A new class between theta open sets and theta omega open sets

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ABSTRACT

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We define \( \theta \omega \)-closure operator as a new topological operator which lies between the \( \theta \)-closure and the \( \theta \omega \)-closure. Some relationships between this new operator and each of \( \theta \)-closure, \( \theta \omega \)-closure, and usual closure are obtained. Via \( \theta \omega \)-closure operator, we introduce \( \theta \omega \)-open sets as a new topology. Some mapping theorems related to the new topology are given. \( T_{2} \) topological spaces are characterized in terms of \( \theta \omega \)-closure operator. Also, we use \( N \)-open sets to define \( N \)-regularity as a new separation axiom which lies strictly between \( \omega \)-regularity and regularity. For a given topological space \(( Y, \sigma )\), we show that \( N \)-regularity is equivalent to the condition \( \sigma = \sigma_{\omega} \). Finally, \( \theta \omega \)-continuity, \( N \omega \)-continuity, weak \( \theta \omega \)-continuity, and faint \( \theta \omega \)-continuity are introduced and studied.

1. Introduction

Throughout this paper, \( \text{ts} \) and \( \text{ts}'s \) will denote respectively topological space and topological spaces. The notions of \( \theta \)-closure, \( \theta \)-closed sets, and \( \theta \)-open sets are introduced by Velicko [1] in which he used them in studying \( H \)-closed spaces. The work of Velicko was continued by many topologists. Recently, authors in [2, 3, 4, 5, 6, 7] have obtained several interesting results related to these sets. The notions of \( \omega \)-closed sets and \( \omega \)-open sets are introduced by Hedeib [8]. Area of research related to \( \omega \)-open sets was and still is hot, authors in [9, 10, 11, 12, 13, 14, 15] have obtained several interesting results related to these sets. Authors in [16, 17, 18] introduced \( \omega \)-open sets in fuzzy topological spaces and soft \( \text{ts}'s \). Recently by means of open sets and \( \omega \)-open sets, authors in [19] introduced the notion of \( \theta \omega \)-open sets as a class between open sets and \( \theta \)-open sets, then authors in [20, 21] continued the study of [19]. Authors in [22] introduced the notion of \( N \)-open sets as a strong form of \( \omega \)-open sets, in which they used this notion to give new characterizations of strongly compact topological spaces, and authors in [23, 24, 25, 26] continued the study of [22]. This work is devoted to introduce and investigate the notions of \( \theta \omega \)-closure, \( \theta \omega \)-closed sets, and \( \theta \omega \)-open sets, by means of open sets and \( N \)-open sets. The class of \( \theta \omega \)-open sets lies between \( \theta \)-open sets and \( \theta \omega \)-open sets. As in many other topological concepts, we hope that this will open the door for a number of future related studies. This work is organized as follows: In section 2, we introduce some basic definitions and results which we use in this research, also we give a characterization of compactness via \( N \)-open sets and as a consequence we improve the fact that closed subsets of a compact set is compact; moreover, we improve a well known mapping theorem related to compactness. In section 3, we define and investigate \( \theta \omega \)-closure operator as a new topological operator which lies between the \( \theta \)-closure and the \( \theta \omega \)-closure operators. In section 4, via \( \theta \omega \)-closure operator, we introduce and study \( \theta \omega \)-open sets as a new class between \( \theta \)-open sets and \( \theta \omega \)-open sets. In section 5, we characterize \( T_{2} \) ts's in terms of \( \theta \omega \)-closure operator. Also, we use \( N \)-open sets to define \( N \)-regularity as a new separation axiom which lies between \( \omega \)-regularity and regularity. We prove that a ts \(( Y, \sigma )\) is \( N \)-regular if and only if \( \sigma = \sigma_{\omega} \). In section 6, we introduce and investigate \( \theta \omega \)-continuity and \( N \omega \)-continuity as two new forms of continuity. In section 7, we introduce and investigate weak \( \theta \omega \)-continuity and faint \( \theta \omega \)-continuity as two new forms of continuity.

In this paper, \( \mathbb{R} \), \( \mathbb{Q} \), and \( \mathbb{N} \) denote, respectively the set of real numbers, the set of rational numbers, and the set of natural numbers. And, on any non-empty set \( X \), \( \tau_{\text{ind}} \), \( \tau_{\text{cof}} \) denote respectively, the indiscrete topology and the cofinite topology.

2. Preliminaries

The following sequence of definitions, propositions, and notations will be used in the sequel:

Definition 2.1. [1] Let \( B \) be a subset of a ts \(( Y, \sigma )\).

a. The \( \theta \)-closure of \( B \) is denoted by \( C_{\theta}(B) \) and defined by

\[
C_{\theta}(B) = \left\{ y \in Y : \overline{\bigcup_{V \in \sigma \text{ with } B \subseteq V} V} \cap B \neq \emptyset, V \in \sigma \text{ and } y \in V \right\}.
\]

b. \( B \) is called \( \theta \)-closed if \( C_{\theta}(B) = B \).

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c. $B$ is $\theta$-open if $Y - B$ is $\theta$-closed.

d. The family of all $\theta$-open sets in $(Y, \sigma)$ is denoted by $\sigma_\theta$.

For any ts $(Y, \sigma)$, it is well known that $\sigma_\theta$ forms a topology on $Y$, $\sigma_\theta \subseteq \sigma$, and $\sigma_\theta \neq \sigma$ in general.

**Definition 2.2.** [8] Let $(Y, \sigma)$ be a ts and let $B \subseteq Y$.

a. The set of condensation points of $B$ is denoted by $\text{Cond}(B)$ and defined by

$$\text{Cond}(B) = \{ y \in Y : V \cap B \text{ is uncountable, } V \in \sigma \text{ and } y \in V \}.$$  

b. $B$ is $\omega$-closed if $\text{Cond}(B) \subseteq B$.

c. $B$ is $\omega$-open if $Y - B$ is $\omega$-closed.

d. The family of all $\omega$-open sets in $(Y, \sigma)$ is denoted by $\sigma_\omega$.

**Proposition 2.3.** [27] Let $(Y, \sigma)$ be a ts and $B \subseteq Y$. Then $B \in \sigma_\omega$ if and only if for each $y \in B$, there are $V \in \sigma$ and a countable subset $D \subseteq Y$ such that $y \in V - D \subseteq B$.

**Proposition 2.4.** [28] Let $(Y, \sigma)$ be a ts. Then $\sigma_\omega$ is a topology on $Y$, $\sigma \subseteq \sigma_\omega$, and $\sigma_\omega \neq \sigma$ in general.

**Notation 2.5.** Let $(Y, \sigma)$ be a ts and $B \subseteq Y$. The closure of $B$ in $(Y, \sigma_\omega)$ is denoted by $\overline{B}$.

**Proposition 2.6.** [27] Let $(Y, \sigma)$ be a ts and $B \subseteq Y$. Then $\overline{B} \subseteq \overline{B}$, and $\overline{B} \neq \overline{B}$ in general.

**Definition 2.7.** [19] Let $(Y, \sigma)$ be a ts and $B \subseteq Y$.

a. The $\theta_\omega$-closure of $B$ is denoted by $\text{Cl}_{\theta_\omega}(B)$ and defined by

$$\text{Cl}_{\theta_\omega}(B) = \{ y \in Y : V \cap B \neq \emptyset, V \in \sigma \text{ and } y \in V \}.$$  

b. $B$ is called $\theta_\omega$-open if $\text{Cl}_{\theta_\omega}(B) = B$.

c. $B$ is $\theta_\omega$-open if the complement of $B$ is $\theta_\omega$-closed.

d. The family of all $\theta_\omega$-open sets in $(Y, \sigma)$ is denoted by $\sigma_{\theta_\omega}$.

**Definition 2.8.** [22] Let $(Y, \sigma)$ be a ts and $B \subseteq Y$. Denote the set

$$\{ y \in Y : V \cap B \text{ is infinite, } V \in \sigma \text{ and } y \in V \}$$  

by $\mathcal{N}(B)$. Then

a. $B$ is $\mathcal{N}$-closed if $\mathcal{N}(B) \subseteq B$.

b. $B$ is $\mathcal{N}$-open if $Y - B$ is $\mathcal{N}$-closed.

c. The family of all $\mathcal{N}$-open sets in $(Y, \sigma)$ is denoted by $\sigma_\mathcal{N}$.

**Proposition 2.9.** [25] Let $(Y, \sigma)$ be a ts and $B \subseteq Y$. Then $B \in \sigma_\mathcal{N}$ if and only if for each $y \in B$, there is $V \in \sigma$ and a finite set $F \subseteq Y$ such that $y \in V - F \subseteq B$.

Recall that a topological space $(Y, \sigma)$ is $T_1$ if every singleton is closed in $(Y, \sigma)$.

**Proposition 2.10.** [23] Let $(Y, \sigma)$ be a topological space. Then

a. $\sigma_\mathcal{N}$ is a topology on $Y$.

b. $\sigma \subseteq \sigma_\mathcal{N} \subseteq \sigma_{\theta_\omega}$, $\sigma_\mathcal{N} \neq \sigma$ in general, and $\sigma_\mathcal{N} \neq \sigma_{\theta_\omega}$ in general.

c. $\sigma = \sigma_\mathcal{N}$ if and only if $(Y, \sigma)$ is $T_\omega$.

d. $(Y, \sigma_\mathcal{N})$ is $T_1$.

**Notation 2.11.** Let $(Y, \sigma)$ be a ts and $B \subseteq Y$. The closure of $B$ in $(Y, \sigma_\mathcal{N})$ is denoted by $\overline{B}_\mathcal{N}$.

**Proposition 2.12.** Let $(Y, \sigma)$ be a ts and $B \subseteq Y$. Then

a. $[23] \overline{B}_\mathcal{N} \subseteq \overline{B}_\omega \subseteq \overline{B}_\sigma$ in general, and $\overline{B}_\mathcal{N} \neq \overline{B}_\sigma$ in general.

b. $[19] \overline{B} \subseteq \text{Cl}_{\theta_\omega}(B)$.

**Definition 2.13.** A ts $(Y, \sigma)$ is said to be

a. $[29]$ locally indiscrete if every open set in $(Y, \sigma)$ is closed.

b. $[19]$ $\omega$-locally indiscrete if every open set in $(Y, \sigma)$ is $\omega$-closed.

c. $[30]$ locally countable if $(Y, \sigma)$ has a base consists of countable sets.

d. $[28]$ anti-locally countable if each $V \in \sigma - \{ \emptyset \}$ is uncountable.

e. $[31]$ locally finite if $(Y, \sigma)$ has a base consists of finite sets.

f. $[23]$ anti-locally finite if each $V \in \sigma - \{ \emptyset \}$ is infinite.

**Proposition 2.14.** $[28]$ If $(Y, \sigma)$ is an anti-locally countable, then $\overline{B} = \overline{B}$ for every $B \in \sigma_\omega$.

b. $[27]$ If $(Y, \sigma)$ is locally countable, then $\sigma_\omega$ is the discrete topology.

c. $[23]$ If $(Y, \sigma)$ is anti-locally finite, then $\overline{B} = \overline{B}$ for every $B \in \sigma_N$.

d. $[23]$ If $(Y, \sigma)$ is locally finite, then $\sigma_\mathcal{N}$ is the discrete topology.

e. $[19]$ Every locally indiscrete is $\omega$-locally indiscrete.

f. $[19]$ Every locally countable ts is $\omega$-locally indiscrete.

**Proposition 2.15.** $[19]$ Let $(Y, \sigma)$ be anti-locally countable and $B \subseteq Y$.

Then

a. $\text{Cl}_{\theta_\omega}(B) = \text{Cl}_{\omega_\omega}(B)$.

b. If $B$ is $\theta_\omega$-closed in $(Y, \sigma)$, then $B$ is $\theta$-closed in $(Y, \sigma)$.

**Proposition 2.16.** [1] For any ts $(Y, \sigma)$ and for any $B \in \sigma$, $\text{Cl}_{\theta_\omega}(B) = \overline{B}$.

**Proposition 2.17.** [1] For any ts $(Y, \sigma)$, we have the following:

a. $\sigma_\theta$ forms a topology on $Y$.

b. $\sigma_\theta \subseteq \sigma$ and $\sigma_\theta \neq \sigma$ in general.

**Proposition 2.18.** [19] For any ts $(Y, \sigma)$, we have the following:

a. $(Y, \sigma_\omega)$ is a ts.

b. $\sigma_\theta \subseteq \sigma_\omega \subseteq \sigma$, $\sigma_\omega \neq \sigma_\omega$ in general and $\sigma_\omega \neq \sigma$ in general.

**Proposition 2.19.** [1] Let $(Y, \sigma)$ be a ts and $B \subseteq Y$. Then

a. $B$ is $\theta$-open in $(Y, \sigma)$ if and only if for any $y \in B$, there exists $V \in \sigma$ such that $y \in V$ and $V \subseteq B$.

b. $(Y, \sigma)$ is regular if and only if $\sigma = \sigma_\theta$.

**Definition 2.20.** [19] A ts $(Y, \sigma)$ is called $\omega$-$T_\omega$ if for every $y, z \in Y$ with $y \neq z$ there exist an open set $V$ containing $y$ and an $\omega$-open set $W$ containing $z$ such that $V \cap W = \emptyset$.

Recall that a ts $(Y, \sigma)$ is $T_1$ if for every $y, z \in Y$ with $y \neq z$ there exist an open set $V$ containing $y$ and an open set $W$ containing $z$ such that $V \cap W = \emptyset$.

**Proposition 2.21.** [19] a. Every $T_\omega$ is $\omega$-$T_2$, but not conversely.

b. [19] Every $\omega$-$T_2$ ts is $T_1$, but not conversely.

**Definition 2.22.** [27] A ts $(Y, \sigma)$ is called $\omega$-regular if given any closed set $C$ in $(Y, \sigma)$ and any point $y$ that does not belong to $C$, there are $V \in \sigma$ and $W \in \sigma_\omega$ such that $y \in V, C \subseteq W$ and $V \cap W = \emptyset$.

**Theorem 2.23.** [27] A ts $(Y, \sigma)$ is $\omega$-regular if and only if for any $V \in \sigma$ and any $y \in V$ there exists $W \in \sigma$ such that $y \in W$ and $\overline{W} \subseteq V$. 



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Definition 2.24. A function \( g : (Y, \sigma) \to (Z, \delta) \) is said to be \( \theta \)-continuous [32] (resp. \( \theta_\omega \)-continuous [21]) if for every \( y \in Y \) and every \( W \in \delta \) such that \( g(y) \in W \), there exists \( V \in \sigma \) such that \( y \in V \) and \( g(V) \subseteq W \) (resp. \( g(V) \subseteq W^\omega \)).

Proposition 2.32. [32] Every continuous function is \( \theta \)-continuous.

Definition 2.26. A function \( g : (Y, \sigma) \to (Z, \delta) \) is called weakly \( \omega \)-continuous [33] (resp. \( \omega \)-\( \theta \)-continuous [21]) if for every \( y \in Y \) and every \( W \in \delta \) such that \( g(y) \in W \), there exists \( V \in \sigma \) such that \( y \in V \) and \( g(V) \subseteq W \) (resp. \( g(V) \subseteq W^\omega \)).

Definition 2.27. [21] A function \( g : (Y, \sigma) \to (Z, \delta) \) is called weakly \( \theta_\omega \)-continuous if for every \( y \in Y \) and every \( W \in \delta \) such that \( g(y) \in W \), there is \( V \in \sigma \) such that \( g(V) \subseteq W^\omega \).

Definition 2.28. A function \( g : (Y, \sigma) \to (Z, \delta) \) is called faintly continuous [34] (resp. faintly \( \theta_\omega \)-continuous [21]) if for every \( y \in Y \) and \( W \in \delta_\omega \) (resp. \( W \in \delta_\omega \)) with \( g(y) \in W \), there exists \( V \in \sigma \) such that \( y \in V \) and \( g(V) \subseteq W \).

Theorem 2.29. A ts \((Y, \sigma)\) is compact if and only if \((Y, \sigma_N)\) is compact.

Proof. Assume that \((Y, \sigma)\) is compact. Consider the base \( B = \{U \subseteq F \in \sigma \text{ is a finite subset of } Y \} \). Let \( U' \) be a cover of \( Y \) with \( U' \subseteq B \), say \( U' = \{U_{-1} - F_0 : U_a \in \tau \text{ and } F_a \in \text{ a finite subset of } X_0, a \in \Delta \} \). Then \( \{U_a : U_a \in \sigma, a \in \Delta \} \) is an open cover of \((Y, \sigma)\). Since \((Y, \sigma)\) is compact, then it has a finite subcover, \( \{U_{a_1}, U_{a_2}, \ldots, U_{a_n} \} \).

Corollary 2.30. An \( N \)-closed subset of compact ts is compact.

Proof. Let \((Y, \sigma)\) be a compact space and let \( A \) be an \( N \)-closed subset of \((Y, \sigma)\). By Theorem 2.29, we know that \((Y, \sigma_\gamma)\) is compact. Since \( A \) is closed in the compact space \((Y, \sigma_\gamma)\), it follows that \( A \) is compact as a subset of \((Y, \sigma_\gamma)\). Since \( \sigma \subseteq \sigma_\gamma \), then \( A \) is compact as a subset of \((Y, \sigma)\).

Definition 2.31. A function \( g : (Y, \sigma) \to (Z, \delta) \) is called \( N \)-closed function if it maps closed sets onto \( N \)-closed sets.

It is clear that every closed function is \( N \)-closed, however, the following example shows that the converse is not true in general.

Example 2.32. Let \( g : [a, b] \to [a_1, b_1] \), where \( g(x) = x \). Then \( g \) is \( N \)-closed but not closed.

The following important theorem is well known between topologists and can be found in any general topology book:

Theorem 2.33. If \( g : (Y, \sigma) \to (Z, \delta) \) is a closed function such that for each \( z \in Z \), \( g^{-1}(\{z\}) \) is a compact subset of \((Y, \sigma)\), then \((Y, \sigma)\) is compact.

Theorem 2.34. If \( g : (Y, \sigma) \to (Z, \delta) \) is a \( \theta \)-closed function such that for each \( z \in Z \), \( g^{-1}(\{z\}) \) is a compact subset of \((Y, \sigma)\), then \((Y, \sigma)\) is compact.

Proof. Let \( \{U_a : a \in \Delta \} \) be an open cover of \((Y, \sigma)\). For each \( z \in Z \), \( g^{-1}(\{z\}) \) is a compact subset of \((Y, \sigma)\) and there exits a finite set \( \Delta(z) \subseteq \Delta \) such that \( g^{-1}(\{z\}) \subseteq \bigcup\{U_a : a \in \Delta(z)\} \). For each \( z \in Z \), put \( U(z) = \bigcup\{U_a : a \in \Delta(z)\} \) and \( V(z) = Z - g(Y - U(z)) \). Since \( g \) is \( N \)-closed, then for each \( z \in Z \), \( V(z) \) is \( N \)-open in \((Z, \delta)\) with \( z \in Z \) and \( g^{-1}(V(z)) \subseteq U(z) \). Since \( V(z) \) is \( N \)-open in \((Z, \delta)\), there exist an open set \( W(z) \) in \((Z, \delta)\) and a finite subset \( F(z) \) of \( Z \) such that \( z \in W(z) \) and \( W(z) - V(z) \) is finite. For each \( z \in Z \), we have \( W(z) \subseteq (W(z) - V(z)) \cup U(z) \) and so

\[
g^{-1}(W(z)) \subseteq g^{-1}((W(z) - V(z)) \cup U(z))
\]

Since \( W(z) - V(z) \) is a finite and \( g^{-1}(\{z\}) \) is a compact subset of \((Y, \sigma)\), there exists a finite subset \( \Delta(z) \) of \( \Delta \) such that \( g^{-1}(W(z) - V(z)) \subseteq \bigcup\{U_a : a \in \Delta(z)\} \) and hence

\[
g^{-1}(W(z)) \subseteq \bigcup\{U_a : a \in \Delta(z)\} \cup U(z)
\]

Since \( W(z) - V(z) \) is a finite and \( g^{-1}(\{z\}) \) is a compact subset of \((Y, \sigma)\), there exists a finite subset \( \Delta(z) \) of \( \Delta \) such that \( g^{-1}(W(z) - V(z)) \subseteq \bigcup\{U_a : a \in \Delta(z)\} \) and hence

\[
g^{-1}(W(z)) \subseteq \bigcup\{U_a : a \in \Delta(z)\} \cup U(z)
\]

This shows that \((Y, \sigma)\) is compact.

3. \( \sigma_N \)-closure operator and \( \sigma_N \)-closed sets

In this section, we define and investigate \( \sigma_N \)-closure operator as a new topological operator which lies between the \( \theta \)-closure and the \( \theta \)-closure operators. Via \( \sigma_N \)-closure, we define \( \sigma_N \)-closed sets and \( \sigma_N \)-open sets as two new classes of sets. We introduce several properties and relationships related to \( \sigma_N \)-closure and \( \sigma_N \)-closed sets.

We start by the main definition of this paper.

Definition 3.1. Let \( B \) be a subset of a ts \((Y, \sigma)\), the set

\[
\{ y \in Y : \forall \delta \subseteq Y \text{ such that } y \in V \}
\]

will be called the \( \sigma_N \)-closure of \( B \) and will be denoted by \( C\sigma_N(B) \). \( B \) is called \( \sigma_N \)-closed if \( C\sigma_N(B) = B \). Complements of \( \sigma_N \)-closed sets will be called \( \sigma_N \)-open sets.

The collection of the \( \sigma_N \)-open sets of a ts \((Y, \sigma)\) will be denoted by \( \sigma_N \).

The following theorem shows the \( \sigma_N \)-closure lies between \( \theta \)-closure and \( \theta \)-closure, also it shows that the class of \( \sigma_N \)-closed sets lies between the class of \( \theta \)-closed sets and the class of \( \theta \)-closed:

Theorem 3.2. Let \( B \) be a subset of a ts \((Y, \sigma)\). Then

a. The \( \sigma_N \)-closure of \( B \) is contained in \( \theta \)-closure of \( B \).

b. The \( \theta \)-closure of \( B \) is contained in \( \theta \)-closure of \( B \).

c. Every \( \theta \)-closed set is \( \theta \)-closed.

d. Every \( \theta \)-closed set is \( \theta \)-closed.

Proof. a. Let \( y \in C\sigma_N(B) \), then for any \( V \in \sigma \) we have \( V \cap B \neq \emptyset \), and by Proposition 2.12, \( V \cap B \neq \emptyset \). It follows that \( y \in C\sigma_N(B) \).

b. Similar to the proof of (a).
c. Let $M$ be $\theta$-closed, then $\text{Cl}_\theta(M) = M$. Thus by (b), $\text{Cl}_\theta(\omega)(M) = M$. So, $M$ is $\theta_\omega$-closed.

d. Let $M$ be $\theta_\omega$-closed, then $\text{Cl}_\theta(\omega)(M) = M$. Thus by (a), $\text{Cl}_\theta(\omega)(M) = M$, and hence $M$ is $\theta_\omega$-closed. □

The following definition will be used throughout this paper:

**Definition 3.3.** We call a ts $\mathcal{N}$-locally indiscrete if it's open sets are $\mathcal{N}$-closed.

The following result introduces relationships between $\mathcal{N}$-local indiscreteness and some other known topological concepts:

**Theorem 3.4.** a. Locally indiscrete ts's are $\mathcal{N}$-locally indiscrete.

b. $\mathcal{N}$-locally indiscrete ts's are $\omega$-locally indiscrete.

c. Locally finite ts's are $\mathcal{N}$-locally indiscrete.

**Proof.** The proofs of (a) and (b) follow directly from the definitions and Proposition 2.10 (b).

(c) If $(Y, \sigma)$ is locally finite, then by Proposition 2.14 (d), $\tau_{\mathcal{N}}$ is the discrete topology and thus $(Y, \sigma)$ is $\mathcal{N}$-locally indiscrete. □

In the following example, we show that each of the implications in parts (a) and (c) of the above theorem is not reversible:

**Example 3.5.** Let $Y = \mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, \{2\}\}$. By Proposition 2.10 (d), $\{2\}$ is $\mathcal{N}$-closed and so $(Y, \sigma)$ is $\mathcal{N}$-locally indiscrete. Also, $(Y, \sigma)$ is not locally indiscrete because $\{2\}$ is not closed but open. Finally, it is not difficult to check that $(Y, \sigma)$ is not locally finite.

In the following example, we show that the implication in part (b) of Theorem 3.4 is not reversible:

**Example 3.6.** Let $Y = \mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{N}\}$. $(Y, \sigma)$ is not $\mathcal{N}$-locally indiscrete because $\mathbb{N}$ is not $\mathcal{N}$-closed but open. Also, it is clear that $(Y, \sigma)$ is $\omega$-locally indiscrete.

In the following result, we show that $\mathcal{N}$-local indiscreteness is a sufficient condition for the equivalence between several closure operators:

**Theorem 3.7.** For any subset $B$ of an $\mathcal{N}$-locally indiscrete ts $(Y, \sigma)$, we have the following:

a. $\overline{B}^\mathcal{N} = \overline{\overline{B}} = \overline{B}$.

b. $\overline{B} = \text{Cl}_{\theta_\omega}(B) = \text{Cl}_{\theta_\omega}(B)$.

**Proof.** a. By definition $\mathcal{N}$-local indiscreteness, we have $\overline{B}^\mathcal{N} = \overline{B}$. Also, by Theorem 3.4 (b) and definition of $\omega$-local indiscreteness we have $\overline{B} = \overline{B}$.

b. By Proposition 2.12 (b) and Theorem 3.2 (a), $\overline{B} \subseteq \text{Cl}_{\theta_\omega}(B) \subseteq \text{Cl}_{\theta_\omega}(B)$. Showing that $\text{Cl}_{\theta_\omega}(B) \subseteq \overline{B}$ will end the proof, let $y \in \text{Cl}_{\theta_\omega}(B)$, $V \in \sigma$ with $y \in V$. Then $B \cap V^\mathcal{N} \neq \emptyset$. By (a), $V^\mathcal{N} = V$ and hence $B \cap V \neq \emptyset$. It follows that $y \in \overline{B}$. □

The following three corollaries follow directly from Theorems 3.4 and 3.7:

**Corollary 3.8.** For any subset $B$ of a locally indiscrete ts $(Y, \sigma)$, we have the following:

a. $B = \overline{B} = \overline{B}^\mathcal{N}$.

b. $\text{Cl}_{\theta_\omega}(B) = \overline{B} = \text{Cl}_{\theta_\omega}(B)$.

**Corollary 3.9.** For any subset $B$ of a locally finite ts $(Y, \sigma)$, we have the following:

a. $B = \overline{B} = \overline{B}^\mathcal{N}$.

b. $\text{Cl}_{\theta_\omega}(B) = \overline{B} = \text{Cl}_{\theta_\omega}(B)$.

**Corollary 3.10.** For any subset $B$ of an $\mathcal{N}$-locally indiscrete ts $(Y, \sigma)$, the following are equivalent:

a. $B$ is $\theta_\omega$-closed in $(Y, \sigma)$.

b. $B$ is $\theta_\omega$-closed in $(Y, \sigma)$.

c. $B$ is closed in $(Y, \sigma)$.

**Proof.** Follows from Corollary 3.8. □

**Corollary 3.11.** For any subset $B$ of a locally indiscrete ts $(Y, \sigma)$, the following are equivalent:

a. $B$ is $\theta_\omega$-closed in $(Y, \sigma)$.

b. $B$ is $\theta_\omega$-closed in $(Y, \sigma)$.

c. $B$ is closed in $(Y, \sigma)$.

**Proof.** Follows from Corollary 3.9. □

The following theorem shows that anti-locally countable is a sufficient condition for the equivalence between the new operator $\theta_{\mathcal{N}}$-closure and the known operators $\theta$-closure and $\theta_\omega$-closure:

**Theorem 3.13.** For any subset $B$ of an anti-locally countable ts $(Y, \sigma)$, $\text{Cl}_{\theta}(B) = \text{Cl}_{\theta_{\mathcal{N}}}(B) = \text{Cl}_{\theta_\omega}(B)$.

**Proof.** Follows from Proposition 2.15 (a) and Theorem 3.2 (a) and (b). □

**Corollary 3.14.** For any subset $B$ of an anti-locally countable ts $(Y, \sigma)$, the following are equivalent:

a. $B$ is $\theta$-closed in $(Y, \sigma)$.

b. $B$ is $\theta_{\mathcal{N}}$-closed in $(Y, \sigma)$.

c. $B$ is $\theta_\omega$-closed in $(Y, \sigma)$.

**Proof.** Follows from Proposition 2.15 (b) and Theorem 3.2 (c) and (d). □

The following theorem shows that each of anti-locally finite and $T_1$ properties is a sufficient condition for the equivalence between the $\theta_{\mathcal{N}}$-closure and the $\theta$-closure:

**Theorem 3.15.** For any subset $B$ of a ts $(Y, \sigma)$ that is anti-locally finite or $T_1$, $\text{Cl}_{\theta}(B) = \text{Cl}_{\theta_{\mathcal{N}}}(B)$.

**Proof.** Assume that $(Y, \sigma)$ is anti-locally finite and let $B \subseteq Y$. By Theorem 3.2 (b), $\text{Cl}_{\theta_{\mathcal{N}}}(B) \subseteq \text{Cl}_{\theta}(B)$. To show that $\text{Cl}_{\theta}(B) \subseteq \text{Cl}_{\theta_{\mathcal{N}}}(B)$, let $y \in \text{Cl}_{\theta}(B)$ and $V \in \sigma$ with $y \in V$. Then $B \cap V^\mathcal{N} \neq \emptyset$. By Proposition 2.14 (c), $V^\mathcal{N} = V$. Thus, $B \cap V^\mathcal{N} \neq \emptyset$ and hence $y \in \text{Cl}_{\theta_{\mathcal{N}}}(B)$.
Assume that \((Y, \sigma)\) is \(T_1\), and let \(B \subseteq Y\). Then by Proposition 2.10 (c), \(\sigma = \sigma_N\) and so \(\overline{V}^N = \overline{V}^N\) for any \(V \subseteq Y\). Thus, similar way to that used in the above case we can show that \(\text{Cl}_{\theta}(B) = \text{Cl}_{\theta_N}(B)\). □

**Corollary 3.16.** Let \((Y, \sigma)\) be anti-locally finite or \(T_1\), and \(B \subseteq Y\). Then \(B\) is \(\theta_N\)-closed in \((Y, \sigma)\) if and only if \(B\) is \(\theta\)-closed in \((Y, \sigma)\).

The following theorem introduces the main properties of the \(\theta_N\)-closure operator:

**Theorem 3.17.** Let \(B, C\) be two subsets of \(a\) ts \((Y, \sigma)\). Then

- a. \(\text{Cl}_{\theta_N}(B) \subseteq \text{Cl}_{\theta_N}(C)\) whenever \(B \subseteq C \subseteq Y\).
- b. \(\text{Cl}_{\theta_N}(B \cup C) = \text{Cl}_{\theta_N}(B) \cup \text{Cl}_{\theta_N}(C)\).
- c. \(\text{Cl}_{\theta_N}(B)\) is closed in \((Y, \sigma)\).
- d. \(\text{Cl}_{\theta}(B) = \text{Cl}_{\theta_N}(B)\) for all \(B \in \sigma_N\).
- e. \(\text{Cl}_{\theta}(B) = \text{Cl}_{\theta_N}(B) = \overline{B}^\sigma\) for all \(B \in \sigma\).

**Proof.** Let \(B \subseteq C \subseteq Y\). We have \(V \in \text{Cl}_{\theta_N}(B)\) if and only if \(\exists y \in V\), \(y \in \text{Cl}_{\theta_N}(B)\). Then \(V = \bigcup_{y \in V} \text{Cl}_{\theta_N}(B)\). This shows the main property of \(\theta_N\)-closed sets in the following theorem:

**Theorem 3.18.** Let \((Y, \sigma)\) be a ts. Then

- a. \(\emptyset\) and \(Y\) are \(\theta_N\)-closed sets.
- b. For any two \(\theta_N\)-closed sets \(B, C\) in \((Y, \sigma)\), \(B \cup C\) is \(\theta_N\)-closed.
- c. For any family \(\{B_a : a \in A\}\) of \(\theta_N\)-closed sets in \((X, \sigma)\), \(\bigcap\{B_a : a \in A\}\) is \(\theta_N\)-closed.

**Proof.** (a) Obvious.

(b) If \(B\) and \(C\) are \(\theta_N\)-closed in \((Y, \sigma)\), then \(B \cup C = \text{Cl}_{\theta}(B) \cup \text{Cl}_{\theta}(C)\) and \(\text{Cl}_{\theta}(B) = \text{Cl}_{\theta}(B) \cap \text{Cl}_{\theta}(C)\) which shows that \(B \cup C\) is \(\theta_N\)-closed.

(c) Let \(\{B_a : a \in A\}\) be a family of \(\theta_N\)-closed sets in \((Y, \sigma)\), then for all \(a \in A\), \(\text{Cl}_{\theta}(B_a) = \overline{B_a} = B_a\). It is sufficient to show that \(\text{Cl}_{\theta}(\bigcap\{B_a : a \in A\}) = \bigcap\{B_a : a \in A\}\). Let \(y \in \text{Cl}_{\theta}(\bigcap\{B_a : a \in A\})\) and let \(V \in \sigma\) such that \(y \in V\). Then \(\overline{V}^N \cap (\bigcap\{B_a : a \in A\}) \neq \emptyset\) and so \(\overline{V}^N \cap B_a \neq \emptyset\) for all \(a \in A\). Therefore, \(y \in (\bigcap\{B_a : a \in A\}) \cap \bigcap\{B_a : a \in A\}\). □

4. The topology of \(\theta_N\)-open sets

In this section, for a given ts \((Y, \sigma)\) we will show that the family \(\sigma_{\theta_N}\) forms a topology on \(Y\) which lies strictly between \(\sigma_\theta\) and \(\sigma_{\theta_N}\) and \(\sigma\), and we shall also give three mapping theorems.

The following result shows that the collection of \(\theta_N\)-open sets in a ts forms a topology:

**Theorem 4.1.** If \((Y, \sigma)\) is a ts, then \((Y, \sigma_{\theta_N})\) is a ts.

**Proof.** By Theorem 3.18 (a), we have \(\emptyset, Y \in \sigma_{\theta_N}\). If \(B, C \in \sigma_{\theta_N}\), then by Theorem 3.18 (b), \(Y = (B \cap C) = (Y - B) \cup (Y - C)\) is \(\theta_N\)-closed, and so \(B \cap C \in \sigma_{\theta_N}\). Hence \(Y - \emptyset = \emptyset\) and \(Y - \sigma_{\theta_N}\) is a family of \(\theta_N\)-closed sets in \((Y, \sigma)\) and by Theorem 3.18 (c) \(\cap\{Y - B_a : a \in A\} = Y - \bigcup\{B_a : a \in A\}\) is \(\theta_N\)-closed, hence \(\bigcup\{B_a : a \in A\} \in \sigma_{\theta_N}\). □

The following theorem gives the relationships between the new topology introduced in Theorem 4.2 and some other known topologies:

**Theorem 4.2.** For any ts \((Y, \sigma)\), \(\sigma_\theta \subseteq \sigma_{\theta_N} \subseteq \sigma_{\theta_N} \subseteq \sigma\).

**Proof.** To show that \(\sigma_\theta \subseteq \sigma_{\theta_N}\), let \(B \in \sigma_\theta\). Then by Theorem 3.2 (c), \(Y - B\) is \(\theta_N\)-closed and so \(B \in \sigma_{\theta_N}\). To show that \(\sigma_{\theta_N} \subseteq \sigma_{\theta_N}\), let \(B \in \sigma_{\theta_N}\). Then \(Y - B\) is \(\theta_N\)-closed and by Theorem 3.2 (d), \(Y - B\) is \(\theta_N\)-closed and so \(B \in \sigma_{\theta_N}\). For \(\sigma_{\theta_N} \subseteq \sigma\), just see Proposition 2.18.

Examples 4.10 and 4.11 will show that \(\sigma_{\theta_N} \neq \sigma_\theta\) and \(\sigma_{\theta_N} \neq \sigma_{\theta_N}\) in general. However, Theorems 4.3 and 4.5, 4.6, give results discussed the equality between two of \(\sigma_{\theta_N}, \sigma_\theta, \sigma_{\theta_N}\) under some conditions.

**Theorem 4.3.** For any \(\theta_N\)-locally indiscrete ts \((Y, \sigma)\), \(\sigma = \sigma_{\theta_N} = \sigma_{\theta_N}\).

**Proof.** Follows from Corollary 3.10. □

**Corollary 4.4.** For any locally indiscrete ts \((Y, \sigma)\), \(\sigma = \sigma_{\theta_N} = \sigma_{\theta_N}\).

**Proof.** Follows from Corollary 3.11. □

**Corollary 4.5.** For any locally finite ts \((Y, \sigma)\), \(\sigma = \sigma_{\theta_N} = \sigma_{\theta_N}\).

**Proof.** Follows from Corollary 3.12. □

**Theorem 4.6.** a. For any anti-locally countable ts \((Y, \sigma)\), \(\sigma_{\theta_N} = \sigma_{\theta_N} = \sigma_{\theta_N}\).
   b. If \((Y, \sigma)\) is anti-locally finite or \(T_1\), then \(\sigma_{\theta_N} = \sigma_\theta\).

**Proof.** a. Follows from Corollary 3.14. □
   b. Follows from Corollary 3.16. □

Now we characterize \(\theta_N\)-open sets.

**Theorem 4.7.** Let \((Y, \sigma)\) be a ts and \(B \subseteq Y\). Then \(B\) is \(\theta_N\)-open in \((Y, \sigma)\) if and only if for each \(y \in B\), there exists an open set \(V\) in \((Y, \sigma)\) containing \(y\) such that \(\overline{V}^N \subseteq B\).

**Proof.** Necessity. Suppose that \(B\) is \(\theta_N\)-open and let \(y \in B\). Then \(X - B\) is \(\theta_N\)-closed and \(y \notin X - B = \text{Cl}_{\theta}(Y - B)\). Choose and open set \(V\) such that \(y \in V\) and \(\overline{V}^N \cap (Y - B) = \emptyset\). So, \(\overline{V}^N \subseteq B\).

**Sufficiency.** Suppose to the contrary that \(\emptyset \notin \sigma_{\theta_N}\), then there is \(y \in \text{Cl}_{\theta}(Y - B)\). By assumption, we can choose an open set \(V\) containing \(y\) such that \(\overline{V}^N \subseteq B\) and so \(\overline{V}^N \cap (Y - B) = \emptyset\). This shows that \(\emptyset \notin \text{Cl}_{\theta}(Y - B)\), a contradiction. □
Theorem 4.8. Open $\mathcal{N}$-closed sets of a ts $(Y, \sigma)$ are $\sigma_{\mathcal{N}}$-open sets.

Proof. Let $B$ be open and $\mathcal{N}$-closed set in $(Y, \sigma)$. We apply Theorem 4.7. Let $y \in B$, then $B \in \sigma_{\mathcal{N}}$ and $B = B \subseteq B$. This shows that $B \in \sigma_{\mathcal{N}}$. \qed

Corollary 4.9. Finite open subsets of a ts $(Y, \sigma)$ are $\sigma_{\mathcal{N}}$-open sets.

Proof. It follows from Theorem 4.8 and Proposition 2.10 (d). \qed

In general, $\sigma_{\mathcal{N}} \neq \sigma_0$.

Example 4.10. Let $Y = [1, 2]$ and $\sigma = \{\emptyset, Y, \{1\}\}$. Then by Corollary 4.5, $\sigma_{\mathcal{N}} = \sigma$. On the other hand, it is clear that $(Y, \sigma)$ is not regular and by Proposition 2.19 (b), it follows that $\sigma_0 \neq \sigma = \sigma_{\mathcal{N}}$.

In general, $\sigma_{\mathcal{N}} \neq \sigma_0$.

Example 4.11. Let $Y = \mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, [a, b], [a, b), (a, b), (a, b], [a, b)\}$. It is proved in [19] that $\sigma_{\mathcal{N}} = \{\emptyset, \mathbb{R}, [a, b], [a, b), (a, b), (a, b], [a, b)\}$ and $\sigma_0 = \{\emptyset, \mathbb{R}, [a, b], [a, b), (a, b), (a, b], [a, b)\}$. On the other hand, since $(Y, \sigma)$ is anti-locally finite, then by Theorem 4.6 (b), $\sigma_{\mathcal{N}} = \sigma_0$.

If $(Y, \sigma)$ and $(Z, \delta)$ are two ts's, then $\sigma \times \delta$ will denote the product topology on $Y \times Z$, also $\sigma_1$ and $\delta_1$ will denote the projection functions on $Y$ and $Z$, respectively.

The following proposition will be used in the next result:

Proposition 4.12. Let $(Y, \sigma)$ and $(Z, \delta)$ be two ts's.

a. $(\sigma \times \delta)_1 \subseteq (\sigma_1 \times \delta_1)$.

b. If $B \subseteq Y$ and $C \subseteq Z$, then $\overline{B \times C}^C \subseteq \overline{B \times C}^\mathcal{N}$.

Proof. a. Let $G \in (\sigma \times \delta)$, and let $(y, z) \in G$. By Proposition 2.9, there are $H \in \sigma$ and a finite subset $F \subseteq Y \times Z$ such that $(y, z) \in H - F \subseteq G$. Choose $V \in \sigma$ and $W \in \delta$ such that $(y, z) \in V \times W \subseteq H$. By Proposition 2.9 we have $V - \pi_1(F) \subseteq \sigma_1$ and $W - \pi_2(F) \subseteq \delta_1$. Also, $(y, z) \in (V - \pi_1(F)) \times (W - \pi_2(F)) \subseteq (V \times W) - (\pi_2(F) \times \pi_1(F)) \subseteq H - F \subseteq G$.

Thus by Proposition 2.9, $G \in (\sigma \times \delta)_1$.

b. Let $(y, z) \in \overline{B \times C}^C$ and let $G \in (\sigma \times \delta)$ such that $(y, z) \in G$. By (a), $G \in (\sigma \times \delta)_1$, so that $(y, z) \in G$. By (a), $G \in (\sigma_1 \times \delta_1)$ such that $(y, z) \in G$. Since $y \in \overline{B \times N} \cap M$, then $B \cap M \neq \emptyset$ (resp. $C \cap N \neq \emptyset$). But $(B \cap M) \times (C \cap N) = (B \times C) \cap (M \times N) \subseteq (B \times C) \cap (M \times N)$.

It follows that $(y, z) \in \overline{B \times C}^\mathcal{N}$. \qed

Theorem 4.13. Let $(Y, \sigma)$ and $(Z, \delta)$ be two ts's. Then the projection functions $\sigma_1 : (Y \times Z, (\sigma \times \delta)_1) \rightarrow (Y, \sigma_1)$ and $\delta_1 : (Y \times Z, (\sigma \times \delta)_1) \rightarrow (Z, \delta_1)$ are open functions.

Proof. Let $M \in (\sigma \times \delta)_1$ and $y \in \sigma_1(M)$. Pick $z \in Z$ such that $(y, z) \in M$. By Theorem 4.7, we find $K \in \sigma \times \delta$ such that $(y, z) \in K \subseteq \overline{K}^\mathcal{N} \subseteq M$. There are $V \in \sigma$ and $W \in \delta$ such that $(y, z) \in V \times W \subseteq K$. So, by Proposition 4.12 (b),

$$(y, z) \in V \times W \subseteq \overline{V \times W}^\mathcal{N} \subseteq \overline{V \times W}^\mathcal{N} \subseteq \overline{K}^\mathcal{N} \subseteq M$$

and so $y \in V \subseteq \overline{V}^\mathcal{N} \subseteq \sigma_1(M)$.

Therefore, $\sigma_1(M) \in (\sigma \times \delta)_1$. It follows that $\sigma_1 : (Y \times Z, (\sigma \times \delta)_1) \rightarrow (Y, \sigma_1)$ is an open function. Similarly, we can show that $\delta_1 : (Y \times Z, (\sigma \times \delta)_1) \rightarrow (Z, \delta_1)$ is an open function. \qed

Remark 4.14. If $g : (Y, \sigma) \rightarrow (Z, \delta)$ is a closed function, then $g : (Y, \sigma) \rightarrow (Z, \delta_1)$ is closed.

The implication in Remark 4.14 is not reversible in general as we can see from the following example:

Example 4.15. Define $g : (\mathbb{R}, \tau_1) \rightarrow (\mathbb{R}, \tau_1)$ by $g(y) = 1$ for all $y \in \mathbb{R}$. Then clearly that $g : (\mathbb{R}, \tau_1) \rightarrow (\mathbb{R}, \tau_1)$ is not a closed function. On the other hand, if $C$ is a closed subset of $(\mathbb{R}, \tau_1)$, then $(C, \tau_C)$ is $\mathcal{N}$-closed. Thus, $g : (\mathbb{R}, \tau_1) \rightarrow (\mathbb{R}, \tau_C)$ is a closed function.

By the end of this section, we introduce three mapping theorems.

Theorem 4.16. If $g : (Y, \sigma) \rightarrow (Z, \delta)$ is an open function such that $g : (Y, \sigma_1) \rightarrow (Z, \delta_1)$ is a closed function, then $g : (Y, \sigma_1) \rightarrow (Z, \delta_2)$ is an open function.

Proof. Let $B \in \sigma_1$. Let $z \in g(B)$. Pick $y \in B$ with $z = g(y)$. By Proposition 2.19 (a), we find $V \in \sigma$ such that $y \in V$ and $\overline{V} \subseteq B$. By assumptions, $z \in g(V) \in \delta$ and $g(V) \subseteq \overline{M}$, so $g(V) \subseteq g(B) = B$. By Theorem 4.7 it follows that $g(B) \in \delta_2$. \qed

Theorem 4.17. If $g : (Y, \sigma) \rightarrow (Z, \delta)$ is an open function such that $g : (Y, \sigma_1) \rightarrow (Z, \delta_1)$ is a closed function, then $g : (Y, \sigma_1) \rightarrow (Z, \delta_2)$ is an open function.

Proof. Let $B \in \sigma_1$. Let $z = g(y) \in g(B)$ with $y \in B$. By Theorem 4.7, there exists $O \in \sigma$ such that $y \in O \subseteq \overline{O} \subseteq B$ and so $z \in g(O) \subseteq g(O) = B$. Since $g(O) \subseteq \overline{O}$ is open, then $g(O) \subseteq \overline{O}$. Since $g(y) = g(O) \subseteq \overline{O}$ is closed, then $g(O) \subseteq \overline{O}$ is $\mathcal{N}$-closed. Thus, $g(O) \subseteq \overline{O}$. Hence, $g(B) \in \delta_2$. \qed

Theorem 4.18. If $g : (Y, \sigma) \rightarrow (Z, \delta)$ is a continuous function such that $g : (Y, \sigma_1) \rightarrow (Z, \delta_1)$ is also continuous, then $g : (Y, \sigma_1) \rightarrow (Z, \delta_2)$ is continuous.

Proof. Let $H \in \delta_2$ and let $y \in g^{-1}(H)$. Then $g(y) \in H$, and so by Theorem 4.7 there exists $O \in \delta$ such that $g(y) \in O \subseteq \overline{O} \subseteq H$. Thus, we have $y \in g^{-1}(O) \subseteq \overline{g^{-1}(O)} \subseteq \overline{g^{-1}(H)}$. By assumptions, $g^{-1}(O) \in \sigma$ and $g^{-1}(O) \subseteq \overline{g^{-1}(O)} \subseteq \overline{g^{-1}(H)}$. It follows that $g^{-1}(H) \in \sigma_1$. \qed

5. Separation axioms

In this section, we will obtain a new characterization of $T_2$ ts's via $\delta_1$-closure operator. Also, we will introduce $\mathcal{N}$-regularity as a new...
separation axiom. We will introduce several implications and characterizations of $\mathcal{N}$-regularity.

We start by defining $\mathcal{N}$-T$_2$ ts’s.

**Definition 5.1.** A ts $(Y, \sigma)$ is called $\mathcal{N}$-T$_2$ if for every $y, z \in Y$ with $y \neq z$ there exists an open set $V$ containing $y$ and an $\mathcal{N}$-open set $W$ containing $z$ such that $V \cap W = \emptyset$.

Now, we characterize $\mathcal{N}$-T$_2$ ts’s.

**Theorem 5.2.** A ts $(Y, \sigma)$ is $\mathcal{N}$-T$_2$ if and only if for each $y \in Y$, $Cl_{\sigma, Y}^{\mathcal{N}}(\{y\}) = \{y\}$.

**Proof.** Necessity. Suppose that $(Y, \sigma)$ is $\mathcal{N}$-T$_2$ and suppose to the contrary that there $y, z \in Y$ such that $z \in Cl_{\sigma, Y}^{\mathcal{N}}(\{y\}) - \{y\}$. Then there exist $V \in \sigma$ and $W \in \sigma$ such that $y \in V \subseteq W$, and $V \cap W = \emptyset$. Since $z \in W$, $z \in Cl_{\sigma, Y}^{\mathcal{N}}(\{y\})$, then $\overline{W} \subseteq \{y\} \neq \emptyset$. Therefore, we have $y \in V \subseteq \sigma$, and $y \in \overline{V}$ which implies that $V \cap W \neq \emptyset$, a contradiction.

Sufficiency. Let $y, z \in Y$ with $y \neq z$. Then by assumption, $Cl_{\sigma, Y}^{\mathcal{N}}(\{y\}) = \{y\}$ and $y \notin Cl_{\sigma, Y}^{\mathcal{N}}(\{z\})$. So, there exists $V \in \sigma$ such that $y \in V \cap \overline{V} \cap \{z\} = \emptyset$. Put $W = Y - \overline{V}$. Then we have $y \in W \cap \sigma$, and $V \cap W = \emptyset$. It follows that $(Y, \sigma)$ is $\mathcal{N}$-T$_2$.

**Theorem 5.3.** Every $\mathcal{N}$-T$_2$ ts is $\omega$-T$_2$.

**Proof.** Follows because for any ts $(Y, \sigma)$, $\sigma \subseteq \sigma, \omega$.

**Corollary 5.4.** Every $\mathcal{N}$-T$_2$ ts is $T_1$.

**Proof.** Follows from Theorem 5.3 and Proposition 2.21 (b).

$\mathcal{N}$-T$_2$ and $T_2$ topological properties are equivalent:

**Theorem 5.5.** A ts $(Y, \sigma)$ is $\mathcal{N}$-T$_2$ if and only if it is $T_2$.

**Proof.** Necessity. Assume $(Y, \sigma)$ is $\mathcal{N}$-T$_2$. Let $y, z \in Y$ with $y \neq z$. Then by the definition of $\mathcal{N}$-T$_2$, there are $V \in \sigma, W \in \sigma, \chi$ such that $y \in \chi \subseteq X$, and $V \cap W = \emptyset$. Since $(Y, \sigma)$ is $\mathcal{N}$-T$_2$, then by Corollary 5.4 $(Y, \sigma)$ is $T_2$. So, by Proposition 2.10 (c) $\sigma = \sigma, \omega$. It follows that $(Y, \sigma)$ is $T_2$.

Sufficiency. Obvious.

The following nice characterization of $T_2$ ts’s is one of the main results in this section:

**Corollary 5.6.** A ts $(Y, \sigma)$ is $T_2$ if and only if $Cl_{\sigma, Y}^{\mathcal{N}}(\{y\}) = \{y\}$ for all $y \in Y$.

**Proof.** Follows from Theorems 5.2 and 5.5.

Now, we define $\mathcal{N}$-regular ts’s.

**Definition 5.7.** A topological space $(Y, \sigma)$ is called $\mathcal{N}$-regular if given any closed set $C$ in $(Y, \sigma)$ and any point $y$ that does not belong to $C$, there are $V \in \sigma$ and $W \in \sigma, \chi$ such that $y \in V \subseteq \chi \subseteq W$ and $V \cap W = \emptyset$.

Theorems 5.8 and 5.9 introduce characterizations of $\mathcal{N}$-regular ts’s.

**Theorem 5.8.** Let $(Y, \sigma)$ be a ts. Then $(Y, \sigma)$ is $\mathcal{N}$-regular if and only if for each $V \in \sigma$ and each $y \in V$ there is $W \in \sigma$ such that $y \in W \subseteq \overline{W} \subseteq V$.

**Proof.** Necessity. Suppose that $(Y, \sigma)$ is $\mathcal{N}$-regular. Let $V \in \sigma$ and let $y \in V$. Then $y - V$ is closed in $(Y, \sigma)$ and $y \notin Y - V$. By $\mathcal{N}$-regularity, there exist $W \in \sigma$ and $H \in \sigma, \chi$ such that $y \in W \subseteq Y - V \subseteq H$, and $W \cap H = \emptyset$. Consequently, $W \in \sigma$ and $y \in W \subseteq \overline{W} \subseteq Y = H \subseteq V$ which completes the proof.

Sufficiency. Let $C$ be closed in $(Y, \sigma)$ and $y \in Y - C$. Since $y \in Y - C \subseteq \sigma$, then by assumption there is $W \in \sigma$ such that $x \in W \subseteq \overline{W} \subseteq Y - C$. Put $H = Y - \overline{W}$. Then we have $W \in \sigma$, $H \in \sigma, \chi$, $y \in W, C \subseteq Y - \overline{W} = H$, and $W \cap H = W \cap (Y - \overline{W}) = \emptyset$. It follows that $(Y, \sigma)$ is $\mathcal{N}$-regular.

**Theorem 5.9.** The following are equivalent for any ts $(Y, \sigma)$:

a. $(Y, \sigma)$ is $\mathcal{N}$-regular.

b. $\sigma = \sigma, \omega$.

c. $Cl_{\sigma, Y}^{\mathcal{N}}(B) = \overline{B}$, for every $B \subseteq Y$.

**Proof.** (a) $\implies$ (b): Suppose that $(Y, \sigma)$ is $\mathcal{N}$-regular. To see that $\sigma \subseteq \sigma, \omega$, let $B \in \sigma$ and let $y \in B$. Then by Theorem 5.8, there is $V \in \sigma$ such that $y \in V \subseteq \overline{V} \subseteq B$. By Theorem 4.7, it follows that $B \in \sigma, \omega$. This ends the proof that $\sigma \subseteq \sigma, \omega$. On the other hand, by Theorem 4.2 we have $\sigma, \omega \subseteq \sigma$.

(b) $\implies$ (c): Let $B \subseteq Y$. By Proposition 2.12 and Theorem 3.2 (a), $\overline{B} \subseteq Cl_{\sigma, Y}^{\mathcal{N}}(B)$. To see that $Cl_{\sigma, Y}^{\mathcal{N}}(B) \subseteq \overline{B}$, we show that $Y - B \subseteq Y - Cl_{\sigma, Y}^{\mathcal{N}}(B)$. Let $y \in Y - B$. Since $Y - B \in \sigma$, (by (b)) $y \in \sigma, \omega$. By Theorem 4.7, there is $V \in \sigma$ such that $y \in V \subseteq \overline{V} \subseteq Y - B$. Thus we have $y \in V \subseteq \sigma$ and $\overline{V} \cap B \subseteq \overline{V} \cap B = \emptyset$. This implies that $y \in Y - Cl_{\sigma, Y}^{\mathcal{N}}(B)$.

(c) $\implies$ (a): Suppose for each subset $B \subseteq Y$, $Cl_{\sigma, Y}^{\mathcal{N}}(B) = \overline{B}$. Let $C$ be closed in $(Y, \sigma)$ and $y \in Y - C$. By (c), we have $Cl_{\sigma, Y}^{\mathcal{N}}(C) = \overline{C} = C$. Thus, $y \in Y - Cl_{\sigma, Y}^{\mathcal{N}}(C)$ and so there exists $H \in \sigma$ such that $y \in H$ and $\overline{V} \cap C = \emptyset$. Set $W = Y - \overline{V}$. Then $W \in \sigma, \chi$ with $C \subseteq W$ and $W \cap H = (Y - \overline{V}) \cap H = \emptyset$. It follows that $(Y, \sigma)$ is $\mathcal{N}$-regular.

In the rest of this section, we will study the validity of several implications related to $\mathcal{N}$-regularity.

**Corollary 5.10.** $\mathcal{N}$-locally indiscrete ts’s are $\mathcal{N}$-regular.

**Proof.** Follows from Theorems 3.7 and 5.9.

**Corollary 5.11.** Locally indiscrete ts’s are $\mathcal{N}$-regular.

**Proof.** Follows from Corollary 5.10 and Theorem 3.4 (a).

**Corollary 5.12.** Locally finite ts’s are $\mathcal{N}$-regular.

**Proof.** Follows from Corollary 5.10 and Theorem 3.4 (c).

**Theorem 5.13.** a. Regular ts’s are $\mathcal{N}$-regular.

b. $\mathcal{N}$-regular ts’s are $\omega$-regular.

c. $\mathcal{N}$-regular $T_1$ ts’s are regular.

d. Every $\mathcal{N}$-regular anti-locally finite ts is regular.

**Proof.** a. Follows from the definitions and Proposition 2.10 (a).
b. Follows from the definitions and Proposition 2.10 (a).
c. Follows from the definitions and Proposition 2.10 (c).
d. Follows from Proposition 2.19 (b) and Theorems 4.6 (b) and 5.9.
The implication in Theorem 5.13 (a) is not reversible: Consider \((Y, \sigma)\) as in Example 4.10. Since \(\sigma_{\omega} = \sigma\) and \(\sigma_\theta \neq \sigma\), then by Theorem 5.9 and Proposition 2.19 (b) \((Y, \sigma)\) is \(\mathcal{N}\)-regular but not regular.

Authors in [19] gave \((\mathbb{N}, \tau_{cof})\) as an example on an \(\omega\)-regular topological space that is not regular. On the other hand, since \((\mathbb{N}, \tau_{cof})\) is \(T_1\) but not regular, then by Theorem 5.13 (c) it is not \(\mathcal{N}\)-regular. Therefore, the implication in Theorem 5.13 (b) is not reversible.

6. \(\theta_\mathcal{N}\)-continuity and \(\mathcal{N}\)-\(\theta\)-continuity

In this section, we will introduce \(\theta_\mathcal{N}\)-continuity and \(\mathcal{N}\)-\(\theta\)-continuity as two new forms of continuity. We will study relationships between them and some other related continuity concepts.

We start by defining \(\theta_\mathcal{N}\)-continuity.

Definition 6.1. A function \(g : (Y, \sigma) \rightarrow (Z, \delta)\) is called \(\theta_\mathcal{N}\)-continuous if for every \(y \in Y\) and every \(W \in \delta\) such that \(g(y) \in W\), there exists \(V \in \sigma\) such that \(y \in V\) and \(g(V) \subseteq \overline{W}^\mathcal{N}\).

The following theorem and Examples 6.9 and 6.10 will show that \(\theta_\mathcal{N}\)-continuity lies strictly between \(\theta_\omega\)-continuity and \(\theta\)-continuity.

Theorem 6.2. a. \(\theta_\omega\)-continuous functions are \(\theta_\mathcal{N}\)-continuous.
   b. \(\theta_\mathcal{N}\)-continuous functions are \(\theta\)-continuous.

Proof. a. Let \(g : (Y, \sigma) \rightarrow (Z, \delta)\) be a \(\theta_\omega\)-continuous function. Let \(y \in Y\) and \(W \in \delta\) such that \(g(y) \in W\). By \(\theta_\omega\)-continuity of \(g\), we find \(V \in \sigma\) such that \(y \in V\) and \(g(V) \subseteq \overline{W}\). This shows that \(g\) is \(\theta_\mathcal{N}\)-continuous.

   b. Let \(g : (Y, \sigma) \rightarrow (Z, \delta)\) be a \(\theta_\mathcal{N}\)-continuous function. Let \(y \in Y\) and \(W \in \delta\) such that \(g(y) \in W\). By \(\theta_\mathcal{N}\)-continuity of \(g\), we find \(V \in \sigma\) such that \(y \in V\) and \(g(V) \subseteq \overline{W}\). This shows that \(g\) is \(\theta\)-continuous.

The following two theorems show that \(\theta\)-continuity and \(\theta_\mathcal{N}\)-continuity are equivalent for functions whose co-domain is anti-locally finite or \(T_1\).

Theorem 6.3. If \(g : (Y, \sigma) \rightarrow (Z, \delta)\) is a \(\theta\)-continuous function with \((Z, \delta)\) is anti-locally finite, then \(g\) is \(\theta_\mathcal{N}\)-continuous.

Proof. Follows from the definitions and Proposition 2.14 (c).

Theorem 6.4. If \(g : (Y, \sigma) \rightarrow (Z, \delta)\) is a \(\theta\)-continuous function with \((Z, \delta)\) is \(T_1\), then \(g\) is \(\theta_\mathcal{N}\)-continuous.

Proof. Let \(y \in Y\) and \(W \in \delta\) such that \(g(y) \in W\). By \(\theta\)-continuity of \(g\), there is \(V \in \sigma\) such that \(y \in V\) and \(g(V) \subseteq \overline{W}\). Since \((Z, \delta)\) is \(T_1\), then Proposition 2.10 (c), \(\overline{W} = \overline{W}^\mathcal{N}\) and thus \(g(V) \subseteq \overline{W}^\mathcal{N}\). This shows that \(g\) is \(\theta_\mathcal{N}\)-continuous.

The following theorem shows that \(\theta_\mathcal{N}\)-continuity and \(\theta_\omega\)-continuity are equivalent for functions whose co-domain is anti-locally finite \(\mathcal{N}\)-locally indiscrete.

Theorem 6.5. For any function \(g : (Y, \sigma) \rightarrow (Z, \delta)\) with \((Z, \delta)\) is \(\mathcal{N}\)-locally indiscrete, the following statements are equivalent:
   a. \(g\) is \(\theta_\mathcal{N}\)-continuous.
   b. \(g\) is \(\theta_\omega\)-continuous.

Proof. (a) \(\rightarrow\) (b): Follows from Theorem 6.2 (a).

   (b) \(\rightarrow\) (a): Let \(y \in Y\) and \(W \in \delta\) such that \(g(y) \in W\). By \(\theta_\omega\)-continuity of \(g\), we find \(V \in \sigma\) such that \(y \in V\) and \(g(V) \subseteq \overline{W}\). By \((Z, \delta)\) is \(\mathcal{N}\)-locally indiscrete, \(\overline{W} = W\) and thus \(W \subseteq \overline{W} \subseteq \overline{W}^\mathcal{N}\). Therefore, \(\overline{W} \subseteq \overline{W}^\mathcal{N}\) and hence \(g(V) \subseteq \overline{W}^\mathcal{N}\). Therefore, \(g\) is \(\theta_\omega\)-continuous. \(\square\)

Corollary 6.6. For any function \(g : (Y, \sigma) \rightarrow (Z, \delta)\) with \((Z, \delta)\) is locally indiscrete, the following statements are equivalent:
   a. \(g\) is \(\theta_\omega\)-continuous.
   b. \(g\) is \(\theta_\mathcal{N}\)-continuous.

Proof. Follows from Theorem 6.5 and Theorem 3.4 (a).

Corollary 6.7. For any function \(g : (Y, \sigma) \rightarrow (Z, \delta)\) with \((Z, \delta)\) is locally finite, the following statements are equivalent:
   a. \(g\) is \(\theta_\mathcal{N}\)-continuous.
   b. \(g\) is \(\theta_\omega\)-continuous.

Proof. Follows from Theorem 6.5 and Theorem 3.4 (c).

The following theorem shows that \(\theta_\mathcal{N}\)-continuity, \(\theta_\omega\)-continuity and \(\theta\)-continuity are equivalent for functions whose co-domain is anti-locally countable.

Theorem 6.8. For any function \(g : (Y, \sigma) \rightarrow (Z, \delta)\) with \((Z, \delta)\) is anti-locally countable, the following statements are equivalent:
   a. \(g\) is \(\theta_\omega\)-continuous.
   b. \(g\) is \(\theta_\mathcal{N}\)-continuous.
   c. \(g\) is \(\theta\)-continuous.

Proof. (a) \(\rightarrow\) (b) and (b) \(\rightarrow\) (c) are just Theorem 6.2.
   (c) \(\rightarrow\) (a): Follows from Theorem 2.5 of [21]. \(\square\)

\(\theta_\mathcal{N}\)-continuous functions are not \(\theta_\omega\)-continuous, in general as it can be seen from the following example:

Example 6.9. We utilize Example 2.4 of [21]. Consider the function \(g : (\mathbb{N}, \tau_{cof}) \rightarrow (\mathbb{N}, \tau_{cof})\) defined as \(g(x) = x\). It is proved [21] that \(g\) is \(\theta\)-continuous but not \(\theta_\omega\)-continuous. Since \((\mathbb{N}, \tau_{cof})\) is \(T_1\), then by Theorem 6.4 \(f\) is \(\theta_\mathcal{N}\)-continuous.

\(\theta\)-continuity does not imply \(\theta_\mathcal{N}\)-continuity, in general as it can be seen from the following example:

Example 6.10. Let \(X = \{1, 2\}\) and \(\sigma = \{\emptyset, X, \{1\}\}\). Consider the function \(g : (X, \tau_{cof}) \rightarrow (X, \sigma)\) defined by \(g(x) = x\). Then
   a. \(g\) is \(\theta\)-continuous.
   b. \(g\) is not \(\theta_\mathcal{N}\)-continuous.

Proof. a. Let \(x \in X\) and \(V \in \sigma\) such that \(g(x) \in V\). If \(x = 1\) and \(V = \{1\}\), then choose \(U = X\), so \(U \in \tau_{cof}\), \(x \in U\), and \(g(U) = f(X) = X = \overline{V}\). The other cases where \(V = X\) in each of them are trivial.

   b. Suppose that \(g\) is \(\theta_\mathcal{N}\)-continuous. Let \(x = 1\) and \(V = \{1\}\). Then \(V \in \sigma\) with \(g(1) = 1 \in V\) and so there exists \(U \in \tau_{cof}\) such that \(1 \in U \in \tau_{cof}\) and \(g(U) \subseteq \overline{V}\). Since \((X, \sigma)\) is locally finite, then \(\overline{V} = V\). Also, it is clear that \(U = X\). Thus we have \(X = g(U) = \overline{U} \subseteq \overline{V} = \{1\}\) and hence \(X = \{1\}\), a contradiction. \(\square\)
Authors in [21] introduced a $\omega$-continuous function (and by Theorem 6.2 (a) it is $\alpha$-continuous) that is not continuous. So, $\alpha$-continuity does not imply continuity in general. Moreover, the following example shows that continuity does not imply $\theta$-continuity, in general:

**Example 6.11.** Let $Y = \{1, 2\}$ and $\sigma = \{\emptyset, Y, \{1\}\}$. Define $g : (Y, \sigma) \to (Y, \sigma)$ by $g(\{y\}) = y$. Then $g$ is continuous. If $g$ is $\theta$-continuous, then there exists $V \in \sigma$ such that $1 \in V$ and $g(\overline{V}) \subseteq \overline{1} = \{1\}$. But $\overline{V} = Y$ and so $g(\overline{Y}) = f(\overline{1}) \subseteq \overline{\{1\}} = \{1\}$. Therefore, $g$ is not $\theta$-continuous.

We see above that continuity and $\theta$-continuity are independent concepts, in general. However, $\theta$-continuity implies continuity for those functions whose co-domain is $\mathcal{N}$-regular, as the following theorem shows:

**Theorem 6.12.** If $g : (Y, \sigma) \to (Z, \delta)$ is a $\theta$-continuous such that $(Z, \delta)$ is $\mathcal{N}$-regular, then $g$ is continuous.

**Proof.** Let $y \in Y$ and $W \in \delta$ such that $g(y) \in W$. By Theorem 5.8, there is $H \in \delta$ such that $g(y) \in H \subseteq \overline{W} \subseteq W$. By $\theta$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq \overline{H} \subseteq \overline{W}$. Therefore, $g(V) \subseteq g(\overline{V}) \subseteq \overline{W} \subseteq W$.

Hence, $g$ is continuous.

Now, we introduce the next main concept of this section:

**Definition 6.13.** A function $g : (Y, \sigma) \to (Z, \delta)$ is called $\mathcal{N}$-continuous if for every $y \in Y$ and every $W \in \delta$ with $g(y) \in W$, there is $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq \overline{W}$.

**Theorem 6.14.** Remark 6.22, and Example 6.23 will show that $\mathcal{N}$-continuity lies strictly between $\theta$-continuity and $\omega$-continuity.

**Theorem 6.14.**

a. $\mathcal{N}$-continuous functions are $\mathcal{N}$-continuous.

b. $\mathcal{N}$-continuous functions are $\omega$-continuous.

d. $\mathcal{N}$-continuous functions are $\omega$-continuous.

e. $\mathcal{N}$-continuous functions are $\omega$-continuous.

**Proof.** a. Let $g : (Y, \sigma) \to (Z, \delta)$ be a $\theta$-continuous function. Let $y \in Y$ and $W \in \delta$ with $g(y) \in W$. By $\theta$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq \overline{W}$ and thus $g(\overline{V}) \subseteq g(\overline{W}) \subseteq \overline{W}$. Therefore, $g$ is $\mathcal{N}$-continuous.

b. Let $g : (Y, \sigma) \to (Z, \delta)$ be an $\mathcal{N}$-continuous function. Let $y \in Y$ and $W \in \delta$ with $g(y) \in W$. By $\mathcal{N}$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(\overline{V}) \subseteq \overline{W}$ and thus $g(\overline{V}) \subseteq g(\overline{W}) \subseteq \overline{W}$. Therefore, $g$ is $\omega$-continuous.

By Theorems 6.14 (b) and 2.21 of [21], $\mathcal{N}$-continuous functions are weakly continuous. Theorems 6.15 and 6.18 will show that $\mathcal{N}$-continuity, $\omega$-continuity, and weak continuity are equivalent for those functions whose domain is $\mathcal{N}$-locally indiscrete or $\mathcal{N}$-regular:

**Theorem 6.15.** Every weakly continuous function with $\mathcal{N}$-locally indiscrete domain is $\mathcal{N}$-continuous.

**Proof.** Let $g : (Y, \sigma) \to (Z, \delta)$ be a function with $(Y, \sigma)$ is $\mathcal{N}$-locally indiscrete. Let $y \in Y$ and let $W$ with $g(y) \in W$. By weak continuity of $g$, we find $V \in \sigma$ such that $g(V) \subseteq \overline{W}$. By $\mathcal{N}$-local indiscreteness of $(Y, \sigma)$ and Theorem 3.7 (a), $\overline{V} = V$. So, $g(\overline{V}) = g(V) \subseteq \overline{W}$. Hence, $g$ is $\mathcal{N}$-continuous.

**Corollary 6.16.** Every weakly continuous function with locally indiscrete domain is $\mathcal{N}$-continuous.

**Proof.** Follows from Theorem 6.15 and Theorem 3.4 (a).

**Corollary 6.17.** Every weakly continuous function with locally finite domain is $\mathcal{N}$-continuous.

**Proof.** Follows from Theorem 6.15 and Theorem 3.4 (c).

**Theorem 6.18.** Every weakly continuous function with $\mathcal{N}$-regular domain is $\mathcal{N}$-continuous.

**Proof.** Let $g : (Y, \sigma) \to (Z, \delta)$ be weakly continuous with $(Y, \sigma)$ is $\mathcal{N}$-regular. Let $y \in Y$ and $W \in \delta$ with $g(y) \in W$. By weak continuity of $g$, we find $H \in \sigma$ such that $y \in H$ and $g(H) \subseteq \overline{W}$. By $\mathcal{N}$-regularity of $(Y, \sigma)$, we find $V \in \sigma$ such that $y \in V$ and $\overline{V} \subseteq H$. Therefore, $g(\overline{V}) = g(H) \subseteq \overline{W}$. This shows that $g$ is $\mathcal{N}$-continuous.

Theorems 6.19 and 6.20 will show that $\theta$-continuity and $\mathcal{N}$-continuity are equivalent for those functions whose domain is anti-locally finite or $T_1$:

**Theorem 6.19.** If $g : (Y, \sigma) \to (Z, \delta)$ is a function with $(Y, \sigma)$ is anti-locally finite, then the following are equivalent:

a. $g$ is $\theta$-continuous.

b. $g$ is $\mathcal{N}$-continuous.

**Proof.** (a) $\to$ (b) is just Theorem 6.14 (a).

(b) $\to$ (a): Let $y \in Y$ and let $W \in \delta$ with $g(y) \in W$. By $\mathcal{N}$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq \overline{W}$. By anti-local finiteness of $(Y, \sigma)$ and Proposition 2.14 (c), $\overline{\overline{V}} = \overline{V}$. So, $g(\overline{V}) = g(\overline{\overline{V}}) \subseteq \overline{\overline{W}}$. Hence, $g$ is $\theta$-continuous.

**Theorem 6.20.** If $g : (Y, \sigma) \to (Z, \delta)$ is a function with $(Y, \sigma)$ is $T_1$, then the following are equivalent:

a. $g$ is $\theta$-continuous.

b. $g$ is $\mathcal{N}$-continuous.

**Proof.** (a) $\to$ (b) is just Theorem 6.14 (a).

(b) $\to$ (a): Let $y \in Y$ and $W \in \delta$ with $g(y) \in W$. By $\mathcal{N}$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(\overline{V}) \subseteq \overline{\overline{W}}$. Since $(Y, \sigma)$ is $T_1$, then by Proposition 2.10 (c), $\overline{\overline{V}} = \overline{V}$. So, $g(\overline{V}) = g(\overline{\overline{V}}) \subseteq \overline{\overline{W}}$. Hence, $g$ is $\theta$-continuous.

The following theorem will show that $\theta$-continuity, $\mathcal{N}$-continuity, and $\omega$-continuity are equivalent for those functions whose domain is anti-locally countable:

**Theorem 6.21.** If $g : (Y, \sigma) \to (Z, \delta)$ is a function with $(Y, \sigma)$ is anti-locally countable, then the following are equivalent:

a. $g$ is $\theta$-continuous.
b. $g$ is $\mathcal{N}$-θ-continuous.
c. $g$ is $\omega$-continuous.

Proof. (a) $\rightarrow$ (b) is just Theorem 6.14 (a).
(b) $\rightarrow$ (c) is just Theorem 6.14 (b).
(c) $\rightarrow$ (a): Follows from Theorem 2.18 of [21]. □

Remark 6.22. Example 2.20 in [21] contains a function $f : (Y, \sigma) \rightarrow (Z, \delta)$ with $(Y, \sigma)$ is locally finite and $f$ is $\omega$-θ-continuous but not θ-continuous. And by Corollary 6.17, $f$ is $\mathcal{N}$-θ-continuous. It follows that the implication in Theorem 6.14 (a) is not reversible.

The following is an example to show the converse of Theorem 6.14 (b) is not true, in general:

Example 6.23. Let $X = \mathbb{R}$, $Y = \{a, b, c\}$, $r = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : Q \subseteq U\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Define $f : (X, r) \rightarrow (Y, \sigma)$ by

$$f(x) = \begin{cases} 
a & \text{if } x \in (-\infty, 0) \cap (\mathbb{R} - Q), 
b & \text{if } x \in (0, \infty) \cap (\mathbb{R} - Q), 
c & \text{if } x \in Q.
\end{cases}$$

Then

a. $f$ is $\omega$-θ-continuous.
b. $f$ is not $\mathcal{N}$-θ-continuous.

Proof. a. Let $x \in X$ and $\sigma \in \sigma$ such that $f(\sigma) \in Y$. Since $f$ is $\omega$-θ-continuous, then $f(\sigma) \in Y$.

Case 1. $x \in (-\infty, 0) \cap (\mathbb{R} - Q)$. Choose $U = Q \cup \{x\}$. Then $x \in U \in r$.

Since $U$ is countable, then $\overline{U} = U$ and $f(\overline{U}) = [a, c] \subseteq V$.

Case 2. $x \in (0, \infty) \cap (\mathbb{R} - Q)$. Choose $U = Q \cup \{x\}$. Then $x \in U \in r$.

Since $U$ is countable, then $\overline{U} = U$ and $f(\overline{U}) = [b, c] \subseteq V$.

Case 3. $x \in Q$. Choose $U = Q$. Then $x \in U \in r$. Since $U$ is countable, then $\overline{U} = U$ and $f(\overline{U}) = [c] \subseteq V$.

b. Take $x = \sqrt{2}$ and $V = \{a\}$. Then $V \in \sigma$ with $f(\sigma) = a \in V$. Let $U \in r$ with $x \in U$, then $Q \cup \{\sqrt{2}\} \subseteq U$ and so $f(\overline{Q \cup \{\sqrt{2}\}}) \subseteq f(\overline{U}^\mathcal{N})$. Since $(X, r)$ is anti-locally finite, then by Proposition 2.14 (c), $Q \cup \{\sqrt{2}\} = \mathbb{R}$. So, $Y \not\subseteq f(\overline{U}^\mathcal{N})$ and hence $f(\overline{U}^\mathcal{N}) = Y$ which is not a subset of $V$. □

7. Weakly $\theta_N$-continuity and faintly $\theta_N$-continuity

In this section, we will introduce weakly $\theta_N$-continuity and faintly $\theta_N$-continuity as two new forms of continuity. We will study relationships between them and some other related continuity concepts.

We start by defining weak $\theta_N$-continuity.

Definition 7.1. A function $g : (Y, \sigma) \rightarrow (Z, \delta)$ is called weakly $\theta_N$-continuous if for every $y \in Y$ and every $W \in \delta$ such that $g(y) \in W$, there exists $V \in \sigma$ such that $g(V) \subseteq W^\mathcal{N}$.

Theorem 7.2. Remark 6.22, Example 7.8, and Remark 7.9 will show that weak $\theta_N$-continuity lies strictly between Weak $\theta_o$-continuity and weak continuity.

Theorem 7.2. a. Weak $\theta_N$-continuity implies weak $\theta_N$-continuity.
b. Weak $\theta_N$-continuity implies weak continuity.

Proof. a. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be weakly $\theta_N$-continuous. Let $y \in Y$ and let $W \in \delta$ with $g(y) \in W$. By weak $\theta_N$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq W^\mathcal{N}$. Therefore, $g$ is weakly $\theta_N$-continuous.

b. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be weakly $\theta_N$-continuous. Let $y \in Y$ and let $W \in \delta$ with $g(y) \in W$. By weak $\theta_N$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq W^\mathcal{N} \subseteq W$. Thus, $g$ is weakly continuous. □

The following theorem shows that weak $\theta_o$-continuity, weak $\theta_N$-continuity, and weak continuity are equivalent for functions whose co-domain is anti-locally countable:

Theorem 7.3. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be a function such that $(Z, \delta)$ is anti-locally countable. Then the following are equivalent:

a. $g$ is weakly $\theta_o$-continuous.
b. $g$ is weakly $\theta_N$-continuous.
c. $g$ is weakly continuous.

Proof. (a) $\rightarrow$ (b) and (c) $\rightarrow$ (d) are just Theorem 7.2.
(c) $\rightarrow$ (a): is just Theorem 2.24 of [21]. □

The following theorem shows that weak $\theta_N$-continuity and weak continuity are equivalent for functions whose co-domain is anti-locally finite:

Theorem 7.4. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be a function with $(Z, \delta)$ being anti-locally finite. Then the following are equivalent:

a. $g$ is weakly $\theta_N$-continuous.
b. $g$ is weakly continuous.

c. $g$ is weakly continuous.

Proof. (a) $\rightarrow$ (b) is just Theorem 7.2 (b).
(b) $\rightarrow$ (a): Let $y \in Y$ and $W \in \sigma$ with $g(y) \in W$. By weak continuity of $g$, there is $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq W$. Since $(Z, \delta)$ is anti-locally finite, then by Proposition 2.14 (c), $W^\mathcal{N} = W$, and so $g(V) \subseteq W^\mathcal{N} = W$. Thus, $g$ is weakly $\theta_N$-continuous. □

The following theorem shows that continuity, weak $\theta_o$-continuity, and $\theta_N$-weak continuity are equivalent for functions whose co-domain is $\mathcal{N}$-locally indiscrete:

Theorem 7.5. Let $f : (Y, \sigma) \rightarrow (Z, \delta)$ be a function such that $(Z, \delta)$ is $\mathcal{N}$-locally indiscrete. Then the following are equivalent:

a. $f$ is continuous.
b. $f$ is weakly $\theta_o$-continuous.
c. $f$ is weakly $\theta_N$-continuous.

Proof. (a) $\rightarrow$ (b) is just Theorem 2.2.5 of [21].
(b) $\rightarrow$ (c) is just Theorem 7.2 (a).
(c) $\rightarrow$ (a): Let $y \in Y$ and let $W \in \delta$ with $g(y) \in W$. By weak $\theta_N$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq W^\mathcal{N}$. Since $(Z, \delta)$ is $\mathcal{N}$-locally indiscrete, then by Theorem 3.7 (a), $W^\mathcal{N} = W$, and so $g(V) \subseteq W = W$. It follows that $g$ is continuous. □

Corollary 7.6. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be a function with $(Z, \delta)$ being locally indiscrete. Then the following are equivalent:

a. $g$ is continuous.
b. $g$ is weakly $\theta_o$-continuous.
c. $g$ is weakly $\theta_N$-continuous.
Theorem 7.13. a. Every faintly $\theta_w$-continuous function is faintly $\theta_N$-continuous.
   b. Every faintly $\theta_N$-continuous function is faintly continuous.
   c. Every weakly $\theta_N$-continuous function is faintly $\theta_N$-continuous.

Proof. a. and b. Follow from the definitions and Theorem 4.2.
   c. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be a weakly $\theta_N$-continuous function. Let $y \in Y$ and $W \in \delta_y$ with $g(y) \in W$. Choose $H \in \delta$ such that $g(y) \in H \subseteq TF^N \subseteq W$. By weak $\theta_N$-continuity of $g$, we find $V \in \sigma$ such that $y \in V$ and $g(V) \subseteq TF^N \subseteq W$. Therefore, $g$ is faintly $\theta_N$-continuous. □

The following is an example to show that the implication in Theorem 7.13 (a) is not reversible in general:

Example 7.14. Consider the identity function $f : (\mathbb{R}, \sigma) \rightarrow (\mathbb{R}, \tau)$ where $\sigma = \{\emptyset, \mathbb{R}\}$ and $\tau = \{\emptyset, \mathbb{R}, [1]\}$. Therefore, $f$ is faintly $\theta_N$-continuous but not faintly $\theta_w$-continuous.

The following is an example to show that the implication in Theorem 7.13 (b) is not reversible in general:

Example 7.15. Let $X = \{1, 2\}$. Consider the identity function $f : (X, \sigma) \rightarrow (X, \tau)$ where $\sigma = \{\emptyset, X\}$ and $\tau = \{\emptyset, X, \{1\}\}$ as in Example 4.10. By Example 4.10, $\tau_{X_1} = \{\emptyset, \mathbb{R}, \{1\}\}$ and $\tau_{X_2} = \{\emptyset, \mathbb{R}\}$. Therefore, $f$ is faintly continuous but not faintly $\theta_N$-continuous.

The following is a remark to show that the implication in Theorem 7.13 (c) is not reversible in general:

Remark 7.16. The function in Example 2.38 of [21] is faintly $\theta_w$-continuous but not weakly continuous. So, by Theorems 7.13 (a) and 7.2 (b) it is faintly $\theta_N$-continuous but not weakly $\theta_N$-continuous.

In the rest of this section, we give sufficient conditions for the equivalence between faint $\theta_N$-continuity and some other continuity concepts.

Theorem 7.17. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be a function with $(Z, \delta)$ being $\mathcal{N}$-locally indiscrete. Then the following are equivalent:
   a. $g$ is continuous.
   b. $g$ is faintly $\theta_w$-continuous.
   c. $g$ is faintly $\theta_N$-continuous.

Proof. Follows from Theorem 4.3 and the definitions. □

Corollary 7.18. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be a function with $(Z, \delta)$ being locally indiscrete. Then the following are equivalent:
   a. $g$ is continuous.
   b. $g$ is faintly $\theta_w$-continuous.
   c. $g$ is faintly $\theta_N$-continuous.

Proof. Follows from Theorems 3.4 (a) and 7.17. □

Corollary 7.19. Let $g : (Y, \sigma) \rightarrow (Z, \delta)$ be a function with $(Z, \delta)$ being locally finite. Then the following are equivalent:
   a. $g$ is continuous.
   b. $g$ is faintly $\theta_w$-continuous.
   c. $g$ is faintly $\theta_N$-continuous.

Proof. Follows from Theorems 3.4 (c) and 7.17. □
Theorem 7.20. Let \( g : (Y, \sigma) \to (Z, \delta) \) be a function with \((Z, \delta)\) being anti-locally countable. Then the following are equivalent:

a. \( g \) is faintly \( \theta_{\omega} \)-continuous.

b. \( g \) is faintly \( \theta_{\omega} \)-continuous.

c. \( g \) is faintly continuous.

Proof. Follows from Theorem 4.6 (a) and the definitions.

Theorem 7.21. Let \( g : (Y, \sigma) \to (Z, \delta) \) be a function with \((Z, \delta)\) being anti-locally finite or \( T_1 \). Then \( g \) is faintly \( \theta_{\omega} \)-continuous if and only if \( g \) is faintly continuous.

Proof. Follows from Theorem 4.6 (b) and the definitions.

8. Conclusion

Recently, the use of set and functions for topological spaces has significantly evolved in data analysis, information systems, digital topology, and others. This paper deals mainly with topological spaces which are not \( T_1 \). This paper contains a new class of sets with some related functions and separation axioms. Like \( \theta \)-open sets, \( \theta_{\omega} \)-open sets can be used in defining several new topological spaces as well as several types of mappings. Our new results are expected to have applications in general topology (specially in characterizing some compactness notions), and may other sciences.

Declarations

Author contribution statement

S. Al Ghou and S. Al-Zoubi: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

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The authors declare no conflict of interest.

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