On second order \( q \)-difference equations for high–order Sobolev-type \( q \)-Hermite orthogonal polynomials

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Abstract

The \( q \)-Hermite I-Sobolev type polynomials of higher order are consider for their study. Their hypergeometric representation is provided together with further useful properties such as several structure relations which give rise to a three–term recurrence relation of their elements. Two different \( q \)-difference equations satisfied by the \( q \)-Hermite I-Sobolev type polynomials of higher order are also established.

Key words and phrases. Orthogonal polynomials, discrete Sobolev polynomials, \( q \)-Hermite polynomials, \( q \)-difference equation.

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1 Introduction

The \( q \)-Hermite \( I \) polynomials or (continuous) \( q \)-Hermite polynomials of degree \( n \), usually denoted in the literature as \( H_n(x; q) \), are a family of \( q \)-hypergeometric polynomials introduced at the end of the nineteenth century by L. J. Rogers, in his memoirs on expansions of certain infinite products (see [30], [31], and [32]). In this series of papers, and along with these polynomials, Rogers introduced the \( q \)-ultraspherical polynomials (also known as Rogers or Rogers–Askey–Ismail polynomials), and he used both families to prove the celebrated Rogers-Ramanujan identities. The \( q \)-Hermite polynomials are also related to the Rogers-Szegő polynomials (see [35]), and to other important families such as the Al-Salam-Chihara polynomials (see [1]). Throughout the twentieth century, these polynomials were studied by Szegő (see [35]) and Carlitz (see [11], [12], and [13]), and now they have received a strong impulse due to their deep involvement in different fields, appearing in classical probability theory ([8], [9], [10], [27]), non-commutative probability
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(see [2], [7], [36]), combinatorics (see [22], [23], and [24]) and quantum-physics (see [16], [17], [20], [28], [29]). For a recent and comprehensive survey on the subject, see [34].

The q-Hermite polynomials are usually defined by means of their generating function

\[
\sum_{n=0}^{\infty} H_n(x; q) \frac{t^n}{(q; q)_n} = \prod_{n=0}^{\infty} \frac{1}{1 - 2xtq^n + t^2q^{2n}},
\]

where q stands for their unique parameter, for which we assume that 0 < q < 1, which means that they belong to the class of orthogonal polynomial solutions of certain second order q-difference equations, known in the literature as the Hahn class (see [19], [25]). A recent study of their zeros and other interesting properties has been carried out in [33]. This sequence of polynomials lie at the bottom of the Askey-scheme of hypergeometric orthogonal polynomials, and they are orthogonal with respect to the measure \( d\mu = (qx, -qx; q)_\infty dx \). For \( q = 1 \) one recovers the classical Hermite polynomials and, for \( q = 0 \) one recovers the re-scaled Chebyshev polynomials of the second kind.

On the other hand, the so called Sobolev orthogonal polynomials refer to those families of polynomials orthogonal with respect to inner products involving positive Borel measures supported on infinite subsets of the real line, and also involving regular derivatives. When these derivatives appear only on function evaluations on a finite discrete set, the corresponding families are called Sobolev-type or discrete Sobolev orthogonal polynomial sequences. For a recent and comprehensive survey on the subject, we refer to [25] and the references therein. At the end of the twentieth century, H. Bavinck introduced the study of inner products involving differences (instead of regular derivatives) in uniform lattices on the real line (see [26], [27], [28], and also [29] for recent results on this topic). By analogy with the continuous case, these are also called Sobolev-type or discrete Sobolev inner products. We also refer to [4], [5], [6] and [21].

In the present work, we focus on a Sobolev type family of orthogonal polynomials associated to the q-Hermite I polynomials. More precisely, we study the sequence of q-Hermite I-Sobolev type polynomials of higher order \( \{H_n(x; q)\}_{n \geq 0} \), which are orthogonal with respect to Sobolev-type inner product

\[
\langle f, g \rangle_\lambda = \langle f, g \rangle + \lambda(D^j_q f)(\alpha)(D^j_q g)(\alpha),
\]

where \( \langle \cdot, \cdot \rangle \) stands for the inner product associated to the orthogonality related to the monic q-Hermite I polynomials, \( \lambda \) stands for a fixed positive real number, \( \alpha \) is a real number outside the support of the measure \( d\mu = (qx, -qx; q)_\infty dx \). For \( q = 1 \) one recovers the classical Hermite polynomials and, for \( q = 0 \) one recovers the re-scaled Chebyshev polynomials of the second kind.

The paper is structured as follows. In Section 2, we recall the basic facts and definitions from q-calculus and some of the main elements associated to the q-Hermite I polynomials. In Section 3...
we define the $q$-Hermite I-Sobolev type polynomials of higher order and state some properties relating them to the $q$-Hermite I polynomials via connection formulas, and also providing their hypergeometric representation. In Section 4 we state the main results of the present work, namely the formulation of a three term recurrence relation for the $q$-Hermite I-Sobolev type polynomials of higher order (Theorem 1), together with two different second order $q$-difference equations satisfied by these polynomials (Theorem 2 and Theorem 3). The paper concludes with a section of conclusions, further remarks and examples.

2 Preliminaries

This preliminary section is devoted to recall the main results from $q$-calculus, and the definition and main properties of the $q$-Hermite I polynomials. We have decided to include them for the sake of completeness.

2.1 Results from $q$-calculus

We begin by providing some background and definitions from $q$-calculus, for the sake of completeness. We refer to [25] for further details.

Given $n \in \mathbb{N}$, the $q$-number $[n]_q$, is given by

$$[n]_q = \begin{cases} 0, & \text{if } n = 0, \\ \frac{1 - q^n}{1 - q} \sum_{0 \leq k \leq n-1} q^k, & \text{if } n \geq 1. \end{cases}$$

It makes sense to extend the previous definition to $n \in \mathbb{Z}$ by $[n]_q = (1 - q^n)/(1 - q)$. From the previous definition, a $q$-analogue of the factorial of $n$ can be stated by

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [n]_q[n-1]_q \cdots [2]_q[1]_q, & \text{if } n \geq 1. \end{cases}$$

In addition to this, we will make use of a $q$-analogue of the Pochhammer symbol, or shifted factorial, which is given by

$$(a; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{0 \leq j \leq n-1} (1 - aq^j), & \text{if } n \geq 1, \end{cases}$$

$$(a; q)_\infty = \prod_{j \geq 0} (1 - aq^j), \text{ if } n = \infty \text{ and } |a| < 1.$$ 

Let us fix the following notation

$$(a_1, \ldots, a_r; q)_k = \prod_{1 \leq j \leq r} (a_j; q)_k.$$
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Given two finite sequences of complex numbers \( \{a_i\}_{i=1}^r \) and \( \{b_j\}_{j=1}^s \) under the assumption that \( b_j \neq q^{-n} \) with \( n \in \mathbb{N} \), for \( j = 1, 2, \ldots, s \), the basic hypergeometric, or \( q \)-hypergeometric, \( \phi \) series with variable \( z \) is defined by

\[
r\phi_s \left( \frac{a_1, a_2, \ldots, a_r}{b_1, b_2, \ldots, b_s}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} (-1)^k q^{\binom{k}{2}} (1 + s - r) z^k \left( \frac{q}{q; q} \right)^k.
\]

Following [3], the \( q \)-falling factorial is defined by

\[
[n]_q^{(n)} = \frac{(q^{-s}; q)_n}{(q - 1)_n q^{ns - \binom{s}{2}}}, \quad n \geq 1.
\]

We observe that \([s]_q^{(1)}\) coincides with the \( q \)-number \([s]_q\). Departing from the repvious defini-
tion, one can state the \( q \)-analog of the binomial coefficients (see [25]). Namely, the \( q \)-binomial coefficient is defined by

\[
\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad n = 0, 1, \ldots, n.
\]

where \( n \) denotes a nonnegative integer.

Following [15], the Jackson-Hahn-Cigler \( q \)-subtraction is defined by

\[
(x \sqcup y)^n = \sum_{0 \leq k \leq n} \binom{n}{k}_q q^k (y)^k x^{n-k}.
\]

It will be useful for determining a more compact writing for the derivatives of the reproducing kernel of the sequence of \( q \)-Hermite I polynomials, and all the elements determined from it.

At this point, we define the \( q \)-derivative, or the Euler–Jackson \( q \)-difference operator, by

\[
(D_q f)(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & \text{if } z \neq 0, \ q \neq 1, \\ f'(z), & \text{if } z = 0, \ q = 1, \end{cases} \quad (2)
\]

where \( D_q^0 f = f, \ D_q^n f = D_q(D_q^{n-1} f) \), with \( n \geq 1 \). We observe that

\[
\lim_{q \to 1} D_q f(z) = f'(z).
\]

The previous \( q \)-analog of the derivative operator will determine two functional equations satisfied by the Sobolev type polynomials defined in the present work. Moreover, the \( q \)-derivative satisfies the following algebraic statements:

\[
D_q[f(\gamma x)] = \gamma (D_q f)(\gamma x), \quad \forall \gamma \in \mathbb{C}. \quad (3)
\]

\[
D_q f(z) = D_{q^{-1}} f(qz). \quad (4)
\]

\[
D_q[f(z)g(z)] = f(qz)D_q g(z) + g(z)D_q f(z). \quad (5)
\]

\[
D_q(D_q^{-1} f)(z) = q^{-1} D_{q^{-1}}(D_q f)(z). \quad (6)
\]
2.2 \(q\)-Hermite I orthogonal polynomials

After the above section recalling the main elements of \(q\)-calculus, we continue by giving several aspects and properties of the \(q\)-Hermite I polynomials \(\{H_n(x; q)\}_{n \geq 0}\).

Departing from the generating function \(\Phi_1\), one has that the monic \(q\)-Hermite I polynomials can be explicitly given by

\[
H_n(x; q) = q^{n^2/2} \Phi_1 \left( q^{-n}, x^{-1}; 0, -qx \right),
\]

satisfying the orthogonality relation

\[
\int_{-1}^{1} H_m(x; q) H_n(x; q) (qx, -qx; q)_\infty d_q x = (1 - q)(q; q)_{n}(q, -1, -q; q)_{\infty} q^{\binom{n}{2}} \delta_{m,n},
\]

where \(\delta_{m,n}\) stands for Kronecker delta. In the previous definition, the \(q\)-integral is defined for every real function \(f\) defined in \([-1, 1]\) by

\[
\int_{-1}^{1} f(t) d_q t = (1 - q) \left( \sum_{n \geq 0} f(q^n) q^n + \sum_{n \geq 0} f(-q^n) q^n \right).
\]

We observe the evaluations of \(f\) make sense due to \(0 < q < 1\), and we refer to (1.15.7) in [25] for an expression derived from that formula.

The following statements can be derived from the previous definitions (see [25] for further details).

**Proposition 1** Let \(\{H_n(x; q)\}_{n \geq 0}\) be the sequence of \(q\)-Hermite I polynomials of degree \(n\). Then following statements hold.

1. **Recurrence relation.** The recurrence relation holds for every integer \(n \geq 0\)

\[
x H_n(x; q) = H_{n+1}(x; q) + \gamma_n H_{n-1}(x; q),
\]

with initial conditions \(H_{-1}(x; q) = 0\) and \(H_0(x; q) = 1\). Here, \(\gamma_n = q^{n-1}(1 - q^n)\).

2. **Squared norm.** For every \(n \in \mathbb{N}\),

\[||H_n||^2 = (1 - q)(q; q)_{n}(q, -1, -q; q)_{\infty} q^{\binom{n}{2}}.\]

3. **Forward shift operator.**

\[
\mathcal{D}_q^n H_n(x; q) = [n]_q^{(k)} H_{n-k}(x; q),
\]

where we recall that

\[ [n]_q^{(k)} = \frac{(q^{-n}; q)_k q^{kn-\binom{k}{2}}}{(q-1)^k q^{-\binom{k}{2}}}, \]

denotes the \(q\)-falling factorial.

4. **Second-order \(q\)-difference equation.**

\[
\sigma(x) \mathcal{D}_q \mathcal{D}_q^{-1} H_n(x; q) + \tau(x) \mathcal{D}_q H_n(x; q) + \lambda_{n,q} H_n(x; q) = 0,
\]

where \(\sigma(x) = x^2 - 1\), \(\tau(x) = (1 - q)^{-1} x\) and \(\lambda_{n,q} = [n]_q ([1 - n]_q \sigma'' / 2 - \tau')\).
It is worth mentioning that the previous \( q \)-difference equation appears in [23] in an equivalent form in equation (14.28.5).

**Proposition 2 (Christoffel-Darboux formula)** Let \( \{H_n(x; q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I polynomials. If we denote the \( n \)-th reproducing kernel by

\[
K_{n,q}(x, y) = \sum_{k=0}^{n} \frac{H_k(x; q)H_k(y; q)}{||H_k||^2}.
\]

Then, for all \( n \in \mathbb{N} \), it holds that

\[
K_{n,q}(x, y) = \frac{H_{n+1}(x; q)H_n(y; q) - H_{n+1}(y; q)H_n(x; q)}{(x - y) ||H_n||^2}. 
\] (10)

Let us fix the following notation for the partial \( q \)-derivatives of \( K_{n,q}(x, y) \):

\[
K_{n,q}^{(i,j)}(x, y) = \partial_q^j (\partial_q^i K_{n,q}(x, y)) = \sum_{k=0}^{n} \frac{\partial_q^j H_k(x; q)\partial_q^i H_k(y; q)}{||H_k||^2}.
\]

Christoffel-Darboux formula allow us to give a more precise writing for the derivatives of the \( n \)-th reproducing kernel associated to the sequence of \( q \)-Hermite I polynomials in the next result. Its proof follows an analogous technique as that of Lemma 1 [3]. Therefore, we omit the proof.

**Proposition 3** Let \( \{H_n(x; q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I polynomials of degree \( n \). Then following statements hold, for all \( n, j \in \mathbb{N} \) one has

\[
K_{n-1,q}^{(0,j)}(x, y) = A_n^{(j)}(x, y)H_n(x; q) + B_n^{(j)}(x, y)H_{n-1}(x; q), 
\] (11)

where

\[
A_n^{(j)}(x, y) = \frac{[j]_q!}{||H_{n-1}||^2 (x \boxplus_q y)^{j+1}} \sum_{k=0}^{j} \frac{\partial_q^k H_{n-1}(y; q)}{[k]_q!} (x \boxplus_q y)^k,
\]

and

\[
B_n^{(j)}(x, y) = -\frac{[j]_q!}{||H_{n-1}||^2 (x \boxplus_q y)^{j+1}} \sum_{k=0}^{j} \frac{\partial_q^k H_{n}(y; q)}{[k]_q!} (x \boxplus_q y)^k.
\]

**Proposition 4** Let \( \{H_n(x; q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I polynomials of degree \( n \). Then following statements hold, for all \( n, j \in \mathbb{N} \),

\[
K_{n-1,q}^{(1,j)}(x, y) = C_{1,n}(x, y)H_n(x; q) + D_{1,n}(x, y)H_{n-1}(x; q), 
\] (12)

\[
K_{n-1,q}^{(2,j)}(x, y) = C_{2,n}(x, y)H_n(x; q) + D_{2,n}(x, y)H_{n-1}(x; q), 
\] (13)

where

\[
C_{1,n}(x, y) = \partial_q A_n^{(j)}(x, y) - [n-1]_q \gamma_{n-1}^{-1} B_n^{(j)}(q,x,y),
\]

\[
D_{1,n}(x, y) = [n]_q A_n^{(j)}(qx,y) + [n-1]_q \gamma_{n-1}^{-1} x B_n^{(j)}(qx,y) + \partial_q B_n^{(j)}(x,y),
\]

\[
C_{2,n}(x, y) = \partial_q C_{1,n}(x, y) - [n-1]_q \gamma_{n-1}^{-1} D_{1,n}(qx,y),
\]

and

\[
D_{2,n}(x, y) = [n]_q C_{1,n}(qx,y) + [n-1]_q \gamma_{n-1}^{-1} x D_{1,n}(qx,y) + \partial_q D_{1,n}(x,y).
\]
Thus the property (5) yields

Therefore, we obtain (15).

which entails after evaluation at $x$

Applying the operator $D_q$ with respect to $x$ variable to $\{H_n(x; q)\}_{n \geq 0}$, together with the property (5) yields

$$K_{n-1,q}^{(1,j)}(x, y) = A_{n}^{(j)}(qx)D_q H_n(x; q) + H_n(x; q)D_q A_{n}^{(j)}(x)$$

$$+ \mathcal{E}_n^{(j)}(qx)D_q H_{n-1}(x; q) + H_{n-1}(x; q)D_q \mathcal{E}_n^{(j)}(x).$$

Using (5), (8) and (9) we deduce (12). We obtain (13) in an analogous way departing from (12) instead of (11). □

3 Connection formulas and hypergeometric representation

In this section we introduce the $q$-Hermite I-Sobolev type polynomials of higher order $\{H_n(x; q)\}_{n \geq 0}$, which are orthogonal with respect to the Sobolev-type inner product

$$\langle f, g \rangle_\lambda = \int_1^1 f(x; q)g(x; q)(qx, -qx; q)_\infty d_q x + \lambda (D_q^2 f)(\alpha; q)(D_q^2 g)(\alpha; q),$$

where $\alpha \in \mathbb{R} \setminus [-1, 1]$, $\lambda \in \mathbb{R}^+$ and $j \in \mathbb{N}$. In addition, we express the $q$-Hermite I-Sobolev type polynomials of higher order $\{H_n(x; q)\}_{n \geq 0}$ in terms of the $q$-Hermite polynomials $\{H_n(x; q)\}_{n \geq 0}$, the kernel polynomials and their corresponding derivatives. Moreover, we obtain a representation of the proposed polynomials as hypergeometric functions.

**Proposition 5** Let $\{H_n(x; q)\}_{n \geq 0}$ be the sequence of $q$-Hermite I-Sobolev type polynomials of degree $n$. Then, the following statements hold for $n \geq 1$:

$$H_n(x; q) = H_n(x; q) - \lambda \frac{\alpha[n]_{q} H_{n-j}(\alpha; q)}{1 + \lambda K_{n-1,q}^{(0,j)}(\alpha, \alpha)} K_{n-1,q}^{(0,j)}(x, \alpha).$$

**Proof.** Taking into account the Fourier expansion

$$H_n(x; q) = H_n(x; q) + \sum_{0 \leq k \leq n-1} a_{n,k} H_k(x; q),$$

one can apply the property (11) and consider the orthogonality properties for $H_n(x; q)$. Therefore, the coefficients in the previous expansion are given by

$$a_{n,k} = \frac{\lambda D_q^j H_n(x; q)}{\|H_k\|^2}, \quad 0 \leq k \leq n - 1.$$
Corollary 1 Let \( \{\mathbb{H}_n(x;q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I-Sobolev type polynomials of degree \( n \). Then, the following statements hold,

\[
\mathcal{D}_q \mathbb{H}_n(x; q) = [n]_q H_{n-1}(x; q) - \lambda \frac{[n]_q^{(j)} H_{n-j}(\alpha; q)}{1 + \lambda K_{n-1,q}^{(j,j)}(\alpha, \alpha)} K_n^{(1,j)}(x, \alpha).
\]

and

\[
\mathcal{D}_q^2 \mathbb{H}_n(x; q) = [n]_q^{(2)} H_{n-2}(x; q) - \lambda \frac{[n]_q^{(j)} H_{n-j}(\alpha; q)}{1 + \lambda K_{n-1,q}^{(j,j)}(\alpha, \alpha)} K_n^{(2,j)}(x, \alpha).
\]

Lemma 1 Let \( \{\mathbb{H}_n(x;q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I-Sobolev type polynomials of degree \( n \). Then, the following statements hold,

\[
\mathbb{H}_n(x; q) = \mathcal{E}_{1,n}(x) H_n(x; q) + \mathcal{F}_{1,n}(x) H_{n-1}(x; q),
\]

where

\[
\mathcal{E}_{1,n}(x) = 1 - \lambda \frac{[n]_q^{(j)} H_{n-j}(\alpha; q)}{1 + \lambda K_{n-1,q}^{(j,j)}(\alpha, \alpha)} A_n^{(j)}(x, \alpha),
\]

and

\[
\mathcal{F}_{1,n}(x) = -\lambda \frac{[n]_q^{(j)} H_{n-j}(\alpha; q)}{1 + \lambda K_{n-1,q}^{(j,j)}(\alpha, \alpha)} B_n^{(j)}(x, \alpha).
\]

Proof. From [15] and Proposition 3 we conclude the result. \( \blacksquare \)

On the other hand, from the previous Lemma and recurrence relation [8] we have the following result

\[
\mathbb{H}_{n-1}(x; q) = \mathcal{E}_{2,n}(x) H_n(x; q) + \mathcal{F}_{2,n}(x) H_{n-1}(x; q),
\]

where

\[
\mathcal{E}_{2,n}(x) = -\frac{\mathcal{F}_{1,n-1}(x)}{\gamma_{n-1}},
\]

and

\[
\mathcal{F}_{2,n}(x) = \mathcal{E}_{1,n-1}(x) - x \mathcal{E}_{2,n}(x).
\]

Lemma 2 Let \( \{\mathbb{H}_n(x;q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I-Sobolev type polynomials of degree \( n \). Then, the following statements hold,

\[
\Xi_{1,n}(x) H_n(x; q) = \begin{vmatrix} \mathcal{E}_{1,n}(x) & \mathcal{E}_{2,n}(x) \\ \mathcal{F}_{1,n}(x) & \mathcal{F}_{2,n}(x) \end{vmatrix},
\]

and

\[
\Xi_{1,n}(x) H_{n-1}(x; q) = - \begin{vmatrix} \mathcal{E}_{1,n}(x) & \mathcal{E}_{2,n}(x) \\ \mathcal{F}_{1,n}(x) & \mathcal{F}_{2,n}(x) \end{vmatrix},
\]

where

\[
\Xi_{1,n}(x) = \begin{vmatrix} \mathcal{E}_{1,n}(x) & \mathcal{E}_{2,n}(x) \\ \mathcal{F}_{1,n}(x) & \mathcal{F}_{2,n}(x) \end{vmatrix}.
\]
Proof. Let us multiply (16) by $F_{2,n}(x)$ and (17) by $-F_{1,n}(x)$. Adding and simplifying the resulting equations, we deduce (18). In addition, we can proceed analogously to get (19). ■

Finally, we will focus our attention in the representation of $H_n(x;q)$ as hypergeometric functions. We omit the details of the proof that follows a similar analysis to that carried out in [14] [18].

**Proposition 6 (Hypergeometric character)** The \(q\)-Hermite I-Sobolev type polynomials of higher order \(H_n(x;q)\) \(\forall n \geq 0\), have the following hypergeometric representation:

\[
 H_n(x;q) = -\frac{F_{1,n}(x)(1 - \psi_n(x)q^{-1})q^{\frac{n}{2}} - n^2}{[n]q\psi_n(x)(1 - q)} \times _3\Phi_2 \left( \begin{array}{l} q^{-n}, x^{-1}, \psi_n(x) \\ 0, \psi_n(x)q^{-1} \end{array} ; q, -qx \right)
\]  

(20)

where \(\psi_n(x) = ((1 - q)\vartheta_n(x) + 1)^{-1}\) and

\[
 \vartheta_n(x) = -\frac{q^{n-2}[n]q\mathcal{E}_{1,n}(x)}{F_{1,n}(x)} - [n - 1]q.
\]

**Lemma 3** Let \(\{H_n(x;q)\}_{n \geq 0}\) be the sequence of \(q\)-Hermite I-Sobolev type polynomials of degree \(n\). Then, the following statements hold,

\[
 D_qH_n(x;q) = \mathcal{E}_{3,n}(x)H_n(x;q) + F_{3,n}(x)H_{n-1}(x;q),
\]

(21)

where

\[
 \mathcal{E}_{3,n}(x) = -\lambda \frac{[n]q^{\binom{j}{2}}H_{n-j}(\alpha;q)}{1 + \lambda K_{n-j,q}(\alpha,\alpha)} \mathcal{C}_{1,n}(x,\alpha),
\]

and

\[
 F_{3,n}(x) = [n]q - \lambda \frac{[n]q^{\binom{j}{2}}H_{n-j}(\alpha;q)}{1 + \lambda K_{n-j,q}(\alpha,\alpha)} \mathcal{D}_{1,n}(x,\alpha).
\]

**Proof.** From Corollary 1 and Proposition 4 we deduce the desired result. ■

**Proposition 7** The \(q\)-Hermite I-Sobolev type polynomials \(H_n(x;q)\) of degree \(n\) satisfy the following structure relation,

\[
 \Xi_{1,n}(x)D_qH_n(x;q) = \mathcal{E}_{4,n}(x)H_n(x;q) + F_{4,n}(x)H_{n-1}(x;q),
\]

where

\[
 \mathcal{E}_{4,n}(x) = -\begin{bmatrix} \mathcal{E}_{2,n}(x) & \mathcal{E}_{3,n}(x) \\ F_{2,n}(x) & F_{3,n}(x) \end{bmatrix},
\]

and

\[
 F_{4,n}(x) = \begin{bmatrix} \mathcal{E}_{1,n}(x) & \mathcal{E}_{3,n}(x) \\ F_{1,n}(x) & F_{3,n}(x) \end{bmatrix}.
\]

**Proof.** Using Lemma 2 and Lemma 3, respectively, we deduce that

\[
 \begin{bmatrix} H_n(x;q) & H_{n-1}(x;q) \\ F_{1,n}(x) & F_{2,n}(x) \end{bmatrix} \mathcal{E}_{3,n}(x) = \begin{bmatrix} H_n(x;q) & H_{n-1}(x;q) \\ F_{1,n}(x) & F_{2,n}(x) \end{bmatrix} F_{3,n}(x)
\]
This concludes the result. ■

**Lemma 4** Let \( \{\mathbb{H}_n(x; q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I-Sobolev type polynomials of degree \( n \). Then, the following statements hold

\[
\mathcal{D}_q^2 \mathbb{H}_n(x; q) = \mathcal{E}_{5,n}(x) H_n(x; q) + \mathcal{F}_{5,n}(x) H_{n-1}(x; q),
\]

where

\[
\mathcal{E}_{5,n}(x) = -[n]_q^{(2)} \gamma_{n-1} - \lambda \frac{[n]_q^{(j)} H_{n-j}(\alpha; q)}{1 + \lambda K_{n-1,q}^{(j)}} C_{2,n}(x, \alpha),
\]

and

\[
\mathcal{F}_{5,n}(x) = [n]_q^{(2)} \gamma_{n-1} x - \lambda \frac{[n]_q^{(j)} H_{n-j}(\alpha; q)}{1 + \lambda K_{n-1,q}^{(j)}} D_{2,n}(x, \alpha).
\]

**Proof.** It is straightforward to check the result departing from Corollary 1 the recurrence relation (8) and Proposition 4, deducing (22). ■

**Proposition 8** The \( q \)-Hermite I-Sobolev type polynomials \( \mathbb{H}_n(x; q) \) of degree \( n \) satisfy the following relation,

\[
\Xi_{1,n}(x) \mathcal{D}_q^2 \mathbb{H}_n(x; q) = \mathcal{E}_{6,n}(x) \mathbb{H}_n(x; q) + \mathcal{F}_{6,n}(x) \mathbb{H}_{n-1}(x; q),
\]

where

\[
\mathcal{E}_{6,n}(x) = - \left| \frac{\mathcal{E}_{2,n}(x) \mathcal{E}_{5,n}(x)}{\mathcal{F}_{2,n}(x) \mathcal{F}_{5,n}(x)} \right|,
\]

and

\[
\mathcal{F}_{6,n}(x) = \left| \frac{\mathcal{E}_{1,n}(x) \mathcal{E}_{5,n}(x)}{\mathcal{F}_{1,n}(x) \mathcal{F}_{5,n}(x)} \right|.
\]

**Proof.** From the application of Lemma 2 and Lemma 4 we reach the conclusion. ■

### 4 Recurrence relation and \( q \)-difference equations

This main section of the present work states a three-term recurrence relation for the sequence of polynomials \( \{\mathbb{H}_n(x; q)\}_{n \geq 0} \), together with the establishment of two \( q \)-difference equations satisfied by the polynomials in \( \{\mathbb{H}_n(x; q)\}_{n \geq 0} \).

**Theorem 1** Let \( \{\mathbb{H}_n(x; q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I-Sobolev type polynomials of degree \( n \). Then, \( \mathbb{H}_n(x; q) \) satisfies the following three-term recurrence relations,

\[
\Xi_{2,n}(x) \mathbb{H}_{n+1}(x; q) = \alpha_n(x) \mathbb{H}_n(x; q) + \beta_n(x) \mathbb{H}_{n-1}(x; q),
\]
where
\[ \Xi_{2,n}(x) = \Xi_{1,n}(x) \mathcal{E}_{4,n+1}(x), \]
\[ \alpha_n(x) = \Xi_{1,n+1}(x) \mathcal{E}_{8,n}(x) - \Xi_{1,n}(x) \mathcal{F}_{4,n+1}(x), \]
and
\[ \beta_n(x) = \Xi_{1,n+1}(x) \mathcal{F}_{8,n}(x). \]

**Proof.** Shifting the index in (21) from \( n \) to \( n+1 \) and using the recurrence relation (8), yields
\[ \mathcal{D}_q \mathcal{H}_{n+1}(x; q) = \mathcal{E}_{7,n}(x) \mathcal{H}_n(x; q) + \mathcal{F}_{7,n}(x) \mathcal{H}_{n-1}(x; q), \]
where
\[ \mathcal{E}_{7,n}(x) = x \mathcal{E}_{3,n+1}(x) + \mathcal{F}_{3,n+1}(x), \quad \text{and} \quad \mathcal{F}_{7,n}(x) = -\gamma_n \mathcal{E}_{3,n+1}(x). \]
Then, using Lemma 2, we deduce
\[ \Xi_{1,n}(x) \mathcal{D}_q \mathcal{H}_{n+1}(x; q) = \mathcal{E}_{8,n}(x) \mathcal{H}_n(x; q) + \mathcal{F}_{8,n}(x) \mathcal{H}_{n-1}(x; q), \]
where
\[ \mathcal{E}_{8,n}(x) = -\left[ \begin{array}{cc} \mathcal{E}_{2,n}(x) & \mathcal{E}_{7,n}(x) \\ \mathcal{F}_{2,n}(x) & \mathcal{F}_{7,n}(x) \end{array} \right], \]
and
\[ \mathcal{F}_{8,n}(x) = \left[ \begin{array}{cc} \mathcal{E}_{1,n}(x) & \mathcal{E}_{7,n}(x) \\ \mathcal{F}_{1,n}(x) & \mathcal{F}_{7,n}(x) \end{array} \right]. \]

And the other hand, from Proposition 7 we have
\[ \Xi_{1,n+1}(x) \Xi_{1,n}(x) \mathcal{D}_q \mathcal{H}_{n+1}(x; q) = \]
\[ \Xi_{1,n}(x) \mathcal{E}_{4,n+1}(x) \mathcal{H}_{n+1}(x; q) + \Xi_{1,n}(x) \mathcal{F}_{4,n+1}(x) \mathcal{H}_{n}(x; q). \]
Finally, substituting (23) in the previous expression, we get the desired result. ■

**Theorem 2 (Second order difference equation, I)** Let \( \{\mathcal{H}_n(x; q)\}_{n \geq 0} \) be the sequence of \( q \)-Hermite I-Sobolev type polynomials defined by (20). Then, the following statement holds,
\[ \mathcal{R}_n(x) \mathcal{D}_q^2 \mathcal{H}_n(x; q) + \mathcal{S}_n(x) \mathcal{D}_q \mathcal{H}_n(x; q) + \mathcal{T}_n(x) \mathcal{H}_n(x; q) = 0, \quad n \geq 0, \]
where
\[ \mathcal{R}_n(x) = \mathcal{F}_{4,n}(x) \Xi_{1,n}(x), \]
\[ \mathcal{S}_n(x) = -\mathcal{F}_{6,n}(x) \Xi_{1,n}(x), \]
and
\[ \mathcal{T}_n(x) = \mathcal{E}_{4,n}(x) \mathcal{F}_{6,n}(x) - \mathcal{E}_{6,n}(x) \mathcal{F}_{4,n}(x). \]

**Proof.** We have from Proposition 7 that
\[ \mathcal{F}_{4,n}(x) \mathcal{H}_{n-1}(x; q) = \Xi_{1,n}(x) \mathcal{D}_q \mathcal{H}_n(x; q) - \mathcal{E}_{4,n}(x) \mathcal{H}_n(x; q). \]
The application of Proposition 8 yields
\[ \mathcal{F}_{4,n}(x) \Xi_{1,n}(x) \mathcal{D}_q^2 \mathcal{H}_n(x; q) = \]
\[ 
\mathcal{E}_{6,n}(x) \mathcal{F}_{4,n}(x) \mathbb{H}_n(x; q) + \mathcal{F}_{6,n}(x) \mathcal{F}_{4,n}(x) \mathbb{H}_n(x; q) = 0. 
\]

Then, from (25) we get
\[ 
\mathcal{F}_{4,n}(x) \mathcal{E}_{1,n}(x) \mathcal{D}_q^2 \mathbb{H}_n(x; q) = \mathcal{E}_{6,n}(x) \mathcal{F}_{4,n}(x) \mathbb{H}_n(x; q) + \mathcal{F}_{6,n}(x) \mathcal{E}_{1,n}(x) \mathcal{D}_q \mathbb{H}_n(x; q) - \mathcal{E}_{1,n}(x) \mathcal{H}_n(x; q). 
\]

Thus
\[ 
\mathcal{F}_{4,n}(x) \mathcal{E}_{1,n}(x) \mathcal{D}_q^2 \mathbb{H}_n(x; q) - \mathcal{E}_{6,n}(x) \mathcal{F}_{4,n}(x) \mathbb{H}_n(x; q) 
- \mathcal{F}_{6,n}(x) \mathcal{E}_{1,n}(x) \mathcal{D}_q \mathbb{H}_n(x; q) + \mathcal{E}_{4,n}(x) \mathcal{F}_{6,n}(x) \mathbb{H}_n(x; q) = 0. 
\]

Therefore, reagrouping the terms yields the conclusion
\[ 
\mathcal{F}_{4,n}(x) \mathcal{E}_{1,n}(x) \mathcal{D}_q^2 \mathbb{H}_n(x; q) - \mathcal{F}_{6,n}(x) \mathcal{E}_{1,n}(x) \mathcal{D}_q \mathbb{H}_n(x; q) 
+ \mathcal{E}_{4,n}(x) \mathcal{F}_{6,n}(x) \mathbb{H}_n(x; q). 
\]

\[ \blacksquare \]

A similar analysis to that carried out in (18) yields the following result.

**Theorem 3 (Second order difference equation II)** Let \( \mathbb{H}_n(x; q) \) be the sequence of \( q \)-Hermite I-Sobolev type polynomials defined by (12). Then, the following statement holds,
\[ 
\overline{R}_n(x) \mathcal{D}_q^{-1} \mathcal{D}_q \mathbb{H}_n(x; q) + \mathcal{S}_n(x) \mathcal{D}_q^{-1} \mathbb{H}_n(x; q) + \overline{T}_n(x) \mathbb{H}_n(x; q) = 0, \quad n \geq 0, 
\]

where
\[ 
\overline{R}_n(x) = R_n(q^{-1}x), 
\]
\[ 
\overline{S}_n(x) = S_n(q^{-1}x) + (q^{-1} - 1)x T_n(q^{-1}x), 
\]

and
\[ 
\overline{T}_n(x) = T_n(q^{-1}x). 
\]

**Proof.** Combining (4) with (24), and then using (3) we get
\[ 
q \overline{R}_n(x) \mathcal{D}_q \mathbb{H}_n(x; q) 
+ \mathcal{S}_n(x) \mathcal{D}_q^{-1} \mathbb{H}_n(x; q) + \overline{T}_n(x) \mathbb{H}_n(x; q) = 0, \quad n \geq 0, 
\]

Then, using (6), yields
\[ 
qq^{-1} \overline{R}_n(x) \mathcal{D}_q^{-1} \mathcal{D}_q \mathbb{H}_n(x; q) 
+ \mathcal{S}_n(x) \mathcal{D}_q^{-1} \mathbb{H}_n(x; q) + \overline{T}_n(x) \mathbb{H}_n(x; q) = 0, \quad n \geq 0, 
\]

Thus
\[ 
\overline{R}_n(x) \mathcal{D}_q^{-1} \mathcal{D}_q \mathbb{H}_n(x; q) 
+ \mathcal{S}_n(x) \mathcal{D}_q^{-1} \mathbb{H}_n(x; q) + \overline{T}_n(x) \mathbb{H}_n(x; q) = 0, \quad n \geq 0, 
\]

Next, replacing \( x \) by \( q^{-1}x \) we have
\[ 
\overline{R}_n(q^{-1}x) \mathcal{D}_q^{-1} \mathcal{D}_q \mathbb{H}_n(x; q) 
+ \mathcal{S}_n(q^{-1}x) \mathcal{D}_q^{-1} \mathbb{H}_n(x; q) + \overline{T}_n(q^{-1}x) \mathbb{H}_n(x; q) = 0, \quad n \geq 0. 
\]

Finally, a rearrangement of the terms involved gives
\[ 
\overline{R}_n(q^{-1}x) \mathcal{D}_q^{-1} \mathcal{D}_q \mathbb{H}_n(x; q) 
+ [\mathcal{S}_n(q^{-1}x) + (q^{-1} - 1)x T_n(q^{-1}x)] \mathcal{D}_q^{-1} \mathbb{H}_n(x; q) + \overline{T}_n(q^{-1}x) \mathbb{H}_n(x; q) = 0, \quad n \geq 0. 
\]

This allows to conclude the result. \( \blacksquare \)
5 Conclusions and further remarks

In this paper, we have considered the $q$-Hermite I-Sobolev type polynomials of higher order, and described important properties related to them such as several structure relations, their hypergeometric representation derived from the $q$-Hermite I polynomials, together with a three-term recurrence relation of their elements and two $q$-difference equation satisfied by their elements.

In this last section we present some examples and further remarks on these families of polynomials.

First, let us fix the parameters defining one of the previous families of orthogonal polynomials with $\alpha = 3$, $q = 3/5$. We also choose $j = 2$ which entails that the $q$-derivatives of second order evaluated at $\alpha$ are involved in the definition of the sequence of polynomials.

The first elements in $H_n(x; q)$ are given by

$$H_0(x; q) = 1, \quad H_1(x; q) = x,$$

$$H_2(x; q) = x^2 - \frac{2}{5}, \quad H_3(x; q) = x^3 - \frac{98}{125} x,$$

$$H_4(x; q) = x^4 - \frac{3332}{3125} x^2 + \frac{1764}{15625},$$

$$H_5(x; q) = x^5 - \frac{97988}{78125} x^3 + \frac{2541924}{9765625} x,$$

whereas for every $\lambda > 0$, the first elements in $\{H_n(x; q)\}_{n \geq 0}$ are given by

$$H_0(x; q) = H_0(x; q), \quad H_1(x; q) = H_1(x; q),$$

$$H_2(x; q) = H_2(x; q), \quad H_3(x; q) \approx H_3(x; q) - \frac{9.408 \lambda (8.707 x^2 - 3.483)}{17.415 \lambda + 1},$$

$$H_4(x; q) \approx H_4(x; q) - \frac{36.679 \lambda (277.663 x^3 + 8.707 x^2 - 217.687 x - 3.483)}{5015.349 \lambda + 1},$$

$$H_5(x; q) \approx H_5(x; q) - \frac{123.658 \lambda (8686.316 x^4 + 277.663 x^3 - 9252.990 x^2 - 217.687 x + 977.167)}{924614.128 \lambda + 1}.$$
Figure 1: The polynomials $\mathbb{H}_k(x; q)$ for $k = 0, \ldots, 5$ under the previous settings

**Proof.** It is sufficient to prove the result for the monomials $p_k(x) = x^k$ for $k = 0, 1, \ldots, j - 1$. We observe that $D_q^j 1 \equiv 0$ and

$$D_q x^k = \frac{1 - q^k}{1 - q} x^{k-1}, \quad k \geq 1.$$ 

This allows to conclude the result. \[\blacksquare\]

Applying Lemma 5, we observe from the definition of the $n$-th reproducing kernel associated to the $q$-Hermite I polynomials that $K_{j-1,q}^{(0,j)}(x, \alpha) \equiv 0$. Therefore, in view of (15) we also obtain that $\mathbb{H}_j(x; q) = H_j(x; q)$.

As a consequence, we have the following result.

**Corollary 2** $\mathbb{H}_k(x; q) \equiv H_k(x; q)$ for every $0 \leq k \leq j$.

Finally, we observe that the $q$-Hermite I polynomials are recovered when $\lambda = 0$ for the $q$-Hermite I-Sobolev polynomials. Indeed, this can be checked at many stages described in the work. It is direct to check that for every $n \in \mathbb{N}$ the polynomial $\mathbb{H}_n(x; q)$ tends to $H_n(x; q)$ for $\lambda \to 0$ in view of (15). Also, the asymptotic behavior of the elements involved in the connection formulas when $\lambda$ approaches to 0 yields the result. It is straightforward to check the asymptotic behavior of $\mathcal{E}_{k,n}, \mathcal{F}_{k,n}$ for $k = 1, 2$, and therefore of $\psi_n(x)$ for $\lambda \to 0$. In addition to that, this asymptotic behavior can also be observed from the hypergeometric representation of such polynomials.

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