STEENROD OPERATIONS AND HOCHSCHILD HOMOLOGY.

by

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Abstract. Let $X$ be a simply connected space and $F_p$ be a prime field. The algebra of normalized singular cochains $N^*(X; F_p)$ admits a natural homotopy structure which induces natural Steenrod operations on the Hochschild homology $HH^*(N^*(X; F_p))$ of the space $X$. The primary purpose of this paper is to prove that the J. Jones isomorphism $HH^*(N^*(X; F_p)) \cong H^*(X^{S^1}; F_p)$ identifies these Steenrod operations with those defined on the cohomology of the free loop space with coefficients in $F_p$. The other goal of this paper is to describe a theoritic model which allows to do some computations.

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Introduction.

Let $A = \{A^i\}_{i \geq 0}$ be a augmented differential graded algebra over the field $F_p$. The homology of the normalized Hochschild chain complex $\varepsilon_* A$ is called the Hochschild homology (with coefficients in $A$) of $(A, d_A)$ and is denoted by $HH_* A$. Let $X$ be a simply connected space. In 1987, J.D.S. Jones [10] constructed an isomorphism of graded vector spaces $HH_* N^*(X; F_p) \cong H^*(X^{S^1}; F_p)$, where $X^{S^1}$ denotes the free loop space and $N^* X$ denotes singular cochains with coefficients in $F_p$. In view of this result, one may ask the following question:

Does there exist a subcategory $DA'$ of the category $DA$ of differential graded algebras such that:

(Q.1) for any connected space $X$, $N^* X$ is an object of $DA'$,
(Q.2) if $A$ is an object of $DA'$ then $HH_* A$ is an $B_p$-unstable algebra,
(Q.3) the Jones isomorphism is an homomorphism of $A_p$-unstable algebras?

Here $B_p$ denotes the large mod $p$-Steenrod algebra and $A_p = B_p/(P^0 = id)$ denotes the usual mod $p$-Steenrod algebra.

In this paper we give an affirmative answer to (Q.1) and (Q.3) and a partial answer to (Q.2). For this purpose we introduce the notion of $\pi$-shc algebra where $\pi$ denotes the cyclic group of order a fixed prime $p$ (1.6). More precisely we prove:

**Theorem A.** Let $p$ be a prime and $A$ be the mod $p$ reduction of a graded module over $\mathbb{Z}$ and denote by $\beta$ the Bockstein homomorphism. If $A$ is a $\pi$-shc algebra over $F_p$ then for any $i \in \mathbb{Z}$ and any $x \in (HH_* A)^n$ there exist well defined homology classes $Sq^i(x) \in (HH_* A)^{n+i}$
if \( p = 2 \) and \( P^i(x) \in (HH_\ast A)^{n + 2i(p - 1)} \) if \( p > 2 \) such that: \( Sq^1(x) = P^i(1_{HA}) = 0, i \neq 0 \) and \( Sq^i(x) = \begin{cases} 0 & \text{if } i > n \\ x^2 & \text{if } i = n \end{cases} \) and \( \beta^p P^i(x) = \begin{cases} 0 & \text{if } 2i + \epsilon > n, \epsilon = 0, 1 \\ x^p & \text{if } n = 2i, \epsilon = 0 \end{cases} \)

Moreover these classes are natural with respect to homomorphisms of \( \pi \)-shc algebras and satisfy the Cartan formula.

**Theorem B.**

1) The algebra \( N^\ast X \) is naturally a \( \pi \)-shc algebra. Moreover, any two natural structural maps defining a \( \pi \)-shc structure on \( N^\ast X \) are \( \pi \)-homotopic.

2) If \( X \) is 1-connected then, the Jones’ quasi-isomorphism \( e N^\ast X \to C^\ast(X^{S^1}) \) identifies the algebraic Steenrod operations, defined on \( HH_\ast N^\ast X \) by theorem A, with the topological Steenrod operations on \( H^\ast(X^{S^1}; \mathbb{F}_p) \).

In §3 we develop the construction of a convenient model for a \( \pi \)-shc algebra. But, at it can be easily imagined, explicit computations with this model are not very tractable in general (cf. Example 3.9).

These two theorems are a generalization of theorem A and B, in [2]. In fact, we proved that the Jones’ homomorphism induces a commutative diagram

\[
\begin{array}{ccc}
N^\ast X & \rightarrow & e N^\ast X \\
\downarrow \ | & \ | \downarrow J_X & \downarrow J_X \\
N^\ast X & \rightarrow & N^\ast X^{S^1}
\end{array}
\]

where the horizontal arrows in the lower lines are induced from the canonical fibration \( \Omega X \to X^{S^1} \to X \) and \( HJ_X \) (resp. \( HJ^\ast_X \)) is an isomorphism of commutative graded algebras (resp. of commutative graded Hopf algebras). Now following the lines of the proof of theorem B and of [2]-II-§3 we obtain:

**Theorem C.** If \( X \) is 1-connected then the homomorphisms, \( H\iota, H\rho, HJ_X \) and \( HJ^\ast_X \) respect Steenrod operations.

The paper will be organized as follows:

- § 1 - Proof of theorem A.
- § 2 - Proof of theorem B
- § 3 - \( \pi \) shc model.
- Appendix A - Technicalities on shc algebras.
- Appendix B - Equivariant acyclic model theorem.

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### §1 Proof of theorem A.

#### 1.1 Preliminaries.

Throughout the paper, \( p \) is a fixed prime, and we work over a field \( \mathbb{F}_p \). If \( \pi \) is any finite group, the group ring \( \mathbb{F}_p[\pi] \) is a augmented algebra. If two \( \pi \)-linear maps \( f \) and \( g \) are \( \pi \)-linear homotopic we write \( f \simeq \pi g \).

#### 1.2. Algebraic Steenrod operations.

The material involved in this section is contained in [L3]. Let \( \pi = \{1, \tau, \ldots, \tau^{p - 1}\} \) be the cyclic group of order \( p \). We identify \( \tau \) with the
p-cycle \((p,1,2,...,p-1)\) of \(\mathfrak{S}_p\) thus \(\pi\) acts on \(A^{\otimes p}\). Let \(W^{\otimes p} \to \mathbb{F}_p\) be a projective resolution of \(\mathbb{F}_p\) over \(\mathbb{F}_p[\pi]\): \(W = \{W_i\}_{i \geq 0}, \quad \partial : W_i \to W_{i-1}, W_0 \cong \mathbb{F}_p[\pi]\) where each \(W_i\) is a right projective \(\pi\)-module and \(\partial\) is \(\pi\)-linear. We choose a linear map \(\eta : \mathbb{F}_p \to W\) such that \(\varepsilon_W \circ \eta = id_{\mathbb{F}_p}\). Necessarily, such \(\eta\) satisfies also \(\eta \circ \varepsilon_W \simeq id_{W}\).

Let \(A = \{A_i\}_{i \in \mathbb{Z}}\) be a differential graded algebra (not necessarily associative). We denote by \(m^{(p)}\) (resp. \((Hm)^{(p)}\)) the iterated product \(a_1 \otimes a_2 \otimes ... \otimes a_p \mapsto a_1(a_2(...a_p)...)(\text{resp. the iterated product induced on } HA \text{ by } m^{(p)}\)). Assume that \(\pi\) acts trivially on \(A\) and diagonally on \(W \otimes A^{\otimes p}\). If there exists a \(\pi\)-chain map \(\theta\) such that the left hand diagram induces the right hand diagram which is commutative:

\[
\begin{array}{cccccc}
W \otimes A^{\otimes p} & \xrightarrow{\theta} & A & \xrightarrow{H(W \otimes A^{\otimes p})} & HA \\
\eta_W \otimes id & \uparrow & m^{(p)} & \uparrow & (Hm)^{(p)} \\
A^{\otimes p} & \xrightarrow{\cong} & (HA)^{\otimes p}
\end{array}
\]

then for any \(i \in \mathbb{Z}\) and any \(x \in H^n A\) there exist well defined homology classes

\[
P^i(x) \in \begin{cases} H^{n+1} A & \text{if } p = 2 \\ H^{n+2(p-1)+1} A & \text{if } p > 2 \end{cases}, \quad \text{and if } p > 2, \quad P^i(x) \in H^{n+2(p-1)+1} A,
\]

such that:

1) \(P^i(1_{HA}) = 0, i \neq 0,\)
2) if \(p = 2,\) \(P^i(x) = 0\) if \(i > n\) and \(P^n(x) = x^2,\)
3) if \(p > 2,\) \(P^i(x) = 0\) if \(2i > n\) and \(P^i(x) = x^p\) if \(n = 2i,\) and \(P^i(x) = 0\) if \(2i \geq n,\)
4) If \(A\) is the mod \(p\) reduction of a free graded module over \(\mathbb{Z}\) then \(\theta \circ P^{i-1} = iP^i\) if \(p = 2\) and if \(p > 2\) \(\theta = \beta \circ P^i\).

Moreover these classes do not depend on the choice of \(W\) neither on \(\eta\), and are natural with respect to homomorphisms of algebras commuting with the structural maps \(\theta\). These algebraic Steenrod operations do not in general satisfy: \(P^i(x) = 0\) if \(i < 0,\) \(P^0(x) = x,\) the Cartan formula and the Adem relations.

### 1.3 Cartan formula.
Assume \(A^i = 0, i < 0.\) The structural map \(\theta : W \otimes A^{\otimes p} \to A\) induces \(\bar{\theta} : A^{\otimes p} \to Hom(W, A), \quad \bar{\theta}(u)(w) = (-1)^{|u||w|}\theta(w \otimes u), \quad u \in A^{\otimes p}, w \in W.\) Let us precise that if \(f \in Hom^k(W, A) = \prod_{i \geq 0} Hom(W_i, A^{k-i}) = \bigoplus_{i=0}^{k} Hom(W_i, A^{k-i})\) then: \(Df = d \circ f - (-1)^{k} f \circ \partial,\)

\((\sigma f)(w) = f(w\sigma), \quad \sigma \in \pi, w \in W.\) A direct verification shows that \(Hom^k(W, A)\) is a \(\pi\)-complex and that \(\bar{\theta}\) is a \(\pi\)-chain map. If \(ev_0 : Hom(W, A) \to A\) denotes the evaluation map on the generator \(e_0 = \eta(1)\) of \(W_0\), the left hand diagram induces the right hand diagram which is commutative.

\[
\begin{array}{cccccc}
A^{\otimes p} & \xrightarrow{\bar{\theta}} & Hom(W, A) & \xrightarrow{(HA)^{\otimes p}} & H(\text{Hom}(W, A)) \\
\downarrow m^{(p)} & & \downarrow ev_0 & & \downarrow H(ev_0) \\
A & \xrightarrow{HA} & HA
\end{array}
\]

Let \(\psi : W \to W \otimes W\) be a diagonal approximation and let \(m\) denotes the product in the algebra \(A.\) The formula \(f \cup g = m \circ (f \otimes g) \circ \psi\) defines cup product \(Hom^k(W, A) \otimes Hom^l(W, A) \to Hom^{k+l}(W, A), \quad f \otimes g \mapsto f \cup g\) and \(Hom(W, A)\) is (non associative) differential graded algebra.

### 1.4 Proposition.
Let \((A, \bar{\theta})\) be as above. If we assume that \(H^*\bar{\theta}\) respects products then the algebraic Steenrod operations defined by \(\bar{\theta}\) satisfy the Cartan formula: \(P^i(xy) = \Sigma_{j+k=i} P^j(x)P^k(y), \quad x, y \in H^* A.\)
Proof. We consider the standard small free resolution of a finite cyclic group \([3]\): the (right) \(\pi\)-free acyclic chain complex defined by: \(W = \{W_i\}_{i \geq 0}, W_i = e_i \mathbb{F}_p[\pi]\), \(\partial : W_i \to W_{i-1}\) and if \(\tau\) is a fixed generator of \(\pi\) then \(\partial e_{2i-1} = (1 + \tau)e_{2i}, \partial e_{2i} = (1 + \tau + \ldots + \tau^{p-1})e_{2i-1}\). The augmentation \(\epsilon_W : W \to \mathbb{F}_p\) is defined by \(\epsilon_W(e_i) = 0, i > 0\) and \(\epsilon_W(e_0) = 1\). Let \(\Delta A \subset A^{\otimes p}\) be the subspace generated by the image of the diagonal map \(\Delta : A \to A^{\otimes p}\) \(a \mapsto a^{\otimes p}\). Thus \(\pi\) acts trivially on the sub-algebra \(\Delta A\), and there exists a \(\pi\)-module \(M \subset A^{\otimes p}\) such that, \([3]\)-Lemma 1.3, \(H(W \otimes \pi A^{\otimes p}) = (\bigoplus_{i=0}^{\infty} \mathbb{F}_p e_i \otimes \Delta A) \oplus (\mathbb{F}_p e_0 \otimes M)\). The algebraic Steenrod operations are defined by \([13]\)-Proposition 2.3: for \(H\) algebraic Steenrod operations are defined by \([13]\)-Proposition 2.3) : for \(X\) a reader that the linear map \(P^i(x) = (e_{n-i} \otimes x^{\otimes p}) := Sq^i(x)\) and if \(p\) is odd by \(P^i(x) = (\epsilon(n) e_{(n-2i)(p-1)} \otimes x^{\otimes p})\), \(\overline{P}^i(x) = (1)^i \nu(n) \theta^*(e_{(n-2i)(p-1)} \otimes x^{\otimes p})\). Here \(\nu(n) = (1)^i \left(\frac{p-1}{2}\right)\) if \(n = 2j + \epsilon, \epsilon = 0, 1\) and \(\theta^*\) denotes the map induced by \(\theta\) on \(H(W \otimes \pi A^{\otimes p})\). A diagonal approximation is explicitly given by, \([3]\) -§31-Chap. XII: \(\psi(e_i) = \psi(e_i) \theta^*, \psi(\epsilon_{2l+1}) = \sum e_{2l + 1} \otimes e_{2k} + \sum_{j+k+l} e_{2j+1} \otimes e_{2k + 1} - \sum_{j+k+l} e_{2j+1} \otimes e_{2k + 1}\). For simplicity, let us assume that \(p = 2\). First observe that the restriction of \(\theta\) on \(\Delta A \subset A^{\otimes 2}\) factorizes through a map \(\tilde{\theta} : \Delta A \to \Omega\pi(W, A)\) where \(\Omega\pi(W, A)\) denotes the sub-complex of \(\pi\) linear maps. Secondly, note that \(H^*(\Omega\pi(W, A)) \cong H^*(\Omega\pi(W, \Omega^2 A))\) and that the map \(H^*\theta\) factorizes through a map \((H^*\tilde{\theta}) : K \cap ker d \to \Omega\pi(W, H^* A)\). Let us denote by \(j : \Omega\pi(W, A) \to \Omega\pi(W, A)\) the natural inclusion of complexes. By definition, if \(x \in H^* A\) and \(a\) denotes any cocycle such that \(x = d(a)\) : \(Sq^i(x) = H^* j \circ ((H^* \tilde{\theta}) \pi(a^{\otimes 2}))(e_{n-i}) \in \mathbb{F}_p\). If \(y = \mathbb{F}_2(b) \in H^* A\) then: \(Sq^i(xy) = \tilde{\theta}(ab \otimes b)(e_{n-i})\), \(\tilde{\theta}(a \otimes b)(e_{n-i} - j)\). We set \(\epsilon((\mathbb{F}_2, \mathbb{F}_p) = (\mathbb{F}_2, \mathbb{F}_p)\). We set \(\epsilon(i) = \epsilon_i\), \(\epsilon(1) = \epsilon_1\) and \(\epsilon(2) = \epsilon_2\). we extend \(\epsilon_i\) to \(\epsilon_0\) \(\epsilon(1) = \epsilon_1\) and \(\epsilon(2) = \epsilon_2\) are defined by: \(\epsilon_0(a) = a, \epsilon_1(a) = a + a\). For confluence and \(\epsilon(1) = \epsilon_1\) and \(\epsilon(2) = \epsilon_2\) are defined by: \(\epsilon_0(a) = a, \epsilon_1(a) = a + a\). We set \(\epsilon_0(a) = a, \epsilon_1(a) = a + a\). We set \(\epsilon_0(a) = a, \epsilon_1(a) = a + a\). We set \(\epsilon_0(a) = a, \epsilon_1(a) = a + a\). We set \(\epsilon_0(a) = a, \epsilon_1(a) = a + a\).
graded algebra is not commutative but only strongly homotopy commutative (a shc algebra for short), it seems natural for our purpose, to consider shc algebra in order to define a product on εA.

A shc algebra is a pair \((A,\mu)\) such that \(\mu : \Omega B(A^\otimes 2)\to \Omega BA\) is a homomorphism of differential graded algebras where \(\alpha \circ \mu \circ i_{A\otimes A} = m_A\) (\(m_A\) denotes the product in \(A\)). Moreover, \(\mu\) satisfies the unity axiom, the associativity axiom and the commutativity axiom as described in [15]-4.1. Recall that two homomorphisms of differential graded algebras \(f, g : A \to A'\) are homotopic (we write \(f \simeq_{DA} g\), even if \(A\) and \(A'\) are not associative) if there exists a \((f,g)\)-derivation \(h\) such that \(f - g = dh + hd\). Let \((A,d,\mu)\) and \((A',d',\mu')\) be two shc algebras. A strict shc homomorphism \(f : (A,d,\mu) \to (A',d',\mu')\) is a morphism \(f \in DA(A,A')\) such that \(\Omega Bf \circ \mu \simeq_{DA} \mu' \circ \Omega B(f \otimes f)\).

The chain map \(\varepsilon(\alpha_A)\) is surjective and \(H\varepsilon(\alpha_A) = HH_*(\alpha_A)\) is an isomorphism. We denote by \(s_A : \varepsilon A \to \varepsilon \Omega BA\) any chain map such that \(\varepsilon(\alpha_A) \circ s_A = id\) and \(s_A \circ \varepsilon(\alpha_A) \simeq id\). When \((A,\mu)\) is a shc algebra a product \(m_{\varepsilon A} : \varepsilon A \otimes \varepsilon A \to \varepsilon A\) is defined by the composite 
\[
\varepsilon A \otimes \varepsilon A \xrightarrow{sh} \varepsilon(A \otimes A) \xrightarrow{\varepsilon A \circ \varepsilon A} \varepsilon \Omega BA(A \otimes A) \xrightarrow{\varepsilon A} \varepsilon \Omega BA \xrightarrow{\varepsilon A} \varepsilon A.
\]
where \(sh\) denotes the shuffle product, [2]. More precisely, we proved: If \(A\) is a shc algebra then

1) \(BA\) is a differential graded Hopf algebra such that \(HBA\) is a commutative graded Hopf algebra,

2) \(\varepsilon A\) is a (non associative) differential graded algebra such that \(HH_*(A) := H\varepsilon A\) is a commutative graded algebra,

3) If \(A'\) is another shc algebra and if \(f\) is a homomorphism of shc algebras then there is a commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & \varepsilon A \\
n & \downarrow & \varepsilon f \\
A' & \rightarrow & \varepsilon A'
\end{array}
\]

where the maps \(\iota, \iota', \rho, \rho'\) and \(\varepsilon f\) are homomorphisms of differential graded algebras, \(Bf\) is a homomorphism of differential graded Hopf algebras.

If \(A\) is a shc algebra and if \(p = 2\), then both \(Sq^n\) and \(Sq^{n-1}\) are determined on \(H^nA\), \([8]-6.6\) or \([13]-4.8\), while \(Sq^{n-2}\) is not determined on \((HH_*)^n\), (cf. Example 3.8). Therefore, we need to enrich the notion of shc algebra.

1.6 \(\pi\)-shc algebra. Let \((A,d,\mu)\) be a shc algebra. For any \(n \geq 2\), by Lemma A.3, there exists a homomorphism of differential graded algebras, called the shc iterated structural map \(\mu^{(n)} : \Omega B(A^\otimes n) \to \Omega BA\) such that: \(\mu^{(2)} = \mu\) and \(\alpha_A \circ \mu^{(n)} \circ i_{A^\otimes n} \simeq m_A^n\). Let \(W,\psi\) as in 1.3.

A shc algebra \((A,\mu)\) is, a \(\pi\)-shc algebra if there exists a map \(\tilde{k}_A : \Omega B(A^\otimes p) \to \text{Hom}(W,A)\) which is both a \(\pi\)-linear map and a homomorphism of differential graded algebras such that the next diagram commutes, up to a derivation homotopy.

\[
\begin{array}{ccc}
\Omega B(A^\otimes p) & \xrightarrow{\tilde{k}_A} & \text{Hom}(W,A) \\
\downarrow_{\alpha_A \circ \mu^{(p)}} & \downarrow & \downarrow_{\psi_0} \\
A & \rightarrow & A
\end{array}
\]

Here \(\sigma \in \varepsilon_p\) acts on \(B(A^\otimes p)\) by the rule \(\sigma[x_1|x_2|...|x_k] = [\sigma x_1|\sigma x_2|...|\sigma x_k]\), \(x_i \in A^\otimes p\) and on \(\Omega B(A^\otimes p)\) by the rule \(\sigma \in \pi\), \(\sigma < y_1|y_2|...|y_i >= <\sigma y_1|\sigma y_2|...|\sigma y_i>\), \(y_j \in B(A^\otimes p)\).

A strict \(\pi\)-shc homomorphism \(f : (A,\mu_A,\tilde{k}_A) \to (A',\mu_A',\tilde{k}_A')\) is a strict shc homomor-
coalgebra then there exists a natural chain map \( \phi \), \( k > 1 \) denotes the suspension of degrees. Obviously, \( \kappa_A \circ \Omega B(f^{\otimes p}) \simeq_{\pi-DA} \text{Hom}(W,f) \circ \tilde{\kappa}_A \).

If \( (A,\mu,\tilde{\kappa}_A) \) is a \( \pi \)-sch algebra then the next diagram commutes, up to a \( \pi \)-linear homotopy:

\[
\begin{array}{ccc}
\Omega B(A^{\otimes p}) & \xrightarrow{\tilde{\kappa}_A} & \text{Hom}(W,A) \\
\alpha_{A^{\otimes p}} & \downarrow & \downarrow ev_0 \\
A^{\otimes p} & \xrightarrow{m(p)} & A
\end{array}
\]

and the Steenrod operations, defined for \( x = cl(a) \in H^n A \) by: \( Sq^i(x) = cl(\tilde{\kappa}(iA^{\otimes 2})(a^{\otimes 2}))(e_{n-i}) \) if \( p = 2 \) and by \( P^i(x) = (-1)^i \nu(n)cl(\tilde{\kappa}(iA^{\otimes p})(a^{\otimes p}))(e_{(n-2p)(p-1)}) \) if \( p > 2 \), satisfy the Cartan formula (see Proof of 1.4 for notations).

In order to construct Steenrod operations on \( \mathfrak{e}(A) \) we need to compare \( \text{Hom}(W,\mathfrak{e}A) \) and \( \mathfrak{e}\text{Hom}(W,A) \). But this last expression is not defined unless \( \text{Hom}(W,A) \) is associative. If \( p \neq 2 \), the diagonal approximation considered in the proof of Proposition 1.4 is not coassociative. Thus, for \( p \) odd, we will consider the standard resolution of a finite cyclic group \( \pi \), \([1], [4] \). The Alexander-Whitney diagonal approximation \( \psi_W \), admits the counity \( \epsilon \) and is strictly coassociative.

1.7. Proposition. If \( A \) is a differential graded algebra and if \( (W,\psi_W) \) is a coassociative coalgebra then there exists a natural chain map \( \phi_A \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathfrak{e}\text{Hom}(W,A) & \xrightarrow{\phi_A} & \text{Hom}(W,\mathfrak{e}A) \\
\mathfrak{e}ev_0 & \searrow & ev_0 \\
\mathfrak{e}A & \nearrow & \mathfrak{e}A
\end{array}
\]

Moreover, \( \phi \) is \( \pi \)-linear (\( \pi \) acts trivially on \( B = A,\mathfrak{e}A \), diagonally on \( \text{Hom}(W,B) \)).

Proof. We define \( \phi_A \) by:

\[
\phi_A(f_0[]) = f_0 \quad \text{and if } k > 0 \quad \phi_A(f_0[f_1|f_2|...|f_k]) = (id \otimes s^{\otimes k}) \circ (f_0 \otimes f_1 \otimes f_2 \otimes ... \otimes f_k) \circ \psi_W^{(k)}
\]

where \( \psi_W^{(k)} \) denotes the iterated diagonal: \( \psi_W^{(1)} = \psi_W \), \( \psi_W^{(k+1)} = (id \otimes \psi_W) \circ \psi_W^{(k)} \) and \( s : A \to sA \) denotes the suspension of degrees. Obviously, \( \phi_A \) is natural in \( A \). Let us check in detail that \( \phi_A \) commutes with the differentials. ( \( \sigma_k \) denotes the cycle \( (1,...,k) \to (k,1,...,k-1) \))

\[
\phi_A(d^1(f_0[f_1|f_2|...|f_k])) = \phi_A \left( df_0[f_1|f_2|...|f_k] - \sum_{i=1}^{k} (-1)^i f_0[f_1|...|df_i|...|f_k] \right)
\]

\[
= (id \otimes s^{\otimes k}) \left( df_0 \otimes f_1 \otimes f_2 \otimes ... \otimes f_k - \sum_{i=1}^{k} (-1)^i f_0 \otimes f_1 \otimes ... \otimes df_i \otimes ... \otimes f_k \right) \circ \psi_W^{(k)}
\]

\[
= (id \otimes s^{\otimes k}) \left( (dA \circ f_0 - (-1)^{|f_0|} f_0 \circ \partial) \otimes f_1 \otimes f_2 \otimes ... \otimes f_k 
- \sum_{i=1}^{k} (-1)^i f_0 \otimes f_1 \otimes ... \otimes (dA \circ f_i - (-1)^{|f_i|} f_i \circ \partial) \otimes ... \otimes f_k \right) \circ \psi_W^{(k)}
\]

\[
= (id \otimes s^{\otimes k}) \left( (dA \circ id - id \circ dA_{\otimes k}) \circ (f_0 \otimes f_1 \otimes f_2 \otimes ... \otimes f_k) \circ \psi_W^{(k)}
- (-1)^{|f_0|} f_0 \otimes f_1 \otimes f_2 \otimes ... \otimes f_k \circ \partial \psi_W^{(k)} \right) \circ \psi_W^{(k)}
\]

\[
= d_A^{\otimes A} \circ \phi_A((f_0[f_1|f_2|...|f_k]) - (-1)^{\xi_k} \phi_A((f_0[f_1|f_2|...|f_k]) \circ \partial
\phi_A(d^2(f_0[f_1|f_2|...|f_k])) = \phi_A \left( (-1)^{|f_0|} (f_0 \cup f_1)[f_2|...|f_k] + \sum_{i=2}^{k} (-1)^i f_0[f_1|...|f_i \cup f_{i+1}|...|f_k] 
+ (-1)^{|f_0|} f_0[f_1|f_2|...|f_{k-1}] \right)
\]

\[
= (id \otimes s^{\otimes k-1}) \circ \left( (-1)^{|f_0|} (m_A \circ (f_0 \otimes f_1) \circ \psi_W) \circ (f_2 \otimes ... \otimes f_k) \right)
\]
1.8 End of the proof of Theorem A. One also define, for $n \geq 3$, $(k_1, k_2, \ldots, k_n)$-shuffles, the $n$-iterated shuffle map $sh^{(n)}: (CE)^{\otimes n} \to (CE)^{\otimes n}$ and the iterated product $m^{(p)}_{CE}: C^{\otimes p} \to CE$:

$$\cA^{\otimes p} \xrightarrow{sh^{(p)}} \cA(CE)^{\otimes p} \xrightarrow{s_{A^{\otimes p}}} \cC(\Omega B(CE)^{\otimes p}) \xrightarrow{\varepsilon^{\mu^{(p)}}} \cC(\Omega B \cA) \xrightarrow{\varepsilon^{\mu^{(p)}}} \cC A.$$ (compare with 1.5).

Consider $\phi_A$ as defined above, then applying the functor $\cC$ to the diagram that appear in the definition of a $\pi$-shc algebra (1.6), we obtain the following diagram:

$$\begin{array}{c}
\cC(\Omega B(CE)^{\otimes p}) & \xrightarrow{\phi_{\cC A}} & \text{Hom}(W, \cC A) \\
\downarrow & & \downarrow ev_0 \\
\cC(CE)^{\otimes p} & \xrightarrow{m_{CE}^{(p)}} & \cC A.
\end{array}$$

Since $ev_0 \circ \bar{k}_A \simeq_{DA} \alpha_A \circ \mu_{A}^{(p)}$, Lemma A.6 implies that $ev_0 \circ \phi_A \circ \bar{k}_A \simeq \cC \mu^{(p)}$. Now observe that in the sequence $\cC(\Omega B(CE)^{\otimes p}) \xrightarrow{\phi_{\cC A}} \cC(CE)^{\otimes p}$ of chain maps:

a) $\cC(\alpha_{CE^{\otimes p}})$ is $\pi$-linear. Indeed $\sigma \in \mathfrak{S}_p$ acts on $\cC(CE)^{\otimes p}$ (resp. on $\cC(\Omega B(CE)^{\otimes p})$ by the rule

$$\sigma(x_0[x_1|x_2|\ldots|x_k]) = \sigma x_0[\sigma x_1|\sigma x_2|\ldots|\sigma x_k], \quad x_i \in CE^{\otimes p} \text{ (resp. } x_i \in \Omega B CE^{\otimes p}).$$

Then, $\cC(\Omega B(CE)^{\otimes p})$ and $\cC(CE)^{\otimes p}$ are $\mathfrak{S}_p$-complexes, $\alpha_{CE^{\otimes p}}$ and thus $\cC \alpha_{CE^{\otimes p}}$ are $\mathfrak{S}_p$-linear.

b) $sh^{(p)}$ is also $\mathfrak{S}_p$-linear. Indeed, one simply writes: $sh^{(p)}(a_0^{(p)}|a_1^{(p)}|\ldots|a_n^{(p)}) = (a_0^{(p)}|a_1^{(p)}|\ldots|a_n^{(p)}) \sum_{\sigma} \sigma[a_1^{(p)} \otimes 1 \otimes 1 \ldots \otimes 1 \ldots |a_1^{(p)} \otimes 1 | \ldots |a_1^{(p)} \otimes 1 | \ldots |a_1^{(p)} \otimes 1 | \ldots |a_1^{(p)} \otimes 1 | \ldots |a_1^{(p)} \otimes 1 | \ldots |a_1^{(p)} \otimes 1 | \ldots |a_1^{(p)} \otimes 1 | \ldots |a_1^{(p)} \otimes 1].$ where $\sigma$ ranges over all $(n_1, n_2, \ldots, n_p)$-shuffles and observe that the set of all $(n_1, n_2, \ldots, n_p)$-shuffles coincides with the set of all $(\lambda(n_1), \lambda(n_2), \ldots, \lambda(n_p))$-shuffles when $\lambda \in \mathfrak{S}_p$.

c) $\cC(\alpha_{CE^{\otimes p}})$ is a homomorphism of differential graded algebras. This follows directly from F4, F6 and F8.

From, properties a), b), c) and following the lines of the proof on [13]—Proposition 2.3, (sketched in the proof of Proposition 1.4), using the section $s_{CE^{\otimes p}}$, this diagram defines natural Steenrod operations on $HH_*(A)$. In order to establish the Cartan formula it is enough, by Proposition 1.4, to prove that $H^* \bar{k}_A$ respects products. This is Lemma A.5. Naturality follows directly from Lemma A.6.

§2 - Proof of theorem B.

2.1 Twisting cochains. In [15]—4.7, H.J. Munkholm has established the existence of a natural transformation $\mu_X : \Omega B(N^*X \otimes N^*X) \to \Omega B N^*X$ such that $\alpha_{N^*X} \circ \mu_X \circ i_{N^*X \otimes N^*X}$ is the usual cup product $N^*X \otimes N^*X \to N^*X$. Moreover, $(N^*X, \mu_X)$ is a $shc$-algebra. Let us denote by $\mu_X^{(p)} : \Omega B((N^*X)^{\otimes p}) \to \Omega B N^*X$ the iterated $shc$ structural map and by
the twisting cochain associated to $\alpha_{N^*X} \circ \mu^{(p)}_X: t_X \in \operatorname{Hom}^1 (B(\mathbb{N}^X)^{\otimes p})$, $t \in t = D t$, $t(x)(e_q) = \mu (x > x)$, $x \in B_\mathbb{N}^X$, $t([i]) = 0$ where $D$ denotes the differential in $\operatorname{Hom}(B((\mathbb{N}^X)^{\otimes p}), N^*X)$ (See [15]-1.8 for more details on twisting cochains). We want to construct a twisting cochain $t_X' \in \operatorname{Hom}^1 (B((\mathbb{N}^X)^{\otimes p}), \operatorname{Hom}(W, N^*X))$, $t' \cup t' = D t'$ such that: $e_{\psi} \circ t_X' = t_X$, $t_X'$ is natural in $X$ and $t_X'$ is $\pi$-linear. (Here $D'$ denotes the differential in $\operatorname{Hom}(B((\mathbb{N}^X)^{\otimes p}), \operatorname{Hom}(W, N^*X))$.) Such a twisting cochain $t_X'$ will determine a homomorphism of differential graded algebras $\tilde{\kappa}_X: \Omega B((\mathbb{N}^X)^{\otimes p}) \to \operatorname{Hom}(W, N^*X)$ such that $(\mathbb{N}^X, \mu_X, \tilde{\kappa}_X)$ is a $\pi$-shc algebra. Moreover, by Corollary B.4, $\tilde{\kappa}_X$ is well determined up to a $\pi$-linear homotopy, and so the first part of theorem B will be proved.

2.2 Proof of part 1. We construct $t'$ by induction on the degree in $W = \{W_i\}_{i \geq 0}$ and on the bar degree: $B((\mathbb{N}^X)^{\otimes p}) = \{B_k((\mathbb{N}^X)^{\otimes p})\}_{k \geq 1}$. We denote by $\Delta$ the coproduct in $B((\mathbb{N}^X)^{\otimes p})$. For simplicity, if $x \in B_k(N^*X^{\otimes p})$ we write $\Delta x = 1 \otimes x + x \otimes 1 + \Delta x = x' \otimes x''$ where $\Delta x \in \oplus_{p',p''=k} B_{p'}((\mathbb{N}^X)^{\otimes p}) \otimes B_{p''}((\mathbb{N}^X)^{\otimes p})$ with $l'$ (resp. $l''$) such that $0 < l' < k$ (resp. $0 < l'' < k$).

First, if $e_{\psi}$ denotes a generator of $W_0$, we define $t'$ on $W_0 \otimes \overline{B((\mathbb{N}^X)^{\otimes p})}$ by setting:

$$(\sigma^*) \quad t'(x)(e_{\psi}) = t(\sigma^{-1}x), \quad \sigma \in \pi, \quad x \in B_k(N^*X).$$

$t'$ is a $\pi$-linear map, that is: if $\sigma \in \pi$ and $x \in B((\mathbb{N}^X)^{\otimes p})$ then $t'(\sigma x) = \sigma(t'(x))$. It suffices to prove this last formula for the elements $e_{\psi} \tau^j$ in $W_0$:

$$(\sigma^j) \quad t'(x)(e_{\psi} \tau^j) = t'(x)(e_{\psi} \tau^j) \quad t'(x)(e_{\psi} (\sigma^j)^{(-1)}) = t'(\sigma^j) x(e_{\psi}) \quad t'(\sigma^j)(x)(e_{\psi}) = t'(\sigma x)(e_{\psi} \tau^j).$$

Precise that $\pi$ acts diagonally on $B((\mathbb{N}^X)^{\otimes p}) \otimes B((\mathbb{N}^X)^{\otimes p})$ and on $W \otimes W$ and that $\Delta$ (resp. $\psi$) are $\pi$-linear. We check in detail that: $D t'(x)(w) = (t' \cup t')(x)(w)$, $w \in W_0$, $x \in B((\mathbb{N}^X)^{\otimes p})$. Once again, it suffices to prove this formula for the elements $e_{\psi} \sigma \in W_0$:

$$(\sigma^j) \quad t'(x)(e_{\psi} \sigma) = D t'(x)(e_{\psi} \sigma) = t'(x)(e_{\psi} \sigma) = t'(x)(e_{\psi} \sigma) = t'(x)(e_{\psi} \sigma) = t'(x)(e_{\psi} \sigma).$$

Secondly, for an arbitrary generator $e_i \in W_i$, $i \geq 1$ and for $[x] \in B_1((\mathbb{N}^X)^{\otimes p})$, we define

$$(\sigma^j) \quad t'(x)(e_i) = \begin{cases} 0 & \text{if} \quad x \in V_k, \\ \lambda(\sigma^{-1}x)(e_i) & \text{if not}, \quad \text{where} \quad V_k = \mathbb{F}_p^{\otimes p-k} \otimes \mathbb{N}^X \otimes \mathbb{F}_p^{\otimes p-k} \text{and} \lambda \text{is a natural } \pi \text{-linear transformation of degree } 0 \text{ such that the following diagram commutes up to homotopy} \end{cases}$$

$$\xymatrix{(\mathbb{N}^X)^{\otimes p} \ar[r]^-{\lambda} & \operatorname{Hom}(W, \mathbb{N}^*X) \ar[d]^-{e_{\psi}} \ar[r]_-{N^*_X} & \mathbb{N}^*X \ar[d]^-{p} \ar[r]_-{N^*_X} & \operatorname{Hom}(W, \mathbb{N}^*X) \ar[r]_-{e_{\psi}} & \mathbb{N}^*X.}$$

Such a transformation exists for Corollary B.4 applies. Observe that if $w \in W_i$ and $x \in ((\mathbb{N}^X)^{\otimes p})$ then $(d_{\mathbb{N}^X}(\lambda_X(x) - \Delta_X(dx))(w) = (-1)^{[x]}(x)(\partial w)$. The formula $(\sigma^j)$ defines a $\pi$-linear map $t': B_1((\mathbb{N}^X)^{\otimes p}) \to \operatorname{Hom}(W, \mathbb{N}^*X)$ and since $\Delta x = 0$ it follows that $D t'(x) = 0 = (t' \cup t')[x]$, $[x] \in B_1((\mathbb{N}^X)^{\otimes p})$.

Thirdly, we assume that, for some $k \geq 1$, there is a $\pi$-linear map $t': B_{i<k}((\mathbb{N}^X)^{\otimes p}) \to \operatorname{Hom}(W, \mathbb{N}^*X)$, which extends to a $\pi$-linear map $t': B_k((\mathbb{N}^X)^{\otimes p}) \to \operatorname{Hom}(W_{<i}, \mathbb{N}^*X)$, $i \geq 1$ and satisfying $D t'(y) = (t' \cup t')(y), y \in B_{i<k}((\mathbb{N}^X)^{\otimes p})$. Let $d_1$ be the internal part of the differential in the bar construction $d = d_1 + d_2$. The contravariant
functor $X \mapsto FX = (\mathbb{F}_p \oplus s^i B_k ((N^*X)^{\otimes p}), d)$ where $d(s^i y) = s^i dy$, $y \in B_k((N^*X)^{\otimes p})$ is acyclic on the models $\{\Delta^n\}_{n \geq 0}$. We define the natural transformation: $T_{i,X} : FX \to N^* X$, $y \mapsto T_{i,X}(y) = (t' \cup t') (y)(e_i) - (-1)^{|t|} t'(y)(\partial e_i) - t'(d^2 y)(e_i)$. (The inductive hypothesis ensures us that $T_{i,X}(y)$ is well defined.) Let us check in detail that $D T_{i,X} = 0$ when $D$ denotes the differential in $\text{Hom}(F(X, N^*X))$. Observe that, for any generator $e_i$ of $W_i$ and if $y \in FX$, the inductive hypothesis implies: $(D t')(y)(\partial e_i) = (t' \cup t')(\partial e_i)$ and $(D t')(d^2 y)(e_i) = (t' \cup t')(d^2 y)(e_i)$. Thus we obtain:

$$\left( DT_{i,X} \right) (y) = d_{N^*X} ((t' \cup t')(y)(e_i) - (-1)^{|t'|} (D t')(y)(\partial e_i))$$

$$= d_{N^*X} ((t' \cup t')(y)(e_i) - (-1)^{|t'|} (D t')(d^2 y)(e_i) + (-1)^{|t'|} t'(d^2 y)(e_i))$$

$$= D t'(y)(e_i) - (-1)^{|t'|} t'(d^2 y)(e_i) - t'(d^2 y)(e_i)$$

Now $(DT_{i,X})(y) = 0$, since each line in the last equality is 0.

Theorem B.4, establishes the existence of a natural transformation $S_{i,X} : FX \to N^* X$ such that $D S_{i,X} = T_{i,X}$ that is: if $z \in FX$ then $T_{i,X}(z) = d_{S_{i,X}}(z) - (-1)^{|S_{i,X}|} d^2 z$. We set for $y \in B_k((N^*X)^{\otimes p})$, $t'(y)(e_i) = S_{i,X}(\sigma^{-1} y)$. This formula defines a $\pi$-linear map $t' : B_k((N^*X)^{\otimes p}) \to \text{Hom}(W_{<+1}, N^* X)$. It remains to check that $D t' = t' \cup t'$. Let $e_i$ be any generator of $W_i$, and if $y \in FX$,

$$(t' \cup t')(y)(e_i) = T_{i,X} (s^i y) - (-1)^{|t'|} t'(d^2 y)(e_i) = d_{S_{i,X}} (s^i y) - (1)^{|S_{i,X}|} d_{S_{i,X}} (s^i y) + (-1)^{|t'|} t'(y)(\partial e_i) + t'(d^2 y)(e_i)$$

$$= d(t'(y)(e_i)) + (t'(d^2 y)(e_i)) + (1)^{|t'|} t'(y)(\partial e_i)$$

$$= (D t') (y)(e_i).$$

We have thus constructed a twisting cochain $t'_X \in \text{Hom}^1 (B ((N^*X)^{\otimes p}), \text{Hom}(W, N^* X))$ such that $t'_X$ is natural in $X$ and $t'_X$ is $\pi$-linear.

### 2.3 Proof of part 2.

Recall the notation introduced in [2]-Part II-§3. In particular, consider the simplicial model $K$ of $S^1$, and the cosimplicial space $\underline{X}$ defined as $\underline{X}(n) = \text{Map}(K(n), X)$ whose geometric realization $|\underline{X}|$ is homeomorphic to $X^{S^1}$. Moreover, there is a natural equivalence $\psi_X : \text{Tot} C^* \underline{X} \rightarrow N^*|\underline{X}|$, where $\text{Tot} C^* \underline{X}$ denotes the total complex of the simplicial complex $C^* \underline{X}$. On the other hand, J. Jones, [10]-6.3, has proved that there is natural chain map $\theta_X : \mathfrak{c} N^* X \rightarrow \text{Tot} C^* \underline{X}$ such that, if $X$ is 1-connected, the composite $J_X = \psi_X \circ \theta_X$ induces an isomorphism $HH_* N^* X \cong H_* X^{S^1}$. We obtain the following diagram in the category of $\pi$-differential graded modules:

$$\begin{array}{ccc}
\mathfrak{c} & (\Omega B(N^*X)^{\otimes p}) & \to \text{Hom}(W, \mathfrak{c} N^*X) \\
\downarrow \phi_{\mathfrak{c} X} & & \downarrow \text{Hom}(W, \theta_X \circ \psi_X) \\
\mathfrak{c}(\alpha_{(N^*X)^{\otimes p}}) & \downarrow \text{Hom}(W, N^*|\underline{X}|) \\
\mathfrak{c}((N^*X)^{\otimes p}) & \to \text{Hom}(W, N^*|\underline{X}|) \\
sh^{(p)} & \uparrow \Gamma_X \\
(\mathfrak{c} N^* X)^{\otimes p} & \psi_X^{(p)} & (N^*|\underline{X}|)^{\otimes p}
\end{array}$$

where $\Gamma_X$ denotes any natural structural map defining the usual Steenrod operations, [13].

The functor $X \mapsto \mathfrak{c}((N^*X)^{\otimes p})$ preserves the units and is acyclic and the functor $X \mapsto \text{Hom}(W, N^*|\underline{X}|)$ is corepresentable on the models $\mathcal{M}$, (see [2]-II-lemma 3.9). Therefore, by Corollary B.4, there exists a $\pi$-linear natural transformation $T_X : \mathfrak{c}((N^*X)^{\otimes p}) \to \text{Hom}(W, N^*|\underline{X}|)$ such that $T_X \circ \mathfrak{c} \alpha_{(N^*X)^{\otimes p}} \simeq_{\pi} \text{Hom}(W, \theta_X \circ \psi_X \circ \phi \circ \mathfrak{c} \kappa_X)$ and $T_X \circ sh^{(p)} \simeq_{\pi}$
\[ \tilde{\Gamma}_X \circ \psi_X^\otimes \]. Consequently, the second part of theorem B is proved.

\section{\pi-shc models.}

We continue the study of the homotopy category of \pi-shc algebras by constructing a convenient model of these algebras. As it would be expected this model is rather complicated. Nevertheless, in the topological case, it provides a theoretical object simpler than the singular cochain algebra.

### 3.1 Minimal algebra

Let \( V = \{ V^i \}_{i \geq 1} \) be a graded vector space and let \( TV \) denotes the free graded algebra generated by \( V \): \( T^n V = V \otimes V \otimes \ldots \otimes V \) (\( r \)-times) and \( v_1 v_2 \ldots v_k \in (TV)^n \) if \( \sum_{i=1}^k |v_i| = n \). The differential \( d \) on \( TV \) is the unique degree 1 derivation on \( TV \) defined by a given linear map \( V \to TV \) and such that \( d_V \circ d = 0 \). The differential \( d_V : TV \to TV \) decomposes as \( d = d_0 + d_1 + \ldots \) with \( d_k V \subset T^{k+1} V \). In particular \((V,d)\) is a differential graded vector space. By the discussion above, for each \( n \geq 0 \), we obtain a sequences:

\[ (TV,d) \to (TV,d_V) \to (TU,d_V) \]

where \((TV,d_V)\) denotes a 1-connected minimal algebra, \( p_V \circ \varphi = id \), \( \varphi \circ p_V \cong \Id \) and \( V \) such that \( V \cong H(U,d) \). Moreover, \( \left[ \begin{array}{c} H \end{array} \right] \), \( \left[ \begin{array}{c} V \end{array} \right] \), \( \left[ \begin{array}{c} A \end{array} \right] \), \( \left[ \begin{array}{c} U \end{array} \right] \), \( \left[ \begin{array}{c} F \end{array} \right] \) is unique up to isomorphisms.

### 3.2 Minimal model of a product

Assume that \((A,d_A)\) is a differential graded algebra such that \( H^0(A,d_A) = \mathbb{F}_p \) and \( H^1(A,d_A) = 0 \), and let \((TU,d) = \Omega ((BA) \otimes \), \( n \geq 1 \).

By the discussion above, for each \( n \geq 1 \), we obtain a sequences:

\[ (TU[n],d_U[n]) = \Omega ((BA) \otimes , n \geq 1) \]

with \( V[n] = s^{-1} (H ((BA) \otimes )) \cong s^{-1} ((H(BA) \otimes )) = s^{-1} (\mathbb{F}_p \otimes sV) \otimes \)

\[ \cong \left( \bigoplus_{k=1}^n \left( (\mathbb{F}_p \otimes sV) \otimes \right) \right) \]

For \( n = 1 \), \( V[1] = V = s^{-1} H(BA) \) and the composite \( \psi_V = \alpha_A \circ \varphi_V : (TV,d_V) \to A \) is a quasi-isomorphism. The algebra \((TV,d_V)\) is called the 1-connected minimal model of \( A \).

For \( n \geq 2 \), consider the homomorphism \( q_V[n] : (TV[n],d_V[n]) \to (TV,d_V) \otimes sV \) defined by \( q_V[n](y) = 1 \otimes y \otimes 1 \) if \( y \in V_k := \mathbb{F}_p \otimes sV \otimes \mathbb{F}_p \otimes , k = 1,2,\ldots,n \) et \( q_V[n](y) = 0 \) if \( y \in V[n] - \bigoplus_{i=1}^n V_i \). The composite \((TV[n],d_V[n]) \to (TV,d_V) \otimes sV \to (A,d_A) \otimes sV \) is a quasi-isomorphism \( (\mathbb{F}_p \otimes \mathbb{F}_p) \)

Therefore \((TV[n],d_V[n])\) is the minimal model of \( A \).

### 3.3 Lemma

(Equivariant lifting lemma.) Let \( \pi \) any finite group and consider

i) a graded free \( \pi \)-module, \( U = \{ U^i \}_{i \geq 2} \),

ii) a minimal model, \((TU,d)\), with \( \pi \) acting by the rule \( \sigma \cdot u_1 u_2 \ldots u_k = \sigma u_1 \ldots u_k \), so that \((TU,d)\) is a \( \pi \)-complex,

iii) two differential graded algebras, \( A \) and \( B \), which are also \( \pi \)-complexes,

iv) a homomorphism of differential graded algebras, \( f : (TU,d) \to B \), which is \( \pi \)-equivariant,

v) a surjective quasi-isomorphism, \( \varphi : A \to B \), of differential graded algebras which is \( \pi \)-equivariant.
Then there exists a homomorphism of differential graded algebras \( g : (TU, d) \to B \) which is \( \pi \)-equivariant and such that \( \varphi \circ g = f \).

**Proof.** We choose a homogeneous linear basis \( \{ u_i \}_{i \in I} \) of the \( \pi \)-free graded module \( U \) with \( I \) a well-ordered set. We denote by \( U_{<i} \) the sub-\( \pi \)-module generated by the elements \( u_i, \ j < i \). The minimality condition together with the 1-connectivity condition imply that \( du_i \in T(U_{<i}) \). Suppose that \( g : (T(U_{<i}), d) \to B \) has been constructed so that \( g \) is \( \pi \)-equivariant and \( \varphi \circ g = f \). Then \( g(du_i) \in B \cap \text{Ker} \ d_B \) and \( \varphi \circ g(du_i) = d_A f(e_i) \). Therefore there exists \( b_i \in B \) such that \( g(du_i) = d_B b_i \). Moreover, \( d_A f(du_i) = d_A \varphi b_i \). For \( \sigma \in \pi \), we set \( g(\sigma u_i) = \sigma(b_i + z_i) \), and we check that \( \varphi \circ g(\sigma u_i) = f(\sigma u_i) \) and \( dg(\sigma u_i) = g(d\sigma u_i) \).

### 3.4 \( \pi \)-shc-minimal model

For any integer \( n > 1 \) the group \( S_n \) acts on \( V[n] \subset s^{-1}(H(BA))(\otimes n) \). This action extends diagonally on \( TV[n] \) so that the differential \( d_{V[n]} \) and the homomorphism \( (\psi_V)(\otimes n) \circ q_{V[n]} \) are \( S_n \)-linear. Since the natural map \( \alpha_{A^{\otimes n}} \) is a \( S_n \)-equivariant surjective quasi-isomorphism, Lemma 3.3 implies that the composite \( (\psi_V)(\otimes n) \circ q_{V[n]} \) lifts to a homomorphism of differential graded algebras \( L : TV[n] \to \Omega B(A^{\otimes n}) \) which is \( S_n \)-equivariant and \( \alpha_{A^{\otimes n}} \circ L = (\psi_V)(\otimes n) \circ q_{V[n]} \). Let \( (A, d_A, \mu_A) \) be an augmented shc algebra and assume that \( H^0(A, d_A) = \mathbb{F}_p \) and that \( H^1(A, d_A) = 0 \). The composite, \( \mu_V^{(n)} := p_V \circ \mu_A^{(n)} \) is an \( S_n \)-equivariant homomorphism of algebras \( \tilde{\kappa}_V = (TV[n], d_{V[n]}) \to (T(V), d_V) \) is a homomorphism of differential graded algebras. If \( n = 2 \), \( \mu_V^{(2)} := \mu_V : (TV[2], d_{V[2]}) \to (TV, d_V) \). The triple \( (TV, d_V, \mu_V) \) is called a shc-minimal model for \( (A, d_A, \mu_A) \) (See [2]-I-$\S$6 for more details.)

Let \( (A, d_A, \mu_A, \tilde{\kappa}_A) \) be a \( \pi \)-shc algebra such that \( H^0A = \mathbb{F}_p \) and \( H^1A = 0 \). By Lemma 3.3, \( \tilde{\kappa}_A \circ L \) lifts to a \( \pi \)-equivariant homomorphism of algebras \( \tilde{\kappa}_V = (TV[p], d_{V[p]}) \to \text{Hom}(W, \Omega B A) \). Since \( \pi \) is assumed to act trivially on \( TV \) the composite \( \tilde{\kappa}_V = \text{Hom}(W, p_V) \circ \tilde{\kappa}_A : TV[p] \to \text{Hom}(W, TV) \) is a \( \pi \)-equivariant map and we obtain the commutative diagram:

\[
\begin{array}{ccc}
\Omega B(A^{\otimes p}) & \xrightarrow{\tilde{\kappa}_A} & \text{Hom}(W, A) \\
\downarrow L & & \downarrow \text{Hom}(W, \alpha_A) \\
(TV[p], d_{V[p]}) & \xrightarrow{\tilde{\kappa}_V} & \text{Hom}(W, \Omega B A) \\
& \searrow & \downarrow \text{Hom}(W, p_V) \\
& & \text{Hom}(W, TV)
\end{array}
\]

The triple \( (TV, d_V, \mu_V, \tilde{\kappa}_V) \) is called a \( \pi \)-shc minimal model for the \( \pi \)-shc algebra \( (A, d_A, \mu_A, \tilde{\kappa}_A) \).

### 3.5 Proposition

Let \( (A, d_A, \mu_A, \tilde{\kappa}_A) \) and \( (TV, d_V, \mu_V, \tilde{\kappa}_V) \) be as above then:

1. \((TV, d_V, \mu_V, \tilde{\kappa}_V)\) is a \( \pi \)-shc algebra,
2. \( \psi_V : (TV, d) \to A \) is a strict \( \pi \)-shc homomorphism and a quasi-isomorphism,
3. the canonical maps \( (TV, d_V) \xrightarrow{\psi_V} \mathcal{C}(TV, d_V) \xrightarrow{p_V} B(TV, d_V) \) respect the Steenrood operation in homology,
4. the quasi-isomorphism \( \psi_V : (TV, d_V) \to (A, d_A) \) induces a commutative diagram

\[
\begin{array}{ccc}
(TV, d_V) & \xrightarrow{\psi_V} & \mathcal{C}(TV, d_V) \\
\downarrow & & \downarrow \psi_V \\
(A, d_A) & \xrightarrow{\psi_V} & B(A, d_A)
\end{array}
\]

in which all vertical arrows induce isomorphisms in homology which respect the Steenrood operations.
Proof. From the definition of $\tilde{\kappa}_V$, we deduce that $ev_0 \circ \tilde{\kappa}_V \simeq_{DA} \mu_V$. This implies that $(TV, \mu_V, \tilde{\kappa}_V)$ is a $\pi$-shc algebra. Since $p_V \circ \varphi_V = id_{TV}$, $\varphi_V \circ p_V \simeq_{DA} id_{BA}$ then $\text{Hom}(W, \varphi_V) \circ \tilde{\kappa}_V \simeq_{DA} \tilde{\kappa}_A$. Indeed, it is straightforward to check that if $\pi$ acts trivially on $A$ and $A'$ and if $f, g : A \to A'$ are homotopic in $DA$, then $\text{Hom}(W, f)$ and $\text{Hom}(W, g)$ are $\pi$-homotopic in $DA$. It remains to show that $\psi_V$ is a homomorphism $\pi$-shc algebras. This follows directly from the commutativity up to $\pi$- homotopy in $DA$ of the diagram

$$\begin{array}{ccc}
\Omega (\langle (BA) \otimes^p \rangle) & \xrightarrow{\kappa_A} & \text{Hom}(W, A) \\
L \uparrow & & \uparrow \text{Hom}(W, \psi_V) \\
TV[p] & \xrightarrow{\tilde{\kappa}_V} & \text{Hom}(W, TV).
\end{array}$$

3.6. Proposition. Let $(A, d_A, \mu_A, \tilde{\kappa}_A)$ and $(TV, d_V, \mu_V, \tilde{\kappa}_V)$ be as above. Assume $p = 2$ and set $\tilde{V} = V[2]$, $V' = V \otimes \mathbb{F}_p$, $v'' = \mathbb{F}_p \otimes V$ and $V' \# v'' = s^{-1}(sV \otimes sV)$, so that $\tilde{V} = V' \oplus V'' \oplus V' \# v''$. If $\tilde{\kappa} : \Omega BA \otimes \Omega BA \to \text{Hom}(W, A)$ satisfies:

$(*) \quad \tilde{\kappa}([a \otimes b]) = 0$ if $[a \otimes b] \in B_1(A \otimes A)$ and $\epsilon_A(a) = 0$ or $\epsilon_A(b) = 0$,

then, for any $v \in V' \cap \ker d_V$, $Sq^i(cl(v)) = \tilde{\kappa}_V(v \# v)(\epsilon_{n-i-1}).$

Observe that if $A = N^*X$, condition $(*)$ is satisfied (see 2.2, $(*)_i$).

Proof. Define $K_A : \Omega(BA \otimes BA) \to \Omega BA \otimes \Omega BA$ as follows: we write $\Omega(BA \otimes BA) = (T(s^{-1}(B + A \otimes_B \mathbb{F}_p)) \otimes s^{-1}(F \otimes B + (B \otimes B + B)))$, $D$ and we set $K_A(s^{-1}[a_1][a_i] \otimes 1) = s^{-1}[a_1][a_i] \otimes 1$, $K_A(1 \otimes s^{-1}[a_1][a_i]) = 1 \otimes s^{-1}[a_1][a_i]$, $K_A(s^{-1}[a_1][a_i] \otimes [b_1][b_2]) = 0, i, j > 0$. One easily check (3.1-6.4) that $K_A$ commutes with the differentials. Since, $(\alpha_A \otimes \alpha_A) \circ K_A = \alpha_A \otimes \alpha_A \circ \Omega \tilde{\kappa}$, the homomorphism $K_A$ is a surjective quasi-isomorphism. The map $S : W \otimes_{\pi}(A \otimes A) \to W \otimes \pi \Omega(BA \otimes BA)$ defined by $S(e_0 \otimes a \otimes b) = e_0 \otimes ([a] \otimes 1)(1 \otimes [b])$ and if $i > 0$ by $S(e_i \otimes a \otimes b) = e_i \otimes ([a] \otimes 1)(1 \otimes [b]) + e_{i-1} \otimes ([b] \otimes [a])$, satisfies: $id_W \otimes (\alpha_A \otimes \alpha_A) \circ id_W \circ K_A \circ S = id_{W \otimes_{\pi} A}$, and $Sd = dS$. As in 1.6, we deduce:

$Sq^i cl([\tilde{\kappa}(i_{A \otimes a}(a \otimes a))(e_{n-i})]) = cl([\kappa(e_{n-i}) \otimes i_{A \otimes a}(a \otimes a)]) = cl([\kappa/e_{n-i} \otimes i_{A \otimes a}(a \otimes a)])$

$= cl(\kappa(\overline{\Omega \tilde{\kappa}}(\overline{[a] \otimes [a]})))(e_{n-i-1}).$

The last equality comes from the condition $(*)$ and the fact that $\Omega \overline{\tilde{\kappa}}$ is $\pi$-linear.

Now observe that $\varphi_V : T(V' \oplus V'' \oplus V' \# V'') \to T(s^{-1}BA \otimes BA)$ identifies $v' \in V'$ with $[a] \otimes 1$, $v'' \in V''$ with $1 \otimes [a]$ and $v' \# v'' \in V' \# V''$ with $([a] \otimes [a])$. Thus if $v \in V' \cap \ker d_V$ and $x = cl(v)$ then $H \psi_V \circ Sq^i(cl(v)) = cl(\tilde{\kappa}(\overline{\Omega \tilde{\kappa}} \circ \varphi_V(v'))(v' \# v''))$. The formula now follows from the definition of $\tilde{\kappa}_V$.

3.7 shc-equivalence. Two shc algebras (resp. $\pi$-shc algebras) $A$ and $A'$ are shc equivalent, $A \simeq_{shc} A'$ (resp. $\pi$-shc equivalent, $A \simeq_{\pi-shc} A'$) if there exists a sequence of strict shc (resp. $\pi$-shc) homomorphism $A \leftarrow A_1 \rightarrow \ldots \rightarrow A'$ inducing isomorphisms in homology. If $A \simeq_{\pi-shc} A'$, then $H(A) \cong H(A')$, and $A, A'$ have the same $\pi$-shc minimal model. Two spaces $X$ and $Y$ are shc equivalent (resp. $\pi$-shc equivalent) if the differential graded algebras $N^*X$ and $N^*Y$ are shc equivalent (resp. $\pi$-shc equivalent).

3.8 Example. We exhibit two spaces $X$ and $Y$ which are shc equivalent but not $\pi$-shc equivalent. Let us consider $X = \Sigma^2CP^2$ and $Y = S^4 \vee S^6$. The spaces $X$ and $Y$ have the same Adams-Hilton model namely $(T(x_3, x_5), 0)$. This shows, in particular, that $H_*X \cong H_*Y$, as graded Hopf algebras, and that $X$ and $Y$ are $\mathbb{F}_2$-formal.
Furthermore, $H^* = H^*X = F_2a_4 \oplus F_2a_6 \cong H^*Y$ as graded algebras (with trivial products). Hence $X$ and $Y$ have the minimal model say $(TV, d_V)$ with $V = s^{-1}H_*\Omega X = s^{-1}T^*(x_3, x_5)$, $d_V s^{-1}x_3 = d_V s^{-1}x_5 = 0$. The map $\psi : (TV, d_V) \to H^*$ defined by $\psi(s^{-1}x_3) = a_4$, $\psi(s^{-1}x_5) = a_6$ and $\psi = 0$ on $s^{-1}T^{\geq 2}(x_3, x_5)$ is a surjective quasi-isomorphism. Let $(TV, d_V, \mu_1, \kappa_1)$ (resp. $(TV, d_V, \mu_2, \kappa_2)$) be a $\pi$-$shc$ minimal model of $X$ (resp. $Y$), then for $i = 1, 2$ $\mu_i : (TV, d_V) \to (TV, d_V)$ identifies the two copies of $V$. Moreover, $|\mu_i(x\# y)| > 6$ for any $x, y \in V$. Therefore $\psi \circ \mu_i(x\# y) = 0$. This yields the commutative diagram

$$
\begin{array}{ccc}
T^V & \xrightarrow{\psi \otimes \phi} & H^* \otimes H^* \\
\mu_1 \downarrow & & \downarrow m_{H^*} \\
TV & \xrightarrow{\psi} & TV
\end{array}
$$

This proves that $X$ and $Y$ are $shc$ equivalent. Now we prove that $X$ and $Y$ are not $\pi$-$shc$ equivalent. Recall that if $a_4 \in H^4(\Sigma^2 CP^2)$ then $Sq^2a_4 = a_6$ while if $a_4 \in H^4(S^4 \vee S^6)$ then $Sq^2a_4 = 0$. Thus, by Proposition 3.6, $\tilde{\kappa}_1(a_4 \# a_4)(e_1) = a_6$ while $\tilde{\kappa}_2(a_4 \# a_4)(e_1) = 0$. Now $\tilde{\kappa}_1, \tilde{\kappa}_2$ are not $\pi$-homotopic, and hence $X$ and $Y$ are not $\pi$-$shc$ equivalent.

3.9. **Example.** We compute the Steenrod operations on $HH_*A$ when $A = F_2[u] = T(u), |u| = 2$.

First, recall that, endowed with the shuffle product, $cT(u) \xrightarrow{\phi} \c(T(u) \otimes T(u)) \cong \cT(u)$, the complex $\cT(u)$ is a commutative differential graded algebra. We consider the homomorphism of differential graded algebras $\rho : (T(u) \otimes \Lambda u, 0) \to \cT(u)$ defined by $\rho(u \otimes 1) = u[1], \rho(1 \otimes su) = 1[u]$. As proved in Proposition 3.1, $\rho$ is a quasi-isomorphism of differential graded algebras. In particular, $T(u) \otimes \Lambda u \cong HH_*T(u)$ as commutative graded algebras.

Secondly, we define a structure of $\pi$-$shc$ algebra on $T(u)$. Let $V = uF_2$ and set $\tilde{V} = uF_2 \oplus uF_2 \oplus u\# uF_2$. We define $\tilde{\kappa} : T(\tilde{V}) \to \Hom(W, T(u))$ by $\tilde{\kappa}(u')(e_i) = \tilde{\kappa}(u')(e_i) = \tilde{\kappa}(u')(e_i) = u$ if $i = 0$ and $= 0$ if $i > 0$. As in the proof of Proposition 3.6, $\tilde{\kappa}$ defines the Steenrod operations on $HT(u) = T(u)$:

$$
Sq^i(u) = u, \quad Sq^2 = u^2 \text{ and } Sq^i = 0 \text{ if } i \neq 0, 2.
$$

By Proposition 3.6 one deduces that $\tilde{\kappa}$ is the unique $\pi$-$shc$ structural map on $T(u)$ with the Steenrod operations defined above.

Now in order to compute the Steenrod operations on $HH_*T(u)$ we consider the structural map $\bar{\rho} = \phi \circ \tilde{\kappa} : \c(\tilde{T}) \to \Hom(W, \cT(u))$, and a linear section $S$ of $id_W \otimes cq_U : W \otimes \pi \c(\tilde{T}) \to W \otimes \pi \c(T(u) \otimes T(u))$. Then Steenrod operations are defined by:

$$
Sq^i(x) = cl(\theta(S \circ (id_W \otimes sh)(e_{n-i} \otimes x \otimes x)), \quad x \in HH^nT(u).
$$

Finally observe that $S$ is uniquely determined in low degrees as follows:

$$
\begin{array}{llll}
S(e_i \otimes 1[1]) & = e_i \otimes 1[1] & \quad & S(e_i \otimes (u \otimes 1[1]) = e_i \otimes u'[1]
\\
S(e_i \otimes (1 \otimes u)[1]) & = e_i \otimes u'[1] & \quad & S(e_i \otimes 1[1 \otimes u] = e_i \otimes 1[u']
\\
S(e_i \otimes (u \otimes u)[1]) & = e_i \otimes u''[1] + e_{i-1} \otimes u' \# u''[1] & \quad & S(e_i \otimes 1[1 \otimes u] = e_i \otimes 1[u'']
\\
S(e_i \otimes (u \otimes 1)[u \otimes 1]) & = e_i \otimes u'[u'] & \quad & S(e_i \otimes (1 \otimes u)[1 \otimes u] = e_i \otimes u''[u']
\\
S(e_i \otimes (1 \otimes u)[u \otimes 1]) & = e_i \otimes u''[u'] + e_i \otimes u' \# u''[1] & \quad & S(e_i \otimes 1[1 \otimes u] = e_i \otimes 1[u' \# u'']
\\
S(e_i \otimes (u \otimes 1)[1 \otimes u]) & = e_i \otimes u''[u'] + e_i \otimes u' \# u''[1] & \quad & S(e_i \otimes 1[1 \otimes u] = e_i \otimes 1[u'' \# u']
\\
S(e_i \otimes 1[u \otimes u]) & = e_i \otimes 1[u' \# u''] + e_{i-1} \otimes 1[u \# u''][1] & \quad & S(e_i \otimes 1[1 \otimes u] = e_i \otimes 1[u' \# u'']
\\
S(e_i \otimes 1[1 \otimes u\otimes 1]) & = e_i \otimes 1[u'' \# u'] + e_i \otimes 1[1 \otimes u' \# u''][1]
\end{array}
$$

(Here we make the convention $e_{-1} = 0$.) Therefore, since $\bar{\theta}$ respects the products
and with the aid of Proposition 3.6, we can do the following computations:

\[ Sq^0(u[\cdot]) = cl(\theta \circ S \circ sh(e_2 \times u[\cdot] \otimes u[\cdot])) = cl(\bar{\theta}(e_2 \times u'_{u''}[\cdot] + e_1 \times u'_{u''}[\cdot])) \]

\[ = cl(\bar{\theta}(u'_{u''}(e_2)(\cdot))) + cl(\bar{\theta}(\bar{\theta}(u'_{u''}'))) = cl(\bar{\kappa}(u'_{u''}')(\cdot)) = Sq^0(u[\cdot]) = u[\cdot] \]

\[ Sq^1(u[\cdot]) = cl(\theta \circ S \circ sh(e_1 \times u[\cdot] \otimes u[\cdot])) = cl(\theta(e_1 \times u'_{u''}[\cdot] + e_0 \otimes 1[u'_{u''}][\cdot])) \]

\[ = cl(\bar{\kappa}(u'_{u''}')(\cdot)) = 0 \]

\[ Sq^2(u[\cdot]) = cl(\theta \circ S \circ sh(e_0 \times u[\cdot] \otimes u[\cdot])) = cl(\bar{\kappa}(u'_{u''}')(e_0)) = u^2. \]

\[ Sq^0(1[u]) = cl(\theta \circ S \circ sh(e_1 \times 1[u] \otimes 1[u])) = cl(\theta(e_1 \times 1[u]_{u''}[\cdot] + e_1 \times 1[u'_{u''}][\cdot])) \]

\[ = cl(\theta(e_0 \otimes 1[u]_{u''}[\cdot] + e_0 \otimes 1[u'_{u''}][\cdot]) + e_0 \otimes 1[u'_{u''}][\cdot]) \]

\[ = slc(\theta(e_0 \otimes 1[u]_{u''}[\cdot]) + e_0 \otimes 1[u'_{u''}][\cdot]) + e_0 \otimes 1[u'_{u''}][\cdot]) = 0. \]

Thus the Steenrod operations are completely defined on \( HH_*T(u) = T(u) \otimes \Lambda s \), by the Cartan formula and the formulas: \( Sq^i(u[\cdot]) = u[\cdot] \), \( Sq^2(u[\cdot]) = u^2[\cdot] \), \( Sq^1(u[\cdot]) = 0 \) for \( i \neq 1, 2 \), \( Sq^0(1[u]) = 1[u] \), \( Sq^0(1[u]) = 0 \) if \( i > 0 \). We recover the topological Steenrod squares on \( H^*(\mathbb{C}P^\infty)^S \) = \( H^*(\mathbb{C}P^\infty) \otimes H^*(S^1) \). The technics developed in this example, can be performed in order to study the case when \( A = F_2[u]/(u^k), k \geq 2 \).

Appendix A - Technicalities on shc-algebras.

In this appendix we lay the material necessary in order to complete the proof of Theorem A.

A.1 Lemma. Let \( A, A' \) be two shc algebras and let \( f, g \in DA(A,A') \) be such that \( f \simeq_{DA} g \). If \( f \) is a shc homomorphism then \( g \) is a shc homomorphism.

Proof Let \( \theta \) be a \( (f,g) \)-derivation such that \( f - g = d_{A'} \circ \theta + \theta \circ d_{A} \). Then \( \theta \) defines a coderivation \( \theta' : BA \to BA' : \theta'(a_1,..,a_k) = \sum_{i=1}^{k}(-1)^{k-i}a_1...a_i-1\theta(a_i)...a_{i+1}|a_k) \) where \( e_i = \sum_{j=1}^{k}a_j + (i - 1) \) satisfying \( BF - BG = d_{BA} \circ \theta + \theta \circ d_{BA} \). Now let \( f_1, f_2 \in DC(C,C') \) and consider the adjoint homomorphism \( f_1 : C \to C' \to B\Omega C' \) \( (i = 1, 2) \). If we assume that \( f_1 \simeq_{DC} f_2 \) then \( \tilde{f}_1 \simeq_{DC} \tilde{f}_2 \) and thus, by [3]-1.11, \( \Omega f_1 \simeq_{DA} \Omega f_2 \). Therefore if \( f \simeq_{DA} g \) then \( \Omega BF \simeq_{DA} \Omega BG \). The lemma follows now, from the obvious relations: \( \mu_{A'} \circ \Omega B(g \otimes g) \simeq_{DA} \mu_{A'} \circ \Omega B(f \otimes f) \simeq_{DA} \Omega BF \circ \mu_A \simeq_{DA} \Omega BG \circ \mu_A \).

A.2 Trivialized extensions. Let \( \text{TEX}_A \) be the category of trivialized extensions of \( A \) in \( DA \) in the sense of [3]-2.1: \( X \overset{\alpha}{\to} A \in \text{TEX} \) if \( \alpha \in DA(X,A) \) and if there exist \( \rho \in DM(A,X), h \in \text{Hom}(X,X) \) such that, \( \alpha \circ \rho = id_A, \rho \circ \alpha - id_X = d_X \circ h + h \circ d_X, \rho \circ \eta_A = \eta_X, \epsilon_X \circ \rho = \epsilon_A, \alpha \circ h = 0, h \circ \rho = 0, h \circ h = 0 \). A (strict) morphism of trivialized extension, \( f : (X \overset{\alpha}{\to} A, \rho, h) \to (X' \overset{\alpha'}{\to} A', \rho', h') \) is a homomorphism \( f \in DA(X,X') \) such that \( f \circ \rho = \rho' \) and \( f \circ h = h' \circ f \).

The following facts are proved in [3] or straightforward to prove.

\[ \text{F1} \quad \text{If } X \overset{f}{\to} A \in \text{TEX}_A \text{ and } Y \overset{g}{\to} X \in \text{TEX}_X \text{, then } Y \overset{fg}{\to} A \in \text{TEX}_A. \]

\[ \text{F2} \quad \text{If } X \overset{f}{\to} A \in \text{TEX}_A \text{ and } X' \overset{f'}{\to} A' \in \text{TEX}_{A'}, \text{ then } X \otimes X' \overset{f \otimes f'}{\to} A \otimes A' \in \text{TEX}_{A \otimes A'}. \]

\[ \text{F3} \quad \text{If } A \text{ is a differential graded algebra then } \Omega BA \overset{a \cdot \delta}{\to} A \text{ in an initial object in } \text{TEX}. \]

\[ \text{F4} \quad \text{If } (A, \mu_A) \text{ and } (A', \mu_{A'}) \text{ are shc algebras then there exists a natural homomorphism} \]
\(\mu_{A \otimes A'}\) such that \((A \otimes A', \mu_{A \otimes A'})\) is a shc algebra. In particular, \(A^{\otimes n}, n \geq 2\) is naturally a shc algebra.

**F5** The homomorphism \(\alpha_{\Omega B} : \Omega B \Omega B A \to \Omega B A \) admits a section \(\theta_{\Omega B} \in DA(\Omega B A, \Omega B \Omega B A)\).

**F6** If \((A, \mu_A)\) is a shc algebra so is \(\Omega B A\).

**F7** \(\Omega B \alpha_A \cong DA \alpha_{\Omega B} A\).

**F8** If \((A, \mu_A)\) is a shc algebra then \(\alpha_A : \Omega B A \to A\) is a strict shc homomorphism.

**A.3 Lemma.** Let \((A, d, \mu)\) be a shc algebra. For any \(n \geq 2\), there exists a homomorphism of differential graded algebras (called the shc iterated structural map) \(\mu^{(n)} : \Omega B (A^{\otimes n}) \to \Omega B A\) such that: \(\mu^{(2)} = \mu\) and \(\alpha_A \circ \mu^{(n)} \circ i_{A^{\otimes n}} \simeq \mu_A^{(n)}\). Moreover, \(\mu^{(n)}\) is a strict homomorphism of shc algebras.

**Proof.** Let \(A_1\) and \(A_2\) be two differential graded algebras. From F3, we deduce that there exists a natural homomorphism of differential graded algebras: \(\theta_{A_1, A_2} : \Omega B (A_1 \otimes A_2) \to \Omega B (A_1 \otimes \Omega B A_2)\) such that: \((1)\) \((\text{id}_{A_1} \otimes \alpha_{A_2}) \circ \alpha_{A_1 \otimes \Omega B A_2} \circ \theta_{A_1, A_2} = \alpha_{A_1 \otimes A_2}\). It follows, from the unicity property, that if \(A_3\) is a differential graded algebra then \((\theta_{A_1, A_2} \otimes \text{id}_{\Omega B A_3}) \circ \theta_{A_1 \otimes A_2, A_3} = (\text{id}_{\Omega B A_1} \otimes \theta_{A_2, A_3}) \circ \theta_{A_1, A_2 \otimes A_3}\). Suppose inductively that \(\mu^{(n)}\) is defined for some \(n \geq 2\). We define \(\mu^{(n+1)}\) as the composite \(\Omega B (A^{\otimes n+1}) \xrightarrow{\theta_{A_1, A_2, A_3}} \Omega B (A \otimes \Omega B (A^{\otimes n})) \xrightarrow{\Omega B (\text{id} \otimes \mu^{(n)})} \Omega B (A \otimes \Omega B (A)) \xrightarrow{\Omega B (\text{id} \otimes \alpha)} \Omega B (A \otimes A) \xrightarrow{\mu} \Omega B A\). If we assume moreover that \(\alpha_A \circ \mu^{(n)} \circ i_{A^{\otimes n}} = \mu_A^{(n)}\) then the identities \((1)\) and \(\alpha \circ \iota = \text{id}\) imply that \(\alpha_A \circ \mu^{(n+1)} \circ i_{A^{\otimes n+1}} = \mu_A^{(n+1)}\). The last statement is Proposition 4.5 in [15].

**A.4 Lemma.** Let \((W, \psi_W)\) be the standard resolution.

a) If \((A, d)\) is a differential graded algebra then \(\operatorname{Hom}(W, A) \xrightarrow{\psi_W} A \in \text{TEX}_A\).

b) If \(X \xrightarrow{\sigma} A \in \text{TEX}_A\) then \(\operatorname{Hom}(W, X) \xrightarrow{\psi(W, \sigma)} \operatorname{Hom}(W, A) \in \text{TEX}_{\operatorname{Hom}(W, A)}\).

c) If \(A\) is a shc algebra then
   i) \(\operatorname{Hom}(W, A)\) is a shc algebra,
   ii) \(ev_0\) is a shc homomorphism,
   iii) Let \(A'\) be a shc algebra. A homomorphism of differential graded algebras \(f : A' \to \operatorname{Hom}(W, A)\) is a shc homomorphism if and only if \(ev_0 \circ f\) is a shc homomorphism.

**Proof.** a) Let us denote \([g_0, g_1, ..., g_n] \in G^* n + 1\) a generator on \(W_n\) and recall the linear homotopy \(k : W \to W\) defined by \(k([g_0, ..., g_n]) = \sum_{i=0}^{n} (-1)^i [g_0, ..., g_{i-1}, 1, g_i, ..., g_n]\). Obviously, \(k \circ \partial W + \partial W \circ k = \text{id}_W - \eta_W \circ \epsilon_W\) and \(k \circ k = 0\). We define \(h : \operatorname{Hom}(W, A) \to \operatorname{Hom}(W, A)\) by \(h(f) = k \circ f\). If \(\rho : A \to \operatorname{Hom}(W, A)\) is defined by \(\rho(a)(w) = a \eta_A \epsilon_W (w), a \in A, w \in W\), a straightforward computation shows that \(\rho \circ D + D \circ \rho = \text{id}_{\operatorname{Hom}(W, A)} - \rho \circ ev_0\) and \(h \circ h = 0\). One also easily checks that \(ev_0 \circ \rho = \text{id}_{A}, ev_0 \circ h = 0, h \circ \rho = 0, \rho \circ \eta_A = \eta_{\operatorname{Hom}(W, A)}\), \(\epsilon_{\operatorname{Hom}(W, A)} \circ \rho = \epsilon_A\), where \(\eta_{\operatorname{Hom}(W, A)} = \eta_A \circ \epsilon_W\) and \(\epsilon_{\operatorname{Hom}(W, A)}(f) = \epsilon_A \circ f \circ \eta_W(1)\).

b) The proof is similar to the end of the proof of part a).

c) (i) Consider the differential linear map \(\psi_A : \operatorname{Hom}(W, A) \otimes \operatorname{Hom}(W, A) \to \operatorname{Hom}(W, A \otimes A)\) defined by \(\psi_A(f \otimes g) = (f \otimes g) \circ \psi_W\). Then, \(\operatorname{Hom}(W, m_A) \circ \psi_A = \cup_{\operatorname{Hom}(W, A)}\) and the following equalities prove that \(\psi_A\) is a homomorphism of differential graded algebras.
\[\psi_A((f_1 \otimes g_1) \cup (f_2 \otimes g_2)) = (-1)^{f_2|g_1}|\psi_A((f_1 \cup f_2) \otimes (g_1 \cup g_2))\]
\[= (-1)^{f_2|g_1}|\psi_A((f_1 \cup f_2) \otimes (g_1 \cup g_2)) \circ \psi_W\]
\[= (-1)^{f_2|g_1}|(m_A \circ (f_1 \otimes f_2) \circ \psi_W) \otimes m_A \circ ((g_1 \otimes g_2) \circ \psi_W) \circ \psi_W\]
\[= m_{A \otimes A} \circ (((f_1 \otimes g_1) \otimes (f_2 \otimes g_2)) \circ \psi_W) \circ \psi_W\]
\[= m_{A \otimes A} \circ (\psi_A(f_1 \otimes g_1) \otimes \psi_A(f_2 \otimes g_2)) \circ \psi_W\]
\[= \psi_A(f_1 \otimes g_1) \cup \psi_A(f_2 \otimes g_2).\]

We define \(\mu_{\text{Hom}(W,A)} : \Omega B (\text{Hom}(W,A)^{\otimes 2}) \to \Omega B \text{Hom}(W,A)\) as the composite \(\Omega B \text{Hom}(W,A \circ A) \circ \theta_{\text{Hom}(W,A \otimes A)} \circ \Omega B\psi\) where \(\theta_{\text{Hom}(W,A \otimes A)}\) denotes the unique homomorphism of differential graded algebras such that \(\theta_{\text{Hom}(W,A \otimes A)} \circ \text{Hom}(W,A \circ A) \circ A_{\text{Hom}(W,A \otimes A)} = A_{\text{Hom}(W,A \otimes A)}\).

Existence and unicity of \(\theta_{\text{Hom}(W,A \otimes A)}\) is a direct consequence of F3, F1, F2 and part b). It turns out that \(\text{Hom}(W,A),\mu_{\text{Hom}(W,A)}\) is a shc algebra.

\(\text{c)}\)-(ii). Consider for any \(A\) in \(\text{DA}\) the commutative diagram

\[
\begin{array}{ccc}
\Omega B A & \xleftarrow{\Omega B(\text{ev}_0)} & \Omega B \text{Hom}(W,A) & \xleftarrow{\alpha_{\text{Hom}(W,A)}} & (\Omega B)^2 \text{Hom}(W,A) \\
\alpha_A \downarrow & & \alpha_{\text{Hom}(W,A)} \downarrow & & \downarrow \Omega B \alpha_{\text{Hom}(W,A)} \\
\text{Hom}(W,A) & \xleftarrow{\alpha_{\text{Hom}(W,A)}} & \Omega B \text{Hom}(W,A) B & \xleftarrow{\alpha_A} & \Omega B \text{ev}_0 \\
\text{ev}_0 \downarrow & & \downarrow \Omega B \text{ev}_0 & & \\
A & \xleftarrow{\alpha_A} & \Omega B A.
\end{array}
\]

\(\text{From F3, we deduce the existence of } \theta'_{\Omega B A} : \Omega B A \to \Omega B \text{Hom}(W,A)\) such that \(\text{ev}_0 \circ \alpha_{\text{Hom}(W,A)} \circ \theta'_{\Omega B A} = \alpha_A\) and \(\theta'_{\Omega B A} \circ \Omega B \text{Hom}(W,A) \circ \text{ev}_0 \simeq_{\text{DA}} \alpha_{\Omega B \text{Hom}(W,A)}\). By F7, \(\Omega B \alpha_{\text{Hom}(W,A)} \simeq_{\text{DA}} \alpha_{\Omega B \text{Hom}(W,A)}\) and by F5, there is a section for \(\alpha_{\Omega B \text{Hom}(W,A)}\) so that \((\star_A)\) \(\theta'_{\Omega B A} \circ \Omega B \text{ev}_0 \simeq_{\text{DA}} \text{id}_{\Omega B \text{Hom}(W,A)}\).

Moreover, from the unicity property in F3, we know that \(\Omega B \text{ev}_0 \circ \theta'_{\Omega B A} = \text{id}_{\Omega B A}\). In the following diagram, the left hand square is commutative,

\[
\begin{array}{ccc}
\Omega B A & \xleftarrow{\Omega B(\text{ev}_0)} & \Omega B \text{Hom}(W,A) & \xleftarrow{\text{id}_{\text{Hom}(W,A)}} & \Omega B A \\
\mu_A \uparrow & & \uparrow \Omega B \text{Hom}(W,A \circ \alpha_A) & & \uparrow \mu_A \\
\Omega B (A^{\otimes 2}) & \xleftarrow{\alpha_{\Omega B (A^{\otimes 2})}} & \Omega B \text{Hom}(W,\Omega B (A^{\otimes 2})) & \xleftarrow{\alpha_{\Omega B (A^{\otimes 2})}} & \Omega B (A^{\otimes 2}).
\end{array}
\]

Since the composite of horizontal maps are the identity maps, relations \((\star_A)\) and \((\star_{A \otimes A})\) imply that the right hand square commutes up to homotopy in \(\text{DA}\). Therefore, in the following diagram the upper square commutes , up to homotopy in \(\text{DA}\).

\[
\begin{array}{ccc}
\Omega B \text{Hom}(W,A) & \xrightarrow{\Omega B(\text{ev}_0)} & \Omega B A & \xrightarrow{\theta'_{\Omega B A} \circ \mu_A} & \Omega B \text{Hom}(W,A) \\
\mu_{\text{Hom}(W,A)} \uparrow & & \uparrow \Omega B \text{Hom}(W,A \circ \alpha_A) & & \uparrow \mu_A \\
\Omega B \text{Hom}(W,A)^{\otimes 2} \downarrow & & \Omega B \text{Hom}(W,\Omega B (A^{\otimes 2})) \downarrow & & \Omega B (A^{\otimes 2}) \\
\Omega B (ev_0 \otimes ev_0) \downarrow & & \Omega B (ev_0) \downarrow & & \Omega B (A^{\otimes 2}) \\
\Omega B (A^{\otimes 2}) & = & \Omega B (A^{\otimes 2}) & = & \Omega B (A^{\otimes 2}).
\end{array}
\]

It is not difficult, if tedious, to check that the other cells in the diagram commutes also up to homotopy in \(\text{DA}\). This shows that \(\Omega B (ev_0) \circ \mu_{\text{Hom}(W,A)} \simeq_{\text{DA}} \mu_A \circ \Omega B (ev_0 \otimes ev_0)\), i.e. \(ev_0\) is a strict shc map.

\(\text{c)}\)-(iii) Let \(f : A' \to \text{Hom}(W,A)\) be a homomorphism of differential graded algebras such that \(\mu_A \circ \Omega (ev_0 \otimes ev_0) \circ \Omega B (f \otimes f) \simeq_{\text{DA}} (\Omega B (ev_0 \circ f)) \circ \mu_{A'}\). Thus in the following diagram the bigger square commutes, up to homotopy in \(\text{DA}\).
A.5 Lemma. If \((A, d_A, \mu_A, \kappa_A)\) is a \(\pi\)-shc algebra then,

a) \(\kappa\) is a strict homomorphism of shc algebras,

b) \(H^*\phi_A : H\epsilon(Hom(W, A)) \to H\epsilon(Hom(W, \epsilon A))\) preserves the natural multiplications.

Proof. a) By lemma A.3, \(\mu^{(p)}\) is a strict homomorphism of shc algebras. Since \(ev_0 \circ \kappa_A \simeq_{DA} \alpha_A \circ \mu^{(p)}\), from Lemma A.4-c)-(iii), we deduce that \(\kappa_A\) is a strict shc homomorphism.

b) We obtain the two commutative diagrams (Diagrams A and B):

\[
\begin{array}{ccccccccc}
\Omega B A' & \xrightarrow{\Omega f} & \Omega B \text{Hom}(W, A) & \xrightarrow{\Omega B ev_0} & \Omega B A \\
\mu_A' & \uparrow & \mu_{\text{Hom}(W, A)} & \uparrow & \mu_A \\
\Omega B (A' \otimes A') & \xrightarrow{\Omega (f \otimes f)} & \Omega B (\text{Hom}(W, A) \otimes \text{Hom}(W, A)) & \xrightarrow{\Omega B (ev_0 \otimes ev_0)} & \Omega B (A \otimes A)
\end{array}
\]

The right hand square commutes, up to homotopy in \(DA\), by part c-(ii) and thus so does the left hand square. This proves that \(f\) is a strict shc homomorphism. Conversely, if \(f\) is a shc map, by part c-(ii), the composite \(f \circ ev_0\) is also a strict shc homomorphism.

Now we choose linear sections of \(\epsilon\alpha_{\text{Hom}(W, A)}(\Omega^2)\) (resp. of \(\text{Hom}(W, \epsilon A)\)) on \(\epsilon\text{Hom}(W, A)\) (resp. on \(\text{Hom}(W, \epsilon A)\)). Then gluing together diagrams A,B,C, and D we deduce that the following diagram
there exists a linear homotopy.

A.6 Lemma. Let \((TV, d_V)\) be a differential graded algebra and assume that a finite group \(\pi\) acts freely on \(V\). Let \(A\) be a \(\pi\)-differential graded algebra and \(f, g \in \pi\text{-DA}(TV, A)\). If \(f \simeq_{\pi\text{-DA}} g\) then \(\mathfrak{e}f \simeq_{\pi} \mathfrak{e}g\).

Proof. The closed model category framework provides a convenient language in which we prove the lemma. For our purpose we define the cylinder \(I(TV, d)\) on \((TV, d)\):

\[
(TV, d) \xrightarrow{\partial_0} I(TV, d) \xrightarrow{p} (TV, d)
\]

with \(I(TV, d) := (T(V_0 \oplus V_1), D), \partial_0 V = V_0, \partial_1 V = V_1, pv_0 = pv_1 = v, psv = 0, D = d\) on \(V_0\) and on \(V_1, Dsv = Sdv\) where \(S\) is the unique \((\partial_0, \partial_1)\)-derivation \(S : TV \to T(V_0 \oplus V_1)\) extending the graded isomorphism \(s : V \to sV\). The free \(\pi\)-action on \(TV\) naturally extends to a free \(\pi\)-action on \(I(TV, d)\) so that \(I(TV, d)\) is a \(\pi\)-algebra and the maps \(p, \partial_0\) and \(\partial_1\) are \(\pi\)-equivariant quasi-isomorphisms. By definition \(f \simeq_C g\) (resp. \(f \simeq_{\pi-C} g\)) if there exists \(H \in \text{DA}(I(TV, d), A)\) (resp. \(H \in \pi\text{-DA}(I(TV, d), A)\)) such that \(H\partial_0 = f\) and \(H\partial_1 = g\). It is straightforward to check that \(f \simeq_C g\) if and only if \(f \simeq_{\pi\text{-DA}} g\), (resp. \(f \simeq_{\pi-C} g\) if and only if \(f \simeq_{\pi\text{-DA}} g\)). Consider the commutative diagram

\[
\begin{align*}
\mathfrak{e}TV \oplus \mathfrak{e}TV & \xrightarrow{\mathfrak{e}id \oplus \mathfrak{e}id} \mathfrak{e}TV \\
\mathfrak{e}\partial_0 \oplus \mathfrak{e}\partial_1 & \downarrow \quad \mathfrak{e}p \uparrow \\
\mathfrak{e}ITV & \xrightarrow{id} \mathfrak{e}ITV.
\end{align*}
\]

Now \(\mathfrak{e}\partial_0 \oplus \mathfrak{e}\partial_1\) is injective and its cokernel is a projective \(\pi\)-module. Hence, \(\mathfrak{e}\partial_0 \oplus \mathfrak{e}\partial_1\) is a cofibration and since \(\mathfrak{e}p\) is a weak equivalence, the complex \(\mathfrak{e}ITV\) is a cylinder object in the closed model category \(\pi\text{-DM}\). Let \(H \in \pi\text{-DA}(I(TV, d), A)\) be a homotopy between \(f\) and \(g\) then, \(\mathfrak{e}f = \mathfrak{e}H \circ \mathfrak{e}\partial_0, \mathfrak{e}g = \mathfrak{e}H \circ \mathfrak{e}\partial_1\) and thus \(\mathfrak{e}f \simeq_{\pi} \mathfrak{e}g\).

Appendix B - Equivariant acyclic model theorem.

The proof of Theorem B relies heavily on the \(\pi\)-equivariant acyclic model theorem for cochain functors. We state and prove this theorem.

B.1 Let \(R\) be a (ungraded) commutative algebra over the field \(\mathbb{F}_p\) and let \(C\) be a category with models \(\mathcal{M}\). Consider a contravariant functor \(F : C \to \text{Coch}_R, A \mapsto FA = \{F^nA\}_{j \in \mathbb{Z}}\) with values in the category of \(R\)-cochain complexes \((R\) acting on the left). See [3], [12] or [2] for the definitions of: \(F\) admits a unit, \(F\) is acyclic on the models and \(F\) is corepresentable for the models \(\mathcal{M}\). The singular cochains functor \(C^n : \text{Top} \to \text{Mod}_R\) is corepresentable on the models \(\{\Delta^k\}_{k \geq 0}\). Observing that a retract of corepresentable functor is corepresentable, we deduce that the functor \(X \mapsto N^*X\) is corepresentable on the models \(\{\Delta^k\}_{k \geq 0}\). For further use it is interesting to remark here that is \(F\) is corepresentable on a family of models and if \(W\) is an \(R\)-free graded module of finite type then so is the functor \(A \mapsto \text{Hom}(W, FA)\). Indeed, in this case each \(\text{Hom}(W_n, FA)\) is a finite sum on
copies of \( FA \) and one concludes using the fact that \( \prod \alpha F_\alpha = \prod \alpha \hat{F}_\alpha \) (see [2]-II for more details).

**B.2** We embed the category \( \text{Coch}_R \) into the category \( \text{Mod} \) of graded (without differential) \( \mathbb{F}_p \)-modules. Then, considering the two contravariant functors \( F, G : C \to \text{Coch}_R \) as functors with values in \( \text{Mod}_R \), we denote by \( \text{Hom}^i(F,G) \), \( i \in \mathbb{Z} \) the sets of natural transformations of degree \( i \). A differential \( D : \text{Hom}^i_R(F,G) \to \text{Hom}^{i+1}(F,G) \) is defined by: \( (DT)_A = d_{\text{G}(A)}T_A - (-1)^i T_A d_{\text{F}(A)}. \) For our purpose \( R = \mathbb{F}_p[\pi] = \mathbb{F}_p[\tau]/(\tau^p - 1) \) then, for any object \( A \) of \( C \), \( \text{Hom}^i(FA,GA) \) is a \( \pi \)-module by the rule \( \sigma \in \pi, T \in \text{Hom}^i_R(F,G), A, C, x \in FA \), \((\sigma T)_A(x) = \sigma T_A(\sigma^{-1} x). \) The \( \mathbb{F}_p \)-module \( \text{Hom}_\pi(FA,GA) \) of \( \pi \)-linear transformations is the fixed point set of \( \text{Hom}(FA,GA) \) under this action. The next theorem has been prove in [3].

**B.3 Theorem.** Let \( C \) be a category with models \( M_R \) and \( F, G : C \to \text{Coch}_R \) two contravariant functors with units such that for any \( A \in C \), \( F^iA = 0 = G^iA \) if \( i < 0 \), \( F \) is acyclic on the models and \( G \) is corepresentable on the models. Then \( H^0(\text{Hom}_\pi(F,G)) = R \) and \( H^0(\text{Hom}_\pi(F,G)) = 0 \) if \( i \neq 0 \).

**B.4 Corollary.** If \( F : \text{Top} \to \text{Coch}_\pi \) is any functor such that:

a) \( \eta : \mathbb{F}_p \to F \) is a unit,

b) for any model \( \Delta^n, n \geq 0 \) there exists a map \( \epsilon_n : F(\Delta^n) \to \mathbb{F}_p \) such that \( \epsilon_n \circ \eta_n = \text{id}_{\mathbb{F}_p} \) and \( \eta_n \circ \epsilon_n \simeq \text{id}_{F(\Delta^n)} \) then, \( H^i(\text{Hom}_\pi(F,\text{Hom}(W,N^*)) = \begin{cases} \mathbb{F}_p[\pi] & i = 0 \\ 0 & i > 0 \end{cases} \).

**Proof** Recall the (right) \( \pi \)-free acyclic complex \( W \) with left \( \pi \)-action defined by \( \sigma w = w \sigma^{-1}, \sigma \in \pi \) and \( w \in W \) and the canonical isomorphism \( \text{Hom}_\pi(\pi, \text{Hom}(W,B)) \simeq \text{Hom}_\pi(W \otimes A, B) \), where \( (B, d_B) \) is a (left) \( \pi \)-complex. We precise that \( \pi \) acts trivially on \( N^*X \) for any space \( X \). The functor \( G = \text{Hom}(W,N^*) : \text{Top} \to \text{Coch}_{\mathbb{F}_p[\pi]} \), admits a unit \( \eta \) defined as follows: for any space \( X \) \( \eta X(\sigma) = \sigma \circ e^0_\sigma \), where \( e^0_\sigma \in \text{Hom}(W,N^*X) \) denotes the dual of \( e_0 \in W_0 \). As remarked in B.1, \( G = \text{Hom}(W,N^*) \) is corepresentable on the models \( \Delta^n \).

The natural transformation \( \eta \) induces a natural transformation, also denoted \( \eta : R = \mathbb{F}_p[\pi] \to F \otimes R. \) Moreover, for any \( n \geq 0 \) there exists \( \epsilon_n = F(\Delta^n) \otimes R \to R \) such that \( \eta_{\Delta^n} \circ \epsilon_{\Delta^n} = \text{id}_{\Delta^n} \) and \( \epsilon_{\Delta^n} \circ \eta_{\Delta^n} \simeq \text{id}_{\Delta^n} \). Since \( \pi \) acts diagonally on \( F \otimes R \), the map \( \varphi \mapsto (\hat{\varphi} : x \otimes y \mapsto \varphi(x)y) \) is an isomorphism of chain complexes \( \text{Hom}_\pi(F,G) \simeq \text{Hom}_\pi(F \otimes R,G) \), for any functor \( G : \text{Top} \to \text{Coch}_\pi \). We apply theorem B.3 to end the proof.

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