Predictive power of renormalisation group flows: a comparison

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Abstract

We study a proper-time renormalisation group, which is based on an operator cut-off regularisation of the one-loop effective action. The predictive power of this approach is constrained because the flow is not an exact one. We compare it to the Exact Renormalisation Group, which is based on a momentum regulator in the Wilsonian sense. In contrast to the former, the latter provides an exact flow. To leading order in a derivative expansion, an explicit map from the exact to the proper-time renormalisation group is established. The opposite map does not exist in general. We discuss various implications of these findings, in particular in view of the predictive power of the proper-time renormalisation group. As an application, we compute critical exponents for $O(N)$-symmetric scalar theories at the Wilson-Fisher fixed point in $3d$ from both formalisms.

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1. Introduction

Renormalisation group methods are an essential ingredient in the study of non-perturbative problems in continuum and lattice formulations of quantum field theory (QFT). A well-established renormalisation group, based on the Wilsonian idea of integrating-out infinitesimal momentum shells, is known as the Exact Renormalisation Group (ERG) [1, 2, 3, 4] (for recent reviews, see Refs. [6] for scalar and Ref. [7] for gauge theories). The fully integrated ERG flow of the effective action covers all momenta. Thus the endpoint of an ERG flow is the full quantum effective action. An important advantage of this formalism is its flexibility, when it comes to approximations or truncations. This makes it an interesting tool when, due to the complexity of the problem at hand, approximations are unavoidable. For non-perturbative effects at strong coupling or for large correlation lengths, such an approach is essentially unavoidable. The fact that this flow is derived from first principles—in combination with the control of approximate solutions [8, 9, 10, 11]—is at the root of reliable approximations and the predictive power of the formalism.

A somewhat similar renormalisation group has been proposed in Refs. [12] and [13] for scalar theories and gauge theories, respectively. It is based on a regularised Schwinger proper-time representation [14, 15] of the one-loop effective action [16]. A variation with respect to the scale leads to a flow equation. The final proper-time renormalisation group (PTRG) equation is obtained by a one-loop improvement. On the technical level, the PTRG is as flexible as the ERG, and approximations schemes or expansions used for the latter can immediately be taken over to the former.

As opposed to the ERG, the conceptual understanding of the PTRG is less well developed. There exists no first principle derivation of the PTRG flow. Furthermore, the solution to a PTRG flow is not the full quantum effective action [17]. Hence, the PTRG is at best an approximation to an exact flow. Currently, it is not known what kind of approximation it represents. Neither is it known how to relate systematic approximations of the PTRG to systematic approximations of the full quantum effective action: at present, the PTRG lacks predictive power.

Here, we take a practitioners point of view in order to shed some light on these issues. We provide a partial control for PTRG flows by comparing them to ERG flows to leading order of the derivative expansion. In this approximation we discuss the map which connects PTRG and ERG. We show that the map exists from ERG to PTRG. We also show that the opposite map does not exist in general, not even to leading order in the derivative expansion. Within a restricted set of PTRG regulator functions, these findings provide the required link between approximations to the PTRG and approximations to the physical theory. We also discuss the content of those PTRG flows, which cannot be mapped on ERG flows. As an application of these results, we compute critical exponents for $3d O(N)$-symmetric scalar theories for both the ERG and the PTRG. For specific regulators, our results compare very well with both experiment and results obtained by other methods. We discuss the dependence of the results on the regularisation, and the applicability of an optimisation condition [8, 10, 11] or a minimum sensitivity condition [18, 19, 20, 11].
2. Exact renormalisation group

Let us start with a brief discussion of the main conceptual ingredients of the ERG. The basic idea leading to an exact flow equation is the step-by-step integrating-out of fluctuations within a path integral formulation of quantum field theory in a Wilsonian sense \[1\]. This can be seen as a continuum version of the earlier block spin actions introduced by Kadanoff. A path-integral formulation of these ideas is due to Polchinski \[2\], while formulations for the effective action have been given in Refs. \[3, 4, 5\]. For recent reviews, see Refs. \[6, 7\].

Within the standard ERG, the integrating-out of fluctuations is achieved by a momentum regulator \(R_k(q^2)\). It modifies the effective propagator of the fields, and depends on a fiducial infra-red scale \(k\). The ERG flow of the effective action \(\Gamma_k\) with respect to \(k\) is given by

\[
\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left[ \frac{\Gamma_k^{(2)}[\phi]}{\Gamma_k[\phi]} + R_k \right] \partial_t R_k
\]

for bosonic fields \(\phi\). Here, the trace denotes a sum over all momenta and indices, \(t = \ln k\) and \(\Gamma_k^{(2)}[\phi]\) stands for the second derivative of \(\Gamma_k\) w.r.t. the field \(\phi\).

The function \(R_k(q^2)\) has to satisfy some constraints in order to provide an infra-red regulator for the effective propagator, and to ensure that the flow (1) interpolates precisely between an initial (classical) action in the UV and the full quantum effective action in the IR. The necessary conditions on \(R_k\) are summarised as

\[
\lim_{q^2/k^2 \rightarrow 0} R_k(q^2) > 0 \quad (2)
\]

\[
\lim_{k^2/q^2 \rightarrow 0} R_k(q^2) = 0 \quad (3)
\]

\[
\lim_{k \rightarrow \Lambda} R_k(q^2) \rightarrow \infty \quad (4)
\]

Eq. (2) guarantees that \(R_k\) provides an IR regularisation, because mass-less modes are effectively cut-off. The second constraint (3) ensures that the regulator is removed in the IR limit \(k \rightarrow 0\). The condition (4) ensures that the correct initial condition is reached for \(\lim_{k \rightarrow \Lambda} \Gamma_k = S_{\Lambda}\), where \(\Lambda\) defines the initial (UV) scale. From now on we will use \(\Lambda = \infty\). Then the regulator \(R_k\) can be rewritten in terms of a dimensionless function \(r(y)\) as

\[
R_k(q^2) = q^2 r(q^2/k^2)
\]

The constraints (2) – (4) on \(R_k\) are sufficient to guarantee that the flow (1) interpolates between the initial UV action and the full quantum effective action \(\Gamma\) for \(k = 0\). In addition, these conditions imply that the insertion \(\partial_t R_k\) in Eq. (1) is peaked as a function of momenta about \(q^2 \approx k^2\). For large momenta, \(\partial_t R_k\) decays rapidly. Thus contributions to the flow from UV modes are suppressed. For small momenta, \(\partial_t R_k\) either diverges, or it approaches a constant limit. This structure explains why the flow (1) integrates-out only a narrow momentum shell around the scale \(k\).

For an explicit computation of the IR effective action \(\Gamma\) based on the ERG approach the specification of the field content, the initial condition \(\Gamma_{\Lambda}\) and the choice of a particular regulator is required. As soon as it comes to (unavoidable) approximations it is very important
that the integrated flow approaches the full quantum effective action. Although approximate solutions to ERG flows may depend spuriously on the IR regulator [8], it has been clarified recently that convergence properties of approximate solutions towards the physical theory are controlled by the IR regulator, and improved for specific optimised choices [9, 10, 11] (see also Ref. [19]). This guarantees that any systematic truncation of the ERG provides an approximation to the full quantum effective action. In particular, systematic truncations can be improved to higher order, leading to better approximations of the physical theory.

3. Proper-time renormalisation group

As opposed to the derivation of an ERG sketched above there is no first principle derivation of the PTRG. Instead, the PTRG follows as an one-loop improvement based on a proper-time regularisation of the one-loop effective action. The heuristic derivation of the flow starts with the well-known expression

$$\Gamma^{1-\text{loop}}[\phi] = S_{\text{cl}}[\phi] + \frac{1}{2} \text{Tr} \ln \frac{\delta^2 S_{\text{cl}}[\phi]}{\delta \phi \delta \phi}$$  \hspace{1cm} (6)$$

for the one-loop effective action $\Gamma^{1-\text{loop}}[\phi]$. The trace in Eq. (6) is ill-defined and requires an UV regularisation and, in the case of massless modes, also an IR regularisation. Oleszczuk proposed an UV regularisation by means of a Schwinger proper-time representation of the trace [16],

$$\Gamma^{1-\text{loop}}[\phi; \Lambda] = S_{\text{cl}} - \frac{1}{2} \int ds f(\Lambda, s) \text{Tr} \exp \left( -s S^{(2)}_{\text{cl}} \right).$$  \hspace{1cm} (7)$$

The regulator function $f(\Lambda, s)$ provides an UV cut-off $\Lambda$ if $\lim_{s \to 0} f(\Lambda, s) = 0$. It implies that $\Gamma^{1-\text{loop}}[\phi; \Lambda]$ depends on the scale $\Lambda$. Sending the UV scale to $\infty$ should reduce Eq. (7) to the standard Schwinger proper-time integral. This happens for the boundary condition $f(\Lambda \to \infty, s) = 1$. A new ingredient has then been added by Liao [12], who noticed that Eq. (7) can be turned into a simple flow equation by also adding an IR scale $k$, replacing $f(\Lambda, s) \to f_k(\Lambda, s)$. Introducing another scale parameter turns $\Gamma^{1-\text{loop}}[\phi; \Lambda]$ into a $k$-dependent functional. A flow equation for some functional $\Gamma_k$ with respect to the infra-red scale $k$ is given by

$$\partial_t \Gamma_k[\phi] = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \left( \partial_t f_k(\Lambda, s) \right) \text{Tr} \exp \left( -s \Gamma^{(2)}_k \right)$$  \hspace{1cm} (8)$$

Here, the classical action has been replaced by the scale-dependent effective action on the right-hand side of Eq. (8). This is the philosophy of an one-loop improvement. Note that in Eq. (8) only the explicit scale dependence due to the regulator term is considered. A total $k$-derivative would require in addition that the scale dependence of $\Gamma^{(2)}_k$ is taken into account. It is of course tempting to identify $\Gamma_k$ as implicitly defined in Eq. (8) with the full scale dependent effective action. However, from its derivation we only can conclude that it is a RG improved approximation to the latter.
The PT regulator has to satisfy some constraints similar to those imposed on $R_k$ \[12, 21\]. We require that
\[
\lim_{s \to \infty} f_{k \neq 0}(\Lambda, s) = 0 \quad (9)
\]
\[
\lim_{k \to \Lambda} f_k(\Lambda, s) = 0 \quad (10)
\]
\[
\lim_{\Lambda \to \infty} f_{k=0}(\Lambda, s) = 1 \quad (11)
\]
The condition (9) ensures that the IR region is suppressed. Notice that the limits $k \to 0$ and $s \to \infty$ do not commute, since $\lim_{s \to \infty} f_{k=0}(\Lambda, s) = 1$. The condition (10) implies that all initial traces regularised with $f_k$ vanish at this point. Thus, at one-loop, we have a trivial initial condition $\Gamma_{k=\Lambda} = S_{cl}$. Finally, the condition (11) ensures that the proper-time regularisation reduces to the usual Schwinger proper time regularisation for $k = 0$. This implies that $\Gamma_{k=0}[\phi]_{1\text{-loop}} = \Gamma_{1\text{-loop}}[\phi; \Lambda]$. Beyond one-loop, $\Gamma_k$ is not given as a closed expression but only by the integrated flow and the initial condition $\Gamma_{\Lambda}$. Thus Eqs. (10) and (11) do not imply $\Gamma_{k=\Lambda} = S_{cl}$ and $\Gamma_{k=0}[\phi] = \Gamma[\phi]$, the full effective action, as opposed to the ERG case. We introduce the dimensionless function $f_{PT}(x)$ as
\[
f_k(\Lambda, s) = f_{PT}(s\Lambda^2) - f_{PT}(sk^2) \quad (12)
\]
for later convenience. It obeys $\partial_t f_k(\Lambda, s) = -\partial_t f_{PT}(sk^2)$ and $\partial_t f_{PT}(x) = 2xf_{PT}'(x)$. In this parametrisation, only the function $f_{PT}(sk^2)$ appears in the flow (8). The conditions (3) – (11) translate into $f_{PT}(x \to \infty) = 1$ and $f_{PT}(x \to 0) = 0$.

The flow (8) describes, at least, a resummation of a subset of perturbative diagrams. Let us denote with $\Gamma_{PT}$ the solution for $k \to 0$ of the flow (8). Since a derivation from first principles is missing, it is not known how $\Gamma_{PT}$ is related to the physical theory, e.g. the full quantum effective action $\Gamma$. Furthermore, $\Gamma_{PT}$ may depend on the specific regulator chosen. These facts are a severe limitation of the PTRG formalism; they restrict the predictive power, because no control over the link of $\Gamma_{PT}$ to the physical theory is yet available.

In order to provide some insight into these issues we investigate the relation of PTRG to ERG. Within the ERG approach it is known by construction that the limit $\lim_{k \to 0} \Gamma_k = \Gamma_{ERG}$ of the ERG flow coincides with the full quantum effective action $\Gamma_{ERG} \equiv \Gamma$. We exploit this piece of information to provide a better understanding of the physical content of the PTRG.

### 4. Derivative expansion

From now on, we restrict ourselves to the study of Eqs. (1) and (8) within the leading order of the derivative expansion. The derivative expansion is the most commonly used systematic approximation scheme. It is most useful for theories which, except for possible modifications due to anomalous dimensions, retain a standard kinetic term in the IR limit. The derivative expansion has been used very successfully for the computation of critical exponents or equations of states for scalar theories.
For later use, we consider an $O(N)$ symmetric scalar field theory in $d$ dimensions and to leading order in the derivative expansion. The model has an effective action

$$\Gamma_k = \int d^dx \left[ U_k(\bar{\rho}) + \frac{1}{2} \partial_\mu \phi^a \partial_\mu \phi_a + \mathcal{O}(\partial^4) \right]$$

(13)

and $\bar{\rho} = \frac{1}{2} \phi^a \phi_a$. We introduce dimensionless variables,

$$u(\rho) = U_k(\bar{\rho}) k^{-d}$$

(14)

$$\rho = \bar{\rho} k^{2-d}.$$  

(15)

Inserting the Ansatz (13) into either Eq. (1) or Eq. (8) leads to

$$\partial_t u + du - (d-2) \rho u' = 2 v_d (N-1) \ell(u') + 2 v_d \ell(u' + 2 \rho u''),$$

(16)

and $v_d^{-1} = 2^{d+1} \pi^{d/2} \Gamma\left(\frac{d}{2}\right)$. All information regarding the regulator function is contained in the function $\ell(\omega)$. For the ERG case, it is given by

$$\ell_{\text{ERG}}(\omega) = \frac{1}{2} \int_0^\infty dy y^{\frac{d}{2}} \frac{\partial_r(y)}{y(1+r) + w}.$$  

(17)

Here, we have used Eq. (11), and $y = q^2/k^2$ is the dimensionless momentum variable, and $\partial_r(y) = -2y r'(y)$. For the PTRG case, $\ell(\omega)$ is given by

$$\ell_{\text{PT}}(\omega) = \frac{1}{2} \Gamma\left(\frac{d}{2}\right) \int_0^\infty \frac{dx}{x^{1+a/2}} \left[ \partial_r f_{\text{PT}}(x) \right] \exp(-x \omega),$$

(18)

the PTRG counterpart of $\ell_{\text{ERG}}(\omega)$. Here, we have used Eq. (12), and $x \equiv sk^2$ is the dimensionless integration variable of the proper-time integral. Notice that the leading order flow (16) is structurally the same for ERG and PTRG. This suggests that the formalisms can be mapped onto each other to leading order in the derivative expansion.

5. From ERG to PTRG

Now we establish the map from ERG $\rightarrow$ PTRG to leading order in the derivative expansion. Specifically, for every given regulator function $R_k(q^2)$, we provide a corresponding function $f_k(s)$. The construction is the following: we assume that a regulator $R_k(q^2)$ has been given. Therefore, the function $\ell_{\text{ERG}}(\omega)$ is known, either (i) explicitly as an analytic function of $\omega$, or (ii) as an expansion in powers of $1/(1+\omega)$, or (iii) as an expansion in powers of $\omega$. Equating the function $\ell_{\text{ERG}}(\omega)$ with $\ell_{\text{PT}}(\omega)$ provides the map from the momentum regulator $R_k(q^2)$ to the proper-time regulator $f_k(\Lambda, s)$.

(i) We begin with the simplest case, where $\ell_{\text{ERG}}(\omega)$ is given as an explicit analytic function of $\omega$. In this case one can read off from (18) that $\ell_{\text{ERG}}(\omega)$ is the Laplace transform of the function

$$g(x) = \Gamma\left(\frac{d}{2}\right) x^{-d/2} f_{\text{PT}}'(x).$$

(19)

Then it suffices to perform the inverse Laplace transformation in order to obtain the corresponding $f_{\text{PT}}'(x)$ and hence $f_{\text{PT}}(x)$. Let us give three examples. We first consider the sharp
cut-off $R_{\text{sharp}} = \lim_{c \to \infty} c\Theta(k^2 - q^2)$. It leads to the threshold function $\ell_{\text{sharp}}(\omega) = -\ln(1 + \omega)$. Considering $\partial_\omega \ell_{\text{sharp}}(\omega)$ (because the Laplace transform of $\ln(1 + \omega)$ does not exist) gives $g_{\text{sharp}}(x) = -x^{-1}e^{-x}$ and the differential equation

$$\partial_t f_{\text{sharp}}(x) = -\frac{2}{\Gamma\left(\frac{d}{2}\right)} x^{d/2} e^{-x},$$

(20)
in agreement with the corresponding result given in Ref. [12]. The second example concerns the power-like regulator $R_{\text{power}}(q^2) = q^2(k^2/q^2)^b$ for $b = 2$ in $d = 3$ dimensions [22]. It leads to $\ell_{\text{power}}(\omega) = 2\pi/\sqrt{2 + \omega}$. The corresponding PT regulator (for $d = 3$) is found as

$$\partial_t f_{\text{power}}(x) = 8x^2 e^{-2x},$$

(21)
in agreement with the result given in Ref. [10].

(ii) Let us assume that $\ell_{\text{ERG}}(\omega)$ is given as the particular series

$$\ell_{\text{ERG}}(\omega) = \frac{2}{d} \frac{1}{1 + \omega} \sum_{n=0}^{\infty} b_n \left(\frac{-1}{1 + \omega}\right)^n$$

(23)

with the expansion coefficients

$$b_n = \int_0^{\infty} dy y^{d/2+1} [-r'(y)][y(1 + r) - 1]^n.$$  

(24)

Such series exist for all ERG regulators that decay more than power-like for large momenta and have a mass-like limit for small momenta [10]. In particular it follows from Eq. (17) or Eq. (24) that $b_0 \neq 0$. The basic set of functions are the monomials $(1 + \omega)^{-n}$ for which their Laplace transform is known; hence

$$g(x) = \frac{2}{d} \sum_{n=0}^{\infty} b_n (-x)^n e^{-x}.$$  

(25)

and

$$\partial_t f_{\text{PT}}(x) = \frac{4}{d} \frac{1}{\Gamma\left(\frac{d}{2}\right)} \sum_{n=0}^{\infty} b_n (-)^n x^{n+\frac{d}{2}+2} e^{-x}.$$  

(26)

Finally, this gives $f_{\text{PT}}(x)$ as an alternating sum

$$f_{\text{PT}}(x) = \frac{4}{d} \frac{1}{\Gamma\left(\frac{d}{2}\right)} \sum_{n=0}^{\infty} b_n (-)^n \Gamma(n + \frac{d}{2} + 2, x)$$

with $b_0 \neq 0$ (27)

where $\Gamma(a, x) = \int_0^{x} dt t^{a-1} e^{-t}$ is the incomplete $\Gamma$-function. Notice that Eq. (20) reduces to Eq. (22) for $b_0 = 1$ and all other $b_n = 0$. 

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(iii) The third case concerns those ERG regulators for which the functions $\ell_{\text{ERG}}(\omega)$ are not expandable as in Eq. (23). A well-known example is given by the sharp cut-off, or power-like regulators $R_k \sim q^2(k^2/q^2)^b$. However, the expansion

$$\ell_{\text{ERG}}(\omega) = \frac{2}{\pi} \sum_{n=0}^{\infty} a_n (-\omega)^n \quad (28)$$

with the expansion coefficients

$$a_n = \int_0^\infty dy y^{d/2+1} \frac{-r'(y)}{[y(1+r)]^n}. \quad (29)$$

always exists for arbitrary regulators \([9]\). In this case the Laplace transform is of no help because the space of functions spanned by \(\{\omega^n\}\) is not Laplace transformable. Still, we can compute all moments of the function $g(x)$, which are given by

$$a_n = \frac{d}{2n! \Gamma(\frac{d}{2})} \int_0^\infty dx^n g(x) \quad (30)$$

The reconstruction problem (i.e. finding $g(x)$ and hence $f_{\text{PT}}(x)$ from the set $\{a_n\}$) is very similar to the reconstruction of structure functions within perturbative QCD. Let us assume that we know the set $\{a_n \leq N\}$ numerically up to an order $N$. This implies that infinitely many functions $g(x)$ can be found which all have the first $n$ moments (30). The map can be made unique only under additional assumptions regarding the small-$x$ and the large-$x$ behaviour of $g(x)$, provided by the large-$n$ behaviour of $a_n$. Here, we make the ad hoc assumption that $g(x) = P_N(x)x^\alpha \exp(-x)$ with $P_N$ a polynomial of order $N$. Then all coefficients of $P_N = \sum_{m=0}^N \frac{1}{m!}p_m x^m$ are determined as $p = M^{-1}a$, with the numerical $\left(\begin{array}{c} N+1 \\ N+1 \end{array}\right)$ matrix $M$ given by

$$(M)_{nm} = \frac{d}{2} \frac{\Gamma(m+n+\alpha+1)}{\Gamma(\frac{d}{2}) n! m!}. \quad (31)$$

The quality of the result would then depend on the assumptions concerning the asymptotic behaviour (like the free parameter $\alpha$). From the explicit solutions (20) and (22) we deduce that $\alpha = -1$ for the sharp cut-off, and $\alpha = 0$ for the optimal regulator. Deriving the correct value for $\alpha$ from the large-$n$ behaviour of $a_n$ seems to be the most difficult part of the reconstruction problem. Notice also that the expansion (23) is much more powerful than the expansion (28), because the corresponding radius of convergence is larger \([10]\).

6. From PTRG to ERG

Here, we show that the inverse map from PTRG to ERG does not exist in general, not even to leading order in the derivative expansion. We will not discuss the specific conditions required for $f_k(s)$ such that this map may exist. Consider the function

$$f_{\text{PT}}(x; m) = \frac{\Gamma(m, x)}{\Gamma(m)} \quad (32)$$
with
\[ \partial_t f_{PT}(x; m) = \frac{2}{\Gamma(m)} x^m e^{-x}. \]  

This set of functions is particularly important since it is the standard set used for analytical considerations \[12, 13\] or numerical applications \[20, 24, 25, 26\] of the PTRG. For \( m \geq \frac{d}{2} \), this function satisfies the basic requirements imposed on \( f_k(\Lambda, s) \). It leads to the function
\[ \ell_{PT}(\omega) = \frac{\Gamma(d/2) \Gamma(m - d/2)}{\Gamma(m)(1 + \omega)^{m - d/2}} \]  

with the asymptotic behaviour \( \ell_{PT}(\omega \to \infty) \sim \omega^{-m+d/2} \) for arbitrary \( m \). However, it follows from Eq. \(17\), that \( \ell_{ERG}(\omega \to \infty) > C \omega^{-1} \), where \( C > 0 \) depends on the regulator \( R_k \). This is easily deduced from Eq. \(17\), if \( r(y) \) is a monotonously decreasing function in \( y \) (and \( \ln k \)). It holds as well for oscillating regulators \( r \), as long as \( y(1 + r) \) is strictly positive. Hence, the asymptotic decay of \( \ell_{ERG}(\omega) \) is at most \( \sim \omega^{-1} \) or weaker. Therefore, it is impossible to find the ERG analogue to Eq. \(32\) once \( m > \frac{d}{2} + 1 \). It is interesting to note that the optimised regulator of Ref. \[10\] corresponds precisely to the boundary case \( m = \frac{d}{2} + 1 \).

Let us now consider \( m > \frac{d}{2} + 1 \). The scale derivative \( \partial_t f_{PT}(x; m) \) as given in Eq. \(13\) decays for both \( x \to 0 \) and \( x \to \infty \) and has its maximum at \( x = m \). Thus the PTRG flow with a regulator \(33\) has an effective IR scale \( k_{\text{eff}} \propto k/m^{1/2} \), since \( x \) (roughly) corresponds to \( k^2/q^2 \). This has already been noted in Ref. \[26\]. It is understood that the factor \( m^{1/2} \) only takes care of the leading \( m \)-dependence relevant for the limit of large \( m \gg \frac{d}{2} + 1 \). Hence, the flow, with increasing \( m \) and fixed \( k \), is peaked at increasingly small momenta with decreasing width. The above explanations make it clear why such a flow, for sufficiently large \( m \), cannot be mapped onto ERG flows at an IR scale \( k \). Moreover the integrated flow (starting at a fixed initial scale \( \Lambda \) and going to \( k = 0 \)) is not covering the whole momentum regime, but only the interval \([\Lambda_{\text{eff}}, 0]\) with \( \Lambda_{\text{eff}} \propto \Lambda/m^{1/2} \). Hence, the initial effective action \( \Gamma_\Lambda \), for consistency, has to contain all quantum effects originating from the momentum interval \([\infty, \Lambda_{\text{eff}}]\). For \( m \to \infty \) at fixed initial scale \( \Lambda \), this means that the starting point is the full quantum effective action.

Based on the vanishing width of the regulator \(33\) for \( m \to \infty \), it has been suggested that it may correspond to a sharp cut-off limit, and that this limit may provide a sensible regulator for both UV and IR modes \[26\]. However, a decreasing – and eventually vanishing – width occurs, by definition, if the effective cut-off scale \( k_{\text{eff}} \) is removed. Again a comparison with the ERG is helpful: any smooth regulator \( R_k(q^2) \) gets a vanishing width for \( k \to 0 \). Moreover the flow at \( k = 0 \), which still depends on the specific form chosen for \( R_k \), is not unique. The same applies to PTRG flows. Hence, the flow at \( m = \infty \) at a finite scale \( k \) is neither sharp, nor exact (see also Ref. \[17\]). It is a flow at vanishing effective IR scale \( k_{\text{eff}} = 0 \).

However, it is more desirable to consider flows where the effective initial scale \( \Lambda_{\text{eff}} \) and the effective infrared scale \( k_{\text{eff}} \) are independent of the choice for the regulator. For the class of regulators given by Eq. \(32\), this is achieved by introducing the effective scales \( k_{\text{eff}} \) and \( \Lambda_{\text{eff}} \) according to \( k = m^{1/2} k_{\text{eff}} \), and hence \( \Lambda = m^{1/2} \Lambda_{\text{eff}} \). The corresponding effective action is \( \Gamma_{\text{eff}} = \Gamma_{m^{1/2} k_{\text{eff}}} \). As has been argued before, it includes all quantum effects of momenta.
larger than $k_{\text{eff}}$. Furthermore, we note that the width of the flow stays finite for any $m$ and fixed effective scales. In order to confirm this picture, we introduce $(\Lambda_{\text{eff}}, k_{\text{eff}})$ as described above, but denote them as $(\Lambda, k)$ for notational simplicity. After these manipulations, and using Eqs. (3), (32) and (33), the flow for $\hat{\Gamma}_k$ becomes

$$
\partial_t \hat{\Gamma}_k = \int_0^\infty ds \frac{(m sk^2)^m e^{-m sk^2}}{\Gamma(m)} \text{Tr} \exp \left( -s \hat{\Gamma}_k^{(2)} \right). 
$$

(35)

The prefactor in front of the trace is $\frac{1}{2} \partial_t f_{\text{PT}}(m x; m)$ with $x = sk^2$, and has the simple limit $\lim_{m \to \infty} \frac{1}{2} f_{\text{PT}}(m x; m) = \delta(x - 1)$. This follows from the asymptotic behaviour of the $\Gamma$ function $\Gamma(m \to \infty) \to \sqrt{2\pi} m^{1/2} m^m e^{-m}$. Thus, for $m \to \infty$ we arrive at

$$
\partial_t \hat{\Gamma}_k = \text{Tr} \exp \left( -\frac{\hat{\Gamma}_k^{(2)}}{k^2} \right).
$$

(36)

Eq. (36) is the closed form of the integral equation (3) for $m = \infty$ at the effective cut-off scale $k_{\text{eff}}$. No approximation to the full flow related to $\partial_t f_{\text{PT}}(x; m = \infty)$ is made. Note also that we could have started with the relation $k \to \alpha m^{1/2} k$ leading to $1/((\alpha^2 k^2)$ in the exponent in (36). This amounts to a redefinition of $\hat{\Gamma}_k$ and displays some lack of information about the initial effective action $\hat{\Gamma}_\Lambda$.

Now we are in the position to compare ERG flows for general regulators $R_k(q^2)$, and PTRG flows based on $f_{\text{PT}}(x; m)$ as defined in Eq. (33) at a fixed cut-off scale relevant for both flows. After a fixed effective scale is taken into account, the functions $\ell_{\text{PT}}(\omega)$ effectively depend on $\omega/m$. They still decay faster than $\omega^{-1}$ for $m > \frac{d}{2} + 1$. This remains so even for $m \to \infty$, where $\ell_{\text{PT}}(\omega) = \Gamma(\frac{d}{2}) e^{-\omega}$. Hence also for $m = \infty$ the function $\ell_{\text{PT}}$ does not meet the decay condition $\ell_{\text{ERG}}(w \to \infty) > C\omega^{-1}$ necessary for having an ERG analogue in the approximation studied here.

7. Comparison of critical exponents

We now turn to the computation of universal critical exponents of $O(N)$-symmetric scalar theories in 3$d$ to leading order in the derivative expansion, based on the $m$-dependent flows (34). The range $\frac{d}{2} \leq m \leq \frac{d}{2} + 1$ corresponds to approximations of exact flows, and the corresponding results can be taken as predictions. In turn, it is not known how the flows for the parameter range $\frac{d}{2} + 1 < m \leq \infty$ relate to approximations of exact flows. Therefore, the corresponding results have to be taken with some reservations.

The physically most interesting cases are the universality classes $N = 0$ (polymers), $N = 1$ (Ising model), $N = 2$ (XY-model), $N = 3$ (Heisenberg model) and $N = 4$, which is expected to be relevant to the thermal QCD phase transition with two light quark flavours. The Wilson-Fisher fixed point corresponds to the non-trivial fixed point solution $\partial_t u_* = 0$. The scaling solution $u_*$ has one unstable eigen-direction with eigenvalue $\lambda_0 < 0$. Its negative inverse is given by the critical index $\nu = -1/\lambda_0$. The smallest positive eigenvalue is denoted as $\omega$. For our computation of $\nu$ and $\omega$ we use the flow (16) with the regulator (32) and...
Figure 1: The dependence of $\nu$ on the parameter $m$. The $m$-axis has been squeezed as $m \to (m - \frac{3}{2})/m$ for display purposes.

$m \geq \frac{3}{2}$. For $d = 3$, the flow is given by

$$\partial_t u + 3u - \rho u' = \frac{(N - 1)\Gamma(m - \frac{3}{2})}{8\pi^{3/2}\Gamma(m)(1 + u')^{m-3/2}} + \frac{\Gamma(m - \frac{3}{2})}{8\pi^{3/2}\Gamma(m)(1 + u' + 2\rho u'')^{m-3/2}}.$$  \hspace{1cm} (37)

The case $m = \frac{3}{2}$ corresponds to the sharp cut-off [1], $m = 2$ to the quartic regulator $R = k^4/q^2$ [22], and $m = \frac{5}{2}$ to the optimal regulator $R_{\text{opt}}$ [10].

For $m \to \infty$ we can rely on Eq. (36) which, within the given approximation, leads to

$$\partial_t u + 3u - \rho u' = (N - 1) \exp(-u') + \exp(-u' - 2\rho u''),$$  \hspace{1cm} (38)

where the variables $u, \rho$ are those of $\hat{\Gamma}_k$ and we have redefined $u \to u/(8\pi^{3/2})$ and $\rho \to \rho/(8\pi^{3/2})$. Of course one also can obtain Eq. (38) from Eq. (37). This is seen as follows. The dimensionless fixed point potential is non-convex (convex) in those regions of field space where $u'$ or $u' + 2\rho u''$ are negative (positive). Hence, increasing $m$ in Eq. (37) has three effects. First, it leads to a parametrical suppression of the right-hand side of Eq. (37), proportional to $\Gamma(m - \frac{3}{2})/\Gamma(m)$. Second, the contributions from the convex part of the potential are strongly suppressed. Third, the contributions from the non-convex region, which shrinks as $\Gamma(m - \frac{3}{2})/\Gamma(m)$, are strongly enhanced. Therefore, we redefine the field variable in Eq. (37) as $\rho \to \rho \Gamma(m - \frac{3}{2})/(8\pi^{3/2}\Gamma(m))$ and the potential as $u \to u \Gamma(m - \frac{3}{2})/(8\pi^{3/2}\Gamma(m))$. It is worth pointing out that this redefinition of $u$ and $\rho$ precisely amount to the redefinition $k \to m^{1/2}k$ within the derivation of Eq. (36) and the finite redefinition of field variables as
Figure 2: The dependence of $\omega$ on the parameter $m$.

explained directly below Eq. (38).

We have computed the critical indices $\nu$ and $\omega$ from Eq. (37) for $N = 0 \cdots 4$ and for $m = \frac{3}{2}$ up to $m = 10^6$. For the case $m = \infty$ we use the flow (38). In practice, instead of solving numerically the flow $\partial_t u$, we found it more convenient to solve the flows for the $\rho$-derivative $\partial_t u'$. The numerical results for $\nu$ and $\omega$ are given in Tab. 1 and Tab. 2, respectively.

Notice that $m = 10^3$ is not yet sufficiently large to determine the first four significant figures for $\nu$ and $\omega$ corresponding to $m = \infty$ for all $N$ considered. However, for $m = 10^6$, the critical indices obtained from Eq. (37) agree to (at least) four significant figures with the result obtained from Eq. (38).

Let us first discuss our results for $\nu$ and $\omega$ as displayed in Figs. 1 and 2 as functions of $m$. The critical index $\nu$ is monotonously decreasing with $m$ for all values of $N$. For $\nu$, the convergence to the asymptotic value at $m = \infty$ is slightly better for smaller values of $N$. In contrast, we notice that the first irrelevant eigenvalue $\omega$ is no longer a monotonous function of $m$, because its $m$-derivative changes sign for $N = 2, 3$ and 4. From Tab. 1 and 2, we deduce that the limit for $m \to \infty$ is approached smoothly. However, the variation with $m$, over the entire scale, is roughly twice as big for $\omega$ as compared to $\nu$.

The results depend on the unphysical parameter $m$. The dependence of physical observables on unphysical parameters is well-known from perturbative QCD, e.g. Ref. [18]. Here,

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3For $m = \frac{3}{2}$ and $m = 2$, and for $N = 1 \cdots 4$, we have checked our numerical code with earlier findings summarised in Ref. [23]. For $2 \leq m \leq 10$ and $N = 1$, we have compared our results with those of Ref. [26]. Notice that our parameter $m$ is related to the one of Ref. [24] by $m = m_{\text{BZ}} + 1$. 

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it is a side effect of unavoidable approximations [8]. It may seem, at first sight, that an unphysical scheme dependence restrains the predictive power. However, in the context of the ERG, it has been shown that this is not the case [9, 10, 11]. Indeed, the convergence of approximate solutions is partly controlled by the regulator. Therefore, an adequate choice of the latter improves the convergence. These findings suggested that the regulator $R_k$ can be used to “optimise” the physical content within a given approximation. For a generic optimisation criterion, based on the regularised inverse propagator, and which improves the convergence towards the physical theory, we refer to Refs. [9, 10]. The result for the optimised regulator Refs. [10, 11] corresponds to $m_{\text{opt}} = \frac{5}{2}$. As stated in Ref. [11], we confirm that the critical exponents for $m_{\text{opt}}$ are indeed the smallest of all values for $\nu$ within the range accessible to the ERG.

| $m$   | $\nu_{N=0}$ | $\nu_{\text{Ising}}$ | $\nu_{\text{XY}}$ | $\nu_{\text{Heisenberg}}$ | $\nu_{N=4}$ |
|-------|-------------|----------------------|-------------------|-----------------------------|-------------|
| $\frac{3}{2}$ | .6066 | .6895 | .7678 | .8259 | .8648 |
| 2     | .5961      | .6604               | .7253             | .7811                        | .8240       |
| $\frac{5}{2}$ | .5921 | .6496 | .7082 | .7611 | .8043 |
| 10    |          | | | | |
| $10^2$ |          | | | | |
| $10^3$ |          | | | | |
| $10^6$ |          | | | | |
| $\infty$ | .5828 | .6260 | .6690 | .7103 | .7485 |

**Table 1:** The critical exponent $\nu$ in $3d$ as a function of $m$ for different values of $N$. The case $m = \frac{3}{2}$ corresponds to the Wegner-Houghton equation, $m = 2$ to an ERG flow with quartic regulator, and $m = \frac{5}{2}$ to the optimised ERG flow of Ref. [10].

In the context of the PTRG, the values $m > \frac{5}{2}$ are also allowed. However, the dependence on $m$ has a qualitatively different aspect: the PTRG flow is not an exact one, which implies that the endpoint $\Gamma_{\text{PT}}$ depends on $m$ as well. Hence, the $m$-dependence of $\nu$ and $\omega$ cannot be understood in the same way as within the ERG; there, we took advantage of the fact that the endpoint of the ERG flow is the full quantum effective action.

Therefore, we employ a principal of minimum sensitivity (PMS) condition [13], in order to single-out specific values for $m$. Hence, we will assume that the physical content of all $m$-dependent flows are equivalent. Only then it is sensible to require that physical observables should not depend on $m$, e.g. $\partial \nu / \partial m = 0$ or $\partial \omega / \partial m = 0$. From Tab. 1, we conclude that $\partial \nu / \partial m < 0$ for all $\frac{3}{2} < m < \infty$. Hence, $\nu$ reaches its extrema at the boundary values $m = \frac{3}{2}$ and $m = \infty$. For $\omega$, we notice that $\partial \omega / \partial m$ changes sign for $N \geq 2$. Hence, for $N < 2$ the boundary values $m = \frac{3}{2}$ and $m = \infty$ are distinguished. For $N \geq 2$ a true minimum

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4Within the PTRG approach, the scheme dependence has been addressed in Ref. [20] for $\frac{3}{2} \leq m \leq \frac{7}{2}$. Within the ERG formalism, a PMS condition has been used in Ref. [19]. In Ref. [11], it has been explained why a PMS condition works for the ERG, and how it relates to the generic optimisation of Ref. [8].
at an intermediate value of $m$ appears, and the boundary value at $m = \infty$ (for $N = 2$) or at $m = \frac{3}{2}$ (for $N = 3, 4$) corresponds to the maximum. We conclude from the facts that a PMS condition does not lead, for all observables, to a unique prescription for $m$. However, the endpoints given by the sharp cut-off $m = \frac{3}{2}$, and by $m = \infty$, are singled out because they represent (at least local) extrema for all observables studied. We emphasise that the optimised case $m_{\text{opt}} = \frac{5}{2}$ – which follows within the ERG formalism, and which is closely linked to a PMS condition within the ERG approach [11] – does not follow from a PMS condition within the PTRG approach.

| $m$   | $\omega_{N=0}$ | $\omega_{\text{Ising}}$ | $\omega_{\text{XY}}$ | $\omega_{\text{Heisenberg}}$ | $\omega_{N=4}$ |
|-------|----------------|--------------------------|-----------------------|-------------------------------|----------------|
| $\frac{3}{2}$ | .5432         | .5952                    | .6732                 | .7458                         | .8007          |
| 2     | .6175         | .6286                    | .6621                 | .7068                         | .7519          |
| $\frac{5}{2}$ | .6579         | .6557                    | .6712                 | .6998                         | .7338          |
| 10    | .7559         | .7376                    | .7250                 | .7200                         | .7231          |
| $10^2$ | .7794         | .7598                    | .7437                 | .7330                         | .7290          |
| $10^3$ | .7816         | .7620                    | .7455                 | .7344                         | .7297          |
| $10^6$ | .7819         | .7622                    | .7457                 | .7346                         | .7298          |
| $\infty$ | .7819         | .7622                    | .7457                 | .7346                         | .7298          |

Table 2: The first irrelevant eigenvalue $\omega$ in 3$d$ as a function of $m$ for different values of $N$.

Finally, we compare the results of Tabs. 1 and 2 with those of other methods (see Ref. [27] for a recent overview). From all ERG flows, the results for $\nu$ with $m = \frac{3}{2}$ are closest to those obtained by other methods. This confirms the conjecture that optimised ERG flows have an improved derivative expansion [11]. For PTRG flows, it is intriguing to realise that the values obtained for $m = \infty$ are in even better agreement with both experimental values and those obtained by other techniques.

8. Conclusions

We investigated the predictive power of the PTRG by providing its link to the ERG at leading order in the derivative expansion. We found that the space of PTRG regulator functions $f_{PT}$ is larger than the space of ERG regulators $R_k$. Given the heuristic derivation of the PTRG flow, there is no additional criteria available which would allow to discard a specific subset of PTRG regulators. The ERG regulators, however, have a simple physical interpretation. With help of the map from ERG to PTRG – even though only in the approximation discussed here – we can identify the subset of PTRG flows which are directly related to momentum shell integrations in the Wilsonian sense. This map cannot be extended to the full flow.

PTRG flows at a given IR scale $k$ for $m > \frac{3}{2} + 1$ cannot be mapped to ERG flows at $k$ even at leading order in the derivative expansion. Still, they have a simple interpretation in
terms of flows at a lower effective IR scale $k_{\text{eff}} \propto k/m^{1/2}$. Increasing $m$ changes the shape of the momentum cut-off and the initial momentum scale $\Lambda_{\text{eff}} \propto \Lambda/m^{1/2}$, and hence the initial action. In terms of the fixed effective scales $\Lambda_{\text{eff}}$ and $k_{\text{eff}}$, the PTRG flow (35) corresponds to a momentum shell integration at the scale $k_{\text{eff}}$ starting at $\Lambda_{\text{eff}}$. It is precisely this picture which stands behind the flow Eq. (36), and the results given for $m = \infty$. However, we still cannot map these flows to ERG flows at the scale $k_{\text{eff}}$. Thus it remains unclear what kind of approximation to flows of the full theory they correspond to.

As an application, we have computed critical exponents for 3d $O(N)$-symmetric scalar theories. Within the ERG, we have seen that the optimisation of Ref. [10] indeed leads to improved results, which correspond to the choice $m = \frac{d}{2} + 1$. Within the PTRG, we found very good results in the limit $m \to \infty$. While our above findings suggest that this limit is sensible, it remains to be seen how approximations to PTRG flows for $m > \frac{d}{2} + 1$ are related to systematic approximations of the physical theory, before these results can be considered as predictions.

Our findings made clear that the precise structure of the inherit approximation of the PTRG still have to be investigated more deeply in order to use this approach in a well-controlled way. These inherit approximations strongly depend on the regulator $f_k$, which makes the task even more difficult. A possible avenue to escape from this problem lies in a structural comparison of ERG and PTRG. If one can cast both equations into similar closed expressions, a full discussion of similarities and differences is possible. Then one could hope to derive quality checks for the PTRG in a closed form. These issues will be discussed in [17].

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Note added
After this work was completed, the preprint [28] appeared, which also treats the limit $m \to \infty$.

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