Matrix String Theory and its Moduli Space

G. Bonelli, L. Bonora, F. Nesti, A. Tomasiello

International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2-4, 34014 Trieste, Italy, and INFN, Sezione di Trieste
E-mail: bonelli@sissa.it, bonora@sissa.it, nesti@sissa.it, tomasiel@sissa.it

Abstract: The correspondence between Matrix String Theory in the strong coupling limit and IIA superstring theory can be shown by means of the instanton solutions of the former. We construct the general instanton solutions of Matrix String Theory which interpolate between given initial and final string configurations. Each instanton is characterized by a Riemann surface of genus $h$ with $n$ punctures, which is realized as a plane curve. We study the moduli space of such plane curves and find out that, at finite $N$, it is a discretized version of the moduli space of Riemann surfaces: instead of $3h-3+n$ its complex dimensions are $2h-3+n$, the remaining $h$ dimensions being discrete. It turns out that as $N$ tends to infinity, these discrete dimensions become continuous, and one recovers the full moduli space of string interaction theory.
1. Introduction

The $\mathcal{N}=(8,8)$ SYM on a cylindrical 2D space-time with gauge group U(N) (hereafter referred to as Matrix String Theory (MST)) represents in the strong coupling limit a theory of type II superstrings. This had been conjectured with various degrees of plausibility in several papers [1, 2, 3, 4] (see also [5, 6, 7] and the review article [8]). A step forward for finding more compelling evidence for such a conjecture was made in refs. [9, 10, 11], where it was pointed out that the MST contains BPS instanton solutions which interpolate between different initial and final string configurations via suitable punctured Riemann surfaces. We
often refer to them as *stringy instantons*. In a recent paper, \[12\], it was shown that, in the strong coupling limit, MST in the background of a given classical BPS instanton solution reduces to the Green-Schwarz superstring theory plus a decoupled Maxwell theory, and that the leading term of the amplitude in such background is proportional to $g_s^{-\chi}$, where $g_s$ is the string coupling constant and $\chi = 2 - 2h - n$ is the Euler characteristic of the Riemann surface of genus $h$ with $n$ punctures, which characterizes the given classical solution. This is the result one expects from perturbative string interaction theory. Needless to say this is a strong confirmation of the abovementioned conjecture.

The results of \[12\], although striking (and confirmed by the present paper), were not complete. They were essentially based on a small subset of instantons, those corresponding to the so-called $\mathbb{Z}_N$ coverings. In this paper we intend to fill the gap by considering any kind of coverings and constructing the corresponding instantons. Any such instanton consists of two ingredients, a *group theoretical factor* and a *core*. The latter corresponds to a branched covering of the cylinder. The group theoretical factor contains fields that satisfy WZNW-like equations with delta-function sources. Inside these instantons Riemann surfaces appear as branched coverings of the base cylinder in the form of *plane curves*, i.e. the zero locus of polynomials of two complex variables of order $N$ in one of them. This implies that, as we consider string amplitudes beyond the tree level, we are bound to meet mostly singular curves, which, in order to be classified, need to be desingularized. One can then set out to study the moduli space of such curves. This problem is of utmost importance, not only because one would like to make sure that the moduli space of type IIA theory is actually recovered within MST, but especially because one would like to know *in what sense* this becomes true.

In MST there is indeed a dependence on the discrete parameter $N$ (see \[13, 14\] for Matrix Theory): it was already noticed in \[12\] that for finite $N$ the moduli space ensuing from MST is only an approximation of the moduli space of type IIA superstring theory. We are now in a position to be more precise on this issue. We show below that, for finite $N$, the instantons of MST reproduce exactly only the tree string amplitudes, while they cover only part of the moduli space of higher genus Riemann surfaces with punctures. More precisely, in a process with $n$ external string states mediated by a Riemann surface of genus $h$, one expects $3h - 3 + n$ complex moduli; at finite $N$, MST reproduces only part of these parameters, and $h$ of them are anyhow discrete. The latter become continuous and we recover the full moduli space of Riemann surfaces only when $N \to \infty$.

In this paper we clarify also another issue of MST: an instanton seems to extend at first sight over four space-time dimensions of type IIA superstring theory. However one can show that the Riemann surfaces corresponding to the string instantons only in particular cases are contained in four dimensions; in general they extend over more physical dimensions and they can actually fill up the ten dimensional space-time of type IIA theory.

The paper is organized as follows. The *second section* is a rather detailed review of previous results, as well as an overview of problems we want to clarify in this paper. It also contains a summary of our main results while the details of their derivation are deferred to the following sections. In section \[3\] we give the general construction of the first
ingredient of a stringy instanton, i.e. its group theoretical factor. The last two sections are instead devoted to a description of the second ingredient, i.e. Riemann surfaces as branched coverings of the cylinder (section 4) and their moduli space (section 5).

2. Matrix String Theory: overview of problems and solutions

2.1 Euclidean MST and Hitchin systems

To start with let us summarize the results of [11, 12]. MST is a theory defined on a cylinder $C$ with coordinates $\sigma$ and $\tau$. Its Euclidean action is

$$S = \frac{1}{\pi} \int_{C} d^2w \text{Tr} \left( D_w X^i D_{\bar{w}} X^i - \frac{1}{4g^2} F_{w\bar{w}}^2 - \frac{g^2}{2} [X^i, X^j]^2 + i(\theta^-_{\tau} D_w \theta^-_{\bar{w}} + \theta^+_{\bar{w}} D_w \theta^+_{\tau}) + ig\theta^T \Gamma_i [X^i, \theta] \right),$$

where we use the notation

$$w = \frac{1}{2}(\tau + i\sigma), \quad \bar{w} = \frac{1}{2}(\tau - i\sigma), \quad A_w = A_\tau - iA_\sigma, \quad A_{\bar{w}} = A_{\bar{\tau}} + iA_{\bar{\sigma}}.$$

Moreover $X^i$ with $i = 1, \ldots, 8$ are hermitean $N \times N$ matrices and $D_w X^i = \partial_w X^i + i[A_w, X^i]$. $F_{w\bar{w}}$ is the gauge curvature. Summation over the $i, j$ indices is understood. $\theta$ represents 16 $N \times N$ matrices whose entries are 2D spinors. It can be written as $\theta^T = (\theta^-, \theta^+)$, where $\pm$ denotes the 2D chirality and $\theta^-_s, \theta^+_c$ are spinors in the $8_s$ and $8_c$ representations of $SO(8)$, while $T$ represents the 2D transposition. The matrices $\Gamma_i$ are the $16 \times 16 SO(8)$ gamma matrices.

The action (2.1) has $\mathcal{N} = (8, 8)$ supersymmetry. In [11] we singled out classical supersymmetric configurations that preserve $(4, 4)$ supersymmetry. In this configurations the fermions are zero, $\theta = 0$, and $X^i = 0$ for all $i$ except two, for definiteness $X^i \neq 0$ for $i = 1, 2$. Introducing the complex notation $X = X^1 + iX^2, \quad \bar{X} = X^1 - iX^2 = X^\dagger$, the conditions to be satisfied for such BPS configurations are

$$F_{w\bar{w}} + ig^2 [X, \bar{X}] = 0 \quad (2.2)$$

$$D_w X = 0, \quad D_{\bar{w}} \bar{X} = 0. \quad (2.3)$$

From a mathematical viewpoint (2.2), (2.3) can be identified with a Hitchin system [17] on a cylinder. Hitchin systems are defined starting from a $U(N)$ vector bundle $V$ over $C$, associated with the fundamental representation of $U(N)$. They consist of couples $(A, X)$ where $A$ is a gauge connection and $X$ a section of $EndV \otimes K$, where $K$ is the canonical bundle of $C$, which satisfy (2.2) and (2.3), [17]. Such systems can be lifted to an $N$-branched covering of $C$, [18, 19, 20, 21]. Such lifting was the basis of all the developments in [12].

\[ \text{In [14, 16] the authors have considered related theories on the torus.} \]
2.2 Construction of instanton solutions

Each solution of (2.2), (2.3) consists of two parts: a branched covering of the cylinder via the relative $X$ characteristic polynomial and a group theoretical factor. A few explicit solutions, based on $\mathbb{Z}_N$ coverings, were presented and discussed in [11, 12]. $\mathbb{Z}_N$ coverings are but a very restricted set of coverings. Consequently, the corresponding instantons are only a limited set of all possible stringy instantons.

The aim of the present paper is to generalize the analysis of [11, 12] by considering the most general possible coverings and constructing the corresponding instantons. The purpose of this section is to provide a mostly qualitative overview of the problems involved and of the main results.

Let us start by recalling our construction of the solutions of eqs. (2.2), (2.3). By this we mean a couple $(X,A_w)$ which are solutions of (2.2), (2.3) and are smooth everywhere on $\mathbb{C}$. We parametrize them as

$$X = Y^{-1}MY, \quad A_w = -iY^{-1}\partial_Y Y.$$  \hfill (2.4)

The group theoretical factor $Y$ takes values in the complex group $SL(N, \mathbb{C})$ while the matrix $M$ determines the branched covering (see below). The dependence on the Yang-Mills coupling constant $g$ is contained in the $Y$ factor, while $M$ does not depend on $g$. In [12] we have shown on several examples that $Y = Y_sY_d$, where $Y_d$, the dressing factor, tends to 1 in the strong coupling limit outside the string interaction points, while $Y_s$ is a special matrix, independent of $g$, endowed with the property that $Y_s^{-1}MY_s$ and $Y_s^\dagger M^\dagger(Y_s^\dagger)^{-1}$ are simultaneously diagonalizable. The construction of $Y_s$ and $Y_d$ in the general case is rather subtle. One first diagonalizes $M$ by means of a matrix $S$ of $SL(N, \mathbb{C})$. Then one introduces a matrix $K$ such that $KS = U$ is unitary. As it turns out, $K$ may have singularities at the points of $\mathbb{C}$ where any two eigenvalues of $M$ coincide: these correspond to the branch points of the spectral covering ($K$ is also allowed to diverge in a prescribed way for $w = \pm \infty$, but we disregard this issue for the time being). Therefore $KMK^{-1}$ is in general singular at these points. We therefore introduce into the game a new matrix $L$, with the purpose of canceling the singularities of $KMK^{-1}$ in such a way that $LKM(K^{-1}L^{-1}$ is smooth and satisfies (2.2), (2.3). In order for this to be true the entries of $L$ must satisfy equations of the WZNW type with delta-function-type sources at the branch points. By construction $K$ is independent of $g$ while $L$ does depend on $g$. We will show that in fact $L \to 1$ as $g \to \infty$. We therefore see that $K^{-1}$ plays the role of $Y_s$ and $L^{-1}$ is to be identified with the dressing factor $Y_d$, so that $LK = Y^{-1}$.

We will deal with the general construction of $Y_s$ and $Y_d$ in detail in section 3. Here we would like to stress the double ‘miracle’ of the above construction: we construct an everywhere smooth solution by means of two non smooth matrices $K$ and $L$, which are such that on the one hand $L \to 1$ as $g \to \infty$ and on the other hand $K$ form with $S$ a unitary matrix $U = KS$. This can be adequately appreciated in the light of ref. [12]. It was shown there that, thanks to these properties, in the strong coupling limit, we can get entirely rid of the non-diagonal background part. In order to deal with the singularities
that are exposed as the dressing factor tends to 1, one cuts out a small disc around any string interaction point (i.e. one introduces a regulator), defines the theory on the cylinder minus such discs (where, in the strong coupling limit, \( Y_d = 1 \)) and get rid of the \( U \) factor by means of a gauge transformation. At this point the analysis of [12] can be conveniently carried out and, eventually, the regulator removed.

2.3 Riemann surfaces as branched coverings of the cylinder

The second ingredient of an instanton solution is a branched covering of the cylinder. In dealing with branched coverings it is however more convenient to map, by the standard mapping \( z = e^{i\bar{w}} \), the infinite cylinder \( \mathcal{C} \) to the punctured complex \( z \)-plane \( \mathbb{C} \setminus \{0\} \), i.e. \( \mathbb{C}^* \). Very often we will refer to it as the Riemann sphere \( \mathbb{C}P^1 \) (with two punctures).

Let us consider the polynomial

\[
P_X(y) = \text{Det}(y-X) = y^N + \sum_{i=0}^{N-1} y^i a_i,
\]

where \( y \) is a complex indeterminate. Due to (2.3), we have \( \partial_\bar{z} a_i = 0 \) which means that the set of functions \( \{a_i\} \) are analytic on the complex plane,\(^2\) although they are allowed to have poles at \( z = 0 \) and \( z = \infty \). Therefore the equation

\[
P_X(y) = 0 \quad \text{(2.5)}
\]

identifies in the \((z,y)\) space (i.e. \( \mathbb{C}^* \times \mathbb{C} \), but often in the following for simplicity we replace it with the affine space \( \mathbb{C}^2 \)) a Riemann surface \( \Sigma \), which is an \( N \)-sheeted branched covering of the cylinder. For later use we recall that eq. \( P_X(y) = 0 \) is tantamount to considering the matrix equation

\[
X^N + a_{N-1}X^{N-1} + \cdots + a_0 = 0. \quad \text{(2.6)}
\]

The explicit form of the covering map is given by the set \( \{x^{(1)}(z), \ldots, x^{(N)}(z)\} \) of eigenvalues of \( X \). Each eigenvalue spans a sheet. The projection map to the base cylinder \( \mathcal{C} \) will be denoted \( \pi : \Sigma \to \mathcal{C} \). The points where two or more eigenvalues coincide are called branch points. The identification cuts in the sheets start or end at these points. We would like to warn the reader that with the term ‘branch point’ we denote, in general, both the points on the covering and their image in \( \mathcal{C} \) under the projection \( \pi \). The context should make clear which exactly we refer to.

A diagonalizable matrix \( M \), solution of eq. (2.6), can always be cast in canonical form

\[
M = \begin{pmatrix}
-a_{N-1} & -a_{N-2} & \cdots & \cdots & -a_0 \\
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}. \quad \text{(2.7)}
\]

The branched covering structure is completely encoded in the \( \{a_i\} \) analytic functions and we have already stressed that it is independent of the coupling \( g \).

\(^2\)Notice the change of convention with respect to [11, 12], which amounts to the exchange \( z \leftrightarrow \bar{z} \). This is in order to refer to the \( a_i \)'s as analytic functions, rather than antianalytic.
2.4 String interpretation

This is a good point for reviewing the string theory interpretation of solutions \((X, A_w)\) of (2.2), (2.3) that we presented in [11, 12], while enriching it with new remarks pertinent to the present paper. We recall that we interpret the Riemann surface defined by the relevant branched covering of the cylinder as the classical carrier of a string process. The branch points at \(z \neq 0, \infty\) represent joining and splitting processes of the string. Generically, when the branch point is simple, we have the joining of two strings to form a unique string or the splitting of one string into two. We may also have multiple branch points in which more then two incoming or outgoing strings are involved. However in this paper, contrary to [11, 12], the emphasis will be on simple branch points.4

The inverse images under \(\pi\) of \(z = 0, z = \infty\) are punctures in \(\Sigma\) with a definite string interpretation: they represent incoming and outgoing strings, respectively. More precisely they represent the points where incoming strings enter (outgoing strings leave) the process represented by the Riemann surface \(\Sigma\).

It has to be kept in mind that, in MST, the counterimages of \(z = 0\) and \(z = \infty\) are distinguished points with an associated physical meaning. This is to be contrasted with the usual mathematical treatment of branched coverings of \(\mathbb{C}P^1\), where these points do not play any particular role. This remark will become extremely important below, in connection with the discussion about moduli space.

Let us discuss further properties of the punctures corresponding to \(z = 0\) (an analogous discussion holds for \(z = \infty\)), see for example [11]. The counterimages of \(z = 0\) by \(\pi\) may be \(N\) distinct points, i.e. the solutions of the algebraic equation (2.5) at \(z = 0\) may be all distinct. In such a case we say we have \(N\) small incoming strings (of length 1 each). However, in general, the inverse image of \(z = 0\) may contain several branch points \(P_1^{in}, \ldots, P_s^{in}\), with multiplicity \(l_1 - 1, \ldots, l_s - 1\), respectively (if \(z = 0\) is a singular point of the eq. (2.5) it has to be desingularized first, see below). In this case the process represented by \(\Sigma\) involve \(s\) incoming strings of length \(l_1, \ldots, l_s\), respectively. The physical interpretation of the string length has been given in [4]. In the framework of the light-cone quantization of type IIA superstring, the string length is identified with the momentum component \(p^+ = p^9 + p^0\) of the string in suitably normalized units. Here 0, 9 are of course the time and longitudinal direction of the ambient space, not explicitly appearing in (2.1).

Let us summarize the string interpretation of the solutions of (2.2), (2.3). Each such solution is characterized by a punctured Riemann surface realized as a branched covering of the cylinder. The punctures represent the sites where strings enter or leave the interaction process. The length of each string is associated to its \(p^+\) momentum component. This is the picture of MST at strong coupling. At finite coupling \(g\) the string interpretation of the

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3The multiplicity or ramification index of a branch point is defined as number of sheets which come together at that point, minus one; therefore branch points that involve only two sheets are called simple.

4This is to simplify the exposition. Multiple branch points at \(z \neq 0, \infty\) can also be analyzed. They define anyhow lower dimensional subspaces of the moduli space and can be obtained as limiting cases of simple branch points.
instantons persists, but the dressing factor $Y_d$ has the effect of blurring it by smearing the string interactions and so on.

Now, the question arises of how to transform this scenario into an effective calculational tool. One definite suggestion has been made in [12]. One considers (2.1) and expands it about a classical solution which spans a given string process (Riemann surface with punctures), then takes the $g \to \infty$ limit. It turns out that the dressing factor of the instanton disappears and the unitary factor can be gauged away. The strong coupling limit turns out to be well defined and, practically, to the lowest order in $1/g$, the background fields disappear, except for the notion of the covering surface $\Sigma$. This can be seen by splitting all the matrix fields into Cartan (diagonal) and non-Cartan modes. A suitable gauge fixing makes it clear that the non-Cartan part can be integrated out, while the Cartan modes can be interpreted as fields on $\Sigma$. The final upshot is that MST at strong coupling reduces to the Green-Schwarz superstring theory plus a decoupled Maxwell theory, and that the leading term of the amplitude in such background is proportional to $g_s^{-\chi}$, where $g_s = g^{-1}$ is the string coupling constant and $\chi = 2 - 2h - n$ is the Euler characteristic of $\Sigma$ if its genus is $h$ and its punctures are $n$.

Let us denote, as in [12], the fields in $\Sigma$ with a tilde: $\tilde{x}, \tilde{a}, \tilde{\theta}, \tilde{c}$ are the small oscillations of $X, A, \theta$ and the ghost $c$, respectively, which survive in the strong coupling limit. Then, in order to describe a complete string process, we can introduce the vertex operators $V_1, \ldots, V_n$ corresponding to $n$ incoming and outgoing strings, expressed in terms of $\tilde{x}, \tilde{\theta}$, possibly the superstring reparametrization ghosts, and of the string transverse momenta, and insert them into the path integral. The amplitude (in the strong coupling limit) will schematically be:

$$
\langle V_1, \ldots, V_n \rangle_h = g_s^{-\chi} \int d^2 \Sigma_{\chi, h} \prod V_1 \ldots V_n e^{-S_{GS} - S_{Maxwell}}, \quad (2.8)
$$

where most of the symbols are the same as in [12]. Here we have singled out the integration over $\mathcal{M}^{(h, n)}_N$, namely over all distinct instantons which underlie the given string process for fixed $N$, that is to say with assigned incoming and outgoing strings and string interactions. In ordinary string interaction theory $\mathcal{M}^{(h, n)}_N$ is nothing but the moduli space of Riemann surfaces of genus $h$ with $n$ punctures, a complex space of dimension $3h - 3 + n$. What actually $\mathcal{M}^{(h, n)}_N$ is in MST is the main subject of the present paper.

2.5 Plane curves and moduli space

So far we have tacitly given for granted that any Riemann surface with $h$ handles and $n$ punctures can be represented as a branched covering of $\mathbb{CP}^1$. This point has to be carefully handled.

Let us start from eq. (2.5). This is the equation of a curve in the affine space $\mathbb{C}^2$ spanned by the two complex coordinates $y$ and $z$. A curve embedded in a two-dimensional affine space is called a plane curve. Therefore MST dynamically engenders Riemann surfaces in the form of plane curves.
An important role in the following is played by singular plane curves. If $P(y, z) = 0$ is the polynomial equation of the curve, a singular point is a point where $P(y, z) = \partial_y P(y, z) = \partial_z P(y, z) = 0$. When no such points are present the curve is smooth. However this can happen only when its genus is $\frac{1}{2}(d - 1)(d - 2)$, where $d$ is the degree of the curves, i.e. the degree of the polynomial $P(y, z)$: we will see later on that in our case the degree is $N$, see section 4. We can lower the genus of the plane curve, while keeping the degree constant, by allowing for singularities. This means two important things: first, for finite $N$ there exists an upper bound ($\frac{1}{2}(N - 1)(N - 2)$) on the genus of the Riemann surfaces which define the core of the stringy instantons; second, far from discarding singular curves, as one would be tempted to do as a first approach, we have to take them into account, they are bound to fill up most of the moduli space of plane curves. As we will see, singular curves are a happy occurrence, not a nuisance.

The formulation of plane curves in terms of the variables $y$ and $z$ is at times ambiguous, especially in connection with singularities. A possible way to resolve such ambiguities is to embed them in the projective space $\mathbb{P}^2$ by introducing three homogeneous coordinates $x_0, x_1, x_2$. Simply set $z = x_1/x_0, y = x_2/x_0$ in eq. (2.5), and multiply by a suitable power of $x_0$. The new equation refers to the same curve embedded in $\mathbb{P}^2$. This is a ‘cleaner’ representation and the one we use in the following. Singular curves can be desingularized, for example by (repeatedly) blowing up the singular points until we reach smooth configurations. However the smooth curves one obtains this way are not plane curves. In order to be represented as algebraic curves they need to be embedded in a larger affine or projective space. It is well-known that any compact Riemann surface can be represented as a smooth algebraic curve embedded in $\mathbb{P}^3$. However, if we try to project it to $\mathbb{P}^2$, i.e. to represent it as a plane curve, we are bound to produce singularities. The lesson we learn is that we have to count mostly on singular plane curves in order to reproduce the moduli space of Riemann surfaces which is needed in string interaction theory. There is nothing arbitrary about this: all the information we need to reproduce string theory (topology and moduli) is contained in the singular curve: if we know the singular curve we can reproduce a smooth counterpart with a standard algorithm. Once this point about plane curves is clarified, we will not talk about singular or regular plane curves but simply about plane curves. Plane curves will be discussed in detail in section 3.

At this point it would seem that we are done: a theorem by Clebsch, [22], guarantees that any (compact) algebraic curve is birationally equivalent to a plane algebraic curve which has at most ordinary double points as singularities. However this is too simplistic. Beside the upper bound for the genus we have mentioned above, we should remember that in our case we do not have to do with compact Riemann surfaces, but with Riemann surfaces with a certain number of punctures. Therefore the above theorem is not conclusive. Actually we will find out in section 4 that the presence of punctures on the Riemann surface entails the consequence that the *moduli space of plane curves of genus $h$ with $n$ punctures is a discretized version of the the moduli space of genus $h$ Riemann surfaces with $n$ punctures*, whose complex dimension is $3g - 3 + n$. A good parametrization of the moduli space, fit for string interaction theory, is provided by Mandelstam’s variables [29, 28, 30, 31]. By
making a comparison with this parametrization, we will find out that \( h \) of the Mandelstam complex parameters are actually discrete for the plane curves that appear in MST.

This point is rather technical and we only have a technical explanation for it (section 3). Its origin can be briefly described as follows. The coordinate \( z \) we have introduced above, can be naturally regarded as a meromorphic function on a given plane curve (it is a realization of the projection \( \pi : \Sigma \rightarrow \mathbb{C} \)). The counterimages of \( z = 0 \) and \( z = \infty \) form a principal divisor in \( \Sigma \). This entails, by Abel’s theorem, \( h \) discretizing conditions on the parameters describing the plane curve. A detailed analysis shows that this imposes \( h \) of the Mandelstam parameters to be discrete. A confirmation of this result comes from an estimate of the moduli space of stringy instantons. Since in the \( Y \) factor there no free parameters, the moduli space of stringy instantons must coincide with the free parameters contained in \( M \), i.e. with the moduli space of plane curves. The estimate carried out in section 3 confirms the above evaluation of the continuous dimension of the moduli space of the latter.

However when \( N \rightarrow \infty \) these discrete parameters become continuous and, in addition, the upper limit on the genus we mentioned above becomes ineffective. Therefore for large \( N \) MST recovers the full moduli space of string theory. We recall that, for finite \( N \), also the \( p^+ \) components of the momenta of the incoming and outgoing strings are discrete and continuity is recovered only for \( N \rightarrow \infty \). Therefore, a complete description of string interaction theory is truly achieved by MST only in the large \( N \) limit. It is nevertheless remarkable that genus 0 processes (with discrete \( p^+ \) components of the external momenta) are exactly described by MST also for finite (but large enough) \( N \).

2.6 Singularities and space-time dimensions

Singularities of plane curves provide a striking suggestion of how to resolve an unsatisfactory aspect of the correspondence between MST and string theory. In (2.1) we see eight ambient space dimensions and two world-sheet dimensions of the cylinder. We should now remember that the correspondence MST — string theory is established in the light-cone gauge. Therefore the two world-sheet dimensions are nothing but representatives of the time and longitudinal dimensions, denoted 0 and 9, which bring the total of physical dimensions to ten. Now, stringy instantons seem to span four out of these ten dimensions. In other words it would seem that MST at strong coupling can only describe four-dimensional string processes. However this is strictly true only for processes which are mediated by smooth plane curves. As we have pointed out, singular curves become smooth if we enlarge the space where they are embedded. For example, curves in \( \mathbb{C}P^2 \) with nodes only, can be smoothed out by embedding them in \( \mathbb{C}P^3 \), that is by adding two dimensions, and so on. We interpret this by saying that the corresponding string process extend over six (instead of four) dimensions. It is not difficult to envisage processes that extend over ten dimensions. One can phrase it as follows: all these high dimensional processes are squeezed to four dimensions in order to fit into the instantons of the 2d field theory (2.1), and this projection gives rise to singularities.
3. The unitary and dressing factors

As explained in the previous section, the MST instantons consist of two pieces: a group theory factor $Y$ and a branched covering (plane curve) parametrized by the matrix $M$. In section 4 and 5 we will discuss the latter. Let us now concentrate on the former. In the derivation of the results of ref. [12] it is of paramount importance that $Y$ splits according to $Y = Y_s Y_d$, where $Y_d \rightarrow 1$ as $g \rightarrow \infty$ outside the string interaction points (branch points at $z \neq 0, \infty$), and $Y_s$ is a matrix independent of $g$ such that $Y_s^{-1} M Y_s$ and $Y_s^\dagger M^\dagger (Y_s^\dagger)^{-1}$ commute.

In this section we prove that the above structure of the $Y$ factor holds for a general covering. In other words, we start from a general matrix $M$ (2.7), construct the corresponding $Y$ factor and show that it satisfies the requirements we have just mentioned.

3.1 The unitary factor

The factor $Y_s$ can be constructed as follows. It is well-known, [9], that the matrix $M$ can be diagonalized

$$M = S D S^{-1}, \quad D = \text{Diag}(\lambda_1, \ldots, \lambda_N)$$

by means of the following matrix $S \in SL(N, \mathbb{C})$:

$$S = \Delta^{-1/2} \begin{pmatrix} \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \cdots & \lambda_N^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

where

$$\Delta = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j).$$

We now introduce a matrix $K$ such that $U = KS$ is unitary. If such a matrix exists then $K M K^{-1} = U D U^{-1}$ and $(K^\dagger)^{-1} M^\dagger K = U D^\dagger U^{-1}$ do commute. Then $K$ can play the role of $Y_s^{-1}$. We refer to $U = KS$ as the unitary factor in the construction of the background solution $X$.

One such $K$ can be constructed with the Gram-Schmidt procedure. The result is the following upper triangular matrix belonging to $SL(N, \mathbb{C})$:

$$K = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ 0 & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{NN} \end{pmatrix},$$

where

$$k_{pp} = \frac{Q^{(N-p)}}{Q^{(N-p+1)}} |\Delta|^{1/2}, \quad 1 \leq p \leq N$$

and

$$k_{pq} = \frac{Q_{p,N-p}^{(N-p+1)}}{Q^{(N-p)}Q^{(N-p+1)}} |\Delta|^{1/2}, \quad 1 \leq p < q \leq N - 1.$$
The symbols in the previous formulae have the following meaning. For $2 \leq k \leq N$ we set

$$Q^{(k)} = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq N} \left( \prod_{1 \leq l \leq k-1} |\lambda_{j_l} - \lambda_{j_{l+1}}|^2 |\lambda_{j_l} - \lambda_{j_{l+2}}|^2 \ldots |\lambda_{j_l} - \lambda_{j_k}|^2 \right),$$

$$Q^{(k)}_{n,m} = (-1)^{n+m} \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq N} S_{k-1-n}(\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_{k-1}}) S_{k-1-m}(\bar{\lambda}_{j_1}, \bar{\lambda}_{j_2}, \ldots, \bar{\lambda}_{j_{k-1}}) \times$$

$$\times \left( \delta_{k,2} + \prod_{1 \leq l \leq k-2} |\lambda_{j_l} - \lambda_{j_{l+1}}|^2 |\lambda_{j_l} - \lambda_{j_{l+2}}|^2 \ldots |\lambda_{j_l} - \lambda_{j_{k-1}}|^2 \right), \quad 0 \leq n \leq m \leq N - 1$$

where $S_p(x_1, \ldots, x_s)$ denotes the elementary symmetric polynomial of order $p$ in $x_1, \ldots, x_s$ $(s \geq p)$:

$$S_p(x_1, \ldots, x_s) = \sum_{1 \leq j_1 < j_2 < \ldots < j_p \leq s} x_{j_1} \ldots x_{j_p}, \quad S_0 = 1. \quad (3.6)$$

Moreover, by definition, $Q^{(0)} = 1, Q^{(1)} = N$. Notice that

$$Q^{(N)} = |\Delta|^2, \quad k_{11}k_{22} \ldots k_{NN} = 1.$$  

The $Q^{(k)}_{n,m}$ are homogeneous polynomials of order $\frac{k(k-1)}{2} - n$ in the variables $\lambda_i$ and of order $\frac{k(k-1)}{2} - m$ in the complex conjugates $\bar{\lambda}_i$. $Q^{(k)}$ are homogeneous polynomials of order $\frac{k(k-1)}{2}$ in both $\lambda_i$ and $\bar{\lambda}_i$.

For example, in the case $N = 3$, the matrix $K$ is given by

$$K = \begin{pmatrix}
\frac{\sqrt{\sum_{i<j} |\lambda_i - \lambda_j|^2}}{|\Delta|^\frac{3}{2}} & -\frac{\sum_{i<j} (\lambda_i - \lambda_j)(\lambda_i - \lambda_j)^2}{\sqrt{\sum_{i<j} |\lambda_i - \lambda_j|^2}|\Delta|^\frac{3}{2}} & \frac{\sum_{i<j} |\lambda_i\lambda_j| |\lambda_i - \lambda_j|^2}{\sqrt{\sum_{i<j} |\lambda_i - \lambda_j|^2}|\Delta|^\frac{3}{2}} \\
0 & \frac{\sqrt{\sum_{i<j} |\lambda_i - \lambda_j|^2}|\Delta|^\frac{3}{2}}{\sum_{i<j} |\lambda_i - \lambda_j|^2} & -\frac{\sum_{i<j} \lambda_i |\lambda_i - \lambda_j|^2}{\sum_{i<j} |\lambda_i - \lambda_j|^2 |\Delta|^\frac{3}{2}} \\n0 & 0 & \frac{|\Delta|^\frac{3}{2}}{\sqrt{3}}
\end{pmatrix}, \quad 1 \leq i, j \leq 3.$$  

This completes the construction of $K$. We remark that generally the entries of $U = KS$ contain as a factor some fractional power of $|\Delta|$. Therefore they may vanish or diverge with some fractional power whenever two of the eigenvalues of $M$ coincide. This corresponds to a simple branch point in the spectral covering, as we have seen above.\footnote{The entries of $K$ contains other factors, beside $\Delta$, that may vanish when more than two eigenvalues coincide. This corresponds to multiple branch points, which we disregard in this paper.} Outside these points the unitary factor $U$ is smooth. It is therefore justified to get rid of it by a gauge transformation, as we have done in \cite{[12]}.  

### 3.2 The dressing factor

Let us come now to the $Y_d$ factor. We have just noted that the entries of $K$ are generically singular whenever two eigenvalues of $M$ coincide, that is at the site of a branch point of the covering. Then $KMK^{-1}$ shares the same singularities and it is not a satisfactory
ansatz for our solution $X$ of \((2.2), (2.3)\), which we want to be everywhere smooth (except perhaps at $w = \pm\infty$). To this end we introduce a new matrix $L$ with the requirement that $L K M K^{-1} L^{-1}$ is smooth and is the desired solution $X$ of \((2.2),(2.3)\). While $K$ is independent of $g$, $L$ will depend on $g$. $L^{-1}$ is our candidate for the dressing factor $Y_d$.

Since $L$ has to smooth out the singularities of $K$, it is enough to take for it an upper triangular matrix belonging to $\text{SL}(N, \mathbb{C})$. A possible parametrization for $L$ is the following

$$
L = \begin{pmatrix}
e^{u_1} & e^{u_1} \psi_{12} & e^{u_1} \psi_{13} & \ldots & e^{u_1} \psi_{1N} \\
0 & e^{u_2-u_1} & e^{u_2-u_1} \psi_{23} & \ldots & e^{u_2-u_1} \psi_{2N} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & e^{u_{N-1}-u_N-2} & e^{u_{N-1}-u_N-2} \psi_{N-1N} \\
0 & 0 & \ldots & 0 & e^{-u_{N-1}}
\end{pmatrix}.
$$

(3.7)

The fields $u$ and $\psi$ have to satisfy certain differential equations in order to comply with the requirements. We just plug the ansatz for $X$ (and the connection) into \((2.2)\) and work out the relevant equations. We will not write them down here. A few examples were given in \([11,12]\). They are all equations of the WZNW type and can be cast in the general form

$$
\partial_w \partial_{\bar{w}} \phi + \ldots \sim \partial_w \partial_{\bar{w}} \ln |\Delta| = \frac{\partial \Delta}{\partial w} \frac{\partial \bar{\Delta}}{\partial \bar{w}} \delta(\Delta),
$$

(3.8)

where $\phi$ denotes any field $u$ or $\psi$, while dots represent all the other terms, which are irrelevant in the cancellation of singularities. Let us refer to these equations as the ‘dressing equations’. On the right-hand side we see the typical delta-function type source which characterizes them. The sources are point-like and located at the zeroes of $\Delta$, that is at the branch points of the covering.

Naturally the solution $X$ exists with the required properties only if the ‘dressing equations’ admit solutions that vanish at $w = \pm\infty$. To our best knowledge, not much is known in the literature concerning the existence of such solutions. Some simple cases were discussed in \([12]\), where we also presented a few numerical solutions. We deem these sufficient for us to assume that the ‘dressing equations’ do admit solutions that vanish at $w = \pm\infty$. Once one assumes this, it is rather easy to argue, on a completely general ground, that in the strong coupling limit, $g \to \infty$, such solutions vanish outside the zeroes of the discriminant.

The argument goes as follows. Consider a candidate solution of \((2.2)\) in which $u = \psi = 0$ outside the zeroes of the discriminant. Then, there, $L = 1$, and $X = KMK^{-1}$. As we have already noticed in section \(2\), in such a situation $[X, \bar{X}] = 0$, since both $X$ and $\bar{X}$ are simultaneously diagonalized by the matrix $U = KS$. This is most welcome since, if $[X, \bar{X}]$ were not to vanish (outside the zeroes of the discriminant), it would be impossible to have any finite solution of \((2.2)\). Next, we have to show that also $F_{\bar{w}w}$ vanishes in the same region when $L = 1$. In fact when $L = 1$,

$$
A_w = -iK \partial_w K^{-1} = -i(KSS^{-1}) \partial_w (SS^{-1}K^{-1}) = -iU(\partial_w + \bar{A}_w)U^{-1},
$$

where $\bar{A}_w = S^{-1} \partial_w S$. But $\partial_w S \equiv 0$ due to holomorphicity of the eigenvalues of $M$. In conclusion \((2.2)\) is identically satisfied by the ansatz $L = 1$ outside the zeroes of the
discriminant. Since the solutions are uniquely determined by their boundary conditions, we can conclude that, as $g \to \infty$, the only solution of the dressing equations outside the zeroes of the discriminant, is the identically vanishing solution. We infer from this argument that the solutions of the dressing equations for large $g$ are concentrated around the branch points and become more and more spiky as $g$ grows larger and larger. Therefore the matrix $L$ is just the dressing factor $Y_d$.

This conclusion, as pointed out in section 2, justifies our approach in [12]. We warn the reader, however, that the above argument is not entirely satisfactory, for it assumes that, as $g \to \infty$, the only possibility is $[X, \bar{X}] = 0$ and $F_{w\bar{w}} = 0$. One can envisage other types of solutions. For instance, let $A_w$ and $X$ be any $g$-dependent solution considered so far. Then call $A_w^{(1)}$, $X^{(1)}$ their value at $g = 1$. The latter satisfy (2.2), (2.3) for $g = 1$. Now set $X_g = X^{(1)}/g$. Then $A_w^{(1)}$, $X_g$ satisfy the same equation, but, as $g \to \infty$, $X_g \to 0$ while $F_{w\bar{w}}^{(1)} \neq 0$. Therefore the eigenvalues of $X_g$ for $g \to \infty$ do not describe any asymptotic string configuration.

We call stringy instantons those solutions of the Hitchin equations (2.2), (2.3) for which at $g = \infty$ we have both $[X, \bar{X}] = 0$ and $F_{w\bar{w}} = 0$. In this paper we consider only this kind of solutions.

4. Branched coverings of Riemann surfaces and plane curves

This section is devoted to the detailed analysis of the stringy instanton core, i.e. of branched coverings of the Riemann sphere. The latter appear in MST as solutions of affine equations

$$P(y, z) \equiv \sum_{p,q} a_{p,q} y^p z^q = 0, \quad (y, z) \in \mathbb{C}^2,$$

where $P$ is a polynomial of degree $N$. Actually this does not follow immediately from what we said in section 2.2. From eq. (2.5) it follows that $P_X$ has degree $N$ in $y$, but the $a_i(z)$’s could be any analytic functions on the punctured Riemann sphere. This means that they could be expressed by means of Laurent series in $z$. However in order to preserve the string interpretation we will limit ourselves to $a_i(z)$’s which are Laurent polynomials and, for simplicity, in this paper we will explicitly consider only $a_i(z)$’s which are polynomials in $z$ in such a way that $P(y, z)$ has overall degree $N$.

The locus in $\mathbb{C}^2$ of the solutions $(y, z)$ of (4.1) is a plane curve. The independent non-vanishing coefficients $a_{p,q}$ can be varied without changing, in general, the topological type $(h,n)$ of the curve. They are the moduli of the plane curve. Counting them is an exercise we have to do in order to see whether the moduli space of MST coincides with the moduli space of IIA superstring theory.

The literature on plane curves, and, more generally, on algebraic curves is vast (see for instance [22, 23, 24]), and we will be using many well-known results. However one should bear in mind a peculiarity of our problem which is not usually considered in the textbooks on the subject. This is the question of punctures, which has already been introduced in section 2.4. Let us discuss it now in more detail.
4.1 Punctures on plane curves

Punctures are the sites on the embedded Riemann surface, that is on the corresponding plane curve, where the incoming strings enter and the outgoing strings exit. They are the counterimages by $\pi$ of $z = 0$ and $z = \infty$, respectively. If any such point on the plane curve is a branch point of multiplicity $l - 1$, then the corresponding incoming or outgoing string has length $l$. Incidentally, since eventually we want to take the large $N$ limit, we are especially interested in the case when $l$ is comparable with $N$. In the ordinary treatment of compact Riemann surfaces, if these points are regular, they are in no way special and must be considered on the same ground as all the other regular points (this can be seen for example by using projective coordinates). In our approach, this is not allowed. As we have already pointed out in section 2 the length of an incoming or outgoing string is interpreted as the $+\,$ component of the momentum in the light-cone framework. Therefore, non only the locations of the branch points in the inverse image of $z = 0, \infty$, but also their multiplicities have a precise physical meaning. Two processes that differ by these multiplicities must be kept distinct, even if, say, the topological type is the same.

Let us see an example. Suppose $y = y_1$ is a branch point of multiplicity $l - 1$ in the counterimage of $z = 0$. This means that we have $l$ roots of (2.5). For example,

$$y^{(i)} \sim y_1 + \eta^i z^{1/l}, \ i = 0, \ldots, l - 1$$

and $\eta = \exp(2\pi i/l)$. In other words $l$ sheets of the covering join along a cut starting at $y_1$. The counterimage of a circle around $z = 0$ in the $z$-plane contains a curve around $y = y_1$ on the covering that closes after crossing the cut $l$ times, i.e. we have an incoming string of length $l$. Therefore an easy rule to compute the length of an asymptotic string at a branch point in the inverse image of $z = 0$ is to count the number of sheets that meet there. Alternatively such length can be seen as the period of the differential $d\ln z$ around the point $y = y_1$ of the covering. In fact $y - y_1$ is a good coordinate near $y_1$ and $d\ln z = k \, d\ln(y - y_1)$. The same conclusion can be drawn if the roots are like $y^{(i)} \sim y_1 + \eta^i z^{j/l}$, where $j$ and $l$ are relatively prime integers. A similar discussion can be carried out for the counterimages of $z = \infty$ as well.

At this point it is convenient to mention the concept of tameness. Tameness was introduced by C.T.Simpson [24, 27]. It means the following. Eq. (2.5) is an algebraic equation of order $N$ in $y$; if one solves for $y$ one gets $N$ (possibly coincident) roots as functions of $z$, whose behaviour near $z = 0$ and $z = \infty$ will be dominated by some (in general, fractional) power of $z$. We say that the curve (more properly, the corresponding Hitchin system) is tame if these roots have at most a simple pole at $z = 0, \infty$. Tameness guarantees the existence of a well-behaved bundle metric in the $V$ bundle mentioned in section 3.1.

If tameness is necessary from a mathematical point of view, there does not seem to be any physical motivation for it. Let us remark that a process is tame if the roots at the punctures behave as $y \sim \text{const} + \zeta^{j/l}$, where $\zeta$ is the appropriate local coordinate at $z = 0, \infty$ and $|j/l| \leq 1$. Now suppose that one of the roots of (2.5) is not branched at $y_1$.

---

6In all the figures below we show the incoming and outgoing strings not as punctures, but as macroscopic strings in order to stress their different lengths.
which lies in the counterimage of \( z = 0 \), and behaves like \( y_1 + z^2 \). This process is not tame but it represents an allowed string configuration. Anyhow, in the light of the large \( N \) limit, the problem of untameness becomes somewhat irrelevant. This is the reason why in this paper we consider only tame curves.

4.2 Some examples

Before we continue the general discussion of plane curves, let us present some concrete examples of cases which are not unfamiliar in the physical literature.

We would like first to describe in detail how the genus zero (tree level) string interactions can be reproduced with a suitable form of the coefficients in the spectral equation (2.5) or (4.1). In the genus zero sector any Riemann surface \( \Sigma \) is a punctured sphere, realized as a \( N \)-fold branched cover of the \( z \)-sphere.

Assume we have \( n \) incoming and \( n' \) outgoing strings of lengths \( l_i \) and \( l'_j \), \( (i = 1 \ldots n, j = 1 \ldots n') \), respectively (see the figure). From a physical point of view, we have seen that the length of a string is interpreted as the + component of its light cone momentum. We recall that the relation

\[
\sum_i l_i = \sum_j l'_j = N,
\]

must hold due to conservation of the momentum. We have also seen that the length of an incoming string \( l_i \) being \( l_i > 1 \) means that the cover has a branch of order \( l_i - 1 \) at \( z = 0 \), and likewise for outgoing strings at \( z = \infty \).

Our aim here is to construct a polynomial \( P \) which underlies such a string process. Let us tackle this problem by studying the \( N \)-fold cover as a holomorphic projection from \( \Sigma \) to \( \mathbb{CP}^1 \). As we have already noticed, the coordinate \( z \) does represent such a projection as a meromorphic function on \( \Sigma \): punctures manifest themselves as zeroes or poles of appropriate orders \( l_i, l'_j \). The condition (4.2) means in this picture that the number of zeroes minus the number of poles, with multiplicity, is zero (this is the degree of the divisor).

Proceeding in this direction, we construct the generic meromorphic function in terms of a global coordinate on \( \Sigma \), which we can take to be \( y \) itself. This is a useful simplification, which is not possible in higher genus cases.

The generic meromorphic function satisfying the above requirements on zeroes and poles is given by the following rational map:

\[
z = K \frac{(y - y_1)^{l_1} (y - y_2)^{l_2} \cdots (y - y_n)^{l_n}}{(y - y'_1)^{l'_1} (y - y'_2)^{l'_2} \cdots (y - y'_{n'})^{l'_{n'}}}.
\]

This map depends on \( n + n' \) parameters, in addition to the constant \( K \): it fixes the \( n + n' \) punctures on \( \Sigma \) to be located at the points \( y_i \) and \( y'_j \). The case of \( y_i \) or \( y'_j = \infty \) is a limiting
case of the above formula when the relevant factor is absent. Let us verify that \( (\ref{4.3}) \) gives the right behaviour at \( z = 0 \) and \( z = \infty \), see \( (\ref{1.1}) \). An example will suffice. Near \( y_1 \) we can write \( z \sim (y - y_1)^{l_1} \), therefore \( y \sim y_1 + z^{1/l_1} \), which is exactly the behaviour considered above.

Now we can make a first exercise of moduli counting. Let us recall that the moduli space of the Riemann sphere with \( p \) punctures is \( p - 3 \). To count the moduli in \( (\ref{4.3}) \), we first notice that we have \( n + n' + 1 \) free parameters. Of these, \( K \) corresponds to a rescaling of the \( z \) coordinate; then we can use \( PSL(2, \mathbb{C}) \) to reabsorb three parameters among the \( y_i, y'_j \). As a result the meromorphic function describes spheres with \( n + n' - 3 \) moduli, as expected.

Now, in order to see whether these curves are reproduced within MST, we try to cast \( (\ref{4.3}) \) in the form \( (\ref{2.6}) \). One sees immediately that \( (\ref{2.6}) \) corresponds to curves where one of the outgoing punctures is at infinity, say \( y'_1 = \infty \). Given that, the above map is indeed of the form of \( (\ref{2.6}) \) with coefficients \( a_i \) which are at most linear in \( z \):

\[
y^N + a_{N-1}y^{N-1} + \cdots + a_0 = 0, \quad a_i = \alpha_i z + \beta_i.
\]

(4.4)

The generic polynomial of this form corresponds to a curve which has all \( l, l' = 1 \), i.e. it has no branches at the \( 2N \) punctures, and depends on \( 2N \) parameters. Of these, three can be ignored, since they correspond to transformations that leave \( y'_1 = \infty \): a rescaling of \( z \); a shift of \( y \) and a rescaling of \( y \). They are the remnant of \( PSL(2, \mathbb{C}) \) which keeps \( y'_1 = \infty \).

Therefore \( (\ref{2.6}) \), or \( (\ref{4.1}) \), contains the right \( 2N - 3 \) moduli of spheres.

The cases when some punctures are branched, are limiting cases of the previous curve when two or more punctures coincide. This can be easily seen from the meromorphic map \( (\ref{4.3}) \). Therefore, for each \( l_i > 1 \), we have to enforce \( l_i - 1 \) conditions on the parameters \( \alpha_i, \beta_i \) of the spectral equation. Thus the free parameters are, as expected:

\[
2N - \sum_{i=1}^{n} (l_i - 1) - \sum_{j=1}^{n'} (l'_j - 1) - 3 = n + n' - 3.
\]

We conclude that at genus zero MST reproduces, via \( (\ref{1.1}) \), the full \( n + n' - 3 \) moduli.

In the case of curves with non-vanishing genus, one would be tempted to proceed in the same way, that is to construct the meromorphic projection \( \Sigma \rightarrow \mathbb{C} \mathbb{P}^1 \) and then invert it. It is rather easy to construct the meromorphic function \( z \) at genus 1. However we come immediately across a novel feature which was absent in genus 0, but has dramatic consequences for the moduli counting.

The point is that the punctures on \( \Sigma \), represented as zeroes and poles of the meromorphic function, cannot be arbitrary. This is a feature of the torus and of higher genus curves. There is a condition that they have to satisfy, which is the price we have to pay to be able to represent the punctured surface as an algebraic curve. Mathematically speaking, the divisor of a meromorphic function, is not a generic divisor of degree zero, but is a principal
one, which amounts to some extra condition on the punctures. The same condition was absent on the Riemann sphere because there every divisor of degree zero is principal.

To see which condition appears, let us represent explicitly the meromorphic function using a coordinate $t$ taking values in the fundamental parallelogram. On a torus a meromorphic function can be written as the ratio of products of “translated” theta functions:

$$z = K \frac{\theta(t-t_1)^{l_1} \theta(t-t_2)^{l_2} \cdots \theta(t-t_n)^{l_n}}{\theta(t-t'_1)^{l'_1} \theta(t-t'_2)^{l'_2} \cdots \theta(t-t'_{n'})^{l'_{n'}}},$$  

(4.5)

(see the end of section 5.1 for details). Now, for $z$ to be single valued, the $t_i$, $t'_j$ have to satisfy a condition, that is the vanishing of the Abel-Jacobi map:

$$\sum_i l_i t_i - \sum_j l'_j t'_j = 0 \mod \Gamma.$$  

(4.6)

where $\Gamma$ is the group of periods, which for the torus is the usual lattice of complex translations: $\Sigma = \mathbb{C}/\Gamma$.

It is instructive to look at the case of the propagator of a long string at genus one. We require the insertion on the torus of an incoming and an outcoming string of length $l$, at two points. By translation we can bring one of them at the origin and the other at, say, $\bar{t}$. The above condition is in this case:

$$l \bar{t} = 0 \mod \Gamma,$$  

(4.7)

and we see that $\bar{t}$ has to lie on the lattice $\Gamma/l$ indicated in the picture.

We can see from here that at finite $N$ we have some limitations on the possible diagrams we can realize, however as $l$ and $N$ become large, the lattice $\Gamma/l$ fills the plane and we recover the continuous modulus.

In section 5 we will discuss in general the limitations of this kind. Therefore we leave this subject at this point and discuss other aspects concerning genus one curves.

The next thing we would like to do is to mimic the genus 0 case by inverting eq. (4.5). This is certainly possible locally, but, unlike the genus 0 case, we will not find in general a polynomial equation of the type (2.5). Therefore constructing the meromorphic projection (4.5) gives us only limited information about plane curves. In fact, what one expects is that the plane curve corresponding to (4.5) is in general in a singular representation (see below).

It is then necessary to study singular plane curves.

### 4.3 Plane curves and their representation

At the beginning of this section we have called plane curves the locus of points which are solution of an equation like (4.1) in $\mathbb{C}^2$. This definition is too generic and lends itself to
ambiguities. For example, we know the coordinates $y$ and $z$ are not on the same footing in MST. A $z$ rescaling (at strong coupling) is a symmetry of any process in MST, but no other $pSL(2, \mathbb{C})$ transformation is a symmetry transformation of a string process ($z \rightarrow 1/z$ is a symmetry transformation of the theory, not of a single process). As for $y$ it is not clear which coordinate transformations are a symmetry.\footnote{For instance, a $y$ inversion is not allowed, since it generates poles violating the holomorphicity the $a_i$’s.}

We resolve this and other ambiguities by embedding our curves in $\mathbb{CP}^2$: we introduce the homogeneous coordinates $x_0, x_1, x_2$ with $z = x_1/x_0, y = x_2/x_0$. By multiplying (2.5) by a suitable power of $x_0$ we obtain the equation of the curve in $\mathbb{CP}^2$ in the form

\[ F(x_0, x_1, x_2) = 0. \] (4.8)

Then the coordinate transformations that do not change the curve are in general those of $PGL(3, \mathbb{C})$. However, as we said above, the points $z = 0, \infty$ should be fixed in MST. This means that $x_0 = 0$ and $x_1 = 0$ should not be modified by any transformation. In conclusion the coordinate transformations that give rise to physically indistinguishable processes in MST, are those of the subgroup $\mathcal{H} \subset PGL(3, \mathbb{C})$ defined by

\[
\begin{pmatrix}
  x_0' \\
  x_1' \\
  x_2'
\end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\
  0 & * & 0 \\
  * & * & *
\end{pmatrix} \begin{pmatrix}
  x_0 \\
  x_1 \\
  x_2
\end{pmatrix}.
\] (4.9)

In terms of $y$ and $z$, these transformations include rescalings of $y$ and $z$ and linear transformations $y \rightarrow y + \alpha z + \beta$, with complex constants $\alpha$ and $\beta$. They are acceptable coordinate transformations which involve 4 complex parameters. This fits in our counting of the independent parameters in the previous subsection.

From now on, although we keep speaking mostly in terms of $y$ and $z$, we always understand the corresponding formulation in terms of $x_0, x_1, x_2$. For example, a transformation like $z \rightarrow 1/z$ must be accompanied by $y \rightarrow y/z$ in order for us to remain within $\mathbb{CP}^2$. The latter is a compact space, therefore embedding the curves in it means compactifying them by filling the punctures with suitable points in $\mathbb{CP}^2$.

Given a curve defined by (4.8), the points in it where all partial first order derivatives vanish are singular points. See Appendix for a short summary on singularities. Singularities will play an important role in the following. For example, eq. (4.5) above, when written in homogeneous coordinates reveals a singularity corresponding to the point $z = \infty$.

The information about the branch points of a curve is contained in the discriminant. The discriminant $\delta$ of (2.3) is proportional to $\Delta^2$, where $\Delta$ was defined in (3.3). For a definition of the discriminant in terms of the $a_i$’s, see for example $\cite{23}$. The zeroes of the discriminant define the branch points and their multiplicity gives the multiplicity of the branch points.

A useful tool in studying plane curves is the Newton Polygon. Let us consider the polynomial $P(y, z)$ in (4.1). We associate to each monomial $z^\alpha y^\beta$ in it a point $p = \alpha, q = \beta$ in a $p, q$ plane. We obtain a set of points called the carrier: its convex hull is by definition
the Newton polygon associated to the curve. From the Newton polygon one can deduce a lot of information concerning the curve. For the curves we consider the Newton polygon always contains the point \((p = 0, q = N)\) and is contained in the equilateral triangle formed by the \(p\) and \(q\)-axis and by the line \(p + q = N\).

If the curve is irreducible (we will always consider only irreducible curves), the Newton polygon must contain at least one point in the \(p\)-axis. Therefore for standard curves the Newton polygon will be formed by an upper broken line and a lower broken line (the latter by definition may contain segments of the \(q\) and \(p\)-axis). For example, if the point \((p = 0, q = 0)\) is not in the carrier it means that the point \(y = z = 0\) belongs to the curve. If, in addition, one of the points \((p = 1, q = 0)\) or \((p = 0, q = 1)\) does not belong to the carrier, the point \(y = z = 0\) is obviously singular. If it is so, the singularity is locally given by the product of the local curves corresponding to the various sides of the lower line. In addition to that, each side can contain a number \(m\) of points of the integer lattice, beside the vertices, which may or may not be present in the carrier. The number of the components corresponding to this side is simply \(m + 1\).

If one wants to control what happens over \(z = \infty\) in a given curve, one has to find what the intersections with the line \(x_0 = 0\) are; the polynomial which describes such intersection is simply given by the points of the carrier which lie on the upper line of the Newton polygon. It is useful to see what the Newton diagram looks like in new affine coordinates around the point. This is easily done by means of transition functions: \(z' = 1/z, y' = y/z\). Figure 3 shows an example: diagonal lines in the first diagram become the new vertical lines in the second one (one has to multiply by \(z^N\) in order to recover a polynomial).

In addition to that, suppose for instance that the point \(p = N, q = 0\) does not belong to the carrier; the Newton polygon around the point \(x_0 = 0, x_2 = 0\), which now is in the curve, after going to the affine chart in which \(x_1 = 1\), is simply given by a linear deformation and a reflection around the line \(p = N\). In particular, in order for this point not to be singular, one of the points \(p = N - 1, q = 0\) and \(p = N - 1, q = 1\) has to be in the carrier.

After this generic information about Newton polygons, let us see some concrete examples of genus 1 processes.

### 4.4 Examples: smooth elliptic curves

We start with the case \(N = 3, q = 1\), for which there is already a good variety of examples. These have the advantage that one can check the results by explicitly solving the cubic algebraic equation by means of Cardano’s formula. We do not write down the algebraic equations, but simply the corresponding polygons. The coefficient of the monomials within
or on the border of the Newton polygon are understood to be generic, unless otherwise specified.

The simplest process one can imagine is the string self-energy. This means that we have to look for a totally branched curve over $z = 0$ and $z = \infty$. Remember that the polynomials giving the solutions over these points are given by the points of the carrier on the $q = 0$ and $p + q = N$ lines respectively. So one simple solution is given by the carrier shown in figure 4; the generic case will be non-singular also at finite $z$ and so the genus will be one. The presence of the points $(1, 0)$ and $(2, 0)$ ensures the nonsingularity of 0 and $\infty$; the local behaviour around them is given by the upper side of the inner (shaded) triangle.

Next we would like to describe a joining of strings. In this case we keep $z = \infty$ totally branched, while we add a point on the $q$ axis in order to have, at 0, a polynomial like $y^3 + y^2$ instead of $y^3$, so that $y = 0$ appears twice as a solution and $y = -1$ once. It is now easy to construct all combinations: figure 5 shows various examples and their Newton polygons — as above, with generic coefficients.

4.5 Examples: singular plane curves

Of course for some choice of the parameters a singularity may appear. In this case one has simply to replace the finite hole in these figures by a hole shrunk to a point (for example see fig. 6); the curve becomes genus zero, i.e. a sphere with two identified points. This singularity is the simplest one, it is characterized by a non-vanishing Hessian and is called a node. All nodes can be viewed as two points identified: blowing up a node amounts to separating the points. For instance, consider our first case (figure 4): the polynomial which
corresponds to the diagram can be written as

\[ P = y^3 + czy + z(z - a), \]  

(4.10)

Imposing that a point be singular, one finds that a necessary (but not sufficient) condition is that its discriminant, \( \delta = z^2[27(z - a)^2 - 4c^3z] \), has a multiple root. The double root at \( z = 0 \) just signals that this point is another triple branch point, as we already knew; imposing that the remaining factor be a square, one finds several values, of which for instance \( c = 0 \) gives a triple branch at \( z = a \) and no singular point, and \( c = -3a^{1/3} \) gives instead a node.

This introduces us to our next task: to show how it is possible to describe low-genus highly branched curves. We will describe in detail the self-energy case. We take \( N = 4 \); since we want total branching we can choose a diagram like that in figure 6. The corresponding polynomial has the coefficients corresponding to the vertices of the polygon, and can also have coefficients corresponding to the points on the sides or in the interior (by the way the latter are always \((N - 1)(N - 2)/2\) in number if there is no singular point at 0 and at \( \infty \) and count the genus of the corresponding smooth curve). Now we can look for singular cases in this family along the lines of the previous example; since already in this case computations become complicated, we restrict ourselves to the biquadratic case. In other words the polynomial we start with is

\[ P = y^4 + bzy^2 + z(d + ez + fz^2); \]  

(4.11)

its discriminant is

\[ \delta = 16z^3(d + ez + f^2z^2)(4d + 4ez + 4fz^2 - b^2z)^2. \]  

(4.12)

As before, the term \( z^3 \) shows that the branching at \( z = 0 \) is of order three, i.e. four sheets meet there. The other two terms mean the following. Solutions of a biquadratic equation are in general \( \pm y_{1,2} \). Its discriminant can vanish in two cases: if \( y_1 = y_2 \) or \( y_1 = -y_2 \) — this is determined by the third term in (4.12) — in which case, at the corresponding value of \( z \), there is a couple of double branch points; if \( y_1 = 0 \) or \( y_2 = 0 \), which is determined by the second term, there is a single node. If we choose the coefficients so that the third term is a fourth power, we have two nodes, and so genus one; if, instead, the coefficients are chosen so that the second term is a square, we have a single node, and so genus two. The situation is shown in figure 6.
If one does not wish to restrict to this particular case, one can still find examples of genus 1 and 2 curves. One notes, for instance, that imposing a node first and a total branch in $z = 0$ afterwards is computationally easier. A totally branched, non biquadratic genus one quartic is for instance given by

$$P = y^4 - 5zy^2 + 3z(z+1)y - \frac{z}{2}(z+1)^2,$$

which has two nodes at $\{z = -1, y = 0\}$ and $\{z = 1, y = 1\}$, and two regular branch points: $\delta = z^3(z+1)^2(z-1)^2(63z^2 + 62z + 63)$.

### 4.6 The role of singularities

We believe the above examples are sufficient to illustrate the problems connected with the representation of Riemann surfaces with punctures by means of plane curves. It is impossible in general to represent Riemann surfaces by means of smooth plane curves embedded in the two complex dimensional space spanned by the coordinates $y$ and $z$. One can say that singular plane curves within stringy instantons are the ordinary tools MST uses in order to reproduce the string interaction configurations required by string theory (actually, as will be seen in the next section, only in the $N \to \infty$ limit is this completely true).

Far from representing a problem, singular plane curves are most welcome. They come with a gratifying bonus: the solution of a serious problem for the identification of MST at strong coupling with string theory. This identification is possible if string theory is formulated in the light-cone gauge. In MST, $[2.1]$, ten dimensions enter into the game, two world-sheet dimensions plus eight transverse dimensions represented by the (diagonal) $X^i$. At first sight they seem to have a different nature, however it is clear that in a light-cone framework the two world-sheet dimensions are to be interpreted as representatives of the time and longitudinal dimensions, denoted 0 and 9, which bring the total of physical dimensions to ten. Now, stringy instantons characterized by a smooth plane curve, extend over four out of these ten dimensions. In other words it would seem that MST at strong coupling can only describe four-dimensional string processes. If this were true it would be hard to justify it in the light of the correspondence MST — string theory.

However here come singular curves to our rescue. Singular curves become smooth if one enlarges the space where they are embedded. The standard way to resolve a singularity is to blow it up (see Appendix), which means that a singular point is replaced by a two-dimensional sphere. For example, we have already pointed out that curves in $\mathbb{CP}^2$ with nodes (a node is the simplest possible type of singularity) only, can be smoothed out by embedding them in $\mathbb{CP}^3$, i.e. by adding two dimensions. It is natural to interpret this by saying that the corresponding string process extend over six (instead of four) dimensions. It is not difficult to imagine processes that extend over more (up to ten) dimensions. To better convince ourselves of this fact we can take the reverse point of view. Suppose we want to embed these higher (than four) dimensional processes within the instantons of the 2d field theory (2.1). The only possibility is to squeeze (project) them to the appropriate
four dimensions: such operation of projecting gives rise to singularities. It goes without saying that the true significance of singular plane curves is given by their representing higher (than four) dimensional processes.

This is in particular true for punctures. Singularities that occur in the counterimage of \( z = 0, \infty \) represent overlapping punctures, and in order to disentangle them one has to enlarge the embedding space. It is clear what happens: a high dimensional string process squeezed to four dimensions may require that the locations of incoming and outgoing strings overlap.

5. The moduli space of MST

We have seen in section 2 and 4 that for finite \( N \) the genus of the plane curves that appear in MST has an upper bound given by \( \frac{1}{2}(N - 1)(N - 2) \). We also anticipated in section 2 that the moduli space of plane curves of genus \( h \) with \( n \) punctures turns out to be a discretized version of the moduli space of Riemann surfaces with the same topological type. In this section we want to examine this point. In the first part we see the origin of the discretization, in the second part we confirm this result by estimating the dimension of the moduli space of stringy instantons on the cylinder.

5.1 Discretization

A very convenient way to proceed is to make a comparison with the Mandelstam parameterization of the moduli space of Riemann surfaces with punctures [29, 30, 31]. To this end, let us first review some basic facts about the realization of Mandelstam diagrams. We refer to [28] for a complete account of the following very quick review, after which, we will examine the consequence of the main new input from MST, that is holomorphicity of the covering map which defines the Mandelstam diagram. The result will be a set of constraints on the kinematical data of the diagram which turn out to be a quantization condition for some of the Mandelstam parameters. In the large \( N \) limit these constraints loosen their effectiveness and allow us to recover the full moduli space of the string diagrams.

Let \( \Sigma \) be a compact Riemann surface of genus \( h \) and let \( \omega_I, I = 1, \ldots, h \) be a set of holomorphic differentials on \( \Sigma \) normalized by \( f_{\alpha J}^{\omega_I} = \delta_{IJ} \), while \( f_{\beta J}^{\omega_I} = \Omega_{IJ} \) is the period matrix. We fix \( n \) punctures \( \{Q_1, \ldots, Q_n\} \) on \( \Sigma \) and define the divisor \( D = Q_1 \cdot \ldots \cdot Q_n \).

We also introduce a set of \( n \) real numbers \( R = \{r_1, \ldots, r_n\} \) such that \( \sum_i r_i = 0 \).

Now, let \( \omega \) be the differential which is holomorphic on \( \Sigma \setminus D \) with simple poles at \( D \) with \( \text{res}_{Q_i} \omega = r_i \) and \( \text{Re} \ f_{\alpha J}^{\omega} = 0 = \text{Re} \ f_{\beta J}^{\omega} \).

In [23] it was shown how the above differential defines a nice procedure which allows us to look at \( \Sigma \) as a topological covering of a cylinder: one can easily decompose \( \Sigma \) into pants along the level lines of the function \( \tau(P) \equiv \text{Re} \int^P \omega \). In this sense, \( \omega \) induces on \( \Sigma \) the structure of a Mandelstam diagram. The Mandelstam parameters are the twist-angles \( \theta_b, b = 1, \ldots, 3h + n - 3 \), along the junctures of the pants decomposition and the relative ‘time’ coordinates \( \tau_a - \tau_0, a = 1, \ldots, 2h + n - 3 \), of the \( 2h + n - 2 \) interaction points. \( h \) additional real parameters are the internal light-cone momenta \( p_I^\tau = \int_{\alpha I}^{\omega} \). Altogether they form a
set of $6h - 6 + 2n$ real parameters. In [28] it was shown that these parameters represent
good coordinates on the moduli space of genus $h$ Riemann surfaces with $n$ punctures $\mathcal{M}_{h,n}$.

To complete the picture we identify the set $R$ with the $+$ components of the external
light-cone momenta of the diagram, i.e. the periods of $\omega$ around the punctures. We
also have the relations $\oint_{\beta} \omega = \frac{i}{2\pi} p^+_K W^J_{Kb} \theta_b$, where $W^J$ are integer-valued matrices which
depends on the pants decomposition of the Riemann surface and its intersections with the
$\alpha$ and $\beta$ cycles.

Our strategy now is the following. We first construct an explicit form for $\omega$, in terms of
the prime-form, the $\omega_I$’s and the period matrix of $\Sigma$. Then we compare this $\omega$ with the one
that comes from MST. The relevant new input consists in the fact that MST induces on $\Sigma$ the structure of a holomorphic
covering of the Riemann sphere (as usual we consider the latter instead of the cylinder). By this we mean that, if $\mathcal{z}: \Sigma \to \mathbb{CP}^1$ is the covering map in
the MST scheme, the coordinate $z$ is a meromorphic function on $\Sigma$. The role of $\omega$ in MST
is played by $d \ln \mathcal{z}$, therefore we have to identify them. This condition becomes a constraint
on the data of the Mandelstam diagram. In fact, it means that $D^R \equiv Q^1_1 \cdots Q^n_n$, being
the divisor of the meromorphic function $\mathcal{z}$, is a principal divisor on $\Sigma$, so that in particular
$r_i \in \mathbb{Z}$. As a consequence, some constraints appear in the data of the Mandelstam diagram and these conditions induce a complex codimension $h$ slicing of the moduli space. This can
be seen as follows.

Let $\omega_{P_+, P_\pm}$ be the holomorphic differential on $\Sigma \setminus \{P_+, P_\pm\}$ with simple poles at $P_\pm$
with residues $\pm 1$ and imaginary periods. It can be written as

$$
\omega_{P_+, P_\pm}(P) = d(P) \ln \left[ \frac{E(P, P_+) \cdot e^{2\pi i \text{Im} \int_{P_+}^{P_-} \omega_I \Omega^{(2)}_{Ij} f^P \omega_J}}{E(P, P_-)} \right] = d(P) \ln H(P, P_+, P_-),
$$

where $E(P, Q)$ is the prime form on $\Sigma$, $\Omega^{(2)}$ is the imaginary part of the period matrix and
$d(P) = dP \cdot \frac{\partial}{\partial P}$.

In terms of the above differentials we can write

$$
\omega = \sum_{l=1}^{n-1} k_l \omega_{Q_l Q_{l+1}},
$$

where $k_i - k_{i-1} = r_i$ and $k_0 = k_{n}$; substituting (5.1) into (5.2) we obtain

$$
\omega(P) = d(P) \ln \tilde{z}(P)
$$

where

$$
\tilde{z}(P) = \prod_{l=1}^{n-1} [H(P, Q_l, Q_{l-1})]^{k_l}.
$$

Now, as anticipated above, we make the identification $\omega = d \ln z$. This requires that
$\tilde{z} = z$ up to a multiplicative constant, which implies that $\tilde{z}$ is a well defined meromorphic

---

8$H(P, P_+, P_-)$ depends also on a base point which is irrelevant in this context and, for the sake of
simplicity, is not specified.
function on $\Sigma$. On the one hand this imposes that the residues $r_l$ be quantized in integer values. On the other hand it requires that the differential $d\tilde{z}$ have vanishing periods along $\alpha$ and $\beta$ cycles. The latter condition is fulfilled iff

$$\sum_{l=1}^{n} r_l \int_{Q_l} \omega_I = m_I + n_I \Omega_{II} \quad (5.4)$$

for some $m_I, n_I \in \mathbb{Z}$. At this point the situation is clear: (5.4) is the vanishing condition for the Abel map and says that the divisor $D^R$ is principal.

Conversely, let $z$ be a meromorphic function on $\Sigma$ and $D^R$ its divisor. By definition (5.4) holds and $\text{res}_{Q_l} d(P) \ln z = r_l$.

Notice that the periods of $\omega$ are quantized in integral values as

$$\oint_{\alpha_I} \omega = 2\pi i n_I \quad \text{and} \quad \oint_{\beta_I} \omega = -2\pi i m_I, \quad (5.5)$$

and this condition is equivalent to (5.4).

Eq. (5.5) means that the internal light-cone momenta of the diagram are quantized and that, in addition, there are $h$ discretizing constraints on the twist-angles of the Mandelstam diagram. Since these variables, together with the relative interaction times which have been left untouched, are the coordinates of the moduli space, we are left with a discrete slicing of the moduli space $\mathcal{M}_{h,n}$, each slice being of complex dimension $2g-3+n$. This discretized moduli space is what we have called $\mathcal{M}^{(h,n)}_N$ in section 3.

In the large $N$ limit, however, the quantization condition disappears in a continuum of values

$$\lim_{N \to \infty} \frac{1}{N} \left[ \mathbb{Z}^h \bigoplus \Omega \mathbb{Z}^h \right] = \mathbb{C}^h. \quad (5.6)$$

Simultaneously, for large $N$ also the bound $\frac{1}{8}(N-1)(N-2)$ on the genus of the plane curves in MST, becomes ineffective, and we recover the full moduli space of string theory.

It is interesting to review the genus 0 and 1 case in detail.

**Genus zero** On the sphere the prime form is simply $E(P,Q) = P - Q$ and then, up to a multiplicative constant,

$$z = \prod_{i=1}^{n} (P - Q_i)^{r_i}. \quad (5.7)$$

Splitting the divisor $D^R = D_0 \cdot D_\infty^{-1}$ into its zero and polar parts — where $D_0 = \prod_{r_i > 0} Q_i^{r_i}$ and $D_\infty = \prod_{r_i < 0} Q_i^{-r_i}$ are its positive and negative parts respectively — we get

$$z = \frac{\prod_{i \mid r_i > 0} (P - Q_i)^{r_i}}{\prod_{i \mid r_i < 0} (P - Q_i)^{-r_i}}. \quad (5.8)$$

\[^9\text{In some sense, the topological reconstruction of the Mandelstam diagram can be seen as an infinitely-sheeted holomorphic covering of the cylinder.}\]
and we recover the result \( \{4.3\} \). The counting of section \( \{4.2\} \) tells us that the independent complex parameters are \( n - 3 \). Therefore in this case there is no moduli quantization at all and the moduli of plane curves cover the full moduli space \( \mathcal{M}_{0,n} \).

These curves correspond to tree level Mandelstam diagrams with \( n \) external strings each of light-cone momentum \( r_i \). Therefore, in this case, MST gives exact results at finite \( N \) (except for the fact that the + components of the external momenta are discrete).

**Genus one.** The first case in which moduli quantization becomes effective is at \( h = 1 \). Let us specialize to the torus the above construction. In this case, the prime form is proportional to the odd theta function \( E(P, Q) \propto \Theta_{\text{odd}}(P - Q|\tau) \) and there is a unique holomorphic differential \( \omega_1 = dP \). Therefore

\[
\omega_{P_+,P_-}(P) = d(P)\ln\left[\frac{\Theta_{\text{odd}}(P - P_+|\tau)}{\Theta_{\text{odd}}(P - P_-|\tau)}\cdot\exp\left(2\pi i \text{Im} (P_- - P_+) (\text{Im}\tau)^{-1}(P - P_0)\right)\right],
\]

and we get

\[
\omega(P) = \sum_{l=1}^{n-1} k_l \omega_{Q_l,Q_{l+1}}(P) = d(P)\ln\prod_{l=1}^{n} [\Theta_{\text{odd}}(P - Q_l|\tau)]^{r_l} \cdot e^{r_l 2\pi i \text{Im}(Q_l)(\text{Im}\tau)^{-1}P} = d(P)\ln z.
\]

Using the standard modular properties of \( \Theta \)-functions one obtains that \( z \) is a well defined meromorphic function on the torus \( T_\tau \equiv \mathbb{C}^{(1,\tau)} \) iff \( r_i \in \mathbb{Z} \) and

\[
\sum_{l=1}^{n} r_l Q_l = m + n\tau
\]

for some integers \( n \) and \( m \). This is the discretizing condition for the the genus one case. It coincides with the condition already found in section \( \{4\} \) eq. \( \{4.6\} \).

One can verify that in all the genus one examples we have considered in secs. \( \{4.4\} \) and \( \{4.5\} \) the counting of independent parameters matches the formula \( 2h + n - 3 \). We believe that, for any topological type \( (h, n) \), one can construct plane curves with \( 2h + n - 3 \) independent parameters.

### 5.2 The dimension of the moduli space of stringy instantons

In this section we verify the correctness of the result obtained above by counting the dimensions of the solution space of Hitchin equations on a cylinder up to gauge transformations in the strong coupling limit. The method is based on zero modes counting, therefore it is sensitive only to continuous dimensions. Since the group factor \( Y \) does not contain free parameters we expect to find a \( 2h + n - 3 \) dimensional space of solutions.

The strategy is the following: first we linearize the system by transferring the calculation to the tangent space, then we calculate the dimension of the subspace of the tangent space which is orthogonal to infinitesimal gauge transformations, in the strong coupling limit \( g \to \infty \). The equations in the \( g \to \infty \) limit are in a form which can be lifted to the relevant spectral curve \( \Sigma \). This way the calculation reduces to a zero modes counting on
The exact result is then obtained by taking into account the redundancy of parameters in the plane curve representation of the spectral curve \( \Sigma \).

Linearizing the Hitchin equations (2.2), (2.3) gives

\[
D_w \delta A - D_{\bar{w}} \delta A - ig^2 [\bar{X}, \delta X] + ig^2 [X, \delta \bar{X}] = 0 \tag{5.10}
\]

\[
D_w \delta X + i [\delta A, X] = 0, \quad D_{\bar{w}} \delta \bar{X} + i [\delta A_{\bar{w}}, \bar{X}] = 0. \tag{5.11}
\]

The condition that tangent vectors be orthogonal to gauge transformations can be written in the form

\[
D_w \delta A_{\bar{w}} + D_{\bar{w}} \delta A_w + ig^2 [\bar{X}, \delta X] + ig^2 [X, \delta \bar{X}] = 0 \tag{5.12}
\]

by making use of the scalar product

\[
< (\delta_1 A, \delta_1 X) | (\delta_2 A, \delta_2 X) > = \int_{Cyl} d^2 w \text{Tr} \left[ \delta_1 A_w \delta_2 A_{\bar{w}} + g^2 \delta_1 \bar{X} \delta_2 \bar{X} + \text{h.c.} \right]. \tag{5.13}
\]

Let us now take the \( g \to \infty \) limit of the above equations. The limit background, labeled with \( \infty \), is then a strong coupling limit instanton characterized by a definite spectral curve \( \Sigma \), as explained in [12] and in section 2 above,

\[
A_w^\infty = -iU \partial_w U, \quad X^\infty = U \bar{X} U^+ \tag{5.14}
\]

and the equations for the tangent space to these instantons become the following ones

\[
D_{\bar{w}}^\infty \delta A_w = 0, \tag{5.15}
\]

\[
[\bar{X}^\infty, \delta X] = 0, \tag{5.16}
\]

\[
D_w^\infty \delta X + i [\delta A_w, X^\infty] = 0, \tag{5.17}
\]

together their hermitian conjugates. We specify that these equations characterize the tangent space to the stringy instantons (see comment at the end of section 3.2), not necessarily to the most general Hitchin solutions. To solve these equations we use the lifting technique we exploited in [12], which we refer to for notation and some technical points.

Let \( t \) be the Cartan subalgebra in \( u(N) \) obtained as \( U t_d U^+ \) where \( t_d \) is the diagonal one. Since \( \bar{X}^\infty \in t \), then, by (5.16), also \( \delta X \in t \). Using (5.14) it is easy to see that also \( D_w^\infty X^\infty \in t \) and, by (5.17), and \( X \in t \), we conclude that \( \delta A_w \in t \).

This means that all the variations are in the Cartan subalgebra \( t \), and the above equations reduce to

\[
D_{\bar{w}}^\infty \delta A_w = 0, \quad D_w^\infty \delta X = 0 \quad \text{where} \quad \delta A_w, \ \delta X \in t, \tag{5.18}
\]

plus their hermitian conjugates.

To count the solutions of eq. (5.18), we lift them to the spectral surface defined by the limit background field (5.14). They can be shown to reduce to

\[
\partial_\bar{z} \delta \tilde{A}_z = 0, \quad \partial_\bar{z} \delta \tilde{X} = 0, \tag{5.19}
\]
where $\delta \tilde{A}_z dz$ is therefore a holomorphic differential on $\Sigma$ and $\delta \tilde{X}$ a holomorphic scalar. Following the doubling trick explained in [12] we get a total number of solutions equal to $\hat{h} + 1 = (2h + n - 1) + 1 = 2h + n$.

Finally, from this number we have to subtract 3; in fact the 3 parameters of the transformations of the variable $y$ belonging to $\mathcal{H}$ (section [3]) appear as genuine moduli in the above counting, while, of course, they are not.

A. Singularities of plane curves

Singularities of plane curves are a classical subject. Here we briefly review the most useful methods for their study, which we partially use in the text.

Let us denote the curve defined by the polynomial $P(y, z) = 0$ with $\Sigma$. A singular point is a point of $\Sigma$, in which $\partial_y P$ and $\partial_z P$ vanish. Suppose, without loss of generality, that the point we are interested in $\{y = 0, z = 0\}$. Let us find the local behaviour of the solutions $y(z)$ around it. This can be obtained from the Puiseux expansion, which in turn can be reconstructed from the Newton polygon. A Newton polygon may in general have several sides: each side represents (locally) a factor of the curve, which can in turn be reducible, depending on the number of points of the lattice that lie on it. Consider now the steepest of these sides near the origin: call it $L_0$, and $\mu_0$ its slope. $L_0$ defines a grading according to which $y$ has the same weight as $z^{\mu_0}$. Now we approximate the solution of the equation by the ansatz $y = t_0 z^{\mu_0}$. If we insert this in $P$, the latter takes the form $P = z^{\mu_0} g(t_0)$ plus other terms which are higher order terms with respect to the grading defined by $L_0$; $g$ is a polynomial in $t_0$. Solving $g(t_0) = 0$ we find in general several values of $t_0$. For each of them we can now substitute $y = t_0(z^{\mu_0} + y_1)$ in $P$ and find a new polynomial $P_1$, together with a new Newton polygon. Looking at its steepest side, say $L_1$, with slope $\mu_1$, we can try a new solution in the form $y_1 = t_1 z^{\mu_1}$, and so on.

This procedure needs not come to an end after a finite number of steps; however, the series

$$y = t_0(z^{\mu_0} + t_1(z^{\mu_1} + \ldots))$$

(A.1)

can be shown to converge to the solutions of $F$ in a sufficiently small neighborhood of the origin. The numbers $\mu_0, \mu_1, \ldots$ are rational and increasing. A series like (A.1) is called a Puiseux expansion for $P$.

Puiseux expansions can now be used to analyse how solutions 'twist around one another', and to classify singularities up to topological equivalence; around $z = 0$ and $z = \infty$ this leads to nothing else than the string interpretation already discussed in the text - although, from a mathematical point of view, there is something more to it. Consider the first term $\mu_0$ of the Puiseux expansion and suppose $\mu_0 > 0$. Let us write $y^{\mu_0} = z^{\mu_0}$ ($\mu_0 = p_0/q_0$), and let us follow the solutions $y$ as $z = \epsilon e^{2\pi i \phi}$ makes a small loop around zero. They lie on a torus $\{|y| = \epsilon^{\mu_0}, |z| = \epsilon\}$, and wind around its two cycles $q_0$ and $p_0$ times respectively. We can represent the union of various circles covered by the solutions in $\mathbb{R}^3$, and it turns out
to be a link of knots. We can simply represent such knots by first drawing a braid with \( q_0 \) threads which permute \( p_0 \) times, and then connect initial with final points, as in figure 7.

![Figure 7: Knots corresponding to \( y^2 = z^2 \) and \( y^3 = z^2 \) singularities.](image)

This knot is not in general the one that describes our singularity; one has to take into account the higher terms in the expansion. If however \( \mu_1 = p_1/q_1 \) and \( q_1 = q_0 \), this does not really change the braid (nor the knot). If it is not so, one is led to a more complicated figure, a knot which winds around a little torus constructed around the previously defined knot; this is called an \textit{iterated torus knot}. This process can be shown to stop - in fact in our case we know that the number of threads of the braid really is \( N \) - and the topological type of the knot can be completely encoded in a finite ordered set of pairs, of which the first is \( \{p_0, q_0\} \), called \textit{Puiseux pairs}. The knot in turn classifies the singularity; it is in fact homeomorphic to the intersection of the curve with a suitably small sphere \( S^3 \).

This latter fact means that the knot which classifies the singularity is in fact the projection of the physical string embedded in the target space, projected down to \( 3 + 1 \) dimensions. What is really of interest to us is the number of component, since all knots and links can be undone in higher dimensions; and this is readily found also without drawing knots.

In relation with our discussion in the text, perhaps the most useful thing one can explore, is the contribution of a given singularity to the genus. For singular curves the genus is defined as that of their \textit{resolution}; given \( \Sigma \) and the set \( \mathcal{S} \) of its singular points, a resolution of \( \Sigma \) is a smooth curve \( \tilde{\Sigma} \), (usually embedded in a larger space than the original curve), together with a holomorphic projection \( p : \tilde{\Sigma} \to \Sigma \), such that its restriction \( \tilde{p} : \tilde{\Sigma} - p^{-1}(\mathcal{S}) \to \Sigma - \mathcal{S} \) is a biholomorphism.

In words, a resolution can be locally achieved by replacing the singular point by some space; Puiseux expansions are resolutions if we replace a singular point with all possible behaviours near it. But this is not a very economical resolution since the resulting space is huge. A handier way is to replace the singular point by a sphere - called \textit{exceptional divisor}. This is precisely the well known procedure of blowing up the singularity, which in addition has the advantage of exhibiting the smooth curve in some \( \mathbb{C} \bar{\mathbb{P}}^k \).

Fortunately one needs not compute explicitly the blown-up curve; every singular point simply gives a contribution that can be computed as follows. If we blow up the singularity, we get an exceptional divisor, which meets the \textit{strict preimage} \( \tilde{\Sigma} - p^{-1}(\mathcal{S}) \) of the curve in a number of points - called \textit{infinitely near}; some of these can be singular again, and we can blow them up in turn, getting new infinitely near points; it can be shown that this procedure eventually comes to an end. Now, if \( h \) is the genus of \( \tilde{\Sigma} \), the genus of the original
curve $\Sigma$ is given by

$$h - \frac{1}{2} \sum_k \nu_k (\nu_k - 1),$$  \hspace{1cm} (A.2)

where $k$ runs over all the infinitely near points $P_k$ of the singular point and $\nu_k$ are their multiplicity, i.e. is the order of the first non-vanishing term of the Taylor expansion of $P(y,z)$ at $P_k$.

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