Timelike Vector Field Dynamics in the Early Universe

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Abstract

We study the dynamics of a timelike vector field when the background spacetime is in an accelerating phase in the early universe. It is shown that a timelike vector field is difficult to realize an inflationary phase, so we investigate the evolution of a vector field within a scalar field driven inflation model. And we calculate the power spectrum of the vector field without considering the metric perturbations. While the time component of the vector field perturbations provides a scale invariant spectrum when $\xi = 0$, where $\xi$ is a nonminimal coupling parameter, both the longitudinal and transverse perturbations give a scale invariant spectrum when $\xi = 1/6$ in the absence of the coupling terms. The deviation of the power spectrum due to the coupling terms are calculated by use of the Greens’ function.
I. INTRODUCTION

A recent remarkable development in observational data reveals many interesting, unusual features which could not be predicted theoretically before such as the present accelerating phase by the supernova data [1], the suppression of the cosmic microwave background (CMB) angular power in the low multipole moments [2] and the large scale anomaly in CMB [3]. These progresses enable us to enter into the precision cosmology theoretically and to seek the new physics. Recently, vector fields are widely investigated to explore inflation [4, 5, 6, 7] and also to explain the dark energy problem [8, 9, 10, 11].

It was known to be difficult to realize a vector field inflation model because the effective mass of a vector field must be order of the Hubble scale [4]. But recently a successful inflation model is achieved in [6]. They have used either a triad (a triplet of orthogonal vector fields) [8] or large $N$ vector fields for an isotropic spacetime and taken into account a nonminimal coupling for a slow-roll phase. In that case, vector field inflation looks like a scalar field chaotic inflation model. A vector field can also play against the cosmic no hair theorem in anisotropic inflation [5]. In a usual scalar field driven inflation model, even if the initial stage starts off from an anisotropic background spacetime (e.g. Bianchi type models), the anisotropy would disappear very soon because of an accelerating expansion. But if there exists a vector field, the anisotropy will remain even if the universe undergoes a period of inflation. But the vector field models [4, 5, 6] which have the standard Maxwell kinetic term are known to be unstable [12] because they contain a ghost. The presence of the ghost is due to the sign that one needs to choose for the nonminimal coupling of the vector field to the curvature.

Although a vector field can provide some interesting properties in inflation or in the dark energy problem, it is difficult to handle, especially with the linear perturbations [8]. Because a non-vanishing vector field breaks spatial isotropy in a background spacetime and further makes it impossible to decompose the perturbation modes [13] - scalar, vector and tensor perturbation - in the linear perturbation theory. One way to resolve spatial anisotropy is to use a triad [8] or large $N$ random vector fields [6]. The other way is to use an anisotropic background spacetime [4, 5, 14]. The difficulty, however, in using decomposition theorem in the linear perturbations causes new obstacles to the calculation of the power spectrum and to the fitting with the observational data. In spite of this difficulty in the linear perturbations,
the gravitational wave spectrum is calculated in [15], in which they used the underlying symmetry in order to eliminate the mode coupling terms. The power spectra of scalar and tensor perturbations of a timelike vector fields are calculated in [17] with a fixed-norm condition. And in [14], the power spectrum of the longitudinal and transverse component of a spacelike vector field are calculated, but the gravitational metric perturbations were not considered. While the longitudinal component is scale invariant for \( m^2 \ll H^2 \), the transverse component is scale invariant when a vector field is coupled to the gravity nonminimally (\( \xi = 1/6 \)) for \( m^2 \ll H^2 \).

In order to avoid the breaking of a spatial isotropic background and the existence of mode coupling between different perturbation modes, we will take into account a timelike vector field instead of a spacelike vector field [4, 5, 6]. Since the Maxwell kinetic energy, \(- F_{\mu\nu} F^{\mu\nu} / 4\), could not present any dynamics for a timelike vector field, we will consider a vector field Lagrangian with the general kinetic energy terms [16, 17] for the nontrivial dynamics of a timelike vector field

\[
\mathcal{L}_A = -\frac{1}{2} \beta_1 \nabla_\mu A_\nu \nabla^\mu A^\nu - \frac{1}{2} \beta_2 (\nabla_\rho A^\rho)^2 - \frac{1}{2} \beta_3 \nabla_\mu A^\mu \nabla_\nu A^\nu - \frac{1}{2} (m^2 - \xi R) A_\mu A^\mu. \tag{1}
\]

In this paper, especially we will only focus on \( \beta_1 = \beta_2 = -\beta_3 = 1 \) case, then the Lagrangian for a vector field can be written as

\[
\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\nabla_\mu A^\mu)^2 - \frac{1}{2} (m^2 - \xi R) A_\mu A^\mu, \tag{2}
\]

where \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \). Unlike in [16, 17], we do not require a fixed-norm condition, \( A_\mu A^\mu = -m^2 \).

This paper is organized as follows: in Section III we describe our model and discuss about the difficulty in realization of successful inflation with a timelike vector field. And we calculate the evolution of a vector field in a scalar field driven inflation model. In Section III we calculate the linear perturbation of a vector field without considering metric perturbations on a scalar field driven inflationary background. And the power spectrum of a vector field is calculated. We discuss about the spectral index of the vector field perturbations and briefly comment about the linear perturbations including gravitational metric perturbations. We conclude in Section IV.
II. BACKGROUND DYNAMICS WITH TIMELIKE VECTOR FIELDS

We start with an action of a massive vector field which is coupled nonminimally to gravity

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\nabla_{\mu} A^{\mu})^2 - \frac{1}{2} (m^2 - \xi R) A_{\mu} A^{\mu} \right] \]

where \( \xi \) is a nonminimal coupling parameter. This nonminimal coupling term can make it possible to occur a successful inflationary period with a spacelike vector field \([6]\).

By varying the action (3) with respect to \( A_{\mu} \), one obtains the equations of motion,

\[ \nabla_{\mu} F^{\mu\nu} + \nabla^{\nu} \nabla_{\rho} A^{\rho} - (m^2 - \xi R) A^{\nu} = 0. \]

(4)

And Einstein equations can be obtained by varying the action with respect to \( g_{\mu\nu} \)

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T^{(A)}_{\mu\nu}, \]

(5)

\[ T^{(A)}_{\mu\nu} = F^{\rho}_\mu F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - A_{\mu} \nabla_{\nu} A^{\rho} - A_{\nu} \nabla_{\rho} A^{\mu} \]

\[ + \frac{1}{2} g_{\mu\nu} (\nabla_\rho A^\rho)^2 + g_{\mu\nu} A^\rho \nabla_\rho A^\sigma + (m^2 - \xi R) A_{\mu} A_{\nu} \]

\[ - \xi R_{\mu\nu} A^\rho + \xi (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) A_{\rho} A^{\rho} - \frac{1}{2} g_{\mu\nu} (m^2 - \xi R) A_{\rho} A^{\rho}. \]

(6)

Since we want a homogeneous and isotropic background spacetime, we consider a timelike vector field, \( A_{\mu} A^{\mu} < 0 \), and choose \( A_{\mu} = (\chi, \vec{0}) \). In the spatially flat FRW metric

\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \]

(7)

the Einstein equations and equation of motion for \( \chi \) can be expressed as

\[ H^2 = \frac{8\pi G}{3} \rho_\chi \]

\[ = \frac{8\pi G}{3} \left[ -\frac{1}{2} \dot{\chi}^2 - 3(1 - 2\xi) H \dot{\chi} - 12\xi \dot{H} \dot{\chi}^2 - \frac{3}{2} (3 + 14\xi) H^2 \chi^2 + \frac{3}{2} m^2 \chi^2 \right], \]

(8)

\[ \dot{H} = -4\pi G (\rho_\chi + p_\chi) \]

\[ = -4\pi G \left[ (1 + 2\xi) m^2 \chi^2 - 2\xi (\dot{\chi}^2 - 5H \chi \dot{\chi} + (1 + 6\xi) H^2 \chi^2 + 6(1 + 2\xi) H^2 \chi^2) \right], \]

(9)

\[ \ddot{\chi} + 3H \dot{\chi} + \left[ 3(1 - 2\xi) \dot{H} - 12\xi H^2 + m^2 \right] \chi = 0, \]

(10)

where we have used \( R = 6(\dot{H} + 2H^2) \).

We need to check if the timelike vector field could generate an accelerating phase. If we define \( m_{\text{eff}}^2 = 3(1 - 2\xi) \dot{H} - 12\xi H^2 + m^2 \) in \([10]\), it is required \( m_{\text{eff}}^2 \ll H^2 \) for a slow-rolling
field and $\dot{H} \ll H^2$ for sufficient inflation, which means $N_e = \int Hdt \geq 50$ to fit to the observational data. From the expression in (8), the energy density of the vector field is not positive definite. The vector field may have a negative energy density in some range. So we need to constrain on $\chi$ to avoid a negative energy density in our discussion.

If we assume that the vector field can generate inflation, one obtains from (8) for $\xi = 0$

$$H^2 \simeq \frac{8\pi}{3m_{pl}^2} \left(1 + 12\pi \left(\frac{\chi}{m_{pl}}\right)^2 \right)^{-1} \left[-4\pi m^2 \chi \left(\frac{\chi}{m_{pl}}\right) + \frac{5}{2} m^2 \chi \right] \chi^2,$$

(11)

where we have neglected the kinetic energy term and used (10) in which we neglect $\ddot{\chi}$. But we keep $\dot{H}$ term when we obtain (11) because it has the same order of magnitude as the potential term from (9) for $\xi = 0$. Here $m_{pl}$ is Planck mass. In order to guarantee the positive energy density, $\chi$ should be constrained by $\chi \ll m_{pl}$. This is different from a scalar field chaotic inflation model in which the initial amplitude of an inflaton should be larger than the Planck mass to have a sufficient inflationary period.

As long as $\chi \ll m_{pl}$, $H^2 \simeq \dot{H}$ and $H^2 < m_{eff}^2$ where we have used $\dot{H} \sim -4\pi m^2 \chi \left(\frac{\chi}{m_{pl}}\right)^2$ when $\xi = 0$. These conditions contradict with those for sufficient inflation. So it is hard to realize inflation by the timelike vector field. Even if the nonminimal coupling is taken into account, $\dot{H}$ is shown to be of the same order of $H^2$ and the potential of the vector field in (8) and (9)

$$H^2 \sim \frac{8\pi}{3m_{pl}^2} \left[3(1 - 8\xi + 4\xi^2)\dot{H}\chi^2 + \frac{1}{2}(5 - 4\xi)m^2 \chi^2 \right],$$

(12)

$$\dot{H} \sim -\frac{4\pi}{m_{pl}^2} \left[4\xi(4\xi - 3)H^2 \chi^2 + \frac{1}{3}(3 - 4\xi)m^2 \chi^2 \right],$$

(13)

where we have used $\chi \ll m_{pl}^2$. As a result, the slow-roll conditions could not be fulfilled.

We calculate (8), (9) and (10) numerically to confirm these analytical arguments. The results are shown in Fig. 1. We set to $\chi_i = 0.05m_{pl}$ and $\dot{\chi} = 0$ as an initial condition. As expected, we could not obtain the inflationary solutions. Even if we consider the nonminimal coupling $\xi$, it could not help to get an accelerating phase. We consider $\xi = 1/6$ and $1/2$ in Fig. 1 but it does not improve the results.

Even if the vector field have failed to generate inflation, it may play a role as a curvaton [14, 19] which can generate the curvature perturbation after the end of inflation. So we need to investigate the dynamics of the vector field during an inflationary period which occurs due to an additional matter such as a scalar field. Then we add a scalar field $\phi$, which drives
FIG. 1: The evolution of $\chi$ is plotted with a different nonminimal coupling parameter $\xi$.

FIG. 2: The evolution of $\chi$ is plotted depending on $m_\chi/m_\phi$. Here a scalar field $\phi$ is responsible for an accelerating expansion. We have used $m_\phi/m_{pl} = 10^{-5}$, $\phi_i = 3m_{pl}$ and $\chi_i = 10^{-3}m_{pl}$.

an accelerating phase, to the vector field action:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} 
- \frac{1}{2} (\nabla_\mu A^\mu)^2 - \frac{1}{2} (m^2_\chi - \xi R) A_\mu A^\mu \right],$$

where we will consider a massive scalar field potential $V(\phi) = \frac{1}{2} m^2_\phi \phi^2$ in the present paper.
One obtains the equations of motion for $\phi$ and $\chi$

\[
\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0, \tag{15}
\]
\[
\ddot{\chi} + 3H\dot{\chi} + \left[3(1 - 2\xi)\dot{H} - 12\xi H^2 + m_\chi^2\right] \chi = 0, \tag{16}
\]

where the Einstein equations become

\[
H^2 = \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) + \rho_\chi\right) \simeq \frac{8\pi G}{3} V(\phi), \tag{17}
\]
\[
\dot{H} = -4\pi G (\dot{\phi}^2 + \rho_\chi + p_\chi), \tag{18}
\]

and $\rho_\chi$ and $p_\chi$ are given in (8) and (9).

Since the scalar field is responsible for an inflationary phase, we can assume $H \simeq \text{const.}$ and use the slow-roll conditions, $|\dot{H}| \ll H^2$, $\frac{1}{2} \dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll 3H\dot{\phi}$ . In addition, in order for the scalar field $\phi$ to be a dominant component, we assume $\rho_\chi \ll \rho_\phi$.

Then one can easily get the solution of (15) for $V = \frac{1}{2} m_\phi^2 \phi^2$

\[
\phi(t) = \phi_i - \frac{m_\phi}{\sqrt{12\pi G}} (t - t_i), \tag{19}
\]

where $\phi_i$ is an initial value at $t = t_i$. The equation for $\chi$ can be expressed for $\xi = 0$

\[
\ddot{\chi} + 3H\dot{\chi} + (m_\chi^2 - m_\phi^2) \chi \simeq 0, \tag{20}
\]

where we have assumed $\frac{3}{2} m_\chi^2 \chi^2 \ll \frac{1}{2} m_\phi^2 \phi^2$ from the slow-roll conditions ($|\dot{H}| \ll H^2$) and $\chi^2 \ll m_{pl}^2$ which can be derived from the condition $\rho_\chi \ll \rho_\phi$. If we set $m_{eff}^2 = m_\chi^2 - m_\phi^2$, then the solution of $\chi$ depends on $m_{eff}^2$. If $\ddot{\chi}$ in (20) can be neglected, $\chi(t)$ is constant for $m_{eff}^2 = 0$ during an inflationary period (see Fig. 2). And if $m_{eff}^2 > 0$, we can obtain

\[
\chi(t) = \chi_i \left(\frac{\phi(t)}{\phi_i}\right)^{m_{eff}^2/m_\phi^2} \approx \chi_i \left(1 - \frac{m_\phi}{\phi_i/m_{pl}} t\right)^{m_{eff}^2/m_\phi^2}. \tag{21}
\]

As the universe expands, $\chi(t)$ decreases. On the contrary, if $m_{eff}^2 < 0$, then (20) becomes

\[
3H\dot{\chi} - |m_{eff}^2| \chi \simeq 0, \tag{22}
\]

and its solution is given by

\[
\chi(t) = \chi_i \left(\frac{\phi(t)}{\phi_i}\right)^{-|m_{eff}^2|/m_\phi^2} \approx \chi_i \left(1 - \frac{m_\phi}{\phi_i/m_{pl}} t\right)^{-|m_{eff}^2|/m_\phi^2}. \tag{23}
\]

Contrary to $m_{eff}^2 > 0$ case, $\chi(t)$ increases as the universe expands.
In order to check these analytic results, we compute numerically the evolutions of $\phi$ and $\chi$. In Fig. 2 we plot the evolution of $\chi$ depending on $m_\chi/m_\phi$. We set to $m_\phi = 10^{-5}m_{pl}$, $\phi_i = 3m_{pl}$ and $\chi_i = 10^{-3}m_{pl}$ for the computation. The evolutions of $\chi$ show much different behavior depending on $m_\chi/m_\phi$. The numerical results are consistent with the analytic results. For $m_\chi = m_\phi$, $\chi$ shows constant behavior during an inflation period and after the end of inflation it begins to oscillate as a scalar field does. If $m_\chi = 10^{-3}m_\phi (m_{eff}^2 < 0)$, $\chi$ increases slowly as time goes on and starts oscillation after the end of inflation. Finally, for $m_\chi = 3m_\phi (m_{eff}^2 > 0)$, $\chi$ is decreasing.

III. LINEAR PERTURBATIONS

The linear perturbation calculations with a vector field is not an easy task. Even if we begin with either an anisotropic spacetime or an isotropic FRW spacetime using a triplet of orthogonal vectors for a spacelike vector field, the non-vanishing background vector fields will make it impossible to use decomposition theorem in the linear perturbations. This means there exist mode couplings between scalar, vector and tensor perturbations. In spite of these problems, gravitational wave spectrum is calculated in [15], in which the coupling terms are eliminated using the underlying symmetry and the metric perturbations for a spacelike vector field are discussed in anisotropic spacetime with the fixed-norm condition in [18]. The power spectrum and non-Gaussianity of the longitudinal and transverse component of a spacelike vector field are calculated without considering the metric perturbations.

Although mode coupling problems do not arise any more if we begin with a timelike vector field, the calculations are too messy and complicated. Linear perturbations of a timelike vector field is discussed in [17]. They also considered the metric perturbations with the fixed-norm constraint. We will discuss about the linear perturbations without taking into account the metric perturbations in this section and we will try to investigate the gravitational metric perturbations in the forthcoming paper [21].
A. Linear perturbations without metric perturbations

We decompose the perturbation of the vector field into the scalar and vector mode perturbations

$$\delta A_\mu(t, x) = (\delta A_0(t, x), \delta A_i(t, x)) \equiv (\delta \chi, a \nabla_i \psi + a S_i),$$

(24)

where $\nabla_i S^i = 0$. Here $\delta \chi$ and $\psi$ are scalars and $S_i$ is a vector perturbation.

With this decomposition of the linearized vector field, we obtain the perturbed equations of motion in momentum space by linearizing [4]

$$\ddot{\delta \chi}_k + 3H \dot{\delta \chi}_k + \left[ \frac{k^2}{a^2} + 3(1 - 2\xi)\dot{H} - 12\xi H^2 + m_\chi^2 \right] \delta \chi_k - 2aH \frac{k^2}{a^2} \dot{\psi}_k = 0,$$

(25)

$$\ddot{\psi}_k + 3H \dot{\psi}_k + \left[ \frac{k^2}{a^2} + (1 - 6\xi)\dot{H} + 2(1 - 6\xi)H^2 + m_\chi^2 \right] \psi_k + \frac{2H}{a} \delta \chi_k = 0,$$

(26)

$$\ddot{S}_{ik} + 3H \dot{S}_{ik} + \left[ \frac{k^2}{a^2} + (1 - 6\xi)\dot{H} + 2(1 - 6\xi)H^2 + m_\chi^2 \right] S_{ik} = 0,$$

(27)

where we have used the Fourier transform

$$\delta \chi(t, x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \delta \chi_k(t)e^{ik \cdot x},$$

(28)

and similarly for $\psi$ and $S_i$. Unlike a scalar field case, the perturbed equations of $\delta \chi$ and $\psi$ are coupled each other. Since it is not easy to calculate above equations analytically because
of the coupling terms, first we calculate numerically and the results are shown in Figs. 3 and 4 when \( k = 5aH \). We also calculate the equations for comparison with and without the coupling terms.

We have used \( m_\phi = 10^{-5}m_{pl}, \phi_i = 3m_{pl}, \chi_i = 10^{-3}m_{pl} \) and \( m_\phi = m_\chi \). We take the initial conditions for \( \delta \chi_{ki} \) and \( \psi_{ki} \) as

\[
\delta \chi_{ki}, \psi_{ki} = \frac{a_i^{-1}}{\sqrt{2k}} \exp (ik/aiH_i),
\]

where the subscript \( i \) denotes the initial value at \( a = a_i \). Because the equation of \( S_{ik} \) has the same form with that of \( \psi_k \) except the coupling term, the behavior of \( S_{ik} \) is similar to that of \( \psi_k \).

In Fig. 3 the evolutions of \( \delta \chi_k \) and \( \psi_k \) are shown for the minimal coupling case (\( \xi = 0 \)). In this diagram \( \delta \chi_k \) shows constant behavior on super-Hubble scales. But \( \psi_k \) as well as \( S_{ik} \) is decaying when the modes are larger than the Hubble horizon. On the contrary, for the nonminimal coupling (\( \xi = 1/6 \)) in Fig. 4 the situations are reverse. While \( \psi_k \) and \( S_{ik} \) show the constant solutions on super-Hubble scales, \( \delta \chi_k \) is increasing as the universe expands. And the amplitude of \( |\psi| \) is enhanced due to the effect of the mixing term in the sub-Hubble scale region. The mixing term in \( \psi \) seems to cause the instability in the small scale.

The coupled equations can be solved by use of the appropriate Green functions

\[
\delta \chi_k(t) = \delta \chi_k(t) + \Delta \chi_k,
\]
\[ \psi_k(t) = \tilde{\psi}_k(t) + \Delta \psi_k \] (31)

where \( \delta \tilde{\chi}_k \) and \( \tilde{\psi}_k \) are the solutions of Eqs. (25) and (26) in the absence of the coupling terms and

\[ \Delta \chi_k(t) = \int_{t_i}^{t} dt' G_{\chi,k}^R(t, t') 2aH \frac{k^2}{a^2} \psi_k(t'), \] (32)
\[ \Delta \psi_k(t) = \int_{t_i}^{t} dt' G_{\psi,k}^R(t, t') \frac{2H}{a} \delta \tilde{\chi}_k(t'). \] (33)

Here \( G_{\chi,k}^R(t, t') \) and \( G_{\psi,t}^R(t, t') \) are the retarded Greens’ functions which are defined by [22]

\[ G_{\chi,k}^R(t, t') = \theta(t - t')i(\delta \tilde{\chi}_k(t) \delta \tilde{\chi}_k(t') - \delta \tilde{\chi}_k(t) \delta \tilde{\chi}_k(t')), \] (34)

where \( \theta(t - t') = 1 \) if \( t > t' \) and \( \theta(t - t') = 0 \) if \( t < t' \) and similarly for \( G_{\psi,k}^R \). Here superscripts + and − denote the positive and negative frequency mode, respectively. These Greens’ functions obey the differential equations

\[ \left[ \partial_t^2 + 3H \partial_t + \frac{k^2}{a^2} - 12\xi H^2 + m^2 \chi \right] G_{\chi,k}^R(t, t') = -\delta(t - t'), \] (35)
\[ \left[ \partial_t^2 + 3H \partial_t + \frac{k^2}{a^2} + 2(1 - 6\xi) H^2 + m^2 \chi \right] G_{\psi,k}^R(t, t') = -\delta(t - t'). \] (36)

First, we will try to analyze the numerical results analytically on super-Hubble scales in the absence of the coupling terms. The temporal behaviors of the linearized vector field perturbations even with the coupling terms can be expected to show similar results on super-Hubble scales from Figs. 3 and 4. If we change the variable \( t \) into the scale factor \( a \), then the equations (25) and (26) can be expressed in the large scale limit as

\[ \frac{d^2 \delta \tilde{\chi}_k}{da^2} + \frac{4d\delta \tilde{\chi}_k}{a da} - \left( 12\xi - \frac{m^2 \chi}{H^2} \right) \frac{\delta \tilde{\chi}_k}{a^2} \approx 0, \] (37)
\[ \frac{d^2 \tilde{\psi}_k}{da^2} + \frac{4d\tilde{\psi}_k}{a da} + \left( 2(1 - 6\xi) + \frac{m^2 \chi}{H^2} \right) \frac{\tilde{\psi}_k}{a^2} \approx 0, \] (38)

where we have assumed \( H \simeq \text{const.} \) and \( |\dot{H}| \ll H^2 \). We can obtain the following solutions

\[ \delta \tilde{\chi}_k \sim C_k a^{p^+} + D_k a^{p^-}, \] (39)
\[ \tilde{\psi}_k \sim C'_k a^{q^+} + D'_k a^{q^-}, \] (40)

where

\[ p_\pm = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + 12\xi - \frac{m^2 \chi}{H^2}}, \quad q_\pm = -\frac{3}{2} \pm \sqrt{\frac{1}{4} + 12\xi - \frac{m^2 \chi}{H^2}}. \] (41)
and $C_k, D_k, C_k', D_k'$ are the constant coefficients depending on $k$.

For $\xi = 0$, if $m_\chi \ll H^2$, the dominant mode solution of $\delta \tilde{\chi}_k$ is constant. But $\tilde{\psi}_k$ shows decaying solutions (See Fig. 3). On the contrary, for $\xi = 1/6$, while $\delta \tilde{\chi}_k \propto a^{1/2}$, the dominant mode solution of $\tilde{\psi}_k$ is constant (See Fig. 4).

Next we calculate the power spectrum of $\delta \chi_k, \psi_k$ and $S_{ik}$. In order to calculate the power spectrum, we need the exact solutions of (25), (26) and (27). If we use $|\dot{H}| \ll H^2$ and $H \simeq \text{const}$ during inflation, $\delta \tilde{\chi}, \tilde{\psi}$ and $S_{ik}$ have the following exact form of the solution in the absence of the coupling terms:

\[
\delta \tilde{\chi}_k(t) = \left( \frac{k}{aH} \right)^{3/2} \left[ c_{1k} H^{(1)}_{\nu_\chi} \left( \frac{k}{aH} \right) + d_{1k} H^{(2)}_{\nu_\chi} \left( \frac{k}{aH} \right) \right], \tag{42}
\]

\[
\tilde{\psi}_k(t) = \left( \frac{k}{aH} \right)^{3/2} \left[ c_{2k} H^{(1)}_{\nu_\psi} \left( \frac{k}{aH} \right) + d_{2k} H^{(2)}_{\nu_\psi} \left( \frac{k}{aH} \right) \right], \tag{43}
\]

\[
S_{ik}(t) = \left( \frac{k}{aH} \right)^{3/2} \left[ c_{3k} H^{(1)}_{\nu_\psi} \left( \frac{k}{aH} \right) + d_{3k} H^{(2)}_{\nu_\psi} \left( \frac{k}{aH} \right) \right], \tag{44}
\]

where $H^{(1)}_{\nu}(x)$ and $H^{(2)}_{\nu}(x)$ are the Hankel function of the first and second kind, respectively, and

\[
\nu_\chi = \sqrt{\frac{9}{4} + 12\xi - \frac{m_\chi^2}{H^2}}, \quad \nu_\psi = \sqrt{\frac{1}{4} + 12\xi - \frac{m_\chi^2}{H^2}}. \tag{45}
\]

In order to determine the coefficients $c_{ik}$ and $d_{ik}$, we need initial conditions when the modes are well within the horizon, $k/aH \to \infty$. Although quantum field theory is not well constructed for the ghost fields, we assume in this paper the initial conditions for the vector field satisfy the WKB-type solution

\[
\lim_{k\eta \to -\infty} \delta \chi_k \approx \frac{a^{-1}}{\sqrt{2}\omega_k} e^{-i \int_{\eta}^{\eta'} \omega d\eta'} \tag{46}
\]

and similarly for $\psi_k$ and $S_{ik}$. When the modes stay well inside of horizon, $\omega$ can be approximated as $\omega \approx k$, then

\[
\lim_{k\eta \to -\infty} \delta \chi_k \approx \frac{a^{-1}}{\sqrt{2}k} \exp(-i k\eta). \tag{47}
\]

Here $\eta$ is a conformal time, $dt = a d\eta$, and for $H = \text{const.}$, $\eta = -\frac{1}{aH}$. Using the asymptotic form of the Hankel functions in the limit $x \gg 1$

\[
H^{(1,2)}_{\nu}(x) \sim \sqrt{\frac{2}{\pi x}} \exp \left[ \pm i x - \left( \nu + \frac{1}{2} \right) \frac{\pi}{2} \right], \tag{48}
\]
we choose $d_{ik} = 0$ for a positive frequency mode and determine $c_{ik}$ through the matching to the initial condition

$$c_{1k} = \frac{\sqrt{\frac{\pi}{4}} e^{i(\nu \chi + 1/2)/2} H}{k^{3/2}}, \quad c_{2k} = c_{3k} = \frac{\sqrt{\frac{\pi}{4}} e^{i(\nu \psi + 1/2)/2} H}{k^{3/2}}. \quad (49)$$

With these coefficients, $\delta \tilde{\chi}_k$ can be expressed in the large scale limit ($k \ll aH$) as

$$\delta \tilde{\chi}_k \simeq \sqrt{\frac{\pi}{a^3 H}} e^{i(\nu \chi)/2} \left( 1 - e^{2i\nu \chi} \right) \left[ \frac{1}{\Gamma(1 + \nu \chi)} \left( \frac{k}{2aH} \right)^\nu - \frac{e^{i\nu \chi}}{\Gamma(1 - \nu \chi)} \left( \frac{k}{2aH} \right)^{-\nu} \right]. \quad (50)$$

and $\tilde{\psi}_k$ and $S_{ik}$ are also obtained by replacing $\nu \chi$ with $\nu \psi$. Here we use the asymptotic form of $H^{(1)}_\nu(x)$ for $x \ll 1$

$$H^{(1)}_\nu(x) \sim \frac{2}{1 - e^{2i\nu \pi}} \left[ \frac{1}{\Gamma(1 + \nu)} \left( \frac{x}{2} \right)^\nu - \frac{e^{i\nu \pi}}{\Gamma(1 - \nu)} \left( \frac{x}{2} \right)^{-\nu} \right]. \quad (51)$$

Second term in (50) becomes a dominant mode and first term is a subdominant mode. These solutions are exactly consistent with the large scale solutions in (39) and (40).

The power spectrum for $\delta \tilde{\chi}_k$, $\tilde{\psi}_k$ and $S_{ik}$ are calculated for the dominant mode

$$\tilde{P}_\chi(k) = \frac{k^3}{2\pi^2} |\delta \tilde{\chi}_k|^2 = \frac{4\pi \csc^2 \nu \chi \pi}{\Gamma^2(1 - \nu \chi)} H^2 \left( \frac{H}{2aH} \right)^2 \left( \frac{k}{2aH} \right)^{3-2\nu \chi}, \quad (52)$$

$$\tilde{P}_\psi(k) = \frac{4\pi \csc^2 \nu \psi \pi}{\Gamma^2(1 - \nu \psi)} H^2 \left( \frac{H}{2aH} \right)^2 \left( \frac{k}{2aH} \right)^{3-2\nu \psi}. \quad (53)$$

and the power spectrum of $S_{ik}$ is same as that of $\tilde{\psi}_k$.

The spectral indexes for $\delta \tilde{\chi}_k$ and $\tilde{\psi}_k$ at late times ($k \ll aH$) are

$$n_{\chi} - 1 \equiv \frac{d \ln \tilde{P}_\chi}{d \ln k} = 3 - 2\nu \chi, \quad (54)$$

$$n_{\psi} - 1 = 3 - 2\nu \psi. \quad (55)$$

From (45), if $m^2_\chi \ll H^2$ and $\xi = 0$, $n_{\chi} = 1$ and $n_{\psi} = 3$. But if $m^2_\chi \ll H^2$ and $\xi = 1/6$, $n_{\chi} \simeq -0.1$ and $n_{\psi} = 1$. In other words, for a light vector field with $m^2_\chi \ll H^2$, $\delta \tilde{\chi}_k$ gives a scale invariant spectrum only when $\xi = 0$, but both $\tilde{\psi}_k$ and $S_{ik}$ are scale invariant only the nonminimal coupling case ($\xi = 1/6$). These results are a little different from those in [14] in which the scale invariant spectrum for the longitudinal perturbations ($\psi_k$) is only possible when $\xi = 0$ but the transverse perturbation ($S_{ik}$) gives a scale invariant spectrum only when $\xi = 1/6$. 
Now we can calculate the power spectrum of $\delta \chi$ using (56)
\[
\mathcal{P}_\chi(k) = \frac{k^3}{2\pi^2} |\delta \chi(k)|^2 = \tilde{\mathcal{P}}_\chi(k) \left(1 + \frac{\delta \mathcal{P}_\chi}{\mathcal{P}_\chi}\right),
\]  
and similarly for $\psi_k$ using (57). The second term in the last expression represents the effect of the coupling term.

Although it is difficult to compute (56), since we are interested in the amplitude of the perturbations well after horizon exit, the change in the power spectrum due to the coupling terms can be calculated in the limit $t \to \infty$ for $\xi = 0$:
\[
\frac{\delta \mathcal{P}_\chi}{\mathcal{P}_\chi}(\infty) \simeq \frac{2\tilde{\chi}_k \text{Im} \Delta \chi_k(\infty)}{|\tilde{\chi}_k(\infty)|^2}, \quad \frac{\delta \mathcal{P}_\psi}{\mathcal{P}_\psi}(\infty) \simeq \frac{2\tilde{\psi}_k \text{Re} \Delta \psi_k(\infty)}{|\tilde{\psi}_k(\infty)|^2}.
\]

The leading order in $\delta \tilde{\chi}$ and $\tilde{\psi}$ can be obtained from (58) and (59) in the late time limit:
\[
\delta \tilde{\chi}_k(\infty) = i \frac{H}{\sqrt{2k^{3/2}}}, \quad \tilde{\psi}_k(\infty) = \frac{a^{-1}}{\sqrt{2k^{1/2}}},
\]
and $\Delta \chi_k(\infty)$ and $\Delta \psi_k(\infty)$ can be calculated from (60) and (61)
\[
\Delta \chi_k(\infty) = -\frac{\pi^{3/2}2^{\nu_\chi-1}}{\Gamma(1-\nu_\chi) \sin \nu_\chi \pi} e^{i\pi(\nu_\psi+1/2)/2} \frac{H^4}{k^{7/2}} \int_0^{x_i} dx' x'^{3/2-\nu_\chi} x'^3 J_{\nu_\chi}(x') H_{\nu_\psi}^{(1)}(x'),
\]
\[
\Delta \psi_k(\infty) = -\frac{\pi^{3/2}2^{\nu_\psi-1}}{\Gamma(1-\nu_\psi) \sin \nu_\psi \pi} e^{i\pi(\nu_\psi+1/2)/2} \frac{H^4}{k^{11/2}} \int_0^{x_i} dx' x'^{3/2-\nu_\psi} x'^3 J_{\nu_\psi}(x') H_{\nu_\chi}^{(1)}(x'),
\]
where $x = \frac{k}{aH}$ and $J_{\nu}(x)$ is the Bessel function of the first kind. In the limit $k \to 0$ ($x_i \to 0$), the leading order terms become
\[
\text{Im} \Delta \chi_k(\infty) \propto \frac{k^{5/2}}{a_0^6 H^2}, \quad \text{Re} \Delta \psi_k(\infty) \propto \frac{k^{3/2}}{a_0^6 H^3},
\]
and then the leading order contributions in the change of the power spectrum are
\[
\frac{\delta \mathcal{P}_\chi}{\mathcal{P}_\chi}(\infty) \propto \frac{k^4}{a_0^6 H^7}, \quad \frac{\delta \mathcal{P}_\psi}{\mathcal{P}_\psi}(\infty) \propto \frac{k^2}{a_0^6 H^3}.
\]

The amplitude of the deviations from the power spectrum of $\delta \tilde{\chi}$ increases as $k^4$ for small $k$, while the deviations from the power spectrum of $\tilde{\psi}$ increases as $k^2$.

For $\xi = 1/6$, the deviations in the power spectrum due to the coupling terms can be calculated in the limit $k \to 0$, then
\[
\frac{\delta \mathcal{P}_\chi}{\mathcal{P}_\chi}(\infty) \simeq \frac{2\tilde{\chi}_k \Delta \chi_k(\infty)}{|\tilde{\chi}_k|^2} \times \frac{k^{1/2+\nu_\chi}}{a_0^{\nu_\psi+5/2} H^{1/2-\nu_\chi}},
\]
\[
\frac{\delta \mathcal{P}_\psi}{\mathcal{P}_\psi}(\infty) \simeq \frac{2\tilde{\psi}_k \text{Im} \Delta \psi_k(\infty)}{|\tilde{\psi}_k|^2} \times \frac{k^{3/2-\nu_\psi}}{a_0^{11/2-\nu_\psi} H^{5/2-\nu_\chi}},
\]
where we have used
\[ \delta\tilde{\chi}_k(\infty) \propto \frac{1}{a^{3/2}H^{1/2}} \left( \frac{k}{aH} \right)^{-\nu_\chi}, \quad \tilde{\psi}_k(\infty) = \frac{iH}{\sqrt{2k^{3/2}}}, \] (65)
and the leading order terms of \( \Delta\chi_k(\infty) \) and \( \text{Im}\Delta\psi_k(\infty) \) are
\[ \Delta\chi_k(\infty) \propto \frac{k^{1/2}}{a^{\nu_\chi+5/2}H^{3/2-\nu_\chi}}, \quad \text{Im}\Delta\psi_k(\infty) \propto \frac{k^{-\nu_\chi}}{a^{-\nu_\chi+11/2}H^{-\nu_\chi+3/2}}. \] (66)
Since \( \nu_\chi = \sqrt{17}/2 \approx 2 \), the deviation of \( \delta\tilde{\chi} \) increases as \( k^{1/2+\nu_\chi} \) and that of \( \tilde{\psi} \) decreases as \( 1/k^{\nu_\chi-3/2} \).

B. Brief discussions about the metric perturbations

We will briefly discuss about the linear perturbations of the vector field including the metric perturbations in this section. It is convenient to use a conformal time, \( \eta \), to treat the metric perturbations, so in this section we will use the conformal time.

We consider the perturbed metric in scalar and vector longitudinal gauge \([8, 23]\) in which the metric takes the form
\[ ds^2 = a^2(\eta) \left[ -(1 + 2\Phi)d\eta^2 - 2B_i d\eta dx^i + (1 - 2\Psi)\gamma_{ij} dx^i dx^j \right], \] (67)
where \( \nabla_i B^i = 0 \).

With this metric, we can derive the perturbed equations of motion of the time component of the vector field by linearizing (4)
\[ \delta\chi'' + 2H\delta\chi' - 3 \left( \mathcal{H}^2 - \mathcal{H}' - \frac{1}{3}(m^2 - \xi R)a^2 \right) \delta\chi - \nabla^2 \delta\chi + 2\mathcal{H}\nabla^2 \psi \
= \chi\Psi'' + 3\chi\Psi' + 3(\chi' - \mathcal{H}\chi)(\Phi' + \Psi') + 8\mathcal{H}\chi\Phi' + 2(\chi'' + 2\mathcal{H}\chi' - 3\mathcal{H}'\chi + 3\mathcal{H}'\chi)\Phi \
+ \xi a^2\chi\delta R, \] (68)
where \( \mathcal{H} = a'/a \) and a prime denotes the derivative with respect to the conformal time, and of the longitudinal (scalar) mode for \( \nu = i \)
\[ \psi'' + 2\mathcal{H}\psi' + \left[ \mathcal{H}' + \mathcal{H}^2 + a^2(m^2 - \xi R) \right] \psi - \nabla^2 \psi + 2\mathcal{H}\delta\chi \
= \chi(\Phi' + \Psi') + 2\chi\Psi' + (2\chi' + 6\mathcal{H}\chi)\Phi, \] (69)
and finally of transverse (vector) mode for \( \nu = i \)
\[ S_i'' + 2\mathcal{H}S_i' + \left[ \mathcal{H}' + \mathcal{H}^2 + a^2(m^2 - \xi R) \right] S_i - \nabla^2 S_i = 0. \] (70)
Since the spatial components of the background vector field vanish, the mode couplings between different perturbation modes do not occur. And the perturbed equation of the transverse components \( (S_i) \), (70), is completely decoupled from the metric perturbations. This implies that the power spectrum of \( S_i \) at late times becomes as in \((53)\)

\[
P_S(k) = \frac{4\pi \csc^2 \nu_\psi \pi}{\Gamma^2(1 - \nu_\psi)} \left( \frac{H}{2\pi} \right)^2 \left( \frac{k}{2aH} \right)^{3-2\nu_\psi},
\]

where \( \nu_\psi \) is given in \((45)\), and hence \( S_i \) is scale invariant when the light vector field is nonminimally coupled to gravity (\( \xi = 1/6 \)) even the metric perturbations are taken into account.

IV. CONCLUSIONS

We have investigated the dynamics of a vector field, which violates Lorentz invariance, during an accelerating phase in the early universe. In order to avoid a spatial anisotropic background, we employ the timelike vector field. We have shown that the timelike vector field is difficult to realize successful inflation since the effective mass of the vector field is order of the Hubble scale and the slow-roll conditions could not be fulfilled. Contrary to the spacelike vector field inflation model \([6]\) in which slow-roll phase can be realized by introducing a nonminimal coupling, it turns out that the nonminimal coupling does not help to generate an accelerating phase.

Although the timelike vector field could not generate inflation, we would expect it to play a role as a curvaton after the end of inflation which is driven by a scalar field. So we calculated the evolution of the vector field during a scalar field driven inflation period. The vector field can roll down slowly enough if the mass of the vector field is similar to that of a scalar field. We need to calculate the linear perturbations in order to check if the vector field can generate a scale invariant power spectrum.

Although we only consider the non-vanishing time component of the vector field on the spatially isotropic background spacetime to avoid anisotropy, the spatial component perturbation as well as the time component perturbations of the vector field should be taken into account. The spatial component perturbations can be decomposed into the longitudinal and transverse part. The time component perturbation and the longitudinal mode of the spatial component perturbation are coupled each other and these coupling terms cause the
instability in sub-Hubble scale. It would be necessary to investigate the instability due to the coupling terms in small scale \[21\].

If the transverse perturbations can give a scale invariant spectrum, it may be responsible for some large scale anomaly \[3\] in CMB power spectrum. We have calculated the power spectrum and spectral indexes of the time and spatial component perturbations of the vector field without considering the gravitational metric perturbations. In the absence of the coupling terms, the time component perturbation of the vector field gives a scale invariant spectrum when $\xi = 0$, but the longitudinal and transverse perturbation of the vector field provide a scale invariant spectrum when the vector field is coupled nonminimally to gravity. The fact that both longitudinal and transverse perturbations have a scale invariant spectrum only when $\xi = 1/6$ is different from the results of \[14\], in which the longitudinal perturbation has a scale invariant spectrum when $\xi = 0$ but the transverse perturbation does when $\xi = 1/6$. Further, we have calculated the amplitude of the deviations from the power spectrum which are calculated in the absence of the coupling terms. The deviations of $\delta \chi$ rise as $k^4$ when $\xi = 0$ and $k^{5/2}$ when $\xi = 1/6$ for small $k$, while the deviations of $\psi$ rise as $k^2$ when $\xi = 0$ and fall like $1/k^{1/2}$ when $\xi = 1/6$.

Note that even if we consider the gravitational metric perturbations, the transverse perturbation of the vector field is not affected by the metric perturbations as shown at the end of Section \[11\]. So the transverse perturbation of the vector field still gives a scale invariant spectrum only when $\xi = 1/6$. But it would be necessary to calculate the power spectrum and spectral indexes of the time component and longitudinal perturbation of the vector field with including the metric perturbations.

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