CUBIC APPROXIMATION TO STURMIAN CONTINUED FRACTIONS

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ABSTRACT. We determine the classical exponents of approximation $w_3(\zeta)$, $w_3^*(\zeta)$, $\lambda_3(\zeta)$ and $\hat{w}_3(\zeta)$, $\hat{w}_3^*(\zeta)$, $\hat{\lambda}_3(\zeta)$ associated to real numbers $\zeta$ whose continued fraction expansions are given by a Sturmian word. We more generally provide a description of the combined graph of the parametric successive minima functions defined by Schmidt and Summerer in dimension three for such Sturmian continued fractions. This both complements similar results due to Bugeaud and Laurent concerning the two-dimensional exponents and generalizes a recent result of the author. As a side result we obtain new information on the spectra of the above exponents. Moreover, we provide some information on the exponents $\lambda_n(\zeta)$ for a Sturmian continued fraction $\zeta$ and arbitrary $n$.

Keywords: geometry of numbers, continued fractions, Sturmian words

Math Subject Classification 2010: 11H06, 11J13, 11J70

1. Exponents of approximation and Sturmian continued fractions

We start with the definition of classical exponents of Diophantine approximation. Let $\zeta$ be a real transcendental number and $n \geq 1$ be an integer. For a polynomial $P(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0$, as usual denote by $H(P) = \max_{0 \leq j \leq n} |a_j|$ its naive height. Let $w_n(\zeta)$ and $\hat{w}_n(\zeta)$ be the supremum of $w \in \mathbb{R}$ such that the system

$$\tag{1} H(P) \leq X, \quad 0 < |P(\zeta)| \leq X^{-w},$$

has a non-zero polynomial solution $P(T) \in \mathbb{Z}[T]$ of degree at most $n$ for arbitrarily large $X$, and all large values of $X$, respectively. For fixed $\zeta$, the sequences $(w_n(\zeta))_{n \geq 1}$ and $(\hat{w}_n(\zeta))_{n \geq 1}$ are obviously non-decreasing as $n$ increases. Dirichlet’s Theorem implies the lower bounds

$$\tag{2} w_n(\zeta) \geq \hat{w}_n(\zeta) \geq n.$$

Following Bugeaud and Laurent [4], we define the exponents of simultaneous approximation $\lambda_n(\zeta)$ and $\hat{\lambda}_n(\zeta)$ as the supremum of $\lambda \in \mathbb{R}$ such that the system

$$\tag{3} 1 \leq |x| \leq X, \quad \max_{1 \leq i \leq n} |\zeta^i x - y_i| \leq X^{-\lambda},$$

has a solution $(x, y_1, y_2, \ldots, y_n) \in \mathbb{Z}^{n+1}$ for arbitrarily large values of $X$, and all large $X$, respectively. In contrast to the polynomial exponents, $(\lambda_n(\zeta))_{n \geq 1}$ and $(\hat{\lambda}_n(\zeta))_{n \geq 1}$ clearly

Research supported by the Schrödinger scholarship J 3824 of the Austrian Science Fund FWF.
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form non-increasing sequences, as the number of estimates that need to be satisfied in (3) grows with \(n\). Another version of Dirichlet’s Theorem yields

\[
\lambda_n(\zeta) \geq \hat{\lambda}_n(\zeta) \geq \frac{1}{n},
\]

for all \(n \geq 1\) and transcendental real \(\zeta\). Khintchine \cite{14} established the connection

\[
\frac{w_n(\zeta)}{(n-1)w_n(\zeta) + n} \leq \lambda_n(\zeta) \leq \frac{w_n(\zeta) - n + 1}{n}.
\]

Finally we define the quantities \(w_n(\zeta)\) and \(\hat{w}_n(\zeta)\). For \(\alpha\) an algebraic number, we put \(H(\alpha) = H(P)\) where \(P\) is its (up to sign) unique minimal polynomial \(P \in \mathbb{Z}[T]\) with coprime coefficients, and call \(H(\alpha)\) the height of \(\alpha\). Then \(w_n(\zeta)\) and \(\hat{w}_n(\zeta)\) are given as the supremum of real \(w^*\) such that

\[
H(\alpha) \leq X, \quad 0 < |\alpha - \zeta| \leq H(\alpha)^{-1}X^{-w^*}
\]

has an algebraic real solution \(\alpha\) of degree at most \(n\) for arbitrarily large \(X\), and all large \(X\), respectively. For \(n = 1\), it is not hard to see that \(w_1(\zeta) = \lambda_1(\zeta) = \hat{\lambda}_1(\zeta) = \hat{w}_1(\zeta) = 1\). Clearly again the definition of the exponents directly imply that the sequences \((w_n(\zeta))_{n \geq 1}\) and \((\hat{w}_n(\zeta))_{n \geq 1}\) are non-decreasing. It is further known that

\[
w_n(\zeta) \leq w_n(\zeta) \leq w_n(\zeta) + n - 1, \quad \hat{w}_n(\zeta) \leq \hat{w}_n(\zeta) \leq \hat{w}_n(\zeta) + n - 1
\]

hold for all integers \(n \geq 1\) and any transcendental real \(\zeta\), see \cite{3} Lemma A.8]. Roughly speaking, for generic numbers \(\zeta\) we expect equalities in the respective left inequalities of (6). Otherwise, in case of strict inequality, for all polynomials with very small absolute values of evaluations at \(\zeta\) (inducing \(w_n(\zeta)\)) the derivatives at \(\zeta\) are unsymptomatically small by absolute value. On the other hand, it is known that \(w_n(\zeta) - w_n(\zeta)\) takes any value in the interval \([0, n/2 + (n - 2)/(4(n - 1)])\) for some real transcendental \(\zeta\), and for \(n \in \{2, 3\}\) any value in the allowed interval \([0, n - 1]\), see \cite{3}. We will discuss certain spectra for \(n = 3\) in Section 2.2.

We follow the definition of Sturmian words in \cite{3} Section 3 and enclose certain properties that are of importance for us. Let \((s_k)_{k \geq 1}\) be a sequence of positive integers and for fixed distinct positive integers \(a, b\) consider the words recursively defined by

\[
m_0 = 0, \quad m_1 = b^{s_1 - 1}a, \quad m_{k+1} = m_k s_{k+1} m_{k-1}, \quad k \geq 1.
\]

Let

\[
\varphi = [0; s_1, s_2, s_3, \ldots], \quad m_\varphi = \lim_{k \to \infty} m_k = b^{s_1 - 1}a \cdots .
\]

Denote by \(\zeta_\varphi = [0; m_\varphi]\) the number whose continued fraction expansion is given by concatenation of 0 and \(m_\varphi\) and let

\[
\sigma_\varphi = \liminf_{k \to \infty} [0; s_k, s_{k-1}, \ldots, s_1] = \frac{1}{\limsup_{k \to \infty} [s_k; s_{k-1}, \ldots, s_1]}.
\]

Throughout the paper let

\[
\gamma = \frac{1 + \sqrt{5}}{2} \approx 1.6180
\]
be the golden ratio. We shall also implicitly consider fixed distinct positive integers \(a, b\) whenever words \(\varphi\) are involved in accordance with the above definitions. As in [4] denote by \(S\) the set of values \(\sigma\) arising in this way, that is the set of all \(\sigma_\varphi\) that arises by all sequences \((s_k)_{k \geq 1}\) of positive integers via (9). It is not hard to see that \(S\) forms a subset of \([0, \gamma^{-1}]\). The set \(S\) has countable intersection with interval \([s, \gamma^{-1}]\), where \(s \approx 0.3867\) is the largest accumulation point of \(S\), but as shown in [4, Theorem 8.2] for any \(s' < s\) the interval \([s', s]\) has uncountable intersection with \(S\). On the other hand, as pointed out in the Remark on page 27 in [4], any interval contained in \([0, s]\) contains a subinterval which has empty intersection with \(S\). For more information on the set \(S\) and the continued fraction expansion of the value \(s\) see [4, Section 8] and also Cassaigne [8] and Fischler [11], [12]. In the special case of a constant sequence \((s_k)_{k \geq 1}\) equal to one, the corresponding Sturmian continued fractions satisfy \(\sigma_\varphi = \gamma^{-1} = \gamma - 1\) and the arising numbers \(\zeta_\varphi = \zeta_{\gamma^{-1}}\) provide examples of extremal numbers defined by Roy [16]. Extremal numbers attain simultaneously the maximum possible values \(\hat{w}_2(\zeta) = \hat{w}_2^*(\zeta) = \gamma + 1\) and \(\hat{\lambda}_2(\zeta) = \gamma - 1\) among real \(\zeta\) which are not rational or quadratic irrational. However, Roy [17, Theorem 2.4] proved the existence of extremal numbers for which no birational equivalent number has continued fraction expansion given by a Sturmian word (observe that birational transformations \(\zeta \rightarrow (a\zeta + b)/(c\zeta + d)\) for \(a, b, c, d \in \mathbb{Q}\) with \(ad - bc \neq 0\) do not affect the classical exponents).

The classical exponents of Diophantine approximation of the Sturmian continued fractions above in dimension \(n = 2\) have been established by Bugeaud and Laurent [4]. The central result of their paper is the following [4, Theorem 3.1].

**Theorem 1.1** (Bugeaud, Laurent). Let \(\zeta = \zeta_\varphi\) with corresponding \(\sigma = \sigma_\varphi\) be as above. Then

\[
\begin{align*}
\lambda_2(\zeta) &= 1, & w_2(\zeta) &= w_2^*(\zeta) = 1 + \frac{2}{\sigma} \\
\hat{\lambda}_2(\zeta) &= \frac{1 + \sigma}{2 + \sigma}, & \hat{w}_2(\zeta) &= \hat{w}_2^*(\zeta) = 2 + \sigma.
\end{align*}
\]

The goal of this paper is to determine the exponents for \(n = 3\).

### 2. The main new results

#### 2.1. Exponents is dimension three. The main result of the paper is the following.

**Theorem 2.1.** Let \(\zeta = \zeta_\varphi\) be a Sturmian continued fraction with corresponding \(\sigma = \sigma_\varphi\) as above. Then we have

\[
\begin{align*}
\lambda_3(\zeta) &= \frac{1}{1 + 2\sigma}, & w_3(\zeta) &= w_3^*(\zeta) = 1 + \frac{2}{\sigma}, & \lambda_3(\zeta) &= \frac{1}{1 + 2\sigma}, \\
\hat{\lambda}_3(\zeta) &= \frac{1}{3}, & \hat{w}_3(\zeta) &= 3, & \hat{\lambda}_3(\zeta) &= \frac{1}{3}, \\
\hat{w}_3^*(\zeta) &= 2 + \sigma.
\end{align*}
\]
For $\sigma = \gamma^{-1}$ we obtain the results for extremal numbers of [19, Theorem 2.1], apart from the new claims for $w_3^*$ and $\hat{w}_3^*$. The proof of Theorem 2.1 will lead to the following generalization of [19, Theorem 2.2].

**Theorem 2.2.** Let $\zeta = \zeta_\varphi$ with corresponding $\sigma = \sigma_\varphi$ be as above and $\epsilon > 0$. Then

\[ |Q(\zeta)| \leq H(Q)^{-3-\epsilon} \]

has only finitely many irreducible solutions $Q \in \mathbb{Z}[T]$ of degree precisely three. In particular

\[ |\zeta - \alpha| \leq H(\alpha)^{-4-\epsilon} \]

has only finitely many algebraic solutions $\alpha$ of degree precisely three. On the other hand, the inequalities

\[ |Q(\zeta)| \leq H(Q)^{-3+\epsilon}, \quad |\zeta - \alpha| \leq H(\alpha)^{-4+\epsilon} \]

have solutions in irreducible polynomials $P$ and algebraic numbers $\alpha$ of degree precisely three respectively, for arbitrarily large $H(Q)$ and $H(\alpha)$. Moreover, there exist arbitrarily large values of $X$ such that

\[ H(Q) \leq X, \quad |Q(\zeta)| \leq X^{-2\sigma-1-\epsilon} \]

has no irreducible solution $Q \in \mathbb{Z}[T]$ of degree precisely three. Consequently, for arbitrarily large values of $X$ the system

\[ H(\alpha) \leq X, \quad |\zeta - \alpha| \leq H(\alpha)^{-1}X^{-2\sigma-1-\epsilon} \]

admits no solution in $\alpha$ an algebraic number of degree precisely three. Conversely, for any large $X$ the estimates

\[ |Q(\zeta)| \leq X^{-2\sigma-1+\epsilon}, \quad |\zeta - \alpha| \leq H(\alpha)^{-1}X^{-2\sigma-1+\epsilon} \]

have solutions in irreducible cubic polynomials $Q \in \mathbb{Z}[T]$ and cubic algebraic numbers $\alpha$ of respective height at most $X$.

The estimate (15) was only conjectured for the special case of extremal numbers in [19], however this estimate as well as (15) are essentially consequences of the method and results of Davenport and Schmidt [9], with some minor modifications explained in Bugeaud [3]. The proof should in fact allow for replacing the $\epsilon$ in the exponents by suitable multiplicative constants throughout. We do not carry this out.

Finally we discuss the exponents $\lambda_n(\zeta_\varphi)$ for $n > 3$. In contrast to Theorem 2.1 and Theorem 2.2, which are generalizations of [19, Theorem 2.1 and Theorem 2.2] with similar proofs, no variant of the next theorem was mentioned in [19]. It is essentially derived from specializing recent results from [22] and [25].

**Theorem 2.3.** Let $\zeta = \zeta_\varphi$ with corresponding $\sigma = \sigma_\varphi$ and $n$ be a positive integer. Then

\[ \max\left\{ \frac{1}{n}, \frac{2 + (3 - n)\sigma}{2 + (n + 1)\sigma} \right\} \leq \lambda_n(\zeta) \leq \frac{1}{1 + 2\sigma}, \quad n \geq 3. \]

In particular, if and only if $\sigma = 0$ we have

\[ \lambda_n(\zeta) = 1, \quad n \geq 1. \]
On the other hand, if \( \sigma > 0 \), we have the refined upper bounds

\[
\lambda_n(\zeta) \leq \min \left\{ \frac{1}{(n-1)\sigma}, \frac{1}{1 + 2\sigma} \right\}, \quad 3 \leq n \leq 2 + \left\lceil \frac{2}{\sigma} \right\rceil,
\]

\[
\lambda_n(\zeta) \leq \frac{1}{2 + \sigma} < \frac{1}{2}, \quad n \geq 2 + \left\lceil \frac{2}{\sigma} \right\rceil,
\]

and

\[
\lim_{n \to \infty} \lambda_n(\zeta) = 0.
\]

For \( n = 3 \) the lower bound in (19) becomes the precise value from Theorem 2.1, but notice there is no equality for \( n = 2 \). Also there is no reason to believe in equality for any Sturmian continued fraction and any \( n \geq 4 \). A better bound for \( n = 4 \) and \( \zeta \) an extremal number was established in [19, Theorem 2.3]. In the deduction the identities \( w_2(\zeta)/\hat{w}_2(\zeta) = \gamma \) and \( \gamma^2 - \gamma - 1 = 0 \) played a crucial role. For general Sturmian continued fractions \( \zeta_\varphi \) this kind of argument seems no longer to work. In particular, the proof of [19, Theorem 2.3] suggested equality in the right transference inequality of (5) for \( \zeta \) an extremal number and \( n = 4 \). There is no reason for this identity to extend to an arbitrary Sturmian continued fraction \( \zeta_\varphi \). The lower bound in (19) is of interest only for \( n \) not too large, unless \( \sigma = 0 \). Indeed, in case of \( \sigma > 0 \), for \( n > 1 + 2/\sigma \) we only get \( \lambda_n(\zeta) \geq 1/n \) which is the trivial bound from (4). Notice also that numbers with the property (20) were constructed in [5, Theorem 4] by a similar word concatenation argument. For \( \zeta \) an extremal number, the estimates (21) and (22) lead to the new information

\[
\lambda_5(\zeta) \leq \frac{\sqrt{5} + 1}{8} \approx 0.4045, \quad \lambda_6(\zeta) \leq \frac{3 - \sqrt{5}}{2} \approx 0.3820.
\]

The bounds are smaller than \( \lambda_3(\zeta) = 1/\sqrt{5} \approx 0.4472 \), however considerably larger than the expected value \((\sqrt{5} - 1)/4 \approx 0.3090\) for \( \lambda_4(\zeta) \), see [19, Theorem 2.3] and the enclosed comments. Concerning (23), we refer to [25, Section 4.3] for a semi-effective minimum rate at which the limit 0 is approached, and analogous results for more general classes of numbers.

2.2. Spectra. We display immediate consequences of Theorem 2.1 concerning the spectra of certain approximation constants. Following [4] we define the spectrum of an exponent of approximation as the set of real values obtained as \( \zeta \) runs through the real transcendental numbers. For convenience we will write \( \text{spec}(\cdot) \), for example \( \text{spec}(\lambda_3) \). Recall the properties of the set \( S \) from Section 1. For a definition of Hausdorff dimension see [10].

**Corollary 2.4.** For any \( \epsilon > 0 \), the sets

\[
\text{spec}(\lambda_3) \cap [1 - \epsilon, 1], \quad \text{spec}(\hat{w}_3^*) \cap [2, 2 + \epsilon], \quad \text{spec}(\hat{w}_3 - \hat{w}_3^*) \cap [1 - \epsilon, 1]
\]

take Hausdorff dimension 1. Moreover the points \( 1/(2s + 1), 2 + s \) and \( 1 - s \) are accumulation points of \( \text{spec}(\lambda_3), \text{spec}(\hat{w}_3^*) \) and \( \text{spec}(\hat{w}_3 - \hat{w}_3^*) \) respectively.

**Proof.** Theorem 2.1 shows that any \( s' \in S \) gives rise to a number \( 1/(2s' + 1) \) in \( \text{spec}(\lambda_3) \), a number \( 2 + s' \) in \( \text{spec}(\hat{w}_3^*) \) and a number \( 1 - s' \) in \( \text{spec}(\hat{w}_3 - \hat{w}_3^*) \). On the other hand,
as essentially shown in [4], the intersection of the set \( S \) with \([0, \epsilon)\) has full Hausdorff dimension. Indeed, the reciprocal of any number with continued fraction expansion of the form \([K; a_1, a_2, \ldots]\) with all \(a_j < K\), belongs to \( S \) as remarked in [4, page 27]. These reciprocal numbers obviously tend to 0 as \( K \to \infty \). On the other hand, as remarked in [4, page 27], Jarník showed that the dimension of the implied sets tends to 1 as \( K \to \infty \). Thus the metrical results follow from the invariance of Hausdorff dimensions under bi-Lipschitz continuous maps. The latter numbers are accumulation points in the respective spectra since \( s \) is an accumulation points of \( S \).

We expect that \( \text{spec}(\lambda_3) \) equals the entire interval \([1/3, \infty)\). This is [5, Problem 1] for \( n = 3 \). However, only the interval \([1, \infty)\) is known to be included and corresponding \( \zeta \) can be constructed, as first noticed by Bugeaud [5, Theorem 2]. In this interval the numbers \( \zeta \) can even be chosen in the Cantor middle third set, see [20, Theorem 4.4] and the more general [21, Theorem 2.9]. Moreover the Hausdorff dimensions of \( \{ \zeta \in \mathbb{R} : \lambda_3(\zeta) = t \} \) for a prescribed value \( t \in [1, \infty) \) are known [20, Corollary 1.8]. On the other hand, very little is known about the remaining interval \([1/3, 1)\). In fact the first explicit constructions of numbers with prescribed value of \( \lambda_3(\zeta) < 1 \) for any \( n \geq 2 \) seem to be extremal numbers \( \zeta \) in [15], and \( n = 3 \). The spectrum of \( \hat{w}_3^* \) is also supposed to contain the interval \([1, 3]\) and \( \text{spec}(\hat{w}_3^* - \hat{w}_3^*) \) to equal the entire interval \([0, 2]\) in which it is contained according to the restriction [9]. However, none of these conjectures has been proved. However, as pointed out to me by Y. Bugeaud, it can be deduced from combination of [5, Corollary 1] and [7, Theorem 2.2] with \( m = 1 \) that \( \text{spec}(\hat{w}_n^*) \) contains \([1, 2 - 1/n]\) and \( \text{spec}(\hat{w}_n - \hat{w}_n^*) \) contains \([n - 2 + 1/n, n - 1]\), for all \( n \geq 2 \). The method shows that a sufficient criterion for the conjectures above is that numbers \( \zeta \) with the property

\[
\hat{w}_1(\zeta) = \hat{w}_2(\zeta) = \cdots = \hat{w}_n(\zeta) = w
\]

exist for all \( w \geq n \). This is only known for \( w \geq 2n - 1 \). Finally we want to investigate the consequences of Theorem 2.1 to \( \text{spec}(\hat{w}_3) \). A metric result of Bernik [2] implies \( \text{spec}(\hat{w}_3) = [3, \infty] \). However, the numbers \( \zeta_\sigma \) with \( \sigma \) sufficiently close to \( \gamma_1 = \gamma - 1 \) provide the first explicit constructions of transcendental real numbers with \( \hat{w}_3(\zeta) \in (3, 5) \). On the other hand the subset of \( \text{spec}(\hat{w}_3) \) induced by some \( \zeta_\sigma \) intersected with \( (3, 5) \) consists only of \( \{2 + \sqrt{5}, 2 + 2\sqrt{2}\} \) and is induced by only countably many \( \zeta_\sigma \), as a closer look at the largest few elements of \( S \) shows, see again [4, page 27]. For the remaining interval \([5, \infty] \) the construction of Bugeaud [5, Corollary 1] indicated above applies and yields uncountably many \( \zeta \) with prescribed value \( \hat{w}_3(\zeta) \).

We want to point out that Roy [18] proved that \( \text{spec}(\hat{w}_2) \) is dense in \([2, \gamma + 1]\), which by Jarnik’s identity [13] also shows that \( \text{spec}(\hat{w}_3) \) is dense in \([1/2, \gamma - 1]\). He constructed suitable numbers \( \zeta \) in a class he called of Fibonacci type [18, Section 7], which provide a more general concept than Sturmian continued fractions. A generalization of Theorem 2.1 to Fibonacci type numbers might lead to density results for \( \text{spec}(\lambda_3), \text{spec}(\hat{w}_3) \) and \( \text{spec}(\hat{w}_3 - \hat{w}_3^*) \) in the respective intervals \([1/\sqrt{5}, 1], [2, 1 + \gamma]\) and \([2 - \gamma, 1]\). We finally remark that \( \text{spec}(\hat{w}_2) \) has countable intersection with \([c, \gamma + 1]\) for some \( c < \gamma + 1 \), in particular it is a proper subset of \([2, \gamma + 1]\).
3. Determination of the uniform exponents

3.1. Proof of (11). The main ingredient for the proof of Theorem 2.1 will be the following Theorem 3.1 which is stated in a more general form then needed in our applications. Roughly speaking it claims that if \( w_{n-1}(\zeta) = n - 1 \) and moreover there are infinitely many pairs of polynomials \( P, Q \in \mathbb{Z}[T] \) of degree \( n \) whose heights do not differ too much and both \( |P(\zeta)| \) and \( |Q(\zeta)| \) are very small, then \( \hat{w}_{2n-1}(\zeta) = 2n - 1 \) and \( \hat{\lambda}_{2n-1}(\zeta) = 1/(2n - 1) \). This looks somehow exotic at first sight, but if \( n = 2 \) and \( \zeta = \zeta_{\varphi} \) the assumptions will be satisfied as can be inferred from results in [4]. In particular the claim (11) will follow as a direct consequence of the results in [4] combined with Theorem 3.1 below, as we will show in this section. This is also an intermediate step for the proof of (10), however the deduction requires additional technical arguments involving parametric geometry of numbers, very similar to the proof of [19, Theorem 2.1].

**Theorem 3.1.** Let \( n \geq 2 \) be an integer and \( \zeta \) be a real number. Assume \( w_{n-1}(\zeta) = n - 1 \). Let \( \epsilon > 0 \) arbitrarily small. Assume there exist (not necessarily disjoint) sequences of polynomials \((P_i)_{i \geq 1}\) and \((Q_i)_{i \geq 1}\) in \( \mathbb{Z}[T] \) of degree at most \( n \) with the following properties:

- \( \{P_i, Q_i\} \) is linearly independent for \( i \geq 1 \)
- \( H(P_1) < H(P_2) < \cdots \) and \( H(Q_1) < H(Q_2) < \cdots \)
- \( H(P_i) < H(Q_i) \) for \( i \geq 1 \)
- For large \( i \geq i_0(\epsilon) \) we have
  \[
  |P_i(\zeta)| \leq H(P_i)^{-2n+1+\epsilon}, \quad |Q_i(\zeta)| \leq H(Q_i)^{-2n+1+\epsilon},
  \]
- if \( \mu_i > 0 \) and \( \nu_i \in (0, 1) \) are defined by
  \[
  |P_i(\zeta)| = H(P_i)^{-\mu_i}, \quad H(Q_i)^{\nu_i} = H(P_i)
  \]
  then for large \( i \geq i_1(\epsilon) \) we have
  \[
  \frac{n}{\nu_i(\mu_i - n + 1)} < 1 + \epsilon.
  \]

Then for large \( i \geq i_2(\epsilon) \) and the parameter \( X_i = H(Q_i) \), the system

\[
H(P) \ll X_i, \quad 0 < |P(\zeta)| \leq X_i^{-2n+1-\epsilon}
\]

has no non-trivial solution \( P \in \mathbb{Z}[T] \) of degree at most \( 2n - 1 \), for a suitable implied constant. In particular

\[
\hat{w}_{2n-1}(\zeta) = 2n - 1, \quad \hat{\lambda}_{2n-1}(\zeta) = \frac{1}{2n - 1}.
\]

**Remark 3.2.** If we weaken the assumptions by only prescribing some upper bound less than \( 2n - 1 \) for \( w_{n-1}(\zeta) \) and/or replacing the right hand side in (25) by a larger number, we still obtain upper bounds for \( \hat{w}_{2n-1}(\zeta) \) and \( \hat{\lambda}_{2n-1}(\zeta) \). Moreover, upon the assumptions of the theorem the condition (25) in fact implies that the left hand side of (25) must tend to 1 as \( i \to \infty \). We could equivalently impose \( |nu_i^{-1}(\mu_i - n + 1)^{-1} - 1| < \epsilon \) for \( i \geq i_0(\epsilon) \) instead of (25). Furthermore the left condition in (24) is in fact redundant as it can be derived from the right condition in (24) combined with (25). We do not carry these issues out.
We will now carry out how to apply Theorem 3.1 to Sturmian words \( \zeta = \zeta_\varphi \) and \( n = 2 \) to deduce (11). The method generalizes the proof of [19, (12) in Theorem 2.1] and leads to a deeper insight why it is true. We have to check that the assumptions of Theorem 3.1 are satisfied in this context. This follows from certain results due to Bugeaud and Laurent [4]. First we introduce some additional notation. Let \( \zeta_\varphi \) be defined as in (8) via a sequence \((s_k)_{k \geq 1}\) and let \((m_k)_{k \geq 1}\) be the corresponding words as in (7). Then for \( k \geq 1 \) let
\[
\eta_k := [s_{k+1}; s_k, \ldots, s_1], \quad \alpha_k := [0; m_k, m_k, \ldots].
\]
Every \( \alpha_k \) is a quadratic irrational number. Let \( W_k \) denote its minimal polynomial over \( \mathbb{Z}[T] \) with coprime coefficients and let \( H(\alpha_k) = H(W_k) \). Concretely the following was shown in the proofs of [4, Corollary 6.1 and Corollary 6.2]. As usual \( a \asymp b \) means both \( a \ll b \) and \( b \ll a \) are satisfied everywhere it occurs in the sequel.

**Lemma 3.3** (Bugeaud, Laurent). Let \( \zeta_\varphi \) be as above. Then
\[
|\zeta_\varphi - \alpha_k| \asymp H(\alpha_k)^{-2-2\eta_k},
\]
and
\[
|W_k(\zeta_\varphi)| \asymp H(W_k)^{-1-2\eta_k}.
\]
Moreover
\[
|\zeta_\varphi - \alpha_k| \asymp H(\alpha_k)^{-1} H(\alpha_{k+1})^{-2-\frac{1}{\eta_k}},
\]
and
\[
|W_k(\zeta_\varphi)| \asymp H(W_{k+1})^{-2-\frac{1}{\eta_k}}.
\]

Since \( \eta_k \) corresponds to \( 1/\sigma \) by (9), Lemma 3.3 readily yields the lower bounds for the exponents in Theorem 1.1. For the upper bounds Liouville inequality was applied to exclude better quadratic approximations, see the proofs of [4] Corollary 6.1 and Corollary 6.2. It was shown that there are no algebraic \( \alpha \) of degree at most two and height less than \( H(\alpha_{k+1}) \) closer to \( \zeta \) than \( \alpha_k \). Similarly \( |P(\zeta_\varphi)| \) is minimized among all linear or quadratic \( P \in \mathbb{Z}[T] \) of height less than \( H(W_{k+1}) \) for \( P = W_k \). We will apply this in the proof of Theorem 2.2. In fact the proof of Theorem 2.1 will confirm it, though. We show that the conditions of Theorem 3.1 are satisfied for \( \zeta_\varphi \) with the polynomials given by
\[
P_i = W_i, \quad Q_i = W_{i+1},
\]
with \( W_i \) as above. Observe that these specializations lead to
\[
\mu_i = 1 + 2\eta_k + o(1), \quad \nu_i = \frac{2 + \frac{1}{\eta_k} + o(1)}{1 + 2\eta_k} = \frac{1}{\eta_k} + o(1)
\]
by (30) and (31). It follows from \( \eta_k > 1 \) that (24) is satisfied and (25) can be readily checked for \( n = 2 \) with (32) as well. The remaining conditions are obviously satisfied. Thus Theorem 3.1 implies the following.

**Corollary 3.4.** Let \( \zeta = \zeta_\varphi \) be as above. Then for every \( \epsilon > 0 \) and large \( i \geq i_0(\epsilon) \) with \( X_i = H(W_i) \) the inequality (26) for \( n = 2 \) has no solution \( P \in \mathbb{Z}[T] \) of degree at most 3. In particular we deduce (11) a consequence of Theorem 3.1.
For the proof \((10)\) in Section 5 we will recall the general assertion of Corollary \(3.4\) in particular that \(X_i\) for the conclusion \((11)\) can be chosen of the special form \(H(W_i)\). In fact we will need the following fact evolving from the proof of Theorem 3.1 in combination with the above observations: for \(\epsilon > 0\) and large \(i \geq i_0(\epsilon)\) there exist \(R_1(T), R_2(T)\) linear integer polynomials such that the derived polynomials
\[
\{G_{1,i}, G_{2,i}, G_{3,i}, G_{4,i}\} = \{R_1 W_i, R_2 W_i, W_{i+1}, TW_{i+1}\}
\]
span the four-dimensional space of polynomials of degree at most three, and with \(X_i := H(W_{i+1})\) satisfy
\[
\max_{1 \leq j \leq 4} H(G_{j,i}) \ll X_i, \quad \max_{1 \leq j \leq 4} |G_{j,i}(\zeta)| \leq X_i^{-3+\epsilon}.
\]
The proof of \((10)\) requires some more preparation from Section 4 below.

3.2. An identity for Sturmian continued fractions. At this point we want to enclose some remarks on the proof in the last section. A special case of \([7, \text{Theorem 2.2}]\) shows that for any real number \(\zeta\) which satisfies the condition \(w_n(\zeta) > w_{n-1}(\zeta)\), we have
\[
\hat{\omega}_n(\zeta) \leq \frac{n w_n(\zeta)}{w_n(\zeta) - n + 1}.
\]
We do not know whether the assumption \(w_n(\zeta) > w_{n-1}(\zeta)\) is really necessary for the conclusion or not, the analogue of \((33)\) for the exponents \(w_n^*\), \(\hat{\omega}_n^*\) holds unconditionally \([7, \text{Theorem 2.4}]\). See also \([24]\) for an unconditioned weaker estimate in the flavor of \((33)\).

Inserting the values of Theorem 1.1, we see that for \(n = 2\) the Sturmian continued fractions \(\zeta_\varphi\) provide equality in \((33)\). Our proof of \((11)\) relied on a parametric version of this identity, concretely \((25)\), that arose from Lemma 3.3. Indeed, if we identify \(\mu_i\) with \(w_2(\zeta_\varphi)\) and \(v_i\) with \(\hat{\omega}_2(\zeta_\varphi)/w_2(\zeta_\varphi)\), then \((25)\) becomes the reverse inequality to \((33)\) as \(\epsilon \to 0\). We see that in fact the quotients \(n/(v_i(\mu_i - n + 1))\) in \((25)\) converge to 1 as \(i \to \infty\) for \(\zeta_\varphi\).

We briefly discuss equality in \((33)\) for \(n > 2\). By \((2)\), equality obviously occurs for any \(U_n\)-number in Mahler’s classification of real numbers, that is a transcendental real number which satisfies \(w_n(\zeta) = \infty\) and \(n\) is the smallest such index. The existence of \(U_n\)-numbers of any prescribed degree \(n \geq 1\) was constructively proved by LeVeque \([15]\).

Most likely equality will hold for all (or at least some) \(U_n\)-numbers with \(m < n\) as well. For \(n = 1\) this is true as a consequence of \([20, \text{Theorem 1.12}]\). However, for \(n \geq 3\) and \(w_n(\zeta) < \infty\) (or \(\hat{\omega}_n(\zeta) > n\)), it is completely open whether equality in \((33)\) can occur. In fact it is not even known if there exist transcendental real \(\zeta\) with \(\hat{\omega}_n(\zeta) > n\) for some \(n \geq 3\). A necessary condition for equality in \((33)\) is \(w_n(\zeta) \geq h_n\), where \(h_n > 2n - 1\) is given by
\[
h_n := \frac{1}{2} \left( \frac{1 + 2n\sqrt{n^2 - 2n + \frac{5}{4}}}{n - 1} + 2n - 1 \right),
\]
and even a slightly larger bound can be given when \(n = 3\). This follows from the proof of \([7, \text{Theorem 2.1}]\) based on the fact that otherwise smaller upper bounds for \(\hat{\omega}_n(\zeta)\) are obtained by Schmidt and Summerer \([27]\). Assuming a conjecture of Schmidt and
Summerer [28] the bound $h_n$ can be replaced by a larger one in case of $n \geq 4$ as well. See [23, Section 5.1], in particular [23, (33)], on how to derive concrete numerical bounds smaller than $h_n$.

4. Parametric geometry of numbers

We recall the concepts of the parametric geometry of numbers for the proof of (10). We follow Schmidt and Summerer [26], [27], very similarly as in [19]. Let $\zeta \in \mathbb{R}$ and an integer $n \geq 1$ be given, and let $Q > 1$ be a parameter. For $1 \leq j \leq n + 1$, define $\psi_{n,j}(Q)$ as the minimum of $\eta \in \mathbb{R}$ such that
\[
|x| \leq Q^{1+\eta}, \quad \max_{1 \leq j \leq n} |\zeta^j x - y_j| \leq Q^{-\frac{1}{n}+\eta}
\]
has (at least) $j$ linearly independent solutions in integer vectors $(x, y_1, \ldots, y_n)$. It is easy to verify that they are non-decreasing in the second parameter and the possible range of every $\psi_{n,j}$ for $Q \in (1, \infty)$ is given by
\[
-1 \leq \psi_{n,1}(Q) \leq \psi_{n,2}(Q) \leq \cdots \leq \psi_{n,n+1}(Q) \leq \frac{1}{n}.
\]
Define the derived quantities
\[
\psi_{n,j} = \liminf_{Q \to \infty} \psi_{n,j}(Q), \quad \overline{\psi}_{n,j} = \limsup_{Q \to \infty} \psi_{n,j}(Q).
\]
These quantities obviously all lie in the interval $[-1/n, 1]$. In this setting, Dirichlet’s Theorem becomes $\psi_{n,1}(Q) < 0$ for all $Q > 1$, thus $\overline{\psi}_{n,1} \leq 0$. Now we define the dual functions $\overline{\psi}_{n,j}^*(Q)$ from [26]. For $1 \leq j \leq n + 1$ and a parameter $Q > 1$, let $\overline{\psi}_{n,j}^*(Q)$ be the minimum of $\eta \in \mathbb{R}$ such that $H(P) \leq Q^{\frac{1}{n}+\eta}$, $|P(\zeta)| \leq Q^{-1+\eta}$ has (at least) $j$ linearly independent solutions in polynomials $P \in \mathbb{Z}[T]$ of degree at most $n$. The range for these functions is given by
\[
\psi_{n,j}^* = \liminf_{Q \to \infty} \psi_{n,j}^*(Q), \quad \overline{\psi}_{n,j}^* = \limsup_{Q \to \infty} \psi_{n,j}^*(Q),
\]
which take values in $[-1/n, 1]$. Moreover $\overline{\psi}_{n,1} \leq 0$ follows again from Dirichlet’s Theorem. For transcendental real $\zeta$, Schmidt and Summerer [27, (1.11)] established the inequalities
\[
j\psi_{n,j}^* + (n + 1 - j)\overline{\psi}_{n,n+1} \geq 0, \quad j\overline{\psi}_{n,j}^* + (n + 1 - j)\overline{\psi}_{n,n+1} \geq 0,
\]
for $1 \leq j \leq n + 1$. The dual inequalities
\[
j\psi_{n,j}^* + (n + 1 - j)\overline{\psi}_{n,n+1} \geq 0, \quad j\overline{\psi}_{n,j}^* + (n + 1 - j)\overline{\psi}_{n,n+1} \geq 0,
\]
hold as well for the same reason, as pointed out in [19]. Mahler’s duality implies
\[
\psi_{n,j} = -\overline{\psi}_{n,n+2-j}^*, \quad \overline{\psi}_{n,j} = -\psi_{n,n+2-j}^*, \quad 1 \leq j \leq n + 1.
\]
For \( q = \log Q > 0 \) we further define the derived functions
\[
L_{n,j}(q) = q\psi_{n,j}(Q), \quad L^*_{n,j}(q) = q\psi^*_{n,j}(Q).
\]
These functions are piecewise linear with slopes among \( \{-1, 1/n\} \) and \( \{-1/n, 1\} \) respectively. Further for any \( \underline{x} = (x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1} \) define the function
\[
L_{\underline{x}}(q) = \max \left\{ \log |x| - q, \max_{1 \leq j \leq n} \log |\zeta^j x - y_j| + \frac{q}{n} \right\}.
\]
Then by construction \( L_{n,j}(q) = L_{\underline{x}}(q) \) for every \( q > 0 \) and some uniquely defined \( \underline{x} = \underline{x}(j, q) \). In case of \( j = 1 \), the vector \( \underline{x} \) is chosen such that the expression in (38) is minimized among all non-zero vectors. Similarly, any \( L^*_{n,j} \) coincides at any point \( q \) with some
\[
L^*_P(q) = \max \left\{ \log H(P) - \frac{q}{n}, \log |P(\zeta)| + q \right\}
\]
for some \( P \in \mathbb{Z}[T] \) of degree at most \( n \). Minkowski’s second lattice point Theorem translates into
\[
\left| \sum_{j=1}^{n+1} L_{n,j}(q) \right| \ll 1, \quad \left| \sum_{j=1}^{n+1} L^*_{n,j}(q) \right| \ll 1.
\]
Hence in any interval \( I = (q_1, q_2) \), the sums of the differences \( L_{n,j}(q_2) - L_{n,j}(q_1) \) and \( L^*_{n,j}(q_2) - L^*_{n,j}(q_1) \) over \( 1 \leq j \leq n+1 \), are bounded in absolute value.

We relate \( \psi_{n,j}(Q), \psi^*_{n,j}(Q) \) and the quantities derived from them to another kind of successive minima approximation exponents, which generalize the exponents \( w_{n,j}, \hat{w}_{n,j}, \lambda_{n,j}, \hat{\lambda}_{n,j} \) from Section 1. Fix \( n \geq 1 \) and let \( 1 \leq j \leq n + 1 \). Let \( w_{n,j}(\zeta), \hat{w}_{n,j}(\zeta) \) be the supremum of \( w \in \mathbb{R} \) such that (1) has (at least) \( j \) linearly independent polynomial solutions \( P(T) \in \mathbb{Z}[T] \) of degree at most \( n \) for arbitrarily large \( X \), and all large \( X \), respectively. Similarly, define \( \lambda_{n,j}(\zeta) \) and \( \hat{\lambda}_{n,j}(\zeta) \) as the supremum of \( \lambda \in \mathbb{R} \) such that (3) has (at least) \( j \) linearly independent solutions \( (x, y_1, y_2, \ldots, y_n) \in \mathbb{Z}^{n+1} \) for arbitrarily large \( X \), and all large \( X \), respectively. For \( j = 1 \) just the classic exponents are obtained. The indicated relation between \( \psi_{n,j}(Q), \psi^*_{n,j}(Q) \) and the above successive minima exponents essentially established in [20] is given by
\[
(1 + \lambda_{n,j}(\zeta))(1 + \psi_{n,j}) = (1 + \hat{\lambda}_{n,j}(\zeta))(1 + \psi^*_{n,j}) = \frac{n+1}{n}, \quad 1 \leq j \leq n + 1,
\]
and
\[
(1 + w_{n,j}(\zeta))\left(\frac{1}{n} + \psi^*_{n,j}\right) = (1 + \hat{w}_{n,j}(\zeta))\left(\frac{1}{n} + \psi^*_{n,j}\right) = \frac{n+1}{n}, \quad 1 \leq j \leq n + 1.
\]
As already quoted in [19] from (36), (41) and (42) one may deduce
\[
\lambda_{n,j}(\zeta) = \frac{1}{\hat{w}_{n,n+2-j}(\zeta)}, \quad \hat{\lambda}_{n,j}(\zeta) = \frac{1}{w_{n,n+2-j}(\zeta)}, \quad 1 \leq j \leq n + 1.
\]
We recall [19] Lemma 3.3] in a slightly modified form.
Lemma 4.1. Let ζ be a real transcendental number. Let \( P, Q \in \mathbb{Z}[T] \) be of large heights, \( R = PQ \) and suppose \( R \) has degree at most three. Define the functions \( L_P^\ast, L_R^\ast \) as in (39) with respect to \( n = 3 \). Assume that \((q_1, L_P^\ast(q_1))\) is the local minimum of \( L_P^\ast \) and \((q_2, L_R^\ast(q_2))\) the local minimum of \( L_R^\ast \). Further assume
\[
|Q(\zeta)| \asymp H(Q)^{-1}.
\]
Then, we have
\[
\frac{L_R^\ast(q_2) - L_P^\ast(q_1)}{q_2 - q_1} = \frac{1}{3} + o(1), \quad H(Q) \to \infty.
\]

We omit the proof since it is very similar to the one of [19, Lemma 3.3]. Our error term is even smaller as a consequence of the stronger condition (44) instead of \( |Q(\zeta)| \leq H(Q)^{-1+\delta} \) for small \( \delta \). The stronger condition will be satisfied in our applications to the numbers \( \zeta_\varphi \) for the linear best approximation polynomials \( Q \), since \( \zeta_\varphi \) are badly approximable with respect to one-dimensional approximation. This property is unknown for extremal numbers, only the weaker claim that \( |\zeta - y/x| \gg x^{-2} \max\{2, \log x\}^{-c} \) for some \( c = c(\zeta) \geq 0 \) and all rational numbers \( y/x \) was shown by Roy [16, Theorem 1.3].

5. Proofs of Theorems 3.1, 2.1 and 2.2

For convenience we first provide an easy proposition, which in fact is an immediate consequence of Minkowski’s second lattice point theorem.

Proposition 5.1. Let \( k \geq 1 \) be an integer and ζ be a real number. Then the following assertions are equivalent.

- \( w_{k,k+1}(\zeta) \geq k \)
- \( w_{k,k+1}(\zeta) = k \)
- \( \hat{w}_k(\zeta) \leq k \)
- \( \hat{\lambda}_k(\zeta) = k \)
- \( \hat{\lambda}_k(\zeta) = \frac{1}{k} \)

Proof. It is obvious by (2) that the last two claims are equivalent. From the right relation in (13) we see that \( w_{k,k+1}(\zeta) = \hat{\lambda}_k(\zeta)^{-1} \), and together with (4) the first two and the last claims are all equivalent. To finish the proof, it suffices to show that the first and third relation are equivalent. This is essentially an application of Minkowski’s second Theorem. If \( w_{k,k+1}(\zeta) \geq k \) holds, then (12) implies \( \psi_{k,k+1}^s \leq 0 \). Hence for certain arbitrarily large \( Q \) we have \( \psi_{k,k+1}^s(Q) \leq \epsilon \). It follows from (34), (57) and (40) that \( \psi_{k,1}^s(Q) \geq -k \epsilon + o(1) \) as \( Q \to \infty \). Hence \( \psi_{k,1}^s \geq 0 \) as \( \epsilon \) can be chosen arbitrarily small (and thus actually \( \psi_{k,1}^s = 0 \)). From (12) with \( j = 1 \) we further deduce \( \hat{w}_k(\zeta) \leq k \) as claimed. The reverse implication is performed by reversing the argument, we leave the details to the reader. □
We further recall an estimate often referred to as Gelfond’s lemma, see also [29, Hilfssatz 3]. It asserts that for polynomials $Q_1, Q_2$ with integral coefficients of degree at most $n$, we have

\begin{equation}
H(Q_1Q_2) \asymp_n H(Q_1)H(Q_2).
\end{equation}

Now we can prove Theorem 3.1.

**Proof of Theorem 3.1.** In view of Proposition 5.1 with $k = 2n - 1$, it suffices to show (26). Further by Proposition 5.1 it suffices to find for certain arbitrarily large real $X_i$ a set of $2n$ linearly independent polynomials $G_{1,i}, \ldots, G_{2n,i} \in \mathbb{Z}[T]$ that satisfy

\begin{equation}
H(G_{j,i}) \leq X_i, \quad |G_{j,i}(\zeta)| \leq X_i^{-2n+1+\epsilon}, \quad 1 \leq j \leq 2n, \ i \geq 1.
\end{equation}

Since by assumption $w_{n-1}(\zeta) = n - 1 < n \leq w_n(\zeta)$, for large $i$ the polynomials $P_i$ and $Q_i$ must have degree precisely $n$ and be irreducible. This follows from (16) and the definition of $w_n$ and was essentially carried out in [29] and already used frequently in [7].

Let $X_i := H(Q_i)$. By assumption

\[ H(P_i) = X_i^{\nu_i}, \quad |P_i(\zeta)| = X_i^{-\nu_i\mu_i}. \]

By Proposition 5.1 with $k = n - 1$, our assumption $w_{n-1}(\zeta) = n - 1$ yields $\omega_{n-1,n}(\zeta) = n - 1$. In other words for any large parameter $Y \geq Y_0(\epsilon)$ there exist linearly independent polynomials $R_1, \ldots, R_n \in \mathbb{Z}[T]$ of degree at most $n - 1$ that satisfy

\[ H(R_j) \leq Y, \quad |R_j(\zeta)| \leq Y^{-n+1+\epsilon}, \quad 1 \leq j \leq n. \]

Choose $Y = Y_i$ with the sequence of parameters

\[ Y_i = \frac{X_i}{H(P_i)} = X_i^{1-\nu_i}. \]

For $1 \leq j \leq n$ and $i \geq 1$ denote by $R_{j,i}$ the polynomials $R_j$ above for the parameter $Y_i$.

We show that the polynomials $P_{j,i} := P_iR_{j,i}$ provide $n$ of the $2n$ polynomials in (47) with respect to the parameter $X_i$. Clearly each $P_{j,i}$ has degree at most $n + (n - 1) = 2n - 1$. Moreover by construction and (46) we have

\[ H(P_{j,i}) \ll_n H(P_i) \cdot H(R_{j,i}) \leq X_i, \quad 1 \leq j \leq n, \ i \geq 1. \]

The evaluations of the $P_{j,i}$ at $\zeta$ can be estimated via

\begin{equation}
|P_{j,i}(\zeta)| = |P_i(\zeta)| \cdot |R_{j,i}(\zeta)| \leq X_i^{-\nu_i\mu_i} \cdot X_i^{(1-\nu_i)(-n+1+\epsilon)}, \quad 1 \leq j \leq n, \ i \geq i_1.
\end{equation}

By assumption (25) of the theorem we infer

\[ |P_{j,i}(\zeta)| \leq X_i^{-2n+1+\epsilon_0}, \quad 1 \leq j \leq n, \ i \geq i_1, \]

for $\epsilon_0 = n - \frac{n}{1+\epsilon} + \epsilon(1 - \nu_i)$, which tends to 0 as $\epsilon$ does. For the remaining $n$ polynomials we take $Q_{j,i}(T) = T^{j-1}Q_i(T)$ for $1 \leq j \leq n$ and $i \geq 1$. Obviously $Q_{j,i}$ have degree at most $2n - 1$, height $H(Q_{j,i}) = H(Q_i) = X_i$ and satisfy $|Q_{j,i}(\zeta)| \ll_n |Q_i(\zeta)|$ for $1 \leq j \leq n$. Hence

\begin{equation}
|Q_{j,i}(\zeta)| \ll_{n,\zeta} X_i^{-2n+1+\epsilon}, \quad 1 \leq j \leq n, \ i \geq i_1.
\end{equation}
Let $\epsilon_1 = \max\{\epsilon, \epsilon_0\}$. Summing up, for certain arbitrarily large $X = X_i$ we have found $2n$ polynomials $G := \{G_{1,i}, \ldots, G_{2n,i}\} := \{P_{1,i}, \ldots, P_{n,i}, Q_{1,i}, \ldots, Q_{n,i}\}$ that satisfy

$$H(G_{j,i}) \ll_n X_i, \quad |G_{j,i}(\zeta)| \leq X_i^{-2n+1+\epsilon_1} \quad 1 \leq j \leq 2n, \ i \geq i_2.$$ 

As $\epsilon_1$ clearly can be chosen arbitrarily small, for the proof of (17) it remains to be checked that $G$ is a linearly independent set. The linear independence of $P := \{P_{1,i}, \ldots, P_{n,i}\}$ follows obviously from the linear independence of $\{R_{1,i}, \ldots, R_{n,i}\}$. The linear independence of $Q := \{Q_{1,i}, \ldots, Q_{n,i}\}$ is obvious. Any linear combination of the $P_{j,i}$ is of the form $P_iU$ for some polynomial $U \in \mathbb{Z}[T]$ and any linear combination of the $Q_{j,i}$ is of the form $Q_iV$ for some $V \in \mathbb{Z}[T]$ where $U, V$ have degree at most $n - 1$. Since on the other hand $P_i, Q_i$ are linearly independent, irreducible and of degree $n$, by the unique factorization in $\mathbb{Z}[T]$ we infer that the equation $P_iU + Q_iV = 0$ can only be satisfied if both $U$ and $V$ vanish identically. By the linear independence of $P$ and $Q$ this results in the linear independence of $G$. The proof is finished.

Now we turn to the proof of the following next partial assertion of Theorem 2.1

**Theorem 5.2.** Let $\zeta = \zeta_\varphi$ with corresponding $\sigma = \sigma_\varphi$ be as above. Then

$$w_3(\zeta) = 1 + \frac{2}{\sigma}, \quad \lambda_3(\zeta) = \frac{1}{1 + 2\sigma}.$$ 

We provide a brief outline and preliminaries of the proof of Theorem 5.2. We will establish a rather precise description of the functions $L_{3,1}^*(q), \ldots, L_{3,4}^*(q)$ on $q \in (0, \infty)$ induced by $(\zeta_\varphi, \zeta_\varphi^2, \zeta_\varphi^3)$. Denote by $|I|$ the length of an interval $I$. We will show that any $\zeta_\varphi$ induces a partition of the positive real numbers in half-open successive intervals $I_1, J_1, I_2, J_2, \ldots$ with the following properties.

- $\lim_{k \to \infty} |I_k|/|J_k| = 1$.
- At the left interval end of every $I_k$, all $L_{3,j}^*(q)$ are small (more precisely $o(q)$ as $q \to \infty$) by absolute value. In $I_k$ the functions $L_{3,1}^*(q), L_{3,2}^*(q)$ essentially decay with slope $-1/3$, whereas $L_{3,3}^*(q), L_{3,4}^*(q)$ essentially rise with slope $1/3$.
- At the right interval end of $I_k$, which equals the left interval end of $J_k$, the opposite behavior appears. The functions $L_{3,1}^*(q), L_{3,2}^*(q)$ essentially rise with slope $1/3$ in $J_k$, whereas $L_{3,3}^*(q), L_{3,4}^*(q)$ essentially decay with slope $-1/3$. Ultimately the functions $L_{3,1}^*, \ldots, L_{3,4}^*$ asymptotically meet at the right end of $J_k$, which equals the left interval end of the successive $I_{k+1}$.
- The functions $|L_{3,1}^*(q) - L_{3,2}^*(q)|, |L_{3,3}^*(q) - L_{3,4}^*(q)|$ are of order $o(q)$ as $q \to \infty$.

The word "essentially" in the above description means that the stated behavior might be violated in short intervals only. As for the special case of extremal numbers $\zeta_{\varphi-1}$ in [19], the description applies to the simultaneous approximation functions $L_{3,j}(q)$ as well by (36), and as for the special case of extremal numbers in [19] we have

$$w_{3,1}(\zeta) = w_{3,2}(\zeta), \quad w_{3,3}(\zeta) = w_{3,4}(\zeta), \quad \hat{w}_{3,1}(\zeta) = \hat{w}_{3,2}(\zeta), \quad \hat{w}_{3,3}(\zeta) = \hat{w}_{3,4}(\zeta),$$  

$$\lambda_{3,1}(\zeta) = \lambda_{3,2}(\zeta), \quad \lambda_{3,3}(\zeta) = \lambda_{3,4}(\zeta), \quad \hat{\lambda}_{3,1}(\zeta) = \hat{\lambda}_{3,2}(\zeta), \quad \hat{\lambda}_{3,3}(\zeta) = \hat{\lambda}_{3,4}(\zeta).$$
which refines the claim of Theorem 2.1. Similarly as in the special case, the decay phases of $L_{3,i}^*$ will turn out to be induced by the polynomials $W_k$ from Lemma 3.3. The rising phases are induced by products $W_k E_l$ for fixed $W_k$ and suitable successive best approximation polynomials $E_l$ in dimension 1, defined by the property

$$E_l(\zeta_p) = \min \{|Q(\zeta_p)| : Q \in \mathbb{Z}[T], \deg(Q) = 1, 1 \leq H(Q) \leq H(E_l)\}. \tag{51}$$

By Dirichlet’s Theorem and since on the other hand any $\zeta_p$ is badly approximable, for any parameter $X > 1$ there exist an index $l$ such that $H(E_l) \leq H(E_{l-1}) \leq X$ and

$$|E_l(\zeta_p)| \asymp |E_{l-1}(\zeta_p)| \asymp \zeta_p^{-1} H(E_l)^{-1} \asymp \zeta_p^{-1} H(E_{l-1})^{-1} \asymp \zeta_p^{-1} X^{-1}. \tag{52}$$

In fact one might take arbitrary many successive $l, l + 1, \ldots, l + m$ with this property, but for us considering two suffices. In view of Lemma 4.1, the relation (52) will lead to an asymptotic increase by $1/3$ for graphs of $L_{3,j}^*$ induced by the succession of certain products $W_k E_l, W_k E_{l+1}, \ldots, W_k E_{l+m}$. In contrast to the special case of extremal numbers $\zeta_{l-1}$, where the quotients of $|I_{k+1}|/|I_k|$ and $|J_{k+1}|/|J_k|$ tend to the golden ratio $\gamma$, in general they depend on the sequence $(s_k)_{k \geq 1}$ connected to $\zeta_p$. Our proof of Theorem 5.2 below essentially employs the same method as for the determination of $\lambda_3(\zeta), w_3(\zeta)$ for extremal numbers $\zeta$ in the proof of [19] (11) in Theorem 2.1]. However, we attempt to provide clearer and better readable arguments for some rather sketchy parts of the proof of [19] (11), Theorem 2.1] below. Recall the definitions of $\eta_k$ and $W_k$ from Section 3 associated to $\zeta_p$, the latter essentially replace the polynomials $P_k$ from [19]. We will implicitly use the consequence of [40] that if a polynomial $Q$ of degree not exceeding $n$ factors as $Q = Q_1 Q_2$, then $|Q(\zeta)| \leq H(Q)^{-w}$ implies that either $|Q_1(\zeta)| \ll_n H(Q_1)^{-w}$ or $|Q_2(\zeta)| \ll_n H(Q_2)^{-w}$. This type of argument was already used by Wirsing [29]. Since we deal with polynomials of degree at most three, the implied constant is in fact absolute.

Proof of Theorem 5.2. First we recall and justify the explanations at the end of Section 3.4. Here we denote the index by $k$ instead of $i$. In the proof of Theorem 3.1 with the auxiliary Proposition 5.1 we saw that for $P_k, Q_k$ as given there, for the parameter $X = H(Q_k)$ we have four polynomials $G_{1,k}, \ldots, G_{4,k}$ that satisfy $H(G_{i,k}) \ll X, |G_{i,k}(\zeta)| \leq X^{-3+\epsilon}$. Furthermore the proof shows the $G_{i,k}$ can be chosen of the form

$$\{G_{1,k}, \ldots, G_{4,k}\} = \{R_{1,k} \cdot P_k, R_{2,k} \cdot P_k, Q_k, T \cdot Q_k\}$$

for linear integer polynomials $R_{1,k}(T), R_{2,k}(T)$. In the present situation $P_k$ corresponds to $W_k$ and $Q_k$ to $W_{k+1}$, as carried out in Section 3.4. Hence, for any large $k$, with $X_k = H(W_{k+1})$ we have four linearly independent polynomials

$$\mathcal{G}_k = \{G_{1,k}, \ldots, G_{4,k}\} = \{W_k \cdot R_{1,k}, W_k \cdot R_{2,k}, W_{k+1}, T \cdot W_{k+1}\}, \tag{53}$$

with $R_{1,k}(T), R_{2,k}(T)$ linear integer polynomials, so that

$$\max_{1 \leq i \leq 4} H(G_{i,k}) \ll X_k, \quad \max_{1 \leq i \leq 4} |G_{i,k}(\zeta)| \leq X_k^{-3+\epsilon}.$$

More precisely, essentially the same argument as in the proof of Proposition 5.1 shows the following parametric claim. For arbitrarily small $\epsilon > 0$ and sufficiently large $k$ if we take $q = q_k$ as the solution to

$$L_{W_{k+1}}^*(q) = \log H(W_{k+1}) - \frac{q}{3} = 0,$$
which leads to
\[ q_k = 3 \log H(W_{k+1}), \]
then
\[ |L_{3,i}^*(q_k)| \leq \varepsilon q_k, \quad 1 \leq i \leq 4, \]
will hold for a suitable variation of $\varepsilon$. Indeed since $W_{k+1}$ is one of the involved polynomials whose related function $L_{W_{k+1}}^*(q)$ decays at $q = q_k$, in view of (39) the claim can be readily verified.

By Lemma 3.3, with the constant $\eta_k$ defined there, any polynomial $W_k$ satisfies
\[ \frac{\log |W_k(\zeta)|}{\log H(W_k)} = 1 + 2\eta_k + o(1) > 3, \quad k \to \infty. \] Clearly the same argument applies to $TW_k$ as $H(TW_k) = H(W_k)$ and $|\zeta W_k(\zeta)|$ differs from $|W_k(\zeta)|$ only by the constant factor $\zeta \neq 0$. Hence
\[ \frac{\log |G_{i,k}(\zeta)|}{\log H(G_{i,k})} = 1 + 2\eta_{k+1} + o(1) > 3, \quad i \in \{3,4\}, \quad k \to \infty. \]

Let $b_k$ and $c_k$ respectively be the local minima of the functions $L_{G_{3,k}}^*(q)$, $L_{G_{4,k}}^*(q)$. In view of (39) and since $H(G_{3,k}) = H(TG_{3,k}) = H(G_{4,k})$ these are given as the solutions to
\[ \log H(W_{k+1}) - \frac{b_k}{3} = \log |W_{k+1}(\zeta)| + b_k, \quad \log H(W_{k+1}) - \frac{c_k}{3} = \log |\zeta W_{k+1}(\zeta)| + c_k, \]
which yields
\[ b_k = \frac{3}{4}(\log H(W_{k+1}) - \log |W_{k+1}(\zeta)|), \quad c_k = \frac{3}{4}(\log H(W_{k+1}) - \log |\zeta W_{k+1}(\zeta)|). \]

With (57) it is not hard to check that
\[ q_k < \min\{b_k, c_k\}, \quad |b_k - c_k| = O(1), \quad |L_{G_{3,k}}^*(b_k) - L_{G_{4,k}}^*(b_k)| = O(1). \]

Combining (57), (58) and (59), a short calculation shows
\[ L_{G_{i,k}}^*(b_k) = \log H(W_{k+1}) - \frac{b_k}{3} + O(1) \left( \frac{1 - 3\eta_{k+1} + o(1)}{3(1 + \eta_k)} \right) b_k, \quad i \in \{3,4\}, \quad k \to \infty. \]

Since $G_{3,k}^*$ and $G_{4,k}^*$ are linearly independent, in particular
\[ L_{3,1}^*(b_k) \leq L_{3,2}^*(b_k) \leq \left( \frac{1 - 3\eta_{k+1} + o(1)}{3(1 + \eta_k)} \right) b_k. \]

On the other hand, by (55) we have $L_{3,2}^*(q_k) \geq L_{3,1}^*(q_k) \geq -\varepsilon q_k$. Since we noticed $q_k < b_k$ and all functions $L_{3,j}^*, 1 \leq j \leq 4$, have slope at least $-1/3$, we have $L_{3,1}^*(b_k) \geq -\varepsilon q_k - \frac{1}{3}(b_k - q_k)$. After a short calculation using (54), (57), (58) and (59) again, we end up with the reverse asymptotic inequality
\[ L_{3,1}^*(b_k) \geq \left( \frac{1 - 3\eta_{k+1} + o(1)}{3(1 + \eta_k)} - \varepsilon \right) b_k. \]
we have asymptotically increases with constant slope \( \frac{2}{o} \)

Here and in similar succeeding estimates, the additional \( O(1) \) term is only needed when \( a \) an \( b \) are roughly of the same size. It follows from (61) and (60) that the sum \( L_{3,3}^* + L_{3,4}^* \) asymptotically increases with constant slope \( \frac{2}{3} \) in \( I_k \), that is for any \( q_k \leq a < b \leq b_k \) we have

\[
L_{3,3}^*(b) + L_{3,4}^*(b) - L_{3,3}^*(a) - L_{3,4}^*(a) = (b - a) \left( \frac{2}{3} + o(1) \right) + O(1), \quad k \to \infty.
\]

Assume we have already shown that both \( L_{3,3}^* \) and \( L_{3,4}^* \) increase at most by \( 1/3 \) in the interval \( (q_k, \infty) \), that is for \( q_k \leq a < b \leq b_k \) we have

\[
L_{3,3}^*(b) - L_{3,3}^*(q_k) \leq (b - q_k) \left( \frac{1}{3} + \varepsilon \right), \quad L_{3,4}^*(b) - L_{3,4}^*(q_k) \leq (b - q_k) \left( \frac{1}{3} - \varepsilon \right).
\]

Then by (62) both must have asymptotically slope \( 1/3 + o(1) \) as \( k \to \infty \) in the entire interval \( I_k \), that is for \( q_k \leq a < b \leq b_k \) we have

\[
L_{3,3}^*(b) - L_{3,3}^*(a) = (b - a) \left( \frac{1}{3} + o(1) \right) + O(1), \quad j \in \{3, 4\}, \quad k \to \infty.
\]

To show (63), first note that by (52) and since \( L_{3,4}^* \geq L_{3,3}^* \) it suffices to show

\[
L_{3,4}^*(b) \leq (b - q_k) \left( \frac{1}{3} + o(1) \right) + O(1), \quad k \to \infty.
\]

For any parameter \( Y_k > X_k = H(W_{k+1}) \), consider the two linear polynomials \( E_t, E_{t-1} \) defined by (51), with \( t = t(Y_k) \) chosen largest possible such that

\[
\max\{H(W_k E_t), H(W_k E_{t-1})\} \leq Y_k.
\]

By Gelfond’s Lemma [16] we have \( H(E_t) \asymp H(E_{t-1}) \asymp Y_k / X_{k-1} \). Thus and since \( \zeta \) is badly approximable with respect to one-dimensional approximation, as previously noticed in [52], we have

\[
|E_t(\zeta)| \asymp \zeta |E_{t-1}(\zeta)| \asymp \zeta H(E_t)^{-1} \asymp \zeta H(E_{t-1})^{-1} \asymp \zeta \left( \frac{Y_k}{X_{k-1}} \right)^{-1} = \frac{X_{k-1}}{Y_k}.
\]
Define \( U_{k,1} := W_k E_t \) and \( U_{k,2} := W_k E_{t-1} \) for \( t = t(Y_k) \), depending on \( Y_k \) which is treated as a variable. As \( Y_k \) increases in \((X_k, \infty)\), define the two functions \( L_{k,i}^* \), for \( i = 1, 2 \) as the succession of the functions \( L_{U_{k,i}}^* \) in \((q_k, \infty)\) with the respective polynomials \( U_{k,i} = U_{k,i}(t) = U_{k,i}(Y_k) \). That is by (39) formally

\[
(67) \quad L_{k,i}^*(q) = \max_{t \geq 1} \left\{ -\frac{q}{3} + \log H(W_k E_t), q + \log |W_k(\zeta)E_t(\zeta)| \right\}
\]

and if \( t_0 = t_0(q) \) denotes the minimum index in (67), then

\[
(68) \quad L_{k,2}^*(q) = \min_{t \geq 1} \left\{ -\frac{q}{3} + \log H(W_k E_t), q + \log |W_k(\zeta)E_t(\zeta)| \right\}.
\]

In fact the latter minimum index \( t_1 \) is just \( t_1 = t_0 - 1 \). Since \( H(E_t) \ll_\zeta H(E_{t-1}) \) by (66), application of Lemma 4.1 with the polynomials \( P(T) = W_k(T) \), and \( Q(T) = E_t(T) \) and \( Q(T) = E_{t-1}(T) \) respectively, yields that both induced functions \( L_{f_{k,1}}^*, L_{f_{k,2}}^* \) rise with slope \( 1/3 + o(1) \) in \((q_k, \infty)\), as \( H(E_t) \to \infty \). Since \( H(E_t) \asymp Y_k/X_{k-1} > X_k/X_{k-1} \to \infty \) as \( k \to \infty \), it suffices to assume \( k \to \infty \). Hence for any \( q_k \leq a < b \) the estimate

\[
(69) \quad L_{k,i}^*(b) - L_{k,i}^*(a) = (b - a) \left( \frac{1}{3} + o(1) \right) + O(1), \quad i \in \{1, 2\}, \quad k \to \infty,
\]

holds. On the other hand, the set \( \{W_k E_t, W_k E_{t-1}, W_{k+1}, TW_{k+1}\} \) is linearly independent for any large \( k \), as they span the same space as \( \mathcal{G}_k \). It is not hard to verify that

\[
\max_{i=1,2} \{|W_{k+1}(\zeta)|, |\zeta W_{k+1}(\zeta)|\} < \min_{i=1,2} |U_{k,i}(\zeta)|,
\]

as soon as \( H(U_{k,i}) < H(W_{k+2}) \), thus we have

\[
(70) \quad L_{3,1}^*(b) \leq \max_{i=1,2} L_{f_{k,i}}^*(b), \quad q_k \leq b \leq q_{k+1}.
\]

Consider the polynomials \( W_k R_{1,k} \) and \( W_k R_{2,k} \) with \( R_{t,k} \) from (54). Form the proof of Theorem 3.1 \( R_{1,k}, R_{2,k} \) can be chosen \( E_t, E_{t-1} \). Therefore, by (55), for the parameter \( q = q_k \) we conclude

\[
(71) \quad \max_{i=1,2} L_{f_{k,i}}^*(q_k) \leq \varepsilon q_k.
\]

Finally (65) follows from (69) with \( a = q_k \), (70) and (71). The relation (64) is implied.

We have just shown in (61) and (64) that in the interval \( I_k \), the first two successive minima asymptotically decay with slope \(-1/3\) and the third and fourth asymptotically increase with slope \(1/3\). Since at \( q = q_k \) all \( |L_{3,1}^*(q)| \) are of order \( o(q) \) by (55), we have \( L_{3,1}^*(b_k) = -L_{3,1}^*(b_k) + o(b_k) \) and thus \( \psi_{3,1}(B_k) = -\psi_{3,1}(B_k) + o(1) \) for \( i \in \{1, 2\} \) and \( j \in \{3, 4\} \), as \( k \to \infty \). We conclude that relation (60) can be complemented to

\[
(72) \quad \psi_{3,1}(B_k) = \frac{1 - \eta_k}{3(1 + \eta_k)} + o(1), \quad j \in \{1, 2\}, \quad k \to \infty,
\]

\[
(73) \quad \psi_{3,1}(B_k) = -\frac{1 - \eta_k}{3(1 + \eta_k)} + o(1), \quad j \in \{3, 4\}, \quad k \to \infty.
\]
Let $J_k := [b_k, q_{k+1})$. We first show that $I_k$ and $J_k$ have roughly the same length, that is $b_k$ should lie roughly in the middle between $q_k$ and $q_{k+1}$ and $\lim_{k \to \infty} |I_k|/|J_k| = 1$. More precisely we establish $q_{k+1} - b_k = b_k - q_k + o(q_k)$, or equivalently

\begin{equation}
q_k + q_{k+1} \over 2 = b_k + o(q_k).
\end{equation}

Recall combination of (50) and (71) yields

\begin{equation}
\log H(W_{k+1}) \over \log H(W_k) = \eta_{k+1} + o(1), \quad k \to \infty.
\end{equation}

Combination of (57), (58) and (75) yields

\begin{align*}
b_k &= \frac{3}{4} (\log H(W_{k+1}) - \log |W_{k+1}(\zeta)|) = \\
&= \frac{3}{4} (\log H(W_{k+1}) + (1 + 2\eta_{k+1} + o(1)) \log H(W_{k+1})) = \\
&= \frac{3}{2} (1 + \eta_{k+1} + o(1)) \log H(W_{k+1}).
\end{align*}

This is indeed as the same value we obtain from combining (53), (75) and the hypothesis (74), as the calculation

\begin{equation}
\frac{q_k + q_{k+1}}{2} = \frac{3 \log H(W_{k+1}) + 3 \log H(W_{k+2})}{2} = \frac{3}{2} (1 + \eta_{k+1}) \log H(W_{k+1}),
\end{equation}

shows. Thus indeed (74) must hold and $|I_k|/|J_k| = 1 + o(1)$ as $k \to \infty$.

We now conclude that in the interval $J_k$, the functions $L^*_{3,1}, L^*_{3,2}$ have asymptotic slope 1/3, whereas the functions $L^*_{3,3}, L^*_{3,4}$ have asymptotic slope $-1/3$, until they all meet (asymptotically) at $q_{k+1}$. Recall we have established that $L^*_{3,3}$ and $L^*_{3,4}$ both rise in $I_k$ with asymptotic slope 1/3. If we let $b = b_k$ and $a = q_k$ in (64), since $|L^*_{3,j}(q)|$ are all small at $q = q_k$ in view of (53) and $q_k \approx b_k$, this means

\begin{equation}
\frac{L^*_{3,j}(b_k) - L^*_{3,j}(q_k)}{b_k - q_k} = \frac{L^*_{3,j}(b_k)}{b_k - q_k} + o(1) = \frac{1}{3} + o(1), \quad j \in \{3, 4\}, k \to \infty.
\end{equation}

In combination with the fact that $q_{k+1} - b_k$ and $b_k - q_k$ are roughly equal by (74) and since $|L^*_{3,j}(q)|$ are all small at $q = q_{k+1}$ again by (53) with index shift $k \to k + 1$ and $q_{k+1} \approx b_k$, we infer

\begin{equation}
\frac{L^*_{3,j}(q_{k+1}) - L^*_{3,j}(b_k)}{q_{k+1} - b_k} = -\frac{L^*_{3,j}(b_k)}{q_{k+1} - b_k} + o(1) = -\frac{1}{3} + o(1), \quad j \in \{3, 4\}, k \to \infty.
\end{equation}

On the other hand the functions $L^*_{3,j}$ have slope at least $-1/3$, so $L^*_{3,3}, L^*_{3,4}$ must each decay asymptotically with slope $-1/3 + o(1)$ in $J_k$, that is for $b_k \leq a < b \leq q_{k+1}$ we must have

\begin{equation}
L^*_{3,j}(b) - L^*_{3,j}(a) = \left(-\frac{1}{3} + o(1)\right) (b - a) + O(1), \quad j \in \{3, 4\}, k \to \infty.
\end{equation}
From (10) we further deduce that the sum $L^*_3,1 + L^*_3,2$ must asymptotically increase with slope $2/3$ in $J_k$. By this again we mean that for $b_k \leq a < b \leq q_{k+1}$ we have

$$(78) \quad L^*_3,1(b) + L^*_3,2(b) - L^*_3,1(a) - L^*_3,2(a) = \left( \frac{2}{3} + o(1) \right) \cdot (b - a) + O(1), \quad k \to \infty.$$ 

Now from a very similar argument as for (65) we derive that both $L^*_3,1$ and $L^*_3,2$ increase with slope $1/3 + o(1)$ as $k \to \infty$ in (any not too small subinterval of) $J_k$. On the one hand, since $L_{3,i}(b_k) = -L_{3,i}(b_k)$ for $i \in \{1, 2\}$, $j \in \{3, 4\}$ by (72), (73), similarly to (76) we calculate

$$(79) \quad \frac{L^*_3,j(q_{k+1}) - L^*_3,j(b_k)}{q_{k+1} - b_k} = -\frac{L^*_3,j(b_k)}{q_{k+1} - b_k} + o(1) = \frac{1}{3} + o(1), \quad j \in \{1, 2\}, \quad k \to \infty.$$ 

This shows that on average $L^*_3,1$ and $L^*_3,2$ increase with slope $1/3$ in $J_k$. For the refined local version (in any subinterval of $J_k$), notice that (79) implies that both $L^*_3,1, L^*_3,2$ increase on average by $1/3$ in $J_k$. However, by essentially the same argument as for (63), (65), each of $L^*_3,1, L^*_3,2$ can increase with slope at most $1/3$ in any not too small subinterval of $J_k$. Indeed, considering the polynomials $W_{k+1} E_1, W_{k+1} E_{k-1}$ and the derived functions $f^*_k + i_1, i_2$ from (67) and (68), essentially Lemma 1.1 yields the estimate

$$(80) \quad L^*_3,1(b) \leq L^*_3,2(b) \leq (b - b_k) \left( \frac{1}{3} + o(1) \right) + O(1), \quad b > b_k, \quad k \to \infty,$$

similarly to (65). By combination of (78) and (80) for $b_k \leq a < b \leq q_{k+1}$ we indeed derive

$$(81) \quad L^*_3,j(b) - L^*_3,j(a) = \left( \frac{1}{3} + o(1) \right) (b - a) + O(1), \quad j \in \{1, 2\}, \quad k \to \infty,$$

as claimed.

The claims (77) with (81) for $b = q_{k+1}, a = b_k$, can in view of (65) be stated as

$$\lim_{k \to \infty} \frac{L^*_3,1(b_k)}{q_{k+1} - b_k} = \lim_{k \to \infty} \frac{L^*_3,2(b_k)}{q_{k+1} - b_k} = \lim_{k \to \infty} \frac{L^*_3,1(b_k)}{q_{k+1} - b_k} = \lim_{k \to \infty} \frac{L^*_3,4(b_k)}{q_{k+1} - b_k} = \frac{1}{3}.$$ 

We eventually put the above findings together to determine the exponents. Fix any interval $[q_k, q_{k+1}] = I_k \cup J_k$, and let $q \in [q_k, q_{k+1}]$. From (61), (64), (77), and (80), no matter if we have $q \in [q_k, b_k)$ or $q \in [b_k, q_{k+1})$, as $k \to \infty$ we derive

$$L^*_3,j(q) \geq L^*_3,j(b_k) + \left( \frac{1}{3} + o(1) \right) |q - b_k| + O(1) \geq L^*_3,j(b_k) + o(q), \quad j \in \{1, 2\},$$

$$L^*_3,j(q) \leq L^*_3,j(b_k) - \left( \frac{1}{3} + o(1) \right) |q - b_k| + O(1) \leq L^*_3,j(b_k) + o(q), \quad j \in \{3, 4\}.$$ 

It is easy to check with (72) and (73) that for $Q_k := e^{\eta_k}$ and $M_k = [Q_k, Q_{k+1}]$, these results imply

$$(82) \quad \psi^*_3,j(Q) \geq \psi^*_3,j(B_k) + o(1) = \frac{1 - \eta_k}{3(1 + \eta_k)} + o(1), \quad Q \in M_k, \quad j \in \{1, 2\},$$

$$(83) \quad \psi^*_3,j(Q) \leq \psi^*_3,j(B_k) + o(1) = -\frac{1 - \eta_k}{3(1 + \eta_k)} + o(1), \quad Q \in M_k, \quad j \in \{3, 4\},$$
as \( k \to \infty \). Observe that the right interval end of \( J_k \) and the left interval end of the successive \( I_{k+1} \) both equal \( q_{k+1} \), and hence \( I_1, J_1, I_2, J_2, \ldots \) forms a partition of \([q_2, \infty)\). Hence \( \cup_{k \geq 1} M_k \) form a partition of \([Q_2, \infty)\), and \([82], [83]\) yield for \( j \in \{1, 2\} \)

\[
\psi_{\lambda,j}^* = \liminf_{Q \to \infty} \psi_{\lambda,j}^*(Q) = \liminf_{k \to \infty} \inf_{Q \in M_k} \psi_{\lambda,j}^*(Q) = \liminf_{k \to \infty} \psi_{3,j}^*(B_k) = \liminf_{k \to \infty} \frac{1 - \eta_k}{3(1 + \eta_k)},
\]

and for \( j \in \{3, 4\} \) the estimates

\[
\psi_{3,j}^* = \limsup_{Q \to \infty} \psi_{3,j}^*(Q) = \limsup_{k \to \infty} \sup_{Q \in M_k} \psi_{3,j}^*(Q) = \limsup_{k \to \infty} \psi_{3,j}^*(B_k) = \limsup_{k \to \infty} -\frac{1 - \eta_k}{3(1 + \eta_k)}.
\]

Thus since \( \limsup_{k \to \infty} \eta_k = \sigma^{-1} \) by \([9]\), we infer

\[
\psi_{3,1}^* = \psi_{3,2}^* = \frac{\sigma - 1}{3(\sigma + 1)}, \quad \psi_{3,3}^* = \psi_{3,4}^* = -\frac{\sigma - 1}{3(\sigma + 1)}.
\]

Application of \([39], [41]\) and \([42]\) yields

\[
(84) \quad w_3(\zeta) = w_{3,2}(\zeta) = 1 + \frac{2}{\sigma}, \quad \lambda_3(\zeta) = \lambda_{3,2}(\zeta) = \frac{1}{1 + 2 \sigma},
\]

containing the claims of the theorem. Furthermore, we point out the method shows that both functions \(|L_{3,1}(q) - L_{3,2}(q)|\) and \(|L_{3,3}(q) - L_{3,4}(q)|\) differ at most by \( o(q) \) as \( q \to \infty \), and that very similarly as for extremal numbers \([19]\) Remark 4.2 and (50),(51)], we obtain

\[
(85) \quad w_{3,3}(\zeta) = w_{3,4}(\zeta) = \hat{w}_3(\zeta) = \hat{w}_{3,2}(\zeta) = 3, \quad 
\lambda_{3,3}(\zeta) = \lambda_{3,4}(\zeta) = \hat{\lambda}_3(\zeta) = \hat{\lambda}_{3,2}(\zeta) = \frac{1}{3},
\]

and

\[
(86) \quad \hat{w}_{3,3}(\zeta) = \hat{w}_{3,4}(\zeta) = 1 + 2 \sigma, \quad \hat{\lambda}_{3,3}(\zeta) = \hat{\lambda}_{3,4}(\zeta) = \frac{\sigma}{\sigma + 2}.
\]

We infer \( w_3^*(\zeta) = 1 + 2/\sigma \) for \( \zeta = \zeta_\rho \) from the monotonicity of the sequence \((w_n^*(\zeta))_{n \geq 1}\), inequality \([6]\), Theorem \([11]\) and Theorem \([5,2]\) via

\[
1 + \frac{2}{\sigma} = w_2^*(\zeta) \leq w_3^*(\zeta) \leq w_3(\zeta) = 1 + \frac{2}{\sigma}.
\]

Before we establish the last claim \([12]\) of Theorem \([2,1]\) we turn towards Theorem \([2,2]\). As in the case of extremal numbers, the description of the combined graph of the functions \(L_{3,j}(q)\) from Theorem \([2,1]\) allows for deducing Theorem \([2,2]\). The proof is again similar to the one of \([19]\) Theorem 2.2. However, application of results of Davenport and Schmidt \([9]\) and Bugeaud \([8]\) lead almost directly to a proof of \([13]\), compared to the rather technical and lengthy proof of the corresponding claim in \([19]\) Theorem 2.2. These results enable us to establish the new claim \([18]\) as well.

**Proof of Theorem \([2,2]\)**. It follows from the proof of Theorem \([5,2]\) that for any large \( q \), the first two successive minima functions of the linear form problem related to \( \psi_{3,1}^*(q) \), \( \psi_{3,2}^*(q) \) are induced by polynomial multiples \( W_kE_1, W_kE_{r-1} \) (or \( W_{k+1}, TW_{k+1} \), the argument below remains essentially the same) of some polynomial \( W_k \). For each \( k \geq 1 \) they obviously
span the same space as \( \{W_k, TW_k\} \). Since the degree of \( W_k \) is two, there is no irreducible polynomial of degree exactly three which lies in this space. Thus the optimal exponent in \( (13) \) cannot exceed \( w_{3,3}(\zeta) \). On the other hand it follows from the proof of Theorem 5,2 that \( \hat{w}_{3,3}^* = \hat{w}_{3,4}^* = 0 \), or equivalently \( w_{3,3}(\zeta) = w_{3,4}(\zeta) = 3 \) by \( (12) \), as already noticed in \( (85) \). Thus indeed \( (13) \) has only finitely many solutions in \( Q \in \mathbb{Z}[T] \) an irreducible polynomial of degree precisely three. From \( (13) \) we infer \( (14) \) by a standard argument, namely if \( \alpha \) is a root of a polynomial \( P \) of degree \( n \) and close to some real number \( \zeta \), then

\[
|P(\zeta)| \ll_{\zeta,n} H(P)|\zeta - \alpha|.
\]

Next we show \( (16) \) and \( (17) \). From essentially the vector space argument from the proof of \( (13) \) we obtain similarly that \( \hat{w}_{3,3}(\zeta) + \epsilon \) is an upper bound for the exponent in \( (16) \) for certain arbitrary large \( X \). On the other hand, the proof of Theorem 5.2 and \( (13) \) shows \( \hat{w}_{3,3}(\zeta) = 1 + 2\sigma \), as pointed out in \( (86) \). We conclude \( (16) \), and further deduce \( (17) \) very similarly as \( (14) \) from \( (13) \).

The claims \( (15) \), \( (18) \) follow from \( (11) \), essentially using an argument of Davenport and Schmidt \( [9] \), and variants due to Bugeaud \( [3] \). Indeed \( [9, \text{Lemma 1}] \) claims, in our notation, that for any real transcendental \( \zeta \), the estimate

\[
|\zeta - \alpha| \leq H(\alpha)^{-1/\hat{\lambda}_n(\zeta) - 1 + \epsilon}
\]

has infinitely many solutions in real algebraic integers \( \alpha \) of degree at most \( n + 1 \). For \( n = 3 \), since we have shown \( \hat{\lambda}_3(\zeta) = 1/3 \) for Sturmian continued fractions, the right estimate of \( (15) \) follows for algebraic integers of degree at most \( 4 \) in place of algebraic numbers of degree precisely three. To make the transition to cubic algebraic numbers, we essentially proceed as in \([3, \text{Theorem 2.11}]\), incorporating also the comments below \([3, \text{Theorem 2.11}] \). As pointed out the method of \([3, \text{Theorem 2.11}] \) with \( n = d - 1 \) leads to a new proof of \([3, \text{Theorem 2.9}] \), which states that for a real algebraic number \( \zeta \) of degree \( d \), we have \( |\zeta - \alpha| \ll_{\zeta} H(\alpha)^{-d} \) for infinitely many real algebraic numbers \( \alpha \) of degree precisely \( n = d - 1 \). When we let \( d = 4, n = d - 1 = 3 \), it becomes that for an algebraic number \( \zeta \) of degree four we have \( |\zeta - \alpha| \ll_{\zeta} H(\alpha)^{-4} \), the estimate in \( (15) \), for infinitely many real numbers \( \alpha \) of degree precisely three. We have to adapt the method slightly since we deal with transcendental \( \zeta \). We claim that the same argument yields that for any real transcendental \( \zeta \) the estimate \( |\zeta - \alpha| \leq H(\alpha)^{-n - 1 + \epsilon} \) has infinitely many real algebraic \( \alpha \) of degree precisely \( n \), as soon as \( \hat{w}_n(\zeta) = n \). The only time where \( \zeta \) being algebraic was required in the proof of \([3, \text{Theorem 2.11}] \) is at the start when the Liouville inequality \([3, \text{Theorem A.1}] \) is applied. Its purpose is to guarantee that for certain large \( X \) we have

\[
|P(\zeta)| \gg_{d,\zeta} H(P)^{-d + 1} \geq X^{-d + 1}
\]

for all non-zero integer polynomials \( P \) of degree at most \( n \) and \( H(P) \leq X \). However, for transcendental \( \zeta \), by the definition of \( \hat{w}_n(\zeta) \) and \( n = d - 1 \), the estimate \( (89) \) holds as soon as \( \hat{w}_n(\zeta) = n \), up to the multiplicative constant replaced by an \( \epsilon \) in the exponent. When \( n = 3 \), note in Theorem 2.11 we have indeed verified the condition \( \hat{w}_3(\zeta) = 3 \) for Sturmian continued fractions \( \zeta \). From this point on we proceed as in the proof of \([3, \text{Theorem 2.11}] \) to obtain the right estimate of \( (15) \). The left estimate of \( (15) \) is further implied by \( (87) \).
For (18) we kind of dualize the argument above, in the sense of exchanging best and uniform approximation accordingly. In fact in the proof of [9, Lemma 1] deals with the inequality $|\zeta - \alpha| \leq H(\alpha)^{-n_{\alpha+1}(\zeta)}$ instead of (88), and the highlighted result (88) is deduced by Mahler’s duality. The proof can be readily altered to show the analogous uniform version, that is the estimate $|\zeta - \alpha| \leq H(\alpha)^{-1}X^{-\tilde{w}_{n,n+1}(\zeta)+\epsilon}$ has a real algebraic integer solution $\alpha$ of degree precisely $n$ for all large $X$. We apply it to $n = 3$ again. Since $\tilde{w}_{3,4}(\zeta) = \lambda_3(\zeta)^{-1} = 2\sigma + 1$ as shown in Theorem 2.11, the right claim in (18) holds for algebraic integers $\alpha$ of degree at most 4. Again the transition to cubic algebraic numbers can be carried out as in [3, Theorem 2.11], and yields the right estimate in (18). The left is inferred from (87) again.

We now show the final claim (12) of Theorem 2.1. From Theorem 1.1 we see that $\tilde{w}_3^*(\zeta) \geq \tilde{w}_3^*(\zeta) = 2 + \sigma$ for any $\zeta = \zeta_\alpha$. We need to show the reverse inequality. The key idea is to derive from the proof of Theorem 5.2 that for parameters $X$ slightly smaller than $H(W_{k+1})$ for large $k$, the best algebraic approximation $\alpha$ of degree three or less to $\zeta$ is given by the root $\alpha_k$ of the quadratic polynomial $W_k$ closer to $\zeta$. Then the claim follows from Lemma 3.3. We only sketch some arguments derived from the proof of Theorem 5.2 to avoid repeating cumbersome computations.

Let $\epsilon > 0$ small. From Theorem 5.2 and its proof we know that for large $k$ and certain $Z_k$ roughly of size $H(W_{k+1})$, the inequality

$$H(P) \leq Z_k, \quad |P(\zeta)| \leq Z_k^{-1-\frac{2}{\eta_k}-\epsilon}$$

has no three linearly independent solutions in at most cubic polynomials $P$. More precisely we certainly have $H(W_{k+1})^{1-\delta} \leq Z_k < H(W_{k+1})^{1+\delta}$ for arbitrarily small $\delta > 0$ and $k \geq k_0(\delta)$. This follows basically from the fact that the essential local maxima of $\psi_{3,3}(Q)$ (or $L_{3,3}(q)$) are attained close to $Q = B_k$ (or $q = b_k$), see (73), and at these points they catch up with the falling slope $\log H(W_{k+1}) - q$ of $L_{W_{k+1}}^*(q)$. We spare the details. On the other hand, as noticed in the proof of Theorem 2.2, the two linearly solutions to (90) that do exist span the same space as $\{W_k, TW_k\}$, which contains only multiples of $W_k$ and thus no irreducible cubic polynomial. Hence for any irreducible cubic integer polynomial of height at most $Z_k$ we have the reverse inequality for $|P(\zeta)|$. Thus, choosing $\delta$ small enough compared to $\epsilon$, (87) yields that for large $k$ the estimate

$$H(\beta) \leq Z_k, \quad |\zeta - \beta| \ll n_{\zeta} H(\beta)^{-1}Z_k^{-1-\frac{2}{\eta_k}-\epsilon}$$

has no solution in real cubic algebraic numbers $\beta$, with the implied constant from (87). On the other hand, since $Z_k$ is essentially of size $H(W_{k+1})$ and we can decrease it just a little to satisfy $Z_k < H(W_{k+1})$ if necessary, from Lemma 3.3 we know that for large $k$

$$H(\beta) \leq Z_k, \quad |\zeta - \beta| \leq H(\beta)^{-1}Z_k^{-2-\frac{2}{\eta_k}-\epsilon}$$

has no solution among real algebraic numbers $\beta$ of degree at most 2. Combination and $1/\eta_k < 1$ yields that (91) has no solution in algebraic numbers of degree at most three. Now recall by (9), (28) and Lemma 3.3 we have $\limsup_{k \to \infty} \eta_k = \sigma^{-1}$. Thus we find
arbitrarily large values of $k$ for which $\eta_k^{-1}$ is close to $\sigma$, more precisely
\begin{equation}
\eta_k^{-1} = \sigma + o(1), \quad k \to \infty.
\end{equation}
Choosing such a sequence of $k$ as in (92), from (91) and letting $\epsilon \to 0$, we see that
\[ \frac{\lambda_2^*(\zeta)}{\hat{W}_2^*(\zeta)} \leq 2 + \sigma = \frac{1}{\hat{W}_2^*(\zeta)}. \]
The proof is finished.

We conclude with Theorem 2.3. The proof of (19) will be rather simple and only relies on the knowledge of the value of $w_2(\zeta_\varphi)$ from Theorem 1.1 and duality arguments. The refined upper bounds in (21) and (22) will be derived from the recent result [22, Theorem 2.9]. For $n = 2$ in the notation of [22], its claim becomes
\begin{equation}
\lambda_n(\zeta) \leq \max \left\{ \frac{w_2(\zeta)}{\hat{W}_2(\zeta)\hat{W}_1(\zeta)^{-1}}, \frac{1}{\hat{W}_2(\zeta)} \right\}, \quad n \geq 2,
\end{equation}
for any transcendental real $\zeta$ that satisfies $w_1(\zeta) < 2$.

Proof of Theorem 2.3. Since the sequence $(\lambda_n(\zeta))_{n \geq 1}$ is non-increasing, the upper bound in (19) follows from (10) and the hypothesis $n \geq 3$. The lower bound $1/n$ comes from (4).

We have to prove the remaining lower bound. From Theorem 1.1 we know that there exist $P$ of degree two and arbitrarily large height $H(P)$ such that $|P(\zeta)| \leq H(P)^{-1 - 2/\sigma + \epsilon}$. For any such $P$ the $n - 1$ polynomials $P_0(T) = P, P_1(T) = TP(T), \ldots, P_{n-2}(T) = T^{n-2}P(T)$ have degree at most $n$ and share the same estimates since obviously $H(P_0) = H(P_1) = \cdots = H(P_{n-2})$ and $P_0(\zeta) \approx \zeta P_1(\zeta) \approx \zeta \cdots \approx \zeta P_{n-2}(\zeta)$. Hence
\[ w_{n,n-1}(\zeta) \geq 1 + \frac{2}{\sigma}. \]
With (42) we obtain
\[ \psi_{n,n-1}^* \leq \frac{(n-1)\sigma - 2}{2n(\sigma + 1)}. \]
Hence, by application of (35) and (36), we infer
\[ \psi_{n,1} = -\psi_{n,n+1} \leq \frac{n-1}{2} \psi_{n,n-1}^* \leq \frac{(n-1)^2\sigma - 2(n-1)}{4n(\sigma + 1)}. \]
Inserting in (41) we obtain the right expression in the maximum as a lower bound for $\lambda_n$, and thus have established (19).

The identity (20) is true for $n \in \{1, 2\}$ since any $\zeta_\varphi$ has bounded partial quotients and $\lambda_2(\zeta_\varphi) = 1$ as well by Theorem 1.1. For $n \geq 3$, the identity follows from (19) if and only if $\sigma = 0$. For the estimates (21) and (22), recall (93) applies since $w_1(\zeta_\varphi) = 1 < 2$. Moreover, if $\sigma_\varphi > 0$, then $w_2(\zeta) = 1 + 2/\sigma_\varphi < \infty$. The left expression bound in (21) follows by inserting the corresponding values from Theorem 1.1 and $\hat{W}_{n-1}(\zeta) \geq n - 1$, and checking that it the left expression in (93) is larger for $n$ in the given range. The bound in (22) follows similarly checking that the right expression is the larger one for larger $n$. The right expression bound in (21) only reproduces (19), derived from (10). Finally, as shown in [25, Theorem 2.1], the property (23) holds for any real number which is not a $U$-number in Mahler’s classification of real numbers, and it follows from Adamczewski and Bugeaud [1] that any Sturmian continued fraction with $\sigma > 0$ is not a $U$-number. □
The author warmly thanks the referee for pointing out inaccuracies and giving several other valuable advices.

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