MULTIDIMENSIONAL DIVISOR FUNCTION ON AVERAGE OVER VALUES OF QUADRATIC POLYNOMIAL

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ABSTRACT. Let $F(x) = x^t Q_m x + b^t x + c \in \mathbb{Z}[x]$ be a quadratic polynomial in $\ell(\geq 3)$ variables $x = (x_1, \ldots, x_k)$, where $F(x)$ is positive when $x \in \mathbb{R}_{\geq 1}^k$, $Q_m \in \mathbb{M}_\ell(\mathbb{Z})$ is an $\ell \times \ell$ matrix and its discriminant $\text{det}(Q_m^t + Q_m) \neq 0$. It gives explicit asymptotic formulas for the following sum

$$T_{k,F}(x) = \sum_{x \in \mathbb{Z}^{(k)}} \tau_k(F(x))$$

with the help of circle method. Here $\tau_k(n) = \#\{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k : n = x_1 x_2 \ldots x_k\}$ with $k \in \mathbb{Z}_{\geq 2}$ is the multidimensional divisor function.

1. Introduction

The multidimensional divisor functions are generalisations of the divisor function $\tau(n) = \sum_{d|n} 1$, defined by

$$\tau_k(n) = \#\{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k : n = x_1 x_2 \ldots x_k\},$$

and counting the number of ways that $n$ can be written as a product of $k$ positive integer numbers. Understand the average order of $\tau_k(n)$, as it ranges over the values taken by polynomials is an important topic in analytic number theory. The behavior of $\tau_k(n)$ is far less than perfectly understood even for $k = 3$. For example, so far there are no asymptotic formulas for the sum $\sum_{m \leq x} \tau_3(m^2 + 1)$. If one considers the sum

$$\sum_{|F(x)| \leq x} \tau_k(|F(x)|),$$

where $F(x) \in \mathbb{Z}[x_1, x_2]$ is a binary form. For $k = 2$ and $F(x)$ is irreducible cubic form, Greaves [3] showed that there exists constants $c_0, c_1 \in \mathbb{R}$ with $c_0 > 0$ depending only on $F$, such that

$$\sum_{|F(x)| \leq X} \tau(|F(x)|) = c_0 X^{\frac{7}{4}} \log X + c_1 X^{\frac{7}{4}} + O_{F}(X^{\frac{7}{4} + \varepsilon}),$$

holds for any $\varepsilon > 0$. If $F$ is an irreducible quartic form, Daniel [1] showed that

$$\sum_{|F(x)| \leq X} \tau(|F(x)|) = c_2 X^{\frac{3}{2}} \log X + O_{F}(X^{\frac{3}{2} \log \log X}),$$

where $c_2$ is a constant depending only on $F$. However, if $k \geq 3$, this kind of problems will become more complicated. There are few results in this direction. For $\tau_3(n)$, Friedlander and Iwaniec [2] showed that

$$\sum_{n_1^2 + n_2^2 \leq x} \tau_3(n_1^2 + n_2^2) = c x^{\frac{3}{2}} (\log x)^2 + O \left( x^{\frac{3}{2}} (\log x)^{\frac{3}{2}} (\log \log x)^{\frac{5}{2}} \right),$$

where $c$ is a constant and $*$ means that $(n_1, n_2) = 1$. If $F(x)$ is positive definite quadratic form with $\ell \geq 2$ variables, then it is easy to obtain the sum

$$\sum_{F(x) \leq X} \tau_k(F(x)) = \sum_{m \leq x} \tau_k(m) r_F(m)$$

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by classical results of quadratic form, where \( r_F(m) = \#\{ x \in \mathbb{Z}^2 : m = F(x) \} \). But this method not directly applies for the sum

\[
T_{k,F}(X,B) = \sum_{x \in B_l(X)} \tau_k(F(x))
\]

when the number set \( B_l(X) \) is usual. For example, let \( B_3(X) = [1, X]^3 \cap \mathbb{Z}^3 \), Sun and Zhang [7] obtained the following asymptotic formula

\[
\sum_{1 \leq m_1, m_2, m_3 \leq X} \tau_3(m_1^2 + m_2^2 + m_3^2) = c_4 X^3 \log^2 X + c_5 X^3 \log X + c_6 X^3 + O(X^{\frac{11}{2} + \varepsilon}),
\]

where \( c_4, c_5, c_6 \) are constants and \( \varepsilon \) is an arbitrarily positive number. Furthermore, nothing of the cases that \( F \) is indefinite known for \( \tau_k(n) \) with \( k \geq 3 \).

Let \( F(x) \) be a quadratic polynomial with \( \ell(\geq 3) \) variables \( x_1, x_2, ..., x_\ell \) and integer coefficients. Unless stated explicitly otherwise, we shall write \( x \) for the vector \( (x_1, x_2, ..., x_\ell) \in \mathbb{Z}^\ell \), and denote \( B_\ell(X) = [1, X]^\ell \) as a box for some sufficiently large positive number \( X \). We assume quadratic polynomial \( F(x) \) satisfies

\[
F(x) = x^t Q_m x + b^t x + c,
\]

where \( Q_m \in M_\ell(\mathbb{Z}) \) is an \( \ell \times \ell \) matrix with entries \( a_{ij} \), vector \( b = (b_1, ..., b_\ell) \in \mathbb{Z}^\ell \), \( c \in \mathbb{Z} \) and suppose those coefficients satisfies the following conditions

\[
\begin{align*}
\min_{x \in B_\ell(X)} F(x) &> 0 \\
\Delta_F = \det (Q_m + Q_m) &\neq 0.
\end{align*}
\]

Thus \( F(x) \) has a maximum value \( N_F(X) \) in the box \( B_\ell(X) \) when \( X \) is sufficiently large, say

\[
N_F(X) = X^2 \sum_{1 \leq i,j \leq \ell} a_{ij} + X \sum_{1 \leq r \leq \ell} b_r + c.
\]

In present paper we will prove an asymptotic formula of (1.1) with \( B_\ell(X) = [1, X]^\ell \cap \mathbb{Z}^\ell \), \( \ell \geq 3 \) and all \( k \geq 2 \). Exactly, we shall prove the following theorem.

**Theorem 1.1.** Let \( F, B(X) \) defined as above, \( k \geq 2 \) and \( \ell \geq 3 \). For any \( \varepsilon > 0 \) there exists constants \( H_{k,0}(F), H_{k,2}(F), ..., \) and \( H_{k,k-1}(F) \), such that

\[
T_{k,F}(X) = \sum_{r=0}^{k-1} H_{k,r}(F) \int_{[1,X]^\ell} (\log F(t))^r dt + O_{k,F}(X^{\ell-\frac{2}{\ell+2} \min(1, \frac{\ell}{1+\ell})}+\varepsilon)
\]

and

\[
H_{k,r}(F) = \frac{1}{r!} \sum_{t=0}^{k-r-1} \frac{1}{t!} \left( \frac{d^r L(s;k,F)}{ds^r} \right)_{s=1} \text{Res} \left( (s-1)^{r+1} \zeta(s)^k; s = 1 \right),
\]

where function \( L(s;k,F) \) is given in Lemma 4.1.

**Notation.** The symbols \( \mathbb{Z} \) and \( \mathbb{R} \) denote the integers and the real numbers, respectively. \( \epsilon(z) = e^{2\pi i z} \), the letter \( p \) always denotes a prime, \( M^t \) is transpose operation of matrix \( M \). The symbol \( \mathbb{Z}_q \) represents shorthand for the groups \( \mathbb{Z}/q\mathbb{Z} \). Also, the shorthand for the multiplicative group reduced residue classes \( (\mathbb{Z}/q\mathbb{Z})^* \) is \( \mathbb{Z}_q^* \). Occasionally we make use of the \( \varepsilon \)-convention: whenever \( \varepsilon \) appears in a statement, it is asserted that the statement is true for all real \( \varepsilon \). This allows us to write \( x^\varepsilon \log x \ll x^\varepsilon \), for example.
2. Primaries

Lemma 2.1 (R. A. Smith). Let \( 1 \leq h \leq q \), \((q, h) = \delta\). Then for \( q \leq x^{2/3 + \varepsilon} \), we have

\[
\sum_{\substack{m \leq x \\ m \equiv h \pmod{q}}} \tau_k(m) = M_k(x; h, q) + O_k(x^{1 - \frac{2}{3}\varepsilon + \varepsilon}),
\]

where

\[
M_k(x; h, q) = \text{Res} \left( \frac{s^k}{\zeta(s)} f_k(q, \delta, s); s = 1 \right)
\]

with

\[
f_k(q, \delta, s) = \frac{1}{\varphi(q/\delta) \delta^s} \prod_{p \mid (q/\delta)} \left( 1 - \frac{1}{p^s} \right)^k \sum_{d_1, d_2, \ldots, d_k = \delta} \prod_{i=1}^{k-1} \prod_{d_i, d_{i+1} > 0} \left( 1 - \frac{1}{p^s} \right).
\]

Proof. This lemma is essentially made by Smith [6], and we just change the form as needed. Firstly, by the equation (30) of [6], we get

\[
A_k(x; h, q) = \sum_{d_1, d_2, \ldots, d_k = \delta} \sum_{t_1 \mid \prod_{i=1}^{l} d_r \text{ and } l \leq k} \mu(t) A_k \left( \frac{x}{\delta t_1 t_2 \ldots t_k}; \frac{h}{\delta}, \frac{q}{\delta} \right)
\]

where the notations be followed. Theorem 3 of this paper yields

\[
A_k(x; h, q) = M_k(x; h, q) + \Delta_k(x; h, q),
\]

where

\[
M_k(x; h, q) = \sum_{d_1, d_2, \ldots, d_k = \delta} \sum_{t_1 \mid \prod_{i=1}^{l} d_r \text{ and } l \leq k} \mu(t) \frac{x}{\delta t_1 t_2 \ldots t_k} P_k \left( \log \left( \frac{x}{\delta t_1 t_2 \ldots t_k} \right); \frac{q}{\delta} \right)
\]

and

\[
\Delta_k(x; h, q) = \sum_{d_1, \ldots, d_k = \delta} \sum_{t_1 \mid \prod_{i=1}^{l} d_r \text{ and } l \leq k} \mu(t) \left( D_k \left( 0; \frac{t_1, \ldots, t_k h}{\delta} ; \frac{q}{\delta} \right) + O \left( \frac{\tau_k(q/\delta) x^{\frac{k-1}{k+1}} \log^k x}{(\delta t_1 t_2 \ldots t_k)^{\frac{k-1}{k+1}} \log x} \right) \right).
\]

By the definition of \( P_k \log x, q \), namely (13),(21) and relatively talking about (21) of [6]. It is easily seen that

\[
x P_k \log x = \frac{1}{\varphi(q)} \text{Res} \left( \left( \frac{s^k}{\zeta(s)} \sum_{d \mid q} d^{-s} \mu(d) \right) \frac{x^s}{s}; s = 1 \right).
\]
Hence we obtain that
\[
M_k(x; h, q) = \sum_{d_1, \ldots, d_k = \delta} \sum_{d_1, \ldots, d_k > 0} \mu(t) \varphi(q/\delta) \text{Res} \left( \left( \zeta(s) \sum_{d_1, d_2, \ldots, d_k > 0} \mu(d) \frac{x^s}{d^s} \right)^k \frac{1}{s(\delta t_1 \cdots t_k)^s} ; s = 1 \right)
\]
\[
= \text{Res} \left( \frac{x^s}{s} \zeta(s)^k \prod_{p | (q/\delta)} \left( 1 - \frac{1}{p^s} \right)^k \sum_{d_1, \ldots, d_k > 0} \mu(t) \frac{1}{(t_1 \cdots t_k)^s} ; s = 1 \right)
\]
\[
= \text{Res} \left( \zeta(s)^k \frac{x^s}{s} f_k(q, \delta, s) ; s = 1 \right),
\]
where
\[
f_k(q, \delta, s) = \frac{1}{\varphi(q/\delta)^s} \prod_{p | (q/\delta)} \left( 1 - \frac{1}{p^s} \right)^k \sum_{d_1, \ldots, d_k > 0} \mu(t) \frac{1}{(t_1 \cdots t_k)^s}.
\]

Smith [6] conjectured the validity of the estimate \( D(0, h, q) \ll q^{-\frac{1}{2}+\varepsilon} \) for any \((q, h) = 1\) and proved by Matsumoto [4] implies the trivially bound
\[
\Delta_k(x; h, q) \ll \sum_{d_1, \ldots, d_k = \delta} \sum_{t_1, \ldots, t_k > 0} |\mu(t)| \left( \frac{1}{\delta} + q^{-\frac{k+1}{2}} \right) \ll_k \left( q^{-\frac{k+1}{2}} + x^{-\frac{k+1}{2}+\varepsilon} \right) \sum_{d_1, \ldots, d_k = \delta} \tau(\delta)^{k-1} \ll_k x^{1-\frac{2}{k+1}+\varepsilon},
\]
where the condition \( q \leq x^{\frac{2}{k+1}} \) be used. Which complete the proof of the lemma.

**Lemma 2.2.** Let \( q \geq 1 \) be an integer, \((a, q) = 1\) and denote \((h, q) = 1\). Also let \( f(q, \delta, s) \) defined as Lemma 2.1. Define
\[
F_{k,a}(q, s) = \sum_{h \in \mathbb{Z}_q} e \left( -\frac{ah}{q} \right) f_k(q, \delta, s).
\]

Then \( F_{k,a}(q, s) \) independent on \( a \) and we may write it as \( F_k(q, s) \). Furthermore, \( F_k(q, s) \) is multiplicative function and
\[
\frac{d^r F_k(q, 1)}{ds^r} \ll_k q^{-1+\varepsilon}
\]
holds for any integer \( r = 0, 1, \ldots, k - 1 \).

**Proof.** First, we have
\[
F_{k,a}(q, s) = \sum_{\delta \mid q} \sum_{h \in \mathbb{Z}_q \atop (h, q) = \delta} e \left( -\frac{ah}{q} \right) f_k(q, \delta, s) = \sum_{\delta \mid q} f_k(q, \delta, s) \sum_{h_1 \in \mathbb{Z}_q^{*}} e \left( -\frac{ah_1}{q/\delta} \right)
\]
\[
= \sum_{\delta \mid q} c_\delta(a) f_k(q, \delta, s) = \sum_{\delta \mid q} \mu(\delta) f_k(q, \delta, s),
\]
where \(c_\delta(a)\) is the Ramanujan’s sum and the fact that if \((a, \delta) = 1\) then \(c_\delta(a) = \mu(\delta)\) be used. This result yields \(F_{k, \alpha}(q, s)\) independent on \(a\). Suppose that positive integers \(q_1\) and \(q_2\) are coprime, then

\[
F_k(q_1, s)F_k(q_2, s) = \sum_{\delta_1|q_1, \delta_2|q_2} \mu(\delta_1)\mu(\delta_2) f_k(q_1, q_1/\delta_1, s) f_k(q_2, q_2/\delta_2, s)
\]

hence we just need to show

\[
f_k(q_1, q_1/\delta_1, s)f_k(q_2, q_2/\delta_2, s) = f_k(q_1q_2, q_1q_2/(\delta_1\delta_2), s)
\]

whenever \(\delta_1|q_1\) and \(\delta_2|q_2\). By the definition of \(f_k(q, q/\delta, s)\), say

\[
f_k(q, q/\delta, s) = \frac{\varphi(\delta)}{\varphi(q)} \prod_{p|\delta} \left(1 - \frac{1}{p^s}\right)^k \prod_{d_1d_2...d_k=q/\delta} \left(1 - \frac{1}{p^s}\right),
\]

Furthermore,

\[
f_k(q, q/\delta, s) \ll \frac{\varphi(\delta)}{\varphi(q)} \prod_{p|\delta} \left(1 - \frac{1}{p^s}\right)^k \prod_{d_1d_2...d_k=q/\delta} \left(1 - \frac{1}{p^s}\right),
\]

where \(\sigma = \text{Re}(s)\). It is easily seen that if \(s = 1 + \rho e(\theta)\) with \(\theta \in [0, 1)\), then

\[
f_k(q, q/\delta, s) \ll \frac{\varphi(\delta)}{\varphi(q)} 2^{k\omega(\theta)} T_k(q) 2^{(k-1)\omega(q)} \ll q^\sigma \frac{\varphi(\delta)}{\varphi(q)}.
\]

Thus we have

\[
F_k(q, s) \ll q^\sigma \sum_{\delta|q} |\mu(\delta)| \frac{\varphi(\delta)}{\varphi(q)} = q^{-\sigma+\varepsilon} \prod_{p|q} \left(1 + \frac{p^\sigma}{p-1}\right) \ll q^{-\sigma+\varepsilon} \prod_{p|q} \left(1 + \frac{p^\sigma}{p}\right).
\]

On the other hand

\[
q^{-\sigma} \prod_{p|q} \left(1 + \frac{p^\sigma}{p}\right) \ll \begin{cases} q^{-\sigma+\varepsilon} & \sigma \in (0, 1] \\ q^{-\sigma+\varepsilon} \prod_{p|q} p^{-1+\sigma} & \sigma \in (1, 2). \end{cases}
\]

Therefore

\[
(2.2) \quad F_k(q, s) \ll q^{-\min(\sigma, 1)+\varepsilon}.
\]

It is obviously that \(F_k(q, s)\) is analytic in \(\mathbb{C}\) for every \(q\) which concerned. Hence one can use Cauchy estimate, say

\[
(2.3) \quad \frac{d^r F_k(q, s)}{ds^r} \bigg|_{s=1} = \frac{r!}{2\pi i} \int_{|\xi-1|=\rho} \frac{F_k(q, \xi)}{(\xi - 1)^{r+1}} d\xi \ll \frac{r!}{\rho^{r}} \max_{\theta \in [0, 1]} |F_k(q, 1 + \rho e(\theta))|,
\]

where \(\rho \in (0, 1)\). Hence combine with (2.2), we obtain that

\[
\frac{d^r F_k(q, 1)}{ds^r} \ll \frac{r!}{\rho^{r}} q^{-(1-\rho)+\varepsilon} \ll_k q^{1+\varepsilon}.
\]

Thus complete the proof of the lemma.

\[\square\]

**Lemma 2.3.** Let \(\alpha = a/q + \beta\) with \(q \leq X^{\frac{2}{\pi+\alpha}}\) be an positive integer and \((a, q) = 1\). Define

\[
J_k(\alpha, X) = \sum_{m \leq X} \tau_k(m)e(m\alpha).
\]
Then
\[ J_k(\alpha, X) = \int_1^X e(\alpha \beta) \text{Res} \left( \zeta(s)^k F_k(q, s) u^{s-1}; s = 1 \right) \, du + O_k \left( q(1 + |\beta|X) X^{1-\frac{2}{\gamma^2} + \epsilon} \right), \]
where \( F_k(q, s) \) defined as Lemma 2.2.

Proof. First, by Lemma 2.3 we have
\[
J_k(\alpha, X) = \sum_{h \in \mathbb{Z}_q} e \left( \frac{ah}{q} \right) \sum_{m \equiv h \pmod{q}} \tau_k(m) e(m\beta) \\
= \sum_{h \in \mathbb{Z}_q} e \left( \frac{ah}{q} \right) \int_1^X e(\alpha \beta) d \left( M_k(u; h, q) + O_k \left( u^{1-\frac{2}{\gamma^2} + \epsilon} \right) \right) \\
= \sum_{h \in \mathbb{Z}_q} e \left( \frac{ah}{q} \right) \int_1^X e(\alpha \beta) M'(u; h, q) du + O_k \left( q(1 + |\beta|X) X^{1-\frac{2}{\gamma^2} + \epsilon} \right).
\]

On the other hand,
\[
\sum_{h \in \mathbb{Z}_q} e \left( \frac{ah}{q} \right) M'(u; h, q) = \sum_{h \in \mathbb{Z}_q} e \left( \frac{ah}{q} \right) \text{Res} \left( \zeta(s)^k u^{s-1} f_k(q, \delta, s); s = 1 \right),
\]
where \( \delta = (q, h) \) and use Lemma 2.2 we complete the proof of the lemma.

The Riemann zeta function is meromorphic with a single pole of order one at \( s = 1 \). It can therefore be expanded as a Laurent series about \( s = 1 \), say
\[
\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{n!} (s-1)^n,
\]
where
\[
\gamma_n = \lim_{M \to \infty} \left( \sum_{\ell=1}^M \log \frac{n}{\ell} - \frac{\log^{n+1} M}{n+1} \right), \quad n \in \mathbb{N}
\]
are the Stieltjes constants. Therefore there exists constants \( \alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,k} \) and a holomorphic function \( h_k(s) \) on \( \mathbb{C} \) such that
\[
(2.4) \quad \zeta(s)^k = \sum_{r=1}^{k} \frac{\alpha_{k,r}}{(s-1)^r} + h_k(s).
\]

Furthermore, we obtain that
\[
(2.5) \quad \zeta(s)^k x^{s-1} = \sum_{r=1}^{k} \frac{1}{(s-1)^r} \sum_{r_1=0}^{k-r} \alpha_{k,r_1+r} \frac{\log^{r_1} x}{r_1 !} + g_{k,x}(s),
\]
for any \( x > 0 \), where \( h_{k,x}(s) \) is a holomorphic function on \( \mathbb{C} \) about \( s \). On the other hand, we also have a Taylor series for \( F_k(q, s) \) at \( s = 1 \), say
\[
F_k(q, s) = \sum_{\ell=0}^{\infty} \frac{F_k^{(\ell)}(q, 1)}{\ell !} (s-1)^\ell.
\]

Therefore the residue of \( \zeta(s)^k x^{s-1} F_k(q, s) \) at \( s = 1 \) is
\[
(2.6) \quad \sum_{\ell, r \in \mathbb{N}, 1 \leq r \leq k} \sum_{r_1=0}^{k-r} \frac{F_k^{(\ell)}(q, 1)}{\ell !} \frac{\alpha_{k,r_1+r} \log^{r_1} x}{r_1 !} = \sum_{r=1}^{k} \frac{\log^{r-1} x}{(r-1)!} \sum_{t=0}^{k-r} \frac{F_k^{(t)}(q, 1)}{t !} \alpha_{k,r+t}.
\]
We Define
\[ \beta_{k,r}(q) = \frac{1}{r!} \sum_{t=0}^{k-r-1} \frac{\alpha_{k,r+t}}{t!} \left( \frac{d^t F_k(q,s)}{ds^t} \right)_{s=1}^{\ell}. \]

Then by Lemma 2.2 we have \( \beta_{k,r}(q) \ll q^{-1+\varepsilon} \) and the results of Lemma 2.3 rewritten as

**Lemma 2.4.** Let \( \alpha = a/q + \beta \) with \( q \leq X^{2/7} \) be an positive integer and \( (a,q) = 1 \). Then, we have
\[
J_k(\alpha, X) = \sum_{r=0}^{k-1} \beta_{k,r}(q) \int_1^X (\log u)^r e(u\beta)du + O_k \left( q(1+|\beta|X)X^{1-2/7+1/7} \right).
\]
where
\[
\beta_{k,r}(q) = \frac{1}{r!} \sum_{t=0}^{k-r-1} \frac{\alpha_{k,r+t}}{t!} \left( \frac{d^t F_k(q,s)}{ds^t} \right)_{s=1}^{\ell}.
\]

with
\[ \alpha_{k,r} = \text{Res} \left( (s-1)^{-1} \zeta(s)^{k}; s = 1 \right) \]\nand where \( F_k(q,s) = \sum_{\delta/q} \mu(q/\delta) f_k(q,\delta,s) \)

**Lemma 2.5.** Let \( \alpha = a/q + \beta \) with \( q \) be an positive integer and \( (a,q) = 1 \). Define
\[ I_F(\alpha, X) = \sum_{x \in B_\ell(X)} e(F(x)\alpha), \]
where \( B_\ell(X) = [1,X]^{\ell} \cap \mathbb{Z}^\ell \). Then
\[
I_F(\alpha, X) = q^{-\ell} S_F(q,a) \int_{[1,X]^\ell} e(F(t)\beta) dt + O_F \left( q(1+|\beta|X^2)X^{\ell-1} \right),
\]
where
\[ S_F(q,a) = \sum_{h \in (\mathbb{Z}_q)^\ell} e \left( \frac{a}{q} F(h) \right). \]

**Proof.** Firstly,
\[
I_F(\alpha, X) = \sum_{h \in (\mathbb{Z}_q)^\ell} e \left( \frac{a}{q} F(h) \right) \sum_{x \in B_\ell(X)} \sum_{x \equiv h \ (\text{mod } q)} e(F(x)\beta).
\]

We shall prove
\[
\sum_{x \in B_\ell(X)} e(F(x)\beta) - \frac{1}{q^\ell} \int_{[1,X]^\ell} e(F(t)\beta) dt \ll_F (1+|\beta|X^2)(X/q)^{\ell-1},
\]
which immediately yields the proof. For any \( a, b \in \mathbb{R} \) with \( a \ll_F 1 \) and \( b \ll_F X \), let us consider the follows estimate
\[
\sum_{m \leq X \atop m \equiv h \ (\text{mod } q)} e((am^2 + bm)\beta) - q^{-1} \int_1^X e((at^2 + bt)\beta)dt \ll_F 1 + |\beta|X^2,
\]
which obtained by part integral directly. We apply the fact successively for each variables \( x_i \equiv h_i \ (\text{mod } q) \) yields the result. \( \square \)

**Lemma 2.6.** Let \( S_F(q,a) \) defined as Lemma 2.5 with \( (q,a) = 1 \). Then for any \( F(x) \) we concerned, thus \( F \) be a nonsingular quadratic polynomial, we have
\[ S_F(q,a) \ll_F q^{\frac{6}{7}}. \]
Proof. Firstly,
\[ |S_F(q, a)|^2 = S_F(q, a) \overline{S_F(q, a)} = \sum_{h \in (\mathbb{Z}_q)^\ell} e \left( \frac{a}{q} F(h) \right) \sum_{k \in (\mathbb{Z}_q)^\ell} e \left( -\frac{a}{q} F(k) \right) \]
\[ = \sum_{k \in (\mathbb{Z}_q)^\ell} \sum_{h+k \in (\mathbb{Z}_q)^\ell} e \left( \frac{a}{q} (F(h+k) - F(k)) \right). \]

It is easily seen that
\[ F(h+k) - F(k) = (h+k)^t Q_m(h+k) - k^t Q_m k + b^t(h+k-k) \]
\[ = h^t Q_m h + b^t h + h^t Q_m s k = F(h) - c + h^t Q_m s k. \]

Hence we deduce that
\[ |S_F(q, a)|^2 = \sum_{k \in (\mathbb{Z}_q)^\ell} \sum_{h \in (\mathbb{Z}_q)^\ell} e \left( \frac{a}{q} (F(h) - c + h^t Q_m s k) \right) \]
\[ = \sum_{h \in (\mathbb{Z}_q)^\ell} e \left( \frac{a}{q} (F(h) - c) \right) \sum_{k \in (\mathbb{Z}_q)^\ell} e \left( \frac{a}{q} k^t Q_m s h \right) \]
\[ = q^\ell \sum_{h \in (\mathbb{Z}_q)^\ell : Q_m s h \equiv 0 (\text{mod} q)} e \left( \frac{a}{q} (F(h) - c) \right) \]
\[ \ll q^\ell \# \{ h \in (\mathbb{Z}_q)^\ell : Q_m s h \equiv 0 \text{ mod } q \}. \]

Since \( Q_m s \) is nonsingular, hence
\[ |S_F(q, a)|^2 \ll q^\ell \# \{ h \in (\mathbb{Z}_q)^\ell : Q_m s h \equiv 0 \text{ mod } q \} \ll F q^\ell. \]

This completes the proof. \( \square \)

To give a good estimate for \( I_F(\alpha, X) \) in the minor arcs of the circle method, we need the following lemma.

**Lemma 2.7.** Let \( A \in M_\ell(\mathbb{Z}) \) be a nonsingular matrix with column vectors are \( a_1, ..., a_\ell \). Also let \( \alpha = a/q + \beta \) with \( q \) be an positive integer, \( (a, q) = 1 \) and \( |\beta| \leq q^{-2} \). Define
\[ H(X, A, \alpha) = \sum_{x \in B(X)} \prod_{1 \leq t \leq \ell} \min \left( X, \| a^t X \alpha \|^{-1} \right), \]
where \( B(X) = [1, X]^\ell \cap \mathbb{Z}^\ell \). Then
\[ H(X, A, \alpha) \ll A X^2 q^{-\ell} + X^\ell \log^\ell q + q^\ell \log^\ell q. \]

**Proof.** Firstly we have
\[ \sum_{x \in B(X)} \prod_{1 \leq t \leq \ell} \min \left( X, \| a^t x \alpha \|^{-1} \right) \leq \sum_{x \in B(X/q)} \sum_{h \in (\mathbb{Z}_q)^\ell} \prod_{1 \leq t \leq \ell} \min \left( X, \| a^t(q x + h) \alpha \|^{-1} \right). \]

For the inner sum above
\[ U(X, A, x, q) = \sum_{h \in (\mathbb{Z}_q)^\ell} \prod_{1 \leq t \leq \ell} \min \left( X, \| a^t(q x + h) \alpha \|^{-1} \right) \]
(2.7)
notes that \( \alpha = a/q + \beta \), then
\[ a^t(q x + h) \alpha = \frac{a a^t h}{q} + a^t(q x + h) \beta \pmod{1}. \]
On the other hand, for each \( v = 1, 2, ..., \ell \), there exists some \( t_v \in [0, 1 - \frac{1}{q}] \) and uniquely \( h^v = (h^v_1, ..., h^v_\ell) \in \mathbb{Z}^\ell \) such that
\[ \{ a^t(q k + h) \alpha \} = a a^t h/q + a^t(q x + h) \beta - h^v \in [t_v, t_v + 1/q], \]
In this case, it has uniformly \( \theta_i \) integers on above interval bounded by \( O \left( \frac{4}{q^2} \right) \), namely

\[
\sum_{0 \leq m \leq \ell} \prod_{1 \leq u \leq m} \min \left( X, \left\| a_i''(qx + h) \alpha \right\|^{-1} \right)
\]

(2.9) \( A_h - qh_v' = N_v \),

then \( (a, q) = 1 \) and \( A \) nonsingular implies

\[
\# \{ h \in (\mathbb{Z}_q)^{\ell} : aAh \equiv N_v \mod q \} \ll A 1.
\]

Hence the number of \( h \in (\mathbb{Z}_q)^{\ell} \) satisfies (2.9) bounded by \( O_A(1) \). Furthermore, for all \( (t_1, t_2, \ldots, t_\ell) \in [0,1 - 1/q]^\ell \), the number of \( h \in (\mathbb{Z}_q)^{\ell} \) satisfy the condition

\[
\left( \{ a_i''(qx + h) \alpha \}, \ldots, \{ a_i''(qx + h) \alpha \} \right) \in [t_1, t_1 + 1/q] \times \ldots \times [t_\ell, t_\ell + 1/q]
\]

bounded by \( O_A(1) \). On the other hand \( \| a_i''(qx + h) \alpha \| \in [t_v, t_v + 1/q] \) if and only if

\[
\{ a_i''(qx + h) \alpha \} \in [t_v, t_v + 1/q] \text{ or only if } 1 - \{ a_i''(qx + h) \alpha \} \in [t_v, t_v + 1/q].
\]

Hence for all \( (t_1, t_2, \ldots, t_\ell) \in [0,1 - 1/q]^\ell \), the number of \( h \in (\mathbb{Z}_q)^{\ell} \) satisfies \( \| a_i''(qx + h) \alpha \| \in [t_v, t_v + 1/q] \) bounded by \( O_A(1) \).

For the convenience of discussion, \( \forall s_1, s_2, \ldots, s_\ell \in [0, q/2) \cap \mathbb{Z} \), we denote

\[
K(s) = \left[ \frac{s_1}{q}, \frac{s_1 + 1}{q} \right] \times \left[ \frac{s_2}{q}, \frac{s_2 + 1}{q} \right] \times \ldots \times \left[ \frac{s_\ell}{q}, \frac{s_\ell + 1}{q} \right]
\]

and

\[
A(q, x, h) = \left( \| a_i''(qx + h) \alpha \|, \ldots, \| a_i''(qx + h) \alpha \| \right).
\]

Then

\[
\# \{ h \in (\mathbb{Z}_q)^{\ell} : A(q, x, h) \in K(s) \} \ll A 1.
\]

Thus the sum (2.7) can be rewritten as

\[
U(X, A, q) \ll \sum_{s \in [0, q/2)^{\ell} \cap \mathbb{Z}^\ell} \sum_{A(q, x, h) \in K(s)} \prod_{1 \leq u \leq \ell} \min \left( X, \left\| a_i''(qx + h) \alpha \right\|^{-1} \right)
\]

\[
\ll \sum_{0 \leq m \leq \ell} \sum_{s \in [0, q/2)^{\ell} \cap \mathbb{Z}^m} \prod_{1 \leq u \leq m} \min \left( X, \left\| a_i''(qx + h) \alpha \right\|^{-1} \right)
\]

\[
\ll A \sum_{0 \leq m \leq \ell} \sum_{s \in [1, q/2)^{\ell} \cap \mathbb{Z}^m} X^{\ell - m} \prod_{1 \leq u \leq m} \min \left( X, \frac{q}{s_v} \right)
\]

\[
\ll X^\ell \sum_{0 \leq m \leq \ell} \sum_{s \in [1, q/2)^{\ell} \cap \mathbb{Z}^m} \prod_{1 \leq u \leq m} \frac{q}{s_v} \ll X^\ell \sum_{0 \leq m \leq \ell} \left( \frac{q}{X} \right)^m \left( \sum_{1 \leq s \leq q} \frac{1}{s} \right)^m
\]

\[
\ll X^\ell \sum_{0 \leq m \leq \ell} \left( \frac{q \log q}{X} \right)^m \leq \ell (X^\ell + q^\ell \log^\ell q).
\]

Therefore we obtain that

\[
H(X, A, \alpha) \ll A \sum_{x \in B_{q}(X/q)} \left( X^\ell + q^\ell \log^\ell q \right) \ll \left( 1 + \frac{X}{q} \right)^\ell \left( X^\ell + q^\ell \log^\ell q \right).
\]

which completes the proof of the lemma.

By this lemma, we have a nontrivial estimate for \( I_F(\alpha, X) \) as follows.
Lemma 2.8. Let $F(x)$ defined by (1.2) and (1.3). Also let let $\alpha = a/q + \beta$ with $q$ be an positive integer, $(a, q) = 1$ and $|\beta| \leq q^{-2}$. Then, we have

$$I_F(\alpha, X) \ll_F X^{\ell} q^{-\frac{\ell}{2}} + X^{\ell} \log^2 q + q^{\ell} \log^2 q.$$  

Proof. First of all, we have that

$$|I_F(X, \alpha)|^2 = \sum_{x, y \in B_r(X)} e((F(x) - F(y))\alpha) = \sum_{x \in \mathbb{Z}^\ell \cap (-X, X)^\ell} \sum_{y, y + x \in B_r(X)} e((F(x + y) - F(y))\alpha) = \sum_{x \in \mathbb{Z}^\ell \cap (-X, X)^\ell} \sum_{y, y + x \in B_r(X)} e((F(x) - c + y^t(Q_m^t + Q_m)x)\alpha)$$

$$\ll \sum_{h \in \mathbb{Z}^\ell \cap (-X, X)^\ell} \sum_{y, x + y \in B_r(X)} e(y^t(Q_m^t + Q_m)x\alpha).$$

Now we write the symmetric matrix $Q_m S = (Q_1, Q_2, ..., Q_\ell)$ ($Q_j \in \mathbb{Z}^\ell$, $j = 1, ..., \ell$). Then, using the fact that

$$\sum_{a < n \leq b} e(n\alpha) \ll \min(b - a + 1, \|2\alpha\|^{-1})$$

we obtain that

$$\sum_{y, x + y \in B_r(X)} e(y^t(Q_m^t + Q_m)x\alpha)$$

$$\ll \prod_{1 \leq v \leq \ell} \min(X - |h_v|, \|2Q_v x\alpha\|^{-1}) \ll \prod_{1 \leq v \leq \ell} \min(X, \|2Q_v x\alpha\|^{-1}).$$

Using the same method as in the proof of Lemma 2.8, we can derive that

$$\sum_{x \in \mathbb{Z}^\ell \cap (-X, X)^\ell} \prod_{1 \leq v \leq \ell} \min(X, \|2Q_v x\alpha\|^{-1}) \ll_F X^{2\ell} q^{-\ell} + X^{\ell} \log^2 q + q^{\ell} \log^2 q.$$  

This completes the proof of the lemma. \qed

3. Singular integral

The well known results says that the gaussian integral

$$\int_{\mathbb{R}^\ell} dx \exp(-x^t Ax)$$

converges if $A$ is a symmetric complex matrix with the real part of $A$ is non-negative and no eigenvalue $\alpha_i$ of $A$ vanishes. Hence we obtain that

$$\int_{[0, 1]^\ell} e\left((t^t Q_m t + b^t t/X + c/X^2) \lambda\right) dt$$

$$= |\lambda|^{\ell} \int_{[0, \sqrt{\lambda}]^\ell} e\left(u^t Q_m u + b^t \sqrt{|\lambda|} u + c|\lambda| X \lambda X^t \right) \text{sign}(\lambda) \, du \ll_F |\lambda|^{-\frac{\ell}{2}},$$

where $\text{sign}(\lambda)$ is general symbol function and $\lambda \neq 0$. We have the follows lemma.
Lemma 3.1. We have
\[
\int_{[\beta \leq \frac{Q}{X^2} \cdot [1, X]^\ell]} d\beta \alpha \int \left(\frac{F(t)}{u} \right)^{r\beta} \left(\log u\right)^{r\beta} du - \int_{[1, X]^\ell} dt \left(\log(F(t))\right)^{r\beta} \approx_{k,F} X^{\ell + \epsilon} \left(\frac{q}{Q}\right)^{\frac{\ell}{2}}.
\]

Proof. Firstly, we have
\[
\int_{[\beta \leq \frac{Q}{X^2} \cdot [1, X]^\ell]} d\beta \int_{[1, X]^\ell} e\left(F(t)\beta\right) dt \int_{[1, X]^\ell} (\log u)^{r\beta} e\left(-u\beta\right) du
\]
\[
= X^\ell \int_{[\beta \leq \frac{Q}{X^2} \cdot [1, X]^\ell]} d\beta \int_{[1, X]^\ell} e\left(F(t)\beta\right) dt \int_{[1, X]^\ell} (\log(X^2u))^\ell e\left(-u\beta\right) du,
\]
where \(F(t, X) = t^\ell Q_m t + b^\ell t/X + c/X^2\). If \(|\alpha| > Q/q \geq 1\), using (3.1) then
\[
I_{r,F}(\mu, X) = \int_{[1, X]^\ell} \left(\log(uX^2)\right)^\ell e\left(-u\mu\right) du \int_{[1, X]^\ell} e\left(F(t, X)\mu\right) dt
\]
\[
\approx_{f,F} \left|\int_{[\alpha \leq \frac{Q}{X^2} \cdot [1, X]^\ell]} \left(\log u + \log|\alpha| + 2\log X\right)^\ell e\left(-u\beta\right) du\right|
\]
\[
\approx_{k,F} |\alpha|^{-\frac{\ell+2}{4} + \epsilon} \log X \sum_{t=0}^{k} \int_{[\alpha \leq \frac{Q}{X^2} \cdot [1, X]^\ell]} \left(\log u\right)^e\left(-u\beta\right) du
\]
\[
\approx_{k,F} |\alpha|^{-\frac{\ell+2}{4} + \epsilon} \log X \log^k(\alpha, X) \approx_{k,F} |\alpha|^{-\frac{\ell+2}{4} + \epsilon} X^\epsilon.
\]
The above result implies that
\[
(3.2)
\int_{[\alpha \leq Q/q]} I_{r,F}(\mu, X) d\mu = \int_{[1, X]^\ell} I_{r,F}(\mu, X) d\mu + O_{k,F} \left(X^\epsilon(Q/q)^{-\frac{3}{4}}\right).
\]
On the other hand,
\[
I_{r,F}(X) = \int_{[1, X]^\ell} d\mu \int_{[1, X]^\ell} \left(\log(uX^2)\right)^\ell e\left(-u\mu\right) du \int_{[1, X]^\ell} e\left(F(t, X)\mu\right) dt
\]
\[
= 2\int_{[1, X]^\ell} d\mu \int_{[1, X]^\ell} \left(\log(uX^2)\right)^\ell du \int_{[1, X]^\ell} dt \cos\left(2\pi u - F(t, X)\mu\right)
\]
\[
= \frac{1}{\pi} \int_{[1, X]^\ell} dt \int_{[1, X]^\ell} d\mu \int_{[1, X]^\ell} \left(\log(uX^2)\right)^\ell d\left(\frac{\sin\left(2\pi u - F(t, X)\mu\right)}{\mu}\right)
\]
\[
= \frac{1}{\pi} \int_{[1, X]^\ell} dt \int_{[1, X]^\ell} \left(\log(uX^2)\right)^\ell d\left(\frac{\pi}{2} \text{sign}(u - F(t, X))\right),
\]
where we have used the fact: \(\int_0^\infty \frac{\sin(\alpha x)}{x} dx = \frac{\pi}{2} \text{sign}(\alpha)\). Note that
\[
\text{sign}(\alpha) = \begin{cases} 
\frac{\alpha}{|\alpha|} & \alpha \neq 0 \\
0 & \alpha = 0,
\end{cases}
\]
then part integration yields

\[
I_{r,F}(X) = \frac{1}{2} \int_{[1/X,1]} dt \int_{1/X^2}^{N_F(X)} \frac{du}{X^2} \left( \log(uX^2) \right)^r d \left( \text{sign}(u - F(t, X)) \right)
\]

\[
= \frac{1}{2} \int_{[1/X,1]} dt \int_{|u - F(t, X)| \leq \varepsilon} \frac{1}{X^2} \left( \log(uX^2) \right)^r d \left( \text{sign}(u - F(t, X)) \right)
\]

\[
= \frac{1}{2} \int_{[1/X,1]} dt \left( 2 \left( \log(X^2F(t, X)) \right)^r + O_k,F(\varepsilon \log X) \right)
\]

\[
= \int_{[1/X,1]} dt \left( \log(F(Xt)) \right)^r = X^{-r} \int_{[1/X,1]} dt \left( \log(F(t)) \right)^r.
\]

Together with (3.2) and above, we get the proof the lemma. \(\square\)

4. The Proof of Main Theorem

Where we refer the methods of Pleasants [5] to deal with the minor arcs. Firstly, let \(j \in \mathbb{N}\) and define

\[
\mathfrak{M}(2^j Q) = \{ \alpha \in [0,1] : |\alpha - a/q| \leq 2^j Q/(qX^2), \text{ with } q \leq 2^j Q, \ a \in \mathbb{Z}_q^* \}.
\]

It is obviously that,

\[
\mathfrak{M}(2^j Q) \subseteq \mathfrak{M}(2^{j+1} Q)
\]

and \(\mathfrak{M}(2^j Q) = [0,1]\) when \(j > \lfloor (\log(X/Q))/(\log 2) \rfloor := N\) by well know Dirichlet’s approximation theorem. If we define

\[
(4.1) \quad \mathcal{F}_j(Q) = \mathfrak{M}(2^{j+1} Q) \setminus \mathfrak{M}(2^j Q)
\]

for \(j = 0, 1, 2, \ldots, N\), then for all \(i \neq j (i, j = 0, 1, 2, \ldots, N)\) one has

\[
[0,1] = \mathfrak{M}(Q) \cup \left( \bigcup_{0 \leq j \leq N} \mathcal{F}_j(Q) \right), \quad \mathcal{F}_i(Q) \cap \mathcal{F}_j(Q) = \emptyset \text{ and } \mathcal{F}_i(Q) \cap \mathfrak{M}(Q) = \emptyset.
\]

We take \(\mathfrak{M}(Q)\) as the major arcs, and the minor arcs is \(\mathfrak{m}(Q) = [0,1] \setminus \mathfrak{M}(Q)\). As we all know, \(\mathfrak{M}(2^j Q)\) is the union of all disjoint small intervals \(\mathfrak{M}_j(q,a)\) with \(1 \leq q \leq 2^j Q\) and \((a,q) = 1\), where \(\mathfrak{M}_j(q,a) = [a/q - 2^j Q(qX^2)^{-1}, a/q + 2^j Q(qX^2)^{-1}]\). Thus we have

\[
\mathfrak{M}(2^j Q) = \bigcup_{1 \leq q \leq 2^j Q} \bigcup_{a \in \mathbb{Z}_q^*} \mathfrak{M}_j(q,a)
\]

for all \(j = 0, 1, \ldots, N\) and \(\mathfrak{m}(Q) = \bigcup_{j=0}^N \mathcal{F}_j(Q)\). Therefore

\[
T_{k,F}(X) = \int_0^1 I_F(\alpha, X)J_k(-\alpha, N_F(X))d\alpha
\]

\[
= \left\{ \int_{\mathfrak{M}(Q)} + \int_{\mathfrak{m}(Q)} \right\} I_F(\alpha, X)J_k(-\alpha, N_F(X))d\alpha := T_{\mathfrak{M},k,F}(X) + T_{\mathfrak{m},k,F}(X).
\]

Applying the Cauchy-Schwarz inequality give an estimate for the minor arcs integral as follows

\[
T_{\mathfrak{m},k,F}(X) = \int_{\mathfrak{m}(Q)} I_F(\alpha, X)J_k(-\alpha, N_F(X))d\alpha = \sum_{j=0}^N \int_{\mathcal{F}_j(Q)} I_F(\alpha, X)J_k(-\alpha, N_F(X))d\alpha
\]

\[
\leq \sum_{j=0}^N \left( |\mathcal{F}_j(Q)|^{\frac{1}{2}} \sup_{\alpha \in \mathcal{F}_j(Q)} |I_F(\alpha, X)| \right) \left( \int_{\mathcal{F}_j(Q)} |J_k(\alpha, N_F(X))|^2 d\alpha \right)^{\frac{1}{2}}.
\]
where \(|F_j(Q)|\) is the Lebesgue measure of the set \(F_j(Q)\). By (4.1) one has

\[
|F_j(Q)| \leq |\mathcal{W}_{j+1}(Q)| \leq \sum_{q \leq 2^{j+1}Q} \varphi(q) \int_{|\lambda| \leq \frac{2^{j+1}Q}{qX^2}} d\lambda \leq 4(2^jQX^{-1})^2.
\]

For \(j \leq N\), notes that \(2^jQ \leq X\) and Lemma 2.8 one has

\[
\sup_{\alpha \in F_j(Q)} |I(F; X, \alpha)| \ll_X X^\ell(2^jQ)^{-\frac{\ell}{2}} + X^{\frac{\ell}{2}} \log^\ell X.
\]

Hence by (4.3), (4.4) and \(\ell \geq 3\) implies

\[
\sum_{j=0}^{N} \left( |F_j(Q)|^{\frac{3}{2}} \sup_{\alpha \in F_j(Q)} |I_F(\alpha, X)| \right) \ll_F \sum_{j=0}^{N} 2^j Q \left( X^\ell(2^jQ)^{-\frac{\ell}{2}} + X^{\frac{\ell}{2}} \log^\ell X \right)
\]

\[
\ll X^{\ell-1} Q^{-\frac{\ell-2}{2}} + X^{\frac{\ell}{2}} \log^\ell X.
\]

On the other hand

\[
\int_0^1 |J(\alpha, N_F(X))|^2 d\alpha = \sum_{m \leq N_F(X)} \tau_k(m)^2 \ll_{k,F} X^{2+\varepsilon},
\]

hence together it with (4.2) and (4.5) we obtain that

\[
T_{m,k,F}(X) \ll_{k,F} \left( Q^{-\frac{4\varepsilon}{2\ell+4}} + X^{-\frac{4\varepsilon}{2\ell+4}} \right) X^\ell + \varepsilon.
\]

For the major arc, by Lemma 2.4 and Lemma 2.5, we have

\[
T_{2R(k,F)}(X) = \int_{\mathcal{W}_1(Q)} d\alpha I_F(\alpha, X) J_k(-\alpha, N_F(X))
\]

\[
= \sum_{q \leq Q} \sum_{\alpha \in \mathbb{Z}[\beta] |\beta| \leq \frac{Q}{qX^2}} \int \beta \left( \frac{S_F(q, \alpha)}{q^\ell} \int_{[1, X]^{\ell}} e(F(t)\beta) dt + O_F(q(1 + |\beta|X^2)X^{\ell-1}) \right)
\]

\[
 \times \left( \sum_{r=0}^{k-1} \beta(r) \int_1^{N_F(X)} (\log u)^r e(-u\beta) du + O_{k,F} \left( q(1 + |\beta|X^2)X^{2-\frac{4\varepsilon}{2\ell+4}} \right) \right).
\]

Note that Lemma 2.6 and \(\beta_{k,r} \ll q^{-1+\varepsilon}\), we obtain that

\[
T_{2R(k,F)}(X) = \sum_{r=0}^{k-1} \sum_{q \leq Q} \beta_{k,r}(q) S_F(q) \int \beta \left( \frac{S_F(q, \alpha)}{q^\ell} \int_{[1, X]^{\ell}} e(F(t)\beta) dt \int_{[1, X]^{\ell}} (\log u)^r e(-u\beta) du + O_{k,F} \left( q^2 X^{\ell-1+\varepsilon} \right) \right),
\]

where

\[
S_F(q) = \sum_{\alpha \in \mathbb{Z}_q^\times} q^{-\ell} S_F(q, \alpha).
\]
On the other hand, by Lemma 3.1 we have

\[ T_{2k,F}(X) = \sum_{r=0}^{k-1} \left( \frac{H_{k,r}(F)}{[1,X]^\ell} \left( \log F(t) \right)^r dt + \sum_{q>Q} \beta_{k,r}(q) S_F(q) \frac{\log F(t)}{[1,X]^\ell} dt \right) \]

\[ + O_{k,F} \left( X^{\ell+\epsilon} Q^{\frac{\ell}{\ell+2}} + Q^2 X^{\ell-\frac{\ell}{\ell+2}+\epsilon} + Q^2 X^{\ell-1+\epsilon} + Q^4 X^{\ell-\frac{\ell}{\ell+1}+\epsilon} \right) \]

\[ = \sum_{r=0}^{k-1} H_{k,r}(F) \frac{\log F(t)}{[1,X]^\ell} dt \]

\[ + O_{k,F} \left( X^{\ell+\epsilon} \left( Q^{\frac{\ell}{\ell+2}} + Q^2 X^{-\frac{\ell}{\ell+1}} + Q^2 X^{-1} + Q^4 X^{-\frac{\ell}{\ell+1}} \right) \right), \]

where

(4.8) \[ H_{k,r}(F) = \sum_{q=1}^{\infty} S_F(q) \beta_{k,r}(q). \]

It is easily seen that when \( Q = X^{\min(1,4/(k+1))/(\ell+2)} \), one has the optimal estimate

\[ T_{k,F}(X) = \sum_{r=0}^{k-1} H_{k,r}(F) \frac{\log F(t)}{[1,X]^\ell} dt + O_{k,F} \left( X^{\ell-\frac{\ell}{\ell+2} \min(1, \frac{1}{k+1}) + \epsilon} \right). \]

We define

\[ L(s; k, F) = \sum_{q=1}^{\infty} S_F(q) F_k(q, s), \]

then combine Lemma 2.4 and (4.8) we obtain that

\[ H_{k,r}(F) = \frac{1}{r!} \sum_{t=0}^{k-r-1} \frac{1}{t!} \left( \frac{d^t L(s; k, F)}{ds^t} \right) \bigg|_{s=1} \text{Res} \left( (s-1)^{\ell+t} \zeta(s)^k; s = 1 \right). \]

We next try to give an explicit expression for \( L(s; k, F) \).

**Lemma 4.1.** The function \( L(s; k, F) \) have the Euler product as follows

\[ L(s; k, F) = \prod_p \left( 1 + \sum_{m=1}^{\infty} S_F(p^m) F_k(p^m, s) \right), \]

where

\[ S_F(p^m) = \varphi(p)^{m-1} \varphi_F(p^m) - p^{-(\ell-1)m} \varphi_F(p^{m-1}), \]

\[ \varphi_F(n) = \# \{ h \in (\mathbb{Z}/n)^\ell : F(h) \equiv 0 \mod n \} \]

for \( n \in \mathbb{N}_{\geq 1} \) and

\[ F_k(p^m, s) = (1 - p^{-s})^{k-1} \frac{\tau_k(p^m)}{p^{ms}} + p^{-s} \sum_{v=1}^{k-1} (1 - p^{-s})^{u-1} \tau_v(p^m) \frac{p^{ms}}{p^m} - \varphi(p)^{-1} (1 - p^{-s})^k \frac{\tau_k(p^{m-1})}{p^{(m-1)s}}. \]

**Proof.** It is easily seen that

\[ S_F(q) = q^{-\ell} \sum_{a \in \mathbb{Z}_q} \sum_{h \in (\mathbb{Z}/q)^\ell} e\left( \frac{q F(h)}{q} \right) \]

is real and multiplicative. When \( q = p^m \) be a prime power with integer \( m \geq 1 \). It is easily seen that

\[ S_F(p^m) = p^{-(\ell-1)m} \varphi_F(p^m) - p^{-(\ell-1)(m-1)} \varphi_F(p^{m-1}). \]
On the other hand, by Lemma 2.2 we shown that $F_k(q, s)$ is also multiplicative. Thus above implies the Euler product of $L(s; k, F)$. Applying Lemma 2.1 and Lemma 2.2, we have

$$F_k(p^m, s) = f_k(p^m, p^m, s) - f_k(p^m, p^{m-1}, s)$$

$$= p^{-ms} \sum_{d_1 \ldots d_k = p^m} \prod_{i=1}^{k-1} q(i(q^{\prod_{r=i+1}^k d_r}) q \text{ prime} (1 - \frac{1}{q^s}) - \frac{p^s}{\varphi(p)p^{ms}} (1 - \frac{1}{p^s})^k \tau_k(p^{m-1}).$$

For the first term above, denote by

$$I_k = \sum_{d_1 \ldots d_k = p^m} \prod_{i=1}^{k-1} q(i(q^{\prod_{r=i+1}^k d_r}) q \text{ prime} (1 - \frac{1}{q^s}).$$

Then

$$I_k = \sum_{v=0}^{m} \sum_{d_1 d_2 \ldots d_{k-1} = p^m - v} \prod_{i=1}^{k-1} q(i(q^{\prod_{r=i+1}^k d_r}) q \text{ prime} (1 - \frac{1}{q^s})$$

$$= \sum_{d_1 d_2 \ldots d_{k-1} = p^m} \prod_{i=1}^{k-2} q(i(q^{\prod_{r=i+1}^k d_r}) q \text{ prime} (1 - \frac{1}{p^s}) + \sum_{v=1}^{m} \sum_{d_1 d_2 \ldots d_{k-1} = p^m - v} \prod_{i=1}^{k-1} (1 - \frac{1}{p^s})^{k-1}$$

$$= I_{k-1} + (1 - p^{-s})^{k-1} \sum_{v=1}^{m} \tau_{k-1}(p^{m-v}) = I_{k-1} + (1 - p^{-s})^{k-1} (\tau_k(p^m) - \tau_{k-1}(p^m))$$

$$= (1 - p^{-s})^{k-1} \tau_k(p^m) + p^{-s} \sum_{v=1}^{k-1} (1 - p^{-s})^{v-1} \tau_v(p^m).$$

Hence

$$F_k(p^m, s) = (1 - p^{-s})^{k-1} \frac{\tau_k(p^m)}{p^{ms}} + p^{-s} \sum_{v=1}^{k-1} (1 - p^{-s})^{v-1} \frac{\tau_v(p^m)}{p^{ms}} - \varphi(p)^{-1} (1 - p^{-s})^k \frac{\tau_k(p^{m-1})}{p^{(m-1)s}}. \square$$

Combining above estimates and calculations, we obtain the proof of the main theorem.

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