A splitting algorithm for stochastic partial differential equations driven by linear multiplicative noise

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Abstract
We study the convergence of a Douglas-Rachford type splitting algorithm for the infinite dimensional stochastic differential equation
\[ dX + A(t)(X)dt = X dW \text{ in } (0,T); \ X(0) = x, \]
where \( A(t) : V \rightarrow V' \) is a nonlinear, monotone, coercive and demicontinuous operator with sublinear growth and \( V \) is a real Hilbert space with the dual \( V' \). \( V \) is densely and continuously embedded in the Hilbert space \( H \) and \( W \) is an \( H \)-valued Wiener process. The general case of a maximal monotone operators \( A(t) : H \rightarrow H \) is also investigated.

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1 Introduction
We consider here the stochastic differential equation
\[ dX(t) + A(t)X(t)dt = X(t)dW(t), \ t \in (0, T), \]
\[ X(0) = x, \]
in a real separable Hilbert space $H$, whose elements are functions or distributions on a bounded and open set $O \subset \mathbb{R}^d$ with smooth boundary $\partial O$.

In particular, $H$ can be any of the spaces $L^2(O)$, $H^1_0(O)$, $H^{-1}(O)$, $H^k(O)$, $k = 1, 2, \ldots,$ with the corresponding Hilbertian structure. Here $H^1_0(O)$, $H^k(O)$ are the standard $L^2$-Sobolev spaces on $O$, and $W$ is a Wiener process of the form

$$W(t, \xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi) \beta_j(t), \ \xi \in O, \ t \geq 0, \quad (1.2)$$

where $\{\beta_j\}_{j=1}^{\infty}$ is an independent system of real-valued Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with natural filtration $(\mathcal{F}_t)_{t \geq 0}$. Here, $e_j \in C^2(O) \cap H$, $j \in \mathbb{N}$, is an orthonormal basis in $H$, and $\mu_j \in \mathbb{R}$, $j = 1, 2, \ldots$.

The following hypotheses will be in effect throughout this work.

(i) There is a Hilbert space $V$ with dual $V'$ such that $V \subset H$, continuously and densely. Hence $V \subset H'$ continuously and densely.

(ii) $A : [0, T] \times V \times \Omega \to V'$ is progressively measurable, i.e., for every $t \in [0, T]$, this operator restricted to $[0, t] \times V \times \Omega$ is $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$-measurable.

(iii) There is $\delta \geq 0$ such that, for each $t \in [0, T]$, $\omega \in \Omega$, the operator $u \mapsto \delta u + A(t, \omega)u$ is monotone and demicontinuous (that is, strongly-weakly continuous) from $V$ to $V'$.

Moreover, there are $\alpha_i, \gamma_i \in \mathbb{R}$, $i = 1, 2, 3$, $\alpha_1 > 0$, such that, $\mathbb{P}$-a.s.,

$$\langle A(t, \omega)u, u \rangle \geq \alpha_1 |u|^2_V + \alpha_2 |u|^2_H + \alpha_3, \ \forall u \in V, \ t \in [0, T], \quad (1.3)$$

$$|A(t, \omega)u|_{V'} \leq \gamma_1 |u|_V + \gamma_2, \ \forall u \in V, \ t \in [0, T]. \quad (1.4)$$

(iv) $e^{\pm W(t)}$ is, for each $t$, a multiplier in $V$ and a multiplier in $H$ such that there exists an $(\mathcal{F}_t)$-adapted, $\mathbb{R}_+$-valued process $Z(t)$, $t \in [0, T]$, with

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |Z(t)| \right] < \infty \text{ for all } r \in [1, \infty) \text{ and such that, } \mathbb{P}-a.s.,$$

$$|e^{\pm W(t)} y|_V \leq Z(t)|y|_V, \ \forall t \in [0, T], \ \forall y \in V,$n
$$|e^{\pm W(t)} y|_H \leq Z(t)|y|_H, \ \forall t \in [0, T], \ \forall y \in H. \quad (1.5)$$
One assumes also that, for each $\omega \in \Omega$, the function $t \rightarrow e^{\pm W(t)}$ is $H$-valued continuous on $[0,T]$.

Throughout in the following, $|\cdot|_V$ and $|\cdot|_{V'}$ denote the norms of $V$ and $V'$, respectively, and by $\langle \cdot, \cdot \rangle$ we denote the duality pairing between $V$ and $V'$ with $H$ as pivot space; on $H \times H$, $\langle \cdot, \cdot \rangle$ is just the scalar product of $H$. The norms of $V$ and $V'$ are denoted by $|\cdot|_H$ and $|\cdot|_V$, $|\cdot|_{V'}$, respectively, $\mathcal{B}(H)$, $\mathcal{B}(V)$ etc. are the classes of Borel sets in the corresponding spaces.

As regards the orthonormal basis $\{e_j\}_{j=1}^{\infty}$ in (1.2), we assume that there exist $\tilde{\gamma}_j \in [1,\infty)$ such that

$$|ye_j|_H \leq \tilde{\gamma}_j|y|_H, \forall y \in H, j = 1, 2, ..., \nu := \sum_{j=1}^{\infty} \mu_j^2 \tilde{\gamma}_j^2 |e_j|_H^2 < \infty. \tag{1.6}$$

and we assume also that

$$\mu := \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 e_j^2 \tag{1.7}$$

is a multiplier in $V$, $V'$ and $H$.

It should be noted that $X \, dW = \sigma(X) \, d\tilde{W}$ where $\sigma : H \rightarrow L^2(H)$ (the space of Hilbert-Schmidt operators on $H$) is defined by

$$\sigma(u)v = \sum_{j=1}^{\infty} \mu_j u \langle v, e_j \rangle e_j, \forall v \in H,$$

and so, $\tilde{W} = \sum_{j=1}^{\infty} e_j \beta_j$ is a cylindrical Wiener process on $H$ (see [5]).

**Definition 1.1.** By a solution to (1.1) for $x \in H$, we mean an $(\mathcal{F}_t)_{t \geq 0}$-adapted process $X : [0,T] \rightarrow H$ with continuous sample paths which satisfies

$$X(t) + \int_0^t A(s)X(s)ds = x + \int_0^t X(s)dW(s), t \in [0,T], \tag{1.8}$$

$$X \in L^2((0,T) \times \Omega; V). \tag{1.9}$$

The stochastic integral arising in (1.8) is considered in Itô’s sense.

In [3], the authors developed an operatorial approach to (1.1) under the more general hypotheses than (i)–(iv) above. As a special case (see Theorem 3.1 in [3]), we have

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Theorem 1.2. Under Hypotheses (i)–(iv), for each $x \in H$, equation (1.1) has a unique solution $X$ (in the sense of Definition 1.1). Moreover, the function $t \mapsto e^{-W(t)}X(t)$ is $V'$-absolutely continuous on $[0,T]$ and
\[
\mathbb{E} \int_0^T \left| \frac{d}{dt} (e^{-W(t)}X(t)) \right|^2_{V'} \, dt < \infty. \tag{1.10}
\]

In a few words, the method developed in [3] is the following. By the transformation
\[
X(t) = e^{W(t)}y(t), \quad t \geq 0, \tag{1.11}
\]
one reduces equation (1.1) to the random differential equation
\[
\frac{dy}{dt}(t) + e^{-W(t)}A(t)(e^{W(t)}y(t)) + \mu y(t) = 0, \quad \text{a.e. } t \in (0,T),
\]
y(0) = x, \tag{1.12}
and treat (1.12) as an operatorial equation of the form
\[
By + Ay = 0 \tag{1.13}
\]
in a suitable Hilbert space $H$ of stochastic processes on $[0,T]$. Here, $A$ and $B$ are maximal monotone operators suitably defined from $V$ to $V'$, where $(V, V')$ is a dual pair of spaces such that $V \subset H \subset V'$ with dense and continuous embeddings.

The operatorial form (1.13) of equation (1.12) suggests to approximate the solution $y$ by the Douglas–Rachford splitting algorithm ([6]–[8]).

The exact form and convergence of the corresponding splitting algorithm for equation (1.13) will be given below in Section 2. As seen later on in Theorem 2.1, it leads to a convergent splitting algorithm for the stochastic differential equation (1.1).

In this way, the operator theoretic approach to equation (1.1) written in the form (1.13) allows to design a convergent splitting scheme for equation (1.1) inspired by the Rockafellar [9] proximal point algorithm for nonlinear operatorial equations (on these lines see also [4]). By our knowledge, the splitting algorithm obtained here for the stochastic equation is new and might have implications in numerical approximation of stochastic PDEs.

**Notations.** If $U$ is a Banach space, we denote by $L^p(0,T;U)$, $1 \leq p \leq \infty$, the space of all $L^p$-integrable $U$-valued functions on $(0,T)$. The space
is defined similarly. We refer to [2] for notation and standard results of the theory of maximal monotone operators in Banach spaces. If $O$ is an open domain of $\mathbb{R}^d$, we denote by $W^{1,p}(O)$, $1 \leq p \leq \infty$ and $H^1(O)$, $H^{-1}(O)$ the standard Sobolev spaces on $O$.

2 Main results

Without loss of generality, we may assume that, besides assumptions (i)–(iii), $A(t)$ satisfies also the strong monotonicity condition

$$\langle A(t)u - A(t)v, u - v \rangle \geq \nu|u - v|^2_H, \ \forall u, v \in V,$$

where $\nu > 0$ is given by (1.6). (In fact, as easily seen, by the substitution $X \to \exp(-(\nu + \delta)t)X$ with a suitable $\delta$, equation (1.1) can be rewritten as

$$dX + \tilde{A}(t)X dt = X dW,$$

where the operator $X \to \tilde{A}(t)X = e^{-(\nu + \delta)t}A(t)(e^{(\nu + \delta)t}X) + (\nu + \delta)X$ satisfies conditions (i)–(iii) and (2.1).)

We associate with equation (1.1) the following splitting algorithm

$$\lambda dZ_{n+1} + J(Z_{n+1}) dt + \lambda \nu Z_{n+1} dt = \lambda Z_{n+1} dW - \lambda A(t)X_n dt + \lambda \nu X_n dt + J(X_n) dt, \ t \in (0, T), (2.2)$$

$$Z_{n+1}(0) = x, \ n = 0, 1, ...$$

$$\lambda A(t)X_{n+1}(t) + J(X_{n+1}(t)) - \lambda \nu X_{n+1}(t) = J(Z_{n+1}(t)) + \lambda A(t)X_n(t) - \lambda \nu X_n(t), (2.3)$$

where $X_0 \in L^2((0, T) \times \Omega; V)$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted and arbitrary. Here, the parameter $\lambda > 0$ is arbitrary but fixed and $J : V \to V'$ is the canonical isomorphism of the space $V$ onto its dual $V'$.

Taking into account assumptions (i)–(iii) and (2.1), which, in particular, implies that the operator $\Gamma_0 : L^2(0, T; V) \to L^2(0, T; V')$, $\Gamma_0 u = \lambda A(t)u + J(u) - \lambda \nu u$, $u \in L^2(0, T; V)$, is demicontinuous, locally bounded, and with inverse continuous, we see that the sequence $(Z_n, X_n)$ is well defined by (2.2), (2.3) and we have also

$$X_n, Z_n \in L^2((0, T) \times \Omega; V) \text{ and } Z_n \in L^2(\Omega; C([0, T]; H)), \ n = 1, 2, ... \ (2.4)$$

Moreover, the processes $X_n, Z_n$ are $(\mathcal{F}_t)_{t \geq 0}$-adapted on $[0, T]$.

Theorem 2.1 is the main result.
Theorem 2.1. Under Hypotheses (i)–(iv) and (2.1), assume that \( x \in V \) and \( \lambda > 0 \). If \( (X_n, Z_n) \) is the sequence defined by (2.2), (2.3), we have for \( n \to \infty \)

\[
X_n \to X \text{ weakly in } L^2((0,T) \times \Omega; V),
\]

where \( X \) is the solution to equation (1.1) given by Theorem 1.2. Assume further that the operator \( u \to A(t)u \) is odd, that is, \( A(t)(-u) = -A(t)u, \forall u \in V. \) Then, for \( n \to \infty \),

\[
X_n \to X \text{ strongly in } L^2((0,T) \times \Omega; V). \quad (2.6)
\]

The splitting scheme (2.2)–(2.3) reduces the approximation of problem (1.1) to a sequence of simpler linear equations. In fact, at each step \( n \), one should solve a linear stochastic differential equation of the form

\[
dZ_{n+1} + \frac{1}{\lambda} J(Z_{n+1}) dt + \nu Z_{n+1} dt = Z_{n+1} dW + F_n dt, \quad t \in (0,T),
\]

\[
Z_{n+1}(0) = x,
\]

and the stationary random equation (2.3), where

\[
F_n = -\lambda A(t)X_n + \lambda \nu X_n + J(X_n).
\]

By Itô’s formula (see, e.g., [3]), equation (2.7) has, for each \( n \), the solution

\[
Z_{n+1} = e^W z_{n+1},
\]

where \( z_{n+1} \) is the solution to the random differential equation

\[
\frac{d}{dt} z_{n+1} + \frac{1}{\lambda} e^{-W} J(e^W z_{n+1}) + (\mu + \nu) z_{n+1} = e^{-W} F_n,
\]

\[
z_{n+1}(0) = x.
\]

If \( F : L^2((0,T) \times \Omega; V) \to L^2((0,T) \times \Omega; V) \) is the linear continuous operator defined by

\[
F(f) = Y,
\]

where \( Y \) is the solution to the stochastic equation

\[
dY + \frac{1}{\lambda} J(Y) dt + \nu Y dt = Y dW + f dt; \quad Y(0) = x,
\]
then we may rewrite (2.2)–(2.3) as

\[ X_{n+1} = (\lambda(A - \nu I) + J)^{-1}[JF((\lambda(\nu I - A) + J)(X_n)) + \lambda(A - \nu I)X_n], \quad n = 0, 1, \ldots \]

Equivalently,

\[ X_{n+1} = \Gamma^n X_0, \quad \forall n \in \mathbb{N}, \tag{2.9} \]

where \( \Gamma : L^2((0, T) \times \Omega; V) \to L^2((0, T) \times \Omega; V) \) is the Lipschitzian and given by

\[ \Gamma = (\lambda(A - \nu I) + J)^{-1}[JF(\lambda(\nu I - A) + J) + \lambda(A - \nu I)]. \tag{2.10} \]

Then, by Theorem 2.1, we get

**Corollary 2.2.** Under assumptions (i)-(iv), (2.1), for each \( \lambda > 0 \) the solution \( X \) to (1.1) is expressed as

\[ X = w - \lim_{n \to \infty} \Gamma^n X_0 \text{ in } L^2((0, T) \times \Omega; V), \tag{2.11} \]

where \( X_0 \in L^2((0, T) \times \Omega; V) \) is an arbitrary \((\mathcal{F}_t)_{t \geq 0}\)-adapted process.

Here \( w - \lim \) indicates the weak limit.

## 3 Proof of Theorem 2.1

Proceeding as in [3], we consider the spaces \( \mathcal{H}, \mathcal{V} \) and \( \mathcal{V}' \), defined as follows. \( \mathcal{H} \) is the Hilbert space of all \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes \( y : [0, T] \to H \) such that

\[ |y|_\mathcal{H} = \left( \mathbb{E} \int_0^T |e^{W(t)}y(t)|_H^2 dt \right)^{\frac{1}{2}} < \infty, \]

where \( \mathbb{E} \) denotes the expectation in the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The space \( \mathcal{H} \) is endowed with the norm \( | \cdot |_\mathcal{H} \) generated by the scalar product

\[ \langle y, z \rangle_\mathcal{H} = \mathbb{E} \int_0^T \langle e^{W(t)}y(t), e^{E(t)}y(t) \rangle dt. \]

\( \mathcal{V} \) is the space of all \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes \( y : [0, T] \to V \) such that

\[ |y|_\mathcal{V} = \left( \mathbb{E} \int_0^T |e^{W(t)}y(t)|_V^2 dt \right)^{\frac{1}{2}} < \infty. \]
\( \mathcal{V}' \) (the dual of \( \mathcal{V} \)) is the space of all \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes \( y : [0, T] \to \mathcal{V}' \) such that
\[
|y|_{\mathcal{V}'} = \left( \mathbb{E} \int_0^T |e^{W(t)}y(t)|^2_{\mathcal{V}'} dt \right)^{\frac{1}{2}} < \infty.
\]
We have \( \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}' \) with continuous and dense embeddings. Moreover,
\[
\mathcal{V}' \langle u, v \rangle_{\mathcal{V}} = \mathbb{E} \int_0^T \langle e^{W(t)}u(t), e^{W(t)}v(t) \rangle \, dt, \ v \in \mathcal{V}, \ u \in \mathcal{V}',
\]
is the duality pairing between \( \mathcal{V} \) and \( \mathcal{V}' \), with the pivot space \( \mathcal{H} \), that is,
\[
\mathcal{V}' \langle u, v \rangle_{\mathcal{V}} = \langle u, v \rangle_{\mathcal{H}}, \ \forall u \in \mathcal{H}, \ v \in \mathcal{V}.
\]
Now, for \( x \in H \), define the operators \( \mathcal{A} : \mathcal{V} \to \mathcal{V}' \) and \( \mathcal{B} : D(\mathcal{B}) \subset \mathcal{V} \to \mathcal{V}' \) as follows:
\[
(\mathcal{A}y)(t) = e^{-W(t)}A(t)(e^{W(t)}y(t)) - \nu y(t), \ \text{a.e.} \ t \in (0, T), \ y \in \mathcal{V},
\]
\[
(\mathcal{B}y)(t) = \frac{dy}{dt}(t) + (\mu + \nu)y(t), \ \text{a.e.} \ t \in (0, T), \ y \in D(\mathcal{B}),
\]
\[
D(\mathcal{B}) = \left\{ y \in \mathcal{V} : y \in AC([0, T]; V') \cap C([0, T]; \mathcal{H}), \mathbb{P}\text{-a.s.,} \right. \left. \frac{dy}{dt} \in \mathcal{V}', \ y(0) = x \right\}.
\]
Here, \( AC([0, T]; V') \) is the space of all absolutely continuous \( V' \)-valued functions on \([0, T]\). If \( y \in D(\mathcal{B}) \), then \( y \in C([0, T]; \mathcal{H}) \) and \( \frac{dy}{dt} \) is the derivative of \( y \) in the sense of \( V' \)-valued distributions on \((0, T)\). Then, equation (1.12) can be expressed as
\[
\mathcal{B}y + \mathcal{A}y = 0.
\]
Then, the map \( \Lambda : \mathcal{V} \to \mathcal{V}' \) defined by
\[
\Lambda v = e^{-W}J(e^Wv), \ v \in \mathcal{V},
\]
is the canonical isomorphism of \( \mathcal{V} \) onto \( \mathcal{V}' \) and the scalar product \( \mathcal{V} \langle \cdot, \cdot \rangle_{\mathcal{V}} \) of the space \( \mathcal{V} \) can be expressed as
\[
\mathcal{V} \langle v, \bar{v} \rangle_{\mathcal{V}} = \mathcal{V} \langle v, \Lambda \bar{v} \rangle_{\mathcal{V}'}, \ \forall v, \bar{v} \in \mathcal{V}.
\]
We set
\[(A^* u)(t) = \Lambda^{-1} A u(t) = e^{-W J^{-1}(A(t)(e^W u) - \nu e^W u)}, \forall u \in \mathcal{V}, \quad (3.6)\]
\[(B^* u)(t) = \Lambda^{-1} B u(t) = e^{-W J^{-1}\left(e^W \left(\frac{du}{dt} + (\mu + \nu)u\right)\right)}, \forall u \in D(B^*) = D(B). \quad (3.7)\]

Since the operators \(A\), \(B\) and \(A + B\) are maximal monotone in \(\mathcal{V} \times \mathcal{V}'\) ([3], Lemma 4.1, Lemma 4.2), it is easily seen by (3.6)-(3.7) that \(A^*, B^*\) and \(A^* + B^*\) are maximal monotone in \(\mathcal{V} \times \mathcal{V}\).

On the other hand, by (3.3) we can rewrite equation (3.3) as
\[B^* y + A^* y = 0. \quad (3.8)\]

Let \(y \in D(B)\) be the unique solution to equation (3.3) (see [3], Proposition 3.3). Then, \(y\) is also the solution to (3.8) and so, by Theorem 1 in [8] (see, also, Corollary 6.1 in [7]), we have that
\[y = \lim_{n \to \infty} (I + \lambda A^*)^{-1} v_n \text{ weakly in } \mathcal{V} \text{ as } n \to \infty, \quad (3.9)\]
where \(\{v_n\} \subset \mathcal{V}\) is, for \(n \geq 0\), defined by
\[v_{n+1} = (I + \lambda B^*)^{-1}(2(I + \lambda A^*)^{-1} v_n - v_n) + (I - (I + \lambda A^*)^{-1}) v_n, \quad (3.10)\]
and \(v_0\) is arbitrary in \(\mathcal{V}\). Here, \(I\) is the identity operator in \(\mathcal{V}\).

The splitting algorithm (3.9)-(3.10) is just the Douglas–Rachford algorithm ([6]) for equation (3.8) and it can be equivalently expressed as
\[y = \lim_{n \to \infty} y_n \text{ weakly in } \mathcal{V}, \quad (3.11)\]
\[y_n = (I + \lambda A^*)^{-1} v_n, \quad n = 0, 1, ..., \quad (3.12)\]
\[y_{n+1} + \lambda A^* y_{n+1} = z_{n+1} + v_n - y_n, \quad (3.13)\]
\[z_{n+1} + \lambda B^* z_{n+1} = 2 y_n - v_n, \quad (3.14)\]
where \(v_0 \in \mathcal{V}\). (To get (3.12)-(3.14) from (3.10), we have used the identity \((I + \lambda B^*)^{-1}(v + \lambda B^* v) = v, \forall v \in D(B^*)\) and the linearity of \(B^*\).)

In fact, the weak convergence of \(\{v_n\}\) in the space \(\mathcal{V}\) is also a consequence of the convergence of the Rockafellar proximal point algorithm [9] for the maximal monotone operator \(v \to G^{-1}(v) - v\), where
\[G(z) = (I + \lambda B^*)^{-1}(2(I + \lambda A^*)^{-1} z - z) + z - (I + \lambda A^*)^{-1} z, \forall z \in \mathcal{V}. \quad (3.15)\]
(See [7], Theorem 4.) Taking into account (3.6), (3.7), (3.12) we rewrite (3.14) as

\[ e^{-WJ(e^Wz_{n+1})} + \lambda \left( \frac{dz_{n+1}}{dt} + (\mu + \nu)z_{n+1} \right) \]

\[ = e^{-WJ(e^W(2y_n - v_n))} = e^{-WJ(e^W(-\lambda A^*y_n + y_n))} \]

\[ = -\lambda e^{-W}A(t)(e^Wy_n) + \lambda \nu y_n + e^{-W}J(e^Wy_n) \]

and (3.13) as

\[ J(e^Wy_{n+1}) + \lambda A(t)(e^Wy_{n+1}) - \lambda \nu e^Wy_{n+1} \]

\[ = J(e^W(z_{n+1} + v_n - y_n)). \]  

(3.17)

We set

\[ X_n = e^Wy_n, \quad Z_n = e^Wz_n. \]

Then, by (3.16), we get via Itô’s formula (see [3] and (2.8), (2.7))

\[ \lambda dZ_{n+1} + J(Z_{n+1})dt + \lambda \nu Z_{n+1}dt = \lambda Z_{n+1}dW - \lambda A(t)X_ndt \]

\[ + \lambda \nu X_n dt + J(X_n)dt, \]

\[ Z_{n+1}(0) = x. \]

By (3.17) and (3.12), we also get that

\[ \lambda A(t)X_{n+1}(t) + J(X_{n+1}(t)) - \lambda \nu X_{n+1}(t) \]

\[ = J(Z_{n+1}(t)) + \lambda A(t)X_n(t) - \lambda \nu X_n(t), \quad t \in (0,T), \]

which are just equations (2.2), (2.3). Moreover, by (3.11), we see that (2.5) holds.

Assume now that \( A(t) : V \rightarrow V' \) is odd. Then so is \( A^* : V \rightarrow V \) and also the operator \( G \) defined by (3.15). Then, according to a result of J. Baillon [1], the sequence \( \{v_n\} \) defined by (3.10), that is \( v_{n+1} = G(v_n) \), is strongly convergent in \( V \). Recalling (3.9), we infer that so is the sequence \( \{y_n\} \) and, consequently, (2.6) holds. This completes the proof of Theorem 2.1.

**Remark 3.1.** One might expect that a similar splitting scheme can be constructed for nonlinear monotone operators \( A(t) : V \rightarrow V' \), where \( V \) is a
reflexive Banach space and $A(t)$ are demicontinuous coercive and with polynomial growth as in [3]. In fact, in this case, one might replace (2.2) by

$$
\lambda dZ_{n+1} + Z_{n+1}dt + \lambda \nu Z_{n+1}dt = Z_{n+1}dW - \lambda A_H(t)X_n dt + \lambda \nu X_n dt + X_n dt,
$$

$$
t \in (0, T),
$$

$$
\lambda A_H(t)X_{n+1} + X_{n+1} - \lambda \nu X_{n+1} = Z_{n+1} + \lambda A_H(t)X_n - \lambda \nu X_n,
$$

where $A_H(t)u = A(t)u \cap H$. This question will be addressed in Section 5 below (see Remark 5.2).

4 Examples

We shall illustrate here the splitting algorithm (2.2)–(2.3) for a few parabolic stochastic differential equations.

Example 4.1. Nonlinear stochastic parabolic equations.

Consider the reaction-diffusion stochastic equation in $\mathcal{O} \subset \mathbb{R}^d$,

$$
dX - \text{div}(a(t, \xi, \nabla X))dt + \nu X dt + \psi(X)dt = X dW \text{ in } (0, T) \times \mathcal{O},
$$

$$
X = 0 \text{ on } (0, T) \times \partial \mathcal{O}, \quad X(0) = x \text{ in } \mathcal{O}.
$$

(4.1)

Here, $a : (0, T) \times \mathcal{O} \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable in $(t, \xi, r)$ continuous in $r$ on $\mathbb{R}^d$, $a(t, \xi, 0) = 0$. (The more general case, when $a : (0, T) \times \mathcal{O} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is progressively measurable, could also be considered.) We assume also that

$$
(a(t, \xi, r_1) - a(t, \xi, r_2)) \cdot (r_1 - r_2) \geq 0, \quad \forall r_1, r_2 \in \mathbb{R}^d, \quad (t, \xi) \in (0, T) \times \mathcal{O},
$$

$$
a(t, \xi, r) \cdot r \geq a_1 |r|^2 + a_2, \quad \forall r \in \mathbb{R}^d, \quad (t, \xi) \in (0, T) \times \mathcal{O},
$$

$$
|a(t, \xi, r)|_d \leq c_1 |r|_d + c_2, \quad \forall r \in \mathbb{R}^d, \quad (t, \xi) \in (0, T) \times \mathcal{O},
$$

where $a_1, c_1, \nu > 0$, $a_2, c_2 \in \mathbb{R}$, are independent of $(t, \xi)$, and $\psi : \mathbb{R} \to \mathbb{R}$ is a continuous and monotonically nondecreasing function such that $\psi(0) = 0$ and $|\psi(r)| \leq C(|r|^\frac{2\nu}{\nu+2} + 1), \forall r \in \mathbb{R}$. Here $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open subset with smooth boundary $\partial \mathcal{O}$, and $|\cdot|_d$ is the Euclidean norm of $\mathbb{R}^d$.

If $H = L^2(\mathcal{O})$, $V = H_0^1(\mathcal{O})$, $V' = H^{-1}(\mathcal{O})$ and $\cdot, \cdot$ for $t \in (0, T)$, the operator $A(t) : V \to V'$ is defined by

$$
V \langle A(t)y, \varphi \rangle_V = \int_{\mathcal{O}} (a(t, \xi, \nabla y) \cdot \nabla \varphi + \psi(y)\varphi) d\xi, \quad \forall \varphi \in H_0^1(\mathcal{O}), \; y \in H_0^1(\mathcal{O}),
$$

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then Hypotheses (i)–(iii) are satisfied. As regards the Wiener process $W$, we assume here that, besides \( \text{(1.6)} \), the following condition holds:

$$
\sum_{j=1}^{\infty} \mu_j^2 |\nabla e_j|^2_{\infty} < \infty.
$$

Then, by Theorem 2.1, where $H$, $V$ and $A(t)$ are defined above and $J = -\Delta$ with Dirichlet homogeneous boundary conditions, if $x \in H^1_0(\mathcal{O})$, the solution $X \in L^2(\Omega; C([0, T]; L^2(\mathcal{O})) \cap L^2((0, T) \times \Omega; H^1_0(\mathcal{O})))$ to \( \text{(1.1)} \) can be obtained as

$$
X = w - \lim_{n \to \infty} X_n \quad \text{in} \quad L^2((0, T) \times \Omega; H^1_0(\mathcal{O})),
$$

where $(X_n, Z_n) \in L^2((0, T) \times \Omega; H^1_0(\mathcal{O}))$ is the solution to the system (we take $\lambda = 1$)

$$
dZ_{n+1} - \Delta Z_{n+1}dt + \nu Z_{n+1}dt = Z_{n+1}dW + \text{div}(a(t, \xi, \nabla X_n))dt - \Delta X_n dt \\
Z_{n+1}(0) = x \quad \text{in} \quad \mathcal{O},
$$

$$
Z_{n+1} = 0 \quad \text{in} \quad (0, T) \times \partial \mathcal{O},
$$

$$
\text{div} a(\nabla X_{n+1}) + \Delta X_{n+1} = \Delta Z_{n+1} + \text{div}(a(t, \xi, \nabla X_n)) \quad \text{in} \quad (0, T) \times \mathcal{O},
$$

where $X_0 \in L^2((0, T) \times \Omega; H^1_0(\mathcal{O}))$ is arbitrary but $\mathcal{F}_t$-adapted. Moreover, if $a(t, \xi, -r) \equiv -a(t, \xi, r)$, $\forall r \in \mathbb{R}^d$, then the convergence \( \text{(4.2)} \) is strong in $L^2((0, T) \times \Omega; H^1_0(\mathcal{O}))$.

**Example 4.2. Stochastic porous media equations.**

Consider the stochastic equation

$$
dX - \Delta \psi(t, \xi, X)dt - \nu \Delta X dt = XdW \quad \text{in} \quad (0, T) \times \mathcal{O},
$$

$$
X(0, \xi) = x(\xi) \quad \text{in} \quad \mathcal{O},
$$

$$
\psi(t, \xi, X(t, \xi)) = 0 \quad \text{on} \quad (0, T) \times \partial \mathcal{O},
$$

where $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, $\nu > 0$, the function $\psi : [0, T] \times \overline{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ is continuous, $r \to \psi(t, \xi, r)$ is monotonically increasing in $r$, and there exist

\(12\)
We shall write equation (4.4) under the form (1.1) with $H = H^{-1}(\mathcal{O})$. Namely, we take $V = L^2(\mathcal{O})$, $H = H^{-1}(\mathcal{O})$, and $V'$ is the dual of $V$ with the pivot space $H^{-1}(\mathcal{O})$. Then, $V \subset H \subset V'$ and

$$V' = \{ \theta \in \mathcal{D}'(\mathcal{O}) : \theta = -\Delta v, v \in L^2(\mathcal{O}) \},$$

where $\Delta$ is taken in the sense of distributions on $\mathcal{O}$. (Here $\mathcal{D}'(\mathcal{O})$ is the space of Schwartz distributions on $\mathcal{O}$.) The duality $V' \langle \cdot, \cdot \rangle_V$ is defined as

$$V' \langle \theta, u \rangle_V = \int_{\mathcal{O}} \widetilde{\theta} u d\xi, \quad \widetilde{\theta} = (-\Delta)^{-1}\theta,$$

where $\Delta$ is the Laplace operator with homogeneous Dirichlet boundary conditions on $\partial \mathcal{O}$. The duality mapping $J : V \rightarrow V'$ is just the operator $-\Delta$ defined from $L^2(\mathcal{O})$ to $V' \subset \mathcal{D}'(\mathcal{O})$ by

$$\Delta u(\varphi) = \int_{\mathcal{O}} u\Delta \varphi d\xi, \quad \forall \varphi \in H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}).$$

The operator $A(t) : V \rightarrow V'$ is defined by

$$V' \langle A(t)y, v \rangle_V = \int_{\mathcal{O}} \psi(t, \xi, y) v d\xi, \quad \forall y, v \in V = L^2(\mathcal{O}), \ t \in [0, T].$$

Then, Hypotheses (i)–(iv) hold and so, if $x \in L^2(\mathcal{O})$, by Theorem 2.1, the solution $X \in L^2(\Omega; C([0, T]; H^{-1}(\mathcal{O}))) \cap L^2((0, T) \times \Omega; L^2(\mathcal{O}))$ to (1.4) is given by

$$X = w - \lim_{n \rightarrow \infty} X_n \text{ in } L^2((0, T) \times \Omega; L^2(\mathcal{O})),$$

where

$$dZ_{n+1} - \Delta Z_{n+1} dt + \nu Z_{n+1} dt = Z_{n+1} dW - \Delta \psi(t, \cdot, X_n) dt - \Delta Z_n dt$$

in $(0, T) \times \mathcal{O}$,

$$Z_{n+1}(0) = x \in L^2(\mathcal{O}), \ n = 0, 1, \ldots,$$

$$\Delta \psi(t, \cdot, X_{n+1}) + X_{n+1} = Z_{n+1} + \Delta \psi(t, \cdot, X_n), \text{ in } \mathcal{O},$$

$$\psi(t, \cdot, X_{n+1}(t, \cdot)) = 0 \text{ on } \partial \mathcal{O},$$

$$n = 0, 1, \ldots, X_0 \in L^2(0, T; L^2(\Omega; L^2(\mathcal{O}))).$$
If $\psi(t, \xi, r) = -\psi(t, \xi, -r)$, $\forall r \in \mathbb{R}$, then the convergence of the sequence $\{X_n\}$ is strong in $L^2((0, T) \times \Omega; L^2(\mathcal{O}))$.

5 The case where $A(t)$ is maximal monotone in $H \times H$

Consider now equation (1.1) under the following assumptions on $A$:

(j) $A : [0, T] \times H \times \Omega \to H$ is progressively measurable and, for each $(t, \omega) \times [0, T] \times \Omega$ the operator $u \to A(t, \omega, u)$ is maximal monotone in $H \times H$. Moreover, there is $f \in L^2((0, T) \times \Omega; H)$ such that

$$(I + A(t))^{-1}f(t) \in L^2((0, T) \times \Omega; H).$$

(5.1)

We assume also that condition (2.1) holds.

It should be noted that, if $A(t) : V \to V'$ satisfies assumptions (i)-(ii), where $V$ is a reflexive Banach space, then the operator $A(t) : H \to H$, defined by

$$A(t)_{H}u = A(t)u \cap V,$$

satisfies assumption (j). However, the class of the operators $A$ satisfying (j) is considerably larger.

We consider the splitting scheme (which is well defined by strong monotonicity of $A^*_1 + B^*_1$)

$$\lambda dY_{n+1} + (1 + \lambda \nu)Y_{n+1}dt = \lambda Y_{n+1}dW$$

$$+ (V_n - ((1 - \lambda \nu)I - \lambda A(t))^{-1}V_n)dt,$$

$$Y_{n+1}(0) = x \text{ in } (0, T),$$

$$V_{n+1} = Y_{n+1} + V_n - ((1 - \lambda \nu)I - \lambda A(t))^{-1}V_n,$$

(5.2)

where $V_0 \in L^2((0, T) \times \Omega; H)$ is an $(\mathcal{F}_t)_{t \geq 0}$-adapted process such that $A(t)V_0 \in L^2((0, T) \times \Omega; H)$. We have

**Theorem 5.1.** Assume that $x \in H$ and that equation (1.1) has a solution $X \in L^2(\Omega; C([0, T]; H))$ such that $A(t)X \in L^2((0, T) \times \Omega; H)$.

Then, for $n \to \infty$,

$$V_n \to V \text{ weakly in } L^2((0, T) \times \Omega; H),$$

(5.3)
where $X = ((1 - \lambda \nu)I + A(t))^{-1}V$ is the solution to (1.1).

If $A(t)$ is odd, then the convergence (5.3) is strong.

**Proof.** The operators $A_1^*$ and $B_1^*$ defined by

$$(A_1^*u)(t) = e^{-W}A(t)(e^Wu) - \nu u, \quad \forall u \in D(A_1^*),$$

$$(B_1^*u)(t) = du + (\mu + \nu)u, \quad \forall u \in D(B_1^*),$$

with the domains

$$D(A_1^*) = \{u \in \mathcal{H}; e^{-W}A(t)(e^Wu) - \nu u \in \mathcal{H}\},$$

$$D(B_1^*) = \{u \in \mathcal{H}; u \in W^{1,2}([0,T];\mathcal{H}) \cdot \mathbb{P}\text{-a.s.}, u(0) = x\}$$

are, by the above hypotheses, maximal monotone in $\mathcal{H} \times \mathcal{H}$ (see also [4]). Moreover, there is at least one solution $y^*$ to the equation

$$A_1^*y^* + B_1^*y^* = 0. \quad (5.4)$$

Then, again by [8], it follows that the sequence $\{v_n\} \subset \mathcal{H}$ defined by

$$v_{n+1} = (I + \lambda B_1^*)^{-1}(2(I + \lambda A_1^*)^{-1}v_n - v_n) + v_n - (I + \lambda A_1^*)^{-1}v_n, \quad n = 0, 1, \ldots \quad (5.5)$$

is weakly convergent in $\mathcal{H}$ to $v^*$, where $(1 + \lambda A_1^*)^{-1}v^* = y^*$ is the solution to equation (5.4).

We set

$$\tilde{z}_{n+1} = v_{n+1} - v_n + (I + \lambda A_1^*)^{-1}v_n \quad (5.6)$$

and, by (5.5), we have

$$\tilde{z}_{n+1} + \lambda B_1^*z_{n+1} = v_n - (I + \lambda A_1^*)^{-1}v_n. \quad (5.7)$$

Then, if $Y_n = e^W\tilde{z}_n$ and $V_n = e^Wv_n$, we can rewrite (5.6)-(5.7) as (5.2) and get (5.3), as claimed.

**Remark 5.2.** The convergence of the splitting algorithm (5.1)-(5.2) does not require conditions of the form (ii)-(iii) for the operator $A(t)$ but in change it requires the existence of a sufficiently regular solution $X$ for equation (1.1)
\((A(t)x \in L^2((0,T) \times \Omega; H))\) which is not the case for Examples 4.1, 4.2. Such a condition holds, however, for the stochastic reaction-diffusion equation
\[
dX - \Delta X \, dt + \Psi(X) \, dt = X \, dW \text{ in } (0,T) \times \mathcal{O},
\]
\[
X = 0 \text{ on } (0,T) \times \partial \mathcal{O},
\]
\[
X(0) = x,
\]
if \(x \in H^1_0(\mathcal{O})\) and \(\Psi : \mathbb{R} \to \mathbb{R}\) is continuous and monotonically increasing and for other stochastic parabolic equations as well.

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