AN INVERTIBILITY CRITERION IN A C*-ALGEBRA ACTING ON THE HARDY SPACE WITH APPLICATIONS TO COMPOSITION OPERATORS

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Abstract. In this paper we prove an invertibility criterion for certain operators which is given as a linear algebraic combination of Toeplitz operators and Fourier multipliers acting on the Hardy space of the unit disc. Very similar to the case of Toeplitz operators we prove that such operators are invertible if and only if they are Fredholm and their Fredholm index is zero. As an application we prove that for “quasi-parabolic” composition operators the spectra and the essential spectra are equal.

1. Introduction

In this paper we investigate the invertibility of elements in the C*-algebra \( \Psi \) generated by Toeplitz operators and Fourier multipliers. The C*-algebra \( \Psi \) is defined to be

\[
\Psi = \Psi(QC, C([0, \infty])) = C^*([T_\varphi : \varphi \in QC] \cup \{D_\vartheta : \vartheta \in C([0, \infty])\})
\]

the C*-algebra generated by Toeplitz operators with QC symbols and Fourier multipliers with continuous symbols. This C*-algebra was introduced by the first author in \cite{3} in order to study the spectral properties of a class of composition operators. In \cite{3}, the first author showed that \( \Psi/K(H^2) \) is a commutative C*-algebra with identity and determined its maximal ideal space. The maximal ideal space \( M \) of \( \Psi/K(H^2) \) is found to be homeomorphic to a certain subset of \( M(QC) \times [0, \infty] \) which can be described as

\[
M \cong (M_\infty(QC(\mathbb{R})) \times [0, \infty]) \cup (M(QC(\mathbb{R})) \times \{\infty\})
\]

where \( M_\infty(QC) \) is the fiber of \( M(QC) \) at infinity.

In this paper we show that if \( T = \sum T_\varphi D_{\vartheta} + \sum D_\vartheta T_\psi \in \Psi \) is written as a finite sum or as an infinite sum converging in the operator norm where \( \psi, \varphi \in QC \) and \( \vartheta, \psi \in C([0, \infty]) \) then \( T \) is invertible if and only if \( T \) is Fredholm and has Fredholm index zero. We do this through constructing a homotopy \( H : [0, 1] \to \Psi \) which is defined as

\[
H(w) := \sum T_\varphi D_{\vartheta} + \sum D_\vartheta T_\psi
\]

where \( \nu^w_j(t) := \vartheta_j(t - \ln w), \psi^w_j(t) := \vartheta_j(t - \ln w) \) and \( H(0) := T_\varphi \) where \( \varphi := \sum \lambda_j \varphi_j + \sum \mu_j \psi_j \) with \( \lambda_j := \lim_{t \to \infty} \vartheta_j(t) \) and \( \mu_j := \lim_{t \to \infty} \psi_j(t) \). We observe that this homotopy acts continuously on finite sums hence keeps on to act continuously on infinite sums which converge in operator norm. We apply this result to show

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that the class of composition operators that the first author studied in [3] have spectra equal to their essential spectra. The class of composition operators that was studied in [3] is the class of composition operators with symbols \( \varphi \) which have upper half-plane re-incarnation

\[
\mathcal{C}^{-1} \circ \varphi \circ \mathcal{C}(z) = z + \psi(z)
\]

for a bounded analytic function \( \psi \) satisfying \( \Im(\psi(z)) > \epsilon > 0 \) for all \( z \in \mathbb{H} \). We call this class of composition operators “quasi-parabolic”.

2. PRELIMINARIES

In this section we fix the notation that we will use throughout and recall some preliminary facts that will be used in the sequel.

Let \( S \) be a compact Hausdorff topological space. The space of all complex valued continuous functions on \( S \) will be denoted by \( C(S) \). For any \( f \in C(S) \), \( \|f\|_\infty \) will denote the sup-norm of \( f \), i.e.

\[
\|f\|_\infty = \sup \{ |f(s)| : s \in S \}.
\]

For a Banach space \( X \), \( K(X) \) will denote the space of all compact operators on \( X \) and \( B(X) \) will denote the space of all bounded linear operators on \( X \). The open unit disc will be denoted by \( D \), the open upper half-plane will be denoted by \( H \), the real line will be denoted by \( \mathbb{R} \) and the complex plane will be denoted by \( \mathbb{C} \). The one point compactification of \( \mathbb{R} \) will be denoted by \( \dot{\mathbb{R}} \) which is homeomorphic to \( \mathbb{T} \).

For any \( z \in \mathbb{C} \), \( \Re(z) \) will denote the real part, and \( \Im(z) \) will denote the imaginary part of \( z \), respectively. For any subset \( S \subset B(H) \), where \( H \) is a Hilbert space, the C*-algebra generated by \( S \) will be denoted by \( C^*(S) \). The Cayley transform \( \mathcal{C} \) will be defined by

\[
\mathcal{C}(z) = z - \frac{i}{z + i}.
\]

For any \( a \in L^\infty(\mathbb{R}) \) (or \( a \in L^\infty(\mathbb{T}) \)), \( M_a \) will be the multiplication operator on \( L^2(\mathbb{R}) \) (or \( L^2(\mathbb{T}) \)) defined as

\[
M_a(f)(x) = a(x)f(x).
\]

For convenience, we remind the reader of the rudiments of the theory of Toeplitz operators and commutative C*-algebras.

Let \( A \) be a commutative Banach algebra. Then its maximal ideal space \( M(A) \) is defined as

\[
M(A) = \{ x \in A^* : x(ab) = x(a)x(b) \quad \forall a, b \in A \}
\]

where \( A^* \) is the dual space of \( A \). If \( A \) has identity then \( M(A) \) is a compact Hausdorff topological space with the weak* topology. The Gelfand transform \( \Gamma : A \to C(M(A)) \) is defined as

\[
\Gamma(a)(x) = x(a).
\]

If \( A \) is a commutative C*-algebra with identity, then \( \Gamma \) is an isometric *-isomorphism between \( A \) and \( C(M(A)) \). If \( A \) is a C*-algebra and \( I \) is a two-sided closed ideal of \( A \), then the quotient algebra \( A/I \) is also a C*-algebra (see [6] and [5]).

For a Banach algebra \( A \), we denote by \( \text{com}(A) \) the closed ideal in \( A \) generated by the commutators \( \{ a_1a_2 - a_2a_1 : a_1, a_2 \in A \} \). It is an algebraic fact that the quotient
algebra $A/\text{com}(A)$ is a commutative Banach algebra. For $a \in A$ the spectrum $\sigma_A(a)$ of $a$ on $A$ is defined as

$$\sigma_A(a) = \{ \lambda \in \mathbb{C} : \lambda e - a \text{ is not invertible in } A \},$$

where $e$ is the identity of $A$. In particular the spectrum $\sigma(T)$ of a linear bounded operator $T : X \to X$ where $X$ is a Banach space is defined as $\sigma(T) := \sigma_{B(X)}(T)$. Recall that a bounded linear operator $T$ on a Hilbert space $H$ is called Fredholm if the range of $T$ is closed, $\dim \ker(T)$ and $\dim \ker(T^*)$ are finite. The Fredholm index $\text{ind}$ is defined as

$$\text{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*))$$

It is a very well known fact that ([5]) when the set of Fredholm operators $F \subset B(H)$ is equipped with operator norm topology and $\mathbb{Z}$ is equipped with discrete topology, the index function $\text{ind} : F \to \mathbb{Z}$ is continuous. The essential spectrum $\sigma_e(T)$ of an operator $T$ acting on a Banach space $X$ is the spectrum of the coset of $T$ in the Calkin algebra $B(X)/K(X)$, the algebra of bounded linear operators modulo compact operators. The following Atkinson’s characterization for Fredholm operators is also well known:

**Theorem 1.** ([5] p.28, Theorem 1.4.16) A bounded linear operator $T$ on a Hilbert space $H$ is Fredholm if and only if $T + K(H)$ is invertible in the quotient algebra $B(H)/K(H)$, where $K(H)$ is the algebra of all compact operators on $H$.

For $1 \leq p < \infty$ the Hardy space of the unit disc will be denoted by $H^p(\mathbb{D})$ and the Hardy space of the upper half-plane will be denoted by $H^p(\mathbb{H})$.

The two Hardy spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$ are isometrically isomorphic. An isometric isomorphism $\Phi : H^2(\mathbb{D}) \to H^2(\mathbb{H})$ is given by

$$\Phi(g)(z) = \left( \frac{1}{\sqrt{\pi(z + i)}} \right) g \left( \frac{z - i}{z + i} \right)$$

(1)

The mapping $\Phi$ has an inverse $\Phi^{-1} : H^2(\mathbb{H}) \to H^2(\mathbb{D})$ given by

$$\Phi^{-1}(f)(z) = \frac{e^{\Phi}(4\pi)^{\frac{1}{2}}}{(1 - z)} f \left( \frac{i(1 - z)}{1 - z} \right)$$

Using the isometric isomorphism $\Phi$, one may transfer Fatou’s theorem in the unit disc case to upper half-plane and may embed $H^2(\mathbb{H})$ in $L^2(\mathbb{R})$ via $f \to f^*$ where $f^*(x) = \lim_{y \to 0} f(x + iy)$. This embedding is an isometry.

Throughout the paper, using $\Phi$, we will go back and forth between $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$. We use the property that $\Phi$ preserves spectra, compactness and essential spectra i.e. if $T \in B(H^2(\mathbb{D}))$ then

$$\sigma_{B(H^2(\mathbb{D}))}(T) = \sigma_{B(H^2(\mathbb{H}))}(\Phi \circ T \circ \Phi^{-1}),$$

$K \in K(H^2(\mathbb{D}))$ if and only if $\Phi \circ K \circ \Phi^{-1} \in K(H^2(\mathbb{H}))$ and hence we have

$$\sigma_e(T) = \sigma_e(\Phi \circ T \circ \Phi^{-1}).$$

(2)

We also note that $T \in B(H^2(\mathbb{D}))$ is essentially normal if and only if $\Phi \circ T \circ \Phi^{-1} \in B(H^2(\mathbb{H}))$ is essentially normal.

The Toeplitz operator with symbol $a$ is defined as

$$T_a = PM_a|_{H^2},$$
where $P$ denotes the orthogonal projection of $L^2$ onto $H^2$. A good reference about Toeplitz operators on $H^2$ is Douglas’ treatise ([2]). Although the Toeplitz operators treated in [2] act on the Hardy space of the unit disc, the results can be transferred to the upper half-plane case using the isometric isomorphism $\Phi$ introduced by equation (1). In the sequel the following identity will be used:

$$\Phi^{-1} \circ T_a \circ \Phi = T_{a \circ C^{-1}},$$

where $a \in L^\infty(\mathbb{R})$. We also employ the fact

$$\|T_a\|_e = \|T_a\| = \|a\|_\infty$$

for any $a \in L^\infty(\mathbb{R})$, which is a consequence of Theorem 7.11 of [2] (pp. 160–161) and equation (3). For any subalgebra $A \subseteq L^\infty(\mathbb{R})$ the Toeplitz C*-algebra generated by symbols in $A$ is defined to be $T(A) = C^*(\{T_a : a \in A\})$.

It is a well-known result of Sarason (see [7]) that the set of functions $H^\infty + C = \{f_1 + f_2 : f_1 \in H^\infty(\mathbb{D}), f_2 \in C(\mathbb{T})\}$ is a closed subalgebra of $L^\infty(\mathbb{T})$. The following theorem of Douglas [2] will be used in the sequel.

**Theorem 2 (Douglas’ Theorem).** Let $a, b \in H^\infty + C$ then the semi-commutators

$$T_{ab} - T_a T_b \in K(H^2(\mathbb{D})), \quad T_{ab} - T_b T_a \in K(H^2(\mathbb{D})),
$$

and hence the commutator

$$[T_a, T_b] = T_a T_b - T_b T_a \in K(H^2(\mathbb{D}))$$

is compact.

Let $QC$ be the C*-algebra of functions in $H^\infty + C$ whose complex conjugates also belong to $H^\infty + C$. Let us also define the upper half-plane version of $QC$ as the following:

$$QC(\mathbb{R}) = \{\varphi \in L^\infty(\mathbb{R}) : \varphi \circ C^{-1} \in QC\}.$$ 

Going back and forth with Cayley transform one can deduce that $QC(\mathbb{R})$ is a closed subalgebra of $L^\infty(\mathbb{T})$. By Douglas’ theorem and equation (3), if $a, b \in QC(\mathbb{R})$, then

$$T_{ab} - T_a T_b \in K(H^2(\mathbb{D})).$$

Let $scom(QC(\mathbb{R}))$ be the closed ideal in $T(QC(\mathbb{R}))$ generated by the semi-commutators $\{T_a T_b - T_{ab} : a, b \in QC(\mathbb{R})\}$. Then we have

$$com(T(QC(\mathbb{R}))) \subseteq scom(QC(\mathbb{R})) \subseteq K(H^2(\mathbb{D})).$$

By Proposition 7.12 of [2] and equation (3) we have

$$com(T(QC(\mathbb{R}))) = scom(QC(\mathbb{R})) = K(H^2(\mathbb{D})).$$

Now consider the symbol map

$$\Sigma : QC(\mathbb{R}) \to T(QC(\mathbb{R}))$$

defined as $\Sigma(a) = T_a$. This map is linear but not necessarily multiplicative; however if we let $q$ be the quotient map

$$q : T(QC(\mathbb{R})) \to T(QC(\mathbb{R}))/scom(QC(\mathbb{R})), $$
then \( q \circ \Sigma \) is multiplicative; moreover by equations (4) and (5), we conclude that 
\( q \circ \Sigma \) is an isometric \(*\)-isomorphism from \( QC(\mathbb{R}) \) onto 
\( T(QC(\mathbb{R}))/K(H^2(\mathbb{H})) \). The maximal ideal space \( M(QC(\mathbb{R})) \) is fibered over \( \mathbb{R} \) in the following way: For any 
\( x \in M(QC(\mathbb{R})) \) consider \( \hat{x} = x|_{C(\mathbb{R})} \) then \( \hat{x} \in M(C(\mathbb{R})) = \hat{\mathbb{R}} \). Hence \( M(QC(\mathbb{R})) \) is fibered over \( \hat{\mathbb{R}} \), i.e.
\[
M(QC(\mathbb{R})) = \bigcup_{t \in \hat{\mathbb{R}}} M_t(QC),
\]
where
\[
M_t(QC) = \{ x \in M(QC(\mathbb{R})) : \hat{x} = x|_{C(\mathbb{R})} = \delta_t, \delta_t(f) = f(t) \}.
\]
We also remind the reader about the very important fact in the theory of Toeplitz operators that any Toeplitz operator \( T_\varphi \) with symbol \( \varphi \in L^\infty \) is invertible if and only if \( T_\varphi \) is Fredholm and the Fredholm index \( ind(T_\varphi) = 0 \) is zero. The proof of this fact can also be found in [2]. This fact will be used in the proof of our main result in this paper.

Let \( \varphi : \mathbb{D} \to \mathbb{D} \) or \( \varphi : \mathbb{H} \to \mathbb{H} \) be a holomorphic self-map of the unit disc or the upper half-plane. The composition operator \( C_\varphi \) on \( H^p(\mathbb{D}) \) or \( H^p(\mathbb{H}) \) with symbol \( \varphi \) is defined by
\[
C_\varphi(g)(z) = g(\varphi(z)), \quad z \in \mathbb{D} \quad \text{or} \quad z \in \mathbb{H}.
\]
Composition operators of the unit disc are always bounded [1] whereas composition operators of the upper half-plane are not always bounded. For the boundedness problem of composition operators of the upper half-plane see [3]. The composition operator \( C_\varphi \) on \( H^2(\mathbb{D}) \) is carried over to \( (\overline{\varphi(z)+\psi})C_\varphi \) on \( H^2(\mathbb{H}) \) through \( \Phi \), where \( \hat{\varphi} = \mathcal{C} \circ \varphi \circ \mathcal{C}^{-1} \), i.e. we have
\[
\Phi C_\varphi \Phi^{-1} = T_{\hat{\varphi}(z)}C_{\hat{\varphi}},
\]
(6)
However this gives us the boundedness of \( C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H}) \) for
\[
\varphi(z) = pz + \psi(z),
\]
where \( p > 0, \psi \in H^\infty \) and \( \Im(\psi(z)) > \epsilon > 0 \) for all \( z \in \mathbb{H} \):

Let \( \hat{\varphi} : \mathbb{D} \to \mathbb{D} \) be an analytic self-map of \( \mathbb{D} \) such that \( \varphi = \mathcal{C}^{-1} \circ \hat{\varphi} \circ \mathcal{C} \), then we have
\[
\Phi C_\varphi \Phi^{-1} = T_{\tau}C_\varphi
\]
where
\[
\tau(z) = \frac{\varphi(z) + i}{z + i}.
\]
If
\[
\varphi(z) = pz + \psi(z)
\]
with \( p > 0, \psi \in H^\infty \) and \( \Im(\psi(z)) > \epsilon > 0 \), then \( T_\varphi \) is a bounded operator. Since \( \Phi C_\varphi \Phi^{-1} \) is always bounded we conclude that \( C_\varphi \) is bounded on \( H^2(\mathbb{H}) \).

The Fourier transform \( \mathcal{F}f \) of \( f \in S(\mathbb{R}) \) (the Schwartz space, for a definition see [3]) is defined by
\[
(\mathcal{F}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-itx} f(x) dx.
\]
The Fourier transform extends to an invertible isometry from \( L^2(\mathbb{R}) \) onto itself with inverse
\[
(\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{itx} f(x) dx.
\]
The following is a consequence of a theorem due to Paley and Wiener [6]. Let $1 < p < \infty$. For $f \in L^p(\mathbb{R})$, the following assertions are equivalent:

(i) $f \in H^p$,  
(ii) $\text{supp}(f) \subseteq [0, \infty)$

A reformulation of the Paley-Wiener theorem says that the image of $H^2(\mathbb{H})$ under the Fourier transform is $L^2((0, \infty))$.

By the Paley-Wiener theorem we observe that the operator $D_\vartheta = F^{-1}M_\vartheta F$ for $\vartheta \in C([0, \infty])$ maps $H^2(\mathbb{H})$ into itself, where $C([0, \infty])$ denotes the set of continuous functions on $[0, \infty)$ which have limits at infinity. Since $F$ is unitary we also observe that

$$|| D_\vartheta || = || M_\vartheta || = || \vartheta ||_{\infty}$$

Let $F$ be defined as

$$F = \{ D_\vartheta \in B(H^2(\mathbb{H})) : \vartheta \in C([0, \infty]) \}.$$  

We observe that $F$ is a commutative $C^*$-algebra with identity and the map $D : C([0, \infty)) \to F$ given by

$$D(\vartheta) = D_\vartheta$$

is an isometric *-isomorphism by equation above. Hence $F$ is isometrically *-isomorphic to $C([0, \infty])$. The operator $D_\vartheta$ is usually called a “Fourier Multiplier.” We will also need the fact that, under the Fourier transform the Beurling type invariant subspace $e^{i\eta z}H^2(\mathbb{H})$, for $\eta > 0$, is mapped onto $L^2_\eta((0, \infty)) := \{ f \in L^2(0, \infty) : f(t) = 0 \text{ for a.e. } t \in (0, \eta) \}$ for all $\eta > 0$ and $F T_{e^{i\eta z}} F^{-1} = S_\eta$ where $S_\eta : L^2((0, \infty)) \to L^2((0, \infty))$ is defined as $S_\eta f(t) := f(t-\eta)$ if $t \geq \eta$ and $S_\eta f(t) = 0$ if $0 \leq t < \eta$. Similarly we have $F T_{e^{-i\eta z}} F^{-1} = S^{*}_\eta$ where $S^{*}_\eta f(t) = f(t+\eta)$. Here $T_{e^{i\eta z}}$ and $T_{e^{-i\eta z}}$ are Toeplitz operators with symbols $e^{i\eta z}$ and $e^{-i\eta z}$ respectively. Since $\cup_{\eta > 0} L^2_\eta((0, \infty))$ is dense in $L^2((0, \infty))$, $\cup_{\eta > 0} e^{i\eta z} H^2(\mathbb{H})$ is also dense in $H^2(\mathbb{H})$.

In [3] the first author studied the $C^*$-algebra $\Psi$ generated by Toeplitz operators with $QC$ symbols and Fourier multipliers with continuous symbols. He proved that the commutator $[T_\varphi, D_\vartheta] = T_\varphi D_\vartheta - D_\vartheta T_\varphi \in K(H^2)$ of any Toeplitz operator with $QC$ symbol and a Fourier multiplier with continuous symbol is compact which implies that the Calkin algebra $\Psi/K(H^2)$ is a commutative $C^*$-algebra with identity. The maximal ideal space $\mathcal{M}$ of $\Psi/K(H^2)$ is also studied in [3] and is found to be

$$\mathcal{M} \cong (M_\infty(QC(\mathbb{R})) \times [0, \infty]) \cup (M(QC(\mathbb{R})) \times \{ \infty \}) \subset M(QC) \times [0, \infty]$$

where $M_\infty(QC) := \{ x \in M(QC) : x|_{C^{(\infty)}} = \delta_\infty, \delta_\infty(f) = \lim_{t \to \infty} f(t) \}$ is the fiber of $M(QC)$ at infinity. The Gelfand transform $\Gamma$ of $\Psi/K(H^2)$ looks like

$$\Gamma \left( \left[ \sum T_{\varphi_j} D_{\vartheta_j} \right] \right) (x, t) = \begin{cases} \sum \hat{\varphi}_j(x) \hat{\theta}_j(t) & \text{if } x \in M_\infty(QC(\mathbb{R})) \\ \sum \hat{\varphi}_j(x) \hat{\theta}_j(\infty) & \text{if } t = \infty \end{cases}$$

3. THE MAIN RESULT

In this section we prove the main result of this paper which asserts that any sum $T = \sum T_{\varphi_j} D_{\vartheta_j} + \sum D_{\vartheta_j} T_{\varphi_j} \in \Psi$ convergent in the operator norm is invertible if and only if $T$ is Fredholm and has Fredholm index zero. This may be regarded as a generalization of the fact that any Toeplitz operator $T_\varphi$ with a bounded symbol $\varphi \in L^\infty$ is invertible if and only if $T_\varphi$ is Fredholm and has Fredholm index zero. In
proving this fact our main technical tool will be a homotopy $H : [0, 1] \to \Psi$ which carries $T$ to a Toeplitz operator. The homotopy $H$ for $T = \sum T_{\varphi_j}D_{\vartheta_j} + \sum D_{\nu_j}T_{\psi_j}$ is defined as follows:

$$H(w) := \sum T_{\varphi_j}D_{\vartheta_j} + \sum D_{\nu_j}T_{\psi_j}$$

where $w \in (0, 1]$, $\vartheta_j(w)(t) = \vartheta_j(t - \ln w)$, $\nu_j(w)(t) = \nu_j(t - \ln w)$ and for $w = 0$, $H(0) := T_{\varphi}, \varphi = \sum \lambda_j \varphi_j + \sum \mu_j \psi_j$ and $\lambda_j = \lim_{t \to \infty} \vartheta_j(t)$, $\mu_j = \lim_{t \to \infty} \nu_j(t)$. It is easily seen that when $T = \sum T_{\varphi_j}D_{\vartheta_j} + \sum D_{\nu_j}T_{\psi_j}$ consists of finite sums, $H$ is continuous. And thus $H$ keeps on being continuous when $T$ consists of infinite sums which are both convergent in operator norm. In the proof of this fact we will always work with finite sums since it is enough to prove it for finite sums. In this section, unless otherwise stated, $H^2$ will always be understood as $H^2(\mathbb{H})$ and $QC$ will always be understood as $QC(\mathbb{R})$.

Here is our main theorem:

**Theorem 3.** Let $T = \sum T_{\varphi_j}D_{\vartheta_j} + \sum D_{\nu_j}T_{\psi_j} \in \Psi$ be such that $\nu_j, \vartheta_j \in C([0, \infty])$, $\psi_j, \varphi_j \in QC$. Then $T \in \Psi$ is invertible $\iff$ $T$ is Fredholm and $\text{ind}(T) = 0$

**Proof.** ($\Rightarrow$): Let $T$ be Fredholm and $\text{ind}(T) = 0$. Let $H : [0, 1] \to \Psi$ be the homotopy constructed above. Since $\sigma_x(T) = \{ \sum \lambda_j \varphi_j(x) + \sum \mu_j \psi_j(x) : x \in M_\infty(QC), t \in [0, \infty]\} \cup \{ \sum \lambda_j \varphi_j(x) + \sum \mu_j \psi_j(x) : x \in M(QC)\}$ and $\sigma_x(H(w)) = \{ \sum \lambda_j \varphi_j(x) + \sum \mu_j \psi_j(x) : x \in M(QC)\}$, we have $\sigma_x(H(w)) \subseteq \sigma_x(T)$ for all $w \in [0, 1]$. Hence if $T$ is Fredholm then $0 \not\in \sigma_x(T) \Rightarrow 0 \not\in \sigma_x(H(w))$ for all $w \in [0, 1]$ which implies that $H(w)$ is Fredholm for all $w \in [0, 1]$. Since $H$ is continuous and $\text{ind}$ is continuous on the set of Fredholm operators, we have

$$\text{ind}(H(0)) = \text{ind}(H(1)) = \text{ind}(T) = 0 \quad \forall w \in [0, 1].$$

Hence $\text{ind}(H(0)) = \text{ind}(H(w)) = \text{ind}(T) = 0 \quad \forall w \in [0, 1].$

For $\eta = \ln(w_0) - \ln(w)$, where $w_0 > w$, we have $M_{\gamma(w)} = S^{*}_\eta M_{\gamma(0)}S_\eta$ where $M_\gamma : L^2((0, \infty)) \to L^2((0, \infty))$, $M_\gamma f(t) := g(t)f(t)$ is the multiplication operator. Hence we have

$$D_\gamma = (F^{-1}S^{*}_\eta F)D_{\gamma(0)}(F^{-1}S_\eta F) = T_{e^{-\eta w_0}}D_{\gamma(0)}T_{e^{\eta w_0}}.$$ 

Since $\varphi_j, \varphi_j \in QC$ we have $T_{\varphi_j}T_{e^{-\eta w_0}} - T_{e^{-\eta w_0}}T_{\varphi_j} \in K(H^2)$ and $T_{\varphi_j}T_{e^{\eta w_0}} - T_{e^{\eta w_0}}T_{\varphi_j} \in K(H^2)$ for all $\gamma > 0$(see [2]). Hence we have

$$H(w) = \sum T_{\varphi_j}D_\gamma + \sum D_{\gamma}T_{\psi_j} =$$

$$\sum T_{\varphi_j}T_{e^{-\eta w_0}}D_\gamma T_{e^{\eta w_0}} + \sum T_{e^{-\eta w_0}}D_{\gamma}T_{e^{\eta w_0}}T_{\psi_j}$$

$$= T_{e^{-\eta w_0}}(\sum T_{\varphi_j}D_{\gamma(0)} + \sum D_{\gamma(0)}T_{\psi_j})T_{e^{\eta w_0}} + K(w, w_0)$$

$$= T_{e^{-\eta w_0}}H(w_0)T_{e^{\eta w_0}} + K(w, w_0).$$

where $K(w, w_0) \in K(H^2)$ is a compact operator.
Hence we have

$$H(w) = T_{e^{-i\eta x}}H(w_0)T_{e^{i\eta x}} + K(w, w_0)$$

for some compact operator $K(w, w_0) \in K(H^2)$. Since Fredholm index is stable under compact perturbations this implies that the operator $T_{e^{-i\eta x}}H(w_0)T_{e^{i\eta x}}$ is Fredholm with index 0 for all $\eta > 0$. Hence if $H(w_0)$ is non-invertible then $\ker(T_{e^{-i\eta x}}H(w_0)T_{e^{i\eta x}}) \neq \{0\}$ for some $\eta > 0$ since $T_{e^{-i\eta x}}H(w_0)T_{e^{i\eta x}}$ is Fredholm with index 0. If this is not the case i.e. if we have $\ker(T_{e^{-i\eta x}}H(w_0)T_{e^{i\eta x}}) = \{0\}$ for all $\eta > 0$, since $T_{e^{-i\eta x}}H(w_0)T_{e^{i\eta x}}$ is Fredholm with index 0, $T_{e^{-i\eta x}}H(w_0)T_{e^{i\eta x}}$ is invertible for all $\eta > 0$. And this implies that $H(w_0)$ maps each $e^{i\eta x}H^2$ onto itself in a one to one manner. Hence $H(w_0)$ maps $\cup_{\eta > 0}e^{i\eta x}H^2$ onto itself in a one to one manner. Now suppose that $\psi_j, \varphi_j \in H^\infty$ for all $j \in \mathbb{N}$. Since $\psi_j, \varphi_j \in H^\infty$ for all $j$, we have $H(w_0)(e^{i\eta x}H^2) \subseteq (e^{i\eta x}H^2)$ for all $\eta > 0$ and $\ker(T_{e^{-i\eta x}}) = (e^{i\eta x}H^2)^\perp$, we have $H(w_0)(e^{i\eta x}H^2) \cap \ker(T_{e^{-i\eta x}}) = \{0\}$ for all $\eta > 0$. So if $H(w_0)$ is non-invertible we should have $\ker(H(w_0)) \cap (e^{i\eta x}H^2) \neq \{0\}$. Since $H(w_0)$ is Fredholm, $\ker(H(w_0))$ is finite dimensional an $\eta_0 := \max \{\eta > 0 : \ker(H(w_0)) \cap (e^{i\eta x}H^2) \neq \{0\}\}$ exists. For such $\eta_0 > 0$, let $H_0 := T_{e^{-i\eta_0 x}}H(w_0)T_{e^{i\eta_0 x}}$. Then since $\ker(H(w_0)) \cap (e^{i\eta_0 x}H^2) \neq \{0\}$, $H_0$ is non-invertible. But since $\ker(H(w_0)) \cap (e^{i(\eta_0 + \delta)x}H^2) = \{0\}$ for all $\delta > 0$ and $H(w_0)(e^{i(\eta_0 + \delta)x}H^2) \subseteq e^{i(\eta_0 + \delta)x}H^2$ we have $\ker(T_{e^{-i(\eta_0 + \delta)x}}H(w_0)T_{e^{i(\eta_0 + \delta)x}}) = \ker(T_{e^{-i\delta x}}H_0T_{e^{i\delta x}}) = \{0\}$ and this implies that $T_{e^{-i\delta x}}H_0T_{e^{i\delta x}}$ is invertible for all $\delta > 0$. This again implies that $H_0$ maps $e^{i\delta x}H^2$ onto itself in a one to one manner which implies that $H_0$ should be invertible. This contradicts our assumption that $H(w_0)$ is non-invertible. Hence $H(w_0)$ should be invertible. Therefore, in this case where $\psi_j, \varphi_j \in H^\infty$ for all $j$, if $H(w)$ is invertible for all $0 \leq w < w_0$ then $H(w_0)$ is also invertible. So by transfinite induction $H(w)$ is invertible for all $w \in [0, 1]$ and in particular $H(1) = T$ is invertible.

Now suppose that $\varphi_j$ and $\psi_j$ are continuous for all $j$. Since trigonometric polynomials are dense in continuous functions, it is enough to prove the claim when $\varphi_j(z) := \sum_{k=-m}^{m}a_kz^k$ and $\psi_j(z) = \sum_{k=-m}^{m}b_kz^k$ are trigonometric polynomials for all $j$. One can write $\varphi_j$ and $\psi_j$ in the form

$$\varphi_j(z) = z^{-m}\sum_{k=0}^{m'}a_kz^k = z^{-m}q_j(z), \quad \psi_j(z) = z^{-m}p_j(z)$$

where $z = (z, 1)$ and $q_j$ and $p_j$ are analytic polynomials. In this case we have

$$H(w) = T_{e^{-i\eta x}}H(w_0)T_{e^{i\eta x}} + K(w, w_0)$$

$$= T_{e^{-i\eta x}}(\sum T_{\varphi_j}D_{\varphi_j}w_0 + \sum D_{\psi_j}w_0T_{\varphi_j})T_{e^{i\eta x}} + K(w, w_0)$$

$$= T_{e^{-i\eta x}}(\sum T_{z^{-m}\varphi_j}D_{\varphi_j}w_0 + \sum D_{\psi_j}w_0T_{z^{-m}\varphi_j})T_{e^{i\eta x}} + K(w, w_0)$$

$$= T_{z^{-m}}T_{e^{-i\eta x}}(\sum T_{\varphi_j}D_{\varphi_j}w_0 + \sum D_{\psi_j}w_0T_{\varphi_j})T_{e^{i\eta x}} + K_0 + K(w, w_0)$$
for some $K_0 \in K(H^2)$ since $D_{\nu_j}T_{z-m} - T_{z-m}D_{\nu_j} \in K(H^2)$ \ \forall j, \ T_{z-m}T_{e^{-\nu x}} = T_{e^{-\nu x}}T_{z-m}$ and the sum is finite. Let $\tilde{H}(w_0) = \sum T_{\nu_j}D_{\theta_j}w_0 + \sum D_{\nu_j}w_0T_{\varphi_j}$. Then we have

$$H(w) = T_{e^{-\nu x}}T_{z-m}\tilde{H}(w_0)T_{e^{\nu x}} + \tilde{K}(w, w_0)$$

where $\tilde{K}(w, w_0) = K_0 + K(w, w_0) \in K(H^2)$. Since $T_{z-m}T_{e^{-\nu x}} = T_{e^{-\nu x}}T_{z-m}$ we have $\ker(\tilde{H}(w_0)) \cap e^{\nu x}H^2 \neq \{0\}$ for some $\eta > 0$. If this is not the case i.e. if $\ker(\tilde{H}(w_0)) \cap e^{\nu x}H^2 = \{0\}$ then $\forall \eta > 0$ then $H(w) = \tilde{K}(w, w_0) = T_{z-m}T_{e^{-\nu x}}H(w_0)T_{e^{\nu x}}$ is invertible for all $\eta > 0$. And this implies that $\tilde{H}(w_0)(e^{\nu x}H^2) = z^m(e^{\nu x}H^2) \ \forall \eta > 0$. Since $\text{ran}(\tilde{H}(w_0)) \subseteq H^2$ is closed, this implies that $\text{ran}(\tilde{H}(w_0)) = z^mH^2$ and this in turn implies that $H(w_0) = T_{z-m}\tilde{H}(w_0)$ is invertible which contradicts our assumption. Since $\ker(\tilde{H}(w_0))$ is finite dimensional, an $\eta_0 := \max\{\eta > 0 : \ker(\tilde{H}(w_0)) \cap (e^{\nu x}H^2) \neq \{0\}\}$ exists. Now let $\tilde{H}_0 := T_{e^{-\nu x}}\tilde{H}(w_0)T_{e^{\nu x}}$, then $\ker(\tilde{H}_0) \neq \{0\}$. Since $\ker(\tilde{H}(w_0)) \cap (e^{(\eta_0 + \delta)x}H^2) = \{0\}$ \ \forall $\delta > 0$, we have

$$T_{z-m}T_{e^{-\nu x}}\tilde{H}_0T_{e^{\nu x}} = T_{e^{-\nu x}}T_{z-m}\tilde{H}_0T_{e^{\nu x}}$$

is invertible for all $\delta > 0$. This implies that $\tilde{H}_0(e^{\nu x}H^2) = z^m(e^{\nu x}H^2)$ for all $\delta > 0$. Since $\text{ran}(\tilde{H}_0) \subseteq H^2$ is closed, this implies that $\text{ran}(\tilde{H}_0) = z^mH^2$. Since $\text{ind}(T_{z-m}\tilde{H}_0) = 0$ this implies that $\text{ind}(\tilde{H}_0) = -m$ which in turn implies that $\ker(\tilde{H}_0) = \{0\}$. This contradiction implies that $H(w_0)$ is invertible.

Hence if $\psi_j, \varphi_j \in H^\infty$ or $\psi_j, \varphi_j$ are continuous for all $j$ then $T = \sum T_{\varphi_j}D_{\theta_j} + \sum D_{\nu_j}T_{\psi_j}$ is Fredholm with index zero implies that $T$ is invertible. So if $\psi_j, \varphi_j \in (H^\infty + C) \cap QC = QC$ for all $j$ then $T = \sum T_{\varphi_j}D_{\theta_j} + \sum D_{\nu_j}T_{\psi_j}$ is Fredholm with index zero implies that $T$ is invertible.

A reinterpretation of this theorem would be relating the invertibility of a generic element in $\Psi$ to the invertibility of a related Toeplitz operator which is the corollary below:

**Corollary 4.** For any $T = \sum T_{\varphi_j}D_{\theta_j} + \sum D_{\nu_j}T_{\psi_j} \in \Psi$ such that $T$ is Fredholm $\psi_j \varphi_j \in QC$ and $\nu_j, \theta_j \in C([0, \infty)) \ \forall j \in \mathbb{N}$, $T$ is invertible if and only if $T_\varphi \in \Psi$ is invertible where $\varphi = \sum \lambda_j \varphi_j + \sum \mu_j \psi_j$ and $\mu_j = \lim_{t \to \infty} \nu_j(t), \lambda_j = \lim_{t \to \infty} \theta_j(t)$.

**Proof.** If $T$ is invertible then $T$ is Fredholm with $\text{ind}(T) = 0$. Using the homotopy $H : [0, 1] \to \Psi$ constructed in the beginning of this section, since $\text{ind}$ is continuous we have $H(w)$ is Fredholm $\forall w \in [0, 1]$ and $\text{ind}(H(w)) = \text{ind}(T) = 0 \ \forall w \in [0, 1]$. In particular $H(0) = T_\varphi$ is Fredholm and $\text{ind}(T_\varphi) = 0$, since any Fredholm Toeplitz operator $T_\varphi$ with $\text{ind}(T_\varphi) = 0$ is invertible, $T_\varphi$ is invertible.

On the other hand if $T_\varphi$ is invertible, since $T$ is Fredholm, by the proof of Theorem 3, $H(w)$ is invertible $\forall w \in [0, 1]$, in particular $H(1) = T$ is invertible.

4. APPLICATIONS OF THE MAIN RESULTS

An immediate application of Corollary 4 shows that the essential spectrum and the spectrum of a quasi-parabolic composition operator coincide:

**Theorem 5.** Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\varphi : \mathbb{H} \to \mathbb{H}$ be such that $\varphi(z) = \frac{2iz + \eta(z)(1-z)}{2i + \eta(z)(1-z)}$ and $\tilde{\varphi}(w) = w + \psi(w)$ where $\eta \in QC(T) \cap H^\infty, \Im(\eta(z)) > \delta > 0$ for all $z \in \mathbb{D}$, $\psi \in QC(\mathbb{R}) \cap H^\infty, \Im(\eta(w)) > \delta > 0$ for all $w \in \mathbb{H}$. Then $C_{\tilde{\varphi}} : H^2(\mathbb{H}) \to H^2(\mathbb{H})$ is bounded and $\sigma_e(C_{\varphi}) = \sigma(C_{\tilde{\varphi}})$. We also have $\sigma_e(C_{\varphi}) = \sigma(C_{\tilde{\varphi}})$. 

Proof. The boundedness of $C_{\tilde{\varphi}}$ was shown in [3]. In particular we have

$$C_{\tilde{\varphi}} = \sum_{n=0}^{\infty} T_{\tau^n} D_{\vartheta_n}$$

where $\tau(x) = i\alpha - \psi(x)$ and $\vartheta_n(t) = \frac{(-it)^n e^{-at}}{n!}$ for some $\alpha > 0$. So for any $\lambda \notin \sigma_e(C_{\tilde{\varphi}})$ we have

$$\lambda - C_{\tilde{\varphi}} = \lambda - \sum_{n=0}^{\infty} T_{\tau^n} D_{\vartheta_n}$$

where the series on the Right Hand Side converges in the operator norm. Since $\lambda_n := \lim_{t \to \infty} \vartheta_n(t) = 0$, by Corollary 1 we have $\lambda - C_{\tilde{\varphi}}$ is invertible if and only if $\lambda$ is invertible which is certainly the case since $\lambda \neq 0$. Hence $\lambda \notin \sigma(C_{\tilde{\varphi}}) \Rightarrow \sigma(C_{\tilde{\varphi}}) \subseteq \sigma_e(C_{\tilde{\varphi}}) \Rightarrow \sigma(C_{\tilde{\varphi}}) = \sigma_e(C_{\tilde{\varphi}})$. The same argument applies to $C_{\varphi}$ since

$$\Phi \circ C_{\varphi} \circ \Phi^{-1} = T_{\frac{1}{\sqrt{\pi(z+i)}}} \sum_{n=0}^{\infty} T_{\tilde{\tau}^n} D_{\vartheta_n}$$

where $\Phi : H^2(D) \to H^2(H)$ is the isometric isomorphism

$$\Phi(f)(z) = \left( \frac{1}{\sqrt{\pi(z+i)}} \right) f \left( \frac{z-i}{z+i} \right)$$

$\Phi$ is the Cayley transform and $\tilde{\tau}(x) = i\alpha - \eta \circ \mathcal{C}(x)$. Hence we also have $\sigma(C_{\varphi}) = \sigma_e(C_{\varphi}).$ \qed

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