WIENER TYPE REGULARITY OF A BOUNDARY POINT FOR THE 3D LAMÉ SYSTEM

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Abstract

In this paper, we study the 3D Lamé system and establish its weighted positive definiteness for a certain range of elastic constants. By modifying the general theory developed in [4], we then show, under the assumption of weighted positive definiteness, that the divergence of the classical Wiener integral for a boundary point guarantees the continuity of solutions to the Lamé system at this point.

1. The Main Results

In our previous work [1], we studied weighted integral inequalities of the type

\( \int_{\Omega} Lu \cdot \Psi u \, dx \geq 0 \)

for general second order elliptic systems \( L \) in \( \mathbb{R}^n \) \((n \geq 3)\). For weights that are smooth and positive homogeneous of order \( 2 - n \), we have shown that \( L \) is positive definite in the sense of (1) only if the weight is the fundamental matrix of \( L \), possibly multiplied by a semi-positive definite constant matrix.

A question that arises naturally is under what conditions are elliptic systems indeed positive definite with such weights. Although it is difficult to answer this question in general, we study, as a special case, the 3D Lamé system

\( Lu = -\Delta u - \alpha \mathrm{grad} \, \mathrm{div} u \), \quad u = (u_1, u_2, u_3)^T \)

in this paper, deriving sufficient conditions for its weighted positive definiteness and showing that some restrictions on the elastic constants are inevitable. By modifying the general theory developed in [4], we then show that the divergence of the classical Wiener integral for a boundary point guarantees the continuity of solutions to the Lamé system at this point, assuming the weighted positive definiteness.

We first recall the following definition.

Definition 1.1. Let \( L \) be the 3D Lamé system

\( Lu = -\Delta u - \alpha \mathrm{grad} \, \mathrm{div} u = -D_{kk}u_i - \alpha D_{ki}u_k \quad (i = 1, 2, 3) \),

where as usual repeated indices indicate summation. The system \( L \) is said to be positive definite with weight \( \Psi(x) = (\Psi_{ij}(x))_{i,j=1}^3 \) if

\( \int_{\mathbb{R}^3} (Lu)^T \Psi u \, dx = -\int_{\mathbb{R}^3} \left[ D_{kk}u_i(x) + \alpha D_{ki}u_k(x) \right] u_j(x) \Psi_{ij}(x) \, dx \geq 0 \)

for all real valued, smooth vector functions \( u = (u_i)_{i=1}^3 \), \( u_i \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \). As usual, \( D \) denotes the gradient \( (D_1, D_2, D_3)^T \) and \( Du \) is the Jacobian matrix of \( u \).
Remark. The 3D Lamé system satisfies the strong elliptic condition if and only if $\alpha > -1$, and we will make this assumption throughout this paper.

The fundamental matrix of the 3D Lamé system is given by

$$
\Phi = (\Phi_{ij})_{i,j=1}^3,
$$

where

$$
\Phi_{ij} = c_{ij}r^{-1} \left( \delta_{ij} + \frac{\alpha}{\alpha + 2} \omega_i \omega_j \right) \quad (i, j = 1, 2, 3),
$$

$$
c_{ij} = \alpha + 2 \frac{\pi}{\alpha + 1} > 0.
$$

As usual, $\delta_{ij}$ is the Kronecker delta, $r = |x|$ and $\omega_i = x_i / |x|$.

The first result we shall prove in this paper is the following

**Theorem 1.2.** The 3D Lamé system $L$ is positive definite with weight $\Phi$ when $\alpha_- < \alpha < \alpha_+$, where $\alpha_- \approx -0.194$ and $\alpha_+ \approx 1.524$. It is not positive definite with weight $\Phi$ when $\alpha < \alpha_-(c)$ or $\alpha > \alpha_+(c) \approx 39.450$.

The proof of this theorem is given in Section 2.

Let $\Omega$ be an open set in $\mathbb{R}^3$ and consider the Dirichlet problem

$$
Lu = f, \quad f_i \in C^\infty_0(\Omega), \quad u_i \in \hat{H}^1(\Omega).
$$

As usual, $\hat{H}^1(\Omega)$ is the completion of $C^\infty_0(\Omega)$ in the Sobolev norm:

$$
\|f\|_{H^1(\Omega)} = \left[ \|f\|^2_{L^2(\Omega)} + \|Df\|^2_{L^2(\Omega)} \right]^{1/2}.
$$

**Definition 1.3.** The point $P \in \partial \Omega$ is regular with respect to $L$ if, for any $f = (f_i)_{i=1}^3$, $f_i \in C^\infty_0(\Omega)$, the solution of

$$
\lim_{\Omega \ni x \to P} u_i(x) = 0 \quad (i = 1, 2, 3).
$$

**Definition 1.4.** The classical harmonic capacity of a compact set $K$ in $\mathbb{R}^3$ is given by:

$$
cap(K) = \inf \left\{ \int_{\mathbb{R}^3} |Df(x)|^2 \, dx : f \in A(K) \right\},
$$

where

$$
A(K) = \left\{ f \in C^\infty_0(\mathbb{R}^3) : f = 1 \text{ in a neighborhood of } K \right\}.
$$

Note that an equivalent definition of $\cap(K)$ can be obtained by replacing $A(K)$ with $A_1(K)$ where

$$
A_1(K) = \left\{ f \in C^\infty_0(\mathbb{R}^3) : f \geq 1 \text{ on } K \right\}.
$$

( [3], sec. 2.2.1).

Using Theorem 1.2, we will prove that the divergence of the classical Wiener integral for a boundary point $P$ guarantees its regularity with respect to the Lamé system. To simplify notations we assume, without loss of generality, that $P = O$ is the origin of the space.

**Theorem 1.5.** Suppose the 3D Lamé system $L$ is positive definite with weight $\Phi$. Then $O \in \partial \Omega$ is regular with respect to $L$ if

$$
\int_0^1 \cap(B_p \setminus \Omega) \rho^{-2} \, d\rho = \infty.
$$

As usual, $B_p$ is the open ball centered at $O$ with radius $p$.

The proof of this theorem is given in Section 3.
2. Proof of Theorem 1.2

We start the proof of Theorem 1.2 by rewriting the integral
\[ \int_{\mathbb{R}^3} (Lu)^T \Phi u \, dx = - \int_{\mathbb{R}^3} (D_{kk} u_i + \alpha D_{ki} u_k) u_j \Phi_{ij} \, dx \]
into a more revealing form. In the following, we shall write \( \int f \, dx \) instead of \( \int_{\mathbb{R}^3} f \, dx \), and by \( u_{ii} \) we always mean \( \sum_{i=1}^{3} u_{ii}^2 \); to express \( (\sum_{i=1}^{3} u_{ii})^2 \) we will write \( u_{ii} u_{jj} \) instead. Furthermore, we always assume \( u_i \in C_0^\infty(\mathbb{R}^3) \) unless otherwise stated.

**Lemma 2.1.**

(7) \( \int (Lu)^T \Phi u \, dx = \frac{1}{2} |u(0)|^2 + \mathcal{B}(u, u) \)

where
\[
\mathcal{B}(u, u) = \frac{\alpha}{2} \int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij} \, dx \quad + \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{ij} \, dx.
\]

**Proof.** By definition,
\[
\int (Lu)^T \Phi u \, dx = - \int D_{kk} u_i \cdot u_j \Phi_{ij} \, dx - \alpha \int D_k u_i \cdot u_j \Phi_{ij} \, dx =: I_1 + I_2.
\]
Since \( \Phi \) is symmetric, we have \( \Phi_{ij} = \Phi_{ji} \) and
\[
I_1 = - \int D_{kk} u_i \cdot u_j \Phi_{ij} \, dx = - \frac{1}{2} \int \left[ D_{kk}(u_i u_j) - 2D_k u_i D_k u_j \right] \Phi_{ij} \, dx = - \frac{1}{2} \int u_i u_j D_{kk} \Phi_{ij} \, dx + \int D_k u_i D_k u_j \cdot \Phi_{ij} \, dx.
\]
On the other hand, \( \Phi \) is the fundamental matrix of \( L \), so we have
\[
-D_{kk} \Phi_{ij} - \alpha D_{ki} \Phi_{kj} = \delta_{ij} \delta(x),
\]
and
\[
-\frac{1}{2} \int u_i u_j D_{kk} \Phi_{ij} \, dx = \frac{1}{2} \int u_i u_j \left[ \delta_{ij} \delta(x) + \alpha D_k \Phi_{kj} \right] \, dx = \frac{1}{2} |u(0)|^2 - \frac{\alpha}{2} \int (D_i u_i \cdot u_j + u_i D_i u_j) D_k \Phi_{kj} \, dx = \frac{1}{2} |u(0)|^2 - \frac{\alpha}{2} \int (D_k u_k \cdot u_j + u_k D_k u_j) D_i \Phi_{ij} \, dx.
\]
Now \( I_2 \) can be written as
\[
I_2 = \alpha \int D_k u_k (D_i u_j \cdot \Phi_{ij} + u_j D_i \Phi_{ij}) \, dx,
\]
and the lemma follows by adding up the results. \( \square \)
Remark. With $\Phi(x)$ replaced by $\Phi_y(x) := \Phi(x - y)$, we have
\[
\int (Lu)^T \Phi_y u \, dx = \int (Lu_y)^T \Phi u_y \, dx \quad (u_y(x) = u(x + y))
\]
\[
= \frac{1}{2} |u_y(0)|^2 + \mathcal{B}(u_y, u_y) =: \frac{1}{2} |u(y)|^2 + \mathcal{B}_y(u, u),
\]
where
\[
\mathcal{B}_y(u, u) = \frac{\alpha}{2} \int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{y, ij} \, dx
\]
\[
+ \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{y, ij} \, dx.
\]

To proceed, we introduce the following decomposition for $C_0^\infty(\mathbb{R}^3)$ functions:
\[
f(x) = \tilde{f}(r) + g(x), \quad \tilde{f} \in C_0^\infty[0, \infty), \ g \in C_0^\infty(\mathbb{R}^3),
\]
where
\[
\tilde{f}(r) = \frac{1}{4\pi} \int_{S^2} f(r\omega) \, d\sigma.
\]
Note that
\[
\int_{S^2} g(r\omega) \, d\sigma = 0, \quad \forall r \geq 0,
\]
so we may think of $\tilde{f}$ as the “0-th order harmonics” of the function $f$. We shall show below in Lemma 2.2 that all 0-th order harmonics in (7) are canceled out, so it is possible to control $u$ by $Du$.

Lemma 2.2. With the decomposition
\[
u_i(x) = \bar{u}_i(r) + v_i(x) \quad (i = 1, 2, 3)
\]
where
\[
\begin{aligned}
\bar{u}_i(r) &= \frac{1}{4\pi} \int_{S^2} u_i(r\omega) \, d\sigma \\
v_i(r\omega) &= 0 \quad \forall r \geq 0 \quad (i = 1, 2, 3),
\end{aligned}
\]
we have
\[
\int (Lu)^T \Phi u \, dx = \frac{1}{2} |u(0)|^2 + \mathcal{B}^*(u, u)
\]
where
\[
\mathcal{B}^*(u, u) = \frac{\alpha}{2} \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij} \, dx
\]
\[
+ \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{ij} \, dx.
\]

Proof. By Lemma 2.1 it is enough to show
\[
\int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij} \, dx = \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij} \, dx.
\]
Since
\[
\int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij} \, dx
\]
\[
= \int (\bar{u}_j D_k \bar{u}_k - \bar{u}_k D_k \bar{u}_j) D_i \Phi_{ij} \, dx + \int (\bar{u}_j D_k v_k - \bar{u}_k D_k v_j) D_i \Phi_{ij} \, dx
\]
\[
+ \int (v_j D_k \bar{u}_k - v_k D_k \bar{u}_j) D_i \Phi_{ij} \, dx + \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij} \, dx
\]
\[=: I_1 + I_2 + I_3 + I_4,\]

it suffices to show \( I_1 = I_2 = I_3 = 0. \) Now
\[
D_i \Phi_{ij} = D_i \left[ c_\alpha r^{-1} \left( \delta_{ij} + \frac{\alpha}{\alpha + 2} \omega_{ij} \right) \right]
\]
\[= -c_\alpha r^{-2} \omega_i \delta_{ij} + \frac{c_\alpha \alpha}{\alpha + 2} r^{-2} \left[ -\omega_i^2 \omega_j + (\delta_{ij} - \omega_i^2) \omega_j + (\delta_{ij} - \omega_j \omega_i) \omega_i \right]
\]
\[= -c_\alpha r^{-2} \omega_j + \frac{c_\alpha \alpha}{\alpha + 2} r^{-2} \omega_j =: d_\alpha r^{-2} \omega_j,
\]
where
\[d_\alpha = \frac{-2c_\alpha}{\alpha + 2} = -\frac{1}{4\pi(\alpha + 1)}.\]

We have
\[I_1 = d_\alpha \int r^{-2} \omega_j \left( \bar{u}_j D_r \bar{u}_k \cdot \omega_k - \bar{u}_k D_r \bar{u}_j \cdot \omega_k \right) \, dx\]
\[= d_\alpha \int r^{-2} \left( \bar{u}_j D_r \bar{u}_k \cdot \omega_j \omega_k - \bar{u}_k D_r \bar{u}_j \cdot \omega_j \omega_k \right) \, dx = 0,
\]
\[I_3 = d_\alpha \int r^{-2} \left( v_j D_r \bar{u}_k \cdot \omega_j \omega_k - v_k D_r \bar{u}_j \cdot \omega_j \omega_k \right) \, dx = 0.
\]

As for \( I_2, \) we obtain
\[I_2 = d_\alpha \int r^{-2} \left( \bar{u}_j D_k v_k \cdot \omega_j - \bar{u}_k D_k v_j \cdot \omega_j \right) \, dx
\]
\[= d_\alpha \int r^{-2} \left( \bar{u}_j D_k v_k \cdot \omega_j - \bar{u}_j D_k v_k \cdot \omega_k \right) \, dx
\]
\[= -\lim_{\varepsilon \to 0^+} d_\alpha \int_{S^2} \left[ \bar{u}_j(\varepsilon) v_k(\varepsilon) \omega_j \omega_k - \bar{u}_j(\varepsilon) v_k(\varepsilon) \omega_j \omega_k \right] \, d\sigma
\]
\[- \lim_{\varepsilon \to 0^+} d_\alpha \int_{S^2 \setminus B_r} \left\{ v_k r^{-3} \left[ -2 \bar{u}_j \omega_j \omega_k + r D_r \bar{u}_j \cdot \omega_j \omega_k \bar{u}_j \delta_{jk} - \omega_j \omega_k \right] \right\} \, d\sigma = 0.
\]

The result follows. \( \Box \)

**Remark.** With \( \Phi(x) \) replaced by \( \Phi_y(x) := \Phi(x - y) \) and \( \mathbb{S} \) replaced by
\[u_i(x) = \bar{u}_i(r_y) + v_i(x) \quad (i = 1, 2, 3),\]
where \( r_y = |x - y| \) and
\[
\begin{align*}
\bar{u}_i(r_y) &= \frac{1}{4\pi} \int_{S^2} u_i(y + r_y \omega) \, d\sigma \quad \forall r_y \geq 0 \quad (i = 1, 2, 3), \\
\int_{S^2} v_i(y + r_y \omega) \, d\sigma &= 0
\end{align*}
\]
we have
\[ \int (Lu)^T \Phi y u \, dx = \frac{1}{2} |u(y)|^2 + B^* u, u \]
where
\[ B^* u! u = \frac{\alpha}{2} \int (v_j D_k v_k - v_k D_j v_j) D_i \Phi y_{ij} \, dx \]
\[ + \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi y_{ij} \, dx. \]

\[ \Box \]

In the next Lemma, we use the definition of \( \Phi \) and derive an explicit expression for the bilinear form \( B^* u, u \) defined in (10).

**Lemma 2.3.**

(12) \[ B^* u, u = c_{\alpha} \int \left\{ \frac{\alpha}{\alpha + 2} r^{-2} \left[ v_k (D_k v) \cdot \omega - (\text{div } v) (v \cdot \omega) \right] \right. \]
\[ + r^{-1} \left[ |D_r \bar{u}|^2 + \alpha \frac{2\alpha + 3}{\alpha + 2} (D_r \bar{u})^2 \omega_i^2 + |Dv|^2 + (\text{div } v)^2 \right. \]
\[ + \alpha \frac{\alpha + 2}{\alpha + 2} |D_k v \cdot \omega|^2 + \frac{\alpha^2}{\alpha + 2} (\text{div } v) [\omega_i (D_i v) \cdot \omega] \]
\[ \left. + \frac{3\alpha + 4}{\alpha + 2} (D_r \bar{u} \cdot \omega) (\text{div } v) + \alpha (D_r \bar{u} \cdot \omega) [\omega_i (D_i v) \cdot \omega] \right\} \, dx. \]

Before proving this lemma, we need a simple yet important observation that will be useful in the following computation.

**Lemma 2.4.** Let \( g \in C^\infty_c(\mathbb{R}^3) \) be such that
\[ \int_{S^2} g(r\omega) \, d\sigma = 0, \quad \forall r \geq 0. \]
Then
\[ \left\{ \begin{array}{l}
\int f(r) g(x) \, dx = 0 \\
\int r^{-1} Df(r) \cdot Dg(x) \, dx = 0
\end{array} \right\} \quad \forall f \in C^\infty_c(0, \infty). \]

**Proof.** By switching to polar coordinates, we easily see that
\[ \int f(r) g(x) \, dx = \int_0^\infty r^2 f(r) \, dr \int_{S^2} g(r\omega) \, d\sigma = 0. \]
On the other hand,
\[ \int r^{-1} Df(r) \cdot Dg(x) \, dx = \int r^{-1} D_r f D_i g \cdot \omega_i \, dx \]
\[ = - \int g \left[ -r^{-2} (D_r f) \omega_i^2 + r^{-1} (D_{rr} f) \omega_i^2 + r^{-2} D_r f (\delta_{ii} - \omega_i^2) \right] \, dx \]
\[ = - \int g (r^{-2} D_r f + r^{-1} D_{rr} f) \, dx = 0, \]
where the last equality follows by switching to polar coordinates. \( \Box \)
Proof of Lemma 2.3. By definition,
\[ B^*(u, u) = \frac{\alpha}{2} \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij} \, dx \]
\[ + \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{ij} \, dx =: I_1 + I_2. \]

We have shown in Lemma 2.2 that (see (11))
\[ I_1 = 2^{-1} \alpha d_\alpha \int r^{-2} \omega_j (v_j D_k v_k - v_k D_k v_j) \, dx \]
\[ = \frac{c_\alpha \alpha}{\alpha + 2} \int r^{-2} \left[ v_k (D_k v) \cdot \omega - (\text{div} v)(v \cdot \omega) \right] \, dx. \]

On the other hand,
\[ I_2 = c_\alpha \int r^{-1} D_k u_i D_k u_i \, dx + \frac{c_\alpha \alpha}{\alpha + 2} \int r^{-1} D_k u_i D_k u_j \cdot \omega_i \omega_j \, dx \]
\[ + c_\alpha \alpha \int r^{-1} D_k u_k D_i u_i \, dx + \frac{c_\alpha \alpha^2}{\alpha + 2} \int r^{-1} D_k u_k D_i u_j \cdot \omega_i \omega_j \, dx \]
\[ =: I_3 + I_4 + I_5 + I_6. \]

Substituting \( u_i = \bar{u}_i + v_i \) into \( I_3 \) and using Lemma 2.3 yields
\[ I_3 = c_\alpha \int r^{-1} \left( D_r \bar{u}_i D_r \bar{u}_i \cdot \omega_k \omega_i + 2 D_i v_i D_r \bar{u}_k \cdot \omega_k + D_k v_k D_i v_i \right) \, dx \]
\[ = c_\alpha \int r^{-1} \left( |D_r \bar{u}_i|^2 + |Dv|^2 \right) \, dx. \]

Next,
\[ I_5 = c_\alpha \alpha \int r^{-1} \left( D_r \bar{u}_k D_r \bar{u}_i \cdot \omega_k \omega_i + 2 D_i v_i D_r \bar{u}_k \cdot \omega_k + D_k v_k D_i v_i \right) \, dx. \]

Note that for \( k \neq i \),
\[ \int r^{-1} D_r \bar{u}_k D_r \bar{u}_i \cdot \omega_k \omega_i \, dx = \int_0^\infty r D_r \bar{u}_k D_r \bar{u}_i \, dr \int_{S^2} \omega_k \omega_i \, d\sigma = 0, \]
and therefore
\[ I_5 = c_\alpha \alpha \int r^{-1} \left[ (D_r \bar{u}_i)^2 \omega_i^2 + 2(\text{div} v)(D_r \bar{u} \cdot \omega) + (\text{div} v)^2 \right] \, dx. \]

As for \( I_4 \),
\[ I_4 = \frac{c_\alpha \alpha}{\alpha + 2} \int r^{-1} D_k (\bar{u}_i + v_i) D_k (\bar{u}_j + v_j) \cdot \omega_i \omega_j \, dx \]
\[ = \frac{c_\alpha \alpha}{\alpha + 2} \int r^{-1} \left( D_r \bar{u}_i D_r \bar{u}_j \cdot \omega_i \omega_j \omega_k^2 + D_r \bar{u}_i D_k v_j \cdot \omega_i \omega_j \omega_k \right) \]
\[ + D_k v_i D_r \bar{u}_j \cdot \omega_i \omega_j \omega_k + D_k v_i D_k v_j \cdot \omega_i \omega_j \, dx \]
\[ = \frac{c_\alpha \alpha}{\alpha + 2} \int r^{-1} \left( (D_r \bar{u}_i)^2 \omega_i^2 + 2(D_r \bar{u} \cdot \omega)[\omega_k (D_k v) \cdot \omega] + |D_k v \cdot \omega|^2 \right) \, dx. \]
Similarly,

\[ I_6 = \frac{c_\alpha \alpha^2}{\alpha + 2} \int r^{-1} D_k(\bar{u}_k + v_k) D_l(\bar{u}_j + v_j) \cdot \omega_i \omega_j \, dx \]

\[ = \frac{c_\alpha \alpha^2}{\alpha + 2} \int r^{-1} (D_r \bar{u}_k D_r \bar{u}_j \cdot \omega_i^2 \omega_j + D_r \bar{u}_k D_l v_j \cdot \omega_i \omega_j \omega_k + D_r \bar{u}_j D_k v_k \cdot \omega_i^2 \omega_j + D_l v_k D_l v_j \cdot \omega_i \omega_j) \, dx \]

\[ = \frac{c_\alpha \alpha^2}{\alpha + 2} \int r^{-1} \left[ (D_r \bar{u}_j)^2 \omega_i^2 + (D_r \bar{u}_j \cdot \omega_i) \omega_l (D_l \bar{u}_j) \cdot \omega_l \right] \, dx. \]

The lemma follows by adding up all these integrals. \qed

With the help of Lemma 2.3 we now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.2 and 2.3,

\[ -c_\alpha^{-1} \int (Lu)^T \Phi u \, dx = \frac{1}{2} c_\alpha^{-1} |u(0)|^2 + I_1 + I_2 + I_3, \]

where

\[ I_1 = \int r^{-1} \left[ \frac{\alpha^2}{\alpha + 2} (\text{div} v) [\omega_i (D_l \bar{u}_j) \cdot \omega_l] + \frac{3 \alpha}{\alpha + 2} (D_r \bar{u}_j \cdot \omega_l) (\text{div} v) \right] \, dx, \]

\[ I_2 = \int r^{-1} \left[ \frac{\alpha^2}{\alpha + 2} (\text{div} v) + \frac{\alpha}{\alpha + 2} (D_r \bar{u}_j \cdot \omega_l) (\text{div} v) \right] \, dx, \]

\[ I_3 = \int \frac{\alpha}{\alpha + 2} r^{-2} \left[ v_k (D_l \bar{u}_j) \cdot \omega_l (\text{div} v) \cdot \omega_l \right] \, dx. \]

Consider first the case \( \alpha \geq 0 \). By switching to polar coordinates, we have

\[ I_1 \geq \int r^{-1} \left[ |D_r \bar{u}_j|^2 + \frac{2 \alpha + 3}{\alpha + 2} (D_r \bar{u}_j)^2 \omega_i^2 + |D_l \bar{u}_j|^2 + \alpha (\text{div} v)^2 \right] \, dx \]

\[ = \int_0^\infty r \left[ 1 + \frac{\alpha}{3} \left( \frac{2 \alpha + 3}{\alpha + 2} \right) \right] \|D_r \bar{u}_j\|_\omega^2 + \|D_l \bar{u}_j\|_\omega^2 + \|\text{div} v\|_\omega^2 \right] \, dr, \]

where we have written \( \| \cdot \|_\omega \) for \( \| \cdot \|_{L^2(S^2)} \) and used the fact that

\[ \int_{S^2} (D_r \bar{u}_j)^2 \omega_i^2 \, d\sigma = \frac{4 \pi}{3} \sum_{i=1}^3 (D_r \bar{u}_i)^2 = \frac{1}{3} \int_{S^2} |D_r \bar{u}|^2 \, d\sigma = \frac{1}{3} \|D_r \bar{u}\|_\omega^2. \]
Next,

\[
|I_2| \leq \int_{-1}^{1} \left[ \frac{\alpha^2}{\alpha + 2} |\text{div} v||Dv| + \frac{3\alpha + 4}{\alpha + 2} |D_r \bar{u} \cdot \omega||\text{div} v| + \alpha |D_r \bar{u} \cdot \omega||Dv| \right] dx
\]

\[
\leq \int_0^\infty r \left[ \frac{\alpha^2}{\alpha + 2} \|\text{div} v\|\omega \|Dv\|\omega + \frac{\alpha}{\sqrt{3}} \frac{3\alpha + 4}{\alpha + 2} \|D_r \bar{u} \|\omega \|\text{div} v\|\omega \\
+ \alpha \sqrt{\frac{\alpha}{\sqrt{3}}} \|D_r \bar{u} \|\omega \|Dv\|\omega \right] dr,
\]

where we have used

\[
\|D_r \bar{u} \cdot \omega\|_\omega^2 = \int_{S^2} D_r \bar{u}_i D_r \bar{u}_j \cdot \omega_i \omega_j d\sigma
\]

\[
= D_r \bar{u}_i D_r \bar{u}_j \cdot \frac{4\pi}{3} \delta_{ij} = \frac{4\pi}{3} \sum_{i=1}^{3} (D_r \bar{u}_i)^2 = \frac{1}{3} \|D_r \bar{u}\|_\omega^2.
\]

As for \(I_3\), we note that

\[
|I_3| \leq \frac{\alpha}{\alpha + 2} \int_{-1}^{1} r^{-2} (|v||Dv| + |v||\text{div} v|) dx
\]

\[
\leq \frac{\alpha}{\alpha + 2} \int_0^\infty \|v\|\omega (\|Dv\|\omega + \|\text{div} v\|\omega) dr.
\]

Since 2 is the first non-trivial eigenvalue of the Laplace-Beltrami operator on \(S^2\), we have

\[
\|v\|_\omega^2 = \int_{S^2} |v(r\omega)|^2 d\sigma \leq \frac{1}{2} \int_{S^2} |D_\omega [v(r\omega)]|^2 d\sigma
\]

\[
= \frac{r^2}{2} \int_{S^2} |(D_\omega v)(r\omega)|^2 d\sigma \leq \frac{r^2}{2} \|Dv\|_\omega^2,
\]

where \(D_\omega\) is the gradient operator on \(S^2\). Thus

\[
|I_3| \leq \frac{1}{\sqrt{2}} \cdot \frac{\alpha}{\alpha + 2} \int_0^\infty r \left[ \|Dv\|_\omega^2 + \|Dv\|_\omega \|\text{div} v\|_\omega \right] dr,
\]

and by putting all pieces together we obtain

\[
(15) \quad I_1 + I_2 + I_3 \geq \int_0^\infty r (w^T B_+ w) dr,
\]

where

\[
w = \left( \|D_r \bar{u}\|_\omega, \|Dv\|_\omega, \|\text{div} v\|_\omega \right)^T,
\]

\[
B_+ = \begin{bmatrix}
1 + \frac{\alpha}{3} & \frac{2\alpha + 3}{\alpha + 2} & -\frac{\alpha}{2\sqrt{3}} & -\frac{\alpha}{\sqrt{3}} & \frac{3\alpha + 4}{\alpha + 2} \\
-\frac{\alpha}{2\sqrt{3}} & 1 & \frac{\alpha}{\alpha + 2} & \frac{\alpha}{\alpha + 2} & \frac{3\alpha + 4}{\alpha + 2} \\
-\frac{\alpha}{\alpha + 2} & \frac{3\alpha + 4}{\alpha + 2} & -\frac{\alpha}{\alpha + 2} & \frac{3\alpha + 4}{\alpha + 2} & \frac{\alpha}{\alpha + 2} \\
\end{bmatrix}.
\]
The positive definiteness of \( B \) of the determinants of all leading principal minors of \( I \):

\[
(17)
\]

Hence with the help of computer algebra packages, we find that (16) holds for \( 0 < \alpha < \alpha_+ \), where \( \alpha_+ \approx 1.524 \) is the largest real root of \( p_{-2,3}(\alpha) \).

The estimates of \( I_1, I_2, \) and \( I_3 \) are slightly different when \( \alpha < 0 \), since now the quadratic term \( \alpha \| \text{div} v \|^2 \omega \) in \( I_1 \) is negative. This means that it is no longer possible to control the \( \| \text{div} v \|^2 \omega \) terms in \( I_2, I_3 \) by \( \alpha \| \text{div} v \|^2 \omega \), and in order to obtain positivity we need to bound \( \| \text{div} v \|^2 \omega \) by \( \| Dv \|^2 \omega \) as follows:

\[
\| \text{div} v \|^2 \omega \leq 3 \| Dv \|^2 \omega.
\]

This leads to the following revised estimates:

\[
I_1 \geq \int_0^\infty r \left[ 1 + \frac{\alpha}{3} \cdot \frac{2\alpha + 3}{\alpha + 2} \right] \| D_r \! \bar{u} \|_\omega^2 \! + \| Dv \|_\omega^2 \! + 3\alpha \| Dv \|_\omega^2 \! + \frac{\alpha}{\alpha + 2} \| Dv \|_\omega^2 \! \right] dr,
\]

\[
|I_2| \leq \int_0^\infty r \left[ \frac{\sqrt{3} \alpha^2}{\alpha + 2} \| Dv \|_\omega^2 \! - \alpha \frac{3\alpha + 4}{\alpha + 2} \| D_r \! \bar{u} \|_\omega \| Dv \|_\omega \! - \frac{\alpha}{\sqrt{3}} \| D_r \! \bar{u} \|_\omega \| Dv \|_\omega \! \right] dr,
\]

\[
|I_3| \leq \frac{1}{\sqrt{2}} \frac{\alpha}{\alpha + 2} \int_0^\infty r \left[ \| Dv \|_\omega^2 \! + \sqrt{3} \| Dv \|_\omega^2 \! \right] dr.
\]

Hence

\[
(17) \quad I_1 + I_2 + I_3 \geq \int_0^\infty r (w^T B_- w) \ dr,
\]

where

\[
w = \left( \| D_r \! \bar{u} \|_\omega, \| Dv \|_\omega \right)^T, \\
B_- = \begin{bmatrix}
1 + \frac{\alpha}{3} \cdot \frac{2\alpha + 3}{\alpha + 2} & \frac{\alpha}{\alpha + 2} \cdot \frac{3\alpha + 4}{\alpha + 2} + \frac{\alpha}{\sqrt{3}} \\
\frac{\alpha}{\alpha + 2} \cdot \frac{3\alpha + 4}{\alpha + 2} + \frac{\alpha}{\sqrt{3}} & 1 + \frac{\alpha}{\alpha + 2} \left( 1 + \frac{1 + \sqrt{3}}{\sqrt{2}} - \sqrt{3} \alpha \right)
\end{bmatrix}.
\]
The positive definiteness of $B_-$ is equivalent to:

\[(18a) \quad p_{-1}(\alpha) = \frac{2\alpha^2 + 6\alpha + 6}{3(\alpha + 2)} > 0,\]

\[(18b) \quad p_{-2}(\alpha) = \frac{1}{6(\alpha + 2)^2} \left[ -(2 + 7\sqrt{3})\alpha^4 + 2(15 + \sqrt{2} - 11\sqrt{3} + \sqrt{6})\alpha^3 
+ 2(57 + 3\sqrt{2} - 10\sqrt{3} + 3\sqrt{6})\alpha^2 + 6(20 + \sqrt{2} + \sqrt{6})\alpha + 24 \right] > 0,\]

and \[(18)\] holds for $\alpha_- < \alpha < 0$, where $\alpha_- \approx -0.194$ is the smallest real root of $p_{-2}$.

Now we show that the 3D Lamé system is not positive definite with weight $\Phi$ when $\alpha$ is either too close to $-1$ or too large. By Proposition 3.11 in [1], the 3D Lamé system is positive definite with weight $\Phi$ only if

\[
\sum_{i,\beta,\gamma} A_{ij}^{\beta\gamma} \xi_i \xi_j \Phi_{ij}(\omega) \geq 0, \quad \forall \xi \in \mathbb{R}^3, \forall \omega \in S^2 \quad (p = 1, 2, 3),
\]

where

\[
A_{ij}^{\beta\gamma} = \delta_{ij}\delta_{\beta\gamma} + \frac{\alpha}{2}(\delta_{ij}\delta_{\beta\gamma} + \delta_{ij}\delta_{\beta\gamma})
\]

and (see equation 3)

\[
\Phi_{ij}(\omega) = c_\alpha r^{-1} \left( \delta_{ij} + \frac{\alpha}{\alpha + 2}\omega_i\omega_j \right) \quad (i, j = 1, 2, 3).
\]

This means, in particular, that the matrix

\[
A(\omega; \alpha) := \left( \sum_{p=1}^{3} A_{ij}^{p\gamma} \Phi_{ij}(\omega) \right)_{\beta,\gamma=1}^{3} = \frac{c_\alpha r^{-1}}{2(\alpha + 2)^2} \begin{bmatrix}
2(\alpha + 1)(\alpha + 2 + \alpha\omega_1^2) & \alpha^2\omega_1\omega_2 & \alpha^2\omega_1\omega_3 \\
\alpha^2\omega_1\omega_2 & 2(\alpha + 2 + \alpha\omega_1^2) & 0 \\
\alpha^2\omega_1\omega_3 & 0 & 2(\alpha + 2 + \alpha\omega_1^2)
\end{bmatrix}
\]

is semi-positive definite for any $\omega \in S^2$ if the 3D Lamé system is positive definite with weight $\Phi$. But $A(\omega; \alpha)$ is semi-positive definite only if the determinant of its leading principal minor

\[
d_2(\omega; \alpha) := \text{det} \begin{bmatrix}
2(\alpha + 1)(\alpha + 2 + \alpha\omega_1^2) & \alpha^2\omega_1\omega_2 \\
\alpha^2\omega_1\omega_2 & 2(\alpha + 2 + \alpha\omega_1^2)
\end{bmatrix}
\]

is non-negative, and elementary estimate shows that

\[
\min_{\omega \in S^2} d_2(\omega; \alpha) \leq d_2(2^{-1/2}, 2^{-1/2}, 0; \alpha)
\]

\[
= (\alpha + 1)(3\alpha + 4)^2 - \frac{\alpha^4}{4} =: q(\alpha).
\]

It follows that the 3D Lamé system is not positive definite with weight $\Phi$ when $q(\alpha) < 0$, which holds for $\alpha < \alpha_c \approx -0.902$ or $\alpha > \alpha_c \approx 39.450$.

\[\square\]

**Remark.** We have in fact shown that, for $\alpha_- < \alpha < \alpha_+$ and some $c > 0$ depending on $\alpha$,

\[
\int (Lu)^T \Phi u \, dx \geq \frac{1}{2} |u(0)|^2 + c \int |Du(x)|^2 \frac{dx}{|x|}.
\]
If we replace $\Phi(x)$ by $\Phi_y(x) := \Phi(x - y)$, then
\[
\int (Lu)^T \Phi_y u \, dx = \int [Lu(x + y)]^T \Phi u(x + y) \, dx \\
\geq \frac{1}{2} |u(y)|^2 + c \int |Du(x + y)|^2 dx \frac{2}{|x|} \\
\geq \frac{1}{2} |u(y)|^2 + c \int |Du(x)|^2 dx \frac{2}{|x - y|}.
\]
(19)

\[\square\]

3. Proof of Theorem 1.5

In the next lemma and henceforth, we use the notation
\[
\begin{align*}
m_p(u) &= \rho^{-3} \int_{\Omega \cap S_p} |u(x)|^2 \, dx, \\
S_p &= \{x : \rho < |x| < 2\rho\}, \\
M_p(u) &= \rho^{-3} \int_{\Omega \cap B_p} |u(x)|^2 \, dx.
\end{align*}
\]

Lemma 3.1. Suppose $L$ is positive definite with weight $\Phi$, and let $u = (u_i)_{i=1}^3$, $u_i \in H^1(\Omega)$ be a solution of
\[Lu = 0 \quad \text{on } \Omega \cap B_{2\rho}.
\]

Then
\[
\int_{\Omega} [Lu \eta_p]^T \Phi_y \eta_p \, dx \leq cm_p(u), \quad \forall y \in B_p,
\]
where $\eta_p(x) = \eta(x/\rho)$, $\eta \in C_0^\infty(B_{5/3})$, $\eta = 1$ on $B_{4/3}$, and $\Phi_y(x) = \Phi(x - y)$.

Proof. By definition of $u$,
\[
\int_{\Omega} [Lu \eta_p]^T \Phi_y \eta_p \, dx = \int_{\Omega} [Lu \eta_p]^T \Phi_y \eta_p \, dx - \int_{\Omega} (Lu)^T \Phi_y \eta_p^2 \, dx,
\]
where the second integral on the right side vanishes and the first one equals
\[-\int \left[2D_i u_i D_k \eta_p + u_i D_k \eta_p + \alpha \left(D_i u_k D_k \eta_p + D_k u_k D_i \eta_p + u_k D_i \eta_p \right) \right] \eta_p \Phi_{ij} \eta_p \, dx.
\]
Note that $D \eta_p$, $D^2 \eta_p$ have compact support in $R := B_{5\rho/3} \setminus B_{4\rho/3}$, $|D^k \eta_p| \leq c\rho^{-k}$, and $\Phi_{ij}(x) \leq \frac{c}{|x - y|} \leq c\rho^{-1}$, $\forall x \in R$, $\forall y \in B_p$.

Thus
\[
\int_{\Omega} [Lu \eta_p]^T \Phi_y \eta_p \, dx \leq c \int_{\Omega \cap R} \rho^{-2} |u| |Du| \, dx + c \int_{\Omega \cap R} \rho^{-3} |u|^2 \, dx
\]
\[
\leq c \left[ \rho^{-3} \int_{\Omega \cap S_\rho} |u|^2 \, dx \right]^{1/2} \left[ \rho^{-1} \int_{\Omega \cap R} |Du|^2 \, dx \right]^{1/2} + c \rho^{-3} \int_{\Omega \cap S_\rho} |u|^2 \, dx.
\]

The lemma then follows from the well-known local energy estimate [5]
\[
\rho^{-1} \int_{\Omega \cap R} |Du|^2 \, dx \leq \rho^{-3} \int_{\Omega \cap S_\rho} |u|^2 \, dx.
\]

\[\square\]
Combining (19) (with $u$ replaced by $u\eta_{\rho}$) and Lemma 3.1, we arrive at the following local estimate.

**Corollary 3.2.** Let the conditions of Lemma 3.1 be satisfied. Then

$$|u(y)|^2 + \int_{\Omega \cap B_\rho} \frac{|Du(x)|^2}{|x-y|} \, dx \leq cm_\rho(u), \quad \forall y \in \Omega \cap B_\rho.$$  

To proceed, we need the following Poincaré-type inequality (see [2]).

**Lemma 3.3.** Let $u = (u_i)_{i=1}^3$ be any vector function with $u_i \in \mathcal{H}^1(\Omega)$. Then for any $\rho > 0$,

$$m_\rho(u) \leq \frac{c}{\text{cap}(\mathcal{S}_\rho \setminus \Omega)} \int_{\Omega \cap S_\rho} |Du|^2 \, dx$$

where $c$ is independent of $\rho$.

The next corollary is a direct consequence of Corollary 3.2 and Lemma 3.3.

**Corollary 3.4.** Let the conditions of Lemma 3.1 be satisfied. Then

$$|u(y)|^2 + \int_{\Omega \cap B_\rho} \frac{|Du(x)|^2}{|x-y|} \, dx \leq \frac{c}{\text{cap}(\mathcal{S}_\rho \setminus \Omega)} \int_{\Omega \cap S_\rho} |Du|^2 \, dx, \quad \forall y \in \Omega \cap B_\rho.$$  

We are now in a position to prove the following lemma, which is the key ingredient in the proof of Theorem 1.5.

**Lemma 3.5.** Suppose $L$ is positive definite with weight $\Phi$, and let $u = (u_i)_{i=1}^3$ be a solution of $Lu = 0$ on $\Omega \cap B_{2R}$. Then, for all $\rho \in (0, R)$,

$$(20) \quad \sup_{x \in \Omega \cap B_\rho} |u(x)|^2 + \int_{\Omega \cap B_\rho} \frac{|Du(x)|^2}{|x-y|} \, dx \leq c_1 M_{2R}(u) \exp \left[ -c_2 \int_\rho^R \text{cap}(\mathcal{B}_r \setminus \Omega) r^{-2} \, dr \right],$$

where $c_1$, $c_2$ are independent of $\rho$.

**Proof.** Define

$$\gamma(r) := r^{-1} \text{cap}(\mathcal{S}_r \setminus \Omega).$$

We first claim that $\gamma(r)$ is bounded from above by some absolute constant $A$. Indeed, The monotonicity of capacity implies that

$$\text{cap}(\mathcal{S}_r \setminus \Omega) \leq \text{cap}(\mathcal{B}_r).$$

By choosing smooth test functions $\eta_r(x) = \eta(x/r)$ with $\eta \in C_0^\infty(B_2)$ and $\eta = 1$ on $B_{3/2}$, we also have

$$\text{cap}(\mathcal{B}_r) \leq \int_{\mathbb{R}^3} |D\eta_r|^2 \, dx \leq \sup_{x \in \mathbb{R}^3} |D\eta(x)|^2 \int_{B_2} r^{-2} \, dx$$

$$= \frac{32}{3} \pi \sup_{x \in \mathbb{R}^3} |D\eta(x)|^2 \pi r.$$  

Hence the claim follows.

We next consider the case $\rho \in (0, R/2]$. Denote the first and second terms on the left side of (20) by $\varphi_{\rho}$ and $\psi_{\rho}$, respectively. From Corollary 3.4 it follows that for $r \leq R$,

$$\varphi_{\rho} + \psi_{\rho} \leq \frac{c}{\gamma(r)} (\psi_{2r} - \psi_r) \leq \frac{c}{\gamma(r)} (\psi_{2r} - \psi_r + \varphi_{2r} - \varphi_r),$$

where $c$ is independent of $\rho$. 

We then define $\gamma^*(r)$ as the minimum of $\gamma(r)$ and $\gamma^*(r)$. This completes the proof of Lemma 3.5.
which implies that
\[ \varphi_r + \psi_r \leq \frac{c}{c + \gamma(r)}(\varphi_{2r} + \psi_{2r}) = \frac{ce^{c_0\gamma(r)}}{c + \gamma(r)}e^{-c_0\gamma(r)}(\varphi_{2r} + \psi_{2r}), \quad \forall c_0 > 0. \]

Since \( \gamma(r) \leq A \) and
\[ \sup_{s \in [0, A]} \frac{ce^{c_0s}}{c + s} \leq \max \left\{ 1, \frac{ce^{c_0A}}{c + A}, cc_0e^{1-cc_0} \right\}, \]
it is possible to choose \( c_0 > 0 \) sufficiently small so that
\[ \sup_{r > 0} \frac{ce^{c_0\gamma(r)}}{c + \gamma(r)} \leq 1. \]

It follows, for \( c_0 \) chosen this way, that
\[ (21) \quad \varphi_r + \psi_r \leq e^{-c_0\gamma(r)}(\varphi_{2r} + \psi_{2r}). \]

By setting \( r = 2^{-l}R \) (\( l \in \mathbb{N} \)) and repeatedly applying (21), we obtain
\[ \varphi_{2^{-l}R} + \psi_{2^{-l}R} \leq \exp \left[ -c_0 \sum_{j=1}^{l} \gamma(2^{-j}R) \right] (\varphi_R + \psi_R). \]

If \( l \) is such that \( l \leq \log_2(R/\rho) < l + 1 \), then \( \rho \leq 2^{-l}R < 2\rho \) and
\[ \varphi_{\rho} + \psi_{\rho} \leq \varphi_{2^{-l}R} + \psi_{2^{-l}R} \leq \exp \left[ -c_0 \sum_{j=1}^{l} \gamma(2^{-j}R) \right] (\varphi_R + \psi_R). \]

Note that by Corollary 3.2
\[ \varphi_R + \psi_R \leq cm_R(u) \leq cM_2R(u). \]

In addition, the subadditivity of the harmonic capacity implies that
\[
\sum_{j=1}^{l} \gamma(2^{-j}R) \geq \sum_{j=1}^{l} \frac{\text{cap}(\bar{B}_{2^{-j}R} \setminus \Omega) - \text{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R} \\
= \left[ \frac{\text{cap}(\bar{B}_R \setminus \Omega)}{2^{-1}R} - \frac{\text{cap}(\bar{B}_{2^{-1}R} \setminus \Omega)}{2^{-1}R} \right] + \sum_{j=1}^{l-1} \frac{\text{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R} \\
= \frac{1}{2} \cdot \frac{\text{cap}(\bar{B}_R \setminus \Omega)}{R} - 2 \frac{\text{cap}(\bar{B}_{2^{-1}R} \setminus \Omega)}{2^{-1}R} + \sum_{j=1}^{l-1} \frac{\text{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R} \\
\geq -2 \frac{\text{cap}(\bar{B}_{2^{-1}R} \setminus \Omega)}{2^{-1}R} + \frac{1}{2} \sum_{j=0}^{l-1} \frac{\text{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R}. 
\]
Since
\[
\frac{\text{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R} \leq A,
\]
\[
\sum_{j=0}^{l} \frac{\text{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R} \geq \frac{1}{2} \sum_{j=1}^{l+1} \frac{\text{cap}(\bar{B}_{2^{-1}R} \setminus \Omega)}{(2^{-1}R)^2} \cdot 2^{-j}R
\]
\[
\geq \frac{1}{2} \sum_{j=1}^{l+1} \int_{2^{-j}R}^{2^{-1}R} \text{cap}(\bar{B}_r \setminus \Omega)r^{-2} \, dr
\]
\[
\geq \frac{1}{2} \int_{\rho}^{R} \text{cap}(\bar{B}_r \setminus \Omega)r^{-2} \, dr,
\]
we have
\[
\exp\left[-c_0 \sum_{j=1}^{l} \gamma(2^{-j}R)\right] \leq \exp\left[-\frac{c_0}{4} \int_{\rho}^{R} \text{cap}(\bar{B}_r \setminus \Omega)r^{-2} \, dr + 2c_0 A\right].
\]
Hence (20) follows with \(c_1 = c_0 c_2 A\) and \(c_2 = c_0/4\).

Finally we consider the case \(\rho \in (R/2, R)\). By Corollary 3.2,
\[
|u(y)|^2 + \int_{\Omega \cap B_{\rho}} \frac{|Du(x)|^2}{|x-y|} \, dx \leq c_m \rho(u), \quad \forall y \in \Omega \cap B_{\rho},
\]
which implies that
\[
\sup_{y \in \Omega \cap B_{\rho}} |u(y)|^2 + \int_{\Omega \cap B_{\rho}} \frac{|Du(x)|^2}{|x|} \, dx \leq c M_2 R(u).
\]
In addition,
\[
\int_{\rho}^{R} \text{cap}(\bar{B}_r \setminus \Omega)r^{-2} \, dr \leq A \int_{R/2}^{R} r^{-1} \, dr = A \log 2,
\]
so
\[
\left[ \sup_{y \in \Omega \cap B_{\rho}} |u(y)|^2 + \int_{\Omega \cap B_{\rho}} \frac{|Du(x)|^2}{|x|} \, dx \right] \exp\left[c_2 \int_{\rho}^{R} \text{cap}(\bar{B}_r \setminus \Omega)r^{-2} \, dr\right] \leq c_1 M_2 R(u)
\]
provided that \(c_1 \geq c c_2 A \log 2\).

Proof of Theorem 1.5. Consider the Dirichlet problem
\[
Lu = f, \quad f_i \in C_0^\infty(\Omega), \quad u_i \in H^1(\Omega).
\]
Since \(f\) vanishes near the boundary, there exists \(R > 0\) such that \(f = 0\) in \(\Omega \cap B_{2R}\).

By Lemma 3.5
\[
\sup_{x \in \Omega \cap B_{\rho}} |u(x)|^2 \leq c_1 M_2 R(u) \exp\left[-c_2 \int_{\rho}^{R} \text{cap}(\bar{B}_r \setminus \Omega)r^{-2} \, dr\right],
\]
and in particular,
\[
\limsup_{x \to 0} |u(x)|^2 \leq c_1 M_2 R(u) \exp\left[-c_2 \int_{0}^{R} \text{cap}(\bar{B}_r \setminus \Omega)r^{-2} \, dr\right] = 0,
\]
where the last equation follows from the divergence of the Wiener integral
\[
\int_{1}^{0} \text{cap}(\bar{B}_r \setminus \Omega)r^{-2} \, dr = \infty.
\]
Hence $O$ is regular with respect to $L$. □

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