Two Dimensional Yang-Mills,

Black Holes

and Topological Strings

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We show that topological strings on a class of non-compact Calabi-Yau threefolds is equivalent to two dimensional bosonic $U(N)$ Yang-Mills on a torus. We explain this correspondence using the recent results on the equivalence of the partition function of topological strings and that of four dimensional BPS black holes, which in turn is holographically dual to the field theory on a brane. The partition function of the field theory on the brane reduces, for the ground state sector, to that of 2d Yang-Mills theory. We conjecture that the large $N$ chiral factorization of the 2d $U(N)$ Yang-Mills partition function reflects the existence of two boundaries of the classical $AdS_2$ geometry, with one chiral sector associated to each boundary; moreover the lack of factorization at finite $N$ is related to the transformation of the vacuum state of black hole from a pure state at all orders in $1/N$ to a state which appears mixed at finite $N$ (due to $O(e^{-N})$ effects).

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1. Introduction

In a recent paper [1], following earlier work [2], a relation was proposed between the partition function of 4d BPS black holes and the topological string partition function:

\[ Z_{BH} = |Z_{top}|^2. \]

On the other hand the black hole partition function is given by the partition function of a quantum field theory living on the brane which produces the black hole:

\[ Z_{BH} = Z_{brane} \]

which in turn leads to

\[ Z_{BH} = Z_{brane} = |Z_{top}|^2. \] (1.1)

In some cases, as was noted in [1], the relevant brane theory is given by a topologically twisted maximally supersymmetric Yang-Mills theory. The charge of the magnetic D-branes \( p^I \) and the chemical potentials for the electric D-branes \( \phi^I \) fix the homogeneous coordinates on the moduli of the Calabi-Yau by

\[ X^I = p^I + i \frac{\phi^I}{\pi} \]

where in the A-model topological string (in a suitable gauge) we have the topological string coupling constant \( g_s \) given by

\[ g_s = \frac{4\pi i}{X^0}. \]

The aim of this paper is to give a concrete realization of (1.1). In fact we shall see that an analogy pointed out in [1] can be concretely realized in this context: It was noted in [1] that the relation (1.1) is analogous to the relation between the large \( N \) expansion of the partition function of 2d Yang-Mills theory on a Riemann surface and the existence of an almost factorized structure into holomorphic and anti-holomorphic maps to the Riemann surface [3,4,5,6]. Here we construct a class of non-compact Calabi-Yau threefolds for which the \( Z_{brane} \) is equivalent to a topologically twisted 4 dimensional \( \mathcal{N} = 4 \) supersymmetric \( U(N) \) Yang-Mills theory which in turn can be mapped to the two dimensional bosonic Yang-Mills on a two dimensional torus. We thus end up identifying the topological string theory encountered in the large \( N \) expansion of Yang-Mills theory with that of topological A-model on a local Calabi-Yau threefold. This is consistent with the observations of
similarities between large $N$ 't Hooft expansion of 2d Yang-Mills and topological strings on 3-folds. For example it was already shown in [7] that the large $N$ expansion of 2d Yang-Mills theory on $T^2$ does satisfy the holomorphic anomaly equation of [8] which is a property expected for topological strings on threefolds. Moreover we show using direct computation of the corresponding topological string that the two sides are equal to all orders in string perturbation theory.

In fact this example also shows the more general meaning of (1.1): It is known that for finite $N$ the 2d Yang-Mills theory does not factorize to a holomorphic square. This is only the structure as $N \to \infty$. Nevertheless there is a precise finite answer for finite $N$. What this means is that the right hand side of (1.1) is the absolute value square of the holomorphic function to all orders in string perturbative expansion (i.e. to all orders in $1/N$) but at finite $N$ this structure loses its meaning. In other words there is no sense to the non-perturbative meaning of $Z_{\text{top}}$. It is only the “absolute value squared” $|Z_{\text{top}}|^2$, i.e. a density operator which may have a non-perturbative meaning. In the quantum mechanical correspondence discussed in [1] this would map to the statement that the state corresponding to the black hole is a pure state to all orders in the $1/N$ expansion which becomes mixed at order $O(\exp(-N))$. Here $N$ can be viewed as being proportional to the radius of $AdS_2$. Moreover we conjecture that to all orders in the $1/N$ expansion, where $Z_{\text{top}}$ and $\overline{Z}_{\text{top}}$ are well defined, they can be identified with the two boundaries of $AdS_2$, thus proposing a single dual theory for $AdS_2$.

The organization of this paper is as follows: In section 2 we review the results for the partition function for $U(N)$ and $SU(N)$ Yang-Mills on $T^2$. In section 3 we discuss a particular class of local threefolds (involving the total space of a direct sum of a line bundle and its inverse on $T^2$) and show, using the ideas of topological vertex, that the perturbative amplitudes of topological strings matches the large $N$ expansion for amplitudes of the $U(N)$ bosonic Yang-Mills on $T^2$. In section 4 we use the ideas in [1] to relate the topological string theory amplitudes to the dual brane theory, which turns out to be four dimensional $\mathcal{N} = 4$ topologically twisted Yang-Mills theory on a line bundle over $T^2$. In section 5 we show that this topologically twisted theory reduces to a two dimensional topologically twisted theory on $T^2$ which in turn is equivalent to 2d bosonic Yang-Mills on $T^2$, thus completing the circle of ideas. In section 6 we discuss some implications for the black hole physics.
2. Yang-Mills on the two dimensional torus

Two dimensional Yang-Mills theory was solved in [9,10,11,12,13]. Moreover this theory was studied at large \( N \) beginning with [3] from a number of viewpoints [4,5,6,14,15,16]. Here we will only need the results for the partition function of \( U(N) \) and \( SU(N) \) Yang-Mills theory on \( T^2 \). Consider the 2d Yang-Mills action for group \( G \):

\[
S = \frac{1}{2g^2_{YM}} \int_{T^2} [TrF^2 + \theta TrF]
\]

where the latter term is non-zero only for \( U(N) \). For simplicity of expression we often take the area of \( T^2 \) to be 1. The area dependence can be restored by dimensional analysis \((g^2_{YM} \rightarrow g^2_{YM} A)\). The partition function for this theory can be obtained as a sum over representations \( R \):

\[
\sum_{R} \exp(-\frac{1}{2}g^2_{YM}C_2(R) + i\theta C_1(R))
\]

For \( SU(N) \) the representations \( R \) are given by 2d Young diagrams. Consider a Young diagram where the \( k \)-th row has \( n_k \) boxes with \( k = 1, \ldots, N \), where \( n_i \geq n_j \) for \( i > j \). In this case the \( C_2(R) \) is given by

\[
C_2(R) = \sum_{i=1}^{N} n_i(n_i + N + 1 - 2i) - \frac{n^2}{N}
\]

where

\[
n = \sum_{i=1}^{N} n_i.
\]

For the \( SU(N) \) theory the \( C_1(R) = 0 \). The representations of the \( U(N) \) theory can be obtained from those of the \( SU(N) \) theory by noting the fact that

\[
U(N) = SU(N) \times U(1)/\mathbb{Z}_N.
\] (2.1)

In other words, by decomposing the representation in terms of a representaion \( R \) of \( SU(N) \) and a charge \( q \) of \( U(1) \) we have \( q = n + Nr \) reflecting the relation (2.1).

For the \( U(N) \) representation the casimirs are given by

\[
C_2(R, q) = C_2(R) + \frac{q^2}{N}
\] (2.2)

\[
C_1(R, q) = q
\]
At large $N$ the $SU(N)$ partition functions splits up essentially to a product of a holomorphic and an anti-holomorphic function involving certain chiral blocks $Z_{\pm}$ of the $U(N)$ theory. We have

\[
Z_+ = \sum_{R_+} \exp\left( -\frac{1}{2} g^2 Y M C_2(R_+) + i\theta |R_+| \right)
\]

\[
Z_- = Z_+^*
\]

where by the sum over $R_+$ we mean summing over all Young diagrams (i.e. labelling representations of $SU(\infty)$) and $|R_+|$ denotes the total number of boxes. We define a new casimir $\kappa(R_+)$ by

\[
C_2(R_+) = \kappa(R_+) + N|R_+|
\]

which is given by

\[
\kappa(R_+) = \sum_{i=1}^{\infty} n_i(n_i + 1 - 2i).
\]

Note that $\kappa(R_+)$ does not have any explicit dependence on $N$. We can write $Z_+$ as

\[
Z_+ = \sum_{R_+} \exp\left( -\frac{1}{2} g^2 Y M \kappa(R_+) - t|R_+| \right)
\]

where

\[
t = \frac{1}{2} Ng^2 Y M - i\theta.
\]

In terms of these chiral blocks the $SU(N)$ partition function at large $N$ takes the almost factorized form

\[
Z_{SU(N)} = \exp\left[ \frac{g^2 Y M}{2N} (\partial_t - \overline{\partial}_t)^2 \right] Z_+(t) Z_-(-\overline{t})
\]

where we set $t = \overline{t}$ after taking the derivative. This arises by splitting the 2d Young diagrams as excitations with columns with few boxes (giving $R_+$) combined with columns of order of $N$ boxes (whose complement gives $R_-$).

It is also useful for us to expand the $U(N)$ partition function at large $N$. In this case we have an extra sum over the $U(1)$ charges. Using the relation (2.2) we can relate this to the $Z_{SU(N)}$ as follows: Let $n_\pm$ denotes $|R_\pm|$. Then the total number of boxes of the $SU(N)$ representation is $Nl_- + (n_+ - n_-)$ where $l_-$ is the number of boxes of the first row of $R_-$. Since the $U(1)$ charge $q$ is equal to the number of boxes of the Young diagram mod $N$ we have

\[
q = Nr + Nl_- + (n_+ - n_-) = Nl + (n_+ - n_-)
\]
where we have defined a new variable \( l = r + l_- \). Here \( l \) runs over arbitrary positive and negative integers. We thus have to add to partition function of \( SU(N) \) a multiplicative term

\[
\exp(-g_{YM}^2 q^2/2N) = \exp\left[\frac{-g_{YM}^2 (Nl + n_+ - n_-)^2}{2N}\right]
\]

Note that this multiplicative term cancels the \( g_{YM}^2 (n_+ - n_-)^2 / 2N \) term which mixes the two chiral block and instead gives

\[
\exp(-g_{YM}^2 [l(n_+ - n_-)] - \frac{1}{2} Nl^2 g_{YM}^2) = \exp(g_{YM}^2 l(\partial_t - \partial_t) - \frac{1}{2} Nl^2 g_{YM}^2)
\]

This acting on the two blocks just shifts the arguments together with the extra prefactor

\[
\exp(-\frac{1}{2} (t + \bar{t})(\bar{t} - t)^2) \cdot Z_+ (t + lg_{YM}^2)Z_- (\bar{t} - lg_{YM}^2)
\]

We also have to add the \( i\theta q \) term to this. In order to do this note that if we do not set \( t = \bar{t} \) the above expression would have given \( i\theta (n_+ - n_-) \). This is almost \( q \) given by (2.3). To correct it we need to also add an extra factor

\[
\exp[-\frac{1}{2} (t + \bar{t})(t - \bar{t})l/2] = \exp\left[\frac{-1}{g_{YM}^2} (t + \bar{t})(t - \bar{t})l/2\right] = \exp\left[\frac{-1}{2g_{YM}^2} (t^2 - \bar{t}^2)l\right]
\]

Note that the prefactors can be neatly combined as follows: Consider a modified

\[
Z_+ (t) \to Z_+ (t) \cdot \exp\left[\frac{-t^3}{3!g_{YM}^4}\right]
\]

It is also natural to further multiply \( Z_+ (t) \to Z_+ (t) \cdot \exp(t/24) \) (this will give the usual prefactor for the one loop term for the Dedekin eta function). Similarly it is natural to add a ‘casimir’ term to the \( U(N) \) Yang-Mills:

\[
Z_{U(N)} \to \exp\left[-\frac{t^3 + \bar{t}^3}{3!g_{YM}^4} + \frac{t + \bar{t}}{24}\right]Z_{U(N)}
\]

This amounts to adding a term \( N(N^2 - 1)/12 \) term to the \( C_2(R) \) (when \( \theta = 0 \)) which is natural in the fermionic formulation of 2d Yang-Mills \([14,15]\). In the topological string context the \( N^3 \) terms corresponds to the classical amplitude at genus 0 and the \( N \) term corresponds to the classical contribution at genus 1. Then we have the full \( U(N) \) partition function as being given at large \( N \) by

\[
Z_{U(N)} = \sum_{l=-\infty}^{+\infty} Z_+ (t + lg_{YM}^2)Z_- (\bar{t} - lg_{YM}^2)
\]

(2.4)

It is this factorized form which we will need to make contact with the predictions of [1]. Note that non-perturbatively, i.e. for finite \( N \), the expression (2.4) would not be true. The full answer of course exists for finite \( N \) but does not take a factorized form as is suggested in this perturbative version. Later we will interpret the sum over \( l \) as a sum over RR-fluxes on \( T^2 \) in the type IIA setup.
3. Local Elliptic Threefold and $U(N)$ Yang-Mills on $T^2$

Topological A-model amplitudes on local toric threefolds can be effectively computed using topological vertex [17]. This involves using the toric graph of the threefold as a Feynman diagram with cubic vertices, with edges labeled by arbitrary 2d Young-diagrams. Each edge has a ‘Schwinger time’ $t_i$ (which corresponds to Kähler classes of the 3-fold) and a propagator factor given by $e^{-t_i|R|}$ where $|R|$ is the number of boxes of the Young diagram $R$. In addition each oriented edge can be identified with a primitive 1-cycle $v$ of a two torus. With a given choice of basis this can be represented by two relatively prime integers $(p, q) = v$. In addition each edge comes with a choice of a ‘framing’ $f$ which is another choice of a 1-cycle of $T^2$ with the property that $(f, v)$ forms a good basis for the integral 1-cycles of $T^2$. This means $f \cap v = 1$. Note that $f$ is unique up to an integer shift $m$ times $v$

$$f \rightarrow f + mv$$

For any three edges incoming to a vertex we have

$$\sum_{i=1}^{3} v_i = \sum_{i=1}^{3} (p_i, q_i) = (0, 0)$$

The interactions are cubic and given by a vertex $C_{R_1R_2R_3}$. More precisely the vertex does depend on the choice of the three framings $f_i$. Changing the framing $f_i$ by an integer amount $f_i \rightarrow f_i + m_i v_i$ changes

$$C_{R_1R_2R_3} \rightarrow q \sum_{i}^{3} m_i \kappa(R_i)/2 C_{R_1R_2R_3}$$

where $q = e^{-g_s}$ and $\kappa(R)$ denotes a second casimir of $R$ with the unique property that $\kappa(R^t) = -\kappa(R)$, where $R^t$ denotes the Young diagram which is the transpose of $R$.

Fixing a canonical framing and gluing the vertices with compatible framing leads to arbitrary toric threefolds. It is possible to choose different framings on each edge without affecting the geometry as long as the framing on both vertices are changed in opposite directions. The effect of framing would then cancel. If however one changes the relative framing between vertices which are being glued together one changes the intersection numbers of the Calabi-Yau. This can be inferred for example from [17,18]. Let $C_2$ denote the two cycle defined by the edge and $C_4$ define a 4-cycle whose projection
on toric base contains the same edge. Then changing the relative framing of the edge by $m$ units affects the intersection of $C_2$ and $C_4$ by

$$\Delta(C_2 \cap C_4) = m$$

There is a sense of orientation here. In particular there are two natural toric 4-cycles containing $C_2$, given by one of the two faces bordering the edge. Let us denote them by $C_4, C'_4$. $C_4$ is determined by the direction of the framing vector $f$. For $C'_4$ we have

$$\Delta(C_2 \cap C'_4) = -m.$$ 

The elements involved in the definition of the topological string amplitudes for toric threefold are very similar to that of 2d Yang-Mills. Here we wish to make contact with 2d $U(N)$ Yang-Mills on $T^2$, whose chiral partition function is given, as discussed in the previous section, by

$$Z_{YM}^+ = \sum_R Q^{C_2(R)/2} \exp(i\theta C_1(R))$$

where $Q = \exp[-g^2_{YM} A]$ where $A$ is the area of torus and $g_{YM}$ is the Yang-Mills coupling and $\theta$ is the theta angle in 2d. For simplicity of notation we set $A = 1$ (by dimensional analysis it can always be restored). This is a very similar object to the propagator of the topological strings, where $C_2(R)$ and $\kappa(R)$ differ by terms involving the first casimir $C_1(R) = |R|$. In fact this can also be rewritten as discussed in the last section as

$$Z_{YM}^+ = \sum_R Q^{\kappa(R)/2} \exp(-\frac{1}{2}N g^2_{YM} + i\theta)|R|$$

Let us define

$$t_{YM} = \frac{1}{2}N g^2_{YM} - i\theta$$

Then we have

$$Z_{YM}^+ = \sum_R Q^{\kappa(R)/2} \exp(-t_{YM}|R|). \quad (3.1)$$

This is very similar to the expressions of topological strings on toric threefolds and we ask whether or not it corresponds to a particular threefold? The fact that there is a trace over all representations suggests that we glue an edge to itself, giving a ‘loop’ for the

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1 This has been previously observed by R. Gopakumar and M. Marino in the context of finding a closed string dual to 2d Yang-mills on the sphere. See also the related work.
toric Feynman diagram. This is only possible if the plane of toric diagram is periodically identified. It is possible to interpret this geometrically as was done in [17] and [21]. In particular if we just consider gluing an edge to itself we obtain a torus whose Kahler class is given by the length of the edge $t$. Note that this is consistent (up to classical polynomial terms in $t$ that topological vertex does not compute) with the fact that

$$\sum_R e^{\exp(-t|R|)} = 1/\prod_{n=1}^{\infty} (1 - e^{-nt})$$

and that the topological string amplitude on $\mathbb{C} \times \mathbb{C} \times T^2$ is given by the inverse of Dedekind eta function (assuming that euler character of $\mathbb{C}^2$ is one) [22].

To obtain the full structure of the 2d Yang-Mills (3.1) we also need to include framing. In particular before gluing the vertex back to itself we perform a change of framing by $m$ units and then glue. In this way we obtain

$$Z_{top} = \sum_R q^{m\kappa(R)/2}e^{\exp(-t|R|)}$$

We see that $Z_{top} = Z_{YM}$ provided that we identify

$$Q = q^m \to g_{YM}^2 = mg_s$$

$$t_{YM} = t \to \frac{1}{2}Ng_{YM}^2 - i\theta = t \to \frac{1}{2}Nmgs - i\theta = t$$

Note that the classical triple intersection number for this geometry would lead to the natural prefactor we introduced for the $U(N)$ Yang-Mills theory if we assign it to be

$$\frac{F_0(t)}{g_{YM}^2} + F_1(t) = -\frac{t^3}{3!m^2g_{YM}^2} + \frac{t}{24} = -\frac{t^3}{3!g_{YM}^4} + \frac{t}{24}.$$ 

Due to the non-compactness of the geometry this is a bit difficult to apriori justify; nevertheless it is gratifying to see that the Yang-Mills Casimir prefactors have a natural geometric interpretation in terms of a choice of triple intersection number (entering $F_0$) and the second chern class (entering $F_1$).

We next turn to what is the non-compact threefold geometry defined by this framing operation.
3.1. The elliptic threefold geometries

As discussed before the framing operation affects the intersection numbers. To begin with we can view the local geometry before twisting by the framing factor, as a direct sum of two trivial line bundles on $T^2$:

$$\mathcal{O}(0) \oplus \mathcal{O}(0) \to T^2$$

The two faces of the toric diagram correspond to the two cycles $C_4$ and $C_4'$ which in this case are both isomorphic to

$$C_4 = C_4' = \mathcal{O}(0) \to T^2$$

Note that

$$C_4 \cap T^2 = 0$$

After twisting by the framing the two line bundles have changed in such a way that their intersection numbers with $T^2$ change to $m$ and $-m$. Moreover their line bundles are canonically inverse of one another (for the threefold to have trivial canonical line bundle). We will now show that this implies that the local threefold is given by

$$\mathcal{L}^{-m} \oplus \mathcal{L}^m \to T^2$$

where $\mathcal{L}^m$ is a degree $m$ line bundle over $T^2$. This is unique up to tensoring by a flat bundle. $\mathcal{L}^m$ is characterized by the statement that a holomorphic section of this bundle has a divisor of degree $m$ on $T^2$ which denotes the zeros of the corresponding holomorphic section. Moving the $m$ divisors on $T^2$ corresponds to choosing a different holomorphic section. $\mathcal{L}^{-m}$ is the inverse degree $-m$ bundle. Each meromorphic section of it will have at least $m$ poles. In this context we have

$$C_4 = \mathcal{L}^{-m} \to T^2$$

$$C_4' = \mathcal{L}^m \to T^2.$$

We will now show that

$$C_4 \cap T^2 = m.$$

To show this deform $T^2$ to $T^2'$ by using a holomorphic section of $\mathcal{L}^m$. As long as we are away from zeros of this section this does not intersect $C_2$. The zeros of the holomorphic section are precisely where it intersects $C_4$ transversally. Thus the intersection number
is \( m \) because there are \( m \) zeros for the holomorphic section. Note that this uniquely defines the geometry of the normal bundles. We have thus shown that the local threefold corresponding to the case where \( Z_{\text{top}} = Z_{YM} \) is given by the sum of a degree \( m \) line bundle and its inverse on \( T^2 \). This local geometry we shall call \( E_m \):

\[
E_m = \mathcal{L}^{-m} \oplus \mathcal{L}^m \to T^2.
\]

4. Black Hole and Elliptic Threefold

Recently it was suggested in [1] that there is a deep connection between 4d BPS black holes and topological strings, namely the partition function of 4d black holes is given by the square of the topological string partition function:

\[
Z_{BH} = |Z_{\text{top}}|^2
\]

where the moduli of the Calabi-Yau is fixed by the attractor mechanism. Moreover

\[
Z_{BH} = Z_{\text{brane}}
\]

is the partition function on the brane defining the black hole, which leads to

\[
Z_{\text{brane}} = |Z_{\text{top}}|^2.
\]

This is roughly the structure we have here. Namely if somehow the brane theory is equivalent to \( U(N) \) 2d Yang-Mills, since

\[
Z^{YM} \sim Z_{+}^{YM} Z_{-}^{YM} = Z_{\text{top}} Z_{\text{top}}
\]

we would obtain \( Z_{\text{brane}} = |Z_{\text{top}}|^2 \). This is almost the structure we found for \( U(N) \) 2d Yang-Mills theory at large \( N \). However we found in addition an extra sum over integer \( l \):

\[
Z^{YM} = \sum_l Z_{+}^{YM}(t + lg_{YM}) Z_{-}^{YM}(\bar{t} - lg_{YM})
\]

This extra sum was related to the \( U(1) \subset U(N) \) charge. Viewed from the topological string side this would lead to the statement

\[
Z_{\text{brane}} = \sum_l Z_{\text{top}}(t + mlg_s) Z_{\text{top}}(\bar{t} - mlg_s)
\]

(4.1)
We could have restricted the sum to a fixed \( l = 0 \) and interpret it as ‘freezing’ of the \( U(1) \) center of mass dynamics of the black hole. However, given that the non-perturbative completion of the \( U(N) \) theory requires this sum, it is important to better understand the physical meaning of this sum for the black hole partition function. Our local geometry \( E_m \) is to be viewed as the attractor fixed point for the black hole geometry. However the full partition function of the theory should include a sum over all finite energy configurations, and in this case this includes the sum over \( RR \) flux on the compact cycle \( T^2 \). Note that this can preserve the same amount of supersymmetry as the BPS black hole and it would be natural to expect that it enters the full black hole partition function. It is also natural to consider turning on RR-flux on 2-cycles. As discussed in \([23]\) this has the effect of shifting

\[
Z_{\text{top}} \to Z_{\text{top}}(t + l m g_s)
\]

where the coefficient \( m \) enters here because of the intersection of \( T^2 \) with \( C_4 \) as was explained in \([23]\). The same reasoning as in \([23]\) would lead to the statement that for anti-topological theory \( Z_{\text{top}} \to Z_{\text{top}}(\bar{t} - l m g_s) \). This means that the dynamics of the \( U(1) \) sector mixes with the \( RR \) 2-form flux (which is natural from the viewpoint of the CS coupling of the \( U(1) \) to the gravitational sector). We thus end up with a simple interpretation of \((2.4)\) as the \textit{full} partition function for the black hole background including a sum over \( RR \) fluxes through \( T^2 \). Thus \((4.1)\) is consistent with the results of \([1]\) which focused on the zero flux sector \( l = 0 \).

4.1. Setting up the map between 4d and 2d Yang-Mills

To complete the circle of ideas we need to show that the brane gauge theory which is the four dimensional \( \mathcal{N} = 4 \) topologically twisted \( U(N) \) Yang-Mills theory on \( C_4 \) is equivalent to bosonic 2d \( U(N) \) Yang-Mills theory on \( T^2 \) with

\[
g_{YM}^2 = m g_s \quad \theta_{YM} = \theta.
\]

The relation between the charges and the moduli in the type IIA setup is given by \([1]\)

\[
t^i = 2 \pi i X^i / X^0
\]

\[
g_s = 4 \pi i / X^0
\]
where \( i = 1, \ldots, h^{1,1} \) and

\[
X^I = p^I + i \frac{\phi^I}{\pi}
\]

where \( p^i \) denotes the magnetic charge (given by D4 branes wrapped over 4-cycles) dual to the electric charges (given by the D2 branes wrapped on dual 2-cycles) and \( \phi^i \) are the chemical potentials for the electric charge. Moreover \( p^0 \) denotes the number of D6 branes and \( \phi^0 \) is the chemical potential for the 0-branes.

Even though the main discussion in [1] was in the context of compact Calabi-Yau’s, it is natural to expect that this relation continues to hold (with suitable boundary conditions) also for the case of non-compact Calabi-Yau. This is natural to expect because the right hand side does make sense for the non-compact Calabi-Yau’s. In fact this can be viewed as a limit of the considerations of [1] where we take a large volume limit of a compact Calabi-Yau and focus on a particular sector of the theory localized near some compact cycles.

So let us consider our elliptic threefold \( E_m \). There is only one compact 2-cycle \( T^2 \) which we identify with an electric charge. Here we only have \( X^0 \) and \( X^1 \) and

\[
t = 2\pi i \frac{X^1}{X^0}
\]

For the magnetic charge dual to the electric 2-cycle charge we can consider the four cycle \( C_4 \). However note that the quantum of magnetic charge is \( m \) times bigger on this cycle because \( C_4 \cap T^2 = m \). Thus if we consider \( N \) D4 branes wrapping \( C_4 \) it will have magnetic charge \( p^1 = Nm \).

To make contact with the parameters defined in the previous section we see that we have to set

\[
g_s = \frac{4\pi i}{X^0} = \frac{4\pi i}{p^0 + i\frac{\phi^0}{\pi}}
\]

which for real \( g_s \) implies \( p^0 = 0 \), i.e. we have no D6 branes. We thus set

\[
g_s = \frac{4\pi^2}{\phi^0} \rightarrow \phi^0 = \frac{4\pi^2}{g_s}
\]

where \( \phi^0 \) is the chemical potential for the 0-brane. Similarly we have

\[
X^1 = \frac{1}{2\pi i} X^0 t = \frac{2}{g_s} \left( \frac{1}{2} N mg_s + i\theta \right) = Nm + \frac{2i\theta}{g_s}
\]
which implies

\[ p^1 = Nm \quad \phi^1 = \frac{2\pi}{g_s} \theta \]

We now ask what is the gauge theory on the brane? We have no D6 branes but we do have \( Nm \) magnetic branes. This is reassuring as it is naturally quantized in multiples of \( m \): We just take \( N \) D4 branes wrapped around \( C_4 \) and this gives magnetic charge \( Nm \) as discussed before. This gives a topologically twisted \( \mathcal{N} = 4 \), \( U(N) \) gauge theory on \( C_4 \). In fact the twisting is exactly the one studied in [24]. All we have to do now is to induce the corresponding chemical potentials for the D0 and D2 branes. To induce a D0 brane charge we have to use an interaction term \( \frac{1}{4\pi^2} \int_{C_4} \frac{1}{2} \text{tr} F \wedge F \) which means that we introduce the term

\[ \frac{\phi^0}{4\pi^2} \int_{C_4} \frac{1}{2} \text{tr} F \wedge F = \frac{1}{2g_s} \int_{C_4} \text{tr} F \wedge F \]

To induce the electric charge corresponding to D2 wrapping \( T^2 \) we note that the charge is measured by \( \frac{1}{2\pi} \int_{C_4} F \wedge k \) where \( k \) is the unit volume 2-form on \( T^2 \). We thus add the additional term to the action

\[ \phi^1 \int \frac{F \wedge k}{2\pi} = \frac{\theta}{g_s} \int F \wedge k \]

We thus add to the topologically twisted \( \mathcal{N} = 4 \) Yang-Mills on \( C_4 \) the term

\[ \frac{1}{2g_s} \int_{C_4} \text{tr} F \wedge F + \frac{\theta}{g_s} \int F \wedge k \] (4.2)

All we have to do now is to show that the partition function of this theory is equivalent to that of 2d \( U(N) \) Yang-Mills theory on \( T^2 \) with \( g_{YM}^2 = mg_s \) and \( \theta_{YM} = \theta \). We will show in the next section that, modulo some plausible assumptions, this is indeed the case.

5. \( \mathcal{N} = 4 \) topological Yang-Mills and 2d Yang-Mills

The \( \mathcal{N}=4 \) theory on the \( N \) D-branes wrapping \( C_4 \) is the twisted theory of the type studied in [24]. To see this it essentially suffices to study the spin content of the 3 complex adjoint fields \( T, U, V \). In the theory studied in [24], for Kähler manifolds, two of them \( T, U \) are ordinary scalars but one of them \( V \) is a section of \( (2, 0) \) line bundle. As noted in [25] to find the twisted theory and in particular the spin content of the scalars we have to look at the normal bundle to our brane, i.e. the normal bundle to \( C_4 \). Four of them correspond to moving in spacetime and these correspond to scalars on \( C_4 \). These we identify with
the \( T, U \). The other one \( V \) corresponds to moving \( C_4 \) inside the Calabi-Yau and thus this corresponds to a section of \( V \in S(\mathcal{L}^m) \). But that is precisely the \((2,0)\) bundle on \( C_4 \). To see this note that \( Tdwdz \) would be well defined if \( w \) denotes the coordinate of \( \mathcal{L}^{-m} \) bundle over \( T^2 \) and \( z \) denotes the coordinates of \( T^2 \).

To simplify the analysis of this theory we add mass terms to the adjoint fields as in [26] and [24]. This corresponds to addition of a superpotential

\[
W = mUV + \omega T^2
\]

where as explained in [26] \( W \) is a section of \((2,0)\) bundle on \( C_4 \) which is satisfied if \( m \) is a constant and \( \omega \) is a holomorphic section of the \((2,0)\) line bundle. Addition of this term does not affect the \( \mathcal{N} = 4 \) topological theory. The simplification for doing this is that the theory is simpler with \( \mathcal{N} = 1 \) and in particular has a mass gap. The basic strategy of [24] was to study the symmetry structure of this theory. The simplest situation would arise if \( \omega \) has no zeroes. In this case we have a purely \( \mathcal{N} = 1 \) theory in the IR and this structure and topological invariance was enough to fix all the parameters (with the help of a few computations for special examples) [24]. If \( \omega \) has zeroes, it will appear at complex codimension one, i.e. on ‘cosmic string’ loci in the four manifold. This would then give additional contributions to the bulk, coming from the cosmic strings. Again some general facts about what can appear there and the assumption of Montonen-Olive duality essentially fixed the answer in [24].

The case at hand has one major difference with most of the cases considered in [24]: For us the four manifold \( C_4 \) is non-compact. For such a case no general strategy was proposed in [24] but some special cases related to the work of Nakajima on instantons on ALE spaces was studied as a check of Montonen-Olive conjecture. A crucial role was played there by the choice of a flat connection at infinity, which relates to the choice of the holonomy of the \( U(N) \) gauge theory at infinity, i.e. a map \( \pi_1(\text{Boundary}) \to U(N) \).

The approach we follow here is to add the mass term, as was done in [24] for the compact cases, but also use the symmetries of the non-compact theory to constrain our answer. For us \( \omega \) is a section of \( \mathcal{L}^m \). This means that each holomorphic section will have \( m \) zeroes localized on \( m \) cosmic strings which span \((z_i, w)\) where \( z_i \) denotes the zero of the section \( \omega \) as a function of the point \( z \) on \( T^2 \), and \( w \) denotes the coordinate of the holomorphic line bundle \( \mathcal{L}^{-m} \) over each zero \( z_i \). First let us ignore these zeroes—we will return to them after we have discussed the bulk physics.
The fact that the space is non-compact means that we cannot sum over all the allowed vacua as in the compact case of [24] but rather we have to select which vacuum we are in. We take this to be the vacuum corresponding to preserving the $U(N)$ symmetry and which classically corresponds to setting $T = U = V = 0$.

Since our space is non-compact we have to deal with boundary conditions. In particular for each point $z$ on the torus we have a complex plane $w$ representing the fiber of $\mathcal{L}^{-m}$. For each such plane we have to specify the boundary condition. Since the boundary of the complex plane is a circle, all we have to do is to specify the holonomy $U$ at a circle at infinity. Let for each $z$, $\Phi(z)$ denote the generator of this holonomy, i.e., $U = e^{\Phi}$. In other words in a suitable gauge

$$\Phi(z) = \oint_{S^1_{w, |w| = \infty}} A. \tag{5.1}$$

What can be said about the field configurations as we come inwards on each fiber? We note that $C_4$ enjoys a $U(1)$ symmetry corresponding to phase rotations of the line bundle $\mathcal{L}^{-m}$:

$$(z, w) \to (z, e^{i\theta}w)$$

We will make the assumption that the path integral for the topologically twisted Yang-Mills can be localized to $U(1)$ invariant configurations. To argue why this can be assumed note that the Kähler metric on each fiber of $C_4$ can be viewed as $d|w|^2 \wedge d\theta$ which is a half-line times a circle. That we can restrict the configurations of the supersymmetric theory to be independent of $\theta$ is plausible as is familiar from Witten’s index $\text{Tr}(-1)^F \exp(-\beta H)$: The index is independent of $\beta$ and considering $\beta \to 0$ gives time independent field configurations. The main difference in our case is that our space is a half-line and so the configurations can become somewhat singular near $|w| = 0$. We will assume that the nature of the singularity is mild and compatible with the $\theta$ invariance for $|w| > 0$. In particular being $\theta$ independent means that for each fiber (in a suitable gauge)

$$\int_{\text{fiber}} F_{w\bar{w}}(z, w)dw d\bar{w} = \Phi(z)$$

Note that this circle invariance has effectively reduced our $\mathcal{N} = 1$ gauge theory on $C_4$ to a topologically twisted $(2, 2)$ supersymmetric gauge theory on $T^2$. Under this reduction the observables in (4.2) map to

$$\int_{T^2} \left[ \frac{1}{g_s} \text{Tr}F\Phi + \frac{\theta}{g_s} \text{Tr}\Phi \right]$$
So far we have ignored the $m$ zeroes of $\omega$. At these points $T$ becomes massless and so integrating $T$ out for the effective $(2, 2)$ theory on $T^2$ should introduce some topologically invariant point-like observables on the effective 2d theory at these points. The topologically invariant point-like observables at a point $z$ are given by $Tr\Phi^r(z)$ for some positive integer $r$. We will now argue that only the $r = 2$ term is generated and with strength $1/2g_s$. In particular we will now show that the term

$$\sum_{i=1}^{m} \frac{1}{2g_s} Tr\Phi^2(z_i) \quad (5.2)$$

is generated. We will argue this by consistency as follows: When we describe the field $\Phi$ as the holonomy about the circle at the infinity of the $L^{-m}$ fiber, this assumes a trivialization of this circle bundle. Let $1/\vartheta(z, \tau)$ denote a holomorphic section of the $L^{-m}$ bundle where $\vartheta(z, \tau)$ is a suitable theta function. This section will have $m$ poles at $z_i$, the zeroes of $\vartheta$. We can use the argument of $1/\vartheta$ to define a trivialization of the circle bundle on $T^2 - \{z_i\}$. This gives a global definition of holonomy $\Phi$ for $T^2 - \{z_i\}$. We then need to glue this to the circle bundles over each small neighborhood $U_i$ of $z_i$. However in this gluing we will need to do a ‘surgery’ because of the poles of $1/\vartheta$: The circle on $T^2$ going around each $z_i$ is identified with the circle at infinity over $z_i$. Let $S^1_i$ denote an infinitesimal circle on $T^2$ going around $z_i$, and let $S^1_{\infty i}$ denote the circle over $z_i$ from the line bundle. Then we have to identify these two classes:

$$[S^1_i] = [S^1_{\infty i}].$$

If we consider the holonomy around this circle we learn that

$$\int_{\text{neighborhood } z_i} F(z) = \int_{S^1_i} A = \int_{S^1_{\infty i}} A = \Phi(z_i)$$

This is exactly consistent with the insertion of the term $\frac{1}{g_s} Tr\Phi^2(z_i)$, because from the term $\frac{1}{g_s} Tr\Phi dA$ we see that $A$ and $\Phi$ are conjugate variables so if we consider

$$\frac{1}{2g_s} \left< \ldots \oint_{z_i} A \ Tr\Phi^2(z_i) \ldots \right> = \left< \ldots \Phi(z_i) \ldots \right>$$

reproducing the above geometric fact. Summing up all these terms for each point we obtain the term given in (5.2).
Since the point observables are independent of which point they are inserted we can put them at any point and integrate over them with the unit area. We have thus ended up with the action

$$\int_{T^2} \left[ \frac{1}{g_s} \text{Tr} F \Phi + \frac{\theta}{g_s} \text{Tr} \Phi + \frac{m}{2g_s} \text{Tr} \Phi^2 \right]$$

As argued in [27], this topologically twisted theory is equivalent to the bosonic 2d Yang-Mills theory upon integrating out $\Phi$ and the fermions. In particular we get

$$- \int_{T^2} \frac{1}{m g_s} \left[ \frac{1}{2} \text{Tr} F^2 + \theta \text{Tr} F \right]$$

Leading to the identification

$$g_{YM}^2 = mg_s \quad \theta_{YM} = \theta.$$

This concludes what we wished to show.

6. Implications/speculations for the black hole physics

It was conjectured in [1] that a relation of the form $Z_{BH} = |Z_{top}|^2$ exists for all BPS black holes in 4 dimensions. Note that our geometry is given by $AdS_2 \times X$ where $X = S^2 \times CY$. What is the meaning of this squared structure $|...|^2$? In this context it is natural to recall a related puzzle [28,29,30]: Unlike higher dimensional $AdS_p$ geometries, $AdS_2$ has two boundaries. This has raised the question of whether we should have one or two $CFT_1$ duals. In the present case, as noted in [1] the $AdS_2$ geometry should be holographic dual to the $D4$ brane worldvolume theory, which as we have noted reduces, for the ground state sector, to the dynamics of 2d Yang-Mills. The geometry of $AdS_2$ is reminiscent of the structure of 2d $U(N)$ Yang-Mills theory which at $N >> 1$ splits to two chiral sectors which don’t talk with one another. It is natural to identify one chiral sector with one boundary of $AdS_2$ and the conjugate sector with the opposite boundary. At infinite $N$ the two boundaries don’t talk with one another just as the two chiral sectors of the $U(N)$ Yang-Mills don’t couple. In terms of our discussion we conjecture that classically (as $N \rightarrow \infty$) we identify $Z_{top}$ with the wave function at the left boundary of $AdS_2$ and $\overline{Z}_{top}$ with the wave function at the right boundary. There is some evidence that this is the correct picture [31], where one identifies the dynamics near the two fermi surfaces of the fermionic formulation of 2d Yang-Mills [14,15] with the two boundaries of $AdS_2$. 

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However what is the significance of the lack of the splitting\(^2\) of the two chiral sectors at finite \(N\) as we have discovered here? It was noted that \(Z_{\text{top}}\) should be viewed as a state in a Hilbert space \([33]\) which would be natural by associating each to one of the two boundaries of \(AdS_2\). To gain a better understanding of the mixing let us rewrite the result for \(Z_{BH}\) as

\[
Z_{BH} = \langle Z_{\text{top}} | Z_{\text{top}} \rangle = \text{Tr} | Z_{\text{top}} \rangle \langle Z_{\text{top}} | = \text{Tr} \rho
\]

where \(\rho\) is a density matrix. In this context the lack of factorization at finite \(N\) means that in this context there is no \(| Z_{\text{top}} \rangle \) which satisfies the above equation. In other words \(\rho\) is not a pure state at finite \(N\). Associating this to the vacuum state of the black hole would suggest that for finite \(N\) the black hole vacuum is not a pure state. These ideas are currently under investigation \([31]\).

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\(^2\) For a different string theory application of lack of this factorization see \([32]\).
References

[1] H. Ooguri, A. Strominger and C. Vafa, “Black hole attractors and the topological string,” arXiv:hep-th/0405146.
[2] G. Lopes Cardoso, B. de Wit and T. Mohaupt, “Corrections to macroscopic supersymmetric black-hole entropy,” Phys. Lett. B 451, 309 (1999) [arXiv:hep-th/9812082].
[3] D. J. Gross, “Two-dimensional QCD as a string theory,” Nucl. Phys. B 400, 161 (1993) [arXiv:hep-th/9212149].
[4] D. J. Gross and W. I. Taylor, “Two-dimensional QCD is a string theory,” Nucl. Phys. B 400, 181 (1993) [arXiv:hep-th/9301068].
[5] G. W. Moore, “2-D Yang-Mills theory and topological field theory,” arXiv:hep-th/9409044.
[6] S. Cordes, G. W. Moore and S. Ramgoolam, “Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories,” Nucl. Phys. Proc. Suppl. 41, 184 (1995) [arXiv:hep-th/9411210].
[7] R. Dijkgraaf, “Chiral deformations of conformal field theories,” Nucl. Phys. B 493, 588 (1997) [arXiv:hep-th/9609022].
[8] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” Commun. Math. Phys. 165, 311 (1994) [arXiv:hep-th/9309140].
[9] A. A. Migdal, “Recursion Equations In Gauge Field Theories,” Sov. Phys. JETP 42, 413 (1975) [Zh. Eksp. Teor. Fiz. 69, 810 (1975)].
[10] B. E. Rusakov, “Loop Averages And Partition Functions In U(N) Gauge Theory On Two-Dimensional Manifolds,” Mod. Phys. Lett. A 5, 693 (1990).
[11] D. S. Fine, “Quantum Yang-Mills On The Two-Sphere,” Commun. Math. Phys. 134, 273 (1990).
[12] E. Witten, “On Quantum Gauge Theories In Two-Dimensions,” Commun. Math. Phys. 141, 153 (1991).
[13] M. Blau and G. Thompson, “Quantum Yang-Mills theory on arbitrary surfaces,” Int. J. Mod. Phys. A 7, 3781 (1992).
[14] J. A. Minahan and A. P. Polychronakos, “Equivalence of two-dimensional QCD and the C = 1 matrix model,” Phys. Lett. B 312, 155 (1993) [arXiv:hep-th/9303153].
[15] M. R. Douglas, “Conformal field theory techniques in large N Yang-Mills theory,” arXiv:hep-th/9311130.
[16] R. E. Rudd, “The String partition function for QCD on the torus,” arXiv:hep-th/9407176.
[17] M. Aganagic, A. Klemm, M. Marino and C. Vafa, “The topological vertex,” arXiv:hep-th/0305132.
[18] A. Iqbal, to appear
[19] R. Gopakumar and M. Marino (unpublished), 2003.
[20] S. de Haro and M. Tierz, “Brownian Motion, Chern-Simons Theory, and 2d Yang-Mills,” to appear.
[21] T. J. Hollowood, A. Iqbal and C. Vafa, “Matrix models, geometric engineering and elliptic genera,” arXiv:hep-th/0310272.
[22] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Holomorphic anomalies in topological field theories,” Nucl. Phys. B 405, 279 (1993) arXiv:hep-th/9302103.
[23] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, “Topological strings and integrable hierarchies,” arXiv:hep-th/0312083.
[24] C. Vafa and E. Witten, “A Strong coupling test of S duality,” Nucl. Phys. B 431, 3 (1994) arXiv:hep-th/9408074.
[25] M. Bershadsky, V. Sadov and C. Vafa, “D-Branes and Topological Field Theories,” Nucl. Phys. B 463, 420 (1996) arXiv:hep-th/9511222.
[26] E. Witten, “Supersymmetric Yang-Mills theory on a four manifold,” J. Math. Phys. 35, 5101 (1994) arXiv:hep-th/9403195.
[27] E. Witten, “Two-dimensional gauge theories revisited,” J. Geom. Phys. 9, 303 (1992) arXiv:hep-th/9204083.
[28] A. Strominger, “AdS(2) quantum gravity and string theory,” JHEP 9901, 007 (1999) arXiv:hep-th/9809027.
[29] J. M. Maldacena, J. Michelson and A. Strominger, “Anti-de Sitter fragmentation,” JHEP 9902, 011 (1999) arXiv:hep-th/9812073.
[30] V. Balasubramanian, A. Naqvi and J. Simon, “A multi-boundary AdS orbifold and DLCQ holography: A universal holographic description of extremal black hole horizons,” arXiv:hep-th/0311237.
[31] R. Dijkgraaf, R. Gopakumar, H. Ooguri and C. Vafa, work in progress.
[32] T. Matsuo and S. Matsuura, “String theoretical interpretation for finite N Yang-Mills theory in two-dimensions,” arXiv:hep-th/0404204.
[33] E. Witten, “Quantum background independence in string theory,” arXiv:hep-th/9306122.