A SOPHISTICATED PROOF OF THE MULTIPLICATION FORMULA FOR MULTIPLE WIENER INTEGRALS

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Abstract. We prove that the formula which gives the Wiener chaos decomposition of the multiplication of two multiple Wiener integrals with symmetric kernels is a straightforward application of the Leibniz’ formula.

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1. Introduction

Let \((W, H, \mu)\) be the classical Wiener space, i.e., \(W = C_0([0, 1], \mathbb{R}^d)\), \(H\) is the corresponding Cameron-Martin space consisting of \(\mathbb{R}^d\)-valued absolutely continuous functions on \([0, 1]\) with square integrable derivatives, which is a Hilbert space under the norm \(|h|^2_H = \int_0^1 |\dot{h}(s)|^2 ds\), where \(\dot{h}\) denotes the Radon-Nikodym derivative of the absolutely continuous function \(t \mapsto h(t)\) w.r.t. the Lebesgue measure on \([0, 1]\). Denote by \((F_t, t \in [0, 1])\) the filtration of the canonical Wiener process, completed w.r.t. \(\mu\)-negligeable sets. For \(f \in H^{\otimes p}\), i.e., the \(p\)-th order symmetric tensor product of \(H\), we write \(I_p(f)\) the multiple Wiener integral of \(f\), which is defined as the iterated integral:

\[ I_p(f) = p! \int_{t_1 < t_2 < \ldots < t_p < 1} f(t_1, \ldots, t_p) dW_{t_1} \ldots dW_{t_p}. \]

Let \(g \in H^{\otimes q}\), then the following multiplication formula, which is due to Shigekawa (cf. [5]) is extremely important.

Theorem 1. We have

\[ I_p(f)I_q(g) = \sum_{i=0}^{p \wedge q} \frac{p!q!}{i!(p-i)!(q-i)!} I_{p+q-2i}(f^{\otimes_i}g) \mu\text{-a.s.}, \]

where \(f \otimes_i g\) denotes the tensor \(f \otimes g\) which is contracted in its \(2i\) components, i.e.,

\[ f \otimes_i g(t_1, \ldots, t_{p-i}, s_1, \ldots, s_{q-i}) = \int_{[0, 1]^i} f(t_1, \ldots, t_{p-i}, u_1, \ldots, u_i) g(s_1, \ldots, s_{q-i}, u_1, \ldots, u_i) du_1 \ldots du_i \]

and \(f^{\otimes_i}g\) is the symmetrization of \(f \otimes_i g\) in its remaining \(p + q - 2i\) variables.

This theorem has been proved in 1980 by Shigekawa using Itô formula and induction. It is astonishing that no other proof has been seen in the mathematical literature in spite of all the new techniques developed through the applications and the extensions of the Malliavin calculus. We shall give here a completely new proof by relating the formula given above to the Leibniz formula for \(n\)-th order derivative of the multiplication of two smooth functions which is given below, whose proof follows from its one-dimensional version:

Lemma 1. Assume that \(F, G\) are two real-valued polynomials on \(W\), then, for any \(n \in \mathbb{N}\), we have

\[ \nabla^n(FG) = \sum_{i=0}^{n} \binom{n}{i} \nabla^i F \otimes \nabla^{n-i} G. \]

almost surely.
To make this note self-contained, we shall give also some results connecting the Meyer distributions on Wiener space to the Itô-Wiener chaos decomposition of the elements of $L^2(\mu)$. For this we need some notations which are explained in the next section.

2. Notations

We denote by $\nabla$ the Sobolev derivative on $(W, H, \mu)$ in the direction of the Cameron-Martin space extended to $L^p(\mu)$, $p > 1$, the corresponding Sobolev spaces of real-valued functions are denoted by $D_{p,k}$, $p > 1$, $k \in \mathbb{N}$, where $k$ denotes the degree of differentiability and $p$ denotes the degree of invertibility. For vector valued functions, we use the notation $D_{p,k}(X)$, where $X$ is the range space. Note that, for any $F \in D_{p,k}$, $\nabla F$ is an element of $D_{p,k-1}(H)$. The formal adjoint of $\nabla$ w.r.t. $\mu$ is denoted as $\delta$ and called the divergence operator. It is easy to see that $D_{p,1}(H)$ is in the domain of $\delta$ and for a $\xi \in D_{p,1}(H)$, $\delta \xi$ coincides with the Itô integral of $\xi$ if the latter is adapted to the Wiener filtration, here $\xi$ is the Sobolev derivative of $t \rightarrow \xi(t, w) \in H$. We define the Ornstein-Uhlenbeck operator as $L = \delta \circ \nabla$, it follows from Meyer inequalities that the seminorms defined by $\| (I + L)^{k/2} F \|_{L^p(\mu, X)}$ are equivalent to the Sobolev norms explained above for each $p > 1$ and $k \in \mathbb{N}$. Since the seminorms defined with $L$ are also extendable to the case $k \in \mathbb{R}$ and $p > 1$, we obtain a scale of Banach spaces, still denoted by the same notation $D_{p,k}(X)$, for $p > 1$ and $k \in \mathbb{R}$. This construction implies that $\nabla$ has a continuous extension as a map from $D'(X) \rightarrow D'(X \otimes H)$, where

$$D'(X) = \bigcup_{p>1, k \in \mathbb{R}} D_{p,k}(X)$$

and that $\delta$ has a continuous extension from $D'(X \otimes H) \rightarrow D'(X)$, where the unions are all equipped with their inductive limit topologies. We denote by $D(X)$ the intersection of the Sobolev spaces $(D_{p,k}(X) : p > 1, k \in \mathbb{R})$ equipped with the projective topology.

As an example of these considerations let $\phi \in D$ be a polynomial (i.e., $X = \mathbb{R}$), then for any $h \in H$, denoting by $\rho(\delta h)$ the Wick exponential $\exp(\delta h - 1/2|h|_H^2)$, due to Cameron-Martin theorem, we have

$$E[\phi \rho(\delta h)] = E[\phi(\cdot + h)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} E[(\nabla^n \phi, h^{\otimes n})_{H^{\otimes n}}]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} E[\nabla^n \phi, h^{\otimes n})_{H^{\otimes n}}]$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} E[I_n(E[\nabla^n \phi])I_n(h^{\otimes n})]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} E[I_n(E[\nabla^n \phi])\rho(\delta h)] .$$

Since the linear combinations of the Wick exponentials are dense in any $L^p(\mu)$, $p > 1$, we deduce from the above calculations, on the one hand that

$$\phi = E[\phi] + \sum_{n=1}^{\infty} \frac{I_n(E[\nabla^n \phi])}{n!} \quad \mu\text{-a.s.,}$$

and on the other hand that

$$\delta^n h^{\otimes n} = I_n(h^{\otimes n}) \quad \mu\text{-a.s.}$$

By density of the linear combinations of $\{h^{\otimes n}, h \in H\}$ in $H^{\otimes n}$, we deduce that

$$\delta^n \eta = I_n(\eta)$$
\(\mu\text{-a.s. for any } \eta \in H^{\otimes n}, \text{ where } \delta^n \text{ denotes the adjoint of } \nabla^n \text{ w.r.t. the measure } \mu. \) Furthermore the relation (2.1) has been given in an informal manner by Mc. Kean, \([2]\), later D. Stroock has remarked that this expression extends to \(L^n\) functionals since

\[
(E[\nabla^n \phi], h^{\otimes n})_{H^{\otimes n}} = \langle \nabla^n \phi, h^{\otimes n} \rangle,
\]

consequently

\[
|E[\nabla^n \phi]|_{H^{\otimes n}} \leq \sqrt{n!} \|\phi\|_{L^n(\mu)}.
\]

Let summarize what we have explained:

**Theorem 2** (McKean-Stroock). *The map defined on the smooth functions with values in \(H^{\otimes n}, \text{ defined as } \phi \mapsto E[\nabla^n \phi] \text{ has a unique bounded (linear) extension to the whole space } L^n(\mu) \text{ for any } n \geq 1 \text{ and if we denote it again with the same notation, then the following identity holds true}*

\[
\phi = \sum_{i=1}^{\infty} \frac{1}{i!} E[\nabla^n \phi]
\]

where the sum converges in \(L^n(\mu)\).

\[=\]

### 3. Proof of the Multiplication formula

Suppose that \(p > q\) and let \(\phi \in \mathbb{D}\), using the identity \(\delta^p f = I_p(f)\) and the fact that \(\delta^p\) is the adjoint of the operator \(\nabla^p\), we get, from Lemma [1]

\[
E[I_p(f)I_q(g)\phi] = E[(f, \nabla^p(I_q(g)\phi))]_{H^{\otimes r}}
\]

\[
= E \left[ \sum_{i=0}^{p} \binom{p}{i} (f, \nabla^i I_q(g) \hat{\otimes} \nabla^{p-i} \phi)_{H^{\otimes r}} \right]
\]

\[
= E \left[ \sum_{i=0}^{p} \binom{p}{i} \frac{q!}{(q-i)!} (f, I_{q-i}(g) \otimes \nabla^{p-i} \phi)_{H^{\otimes r}} \right]
\]

\[
= \sum_{i=0}^{p} \binom{p}{i} \frac{q!}{(q-i)!} E[(f, I_{q-i}(g) \otimes \nabla^{p-i} \phi)_{H^{\otimes r}}]
\]

\[
= \sum_{i=0}^{p} \binom{p}{i} \frac{q!}{(q-i)!} E[(I_{q-i}(g) \otimes I_i f, \nabla^{p-i} \phi)_{H^{\otimes (p-i)}}]
\]

\[
= \sum_{i=0}^{p} \binom{p}{i} \frac{q!}{(q-i)!} E[(g \otimes I_i f, \nabla^{q-i} \nabla^{p-i} \phi)_{H^{\otimes (p+q-2i)}}]
\]

\[
= \sum_{i=0}^{p} \binom{p}{i} \frac{q!}{(q-i)!} E[I_{p+q-2i}(g \otimes_i I_i f, \phi)]
\]

in the third equality we have used the fact that \(f\) is a symmetric tensor and the proof of Theorem [1] follows. \(\square\)
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