A RELATIVELY SHORT PROOF OF ITÔ’S FORMULA FOR SPDES AND ITS APPLICATIONS

N.V. KRYLOV

Abstract. We give a short proof of Itô’s formula for stochastic Hilbert-space valued processes in the setting \( V \subset H \subset V^* \) based on the possibility to lift the stochastic differentials, which are originally in \( V^* \), into \( H \). Using this result we also prove the maximum principle for second-order SPDEs in arbitrary domains.

Itô’s formula is one of the main tools in Stochastic Analysis and, in particular, in the theory of stochastic partial differential equations (SPDEs) of Itô type. E. Pardoux ([14]) was the first to consider the most general SPDEs with deterministic and stochastic terms containing the unknown function and its derivatives from an abstract point of view of stochastic Itô equations in the setting symbolically described as \( V \subset H \subset V^* \), where \( H \) is a Hilbert space and \( V,V^* \) are Banach spaces (see [14] for references to previous results). One of the main steps in treating SPDEs consists of establishing Itô’s formula for the square of the \( H \)-norm of solutions.

In the deterministic case (without any stochastic terms) we deal with a function \( v_t \), that is in \( V \) for almost all \( t > 0 \) and its time derivative is in \( V^* \), which is dual to \( V \), for almost all \( t > 0 \) (cf. Remark 2.3). The goal is to show that there is a modification \( u_t \) of \( v_t \) which is an \( H \)-valued continuous function and \( \|u_t\|_H^2 \) is an absolutely continuous function admitting a natural formula for its time derivative. Even in this case and even if \( V,V^* \) are Hilbert spaces the formula is not completely trivial. For instance, the proof of Theorem 3 on page 287 in [5] still has a tiny gap since the continuity of \( \|u_t\|_H^2 \) at \( t = 0 \) is not proved.

In the stochastic case the proof given in [14] and [15] is rather involved and consists of many steps. In particular, it is based on the deterministic version of Itô’s formula in Banach spaces with reference to [13] where in Remark 1.2 on page 156 and Remark 7.9 on page 236 one indeed finds the statement that the result is true but neither a proof or a reference to a proof is given. In [9] an approach not using the deterministic result was suggested for equations driven by continuous martingales. In contrast with the deterministic case or, for that matter, with [14] and [15], where

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the deterministic case was the starting point, the method of [9] is based on
discretization of the time variable and some arithmetical manipulations. The
method of [9] is generalized in [6] for the case of arbitrary cadlag martingales,
which required a dramatic increase in what concerns arithmetics.

Most likely there is no simple proof of Itô’s formula in the case of general
Banach spaces and this makes it hard for a person interested in SPDEs to
enter the area. On the other hand, the case that \( V, H \), and \( V' \) are Hilbert
spaces is the most common in applications and it turns out that in this case
there is a simple proof presented here of the continuity of the process \( \|u_t\|_H^2 \)
and of Itô’s formula for it. The method seems to be much simpler than the
ones previously known.

Basically, we lift \( du_t \) into \( H \) by using resolvent operators \( R_\lambda \) constructed
in Section 1, write Itô’s formula for Hilbert space-valued processes, and then
pass to the limit as \( \lambda \to \infty \) in Section 2. In the end of Section 2 we give a
version of Theorem 2.1 for SPDEs when \( V \) is a Sobolev-Hilbert space with
integral numbers of derivatives. In Section 3 we prove Itô’s formula not for
the squared norm in \( H \) but for more general functions as in [14], [15]. In our
view the proofs given here are more straightforward than previously known.
In the final Section 4 we apply the result of Section 3 to prove the maximum
principle for the second-order SPDEs in arbitrary domains. To the best of
the author’s knowledge, in what concerns the arbitrariness of the domain
and the structure of the equation this is the most general result known so
far.

It seems that the maximum principle for general second order SPDEs was
first proved in [8] (see also [10] for the case of random coefficients) for SPDEs
in the whole space by the method of random characteristics introduced there
and also in [12] (a particular case of the maximum principle appeared already
in [14]). Later the method of random characteristics was used in many
papers for various purposes, for instance, to prove smoothness of solutions
(see, for instance, [1], [2], [3], [16] and the references therein). It was very
tempting to try to use this method for proving the maximum principle for
SPDEs in domains. However, the implementation of the method turns out
to become extremely cumbersome and inconvenient if the coefficients of the
equation are random processes. Also, it requires more regularity of solutions
than actually needed.

In [7] a new method was suggested based on Itô’s formula for \( \|u_t^+\|_H^2 \)
which was derived by using mollifications in space variable. This required
the underlying domain to be in \( C^1_{loc} \). In the present article the domain is
an arbitrary open set. On the other hand, the results of [7] are much more
general in other respects. In particular, they were applied in [7] to prove the
Hölder continuity of solutions up to the boundary and the results of the
present article seem not to be applicable for this purpose. Still they can be
applied in the same way as in [11] for investigating filtering problems in the
situation of partially observable diffusion processes when the observations
are only available until the unobservable component exits from a given domain. This would show that the result of [11] about filtering density before the exit time occurs is true in case of arbitrary domains.

Finally, we mention that there are many situations in which Itô’s formula is known for Banach space valued processes. See, for instance, [4] and the references therein. These formulas could be more general in some respects but they do not cover our situation when the stochastic differential leaves in $V^*$.

1. **Resolvent operator**

It is highly unlikely that the results of this section are new. However, the author’s several attempts to find them in the literature failed and were abandoned for the reason that it takes less time to prove the results than to find them published somewhere. In addition, the proof only takes two pages.

Let $V$ and $H$ be two Hilbert spaces with scalar products and norms $(\cdot,\cdot)_V$, $\|\cdot\|_V$ and $(\cdot,\cdot)_H$, $\|\cdot\|_H$, respectively. Assume that $V \subset H$ and $V$ is dense in $H$ (in the metric of $H$) and $\|u\|_H \leq \|u\|_V$ for any $u \in V$.

The norm in $V$ is obviously equivalent to

$$
(\lambda\|u\|_H^2 + \|u\|_V^2)^{1/2},
$$

where $\lambda \geq 0$ is any fixed number. Then take an $f \in H$ and observe that the linear functional $(f,u)_H$ is bounded as a linear functional on $V$. By Riesz’s representation theorem there exists a unique $v =: R_\lambda f \in V$ such that

$$(f,u)_H = \lambda(v,u)_H + (v,u)_V \quad \forall u \in V,$$  \hspace{2cm} (1.1)

that is

$$(f,u)_H = \lambda(R_\lambda f,u)_H + (R_\lambda f,u)_V, \quad \forall u \in V; \hspace{2cm} (1.2)$$

If $f, g \in H$, then (1.1) with $u = R_\lambda g$ reads

$$(f,R_\lambda g)_H = \lambda(R_\lambda f,R_\lambda g)_H + (R_\lambda f,R_\lambda g)_V,$$

where the right-hand side is symmetric in $f, g$. So is the left-hand side implying that $R_\lambda$ is a symmetric operator in $H$. After that (1.1) shows that $(R_\lambda f,u)_V = (f,R_\lambda u)_V$ if $f, u \in V$, so that $R_\lambda$ is also a symmetric operator in $V$.

Then observe that for $u = R_\lambda f$ equation (1.1) implies that

$$
\lambda\|R_\lambda f\|_H^2 + \|R_\lambda f\|_V^2 = (f,R_\lambda f)_H \leq \|f\|_H \|R_\lambda f\|_H, \hspace{2cm} (1.3)
$$

which yields the energy estimates

$$
\lambda\|R_\lambda f\|_H \leq \|f\|_H, \quad \|R_\lambda f\|_V \leq \|f\|_H. \hspace{2cm} (1.4)
$$
\textbf{Theorem 1.1.} (i) The norms of the operator $\lambda R_\lambda$ as an operator from $H$ into $H$ as well as an operator from $V$ into $V$ are less than one;
(ii) If $f \in H$, $\lambda \geq 0$, and $\lambda R_\lambda f = f$, then $f = 0$;
(iii) The set $R_\lambda H$ is dense in $V$ in the metric of $V$;
(iv) For any $f \in H$ we have
\[ \lim_{\lambda \to \infty} \| f - \lambda R_\lambda f \|_H = 0; \] (1.5)
(v) For $f \in V$ we have
\[ \lim_{\lambda \to \infty} \| f - \lambda R_\lambda f \|_V = 0. \] (1.6)

\textbf{Proof.} (i) We get the first part of the assertion from (1.4). Next, if $f \in V$, then for $u = f$ we get from (1.2) that
\[ (R_\lambda f, f)_V = ((1 - \lambda R_\lambda)f, (1 - \lambda R_\lambda)f)_H + ((1 - \lambda R_\lambda)f, \lambda R_\lambda f)_H \]
\[ = \| f - \lambda R_\lambda f \|^2_H + (\lambda R_\lambda f, f)_H - \| \lambda R_\lambda f \|^2_H, \]
where according to (1.3) we have $(\lambda R_\lambda f, f)_H - \| \lambda R_\lambda f \|^2_H = \lambda \| R_\lambda f \|^2_V$, so that
\[ (R_\lambda f, f)_V = \| f - \lambda R_\lambda f \|^2_H + \lambda \| R_\lambda f \|^2_V, \]
\[ \lambda \| R_\lambda f \|^2_V \leq (R_\lambda f, f)_V, \quad \| \lambda R_\lambda f \|_V \leq \| f \|_V. \] (1.7)
(ii) Under given conditions we have $f \in V$ and equation (1.1) implies that
$$(R_\lambda f, u)_V = 0$$ for all $u \in V$. Hence $R_\lambda f = 0$ and $f = 0$ indeed.
(iii) Assume the contrary. Then there exists $u \in V$, $u \neq 0$, such that
$$(R_\lambda f, u)_V = 0$$ for all $f \in H$. Then (1.1) shows that $(f, u)_H = (\lambda R_\lambda f, u)_H = (f, \lambda R_\lambda u)_H$ for all $f \in H$. It follows that $\lambda R_\lambda u = u$ and $u = 0$ by (ii), which is the desired contradiction.
(iv) If $f \in V$, this assertion follows from (1.7) after we let $\lambda \to \infty$, and use (i). In the general case it suffices to use the denseness of $V$ in $H$ and assertion (i).
(v) Owing to (iii) and (i) while proving (1.6) we may concentrate on $f = R_1 g$, where $g \in H$. Next, we observe that taking $u = (1 - \lambda R_\lambda)f$ in (1.2) leads to
\[ (R_\lambda f, (1 - \lambda R_\lambda)f)_V = ((1 - \lambda R_\lambda)f, (1 - \lambda R_\lambda)f)_H, \]
which implies that
\[ \|(1 - \lambda R_\lambda)f\|_V^2 = (f, (1 - \lambda R_\lambda)f)_V - (\lambda R_\lambda f, (1 - \lambda R_\lambda)f)_V \]
\[ = (f, (1 - \lambda R_\lambda)f)_V - \lambda \|(1 - \lambda R_\lambda)f\|_H^2 \leq (f, (1 - \lambda R_\lambda)f)_V. \]
Here, in light of (1.2), the last expression is
\[ (R_1 g, (1 - \lambda R_\lambda)f)_V = ((1 - R_1)g, (1 - \lambda R_\lambda)f)_H, \]
which tends to zero as $\lambda \to \infty$ by (iv). The theorem is proved.

As a justification of the notation $R_\lambda$ and its name as a resolvent operator consider the following situation.
Let $G$ be an open set in $\mathbb{R}^d = \{x = (x^1, ..., x^d) : x^i \in \mathbb{R}\}$ and let $m \geq 1$ be an integer. Define

$$H_2^m = H_2^m(G)$$

as the closure of $C_0^\infty = C_0^\infty(G)$ with respect to the norm

$$\|u\|_{H_2^m} := \left( \sum_{|\alpha| \leq m} \int_G |D^\alpha u|^2 \, dx \right)^{1/2},$$

where as usual for any multi-index $\alpha = (\alpha_1, ..., \alpha_d)$

$$D^\alpha = D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d}, \quad D_i = \partial / \partial x^i, \quad |\alpha| = \alpha_1 + ... + \alpha_d.$$

The space $H_2^m$ is a Hilbert space with scalar product

$$(u, \phi)_{H_2^m} = \sum_{|\alpha| \leq m} \int_G (D^\alpha u)D^\alpha \phi \, dx.$$

If above we take $H = L_2 = L_2(G)$ and $V = H_2^m$ then our hypotheses about $V$ and $H$ are satisfied and (1.1) becomes

$$\int_G f u \, dx = \lambda \int_G u R_\lambda f \, dx + \sum_{|\alpha| \leq m} \int_G (D^\alpha R_\lambda f)D^\alpha u \, dx,$$

which in the sense of generalized functions shows that $R_\lambda f$ is a solution of the equation

$$f = \lambda v - Lv,$$

where

$$L = - \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{2\alpha}.$$

Hence, $R_\lambda$ is indeed a resolvent operator for $L$. By the way, this example shows that, generally, $R_\lambda H \not= V$.

2. Itô’s Formula for the Squared Norm

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\{\mathcal{F}_t, t \geq\}$ be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$, which are complete with respect to $\mathcal{F}, P$.

In order to avoid unimportant complications we assume that $(V, (\cdot, \cdot)_V)$ is a separable Hilbert space, which is the case in many applications. Then $(H, (\cdot, \cdot)_H)$ is also separable. It is convenient that under this assumption there is no difference between weak and strong measurability.

Assume that we are given $V$-valued processes $v_t, v_t^*$, $t > 0$, which are predictable and satisfy

$$E \int_0^T \|v_t, v_t^*\|^2_V \, dt < \infty \quad (2.1)$$
for any \( T \in (0, \infty) \). Also let \( m_t, t \geq 0 \), be an \( H \)-valued continuous martingale starting at the origin with
\[
d\langle m \rangle_t \leq dt. \tag{2.2}
\]
The theory of integrating predictable Hilbert-space valued processes with respect to continuous same space-valued martingales is quite parallel to that in case the Hilbert space is just \( \mathbb{R}^d \). This theory implies that under the above conditions the stochastic integral
\[
h_t := \int_0^t (v_s, dm_s)_H \tag{2.3}
\]
is well defined and is a continuous real-valued martingale with
\[
\langle h \rangle_t \leq \int_0^t \| v_s \|_H^2 \, ds.
\]
Suppose that \( v_0 \) is an \( H \)-valued \( \mathcal{F}_0 \)-measurable random vector. Finally, assume that for any \( \phi \in V \) we have
\[
(\phi, v_t)_H = (\phi, v_0)_H + \int_0^t (\phi, v_s^*)_V \, ds + (\phi, m_t)_H \tag{2.4}
\]
for almost all \((\omega, t)\).

**Theorem 2.1.** Under the above assumptions there exists a continuous \( H \)-valued \( \mathcal{F}_t \)-adapted process \( u_t \) and a set \( \Omega' \subset \Omega \) of full probability such that
(i) \( u_t = v_t \) for almost all \((\omega, t)\), so that
\[
E \int_0^T \| u_t \|_V^2 \, dt < \infty
\]
for any \( T \in (0, \infty) \),
(ii) for all \( \omega \in \Omega' \), all \( \phi \in V \), and all \( t \geq 0 \) we have
\[
(\phi, u_t)_H = (\phi, v_0)_H + \int_0^t (\phi, v_s^*)_V \, ds + (\phi, m_t)_H, \tag{2.5}
\]
(iii) for all \( \omega \in \Omega' \) and all \( t \geq 0 \) we have
\[
\| u_t \|_H = \| v_0 \|_H + 2 \int_0^t (u_s, v_s^*)_V \, ds + \langle m \rangle_t + 2 \int_0^t (v_s, dm_s)_H. \tag{2.6}
\]

Proof. Inspired by (1.2) for \( n = 1, 2, \ldots \) define \( S_n = nR_n \) and
\[
u^n_t = S_nv_0 + \int_0^t n(1 - S_n)v_s^* \, ds + S_nm_t. \tag{2.7}
\]
Here the integral makes sense as the integral of an \( H \)-valued function. Furthermore, \( u^n_t \) is obviously continuous as an \( H \)-valued function.

Also observe that (2.4) with \( \phi = S_n\psi, \psi \in H \), and (1.2) yield that for almost all \((\omega, t)\)
\[
(\psi, S_nv_t)_H = (\psi, S_nv_0)_H + \int_0^t (\psi, n(1 - S_n)v_s^*)_H \, ds
\]
\[ +(\psi, S_n m_t)_H = (\psi, u^n_t)_H. \]  

This and the separability of \( H \) shows that

\[ u^n_t = S_n v_t \]

for almost all \((\omega, t)\).

Next, from Doob’s inequality it follows that for any \( T \in [0, \infty) \)

\[ E \sup_{t \leq T} \|u^n_t\|^2_H < \infty. \]

By Itô’s formula for integrals of Hilbert-space valued processes we have (a.s.)

\[ \|u^n_t\|^2_H = \|S_n v_0\|^2_H + 2 \int_0^t (u^n_s, n(1 - S_n) v^*_s)_H \, ds \]

\[ + (S_n m)_t + 2 \int_0^t (S_n u^n_s, dm)_H, \tag{2.9} \]

\[ \|u^n_t - u^k_t\|^2_H = \|(S_n - S_k)v_0\|^2_H + ((S_n - S_k)m)_t + 2 \int_0^t (S_n u^n_s - S_k u^k_s, dm)_H \]

\[ + 2 \int_0^t (u^n_s - u^k_s, [n(1 - S_n) - k(1 - S_k)]v^*_s)_H \, ds \tag{2.10} \]

for all \( t \geq 0 \).

Observe that there is \( \Omega' \) with \( P(\Omega') = 1 \) such that

\[ u^n_t = S_n v_t, \quad n = 1, 2, \ldots, \quad v_t, v^*_t \in V \tag{2.11} \]

for almost all \( t \) on \( \Omega' \). It follows that in the integrands in (2.9) and (2.10) we can replace \( u^n_t \) with \( S_n v_s \) if \( \omega \in \Omega' \) and use (1.2). Then for \( \omega \in \Omega' \) and \( t \) such that (2.11) holds we have

\[ (u^n_s - u^k_s, [n(1 - S_n) - k(1 - S_k)]v^*_s)_H = \]

\[ (S_n v_s - S_k v_s, n(1 - S_n)v^*_s)_H - (S_n v_s - S_k v_s, k(1 - S_k)v^*_s)_H \]

\[ = (S_n v_s - S_k v_s, S_n v^*_s)_V - (S_n v_s - S_k v_s, S_k v^*_s)_V \]

\[ = (S_n v_s - S_k v_s, S_n v^*_s - S_k v^*_s)_V. \]

Hence, for \( \omega \in \Omega' \) and all \( n, k \geq 1 \) and \( t \geq 0 \) we get that

\[ \|u^n_t\|^2_H = \|S_n v_0\|^2_H + 2 \int_0^t (S_n v_s, v^*_s)_V \, ds \]

\[ + (S_n m)_t + 2 \int_0^t (S_n u^n_s, dm)_H, \]

\[ \|u^n_t - u^k_t\|^2_H = \|(S_n - S_k)v_0\|^2_H + ((S_n - S_k)m)_t + 2 \int_0^t (S_n^2 v_s - S_k^2 v_s, dm)_H \]

\[ + 2 \int_0^t (S_n v_s - S_k v_s, S_n v^*_s - S_k v^*_s)_V \, ds. \]
Furthermore, by Doob’s inequality for any $T \in [0, \infty)$
\[
E \sup_{t \leq T} \|u^n_t - u^k_t\|_H^2 \leq I_{nk}^1 + 2I_{nk}^2 + I_{nk}^3 + 4(I_{nk}^4)^{1/2},
\]
(2.12)
where
\[
I_{nk}^1 = E\| (S_n - S_k)v_0\|_H^2,
\]
\[
I_{nk}^2 = E\int_0^T |((S_n - S_k)v_s, (S_n - S_k)v_s^*)_V| \, ds,
\]
\[
I_{nk}^3 = E\langle (S_n - S_k)m \rangle_T = E\| (S_n - S_k)m_T\|_H^2,
\]
\[
I_{nk}^4 = E\int_0^T \| S_n^2 v_s - S_k^2 v_s \|_H^2 \, ds.
\]

By using the dominated convergence theorem, Theorem 1.1, and the inequality
\[
|(S_n - S_k)v_s, (S_n - S_k)v_s^*)_V| \leq \|(S_n - S_k)v_s\|_V^2 + \|(S_n - S_k)v_s^*\|_V^2,
\]
we easily conclude that $I_{nk}^1 + 2I_{nk}^2 + I_{nk}^3 \to 0$ as $n, k \to \infty$. Furthermore,
\[
S_n^2 - S_k^2 = (S_n + S_k)(S_n - S_k)
\]
so that
\[
I_{nk}^4 \leq 4E\int_0^T \| (S_n - S_k)v_s\|_H^2 \, ds,
\]
which by the dominated convergence theorem implies that $I_{nk}^4 \to 0$ as $n, k \to \infty$ as well.

We now conclude from (2.12) that its left-hand side tends to zero. Furthermore,
\[
E\int_0^T \| u^n_t - v_t\|_V^2 \, dt = E\int_0^T \| (S_n - 1)v_t\|_V^2 \, dt \to 0.
\]

Hence $u^n_t$ converges to $v_t$ in $H(\Omega \times (0, T), V)$ and converges uniformly on $[0, T]$ as $H$-valued functions in probability. The latter limit we denote by $u_t$ and show that this function is the one we want. Of course, $u_t$ is a continuous $H$-valued functions, it is $\mathcal{F}_t$-adapted, and $u_t = v_t$ for almost all $(\omega, t)$.

One easily obtains that for each $t$ equation (2.6) holds with probability one by passing to the limit in (2.9). Since both parts of (2.6) are continuous in $t$, it holds on a set of full probability for all $t$.

Obviously (2.4) will hold for almost all $(\omega, t)$ if we replace $v_t$ with $u_t$, that is, (2.5) holds for any $\phi \in V$ for almost all $(\omega, t)$. The continuity of both parts of (2.5) with respect to $t$ and $\phi \in V$ and the separability of $V$ then imply that there is a set $\Omega'$ of full probability such that assertion (iii) holds.

The theorem is proved.

Remark 2.2. The reader understands, of course, that condition (2.1) can be replaced with the same condition but without expectation sign. This generalization is easily achieved by using appropriate stopping times.
Next in the setting described in the end of Section 1 suppose that we are given an $\tilde{H}_2^m$-valued process $v_t$ and $L_2$-valued processes $f_t^\alpha$, $|\alpha| \leq m$. We assume that all these processes are predictable and such that

$$E \int_0^T \left[ \|v_t\|_{\tilde{H}_2^m}^2 + \sum_{|\alpha| \leq m} \|f_t^\alpha\|_{L_2}^2 \right] dt < \infty$$

for any $T \in (0, \infty)$. We also assume that we are given a continuous $L_2$-valued martingale $m_t$ satisfying (2.2) and $v_0$ is an $L_2$-valued $\mathcal{F}_0$-measurable random function. Finally, suppose that for any $\phi \in C_0^\infty$ we have

$$\int_G \phi v_t \, dx = \int_G \phi v_0 \, dx + \int_0^t \int_G \sum_{|\alpha| \leq m} f_s^\alpha D^\alpha \phi \, dx \, ds + \int_G \phi m_t \, dx \quad (2.13)$$

for almost all $(\omega, t)$.

**Remark 2.3.** Formally (2.13) can be expressed as

$$dv_t = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_t^\alpha \, dt + dm_t,$$

which shows that $dv_t$ lives in $H_{-m}^2 := (\tilde{H}_2^m)^*$. 

**Theorem 2.4.** Under the above assumptions there exists a continuous $L_2$-valued $\mathcal{F}_t$-adapted process $u_t$ and a set $\Omega' \subset \Omega$ of full probability such that

(i) $u_t = v_t$ for almost all $(\omega, t)$, so that

$$E \int_0^T \|u_t\|_{\tilde{H}_2^m}^2 dt < \infty$$

for any $T \in (0, \infty)$,

(ii) for all $\omega \in \Omega'$, all $\phi \in \tilde{H}_2^m$, and all $t \geq 0$ we have

$$(\phi, u_t)_{L_2} = (\phi, v_t)_{L_2} + \int_0^t \sum_{|\alpha| \leq m} (D^\alpha \phi, f_s^\alpha)_{L_2} \, ds + (\phi, m_t)_{L_2}, \quad (2.14)$$

(iii) for all $\omega \in \Omega'$ and all $t \geq 0$ we have

$$\|u_t\|_{L_2} = \|v_0\|_{L_2} + 2 \int_0^t \sum_{|\alpha| \leq m} (D^\alpha u_s, f_s^\alpha)_{L_2} \, ds + \langle m \rangle_t + 2 \int_0^t (v_s, dm_s)_{L_2}.$$ 

Proof. To derive this result from Theorem 2.1, we first observe that in light of the denseness of $C_0^\infty$ in $\tilde{H}_2^m$ equation (2.13) also holds for any $\phi \in \tilde{H}_2^m$. Then notice that for each $(\omega, s)$

$$F_s(\phi) := \int_G \sum_{|\alpha| \leq m} f_s^\alpha D^\alpha \phi \, dx$$


is a bounded linear functional on \( \hat{H}_2^m \) with
\[
|F_s(\phi)| \leq \|\phi\|_{H_2^m} \left( \sum_{|\alpha| \leq m} \|f^\alpha_s\|_L^2 \right) \frac{1}{2}.
\]

It follows by Riesz’s representation theorem that there exists a unique \( v^*_s \in \hat{H}_2^m \) such that
\[
F_s(\phi) = (\phi, v^*_s)_{H_2^m}, \quad \|v^*_s\|_2 \leq \sum_{|\alpha| \leq m} \|f^\alpha_s\|_L^2.
\]

These relations imply that \( v^*_s \) is weakly predictable and, since \( \hat{H}_2^m \) is separable, it is (just) predictable. Also we have that condition (2.1) is satisfied. Hence one can rewrite (2.13) in form (2.4) and then all assertion of the present theorem follow directly from Theorem 2.1. The theorem is proved.

3. A More General Itô’s Formula

We suppose that all assumptions stated in Section 2 are satisfied.

Let \( \phi(h) \) be a real-valued function on \( H \). Assume that

(i) for any \( h, \xi \in H \) the functions \( \phi(h + t\xi) \) is twice continuously differentiable as a function of \( t \) and the functions
\[
\phi_t(h) := \frac{\partial}{\partial t}\phi(h + t\xi)_{t=0}, \quad \phi_{tt}(h) := \frac{\partial^2}{(\partial t)^2}\phi(h + t\xi)_{t=0}
\]
are continuous as functions of \( (h, \xi) \in H \times H \);

(ii) For any \( R \in (0, \infty) \) there exists a \( K(R) \) such that for all \( h, \xi \in H \) satisfying \( \|h\|_H \leq R \) we have
\[
\|\phi_t(h)\|_H \leq K(R)\|\xi\|_H, \quad \|\phi_{tt}(h)\|_H \leq K(R)\|\xi\|^2_H.
\]

In this situation for any \( h \in H \) the function \( \phi_t(h) \) as a function of \( \xi \in H \) is a continuous linear functional and by Riesz’s representation theorem there exists an element \( \phi_t(h) \in H \) such that
\[
\phi_t(h) = (\phi_t(h), \xi)_H, \quad \|\phi_t(h)\|_H \leq K(\|h\|_H).
\]

Next, we assume that,

(iii) If \( h \in V \), then \( \phi_t(h) \in V \) and
\[
\|\phi_t(h)\|_V \leq K(1 + \|h\|_V),
\]
where \( K \) is a fixed constant;

(iv) For any \( v^* \in V \) the function \( (\phi_t(v), v^*)_V \) is a continuous function on \( V \) (in the metric of \( V \)).

Let \( w_1^1, w_2^2, \ldots \) be a finite or infinite sequence of independent Wiener processes on \( (\Omega, \mathcal{F}, P) \), which are Wiener processes with respect to \( \{\mathcal{F}_t\} \). We
assume that we are given a sequence of predictable $H$-valued processes $\sigma^i_t$ such that for any $T \in (0, \infty)$

$$\sum_k E \int_0^T \|\sigma^k_t\|^2_H dt < \infty.$$ 

Under this assumption it is well known that the series

$$\sum_k \int_0^t \sigma^k_s dw^k_s$$

converges in $H$ uniformly on finite time intervals in probability and we assume that the series converges to $m_t$. From the continuity of the scalar product in $H$ it follows also that for any $h \in H$ we have (a.s.) for all $t$

$$(h, m_t)_H = \sum_k \int_0^t (h, \sigma^k_s)_H dw^k_s,$$

where the series converges uniformly on finite time intervals in probability. Then equation (2.14) is equivalent to saying that the function $u_t$ satisfies

$$(\phi, u_t)_H = (\phi, v_0)_H + \int_0^t (\phi, v^*_s)_V ds + \sum_k \int_0^t (\phi, \sigma^k_s)_H dw^k_s$$

(3.1)

for each $\phi \in V$ (a.s.) for all $t$.

The following result can be found in [14] in a more general situation. Our innovation is a different and shorter proof.

**Theorem 3.1.** Under the above assumptions (a.s.) for all $t$

$$\phi(u_t) = \phi(u_0) + \sum_k \int_0^t \phi(\sigma^k_s)(u_s) dw^k_s$$

$$+ \int_0^t \left[(\phi(\cdot)(u_s), v^*_s)_V + (1/2) \sum_k \phi(\sigma^k_s)\sigma^k_s(u_s)\right] ds, \quad (3.2)$$

where $u_t$ is taken from Theorem 2.1 and the series of stochastic integrals converges uniformly on finite time intervals in probability.

**Proof.** The last assertion of the theorem follows from the fact that the series of quadratic variations of the stochastic integrals in (3.2) converges:

$$\sum_k \int_0^t |\phi(\sigma^k_s)(u_s)|^2 ds \leq \sum_k \int_0^t \|\phi(\cdot)(u_s)\|^2_H \|\sigma^k_s\|^2_H ds$$

$$\leq K^2 \left(\sup_{s \leq t} \|u_s\|_H\right) \sum_k \int_0^t \|\sigma^k_s\|^2_H ds < \infty.$$

It is also worth noting that other terms in (3.2) make sense as well. Indeed,

$$\int_0^t \sum_k |\phi(\sigma^k_s)(\sigma^k_s)(u_s)| ds \leq K^2 \left(\sup_{s \leq t} \|u_s\|_H\right) \sum_k \int_0^T \|\sigma^k_t\|^2_H dt < \infty,$$
\[ \int_0^t |(\phi(\cdot), v^*_s)_V| \, ds \leq K \int_0^t (1 + \|u_s\|_V)\|v^*_s\|_V \, ds < \infty. \]

This argument shows that the right-hand side of (3.2) is a continuous process (a.s.). So is its left-hand side and, to prove that (3.2) holds (a.s.) for all \( t \), it suffices to prove that (3.2) holds for each \( t \) (a.s.).

The rest of the proof we split into a few steps.

**Step 1.** Consider the case that \( V = H \) and the number of the Wiener processes is finite, say, equal to \( p \). Take an orthonormal basis \( \{e_i\} \) in \( H \), denote by \( \Pi_n \) the orthogonal projection operator on \( \text{Span} \{e_1, ..., e_n\} \), and set

\[ u^n_t := \Pi_n u_t = \Pi_n u_0 + \int_0^t \Pi_n v^*_s \, ds + \sum_{k \leq p} \int_0^t \Pi_n \sigma^k_s \, dw^k_s, \quad \phi_n(h) = \phi(\Pi_n h). \]

The function \( \phi_n \), as a continuous function on a finite-dimensional Euclidean space, has two continuous directional derivatives in any direction. Therefore, it is twice continuously differentiable and by the classical Itô’s formula

\[ \phi(u^n_t) = \phi(u^n_0) + \sum_{k \leq p} \int_0^t \phi(\Pi_n \sigma^k_s)(\Pi_n u_s) \, dw^k_s \]

\[ + \int_0^t \left[ \phi(\Pi_n v^*_s)(\Pi_n u_s) + (1/2) \sum_{k \leq p} \phi(\Pi_n \sigma^k_s)(\Pi_n \sigma^k_s)(\Pi_n u_s) \right] \, ds. \tag{3.3} \]

Here

\[ |\phi(\Pi_n \sigma^k_s)(\Pi_n u_s)| \leq \|\sigma^k_s\|_H^2 K \left( \max_{\|u_s\|_H} \right) \]

and on an event of full probability on which \( u_s \) is an \( H \)-valued continuous function

\[ \phi(\Pi_n \sigma^k_s)(\Pi_n u_s) \to \phi(\sigma^k_s)(\sigma^k_s)(u_s) \]

for all \( s \) and \( k \). It follows by the dominated convergence theorem that

\[ \int_0^t \phi(\Pi_n \sigma^k_s)(\Pi_n u_s) \, ds \to \int_0^t \phi(\sigma^k_s)(\sigma^k_s)(u_s) \, ds \]

for any \( t \) and \( k \) (a.s.). Similarly, for any \( t \) (a.s.)

\[ \int_0^t \phi(\Pi_n v^*_s)(\Pi_n u_s) \, ds \to \int_0^t \phi(v^*_s)(u_s) \, ds. \]

Finally, by the same reasons as above

\[ \int_0^t |\phi(\Pi_n \sigma^k_s)(\Pi_n u_s) - \phi(\sigma^k_s)(u_s)|^2 \, ds \to 0 \]

for any \( t \) and \( k \) (a.s.).

This allows us to pass to the limit in (3.3) and conclude that

\[ \phi(u_t) = \phi(u_0) + \sum_{k \leq p} \int_0^t \phi(\sigma^k_s)(u_s) \, dw^k_s \]
for any $t$ (a.s.).

**Step 2.** Again let $V = H$ but suppose that the number of the Wiener processes is infinite. Then introduce

$$u^n_t = u_0 + \int_0^t v^*_s ds + \sum_{k \leq n} \int_0^t \sigma^k_s dw_s$$

and observe that, as we pointed out before the theorem, for any (finite) $t, \varepsilon > 0$,

$$P(\sup_{s \leq t} \|u_s - u^n_s\|_H > \varepsilon) \to 0 \quad (3.5)$$

as $n \to \infty$. By the result of Step 1

$$\phi(u^n_t) = \phi(u_0) + \sum_{k \leq n} \int_0^t \phi(\sigma^k_s)(u^n_s) dw^k_s$$

$$+ \int_0^t [\phi(v^*_s)(u^n_s) + (1/2) \sum_{k \leq n} \phi(\sigma^k_s)(u^n_s)] ds \quad (3.6)$$

for any $t$ (a.s.).

Next, owing to (3.5) there is a subsequence $n(j) \to \infty$ as $j \to \infty$ such that for any $t \in (0, \infty)$ (a.s.)

$$\sup_{s \leq t} \|u_s - u^{n(j)}_s\|_H \to 0.$$

Then, of course,

$$\int_0^t \phi(v^*_s)(u^{n(j)}_s) ds \to \int_0^t \phi(v^*_s)(u_s) ds \quad (a.s.).$$

Furthermore, (a.s.)

$$\phi(\sigma^k_s)(u^{n(j)}_s) \to \phi(\sigma^k_s)(u_s)$$

because of the continuity of $\phi(\xi)(h)$ on $H \times H$. In addition,

$$\sum_k |\phi(\sigma^k_s)(u^{n(j)}_s)|^2 \leq K^2 \left( \sup_{s \leq t, r \geq 1} \|u^{n(r)}_s\|_H \right) \sum_k \|\sigma^k_s\|_H^2 ds$$

and the right-hand side has a finite integral over $[0, t]$ (a.s.). It follows by the dominated convergence theorem that the quadratic variation at time $t$ of the difference

$$\sum_{k \leq n} \int_0^t \phi(\sigma^k_s)(u^{n(j)}_s) dw^k_s - \sum_k \int_0^t \phi(\sigma^k_s)(u_s) dw^k_s$$

tends to zero (a.s.) as $k \to \infty$ and the difference itself goes to zero in probability.
For similar reasons
\[ \sum_k \int_0^t |\phi(\sigma^k_s)(\sigma^k_u)(u^n_s) - \phi(\sigma^k_s)(\sigma^k)(u_s)| \, ds \to 0 \]
(a.s.) and we conclude from (3.6) that (3.2) holds (a.s.).

Step 3. Now we consider the general case. As in the proof of Theorem 2.1 we introduce \( u^*_n \) by (2.7) and observe that the computation (2.8) shows that (a.s.) \( u^*_n = S_n u_t \) for all \( t \). According to Step 2 for any \( t \) (a.s.)
\[ \phi(S_n u_t) = \phi(S_n u_0) + \sum_k \int_0^t \phi(S_n \sigma^k_s)(S_n u_s) \, dw^k_s \]
\[ + \int_0^t [\phi(\cdot)(S_n u_s), n(1 - S_n) v^*_s]^H + (1/2) \sum_k \phi(S_n \sigma^k_s)(S_n u_s) \, ds. \] (3.7)

Here (a.s.) for all \( s \)
\[ \phi(S_n \sigma^k_s)(S_n u_s) \to \phi(\sigma^k_s)(u_s) \]
as \( n \to \infty \) because of the continuity of \( \phi(\cdot)(h) \) on \( H \times H \). Furthermore,
\[ |\phi(S_n \sigma^k_s)(S_n u_s)| \leq K(\|u_s\|_H)\|S_n \sigma^k_s\|_H \leq K(\|u_s\|_H)\|\sigma^k_s\|_H. \]
As before this implies that the series of stochastic integrals in (3.7) converges to that in (3.2) in probability as \( n \to \infty \).

Next, owing to (1.2) and the fact that \( S_n u_s \in V \) and \( \phi(\cdot)(S_n u_s) \in V \)
\[ (\phi(\cdot)(S_n u_s), n(1 - S_n) v^*_s)^H = (\phi(\cdot)(S_n u_s), S_n v^*_s)^V. \]
With probability one \( u_s \in V \) for almost all \( s \) for which also \( \phi(\cdot)(S_n u_s) \to \phi(\cdot)(u_s) \) weakly in \( V \), owing to assumption (iv), whereas \( S_n v^*_s \to v^*_s \) strongly in \( V \). Hence with probability one for almost all \( s \)
\[ (\phi(\cdot)(S_n u_s), n(1 - S_n) v^*_s)^H \to (\phi(\cdot)(u_s), v^*_s)^V. \]
as \( n \to \infty \). Furthermore,
\[ |(\phi(\cdot)(S_n u_s), n(1 - S_n) v^*_s)^H| = |(\phi(\cdot)(S_n u_s), S_n v^*_s)^V| \leq K(1 + \|u_s\|_V)\|v^*_s\|_V \]
by assumption (iii). It follows by the dominated convergence theorem that (a.s.)
\[ \int_0^t (\phi(\cdot)(S_n u_s), n(1 - S_n) v^*_s)^H \, ds \to \int_0^t (\phi(\cdot)(u_s), v^*_s)^V \, ds. \]

Finally, (a.s.) for all \( s \)
\[ \phi(S_n \sigma^k_s)(S_n \sigma^k_u)(S_n u_s) \to \phi(\sigma^k_s)(\sigma^k)(u_s) \]
because of assumption (i) and
\[ |\phi(S_n \sigma^k_s)(S_n \sigma^k_u)(S_n u_s)| \leq K(\sup_{s \leq t} \|u_s\|_H)\|\sigma^k_s\|_H^2 \]
in light of assumption (ii). This allows us to pass to the limit in the remaining expression in (3.7) and brings the proof of the theorem to an end.
4. The maximum principle for second-order SPDEs

In Section 3 take a domain $G \subset \mathbb{R}^d$, $V = \dot{H}_2^1 = \dot{H}_2^1(G)$, and $H = L_2 = L_2(G)$.

Take an infinitely differentiable function $r(x)$, $x \in \mathbb{R}$, such that $|r(x)| \leq N|x|^2$, $|r'(x)| \leq N|x|$, and $|r''| \leq N$, where $N$ is a constant. For $h \in L_2$ define

$$\phi(h) = \int_G r(h(x)) \, dx.$$ 

As is easy to see, assumptions (i) and (ii) of Section 3 are satisfied and for $h, \xi \in L_2$

$$\phi(\xi)(h) = \int_G r'(h(x))\xi(x) \, dx, \quad \phi(h) = r'(h(x)),$$

$$\phi(\xi)(h) = \int_G r''(h(x))\xi^2(x) \, dx.$$ 

Furthermore, if $h \in \dot{H}_2^1$, then there exists a sequence of $h_n \in C_0^\infty$ such that $h_n \to h$ in the norm of $\dot{H}_2^1$. Almost obviously $\phi(\cdot)(h_n) = r'(h_n(x)) \in C_0^\infty$ and $r'(h_n(x)) \to r'(h(x))$ in the $\dot{H}_2^1$-norm. Hence $\phi(\cdot)(h) \in \dot{H}_2^1$ if $h \in \dot{H}_2^1$.

One can also easily verify that

$$\|\phi(\cdot)(h)\|_{\dot{H}_2^1} \leq N\|h\|_{\dot{H}_2^1},$$

where $N$ is the constant from above, so that assumption (iii) of Section 3 is satisfied as well. Finally, it is not hard to check that for $v^* \in \dot{H}_2^1$

$$(r'(h), v^*)_{\dot{H}_2^1} = \int_G r'(h(x))v^*(x) \, dx + \sum_{|\alpha|=1}^\infty \int_G r''(h(x))(D_\alpha h(x))D_\alpha v^*(x) \, dx$$

is continuous as a function of $h$ on the space $\dot{H}_2^1$ and this is what is required in assumption (iv) of Section 3.

By Theorem 3.1 we now conclude that

$$\int_G r(u_t) \, dx = \int_G r(u_0) \, dx + \sum_k \int_0^t \int_G r'(u_s)\sigma_s^k \, dx \, dw_s^k$$

$$+ \int_0^t \int_G r'(u_s)v_s^* \, dx \, ds + \sum_{|\alpha|=1}^\infty \int_0^t \int_G r''(u_s)(D_\alpha u_s)D_\alpha v_s^* \, dx \, ds$$

$$+(1/2) \int_0^t \int_G r''(u_s) \sum_k |\sigma_s^k|^2 \, dx \, ds. \quad (4.1)$$

Next, we generalize this formula for $r$ from a wider class. Denote by $\mathcal{R}$ the set of real-valued functions $r(x)$ on $\mathbb{R}$ such that

(i) $r$ is continuously differentiable, $r(0) = r'(0) = 0$, ....
where \( \alpha \) is a left-continuous function with which is usual \( r'' \) which exists almost everywhere is bounded and there is a left-continuous function with which \( r'' \) coincides almost everywhere.

For \( r \in \mathcal{R} \) by \( r'' \) we will always mean the left-continuous modification of the usual second-order derivative of \( r \).

It turns out (see Remark 2.1 in [7]) that for any \( r \in \mathcal{R} \) there exists a sequence \( r_n \in \mathcal{R} \) of infinitely differentiable functions such that \( |r_n(x)| \leq N|x|^2 \), \( |r'_n(x)| \leq N|x| \), and \( |r''_n| \leq N \) with \( N < \infty \) independent of \( x \in \mathbb{R} \) and \( n \), and \( r_n, r'_n, r''_n \to r, r', r'' \) on \( \mathbb{R} \). By using this fact one easily shows that \( r(u) \in \dot{H}^1_2 \) if \( u \in \dot{H}^1_2 \) and (4.1) also holds for \( r \in \mathcal{R} \).

In particular, we can apply (4.1) with \( r(x) = (x^+)^2 \) and by using the well-known fact that

\[
D_i(u^+) = I_{u > 0}D_iu \quad \text{(4.2)}
\]

we then obtain that

\[
\int_G (u_i^+)^2 \, dx = \int_G (u_i^+)^2 \, dx + 2 \sum_k \int_0^t \int_G u_s^+ \sigma^k_s \, dx \, dw^k_s \\
+ 2 \int_0^t \int_G u_s^+ v_s^+ \, dx \, ds + 2 \sum_{i=1}^d \int_0^t \int_G (D_i u_s^+) D_i v_s^+ \, dx \, ds \\
+ \int_0^t I_{u_i > 0} \sum_k |\sigma^k|^2 \, dx \, ds = \int_G (u_0^+)^2 \, dx + 2 \sum_k \int_0^t (u_s^+, \sigma^k_s)_{L_2} \, dw^k_s \\
+ 2 \int_0^t (u_i^+, v_i^+)_{H^1_2} \, ds + \int_0^t I_{u_i > 0} \sum_k |\sigma^k|^2 \, dx \, ds. \quad \text{(4.3)}
\]

Next, assume that in addition to (3.1) we have that for any \( \phi \in H^1_2 \) (a.s.) for all \( t \)

\[
(\phi, u_t) = (\phi, u_0) + \int_0^t (\phi, \sigma^i_s D_i u_s + v^i_s u_s)_{L_2} \, dw^k_s \\
+ \int_0^t [(D_i \phi, -a^{ij}_s D_j u_s - a^i_s u_s)_{L_2} + (\phi, b^i_s D_i u_s + c_s u_s + f_s)_{L_2}] \, ds, \quad \text{(4.4)}
\]

where the summation with respect to repeated indices is understood. We assume that \( a^{ij}_s(x), b^i_s(x), a^i_s(x), c_t(x), f_t(x), \sigma^i_k(x), \) and \( v^i_s(x) \) are real-valued functions defined for \( i, j = 1, ..., d, \ k = 1, 2, ..., \ t \in [0, \infty), \ x \in \mathbb{R}^d \) and also depending on \( \omega \in \Omega \).

**Assumption 4.1.** For all values of the arguments

(i) \( \sigma^i := (\sigma^{i1}, \sigma^{i2}, ...) \), \( \nu := (\nu^1, \nu^2, ...) \in \ell_2 \);

(ii) for all \( \lambda \in \mathbb{R}^d \)

\[
(2a^{ij} - \alpha^{ij})\lambda^i \lambda^j \geq 0,
\]

where \( \alpha^{ij} = (\sigma^i, \sigma^j)_{\ell_2} \).

Assumption 4.1 (ii) is just the usual parabolicity assumption. We need one more function \( K_t \geq 0 \) defined on \( \Omega \times [0, \infty) \).
Assumption 4.2. (i) The functions \( a^i_1(x), b^i_1(x), a^i_1(x), c_i(x), \sigma^i_1(x), \nu^i_1(x), \) and \( K(t) \) are measurable with respect to \((\omega, t, x)\) and \( \mathcal{F}_t \)-adapted for each \( x \);
(ii) the functions \( a^i_2(x), b^i_2(x), a^i_2(x), c_i(x), \sigma^i_2(x), \) and \( \nu^i_2(x) \) are bounded;
(iii) for each \( \omega, t \) the functions
\[
\eta^i_t := a^i_t - b^i_t - (\sigma^i_t, \nu_t)\ell_2
\]
are once continuously differentiable on \( D \), have bounded derivatives, and satisfy
\[
D_t \eta^i_t + 2c + |\nu^2_t|_{\ell_2} \leq K
\]
for all values of arguments;
(iv) the process \( f_t \), is \( L_2 \)-valued \( \mathcal{F}_t \)-adapted and jointly measurable; and
for all \( T \in [0, \infty) \)
\[
E \int_0^T \left( ||f_s||^2_{L_2} + K_s \right) ds < \infty.
\]

Under these assumptions (and the assumption that \( u_t \) is taken from Section 3 corresponding to some \( v^*_t \) and \( m_t \)) the stochastic integrals in (4.4) have exactly the same form as in (3.1) if in the latter we replace \( \sigma^k_s \) with
\[
\sum_{i=1}^{d} \sigma^i_s D_i u_s + \nu^k_s u_s,
\]
for which
\[
\sum_k \left| \sum_{i=1}^{d} \sigma^i_s D_i u_s + \nu^k_s u_s \right|^2 = \alpha^i_s (D_i u_s) D_j u_s + 2(\sigma^i_s, \nu_s)\ell_2 u_s D_i u_s + |\nu_s|^2_{\ell_2} u_s^2.
\]
The processes (4.6) are predictable \( L_2 \)-valued processes satisfying
\[
E \sum_k \int_0^T \left\| \sum_i \sigma^i_s D_i u_s + \nu^k_s u_s \right\|^2_{L_2} ds
\leq NE \sum_i \int_0^T \int_G \alpha^i_s (D_i u_s)^2 dx ds + NE \int_0^T \int_G |\nu_s|^2_{\ell_2} u_s^2 dx ds < \infty
\]
for any \( T \in (0, \infty) \), where \( N \) are absolute constants.

At the first sight, the usual integral in (4.4) does not look like the one in (3.1). However, observe that on \( A := \{ (\omega, s) : u_s \in \overset{0}{H}_2^1 \} \) the function
\[
(D_i \phi, -a^i_s D_j u_s - a^i_2 u_s)_{L_2} + (\phi, b^i_s D_i u_s + c_s u_s + f_s)_{L_2}
\]
as a function on \( \overset{0}{H}_2^1 \) is continuous. By Riesz’s representation theorem there exists a unique \( v^* \) such that
\[
I_A(D_i \phi, -a^i_s D_j u_s - a^i_2 u_s)_{L_2} + (\phi, b^i_s D_i u_s + c_s u_s + f_s)_{L_2} = (\phi, v^*_{H_2^1})
\]
Since \( u_t \) is an \( L_2 \)-continuous process and \( \overset{0}{H}_2^1 \) is a Borel subset of \( L_2 \), the set \( A \) is predictable. Also notice that \( D_j u_s \) could be defined as the limits
of finite differences. Hence, $I_A D_j u_s$ are also predictable and formula (4.8) (along with the separability of $H^1_2$) shows that $v^*_s$ is an $H^1_2$-valued predictable process. Furthermore, the absolute value of the left-hand side of (4.8) is obviously less than

$$NI_A \| \phi \|_{H^1_2} (\| u_s \|_{H^1_2} + \| f_s \|_{L_2}),$$

where $N$ depends only on $d$ and the sup norms of the coefficients. It follows that

$$E \int_0^T \| v^*_s \|_{H^1_2}^2 \, ds < \infty$$

for any $T$.

Summing up all the above comments on equation (4.4) we conclude that, our assumption that $u_t$ satisfies it, is justified if it satisfies (3.1) with $\sigma$ and $v^*$ specified above. We are not going to discuss the possibility of existence of such $u_t$, that is the existence of solutions of (4.4) in the class of functions $u_t$ as in Theorem 2.1. By the way, generally, such solutions may not even exist. For instance, if all the coefficients and $f_s$ in (4.4) vanish identically, we have $u_t = u_0$ and, if $u_0 \notin H^1_2$, we do not have $u_t \in H^1_2$ for almost all $(\omega, t)$.

We will just assume that we are given a continuous $L_2$-valued predictable process $u_t$ such that

$$E \int_0^T \| u_t \|_{H^1_2}^2 \, dt < \infty$$

for any $T \in [0, \infty)$ and equation (4.4) holds for any $\phi \in H^1_2$ (a.s.) for all $t$.

A very particular case of the following theorem can be found in [14] (see also the references therein).

**Theorem 4.3** (maximum principle). Under the above assumptions suppose that $u_0 \leq 0$ and $f_t \leq 0$ for almost all $(\omega, t)$. Then (a.s.) for all $t$ we have $u_t \leq 0$.

**Proof.** According to what has been explained before the theorem and formulas (4.3), (4.7), and (4.8) we have that (a.s.) for all $t$

$$\| u^+_t \|_{L_2} = M_t + 2 \int_0^t \left[ (D_i u^+_s, -a^i_s D_j u_s - a^i_s u_s)_{L_2} + (u^+_s, b^i_s D_i u_s + c_s u_s + f_s)_{L_2} \right] ds$$

$$+ \int_0^t \int_G \alpha^i_s (D_i u_s) D_j u_s + 2 (\sigma^i_s, \nu_s)_{L_2} u_s D_i u_s + |\nu_s|^2_{L_2} u_s^2 \, dx ds$$

(4.9)

where $M_t$ is a martingale.

According to (4.2)

$$(a^i_s (D_i u^+_s), D_j u_s)_{L_2} = (a^i_s (D_i u^+_s), D_j u^+_s)_{L_2}$$

$$(I_{u_t>0} a^i_s (D_i u_s), D_j u_s)_{L_2} = (a^i_s (D_i u^+_s), D_j u^+_s)_{L_2},$$
so that
\[-2 \langle a^{ij}_s (D_i u^+_s), D_j u^+_s \rangle_{L^2} + \langle I_{u^+_s > 0} a^{ij}_s (D_i u^+_s), D_j u^+_s \rangle_{L^2} \leq 0\]
in light of Assumption 4.1 (ii). Furthermore, at points where \( u^+_s \in H^1_2 \), we have
\[I_s := -(D_i u^+_s, a^{ij}_s u^+_s)_{L^2} + (u^+_s, b^j_s D_j u^+_s)_{L^2} + ((\sigma^i_s, \nu^i_s)_{L^2} u^+_s, D_i u^+_s)_{L^2}\]
\[= -(u^+_s, D_i u^+_s)_{L^2} = (D_i \eta^i_s u^+_s, u^+_s)_{L^2},\]
where the last equality is obtained by integrating by parts, which is justified by approximating \( u^+_s \in H^1_2 \) by \( C^\infty_0 \)-functions and passing to the limit. At this point the reader can understand that, actually, we only need \( \eta^i_t \) to be Lipschitz continuous rather than continuously differentiable. In any case by also observing that \( u^+_s f_s \leq 0 \) and using Assumption 4.2 (iii) we conclude from (4.9) that (a.s.)
\[d \| u^+_t \|_{L^2} \leq d M_t + K_t \| u^+_t \|_{L^2},\]
\[d \left[ \| u^+_t \|_{L^2} \exp \left( - \int_0^t K_s ds \right) \right] \leq \exp \left( - \int_0^t K_s ds \right) d M_t,\]
\[\| u^+_t \|_{L^2} \exp \left( - \int_0^t K_s ds \right) \leq \int_0^t \exp \left( - \int_0^r K_s dr \right) d M_r.\]
In the last relation the left-hand side is nonnegative and the right-hand side is a martingale starting from zero. It follows that, with probability one, the martingale is zero and so is \( \| u^+_t \|_{L^2} \) which proves the theorem.

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127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455
E-mail address: krylov@math.umn.edu