ON THE HYPERGRAPH CONNECTIVITY OF SKELETA OF POLYTOPES

DANIEL HATHCOCK AND JOSEPHINE YU

Abstract. We show that for every $d$-dimensional polytope, the hypergraph whose nodes are $k$-faces and whose hyperedges are $(k + 1)$-faces of the polytope is strongly $(d - k)$-vertex connected, for each $0 \leq k \leq d - 1$.

1. Introduction

Balinski proved that the edge graph of any $d$-dimensional polytope is $d$-vertex connected [Bal61]. That is, removing fewer than $d$ of the vertices leaves the remaining vertices connected via edges. A number of natural generalizations of this result have since been investigated. Sallee found bounds for several different notions of connectivity of incidence graphs between $r$-faces and $s$-faces of a polytope [Sal67]. More recently, Athanasiadis considered the graphs $G_k(P)$ for a convex polytope $P$, whose nodes are the $k$-faces of $P$, and with two nodes adjacent if the corresponding $k$-faces are both contained in the same $(k + 1)$-face. Vertex connectivity of $G_k(P)$ is equivalent to one of the connectivity notions on the incidence graphs considered by Sallee. Athanasiadis described exactly the minimum vertex connectivity of $G_k(P)$ over all $d$-polytopes for every $k$ and $d$ [Ath09].

Let $P$ be a convex $d$-dimensional polytope. We denote by $H_k(P)$ the hypergraph whose nodes are the $k$-faces of polytope $P$, and whose hyperedges correspond naturally to the $(k + 1)$-faces of $P$. We say a hypergraph is strongly $\alpha$-vertex connected if removing fewer than $\alpha$ nodes along with all hyperedges incident to each removed node leaves the remaining nodes connected. Using tropical geometry, Maclagan and the second author showed that for every rational $d$-polytope, $H_k(P)$ is strongly $(d - k)$-vertex connected [MY19]. Our main result is generalizing this statement to all polytopes:

Theorem 1. For every $d$-polytope $P$, the hypergraph $H_k(P)$ is strongly $(d - k)$-vertex connected, for each $0 \leq k \leq d - 1$.

The result is tight. For simple polytopes, each $k$-face is contained in exactly $d - k$ of the $(k + 1)$-faces, so the hypergraph $H_k(P)$ cannot have higher connectivity.

2. Proof of the result

We say that a pure $k$-dimensional polyhedral complex is $c$-connected through codimension one if after removing fewer than $c$ closed maximal faces, the remaining maximal faces are connected via paths through faces of dimension $k - 1$. That is, for any two remaining
maximal faces $F, F'$, there remains a sequence $F = G_1, \ldots, G_\ell = F'$ of maximal faces such that for each $i$, $G_i \cap G_{i+1}$ is a face of dimension $k - 1$ not belonging to a removed face. The $m$-skeleton of a polytope $Q$ is the polyhedral complex whose maximal faces are the $m$-dimensional faces of $Q$. Then Theorem 1 can be rephrased as the following equivalent form on the polar dual $Q = P^\Delta$.

**Theorem 2.** For every $d$-polytope $Q$, the $(d - k - 1)$-skeleton is $(d - k)$-connected through codimension one, for each $0 \leq k \leq d - 1$. Equivalently, the $k$-skeleton of $Q$ is $(k + 1)$-connected through codimension one for each $0 \leq k \leq d - 1$.

We will need some lemmas before proceeding with the proof by induction on dimension.

**Lemma 3.** Let $F, G, R$ be three distinct $k$-faces of a $d$-polytope $Q$, for some $1 \leq k \leq d - 1$. Then there is a hyperplane intersecting $F$ and $G$ and avoiding $R$. Moreover, the hyperplane can be chosen to avoid all vertices of $Q$.

**Proof.** Let $f \in F$ and $g \in G$ be relative interior points, and let $L$ be the line through $f$ and $g$. Let $Q'$ be the smallest face of $Q$ containing $F \cup G$. By convexity, $L \cap Q \subset Q'$ and $L$ meets the boundary of $Q'$ only at the two points $f$ and $g$. In particular $L$ does not meet $R$ or any other face of dimension $\leq k$.

We may assume that $Q$ is a $d$-dimensional polytope in $\mathbb{R}^d$. Let $\pi$ be a corank one linear map from $\mathbb{R}^d$ to $\mathbb{R}^{d-1}$ such that the image of $L$ is a point. Then the image $R' = \pi(R)$ does not contain $\pi(L)$, and each vertex $v_1, \ldots, v_n$ of $Q$ has $v_i = \pi(v_i) \neq \pi(L)$ since $L$ does not contain any of the vertices.

Since $R'$ is convex and does not contain $\pi(L)$, there is a hyperplane through $\pi(L)$ which does not meet $R'$. Since $R'$ is compact, the set of normal vectors of such hyperplanes form a full dimensional open set in $\mathbb{R}^{d-1}$. (More precisely, it is the interior of the dual cone, and its negative, of the pointed cone generated by $R'$ after a translation that sends $\pi(L)$ to the origin.) On the other hand, the condition that such a hyperplane contains each $v'_i$ is a codimension one closed condition. Thus, as there are finitely many $v'_i$, the cone of such normal vectors restricted to those whose hyperplane does not contain any $v'_i$ is non-empty. In particular, there is a hyperplane $H'$ through $\pi(L)$ which does not meet $R'$ or any of the $v'_i$. Its preimage $\pi^{-1}(H)$ is a desired hyperplane. \qed

**Lemma 4.** Let $Q$ be a polytope and $H$ a hyperplane intersecting $Q$ but not containing any vertices of $Q$. The map $\phi : F \mapsto F \cap H$ is a poset isomorphism from the poset of faces of $Q$ that meet $H$ to the face poset of $Q \cap H$.

**Proof.** For any face $F$ of $Q$ which meets $H$, since $H$ does not contain any vertices of $F$, $F$ is not contained in $H$ and $H$ meets the relative interior of $F$, so $\dim(F \cap H) = \dim(F) - 1$. Moreover, $F \cap H$ is indeed a face of $Q \cap H$: any supporting hyperplane for $F$ in $Q$ is also a supporting hyperplane for $F \cap H$ in $Q \cap H$. On the other hand, for any face $F'$ of $Q \cap H$, let $x \in F'$ be a relative interior point in $F'$, and let $F$ be the unique face of $Q$ for which $x$ is a relative interior point. Then $x$ is also in the relative interior of $F \cap H$. Since $F'$ and $F \cap H$ are two faces of $Q \cap H$ that meet in their relative interiors, we have $F \cap H = F'$. So $\phi$ is a surjective map between the desired sets. If $F \cap H = G \cap H$ for $k$-faces $F, G$ meeting $H$, then $F$ and $G$ would have a common relative interior point, which implies $F = G$. Thus $\phi$ is injective. It is clear that $\phi$ preserves the inclusion relation. \qed
Proof of Theorem 2. We will use induction on $k$. The statement is trivial for $k = 0$, as we are not removing any faces, and the vertices of a polytope are connected through the empty face. The case when $k = 1$ is clear, as removing a single edge does not disconnect the vertex-edge graph of any polytope.

Suppose $2 \leq k \leq d - 1$. Let $Q$ be a $d$-polytope and $B$ be any set of $k$ $k$-faces of $Q$ to remove. We need to find a path between any two $k$-faces $F, G \notin B$, through codimension-one faces, which we will call ridge paths. Arbitrarily choose any $R \in B$. Lemma 3 gives a hyperplane $H$ intersecting $F$ and $G$, and avoiding $R$ and vertices of $Q$. Let $Q' = Q \cap H$. Since $H$ intersects $F$ and $G$, $F' = F \cap H$ and $G' = G \cap H$ are two $(k - 1)$-faces of $Q'$ by Lemma 4. Moreover, each face in $B \setminus \{R\}$ corresponds to at most one $(k - 1)$-dimensional face in $Q'$. Call these faces $B'$. As $|B'| \leq k - 1$, by induction there is a ridge path in $Q'$ connecting $F'$ to $G'$ and avoiding each face in $B'$. Using Lemma 4, we can lift this path back up to a ridge path connecting $F$ to $G$ in $Q$ avoiding $B$.

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DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH PA, US

Email address: dhathcoc@andrew.cmu.edu

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA GA, US

Email address: jyu@math.gatech.edu