
$$

is a Fredholm operator if $s$ is a real number with $1/p - 1 < s - \text{Re} \mu_0 < 1/p$.

In the proof one observes that it suffices to prove the a priori estimate for smooth functions

$$
(3.6) \quad \|u\|_s \leq C(\|r_\Omega Pu\|_{s-\text{Re} m_0} + \|u\|_{s-1}), \quad u \in \hat{H}_p^s(\Omega),
$$

together with an analogous estimate for the adjoint $^tP$. This can be reduced to the study of “constant-coefficient” symbols $p_0(x_0, \xi)$ for $x_0 \in \partial \Omega$ in the case $\Omega = \mathbb{R}_+^n$. Here there is a factorization

$$
(3.7) \quad p_0(x_0, \xi) = p_-(x_0, \xi)p_+(x_0, \xi)
$$

with $p_\pm$ of degree $\mu_0$ resp. $m - \mu_0$, extending as analytic functions of $\xi_n$ to $\mathbb{C}_-$ resp. $\mathbb{C}_+$, hence defining operators preserving support in $\mathbb{R}_+^n$ resp. $\mathbb{R}_-^n$. Details on the factorization and its application to obtain the estimates are found e.g. in [E81] §6, 7, 19, extended to $L_p$-spaces in [S94]. (See (1.10)ff. concerning sign conventions.) Those works moreover treat systems $P$ and cases where $\mu_0$ depends on $x \in \partial \Omega$; then the interval where $s$ runs has a smaller length.

**Example 3.2.** When $A^\nu$ is defined as in Lemma 2.9, the principal symbol at a boundary point $(x', 0)$ has the factorization

$$a_0(x', 0, \xi', \xi_n)^\nu = s_0(x')^{\nu}(m^+(x', \xi') - \xi_n)^\nu(m^-(x', \xi') - \xi_n)^\nu,
$$

where $m^\pm$ are the roots in $\mathbb{C}_\pm$, respectively, of the characteristic polynomial of degree 2. Here $(m^\pm(x', \xi') - \xi_n)^\nu$ extends analytically to $\mathbb{C}_\pm$, respectively. Thus the factorization index equals $\nu$, and Theorem 3.1 applies with $s - \text{Re} \nu \in ]-1/p', 1/p[.$

Let $\nu = a \in \mathbb{R}_+$. In the application of the theorem, $s \in a+\lfloor -1/p', 1/p[,$ so regardless of how regular $r_\Omega Pu$ is, this gives at best $u \in \hat{H}_p^{a+1/p-0}(\Omega)$. When $p > n/a$, Sobolev embedding gives $u \in C^{a+1/p-n/p-0}(\Omega)$ with boundary value zero. For $p \to \infty$ we get $u \in C^{a-0}(\Omega)$. It is pointed out in Ros-Oton and Serra [RS14] for $(-\Delta)^a$ with $a \in [0, 1]$ that the exponent $a - 0$ cannot in general be lifted to values $> a$.

There are similar considerations for strongly elliptic $2m$-order differential operators. Here the principal symbol at the boundary factors into two polynomials in $\xi_n$ of degree $m$ with roots in $\mathbb{C}_\pm$, respectively. The $\nu'$th power is then of order $2\nu m$ and type $\nu m$, and has factorization index $\nu m$.

More generally, let $P$ be of order $m \in \mathbb{C}$ with an even symbol, that is, $p_j(x, -\xi) = (-1)^jp_j(x, \xi)$ for all $j \geq 0$. Then in view of the homogeneity of each $p_j$, $p$ satisfies (2.11) with $\mu = m/2$. For the principal symbol, $a_+ = a_-$ in (3.3)–(3.4), so the factorization index is $m/2$. (One can also include a skew factor $e^{i\pi \theta}$.)
The integral operators treated in the recent work of Ros-Oton and Serra [RS14] have these properties, with \( m = 2s \), when the kernel is smooth (outside 0).

Note that (2.11) is only required for the interior normal \( N(x) \) to a given smooth subset \( \Omega \); the above examples have the property with respect to all directions.

The new task is to characterize the regularity of \( u \) when \( Pu \) is given in more smooth spaces. There is a preparatory result in [H65] on “tangential regularity” which follows by classical arguments due to Nirenberg.

Let \( \Omega \) be the half ball \( \{ x \in \mathbb{R}^n \mid |x| < 1, x_n > 0 \} \). The unit ball we denote by \( \tilde{\Omega} \). By \( \dot{H}^{s, t}_p(\Omega') \) and \( \overline{H}^{s, \text{Re} m}_p(\Omega) \) we denote the distributions which multiplied with functions in \( C_0^\infty(\tilde{\Omega}) \) give elements in the analogous spaces in \( \mathbb{R}^n_+ \). Here \( \Omega' = \{ x \in \mathbb{R}^n \mid |x| < 1, x_n \geq 0 \} \).

**Theorem 3.3.** Let \( P \) satisfy the hypotheses of Theorem 3.1. If \( -1/p' < s - \text{Re} \mu < 1/p \) and \( t_0, t_1 \) are real numbers, then

\[
(3.8) \quad u \in \dot{H}^{s, t_0}_p(\Omega'), r_{\Omega} Pu \in \overline{H}^{s, \text{Re} m, t_1}_p(\Omega)
\]

implies that

\[
(3.9) \quad u \in \dot{H}^{s, t_1}_p(\Omega'),
\]

**Proof.** It is no restriction to assume that \( t_1 - t_0 \) is a positive integer, for we may always decrease \( t_0 \). It suffices to prove the theorem when \( t_1 - t_0 = 1 \). Now we claim that for every compact subset \( K \) of \( \Omega \), and every real number \( t \) there is a constant \( C \) such that

\[
(3.10) \quad \|u\|_{s, t} \leq C(\|r_{\Omega} Pu\|_{s - \text{Re} m, t} + \|u\|_{s-1, t})
\]

for all \( u \in C_0^\infty(K) \), hence for all \( u \in \dot{H}^{s, t} \) with support in \( K \). In fact, this follows from by applying (3.6) to \( |D|^t u \), cut off conveniently. We may replace the last term in (3.10) by the larger quantity \( \|u\|_{s, t-1} \). Now assume that (3.8) is fulfilled with \( t_0 = t, t_1 = t + 1 \). Then \( \varphi u \) satisfies the same hypothesis if \( \varphi \in C_0^\infty(\tilde{\Omega}) \). Let therefore \( u \) have compact support in \( \Omega' \). Denote by \( u_h \) the convolution of \( u \) by the Dirac measure at \((h_1, \ldots, h_{n-1}, 0) = h \), that is, \( u_h \) is a tangential translation of \( u \). Let \( P_h \) be the analogous translation of \( P \). Then

\[
(3.11) \quad P(u_h - u)/|h| = (f_h - f)/|h| + (P - P_h)/|h| u,
\]

where \( f = Pu \). Since

\[
\|(f - f_h)/|h|\|_{s, t} \leq \|f\|_{s, t+1},
\]

and since \( (P - P_h)/|h| \) is continuous from \( H^{s, t}_p \) to \( H^{s - \text{Re} m, t}_p \) uniformly when \( h \to 0 \), we conclude using (3.10) that \( \|(u_h - u)/|h|\|_{(s, t)} \) is bounded when \( h \to 0 \). Hence \( \|D_j u\|_{(s, t)} < \infty \) when \( j < n \), which proves that \( u \in \dot{H}_{(s, t+1)} \).

4. Solvability of homogeneous problems

For the study of solvability, we first set the \( H^\mu(s)_p \)-spaces in relation to \( E_\mu \). In the following we assume that \( \overline{\Omega} \) is compact, unless otherwise mentioned.
Proposition 4.1. 1° Let \( s > \Re \mu - 1/p' \). For any compact \( K \), \( u \in \mathcal{E}_\mu(\mathbb{R}^n_+) \cap \mathcal{E}'(K) \) implies \( u \in H^\mu_p(\mathbb{R}^n_+) \) Similarly, \( \mathcal{E}_\mu(\Omega) \subset H^\mu_p(\Omega) \).

2° We have that \( \bigcap_s H^\mu_p(\mathbb{R}^n_+) \subset \mathcal{E}_\mu(\mathbb{R}^n_+) \), and that

\[
(4.1) \quad \bigcap_s H^\mu_p(\Omega) = \mathcal{E}_\mu(\Omega).
\]

3° Moreover, \( \mathcal{E}_\mu(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n_+) \), resp. \( \mathcal{E}_\mu(\Omega) \), is dense in \( H^\mu_p(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n_+) \) resp. \( H^\mu_p(\Omega) \), when \( s > \Re \mu - 1/p' \).

Proof. 1° Let \( u \in \mathcal{E}_\mu(\mathbb{R}^n_+) \cap \mathcal{E}'(K) \). Then by (2.8), we have for \( |\xi| > 1 \), \( M \in \mathbb{N} \), and any \( N \),

\[
(4.2) \quad \hat{u}(\xi) = \sum_{j=0}^{M-1} \hat{u}_j(\xi') (\xi_n^-)^{-\mu-j-1} + O( [|\xi'|^{-N}|\xi_n|^{-\Re \mu-M-1} ) ,
\]

where the \( \hat{u}_j \) are in \( \mathcal{S}(\mathbb{R}^{n-1}) \). To estimate \( \Xi^\mu_+ u \), we shall calculate \( \hat{u}(\xi) (\xi_n - i[\xi'])^\mu \), where we note that \( (\xi_n - i[\xi'])^\mu = (-i)^\mu(\xi') + i\xi_n^\mu = (-i)^\mu \chi^\mu_+ \). There are Taylor expansions (for \( |\xi| > 1 \), say)

\[
(4.3) \quad (\xi_n - i[\xi'])^\mu = (\xi_n^-)^\mu + c_1 \xi'^\mu (\xi_n^-)^{-\mu-1} + \cdots + c_{l-1} \xi'^{l-1} (\xi_n^-)^{-\mu-l+1} + O( [|\xi'|^{l+\Re \mu-l} |\xi_n|^{\Re \mu-l} ).
\]

Insertion gives (with \( c_0 = 1 \)):

\[
(4.4) \quad \mathcal{F}(\Xi^\mu_+ u)(-i)^\mu = \hat{u}(\xi) (\xi_n - i[\xi'])^\mu
\]

\[
= \sum_{j=0}^{M-1} \hat{u}_j(\xi') (\xi_n^-)^{-\mu-j-1} \left[ \sum_{l=0}^{M-j-1} c_l \xi'^l (\xi_n^-)^{-\mu-l} + O( [|\xi'|^{M-j+\Re \mu-M+j} |\xi_n|^{\Re \mu-M+j} ] ) \right]
\]

\[
+ O( [|\xi'|^{-N}|\xi_n|^{-M-1} )
\]

\[
= \sum_{j=0}^{M-1} \sum_{l=0}^{M-j-1} \hat{u}_j(\xi') c_l \xi'^l (\xi_n^-)^{-\mu-l-1} + O( [|\xi'|^{-N}|\xi_n|^{-M-1} )
\]

\[
= \sum_{j=0}^{M-1} \sum_{k=0}^j c_{jk} \hat{u}_k(\xi') [\xi'^j]^k \xi_n^{-j-1} + O( [|\xi'|^{-N}|\xi_n|^{-M-1} )
\]

In the last step we replaced \( l, j \) by \( j' = l + j \) and \( k' = j \), and removed the primes. The \( c_{jk} \) are constants, with \( c_{jj} = 1 \). (It is also for later purposes that we account for this in detail.)

The terms in the sum are Fourier transforms of functions in \( \mathcal{S}(\mathbb{R}^n_+) \), and the remainder is bounded by \( \langle \xi \rangle^{-N'} \) for \( N' \leq \min\{N,M+1\} \), so by letting \( N, M \to \infty \), we see that any \( \overrightarrow{T}_p(\mathbb{R}^n_+) \)-norm of \( \Xi^\mu_+ u \) is bounded.

The result for \( \overrightarrow{T}_p(\mathbb{R}^n_+) \) follows by using the above in local coordinate patches where \( d(x) = x_n \).
2°. Now let \( u \in \bigcap_s H^\mu(s)(\mathbb{R}^n_+) \). Then \( v = r^+ \Xi^\mu u \in \bigcap_t \mathcal{H}_p^t(\mathbb{R}^n_+) \), which consists of \( C^\infty(\mathbb{R}^n_+) \)-functions with all \( L_p \)-norms of derivatives bounded. In view of Proposition 1.7, \( u = \Xi^\mu e^+ v \). By Lemma 2.1, \( v \) has an expansion as in (2.8) with \( \mu = 0 \), and the multiplication by \((\xi' + i\xi_n)^{-\mu} = i^{-\mu}(\xi_n - i[\xi'])^{-\mu}\) gives a function with an expansion (2.8) with the actual \( \mu \), so we conclude from Lemma 2.1 with (2.8) that \( u \in \mathcal{E}_\mu(\mathbb{R}^n_+) \).

For \( \overline{\Omega} \) we find from this by localization that \( \bigcap_s H^\mu(s)(\overline{\Omega}) \subseteq \mathcal{E}_\mu(\overline{\Omega}) \); here there is equality in view of 1°.

3°. To show that \( \mathcal{E}_\mu \cap \mathcal{E}'(\mathbb{R}^n_+) \) is dense in the set of all \( u \in \mathcal{S}'(\mathbb{R}^n_+) \) satisfying (1.24), we first take a sequence \( v_j \in C^\infty(\mathbb{R}^n_+) \) of compactly supported functions approximating \( \mathcal{F}^{-1}(\xi_n - i[\xi'])^{-\mu} \hat{u} \) in the norm \( \| \mathcal{F}_p^{-1} - \mathcal{F}_p \| \), and also in the topology of \( \mathcal{S} \) outside a neighborhood of \( \text{supp} \ u \) (which is possible since the function to approximate agrees with a function in \( \mathcal{S} \) there). Define \( v_j = 0 \) in \( \mathbb{R}^- \). Set \( u_j = \mathcal{F}^{-1}(\xi_n - i[\xi'])^{-\mu} \hat{v}_j \). This is an element of \( \mathcal{E}_\mu \) in view of Lemma 2.2 (the Fourier transform is the product of that of \( v_j \) and \( (\xi_n - i[\xi'])^{-\mu} \)), and the behavior of the Fourier transform of \( v_j \) is described by Lemma 2.3 with \( \mu = 0 \).

Then by Proposition 1.7, \( u_j \to u \) in the norm in (1.24), and also in the topology of \( \mathcal{S} \) outside a neighborhood of \( \text{supp} \ u \). Hence we can cut off \( u_j \) there without disturbing the convergence in order to obtain an approximating sequence with compact supports.

The statement for \( H^\mu(s)(\overline{\Omega}) \) follows by localization. \( \square \)

In the next theorems we use the order-reduction operators to reach situations where we can draw on results from the Boutet de Monvel calculus. The calculus was established in [B71] and is moreover presented in detail e.g. in [G96, G09], see also [G90].

**Theorem 4.2.** Let the \( \psi \)-do \( P \) on \( \mathbb{R}^n \) be of order \( m \), and type \( \mu \) relative to \( \mathbb{R}^n_+ \), and compactly supported. Then for \( s > \text{Re} \mu - 1/p' \) and \( u \in H^\mu(s)(\mathbb{R}^n_+) \),

\[
\| r^+ Pu \|_{\mathcal{H}_p^{-\text{Re}m}} \leq C \| r^+ \Xi^\mu u \|_{\mathcal{H}_p^{-\text{Re} \mu}}, \quad \| r^+ Pu \|_{\mathcal{H}_p^{-\text{Re}m}} \leq C' \| r^+ \Lambda^\mu u \|_{\mathcal{H}_p^{-\text{Re} \mu}}.
\]

Similarly, for a \( \psi \)-do \( P \) on the manifold \( \Omega_1 \) of order \( m \), and type \( \mu \) on \( \Omega \), one has for \( u \in H^\mu(s)(\overline{\Omega}) \),

\[
\| r_\Omega Pu \|_{\mathcal{H}_p^{-\text{Re}m}(\Omega)} \leq C \| r_\Omega \Lambda^\mu u \|_{\mathcal{H}_p^{-\text{Re} \mu}(\Omega)}.
\]

In other words, \( r^+ P \) maps \( H^\mu(s)(\Omega) \) continuously into \( \mathcal{H}_p^{s-\text{Re}m} \) when \( s > \text{Re} \mu - 1/p' \).

**Proof.** By definition, \( v = r^+ \Lambda^\mu u \in \mathcal{H}_p^{-\text{Re} \mu} \), and by Proposition 1.7, \( u = \Lambda^\mu e^+ v \) then. Thus we can write

\[
r^+ Pu = r^+ P \Lambda^\mu v.
\]

Moreover, by (1.15),

\[
\| r^+ Pu \|_{\mathcal{H}_p^{-\text{Re}m}} \simeq \| \Lambda^\mu u \|_{\mathcal{H}_p^{-\text{Re} \mu}},
\]

where

\[
\Lambda^\mu u = r_+ \Lambda^\mu Pu
\]
Proposition 4.3. \( s > \text{Re} \mu - 1/p' \). Both for spaces over \( \mathbb{R}^n_+ \) and over \( \overline{\Omega} \), we have that
\[
H^\mu(s) \subset H^\mu(s-1)(s),
\]
and the norms are equivalent on \( H^\mu(s) \).

Proof. When \( u \in H^\mu(s)(\mathbb{R}^n_+) \) for some \( s > \text{Re} \mu - 1/p' \), then
\[
\| r^+ \Xi^\mu u \|_{H^\mu(s)} \approx \sum_{j \leq n} \| D_j r^+ \Xi^\mu u \|_{H^\mu(s)},
\]
where we could use Theorem 4.2 in the last step, since \( D_j \Xi^\mu \) is of type \( \mu \) and order \( \mu \). On the other hand, since \( r^+ \Xi^\mu = r^+((D') + iD)(D^\mu)^{-1} u \),
\[
\| r^+ \Xi^\mu u \|_{H^\mu(s)} \leq C \| r^+ \Xi^\mu u \|_{H^\mu(s-1)},
\]
 Altogether, (4.8) holds, with equivalent norms on \( H^\mu(s) \). Moreover, \( \Xi^\mu \) can be replaced by \( \Lambda^\mu_{\pm} \) in the inequalities in view of Lemma 1.2.

The statements carry over to the manifold situation by localization. \( \square \)

The \( H^\mu(s) \)-spaces serve the purpose of describing the regularity of solutions with data in more regular Sobolev spaces than the result of Vishik and Eskin (Theorem 3.1) allows. We can now show the main regularity result for homogeneous boundary problems (proved for \( p = 2 \) in [H65]), obtaining moreover a formula for a parametrix:
Theorem 4.4. Let $P$ be classical elliptic of order $m \in \mathbb{C}$ on $\Omega$ and of type $\mu_0 \in \mathbb{C}$ relative to $\Omega$, and with factorization index $\mu_0$. Let $s > \text{Re} \mu_0 - 1/p'$, and let $u \in H_p^{\sigma}(\Omega)$ for some $\sigma > \text{Re} \mu_0 - 1/p'$. If $r^+ Pu \in \overline{H}^{s-\text{Re} \mu_0}(\Omega)$, then $u \in H^{\mu_0}(\overline{\Omega})$. The mapping

$$H^{\mu_0}(\overline{\Omega}) \ni u \mapsto r^+ Pu \in \overline{H}^{s-\text{Re} \mu_0}(\Omega)$$

is Fredholm, and has the parametrix

$$R = \Lambda^1(-\mu_0)e^+ \overline{Q} + \Lambda^1(\mu_0 - m) : \overline{H}^{s-\text{Re} \mu_0}(\Omega) \rightarrow H^{\mu_0}(\overline{\Omega}),$$

where $\overline{Q}$ is a parametrix of $Q = r^+ Q e^+$, with

$$Q = \Lambda^1(\mu_0 - m) P \Lambda^1(-\mu_0),$$

elliptic of order and type $0$, with factorization index $0$.

In particular, if $r^+ Pu \in C^\infty(\overline{\Omega})$, then $u \in \mathcal{E}_{\mu_0}(\overline{\Omega})$. The mapping

$$\mathcal{E}_{\mu_0}(\overline{\Omega}) \ni u \mapsto r^+ Pu \in C^\infty(\overline{\Omega})$$

is Fredholm.

Proof. Note first that there is a $\sigma_0 \le \min\{s, \sigma\}$ with $\sigma_0 \in \text{Re} \mu_0 + \] - 1/p', 1/p[. Theorem 3.1 (by Vishik-Eskin-Shargorodsky) applies with $s$ replaced by $\sigma_0$ to show the Fredholm solvability of $r^+ Pu = f$ in $\overline{H}^{\sigma_0-\text{Re} \mu_0}(\Omega)$ with solution $u \in \dot{H}^{\sigma_0}(\Omega)$. We must show that this solution lies in $H^{\mu_0}(\Omega)$. It already lies in $H^{\mu_0}(\sigma_0)$, since $\Lambda u \in \overline{H}^{\sigma_0-\text{Re} \mu_0} \subset \dot{H}^{-1/p'+0}$.

To discuss the solvability of

$$r^+ Pu = f \in \overline{H}^{s-\text{Re} \mu_0}(\Omega),$$

in spaces with general $s$ we prefer to start from scratch, using devices from Theorem 4.2. Compose to the left with $\Lambda^1(\mu_0 - m)$; this gives the equivalent problem

$$\Lambda^1(\mu_0 - m) r^+ Pu = g, \quad \text{where} \quad g = \Lambda^1(\mu_0 - m) f \in \overline{H}^{s-\text{Re} \mu_0}(\Omega),$$

when we recall (1.20). Note that $f = \Lambda^1(\mu_0 - m) g$. Moreover, in view of Remark 1.1,

$$\Lambda^1(\mu_0 - m) r^+ Pu = r^+ \Lambda^1(\mu_0 - m) P u.$$

Now set $v = r^+ \Lambda^1(\mu_0) u$; then $u = \Lambda^1(-\mu_0) e^+ v$ by Proposition 1.7. Expressed in terms of $g$ and $v$, equation (4.13) becomes

$$Q_+ v = g; \quad g \text{ given in } \overline{H}^{s-\text{Re} \mu_0}(\Omega),$$

where we have defined $Q$ by (4.11).
The properties of $P$ imply that $Q$ is elliptic of order 0 and type 0 and has factorization index 0; in particular, it belongs to the Boutet de Monvel calculus. The principal symbol at the boundary $q(x', 0, \xi)$ has a factorization $q = q^+ q^-$, in symbols $q^\pm(x', \xi)$ of plus/minus type and order 0. (We here use upper indices $\pm$ to avoid confusion with the lower plus-index for truncation.) The associated operators on $L_2(\mathbb{R})$ satisfy

$$q_+ = r^+ q(x', 0, \xi', D_n) e^+ = r^+ q^-(x', \xi', D_n)(e^+ e^+ + e^- e^-) q^+(x', \xi', D_n) e^+ = q^- q^+,$$

since $r^+ q^- e^-$ and $r^- q^+ e^+$ are zero. Let $\tilde{q}(x', \xi) = 1/q(x', \xi)$, it likewise has a factorization $\tilde{q} = \tilde{q}^+ \tilde{q}^-$ in plus/minus symbols, with $\tilde{q}^\pm = 1/q^\pm$. Now for the associated operators on $\mathbb{R}$,

$$r^+ q^+ e^+ e^+ \tilde{q}^+ e^+ = r^+ q^+ \tilde{q}^+ e^+ - r^+ q^+ e^- r^- \tilde{q}^+ e^+ = I_{\mathbb{R}}^+,$$

since $\tilde{q}^+$ preserves support in $\mathbb{R}_+$ so that $r^- \tilde{q}^+ e^+ = 0$. One checks similarly that $r^+ \tilde{q}^- e^+ r^+ q^+ e^+ = I_{\mathbb{R}}^+$, and that also $r^+ q^- e^+ r^- \tilde{q}^- e^+ = I_{\mathbb{R}}^+$, $r^+ \tilde{q}^- e^+ r^+ q^- e^+ = I_{\mathbb{R}}^+$. In other words,

$$q^\pm(x', \xi', \xi_n)_+$$

has the inverse $\tilde{q}^\pm(x', \xi', \xi_n)_+$ in $L_2(\mathbb{R}_+)$. \n
In view of (4.16), $q(x', 0, \xi', D_n)_+$ therefore has the inverse

$$\tilde{q}^+ = (q^- q^+)^{-1} = \tilde{q}^+ \tilde{q}^-.$$
Example 4.5. Let us check how this looks in the well-known case of the Laplace-Beltrami operator, \( P = \Delta \). It is of order 2 and type 0, and has factorization index 1 (cf. Example 3.2). Let \( s > 1 - 1/p' = 1/p \), so \( f \) is given in \( \overline{\mathcal{H}}_{p}^{-2} \) with \( s > -2 + 1/p \). From Example 1.6 with \( m = 1 \) we have that \( \mathcal{H}_{p}^{1(s)} = \{ u \in \overline{\mathcal{H}}_{p}^{0} \mid \gamma_{0}u = 0 \} \). Thus \( u \) is the solution of the homogeneous Dirichlet problem: \( \Delta u = f \) in \( \Omega \), \( \gamma_{0}u = 0 \).

Remark 4.6. Not all elliptic \( \psi \)do’s \( P \) of order and type 0 have \( P_{+} \) elliptic without supplementing trace or Poisson operators. For example, \( P = \Lambda_{-}^{(1)} \Lambda_{+}^{(-1)} \) has \( P_{+} = \Lambda_{-}^{(1)} \Lambda_{+}^{(-1)} \) (in view of Remark 1.1); here \( \Lambda_{-}^{(1)} : \hat{H}_{p}^{0} \cong \hat{H}_{p}^{1} \), but since \( \Lambda_{-}^{(1)} : \overline{\mathcal{H}}_{p}^{0} \cong \overline{\mathcal{H}}_{p}^{1} \), it maps the subspace \( \hat{H}_{p}^{1} \) onto a subspace of \( \overline{\mathcal{H}}_{p}^{0} \) with infinite codimension.

Applications to fractional powers \( A^{\mu} \) will be given below in Section 7.

5. The \( \mathcal{H}_{p}^{\mu(s)} \)-Spaces and their Boundary Values

It will now be shown that the \( \mathcal{H}_{p}^{\mu(s)} \)-spaces admit a special definition of \( \mu \)-boundary values.

Let \( M \) be a positive integer. First we consider \( \mathcal{E}_{\mu} \) and \( \mathcal{E}_{\mu+M} \) for a smooth subset \( \Omega \) of a paracompact manifold \( \Omega_{1} \) as in Section 2.

Let us introduce the natural mapping

\[
\varrho_{\mu,M} : \mathcal{E}_{\mu} \to \mathcal{E}_{\mu}/\mathcal{E}_{\mu+M}.
\]

The first step is to represent \( \mathcal{E}_{\mu}/\mathcal{E}_{\mu+M} \) as the space of sections of a trivial bundle and introduce norms in it. To do so we first choose a Riemannian metric in \( \Omega_{1} \) and then a \( C^{\infty} \) function \( d \) in \( \overline{\Omega} \) which is equal to the distance from \( \partial \Omega \) sufficiently close to the boundary and is positive and \( C^{\infty} \) throughout \( \Omega \). Set

\[
I^{\mu}(x) = d(x)^{\mu}/\Gamma(\mu + 1) \text{ in } \overline{\Omega}, \text{ and } I^{\mu} = 0 \text{ in } \partial \Omega,
\]

when \( \text{Re} \mu > -1 \) (consistently with (2.2)). This definition can be uniquely extended modulo \( C_{0}^{\infty}(\Omega) \) to arbitrary values of \( \mu \) so that \( \partial_{a}I^{\mu} = I^{\mu-1} \), where \( \partial_{a} \) denotes differentiation along the geodesics perpendicular to \( \partial \Omega \), sufficiently close to \( \partial \Omega \), and is defined as a \( C^{\infty} \) function elsewhere. By our definition of \( \mathcal{E}_{\mu} \) it follows easily that every class in \( \mathcal{E}_{\mu}/\mathcal{E}_{\mu+1} \) contains an element of the form \( I^{\mu}(x)f \) where \( f \in C^{\infty}(\overline{\Omega}) \), and that such elements are congruent to 0 if and only if \( f = 0 \) on the boundary. By repeated application of this fact we conclude that any element \( u \in \mathcal{E}_{\mu} \) can be written

\[
u = u_{0}I^{\mu} + u_{1}I^{\mu+1} + \cdots + u_{M-1}I^{\mu+M-1} + v,
\]

where the \( u_{j} \in C^{\infty}(\overline{\Omega}) \) are constant close to \( \partial \Omega \) on normal geodesics, and \( v \in \mathcal{E}_{\mu+M} \). The boundary values of \( u_{j} \) are uniquely determined by \( u \), and it is natural to write

\[
\gamma_{\mu,j}u = u_{j}\big|_{\partial \Omega}.
\]

Note that

\[
\gamma_{\mu,j}u = \gamma_{\mu+j,0}u, \text{ when } u \in \mathcal{E}_{\mu+j};
\]

\[
\gamma_{\mu,0}u = \Gamma(\mu + 1)\gamma_{0}d(x)^{-\mu}u, \text{ when } u \in \mathcal{E}_{\mu} \text{ with } \text{Re} \mu > -1.
\]
When $\Omega = \mathbb{R}^n$, and $u(x)$ is written as $I^\mu w$ with $I^\mu (x_n) = x_n^\mu / \Gamma (\mu + 1)$ and $w(x) \in C^\infty (\mathbb{R}^n)$, then $u_j (x') = \partial_{x_j} w(x', 0) / (x_j^\mu)$, where $(x_j^\mu) = \Gamma (\mu + j + 1) / (j! \Gamma (\mu + 1))$.

The mapping

\begin{equation}
\varrho_{\mu, M} : u \mapsto \{ \gamma_{\mu, j} u \}_{j=0}^{M-1}
\end{equation}

has nullspace $\mathcal{E}_{\mu + M}$ and identifies $\mathcal{E}_{\mu} / \mathcal{E}_{\mu + M}$ with $C^\infty (\partial \Omega)^M$; the mapping identifies with the mapping in (5.1). The identification depends of course on the choice of the Riemannian structure but we shall keep it fixed in all that follows. We can now think of $\varrho_{\mu, M}$ as a mapping of $\mathcal{E}_\mu$ onto $C^\infty (\partial \Omega)^M$.

**Theorem 5.1.** Let $s > \Re \mu + M - 1/p'$, and let $\Omega$ equal $\mathbb{R}^n_+$ or a compact smooth manifold with boundary. The mapping $\varrho_{\mu, M}$ in (5.6) extends by continuity to a continuous mapping, also denoted $\varrho_{\mu, M}$,

\begin{equation}
\varrho_{\mu, M} : H^\mu_p (\Omega) \to \prod_{0 \leq j < M} B_p^{s - \Re \mu - j - 1/p} (\partial \Omega);
\end{equation}

surjective and with kernel $H^\mu_{\mu + M} (\Omega)$. In other words, $\varrho_{\mu, M}$ defines a homeomorphism of $H^\mu_p (\Omega) / H^\mu_{\mu + M} (\Omega)$ onto $\prod_{0 \leq j < M} B_p^{s - \Re \mu - j - 1/p} (\partial \Omega)$.

**Proof.** We want to introduce in $\mathcal{E}_\mu / \mathcal{E}_{\mu + M}$ the quotient of the topology of $H^\mu_p (\Omega)$. When discussing the quotient topology it is sufficient to consider sections with support in a local coordinate patch.

Thus let $u \in \mathcal{E}_\mu (\mathbb{R}^n_+) \cap \mathcal{E}' (K)$ where $K$ is a compact set, and let $d(x) = x_n$. Writing $u$ in the form (5.3) we have for $|\xi_n| > 1$, say, and any $N$,

$$
\hat{u}(\xi) = \sum_{j=0}^{M-1} b_j \hat{u}_j (\xi') (\xi_n)^{-\mu - j - 1} + O (|\xi'|^{-N} |\xi_n|^{-\Re \mu - M - 1}) , \text{ where } b_j = i^{-(\mu + j + 1)},
$$

cf. (2.4). This is similar to the formula (4.2), except that the nonzero factors $b_j$ were incorporated in $\hat{u}_j$ in (4.2). Then we can use the calculation in (4.4) to obtain:

$$
\mathcal{F} (\Xi^\mu_n u) = i^\mu \hat{u}(\xi) (\xi_n - i [\xi'])^\mu = i^\mu \sum_{j=0}^{M-1} \sum_{k=0}^{j} c_{jk} b_k \hat{u}_k (\xi') [\xi']^{j-k} \xi_n^{-j} + O (|\xi'|^{-N} |\xi_n|^{-M - 1})
$$

$$
= \sum_{j=0}^{M-1} \sum_{k=0}^{j} c_{jk} i^{j-k-1} \hat{u}_k (\xi') [\xi']^{j-k} \xi_n^{-j-1} + O (|\xi'|^{-N} |\xi_n|^{-M - 1}),
$$

where the $c_{jj}$ equal 1. Moreover, when $l < M$,

$$
\mathcal{F} (\partial^l_n \Xi^\mu_n u) = (i \xi_n)^l \mathcal{F} (\Xi^\mu_n u) = \sum_{j=0}^{M-1} \sum_{k=0}^{j} c_{jk} i^{j-k-1} \hat{u}_k (\xi') [\xi']^{j-k} \xi_n^{-j-1} + O (|\xi'|^{-N} |\xi_n|^{-2}).
$$
To calculate the boundary value \( \gamma_0 \partial_n^l \Xi^\mu u \) from \( \mathbb{R}^n_+ \), note that for \( l - j - 1 \geq 0 \) the terms contribute with distributions supported by \( x_n = 0 \), and for \( l - j - 1 < 0 \) it is the coefficient of \( \xi_n^{l-1} \) that gives the boundary value at \( x_n = 0 \), cf. (2.9), so only \( l = j \) contributes:

\[
(5.8) \quad \gamma_0 \partial_n^l \Xi^\mu u = \gamma_0 \mathcal{F}^{-1} i \sum_{k=0}^{j} c_{jk} i^{j-k-1} \hat{u}_k(\xi') [\xi']^{j-k} = \sum_{k=0}^{j} c'_{jk} [D']^{j-k} u_k,
\]

with \( c'_{jj} = 1 \) for all \( j \). In other words, with \( \gamma_j = \gamma_0 \partial_n^j \), the boundary values \( \gamma_j \Xi^\mu u \) satisfy

\[
(5.9) \quad \begin{pmatrix}
\gamma_0 \Xi^\mu u \\
\gamma_1 \Xi^\mu u \\
\vdots \\
\gamma_{M-1} \Xi^\mu u \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
c'_{10} [D'] & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
c'_{M-1,0} [D']^{M-1} & c'_{M-1,1} [D']^{M-2} & \cdots & 1 \\
\end{pmatrix} \begin{pmatrix}
\gamma_{\mu,0} u \\
\gamma_{\mu,1} u \\
\vdots \\
\gamma_{\mu,M-1} u \\
\end{pmatrix} = \Phi \varrho_{\mu,M} u,
\]

with an invertible triangular transition matrix \( \Phi \).

Now we have from the well-known continuity properties of \( \varrho_M = \{ \gamma_0, \ldots, \gamma_{M-1} \} \) (cf. (1.6)) that

\[
\sum_{j=0}^{M-1} \| \gamma_j \Xi^\mu u \|_{B_p^{s-Re \mu -j-1/p} (\mathbb{R}^{n-1})} \leq C \| r^+ \Xi^\mu u \|_{H_p^{s-Re \mu} (\mathbb{R}^n_+)} = C \| u \|_{\mu(s)}.
\]

Moreover, \( \Phi \) is clearly a homeomorphism in \( \prod_{0 \leq j < M} B_p^{s-Re \mu -j-1/p} (\mathbb{R}^{n-1}) \), so by (5.9), we likewise have

\[
(5.10) \quad \sum_{j=0}^{M-1} \| \gamma_{\mu,j} u \|_{B_p^{s-Re \mu -j-1/p} (\mathbb{R}^{n-1})} \leq C \| u \|_{\mu(s)}.
\]

Thus the mapping \( \varrho_{\mu,M} \) extends by continuity as asserted.

Finally, the extended map is surjective: For a given vector \( \varphi = \{ \varphi_0, \ldots, \varphi_{M-1} \} \in \prod_{0 \leq j < M} B_p^{s-Re \mu -j-1/p} (\mathbb{R}^{n-1}) \), let \( g \in \mathcal{T}_p^{\mu} (\mathbb{R}^{n}) \) be an element of \( \mathcal{T}_p^{\mu} (\mathbb{R}^{n}_+) \) with \( \varrho_M g = \Phi \varphi \), e.g. \( g = K_M \Phi \varphi \) with \( K_M \) defined in Section 1.1, cf. (1.7). Set \( u = \Xi^\mu u^+ g \). By Proposition 1.7, it has the desired properties. \( \square \)

One can replace \( [\xi'] \) by \( \langle \xi' \rangle \) throughout the proof if convenient.

Note that on the space \( H_p^{\mu(s)} (\overline{\Omega}) \), all the boundary operators \( \gamma_{\mu,j}, j = 0, 1, \ldots, M-1 \), are defined when \( s > Re \mu + M - 1/p' \). They are local, in the sense that they are extensions by continuity of local operators of the form: \( \gamma_0 \) composed with multiplication and differential operators. For this extended definition, the first line in (5.5) is valid on \( H^{(\mu+j)(s)} (\overline{\Omega}) \), and the second line holds on \( H^{\mu(s)} (\overline{\Omega}) \) when \( Re \mu > -1 \).

**Remark 5.2.** In the course of the above proof we have in fact constructed an explicit right inverse to \( \varrho_{\mu,M} \) in the case \( \Omega = \mathbb{R}^n_+ \), namely

\[
(5.11) \quad K_{\mu,M} = \Xi^\mu u^+ K_M \Phi.
\]

We observe in particular from (5.9) that \( \Phi = I \) when \( M = 1 \), and hence \( \gamma_0 \Xi^\mu u = \gamma_{\mu,0} u \). For the case \( M = 1 \) we consequently have:
Corollary 5.3. When \( s > \text{Re} \mu + 1/p \), the mapping \( \gamma_{\mu,0} \) is continuous and surjective from \( H^{\mu(s)}_p(\mathbb{R}^n_+) \) to \( B^{s-\text{Re} \mu - 1/p}_p(\mathbb{R}^{n-1}) \) with nullspace \( H^{(\mu+1)(s)}_p(\mathbb{R}^n_+) \). It coincides with \( \gamma_0 \Xi^\mu_+ \). A right inverse is \( K_{\mu,0} = \Xi^\mu_+ e^+ K_0 \), where \( K_0 : B^{t-1/p}_p(\mathbb{R}^{n-1}) \rightarrow \overline{H}^t_p(\mathbb{R}^n_+) \) is a right inverse of \( \gamma_0 \).

As an example, let us also do the calculation of \( \Phi \) in detail in the case \( M = 2 \). For \( u \in \mathcal{E}_\mu(\mathbb{R}^n_+) \cap \mathcal{E}'(K) \),

\[
\begin{align*}
    u(x', x_n) = u_0(x') I^\mu(x_n) + u_1(x') I^{\mu+1}(x_n) + \text{remainder},
\end{align*}
\]

so we have for \( |\xi_n| \geq 1 \) (assumed in the following):

\[
\hat{u}(\xi) = i^{-\mu-1} \hat{u}_0(\xi') (\xi_n^-)^{-\mu-1} + i^{-\mu-2} \hat{u}_1(\xi') (\xi_n^-)^{-\mu-2} + O(\xi_n^\mu) .
\]

Denote \( [\xi'] = \sigma \). The function \( (\sigma + i\xi_n)^\mu \) is Taylor expanded:

\[
(\sigma + i\xi_n)^\sigma = i^\mu (\xi_n - i\sigma)^\mu = i^\mu (\xi_n^-)^\mu - i^{\mu-1} \mu \sigma (\xi_n^-)^{\mu-1} + O(\xi_n^{\mu-2}) .
\]

Hence

\[
(\sigma + i\xi_n)^\sigma \hat{u}(\xi) = i^{-1} \hat{u}_0(\xi') \xi_n^{-1} + i^{-2} \sigma \hat{u}_0(\xi') \xi_n^{-2} + i^{-2} \hat{u}_1(\xi') \xi_n^{-2} + O(\xi_n^{-3}) .
\]

In view of (2.9),

\[
\gamma_0 \Xi^\mu_+ u = u_0 .
\]

Moreover,

\[
i\xi_n (\sigma + i\xi_n)^\sigma \hat{u}(\xi) = \hat{u}_0(\xi') + i^{-1} \mu \sigma \hat{u}_0(\xi') \xi_n^{-1} + i^{-1} \hat{u}_1(\xi') \xi_n^{-1} + O(\xi_n^{-2}) ,
\]

so since \( \mathcal{F}^{-1}_{\xi \rightarrow x} \hat{u}_0(\xi') = u_0(x') \otimes \delta_0(x_n) \) does not contribute to the boundary value from \( \mathbb{R}^n_+ \),

\[
\gamma_0 \partial_n \Xi^\mu_+ u = \mu \sigma (D') u_0 + u_1 .
\]

Thus

\[
(5.12) \quad \begin{pmatrix} \gamma_0 \Xi^\mu_+ u \\ \gamma_1 \Xi^\mu_+ u \end{pmatrix} = \begin{pmatrix} 1 \\ \mu [D'] \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} \gamma_{\mu,0} u \\ \gamma_{\mu,1} u \end{pmatrix} , \quad \text{and} \quad \Phi = \begin{pmatrix} 1 \\ \mu [D'] \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} .
\]

If \( \sigma \) is taken equal to \( \langle \xi' \rangle \) instead of \( [\xi'] \), we get of course \( \Phi \) of the above form with \( [D'] \) replaced by \( \langle D' \rangle \).

By use of concrete formulas from the Boutet de Monvel calculus we can show that not only the boundary operators from \( H^{\mu(s)}_p(\mathbb{R}^n_+) \) carry a \( \mu \)'th power of \( x_n \), but also the functions on \( \mathbb{R}^n_+ \) themselves do so.
Theorem 5.4. When \( s > \text{Re} \mu + M - 1/p' \), and \( u \in H^\mu_p(\mathbb{R}^n) \), then with \( \mathcal{K} = \mathcal{K}_\mu,M \) taken as in (5.11),

\[
(5.13) \quad u = v + w, \quad \text{where } v = \mathcal{K}_\mu,M \varrho u \text{ and } w = H^{(\mu+M)}(\mathbb{R}^n).
\]

Here if \( \text{Re} \mu > -1 \), \( v = \Xi u^+ e^+ \mathcal{K}_M \varrho \Xi u \) has the form

\[
(5.14) \quad v = \sum_{j=0}^{M-1} c_j x_n^{\mu+j} K_0(\gamma u_j) = e^+ x_n^\mu v_0,
\]

with \( v_0 \in \overline{H}^{s-\text{Re} \mu}(\mathbb{R}^n) \), \( K_0 \) as in (1.7).

Thus one has for \( \text{Re} \mu > -1, s > \text{Re} \mu - 1/p' \), with \( M \in \mathbb{N} \):

\[
H^\mu_p(\mathbb{R}^n) \subseteq e^+ x_n^\mu \mathcal{H}_p^{s-\text{Re} \mu}(\mathbb{R}^n) + \begin{cases}
\hat{H}_p^s(\mathbb{R}^n) & \text{if } s - \text{Re} \mu \in ]-1/p', 1/p[,
\hat{H}_p^{s-0}(\mathbb{R}^n) & \text{if } s - \text{Re} \mu = 1/p.
\end{cases}
\]

(5.15)

The inclusions (5.15) also hold in the manifold situation, with \( \mathbb{R}^n \) replaced by \( \Omega \) and \( x_n \) replaced by \( d(x) \).

Proof. The decomposition (5.13) is an immediate consequence of Theorem 5.1; here \( w \in H^{(\mu+M)}(\mathbb{R}^n) \) since \( \varrho u = 0 \). In the next statements we take \( \text{Re} \mu > -1 \) in order to identify \( I^\mu \) with the locally integrable function \( e^+ r^+ x_n^\mu / \Gamma(\mu + 1) \). Distributional formulations can be made for lower \( \mu \).

For the description in (5.14), note that the first equality follows from (5.9) and (5.11). For the next equality, consider first the case \( M = 1 \), where simply \( v = K_{\mu,0} \gamma u \).

Recall from (1.7) that \( K_0 \) is the elementary Poisson operator of order 0

\[
\varphi \mapsto \mathcal{F}^{-1}_{\xi \rightarrow x'}(\hat{\varphi}(\xi') e^+ r^+ e^{-|\xi'|^2} x_n) = \mathcal{F}^{-1}_{\xi \rightarrow x}(\hat{\varphi}(\xi') (|\xi'| + i\xi_n)^{-1}).
\]

Construing \( K_{\mu,0} \) as in Corollary 5.3 we have, cf. (2.5),

\[
(5.16) \quad K_{\mu,0} \varphi = \mathcal{F}^{-1}_{\xi \rightarrow x}((|\xi'| + i\xi_n)^{-\mu} \hat{\varphi}(\xi') (|\xi'| + i\xi_n)^{-1}) = c_\mu \mathcal{F}^{-1}_{\xi \rightarrow x}(e^+ r^+ x_n^\mu e^{-|\xi'|^2} x_n \hat{\varphi}(\xi')) = c_\mu e^+ x_n^\mu K_0 \varphi.
\]

Hence since \( \gamma_{\mu,0} u \in B_{p}^{s-\text{Re} \mu - 1/p}(\mathbb{R}^n) \),

\[
(5.17) \quad v = c_\mu e^+ x_n^\mu K_0 \gamma_{\mu,0} u \in e^+ x_n^\mu \mathcal{H}_p^{s-\text{Re} \mu}(\mathbb{R}^n),
\]

by the mapping properties of Poisson operators shown in [G90].

For general \( M \) we have that \( v = K_{\mu,0} \gamma_{\mu,0} u + \cdots + K_{\mu,M-1} \gamma_{\mu,M-1} u \), and we have to account for the general term \( K_{\mu,j} \gamma_{\mu,j} u \). Here \( \varphi_j = \gamma_{\mu,j} u \in B_{p}^{s-\text{Re} \mu - j - 1/p}(\mathbb{R}^n) \). By (1.7), \( K_j \) acts as

\[
\varphi_j \mapsto \mathcal{F}^{-1}_{\xi \rightarrow x}(\hat{\varphi}_j(\xi') j^{-1} d^j x_n (|\xi'| + i\xi_n)^{-1}) = \mathcal{F}^{-1}_{\xi \rightarrow x}(\hat{\varphi}_j(\xi') j^j (|\xi'| + i\xi_n)^{-j-1}).
\]
Then

\[ K_{\mu,j} \varphi_j = \mathcal{F}_{\xi' \rightarrow x'}^{-1} \left( \left( [\xi'] + i\xi_n\right)^{-\mu} \hat{\varphi}_j(\xi') \right) \]

\[ = c_{\mu,j} \mathcal{F}_{\xi' \rightarrow x'}^{-1} \left( e^{+r^{-1}} x_n^{\mu+j} e^{-i\xi_n x_n} \hat{\varphi}_j(\xi') \right) = c_{\mu,j} e^{+x_n^{\mu+j} K_0 \varphi_j}. \]

By the rules of the Boutet de Monvel calculus, \( x_n^0 K_0 \) is a Poisson operator of order \(-j\), so the mapping properties from [G90] assure that \( x_n^j K_0 \varphi_j \in \overline{H}^{s - \Re \mu}(\mathbb{R}_+^n) \). Thus

\[ K_{\mu,j} \gamma_{\mu,j} u \in e^{+x_n^\mu} \overline{H}^{s - \Re \mu}(\mathbb{R}_+^n). \]

The first line in (5.15) is shown in (1.26) when \( s - \Re \mu < 1/p \), and when \( s - \Re \mu = 1/p \), it follows in view of (1.31). The second line in (5.15) follows from (5.13) and (5.14), when \( s - \Re \mu \in ] - 1/p', 1/p[ \), since \( H^{(\mu + M)(s)}(\mathbb{R}_+^n) \) then is as in the first line.

The conclusions in (5.15) carry over to the manifold situation by use of local coordinates. □

The formulas (5.17), (5.18) are of interest in themselves.

**Corollary 5.5.** Let \( \Re \mu \geq 0, s > \Re \mu + n/p \). Then

\[ H_p^{(s)}(\overline{\Omega}) \subset e^{+d(x)^\mu} C^{s - \Re \mu - n/p - 0}(\overline{\Omega}), \]

where \(-0\) can be left out when \( s - \Re \mu - n/p, s - n/p \) and \( s - \Re \mu - 1/p \) are noninteger.

**Proof.** We use the description by two terms in (5.15). By (1.23),

\[ e^{+d(x)^\mu} \overline{H}_p^{s - \Re \mu}(\Omega) \subset e^{+d(x)^\mu} C^{s - \Re \mu - n/p - 0}(\overline{\Omega}), \]

where \(-0\) can be left out when \( s - \Re \mu - n/p \) is not integer. When \( u \in \hat{H}_p^{s}(\overline{\Omega}) \), it belongs to \( C^{s - n/p - 0}(\Omega) \) and is supported in \( \overline{\Omega} \); here \(-0\) can be left out when \( s - n/p \) is not integer. Since \( s > 1/p, \gamma_0 u = 0 \); then in view of the Hölder continuity, \( u \in e^{+d(x)^\mu} C^{s - \Re \mu - n/p - 0}(\overline{\Omega}) \), since \( s - n/p > \Re \mu \geq 0 \). This extends to \( \hat{H}_p^{s - 0}(\overline{\Omega}) \) when \( s - \Re \mu - 1/p \) is integer; the \(-0\) is needed then in view of (5.15). Hereby the assertion is verified for the two terms in (5.15). □

6. NONHOMOGENEOUS BOUNDARY VALUE PROBLEMS, PARAMETRICES

The problems treated in Theorem 4.4 can be regarded as homogeneous boundary problems, when we see them in the following perspective.

Consider again our operator \( P \) satisfying the hypotheses of Theorem 4.4, with the factorization index \( \mu_0 \in \mathbb{C} \). For a positive integer \( M \) let \( \mu = \mu_0 - M \). We have from Theorem 5.1 that when \( s > \Re \mu + M - 1/p = \Re \mu_0 - 1/p \), then \( \varrho_{\mu,M} \) defines a homeomorphism

\[ \varrho_{\mu,M} : H_p^{(\mu_0)}(\overline{\Omega})/H_p^{(\mu_0)}(\overline{\Omega}) \sim \prod_{0 \leq j < M} B_p^{s - \Re \mu - j - 1/p}(\partial \Omega). \]

Combining this with the Fredholm property of

\[ r^+ P : H_p^{(\mu_0)}(\overline{\Omega}) \rightarrow \overline{H}_p^{s - \Re \mu}(\Omega), \]

we have immediately:
Theorem 6.1. Let $P$ satisfy the hypotheses of Theorem 4.4, and let $\mu = \mu_0 - M$ for a positive integer $M$. Then when $s > \text{Re} \mu_0 - 1/p'$, \{r^{+}P, g_{\mu, M}\} defines a Fredholm operator

$$
(6.3) \quad \{r^{+}P, g_{\mu, M}\}: H_{p}^{s}(\Omega) \to \overline{H}_{p}^{s-\text{Re} \mu}(\Omega) \times \prod_{0 \leq j < M} B_{p}^{s-\text{Re} \mu - j - 1/p} (\partial \Omega).
$$

This is a solvability result for the following inhomogeneous “Dirichlet problem” for $P$:

$$
(6.4) \quad r^{+}Pu = f, \quad g_{\mu, M}u = \varphi,
$$

where $\varphi$ is an $M$-vector $\{\varphi_0, \ldots, \varphi_{M-1}\}$ of boundary data.

We can in particular take $M = 1$; this gives:

**Corollary 6.2.** With $P$ as in Theorem 5.1, let $\mu = \mu_0 - 1$. Then

$$
(6.5) \quad \{r^{+}P, \gamma_{\mu, 0}\}: H_{p}^{s}(\Omega) \to \overline{H}_{p}^{s-\text{Re} \mu}(\Omega) \times B_{p}^{s-\text{Re} \mu - 1/p} (\partial \Omega)
$$

is Fredholm when $s > \text{Re} \mu + 1 - 1/p' (= \text{Re} \mu_0 - 1/p')$.

This shows a solvability result for the problem

$$
(6.6) \quad r^{+}Pu = f, \quad \gamma_{\mu, 0}u = \varphi_0.
$$

with just $\gamma_{\mu, 0}u$ prescribed, $\mu = \mu_0 - 1$.

**Example 6.3.** For the Laplace-Beltrami operator, $\mu_0 = 1$, so Corollary 6.2 is applicable with $\mu = 0$. Here $H_{p}^{0}(\Omega) = \overline{H}_{p}^{s}$ and $\gamma_{0, 0} = \gamma_0$, so it gives the Fredholm property of the mapping

$$
\{\Delta, \gamma_0\}: \overline{H}_{p}^{s}(\Omega) \to \overline{H}_{p}^{s-2}(\Omega) \times B_{p}^{s-1/p} (\partial \Omega)
$$

for $s > 1/p$, which is well-known as the inhomogeneous Dirichlet problem for $\Delta$.

For $M = 2, \mu = \mu_0 - M = -1$ and $g_{\mu, M} = \{\gamma_{-1, 0}, \gamma_{-1, 1}\}$. When $u \in \mathcal{E}_{-1}(\mathbb{R}^{n})$,

$$
\quad u = u_0(x')\delta(x_n) + u_1(x') + v, \quad v \in \mathcal{E}_{1}(\mathbb{R}^{n}), u_0 \text{ and } u_1 \in C^{\infty}(\mathbb{R}^{n-1}),
$$

according to (5.3); then $\gamma_{-1, 0}u = u_0(x')$ and $\gamma_{-1, 1}u = u_1(x')$. We get a solvability result for $\Delta$ where the term $u_0(x')\delta(x_n)$ can be prescribed arbitrarily. This is a point of view on boundary problems related to the works of Roitberg and Sheftel’ [RS69], [R96], going beyond the ordinary concept of boundary value problems.

**Remark 6.4.** Since the distributions $I^{\mu}(x_n)$ are locally integrable functions $e^{+r+c_{\mu}x_n}$ only when $\text{Re} \mu > -1$, the trace maps $\gamma_{\mu, 0}$ are somewhat “wild” when $\text{Re} \mu \leq -1$. In the interpretations of concrete cases we shall in this paper only consider situations where the entering trace operators have $\text{Re} \mu > -1$; e.g. in applications of Theorem 6.1 we only take $M < \text{Re} \mu_0 + 1$.

We shall finally show that a parametrix of the nonhomogeneous boundary problem considered in Corollary 6.2 can be obtained by a combination of the knowledge from the type 0 calculus and the special operators used here. The construction of $K$ “from scratch” takes up much effort in [H65].
Theorem 6.5. Let $P$ be a globally estimated $\psi$do of order $m \in \mathbb{C}$ and type $\mu_0 \in \mathbb{C}$, and factorization index $\mu_0$, relative to the domain $\mathbb{R}_n^+$. Let $s > \text{Re} \mu_0 - 1/p'$.

For the problem considered in Corollary 6.2:

\begin{equation}
(6.7) \quad r^+ Pu = f, \quad \gamma_{\mu_0-1,0} u = \varphi,
\end{equation}

with $f$ given in $H^{s-\text{Re} m}_p(\mathbb{R}_n^+)$ and $\varphi$ given in $B^{s-\mu_0+1-1/p}_p(\mathbb{R}^{n-1})$, a parametrix is

\begin{equation}
(6.8) \quad \left( R \quad K \right): \quad H^{s-\text{Re} m}_p(\mathbb{R}_n^+) \times B^{s-\mu_0+1-1/p}_p(\mathbb{R}^{n-1}) \rightarrow H^{(\mu_0-1)(s)}_p(\mathbb{R}_n^+),
\end{equation}

where $R$ is as in Theorem 4.4, and $K$ is of the form

\begin{equation}
(6.9) \quad K = \Xi_1^{1-\mu_0} e^+ K', = \Lambda_1^{1-\mu_0} e^+ K'',
\end{equation}

with Poisson operators $K'$ and $K''$ of order 0 in the Boutet de Monvel calculus.

Proof. As a parametrix for the problem (6.7) with $\varphi = 0$ we can use $R$ introduced in Theorem 4.4, since $H^{(\mu_0-1)(s)}_p(\mathbb{R}_n^+)$ is the subspace of $H^{(\mu_0-1)(s)}_p(\mathbb{R}_n^+)$ where $\gamma_{\mu_0-1,0} u = 0$. Note that $P$ is expressed in terms of $Q$ by

\begin{equation}
(6.10) \quad P = \Lambda_1^{m-\mu_0} Q \Lambda_1^{\mu_0}.
\end{equation}

It remains to solve problem (6.7) when $f = 0$. Consider

\begin{equation}
(6.11) \quad r^+ Pu = 0, \quad \gamma_{\mu_0-1,0} u = \varphi,
\end{equation}

with $\varphi$ given in $B^{s-\mu_0+1-1/p}_p(\mathbb{R}^{n-1})$. On $\mathbb{R}_n^+$ we have explicit formulas for the elementary Poisson-like operators $K_{\mu,M}$. Here

\begin{equation}
(6.12) \quad K_{\mu_0-1,0} = \Xi_1^{1-\mu_0} e^+ K_0,
\end{equation}

cf. Corollary 5.3. To solve (6.11), let

\[ z = \Xi_1^{1-\mu_0} e^+ K_0 \varphi, \]

and form $w = u - z$; it must solve

\begin{equation}
(6.13) \quad r^+ Pw = -r^+ P \Xi_1^{1-\mu_0} e^+ K_0 \varphi, \quad \gamma_{\mu_0-1,0} w = 0.
\end{equation}

By Theorem 4.4, this problem has the solution in a parametrix sense:

\[ w = -r^+ P \Xi_1^{1-\mu_0} e^+ K_0 \varphi = -\Lambda_+^{1-\mu_0} e^+ \widetilde{Q} \Lambda_1^{\mu_0-m} r^+ \Lambda_1^{m-\mu_0} Q \Lambda_1^{\mu_0} \Xi_1^{1-\mu_0} e^+ K_0 \varphi \]
\[ = -\Lambda_+^{1-\mu_0} e^+ \widetilde{Q} r^+ Q \Lambda_1^{\mu_0} \Xi_1^{1-\mu_0} e^+ K_0 \varphi, \]

when we take (6.10) into account, using also Remark 1.1.
We now observe, recalling the definition of $Y^\mu_+$ from (1.16)ff., that
\[
\Lambda_+^{\mu_0} \Xi_+^{1-\mu_0} e^+ K_0 = \Lambda_+^{1} e^+ r^+ \text{OP}(\lambda^{\mu_0-1}_+ \chi^{1-\mu_0}_+) e^+ K_0 = \Lambda_+^{1} e^+ Y^{\mu_0-1}_+ K_0.
\]
An application of Lemma 6.6 below gives that $Y^{\mu_0-1}_+ K_0$ is a Poisson operator of order 0 in the Boutet de Monvel calculus. Hence $\tilde{Q}_+ r^+ Q \Lambda_+^{1} e^+ Y^{\mu_0-1}_+ K_0$ is a Poisson operator $K_1$ of order 1 in the Boutet de Monvel calculus, and
\[(6.14)\]
\[w = -\Lambda_+^{\mu_0} e^+ K_1 \varphi.\]
This can be rewritten, using again Lemma 6.6, as
\[w = -\Xi_+^{1-\mu_0} e^+ Y^{\mu_0}_+ K_1 \varphi,
\]
where $Y^{\mu_0}_+ K_1$ is another Poisson operator of order 1. Thus $u = z + w$ has the structure
\[u = \Xi_+^{1-\mu_0} e^+ K'_1 \varphi,
\]
with a Poisson operator $K'$ of order 1. This shows the first formula in (6.9). For the second formula, we keep $w$ in the form (6.14) and instead rewrite
\[z = \Xi_+^{1-\mu_0} e^+ K_0 \varphi = \Lambda_+^{\mu_0} e^+ \Lambda_+^{1} + Y^{\mu_0-1}_+ K_0 \varphi,
\]
where $\Lambda_+^{1} + Y^{\mu_0-1}_+ K_0$ is a Poisson operator of order 1 in view of Lemma 6.6 and the composition rules. □

Analogous constructions can be made in case $M > 1$.

The following lemma shows a case where the composition of a Poisson operator with certain generalized $\psi$do’s defined from symbols that only satisfy some of the estimates required for the $S^d_{1,0}$ classes, is again a Poisson operator (similarly to some cases considered in Sect. 3.2 of [GK93]).

**Lemma 6.6.** Let $K$ be a Poisson operator on $\mathbb{R}^d_+$ of order $m$, with symbol $k(x', \xi)$, let $s(x', \xi)$ be a Poisson symbol of order 0, and let $S = \text{OP}(s(x', \xi))$ be the generalized $\psi$do with symbol $s$ (defined as in (1.2)). Then the composed operator $S_+ K$ is a Poisson operator of order 0 with symbol $k' = s \circ k \sim \sum_{\alpha \in \mathbb{N}} 1/\alpha! D_{\xi}^\alpha s \partial_x^\alpha k$.

The result applies in particular when $S = Y^\mu_+ - 1 = \text{OP}(\eta^\mu_+) - 1$ as defined in (1.6)ff.

**Proof.** When $k$ is independent of $x'$, so that $e^+ K u = \mathcal{F}^{-1}(k(\xi) \hat{u}(\xi'))$, we can move $k(\xi) \hat{u}$ inside the integral defining the action of $S$, and the result follows since $sk$ is a Poisson symbol of order $m$ (a product of functions in $\mathcal{H}^+$ is in $\mathcal{H}^+$). In the $x'$-dependent case, there is a standard procedure of replacing $k$ by a $y'$-form symbol; it can then be moved inside the integral as above, and the resulting symbol in $(x', y')$-form reduced to $x'$-form as an asymptotic series.

For the last statement, we recall that
\[
\lambda_+^{1}/\chi_+^{1} = 1 + q_+^{1}(\xi), \quad q_+^{1} = [\xi']^{1}([\xi_n/a[\xi']]) - 1)/([\xi'] + i\xi_n) \in \mathcal{H}^+
\]
as a function of $\xi_n$ for all $\xi'$, and $|q_+^{1}(\xi)| \leq 1/2$ (recall that $a$ is taken large). Then
\[
\eta^\mu_+(\xi) = (1 + q_+^{1})^\mu = 1 + \mu q_+^{1} + \mu(\mu - 1)\frac{1}{2}(q_+^{1})^2 + \cdots = 1 + q
\]
as a convergent Taylor series, where $(q_+^{1})^k \in \mathcal{H}^+_{-k}$ as a function of $\xi_n$, so that $q \in \mathcal{H}^+$ for all $\xi'$; moreover, it is homogeneous of degree 0 for $|\xi'| \geq 1$. □
Corollary 6.7. The operator $K$ in Theorem 6.5 has the property that when $B$ is a $\psi$do of type $\mu_0$ and order $m_0 + \mu_0$, $m_0 \in \mathbb{Z}$, then $\gamma_0 r^+ BK$ is a $\psi$do on $\mathbb{R}^{n-1}$ of order $m_0 + 1$.

Proof. Let $B' = B\Lambda^{1-\mu_0}_+$; then $B'$ is of order $m_0 + 1$ and type 0, hence belongs to the Boutet de Monvel calculus. From the rules there we conclude, using (6.9), that $\gamma_0 r^+ BK = \gamma_0 B'_+ K''$ is a $\psi$do of order $m_0 + 1$. $\square$

7. Applications to fractional powers of elliptic operators

We here show some consequences for fractional powers of differential operators. Let $A$ be a second-order strongly elliptic operator with $C^\infty$-coefficients on $\Omega_1$ (that can be taken compact), and consider the fractional powers $P_a = A^a$ for $a > 0$. By Lemma 2.9 and Example 3.2, they are classical $\psi$do's of order $2a$, having type $a$ and factorization index $\mu_0 = a$ relative to $\Omega$. This holds in particular for $(-\Delta)^a$, where $\Delta$ is the Laplace-Beltrami operator on $\Omega_1$. See also Remark 2.10.

We have as an immediate corollary of Theorems 4.4 and 6.1:

**Theorem 7.1.** Let $1 < p < \infty$, and let $s > a - 1/p' = a - 1 + 1/p$.

1° Let $u \in H^s_p(\bar{\Omega})$ for some $\sigma > a - 1/p'$. If $r^+ P_a u \in H^{s-2a}_p(\Omega)$, then $u \in H^a_p(s)(\bar{\Omega})$.

The mapping $r^+ P_a$ is Fredholm:

$$r^+ P_a : H^a_p(s)(\bar{\Omega}) \rightarrow H^{s-2a}_p(\Omega). \tag{7.1}$$

2° In particular, if $r^+ P_a u \in C^\infty(\bar{\Omega})$, then $u \in E_a(\bar{\Omega})$, and the mapping $r^+ P_a$ is Fredholm:

$$r^+ P_a : E_a(\bar{\Omega}) \rightarrow C^\infty(\bar{\Omega}). \tag{7.2}$$

3° Moreover, when $M$ is a positive integer, the operator $\{r^+ P_a, g_a - M, M\}$ is Fredholm:

$$\{r^+ P_a, g_a - M, M\} : H^{a-M(s)}_p(\bar{\Omega}) \rightarrow H^{s-2a}_p(\Omega) \times \prod_{0 \leq j < M} B^{s-a+M-j-1/p}(\partial \Omega), \tag{7.3}$$

$$\{r^+ P_a, g_a - M, M\} : E_a - M(\bar{\Omega}) \rightarrow C^\infty(\bar{\Omega}) \times C^\infty(\partial \Omega)^M.$$

As mentioned in Remark 6.4, we shall here only discuss 3° when $M < a + 1$.

**Example 7.2.** Let us describe the domain of the Dirichlet realization for $p = 2$ in this context. Define it as the space of solutions of $r^+ P_a f = u$ with $f \in L_2(\Omega)$ according to the above theorem:

$$D(P_{a,\text{Dir}}) = \{u \in H^{a-\frac{1}{2}+0}_2(\bar{\Omega}) \mid r^+ P_a u \in L_2(\Omega)\}.$$

The order of $P_a$ is $2a$, so the range space in Theorem 7.1 1° equals $L_2(\Omega)$ when $s = 2a$. Then $D(P_{a,\text{Dir}}) = H^{2a}_2(\bar{\Omega})$, where $r^+ P_a$ is Fredholm. This is a precise and seemingly new result when $a \geq \frac{1}{2}$, the case $a < \frac{1}{2}$ being covered by Vishik and Eskin’s theorem.

Note that

$$2a \in a + \frac{1}{2}, \frac{1}{2} \text{ when } a < \frac{1}{2}, \quad 2a \in a + 1 + \frac{1}{2}, \frac{1}{2} \text{ when } \frac{1}{2} \leq a < \frac{3}{2}, \text{ etc.}$$
Then we have by Theorem 5.4,
\[
D(P_{a,\text{Dir}}) = \begin{cases} 
\dot{H}_{2}^{2a}(\Omega), & \text{when } 0 < a < \frac{1}{2}, \\
\dot{H}_{2}^{\frac{1}{2}(1)}(\Omega) \subset \dot{H}_{2}^{1-0}(\Omega) & \text{when } a = \frac{1}{2}, \\
\subset e^{+}d(x)^{a}\dot{H}_{2}^{a}(\Omega) + \dot{H}_{2}^{2a}(\Omega) & \text{when } \frac{1}{2} < a < \frac{3}{2}, \text{ etc.}
\end{cases}
\]

For \( a > \frac{1}{2} \), the structure of the contribution from \( d(x)^{a}\dot{H}_{2}^{a} \) is described in (5.14), (5.17).

We remark that the operator \( P_{a,\text{Dir}} \) for \( A = -\Delta \) is not the same as the operator \( B_{a} = (-\Delta_{\text{Dir}})^{a} \) defined by \( L_{2} \) spectral theory from the Dirichlet realization \( \Delta_{\text{Dir}} \) of the Laplacian when \( 0 < a < 1 \). Here \( D(B_{a}) \) is the interpolation space between \( \dot{H}_{2}^{2a}(\Omega) \cap \dot{H}_{2}^{1}(\Omega) \) and \( L_{2}(\Omega) \), equal to \( \{ u \in \dot{H}_{2}^{2a}(\Omega) \mid \gamma_{0}u = 0 \} \) when \( a > \frac{1}{4} \) and to \( \dot{H}_{2}^{2a}(\Omega) \) when \( a < \frac{3}{4} \).

Now we want to see what the result gives in terms of bounded or Hölder continuous functions. It has been shown by Ros-Oton and Serra in [RS14] for \( 0 < a < 1, \Omega \subset \mathbb{R}^{n}, \) that solutions of \( r^{+}(-\Delta)^{a}u = f \in L_{\infty}(\Omega) \) with \( u \in \dot{H}^{a}(\Omega) \) are in \( d(x)^{a}C^{\alpha}(\Omega) \) for some \( \alpha < \min\{a, 1 - a\} \), when \( \Omega \) is \( C^{1,1} \). (See [RS14] for further references to contributions to the problem.)

Let us study the solutions of the homogeneous Dirichlet problem
\[
r^{+}P_{a}u = f,
\]
where \( f \) is given in \( \dot{H}_{p}^{t}(\Omega) \) with \( t \geq 0, \) for \( u \in \dot{H}_{p}^{a-1/p'+0}(\Omega) \). By Theorem 7.1 \( 1^{o} \) with \( s = t + 2a \), \( u \) belongs to \( H_{p}^{a(t+2a)}(\Omega) \). By Corollary 5.5,
\[
H_{p}^{a(t+2a)}(\Omega) \subset e^{+}d(x)^{a}C^{t+a-n/p-0}(\Omega),
\]
when \( p \) is so large that \( a > n/p \) (for then \( t + 2a > a + n/p \)); here \( -0 \) can be left out except at certain values of \( t \). The ellipticity of \( P_{a} \) moreover assures that \( u \in H_{p,\text{loc}}^{t+2a}(\Omega) \), which is contained in \( C^{t+2a-n/p-0}(\Omega) \). We conclude that
\[
u \in e^{+}d(x)^{a}C^{t+a-n/p-0}(\Omega) \cap C^{t+2a-n/p-0}(\Omega).
\]

Note that the prerequisite \( u \in \dot{H}_{p}^{a-1/p'+0}(\Omega) \) is satisfied if (cf. (1.23))
\[
u \in \begin{cases} 
e^{+}L_{p}(\Omega), & \text{when } a < 1/p', \\
C^{a-1/p'+0}(\Omega), & \text{when } a \geq 1/p'.
\end{cases}
\]

For \( t = 0 \) we have found in particular:
\[
f \in L_{p}(\Omega) \implies u \in e^{+}d(x)^{a}C^{a-n/p-0}(\Omega) \cap C^{2a-n/p-0}(\Omega),
\]
where \( -0 \) can be omitted when \( a - n/p, 2a - n/p \) and \( a - 1/p \) are not integer. For \( p \to \infty, \) \( a - n/p \to a, \) and (7.9) gives, since \( L_{\infty}(\Omega) \subset L_{p}(\Omega) \) for all \( p, \)
\[
f \in L_{\infty}(\Omega) \implies u \in e^{+}d(x)^{a}C^{a-0}(\Omega) \cap C^{2a-0}(\Omega).
\]
(It suffices that $u \in \dot{H}^{a-1/p'_0} p_0 (\Omega)$ for some $p_0$.)

This shows an improvement of Th. 1.2 of Ros-Oton and Serra [RS14], in higher generality concerning the studied operator and the data, when the boundary is smooth.

For general higher $t$, we similarly find, noting that $C^{t+0} (\overline{\Omega}) \subset \mathcal{H}^t(\Omega)$ and letting $p \to \infty$:

\[(7.11) \quad f \in C^{t+0} (\overline{\Omega}) \implies u \in e^+ d(x)^a C^{t+a-0} (\overline{\Omega}) \cap C^{t+2a-0} (\Omega).\]

Recall also that Theorem 7.1.2\(^\circ\) shows:

\[(7.12) \quad f \in C^\infty (\overline{\Omega}) \iff u \in e^+ d(x)^a C^\infty (\overline{\Omega}) \left( = \mathcal{E}_a (\overline{\Omega}) \right),\]

with Fredholm solvability, when $u \in \dot{H}^{a-1/p'_0} p_0 (\Omega)$ for some $p$.

This extends results of [RS14] to arbitrarily smooth spaces. The Fredholm property of (7.1) implies that in each of the cases (7.9)–(7.11), there is solvability for $f$ in the indicated space, subject to a finite dimensional linear condition.

We have hereby obtained:

**Theorem 7.3.** Let $A$ be a second-order strongly elliptic differential operator on $\Omega$ with smooth coefficients, and let $P_a = A^a$ for some $a > 0$, a PDO of order $2a$ by Seeley’s construction. Let $d(x) > 0$ on $\Omega$, $d \in C^\infty (\overline{\Omega})$ and proportional to $\text{dist}(x, \partial \Omega)$ near $\partial \Omega$.

Consider the homogeneous Dirichlet problem (7.5), taking $u \in \dot{H}^{a-1/p'+0} p_0 (\Omega)$ for some $p$, cf. also (7.8).

Let $p > n/a$. Then (7.5) is solvable when $f$ is in a subspace of $L_p(\Omega)$ with finite codimension, and the solutions satisfy (7.9)ff.

A similar statement holds for $f \in L_\infty (\Omega)$ with solutions satisfying (7.10), and for $f \in C^{t+0} (\overline{\Omega})$ with solutions satisfying (7.11). Moreover, (7.12) holds with Fredholm solvability.

Since $a > 0$, we can also apply Theorem 7.1.3\(^\circ\) with $M = 1$. Recall that $\gamma_{a-1,0} u$ is a constant times $\gamma_0 (d(x)^{1-a} u)$. According to the theorem, the nonhomogeneous Dirichlet problem

\[(7.13) \quad r^+ P_a u = f, \quad \gamma_0 d(x)^{1-a} u = \varphi,\]

is, when $s > a - 1/p'$, Fredholm solvable for $f \in \mathcal{H}^{s-2a} (\Omega)$, $\varphi \in D^{s-a+1-1/p}(\partial \Omega)$, with solution $u \in H^{a-1(s)} p_0 (\Omega)$.

Since $s > (a - 1) + 1 - 1/p'$, and $a - 1 > -1$, Theorem 5.4 and its corollary apply to show that when $s > n/p$,

\[(7.14) \quad H^{a-1(s)} p_0 (\Omega) \subset e^+ d(x)^a \mathcal{H}^{s-a+1} p_0 (\Omega) + \dot{H}^{s-0} p_0 (\Omega) \]
\[\subset e^+ d(x)^a C^{s-a+1-n/p-0} p_0 (\Omega) + \dot{C}^{s-n/p-0} p_0 (\Omega),\]

where $-0$ can be left out except at certain values of $t$. (The $\dot{C}$-term is needed when $a < 1$.)

Here we find:

\[(7.15) \quad f \in L_p(\Omega), \varphi \in C^{a+1-1/p+0} (\partial \Omega) \implies u \in e^+ d(x)^a \mathcal{H}^{s-a+1-n/p-0} p_0 (\Omega) \cap C^{2a-n/p-0} p_0 (\Omega) + \dot{C}^{2a-n/p-0} p_0 (\Omega),\]
when $p > n/(a + 1)$; the $-0$ can be left out when $a - n/p, 2a - n/p$ and $a - 1/p$ are not integer. For $p \to \infty$ this gives, since $L_\infty(\Omega) \subset L_p(\Omega)$ and $C^{a+1}(\partial \Omega) \subset C^{a+1-1/p+0}(\partial \Omega)$ for all $p$,

\begin{equation}
(7.16) \quad f \in L_\infty(\Omega), \varphi \in C^{a+1}(\partial \Omega) \implies u \in e^+ d(x)^{a-1} C^{a+1-0}(\Omega) \cap C^{2a-0}(\Omega) + \dot{C}^{2a-0}(\Omega).
\end{equation}

For $t \geq 0$ we likewise find

\begin{equation}
(7.17) \quad f \in C^{t+0}(\Omega), \varphi \in C^{t+a+1}(\partial \Omega) \implies u \in e^+ d(x)^{a-1} C^{t+a+1-0}(\Omega) \cap C^{t+2a-0}(\Omega) + \dot{C}^{t+2a-0}(\Omega).
\end{equation}

In each of these situations, there is solvability when the data \{\(f, \varphi\)\} are subject to a finite dimensional linear condition. We recall moreover from Theorem 7.1 3° that

\begin{equation}
(7.18) \quad f \in C^{t+a}(\Omega), \varphi \in C^{t}(\partial \Omega) \iff u \in e^+ d(x)^{a-1} C^{t+a}(\Omega) (\text{or } \mathcal{E}_{a-1}(\Omega)),
\end{equation}

with Fredholm solvability, when $u \in H_p^{(a-1)(s)}(\Omega)$ for some $s, p$ with $s > a - 1/p'$.

We have then obtained:

**Theorem 7.4.** Hypotheses as in Theorem 7.3. Consider the nonhomogeneous Dirichlet problem (7.13).

Let $p > n/(a + 1)$. For $u \in H^{(a-1)(\sigma)}(\Omega)$ with $\sigma > \max \{a - 1/p', n/p\}$, cf. also (7.14), (7.13) is solvable when $f \in L_p(\Omega), \varphi \in C^{a+1-1/p+0}(\partial \Omega)$, subject to a finite dimensional linear condition, with solutions satisfying (7.15)ff.

A similar statement holds when $f \in L_\infty(\Omega), \varphi \in C^{a+1}(\partial \Omega)$, with solutions satisfying (7.16), and when $f \in C^{t+0}(\Omega), \varphi \in C^{t+a+1}(\partial \Omega)$, with solutions satisfying (7.17).

Moreover, (7.18) holds with Fredholm solvability.

Note that since $a$ can be any positive number, this covers powers between 0 and 1 of $\Delta^2, \Delta^3$, etc. When $a > 1$, we can also apply Theorem 7.1 3° for larger $M$ (namely for $M < a + 1$), which gives natural extensions of Theorem 7.4. Details are left to the reader.

The theory moreover applies to $a$’th powers of $2m$-order strongly elliptic differential operators, since they are of order $2am$ and type $am$, and have factorization index $am$, cf. Example 3.2. The power $a$ can also be taken complex.

Other boundary operators (e.g. the Neumann operator $\gamma_{a-1,1}$ in lieu of $\gamma_{a-1,0}$ in (7.13), and more generally combinations of $\varrho_{\mu,M}$ with suitable $\psi$do’s) can also be investigated, and one can make applications to mixed problems and transmission problems, and to spectral asymptotics. The solvability properties in Hölder spaces can be sharpened slightly by applying the $\psi$do techniques directly to scales of Hölder-Zygmund spaces $B^s_{\infty,\infty}$. We shall return to these subjects in subsequent works.

**Remark 7.5.** The notes [H65], labeled Chapter II, were given to me by Lars Hörmander in 1980, but I have only studied them in depth recently. They have been given to a number of people, but those colleagues that I have asked (in order to find the missing Chapter I) have lost track of them. I have typed the text in TeX (with comments on misprints etc.), and am willing to send it to interested readers; it can also be found on my homepage.
References

[AM09]. P. Albin and R. B. Melrose, *Fredholm realizations of elliptic symbols on manifolds with boundary*, J. Reine Angew. Math. **627** (2009), 155–181.

[B69]. L. Boutet de Monvel, *Opérateurs pseudo-différentiels elliptiques et problèmes aux limites*, Ann. Inst. Fourier **19** (1969), 169–268.

[B71]. L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[B79]. L. Boutet de Monvel, *Lacunas and transmissions*, Annals of Math. Studies, vol. 91, Princeton, 1979, pp. 209–218.

[BGR10]. K. Bogdan, T. Grzywny and M. Ryznar, *Heat kernel estimates for the fractional Laplacian with Dirichlet conditions*, Ann. of Prob. **38** (2010), 1901–1023.

[CD01]. O. Chkadua and R. Duduchava, *Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotics*, Math. Nachr. **222** (2001), 79–139.

[E81]. G. Eskin, *Boundary value problems for elliptic pseudodifferential equations*, Amer. Math. Soc., Providence, R.I., 1981.

[FG14]. R. Frank and L. Geisinger, *Refined semiclassical asymptotics for fractional powers of the Laplace operator*, arXiv:1105.5181, to appear in J. Reine Angew. Math.

[F86]. J. Franke, *Elliptische Randwertprobleme in Besov-Triebel-Lizorkin-Raumen* (1986), Dissertation, Friedrich-Schiller-Universität Jena.

[GMS09]. M. Gonzalez, Rafe Mazzeo and Y. Sire, *Singular solutions of fractional order conformal Laplacians*, J. Geom. Anal. **22** (2012), 845-863.

[G90]. G. Grubb, *Pseudo-differential boundary problems in $L_p$-spaces*, Comm. Part. Diff. Eq. **13** (1990), 289–340.

[G96]. G. Grubb, *Functional calculus of pseudodifferential boundary problems*. Progress in Math. vol. 65, Second Edition, Birkhäuser, Boston, 1996, first edition issued 1986.

[G09]. G. Grubb, *Distributions and operators*. Graduate Texts in Mathematics, 252, Springer, New York, 2009.

[GH90]. G. Grubb and L. Hörmander, *The transmission property*, Math. Scand. **67** (1990), 273–289.

[GK93]. G. Grubb and N. J. Kokholm, *A global calculus of parameter-dependent pseudodifferential boundary problems in $L_p$ Sobolev spaces*, Acta Math. **171** (1993), 165–229.

[HS08]. G. Harutyunyan and B.-W. Schulze, *Elliptic mixed, transmission and singular crack problems*. EMS Tracts in Mathematics, 4, European Mathematical Society (EMS), Zürich, 2008.

[HP79]. A. Hirschowitz and A. Piriou, *Propriétés de transmission pour les distributions intégrales de Fourier*, Comm. Part. Diff. Eq. **4** (1979), 113-217.

[H63]. L. Hörmander, *Linear partial differential operators*, 1963.

[H65]. L. Hörmander, *Ch. II, Boundary problems for “classical” pseudo-differential operators*, 1965, photocopied lecture notes at Inst. Adv. Study, Princeton.

[H83]. L. Hörmander, *The analysis of linear partial differential operators, I*, Springer Verlag, Berlin, New York, 1983.

[H85]. L. Hörmander, *The analysis of linear partial differential operators, III*, Springer Verlag, Berlin, New York, 1985.

[M93]. R. B. Melrose, *The Atiyah-Patodi-Singer index theorem*, A. K. Peters, Wellesley, MA, 1993.

[RS84]. S. Rempel and B.-W. Schulze, *Complex powers for pseudo-differential boundary problems II*, Math. Nachr. **116** (1984), 269–314.

[R96]. Y. Roitberg, *Elliptic boundary value problems in the spaces of distributions*. Mathematics and its Applications, vol. 384, Kluwer Academic Publishers Group, Dordrecht, 1996, pp. 415 pp.

[RS69]. Y.A. Roitberg and V. Sheftel, *A homeomorphism theorem for elliptic systems, and its applications*, Mat. Sb. (N.S.) **78** (1969), 446-472.

[RS14]. X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian*, J. Math. Pures Appl. **101** (2014), 275-302.

[S67]. R. T. Seeley, *Complex powers of an elliptic operator*, Amer. Math. Soc. Proceedings Symposia Pure Math. **10** (1967), 288–307.

[S94]. E. Shargorodsky, *An $L_p$-analogue of the Vishik-Eskin theory*, Memoirs on Differential Equations and Mathematical Physics, Vol. 2, Math. Inst. Georgian Acad. Sci., Tblisi, 1994, pp. 41–146.
[T95]. H. Triebel, *Interpolation theory, function spaces, Differential operators (2nd edition)*, J. A. Barth, Leipzig, 1995.

[VE65]. M. I. Vishik and G. I. Eskin, *Convolution equations in a bounded region*, Uspehi Mat. Nauk 20 (1965), 89–152; English translation in Russian Math. Surveys 20 (1965), 86–151.

[VE67]. M. I. Vishik and G. I. Eskin, *Convolution equations of variable order*, Trudy Mosk. Mat. Obsc. 16 (1967), 25–50; English translation in Trans. Moscow Math. Soc. 16 (1967), 27–52.