Pseudogroup structures on Spencer manifolds

Stancho Dimiev

e-mail address: sdimiev@math.bas.bg

Abstract

This note is a continuation of the paper [2] (see references). We describe some natural pseudogroup structures on almost complex manifolds of type $m$. A kind of coherency is discussed for the sheaf of almost holomorphic functions.

1 Recall of definitions

Let $(M, J)$ be an almost complex manifold equipped with a smooth coordinates $(x^j)$, $j = 1, 2, ..., 2n$. In [1] complex self-conjugate coordinates $(z^j, \overline{z}^j)$, $j = 1, 2, ..., n$, are used, where $z^j = x^{2j-1} + ix^{2j}$ and $\overline{z}^j = x^{2j-1} - ix^{2j}$. A function $f : U \to \mathbb{C}$, where $U$ is an open subset of $M$, is called almost holomorphic if $J^* df = idf$. Here by $J^*$ the action of $J$ on differential forms of $M$ is denoted, i.e. $J^*(\omega) X := \omega(X)$, where $\omega$ is a differential form on $M$, and $X$ is a vector field on $M$.

Let $m$ be an integer, $m \leq n$. We say that a local Spencer coordinate system is defined on $(M, J)$, iff

1. there exist an open subset $U$ of $M$ and $m$ different functionally independent almost holomorphic functions $f_j : U \to \mathbb{C}$, $j = 1, ..., m$, such that
2. the sequence $f_1, ..., f_m$ is a maximal sequence of functionally independent on $U$ almost-holomorphic functions
3. the sequence $(U, f_1, ..., f_m, z^{m+1}, ..., z^n, \overline{f}_1, ..., \overline{f}_m, \overline{z}^{m+1}, ..., \overline{z}^n)$ determines a local self-conjugate coordinate system on $(M, J)$.

The condition (3) can be reformulate in terms of smooth coordinates as follows:

$$(3')$$ The sequence $(U, \text{Re} f_1, \text{Im} f_1, ..., \text{Re} f_m, \text{Im} f_m, x^{m+1}, y^{m+1}, ..., x^{2n}, y^{2n})$ is a local coordinate system in smooth coordinates on $(M, J)$.

Setting $u_j = \text{Re} f_j$, $v_j = \text{Im} f_j$, we have $dv_j = J^* du_j$, $j = 1, ..., m$. These are Cauchy-Riemann equations for $f_j$.

Clearly, each Spencer coordinate system on $M$ defines a diffeomorphism of $U$ in $\mathbb{C}^n \times \overline{\mathbb{C}}^n$, or equivalently in $\mathbb{R}^{2n}$. Complexifying the first $2m$ coordinate we can take $\mathbb{C}^m \times \mathbb{R}^{2n-2m}$, i.e. $\mathbb{R}^{2n} \equiv \mathbb{C}^m \times \mathbb{R}^{2n-2m}$.

We shall consider the smooth submersion $\mathbb{R}^{2n} \to \mathbb{C}^m$ defined as a composition of the above mentioned diffeomorphism of $U$ on $\mathbb{R}^{2n}$ and the projection of
\( \mathbb{R}^{2n} \) on \( C^m \)

\[
(f_1, ..., f_m, z^{m+1}, ..., z^n, \bar{f}_1, ..., \bar{f}_m, \bar{z}^{m+1}, ..., \bar{z}^n) \rightarrow (f_1, ..., f_m).
\]

This composition is denoted by \( f_U \) [more precisely by \((f_1, ..., f_m)U\)] and the image of \( U \) by \( f_U \) is denoted by \( U^{c_m} \).

As it was proved in [2], each almost-holomorphic function \( h : U \rightarrow C \) is a superposition of a holomorphic function \( H : U^{c_m} \rightarrow C \) and the almost-holomorphic functions \( f_j, j = 1, ..., m \), i.e. \( h = H(f_1, ..., f_m) \), where \( H = H(w_1, ..., w_m) \in O(U^{c_m}) \).

It was proved also that for each two systems of almost-holomorphic functions \((f_1, ..., f_m)\) and \((g_1, ..., g_m)\) defined on \( U \cap V \), there exists a bijective holomorphic transition mapping between the corresponding Spencer coordinate systems.

As a consequence one obtain a theorem formulated by spencer [1], namely

**Theorem.** On each paracompact almost-complex manifold \((M, J)\) of constant Spencer type \( m \) there exists a locally finite covering \( \{U_\alpha\} \) constituted by self-conjugate Spencer’s coordinate systems \((U_\alpha, w^j_\alpha, ...)\), \( j = 1, ..., m \), such that on every intersection \( U_\alpha \cap V_\beta \) the Spencer coordinates \((w^j_\alpha)\) change biholomorphically in the other Spencer’s coordinates \((w^j_\beta)\).

### 2 Pseudogroup structures

By \( X \) a topological space is denoted, respectively differentiable manifold \((X = M)\), or vector space \((X = \mathbb{R}^n, X = \mathbb{C}^n)\). We recall the notion of pseudogroup \( \Gamma \) of local homeomorphisms of \( X \), respectively - local diffeomorphisms of \( M \), or local biholomorphisms of \( \mathbb{C}^m \). Let \( U \) e an open subset of \( X \), respectively of \( M \), or of \( \mathbb{C}^m \), and \( \varphi : U \rightarrow X \) be a local mapping (homeomorphism, diffeomorphism, biholomorphism). The open set \( U \) is called the source of \( \varphi \), and the image of \( \varphi(U) \) (which is also an open set in \( X \)) is called a target of \( \varphi \).

A family of local mappings \( \{(U, \varphi) : U \text{ varying in a part of the set of open subsets of } X\} \) is by definition a pseudogroup of local mappings (homeomorphisms, diffeomorphisms, biholomorphisms) if the following axioms are valid:

1. If \((U, \varphi), (V, \psi) \in \Gamma \) and \( \varphi(V) \subset U \), then \((V, \varphi \circ \psi) \in \Gamma \),
2. If \((U, \varphi) \in \Gamma \) and \( V = \varphi(U) \), then \((V, \varphi^{-1}) \in \Gamma \),
3. If \((U, \varphi) \in \Gamma \) and \( V \subset U \), then \((V, \varphi | V) \) (the restriction remains in \( \Gamma \)),
4. If \((U, \varphi) \in \Gamma \) and every point of \( U \) admits a neighborhood on which the restriction of \( \varphi \) is in \( \Gamma \), then \( \varphi \) is in \( \Gamma \),
5. The restriction of the identity on every source of an element of \( \Gamma \) is in \( \Gamma \).

Let \((X, \Gamma_X)\) be a topological space \( X \) equipped with a pseudogroup of local homeomorphisms \( \Gamma_X \), and \((Y, \Gamma_Y)\) is another topological space equipped with a pseudogroup of local homeomorphisms \( \Gamma_Y \). We say that \((X, \Gamma_X)\) is defined over \((Y, \Gamma_Y)\) if for every source \( U \) of an element of \( \Gamma_X \), \((U, \varphi) \in \Gamma_X \), there exists continuous mappings \( f_U, f_{\varphi(U)} \) and \( \psi \) such that the following diagram should be commutative
$U \xrightarrow{\xi} \varphi(U)$

$\downarrow f_U \quad \downarrow f_{\varphi(U)}$

$V \xrightarrow{\psi} \psi(V)$

where $V = f_U(U)$ is a source of $(V, \psi) \in \Gamma_Y$ and $\psi(V) = f_{\varphi(U)}(\varphi(U))$, i.e.

$f_{\varphi(U)} \circ \varphi = \psi \circ f_U$.

By $\Gamma_d(M)$ the transitive pseudogroup of all local diffeomorphisms of the differentiable manifold $M$ is denoted. In the case $M$ is an almost-complex manifold, $(M, J)$, we shall consider local almost-holomorphic diffeomorphisms of $M$, i.e. the diffeomorphisms $f : U \to M$, $U$ being an open subset of $M$, and $f$ satisfying the condition

$f_s \circ J = J \circ f_s$, where $f_s$ is the tangent mapping (the differential) of the mapping $f$.

The minimal transitive pseudogroup of local almost-holomorphic mappings will be denoted $\Gamma_{ahd}(M)$. This pseudogroup is a subpseudogroup of $\Gamma_d(M)$. It is to recall here that the composition of two almost-holomorphic mappings is also an almost-holomorphic mapping where it is defined. The same is true for the inverse of an almost-holomorphic mapping.

Accordingly to the theorem formulated in previous paragraph, we shall consider all local almost-holomorphic diffeomorphisms $\Phi_{\alpha\beta} : U_{\alpha} \to U_{\beta}$, and corresponding biholomorphisms $\varphi_{\alpha\beta} : (U_{\alpha})^c_m \to (U_{\beta})^c_m$ in $C^m$, where $\{U_{\alpha}\}$ are the sources of local Spencer coordinates systems. The family $\{\Phi_{\alpha\beta}\}$ generates a subpseudogroup of $\Gamma_{ahd}(M)$ which will be denoted by $\Gamma_{spd}(M)$. Denoting by $\Gamma_h(C^m)$ the pseudogroup of all local biholomorphisms in $C^m$, we consider the family $\{\varphi_{\alpha\beta}\}$ and the generated subpseudogroup of $\Gamma_h(C^m)$ which will be denoted by $\Gamma_{sph}(C^m)$. The pseudogroup $\Gamma_{spd}(M)$ is over the pseudogroup $\Gamma_{sph}(C^m)$ according to the above introduced definition. The pseudogroups $\Gamma_{spd}(M)$ and $\Gamma_{sph}(C^m)$ are called Spencer pseudogroup of the almost complex manifold $(M, J)$ of type $m$. This means that for every $\Phi : U \to V$, $(U, V) \in \Gamma_{spd}(M)$, we have $f_V \circ \Phi = \varphi \circ f_U$, where $(U^c_m, \varphi) \in \Gamma_{sph}(C^m)$ and also $\Phi_s \circ J = J \circ \Phi_s$. The following diagram is commutative

$U \xrightarrow{f_U} V$

$U^c_m \xrightarrow{f_V} V^c_m$

### 3 Γ– manifolds, integrability of $(M, J)$

Let $\Gamma$ be a pseudogroup of local diffeomorphisms of one differentiable manifold $M$. It is said that a $\Gamma$-structure on the manifold $M$ is defined if an atlas
of local coordinates systems is introduced in such a way that all transition transformations between them to belong to the pseudogroup $\Gamma$.

Let $(M, J)$ be an almost-complex manifold of type $m$. We shall consider the corresponding Spencer pseudogroups $\Gamma_{spd}(M)$ and $\Gamma_{sph}(\mathbb{C}^m)$. Let us suppose that $N$ is an orientable differentiable $(2m)$-manifold equipped with a $\Gamma_{sph}(\mathbb{C}^m)$-structure. We remark that the problem of the existence of $\Gamma$-structures on a given manifold, especially $\Gamma_{sph}(\mathbb{C}^m)$-structures, is a difficult problem.

Denoting $f = \{f_U\}$ as the family of all local $m$-projections $f_U: U \to \mathbb{C}^m$ defined by almost-holomorphic coordinates $f_1, f_2, \ldots, f_m$, we obtain the following diagram

$$
\begin{array}{ccc}
M & \equiv & M \\
\downarrow f & \downarrow F \\
\mathbb{C}^m & \overset{\theta_{\text{atlas}}}{\leftarrow} & N
\end{array}
$$

where $F$ is defined locally as follows:

$$F(U) = \theta^{-1} \circ f(U),$$

$U$ being an open subset of $M$.

In the case $m = n$ [$M$ is a $(2n)$-manifold] the $\Gamma_{sph}(\mathbb{C}^n)$-structure on $N$ is a structure of a complex manifold, and $F$ is a diffeomorphism. So the almost complex manifold $(M, J)$ is diffeomorphic to a complex manifold $N$. Taking $N = \mathbb{C}^n$ as differentiable manifold, we obtain that $\Gamma_{sph}(\mathbb{C}^n)$ defines an atlas of biholomorphisms on $M$. This implies that $(M, J)$ is an integrable manifold.

In the case of 4-dimensional almost-complex manifold $M$ of type $m = 1$ there are Spencer pseudogroup structures $\Gamma_{spd}(M)$ and $\Gamma_{sph}(\mathbb{C})$ The corresponding 2-dimensional manifold $N$, equipped with a $\Gamma_{sph}(\mathbb{C})$ must be a Riemann surface.

4 $\Gamma$-coherency

The notion of coherent sheaf is local. Having a pseudogroup $\Gamma$ on a manifold $M$ we can assign to each source $U$ of $\Gamma$ the set $\Gamma(U)$ of different $\Gamma$-objects (functions, vector fields) defined on $U$. The mapping $U \to \Gamma(U)$ defines a sheaf.

Our purpose is to discuss a kind of coherency on almost complex manifolds using the introduced Spencer pseudogroup structures of type $m$. It is not difficult to see that almost-holomorphic functions on $(M, J)$ define a sheaf. We denote this sheaf by $O_{ah}(M)$ and, respectively, by $O_h(\mathbb{C}^m)$ - the sheaf of all holomorphic functions on $\mathbb{C}^m$. According to the famous Oka theorem $O_h(\mathbb{C}^n)$ is a coherent sheaf, but the same is not true in general for the subsheaf $O_h(U)$, where $U$ is an open subset of $\mathbb{C}^m$.

**Proposition.** For every source $U \in \Gamma_{spd}(M)$, the sheaf $O_{ah}(U)$ is a coherent sheaf of almost holomorphic functions if and only if the corresponding sheaf of holomorphic functions $O_h(U^c_{m})$ is finitely generated as a subsheaf of $O_h(\mathbb{C}^m)$.
Proof. It is enough to remark that $O_{ab}(U)$ is an inverse image of $O_h(U^c_m)$. Being a finitely generated subsheaf of $O_h(C^m)$ the sheaf $O_h(U^c_m)$ is a coherent sheaf too. □

5 Perspectives

It seems that one can develop a deformation theory for almost complex manifolds of type $m$ following some ideas of Donald Spencer (see [2]).

References

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[3] D. C. Spencer, Deformations of structures on manifolds by transitive continuous pseudogroups. Ann. of Math. 76 (1962), 306–445.