Non-rigidity degrees of root lattices and their duals

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Abstract

Non-rigidity degree of a lattice \( L \), \( nrd_L \), is dimension of the L-type domain to which \( L \) belongs. We complete here the table of \( nrd \)'s of all root lattices and their duals; namely, the hardest remaining case of \( D_n^* \), and the case of \( E_7^* \) are decided.

We describe explicitly the \( L \)-type domain \( \mathcal{D}(D_n^*) \), \( n \geq 4 \). For \( n \) odd, it is a non-simplicial polyhedral open cone of dimension \( n \). For \( n \) even, it is one-dimensional, i.e. for even \( n \), \( D_n^* \) is an edge form.

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1 Introduction

Voronoi [Vo1908] defined the partition of the cone \( \mathcal{P}_n \) of positive semidefinite \( n \)-ary quadratic forms into \( L \)-type domains, that we call here \( L \)-domains. Forms of the same \( L \)-domain correspond to lattices that determine affinely equivalent Voronoi partitions of \( \mathbb{R}^n \), i.e. partition into Voronoi polytopes. The partition of \( \mathbb{R}^n \), which is (both, combinatorially and affinely) dual to a Voronoi partition, is called Delaunay partition and consists of Delaunay polytopes. In other words, two lattices have the same \( L \)-type if and only if the face posets of their Voronoi polytopes are isomorphic. (The face poset of a polytope \( P \) is the set of all faces of \( P \) of all dimensions ordered by inclusion.)

If the \( L \)-type of a lattice changes, then either some Delaunay polytopes are glued into a new Delaunay polytope, or a Delaunay polytope is partitioned into several new Delaunay polytopes. Recall that the center of a Delaunay polytope is a vertex of a Voronoi polytope. Hence if the \( L \)-type of a lattice changes, then for each Voronoi polytope either some vertices are glued into one vertex, or a vertex splits into several new vertices.

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Voronoi proved that each $L$-domain is an open polyhedral cone of the dimension $k$, $1 \leq k \leq N$, where $N = \frac{n(n+1)}{2}$ is the dimension of $\mathcal{P}_n$, i.e. dimension of the space of coefficients of $f \in \mathcal{P}_n$. An $L$-type having an $N$-dimensional domain, is called generic. Otherwise, $L$-type is called special.

In [BG01], a notion of non-rigidity degree of a form $f$ and the corresponding lattice $L(f)$ is introduced. It is denoted by $\text{nrd} f$ and is equal to dimension of the $L$-domain containing $f$. It is shown in [BG01] that $\text{nrd} f$ is equal to corank of a system of equalities connecting the norms of minimal vectors of cosets $2L$ in $L$. In fact, $\text{nrd}$ is the number of degrees of freedom, such that the Delaunay partition (more precisely, its star) has, when one deforms it affinely, so that the result remains an Delaunay partition. Clearly, the maximal $\text{nrd}$ is $\binom{n+1}{2}$ and it is realized only by simplicial Delaunay partitions.

So, $1 \leq \text{nrd} f \leq N$, and $\text{nrd} f = N$ if $f$ belongs to a generic $L$-domain. A form $f$ and the corresponding lattice $L(f)$ are called rigid if $\text{nrd} f = 1$. This name was used because any, distinct from a scaling, affine transformation of a rigid lattice changes its $L$-type. (Sometimes, a rigid form is called an edge form, since it lies on an extreme ray of the closure of an $L$-domain.) Clearly, any 1-dimensional lattice is trivially rigid. In [BG01], were given 7 examples of rigid lattices of dimension 5 and shown that $D_4$ is unique such a lattice of dimension $n$, $2 \leq n \leq 4$.

The set of all $L$-domains is partitioned into classes of unimodularly equivalent domains. For $n \leq 3$, there is only one class, i.e. only one generic $L$-type. For $n = 4$, there are 3 generic $L$-types.

The $L$-domain $\mathcal{A}_n$ of the lattice $A_n^*$ which is dual to the root lattice $A_n$ is well known. $\mathcal{A}_n$ has dimension $N$, i.e. it is generic. It is the unique $L$-domain such that all the extreme rays of its closure span forms of rank 1. All these facts were known to Voronoi. He called the domain $\mathcal{A}_n$ the first type domain and one of the forms of $A_n^*$ the principal form of the first type.

The $L$-domain of the lattice $A_n$, $n \geq 2$, is a simplicial $(n + 1)$-dimensional cone; it is described in [BG01]. So, $\text{nrd} A_n = n + 1$ for $n \geq 2$. Also $\text{nrd} Z_n = n$ for $n \geq 1$ and $A_1$, $D_2$, $D_3$ are scalings of $Z_1$, $Z_2$, $A_3$, respectively. So, $\text{nrd}(A_1 = A_1^*) = 1$, $\text{nrd}(D_2 = D_2^*) = 2$ and $\text{nrd} D_3 = 4$, $\text{nrd} D_3^* = 6$.

It is proved in [BG01] that the lattice $D_n$ is rigid for $n \geq 4$.

The lattices $E_6$, $E_6^*$, $E_7$, $E_7^*$ and $E_8 = E_8^*$ are rigid, i.e. their $L$-domains are one-dimensional. The rigidity of root lattices $E_6$, $E_7$ and $E_8$ is shown in [BG01]. The rigidity of the lattice $E_6^*$ was proved independently by Engel and Erdahl (personal communications). The rigidity of $E_7^*$ can be proved easily by using a nice symmetric quadratic form of $\Gamma(A) = E_7^*$ given in [Ba94] (see formula (2) there).

The rigid lattices $A_1$, $E_6$ and $E_7$ are first instances of strongly rigid lattices, i.e. such that amongst of their Delaunay polytopes there are extreme ones (see [DGL92]), i.e. such that any, distinct from a scaling, its affine transformation is not a Delaunay polytope. The 1-simplex and unique Delaunay polytope of $E_6$ are only such polytopes of dimension at most 6 (see [DD01]). $A_1$, $D_4 = D_4^*$, $E_6$ and $E_7^*$ are rigid lattices, having unique type of Delaunay polytope, but only for $A_1$ and $E_6$ this polytope is extreme. In [DGL92] (see also Chapter 16 of [DL97]) were given 10 examples of extreme Delaunay polytopes: by
In the basis \( \{ D_n \} \) was considered the general notion of rank of a Delaunay polytope, i.e. the number of degrees of freedom, that it has, when one deforms it affinely, so that the result remains an Delaunay polytope. Clearly, the maximal rank is (as well as maximal nrd) \( \binom{n+1}{2} \) and it is realized only by \( n \)-simplices.

In this note we describe explicitly the \( L \)-domain for the lattice \( D_n^* \) which is dual to the root lattice \( D_n \). This \( L \)-domain is special and has dimension \( n \) for even \( n \) and dimension 1 for even \( n \). This special \( L \)-domain is a facet of the closure of several generic \( L \)-domains.

This work completes computation of non-rigidity degree of root lattices and their duals. In a sense, it is an addition to the work \([\text{CS}91]\), where Delaunay and Voronoi polytopes of the root lattices and their duals are enumerated. We present the values of nrd for root lattices \( L \) and their duals in the table below.

| \( L \)  | \( A_1 = A_1^* \) | \( A_n \) \( n \geq 2 \) | \( A_n^* \) \( n \geq 1 \) | \( D_n \) \( n \geq 4 \) | \( D_{2m+1}^* \) \( m \geq 2 \) | \( D_{2m}^* \) \( m \geq 2 \) | \( E_6 \) | \( E_6^* \) | \( E_7 \) | \( E_7^* \) | \( E_8 = E_8^* \) |
|-------|-----------------|----------------|-----------------|----------------|----------------|----------------|-----|-----|-----|-----|-----|
| nrdL  | 1               | \( n + 1 \)    | \( \frac{n(n+1)}{2} \) | 1               | \( 2m + 1 \)    | 1               | 1   | 1   | 1   | 1   | 1   |

### 2 The cone \( \mathcal{G}_n \)

Let \( \{ e_i : i \in I_n \} \) be a set of mutually orthogonal vectors of norms (i.e. of squared lengths) \( e_i^2 = 2\gamma_i \), where \( I_n = \{ 1, 2, ..., n \} \). For \( S \subseteq I_n \), let \( e(S) = \sum_{i \in S} e_i \) and \( \gamma(S) = \sum_{i \in S} \gamma_i \). We introduce the vector \( b \) of norm \( \alpha \) as follows:

\[
b = \frac{1}{2} \sum_{i \in I_n} e_i = \frac{1}{2}e(I_n), \quad \text{where} \quad b^2 = \frac{1}{2} \sum_{i \in I_n} \gamma_i = \frac{1}{2} \gamma(I_n).
\]  

(1)

Let \( \gamma \) be the vector with the coordinates \( \{ \gamma_i : 1 \leq i \leq n \} \). Consider the lattice \( L(\gamma) \) generated by the vector \( b \) and any \( n-1 \) vectors \( e_i \). If \( \gamma_i = 1 \) for all \( i \), and \( n \geq 4 \), then \( L(\gamma) = D_n^* \).

We take as a basis of \( L(\gamma) \) the vector \( b \) and the vectors \( e_i \) for \( 1 \leq i \leq n-1 \). Then the coefficients of the quadratic form \( f_\gamma \) corresponding to this basis are as follows:

\[
a_{ii} = e_i^2 = 2\gamma_i, \quad 1 \leq i \leq n - 1, \quad a_{ij} = e_i e_j = 0, \quad 1 \leq i, j \leq n - 1, i \neq j, \quad (2)
\]

\[
a_{nn} = b^2 = \alpha, \quad a_{in} = e_i b = \gamma_i, \quad 1 \leq i \leq n - 1. \quad (3)
\]

This form has the following explicit expression

\[
f_\gamma(x) = (x_n b + \sum_{i=1}^{n-1} x_i e_i)^2 = \alpha x_n^2 + 2 \sum_{i=1}^{n-1} \gamma_i x_i^2 + 2 \sum_{i=1}^{n-1} \gamma_i x_i x_n. \quad (4)
\]

In the basis \( \{ e_i : i \in I_n \} \), each vertex of \( L(\gamma) \) has integer or half-integer coordinates.

Let \( n \) be odd, say \( n = 2m + 1 \). Suppose that the parameters \( \gamma_i \) satisfy the following \( \binom{n}{m} \) inequalities

\[
\sum_{i \in S} \gamma_i < \alpha, \quad S \subset I_n, \quad |S| = m. \quad (5)
\]
Denote by \( G_n \) the \( n \)-dimensional domain determined in the space of variables \( \gamma_i, \ i \in I_n \), by the inequalities (3). Since these inequalities are linear and homogeneous (recall that \( \alpha = \frac{1}{2} \gamma(I_n) \)), \( G_n \) is an open polyhedral cone. Since \( \gamma(S) + \gamma(I_n - S) = 2\alpha \), the inequalities (3) imply the following inequalities

\[
\gamma(T) > \alpha, \ T \subset I_n, \ |T| = m + 1.
\]

(6)

For a set \( T \) of cardinality \( |T| = m + 1 \), let \( T = S \cup \{i\} \), where \( |S| = m \). Then (3) and (3) imply

\[
\alpha < \gamma(T) = \gamma(S) + \gamma_i, \text{ i.e. } \gamma_i > \alpha - \gamma(S) > 0.
\]

Hence the cone \( G_n \) lies in the positive orthant of \( \mathbb{R}^n \).

Consider the closure \( \text{cl} G_n \) of the cone \( G_n \). Obviously, \( \text{cl} G_n \) is defined by the non-strict version of inequalities (3). So, using that \( 2\alpha = \gamma(I_n) \), we have

\[
\text{cl} G_n = \{ \overline{\gamma} : \gamma(S) - \gamma(I_n - S) \leq 0, \ S \subset I_n, |S| = m \}.
\]

(7)

Note that the zero vector belongs to \( \text{cl} G_n \). The automorphism group of \( \text{cl} G_n \) is isomorphic to the group of all permutations of the set \( I_n \).

Obviously, the hyperplanes supporting facets of \( \text{cl} G_n \) are contained among the hyperplanes defined by the equalities

\[
\gamma(S) = \gamma(I_n - S) = \alpha = \frac{1}{2} \gamma(I_n), \ S \subset I_n, \ |S| = m.
\]

(8)

Note that the equality \( \gamma(S_1) = \alpha \) can be transposed into the equality \( \gamma(S_2) = \alpha \) by the automorphism group, for any \( S_1, S_2 \subset I_n \) with \( |S_1| = |S_2| = m \). Hence each of the equations of (3) determines a facet of \( \text{cl} G_n \).

**Proposition 1** Let \( n \) be odd and \( n = 2m + 1 \geq 5 \), i.e. \( m \geq 2 \). Then the closure of \( G_n \) has the following \( 2n \) extreme rays

\[
\overline{\gamma}_q^k = \{ \gamma_i = \gamma \geq 0, i \in I_n - \{k\}, \gamma_k = 2q\gamma \}, \ q = 0, 1, \ k \in I_n.
\]

**Proof.** Let \( \overline{\gamma} \in \text{cl} G_n \) be fixed. Then \( \overline{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_n) \) defines a partition of the set \( I_n \) as follows. Let the coordinates \( \gamma_i \) take \( k \) distinct values \( 0 \leq \beta_1 < \beta_2 < ... < \beta_k \), where \( k \) is an integer between 1 and \( n \). For \( 1 \leq j \leq k \), set \( S_j = \{ i \in I_n : \gamma_i = \beta_j \} \) and \( s_j = |S_j| \). Then \( \sum_{j=1}^k s_j = n = 2m + 1 \) and \( I_n = \bigcup_{j=1}^k S_j \) is the above mentioned partition.

Consider the values of \( \gamma(S) \) for \( S \subset I_n, \ |S| = m \). \( \gamma(S) \) takes a maximal value for the following sets \( S \). Let \( j_0 \) be such that \( \sum_{j=1}^{j_0} s_j < m + 1 \), but \( \sum_{j=1}^{j_0} s_j \geq m + 1 \). Then \( \sum_{j=j_0+1}^k s_j \leq m \). Let \( S_{\max}(T) = T \cup_{j=j_0+1}^k S_j, \) where \( T \subseteq S_{j_0}, \ |T| = t_0 \) and \( t_0 := m - \sum_{j=j_0+1}^k s_j \). Obviously, \( \gamma(S) \) takes the maximal value \( t_0 \beta_{j_0} + \sum_{j=j_0+1}^k s_j \beta_j \) if \( S = S_{\max}(T) \) for any \( T \subseteq S_{j_0} \) of cardinality \( |T| = t_0 \).

For given \( \overline{\gamma} \in \text{cl} G_n \), let \( S_{\max}(\overline{\gamma}) \) be the system of equations of type (3), where \( S = S_{\max}(T) \) for all \( T \subseteq S_{j_0} \) with \( |T| = t_0 \). If \( \overline{\gamma} \) is an extreme ray of \( \text{cl} G_n \), then \( S_{\max}(\overline{\gamma}) \)
determines uniquely up to a multiple the vector $\gamma$. It is not difficult to see that $S_{\text{max}}(\gamma)$ can uniquely determine $\gamma$ only if $k = 2$. Consider this case in detail.

If $k = 2$, we have $I_n = S_1 \cup S_2$ and $n = 2m + 1 = s_1 + s_2$. Let $s_2 \leq m$, i.e. $j_0 = 1$. Then $S_{\text{max}}(\gamma)$ consists of the following equations

$$\gamma(T \cup S_2) = \gamma(T) + \gamma(S_2) = \alpha = \gamma(S_1 - T), \quad T \subseteq S_1, \quad |T| = m - s_2.$$

Note that $\gamma_i$ for $i \in S_2$ belongs to the above system only as a member of the sum $\gamma(S_2) = \sum_{i \in S_2} \gamma_i$. Hence such a system can determine the coordinates $\gamma_i, i \in S_2$, only if $s_2 = 1$.

Now the above system implies that $\gamma_i$ takes the same value, say $\gamma$, for all $i \in S_1$. In fact, let $i_1, i_2 \in S_1, i_1 \in T_1, i_1 \not\in T_2, i_2 \not\in T_1, i_2 \in T_2$, for some $T_1, T_2 \subset S_1$ of cardinality $m - 1$. Such $T_1$ and $T_2$ exist, since $|T_j| = m - 1 \geq 1$. Subtracting the equation of the above system for $T = T_2$ from the equation for $T = T_1$, we obtain the equality $\gamma_{i_1} - \gamma_{i_2} = \gamma_{i_2} - \gamma_{i_1}$, i.e. $\gamma_{i_1} = \gamma_{i_2}$.

In this case, the above system, where $S_2 = \{k\}$, gives $\gamma_k = \gamma(S_1) - 2\gamma(T) = s_1 \gamma - 2(m - 1)\gamma = 2\gamma$. We obtain the extreme ray $\gamma_{i_1}$.

Now, let $s_2 > m$, i.e. $s_1 < m + 1$ and $j_0 = 2$. A similar analysis shows that $s_1 = 1$, say $S_1 = \{k\}$, and $\gamma_i$ take the same value, say $\gamma$, for all $i \in S_2$. This gives $\gamma_k = 0$, and we obtain the extreme ray $\gamma_{i_2}$. The result follows.

The facet defined by the equation $\gamma(S) = \alpha, |S| = m$, contains the following $n = 2m + 1$ extreme rays: $\gamma_{i_1}, k \not\in S, \gamma_{i_2}, k \in S$. Each facet has the following geometrical description. The $m + 1$ rays $\gamma_{i_1}, k \not\in S$, form an $(m + 1)$-dimensional simplicial cone. Similarly, the $m$ rays $\gamma_{i_2}, k \in S$, form an $m$-dimensional cone. Both these cones intersect by the ray $\{\gamma : \gamma_i = (m + 1)\gamma, i \in S, \gamma_i = m\gamma, i \not\in S\}$. Hence the cone hull of these two cones is a cone of dimension $(m + 1) + m - 1 = 2m$. This cone is just a facet of $\mathcal{G}_n$ for $n = 2m + 1$.

Let $n$ be even, $n = 2m$. In this case, all the inequalities (3) imply the following set of inequalities

$$\gamma(I_n - S) = \gamma(T) > \alpha, \quad T = I_n - S \subset I_n, \quad |T| = m.$$

We see that this system of inequalities contradicts to the system (5). This means that the open cone $\mathcal{G}_n$ for even $n = 2m$ is empty. But the solution of the set of equalities (8) is not empty. Namely, it has the solution $\gamma_i = \gamma \geq 0$ for all $i \in I_n$. In other words, $\text{cl}\mathcal{G}_n$ is the following ray

$$\text{cl}\mathcal{G}_{2m} = \{\gamma : \gamma_i = \gamma \geq 0, i \in I_{2m}\}.$$

3 The domain $\mathcal{D}_n$

Denote by $\mathcal{D}_n$ the domain of forms $f_\gamma$, where $\gamma$ belongs to $\mathcal{G}_n$.

We prove the following theorem.

Theorem 1 Let $n$ be odd, $n = 2m + 1$. The domain $\mathcal{D}_n$ is an $L$-domain. It lies in an $n$-dimensional space which is an intersection of $\binom{n}{2}$ hyperplanes given by the following
equalities
\[ a_{ij} = 0, 1 \leq i < j \leq n - 1, 2a_{in} = a_{ii}, 1 \leq i \leq n - 1. \] (9)

The domain \( D_n \) is cut from this space by the following inequalities
\[ \sum_{i \in S} a_{ii} < 2a_{nn}, \ S \subset I_{n-1}, |S| = m, \] (10)
\[ 2a_{nn} < \sum_{i \in T} a_{ii}, \ T \subset I_{n-1}, |T| = m + 1. \] (11)

There is a one-to-one correspondence between \( D_n \) and the cone \( G_n \) given by the equalities (2) and (3).

In particular, the closure of \( D_n \) has 2n extreme rays \( f_0^k, f_1^k, k \in I_n \), with the coefficients \( a_{ij} \) of these forms defined as follows (where the term \( a_{kk}(f_{0,1}^k) \) should be omitted if \( k = n \)):
\[ a_{ii}(f_0^k) = 2\gamma, i \in I_{n-1}, i \neq k; \ a_{kk}(f_0^k) = 0, a_{nn}(f_0^k) = m\gamma; \]
\[ a_{ii}(f_1^k) = 2\gamma, i \in I_{n-1}, i \neq k; \ a_{kk}(f_1^k) = 4\gamma, a_{nn}(f_1^k) = (m + 1)\gamma; \]
\[ a_{ij}(f_{0,1}^k) \text{ for } i \neq j \text{ are defined by the equations } (9). \]

The inequalities (10) and (11) define facets of the closure \( \text{cl}D_n \). All facets are domains of equivalent \( L \)-types, each having \( n \) extreme rays \( f_1^k, k \in S, f_0^k, k \notin S, S \subset I_{n-1}, |S| = m, \) or \( S = I_n - T \) and \( T \) is as in (7).

If \( n \) is even, \( n = 2m \), then \( \text{cl}D_n \) is one dimensional. The ray \( \text{cl}D_{2m} \) is the intersection of the \( \binom{n}{2} \) hyperplanes (3) and the \( n - 1 \) hyperplanes given by the following equalities
\[ 2a_{nn} = m a_{ii}, 1 \leq i \leq n - 1. \] (12)

Proof of Theorem 1 will be proceeded as follows. For a function \( f_\gamma \) given by (4), we find the Voronoi polytope. Take attention that the inequalities (10) and (11) in terms of the parameters \( \alpha \) and \( \gamma_i \) take the form (3) for \( n \notin S \) and \( n \in S \), respectively. We show that the face poset of the Voronoi polytope does not change if the parameters of \( f_\gamma \) change such that they satisfy (3).

On the other hand, we show if at least one of inequalities (3) holds as equality for parameters of a function \( f_\gamma \), then the \( L \)-type of \( f_\gamma \) differs from the \( L \)-type of \( f_\gamma \in D_n \). This will mean that \( D_n \) is an \( L \)-domain.

For to find the Voronoi polytope of \( f_\gamma \) given by (4), consider the cosets of \( 2L \) in the lattice \( L = L(\gamma) \). Let \( v = x_n b + \sum_{i \in I_{n-1}} x_i e_i \) be a vector of \( L(\gamma) \). Then this vector belongs to the coset \( Q(S, z) \), where \( S \subset I_{n-1} \) is the set of indices of odd coordinates \( x_i \) and the number \( z \in \{0, 1\} \) indicates the parity of the \( b \)-coordinate \( x_n \) of the vector \( v \). Note that the vector \( e(I_n) = 2b \) belongs to the trivial coset \( Q(\emptyset, 0) = 2L \). Hence the vectors \( e(S) \) and \( e(I_n - S) \) belong to the same coset for any \( S \subset I_n \). This coset is \( Q(S, 0) \) if \( n \notin S \), and \( Q(I_n - S, 0) \) if \( n \in S \). In particular, \( e_n \) belongs to \( Q(I_{n-1}, 0) \), and it is minimal in this coset. Moreover, we have \( b - e(S) = -(b - e(I_n - S)) \). So the \( 2^n \) vectors \( b - e(S) \), \( S \subset I_n \), are partitioned into \( 2^{n-1} \) pairs of opposite vectors.
Note that \( e^2(S) = \sum_{i \in S} e_i^2 = 2\gamma(S) \) and, according to (11), \( \gamma(S) + \gamma(I_n - S) = 2\alpha \). Recall that \( D_n \) is the domain of \( f_\gamma \), where \( \gamma \) belongs to \( G_n \). Hence \( \gamma_i, i \in I_n \), satisfy (4).

Taking in attention (3), we see that, for \( |S| \leq m \), the norm of \( e(S) \) is less than the norm of \( e(I_n - S) = e(T) \) for \( |T| \geq m + 1, S, T \subset I_n \).

If \( f_\gamma \) go to the boundary of \( D_n \), then the sets of minimal vectors of some cosets change. At first we describe the simple cosets which are constant on the closure of \( D_n \). The norm of minimal vectors of a coset is called also norm of the coset. These are the following cosets:

The \( n \) cosets \( Q(\{i\}, 0), i \in I_{n-1} \), and \( Q(I_{n-1}, 0) \) of norms \( 2\gamma_i, i \in I_n \), with minimal vectors \( e_i, i \in I_{n-1} \) and \( e_n = 2b - e(I_{n-1}) \), respectively.

The \( 2^{n-1} \) cosets \( Q(S, 1) \) of norm \( \alpha \) with minimal vectors \( b - e(S), S \subset I_{n-1} \).

The \( 2^{n-1} - n \) non-simple cosets \( Q(S, 0), S \subset I_n, 1 < |S| \leq m \), have norms \( \gamma(S) \) with minimal vectors \( \sum_{i \in S} \varepsilon_i e_i \), where \( \varepsilon_i \in \{ \pm 1 \} \).

If \( \alpha = \gamma(S) \), then \( |S| = m \) and the coset \( Q(S, 0) \) contains also the vector \( \sum_{i \in I_n - S} \varepsilon_i e_i \).

Recall that the minimal vectors of simple cosets determine facets of the Voronoi polytope. Consider a point \( x \in \mathbb{R}^n \) in the basis \( \{ e_i : i \in I_n \} \), \( x = \sum_{i \in I_n} x_i e_i \). Then \( x \) belongs to the Voronoi polytope \( P \) of \( L(\gamma) \) if the inequalities

\[
-\frac{v^2}{2} \leq xv \leq \frac{v^2}{2}
\]

hold for all minimal vectors \( v \) of simple cosets of \( L(\gamma) \). Using (2), (3) and the identity \( b = \frac{1}{2} \sum_{i \in I_n} e_i \), we obtain the following system of inequalities describing the Voronoi polytope of \( L(\gamma) \):

\[
-\frac{1}{2} \leq x_i \leq \frac{1}{2}, i \in I_n, \tag{13}
\]

\[
-\frac{1}{2} \alpha \leq \sum_{i \in I_n} \gamma_i \varepsilon_i x_i \leq \frac{1}{2} \alpha, \varepsilon_i \in \{ \pm 1 \}, i \in I_n. \tag{14}
\]

Here the inequality (14) is given by minimal vectors of \( Q(S, 1) \) such that \( \varepsilon_i = -1 \) if \( i \in S \), and \( \varepsilon_i = 1 \) if \( i \not\in S \).

Note that \( \sum_{i \in I_n} \gamma_i \varepsilon_i x_i \) is the linear function on \( \varepsilon_i x_i \) taking maximal value if \( \varepsilon_i x_i \geq 0 \) for all \( i \in I_n \). Hence the right hand inequality in (14) holds as equality for a vertex \( x \) only if \( \varepsilon_i x_i > 0 \) for \( x_i \neq 0 \).

An analysis of the system (13), (14) shows, that for each vertex \( x \) there is the opposite vertex \(-x\), and \( x \) has the following coordinates

\[
x_i = \frac{1}{2} \varepsilon_i, i \in S \subseteq I_n, |S| = m, x_k = \frac{\varepsilon_k}{2\gamma_k} (\alpha - \gamma(S)), x_l = 0, \text{ for } l \in I_n - (S \cup \{k\}). \tag{15}
\]

There are \( \binom{n}{m} \) positive vertices of this type. Taking in attention signs, we obtain \( 2^{m+1} \binom{n}{m} \) vertices of the Voronoi polytope. Denote the vertex (15) by \( x(k; S) \).
The form of the vertex $x(k; S)$ shows that some vertices can be glued if and only if the equality $\alpha = \gamma(S)$ holds for some set $S$. If $\alpha = \gamma(S)$, then $x_k(k; S) = 0$, and the $m + 1$ vertices $x(l; S)$, $l \in I_n - S$, are glued into one vertex. This means that if $\alpha = \gamma(S)$ for some $S$, then $L$-type of $f_\gamma$ changes. So, we proved that the inequalities (8), i.e. the inequalities (8) and (9) hold for $f \in D_n$.

Now, we show that $D_n$ lies in the intersection of the hyperplanes (8). It is proved in [BG01] that the equations of the hyperplanes in the intersection of which an $L$-type domain lies are given by some linear forms on norms of minimal vectors of cosets of 2$L$ in $L$. Some of such linear forms are obtained by equating norms of minimal vectors of a non-simple coset. There are $L$-type domains for which linear forms of last type are sufficient for to describe the space, where this $L$-type domain lies. This is so in our case.

In fact, it is sufficient to consider the non-simple cosets $Q(S, 0)$ for $|S| = 2$, i.e. to equate the norms of vectors $e_i + e_j$ and $e_i - e_j$, $i, j \in I_n$. The equality $(e_i + e_j)^2 = (e_i - e_j)^2$ implies $e_i e_j = 0$, i.e. $a_{ij} = 0$, $0 \leq i < j \leq n - 1$. We obtain the first equalities in (8). For $j = n$, we have $e_i e_n = -e_i(2b - e(I_n - 1)) = 0$. Since $e_i e_j = 0$, this equality is equivalent to $2be_i = e_i^2$. We obtain the second equalities in (8). If we set $be_i = \gamma_i$, $b^2 = \alpha$, we obtain the original function $f_\gamma$.

Now let $n = 2m$ be even. In this case, for $f \in cl D_n$, all cosets of 2$L$ in $L$ (excluding the cosets $Q(S, 0)$ for $|S| = m$) are the same as in the odd case. But, for $|S| = m$, $Q(S, 0)$ contains besides the vectors $\sum_{i \in S} e_i e_i$ also the vectors $\sum_{i \in I_n - S} e_i e_i$. Since norms of these vectors are $\gamma(S)$ and $\gamma(I_n - S)$, respectively, the equating of these norms gives the system (8). This system has the unique solution $\gamma_i = \gamma$ for all $i \in I_n$.

Hence, for $n = 2m$, $\alpha = \frac{1}{2}\gamma(I_n) = \frac{1}{2}(\gamma(S) + \gamma(I_n - S)) = \gamma(S) = m\gamma$. Taking in attention (2) and (3), we can rewrite this equality as $2a_{nn} = m\alpha_{\gamma}$ for any $i \in I_n$. So, we obtain (12). This means that $cl D_n$ is a ray, which lies in the intersection of the hyperplanes given by the equations (11), (11) and (12). Any $f \in cl D_n$ is a rigid (i.e. edge) form.

So, Theorem 1 is proved.

Recall that $L(\gamma) = D_n$ if $\gamma_i = 1$ for all $i \in I_n$. Since, for this $\gamma$, the parameters of $f_{\gamma}$ satisfy (3), where $\alpha = m + \frac{1}{2}$, this implies the following

**Corollary** $D_n$ is the $L$-domain of $D_n^*$. In particular, the lattice $D_{2m}^*$, $m \geq 2$, is rigid.

**Remark.** Note that, for $n$ odd, $n = 2m + 1$, the extreme rays $f_{0}^{k}$ have rank $n - 1 = 2m$. These forms are forms of lattices isomorphic to $\gamma D_{n-1}^* = \gamma D_{2m}^*$.

We saw in the proof of Theorem 1 that the Voronoi polytope of the lattice $D_n^*$ is an $n$-cube whose vertices are cut by hyperplanes (11). A description of the Voronoi polytope of $D_n^*$ can be found in [CS91].

Theorem 1 is a generalization of the result of [EG01], where the $L$-domain of the lattice $D_5^*$ is described in detail.
References

[Ba94] E.P.Baranovskii, *The perfect lattices \( \Gamma(A^n) \) and the covering density of \( \Gamma(A^9) \),* Europ.J.Combinatorics 15 (1994) 317–323.

[BG01] E.P.Baranovskii and V.P.Grishukhin, *Non-rigidity degree of a lattice and rigid lattices*, Europ.J.Combinatorics 22 (2001) 921–935.

[CS91] J.H.Conway and N.J.A.Sloane, *The cell structures of certain lattices*, in: Miscel-
lanea Mathematica, eds. P.Hilton, F.Hurzebruch, R.Remmert, Springer Verlag, New York 1991, pp.71–107.

[DD01] M.Dutour and M.Deza, *The hypermetric cone on seven vertices*, papers/math.MG/0108177 of LANL archive (2001).

[DGL92] M.Deza, V.P.Grishukhin and M.Laurent, *Extreme hypermetrics and L-polytopes*, in: Sets, Graphs and Numbers, Budapest (Hungary) 1991; vol.60 of Colloquia Mathematica Societatis Jánosh Bolyai, G.Halásh at al.(eds), 1992, 157–209.

[DL97] M.Deza and M.Laurent, *Geometry of cuts and metrics*, Algorithms and Combinatorics 15, Springer-Verlag, Berlin, 1997.

[EG01] P.Engel and V.Grishukhin, *An example of a non-simplicial L-type domain*, Europ.J.Combinatorics 22 (2001) 491–496.

[Vo1908] G.F.Voronoi, *Nouvelles applications des paramètres continus à la théorie de formes quadratiques - Deuxième mémoire*, J. für die reine und angewandte Mathematik, 134 (1908) 198–287 and 136 (1909) 67–178.