Adrian Vasiu

Surjectivity Criteria for p-adic Representations, Part I

Received: 10.09.2002 Revised version: 27.01.2003 Manuscripta Mathematica, MM-No:577
Published online: 15.10.2003

ABSTRACT. We prove general surjectivity criteria for $p$-adic representations. In particular, we classify all adjoint and simply connected group schemes $G$ over the Witt ring $W(k)$ of a finite field $k$ such that the reduction epimorphism $G(W_2(k)) \to G(k)$ has a section.

§1. Introduction

Let $p$ be a rational prime. Let $d, r \in \mathbb{N}$. Let $q := p^r$. Let $A$ be an abelian variety of dimension $d$ over a number field $E$. The action of $\text{Gal}(\overline{E}/E)$ on the $\overline{E}$-valued points of $A$ of $p$-power order defines a $p$-adic representation

$$\rho_{A,p} : \text{Gal}(\overline{E}/E) \to \text{GL}(T_p(A))(\mathbb{Z}_p) = \text{GL}_{2d}(\mathbb{Z}_p),$$

where $T_p(A)$ is the Tate module of $A$. If $A$ is a semistable elliptic curve over $\mathbb{Q}$ (so $d = 1$ and $E = \mathbb{Q}$), then Serre proved that for $p \geq 3$ the homomorphism $\rho_{A,p}$ is surjective iff its reduction mod $p$ is irreducible (cf. [23, Prop. 21], [24, p. IV-23-24] and [26, p. 519]).

Similar results are expected to hold for many homomorphisms of the form

$$\rho : \text{Gal}(E) \to G(W(k)),$$

with $G$ a reductive group scheme over the Witt ring $W(k)$ of $k := \mathbb{F}_q$ (like when the Zariski closure $G_{A,p}$ of $\text{Im}(\rho_{A,p})$ in $\text{GL}(T_p(A))$ is a reductive group scheme and $\rho$ is the factorization of $\rho_{A,p}$ through $G_{A,p}(\mathbb{Z}_p)$). Cases involving $q = p$ and groups $G$ which are extensions of $\mathbb{G}_m$ by products of Weil restrictions of $\text{SL}_2$ groups were treated for $p \geq 5$ in [22]. If $f : G \hookrightarrow \text{GL}_n$ is a monomorphism of interest, then in general the assumption that the reduction mod $p$ of $f(W(k)) \circ \rho$ is an irreducible representation over $k$, is too weak to produce significant results. So the goal of this paper is to get criteria under which $\rho$ is surjective if its reduction mod $p$ (or occasionally mod 4 for $p = 2$) is surjective.

Adrian Vasiu, Mathematics Department, University of Arizona, 617 N. Santa Rita, P.O. Box 210089, Tucson, AZ-85721, USA. e-mail: adrian@math.arizona.edu

Mathematics Subject Classification (2000): Primary 11S23, 14L17, 17B45, 20G05 and 20G40
A reductive group scheme $F$ over a connected affine scheme Spec$(R)$ is assumed to have connected fibres. Let $F^\text{der}$, $Z(F)$, $F^\text{ab}$ and $F^\text{ad}$ be the derived group, the center, the maximal commutative quotient and respectively the adjoint group of $F$. So $F^\text{ad} = F/Z(F)$ and $F/F^\text{der} = F^\text{ab}$. Let $Z^0(F)$ be the maximal torus of $Z(F)$. Let $F^\text{sc}$ be the simply connected semisimple group cover of $F^\text{der}$. Let $c(F^\text{der})$ be the degree of the central isogeny $F^\text{sc} \to F^\text{der}$. If $S$ is a closed subgroup of $F$, then Lie$(S)$ is the $R$-Lie algebra of $S$. We now review the contents of this paper.

1.1. Adjoint representations. If the reduction mod $p$ of $\rho$ is surjective, then the study of the surjectivity of $\rho$ is intimately interrelated with the study of the adjoint representation $AD_{G_k}$ of $G(k)$ on Lie$(G_k)$. Though not stated explicitly, this principle is present in disguise in [24, p. IV-23-24] and [21, 2.1]. In order to apply it in §4, in §3 we assume $G$ is semisimple, we work just with $G_k$ (without mentioning $G$), and we deal with the classification of subrepresentations of $AD_{G_k}$. If $G_k$ is split, simply connected and $G_k^{\text{ad}}$ is absolutely simple, then such a classification was obtained in [13]. The main goal of §3 is to extend loc. cit. to the more general context of Weil restrictions of scalars (see [2, 7.6]) of semisimple groups having absolutely simple adjoints (see 3.4, 3.10 and 3.11). The methods we use are similar to the ones of [13] except that we rely more on the work of Curtis and Steinberg on representations over $\bar{k}$ of finite groups of Lie type and on the work of Humphreys and Hogeweij on ideals of Lie$(G_k^{\text{der}})$ (see [17] and [14]; see also [18, 0.13] and [20, §1]). Whenever possible we rely also on [13]. Basic properties of Lie algebras and Weil restrictions are recalled in 2.2 and 2.3.

1.2. The problem. For $s \in \mathbb{N}$ let $W_s(k) := W(k)/p^sW(k)$. Let Lie$_{\mathbb{F}_p}(G_k)$ be Lie$(G_k)$ but viewed just as an abelian group identified with Ker$(G(W_2(k)) \to G(k))$. So if $H$ is a normal subgroup of $G_k$, then we also view Lie$_{\mathbb{F}_p}(H)$ as a subgroup of $G(W_2(k))$. Let $K$ be a closed subgroup of $G(W(k))$ surjecting onto $(G/Z^0(G))(k)$ (we think of it as the image of some $\rho$). Let $K_2 := \text{Im}(K \to G(W_2(k)))$. The problem we deal with is to find conditions which imply that $K$ surjects onto $(G/Z^0(G))(W(k))$. It splits into two cases: $G$ is or is not semisimple. In this Part I we deal with the first case and in Part II we will deal with the second case and with applications of it to abelian varieties. The group $G$ is semisimple iff the torus $Z^0(G)$ is trivial. If $G$ is semisimple and $p|c(G)$, then there are proper, closed subgroups of $G(W(k))$ surjecting onto $G(k)$ (cf. 4.1.1). On the other hand, we also have the following general result:

1.3. Main Theorem. We assume that $G$ is semisimple, that g.c.d.$(p, c(G)) = 1$ and that $K$ surjects onto $G(k)$. We also assume that one of the following five conditions holds:

a) $q \geq 5$;

b) $q = 3$ and for each normal subgroup $H$ of $G_k$ which is a PGL$_2$ or an SL$_2$ group we have Lie$_{\mathbb{F}_3}(H) \cap K_2 \neq \{0\}$;

c) $q = 4$ and $G_k^{\text{ad}}$ has no simple factor which is a PGL$_2$ group;

d) $q = 2$ and the following two additional conditions hold:

- no simple factor of $G_k^{\text{ad}}$ is a PGL$_2$ group, a Weil restriction from $\mathbb{F}_4$ to $k$ of a PGL$_2$ group, or an SU$_4^{\text{ad}}$ group;
for each normal subgroup \( H \) of \( G_k \) whose adjoint is a \( \mathrm{PGL}_3 \) group, a \( \mathrm{SU}^\mathrm{ad}_3 \) group or split of \( G_2 \) Lie type, we have \( \mathrm{Lie}_{\mathbb{F}_p}(H) \cap K_2 \neq \{0\} \);

e) \( p = 2 \) and \( K_2 = G(W_2(k)) \).

Then we have \( K = G(W(k)) \).

The case \( G = \mathrm{SL}_2 \) for \( p \geq 3 \) is due to Lenstra (unpublished computations with \( 3 \times 3 \) matrices). Serre proved the case \( p \geq 5 \) for \( SL_n \) and \( \mathrm{Sp}_{2n} \) groups over \( \mathbb{Z}_p \) and the mod 8 variant of e) for \( SL_n \) groups over \( \mathbb{Z}_2 \) (see [24, IV] and [26, p. 52]). Most of the extra assumptions of b) to d) were known to be needed before (for instance, cf. [12, Sect. 4] in connection to d) for the \( G_2 \) Lie type). Though the case \( q >> 0 \) of 1.3 is considered well known, we do not know any other concrete literature pertaining to 1.3.

**1.4. On the proof of 1.3.** The proof of 1.3 is presented in 4.7. Its main ingredients are 4.3 to 4.5 and most of §3. In 4.5 we list all isomorphism classes of adjoint groups \( G = G^\mathrm{ad} \) for which the short exact sequence \( 0 \to \mathrm{Lie}_{\mathbb{F}_p}(G_k) \to G(W_2(k)) \to G(k) \to 0 \) of abstract groups has a section, i.e. the epimorphism \( G(W_2(k)) \to G(k) \) has a right inverse. They are the eight ones showing up concretely in 1.3 b) to d). Though most of them are well known, we were not able to trace a reference pertaining to the complete classification of 4.5. The main idea of 4.5 is the following approach of inductive nature (see 4.4). Let \( \gamma_G \in H^2(G(k), \mathrm{Lie}_{\mathbb{F}_p}(G_k)) \) be the class defining the mentioned short exact sequence, with \( \mathrm{Lie}_{\mathbb{F}_p}(G_k) \) viewed as a left \( G(k) \)-module via \( AD_{G_k} \). If \( \gamma_G = 0 \), then all images of restrictions of \( \gamma_G \) are 0 classes and so \( \gamma_{G^0} = 0 \) for any semisimple subgroup \( G_0 \) of \( G \) normalized by a maximal torus of \( G \) (see 4.3.4). But if there is a simple factor of \( G \) whose isomorphism class is not in the list, then we can choose \( G_0 \) such that the direct computations of 4.4 show that \( \gamma_{G^0} \neq 0 \); so \( \gamma_G \neq 0 \).

We now detail how 4.5 and §3 get combined to prove 1.3 for \( p \geq 5 \). Serre’s method of [24, IV] can be adapted to get that for \( p \geq 5 \) it is enough to show that \( K_2 = G(W_2(k)) \) (see 4.1.2). Based on 4.5 we know that \( \mathrm{Lie}_{\mathbb{F}_p}(G_k) \cap K_2 \) is not included in \( \mathrm{Lie}_{\mathbb{F}_p}(Z(G_k)) \). So \( \mathrm{Lie}_{\mathbb{F}_p}(G_k) \cap K_2 = \mathrm{Lie}_{\mathbb{F}_p}(G_k) \), cf. 3.7.1 and 3.10.2. So \( K_2 = G(W_2(k)) \).

Though §3 and §4 handle also the cases \( p = 2 \) and \( p = 3 \), §4 does not bring anything new to Serre’s result recalled before 1.1; however, one can adapt our results to elliptic curves or to [22] in order to get meaningful results in mixed characteristics (0, 2) and (0, 3) (for instance, cf. 1.3 c) to e)). This Part I originated from seminar talks in Berkeley of Ribet and Lenstra; the first draft of §3 was a letter to Lenstra.

**§2. Preliminaries**

In 2.1 we list our notations and conventions. In 2.2 and 2.3 we recall simple properties of Lie algebras and respectively of Weil restrictions of scalars.

**2.1. Notations and conventions.** Always \( n \in \mathbb{N} \). We denote by \( k_1 \) a finite field extension of \( k = \mathbb{F}_q \). We abbreviate absolutely simple as a.s. and simply connected as s.c. Let \( R, F \) and \( S \) be as in §1. We say \( F^\mathrm{ad} \) is simple (resp. is a.s.) if (resp. if each geometric fibre of) it has no proper, normal subgroup of positive relative dimension. If
$M$ is a free $R$-module of finite rank, then $GL(M)$ ($SL(M)$, etc.) are viewed as reductive group schemes over $R$. So $GL(M)(R)$ is the group of $R$-linear automorphisms of $M$.

Let $F$ be semisimple and Spec$(R)$ connected. Let $o(F)$ be the order of $Z(F)$ as a finite, flat, group scheme. So $o(SL_2) = 2$. We have $c(F) = o(F^{\text{sc}})/o(F)$. See item (VIII) of [5, pl. I to IX] for $o(F)$’s of s.c. semisimple groups having a.s. adjoints. If $X_*$ (resp. $X_{R_1}$ or $X$) is a scheme over Spec$(R_1)$, then $X_{R_1}$ (resp. $X_R$) is its pull back to Spec$(R)$. All modules are left modules. A representation of a Lie algebra or a finite group $X$ is an arbitrary Spec$(R/R)$-scheme. Based on 2.2 and (1) we get a canonical and functorial identification $Y$ where the $\text{Res}_{R/R}$ is of (or has) isotypic $\text{DT}$ of $\text{Spec}(R)$ but viewed as an affine group scheme over $\text{Spec}(k)$.

Let Lie$(S)$ be the connected Dynkin diagram of any simple factor of every geometric fibre of $F$ if $F$ is DT; if $F$ has a.s. isotypic $\text{Spec}(R)$ is an arbitrary Spec$(R)$-scheme. Based on 2.2 and (1) we get a canonical and functorial identification $Y$.

2.2. Lie algebras. Let $x$ be an independent variable. As an $R$-module, we identify Lie$(S)$ with the tangent space of $S$ at the identity section, i.e. with $\text{Ker}(S(R[x]/x^2) \rightarrow S(R))$, where the $R$-epimorphism $R[x]/(x^2) \twoheadrightarrow R$ takes $x$ into 0. If $y, z \in \text{Lie}(S)$, then the Lie bracket $[y, z]$ is $zyz^{-1}z^{-1}$, the product being taken inside $\text{Ker}(S(R[x]/x^2) \rightarrow S(R))$. We now assume that $S$ is a smooth group scheme over $R$. So Lie$(S)$ is a free $R$-module of rank equal to the relative dimension of $S$. The representation of $S$ on $\text{GL}(\text{Lie}(S))$ defined by inner conjugation is called the adjoint representation. Let

$$AD_S : S(R) \rightarrow \text{GL}((\text{Lie}_R(S)))(R)$$

be the adjoint representation evaluated at $R$.

2.3. Weil restrictions of scalars. Let $i_1 : R_1 \hookrightarrow R$ be a finite, flat $Z$-monomorphism. Let $\text{Lie}_{R_1}(S)$ be Lie$(S)$ but viewed as an $R_1$-Lie algebra. If $R_1 = \mathbb{F}_p$, then we often view $\text{Lie}_{\mathbb{F}_p}(S)$ just as an abelian group; if also $R = \mathbb{F}_p$, then we often drop the lower right index $\mathbb{F}_p$. Let $\text{Res}_{R/R_1}S$ be the affine group scheme over $R_1$ obtained from $S$ through the Weil restriction of scalars (see [2, 7.6] and [7, 1.5]). So $\text{Res}_{R/R_1}S$ is defined by the functorial group identification

$$\text{Hom}_{\text{Spec}(R)}(Y, \text{Res}_{R/R_1}S) = \text{Hom}_{\text{Spec}(R)}(Y \times_{\text{Spec}(R_1)} \text{Spec}(R), S),$$

where $Y$ is an arbitrary Spec$(R_1)$-scheme. Based on 2.2 and (1) we get a canonical and functorial identification

$$\text{Lie}(\text{Res}_{R/R_1}S) = \text{Lie}_{R_1}(S).$$
If \( i_2 : R_2 \hookrightarrow R_1 \) is a second finite, flat \( \mathbb{Z} \)-monomorphism, then we have a canonical and functorial identification

\[
\text{Res}_{R_1/R_2} \text{Res}_{R/R_1} S = \text{Res}_{R/R_2} S.
\]

**2.3.1. Proposition.** We assume there is a finite subgroup \( C \) of \( \text{Aut}_{R_1}(R) \) such that we have an \( R \)-isomorphism \( R \otimes_{R_1} R = \prod_{c \in C} R \otimes_{R} cR \). Then we have

\[
(\text{Res}_{R/R_1} S)_R = \prod_{c \in C} S \times_{\text{Spec}(R)} \text{Spec}(R).
\]

If \( S_1 \) is a closed (resp. smooth, flat, semisimple or reductive) subgroup of \( S \), then \( \text{Res}_{R/R_1} S_1 \) is a closed (resp. smooth, flat, semisimple or reductive) subgroup of \( \text{Res}_{R/R_1} S \). If \( S \) is a reductive group scheme, then any maximal torus (resp. Borel subgroup) \( J \) of \( \text{Res}_{R/R_1} S \) is of the form \( \text{Res}_{R/R_1} T \) (resp. \( \text{Res}_{R/R_1} B \)), where \( T \) (resp. \( B \)) is a maximal torus (resp. Borel subgroup) of \( S \).

*Proof:* Formula (3) follows from (1) and our hypothesis. See [2, Prop. 5 of 7.6] for the closed subgroup, smooth or flat part. Let now \( S \) be a reductive group scheme. We know that \( \text{Res}_{R/R_1} S \) is affine, flat and smooth. So in order to show that it is a reductive group scheme, it is enough to show that its geometric fibres are so. So it is enough to show that the fibres of \( (\text{Res}_{R/R_1} S)_R \) are reductive group schemes. But this is so as \( (\text{Res}_{R/R_1} S)_R \) is a reductive group scheme, cf. (3). Similarly we argue that \( \text{Res}_{R/R_1} S \) is semisimple if \( S \) is so.

To check the last part, we first remark that \( J_R \) is a maximal torus (resp. Borel subgroup) of \( (\text{Res}_{R/R_1} S)_R \). So based on (3), it is uniquely determined by the projection \( T \) (resp. \( B \)) of \( J_R \) on the factor \( S = S \times_{\text{Spec}(R)} \text{Spec}(R) \) of \( (\text{Res}_{R/R_1} S)_R \). So \( J = \text{Res}_{R/R_1} T \) (resp. \( J = \text{Res}_{R/R_1} B \)). This ends the proof.

The next structure Theorem (see [29, 3.1.2]) will play important roles in §3 and §4.

**2.3.2. Theorem.** We assume \( R \) is a field. Then any adjoint (resp. s.c.) group \( F \) over \( R \) is isomorphic to a product of adjoint (resp. s.c.) groups of the form \( \text{Res}_{R/R} \tilde{F} \), where \( \tilde{R} \) is a separable finite field extension of \( R \) and \( \tilde{F} \) is an adjoint (resp. s.c.) group over \( \tilde{R} \) having an a.s. adjoint. So if \( F_{\text{ad}} \) is simple, then \( F \) is of isotypic Dynkin type.

### §3. Adjoint representations over finite fields

Let \( H \) be a semisimple group over \( k = \mathbb{F}_q \). In §3 we study \( H, \text{Lie}(H) \) and \( AD_H \). In 3.3 we recall all cases when \( \text{Lie}(H) \) is simple, i.e. it has no proper ideal defined over \( k \). These cases are precisely the ones when \( AD_H \) is irreducible, cf. 3.4. In 3.5 to 3.11 we work in the context of Weil restrictions. In 3.1, 3.2 and 3.5 to 3.8 we include preliminary material on \( H \) and \( \text{Lie}(H) \). See 3.7.1 for a refinement of 3.4 obtained by working with

\[
\mathcal{L}_H := \text{Im}(\text{Lie}(H^\text{sc}) \to \text{Lie}(H))
\]

instead of \( \text{Lie}(H) \). In 3.9 we list semisimple subgroups of \( H \) to be used in 3.10, 3.11 and §4. In 3.10 and 3.11 we study subrepresentations of \( AD_H \); these sections extend [13].
3.1. On $Z(H)$. The finite group scheme $Z(H)$ is of multiplicative type, cf. [27, Vol. III, 4.1.7 of p. 173]. So the finite group $Z(H)(k)$ is of order prime to $p$. If $p|o(H)$, then $\text{Lie}(Z(H))$ is a proper ideal of $\text{Lie}(H)$ normalized by $H$ and so $AD_H$ is reducible. If $g.c.d.(p,o(H)) = 1$, then the central isogeny $H \to H^{ad}$ is étale and so by identifying the tangent spaces in the origin we get $\text{Lie}(H^{ad}) = \text{Lie}(H)$. Similarly, if $g.c.d.(p,c(H)) = 1$ we have $\text{Lie}(H^{sc}) = \text{Lie}(H)$.

3.2. Lemma. We assume that $H^{ad}$ is a.s. and $\text{Lie}(H^{ad})$ is not a.s. Then $\text{Lie}(H^{ad})$ has a proper, characteristic ideal and so $AD_H$ is reducible.

Proof: If $p \geq 5$, or if $p = 3$ and $H^{ad}$ is not of $G_2$ Lie type, or if $p = 2$ and $H^{ad}$ is not of $F_4$ Lie type, then $[\text{Lie}(H^{ad}), \text{Lie}(H^{ad})]$ is a characteristic ideal of $\text{Lie}(H^{ad})$ of codimension 1 or 2 (cf. [18, 0.13] applied to $H^{ad}_k$). In the excluded two cases the group $H^{ad}$ is split and $\text{Lie}(H^{ad})$ has a unique proper ideal: it corresponds to root vectors of short roots. So $\text{Lie}(H^{ad})$ has a proper, characteristic ideal and so $AD_H$ is reducible.

3.3. Proposition. The Lie algebra $\text{Lie}(H)$ is not simple iff one of the following four conditions holds:

i) $H^{ad}$ is not simple;

ii) $H^{ad}$ is simple of isotypic $A_{p-1}$ Dynkin type;

iii) $p = 2$ and $H^{ad}$ is simple of an isotypic Dynkin type belonging to the union $\{B_n, C_n | n \in \mathbb{N}\} \cup \{D_n | n \geq 3\} \cup \{E_7, F_4\}$;

iv) $p = 3$ and $H^{ad}$ is simple of isotypic $E_6$ or $G_2$ Dynkin type.

Proof: If i) to iv) do not hold, then $g.c.d.(p,o(H)) = 1$ and so $\text{Lie}(H) = \text{Lie}(H^{ad})$. If $\text{Lie}(H)$ is simple, then $g.c.d.(p,o(H)) = 1$ and so $\text{Lie}(H) = \text{Lie}(H^{ad})$ (cf. 3.1) and $H$ is simple (cf. 2.3.2). So it suffices to prove the Proposition under the extra assumption that $H$ is adjoint and simple. Let $k_1$ be such that $H = \text{Res}_{k_1/k}H_1$, where $H_1$ is an a.s. adjoint group over $k_1$ (cf. 2.3.2). We have $\text{Lie}(H) = \text{Lie}_k(H_1)$, cf. (2). We show that $\text{Lie}(H_1)$ is simple iff $\text{Lie}(H)$ is simple. The “if” part is obvious. To check the “only if” part let $I$ be a non-zero ideal of $\text{Lie}(H)$. So $I \otimes_k k_1$ is a non-zero ideal of

$$\text{Lie}(H) \otimes_k k_1 = \text{Lie}_k(H_1) \otimes_k k_1 \to \bigoplus_{\tau \in \text{Gal}(k_1/k)} \text{Lie}(H_1) \otimes_{k_1} \tau k_1$$

and so a direct sum of some of these direct summands indexed by elements of $\text{Gal}(k_1/k)$. But the only non-zero such direct sum defined over $k$ is $\text{Lie}_k(H_1) \otimes_k k_1$. So $I = \text{Lie}(H)$.

Also, $\text{Lie}(H_1)$ is a.s. iff it is simple (cf. 3.2 for the if part). So $\text{Lie}(H)$ is simple iff $\text{Lie}(H_1)$ is a.s. So the Proposition follows from [18, 0.13].

3.4. Proposition. Let $H$ be a semisimple group over $k$. Then $AD_H$ is irreducible iff $\text{Lie}(H)$ is simple (i.e. iff none of the four conditions of 3.3 holds).

Proof: We assume $AD_H$ is irreducible. So $g.c.d.(p,o(H)) = 1$ and $\text{Lie}(H) = \text{Lie}(H^{ad})$, cf. 3.1. Moreover $H^{ad}$ is simple, cf. 2.3.2. So it suffices to show that $\text{Lie}(H)$ is simple under the extra assumption that $H^{ad}$ is a.s., cf. the proof of 3.3. So $\text{Lie}(H)$ is a.s., cf. 3.2.

We now assume $\text{Lie}(H)$ is simple. So $H^{ad}$ is simple and $g.c.d.(p,o(H^{sc})) = 1$, cf. 3.3. So to show that $AD_H$ is irreducible we can assume that $H = \text{Res}_{k_1/k}H_1$, with $H_1$ a s.c.
semisimple group over $k_1$ having an a.s. adjoint. So $\text{Lie}(H_1) = \text{Lie}(H_1^{\text{ad}})$ are a.s., cf. 3.3. Let $DT$ be the Dynkin type of $H_1$. The maximal weight $\varpi$ of the adjoint representation of the complex, simple Lie algebra of $DT$ Dynkin type is the maximal root. So if $DT$ is $A_n$ (resp. is $B_n$ with $n \geq 3$, $C_n$ with $n \geq 2$, $D_n$ with $n \geq 4$, $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$), then with the notations of [5, planche I to IX] $\varpi$ is $\varpi_1 + \varpi_n$ (resp. is $\varpi_2, 2\varpi_1, \varpi_2, \varpi_1, \varpi_8, \varpi_1$ or $\varpi_2$). From Curtis and Steinberg theory and the last two sentences we get that $AD_{H_1}$ is absolutely irreducible. Let $I$ be a non-trivial, irreducible subrepresentation of $AD_H$. The extension of $AD_H$ to $k_1$ is a direct sum of $[k_1 : k]$ copies of $AD_{H_1}$, cf. (4). So due to the absolute irreducibility of $AD_{H_1}$, $I \otimes_k k_1$ is a Lie($H_1$)-submodule of Lie($H$) $\otimes_k k_1$. So as $(I \otimes_k k_1) \cap \text{Lie}(H) = I$, $I$ is an ideal of Lie($H$). So $I = \text{Lie}(H)$. This ends the proof.

3.5. The basic setting. Until §4 we assume that

$$H = \text{Res}_{k_1/k} H_1,$$

where $H_1$ is a semisimple group over $k_1$ such that $H_1^{\text{ad}}$ is a.s. So $\text{Lie}(H) = \text{Lie}_k(H_1)$. Let

$$DT$$

be the Dynkin type of $H_1$. Let $k_2$ and $k_3$ be the quadratic and respectively the cubic field extension of $k_1$. The group $H_1$ has a Borel subgroup $B_1$ (this is part of Lang theorem). Let $T_1$ be a maximal torus of $B_1$, cf. [4, (i) of Theorem 18.2]. The group $H_1$ is split iff $T_1$ is. So there is a smallest field extension $l$ of $k_1$ such that $H_{1l}$ is split. Let

$$\text{Lie}(H_{1l}) = \text{Lie}(T_{1l}) \bigoplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the Weyl decomposition of $\text{Lie}(H_{1l})$ with respect to $T_{1l}$. So each $\mathfrak{g}_\alpha$ is a 1 dimensional $l$-vector space normalized by $T_{1l}$ and $\Phi$ is an irreducible root system of characters of $T_{1l}$. Let $\Phi^+ := \{ \alpha \in \Phi \mid \mathfrak{g}_\alpha \subset \text{Lie}(B_1) \}$. Let $\Delta$ be the basis of $\Phi$ included in $\Phi^+$. We use [5, planches I to IX] to denote the elements of $\Delta$, $\Phi^+$ and $\Phi$; so $\Delta = \{ \alpha_1, \ldots, \alpha_{|\Delta|} \}$, where $|\Delta|$ is the number of elements of $DT$. Warning: if $l = k_1$ (resp. $l = k$), then in connection to (5) we often drop the lower right index $l$ (resp. 1) and so also $T_1$ becomes $T$.

We denote also by $\Delta$ the Dynkin diagram of $DT$ defined by the elements of $\Delta$. Let $\phi$ be the Frobenius automorphism of $\bar{l}$ fixing $k_1$. It acts on $\Phi$ via the rule: if $\alpha \in \Phi$, then we have $\mathfrak{g}_{\phi(\alpha)} = (\text{Lie}(H_{1l}) \otimes \phi)(\mathfrak{g}_\alpha)$. Let $e \in \mathbb{N}$ be the order of the permutation of $\Delta$ defined by $\phi$. For $m \in \mathbb{N}$, $\phi^m$ fixes $\text{Lie}(T_{1l})$ iff $e|m$. So $[l : k_1] = e$. But $e$ is the order of an element of $\text{Aut}(\Delta)$ and so $e \in \{1, 2, 3\}$. More precisely, we have $l = k_1$ if $e \in \{B_n, C_n|n \geq 1\} \cup \{E_7, E_8, F_4, G_2\}$, $l \in \{k_1, k_2\}$ if $DT \in \{A_n|n \geq 2\} \cup \{D_n|n \geq 5\} \cup \{E_6\}$ and $l \in \{k_1, k_2, k_3\}$ if $LT = D_4$. If $e = 1$ let $LT := DT$. If $e \in \{2, 3\}$ let $LT := e DT$. We recall that the pair $(\Delta, \phi)$ or just

$$LT$$

is called the Lie type of $H_1$ and that $H_1^{\text{ad}}$ and $H_1^{\text{sc}}$ are uniquely determined by it (see [29]).

For $\alpha \in \Phi$ let $H(|\alpha|)$ be the semisimple subgroup of $H_{1l}$ generated by the two $\mathbb{G}_a$ subgroups $\mathbb{G}_{\alpha, \alpha}$ and $\mathbb{G}_{\alpha, -\alpha}$ of $H_{1l}$ normalized by $T_{1l}$ and having $\mathfrak{g}_\alpha$ and respectively $\mathfrak{g}_{-\alpha}$ as
their Lie algebras. So \( H(|\alpha|)^{\text{ad}} \) is a \( \text{PGL}_2 \) group and \( H_1 \) is generated by all these \( H(|\alpha|)'s. All these can be deduced from [4, §13 of Ch. IV] via descent from \( \bar{l} \) to \( l \).

3.6. Proposition. Let \( \beta \in \Phi \setminus \{-\alpha, \alpha\}. \) Let \( \Psi_1(\alpha, \beta) := \{i\alpha + \beta | i \in \mathbb{N}, i\alpha + \beta \in \Phi\}. \) If \( \alpha + \beta \notin \Phi \) let \( g_{\alpha+\beta} := \{0\}. \) Let \( g \in \mathcal{G}_{\alpha,\alpha}(l) \) be a non-identity element. Let \( x \in g_{\beta} \setminus \{0\} \).

a) We have \( AD_{H_1}(g)(x) - x \in \oplus_{\delta \in \Psi_1(\alpha, \beta)} g_{\delta}. \)

b) We always have an inclusion \( [g_{\alpha}, g_{\beta}] \subset g_{\alpha+\beta}. \) It is not an equality iff one of the following disjoint three conditions holds:

\begin{enumerate}
    \item If \( p = 3, DT = G_2, \) and \( \alpha \) and \( \beta \) are short roots forming an angle of \( 60^\circ; \)
    \item \( p = 2, DT = G_2, \) and \( \alpha \) and \( \beta \) are short roots forming an angle of \( 120^\circ; \)
    \item If none of the three conditions of b) holds and if \( \alpha + \beta \in \Phi, \) then the component of \( AD_{H_1}(g)(x) - x \) in \( g_{\alpha+\beta} \) is non-zero.
\end{enumerate}

Proof: Let \( \kappa(\alpha, \beta) := \Phi \cap \{i\alpha + j\beta | (i, j) \in \mathbb{N}^2\}. \) It is known that there are isomorphisms \( x_\delta : \mathcal{G}_\alpha \rightarrow \mathcal{G}_{\alpha, \delta} (\delta \in \kappa(\alpha, \beta)), \) an ordering on \( \mathbb{N}^2, \) and integers \( m_{\alpha, \beta; i, j} ((i, j) \in \mathbb{N}^2) \) such that for all \( s, t \in \mathcal{G}_\alpha(l) = l \) we have (cf. [27, Vol. III, p. 321] or [8, 4.2])

\[
x_{\alpha}(s)x_{\beta}(t)x_{\alpha}(s^{-1})x_{\beta}(t^{-1}) = \prod_{(i, j) \in \mathbb{N}^2, \alpha + j\beta \in \kappa(\alpha, \beta)} x_{i\alpha + j\beta}(m_{\alpha, \beta; i, j}s^it^j).
\]

Taking the derivative in (6) with respect to \( t \) at the identity element, we get that the difference \( AD_{H_1}(x_{\alpha}(s))(x) - x \) belongs to \( \oplus_{l \in \mathbb{N}, i\alpha + \beta \in \Psi_1(\alpha, \beta)} g_{\alpha+\beta} \otimes_l \mathbb{I} \) and its component in \( g_{i\alpha + \beta} \otimes_l \mathbb{I} \) is a fixed generator of \( g_{i\alpha + \beta} \otimes_l \mathbb{I} \) times \( m_{\alpha, \beta; i, 1}s^i \) (cf. [4, 3.16 of Ch. 1]). Similarly, taking the derivative in (6) with respect to \( s \) and \( t \) at the identity element, we get that \( [g_{\alpha}, g_{\beta}] = m_{\alpha, \beta; 1, 1}g_{\alpha+\beta} \subset g_{\alpha+\beta}. \) But \( m_{\alpha, \beta; 1, 1} = \pm (m + 1), \) where \( m \in \mathbb{Z} \) is the greatest integer such that \( \beta - ma \in \Phi \) (see [27, Vol. III, p. 321] or [8, 4.2 and 4.3]). So the second part of b) and c) follow from the fact that we have \( p|m + 1 \) iff one of the three conditions of b) holds (see loc. cit.). This ends the proof.

Let \( B_1^{\text{opp}} \) be the Borel subgroup of \( H_1 \) which is the opposite of \( B_1 \) with respect to \( T_1. \) Let \( U_1 \) and \( U_1^{\text{opp}} \) be the unipotent radicals of \( B_1 \) and respectively of \( B_1^{\text{opp}}. \) Let \( L_H \) be the Lie subalgebra of \( \text{Lie}(H) = \text{Lie}_k(H_1) \) generated by \( \text{Lie}_k(U_1) \) and \( \text{Lie}_k(U_1^{\text{opp}}). \)

3.7. Proposition. 1) The Lie algebra \( L_H \) is an ideal of \( \text{Lie}(H) \) containing \([\text{Lie}(H), \text{Lie}(H)], \)

If \( p > 2 \) and \( l = k_1, \) then \( L_H = \sum_{\alpha \in \Phi^+} \text{Lie}_k(H(|\alpha|)). \)

2) Always \( L_H = L_{\mathcal{L}}. \) Also, if \( p > 2 \) or if \( p = 2 \) and \( DT \notin \{C_n|n \in \mathbb{N}\}, \) then \( L_H = [\text{Lie}(H), \text{Lie}(H)]. \)

3) We have \( \mathcal{L}_{H^{\text{ad}}} = L_{H^{\text{ad}}} = [\text{Lie}(H^{\text{ad}}), \text{Lie}(H^{\text{ad}})]. \)

Proof: Tensoring with \( l \) we can assume \( k = k_1 = l; \) so \( H = H_1 \) is split. The first part of 1) is obvious. If \( p \geq 3, \) then \( \text{Lie}(H(|\alpha|)) = g_\alpha \oplus g_{-\alpha} \oplus [g_\alpha, g_{-\alpha}]. \) So the last part of 1) follows from the inclusions of 3.6 b). We now prove 2). Both \( U_1 \) and \( U_1^{\text{opp}} \) are naturally identified with unipotent subgroups of \( H_1^{\text{sc}} \) and so \( L_H \) makes sense if \( H \rightarrow H \) is an arbitrary central isogeny. Always \( L_{\bar{H}} \) surjects onto \( L_H \) and so to prove 2) we can assume that \( H \) is s.c.
If $p > 2$ or if $p = 2$ and $DT \not\in \{C_n | n \in \mathbb{N}\}$, then $\text{Lie}(H) = [\text{Lie}(H), \text{Lie}(H)]$ (cf. [18, 0.13]) and so 2) follows from 1). So we are left to show that $L_H = \text{Lie}(H)$ if $p = 2$ and $DT = C_n$. But then $H$ has a subgroup $SL(H)$ which is a product of $SL_2$ groups and has $T$ as a maximal torus; so $\text{Lie}(T) \subset L_H$ and so $L_H = \text{Lie}(H)$. This proves 2).

Based on 2) and its proof, its suffices to prove 3) under the extra assumptions that $p = 2$ and $DT = C_n$. As $L_{H_{\text{ad}}} = L_{H_{\text{ad}}}$, the ideal $L_{H_{\text{ad}}}$ of $\text{Lie}(H_{\text{ad}})$ is of codimension 1 and contains $[\text{Lie}(H_{\text{ad}}), \text{Lie}(H_{\text{ad}})]$. By reasons of dimensions we get $L_{H_{\text{ad}}} = [\text{Lie}(H_{\text{ad}}), \text{Lie}(H_{\text{ad}})]$ (see [18, 0.13]). This ends the proof.

3.7.1. Corollary. We assume $\text{g.c.d.}(p, o(H)) = 1$. If $p = 3$ we also assume that $DT \neq G_2$ and if $p = 2$ we also assume that $DT \notin \{B_n, C_n | n \geq 2 \} \cup \{F_4\}$. Then the natural representation of $H^{sc}(k) = H^{sc}(k_1)$ on $L_{H_{\text{ad}}} = L_H = [\text{Lie}(H), \text{Lie}(H)]$ is irreducible.

Proof: The $H_1$-module $L_{H_1}$ is a.s. Argument: the case $DT = A_1$ is trivial and the case $DT \neq A_1$ is a consequence of the fact that $L_{H_1}$ is an a.s. $\text{Lie}(H_1)$-module (see [18, 0.13]). The maximal weight of the representation of $H_{1l}$ on $L_{H_{1l}}$ is $\varpi$ of the proof of 3.4. The rest of the proof is entirely the same as the proof of 3.4.

We come back to (5). Let $\alpha \in \Phi$. Let $T(|\alpha|)$ be the maximal subtorus of $T_{1l}$ centralizing $g_{\alpha}$. So $[g_{\alpha}, \text{Lie}(T(|\alpha|))] = \{0\}$. We have a short exact sequence

$$0 \to T(|\alpha|) \to TH(|\alpha|) \to \tilde{H}(|\alpha|) \to 0,$$

where $TH(|\alpha|)$ is the reductive subgroup of $H_{1l}$ generated by $T(|\alpha|)$ and $H(|\alpha|)$ and where $\tilde{H}(|\alpha|)$ is either $H(|\alpha|)$ or its adjoint.

3.8. Lemma. The group $H(|\alpha|)$ is a $\text{PGL}_2$ group iff $DT = B_n$, $H_1$ is adjoint and $\alpha$ is short. Also, if $H_1$ is adjoint, then $\tilde{H}(|\alpha|) = H(|\alpha|)$. Proof: For $\beta \in \Phi$ let $ST(\alpha, \beta) := \{i\alpha + \beta | i \in \mathbb{Z}\} \cap \Phi$ be the $\alpha$-string through $\beta$. It is of the form $\{i\alpha + \beta | i \in \{-s, -s+1, ..., t\}\}$, for some $s, t \in \mathbb{N} \cup \{0\}$ with $s + t \leq 3$ (see [15, p. 45]). The set $\Phi \setminus \{\alpha, -\alpha\}$ is a disjoint union of $\alpha$-strings. So let $\Phi(\alpha)$ be a subset of $\Phi \setminus \{\alpha, -\alpha\}$ such that we have a disjoint union $\Phi \setminus \{\alpha, -\alpha\} = U_{\beta \in \Phi(\alpha)} ST(\alpha, \beta)$. For $\beta \in \Phi(\alpha)$ let $V_{\alpha, \beta} := \oplus_{\delta \in ST(\alpha, \beta)} g_\delta$. It is an $H(|\alpha|)$-module, cf. 3.6 a). We get a direct sum decomposition of $H(|\alpha|)$-modules $H_{1l} = \text{Lie}(TH(|\alpha|)) \oplus \oplus_{\beta \in \Phi(\alpha)} V_{\alpha, \beta}$.

If $H(|\alpha|)$ is a $\text{PGL}_2$ group, then $\dim(V_{\alpha, \beta}) \in \{1, 3\}, \forall \beta \in \Phi(\alpha)$, and the converse holds if $H_1$ is adjoint. Using [5, planches I to IX] and [15, Lemma C of p. 53] we easily get that $\dim(V_{\alpha, \beta}) \in \{1, 3\}, \forall \beta \in \Phi(\alpha)$, iff $DT = B_n$ and $\alpha$ is short. But if $H_1$ is s.c., then $H(|\alpha|)$ is an $SL_2$ group as we can check by reduction to the $B_1$ and $B_2 = C_2$ Dynkin types. This proves the first part.

If $H_1$ is adjoint, then the subgroup of $T_{1l}$ fixing $g_\alpha$ is a subtorus of $T_{1l}$ of codimension 1 and so it is $T(|\alpha|)$. So $Z(H(|\alpha|))$ is a subgroup of $T(|\alpha|)$ and so $\tilde{H}(|\alpha|) = H(|\alpha|)$.
(resp. \( DT_2 = G_2 \)), then \( \Psi_1(\alpha, \beta) \) has at most 2 (resp. 3) elements and so \( \text{ad}(x)^3 \) (resp. \( \text{ad}(x)^4 \)) annihilates \( V_{\alpha, \beta} \).

### 3.9. Subgroups.
We list semisimple subgroups of \( H_1 \) normalized by \( T_1 \). By taking the \( \text{Res}_{k_1/k} \) of them we get semisimple subgroups of \( H \). So except for the first two Cases below (which are samples) we will just mention semisimple subgroups of \( H_1 \) and not of \( H \).

Let \( \Phi_0 \) be a subset of \( \Phi \) having the following three properties:

1. it is invariant under \( \phi \) of 3.5;
2. it is stable under additions, i.e. we have \( \Phi \cap \{ \alpha + \beta | \alpha, \beta \in \Phi_0 \} \subset \Phi_0 \);
3. it is symmetric, i.e. we have \( \Phi_0 = -\Phi_0 \).

Let \( H_{0l} \) be the subgroup of \( H_{1l} \) generated by \( H(|\alpha|) \), where \( \alpha \in \Phi_0 \). It is invariant under \( \phi \) and \( T_1 \) and so it the extension to \( l \) of a subgroup \( H_0 \) of \( H_1 \) normalized by \( T_1 \). The subgroup \( TH_0 \) of \( H_1 \) generated by \( T_1 \) and \( H_0 \) is reductive, cf. [27, Vol. III, 5.4.7 and 5.10.1 of Exp. XXII] applied to \( TH_{0l} \). But \( H_0 \) is a subgroup of \( (TH_0)^{\text{der}} \) and contains the unipotent radical of any Borel subgroup of \( TH_0 \) normalized by \( T_1 \). So \( H_0 = (TH_0)^{\text{der}} \) and \( H_{0l} \) is semisimple. A subset \( \Phi^+_0 \) of \( \Phi_0 \cap \Phi^+ \) is said to generate \( \Phi_0 \), if each element of \( \Phi_0 \cap \Phi^+ \) is a linear combination with non-negative integer coefficients of elements of \( \Phi^+_0 \). We refer to \( H_{0l} \) as the semisimple subgroup of \( H_1 \) associated to \( \Phi_0 \) or generated by \( \Phi^+_0 \).

In this paragraph we assume \( l = k_1 \). If \( DT \notin \{A_1, B_2, G_2 \} \), then we can choose \( \Phi_0 \) such that \( H_0 \) is of \( A_2 \) Lie type. If \( DT \notin \{A_1, A_2, B_2, B_3, C_3, D_4, F_4, G_2 \} \) (resp. \( DT \in \{B_n, C_n|n \geq 2 \} \cup \{F_4\} \)), then we can choose \( \Phi_0 \) such that \( H_0 \) is of \( A_3 \) (resp. \( C_2 \)) Lie type. We assume now that \( DT \in \{B_n, C_n|n \geq 2 \} \cup \{F_4, G_2\} \). Let \( H_{1l}^{\text{long}} \) be the semisimple subgroup of \( H_1 \) generated by \( \{\alpha \in \Phi^+_0 | \alpha \text{ is long}\} \). Warning: the group \( H_{1l}^{\text{long}} \) as well as most of the below semisimple subgroups of \( H_1 \) are not generated by subsets of \( \Delta \). If \( n \geq 3 \) and \( DT = B_n \), then \( H_{1l}^{\text{long}} \) is of \( D_n \) Lie type. If \( n \geq 2 \), \( H_1 \) is s.c. and \( DT = C_n \), then \( H_{1l}^{\text{long}} \) is an \( SL_2 \) group. If \( DT = F_4 \) (resp. \( DT = G_2 \)), then \( H_{1l}^{\text{long}} \) is of \( A_3 \) (resp. \( A_2 \)) Lie type. These Lie types can be read out from [5, planches II, III, VIII and IX].

Until 3.10 we use (5) with \( l \in \{k_2, k_3\} \). If \( \varpi \) is as in the proof of 3.4, then the subgroup \( H(|\varpi|) \) of \( H_{1l} \) is the pull back of an \( SL_2 \) subgroup of \( H_1 \) (cf. also 3.8 applied to \( H_{1l} \)). If \( LT = 2A_{n+4} \) and \( \Phi^+_0 = \{\alpha_0, \alpha_2, \alpha_{n+3}, \alpha_{n+4}\} \), then \( H_0 \) is isogenous to the \( \text{Res}_{k_2/k_1} \) of an \( SL_3 \) group. If \( LT = 2A_4 \) and \( \Phi^+_0 = \{\alpha_1, \alpha_4\} \), then \( H_0 \) is the \( \text{Res}_{k_2/k_1} \) of an \( SL_2 \) group. We assume now that \( LT = 2D_{n+3} \). If \( n = 1 \) we assume \( \Phi \) permutes \( \alpha_3 \) and \( \alpha_4 \). If \( n > 1 \) (resp. \( n = 1 \)) and \( \Phi^+_0 = \{\alpha_1, \alpha_2, \alpha_3\} \) (resp. \( \Phi^+_0 = \{\alpha_1, \alpha_2, \alpha_3 + \alpha_4\} \)), then \( H_0 \) is a \( PGL_4 \) group. To introduce other semisimple subgroups of \( H_1 \) we consider five Cases.

**Case 1:** \( LT = 2A_{2n+1} \). Let \( H_1(n+1) \) be the \( SL_2 \) subgroup of \( H_1 \) such that \( H_1(n+1)_l = H(|\alpha_{n+1}|) \). So \( H(n+1) := \text{Res}_{k_1/k} H_1(n+1) \) is a subgroup of \( H \). Let \( i \in \{1, \ldots, n\} \). Let \( \alpha_1^i := \sum_{j=i}^{n+1} \alpha_j \) and \( \alpha_2^i := \sum_{j=n+2}^{2n+2-i} \alpha_j \). The subgroups \( H(|\alpha_1^i|) \) and \( H(|\alpha_2^i|) \) of \( H_{1k_2} \) are \( SL_2 \) groups and commute. We first assume that either \( n > 1 \) or \( H_1 = H_1^{\text{sc}} \). The product \( H(|\alpha_1^i|) \times_{k_2} H(|\alpha_2^i|) \) is naturally a subgroup of \( H_{1k_2} \), invariant under the action of \( \text{Gal}(k_2/k_1) \) on \( H_{1k_2} \). So it is the extension to \( k_2 \) of a subgroup \( H_1(i) \) of \( H_1 \). As \( \text{Gal}(k_2/k_1) \) takes \( \text{Lie}(H(|\alpha_1^i|)) \) into \( \text{Lie}(H(|\alpha_2^i|)) \), the group \( H_1(i) \) is the \( \text{Res}_{k_2/k_1} \) of an \( SL_2 \) group. So \( H(i) := \text{Res}_{k_1/k} H_1(i) \) is a subgroup of \( H \) which is the \( \text{Res}_{k_2/k} \) of an \( SL_2 \) group. If \( n = 1 \) and \( H_1 \neq H_1^{\text{sc}} \), then let \( H_1(1) \) be the semisimple subgroup of \( H_1 \) such that we have
a natural central isogeny \( H(|\alpha_1|) \times_k 2 H(|\alpha_2|) \to H_1(1)_{k_2} \) of degree 2; the adjoint of the subgroup \( H(1) := \text{Res}_{k_1/k} H_1(1) \) of \( H \) is the \( \text{Res}_{k_2/k} \) of a \( \text{PGL}_2 \) group.

**Case 2:** \( LT = ^2A_{2n} \). For \( i \in \{1, \ldots, n-1\} \) let \( \alpha^1_i := \sum_{j=i}^{n+1} \alpha_j \) and \( \alpha^2_i := \sum_{j=n-i}^{2n-1} \alpha_j \).

Let \( H_1(i) \) and \( H_1(n+1) \) be subgroups of \( H_1 \) such that \( H_1(i)_{k_2} = H(|\alpha^1_i|) \times_k 2 H(|\alpha^2_i|) \) and \( H_1(n+1)_{k_2} = H(|\alpha_n + \alpha_{n+1}|) \). The adjoint of the semisimple subgroup \( H_1(n) \) of \( H_1 \) generated by \( \{\alpha_n, \alpha_{n+1}\} \) is a \( \text{PGU}_3 \) group. So for \( i \in \{1, \ldots, n+1\} \), the group \( H(i) := \text{Res}_{k_1/k} H_1(i) \) is a subgroup of \( H \). If \( i \in \{1, \ldots, n-1\} \) (resp. if \( i = n+1 \)), then \( H(i) \) is the \( \text{Res}_{k_2/k} \) (resp. \( \text{Res}_{k_1/k} \)) of an \( SL_2 \) group.

**Case 3:** \( LT = ^2E_6 \). Let \( H_1(1)' \) (resp. \( H_1(4) \)) be the semisimple subgroup of \( H_1 \) generated by \( \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \) (resp. \( \{\alpha_2\} \)). So \( H_1(1)' \) is of \( ^2A_5 \) Lie type and \( H_1(4) \) is an \( SL_2 \) group, cf. 3.8. We apply Case 1 to \( H_1(1)' \): we get semisimple subgroups \( H_1(1), H_1(2) \) and \( H_1(3) \) of \( H_1(1)' \) and so also of \( H_1 \).

**Case 4:** \( LT = ^2D_{n+3} \). If \( n = 1 \) we assume \( \phi \) permutes \( \alpha_3 \) and \( \alpha_4 \). For \( i \in \{1, \ldots, n+2\} \) let \( H_1(i) \) be the semisimple subgroup of \( H_1 \) which over \( k_2 \) is generated by \( H(|\alpha|) \), with \( \alpha \in \{\sum_{j=i}^{n+2} \alpha_j, \alpha_{n+3} + \sum_{j=i}^{n+1} \alpha_j\} \). So \( H_1(i) \) is the \( \text{Res}_{k_2/k_1} \) of an \( SL_2 \) group.

**Case 5:** \( LT = ^3D_4 \). Let \( H_1(1) \) (resp. \( H_1(2) \)) be the semisimple subgroup of \( H_1 \) which over \( k_3 \) is generated by \( H(|\alpha|) \), with \( \alpha \in \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\} \) (resp. with \( \alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \)). So \( H_1(2) \) is an \( SL_2 \) group (cf. 3.8) and \( H_1(1) \) is the \( \text{Res}_{k_3/k_1} \) of an \( SL_2 \) group.

**3.9.1. Lemma.** We refer to Case 4 (resp. Case 5). If \( H = H^{\text{sc}} \), then each \( H_1(i) \) (resp. then \( H_1(1) \)) are (resp. is) s.c.

**Proof:** To check this we can work with \( H_1 \). Even better, based on the existence of Chevalley group schemes over \( \mathbb{Z} \) we can work over \( \mathbb{C} \). Based on [6, §13 of Ch. VIII] we need to show that any standard (resp. standard composite) monomorphism \( SO_4 \hookrightarrow SO_{2n+6} \) (resp. \( SL_2 \times SO_4 \leftrightarrow SO_4 \times SO_4 \leftrightarrow SO_8 \) over \( \mathbb{C} \), lifts to a monomorphism \( Spin_4 \hookrightarrow Spin_{2n+6} \) (resp. \( SL_2 \times Spin_4 \leftrightarrow Spin_8 \)). To check this last statement we can work just in the context of \( SO_{2n+6} \). The restriction of the spin representation of \( \text{Lie}(SO_{2m}) \) to \( \text{Lie}(SO_{2m-2}) \) via a standard monomorphism \( SO_{2m-2} \hookrightarrow SO_{2m} \) is a disjoint union of two spin representations of \( \text{Lie}(SO_{2m-2}), \forall m \in \mathbb{N}, m \geq 4 \) (see the description of spin representations in loc. cit.). From this the statement on lifts follows. This ends the proof.

**3.9.2. Lemma.** The Lie subalgebra \( LS_H \) of \( \text{Lie}_k(H_1) = \text{Lie}(H) \) generated by all intersections \( \text{Lie}_k(H_1(i) \cap U_1) \)'s and \( \text{Lie}_k(H_1(i) \cap U_1^\text{opp}) \)'s is \( \mathfrak{S}_H = L_H \).

**Proof:** Tensoring with \( l \) over \( k \), this follows easily from 3.6 b).

**3.9.3. Simple properties.** Let \( TH_1(i) \) be the reductive subgroup of \( H_1 \) generated by \( T_1 \) and \( H_1(i) \). Let \( T_1(i) \) be the maximal torus of \( Z(TH_1(i)) \). We have a short exact sequence

\[
0 \to \text{Lie}(T_1(i)) \to \text{Lie}(TH_1(i)) \to \text{Lie}(\bar{H}_1(i)) \to 0,
\]

where \( \bar{H}_1(i) \) is the quotient of \( H_1(i) \) by \( Z(H_1(i)) \cap T_1(i) \). If \( H \) is adjoint, then as in the proof of 3.8 we argue that in fact \( H_1(i) = H_1(i) \text{ad} \). Let \( S_1(i) := \text{Lie}(TH_1(i)) \). Let

\[
S_2(i) := \text{Lie}(H_1) \cap (\oplus_{\alpha \in \Phi, \mathfrak{g}_\alpha \not\subset \text{Lie}(H_1(i))_1})\mathfrak{g}_\alpha).
\]
We have a direct sum decomposition \( \text{Lie}(H) = S_1(i) \oplus S_2(i) \) of \( H(1) \)-modules.

We assume now that \( H_1(i)^{ad} \) is the \( \text{Res}_{l/k_1} \) of a \( PGL_2 \) group. Let \( x \in \text{Lie}(U_1) \cap \text{Lie}(H(1)) \). In the first four Cases (resp. in Case 5) we can write \( x = x_1 + x_2 \) (resp. \( x = x_1 + x_2 + x_3 \)), where \( x_i \in g_{\alpha_i} \), for some \( \alpha_i \in \Phi^+ \). This writing is unique up to ordering. The restriction of \( \text{ad}(x_i)^2 \) to \( S_2(i) \) is 0, cf. the \( DT_2 = A_2 \) part of 3.8.1. As \( H_1(i)^{ad} \) is the \( \text{Res}_{l/k_1} \) of a \( PGL_2 \) group we have \( \text{ad}(x_1)\text{ad}(x_2) = \text{ad}(x_2)\text{ad}(x_1) \) (resp. \( \text{ad}(x_i)\text{ad}(x_j) = \text{ad}(x_j)\text{ad}(x_i) \), \( \forall i, j \in \{1, 2, 3\} \)). Also in Case 5 we have \( \text{ad}(x_1)\text{ad}(x_2)\text{ad}(x_3) = 0 \). So

\[
h_x := \sum_{s=0}^{\infty} \frac{\text{ad}(x)^s}{s!} = \sum_{s=0}^{2} \frac{\text{ad}(x)^s}{s!}
\]

is a well defined automorphism of \( S_2(i) \). As in characteristic 0 we can identify \( h_x \) with the automorphism of \( S_2(i) \) defined by an element of \( H_1(i)(k_1) \).

We assume now that \( LT = 2A_{2n} \). So \( H_1(n) \) is of \( 2A_2 \) Lie type. Let \( x \in \text{Lie}(H_1(n)) \cap g_{\alpha_n} \oplus g_{\alpha_{n+1}} \). We write \( x = x_1 + x_2 \), with \( x_1 \in g_{\alpha_n} \) and \( x_2 \in g_{\alpha_{n+1}} \). The restrictions of \( \text{ad}(x_1) \) and \( \text{ad}(x_2) \) to \( S_2(n) \) do not commute but \( \text{ad}(x_{i_1})\text{ad}(x_{i_2})\text{ad}(x_{i_3}) \) restricted to \( S_2(n) \) is 0, \( \forall i_1, i_2, i_3 \in \{1, 2\} \). So for \( p \geq 3 \) we can define \( h_x \) as above.

3.10. Theorem. 1) If \( g.c.d.(p, o(H)) = 1 \), then no 1 dimensional \( k \)-vector subspace of \( \text{Lie}(H) \) is normalized by \( \text{AD}_H(\text{Im}(H^{sc}(k) \to H(k))) \).

2) We assume that \( p|o(H) \) and \( g.c.d.(p, c(H)) = 1 \). If \( q = 2 \), we also assume that \( H \) is not an \( SL_2 \) group. Then there is no proper \( k \)-vector subspace \( V \) of \( \text{Lie}(H) \) normalized by \( \text{AD}_H(\text{Im}(H^{sc}(k) \to H(k))) \) and such that \( \text{Lie}(H) = V + \text{Lie}(Z(H)) \).

3) We assume that \( DT = D_{2n+2} \), that \( o(H_1) = 2 \) and that \( p = 2 \). Then there is no proper \( k \)-vector subspace \( V_0 \) of \( \text{Lie}(H), \text{Lie}(H) \) normalized by \( \text{AD}_H(\text{Im}(H^{sc}(k) \to H(k))) \) and such that \( \text{[Lie(H), Lie(H)]} = V_0 + \text{Lie}(Z(H)) \).

4) We assume that \( p \) divides both \( c(H) \) and \( o(H) \). If \( p = 2 \) we also assume that \( DT \neq D_{2n+2} \). Then the adjoint representation of \( H^{sc}(k) \) on \( \text{Lie}(H) \) is the direct sum of an irreducible, non-trivial representation and of a trivial representation.

5) If \( p = 2 \) and \( DT = C_n \) we assume that \( H_1 \) is adjoint. Then the \( H_1(k_1) \)-module \( \text{Lie}(H_1)/\text{[Lie}(H_1), \text{Lie}(H_1)] \) is trivial and of dimension over \( k_1 \) at most 2.

Proof: If \( l \neq k_1 \) we use the notations of 3.9. To prove 1) we can assume \( H \) is adjoint and \( k_1 = k \). Let \( u \in \text{Lie}(H) \) be such that \( ku \) is normalized by \( \text{AD}_H(\text{Im}(H^{sc}(k) \to H(k))) \). The case \( DT = A_1 \) is well known (we can assume that \( q = p \) and cf. 3.4 that \( p = 2 \)). So we can assume \( DT \neq A_1 \). We first assume \( l \neq k \). Let \( x \in \text{Lie}(H_1(i) \cap U_1) \), with \( H_1(i) \) a semisimple subgroup of \( H = H_1 \) listed in the five cases of 3.9. We write \( u = u_1 + u_2 \), where \( u_s \in S_s(i) \) (\( s \in \{1, 2\} \)). The group \( H_1(i)(k_1) \) normalizes \( ku_1 \) and \( ku_2 \), cf. 3.9.3.

We assume now that \( DT \neq A_{2n} \). So \( H_1(i) \) is of isotypic \( A_1 \) Dynkin type. We show that \( [x, u_2] = 0 \). Let \( a \in l \setminus \{0, 1\} \). If \( H_1(i)^{ad} \) is a \( PGL_2 \) group, then the restriction of \( \text{ad}(x)^2 \) to \( S_2(i) \) is 0 (cf. 3.8.1 applied to \( H_1(i) \)) and so the automorphism of \( S_2(i) \) induced by \( \sum_{s=0}^{1} \text{ad}(x)^s \) is defined by an element of \( H_1(i)(k) \). So \( u_2 + [x, u_2] \in ku_2 \) and so as \( \text{ad}(x) \) is nilpotent we get \( [x, u_2] = 0 \). If \( H_1(i)^{ad} \) is the \( \text{Res}_{l/k} \) of a \( PGL_2 \) group, then as \( h_x \) and \( h_{ax} \) (of 3.9.3) normalize \( ku_2 \) we similarly get that \( [x, u_2] = 0 \). We now show that \( [x, u_1] = 0 \).
We use (8). If \( p > 2 \), then from 3.4 we get that the image of \( u_1 \) in \( \text{Lie}(\tilde{H}_1(i)) \) is 0; so \([x, u_1] = 0\). If \( p = 2 \), then \( \tilde{H}_1(i)(k_1) = H_1(i)_{\text{ad}}(k_1) \) and as \( \text{Lie}(H_1(i)_{\text{ad}}) \) has no non-zero elements normalized by \( H_1(i)_{\text{ad}}(k_1) \) (cf. the case \( DT = A_1 \)) we get \([x, u_1] = 0\).

So \([x, u] = 0\). Similarly we get \([y, u] = 0\), where \( y \in \text{Lie}(H_1(i) \cap U_1^{\text{opp}}) \). So \( u \) is annihilated by \( \mathcal{L}_H \), cf. 3.9.2. So \( u \) as an element of \( \text{Lie}(H_1) \) annihilates \( g_{\alpha} \), \( \forall \alpha \in \Phi \). As \( l \neq k \) none of the three conditions of 3.6(b) holds and so from 3.6(b) we get that the component of \( u \) in \( g_{\alpha} \) with respect to (5) is 0, \( \forall \alpha \in \Phi \). So \( u \in \text{Lie}(T_1) \). But as \( H \) is adjoint, no element of \( \text{Lie}(T_1) \) annihilates \( g_{\alpha} \), \( \forall \alpha \in \Phi \). So \( u = 0 \).

We assume now that \( DT = A_{2n} \). If g.c.d.\((p, 2n + 1) = 1\), then the equality \( u = 0 \) is implied by 3.4. So we can assume \( p \geq 3 \). If \( i \neq n \), then as above we get \([x, u] = 0\). If \( i = n \) we assume that \( x \) is as in the end of 3.9.3. As \( h_x \) and \( h_{2x} \) normalize \( ku_2 \), as above we get \([x, u_2] = 0\). So as in the previous paragraph to show that \( u = 0 \) it is enough to show that \([x, u_1] = 0\). To show this we can assume \( n = 1 \) (so \( \text{Lie}(H) = S_1(1) \) and \( u = u_1 \)). As \( u \) is annihilated by \( \pm g_{\alpha_1 + \alpha_2} \) (cf. the case \( i = n + 1 \)) from 3.6(b) we get that the component of \( u \) in \( g_{\alpha} \) with respect to (5) is 0, \( \forall \alpha \in \Phi \). So \( u \in \text{Lie}(T) \). Let \( g \in U_1(k_1) \setminus G_{a, \alpha_1 + \alpha_2}(k_1) \). As \( AD_H(g)(ku) \subset ku \), we get that \( u \) is annihilated by \( g_{\alpha_1} \) and \( g_{\alpha_2} \). So \( u = 0 \).

We are left to prove 1) in the case when \( l = k \). If all roots of \( \Phi \) have equal lengths, then the above part involving \( u_1 \) and \( u_2 \) for \( DT \neq A_{2n} \) applies entirely. If \( DT \) is \( F_4 \) or \( G_2 \), then \( H_{\text{ad}} = H_{\text{sc}} \) and so [13, Hauptsatz] applies. So to prove 1) for \( l = k \), we can assume \( DT \in \{B_n, C_n\} \) \((n \geq 2)\). We can also assume \( p = 2 \), cf. 3.4. As in the mentioned part involving \( u_1 \) and \( u_2 \) we get that \([u, g_\beta] = \{0\} \), provided \( \beta \in \Phi \) is long. For any short root \( \alpha \in \Phi \) there is a long root \( \beta \in \Phi \) such that \( \alpha + \beta \in \Phi \); so the component of \( u \) in \( g_{\alpha} \) with respect to (5) is 0, \( \forall \alpha \in \Phi \) (3.6(b)). For any long root \( \beta \in \Phi \) there is a short root \( \alpha \in \Phi \) such that \( \alpha + \beta \in \Phi \) is also short. So the component of \( u \) in \( g_\beta \) is also 0, as otherwise the component of \( AD_H(g)(u) - u \) in \( g_{\alpha + \beta} \) is non-zero (cf. 3.6(c)); here \( g \in G_{a, \alpha}(k_1) \) is an arbitrary non-identity element. So \( u \in \text{Lie}(T) \). As \( H \) is adjoint, all \( \tilde{H}(|\alpha|) \) groups are \( PGL_2 \) groups (cf. 3.8). So from (7) and the \( DT = A_1 \) case we get that \( u \in \text{Lie}(T(|\alpha|)) \), \( \forall \alpha \in \Phi \). As \( H \) is adjoint we have \( \cap_{\alpha \in \Phi} \text{Lie}(T(|\alpha|)) = \{0\} \). So \( u = 0 \). This ends the proof of 1).

To prove 2) we can assume \( H \) is s.c. We assume such a \( V \) does exist and we show that this leads to a contradiction. We first show that we can assume \( k_1 = k \). We write \( H_{k_1} = \tilde{H}_1 \times_{k_1} \tilde{H}_2 \), where \( \tilde{H}_1 \) and \( \tilde{H}_2 \) are normal, s.c. subgroups of \( H_{k_1} \), the adjoint of \( \tilde{H}_1 \) being simple. Let \( \tilde{V}_1 := \text{Ker}(V \otimes k_1 \to \text{Lie}(H_{2, \text{ad}})) \). Any trivial \( \tilde{H}_1(k_1) \)-submodule of \( \tilde{V}_1 \) is contained in \( \text{Lie}(Z(H_{k_1})) \), cf. 1). We get \( \tilde{V}_1 = \tilde{V}_1 \cap \text{Lie}(\tilde{H}_1) + \tilde{V}_1 \cap \text{Lie}(Z(H_{k_1})) \). From this and the inclusion \( \text{Lie}(\tilde{H}_1) \subset \tilde{V}_1 + \text{Lie}(Z(H_{k_1})) \) we get \( \text{Lie}(\tilde{H}_1) = \tilde{V}_1 \cap \text{Lie}(\tilde{H}_1) + \text{Lie}(Z(\tilde{H}_1)) \). So if we have \( \text{Lie}(\tilde{H}_1) = \tilde{V}_1 \cap \text{Lie}(\tilde{H}_1) \) for any such \( \tilde{H}_1 \), then \( V \otimes k_1 \) is \( \text{Lie}(H_{k_1}) \) and this contradicts the fact that \( V \) is a proper \( k \)-vector subspace of \( \text{Lie}(H) \). So we can assume \( k_1 = k \); so \( H = H_1 \). If \( DT \) is not (resp. is) \( D_{2n+2} \), then \( \dim_k(\text{Lie}(Z(H))) \) is 1 (resp. is 2). So the number \( \dim_k(\text{Lie}(H)/V) \) is 1 (resp. is 1 or 2). We can assume \( \dim_k(\text{Lie}(H)/V) = 1 \).

We consider first the case \( p \geq 3 \). So either \( DT = A_{pm-1} \) or \( p = 3 \) and \( DT = E_6 \). If \( l = k \), then \( \dim_k(\text{Lie}(H(|\alpha|)) \cap V) \geq 2 \) and so \( \text{Lie}_k(H(|\alpha|)) \cap V = \text{Lie}_k(H(|\alpha|)) \) (cf. 3.4), \( \forall \alpha \in \Phi \). So \( \text{Lie}(H) = \mathcal{L}_H \subset V \), cf. 3.7 1 and 2). Contradiction. If \( l \neq k \), then \( l = k_2 \). Using similar intersections \( V \cap \text{Lie}_k(H_1(i)) \) we get (cf. Cases 1, 2 and 3 of 3.9) that to prove \( V = \text{Lie}(H) \) we can assume that \( DT = A_2, p = 3 \) and we just need to show
that Lie(H₁(2)) ⊂ V. The H₁(2)(k)-module Lie(H) is semisimple: it is the direct sum of Lie(H₁(2)) with Lie(Z(H)) and with two k-vector spaces of dimension 2 included in g−α₁ ⊕ g−α₂ ⊕ gα₁ ⊕ gα₂. So as V is normalized by H₁(2)(k), by reasons of dimensions we get that V contains these two k-vector spaces and Lie(H₁(2)). Let g ∈ (U₁ \ H₁(2))(k). Let x ∈ V ∩ (g−α₁ ⊕ g−α₂ ⊕ gα₁ ⊕ gα₂ \ {0}). It is easy to see that we can choose g and x such that the component of AD_H(g)(x) ∈ Lie(Z(H)) is non-zero. So as AD_H(g)(x) − x ∈ V we get Lie(Z(H)) ⊂ V. So Lie(H₁(2)) ⊂ V. So V = Lie(H). Contradiction.

We are left with the case p = 2. The case DT ∈ {A₁, E₇} follows from [13, Hauptsatz]. So we can assume DT ∈ {A₂n−1, Bₙ, Cₙ | n ≥ 2} ∪ {Dₙ₊₃ | n ∈ N}. As in the previous paragraph we just need to show that Lie_k(H₁(i)) ∩ V = Lie_k(H₁(i)), for any H₁(i) as in 3.9. The case l = k ≠ F₂ is as above. If l = k = F₂, then 2) follows from loc. cit. So we can assume l ≠ k. So DT is A₂n−1 or Dₙ₊₃ and we are in one of the Cases 1, 4 or 5 of 3.9. In Case 4 all H₁(i)’s are Res_k₂/k of SL₂ groups, cf. 3.9.1; so as l = k ≠ F₂ we get Lie_k(H₁(i)) ∩ V = Lie_k(H₁(i)). The same applies to Cases 1 and 5, except for the situation when l = k₃ and k₁ = k = F₂; so DT = D₄. We now refer to this situation and we use the notations of Case 5 of 3.9 to show that V ∩ Lie(H₁(2)) = Lie(H₁(2)). As H₁(2) is an SL₂ group (cf. 3.9.1), we have ∅, x + h, y + h, x + y) ⊂ V ∩ Lie(H₁(2)), where x, y and h are the non-zero elements of Lie(H₁(2)) such that x ∈ gα₁+α₂, y ∈ g−α₁−α₂, h ∈ Lie(T). If g ∈ Gα₁,α₂(l) ∩ H(k) is a non-identity element, then u := AD_H(g)(x+y)-(x+y) is the non-zero element of gα₁+α₂ or g−α₁−α₂ ∩ Lie(H) (cf. 3.6 c)). Similarly, if u ∈ Gα₁,α₂(l) ∩ H(k) is the non-identity element, then x = AD_H(u)(u)−u. So V ∩ Lie(H₁(2)) contains x. So V ∩ Lie(H₁(2)) = Lie(H₁(2)). This ends the proof of 2).

We prove 3). As Z(H₁sc) = μ²₂ and Z(H₁) = μ₂ we have dim_k₁(Lie(H₁sc)/L₁sc) = 2 and dim_k₁(Lie(H₁)/L₁sc) = 1. So Lie(Z(H₁)) ⊂ L₁sc. So 3) follows from 2) applied to the inverse image V of V₀ in Lie(Hsc). We prove 4). Either DT = A₉_{2n−1} or p = 2 and DT = D₂n−3. We have a direct sum decomposition of H-modules Lie(H) = Lie(Z(H)) ⊕ [Lie(H), Lie(H)] = Lie(Z(H)) ⊕ [Lie(Hsc), Lie(Hsc)]. Hence, cf. [18, 0.13], 4) follows from 3.7.1. To prove 5) we can assume l = k. We have dim_k(Lie(H)/L₁sc) ≤ 2 and the equality can take place only for p = 2, cf. loc. cit. As dim_k(H₁) ∪ 3 and as for p = 2 the group H is not an SL₂ group, the representation of H on Lie(H)/L₁sc is trivial. So H(k) acts trivially on Lie(H)/L₁sc. This ends the proof.

3.11. On exceptional ideals. Until 3.12 we assume that H is adjoint and that either p = 2 and DT ∈ {Bₙ, Cₙ | n ∈ N} ∪ {F₁} or p = 3 and DT = G₂. So l = k₁. It is known that H₁ is the extension to k₁ of a split, adjoint group H₀ over k, cf. [27, Vol. III, p. 410]. Let I (resp. Isc) be the ideal of Lie(H₁) (resp. Lie(H₁sc)) generated by gα, with α ∈ Φ short. We have I = I₀ ⊗ k₁ k₁ (resp. Isc = I₀sc ⊗ k₁ k₁), with I₀ (resp. I₀sc) as the similarly defined ideal of Lie(H₀) (resp. Lie(H₀sc)). As Isc and I are normalized by H₁ we have I = Im(Isc → Lie(H₁)). So I is an a.s. H₁(k₁)-module, cf. [13, Hauptsatz]. Similarly, I₀ is an a.s. H₀(k₀)-module. Let I₁ (resp. I₁sc) be I (resp. Isc) but viewed as a k₁-ideal or as a k-vector subspace of Lie(H₁).

3.11.1. Proposition. The only simple H(k₁)-submodule of Lie(H) is I₁.

Proof: Let I₁₀ ⊗ k₁ k₁ : H₀(k) → GL(I₀ ⊗ k₁ k₁)(k) be the representation defined by the restriction of AD_H to H₀(k) ⊂ H₀(k₁) = H₁(k₁) = H(k). It is semisimple, being a direct
3.11.2. Theorem. 1) If DT is $B_n$ or $C_n$ we assume $n$ is odd and at least 3. Then the H(k)-module $\mathfrak{g}_H/\mathfrak{i}_k$ is simple and non-trivial. Moreover, the short exact sequence of H(k)-modules $0 \rightarrow \mathfrak{i}_k \rightarrow \mathfrak{g}_H \rightarrow \mathfrak{g}_H/\mathfrak{i}_k \rightarrow 0$ does not split.

2) If DT = $C_n$ with $n$ odd and at least 3, then there are no 1 dimensional k-vector subspaces of $\text{Lie}(H)/\mathfrak{i}_k$ normalized by H(k).

3) Let $H_1^{\text{slong}}$ be the analogue of $H_1^{\text{long}}$ of 3.9 but for $H_1^{\text{sc}}$ instead of H_1. We assume DT = $C_n$ (resp. DT = $B_n$). If $n = 1$ we also assume $k \neq \mathbb{F}_2$. Then $[\text{Lie}(H^{\text{sc}}), \text{Lie}(H^{\text{sc}})]$ (resp. $\mathfrak{i}_k + \text{Lie}_k(Z(H_1^{\text{slong}}))$) is the only maximal H(k)-submodule of $\text{Lie}(H^{\text{sc}})$.

Proof: The first part of 1) is argued as in 3.11.1 for $\mathfrak{i}_k$. The second part of 1) follows from 3.11.1. To prove 2) we can assume $k = k_1$. So $H = H_1$ and $I = \mathfrak{i}_k$. The group $H^{\text{long}} := H_1^{\text{long}}$ is the quotient of the group $SL(H) = SL^n$ of the proof of 3.7 by $Z(H^{\text{sc}})$. Let $T^{\text{sc}}$ be the inverse image of $T$ in $H^{\text{sc}}$. We have $\dim_k(\text{Lie}(H^{\text{sc}})/I^{\text{sc}}) = 2n$, cf. [13, Hauptsatz]. This implies Lie($T^{\text{sc}}$) $\subset T^{\text{sc}}$ and so Lie($H)/I$ (resp. $\mathfrak{g}_H/I$) is 2n + 1 (resp. 2n) dimensional. So Lie($H)/I$ is identified naturally with a Lie subalgebra of Lie($H^{\text{long}}$) in such a way that $\mathfrak{g}_H/I$ corresponds to $\mathfrak{g}_{(H^{\text{long}})^{\text{ad}}}$. But Lie($H^{\text{long}}$) has no 1 dimensional k-vector subspace normalized by the subgroup $H^{\text{long}}(k)$ of H(k), cf. 3.10 1). So 2) holds.

We now prove 3). We deal just with the DT = $C_n$ case as the DT = $B_n$ case is entirely the same. The H(k)-module $\text{Lie}(H^{\text{sc}})/[\text{Lie}(H^{\text{sc}}), \text{Lie}(H^{\text{sc}})]$ is simple. Argument: the case $n = 1$ follows from 3.7.1 and the case $n > 1$ is argued in the same way 3.11.1 was, the only difference being that $T_1 \rightarrow \mathbb{G}_m$ defines this time a long root of $\Phi$. Let $M$ be an $\text{H}(k)$-submodule of $\text{Lie}(H^{\text{sc}})$ not contained in $[\text{Lie}(H^{\text{sc}}), \text{Lie}(H^{\text{sc}})]$. It is enough to show that $M = \text{Lie}(H^{\text{sc}})$. We show that the assumption $M \neq \text{Lie}(H^{\text{sc}})$ leads to a contradiction. Let $\bar{M}$ be a maximal $H_1(k_1)$-submodule of $\text{Lie}(H^{\text{sc}})_{k_1} = \text{Lie}(H^{\text{sc}}) \otimes_k k_1$ containing $M \otimes_k k_1$.

From (4) and loc. cit. we get that $[\text{Lie}(H^{\text{sc}}), \text{Lie}(H^{\text{sc}})] \otimes_k k_1 \subset \bar{M}$. As $M$ surjects onto $\text{Lie}(H^{\text{sc}})/[\text{Lie}(H^{\text{sc}}), \text{Lie}(H^{\text{sc}})]$, $\bar{M}$ surjects onto $(\text{Lie}(H^{\text{sc}})/[\text{Lie}(H^{\text{sc}}), \text{Lie}(H^{\text{sc}})]) \otimes_k k_1$. So $\bar{M} = \text{Lie}(H^{\text{sc}})_{k_1}$. Contradiction. So $M = \text{Lie}(H^{\text{sc}})$. This ends the proof.

From 3.7.1 and 3.11.2 3) we get:

3.12. Corollary. In general, the $H^{\text{sc}}(k)$-module $\mathfrak{g}_{H^{\text{ad}}}$ has a unique maximal submodule $\mathcal{J}_{H^{\text{ad}}}$. Moreover, the $H^{\text{sc}}(k)$-module $\mathfrak{g}_{H^{\text{ad}}}/\mathcal{J}_{H^{\text{ad}}}$ is non-trivial.

§4. The $p$-adic context
Let $k = \mathbb{F}_q$ and $s \in \mathbb{N}$. Let $G$ be a reductive group scheme over $W(k)$. Let $\mathbb{A}^1$ be the affine line over $B(k) := W(k)[t]$. The group $G(B(k))$ is endowed with the coarsest topology making all maps $G(B(k)) \to \mathbb{A}^1(B(k)) = B(k)$ induced by morphisms $G_B(k) \to \mathbb{A}^1$ to be continuous. Let $K$ be a closed subgroup of $G(W(k))$.

**4.1. Problem.** Find practical conditions on $G$, $K$ and $p$ which imply $K = G(W(k))$.

This Problem was first considered in the context of $SL_n$ groups over $\mathbb{Z}_p$ in [24, Ch. IV]. After mentioning two general properties (see 4.1.1 and 4.1.2), in 4.1.3 and 4.1.4 we identify a key subpart (question) of this Problem. In 4.2 we list some simple properties of $G$. In 4.3 we include an approach of inductive nature which allows to get information on $\text{Im}(K \to G(W_2(k)))$ from information on semisimple subgroups of $G$. In 4.4 to 4.7 we assume $G$ is semisimple. Section 4.5 solves question 4.1.4 for adjoint and s.c. groups $G$. The computations needed for this are gathered in 4.4. In 4.6 we include a supplement to 4.5 related to the exceptional ideals. In 4.7 we prove 1.3.

We now point out that in general we do need some conditions on $G$, $K$ and $p$.

**4.1.1. Example.** We assume there is an isogeny $f : G^1 \to G$ over $W(k)$ of order $p^n$. So $\text{Ker}(f)$ is a finite, flat subgroup of $Z(G^1)$ of multiplicative type (see [27, Vol. III, 4.1.7 of p. 173]). Being of order $p^n$ it is connected. So the Lie homomorphism $\text{Lie}(G^1) \to \text{Lie}(G_k)$ has non-trivial kernel and so it is not an epimorphism. Moreover, the group $\text{Ker}(f)(k)$ is trivial and so the homomorphism $f(k) : G^1(k) \to G(k)$ is an isomorphism. So the image of the homomorphism $f(W(k)) : G^1(W(k)) \to G(W(k))$ is a proper subgroup of $G(W(k))$ surjecting onto $G(k)$.

We identify $\text{Ker}(G(W_{s+1}(k)) \to G(W_s(k)))$ and $\text{Lie}_p(G_k)$ as $G(k)$-modules. Let

$$L_{s+1} := \text{Im}(K \to G(W_{s+1}(k))) \cap \text{Ker}(G(W_{s+1}(k)) \to G(W_s(k))).$$

**4.1.2. Lemma.** If $p = 2$ (resp. $p > 2$) let $m := 3$ (resp. $m := 2$). If $K$ surjects onto $G(W_m(k))$, then $K = G(W(k))$.

*Proof:* This is just the generalization of [24, Exc. 1 p. IV-27]. We show that in the abstract context of $G$ we can appeal to matrix computations as in loc. cit. Let $\rho_{B(k)} : G_{B(k)} \hookrightarrow GL(W)$ be a faithful representation, with $W$ a finite dimensional $B(k)$-vector space. So $\rho_{B(k)}$ extends to a representation $\rho : G \to GL(L)$, with $L$ a $W(k)$-lattice of $W$ (cf. [19, 10.4 of Part I]). The morphism $\rho$ is a closed embedding, cf. [30, c) of Proposition 3.1.2.1].

So each element $g \in \text{Ker}(G(W(k)) \to G(W_s(k)))$ mod $p^{s+1}$ is congruent to $1_L + p^s x$, where $x \in \text{Lie}(G) \subset \text{End}(L)$. So if $s \geq 3$ (resp. $s \geq 2$), then the image of $g$ in $G(W_{s+1}(k))$ is the $p$-th power of any element of $G(W_{s+1}(k))$ whose image in $G(W_s(k))$ is the same as the one of $1_L + p^{s-1} x$ mod $p^s$. So by induction on $s \geq 3$ (resp. on $s \geq 2$) we get that $L_{s+1} = \text{Ker}(G(W_{s+1}(k)) \to G(W_s(k)))$ and so that $K$ surjects onto $G(W_{s+1}(k))$. As $s$ is arbitrary and $K$ is compact we get $K = G(W(k))$. This ends the proof.

If $p = 2$, then 4.1.2 does not hold in general if we replace $m = 3$ by $m = 2$. Example: $G = \mathbb{G}_{m,n}$ and $q = 2$. See 4.7.1 below for more examples.

**4.1.3. The class $\gamma_G$.** We view $\text{Lie}_p(G_k)$ as a $G(k)$-module via $AD_{G_k}$. Let

$$\gamma_G \in H^2(G(k), \text{Lie}_p(G_k)).$$

16
be the class defining the standard short exact sequence

\[(9) \quad 0 \to \text{Lie}_p(G_k) \to G(W_2(k)) \to G(k) \to 0.\]

Lemma 4.1.2 points out that for the study of 4.1 we need to understand what possibilities for \(L_2\) (resp. for \(L_2\) and \(L_3\)) we have if \(p > 2\) (resp. if \(p = 2\)). The study of \(L_2\) and so of 4.1 is intimately interrelated to the following question.

4.1.4. **Question.** When does the short exact sequence (9) have a section (i.e. when is \(\gamma_G = 0\))?

4.2. **Proposition.** 1) Any maximal torus \(T_k\) of a Borel subgroup \(B_k\) of \(G_k\) lifts to a maximal torus \(T\) of \(G\). Moreover, there is a unique Borel subgroup \(B\) of \(G\) containing \(T\) and having \(B_k\) as its special fibre.

2) The map \(G \to G_k\) establishes a bijection between isomorphism classes of reductive (resp. semisimple) group schemes over \(W(k)\) and isomorphism classes of reductive (resp. semisimple) group schemes over \(k\).

3) If \(G\) is an adjoint (resp. a s.c. semisimple) group, then it is a product of Weil restrictions of a.s. adjoint groups (resp. of s.c. semisimple groups having a.s. adjoints) over Witt rings of finite field extensions of \(k\). So if \(G^{\text{ad}}\) is simple, then \(G\) is of isotypic Dynkin type.

**Proof:** As \(W(k)\) is complete, the first part of 1) follows from [27, Vol. II, p. 47–48]: by induction on \(s \in \mathbb{N}\) we lift \(T_k\) to a maximal torus \(T_{W_s(k)}\) of \(G_{W_s(k)}\). To prove the second part of 1) we can assume that \(T\) is split and so the existence and the uniqueness of \(B\) follow from [27, Vol. III, 5.5.1 and 5.5.5 of Exp. XXII]. See [27, Vol. III, Prop. 1.21 of p. 336–337] for 2). Part 3) follows from 2.3.2 and 2).

4.3. **An inductive approach.** Until end let \(T\) and \(B\) be as in 4.2 1). Let \(G_0k\) be a semisimple subgroup of \(G_k\) normalized by \(T_k\). Let \(T_{0k}\) (resp. \(T_{00k}\)) be the maximal subtorus of \(T_k\) which is a torus of \(G_{0k}\) (resp. which centralizes \(G_{0k}\)). It lifts uniquely to a subtorus \(T_0\) (resp. \(T_{00}\)) of \(T\), cf. [27, Vol. II, p. 47–48].

4.3.1. **Lemma.** The group \(G_0k\) lifts to at most one closed, semisimple subgroup \(G_0\) of \(G\) normalized by \(T\).

**Proof:** Let \(l\) be the smallest field extension of \(k\) such that \(T_l\) is split. So \(T_{W(l)}\) is also split. We consider the Weyl decomposition

\[(10) \quad \text{Lie}(G_{W(l)}) = \text{Lie}(T_{W(l)}) \bigoplus_{\alpha \in \Phi} \Phi(\mathfrak{g}_\alpha)\]

of \(\text{Lie}(G_{W(l)})\) with respect to \(T_{W(l)}\). Warning: whenever \(T\) is split (i.e. \(k = l\)) we drop the left index \(W(l)\). Let \(\Phi_0 := \{\alpha \in \Phi | \mathfrak{g}_\alpha \otimes W(l) \subseteq \text{Lie}(G_0l)\}\). It suffices to prove the Lemma under the extra assumptions that \(T\) is split and that \(T_{00}\) is trivial (otherwise we replace \(G\) by the derived subgroup of the centralizer of \(T_{00}\) in \(G\)). So \(G\) is semisimple and the kernel of the natural homomorphism from \(T_k\) into the identity component \(\text{Aut}^0(G_0k)\) of the group scheme of automorphisms of \(G_0k\) is finite. We have \(\text{Aut}^0(G_0k) = G_0^{\text{ad}}\), cf. [27, Vol. III, Prop. 2.15 of p. 343]. So \(T_k\) and \(G_0k\) have equal ranks. More precisely, \(T_k\) is the
maximal torus of $G_{0k}$ whose image in $\text{Aut}^0(G_{0k})$ is the same as of $T_k$. So $T_0 = T$. As $G_{0k}$ is a semisimple group having $T_k$ as a maximal torus, $\Phi_0$ is symmetric and invariant under the natural action of the group $\text{Aut}_{W(k)}(W(l))$ on $\Phi$. If $G_0$ exists, then it is generated by $T$ and by the $G_0$ subgroups of $G$ normalized by $T$ and whose Lie algebras are $g_\alpha$, with $\alpha \in \Phi_0$. So as $\Phi_0$ is determined uniquely by $G_{0k}$, the Lemma follows.

4.3.2. Proposition. There is a closed, semisimple subgroup $G_0$ of $G$ lifting $G_{0k}$ iff $\Phi_0$ is closed under addition, i.e. we have $\Phi \cap \{\alpha + \beta | \alpha, \beta \in \Phi_0\} \subset \Phi_0$. So $G_0$ exists if for any simple factor $H$ of $G_{0k}$ none of the three conditions of 3.6 b) holds.

Proof: We can assume that $T$ is split and $T_{00}$ is trivial. So $T$ is a maximal torus of $G_0$. If $G_0$ exists, then the direct sum $\text{Lie}(T) \oplus \oplus_{\alpha \in \Phi_0} g_\alpha$ is a Lie subalgebra of $\text{Lie}(G)$. From Chevalley rule in characteristic 0 (see [27, Vol. III, 6.5 of p. 322]) we get that $\Phi_0$ is closed under addition. We assume now that $\Phi_0$ is closed under addition. So $G_0$ exists and is a reductive group scheme, cf. [27, Vol. III, 5.3.4, 5.4.7 and 5.10.1 of Exp. XXII]. So $G_0$ is semisimple as $G_{0k}$ is so. We assume now that for any simple factor $H$ of $G_{0k}$ none of the three conditions of 3.6 b) holds. So as $\text{Lie}(G_{0k})$ is a Lie subalgebra of $\text{Lie}(G_k)$, from the equality part of 3.6 b) we get that $\Phi_0$ is closed under addition. So $G_0$ exists. This ends the proof.

We assume $G_0$ exists. So $\Phi_0 = -\Phi_0$ is closed under addition. Let $TG_0$ be the closed subgroup of $G$ generated by $T$ and $G_0$. It is a reductive group scheme, cf. loc. cit.

4.3.3. Lemma. We have a unique direct sum decomposition $\text{Lie}(G) = \text{Lie}(TG_0) \oplus V_0$ of $TG_0$-modules. The resulting direct sum decomposition $\text{Lie}_{\mathbb{F}_p}(G_k) = \text{Lie}_{\mathbb{F}_p}(TG_{0k}) \oplus V_0/pV_0$ is $G_0(k)$-invariant.

Proof: To check the first part we can assume that $T$ is split and $G$ is semisimple. As $\Phi_0$ is closed under addition, $V_0 := \oplus_{\alpha \in \Phi \setminus \Phi_0} g_\alpha$ is a $\text{Lie}(TG_0)$-module and so a $TG_0$-module. But $V_0$ is the unique supplement of $\text{Lie}(TG_0)$ in $\text{Lie}(G)$ normalized by $T$, cf. (10). So the first part holds. The second part follows from the first part. This ends the proof.

4.3.4. Key Lemma. If $\gamma_{G_0} \neq 0$, then $\gamma_G \neq 0$.

Proof: It is enough to show that if $\gamma_G = 0$, then $\gamma_{G_0} = 0$. Let $\gamma_{G_0,G}^0 \in H^2(G_0(k), \text{Lie}_{\mathbb{F}_p}(G_k))$ be the restriction of $\gamma_G$ via the monomorphism $G_0(k) \hookrightarrow G(k)$. The component of $\gamma_{G_0,G}$ in $H^2(G_0(k), \text{Lie}_{\mathbb{F}_p}(TG_{0k}))$ with respect to the decomposition of the second part of 4.3.3, is the 0 class. So $TG_0(W_2(k))$ has a subgroup $S_0$ mapping isomorphically into $G_0(k)$. The order $o$ of $Z(G_0)(k)$ is prime to $p$. So the image of $S_0$ in $(TG_0)^{\text{ad}}(W_2(k)) = G_0^{\text{ad}}(W_2(k))$ is a subgroup mapping isomorphically into $\text{Im}(G_0(k) \rightarrow G_0^{\text{ad}}(k))$. The index of this last group in $G_0^{\text{ad}}(k)$ is $o$. So as $\text{Lie}_{\mathbb{F}_p}(G_0^{\text{ad}})$ is a $p$-group, we get $\gamma_{G_0} = 0$. This ends the proof.

4.3.5. Lemma. We assume that $q = 2$ and $\gamma_G \neq 0$. Then $\gamma_{G_{W(\mathbb{F}_4)}} \neq 0$.

Proof: Let $a \in \mathbb{F}_4 \setminus k$. We have a direct sum decomposition $\text{Lie}_{\mathbb{F}_2}(G_{\mathbb{F}_4}) = \text{Lie}(G_k) \oplus a\text{Lie}(G_k)$ of $G(k)$-modules. So the restriction of $\gamma_{G_{W(\mathbb{F}_4)}}$ via the monomorphism $G(k) \hookrightarrow G(\mathbb{F}_4)$ is the direct sum of $\gamma_G$ and of a class in $H^2(G(k), a\text{Lie}(G_k))$. So $\gamma_{G_{W(\mathbb{F}_4)}} \neq 0$.

4.4. Computations. Until we assume $G$ is semisimple. For $y \in W_2(k)$ let $\bar{y}$ be its reduction mod $p$. For $\alpha \in k^*$ let $t_\alpha \in W_2(k)$ be the reduction mod $p^2$ of the Teichmüller
lift of $\alpha$. We start answering 4.1.4 for adjoint and s.c. groups. In 4.4.1 we include a general Proposition which among other things allows us to assume that $q \leq 4$. In 4.4.2 (resp. 4.4.3 to 4.4.9) we deal with four (resp. seven) special cases corresponding to $q \leq 4$ and s.c. (resp. a.s. adjoint) groups $G$ of small rank. All of them are minimal cases in the sense that 4.4.1 does not apply to them and moreover for any closed, semisimple subgroup $G_0$ of $G$ normalized by $T$ and different from $G$, we have $\gamma_{G_0} = 0$. Sections 4.3 and 4.4 allow us to list in 4.5 all cases when $\gamma_G = 0$ and $G$ is adjoint or s.c. In other words, all other minimal such cases are “handled” by 4.4.5, 4.3.5 and the available literature (see proof of 4.5).

The identity (resp. zero) $n \times n$ matrix is denoted as $I_n$ (resp. as $0_n$). For $i, j \in \{1, \ldots, n\}$ let $E_{ij}$ be the $n \times n$ matrix whose all entries are 0 except the $ij$ entry which is 1. We list matrices by rows in increasing order (the first row, then the second row, etc.).

**4.4.1. Proposition.** We have $\gamma_G \neq 0$ if any one of the following three conditions hold:

- $q \geq 5$;
- $q = 3$ and $G_k^\text{ad}$ has a non-split simple factor which is not a PGU$_3$ group;
- $q \in \{2, 4\}$ and $G_k^\text{ad}$ has simple factors which do not split over $\mathbb{F}_4$ and are not the $\text{Res}_{W(\mathbb{F}_4)/W(k)}$ of a PGU$_3$ group.

**Proof:** We can assume $G^\text{ad}$ is adjoint and simple. Our hypotheses imply that $G$ has a closed subgroup $G_0$ normalized by $T$ and isogenous to the Weil restriction of an $SL_2$ group over the Witt ring of a finite field with at least 5 elements. Argument: we can assume $G$ is a.s. (cf. 4.2.3) and we can deal just with special fibres (cf. 4.2.2 and 4.3.2); so the statement follows from 3.9 and 4.3.2. So based on 4.3.4, if $p \geq 3$ (resp. $p = 2$) it suffices to show that $\gamma_G \neq 0$ for $q \geq 5$ and $G$ an $SL_2$ (resp. a PGL$_2$) group. If $p \geq 5$ and $y \in M_2(W_2(k))$, then $(I_2 + E_{12} + py)^p = I_2 + pE_{12} \neq I_2$. So $G(W_2(k))$ has no element of order $p$ specializing to a non-identity $k$-valued point of a $G_a$ subgroup of $G$. So $\gamma_G \neq 0$ if $p \geq 5$.

Let now $p = 3$; so $r \geq 2$. For $\alpha \in k^*$ let $X_\alpha := I_2 + t_\alpha E_{12} + 3x_\alpha E_{21} + 3Y_\alpha \in M_2(W(k))$, with $x_\alpha \in W_2(k)$ and with $Y_\alpha \in M_2(W_2(k))$ such that its 21 entry is 0. Let $\beta \in k^*$. The matrix equations $X_\alpha^3 = X_\beta^3 = I_2$ and $X_\alpha X_\beta = X_\beta X_\alpha$ inside $M_2(W_2(k))$, get translated into equations with coefficients in $k$ involving the reductions mod $p$ of $x_\alpha, x_\beta$ and of the entries of $Y_\alpha$ and $Y_\beta$. We need just few of these equations. Identifying the 12 (resp. 11) entries of the equations $X_\alpha^3 = X_\beta^3 = I_2$ (resp. $X_\alpha X_\beta = X_\beta X_\alpha$) we get $1 + \alpha x_\alpha = 1 + \beta x_\beta = 0$ (resp. $\alpha \beta = \beta \alpha$). So $\alpha^2 = \beta^2$. But as $r \geq 2$ there are elements $\alpha, \beta \in k^*$ such that $\alpha^2 \neq \beta^2$. So $\gamma_G \neq 0$ if $G$ is an $SL_2$ group, $p = 3$ and $r \geq 2$.

Let now $p = 2$; so $r \geq 3$. As the homomorphism $GL_2(W_2(k)) \to PGL_2(W_2(k))$ is an epimorphism, we can use again matrix computations. For $\delta \in k$ let

$$X_\delta := I_2 + t_\delta E_{12} + 2Y_\delta,$$

with $Y_\delta \in M_2(W_2(k))$. Let $(a_\delta, b_\delta)$ and $(c_\delta, d_\delta)$ be the rows of $Y_\delta$. As $r \geq 3$ we can choose $\alpha, \beta \in k \setminus \{0, 1\}$ such that $1 + \alpha + \beta \neq 0$ and $\alpha \neq \beta$. The conditions that the images of $X_1, X_\alpha, X_\beta$ in $PGL_2(W_2(k))$ are of order 2 and commute with each other imply that we have the power 2 equations $X_1^2 = (1 + 2v_1)I_2$, $X_\alpha^2 = (1 + 2v_\alpha)I_2$ and $X_\beta^2 = (1 + 2v_\beta)I_2$ and the commuting equations $X_1X_\alpha = (1 + 2v_1\alpha)X_\alpha X_1$, $X_1X_\beta = (1 + 2v_1\beta)X_\beta X_1$ and

19
\( X_\alpha X_\beta = (1 + 2v_{\alpha \beta})X_\beta X_\alpha \), where \( v_1, v_\alpha, v_\beta, v_{1\alpha}, v_{1\beta} \) and \( v_{\alpha \beta} \) belong to \( W_2(k) \). Among the equations we get are the following ones:

\begin{equation}
\bar{a}_1 + \bar{d}_1 = \bar{a}_\alpha + \bar{d}_\alpha = \bar{a}_\beta + \bar{d}_\beta = 1; \tag{11}
\end{equation}

\begin{equation}
\bar{v}_{1\alpha} = \bar{v}_{1\beta} = \bar{v}_{\alpha \beta} = 1; \tag{12}
\end{equation}

\begin{equation}
1 + \bar{c}_\alpha + \alpha \bar{c}_1 = 1 + \bar{c}_\beta + \beta \bar{c}_1 = 1 + \alpha \bar{c}_\beta + \beta \bar{c}_\alpha = 0. \tag{13}
\end{equation}

The equations (11) are obtained by identifying the 12 entries of the power 2 equations (we have \( X_3^2 = I_2 + 2t_3E_{12} + 2t_3[Y_3, E_{12}] \), etc.). The equations (12) are obtained by identifying the 12 entries of the commuting equations and by inserting (11) into the resulting equations. The equations (13) are obtained by identifying the 11 entries of the commuting equations and by inserting (12) into the resulting equations. As \( 1 + \alpha + \beta \neq 0 \), the system (13) of equations in the variables \( c_1, \bar{c}_\alpha \) and \( \bar{c}_\beta \) has no solution in \( k \). So \( \gamma_G \neq 0 \) if \( p = 2 \), \( r \geq 3 \) and \( G \) is a \( PGL_2 \) group. This ends the proof.

4.4.2. Applications. Let \( W_3 := W(\mathbb{F}_4)^2 \). We refer to the proof of 4.4.1 with \( q = 4 \), \( \alpha \in k \setminus \mathbb{F}_2 \) and without mentioning \( \beta \). The equations \( X_1^2 = X_2^2 = I_2 \) and \( X_1X_\alpha = X_\alpha X_1 \) have no solution, cf. the value of \( v_{1\alpha} \) in (12). So the class \( \gamma_4 \in H^2(SL(W_0)(k), \text{Lie}_{\mathbb{F}_2}(GL(W_0/pW_0))) \) defined naturally by the inverse image of \( SL(W_0)(k) \) in \( GL(W_0)(W_2(k)) \) is non-zero. So also \( \gamma_G \neq 0 \) for \( G \) an \( SL_2 \) group over \( W(k) \). For most applications 4.3.4 suffices. We now use \( \gamma_4 \) to get a variant of 4.3.4 in two situations to be used later on.

First we assume that \( q = 2 \) and that \( G \) is an \( SU_4 \) (resp. \( SU_5 \)) group. Let \( G_0 \) be such that \( G_{0W(\mathbb{F}_4)} \) is generated by the \( G_\alpha \) subgroups of \( G_{W(\mathbb{F}_4)} \) normalized by \( T_{W(\mathbb{F}_4)} \) and corresponding as in Case 2 (resp. Case 1) of 3.9 to the roots \( \pm \alpha_1 \) and \( \pm \alpha_3 \) (resp. \( \pm \alpha_1 \) and \( \pm \alpha_4 \)) of \( \Phi \). It is the \( \text{Res}_{W(\mathbb{F}_4)/W(k)} \) of an \( SL_2 \) group \( G_{02} \). The composite monomorphism \( G_0(W(k)) = G_{02}(W(\mathbb{F}_4)) \hookrightarrow G(W(k)) \hookrightarrow G(W(\mathbb{F}_4)) \) corresponds to the direct sum \( W_0 \) of two copies of the standard rank 2 representation \( W_0 \) of \( G_{02} \) with the trivial rank 0 (resp. 1) representation of \( G_0 \) over \( W(\mathbb{F}_4) \). We show that the assumption \( \gamma_G \neq 0 \) leads to a contradiction. The image \( \gamma^1_{G_0, G} \) of \( \gamma_{G_0, G}^0 \) in \( H^2(G_{02}(\mathbb{F}_4), \text{Lie}_{\mathbb{F}_2}(GL(W/2W))) \) is also 0. This image is defined by the natural monomorphisms \( \text{Lie}(G_k) \hookrightarrow \text{Lie}_{\mathbb{F}_2}(G_{k'}) \hookrightarrow \text{Lie}_{\mathbb{F}_2}(GL(W/2W)) \) of \( G_0(k) = G_{02}(\mathbb{F}_4) \)-modules. But the component of \( \gamma^1_{G_0, G} \) in \( H^2(G_{02}(\mathbb{F}_4), (\text{Lie}_{\mathbb{F}_2}(GL(W_0/2W_0)))^4 \) can be identified with four copies of \( \gamma_4 \) and so it is non-zero. Contradiction. So \( \gamma_G \neq 0 \).

Second we assume that \( q = 4 \) and \( G \) is an \( SU_3 \) group. Let \( G_0 \) be the unique \( SL_2 \) subgroup of \( G \) normalized by \( T \). The composite monomorphism \( G_0(W(k)) \hookrightarrow G(W(\mathbb{F}_4)) = SL_3(W(\mathbb{F}_4)) \) corresponds to the direct sum of \( W_0 \) with the trivial rank 2 representation of \( G_0 \) over \( W(k) \). So as in the previous paragraph we get that \( \gamma_G \neq 0 \).

4.4.3. On \( PGL_3 \) over \( \mathbb{Z}_3 \). We assume that \( q = 3 \) and \( G \) is a \( PGL_3 \) group. We show that \( \gamma_G \neq 0 \). We work inside \( M_3(W_2(k)) \). Let \( X_1 := I_3 + E_{12} + 3Y_1 \) and \( X_2 := I_3 + E_{23} + 3Y_2 \), with \( Y_1, Y_2 \in M_3(W_2(k)) \). Let \( X_3 := X_1X_2X_2X_2 \). We have \( X_3 = I_3 + E_{13} + 3Y_3 \), with \( Y_3 \in M_3(W_2(k)) \). It is enough to show that we can not choose \( Y_1 \) and \( Y_2 \) such that \( X_3^3 \) is of the form \( \beta I_3 \), with \( \beta \in \text{Ker} (G_m(W_2(k)) \to G_m(k)) \). As \( X_3^3 = I_3 + 3E_{13} + 3E_{13}Y_3E_{13} \), it is enough to show that the 31 entry of \( X_3 \) (and so also of \( Y_3 \) mod 3) is 0.
Let $=_{31}$ be the equivalence relation on $M_3(W_2(k))$ such that two matrices are in relation $=_{31}$ iff their 31 entries are equal. For $Z \in M_3(W_2(k))$ we have

\[(14) \quad E_{12}Z =_{31} E_{13}Z =_{31} E_{23}Z =_{31} ZE_{12} =_{31} ZE_{23} =_{31} 03.\]

So we compute

\[
X_3 =_{31} (I_3+E_{12}+E_{23}+3E_{12}Y_2+6E_{12}E_{23}+3Y_1+3Y_2+3Y_1E_{23})(I_3+2E_{12}+3E_{12}Y_1+6Y_1+3Y_1E_{12})
\]

\[
(I_3+2E_{23}+3E_{23}Y_2+6Y_2+3Y_2E_{23}) =_{31} (I_3+3Y_1+3Y_2+3Y_1E_{23})(I_3+2E_{12}+3E_{12}Y_1+6Y_1+3Y_1E_{12})
\]

\[
(I_3+3E_{23}Y_2+6Y_2) =_{31} (I_3+3Y_1+3Y_2+3Y_1E_{23})(I_3+3E_{23}Y_2+6Y_2+6E_{12}E_{23}Y_2+3E_{12}Y_2+
\]

\[-+3E_{12}Y_1+6Y_1+3Y_1E_{12}) =_{31} (I_3+3Y_1+3Y_2+3Y_1E_{23})(I_3+3E_{23}Y_2+6Y_2+6E_{13}Y_2+3E_{12}Y_2+
\]

\[-+3E_{12}Y_1+6Y_1) =_{31} I_3+3E_{23}Y_2+6Y_2+6E_{13}Y_2+3E_{12}Y_2+3E_{12}Y_1+6Y_1+3Y_1+3Y_1E_{23}.
\]

So $X_3 =_{31} I_3 + 9Y_1 + 9Y_2 =_{31} I_3$ and so $\gamma_G \neq 0$.

4.4.4. On $\text{PGSp}_4$ over $\mathbb{Z}_3$. We assume that $q = 3$ and $G$ is a $\text{PGSp}_4$ group. To show that $\gamma_G \neq 0$ we follow the pattern of 4.4.3. We choose the alternating form $\psi$ on $W(k)^4$ such that for the standard $W(k)$-basis $\{e_1, ..., e_4\}$ of $W(k)^4$ and for $i, j \in \{1, ..., 4\}$, $j > i$, we have $\psi(e_i, e_j) = 1$ if $j - i = 2$ and $\psi(e_i, e_j) = 0$ if $j - i \neq 2$. We take $X_1 := I_4+E_{12}E_{43}+3Y_1, X_2 := I_3-E_{14}E_{23}+3Y_2, \text{ with } Y_1 \text{ and } Y_2$ as in 4.4.3. Defining $X_3$ as in 4.4.3, we have $X_3 = I_3+E_{13}+3Y_3$. As in 4.4.3 it is enough to show that the 31 entry of $X_3$ is 0. But the computations of 4.4.3 apply, once we remark that similar to (14) for $Z \in M_4(W_2(k))$ and for the equivalence relation $=_{31}$ on $M_4(W_2(k))$ defined as in 4.4.3, we have $(E_{12}E_{43})Z =_{31} (-E_{14}E_{23})Z =_{31} Z E_{12}E_{43} =_{31} Z(-E_{14}E_{23}) =_{31} 04.$

So $\gamma_G \neq 0$. Warning: here $X_1, X_2$ and $X_3$ are “related” to the roots $\alpha_1, \alpha_1 + \alpha_2$ and respectively $2\alpha_1 + \alpha_2$ of the $C_2$ Dynkin type; the similar computations for the roots $\alpha_1, \alpha_2$ and $\alpha_1 + \alpha_2$ do not imply that $\gamma_G \neq 0$.

4.4.5. On $\text{PGSp}_4$ over $\mathbb{Z}_2$. We assume that $q = 2$ and $G$ is a $\text{PGSp}_4$ group. Let $\psi$ be as in 4.4.4 (but with $q = 2$). Let $GSp_4 := GSp(W(k)^4, \psi).$ We show that $\gamma_G \neq 0$. The group $GSp_4(k)$ is the symmetric group $S_6$ (see [1]). It can be checked that $G(W(k))$ has a subgroup isomorphic to $A_6$ and so in this case the following computations are more involved. Let $I := [\text{Lie}(G_k^{sc}), \text{Lie}(G_k^{sc})].$ We view it as a $GSp_4(k)$-module and so also as a normal subgroup of $GSp_4(W_2(k)).$ As $GSp_4(k) = G(k)$ and as Lie($Z(GSp_4)_k) \subset I,$ it is enough to show that the short exact sequence

\[(15) \quad 0 \rightarrow \text{Lie}(GSp_4)_k / I \rightarrow GSp_4(W_2(k)) / I \rightarrow GSp_4(k) \rightarrow 0\]

does not have a section. We show that the assumption that (15) has a section leads to a contradiction. Let $S$ be a subgroup of $GSp_4(W_2(k)) / I$ mapping isomorphically into $GSp_4(k)$.

If $U \in GSp_4(W_2(k))$ let $\bar{U}$ be the reduction mod $p$ of $U$ and let $\tilde{U}$ be the image of $U$ in $GSp_4(W_2(k)) / I.$ Below $x_i, y_i, z_i, w_i \in W_2(k)$, where $i \in \{1, ..., 5\}.$ Let $X \in GSp_4(W_2(k))$ whose rows are $(1 + 2x_5, 0, 2x_1, 0), (0, 1 + 2x_5, 0, 1 + 2x_2 + 2x_5), (2x_3, 0, 1, 0)$ and $(0, 2x_4, 0, 1 +
Let \( Y \in GSp_4(W_2(k)) \) whose rows are \((1 + 2y_5 0 1 + 2y_1 + 2y_5 0), (0 1 + 2y_5 0 2y_2), (2y_3 0 1 + 2y_3 0) \) and \((0 2y_4 0 1)\). Let \( Z \in GSp_4(W_2(k)) \) whose rows are \((1 + 2z_5 0 2z_1 2z_1), (-1 + 2z_5 1 + 2z_5 0 2z_2), (2z_3 0 1 1) \) and \((2z_4 2z_4 0)\). Let \( W \in GSp_4(W_2(k)) \) whose rows are \((1 + 2w_5 0 2w_1 1 + 2w_5 0), (0 1 + 2w_5 1 + 2w_5 2w_2), (2w_3 0 1 2w_3) \) and \((0 2w_4 2w_4 1)\). The subgroup of \( GSp_4(k) \) generated by \( X, Y, Z \) and \( W \) is a 2-Sylow subgroup.

The subgroup of \( \text{Lie}(GSp_4(k)) \) generated by \( E_{13}, E_{24}, E_{31}, E_{42} \) and \( E_{11} + E_{22} \) is a direct supplement of \( I \). So for \( U \in \{X, Y, Z, W, \bar{U}\} \), \( \bar{U} \) is the general element of \( GSp_4(W_2(k))/I \) lifting \( U \). We now choose \( \bar{X}, \bar{Y}, \bar{Z} \) and \( \bar{W} \) to generate a 2-Sylow subgroup of \( S \). So \( X^2, Y^2, Z^2, W^2, (ZW)^2, (WX)^2, (XY)^2 \) and \((\bar{ZX})^2 \) are all identity elements.

Both \( \bar{Z} \) and \( \bar{W} \) are associated to short roots of the \( C_2 \) Lie type. So from (6) we get that they belong to the commutator subgroup of \( G(k) \). So also \( \bar{Z} \) and \( \bar{W} \) belong to the commutator subgroup of \( S \) and so to \( G^{sc}(W_2(k))/I \). So \( 2z_5 = 2w_5 = 0 \). Looking at the 31 entry of \( Z^2 \) and so of \( \bar{Z}^2 \) we get \( 2z_4 = 0 \). Looking at the 24 entry of \( W^2 \) we get \( 2w_3 = 0 \). Let \( S \) be a subgroup of \( G(W_2(k)) \) mapping isomorphically into \( G(k) \). So \( S \) is a simple group, cf. [11, 2.2.7]. The group \( G(W_2(k))/\mathcal{L}_{G_h} \) is isomorphic to \( S \times Z/2Z \), cf. 3.10 5). As there is no epimorphism from \( S \) onto \( Z/2Z \), the images of \( S \) and \( G^{sc}(W_2(k)) \) in \( G(W_2(k))/\mathcal{L}_{G_h} \) coincide. So the inverse image \( S_1 \) of \( S \) in \( G^{sc}(W_2(k)) \) surjects onto \( S \). Let \( SP := \{(i, j)|i, j \in \{1, 2, 3, 4\} \text{ and } i \neq j\} \). For \((i, j) \in SP \) let \( u_{ij} \in S_1 \) be such that its image in \( S \) is the reduction mod \( p \) of \( I_4 + E_{ij} \). We have \( u_{ij}^2 = \pm I_4 \). If \( s \in \{1, 2, 3, 4\} \setminus \{i, j\} \), then the ss component of \( u_{ij}^2 \) is 1. So \( u_{ij}^2 = I_4 \). If \((i, j), (s, t) \in SP \) and \( v \in \{1, 2, 3, 4\} \setminus \{i, j, s, t\} \), then similarly by identifying the vv components we get \( u_{ij}u_{st} = u_{st}u_{ij} \).

Let \( X_1 := u_{23}u_{34}u_{23}u_{34} = I_4 + E_{24} + 2Y_1 \) and \( X_2 := u_{13} = I_4 + E_{13} + 2Y_2 \), where \( Y_1, Y_2 \in M_2(W_2(k)) \). We know that \( X_1, X_2 \in SL_4(W(k)^4) \), \( X_1^2 = X_2^2 = I_4 \) and \( X_1X_2 = X_2X_1 \). We take \( x_{y} \in W_2(k) \), where \( x \in \{a, b, ..., p\} \) and \( y \in \{1, 2\} \). As \( X_1^2 = I_4 + 2E_{24} + 2[Y_1, E_{24}] = I_4 \), the rows of \( X_1 \) are \((1 + 2a_1 0 2c_1 2d_1), (2c_1 1 + 2f_1 2g_1 1 + 2h_1), (2i_1 0 1 + 2k_1 2l_1) \) and \((0 0 0 3 + 2f_1)\). Similarly, as \( X_2^2 = I_4 \) we get that the rows of \( X_2 \) are \((1 + 2a_2 2b_2 1 + 2c_2 2d_2), (0 1 + 2f_2 2g_2 2h_2), (0 0 3 + 2a_3 0) \) and \((0 2a_2 2o_2 1 + 2p_2)\). Identifying the 24 entries of \( X_2X_1 \) and \( X_1X_2 \) we get \((1 + 2h_1)(1 + 2f_2) + 2h_2(3 + 2f_1) = (1 + 2f_1)2h_2 + (1 + 2h_1)(1 + 2p_2) \). So \( 2f_2 = 2p_2 \). As \( 2f_2 = 2p_2 \) we have \( \det(X_2) = (1 + 2a_2)(3 + 2a_2)(1 + 2f_2)(1 + 2p_2) = 3 \). So \( X_2 \notin G^{sc}(W_2(k)) \). Contradiction. So \( \gamma_G \neq 0 \).
4.4.8. On $PGU_3$ over $\mathbb{Z}_2$. We show that $\gamma_G = 0$ if $q = 2$ and $G^{\text{ad}}$ is a $PGU_3$ group. We can assume $G$ is s.c. Let $U$ be the unipotent radical of $B$. As $G_{W(\mathbb{F}_4)}$ splits, it is the $SL$ group of $M := W(\mathbb{F}_4)^3$ and $T_{W(\mathbb{F}_4)}$ splits. So we can choose a $W(\mathbb{F}_4)$-basis $B = \{e_1, e_2, e_3\}$ of $M$ such that the $W(\mathbb{F}_4)$-spans $< e_1 >$ and $< e_1, e_2 >$ are normalized by $B_{W(\mathbb{F}_4)}$ and the $W(\mathbb{F}_4)$-spans $< e_1 >$, $< e_2 >$ and $< e_3 >$ are normalized by $T_{W(\mathbb{F}_4)}$. In what follows the matrices of elements of $G(W(\mathbb{F}_4))$ are computed with respect to $B$. We can assume that $B$ is such that the automorphism of $U_{W(\mathbb{F}_4)}$ defined by the non-identity element $\tau$ of $\text{Gal}(\mathbb{F}_4/k) = Gal(B(\mathbb{F}_4)/B(k))$ takes $A \in U(W(\mathbb{F}_4))$ whose rows are $(1 x y)$, $(0 1 z)$ and $(0 0 1)$ into the element $\tau(A) \in U(W(\mathbb{F}_4))$ whose rows are $(1 \tau(z) \tau(xz - y))$, $(0 1 \tau(x))$ and $(0 0 1)$. We have $U(W(3)) := \{A \in U(W(\mathbb{F}_4))|\tau(A) = A\}$. As the $G(k)$-submodule $\text{Lie}(G_k)$ of $\text{Lie}_{\mathbb{F}_2}(G_{\mathbb{F}_4})$ has a direct supplement and as $U(k)$ is a Sylow 2-subgroup of $G(k)$, it is enough to show that the pull back of the standard short exact sequence $0 \rightarrow \text{Lie}_{\mathbb{F}_2}(G_{\mathbb{F}_4}) \rightarrow G(W_2(\mathbb{F}_4)) \rightarrow G(\mathbb{F}_4) \rightarrow 0$ via the composite monomorphism $U(k) \hookrightarrow G(k) \hookrightarrow G(\mathbb{F}_4)$ has a section. Even better, as the $G(k)$-submodule $\text{Lie}_{\mathbb{F}_2}(G_{\mathbb{F}_4})$ of $\text{Lie}_{\mathbb{F}_2}(GL(M/2M))$ has $\text{Lie}_{\mathbb{F}_2}(Z(GL(M/2M)))$ as a direct supplement, it is enough to deal with the image of the resulting short exact sequence via the monomorphism $\text{Lie}_{\mathbb{F}_2}(G_{\mathbb{F}_4}) \hookrightarrow \text{Lie}_{\mathbb{F}_2}(GL(M/2M))$.

Let $a \in \mathbb{F}_4$ be such that $\mathbb{F}_4 = \{0, 1, a, a + 1\}$. Let $\widetilde{t}_a$ and $\widetilde{t}_{a+1}$ be elements of $W_2(\mathbb{F}_4)$ lifting $a$ and respectively $a + 1$ and such that we have the following two identities

$$\begin{equation}
\widetilde{t}_{a+1} + \widetilde{t}_a = \widetilde{t}_a \widetilde{t}_{a+1} = 1.
\end{equation}$$

Let $X_1$, $X_2$ and $X_3 \in GL(M)(W_2(\mathbb{F}_4))$ be defined as follows. The rows of $X_1$ (resp. of $X_2$) are $(1 \; 1 \; \widetilde{t}_a)$, $(2 + 2 \; 2 \; \widetilde{t}_a)$ and $(0 \; 0 \; 1)$ (resp. are $(1 \; \widetilde{t}_a \; \widetilde{t}_a)$, $(2 + 2 \; \widetilde{t}_a \; 1 \; \widetilde{t}_{a+1})$ and $(0 \; 0 \; 3 \; 2 \; \widetilde{t}_a)$). The rows of $X_3$ are $(3 \; 2 \; \widetilde{t}_a \; 3 \; 2 \; \widetilde{t}_a)$, $(0 \; 3 \; 0)$ and $(0 \; 0 \; 1)$. It is easy to see that $X_3^2 = I_3$ and $X_1^2 = X_2^2 = (X_1 X_2)^2 = X_3$. We include here just the only two computations which appeal to (16). The 13 entry of $X_2^2$ is $\widetilde{t}_a + \widetilde{t}_a \widetilde{t}_{a+1} + 3 \widetilde{t}_a + 2 \widetilde{t}_a = 1 + (2 \; \widetilde{t}_a)^2 = 3 + 2 \; \widetilde{t}_a$. The rows of $X_1 X_2$ are $(3 + 2 \; \widetilde{t}_a \; 1 + \widetilde{t}_a 2 \; \widetilde{t}_a^2 + \widetilde{t}_{a+1})$, $(2 \; \widetilde{t}_a \; 3 \; 3 + 3 \; \widetilde{t}_{a+1})$ and $(0 \; 0 \; 3 \; 2 \; \widetilde{t}_a)$. So the 13 entry of $(X_1 X_2)^2$ is $2(3 + 2 \; \widetilde{t}_a)[2 \; \widetilde{t}_a^2 + 2 \; \widetilde{t}_{a+1}] + (1 + \widetilde{t}_a)(3 + 3 \; \widetilde{t}_{a+1}) = \widetilde{t}_{a+1} + 3 + 3 \; \widetilde{t}_a + 3 \; \widetilde{t}_a \; \widetilde{t}_{a+1} = 3 + 2 \; \widetilde{t}_a$.

So the subgroup of $GL(W)(W_2(\mathbb{F}_4))$ generated by $X_1$, $X_2$, $X_3$ is a quaternion group of order 8. Its reduction mod 2 is $U(k)$. So $\gamma_G = 0$ for the present case.

4.4.9. On $PGU_3$ over $\mathbb{Z}_3$. We assume that $q = 3$ and $G^{\text{ad}}$ is a $PGU_3$ group. We show that $\gamma_G \neq 0$. Let $M$, $B$, $U$, $\tau$ be as in 4.4.8 but with $q = 3$ and with $\mathbb{F}_4$ replaced by $\mathbb{F}_9$. Let $a \in \mathbb{F}_9$ be such that $a^2 + 1 = 0$. Let $X_1$, $X_2$, $X_3$, $Y_1$, $Y_2$, $Y_3 \in M_3(W(\mathbb{F}_9))$ be such that $X_1 = I_3 + t_a E_{12} + t_{a+1} E_{23} + 2 E_{13} + 3 Y_1$, $X_2 = I_3 + t_{a+1} E_{12} + t_{a+1} E_{23} + E_{13} + 3 Y_2$ and $X_3 = X_1 X_2 X_1^2 X_3^2 = I_3 + t_{a} E_{13} + 3 Y_3$. The reductions mod 3 of $X_1$, $X_2$ and $X_3$ define elements of $U(k)$. As in 4.4.3 and 4.4.4 we get that the 31 entry of $Y_3$ mod 3 is 0. So $X_3^3$ is not a scalar multiple of $I_3$. So $\gamma_G \neq 0$.

4.5. Theorem. We assume $G$ is adjoint. Then $\gamma_G = 0$ (resp. $\gamma_G^{\text{sc}} = 0$) iff $q \leq 4$ (resp. $q \leq 3$) and $G$ (resp. $G^{\text{sc}}$) is a product of adjoint (resp. s.c.) groups of the following type:

- **F2** $\text{PGL}_2$, $\text{PGL}_3$, $\text{Res}_{W(\mathbb{F}_4)/\mathbb{Z}_2}\text{PGL}_2$, $\text{PGU}_3$, $\text{PGU}_4$ and split of $G_2$ Dynkin type (resp. $SL_3$, $SU_3$ and split of $G_2$ Dynkin type) if $q = 2$;
- **F3** $\text{PGL}_2$ (resp. $SL_2$) if $q = 3$;
- **F4** $\text{PGL}_2$ if $q = 4$.
Proof: If $\gamma_{G^{sc}} = 0$, then $\gamma_G = 0$. It suffices to prove the Theorem for a simple, adjoint group $G$, cf. 4.2.3) and 4.3.4. We first assume $\gamma_G = 0$. Let $k_1$ be such that $G = \text{Res}_{W(k_1)/W(k)}G_1$, where $G_1$ is an a.s. adjoint group (cf. 4.2.3)). Let $T_1$ (resp. $B_1$) be the maximal torus (resp. Borel subgroup) of $G_1$ such that $\text{Res}_{W(k_1)/W(k)}T_1 = T$ (resp. $\text{Res}_{W(k_1)/W(k)}B_1 = B$), cf. 2.3.1. Let $G_{01k_1}$ be a closed, semisimple subgroup of $G_{1k_1}$ normalized by $T_{1k_1}$ and obtained as in 3.9 (i.e. working with respect to $B_{1k_1}$). It lifts to a semisimple subgroup $G_{01}$ of $G_1$ normalized by $T_1$, cf. 4.3.2 and 3.9 (2). So $G_0 := \text{Res}_{W(k_1)/W(k)}G_{01}$ is a closed, semisimple subgroup of $G$ (cf. 2.3.1) normalized by $T$.

As $\gamma_G = 0$ we have $p \leq 3$, cf. 4.4.1. If $p = 3$, then $q = 3$ and $G$ is split (cf. 4.4.1 and 4.4.9). We show that $G$ has only a finite number of $\gamma$ over $\gamma$ in $W(k_1)/W(k)$. If it is not, then we can take $G_0$ to be split of $A_2$ or $C_2$ Dynkin type (even if $G$ is of $G_2$ Dynkin type). So $\gamma_G^{ad} \neq 0$ (cf. 4.4.3 and 4.4.4) and so $\gamma_G \neq 0$ (cf. 4.3.4). We reached a contradiction. So $G$ is a $PGL_2$ groups. But $\gamma_G = 0$ and $\gamma_G^{sc} = 0$ for such a $G$, cf. [24, p. IV-27-28].

Until end of the proof we assume $p = 2$. As $\gamma_G = 0$, $k$ is a subfield of $\mathbb{F}_4$ and either $G$ splits over $W(\mathbb{F}_4)$ or is the $W_4(k)$ of a $PGU_3$ group (cf. 4.4.1).

i) The group $G_0^{ad}$ can not be a $PGL_4$ group over $\mathbb{Z}_2$, cf. 4.4.6 and 4.3.4. So also $G_0^{ad}$ can not be the $W_4(\mathbb{F}_4)/W(k)$ of a $PGL_4$ group, cf. 4.3.4 and 4.3.5. So $G$ is not the Weil restriction of a split group of $E_6$ Dynkin type, cf. 3.9 and 4.3.2. Also $G$ is not a $PGU_{n+7}$ group, a $PGL_{n+3}$ group, or of isotypic $D_n$, $E_7$ or $E_8$ Dynkin type (cf. 3.9 and 4.3.2).

ii) The group $G_0^{ad}$ can not be a $PGSp_4$ group over $\mathbb{Z}_2$, cf. 4.4.5 and 4.3.4. So also $G_0^{ad}$ can not be the $W_4(\mathbb{F}_4)/W(k)$ of a $PGSp_4$ group, cf. 4.3.4 and 4.3.5. So $G$ is not of isotypic $B_n$, $C_n$ with $n \geq 2$, $G_2$ with $n \geq 2$ or $F_4$ Dynkin type (cf. 3.9 and 4.3.2).

iii) The group $G_0^{ad}$ can not be the $W_4(\mathbb{F}_4)/\mathbb{Z}_2$ of a $PGL_3$ group, cf. 4.4.7 and 4.3.4. So $G$ is not a $PGU_6$ or a $PGU_7$ group, cf. 3.9 and 4.3.2.

iv) The group $G_0$ can not be a $PGU_6$ group over $\mathbb{Z}_2$, cf. iii) and 4.3.4. So $G$ is not non-split of $E_6$ Dynkin type, cf. Case 3 of 3.9 and 4.3.2.

v) The groups $G_0$ and $G$ are not the $W_4(\mathbb{F}_4)/W(k)$ of a $PGL_3$ group, cf. 4.4.7 and 4.3.4. So $G$ is also not $W_4(\mathbb{F}_4)/W(k)$ of a split group of $G_2$ Dynkin type, cf. 3.9 and 4.3.2.

vi) The group $G$ is not a $PGU_5$ group or the $W_4(\mathbb{F}_4)/W(k)$ of a $PGU_3$ group, cf. 4.4.2 and the fact that $c(G)$ is odd.

So $G$ is among the adjoint groups listed in F2 and F4. If $q = 2$ and $G$ is split of $G_2$ Dynkin type, then the epimorphism $G(W(k)) \to G(k)$ has a right inverse (cf. [12, Sect. 4]). It is well known that this also holds if $q = 2$ (resp. $q = 4$) and $G$ is a $PGL_n$ group with $n \in \{2, 3\}$ (resp. $n = 2$). Let $SO_n^-$ be the semisimple group over $\mathbb{Z}_2$ which is an isogeny cover of degree 2 of a $PGU_4$ group. The epimorphism $SO_n^-(\mathbb{Z}_2) \to SO_n^-(\mathbb{F}_2)$ has a right inverse (see [1, p. 26]; $SO_6^-(\mathbb{F}_2)$ is the Weyl group $W_{E_6}$). Based on all these and 4.4.8, we get that F2 and F4 list all cases when $p = 2$, $G$ is a simple adjoint group and $\gamma_G = 0$.

The passage from adjoint groups to s.c. groups for $p = 2$ is easy. It is well known that $\gamma_{G^{sc}} \neq 0$ if $q \in \{2, 4\}$ and $G$ is a $PGL_2$ group (see 4.4.2 for $q = 4$). So $\gamma_{G^{sc}} = 0$ iff $q = 2$ and $G^{sc}$ is a product of s.c. semisimple groups having a.s. adjoints of $A_2$ or $G_2$ Dynkin type (cf. 4.4.2 for the exclusion of $SU_4$ groups for $q = 2$). This ends the proof.

4.5.1. Remark. We have $\gamma_G = 0$ if $q = 2$ and $G$ is the quotient of $W_4(\mathbb{F}_4)/\mathbb{Z}_2 SL_2$ by $\mu_2$ (this can be deduced either by just adapting (11) to (13) or from [1, p. 26] via the
standard embedding of $G$ into $SO^-_G$).

4.6. A supplement to 4.5. Until 4.7 we assume $G$ is split, s.c. and $G^{\text{ad}}$ is a.s. We also assume that either $p = 3$ and $G$ is of $G_2$ Dynkin type or $p = 2$ and $G$ is of $B_n$, $C_n$ or $F_4$ Dynkin type. If $G$ is an $SL_2$ group, then we also assume that $q \geq 8$. Let $I$ be the maximal $G(k)$-submodule of $\text{Lie}_{\mathbb{Q}_p}(G_k)$, cf. 3.11.2 1) and 3). We refer to (10). Let $G_0$ be the closed, semisimple subgroup of $G$ generated by the $\mathbb{G}_a$ subgroups of $G$ having $\mathfrak{g}_\alpha$'s as their Lie algebras, with $\alpha \in \Phi$ a long root (see 3.9 and 4.3.2).

4.6.1. Lemma. We have identifications of Lie algebras

$$(17) \quad \text{Lie}(G_k)/I = \text{Lie}(G_{0k})/\text{Lie}(Z(G_{0k})) = \Sigma_{G^{\text{ad}}_{0k}}.$$  

Proof: The intersection $I \cap \text{Lie}(G_{0k})$ is fixed by any maximal torus of $G_{0k}$ and so also by $G_{0k}$. So $I \cap \text{Lie}(G_{0k}) \subseteq \text{Lie}(Z(G_{0k}))$, cf. 3.10 1) applied to $G^{\text{ad}}_{0k}$. A simple computation involving $\dim_k(I)$ (see [13, p. 409]) and $\dim_k(G_{0k})$ (see 3.9 for the Lie type of $G^{\text{ad}}_{0k}$) shows that $\dim_k(I \cap \text{Lie}(G_{0k})) \geq \dim_k(Z(G^{\text{sc}}_{0k}))$. Example: if $G$ is of $C_n$ Dynkin type, then $\dim_k(I \cap \text{Lie}(G_{0k})) \geq \dim_k(T_k) = n = \dim_k(Z(G^{\text{sc}}_{0k}))$. As $\dim_k(Z(G^{\text{sc}}_{0k})) \geq \dim_k(Z(G_{0k}))$, we get $I \cap \text{Lie}(G_{0k}) = \text{Lie}(Z(G_{0k}))$ and $\dim_k(Z(G_{0k})) = \dim_k(Z(G^{\text{sc}}_{0k}))$. So $\text{Lie}(G_{0k})/\text{Lie}(Z(G_{0k})) = \text{Lie}(G^{\text{sc}}_{0k})/\text{Lie}(Z(G^{\text{sc}}_{0k})) = G^{\text{ad}}_{0k}$ (cf. also proof of 3.10 4) for when $p = 2$ and $G_{0k}$ is of $A_3$ or $D_{2n+3}$ Lie type). By reason of dimensions the monomorphism $\text{Lie}(G_{0k})/\text{Lie}(Z(G_{0k})) = \text{Lie}(G_{0k})/I \cap \text{Lie}(G_{0k}) \hookrightarrow \text{Lie}(G_{0k})/I$ is onto. So (17) holds. This ends the proof.

4.6.2. Proposition. We view $I$ as a normal subgroup of $G(W_2(k))$. Then the short exact sequence

$$(18) \quad 0 \rightarrow \text{Lie}_{\mathbb{Q}_p}(G_k)/I \rightarrow G(W_2(k))/I \rightarrow G(k) \rightarrow 0$$  

does not have a section.

Proof: Its pull back via the monomorphism $G_0(k) \hookrightarrow G(k)$ is the short exact sequence

$$(19) \quad 0 \rightarrow \text{Lie}_{\mathbb{Q}_p}(G_{0k})/\text{Lie}_{\mathbb{Q}_p}(Z(G_{0k})) \rightarrow \text{Im}(G_0(W_2(k)) \rightarrow G^{\text{ad}}_0(W_2(k))) \rightarrow G_0(k) \rightarrow 0,$$  

cf. (17). We show that the assumption (18) has a section leads to a contradiction. So (19) has a section too. So $\gamma_{G^{\text{ad}}_0} = 0$. So $q \leq 4$ and $G$ is of isotropic $C_n$ Dynkin type with $n \geq 2$ (cf. 4.5; see 3.9 for the structure of $G^{\text{ad}}_{0k}$). To reach a contradiction we can assume $q = 2$ (to be compared with 4.3.5 and the proof of 3.11.1). Using a standard monomorphism $\mathbb{G}_m^{n-2} \times \text{Sp}_4 \hookrightarrow \text{Sp}_{2n}$ over $W(k)$, similar to 4.3.3 and 4.3.4 we argue that we can assume $n = 2$. But as (15) does not have a section (see 4.4.5), (18) does not have a section for $q = 2$ and $G$ an $\text{Sp}_4$ group. Contradiction. This ends the proof.

4.7. Proof of 1.3. We now prove 1.3. Let $G$ and $K$ be as in 1.3. Let $L_2$ and $L_3$ be as before 4.1.2. As $K$ surjects onto $G(k)$, both $L_2$ and $L_3$ are $G(k)$-submodules of $\text{Lie}_{\mathbb{Q}_p}(G_k)$. Let $K^{\text{sc}}$ be the inverse image of $K$ in $G^{\text{sc}}(W(k))$. We have $\text{Im}(K^{\text{sc}} \rightarrow G(k)) = \text{Im}(G^{\text{sc}}_0(k) \rightarrow G(k))$. So $\text{Im}(K^{\text{sc}} \rightarrow G^{\text{sc}}_0(k))$ is a normal subgroup of $G^{\text{sc}}_0(k)$ of index dividing the order of $Z(G^{\text{sc}}_0(k))$ and so prime to $p$. So as $G^{\text{sc}}_0(k)$ is generated by elements of order $p$ (cf. [11,
2.2.6 (f)), \( K^{sc} \) surjects onto \( G^{sc}(k) \). As \( \gcd(p, c(G)) = 1 \), the isogeny \( G^{sc} \to G \) is étale. So \( \text{Ker}(G^{sc}(W(k)) \to G^{sc}(k)) = \text{Ker}(G(W(k)) \to G(k)) \). So if \( K^{sc} = G^{sc}(W(k)) \), then \( K = G(W(k)) \). So to show that \( K = G(W(k)) \) we can assume \( G = G^{sc} \). Let \( G^0 \) be a direct factor of \( G \) such that \( G^0^{ad} \) is simple.

We first assume that 1.3 e) holds; so \( p = 2 \) and \( L_2 = \text{Lie}_{\mathbb{F}_2}(G_k) \). To prove that \( K = G(W(k)) \) it suffices to show that \( L_3 \) is \( \text{Lie}_{\mathbb{F}_2}(G_k) = \text{Ker}(G(W_3(k)) \to G(W_2(k))) \), cf. 4.1.2. We consider a faithful representation \( \rho : G \to GL(L) \) as in the proof of 4.1.2. Let \( x, y \in \text{Lie}(G) \). Let \( h_x \) and \( h_y \in K \) be such that we have \( h_x = 1_L + 2x + 4\bar{x} \) and \( h_y = 1_L + 2y + 4\bar{y} \), where \( \bar{x}, \bar{y} \in \text{End}(L) \). We compute

\[
(20) \quad h_x h_y h_x^{-1} h_y^{-1} = (1_L + 2x + 4\bar{x} + 2y + 4xy + 4\bar{y})(1_L + 2x + 4\bar{x})^{-1}(1_L + 2y + 4\bar{y})^{-1} =
= (1_L + 2y + 4xy - 4yx + 4\bar{y})(1_L + 2y + 4\bar{y})^{-1} = 1_L + 4(xy - yx).
\]

So \( L_3 \) contains \( [\text{Lie}_{\mathbb{F}_2}(G_k), \text{Lie}_{\mathbb{F}_2}(G_k)] \). So \( L_3 \) contains also \( \text{Lie}_{\mathbb{F}_2}(G_k^0) \) except when \( G^0 \) is of isotypic \( C_\alpha \) Dynkin type, cf. 3.7.2. Let now \( G^0 \) be of isotypic \( C_\alpha \) Dynkin type. Let \( k_1 \) be such that \( G^0 = \text{Res}_{W(k_1)/W(k_3)} G_1 \), with \( G_1 \) a s.c. semisimple group over \( W(k_1) \) whose adjoint is a.s. (cf. 4.2.3)). The group \( G^0_1 \) is split.

Let \( G_1^{10} \) be a closed \( SL_2 \) subgroup of \( G_1^0 \) normalized by a maximal split torus \( T_1^0 \) of \( G_1^0 \). We identify \( G_1^{10}(W_3(k_1)) \) with a subgroup of \( GL_2(W_3(k_1)) \). We denote also by \( t_\alpha \) the reduction mod 8 = \( p^3 \) of the Teichmüller lift of \( \alpha \in k_1 \setminus \{0\} \). Let \( j_2 \) (resp. \( j_4 \)) be the image in \( GL(L)(W_3(k)) \) of \( \rho \) \( \pi \) of the element \( I_2 + 2t_\alpha E_1 \) (resp. \( I_2 + 4t_\alpha E_2 \)) of \( G_1^{10}(W_3(k)) \). For \( y \in \text{End}(L/8L) \) we have \( (j_2 + 4y)^2 = j_4 \neq I_2 \). So \( L_3 \) contains the Lie algebra of any \( \mathbb{G}_a \) subgroup of \( G_1^{10}(k_1) \) normalized by \( T_1^0(k_1) \). So the intersection \( L_3 \cap \text{Lie}_{\mathbb{F}_2}(G_1^{10}(k_1)) \) is an arbitrary direct factor of \( G(k) \)-submodule of \( \text{Im}(\text{Lie}_{\mathbb{F}_2}(G_k) \to \text{Lie}_{\mathbb{F}_2}(G_k)) \) (see 3.12). So \( \tilde{L}_2 = \text{Im}(\text{Lie}_{\mathbb{F}_2}(G_k) \to \text{Lie}_{\mathbb{F}_2}(G_k)) \). So \( L_2 = \text{Lie}_{\mathbb{F}_2}(G_k) \), cf. 3.10.2). So \( K = G(W(k)) \).

We now treat the general case; so \( G^{ad} \) is not simple. We know that \( K \) projects onto \( G^0(W(k)) \) and that \( L_2 \) projects onto \( \text{Lie}_{\mathbb{F}_2}(G^0) \). If \( x \in L_2 \) and \( g \in G^0(k) \), then \( AD_{G^0}(g)(x) - x \in L_2 \cap \text{Lie}_{\mathbb{F}_2}(G^0) \). From this and 3.12 we get that there is a \( G(k) \)-submodule of \( L_2 \cap \text{Lie}_{\mathbb{F}_2}(G^0) \) surjecting onto \( \text{Im}(\text{Lie}_{\mathbb{F}_2}(G^0) \to \text{Lie}_{\mathbb{F}_2}(G^0)) \). From 3.10 2) applied to \( \text{Res}_{k/\mathbb{F}_2}(G^0_k) \) we get that the direct factor \( \text{Lie}_{\mathbb{F}_2}(G^0_k) \) of \( \text{Lie}_{\mathbb{F}_2}(G_k) \) is contained in \( L_2 \). As \( G^0 \) is an arbitrary direct factor of \( G \) having a simple adjoint, we get that \( L_2 \) contains \( \text{Lie}_{\mathbb{F}_2}(G_k) \) and so it is \( \text{Lie}_{\mathbb{F}_2}(G_k) \). This ends the proof of 1.3.

**4.7.1. Remark.** We assume that \( p = 2 \), \( G = G^{ad} \) and \( 2|c(G) \). Then there are proper, closed subgroups of \( (W(k)) \) surjecting onto \( G(W_2(k)) \). One checks this using a short exact sequence \( 0 \to \text{Lie}_{\mathbb{F}_2}(G_k)/[\text{Lie}_{\mathbb{F}_2}(G_k), \text{Lie}_{\mathbb{F}_2}(G_k)] \to G(W_3(k))/I \to G(W_2(k)) \to 0 \).
and the fact that for any torus $\tilde{T}$ over $W(k)$ the square homomorphism $\text{Ker}(\tilde{T}(W(k)) \to \tilde{T}(k)) \to \text{Ker}(\tilde{T}(W_3(k)) \to \tilde{T}(W_2(k)))$ is not surjective.

Acknowledgements. We would like to thank Serre for two e-mail replies; the first one was the starting point of 4.4 and 4.5 which prove all its expectations and the second one led to a better presentation and to precise references in the proof of 4.5. We would also like to thank D. Ulmer and the referee for many useful comments and U of Arizona for good conditions for the writing of the paper.

References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of finite groups, Oxford Univ. Press, Eynsham, 1985
[2] S. Bosch, W. Lüttkebohmert, M. Raynaud, Néron models, Springer–Verlag, 1990
[3] A. Borel, Properties and linear representations of Chevalley groups, LNM 131, Springer–Verlag, 1970, pp. 1–55
[4] A. Borel, Linear algebraic groups, Grad. Text Math. 126, Springer–Verlag, 1991
[5] N. Bourbaki, Groupes et algèbres de Lie, Chapitre 4–6, Hermann, 1968
[6] N. Bourbaki, Groupes et algèbres de Lie, Chapitre 7–8, Hermann, 1975
[7] F. Bruhat and J. Tits, Groupes réductifs sur un corps local, Publ. Math. I.H.E.S. 60, 5–184 (1984)
[8] A. Borel and J. Tits, Homomorphismes “abstraits” de groupes algébriques simples, Ann. of Math. (3) 97, 499–571 (1973)
[9] C. W. Curtis, Representations of Lie algebras of classical Lie type with applications to linear groups, J. Math. Mech. 9, 307–326 (1960)
[10] C. W. Curtis, On projective representations of certain finite groups, Proc. Am. Math. Soc. 11, 852–860, A. M. S. 1960
[11] D. Gorenstein, R. Lyons and R. Soloman, The classification of the finite simple groups, Number 3, Math. Surv. and Monog., Vol. 40, A. M. S. 1994
[12] B. Gross, Groups over $\mathbb{Z}$, Inv. Math. 124, 263–279 (1996)
[13] G. Hiss, Die adjungierten Darstellungen der Chevalley-Gruppen, Arch. Math. 42, 408–416 (1982)
[14] G. M. D. Hogeweij, Almost-classical Lie algebras, Indag. Math. 44, I. 441–452, II. 453–460 (1982)
[15] J. E. Humphreys, Introduction to Lie algebras and representation theory, Grad. Texts Math. 9, Springer–Verlag, 1975
[16] J. E. Humphreys, Linear algebraic groups, Grad. Texts Math. 21, Springer–Verlag, 1975
[17] J. E. Humphreys, Algebraic groups and modular Lie algebras, Mem. Am. Math. Soc., No. 71, A. M. S., 1976
[18] J. E. Humphreys, Conjugacy classes in semisimple algebraic groups, Math. Surv. and Monog., Vol. 43, A. M. S., 1995
[19] J. C. Jantzen, Representations of algebraic groups, Academic Press 1987
[20] R. Pink, Compact subgroups of linear algebraic groups, J. of Algebra 206, 438–504 (1998)
[21] K. Ribet, On l-adic representations attached to modular forms, Inv. Math., Vol. 28, 245–275 (1975)
[22] K. Ribet, Images of semistable Galois representations, Pac. J. of Math., Vol. 3 (3), 277–297 (1997)
[23] J. -P. Serre, Propriétés galoisiennes des points d’ordre fini des courbes elliptiques, Inv. Math. 15, 259–331 (1972)
[24] J. -P. Serre, Abelian l-adic representations and elliptic curves, Addison–Wesley Publ. Co. 1989
[25] J. -P. Serre, Galois Cohomology, Springer–Verlag 1997
[26] J. -P. Serre, Collected papers, Vol. IV, Springer–Verlag 2000
[27] M. Demazure, A. Grothendieck, et al. Schemes en groupes, Vol. I to III, LNM 151–153, Springer–Verlag 1970
[28] R. Steinberg, Representations of algebraic groups, Nagoya Math. J. 22, 33–56 (1963)
[29] J. Tits, Classification of algebraic semisimple groups, Proc. Sympos. Pure Math., Vol. 9, 33–62, A. M. S. 1966
[30] A. Vasiu, Integral canonical models of Shimura varieties of preabelian type, Asian J. Math., Vol. 3 (2), 401–518 (1999)