Volume Effects for Pion Two-Point Functions in Constant Electric and Magnetic Fields

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(Dated: January 8, 2009)

Abstract

We compute finite volume effects relevant for lattice QCD simulations using background fields. Focusing on constant electric and magnetic fields on a periodic lattice, we determine volume corrections to pion two-point functions using chiral perturbation theory. Among such corrections are the finite volume shifts to the electric and magnetic polarizabilities, which are numerically shown to be non-negligible. We additionally find that all terms in the single-particle effective action can be renormalized by infrared effects. This includes Born couplings to the pion current and total charge-squared, which can be renormalized due to the nature of gauge invariance on a torus.

PACS numbers: 12.38.Gc, 12.39.Fe
I. OVERVIEW

Lattice gauge theory simulations are making dramatic progress towards addressing quantitatively the non-perturbative dynamics of quarks and gluons [1]. Despite its successes, lattice QCD suffers from systematic errors due to a number of sources. Understanding and accounting for these systematic errors is necessary for accurate determination of hadronic observables. In particular, as the lattice pion masses approach the physical pion mass on a fixed-size lattice, the effect of finite volume will become pronounced. This is increasingly important to note as lattice calculations are in reach of the physical point [2]. Certain observables are more susceptible to volume corrections; thus, a practical application of field theories in finite volume is the study of finite-size scaling of observables calculated using lattice QCD simulations.

The long-range physics of low-energy QCD is a consequence of the lightest degrees of freedom, the pseudo-scalar pions. In finite volume, hadronic observables are altered due to modified pion dynamics arising from boundary conditions imposed on the underlying lattice action. These finite-size effects can be treated systematically using chiral perturbation theory [3, 4, 5]. In this theory, the light pions emerge as a consequence of spontaneous chiral symmetry breaking, and the smallness of the up and down quark masses compared to the QCD scale. For QCD in the presence of external sources, low-energy hadronic properties are encoded in multipole moments, radii, and polarizabilities. These quantities are accessible with lattice QCD correlation functions using two different techniques: the source insertion method and the classical external field method. The source insertion method does not directly probe such observables: lattice data additionally require momentum extrapolation. Furthermore, at finite volume the multipole decomposition into momentum-transfer dependent form factors is no longer valid. Finite volume studies of hadronic current matrix elements at non-vanishing momentum transfer and their generalizations have been studied using chiral perturbation theory [6, 7, 8, 9].

In this work, we address finite volume effects relevant for QCD simulations in background fields, see e.g. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Focusing on constant electric and magnetic fields, we use chiral perturbation theory to calculate pion two-point functions in the presence of classical external fields. These results can be utilized to analyze lattice correlation functions calculated in background field simulations. Our computations demonstrate that the nature of gauge invariance on a torus leads to finite volume artifacts. A striking manifestation of this is the infrared renormalization of pion currents and charge-squared couplings. Identical effects have been found from momentum space calculations of current matrix elements [22, 23]. Consequently two-point functions for both charged and neutral particles are modified at finite volume; for example, neutral particles propagating in external fields no longer have correlation functions with a simple exponential falloff at long times. On general grounds, one expects the finite volume renormalization of all terms in the effective action for a particle in a background field.

Our presentation is organized as follows. First in Section II we consider the case of pions propagating in a background electric field. Here the field is treated in Euclidean space. Corrections to the infinite volume single-particle effective action are determined using spatially periodic boundary conditions. In Section III we consider the case of pions propagating in a background magnetic field. The effects of finite volume are determined on the single-particle effective action for this case as well. For both electric and magnetic cases, we provide expressions for the volume effects under the lattice approximation of omitting
disconnected quark contractions. Finally in Section [IV] we summarize our findings, make important clarifications, and discuss further applications of our work.

II. MESONS IN FINITE VOLUME: ELECTRIC CASE

Pions propagating in a background electric field at finite volume can be handled using Schwinger’s proper time method [24]. We choose to implement the field with a time-dependent gauge potential

\[ A_\mu(x) = (0, 0, -E x_4, 0), \tag{1} \]

which corresponds to a Euclidean electric field, \( \mathcal{E} \), in the \( z \)-direction. As we are interested in quantities perturbative in the field strength, we can perform the analytic continuation to Minkowski space trivially [25]. The lattice simulations, moreover, are performed in Euclidean space for which the choice in Eq. (1) is natural. Throughout, we work with theories defined on a spatially periodic torus of infinite time extent. The Euclidean space action has the form

\[ S = \int_{-\infty}^{\infty} dx_4 \int_0^L d\bm{x} \mathcal{L}(x). \tag{2} \]

The length of each spatial direction is given by \( L \). As the field is implemented by a time-dependent gauge potential, there is no conflict between Eq. (1) and the spatial periodicity of the underlying lattice action.\(^1\)

The low-energy dynamics of pions is described by chiral perturbation theory [30], in which pions are the Goldstone modes emerging from spontaneous chiral symmetry breaking.\(^2\) These modes are non-linearly realized in \( \Sigma \), where \( \Sigma = \exp(2i \phi / f) \), and \( \phi \) is an \( SU(2) \) matrix of pion fields. The dimensionful parameter \( f \) can be identified with the chiral limit value for the pion decay constant, and \( f = 130 \text{ MeV} \) in our normalization. The chiral Lagrangian appears as

\[ \mathcal{L} = \frac{f^2}{8} < D_\mu \Sigma \Gamma D_\mu \Sigma > - \frac{\lambda}{2} < m_Q (\Sigma^\dagger + \Sigma) >, \tag{3} \]

where we have used a bracketed notation to denote flavor traces: \(< A > = \text{tr}(A)\). The quark mass matrix is \( m_Q \), where \( m_Q = \text{diag}(m_u, m_d) \), and the action of the covariant derivative is specified by

\[ D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu [Q, \Sigma], \tag{4} \]

where \( Q \) is the quark electric charge matrix, \( Q = \text{diag}(2/3, -1/3) \). In writing this Lagrangian, we have included only the lowest-order terms. These terms scale as \( \mathcal{O}(\varepsilon^2) \), where \( \varepsilon \) is a small dimensionless number. We assume the power counting: \( k^2 / \Lambda_\chi^2 \sim m_\pi^2 / \Lambda_\chi^2 \sim \varepsilon^2 \). Here \( \Lambda_\chi \) is the chiral symmetry breaking scale, \( \Lambda_\chi = 2\sqrt{2}\pi f \), \( k \) is a typical momentum,

\(^1\) In actual lattice simulations, the length of the time direction is longer than the spatial directions, but obviously not infinite. To arrive at a periodic action in this case, the field strength must be quantized in the form \( q \mathcal{E} = 2\pi n / \beta L \), where \( \beta \) is the temporal extent of the lattice, and \( n \) is an integer [26, 27, 28, 29]. We assume this quantization condition has been met, but that \( \beta \geq 2L \) so that effects from the finite temporal extent of the lattice are comparatively quite small.

\(^2\) At finite volume, we must also specify that the volume is large enough to prevent the pion zero modes from conspiring to restore chiral symmetry. With this assumption, we work in the so-called \( p \)-regime [3, 4, 5].
FIG. 1: Pion propagator in strong external fields. Depicted are the Born couplings to the charged pion which have been summed to arrive at Eq. (5).

FIG. 2: Diagrams for the pion two-point function in a non-perturbative external field. The dashed lines represent pion propagation in the background field. The wiggly lines terminating in crosses represent couplings of the vertices to the background field.

and $m_{\pi}$ is the pion mass. The leading-order pion mass can be determined by expanding the Lagrangian in Eq. (3) to quadratic order in the meson fields. Using the strong isospin limit, $m_u = m_d = m$, we have $f^2 m_{\pi}^2 = 4 \lambda m$.

At leading order, the neutral pion propagator maintains a Klein-Gordon form due to the absence of charge couplings to the external field. For the charged pions, there are additional interactions. Although we shall be interested in small fields, i.e. $e F_{\mu\nu} / m_{\pi}^2 \sim \varepsilon^2$, it is actually easiest first to treat this ratio as $O(\varepsilon^0)$, and then expand quantities to the desired order in the external field strength. Summing the Born couplings to all orders, see Figure 1, produces the propagator of the positively charged pion

$$D(x', x) = \frac{1}{2L^3} \int_0^\infty ds \sum_n e^{i k \cdot (x' - x)} \left< x_4' - \frac{k_3}{eE}, s \right| x_4 - \frac{k_3}{eE}, 0 \right> e^{-\frac{1}{2} s E_{k,\perp}^2}, \quad (5)$$

where $E_{k,\perp}^2 = k_1^2 + k_2^2 + m_{\pi}^2$, and the periodic momentum modes are given in terms of triplets of integers $n$, by the relation $k = 2\pi n / L$. This propagator depends on the quantum mechanical harmonic oscillator propagator evolved in proper-time $s$,

$$\left< X', s \right| X, 0 \right> = \sqrt{\frac{eE}{2\pi \sinh eEs}} \exp \left\{ -\frac{eE}{2\sinh eEs} \left[ (X'^2 + X^2) \cosh eEs - 2X'X \right] \right\}. \quad (6)$$

To compute finite volume corrections to pion two-point functions in background fields, we use the charged pion propagator above in loop diagrams that arise in the effective theory, see Figure 2. The neutral pion contributions to such loop diagrams only lead to field independent modifications. These modifications are part of the finite volume corrections to the pion mass. Computation of charged pion loop diagrams with Eq. (5) leads to ultraviolet divergences, which we regulate using a simple subtraction

$$\sum \int \frac{dk}{(2\pi)^3} \equiv \frac{1}{L^3} \sum_n - \int \frac{dk}{(2\pi)^3}, \quad (7)$$

that produces only the infrared corrections we seek.

It is instructive to consider the simplest loop contribution explicitly. The four-pion vertex arising from the mass term in Eq. (3) makes contributions to diagram C depicted in the
figure. This mass-insertion tadpole diagram depends on the self-contracted propagator. Denoting the space-time coordinate of the vertex as $y_\mu$, we have the ultraviolet regulated pion self-contraction

$$D(y, y) \xrightarrow{\text{UV reg.}} \delta_L[D(y, y)] = \frac{1}{2} \int_0^\infty ds \sum_k \left\langle y_4 - \frac{k_3}{e\mathcal{E}}, s \middle| y_4 - \frac{k_3}{e\mathcal{E}}, 0 \right\rangle e^{-\frac{1}{2} s E_k^2}$$

(8)

$$= \ldots + \frac{e^2 \mathcal{E}^2}{6} \int_0^\infty ds \sum_k \sqrt{\frac{s}{\pi}} [-3y_4^2 + s(-1 + 6k_3^2y_4^2) + s^2k_3^2] e^{-s(k^2 + m_\pi^2)} + \ldots,$$

(9)

where in the second line we have series expanded in powers of the field strength $\mathcal{E}$, keeping only the second-order term. The zeroth-order term in $\mathcal{E}$ combines with the corrections from neutral pion loops to give the finite volume pion mass shift. We assume that the background field strengths employed on the lattice are sufficiently small so that the fourth order and higher-order terms can be neglected. The sums in Eq. (9) are standard in the study of finite volume modifications to meson properties. Using the catalog of finite volume functions of [22], we can write the finite volume modification at $O(\mathcal{E}^2)$ as

$$\delta_L[D(y, y)] = \frac{1}{4} e^2 \mathcal{E}^2 \left\{ y_4^2 \left[ J_{5/2}(m_\pi) - I_{3/2}(m_\pi) \right] + \frac{5}{12} J_{7/2}(m_\pi) - \frac{1}{2} I_{5/2}(m_\pi) \right\}.$$  

(10)

The last two terms give rise to a field-dependent shift in the effective mass squared. These are thus contributions to the finite volume electric polarizability.

Notice the remaining dependence on the vertex time, $y_4$, in the first terms of Eq. (10). In infinite volume, such dependence disappears owing to an allowed shift of the $k_3$ integrand:

$$\int_{-\infty}^{\infty} dk_3 e^{-\alpha(k_3 - e\mathcal{E}y_4)^2} = \int_{-\infty}^{\infty} dk_3 e^{-\alpha k_3^2}.$$  

At finite volume, discrete translational invariance is not enough to remove the time-dependence.\(^3\) This time dependence leads to an infrared renormalization of the single-pion effective action, as we now demonstrate. Consider the full pion two-point function $G(x', x)$.

Because we have discrete translational invariance in space, we can choose to locate the pion source at $x = 0$. Furthermore, because the three momentum is a good quantum number, we can project onto a final-state pion at rest. This leads us to

$$G(t, 0) \equiv \int_0^L dx \ G(x, t; 0, 0),$$

(11)

and similarly for the perturbative propagator, $D(t, 0)$. This notation allows us to focus on the time-dependence. A perturbative contribution to $G(t, 0)$ with time-dependence as in Eq. (10) has the form

$$G(t, 0) = D(t, 0) - \mathcal{A} \mathcal{E}^2 \int dy_4 D(t, y_4) y_4^2 D(y_4, 0).$$

(12)

\(^3\) There is one special case: for a discrete torus with field strengths quantized in the form $e\mathcal{E} = 2\pi n/L$ the time-dependence can be shifted away by reindexing the sum on momentum modes. As these field strengths are prohibitively large, however, we will not separately address this case for which we would need an entirely different treatment.
The coefficient $A$ arises from the one-loop graphs; and, as such, can be treated as a perturbation. Using a quantum mechanical notation, we have

$$\langle t|G|0 \rangle = \langle t|D|0 \rangle - A \mathcal{E}^2 \int dy_4 \langle t|D|y_4 \rangle y_4^2 \langle y_4|D|0 \rangle$$

$$= \langle t|[D - A \mathcal{E}^2 D^2 T^2] |0 \rangle = \langle t|D \frac{1}{1 + A \mathcal{E}^2 T^2 D} |0 \rangle + \ldots,$$

(13)

where $T|t\rangle = t|t\rangle$. Applying the operator identity

$$\frac{1}{1 + AD} = D^{-1} [D^{-1} + A]^{-1},$$

(14)

we see that the inverse propagator $G^{-1}$ has been additively renormalized

$$G^{-1} = D^{-1} + A \mathcal{E}^2 T.$$

(15)

For the case of charged and neutral pions projected onto vanishing three-momentum, the inverse perturbative propagator has the form

$$D^{-1} = -\frac{\partial^2}{\partial T^2} + m^2 + Q^2 \mathcal{E}^2 T^2,$$

(16)

where $Q$ is the pion charge. Hence the perturbation by $A$ corresponds to an infrared renormalization of the field-squared coupling: $Q^2 \rightarrow Q^2 + A$. This is the coordinate space analog of the infrared renormalization of the zero-frequency Compton scattering tensor, originally found by [22]. Gauge invariance is maintained, however, because Ward identities no longer protect zero-frequency scattering [23]. A gauge invariant operator that gives rise to a shift in the charge-squared coupling of the neutral pion, for example, is

$$\mathcal{L} = \frac{1}{2L^2} \mathcal{A}(L) \pi^0 \pi^0 W^{(-)} \cdot W^{(-)}.$$  

(17)

The $W^{(-)}$ is defined in terms of Wilson lines, $W^{(-)} = \frac{1}{2i} \left( W_i - W_i^\dagger \right)$, where $W_i$ is given by

$$W_i = \exp \left( i \int_0^L dx_i A_i(x) \right).$$  

(18)

The Einstein summation convention has been suspended in this equation. Because the space is compact, the Wilson line $W_i$ is actually a gauge invariant loop. The coefficient of this operator, $\mathcal{A}(L)$, runs to zero exponentially fast as the volume is taken to infinity. Expanding the operator out to second order in the background field, we have

$$\mathcal{L} = \frac{1}{2} \mathcal{A}(L) \pi^0 \pi^0 \mathcal{E}^2 t^2 + \mathcal{O}(\mathcal{E}^4),$$  

(19)

which has precisely the form needed above to describe renormalization of the charged-squared coupling.

The remaining contributions to the finite volume two-point function arise from expanding out the kinetic term in Eq. (3) to fourth order in the pion fields. The two gauge covariant derivatives generate all three diagrams shown in Figure 2. The gauge part of the derivative
leads to an explicit power of the external field at the vertex. Accordingly we can evaluate each diagram by using the propagator in Eq. (5) in the appropriate contractions, and expand the result to second order in the field strength. Unlike the infinite volume calculation, there are no particularly useful cancellations to be aware of. Focusing on the sector with vanishing three-momentum, the resulting single-particle effective action to $O(E^2)$ has the form

$$S_{\text{eff}} = \int_{-\infty}^{\infty} dt \, \pi^\dagger(t) \left[ -\frac{\partial^2}{\partial t^2} + m_\pi^2(L) + 4\pi m_\pi \alpha_E(L) E^2 + Q^2(L) e^2 E^2 t^2 \right] \pi(t). \quad (20)$$

Here we have used $\alpha_E$ to denote the electric polarizability which gives a positive contribution to the effective mass due to our Euclidean space treatment. Each of the renormalized couplings is a sum of the infinite volume piece plus a finite volume correction,

$$m_\pi^2(L) = m_\pi^2 + \delta_L[m_\pi^2],$$
$$\alpha_E(L) = \alpha_E + \delta_L[\alpha_E],$$
$$Q^2(L) = Q^2 + \delta_L[Q^2]. \quad (21)$$

With the exception of the pion mass, which we do not consider here, the finite volume corrections are isospin dependent. For the neutral pion, we find

$$\delta_L[Q_{\pi^0}^2] = -\frac{1}{2f^2} m_\pi^4 I_{5/2}(m_\pi), \quad (22)$$
$$\delta_L[\alpha_{E\pi^0}] = \frac{5\alpha_{f.s.}}{24f^2 m_\pi} m_\pi^2 \left[ 2I_{5/2}(m_\pi) + J_{7/2}(m_\pi) \right], \quad (23)$$

while for the charged pions, we have

$$\delta_L[Q_{\pi^\pm}^2] = -\frac{2}{3f^2} \left[ 2I_{1/2}(m_\pi) + m_\pi^2 I_{3/2}(m_\pi) \right], \quad (24)$$
$$\delta_L[\alpha_{E\pi^\pm}] = \frac{\alpha_{f.s.}}{3f^2 m_\pi} m_\pi^2 I_{5/2}(m_\pi). \quad (25)$$

Here $\alpha_{f.s.} = e^2/4\pi$ is the fine-structure constant. For both charged and neutral pions, the values of $\delta_L[Q^2]$ found here agree with those deduced from the zero-frequency Compton scattering tensor [23]. Focusing on the charged pion electric polarizability, we recall that delicate cancellations lead to only neutral pion loop contributions in infinite volume. These contributions do not depend on the background field; thus, there are no one-loop effects contributing to the polarizability. In finite volume, however, such cancellations are spoiled due to the lack of SO(4) invariance. Consequently the value of $\delta_L[\alpha_{E\pi^\pm}^\pm]$ is non-vanishing.

To investigate the size of the finite volume corrections, we consider the relative change in each of the quantities. For the charged pion, the relative change in charge squared is just the finite volume effect, $\Delta Q_{\pi^\pm}^2 = \delta_L[Q_{\pi^\pm}^2]$, because in infinite volume $Q^2 = 1$. For the neutral pion, we must define the relative change, since the infinite volume charge-squared coupling is zero. We choose the simplest possible definition, $\Delta Q_{\pi^0}^2 = \delta_L[Q_{\pi^0}^2]$. The charge-squared volume effects are shown in Figure 3. To consider the finite volume effects for the electric polarizabilities, we must recall the one-loop infinite volume results [31, 32, 33]:

$$\alpha_{E\pi^\pm} = 8\alpha_{f.s.} \frac{\alpha_9 + \alpha_{10}}{f^2 m_\pi}, \quad \text{and} \quad \alpha_{E\pi^0} = -\frac{\alpha_{f.s.}}{3(4\pi f)^2 m_\pi}. \quad (26)$$
FIG. 3: Relative change in charge-squared couplings. Plotted versus $L$ for a few values of the pion mass are the finite volume effects $\Delta Q^2$ for both neutral and charged pions.

Because both of these are non-vanishing, we can define the relative change due to volume effects in the ordinary manner, $\Delta \alpha_E = \frac{\alpha_E(L) - \alpha_E}{\alpha_E}$. To determine the low-energy constants required for the charged pion polarizability, we use [34]

$$\alpha_9 + \alpha_{10} = \frac{1}{32\pi^2} f_A / f_V,$$

(27)

where the form factors are given by $f_A = 0.0115$ [35], and $f_V = 0.0262$ [36]. The latter value is consistent with the determination of $f_V$ assuming conservation of vector current, $f_V = 0.0259$. Notice the combination $\alpha_9 + \alpha_{10}$ is renormalization scale independent. The finite volume effect on the electric polarizabilities is show in Figure 4, which depicts non-negligible corrections. The relative change in neutral pion polarizability is more sensitive to volume effects than the charged pion. One reason for this difference is the scaling by the infinite volume result. The infinite volume polarizability is a factor of five times smaller for the neutral pion compared to the charged pion, while the finite volume modifications are roughly the same size.

The neutral pion polarizability is already challenging to calculate on the lattice due to disconnected quark contractions (which have so far been neglected in simulations). In infinite volume, the one-loop expression for the neutral pion polarizability stems entirely
from such disconnected quark contractions [8]. The electric polarizability of the charged pion can be determined from the lattice. Reports of a preliminary calculation have been given in [20]. At this stage, contributions from background field couplings to the sea quarks have been omitted due to computational restrictions. To address the omission of disconnected quark contractions for neutral and charged pions, we turn to a partially quenched theory to differentiate between valence and sea quarks [37, 38]. We perform the partially quenched finite volume analysis with degenerate valence and sea quarks using a partially quenched charge matrix that distinguishes between valence and sea quark charges [39, 40]. The latter are set to zero. Additionally for the neutral pion, we perform the computation for only the connected part of the two-point function. The results are as follows: the charge-squared couplings are renormalized by

$$\delta L [Q^2_{\pi^0}]_{\text{QEM}} = 0, \quad \text{and} \quad \delta L [Q^2_{\pi^\pm}]_{\text{QEM}} = \delta L [Q^2_{\pi^\pm}],$$

(28)

where the subscript QEM denotes that we have quenched the electromagnetic charges, and the superscript connected denotes that only the connected part of the correlator has been retained. The shifts in polarizabilities are given by

$$\delta L [\alpha_{\pi^0}]_{\text{QEM}} = \delta L [\alpha_{\pi^\pm}]_{\text{QEM}} = \frac{5\alpha_{f.s.}}{2\pi f^2 m_\pi} m_\pi^2 I_{5/2}(m_\pi),$$

(29)

and are thus of comparable size to those with unquenched charges. Finite volume results for pion electric polarizabilities suggest that as the pion mass is brought down on fixed sized lattices, it will become increasingly challenging to isolate the infinite volume physics. A thorough study of the pion mass and volume dependence of lattice data will be required to extract pion polarizabilities.

### III. MESONS IN FINITE VOLUME: MAGNETIC CASE

Now we investigate the finite volume modifications to pion two-point functions for the case of a magnetic field. To be concrete, we implement the background field with the gauge potential

$$A_\mu(x) = (-Bx_2, 0, 0, 0),$$

(30)

which corresponds to a magnetic field, $B$, in the $z$-direction. On a spatial torus, the implementation of Eq. (30) generally breaks the periodicity of the lattice. To maintain discrete translational invariance of the lattice action, the field strength must be quantized in the form [26, 27, 28, 29]

$$qB = \frac{2\pi n}{L^2},$$

(31)

where $n$ is an integer. Provided this condition is met, we can expand hadronic field operators in periodic momentum modes, and treat them in a constant background field.

The Schwinger proper-time method cannot be directly utilized for the magnetic case because the $y$-component of momentum is missing in the resummed propagator. Pions propagating in a background magnetic field at finite volume, however, can be handled using the free particle propagator in coordinate space. We return to Eq. (3) and treat the external field as a small perturbation, $B/m_\pi^2 \sim \varepsilon^2$. Thus to leading order, the external field
dependence can be neglected, and we have the coordinate space propagator

\[
D(x', x) = \frac{1}{L^3} \sum_n \int \frac{dk_4}{2\pi} \frac{e^{i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})} e^{i\mathbf{k}_4 \cdot (\mathbf{x}'_4 - \mathbf{x}_4)}}{k^2 + k_4^2 + m_\pi^2}
\]  

(32)

for both charged and neutral pions. The spatial momentum modes are again quantized in the form \( k = 2\pi n / L \). The remaining field-dependent Born terms in Eq. (3) must then be treated as coordinate-space interactions in perturbation theory. There are two types of interactions: a two-pion, linear background field coupling; and, a two-pion, quadratic background field coupling.

To calculate one-loop corrections to pion two-point functions, we expand the Lagrangian in Eq. (3) to fourth order in the pion fields. We then generate all possible one-loop diagrams up to quadratic order in the external field \( B \). Due to symmetry, diagrams linear in the external field vanish.\(^4\) The non-vanishing diagrams are shown in Figure 5. It is particularly illustrative to consider a simple contribution explicitly. Consider the four-pion vertex with mass insertion derived from Eq. (3). Using this vertex along with the quadratic background field coupling, we arrive at a contribution to diagram \( C \) in the figure. Let us denote the location of the four-pion vertex as \( y_\mu \) and the location of the quadratic field insertion as \( z_\mu \).

Then the correction to the propagator, \( \delta G(x', x) \), from this diagram is of the form

\[
\delta G(x', x) = -\int_{-\infty}^{\infty} dy_4 \int_0^L dy D(x', y) \Delta(y) D(y, x),
\]

(33)

where, up to overall constants,

\[
\Delta(y) = -\int_{-\infty}^{\infty} dz_4 \int_0^L dz D(y, z) z_4^2 B^2 D(z, y).
\]

(34)

\(^4\) This is true when the action is projected onto vanishing \( x \)-component of momentum, \( k_1 = 0 \). For a general momentum, the infinite volume action contains a term proportional to \( 2iBx_2k_1 \), which acquires infrared renormalization in finite volume. This corresponds to screening of charged particle currents.\(^2\) Charge conjugation invariance forbids this effect for the neutral pion.
Inserting the perturbative propagator, Eq. (32), into the expression for $\Delta(y)$, yields three trivial integrations that can be performed: those over $z_1$, $z_3$, and $z_4$. Clearly the $z_2$ integration is non-trivial given the coordinate dependence introduced by the gauge potential. After performing trivial integrations, we have

$$\Delta(y) = -\frac{1}{L^4} B^2 \int_0^L dz_2 \int \frac{dk_4}{2\pi} \sum_{n,m} \frac{z_2^2 e^{ik_2(y_2-z_2)} e^{ik'_2(z_2-y_2)}}{[k^2 + k_4^2 + m_\pi^2][k_1^2 + k_2^2 + k_3^2 + k_4^2 + m_\pi^2]}
$$

$$= \frac{1}{L^4} B^2 \int_0^L dz_2 \int \frac{dk_4}{2\pi} \sum_{n,m} \frac{e^{-ik_2k'_2z_2}}{[k_1^2 + k_2^2 + k_3^2 + k_4^2 + m_\pi^2]}
$$

$$\left( \frac{\partial^2}{\partial k_2^2} \frac{e^{ik_2y_2}}{[k^2 + k_4^2 + m_\pi^2]} \right) e^{-ik'_2y_2},$$

with $k = 2\pi n/L$, and $k'_2 = 2\pi m/L$. We then differentiate, perform the integrals over $z_2$ and $k'_2$ that have been rendered trivial, and drop terms that vanish by parity. Regulating the sum over momentum modes, we are left with

$$\Delta(y) = B^2 \int \frac{dk_4}{2\pi} \sum_k \left[ -\frac{y_2^2}{[k^2 + k_4^2 + m_\pi^2]^2} - \frac{2}{[k^2 + k_4^2 + m_\pi^2]^3} + \frac{8k_2^2}{[k^2 + k_4^2 + m_\pi^2]^4} \right]
$$

$$= B^2 \left[ -\frac{1}{4} y_2^2 I_{3/2}(m_\pi) - \frac{3}{8} I_{5/2}(m_\pi) + \frac{5}{12} I_{7/2}(m_\pi) \right].$$

The last two terms are finite volume corrections to the magnetic polarizability, while the first term is a finite volume correction to the charge-squared coupling. The argument exposing this fact follows, mutatis mutandis, from that for the electric case.

The remaining one-loop contributions to the pion two-point function can be evaluated similarly. The net result is that the pions are described by an effective action of the form

$$S_{\text{eff}} = \int_{-\infty}^\infty dt \int_0^L dy \pi^1(y, t) \left[ -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + m_\pi^2(L) - 4\pi m_\pi \beta_M(L) B^2 + Q^2(L) e^2 B^2 y^2 \right] \pi(y, t),$$

up to $O(B^4)$ corrections. In writing this effective action, we have projected onto the sector with $k_1 = k_3 = 0$. The modification to the charge-squared term of the action is allowed by gauge invariance due to exactly the same operators as in the electric case. The single-particle effective theory consequently demands that the charge-squared couplings be identical to those determined above [23]. The results of the explicit finite volume calculations indeed show that the $Q^2(L)$ in Eq. (37) are identical to those in Eq. (20) for both $\pi^0$ and $\pi^\pm$. The same is additionally true for the case where disconnected quark contractions are omitted.

The term in the effective action with coefficient $\beta_M(L)$ denotes the magnetic polarizability contribution. This coefficient is a sum of the infinite volume polarizability, and the finite volume modification

$$\beta_M(L) = \beta_M + \delta_L[\beta_M].$$

The former is given by $\beta_M = -\alpha_E$ for charged and neutral pions, due to the helicity structure of the one-loop diagrams, while the latter are given by

$$\delta_L[\beta_M^0] = -\frac{\alpha_{f.s.}}{24 f^2 m_\pi} m_\pi^2 \left[ 2I_{5/2}(m_\pi) + 7K_{9/2}(m_\pi) \right],$$

$$\delta_L[\beta_M^\pm] = -\frac{5 \alpha_{f.s.}}{12 f^2 m_\pi} m_\pi^4 I_{7/2}(m_\pi).$$

\(11\)
The relative size of finite volume corrections to the magnetic polarizability, $\Delta \beta_M = \frac{\beta_M(L) - \beta_M}{\beta_M}$, is investigated in Figure 6. The figure depicts non-negligible finite volume corrections. The corrections are of comparable size when disconnected diagrams are neglected. Carrying out the partially quenched computation, we find

$$\delta_L[\beta_{\pi^+}]_{QEM} = \delta_L[\beta_{\pi^0}]_{QEM} = -\frac{25}{108} \frac{\alpha_{f.s.}}{f^2 m^4} \frac{m^4}{\pi} I_{7/2}(m^2).$$

(41)

IV. SUMMARY AND DISCUSSION

Above we have investigated finite size effects on pion two-point correlation functions calculated in constant background fields. Treating the external field strength as weak, couplings to pions can be considered as perturbations in coordinate space. One then uses coordinate-space perturbation theory on a torus to ascertain the two-point correlation function in finite volume. In the case of electric fields, we utilized a time-dependent gauge potential which allowed us to sum the couplings to the external field to all orders. The resulting Schwinger proper-time Green’s functions were used to deduce volume corrections at one-loop order in chiral perturbation theory. The effect of such corrections is to renormalize parameters of the single-particle effective action in the infrared. To second order in the electric field, we found the electric polarizabilities and Born-level charge-squared couplings are renormalized. In the case of magnetic fields, we treated the external field perturbatively from the outset, and derived the single-particle effective action to second order in the magnetic field. For this case, we found the infrared effect on the magnetic polarizabilities, and an identical renormalization of charged-squared couplings as in the electric case. Numerical estimates of the finite size effects on polarizabilities show non-negligible dependence on the lattice size for pion masses $\lesssim 350 \text{ MeV}$. Depending on the sign of such corrections relative to the infinite volume value, signals for polarizabilities will either be sizably enhanced, or sizably diminished. Based on one-loop chiral perturbation theory, the field-squared shifts in the effective mass of charged pions will be enhanced in both electric and magnetic fields. On the other hand, the shifts for neutral pions will be diminished in both electric and magnetic fields. Results were also presented for two-point functions calculated using the lattice approximation of omitting the disconnected quark contractions. For the charged and neutral pions this approximation
amounts to quenching the electric charges of the sea quarks; while for the neutral pion, the annihilation contraction is additionally neglected.

There are three clarifications that must be made about our results. Firstly and quite obviously, the gauge invariance of the single-particle effective actions is not manifest because we have expanded results to second order in the external field. In infinite volume, the charge couplings are exactly fixed by Ward identities, while in a compact space additional gauge invariant terms lead to renormalization of charges and currents. Because we treat the time direction as infinite, charge is not renormalized. The possibility to renormalize spatial Born-level couplings in a gauge invariant manner has been thoroughly detailed using a momentum-space analysis [23], and we reproduce the findings of that work using a complementary coordinate-space method. Finite volume corrections to polarizabilities show up as coefficients to field squared terms, \( E^2 \) and \( B^2 \). These terms, moreover, can be further split into various gauge invariant terms. For example, at second order in the electric field strength one cannot distinguish between operators of the form

\[
O_1 = E^2, \quad \text{and} \quad O_2 = \frac{\partial}{\partial x_4} W(-) \cdot E,
\]

because both terms have the form of polarizability operators at second order. Another way to implement a constant electric field on a torus is to use the vector potential

\[
A_\mu(x) = (0, 0, 0, E x_3),
\]

for which the operator \( O_2 \) vanishes. On a torus, however, the vector potentials, Eqs. (41) and (43), are not related by a gauge transformation. Consequently the finite volume shift of the polarizability (defined as the coefficient of the \( E^2 \) term of the effective action) will be different. A direction for future investigation is to classify all possible gauge invariant, single-particle operators allowed on a torus, and determine the running of their coefficients with the lattice volume. In this way, one could determine how to implement the external field with minimal volume corrections.

Secondly the finite volume renormalization of charge-squared couplings must be addressed to properly fit hadron correlation functions. Let us focus on the electric case. For the charged pion, this is relatively straightforward. The two-point function arising from the effective action in Eq. (20) has precisely the form of Eq. (5) projected onto zero three-momentum with the mass replaced by the effective mass, namely \( m_\pi^2 \rightarrow m_\pi^2 + 4\pi\alpha^2_E(L) E^2 \), and the charge-squared replaced by the renormalized charge-squared. Because Eq. (5) at zero momentum is an even function of the charge \( e \), the latter can be achieved simply by the trick \( e \rightarrow e (1 + \frac{1}{2} \delta_L[Q_{\pi}^2]) \). While chiral perturbation theory estimates the latter at the few percent level, this expectation can be confirmed by treating the charge-squared as a free parameter in fits to charged pion correlation functions.\(^5\) The neutral pion, however, presents difficulty due to the sign of the renormalized charge-squared coupling. The two-point function derived

\(^5\) We stress that the charge (calculated from the time component of the current) has not been renormalized, as finite volume calculations indeed confirm [23]. For the charged pion with non-zero momentum, the spatial current is screened in finite volume, and the effective charge arising from the spatial current is what appears in Eq (5). The square of this effective charge has been determined above by projecting onto zero three-momentum. For the neutral pion, however, charge conjugation invariance forbids renormalization of (odd powers of) the current operator(s). In the absence of current screening, however, a renormalization of
from Eq. (20) for the neutral pion appears as Eq. (5) projected onto zero three-momentum, with $m_{\pi}^2 \rightarrow m_{\pi}^2 + 4\alpha_E^0(L)\mathcal{E}^2$, and $e \rightarrow i|Q_{\pi^0}(L)|$ necessitating analytic continuation. The two-point function acquires an imaginary part due to the instability in Eq. (20) for negative charge squared. This imaginary part is exponentially small provided either the electric field strength is small, $\mathcal{E}/m_{\pi}^2 \ll 1$, or the pion Compton wavelength is small compared to the lattice size, $m_{\pi}L \gg 1$. In practice, this should thus be a very small effect. All neutral particles potentially have this difficulty. If the renormalized charge-squared is positive, we merely use Eq. (5) with $Q(L)$, and their correlation functions should look like the zero-momentum projection of charged particle correlation functions. In particular, they will not be simple exponentials at long times. If, on the other hand, the renormalized charge-squared is negative (as is the case for the neutral pion), our only recourse is to attempt to fit the correlation function using the real part of the analytically continued two-point function. This procedure relies on the hope that the imaginary part is numerically insignificant to allow treatment in Euclidean space. Notice for the connected part of the neutral pion two-point function, this issue does not arise because $\delta_L[Q_{\pi^0}^2]_{\text{connected}} = 0$.

Thirdly the choice of coordinate origin is ordinarily arbitrary. In backgournd fields on a torus, however, this is no longer the case, as was pointed out for an electric field, see e.g. [19].

Focusing on the electric case and the gauge potential in Eq. (1), we observe that two-point correlation functions for both charged and neutral particles at finite volume will depend on the source time. In infinite volume this was already the case for charged particles via Eq. (5). For neutral particles, charge-squared couplings are generated from infrared effects, and a neutral hadron source will thus break discrete time-translational invariance. Another way to shift the time is through the gauge potential in Eq. (1). We are free to translate $x_4 \rightarrow x_4 + c$ in infinite volume as it corresponds to a gauge transformation. In finite volume, this shift in time leads to an additional coupling of the matter fields to a constant gauge potential

$$A_\mu^c(x) = (0, 0, -\mathcal{E}c, 0),$$

that cannot be gauged away on a torus. The effect is equivalent to a twisted boundary condition on the matter field, see e.g. [41]. Consequently a hadron with charge $Q$ acquires kinematic momentum, $k = -Q\mathcal{E}c\hat{z}$. One can extend the analysis formulated here to this flavor-twisted case by working at non-zero spatial momentum. Volume effects can be computed [42], however, the expectation is that broken symmetries will lead to extra finite volume terms [43, 44, 45]. This complication notwithstanding, effects on the two-point function can be determined. On general grounds, we expect all terms in the single particle effective action to be renormalized including the current, or equivalently the momentum. As was the case for periodic boundary conditions, neutral particle correlation functions will no longer have a simple exponential falloff at long times.

Using the methods established here, many subsequent investigations are possible. Of particular interest is the calculation of finite volume effects for baryon electromagnetic prop-

the Thomson cross section gives rise to a charge-squared coupling. This is permitted because two current operators are involved. The finite volume neutral pion propagator at non-zero momentum is then given by

$$D(x', x) = \frac{1}{2L^3} \int_0^\infty ds \sum_n e^{i(k \cdot (x' - x))} \langle x'_4, s | x_4, 0 \rangle e^{-\frac{1}{2}s(k^2 + m_{\pi}^2)},$$

where $\langle x'_4, s | x_4, 0 \rangle$ depends on the effective charge squared $\delta_L[Q_{\pi^0}^2]$. 

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erties. These too can be addressed using the present framework. In order to reliably extract these quantities from lattice QCD simulations in background fields, one must ascertain whether volume corrections are as sizable as in the meson sector.

Acknowledgments

This work is supported in part by the U.S. Department of Energy, under Grant No. DE-FG02-93ER-40762.

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