Factorisation of Macdonald polynomials

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1. Macdonald polynomials

Macdonald polynomials \( P_\lambda(x; q, t) \) are orthogonal symmetric polynomials which are the natural multivariable generalisation of the continuous \( q \)-ultraspherical polynomials \( C_n(x; \beta|q) \) [2] which, in their turn, constitute an important class of hypergeometric orthogonal polynomials in one variable. Polynomials \( C_n(x; \beta|q) \) can be obtained from the general Askey-Wilson polynomials [3] through a specification of their four parameters (see, for instance, [9]), so that \( C_n(x; \beta|q) \) depend only on one parameter \( \beta \), apart from the degree \( n \) and the basic parameter \( q \). In an analogous way, the Macdonald polynomials \( P_\lambda(x; q, t) \) with one parameter \( t \) could be obtained as a limiting case of the 5-parameter Koornwinder’s multivariable generalisation of the Askey-Wilson polynomials [10].

The main reference for the Macdonald polynomials is the book [17], Ch. VI, where they are called symmetric functions with two parameters. Let \( K = \mathbb{Q}(q, t) \) be the field of rational functions in two indeterminants \( q, t \); \( K[x] = K[x_1, \ldots, x_n] \) be the ring of polynomials in \( n \) variables \( x = (x_1, \ldots, x_n) \) with coefficients in \( K \); and \( K[x]^W \) be the subring of all symmetric polynomials. The Macdonald polynomials \( P_\lambda(x) = P_\lambda(x; q, t) \) are symmetric polynomials labelled by the sequences \( \lambda = \{0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n\} \) of integers (dominant weights). They form a \( K \) basis of \( K[x]^W \) and are uniquely characterised as joint eigenvectors of the commuting \( q \)-difference operators \( H_k \)

\[
H_k P_\lambda = h_{k; \lambda} P_\lambda, \quad k = 1, \ldots, n, \tag{1.1}
\]

normalised by the condition

\[
P_\lambda = \sum_{\lambda' \preceq \lambda} \kappa_{\lambda\lambda'} m_{\lambda'}, \quad \kappa_{\lambda\lambda'} = 1, \quad (\kappa_{\lambda\lambda'} \in K) \tag{1.2}
\]
where, for each $\mu$, $m_\mu(x)$ stands for the monomial symmetric function: $m_\mu(x) = \sum x_1^{\nu_1} \cdots x_n^{\nu_n}$ with the sum taken over all distinct permutations $\nu$ of $\mu$, and $\preceq$ is the dominance order on the dominant weights

$$\lambda' \preceq \lambda \iff \left\{ |\lambda'| = |\lambda| ; \sum_{j=k}^{n} \lambda'_j \leq \sum_{j=k}^{n} \lambda_j , \quad k = 2, \ldots, n \right\} .$$

The commuting operators $H_i$ have the form:

$$H_i = \sum_{j \in \{1, \ldots, n\} \setminus \{i\}} \left( \prod_{k \in \{1, \ldots, n\} \setminus \{j\}} v_{jk} \right) \left( \prod_{j \in J} T_{q,x_j} \right) , \quad i = 1, \ldots, n , \quad (1.3)$$

where

$$v_{jk} = \frac{t^{\frac{1}{2}} x_j - t^{-\frac{1}{2}} x_k}{x_j - x_k} , \quad |q| < 1 , \ |t| < 1 . \quad (1.4)$$

The $T_{q,x_j}$ stands for the $q$-shift operator in the variable $x_j$: $(T_{q,x_j} f)(x_1, \ldots, x_n) = f(x_1, \ldots, q x_j, \ldots, x_n)$. The operators (1.3) were introduced for the first time in [21] and are the integrals of motion for the quantum Ruijsenaars model, which is a relativistic (or $q$-) analog of the trigonometric Calogero-Moser-Sutherland model. They are called sometimes Macdonald operators in the mathematical literature.

The corresponding eigenvalues $h_{\mu;\lambda}$ in (1.3) are

$$h_{\mu;\lambda} = \sum_{j_1 < \cdots < j_k} \mu_{j_1} \cdots \mu_{j_k} , \quad \mu_j = q^{\lambda_j} p^{j - \frac{1}{2} j^2} \quad (1.5)$$

The polynomials $P_{\lambda}$ are orthogonal

$$\frac{1}{(2\pi i)^n} \oint_{|x_1|=1} \cdots \oint_{|x_n|=1} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} P_{\lambda}(x; q, t) P_{\lambda'}(x; q, t) \Delta(x) = 0 , \quad \lambda \neq \lambda' \quad (1.6)$$

with respect to the weight

$$\Delta(x_1, \ldots, x_n) = \prod_{j \neq k} \frac{(x_j x_k^{-1}; q)_\infty}{(tx_j x_k^{-1}; q)_\infty} . \quad (1.7)$$

For instance, for $n = 3$,

$$m_{000} = 1 , \quad m_{001} = x_1 + x_2 + x_3 , \quad m_{011} = x_1 x_2 + x_1 x_3 + x_2 x_3 , \quad m_{002} = x_1^2 + x_2^2 + x_3^2 ,$$

$$m_{111} = x_1 x_2 x_3 , \quad m_{012} = x_1 x_2^2 + x_1^2 x_2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + x_3^2 x_2 ,$$

$$m_{112} = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 , \quad m_{022} = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 , \quad m_{003} = x_1^3 + x_2^3 + x_3^3 ,$$

$$P_{000} = m_{000} , \quad P_{001} = m_{001} , \quad P_{011} = m_{011} , \quad P_{002} = m_{002} + \frac{(1 - t)(1 + q)}{1 - qt} m_{011} ,$$

$$P_{111} = m_{111} , \quad P_{012} = m_{012} + \frac{(1 - t)(q(2t + 1) + 1)}{1 - qt^2} m_{111} ,$$

$$P_{112} = m_{112} , \quad P_{022} = m_{022} + \frac{(1 - t)(1 + q)}{1 - qt} m_{112} ,$$

$$P_{122} = m_{122} , \quad P_{222} = m_{222} + \frac{(1 - t)(1 + q)}{1 - qt} m_{112} ,$$

$$P_{123} = m_{123} , \quad P_{223} = m_{223} + \frac{(1 - t)(1 + q)}{1 - qt} m_{112} ,$$

$$P_{124} = m_{124} , \quad P_{224} = m_{224} + \frac{(1 - t)(1 + q)}{1 - qt} m_{112} .$$
\[ P_{003} = m_{003} + \frac{(1-t)(1+q+q^2)}{1-q^2t} \ m_{012} + \frac{(1-t)^2(1+q)(1+q+q^2)}{(1-q)(1-q^2t)} \ m_{111}. \]

Rodrigues type of formula for Macdonald polynomials was recently found in \cite{11, 12, 16}. In the limit \( q \uparrow 1 \) Macdonald polynomials turn into Jack polynomials \cite{17}.

A multivariable function/polynomial can be called special function if it is some recognised (classical) special function in the case of one variable and if it is common eigenfunction of a complete set of commuting linear partial differential/difference operators defining a quantum integrable system. Macdonald polynomials are special functions in the above-mentioned sense. They diagonalise the integrals of motion \( H_j \) of the quantum Ruijsenaars system.

For any special function in many variables one can set up a general problem of its factorisation in terms of one-variable (special) functions. For the Macdonald polynomials \( P_\lambda \) this would mean finding a factorising integral operator \( M \) such that

\[ M : P_\lambda(x; q, t) \mapsto \prod_{j=1}^n \Phi_{\lambda,j}(y_j). \]

For some particular special functions (or, in other words, for some particular quantum integrable systems) such an operator might simplify to a local transform which is a simple change of variables, from \( x \) to \( y \). This happens for example in the case of ellipsoidal harmonics in \( \mathbb{R}^n \) (see \cite{18} and \cite{23} for \( n = 3 \)), correspondingly, in the case of quantum Neumann system, where the transform \( x \mapsto y \) is the change of variables, from Cartesian to ellipsoidal.

As it is shown further on, in Section 3, the factorising operator for the (symmetric) Macdonald polynomials has to be non-local, i.e. to be some linear integral operator. Moreover, it is explicitly described in the first two non-trivial cases (when \( n = 2 \) and \( n = 3 \)) in terms of the Askey-Wilson operator.

The formulated factorisation problem also makes sense in the limit to the Liouville integrable systems in classical mechanics, becoming there the well-known problem of separation of variables (SoV) in the Hamilton-Jacobi equation.

The kernel of the factorising integral operator \( M \) turns in this limit into the generating function of the separating canonical transformation.

\section{2. Hypergeometric polynomials \( f_\lambda \) and \( \varphi_\lambda \) in one variable}

In this Section we introduce the hypergeometric polynomials \( f_\lambda / \varphi_\lambda \) of one variable each constituting a basis which is conjugated to Jack/Macdonald polynomials with respect to the factorising integral transform \( M \). First of all, we describe the procedure of lowering the order of (basic) hypergeometric functions which will lead us to these new sets of interesting polynomials in one variable. We will use (becoming already) standard notations of \cite{8} for the (basic) hypergeometric series and other formulas of the \( q \)-analysis.
In 1927 Fox [4] found an interesting relation between hypergeometric functions:

\[ pF_q \left[ \begin{array}{c} a_1 + m_1, a_2, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; y \right] = \sum_{j=0}^{\infty} \frac{(-y)^j(-m_1)\ldots(-a_p)}{j!} \frac{(a_2)\ldots(a_p)}{(b_1)\ldots(b_q)} pF_q \left[ \begin{array}{c} a_1 + j, \ldots, a_p + j \\ b_1 + j, \ldots, b_q + j \end{array}; y \right]. \]  

(2.1)

When \( a_1 = b_1 \) (2.1) gives the expansion of \( pF_q \left[ \begin{array}{c} a \\ b \end{array}; y \right] \) in terms of functions of the type \( p-1F_{q-1} \left[ \begin{array}{c} a \\ b \end{array}; y \right] \). When \( m_1 \) is a positive integer, the series on the right of (2.1) terminates, and we have (for the case of \( a_1 = b_1 \) and \( q = p-1 \)) the following relation

\[ pF_{p-1} \left[ \begin{array}{c} b_1 + m_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_{p-1} \end{array}; y \right] = \sum_{j=0}^{m_1} \frac{(-y)^j(-m_1)\ldots(-a_p)}{j!} \frac{(a_2)\ldots(a_p)}{(b_1)\ldots(b_{p-1})} pF_{p-2} \left[ \begin{array}{c} a_2 + j, \ldots, a_p + j \\ b_2 + j, \ldots, b_{p-1} + j \end{array}; y \right]. \]  

(2.2)

Supposing that \( a_2 = b_2 + m_2, a_3 = b_3 + m_3, \ldots, a_{p-1} = b_{p-1} + m_{p-1} \) with some non-negative integers \( m_k, k = 2, \ldots, p-1 \), we then iterate the relation (2.2) further on and, finally, using the binomial theorem,

\[ _1F_0 \left[ \begin{array}{c} a \\ -1 \end{array}; y \right] = (1 - y)^{-a}, \]

conclude that the function

\[ f_{m_1, \ldots, m_{p-1}} \equiv (1 - y)^{a_p + \sum_{j=1}^{p-1} m_j} pF_{p-1} \left[ \begin{array}{c} b_1 + m_1, b_2 + m_2, \ldots, b_{p-1} + m_{p-1}, a_p \\ b_1, b_2, \ldots, b_{p-1} \end{array}; y \right] \]

is a polynomial in \( y \) of the cumulative degree \( \sum_{j=1}^{n-1} m_j \).

As for a \( q \)-analog of the reduction formula (2.2), we refer to (1.9.4) in [8] (see also [4]). It was proved for the first time in [4] in a more general case (although in different notations than ones in [8] which are adopted here). Following [8], consider the \( q \)-analog of the Vandermonde formula in the form

\[ _2\phi_1 \left[ \begin{array}{c} q^{-n}, q^{-m} \\ b_{p-1} \end{array}; q, q \right] = \frac{(b_{p-1}q^m; q)_n}{(b_{p-1}; q)_n} q^{-mn} \]

(2.3)

where \( m \) and \( n \) are non-negative integers such that \( m \geq n \), then we can reduce the order of the basic hypergeometric function through the following equalities (\(|y| < 1\):

\[ p\phi_{p-1} \left[ \begin{array}{c} a_1, \ldots, a_{p-1}, b_{p-1}q^m \\ b_1, \ldots, b_{p-2}, b_{p-1} \end{array}; q, y \right] \]

(2.4)

\[ = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{p-1}; q)_n}{(q,b_1,\ldots,b_{p-2}; q)_n} y^n \sum_{k=0}^{n} \frac{(q^{-n}, q^{-m}; q)_k}{(q,b_{p-1}; q)_k} q^{mn+k} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{p-1}; q)_n}{(b_1, \ldots, b_{p-2}; q)_n, (q,q)_{n-k}(q,b_{p-1}; q)_k} y^n \left( -1 \right)^k q^{mn+k-nk+\binom{k}{2}} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{-m}, a_1, \ldots, a_{p-1}; q)_k}{(q,b_1,\ldots,b_{p-1}; q)_k} \left( -yq^m \right)^k \left( \frac{1}{2} \right)^k \]

\[ \cdot p-1\phi_{p-2} \left[ \begin{array}{c} a_1q^k, \ldots, a_{p-1}q^k \\ b_1q^k, \ldots, b_{p-2}q^k \end{array}; q, yq^{m-k} \right]. \]
Then again, iterating it and using the $q$-binomial theorem,

$$1\phi_0\left[ a; q, y \right] = \frac{(ay; q)_\infty}{(y; q)_\infty}, \quad |y| < 1, \quad |q| < 1,$$

we conclude that, for non-negative integers $m_k$, $k = 1, \ldots, p - 1$, the function

$$\varphi_{m_1, \ldots, m_{p-1}} \equiv \frac{(y; q)_\infty}{(ya_p q^{m_1 + \ldots + m_{p-1}}; q)_\infty} p\phi_{p-1}\left[ b_1 q^{m_1}, \ldots, b_{p-1} q^{m_{p-1}}, a_p; q, y \right]$$

is a polynomial in $y$ of the cumulative degree $\sum_{j=1}^{n-1} m_j$.

Now we can connect integers $m_j$ to a partition $\lambda = \{\lambda_1, \ldots, \lambda_n\}$, namely:

$$m_j = \lambda_{j+1} - \lambda_j \quad (\geq 0), \quad j = 1, \ldots, n - 1.$$

Put also $(g \in \mathbb{R})$

$$b_j = \lambda_1 - \lambda_{j+1} + 1 - jg, \quad a_j = b_j + m_j, \quad j = 1, \ldots, n - 1, \quad a_n = \lambda_1 - \lambda_n + 1 - ng.$$

Let us define the following function

$$f_\lambda(y) := y^{\lambda_1} (1 - y)^{1-ng} nF_{n-1}\left[ b_1 + m_1, \ldots, b_{n-1} + m_{n-1}, a_n; y \right].$$

Obviously, this function is a polynomial in $y$ of the form

$$\sum_{k=\lambda_1}^{\lambda_n} \chi_k y^k. \quad (2.5)$$

The polynomials $f_\lambda(y)$ were introduced in [13] and they are factorised polynomials of one variable for the multivariable Jack polynomials. In [22, 13] the corresponding sine-kernel (of Gegenbauer type) factorising the $A_2$ Jack polynomials was described in detail. In the sequel we concentrate on the $q$-anals of these polynomials, the $\varphi_\lambda(y)$, which were introduced in [14].

Put again $m_j = \lambda_{j+1} - \lambda_j, \quad j = 1, \ldots, n - 1$, but

$$b_j = q^{\lambda_1-\lambda_{j+1}+1-jg}, \quad a_j = b_j q^{m_j}, \quad j = 1, \ldots, n - 1, \quad a_n = q^{\lambda_1-\lambda_n+1-ng}.$$

We will also use the parameter $t$ which is connected to $g$ in the following way

$$t := q^g.$$

Then the polynomials $\varphi_\lambda$ are defined as follows:

$$\varphi_\lambda(y) := y^{\lambda_1} (y; q)_{1-ng} n\phi_{n-1}\left[ b_1 q^{m_1}, \ldots, b_{n-1} q^{m_{n-1}}, a_n; q, y \right], \quad (2.6)$$

and, like $f_\lambda(y)$, expand in $y$ as (2.5).
In \([14]\) we have found several useful representations for these polynomials in addition to their definition \((2.6)\). Let us list here some of them. Introduce notations:

\[
\lambda_{ij} := \lambda_i - \lambda_j, \quad |\lambda| = \sum_{j=1}^{n} \lambda_j.
\]

First of all, the coefficients \(\chi_k\) in \((2.3)\) have the following explicit representation:

\[
\chi_k = \left( q^{-1}t^n \right)^{\lambda_{1-k}} \frac{(q^{-1}t^n ; q)_{k-\lambda_1}}{(q ; q)_{k-\lambda_1}} n+1 \phi_n \left[ q^{\lambda_{1-k}} ; a_1, \ldots, a_n ; q^{\lambda_{1-k}+2t-n}, b_1, \ldots, b_{n-1} ; q, q \right]. \tag{2.7}
\]

It is easy to give simpler expressions for some of \(\chi_k\) such as

\[
\chi_{\lambda_1} = 1, \quad \chi_{\lambda_n} = t^n \lambda_1 - |\lambda| \prod_{j=1}^{n-1} \frac{(t_j ; q)_{\lambda_j - \lambda_1} (t_j^* ; q)_{\lambda_n - \lambda_{n-j}}}{(t_j ; q)_{\lambda_{j+1} - \lambda_1} (t_j^* ; q)_{\lambda_n - \lambda_{n-j+1}}}. \tag{2.8}
\]

Let us give few first polynomials in the case \(n = 3\)

\[
\varphi_{000} = 1, \quad \varphi_{001} = 1 + \frac{1}{(1+t)} y, \quad \varphi_{011} = 1 + \frac{1+ty}{t^2} y,
\]

\[
\varphi_{002} = 1 + \frac{(1+q)(1-t)}{t(1-qt^2)} y + \frac{1-qt}{t^2(1-qt^2)(1+t)} y^2,
\]

\[
\varphi_{012} = 1 + \frac{1+qt+qt^2-qt^3}{t^2(1-qt)} y + \frac{1-qt}{t^2(1-qt)} y^2,
\]

\[
\varphi_{022} = 1 + \frac{(1+q)(1-t)}{t^2(1-qt)} y + \frac{t(1-qt^2)}{t^2(1+qt)(1-qt^2)} y^2.
\]

There is also a simple formula for \(\varphi_{\lambda}(t^n)\)

\[
\varphi_{\lambda}(t^n) = t^n \lambda_1 - |\lambda| \prod_{j=1}^{n-1} \frac{(t_j ; q)_{\lambda_j - \lambda_1}}{(t_j ; q)_{\lambda_{j+1} - \lambda_1}}. \tag{2.9}
\]

There is a nice representation of the polynomials \(\varphi_{\lambda}\) in terms of the \(\phi_D\) \(q\)-Lauricella function \(\text{(19)a}, \text{(19)b}\)

\[
\varphi_{\lambda}(y) = y^{\lambda_1} \frac{(qt^{-n}q^{\lambda_1+n}\lambda_1)_{\lambda_1}}{\prod_{j=1}^{n} (q^{\lambda_{1-j+1}+1-t-j} ; q)_{\lambda_{n-j+1}+\lambda_1-j}} \phi_D \left[ y ; b'_1, \ldots, b'_n ; q; a'_1, \ldots, a'_n \right],
\]

with

\[
a'_j = qt^{-n} q^{\lambda_1-j}, \quad b'_j = q^{\lambda_n-j-n_{j+1}}, \quad j = 1, \ldots, n-1. \tag{2.10}
\]

The \(\phi_D\) \(q\)-Lauricella function of \(n-1\) variables \(z_i\) is a multivariable generalisation of the basic hypergeometric series and is defined by the formula

\[
\phi_D \left[ a'_j ; b'_j, \ldots, b'_{n-1} ; q, z_1, \ldots, z_{n-1} \right] := \sum_{k_1, \ldots, k_{n-1}=0}^{\infty} \frac{(a'_j ; q)_{k_1+\ldots+k_{n-1}}}{(c ; q)_{k_1+\ldots+k_{n-1}}} \prod_{j=1}^{n-1} \frac{(b'_j ; q)_{k_1+\ldots+k_{n-1}}}{(q ; q)_{k_1+\ldots+k_{n-1}}},
\]

\]
using which we get the most explicit representation of our polynomials \( \varphi \)

\[
\varphi(y) = y^{\lambda_1} \left( \prod_{j=1}^{n-1} (q^{\lambda_1 - \lambda_{n-j+1} + 1} y^{j-n}; q)_{\lambda_{n-j+1} - \lambda_j} \right)^{-1} \times
\]

\[
\times \sum_{k_1=0}^{\lambda_n - \lambda_{n-1}} \cdots \sum_{k_{n-1}=0}^{\lambda_2 - \lambda_1} (qt^{-n} q^{\lambda_1 n + k_1 + \ldots + k_{n-1} - \lambda_n - k_{n-1}} y; q)_{\lambda_{n-1} - k_1 - \ldots - k_{n-1}} (y; q)_{k_1 + \ldots + k_{n-1}} \times
\]

\[
\prod_{j=1}^{n-1} (q^{\lambda_{n-j} - \lambda_{n-j+1}}; q)_{k_j} (qt^{j-n} q^{\lambda_1 - \lambda_{n-j}}; q)_{k_j} \frac{1}{(q; q)_{k_j}} .
\]

(2.11)

Finally, these polynomials satisfy the following \( q \)-difference equation

\[
\sum_{k=0}^{n} (-1)^k t^{-\frac{n-1}{2}} (1 - q^k t^{-k} y) (y; q)_k (q^{k+1} t^{-n} y; q)_{n-k} h_{n-k; \lambda} \varphi(y) = 0 \quad (2.12)
\]

where \( h_{k; \lambda} \) are given by (1.3) and we assume \( h_{0; \lambda} = 1 \).

Let us give few remarks about this new class of basic hypergeometric polynomials in one variable.

**Remark 1.** First of all, we stress that our way of extracting polynomials from hypergeometric series is quite different from the usual one. Indeed, all classical orthogonal \( q \)-polynomials of one variable are obtained by just terminating the corresponding hypergeometric series. For instance, the generic Askey-Wilson orthogonal \( q \)-polynomials which appear on the \( _4 \phi_3 \) level are defined as follows

\[
p_n(x; a, b, c, d|q) = \text{const} \cdot _4 \phi_3 \left[ q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \mid ab, ac, ad \right] q, q , \quad x = \cos \theta
\]

In contrast, to extract the polynomials \( \varphi \) at the level \( _n \phi_{n-1} \) we use the procedure of order reduction (2.4) of the basic hypergeometric functions with the specific choice of upper and lower parameters, namely: when the ratio of an upper parameter and one of the lower parameters is equal to \( q^{m_i} \) where \( m_i \) are non-negative integers. The procedure of order reduction is thus an important second possibility (in addition to simple termination) in order to get polynomials out of hypergeometric series.

**Remark 2.** The importance of polynomials \( \varphi \) stems from the fact that they are the *factorised polynomials* of one variable for the multivariable Macdonald polynomials. Notice that the polynomials \( \varphi \), as well as Macdonald polynomials \( P_\lambda \), are labelled by the dominant weights. The multivariable polynomials \( \Phi_\lambda \) combined from the one-variable polynomials \( \varphi \):

\[
\Phi_\lambda(y_1, \ldots, y_n) := y_n^{\lambda_1} \prod_{k=1}^{n-1} \varphi(y_k)
\]

satisfy the following multiparameter spectral problem (cf. (2.12))

\[
\Phi_\lambda(y_1, \ldots, y_{n-1}, qy_n) = h_{n; \lambda} \Phi_\lambda(y_1, \ldots, y_n) \quad (2.13)
\]

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\[
\sum_{k=0}^{n} (-1)^k t^{-\frac{n+1}{2}} k (1 - q^k t^{-k} y_j) (y_j; q)_k (q^{k+1} t^{-n} y_j; q)_{n-k} h_{n-k; \lambda} \times
\]
\[
\Phi_{\lambda}(y_1, \ldots, q^k y_j, \ldots, y_n) = 0, \quad j = 1, \ldots, n - 1,
\]
with the same set of spectral parameters \((h_1; \lambda, \ldots, h_n; \lambda)\) as in the spectral problem (1.1) for the Macdonald polynomials \(P_{\lambda}\). Hence, one can introduce the commuting operators \(H_i^{(y)}\), \(i = 1, \ldots, n\) defined by their eigenfunctions \(\Phi_{\lambda}(y)\) and eigenvalues \(h_{k; \lambda}\). Since the spectrum \((h_1; \lambda, \ldots, h_n; \lambda)\) coincides with that (1.5) of the Macdonald polynomials, the two sets of commuting operators are isomorphic and there has to exist an intertwining linear operator \(M\)

\[
M H_i^{(x)} = H_i^{(y)} M \quad (H_i^{(x)} \equiv H_i).
\]

Actually, to any choice of the normalisation coefficients \(c_\lambda\) in

\[
M : P_\lambda(x) \mapsto c_\lambda \Phi_{\lambda}(y)
\]

there corresponds some intertwiner \(M\). The problem is to select a factorising operator \(M\) having an explicit description in terms of its integral kernel or its matrix in some basis in \(\mathbb{K}[x]^W\). In [14, 15] we have found such an operator as well as its inversion in the first two non-trivial cases, when \(n = 2\) and \(n = 3\). It appears that the factorising operator \(M\) can be expressed in those cases through the Askey-Wilson operator.

### 3. The cases \(n = 2\) and \(n = 3\)

Let us first describe the integral operator \(M_\xi\) performing the separation of variables in the \(A_1\) Macdonald polynomials (we skip the trivial case of the purely coordinate SoV \(x_{1,2} \rightarrow x_\pm \equiv (x_1 x_2^{\pm 1})^{1/2}\)). Our main technical tool is the famous Askey-Wilson integral identity [3, 8]

\[
\frac{1}{2\pi i} \int_{\Gamma_{abcd}} \frac{dx}{x} \frac{(x^2, x^{-2}, q)_\infty}{(ax, ax^{-1}, bx, bx^{-1}, cx, cx^{-1}, dx, dx^{-1}; q)_\infty} = \frac{2(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.
\]  

(3.1)

The cycle \(\Gamma_{abcd}\) depends on complex parameters \(a, b, c, d\) and is defined as follows. Let \(C_{z,r}\) be the counter-clockwise oriented circle with the center \(z\) and radius \(r\). If \(\max(|a|, |b|, |c|, |d|, |q|) < 1\) then \(\Gamma_{abcd} = C_{0,1}\). The identity (3.1) can be continued analytically for the values of parameters \(a, b, c, d\) outside the unit circle provided the cycle \(\Gamma_{abcd}\) is deformed appropriately. In general case

\[
\Gamma_{abcd} = C_{0,1} + \sum_{z=a,b,c,d} \sum_{k \geq 0} \frac{(C_{zq^k, \varepsilon} - C_{z^{-1}q^{-k}, \varepsilon})}{|z|q^k > 1},
\]

\(\varepsilon\) being small enough for \(C_{z^{\pm 1}q^{\pm k}, \varepsilon}\) to encircle only one pole of the denominator.
Put
\[ a = yq^\frac{\alpha}{2}, \quad b = y^{-1}q^\frac{\alpha}{2}, \quad c = rq^\frac{\beta}{2}, \quad d = r^{-1}q^\frac{\beta}{2}. \]
We will use the notation \( \Gamma_{\alpha\beta}^{xy} \) for the contour obtained from \( \Gamma_{abcd} \) by these substitutions. Introduce also the following useful notation
\[ \mathcal{L}_q(\nu; x, y) := (\nu xy, \nu xy^{-1}, \nu x^{-1} y, \nu x^{-1} y^{-1}; q)_{\infty}. \] (3.2)

Now let
\[ \alpha = \beta = g, \quad y = y_-, \quad x = x_-; \quad r = t^{-1}y_+, \quad x_\pm \equiv (x_1x_2^{\pm1})^{\frac{1}{2}}, \quad y_\pm \equiv (y_1y_2^{\pm1})^{\frac{1}{2}}. \]

Introduce the kernel \( \mathcal{M}(y_+, y_-|x_-) \):
\[ \mathcal{M}(y_+, y_-|x_-) = \frac{(1 - q)(q; q)_{\infty}^2 (x_-^2, x_-^{-2}; q)_{\infty} \mathcal{L}_q(t; y_-, t^{-1}y_+)}{2B_q(g, g) L_q(t^{\frac{1}{2}}; y_-, y_-) L_q(t^{\frac{1}{2}}; x_-, t^{-1}y_+)} \quad (3.3) \]

Assuming \( \xi \) to be an arbitrary complex parameter, we introduce the integral operator \( M_\xi \) acting on functions \( f(x_1, x_2) \) by the formula
\[ (M_\xi f)(y_1, y_2) \equiv \frac{1}{2\pi i} \int_{t^{-1}y_+,y_-} dz \mathcal{M}(y_+, y_-|z) f(t^{-\frac{1}{2}}\xi y_+z, t^{-\frac{1}{2}}\xi y_+z^{-1}). \] (3.4)

**Theorem 1** ([15]) The operator \( M_\xi \) (3.3) performs the factorisation of (or, in other words, the SoV for) the \( A_1 \) Macdonald polynomials:
\[ M_\xi : P_\lambda(x_1, x_2) \to c_{\lambda, \xi} \varphi_\lambda(y_1) \varphi_\lambda(y_2), \] (3.5)
where \( \varphi_\lambda(y) \) is the factorised (or separated) polynomial and the normalisation coefficient \( c_{\lambda, \xi} \) is equal to
\[ c_{\lambda, \xi} = t^{-2\lambda_1 + \lambda_2 + \lambda_2} \frac{(t; q)_{\lambda_21}}{(t^2; q)_{\lambda_21}}. \] (3.6)

Note that the relation (3.5) is equivalent [13] to the product formula for the continuous \( q \)-ultraspherical polynomials [20, §].

The kernel of the inverse operator \( M_\xi^{-1} \) has the form
\[ \tilde{M}(x_+, x_-|y_-) = \frac{(1 - q)(q; q)_{\infty}^2 (y_-^2, y_-^{-2}; q)_{\infty} \mathcal{L}_q(t^{\frac{1}{2}}; x_-, t^{-\frac{1}{2}}\xi^{-1}x_+)}{2B_q(-g, 2g) L_q(t^{-\frac{1}{2}}; y_-, y_-) L_q(t; y_-, t^{-\frac{1}{2}}\xi^{-1}x_+)} \]
with the following substitutions for the contour \( \Gamma \):
\[ \alpha = -g, \quad \beta = 2g, \quad x = y_-, \quad y = x_-, \quad r = t^{-\frac{1}{2}}\xi^{-1}. \]
Theorem 2 ([15]) The inversion of the operator $M_\xi$ (3.4) is given by the formula

$$
(M_\xi^{-1} f)(x_1, x_2) = \frac{1}{2\pi i} \int_{\Gamma_{-g,2g}} \frac{dz}{_2\tilde{F}_1(x_+, x_-|z)} f(t^{1/2}_\xi^{-1}x_+, t^{1/2}_\xi^{-1}x_-). \tag{3.7}
$$

The operator $M_\xi^{-1}$ provides an integral representation for the $A_1$ Macdonald polynomials in terms of the factorised polynomials $\varphi_{\lambda_1, \lambda_2}(y)$

$$
M_\xi^{-1} : c_{\lambda, \xi} \varphi_{\lambda}(y_1) \varphi_{\lambda}(y_2) \rightarrow P_{\lambda}(x_1, x_2). \tag{3.8}
$$

In contrast to the formula (3.3) which paraphrases an already known result, the formula (3.8) leads to a new integral relation for the $q$-ultraspherical polynomials. Note that for positive integer $g$ the operator $M_\xi^{-1}$ becomes a $q$-difference operator of the order $g$ (cf. [14] and [13]).

Now we describe the factorising operator $M$ and its inversion in the case of $A_2$ Macdonald polynomials. Introduce the following operator $M$ acting on functions $f(x_1, x_2, x_3)$ by the formula

$$
(M f)(y_1, y_2, y_3) = \frac{1}{2\pi i} \int_{\Gamma_{-2g, 2g}} \frac{dx_-}{x_-} M((y_1/y_2)^{1/2}, (y_1/y_2)^{1/2} | x_-) \times
$$

$$
\times f(t^{-3/2}_q y_3(y_1/y_2)^{1/2} x_-, t^{-3/2}_q y_3(y_1/y_2)^{1/2} x_-; y_3)
$$

with the kernel

$$
M(y_+, y_- | x_-) = \frac{(1 - q)(q; q)_\alpha^2 (x_-, x_-^2; q)_\infty \mathcal{L}_q(t^{1/2}_q y_-, y_+ t^{-2}_q)}{2B_q(g, 2g) \mathcal{L}_q(t^{1/2}_q y_-, y_-) \mathcal{L}_q(t; x_-, y_+ t^{-2}_q)}
$$

and the following substitutions: $\alpha = g, \beta = 2g, r = t^{-1/2}_q y_+, y = y_-, x = x_- \equiv (x_1 x_2^{-1})^{1/2}, x_+ \equiv (x_1 x_2)^{1/2}, y_+ \equiv (y_1 y_2^{-1})^{1/2}$.

Theorem 3 ([14]) The operator $M$ transforms any $A_2$ Macdonald polynomial $P_\lambda$ into the product

$$
M : P_\lambda(x_1, x_2, x_3; q, t) \rightarrow c_{\lambda} y_3^{\lambda_1 + \lambda_2 + \lambda_3} \varphi_{\lambda}(y_1) \varphi_{\lambda}(y_2)
$$

of factorised polynomials $\varphi_{\lambda_1, \lambda_2, \lambda_3}(y)$ of one variable

$$
\varphi_{\lambda}(y) = y^{\lambda_1} (y; q)_{1-3g} 3\phi_2 \left[ \begin{array}{c} t^{-3}q^{-1-\lambda_1}, t^{-2}q^{-1-\lambda_2}, t^{-1}q^{-1-\lambda_3} \\ t^{-2}q^{-1-\lambda_1}, t^{-1}q^{-1-\lambda_2} \\ q, y \end{array} \right].
$$

The normalisation coefficient $c_{\lambda}$ equals

$$
c_{\lambda} = t^{2\lambda_2 - 4\lambda_1} \frac{(t^2; q)_{\lambda_1}(t^2; q)_{\lambda_2}(t; q)_{\lambda_3}}{(t^3; q)_{\lambda_1}(t; q)_{\lambda_2}(t^2; q)_{\lambda_3}}.
$$
Theorem 4 ([14]) The inverse operator $M^{-1}$ is the integral operator

$$
(M^{-1}f)(x_1, x_2, x_3) = \frac{1}{2\pi i} \int_{x_1+g, x_3} dy_- \left[ \overline{M} \left( \left( \frac{x_1 x_2}{x_3} \right)^{\frac{1}{2}}, \left( \frac{x_1}{x_2} \right)^{\frac{1}{2}} \right) \right] f \left( \frac{t^2(x_1 x_2)^{\frac{1}{2}} y_-}{x_3}, \frac{t^2(x_1 x_2)^{\frac{1}{2}}}{x_3 y_-}, x_3 \right)
$$

with the kernel

$$
\overline{M}(x_+, x_- | y_-) = \frac{(1-q)(q; q)_{\infty}^2 (y_2^2, y_-^2; q)_{\infty} \mathcal{L}_q(t; x_-, x_+)}{2B_q(-g, 3g) \mathcal{L}_q(t^{-\frac{1}{2}}; y_-, x_-) \mathcal{L}_q(t^2; y_-, x_+)}.
$$

It provides a new integral representation for the $A_2$ Macdonald polynomials in terms of the factorised polynomials $\varphi_{\lambda_1, \lambda_2, \lambda_3}(y)$ of one variable

$$
M^{-1} : c_{\lambda} y_3^{\lambda_1 + \lambda_2 + \lambda_3} \varphi_{\lambda}(y_1) \varphi_{\lambda}(y_2) \mapsto P_{\lambda}(x_1, x_2, x_3; q, t).
$$

For positive integer $g$ the operator $M^{-1}$ turns into the $q$-difference operator of order $g$:

$$
M^{-1} : f(y_1, y_2, y_3) \mapsto \sum_{k=1}^{g} \xi_k \left( \left( \frac{x_1 x_2}{x_3} \right)^{\frac{1}{2}}, \left( \frac{x_1}{x_2} \right)^{\frac{1}{2}} \right) f \left( \frac{q^{g+k} x_1}{x_3}, q^{2g-k} \frac{x_2}{x_3}, x_3 \right)
$$

where $\xi_k(r, y)$ is given by

$$
\xi_k(r, y) = (-1)^k q^{\frac{k(k-1)}{2}} \left[ \begin{array}{c} q^g \\frac{y^{-2k}(1 - q^{g-2k} y^2)}{(t^2 q^{g})_{\infty} (q^{-k} y^{-2}; q)_{\infty} (r y^{-1}; q)_{\infty} (r y^{-1}; q)_{\infty}} \end{array} \right].
$$

Remark 3. As was found in [14, 15] the operators $M_{\xi}$ and $M$ are closely related to a slightly more general integral operator $M_{\alpha\beta}$ which, in turn, is closely related to the fractional $q$-integration operator $I^\alpha$.

Remark 4. The apparent similarity of formulas for the operator $M$ in cases $n = 2$ and $n = 3$ is not a coincidence. Its explanation relies on the reduction $gl(2) \subset gl(3)$, see [15].

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