Energy-momentum conservation laws in Finsler/Kawaguchi Lagrangian formulation

Takayoshi Ootsuka\textsuperscript{1,4}, Ryoko Yahagi\textsuperscript{1}, Muneyuki Ishida\textsuperscript{2} and Erico Tanaka\textsuperscript{3}

\textsuperscript{1}Physics Department, Ochanomizu University, 2-1-1 Ohtsuka Bunkyo-ku, Tokyo 112-8610, Japan
\textsuperscript{2}Department of Physics, Meisei University, 2-1-1 Hodokubo, Hino, Tokyo 191-8506, Japan
\textsuperscript{3}Department of Mathematics and Computer Science, Kagoshima University, 1-21-35 Kōrimoto Kagoshima, Kagoshima, Japan

E-mail: ootsuka@cosmos.phys.ocha.ac.jp, yahagi@hep.phys.ocha.ac.jp, ishida@phys.meisei-u.ac.jp and erico@sci.kagoshima-u.ac.jp

Received 4 February 2015, revised 12 June 2015
Accepted for publication 22 June 2015
Published 30 July 2015

Abstract
We reformulate the standard Lagrangian formulation to a reparameterization invariant Lagrangian formulation by means of Finsler and Kawaguchi geometry. In our formulation, various types of symmetries that appear in theories of physics are expressed geometrically by symmetries of the Finsler (Kawaguchi) metric, and the conservation laws of energy momentum arise as a part of the Euler–Lagrange equations. The Euler–Lagrange equations are given geometrically in reparameterization invariant form, and the conserved energy-momentum currents can be obtained more easily than by the conventional Lagrangian formulation. The applications to scalar field, Dirac field, electromagnetic field and general relativity are introduced. In particular, we propose an alternative definition of the energy-momentum current of gravity, which satisfies gauge invariance under on-shell conditions.

Keywords: classical field theories, Lagrangian formulation, covariant formulation, Finsler geometry, Kawaguchi geometry

1. Introduction
It is essential for an integral action to be defined independent of parameters so that the variational principle (Hamilton’s least action principle) becomes a geometrical expression.

\textsuperscript{4} Author to whom any correspondence should be addressed.
Namely, the Lagrangian of the system needs to be reparameterization invariant. The standard way to derive the conservation law of the energy (energy-momentum current for field theory) is by Noether’s theorem in accord with the translational symmetry. However, in the reparameterization invariant system, it appears as a part of the Euler–Lagrange equations.

Let us take an example of a particle moving in the Schwarzschild spacetime:

\[
L(x^\mu, \dot{x}^\mu) = mc \sqrt{g_{\mu\nu}(x)} \dot{x}^\mu \dot{x}^\nu,
\]

\[
g = \left(1 - \frac{a}{r}\right)(dx^0)^2 - \frac{(dr)^2}{1 - a/r} - r^2 \left\{(d\theta)^2 + \sin^2 \theta (d\phi)^2\right\}.
\]

The Euler–Lagrange equations are given by

\[
0 = \frac{d}{dr} \left( \frac{mc g_{\mu\nu} \dot{x}^\mu}{\sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}} \right),
\]

\[
0 = \frac{mc}{2} \frac{\partial g_{\mu\nu}}{\partial r} \dot{x}^\mu \dot{x}^\nu \sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} - \frac{d}{dr} \left( \frac{mc g_{\mu\nu} \dot{x}^\mu}{\sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}} \right),
\]

with \(i = 1, 2, 3\). Notice that the first equation is what we call the energy conservation law of a relativistic particle. This happens because the action of a relativistic particle is reparameterization invariant, and only three equations out of four are independent. There is no reason that we should not choose this first equation as the equation of motion. The energy conservation law and the equations of motion are equivalent, in this sense.

We see the same mechanism in the model of a free bosonic string. The Lagrangian of Nambu-Goto action is

\[
L(X^\mu, X'^\mu, X'^\mu) = \kappa_0 \sqrt{\left(X'_\mu X'^\mu\right)^2 - \left(X'_\mu X'^\mu\right)(X'_\nu X'^\nu)},
\]

with \(X'^\mu = \frac{\partial X^\mu}{\partial \tau}, X'^\mu = \frac{\partial X^\mu}{\partial \sigma}\). The Euler–Lagrange equations are

\[
0 = \frac{\partial}{\partial \tau} \left\{ \kappa_0 \left( \frac{(X'^\mu)^2 X'_\mu - (X' \cdot X') X'_\mu}{\sqrt{(X' \cdot X')^2 - (X'^\mu)^2}} \right) \right\} + \frac{\partial}{\partial \sigma} \left\{ \kappa_0 \left( \frac{(X'^\mu)^2 X'_\mu - (X' \cdot X') X'_\mu}{\sqrt{(X' \cdot X')^2 - (X'^\mu)^2}} \right) \right\},
\]

with \(\mu = 0, 1, \ldots, N\). These equations contain the conservation law of the energy-momentum current. We can see this by taking the spacetime parameters, \(\tau = X^0, \sigma = X^i\), then the equation for \(\mu = 0\) (\(\mu = 1\)) becomes the conservation law of energy (momentum) current. This is also because the Nambu-Goto action is reparameterization invariant.

However, these are specific results when the action is reparameterization invariant, and without this invariance, such equations do not appear as Euler–Lagrange equations, even if the system is conserved. Nevertheless, it is known that any Lagrangian system of finite degrees of freedom can be rewritten in a reparameterization invariant form without affecting its physical contents [14, 20, 21, 23, 25].

In this paper, we will further extend these results and show how to consider every Lagrangian system of standard physical theory in the framework of reparameterization invariant Lagrangian formulations. Conventionally, a Lagrangian system is described by a pair of configuration space and Lagrangian \((Q, L)\), but in general, this \((Q, L)\) is not a geometric space. In the reparameterization invariant Lagrangian formulation, we will use the
extended configuration space $M = \mathbb{R}^{n+1} \times Q$ instead of $Q$, and Finsler metric $F$ (or Kawaguchi metric (areal metric) $K$ for field theory) as a Lagrangian. The pair $(M, F)$ (for field theory, $(M, K)$) becomes a geometrical space, i.e., a space endowed with a *length (area)*, that is invariant under reparameterization. The solution obtained by taking the variation of the action becomes an oriented curve (oriented $k$-dimensional submanifold) in the Finsler (Kawaguchi) manifold. Since the action is given by taking the integral of the Finsler (Kawaguchi) metric over the oriented curve (oriented $k$-dimensional submanifold), the Euler–Lagrange equations derived from this action are apparently reparameterization invariant, and therefore the energy (energy-momentum) conservation law appears as their part. Thus, the previous examples could be reinterpreted as follows.

In the first example, a relativistic particle moving in Schwarzschild spacetime is described by the Finsler manifold,

$$M = \mathbb{R} \times \mathbb{R}_+ \times S^2, \quad F = m \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu},$$

and the Nambu-Goto string is described by the Kawaguchi manifold [10],

$$M = \mathbb{R}^{N+1}, \quad K = \kappa_0 \sqrt{-\frac{1}{2}} \left( dX_\mu \wedge dX_\nu \right) \left( dX^\mu \wedge dX^\nu \right), \quad \left( dX_\mu = \eta_{\mu\nu} dX^\nu \right).$$

As we mentioned, these are the special cases where the Lagrangian already has the property of reparameterization invariance. However, our formulation is not restricted to such special cases, and we will later show some examples for the Lagrangian without this property.

In the next section, we will give the definition of the Finsler and Kawaguchi manifold used in our formulation. The Finsler–Kawaguchi Lagrangian formulation is described in section 3, and the examples of a point particle, scalar field, Dirac field, and electromagnetic field are introduced successively in section 4. We show that energy-momentum conservation law in the standard sense appears in the Euler–Lagrange equations. In section 5, we apply the theory to general relativity and Palatini $(\mathcal{R})$ gravity, and derive conserved currents naturally in the same way as introduced in section 4. We propose that such quantities could be interpreted as energy-momentum currents of gravitational theories.

### 2. Finsler and Kawaguchi manifold

A Finsler manifold $(M, F)$ is a natural extension of a Riemannian manifold. $M$ is a differentiable manifold and the function $F$ defined by

$$F: D(F) \subset TM \to \mathbb{R}, \quad F: v \in D(F) \mapsto F(v) \in \mathbb{R}, \quad F(\lambda v) = \lambda F(v), \quad \forall \lambda > 0,$$  

is called the Finsler metric or the Finsler function [2, 5, 17]. $D(F)$ is a sub-bundle of the tangent bundle $TM$ where the Finsler function is well defined. Usually, in mathematical literatures, a slit tangent bundle $TM^0 = TM \setminus \{0\}$ is taken for this sub-bundle $D(F)$. However, from the viewpoint of physics, we need to consider it as a more general sub-bundle of $TM$, since it is not guaranteed that we will always have the function $F$ on the whole $TM^0$. The last condition in (1) is called the *homogeneity condition*. The Finsler function gives a vector a geometrically well-defined norm, due to this condition.

In this paper, we formulate the application of Finsler geometry to a Lagrangian system and derive its equations of motion and conserved currents. Since the standard Lagrangians of physics are given in local coordinates, we will also give the definition of a Finsler manifold $(M, F)$ in local coordinates. Let $M$ be an $(n + 1)$-dimensional differentiable manifold and $U$ be a subset of $M$. The Finsler metric is written as a function of the coordinates $x^\mu$ and the
1-forms $dx^\mu$ with ($\mu = 0, 1, \ldots, n$), on $U$. The latter $dx^\mu$ could also be regarded as adapted coordinates on $TM$. Throughout this paper, we have introduced definitions so as to make these two interpretations completely interchangeable. In this way, several concepts (such as prolongation) that usually require lengthy descriptions become quite simple. The homogeneity condition is expressed by

$$F(x^\mu, \lambda dx^\mu) = \lambda F(x^\mu, dx^\mu), \quad \forall \lambda > 0. \quad (2)$$

The Finsler metric gives a tangent vector $v \in D(F)_p \subset T_p M$ its norm by

$$F(x^\mu(p), \, dx^\mu(v)) = F(x^\mu(p), \, v^\mu) \in \mathbb{R}. \quad (3)$$

Standard literatures of mathematics also assume the following conditions:

(i) (positivity) $F(v) > 0$ and
(ii) (regularity) $g_{\mu\nu}(x, \, dx) = \det\left(g_{\mu\nu}(x, \, dx)\right) \neq 0$.

However, for our motivation, these conditions are not necessary. The only requirement for our theory is the homogeneity condition (2).

Next, we will define the Finsler length of an oriented curve $c$ on $M$ by

$$A[c] = \int_c F = \int_\rho^s F\left(x^\mu(s), \frac{dx^\mu(s)}{ds}\right) ds, \quad (4)$$

where $c: [s^0, s^1] \to M$, is called a parametrization and $x^\mu(s) = x^\mu(c(s)), \frac{dx^\mu(s)}{ds} = \frac{dx^\mu(c(s))}{ds}$. Notice that we used the same symbol, $c$, for both submanifold and parameterisation, using bold face for the former. The pullback of $F = F(x^\mu, \, dx^\mu)$ by the map $c$ is naturally considered as $c^*F := F(c^*x^\mu, c^*dx^\mu)$; then, the Finsler length $A[c]$ becomes an integration of a 1-form $c^*F$ over the interval $[s^0, s^1]$. $A[c]$ does not depend on its choice of parameter owing to the homogeneity condition (2). In this sense, it is a well-defined geometrical length for the oriented curve $c$.

The field theory can be also formulated by the infinite dimensional Finsler manifold. In this case, the theory is reparameterization invariant only with respect to the ‘time’ parameter. However, we will show that the mathematical structure becomes more simple if we use the Kawaguchi manifold. It introduces us to a finite dimensional configuration space formulation.

A Kawaguchi manifold $(M, K)$ is a natural generalisation of Finsler manifold to a multi-dimensional parameter space. It is also called the $k$-dimensional areal space [11]. Here, $M$ is an $N$-dimensional differentiable manifold and $K$ is called the Kawaguchi metric. $K$ defines a $k$-dimensional area for an oriented $k$-dimensional submanifold of $M$ ($1 < k \leq N$). We can construct its definition in parallel to Finsler geometry. A Kawaguchi metric (or Kawaguchi function) $K$ is a function that satisfies:

$$K: D(K) \subset \Lambda^k TM \to \mathbb{R}, \quad K: v^{[k]} \mapsto K\left(v^{[k]}\right), \quad K\left(\lambda v^{[k]}\right) = \lambda K\left(v^{[k]}\right), \quad \forall \lambda > 0, \quad (5)$$

where $D(K)$ is assumed to be a sub-bundle of $\Lambda^k TM$, and $v^{[k]} = \frac{1}{k!} \frac{\partial}{\partial x^{\mu_1}} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \in \Lambda^k T_p M$ is a $k$-vector, which express the $k$-dimensional oriented surface element at point $p \in M$. The last condition in (5) is called the homogeneity condition of Kawaguchi metric. Again, since the usual field Lagrangians are given in coordinate expression, we also give the definition of a Kawaguchi manifold $(M, K)$ in local coordinates.
Let $x^\mu (\mu = 1, \ldots, N)$ be the local coordinates of $M$. We define the Kawaguchi metric as the function of $x^\mu$ and $k$-form $dx^{\mu_1 \mu_2 \cdots \mu_k} := dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_k} (\mu_i = 1, 2, \ldots, N, i = 1, 2, \ldots, k)$. The latter $k$-forms could be also regarded as coordinate functions on $\Lambda^k TM$, but as in the case of Finsler, we opt to consider them as some variables on $M$, expressing the first-order derivatives. In these local coordinates, the homogeneity condition becomes

$$K(x^\mu, \lambda dx^{\mu_1 \mu_2 \cdots \mu_k}) = \lambda K(x^\mu, dx^{\mu_1 \mu_2 \cdots \mu_k}), \quad \forall \lambda > 0.$$  

(6)

As a generalisation of the Finsler metric, the Kawaguchi metric gives a geometric norm to a $k$-vector $v$ by

$$K(x^\mu(p), dx^{\mu_1 \mu_2 \cdots \mu_k}(v^{[k]})) = K(x^\mu(p), v^{\mu_1 \mu_2 \cdots \mu_k}) \in \mathbb{R},$$

(7)

and by integration, defines the $k$-dimensional area to a $k$-dimensional oriented submanifold $\sigma$ by

$$A[\sigma] = \int_\sigma K \left( x^\mu(s, s^1, s^2, \ldots, s^k), \frac{\partial(x^{\mu_1}, x^{\mu_2}, \ldots, x^{\mu_k})}{\partial(s^1, s^2, \ldots, s^k)} \right) ds^1 \wedge ds^2 \wedge \cdots \wedge ds^k.$$  

(8)

We define the pullback of the Kawaguchi function $K$ by the map $\sigma$ as $\sigma^*K := K(\sigma^*x^\mu, \sigma^*dx^{\mu_1 \mu_2 \cdots \mu_k})$. Then, by using the homogeneity condition,

$$\sigma^*K = K \left( x^\mu(s, s^1, \ldots, s^k), \frac{\partial(x^{\mu_1}, \ldots, x^{\mu_k})}{\partial(s^1, \ldots, s^k)} \right) ds^1 \wedge \cdots \wedge ds^k$$

becomes a $k$-form on $W$. Consequently, $A[\sigma]$ becomes a reparameterization invariant area of $\sigma$.

3. Covariant Lagrangian formulation

Finsler geometry originated when considering the geometry of calculus of variations. Therefore, it is a natural setting for formulating the variational principle considered in physics.

First, we will explain how to handle the Lagrangian system with finite degrees of freedom in terms of Finsler geometry. It would be ideal if we could start from the definition of a Finsler manifold $(M, F)$ completely in a covariant fashion, namely, without any specific
choice of $M$. Physicists, however, always fix the ‘time’ parameter during their experiments, and it is this viewpoint that we must take into account. So, we will start our discussion with the pair of configuration space and Lagrangian: $(Q, L)$. Note that this implies that we have already selected a certain ‘time’ parameter, and have chosen the theoretical model as $L(q^i, \dot{q}^i, t)$. We will construct our Finsler manifold $(M, F)$ in accord to this model $(Q, L)$, and it is given by the following \[14, 23\]:

$$M := \mathbb{R} \times Q, \quad F(x^\mu, dx^\mu) := L\left(x^i, \frac{dx^i}{dx^0} \cdot x^0\right)dx^0,$$

(9)

with $\mu = 0, 1, \ldots, n, i = 1, 2, \ldots, n$. $M$ is the product space of time and configuration space $Q$, and is called the extended configuration space. It is easy to check that the above $F(x^\mu, dx^\mu)$ satisfies the homogeneity condition (2), and therefore is a Finsler metric. By the reparameterization invariant property of the Finsler metric, the choice of the ‘time’ parameter does not affect its physical meaning. We will call this Finsler metric a covariant Lagrangian and our method a covariant Lagrangian formulation.

The trajectory of a point particle (an oriented curve $c$ which satisfies the equations of motion) in the extended configuration space is determined by the principle of least action. The action integral is given by $A[c] = \int_c F$.

The Euler–Lagrange equations determine the extremal curve $c$. We set the initial point $p_0$ and the final point $p_1$ on $M$, and consider a differentiable map $\varphi: [-\varepsilon_0, \varepsilon_1] \times M \rightarrow M$, $\varphi(\varepsilon, \cdot) := q_\varepsilon: M \rightarrow M$. The map $q_\varepsilon$ satisfies the conditions $q_{0\varepsilon} = \text{id}_M$, $q_{1\varepsilon} = p_1$, $q_{\varepsilon_p} \circ q_\varepsilon = q_{\varepsilon_p}^{-1}$, $q_{\varepsilon_p}$ is called a flow on $M$. Let the vector field $X \in \Gamma(TM)$ be a generator: $q_\varepsilon = \text{Exp}(\varepsilon X)$. We define the variation of the curve by $\delta c = \left. \frac{d}{d\varepsilon}\right|_{\varepsilon = 0} q_\varepsilon(c)$. The principle of least action is described by

$$0 = \delta A[c] := \left. \frac{d}{d\varepsilon}\right|_{\varepsilon = 0} A[q_\varepsilon(c)] = \frac{d}{d\varepsilon}\left|_{\varepsilon = 0} \int_c F \right. = \frac{d}{d\varepsilon}\left|_{\varepsilon = 0} \int_c q_\varepsilon^* F. \right.$$  

(10)

Now, choose a parameterization of the curve as $c: [s^0, s^1] \rightarrow M$, $c(s^0) = p_0$, $c(s^1) = p_1$. Then (10) becomes,

$$\frac{d}{d\varepsilon}\left|_{\varepsilon = 0} \int_c q_\varepsilon^* F = \int_{s^0}^{s^1} c^\varepsilon q^*_\varepsilon F = \int_{s^0}^{s^1} \left[ \left. \frac{d}{d\varepsilon}\right|_{\varepsilon = 0} F \right](x^\mu(q_\varepsilon(c(s)))) , \quad \frac{d}{d\varepsilon}(q_\varepsilon(c(s))))\right].$$

(11)

The integrand of the last part of (11) is evaluated as,

$$c^\varepsilon \delta F = \delta x^\mu c^\varepsilon \left( \frac{\partial F}{\partial x^\mu} \right) + \frac{d}{d\varepsilon} \delta x^\mu c^\varepsilon \left( \frac{\partial F}{\partial dx^\mu} \right) + \delta x^\mu \left( \frac{\partial F}{\partial dx^\mu} \right) - d \left\{ c^\varepsilon \left( \frac{\partial F}{\partial dx^\mu} \right) \right\}. $$

(12)

Here we used the notation $\delta x^\mu = \left. \frac{d}{d\varepsilon}\right|_{\varepsilon = 0} x^\mu(q_\varepsilon(c(s)))) = c^\varepsilon L_X x^\mu = c^\varepsilon X^\mu$, and $L_X$ is the Lie derivative by the vector field $X = X^\mu \frac{\partial}{\partial x^\mu}$. The term $\frac{\partial F}{\partial dx^\mu}$ is considered as a function of $x^\mu$ and $dx^\mu$, so $c^\varepsilon \left( \frac{\partial F}{\partial dx^\mu} \right) = \left( \frac{\partial F}{\partial dx^\mu} \right)(c^\varepsilon x^\mu, c^\varepsilon dx^\mu).$ In the calculation of $\int \delta F$, the contribution from the first term of (12) vanishes, because the vector field $X$ has the condition $X(c(s^0)) = X(c(s^1)) = 0$ at the end points. For the other points, there are no restrictions for
\[ \delta x^\mu = c^\mu X^\mu. \] Therefore, the condition that the action is at its extremum becomes

\[ 0 = c^\mu \left\{ \frac{\partial F}{\partial x^\mu} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}^\mu} \right) \right\}, \quad (\mu = 0, 1, \ldots, n), \] (13)

and such a curve \( c \) is called the extremal of the action \( \mathcal{A}[c] \). We call (13) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F := \mathcal{L}_v x^\mu \frac{\partial F}{\partial x^\mu} + \mathcal{L}_v \dot{x}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}. \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}, \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}, \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}, \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}, \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}, \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}, \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}, \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.

We also comment on the Noether’s theorem. Let us assume that the system has a certain symmetry. It is convenient to use the generalized expression of the Lie derivative, and its action to the Finsler metric \( F \) with respect to the vector field \( v \) is given by

\[ \mathcal{L}_v F = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = v^\mu \frac{\partial F}{\partial x^\mu} + \dot{v}^\mu \frac{\partial F}{\partial \dot{x}^\mu}, \] (14)

The vector field \( v \) which satisfies \( \mathcal{L}_v F = 0 \) is called the symmetry of \( F \). We call (14) the Euler–Lagrange equations. These equations are reparameterization invariant, since they are derived from a reparameterization invariant action integral. Notice that the reparameterization invariant property makes them dependent on each other.
equivalent to our \( L, F = 0 \), but notice that our expression is much simpler, due to the fact that we consider no fibration. In practice, considering \( L \) as the standard Lie derivative and calculations such as

\[
L_v \frac{dx^\mu}{d} = d(L_v, x^\mu) = dv^\mu, \quad L_v, F = (L_v, x^\mu) \frac{\partial F}{\partial x^\mu} + (L_v, dx^\mu) \frac{\partial F}{\partial dx^\mu}
\]  

(20)

are sufficient to obtain the same result as in the conventional framework. From the point of calculational efficiency, this is also a notable result.

Considering

\[
L_v, F = d\left[ v^\mu \frac{\partial F}{\partial x^\mu} \right] + v^\mu \left\{ \frac{\partial F}{\partial x^\mu} - d\left( \frac{\partial F}{\partial dx^\mu} \right) \right\},
\]

under the assumption that Euler–Lagrange equations are satisfied (namely, that the curve \( \mathbf{c} \) is the extremal), we obtain a conservation law:

\[
c^\mu d\left[ v^\mu \left( \frac{\partial F}{\partial x^\mu} \right) \right] = 0, \tag{21}
\]

This is the expression of Noether’s theorem by our formalism.

If the Lagrangian \( L \) does not contain \( x^0 \) explicitly (i.e., a conserved system), the Finsler metric constructed by (9) also does not include \( x^0 \). In this case, \( x^0 \) is called a cyclic coordinate, and its Euler–Lagrange equation for \( x^0 \) represents the energy conservation law of this system. On the other hand, the conservation law is also obtained directly by inserting the generator \( v^0 = \frac{\partial}{\partial x^0} \) to (21). Either method leads to the same expression.

Second we will move on to the field theory, that is the Lagrangian system with infinite degrees of freedom. As we mentioned in the beginning of this section, it would be better if we could start from the definition of the Kawaguchi manifold \((\mathcal{M}, F)\), with general \( \mathcal{M} \). However, under normal circumstances, we can observe the nature only by fixing the ‘spacetime’, namely the parameter space \( \mathcal{W} \), as we have fixed the ‘time’ parameter for the case of dynamical systems. Therefore, we will start by considering the standard Lagrangian system \((E \rightarrow \mathcal{W}, Q, L)\), where \( E \rightarrow \mathcal{W} \) is the vector bundle and its fibre \([4, 19]\). We choose the total space \( E \) to be our Kawaguchi manifold \( \mathcal{M}, \dim \mathcal{M} = \dim \mathcal{W} + \dim Q \). The Kawaguchi metric \( K \) is constructed from the Lagrangian \((u^A, \frac{dx^\mu}{dx^\sigma})\) as follows \([20, 25]\)

\[
K(\z^a, dz^{abcd}) = L(u^A, \frac{dx^\mu}{dx^\sigma} \wedge \frac{du^A}{dx^{0123}})dx^{0123}. \tag{22}
\]

Here, \((\z^a) := (x^a, u^A), a = 0, 1, \ldots, D + 3, \mu = 0, 1, 2, 3, A = 1, 2, \ldots, D, \) where \( D \) is the degree of freedom of fields. The totally anti-symmetric Levi-Civita symbol \( \varepsilon_{\mu
u\rho\sigma} \) (\( \mu, \nu, \rho, \sigma = 0, 1, 2, 3 \)) has the convention \( \varepsilon_{0123} = -1 \). Note that the field variables \( u^A \) are treated as independent variables, just as the spacetime coordinates \( x^\mu \) are. This is the major difference from the standard Lagrangian formulation. The \( K \) constructed in this way satisfies the homogeneity condition (6), and we obtain our Kawaguchi manifold, \((\mathcal{M}, K)\). The second argument of (22) may look a little complicated; nevertheless, its pullback with respect to the spacetime parameters, namely \( x^\mu \), gives the standard variables, \( \frac{dx^\mu}{dx^\sigma} \). The action integral is given by \( A[\sigma] = \int K \), where \( \sigma \) is a 4-dimensional oriented submanifold in \( \mathcal{M} \). As before, the least action principle is described by the map \( q_l = \text{Exp}(\epsilon X) \) on \( \mathcal{M} \), which is fixed on the boundary. Then the variation of \( K \) by \( X \) becomes
\[
\sigma^\flat \delta K = \delta z^a \sigma^\flat \left( \frac{\partial K}{\partial \sigma^a} \right) + \frac{1}{3!} \delta \zeta^a \wedge \delta z^{bcd} \sigma^\flat \left( \frac{\partial K}{\partial \delta \zeta^{abc}} \right) \\
= d \left[ \delta z^a \sigma^\flat \left( \frac{1}{3!} \frac{\partial K}{\partial \delta \zeta^{abc}} \delta z^{bcd} \right) \right] + \delta \zeta^a \left[ \sigma^\flat \left( \frac{\partial K}{\partial \delta \zeta^a} \right) - d \left\{ \frac{1}{3!} \sigma^\flat \left( \frac{\partial K}{\partial \delta \zeta^{abc}} \delta \zeta^{bcd} \right) \right\} \right]. \tag{23}
\]

where we had taken arbitrary spacetime parameterization \( \sigma : W \to \sigma \subset M \). Next, we set \( z^a(s) = z^a(\sigma(s)) \), and differentiate \( \sigma^\flat q^a K = K(\zeta^a(s), \delta z^{ij}(s) \wedge \delta z^a(s)[3] \wedge \delta z^o(s)) \) with respect to \( \epsilon \). By considerations similar to those in the case of Finsler, we obtain the Euler–Lagrange field equations:

\[
0 = \sigma^\flat \left\{ \frac{\partial K}{\partial \sigma^a} - d \left( \frac{1}{3!} \frac{\partial K}{\partial \delta \zeta^{abc}} \delta \zeta^{bcd} \right) \right\}. \tag{24}
\]

These equations are reparameterization invariant, and again, at least four of them are dependent on each other.

Noether’s theorem could be also obtained for the field theory. The expression of the generalized Lie derivative of the Kawaguchi metric \( K \) with respect to the vector field \( v = v^a \frac{\partial}{\partial x^a} \) on \( M \) is now given by

\[
\mathcal{L}_v K := \mathcal{L}_v \delta z^a \frac{\partial K}{\partial \sigma^a} + \mathcal{L}_v \delta z^{abcd} \frac{1}{4!} \frac{\partial K}{\partial \delta \zeta^{abc}} = v^a \frac{\partial K}{\partial \sigma^a} + \frac{1}{3!} \delta \zeta^a \wedge \delta z^{bcd} \frac{\partial K}{\partial \delta \zeta^{abcd}}. \tag{25}
\]

The vector field \( v \) such that satisfies \( \mathcal{L}_v K = 0 \) is called the symmetry of \( K \), or the Killing vector field of \( K \). Under the condition that the system satisfies the Euler–Lagrange equations, we obtain

\[
\sigma^\flat v^a \left( \frac{1}{3!} \frac{\partial K}{\partial \delta \zeta^{abcd}} \delta \zeta^{bcd} \right) = 0 \tag{26}
\]

as a conservation law.

If the Lagrangian \( L \) does not include the coordinates \( (x^0, x^1, x^2, x^3) \) explicitly, they become cyclic coordinates, and the equations (24) for \( a = 0, 1, 2, 3 \), will represent the conservation law of energy-momentum. The same conservation law could be also derived by inserting the Killing vector \( v = \frac{\partial}{\partial x^a}, a = 0, 1, 2, 3 \) to (26).

### 4. Examples

From this section onward, we will omit the pullback symbol \( c^* (\sigma^* \text{ for Kawaguchi}) \) unless we need to emphasize, for notational simplicity. However, it is important to keep in mind that these equations hold only on the submanifold, \( c (\sigma) \).

#### 4.1. Newtonian mechanics

We begin with an example of Newtonian mechanics, using the Lagrangian formulation of Finsler geometry. Let \( L \) be the Lagrangian of a potential system for an \( n \)-dimensional space:

\[
L = \sum_{i=1}^{m} \frac{1}{2} m (q^i)^2 - V(q^1, q^2, \ldots, q^n). \]

Here, \( m \) is the mass of the particle. We define the Finsler manifold \((M, F)\) by
\[ M = \left\{ \left( x^0, x^1, \ldots, x^n \right) \right\} \cong \mathbb{R}^{n+1}, \quad F(x^\mu, dx^\mu) = \sum_{i=1}^{n} \frac{m}{2} \frac{\left( dx^i \right)^2}{dx^0} - V(x^1, \ldots, x^n) dx^0. \]  

(27)

Note that this \( F \) is defined only on the sub-bundle \( D(F) = TM \setminus \{ dx^0 = 0 \} \). The Euler–Lagrange equations become

\[
0 = -d \left( \frac{\partial F}{\partial dx^0} \right) + d \left[ \frac{n}{2} \left( \frac{dx^i}{dx^0} \right)^2 \right] + V(x^i),
\]

(28)

\[
0 = \frac{\partial F}{dx^i} - d \left( \frac{\partial F}{dx^0} \right) = -\frac{dV}{dx^0} - d \left( \frac{m}{2} \frac{dx^i}{dx^0} \right), \quad (i = 1, 2, \ldots, n).
\]

(29)

The reparameterization invariance gives us the freedom to choose the time parameter \( s \), \( c: [s^0, s^1] \to M \). The standard choice is to take \( s = x^0 \), that is, \( c^s x^0 = s \), \( c^s dx^0 = ds \), \( c^s x^i = x^i(s) \), \( c^s dx^i = dx^i(s) \), and one can verify that (28), (29) gives the conventional conservation law of energy and equations of motion. However, from the perspective of the covariant Finsler formulation, such choice of parameterization is not obligatory, and we may take a parameterization such as \( s = x^1 \), under the assumption that we are only considering the local coordinate system. This is one of the significant results of our formalism.

The conservation law (28) can be also derived from Noether’s theorem, namely

\[
\mathcal{L}_{\frac{\partial F}{dx^0}} = \frac{\partial F}{dx^0} = 0 \quad \Rightarrow \quad d \left( \frac{\partial F}{\partial dx^0} \right) = 0.
\]

(30)

4.2. Scalar field theory

The first scalar example of field theory is the real scalar field theory on 4-dimensional Minkowski spacetime \((\mathbb{R}^4, \eta)\). In the local coordinate system, \( \eta = \eta_{\mu \nu} dx^\mu \otimes dx^\nu \), \( \eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1 \) and \( \eta_{\mu \nu} = 0 \), \( (\mu \neq \nu) \). The conventional Lagrangian is \( L = \frac{1}{2} \partial^\mu \phi \partial^\nu \phi - V(\phi) \), where \( V(\phi) \) is the potential term. The Kawaguchi manifold obtained from this Lagrangian becomes

\[
M = \left\{ \left( x^\mu, \phi \right) \right\} \cong \mathbb{R}^4 \times \mathbb{R}, \quad K = -\frac{(dx^\mu \wedge d\phi)(dx^\nu \wedge d\phi)}{2 \cdot 3! dx^{0123}} - V(\phi) dx^{0123},
\]

(31)

\( M \) is the extended configuration space, and we use abbreviations and notations such as \( dx^{\mu \nu} := dx^\mu \wedge dx^\nu \wedge dx^\rho \), \( \xi_\mu := \eta_{\mu \nu} dx^\nu \). By (31), \( D(K) = \Lambda^4 TM \setminus \{ dx^{0123} = 0 \} \). The Euler–Lagrange equations are derived by using (24),

\[
0 = d \left[ \frac{dx^\mu \wedge d\phi}{2! dx^{0123}} - dx^\nu \wedge d\phi - \frac{(dx^\mu \wedge d\phi)(dx^\nu \wedge d\phi)}{2 \cdot 3! dx^{0123}^2} \right] + V(\phi) \frac{1}{3!} \epsilon_{\mu \nu \rho \sigma} dx^\rho \sigma.
\]

(32)
It is also possible to derive these equations by directly calculating the variation, (23). Usually, for more complex systems, the calculation is more simple by the latter method. We can naturally define an energy-momentum current in the covariant form,

\[
\mathcal{J}_\mu := \frac{\mathrm{d}x_{\mu\rho} \wedge \mathrm{d}\phi^\rho - \mathrm{d}x^\mu \wedge \mathrm{d}\phi}{2! \mathrm{d}x_{0123}^{0123}} - \frac{1}{2 \cdot 3!} \left( \frac{\mathrm{d}x_{\mu\rho} \wedge \mathrm{d}\phi^\rho - \mathrm{d}x^\mu \wedge \mathrm{d}\phi}{\left(\mathrm{d}x_{0123}^{0123}\right)^2} + V(\phi) \right) \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \mathrm{d}x^{\nu\rho\sigma},
\]

for \( \mu = 0, 1, 2, 3 \). To avoid confusion, the quantities on the Kawaguchi manifolds are denoted with a tilde \( \tilde{\cdot} \). The four equations of motion (32) indicates that these currents are conserved, namely \( \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{J}_\mu = 0 \). This means \( a \left( \sigma^2 \mathcal{J}_\mu \right) = 0 \) for arbitrary spacetime parameterization \( \sigma \).

As in the previous example, the coordinates \( x^\mu, (\mu = 0, 1, 2, 3) \) are cyclic coordinates, and therefore it is possible to see the conservation law directly as a part of the Euler–Lagrange equations.

Now we will look into the details of this simple example of scalar field theory. From our point of view, the conventional theory in the framework of Minkowski spacetime corresponds to the case where a specific parameterization is chosen in the setup of a Kawaguchi manifold. We rewrite the coordinate functions of Kawaguchi spacetime as \( z^a, (a = 0, 1, \ldots, 4) \), where \( (z^a) := (x^\mu, \phi) \). The ordinary choice of parameterization \( \sigma \) is expressed by \( \sigma(x) : W \subset \mathbb{R}^4 \rightarrow M, \sigma^a x^\mu = x^\mu, \sigma^a z^4 = \phi(x) \). This means that we are simply taking the coordinates of Minkowski spacetime as parameters. The pullback of the Kawaguchi metric to the parameter space becomes

\[
\sigma(x)^* K = - \left( \frac{\mathrm{d}x_{\mu\rho} \wedge \mathrm{d}x^\mu}{2 \cdot 3! \mathrm{d}x_{0123}^{0123}} - \frac{1}{2} \left( \frac{\epsilon_{\mu\nu\rho\sigma} \partial\phi^\rho \partial\phi^\nu}{\left(\mathrm{d}x_{0123}^{0123}\right)^2} + V(\phi) \right) \right) \mathrm{d}x_{0123}^{0123} = \frac{1}{2} \partial\phi^\rho \partial\phi^\nu - V(\phi) \mathrm{d}x_{0123}^{0123},
\]

which is just the conventional Lagrangian function times the volume form of Minkowski spacetime. The second equality is obtained by the cancelation of \( \mathrm{d}x^\mu \), which appears by the pullback on the numerator.

Next, we will also pullback the Euler–Lagrange equations by this specific parameterization, \( \sigma(x) \). Consider \( \phi(x) \) as a function of \( x^a \), and treating \( d \) as an exterior derivative, we get

\[
\frac{\mathrm{d}x_{0123}^{0123}}{\mathrm{d}x_{0123}} \wedge \mathrm{d}x_{123}^{123} = \frac{\mathrm{d}x_{0123}^{0123}}{\mathrm{d}x_{0123}} \wedge \mathrm{d}x_{123}^{123} = \partial_\phi \mathrm{d}x_{0123},
\]

therefore, the pullback of (33) by \( \sigma(x) \) becomes

\[
0 = - V(\phi) \mathrm{d}x_{0123}^{0123} + d \left( - \partial_\phi \phi \mathrm{d}x_{123}^{123} - \partial_\phi \phi \mathrm{d}x_{023}^{023} - \partial_\phi \phi \mathrm{d}x_{123}^{123} - \partial_\phi \phi \mathrm{d}x_{0123}^{0123} - \partial_\phi \phi \mathrm{d}x_{0123}^{0123} \right) = - V(\phi) - \partial_\phi^2 \phi + \partial_\phi^2 \phi + \partial_\phi^2 \phi + \partial_\phi^2 \phi \mathrm{d}x_{0123}^{0123},
\]

which is the standard wave equation of \( \phi \). Similarly, the pullback of the energy-momentum current (34) for \( \mu = 0, 1 \) becomes,
\[ J_0 = \left( \partial_0 \phi \, dx^{23} + \partial_2 \phi \, dx^{31} + \partial_3 \phi \, dx^{12} \right) \wedge d\phi + \left\{ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + V(\phi) \right\} \right\} \, dx^{123} \]

\[ + \partial_0 \phi \partial_1 \phi \, dx^{023} + \partial_0 \phi \partial_2 \phi \, dx^{031} + \partial_0 \phi \partial_3 \phi \, dx^{012}, \]

\[ J_1 = \left( \partial_0 \phi \, dx^{23} - \partial_3 \phi \, dx^{02} + \partial_2 \phi \, dx^{03} \right) \wedge d\phi - \left\{ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + V(\phi) \right\} \, dx^{023} \]

\[ + \partial_0 \phi \partial_1 \phi \, dx^{123} + \left\{ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \left( \partial_2 \phi \right)^2 - \left( \partial_3 \phi \right)^2 - V(\phi) \right\} \right\} \, dx^{023} \]

which is also the well-known definition of the standard energy-momentum current.

A well-established approach to deal with field theory by means of geometry is to use a fibre bundle (normally a vector bundle) structure, where the base manifold is the 4-dimensional spacetime, and the fields are described by the section of the bundle. The theory formulated on such a structure does not depend on the coordinates of the base manifold, meaning that we can use arbitrary spacetime coordinates \( f^\mu (x^\nu) \) as spacetime parameters. This is the standard meaning of covariance. On the contrary, we have formulated the field theory on a Kawaguchi manifold, without any reference to fibred structures. In such an approach, the spacetime coordinates and field variables are treated equally as coordinate functions of the Kawaguchi manifold, and coordinate transformations of the type \( \tilde{f}^\mu (\tilde{x}^\nu, \phi) \), \( \phi \) denoting the field, do not change the theory. We call such a property an extended covariance.

To see this more clearly, first consider a free relativistic particle moving in a Minkowski spacetime \( (M, \eta) \). By special relativity, we know that there is no specific time coordinate for \( M \), and this means we can always choose an appropriate time parameter to describe the trajectory of the particle. The geodesic is obtained as an extremum of the action, by performing the calculus of variation. In this case, \( (M, F) \), \( F = \sqrt{\eta_{\mu\nu} \, dx^\mu \, dx^\nu} \) is the Finsler manifold where we constructed the Lagrangian formulation, \( F \) is the Lagrangian, and the geodesic is a submanifold of \( M \). On the other hand, the non-relativistic description of the free particle is to consider a fibre bundle where the base space \( U \) is the space of time parameter, usually a subset of \( \mathbb{R} \), and total space is a direct product, i.e., \( M = U \times \mathbb{R}^3 \). In this case, the geodesic of the particle is given by a section of this fibre bundle. In this sense, the general covariance is a property arising from the deletion of the bundle structure. The construction of a Lagrangian formulation on a Kawaguchi manifold \( (M, K) \) does the same for the case of field theory. We have removed the fibre bundle structure of the standard field theory (where the base space was given by \( U \in \mathbb{R}^4 \) and total space by \( M = U \times \Sigma \)), and instead of considering the field configuration as its section, gave it as a 4-dimensional submanifold of the total space \( M \). Performing calculus of variation determines the spacetime as the extremal. The extended covariance is a property obtained by deleting the bundle structure.

### 4.3. Dirac field theory

The next example is the theory of free Dirac field. The conventional Lagrangian is given by \( L = \frac{1}{2} \left( \bar{\psi} \gamma^\mu \partial_\mu \psi + \bar{\psi} \gamma^\nu \gamma^\nu \right) - m \bar{\psi} \psi \), where \( \psi \) is a spinor, and \( \bar{\psi} := \psi^0 \) is its Dirac conjugate. We also suppressed the indices, such as \( \psi = (\psi^A) \), \( \bar{\psi} = \psi^0 = (\bar{\psi}_A) \), \( \gamma^\mu \psi =
The Kawaguchi manifold becomes

\[ M = \{ (x^\mu, \psi) \} \cong \mathbb{R}^4 \times \mathbb{C}^4, \]

\[ K = \frac{1}{2 \cdot 3!} \left( \bar{\psi} \gamma_{\mu \rho} \partial x^{\mu \rho} \wedge \partial \psi - d \bar{\psi} \wedge d x^{\mu \rho} \gamma_{\mu \rho} \gamma^5 \psi \right) - m \bar{\psi} \psi d x^{0123}, \]

with the convention \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \) and \( \gamma_{\mu \rho} = \gamma_\mu \gamma_\rho \) (\( \gamma_012 = \gamma_0 \gamma_1 \gamma_2 \), \( \gamma_{011} = 0 \) etc). The Euler–Lagrange equations are derived by using (24),

\[ 0 = d \left\{ - \frac{\bar{\psi} \gamma_{\mu \rho} \partial x^{\mu \rho} \wedge \partial \psi + d \bar{\psi} \wedge d x^{\mu \rho} \gamma_{\mu \rho} \gamma^5 \psi}{2 \cdot 2!} + \frac{1}{3!} \epsilon_{\mu \nu \rho \sigma} m \bar{\psi} \psi d x^{\mu \rho \sigma} \right\}, \]

(35)

\[ 0 = \frac{1}{3!} \gamma_{\mu \rho} \partial x^{\mu \rho} \wedge \partial \psi - m \psi d x^{0123}, \]

(36)

\[ 0 = - \frac{1}{3!} d \bar{\psi} \wedge d x^{\mu \rho} \gamma_{\mu \rho} \gamma^5 - m \bar{\psi} d x^{0123}. \]

Equation (36) represents the conservation law of the energy-momentum current of electromagnetic fields, and we can define the current by

\[ \text{Equation (40) represents the conservation law of the energy-momentum current of electromagnetic field, and we can define the current by} \]

Since spinors are Grassmann variables, note that differentiation with respect to \( \psi (\bar{\psi}) \) must be taken by the right (left) derivatives. Equations (36) are the conservation laws of energy-momentum currents. As in the previous examples, the coordinates \( x^\mu, (\mu = 0, 1, 2, 3) \) are cyclic coordinates, and this is the reason we can see the conservation law directly as a part of the Euler–Lagrange equations. Similar discussions will follow for the choice of arbitrary parameters and the relation to the conventional theory.

### 4.4. Electromagnetic field theory

From the conventional Lagrangian of free electromagnetic fields, \( L = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} \), we obtain our Kawaguchi manifold as

\[ M = \{ (x^\mu, A_\mu) \} \cong \mathbb{R}^4, \quad K = \frac{(\tilde{F} \wedge d x^{\rho \sigma})(\tilde{F} \wedge d x^{\rho \sigma})}{4 d x^{0123}}, \quad \left( d x^{0123} \neq 0 \right), \]

(39)

where \( \tilde{F} = d A_\mu \wedge d x^\mu \). The Euler–Lagrange equations are derived as

\[ 0 = d \left\{ \frac{\tilde{F} \wedge d x^{\rho \sigma} \tilde{F} \wedge d x^\rho + \epsilon_{\mu \rho \sigma} \left( \tilde{F} \wedge d x^{\mu \sigma} \right) \left( \tilde{F} \wedge d x^{\mu \sigma} \right)}{4 \cdot 3! \left( d x^{0123} \right)^2} d x^{\mu \rho \sigma} + \frac{\tilde{F} \wedge d x_{\mu \rho} d A_\sigma \wedge d x^{\mu \rho}}{2 d x^{0123}} \right\}, \]

(40)

\[ 0 = d \left( \frac{\tilde{F} \wedge d x^{\rho \sigma}}{2 d x^{0123}} d x^{\rho \sigma} \right). \]

Equation (40) represents the conservation law of the energy-momentum current of electromagnetic field, and we can define the current by
The pullback of equations (40) and (41) to the parameter space by \( \sigma(x) \) is
\[
0 = d \left( -\frac{1}{4} \epsilon^{\rho\sigma}_{\mu\nu} F_{\rho \mu} F_{\nu \sigma} dx^{\rho \sigma} + \frac{1}{4} \cdot \frac{3!}{4} \epsilon^{\rho\sigma}_{\mu\nu} F_{\rho \mu} F_{\nu \sigma} dx^{\rho \sigma} + \frac{1}{4} \epsilon^{\rho\sigma}_{\mu\nu} F_{\rho \mu} F_{\nu \sigma} dA_{\mu} \wedge d\chi^{\rho \sigma} \right).
\]
(43)
\[
0 = -\partial_{\mu} F^{\mu \nu} dx^{0123}.
\]
(44)
The last term of the pulled-back current (43) is not gauge invariant with respect to the usual gauge transformation \( \chi \rightarrow A_{\mu} + \partial_{\mu} \chi \). However, by using (44), this term becomes an exact term,
\[
\frac{1}{4} \epsilon^{\rho\sigma}_{\mu\nu} F_{\rho \mu} F_{\nu \sigma} A_{\mu} \wedge d\chi^{\rho \sigma} = d \left( \frac{1}{4} \epsilon^{\rho\sigma}_{\mu\nu} F_{\rho \mu} F_{\nu \sigma} A_{\mu} \wedge d\chi^{\rho \sigma} \right).
\]

4.5. Maxwell–Dirac field theory

Here we will combine the last two examples, and consider the Dirac field interacting with the electromagnetic field. The Kawaguchi manifold is given by
\[
M = \left\{ (x^{\mu}, A_{\mu}, \psi, \bar{\psi}) \right\} \simeq \mathbb{R}^{8} \times \mathbb{C}^{4}, \quad K = K_{\text{Maxwell}} + K_{\text{Dirac}},
\]
(45)
where
\[
K_{\text{Maxwell}} := \frac{(\vec{F} \wedge dx^{\rho}) (\vec{F} \wedge dx^{\rho})}{4 dx^{0123}}, \quad (dx^{0123} \neq 0).
\]
(46)
\[
K_{\text{Dirac}} := \frac{1}{2 \cdot 3!} \left( \bar{\psi} \gamma^{\lambda}_{\mu\nu\rho} dx^{\mu \nu \rho} \wedge D\psi - D\bar{\psi} \wedge dx^{\mu \nu \rho} \gamma^{\lambda}_{\mu\nu\rho} \right) - m \bar{\psi} \psi dx^{0123}.
\]
(47)
The covariant derivatives are defined by
\[
D \psi = d\psi - ieA_{\mu} dx^{\mu} \psi \quad \text{and} \quad D \bar{\psi} = d\bar{\psi} + ieA_{\mu} dx^{\mu} \bar{\psi}.
\]
The Euler–Lagrange equations become
\[
0 = d \left\{ -\frac{\vec{F} \wedge dx^{\mu}}{dx^{0123}} \vec{F} \wedge dx^{\rho} - \epsilon^{\rho\sigma}_{\mu\nu} \frac{(\vec{F} \wedge dx^{\rho})(\vec{F} \wedge dx^{\sigma})}{4 \cdot 3! \left(dx^{0123}\right)^2} dx^{\rho \sigma} + \frac{\vec{F} \wedge dx^{\rho}}{2 dx^{0123}} dA_{\mu} \wedge dx^{\rho} \right\} - \frac{\bar{\psi} \gamma^{\lambda}_{\mu\nu\rho} dx^{\mu \nu \rho} \wedge D\psi + D\bar{\psi} \wedge dx^{\mu \nu \rho} \gamma^{\lambda}_{\mu\nu\rho}}{2 \cdot 2!} - \frac{1}{3!} \epsilon^{\rho\sigma}_{\mu\nu} m \bar{\psi} \psi dx^{\rho \sigma} + \frac{1}{3!} ie\bar{\psi} \gamma^{\lambda}_{\mu\nu\rho} \gamma^{5} \gamma^{\lambda}_{\mu\nu\rho} \psi dx^{\rho \sigma}
\]
(48)
\[
0 = \frac{1}{3!} ie\bar{\psi} \gamma^{\lambda}_{\mu\nu\rho} dx^{\mu \nu \rho} \psi - d \left\{ \frac{\vec{F} \wedge dx^{\rho}}{2 dx^{0123}} dx^{\rho} \right\}.
\]
(49)
Equation (48) expresses the energy-momentum conservation law of Maxwell–Dirac field theory. This Kawaguchi metric has a gauge symmetry described by the vector field,

\[ \mathcal{G} = \frac{\partial}{\partial \psi} (ie \Lambda \psi) - ie \psi \Lambda \frac{\partial}{\partial \psi} + \frac{\partial A_{\mu}}{\partial x_{\mu}}, \]  

where \( \Lambda = \Lambda(x^\mu) \) is an arbitrary function of \( x^\mu \). The corresponding transformation is the usual gauge transformation we are familiar with:

\[ \delta \psi = i e \Lambda \psi \delta \psi + \frac{\partial A_{\mu}}{\partial x_{\mu}} \delta x_{\mu}, \]

\[ \delta F = \frac{\partial F}{\partial x_{\mu}} \delta x_{\mu} + \frac{\partial \Lambda}{\partial x_{\mu}} \delta x_{\mu} \]

\[ \delta \bar{F} = - \frac{\partial \bar{F}}{\partial x_{\mu}} \delta x_{\mu} + \frac{\partial \Lambda}{\partial x_{\mu}} \delta x_{\mu}, \]

\[ \delta \bar{F}_0 = \frac{\partial \bar{F}_0}{\partial x_{\mu}} \delta x_{\mu} + \frac{\partial \Lambda}{\partial x_{\mu}} \delta x_{\mu}. \]

Taking the variation of the Kawaguchi metric by the vector field \( \mathcal{G} \) under on-shell conditions generates a conserved current:

\[ J_x^\mu = \frac{\mathcal{L}_C \psi^2 \gamma_{\mu\rho\sigma}^x + \psi_{\mu\rho\sigma}^x \mathcal{L}_C \psi^2}{2 \cdot 3!} d x^{\mu\rho\sigma} + \mathcal{L}_C A_{\mu} \frac{\bar{F}}{2} \wedge d x^{\rho\sigma} + \frac{\partial A_{\mu}}{\partial x_{\mu}} \frac{3! d x^{\mu\rho\sigma}}{2}. \]

Its exterior derivative becomes

\[ 0 = d J_x^\mu = \Lambda d \left\{ \frac{\psi \gamma_{\mu\rho\sigma}^x \psi^2}{3!} d x^{\mu\rho\sigma} \right\} + \frac{\partial A_{\mu}}{\partial x_{\mu}} \left\{ \frac{\psi \gamma_{\mu\rho\sigma}^x \psi^2}{3!} d x^{\mu\rho\sigma} \right\} \]

\[ + d \left( \frac{\bar{F}}{2} \wedge d x^{\rho\sigma} + \frac{\partial A_{\mu}}{\partial x_{\mu}} \frac{3! d x^{\mu\rho\sigma}}{2} \right). \]

This is Noether’s theorem. Since the functions \( \Lambda \) and \( \frac{\partial A_{\mu}}{\partial x_{\mu}} \) are arbitrary, we have the electric charge conservation law,

\[ d J_x = 0, \quad J_x = - i e \frac{\psi \gamma_{\mu\rho\sigma}^x \psi^2}{3!} d x^{\mu\rho\sigma}, \]

and Maxwell equations (50).

5. Application to general relativity

Application to the Hilbert action of Einstein’s general relativity requires a more generalized Kawaguchi manifold; higher-derivative areal space, since the action includes second-order derivatives. It takes two steps to define higher-derivative areal space: a higher-derivative extension of Finsler metric, and an areal extension of the former.

A higher-derivative Kawaguchi metric defines the length of the oriented curve on manifold \( M \) as a function of higher-order derivatives. A second-order Finsler metric, which is usually called a Kawaguchi metric, \( F(x^\mu, d x^\mu, d^2 x^\mu) \), for instance, is defined as a function of \( x^\mu, d x^\mu, d^2 x^\mu \) (\( \mu = 0, 1, 2, ..., n \)), which satisfies the second-order homogeneity condition,
Differentiation of the above condition with respect to $\lambda$ and $\xi$ gives
\[
\frac{\partial F}{\partial x^\mu} + 2 \frac{\partial F}{\partial x^\nu} \frac{d^2 x^\mu}{d\lambda^2} = F, \tag{58}
\]
\[
\frac{\partial F}{\partial d^2 x^\mu} \frac{d x^\mu}{d\xi} = 0, \tag{59}
\]
after setting $\lambda = 1$, $\xi = 0$, respectively. This is the the well-known Zermelo’s condition \[12, 18, 26, 27\].

Condition (57) implies Zermelo’s condition. A parameterization $c(s) : [s_0, s_1] \rightarrow c \subset M$ of the oriented curve $c$ determines the pullback of $x^\mu, \frac{dx^\mu}{ds}, \frac{d^2 x^\mu}{d\lambda^2}$ as
\[
c(s)^n x^\mu = x^\mu(c(s)) := x^\mu(s), \quad c(s)^n \frac{dx^\mu}{ds} = \frac{dx^\mu(s)}{ds}, \quad c(s)^n \frac{d^2 x^\mu}{d\lambda^2} = \frac{d^2 x^\mu(s)}{d\lambda^2} ds^2, \tag{60}
\]
and the pullback of $F$ is given by
\[
c^s F := F\left(c^s x^\mu, c^s \frac{dx^\mu}{ds}, c^s \frac{d^2 x^\mu}{d\lambda^2}\right) = F\left(\frac{dx^\mu(s)}{ds}, \frac{d^2 x^\mu(s)}{d\lambda^2} ds^2\right) ds. \tag{61}
\]
The length of the oriented curve $c$ is then defined by
\[
A[c] = \int_{s_0}^{s_1} F\left(\frac{dx^\mu(s)}{ds}, \frac{d^2 x^\mu(s)}{d\lambda^2} ds^2\right) ds. \tag{62}
\]
In fact, another parameterization $\tilde{c}(t) : [t_0, t_1] \rightarrow c$ yields
\[
\int_{t_0}^{t_1} F\left(\frac{dx^\mu(t)}{dt}, \frac{d^2 x^\mu(t)}{dt^2}\right) dt = \int_{t_0}^{t_1} F\left(\frac{dx^\mu(t)}{dt}, \frac{d^2 x^\mu(t)}{dt^2}\right) dt,
\]
which corresponds to (63), thanks to the homogeneity condition (57).

The second-order Kawaguchi metric (second-order areal metric) $K_{\alpha\beta\gamma\delta}$ is a function of $z^a$, $dz^{abcd}$, $dz_{efg} \wedge dz_{abcd}$, where $z^a$ are coordinate functions of a differentiable manifold $M$. The last argument expresses the second-order derivatives by our notation. In order to keep the reparameterization invariance, it is necessary that the Kawaguchi metric satisfy the following condition
\[
K(z^a, dz^{abcd}, \lambda dz_{efg} \wedge dz_{abcd} + \mu efgh dz_{abcd}) = \lambda K(z^a, dz^{abcd}, dz_{efg} \wedge dz_{abcd}), \quad \forall \lambda > 0, \forall \mu efgh \in \mathbb{R}. \tag{65}
\]
Here, $\lambda > 0$ is the arbitrary constant that also appeared in the case the of first-order homogeneity condition, and $\mu efgh$ are the constants in accord with the second-order derivatives. They are completely antisymmetric in the superscripts.
We call the pair \((M, K)\) a second-order Kawaguchi manifold, and \((65)\) the second-order homogeneity condition. The second-order homogeneity condition guarantees the Kawaguchi manifold the important property of reparameterization invariance.

From \((65)\), we can obtain the generalized version of Zermelo’s condition to areal spaces simply by differentiating the equation \((65)\) with respect to \(\lambda\) and \(\varepsilon_{\mu}^g\) and then setting \(\lambda = 1, \mu^g = 0;\)

\[
\frac{1}{4!} \frac{\partial K}{\partial z_{abcd}} d_{z_{abcd}} + \frac{2}{3!4!} \frac{\partial K}{\partial z_{g}^e} \wedge d_{z_{abcd}}^2 = K, \tag{66}
\]

\[
\frac{\partial K}{\partial z_{g}^e} \wedge d_{z_{abcd}}^2 = 0. \tag{67}
\]

Let \(\sigma\) be an oriented 4-dimensional submanifold embedded in \(M\), and its parameterization given by \(\sigma_0(s^0, s^1, s^2, s^3): W_0 \subset \mathbb{R}^4 \to \sigma \subset M\). Our second-order variable \(z_{g}^e\) is related to the standard second-order derivative by the pullback of \(\sigma_0\) defined by

\[
\sigma_0^*\left( z_{g}^e \wedge d_{z_{abcd}}^2 \right) = \frac{\partial}{\partial (s^0, s^1, s^2, s^3)} (ds^0)^2. \tag{68}
\]

Now, let \(\sigma_1(t^0, t^1, t^2, t^3): W_1 \subset \mathbb{R}^4 \to \sigma\) be another parameterization of \(\sigma\), and suppose we have an orientation preserving diffeomorphism \(f: W_1 \to W_0\), such that \(\sigma_1 = \sigma_0 \circ f\). Then, the pullback of \(\sigma_0^*\left( z_{g}^e \wedge d_{z_{abcd}}^2 \right)\) by \(f\) becomes

\[
f^* \circ \sigma_0^*\left( z_{g}^e \wedge d_{z_{abcd}}^2 \right) = \frac{\partial}{\partial (t^0, t^1, t^2, t^3)} (dt^0)^2 + \frac{\partial}{\partial (t^0, t^1, t^2, t^3)} (dt^0)^2 \tag{69}
\]

The rhs is equal to \(\sigma_1^*\left( z_{g}^e \wedge d_{z_{abcd}}^2 + \mu^g d_{z_{abcd}}^2 \right)\), i.e., the standard relation \(\sigma_1^* = f^* \circ \sigma_0^*\) does not hold for this variable, due to the non-linearity of \(z_{g}^e\) and \(d_{z_{abcd}}^2\). Next, we will define the pullback of the second-order Kawaguchi metric by \(\sigma_0(s)\) such that, \(\sigma_0^* K := K\left( \sigma_0^* z_{g}^e, \sigma_0^* d_{z_{abcd}}^2, \sigma_0^* z_{g}^e \wedge d_{z_{abcd}}^2 \right)\). This is a 4-form on \(W_0\). We will further pullback this \(\sigma_0^* K\) to a 4-form on \(W_1\) by \(f\). By considering the homogeneity condition \((65)\) of \(K\) and the relation \((69)\), we find
\[ f^* \circ \sigma_0 (s)^* K = \sigma_0(t)^* K, \]  
(70)
despite the non-linearity of the second-order variables. This property indicates that, as in the case of Finsler or the first-order Kawaguchi metric, the integration of the second-order Kawaguchi metric \( K \) over \( \sigma \) also gives a reparameterization-invariant area for an oriented 4-dimensional submanifold of \( M \):

\[ A[\sigma] = \int_\sigma K = \int_W K \left( \sigma^* z^a, \sigma^* \left( d_z^{abcd} \right), \sigma^* \left( d_{\varepsilon^{efg}} \wedge d^2 z^{abcd} \right) \right). \]  
(71)

Suppose we are given the usual Lagrangian of second-order field theory,

\[ L(u^a, du^a, \frac{d^2 u^a}{dx^a}) \]  
then we can construct the second-order Kawaguchi metric by

\[ K(z^a, dz^{abcd}, d_{\varepsilon^{efg}} \wedge d^2 z^{abcd}) = \frac{L}{3!} \frac{\epsilon_{\mu
u\rho\sigma}}{3!} \frac{du^\mu}{dx^{0123}} \frac{du^\nu}{dx^{0123}} \frac{du^\rho}{dx^{0123}} \frac{du^\sigma}{dx^{0123}} \left( \frac{d}{dx^{0123}} \right) \left( \frac{d}{dx^{0123}} \right)^2, \]  
(72)

where \( \epsilon^{0123} = 1 \), \( \epsilon_{0123} = -1 \), and \( (x^a) = (x^\mu, u^a) \). The second-order variable in (72) is a short expression:

\[ dx^{\varepsilon^{efg}} \wedge d \left( \frac{du^{op}}{dx^{0123}} \wedge du^a \right) = dx^{\varepsilon^{efg}} \wedge d \left( dx^{0123} \right) - \left( dx^{op} \wedge du^a \right) dx^{\varepsilon^{efg}} \wedge d^2 x^{0123} \]

One can check that the Kawaguchi metric constructed in this way satisfies the homogeneity condition (65), and together with \( M = \{ (x^\mu, u^a) \} \), we obtain the second-order Kawaguchi manifold, \((M, K)\).

The Lagrangian of the vacuum general relativity with cosmological constant \( \lambda \) is given by

\[ L = \sqrt{-g} \left( -\frac{r}{2\kappa} - \frac{\lambda}{\kappa} \right), \]  
(73)

where \( \kappa = \frac{8G}{c^4} \), \( R_{\mu\nu} = R^\rho_{\mu\nu\rho} \), \( r = g^{\mu\nu} R_{\mu\nu} = R^\rho_{\mu\nu\rho} \), with all Greek indices running from 0 to 3. The Kawaguchi manifold \((M, K)\) constructed from this Lagrangian is

\[ M = \{ (x^\mu, g^{\mu\nu}) \} \approx \mathbb{R}^{14}, \]  
(74)

\[ K(z^a, dz^{abcd}, d_{\varepsilon^{efg}} \wedge d^2 z^{abcd}) = \frac{1}{4\kappa} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} \tilde{R}^{\mu\nu} \wedge dx^{\rho\sigma} - \frac{\lambda}{\kappa} \sqrt{-g} dx^{0123}, \]  
(75)

\[ \tilde{R}^{\mu\nu} := g^{\xi\zeta} \tilde{R}^\xi_{\zeta}, \]  
\[ \tilde{R}^\xi_{\zeta} := d\tilde{F}^\xi_{\zeta} + \tilde{F}^\lambda_{\zeta} \wedge \tilde{F}^\lambda_{\xi}, \]  
\[ \tilde{F}^\mu_{\xi} := \epsilon_{\xi\eta\zeta} dx^{\xi\eta\zeta} \]  
(76)

Latin indices runs from 0 to 13, and if we use the unified coordinate system, then \( \{ z^a \} \), \( (dz^{abcd}) \) denotes \( (dx^{0123}, dx^{op} \wedge dg^\mu_{\xi}) \) and \( (dz^{efg} \wedge d^2 z^{abcd}) \) denotes \( (dx^{\varepsilon^{efg}} \wedge d^2 x^{0123}, dx^{op} \wedge dg^\mu_{\xi}) \).

\[ \left( dx^{\varepsilon^{efg}} \wedge d^2 x^{0123}, dx^{op} \wedge dg^\mu_{\xi} \right) \]  
(77)
Here we emphasize that in our framework of the Kawaguchi manifold, the variable \( g^{\mu\nu} \) which conventionally corresponds to the metric of the Riemannian manifold, is merely treated as a coordinate function, similar to the spacetime coordinates \( x^\mu \). Each of the 10 components of the symmetric matrix \( g^{\mu\nu} \) represents an independent coordinate function, and the variable \( g_{\mu\nu} \) is the inverse of this symmetric matrix \( g^{\mu\nu} \). In this sense, \( g^{\mu\nu} \) does not represent any geometric structure. The Kawaguchi metric is the only geometrical structure we need.

Before proceeding, let us check if whether this Kawaguchi metric is a plausible one. We pullback \( K \) by the spacetime parameterization \( \sigma(x) \), which we used to verify the case of scalar field theory. The pullback by \( \sigma(x) \) actually corresponds to consideration of the variables \( g^{\mu\nu} \) as dependent variables of \( x^{\mu} \). In this way, the pullback of (76) becomes the usual curvature tensor, \( \sigma^{\mu} K = \sqrt{-g} \left( -\frac{r}{2\kappa} - \frac{\lambda}{\kappa} \right) dx^{0123} \), and the Kawaguchi metric becomes

\[
\delta K = \frac{1}{4\kappa} \sqrt{-g} \left( \frac{1}{2} \epsilon_{\mu\rho\sigma} g_{\xi\eta} \tilde{R}^{\rho\sigma} \wedge dx^{\xi\eta} + \epsilon_{\mu\rho\sigma} \tilde{R}^{\rho\sigma} \wedge dx^{\xi\eta} + 2\lambda g_{\xi\eta} dx^{0123} \right) \delta g^{\xi\eta}
\]

which is the standard Einstein–Hilbert Lagrangian 4-form.

The general expressions of Euler–Lagrange equations can be obtained by considering the variational principle. However, in some cases, it is much easier to take the variation of the concrete Kawaguchi action directly, and we will take this approach. Remember, that in the covariant Lagrangian formulation, taking the variation \( \delta \) means to take the Lie derivative with respect to arbitrary \( X \in \Gamma(TM) \), and the Lie derivative is commutative with \( d \). For visibility, we will omit the pullback symbol \( \star \) in the following discussion.

The variation of \( K \) becomes

\[
\delta K = \frac{1}{4\kappa} \sqrt{-g} \left( \frac{1}{2} \epsilon_{\mu\rho\sigma} g_{\xi\eta} \tilde{R}^{\rho\sigma} \wedge dx^{\xi\eta} + \epsilon_{\mu\rho\sigma} \tilde{R}^{\rho\sigma} \wedge dx^{\xi\eta} + 2\lambda g_{\xi\eta} dx^{0123} \right) \delta g^{\xi\eta}
\]

\[
+ d \left( \frac{1}{4\kappa} \epsilon_{\mu\rho\sigma} \sqrt{-g} g^{\xi\delta} \delta \tilde{R}^{\rho\sigma} \wedge dx^{\xi\delta} \right)
\]

\[
+ \sqrt{-g} \left( \epsilon_{\mu\rho\sigma} g^{\xi\eta} \tilde{F}^{\rho\sigma} \wedge dx^{\xi\eta} - \epsilon_{\mu\rho\sigma} g^{\xi\eta} \tilde{F}^{\rho\sigma} \wedge dx^{\xi\eta} \right)
\]

\[
- d \left( \frac{1}{2\kappa} \epsilon_{\mu\rho\sigma} \sqrt{-g} \left( \tilde{R}^{\rho\sigma} \wedge dx^{\rho} + \frac{2\lambda}{3!} dx^{\mu\rho}\right) \delta x^{\rho} \right)
\]

\[
+ d \left( \frac{1}{2\kappa} \epsilon_{\mu\rho\sigma} \sqrt{-g} \left( \tilde{R}^{\rho\sigma} \wedge dx^{\rho} + \frac{2\lambda}{3!} dx^{\mu\rho}\right) \delta x^{\rho} \right).\]

The Euler–Lagrange equations described by the pullback of the parameterization \( \sigma: W \subset \mathbb{R}^3 \rightarrow M \) are the conditions for 4-dimensional submanifold \( \sigma \) to be an extremal submanifold of \( A[\sigma] \). We can use the following conditions to simplify the terms of \( \delta K \):

\[
\tilde{F}^{\mu}_{\rho\nu} = \tilde{F}^{\nu}_{\rho\mu}, \quad d g^{\rho\nu} - g^{\rho}_{\nu\sigma} \tilde{F}^{\sigma}_{\mu} - g^{\rho}_{\mu\sigma} \tilde{F}^{\sigma}_{\nu} = 0,
\]

where the sign \( \equiv \) means the equality on the 4-dimensional submanifold \( \sigma \) embedded in \( M \) (ref. Appendix A), and the second equality holds by
\[
g_{\alpha\beta} \tilde{F}_\mu^\alpha + g_{\mu\alpha} \tilde{F}_\nu^\alpha = \tilde{F}_{\mu\alpha} + \tilde{F}_{\nu\mu} = \left( \tilde{F}_{\mu\nu} + \tilde{F}_{\nu\mu} \right) \mathrm{d}x^\rho = \frac{\varepsilon_{\mu\nu\rho} \mathrm{d}x^{\rho}}{3!} \wedge \frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}x^{0123}} \wedge \mathrm{d}x^\rho
\]

\[
\varepsilon = -\frac{\varepsilon_{\mu\rho\nu}}{3!} \left( \frac{\mathrm{d}g_{\mu\nu} \wedge \mathrm{d}x^\rho}{\mathrm{d}x^{0123}} - \frac{\mathrm{d}x^\rho \wedge \mathrm{d}g_{\mu\nu}}{\mathrm{d}x^{0123}} + \frac{\varepsilon_{\mu\rho\nu} \mathrm{d}g_{\mu\nu} \wedge \mathrm{d}x^\rho}{\mathrm{d}x^{0123}} \right)
\]

\[
+ \frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}x^{0123}} \frac{\varepsilon_{\mu\rho\nu}}{3!} \left( \frac{\mathrm{d}g_{\mu\nu} \wedge \mathrm{d}x^\rho}{\mathrm{d}x^{0123}} - \frac{\mathrm{d}x^\rho \wedge \mathrm{d}g_{\mu\nu}}{\mathrm{d}x^{0123}} \right) + \frac{\varepsilon_{\mu\rho\nu}}{4!} \frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}x^{0123}} \frac{\varepsilon_{\mu\rho\nu}}{\mathrm{d}g_{\mu\nu}}.
\]

(81)

The term \( \delta \tilde{F}_{\mu}^\alpha \) in equation (79) becomes zero under these conditions, since

\[
\varepsilon_{\mu\nu\rho} \left( \frac{1}{\sqrt{-g}} g_{\alpha\beta} \varepsilon^{\alpha\beta} \mathrm{d}x^\rho \right) + \frac{1}{\sqrt{-g}} \left( \varepsilon_{\mu\nu\rho\sigma} g_{\alpha\beta} \tilde{F}_{\sigma}^{\alpha} - \varepsilon_{\mu\rho\sigma\nu} g_{\alpha\beta} \tilde{F}_{\sigma}^{\alpha} \right) \wedge \mathrm{d}x^\sigma
\]

\[
= \varepsilon_{\mu\nu\rho} \frac{1}{\sqrt{-g}} \left( \frac{1}{2} g_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} - g_{\mu\nu} g_{\alpha\beta} g^{\alpha\beta} \right) \wedge \mathrm{d}x^\sigma
\]

\[
+ \frac{1}{\sqrt{-g}} \left( \varepsilon_{\mu\rho\nu} g_{\alpha\beta} \tilde{F}_{\alpha}^{\nu} - \varepsilon_{\mu\nu\rho} g_{\alpha\beta} \tilde{F}_{\alpha}^{\nu} \right) \wedge \mathrm{d}x^\sigma = 0,
\]

where we have used (80), substituted \( \tilde{F}_{\mu}^\alpha = \tilde{F}_{\mu}^{\alpha} \mathrm{d}x^\alpha \), and then used the Hodge star relation,

\[
(\ast \mathrm{d}x^\sigma) := \frac{1}{3!} \sqrt{-g} g^{\alpha\beta} \varepsilon_{\alpha\beta} \mathrm{d}x^\sigma, \quad \mathrm{d}x^{\mu\rho} = \frac{1}{\sqrt{-g}} \varepsilon_{\mu\rho\sigma} g^{\sigma\delta} (\ast \mathrm{d}x^\delta).
\]

Consequently, we obtain the Euler–Lagrange equations as

\[
0 = \frac{1}{2 \kappa} \varepsilon_{\mu\rho\sigma} \sqrt{-g} \left( \tilde{R}_{\rho\sigma} - \frac{2}{3!} \mathrm{d}x^{\mu\rho\sigma} \right)
\]

(84)

\[
0 = -\frac{1}{2} \left( \varepsilon_{\mu\rho\sigma} g^{\sigma\delta} \tilde{R}_{\delta}^{\rho} \right) \wedge \mathrm{d}x^\sigma + 2 \left( \varepsilon_{\mu\rho\sigma} \tilde{R}_{\delta}^{\rho} + \varepsilon_{\mu\sigma\rho} \tilde{R}_{\rho}^{\sigma} \right) \wedge \mathrm{d}x^\sigma + 2 g_{\rho\sigma} \lambda \mathrm{d}x^{0123}.
\]

(85)

The pullback of these equations by \( \sigma (x) \) are

\[
0 = \frac{1}{\kappa} \left( G_{\alpha\beta} - \lambda g_{\alpha\beta} \right) \left( \ast \mathrm{d}x^\gamma \right), \quad G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R,
\]

(86)

\[
0 = \left( r g_{\sigma\eta} - 2 R_{\sigma\eta} + 2 \lambda g_{\sigma\eta} \right) \mathrm{d}x^{0123} = -2 \left( G_{\sigma\eta} - \lambda g_{\sigma\eta} \right) \mathrm{d}x^{0123},
\]

(87)

where \( \ast \) is the Hodge operator (83).

Equation (87) is the usual Einstein equation, and therefore, we may say that (85) is the Einstein equation with extended covariance. By the discussions in the previous section, equation (84) coming from the variation with respect to \( x^\mu \) should be considered as a conservation law of the energy-momentum current. Let us define by \( J^G \) the energy-momentum current of the gravitational field, and by \( J^J \), that of the cosmological term, namely

\[
J^G := \frac{1}{2 \kappa} \varepsilon_{\mu\rho\sigma} \sqrt{-g} \tilde{R}_{\rho\sigma} \wedge \mathrm{d}x^\sigma, \quad J^J := \frac{1}{3 \kappa} \lambda \varepsilon_{\mu\rho\sigma} \sqrt{-g} \mathrm{d}x^{\mu\rho\sigma}.
\]

(88)
Then, the equation (84) says that the total energy-momentum current, \( J_\sigma = J_\sigma^G + J_\sigma^i \), satisfies the covariant energy-momentum conservation law,\( 0 = dJ_\sigma \). To consider on the parameter space, namely, in the \( x^\mu \) coordinates, take the pullback by \( \sigma (x) \),

\[
0 = d \left( J_\sigma^G + J_\sigma^i \right), \quad J_\sigma^G := \sigma^* J_\sigma^G \equiv \frac{1}{\kappa} G_{\alpha \xi} \left( \ast dx^\xi \right), \quad J_\sigma^i := \sigma^* J_\sigma^i = - \frac{\lambda}{\kappa} g_{\alpha \xi} \left( \ast dx^\xi \right). \quad (89)
\]

The above expression of the energy-momentum of general relativity is one of the main results of the application of the covariant Lagrangian formulation.

Among these equations (86) and (87), six equations are mutually independent, and the conventional view is to choose them from the Einstein equations (87). Actually, when the Einstein equation (87) holds, the total energy-momentum current \( J_\sigma \) is zero, and its conservation equation (86) is automatically satisfied. Does this mean that the equation (89) is a tautology? We claim that this is not the case. Remember that the conservation law was obtained as a part of the Euler–Lagrange equations. In the theory of extended covariance, there are no differences in their importance.

In such an extended covariant perspective, Einstein’s general relativity was just one case where a specific choice of parameterization was made. The same goes for the choice of equation of motions. The equations (85) that correspond to the balancing of the stress energy-momentum tensor were merely one choice for the fundamental equations, and there is no reason not to choose the others (84). Actually, by using the relations

\[
g_{\alpha \beta} = g_{\alpha \xi} \Gamma^\xi_{\alpha \beta} + g_{\alpha \xi} \Gamma^\xi_{\beta \alpha}, \quad dR^\mu \nu + \Gamma^\mu_{\xi \nu} \wedge R^\xi_{\lambda} + \Gamma^\xi_{\nu \lambda} \wedge R^\mu_{\lambda} = 0, \quad dR^\mu_{\nu \rho} + \Gamma^\mu_{\lambda \rho} \wedge R^\lambda_{\nu \lambda} = 0,
\]

the equation (86) becomes

\[
d \left\{ \epsilon_{\mu \nu \rho \sigma} \sqrt{-g} \left( R^\mu_{\nu \alpha} \wedge dx^\rho + \frac{2 \lambda}{3!} dx^\mu_{\nu \rho} \right) \right\} \\
= \epsilon_{\mu \nu \rho \sigma} \sqrt{-g} d g_{\alpha \beta} \wedge \left( R^\mu_{\nu \alpha} \wedge dx^\rho + \frac{2 \lambda}{3!} dx^\mu_{\nu \rho} \right) + \epsilon_{\mu \nu \rho \sigma} \sqrt{-g} dR^\mu_{\nu \rho} \wedge dx^\sigma, \\
= -2 \sqrt{-g} \Gamma_{\nu \xi}^\mu \left( R_{\mu \xi} - \frac{1}{2} g_{\alpha \beta} g_{\mu \xi} \right) dx^{0123}, \quad (90)
\]

which is just a linear combination of the standard Einstein equations, and is equivalent to the four degrees of freedom (87).

5.1. Gauge symmetry

In our formulation, the general coordinate transformation is simply represented as a geometrical symmetry of the Kawaguchi metric. Let us consider a vector field,

\[
G = f^\mu \frac{\partial}{\partial x^\mu} + \left( \frac{\partial f^\mu}{\partial x^\nu} g^{\nu \rho} + \frac{\partial f^\nu}{\partial x^\rho} g^{\mu \rho} \right) \frac{\partial}{\partial g^{\mu \rho}}, \quad (91)
\]

where \( f^\mu \) are functions of \( x^\mu \). This is a generator of the gauge transformation of the Kawaguchi metric. We can show easily that \( g_{\alpha \beta} dx^\alpha dx^\beta \) is gauge invariant under this transformation, i.e., \( \mathcal{L}_G (g_{\alpha \beta} dx^\alpha dx^\beta) = 0 \). We obtain the following transformation laws:
\[ \mathcal{L}_G \tilde{F}^{\mu \xi}_{\xi} = \left( \mathcal{L}_G g^{\mu \xi} \right) \tilde{F}^{\mu \xi}_{\zeta \eta} \, dx^\eta + g^{\mu \xi} \left( \mathcal{L}_G \tilde{F}^{\mu \xi} \right) \, dx^\eta + g^{\mu \xi} \tilde{F}^{\mu \xi}_{\zeta \eta} \, dL_G(x) \]

\[ \mathcal{L}_G \tilde{F}^{\mu \xi}_{\zeta \eta} = - \left( \partial_\xi \tilde{f}^{\sigma \nu} \right) g^{\sigma \nu} - \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{F}^{\mu \xi}_{\zeta \eta} - \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{F}^{\mu \xi}_{\zeta \eta} \]

\[ \mathcal{L}_G \tilde{F}^{\mu \xi}_{\eta} = - \partial_{\xi f}^{\sigma \nu} \, dx^\eta + \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{F}^{\mu \xi}_{\eta} - \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{F}^{\mu \xi}_{\eta} . \]

Then, we can calculate the transformation of \( \tilde{R}^{\mu \nu} \),

\[ \mathcal{L}_G \tilde{R}^{\mu \nu}_{\xi} = d \left( \mathcal{L}_G \tilde{F}^{\mu \nu}_{\xi} \right) + \left( \mathcal{L}_G \tilde{R}^{\mu \nu}_{\xi} \right) \wedge \tilde{F}^{\lambda \xi} + \tilde{F}^{\mu \lambda} \wedge \tilde{F}^{\nu \xi} \]

\[ = \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\sigma \nu}_{\xi} - \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\sigma \nu}_{\xi} \]

\[ \mathcal{L}_G \tilde{R}^{\mu \nu}_{\xi} = \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\sigma \nu}_{\mu} + \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\mu \nu} . \]

This is equivalent to the standard transformation law of the Riemann curvature. Then, the condition \( \mathcal{L}_G K = 0 \) can be checked as follows:

\[ \mathcal{L}_G K = \left( \mathcal{L}_G \frac{1}{4\kappa} \sqrt{-g} \, \epsilon_{\mu \nu \rho \sigma} \, dx^{\rho \sigma} \right) \wedge \tilde{R}^{\mu \nu} + \frac{1}{4\kappa} \sqrt{-g} \, \epsilon_{\mu \nu \rho \sigma} \, dx^{\rho \sigma} \wedge \left( \mathcal{L}_G \tilde{R}^{\mu \nu} \right) \]

\[ \sigma \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\sigma \nu}_{\mu} = \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\sigma \nu}_{\mu} - 2 \delta_{\mu \nu} \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\sigma \nu}_{\xi} \]

\[ = \sqrt{-g} \{ 2 \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\mu \nu}_{\mu} + 2 \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\mu \nu}_{\nu} - 4 \left( \partial_{\xi f}^{\sigma \nu} \right) \tilde{R}^{\mu \nu}_{\nu} \} \, dx^{0123} \]

where we have used

\[ \tilde{R}^{\mu \nu}_{\xi} = \frac{1}{2} \tilde{R}^{\mu \nu}_{\alpha} dx^{\xi \alpha} , \quad (92) \]

\[ \tilde{R}^{\mu \nu}_{\alpha \beta} = \frac{\epsilon_{\alpha \beta \gamma \delta} \, dx^{\gamma \delta} \wedge \tilde{F}^{\mu \nu}_{\gamma \delta}}{3! \, dx^{0123}} - \frac{\epsilon_{\beta \gamma \delta \alpha} \, dx^{\gamma \delta} \wedge \tilde{F}^{\mu \nu}_{\beta \gamma \delta}}{3! \, dx^{0123}} + \tilde{F}^{\mu \nu}_{\alpha \beta} \tilde{F}^{\alpha \beta} - \tilde{F}^{\mu \nu}_{\alpha \beta} \tilde{F}^{\alpha \beta} \alpha , \quad (93) \]

It is easy to see that the conservation law of the Noether current becomes

\[ 0 = d \left[ \frac{1}{2\kappa} \epsilon_{\mu \nu \rho \sigma} \sqrt{-g} \left( f^{\sigma \nu} \tilde{R}^{\mu \nu}_{\xi} \wedge dx^{\rho \sigma} + \frac{2\lambda}{3!} dx^{\rho \sigma} \right) - \frac{1}{2} \left( \frac{df^{\mu \nu}_{\mu \nu}}{dx^{\xi \eta} \tilde{F}^{\xi \eta}_{\xi \eta}} - \frac{df^{\xi \nu}_{\xi \nu}}{dx^{\xi \eta} \tilde{F}^{\xi \nu}_{\xi \nu}} \right) \wedge dx^{\rho \sigma} \right] \]

\[ + \frac{1}{2} \frac{df^{\mu \nu}_{\mu \nu}}{dx^{\xi \eta} \tilde{F}^{\xi \nu}_{\xi \nu}} dx^{\rho \sigma} \left] \right\} \]

\[ \left(94\right) \]

Here, we used \( \delta K = \mathcal{L}_G K = 0 \) and the covariant Euler–Lagrange equations, (84) and (85). This conservation law can be rewritten as
\[ 0 = f^\sigma d \left\{ \frac{1}{2\kappa} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} \left( R_{\mu\nu} \land dx^\rho + \frac{2\lambda}{3!} dx_{\mu\nu}\right) \right\} \]
\[ - \frac{d^\sigma}{dx^\xi} \left[ \frac{1}{2\kappa} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} \left( R_{\mu\nu} \land dx^\rho + \frac{2\lambda}{3!} dx_{\mu\nu}\right) \right] \]
\[ + \frac{1}{4\kappa} d \left\{ \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} \left( \delta_{\mu}^\nu P_{\sigma}^\rho - g_{\nu}^{\sigma} \left( \right) \right) \land dx^{\rho\sigma} \right\} \]
\[ + \frac{1}{2} \frac{\partial^2 f^\sigma}{\partial x^\xi} \left[ \frac{1}{2\kappa} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} \left( \delta_{\mu}^\nu P_{\sigma}^\rho - g_{\nu}^{\sigma} \left( \right) \right) \land dx^{\rho\sigma} \right] \]
\[ + \frac{1}{2\kappa} d \left\{ \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g_{\nu}^{\sigma} d x^{\rho\sigma} \right\}. \] (95)

Since \( f^\sigma (x^\mu) \) are arbitrary functions of \( x^\mu \), \( f^\sigma \) and its derivative terms must vanish separately. This is the second Noether’s theorem. It means that we can also obtain the conservation law of the energy-momentum current \( J_\mu = J^\mu_0 + J^\mu_\sigma \) from the gauge symmetry (diffeomorphism invariance of general relativity). This is similar to the mechanism we used when we derived the charge conservation law in the Maxwell–Dirac theory from the \( U(1) \) gauge symmetry.

The gauge transformation of the energy-momentum current is
\[ \mathcal{L}_G J^\mu_\sigma = \frac{1}{2\kappa} \epsilon_{\mu\nu\rho\sigma} \left\{ \left( \mathcal{L}_G \sqrt{-g} \right) R_{\mu\nu} \land dx^\rho + \sqrt{-g} \left( \mathcal{L}_G R_{\mu\nu} \right) \land dx^\rho + \sqrt{-g} R_{\mu\nu} \land (\mathcal{L}_G dx^\rho) \right\} \]
\[ = - \frac{1}{\kappa} \left( \partial_\rho f^\sigma \right) \tilde{G}_{\mu\nu} \left( * dx^\rho \right), \] (96)

\[ \mathcal{L}_G J^\mu_\sigma = \frac{1}{3\kappa} \frac{2\epsilon_{\mu\nu\rho\sigma}}{\kappa} \left\{ \left( \mathcal{L}_G \sqrt{-g} \right) dx_{\mu\nu\rho\sigma} + \sqrt{-g} \left( \mathcal{L}_G \left( dx_{\mu\nu\rho\sigma} \right) \right) \right\} \]
\[ = \frac{2}{\kappa} \left( \partial_\rho f^\sigma \right) g_{\rho\sigma} \left( * dx^\rho \right). \] (97)

where we set
\[ \tilde{G}_{\mu\nu} := - \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \tilde{R}^\rho_{\mu\nu} \land dx^\rho + \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \tilde{R}^\rho_{\mu\nu} \land dx^\rho + \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \tilde{R}^\rho_{\mu\nu} \land dx^\rho g_{\rho\sigma}. \] (98)

We obtain \( \mathcal{L}_G J_\mu = \mathcal{L}_G (J^\mu_0 + J^\mu_\sigma) \equiv - \frac{1}{\kappa} \left( \partial_\rho f^\sigma \right) \left( G_{\mu\nu} - \lambda R_{\mu\nu} \right) \left( * dx^\rho \right) \). The energy-momentum current is gauge invariant on the 4-dimensional submanifold \( \sigma \), satisfying the equation of motion (87).

5.2. Einstein-scalar field theory

Here we will combine Einstein’s general relativity and the scalar field theory. We will use the Kawaguchi metric,
\[ K = \frac{1}{4\kappa} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g_{\mu\nu} \tilde{R}_\rho^\sigma \land dx^\rho - \frac{1}{\sqrt{-g}} \left( \frac{d\phi \land dx_{\mu\nu}}{2} \right) \frac{\left( d\phi \land dx_{\mu\nu} \right)}{2 \cdot \sqrt{dx_{123}}} \]
\[ - V(\phi) \sqrt{-g} dx_{123}, \] (99)

where we have defined \( dx_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} dx^{\alpha\beta} \), and the cosmological term is absorbed in the potential term \( V(\phi) \). As in the previous discussion, the variables \( g_{\mu\nu} \) and \( g_{\mu\nu} \) simply represent coordinate functions and functions described by coordinate functions, respectively. The variation of \( K \) now becomes
\[
\delta K = \frac{\sqrt{-G}}{2} \left\{ -\frac{1}{\kappa} \bar{G}_{\eta\eta} + \bar{T}^\phi_{\eta\eta} \right\} \delta g^\eta\eta + d \left[ -\frac{1}{4\kappa} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\zeta} \delta \bar{F}^\mu_{\eta} \wedge dx^{\nu\rho\sigma} \right]\ 
\]
\[
+ \delta \bar{F}^\mu_{\eta} \wedge \left( \sigma^\phi \text{ vanishing term} \right) - d \left[ \left( J^G_{\eta} + J^\phi_{\eta} \right) \delta x^\xi \right] + d \left( J^G_{\eta} + J^\phi_{\eta} \right) \delta x^\xi 
\]
\[
- d \left[ \frac{1}{\sqrt{-g}} \frac{d\phi \wedge dx_{\mu\sigma\rho}}{2!dx^{0123}} - dx_{\mu\rho\sigma} \right] + \delta \phi \left\{ d \left( \frac{1}{\sqrt{-g}} \frac{d\phi \wedge dx_{\mu\rho\sigma}}{3!dx^{0123}} dx_{\mu\rho\sigma} \right) 
\]
\[
- \sqrt{-g} V d x^{0123} \right\}. \tag{100}
\]

where we set
\[
\bar{T}^\phi_{\eta\eta} = \frac{\left( d\phi \wedge dx_{\mu\eta} \right) \left( d\phi \wedge dx_{\eta\nu} \right)}{\left( -g \right) 2!dx^{0123}} + \left\{ -\frac{\left( d\phi \wedge dx_{\mu\rho\sigma} \right) \left( d\phi \wedge dx_{\nu\mu\rho\sigma} \right)}{\left( -g \right) 2 \cdot 3!} \left( dx^{0123} \right)^2 \right\} + V(\phi) g_{\eta\eta} \tag{101}
\]
\[
J^\phi_{\eta} := -\frac{1}{\sqrt{-g}} \left\{ \frac{d\phi \wedge dx_{\mu\rho\sigma}}{2!dx^{0123}} - dx_{\mu\rho\sigma} \right\} + \delta \phi \left\{ \frac{\epsilon_{\mu\rho\sigma} d\phi dx_{\mu\rho\sigma}}{3!} \right\}. \tag{102}
\]

The Euler–Lagrange equations are obtained as
\[
0 = -\frac{1}{\kappa} \bar{G}_{\eta\eta} + \bar{T}^\phi_{\eta\eta} \tag{103}
\]
\[
0 = -\sqrt{-g} V dx^{0123} + d \left( \frac{1}{\sqrt{-g}} \frac{d\phi \wedge dx_{\mu\rho\sigma}}{3!dx^{0123}} dx_{\mu\rho\sigma} \right). \tag{104}
\]
\[
0 = d \left( J^G_{\eta} + J^\phi_{\eta} \right). \tag{105}
\]

5.3. Einstein–Maxwell field theory

The Einstein–Maxwell field theory is described by
\[
M = \left\{ \left( x^\mu, g_{\mu\nu}, A_\mu \right) \right\} \approx \mathbb{R}^{18}, \quad K = K_{\text{Einstein}} + K_{\text{Maxwell}}, \tag{106}
\]
\[
K_{\text{Einstein}} = \frac{1}{4\kappa} \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \bar{g}^{\xi\zeta} \bar{F}_{\eta}^\mu \wedge dx^{\nu\rho\sigma} - \frac{\bar{F}_{\eta}^\mu}{\kappa \sqrt{-g}} dx^{0123}, \tag{107}
\]
\[
K_{\text{Maxwell}} = \frac{1}{\sqrt{-g}} \left( \bar{F} \wedge dx_{\mu\rho\sigma} \right) \left( \bar{F} \wedge dx^{\mu\rho\sigma} \right) \frac{dx^{0123}}{4dx^{0123}}, \quad \left( dx^{0123} \neq 0 \right). \tag{108}
\]
where we have defined $dx_{\rho\sigma} = g_{\rho\sigma} g_{\rho\sigma} dx^{\rho\sigma}$. The Levi-Civita symbol is defined by $\varepsilon^{0123} = 1$ and $\varepsilon_{0123} = -1$. The variation of $K$ becomes

$$
\delta K = \frac{\sqrt{-g}}{2} \left\{ -\frac{1}{\kappa} G_{\gamma\eta} + \frac{1}{\kappa} \delta g_{\gamma\eta} dx^{0123} + T_{\gamma\eta}^{EM} \right\} \delta g^{\gamma\eta} + d \left[ \frac{1}{4\kappa} \sqrt{-g} g^{\mu\rho\sigma\tau} \varepsilon^{\mu\rho\sigma\tau} \delta F^\rho \wedge dx^{\rho\sigma\tau} \right] + \delta F^\rho \left( \sigma^\rho \text{ vanishing term} \right) - d \left[ \left( J_0^G + J_1^A + J_0^M + J_0^A \right) \delta x^0 \right] + \left\{ d \left( J_0^G + J_1^A + J_0^M + J_0^A \right) \right\} \delta x^0 + d \left[ \delta A_\mu \frac{1}{\sqrt{-g}} F \wedge dx_{\mu\nu} dx^{\nu\rho} \right] - \delta A_\mu d \left( \frac{1}{\sqrt{-g}} F \wedge dx_{\mu\nu} dx^{\nu\rho} \right),
$$

(109)

where we set

$$
\bar{J}_M^a = \frac{1}{\sqrt{-g}} \left\{ \bar{F} \wedge dx_{\rho\sigma} \bar{F} \wedge dx^\rho + \varepsilon_{\mu\rho\sigma\tau} \frac{\left( \bar{F} \wedge dx_{\rho\sigma} \right) \left( \bar{F} \wedge dx^\rho \right)}{4 \cdot 3! \left( dx^{0123} \right)^2} dx^{\mu\rho\sigma\tau} \right\},
$$

(110)

$$
\bar{J}_A^a = \frac{1}{\sqrt{-g}} \frac{\bar{F} \wedge dx_{\mu\nu} dx^{\mu\nu}}{2 dx^{0123}} \wedge da_x,
$$

(111)

$$
\bar{T}_{\gamma\eta}^{EM} = \frac{1}{g} \left\{ \frac{\left( \bar{F} \wedge dx_{\gamma\eta} \right) \left( \bar{F} \wedge dx^\gamma \right)}{dx^{0123}} - g_{\gamma\eta} \left( \bar{F} \wedge dx_{\rho\sigma} \right) \left( \bar{F} \wedge dx^{\rho\sigma} \right) \right\}.
$$

(112)

The Euler–Lagrange equations become

$$
0 = d \left( \bar{J}_G^a + \bar{J}_A^a + \bar{J}_M^a + \bar{J}_A^a \right),
$$

(113)

$$
0 = -\frac{1}{\kappa} G_{\gamma\eta} + \frac{1}{\kappa} \delta g_{\gamma\eta} dx^{0123} + \bar{T}_{\gamma\eta}^{EM},
$$

(114)

$$
0 = d \left( \frac{1}{\sqrt{-g}} \frac{\bar{F} \wedge dx_{\rho\sigma} dx^{\rho\sigma}}{2 dx^{0123}} \right).
$$

(115)

Since $dx^{0123} \neq 0$ is assumed for $K_{\text{Maxwell}}$, $\frac{\partial (x^0, x^1, x^2, x^3)}{\partial (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)} \neq 0$ is satisfied for any parameterization $\sigma(s)$. Therefore, the matrix $\left( \frac{\partial \tilde{x}^r}{\partial x^0} \right)$ is invertible. Thus, from (115),
and if \( \sigma \) is the 4-dimensional submanifold satisfying the Euler–Lagrange equations, we get
\[
\sigma^* J_{\alpha}^A = d \left\{ \sigma^* \left( \frac{\tilde{F} \wedge dx^\mu}{\sqrt{-g}} \right) dx^\alpha \right\}.
\]

which is the exact form. We obtain two conservation laws:
\[
d \left( \sigma^* J_{\alpha}^A \right) = 0,
\]
\[
d \left( \sigma^* J_{\alpha}^G + \sigma^* J_{\alpha}^\lambda + \sigma^* J_{\alpha}^M \right) = 0.
\]

However, this happens due to the specific form of \( K_{\text{Maxwell}} \), and for more general models such as Born-Infeld, such a separation of conserved currents does not occur.

### 5.4. Einstein coupled to perfect fluid

Following a standard theory such as [6, 14], we construct the extended configuration space for the perfect fluid under gravitational force as \( M = \{ (x^\mu, g^{\mu\nu}, \pi_\mu, \rho) \} \) with the Kawaguchi metric
\[
K = K_{\text{Einstein}} + K_{\text{P.F.}}, \quad K_{\text{P.F.}} = (\rho + p) \sqrt{-g} dx^{0123} + \int \delta K_p,
\]

\[
\delta K_p := \rho \delta \left( \sqrt{-g} dx^{0123} \right) + p \delta \left( -\frac{1}{3!} e^{\mu\rho\sigma} \sqrt{-g} dx^{\rho\sigma} \right) dx^\sigma,
\]

where \( K_{\text{Einstein}} \) is given by (75) and the integral in (118) denotes a formal functional integral such that it gives (119) when the variation is taken. The \( \rho \) and \( p \) are the usual energy density and pressure density, but since we prefer to work with the variable \( \pi_\mu \) following [8], \( \rho \) is instead defined by this co-vector field \( \pi_\mu \) as
\[
(\rho + p) \sqrt{-g} := g^{\mu\nu} \pi_\mu \pi_\nu.
\]

The commonly used 4-velocity field \( u \) is related to \( \pi_\mu \) by \( \pi_\mu = (\rho + p) u_\mu \). By this definition, we automatically get \( g^{\mu\nu} u_\mu u_\nu = 1 \).

The variation of \( \pi_\mu \) is defined as the following to satisfy the generalized Lin constraints [16]:
\[
\delta \pi_\mu = \pi^\alpha \frac{\partial \delta \xi^\mu}{\partial x^\alpha} = \tilde{F}^{\alpha}_{\mu\rho} \delta \xi^\rho \pi^\beta + \sqrt{-g} g^{\alpha\beta} \delta \xi^\mu \frac{\partial}{\partial x^\rho} u_\rho, \quad \pi^\mu := g^{\mu\nu} \pi_\nu.
\]

Here the vectors \( \delta \xi^\mu := \delta \mu^\mu \left( \frac{\partial}{\partial x^\rho} \right) \delta \xi^\nu \) represent the infinitesimal displacement of the fluid element, which satisfies the relation
\[
\delta \pi_\mu = -\left( L_{\delta \xi} \pi \right)_\mu, \quad \pi := \pi_\rho dx^\rho.
\]
We can now calculate the variation of $K_{P.F.}$ in terms of this $\delta \pi_{\mu}$, together with $\delta x^\mu$ and $\delta g^{\mu \nu}$:

$$\delta K_{P.F.} = \frac{\partial}{\partial x^u} \left\{ (\rho + p) \sqrt{-g} u^a u^b \delta \xi^a \right\} dx^{0123}$$

$$+ \delta \xi^a \left[ g_{\mu \nu} \frac{dp}{dx^\nu} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} \left\{ \sqrt{-g} (\rho + p) u^a u^b \right\} - (\rho + p) F_{\alpha \beta}^\mu u^\alpha u^\beta \right] \sqrt{-g} dx^{0123}$$

$$+ \frac{\delta g^{\mu \nu}}{2} \left\{ (\rho + p) u_\mu u_\nu - p g_{\mu \nu} \right\} \sqrt{-g} dx^{0123}$$

$$+ d \left\{ \frac{1}{3!} (\rho + p) \sqrt{-g} \epsilon_{\mu \nu \rho \sigma} dx^{\mu \nu} \delta x^\sigma + \frac{1}{3!} \sqrt{-g} \epsilon_{\mu \nu \rho \sigma} dx^{\mu \nu} \delta x^\sigma \right\}$$

$$- \left[ dp \wedge \left( -\frac{1}{3!} \sqrt{-g} \epsilon_{\mu \nu \rho \sigma} dx^{\mu \nu} \right) \right] \delta x^\sigma.$$  

(123)

The Euler–Lagrangian equations pulled back by $\sigma(x)$ are then

$$dp \wedge J^\mu_\sigma = d \left( J^\mu_\sigma^G + J^\mu_\sigma^P + \tilde{J}^\mu_\sigma^{P.F.} \right),$$  

(124)

$$0 = -\frac{1}{\kappa} \left( G_\mu^\nu - \lambda g_\mu^\nu + T_\mu^\nu \right),$$  

(125)

$$g^{\mu \nu} \frac{dp}{dx^\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} \left\{ \sqrt{-g} (\rho + p) u^a u^b \right\} + \Gamma^\mu_{\alpha \beta} (\rho + p) u^\alpha u^\beta,$$  

(126)

where we set

$$J^\mu_\sigma^{P.F.} \equiv -\frac{1}{3!} \epsilon_{\mu \nu \rho \sigma} (\rho + 3p) \sqrt{-g} dx^{\mu \nu \rho \sigma}, \quad \tilde{J}^\mu_\sigma^{P.F.} \equiv -\frac{1}{3!} \epsilon_{\mu \nu \rho \sigma} \sqrt{-g} dx^{\mu \nu \rho \sigma},$$  

(127)

$$T_\mu^\nu \equiv (\rho + p) u_\mu u_\nu - p g_\mu^\nu.$$  

(128)

The equation (126) can be rewritten as

$$g^{\mu \nu} \frac{dp}{dx^\nu} = \frac{1}{\sqrt{-g}} \partial_a \left\{ \sqrt{-g} (\rho + p) u^a u^b \right\} + \Gamma^\mu_{\alpha \beta} (\rho + p) u^\alpha u^\beta$$  

(129)

$$= \Gamma^\mu_{\alpha \beta} (\rho + p) u^\alpha u^\beta + \partial_a \left\{ (\rho + p) u^a \right\} u^\mu + (\rho + p) \left( u^a \partial_a u^\mu + \Gamma_{\alpha \beta}^\mu u^\alpha u^\beta \right)$$  

(130)

$$= \nabla^\alpha \left\{ (\rho + p) u^\alpha \right\} u^\mu + (\rho + p) \nabla^\alpha u^\alpha,$$  

(131)

and contracting with $u_\mu$, we get

$$u^\nu \frac{dp}{dx^\nu} = \nabla^\alpha \left\{ (\rho + p) u^\alpha \right\}.$$  

(132)

Then, substituting back, we get the relativistic Euler equation for the perfect fluid,

$$(\rho + p) \nabla^\alpha u^\alpha = g^{\mu \nu} \frac{dp}{dx^\nu} - u^\nu \frac{dp}{dx^\nu} u^\mu.$$  

(133)

Actually, it is easy to determine that the pullback Euler–Lagrange equation (126) is equivalent to $\nabla^\nu \left( T^{P.F.}\right)_{\mu \nu} = 0$.  

27
In our framework, the energy-momentum relation (124) appears as a part of the Euler–Lagrange equations. Here, $J^{\text{P.F.}}_F$ stands for the energy-momentum current of the fluid and the left-hand side $dp \wedge J^F_\sigma$ expresses its dissipation. Taking the differentiation of $J^{\text{P.F.}}_F$,
\begin{align*}
\frac{dJ^{\text{P.F.}}_\sigma}{d\sigma} &= -\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma} \frac{\partial}{\partial x^\nu} \left\{ (\rho + 3p) \sqrt{-g} \right\} dx^{\mu\rho} = -\frac{\partial}{\partial x^\sigma} \left\{ (\rho + 3p) \sqrt{-g} \right\} dx^{0123} \\
&= -\frac{\partial}{\partial x^\sigma} \left\{ \frac{g^{\mu\nu} \mu_\sigma \pi_\nu}{\sqrt{-g}} \right\} dx^{0123} = 2\frac{\partial (\rho - \pi)}{\partial x^\sigma} dx^{0123},
\end{align*}
then applying the Lin constraints (121) with $\delta^F_\sigma = \delta^0_\sigma$, we can rewrite the first term in (134),
\begin{align*}
\frac{\partial}{\partial x^\sigma} \left( \frac{\partial g_{\mu\nu}}{\partial \mu} + T_{\mu\nu} \pi^\mu \right) - \sqrt{-g} \frac{dp}{dx^\sigma} u^\mu &= \pi^\mu \partial_\sigma g_{\mu\nu} + \frac{1}{2} \partial_\sigma g^{\mu\nu} \pi_\mu \pi_\nu \\
&= -\pi^\mu \left( \Gamma_{\mu\sigma} + \Gamma_{\sigma\mu} \right) + \Gamma_{\mu\nu} \pi^\nu - \sqrt{-g} \frac{dp}{dx^\sigma} u^\mu = -\Gamma_{\mu\sigma} \pi^\mu - \sqrt{-g} \frac{dp}{dx^\sigma} u^\mu. \quad (135)
\end{align*}
Equation (134) becomes
\begin{equation}
\frac{dJ^{\text{P.F.}}_\sigma}{d\sigma} = \left\{ 2\sqrt{-g} \Gamma^{\mu\nu}\sigma (\rho + p) u^\mu u^\nu + \sqrt{-g} \frac{dp}{dx^\sigma} u^\mu - 2\frac{\partial (\rho - \pi)}{\partial x^\sigma} \right\} dx^{0123}, \quad (136)
\end{equation}
and together with $dp \wedge J^F_\sigma = -\frac{dp}{dx^\sigma} \sqrt{-g} dx^{0123}$,
\begin{equation}
-\frac{dp}{dx^\sigma} + \frac{dJ^{\text{P.F.}}_\sigma}{d\sigma} = 2\sqrt{-g} \Gamma^{\mu\nu}\sigma \left\{ (\rho + p) u^\mu u^\nu - pg_{\mu\nu} \right\} dx^{0123} = 2\sqrt{-g} \Gamma^{\mu\nu}\sigma T^{\text{P.F.}}_{\mu\nu} dx^{0123}. \quad (137)
\end{equation}
Finally, the Euler–Lagrange equation (124) turns out to be
\begin{equation}
0 = -\frac{dp}{dx^\sigma} + \frac{dJ^{\text{P.F.}}_\sigma}{d\sigma} \left\{ \frac{1}{2} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \lambda - \frac{\lambda}{4} \right) \right\} dx^{0123}, \quad (138)
\end{equation}
which is a generalisation of (90).

5.5. Energy density of Schwarzschild spacetime

We will give an example of Schwarzschild spacetime for the energy current (88). In local coordinate system $(t, x^1, x^2, x^3) = (t, r, \theta, \phi)$, the Schwarzschild metric is given by
\begin{equation}
dx^2 = \left( 1 - \frac{r_s}{r} \right) c^2 dt^2 - \left( 1 - \frac{r_s}{r} \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (139)
\end{equation}
where $r_s = 2GM/c^2$ is the Schwarzschild radius. Since there is a singularity at $r = 0$, we used the distributional techniques to regularize $1/r$ following [1, 13],
\begin{equation}
\frac{1}{r} = \lim_{\varepsilon \to 0} \frac{1}{(r^2 + \varepsilon^2)^{3/2}}, \quad \left( \frac{1}{r} \right)' = \lim_{\varepsilon \to 0} \frac{-r}{(r^2 + \varepsilon^2)^{3/2}}. \quad (140)
\end{equation}
Using expression (88),
\[ J^{G}_{0} = \frac{2r}{2\kappa} \frac{e^{2}}{r^{2}(r^{2} + \epsilon^{2})^{2}} r^{2} \sin \theta dr \wedge d\theta \wedge d\varphi, \]
(141)
and taking the limit \( \epsilon \to 0 \), we get
\[ J^{G}_{0} = -\sqrt{-g} \frac{M_{c}^{2}}{2} \delta^{(3)}(x) dx^{123}, \]
(142)
where we have used the following formulae [13],
\[ \frac{r_{e}^{2}}{r^{2}(r^{2} + \epsilon^{2})^{2}} \rightarrow 4\pi r_{e} \delta^{(3)}(x). \]
(143)
The energy density (142) takes negative value, which is exactly cancelled out by the matter energy density, \( J^{\text{matter}}_{0} = \sqrt{-g} M_{c}^{2} \delta^{(3)}(x) dx^{123} \), so that \( J^{G}_{0} + J^{\text{matter}}_{0} = 0 \) on-shell. This vanishing energy density comes from our specific but gauge-invariant definition of the energy-momentum current of gravity, given in (88) and (89).

Here we mention a relation of our energy-momentum currents of gravity (88) to those by Dirac [8], where the calculation is carried out by first-order formulation. For comparison, we also calculate by first-order in the Kawaguchi setting.

The first-order Kawaguchi metric of general relativity is given by
\[ K^{1\text{st.}} = \frac{1}{4\kappa} \varepsilon_{\mu\rho\sigma} \sqrt{-g} g^{i\xi} \hat{F}_{i \lambda} \wedge \hat{F}^{\lambda}_{\xi} \wedge dx^{\rho\sigma}, \]
(144)
and from \( K^{1\text{st.}} \), the energy-momentum current is derived as
\[ J^{1\text{st.}}_{\rho} = (\frac{1}{2\kappa} \varepsilon_{\mu\rho\sigma} \sqrt{-g} g^{i\xi} dx^{\rho}) \wedge \hat{F}_{i \lambda} + \frac{1}{2\kappa} \varepsilon_{\mu\rho\sigma} \sqrt{-g} g^{i\xi} \hat{F}^{\lambda}_{i} \wedge \hat{F}^{\lambda}_{\xi} \wedge dx^{\rho}. \]
(145)
The derivation details are given in appendix B. The energy current similarly calculated by the distributional technique becomes
\[ J^{1\text{st.}}_{0} = \frac{1}{2\kappa} \left\{ \frac{-2 + 2r_{e}^{2}}{r^{2}(r^{2} + \epsilon^{2})^{2}} \right\} \sin \theta dr \wedge d\theta \wedge d\varphi \rightarrow -\frac{1}{2\kappa r^{2}} dx^{123} \]
\[ + \frac{M_{c}^{2}}{2} \sqrt{-g} \delta^{(3)}(x) dx^{123}. \]
(146)
Taking the integral over the \( t = 0 \) hypersurface, the first term gives a trivial divergence and is neglected. The second term gives a positive value for energy, \( M_{c}^{2} \). However, since the surface term is dropped in (144), this expression is essentially coordinate-dependent. On the other hand, our energy current of gravity (88) is gauge invariant and reparameterization invariant. Thus, it has a physical meaning (i.e., observable) of spacetime.

5.6. Palatini \( f(R) \) gravity

In general relativity, we found that the energy-momentum current is zero under the on-shell conditions. This property holds even in other theories of gravity such as Palatini \( f(R) \) gravity,
\[ L = -\frac{\sqrt{-g}}{2\kappa} f(R), \quad R = g^{\nu\rho} R_{\nu\rho}, \quad R_{\mu}^{\lambda} = \frac{1}{2} R_{\nu\mu\rho}dx^{\nu\rho} = d\Gamma_{\mu}^{\lambda} + \Gamma_{\mu}^{\lambda} \wedge \Gamma_{\nu}^{\nu}. \]
(147)
where the metric $g_{\mu\nu}$ and the linear connection $\Gamma^{\lambda}_{\mu\nu}$ are independent. In other words the metricity condition (77) does not hold. We will assume the torsion-free connection and keep the last two indices of $\Gamma^{\lambda}_{\mu\nu}$ symmetrical, as in [22]. The Kawaguchi metric is given by

$$K = -\frac{\sqrt{-g}}{2\kappa} \left( -\frac{e_{\mu\nu\rho\sigma} \tilde{R}^{\rho\sigma} \wedge dx^{\mu\nu}}{2dx^{0123}} \right) dx^{0123}.$$  

(148)

We obtain the variation of $K$ as follows:

$$\delta K = \frac{1}{4\kappa} \sqrt{-g} \left( f g_{\xi\psi} dx^{0123} + f e_{\mu\xi\psi} \tilde{R}^{\mu\xi\psi} \wedge dx^{\mu\xi\psi} \right) \delta g_{\xi\psi}$$

$$+ \frac{1}{4\kappa} f \left( \frac{e_{\mu\xi\psi}}{\sqrt{-g}} g^{\xi\psi} \delta \tilde{R}^{\mu\xi\psi} \wedge dx^{\mu\xi\psi} \right)$$

$$- \frac{1}{4\kappa} \delta \tilde{R}^{\mu\xi\psi} \left[ d \left( \frac{1}{\sqrt{-g}} f e_{\mu\xi\psi} g^{\xi\psi} \delta \tilde{R}^{\mu\xi\psi} \wedge dx^{\mu\xi\psi} \right) \right]$$

$$+ \sqrt{-g} f' \left( e_{\mu\xi\psi} g^{\xi\psi} \tilde{F}^{\mu\xi\psi} \wedge dx^{\mu\xi\psi} - e_{\xi\psi\mu\nu} g^{\xi\psi} \tilde{F}^{\xi\psi\mu\nu} \wedge dx^{\xi\psi\mu\nu} \right)$$

$$- \frac{1}{2} \frac{\tilde{F}^{\mu\xi\psi}}{dx^{\mu\xi\psi}} \left[ d \left( \frac{1}{\sqrt{-g}} g^{\xi\psi} f \delta e_{\mu\xi\psi} \right) \right]$$

$$+ \sqrt{-g} f' \left( g^{\xi\psi} \tilde{F}^{\xi\psi} \delta e_{\mu\xi\psi} - g^{\xi\psi} \tilde{F}^{\xi\psi} \delta e_{\xi\psi\mu\nu} \right) \frac{dx^{\mu\xi\psi}}{dx^{\mu\xi\psi}}$$

$$= 0.$$  

(149)

where the round brackets denote symmetrization: $A^{\mu\nu} = \frac{1}{2} (A^{\mu\nu} + A^{\nu\mu})$. The Euler–Lagrange equations become

$$0 = f g_{\xi\psi} dx^{0123} + f e_{\mu\xi\psi} \tilde{R}^{\mu\xi\psi} \wedge dx^{\mu\xi\psi}$$

(150)

$$0 = d \left( \frac{1}{\sqrt{-g}} f e_{\mu\xi\psi} \delta \tilde{R}^{\mu\xi\psi} \wedge dx^{\mu\xi\psi} \right)$$

$$+ \sqrt{-g} f' \left( e_{\mu\xi\psi} g^{\xi\psi} \tilde{F}^{\mu\xi\psi} \wedge dx^{\mu\xi\psi} - e_{\xi\psi\mu\nu} g^{\xi\psi} \tilde{F}^{\xi\psi\mu\nu} \wedge dx^{\xi\psi\mu\nu} \right)$$

(151)

$$0 = df' \tilde{R}^{(\mu\xi\psi)}.$$  

(152)
where $J^f_{\alpha}(R)$ is the energy-momentum current of the Palatini $f(R)$ gravity defined by

$$J^f_{\alpha}(R) = \frac{1}{2\kappa} \left\{ f' R^{\mu\nu} \wedge dx^\nu + \frac{1}{3!} f + f' \frac{\epsilon_{\rho\sigma\tau\delta} R^{\rho\sigma}}{2dx^{0123}} \right\} dx^\mu x^\nu,$$

$$- \frac{1}{2} \frac{\Gamma^\mu_{\rho\sigma}}{\epsilon_{\rho\sigma\tau\delta}} \left( d \left( \sqrt{-g} g^{\nu\rho} f' \epsilon_{\mu\rho\sigma\nu} dx^\rho \right) + \sqrt{-g} f' \left( g^{\nu\rho} \tilde{F}^{\nu\sigma\rho}_{\mu\sigma} - g^{\nu\rho} F^{\nu\sigma}_{\mu\sigma} \right) dx^\rho \right).$$

(153)

We recover the standard equations from the pullback of (150) and (151):

$$0 = \left[ f'(R) R_{\xi\eta} - \frac{1}{2} f(R) g_{\xi\eta} \right] dx^{0123}.$$

(154)

$$0 = \left[ V_\mu \left\{ \sqrt{-g} f' g^{\xi\eta} \right\} - V_\xi \left\{ \sqrt{-g} f' g^{\xi\eta} \right\} \right] dx^{0123}.$$

(155)

The pullback of (153) becomes

$$J^f_{\alpha}(R) = \frac{1}{2\kappa} \left\{ f'(R) \left\{ R^{\xi\eta} - R g_{\xi\eta} - R_{\mu\nu} \epsilon_{\mu\nu\xi\eta} \right\} - g_{\xi\eta} \left( f(R) - f'(R) R \right)$$

$$+ \frac{1}{\sqrt{-g}} \Gamma^\mu_{\rho\sigma} \left\{ V_\mu \left( \sqrt{-g} g^{\xi\eta} f'(R) \right) g_{\xi\eta} - V_\xi \left( \sqrt{-g} g^{\xi\eta} f'(R) g_{\xi\eta} \right) \right\} \right\} dx^\xi dx^\eta.$$

(156)

As shown in [22], equation (155) can be reduced to

$$V_\mu \left\{ \sqrt{-g} f' g^{\xi\eta} \right\} = 0.$$

(157)

It is equivalent to

$$V_\mu \left( \sqrt{-g} h^{\xi\eta} \right) = 0,$$

(158)

with new metric $h_{\mu\nu} = f'(R) g_{\mu\nu}$. Equation (158) is the metricity condition of $h_{\mu\nu}$, and it is solved as

$$\Gamma^\xi_{\mu\nu} = \frac{1}{2} h^{\xi\eta} \left( \partial_\mu h_{\nu\eta} + \partial_\nu h_{\mu\eta} - \partial_\eta h_{\mu\nu} \right).$$

(159)

Therefore, we have the same symmetries for the curvature tensor as for general relativity: $R_{\mu\nu\rho\sigma} = -R_{\nu\rho\mu\sigma}$, etc. Consequently, (156) becomes

$$J^f_{\alpha}(R) = \frac{1}{2\kappa} \left\{ 2f'(R) R_{\xi\eta} - f(R) g_{\xi\eta}$$

$$+ \frac{1}{\sqrt{-g}} \Gamma^\mu_{\rho\sigma} \left\{ V_\mu \left( \sqrt{-g} h^{\xi\eta} \right) g_{\xi\eta} - V_\xi \left( \sqrt{-g} h^{\xi\eta} \right) g_{\xi\eta} \right\} \right\} dx^\xi dx^\eta = 0,$$

(160)

by the use of (154) and metricity derived from (155). The generator of gauge transformations in this case is given by
\[ G = f^\mu \frac{\partial}{\partial x^\mu} + \left( \frac{\partial f^\nu}{\partial x^\rho} g^{\rho\nu} + \frac{\partial f^\nu}{\partial x^\rho} g^{\rho\mu} \right) \frac{\partial}{\partial x^\mu} + \left( -\frac{\partial^2 f^\lambda}{\partial x^\rho \partial x^\nu} + \frac{\partial f^\rho}{\partial x^\mu} \Gamma^\lambda_{\mu \rho} - \frac{\partial f^\rho}{\partial x^\mu} \Gamma^\lambda_{\rho \mu} = \frac{\partial f^\rho}{\partial x^\mu} \Gamma^\lambda_{\rho \mu} \right) \frac{\partial}{\partial x^\mu}. \] (161)

The energy-momentum current (160) is invariant with respect to the above gauge transformation. Despite its non-trivial appearance, the energy-momentum current of Palatini gravity eventually vanishes, as we mentioned at the beginning of this section.

6. Discussions

We have constructed the theory of reparameterization-invariant Lagrangian formulation in the setting of Kawaguchi geometry, i.e., geometrisation of the variational principle for field theories, and considered its application to several concrete models of field theory. In this formulation, we have shown that the conservation law of the energy-momentum currents appear as a part of the Euler–Lagrange equations. Mathematically, this result is due to the fact that the Kawaguchi manifold is an extended configuration space including spacetime, and therefore the spacetime coordinates becomes cyclic for the field theory that usually does not have explicit dependency on spacetime coordinates. Physically, the Kawaguchi Lagrangian formulation tells us that the conservation law of energy-momentum currents and the conventional equations of motions are on an equal level. For example, instead of considering the Maxwell equations or Einstein’s field equations, we can start the same discussion by choosing the conservation law. One particular advantage of this formulation is that we were able to propose a new way of understanding the energy-momentum conservation law of general relativity. As in the case of other conventional field theories, it is derived as a part of the Euler–Lagrange equations. In previous studies, the energy-momentum currents of general relativity were defined as pseudo-tensors [3, 9, 15], but in our result, they are derived as geometric quantities including second-order derivatives of \( g^{\mu\nu} \). In Einstein’s theory, the energy-momentum current of gravity becomes zero under on-shell condition \( J^\mu_0 = 0 \). However, it has a property of on-shell gauge invariance, \( L_0 J^\alpha_\mu = 0 \). There exist various definitions and interpretations for the energy-momentum current of gravity [7, 24]. While many of them do not satisfy gauge invariance, our definition is gauge invariant and reparameterization invariant. Therefore, it is physically observable in real spacetime, even if it vanishes.

The Lagrangian formulation using the Finsler/Kawaguchi manifold also simplified Noether’s theorem. The symmetry of the system is exactly the symmetry of the Lagrangian, which was the metric of Finsler/Kawaguchi manifold. And since our Finsler/Kawaguchi manifold encapsulates both spacetime and fields equivalently, no fibered structure was required and, consequently, the description of symmetries was simplified; in other words, no distinctions such as inner or exterior symmetries were introduced. This is a notable calculational efficiency. For future developments, we propose that the Kawaguchi-Lagrangian formulation is a good candidate to construct models such as irreversible processes and systems that demonstrate hysteresis, owing to the nature of Finsler and Kawaguchi geometry. Also, for the Finsler-Lagrangian case, changing the time coordinate generates a non-perturbative transformation such that a harmonic oscillator turns into a free particle [21]. Similarly, we expect that a spacetime coordinate change in a Kawaguchi-Lagrangian formulation would offer us a non-perturbative transformation for field theory.
In the standard formulation, there are two main approaches to deal with the field theory: to consider the infinite dimensional configuration space and construct formal expressions, or to consider the finite dimensional configuration space with a jet bundle structure. The first expression is simple, but concrete problems are difficult to handle; the second is applicable to concrete problems, but the structures and notations maybe sometimes difficult for physicists to handle. Our formulation is, in a sense, a mixture of both, with the simplicity of the former and the applicability of the latter. The actual calculations for concrete problems are accessible for most physicists, as we have shown in the examples, and we hope this formulation will be helpful in understanding both past and future problems of physics.

Acknowledgments

We thank Lajos Tamássy, László Kozma, Masahiro Morikawa, Ken-ichi Nakao, and Akio Sugamoto for creative discussions. T Ootsuka and E Tanaka thank JSPS Institutional Program for Young Researcher Overseas Visits. E Tanaka is grateful for a SAIA grant and the use of the Yukawa Institute Computer Facility. M Ishida acknowledges the grant-in-aid KAKENHI 25400272. This work was greatly inspired by the late Yasutaka Suzuki.

Appendix A

In the covariant Lagrangian formulation, we frequently used the sign $\sigma$, which means the equality on the 4-dimensional submanifold embedded in $M$.

With this symbol we mean

$$A(x, dx) \equiv B(x, dx) \iff \sigma^A(x, dx) = \sigma^B(x, dx),$$

(162)

where $A$ and $B$ are the functions of $x^\mu$ and $dx^{\mu_1...\mu_k}$ ($1 \leq k \leq N$) and $\dim M = N$.

It is related to the ambiguity of the notations such as $dx^{\mu_1...\mu_k}$ and $dx^{ij} \wedge d^2x^{\mu_1...\mu_k}$. The pullback of these quantities by parameterization $\sigma = \sigma(s)$ is defined by

$$\sigma^A dx^{\mu_1...\mu_k} = \frac{\partial(x^{\mu_1}, x^{\mu_2}, x^{\mu_3}, x^{\mu_4})}{\partial(s^0, s^1, s^2, s^3)} dx^{0123},$$

(163)

$$\sigma^A dx^{ij} \wedge d^2x^{\mu_1...\mu_k} = \frac{\partial(x^i, x^j, x^\ell, x^\zeta)}{\partial(s^0, s^1, s^2, s^3)} \left( dx^{0123} \right)^2.$$  

(164)

If we always treat these variables by their pullbacks, as above, no ambiguity will enter into the formulae. However, we also used them as first- and second-order differential forms on $M$. For instance, the Lie derivative $\mathcal{L}_X$ is defined by

$$\mathcal{L}_X dx^{\mu_1...\mu_k} = (\mathcal{L}_X dx^\mu) \wedge dx^{\nu_1...\nu_{k-1}} - (\mathcal{L}_X dx^\nu) \wedge dx^{\mu_1...\mu_k} + (\mathcal{L}_X dx^\sigma) \wedge dx^{\mu_1...\mu_k} - (\mathcal{L}_X dx^\sigma) \wedge dx^{\mu_1...\mu_k}$$

$$= dx^\nu \wedge dx^{\nu_1...\nu_{k-1}} - dx^\nu \wedge dx^{\mu_1...\mu_k} + dx^\sigma \wedge dx^{\mu_1...\mu_k} - dx^\sigma \wedge dx^{\mu_1...\mu_k}. $$

(165)

In other words, we considered $dx^{\mu_1...\mu_k}$ as a 4-form on $M$, rather than the coordinate function on $A^4 TM$. The meaning of the higher-order differential form is not something new but only notational. As we treat $dx^{\mu_1...\mu_k}$ as a 4-form (first-order) on $M$, it acts on a 4-vector field $v = \frac{1}{2^4!} \frac{\partial^4 \phi}{\partial x^{\mu_1} \partial x^{\mu_2} \partial x^{\mu_3} \partial x^{\mu_4}}$ over $M$, whose actions we define as
We define the notation of the second-order differential form $\text{d}x^{\alpha\beta\gamma} \wedge \text{d}^2x^{\mu\nu\rho\sigma}$ by a recursive action of this first-order form,

\[
\text{d}x^\alpha \wedge \text{d}^2x^{\mu\nu\rho\sigma} (v) = \left\{ \text{d}x^\alpha \wedge \left( \text{d} \left( \text{d}x^{\mu\nu\rho\sigma} (v) \right) \right) \right\} (v)
= \text{d}x^\alpha \wedge \text{d}^2x^{\mu\nu\rho\sigma} (v) = v^\alpha \gamma \frac{\partial x^\mu}{\partial x^\gamma}.
\]

Such an operation allows us to simplify the calculation (for example, by taking the variation of the Kawaguchi metric) by using the standard computation technique of the exterior and Lie derivatives, without being aware of further details such as the background mathematical structures. While, given the 4-dimensional submanifold, a 4-vector field could be defined as an oriented surface element on each point of $M$, the converse is not always true. This problem of the integrability of the vector field is the source of the ambiguity. In other words, the formula (such as $K$, $L \kappa K$) expressed by variables on $M$, when pulled back to the 4-dimensional integral submanifold, may give the same value for different expressions. For example, there are identities such as

\[
\sigma^\alpha \text{d}x^\alpha [\delta \text{d}x^{\mu\nu\rho\sigma}] = 0,
\sigma^\alpha \text{d}x^\alpha \wedge \text{d}^2x^{\mu\nu\rho\sigma} = 0.
\]

Nevertheless, the variational principle is given by the pullback equation, $\sigma^\alpha \delta K = 0$, which, as we mentioned previously, does not include such ambiguity, and knowing that this pullback by $\sigma$ removes the ambiguity, we can safely use the symbol $\equiv$ to indicate the equivalence implied under the relation (168).

**Appendix B**

In this appendix, we will derive the first-order energy-momentum current (145) from the first-order Kawaguchi metric of general relativity (144),

\[K_{1\text{st.}} = - \frac{1}{4\kappa^2} \epsilon_{\mu
u\rho\sigma} \sqrt{\gamma^4 g} \gamma^4 \tilde{F}_\lambda \wedge \tilde{F}_\xi \wedge \text{d}x^{\rho\sigma}.\]

The variation of $K_{1\text{st.}}$ with respect to $x^\mu$, $\delta_i K_{1\text{st.}}$, is calculated as follows:

\[
\delta_i K_{1\text{st.}} = - \frac{1}{4\kappa^2} \left\{ \epsilon_{\mu\nu\rho\sigma} \sqrt{\gamma^4 g} \gamma^4 \delta \left( \tilde{F}_\lambda \right) - \epsilon_{\mu\nu\rho\sigma} \sqrt{\gamma^4 g} \gamma^4 \delta \left( \tilde{F}_\xi \right) \right\} \wedge \text{d}x^{\rho\sigma} \wedge \tilde{F}_\xi
- \frac{1}{4\kappa^2} \epsilon_{\mu\nu\rho\sigma} \sqrt{\gamma^4 g} \gamma^4 \tilde{F}_\lambda \wedge \tilde{F}_\xi \wedge \delta \left( \text{d}x^{\rho\sigma} \right)
= - \frac{1}{4\kappa} \epsilon_{\mu\nu\rho\sigma} \sqrt{\gamma^4 g} \gamma^4 \tilde{F}_\lambda \wedge \delta \left( \text{d}x^{\rho\sigma} \right) \wedge \tilde{F}_\xi
+ \frac{1}{4\kappa} \left( \epsilon_{\mu\nu\rho\sigma} \sqrt{\gamma^4 g} \gamma^4 \tilde{F}_\lambda \wedge \delta \left( \text{d}x^{\rho\sigma} \right) \wedge \tilde{F}_\xi
- \frac{1}{4\kappa} \epsilon_{\mu\nu\rho\sigma} \sqrt{\gamma^4 g} \gamma^4 \tilde{F}_\lambda \wedge \tilde{F}_\xi \wedge \delta \left( \text{d}x^{\rho\sigma} \right) \right).\]
\[ d \left\{ \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} d \left( \sqrt{-g} g^{\xi\eta} \right) \right\} \wedge \hat{F}_{\xi}^\mu \wedge dx^\rho \delta x^\sigma + \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} d \left( \sqrt{-g} g^{\xi\eta} \right) \wedge \hat{F}_{\xi}^\mu \wedge dx^\rho \delta x^\sigma + \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} d \left( \sqrt{-g} g^{\xi\eta} \right) \wedge \hat{F}_{\xi}^\mu \wedge dx^\rho \delta x^\sigma \right\} \delta x^\sigma. \tag{169} \]

For the second equality, we used the variation of the relation (82), i.e.,
\[ \epsilon_{\mu\nu\rho\sigma} d \left( \sqrt{-g} g^{\xi\eta} \right) \wedge \delta (dx^\rho) \approx \left\{ \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \delta \left( \hat{F}_{\xi}^\mu \right) - \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \delta \left( \hat{F}_{\xi}^\eta \right) \right\} \wedge dx^\rho + \left( \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \hat{F}_{\xi}^\mu - \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \hat{F}_{\xi}^\eta \right) \wedge \delta (dx^\rho). \tag{170} \]

Then we have the first-order energy-momentum current (145),
\[ \mathcal{J}^\text{1st.}_\sigma = \hat{d} \left( \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} dx^\rho \right) \wedge \hat{F}_{\xi}^\mu + \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \hat{F}_{\xi}^\mu \wedge \hat{F}_{\xi}^\eta \wedge dx^\rho. \]

The consistency with the second-order energy-momentum current (88) can be also checked by
\[ \mathcal{J}^\text{2nd.}_\sigma = \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \hat{R}_{\xi}^\mu \wedge dx^\rho \]
\[ = d \left\{ \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \hat{F}_{\xi}^\mu \wedge dx^\rho \right\} - d \left( \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \right) \wedge \hat{F}_{\xi}^\mu \wedge dx^\rho \]
\[ + \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \hat{F}_{\xi}^\mu \wedge \hat{F}_{\xi}^\eta \wedge dx^\rho \]
\[ = \mathcal{J}^\text{1st.}_\sigma + d \left\{ \frac{1}{2k} \epsilon_{\mu\nu\rho\sigma} \sqrt{-g} g^{\xi\eta} \hat{F}_{\xi}^\mu \wedge dx^\rho \right\}. \tag{171} \]

References

[1] Balasin H and Nachbagauer H 1993 The energy-momentum tensor of a black hole, or what curves the Schwarzschild geometry? Class. Quantum Grav. 10 2271–8
[2] Bao D, Chern S S and Shen Z 2000 An Introduction to Riemann-Finsler Geometry (Berlin: Springer)
[3] Bergmann P G and Thomson R 1953 Spin and angular momentum in general relativity Phys. Rev. 89 400–7
[4] Binz E, Sniatycki J and Fischer H 1988 Geometry of Classical Fields (New York: Dover)
[5] Bucataru I and Miron R 2007 Finsler-Lagrange Geometry; Applications to Dynamical Systems (Editura Academiei Române)
[6] Choquet-Bruhat Y 2009 General Relativity and the Einstein Equations (Oxford Mathematical Monographs) (Oxford: Oxford University Press)
[7] Chang C, Nester J M and Chen C 1999 Pseudotensors and quasilocal energy-momentum Phys. Rev. Lett. 83 1897–901
[8] Dirac P A M 1975 General Theory of Relativity (New York: Wiley)
[9] Dubois-Violette M and Madore J 1987 Conservation laws and integrability conditions for gravitational and Yang-Mills field equations Commun. Math. Phys. 108 213–23
[10] Ingarden R S 1987 On physical interpretations of finsler and kawaguchi geometries and the barthel nonlinear connection Tensor, N. S 46 354–60
[11] Kawaguchi A 1964 On the theory of areal spaces Bull Calcutta Math. Soc. 56 91–107
[12] Kawaguchi A 1976 On the concepts and theories of higher order spaces Period. Math. Hung. 7 291–9
[13] Kawai T and Sakane E 1997 Distributional energy-momentum densities of schwarzschild space-time Prog. Theor. Phys. 98 69–86
[14] Lanczos C 1986 The Variational Principles of Mechanics (Dover Books on Physics)
[15] Landau L D and Lifshitz E M 1962 The Classical Theory of Fields (Reading, MA: Addison-Wesley)
[16] Marsden J E and Ratiu T S 1999 Introduction to Mechanics and Symmetry (Berlin: Springer)
[17] Matsumoto M 1986 Foundations of Finsler geometry and special Finsler spaces (Kaiseisha)
[18] Miron R 1997 The Geometry of Higher-Order Lagrange Spaces (Dordrecht: Kluwer)
[19] Olver P J 1993 Applications of Lie Groups to Differential Equations (Berlin: Springer)
[20] Ootsuka T 2012 New covariant Lagrange formulation for field theories arXiv:1206.6040v1
[21] Ootsuka T and Tanaka E 2010 Finsler geometrical path integral Phys. Lett. A 374 1917–21
[22] Sotiriou T P and Faraoni V 2010 f(R) theories of gravity Rev. Mod. Phys. 82 451–97
[23] Suzuki Y 1956 Finsler geometry in classical physics J. College Arts Scie. Chiba Univ. 2 12–16
[24] Szabados L B 2009 Quasi-local energy-momentum and angular momentum in general relativity Living Rev. Relativity 12
[25] Tanaka E 2013 Parameter invariant lagrangian formulation of Kawaguchi geometry arXiv:1310.4450v1
[26] Urban Z and Krupka D 2013 The Zermelo conditions and higher order homogeneous functions Publ. Math. Debrecen 82 59–76
[27] Yajima T and Nagahama H 2007 Kawaguchi space, zermelo’s condition and seismic ray path Nonlinear Analysis: Real World Applications 8 130–5