Path integral quantization of Yang- Mills theory

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Summary.- Path integral formulation based on the canonical method is discussed. Path integral for Yang-Mills theory is obtained by this procedure. It is shown that gauge fixing which is essential procedure to quantize singular systems by Faddeev’s and Popov’s method is not necessary if the canonical path integral formulation is used.

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1 Introduction

Quantization of classical systems can be achieved by the canonical quantization method [1]. If we ignore the ordering problems, it consists in replacing the classical Poisson brackets, by quantum commutators when classically all the states on the phase space are accessible. This is no longer correct in the presence of constraints. An approach due to Dirac [2] is widely used for quantizing the constrained Hamiltonian systems [3,4]. Physicists start to use the canonical method because of some important properties of quantum theory, such as unitarity and positive definiteness of the metric, may be deduced easily.

The alternative quantization scheme for constrained systems is the path integral quantization. It is important because it serves as a basis to develop perturbation theory and to find out the Fynman rules. The path integral quantization of singular theories with first class constrains in canonical gauge was given by Faddeev and Popov [5,6]. The generalization of the method to theories with second class constraints is given by Senjanovic [7]. Moreover, Fradkin and Vilkovisky [8,9] considered quantization to bosonic theories with first class constraints and it is extension to include fermions in the canonical gauge. More, Gitman and Tyutin [4] discussed the canonical quantization of singular systems as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge.

When the dynamical system possesses some second class constraints there exists another method given by Batalain and Fradkin [10]: the BFV-BRST operator quantization method. One enlarges the phase space such that the original second class constraints became converted into the first class ones, so that the number of degrees of freedom remain unaltered.

Recently another method based on the canonical method [11-14] is introduced to obtain the path integral quantization of singular systems [15-16]. The starting point of this method is the variational principle. The Hamiltonian treatment of the constrained systems leads to obtain the equations of motion as total differential equations in many variables which require the investigation of integrability conditions. The equations of motion are integrable if the corresponding system of partial differential equations is a Jacobi system [11]. In this case one can construct a valid and a canonical phase space coordinates. The path integral then obtained as an integration
over the the canonical phase space coordinates with the action is obtained
directly from the equations of motion. Some applications of the canonical
path integral method are given in references [17-19] and it shown that gauge
fixing is not necessary to quantize singular systems if the canonical method
is used, no need to enlarge the phase space, no need to introduce delta func-
tions, as well as no ambiguous determinant will appear. This new approach,
due to its very recent development, has been applied to very few examples
[17-19] and a better understanding of its features, its advantages in the
study of singular systems when compared to other conventional methods
[7-9] is still necessary.

This paper is arranged as follows: A brief information on the canonical
path integral method is given section 2. Path integral of a pure Yang-Mills
theory is worked out in section 3.

2 A summary of the canonical path integral
formulation

As was discussed in previous papers [15,16], gauge fixing is not necessary to
quantize singular systems if the canonical path integral formulation is used.
Thus, it will be instructive to give a brief discussion of this method.

The canonical method gives the set of Hamilton - Jacobi partial differ-
ential equations [HJPDE] as

\[ H_\alpha'(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_a}) = 0, \]
\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r, \]

(1)

where

\[ H_\alpha' = H_\alpha(t_\beta, q_a, p_a), \]

(2)

and \( H_0 \) is defined as

\[ H_0 = p_\mu w_\mu + p_\mu q_\mu \big|_{p_\mu = H_\mu} - L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a), \]
\[ \mu, \nu = n - r + 1, ..., n. \]

(3)

The equations of motion are obtained as total differential equations in
many variables as follows:
\[ dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad dp_\beta = -\frac{\partial H'_\alpha}{\partial t_\beta} dt_\alpha. \tag{4} \]

\[ dz = (-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha; \tag{5} \]

\[ \alpha, \beta = 0, n - r + 1, ..., n, \quad a = 1, ..., n - r \]

where \( z = S(t_\alpha; q_a) \). The set of equations (4,5) is integrable [11] if

\[ dH'_0 = 0, \tag{6} \]

\[ dH'_\mu = 0, \quad \mu = n - p + 1, ..., n. \tag{7} \]

If condition (6,7) are not satisfied identically, one considers them as new constraints and again testes the consistency conditions. Hence, the canonical formulation leads to obtain the set of canonical phase space coordinates \( q_a \) and \( p_a \) as functions of \( t_\alpha \), besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians \( H'_\alpha \) are considered as the infinitesimal generators of canonical transformations given by parameters \( t_\alpha \) respectively. In this case, the path integral representation may be written as [15,16]

\[ \langle \text{Out} | S | \text{In} \rangle = \int dq^a dp^a [\exp i\{ \int_{t_\alpha}^{t'_\alpha} (-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha \}], \]

\[ a = 1, ..., n - p, \quad \alpha = 0, n - p + 1, ..., n. \tag{8} \]

One should notice that the path integral (8) is an integration over the canonical phase-space coordinates \( q_a \) and \( p_a \).

### 3 Quantization of Yang-Mills theory

In this section we shall consider the path integral quantization of Yang-Mills theory and demonstrate the fact that gauge fixing problem is solved if the
canonical path integral method is used. The action of this theory is given as
\[ S[A_\mu] = \int \mathcal{L} \, d^4x, \]  
where the Lagrangian density is
\[ \mathcal{L} = -\frac{1}{4} F_{\mu
u}^a F_{\mu\nu}^a, \quad \mu, \nu = 0, 1, 2, 3, \]  
with the field strength \( F_{\mu\nu}^a \) defined as
\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \]  
where \( f^{abc} \) are the structure constants of the Lie-Algebra and \( g \) represents the coupling constant. This action is invariant under gauge transformations.

The momenta conjugated to the fields \( A_\mu^a \) are defined as
\[ \pi_\mu^a = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_\mu^a)} = F_{\mu 0}^a. \]  
The non-vanishing Poisson brackets are
\[ \{ A_\mu^a(x), \pi_\nu^b(x') \} = \epsilon_\mu^\nu \delta^a_b \delta^{(3)}(x' - x). \]  
Upon quantization, these brackets have to be converted into proper commutators.

The spatial components read as
\[ \pi_i^a = F_{i 0}^a, \quad i = 1, 2, 3, \]  
and the time component
\[ \pi_0^a = 0, \]  
is the primary constraint. Defining the covariant derivatives \( D_i \) as
\[ D_i x^a = \partial_i x^a + g f^{abc} A_i^b x^c, \]  
in this case Eqn. (14) leads us to express the velocities \( \dot{A}_i^a \) in terms of the momenta \( \pi_i^a \) as
\[ \dot{A}_i^a = - (\pi_i^a - D_i A_0^a). \]
The total canonical Hamiltonian takes the form
\[ H_0 = \int \left[ \frac{1}{4} F_{ij}^a F^{ij}_a - \frac{1}{2} \pi^a_i \pi^a_i - D_i \pi^a_i A^a_0 + \partial_t (\pi^a_i A^a_0) \right] d^3x. \] (18)

Starting from the Hamiltonian defined in (18) and making use of (2), (15), the set of Hamilton-Jacobi partial differential equations reads as
\[ H'_0 = \pi^4 + H_0, \quad \pi^4 = \frac{\partial S}{\partial t}, \] (19)
\[ H'_{a} = \pi^a_0 = 0, \quad \pi^a_0 = \frac{\partial S}{\partial A^a_0}. \] (20)

The total differential equations can be written as
\[ dA^a_i = - (\pi^a_i - D^i A^a_0) dt, \] (21)
\[ d\pi^i_a = (D_j F^{ji}_a + g f_{abc} \pi^a_b A^c_0) dt, \quad j = 1, 2, 3, \] (22)
\[ d\pi^0_a = (D_i \pi^i_a) dt, \] (23)
\[ d\pi^4 = 0. \] (24)

To check whether the set of equations (21-24) is integrable or not, let us consider the variation of (19) and (20). In fact
\[ dH'_0 = F^a_1 dA^a_0, \] (25)
where
\[ F^a_1 = - D_i \pi^i_a. \] (26)
Since \( F^a_1 \) is not identically zero, we consider it as a new constraint, and one should consider the variation of \( F^a_1 \) too. Calculation shows that it lead to the constraint
\[ g f_{abc} A^b_0 D_i \pi^i_c = 0. \] (27)

The variation of \( H'_a \) is zero simply because it is equal to \( -F^a_1 \).

The set of equations (21-24) is integrable, Hence, the canonical phase-space coordinates \( A^a_i \) and \( \pi^i_a \) are obtained in terms of parameters \( t \) and \( A^a_0 \). Making use of equation (5), the canonical action integral is calculated as
\[ z = \int \left[ - \frac{1}{4} F_{ij}^a F^{ij}_a + \frac{1}{2} \pi^a_i \pi^a_i + D_i \pi^i_a A^a_0 - \partial_t (\pi^a_i A^a_0) + \pi^i_a \partial_0 A^a_1 \right] d^3x. \] (28)
Making use of (28) and (8) we obtain the path integral as

\[ \langle \text{Out}|S|\text{In} \rangle = \int \prod_{i,a} DA_a^i \exp \{ i \left[ \int \left[ -\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} \pi_i^a \pi_i^a + D_i \pi_i^a A_0^a - \partial_t (\pi_i^a A_0^a) + \pi_i^a \partial_0 A_i^a \right] \right] \} \].

\[ (29) \]

One should notice that the path integral (28) has no singular nature. In fact the path integral expression (28) is an integration over the canonical phase space coordinates \( A_a^i \) and \( \pi_i \).

We know that for a system with \( n \) degrees of freedom and with \( r \) first class constraints \( \phi^\alpha \) the path integral representation is given by Faddeev [5,6] as

\[ \langle \text{Out}|S|\text{In} \rangle = \int \prod_t d\mu(q_j, p_j) \exp \{ \int_{-\infty}^{\infty} dt (p_j \dot{q_j} - H_0) \}, \quad j = 1, ..., n, \]

\[ (30) \]

where the measure of integration is given as

\[ d\mu = \det \{ \phi^\alpha, \chi^\beta \} \prod_{\alpha=1}^{r} \delta(\chi^\alpha) \delta(\phi^\alpha) \prod_{j=1}^{n} dq_j dp_j, \]

\[ (31) \]

and \( \chi^\alpha \) are \( r \)-gauge constraints.

If we perform now the usual path integral quantization [5,6] using (30) for system (9), one must choose two gauge fixing conditions to obtain the path integral quantization over the canonical phase space coordinates \( A_a^i \) and \( \pi_i^a \).

### 4 Conclusion

Path integral quantization of Yang-Mills theory is obtained using the canonical path integral formulation [15,16]. In this approach, since the integrability conditions \( dH'_0 = 0, dH'_a = 0 \) are satisfied, this system is integrable. Hence, the canonical phase space coordinates \( A_a^i, \pi_i^a \) are obtained in terms of parameters \( (t, A_0^a) \). In this case the path integral, then follows directly as given in (29) without using any gauge fixing conditions. In the usual formulation [5-10], one has to fix a gauge to obtain the path integral over the canonical variables.
As a conclusion, it is obvious that one does not fix any gauge if the canonical path integral formulation is used. If the system is integrable, one can construct canonical phase space coordinates $q_a$ and $p_a$ in terms of parameters $t_\alpha$. Besides the canonical action integral is naturally obtained from the equations of motion without introducing Lagrange multipliers. In this case the path integral quantization is obtained directly as an integration over the canonical phase space coordinates.

Another point to be specified is that as we mentioned in the introduction, the canonical path integral approach is a new method and we still lack a complete analysis of relation between the procedure in this method for constrained systems and the traditional ones, especially with Faddeev’s and Senjanov’s methods.

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