Frequency Switching of Quantum Harmonic Oscillator with time-dependent frequency

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Abstract
An explicit solution of the equation for the classical harmonic oscillator with smooth switching of the frequency has been found. A detailed analysis of a quantum harmonic oscillator with such frequency has been done on the base of the method of linear invariants. It has been shown that such oscillator possesses cofluctuant states, different from widely studied Glauber’s coherent and "ideal" squeezed states.

Oscillator models are widely used in many branches of physics, such as quantum optics, atomic, molecular, and solid state physics. Small vibration of dynamic system can be describe in terms of harmonic oscillators in both quantum and classical mechanics. To include surrounding influences on the vibration, or to simulate the coupling of the vibration with other degree of freedom, one can consider time-dependent parameters specifying the Hamiltonian of a harmonic oscillator, for example, mass and frequency. Besides, non-stationary oscillator models show essentially nonclassical effects, such as squeezing and covariance of their quantum fluctuations. Some examples for such phenomena are: motion of one ion in Paul trap, which is precisely described by harmonic oscillator, with periodically time-dependent frequency [1], also Berry phase can be achieved when parameters of oscillator undergoes a cyclic change [2]-[7]. Agarwal and Kumar have shown that a nonstationary oscillator with linear sweep of the restoring force owns nonclassical states [8]. In the present Letter we study the switching of the frequency of a quantum nonstationary oscillator by using the method suggested in [11, 12].

The Hamiltonian of a harmonic oscillator is given by

$$\hat{H} = \frac{1}{2m}p^2 + \frac{m\Omega^2(t)}{2}q^2,$$

where the constants $m$ and $\Omega(t)$ are mass and the frequency of the quantum harmonic oscillator. The case $M = M(t)$ can be reduced to the case $M = m$ (see for example [11], eq. 122).

We recall the method of linear invariants, developed in series of papers [9]-[12], which we apply to one dimensional case: For each quantum system, described by a quadratic Hamiltonian, there is a classical two- dimensional isotropic nonstationary harmonic oscillator with a Lagrangian (classical)

$$L = \frac{m}{2}(\epsilon_1^2 + \epsilon_2^2) - \frac{m}{2}\Omega^2(t)(\epsilon_1^2 + \epsilon_2^2).$$
and equations of motion
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\epsilon}_k} - \frac{\partial L}{\partial \epsilon_k} = 0, \quad k = 1, 2.
\] (3)

As it is shown in [13], these two real equations are equivalent to one complex classical equation of nonstationary harmonic oscillator
\[
\ddot{\epsilon}(t) + \Omega^2(t)\epsilon(t) = 0,
\] (4)

where complex function \(\epsilon(t) = \epsilon_1(t) + i\epsilon_2(t)\) completely describes the quantum evolution of the system [12], in particular with Hamiltonian [11].

Despite of various investigations of nonstationary harmonic oscillator, smooth switching of the frequency for finite interval of time, was not presented in the literature. Here we study the behaviour of the nonstationary harmonic oscillator with varying frequency and constant mass for finite interval of time. We will show that the found classical solution for this case completely determines the quantum evolution of the corresponding quantum oscillator too.

Let we consider switching of the frequency \(\Omega(t)\) in the form:
\[
\Omega(t) = \begin{cases} 
\omega \sqrt{1 - \frac{\alpha \omega}{(1 + \alpha \omega)^2}} = \Omega_0 & \infty < t < 0 \\
\omega \sqrt{1 - \frac{\alpha \omega}{(1 + \alpha \omega \cos(\omega t))^2}} & 0 \leq t \leq \frac{\pi}{2\omega} \\
\omega \sqrt{1 - \alpha \omega} & \frac{\pi}{2\omega} < t < \infty,
\end{cases}
\] (5)

where \(\alpha\) and \(\omega\) are real constants, with dimension of time and frequency, respectively. The shape of the frequency is shown on figure 1. By direct calculation we can check that the function
\[
\epsilon(t) = \begin{cases} 
\omega \sqrt{1 - \frac{\alpha \omega}{(1 + \omega \cos(\omega t))^2}} + i \int_0^t \frac{\omega}{(1 + \alpha \omega \cos(\omega t))^2} dt \\
\frac{\omega}{(1 + \alpha \omega \cos(\omega t))^2} e^{i \int_0^t \frac{\omega}{(1 + \alpha \omega \cos(\omega t))^2} dt} \\
e^{i \int_0^t \frac{\omega}{(1 + \alpha \omega \cos(\omega t))^2} dt} \left( \frac{1}{\sqrt{\omega}} \cos(\omega \sqrt{1 - \alpha \omega} (t - \frac{\pi}{2\omega})) + i \frac{1}{\sqrt{\omega(1 - \alpha \omega)}} \sin(\omega \sqrt{1 - \alpha \omega} (t - \frac{\pi}{2\omega})) \right)
\end{cases}
\] (6)

is a solution of the subsidiary classical equation (4) of two-dimensional harmonic oscillator for the same time-intervals as in (5). The complex-conjugate of \(\epsilon^*(t)\) is the other linear-independent solution of (4). Figure 2 shows the parametric plot of the real and imaginary parts of \(\epsilon(t)\).

As it is shown in Appendix 1, Wronsky determinant is one classical integral of motion for the equation (4):
\[
D_W(t) = \epsilon(t) \dot{\epsilon}^*(t) - \dot{\epsilon}(t) \epsilon^*(t) = -2i.
\] (7)

We will use this invariant beyond to study the properties of quantum integral of motion, especially at the calculation of their commutator.

So we have solved completely the classical problem of nonstationary harmonic oscillator with switching the frequency (5). Let we consider the quantum problem of nonstationary harmonic oscillator with Hamiltonian (1), when the frequency is switching by the same way as classical oscillator (3). The quantum system evolves in time according to the Schrödinger equation
\[
i\hbar \frac{\partial}{\partial t} \Psi(q, t) = \hat{H}(t) \Psi(q, t),
\] (8)
where \( \Psi(q,t) = \langle q | \Psi, t > \) is a function of the state in coordinate representation. We define creation and annihilation operators (\( \Omega_0 = \Omega(0) \))

\[
\hat{a} = \left[ \frac{1}{2\hbar m\Omega_0}\right]^{\frac{1}{2}}(m\Omega_0\hat{q} + i\hat{p}) \quad \hat{a}^\dagger = \left[ \frac{1}{2\hbar m\Omega_0}\right]^{\frac{1}{2}}(m\Omega_0\hat{q} - i\hat{p})
\]

(9)

and it is easy to show that \( [a,a^\dagger] = 1 \), because quantum-mechanical position and momentum operators \( \hat{q} \) and \( \hat{p} \) obey the usual commutation relation \( [\hat{q},\hat{p}] = i\hbar \). According to the general method \[3,13\] presented also in \[13\] (see formula (6) therein), the linear invariant which corresponds to our particular Hamiltonian has the form;

\[
A(t) = \frac{1}{2} \left[ - \sqrt{\Omega_0 \epsilon(t)} + i \sqrt{\Omega_0} \dot{\epsilon}(t) \right] \hat{a}^\dagger + \left( \sqrt{\Omega_0} \epsilon(t) - i \sqrt{\Omega_0} \dot{\epsilon}(t) \right) \hat{a}
\]

(10)

Operator \( \hat{A}^\dagger(t) \) is Hermitian conjugate to \( \hat{A}(t) \) and has the form;

\[
\hat{A}^\dagger(t) = \frac{1}{2} \left[ \sqrt{\Omega_0} \epsilon^*(t) + i \sqrt{\Omega_0} \dot{\epsilon}^*(t) \right] \hat{a}^\dagger - \left( \sqrt{\Omega_0} \epsilon^*(t) - i \sqrt{\Omega_0} \dot{\epsilon}^*(t) \right) \hat{a}
\]

(11)

It is also an integral of motion for the quantum system. Using \( [a,a^\dagger] = 1 \) and Wronsky determinant \( (\parallel) \) it is easy to show, that these invariants obey commutation relation

\[
[A(t), \hat{A}^\dagger(t)] = 1,
\]

(12)

i.e. their commutator does not depends on time, either. Thus the operators \( \hat{A}(t) \) and \( \hat{A}^\dagger(t) \) satisfy boson commutation relation in every moment \( t \). The Hermitian and anty-Hermitian parts of \( \hat{A}(t) \) and \( \hat{A}^\dagger(t) \) are also integrals of motion, so that one can construct a new pair integrals of motion

\[
\hat{Q}_0(t) = \sqrt{\hbar} m\Omega_0 \left( \hat{A}^\dagger(t) + \hat{A}(t) \right) = \frac{1}{\sqrt{\Omega_0}}(\Im \dot{\epsilon}(\hat{q}) - \frac{\Im \epsilon}{m} \hat{p}),
\]

(13)

\[
\hat{P}_0(t) = i\sqrt{\frac{\hbar m\Omega_0^2}{2}}(\hat{A}^\dagger(t) - \hat{A}(t)) = \sqrt{\Omega_0}(-m\Re \dot{\epsilon}(\hat{q}) + \Re \epsilon(\hat{p})).
\]

(14)

By direct calculation using the found expression for \( \epsilon(t) \) \( (\parallel) \), and the equality \( (\parallel) \) one can check that \( \frac{d\hat{Q}_0}{dt} = \frac{d\hat{P}_0}{dt} - \frac{\hbar}{\Omega_0}[\hat{Q}_0, \hat{H}] = 0 \), respectively \( \frac{d\hat{Q}_0}{dt} = \frac{d\hat{P}_0}{dt} - \frac{\hbar}{\Omega_0}[\hat{P}_0, \hat{H}] = 0 \). The physical sense of these two integrals of motion is that the operators \( \hat{Q}_0(t) \) and \( \hat{P}_0(t) \) are the initial operators of coordinate and momentum of the quantum oscillator, scaled with a factor \( \sqrt{\Omega_0 \epsilon(0)} \) ( \( \dot{\epsilon}(0) = -\dot{\epsilon}^*(0) = \frac{i}{\epsilon(0)} \Rightarrow \Im \dot{\epsilon}(0) = \frac{1}{\epsilon(0)} \));

\[
\hat{Q}_0(t) = \hat{Q}_0(0) = \frac{1}{\sqrt{\Omega_0 \epsilon(0)}} \hat{q}
\]

(15)

\[
\hat{P}_0(t) = \hat{P}_0(0) = \sqrt{\Omega_0 \epsilon(0)} \hat{p},
\]

(16)

where the initial condition for \( \epsilon(0) \) and \( \dot{\epsilon}(0) \) were used ( see \( (\parallel) \) and \( (\parallel) \) in Appendix 1, respectively ).

\( \hat{Q}_0(t) \) and \( \hat{P}_0(t) \) for an arbitrary quantum system are related with the initial operators of coordinate and momentum \( \hat{q}(0) \) and \( \hat{p}(0) \) in the time moment \( t = 0 \). The evolutions of
\( \dot{q} \) and \( \dot{p} \) for the quantum system are compensated in proper way with \( Re \) and \( Im \) parts of \( \epsilon(t) \) and \( \dot{\epsilon}(t) \) to keep \( Q_0(t) \) and \( P_0(t) \) constant. The special choice of operators \( A(t) \) and \( A^\dagger(t) \), which are expressed via solutions of classical harmonic oscillator \([3]\) provides this property of the quantum integrals of motion in the method of the linear invariants \([12]\).

In particular this is true for the present quantum oscillator whose motion is determined in terms of the found solution \([3]\). Indeed, the quantum mean value of the integrals of motion \( Q_0(t) \) and \( P_0(t) \) are constants, as we will convince later.

Let us determine the evolution of the first and second moments of the operators \( \hat{q} \) and \( \hat{p} \) for the particular oscillator with frequency \([3]\). In \([12]\) was proved the following theorem; the necessary an sufficient condition one quantum systems to preserve an equality in the Schrödinger uncertainty relation is the states of the system to be eigenstates of the operator \( \hat{A} = u\hat{a} + v\hat{a}^\dagger + w \). Here \( u, v \) and \( w \) are arbitrary complex numbers, with one connection; \(|u|^2 - |v|^2 = 1 \), and the states are called Schrödinger Minimum Uncertainty States. The invariants \([14], [11]\) obey this condition (the proof is based again on the expression of the Wronsky determinant \([3]\)). Let us consider the oscillator in such states, i.e. \(|\Psi, t > = \mid SMUS > \) and \( z \) is the corresponding eigenvalue; \( A|SMUS > = z|SMUS > \). At this point, there are two ways to solve the problem; to express \( \hat{q} \) and \( \hat{p} \) in terms of \( \hat{A} \) and \( \hat{A}^\dagger \), respectively \( u, v \) in terms of \( \epsilon(t), \dot{\epsilon}(t) \) \([3]\) and to find the first and second moments for this specific Hamiltonian \([11]\). The second way is to use the method of linear invariants where these moments are obtained in general form, and to establish the connection with particular quantum harmonic oscillator with a switching frequency. We chose the second way, following the idea of the method for one-dimensional case from \([12], [13]\).

Presenting \( \hat{A} \) and \( \hat{A}^\dagger \) from \([15]\) in terms of \( \hat{q} \) and \( \hat{p} \) and solving this system about \( \hat{q} \) and \( \hat{p} \) we receive their quantum evolutions, expressed by the solutions of the classical two-dimensional harmonic oscillator \( (\epsilon = \epsilon_1 + i\epsilon_2) \)

\[
\hat{q} = \sqrt{\hbar a(t)}(\epsilon(t)\hat{A}^\dagger + \epsilon^*(t)\hat{A})
\]

\[
\hat{p} = -\sqrt{\frac{\hbar}{a(t)}} \left[ \left( be(t) - \frac{\dot{\epsilon}(t)}{2} - \frac{1}{4}\frac{\dot{a}(t)}{a(t)}\epsilon(t) \right) \hat{A}^\dagger + \left( be^*(t) - \frac{\dot{\epsilon}^*(t)}{2} - \frac{1}{4}\frac{\dot{a}(t)}{a(t)}\epsilon^*(t) \right) \hat{A} \right],
\]

where \( a(t), b(t) \) and \( c(t) \) are the time-dependent coefficients in the general quadratic Hamiltonian \([12], [15]\) in front of \( \hat{p}^2, \hat{q}\hat{p} + \hat{p}\hat{q} \) and \( \hat{q}^2 \). Taking quantum mean value of \( \hat{q} \) and \( \hat{p} \) we obtain the behaviour of their first moments

\[
< \hat{q} > = \sqrt{\hbar a(t)}(\epsilon(t)z^* + \epsilon^*(t)z)
\]

\[
< \hat{p} > = -\sqrt{\frac{\hbar}{a(t)}} \left[ \left( b(t)\epsilon(t) - \frac{\dot{\epsilon}(t)}{2} - \frac{1}{4}\frac{\dot{a}(t)}{a(t)}\epsilon(t) \right) z^* + \left( b(t)\epsilon^*(t) - \frac{\dot{\epsilon}^*(t)}{2} - \frac{1}{4}\frac{\dot{a}(t)}{a(t)}\epsilon^*(t) \right) z \right].
\]

For our case \([11]\) the Hamiltonian’s coefficients are \( a(t) = \frac{1}{2m}, b(t) = 0 \) and \( c(t) = \frac{m\dot{\epsilon}^2(t)}{2} \). Consequently, the evolutions of quantum mean value of the coordinate \( < \hat{q} > \) and the momentum \( < \hat{p} > \) has the forms;

\[
< \hat{q} > = \sqrt{\frac{\hbar}{2m}}(\epsilon(t)z^* + \epsilon^*(t)z)
\]

\[
< \hat{p} > = \sqrt{\frac{\hbar m}{2}}(\dot{\epsilon}(t)z^* + \dot{\epsilon}^*(t)z).
\]
Here $z$ is the corresponding eigenvalue of the eigenstates $|SMUS>$, in whose states the quantum values are obtained. The phase diagram shown on figure 3 presents the evolution of the quantum harmonic oscillator with a frequency (3) ($\hbar = 1 J s$, $m = 1 g$, $\alpha = .5 s$, $\omega = 1 H z$, $z = 1 + i 0.2$). The oscillator evolves as an ellipse when the frequency is a constant ($t < 0$ and $t > \frac{\pi}{2 \omega}$). The ellipse with the smaller horizontal axis corresponds to the region $t < 0$. The bold curve presents the region of the switching frequency. The point in the first quadrant corresponds to the quantum mean value of the pair invariants ($<Q_0(t)>$, $<P_0(t)>$), (13) and (14) respectively.

There are three second independent moments $\sigma_q$, $\sigma_p$ and $c_{qp}$ ($c_{pq} = c_{qp}$). The quantum deviations $\sigma_q^2$, $\sigma_p^2$ as expressions of $\epsilon(t)$ and $\dot{\epsilon}(t)$ were found for first time in general case of the method of linear invariants in [10];

$$\sigma_q^2(t) = \hbar a(t) |\epsilon(t)|^2,$$

$$\sigma_p^2(t) = \frac{\hbar}{a(t)} \left[ \frac{1}{4|\epsilon(t)|^2} + \left( b(t)|\epsilon(t)| - \frac{1}{2} \frac{d|\epsilon(t)|}{dt} - \frac{\dot{a}(t)}{4a(t)}|\epsilon(t)| \right)^2 \right].$$

The third second moment, cofluctuation $c_{qp}$, in terms of $u(t)$ and $v(t)$ was found in [14], but as an expression of the solution $\epsilon(t)$ and its first derivative $\dot{\epsilon}(t)$ of the equation (4) in [13];

$$c_{qp}^2(t) = \hbar^2 |\epsilon(t)|^2 \left( b(t)|\epsilon(t)| - \frac{1}{2} \frac{d|\epsilon(t)|}{dt} - \frac{\dot{a}(t)}{4a(t)}|\epsilon(t)| \right)^2.$$  

As far as we have $\epsilon(t)$ in explicit form (1) and $a(t) = \frac{1}{2m}$, $b(t) = 0$ all second moments for the quantum harmonic oscillator with Hamiltonian (1) and frequency (3) are determined completely;

$$\sigma_q^2(t) = \frac{\hbar}{2m} |\epsilon(t)|^2,$$

$$\sigma_p^2(t) = \frac{\hbar m}{2} \left[ \frac{1}{|\epsilon(t)|^2} + \left( \frac{d|\epsilon(t)|}{dt} \right)^2 \right].$$

$$c_{qp}^2(t) = \frac{\hbar^2}{4} |\epsilon(t)|^2 \left( \frac{d|\epsilon(t)|}{dt} \right)^2.$$  

Using these formulas we have been plotted the evolutions of the quantum mean values $\sigma_q^2$, $\sigma_p^2$ and cofluctuation $c_{qp}^2$, presented on figure 3. We can observe that the quantum oscillator possess all kind Schrödinger minimum uncertainty states $|SMUS>$; coherent, squeezed and cofluctuant states. The term subfluctuant has been suggested by Glauber [10] as more accurate than squeezed, but we will use the traditional squeezed for the states with a subfluctuation below the vacuum in a coherent state and cofluctuant for the states with nonzero cofluctuation. For example the oscillator is in a squeezed-cofluctuant state in the region $t < 0$, as $c_{qp} \neq 0$. They are ideal squeezed [17, 18] only when the cofluctuation $c_{qp} = 0$. In this situation the Schrödinger uncertainty relation devolves in Heisenberg one. In the region of the switching frequency the state is squeezed-cofluctuant, too. The same situation exist in the region $t > \frac{\pi}{2 \omega}$ except in the time moments

$$t_n = \frac{\pi}{2 \omega} + \left( \frac{1}{2} + n \right) \frac{\pi}{4 \omega \sqrt{1 - \frac{\alpha \omega}{(1 + \alpha \omega)^2}}},$$

(29)
\( n = 1, 2, 3, \ldots \) were the state is coherent (no squeezing, no cofluctuations - dimensionless fluctuations are equal; \( m\Omega(t_n)\sigma_q^2 = \sigma_p^2/m\Omega(t_n) = \hbar/2 \), \( z = \alpha ) \). The analysis shows that in the moment \( t = 0 \) for the initial conditions \( \epsilon(0) = \sqrt{1+\alpha^2} \) and \( \dot{\epsilon}(0) = i\sqrt{1+\alpha^2} \) the oscillator is in the ideal squeezed state.

An other possible description of the quantum problem is based on the important Wigner function, which we are going to concern here. The all information on the quantum system is contained in the time-dependent density operator \( \hat{\rho}(t) \), satisfying the conditions of hermiticity, normalization and nonnegativity.

For pure states, the density operator which is projector on the state \( |\Psi, t \rangle \), i.e.

\[
\hat{\rho}(t) = |\Psi, t \rangle \langle \Psi |
\]

satisfies the extra condition

\[
\hat{\rho}^2(t) = \hat{\rho}(t),
\]

and consequently,

\[
Tr\hat{\rho}^2(t) = 1.
\]

The density operator (30) obeys the equation for invariants \( \frac{d\hat{\rho}(t)}{dt} = 0 \) (which differs from Heisenberg equation of motion, see for example [13])

\[
\frac{\partial\hat{\rho}(t)}{\partial t} + \frac{1}{i\hbar}[\hat{\rho}(t), \hat{H}(t)] = 0
\]

Instead of density matrix one can consider Wigner function [21]. Such formulation of the quantum problem by means of functions on phase space is very convenient for quantum systems having classical analog. In Weil representation mean value of every physical variable is an integral of this variable with the distribution function over all phase space. This procedure is a full analog to the classical one.

The Wigner function is Fourier-transformation of the coordinate representation of density operator.

\[
W(q, p) = \int_{-\infty}^{\infty} \rho(q + \frac{v}{2}, q - \frac{v}{2}) exp(-ipv) dv
\]

where \( \rho(q, q', t) = \langle q | \hat{\rho}(t) | q' \rangle \) is the density matrix.

With the help of (30), one can calculate the Wigner function (33) in states \( |SMUS \rangle \);

\[
W(q, p, \epsilon(t), \frac{d\epsilon(t)}{dt}) = \frac{2}{\pi\hbar} \exp\left\{-\frac{2}{\hbar^2} \left[ \sigma_q^2 (p-<\hat{p}>)^2 - 2\sigma_{qp} (p-<\hat{p}>)(q-<\hat{q}>) + \sigma_p^2 (q-<\hat{q}>)^2 \right]\right\},
\]

where for the determinant of the variances matrix \( det(\sigma(t)) \) we have used the following expression:

\[
det(\sigma(t)) = \sigma_p^2 \sigma_q^2 - \sigma_{qp}^2 = \frac{\hbar^2}{4}.
\]

As far as \( \sigma_p, \sigma_q \) and \( c_{qp} \) (24-28) are functions of the solution of the equation (1), we have obtained the evolution of Wigner function in terms of \( \epsilon(t) \) and \( \frac{d\epsilon(t)}{dt} \) for our particular case of frequency (3). On the figure 3 Wigner function in the time moment \( t = 0 \) is shown. It corresponds to the initial point of the bold curve on figure 3, where the phase-space diagram of the quantum harmonic oscillator is presented.
In conclusion, an explicit solution of the equation for the classical harmonic oscillator, with smooth switching frequency has been found. A detailed analysis of a quantum harmonic oscillator with such frequency has been done on the base of the method of linear invariants. It has been shown that such oscillator also possesses cofluctuant states, different from widely studied Glauber coherent and ”ideal” squeezed states. It has been found the evolution of the Wigner function of such quantum oscillator with a switching frequency.

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1 Appendix 1

Here we will calculate the Wronsky determinant for the classical equation of the two-dimensional oscillator (4). To derive the Wronskian we need to know the first derivative of the function \( \epsilon(t) \) in the three regions;

\[
\frac{d\epsilon}{dt} = \begin{cases} 
-\sqrt{\frac{\omega(1+\alpha\omega+\alpha^2\omega^2)}{1+\alpha\omega}} \sin(\omega \sqrt{1-\frac{\alpha\omega}{(1+\alpha\omega)^2}} t) + i\sqrt{\frac{\omega}{1+\alpha\omega}} \cos(\omega \sqrt{1-\frac{\alpha\omega}{(1+\alpha\omega)^2}} t) \\
\frac{e^{i \int_0^t \frac{dt}{\sqrt{\frac{1}{\omega} + \alpha (\cos(\omega t))}}}}{\sqrt{\frac{1}{\omega} + \alpha (\cos(\omega t))}} (-\alpha \omega \sin(2\omega t) + i) \\
\frac{e^{i \int_0^t \frac{dt}{\sqrt{\frac{1}{\omega} + \alpha (\cos(\omega t))}}}}{\sqrt{\frac{1}{\omega} + \alpha (\cos(\omega t))}} (-\sqrt{\omega(1-\alpha\omega)} \sin(\omega \sqrt{1-\alpha\omega} (t - \frac{\pi}{2\omega})) + i\sqrt{\omega} \cos(\omega \sqrt{1-\alpha\omega} (t - \frac{\pi}{2\omega})))
\end{cases}
\]

By direct calculation using (36) we receive the Wronsky determinant;

\[
D_W(t) = \left| \frac{\epsilon(t)}{\frac{d\epsilon}{dt}} \frac{\epsilon^*(t)}{\frac{d\epsilon^*}{dt}} \right| = \epsilon(t)\dot{\epsilon}^*(t) - \epsilon^*(t)\dot{\epsilon}(t) = -2i.
\]

Obviously \( D_W(\epsilon(t), \dot{\epsilon}(t)) \) is an ( classical ) integral of motion for equation (4).

References

[1] W.Paul, Rev. Mod. Phys., 62 (1990) 531;
[2] D.A.Moralles, J. Phys. A, 21 (1988) 889;
[3] J.M.Cervero, J.D.Lejarreta, J. Phys. A, 22 (1989) L663;
[4] X.C. Gao, J.B.Xu, T.Z.Qian, Ann. Phys., 204 (1990) 235;
[5] S.K.Soui, J. Phys. A, 23 (1990) L951;
[6] S.K.Bose, B. Dutta-Roy, Phys. Rev. A, 43 (1991) 3217;
[7] W.Ditrich, M.Reuter, Phys. Lett., 155A (1991) 94;
[8] G.S.Agarwal, S.A.Kumar, Phys.Rev. Lett., 67 (1991) 3665;
[9] D.A.Trifonov, Phys. Lett. 48A (1974) 165;
[10] D.A.Trifonov, Bulg. J. Phys, 2 (1975) 303; D.A.Trifonov, Preprint ICTP IC/75/2 (1975). The quantum fluctuations $\sigma_q^2$ and $\sigma_p^2$ should be replaced in their expressions in formulas (18) and (15) in the references respectively;

[11] I.A.Malkin, V.I.Manko, D.A.Trifonov, Phys.Rev. D, 2 (1970) 1371;

[12] I.A.Malkin, V.I.Manko, D.A.Trifonov, J.Math. Phys., 14 (1973) 576;

[13] H.R. Lewis, W.B. Risenfeld, J.Math. Phys., v.10, p.1458 (1969);

[14] H.P. Yuen, Phys.Rev. A, 13. (1976) p.2226

[15] A.K. Angelow, Physica A, 1998 (accepted)

[16] R.J. Glauber, M. Lewenstein, Phys.Rev. A, v.43, no.1 (1991) 467-491;

[17] C.Caves, Lectures on Quantum Optics, University of Southern California, Los Angeles (1989);

[18] C.Caves, Phys. Rev. D, vol.23, no.8, pp.1693-1708 (1981);

[19] D.A.Trifonov, Completeness and geometry of Schrödinger minimym uncertainty states, J.Math.Phys., 34, 1, (1993) 100-110;

[20] G.A.Korn, T.M.Korn, Mathematical Handbook, MGraw-Hill Book Company, Inc., New York Toronto London (1961);

[21] E.Wigner, Phys.Rev. 40, no.5 (1932) 749-759;
