FRACTIONAL PARTS OF NON-INTEGER POWERS OF PRIMES. II

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Abstract. We continue to study the distribution of prime numbers $p$, satisfying the condition $\{p^\alpha\} \in I \subset [0; 1)$, in arithmetic progressions. In the paper, we prove an analogue of Bombieri-Vinogradov theorem for $0 < \alpha < 1/9$ with the level of distribution $\theta = 2/5 - (3/5)\alpha$, which improves the previous result corresponding to $\theta \leq 1/3$.

1. Introduction

As in the previous work [16] let us denote by $E \subset \mathbb{N}$ the subsequence of natural numbers

$$\{n \in \mathbb{N} : \{n^\alpha\} \in I\},$$

where $\alpha > 0$ is any fixed non-integer, $I$ is any subinterval of $[0; 1)$. The distribution of primes from $E$ was studied by a number of authors including Vinogradov, Linnik, Kaufman, Gritsenko, Balog, Harman, Tolev and many others (see [1]–[15]). One of the main results of this investigation is the asymptotic formula for the proportion of such primes:

$$\sum_{\substack{p \leq X \in \mathbb{P} \cap E}} 1 = |I| \cdot \pi(X) + O(X^{1-\vartheta(\alpha)}),$$

where the exponent $0 < \vartheta(\alpha) < 1$ had been sharpening for different values of $\alpha$ until at least 2006. For $0 < \alpha < 1$ see [1], [2], [4], [5], [6], [7]. The case $\alpha > 1$, $\alpha \notin \mathbb{N}$ is covered in [8], [9], [10], [11], [12], [13].

The other direction concerns the existence of infinite number of primes from a very thin subset of integers of the form $\{n \in \mathbb{N} : \{\sqrt{n}\} < n^{-c+\varepsilon}\}$ for fixed $c > 0$ and arbitrary small $\varepsilon > 0$. The first such result is due to Vinogradov, who proved this for all $c \leq 1/10$ (see [22, Ch. 4]). Later it was improved in the work of Kaufman [3] to all $c \leq 0.16310\ldots$ and in the unpublished work of Harman to $c \leq 0.2139\ldots$. In [3] Kaufman also showed that the Riemann Hypothesis implies this result for all $c \leq 1/4$. Finally, it was proved for all $c \leq 1/4$ unconditionally in the papers of Balog [4] and Harman [5].
We focus on the distribution of $p \in \mathcal{E}$ in the arithmetic progressions of the form $qn + a$, $(a, q) = 1$. An analogue of Bombieri-Vinogradov theorem for such subset is usually given by the inequality

$$
\sum_{q \leq Q} \max_{(a, q) = 1} \left| \sum_{p \leq X \atop p \equiv a \pmod{q}} \frac{1}{\varphi(q)} \sum_{p \leq X \atop p \in \mathcal{E}} 1 \right| \ll A \frac{X}{(\log X)^A},
$$

where $A > 0$ can be arbitrarily large, $\varepsilon > 0$ is arbitrarily small, $Q = X^{\theta - \varepsilon}$ and $\theta$ is called the “level of distribution”.

Tolev [14] showed (1) for $\alpha = 1/2$ and all $\theta \leq 1/4$. Later Gritsenko and Zinchenko [15] extended this result to all $1/2 \leq \alpha < 1$ and $\theta \leq 1/3$. In [16] the author further extended it to all $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $\theta \leq 1/3$. In the present paper we improve this result for small $\alpha$, namely we show (1) for all $\theta \leq 2/5 - (3/5)\alpha$, which goes beyond the range $\theta \leq 1/3$ if $0 < \alpha < 1/9$.

The proof is based on the estimation of the exponential sum of the form

$$
\sum_{X \leq p < 2X \atop p \equiv a \pmod{q}} e(hp^\alpha),
$$

where $e(x) := e^{2\pi ix}$. The desired upper bound is given by

**Theorem 1.** Suppose that $0 < \alpha < 1/9$ is fixed non-integer, $\theta, \varepsilon, C$ are fixed constants satisfying the conditions $0 < \varepsilon < \alpha/100$, $\varepsilon < \theta < 2/5 - (3/5)\alpha$, $C \geq 1$, and suppose that $1 \leq h \leq (\log X)^C$, $2 < q \leq X^{\theta - \varepsilon}$, $1 \leq a \leq q - 1$, $(a, q) = 1$. Then the sum

$$
T = \sum_{X \leq p < 2X \atop p \equiv a \pmod{q}} e(hp^\alpha)
$$

satisfies the estimate

$$
T \ll \frac{X}{q} (\log X)^{-A}
$$

with an arbitrarily large $A > 0$.

**Corollary 1.** Let $0 < \alpha < 1/9$ be a fixed non-integer, $\varepsilon > 0$ is arbitrary small number. Then for any $q \leq X^{2/5 - (3/5)\alpha - \varepsilon}$, $a$, $(a, q) = 1$, and any given subinterval $I \subset [0; 1)$ the following asymptotic formula holds true:

$$
\pi_I(X; q, a) := \sum_{p \leq X \atop \{p^\nu\} \in I} 1 = |I| \cdot \pi(X; q, a) + O\left(\frac{\pi(X; q, a)}{(\log X)^A}\right)
$$
The next corollary is the analogue of Bombieri-Vinogradov theorem:

**Corollary 2.** Let \( 0 < \alpha < 1/9 \) be fixed, \( I = [c_1; d] \subset [0; 1) \), \( E = \{ n \in \mathbb{N} : \{ n^\alpha \} \in I \} \), and let \( \theta, \varepsilon, A \) be fixed constants such that \( 0 < \varepsilon < \theta < 2/5 - (3/5)\alpha \), \( \varepsilon < \alpha/100 \), \( A > 0 \). Next, let \( 2 < Q \leq X^{\theta - \varepsilon} \). Then the following inequality holds true:

\[
\sum_{q \leq Q} \max_{(a, q) = 1} \left| \sum_{p \leq X} \frac{1}{\varphi(q)} \sum_{p \leq X} 1 \right| \leq \frac{X}{(\log X)^A}.
\]

**Remark 1.** In the analogue of Bombieri-Vinogradov theorem one can apparently go beyond the level of distribution \( \theta = 2/5 - (3/5)\alpha \) using the large sieve. This is a work in progress.

The way one deduces corollaries 1 and 2 from Theorem 1 is explicated in the Section 2 of [16]. In this work we only focus on the proof of Theorem 1. The main difference between this proof and the proof of Theorem 1 from [16] is the application of Heath-Brown identity [17] in place of Vaughan identity [24, Ch. 13]. A key new ingredient is a combinatorial decomposition of the initial sum

\[
W := \sum_{n \equiv a (\mod q)} \Lambda(n)e(hn^\alpha), \quad X < Y \leq 2X,
\]

into sums of three types. This decomposition is given by Lemma 1 in Section 2 (see also [18, Lemma 3.1]). Then \( W \) splits into

\[
W_I = \sum_{m \sim M} a_m \sum_{n \sim N \atop mn \equiv a (\mod q)} f(n)e(hmn^\alpha),
\]

\[
W_{II} = \sum_{m \sim M} \beta_m \sum_{n \sim N \atop mn \equiv a (\mod q)} \gamma_n e(hmn^\alpha),
\]

\[
W_{III} = \sum_{m \sim M} f_1(m) \sum_{n \sim N \atop mn \equiv a (\mod q)} f_2(n) \sum_{k \sim K \atop mkn \equiv a (\mod q)} f_3(k)e(hmnk^\alpha),
\]

where \( a_m, \beta_m, \gamma_n \) are real coefficients, \( f, f_1, f_2, f_3 \) are smooth functions and “\( x \sim X \)” means \( X^{-1} \Theta \leq x \leq X \Theta \) for some fixed \( \Theta \), \( 1 < \Theta < 2 \). All of the coefficients \( |a_m|, |\beta_m|, |\gamma_n|, |f|, |f_1|, |f_2|, |f_3| \) do not exceed \( X^\varepsilon \). Since the number of sums of each type is bounded by a constant, we have

\[
W \ll |W_I| + |W_{II}| + |W_{III}|.
\]
We treat $W_I, W_{II}, W_{III}$ separately.

The estimation of type I and type II sums is very similar to the one in [16], and it only requires the classical van der Corput second derivative test [23, Ch. 1, Theorem 5] due to the small size of $\alpha$. This estimation is carried out in Section 3 and Section 4.

The estimation of type III sum is contained in Section 5. It is a little more delicate. In this case all three variables are roughly of the same size $K \approx N \approx M \approx X^{1/3}$. This implies that for large values of $q$, precisely, $q \geq X^{1/3}$, the sum over $k, mnk \equiv a \pmod{q}$ might be empty or contain only one term, so one cannot get a cancellation from the inner sum. This is the main reason why the previous method does not work in this case. The new idea is to remove the congruence condition $mnk \equiv a \pmod{q}$ using orthogonality of Dirichlet characters and then apply Poisson summation formula to any two of three sums over $m, n, k$. In this way we replace them by two shorter sums, and the new expression for $W_{III}$ would have the form

$$W_{III} = \sum_{m \sim M} f_4(m) \sum_{u \sim U} \sum_{v \sim V} f_5(uv) S_q(u, v),$$

where $UV \ll KN$, $f_4(m)$ and $f_5(uv)$ will be specified in the end of Section 5. We then estimate $|f_4(m)|$ and $|f_5(uv)|$ trivially and apply Weil’s bound $|S_q(u, v)| \leq \sqrt{q \tau(q)}(u, v, q)^{1/2}$ for Kloosterman sum (see, for example, [26]) to obtain the upper bound for $|W_{III}|$.

The application of Poisson summation requires a smoothed sum. So before estimating $|W_I|$, $|W_{II}|$ and $|W_{III}|$ we would slightly adjust $W$ in Section 2. Precisely, we remove the sharp bounds for $m, n, k$ using the smooth partition of unity, which is also described in [18, Section 3]. To deal with the oscillating integrals arising after Poisson summation we apply the method of stationery phase. The necessary tools are given by Lemma 2 and Lemma 3 in Section 5. They are proven in [21].

2. Initial steps. Heath-Brown identity. Smooth partition of unity

In this section we adjust the initial sum $W$ to simplify the estimation of type III in Section 5. This technique is also described in [18, Section 3]. Suppose that $1 \leq a < q \leq Q$, $(a, q) = 1$. We consider the sum

$$W = W(Y) = \sum_{\substack{n \equiv a \\ n \equiv 0 \pmod{q}}} \Lambda(n) e(hn^\alpha), \quad X < Y \leq 2X.$$ 

Let us denote $y = Y/X > 1$. Fix $B_0 > 0$ and choose $\Delta = (\log X)^{-B_0}$. There exists function $\psi(x) \in C^\infty$, such that $\psi(x) = 1$ if $1 \leq x \leq y$, $0 \leq \psi(x) \leq 1$ if $1 - \Delta \leq x \leq 1$ or $y \leq x \leq y + \Delta$ and $\psi(x) = 0$ otherwise, and its derivatives satisfy the estimates
\( \psi^{(j)}(x) \ll_{j} (\log X)^{jB_0} \). See, for example, \([19]\) or \([20]\). Then \( W \) can be rewritten as

\[
W = \sum_{n=1}^{+\infty} \frac{\psi(n)}{X} n \Lambda(n) e(hn^\alpha) + O\left( \frac{X(\log X)^{-B_0+1}}{q} \right).
\]

By partial summation to prove Theorem 1 it is enough to show that the sum in (3) is bounded by \( X(\log X)^{-A} \). Thus, one can take \( B_0 = A + 1 \).

Applying Heath-Brown identity with \( k = 5, V = X^{1/5} \) ([17, Lemma 1]), we get

\[
W = \sum_{j=1}^{5} (-1)^{j-1} \binom{5}{j} W_j,
\]

where

\[
W_j = \sum_{d_1, \ldots, d_{2j} \equiv \alpha \pmod{q}, d_{j+1} \ldots, d_{2j} \in V} \mu(d_{j+1}) \ldots \mu(d_{2j}) \psi\left( \frac{d_1 \ldots d_{2j}}{X} \right) e(h d_1 \ldots d_{2j}^\alpha).
\]

The statement of the theorem clearly follows from the estimates \( W_j \ll X(\log X)^{-A} \) for each \( 1 \leq j \leq 5 \). We only provide the details for \( W_5 \). The sums \( W_1, \ldots, W_4 \) can be treated similarly.

We first split the summation over \( d_1, \ldots, d_{10} \) to the “refined” dyadic intervals following the technique from \([18]\). Fix \( A_0 > 0 \) and \( \Theta = 1 + (\log X)^{-A_0} \). Let \( \Psi(x) \) be \( C^\infty \) function supported on \([-\Theta; \Theta]\) such that \( \Psi(x) = 1 \) on \([-1; 1]\) and \( |\Psi^{(j)}(x)| \ll \log^{jA_0} x \) for all \( j \geq 0 \). For all \( x \geq 1 \) we have

\[
1 = \sum_{D \in \mathcal{G}} \Psi_D(x),
\]

where

\[
\mathcal{G} = \{ \Theta^l, l \in \mathbb{N} \cup \{0\} \}, \quad \Psi_D(x) = \Psi\left( \frac{x}{D} \right) - \Psi\left( \frac{\Theta x}{D} \right).
\]

Indeed, if \( x \geq 1 \), then

\[
\sum_{D \in \mathcal{G}} \Psi_D(x) = \lim_{m \to +\infty} \left( \Psi(x) - \Psi(\Theta x) + \Psi\left( \frac{x}{\Theta} \right) - \Psi(x) + \Psi\left( \frac{x}{\Theta^2} \right) - \Psi\left( \frac{x}{\Theta} \right) + \ldots \right.
\]

\[
\ldots + \Psi\left( \frac{x}{\Theta^m} \right) - \Psi\left( \frac{x}{\Theta^{m-1}} \right) = \lim_{m \to +\infty} \left( -\Psi(\Theta x) + \Psi\left( \frac{x}{\Theta^m} \right) \right) = -0 + 1 = 1.
\]
The function $\Psi_D$ is supported on $[\Theta^{-1}D; \Theta D]$. Thus,

\begin{equation}
W_5 = \sum_{D_1, \ldots, D_{10} \in \mathcal{G}} \sum_{d_1, \ldots, d_{10} = 1}^{+\infty} \log(d_1)\mu(d_6) \ldots \\
\ldots \mu(d_{10})\Psi_D(d_1) \ldots \Psi_{D_{10}}(d_{10})\psi\left(\frac{d_1 \ldots d_{10}}{X}\right) e(h(d_1 \ldots d_{10})^\alpha). \tag{4}
\end{equation}

The non-zero contribution to $W_5$ is only coming from the terms satisfying

\begin{equation}
(1 - \Delta)X \leq d_1 \ldots d_{10} \leq (y + \Delta)X, \quad \frac{D_i}{\Theta} \leq d_i \leq D_i\Theta \tag{5}
\end{equation}

for $i = 1, \ldots, 10$. From (5) we conclude that the non-zero contribution corresponds to the tuples $D = \{D_1, \ldots, D_{10}\}$ satisfying the inequality

$$X_1 \leq D_1 \ldots D_{10} \leq Y_1,$$

where $X_1 = (1 - \Delta)\Theta^{-1}X$, $Y_1 = (y + \Delta)\Theta^{10}X$,

and also satisfying

$$D_i \leq V\Theta \quad \text{for} \quad i = 6, \ldots, 10.$$

To split each $W_j$ to the sums of three types we need to following auxiliary lemma:

**Lemma 1 ([18, Lemma 3.1]).** Let $1/10 < \sigma < 1/2$, and let $t_1, \ldots, t_n$ be non-negative real numbers such that $t_1 + \ldots + t_n = 1$. Then at least one of the following three statements holds:

**Type I:** There is a $t_i$ with $t_i \geq 1/2 + \sigma$;

**Type II:** There is a partition $\{1, \ldots, n\} = S \cap T$ such that

$$\frac{1}{2} - \sigma \leq \sum_{i \in S} t_i \leq \sum_{i \in T} t_i < \frac{1}{2} + \sigma;$$

**Type III:** There exist distinct $i, j, k$ with $2\sigma \leq t_i \leq t_j \leq t_k \leq 1/2 - \sigma$ and

$$t_i + t_j, \ t_i + t_k, \ t_j + t_k \geq \frac{1}{2} + \sigma.$$

If $\sigma > 1/6$, then the type III situation is impossible.

Applying this lemma with $\sigma = 1/10 + \varepsilon_1$, $\varepsilon_1 < 3\alpha/5$, to $W_5$ we get

$$W_5 \ll |W_I| + |W_{II}| + |W_{III}|,$$

where the sums correspond to the following cases:

**Type I sum:** there is one index $1 \leq i \leq 5$ such that $D_i \geq X_1^{3/5 + \varepsilon_1}$. 
Type II sum: there is a partition $S \cup T = \{1, \ldots, 10\}$ such that

$$X_1^{2/5-\varepsilon_1} < \prod_{i \in S} D_i < X_1^{3/5+\varepsilon_1}.$$  

Type III sum: there are three distinct indices $i, j, k \in \{1, \ldots, 5\}$ such that

$$X_1^{1/5+2\varepsilon_1} \leq D_i \leq D_j \leq D_k \leq X_1^{2/5-\varepsilon_1},$$  

$$D_i D_j, D_i D_k, D_j D_k \geq X_1^{3/5+\varepsilon_1}.$$  

Remark 2. Note that in the expression analogous to (4) for $W_1$ and $W_2$ the type III sum is empty.

3. The estimation of type I sums

For simplicity we only consider the case $D_1 \geq X_1^{3/5+\varepsilon_1}$. The corresponding sum has the form

$$W_I = \sum_{U \leq Y_1 X_1^{-3/5-\varepsilon_1}} \sum_{D_1 \leq \cdots \leq D_{10} \in \mathbf{G}} \sum_{D_{11} \leq \cdots \leq D_{10} \in \mathbf{G}} W(D), \quad D = \{D_1, \ldots, D_{10}\},$$

$$W(D) = \sum_{U \Theta^{-9} \leq u \leq U \Theta^9} b(u) \sum_{u d_1 \equiv a \pmod{q}} f(d_1) e(h(ud_1)^{\alpha}),$$

where

$$b(u) = \frac{\mu(d_6) \cdots \mu(d_{10}) \Psi_{D_2}(d_2) \cdots \Psi_{D_{10}}(d_{10})}{d_1 \cdots d_{10} \leq Y} \quad \text{and} \quad |b(u)| \leq \tau_9(u),$$

$$f(d_1) = (\log d_1) \Psi_{D_1}(d_1) \psi\left(\frac{ud_1}{X}\right).$$

Note that the sum over $U \in \mathbf{G}$ contains only $O((\log X)^{A_0+1})$ terms. We have

$$|W(D)| \leq ||b||_{\infty} \sum_{U \Theta^{-9} \leq u \leq U \Theta^9} \left| \sum_{R_1 < d_1 \leq R_2} f(d_1) e(h(ud_1)^{\alpha}) \right|,$$

where

$$||b||_{\infty} = \max_{n \leq Y_1} |b(n)|, \quad R_1 = \max\left((1-\Delta)\frac{X}{u}, D_1 \Theta^{-1}\right), \quad R_2 = \min\left((y+\Delta)\frac{X}{u}, D_1 \Theta\right).$$
By partial summation,
\[ |W(D)| \leq \|b\|_\infty \sum_{U^{\Theta^{-9}} \leq u \leq U^{\Theta^9}} |f(R_2) \sum_{R_1 < d_1 \leq R_2 \atop u_1 \equiv a \pmod{q}} e(h(u_1 \alpha)| - \]
\[ \int_{R_1}^{R_2} \left( \sum_{R_1 < d_1 \leq v \atop u_1 \equiv a \pmod{q}} e(h(u_1 \alpha)) \right) \frac{df(v)}{dv} dv \].

Next,
\[ \frac{d}{dv} \left( \log(v) \Psi_D(v) \psi\left(\frac{uv}{X}\right) \right) \ll \frac{1}{v} + \frac{\log v}{D_1} (\log X)^{A_0} + \log(v) \frac{u}{X} (\log X)^{B_0} \ll \]
\[ \frac{1}{D_1} (\log X)^{\max(A_0,B_0)+1} , \]
and therefore
\[ \int_{R_1}^{R_2} \left( \sum_{R_1 < d_1 \leq v \atop u_1 \equiv a \pmod{q}} e(h(u_1 \alpha)) \right) \frac{df(v)}{dv} dv \ll \]
\[ (\log X)^{\max(A_0,B_0)+1} \left| \sum_{R_1 < d_1 \leq R_3 \atop u_1 \equiv a \pmod{q}} e(h(u_1 \alpha)) \right| , \]
where \( R_1 < R_3 \leq R_2 \). Thus, by the triangle inequality,
\[ (6) \quad |W(D)| \ll \|b\|_\infty (\log X)^{\max(A_0,B_0)+1} \sum_{U^{\Theta^{-9}} \leq u \leq U^{\Theta^9}} \left| \sum_{R_1 < d_1 \leq R_3 \atop u_1 \equiv a \pmod{q}} e(h(u_1 \alpha)) \right| . \]

Due to the congruence restriction \( u_1 \equiv a \pmod{q} \) we can assume \((u,q) = 1\) and define \( l_1 \equiv au^* \pmod{q}, 1 \leq l_1 < q - 1\). Setting \( d_1 = qr_1 + l_1 \), we obtain
\[ \frac{R_1}{q} \leq r_1 + \xi < \frac{R_3}{q}, \quad \xi = \frac{l_1}{q} . \]

The inner sum over \( d_1 \) in (6) takes the form
\[ \sum_{R_1 - \xi \leq r_1 < R_3 - \xi} e(f_I(r_1)) , \]
where \( f_I(x) = h(uq)^\alpha (x + \xi)^\alpha \). Then, for \( R_1 - \xi \leq x < R_3 - \xi \),
\[ |f_I''(x)| \asymp \frac{hu^\alpha q^2}{R_1^{2-\alpha}} =: \lambda_2 . \]
By van der Corput second derivative test [23, Ch. 1, Theorem 5], we obtain
\[
\left| \sum_{R_1 - \xi < r < R_3 - \xi} e(f_1(r)) \right| \ll (R_3 - R_1) \lambda_2^{1/2} + \lambda_2^{-1/2} = \frac{R_1}{q} \cdot \sqrt{h} \frac{u^{\alpha/2}q}{R_1^{1-\alpha/2}} + \frac{R_1^{1-\alpha/2}}{\sqrt{hq}u^{\alpha/2}q}.
\]
Since \((\log X)^{\max(A_0,B_0)+1}\|b\|_{\infty} \ll \delta_1 X^{\delta_1}\) for arbitrary small \(\delta_1 > 0\), we get
\[
|W(D)| \ll_{\delta_1} X^{\delta_1} \sum_{U \Theta^{-9} \leq u \leq U \Theta^9} \left( \frac{R_1}{q} \sqrt{h} \frac{u^{\alpha/2}q}{R_1^{1-\alpha/2}} + \frac{R_1^{1-\alpha/2}}{\sqrt{hq}u^{\alpha/2}q} \right) \ll
X^{\delta_1} \sqrt{h} R_1^{\alpha/2} \sum_{U \Theta^{-9} \leq u \leq U \Theta^{-9}} u^{\alpha/2} + \frac{X^{\delta_1} R_1^{1-\alpha/2}}{\sqrt{hq}} \sum_{U \Theta^{-9} \leq u \leq U \Theta^{-9}} \frac{1}{u^{\alpha/2}} \ll
X^{\delta_1} \sqrt{h} D_1^{\alpha/2} U^{\alpha/2+1} + \frac{X^{\delta_1} D_1^{1-\alpha/2} U^{1-\alpha/2}}{\sqrt{hq}}.
\]
Thus,
\[
W_I \ll \sum_{U \leq Y_1 X_1^{-3/5+\epsilon_1}} \sum_{U \in G} \sum_{D_2 \cdots D_{10} = U} \sum_{D_2, \ldots, D_{10} \in G} X^{\delta_1} \sqrt{h} D_1^{\alpha/2} U^{\alpha/2+1} + \frac{X^{\delta_1} D_1^{1-\alpha/2} U^{1-\alpha/2}}{\sqrt{hq}} \ll \sum_{U \leq Y_1 X_1^{-3/5+\epsilon_1}} \sum_{U \in G} \sum_{D_2 \cdots D_{10} = U} \sum_{D_2, \ldots, D_{10} \in G} \left( \sqrt{h} X^{\delta_1+\alpha/2} U \log(Y_1/U)^{A_0+1} + \frac{X^{\delta_1+1-\alpha/2}}{\sqrt{hq}} \log(Y_1/U)^{A_0+1} \right).
\]
Finally, for fixed \(U = \Theta^k\), \(k \leq \log(Y_1 X_1^{-3/5})/\log \Theta\), using the trivial bound
\[
\sum_{D_2 \cdots D_{10} = U} 1 \leq \sum_{k_2 + \cdots + k_{10} = k} 1 \leq k^9 \ll (\log X)^{9(A_0+1)},
\]
we get
\[
\sum_{D_2 \cdots D_{10} = U} 1 \leq \sum_{k_2 + \cdots + k_{10} = k} 1 \leq k^9 \ll (\log X)^{9(A_0+1)},
\]
where
\[
W_I \ll \sum_{U \leq Y_1 X_1^{-3/5+\epsilon_1}} \sum_{U \in G} X^{\delta_1} (\log X)^{10(A_0+1)} \left( \sqrt{h} U X^{\alpha/2} + \frac{X^{1-\alpha/2}}{\sqrt{h}} \right) \ll
X^{2\delta_1} \left( X^{2/5+\alpha/2-\epsilon_1} + \frac{X^{1-\alpha/2}}{q} \right).
\]
4. The estimation of type II sums

For a fixed partition $S \cup T = \{1, \ldots, 10\}$ we use the notation

$$m = \prod_{i \in S} d_i, \quad n = \prod_{i \in T} d_i, \quad M = \prod_{i \in S} D_i, \quad N = \prod_{i \in T} D_i.$$ 

Note that $MN \asymp X$. Then type II sum can be written as

$$W_{II} = \left( \sum_{X_1^{2/3+\epsilon_1} \leq M \leq X_1^{3/5+\epsilon_1}} \sum_{X_1^{1/5} \leq N \leq Y_1/N} \sum_{\prod_{i \in S} D_i = M} \sum_{\prod_{i \in T} D_i = N} W(D), \right)$$

where

$$W(D) = \sum_{M \Theta^{-|S|} \leq m \leq M \Theta^{|S|}} \gamma(m) \sum_{n=1}^{+\infty} \beta(n) \psi\left(\frac{mn}{X}\right) e\left(h(mn)^{\alpha}\right),$$

$$\gamma(m) = \prod_{\prod_{i \in S} d_i = m} \prod_{d_i \leq V \text{ for } i \geq 6, i \in S} \left( \prod_{i \in T} a_i(d_i) \Psi_{D_i}(d_i) \right),$$

$$|\gamma(m)| \leq (\log X) \sum_{\prod_{i \in S} d_i = m} 1 \leq (\log X) \tau_S(m),$$

$$\beta(n) = \prod_{\prod_{i \in T} d_i = n} \prod_{d_i \leq V \text{ for } i \geq 6, i \in T} \left( \prod_{i \in S} a_i(d_i) \Psi_{D_i}(d_i) \right),$$

$$|\beta(n)| \leq (\log X) \tau_T(n),$$

$$a_1(d) = \log d, \quad a_2(d) = \ldots = a_5(d) = 1, \quad a_6(d) = \ldots = a_{10}(d) = \mu(d).$$

By definition of $\beta(n)$ we have

$$W(D) = \sum_{M_1 \leq m \leq M_2} \gamma(m) \sum_{n_1 \leq n \leq n_2} \beta(n) \psi\left(\frac{mn}{X}\right) e\left(h(mn)^{\alpha}\right),$$

$$M_1 = M \Theta^{-|S|}, \quad M_2 = M \Theta^{|S|}, \quad N_1 = N \Theta^{-|T|}, \quad N_2 = N \Theta^{|T|}.$$ 

Cauchy’s inequality yields:

$$|W(D)|^2 \leq \left( \sum_{M_1 \leq m \leq M_2} |\gamma(m)|^2 \right) \left( \sum_{M_1 \leq m \leq M_2} \sum_{n_1 \leq n \leq n_2} \beta(n) \psi\left(\frac{mn}{X}\right) e\left(h(mn)^{\alpha}\right) \right)^2.$$
Next, by Mardzhanishvili’s inequality [25] we get

\[
(W(D))^2 \ll M_1 (\log X)^2 + \kappa \left( \sum_{M_1 \leq m \leq M_2} \sum_{\substack{N_1 \leq n \leq N_2 \mod q \atop \text{mn} \equiv a}} \beta(n) \psi \left( \frac{mn}{X} \right) e(h(mn)^\alpha) \right)^2,
\]

where \(\kappa = |S|^2 - 1\). Rewrite the second factor as follows:

\[
\sum_{M_1 \leq m \leq M_2} \sum_{\substack{N_1 \leq n \leq N_2 \mod q \atop \text{mn} \equiv a}} \beta(n_1) \beta(n_2) \psi \left( \frac{mn_1}{X} \right) \psi \left( \frac{mn_2}{X} \right) e(hm^\alpha(n_1^\alpha - n_2^\alpha)) =
\]

\[
\sum_{M_1 \leq m \leq M_2} \sum_{\substack{N_1 \leq n \leq N_2 \mod q \atop \text{mn} \equiv a}} \beta^2(n) \psi^2 \left( \frac{mn}{X} \right) + 2 \text{Re}(S(M, N)),
\]

where

\[
S(M, N) = \sum_{M_1 \leq m \leq M_2} \sum_{\substack{N_1 \leq n \leq N_2 \mod q \atop \text{mn} \equiv a}} \beta(n_1) \beta(n_2) \psi \left( \frac{mn_1}{X} \right) \psi \left( \frac{mn_2}{X} \right) e(hm^\alpha(n_1^\alpha - n_2^\alpha)).
\]

The diagonal term does not exceed

\[
\sum_{M_1 \leq m \leq M_2} \sum_{\substack{N_1 \leq n \leq N_2 \mod q \atop \text{mn} \equiv a}} \beta^2(n) \ll (\log X)^2 \frac{MN}{q} (\log X)^{|T|^2 - 1} \ll \frac{X}{q} (\log X)^{|T|^2 + 1}.
\]

Setting \(m = qr + l\), we get

\[
R_1 - \eta \leq r \leq R_2 - \eta, \quad \eta = \frac{l}{q},
\]

for given \(l, (l, q) = 1\), \(R_1 = M_1/q\), \(R_2 = M_2/q\). Hence,

\[
S(M, N) = \sum_{l=1}^{q} \sum_{(l, q) = 1}^{R_1 - \eta \leq r \leq R_2 - \eta} \sum_{\substack{N_1 \leq n_1 \leq N_2 \mod q \atop n_1, n_2 \equiv e}} \beta(n_1) \beta(n_2) \psi \left( \frac{(qr + l)n_1}{X} \right) \psi \left( \frac{(qr + l)n_2}{X} \right) e(h(n_1^\alpha - n_2^\alpha)q^\alpha(r + \eta)^\alpha),
\]
where $e = al^*$ (mod $q$). Changing the order of summation, we estimate $S(M, N)$ as follows:

$$|S(M, N)| \leq \sum_{l=1}^{q} \sum_{N_1 \leq n_1 < n_2 \leq N_2 \atop (l.q) = 1, n_1, n_2 \equiv l \pmod{q}} |\beta(n_1)||\beta(n_2)| \cdot \left| \sum_{R_1 - \eta \leq r < R_2 - \eta} \psi \left( \frac{(qr + l)n_1}{X} \right) \psi \left( \frac{(qr + l)n_2}{X} \right) e(f_{II}(r)) \right|,$$

where $f_{II}(x) = h(n_1^q - n_2^q)q^\alpha(x + \eta)^\alpha$. Using the conditions $n_1 < n_2$, $n_1 \equiv n_2 \equiv e$ (mod $q$), we set $n_2 = n_1 + qs$ with $s \geq 1$. On the other hand, $n_2 \leq N_2$ implies $n_1 + qs \leq N_2$. Hence, $s < (N_2 - N_1)/q = t$, and therefore

$$|S(M, N)| \leq \sum_{l=1}^{q} \sum_{1 \leq s < t} \sum_{N_1 \leq n \leq N_2 \atop n \equiv e \pmod{q}} |\beta(n)||\beta(n + qs)| \cdot \left| \sum_{R_1 \leq r \leq R_2} \psi \left( \frac{(qr + l)n_1}{X} \right) \psi \left( \frac{(qr + l)n_2}{X} \right) e(f_{II}(r)) \right|.$$ 

By partial summation,

$$|S(M, N)| \leq (\log X)^{B_0} \sum_{l=1}^{q} \sum_{1 \leq s < t} \sum_{N_1 \leq n \leq N_2 \atop n \equiv e \pmod{q}} |\beta(n)||\beta(n + qs)| \left| \sum_{R_1 \leq r \leq R_2} e(f_{II}(r)) \right|.$$

Next,

$$f_{II}''(x) = \frac{\alpha(\alpha - 1)h(n_2^q - n_1^q)q^\alpha}{(x + \eta)^{2-\alpha}}.$$

whence $|f_{II}'(x)| \asymp h(n_2^q - n_1^q)q^\alpha \left( \frac{q}{M} \right)^{2-\alpha}$.

By Lagrange’s mean value theorem,

$$hq^\alpha((n + qs)^\alpha - n^\alpha) = \alpha hsq^{\alpha+1}(n + qs\theta')^{\alpha-1} \asymp hsq^{\alpha+1}N^{\alpha-1} \asymp hsq^{\alpha+1}(\frac{X}{M})^{\alpha-1},$$

where $|\theta'| \leq 1$. Hence,

$$|f_{II}''(x)| \asymp \frac{hsq^2}{X^{1-\alpha}} \frac{q}{M}.$$

Applying van der Corput second derivative test [23, Ch. 1], we get

$$\sum_{R_1 \leq r \leq R_3} e(f_{II}(r)) \ll \frac{M}{q} \left( \frac{hsq^2}{X^{1-\alpha}} \frac{q}{M} \right)^{1/2} + \left( \frac{X^{1-\alpha}}{hsq^2} \frac{M}{q} \right)^{1/2}.$$
The factor \(|\beta(n)| \cdot |\beta(n + qs)|\) is bounded from above by \((X/q)^{\delta_2}\) for arbitrary small \(\delta_2 > 0\). The summation over \(n \equiv e \pmod{q}\) for \(N_1 \leq n \leq N_2\) contributes the factor of at most \(X/(Mq) > 1\) (since \(M \ll X^{3/5 + 3\alpha/5}\), \(q \ll X^{2/5 - 3\alpha/5}\)). Thus,

\[
|S(M, N)| \ll \left(\frac{X}{q}\right)^{\delta_2} \sum_{\substack{l = 1 \atop (l, q) = 1}}^{q-1} \sum_{1 \leq s < t} \frac{X}{Mq} \left(\left(\frac{hsqM}{X^{1-\alpha}}\right)^{1/2} + \left(\frac{MX^{1-\alpha}}{hsq^3}\right)^{1/2}\right) \ll \\
\left(\frac{X}{q}\right)^{\delta_2} \left(\frac{\sqrt{h}X^{2+\alpha/2}}{qM^2} + \frac{X^{2-\alpha/2}}{\sqrt{hq^2M}}\right).
\]

Combining (8), (9) and (10), we get

\[
|W(D)| \ll \sqrt{M}(\log X)^{1+\kappa/2} \left(\frac{X}{q}(\log X)^{|T|^2+1} + \left(\frac{X}{q}\right)^{\delta_2} \left(\frac{\sqrt{h}X^{2+\alpha/2}}{qM^2} + \frac{X^{2-\alpha/2}}{\sqrt{hq^2M}}\right)^{1/2} \right) \ll X^{\delta_3},
\]

where we have used the inequality

\[
\max((\log X)^{\kappa/2 + |T|^2 + 1}, (X/q)^{\delta_2} \sqrt{h}) \leq X^{\delta_3}
\]

for some \(\delta_3 \geq \delta_2\). For fixed \(M = \Theta^k\) and \(N = \Theta^l\) with \(k + l = 10\) the number of corresponding tuples \(S\) and \(T\) does not exceed

\[
\sum_{i+j=10 \atop i+j \geq 1} k^i l^j \ll \left(\frac{\log X}{\log \Theta}\right)^{10} \ll (\log X)^{10(A_0+1)}.
\]

Thus,

\[
W_{II} \ll \sum_{X_{12/5-\epsilon_1} M \in \mathcal{M} \subseteq \mathcal{X}_1^{3/5+\epsilon_1} X_{11/2} / M \leq N \leq N_1 / N} \sum_{\gamma \in \mathcal{S}, T} \prod_{\gamma \in \mathcal{S}, T} D_i = M N \prod_{\Pi \in S, T} D_i = N \sum_{\Pi \in S, T} |W(D)| \ll \\
(\log X)^{10(A_0+1)} \sum_{X_{12/5-\epsilon_1} M \in \mathcal{M} \subseteq \mathcal{X}_1^{3/5+\epsilon_1} X_{11/2} / M \leq N \leq N_1 / N} \sum_{\gamma \in \mathcal{S}, T} X^{\delta_3} \left(\left(\frac{X}{q}\right)^{1/2} + \frac{X^{1+\alpha/4}}{(qM)^{1/2}} + \frac{X^{1-\alpha/4}}{q}\right) \ll \\
X^{2\delta_3} \left(\frac{X^{4/5+\epsilon_1/2}}{\sqrt{q}} + \frac{X^{4/5+\alpha/4+\epsilon_1/2}}{\sqrt{q}} + \frac{X^{1-\alpha/4}}{q}\right) \ll \frac{X^{4/5+\alpha/4+\epsilon_1/2+2\delta_3}}{\sqrt{q}} + \frac{X^{1-\alpha/4+2\delta_3}}{q}.
\]
5. The estimation of type III sums

We apply the method of stationery phase to treat the type III sum. To deal with the oscillatory integrals arising after the Poisson summation we use two auxiliary lemmas given below:

**Lemma 2** *(Proposition 8.1, [21]).* Let \( Y_I \geq 1, X_I, Q_I, V_I, R_I > 0 \), \( w(t) \) is a smooth function supported on some finite interval \( \mathbb{J} \subset \mathbb{R} \) such that
\[
w^{(j)}(t) \ll X_I V_I^{-j}
\]
for all \( j \geq 0 \). Suppose that \( g(t) \) is a smooth function such that \( |g'(t)| \geq R_I \), \( g^{(j)}(t) \ll_j Y_I Q_I^{-j} \) for \( j \geq 2 \), \( t \in \mathbb{J} \). Then the integral \( I \) defined by
\[
I = \int_{-\infty}^{+\infty} \! w(t) e(g(t)) \, dt
\]
satisfies
\[
I \ll_A \mathbb{J} |X_I((Q_I R_I/\sqrt{Y_I})^{-A_I} + (R_I V_I)^{-A_I})
\]
with any fixed real \( A_I > 0 \).

This result gives a non-trivial upper bound for the integral \( I \) in the case if \( R_I V_I \) and \( Q_I R_I Y_I^{-1/2} \) are much bigger than 1.

**Lemma 3** *(Lemma 8.2, [21]).* Let \( 0 < \delta_I < 1/10 \), \( X_I, Y_I, V_I, \bar{V}_I, Q_I > 0 \), \( Z_I = Q_I + X_I + Y_I + \bar{V}_I + 1 \), and assume that \( Y_I \geq Z_I^{3\delta_I} \),
\[
\bar{V}_I \geq V_I \geq \frac{Q_I Z_I^{3\delta_I/2}}{Y_I^{1/2}}.
\]
Suppose that \( w(t) \) is a smooth function supported on an interval \( \mathbb{J} \) of length \( \bar{V}_I \) satisfying
\[
w^{(j)}(t) \ll_j X_I V_I^{-j}
\]
for all \( j \geq 0 \). Suppose that \( g(t) \) is a smooth function such that there is unique point \( t_0 \in \mathbb{J} \) such that \( g'(t_0) = 0 \). Further, \( g(t) \) satisfies the estimates \( g''(t) < 0 \), \( g''(t) \gg Y_I Q_I^{-2} \), \( g^{(j)}(t) \ll_j Y_I Q_I^{-j} \), for all \( j \geq 1 \), \( t \in \mathbb{J} \). Then the integral
\[
I = \int_{-\infty}^{+\infty} \! w(t) e(g(t)) \, dt
\]
has an asymptotic expansion of the form

\[ I = \frac{e(g(t_0))}{|g''(t_0)|^{1/2}} \sum_{0 \leq n \leq 3^{1/2}} p_n(t_0) + O_{A_I, \delta_I}(Z_I^{-A_I}), \]

\[ p_n(t_0) = \sqrt{2\pi e^{-\pi/4}} \frac{(2i)^{-n}}{n!} |g''(t_0)|^{n/2} G^{(2n)}(t_0), \]

where \( A_I > 0 \) is arbitrary, and

\[ G(t) = w(t)e(H(t)), \quad H(t) = g(t) - g(t_0) - \frac{1}{2} g''(t_0)(t - t_0)^2. \]

Later in this section we will use the following notation:

\[ \beta = \frac{2 - \alpha}{1 - \alpha}, \quad \gamma = \frac{\alpha}{1 - \alpha}, \quad \delta = \frac{1}{1 - \alpha}, \]

\[ \xi = \frac{1}{1 - \gamma} = \frac{1 - \alpha}{1 - 2\alpha}, \quad \eta = \frac{\alpha}{1 - 2\alpha}, \quad \omega = \xi(2 - \gamma) = \frac{2 - 3\alpha}{1 - 2\alpha}. \]

Let us denote as \( M, N, K \) the three indices from \( \{D_1, \ldots, D_5\} \) satisfying type III conditions, as \( m, n, k \) the corresponding indices from \( \{d_1, \ldots, d_5\} \), as \( i_1, i_2, i_3 \) the corresponding indices from \( \{1, \ldots, 5\} \), and let \( I \) be the set of all remaining indices \( \{1, \ldots, 10\}\backslash\{i_1, i_2, i_3\} \). Also let

\[ U = \prod_{i \in I} D_i, \quad u = \prod_{i \in I} d_i. \]

We get the sum of the form

\[ W_{III} = \sum' M, N, K \in \mathbf{G} \sum' U \sum_{U \Theta - \tau \not\equiv u \Theta \mod q} \sum_{m, n, k = 1}^{+\infty} f_1(m)f_2(n)f_3(k) \cdot F(U, u) \sum_{mnk \equiv a \mod q} f_1(m)f_2(n)f_3(k) \cdot \Psi_M(m)\Psi_N(n)\Psi_K(k) \psi\left(\frac{umnk}{X}\right) e(h(umnk)^\alpha), \]

where

\[ F(U, u) = \left( \sum_{i \in I} \cdots \sum_{i \in I} \right) \left( \sum_{i \in I} \cdots \sum_{i \in I} \right) \left( \prod_{i \in I} a_i(d_i)\Psi_{D_i}(d_i) \right), \]

\[ a_1(d) = \log d, \quad a_2(d) = \ldots = a_5(d) = 1, \quad a_6(d) = \ldots = a_{10}(d) = \mu(d), \]

\( f_i(x) \) are smooth functions such that \( f_i(x) \equiv 0 \) if \( x \leq 0 \) and \( f_i(x) = 1 \) or \( f_i(x) = \log x \) for \( x \geq 1 \), \( \sum' \) denotes the summation over \( M, N, K, U \in \mathbf{G} \) satisfying the type
III conditions. Without loss of generality we can assume $M \leq N \leq K$. Then rewrite (12) in the following way:

(13) $W_{III} = \sum_{U \in \mathbb{G}} \sum_{M_1 \leq M \leq M_2} \sum_{N_1 \leq N \leq N_2} \sum_{K_1 \leq K \leq K_2} W(M, N, K)$, where

$$M_1 = X^{1/5+2\varepsilon_1}, \quad M_2 = (Y_1 U^{-1})^{1/3},$$

$$N_1 = \max(M, X^{3/5+\varepsilon_1} M^{-1}), \quad N_2 = \min\left(X^{2/5-\varepsilon_1}, \left(\frac{Y}{MU}\right)^{1/2}\right),$$

$$K_1 = N, \quad K_2 = \min\left(X^{2/5-\varepsilon_1}, \frac{Y}{UMN}\right),$$

and

$$W(M, N, K) = \sum_{m,n,k=1}^{+\infty} \sum_{mnku \equiv a \pmod{q}} f_1(m) f_2(n) f_3(k) \Psi_M(m) \Psi_N(n) \Psi_K(k) \psi\left(\frac{umnk}{X}\right) e\left(h(umnk)^\alpha\right).$$

Note that $f_i(\cdot) \Psi_D(\cdot) \psi(\cdot)$ is smooth on $(0; +\infty)$, so one can apply Poisson summation to any of the sums over $n, m, k$. We also note that the number of terms in each sum over $U, M, N, K \in \mathbb{G}$ in (13) is $O((\log X)^{A_0+1})$ and $|F(U, u)|$ can be bounded as follows:

$$|F(U, u)| \ll (\log X) \tau_7(u) \cdot \#\left\{ (e_1, \ldots, e_7) \in \mathbb{Z}_{\geq 0}^7 : e_1 + \ldots + e_7 = \frac{\log U}{\log \Theta}\right\} \ll (\log X) \tau_7(u) \left(\frac{\log U}{\log \Theta}\right)^6 \ll \tau_7(u) (\log X)^6(A_0+1)^{1+1}.$$

**First iteration of Poisson summation.** We first apply Poisson summation to the longest sum over $k$. By orthogonality of characters,

$$W(M, N, K) = \frac{1}{\varphi(q)} \sum_{\chi \mod{q}} \chi(u a^*) \sum_{m=1}^{+\infty} \chi(m) f_1(m) \Psi_M(m) \sum_{n=1}^{+\infty} \chi(n) f_2(n) \Psi_N(n) W_{m,n,\chi},$$

where

$$W_{m,n,\chi} = \sum_{k=1}^{+\infty} \chi(k) f_3(k) \Psi_K(k) \psi\left(\frac{umnk}{X}\right) e\left(h(umnk)^\alpha\right).$$
To remove the factor $\chi(k)$ in the last sum we substitute $k = qr + l$:

$$W_{m,n,\chi} = \sum_{l=1}^{q-1} \chi(l) \sum_{r=-\infty}^{+\infty} f_3(qr + l) \Psi_K(qr + l) \psi\left(\frac{umn(qr + l)}{X}\right) e\left(h(umn(qr + l))^{\alpha}\right).$$

The function $(qr + l)^{\alpha}$ is extended by zero for $r < -l/q$. By Poisson summation,

$$W_{m,n,\chi} = \sum_{l=1}^{q-1} \chi(l) \sum_{s=\infty}^{+\infty} \int_{-\infty}^{+\infty} f_3(qv + l) \Psi_K(qv + l) \psi\left(\frac{umn(qv + l)}{X}\right) e\left(h(umn(qv + l))^{\alpha}\right) e(-vs) dv.$$

We can reduce the range of integration to $(-l/q; +\infty)$ due to the fact that $f_3(x) = 0$ for $x \leq 0$. Then substituting $t = \frac{umn(qv + l)}{X}$ we get

$$W_{n,m,\chi} = \frac{X}{umn} \sum_{s=\infty}^{+\infty} \tau(\chi; s) I_{m,n}(s),$$

where

$$I_{m,n}(s) = \int_{0}^{+\infty} f_3\left(\frac{Xt}{umn}\right) \Psi_K\left(\frac{Xt}{umn}\right) \psi(t) e\left(h(Xt)^{\alpha} - \frac{Xst}{umn}\right) dt,$$

$$\tau(\chi; s) = \sum_{l=1}^{q-1} \chi(l) e\left(\frac{sl}{q}\right)$$

is a Gauss sum.

Next, we verify the conditions of Lemma 2 and Lemma 3. Let

$$w(t) = f_3\left(\frac{Xt}{umn}\right) \Psi_K\left(\frac{Xt}{umn}\right) \psi(t),$$

$$g(t) = \begin{cases} h(Xt)^{\alpha} - \frac{Xst}{umn}, & \text{if } 1 - \Delta \leq t \leq y + \Delta, \\ 0 & \text{if } t \leq 1 - 2\Delta \text{ or } t \geq y + 2\Delta, \end{cases}$$

and extend $g(t)$ to a smooth function on $[1 - 2\Delta, 1 - \Delta]$ and $[y + \Delta, y + 2\Delta]$. We now evaluate the derivatives. First, if $1 - \Delta \leq t \leq y + \Delta$ and $j \geq 2$, then we have

$$g^{(j)}(t) = \frac{(\alpha)_j hX^{\alpha}}{t^{j-\alpha}}, \quad (\alpha)_j = \prod_{i=1}^{j}(\alpha - i + 1), \quad |g^{(j)}(t)| \lesssim_{\alpha,j} hX^{\alpha}.$$
Thus, one can take $Y_I = hX^\alpha$, $Q_I = 1$. Now let us estimate $w^{(j)}(t)$ on $\mathbb{J}$. We have
\[
\frac{d^j w(t)}{dt^j} = \sum_{j_1 + j_2 + j_3 = j} \left( \frac{j}{j_1, j_2, j_3} \right) \frac{d^{j_3} f_3}{dt^{j_1}} \left( \frac{X t}{umn} \right) \frac{d^{j_2} \Psi_K}{dt^{j_2}} \left( \frac{X t}{umn} \right) \frac{d^{j_3} \psi(t)}{dt^{j_3}}.
\]
Next,
\[
\frac{d^{j_1} f_3}{dt^{j_1}} \left( \frac{X t}{umn} \right) \ll \log X,
\]
\[
\frac{d^{j_2} \Psi_K}{dt^{j_2}} \left( \frac{X t}{umn} \right) \ll \left( \frac{X}{Kum} \right)^{j_2} \left( \log X \right)^{j_2 A_0} \ll \left( \log X \right)^{j_2 A_0},
\]
\[
\frac{d^{j_3} \psi(t)}{dt^{j_3}} \ll \left( \log X \right)^{j_3 B_0}.
\]
Thus, we find
\[
(14) \quad w^{(j)}(t) \ll \left( \log X \right) \sum_{j_1 + j_2 + j_3 = j} \left( \frac{j}{j_1, j_2, j_3} \right) \left( \log X \right)^{j_2 A_0} \left( \log X \right)^{j_3 B_0} \ll \left( \log X \right) \left( 1 + \left( \log X \right)^{A_0} + \left( \log X \right)^{B_0} \right)^j \ll \left( \log X \right)^{C_0 j + 1},
\]
where $C_0 = \max(A_0, B_0)$. So one can take $X_I = \log X$, $V_I = \left( \log X \right)^{-C_0}$. From $Z_I = Q_I + X_I + Y_I + V_I + 1$ we get $Z_I \asymp Y_I \asymp hX^\alpha$, whence for any fixed $\delta_I$, $0 < \delta_I < 1/10$, we have
\[
V_I = \left( \log X \right)^{-C_0} \geq \frac{Q_I Z_I^{\delta I/2}}{\sqrt{Y_I}} \asymp (hX^{\alpha})^{-1/2 + \delta I/2}.
\]
Now set
\[
T_1 = \frac{\alpha humNq}{4 X^{1-\alpha}}, \quad T_2 = \frac{\alpha humNq}{X^{1-\alpha}}
\]
and split the sum $W_{m,n,\chi}$ in the following way:
\[
W_{m,n,\chi} = \frac{X}{qumn} \left\{ \sum_{T_1 \leq s \leq T_2} + \sum_{|s| > T_2} + \sum_{-T_2 \leq s < T_1} \right\} \tau(\chi; s) I_{m,n}(s) =: \frac{X}{qumn} (S_1 + S_2 + S_3).
\]
For $S_2$ and $S_3$ we apply Lemma 2 to estimate $I_{m,n}(s)$; for $S_1$ we compute $I_{m,n}(s)$ asymptotically using Lemma 3. We have
\[
g'(t) = \alpha hX^\alpha t^{\alpha - 1} - \frac{X s}{qumn}.
\]
If $|s| > T_2$, then
\[
|g'(t)| \geq \frac{|s|}{2 qmn} \left( 1 - \frac{\alpha h^{\alpha-1} qmn}{X^{1-\alpha T_2}} \right) \geq |g'(t)| \geq \frac{|s|}{2 qmn}.
\]

If $-T_2 \leq s \leq 0$, then
\[
g'(t) = \alpha h X^{\alpha t^{\alpha-1}} + \frac{|s|}{qmn} \geq \alpha h X^{\alpha t^{\alpha-1}} \geq \alpha h X^{\alpha}.
\]

Finally, if $1 \leq s < T_1$, then
\[
g'(t) \geq \alpha h X^{\alpha t^{\alpha-1}} \left( 1 - \frac{XT_1}{qmn \alpha h X^{\alpha}} \right) \alpha h X^{\alpha t^{\alpha-1}} \left( 1 - \frac{5}{8} \right) \geq \alpha h X^{\alpha}.
\]

Thus, one can choose
\[
R_I = \begin{cases} 
\frac{|s|}{2 qmn} & \text{if } |s| > T_2, \\
\frac{\alpha h X^{\alpha}}{6} & \text{if } -T_2 \leq s < T_1.
\end{cases}
\]

In the case $|s| > T_2$ we set
\[
\Delta_1 = \frac{Q_I R_I}{\sqrt{Y_I}}, \quad \Delta_2 = R_I V_I,
\]

and get
\[
\Delta_1 = \frac{|s|}{2 qmn} \frac{1}{\sqrt{h X^{\alpha}}} \geq \frac{X^{1-\alpha T_2}}{2 \sqrt{\alpha h qmn}} \geq X^{\alpha/2} \cdot 2 \alpha \sqrt{\frac{N}{n}} \geq X^{\alpha/2},
\]
\[
\Delta_2 = \frac{|s|}{2 qmn} (\log X)^{-C_0} \geq \frac{XT_2 (\log X)^{-C_0}}{2 qmn} \geq X^{\alpha} \cdot 2 \alpha N \frac{\log X}{n} \geq X^{\alpha/2}.
\]

If $-T_2 \leq s < T_1$, then
\[
\Delta_1 = \frac{\alpha h}{6} \frac{X^{\alpha}}{\sqrt{h X^{\alpha}}} \geq \alpha X^{\alpha/2}, \quad \Delta_2 = \frac{\alpha h}{6} X^{\alpha} (\log X)^{-C_0} \geq X^{\alpha/2}.
\]

Thus, by Lemma 2,
\[
I_{m,n}(s) \ll_{\alpha} (\log X) \left\{ \left( \frac{|s|}{2 qmn \sqrt{h X^{\alpha}}} \right)^{-A_I} + \left( \frac{|s|}{2 qmn \log X^{C_0}} \right)^{-A_I} \right\} \ll_{\alpha} (\log X) \left( \frac{2 qmn \sqrt{h}}{X^{1-\alpha/2|s|}} \right)^{A_I}
\]
for $|s| \geq T_2$, and

$$I_{m,n}(s) \ll_{\alpha} (\log X) \left\{ \left( \frac{\alpha \sqrt{h} X^{\alpha/2}}{6} \right)^{-\delta_I} + \left( \frac{\alpha h}{6} X^{\alpha (\log X) - C_0} \right)^{-\delta_I} \right\} \ll_{\alpha} (\log X) X^{-\alpha A_I/2} \ll (\log X) X^{-\alpha D_0 - \alpha/2}$$

if $-T_2 \leq s \leq T_1$. Going back to $S_2$ and $S_3$, we get

$$S_2 \ll \sum_{|s|>T_2} (\log X) \left( \frac{2qumn \sqrt{h}}{X^{1-\alpha/2}} \right)^{\delta_I} \frac{1}{|s|^{\delta_I}} \ll \frac{umNq}{X^{1-\alpha}} X^{-\alpha D_0},$$

$$S_3 \ll (T_1 + T_2 + 1) X^{-\alpha D_0 - \alpha/2} (\log X) \ll \left( \frac{umNq}{X^{1-\alpha}} + 1 \right) X^{-\alpha D_0}$$

with a large enough fixed $D_0 = D_0(\alpha) > 1$. From $qumn \gg X K^{-1}$ we find

$$W_{m,n,\chi} = \frac{X}{qumn} S_1 + O \left( \frac{X}{qumn} \left( \frac{umNq}{X^{1-\alpha}} + 1 \right) X^{-\alpha D_0} \right) =$$

$$\frac{X}{qumn} S_1 + O \left( X^{-\alpha (D_0 + 1)} + KX^{-\alpha D_0} \right) = \frac{X}{qumn} S_1 + O \left( KX^{-\alpha D_0} \right).$$

Now we compute $S_1$. Choosing $\delta_I = 1/20$, $A_I = E_0$ with a large enough fixed $E_0 = E_0(\alpha)$, we apply Lemma 3 to $I(s)$ when $T_1 < s \leq T_2$. Let $g'(t_0) = 0$. Then

$$t_0 = \frac{1}{X} \left( \frac{\alpha qumn}{s} \right)^{\frac{1}{1-\alpha}}.$$

Notice that for any $T_1 \leq s \leq T_2$ the point $t_0$ lies in $\mathbb{J} = [10^{-1}; 10]$. Thus,

$$I_{m,n}(s) = e \left( g(t_0) - \frac{1}{8} \right) \sum_{0 \leq \nu \leq \nu_1} \frac{\sqrt{2\pi}}{\nu!} (2\nu)^{-\nu} \left| g''(t_0) \right|^{\nu+1/2} G^{(2\nu)}(t_0) + O \left( X^{-\alpha D_0} \right),$$

where

$$G(t) = w(t)e(H(t)), \quad \nu_1 = 60D_0, \quad H(t) = g(t) - g(t_0) - \frac{1}{2} g''(t_0) (t - t_0)^2.$$

One can easily verify the identities

$$g(t_0) = (1 - \alpha)(\alpha^\alpha h)^\delta \left( \frac{qumn}{s} \right)^\gamma, \quad \left| g''(t_0) \right| = \alpha(1 - \alpha) h X^2 \left( \frac{s}{\alpha qumn} \right)^\beta,$$
where $\gamma = \alpha/(1 - \alpha), \delta = 1/(1 - \alpha), \beta = (2 - \alpha)/(1 - \alpha)$. Then, if $1 - \Delta \leq t_0 \leq y + \Delta$, we get

\begin{equation}
I_{m,n}(s) = e\left((1 - \alpha)(\alpha^\delta h)\left(\frac{qumn}{s}\right)\right) \sum_{0 \leq \nu \leq \nu_1} \frac{c_\nu(\alpha)}{(hX^2)^{\nu+1/2}} \left(\frac{hqumn}{s}\right)^{\beta(\nu+1/2)} \cdot G^{(2\nu)}\left(\frac{1}{X}\left(\frac{\alpha qumn}{s}\right)^\delta\right) + O\left(X^{-\alpha D_0}\right),
\end{equation}

with

\[
c_\nu(\alpha) = \sqrt{2\pi} \frac{(2i)^{-\nu}e^{-\pi i/4}}{\nu!} \left(\frac{\alpha - 1}{\nu+1/2}\right)^{\alpha \beta(\nu+1/2)}.
\]

Notice that (15) remains valid if $t_0 \notin [1 - \Delta; y + \Delta]$ since $w(t) \equiv 0, G(t) \equiv 0$ for $t$ close to $t_0$.

Going back to the sum $W_{m,n,\chi}$, we have

\[
W_{m,n,\chi} = \frac{X}{qumn}S_1 + O(KX^{-\alpha D_0}) = \frac{X}{qumn} \sum_{T_1 < s < T_2} \tau(\chi; s) \left\{ e\left((1 - \alpha)(\alpha^\delta h)\left(\frac{qumn}{s}\right)\right) \right\}.
\]

The contribution from the error terms can be made arbitrarily small with the appropriate choice of $D_0$. The main term takes the form

\[
\frac{1}{\varphi(q)} \sum_{\chi \mod q} \chi(u^*) \sum_{m=1}^{+\infty} \chi(m)f_1(m)\Psi_M(m) \sum_{n=1}^{+\infty} \chi(n)f_2(n)\Psi_N(n).
\]

We also note that for the small values of $q$ it is possible to get $T_2 < 1$. This case is not a problem since the sum $S_1$ is empty and the only contribution to the upper bound is coming from Lemma 2.
Second iteration of Poisson summation. We have:

\[(16) \quad W(M, N, K) = \frac{(qu)^{-1}}{\varphi(q)} \sum_{\chi \mod q} \chi(ua^*) \sum_{m=1}^{+\infty} \chi(m) f_1(m) \Psi_M(m) \sum_{T_1<s<T_2} \tau(\chi; s)V_{\chi,m,s} + O\left(X^{-\alpha D_0/2}\right),\]

where

\[V_{\chi,m,s} = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n} \frac{f_2(n)}{\Psi_N(n)} \sum_{0 \leq \nu \leq \nu_1} c_{\nu}(\alpha) h^{\delta(\nu+1/2)} \left(\frac{qumn}{s}\right)^{\beta(\nu+1/2)} G^{(2\nu)} \left(\frac{1}{X} \left(\frac{ahqumn}{s}\right)^{\delta}\right) e \left\{ (1-\alpha)(\alpha h)^{\delta} \left(\frac{qumn}{s}\right)^{\gamma} \right\}.\]

Setting \(n = q\rho + \lambda\), we get

\[V_{\chi,m,s} = \sum_{0 \leq \nu \leq \nu_1} c_{\nu}(\alpha) h^{\delta(\nu+1/2)} \left(\frac{qum}{s}\right)^{\beta(\nu+1/2)} q^{-1} \sum_{\lambda=1}^{+\infty} \chi(\lambda) \sum_{\rho=-\infty}^{+\infty} f_2(q\rho + \lambda) \Psi_N(q\rho + \lambda) \cdot (q\rho + \lambda)^{\beta(\nu+1/2)-1} G^{(2\nu)} \left(\frac{1}{X} \left(\frac{ahqum(q\rho + \lambda)}{s}\right)^{\delta}\right) e \left\{ (1-\alpha)(\alpha h)^{\delta} \left(\frac{qum(q\rho + \lambda)}{s}\right)^{\gamma} \right\}.\]

Applying Poisson summation again, we obtain

\[(17) \quad V_{\chi,m,s} = \sum_{0 \leq \nu \leq \nu_1} c_{\nu}(\alpha) h^{\delta(\nu+1/2)} \left(\frac{qum}{s}\right)^{\beta(\nu+1/2)} \frac{1}{X} \int_{-\infty}^{+\infty} f_2(qv + \lambda) \Psi_N(qv + \lambda) (qv + \lambda)^{\beta(\nu+1/2)-1} \cdot G^{(2\nu)} \left(\frac{1}{X} \left(\frac{ahqum(qv + \lambda)}{s}\right)^{\delta}\right) e \left\{ (1-\alpha)(\alpha h)^{\delta} \left(\frac{qum(qv + \lambda)}{s}\right)^{\gamma} - \sigma v \right\} dv.\]

Next, we substitute

\[\tau = \frac{ahqum(qv + \lambda)}{sX^{1-\alpha}}.\]

This implies

\[t_0 = \frac{1}{X} \left(\frac{ahqum(qv + \lambda)}{s}\right)^{\delta} = \tau^{\delta}, \quad \left(\frac{qum(qv + \lambda)}{s}\right)^{\gamma} = \left(\frac{X^{1-\alpha}\tau}{\alpha h}\right)^{\gamma}, \quad (\alpha h)^{\delta} \left(\frac{qum(qv + \lambda)}{s}\right)^{\gamma} = h X^{\alpha \tau^{\gamma}}.\]
Then the integral in (17) takes the form

\[
\frac{1}{q} \left( \frac{X^{1-\alpha}}{\alpha h q \mu m} \right)^{\beta(\nu+1/2)} \mathcal{L} \left( \frac{X^{1-\alpha}}{\alpha h q \mu m} \right) \Psi_N \left( \frac{X^{1-\alpha} s \tau}{\alpha h q \mu m} \right) \tau^{\beta(\nu+1/2)-1} G^{(2\nu)} \left( \tau^\delta \right) \times e \left\{ (1 - \alpha) h X^\alpha \tau - \frac{X^{1-\alpha} s \sigma \tau}{\alpha h q^2 \mu m} \right\} d\tau.
\]

Hence,

\[
V_{\chi, m, s} = \sum_{0 \leq \nu \leq \nu_1} \frac{c_\nu(\alpha)}{X^{2\nu}} h^\delta(\nu+1/2) \frac{1}{q} \left( \frac{X^{1-\alpha}}{\alpha h} \right)^{\beta(\nu+1/2)} \sum_{\sigma = -\infty}^{+\infty} \tau(\chi; \sigma) J(\sigma),
\]

where the meaning of \( J(\sigma) \) is clear. We further simplify the last expression by setting

\[
b_\nu(\alpha) = \frac{c_\nu(\alpha)}{\alpha^\beta(\nu+1/2)},
\]

which gives

\[
V_{\chi, m, s} = \frac{X}{q} \sum_{0 \leq \nu \leq \nu_1} b_\nu(\alpha) \left( \frac{h}{X^\alpha} \right)^{\nu+1/2} \sum_{\sigma = -\infty}^{+\infty} \tau(\chi; \sigma) J(\sigma).
\]

Let us denote

\[
T_3 = \frac{(\alpha h q)^2 \mu m}{4 s X^{1-2\alpha}}, \quad T_4 = 16 T_3 = \frac{4(\alpha h q)^2 \mu m}{s X^{1-2\alpha}},
\]

and split the sum \( V_{\chi, m, s} \) as follows:

\[
V_{\chi, m, s} = \frac{X}{q} \sum_{0 \leq \nu \leq \nu_1} b_\nu(\alpha) \left( \frac{h}{X^\alpha} \right)^{\nu+1/2} \left( \sum_{T_3 < \sigma < T_4} + \sum_{|\sigma| > T_4} + \sum_{-T_3 < \sigma < T_3} \right) \tau(\chi; \sigma) J(\sigma) =:\n\]

\[
\frac{X}{q} \sum_{0 \leq \nu \leq \nu_1} b_\nu(\alpha) \left( \frac{h}{X^\alpha} \right)^{\nu+1/2} \left( C_1 + C_2 + C_3 \right).
\]

Similarly to above, we apply Lemma 2 to the integrals \( J(\sigma) \) in \( C_2 \) and \( C_3 \) to estimate them from above and use Lemma 3 to compute \( J(\sigma) \) in \( C_1 \). If \( q \) is small enough and \( T_4 < 1 \), the whole sum \( V_{\chi, m, s} \) is estimated by Lemma 2.
Next, we verify the conditions of Lemma 2 and Lemma 3. Put
\[
w_1(\tau) = f_2\left(\frac{X^{1-\alpha}s\tau}{\alpha h q u m}\right)\Psi_N\left(\frac{X^{1-\alpha}s\tau}{\alpha h q u m}\right)\tau^{\beta(\nu+1/2)-1} G^{(2\nu)}(\tau^\delta),
\]
\[
g_1(\tau) = \begin{cases} (1-\alpha)h X^\alpha \tau^\gamma - \frac{X^{1-\alpha}s\sigma\tau}{\alpha h q^2 u m}, & \text{if } \tau \in [(1-\Delta)^{1/\beta}; (y+\Delta)^{1/\delta}], \\ 0, & \text{if } \tau \leq (1-2\Delta)^{1/\delta} \text{ or } \tau \geq (y+2\Delta)^{1/\delta}
\end{cases}
\]
and define \(g_1(\tau)\) on \((1-2\Delta)^{1-\alpha} \leq \tau \leq (1-\Delta)^{1-\alpha}\) and \((y+\Delta)^{1-\alpha} \leq \tau \leq (y+2\Delta)^{1-\alpha}\) appropriately.

Now we estimate \(w_1^{(j)}(\tau)\) on \((1-\Delta)^{1-\alpha} \leq \tau \leq (y+\Delta)^{1-\alpha}\). We have
\[
w_1^{(j)}(\tau) = \sum_{j_1+j_2+j_3+j_4=j} \binom{j}{j_1, j_2, j_3, j_4} d_{\tau}^{j_1} f_2\left(\frac{X^{1-\alpha}s\tau}{\alpha h q u m}\right) d_{\tau}^{j_2} \Psi_N\left(\frac{X^{1-\alpha}s\tau}{\alpha h q u m}\right) \sum_{j_3} \frac{d_{\tau}^{j_3} (\tau^{\beta(\nu+1/2)-1})}{d_{\tau}^{j_3}} G^{(2\nu)}(\tau^\delta).
\]
Next,
\[
\frac{d_{\tau}^{j_1} f_2\left(\frac{X^{1-\alpha}s\tau}{\alpha h q u m}\right)}{d_{\tau}^{j_2}} \ll \log X,
\]
\[
\frac{d_{\tau}^{j_2} \Psi_N\left(\frac{X^{1-\alpha}s\tau}{\alpha h q u m}\right)}{d_{\tau}^{j_3}} \ll \left(\frac{X^{1-\alpha}s}{N\alpha h q u m}\right)^{j_2} (\log X)^{A_{j_2}} \ll_{j_2} (\log X)^{A_{j_2}},
\]
\[
\frac{d_{\tau}^{j_3} (\tau^{\beta(\nu+1/2)-1})}{d_{\tau}^{j_3}} = \binom{\beta(\nu+1/2)-1}{j_3} \ll_{j_3} 1.
\]
By Faa di Bruno’s formula
\[
\frac{d^r G^{(2\nu)}(\tau^\delta)}{d\tau^r} = \sum_{m_1 + m_2 + \ldots + m_r = r} \frac{r!}{m_1! \ldots m_r!} G^{(2\nu+m_1+\ldots+m_r)}(t_0)^{m_1} \left(\frac{r!}{2!}\right)^{m_2} \ldots \ldots \frac{m_r}{r!} \tau^{\delta(m_1+\ldots+m_r)-r},
\]
which implies
\[
\frac{d^r G^{(2\nu)}(\tau^\delta)}{d\tau^r} \ll \max_{\mu=m_1+\ldots+m_r} \left|G^{(2\nu+\mu)}(t_0)\right|.
\]
Using the notation
\[
\tilde{H}(t) = 2\pi i H(t) = 2\pi i (g(t) - g(t_0) - \frac{1}{2} g''(t_0)(t-t_0)^2),
\]
we deduce
\[
\left. \frac{d^\nu e^{\tilde{H}(t)}}{dt^\nu} \right|_{t=t_0} = \sum_{m_1+2m_2+\ldots+\nu m_\nu=\nu} \frac{\nu!}{m_1! \ldots m_\nu!} e^{\tilde{H}(t_0)} \left( \frac{\tilde{H}'(t_0)}{1!} \right)^{m_1} \left( \frac{\tilde{H}''(t_0)}{2!} \right)^{m_2} \ldots \left( \frac{\tilde{H}^{(\nu)}(t_0)}{\nu!} \right)^{m_\nu}.
\]

Clearly, \( \tilde{H}(t_0) = \tilde{H}'(t_0) = \tilde{H}''(t_0) = 0 \), so the only non-zero contribution is coming from the tuples \( (m_1, \ldots, m_\nu) \) with \( m_1 = m_2 = 0 \):
\[
\left. \frac{d^\nu e^{\tilde{H}(t)}}{dt^\nu} \right|_{t=t_0} = \sum_{3m_3+4m_4+\ldots+\nu m_\nu=\nu} \frac{\nu!}{m_1! \ldots m_\nu!} \left( \frac{\tilde{H}^{(3)}(t_0)}{3!} \right)^{m_3} \left( \frac{\tilde{H}^{(4)}(t_0)}{4!} \right)^{m_4} \ldots \left( \frac{\tilde{H}^{(\nu)}(t_0)}{\nu!} \right)^{m_\nu}.
\]

If \( k \geq 3 \), then \( \tilde{H}^{(k)}(t_0) = 2\pi i g^{(k)}(t_0) \ll k h X^\alpha \). That means
\[
\left. \frac{d^\nu e^{\tilde{H}(t)}}{dt^\nu} \right|_{t=t_0} \ll \sum_{3m_3+4m_4+\ldots+\nu m_\nu=\nu} (hX^\alpha)^{m_3+m_4+\ldots+m_\nu}.
\]

But since \( 3m_3 + 3m_4 + \ldots + 3m_\nu \leq 3m_3 + 4m_4 + \ldots + \nu m_\nu \leq \nu \), we get
\[
(18) \quad m_3 + m_4 + \ldots + m_\nu \leq \frac{\nu}{3} \quad \text{and} \quad \left. \frac{d^\nu e^{\tilde{H}(t)}}{dt^\nu} \right|_{t=t_0} \ll (hX^\alpha)^{\nu/3}.
\]

Hence, from (14),
\[
G^{(l)}(t_0) = \sum_{\nu=0}^{l} \binom{l}{\nu} \left. \frac{d^\nu e^{\tilde{H}(t)}}{dt^\nu} \right|_{t=t_0} w^{(l-\nu)}(t_0) \ll \sum_{\nu=0}^{l} (hX^\alpha)^{\nu/3} (\log X)^{C_5(l-\nu)+1} \ll (\log X)(hX^\alpha)^{l/3},
\]

whence
\[
\frac{d^r G^{(2\nu)}}{dt^r}(t^\delta) \ll \max_{m_1+2m_2+\ldots+\nu m_\nu=r} (\log X)(hX^\alpha)^{(2\nu+m_1+\ldots+m_\nu)/3} \ll (\log X)(hX^\alpha)^{(2\nu+r)/3}.
\]

Particularly,
\[
\frac{d^{j_4} G^{2\nu}(t^\delta)}{dt^{j_4}} \ll (\log X)(hX^\alpha)^{(2\nu+j_4)/3}.
\]
Finally, we find

\[(19) \quad w^{(j)}_{1}(t) \ll (\log X)^{2}(hX^{\alpha})^{2
u/3} \sum_{j_{1}+\ldots+j_{4}=j} \left( \frac{j}{j_{1},\ldots,j_{4}} \right) (\log X)^{A_{0}j_{2}}(hX^{\alpha})^{j_{4}/3} \ll (\log X)^{2}(hX^{\alpha})^{2
nu/3+j/3}.\]

So the inequality \(w^{(j)}_{1}(t) \ll X_{I}Y_{I}^{-j}\) holds with \(X_{I} = (\log X)^{2}(hX^{\alpha})^{2

\nu/3}, \ V_{I} = (hX^{\alpha})^{-1/3}\). Next, we have

\[g'_{1}(\tau) = \gamma(1-\alpha)hX^{\alpha}\tau^{-1} - \frac{X^{1-\alpha}s\sigma}{\alpha h q^{2} u m}, \quad g''_{1}(\tau) = \gamma(\gamma - 1)(1-\alpha)hX^{\alpha}\tau^{-2},\]

\[g^{(j)}_{1}(\tau) = (\gamma)_{j}(1-\alpha)hX^{\alpha}\tau^{-j} \approx hX^{\alpha},\]

so one can take \(Y_{I} = hX^{\alpha}, Q_{I} = 1\). Put \(J = [10^{-1}; 10], \ \tilde{V}_{I} = |J| \) and \(Z_{I} = Q_{I} + X_{I} + Y_{I} + \tilde{V}_{I} + 1 \approx (\log X)^{2}(hX^{\alpha})^{2
\nu/3} + hX^{\alpha}\), which implies

\[Z_{I} \approx \begin{cases} hX^{\alpha}, & \text{if } \nu = 0,1, \\ (\log X)^{2}(hX^{\alpha})^{2\nu/3}, & \text{if } \nu \geq 2. \end{cases}\]

We choose the constant \(\delta_{I} > 0\) such that

\[
\frac{Q_{I}Z_{I}^{3\delta_{I}/2}}{\sqrt{Y_{I}}} \leq V_{I} = (hX^{\alpha})^{-1/3}.
\]

If \(\nu = 0,1\), we get \((hX^{\alpha})^{-1/2+\delta_{I}/2} \leq (hX^{\alpha})^{-1/3}\), which holds true for all \(\delta_{I} < 1/3\). If \(\nu \geq 2\), then

\[(\log X)^{3\delta_{I}}(hX^{\alpha})^{3\nu/3-1/2} \leq (hX^{\alpha})^{-1/3},\]

so one can take \(\delta_{I} = 1/(121D_{0})\). It is easy to check that \(Y_{I} > Z_{I}^{3\delta_{I}}\) holds true for all \(\nu\).

Next, if \(|\sigma| \geq T_{4}\), then

\[|g'_{1}(\tau)| = \left| \frac{X^{1-\alpha}s\sigma}{\alpha h q^{2} u m} - \gamma(1-\alpha)hX^{\alpha}\tau^{-1} \right| \geq \frac{X^{1-\alpha}s|\sigma|}{\alpha h q^{2} u m} \left( 1 - \frac{\alpha^{2}h^{2}q^{2}um\tau^{-1}}{X^{1-2\alpha}sT_{4}} \right) \geq \frac{2X^{1-\alpha}s|\sigma|}{3\alpha h q^{2} u m}.
\]

If \(-T_{4} < \sigma \leq 0\), then

\[g'_{1}(\tau) = \gamma(1-\alpha)hX^{\alpha}\tau^{-1} - \frac{X^{1-\alpha}s\sigma}{\alpha h q^{2} u m} = \alpha h X^{\alpha}\tau^{-1} + \frac{X^{1-\alpha}s|\sigma|}{\alpha h q^{2} u m} \geq \alpha h X^{\alpha}\tau^{-1} \geq \frac{1}{2}\alpha h X^{\alpha}.
\]
Finally, if $1 \leq \sigma \leq T_3$, then

$$g'_1(\tau) = ahX^\alpha \tau^{\gamma - 1} \left(1 - \frac{X^{1-2\alpha} s \tau^{1-\gamma}}{(ahq)^2 um}\right) \geq ahX^\alpha \tau^{\gamma - 1} \left(1 - \frac{3}{5}\right) \geq \frac{1}{6} ahX^\alpha.$$

So one can choose

$$R_I = \begin{cases} 
\frac{2 X^{1-\alpha} s |\sigma|}{3 ahq^2 um} & \text{if } |\sigma| \geq T_4, \\
\frac{1}{6} ahX^\alpha & \text{if } -T_4 < \sigma \leq T_3.
\end{cases}$$

Again, setting

$$\Delta_1 = \frac{Q_I R_I}{\sqrt{Y_I}}, \quad \Delta_2 = R_IV_I,$$

we show that $\Delta_1, \Delta_2 > 1$. Indeed, in the case $|\sigma| \geq T_4$, we have

$$\Delta_1 = \frac{2 X^{1-\alpha} s |\sigma|}{3 ahq^2 um} \geq \frac{2}{3} \frac{X^{1-3\alpha/2} s}{\sqrt{h} ahq^2 um} T_4 = \frac{8}{3} \alpha \sqrt{h} X^{\alpha/2} > 1,$$

$$\Delta_2 = \frac{2 X^{1-\alpha} s |\sigma|}{3 ahq^2 um} (hX^\alpha)^{-1/3} \geq \frac{2}{3} \frac{X^{1-\alpha} s}{ahq^2 um} T_4 h^{-1/3} X^{-\alpha/3} = \frac{8}{3} \alpha (hX^\alpha)^{2/3} > 1.$$

If $-T_4 < \sigma \leq T_3$, then

$$\Delta_1 = \frac{1}{6} ahX^\alpha \frac{1}{\sqrt{hX^\alpha}} = \frac{\alpha}{6} \sqrt{h} X^{\alpha/2} > 1,$$

$$\Delta_2 = \frac{1}{6} ahX^\alpha (hX^\alpha)^{-1/3} = \frac{\alpha}{6} (hX^\alpha)^{2/3} > 1.$$

Thus, applying Lemma 2 with a large enough $F_0 = F_0(\alpha) > 1$, $J = [(1 - \Delta)^{1/3}; (y + \Delta)^{1/3}]$, for $|\sigma| \geq T_4$ we find

$$J(\sigma) \ll |J| X_I (\Delta_1^{-F_0} + \Delta_2^{-F_0}) \ll \left(\log X\right)^2 (hX^\alpha)^{2\nu/3} \left\{ \left(\frac{ahq^2 um \sqrt{hX^\alpha}}{2X^{1-\alpha} s |\sigma|}\right)^{F_0} + \left(\frac{3 ahq^2 um (hX^\alpha)^{1/3}}{2X^{1-\alpha} s |\sigma|}\right)^{F_0} \right\} \ll \left(\log X\right)^2 (hX^\alpha)^{2\nu/3} \left(\frac{3\alpha}{2} h^{3/2} q^2 um \frac{1}{X^{1-3\alpha/2} s |\sigma|}\right)^{F_0},$$

and for $-T_4 < \sigma \leq T_3$ we get

$$J(\sigma) \ll \left(\log X\right)^2 (hX^\alpha)^{2\nu/3} \left\{ \left(\frac{6}{\alpha \sqrt{h} X^{\alpha/2}}\right)^{F_0} + \left(\frac{6}{\alpha h^{2/3} X^{2\alpha/3}}\right)^{F_0} \right\} \ll \left(\log X\right)^2 (hX^\alpha)^{2\nu/3} \left(\frac{6}{\alpha \sqrt{h} X^{-\alpha/2}}\right)^{F_0}. $$
Lemma 3. Let $\lambda^\mu \implies \lambda^\nu$ implies $\omega(\theta; \alpha, \beta) = 0$.

Then $\lambda^\mu \implies \lambda^\nu$ where $\omega = 1 - \frac{1}{\xi(\theta; \alpha, \beta)}$.

We are now ready to compute $C_1$ using Lemma 3. Let $T_3 < \sigma < T_4$, $\eta_1(\tau_0) = 0$. Then $\tau_0 \in J = [10^{-1}; 10]$. We find

$$\tau_0 = \frac{1}{X^{1-\alpha}} \left\{ \frac{(\alpha h)q^2 u m}{s \sigma} \right\}^{\xi},$$

where $\xi = 1/(1 - \gamma) = (1 - \alpha)/(1 - 2\alpha)$;

$$g_1(\tau_0) = (1 - \alpha)hX^\alpha \tau_0^\gamma - \frac{X^{1-\alpha} s \sigma \tau_0}{\alpha h q^2 u m} = (1 - 2\alpha)h \left\{ \frac{(\alpha h)q^2 u m}{s \sigma} \right\}^{\eta},$$

where $\eta = \alpha/(1 - 2\alpha)$;

$$g_1''(\tau_0) = -\frac{\alpha(1 - 2\alpha)}{1 - \alpha}hX^{2(1-\alpha)} \left( \frac{s \sigma}{(\alpha h)q^2 u m} \right)^{\omega},$$

where $\omega = \xi(2 - \gamma) = (2 - 3\alpha)/(1 - 2\alpha)$. Finally, take

$$G_1(\tau) = w_1(\tau) e^{2\pi i H_1(\tau)}, \quad H_1(\tau) = g_1(\tau) - g_1(\tau_0) - \frac{g_1''(\tau_0)}{2}(\tau - \tau_0)^2.$$

Then Lemma 3 implies

$$J(\sigma) = e \left( g_1(\tau_0) - \frac{1}{8} \sum_{0 < \mu < 1 \mu_1} \frac{\sqrt{2\pi}}{\mu!} \frac{(2i)^{\mu}}{|g''(\tau_0)|^{\mu+1/2}} G_1^{(2\mu)}(\tau_0) + O(X^{-\alpha F_0}) =

\left[ e \left( (1 - 2\alpha)h \left\{ \frac{(\alpha h)q^2 u m}{s \sigma} \right\}^{\eta} - \frac{1}{8} \sum_{0 < \mu < 1 \mu_1} \frac{\sqrt{2\pi}}{\mu!} \frac{(2i)^{\mu}}{\mu^{\mu+1/2}} \right) \right. \left. \frac{1}{h X^{2(1-\alpha)} \mu^{1+1/2}} \left( \frac{(\alpha h)q^2 u m}{s \sigma} \right)^{\omega(\mu+1/2)} G_1^{(2\mu)} \left( \frac{1}{X^{1-\alpha}} \left\{ \frac{(\alpha h)q^2 u m}{s \sigma} \right\}^{\xi} \right) + O(X^{-\alpha F_0}) \right.$$

with $\mu_1 = 3F_0/\delta_I = 363 D_0 F_0$. Setting

$$c_\mu(\alpha) = e \left( \frac{1}{8} \sqrt{2\pi} \frac{2\mu}{\mu!} (2i)^{-\mu} \left( \frac{1 - \alpha}{\alpha(1 - 2\alpha)} \right)^{\mu+1/2} \alpha^{2\omega(\mu+1/2)} \right),$$

Thus, the contribution from $C_2$, $C_3$ to $V_{x,m,s}$ can be made small enough with the appropriate choice of $F_0$. We get the formula

$$V_{x,m,s} = \frac{X}{q} \sum_{0 < \mu < 1 \mu_1} b_\mu(\alpha) \left( \frac{h}{X^\alpha} \right)^{\nu+1/2} C_1 + O(X^{-\alpha F_0/10}),$$

$$C_1 = \sum_{T_3 < \sigma < T_4} \tau(\chi; \sigma) J(\sigma).$$
we get

\[ J(\sigma) = e \left( \left( 1 - 2\alpha \right) h \left\{ \frac{(\alpha h q)^2 \mu m}{s \sigma} \right\} \right) \sum_{0 \leq \mu \leq \mu_1} c_\mu(\alpha) \left( \frac{h X^{2(1-\alpha)}}{2} \right)^{\mu+1/2} \left( \frac{(\alpha h q)^2 \mu m}{s \sigma} \right)^{\omega(\mu+1/2)} \cdot \]

\[ G_1^{(2\mu)} \left( \frac{1}{X^{1-\alpha}} \left\{ \frac{(\alpha h q)^2 \mu m}{s \sigma} \right\} \right)^{\xi} + O \left( X^{-\alpha F_0} \right). \]

Again, it is not hard to see that the \(O\)-term contributes at most \(O \left( X^{-\alpha F_0/10} \right)\) to the sum \(V_{\chi,m,s}\). This contribution can be made arbitrarily small. Hence, from (20) we have

\[ V_{\chi,m,s} = \]

\[ \frac{X}{q} \sum_{T_3 < \sigma \leq T_4} \sum_{0 \leq \nu \leq \nu_1} b_\nu(\alpha) \left( \frac{h}{X^\alpha} \right)^{\nu+1/2} \sum_{0 \leq \mu \leq \mu_1} c_\mu(\alpha) \left( \frac{h X^{2(1-\alpha)}}{2} \right)^{\mu+1/2} \left( \frac{(\alpha h q)^2 \mu m}{s \sigma} \right)^{\omega(\mu+1/2)} \cdot \]

\[ G_1^{(2\mu)} \left( \frac{1}{X^{1-\alpha}} \left\{ \frac{(\alpha h q)^2 \mu m}{s \sigma} \right\} \right)^{\xi} \left( 1 - 2\alpha \right) h \left\{ \frac{(\alpha h q)^2 \mu m}{s \sigma} \right\}^{\eta} + O \left( X^{-\alpha F_0/10} \right). \]

Substituting this expression into (16) and changing the order of summation, we get

(21) \[ W(M,N,K) = \]

\[ \frac{X}{q^2} \sum_{m=1}^{+\infty} \frac{f_1(m)}{m} \sum_{T_3 < \sigma < T_4} \sum_{0 \leq \nu \leq \nu_1} b_\nu(\alpha) \left( \frac{h}{X^\alpha} \right)^{\nu+1/2} \sum_{0 \leq \mu \leq \mu_1} c_\mu(\alpha) \left( \frac{h X^{2(1-\alpha)}}{2} \right)^{\mu+1/2} \frac{1}{\left( \frac{h X^{2(1-\alpha)}}{2} \right)^{\mu+1/2}} \cdot \]

\[ \left( \frac{(\alpha h q)^2 \mu m}{s \sigma} \right)^{\omega(\mu+1/2)} G_1^{(2\mu)} \left( \frac{1}{X^{1-\alpha}} \left\{ \frac{(\alpha h q)^2 \mu m}{s \sigma} \right\} \right)^{\xi} \left( 1 - 2\alpha \right) h \left\{ \frac{(\alpha h q)^2 \mu m}{s \sigma} \right\}^{\eta} + O \left( X^{-\alpha F_0/20} \right) + O \left( X^{-\alpha D_0/2} \right). \]

**Bounding Kloosterman sum.** We rewrite the inner sum in (21) as follows:

(22) \[ \frac{1}{\varphi(q)} \sum_{\chi \mod q} \chi(mua)^* \tau(\chi; s) \tau(\chi; \sigma) = \sum_{l,r=1}^q e \left( \frac{l s + r \sigma}{q} \right) \frac{1}{\varphi(q)} \sum_{\chi \mod q} \chi(lr mua^*). \]

By orthogonality of characters the sum in the right hand side of (22) transforms into Kloosterman sum

\[ \sum_{l=1}^q e \left( \frac{s l + \sigma a(mu)^*}{q} \right) = S_q(s, \sigma a(mu)^*). \]
Thus,

\[
W(M, N, K) = \frac{X}{u q^2} \sum_{m=1}^{\infty} \frac{f_1(m)}{m} \psi_M(m) \sum_{T_1 < s < T_2} \sum_{T_3 < \sigma < T_4} b_\nu(\alpha) c_\mu(\alpha) \left( \frac{h}{X^\alpha} \right)^{\nu + 1/2} \frac{1}{(hX^{2(1-\alpha)})^{\mu + 1/2}} \left( \frac{(hq)^2 u m}{s \sigma} \right)^{\omega(\mu + 1/2)} G_1^{(2\mu)} \left( \frac{1}{X^{1-\alpha}} \left\{ \frac{(hq)^2 u m}{s \sigma} \right\} \right)^G \cdot e \left( (1 - 2\alpha) h \left\{ \frac{(hq)^2 u m}{s \sigma} \right\} \right) S_q(s, \sigma a(mu)^*) + O(X^{-\alpha F_0/20}) + O(X^{-\alpha D_0/2}).
\]

We can now estimate the multiple sum over \( m, s, \sigma, v \) and \( \mu \). Since \( H_1(\tau_0) = H'_1(\tau_0) = 0 \), similarly to (18), we get the upper bound

\[
\frac{d^n}{d\tau^n} \left( e(H_1(\tau)) \right) \bigg|_{\tau = \tau_0} \ll (hX^\alpha)^{r/3}.
\]

Together with (19) this implies

\[
G_1^{(2\mu)}(\tau_0) \ll (hX^\alpha)^{(2/3)(\nu + \mu)} (\log X)^2.
\]

Next, we apply Weil’s bound:

\[
|S_q(s, \sigma a(mu)^*)| \leq \tau(q) \sqrt{q}(s, \sigma, q)^{1/2}.
\]

Changing the order of summation, we get the inequality

\[
W(M, N, K) \ll \frac{X\tau(q) (\log X)^3}{u q^2 \sqrt{q}} \sum_{M/\Theta < m < M \Theta} \left( \frac{h}{X^\alpha} \right)^{\nu + 1/2} \frac{(hX^\alpha)^{(2/3)(\nu + \mu)}}{(hX^{2(1-\alpha)})^{\mu + 1/2}} (hq)^{\omega(2\mu + 1)} \cdot \sum_{M/\Theta < m < M \Theta} (mu)^{\omega(\mu + 1/2)} \sum_{T_1 < s < T_2} \sum_{T_3 < \sigma < T_4} \frac{(s, \sigma, q)^{1/2}}{(s \sigma)^{\omega(\mu + 1/2)}} + O(X^{-\alpha F_0/20} + X^{-\alpha D_0/2}).
\]

The sums over \( s \) and \( \sigma \) could be bounded as

\[
\sum_{T_1 < s < T_2} \sum_{T_3 < \sigma < T_4} \frac{(s, \sigma, q)^{1/2}}{(s \sigma)^{\omega(\mu + 1/2)}} \ll \left( \frac{X^{1-2\alpha}}{(hq)^2 u m} \right)^{\omega(\mu + 1/2)} \sum_{T_1 < s < T_2} \sum_{T_3 < \sigma < T_4} (s, \sigma)^{1/2}.
\]
The last expression does not exceed

\[
\left( \frac{X^{1-2\alpha}}{(hq)^2um} \right)^{\omega(\mu+1/2)} \sum_{1 \leq d \leq \min(T_2,16T_4)} \sum_{T_1 < s < T_2} \sum_{T_3/16 < \sigma < 16T_4} \sum_{\sigma \equiv 0 \pmod{d}} \sqrt{d} \ll
\]

\[
\left( \frac{X^{1-2\alpha}}{(hq)^2um} \right)^{\omega(\mu+1/2)} \sum_{1 \leq d \leq \min(T_2,16T_4)} \sqrt{d} \frac{T_2 T_4}{d} \ll
\]

\[
T_2 T_6 \left( \frac{X^{1-2\alpha}}{(hq)^2um} \right)^{\omega(\mu+1/2)} \ll \left( \frac{X^{1-2\alpha}}{(hq)^2um} \right)^{\omega(\mu+1/2)-1}.
\]

Next, the summation over \( m \) gives

\[
\sum_{M/\Theta \leq m \leq M\Theta} (mu)^{\omega(\mu+1/2)} \cdot \left( \frac{X^{1-2\alpha}}{(hq)^2mu} \right)^{\omega(\mu+1/2)-1} \ll
\]

\[
M^2 U \left( \frac{X^{1-2\alpha}}{(hq)^2} \right)^{\omega(\mu+1/2)-1} \frac{1}{(\log X)^{A_0}},
\]

whence, if \( D_0 \) and \( F_0 \) are sufficiently large,

\[
W(M, N, K) \ll \frac{X\tau(q) (\log X)^3}{Uq\sqrt{q}} \frac{M^2 U}{\log X} \sum_{0 \leq \mu \leq \mu_1} \frac{h X^\alpha}{X^\alpha} \frac{\nu^{1/2}}{X^{(2/3)(\nu+\mu)}} \frac{(hX^\alpha)^{\nu+\mu}}{(hX^{2(1-\alpha)})^{\mu+1/2}}.
\]

\[
(hq)^{\omega(2\mu+1)} \left( \frac{X^{1-2\alpha}}{(hq)^2} \right)^{\omega(\mu+1/2)-1} \ll \frac{X\tau(q)}{q\sqrt{q}} (\log X)^{3-A_0} \sum_{0 \leq \nu \leq \nu_1} \sum_{0 \leq \mu \leq \mu_1} X^{\kappa_1} h^{\kappa_2} q^2,
\]

where

\[
\kappa_1 = -\frac{\alpha \nu}{3} - \frac{\alpha \mu}{3} + \alpha - 1, \quad \kappa_2 = 2 + \frac{5 \nu}{3} - \frac{\mu}{3}.
\]

Clearly the main contribution comes from the term \( \nu = \mu = 0 \). We get

\[
W(M, N, K) \ll X^\alpha \sqrt{q}\tau(q)(\log X)^{3-A_0} M h^2.
\]

Summing \( W(M, N, K) \) over all admissible \( U, u, M, N, K \), and using Mardzhanishvili’s inequality

\[
\sum_{u \leq 2U} \tau_7(u) \ll U (\log U)^6,
\]
finally, we find

\[(23) \quad W_{III} \ll X^\alpha \sqrt{q\tau(q)}(\log X)^{2C+3-A_0} \sum_{U \in G} \sum'_{U \theta^7} \sum'_{U \leq U \leq U\Theta} |F(u, U)| . \]

\[\sum'_{M_1 \leq M \leq M_2} \sum'_{N_1 \leq N \leq N_2} \sum'_{K_1 \leq K \leq K_2} \sum'_{U \in G} \sum'_{u \leq 2U} \sum'_{M \in G} \sum'_{N \in G} \sum'_{K \in G} 1 \ll \]

\[X^\alpha \sqrt{q\tau(q)}(\log X)^{2C+3-A_0} \sum_{U \in G} \sum'_{u \leq 2U} \tau_7(u)(\log X)^6(A_0+1) + \sum'_{U \leq U \leq U\Theta} (X/U)^{1/3} (\log X)^3(A_0+1) \ll \]

\[X^{1/3+\alpha} \sqrt{q\tau(q)}(\log X)^{2C+3-A_0} + \sum'_{U \leq U \leq U\Theta} U^{2/3}(\log U)^6 \ll \]

\[X^{1/3+\alpha} \sqrt{q\tau(q)}X^{(2/3)(1/10-3\varepsilon_1/2)}(\log X)^{8A_0+2C+19} \ll \]

\[X^{2/5+\alpha-\varepsilon_1} \sqrt{q\tau(q)}(\log X)^{L_0}, \]

where \(L_0 = 8A_0 + 2C + 19.\)

**Final bound.** Combining together type I (7), type II (11) and type III (23) estimates we get

\[W \ll X^{2/5+\alpha/2-\varepsilon_1+2\delta_1} + \frac{X^{1-\alpha/2+2\delta_1}}{q} + \frac{1}{\sqrt{q}} X^{4/5+\alpha/4+\varepsilon_1/2+3\delta_2} + \frac{1}{\sqrt{q}} X^{1-\alpha/4+3\delta_2} + X^{2/5+\alpha-\varepsilon_1} \sqrt{q\tau(q)}(\log X)^{L_0}. \]

Further, the right hand side of the last inequality does not exceed

\[\frac{X}{q} \left( qX^{-3/5+\alpha/2-\varepsilon_1+2\delta_1} + X^{-\alpha/2+2\delta_1} + \sqrt{q} X^{-1/5+\alpha/4+\varepsilon_1/2+3\delta_2} + \sqrt{q} X^{-1/5+\alpha/4+\varepsilon_1/2+3\delta_2} + X^{-\alpha/4+3\delta_2} + q^{3/2} X^{3/5+\alpha-\varepsilon_1+\delta_4} \right) \]

with an arbitrarily small \(\delta_4 > 0.\) Clearly,

\[
\max(X^{-\alpha/2+2\delta_1}, X^{-\alpha/4+3\delta_2}) \ll (\log X)^{-A}
\]

if \(\delta_1\) and \(\delta_2\) are small enough. Next,

\[qX^{-3/5+\alpha/2-\varepsilon_1+2\delta_1} \ll X^{-1/5-\alpha/10+2\delta_1} \ll (\log X)^{-A}.\]

Then

\[W \ll \frac{X}{q} \left( (\log X)^{-A} + \max(\sqrt{q} X^{-1/5+\alpha/4+\varepsilon_1/2+3\delta_2}, q^{3/2} X^{3/5+\alpha-\varepsilon_1+\delta_4}) \right). \]
Thus, $W \ll (X/q)(\log X)^{-A}$ if

$$q \leq \min \left( (\log X)^{-2A} X^{2/5-\alpha/2-\varepsilon_1-6\delta_2}, (\log X)^{-2A} X^{2/5-(2/3)A} X^{2/5-(2/3)\alpha+(2/3)\varepsilon_1-(2/3)\delta_4} \right).$$

The maximum of this bound is reached at $\varepsilon_1 = \alpha/10$. Thus, $q \leq X^{2/5-(3/5)\alpha-\varepsilon}$ with any $\varepsilon < \min(6\delta_2, (2/3)\delta_4)$. Finally, the desired bound (2) follows from partial summation.

**Remark 3.** One can obtain a slightly better level of distribution, $q \leq X^{2/5-\alpha/2-\varepsilon}$, in Theorem 1 by iterating the Poisson summation for the third time (on the sum over $m$) and applying the bound for 2-dimensional Kloosterman sum [27].

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