Constraint propagation in \((N + 1)\)-dimensional space-time

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Higher dimensional space-time models provide us an alternative interpretation of nature, and give us different dynamical aspects than the traditional four-dimensional space-time models. Motivated by such recent interests, especially for future numerical research of higher-dimensional space-time, we study the dimensional dependence of constraint propagation behavior. The \(N + 1\) Arnowitt-Deser-Misner evolution equation has matter terms which depend on \(N\), but the constraints and constraint propagation equations remain the same. This indicates that there would be problems with accuracy and stability when we directly apply the \(N + 1\) ADM formulation to numerical simulations as we have experienced in four-dimensional cases. However, we also conclude that previous efforts in re-formulating the Einstein equations can be applied if they are based on constraint propagation analysis.

I. INTRODUCTION

Higher dimensional space-time models have been investigated from many viewpoints in physics. Current research interests come from brane-world models that try to solve the hierarchical problem in the unified theory (e.g. [1, 2]). Since these models can be probed by future Large Hadron Collider experiments, a lot of research is being undertaken. Even apart from such brane-world models, many new physical results in higher dimensional general relativity are reported. Although we do not have space to list them all, we mention the discoveries of the black-hole solutions in five-dimensional space-time (e.g. [3]) that violate the traditional black-hole no-hair conjecture, the possibility of new stable configurations of black-string models (e.g. [4, 5]), and the modified version of cosmic hoop conjecture (e.g. [6]).

In order to investigate such topics, especially their dynamical and nonlinear behavior, numerical simulations are necessary. Numerical relativity is promising research field, but it is also true that we have not yet obtain the recipe to perform long-term stable and accurate dynamical evolution. Many trial simulations of binary compact objects have revealed that the mathematically equivalent sets of evolution equations are showing different numerical stability in the free evolution schemes. Current research target in numerical relativity is to find out better reformulation of the Einstein equation (see reviews, e.g. [7, 8, 9]).

In this Report, we study the dimensional dependence of constraint propagation in the standard Arnowitt-Deser-Misner (ADM) formulation of the Einstein equation (space-time decomposition) [10, 11]. Reader might think that starting with the ADM equation is old-fashioned since recent large-scale numerical simulations are not using the ADM equation due to its stability problem. However, we still think that ADM is the starting formulation for analyzing the dynamical behavior both analytically and numerically. The plenty of re-formulations of the Einstein equations have been proposed in a last decade. Most of them are starting from the ADM variables. The practical advantages of such re-formulations are extensively under investigation by many groups now, but, in our viewpoint, the essential improvements of them can be explained in a unified way via constraint propagation equations [8]. As we have shown in [12, 13], the stability problem of ADM can be controlled by adjusting constraints appropriately to evolution equations, and that key idea also works in other formulations [14, 15]. Therefore the analysis of the ADM equation is still essential.

The idea of constraint propagation (originally reported in [16, 17]) is a useful tool for calibrating the Einstein equations for numerical simulations. The modifications to the evolution equations change the property of the associated constraint propagation, and several particular adjustments to evolution equations using constraints are expected to diminish the constraint violating modes. We proposed to apply eigenvalue analysis to constraint propagation equations and to judge the property of the constraint violation. The proposed adjusted equations have been confirmed as showing better stability than before by numerical experiments (e.g. [18, 19]). The purpose of
this report is to show this idea is also applicable to all higher dimensional cases.

II. N + 1-DIMENSIONAL ADM EQUATIONS

We start from the \((N + 1)\)-dimensional Einstein equation,

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R + g_{\mu\nu} \Lambda = 8\pi T_{\mu\nu},
\]

and decompose it into \(N\)-dimensional space plus time, using the projection operator \(\perp^\mu\nu\),

\[
\gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu \equiv \perp^\mu_\nu,
\]

where \(n^\mu\) is a unit normal vector of the spacelike hypersurface \(\Sigma\), and we write the metric components,

\[
ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),
\]

where \(\gamma_{ij}\) expresses \(N\)-dimensional intrinsic metric, and \(\alpha\) and \(\beta^j\) the lapse and shift function, respectively. (Greek indices proceed \(0, 1, \cdots, N\), while Latin indices proceed \(1, \cdots, N\).)

The projections of the Einstein equation are the following:

\[
\begin{align*}
G_{\mu\nu} n^\mu n_\nu &= 8\pi T_{\mu\nu} n^\mu n_\nu = 8\pi \rho_H, \quad (2.4) \\
G_{\mu\nu} \perp^\mu_\nu &= 8\pi T_{\mu\nu} \perp^\mu_\nu = -8\pi \rho_P, \quad (2.5) \\
G_{\mu\nu} \perp^\mu_j \perp^\nu_k &= 8\pi T_{\mu\nu} \perp^\mu_j \perp^\nu_k = 8\pi S_{\rho\sigma}, \quad (2.6)
\end{align*}
\]

where we defined

\[
T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \quad (2.7)
\]

which gives \(T = -\rho_H + S_{\ell\ell}\).

To express the decomposition, we introduce the extrinsic curvature \(K_{ij}\) as

\[
K_{ij} \equiv -\perp^i_j \perp^\mu_\nu \nabla_\nu n_\mu \approx \frac{1}{2\alpha} (-\partial_\iota \gamma_{ij} + D_\iota \beta_i + D_i \beta_j), \quad (2.8)
\]

where \(\nabla\) and \(D_\iota\) is the covariant differentiation with respect to \(g_{\mu\nu}\) and \(\gamma_{ij}\), respectively.

The projection of the Einstein equation onto \(\Sigma\) is given using the Gauss equation,

\[
(\text{N}) R^i_{\ jkl} = R^\mu_{\ \nu \rho \sigma} \perp^i_j \perp^\mu_k \perp^\nu_l - K^i_k K_{jl} + K^i_j K_{kl}, \quad (2.9)
\]

and the Codazzi equation,

\[
D_j K_i^j - D_i K = -R_{\rho\sigma} n^\sigma \perp^\rho_i, \quad (2.10)
\]

where \(K = K^i_i\). For later convenience, we contract (2.9) to

\[
\begin{align*}
(\text{N}) R_{ij} &= R^\mu_{\ ijp} (\delta^\rho_j + n^\rho n^j) - K K_{ij} + K^\ell_j K_{i\ell}, \quad (2.11) \\
(\text{N}) R &= R + 2R_{\mu\nu} n^\mu n^\nu - K^2 + K_{ij} K_{ij}, \quad (2.12)
\end{align*}
\]

Eqs. (2.4) and (2.12) give the Hamiltonian constraint,

\[
\mathcal{C}_H \approx 0, \text{ where}
\]

\[
\mathcal{C}_H \equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu \perp^\nu = \frac{1}{2} (\text{N}) R + K^2 - K_{ij} K_{ij} - 8\pi \rho_H - \Lambda, \quad (2.13)
\]

while (2.5) and (2.10) give the momentum constraint,

\[
\mathcal{C}_{Mi} \approx 0, \text{ where}
\]

\[
\mathcal{C}_{Mi} \equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) \perp^\mu_\nu = D_{\ell} K^\ell_i - D_i K - 8\pi J_i. \quad (2.14)
\]

Both (2.13) and (2.14) have the same expression as those of the four-dimensional version.

The evolution equation for \(\gamma_{ij}\) is obtained from (2.8), which is again the same expression as the four-dimensional version.

The evolution equation of \(K_{ij}\) is obtained also from (2.5). The contraction of (2.1) gives

\[
\mathcal{R}_{ij} = 8\pi \left( S_{ij} - \frac{1}{N - 1} \gamma_{ij} T \right) - \frac{2}{1 - N} \gamma_{ij} \Lambda, \quad (2.15)
\]

where we used \(g_{\mu\nu} g^{\mu\nu} = N + 1\). A straightforward calculation of \(\mathcal{R}^\mu_{\ i\rho j} = \partial_\rho \Gamma^\mu_{\ i\rho j} - \partial_j \Gamma^\mu_{\ \rho i} + \Gamma^\mu_{\ \rho\sigma} \Gamma^\sigma_{\ ij} - \Gamma^\mu_{\ \sigma j} \Gamma^\sigma_{\ \rho i}\), where \(\Gamma^\rho_{\ \mu\nu}\) is the Christoffel symbol, gives

\[
\alpha \mathcal{R}^\mu_{\ i\rho j} n^\mu n^\rho = (\partial_i K_{ij}) + (D_i D_j \alpha) - \beta^k (D_k K_{ij}) - (D_j \beta^k) K_{ik} - (D_i \beta^k) K_{kj} + \alpha K_{kj} K^k_i. \quad (2.16)
\]

Substituting (2.15) and (2.16) into (2.11), we obtain

\[
\partial_t K_{ij} = \alpha^{(N)} R_{ij} + \alpha K K_{ij} - 2\alpha K^\ell_j K_{i\ell} - D_i D_j \alpha + \beta^k (D_k K_{ij}) + (D_j \beta^k) K_{ik} + (D_i \beta^k) K_{kj}
\]
The equation $\nabla^\nu S_{\mu\nu} = 0$ can be derived in many ways, and among them the derivation via the Bianchi identity [17] may be the easiest.

In general, we write a $N + 1$-dimensional symmetric tensor $S_{\mu\nu}$ which obeys the Bianchi identity, $\nabla^\nu S_{\mu\nu} = 0$. Let us express $S_{\mu\nu}$ by decomposing as

$$S_{\mu\nu} = X n_\mu n_\nu + Y_\mu n_\nu + Y_\nu n_\mu + Z_{\mu\nu}. \quad (3.1)$$

The normal and spatial projections of $\nabla^\nu S_{\mu\nu}$ become

$$n^\mu \nabla^\nu S_{\mu\nu} = -Z_{\mu\nu}(\nabla^\mu n^\nu) - \nabla^\nu Y_\mu + Y_\mu \nabla^\mu n^\nu - 2Y_\mu n_\nu(\nabla^\nu n^\mu) - X(\nabla^\mu n_\mu) - n_\mu(\nabla^\mu X), \quad (3.2)$$

$$h^\mu \nabla^\nu S_{\mu\nu} = \nabla^\nu Z_{\mu\nu} + Y_\nu(\nabla^\mu n_\mu) + Y_\mu(\nabla^\mu n_\nu) + X(\nabla^\mu n_\nu) n_\mu + n_\mu(\nabla^\mu Y_\nu), \quad (3.3)$$

where we used $\nabla$, while [17] uses different operator. For convenience, we rewrite them

$$\alpha n^\mu \nabla^\nu S_{\mu\nu} = -(\partial_i X) + \alpha K X + (\beta^i(\partial_j X) - \alpha \gamma^{ij}(\partial Y_j) + \alpha(\partial_i \gamma_{mn})(\gamma^{ml} \gamma^{nj} - (1/2)\gamma^{mn}\gamma^{ij})Y_j$$

$$-2\gamma^{im}(\partial_m \alpha)Y_i + \alpha K^i Z_{ij}, \quad (3.4)$$

$$-\alpha h^\mu \nabla^\nu S_{\mu\nu} = -(\partial_i Y_i) - (\partial_i \alpha)X + \alpha K Y_i + \beta^i(\partial_i Y_j) + \gamma^{km}(\partial_i \beta_m)Y_k - \beta^i(\partial_i \gamma_{pj})Y_k$$

$$-\alpha \gamma^{jk}(\partial_i Z_{ij}) - (\partial_i \alpha)Z_{ij} + (1/2)\alpha(\partial_i \gamma_{jk})Z_{kj} + \alpha \gamma^{mk}(\partial_i \gamma_{mk})Z_{ij} - (1/2)\alpha \gamma^{mk}(\partial_j \gamma_{mk})Z_{ij}.$$

respectively.

If we substitute $(S_{\mu\nu}, X, Y_i, Z_{ij}) = (T_{\mu\nu}, \rho_H, J_i, S_{ij})$ into (3.4) and (3.5) and assume $\nabla^\mu T_{\mu\nu} = 0$, then we obtain the matter evolution equations, $\partial_i \rho_H$ and $\partial_i J_i$. If we substitute $(S_{\mu\nu}, X, Y_i, Z_{ij}) = (G_{\mu\nu} - 8\pi T_{\mu\nu}, C_H, C_M, \kappa \gamma_{ij} C_H)$ with $\kappa$ = const. and assume $\nabla^\nu (G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0$ then we obtain the constraint propagation equations, $\partial_i C_H$ and $\partial_i C_M$. The parameter $\kappa$ corresponds to adding a term to (2.17), $+(\kappa - 1)C_H$.

This derivation does not depend on the dimension $N$ at all. Therefore the evolution equations both for the matter and constraints remain the same with those in the traditional four dimensional version.

The constraints include the extrinsic curvature terms, and the evolution equation of $K_{ij}$ changes due to $N$ as we saw in (2.17). Interestingly, however, such changes will be cancelled out and the resultant constraint propagation equations remain the same.

This means that a series of constraint propagation analyses can be directly applied to higher dimensional space-time. That is, the standard ADM evolution equations are likely to fail for long-term stable simulations. However, previously proposed adjustment techniques (e.g. [12, 13]) are also effective.

For example, constraint amplification factors (i.e. the eigenvalues of constraint propagation matrix) in Schwarzschild space-time [eq. (47) in [13]] are $(0, 0, \pm f(r))$ for four-dimensional standard ADM evolution equations, where $f(r)$ is a complex-valued function. In the five-dimensional Schwarzschild or black-string case, they become simply $(0, 0, 0, \pm f(r))$.

### IV. REMARKS

Motivated by the recent interests in higher dimensional space-time, we checked the constraint propagation equations based on the $N + 1$ ADM scheme. The evolution equation has matter terms which depend on $N$, but we show the constraint propagations remain the same as those in the four-dimensional ones. This indicates that there would be problems with accuracy and stability that we directly apply the $N + 1$ ADM formulation
to numerical simulations as we have experienced in four-
dimensional cases. However, we also conclude that pre-
vious efforts in re-formulating the Einstein equations can
be applied if they are based on constraint propagation
analysis. The generality holds for other systems when
their constraints are written in the form of (3.1).

Since we only used the Bianchi identity in the core
discussion, the assertion is also applicable to brane-
world models. In the context of the Randall-Sundrum
brane-world models [2], people study the modified four-
dimensional Einstein equations [20], which are derived
from five-dimensional Einstein equations with a thin-
shell (3-brane) approximation. The terms there addi-
tional to the standard ADM (see eq.(17) in [20]) in-
clude extrinsic curvature (due to shell-normal vector),
cosmological constant(s), and five-dimensional Weyl cur-
vature. These terms, however, can be interpreted as a
single stress-energy tensor which obeys the Bianchi iden-
tity. Therefore the properties of the constraint propa-
gation equations are the same as the above (from the
five-dimensional space-time viewpoint). Our proposals
for the adjustments [12, 13] are also valid in brane-world
models.

We hope this short report helps numerical relativists
for developing their future simulations.

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