THE INTERPLAY BETWEEN 2-AND-3-CALABI–YAU TRIANGULATED CATEGORIES

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Abstract. This short note surveys the constructions of 3-Calabi–Yau triangulated categories with simple-minded collections due to Ginzburg and Kontsevich–Soibelman and the constructions of 2-Calabi–Yau triangulated categories with cluster-tilting objects due to Buan–Marsh–Reineke–Reiten–Todorov and Amiot, and includes a discussion on the normal form of 2-Calabi–Yau triangulated categories with cluster-tilting objects.

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This is a short survey on some ‘recent’ progresses on the interplay between 2-Calabi–Yau and 3-Calabi–Yau triangulated categories. It is based on my talks given in Stuttgart in November 2011 and in Nagoya in May 2013.

2-Calabi–Yau triangulated categories with cluster-tilting objects play a central role in the additive categorification of Fomin–Zelevinsky’s cluster algebras, see for example [10] [11] [14] [15] [16]. A prototypical example of such categories is the cluster category of an acyclic quiver, introduced by Buan–Marsh–Reineke–Reiten–Todorov in [10].

3-Calabi–Yau triangulated categories with simple-minded collections play an important role in algebraic geometry and mathematical physics [12] [11]. There are two constructions of such categories: for a quiver with potential, Ginzburg constructs a 3-Calabi–Yau dg algebra $\Gamma$ [15], whose finite-dimensional derived category $D_{fd}(\Gamma)$ is a 3-Calabi–Yau triangulated category with simple-minded collections; Kontsevich–Soibelman constructs a 3-Calabi–Yau $A_{\infty}$-algebra $A$ [11].
whose perfect derived category $\text{per}(A)$ is a 3-Calabi–Yau triangulated category with simple-minded collections. These two constructions are Koszul-dual to each other.

A direct connection between these two types of categories was observed in [36, 48, 3, 33]. For a quiver with potential, denote by $\Gamma$ the associated Ginzburg dg algebra. It is shown in [3, 33] that the generalised cluster category

$$\mathcal{C} := \text{per}(\Gamma)/\mathcal{D}_{fd}(\Gamma)$$

is a 2-Calabi–Yau triangulated category with cluster-tilting objects. It is conjectured that in characteristic zero all 2-Calabi–Yau triangulated category with cluster-tilting objects are of this form. This conjecture holds true for all known examples [36, 3, 7, 6, 5, 15].

This short note surveys the above constructions due to Buan–Marsh–Reineke–Reiten–Todorov [10], Ginzburg [18], Kontsevich–Soibelman [41], Amiot [3] and Keller [33], and discusses various methods to attack the conjecture that all 2-Calabi–Yau triangulated categories with cluster-tilting objects are generalised cluster categories of quivers with potential.

Throughout, $k$ is an algebraically closed field and $D = \text{Hom}_k(?, k)$ denotes the $k$-dual. All categories are $k$-categories and all algebras are $k$-algebras.

1. Silting objects, simple-minded collections and cluster-tilting objects

Let $\mathcal{C}$ be a triangulated category with suspension functor $\Sigma$. For a set $\mathcal{S}$ of objects of $\mathcal{C}$, let $\text{thick}(\mathcal{S})$ denote the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{S}$.

An object $M$ of $\mathcal{C}$ is a silting object ([37, 8, 1]) of $\mathcal{C}$ if

- $\text{Hom}_\mathcal{C}(M, \Sigma^p M) = 0$ for any $p > 0$,
- $\mathcal{C} = \text{thick}(M)$.

A collection $\{X_1, \ldots, X_n\}$ of objects of $\mathcal{C}$ is simple-minded ([39, 34]) if

- $\text{Hom}_\mathcal{C}(X_i, \Sigma^p X_j) = 0$, $\forall$ $p < 0$,
- $\text{Hom}_\mathcal{C}(X_i, X_j) = \begin{cases} k & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$
- $\mathcal{C} = \text{thick}(X_1, \ldots, X_n)$.

Fix $d \in \mathbb{Z}$. Assume further that $\mathcal{C}$ Hom-finite and Krull–Schmidt. We say that $\mathcal{C}$ is $d$-Calabi–Yau if there is a bifunctorial isomorphism

$$D \text{Hom}_\mathcal{C}(X, Y) \xrightarrow{\cong} \text{Hom}_\mathcal{C}(Y, \Sigma^d X)$$

for any $X, Y \in \mathcal{C}$ ([10, 31]). Note that in [31] such categories are called weakly $d$-Calabi–Yau triangulated categories.

Assume that $\mathcal{C}$ is $d$-Calabi–Yau. An object $T$ of $\mathcal{C}$ is $d$-cluster-tilting ([35, 49, 52, 24]) if

- $\text{Hom}_\mathcal{C}(T, \Sigma^p X) = 0$ for $0 < p < d \iff X$ belongs to $\text{add}(T)$.

2-cluster-tilting objects are usually called cluster-tilting objects.

In this note, we are interested in 3-Calabi–Yau triangulated categories with simple-minded collections and 2-Calabi–Yau triangulated categories with cluster-tilting objects.
2. 3-CalabiYau triangulated categories

The algebraic theory of 3-CY’s has two sides: the Ginzburg side and the Kontsevich–Soibelman side. Both sides are associated to quivers with potential.

2.1. The Ginzburg side. On the Ginzburg side there is a dg algebra.

For a dg algebra $A$, let $\mathcal{D}(A)$ be the derived category of (right) dg $A$-modules, $\text{per}(A) = \text{thick}(A_A)$ be the perfect derived category, and $\mathcal{D}_{fd}(A)$ be the finite-dimensional derived category (i.e. $M \in \mathcal{D}_{fd}(A)$ if and only if $H^*(M)$ is finite-dimensional). See [27, 30].

To a quiver with potential $(Q, W)$, Ginzburg associates a dg algebra $\hat{\Gamma}(Q, W)$, which we call the complete Ginzburg dg algebra of $(Q, W)$, see [18, 38]. Precisely, $\hat{\Gamma}(Q, W)$ is constructed as follows: Let $\tilde{Q}$ be the graded quiver with the same vertices as $Q$ and whose arrows are

- the arrows of $Q$ (they all have degree 0),
- an arrow $a^* : j \to i$ of degree $-1$ for each arrow $a : i \to j$ of $Q$,
- a loop $t_i : i \to i$ of degree $-2$ for each vertex $i$ of $Q$.

The underlying graded algebra of $\hat{\Gamma}(Q, W)$ is the completion of the graded path algebra $k\tilde{Q}$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $\tilde{Q}$. Thus, the $n$-th component of $\hat{\Gamma}(Q, W)$ consists of elements of the form $\sum p, \lambda p$, where $p$ runs over all paths of degree $n$. The completion endows $\hat{\Gamma}(Q, W)$ with a pseudo-compact topology, see [33, 50]. The differential of $\hat{\Gamma}(Q, W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the graded Leibniz rule

$$d(uv) = (du)v + (-1)^p u dv,$$

for all homogeneous $u$ of degree $p$ and all $v$, and takes the following values on the arrows of $\tilde{Q}$:

- $d(a) = 0$ for each arrow $a$ of $Q$,
- $d(a^*) = \partial_a W$ for each arrow $a$ of $Q$, where $\partial_a$ is the cyclic derivative associated to $a$ (see [16]; roughly, it removes $a$ from $W$),
- $d(t_i) = e_i(\sum_{a \in Q_1}[a, a^*])e_i$ for each vertex $i$ of $Q$, where $e_i$ is the trivial path at $i$.

The complete Jacobian algebra of $(Q, W)$ is the 0-th cohomology of $\hat{\Gamma}(Q, W)$:

$$\hat{J}(Q, W) = \hat{k}\tilde{Q}/(\partial_a W : a \in Q_1),$$

where $\overline{I}$ is the closure of $I$. When $W$ is a finite sum, one can define the non-complete Ginzburg dg algebra and Jacobian algebra.

**Theorem 2.1** ([33, Theorem A.17], [32]). Let $(Q, W)$ be a quiver with potential and let $\Gamma = \hat{\Gamma}(Q, W)$ be the complete Ginzburg dg algebra.

(a) $\Gamma$ is topologically homologically smooth, i.e. $\Gamma$ as a dg $\Gamma$-bimodule belongs to $\text{per}(\Gamma^\text{op} \hat{\otimes} \Gamma)$. In particular, the perfect derived category $\text{per}(\Gamma)$ contains the finite-dimensional derived category $\mathcal{D}_{fd}(\Gamma)$.

(b) $\Gamma$ is bimodule 3-Calabi–Yau, i.e. there is an isomorphism $\text{RHom}_{\Gamma^\text{op} \hat{\otimes} \Gamma}(\Gamma, \Gamma^\text{op} \hat{\otimes} \Gamma) \cong \Sigma^{-3} \Gamma$ in $\mathcal{D}(\Gamma^\text{op} \hat{\otimes} \Gamma)$. 


It follows that the triangulated category $\mathcal{D}_{fd}(\Gamma)$ is 3-Calabi–Yau ([33 Section A.15]). Moreover, one checks that the simple $\widehat{J}(Q,W)$-modules form a simple-minded collection of $\mathcal{D}_{fd}(\Gamma)$.

Notice that $\Gamma = \widehat{\Gamma}(Q,W)$ is concentrated in non-positive degrees. It follows that $\Gamma$ is a silting object of $\text{per}(\Gamma)$. We have the converse of Theorem 2.1 due to Van den Bergh, as a special case of [50 Theorems A and B].

**Theorem 2.2** ([51]). Assume that $k$ is of characteristic 0. Let $A$ be a topologically homologically smooth bimodule 3-Calabi–Yau pseudo-compact dg algebra such that $A$ is a silting object in $\text{per}(A)$. Then $A$ is quasi-isomorphic to the complete Ginzburg dg algebra of some quiver with potential.

### 2.2. The Kontsevich–Soibelman side.

On the Kontsevich–Soibelman side, there is an $A_{\infty}$-algebra. An $A_{\infty}$-algebra $A$ is a graded vector space endowed with a family of maps $m_n : A^\otimes n \to A$ homogeneous of degree $2 - n$ satisfying certain conditions, see for example [29]. For an $A_{\infty}$-algebra $A$, one can define the derived category $\mathcal{D}(A)$, the perfect derived category $\text{per}(A)$ and the finite-dimensional derived category $\mathcal{D}_{fd}(A)$ as well.

A cyclic structure of degree $d$ on an $A_{\infty}$-algebra $A$ is a supersymmetric non-degenerate bilinear form of degree $d$ $$(-,-) : A \times A \to k[-d]$$ such that $$(m_n(a_1, \ldots, a_n), a_{n+1}) = (-1)^n(-1)^{|a_1|(|a_2|+\ldots+|a_{n+1}|)}(m_n(a_2, \ldots, a_{n+1}), a_1).$$

To a quiver with potential Kontsevich–Soibelman associates an $A_{\infty}$-algebra $A(Q,W)$, which we call the **Kontsevich–Soibelman $A_{\infty}$-algebra**. Precisely, as a graded vector space $A(Q,W)$ has a basis concentrated in degrees 0, 1, 2, 3:

- the trivial path $e_i$ of $Q$ in degree 0 for each vertex $i$ of $Q$,
- an element $a^*$ in degree 1 for each arrow $a$ of $Q$,
- the arrows of $Q$, in degree 2,
- an element $e_i^*$ in degree 3 for each vertex $i$ of $Q$.

Write $W = \sum_c \lambda_c c$, where the sum is over all cycles $c$ of $Q$ of length $\geq 3$. The maps $m_n$ are given by

- $m_1 = 0$,
- $m_2(e_j \otimes a) = a = m_2(a \otimes e_i)$ and $m_2(e_i \otimes a^*) = a^* = m_2(a^* \otimes e_j)$ if $a : i \to j$ is an arrow of $Q$,
- $m_2(e_i \otimes e_j^*) = 0 = m_2(e_j \otimes e_i)$ and $m_2(e_i \otimes e_j^*) = e_i^* = m_2(e_j^* \otimes e_i)$ if $i$ and $j$ are different vertices of $Q$,
- $m_2(a \otimes a^*) = t_j^*$ and $m_2(a^* \otimes a) = t_i^*$ if $a : i \to j$ is an arrow of $Q$,
- $m_n(\cdots \otimes e_i \otimes \cdots) = 0$ for any vertex $i$ of $Q$, if $n \geq 3$,
- $m_n(a_1^* \otimes \cdots \otimes a_n^*) = - \sum \lambda_{a_n \cdots a_1} a$, where the sum is over all arrows of $Q$.

**Theorem 2.3** ([11]). Assume that $k$ is of characteristic 0.
Let \((Q, W)\) be a quiver with potential. Then \(A(Q, W)\) has a natural cyclic structure of degree 3.

(b) Let \(A\) be an \(A_\infty\)-algebra with a cyclic structure of degree 3 such that the indecomposable direct summands of \(A\) form a simple-minded collection of \(\text{per}(A)\). Then there is a quiver with potential \((Q, W)\) such that \(A\) is \(A_\infty\)-isomorphic to \(A(Q, W)\).

2.3. The connection. The Ginzburg side and the Kontsevich–Soibelman side are related by Koszul duality. Precisely, the complete Ginzburg dg algebra \(\Gamma = \widehat{\Gamma}(Q, W)\) of a quiver with potential \((Q, W)\) can be obtained from the Kontsevich–Soibelman \(A_\infty\)-algebra \(A = A(Q, W)\) by taking the dual bar construction ([33, 44]):

\[
\Gamma = \text{Hom}_K(T_K(A^{\geq 1}[1]), K),
\]

where \(K\) is the semi-simple algebra \(kQ_0\). Conversely, the \(A_\infty\)-Koszul dual of \(\Gamma\) is the Kontsevich–Soibelman \(A_\infty\)-algebra \(A\). This leads to the following triangle equivalences

\[
\text{per}(\Gamma) \cong \text{D}_{fd}(A) \cong \text{per}(A).
\]

3. 2-Calabi–Yau triangulated categories

A typical 2-CY triangulated category is the cluster category associated to an acyclic quiver. Let \(Q\) be an acyclic quiver, i.e. a finite quiver without oriented cycles. Let \(\text{mod} kQ\) denote the category of finite-dimensional modules over the path algebra \(kQ\) and let \(\text{D}^b(\text{mod} kQ)\) denote the corresponding bounded derived category. The derived Nakayama functor \(\nu = L_{kQ} D(kQ)\) is a Serre functor of \(\text{D}^b(\text{mod} kQ)\) ([20 Theorem 4.6]). Define the cluster category \(C_Q\) ([10]) to be the orbit category

\[
C_Q := \text{D}^b(\text{mod} kQ)/\nu \Sigma^{-2}.
\]

Theorem 3.1. (a) ([28 Theorem 1 and Corollary 1]) \(C_Q\) is a 2-CY triangulated category.
(b) ([10 Theorem 3.3(b)]) The image of the \(kQ\) in \(C_Q\) is a cluster-tilting object.

There are many 2-CY triangulated categories with cluster-tilting objects arising from

(1) module categories of preprojective algebras of acyclic quivers ([17, 9]),

and from

(2) categories of maximal Cohen–Macaulay modules of singularities ([13, 21]).

There is the following remarkable recognition theorem due to Keller and Reiten.

Theorem 3.2 ([36 Theorem 2.1]). Let \(\mathcal{C}\) be a 2-Calabi–Yau algebraic triangulated category with a cluster-tilting object whose endomorphism algebra is the path algebra \(kQ\) of an acyclic quiver \(Q\). Then there is a triangle equivalence

\[
C_Q \xrightarrow{\sim} \mathcal{C}.
\]
4. FROM 3-CYS TO 2-CYS

Let \((Q,W)\) be a quiver with potential. The generalised cluster category \(\mathcal{C}_{(Q,W)} := \text{per}(\hat{\Gamma}(Q,W))/\mathcal{D}_{fd}(\hat{\Gamma}(Q,W))\).

The relation among the complete Ginzburg dg algebra \(\hat{\Gamma}(Q,W)\), the Kontsevich–Soibelman \(A_\infty\)- algebra \(A(Q,W)\) and the generalised cluster category \(\mathcal{C}_{(Q,W)}\) is encoded in the following recollement

\[
\begin{array}{c}
\hat{\mathcal{C}}_{(Q,W)} \\
\mathcal{D}(Q,W) \\
\mathcal{D}(A(Q,W))
\end{array}
\]

where \(\hat{\mathcal{C}}_{(Q,W)}\) is an unbounded version of \(\mathcal{C}_{(Q,W)}\) in the sense that \(\hat{\mathcal{C}}_{(Q,W)}\) has infinite direct sums and \(\mathcal{C}_{(Q,W)}\) consists of compact objects in \(\hat{\mathcal{C}}_{(Q,W)}\) ([31, Corollary 3]).

**Theorem 4.1.** If \(\hat{J}(Q,W)\) is finite-dimensional, then \(\mathcal{C}_{(Q,W)}\) is Hom-finite and 2-CY as a triangulated category. Moreover, the image of the object \(\hat{\Gamma}(Q,W)\) in \(\mathcal{C}_{(Q,W)}\) is a cluster-tilting object whose endomorphism algebra is \(\hat{J}(Q,W)\).

This result is a special case of the following more general theorem, which is a ‘topological’ version of [3, Theorems 2.1 and 3.5]. It is generalised in [19, 23].

**Theorem 4.2** ([33, Theorem A.21]). Let \(A\) be a dg algebra satisfying the conditions
- \(H^i(A) = 0\) for \(i > 0\),
- \(H^0(A)\) is finite-dimensional,
- \(A\) is topologically homologically smooth,
- \(A\) is bimodule 3-Calabi–Yau.

Then \(\text{per}(A)/\mathcal{D}_{fd}(A)\) is Hom-finite and 2-CY as a triangulated category. Moreover, the image of \(A\) in \(\text{per}(A)/\mathcal{D}_{fd}(A)\) is a cluster-tilting object whose endomorphism algebra is \(H^0(A)\).

5. FROM 2-CYS TO 3-CYS

Motivated by the Keller–Reiten recognition Theorem [32] we propose the following conjecture (cf. [2, Summary of results, Part 2, Perspectives]).

**Conjecture 5.1.** Assume that \(k\) is of characteristic 0. Let \(\mathcal{C}\) be a 2-Calabi–Yau algebraic triangulated category with a cluster-tilting object \(T\). Then there is a quiver with potential \((Q,W)\) together with a triangle equivalence

\[
\begin{array}{c}
\mathcal{C}_{(Q,W)} \\
\hat{\Gamma}(Q,W)
\end{array} \sim \begin{array}{c}
\mathcal{C} \\
T
\end{array}
\]

If this conjecture holds, then the answer to the following question proposed in Amiot’s ICRA XIV notes is positive.
**Question 5.2** ([4] Question 2.20). Assume that \( k \) is of characteristic 0. Let \( C \) be a 2-Calabi–Yau algebraic triangulated category with a cluster-tilting object \( T \). Then the endomorphism algebra of \( T \) is the complete Jacobian algebra of some quiver with potential.

Note that we have replaced ‘Jacobian algebra’ by ‘complete Jacobian algebra’. The original question has a negative answer, as there are quivers with potentials whose complete Jacobian algebras are not non-complete Jacobian algebras of any quiver with potential, see [47, Example 4.3] for an example.

Moreover, we have put an extra assumption on the characteristic of the field, as when \( k \) is of positive characteristic, the answer to the question is negative. For example, if \( k \) is of characteristic \( p > 0 \), then \( k[x]/(x^p-1) \) is not a Jacobian algebra. However, take \( \Gamma \) as the dg algebra whose underlying graded algebra is \( k\langle\langle x, x^*, t \rangle\rangle \) with \( \deg(x) = 0 \), \( \deg(x^*) = -1 \) and \( \deg(t) = -2 \), and whose differential \( d \) is defined by

\[
d(x) = 0, \quad d(x^*) = x^p - 1, \quad d(t) = xx^* - x^*x.
\]

It is straightforward to check that \( \Gamma \) satisfies the assumptions of Theorem 4.2, so \( k[x]/(x^p-1) \) is the endomorphism of a cluster-tilting object in a 2-Calabi–Yau algebraic triangulated category. More examples can be found in [42].

One approach to attack Conjecture 5.1 is developed by Amiot [3]. For many 2-Calabi–Yau triangulated categories \( C \) arising from (1) and (2) in Section 3, there is a finite-dimensional algebra \( A \) of global dimension 2 with certain conditions such that \( C \) is triangle equivalent to the cluster category of \( A \). On the other hand, the 3-Calabi–Yau completion \( \Pi_3(A) \) of \( A \) in the sense of Keller [32] satisfies the four conditions in Theorem 4.2 (with ‘topologically homologically smooth’ replaced by ‘homologically smooth’), and the cluster category of \( A \) is triangle equivalent to \( \per(\Pi_3(A))/\mathcal{D}_{fd}(\Pi_3(A)) \) ([32, Theorem 6.12(a)]. By [32, Theorem 6.10], there is a quiver with potential \((Q, W)\) such that \( \Pi_3(A) \) is quasi-isomorphic to the \( \Gamma(Q, W) \). Therefore \( C \) is triangle equivalent to \( \per(\Gamma(Q, W))/\mathcal{D}_{fd}(\Gamma(Q, W)) \). Amiot, Iyama, Reiten, Todorov [3, 7, 6, 5] have intensively worked on this topic.

Another promising approach is the from-2CY-to-3CY approach, which we explain in more details below. Let \( \mathcal{E} \) be a stably 2-CY Frobenius category and \( M \in \mathcal{E} \) be a basic cluster-tilting object. Let \( \mathcal{C} \) be the stable category of \( \mathcal{E} \), \( A = \text{End}_\mathcal{E}(M) \) and \( e \in A \) be the idempotent corresponding to the maximal projective-injective direct summand of \( M \). Then \( A/AeA = \text{End}_\mathcal{C}(M) \). The following is a consequence of [35, Proposition 4 (c)].

**Theorem 5.3.** Let \( \mathcal{D}_{fd,A/AeA}(\text{Mod} A) \) be the subcategory of \( \mathcal{D}(\text{Mod} A) \) whose total cohomology is finite-dimensional and supported on \( A/AeA \). Then there is a bifunctorial isomorphism

\[
\mathcal{D} \text{Hom}(X,Y) \cong \text{Hom}(Y, \Sigma^{d+1}X),
\]

where \( X \in \text{mod} A/AeA \) and \( Y \in \mathcal{D}_{fd,A/AeA}(\text{Mod} A) \).

The collection of simple \( A \)-modules supported on \( A/AeA \) is a simple-minded collection in \( \mathcal{D}_{fd,A/AeA}(\text{Mod} A) \). By Lefèvre’s \( A_\infty \)-version of Morita’s theorem for triangulated categories, there is an \( A_\infty \)-algebra \( E \) together with a triangle equivalence \( \per(E) \to \mathcal{D}_{fd,A/AeA}(\text{Mod} A) \).
which takes the direct summands of $E$ to the simple-minded collection above. It follows from Theorem 5.3 that on $E$ there is a supersymmetric non-degenerated bilinear form. Very sadly, however, we cannot apply Theorem 2.3, because it is not clear whether the bilinear form defined a cyclic structure on $E$.

For $\mathcal{E} = \text{MCM}(R)$ the category of maximal Cohen–Macaulay modules over a complete Gorenstein local algebra $R$ with isolated singularity, de Völcsey and Van den Bergh has another approach in [15].

**Theorem 5.4 ([15, Theorem 1.1]).** Assume as above with $\mathcal{E} = \text{MCM}(R)$ for a complete Gorenstein local algebra $R$ with isolated singularity. Assume that $\tilde{A} \to A$ is a cofibrant minimal model of $A$ (with the idempotent $e$ lifted) and let $B = \tilde{A}/\tilde{A}e\tilde{A}$. Then there is a triangle equivalence $\text{per}(B)/\mathcal{D}_{fd}(B) \simeq C$.

It is not hard to check that $\mathcal{D}_{fd}(B) \simeq \mathcal{D}_{fd,A/AeA}(\text{Mod} A)$ ([26, Corollary 2.12(b)]). If $\tilde{A}$ is the complete Ginzburg dg algebra of some quiver with potential, so is $B$. In particular, we obtain a triangle equivalence $\mathcal{C}_{(Q,W)} \cong C$.

Martin Kalck and myself independently started this approach in [26, 25].

**Example 5.5.** As an example, let us consider the 3-dimensional McKay correspondence. Let $G \subset SL_3(k)$ be a finite subgroup. Let $S = k[x,y,z]$ with the induced $G$-action and let $R = S^G$. Assume that $R$ has isolated singularity. Then by a theorem of Iyama [22, Theorem 2.5], as an $R$-module $S$ is a 2-cluster-tilting object in $\text{MCM}(R)$. Its endomorphism algebra $A = \text{End}_R(S)$ is isomorphic to $\hat{J}(Q',W')$, where $Q'$ is the McKay quiver and $W'$ is a canonical potential. See [18, Section 4.4] for the above statement and the construction of $W'$. Moreover, in this case the canonical homomorphism $\hat{\Gamma}(Q',W') \to \hat{J}(Q',W')$ is a quasi-isomorphism, so it is a cofibrant minimal model of $\hat{J}(Q',W')$. Let $e$ be the identity of $R$, considered as an element of $A$. Let 0 be the vertex of $Q'$ corresponding to the trivial $G$-module. Let $(Q,W)$ be the quiver with potential obtained from $(Q',W')$ by deleting the vertex 0. Then $B = \hat{\Gamma}(Q',W')/\hat{\Gamma}(Q',W')e\hat{\Gamma}(Q',W') = \hat{\Gamma}(Q,W)$. As a consequence, the stable category $\text{MCM}(R)$ is triangle equivalent to the generalised cluster category $\mathcal{C}_{(Q,W)}$.

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