LAGRANGIAN SUBCATEGORIES AND BRAIDED TENSOR EQUIVALENCES OF TWISTED QUANTUM DOUBLES OF FINITE GROUPS

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Abstract. We classify Lagrangian subcategories of the representation category of a twisted quantum double $D^\omega(G)$, where $G$ is a finite group and $\omega$ is a 3-cocycle on it. In view of results of [DGNO] this gives a complete description of all braided tensor equivalent pairs of twisted quantum doubles of finite groups. We also establish a canonical bijection between Lagrangian subcategories of Rep($D^\omega(G)$) and module categories over the category Vec$^\omega_G$ of twisted $G$-graded vector spaces such that the dual tensor category is pointed. This can be viewed as a quantum version of V. Drinfeld’s characterization of homogeneous spaces of a Poisson-Lie group in terms of Lagrangian subalgebras of the double of its Lie bialgebra [Dr]. As a consequence, we obtain that two group-theoretical fusion categories are weakly Morita equivalent if and only if their centers are equivalent as braided tensor categories.

1. Introduction

Throughout this paper we will work over an algebraically closed field $k$ of characteristic zero. Unless otherwise stated all cocycles appearing in this work will have coefficients in the trivial module $k^\times$. All categories considered in this work are assumed to be $k$-linear and semisimple with finite-dimensional Hom-spaces and finitely many isomorphism classes of simple objects. All functors are assumed to be additive and $k$-linear.

Let $G$ be a finite group and $\omega$ be a 3-cocycle on $G$. In [DPR1, DPR2] R. Dijkgraaf, V. Pasquier, and P. Roche introduced a quasi-triangular quasi-Hopf algebra $D^\omega(G)$. When $\omega = 1$ this quasi-Hopf algebra coincides with the Drinfeld double $D(G)$ of $G$ and so $D^\omega(G)$ is often called a twisted quantum double of $G$. It is well known that the representation category Rep($D^\omega(G)$) of $D^\omega(G)$ is a modular category [BK, T] and is braided equivalent to the center [K] of the tensor category Vec$^\omega_G$ of finite-dimensional $G$-graded vector spaces with the associativity constraint defined using $\omega$. The category Vec$^\omega_G$ is a typical example of a pointed fusion category, i.e., a finite semisimple tensor category in which every simple object is invertible.

In [DGNO] a criterion for a modular category $\mathcal{C}$ to be braided tensor equivalent to the center of a category of the form Vec$^\omega_G$ for some finite group $G$ and $\omega \in Z^3(G, k^\times)$ is given. Namely, such a braided equivalence exists if and only if $\mathcal{C}$ contains a Lagrangian subcategory, i.e., a maximal isotropic subcategory of dimension $\sqrt{\dim(\mathcal{C})}$. More precisely, Lagrangian subcategories of $\mathcal{C}$ parameterize the classes of braided equivalences between $\mathcal{C}$ and centers of pointed categories, see [DGNO, Section 4]. Note that any Lagrangian subcategory of $\mathcal{C}$ is equivalent (as a symmetric tensor category) to the representation category of some group by the result of P. Deligne [De].

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This means that a description of Lagrangian subcategories of $\text{Rep}(D^\omega(G))$ for all groups $G$ and 3-cocycles $\omega$ is equivalent to a description of all braided equivalences between representation categories of twisted group doubles. Such equivalences for elementary Abelian and extra special groups were studied in [MN] and [GMN]. A motivation for such study comes from a relation between holomorphic orbifolds in the Rational Conformal Field Theory and twisted group doubles observed in [DVVV], [DPR2].

A complete classification of Lagrangian subcategories of $\text{Rep}(D^\omega(G))$ is the principal goal of this paper.

1.1. Main results. Let $G$ be a finite group and let $\omega \in Z^3(G, k^\times)$ be a 3-cocycle on $G$.

**Theorem 1.1.** Lagrangian subcategories of the representation category of the Drinfeld double $D(G)$ are classified by pairs $(H, B)$, where $H$ is a normal Abelian subgroup of $G$ and $B$ is an alternating $G$-invariant bicharacter on $H$.

The proof is based on the analysis of modular data (i.e., the $S$- and $T$-matrices) associated to $D(G)$.

Theorem 1.1 gives a simple classification of Lagrangian subcategories for the untwisted double $D(G)$. In the twisted $(\omega \neq 1)$ case the notion of a $G$-invariant bicharacter needs to be twisted as well, cf. Definition 4.6.

**Theorem 1.2.** Lagrangian subcategories of the representation category of the twisted double $D^\omega(G)$ are classified by pairs $(H, B)$, where $H$ is a normal Abelian subgroup of $G$ such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $B : H \times H \to k^\times$ is a $G$-invariant alternating $\omega$-bicharacter in the sense of Definition 4.6.

Note that bicharacters in the statement of Theorem 1.2 are in bijection with equivalence classes of $G$-invariant cochains $\mu \in C^2(H, k^\times)$ such that $\delta^2 \mu = \omega|_{H \times H \times H}$, see (3).

Let $\mathcal{L}_{(H,B)}$ denote the Lagrangian subcategory of $\text{Rep}(D^\omega(G))$ corresponding to a pair $(H, B)$ in Theorems 1.1 and 1.2. Then there is a group $G'$, defined up to an isomorphism, such that $\mathcal{L}_{(H,B)}$ is equivalent to $\text{Rep}(G')$ as a symmetric tensor category, where the braiding of $\text{Rep}(G')$ is the trivial one [De]. The group $G'$ can be described explicitly in terms of $G$, $H$, and $B$, see Remark 3.7. Note that $G \not\cong G'$ in general.

There is a canonical subcategory $\mathcal{L}_{((1),1)} \cong \text{Rep}(G)$ corresponding to the forgetful functor $\text{Rep}(D^\omega(G)) \cong \mathcal{Z}(\text{Vec}_G^\omega) \to \text{Vec}_G^\omega$. We have $\mathcal{L}_{((1),1)} \cap \mathcal{L}_{(H,B)} \cong \text{Rep}(G/H)$.

In [Na] the first named author classified indecomposable $\text{Vec}_G^\omega$-module categories $\mathcal{M}$ with the property that the dual fusion category $(\text{Vec}_G^\omega)_{\mathcal{M}}$ is pointed. Such module categories $\mathcal{M}$ can be thought of as categorical analogues of homogeneous spaces. The above property gives rise to an equivalence relation on the set of pairs $(G, \omega)$, where $G$ is a finite group and $\omega \in Z^3(G, k^\times)$ with

\[(G, \omega) \approx (G', \omega') \text{ if and only if } (\text{Vec}_G^\omega)_{\mathcal{M}} \cong \text{Vec}_{G'}^{\omega'} \text{ for some } \mathcal{M}.
\]

In other words, $\text{Vec}_G^\omega$ and $\text{Vec}_{G'}^{\omega'}$ are weakly Morita equivalent in the sense of M. Müger [Mu2].

**Theorem 1.3.** There is a canonical bijection between equivalence classes of indecomposable $\text{Vec}_G^\omega$-module categories $\mathcal{M}$ with respect to which the dual fusion category $(\text{Vec}_G^\omega)_{\mathcal{M}}$ is pointed and Lagrangian subcategories of $\text{Rep}(D^\omega(G))$.

Lagrangian subalgebras of the double of a Lie bialgebra $\mathfrak{g}$ were used by V. Drinfeld in [Dr] to describe Poisson homogeneous spaces of the Poisson-Lie group $G$ corresponding to $\mathfrak{g}$. So
Theorem 1.4. Let $C_1$ and $C_2$ be group-theoretical fusion categories. Then $C_1$ and $C_2$ are weakly Morita equivalent if and only if their centers $Z(C_1)$ and $Z(C_2)$ are braided equivalent.

If $C_1$ and $C_2$ are arbitrary (i.e., not necessarily group-theoretical) weakly Morita equivalent finite tensor categories then it was observed by M. Müger in [Mü2] Remark 3.18 that $Z(C_1)$ is braided tensor equivalent to $Z(C_2)$. Also, $Z(C_1)$ being equivalent to $Z(C_2)$ (even in a non-braided way) implies that $C_1 \boxtimes C_1^{\text{rev}}$ is weakly Morita equivalent to $C_2 \boxtimes C_2^{\text{rev}}$, where $C^{\text{rev}}$ denotes the fusion category obtained by reversing the tensor product in $C$. At the moment of writing we do not know if braided equivalence of centers implies weak Morita equivalence of $C_1$ and $C_2$ in a general case.

Note that it was shown by S. Natale [N] that a fusion category $C$ is group-theoretical if and only if $Z(C)$ is braided equivalent to the representation category of some twisted group double $D^\omega(G)$ (this also follows from [O2]).

Combining the explicit description of weak Morita equivalence classes of pointed categories from [Na] and correspondence between braided equivalences of centers and Lagrangian subcategories from [DGNO] one obtains a complete description of braided equivalences of twisted quantum doubles.

Recall that two finite groups $G_1$ and $G_2$ were called categorically Morita equivalent in [Na] if $\text{Vec}_{G_1}$ and $\text{Vec}_{G_2}$ are weakly Morita equivalent. Let us write $G_1 \approx G_2$ for such groups. It follows from Theorem 1.4 that $G_1 \approx G_2$ if and only if the corresponding Drinfeld doubles $D(G_1)$ and $D(G_2)$ have braided tensor equivalent representation categories. By Theorems 1.1 and 1.3 groups categorically Morita equivalent to a given group $G$ correspond to pairs $(H, B)$, where $H$ is a normal Abelian subgroup of $G$ and $B$ is an alternating bicharacter on $H$. Note that such pairs $(H, B)$ for which $B$ is, in addition, a nondegenerate bicharacter were used by P. Etingof and S. Gelaki in [EG] to describe groups isocategorical to $G$, i.e., such groups $G'$ for which $\text{Rep}(G') \cong \text{Rep}(G)$ as tensor categories. This non-degeneracy condition in [EG] is the reason why the categorical Morita equivalence extends isocategorical equivalence. Indeed, since $\text{Rep}(D(G))$ is determined by the tensor structure of $\text{Rep}(G)$, it is clear that isocategorical groups are categorically Morita equivalent. On the other hand, the first author constructed in [Na] examples of categorically Morita equivalent but non-isocategorical groups.

For an Abelian group $H$ let $\hat{H}$ denote the group of linear characters of $H$.

Corollary 1.5. Let $G, G'$ be finite groups, $\omega \in Z^3(G, k^\times)$, and $\omega' \in Z^3(G', k^\times)$. Then the representation categories of twisted doubles $D^\omega(G)$ and $D^{\omega'}(G')$ are equivalent as braided tensor categories if and only if $G$ contains a normal Abelian subgroup $H$ such the following conditions are satisfied:

1. $\omega|_{H \times H \times H}$ is cohomologically trivial,
2. there is a $G$-invariant (see (6)) 2-cochain $\mu \in C^2(H, k^\times)$ such that that $\delta^2 \mu = \omega|_{H \times H \times H}$, and
3. there is an isomorphism $\alpha : G' \cong \hat{H} \times_\nu (H \backslash G)$ such that $\omega \circ (a \times a \times a)$ and $\omega'$ are cohomologically equivalent.
Here \( \nu \) is a certain 2-cocycle in \( \mathbb{Z}^2(H\setminus G, \hat{H}) \) that comes from the \( G \)-invariance of \( \mu \) and \( \varpi \) is a certain 3-cocycle on \( \hat{H} \rtimes \pi(H\setminus G) \) that depends on \( \nu \) and on the exact sequence \( 1 \to H \to G \to H\setminus G \to 1 \) (see [Na, Theorem 5.8] for precise definitions).

Note that in the special case when \( \omega = 1 \) and \( \mu \) is a non-degenerate \( G \)-invariant alternating 2-cocycle on \( H \), our construction of the “dual” group \( G' \) in Corollary 1.5 becomes the construction of a group isocategorical to \( G \) from [EG]. This can be seen by comparing [Na, 4.2] and [EG, Formula (2)].

1.2. Organization of the paper. Section 2 contains necessary preliminary information about fusion categories, module categories, and modular categories. We also recall definitions and results from [DGNO] concerning Lagrangian subcategories of modular categories.

Section 3 (respectively, Section 4) is devoted to classification of Lagrangian categories of the representation category of the Drinfeld double (respectively, twisted double) of a finite group. The reason we prefer to treat untwisted and twisted cases separately is because our constructions in the former case do not involve rather technical cohomological computations present in the latter. We feel that the reader might get a better understanding of our results by exploring the untwisted case first. Of course when \( \omega = 1 \) the results of Section 4 reduce to those of Section 3.

The Sections 3 and 4 contain proofs of our main results stated above. Theorems 1.1 - 1.4 and Corollary 1.5 correspond to Theorems 3.5, 4.12, 4.17, 4.19, and Corollary 4.20.

Section 5 contains examples in which we compute Lagrangian subcategories of Drinfeld doubles of finite symmetry groups. Here we also show that the four non-equivalent non-pointed fusion categories of dimension 8 with integral dimensions of objects are pairwise weakly Morita non-equivalent, and hence their centers are pairwise non-equivalent as braided tensor categories.

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2. Preliminaries

2.1. Fusion categories and their module categories. A fusion category over \( k \) is a \( k \)-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects, finite-dimensional Hom-spaces, and simple neutral object.

In this paper we only consider fusion categories with integral Frobenius-Perron dimensions of simple objects. It was shown in [ENO, Propositions 8.23, 8.24] that any such category is equivalent to the representation category of a semisimple quasi-Hopf algebra and has a canonical spherical structure with respect to which the categorical dimension of any object is equal to its Frobenius-Perron dimension. In particular, all categorical dimensions are positive integers. For any object \( X \) of a fusion category \( \mathcal{C} \) let \( d(X) \) denote its (Frobenius-Perron) dimension.

By a fusion subcategory of a fusion category we will always mean a full fusion subcategory.
A fusion category is said to be pointed if all its simple objects are invertible. A typical example of a pointed category is $\text{Vec}_G^\otimes$ - the category of finite-dimensional vector spaces over $k$ graded by the finite group $G$. The morphisms in this category are linear transformations that respect the grading and the associativity constraint is given by a 3-cocycle $\omega$ on $G$.

Let $\mathcal{C} = (\mathcal{C}, \otimes, 1_{\mathcal{C}}, \alpha, \lambda, \rho)$ be a tensor category, where $1_{\mathcal{C}}$, $\alpha$, $\lambda$, and $\rho$ are the unit object, the associativity constraint, the left unit constraint, and the right unit constraint, respectively. A right module category over $\mathcal{C}$ is a category $\mathcal{M}$ together with an exact bifunctor $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and natural isomorphisms $\mu_{M,X,Y} : M \otimes (X \otimes Y) \to (M \otimes X) \otimes Y$, $\tau_M : M \otimes 1_{\mathcal{C}} \to M$, for all $M \in \mathcal{M}$, $X, Y \in \mathcal{C}$ such that the following two equations hold for all $M \in \mathcal{M}$, $X, Y, Z \in \mathcal{C}$:

$$
\mu_{M \otimes X, Y, Z} \circ \mu_{M, X, Y \otimes Z} \circ (\text{id}_M \otimes \alpha_{X,Y,Z}) = (\mu_{M, X, Y} \otimes \text{id}_Z) \circ \mu_{M, X \otimes Y, Z},
$$

$$(\tau_M \otimes \text{id}_Y) \circ \mu_{M, 1_{\mathcal{C}}, Y} = \text{id}_M \otimes \lambda_Y.
$$

Note that having a $\mathcal{C}$-module structure on a category $\mathcal{M}$ is the same as having a tensor functor from $\mathcal{C}$ to the (strict) tensor category of endofunctors of $\mathcal{M}$. The coherence conditions on a module action follow automatically from those of a tensor functor.

Let $(\mathcal{M}_1, \mu^1, \tau^1)$ and $(\mathcal{M}_2, \mu^2, \tau^2)$ be two right module categories over $\mathcal{C}$. A $\mathcal{C}$-module functor from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ together with natural isomorphisms $\gamma_{M,X} : F(M \otimes X) \to F(M) \otimes X$, for all $M \in \mathcal{M}_1$, $X \in \mathcal{C}$ such that the following two equations hold for all $M \in \mathcal{M}_1$, $X, Y \in \mathcal{C}$:

$$(\gamma_{M,X} \otimes \text{id}_Y) \circ \gamma_{M,Y \otimes X} \circ F(\mu^1_{M,X,Y}) = \mu^2_{F(M),X,Y} \circ \gamma_{M,Y \otimes X},$$

$$\tau^1_{F(M)} \circ \gamma_{M,1_{\mathcal{C}}} = F(\tau^1_M).$$

Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are equivalent if there exists a module functor from $\mathcal{M}_1$ to $\mathcal{M}_2$ which is an equivalence of categories. For two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over a tensor category $\mathcal{C}$ their direct sum is the category $\mathcal{M}_1 \oplus \mathcal{M}_2$ with the obvious module category structure. A module category is indecomposable if it is not equivalent to a direct sum of two non-trivial module categories.

**Example 2.1. Indecomposable module categories over pointed categories.** Let $G$ be a finite group and $\omega \in Z^3(G, k^\times)$. Indecomposable right module categories over $\text{Vec}_G^\otimes$ correspond to pairs $(H, \mu)$, where $H$ is a subgroup of $G$ such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $\mu \in C^2(H, k^\times)$ is a 2-cochain satisfying $\delta^2 \mu = \omega|_{H \times H \times H}$, i.e.,

$$
\mu(h_2, h_3)\mu(h_1h_2, h_3)^{-1}\mu(h_1, h_2h_3)\mu(h_1, h_2)^{-1} = \omega(h_1, h_2, h_3).
$$

for all $h_1, h_2, h_3 \in H$ (see [11]). Let $\mathcal{M} := \mathcal{M}(H, \mu)$ denote the right module category constructed from the pair $(H, \mu)$. The simple objects of $\mathcal{M}$ are given by the set $H \backslash G$ of right cosets of $H$ in $G$, the action of $\text{Vec}_G^\otimes$ on $\mathcal{M}$ comes from the action of $G$ on $H \backslash G$, and the module category structure isomorphisms are induced from the 2-cochain $\mu$. Let $H, H'$ be subgroups of $G$ such that restrictions of $\omega$ are trivial in $H^3(H, k^\times)$ and $H^3(H', k^\times)$. Two pairs $(H, \mu)$ and $(H', \mu')$, where $\delta^2 \mu = \omega|_{H \times H \times H}$ and $\delta^2 \mu' = \omega|_{H' \times H' \times H'}$ give rise to equivalent $\text{Vec}_G^\otimes$-module categories if and only if there is $g \in G$ such that $H' = gHg^{-1}$ and $\mu$ and the $g$-conjugate of $\mu'$ differ by a coboundary. We will say that two elements of $\{\mu \in C^2(H, k^\times) \mid \delta^2 \mu = \omega|_{H \times H \times H}\}$ are equivalent if they differ by a coboundary. Let

$$
\Omega_{H, \omega} := \text{equivalence classes of } \{\mu \in C^2(H, k^\times) \mid \delta^2 \mu = \omega|_{H \times H \times H}\}.
$$

There is an (in general, non-canonical) bijection between $\Omega_{H, \omega}$ and $H^2(H, k^\times)$, i.e., $\Omega_{H, \omega}$ is a (non-empty) torsor over $H^2(H, k^\times)$. Note that $\Omega_{H, 1} = H^2(H, k^\times)$. 

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Let \( M_1 \) and \( M_2 \) be two right module categories over a tensor category \( C \). Let \((F^1, \gamma^1)\) and \((F^2, \gamma^2)\) be module functors from \( M_1 \) to \( M_2 \). A natural module transformation from \((F^1, \gamma^1)\) to \((F^2, \gamma^2)\) is a natural transformation \( \eta : F^1 \to F^2 \) such that the following equation holds for all \( M \in M_1, X \in C \):

\[
(\eta_M \otimes \text{id}_X) \circ \gamma^1_{M, X} = \gamma^2_{M, X} \circ (\eta_M \otimes \text{id}_X).
\]

Let \( C \) be a tensor category and let \( M \) be a right module category over \( C \). The dual category of \( C \) with respect to \( M \) is the category \( C^*_M := \text{Fun}_C(M, M) \) whose objects are \( C \)-module functors from \( M \) to itself and morphisms are \( C \)-module transformations. The category \( C^*_M \) is a tensor category with tensor product being composition of module functors. It is known that if \( C \) is a fusion category and \( M \) is semisimple \( k \)-linear and indecomposable, then \( C^*_M \) is a fusion category \([ENO]\).

Let \( G \) be a finite group and \( \omega \in Z^3(G, k^\times) \). For each \( x \in G \), define \( \Upsilon_x : G \times G \to k^\times \) by

\[
\Upsilon_x(g_1, g_2) := \frac{\omega(xg_1x^{-1}, xg_2x^{-1}, x)\omega(x, g_1, g_2)}{\omega(xg_1x^{-1}, x, g_2)}, \quad \text{for all } g_1, g_2 \in G.
\]

It is straightforward to verify that \( \delta^2 \Upsilon_x = \frac{\omega_x}{\omega_x} \), for all \( x \in G \), where

\[
\omega_x(g_1, g_2, g_3) = \omega(xg_1x^{-1}, xg_2x^{-1}, xg_3x^{-1}),
\]

for all \( g_1, g_2, g_3 \in G \).

For each \( x \in G \), define \( \nu_x : G \times G \to k^\times \) by

\[
\nu_x(g_1, g_2) := \frac{\omega(g_1, g_2, x)\omega(g_2xg_2^{-1}g_1^{-1}, g_1, g_2)}{\omega(g_1, g_2xg_2^{-1}, g_2)}, \quad \text{for all } g_1, g_2 \in G.
\]

It is easy to verify that the following relation holds:

\[
\frac{\Upsilon_{x_1x_2}(g_1, g_2)}{\Upsilon_{x_1}(x_2g_1x_2^{-1}, x_2g_2x_2^{-1})\Upsilon_{x_2}(g_1, g_2)} = \frac{\nu_{g_1}(x_1, x_2)\nu_{g_2}(x_1, x_2)}{\nu_{g_1g_2}(x_1, x_2)}, \quad \text{for all } x_1, x_2, g_1, g_2 \in G.
\]

Let \( H \) be a normal subgroup of \( G \) such that \( \omega|_{H \times H \times H} \) is cohomologically trivial. For any \( x \in G \) and \( \mu \in C^2(H, k^\times) \) such that \( \delta^2 \mu = \omega|_{H \times H \times H} \), define

\[
\mu \cdot x := \mu^x \times \Upsilon_x|_{H \times H},
\]

where \( \mu^x(h_1, h_2) = \mu(xh_1x^{-1}, xh_2x^{-1}) \), for all \( h_1, h_2 \in H \). It is easy to verify that \( \delta^2(\mu\cdot x) = \omega|_{H \times H \times H} \). This induces an action of \( G \) on \( \Omega_{H, \omega} \) (defined in \([Na]\)). Indeed, that this is an action follows from \((5)\). Let \((\Omega_{H, \omega})^G\) denote the set of \( G \)-invariant elements of \( \Omega_{H, \omega} \), i.e.,

\[
(\Omega_{H, \omega})^G := \left\{ \mu \in \Omega_{H, \omega} \mid \frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H} \text{ is trivial in } H^2(H, k^\times), \text{ for all } x \in G \right\}.
\]

Example 2.2. Module categories over \( \text{Vec}_G^w \) with pointed duals. Let \( G \) be a finite group and \( \omega \in Z^3(G, k^\times) \). It is shown in \([Na] \text{ Theorem 3.4}\) that the set of equivalence classes of indecomposable module categories over \( \text{Vec}_G^w \) such that the dual is pointed is in bijection with the set of pairs \((H, \mu)\), where \( H \) is a normal Abelian subgroup of \( G \) such that \( \omega|_{H \times H \times H} \) is cohomologically trivial and \( \mu \in (\Omega_{H, \omega})^G \) (the description in \([Na] \text{ Theorem 3.4}\) is given in somewhat different but equivalent terms).

Two fusion categories \( C \) and \( D \) are said to be weakly Morita equivalent if there exists an indecomposable (semisimple \( k \)-linear) right module category \( \mathcal{M} \) over \( C \) such that the categories \( C^*_\mathcal{M} \) and \( D \) are equivalent as fusion categories. It was shown by M. Müger \([Mu]\) that this is indeed an equivalence relation.
A fusion category $\mathcal{C}$ is said to be group theoretical if it is weakly Morita equivalent to a pointed category.

2.2. Modular categories and centralizers. Let $\mathcal{C}$ be a modular fusion category with braiding $c$, twist $\theta$, and S-matrix $S$ (see [BK]). Let $\mathcal{D}$ be a full (not necessarily tensor) subcategory of $\mathcal{C}$. Its dimension is defined by $\dim(\mathcal{D}) := \sum_{X \in \text{Irr}(\mathcal{D})} d(X)^2$, where $\text{Irr}(\mathcal{D})$ is the set of isomorphism classes of simple objects in $\mathcal{D}$. In [Mu2], M. M"uger introduced the notion of the centralizer of $\mathcal{D}$ in $\mathcal{C}$ as the fusion subcategory $\mathcal{D}' := \{ X \in \mathcal{C} \mid c(Y, X) \circ c(X, Y) = \text{id}_{X \otimes Y}, \text{ for all } Y \in \mathcal{D}\}.$ It was also shown in [Mu2] that if $\mathcal{D}$ is a fusion subcategory then $\mathcal{D}'' = \mathcal{D}$ and

$$\dim(\mathcal{D}) \cdot \dim(\mathcal{D}') = \dim(\mathcal{C}).$$

(7)

Following M. M"uger, we will say that two objects $X, Y \in \mathcal{C}$ centralize each other if $c(Y, X) \circ c(X, Y) = \text{id}_{X \otimes Y}$. For simple $X$ and $Y$ this condition is equivalent to $S(X, Y) = d(X)d(Y)$ [Mu2, Corollary 2.14].

Remark 2.3. If $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ such that all objects in $\mathcal{D}$ centralize each other, i.e., $\mathcal{D} \subseteq \mathcal{D}'$ then $\dim(\mathcal{D})^2 \leq \dim(\mathcal{C})$. Indeed, we have $\dim(\mathcal{D}) \leq \dim(\mathcal{D}')$ and so it follows from (7) that $\dim(\mathcal{D})^2 \leq \dim(\mathcal{C})$. In particular, if $\mathcal{D}$ is a symmetric fusion subcategory of $\mathcal{C}$, then $\dim(\mathcal{D})^2 \leq \dim(\mathcal{C})$.

Lemma 2.4. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ (which is not a priori assumed to be closed under the tensor product or duality) such that $\mathcal{D} \subseteq \mathcal{D}'$. Then the fusion subcategory $\tilde{\mathcal{D}} \subseteq \mathcal{C}$ generated by $\mathcal{D}$ is symmetric.

Proof. We may assume that $\mathcal{D}$ is closed under taking duals. Indeed, it follows from [ENO, Proposition 2.12] that $X$ centralizes $Y$ if and only if $X$ centralizes $Y^*$ for any two simple objects $X, Y \in \mathcal{C}$. Let $Z_1, Z_2$ be simple objects in $\tilde{\mathcal{D}}$. There exist simple objects $X_1, X_2, Y_1, Y_2$ in $\mathcal{D}$ such that $Z_1$ is contained in $X_1 \otimes Y_1$ and $Z_2$ is contained in $X_2 \otimes Y_2$. By [Mu2, Lemma 2.4 (i)], it follows that $Z_1$ centralizes $X_2 \otimes Y_2$, and hence $Z_1, Z_2$ centralize each other.

Corollary 2.5. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ such that $\mathcal{D} \subseteq \mathcal{D}'$ and $\dim(\mathcal{D})^2 = \dim(\mathcal{C})$. Then $\mathcal{D}$ is a symmetric fusion subcategory.

2.3. Lagrangian subcategories and braided equivalences of twisted group doubles. Let $\mathcal{C}$ be a modular category. Recall that we chose the canonical spherical twist for $\mathcal{C}$ with respect to which the categorical dimension of any object of $\mathcal{C}$ is equal to its Frobenius-Perron dimension. This is possible by [ENO, Proposition 8.23, 8.24]. Let us recall some definitions and results from [DGNO].

Definition 2.6. A fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ is said to be isotropic if the twist of $\mathcal{C}$ restricts to identity on $\mathcal{D}$. An isotropic subcategory $\mathcal{D} \subseteq \mathcal{C}$ is said to be Lagrangian if $(\dim(\mathcal{D}))^2 = \dim(\mathcal{C})$.

An isotropic subcategory $\mathcal{D}$ of $\mathcal{C}$ is necessarily symmetric and its objects have positive categorical dimensions. It follows from [De] that there is a (unique up to an isomorphism)
Consider the set of all braided tensor equivalences $F : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{P})$, where $\mathcal{P}$ is a pointed fusion category and $\mathcal{Z}(\mathcal{P})$ denotes its center. There is an equivalence relation on this set defined as follows. We say that $F_1 : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{P}_1)$ and $F_2 : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{P}_2)$ are equivalent if there exists a tensor equivalence $\iota : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ such that $\mathcal{F}_2 \circ F_2 = \iota \circ \mathcal{F}_1 \circ F_1$, where $\mathcal{F}_i : \mathcal{Z}(\mathcal{P}_i) \rightarrow \mathcal{P}_i$, $i = 1, 2$, are the canonical forgetful functors. Let $E(\mathcal{C})$ be the collection of equivalence classes of such equivalences. Informally, $E(\mathcal{C})$ is the set of all “different” braided equivalences between $\mathcal{C}$ and centers of pointed categories, i.e., representation categories of twisted group doubles.

Let $\text{Lagr}(\mathcal{C})$ be the set of all Lagrangian subcategories of $\mathcal{C}$.

In [DGNO] Theorem 4.5 it was proved that there is a bijection

$$f : E(\mathcal{C}) \rightarrow \text{Lagr}(\mathcal{C})$$

defined as follows. Note that each braided tensor equivalence $F : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{P})$ gives rise to the Lagrangian subcategory $f(F)$ of $\mathcal{C}$ formed by all objects sent to multiples of the unit object $1$ under the forgetful functor $\mathcal{Z}(\mathcal{P}) \rightarrow \mathcal{P}$. This subcategory is clearly the same for all equivalent choices of $F$.

In particular, the center of a fusion category $\mathcal{D}$ contains a Lagrangian subcategory if and only if $\mathcal{D}$ is group-theoretical [DGNO].

2.4. The Schur multiplier of an Abelian group. Let $H$ be a normal Abelian subgroup of a finite group $G$. Let $\Lambda^2 H$ denote the Abelian group of alternating bicharacters on $H$, i.e.,

$$\Lambda^2 H := \left\{ B : H \times H \rightarrow k^\times \left| \begin{array}{l}
B(h_1 h_2, h) = B(h_1, h)B(h_2, h), \\
B(h, h_1 h_2) = B(h, h_1)B(h, h_2), \\
B(h, h) = 1, \text{ for all } h, h_1, h_2 \in H
\end{array} \right. \right\}.$$ 

Let $Z^2(H, k^\times)$ be the group of 2-cocycles on $H$. Define a homomorphism $\text{alt} : Z^2(H, k^\times) \rightarrow \Lambda^2 H : \mu \rightarrow \text{alt}(\mu)$ by

$$\text{alt}(\mu)(h_1, h_2) := \frac{\mu(h_2, h_1)}{\mu(h_1, h_2)}, \quad h_1, h_2 \in H.$$ 

It is well known that $\text{alt}$ induces an isomorphism between the Schur multiplier $H^2(H, k^\times)$ of $H$ and $\Lambda^2 H$. By abuse of notation we denote this isomorphism also by $\text{alt}$:

$$\text{alt} : H^2(H, k^\times) \simeq \Lambda^2 H.$$ 

Note that both $H^2(H, k^\times)$ and $\Lambda^2 H$ are right $G$-modules via the conjugation and that $\text{alt}$ is $G$-linear.

3. Lagrangian subcategories in the untwisted case

We fix notation for this Section. Let $G$ be a finite group. For any $g \in G$, let $K_g$ denote the conjugacy class of $G$ containing $g$. Let $R$ denote a complete set of representatives of conjugacy classes of $G$. Let $\mathcal{C}$ denote the representation category $\text{Rep}(D(G))$ of the Drinfeld double of the group $G$:

$$\mathcal{C} := \text{Rep}(D(G)).$$

The category $\mathcal{C}$ is equivalent to $\mathcal{Z}($Vec$)_G$, the center of Vec$_G$. It is well known that $\mathcal{C}$ is a modular category. Let $\Gamma$ denote a complete set of representatives of simple objects of $\mathcal{C}$. The
set \( \Gamma \) is in bijection with the set \( \{(a, \chi) \mid a \in R \text{ and } \chi \text{ is an irreducible character of } C_G(a)\} \),
where \( C_G(a) \) is the centralizer of \( a \) in \( G \) (see [CGR]). In what follows we will identify \( \Gamma \) with the previous set,
\[
(10) \quad \Gamma := \{(a, \chi) \mid a \in R \text{ and } \chi \text{ is an irreducible character of } C_G(a)\}.
\]
Let \( S \) and \( \theta \) be (see, e.g. [BK], [CGR]) the \( S \)-matrix and twist, respectively, of \( C \). Recall that we take the canonical twist. It is known that the entries of the \( S \)-matrix lie in a cyclotomic field. Also, the values of characters of a finite group are sums of roots of unity.

So we may assume that all scalars appearing herein are complex numbers; in particular, complex conjugation and absolute values make sense. We have the following formulas for the \( S \)-matrix, twist and dimensions:
\[
S((a, \chi), (b, \chi')) = \frac{|G|}{|C_G(a)||C_G(b)|} \sum_{g \in G(a,b)} \overline{\chi(gbg^{-1})} \chi'(g^{-1}ag),
\]
\[
\theta(a, \chi) = \frac{\chi(a)}{\deg \chi},
\]
\[
d((a, \chi)) = |K_a| \deg \chi = \frac{|G|}{|C_G(a)|} \deg \chi,
\]
for all \((a, \chi), (b, \chi') \in \Gamma\), where \( G(a, b) = \{ g \in G \mid agbg^{-1} = gbg^{-1}a \}\).

3.1. **Classification of Lagrangian subcategories of** \( \text{Rep}(D(G)) \).

**Lemma 3.1.** Two objects \((a, \chi), (b, \chi') \in \Gamma\) centralize each other if and only if the following conditions hold:

(i) The conjugacy classes \( K_a, K_b \) commute element-wise,

(ii) \( \chi(gbg^{-1}) \chi'(g^{-1}ag) = \deg \chi \deg \chi' \), for all \( g \in G \).

**Proof.** By [Mn2] Corollary 2.14 two objects \((a, \chi), (b, \chi') \in \Gamma\) centralize each other if and only if
\[
S((a, \chi), (b, \chi')) = \deg \chi \deg \chi'.
\]
This is equivalent to the equation
\[
(11) \quad \sum_{g \in G(a,b)} \chi(gbg^{-1}) \chi'(g^{-1}ag) = |G| \deg \chi \deg \chi',
\]
where \( G(a, b) = \{ g \in G \mid agbg^{-1} = gbg^{-1}a \}\). It is clear that if the two conditions of the Lemma hold, then (11) holds since \( G(a, b) = G \).

Now suppose that (11) holds. We will show that this implies the two conditions in the statement of the Lemma. We have
\[
|G| \deg \chi \deg \chi' = \sum_{g \in G(a,b)} |\chi(gbg^{-1})| \chi'(g^{-1}ag)|
\]
\[
\leq \sum_{g \in G(a,b)} \chi'(g^{-1}ag)|\chi'(g^{-1}ag)|
\]
\[
\leq |G| \deg \chi \deg \chi'.
\]
So \( \sum_{g \in G(a,b)} |\chi(gbg^{-1})| \chi'(g^{-1}ag)| = |G| \deg \chi \deg \chi' \). Since
\[
|G(a, b)| \leq |G|, \quad |\chi(gbg^{-1})| \leq \deg \chi, \text{ and } |\chi'(g^{-1}ag)| \leq \deg \chi',
\]

So \( \sum_{g \in G(a,b)} |\chi(gbg^{-1})| \chi'(g^{-1}ag)| = |G| \deg \chi \deg \chi'. \) Since
\[
|G(a, b)| \leq |G|, \quad |\chi(gbg^{-1})| \leq \deg \chi, \text{ and } |\chi'(g^{-1}ag)| \leq \deg \chi',
\]
we must have $G(a, b) = G$, $|\chi(gbg^{-1})| = \deg \chi$, and $|\chi'(g^{-1}ag)| = \deg \chi'$. The equality $G(a, b) = G$ implies that the conjugacy classes $K_a, K_b$ commute element-wise, which is Condition (i) in the statement of the Lemma. Since $|\chi(gbg^{-1})| = \deg \chi$, and $|\chi'(g^{-1}ag)| = \deg \chi'$, there exist roots of unity $\alpha_g$ and $\beta_g$ such that $\chi(gbg^{-1}) = \alpha_g \deg \chi$, and $\chi'(g^{-1}ag) = \beta_g \deg \chi'$, for all $g \in G$. Put this in (11) to get the equation

$$\sum_{g \in G} \alpha_g \beta_g = |G|.$$  

(12)

Note that (12) holds if and only if $\alpha_g \beta_g = 1$, for all $g \in G$. This is equivalent to saying that $\chi(gbg^{-1}) \chi'(g^{-1}ag) = \deg \chi \deg \chi'$, for all $g \in G$ and the Lemma is proved. 

**Lemma 3.2.** Let $E$ be a normal subgroup of a finite group $K$. Let $\text{Irr}(K)$ denote the set of irreducible characters of $K$. Let $\rho$ be a $K$-invariant character of $E$ of degree 1. Then

$$\sum_{\chi \in \text{Irr}(K) : \chi|_E = (\deg \chi) \rho} (\deg \chi)^2 = \frac{|K|}{|E|}.$$  

Proof. Suppose $\chi$ is any irreducible character of $K$. Since $\rho$ is $K$-invariant, by Clifford’s Theorem, if $\rho$ is an irreducible constituent of $\chi|_E$, then

$$\chi|_E = (\deg \chi) \rho.$$  

By Frobenius reciprocity, the multiplicity of any irreducible $\chi$ in $\text{Ind}_E^K \rho$ is equal to the multiplicity of $\rho$ in $\chi|_E$. The latter is equal to $\deg \chi$ if $\chi$ satisfies (3.1) and 0 otherwise. Therefore,

$$\sum_{\chi \in \text{Irr}(K) : \chi|_E = (\deg \chi) \rho} (\deg \chi)^2 = \deg \text{Ind}_E^K \rho = \frac{|K|}{|E|},$$  

as required. 

Let $H$ be a normal Abelian subgroup of $G$ and let $B$ be a $G$-invariant alternating bicharacter on $H$. Then $H = \bigcup_{a \in H \cap R} K_a$. Let

$$\mathcal{L}_{(H, B)} := \text{full Abelian subcategory of } \mathcal{C} \text{ generated by }$$

$$(a, \chi) \in \Gamma \bigg| \ a \in H \cap R \text{ and } \chi \text{ is an irreducible character of } C_G(a) \text{ such that } \chi(h) = B(a, h) \deg \chi, \text{ for all } h \in H \bigg\}.$$  

(13)

**Proposition 3.3.** The subcategory $\mathcal{L}_{(H, B)} \subseteq \text{Rep}(D(G))$ is Lagrangian.

Proof. We have

$$\chi(gbg^{-1}) \chi'(g^{-1}ag) = B(a, gbg^{-1}) \deg \chi B(b, g^{-1}ag) \deg \chi'$$

$$= B(a, gbg^{-1}) B(gbg^{-1}, a) \deg \chi \deg \chi'$$

$$= \deg \chi \deg \chi',$$

for all $(a, \chi), (b, \chi') \in \mathcal{L}_{(H, B)} \cap \Gamma, g \in G$. The second equality above is due to $G$-invariance of $B$ and the third equality holds since $B$ is alternating. By Lemma 3.1, it follows that objects in $\mathcal{L}_{(H, B)}$ centralize each other.

Also, we have $\theta(a, \chi) = \frac{\chi(a)}{\deg \chi} = \frac{B(a, a)}{\deg \chi} \deg \chi = 1$, for all $(a, \chi) \in \mathcal{L}_{(H, B)} \cap \Gamma$. Therefore, $\theta|_{\mathcal{L}_{(H, B)}} = \text{id}$. 

The dimension of $\mathcal{L}(H, B)$ is equal to $|G|$. Indeed,
\[
\dim(\mathcal{L}(H, B)) = \sum_{(a, \chi) \in \mathcal{L}(H, B) \cap \Gamma} d(a, \chi)^2 \\
= \sum_{(a, \chi) \in \mathcal{L}(H, B) \cap \Gamma} |K_a|^2 (\deg \chi)^2 \\
= \sum_{a \in H \cap R} |K_a|^2 \sum_{\chi : (a, \chi) \in \mathcal{L}(H, B) \cap \Gamma} (\deg \chi)^2 \\
= \sum_{a \in H \cap R} |K_a|^2 \left( \frac{|C_G(a)|}{|H|} \right) \\
= \frac{|G|}{|H|} \sum_{a \in H \cap R} |K_a| \\
= |G|.
\]

The fourth equality above is explained as follows. Fix $a \in H \cap R$. Define $\rho : H \to k^\times$ by $\rho(h) := B(a, h)$. Observe that $\rho$ is a $C_G(a)$-invariant character of $H$ of degree 1 and then apply Lemma 3.2.

It follows from Lemma 2.4 that $\mathcal{L}(H, B)$ is a Lagrangian subcategory of $\text{Rep}(D(G))$ and the Proposition is proved.

Now, let $\mathcal{L}$ be a Lagrangian subcategory of $\mathcal{C}$. So, in particular, the two conditions in Lemma 3.1 hold for all simple objects in $\mathcal{L}$. Define
\[
(14) \quad H_\mathcal{L} := \bigcup_{a \in R : (a, \chi) \in \mathcal{L} \text{ for some } \chi} K_a.
\]

Note that $H_\mathcal{L}$ is a normal Abelian subgroup of $G$. Indeed, that $H_\mathcal{L}$ is a subgroup follows from the fact that $\mathcal{L}$ contains the unit object and is closed under tensor products. The subgroup $H_\mathcal{L}$ is normal in $G$ because it is a union of conjugacy classes of $G$. Finally, that $H_\mathcal{L}$ is Abelian follows by Condition (i) of Lemma 3.1.

For each $a \in H \cap R$, define $\xi_a : H_\mathcal{L} \to k^\times$ by
\[
\xi_a(h) := \frac{\chi(h)}{\deg \chi},
\]
for $h \in H_\mathcal{L}$, where $\chi$ is any irreducible character of $C_G(a)$ such that $(a, \chi) \in \mathcal{L} \cap \Gamma$. To see that this definition does not depend on the choice of $\chi$, let $(a, \chi), (a, \chi'), (b, \chi'') \in \mathcal{L} \cap \Gamma$ and apply Condition (ii) of Lemma 3.1 to pairs $(a, \chi), (b, \chi'')$ and $(a, \chi'), (b, \chi'')$ to get
\[
\frac{\chi(gbg^{-1})}{\deg \chi} = \left( \frac{\chi''(g^{-1}ag)}{\deg \chi''} \right)^{-1} \quad \text{and} \quad \frac{\chi'(gbg^{-1})}{\deg \chi'} = \left( \frac{\chi''(g^{-1}ag)}{\deg \chi''} \right)^{-1},
\]
for all $g \in G$. This implies that $\frac{\chi|_{\deg \chi}}{\deg \chi} = \frac{\chi'|_{\deg \chi}}{\deg \chi'}$, for any two pairs $(a, \chi), (a, \chi') \in \mathcal{L} \cap \Gamma$.

For any $a, b \in H_\mathcal{L} \cap R$, by Condition (ii) of Lemma 3.1 $\xi_a$ and $\xi_b$ satisfy the equation:
\[
(15) \quad \xi_a(gbg^{-1}) = \xi_b(g^{-1}ag)^{-1}, \quad \text{for all } g \in G.
\]

Define a map $B_\mathcal{L} : H_\mathcal{L} \times H_\mathcal{L} \to k^\times$ by
\[
(16) \quad B_\mathcal{L}(h_1, h_2) := \xi_a(g^{-1}h_2g),
\]
where \( h_1 = gag^{-1}, g \in G, a \in H_L \cap R \).

**Proposition 3.4.** \( B_L \) is a well-defined \( G \)-invariant alternating bicharacter on \( H_L \).

**Proof.** First, let us show that \( B_L \) is well-defined. Suppose \( gag^{-1} = kak^{-1} \), where \( a \in H_L \cap R, g, k \in G \). Then

\[
B_L(gag^{-1}, lbl^{-1}) = \xi_a((g^{-1}l)b(g^{-1}l))^{-1} \\
= \xi_b((g^{-1}l)^{-1}a(g^{-1}l))^{-1} \\
= \xi_b(l^{-1}(gag^{-1})l)^{-1} \\
= \xi_b(l^{-1}(kak^{-1})l)^{-1} \\
= \xi_a((l^{-1}k)^{-1}b(l^{-1}k)) \\
= \xi_a(k^{-1}(lb)^{-1}k) \\
= B_L(kak^{-1}, lbl^{-1}),
\]

for all \( b \in H_L \cap R, l \in G \). The second and the fifth equalities above are due to (15).

Let \( h_1 = kak^{-1}, h_2 \in H_L, g \in G \), where \( a \in H_L \cap R, k \in G \). Then

\[
B_L(gh_1g^{-1}, gh_2g^{-1}) = B_L(gkak^{-1}g^{-1}, gh_2g^{-1}) \\
= \xi_a((gk)^{-1}(gh_2g^{-1})(gk)) \\
= \xi_a(k^{-1}h_2k) \\
= B_L(kak^{-1}, h_2) \\
= B_L(h_1, h_2).
\]

So, \( B_L \) is \( G \)-invariant.

Now,

\[
B_L(gag^{-1}, gag^{-1}) = B_L(a, a) \\
= \xi_a(a) \\
= \frac{\chi(a)}{\deg \chi} \\
= \theta_{(a, \chi)} \\
= 1,
\]

for all \( a \in H_L \cap R, g \in G \). The first equality above is due to the \( G \)-invariance of \( B_L \). So \( B_L(h, h) = 1 \), for all \( h \in H_L \).

Also, \( B_L(g_1ag_1^{-1}, g_2bg_2^{-1})B_L(g_2bg_2^{-1}, g_1ag_1^{-1}) = \xi_a(g_1^{-1}g_2bg_2^{-1}g_1)\xi_b(g_2^{-1}g_1ag_1^{-1}g_2) = 1 \), for all \( g_1, g_2 \in G, a, b \in H \cap R \). We used (15) in the last equality.

To see that \( B_L \) is a bicharacter, observe first that \( \xi_a \) is a homomorphism, for all \( a \in H_L \cap R \). We have

\[
B_L(gag^{-1}, h_1)B_L(gag^{-1}, h_2) = \xi_a(g^{-1}h_1g)\xi_a(g^{-1}h_2g) \\
= \xi_a(g^{-1}h_1h_2g) \\
= B_L(gag^{-1}, h_1h_2),
\]

for all \( a \in H_L \cap R, g \in G, h_1, h_2 \in H_L \). We conclude that \( B_L \) is a \( G \)-invariant alternating bicharacter on \( H_L \) and the Proposition is proved.
Recall that Lagr(\mathcal{C}) denotes the set of Lagrangian subcategories of a modular category \mathcal{C}.

**Theorem 3.5.** Lagrangian subcategories of the representation category of the Drinfeld double \(D(G)\) are classified by pairs \((H, B)\), where \(H\) is a normal Abelian subgroup of \(G\) and \(B\) is an alternating \(G\)-invariant bicharacter on \(H\).

**Proof.** Let \(\mathcal{E} := \{(H, B) \mid H\) is a normal Abelian subgroup of \(G\) and \(B \in (\Lambda^2 H)^G\}\). Define a map \(\Psi: \mathcal{E} \to \text{Lagr}(\mathcal{C}): (H, B) \mapsto \mathcal{L}_{(H, B)}\), where \(\mathcal{C} = \text{Rep}(D(G))\) and \(\mathcal{L}_{(H, B)}\) is defined in (13). It was shown in Proposition 3.3 that \(\mathcal{L}_{(H, B)}\) is a Lagrangian subcategory.

To see that \(\Psi\) is injective pick any \((H, B), (H', B') \in \mathcal{E}\) and assume that \(\Psi((H, B)) = \Psi((H', B'))\). So in particular we will have \(\mathcal{L}_{(H, B)} \cap \Gamma = \mathcal{L}_{(H', B')} \cap \Gamma\). Note that \(H = \bigcup_{(a, \chi) \in \mathcal{L}_{(H, B)} \cap \Gamma} K_a\) and \(H' = \bigcup_{(a, \chi) \in \mathcal{L}_{(H', B')} \cap \Gamma} K_a\). Since \(\mathcal{L}_{(H, B)} \cap \Gamma = \mathcal{L}_{(H', B')} \cap \Gamma\), it follows that \(H = H'\). Also note that for any \((a, \chi) \in \mathcal{L}_{(H, B)} \cap \Gamma = \mathcal{L}_{(H', B')} \cap \Gamma\), we have \(\chi(h) = B(a, h) \deg \chi = B'(a, h) \deg \chi\), for all \(h \in H = H'\). Since \(B, B'\) are \(G\)-invariant, it follows that \(B = B'\). So \(\Psi\) is injective.

To see that \(\Psi\) is surjective pick any \(\mathcal{L} \in \text{Lagr}(\mathcal{C})\). Consider the pair \((H_{\mathcal{L}}, B_{\mathcal{L}})\), where \(H_{\mathcal{L}}\) and \(B_{\mathcal{L}}\) are defined in (14) and (16), respectively. Proposition 3.3 showed that \((H_{\mathcal{L}}, B_{\mathcal{L}})\) belongs to the set \(\mathcal{E}\). We contend that \(\Psi((H_{\mathcal{L}}, B_{\mathcal{L}})) = \mathcal{L}\). It suffices to show that \(\mathcal{L} \cap \Gamma \subseteq \mathcal{L}_{(H_{\mathcal{L}}, B_{\mathcal{L}})}\). But this hold by definition of \(\mathcal{L}_{(H_{\mathcal{L}}, B_{\mathcal{L}})}\) and the observation that \(\frac{\chi|_{H_{\mathcal{L}}}}{\deg \chi} = \frac{\chi|_{H_{\mathcal{L}}}}{\deg \chi}\), for any two pairs \((a, \chi), (a, \chi') \in \mathcal{L} \cap \Gamma\), \(a \in H_{\mathcal{L}} \cap \Gamma\). So \(\Psi\) is surjective and the Theorem is proved.

3.2. Bijective correspondence between Lagrangian subcategories and module categories with pointed duals. Let \(\mathcal{D}\) be a fusion category and let \(\mathcal{M}\) be an indecomposable \(\mathcal{D}\)-module category. There is a canonical braided tensor equivalence \([\text{Mu1}, \text{EO}]\)

\[
i_M: \mathcal{Z}(\mathcal{D}) \sim \mathcal{Z}(\mathcal{D}_\mathcal{M}^*)
\]

defined by identifying both centers with the category of \(\mathcal{D} \boxtimes (\mathcal{D}_\mathcal{M}^*)^{\text{rev}}\)-module endofunctors of \(\mathcal{M}\).

Let \(f: E(\mathcal{C}) \sim \text{Lagr}(\mathcal{C})\) be the bijection between the set of (equivalence classes of) braided tensor equivalences between \(\mathcal{C}\) and centers of pointed fusion categories and the set of Lagrangian subcategories of \(\mathcal{C}\) defined in \([\text{DGNO}]\), see \([\text{S}]\).

**Theorem 3.6.** The assignment \(\mathcal{M} \mapsto \iota_\mathcal{M}\) restricts to a bijection between the set of equivalence classes of indecomposable \(\text{Vec}_G\)-module categories \(\mathcal{M}\) with respect to which the dual fusion category \((\text{Vec}_G)^*_\mathcal{M}\) is pointed and \(E(\text{Rep}(D(G)))\).

**Proof.** Comparing the result of \([\text{Na}]\) (see Example 2.2) and Theorem 3.5 and taking into account that the isomorphism \(\text{alt}: H^2(H, k^\times) \sim (\Lambda^2 H)\) is \(G\)-linear, we see that the two sets in question have the same cardinality. Thus, to prove the theorem it suffices to check that for \(\mathcal{M} := \mathcal{M}(H, \mu)\) one has \(f(\iota_\mathcal{M}) \subseteq \mathcal{L}_{(H, \text{alt}(\mu))}\), where \(\mathcal{L}_{(H, \text{alt}(\mu))}\) is the Lagrangian subcategory defined in (13).

By definition, \(f(\iota_\mathcal{M})\) consists of all objects \(Z\) in \(\mathcal{C} = \mathcal{Z}(\text{Vec}_G)\) (identified with \(\text{Rep}(D(G))\)) such that the \(\text{Vec}_G\)-module endofunctor \(F_Z: \mathcal{M} \to \mathcal{M}: M \mapsto M \otimes Z\) is isomorphic to a multiple of \(\text{id}_\mathcal{M}\). Note that here we abuse notation and write \(Z\) for both object of the center and its forgetful image.

Let us recall the parameterization of simple objects of \(\mathcal{Z}(\text{Vec}_G)\) in (10). Suppose that a simple \(Z\) corresponds to the conjugacy class \(K_a\) represented by \(a \in R\) and the character afforded by the irreducible representation \(\pi: C_G(a) \to \text{GL}(V_\pi)\). Then as a \(G\)-graded vector
space $Z = \bigoplus_{x \in K_a} V^x_\pi$ and the permutation isomorphism $c_{g, Z} : g \otimes Z \xrightarrow{\sim} Z \otimes g$ is induced from $\pi$, where we identify simple objects of $\text{Vec}_G$ with the elements of the group $G$.

It is clear that $F_Z$ is isomorphic to a multiple of $\text{id}_M$ as an ordinary functor if and only if $K_a \subseteq H$. Note that this implies that $H \subseteq C_G(a)$. Note that for every $\text{Vec}_G$-module functor $F : \mathcal{M} \to \mathcal{M}$ the module structure on $F$ is completely determined by the collection of isomorphisms $F(\text{id}_H \otimes h) \xrightarrow{\sim} F(\text{id}_H) \otimes h$, $h \in H$, where $\text{id}_H$ denotes the trivial coset in $H \backslash G = \text{Irr}(\mathcal{M})$.

For $F = F_Z$ the latter isomorphism is given by the composition

$$(H^1 \otimes h) \otimes Z \xrightarrow{\oplus_{x, h} (h, x)^{-1} \text{id}_V} H^1 \otimes (h \otimes Z) \xrightarrow{\text{id}_{H^1} \otimes c_{h, Z}} H^1 \otimes (Z \otimes h) \xrightarrow{\oplus_{x, h} \mu(x, h) \text{id}_V} (H \otimes Z) \otimes h.$$ 

The restriction of $c_{h, Z}$ to $h \otimes V^a_\pi$ is given by $\pi(h)$, for all $h \in C_G(a)$. If the above composition equals identity, then $\pi(h) = \text{alt}(\mu)(a, h)$ $\text{id}_V$, for all $h \in H$. So $Z \in \mathcal{L}(H, \text{alt}(\mu))$ and, therefore, $f(\mathcal{M}) \subseteq \mathcal{L}(H, \text{alt}(\mu))$, as required.

**Remark 3.7.** Let us explicitly describe the subcategory $\mathcal{L}(H, B)$. By [De] there is a unique up to an isomorphism group $G'$ such that $\mathcal{L}(H, B) \cong \text{Rep}(G')$ as a symmetric category. This group $G'$ is precisely the group of invertible objects in the dual category $(\text{Vec}_G)^*_{\mathcal{M}(H, \mu)}$, where $\mu \in Z^2(H, k^\times)$ is such that $\text{alt}(\mu) = B$. It was shown in [Na] that $G'$ is an extension

$$0 \to \hat{H} \to G' \to H \backslash G \to 0,$$

with the corresponding second cohomology class being the image of the cohomology class of $\mu$ under the canonical homomorphism $H^2(H, k^\times)^G \to H^2(H \backslash G, \hat{H})$, see [Na] for details. Note that in general $G \not\cong G'$, see Section 5 for examples.

### 4. Lagrangian Subcategories in the Twisted Case

In this Section we extend the constructions of the previous Section when the associativity is given by a 3-cocycle $\omega \in Z^3(G, k^\times)$. Note that the results of this Section reduce to the results in Section 3 when $\omega \equiv 1$.

For this Section we follow the notation fixed at the beginning of Section 3. Let $\omega$ be a normalized 3-cocycle on $G$, i.e., $\omega$ is a map from $G \times G \times G$ to $k^\times$ satisfying:

$$\omega(g_2, g_3, g_4)\omega(g_1, g_2g_3, g_4)\omega(g_1, g_2, g_3) = \omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4),$$

$$\omega(g, 1_G, l) = 1,$$

for all $g, l, g_1, g_2, g_3, g_4 \in G$.

Let $\mathcal{C}$ denote the representation category $\text{Rep}(D^\omega(G))$ of the twisted quantum double of the group $G$ [DPR1, DPR2]:

$$\mathcal{C} := \text{Rep}(D^\omega(G)).$$

The category $\mathcal{C}$ is equivalent to $\mathcal{Z}(\text{Vec}_G^\omega)$. It is well known that $\mathcal{C}$ is a modular category. Replacing $\omega$ by a cohomologous 3-cocycle we may assume that the values of $\omega$ are roots of unity.

For all $a, g, h \in G$, define

$$\beta_a(h, g) := \omega(a, h, g)\omega(h, h^{-1}ah, g)^{-1}\omega(h, g, (hg)^{-1}ahg).$$

The $\beta_a$’s satisfy the following equation:

$$\beta_a(x, y)\beta_a(xy, z) = \beta_a(x, yz)\beta_{x^{-1}ax}(y, z), \quad \text{for all } x, y, z \in G.$$
Observe that the restriction of each $\beta_a$ to the centralizer $C_G(a)$ of $a$ in $G$ is a normalized 2-cocycle. Let $\Gamma$ denote a complete set of representatives of simple objects of $\mathcal{C}$. The set $\Gamma$ is in bijection with the set $\{(a, \chi) \mid a \in R$ and $\chi$ is an irreducible $\beta_a$-character of $C_G(a)\}$. In what follows we will identify $\Gamma$ with the previous set:

$$\Gamma := \{(a, \chi) \mid a \in R$ and $\chi$ is an irreducible $\beta_a$-character of $C_G(a)\}.$$  

Let $S$ and $\theta$ be the $S$-matrix and twist, respectively, of $\mathcal{C}$. It is known that the entries of the $S$-matrix lie in a cyclotomic field. Also, the values of $\alpha$-characters of a finite group are sums of roots of unity, so they are algebraic numbers, where $\alpha$ is any 2-cocycle whose values are roots of unity. So we may assume that all scalars appearing herein are complex numbers; in particular, complex conjugation and absolute values make sense. We have the following formulas for the $S$-matrix, twist, and dimensions (see [CGR]):

$$S((a, \chi), (b, \chi')) = \sum_{g \in K_a, g' \in K_b \cap C_G(g)} \frac{\beta_a(x, g')\beta_a(xg', x^{-1})\beta_b(y, g)\beta_b(yg, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \chi(xg'x^{-1}) \chi'(ygy^{-1}),$$

$$\theta(a, \chi) = \frac{\chi(a)}{\deg \chi},$$

$$d((a, \chi)) = |K_a| \deg \chi = \frac{|G|}{|C_G(a)|} \deg \chi,$$

for all $(a, \chi), (b, \chi') \in \Gamma$, where $g = x^{-1}ax, g' = y^{-1}by$.

### 4.1. Classification of Lagrangian subcategories of $\text{Rep}(D^\omega(G))$.

**Remark 4.1.** Let $\rho : K \to GL(V)$ be a finite-dimensional projective representation with 2-cocycle $\alpha$ on the finite group $K$, i.e., $\rho(xy) = \alpha(x, y)\rho(x)\rho(y)$, for all $x, y \in K$. Let $\chi$ be the projective character afforded by $\rho$, i.e., $\chi(x) = \text{Trace}(\rho(x))$, for all $x \in K$. Suppose that the values of $\alpha$ are roots of unity. Then $|\chi(x)| \leq \deg \chi$, for all $x \in K$ and we have equality if and only if $\rho(x) \in k^\times \cdot \text{id}_V$.

**Lemma 4.2.** Two objects $(a, \chi), (b, \chi') \in \Gamma$ centralize each other if and only if the following conditions hold:

(i) The conjugacy classes $K_a, K_b$ commute element-wise,

(ii) $$(\beta_a(x, y^{-1}by)\beta_b(xy^{-1}by, x^{-1})\beta_b(y, x^{-1}ax)\beta_b(x^{-1}ax, y^{-1})) \chi(xy^{-1}byx^{-1}) \chi'(yx^{-1}axy^{-1}) = \deg \chi \deg \chi',$$

for all $x, y \in G$.

**Proof.** Two objects $(a, \chi), (b, \chi') \in \Gamma$ centralize each other if and only if $S((a, \chi), (b, \chi')) = \deg \chi \deg \chi'$. This is equivalent to the equation:

$$\sum_{g \in K_a, g' \in K_b \cap C_G(g)} \frac{\beta_a(x, g')\beta_a(xg', x^{-1})\beta_b(y, g)\beta_b(yg, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \chi(xg'x^{-1}) \chi'(ygy^{-1}) = |K_a||K_b| \deg \chi \deg \chi',$$

where $g = x^{-1}ax, g' = y^{-1}by$. It is clear that if the two conditions of the Lemma hold, then (22) holds since the set over which the above sum is taken is equal to $K_a \times K_b$. 


Now suppose that (22) holds. We will show that this implies the two conditions in the statement of the Lemma. We have

\[ |K_a||K_b| \deg \chi \deg \chi' \]

\[
\leq \sum_{g \in K_a, g' \in K_b \cap C_G(g)} \left| \frac{\beta_a(x, g') \beta_a(xg', x^{-1}) \beta_b(y, g) \beta_b(yg, y^{-1})}{\beta_a(x, x^{-1}) \beta_b(y, y^{-1})} \right| \chi(xg'x^{-1}) \chi'(ygy^{-1}) \]

\[
= \sum_{g \in K_a, g' \in K_b \cap C_G(g)} \left| \chi(xg'x^{-1}) \right| \left| \chi'(ygy^{-1}) \right| \]

So

\[
\sum_{g \in K_a, g' \in K_b \cap C_G(g)} \left| \chi(xg'x^{-1}) \right| \left| \chi'(ygy^{-1}) \right| = |K_a||K_b| \deg \chi \deg \chi'.
\]

Since \(|\{(g, g') \mid g \in K_a, g' \in K_b \cap C_G(g)\}| \leq |K_a||K_b|, \left| \chi(xg'x^{-1}) \right| \leq \deg \chi, \text{ and} \left| \chi'(ygy^{-1}) \right| \leq \deg \chi', \text{we must have} \{|\{(g, g') \mid g \in K_a, g' \in K_b \cap C_G(g)\}| = |K_a||K_b|, \text{ i.e.}\}

\[
\{|\{(g, g') \mid g \in K_a, g' \in K_b \cap C_G(g)\}| = K_a \times K_b, \left| \chi(xg'x^{-1}) \right| = \deg \chi, \text{ and} \left| \chi'(ygy^{-1}) \right| = \deg \chi'.
\]

The equality \(|\{(g, g') \mid g \in K_a, g' \in K_b \cap C_G(g)\}| = K_a \times K_b \text{ implies that} K_b \subseteq C_G(g), \text{ for all} g \in K_a. \text{ This is equivalent to the condition that} K_a, K_b \text{ commute element-wise which is Condition (i) in the statement of the Lemma. Now, (22) becomes:}

(23)

\[
\sum_{(g, g') \in K_a \times K_b} \left| \frac{\beta_a(x, g') \beta_a(xg', x^{-1}) \beta_b(y, g) \beta_b(yg, y^{-1})}{\beta_a(x, x^{-1}) \beta_b(y, y^{-1})} \right| \chi(xg'x^{-1}) \chi'(ygy^{-1}) = |K_a||K_b|,
\]

where \(g = x^{-1}ax, g' = y^{-1}by\). Since \(\left| \chi(xg'x^{-1}) \right| = \deg \chi, \text{ and} \left| \chi'(ygy^{-1}) \right| = \deg \chi', \text{ by Remark 4.1,} \chi_{xg'x^{-1}}\chi'_{ygy^{-1}}\text{ are roots of unity. Note that (23) holds if and only if}

\[
\left| \frac{\beta_a(x, g') \beta_a(xg', x^{-1}) \beta_b(y, g) \beta_b(yg, y^{-1})}{\beta_a(x, x^{-1}) \beta_b(y, y^{-1})} \right| \chi(xg'x^{-1}) \chi'(ygy^{-1}) = \deg \chi \deg \chi',
\]

for all \(g \in K_a, g' \in K_b, \text{ where} g = x^{-1}ax, g' = y^{-1}by\). This is equivalent to Condition (ii) in the statement of the Lemma.

\[\text{Note 4.3. Let} E \text{ be a subgroup of a finite group} K. \text{ Let} \alpha \text{ be a 2-cocycle on} K. \text{ Let} \chi \text{ be a projective} \alpha\text{-character of} E. \text{ For any} x \in K, \text{ define} \chi^x \text{ by}

\[
\chi^x(l) := \alpha(lx, x^{-1})^{-1} \alpha(x, x^{-1}lx^{-1}) \alpha(x, x^{-1}) \chi(x^{-1}lx),
\]

for all \(l \in E\). Then \(\chi^x\) is a projective \(\alpha\text{-character of} E. \text{ Suppose} E \text{ is normal in} K. \text{ Then}\chi \text{ is said to be} \ K\text{-invariant if} \chi^x = \chi, \text{ for all} x \in K.\]

\[\text{Lemma 4.4. Let} E \text{ be a normal subgroup of a finite group} K. \text{ Let} \alpha \text{ be a 2-cocycle on} K. \text{ Let} \text{Irr}(K) \text{ denote the set of irreducible projective} \alpha\text{-characters of} K. \text{ Let} \rho \text{ be a} \ K\text{-invariant projective}\alpha|_{E \times E}\text{-character of} E \text{ of degree} 1. \text{ Then}

\[
\sum_{\chi \in \text{Irr}(K): \chi|_{E} = (\deg \chi) \rho} (\deg \chi)^2 = \frac{|K|}{|E|}.
\]
Proof. The proof is completely similar to the one given in Lemma 3.2 except in this case we apply Clifford’s Theorem [Ka, Theorem 8.1] and Frobenius reciprocity [Ka, Proposition 4.8] for projective characters.

Let $H$ be a normal Abelian subgroup of $G$. Recall that $\omega \in Z^3(G, k^\times)$ gives rise to a collection $[20]$ of 2-cochains $\beta_a, a \in G$.

**Definition 4.5.** We will say that a map $B : H \times H \to k^\times$ is an alternating $\omega$-bicharacter on $H$ if it satisfies the following three conditions:

\begin{align}
\tag{24} & B(h_1, h_2) = B(h_2, h_1)^{-1}, \\
\tag{25} & B(h, h) = 1, \\
\tag{26} & \delta^1 B_h = \beta_h|_{H \times H}.
\end{align}

for all $h, h_1, h_2 \in H$, where the map $B_h : H \to k^\times$ is defined by $B_h(h_1) := B(h, h_1)$, for all $h, h_1 \in H$.

**Definition 4.6.** We will say that an alternating $\omega$-bicharacter $B : H \times H \to k^\times$ on $H$ is $G$-invariant if it satisfies the following condition:

\begin{align}
\tag{27} & B(x^{-1}ax, h) = \frac{\beta_a(x, h)\beta_a(xh, x^{-1})}{\beta_a(x, x^{-1})} B(a, xhx^{-1}), \quad \text{for all } x \in G, a \in H \cap R, h \in H.
\end{align}

Define

\begin{align}
\tag{28} & \Lambda_\omega^2 H := \{ B : H \times H \to k^\times \mid B \text{ is an alternating } \omega - \text{bicharacter on } H \}, \\
\tag{29} & (\Lambda_\omega^2 H)^G := \{ B \in \Lambda_\omega^2 H \mid B \text{ is } G\text{-invariant} \}.
\end{align}

**Remark 4.7.** If $\omega \equiv 1$, then $\Lambda_\omega^2 H^G$ is the Abelian group of $G$-invariant alternating bicharacters on $H$.

**Remark 4.8.** If $B$ is an alternating $\omega$-bicharacter on $H$, then the restriction $\omega|_{H \times H \times H}$ must be cohomologically trivial. Indeed, let $\omega_H := \omega|_{H \times H \times H}$. Then $B$ defines a braiding on the fusion category $\text{Vec}_{H}^{\omega_H}$. The isomorphism $h_1 \otimes h_2 \tilde{\to} h_2 \otimes h_1$ is given by $B(h_1, h_2)$, for all $h_1, h_2 \in H$, where we identify simple objects of $\text{Vec}_{H}^{\omega_H}$ with elements of $H$. It is known (see, e.g., [Q], [FRS]) that in this case $\omega_H$ is an Abelian 3-cocycle on $H$. By a classical result of Eilenberg and MacLane [EM] the third Abelian cohomology group of $H$ is isomorphic to the (multiplicative) group of quadratic forms on $H$. The value of the corresponding quadratic form $q$ on $h \in H$ is given by $q(h) = B(h, h)$. Since $B$ is alternating we have $q \equiv 1$ and so $\omega_H$ must be cohomologically trivial.

Let $B \in (\Lambda_\omega^2 H)^G$ and define:

\begin{align}
\mathcal{L}_{(H, B)} := & \text{ full Abelian subcategory of } \mathcal{C} \text{ generated by } \\
\tag{30} & \left\{ (a, \chi) \in \Gamma \mid a \in H \cap R \text{ and } \chi \text{ is an irreducible } \beta_a\text{-character of } \text{Rep}(C_G(a)) \text{ such that } \chi(h) = B(a, h) \deg \chi, \text{ for all } h \in H \right\}
\end{align}

**Proposition 4.9.** The subcategory $\mathcal{L}_{(H, B)} \subseteq \text{Rep}(D^\omega(G))$ is Lagrangian.
Proof. Pick any \((a, \chi), (b, \chi') \in \mathcal{L}(H, B) \cap \Gamma\). We have

\[
\left( \frac{\beta_a(x, y^{-1}by) \beta_a(xy^{-1}by, x^{-1}) \beta_b(y, x^{-1}ax) \beta_b(yx^{-1}ax, y^{-1})}{\beta_a(x, x^{-1}) \beta_b(y, y^{-1})} \right) \chi(xy^{-1}byx^{-1}) \chi'(yx^{-1}axy^{-1})
\]

\[
= \frac{\beta_a(x, y^{-1}by) \beta_a(xy^{-1}by, x^{-1})}{\beta_a(x, x^{-1})} B(a, xy^{-1}byx^{-1})
\]

\[
\times \frac{\beta_b(y, x^{-1}ax) \beta_b(yx^{-1}ax, y^{-1})}{\beta_b(y, y^{-1})} B(b, yx^{-1}axy^{-1}) \times \deg \chi \deg \chi'
\]

\[
= B(x^{-1}ax, y^{-1}by) B(y^{-1}by, x^{-1}ax) \deg \chi \deg \chi'
\]

\[
= \deg \chi \deg \chi',
\]

for all \(x, y \in G\). The second equality above is due to (27) while the third equality is due to (24). Note that \(K_a, K_b\) commute element-wise since \(H\) is Abelian. By Lemma 4.2 it follows that objects in \(\mathcal{L}(H, B)\) centralize each other.

Also, \(\theta|_{\mathcal{L}(H, B)} = \text{id}\). The proof of this assertion is exactly the one given in Proposition 3.3.

Now, fix \(a \in H \cap R\) and observe that \(B_a\) defines a \(C_G(a)\)-invariant \(\beta_a\)-character of \(H\) of degree 1. Indeed,

\[
(B_a)^x(h) = \frac{\beta_a(x, x^{-1})}{\beta_a(hx, x^{-1}) \beta_a(x, x^{-1}hx)} B(a, x^{-1}hx)
\]

\[
= B(x^{-1}ax, x^{-1}hx)^{-1} B(a, h) B(a, x^{-1}hx)
\]

\[
= B(a, h),
\]

for all \(x \in C_G(a), h \in H\). The second equality above is due to (27).

The dimension of \(\mathcal{L}(H, B)\) is equal to \(|G|\). The proof of this assertion is exactly the one given in Proposition 3.3 except we appeal to Lemma 4.4 in this case.

It follows from Lemma 2.4 that \(\mathcal{L}(H, B)\) is a Lagrangian subcategory of \(\text{Rep}(D^\omega(G))\) and the Proposition is proved.

**Lemma 4.10.** Let \(H\) be a normal Abelian subgroup of \(G\). Let \(B : H \times H \rightarrow k^\times\) be a map satisfying (24), (25), and (27). Suppose \(\delta^1 B_a = \beta_a|_{H \times H}\), for all \(a \in H \cap R\). Then \(B \in (\Lambda^2_H)^G\).
Proposition 4.11. The map $B_{\mathcal{L}}$ defined in (32) is an element of $\left(\Lambda^2_{\mathcal{L}}H\right)^G$.
Lemma 4.2. It follows that bijective correspondence between Lagrangian subcategories and module categories over Vecω categories with pointed duals.

Theorem 4.12. Lagrangian subcategories of the representation category of the twisted double Dω(G) are classified by pairs (H, B), where H is a normal Abelian subgroup of G such that ω|H×H×H is cohomologically trivial and B : H × H → k^ω is a G-invariant alternating ω-bicharacter in the sense of Definition 4.10.

Proof. The proof is completely similar to the one given in Theorem 3.5.

4.2. Bijective correspondence between Lagrangian subcategories and module categories with pointed duals. Recall that equivalence classes of indecomposable module categories over Vecω_G for which the dual is pointed are in bijection with pairs (H, μ),
where $H$ is a normal Abelian subgroup of $G$ such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $\mu \in (\Omega_{H,\omega})^G$ (defined in (33)).

Theorem [1.12] showed that Lagrangian subcategories of $\text{Rep}(D^\omega(G))$ are in bijection with pairs $(H, B)$, where $H$ is a normal Abelian subgroup of $G$ such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $B \in (\Lambda^2 H)^G$ (the latter was defined in (28)).

In this Subsection we will first show that the set of equivalence classes of indecomposable module categories over $\text{Vec}_\omega$ logically trivial and $B_{\mu}$ pairs $(H, \mu)$.

The map $\Lambda$ is a normal Abelian subgroup of $G$ such that $\omega|_{H \times H \times H}$ is cohomologically trivial. We will establish the aforementioned bijection by showing that there is a bijection between $\Omega_{H,\omega}$ (defined in (3)) and $\Lambda^2 H$ that restricts to a bijection between $(\Omega_{H,\omega})^G$ and $(\Lambda^2 H)^G$.

Let $\mu \in C^2(H, k^\times)$ be a 2-cochain satisfying $\delta^2 \mu = \omega|_{H \times H \times H}$. Define $alt'(\mu)$ by

$$alt'(\mu)(h_1, h_2) := \frac{\mu(h_1, h_2)}{\mu(h_1, h_2)} \times \omega(h_1, h_2),$$

for all $h, h_1, h_2 \in H$. In the second, third, and fourth equalities above we used (2) with $(h_1, h_2, h_3) = (h, h_1, h_2), (h_1, h_2, h), (h_1, h, h_2)$, respectively.

The map $alt'$ induces a map between $\Omega_{H,\omega}$ and $\Lambda^2 H$. By abuse of notation we denote this map also by $alt'$:

$$alt' : \Omega_{H,\omega} \rightarrow \Lambda^2 H : \mu \mapsto alt'(\mu).$$

**Lemma 4.14.** The map $alt'$ defined above is a bijection.

**Proof.** First note that $alt'$ is well-defined. Fix $\mu_0 \in C^2(H, k^\times)$ satisfying $\delta^2 \mu_0 = \omega|_{H \times H \times H}$. Let $B_0 := alt'(\mu_0)$. Define bijections $f_1 : \Lambda^2 H \sim \Lambda^2 H : B \mapsto \frac{B}{B_0}$ and $f_2 : \Omega_{H,\omega} \sim H^2(H, k^\times) : \mu \mapsto \left(\frac{\mu}{\mu_0}\right)$. Note that the cardinality of the two sets $\Omega_{H,\omega}$ and $\Lambda^2 H$ are equal. Injectivity, and hence bijectivity, of $alt'$ follows from the equality $f_1 \circ alt' = alt \circ f_2$. 

**Lemma 4.15.** The following relation holds:

$$\frac{Y_x(h_2, h_1)}{\overline{Y}_x(h_1, h_2)} = \frac{\beta_{xh_1x^{-1}}(x, h_2)\beta_{xh_1x^{-1}}(xh_2, x^{-1})}{\beta_{xh_1x^{-1}}(x, x^{-1})},$$

for all $x \in G, h_1, h_2 \in H$. 

Proof. We have

\[
\frac{\Upsilon_x(h_2, h_1)}{\Upsilon_x(h_1, h_2)} = \frac{\beta_{xh_1x^{-1}}(x, x^{-1})}{\beta_{xh_1x^{-1}}(h_2, x^{-1})} = \frac{\omega(xh_2x^{-1}, xh_1x^{-1}, x)}{\omega(xh_2x^{-1}, x, h_1)} \times \frac{\omega(xh_1x^{-1}, x, x^{-1})\omega(x, x^{-1}, xh_1x^{-1})}{\omega(xh_2x^{-1}, x, x^{-1})} = \frac{\omega(xh_2x^{-1}, x, x^{-1})\omega(x, x^{-1}, xh_1x^{-1})}{\omega(xh_2x^{-1}, x, x^{-1})} = \frac{\omega(x, x^{-1}, xh_1x^{-1})}{\omega(xh_2x^{-1}, x, x^{-1})} = 1,
\]

for all \(x \in G, h_1, h_2 \in H\). In the first equality above we used the definition of \(\Upsilon\) and \(\beta\) and canceled some factors. In the second, third, fourth, and fifth equalities we used (18) with \((g_1, g_2, g_3, g_4) = (xh_2x^{-1}, x, h_1, x^{-1}), (xh_2x^{-1}, xh_1x^{-1}, x, x^{-1}), (xh_1x^{-1}, xh_2x^{-1}, x, x^{-1})\), and \((xh_2x^{-1}, x, x^{-1}, xh_1x^{-1})\), respectively.

Lemma 4.16. The map \(\text{alt}'\) defined in (33) restricts to a bijection between \((\Omega_{H,\omega})^G\) and \((\Lambda_2^G)^G\).

Proof. Let us first show that \(\text{alt}'((\Omega_{H,\omega})^G) \subseteq (\Lambda_2^G)^G\). Pick any \(\mu \in (\Omega_{H,\omega})^G\). So \(\text{alt}'(\frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H}) = 1\), for all \(x \in G\). We have

\[
\text{alt}'(\mu)(x^{-1}ax, h) \times \text{alt}'(\mu)(a, xhx^{-1})^{-1} = \frac{\mu(h, x^{-1}ax)\mu(a, xhx^{-1})}{\mu(x^{-1}ax, h)\mu(xh, x^{-1}ax)} = \frac{\mu^x(h, x^{-1}ax)}{\mu^x(x^{-1}ax, h)} \times \frac{\Upsilon_x(h, x^{-1}ax)}{\Upsilon_x(x^{-1}ax, h)} = \beta_a(x, h)\beta_a(xh, x^{-1})\beta_a(x, x^{-1})^{-1},
\]

for all \(x \in G, a \in H \cap R, h \in H\). In the fourth equality above we used the fact that \(\text{alt}'(\frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H}) = 1\) and in the fifth equality we used Lemma 4.15. So \(\text{alt}'((\Omega_{H,\omega})^G) \subseteq (\Lambda_2^G)^G\), as desired.

Now let us show that \((\Lambda_2^G)^G \subseteq \text{alt}'((\Omega_{H,\omega})^G)\). Pick any \(\mu \in \Omega_{H,\omega}\) and suppose that \(\text{alt}'(\mu) \in (\Lambda_2^G)^G\). Suffices to show that \(\text{alt}'(\mu) = 1\), for all \(x \in G\). Let
B := alt'(\mu). We have
\[
alt \left( \frac{\mu^x}{\mu} \times \Upsilon_{x|H \times H} \right) (h_1, h_2) \times \frac{\Upsilon_x(h_1, h_2)}{\Upsilon_x(h_2, h_1)}
\]
\[
= B(xh_1x^{-1}, xh_2x^{-1})B(h_1, h_2)^{-1}
\]
\[
= B((yx^{-1})a(yx^{-1}), xh_2x^{-1})B(y^{-1}ay, h_2)^{-1} \quad \text{(where } h_1 = y^{-1}ay) \]
\[
= \frac{\beta_a(yx^{-1}, xh_2x^{-1})\beta_a(yh_2x^{-1}, xy^{-1})}{\beta_a(yx^{-1}, xy^{-1})} \times \frac{\beta_a(y, y^{-1})}{\beta_a(y, h_2)\beta_a(yh_2, y^{-1})}
\]
\[
= \frac{\beta_{xh_1x^{-1}(xh_2, x^{-1})}\beta_a(y, y^{-1})\beta_a(yx^{-1}, xy^{-1})\beta_a(y, h_2)\beta_a(yh_2, y^{-1})}{\beta_a(yx^{-1}, xh_2x^{-1})\beta_a(yh_2x^{-1}, xy^{-1})\beta_a(y, y^{-1})}
\]
\[
= \frac{\beta_{xh_1x^{-1}(x, h_2)}\beta_{xh_1x^{-1}(xh_2, x^{-1})}\beta_a(yx^{-1}, xy^{-1})\beta_a(y, h_2)\beta_a(yh_2, y^{-1})}{\beta_{xh_1x^{-1}(x, h_2)}\beta_{xh_1x^{-1}(xh_2, x^{-1})}\beta_a(y, x^{-1})\beta_a(yx^{-1}, xy^{-1})\beta_a(yh_2, y^{-1})}
\]
\[
= \frac{\Upsilon_x(h_1, h_2)}{\Upsilon_x(h_2, h_1)} \times \frac{\beta_a(y, y^{-1})\beta_{h_1}(x^{-1}, xy^{-1})}{\beta_a(y, x^{-1})\beta_a(yx^{-1}, xy^{-1})}
\]
for all \(x \in G, h_1, h_2 \in H\). In the fourth through eight equalities above we used (20) with \((x, y, z) = (yx^{-1}, xh_2, x^{-1}), (y^{-1}, x, h_2), (yx^{-1}, x, x^{-1}), (yh_2, x^{-1}, yx^{-1}), \) and \((y, x^{-1}, xy^{-1})\), respectively. It follows that \((\Lambda^2_2 H)^G \subseteq alt'(\Omega_{H, \omega}^G)\) and the Lemma is proved.

Recall that \(E(C)\) denotes the set of (equivalence classes of) braided tensor equivalences between a modular category \(C\) and the centers of pointed fusion categories.

**Theorem 4.17.** The assignment \(\mathcal{M} \mapsto \iota_{\mathcal{M}}\) (defined in (17)) restricts to a bijection between equivalence classes of indecomposable \(\mathrm{Vec}^\omega_G\)-module categories \(\mathcal{M}\) with respect to which the dual fusion category \((\mathrm{Vec}^\omega_G)^*_{\mathcal{M}}\) is pointed and \(E(\text{Rep}(D^\omega(G)))\).

**Proof.** The proof is completely similar to the one given in Theorem 3.6.

**Remark 4.18.** The equivalence type of the symmetric category \(\mathcal{L}_{(H, B)}\) of \(\text{Rep}(D^\omega(G))\) can be explicitly described in a way similar to Remark 3.7 cf. [Na, Theorem 4.5].

**Theorem 4.19.** Let \(C_1, C_2\) be group-theoretical fusion categories. Then \(C_1, C_2\) are weakly Morita equivalent if and only if their centers \(Z(C_1)\) and \(Z(C_2)\) are equivalent as braided fusion categories.

**Proof.** That the “if” part is true for all fusion categories was first observed by M. Müger in [Mu, Remark 3.18]. This follows from the definition of weak Morita equivalence and a theorem of P. Schauenburg [S]. See also [N, OT, EO].

For the “only if” part, let \((G_1, \omega_1), (G_2, \omega_2)\) be two pairs of groups and 3-cocycles such that \(C_1\) is weakly Morita equivalent to \(\text{Vec}^\omega_{G_1}\) and \(C_2\) is weakly Morita equivalent to \(\text{Vec}^\omega_{G_2}\).

If \(Z(C_1) \cong Z(C_2)\) (as braided fusion categories) then \(Z(\text{Vec}^\omega_{G_1}) \cong Z(\text{Vec}^\omega_{G_2})\) (as braided fusion categories) and therefore, \(\text{Vec}^\omega_{G_1}\) and \(\text{Vec}^\omega_{G_2}\) are weakly Morita equivalent by Theorem 4.17 and hence, \(C_1\) and \(C_2\) are weakly Morita equivalent.
Corollary 4.20. Let $G, G'$ be finite groups, $\omega \in Z^3(G, k^\times)$, and $\omega' \in Z^3(G', k^\times)$. Then the representation categories of twisted doubles $D^{\omega}(G)$ and $D^{\omega'}(G')$ are equivalent as braided tensor categories if and only if $G$ contains a normal Abelian subgroup $H$ such the following conditions are satisfied:

1. $\omega|_{H \times H \times H}$ is cohomologically trivial,
2. there is a $G$-invariant (see (3)) 2-cochain $\mu \in C^2(H, k^\times)$ such that $\delta^2 \mu = \omega|_{H \times H \times H}$, and
3. there is an isomorphism $a : G' \to \hat{H} \times_\nu (H\backslash G)$ such that $\omega' \circ (a \times a \times a)$ and $\omega'$ are cohomologically equivalent.

Here $\nu$ is a certain 2-cocycle in $Z^2(H\backslash G, \hat{H})$ coming from the $G$-invariance of $\mu$ and $\omega$ is a certain 3-cocycle on $\hat{H} \times_\nu (H\backslash G)$ depending on $\nu$ and on the exact sequence $1 \to H \to G \to H\backslash G \to 1$ (see [Na] Theorem 5.8 for precise definitions).

5. Examples

5.1. Lagrangian subcategories of the Drinfeld doubles of finite symmetry groups.

Example 5.1. Consider the group of symmetries of a regular $n$-gon, i.e., the dihedral group $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$, $n \geq 2$. Let us describe the Lagrangian subcategories of $\text{Rep}(D(D_{2n})) \cong Z(\text{Vec}_{D_{2n}})$.

Let $n = 2$. Then $D_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. There are six different Lagrangian subcategories of $\text{Rep}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}))$, all of them equivalent to $\text{Rep}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

Let $n = 4$. The group $D_8$ has five normal Abelian subgroups which give rise to seven Lagrangian subcategories of $\mathbb{Z}(\text{Vec}_{D_8})$. With an exception of the Lagrangian subcategory corresponding to the center $\langle r^2 \rangle$ of $D_8$, all Lagrangian subcategories are equivalent to $\text{Rep}(D_8)$. The one corresponding to $\langle r^2 \rangle$ is equivalent to $\text{Rep}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Applying [DGNO] we conclude that for some 3-cocycle $\omega$ on $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ there is a braided tensor equivalence $\text{Rep}(D(D_8)) \cong \text{Rep}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}))$ (existence of such an equivalence is already known to experts, see [CGR] and [GMN]).

If $n = 3$ or $n \geq 5$ then normal Abelian subgroups of $D_{2n}$ are precisely rotation subgroups $\langle r^k \rangle$, where $k$ is a divisor of $n$. Each of these subgroups is cyclic and so has a trivial Schur multiplier. The Lagrangian subcategory of $\mathbb{Z}(\text{Vec}_{D_{2n}})$ corresponding to $\langle r^k \rangle$, $k|n$, is equivalent to $\text{Rep}((\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}))$, where for any Abelian group $A$ we denote $\text{Dih}(A) = A \times \mathbb{Z}/2\mathbb{Z}$ the generalized dihedral group (the action of $\mathbb{Z}/2\mathbb{Z}$ on $A$ is by inverting elements). Consequently, there is a 3-cocycle $\omega$ on $\text{Dih}(\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z})$ such that $\text{Rep}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})) \cong \text{Rep}(D(D_{2n}))$ as braided tensor categories.

Next, consider the symmetry groups of Platonic solids.

Example 5.2. The tetrahedron group $A_4$ has two normal Abelian subgroups: the trivial one and another isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The Schur multiplier of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ with the trivial $A_4$-action. So $\text{Rep}(D(A_4))$ has three Lagrangian subcategories. All three are equivalent to $\text{Rep}(A_4)$.

The double of the cube/octahedron group $S_4$ also has three Lagrangian subcategories (corresponding to the same data as in the case of $A_4$). One can check that corresponding three Lagrangian subcategories of $\text{Rep}(D(S_4))$ are all isomorphic to $\text{Rep}(S_4)$. 
Finally, the group of symmetries of dodecahedron/icosahedron is a simple non-Abelian group $A_5$. It is clear that for any simple non-Abelian group $G$ the category $\text{Rep}(D(G))$ contains a unique Lagrangian subcategory (corresponding to the trivial subgroup of $G$).

Recall [Na] that a group $G$ is called categorically Morita rigid if $\text{Vec}_G$ being weakly Morita equivalent to $\text{Vec}_{G'}$ implies that groups $G$ and $G'$ are isomorphic. One can check that in this case $\omega$ must be cohomologically trivial. Our results imply that $G$ is categorically Morita rigid if and only if all Lagrangian subcategories of $\text{Rep}(D(G))$ are equivalent to $\text{Rep}(G)$ as symmetric categories.

It follows from Examples 5.1 and 5.2 that groups $A_4$, $S_4$, $A_5$, as well as groups $D_{2n}$, where $n$ is a square-free integer, are categorically Morita rigid. It is clear that any group $G$ without non-trivial normal Abelian subgroups is categorically Morita rigid.

Finally, let us consider symmetries of vector spaces.

**Example 5.3.** Let $n$ be a positive integer and let $F_q$ be the finite field with $q$ elements. Let $G = SL(n, q)$ denote the special linear group of $n \times n$ matrices with entries from $F_q$, i.e., matrices having determinant equal to 1.

Let $PSL(n, q) = SL(n, q)/Z(SL(n, q))$, where $Z(SL(n, q))$ is the center of $SL(n, q)$. Let us assume that $(n, q) \neq (2, 2), (2, 3)$. It is known that in this case $PSL(n, q)$ is simple.

Let $d = (n, q−1)$ be the greatest common divisor of $n$ and $q−1$. The group $Z(SL(n, q)) \cong \mathbb{Z}/d\mathbb{Z}$ is cyclic and any normal subgroup of $SL(n, q)$ is contained in $Z(SL(n, q))$. Thus, Lagrangian subcategories of $\text{Rep}(D(SL(n, q)))$ correspond to divisors of $d$. One can easily describe the equivalence types of these subcategories. For instance, the Lagrangian subcategory corresponding to $Z(SL(n, q)) \cong \mathbb{Z}/d\mathbb{Z}$ is equivalent to $\text{Rep}(\mathbb{Z}/d\mathbb{Z} \times PSL(n, q))$. Therefore, $\text{Rep}(D(SL(n, q)))$ is equivalent (as a braided tensor category) to $\text{Rep}(D^\omega(\mathbb{Z}/d\mathbb{Z} \times PSL(n, q)))$ for some 3-cocycle $\omega$.

5.2. **Non-pointed categories of dimension 8.**

**Example 5.4.** It is known [TY] that there are exactly four non-pointed fusion categories of dimension 8 with integral dimensions of objects: $\text{Rep}(D_8)$; $\text{Rep}(Q_8)$, where $Q_8$ is the group of quaternions; $KP$, the representation category of the Kac-Paljutkin Hopf algebra [KacP]; and $TY$, the category of representations of a unique 8-dimensional quasi-Hopf algebra which is not gauge equivalent to a Hopf algebra [TY] (equivalently, $TY$ is the unique non-pointed fusion category of dimension 8 with integral dimensions of objects which does not have a fiber functor).

Let us show that these four categories belong to four different weak Morita equivalence classes and hence, in view of Theorem 4.19, their centers are not equivalent as braided tensor categories.

All proper subgroups of $Q_8$ are normal Abelian and have trivial Schur multipliers. Hence all of them produce pointed duals. So $\text{Rep}(Q_8)$ is the only non-pointed dual of $\text{Vec}_{Q_8}$.

The only non-normal subgroups of $D_8 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^3 \rangle$ are reflection subgroups $\langle s \rangle$, $\langle sr \rangle$, and their conjugates. Such subgroups appear as factors in exact factorizations of $D_8$ (one can take $\langle r \rangle$ as another factor). The corresponding dual categories $(\text{Vec}_{D_8})^*_{\mathcal{M}(s), 1}$ and $(\text{Vec}_{D_8})^*_{\mathcal{M}(sr), 1}$ admit fiber functors by [O2 Corollary 3.1] and so are representations of semisimple Hopf algebras. But Hopf algebras corresponding to exact factorizations of $D_8$ are known to be either commutative or cocommutative. We conclude that

$$(\text{Vec}_{D_8})^*_{\mathcal{M}(s), 1} \cong (\text{Vec}_{D_8})^*_{\mathcal{M}(sr), 1} \cong \text{Rep}(D_8),$$
and, hence, $\text{Rep}(D_8)$ is the unique non-pointed dual of $\text{Vec}_{D_8}$.

Thus, neither $\text{Rep}(Q_8)$ nor $\text{Rep}(D_8)$ is weakly Morita equivalent to any other non-pointed category.

It remains to check that the same is true for $TY$. Let $\omega_0$ be a non-trivial 3-cocycle on $D_8/\langle r^2, s \rangle \cong \mathbb{Z}/2\mathbb{Z}$ (corresponding to the non-zero element of $H^3(\mathbb{Z}/2\mathbb{Z}, k^\times) \cong \mathbb{Z}/2\mathbb{Z}$). Let $\pi : D_8 \to D_8/\langle r^2, s \rangle$ be the canonical projection. Define a 3-cocycle $\omega$ on $D_8$ by $\omega = \omega_0 \circ (\pi \times \pi \times \pi)$. Then $\omega \equiv 1$ on $\langle r^2, s \rangle$ and the restrictions of $\omega$ on each of the subgroups $\langle r \rangle$ and $\langle sr \rangle$ are cohomologically non-trivial. This means that the complete list of equivalence classes of indecomposable module categories over $\text{Vec}_{D_8}^\omega$ consists of $\mathcal{M}(\{1\}, 1)$, $\mathcal{M}(\langle r^2 \rangle, 1)$, $\mathcal{M}(\langle s \rangle, 1)$, and $\mathcal{M}(\langle r^2, s \rangle, \mu)$, where $\mu \in H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, k^\times)$. Therefore, the only non-pointed dual category of $\text{Vec}_{D_8}^\omega$ corresponds to $\mathcal{M}(\langle s \rangle, 1)$, see Example 2.2. It follows from the classification of fiber functors on group-theoretical categories obtained in [O2, Corollary 3.1] that the category $(\text{Vec}_{D_8}^\omega)_{\mathcal{M}(\langle s \rangle, 1)}$ does not have a fiber functor and hence $(\text{Vec}_{D_8}^\omega)_{\mathcal{M}(\langle s \rangle, 1)}^* \cong TY$. Since all other duals of $\text{Vec}_{D_8}^\omega$ are pointed, it follows that $TY$ is not weakly Morita equivalent to any other non-pointed fusion category.

Hence, $\text{Rep}(D_8), \text{Rep}(Q_8), KP$, and $TY$ are pairwise weakly Morita non-equivalent fusion categories. Our claim about their centers follows from Theorem 4.19.

Let us note that there is another 3-cocycle $\eta$ on $D_8$ such that $(\text{Vec}_{D_8}^\eta)_{\mathcal{M}(\langle s \rangle, 1)}^* \cong KP$. Up to a conjugation, such $\eta$ must have a trivial restriction on $\langle r^2, s \rangle$ and $\langle sr \rangle$ (this can be seen from the Kac exact sequence [Kac]).

**Remark 5.5.** The braided tensor equivalence classes of twisted doubles of groups of order 8 were studied in detail in [GMN] using the higher Frobenius-Schur indicators. In particular, it was shown that there are precisely 20 equivalence classes of such non-pointed doubles. In view of results of the present paper this description can be interpreted in terms of weak Morita equivalence classes of pointed categories of dimension 8.

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