Variational formulation of time-fractional parabolic equations *

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Abstract

We consider initial/boundary value problems for time-fractional parabolic PDE of order $0 < \alpha < 1$ with Caputo fractional derivative (also called fractional diffusion equations in the literature). We prove well-posedness of corresponding variational formulations based entirely on fractional Sobolev-Bochner spaces, and clarify the question of possible choices of the initial value.

Key words: Fractional diffusion equation, Initial value/boundary value problem, Well-posedness

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1 Introduction

Physical phenomena based on standard diffusion, where the mean square displacement of a diffusing particle scales linearly with time $\langle x(t)^2 \rangle \sim t$, are typically modeled by partial differential equations involving standard (i.e., integer order) differential operators. So-called anomalous diffusion, on the other hand, is characterized by non-linear scaling. For example, a diversive number of systems exhibit anomalous diffusion which follows the power-law $\langle x(t)^2 \rangle \sim t^\alpha$ with $0 < \alpha < 1$ (subdiffusion) or $1 < \alpha < 2$ (superdiffusion). Systems with such power-laws include ones with constrained pathways such as fractal, disordered, or porous media, polymers, aquifers, and quantum systems, among others. We refer to [18] for an extensive overview on the subject. In the latter work, the authors list various ways how to model anomalous diffusion processes. For problems involving external fields or boundary conditions, the most natural way is to consider partial differential equations involving so-called fractional differential operators. In the work at hand, we consider a time-fractional parabolic initial/boundary value problem of the form

$$\partial_t^\alpha u - \Delta u = f \quad \text{in } (0,T) \times \Omega,$$

$$u = 0 \quad \text{on } (0,T) \times \partial\Omega,$$

$$u = g \quad \text{for } \{0\} \times \Omega,$$

(1)

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where \((0, T)\) is a time interval and \(\Omega \subset \mathbb{R}^n\) a spatial Lipschitz domain. Here, \(-\Delta\) is the spatial Laplacian, \(1/2 < \alpha < 1\), and \(\partial_t^\alpha\) is a fractional time derivative of order \(\alpha\). More specifically, we will use the so-called Caputo derivative, which is defined formally by

\[
\partial_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) \, ds.
\]

Recently, researchers have started to analyze finite element methods with respect to their ability to approximate solutions of fractional differential equations. While this started with classical Galerkin finite element methods for steady-state fractional diffusion equations as in [9, 12], numerical methods for time-dependent fractional partial differential equations include time-stepping methods [8, 13, 14], Discontinuous Galerkin methods [19, 20], as well as methods based on the Laplace transform [17]. It goes without saying that this list is far from being exhaustive. We mention here also the numerical approach from [21] which is based on the extension theory by Caffarelli and Silvestre [4]. The aforementioned numerical methods are usually based on a variational formulation of the equation under consideration. Existing works on variational formulations of time-fractional parabolic partial differential equations are scarce; as to our knowledge, the works [27, 24, 1] are of relevance in connection with our model problem (1) (for Semigroup theory for related Volterra integral equations see [23]). These works have in common that (i) their functional analytic setting is not based exclusively on classical Sobolev regularity in time, but rather involves the operator \(\partial_t^\alpha\), and that (ii) the initial value \(g\) is taken from \(L^2(\Omega)\). The goal of the present work is to derive the well-posedness of variational formulations set up in classical Sobolev-Bochner spaces and to clarify the question of regularity needed for the initial data. Now, as our functional analytic setting is based only on Sobolev regularity, a result of this kind is specifically interesting for numerical analysis of the equation (1). Indeed, approximation results for functions with certain Sobolev regularity are well known and ubiquitous in numerical analysis. The property (i) is owed to the fact that there is no rigorous definition of time-fractional derivatives on fractional Sobolev-Bochner spaces available. It is true that operators defined between real valued Sobolev spaces \(L_2(J) \to L_2(J)\) do extend to vector-valued counterparts \(L_2(J; X) \to L_2(J; X)\) (for \(X\) a Hilbert space, this is a classical result of Marcienkiwicz and Zygmund [16]), but the fact that we are dealing with Sobolev regularity \(H^\alpha(J; X)\) in time needs some care and additional analysis. To that end, we will show first that the fractional Caputo derivative is a linear and bounded operator on a time-fractional Sobolev-Bochner space. This way, we can consider a variational formulation of (1) based exclusively on Sobolev regularity, which resembles classical variational formulations for parabolic equations. Regarding the point (ii), the choice of \(g \in L_2(\Omega)\) as initial value is indeed admissible, but one has to bear in mind the following: While the space \(L_2(J; H^1(\Omega)) \cap L_2(J; H^{-1}(\Omega))\), used in variational formulations of parabolic equations, is continuously embedded in \(C(\bar{J}; L_2(\Omega))\), this is no longer true for the equation (1). We will show that the space of solutions of our variational formulation of (1) is continuously embedded in \(C(\bar{J}; H^{1-1/\alpha-\varepsilon}(\Omega))\) for all \(\varepsilon > 0\). Our main result is then well-posedness of the variational formulation, cf. Theorem 2.
2 Mathematical setting and main results

2.1 Sobolev and Bochner spaces

We denote by \( \Omega \subset \mathbb{R}^d \) a (spatial) Lipschitz domain, and by \( J = (0,T) \) for \( T > 0 \) a temporal interval. We use Lebesgue and Sobolev spaces \( L_2(\Omega) \) and \( \tilde{H}^1(\Omega) \), the tilde denoting vanishing trace on the boundary \( \partial \Omega \). The fractional Sobolev spaces \( \tilde{H}^s(\Omega) \) for \( s \in (0,1) \) are defined by the K-method of interpolation as \( \tilde{H}^s(\Omega) := [L_2(\Omega), \tilde{H}^1(\Omega)]_{s,2} \), cf. \cite{26}, where

\[
[B_0, B_1]_{s,2} := \left\{ u \in B_0 \mid \|u\|_{[B_0, B_1]_{s,2}} < \infty \right\}, \quad \|u\|_{[B_0, B_1]_{s,2}} := \int_0^\infty t^{-2s-1} K_{[B_0, B_1]}(t, u)^2 \, dt
\]

with the K-functional

\[
K_{[B_0, B_1]}(t, u)^2 := \inf_{w \in B_1} \|u - w\|_{B_0}^2 + t^2 \|w\|_{B_1}^2.
\]

The topological dual of a Banach space \( X \) is denoted by \( X' \), and we define \( H^{-s}(\Omega) := \tilde{H}^s(\Omega)' \) as the topological duals with respect to the extended \( L_2(\Omega) \) scalar product \( \langle \cdot, \cdot \rangle \), and duality will be denoted by \( \langle \cdot, \cdot \rangle \). We set \( \tilde{H}^0(\Omega) := L_2(\Omega) \). In time, we additionally use Lebesgue and Sobolev spaces \( L_2(J) \) and \( H^\alpha(J) \), for \( \alpha \in (0,1] \). The \( L_2(J) \) scalar product will also be denoted by \( \langle \cdot, \cdot \rangle \), but it will always be clear which scalar product we are using. We use the notation \( Du \) to denote the distributional derivative of a function \( u \) given on \( J \). For \( \alpha \in (0,1) \) the norm on the space \( H^\alpha(J) \) is given by

\[
\|f\|_{H^\alpha(J)} := \|f\|_{L_2(J)}^2 + |f|_{H^\alpha(J)}^2 < \infty, \quad \text{where} \quad |f|_{H^\alpha(J)}^2 := \int_J \int_J \frac{|f(s) - f(t)|^2}{|s - t|^{2\alpha+1}} \, ds \, dt.
\]

We mention that \( H^\alpha(J) = [L_2(J), H^1(J)]_{\alpha,2} \) with equivalent norms. We set \( H^{-s}(J) := \tilde{H}^s(J)' \) and \( \tilde{H}^{-s}(J) := H^s(J) \) for \( s \in (0,1) \). For a Banach space \( X \), we use the Bochner space \( L_2(J; X) \) of functions \( f : J \to X \) which are strongly measurable with respect to the Lebesgue measure \( ds \) on \( J \) and

\[
\|f\|_{L_2(J; X)}^2 := \int_J \|f(s)\|^2_X \, ds < \infty.
\]

For \( w \) a measurable, positive function on \( J \), we denote by \( L_2(J, w; X) \) the \( w \)-weighted Lebesgue space of strongly measurable functions with norm

\[
\|f\|_{L_2(J, w; X)}^2 := \int_J w(s)^2 \|f(s)\|^2_X \, ds < \infty.
\]

We say that a function \( f \in L_2(J; X) \) has a weak time-derivative \( \partial_t f \in L_2(J; X) \), if

\[
\int_J \partial_t f \varphi = -\int_J f \partial_t \varphi \quad \text{for all} \ \varphi \in C_0^\infty(J).
\]
Note that this last integral has to be understand in the sense of Bochner. We define the space $H^1(J;X)$ as the space of functions $f : J \to X$ with
\[
\|f\|_{H^1(J;X)}^2 := \|f\|_{L^2(J;X)}^2 + \|\partial_t f\|_{L^2(J;X)}^2.
\]
For $0 < \alpha < 1$, we also use the fractional Sobolev-Bochner space $H^\alpha(J;X)$ of $ds$-strongly measurable functions $f : J \to X$ with
\[
\|f\|_{H^\alpha(J;X)}^2 := \|f\|_{L^2(J;X)}^2 + |f|_{H^\alpha(J;X)}^2 < \infty,
\]
where $|f|_{H^\alpha(J;X)}^2 := \int_J \int_J \frac{\|f(s) - f(t)\|_X^2}{|s - t|^{2\alpha + 1}} ds \, dt$.
We will also use these Bochner spaces on $\mathbb{R}$ instead of $J$. For a recent introduction to Bochner spaces, we refer to [11].

### 2.2 Fractional time derivative on Bochner spaces

For $0 < \beta < 1$, we define the left and right-sided Riemann-Liouville fractional integral operators
\[
_0 D^{-\beta}_t u(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} u(s) \, ds \quad \text{and} \quad D^{\beta}_T u(t) := \frac{1}{\Gamma(\beta)} \int_t^T (s - t)^{\beta - 1} u(s) \, ds,
\]
where $\Gamma$ is Euler’s Gamma function. For sufficiently smooth functions $u$, the left-sided Caputo fractional derivative $\partial^\alpha u$ for $\alpha \in (0,1)$ is defined as $\partial^\alpha u := _0 D^{\alpha - 1} D u$. We will show below in Lemma [10] that the tensorised version $_0 D^{\alpha - 1} \otimes I$ defined by $(0 D^{\alpha - 1} \otimes I)(u \otimes x) := (0 D^{\alpha - 1} u) \otimes x$ can be extended uniquely to a linear and bounded operator $0 D^{-\beta} : L_2(J;X) \to H^\beta(J;X)$ for a Hilbert space $X$. This allows us to prove the following result. The proof will be carried out below in Section 3.3.

**Theorem 1.** Let $\partial_t$ be the weak time derivative defined in (2). Then, for $\alpha \in (1/2,1)$, the operator $\partial^\alpha_t := _0 D^{\alpha - 1} \circ \partial_t$ is linear and bounded as $\partial^\alpha_t : H^\alpha(J;H^{-1}(\Omega)) \to L_2(J;H^{-1}(\Omega))$.

### 2.3 Variational formulation and main result

Our variational formulation of (1) is the following: Given $f \in L_2(J;H^{-1}(\Omega))$ and $g \in H^{1-1/\alpha+\delta}(\Omega)$ for some $\delta > 0$, find $u \in L_2(J;H^1(\Omega))$ with $u \in H^\alpha(J;H^{-1}(\Omega))$ such that
\[
\langle \partial^\alpha_t u, v \rangle + (\nabla u, \nabla v) = (f,v) \quad \text{for all } v \in \tilde{H}^1(\Omega),
\] (3)
a почти всюду в $J$, и $u(0, \cdot) = g(\cdot)$. Дуальность $\langle \partial^\alpha_t u, v \rangle$ в (3) имеет смысл благодаря структурным свойствам $\partial^\alpha_t$ из Теоремы 1 и начальный условие имеет смысл, как мы увидим в Приложении 4 ниже, что $L_2(J;\tilde{H}^1(\Omega)) \cap H^\alpha(J;H^{-1}(\Omega))$ непрерывно вложен в $C(J;H^{1-1/\alpha+\delta}(\Omega))$ для всех $\varepsilon > 0$. Настоящий теорема является наш основным результат и будет доказан ниже в Разделе 3.3.
Theorem 2. The variational formulation (3) is well posed: there exists a unique solution \( u \), and

\[
\|u\|_{L^2(J, \tilde{H}^1(\Omega))} + \|u\|_{H^\alpha(J, H^{-1}(\Omega))} \leq C_\delta \left( \|g\|_{H^{1-1/\alpha+\delta}(\Omega)} + \|f\|_{L^2(J, H^{-1}(\Omega))} \right).
\]

The constant \( C_\delta > 0 \) depends only on \( \delta \).

Remark 3. It is textbook knowledge that there holds the continuous embedding

\[
L^2(J, \tilde{H}^1(\Omega)) \cap H^1(J, H^{-1}(\Omega)) \hookrightarrow C(\overline{J}; L^2(\Omega)).
\]

In the present case, we have the embedding

\[
L^2(J, \tilde{H}^1(\Omega)) \cap H^\alpha(J, H^{-1}(\Omega)) \hookrightarrow C(\overline{J}; H^{1-1/\alpha-\varepsilon}(\Omega))
\]
for all \( \varepsilon > 0 \), cf. Lemma \ref{lem:embedding} below. The reason for the missing power of \( \varepsilon \) is that we use the embedding result \( H^{1/2+\varepsilon}(J, X) \hookrightarrow C(\overline{J}, X) \). Furthermore, note that the stability estimate of Theorem 2 involves the \( H^{1-1/\alpha+\delta}(\Omega) \) norm of the initial data \( g \), and the constant \( C_\delta \) is expected to blow up for \( \delta \to 0 \).

3 Technical results

3.1 Fractional integral and differential operators

We have the following results. The first point is an extension of a recent result in \cite{12} and can be found in \cite{7, Lemma 5}, while the second point is part of the proof of \cite{7, Lemma 6}. The third part can be found in \cite{15, Lem. 2.7}.

Lemma 4. (i) For every \( s \in \mathbb{R} \) with \( -\beta \leq s \) and \( \beta > 0 \), the operators \( 0D^{-\beta} \) and \( D_T^{-\beta} \) can be extended to bounded linear operators \( \tilde{H}^s(J) \to H^{s+\beta}(J) \).

(ii) For \( 0 < \beta < 1 \), the operator \( 0D^{-\beta} \) is elliptic on \( H^{-\beta/2}(J) \).

(iii) For \( 0 < \beta < 1 \) and \( u, v \in L^2(J) \) it holds \( (D_0^{-\beta}u, v) = (u, D_T^{-\beta}v) \).

The Mittag-Leffler function arises naturally in the study of fractional differential equations. We refer to \cite{6, Section 18.1} for an overview. It is defined as

\[
E_{n_1, n_2}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kn_1 + n_2)}.
\]

According to \cite{22, Thm. 1.6}, for \( z \in \mathbb{R} \),

\[
E_{n_1, n_2}(z) \lesssim \frac{1}{1 + |z|^\alpha}.
\]
and due to \cite[Thm. 4.3]{5},

\[ 0D^{\alpha-1}DE_{\alpha,1}(\lambda t^\alpha) = \lambda E_{\alpha,1}(\lambda t^\alpha). \]  \hspace{1cm} (5)

Furthermore, by \cite{25}, \( E_{\alpha,1}(-z) \) is completely monotone for \( 0 < \alpha \leq 1 \) and positive \( z \), in particular,

\[ E'_{\alpha,1}(-z) \geq 0 \quad \text{for positive } z. \]  \hspace{1cm} (6)

We will need the following result on fractional seminorms, which combines the \( H^s \) norm and the dual norm of the distributional derivative.

**Lemma 5.** Let \( s \in (0,1) \) be fixed. There holds

\[ |u|_{H^s(J)} \lesssim \|Du\|_{H^{s-1}(J)} \quad \text{for all } u \in H^s(J), \]

where \( Du \) is the distributional derivative of \( u \).

**Proof.** As \( u \in L_2(J) \), it holds \( Du \in H^{-1}(J) \). We can write \( u = D\psi + c \) with \( c \in \mathbb{R} \), where \( \psi \in \bar{H}^1(J) \) is the unique solution of \((Du,D\varphi) = (u,D\varphi)\) for all \( \varphi \in H^1(J) \). Then, \( \|\psi\|_{\bar{H}^1(J)} \lesssim \|u\|_{L_2(J)} \), and due to the definition of the distributional derivative we see

\[ |(u,D\psi)| = |(Du,\psi)| \lesssim \|Du\|_{H^{-1}(J)}\|\psi\|_{\bar{H}^1(J)} \lesssim \|Du\|_{H^{-1}(J)}\|u\|_{L_2(J)}. \]

We conclude that for \( u \in L_2(J) \), it holds

\[ \|u\|^2_{L_2(J)} = (u,D\psi) + (u,c) \lesssim \|Du\|_{H^{-1}(J)}\|u\|_{L_2(J)} + (u,c). \]

Now we apply this estimate to \( u - \bar{u} \), where \( \bar{u} \) denotes the mean value of \( u \), and obtain

\[ \|u - \bar{u}\|_{L_2(J)} \lesssim \|Du\|_{H^{-1}(J)}. \]  \hspace{1cm} (7)

The standard Poincaré inequality states that

\[ \|u - \bar{u}\|_{H^1(J)} \lesssim \|Du\|_{L_2(J)}. \]  \hspace{1cm} (8)

The \( H^s(J) \) norm can equivalently be obtained by the K-method of interpolation via

\[ \|u - \bar{u}\|^2_{H^s(J)} \simeq \|u - \bar{u}\|^2_{[L_2(J),H^1(J)]_{s/2}} = \int_0^\infty t^{-s/2} \left( \inf_{v \in H^1(J)} \|u - \bar{u} - v\|^2_{L_2(J)} + t^2\|v\|^2_{H^1(J)} \right) \frac{dt}{t}. \]

Using (7) and (8), we obtain

\[ \inf_{v \in H^1(J)} \|u - \bar{u} - v\|^2_{L_2(J)} + t^2\|v\|^2_{H^1(J)} \leq \inf_{v \in H^1(J)} \|u - \bar{u} - v\|^2_{L_2(J)} + t^2\|v\|^2_{H^1(J)} \]

\[ \lesssim \inf_{v \in H^1(J)} \|Du - Dv\|^2_{H^{-1}(J)} + t^2\|Dv\|^2_{L_2(J)} \]

\[ \lesssim \inf_{\|v\|_{H^1(J)} = 1} \int_0^\infty t^{-s/2} \left( t^2\|v\|^2_{H^1(J)} + t^2\|v\|^2_{H^1(J)} \right) \frac{dt}{t}. \]
Next we use that for \( w \in L^2(J) \) there is a \( \psi \in H^1(J) \) with \( \overline{\psi} = 0 \) such that \( D\psi = w \). We conclude
\[
\|u - \overline{u}\|_{H^s(J)}^2 \lesssim \int_0^\infty t^{-2s} \left( \inf_{w \in L^2(J)} \|Du - w\|_{H^{-1}(J)}^2 + t^2 \|w\|_{L^2(J)}^2 \right) dt.
\]
By definition, the right-hand side is \( \|Du\|_{H^{s-1}(J)}^2 \), which is equivalent to \( \|Du\|_{H^{s-1}(J)}^2 \).
This concludes the proof.

The next lemma establishes a norm equivalence on a fractional Sobolev space.

**Lemma 6.** Let \( 1/2 < s < 1 \). Then, for all \( u \in H^s(J) \),
\[
|u|_{H^s(J)} \sim \|0D^{s-1}Du\|_{L^2(J)}.
\]

**Proof.** We have
\[
\|0D^{s-1}Du\|_{L^2(J)} \lesssim \|Du\|_{H^{s-1}(J)} \lesssim \|Du\|_{H^{s-1}(J)} \lesssim \|u\|_{H^s(J)}.
\]
Here, the first estimate follows from Lemma 4 and the second one can be found in [10, Lem. 5].
To see the third estimate, recall that \( D \) is the distributional derivative, and hence \( \|Du\|_{H^{-1}(J)} \leq \|u\|_{L^2(J)} \) as well as \( \|Du\|_{L^2(J)} \leq \|u\|_{H^1(J)} \). The third estimate now follows from an interpolation argument. The fact that \( Du = D(u - \overline{u}) \) for the mean value \( \overline{u} \) of \( u \) and Poincare’s inequality show
\[
|u|_{H^s(J)} \gtrsim \|0D^{s-1}Du\|_{L^2(J)}.
\]
To show the converse estimate, we take \( u \in C^\infty(\overline{J}) \) and estimate with Lemmas 5 and 4
\[
|u|_{H^s(J)}^2 \lesssim \|Du\|_{H^{s-1}(J)}^2 \lesssim (0D^{2(s-1)}Du, Du)
= (0D^{s-1}Du, D^{s-1}Du) \leq \|0D^{s-1}Du\|_{L^2(J)} \|D^{s-1}Du\|_{L^2(J)}.
\]
Here, the identity follows from Lemma 3 (iii). Due to Lemma 4 it also holds \( \|D_T^{-1}Du\|_{L^2(J)} \lesssim \|u\|_{H^s(J)} \), where the second estimate was already shown at the beginning of this proof. Applying the whole argument to \( u - \overline{u} \) and using Poincare’s inequality finally shows the statement.

### 3.2 Sobolev and Bochner spaces

For \( s \in (-1, 0] \) we have the interpolation estimate
\[
\|u\|_{H^s(\Omega)} \leq C(s)\|u\|_{H^{-1}(\Omega)}^{(1-s)/2} \|u\|_{H^1(\Omega)}^{(1+s)/2},
\]
with \( C(s) > 0 \) a constant depending only on \( s \). This estimate follows for \( s = 0 \) by duality, and for \( s \in (-1, 0) \) using additionally [26, 1.3.3 (g)] and the fact that duality and interpolation commute,
Proof. The first identity is due to [11, Thm. 2.91], and the second and third identities are well-known results in interpolation theory, cf. [26, 1.11.2]. The last identity is a variant of the first one.

We assume from now on that the Banach spaces $X$ are reflexive; this implies that they have the so-called Radon-Nikodým property, cf. [11] Thm. 1.95], which is sufficient and necessary in order to have that $L_2(J;X)$ is isometrically isomorphic to $L_2(J;X')$, cf. [11] Thm. 1.84]. We can extend $\partial_t$ for $u \in L_2(J;X)$ by defining $\partial_t u \in H^1_0(J;X')$ as

$$-\int J \langle u(s), \partial_t \varphi(s) \rangle \, ds \quad \text{for } \varphi \in H^1_0(J;X').$$

Then, we have that $\partial_t : L_2(J;X) \to H^1_0(J;X')$ is bounded. Furthermore, $\partial_t : H^1(J;X) \to L_2(J;X')$ is bounded, and by interpolation, we have that for $s \in (0,1)$

$$\partial_t : \left[ L_2(J;X), H^1(J;X) \right]_s \to \left[ H^1_0(J;X'), L_2(J;X') \right]_s$$

is bounded. We will need the following results on interpolation of Sobolev-Bochner spaces.

**Lemma 7.** There holds

$$\left[ L_2(\mathbb{R};X), H^1(\mathbb{R};X) \right]_s = H^s(\mathbb{R};X)$$

and

$$\left[ \tilde{H}^1(J;X'), L_2(J;X') \right]_s = \left[ \tilde{H}^1(J;X'), L_2(J;X') \right]_s = \left[ L_2(J;X'), \tilde{H}^1(J;X') \right]_s = \tilde{H}^{1-s}(J;X').$$

**Proof.** The first identity is due to [11] Thm. 2.91], and the second and third identities are well-known results in interpolation theory, cf. [26, 1.11.2]. The last identity is a variant of the first one with bounded interval and zero traces. Using extension theorems, its proof can in fact be reduced to the first identity. In the case of scalar-valued Sobolev spaces, we refer to [3] Thm. 14.2.3] for details.

Next, we will establish continuous embeddings for the function space of our variational formulation.

**Lemma 8.** Suppose that $\alpha \in (0,1)$, $s \in (-1,0]$ and $0 < r$ are such that $r < \alpha(1-s)/2$. Then, we have the continuous embedding

$$L_2(J;\tilde{H}^1(\Omega)) \cap H^\alpha(J;H^{-1}(\Omega)) \to H^r(J;H^s(\Omega)).$$
Proof. It is clear that \( \|u\|_{L^2(J; H^s(\Omega))} \leq \|u\|_{L^2(J; \tilde{H}^1(\Omega))} \). To bound the \( H^r \)-seminorm, we write for \( r < \alpha(1 - s)/2 \)

\[
2r + 1 = \frac{(2\alpha + 1)(1 - s)}{2} + (1 - \varepsilon) \frac{1 + s}{2}
\]

for some \( \varepsilon > 0 \). The interpolation estimate (9) and the inequalities of Cauchy-Schwarz and Young then yield

\[
\int_J \int_J \frac{\|u(s) - u(t)\|_{H^s(\Omega)}^2}{|s - t|^{2r + 1}} dt \, ds \lesssim \int_J \int_J \frac{\|u(s) - u(t)\|_{H^{-1}(\Omega)}^{1-s} \|u(s) - u(t)\|_{\tilde{H}^1(\Omega)}^{1+s}}{|s - t|^{2r+1}} dt \, ds
\]

\[
\lesssim \int_J \left( \int_J \frac{\|u(s) - u(t)\|_{H^{-1}(\Omega)}^2}{|s - t|^{2\alpha + 1}} dt \right)^{(1-s)/2} \left( \int_J \frac{\|u(s) - u(t)\|_{\tilde{H}^1(\Omega)}^2}{|s - t|^{1-\varepsilon}} dt \right)^{(1+s)/2} ds
\]

\[
\lesssim \int_J \int_J \frac{\|u(s) - u(t)\|_{H^{-1}(\Omega)}^2}{|s - t|^{2\alpha + 1}} dt \, ds + \int_J \int_J \frac{\|u(s) - u(t)\|_{\tilde{H}^1(\Omega)}^2}{|s - t|^{1-\varepsilon}} dt \, ds
\]

and as \( \varepsilon > 0 \), the last integral can be bounded by \( \|u\|_{L^2(J; \tilde{H}^1(\Omega))} \).

Corollary 9. Suppose that \( \alpha \in (0, 1) \) and \( s \in (-1, 0] \) are such that \( s < 1 - 1/\alpha \). Then, we have the continuous embedding

\[
L^2(J; \tilde{H}^1(\Omega)) \cap H^\alpha(J; H^{-1}(\Omega)) \hookrightarrow C(J; H^s(\Omega)).
\]

Proof. If \( s < 1-1/\alpha \), then \( 1/2 < \alpha(1-s)/2 \), and according to Lemma 8 there holds the continuous embedding \( L^2(J; \tilde{H}^1(\Omega)) \cap H^\alpha(J; H^{-1}(\Omega)) \hookrightarrow H^{1/2+\varepsilon}(J; H^s(\Omega)) \) for a sufficiently small \( \varepsilon > 0 \). According to [11] Thm. 2.95 there also holds the continuous embedding \( H^{1/2+\varepsilon}(J; H^s(\Omega)) \hookrightarrow C(J; H^s(\Omega)) \), and this proves the statement.

The next lemma shows that the Riemann-Liouville fractional integral operators can be extended in the canonical way (i.e., by tensorisation) to Sobolev-Bochner spaces.

Lemma 10. Suppose that \( X \) is a Hilbert space and \( 0 < \beta < 1/2 \). Then, the operator

\[
_0 D^{-\beta} \otimes I := \begin{cases} 
S(J; X) \rightarrow H^\beta(J; X) \\
\sum_{i=1}^n x_i \chi_{A_i} \mapsto \sum_{i=1}^n (0 D^{-\beta} \chi_{A_i}) x_i
\end{cases}
\]

can be extended uniquely to a linear and bounded operator \( _0 D^{-\beta} : L^2(J; X) \rightarrow H^\beta(J; X) \). The same statement is true for the operator \( D^{-\beta} : L^2(J; X) \rightarrow H^\beta(J; X) \).
Proof. It follow from Lemma 4 (i) that the operator \( \partial D^{-\beta} : L_2(J) \to L_2(J) \) is bounded. Furthermore, it is a positive operator, i.e., \( \partial D^{-\beta}(1_{A_i}) \geq 0 \) on \( J \). It is then easy to see, cf. [11] Thm. 2.3, that

\[
\| \partial D^{-\beta} \otimes Iu \|_{L_2(J,X)} \leq \| \partial D^{-\beta} \|_{L_2(J) \to L_2(J)} \| u \|_{L_2(J,X)} \quad \text{for } u \in S(J;X),
\]

and as \( S(J;X) \) is dense in \( L_2(J;X) \), we obtain boundedness \( \partial D^{-\beta} : L_2(J;X) \to L_2(J;X) \).

Next, we will follow the ideas developed in [12] Thm. 3.1. Denoting by \( \tilde{f} \in L_2(\mathbb{R}) \) the extension of \( f \) by zero, it holds \( \partial D^{-\beta} f(x) = -\partial \partial D^{-\beta} \tilde{f}(x) \). Denote by \( \mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \) the Fourier transformation. Then, the operator \( \mathcal{F} \otimes I \) extends to an isometry \( \mathcal{F} : L_2(\mathbb{R};X) \to L_2(\mathbb{R};X) \): For general operators, this is a classical result by Marcinkiewicz and Zygmund [16], cf. [11] Thm. 2.9, but in the present case of the Fourier transformation it can be seen readily by using density of simple functions \( S(\mathbb{R};X) \) in \( L_2(\mathbb{R};X) \) and the Plancherel theorem for the scalar-valued Fourier transformation. Furthermore, for \( u \in L_2(\mathbb{R};X) \) we have \( \mathcal{F} \mathcal{F} u = \mathcal{P} u \) with \( \mathcal{P} u(x) = u(-x) \) the parity operator. For a function \( \varphi = \sum_{i=1}^{n} \varphi_i \otimes x_i \in S(\mathbb{R};X) \), we conclude

\[
\| \varphi \|_{H^1(\mathbb{R};X)}^2 = \| \mathcal{F} \varphi \|_{L_2(\mathbb{R};X)}^2 + \| \mathcal{F} \partial \varphi \|_{L_2(\mathbb{R};X)}^2 + \| \mathcal{F} \mathcal{P} \varphi \|_{L_2(\mathbb{R};X)}^2 + \| \mathcal{F} \mathcal{P} \partial \varphi \|_{L_2(\mathbb{R};X)}^2
\]

with weight function \( g(\omega) := \sqrt{1 + \omega^2} \). By density, this shows that \( \mathcal{F} \otimes I \) can be extended to an isometry \( \mathcal{F} : H^1(\mathbb{R};X) \to L_2(\mathbb{R}, g^s;X) \). By interpolation and Lemma 7 we conclude that \( \mathcal{F} : H^s(\mathbb{R};X) \to L_2(\mathbb{R}, g^s;X) \) is bounded. To show that this operator is an isometry, consider a decomposition \( \tilde{u}_0 + \tilde{u}_1 = \mathcal{F} u \) with \( \tilde{u}_0 \in L_2(\mathbb{R};X), \tilde{u}_1 \in L_2(\mathbb{R}, g^s;X) \). From \( \| \mathcal{F} \tilde{u}_1 \|_{H^1(\mathbb{R};X)} = \| \mathcal{F} \mathcal{F} \tilde{u}_1 \|_{L_2(\mathbb{R}, g^s;X)} = \| \tilde{u}_1 \|_{L_2(\mathbb{R}, g^s;X)} \) we conclude \( \mathcal{F} \tilde{u}_1 \in H^1(\mathbb{R};X) \) and due to \( \mathcal{F} \tilde{u}_0 + \mathcal{F} \tilde{u}_1 = \mathcal{F} \mathcal{P} u \) we have that \( \mathcal{P} \mathcal{F} \tilde{u}_0 + \mathcal{P} \mathcal{F} \tilde{u}_1 = u \) is a decomposition of \( u \). Hence,

\[
\| \mathcal{F} \tilde{u}_0 \|_{L_2(\mathbb{R};X)}^2 + t \| \mathcal{F} \tilde{u}_1 \|_{L_2(\mathbb{R}, g^s;X)}^2 = \| \mathcal{F} \mathcal{F} \tilde{u}_0 \|_{L_2(\mathbb{R};X)}^2 + t \| \mathcal{F} \mathcal{F} \tilde{u}_1 \|_{H^1(\mathbb{R};X)}^2
\]

which implies \( K_{[L_2(\mathbb{R};X), H^1(\mathbb{R};X)]}(t, u)^2 \leq K_{[L_2(\mathbb{R};X), L_2(\mathbb{R}, g^s;X)]}(t, \mathcal{F} u) \). This shows that \( \mathcal{F} : H^s(\mathbb{R};X) \to L_2(\mathbb{R}, g^s;X) \) is an isometry. Next, for a simple function \( u \in S(\mathbb{R};X) \),

\[
\| \partial D^{-\beta} \otimes Iu \|_{H^s(\mathbb{R};X)}^2 = \int_{\mathbb{R}} g(\omega)^{2s} \| \mathcal{F} \partial D^{\beta} u(\omega) \|_{X}^2 d\omega
\]

\[
\lesssim \int_{|\omega| \leq 1} \| \mathcal{F} \partial D^{\beta} u(\omega) \|_{X}^2 d\omega + \int_{|\omega| > 1} (\omega^{-2} + 1)^{s} \| \mathcal{F} u(\omega) \|_{X}^2 d\omega
\]

\[
\lesssim \int_{\mathbb{R}} \| \mathcal{F} \partial D^{\beta} u(\omega) \|_{X}^2 d\omega + \int_{|\omega| > 1} \| \mathcal{F} u(\omega) \|_{X}^2 d\omega
\]

\[
\lesssim \int_{\mathbb{R}} \| \mathcal{F} \partial D^{\beta} u(s) \|_{X}^2 ds + \int_{\mathbb{R}} \| u(s) \|_{X}^2 ds \lesssim \| u \|_{L_2(\mathbb{R};X)}^2,
\]
and by density we get the desired result. The proof for $D_T^{-\beta}$ follows along the same lines.

**Lemma 11.** The operator $0D^{-\beta}u \otimes I$ has a unique extension as bounded and linear operator

$$0D^{-\beta} \otimes I : H^\beta(J; \tilde{H}^1(\Omega))' \rightarrow L_2(J; H^{-1}(\Omega)).$$

**Proof.** For $u = \sum_{i=1}^n 1_{A_i} \otimes u_i \in S(J; H^{-1}(\Omega))$, $v = \sum_{i=1}^n 1_{A_i} \otimes v_i \in S(J; \tilde{H}^1(\Omega))$, we compute

$$\langle u, D_T^{-\beta} \otimes Iv \rangle = \int_J \langle u(s), D_T^{-\beta} \otimes Iv(s) \rangle \, ds$$

$$= \sum_{i,j=1}^n \langle u_i, v_i \rangle \int_J 1_{A_i}(s)D_T^{-\beta}1_{A_j}(s) \, ds$$

$$= \sum_{i,j=1}^n \langle u_i, v_i \rangle \int_J 0D^{-\beta}1_{A_i}(s)1_{A_j}(s) \, ds$$

$$= (0D^{-\beta} \otimes Iu, v).$$

(12)

As $H^\beta(J; \tilde{H}^1(\Omega))$ is dense in $L_2(J; \tilde{H}^1(\Omega))$, $L_2(J; H^{-1}(\Omega)) = L_2(J; \tilde{H}^1(\Omega))'$ is dense in $H^\beta(J; \tilde{H}^1(\Omega))'$. According to Lemma 10, $D_T^{-\beta} : L_2(J; \tilde{H}^1(\Omega)) \rightarrow H^\beta(J; \tilde{H}^1(\Omega))$ is bounded, and hence the equality (12) shows that $0D^{-\beta}$ can be extended as stipulated. This finishes the proof.

### 3.3 Proof of the main theorems

**Proof of Theorem 1.** Using the boundedness (10) of $\partial_t$ and Lemmas 7 and 11, we conclude that $0D^{\alpha-1} \partial_t : H^\alpha(J; H^{-1}(\Omega)) \rightarrow H^{1-\alpha}(J; \tilde{H}^1(\Omega))' = H^{1-\alpha}(J; \tilde{H}^1(\Omega))' \rightarrow L_2(J; H^{-1}(\Omega))$ is bounded.

**Proof of Theorem 2.** We mimic the proof for parabolic PDE. Take $(w_k)_{k \geq 1}$ the $L_2(\Omega)$-orthonormal basis of eigenfunctions and $(\lambda_k)_{k \geq 1}$ the eigenvalues of $-\Delta$. We make the ansatz

$$u_m(t) := \sum_{k=1}^m d_m^k(t)w_k.$$

Now, we are looking for $d_m^k : J \rightarrow \mathbb{R}$ such that

$$\partial_t^\alpha d_m^k(t) + \lambda_k d_m^k(t) = \langle f(t), w_k \rangle, \quad k = 1, \ldots, m,$$

$$d_m^k(0) = \langle g, w_k \rangle, \quad k = 1, \ldots, m.$$

(13)

According to [2, Thm. 2.1], cf. [5, Thm. 7.2] and [15, Chapter 3.1], the solutions to these equations are given uniquely by

$$d_m^k(t) = \langle g, w_k \rangle E_\alpha(-\lambda_k t^\alpha) + \psi_k(t), \quad k = 1, \ldots, m,$$
where \( \psi_k(t) := \alpha \int_t^1 \langle f(t-s), w_k \rangle s^{\alpha-1} E'_\alpha(-\lambda_k s^\alpha) \, ds \). In order to obtain energy estimates for the \( u_m \), we can extend the calculations carried out in [22]. However, as we aim at weaker initial values, we need a finer analysis. First, using the bound (11), we have for \( \varepsilon \in [0, 1] \)

\[
|E_{\alpha,1}(z)| \lesssim \frac{1}{1+|z|} \leq |z|^{-(1-\varepsilon)}.
\]

Furthermore, \( \alpha \int_J t^{\alpha-1} E'_{\alpha,1}(-\lambda_k t^\alpha) \, dt = \lambda_k^{-1}(1 - E_{\alpha,1}(-\lambda_k T^\alpha)) \), and \( t^{\alpha-1} E'_{\alpha,1}(-\lambda_k t^\alpha) \geq 0 \) due to (13). Hence, we see

\[
\int_J |t^{\alpha-1} E'_{\alpha,1}(-\lambda_k t^\alpha)| \, dt \lesssim \lambda_k^{-1}.
\]

(14)

We conclude that for \( 2\alpha(1-\varepsilon) < 1 \), it holds

\[
\|E_{\alpha,1}(-\lambda_k(\cdot)^\alpha)\|_{L^2(J)}^2 \leq C_\varepsilon \lambda_k^{-2(1-\varepsilon)}.
\]

(15)

Furthermore, due to Lemma 6, the identity (5) and the previous estimate, we also have

\[
|E_{\alpha}(-\lambda_k(\cdot)^\alpha)|_{H^\alpha(J)}^2 \sim \lambda_k^2 \|E_{\alpha,1}(-\lambda_k(\cdot)^\alpha)\|_{L^2(J)}^2 \leq C_\varepsilon \lambda_k^{2\varepsilon}.
\]

(16)

By Young’s inequality and (14),

\[
\|\psi_k\|_{L^2(J)}^2 \lesssim \left( \int_J \langle f(t), w_k \rangle^2 \, dt \right) \cdot \left( \int_J |t^{\alpha-1} E'_{\alpha,1}(-\lambda_k t^\alpha)| \, dt \right) \lesssim \lambda_k^{-2} \int_J \langle f(t), w_k \rangle^2 \, dt.
\]

(17)

According to [22] pp. 140, it holds \( \partial_t^\alpha \psi_k(t) = -\langle f(t), w_k \rangle - \lambda_k \psi_k(t) \). Hence, using Lemma 5 and (17), we also see

\[
|\psi|^2_{H^\alpha(J)} \lesssim \int_J \langle f(t), w_k \rangle^2 \, dt.
\]

(18)

Now choose \( \varepsilon := (2-1/\alpha+\delta)/2 \) and observe that \( 2\alpha(1-\varepsilon) = 1 - \alpha \delta < 1 \) and \( -1+2\varepsilon = 1-1/\alpha+\delta \). Using (15) and (17), we estimate

\[
\|u_m\|_{L^2(J; \tilde{H}^1(\Omega))}^2 = \sum_{k=0}^m \lambda_k \|d_m^k\|_{L^2(J)}^2 \lesssim \sum_{k=0}^m \lambda_k^{-1+2\varepsilon} \|g, w_k\|^2 + \sum_{k=0}^m \lambda_k^{-1} \int_J \langle f(t), w_k \rangle^2 \, dt \lesssim \|g\|^2_{H^{1-1/\alpha+\delta}(\Omega)} + \|f\|^2_{L^2(J; \tilde{H}^1(\Omega))}.
\]

Using (16) and (18), we can analogously estimate

\[
\|u_m\|_{H^\alpha(J; \tilde{H}^{-1}(\Omega))}^2 \lesssim \|g\|^2_{H^{1-1/\alpha+\delta}(\Omega)} + \|f\|^2_{L^2(J; \tilde{H}^{-1}(\Omega))}.
\]

Therefore, \( (u_m)_{m \in \mathbb{N}} \) is a bounded sequence in \( L^2(J; \tilde{H}^1(\Omega)) \) and in \( H^\alpha(J; \tilde{H}^{-1}(\Omega)) \), and we conclude that there is a subsequence \( (u_{m_k})_{k \in \mathbb{N}} \) which converges weakly to some \( u \in L^2(J; \tilde{H}^1(\Omega)) \)
and to some $\tilde{u} \in H^\alpha(J; H^{-1}(\Omega))$. It follows that $u$ also converges weakly in $L_2(J; H^{-1}(\Omega))$ to $u$ as well as to $\tilde{u}$, which yields $u = \tilde{u}$. Taking into account the construction of the $u_m$ and invoking the weak limit, we obtain for all $v \in L_2(J; H^{-1}(\Omega))$

$$\int_J \langle \partial_\alpha^t u, v \rangle + \langle \nabla u, \nabla v \rangle \, dt = \int_J \langle f, v \rangle \, dt.$$ 

Note that due to Corollary 9, $u_{mk}$ also converges weakly to $u$ in $C([0, T]; H^{1-1/\alpha+\delta}(\Omega))$, hence $g = u_{mk}(0) \rightarrow u(0)$. This yields $u(0) = g$, and we conclude that $u$ is a weak solution. As for uniqueness, if $u$ is a weak solution with vanishing data, then the functions $u_k(t) := (u(t), w_k)$ solve the equations (13) with vanishing right-hand side, and hence $u_k(t) = 0$. 

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