The Impact of Inhomogeneous Reionization on Cosmic Microwave Background Anisotropy

Lloyd Knox¹, Román Scoccimarro¹, and Scott Dodelson²

¹ Canadian Institute for Theoretical Astrophysics, Toronto, ON M5S 3H8, CANADA
² NASA/Fermilab Astrophysics Center
Fermi National Accelerator Laboratory, Batavia, IL 60510, USA
(August 23, 2021)

It is likely that the reionization of the Universe did not occur homogeneously. Using a model that associates ionized patches with overdense regions, we find that the resulting cosmic microwave background (CMB) anisotropy power spectrum peaks at angular scales corresponding to the extent of the ionized regions, and has a width that reflects the correlations between them. There is considerable uncertainty in the amplitude. Neglecting inhomogeneous reionization in the determination of cosmological parameters from high resolution CMB maps may cause significant systematic error.

Introduction. Observations of cosmic microwave background (CMB) anisotropy are providing strong constraints on theories of cosmological structure formation. Planned CMB observations may potentially provide constraints on the parameters of these theories at the percent level. One of the reasons the CMB is such a wonderful probe of cosmological parameters is that predictions for a given theory can be made with great precision. This is because linear perturbation theory is an excellent approximation over the relevant length scales at $z \approx 1100$ when the CMB was last tightly coupled to matter.

However, new anisotropies can be generated at late times by the non-linear process of reionization as the ionization field gets much smaller than the scale of variation of the ionization fraction, an effect analogous to the damping and amplification of cosmic variance on large scales, when actually just the opposite is true. The perturbation to the photon temperature distribution function, $\Delta \equiv \delta T/T$, is governed by the Boltzmann equation, which at the late times of interest is

$$\frac{\partial \Delta(x, \eta, \dot{\eta})}{\partial \eta} = n_e \sigma_T a \dot{\eta} \cdot \mathbf{v}(x, \eta)$$

where $a = (1 + z)^{-1}$ is the scale factor of the universe normalized to unity today, $\eta \equiv \int dt/a$ is the conformal time, $n_e$ is the free electron density, $\sigma_T$ is the Thomson cross-section, $\dot{\eta}$ is the integration variable, $\mathbf{v}$ is the electron velocity. Henceforth we work in units where the conformal time today is unity, $\eta_0 = 1$, and assume a flat, matter dominated universe, $a = \eta^2$. The solution to Eq. (1) for the photon perturbations here and now $(x = x_0, \eta = 1)$ is

$$\Delta_0(\dot{\eta}) = 4.06 \times 10^{-5} \Omega_b h \int_{\eta_0}^{\eta_0} \frac{d\eta'}{\eta^3} \mathbf{x}(x, \eta) \cdot \mathbf{v}(x),$$

where, $\mathbf{v}(x, \eta) \equiv -2\eta \mathbf{v}(x)$ (Mpc/h), $n_e = \bar{n}_e \mathbf{x}(x, \eta) a^{-3}$, and $n_e \sigma_T \eta_0 = 0.122 \Omega_b h$, with $\bar{n}_e$ the present mean electron density, $\Omega_b$ the baryon density in units of the critical density and $h$ the Hubble constant in units of 100 km s$^{-1}$ Mpc$^{-1}$. Both $\mathbf{x}$ and $\mathbf{v}$ are evaluated...
at \( \mathbf{x} = \mathbf{x}_0 - \hat{\gamma}(1 - \eta) \), the position at time \( \eta \) of a photon with direction vector \( \hat{\gamma} \) incident on us today. Note that the integral includes all contributions starting from \( \eta_i \), a time far after standard recombination at \( \eta \approx 0.03 \), but before reionization occurs.

We focus on predicting the two-point correlation function, \( C(\theta) \equiv \langle \Delta_0(\hat{\gamma}) \Delta_0(\hat{\gamma}') \rangle \) where \( \cos \theta \equiv \hat{\gamma} \cdot \hat{\gamma}' \) and the angular brackets indicate an average over all locations \( \mathbf{x}_0 \). Using Eq. 2 we obtain the correlation function due to reionization:

\[
C(\eta, \eta', \theta) = 10^{-12} \left( \frac{\Omega_b h}{0.025} \right)^2 \int_{\eta_b}^{\eta_i} \frac{d\eta}{\eta^3} \int_{\eta_b}^{\eta_i} \frac{d\eta'}{\eta'^3} C(\eta, \eta', \theta). \tag{3}
\]

It is therefore determined by an integral over adjacent lines of sight of the four-point correlation function

\[
C(\eta, \eta', \theta) \equiv \hat{\gamma}_i \hat{\gamma}'_j \langle x_{e,i} x'_{e,j} v_i v_j \rangle, \tag{4}
\]

where, for example, \( x'_{e} \equiv x_{e}(x', \eta') \). Different models of reionization will lead to different four-point functions. We now explore several possibilities.

**Homogeneous Reionization.** The simplest possibility is that reionization takes places homogeneously. While not particularly plausible, it is a useful limiting case for demonstrating the SK cancellation. In homogeneous reionization (HR), \( x_e \equiv \bar{x}_e(\eta) \), where \( \bar{x}_e \) is fraction of the Universe that is ionized. Thus we have:

\[
C_{\text{HR}}(\eta, \eta', \theta) = \bar{x}_e \bar{x}'_e C_v(\eta, \eta', \theta) \tag{5}
\]

and we take \( \bar{x}_e \) to be zero at times earlier than \( z_i + \delta z_i \), unity at times later than \( z_i \) and to increase linearly with redshift during the transition. The velocity correlation function can be written as

\[
C_v(\eta, \eta', \theta) = \Psi_\parallel \cos \theta + \left( \Psi_\perp - \Psi_\parallel \right) \frac{\mathbf{r} \cdot \hat{\gamma}}{r^2}, \tag{6}
\]

where \( \mathbf{r} \equiv \mathbf{x} - \mathbf{x}' \). The correlation functions of the velocity components parallel and perpendicular to \( \mathbf{r} \) are

\[
\Psi_\parallel(r) = \int \frac{P(k)}{k^2} \left[ j_0(kr) - 2 \frac{j_1(kr)}{kr} \right] d^3k, \tag{7}
\]

\[
\Psi_\perp(r) = \int \frac{P(k)}{kr} \frac{j_1(kr)}{kr} d^3k, \tag{8}
\]

where \( P(k) \) is the matter power spectrum, throughout assumed to be that of the standard cold dark matter theory \( [1] \). Fig. 1 shows \( C_v(\eta, \eta', 0) \) as a function of \( \eta \) for fixed \( \eta' = 0.188 \). \( C_v \) is strongly peaked at equal times, where the two lines of sight share the same velocity. On the other hand, at large \( |\eta - \eta'| \), \( C_v \) becomes negative due to infall from opposite sides into an overdense region. The integration, in Eq. 8 of this oscillatory function with the slowly-varying \( \bar{x}_e \bar{x}'_e \) leads to a cancellation of Doppler effects, as described by Kaiser \( [2] \) in Fourier space. **Uncorrelated Inhomogeneous Reionization.** The SK cancellation can be avoided if the velocity two-point function \( C_v \) is modulated by the spatial dependence of \( x_e \).

To demonstrate this effect, we adopt a toy model of the reionization process: independent sources turn on randomly and instantaneously ionize a sphere with comoving radius \( R \) and volume \( V_R \), which then remain ionized. The resulting four-point function can be written as:

\[
C_{\text{uc}}(\eta, \eta', \theta) = C_v(\eta, \eta', \theta) \left[ C_{\text{HR}}(\eta, \eta', \theta) \right]. \tag{9}
\]

Since \( \bar{x} \) (where \( \chi \equiv 1 - x_e \)) is equal to the probability that no ionizing source is within volume \( V_R \), the probability that no source is within \( R \) of either \( \mathbf{x} \) or \( \mathbf{x}' \) is (at equal times) \( \chi(\chi') = \bar{x}(V_{\text{in}} - V(r)) / V_{\text{in}} \). Therefore, for the proper generalization to unequal times, we have:

\[
C_{\text{uc}}(\eta, \eta', \theta) = \bar{x} \bar{x}' \left[ \chi(\min(\eta, \eta'))^{-V(r)/V_R} - 1 \right] \tag{10}
\]

\[
C_{\text{uc}}(\eta, \eta', \theta) = \bar{x} \bar{x}' \left[ \chi(\min(\eta, \eta'))^{-V(r)/V_R} - 1 \right] \tag{11}
\]

**FIG. 1.** The mean ionization fraction \( \bar{x}_e \) for HR (\( z_i = 26, \delta z_i = 4 \)), the velocity two-point function \( C_v(\eta, \eta', 0) \) for the SCDM (standard cold dark matter) model with \( \sigma_8 = 1.2 \) where \( \sigma_8 \) is the rms of mass fluctuations in spheres of radius \( 8h^{-1} \) Mpc today, and the ionization fraction two-point function for uncorrelated \( C_{\text{uc}}(\eta, \eta', 0) \) and correlated bubbles \( C_{\text{uc}}(\eta, \eta', 0) \). Dashed lines denote negative values.

The resulting two-point function, \( C_{\text{uc}}(\eta, \eta', 0) \), is sharply peaked at small \( r \approx |\eta - \eta'| \) (see Fig. 1, where \( R = 1 \) Mpc), therefore in Eq. 8 \( C_{\text{uc}}(\theta) \propto C_v(\eta, \eta, \theta) \). Thus, the SK cancellation of homogeneous reionization is avoided; IHR modulates the velocity field, accessing only the region where the velocity field is highly coherent. Note though that the modulation is not particularly effective since the ionization radius \( R \) is typically very small, and thus only a tiny region of \( \eta \) contributes. Therefore, as Gruzinov and Hu (GH) \( [6] \) have pointed out, a model of this type necessarily produces anisotropies which have amplitude proportional to \( R \).

Figure 2 shows the anisotropy spectrum arising from the two-point function of Eq. 11 assuming \( z_i = 26 \) and
\( \delta z_i = 4 \). The GH results for the same parameter choices would each have peaks 30\% lower in \( l \) and two times larger in amplitude due to a slower fall off of \( C_{10}^{\text{GH}} \) with \( r \) and one less factor of \((1 - \eta_c)\) in their version of Eq. 11. Despite these differences, they correctly identified the qualitative features of this type of IHR: (i) amplitude proportional to \( R_c \); (ii) white noise (\( C_{10}^{\text{IHR}} \) constant) on large scales due to lack of correlations at \( r \gg R_c \); and (iii) peak at \( l \sim (1 - \eta_c)/R_c \) where reionization takes place at \( \eta_c \). We now show that the first two of these features break down when more realistic models are considered.

\[
\bar{x}_e = 2E \int_\delta_c^\infty d\delta P_{R,c}(\delta) = E \text{erfc}\left(\frac{\delta_c}{\sqrt{2} \sigma_{R,c}(\eta)}\right),
\]

(12)

\[
C_{\delta e}^0 = \langle x_e x'_e \rangle = \alpha \alpha' \int_\delta_c^\infty d\delta \int_\delta_c^\infty d\delta' \; P_{R,c}(\delta, \delta') 
\approx \frac{\alpha \alpha'}{2\pi} \frac{\Xi_{R,c}(r)}{\sigma_{R,c}(\eta) \sigma_{R,c}(\eta')} \exp\left[ -\frac{\nu^2(\eta) + \nu^2(\eta')}{2} \right],
\]

(13)

where \text{erfc}(x) denotes the complementary error function, \( \alpha = 2E(1 - \bar{x}_e) \) (see below), \( P_{R,c}(\delta) \) and \( P_{R,c}(\delta, \delta') \) are the one and two-point distributions of the smoothed field \( \delta_{R,c} \), respectively; the smoothed correlation function is given by \( \langle \delta \delta' \rangle = \Xi_{R,c}(r), \sigma_{R,c} \equiv \Xi_{R,c}(0), \) and \( \nu(\eta) \equiv \delta_c/\sigma_{R,c}(\eta) \).

Despite a number of studies of the details of reionization \[14\] , we are still far from a satisfactory understanding. Therefore, we take two simple models for \( E \), which can vary by many orders of magnitude (see e.g. table 2 in \[12\]). The requirement that objects more massive than \( M_c(z) \sim 10^{19} M_\odot [1/(1 + z)]^{1/2} \) are necessary for reionization leads to a smoothing scale of \( R_c \approx 0.22 \eta Mpc \[13\] . Our two choices for the efficiency are \( E = 7 \times 10^5 \eta^3 \[13\] \), resulting in \( z_i = 26 \) and \( E = 114 \) (the "middle of the road" model in \[12\] ), which gives \( z_i = 31 \). Note that current scenarios for reionization generally lead to \( E \gg 1 \), i.e. the sources tend to ionize a region much larger than their typical volume (see however \[8\] for a different view).

Fig. 1 shows the results of Eq. 13 for the \( z_i = 26 \) model. The clustering of overdense regions naturally leads to a much wider \( C_{\delta e}^0 \), which falls off only as a power-law, thus more efficiently modulating \( C_v \). For this reason, the correlated models in Fig. 2 (dashed lines) show a much wider distribution of power, and the white noise regime is only reached at scales much larger than the patch size. Over the range of \( \ell \) plotted, the power spectrum of the uncorrelated models with the same values of \( R \) as in the peaks model would be orders of magnitude smaller. The dashed lines in Fig. 2 are the result of assuming \( \langle x_e x'_e v'v' \rangle = \langle x_e x'_e \rangle \langle v'v' \rangle \) and then applying Eq. 13 for \( \langle x_e x'_e \rangle \). For the solid lines, we use an expression for \( \langle x_e x'_e v'v' \rangle \) similar to that for \( \langle x_e x'_e \rangle \) in Eq. 13 but with an integral over the joint probability distribution of \( \delta, \delta', v, v' \[13\] \). This leads to a modification of the white noise behavior at large scales. Cross-correlation between HR and IHR leads to higher-order terms such as \( \bar{x}_e x'_e \delta \delta v'v' \). These are of comparable magnitude to \( \langle x_e x'_e v'v' \rangle \) and will be discussed in \[13\] . Note that the flatness of the power spectra in Fig. 2 above \( \ell \approx 3000 \) reflects the near scale-invariance of the matter power spectrum over the relevant length scales.

**Impact on Parameter Determination.** The anisotropies produced during IHR may significantly affect the exquisite parameter determination anticipated from future CMB observations. Suppose a multi-dimensional fit is performed on the data to extract a set of parameters \( \{p_i\} \). If the fit assumes \( C_\ell(\{p_i\}) \) that ignores the con-
trubution from \( C_i \) of \( n \) at each parameter will be incorrectly estimated by an amount

\[
\Delta p_i = F_{ij}^{-1} \sum_l w_l \frac{\partial C_l}{\partial p_j} C_i^{\text{HR}}
\]

(14)

where the weights \( w_l \) are the inverse of the squares of the errors expected on \( C_l \)’s (the errors are the sum of the errors due to sample variance and those due to noise \( \Omega \)). \( F_{ij} \) is the Fisher matrix \( \sum_l w_l (\partial C_l/\partial p_i) (\partial C_l/\partial p_j) \).

Table 1 shows the ratio of this systematic offset to the statistical uncertainty for Planck [16].

**Sources of Uncertainty.** Even within the context of our peaks model, Eq. [13] is not exact. Since the ionized region is larger than the region that collapsed (i.e., \( E > 1 \)) a more rigorous approach would be to calculate the probability of two points both being within a distance \( R \) of one or more peaks. Instead, we have calculated the probability that two points are both within a peak and then scaled the resulting two-point function by \( EE' \). We expect this approximation to be best for \( r > R \), which covers the range of relevance for the curves in Fig. 2. Because we have not taken the more rigorous approach, we must include suppression factors to correct for overcounting overlapping regions. These are the factors of \( (1 - \bar{x}_c) \), motivated by their appearance in our fully tractable uncorrelated model. Removing them boosts the power spectrum by a factor of about 5.

Uncertainty in \( E \) has a much milder effect on uncertainty in \( C_i^{\text{HR}} \) than one might expect from a quick glance at Eq. [13]. The reason is that increasing \( E \) causes reionization to occur earlier and therefore decreases the exponential factor in Eq. [13]. We find that decreasing \( E \) by a factor of 10 decreases \( C_i^{\text{HR}} \) by a factor of two.

Our peaks model, and the calculations which lead to estimates of its parameters, are highly idealized. Although more sophisticated studies will eventually alter the details of our results, we believe significant power at scales larger than the patch size is a natural consequence of cosmic structure formation via gravitational instability.

**ACKNOWLEDGMENTS**

We thank J.R. Bond, A.H. Jaffe, D. Pogosyan, A. Stebbins and M. Zaldarriaga for useful discussions, and JRB for the \( C_l \) derivatives. SD is supported by the DOE and by NASA Grant NAG 5-7092.

**REFERENCES**

[1] L. Knox, Phys. Rev. D52, 4307 (1995); G. Jungman, M. Kamionkowski, A. Kosowsky & D.N. Spergel, Phys. Rev. Lett. 76, 1007 (1996); *ibid*, Phys. Rev. D54, 1332 (1996); M. Zaldarriaga, D. Spergel & U. Seljak, Astrophys. J. 488, 1 (1997).

[2] J.R. Bond, G. Efstathiou & M. Tegmark, MNRAS 291, L33 (1997).

[3] See e.g. P.J.E. Peebles & J.T. Yu, Astrophys. J. 162, 815 (1970); J.R. Bond & G. Efstathiou, Astrophys. J. 285, L45 (1984); U. Seljak and M. Zaldarriaga, Astrophys. J. 469, 437 (1996).

[4] See e.g. D. Scott and M. White, in the Proceedings of the CWRU CMB Workshop “2 years after COBE” eds. L. Krauss & P. Kerman (1994); S. Hancock, G. Rocha, A. N. Lasenby & C.M. Gutierrez, MNRAS 294, L1 (1998).

[5] R.A. Sunyaev in *Large-Scale Structure of the Universe*, eds. M.S. Longair & J. Einasto (Dordrecht: Reidel), p. 393 (1978); N. Kaiser, Astrophys. J. 282, 374 (1984).

[6] N. Aghanim, F.X. Desert, J.L. Puget, & R. Gispert, Astron. Astrophys. 311, 1 (1996).

[7] A. Gruzinov & W. Hu, *astro-ph/9803188* (1998).

[8] P.J.E. Peebles & R. Juszkiewicz, *astro-ph/9804260* (1998).

[9] J.P. Ostriker & E. Vishniac, Astrophys. J. 306, L51 (1986); E. Vishniac, Astrophys. J. 322, 597 (1987).

[10] J.M. Bardeen, J.R. Bond, & N. Kaiser, & A.S. Szalay, Astrophys. J. 304, 15 (1986).

[11] B.J. Carr, J.R. Bond, & W.D. Arnett, Astrophys. J. 277, 445 (1984); H.M.P. Couchman & M.J. Rees, MNRAS 221, 53 (1986); M. Fukugita & M. Kawasaki, MNRAS 269, 563 (1994); P.R. Shapiro, M.L. Giroux, & A. Babul, Astrophys. J. 427, 25 (1994); J.P. Ostriker & Y.N. Gnedin, Astrophys. J. 486, 581 (1997).

[12] M. Tegmark, J. Silk, & A. Blanchard, Astrophys. J. 434, 395 (1995).

[13] Z. Haiman & A. Loeb, Astrophys. J. 483, 21 (1997); *ibid*, 488, 9710208 (1997).

[14] N. Kaiser, Astrophys. J. 284, L9 (1984).

[15] L. Knox, R. Scoccimarro, & S. Dodelson, in preparation (1998).

[16] http://astro.estec.esa.nl/SA-general/Projects/Planck & http://map.gsfc.nasa.gov

**Table 1. Ratio of systematic offset to statistical uncertainty expected for Planck [10] from the two “peaks” models in Fig. 2.** Since IHR affects very small scales, MAP [10] is much less sensitive to this offset. Variables are, from left to right, the primordial power spectrum index, the ratio of tensor to scalar contributions to \( C_2 \), the optical depth and the contribution to \( h^2 \) from (b)aryons, (cur)vatures, cold dark (m)atter and massive neutrinos. The power spectrum derivatives were evaluated at the parameter values of the SCDM model in [10]. This does not include polarization information; however, since polarization is sourced by the quadrupole moment, not velocity, the polarization IHR power spectrum will be down by roughly a factor of \((10^{-5}/10^{-3})^2\) from the IHR temperature power spectrum.