A CATLIN-TYPE THEOREM FOR GRAPH PARTITIONING AVOIDING PRESCRIBED SUBGRAPHS

YASER ROWSHAN AND ALI TAHERKHANI

Abstract. As an extension of the Brooks theorem, Catlin in 1979 showed that if $H$ is neither an odd cycle nor a complete graph with maximum degree $\Delta(H)$, then $H$ has a vertex $\Delta(H)$-coloring such that one of the color classes is a maximum independent set. Let $G$ be a connected graph of order at least 2. A $G$-free $k$-coloring of a graph $H$ is a partition of the vertex set of $H$ into $V_1, \ldots, V_k$ such that $H[V_i]$ does not contain any subgraph isomorphic to $G$. As a generalization of Catlin’s theorem we show that a graph $H$ has a $G$-free $\left\lceil \frac{\Delta(H)}{\delta(G)} \right\rceil$-coloring for which one of the color classes is a maximum $G$-free subset of $V(H)$ if $H$ satisfies the following conditions; (1) $H$ is not isomorphic to $G$ if $G$ is regular, (2) $H$ is not isomorphic to $K_{\Delta(G)} + 1$, and (3) $H$ is not an odd cycle if $G$ is isomorphic to $K_2$. Indeed, we show even more, by proving that if $G_1, \ldots, G_k$ are connected graphs with minimum degrees $d_1, \ldots, d_k$, respectively, and $\Delta(H) = \sum_{i=1}^{k} d_i$, then there is a partition of vertices of $H$ to $V_1, \ldots, V_k$ such that each $H[V_i]$ is $G_i$-free and moreover one of $V_i$s can be chosen in a way that $H[V_i]$ is a maximum $G_i$-free subset of $V(H)$ except either $k = 1$ and $H$ is isomorphic to $G_1$, each $G_i$ is isomorphic to $K_{\Delta+i}$ and $H$ is not isomorphic to $K_{\Delta(H)+1}$, or each $G_i$ is isomorphic to $K_2$ and $H$ is not an odd cycle.

1. Introduction

In this paper, we are only concerned with simple graphs and we follow [3] for terminology and notations not defined here. For a given graph $G$, we denote its vertex set, edge set, maximum degree, and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$, respectively. For a vertex $v \in V(G)$, we use $\deg_G(v)$ (or simply $\deg(v)$) and $N_G(v)$ to denote the degree and the set of neighbors of $v$ in $G$, respectively. The subgraph of $G$ induced on $X \subseteq V(G)$ is denoted by $G[X]$.

A $k$-coloring of $G$ is an assignment of $k$ colors to its vertices so that no two adjacent vertices receive the same color. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum number $k$ for which $G$ has a $k$-coloring. It is known that for any graph $G$, we have $\chi(G) \leq \Delta(G) + 1$. Brooks showed that if a connected graph $G$ is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$ [5].

The conditional chromatic number $\chi(H, P)$ of $H$, with respect to a graphical property $P$, is the minimum number $k$ for which there is a partition of $V(H)$ into sets $V_1, \ldots, V_k$ such that for each $1 \leq i \leq k$, the induced subgraph $H[V_i]$ satisfies the property $P$. This generalization of graph coloring was introduced by Harary in 1985 [10]. In this sense, for an ordinary graph coloring, the subgraph induced on each $V_i$ of the partition does not contain $K_2$. As another special case, when $P$ is the property of being acyclic, $\chi(H, P)$ is called the vertex arboricity of $H$. In other words, the vertex arboricity of a graph $H$, denoted by $a(H)$, is the minimum number $k$ for which $V(H)$ can be decomposed into subsets $V_1, \ldots, V_k$ so that each subset induces an acyclic subgraph. The vertex arboricity of graphs was first introduced by Chartrand, Kronk, and Wall in [9]. Also, it has been shown that for any arbitrary graph, say $H$, $a(H) \leq \left\lfloor \frac{\Delta(H)+1}{2} \right\rfloor$ [9], while a Brooks-type theorem was proved in [13]. If $H$ is not a cycle or a complete graph of odd order, then we have $a(H) \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$ [13] and for a planar graph $H$, it has been shown that $a(H) \leq 3$ [9,12]. Moreover, for $k \in \{3, 4, 5, 6\}$, and every planar graph $H$ with no subgraph isomorphism to $C_k$, we have $a(H) \leq 2$ [16] (for more results on arboricity see e.g. [1,4,6,9,11,13,16]).
When $P$ is the property of not containing a subgraph isomorphic to $G$, we write $\chi_G(H)$ instead of $\chi(G, P)$ which is called the $G$-free chromatic number, henceforth. In this regard, we say a graph $H$ has a $G$-free $k$-coloring if there is a map $c: V(H) \rightarrow \{1, 2, \ldots, k\}$ such that the subgraph induced on each one of the color classes of $c$ is $G$-free. One can see that an ordinary $k$-coloring is a $K_2$-free coloring of a graph $H$ with $k$ colors. Also, for any graph $H$, one may show that $\chi_G(H) \leq \lceil \frac{\chi(H)}{\delta(G)} \rceil$.

In 1941 Brooks proved that for a connected graph $H$, $\chi(H) \leq \Delta(H)$ when $H$ is neither an odd cycle nor a complete graph. As an extension of Brooks’ theorem, Catlin showed that if $H$ is neither an odd cycle nor a complete graph, then $H$ has a proper $\Delta(H)$-coloring for which one of the color classes is a maximum independent set of $H$ [6]. Here, we prove an extension of Catlin’s result for partitioning of the vertex set of a graph $H$ in a way that each class avoids having a prescribed subgraph. Clearly, in this way, we obtain a Brooks-Catlin-type theorem for the $G$-free chromatic number of a graph $H$ as follows.

**Theorem 1.** Let $k \geq 1$ be a positive integer. Assume that $G_1, \ldots, G_k$ be connected graphs with minimum degrees $d_1, \ldots, d_k$, respectively, and $H$ be a connected graph with maximum degree $\Delta(H)$ where $\Delta(H) = \sum_{i=1}^{k} d_i$. Assume that $G_1, G_2, \ldots, G_k$, and $H$ satisfy the following conditions;

- If $k = 1$, then $H$ is not isomorphic to $G_1$.
- If $G_i$ is isomorphic to $K_{d_i+1}$ for each $1 \leq i \leq k$, then $H$ is not isomorphic to $K_{\Delta(H)+1}$.
- If $G_i$ is isomorphic to $K_2$ for each $1 \leq i \leq k$, then $H$ is neither an odd cycle nor a complete graph.

Then, there is a partition of vertices of $H$ to $V_1, \ldots, V_k$ such that each $H[V_i]$ is $G_i$-free and moreover one of $V_i$s can be chosen in a way that $H[V_i]$ is a maximum induced $G_i$-free subgraph in $H$.

In Theorem 1 if we take $G_i = K_2$ for $1 \leq i \leq k$, then we get Catlin’s result. Also, if for a given graph $G$ and for $1 \leq i \leq k$ we choose $G_i = G$, we obtain the following Brooks-Catlin-type result for $G$-free coloring of graphs.

**Corollary 2.** Let $G$ be a connected graph with minimum degree $\delta(G) \geq 1$. Also, assume that $H$ is a connected graph with maximum degree $\Delta(H)$ while $H$ satisfies the following conditions;

- If $G$ is regular, then $H \cong G$.
- If $G$ is isomorphic to $K_{\delta(G)+1}$, then $H$ is not $K_{k\delta(G)+1}$.
- If $G$ is isomorphic to $K_2$, then $H$ is neither an odd cycle nor a complete graph.

Then, there is an $G$-free $\lceil \frac{\Delta(H)}{\delta(G)} \rceil$-coloring of $H$ such that one of whose color classes is a maximum induced $G$-free subgraph in $H$. In particular,

$$\chi_G(H) \leq \lceil \frac{\Delta(H)}{\delta(G)} \rceil.$$  

An analogue to Catlin’s result for vertex arboricity is due to Catlin and Lai [7]. They proved the following interesting theorem for the vertex arboricity of graphs.

**Theorem A.** [7] Assume that $H$ is neither a cycle nor a complete graph of odd order.

- If $\Delta(H)$ is even, then there is a coloring with $\frac{\Delta(H)}{2}$ colors such that each color class induces an acyclic subgraph and one of those is a maximum induced acyclic subgraph in $H$.
- If $\Delta(H)$ is odd, then there is a coloring with $\lceil \frac{\Delta(H)}{2} \rceil$ colors such that each color class induces an acyclic subgraph. Moreover, this coloring can be chosen to satisfy one of the following properties:
  - (a) one color class is an independent set and one color class is a maximum induced acyclic subgraph in $H$.
  - (b) one color class is a maximum independent set in $H$.  

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Let $\mathcal{G}$ be a family of graphs. For a graph $H$, a subset $W$ of $V(H)$ is said to be $\mathcal{G}$-free if $H[W]$ does not contain any one of the members of $\mathcal{G}$. Therefore, a $\mathcal{G}$-free coloring of graph may be defined similarly. For example, if the family $\mathcal{G}$ consists of all connected graphs with minimum degree at least 2, then the $\mathcal{G}$-free chromatic number of a graph $H$ is equal to the vertex arboricity of $H$. We define the minimum degree of $\mathcal{G}$ by $δ(\mathcal{G}) = \min\{δ(G) | G ∈ \mathcal{G}\}$. In this setup, it is straightforward to a generalization of Theorem 1 as follows.

**Theorem C.** To see this, consider the following example. Set $V$ be decomposed into two subsets $χ$ as a consequence of Theorem C we have $H$ is a connected graph with minimum degree $∆(H)$ where $∆(H) = \sum_{i=1}^{k} d_k$. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$, and $H$ satisfy the following conditions:

- If $k = 1$, then $H \notin \mathcal{G}_1$.
- If $K_{d+1} \in \mathcal{G}_i$ for each $1 ≤ i ≤ k$, then $H$ is not isomorphic to $K_{∆(H)+1}$.
- If $K_2 \in \mathcal{G}_i$ for each $1 ≤ i ≤ k$, then $H$ is neither an odd cycle nor a complete graph.

Then, there is a partition of vertices of $H$ to $V_1, \ldots, V_k$ such that each $H[V_i]$ is $\mathcal{G}_i$-free and moreover one of $V_i$s can be chosen in a way that $H[V_i]$ is a maximum induced $\mathcal{G}_i$-free subgraph in $H$.

A graph $H$ is said to be $p$-degenerate if every subgraph of $H$ has a vertex of degree at most $p$. Let the family $\mathcal{G}_p^p$ consist of all connected graphs with minimum degree at least $p$. One may show that being $p$-degenerate is equivalent to not containing any subgraph isomorphic to any one of the members of $\mathcal{G}_p^{p+1}$. Therefore, the Catlin-Lai theorem and the next theorem due to Matamala are direct consequences of Theorem 3.

**Theorem B.** [15] Let $H$ be a graph with maximum degree $∆(H) ≥ 3$ and $ω(H) ≤ ∆(H)$. If $∆(H) = d_1 + d_2$, then the vertices of $H$ can be partitioned into two sets $V_1$ and $V_2$ such that $H[V_1]$ is a maximum $(d_1 - 1)$-degenerate induced subgraph and $H[V_2]$ is $(d_2 - 1)$-degenerate.

One can easily show that the following result due to Bollobás and Manvel can be extended to their $\mathcal{G}_i$-free versions (instead of $(d_i - 1)$-degeneracy).

**Lemma A.** [2] Let $H$ be a graph with maximum degree $∆(H) ≥ 3$ and $ω(H) ≤ ∆(H)$. If $∆(H) = d_1 + d_2$, then the vertices of $H$ can be partitioned into two sets $V_1$ and $V_2$ such that $∆(H[V_1]) ≤ d_1$, $∆(H[V_2]) ≤ d_2$, $H[V_1]$ is $(d_1 - 1)$-degenerate and $H[V_2]$ is $(d_2 - 1)$-degenerate.

Also, it is worth mentioning the following result of Lovász which has a close relation to the previous result of Bollobás and Manvel.

**Theorem C.** [13] If $d_1 ≥ d_2 ≥ \ldots ≥ d_k$ are positive integers such that $d_1 + d_2 + \ldots + d_k ≥ ∆(H) + 1$, then $V(H)$ can be decomposed into subsets $V_1, V_2, \ldots, V_k$, such that $∆(H[V_i]) ≤ d_i - 1$ for each $1 ≤ i ≤ k$.

Note that if one chooses $k = ∆(H) + 1$ and $d_1 = \cdots = d_k = 1$, then this result implies that $χ(H) ≤ ∆(H) + 1$. Also, it is instructive to note that $∆(H) + 1$ can not be replaced by $∆(H)$ in Theorem C. To see this, consider the following example. Set $H = K_{3,3,3}$ which has maximum degree 6 and assume that $k = 2$ and $d_1 = d_2 = 3$, and note that there is not any decomposition of vertices of $K_{3,3,3}$ to subsets $V_1$ and $V_2$ such that $∆(H[V_i]) ≤ 2$. Of course, one can find some other nontrivial examples, too. Moreover, one may construct a graph $H$ for which $∆(H) = d_1 + d_2$ and $H$ can not be decomposed into two subsets $V_1, V_2$ such that $∆(H[V_i]) ≤ d_i - 1$ for each $i ∈ \{1,2\}$ (see [2]). Also, if $H$ and $G$ are connected graphs with maximum degrees $∆(H)$ and $∆(G)$, respectively, then, as a consequence of Theorem C we have $\chi_G(H) ≤ \lceil \frac{∆(H)+1}{∆(G)} \rceil ≤ \lceil \frac{∆(H)+1}{δ(G)} \rceil$. 3
The following lemma is the main part of the proof of Theorem 1.

**Lemma 4.** Let $G$ and $H$ be two connected graphs, where $G$ has the minimum degree $d$ and $H$ has the maximum degree $\Delta(H)$ where $\Delta(H) \geq d + 1$. Assume that $S \subseteq V(H)$, $H[S]$ is $G$-free and $S$ has the maximum possible size. Suppose that $H \setminus S$ has as few connected $(\Delta(H) - d)$-regular subgraphs as possible. Also, suppose that $H[S]$ has the minimum possible number of connected components. If $H \setminus S$ has a $(\Delta(H) - d)$-regular connected subgraph, say $H_0$, then

(a) for any vertex $v \in V(H_0)$, $|N(v) \cap S| = d$,
(b) the induced subgraph $H[S \cup \{v\}]$ has a unique copy of $G$, say $G_v$, such that $G_v$ is a $d$-regular component of $H[S \cup \{v\}]$, and
(c) Either $G$ is isomorphic to $K_{d+1}$ and $H$ is isomorphic to $K_{\Delta(H)+1}$, $G = K_2$ and $H$ is isomorphic to $C_{2\ell+1}$ for some positive integer $\ell$, or $H$ is isomorphic to $G$.

**Proof.** By the maximality of $S$, for each vertex $v \in V(H) \setminus S$, $H[S \cup \{v\}]$ has a copy of $G$. Therefore, $|N(v) \cap S| \geq d$ and consequently

$$\Delta(H \setminus S) \leq \Delta(H) - d.$$ 

Thus, $H_0$ is a connected component of $H \setminus S$. Hence, for any $v \in V(H_0)$, $|N(v) \cap (V(H) \setminus S)| = \Delta(H) - d$. Consequently, for any $v \in V(H_0)$, we have $|N(v) \cap S| = d$.

To prove Part (b), let $v \in V(H_0)$ and $G_v$ be a copy of $G$ in $H[S \cup \{v\}]$. Since $G$ has minimum degree $d$ and $|N(v) \cap S| = d$, we have $N(v) \cap S \subseteq V(G_v)$.

**Claim 5.** The subgraph $G_v$ is a unique copy of $G$ in $H[S \cup \{v\}]$ and moreover $G_v$ is a $d$-regular graph.

**Proof of Claim 5.** By contradiction suppose that there are two copies of $G$ in $H[S \cup \{v\}]$ such that these two copies of $G$ in $H[S \cup \{v\}]$ have different vertex sets. If $u \in S \cup \{v\}$ and $d_{H[S \cup \{v\}]}(v, u) \leq 1$, then $u$ lies in all copies of $G$ in $H[S \cup \{v\}]$. Now, let $i \geq 1$ be the largest positive integer such that for any vertex $u \in S \cup \{v\}$ with $d_{H[S \cup \{v\}]}(v, u) = i$, we have $u$ lies in all copies of $G$ in $H[S \cup \{v\}]$. Since there exist at least two copies of $G$ with different vertex sets, there exists at least one vertex $w \in S$ and a copy of $G$ in $H[S \cup \{v\}]$, say $G^*$, such that $d_{H[S \cup \{v\}]}(v, w) = i + 1$ and $w \notin V(G^*)$. Therefore, there is at least one neighbor of $w$ in $S$, say $y$, such that $d(v, y) = d(v, w) - 1 = i$. Since $d(v, y) = i$, $y$ lies in all copies of $G$ in $H[S \cup \{v\}]$. Set $S_1 = (S \setminus \{y\}) \cup \{v\}$. Note that $|S_1| = |S|$ and $H[S_1]$ is $G$-free because $y$ lies in all copies of $G$ in $H[S \cup \{v\}]$. Since $y$ is in at least two copies of $G$ in $H[S \cup \{v\}]$ and one of them does not contain $w$, we have $|N(y) \cap (S \cup \{v\})| = |N(y) \cap S_1| \geq d + 1$. Therefore, $|N(y) \cap (V(H) \setminus S_1)| \leq \Delta(H) - d - 1$. As a consequence $y$ does not lie in any $(\Delta(H) - d)$-regular subgraph in $H \setminus S_1$. Hence, the number of $(\Delta(H) - d)$-regular connected subgraphs of $H \setminus S_1$ is less than that of $H \setminus S$, which contradicts the assumption that $H \setminus S$ has as few $(\Delta(H) - d)$-regular subgraphs as possible. Thus, $H[S \cup \{v\}]$ contains only one copy of $G$.

Now assume that all copies of $G$ in $H[S \cup \{v\}]$ have the same vertex set. If there exist at least two distinct copies of $G$ in $H[S \cup \{v\}]$ with the same vertex set, then there is a vertex $u \in V(G_v) \subseteq S$ such that $|N(u) \cap (S \cup \{v\})| \geq d + 1$. Define $S_1 = (S \setminus \{u\}) \cup \{v\}$. Since $H[S_1]$ is $G$-free, $|S_1| = |S|$, and $H \setminus S$ has as few $(\Delta(H) - d)$-regular connected subgraphs as possible, so $u$ must lie in a $(\Delta(H) - d)$-regular subgraph in $H \setminus S_1$. Therefore, $|N(u) \cap (V(H) \setminus (S \cup \{v\}))| \geq \Delta(H) - d$ and consequently $|N(u)| = \deg(u) \geq \Delta(H) + 1$, which is a contradiction.

Assume that $G_v$ is a subgraph of $H[S \cup \{v\}]$ but is not one of its connected components. Thus, there is at least one vertex of $G_v$, say $u$, such that $|N(u) \cap (S \cup \{v\})| \geq d + 1$. Therefore, using the same reasoning as the previous paragraph we can prove that $G_v$ is a component of $H[S \cup \{v\}]$.

**Claim 6.** The subgraph $G_v$ is a component of $H[S \cup \{v\}]$. 

2. **Proofs**
To prove Part (c), set $S_0 = S$. In view of Part (a), we have $H_0$ is a component of $H \setminus S_0$.

Assume that $H_0$ has only one vertex, say $v$. Now by Claims 5 and 6, $H[S_0 \cup \{v\}]$ has a unique copy of $G$, which is a component of $H[S_0 \cup \{v\}]$. Since $H$ is connected and $\Delta(H) = d$, we have $H$ is isomorphic to $G$. Assume that $|V(H_0)| = 2$ and $d = 1$. Then, $H_0 \cong K_2$ and from Claim 5 we have $G \cong K_2$. Consequently, $\Delta(H) = 2$. Since $H$ is connected, we have $H$ is path or cycle. As $S$ independent set of maximum size and $H \setminus S$ has a copy of $K_2$, $H$ must be an odd cycle. Therefore, we may assume that either $|V(H_0)| \geq 3$ or $d \geq 2$.

Let $v \in V(H_0)$. By using Part (b), $H[S_0 \cup \{v\}]$ has a unique copy of $G$, say $G_v$.

**Claim 7.** Let $v$ be a vertex of $V(H_0)$ which is not a cut vertex. If for some $w \neq v$ in $V(H_0)$ we have $G_v \setminus \{v\} = G_w \setminus \{w\}$, then the statement of Part (c) holds.

**proof of Claim 7**. Since $G_v$ and $G_w$ are $d$-regular and $G_v \setminus \{v\} = G_w \setminus \{w\}$, we have $N(v) \cap S_0 = N(w) \cap S_0$. We show that $H[N(v) \cap S_0] \cong K_d$. By contradiction assume that there exist two vertices $y$ and $y'$ in $N(v) \cap S_0$ such that $yy' \notin E(H)$. Define $S^* = (S_0 \setminus \{y, y'\}) \cup \{v, w\}$. One can check that $|S^*| = |S|$ and $|N(v) \cap S^*| = |N(w) \cap S^*| \leq d - 1$ and hence $H[S^*]$ is $G$-free. Since $|N(y) \cap S^*| \geq d + 1$ and $|N(y') \cap S^*| \geq d + 1$, $y$ and $y'$ are not in any $(\Delta(H) - d)$-regular subgraph in $H \setminus S^*$. Hence, the number of $(\Delta(H) - d)$-regular connected subgraphs of $H \setminus S^*$ is less than that of $H \setminus S_0$, which contradicts the assumption that $H \setminus S_0$ has as few $(\Delta(H) - d)$-regular connected subgraphs as possible. Hence, for every two vertices $y$ and $y'$ in $N(v) \cap S_0$, $yy' \notin E(H)$. Therefore, $H[N(v) \cap S_0] \cong K_d$ and moreover $G_v \cong K_{d+1}$.

For every vertex $y \in N(v) \cap S_0$, we shall show that the subgraph induced by $N(y) \setminus S_0$ is isomorphic to the complete graph $K_{\Delta(H) - d+1}$. Define $S_1 = (S_0 \setminus \{y\}) \cup \{v, w\}$. One can check that $|S_1| = |S_0|$ and $H[S_1]$ is $G$-free. Since $H \setminus S_0$ has as few $(\Delta(H) - d)$-regular connected subgraphs as possible, $y$ must lie in a $(\Delta(H) - d)$-regular connected subgraph in $H \setminus S_1$, say $H_1$. Therefore, the number of neighbors of $y$ in $H \setminus S_0$ is $\Delta(H) - d + 1$.

As $N(w) \cap S_0 = N(v) \cap S_0$, we have $y$ is adjacent to $w$ and moreover $w \in V(H_1)$. Since $H_0 \setminus v$ is connected and $w \in V(H_0) \cap V(H_1)$, we have $(V(H_0) \setminus v) \subseteq V(H_1)$; otherwise there is a vertex in $V(H_0) \cap V(H_1)$ which has degree greater than $\Delta(H)$, which is not possible. Since $H_0$ and $H_1$ are $(\Delta(H) - d)$-regular, so $N(v) \setminus S_0 = N(y) \setminus (S_0 \cup \{v\})$. Therefore, $N(y) \setminus S_0$ is a subset of $V(H_0)$. Assume that $v'$ and $v''$ are two neighbors of $y$ in $H \setminus S_0$. We show that $v'v'' \in E(H)$. On the contrary, assume that $v'$ is not adjacent to $v''$. By Part (a), $|N(v') \cap S_0| = |N(v'') \cap S_0| = d$. Define $S_2 = (S_0 \setminus \{y\}) \cup \{v', v''\}$. One can check that $|S_2| = |S_0| + 1$ and $|N(v') \cap S_2| = |N(v'') \cap S_2| = d - 1$. Hence, $H[S_2]$ is $G$-free, which contradicts the maximality of $S_0$. Therefore, $v'v'' \in E(H)$ and consequently the subgraph induced by $N(y) \setminus S_0$ is isomorphic to the complete graph $K_{\Delta(H) - d+1}$. Therefore, $H_0 \cong K_{\Delta(H) - d+1}$.

For any two vertices $y$ and $y'$ in $N(v) \cap S_0$, we shall show $N(y) \setminus S_0 = N(y') \setminus S_0$. The vertex $v$ belongs to $(N(y) \cap N(y')) \setminus S_0$. On the contrary, suppose that there is a vertex $u \in N(y) \setminus S_0$ such that $u \notin N(y') \setminus S_0$. As $v$ is adjacent to $u$ and $v \in N(y') \setminus S_0$, so $v$ has at least $\Delta(H) - d + 1$ neighbors in $H \setminus S_0$. Consequently, $\deg(v) \geq \Delta(H) + 1$ which is not possible. Therefore, every vertex $y \in N(v) \cap S_0$ is adjacent to all vertices of $H_0$. Since $H[N(v) \cap S_0]$ is isomorphic to $K_d$, the subgraph induced by $N(y) \setminus S_0$ is isomorphic to the complete graph $K_{\Delta(H) - d+1}$, and every vertex $y \in N(v) \cap S_0$ is adjacent to all vertices of $H_0$, we conclude that $H[N(v) \cap S_0] \lor H_0 \cong K_{\Delta(H) + 1}$ is a subgraph of $H$. Since $H$ is connected, we have $H \cong K_{\Delta(H) + 1}$.

Now assume that for two vertices $v, w$ in $V(H_0)$ we have $G_v \setminus \{v\} \neq G_w \setminus \{w\}$.

**Claim 8.** If $G_v \setminus \{v\} \neq G_w \setminus \{w\}$, then $N(v) \cap N(w) \cap S_0 = \emptyset$.

**proof of Claim 8**. If $V(G_v \setminus \{v\}) = V(G_w \setminus \{w\})$, then $E(G_v \setminus \{v\}) \setminus E(G_w \setminus \{w\}) \neq \emptyset$ and hence we can find a vertex in $G_w$ with degree greater than $d$. This is a contradiction because from
Part (b), $G_w$ is a $d$-regular component of $H[S_0 \cup \{w\}]$. Therefore, there exists at least one vertex $u \in V(G_v \setminus \{v\}) \setminus V(G_w \setminus \{w\})$.

Suppose, by way of contradiction, that $y \in N(v) \cap N(w) \cap S_0$. Since $H[S_0]$ has the minimum number of connected components, we conclude that $G_v \setminus v$ is connected; otherwise as $G_v$ is a connected component of $H[S_0 \cup \{v\}]$, choose a vertex $v' \in V(G_v)$ such that $G_v \setminus v'$ remains connected. Define $S' = (S_0 \setminus \{v'\}) \cup \{v\}$. One can check that $|S'| = |S_0|$, $H[S']$ is $G$-free, and $H \setminus S'$ contains the same number of $(\Delta(H) - d)$-regular connected subgraphs as $H \setminus S_0$. But the number of connected components of $H[S']$ is less than that of $H[S_0]$, which is impossible. As $G_v \setminus v$ is connected, there is a shortest path $P$ from $y$ to $u$ in $G_v \setminus v$. Let $y'$ be the last vertex of $P$ in $V(G_v \setminus \{v\}) \cap V(G_w \setminus \{w\})$. Define $S^* = (S_0 \setminus \{y'\}) \cup \{w\}$. One can check that $|S^*| = |S_0|$ and $H[S^*]$ is $G$-free. The vertex $y'$ has at least $d + 1$ neighbors in $S^*$, because $y'$ has $d$ neighbors in $G_w$ and $y'$ is adjacent to its immediate successor on $P$ which is not $G_w$. Therefore, the number of neighbors of $y'$ in $H \setminus S^*$ is at most $\Delta(H) - d - 1$. Thus, $y'$ does not lie in any $(\Delta(H) - d)$-regular subgraph in $H \setminus S^*$. This contradicts the assumption that $H \setminus S_0$ has the minimum number of $(\Delta(H) - d)$-regular connected subgraphs.

Suppose that $v_0 \in V(H_0)$ is not a cut vertex of $H_0$. Choose a vertex $y_0 \in V(G_{v_0})$ such that $y_0$ is not a cut vertex in $G_{v_0}$ and $y_0 \neq v_0$. Set $S_i = (S_0 \setminus \{y_0\}) \cup \{v_i\}$. Since $|S_i| = |S_0|$, $H[S_i]$ is $G$-free, and $H \setminus S_i$ has at least $|S_i| - 1$ components of $(\Delta(H) - d)$-regular connected subgraphs as possible, $y_0$ must be in a $(\Delta(H) - d)$-regular subgraph in $H \setminus S_i$, say $H_i$. Also, the number of components of $H[S_i]$ is equal to that of $H[S_0]$. If $V(H_0) \cap V(H_1) \neq \emptyset$, then $V(H_0) \setminus \{v_0\} \subset V(H_1)$; otherwise there is a vertex in $V(H_0) \cap V(H_1)$ has degree greater than $\Delta(H)$, which is not possible. Therefore, $N(v_0) \cap V(H_0) \subseteq V(H_1)$. Using the same reasoning as Claim 7 one can show that the induced subgraph by $N(v_0) \cap V(H_1)$ is a complete graph and consequently $H_0$ and $H_1$ are isomorphic to the complete graph $K_{\Delta(H) - d + 1}$. Since $|V(H_0)| \geq 2$, we can choose a vertex $u$ distinct from $v_0$ in $H_0$ such that $y_0 \in N(v_0) \cap N(u) \cap S_0$. Therefore, by using Claim 8 we have $G_{v_0} \setminus \{v_0\} = G_u \setminus \{u\}$. Hence, Claim 7 implies the statement.

Suppose that $V(H_0) \cap V(H_1) = \emptyset$. For $i \geq 1$, assume that $H_{i-1}$, $S_{i-1}$, $v_{i-1}$, $G_{v_{i-1}}$, and $y_{i-1}$ are chosen such that $v_{i-1}$ is not a cut vertex in $H_{i-1}$, $G_{v_{i-1}}$ is a unique copy of $G$ in $H[S_{i-1} \cup \{v_{i-1}\}]$, and $y_{i-1}$ is not a cut vertex in $G_{v_{i-1}}$. Set $S_i = (S_{i-1} \setminus \{y_{i-1}\}) \cup \{v_{i-1}\}$. The vertex $y_{i-1}$ must be in a $(\Delta(H) - d)$-regular connected subgraph in $H \setminus S_i$, say $H_i$. Also, the number of components of $H[S_i]$ is equal to that of $H[S_{i-1}]$. Choose a vertex $v_i \in V(H_i)$ such that $v_i \neq y_{i-1}$ and $v_i$ is not a cut vertex of $H_i$. Assume that $G_{v_i}$ is a unique copy of $G$ in $H[S_i \cup \{v_i\}]$ and choose $y_i$ such that $y_i$ is not a cut vertex in $G_{v_i}$.

Since $H$ is a finite graph, there is the smallest number $\ell$ such that $V(H_\ell)$ intersects $V(H_j)$ for some $j \leq \ell - 1$. Without loss of generality assume that $j = 0$. As the case $V(H_0) \cap V(H_1) = \emptyset$, one can show that $V(H_0) \setminus \{v_0\} \subset V(H_\ell)$.

Claim 9. We can assume that $V(G_{v_0}) \setminus \{y_0\} \subseteq S_\ell$.

Proof. On the contrary, assume that $V(G_{v_0}) \setminus \{y_0\} \not\subseteq S_\ell$. Let $i$ be the smallest number for which $V(G_{v_0}) \setminus \{y_0\} \not\subseteq S_i$. Therefore, $y_{i-1} \in V(G_{v_0}) \setminus \{y_0\}$. Since $G_{v_0} \setminus \{y_0\}$ is connected in $H[S_{i-1}]$, $G_{v_{i-1}}$ is $d$-regular, and $y_{i-1}$ lies in both $G_{v_0} \setminus \{y_0\}$ and $G_{v_{i-1}}$, we have $V(G_{v_0}) \setminus \{y_0\} \subseteq V(G_{v_{i-1}})$. As $G_{v_0}$ and $G_{v_{i-1}}$ are $d$-regular, it follows that $N(y_0) \cap V(G_{v_0}) = N(v_{i-1}) \cap V(G_{v_{i-1}})$.

If $i = 2$, then $N(y_0) \cap S_1 = N(v_1) \cap S_1$ and consequently $|N(y_0) \cap N(v_1) \cap S_1| = d$. By using Claims 7 and 8 we conclude the statement of Part (c). Therefore, we can assume that $i \geq 3$.

Suppose that $y_0$ is adjacent to $v_{i-1}$. Since $H_{i-1}$ is $(\Delta(H) - d)$-regular, we have $y_0 \in V(H_{i-1})$. Hence, $y_0 \in V(H_1) \cap V(H_{i-1}) = \emptyset$, which contradicts the minimality of $\ell$. Therefore, we can assume that $y_0$ is not adjacent to $v_{i-1}$. Consider the following two cases.

(i) $d \geq 2$.

The vertex $y_0$ has $d$ or $d + 1$ neighbors in $S_{i-1}$. One of them may be $v_1$ and $d$ of them must be in
V(G_{v_0}) \setminus \{y_0\}. We show that $G_{v_i-1}$ is isomorphic to $K_{d+1}$. If two vertices $u, u'$ in $N(v_{i-1}) \cap S_{i-1}$ are not adjacent, then define $S' = (S_{i-1} \setminus \{u, u'\}) \cup \{y_0, v_{i-1}\}$. The vertices $y_0$ and $v_{i-1}$ do not lie in any copy of $H$ in $H[S']$ because $y_0$ have at most $d-1$ and $v_{i-1}$ have $d-2$ neighbors in $S'$, respectively. Thus, $H[S']$ is $G$-free. Both vertices $u$ and $u'$ have $d+1$ neighbors in $S'$. Therefore, $u$ and $u'$ do not lie in any $(\Delta(H) - d)$-regular subgraph in $H[S']$, which contradicts $H \setminus S_{i-1}$ contains the minimum possible number of $(\Delta(H) - d)$-regular connected subgraphs. Then $H_{v_{i-1}}$ is isomorphic to $K_{d+1}$.

For some $u \in N(v_{i-1}) \cap S_{i-1}$, define $S'' = (S_{i-1} \setminus \{u\}) \cup \{y_0, v_{i-1}\}$. If $y_0$ has exactly $d$ neighbors in $S_{i-1}$, then each of $y_0$ and $v_{i-1}$ has $d-1$ neighbors in $S''$ and hence $H[S'']$ is $G$-free, which contradicts the maximality of $S_{i-1}$. Therefore, we can assume that $y_0$ has exactly $d+1$ neighbors in $S_{i-1}$. Therefore, $y_0$ must be adjacent to $v_1$.

The vertex $v_1$ is not adjacent to any vertex of $N(v_{i-1}) \cap S_{i-1} = N(y_0) \cap G_{v_0}$; otherwise if $v_1$ has a neighbor in $N(y_0) \cap G_{v_0}$, then $N(y_0) \cap N(v_1) \cap S_i \neq \emptyset$. Therefore, by Claims 7 and 8 we conclude the statement of Part (c). Since $v_1$ is not adjacent to any vertex in $N(y_0) \cap G_{v_0}$ and $y_0$ has $d-1 \geq 1$ neighbors in $S''$ which are not adjacent to $v_1$, we conclude that $y_0$ cannot lie in a copy of $G \cong K_{d+1}$ in $H[S'']$. Also, $v_{i-1}$ has $d-1$ neighbors in $S''$. Thus, $H[S'']$ is $G$-free, which contradicts the maximality of $S_{i-1}$.

(ii) $d = 1$ and $|V(H_0)| \geq 3$.

Since $d = 1$, we have $G_{v_0} = K_2$ and $v_0 = y_{i-1}$. The vertex $v_0$ is adjacent to $y_0$ and has $\Delta(H) - 1$ neighbours in $V(H_0)$. The vertex $v_{i-1}$ is another neighbour of $v_0 = y_{i-1}$ which is in $H_{i-1}$. Because of $i \geq 3$ we have $v_{i-1}$ is distinct from $y_0 \in H_1$ and the vertices in $H_0$. Therefore, $\deg(v_0) \geq \Delta(H) + 1$ which is not possible.

Since $G_{v_0} \setminus \{y_0\} \subseteq S_\ell$, we have $v_\ell \in S_\ell$. As the case $V(H_0) \cap V(H_1) \neq \emptyset$, one can show that $V(H_0) \setminus \{v_0\} \subset V(H_1)$ and $H_0$ and $H_\ell$ are isomorphic to $K_{\Delta(H) - d + 1}$. If $|V(H_0)| \geq 3$, choose two vertices $w_1, w_2$ in $V(H_0) \setminus \{v_0\}$. Then, $v_0 \in N(w_1) \cap N(w_2) \cap S_1$ and hence Claims 7 and 8 imply the statement.

Assume that $d \geq 2$ and $w \in V(H_0) \cap V(H_\ell)$. Therefore, $w$ is adjacent to $y_{\ell-1}$ and $v_0$. The induced subgraph $H[S_\ell \cup \{w\}]$ contains a unique copy of $G$, say $G_w$. By using Claim 9 and as $G_{v_0} \setminus \{y_0\}$ is connected and $v_0 \in V(G_{v_0}) \cap (V(G_{v_0}) \setminus \{y_0\})$, we have $(V(G_{v_0}) \setminus \{y_0\}) \subseteq V(G_w)$. Consequently, $N(y_0) \cap V(G_{v_0}) = N(w) \cap V(G_{v_0})$. If $w$ is adjacent to $y_0$, then $y_0 \in N(v_0) \cap N(w) \cap S_0$ and Claims 7 and 8 imply the statement. Assume that $y_0$ is not adjacent to $w$. The proof of this case is same as the proof of Claim 9 when $y_0$ is not adjacent to $v_{i-1}$.

Now we are in the position to prove Theorem 11.

**Proof of Theorem 11.** Let $\Delta(H) = \sum_{i=1}^{k} d_k$. The proof is by induction on $k$. The statement trivially holds for $k = 1$. Therefore, we may assume that $k \geq 2$. Let $V_1$ be a subset of $V(H)$ such that $H[V_1]$ is $G_1$-free and $V_1$ has the maximum possible size. Hence, by Lemma 2 (a), we have $\Delta(H \setminus V_1) = \Delta(H) - d_1$. If $H \setminus V_1$ does not contain any $(\Delta(H) - d_1)$-regular components, then from the induction hypothesis, $H \setminus V_1$ can be decomposed into $k-1$ subsets $V_2, \ldots, V_k$ such that $H[V_i]$ is $G_i$-free for each $2 \leq i \leq k$. If $H \setminus V_1$ has a $(\Delta(H) - d_i)$-regular component, then by Lemma 4 (c), we must have either $G \cong K_{d+1}$ and $H \cong K_{\Delta(H)+1}$, $G \cong K_2$ and $H \cong C_{2\ell+1}$ for some positive integer $\ell$, or $H \cong G$, which is not possible.

**References**

[1] D. Bauer, A. Nevo, and E. Schmeichel. Vertex arboricity and vertex degrees. *Graphs Combin.*, 32(5):1699–1705, 2016.

[2] Béla Bollobás and Bennet Manvel. Optimal vertex partitions. *Bull. London Math. Soc.*, 11(2):113–116, 1979.
[3] J. A. Bondy and U. S. R. Murty. *Graph theory with applications*. American Elsevier Publishing Co., Inc., New York, 1976.

[4] Oleg V. Borodin. Cyclic coloring of plane graphs. volume 100, pages 281–289. 1992. Special volume to mark the centennial of Julius Petersen’s “Die Theorie der regulären Graphen”, Part I.

[5] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.*, 37:194–197, 1941.

[6] Paul A. Catlin. Brooks’ graph-coloring theorem and the independence number. *J. Combin. Theory Ser. B*, 27(1):42–48, 1979.

[7] Paul A. Catlin and Hong-Jian Lai. Vertex arboricity and maximum degree. *Discrete Math.*, 141(1-3):37–46, 1995.

[8] G. Chartrand, D. P. Geller, and S. Hedetniemi. A generalization of the chromatic number. *Proc. Cambridge Philos. Soc.*, 64:265–271, 1968.

[9] Gary Chartrand, Hudson V. Kronk, and Curtiss E. Wall. The point-arboricity of a graph. *Israel J. Math.*, 6:169–175, 1968.

[10] Frank Harary. Conditional colorability in graphs. In *Graphs and applications (Boulder, Colo., 1982)*, Wiley-Intersci. Publ., pages 127–136. Wiley, New York, 1985.

[11] Frank Harary and Paul C. Kainen. On triangular colorings of a planar graph. *Bull. Calcutta Math. Soc.*, 69(6):393–395, 1977.

[12] Stephen Hedetniemi. On partitioning planar graphs. *Canad. Math. Bull.*, 11:203–211, 1968.

[13] Hudson V. Kronk and John Mitchem. Critical point-arboritic graphs. *J. London Math. Soc. (2)*, 9:459–466, 1974/75.

[14] L. Lovász. On decomposition of graphs. *Studia Sci. Math. Hungar.*, 1:237–238, 1966.

[15] Martín Matamala. Vertex partitions and maximum degenerate subgraphs. *J. Graph Theory*, 55(3):227–232, 2007.

[16] André Raspaud and Weifan Wang. On the vertex-arboricity of planar graphs. *European J. Combin.*, 29(4):1064–1075, 2008.

Y. Rowshan, Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran

E-mail address: y.rowshan@iasbs.ac.ir

A. Taherkhani, Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran

E-mail address: ali.taherkhani@iasbs.ac.ir