Seshadri constants in finite subgroups of abelian surfaces.

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Abstract

Given an étale quotient $q : X \to Y$ of smooth projective varieties we relate the simple Seshadri constant of a line bundle $M$ on $Y$ with the multiple Seshadri constant of $q^* M$ in the points of the fiber. We apply this method to compute the Seshadri constant of polarized abelian surfaces in the points of a finite subgroup.

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1 Introduction.

The multiple Seshadri constants are a natural generalization of the Seshadri constants at single points defined by Demailly in [5]. If $X$ is a smooth projective variety of dimension $n$, $L$ is an ample line bundle on $X$ and $x_1, \ldots, x_r$ are distinct points in $X$, then the Seshadri constant of $L$ at $x_1, \ldots, x_r$ is:

$$\epsilon(L; x_1, \ldots, x_r) = \sup \{ \epsilon | f^* L - \epsilon \sum_{i=1}^r E_i \text{ is nef } \},$$

where $f$ is the blowing up of $X$ at $x_1, \ldots, x_r$ and $E_1, \ldots, E_r$ are the exceptional divisors. These constants have the upper bound:

$$\epsilon(L; r) \leq \sqrt[n]{\frac{L^n}{r}}.$$

However, explicit values are difficult to obtain even when $r = 1$. General bounds for the simple Seshadri constants on surfaces are given in [2], [7] or [9]. They were computed for simple abelian surfaces by Th. Bauer (see [2]); Ch. Schultz gave values for Seshadri constants on products of two elliptic curves (see [8]).

The case of the multiple Seshadri constants is harder. For example, in the plane the Nagata conjecture is still an open problem (see [10]):
Conjecture 1.1 (Nagata conjecture) Let $x_1, \ldots, x_r$ be $r \geq 10$ be general points in $P^2$ then:

$$\epsilon(O_{P^2}(1); x_1, \ldots, x_r) = \frac{1}{\sqrt{r}}.$$ 

This has been extended for an arbitrary surface. When $r$ is big enough, the value of the Seshadri constant at $r$ very general points is conjectured to be maximal (see [4]). Very interesting lower bounds for multiple Seshadri constants were given by B. Harbourne in [6]. In [11] Tutaj-Gasińska gives bounds for the Seshadri constant of abelian surfaces in half-periods points; in [12], he gives the exact values in two half-periods points.

In this paper, we obtain the exact value of the multiple Seshadri constants of polarized abelian surfaces in points of a finite subgroup. This generalizes the results of [11] and [12] but applying a different method.

If $q : X \rightarrow Y$ is an étale quotient of smooth projective varieties we prove that the simple Seshadri constant of a line bundle $M$ on $Y$ is the same that the multiple Seshadri constant of $q^* M$ in the points of the fiber. We apply this result when $X$ is an abelian surface and $Y = X/G$ is the quotient by a finite subgroup $G$. Since, the simple Seshadri constants on abelian surfaces are known (see [2], [8]), we obtain the multiple Seshadri constants on $G$.

In particular, when $X$ is an abelian surface with Picard number one, we prove the following:

**Theorem 1.2** Let $(X, L)$ be a polarized abelian surface of type $(1, d)$ with $\rho(X) = 1$. Let $x$ be a point of $X$. Let $G$ be a finite subgroup of $X$ of order $g$. Consider the étale quotient:

$$q : X \rightarrow X/G$$

Let $n$ be the minor integer verifying $nL = q^* M$ for some line bundle $M$ on $X/G$. Then:

1. If $\sqrt{2d/g}$ is rational, then $\epsilon(L; x + G) = \sqrt{\frac{2d}{g}}$.

2. If $\sqrt{2d/g}$ is irrational, then

$$\epsilon(L; x + G) = \frac{k_0 2dn}{l_0 g} = \sqrt{1 - \frac{1}{l_0^2}} \sqrt{\frac{L^2}{g}}$$

where $(l_0, k_0)$ is the primitive solution of Pell’s equation $t^2 - \frac{2nd^2}{g} k^2 = 1$.

Note, that the order of $G$ and the degree of $L$ is not sufficient to determine the value of the Seshadri constants. It depends also on the the structure of $G$ (see Remark 3.4 for details).

In general, we also prove:
Theorem 1.3 Let \((X, L)\) be a polarized abelian surface. The multiple Seshadri constant of \(L\) at the points of a finite subgroup is rational.

2 The main theorem.

Theorem 2.1 Let \(q : X \rightarrow Y\) be an étale \(n : 1\) quotient between to smooth projective varieties. Let \(M\) be a line bundle on \(Y\) and \(y \in Y\). Then:
\[ \epsilon(q^*M; q^{-1}(y)) = \epsilon(M, y) \]

Proof: Let \(g : \tilde{Y} \rightarrow Y\) be the blowing up of \(Y\) at \(y\). Let \(f : \tilde{X} \rightarrow X\) be the blowing up of \(X\) at \(q^{-1}(y)\). There is an induced morphism \(\tilde{q} : \tilde{X} \rightarrow \tilde{Y}\) such that the following diagram is commutative:
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{q}} & \tilde{Y} \\
\downarrow f & & \downarrow g \\
X & \xrightarrow{q} & Y \\
\end{array}
\]
If \(E\) is the exceptional divisor of \(g\) and \(E_1, \ldots, E_n\) are the exceptional divisors of \(f\), we have:
\[ f^*q^*M - \epsilon(E_1 + \ldots + E_n) = \tilde{q}^*g^*M - \epsilon \tilde{q}^*E = \tilde{q}^*(g^*M - \epsilon E). \]
Thus,
\[ f^*q^*M - \epsilon(E_1 + \ldots + E_n) \text{ is nef } \iff g^*M - \epsilon E \text{ is nef}. \]

3 Multiple Seshadri constant on polarized abelian surfaces.

We will apply the main theorem to compute the multiple Seshadri constants on polarized abelian surfaces. We will use the results of Bauer to compute the simple Seshadri constant (see [2]):

Theorem 3.1 (Bauer) Let \((Y, M)\) be an abelian surface of type \((1, d')\) with \(\rho(Y) = 1\),
1. If \(\sqrt{2d'}\) is rational, then \(\epsilon(M) = \sqrt{2d'}\).
2. If \(\sqrt{2d'}\) is irrational, then
\[ \epsilon(M) = \frac{k_0}{l_0}2d' = \sqrt{1 - \frac{1}{l_0^2}} \sqrt{M^2} \]
where \((k_0, l_0)\) is the primitive equation of the Pell’s equation \(l^2 - 2dk^2 = 1\).
Let us recall some basic facts about abelian surfaces. We will follow the
notation of [3]. Let $(X, L)$ be a polarized abelian surface of type $(1, d)$. Let $H$
be the first Chern class of $L$. There is a basis $\lambda_1, \lambda_2, \mu_1, \mu_2$, respect to which $E = ImH$ is given by the matrix:

$$
\begin{pmatrix}
0 & D \\
-D & 0
\end{pmatrix}
$$

where $D = diag(1, d)$. In this way, the abelian surface $X$ is the quotient

$$
\pi : V \rightarrow X = V/\Lambda
$$

with

$$
\Lambda = \langle \lambda_1, \lambda_2 \rangle \oplus \langle \mu_1, \mu_2 \rangle.
$$

Let $G$ be a finite subgroup of $X$ of order $g$. Consider the quotient map:

$$
q : X \rightarrow X/G.
$$

The variety $Y = X/G$ is an abelian surface. In particular $Y = V/\Lambda'$, where

$$
\Lambda' = \pi^{-1}(G).$

There is a criterion for a line bundle $L' \in Pic(X)$ to descend
under $q$:

\begin{lemma}
Let $L' = L(H', \chi')$ be a line bundle on $X$. Then, $L' = q^*M$ for
some line bundle $M \in Pic(Y)$ if and only if $ImH'(\Lambda', \Lambda') \subset Z$.
\end{lemma}

\begin{proof}
See Chapter 2, Corollary 4.4 of [3].
\end{proof}

\begin{corollary}
If $L$ is a line bundle on $X$ of type $(1, d)$, then $exp(G)^2L = q^*M$
for some $M \in Pic(Y)$.
\end{corollary}

\begin{proof}
It is a consequence of the previous corollary. The exponent of a group $G$
is the least common multiple of the orders of the elements of $G$. The first
Chern class of $exp(G)^2L$ is $exp(G)^2H$. Let $x, x' \in \pi^{-1}(G)$. We know that

$$
exp(G) \cdot x, exp(G) \cdot x' \in \Lambda.
$$

Thus:

$$
Im (exp(G)^2H)(x, x') = Im H(exp(G) \cdot x, exp(G) \cdot x') \in Z.
$$

\end{proof}

\begin{remark}
We can consider the minor integer $n$ such that $nL$ descend to a
line bundle $M$. The minimality of $n$ implies that $M$ is a primitive line bundle
on $Y$. We know that $1 \leq n \leq exp(G)^2$. However, the number $n$ does not depend
only on the exponent of $G$. For example, if $G$ is the cyclic group generated by
$\pi(\lambda_2/d)$ then $L$ descends to a line bundle $M$ of type $(1, 1)$, so $n = 1$. On the
other hand, if $G$ is the group generated by $\pi(\lambda_1/k), \pi(\mu_1/k)$ for any $k > 1$
then the minor value of $n$ is $n = k^2 = exp(G)^2$.
\end{remark}
The main result on abelian surfaces with Picard number one will be the following:

**Theorem 3.5** Let \((X, L)\) be a polarized abelian surface of type \((1, d)\) with \(\rho(X) = 1\). Let \(x\) be a point of \(X\). Let \(G\) be a finite subgroup of \(X\) of order \(g\). Consider the étale quotient:

\[
q : X \longrightarrow X/G
\]

Let \(n\) be the minor integer verifying \(nL = q^*M\) for some line bundle \(M\) on \(X/G\). Then:

1. If \(\sqrt{2d/g}\) is rational, then \(\epsilon(L; x + G) = \sqrt{2d/g}\).
2. If \(\sqrt{2d/g}\) is irrational, then
   \[
   \epsilon(L; x + G) = \frac{k_0}{l_0} \frac{2dn}{g} = \sqrt{1 - \frac{1}{l_0^2} \frac{L^2}{g}}
   \]
   where \((l_0, k_0)\) is the primitive solution of Pell’s equation \(l^2 - \frac{2n^2d}{g}k^2 = 1\).

**Proof:** Let \(y = q(x)\). By the Theorem 2.1

\[
\epsilon(L; x + G) = \frac{1}{n} \epsilon(nL; x + G) = \frac{1}{n} \epsilon(M, y).
\]

The line bundle \(M\) is a primitive line bundle of type \((1, d')\) with \(d' = n^2d/g\). Thus \(\sqrt{2d/g}\) is rational if and only if \(\sqrt{2d'}\) is rational. Now, the result follows from Theorem 3.1. ■

**Corollary 3.6** Let \((X, L)\) be a polarized abelian surface of type \((1, d)\) with \(\rho(X) = 1\). Let \(x_1, \ldots, x_r\) be \(r\) general points of \(X\). Then:

1. If \(\sqrt{2d/r}\) is rational, then \(\epsilon(L; x_1, \ldots, x_r) = \sqrt{2d/r}\).
2. If \(\sqrt{2d/r}\) is irrational, then
   \[
   \epsilon(L; x_1, \ldots, x_r) \geq 2d \frac{k_0}{l_0} = \sqrt{1 - \frac{1}{l_0^2} \frac{L^2}{r}}
   \]
   where \((l_0, k_0)\) is the primitive solution of Pell’s equation \(l^2 - 2rdk^2 = 1\).

**Proof:** By the semicontinuity of the Seshadri constant,

\[
\epsilon(L; x_1, \ldots, x_r) \geq \epsilon(L; x + G)
\]

for any point \(x \in X\) and any subgroup \(G\) of order \(r\). In particular, taking the cyclic subgroup \(G = \langle \pi(\lambda_r) \rangle\) and applying the previous theorem we obtain the desired bound. ■
Corollary 3.7 Let \((X, L)\) be a polarized abelian surface of type \((1, d)\) with \(\rho(X) = 1\). Let \(x\) be a point of \(X\). Let \(X_m\) be the subgroup of \(m\)-torsion points. Suppose that \(\sqrt{2d^2}\) is not an integer, then:

\[
\epsilon(L; x + X_m) = 2 \frac{d}{m^2} \frac{k_0}{l_0} = \sqrt{1 - \frac{1}{l_0^2}} \sqrt{\frac{L^2}{m^4}}
\]

where \((l_0, k_0)\) is the primitive solution of Pell’s equation \(l^2 - 2dk^2 = 1\).

**Proof:** Note that, in this case, the minor number \(n\) such that \(nL\) descends under \(q\) is \(n = m^2\). Now, it is sufficient to apply the Theorem 1.2.

Remark 3.8 In [11], Tutaj-Gasińska obtains a bound for the Seshadri constant in half-periods of a line bundle on a polarized abelian surface of type \((1, d)\):

\[
\epsilon(L; X_2) \leq 2 \sqrt{1 - \frac{1}{l_0^2}} \sqrt{\frac{L^2}{16}}
\]

where \((l_0, k_0)\) is the primitive solution of Pell’s equation \(l^2 - 32dk^2 = 1\). Here, we see that we exact value appears when we use the Pell’s equation \(l^2 - 2dk^2 = 1\).

Corollary 3.9 Let \((X, L)\) be a polarized abelian surface of type \((1, d)\) with \(\rho(X) = 1\). Let \(e_1, e_2\) be two half periods of \(X\). Suppose that \(\sqrt{d}\) is not an integer.

1. If \(e_1 - e_2 \in K(L)\) then

\[
\epsilon(L; e_1, e_2) = d \frac{k_0}{l_0}
\]

where \((l_0, k_0)\) is the primitive solution of Pell’s equation \(l^2 - dk^2 = 1\).

2. If \(e_1 - e_2 \notin K(L)\) then

\[
\epsilon(L; e_1, e_2) = 2d \frac{k_0}{l_0}
\]

where \((l_0, k_0)\) is the primitive solution of Pell’s equation \(l^2 - 4dk^2 = 1\).

**Proof:** Consider the group \(G = \langle e_1 - e_2 \rangle\). It has order two. Moreover, \(L\) descends to a line bundle in \(X/G\) if and only if \(e_1 - e_2 \in K(L)\). Now we only have to apply the Theorem 1.2.

Remark 3.10 In [12], a similar study is made. However, the Theorem (1, [12]) does not distinguish which are the two half-periods where the Seshadri constant is computed. The problem is the application of Lemma (19, [12]). It says that one can choose a line bundle \(L_c\) such that any two half periods \(e_1, e_2\) have the same parity. However, this changes the number of even and odd half periods of \(L\). From this, the arguments on page (532, [12]) can fail.
We could use similar arguments to compute the multiple Seshadri constants at points of a finite subgroup of line bundles on abelian surfaces with Picard number greater than 1. The simple Seshadri constants of these surfaces were computed by C. Schultz in [8]. Anyway, we know that the simple Seshadri constant of a line bundle on any abelian surface is always rational. From this:

**Theorem 3.11** Let \((X, L)\) be a polarized abelian surface. The multiple Seshadri constant of \(L\) at the points of a finite subgroup is rational.

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