Abstract

The five-dimensional supersymmetric SU(N) gauge theory is studied in the framework of the relativistic Toda chain. This equation can be embedded in two-dimensional Toda lattice hierarchy. This system has the conjugate structure. This conjugate structure corresponds to the charge conjugation.
1 Introduction

Recently there has been a tremendous progress in understanding the non-perturbative behavior of supersymmetric gauge theories in four dimensions.\textsuperscript{[1]} The exact low energy effective action was determined for a large class of $N = 1$ and $N = 2$ theories. The main tool for such determination is the use of holomorphy of various functions and the electric-magnetic duality. They constraint the moduli space.

The solution of the pure $N = 2$ super Yang Mills theory originally is obtained by Seiberg and Witten and and reinterpreted by Grosky et al. in terms of integrable systems. The correspondence between the Seiberg-Witten solution and the elliptic Whitham-KdV equation is pointed out.\textsuperscript{[2]} The observation was soon extended to the solution for the other classical gauge groups. Martinec and Warner noticed that these hyper elliptic curves are spectral curves of affine Toda chain systems.\textsuperscript{[3]} The prepotential $F$ is a Whitham type $\tau$ function.\textsuperscript{[4]}

Consider a five-dimensional supersymmetric gauge theory. Take the space-time manifold to be $M = X \times S_1$, where $X$ is a four manifold and $S_1$ is a circle. Nekrasov pointed out that the solution of five-dimensional theory can be obtained by the relativistic Toda chain (RTC) and determined the exact non-perturbative prepotential of the five-dimensional theory.\textsuperscript{[5]} Recently the relation between five-dimensional gauge theory with matters and Calabi-Yau geometry are studied.\textsuperscript{[6]}

Karchev et al. demonstrated that RTC is a special reduction of the two-dimensional Toda Lattice hierarchy.\textsuperscript{[7]} We extend this system to the periodic case.\textsuperscript{[8], [9]} Since the system has two Lax operators, we can obtain two curves. However the $\tau$-function is one. To describe the five-dimensional moduli space the two curves have to be unified.

This letter is organized as follows. In section 2 we discuss periodic RTC and obtain the two curves. This system has the conjugate structure. In section 3 we study it from the viewpoint of the Whitham-Toda hierarchy.\textsuperscript{[10]-[13]} We can see the inhomogeneity of the two times $t_k$ and $\bar{t}_k$. In section 4 we apply it to the five-dimensional gauge theory. The last section is devoted to the concluding remarks.

2 Lax representation for RTC

We consider the eigenfunctions $\Phi_n$ and $\Phi^*_n$. These eigen functions satisfy the following recurrent relations which give the Lax operators: \textsuperscript{[7]}

$$\Phi_{n+1} - \frac{S_n}{S_{n-1}} \Phi_n = z[\Phi_n - \frac{S_n}{S_{n-1}}(1 - S_nS_{n-1}^*)\Phi_{n-1}],$$  

$$\Phi^*_{n+1} - \frac{S_n^*}{S_{n-1}^*} \Phi^*_n = z^{-1}[\Phi^*_n - \frac{S_n^*}{S_{n-1}^*}(1 - S_nS_{n-1}^*)\Phi^*_{n-1}].$$ (2.1)
with
\[ S_n = S_{n+N}, \quad S_n^* = S_{n+N}^*. \] (2.2)
The superscript * does not mean the complex conjugation. We define the next function:
\[ \frac{h_{n+1}}{h_n} = 1 - S_n S_n^*. \] (2.3)
In terms of the canonically conjugate variables for the RTC \((q_n, p_n, p_n^*)\), we can immediately read off
\[ \frac{S_n}{S_{n-1}} = - \exp(\epsilon p_n), \quad \frac{S_n^*}{S_{n-1}^*} = - \exp(\epsilon p_n^*), \] (2.4)
\[ \frac{h_n}{h_{n-1}} = \epsilon^2 \exp(q_n - q_{n-1}), \] (2.5)
with the periodic condition
\[ q_{n+N} = q_n, \quad p_{n+N} = p_n, \quad p_{n+N}^* = p_n^*. \] (2.6)
Adding this the canonical coordinates satisfy the constraint
\[ \sum_i q_i = \sum_i p_i = \sum_i p_i^* = 0. \] (2.7)
We next consider the equations which describe the time dependence of \(\Phi_n\) and \(\Phi_n^*\). For example \(t_1\) and \(\bar{t}_1\) give the following evolution equations:
\[ \frac{\partial \Phi_n(z)}{\partial t_1} = - \frac{S_n}{S_{n-1}^*} \frac{h_n}{h_{n-1}} (\Phi_n(z) - z \Phi_{n-1}), \] (2.8a)
\[ \frac{\partial \Phi_n(z)}{\partial \bar{t}_1} = \frac{h_n}{h_{n-1}} \Phi_{n-1}(z), \] (2.8b)
\[ \frac{\partial \Phi_n^*(z^{-1})}{\partial t_1} = \frac{h_n}{h_{n-1}^*} \Phi_{n-1}^*(z^{-1}), \] (2.8c)
\[ \frac{\partial \Phi_n^*(z^{-1})}{\partial \bar{t}_1} = - \frac{S_n^*}{S_{n-1}^*} \frac{h_n}{h_{n-1}^*} (\Phi_n^*(z^{-1}) - z^{-1} \Phi_{n-1}^*). \] (2.8d)
The compatibility condition gives the following nonlinear evolution equations:
\[ \frac{\partial S_n}{\partial t_1} = S_{n+1} \frac{h_{n+1}}{h_n}, \quad \frac{\partial S_n}{\partial \bar{t}_1} = - S_{n-1} \frac{h_{n+1}}{h_n}, \] (2.9a)
\[ \frac{\partial S_n^*}{\partial t_1} = - S_{n-1}^* \frac{h_{n+1}}{h_n}, \quad \frac{\partial S_n^*}{\partial \bar{t}_1} = S_{n+1}^* \frac{h_{n+1}}{h_n}, \] (2.9b)
\[ \frac{\partial h_n}{\partial t_1} = S_n S_{n-1} h_n, \quad \frac{\partial h_n}{\partial \bar{t}_1} = S_n^* S_{n-1} h_n. \] (2.9c)
(2.9c) is exactly RTC written in somewhat different form. We can obtain two RTC’s for the two times $t_1$ and $\bar{t}_1$.

Here we define $a_n$, $b_n$ and $b_n^*$:

$$a_n \equiv 1 - S_n S_n^* = \frac{h_n + 1}{h_n},$$  \hspace{1cm} (2.10a)

$$b_n \equiv S_n S_{n-1}^*,$$  \hspace{1cm} (2.10b)

$$b_n^* \equiv S_n^* S_{n-1}.$$  \hspace{1cm} (2.10c)

Notice that from the definitions $a_n$, $b_n$ and $b_n^*$ satisfy the periodic condition:

$$a_n = a_{n+N}, \quad b_n = b_{n+N}, \quad b_n^* = b_{n+N}^*.$$  \hspace{1cm} (2.11)

In terms of $a_n$, $b_n$ and $b_n^*$, (2.9a) and (2.9b) become the two-dimensional Toda equations:

$$\frac{\partial a_n}{\partial t_1} = a_n (b_{n+1} - b_n), \quad \frac{\partial b_n}{\partial t_1} = a_n - a_{n-1},$$  \hspace{1cm} (2.12a)

and

$$\frac{\partial a_n}{\partial \bar{t}_1} = a_n (b_{n+1}^* - b_n^*), \quad \frac{\partial b_n^*}{\partial \bar{t}_1} = a_n - a_{n-1}.$$  \hspace{1cm} (2.12b)

This system has conjugate structures:

$$a_n \rightarrow a_n, \quad b_n \rightarrow b_n^*, \quad b_n^* \rightarrow b_n, \quad t \rightarrow \bar{t}, \quad \bar{t} \rightarrow t.$$  \hspace{1cm} (2.13)

The two-dimensional Toda system with the conjugate structure can be seen is several physical models: the full unitary matrix models \[14\] and $tt$ fusion of the topological sigma models \[15\], \[16\].

We will assume that the eigenfunctions $\Phi_n$ and $\Phi_n^*$ will obey the following quasi-periodic conditions:

$$\Phi_{n+N} = \sqrt{z^N / \prod_{i=1}^{N} c_i \Phi_n}, \quad \Phi_{n+N}^* = \sqrt{z^{-N} / \prod_{i=1}^{N} c_i^* \Phi_n^*},$$  \hspace{1cm} (2.14)

where

$$c_n = -\frac{S_n}{S_{n-1}} (1 - S_n S_n^*), \quad c_n^* = -\frac{S_n^*}{S_{n-1}} (1 - S_n S_n^*).$$  \hspace{1cm} (2.15)

If we set

$$c_n = -f_n^2, \quad c_n^* = -(f_n^*)^2, \quad z = k^2,$$

$$\Phi_n = k^n \alpha_n \varphi_n, \quad \Phi_n^* = k^n \alpha_n^* \varphi_n^*, \quad \alpha_{n-1} = \alpha_n f_n, \quad \alpha_{n-1}^* = \alpha_n^* f_n^*,$$  \hspace{1cm} (2.16)
the new eigenfunctions $\varphi_n$ and $\varphi^*_n$ will fulfill periodic boundary condition,

$$\varphi_{n+N} = \varphi_n, \quad \varphi^*_{n+N} = \varphi^*_n.$$  \hfill (2.17)

The linear problems are transformed into

$$(d_n - k^2)\varphi_n + k(f_n f_{n+1} + f_n f_{n-1}) = 0,$$

$$(d^*_n - k^{-2})\varphi^*_n + k(f_n^* f^-{n+1} + f_n^* f^-{n-1}) = 0,$$  \hfill (2.18)

where

$$d_n = -\frac{S_n}{S_{n-1}}, \quad d^*_n = -\frac{S^*_n}{S^*_{n-1}}.$$  \hfill (2.19)

We have the following $N \times N$ matrix representation of (2.18)

$$L(k, \lambda)\varphi = 0, \quad L^*(k, \lambda^*)\varphi^* = 0,$$  \hfill (2.20)

where

$$L = \begin{pmatrix}
    d_1 - k^2 & k f_1 & 0 & k f_N \lambda \\
    k f_1 & d_2 - k^2 & f_2 & 0 \\
    & \cdots & \cdots & \cdots \\
    k f_N / \lambda & 0 & \cdots & k f_{N-1} \\
\end{pmatrix},$$  \hfill (2.21a)

$$L^* = \begin{pmatrix}
    d^*_1 - k^{-2} & k^{-1} f^*_1 & 0 & k^{-1} f^*_N \lambda^* \\
    k^{-1} f^*_1 & d^*_2 - k^{-2} & f^*_2 & 0 \\
    & \cdots & \cdots & \cdots \\
    k^{-1} f^*_N / \lambda^* & 0 & \cdots & k^{-1} f^*_{N-1} \\
\end{pmatrix}.$$  \hfill (2.21b)

Here $\lambda$ and $\lambda^*$ are the spectral parameters. Notice that $\det L(k, \lambda)$ and $\det L^*(k, \lambda^*)$ are an integral of motion for the two flows $t_1$ and $\bar{t}_1$. The polynomial $\det L(k, \lambda)$ and $\det L^*(k, \lambda^*)$, thought being of degree $2N$ in $k$, have only $N$ functionally independent coefficients respectively. For $N$ even, it is an even function of $k$, while for odd, the only odd surviving power of $k$ is the $N$th power. Using the (2.1) we can obtain the two curves $C$ and $C^*$:

$$\epsilon^{2N} k^{-N} \left( \lambda + \frac{1}{\lambda} \right) = k^{2N} + I_{N-1} k^{2(N-1)} + \cdots + I_1 k^2 \pm 1,$$  \hfill (2.22a)

$$\epsilon^{2N} k^{-N} \left( \lambda^* + \frac{1}{\lambda^*} \right) = k^{-2N} + I^*_{N-1} k^{-2(N-1)} + \cdots + I^*_1 k^{-2} \pm 1.$$  \hfill (2.22b)
For $N$ even, the sign of the last term is $+$, while for odd, it is $-$. If we set $k = e^x$, then we can obtain
\begin{equation}
2^{-N} e^{2N} (\lambda + \frac{1}{\lambda}) = \prod_i [\sinh(x - \hat{\alpha}_i)],
\end{equation}
\begin{equation}
2^{-N} e^{2N} (\lambda^* + \frac{1}{\lambda^*}) = \prod_i [\sinh(-x - \hat{\alpha}_i^*)],
\end{equation}
where
\begin{equation}
\sum_i \hat{\alpha}_i = 0, \quad \sum_i \hat{\alpha}_i^* = 0.
\end{equation}

The two curves have $Z_N$ and $Z_2$ symmetry respectively.
\begin{equation}
\hat{\alpha}_i \rightarrow \hat{\alpha}_i + \frac{2\pi i k}{N}, \quad \hat{\alpha}_i^* \rightarrow \hat{\alpha}_i^* + \frac{2\pi i l}{N}, \quad I_n \rightarrow e^{-2\pi i k (N-n)/N} I_n, \quad I_n^* \rightarrow e^{-2\pi i l (N-n)/N} I_n^*,
\end{equation}
and
\begin{equation}
\lambda \rightarrow -\lambda, \quad \lambda^* \rightarrow -\lambda^*, \quad f_n \rightarrow -f_n, \quad f_n^* \rightarrow -f_n^*.
\end{equation}

### 3 The Whitham-Toda equations

In this section we consider the two-dimensional Toda lattice system (2.12a) and (2.12b) without the periodic condition. This system can be embedded in the two-dimensional Toda hierarchy. The matrix version of the spectral problem is
\begin{equation}
\mathcal{L}_{nk} \phi_k(z) = z \phi_n(z), \quad \mathcal{L}_{nk}^* \phi_k^*(z) = z^{-1} \phi_n^*(z),
\end{equation}
\begin{equation}
\frac{\partial \phi_n(z)}{\partial t_m} = -[(\mathcal{L}^m)_n]_{nk} \phi_k(z), \quad \frac{\partial \phi_n(z)}{\partial \bar{t}_m} = [\mathcal{L}^m]_{nk} \phi_k(z),
\end{equation}
\begin{equation}
\frac{\partial \phi_n^*(z)}{\partial t_m} = [\mathcal{L}^m]_{nk}^* \phi_k^*(z), \quad \frac{\partial \phi_n^*(z)}{\partial \bar{t}_m} = -[\mathcal{L}^m]_{nk}^* \phi_k^*(z), \quad m = 1, 2, \cdots
\end{equation}
where $(A)_+$ is the upper triangular part of the matrix $A$ (including the main diagonal) while $(A)_-$ is strictly the lower triangular part. $\mathcal{L}$ and $\mathcal{L}^*$ are the Lax operators and $\phi_n(z)$ and $\phi_n^*(z)$ are the Baker-Akhiezer (BA) functions.

The two-dimensional Toda lattice system (2.12a) and (2.12b) has infinite number of conserved quantities. The conserved density is obtained from the Lax operators
\begin{equation}
D_1^T = b_n, \quad D_1^* = b_n^*,
\end{equation}
\begin{equation}
D_2^T = \frac{1}{2} b_n^2 - \frac{a_n}{1-a_n} b_n b_{n-1}, \quad D_2^* = \frac{1}{2} b_n^{*2} - \frac{a_n}{1-a_n} b_n^* b_{n-1}^*.
\end{equation}

The conservation laws are given by
\begin{equation}
\frac{\partial D_i}{\partial t_k} = (\Delta - 1) F_i, \quad \frac{\partial D_i}{\partial \bar{t}_k} = (\Delta - 1) \bar{F}_i,
\end{equation}
\begin{equation}
(2.25)
\end{equation}
\[
\frac{\partial D^*_i}{\partial \bar{t}_k} = (\Delta - 1) F^*_i, \quad \frac{\partial D^*_i}{\partial \bar{t}_k} = (\Delta - 1) \tilde{F}^*_i, \tag{3.4b}
\]

where \( \Delta \) is the unit shift operator, i.e., \( \Delta f_n = f_{n+1} \), and \( F_i \) and \( \tilde{F}_i \) are the flows for the time \( t_1 \) and \( \bar{t}_1 \). For example we can obtain

\[
F_1 = F^*_1 = a_n, \quad \tilde{F}_1 = \frac{a_n}{a_n - 1} b_n b_{n-1}, \quad F^*_1 = \frac{a_n}{a_n - 1} b_n b_{n-1},
\]

\[
F_2 = a_n b_n, \quad F^*_2 = a_n b_{n-1}, \quad \tilde{F}_2 = \frac{a_n}{a_n - 1} b_n b_{n-1} b_{n-2} - \frac{a_n}{1 - a_n} b_n b_{n-1}, \quad \tilde{F}^*_2 = \frac{a_n}{a_n - 1} b_n b_{n-1} b_{n-2} - \frac{a_n}{1 - a_n} b_n b_{n-1}. \tag{3.5}
\]

The \( t_1 \)-flow and \( \bar{t}_1 \)-flow of the conserved density are different. There is the inhomogeneity between \( t_k \) and \( \bar{t}_k \).

Here we introduce the 1-form \( dS \) and \( dS^* \) which are meromorphic on \( \mathcal{C} \) and \( \mathcal{C}^* \).

\[
dS \cong \log z \frac{d\lambda}{\lambda}, \quad dS^* \cong \log z \frac{d\lambda^*}{\lambda^*}. \tag{3.6}
\]

To obtain the Whitham equations, we introduce two time scales and then average the flux and density over the fast variables in the conservation laws. We break up the dynamics into slow and fast scales. Let \( n, t_k \) and \( \bar{t}_k \) denote the fast spatial and time variables and let \( X = \varepsilon x, T_k = \varepsilon t_k \) and \( \bar{T}_k = \varepsilon \bar{t}_k \).

\[
\frac{\partial}{\partial T_k} dS = d\Omega_k, \quad \frac{\partial}{\partial T_k} dS^* = d\tilde{\Omega}_k, \quad \frac{\partial}{\partial X} dS = d\Omega_0, \quad \frac{\partial}{\partial X} dS^* = d\tilde{\Omega}_0, \tag{3.7a}
\]

and

\[
\frac{\partial}{\partial \bar{T}_k} dS = d\Omega_k, \quad \frac{\partial}{\partial \bar{T}_k} dS^* = d\tilde{\Omega}_k, \quad \frac{\partial}{\partial X} dS = d\Omega_0, \quad \frac{\partial}{\partial X} dS^* = d\tilde{\Omega}_0. \tag{3.7b}
\]

\( d\Omega_0, d\Omega_k, d\Omega_0^* \) and \( d\Omega_k^* \) are the normalized Abelian differentials. These are normalized by the conditions

\[
\oint_{\alpha_i} d\Omega_0 = \oint_{\alpha_i} d\Omega_k = \oint_{\alpha_i} d\Omega_0^* = \oint_{\alpha_i} d\Omega_k^* = 0, \tag{3.8}
\]

where \( \alpha_i \) and \( \alpha_i^* \) are the standard symplectic basis of homology cycles of the curve (2.23a) and (2.23b).

We can write the Whitham equations in the terms of the Aberian differentials for the general \( T_k \) and \( \bar{T}_k \) as follows

\[
\frac{\partial}{\partial T_k} d\Omega_0 = \frac{\partial}{\partial X} d\Omega_k, \quad \frac{\partial}{\partial T_k} d\Omega_0 = \frac{\partial}{\partial X} d\tilde{\Omega}_k. \tag{3.9a}
\]
and
\[ \frac{\partial}{\partial T_k} d\Omega_0^* = \frac{\partial}{\partial X} d\Omega_k^*, \quad \frac{\partial}{\partial \bar{T}_k} d\bar{\Omega}_0^* = \frac{\partial}{\partial X} d\bar{\Omega}_k^*. \] (3.9b)

This equation is an average form of the conservation law.

## 4 Five-Dimensional Gauge Theory

The prepotential \( \mathcal{F} \) is identified with logarithm of the \( \tau \)-function of the Whitham hierarchy. The system which we considered in the section 2 and 3 has one \( \tau \)-function but has the two curves. To describe the five-dimensional moduli space the two curves have to be unified.

The Coulomb branch of the moduli space is given by \( \Psi = \text{diag}(\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_N) \) with \( \sum \hat{\alpha}_i = 0 \), modulo the Weyl group action, which permutes the \( \hat{\alpha}_i \). It can thus be taken to be the Weyl chamber \( \hat{\alpha}_1 \geq \hat{\alpha}_2 \geq \cdots \geq \hat{\alpha}_N \). To unify \( \text{calC} (2.23a) \) and \( \text{calC}^* (2.23b) \) we set as follows:
\[ \lambda^* = \pm \lambda, \quad \hat{\alpha}_i^* = -\hat{\alpha}_{N-i+1}, \quad I^*_n = I_{N-n+1}. \] (4.1)

For \( N \) even, the sign is +, while for \( N \) odd, it is −.

The effective prepotential on the Coulomb branch has appeared in \[5\] and \[6\]. The prepotential is invariant under the \( \mathbb{Z}_2 \) transformation: \( \hat{\alpha} \rightarrow -\hat{\alpha} \). The gauge group \( SU(N) \) is broken by the Higgs mechanism. However the Weyl group should still be unbroken, so this \( \mathbb{Z}_2 \) symmetry resides. In the physical meaning (4.1) is the charge conjugation. \[6\]

From this conjugation we can obtain next relations:
\[ dS = dS^*, \quad d\Omega_0 = d\Omega_0^*, \quad d\Omega_k = d\bar{\Omega}_k^* \] (4.2)

This result means the homogeneity between \( T_k \) and \( \bar{T}_k \). But the conserved quantity is exchanged as (4.1). For the \( SU(2) \) case the conjugate motions are the same, as \( I_2 = I_2^* \). Then the set of the holomorphic differentials become as follows:
\[ d\omega_n = d\omega_{N-n-1}^*, \] (4.3)

where
\[ d\omega_k = \frac{\partial dS}{\partial I_k}, \quad d\omega_k^* = \frac{\partial dS^*}{\partial I_k^*}. \] (4.4)

## 5 Concluding Remarks

We study the five-dimensional supersymmetric \( SU(N) \) gauge theory and the relativistic Toda (RTC) equations. RTC is described as a particular reduction of the two-dimensional Toda lattice hierarchy. This system has two curves. To describe the moduli space of the
five-dimensional gauge theory, the two curves have to be unified. Then the two times $T_k$ and $\bar{T}_k$ are homogeneous and the conjugate structure becomes the charge conjugation.

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