Non $p$-norm approximated Groups

Alexander Lubotzky

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 91904, Jerusalem, Israel, alex.lubotzky@mail.huji.ac.il

Izhar Oppenheim

Department of Mathematics, Ben-Gurion University of the Negev, Be’er Sheva 84105, Israel, izharo@bgu.ac.il

July 19, 2018

Abstract

It was shown in [8] that there exists a finitely presented group which cannot be approximated by almost-homomorphisms to the Unitary groups $U(n)$ equipped with the Frobenious norms (a.k.a as $L^2$ norm, or the Schatten-2-norm). In his ICM18 lecture, Andreas Thom asks [26, Question 3.11] if this result can be extended to general Schatten-$p$-norms. We show that this is indeed the case for $1 < p < \infty$.

1 Introduction

Let $U(n)$ be the group of unitary $n \times n$ matrices equipped with a bi-invariant metric $d_n$ induced by a Banach norm $\|\|$ on $M_n(\mathbb{C})$, as $d_n(g, h) = \|g - h\|$. Examples of special interest are:

1. The Hilbert-Schmidt norm: $\|T\|_{H.S.} = \sqrt{\frac{1}{n} \text{tr}(T^*T)}$.

2. For $1 \leq p < \infty$, the Schatten $p$-norm: $\|T\|_p = \left(\text{tr}|T|^p\right)^{\frac{1}{p}}$, where $|T| = \sqrt{T^*T}$. When $p = 2$, this is usually called the Frobenius norm: $\|T\|_2 = \|T\|_{Frob} = \sqrt{n\|T\|_{H.S.}}$.

3. The operator norm, $\|T\|_{op} = \max\{\|Tv\| : \|v\| = 1\}$ also known as the Schatten $\infty$-norm.

Whatever $\{d_n\}_{n=1}^\infty$ are, define for $G = (U(n), d_n)$ the following:

Definition 1.1. A finitely generated group $\Gamma$ is called $G$-approximated if there exists an infinite sequence $\{n_k\}_{k=1}^\infty$ of integers and (set-theoretic) maps $\phi = (\phi_{n_k})$, $\phi_{n_k} : \Gamma \to U(n_k)$ such that:
1. \( \forall g, h \in \Gamma, \lim d_{n_k}(\phi_{n_k}(gh), \phi_{n_k}(g)\phi_{n_k}(h)) = 0. \)

2. \( \forall g \in \Gamma, g \neq 1 \) there is \( \varepsilon(g) = \varepsilon > 0 \) such that \( \limsup d_{n_k}(\phi_{n_k}(g), \text{id}_{U(n_k)}) \geq \varepsilon \), where \( \text{id}_{U(n_k)} \) is the \( n_k \times n_k \) identity matrix.

A long standing question is whether there exist groups \( \Gamma \) which are not \( \mathcal{G} \)-approximated. The question for case (1) where \( d_n \) is defined by the Hilbert-Schmidt norm, is equivalent to Alain Connes’ problem whether every group is hyperlinear (see [7], [20] for details), while case (3) is Kirchberg question whether every group has property MF (see [5] for details).

In this paper, a group \( \Gamma \) will be called \( p \)-norm approximated if it is approximated with respect to \( \mathcal{G} = (U(n), \| \cdot \|_p) \).

A recent breakthrough [8] shows that there exist groups that are not Frobenius approximated (i.e., groups that are not 2-norm approximated). Following this, Andreas Thom asks in his ICM 2018 talk [26], if that result can be extended to all Schatten \( p \)-norms. We answer this affirmatively in the case where \( 1 < p < \infty \), and in fact we prove a somewhat stronger result:

**Theorem 1.2.** There exists a finitely presented group \( \Lambda \) which is not \( p \)-norm approximated for any \( 1 < p < \infty \).

The case of \( p = 1 \) is left open, as well as the cases of the Hilbert-Schmidt and the operator norms.

The method of proof follows the one implemented in [8] for \( p = 2 \), but some further cohomology vanishing results are needed.

Let \( \Gamma \) be a finitely presented group \( \Gamma = \langle S | R \rangle \), with \( R \subseteq F_S \) - the free group on \( S \) and \( |R| < \infty \). Any map \( \phi : S \to U(n) \) uniquely determines a homomorphism \( \phi : F_S \to U(n) \) which we will also denote by \( \phi \).

The group \( \Gamma \) is called \( \mathcal{G} = (U(n), d_n) \)-stable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( n \in \mathbb{N} \), if \( \phi : S \to U(n) \) is a map with

\[
\sum_{r \in R} d_n(\phi(r), \text{id}_{U(n)}) < \delta,
\]

then there exists a homomorphism \( \tilde{\phi} : \Gamma \to U(n) \) (or equivalently, a map \( \tilde{\phi} : S \to U(n) \) with \( \sum_{r \in R} d_n(\tilde{\phi}(r), \text{id}_{U(n)}) = 0 \)) with

\[
\sum_{s \in S} d_n(\phi(s), \tilde{\phi}(s)) < \varepsilon.
\]

Below, we will call a group \( \Gamma \) \( p \)-norm stable if it is stable with respect to \( \mathcal{G} = (U(n), \| \cdot \|_p) \).

A well-known observation (see for instance [15], [1], [8]) is that a \( \mathcal{G} \)-approximated \( \mathcal{G} \)-stable group must be residually finite. Thus a non-residually-finite group which is \( \mathcal{G} \)-stable cannot be \( \mathcal{G} \)-approximated.

In [8], a general sufficient criterion for Frobenius stability was given: If \( H^2(\Gamma, V) \neq 0 \) for every unitary representation of \( \Gamma \) on any Hilbert space \( V \), then \( \Gamma \) is Frobenius stable. This was combined then with Graalander’s method [14] (as extended by Ballmann and Świątkowski [2] for general Hilbert spaces) to produce some lattices \( \Gamma_0 \) in some simple \( l \)-adic Lie groups satisfying the desired \( H^2 \) vanishing for every Hilbert space. Then a \( l \)-adic analogue of a result by Deligne [9] was implemented in order to produce some finite central extensions \( \tilde{\Gamma} \).
of $\Gamma_0$ that are not residually finite. These $\tilde{\Gamma}$ are the non-Frobenius approximated groups.

The proof in [8] actually shows more (see Theorem 5.1 and Remark 5.2 there): If $\|\cdot\|$ is any unitarily invariant and submultiplicative norm on $M_n(\mathbb{C})$ (and so is the Schatten $p$-norm for every $1 \leq p \leq \infty$) and if $H^2(\Gamma, V) = 0$ for any $\Gamma$ representation on a Banach space of the form $V = \prod_{k \to U}(M_{n_k}(\mathbb{C}), \|\cdot\|)$, where $V$ is the Banach ultraproduct of $M_{n_k}(\mathbb{C})$ with respect to the norm $\|\cdot\|$ and with respect to any ultrafilter $U$ (see [8] and §2 below for more) then $\tilde{\Gamma}$ is $G$-stable. To get non-$p$-norm approximated groups we need an $H^2$-vanishing result which will work for spaces of the form $V = \prod_{k \to U}(M_{n_k}(\mathbb{C}), \|\cdot\|_p)$, where $\|\cdot\|_p$ is the Schatten $p$-norm.

The technology to extend Garland’s method (or more precisely the methods of Dymara and Januszkiewicz [10]) to a wide class of Banach spaces was developed by the second named author in [19]. More precisely, it is shown there that for certain classes of Banach spaces vanishing of cohomology can be deduced for a $l$-adic Lie group $G$, given a large enough thickness of the affine building on which it acts. Using (a suitable version of) Shapiro’s Lemma, these vanishing results pass to cocompact lattices of $G$. In our context, these methods yield the following Theorem:

**Theorem 1.3.** Let $d \geq 2$ be a fixed integer. For any $1 < p_1 \leq 2 \leq p_2 < \infty$, there exists a natural number $Q = Q(p_1, p_2, d)$ such that if $\Gamma_0$ is a countable group acting properly and cocompactly on a classical affine building of dimension $d$ of thickness greater than $Q$, then $H^i(\Gamma_0, V) = 0$ for every $i = 1, ..., d - 1$ and every Banach space of the form $V = \prod_{k \to U}(M_{n_k}(\mathbb{C}), \|\cdot\|_p)$ where $U$ is any ultrafilter on $\mathbb{N}$ and $p_1 \leq p \leq p_2$.

Most of the paper will be devoted to prove Theorem 1.3. Let us now show how it implies the main Theorem 1.2.

Applying Deligne’s method as in [8], we get a non-residually finite, finite central extension $\tilde{\Gamma} = \Gamma_{p_1, p_2}$ of such a cocompact lattice $\Gamma_0$ in a suitable $l$-adic Lie group $G$. Assuming that the dimension of the affine building associated to $G$ is greater or equal to 3, a standard spectral sequence argument yields that $H^2(\tilde{\Gamma}, V) = 0$ for any $V$ as in Theorem 1.3. Therefore $\tilde{\Gamma}$ is $p$-norm stable for any $p_1 \leq p \leq p_2$ and since it is not residually finite, we deduce by the observation stated above that $\tilde{\Gamma}$ is not $p$-norm approximated for any $p_1 \leq p \leq p_2$.

Recalling now Higman’s Theorem (see [16] Theorem 7.3, page 215)) which asserts that there exists a finitely presented group $\Lambda$ that contains all finitely presented groups. By taking $p_1 \rightarrow 1$ and $p_2 \rightarrow \infty$ and noting that if a group is $G$-approximated, so is every subgroup of it, we deduce that such $\Lambda$ is not $p$-norm approximated for any $1 < p < \infty$ and Theorem 1.2 is proved.

As mentioned above, the cases of $p = 1$ and $p = \infty$ are left open. In both cases (unlike the Hilbert-Schmidt norm) the norms are submultiplicative (see [8] for an explanation of the importance of this property), but at least for $p = \infty$ (which is the case of the operator norm), the method of this paper cannot work: the method applied below shows vanishing of $H^i(\Gamma_0, V) = 0$ for every $i = 1, ..., d - 1$, based on the geometric properties of $V$. Therefore, if the vanishing of cohomology is proved for $p = \infty$, it will be proven for every $\ell^\infty$ Banach space, but it is known that for every discrete group $\Gamma$, $H^i(\Gamma, \ell^\infty(\Gamma)) \neq 0$ (see for instance [8] Section 4)). We note that this type of reasoning excluding
\( p = \infty \) does not hold in the case of \( p = 1 \): in \([2]\), Bader, Gelander and Monod showed that for every group \( \Gamma \) with property (T), \( H^1(\Gamma, L^1(\Omega)) = 0 \) for every measure space \( \Omega \). The methods of \([2]\) are very different from those applied in this paper (and in \([15]\)), but one can ask it those methods can be extended to show the vanishing of the second cohomology for the case \( p = 1 \).

The rest of this paper is devoted to the proof of Theorem 1.3. As noted above, in \([19]\), the second named author proved a similar vanishing of cohomology, but for a \( \ell \)-adic Lie group \( G \) instead of a lattice. Below, we will show how to use the results of \([19]\) together with a version of Shapiro’s Lemma to deduce Theorem 1.3. Unfortunately, the paper \([19]\) was not written with this application in mind and therefore in order to adapt the results of \([19]\) to our setting, a somewhat lengthy exposition regarding the general theory of Banach spaces and group representations on them is needed.

This paper is organized as follows: In \([2]\) we give a number of definitions and results needed to state the results regarding group cohomology with Banach coefficients. In \([3]\) we will prove Theorem 1.3 by combining the results of \([19]\) with Shapiro’s Lemma. Unfortunately, it seems that the version of Shapiro’s Lemma we need (for Banach spaces rather than Hilbert spaces) is not proved in the literature and therefore we will prove it in \([3]\) and then deduce Theorem 1.3 from it.

Acknowledgments. The first named author was supported in part by the ERC and the NSF. This work was done while the authors were visiting the IIAS (Israeli Institute of Advanced Studies) whose great hospitality is warmly acknowledged.

2 Preliminaries

2.1 Strictly \( \theta \)-Hilbertian spaces and Schatten norms

Two Banach spaces \( V_0, V_1 \) form a compatible pair \( (V_0, V_1) \) if they are continuously linear embedded in the same topological vector space. The idea of complex interpolation is that given a compatible pair \( (V_0, V_1) \) and a constant \( 0 \leq \theta \leq 1 \), there is a method to produce a new Banach space \([V_0, V_1]_\theta \) as a “combination” of \( V_0 \) and \( V_1 \). We will not review this method here, and the interested reader can find more information on interpolation in \([4]\).

This brings us to consider the following definition due to Pisier \([21]\): a Banach space \( V \) is called strictly \( \theta \)-Hilbertian for \( 0 < \theta \leq 1 \), if there is a compatible pair \( (V_0, V_1) \) with \( V_1 \) a Hilbert space such that \( V = [V_0, V_1]_\theta \). Examples of strictly strictly \( \theta \)-Hilbertian spaces are \( L^p \) space and non-commutative \( L^p \) spaces (see \([22]\) for definitions and properties of non-commutative \( L^p \) spaces), where in these cases \( \theta = \frac{2}{p} \) if \( 2 \leq p < \infty \) and \( \theta = 2 - \frac{2}{p} \) if \( 1 < p \leq 2 \). We are interested in a very basic case of non-commutative \( L^p \) spaces - namely finite matrices with \( p \)-Schatten norms:

Definition 2.1 (Schatten norm for matrices). Let \( d \in \mathbb{N} \) and let \( M_d(\mathbb{C}) \) be the space of \( d \times d \) complex matrices. For \( A \in M_d(\mathbb{C}) \), recall that \( A^* A \) is always a positive semidefinite matrix and denote \( |A| = \sqrt{A^* A} \). For \( 1 \leq p < \infty \), define the
Schatten $p$-norm on $M_d(\mathbb{C})$ by

$$\|A\|_p = (\text{tr}(|A|^p))^{\frac{1}{p}}.$$ 

### 2.2 Vector valued $L^2$ spaces

Given a measure space $\Omega$ with a finite measure $\mu$ and Banach space $V$, a function $s : \Omega \to V$ is called simple if it is of the form:

$$s(\omega) = \sum_{i=1}^{n} \chi_{E_i}(\omega)v_i,$$

where $\{E_1, ..., E_n\}$ is a partition of $\Omega$ where each $E_i$ is a measurable set, $\chi_{E_i}$ is the indicator function on $E_i$ and $v_i \in V$.

A function $f : \Omega \to V$ is called Bochner measurable if it is almost everywhere the limit of simple functions, i.e., if there is a sequence of simple functions $s_n : \Omega \to V$ such that for almost every $\omega$, $f(\omega) = \lim_{n} s_n(\omega)$. Denote $L^2(\Omega; V)$ to be the space of Bochner measurable functions satisfying:

$$\|f\|_{L^2(\Omega; V)} = \left( \int_{\Omega} \|f(\omega)\|^2_V d\mu(\omega) \right)^{\frac{1}{2}} < \infty.$$

Given a bounded linear operator $T \in B(L^2(\Omega, \mu))$, we can define a bounded linear operator $T \otimes id_V \in B(L^2(\Omega; V))$ by defining it first on simple functions and extending it to the whole space $L^2(\Omega; V)$.

For our use, it will be important to bound the norm of an operator of the form $T \otimes id_V$ given that $V$ is an interpolation space.

**Lemma 2.2.** [24, Lemma 3.1] Let $(V_0, V_1)$ be a compatible pair, $(\Omega, \mu)$ be a finite measure space and $T \in B(L^2(\Omega, \mu))$ be an operator. Then for every $0 \leq \theta \leq 1$,

$$\|T \otimes id_{[V_0, V_1]_\theta}\|_{B(L^2(\Omega; [V_0, V_1]_\theta))} \leq \|T \otimes id_{V_0}\|_{B(L^2(\Omega; V_0))}^{1-\theta} \|T \otimes id_{V_1}\|_{B(L^2(\Omega; V_1))}^\theta,$$

where $[V_0, V_1]_\theta$ is the interpolation of $V_0$ and $V_1$.

This lemma has an immediate corollary for strictly $\theta$-Hilbertian spaces:

**Corollary 2.3.** Let $V$ be a strictly $\theta$-Hilbertian space with $0 < \theta \leq 1$, let $(\Omega, \mu)$ be a measure space and $T \in B(L^2(\Omega, \mu))$ be an operator. Assume that $T \in B(L^2(\Omega, \mu))$ is an operator with the following properties:

1. For every Banach space $V_0$, $\|T \otimes id_{V_0}\|_{B(L^2(\Omega; V_0))} \leq L$.
2. For every Hilbert space $V_1$, $\|T \otimes id_{V_1}\|_{B(L^2(\Omega; V_1))} \leq \delta$.

Then $\|T \otimes id_V\|_{B(L^2(\Omega; V))} \leq L\delta^\theta$.

We will also be interested in how $T \otimes id_V$ behaves under some operations - this is summed up in the following lemmas:
Lemma 2.4. Let $(\Omega, \mu)$ be a measure space with a finite measure, $T$ a bounded operator on $L^2(\Omega, \mu)$ and $C > 0$ constant. Let $\mathcal{E} = \mathcal{E}(C)$ be the class of Banach spaces defined as:

$$\mathcal{E} = \{ V : \| T \otimes id_V \|_{B(L^2(\Omega; V))} \leq C \}.$$ 

Then this class is closed under quotients, subspaces, $l_2$-sums, ultraproducts of Banach spaces, i.e., preforming any of these operations on Banach spaces in $\mathcal{E}$ yield a Banach space in $\mathcal{E}$. Also, for any finite measure space $(\Lambda, \nu)$ and every $V \in \mathcal{E}$, we have that $L^2(\Lambda; V) \in \mathcal{E}$.

Proof. The fact that $\mathcal{E}$ is closed under quotients, subspaces and ultraproducts of Banach spaces was shown in [24, Lemma 3.1]. The fact that $\mathcal{E}$ is closed under $l_2$-sums is straight-forward and left for the reader (we will not make any use of it in this paper).

Let $(\Lambda, \nu)$ be a measure space with a finite measure and $V \in \mathcal{E}$. By our definition of vector valued spaces using simple functions, it is enough to check that the inequality holds for simple functions $s : \Omega \to L^2(\Lambda; V)$. Moreover, it is enough to check for simple functions $s : \Omega \to L^2(\Lambda; V)$ whose values are simple functions in $L^2(\Lambda; V)$. In other word, if we identify $L^2(\Omega; L^2(\Lambda; V))$ with $L^2(\Omega \times \Lambda; V)$, we need to show that the needed inequality holds for functions of the form:

$$s(\omega, \lambda) = \sum_{i=1}^{n} \sum_{j=1}^{m} \chi_{E_i}(\omega) \chi_{F_j}(\lambda) v_{i,j},$$

where $\{E_1, ..., E_n\}$ is a measurable partition of $\Omega$, $\{F_1, ..., F_m\}$ is a measurable partition of $\Lambda$, and $v_{i,j} \in V$. Let $s$ as above, then

$$\|s\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(E_i) \nu(F_j) \| v_{i,j} \|_V^2.$$ 

We recall that for every measurable set $E \subseteq \Omega$ and every $v \in V$, the action of $T \otimes id_V$ on $\chi_{EV}$ is defined by

$$(T \otimes id_V)(\chi_{EV}) = T(\chi_E)v.$$ 

Similarly, for every function $f \in L^2(\Lambda; V)$, the action of $T \otimes id_{L^2(\Lambda; V)}$ of $\chi_{Ef}$ is defined by

$$(T \otimes id_{L^2(\Lambda; V)})(\chi_{Ef}) = T(\chi_E)f.$$ 

Therefore, the action of $T \otimes id_{L^2(\Lambda, V)}$ on $s$ is as follows (we are abusing the notation, since the action of $T \otimes id_{L^2(\Lambda, V)}$ is defined on $L^2(\Omega; L^2(\Lambda; V))$ and not on $L^2(\Omega \times \Lambda; V)$):

$$(T \otimes id_{L^2(\Lambda, V)})s = \sum_{i=1}^{n} T(\chi_{E_i}) \sum_{j=1}^{m} \chi_{F_j} v_{i,j}$$

$$= \sum_{j=1}^{m} \chi_{F_j} \sum_{i=1}^{n} T(\chi_{E_i}) v_{i,j}$$

$$= \sum_{j=1}^{m} \chi_{F_j} \sum_{i=1}^{n} (T \otimes id_V)(\chi_{E_i} v_{i,j})$$

$$= \sum_{j=1}^{m} \chi_{F_j} (T \otimes id_V)(\sum_{i=1}^{n} \chi_{E_i} v_{i,j}).$$
Note that written as above, for every $j$, $\sum_{i=1}^{n}(\chi_{E_{i}})v_{i,j} \in L^{2}(\Omega; V)$ and therefore since $V \in \mathcal{E}$, we have for every $j$ that

$$\| (T \otimes id_{V})(\sum_{i=1}^{n}(\chi_{E_{i}})v_{i,j}) \|^{2} \leq C^{2}\| \sum_{i=1}^{n}(\chi_{E_{i}})v_{i,j} \|^{2}.$$ 

This yields that

$$\| (T \otimes id_{L^{2}(\Lambda; V)})s \|^{2} = \sum_{j=1}^{m}\nu(F_{j})\| (T \otimes id_{V})(\sum_{i=1}^{n}(\chi_{E_{i}})v_{i,j}) \|^{2} \leq \sum_{j=1}^{m}\nu(F_{j})C^{2}\| \sum_{i=1}^{n}(\chi_{E_{i}})v_{i,j} \|^{2} = C^{2}\| s \|^{2},$$

as needed.

### 2.3 Uniformly convex Banach space

We recall the definition of a uniformly convex Banach space:

**Definition 2.5.** A Banach space $V$ is called uniformly convex if for every $0 < \varepsilon \leq 2$, there is $\delta = \delta(\varepsilon) > 0$ such that for every $x, y \in V$, with $\|x\| = \|y\| = 1$, if $\|x - y\| \geq \varepsilon$, then

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

Given a uniformly convex space $V$, the function $\delta_{V} : (0, 2] \to (0, 1]$ defined as

$$\delta_{V}(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in V, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\},$$

is called the modulus of convexity of $V$.

**Lemma 2.6.** Let $\delta_{0} : (0, 2] \to (0, 1]$ be a function and let $\mathcal{E} = \mathcal{E}(\delta_{0})$ be the class of all uniformly convex Banach spaces $V$ such that for every $\varepsilon \in (0, 2]$, $\delta_{V}(\varepsilon) \geq \delta_{0}(\varepsilon)$. Then $\mathcal{E}$ is closed under ultraproducts, subspaces, and $l_{2}$-sums.

**Proof.** The proof is straightforward and left for the reader.

**Proposition 2.7.** Let $0 < \theta_{0} \leq 1$, then for every $\theta$ such that $\theta_{0} \leq \theta \leq 1$ and every strictly $\theta$-Hilbertian space $V$, $V$ is uniformly convex with

$$\delta_{V}(\varepsilon) \geq 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^{\theta_{0}} \right)^{\frac{\theta}{\theta_{0}}}.$$

**Proof.** Let $\theta_{0} \leq \theta \leq 1$ and $V$ be a strictly $\theta$-Hilbertian space. Every strictly $\theta$-Hilbertian space satisfies Clackson inequality (see [21] Section 4): 

$$\forall x, y \in V, \left( \frac{\|x + y\|^{2} + \|x - y\|^{2}}{2} \right)^{\frac{\theta}{2}} \leq \left( \|x\|^{\frac{2}{1+\theta}} + \|y\|^{\frac{2}{1+\theta}} \right)^{\frac{2-\theta}{1+\theta}}.$$

7
Let \( x, y \in V \) such that \( \|x\| = \|y\| = 1 \), then
\[
\left( \frac{\|x + y\| \theta + \|x - y\| \theta}{2} \right)^{\frac{\theta}{2}} \leq 2^{\frac{\theta - 1}{2}},
\]
which implies that
\[
\| \frac{x + y}{2} \|^{\theta} + \| \frac{x - y}{2} \|^{\theta} \leq 1.
\]
Assume also that \( \| x - y \| \geq \varepsilon \), then
\[
\| \frac{x + y}{2} \| \leq \left( 1 - \left( \frac{\varepsilon}{2} \right)^{\theta} \right)^{\frac{\theta}{2}} \leq \left( 1 - \left( \frac{\varepsilon}{2} \right)^{\theta_0} \right)^{\frac{\theta}{2}},
\]
as needed.

**Corollary 2.8.** Let \( \theta_0 > 0 \) and define \( \delta_0 : (0, 2] \to (0, 1] \) by
\[
\delta_0(\varepsilon) = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^{\theta_0} \right)^{\frac{\theta}{2}}.
\]
Let \( \mathcal{E} = \mathcal{E}(\delta_0) \) be the space of all uniformly convex Banach spaces \( V \) such that for every \( \varepsilon \in (0, 2] \), \( \delta_0(\varepsilon) \geq \delta_0(\varepsilon) \). Then \( \mathcal{E} \) contains all strictly \( \theta \)-Hilbertian spaces with \( \theta_0 \leq \theta \leq 1 \) and \( \mathcal{E} \) is closed under ultraproducts, subspaces, and \( l_2 \)-sums.

**Proof.** Combine the above proposition with the preceding lemma.

### 2.4 Group representations in a Banach space

Let \( G \) be a locally compact group and \( V \) a Banach space. Let \( \pi \) be a representation \( \pi : G \to B(V) \), where \( B(V) \) are the bounded linear operators on \( V \). Throughout this paper we shall always assume \( \pi \) is continuous with respect to the strong operator topology without explicitly mentioning it. We recall that given \( \pi \) the dual representation \( \pi^* : G \to B(V^*) \) is defined as
\[
\langle v, \pi^*(g)u \rangle = \langle \pi(g^{-1})v, u \rangle, \forall g \in G, v \in V, u \in V^*.
\]
We remark that \( \pi^* \) might not be continuous for a general Banach space, but it is continuous for a large class of Banach spaces, called Asplund spaces defined below.

Denote by \( S_c(G) \) the group algebra of compactly supported simple functions on \( G \) with convolution. For any \( f \in S_c(G) \) we can define \( \pi(f) \in B(V) \) as
\[
\pi(f).v = \int_G f(g)\pi(g).vd\mu(g), \text{ where } v \in V,
\]
and the above integral is the Bochner integral with respect to the (left) Haar measure \( \mu \) of \( G \).

Next, we will restrict ourselves to the case of compact groups. Let \( K \) be a compact group with a Haar measure \( \mu \) and let \( S(K) = S_c(K) \) defined as above. Let \( V \) be a Banach space and let \( \pi \) be a representation of \( K \) on \( V \). We will show that for every \( f \in S(K) \), we can bound the norm of \( \pi(f) \) using the norm of \( \lambda \otimes id_V \in B(L^2(K; V)) \) (with the notation as in (2.2)).
Proposition 2.9. \[\text{[13, Corollary 2.11]}\] Let \(\pi\) be a representation of a compact group \(K\) on a Banach space \(V\). Then for any real function \(f \in S(K)\),

\[
\|\pi(f)\|_{B(V)} \leq \left(\sup_{g \in K} \|\pi(g)\|\right)^2 \|(\lambda \otimes id_V)(f)\|_{B(L^2(K;V))},
\]

where \(\lambda\) is the left regular representation of \(K\).

In particular, if \(\pi\) is an isometric representation, then \(\|\pi(f)\|_{B(V)} \leq \|(\lambda \otimes id_V)(f)\|_{B(L^2(K;V))}\).

2.5 Asplund spaces

Definition 2.10. A Banach space \(V\) is said to be an Asplund space if every separable subspace of \(V\) has a separable dual.

There are many examples of Asplund spaces - for instance every reflexive space is Asplund (see [27] for a nice exposition on Asplund spaces). The reason we are interested in Asplund spaces is the following theorem of Megrelishvili:

Theorem 2.11. \[\text{[17, Corollary 6.9]}\] Let \(G\) be a topological group and let \(\pi\) be a continuous representation of \(G\) on a Banach space \(V\). If \(V\) is an Asplund space, then the dual representation \(\pi^*\) is also continuous.

Remark 2.12. In this paper, we will be mainly interested in uniformly convex spaces (see Definition 2.5). A standard fact regarding uniformly convex spaces is that they are super-reflexive (and in particular reflexive) and therefore Asplund spaces. Thus, by Theorem 2.11, for every continuous representation \(\pi\) of a topological group \(G\) on a uniformly convex space, \(\pi^*\) is also continuous.

Asplund spaces can be alternatively characterized as Banach spaces that have the Radon-Nikodym property (see definition in [27]). Using this characterization it follows from a result of Sundaresan [25] that the property of being Asplund is preserved when considering vector values \(L^2\)-spaces:

Theorem 2.13. \[\text{[25, Theorem 1]}\] Let \(V\) be a Banach space and let \((\Omega, \mu)\) be a measure space with a finite measure. Then \(V\) is Asplund if and only if \(L^2(\Omega; V)\) is Asplund.

2.6 Group cohomology for groups acting on simplicial complexes

Let \(X\) be an \(n\)-dimensional simplicial complex and let \(G\) be a group acting on \(X\). Denote \(X(k)\) to be the set of \(k\)-faces of \(X\) and \(\tilde{X}(k)\) to be the set of ordered \(k\)-simplices of \(X\). Let \(V\) be a vector space and \(\pi\) a representation of \(G\) on \(V\). Let \(0 \leq k \leq n\) and let \(\phi : \tilde{X}(k) \rightarrow V\). Recall the following definitions:

- \(\phi\) is anti-symmetric if for every permutation \(\tau \in \text{Sym}\{0, ..., k\}\) and every \((v_{i_0}, ..., v_{i_k})\), \(\phi((v_{\tau(i_0)}, ..., v_{\tau(i_k)})) = \text{sgn}(\tau)\phi((v_{i_0}, ..., v_{i_k}))\).

- \(\phi\) is twisted by \(\pi\), if for every \((v_{i_0}, ..., v_{i_k})\) and every \(g \in G\),

\[
\pi(g)\phi((v_{i_0}, ..., v_{i_k})) = \phi(g. (v_{i_0}, ..., v_{i_k})).
\]
For $0 \leq k \leq n$, denote $C^k(X, \pi)$ to be the space of maps $\phi: \vec{X}(k) \to V$ that are anti-symmetric and twisted by $\pi$. Define the differential map $d_k: C^k(X, \pi) \to C^{k+1}(X, \pi)$ in the usual way:

$$(d_k \phi)((v_0, \ldots, v_{k+1})) = \sum_{i=0}^{k+1} (-1)^i \phi((v_0, \ldots, \hat{v}_i, \ldots, v_{k+1})).$$

As in the case of simplicial (untwisted) cohomology, we have that $d_{k+1} \circ d_k = 0$ and $H^k(X, \pi) = \text{Ker}(d_k) / \text{Im}(d_{k-1})$. The next theorem states that under certain conditions, this cohomology is isomorphic to the group cohomology of $G$ under the representation $\pi$:

**Theorem 2.14.** [6, X.1.12] Let $G$ be a topological group and $X$ a contractible, locally finite simplicial complex. Assume that $G$ acts simplicially on $X$ and that this action is cocompact and proper. Assume further, that $V$ is a Banach space and $\pi$ is a continuous representation of $G$ on $V$, then $H^k(G, \pi) = H^k(X, \pi)$.

### 3 Vanishing of cohomology

#### 3.1 Framework

The aim of this section is to prove the cohomology vanishing Theorem stated in the introduction (Theorem 1.3). In order to achieve this, we will prove a version of Shapiro’s Lemma and combine it with the cohomology vanishing result proved in [19].

Below, $X$ will denote an $n$-dimensional pure (i.e., every maximal cell is $n$-dimensional) contractible simplicial complex that is $(n+1)$-colorable (i.e., the vertices of $X$ can be colored by $n+1$ colors and every $n$-dimensional cell of $X$ has a vertex of every color) and locally finite (i.e., every vertex of $X$ is contained in a finite number of simplices). By $G$ we denote a locally compact, unimodular topological group with a Haar measure $\mu$ acting properly and cocompactly on $X$ such that the action preserves the coloring and $G$ acts transitively on the $n$-dimensional simplices of $X$ (note that this implies that $G$ is compactly generated). We denote by $\Delta$ a fixed $n$-dimensional simplex of $X$ that serves as the fundamental domain for the action of $G$. By $\Gamma$ we denote a countable subgroup of $G$ that also acts properly and cocompactly on $X$. So $\Gamma$ is a discrete cocompact subgroup of $G$. The case of interest for us is when $G = G(K)$ - the $K$-points of a simple, rank $n$, $K$-algebraic group $G$ when $K$ is a local non-archimedean field. In this case, $G$ acts properly on the Bruhat-Tits building $X$ associated with it, which is a contractible, pure $n$-dimensional, locally finite, $(n+1)$-colorable simplicial complex and the fundamental domain of the action of $G$ on $X$ is a single $n$-dimensional simplex. In this case, $\Gamma$ is a uniform (= cocompact) lattice, and by Margulis arithmeticity theorem (see [28, Chapter 6]), if $n \geq 2$, it is an arithmetic lattice.

#### 3.2 Shapiro’s Lemma

**Definition 3.1.** Let $G$ and $\Gamma$ as above, $V$ be a Banach space and $\pi$ an isometric representation of $\Gamma$ on $V$. Denote by $\nu$ the invariant measure on $G/\Gamma$ induced
by the Haar measure of $G$. Define the Banach space $L^2(G/\Gamma; V)$ to be the space of Bochner measurable functions $f : G/\Gamma \to V$ with the norm:

$$
\|f\| = \left( \int_{G/\Gamma} \|f(g)\|^2 d\nu(g) \right)^{\frac{1}{2}}.
$$

(1)

By choosing a fundamental domain $D$ to the action of $\Gamma$ on $G$, we can identify functions $L^2(G/\Gamma; V)$ with $L^2(D; V)$, where $D$ is taken with (the restriction of the) measure $\mu$.

The induced representation of $G$ on $\Gamma$, denoted $\text{Ind}_G^G(\pi)_{L^2}$, is defined as follows:

$$
\text{Ind}_G^G(\pi)_{L^2} = \{ f : G \to V : \forall g \in G, h \in \Gamma, f(gh^{-1}) = \pi(h)f(g) \text{ and } f \in L^2(G/\Gamma) \},
$$

where $f \in L^2(G/\Gamma)$ means that $f$ is Bochner measurable when restricted to $D$ and with the norm define in (1) above (the reader should note that $\|f(g)\|$ is well defined on $G/\Gamma$, because $\pi$ is isometric and therefore for every $h \in \Gamma, g \in G$, $\|f(gh^{-1})\| = \|\pi(h)f(g)\| = \|f(g)\|$).

Also, $G$ acts on $\text{Ind}_G^G(\pi)_{L^2}$ by left translation, denoted $\lambda_{\text{Ind}_G^G(\pi)_{L^2}}$, as:

$$
\lambda_{\text{Ind}_G^G(\pi)_{L^2}}(g)f(g') = f(g^{-1}g'), \forall g, g' \in G.
$$

**Remark 3.2.** The induced representation can also be defined as follows: define $\text{Ind}_G^G(\pi)$ to be the vector space

$$
\text{Ind}_G^G(\pi) = \{ f : G \to V \text{ continuous} : \forall g \in G, h \in \Gamma, f(gh^{-1}) = \pi(h)f(g) \},
$$

and complete the vector space with respect to the $L^2$ norm as in the definition of $\text{Ind}_G^G(\pi)_{L^2}$. The equivalence between these definitions is proven in [13, Chapter 4] in the setting of isometric actions on Hilbert spaces, but the proof can be generalized to our setting. We will not make any use this equivalent definition below.

**Proposition 3.3.** Let $G$, $\Gamma$, $V$ and $\pi$ be as above. Then $\text{Ind}_G^G(\pi)_{L^2}$ is a Banach space and the action of $G$ on $\text{Ind}_G^G(\pi)_{L^2}$ by left translation, denoted $\lambda_{\text{Ind}_G^G(\pi)_{L^2}}$ is an isometric continuous representation of $G$ on $\text{Ind}_G^G(\pi)_{L^2}$.

**Proof.** The fact that $\lambda_{\text{Ind}_G^G(\pi)_{L^2}}$ is isometric and continuous when $\pi$ is isometric is straight-forward and left for the reader.

Classically, Shapiro’s Lemma is the equality $H^*(\Gamma, \pi) = H^*(G, \lambda_{\text{Ind}_G^G(\pi)})$. This equality is proven in [6] for $\text{Ind}_G^G(\pi)$ defined in Remark 3.2 but not for $\text{Ind}_G^G(\pi)_{L^2}$ which is a larger space (see Remark 3.2). Below, we will prove the equality $H^*(\Gamma, \pi) = H^*(G, \lambda_{\text{Ind}_G^G(\pi)_{L^2}})$ under the assumptions on $G$ and $\Gamma$ stated in the beginning of this Section. We suspect that this equality is true even without our added assumptions, but in the case we are interested in, the proof of this equality given below is direct and elementary.

**Theorem 3.4** ($L^2$-Shapiro’s Lemma with coefficients in Banach representations). Let $X$, $G$, $\Gamma$ be as above, $V$ a Banach space and $\pi$ an isometric representation of $\Gamma$ on $V$. Then

$$
H^*(\Gamma, \pi) = H^*(G, \lambda_{\text{Ind}_G^G(\pi)_{L^2}}).
$$
Lemma 3.5. Let \( X, G, \Gamma \) be as above, \( V \) a Banach space and \( \pi \) an isometric representation of \( \Gamma \) on \( V \). Given \( \phi \in C^k(X, \pi) \) and \( \sigma \in \bar{X}(k) \), define \( f_{\phi, \sigma} : G \to V \) by

\[
f_{\phi, \sigma}(g) = \phi(g^{-1}.\sigma).
\]

Then \( f_{\phi, \sigma} \in \text{Ind}_{\Gamma}^{G}(\pi)_{L^2} \) and if for some \( g \in G \), \( \phi(g^{-1}.\sigma) \neq 0 \), then \( \|f_{\phi, \sigma}\| > 0 \).

Proof. We note that for every \( h \in \Gamma \) and every \( g \in G \), we have that

\[
f_{\phi, \sigma}(gh^{-1}) = \phi(hg^{-1}.\sigma) = \pi(h)\phi(g^{-1}.\sigma) = \pi(h)f_{\phi, \sigma}(g),
\]

therefore we are left to show that \( f_{\phi, \sigma} \in \text{Ind}_{\Gamma}^{G}(\pi)_{L^2} \) and if for some \( g \in G \), \( \phi(g^{-1}.\sigma) \neq 0 \), then \( \|f_{\phi, \sigma}\| > 0 \). Most of the work in the rest of this proof is choosing a convenient fundamental domain \( D \) for the action of \( \Gamma \) on \( G \).

By our assumptions, \( \Gamma \) acts cocompactly on \( X \) and therefore \( \Gamma \setminus \bar{X}(k) \) is finite. In particular, there are \( \sigma_1, \ldots, \sigma_m \in \bar{X}(k) \) such that

\[
\{g.\sigma : g \in G\} = \bigcup_{i=1}^{m}[h.\sigma_i : h \in \Gamma],
\]

and the union above is disjoint. Fix \( g_i \in G \), \( i = 1, \ldots, m \), such that \( g_i.\sigma = \sigma_i \) (such \( g_i \)'s exist, because we assumed that \( G \setminus X \) is a single colored \( n \)-dimensional simplex). It follows that for every \( g \in G \), there are \( h \in \Gamma \) and a unique \( i \) such that

\[
g^{-1}.\sigma = h.\sigma_i = h g_i.\sigma,
\]

i.e., \( g_i^{-1}h^{-1}.\sigma = \sigma \). If we denote the stabilizer of \( \sigma \) in \( G \) by \( G_{\sigma} \), we deduce that there is \( g_{\sigma} \in G_{\sigma} \) such that \( g^{-1} = h g_{\sigma} \) and therefore

\[
G = \bigcup_{i=1}^{m} G_{\sigma}g_{\sigma}^{-1} = \bigcup_{i=1}^{m} g_{\sigma}^{-1}(\sigma g_{\sigma}^{-1} G_{\sigma}^{-1}) = \bigcup_{i=1}^{m} g_{\sigma}^{-1}(G_{\sigma}^{-1} G_{\sigma} G_{\sigma}^{-1}) = \bigcup_{i=1}^{m} g_{\sigma}^{-1}(G_{\sigma} G_{\sigma}^{-1} G_{\sigma})
\]

and this is a disjoint union.

For every \( i \), denote \( \Gamma_{\sigma_i} = G_{\sigma_i} \cap \Gamma \) and choose \( D_{\sigma_i} \) to be a fundamental domain for the action of \( \Gamma_{\sigma_i} \) on \( G_{\sigma_i} \). We claim that \( D = \bigcup_{i=1}^{m} g_{\sigma_i}^{-1}(D_{\sigma_i}) \) is a fundamental domain for the action of \( \Gamma \) on \( G \). Indeed, \( D_{\sigma_i} \subseteq G_{\sigma_i} \), and therefore \( D \) is defined by a disjoint union and

\[
\left( \bigcup_{i=1}^{m} g_{\sigma_i}^{-1}D_{\sigma_i} \right) \Gamma = \bigcup_{i=1}^{m} g_{\sigma_i}^{-1}D_{\sigma_i} \Gamma_{\sigma_i} \Gamma = \bigcup_{i=1}^{m} g_{\sigma_i}^{-1}G_{\sigma_i} \Gamma = G,
\]

as needed.

With this choice of fundamental domain, it follows that

\[
\int_{G/\Gamma} \|f_{\phi, \sigma}(g)\|_V^2 \, d\nu(g) = \int_{D} \|f_{\phi, \sigma}(g)\|_V^2 \, d\mu(g) = \sum_{i=1}^{m} \int_{g_{\sigma_i}^{-1}D_{\sigma_i}} \|f_{\phi, \sigma}(g)\|_V^2 \, d\mu(g) = \sum_{i=1}^{m} \int_{D_{\sigma_i}} \|f_{\phi, \sigma}(g_{\sigma_i}^{-1} g)\|_V^2 \, d\mu(g) = \sum_{i=1}^{m} \int_{D_{\sigma_i}} \|\phi(g_{\sigma_i}^{-1} g)\|_V^2 \, d\mu(g) = \sum_{i=1}^{m} \int_{D_{\sigma_i}} \|\phi(g_{\sigma_i}^{-1})\|_V^2 \, d\mu(g) = \sum_{i=1}^{m} \|\mu(D_{\sigma_i})\|_V^2 \int_{D_{\sigma_i}} \|\phi(\sigma_i)\|_V^2 \, d\mu(g) = \sum_{i=1}^{m} \mu(D_{\sigma_i}) \|\phi(\sigma_i)\|_V^2.
\]
Note that by the assumption of proper action of $G$ and of $\Gamma$ on $X$, we have for every $1 \leq i \leq m$, that $0 < \mu(G_{\sigma_i}) < \infty$ and $\Gamma_{\sigma_i}$ is a finite group. Therefore $0 < \mu(D_{\sigma_i}) < \infty$ and $f_{\phi,\sigma} \in \text{Ind}_G^\pi(\pi)_{L^2}$. Also note that if for some $g \in G$, $\phi(g^{-1}, \sigma) \neq 0$, then there is $1 \leq i_0 \leq m$, $\phi(\sigma_{i_0}) \neq 0$ and therefore

$$
\int_{G/\Gamma} \|f_{\phi,\sigma}(g)\|^2 d\nu(g) \geq \mu(G_{\sigma_{i_0}}/\Gamma_{\sigma_{i_0}}) \|\phi(\sigma_{i_0})\|^2 > 0.
$$

\[ \square \]

After this Lemma, we can prove Shapiro’s Lemma in our setting:

**Proof.** By Theorem 2.14, it is enough to prove that $H^*(X, \pi) = H^*(X, \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}})$, where $\lambda_{\text{Ind}_G^\pi(\pi)_{L^2}}$ is as above.

We will prove this by finding bijective linear maps $\Phi_k : C^k(X, \pi) \rightarrow C^k(X, \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}})$ for $k = 0, \ldots, n$ such that for every $\phi \in C^k(X, \pi)$, $d_k \Phi_k(\phi) = \Phi_{k+1}(d_k \phi)$. The existence of such maps shows that $H^*(X, \pi) = H^*(X, \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}})$ as needed.

Define $\Phi_k : C^k(X, \pi) \rightarrow C^k(X, \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}})$ by

$$
(\Phi_k(\phi))(\sigma) = f_{\phi,\sigma},
$$

where $f_{\phi,\sigma}$ is defined as in Lemma 3.5.

There are several thing we need to check. First, we need to check that for every $\phi \in C^k(X, \pi)$, it holds that $\Phi(\phi) \in C^k(X, \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}})$. By Lemma 3.5, we have that $f_{\phi,\sigma} \in \text{Ind}_G^\pi(\pi)_{L^2}$. By the definition of $f_{\phi,\sigma}$, it is also clear that $\Phi(\phi)$ is anti-symmetric since $\phi$ is anti-symmetric. $\Phi_k(\phi)$ is also twisted by $\lambda_{\text{Ind}_G^\pi(\pi)_{L^2}}$:

$$
\lambda_{\text{Ind}_G^\pi(\pi)_{L^2}}(g).((\Phi(\phi))(\sigma))(g')
= ((\Phi(\phi))(\sigma))(g^{-1}g')
= f_{\phi,\sigma}(g^{-1}g')
= \phi((g^{-1}g')^{-1}, \sigma)
= \phi((g')^{-1}, (g, \sigma))
= f_{\phi,\sigma}(g')
= ((\Phi(\phi))(g, \sigma))(g'),
$$
as needed. Thus $\Phi_k : C^k(X, \pi) \rightarrow C^k(X, \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}})$ as claimed above.

Second, we note that if $\phi \neq 0$, then for some $\sigma$, $\phi(\sigma) \neq 0$ and therefore by Lemma 3.5 $\Phi(\phi) \neq 0$ and therefore $\Phi$ is injective.

Third, we will check that $\Phi$ is surjective. Let $\psi \in C^k(X, \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}})$, then for every $\sigma \in \tilde{X}(k)$, $\psi(\sigma) \in \text{Ind}_G^\pi(\pi)_{L^2}$. Since $\psi$ is twisted by $\lambda_{\text{Ind}_G^\pi(\pi)_{L^2}}$, we have that for every $g \in G_{\sigma}$,

$$
\psi(\sigma) = \psi(g, \sigma) = \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}}(g)\psi(\sigma).
$$

The above equality is an equality in $\text{Ind}_G^\pi(\pi)_{L^2}$, i.e., for almost every $g' \in G$, $\psi(\sigma)(g') = \lambda_{\text{Ind}_G^\pi(\pi)_{L^2}}(g)\psi(\sigma)(g') = \psi(\sigma)(g^{-1}g')$. In particular, there is $x_\sigma \in V$ such that for almost every $g \in G_{\sigma}$, $\psi(\sigma)(g) = x_\sigma$. Define $\phi_\psi : \tilde{X}(k) \rightarrow V$, by $\phi_\psi(\sigma) = x_\sigma$, where $x_\sigma$ is as above.

We will show that $\phi_\psi \in C^k(X, \pi)$ and that $\Phi(\phi_\psi) = \psi$. The fact that $\phi_\psi$ is anti-symmetric follows directly from the fact that $\psi$ is anti-symmetric. To see
that \(\phi_0\) is twisted by \(\pi\), we note that for every \(h \in \Gamma\) and every \(\sigma \in \tilde{X}(k)\), \(x_{h,\sigma}\) was defined such that for almost every \(g' \in G_{h,\sigma}\), \(\psi(h,\sigma)(g') = x_{h,\sigma}\). Note that \(G_{h,\sigma} = \pi \sigma^{-1} h \pi^{-1}\), and therefore, for almost every \(g \in G_{\sigma}\), \(\psi(h,\sigma)(gh^{-1}) = x_{h,\sigma}\).

Therefore for almost every \(g \in G_{\sigma}\), we have by equivaricance

\[
x_{h,\sigma} = \psi(h,\sigma)(gh^{-1}) = \lambda_{\text{Ind}\, \varphi}|_{\mathcal{L}^2}(h)\psi(\sigma)(gh^{-1}) = \psi(\sigma)(gh^{-1}) = \pi(h)\psi(\sigma)(g),
\]

and since this holds for almost every \(g \in G_{\sigma}\), it follows that

\[
\phi_0(h,\sigma) = x_{h,\sigma} = \pi(h)x_{\sigma} = \pi(h)\phi_0(\sigma),
\]

as needed. To see that \(\Phi(\phi_0) = \psi\), we will show that for almost every \(g \in G\) and every \(\sigma \in \tilde{X}(k)\), \(\Phi(\phi_0)(\sigma)(g) = \psi(\sigma)(g)\). We note that for almost every \(g \in G\) and almost every \(g' \in G_{\sigma}\), \(x_{g^{-1},\sigma} = \psi(g^{-1},\sigma)(g^{-1}g'g)\). Therefore, for almost every \(g \in G\) and almost every \(g' \in G_{\sigma}\),

\[
\Phi(\phi_0)(\sigma)(g) = f_{\phi_0,\sigma}(g) = \phi_0(g^{-1},\sigma) = x_{g^{-1},\sigma} = \psi(g^{-1},\sigma)(g^{-1}g'g) = \lambda_{\text{Ind}\, \varphi}|_{\mathcal{L}^2}(g^{-1})\psi(g^{-1},\sigma)(g) = \psi(\sigma)(g).
\]

Last, one can easily see that \(\Phi\) is linear and by direct computation one can verify that for \(\phi \in C^\infty(X, \pi)\), \(d_k \Phi_k(\phi) = \Phi_{k+1}(d_k \phi)\).

3.3 Vanishing of cohomology result

Let \(G, X\) as in [13,4] and \(\bigtriangleup\) the \(n\)-simplex of \(X\), which was chosen to be the fundamental domain of the action of \(G\) on \(X\).

We recall the notion of angle between subgroups defined in [14] and [19] based on the ideas of Dymara and Januszkiewicz [10], and of Ershov and Jaikin-Zapirain [11]. Denote by \(\mu\) the Haar measure of \(G\). For \(0 \leq i \leq n\), denote \(\bigtriangleup(i)\) to be the set of \(i\)-dimensional simplices of \(\bigtriangleup\) and for any \(\tau \in \bigtriangleup(i)\), denote \(G_\tau\) to be the pointwise stabilizer of \(\tau\) in \(G\). Given \(\sigma, \sigma' \in \bigtriangleup(n-1), \sigma \neq \sigma'\) and a representation \(\pi\) of \(G_{\sigma \cap \sigma'}\), we define the function

\[
k_\sigma = \frac{\chi_{G_\sigma}}{\mu(G_\sigma)}k_{\sigma'} = \frac{\chi_{G_{\sigma'}}}{\mu(G_{\sigma'})}k_{\sigma \cap \sigma'} = \frac{\chi_{G_{\sigma \cap \sigma'}}}{\mu(G_{\sigma \cap \sigma'})},
\]

where for every \(H < G\), \(\chi_H\) denotes the indicator function on \(H\). Recall that above we defined \(S_\sigma(G)\) the group algebra of compactly supported simple functions on \(G\) with convolution and for every such function \(f\), given a representation \(\pi\) of \(G\) on a Banach space \(V\), we defined an operator \(\pi(f) \in B(V)\). We define now the (cosine of the) angle between \(G_\sigma\) and \(G_{\sigma'}\) with respect to the representation \(\pi\) to be

\[
\cos(\angle(\pi(k_\sigma), \pi(k_{\sigma'}))) = \max\{\|\pi(k_\sigma k_{\sigma'} - k_{\sigma \cap \sigma'})\|, \|\pi(k_\sigma k_{\sigma'} - k_{\sigma \cap \sigma'})\|\}.
\]
The motivation for the definition is explained in [18] and [19].

We recall that given a finite graph \( (V, E) \), the \textit{normalized Laplacian} on the graph is an operator: \( \mathcal{L} : \ell^2(V) \to \ell^2(V) \) defined by

\[
(\mathcal{L} \phi)(v) = \phi(v) - \frac{1}{w(v)} \sum_{v \sim u} \phi(u),
\]

where \( w(v) \) is the valency of \( v \). The \textit{spectral gap} of \( \mathcal{L} \) is the smallest eigenvalue of \( \mathcal{L} \) on the space

\[
\ell^2_0(V) = \{ \phi \in \ell^2(V) : \sum_{v \in V} w(v) \phi(v) = 0 \}.
\]

In [18], the following fact is proven:

**Lemma 3.6.** [18 Corollary 4.20] Let \( X \) and \( G \) as above. For every \( \sigma, \sigma' \in \Delta(n-1) \), \( \sigma \neq \sigma' \), if the spectral gap of the normalized Laplacian on the link of \( \sigma \cap \sigma' \) in \( X \) is \( \geq 1 - \delta \), then for every unitary representation \( \pi \) of \( G_{\sigma \cap \sigma'} \), we have that

\[
\cos(\angle(\pi(k_{\sigma}), \pi(k_{\sigma'}))) \leq \delta.
\]

Below, we will need the following corollaries of this Lemma for strictly \( \theta \)-Hilbertian spaces.

**Corollary 3.7.** Let \( X \) and \( G \) as in §3.1 above. Assume that for every \( \tau \in \Delta(n-2) \), the spectral gap of the normalized Laplacian on the link of \( \tau \) in \( X \) is \( \geq 1 - \delta \). Then for every \( 0 < \theta_0 \) and for every \( \theta \) such that \( \theta_0 \leq \theta \leq 1 \), we have for every \( \sigma, \sigma' \in \Delta(n-1) \), \( \sigma \neq \sigma' \) and every strictly \( \theta \)-Hilbertian space \( V \) that

\[
\cos(\angle((\lambda_{G_{\sigma \cap \sigma'}} \otimes id_V)(k_{\sigma}), (\lambda_{G_{\sigma \cap \sigma'}} \otimes id_V)(k_{\sigma'}))) \leq 2\theta_0,
\]

where \( \lambda_{G_{\sigma \cap \sigma'}} \) is the left-regular representation on \( G_{\sigma \cap \sigma'} \).

**Proof.** Note that for every Banach space \( V_0 \),

\[
\|(\lambda_{G_{\sigma \cap \sigma'}} \otimes id_{V_0})(k_{\sigma} k_{\sigma'} - k_{\sigma \cap \sigma'})\| \leq 2,\|(\lambda_{G_{\sigma \cap \sigma'}} \otimes id_{V_0})(k_{\sigma} k_{\sigma'} - k_{\sigma \cap \sigma'})\| \leq 2.
\]

Therefore, by Lemma 2.2 for every strictly \( \theta \)-Hilbertian space \( V \), which is an interpolation of \( V_0, V_1 \), where \( V_1 \) is a Hilbert space, we have that

\[
\|(\lambda_{G_{\sigma \cap \sigma'}} \otimes id_V)(k_{\sigma} k_{\sigma'} - k_{\sigma \cap \sigma'})\| \leq 2,\|(\lambda_{G_{\sigma \cap \sigma'}} \otimes id_V)(k_{\sigma} k_{\sigma'} - k_{\sigma \cap \sigma'})\| \leq 2.
\]

and similarly,

\[
\|(\lambda_{G_{\sigma \cap \sigma'}} \otimes id_V)(k_{\sigma} k_{\sigma'} - k_{\sigma \cap \sigma'})\| \leq 2,\|(\lambda_{G_{\sigma \cap \sigma'}} \otimes id_V)(k_{\sigma} k_{\sigma'} - k_{\sigma \cap \sigma'})\| \leq 2,
\]

and these two inequalities imply that

\[
\cos(\angle((\lambda_{G_{\sigma \cap \sigma'}} \otimes id_V)(k_{\sigma}), (\lambda_{G_{\sigma \cap \sigma'}} \otimes id_V)(k_{\sigma'}))) \leq 2\theta_0.
\]

When the simplicial complex above is an affine Building the above Corollary implies the following:
Corollary 3.8. Let $G$ be a simple, rank $n$, algebraic group $G$ over a non-archimedean local field and $X$ be the associated $n$-dimensional Bruhat-Tits (affine) building on which it acts. For a constant $0 < \theta_0$, denote $\mathcal{E}_{\theta_0}$ to be the class of Banach spaces containing all strictly $\theta$-Hilbertian spaces with $\theta_0 \leq \theta \leq 1$ and let $\overline{\mathcal{E}_{\theta_0}}$ be the closure of $\mathcal{E}_{\theta_0}$ under ultraproducts, subspaces, and $l_2$-sums.

Then for any constant $\gamma > 0$, there is a constant $Q$ such that if the thickness of $X$ is $\geq Q$, then for every $\theta_0 \leq \theta \leq 1$ and every $V \in \overline{\mathcal{E}_{\theta_0}}$,

$$\cos(\zeta((\lambda \otimes id_V)(k_\sigma), (\lambda \otimes id_V)(k_{\sigma'}))) \leq \gamma,$$

for every $\sigma, \sigma' \in \triangle(n-1)$.

Proof. The 1-dimensional links of affine buildings are 1-dimensional spherical buildings. The spectral gap of the normalized Laplacians of 1-dimensional spherical buildings was computed explicitly by Feit and Higman in [12] (see also [14, Proposition 7.1]) and was shown to tend to 1 as the thickness tends to infinity. Therefore applying Corollary 3.7 and Lemma 2.4 completes the proof.

In [19], the second named author used the concept of angle between subgroup to prove the following:

Theorem 3.9. [19, Theorem 4.1, Lemma 4.5] Let $X, G$ as in §3.1 above. Then there are constants $\gamma' = \gamma'(n) > 0, \beta = \beta(n) > 1$ such that for every continuous representation $\pi$ of $G$ on a Banach space $V$, if

$$\sup_{\sigma \in \triangle(n-1)} \|\pi(k_{\sigma})\| \leq \beta, \quad \sup_{\sigma, \sigma' \in \triangle(n-1)} \cos(\zeta(\pi(k_{\sigma}), \pi(k_{\sigma'}))) \leq \gamma',$$

and the representation $\pi^*$ on $V^*$ is continuous, then

$$H^i(G, \pi) = 0$$

for $i = 1, \ldots, n - 1$.

Combining this Theorem with Proposition 2.9 and Theorem 2.11 yields the following:

Corollary 3.10. Let $X, G$ in §3.1 above. For a constant $\gamma$, denote $\mathcal{E}_{\text{Asplund}, \gamma}$ to be the class of all Asplund Banach spaces satisfying

$$\sup_{\sigma, \sigma' \in \triangle(n-1)} \cos(\zeta((\lambda \otimes id_V)(k_{\sigma}), (\lambda \otimes id_V)(k_{\sigma'}))) \leq \gamma.$$

Then there are constants $\gamma = \gamma(n) > 0, \beta = \beta(n) > 1$ such that for every $V \in \mathcal{E}_{\text{Asplund}, \gamma}$ and any continuous representation $\pi$ of $G$ on $V$ such that

$$\forall \tau \in \triangle(n-2), \sup_{g \in \Gamma_{\tau}} \|\pi(g)\| \leq \beta,$$

we have that

$$H^i(G, \pi) = 0$$

for $i = 1, \ldots, n - 1$.

In particular, for every $V \in \mathcal{E}_{\text{Asplund}, \gamma}$ and every continuous isometric representation $\pi$ on $V$, we have that

$$H^i(G, \pi) = 0$$

for $i = 1, \ldots, n - 1$. 16
Proof. By Proposition 2.9, we have for every representation \( \pi \) on a Banach space \( V \), that for every \( \sigma, \sigma' \in \triangle(n-1), \sigma \neq \sigma' \), we have that
\[
\cos(\langle \pi(k_{\sigma}), \pi(k_{\sigma'}) \rangle) \leq \left( \sup_{g \in G_{\sigma \cap \sigma'}} \| \pi(g) \| \right) \cos(\langle \lambda \otimes \text{id}_V)(k_{\sigma}), (\lambda \otimes \text{id}_V)(k_{\sigma'}) \rangle).
\]
Also, by Theorem 2.11 for every continuous representation \( \pi \) on an Asplund space \( V \), \( \pi^* \) is a continuous representation. Let \( \gamma', \beta \) be the constants given in Theorem 3.9 and we take \( \gamma = \frac{1}{\beta} \). Therefore, for every \( V \in E_{\text{Asplund}, \gamma} \) and every representation \( \pi \) on \( V \), if
\[
\forall \tau \in \triangle(n-2), \sup_{g \in G_{\tau}} \| \pi(g) \| \leq \beta,
\]
then
\[
\sup_{\sigma \in \triangle(n-1)} \| \pi(k_{\sigma}) \| \leq \beta,
\]
and hence
\[
\cos(\langle \pi(k_{\sigma}), \pi(k_{\sigma'}) \rangle) \leq \beta \gamma' = \gamma'.
\]
Therefore the conditions of Theorem 3.9 are fulfilled and the corollary follows.

Combining this corollary with Shapiro’s Lemma proven above, yields the following:

**Theorem 3.11.** Let \( X, G, \Gamma \) as in §3.1 above. Also, let \( \gamma \) be the constant as in Corollary 3.10 and \( E_{\text{Asplund}, \gamma} \) be the class of Banach spaces defined that Corollary. Then for every \( V \in E_{\text{Asplund}, \gamma} \) and every isometric representation \( \pi \) of \( \Gamma \) on \( V \),
\[
H^i(\Gamma, \pi) = 0 \text{ for } i = 1, ..., n-1.
\]

**Proof.** Note that by Theorem 2.13 and Lemma 2.4, we have for every finite measure space \( (\Omega, \mu) \), that \( V \in E_{\text{Asplund}, \gamma} \) implies that \( L^2(\Omega; V) \in E_{\text{Asplund}, \gamma} \). In particular, given an an isometric representation \( \pi \) of \( \Gamma \) on \( V \), we have that \( \text{Ind}^G_{\Gamma}(\pi)_{L^2} \in E_{\text{Asplund}, \gamma} \). Therefore
\[
H^i(G, \pi_{\text{Ind}^G_{\Gamma}(\pi)_{L^2}}) = 0 \text{ for } i = 1, ..., n-1,
\]
and by Theorem 3.4 (Shapiro’s Lemma) proven above, we deduce that
\[
H^i(\Gamma, \pi) = 0 \text{ for } i = 1, ..., n-1.
\]

We recall that an affine building is called classical if it arises from a BN-pair of a simple algebraic group over a a non-archimedean local field and a famous theorem of Tits is that all affine building of dimension \( \geq 3 \) are classical - see for instance [23, Chapter 10] (in dimension 2 there are “exotic”, i.e., not classical, affine buildings). Applying the above Theorem in the setting of classical affine buildings and strictly \( \theta \)-Hilbertian Banach spaces yields the following:
Corollary 3.12. Given $0 < \theta_0$, denote $E_{\theta_0}$ to be the class of Banach spaces containing all strictly $\theta$-Hilbertian spaces with $\theta_0 \leq \theta \leq 1$ and let $\overline{E_{\theta_0}}$ be the closure of $E_{\theta_0}$ under ultraproducts, subspaces, and $l_2$-sums.

Let $n \geq 2$ be a constant integer. There is a constant $Q = Q(\theta_0, n)$ such that if $\Gamma$ is a countable group acting cocompactly and properly on an $n$-dimensional classical affine building of thickness $\geq Q$, then for every $V \in \overline{E_{\theta_0}}$, and every isometric representation $\pi$ on $V$,

$$H^i(\Gamma, \pi) = 0$$

for $i = 1, \ldots, n - 1$.

Proof. Let $\gamma$ be the constant of Theorem 3.11. By Corollary 3.8, there is $Q$ such that for every $V \in \overline{E_{\theta_0}}$,

$$\cos(\lambda (\lambda \otimes \text{id}_V)(k_{\sigma} \otimes (\lambda \otimes \text{id}_V)(k_{\sigma'}))) \leq \gamma,$$

for every $\sigma, \sigma' \in \triangle(n - 1)$. Also note that by Corollary 2.8, every $V \in \overline{E_{\theta_0}}$ is uniformly convex and therefore Asplund. It follows that $\overline{E_{\theta_0}} \subseteq \overline{E_{\text{Asplund}, \gamma}}$ and applying Theorem 3.11 completes the proof.

Specializing the above corollary to our setting yields the following Theorem that appeared in the introduction:

Corollary 3.13. Let $n \geq 2$ be a fixed integer. For any $1 < p_1 \leq 2 \leq p_2 < \infty$, there exists a natural number $Q = Q(p_1, p_2, n)$ such that if $\Gamma$ is a countable group acting properly and cocompactly on a classical affine building of dimension $n$ of thickness greater than $Q$, then $H^i(\Gamma, V) = 0$ for every $i = 1, \ldots, n - 1$ and every Banach space of the form $V = \prod_{\nu \in \mathcal{U}} (M_{k_{\nu}(\mathbb{C})}, \|\|_p)$ where $\mathcal{U}$ is any ultrafilter on $\mathbb{N}$ and $p_1 \leq p \leq p_2$.

Proof. Let $\theta_0 = \min\{2 - \frac{2}{p_1}, \frac{2}{p_2}\}$, then $(M_{k_{\nu}(\mathbb{C})}, \|\|_p)$ is strictly $\theta$-Hilbertian with $\theta_0 \leq \theta \leq 1$ and the result follows from the previous corollary.

References

[1] Goulnara Arzhantseva and Liviu Păunescu. Almost commuting permutations are near commuting permutations. *J. Funct. Anal.*, 269(3):745–757, 2015.

[2] U. Bader, T. Gelander, and N. Monod. A fixed point theorem for $L^1$ spaces. *Invent. Math.*, 189(1):143–148, 2012.

[3] W. Ballmann and J. Świątkowski. On $L^2$-cohomology and property (T) for automorphism groups of polyhedral cell complexes. *Geom. Funct. Anal.*, 7(4):615–645, 1997.

[4] Jörn Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[5] Bruce Blackadar and Eberhard Kirchberg. Generalized inductive limits of finite-dimensional $C^*$-algebras. *Math. Ann.*, 307(3):343–380, 1997.
[6] Armand Borel and Nolan Wallach. Continuous cohomology, discrete subgroups, and representations of reductive groups, volume 67 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, second edition, 2000.

[7] Alain Connes. Classification of injective factors. Cases $II_1$, $II_\infty$, $III_\lambda$, $\lambda \neq 1$. Ann. of Math. (2), 104(1):73–115, 1976.

[8] Marcus De Chiffre, Lev Glebsky, Alexander Lubotzky, and Andreas Thom. Stability, cohomology vanishing, and non-approximable groups. https://arxiv.org/abs/1711.10238 2018.

[9] Pierre Deligne. Extensions centrales non résiduellement finies de groupes arithmétiques. C. R. Acad. Sci. Paris Sér. A-B, 287(4):A203–A208, 1978.

[10] Jan Dymara and Tadeusz Januszkiewicz. Cohomology of buildings and their automorphism groups. Invent. Math., 150(3):579–627, 2002.

[11] Mikhail Ershov and Andrei Jaikin-Zapirain. Property (T) for noncommutative universal lattices. Invent. Math., 179(2):303–347, 2010.

[12] Walter Feit and Graham Higman. The nonexistence of certain generalized polygons. J. Algebra, 1:114–131, 1964.

[13] Steven A. Gaal. Linear analysis and representation theory. Springer-Verlag, New York-Heidelberg, 1973. Die Grundlehren der mathematischen Wissenschaften, Band 198.

[14] Howard Garland. $p$-adic curvature and the cohomology of discrete subgroups of $p$-adic groups. Ann. of Math. (2), 97:375–423, 1973.

[15] Lev Glebsky and Luis Manuel Rivera. Sofic groups and profinite topology on free groups. J. Algebra, 320(9):3512–3518, 2008.

[16] Roger C. Lyndon and Paul E. Schupp. Combinatorial group theory. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[17] Michael G. Megrelishvili. Fragmentability and continuity of semigroup actions. Semigroup Forum, 57(1):101–126, 1998.

[18] Izhar Oppenheim. Averaged projections, angles between groups and strengthening of Banach property (T). Math. Ann., 367(1-2):623–666, 2017.

[19] Izhar Oppenheim. Vanishing of cohomology with coefficients in representations on Banach spaces of groups acting on buildings. Comment. Math. Helv., 92(2):389–428, 2017.

[20] Vladimir G. Pestov. Hyperlinear and sofic groups: a brief guide. Bull. Symbolic Logic, 14(4):449–480, 2008.

[21] Gilles Pisier. Some applications of the complex interpolation method to Banach lattices. J. Analyse Math., 35:264–281, 1979.

[22] Gilles Pisier and Quanhua Xu. Non-commutative $L^p$-spaces. In Handbook of the geometry of Banach spaces, Vol. 2, pages 1459–1517. North-Holland, Amsterdam, 2003.
[23] Mark Ronan. *Lectures on buildings*, volume 7 of *Perspectives in Mathematics*. Academic Press, Inc., Boston, MA, 1989.

[24] Mikael de la Salle. Towards strong Banach property (T) for SL(3, R). *Israel J. Math.*, 211(1):105–145, 2016.

[25] Kondagunta Sundaresan. The Radon-Nikodým theorem for Lebesgue-Bochner function spaces. *J. Functional Analysis*, 24(3):276–279, 1977.

[26] Andreas Thom. Finitary approximations of groups and their applications. https://arxiv.org/abs/1712.01052 2018.

[27] David Yost. Asplund spaces for beginners. *Acta Univ. Carolin. Math. Phys.*, 34(2):159–177, 1993. Selected papers from the 21st Winter School on Abstract Analysis (Poděbrady, 1993).

[28] Robert J. Zimmer. *Ergodic theory and semisimple groups*, volume 81 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.