Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking

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Abstract
We have recently solved the inverse spectral problem for integrable partial differential equations (PDEs) in arbitrary dimensions arising as commutation of multidimensional vector fields depending on a spectral parameter $\lambda$. The associated inverse problem, in particular, can be formulated as a nonlinear Riemann–Hilbert (NRH) problem on a given contour of the complex $\lambda$ plane. The most distinguished examples of integrable PDEs of this type, like the dispersionless Kadomtsev–Petviashvili (dKP), the heavenly and the two-dimensional dispersionless Toda equations, are real PDEs associated with Hamiltonian vector fields. The corresponding NRH data satisfy suitable reality and symplectic constraints. In this paper, generalizing the examples of solvable NRH problems illustrated in Manakov and Santini (2009 J. Phys. A: Math. Theor. 42 095203; 2008 J. Phys. A: Math. Theor. 41 055204; 2009 J. Phys. A: Math. Theor. 42 404013), we present a general procedure to construct solvable NRH problems for integrable real PDEs associated with Hamiltonian vector fields, allowing one to construct exact implicit solutions of such PDEs parametrized by an arbitrary number of real functions of a single variable. Then, we illustrate this theory on few distinguished examples for the dKP and heavenly equations. For the dKP case, we characterize a class of similarity solutions, of solutions constant on their parabolic wave front and breaking simultaneously on it, of localized solutions whose breaking point travels with constant speed along the wave front, and of localized solutions breaking in a point of the $(x, y)$ plane. For the heavenly equation, we characterize two classes of symmetry reductions.

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1. Introduction

It was observed long ago [1] that the commutation of multidimensional vector fields can generate integrable nonlinear partial differential equations (PDEs) in arbitrary dimensions. Some of these equations are dispersionless (or quasi-classical) limits of integrable PDEs, having the dispersionless Kadomtsev–Petviashvili (dKP) equation [2, 3] as a universal prototype example; they arise in various problems of mathematical physics and are intensively studied in the recent literature (see, for instance, [4–22]). In particular, an elegant integration scheme applicable, in general, to nonlinear PDEs associated with Hamiltonian vector fields, was presented in [8] and a nonlinear \( \delta \)-dressing was developed in [14]. Special classes of nontrivial solutions were also derived (see, for instance, [13, 16]).

Distinguished examples of PDEs arising as the commutation conditions \([\hat{L}_1(\lambda), \hat{L}_2(\lambda)] = 0\) of pairs of one-parameter families of vector fields, with \(\lambda \in \mathbb{C}\) being the spectral parameter, are the following.

(1) A system of two nonlinear PDEs in 2 + 1 dimensions [23],

\[
\begin{align*}
  u_{tx} + u_{yy} + (u u_x)_x + v_x u_{xy} - v_y u_{xx} &= 0, \\
  v_{tx} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} &= 0,
\end{align*}
\]

arising from the commutation of the two-dimensional vector fields

\[
\begin{align*}
  \hat{L}_1 &= \partial_y + (\lambda + v_x) \partial_x - u_x \partial_\lambda, \\
  \hat{L}_2 &= \partial_t + (\lambda^2 + \lambda v_x + u - v_y) \partial_x + (-\lambda u_x + u_y) \partial_\lambda,
\end{align*}
\]

and describing a general integrable Einstein–Weyl metric [24].

Its \(v = 0\) reduction, the dKP equation

\[
(u_t + u u_x)_x + u_{yy} = 0, \quad u = u(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}
\]

(3)

(1)

(2) The vector nonlinear PDE in 3 + 1 dimensions [27],

\[
\begin{align*}
  \vec{U}_{tx} - \vec{U}_{zy} + (\vec{U}_y \cdot \vec{\nabla}_x) \vec{U}_x - (\vec{U}_x \cdot \vec{\nabla}_y) \vec{U}_z &= \vec{0}, \\
  \vec{U} \in \mathbb{R}^2, \quad \vec{x} = (x, y), \quad \vec{\nabla}_z = (\partial_z, \partial_y),
\end{align*}
\]

(7)

associated with the two-dimensional vector fields:

\[
\begin{align*}
  \hat{L}_1 &= \partial_z + \lambda \partial_x + \vec{U}_y \cdot \vec{\nabla}_z, \\
  \hat{L}_2 &= \partial_t + \lambda \partial_y + \vec{U}_y \cdot \vec{\nabla}_z
\end{align*}
\]

(8)
and its Hamiltonian reduction $\nabla_\xi \cdot \vec{U} = 0$, the celebrated second heavenly equation of Plebanski [28]:

$$\theta_{tx} - \theta_{zy} + \theta_{tx} \theta_{yy} - \theta_{ty} \theta_{xx} = 0, \quad \theta = \theta(x, y, z, t) \in \mathbb{R}, \quad x, y, z, t \in \mathbb{R},$$

(9)
describing self-dual vacuum solutions of the Einstein equations, associated with the following pair of Hamiltonian two-dimensional vector fields:

$$\hat{L}_1 \equiv \partial_z + \lambda \partial_x + \theta_{xy} \partial_x - \theta_{xx} \partial_y,$$

$$\hat{L}_2 \equiv \partial_t + \lambda \partial_y + \theta_{yy} \partial_x - \theta_{xy} \partial_y.$$  

(10)

The inverse spectral transform (IST) for one-parameter families of multidimensional vector fields, developed in [27], has allowed one to construct the formal solution of the Cauchy problem for the nonlinear PDEs (7) and (9) in [27], for equations (1) and (3) in [23], for equation (5) in [29] and for the wave form $(e^{\phi})_t = \phi_{xx} + \phi_{yy}$ of the two-dimensional dispersionless Toda (2dtd) equation [30–32] in [33]. This IST, introducing interesting novelties with respect to the classical IST for soliton equations [34, 26], turns out to be, together with its associated nonlinear Riemann–Hilbert (NRH) dressing, an efficient tool to study several properties of the solution space of the PDE under consideration: (i) the characterization of a distinguished class of spectral data for which the associated NRH problem is linearized and solved, corresponding to a class of implicit solutions of the PDE (for the dKP and 2dtd equations respectively in [35] and in [33], for the Dunajski generalization [37] of the heavenly equation in [36] and for the heavenly equation in [38]); (ii) the construction of the longtime behaviour of the solutions of the Cauchy problem (for the dKP, 2dtd and heavenly equations respectively in [35, 33] and [38]); (iii) the possibility of establishing whether or not the lack of dispersive terms in the nonlinear PDE causes the breaking of localized initial profiles (for the dKP, 2dtd and heavenly equations in [35, 33] and [38], respectively) and, if yes, to investigate in a surprisingly explicit way the analytic aspects of such a wave breaking, as was done for the dKP equation in [35]. Recent results on integrable differential constraints on the hierarchy associated with the nonlinear system (1) and their connection to the associated NRH problems can be found in [39].

In this paper, generalizing the examples of solvable NRH problems illustrated in [33, 35, 38], we present, in section 2, a general procedure to construct solvable NRH problems for integrable PDEs associated with Hamiltonian vector fields, allowing one to construct exact implicit solutions of such PDEs parametrized by an arbitrary number of real functions of a single variable. In section 3, we illustrate this theory on few distinguished examples for the dKP equation, including the similarity solutions, a class of solutions constant on their parabolic wave front and breaking simultaneously on it, and a class of localized solutions breaking in a point of the $(x, y)$ plane or describing a breaking point moving with constant speed along the wave front. In section 4, we briefly apply the theory to the heavenly equation, constructing their similarity solutions and integrating the corresponding nonlinear PDEs.

Since the theory will be illustrated on two basic examples of PDEs associated with Hamiltonian vector fields, the dKP and the heavenly equations, in the remaining part of this introductory section we summarize their NRH dressing formalisms and other useful properties.

1.1. NRH dressing for dKP [23, 35]

Consider the vector nonlinear RH problem on the real line:

$$\psi_1^+ = R_1(\psi_1^-, \psi_2^-),$$

$$\psi_2^+ = R_2(\psi_1^-, \psi_2^-), \lambda \in \mathbb{R},$$

(11)
or, more shortly,
\[ \tilde{\psi}^\pm(\lambda) = \mathcal{R}(\tilde{\psi}^\mp(\lambda)), \]
(12)
where \( \mathcal{R}_j(s_1, s_2), j = 1, 2, \) is a given pair of complex differentiable functions of two arguments, and the solutions \( \tilde{\psi}^\pm(\lambda) = (\psi_1^\pm(\lambda), \psi_2^\pm(\lambda)) \in \mathbb{C}^2 \) are two-dimensional vector functions analytic respectively in the upper and lower halves of the complex \( \lambda \) plane, with the following asymptotics, for \( |\lambda| \gg 1 \) in their analyticity domains:
\[ \psi_1^\pm(\lambda) = -\lambda^2 t - \lambda y + x - 2tq_1^{(1)} + \sum_{n \geq 1} \frac{q_1^{(n)}}{\lambda^n}, \]
\[ \psi_2^\pm(\lambda) = \lambda + \frac{q_2^{(1)}}{\lambda} + \sum_{n \geq 2} \frac{q_2^{(n)}}{\lambda^n}. \]
(13)
If the nonlinear RH problem (11) is uniquely solvable, together with its linearized version
\[ \tilde{v}^\pm = M(\psi_1^\mp, \psi_2^\mp) \tilde{v}^- , \]
\[ M_{jk}(\psi_1^\mp, \psi_2^\mp) = \frac{\partial \mathcal{R}_j(\psi_1^\mp, \psi_2^\mp)}{\partial \psi_k}, \quad j, k = 1, 2, \]
(14)
then \( \psi_1^\pm, \psi_2^\pm \) are solutions of the following pair of vector field equations:
\[ \hat{L}_1 \psi = \psi_y + (\lambda + v_s)\psi_x - u_x \psi_\lambda = 0, \]
\[ \hat{L}_2 \psi = \psi_t + (\lambda^2 + \lambda u_x + u - v_y)\psi_x + (\lambda u_x + u_y) \psi_\lambda = 0, \]
(15)
where
\[ u = q_1^{(1)}, \]
\[ v = -q_1^{(1)} - yq_2^{(1)} - 2tq_2^{(2)}, \]
(16)
and \( u \) and \( v \) solve the system (1) of PDEs.

**Basic non-differential reductions.** There are few basic non-differential reductions of the above NRH problem and, correspondingly, of the integrable system (1).

(R1) If \( \mathcal{R}_2(s_1, s_2) = s_2, \) then \( \psi_2^\pm = \psi_2^\pm = \lambda; \) the nonlinear RH problem becomes scalar and one obtains equation (5), the \( u = 0 \) reduction of the system (1), associated with the non-Hamiltonian–Lax pair of one-dimensional vector fields (6).

(R2) If the transformation \( (s_1, s_2) \to (\mathcal{R}_1(s_1, s_2), \mathcal{R}_2(s_1, s_2)) \) is canonical,
\[ \{\mathcal{R}_1, \mathcal{R}_2\}(s_1, s_2) := \mathcal{R}_{12}(s_1, s_2) - \mathcal{R}_{12}(s_1, s_2) = 1, \]
(17)
then one obtains the \( v = 0 \) reduction of the system (1), the celebrated dKP equation (3), corresponding to the Hamiltonian–Lax pair
\[ \psi_t + \lambda \psi_x - u_x \psi_\lambda = \psi_t + \{H_2, \psi\}_{(x,s)} = 0, \]
\[ \psi_t + (\lambda^2 + u)\psi_x + (-\lambda u_x + u_y) \psi_\lambda = \psi_t + \{H_3, \psi\}_{(x,s)} = 0, \]
(18)
(19)
for the Hamiltonians
\[ H_2 = \frac{\lambda^3}{3} + u, \quad H_3 = \frac{\lambda^3}{3} + \lambda u - \partial_x^{-1} u_y. \]
(20)
We remark that, in this reduction, the eigenfunctions \( \psi_1^\pm, \psi_2^\pm \) are canonically conjugated:
\[ \{\psi_2^\pm, \psi_1^\pm\}_{(x,s)} = 1. \]
(21)
We also recall that the dKP equation is the first member of the dKP hierarchy [6, 5],
\[ H_{m} - H_{n} + \{H_{m}, H_{n}\}_{(\lambda, x)} = 0, \quad m \neq n, m, n \geq 2, \]
(22)
corresponding to the Hamiltonian vector field equations
\[ \psi_{tn} + \{H_{n}, \psi\}_{(\lambda, x)} = 0, \]
(23)
where \( t_2 = y, t_3 = t \) and \((\psi)_+\) is the non-negative (principal) part of the Laurent expansion of \( \psi \) at \( \lambda = \infty \). In particular, if \( m = 2 \) in (22), one obtains the sub-hierarchy
\[ nu_{\alpha} + (\text{Res}_{\infty}(\psi_{2}^{\alpha}))_{\lambda} = 0, \]
(24)
where \( \text{Res}_{\infty}(g) \) is the coefficient of \( \lambda^{-1} \) in the Laurent expansion of \( g(\lambda) \) at \( \lambda = \infty \).

(R3) If
\[ \vec{R}(\vec{R}(\vec{z})) = \vec{z}, \quad \forall \vec{z} \in \mathbb{C}^2, \]
(25)
then one obtains the reality constraint
\[ u, v \in \mathbb{R}, \quad \psi_{j}(\lambda) = \overline{\psi_{j}(\lambda)}, \quad j = 1, 2. \]
(26)
From the integral equations characterizing the solutions of the NRH problem (11), and from the definition \( u = q^{(1)}_{2} \), one obtains the following spectral characterization of the solution \( u \):
\[ u = F(x - 2ut, y, t) \in \mathbb{R}, \]
(27)
where the spectral function \( F \), defined by
\[ F(\xi, y, t) = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\mu}{2\pi i} R_{2}(\pi_{1}^{-}(\lambda; \xi, y, t), \pi_{2}^{-}(\lambda; \xi, y, t)), \]
(28)
is connected with the initial data \( u(x, y, 0) \) via the direct spectral transform developed in [23]. Equation (27) defines the dKP solution implicitly, due to the presence of the \( (x - 2ut) \) term as an argument of \( F \), and describes the wave-breaking features of localized solutions of dKP.

Evaluating the integral equations characterizing the above NRH problem in the spacetime asymptotic region
\[ \xi = 2ut, v_1, v_2 = O(1), \quad v_2 \neq 0, \quad t \gg 1, \]
(29)
\[ x + \frac{y^2}{4t} = \xi, \quad x = \xi + v_1 t, \quad y = v_2 t, \]
one obtains the following longtime behaviour [35]:
\[ u = \frac{1}{\sqrt{t}} G \left( x + \frac{y^2}{4t} - 2ut, \frac{y}{2t} \right) + o \left( \frac{1}{\sqrt{t}} \right), \]
(30)
where
\[ G(\xi, \eta) = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\mu}{2\pi i} R_{2}(\xi - \mu^2 + a_{1}(\mu; \xi, \eta), -\eta + a_{2}(\mu; \xi, \eta)), \]
(31)
and \( a_{1,2}(\mu; \xi, \eta) \) are the unique solutions of the nonlinear integral equations
\[ a_{j}(\mu; \xi, \eta) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\mu'}{2\pi i} (\mu' - (\mu - i0)) R_{j}(\xi - \mu'^2 + a_{1}(\mu'; \xi, \eta), -\eta + a_{2}(\mu'; \xi, \eta)), \quad j = 1, 2, \]
(32)
giving a description of the wave breaking of small initial data in the longtime regime as explicit as for the case of the Riemann–Hopf equation \( u_t + uu_x = 0 \). In particular, one shows that, generically, small and localized initial data will break asymptotically in a point of the \((x, y)\) plane, and the breaking details are given explicitly in terms of the initial data [35, 40].
1.2. NRH dressing for heavenly [27, 38]

Consider the vector NRH problem on the real line
\[ \vec{\psi}^+(\lambda) = \vec{R}(\vec{\psi}^-(\lambda), \lambda), \lambda \in \mathbb{R}, \] (33)
where \( \vec{\psi}^+(\lambda) \) and \( \vec{\psi}^-(\lambda) \in \mathbb{C}^2 \) are two-dimensional analytic vector functions respectively in the upper and lower halves of the complex \( \lambda \) plane, normalized as follows:
\[ \vec{\psi}^\pm(\lambda) = (x - \lambda z, y - \lambda t) + \sum_{n \geq 1} \frac{\vec{q}(n)}{\lambda^n}, \quad |\lambda| \gg 1, \] (34)
and the spectral data \( \vec{R}(\vec{\zeta}, \lambda) = (\vec{R}_1(\zeta_1, \zeta_2, \lambda), \vec{R}_2(\zeta_1, \zeta_2, \lambda)) \in \mathbb{C}^2 \), defined for \( \vec{\zeta} \in \mathbb{C}^2, \lambda \in \mathbb{R} \), satisfy the following properties:
\[ \vec{R}(\vec{R}(\vec{\zeta}, \lambda), \lambda) = \vec{\zeta}, \quad \forall \vec{\zeta} \in \mathbb{C}^2, \text{ reality constraint}, \]
\[ |\vec{R}_1, \vec{R}_2|_{\vec{\zeta}} = 1, \text{ symplectic constraint}. \] (35)
Then, assuming uniqueness of the solution of such a NRH problem and of its linearized version, it follows that \( \vec{\psi}^\pm \) are canonically conjugated solutions \( \{\vec{\psi}_1^\pm, \vec{\psi}_2^\pm\}_{(x,y)} = 1 \) of the linear problems \( \hat{L}_{1,2}\vec{\psi}^\pm = \vec{0} \), where \( \hat{L}_{1,2} \) are defined in (10), and
\[ \left( \begin{array}{c} \theta_y \\ -\theta_x \end{array} \right) = \vec{F}(x, y, z, t) \in \mathbb{R}^2 \] (36)
is the solution of the heavenly equation (9), where
\[ \vec{F}(x, y, z, t) = \int_{\mathbb{R}} \frac{d\lambda}{2\pi i} \vec{R}(\vec{\psi}_1^-(\lambda; x, y, z, t), \vec{\psi}_2^--(\lambda; x, y, z, t), \lambda) \] (37)
and \( \vec{R}(\vec{\zeta}, \lambda) := \vec{R}(\vec{\zeta}, \lambda) - \vec{\zeta} \).

As a consequence of equations \( \hat{L}_{1,2}\vec{\psi}^\pm = \vec{0} \), it follows that, for \( |\lambda| \gg 1 \):
\[ \vec{\psi}_1^\pm = x - \lambda z - \theta_x \lambda^{-1} + \theta_y \lambda^{-2} + O(\lambda^{-3}), \]
\[ \vec{\psi}_2^\pm = y - \lambda t + \theta_y \lambda^{-1} - \theta_x \lambda^{-2} + O(\lambda^{-3}). \] (38)

2. Solvable vector nonlinear RH problems

Since basic nonlinear PDEs like the dKP, heavenly and 2ddT equations are solved via NRH problems, whose data satisfy the symplectic and reality constraints, in this section we present a general procedure to construct solvable vector NRH problems satisfying such constraints.

Consider an autonomous Hamiltonian two-dimensional dynamical system with Hamiltonian
\[ H(\vec{x}) = \mathcal{H}(E(\vec{x})), \] (39)
where \( \vec{x} \equiv (q, p) \) are canonically conjugated coordinates, \( E(\vec{x}) \) is a polynomial function of the coordinates and \( \mathcal{H}(\cdot) \) is an arbitrary function of a single argument, corresponding to the equations of motion
\[ \frac{d\vec{x}}{dt} = \mathcal{H}(E) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \nabla_x E(\vec{x}). \] (40)
Introducing action-angle variables in the usual way:
\[ J \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \rho(q, H) dq, \Rightarrow H = H(J), \]
\[ \theta - \theta_0 \equiv \omega(J)\mathcal{H}(E)(\tau - \tau_0) = \mathcal{H}(E) \int_{\theta_0}^\theta \frac{d\rho(q', H(J))}{dJ} dq', \] (41)
\[ \omega(J) \equiv \frac{\partial H(J)}{\partial J}, \]
the solution can be found inverting the quadrature (41):
\[
\tilde{x} = \tilde{D}(\theta - \theta_0; \tilde{x}_0, J),
\]
where the evolutionary diffeomorphism \(\tilde{D} : \tilde{x}_0 \rightarrow \tilde{x}\) is symplectic,
\[
\{D_1, D_2\}_{\theta_0, p_0} = 1,
\]
and satisfies the properties
\[
\begin{align*}
\tilde{x}_0 &= \tilde{D}(\theta_0 - \theta; \tilde{x}, J), \\
\tilde{D}(\theta_2; \tilde{D}(\theta_1; \tilde{x}_0, J), J) &= \tilde{D}(\theta_1 + \theta_2; \tilde{x}_0, J), \\
\tilde{D}(0; \tilde{x}_0, J) &= \tilde{x}_0.
\end{align*}
\]
Consequently, we have the relation
\[
\tilde{D}(\theta_1 - \theta_0; \tilde{x}, J) = \tilde{D}(\theta_1 - \theta_0; \tilde{x}_0, J)
\]
for any intermediate angle \(\theta_1\), and this relation holds true, at least locally, also in the complex domain.

2.1. Solvable NRH problems

Identifying the solution \(\tilde{x}(\tau)\) at time \(\tau_0\) and at time \(\tau_0 + 1\) with, respectively, \(\tilde{\psi}^- (\lambda)\) and \(\tilde{\psi}^+ (\lambda)\),

(i) equation (42) becomes the two-dimensional vector NRH problem,
\[
\tilde{\psi}^+ = \tilde{D}(\omega(J(\tilde{\psi}^-))) \tilde{\mathcal{H}}'(E(\tilde{\psi}^-); \tilde{x}, J) \equiv \tilde{R}(\tilde{\psi}^-),
\]
connecting the \((-)\) and \((+)\) vector functions through the canonical transformation (17).

(ii) Since \(E(\tilde{x})\) (as well as \(J(\tilde{x})\)) is an invariant of the dynamics (40), \(E(\tilde{x}_0) = E(\tilde{x})\), it follows that \(E(\tilde{\psi}^-) = E(\tilde{\psi}^+)\) is an ‘invariant’ of the NRH problem (46). Since \(E(\tilde{\psi})\) is a polynomial function of its arguments, equations \(E(\tilde{\psi}^-) = E(\tilde{\psi}^+)\) and (13) define a function \(W\) polynomial in \(\lambda\):
\[
E(\tilde{\psi}^-(\lambda)) = E(\tilde{\psi}^+(\lambda)) \equiv W(\lambda; \tilde{q}_1^{N_1}, \tilde{q}_2^{N_2}),
\]
given by the polynomial part of the asymptotic expansion of \(E(\tilde{\psi}^\pm)\) for large \(\lambda\), depending on a finite number of coefficients \(\tilde{q}_1^{N_1} = (q_1^1, \ldots, q_1^{N_1}), \tilde{q}_2^{N_2} = (q_2^1, \ldots, q_2^{N_2})\) of expansions (13) and (34).

In addition, since \(E\) is a real function of its arguments, the reality constraint (26) implies that
\[
\overline{W(\lambda)} = W(\lambda).
\]

(iii) Since \(\tilde{D}\) is also a real function of its arguments and, from (44),
\[
\tilde{D}(z; \tilde{D}(-z; \tilde{\zeta}, J), J) = \tilde{\zeta},
\]
the reality constraint (25) implies that \(\tilde{D}(-z; \tilde{\zeta}, J) = \tilde{D}(\tilde{\zeta}; \tilde{\zeta}, J)\). It follows, from (46), that
\[
\tilde{\mathcal{H}}'(\cdot) = if(\cdot),
\]
where \(f\) is an arbitrary real function of a single argument, for dKP, and of two arguments (the invariant (47) and \(\lambda\)), for heavenly.

(iv) Defining the upper and lower functions
\[
\theta^\pm (\lambda) \equiv \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda \pm i0)} \text{ i}\omega(J(\tilde{\psi}^-(\lambda'))) f(W(\lambda')),
\]
for any intermediate angle \(\theta_1\), and this relation holds true, at least locally, also in the complex domain.
and having the following asymptotic expansion for large $\lambda$:

$$
\theta^\pm(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2}) = -\sum_{n \geq 1} \frac{\langle \lambda^{n-1} \omega f \rangle}{\lambda^n},
$$

(52)

where

$$
\langle \lambda^n g \rangle \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^n g(\lambda) \, d\lambda,
$$

(53)

so that $\omega f = 2^{\theta^+} - 2^{\theta^-}$, equation (45) becomes

$$
\tilde{D}(\theta^+; \psi^+, J(\psi^+)) = \tilde{D}(-\theta^-; \psi^-, J(\psi^-))
$$

(54)

and provides the solution of the problem, at least if $\tilde{D}$ is formally expandable, for large $\lambda$, in the Laurent series with a finite number of positive powers. Indeed, in this case, the left- and right-hand sides of (54) are equal to their polynomial part $\tilde{A}(\lambda)$ at $\lambda \sim \infty$:

$$
\tilde{D}(\theta^+; \psi^+, J(\psi^+)) = \tilde{A}(\lambda).
$$

(55)

Expanding the LHS of (55) for large $\lambda$ and using the asymptotic expansions of the eigenfunctions ([13] for dKP and [34] for heavenly), the polynomial part of such expansion fixes $A$:

$$
\tilde{A}(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2}) \equiv \langle \tilde{D}(\theta^+; \psi^+, J(\psi^+)) \rangle_x.
$$

(56)

Since the negative power part of such expansion is absent, the corresponding coefficients are zero; the first $N_1 + N_2$ of such equations for the first and second components of $\tilde{D}$ define a closed system of algebraic equations for the unknown fields $(\vec{q}_1^{N_1}, \vec{q}_2^{N_2})$, providing the wanted integration of the target nonlinear PDE.

**Remark 1.** Solvable NRHs of this type allow one to construct solutions of the dispersionless PDE characterized by differential reductions of the integrable PDE. To show it, let us consider, for concreteness, the dKP case. Let $g(\lambda)$ be a solution of the NRH problem, eigenfunction of the spectral problem (18), and characterized by the formal Laurent expansion:

$$
g(\lambda) \sim \sum_{n = -\infty}^N g_n \lambda^n, \quad |\lambda| \gg 1.
$$

(57)

Then, the coefficients $g_n$ of $\lambda^n$ satisfy the recursion relations

$$
\begin{align*}
g_{N_1} &= 0, \\
g_{N_2} + g_{N-1} &= 0, \\
g_{N+1} + g_{N-1} &= 0, \quad n < N.
\end{align*}
$$

(58)

Since $W(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2})$, defined by (47), is a polynomial (in $\lambda$) solution of (18), then $Res_{\lambda} W(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2}) = 0$ and equation (58) for $n = 0$ yields the differential constraint

$$
W_{0y} - u_x W_1 = -W_{-1x}(\vec{q}_1^{N_1}, \vec{q}_2^{N_2}) = 0.
$$

(59)

In addition, equations (58) for $n < 0$ imply, as it has to be, that $W_n = 0$ for $n \leq -1$. Equation (59) is the differential constraint satisfied by the solutions of dKP associated with the solvable NRH problem.

**Remark 2.** Starting with any $2n$-dimensional Liouville integrable Hamiltonian system, a generalization of the above construction allows one to obtain a solvable $2n$-dimensional NRH problem $\tilde{\psi}^+(\lambda) = \tilde{R}(\tilde{\psi}^-(\lambda))$, where $\tilde{R} : \tilde{\psi}^{-} \rightarrow \tilde{\psi}^+$ is a symplectic transformation.

**Remark 3.** Also integrable symplectic maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ generate, following the same procedure, solvable $n$-dimensional vector NRH problems.
2.2. Increasing the richness of the solution space

In the previous section, we showed how to construct solvable NRH problems whose spectral data, satisfying the reality and symplectic constraints, have the freedom of an arbitrary real function \( f \), which parametrizes the corresponding space of implicit solutions of the dKP, heavenly and 2ddT equations. In this section, we show how it is possible to increase considerably the richness of the space of implicit solutions. For concreteness, we refer to the dKP equation, but the same considerations apply to all real PDEs associated with Hamiltonian vector fields.

Let \( \psi_{1,2} \) be solutions of the Hamiltonian vector field equations (18) and (19), satisfying the asymptotics (13); then arbitrary differentiable functions of \( \psi_{1,2} \) are also the solutions of (18) and (19). In the first step of our construction, we define new functions of \( \psi_{1,2} \) as follows:

\[
\begin{align*}
\psi_{1}^{(1)\pm} &= \psi_{1}^{\pm} + a_{1} f^{(1)}(\psi_{2}^{\pm}), \\
\psi_{2}^{(1)\pm} &= \psi_{2}^{\pm},
\end{align*}
\]

(60)

where \( a_{1} \in \mathbb{R} \) and \( f^{(1)} \) is an arbitrary real function of one variable. The triangular transformation (60) is clearly invertible and, most important for our purposes, is symplectic and preserves the reality constraint (26). Therefore, \( \psi_{1,2} \) are also canonically conjugated solutions of the vector field equations (18) and (19) satisfying the reality constraint:

\[
\{ \psi_{1}^{(1)\pm}, \psi_{2}^{(1)\pm} \} = 1, \quad \psi_{j}^{(1)\ast}(\lambda) = \overline{\psi_{j}^{(1)-}(\bar{\lambda})}, \quad j = 1, 2,
\]

parametrized by the arbitrary real function \( f^{(1)} \) of a single argument. In the second step of this construction, we use the solutions \( \psi_{1,2}^{(1)\pm} \) to construct new functions \( \psi_{1,2}^{(2)\pm} \) as follows:

\[
\begin{align*}
\psi_{1}^{(2)\pm} &= \psi_{1}^{(1)\pm}, \\
\psi_{2}^{(2)\pm} &= \psi_{2}^{(1)\pm} + a_{2} f^{(2)}(\psi_{1}^{(1)\pm}),
\end{align*}
\]

(61)

where \( a_{2} \in \mathbb{R} \) and \( f^{(2)} \) is another arbitrary real function of one variable (note that, while in (60) the transformation is identical w.r.t. the second eigenfunction, in (61) the transformation is identical w.r.t. the first one). Therefore, we have constructed the family \( \psi_{1,2}^{(2)\pm} \) of canonically conjugated solutions of the vector field equations (18) and (19) satisfying the reality constraint, and now parametrized by the two arbitrary real functions \( f^{(1)}, f^{(2)} \) of a single argument. This procedure goes on, without limitations, alternating transformations of the types (60) and (61), which are identical w.r.t. the second and first eigenfunctions. At the \( m \)th step, one constructs canonically conjugated solutions \( \psi_{1,2}^{(m)\pm} \) of (18) and (19), satisfying the reality constraint:

\[
\{ \psi_{1}^{(m)\pm}, \psi_{2}^{(m)\pm} \} = 1, \quad \psi_{j}^{(m)\ast}(\lambda) = \overline{\psi_{j}^{(m)-}(\bar{\lambda})}, \quad j = 1, 2
\]

(62)

and parametrized by \( m \) arbitrary real functions \( f^{(1)}, \ldots, f^{(m)} \) of a single argument.

Now we consider any of the solvable NRH problems for the family of eigenfunctions \( \psi_{1,2}^{(m)\pm} \); for instance, the NRH problem discussed in section 3.2:

\[
\begin{align*}
\psi_{1}^{(m)+} &= \psi_{1}^{(m)-} + i a f(\psi_{1}^{(m)-} + a \psi_{2}^{(m)-}), \\
\psi_{2}^{(m)+} &= \psi_{2}^{(m)-} - i f(\psi_{1}^{(m)-} + a \psi_{2}^{(m)-}).
\end{align*}
\]

(63)

whose normalization easily follows from the definition of \( \psi_{1,2}^{(m)\pm} \) in terms of \( \psi_{1,2}^{\pm} \), and from the asymptotics (13). Then the solution of this exactly solvable NRH problem and the corresponding family of exact implicit solutions of dKP are parametrized by \( m + 1 \) arbitrary real functions \( f, f^{(1)}, \ldots, f^{(m)} \) of a single argument.
3. Examples for dKP

In this section, we consider some basic examples of solvable NRH problems associated with dKP.

3.1. Example 1. The invariant $\psi^+_1 \psi^+_2$ and the similarity reduction \[35\]

If $E(q, p) = qp$, then equation (42) reads

$$q(\tau) = q_0 e^{i\tilde{f}(E}\tau-\tilde{z}_0), \quad p(\tau) = p_0 e^{-i\tilde{f}(E}\tau-\tilde{z}_0)$$

and corresponds to the NRH problem

$$\psi^+_1 = \psi_1 e^{if(\psi_1, \psi_2^-)}, \quad \psi^+_2 = \psi_2 e^{-if(\psi_1, \psi_2^-)}, \quad \lambda \in \mathbb{R},$$

satisfying the symplectic and reality constraints. Then the invariance equation (47) becomes

$$\psi^+_1 \psi^+_2 = \psi^-_1 \psi^-_2 = -t\lambda^3 - y\lambda^2 + (x-3ut)\lambda - 2yu + 3t \partial^{-1}_x u_y \equiv W(\lambda),$$

and the NRH problem linearizes and decouples:

$$\psi^+_1 = \psi_1 e^{if(W(\lambda))}, \quad \psi^+_2 = \psi_2 e^{-if(W(\lambda))}. \quad (72)$$

Equations (55) become

$$\psi^+_j e^{i(-\tilde{f}(\lambda))} = \psi^-_j e^{i(-\tilde{f}(\lambda))} = A_j(\lambda), \quad j = 1, 2, \quad \text{where the analytic functions} \ f^\pm(\lambda), \text{defined by}$$

$$f^\pm(\lambda) \equiv \frac{1}{2\pi i} \int_{\lambda'} \frac{d\lambda'}{(\lambda - \pm i0)} f(W(\lambda')),$$

exhibit the following asymptotics:

$$f^\pm(\lambda) \sim \frac{i}{\lambda} \sum_{n \geq 1} \langle \lambda^{n-1} f \rangle \langle \lambda^n f \rangle \equiv \frac{1}{2\pi} \int_{\lambda} \lambda^n f(W(\lambda')) d\lambda,$$

if $f$ decays faster than any power. It follows that

$$A_1(\lambda) \equiv -t\lambda^2 - (y + t(f))\lambda + x - 2ut - y(f) - t((\lambda f) + \frac{1}{2}(f)^2), \quad A_2(\lambda) \equiv \lambda - (f),$$

implying the following explicit solution of the NRH problem:

$$\psi^+_j = A_j(\lambda) e^{i(-\tilde{f}(\lambda))}, \quad j = 1, 2. \quad \text{(72)}$$

A characterization of the corresponding solutions of dKP is obtained observing that, since $W$ in (66) depends on the unknowns $u$ and $\partial^{-1}_x u_y$, the $1/\lambda$ terms of the expansions of equations (68) yield the algebraic system

$$q^{(1)}_1 = 2t\partial^{-1}_x u_y - yu = -(x-2ut)(f) + y((f)^2/2 + (\lambda f) + t((\lambda f) + (f)(\lambda f) + (f)^3/6)), \quad q^{(1)}_2 = u = (\lambda f) - (f)^2/2,$$

for the unknowns $u$ and $\partial^{-1}_x u_y$. The constructed solutions of dKP correspond to the following differential reduction (59):

$$3tu_t + xu_x + 2yu_y + 2u = 0. \quad \text{(74)}$$

Substituting its general solution

$$u = t^{-2/3} B(x', y'), \quad x' = \frac{x}{t^{1/3}}, \quad y' = \frac{y}{t^{2/3}}, \quad \text{(75)}$$

into dKP, one obtains the similarity reduction of dKP:

$$B_{x'} + \frac{x'}{3} B_{x'x} + \frac{2}{3} y' B_{x'y} - B_{y'y} - (B B_{x'})_{x'} = 0. \quad \text{(76)}$$

Therefore, the algebraic system (73) characterizes these similarity solutions of dKP.
3.2. Example 2. The invariant $\psi_1^* + a\psi_2^*$

If $E(q, p) = q + ap$, where $a$ is a real parameter, then equation (42) reads

$$q(\tau) = q_0 + a\mathcal{H}(E)(\tau - \tau_0), \quad p(\tau) = p_0 - \mathcal{H}(E)(\tau - \tau_0),$$

(77)

becoming the NRH problem

$$\psi_1^* = \psi_1^- + iaf(\psi_1^- + a\psi_2^-),$$

(78)

$$\psi_2^* = \psi_2^- - if(\psi_1^- + a\psi_2^-), \quad \lambda \in \mathbb{R},$$

satisfying the symplectic (3) and reality (25) constraints.

Due to the invariance equation (47)

$$\psi_1^* + a\psi_2^* = \psi_1^- + a\psi_2^- = -t\lambda^2 - (y - a)\lambda + x - 2ut \equiv W(\lambda),$$

(79)

the NRH problem linearizes and decouples:

$$\psi_1^* = \psi_1^- + iaf(W), \quad \psi_2^* = \psi_2^- - if(W), \quad \lambda \in \mathbb{R},$$

(80)

and equations (55) become

$$\psi_1^*(\lambda) - iaf^\pm(\lambda) = \psi_1^- - iaf^\pm(\lambda) = A_1(\lambda),$$

$$\psi_2^*(\lambda) + if^\pm(\lambda) = \psi_2^- + if^\pm(\lambda) = A_2(\lambda),$$

(81)

where

$$A_1(\lambda) = -t\lambda^2 - y\lambda + x - 2ut, \quad A_2(\lambda) = \lambda.$$

(82)

Then the solution of the NRH problem reads

$$\psi_1^\pm = -t\lambda^2 - y\lambda + x - 2ut + iaf^\pm(\lambda),$$

(83)

$$\psi_2^\pm = \lambda - if^\pm(\lambda).$$

(84)

Expanding equation (81) for large $\lambda$ and using (13), it is possible to express the coefficients $q(n)_{1, 2}$ of the asymptotic expansions in terms of the spectral function $f(W)$ in the following way:

$$q_1^{(n)} = -a(\lambda^{n-1}f), \quad n \geq 1,$$

(85)

$$q_2^{(n)} = (\lambda^{n-1}f), \quad n \geq 1.$$ 

(86)

Since $W$ in (79) is a function of $q_2^{(1)} = u$ only, equation (86) for $n = 1$,

$$u = \frac{1}{\sqrt{t}} F\left(x + \frac{(y - a)^2}{4t} - 2ut \right),$$

(87)

characterizes the family of solutions associated with the above NRH problem, where

$$F(z) = \frac{1}{2\pi} \int_{\mathbb{R}} f(-\mu^2 + z) \, d\mu.$$ 

(88)

Equation (87) describes a family of solutions of dKP, constant on the parabola

$$x + \frac{(y - a)^2}{4t} = \xi,$$

(89)

and breaking simultaneously in all points of it. Therefore, the above exact solutions do not describe the breaking of a localized wave in the $(x, y)$ plane, typical of the dKP evolution of localized waves [35]. In the following subsection, we show how to generalize this solvable case to construct exact solutions describing the breaking of a localized wave in a point of the $(x, y)$ plane.
The corresponding differential constraint (59) reads
\[(y - a)u_t - 2tu_x = 0.\] (90)

Indeed, substituting the general solution
\[u = \psi(\xi, \tau), \quad \xi = x + \frac{(y - a)^2}{4t},\] (91)
of (90) into the dKP equation, one obtains the equation \(\psi_t + \psi/(2t) + \psi\psi_t = 0;\) at last, setting \(\psi = t^{-1/2}v(\xi, \tau), \tau = 2\sqrt{t},\) we finally derive the Burgers–Hopf equation \(v_t + vv_x = 0,\) whose general solution reads \(v = F(\xi - v\tau),\) where \(F\) is an arbitrary function of a single argument. Going back to the original variables, we recover (87). It is interesting to remark that the exact solution (87) can be generalized to the case of the dKP equation in \((n + 1)\) dimensions: \((u_t + uu_x)_x + \sum_{j=1}^{n-1} u_{x_j}y_j = 0,\) allowing us to solve its Cauchy problem for small and localized initial data and investigate the associated breaking features [40].

### 3.3. Example 3: The invariant \(\psi^n_1 + a(\psi^n_2)^n\)

If \(E(q, p) = q + ap^n,\) where \(a\) is a real parameter and \(n \in \mathbb{N}^+\), then equation (42) reads
\[q(\tau) = q_0 + ap^n_0 - a \left(p_0 - \mathcal{H}'(E)(\tau - \tau_0)\right)^n, \quad p(\tau) = p_0 - \mathcal{H}'(E)(\tau - \tau_0),\] (92)
becoming the NRH problem
\[
\begin{align*}
\psi^n_1 &= \psi_1^- + a\psi_2^n - a(\psi_2^- - i f(\psi_1^- + a\psi_2^n))^n, \\
\psi^n_2 &= \psi_2^- - i f(\psi_1^- + a\psi_2^n),
\end{align*}
\] (93)
whose data satisfy the symplectic (3) and reality (25) constraints. Due to the invariance equation (47)
\[\psi^n_1 + a\psi^n_2 = \psi_1^- + a\psi_2^n = -i\lambda^2 - 2yt + x - 2ut + a \left(\psi_2^n\right)_x \equiv W(\lambda),\] (94)
the NRH problem linearizes and decouples and equations (55) become (94) and
\[\psi_2^\pm(\lambda) = -i f^\pm(\lambda) = \psi_2^- + i f^-(\lambda) = \lambda.\] (95)

Then the solution of the NRH problem reads
\[
\begin{align*}
\psi_1^\pm &= W(\lambda) - a\psi_2^\pm, \\
\psi_2^\pm &= \lambda - if^\pm(\lambda).\end{align*}
\] (96)
(97)

Since now \(W\) depends on the \((n - 1)\) unknowns \(u, q_2^{(n)}, n = 2, \ldots, n - 1,\) the system of \((n - 1)\) equations (86), for \(n = 1, \ldots, n - 1,\) is an algebraic system characterizing a family of implicit solutions of dKP parametrized by the arbitrary real function \(f\) on a single variable. The corresponding differential constraint (59) reads, due to (24),
\[\left(\psi_2^{(n)}\right)_{x_1} = yu_x - 2tu_y + anu_x = 0,\] (98)
where \(u_{n_x}\) is the \(n\)th flow of the dKP sub-hierarchy (24).

For \(n = 2,\) the invariant
\[
\begin{align*}
\psi_1^\pm + a\psi_2^\pm &= -\tau_2\lambda^2 - y\lambda + x - 2u\tau_2 \equiv W(\lambda),
\end{align*}
\] (99)
where \(\tau_2 = t - a,\) is equivalent to (79), up to trivial shifts of \(y\) and \(t;\) therefore, the corresponding solution of dKP is essentially the same as that in section 3.2.

Now we show that, if \(n > 2,\) this family of solutions describes the wave breaking of a localized two-dimensional wave evolving according to dKP. To do it, it is convenient to
view (93) as a NRH problem on the real line in the shifted variable \( \mu: \mu = \lambda + y/2t, \ t \neq 0, \) and rewrite the asymptotics (13) in terms of \( \mu: \)

\[
\psi_1^\pm = -t \mu^2 + \xi - 2t q_2^{(1)} + \sum_{n \geq 1} \frac{\tilde{q}_1^{(n)}}{\mu^n},
\]

\[
\psi_2^\pm = \mu - \eta + \sum_{n \geq 1} \frac{\tilde{q}_2^{(n)}}{\mu^n}, \quad |\mu| \gg 1,
\]

where the new space variables \( \xi \) and \( \eta \) are defined by

\[
\xi \equiv x + \frac{y^2}{4t}, \quad \eta \equiv \frac{y}{2t}.
\]

In addition, equation (86) is replaced by

\[
\tilde{q}_2^{(n)} = (\mu^{-1} f)_{\mu} \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \mu^{n-1} f(\tilde{W}(\mu)) \, d\mu,
\]

where \( \tilde{W}(\mu) \equiv \left( \psi_1^+ + a \psi_2^+ \right)_\mu \) is now the non-negative part of the Laurent expansion, in the \( \mu \) variable, of the invariant. If \( n = 3 \), then

\[
\tilde{W}(\mu) = a \mu^3 - \tau_3 \mu^2 + 3a(\eta^2 + u) \mu + \xi - a \eta^3 - 2a \tau_3 + 3q_2^{(2)},
\]

where

\[
\tau_3 = t + 3a \eta.
\]

Since \( \tilde{W} \) depends on the two unknowns \( u, q_2^{(2)} \), the system of two equations (102), for \( n = 1, 2 \), is an algebraic system characterizing a family of implicit solutions of dKP parametrized by the arbitrary real function \( f \) of a single variable.

It is remarkable that, in the longtime regime and for \(|a| \ll 1 \), it is possible to obtain the following explicit asymptotic formulas for such a solution:

\[
u \sim \frac{1}{\sqrt{\tau_3}} F(\xi - a \eta^3 - 2a \tau_3),
\]

\[
\eta = O(1), \quad \xi - a \eta^3 - 2a \tau_3 = O(1), \quad t \gg 1, \quad |a| \ll 1,
\]

where \( F \) is defined by (88).

To derive (105), we first observe that the condition \(|a| \ll 1 \) implies

\[
\int_{\mathbb{R}} f(\tilde{W}(\mu)) \, d\mu \sim \int_{\mathbb{R}} [f(\tilde{W}(\mu)) + a f'(\tilde{W}(\mu)) \tilde{W}^{(1)}(\mu)] \, d\mu
\]

\[
= \int_{\mathbb{R}} [f(\tilde{W}(\mu)) + a f'(\tilde{W}(\mu)) \tilde{W}^{(1)}(\mu)] \, d\mu
\]

\[
\sim \int_{\mathbb{R}} f(\tilde{W}(\mu)) + a \tilde{W}^{(1)}(\mu) \, d\mu,
\]

where \( \tilde{W} = \tilde{W}(\mu) + a \tilde{W}^{(1)} \), with

\[
\tilde{W}(\mu) = (\psi_1^{(1) +})_\mu = -t \mu^2 + \xi - 2t, \quad (\psi_2^{(1) +})_\mu = \mu^3 - 3 \eta \mu^2 + 3(\eta^2 + u) \mu - \eta^3 - 6 \eta u + 3q_2^{(2)},
\]

and \( \tilde{W}^{(1)}(\mu) \) is the even part of \( \tilde{W}^{(1)}(\mu) \):

\[
\tilde{W}^{(1)}(\mu) = -3 \eta \mu^2 - \eta^3 - 6 \eta u + 3q_2^{(2)}.
\]

Second, the conditions \( t \gg 1 \), \( |\eta| = O(1) \) imply that \( \tau_3 \gg 1 \) and suggest the change of the integration variable \( \mu' = \sqrt{\tau_3} \mu \) in (106). Consequently, \( u = O(1/\sqrt{\tau_3}) = O(1/\sqrt{t}) \); the
same change of variables in the integrals (102) implies that \( \tilde{q}^{(n)}_2 = O(t^{-\frac{n-2}{2}}) \). Therefore, \( u \) can be neglected with respect to \( \eta \), and the coefficients \( \tilde{q}^{(n)}_2 \), \( n \geq 2 \) can be neglected with respect to \( u \).

Consequently,

\[
u \sim \frac{1}{2\pi} \int_{R} f(\tilde{W}(\mu) + a\tilde{W}_{\varepsilon}^{(1)}(\mu)) \, d\mu \sim \frac{1}{2\pi} \int_{R} f(-\tau_4 \mu^2 + \xi - a\eta^3 - 2u \tau_3) \]

\[
\quad = \frac{1}{2\pi \sqrt{\tau_3}} \int_{R} f(-\mu^2 + \xi - a\eta^3 - 2u \tau_3) \, d\mu' \quad (109)
\]

and formula (105) is derived.

Knowing the first breaking time \( \tau_b \) from the well-known formula

\[
\tau_b = \frac{1}{4F'(\xi_0)^2} = \min_{\xi \in R} \frac{1}{4F'_{\xi}(\xi)^2}, \quad F'(\xi_0) < 0,
\]

equation (104) implies that, if \( a > 0 \), the first breaking takes place when \( t_b = -\infty \) at \( y_b = -\infty \), outside the asymptotic region (105) of validity of our approximation, travelling towards the inner region (105) along the wave front. Now let \( t \) be close to \( \tau_b \); then, from (104),

\[
y = \frac{2}{3a} f(t_b - t) \sim \frac{2}{3a} \tau_b(t_b - t),
\]

(111)

implying that, in the asymptotic region (105), the breaking point moves approximately with the constant speed \( 2\tau_b/(3a) \) along the wave front.

If \( n = 4 \), then

\[
\tilde{W}(\mu) = -t \mu^2 + \xi - 2ut + a\tilde{\psi}_2^{(4)}(\mu) = -t \mu^2 + \eta \mu \eta^3 - 4\eta^4 - 4\eta \mu^3 + (\eta^4 - 4\eta \mu^3 + 6\mu^2) + a(4\eta^3 - 8\eta^2 + 4\mu^3 - 6\mu^2) \]

(112)

where

\[
\tau_4 = t - 6a\eta^2.
\]

Since now \( \tilde{W} \) depends on the three unknowns \( u, \tilde{q}^{(n)}_2, n = 2, 3 \), the system of three equations (102), for \( n = 1, 2, 3 \), is an algebraic system characterizing a family of implicit solutions of dKP parametrized by the arbitrary real function \( f \) on a single variable.

Following the same derivation as in the previous example, one obtains the following simple one-dimensional implicit asymptotic formula for such a family of solutions:

\[
u \sim \frac{1}{\sqrt{\tau_4}} F(\xi + a\eta^4 - 2u \tau_4), \quad (114)
\]

\[
\eta = O(1), \quad \xi + a\eta^4 - 2u \tau_4 = O(1), \quad t \gg 1, \quad 0 < a \ll 1,
\]

where \( F \) is again defined by (88). It is easy to see that, if the graph of \( F(z) \) is a single positive hump, (114) describes, before breaking, a saddle wave front with a saddle point \( (\zeta_0 + 2F(\zeta_0)/\sqrt{F}, 0) \), where \( \zeta_0 \) is the maximum of the hump: \( F'(\zeta_0) = 0 \) (see figure 1).

Known the first breaking time \( \tau_b \) from (110), equation (113) implies that, if \( a > 0 \), the first (physical) breaking time \( t_b \) is achieved at \( y_b = 0(\tau_b = y_b/2t_b = 0) \) and coincides with \( \tau_b \), while \( x_b \) follows from

\[
x_b = \zeta_b + 2F(\zeta_b)\sqrt{\tau_b}.
\]

(115)

We end this section briefly considering an arbitrary power \( n \in \mathbb{N}^+ \) (less generic and, consequently, less physically relevant than the lowest cases \( n = 3, 4 \)). In this case

\[
\tilde{W}(\mu) = -t \mu^2 + \xi - 2ut + a(\tilde{\psi}_2^{(n)}(\mu))_+.
\]

(116)
Figure 1. The saddle wave front described by (114), before breaking, for \( n = 4, a = 0.3 \), \( F(\zeta) = 0.2e^{-\zeta^2} \) and \( t = t_b - 1 \), where \( t_b = 25e/8 \).
(This figure is in colour only in the electronic version)

Since now \( \tilde{W} \) depends on the \((n-1)\) unknowns \( u, \tilde{q}_2^{(n)}, n = 2, \ldots, n - 1 \), the system of \((n-1)\) equations (102), for \( n = 1, \ldots, n - 1 \), is an algebraic system characterizing a family of implicit solutions of dKP parametrized by the arbitrary real function \( f \) on a single variable.

Proceeding as in the previous examples, one obtains the following simple one-dimensional implicit asymptotic formula for such a family of solutions,

\[
u \sim \frac{1}{\sqrt{\tau_n}} F(\xi + (-1)^n a \eta^n - 2u \tau_n),
\]

\[
\eta = O(1), \quad \xi + (-1)^n a \eta^n - 2u \tau_n = O(1), \quad t \gg 1, \quad 0 < a \ll 1,
\]

where

\[
\tau_n = t - (-1)^n a \left( \frac{n}{n-2} \right) \eta^{n-2}
\]

and \( F \) is defined by (88). Following the same considerations made in the particular case \( n = 4 \), if \( n \) is even and \( a > 0 \), the localized solution breaks first at time \( t_b = \tau_b \), where \( \tau_b \) is defined in (110), in the point \((x_b, 0)\), where \( x_b \) is defined in (115). We remark that formula (117) is an interesting particular case of the generic asymptotic formula (30) obtained in [35].

4. Solvable NRH problems for the heavenly equation

In this section, we consider two examples of solvable NRH problems allowing us to construct two different classes of similarity solutions of the heavenly equation.
4.1. Example 1. The invariant $\psi^+_1\psi^+_2$ and similarity solutions [38]

If $E(q, p) = qp$, then, as in section 3.1, the NRH problem reads

$$\psi^+_1 = \psi^-_1 e^{i f(\psi^-_1, \lambda)}, \quad \psi^+_2 = \psi^-_2 e^{-i f(\psi^-_2, \lambda)}$$

(satisfying the symplectic and reality constraints, where now $f$ is an arbitrary real function of two variables. Then the invariance equation (47) becomes

$$\psi^+_1\psi^+_2 = \psi^-_1\psi^-_2 = z\lambda^2 - (xt + yz)\lambda + xy - z\theta_x + t\theta_y \equiv W(\lambda),$$

and the NRH problem linearizes and decouples:

$$\psi^+_1 = \psi^-_1 e^{i f(W(\lambda), \lambda)}, \quad \psi^+_2 = \psi^-_2 e^{-i f(W(\lambda), \lambda)}.$$ (121)

Equations (55) become

$$\psi^+_j e^{i(-)^j f^+(\lambda)} = \psi^-_j e^{i(-)^j f^-(\lambda)} = A_j(\lambda), \quad j = 1, 2,$$ (122)

where

$$f^\pm(\lambda) \equiv \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda \pm i0)} f(W(\lambda'), \lambda'),$$ (123)

with the asymptotics

$$f^\pm(\lambda) \sim i \sum_{n \geq 1} (\lambda^{n-1} f)\lambda^{-n}, \quad \langle \lambda^n f \rangle \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^n f(W(\lambda), \lambda) d\lambda.$$ (124)

It follows that

$$A_1(\lambda) \equiv x - z\lambda - z\langle f \rangle, \quad A_2(\lambda) \equiv y - t\lambda + t\langle f \rangle,$$ (125)

implying the following explicit solution of the NRH problem:

$$\psi^+_j = A_j(\lambda) e^{i(-)^j f^+(\lambda)}, \quad j = 1, 2.$$ (126)

A characterization of the corresponding solutions of the heavenly equation is obtained observing that, since $W$ in (120) depends on the unknowns $\theta_x$ and $\theta_y$, isolating the $1/\lambda$ terms of equations (122) for $|\lambda| \gg 1$ and using (34), (124), we obtain the algebraic system

$$\theta_y = x(f) - z(\langle \lambda f \rangle + \frac{1}{2}\langle f^2 \rangle), \quad \theta_x = y(f) - t(\langle \lambda f \rangle - \frac{1}{2}\langle f^2 \rangle).$$ (127)

for the unknowns $\theta_x$ and $\theta_y$. The constructed solutions of the heavenly equation, parametrized by an arbitrary real function of two arguments, correspond to the following differential reduction:

$$-W_{-1} = t\theta_x + y\theta_y - x\theta_x - z\theta_z = 0.$$ (128)

Substituting its general solution

$$\theta = B(\tilde{x}, \tilde{y}, \tilde{z}), \quad \tilde{x} = xt, \quad \tilde{y} = y/t, \quad \tilde{z} = zt,$$ (129)

into the heavenly equation, one obtains the following similarity reduction of (9):

$$\tilde{x}B_{\tilde{x}\tilde{x}} - \tilde{y}B_{\tilde{x}\tilde{y}} + \tilde{z}B_{\tilde{x}\tilde{z}} - B_{\tilde{y}\tilde{z}} + B_{\tilde{x}\tilde{x}}B_{\tilde{y}\tilde{y}} - B_{\tilde{x}\tilde{y}}^2 = 0.$$ (130)

Therefore, the algebraic system (127) characterizes the above similarity solutions of heavenly.
4.2. Example 2. The invariant $(\psi_1^+)^2 + (\psi_2^+)^2$ and rotationally invariant solutions

If $E(q, p) = (q^2 + p^2)/2$, then equation (42) reads

$$
q(\tau) = \cos H'(E)(\tau - \tau_0)q_0 + \sin H'(E)(\tau - \tau_0)p_0,
$$

$$
\rho(\tau) = -\sin H'(E)(\tau - \tau_0)q_0 + \cos H'(E)(\tau - \tau_0)p_0,
$$

corresponding to the NRH problem

$$
\psi_1^+ = \cosh f\left(\frac{\psi_1^2 + \psi_2^2}{2}, \lambda\right)\psi_1^- + i \sinh f\left(\frac{\psi_1^2 + \psi_2^2}{2}, \lambda\right)\psi_2^-,
$$

$$
\psi_2^+ = -i \sinh f\left(\frac{\psi_1^2 + \psi_2^2}{2}, \lambda\right)\psi_1^- + \cosh f\left(\frac{\psi_1^2 + \psi_2^2}{2}, \lambda\right)\psi_2^-,
$$

satisfying the symplectic and reality constraints. Then the invariance equation (47) becomes

$$
\psi_1^2 + \psi_2^2 = \psi_1^- + \psi_2^- = z^2 + at^2 - (xz + aty)\lambda + x^2 + ay^2
$$

$$
+ z\theta_y - at\theta_x = W(\lambda),
$$

and the NRH problem linearizes:

$$
\psi_1^+ = \cosh (f(W, \lambda)) \psi_1^- - i \sinh (f(W, \lambda)) \psi_2^-,
$$

$$
\psi_2^+ = i \sinh (f(W, \lambda)) \psi_1^- + \cosh (f(W, \lambda)) \psi_2^-.
$$

Equations (55) become

$$
cosh f^\pm(\lambda)\psi_1^\pm - i \sinh f^\pm(\lambda)\psi_2^\pm = A_1(\lambda),
$$

$$
i \sinh f^\pm(\lambda)\psi_1^\pm + \cosh f^\pm(\lambda)\psi_2^\pm = A_2(\lambda),
$$

where

$$
A_1(\lambda) = -z\lambda + x - t(f),
$$

$$
A_2(\lambda) = -t\lambda + y + z(f).
$$

Therefore, the explicit solution of the NRH problem reads

$$
\psi_1^\pm(\lambda) = \cosh f^\pm(\lambda)A_1(\lambda) + i \sinh f^\pm(\lambda)A_2(\lambda),
$$

$$
\psi_2^\pm(\lambda) = -i \sinh f^\pm(\lambda)A_1(\lambda) + \cosh f^\pm(\lambda)A_2(\lambda).
$$

A characterization of the corresponding solutions of the heavenly equation is obtained observing that, since $W$ in (120) depends on the unknowns $\theta_x$ and $\theta_y$, isolating the $1/\lambda$ terms of equations (135) for $|\lambda| \gg 1$ and using (34) and (124), one obtains the algebraic system

$$
\theta_y = y(f) - t\left(\frac{f}{2}\right),
$$

$$
\theta_x = x(f) - z\left(\frac{f}{2}\right),
$$

for the unknowns $\theta_x$ and $\theta_y$. These solutions, parametrized by an arbitrary real function of two arguments, correspond to the following differential reduction:

$$
-W_{-1}/2 = z\theta_t + x\theta_y - ay\theta_x - at\theta_z = 0.
$$

Substituting its general solution,

$$
\theta = B(\alpha, \beta, \gamma), \quad \alpha = x^2 + ay^2, \quad \beta = z^2 + at^2, \quad \gamma = xt - yz,
$$
into the heavenly equation, one obtains the following reduction of (9):

\[
2B_{\gamma} + y B_{\gamma\gamma} + 2\alpha B_{\alpha\gamma} + 2\beta B_{\beta\gamma} + 4\alpha \gamma B_{\alpha\gamma} + 4\beta B_{\beta\gamma} + 2B_{\gamma}(\beta B_{\gamma\gamma} + 2\alpha B_{\alpha\gamma} + 4\alpha \gamma B_{\alpha\gamma}) \\
+ 4(\alpha \beta - \gamma^2)(B_{\alpha\alpha} B_{\gamma\gamma} - B_{\alpha\gamma}^2) = 0.
\]

(141)

Therefore, the algebraic system (138) characterizes a class of rotationally invariant solutions of heavenly.

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