The role of $SU(2)$ invariants for the classification of multiparty entanglement is discussed and exemplified for the Kempe invariant $I_5$ of pure three-qubit states. It is found to being an independent invariant only in presence of both $W$-type entanglement and threetangle. Furthermore, the present analysis indicates that an $SL^\otimes 3$ orbit of states with equal tangles but continuously varying $I_5$ must exist.

This means that $I_5$ provides no information on the entanglement in the system in addition to that contained in the tangles (concurrences and threetangle) themselves. Together with the numerical evidence that $I_5$ is an entanglement monotone this implies that $SU(2)$ invariance or the monotone property are too weak requirements for the characterization and quantification of entanglement for systems of three qubits, and that $SL(2,\mathbb{C})$ invariance is required. This conclusion can be extended to general multipartite systems (including higher local dimension) because the entanglement classes of three-qubit systems appear as subclasses.

I. INTRODUCTION

The understanding of entanglement is a central issue at the heart of quantum information theory. First insight has been obtained for the archetype of entanglement, namely the bipartite case, and in particular for binary observables of the constituents (e.g. photon polarization, two-level systems) hence termed qubits. Many criteria have been conceived to distinguish disentangled from entangled pure states as the Schmidt rank and the von Neumann entropy followed by varieties of measures for the mixedness of the local reduced density matrix\cite{1,2}. Necessary criteria for any entanglement measure to be fulfilled have been elaborated and have lead to the notion of an entanglement monotone\cite{3}. Crucial requirements are local $SU(2)$ invariance, convexity on the space of density matrices and monotonic diminishment under local SLOCC (Stochastic Local Operations and Classical Communication)\cite{4}. For pure two-qubit states an explicit extension to mixed states – i.e. the convex roof\cite{5} has been found for the concurrence and derived measures\cite{5,6,7}. Indeed, from the convex roof of the concurrence also the convex roof of any other measure of pairwise qubit entanglement can be obtained via its relation to the concurrence for pure states.

The success and the impressive achievements for the bipartite case asked for extensions to the multipartite setting and also to higher local dimensions. However, complications arise from the appearance of various entanglement classes that are inequivalent under SLOCC. It has been understood that SLOCC transformations lead to local transformations in the group $SL(2,\mathbb{C})$ rather than just $SU(2)$. Based on this insight one can proof that only one SLOCC class of global entanglement exists for three qubits\cite{8}. Its representative is the GHZ-state and has genuine three-party entanglement, as measured by the threetangle\cite{9}. A representative of the zero SLOCC class is the $W$ state, which exclusively contains pairwise entanglement, as measured by the concurrence. However, a precise e.g. operational meaning of this classification is still missing; neither a clear analogue to the entanglement of formation exists, nor is it known, whether a finite set of entangled states existed from which all pure states could be generated\cite{10}.

Much work has been done in order to classify multipartite entanglement for more than three qubits\cite{11,12,13,14,15,16,17,18,19,20}, but it is still not clear what conditions are to be imposed on such a classification. The minimal requirement is certainly local $SU(2)$ invariance, but an extension to the local $SL(2,\mathbb{C})$ group shows many appealing advantages. Although it is known in principle how to generate invariants to a certain group\cite{21}, the real problem consists in the distillation of those invariants relevant for entanglement. As to give an example, for four qubits nine different SLOCC classes have been identified\cite{11} as opposed to 19 primary and 1.449.936 secondary $SU(2)$ invariants obtained from the Hilbert series\cite{15,22}. On the other hand, only four inequivalent polynomial $SL(2,\mathbb{C})^\otimes 3$ invariants do exist, three of which vanish on all product states, and three inequivalent maximally entangled states have been singled out, which are all grouped in
one of the SLOCC-class proposed in Ref. [11].

In order to approach an answer to this problem, the three qubit case provides an ideal playground since a complete set of SU(2) invariants is known to contain only 6 independent elements. Only four of them are partially SL(2, C) invariant, and simple normal forms depending on 6 real parameters are available. They have been presented independently in Refs. [23, 24] as multipartite extensions to the Schmidt decomposition. Both have been analyzed with respect to a complete characterization of the local unitary orbits [25, 26]. For this purpose, a canonical form of three qubit states from [23] has been expressed in terms of polynomial SU(2) invariants. Here, focus is given on the connection between the same canonical form and the exhaustive classification of three qubit entanglement in [8]. To this end, we present an expression of the Acín normal form in terms of four (partial) SL(2, C) invariants: the threetangle $\tau_3$ and the concurrences $C_{1,2}$, $C_{1,3}$, and $C_{2,3}$. This expression discriminates the merely SU(2) invariants from SL(2, C) invariants based on entanglement related questions. We argue that SU(2) invariance alone is not sufficient in order to quantify and classify the entanglement pattern of a state, and that SL(2, C) invariance must be present somewhere for this purpose. First weak evidence for this to hold is (i) there is one SU(2) invariant that trivially has nothing to do with the entanglement of the state, namely its modulus. So, for merely SU(2) invariant quantities, in order to be related with entanglement, at least eventual non-local symmetries must be excluded. (ii) The only further invariant for two qubits is the concurrence, which is SL(2, C) invariant. This coincides with one existing entanglement class in this case.

For three qubits there is one further merely SU(2) invariant that is functionally independent from the four tangles: the Kempe invariant $I_5$ [27, 28]

$$I_5 = 3\text{tr} (\rho_i \otimes \rho_j)\rho_{ij} - \text{tr} \rho_i^3 - \text{tr} \rho_j^3 .$$

(1)

$I_5$ distinguishes locally indistinguishable states [28] and is related to the relative entropy of $\rho_{ij}$ and $\rho_i \otimes \rho_j$ of the two-qubit state [27]. It is permutation invariant but not with respect to SL(2, C) on any qubit. We analyze this quantity on pure three qubit states in order to understand the insight it gives into the entanglement of the state. The paper is organized as follows. The next section discusses the functional independence of $I_5$ within a complete set of SU(2) invariants. In Section III we obtain a one-parameter family of states in canonical form with all tangles (concurrences and threetangle) fixed and analyze this one-parameter family showing the behavior of $I_5$ in the interval consistent with the fixed values of the tangles. Section IV is devoted to the conclusions.
II. FUNCTIONAL INDEPENDENCE OF $I_5$ AND THE TANGLES

As in Refs. [25, 26] we start from the canonical form obtained by Acín [23]

$$\lambda_1 |000\rangle + \lambda_0 e^{i\varphi} |100\rangle + \lambda_2 |110\rangle + \lambda_3 |101\rangle + \lambda_4 |111\rangle$$

(2)

with real positive $\lambda_i$ and $\varphi \in [0, \pi]$ [22]. This specific normal form is particularly convenient, since the three-tangle $\tau_3$ assumes the simple form

$$\tau_3 = 4\lambda_1^2\lambda_4^2.$$  (3)

It also permits an easy distinction of the two different SLOCC classes of entanglement. The W-class corresponds to $\tau_3 = 0$, that is iff $\lambda_1 = 0$ or $\lambda_4 = 0$. In the former case, the state is a product.

The opposite extreme, contained in the GHZ class, is when all concurrences vanish. Then, due to the monogamy of entanglement for three qubits [23], all local density matrices are equivalent, since all local tangles $\tau_{i;i} := 4\det \rho_i$ coincides with $\tau_3$. We find $4\det \rho_i = 2\varepsilon_{ijk}\lambda_i^2(\lambda_j^2 + \lambda_k^2) + \tau_3 + f_i$ with $\varepsilon_{ijk}$ the Levi-Civita tensor and $f_i = 4\lambda_0\lambda_4(\lambda_0\lambda_4 - 2\cos \varphi\lambda_2\lambda_3)$ for $i \neq 1$ and $f_1 = 0$. These expressions are all equal to $\tau_3$ if $\lambda_0 = \lambda_2 = \lambda_3 = 0$.

Taking a closer look at the independence of the six $SU(2)$ invariants reveals an interesting connection between the functional independence of $I_5$ and the entanglement pattern of a state. To this end we include the modulus $I_6 := \sum \lambda_i^2$ as the 6th invariant

$$\nabla_I \bar{\nabla} := (\tau_{1,1}, \tau_{1,2}, \tau_{1,3}, \tau_3, I_5, I_6)$$

(4)

and analyze the minors of $\nabla_I \bar{\nabla}$. This yields that $I_5$ is functionally dependent of the remaining invariants if either of the concurrences or the three-tangle vanishes. As two relevant examples we consider the case $C_{12} = 0$ and $\tau_3 = 0$. For $C_{12} = 0$ we find the relation

$$I_5 = |\psi|^2 \left( |\psi|^4 - \frac{3}{4} \tau_{1,2} \right),$$

(6)

which is maximized if $\tau_{1,2} = 0 \Rightarrow I_5 = 1$ and minimized for the GHZ state, i.e. if $\tau_{1,2} = 1 \Rightarrow I_5 = \frac{1}{4}$.

For states from the W-class ($\tau_3 = 0$) we obtain

$$I_5 = |\psi|^6 - \frac{3}{4} |\psi|^2 (C_{12}^2 + C_{13}^2 + C_{23}^2) + \frac{3}{4} C_{12}C_{13}C_{23},$$

(7)

which can be shown to be minimized if all concurrences are equal. The maximum is again $I_5 = 1$ for full product states and the minimum value $I_5 = \frac{2}{9}$ is assumed for the W state ($|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$; this can be demonstrated to be the absolute minimum.

Before going ahead to the analysis of $I_5$ we briefly mention a further invariant discussed in this context, namely the Grassl invariant. It discriminates $|\psi\rangle$ from $|\psi^*\rangle$ [23] and is therefore necessary for a complete characterization of the $SU(2)$ orbits for pure three qubit states [26]. It is a complex $SU(2)$ invariant which, evaluated on the Acín normal form, gives

$$\text{Re } I_G = \tau_3\lambda_1^2 \left[ \cos(2\varphi)(\lambda_0\lambda_2\lambda_3)^2 + \cos \varphi \lambda_0\lambda_2\lambda_3\lambda_4 + \frac{1}{4} \lambda_4^2(1 - 2(\lambda_0^2 + \lambda_4^2)) \right] (1 - 2(\lambda_0^2 + \lambda_4^2))$$

$$\text{Im } I_G = -\tau_3 \sin \varphi \lambda_0\lambda_2\lambda_3 \left[ 2 \cos \varphi \lambda_0\lambda_2\lambda_3 + \lambda_4(1 - 2(\lambda_0^2 + \lambda_4^2)) \right]$$

(8)

It is clear that both real and imaginary part of $I_G$ are functionally dependent of the six other $SU(2)$ invariants including a discrete invariant emphasized on in Ref. [26]. Interestingly $I_G$ vanishes if $\tau_3 = 0$ and is hence only relevant for the GHZ class. Turning around this argument makes immediately visible that for $\tau_3 = 0$, $I_5$ (which is a function of the tangles, the modulus and the Grassl invariant) must be a function of the concurrences and the modulus only.
III. ONE PARAMETER FAMILY OF ACÍN STATES WITH FIXED TANGLES

We now proceed with constructing the one-parameter families of states with all tangles held constant. The variable $\lambda_4$ will be used as the free parameter in what follows. Starting from the relation Eq. (3) for $\tau_3$, we successively obtain the following relations

$$\lambda_1 = \frac{\sqrt{\tau_3}}{2\lambda_4}$$

(10)

$$\lambda_2 = \frac{\lambda_4}{\sqrt{\tau_3}} C_{12}$$

(11)

$$\lambda_3 = \frac{\lambda_4}{\sqrt{\tau_3}} C_{13}$$

(12)

$$\lambda_0 = \sqrt{1 - \frac{\tau_3}{4\lambda_4^2} - \frac{\lambda_4^2}{4\lambda_3^2} (C_{12}^2 + C_{13}^2) - \frac{\lambda_4^4}{4\lambda_3^4}}$$

(13)

$$\cos(\varphi) = \frac{4\lambda_4^2 \tau_3 - 4\tau_3^2 (\tau_3 + C_{23}^2) + \lambda_4^4 (C_{12}^2 (4C_{13}^2 - 2\tau_3) - \tau_3 (6C_{13}^2 + 5\tau_3))}{4\lambda_2^2 \tau_3 C_{12} C_{13} \sqrt{4\lambda_4^4 - 4 \det \rho_{1} \frac{4\lambda_4^2}{\tau_3} - \tau_3}}$$

(14)

Among the concurrences and threetangle, only $C_{23}$ is $\varphi$ dependent. If however, either $C_{12} = 0$ or $C_{13} = 0$, the phase $\varphi$ has no influence, neither on the tangles nor on $I_5$. In fact, the phase can be removed by local phases in both cases (e.g. $e^{-i\varphi}$ in front of the basis element $|1\rangle$ of the first site and $e^{i\varphi}$ in front of the basis element $|1\rangle$ of the second or third site, respectively).

Next we will discuss the interval accessible to $\lambda_4$. Having a look at the square-root in the denominator of eq. (14), we find that

$$\lambda_4^2 \in \frac{\tau_3}{2\tau_{1,1}} \begin{cases} \left[ 1 - \sqrt{1 - \tau_{1,1}}, 1 + \sqrt{1 - \tau_{1,1}} \right] & \text{if } \tau_{1,1} \in \left[ 0, \frac{1}{2} \right] \\ \left[ 1 + \sqrt{1 - \tau_{1,1}}, 1 - \sqrt{1 - \tau_{1,1}} \right] & \text{if } \tau_{1,1} \in \left[ \frac{1}{2}, 1 \right] \end{cases}$$

(15)
Note that \( \tau_1, \tau_2 = C_{12}^2 + C_{13}^2 + \tau_3 \). Additional restrictions come from the zeros of \( \lambda_0 \) leading to

\[
\lambda_4^2 \in \frac{\tau_3}{2(C_{12}^2 + C_{13}^2)} \begin{cases} 
1 - \sqrt{1 - C_{12}^2 - C_{13}^2}, & 1 + \sqrt{1 - C_{12}^2 - C_{13}^2} \\
1 + \sqrt{1 - C_{12}^2 - C_{13}^2}, & 1 - \sqrt{1 - C_{12}^2 - C_{13}^2}
\end{cases}
\]

if

\[
C_{12}^2 + C_{13}^2 \leq \frac{3}{4}
\]

and besides the conditions \( \lambda_i \leq 1 \) for \( i \in \{1, 2, 3\} \), leading to

\[
\lambda_4 \in [\sqrt{\tau_3}/2, \min\{\sqrt{\tau_3}/C_{12}, \sqrt{\tau_3}/C_{13}\}]
\]

In case of three-tangle dominated entanglement, meaning here that \( \tau_3 \) is larger than \( C_{12}^2 \) and \( C_{13}^2 \), the upper bound in (17) is trivial. It is worth noticing that the predominance given to the first qubit is inherent to the chosen normal form, which can as well be defined giving the focus onto another qubit. The corresponding formulae would be the same up to a permutation of the indices.

A further important restriction comes from \(|\cos\phi| \leq 1\); the latter bounds are obtained as the roots of a fourth order polynomial in \( \lambda_4^2 \) (see Eq. (14)); we will show \( \cos\phi(\lambda_4) \) in the figures.

We illustrate our result using a reference state

\[
\psi_\alpha = (|000\rangle + e^{i\alpha}|100\rangle + |101\rangle + |110\rangle + |111\rangle)/\sqrt{5}
\]

for different values of the phase \( \alpha \). It is clear that all these states have the same three-tangle but in general they differ in their distribution of bipartite entanglement. The values for the tangles are \( \tau_3 = 8/25 \), \( C_{12} = C_{13} = 2/5 \), and \( C_{23} = \sqrt{8}(1 - \cos\alpha)/5 \). Then, the Kempe invariant \( I_5 \) is calculated, tuning through the one-parameter family of Acín states with the same entanglement pattern as \( \psi_\alpha \) (i.e. the same three concurrences \( C_{12}, C_{13}, C_{23} \) and the same three-tangle \( \tau_3 \)). This one-parameter family of states is obtained from Eq. (2) subject to the replacements given in (10) - (14). The result is plotted for \( \alpha = \pi \) (fig.1), \( \pi/2 \) (fig.2), \( \pi/4 \) (fig.3), and for \( \alpha = 0 \) (fig.4). In cases where the plot range exceeds the interval accessible to \( \lambda_4 \), its upper bound is indicated by a vertical line. It is nicely seen that all measures for bipartite and tripartite entanglement are constant within this interval. This implies that all states belong to the same non-zero normal form under SLOCC local filtering operations[16]. As to be expected from the known functional independence of the 6
local unitary invariants, $I_5$ varies continuously over the whole range - where also the normalization is preserved. The interval admissible for $\lambda_4$ varies with the initial phase $\alpha$ (see figures). Exceeding the upper bound of the accessible interval for $\alpha = \pi/4$ (fig. 3) it is seen that the tangle $\tau_3$ remains fixed but both norm and some concurrences are no longer constant. This indicates that the states beyond this limit are still $SL(2, C)^{\otimes 3}$ equivalent to the reference state but no longer norm preserving. Within the admissible interval, the states are connected by norm-preserving $SL(2, C)^{\otimes 3}$ transformations (except for figure 4) but inequivalent with respect to local $SU(2)$ operations.

Vice versa, it is clear that keeping $I_5$ fixed admits for a wide range of the remaining tangles. This is seen in fig. 5 where we plot $I_5$ over $\tau_3$ for an ensemble of 5000 random states out of a specific class. The left picture shows random states out of the GHZ class. The full red and green horizontal lines indicate the absolute minimum $I_{5, \text{min}} = 2/9$ and what we called the GHZ bound for $I_5$. The GHZ bound $1/4$ is the lower bound for states with at least one of the concurrences vanishing, and equation (6) applies. Below this bound, i.e. for $I_5 < 1/4$, the states are W-like in the sense that all concurrences are positive. The right picture in fig. 5 shows an ensemble of 5000 random Acín states (2). It is seen that for $\tau_3 = 0$ the Kempe invariant varies over its full range from 2/9 up to 1.

It is clear that local $SL(2, C)$ operations generate a drift in parameter space of the Acín normal form. For norm-preserving transformations and with all tangles larger or equal than some given $\epsilon > 0$, this leads to compact and connected $SL^{\otimes 3}$ orbit for each point in parameter space. Modulo the $SU(2)^{\otimes 3}$ invariance, each $SL(2, C)$ operation can be written in the form

\[
\begin{pmatrix}
  s_1 & r_1 \\
  0 & s_1
\end{pmatrix}
\otimes
\begin{pmatrix}
  s_2 & r_2 \\
  0 & s_2
\end{pmatrix}
\otimes
\begin{pmatrix}
  s_3 & r_3 \\
  0 & s_3
\end{pmatrix}
\]

(19)

with real parameters $s_i, r_i$. For norm preserving transformations, this leads to a five dimensional orbit on the 5-dimensional parameter space of normalized Acín states. Therefore, all states with equal $\tau_3$ and equal characteristic vector $(\Theta(C_{12}), \Theta(C_{13}), \Theta(C_{23}))$ of non-zero concurrences are $SL^{\otimes 3}$ equivalent. As an illustrative example consider the $SL(2)$ transformations $\text{diag} \{ t, t^{-1} \}$ and $\text{diag} \{ s, s^{-1} \}$, with $s, t \in \mathbb{R}$ and the indices indicating the qubit number the matrix acts on. Such diagonal transformations leave the Acín state form-invariant. If we want to keep the state
normalized, this leads to a constraint $s = s(t)$. Namely,

$$|s|^2 = \frac{1}{2} |t|^2 \pm \frac{\sqrt{|t|^4(1 - 4\lambda_1^2(\lambda_2^2 + \lambda_3^2)) - 4(\lambda_2^2 + \lambda_3^2)(\lambda_1^2 + \lambda_3^2)} \lambda_1^2 |t|^4 + \lambda_2^2 + \lambda_3^2}{\lambda_1^2 |t|^4 + \lambda_2^2 + \lambda_3^2};$$

$$|t| \leq \sqrt{\frac{4(\lambda_2^2 + \lambda_3^2)(\lambda_0^2 + \lambda_1^2)}{1 - 4\lambda_1^2(\lambda_2^2 + \lambda_3^2)}}$$

(20)

It is worth mentioning that even if $s$ is complex, a local relative phase on the second qubit restores the original Acín form. Such transformations leave the threetangle and $C_{12}$ constant. When the initial state is taken as the reference state (18), even $C_{23}$ is constant, but this is a coincidence for that particular state. It would be interesting to contract explicitly those orbits with constant tangles, although their existence is clear from dimensional analysis together with the fact that $\tau_3$ is the only continuous polynomial $SL(2, \mathbb{C}) \otimes^3$ invariant. We leave this for future work.

IV. CONCLUSIONS

We have studied the Kempe-invariant $I_5$ for fixed entanglement pattern in a tripartite qubit state. Its known functional independence in particular implies that $I_5$ will vary in general even when the entanglement pattern of the state, as given by the concurrences and the threetangle, is kept fixed. This is indeed what is seen in the figures for some representative examples: for fixed nonzero concurrences and threetangle, $I_5$ varies continuously over a finite interval.

As a further result we find that $I_5$ is functionally independent only for globally distributed pairwise entanglement, as is present for a $W$-state, in coexistence with the threetangle. So it loses its independent justification as soon as one of the tangles is zero. This already admits the conclusion that although $I_5$ is needed for a complete characterization of the local $SU(2)$ orbit of a state, it does not qualify for being an independent measure of entanglement. In particular is $I_5$ not an additional measure of the distribution of pairwise entanglement as surmised in Ref. [27]. Indeed, all entanglement measures for two qubits are equivalent to the concurrence, unless one wants to deviate from the mixed state extension via the convex-roof. This uniqueness of the two-qubit entanglement measure is however also reflected in that all two-qubit states can be generated from a single Bell state [29]. We can also exclude that $I_5$ be an additional measure for the global SLOCC class of entanglement. The key observation to see this is that $I_5$ (besides the modulus of the state) is qualitatively different from the remaining four $SU(2)$ invariants in that it has no $SL(2, \mathbb{C})$ invariance.
on any of the three qubits. Therefore, local filtering operations \[16\] modify the value of \(I_5\). Under suitable (possibly infinitely many) local filtering operations on three qubits the Kempe invariant, \(I_5\), flows to its value for a normal form without concurrence. In this limit it is functionally dependent of the modulus of the state and the three-tangle. In particular do states that differ only by their values of \(I_5\) belong to the same SLOCC-entanglement class \[8\] (having the same normal form).

Furthermore, the \(SL(2, \mathbb{C})\) action leads to compact orbits acting continuously on the Acín normal form, if only we bound all concurrences from below by some \(\epsilon > 0\). So in absence of some hypothetical discrete \(SL(2, \mathbb{C})\) invariant that distinguishes different SLOCC inequivalent GHZ-classes (see \[26\] for \(SU(2)\)), each two states with the same three-tangle are \(SL(2, \mathbb{C})^{\otimes 3}\) inter-convertible. Assuming that \(I_5\) incorporated such a hypothetical discrete \(SL(2, \mathbb{C})\) invariant, it then should vary discontinuously with piecewise constant parts. This is not what we observe. Hence, we can exclude this hypothesis. This means that a continuous family of \(SU(2)^{\otimes 3}\) inequivalent states exist which are inter-convertible by norm preserving \(SL(2, \mathbb{C})^{\otimes 3}\) transformations. Consequently, \(I_5\) is not a measure for entanglement, since it does not carry information about the entanglement structure of a three qubit quantum state (unless it is functionally dependent on the tangles). It is worth adding here that we have statistical numerical evidence for \(I_5\) being even an entanglement monotone: randomly chosen SLOCC operations containing up to 2 Krauss operators on up to two qubits simultaneously did not produce non-monotonic behavior. We therefore believe that \(I_5\) is even an entanglement monotone. This would mean that not even the monotone property \[3\] (which includes \(SU(2)\) invariance) would be a conclusive criterion for a quantity to being useful for the classification and hence quantification of entanglement.

In the light of the peculiarities encountered when dealing with convex-roof extended entanglement measures \[30\] \[51\], we also analyzed a possible connection to the entanglement of assistance \(E_a\) on two qubits which is given by the Uhlmann fidelity, the counterpart of the convex-roof extended concurrence. We find that \(E_a\) can be expressed in terms of the tangles alone as \(E_{a,ij} = \sqrt{C_{ij} + \tau_3}\) without any connection to the Kempe invariant.

In order to clarify possible relevance of \(I_5\) for quantum information processes not related to the entanglement of the state further analysis is needed. The above line of arguments applies in the same way to the Grassl invariant \(I_G\).

Summarizing, a full classification of the local unitary orbits is neither necessary nor sufficient for a classification of entanglement for three qubits. This conclusion can be extended also to more qubits and higher local Hilbert space dimensions, since the classes of entanglement for three qubits appear as subclasses also there. In contrast, those local unitary invariants that are also invariant under the local action of \(SL(2, \mathbb{C})\) have proved to be necessary and sufficient for a full classification for two and three qubit entanglement. It has been demonstrated that \(SL(2, \mathbb{C})\) invariants give access to the classification of entanglement in multi-qubit quantum states \[12\] \[13\]. We have shown here that this requirement can not be relaxed to \(SU(2)\) invariance.

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\[1\] M. A. Nielsen and J. Kempe, Phys. Rev. Lett. 86, 5184 (2001).
\[2\] K. Zyczkowski and I. Bengtsson, Ann. Phys. 195, 115 (2002).
\[3\] G. Vidal, J.Mod.Opt. 47, 355 (2000).
\[4\] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A 63, 012307 (2001).
\[5\] A. Uhlmann, Phys. Rev. A 62, 032307 (2000).
\[6\] S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
9

[7] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[8] W. Dürr, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[9] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[10] S. Ishizaka and M. B. Plenio, Phys. Rev. A 72, 042325 (2005).
[11] F. Verstraete, J. Dehaene, B. D. Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002).
[12] A. Osterloh and J. Siewert, Phys. Rev. A 72, 012337 (2005).
[13] A. Osterloh and J. Siewert, Int. J. Quant. Inf. 4, 531 (2006).
[14] A. Miyake, Phys. Rev. A 67, 012108 (2003).
[15] J.-G. Luque, J.-Y. Thibon, and F. Toumazet, Math. Struct. in Comp. Science 17, 1133 (2007).
[16] F. Verstraete, J. Dehaene, and B. D. Moor, Phys. Rev. A 68, 012103 (2003).
[17] J.-G. Luque and J.-Y. Thibon, Phys. Rev. A 67, 042303 (2003).
[18] J.-G. Luque and J.-Y. Thibon, J. Phys. A 39, 371 (2005).
[19] A. Osterloh and D. Ž. Doković, arXiv:0804.1661.
[20] O. Chterental and D. Ž. Doković, p. 133 (2007), chapter 4 in the book ”Linear Algebra Research Advances”, Nova Science Publishers.
[21] P. Olver, Classical invariant theory (Cambridge University Press, Cambridge, 1999).
[22] E. Briand, J.-G. Luque, and J.-Y. Thibon, J. Phys. A 36, 9915 (2003).
[23] A. Acín, A. Andrianov, L. Costa, E. Jane, J. Latorre, and R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000).
[24] H. A. Carteret, A. Higushi, and A. Sudbery, J. Math. Phys. 41, 7932 (2000).
[25] A. Acín, A. Andrianov, E. Jane, and R. Tarrach, J. Phys. A 34, 6725 (2001).
[26] R. M. Gingrich, Phys. Rev. A 65, 052302 (2002).
[27] A. Sudbery, J. Phys. A 34, 643 (2001).
[28] J. Kempe, Phys. Rev. A 60, 910 (1999).
[29] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A 63, 012307 (2001).
[30] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, Phys. Rev. Lett. 97, 260502 (2006).
[31] C. Eltschke, A. Osterloh, J. Siewert, and A. Uhlmann, New J. Phys. 10, 043014 (2008).
[32] We exchanged $\lambda_0$ with $\lambda_1$ and $\lambda_2$ with $\lambda_3$ respect to [23].
[33] $\Theta$ is the Heaviside step function.