DOMINANT AUSLANDER-GORENSTEIN ALGEBRAS AND KOSZUL DUALITY

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Dedicated to the memory of Hiroyuki Tachikawa

Abstract. We introduce the class of dominant Auslander-Gorenstein algebras as a generalisation of higher Auslander algebras and minimal Auslander-Gorenstein algebras, and give their basic properties. We also introduce mixed (pre)cluster tilting modules as a generalisation of (pre)cluster tilting modules, and establish anAuslander type correspondence by showing that dominant Auslander-Gorenstein (respectively, Auslander-regular) algebras correspond bijectively with mixed precluster (respectively, cluster) tilting modules.

It turns out that dominant Auslander-regular algebras have a remarkable interaction with Koszul duality, namely Koszul dominant Auslander-regular algebras are closed under Koszul duality. As an application, we use the class of dominant Auslander-regular algebras to answer a question by Green about a characterisation of the Koszul dual of an Auslander algebra. We show that the Koszul dual of the Auslander algebras of hereditary or self-injective representation-finite algebras are higher Auslander algebras, and describe the corresponding cluster tilting modules explicitly.

Contents

Introduction 2
1. Preliminaries 5
2. Dominant Auslander-Gorenstein algebras and basic properties 7
2.1. The definition and the first properties 7
2.2. Relation with the Gorenstein condition 8
2.3. The first examples 10
3. Mixed precluster tilting modules and Dominant Auslander-Gorenstein algebras 12
3.1. Dominant Auslander-Solberg correspondence 12
3.2. Examples of mixed precluster tilting modules 14
3.3. Mixed cluster tilting modules and Dominant Auslander correspondence 15
3.4. Examples of mixed cluster tilting modules 16
4. Koszul dual of a dominant Auslander-regular algebra 17
4.1. Our main results 17
4.2. Reminder on Koszul algebras and the proof of the main result 19
4.3. The base algebra of the Koszul dual of an Auslander algebra 20
4.4. Cluster tilting modules over Auslander-Koszul complements 24
4.5. Simple-minded system versus higher cluster tilting
Acknowledgements 28
References 28

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Introduction

The classical Auslander correspondence \([A][ARS]\) is a milestone in representation theory which gives a bijective correspondence between representation-finite artin algebras \(B\) and Auslander algebras \(A\), which are by definition those artin algebras \(A\) with \(\text{gldim } A \leq 2 \leq \text{domdim } A\), where \(\text{domdim } X\) of an \(A\)-module \(X\) with minimal injective coreolution

\[
0 \to X \to I^0 \to I^1 \to \ldots
\]

is the infimum of \(i \geq 0\) such that \(I^i\) is non-projective. This was a starting point of Auslander-Reiten theory developed in the 1970s. It encodes the representation theory of \(B\) into the homological algebra of its Auslander algebra \(A\). In \([ya3]\), this correspondence was generalized to the higher Auslander correspondence for each \(d \geq 1\) by establishing a bijection between so called \(d\)-cluster tilting modules (see \((1.2.2)\)) and \(d\)-Auslander algebras, which are by definition those artin algebras \(A\) with \(\text{gldim } A \leq d + 1 \leq \text{domdim } A\). Finite dimensional algebras admitting \(d\)-cluster tilting modules are called \(d\)-representation finite. These notions play an important role in higher Auslander-Reiten theory of finite dimensional algebras and Cohen-Macaulay representations \([iya3]\), and found several applications and interactions with other fields, including cluster algebras of Fomin-Zelevinsky \([FZ]\) and non-commutative crepant desingularizations in algebraic geometry \([Van]\). We refer to, for example, \([AT, DI, DJL, DJW, GI, IO1, IO2, JKM, H, HS, HI1, HI2, HZ1, HZ2, HZ3, KS, M, P, ST, Vas, Wi, Wu]\) for more results on \(d\)-representation-finite algebras and higher Auslander-Reiten theory.

The defining property of higher Auslander algebras can be regarded as a non-commutative analogue of commutative regular rings of dimension two. In fact, Auslander algebras are typical examples of so called Auslander-regular rings, which play important roles in various areas, including homological algebra, non-commutative algebraic geometry, analytic \(D\)-modules, Lie theory and combinatorics, see for example \([C, IM, VO]\). Recently in \([IS]\), Iyama and Solberg generalised the class of higher Auslander algebras to the realm of infinite global dimension by generalising a classical work by Auslander and Solberg \([AS]\). They define an algebra \(A\) to be \(d\)-minimal Auslander-Gorenstein if \(\text{idim } A \leq d \leq \text{domdim } A\) for some \(d \geq 2\) and generalise several classical results to this more general situation. We refer to, for example, \([CIM, CK, DITW, HKV, Grev, LMZ, LZ, MS, MMZ, Mc, NRTZ, PS, R, Z]\) for results on higher Auslander algebras and minimal Auslander-Gorenstein algebras.

In this paper we give a generalisation of higher Auslander algebras as well as minimal Auslander-Gorenstein algebras. We define an algebra \(A\) to be \(\text{dominant Auslander-Gorenstein}\) if it is Iwanaga-Gorenstein and \(\text{idim } P \leq \text{domdim } P\) for every indecomposable projective \(A\)-module \(P\). A \(\text{dominant Auslander-regular algebra}\) is then defined to be a dominant Auslander-Gorenstein algebra of finite global dimension.

\[
\begin{array}{c}
\text{higher Auslander} \\
\downarrow \\
\text{minimal Auslander-Gorenstein}
\end{array}
\Rightarrow
\begin{array}{c}
\text{dominant Auslander-regular} \\
\downarrow \\
\text{dominant Auslander-Gorenstein}
\end{array}
\Rightarrow
\begin{array}{c}
\text{Auslander-regular} \\
\downarrow \\
\text{Auslander-Gorenstein}
\end{array}
\]

The class of dominant Auslander-Gorenstein algebras is much larger than that of minimal Auslander-Gorenstein algebras, and still enjoys extremely nice homological properties among all Auslander-Gorenstein algebras. One of the advances of dominant Auslander-regular algebras is that they are closed under gluing of quiver algebras (Section 2.3) and Koszul duality (Section 4.1) while higher Auslander algebras are not in general.

One of our main results gives the Auslander correspondence for dominant Auslander-Gorenstein algebras. For this, we generalise the notion of cluster tilting modules as follows: For an algebra \(B\), we consider the higher Auslander-Reiten translates

\[
\tau_n := \tau \Omega^{n-1} \quad \text{and} \quad \tau^\mu_n := \tau^{\mu-1} \Omega^{-(n-1)}.
\]

Then a generator-cogenerator \(M\) of \(B\) is called \(\text{mixed precluster tilting}\) if the following condition is satisfied.

- For each indecomposable non-projective direct summand \(X\) of \(M\), there exists \(c_X \geq 1\) such that \(\text{Ext}_B^i(X, M) = 0\) for all \(1 \leq i < c_X\) and \(\tau_X c_X(X) \in \text{add } M\).

This is equivalent to the following dual condition, where \((d_X)_X\) turns out to be a permutation of \((c_X)_X\), that is, \(c_X = d_Y\) for \(Y = \tau c_X(X)\).
For each indecomposable non-injective direct summand $X$ of $M$, there exists $d_X \geq 1$ such that 
\[ \text{Ext}_B^i(M, X) = 0 \text{ for all } 1 \leq i < d_X \text{ and } \tau_{d_X}(X) \in \text{add } M. \]

In case $d_X = d$ holds for all $X$, we recover the notion of $d$-precluster tilting modules introduced in [IS]. We refer to Section 2.2 for examples of mixed precluster tilting modules. We prove the following Auslander correspondence for dominant Auslander-Gorenstein algebras.

**Theorem 0.1** (Theorem 3.2). There exists a bijection between the following objects.

1. The Morita equivalence classes of dominant Auslander-Gorenstein algebras $A$ with $\text{domdim } A \geq 2$.
2. The Morita equivalence classes of pairs $(B, M)$ of finite dimensional algebras $B$ and mixed precluster tilting modules $M$.

The correspondence from (2) to (1) is given by $(B, M) \mapsto A := \text{End}_B(M)$.

In a forthcoming paper, we will generalize this bijection by dropping the assumption $\text{domdim } A \geq 2$.

For a mixed precluster tilting module $M$, we consider the category $Z(M) := \bigcap X X^\perp = \bigcap X^\perp d_X^{-1}X$, where $X$ runs over all indecomposable direct summands of $M$, and we refer to (1.2.1) of the symbol $\perp$.

We prove in Theorem 3.10 that there exist mutually quasi-inverse equivalences $Z(M) \xrightarrow{\text{Hom}_B(M, -)} \text{CM } A \xleftarrow{- \otimes_A M} Z(M)$, where $\text{CM } A$ denotes the subcategory of maximal Cohen-Macaulay $A$-modules. As an application, we obtain that $A$ is dominant Auslander-regular if and only if $Z(M) = \text{add } M$. Motivated by this result, we call a mixed precluster tilting module $M$ mixed cluster tilting if it satisfies $Z(M) = \text{add } M$. In case $d_X = d$ holds for all $X$, we recover the notion of $d$-cluster tilting modules. Thus we have implications

\[
\begin{array}{ccc}
\text{d-cluster tilting} & \overset{\perp}{\implies} & \text{mixed cluster tilting} \\
\downarrow & & \downarrow \\
\text{d-precluster tilting} & \overset{\perp}{\implies} & \text{mixed precluster tilting}.
\end{array}
\]

We refer to Section 3.4 for examples of mixed cluster tilting modules. Now we are ready to state the following restriction of the correspondence in Theorem 0.1.

**Theorem 0.2** (Theorem 3.12). The bijection in Theorem 0.1 restricts to a bijection between the following objects.

1. The Morita equivalence classes of dominant Auslander-regular algebras $A$ with $\text{domdim } A \geq 2$.
2. The Morita equivalence classes of pairs $(B, M)$ of finite dimensional algebras $B$ and mixed cluster tilting modules $M$.

Koszul duality is a fundamental operation in representation theory and other areas of mathematics which plays important roles in various subjects [BGS, BBK, EL, GM1, GM2, GRS, HH, LV, MV2, PP, S]. In the second half of this paper, we discuss how dominant Auslander-regular algebras and certain subclasses such as higher Auslander algebras behave under taking Koszul duality. The following main result reveals an unexpected property of dominant Auslander-regular algebras.

**Theorem 0.3** (Theorem 4.1). The class of finite-dimensional Koszul dominant Auslander-regular algebras are closed under Koszul duality.

As an application of Theorem 0.3, we give a complete answer to Question 0.4 due to Green [Gre, Section 5]. It was discussed in various papers including [GM1, Guo1, GMT, GW], and is closely related to the study of important classes of algebras including preprojective algebras and zig-zag algebras [BBK, HK].

**Question 0.4** (Green). Suppose $E$ is a Koszul algebra satisfying the following properties:

1. The Loewy length of $E$ is 3.
2. Each indecomposable projective $E$-module of Loewy length 3 is injective.
3. $E$ has positive dominant dimension.

What extra conditions imply that the Koszul dual of $E$ is an Auslander algebra?
We prove that the missing condition that is needed to characterise the Koszul dual of Auslander algebras is exactly the condition to be a dominant Auslander-regular algebra together with the converse of condition (G2). As a modification of (G1) and (G2), for \( \ell \geq 1 \), we call an algebra \( A \) \( \ell \)-stiff if

- the Loewy length of \( A \) is \( \ell \),
- an indecomposable injective \( A \)-module is projective if, and only if, its Loewy length is \( \ell \).

We have the following restriction of the correspondence given in Theorem 0.3.

**Theorem 0.5** (Theorem 4.5). For each \( n \geq 2 \), Koszul duality \( E(E(A)) \cong A \) induces the following correspondence between Koszul algebras:

\[
\{ (n+1)\text{-stiff dominant Auslander-regular algebra} \} \leftrightarrow \{ (n-1)\text{-Auslander algebra} \}.
\]

Moreover, for \( d, n \geq 2 \), this restricts to a correspondence

\[
\{ (n+1)\text{-stiff} \ ((d-1)\text{-Auslander algebra}) \} \leftrightarrow \{ (d+1)\text{-stiff} \ ((n-1)\text{-Auslander algebra}) \}.
\]

Considering the case \( n = 2 \), we obtain the following answer to Question 0.4.

**Corollary 0.6** (Corollary 4.6). Let \( E \) be a ring-indecomposable non-semisimple Koszul algebra. Then the Koszul dual of \( E \) is an Auslander algebra if and only if the following conditions are satisfied.

- (G1) The Loewy length of \( E \) is 3.
- (G2\(^+\)) An indecomposable projective module is injective if and only if it has Loewy length 3.
- (G3\(^+\)) \( E \) is dominant Auslander-regular.

Recall that a representation-finite algebra \( B \) over a field \( K \) is called *standard* if the Auslander algebra \( A \) is isomorphic to the mesh algebra of the Auslander-Reiten quiver of \( B \). This guarantees that \( A \) is Koszul [GM1]. It is known that \( B \) is standard if \( K \) is an algebraically closed field of characteristic not equal to 2 [BGRS] or \( B = KQ \) for a Dynkin quiver \( Q \).

Theorem 0.3 tells us that for a given representation-finite algebra \( A \), we can look at the Koszul dual of its Auslander algebra and obtain a dominant Auslander-regular algebra. Following [KY2], we define the *base algebra* of an algebra \( A \) as the algebra \( eAe \) when \( eA \) is an additive generator in the category of projective-injective \( A \)-modules. A natural question is for which representation-finite algebras \( A \), the Koszul dual of the Auslander algebra of \( A \) is a higher Auslander algebra. While this seems to happen rather rarely in the general case, in the two most important cases of path algebras of quivers and self-injective algebras, it is always true. In the first case, we obtain new classes of \( d \)-representation-finite algebras of global dimension \( d \), which are of central importance in higher Auslander-Reiten theory.

**Theorem 0.7** (Theorem 4.16, Corollary 4.19, Theorem 4.24). Let \( KQ \) be a path algebra of Dynkin type \( Q \) with at least 3 vertices, and \( h_Q \) the Coxeter number of \( Q \).

1. The Koszul dual \( E \) of the Auslander algebra of \( KQ \) is a \( (h_Q - 3) \)-Auslander algebra.
2. The base algebra \( C \) of \( E \) is a \( (h_Q - 3) \)-representation-finite algebra of global dimension \( h_Q - 3 \).
3. The \( (h_Q - 3) \)-cluster tilting \( C \)-module is given by \( C \oplus DC \oplus S \) for a simple \( C \)-module \( S \) (see Theorem 4.24 for the details) if \( Q \) is of linear Dynkin type \( A \), and \( C \oplus DC \) for other cases.
4. \( C \) is a Koszul algebra and its Koszul dual is the stable Auslander algebra of \( KQ \).

Collecting the results so far, \( A \) is the Auslander algebra of \( KQ \) with stable part \( A := A/AeA \), \( E := E(A) \), and \( C := (1 - e)E(1 - e) \) the base algebra of \( E \) (which we call the *Auslander-Koszul complement* of \( KQ \)), we have the following quadratic duality, or \((p,q)\)-Koszul duality due to Brenner-Butler-King [BBK] more specifically, of recollements.
Here, $h := h_Q$ is the Coxeter number and $\ell$ is the length of the longest path in $Q$. A horizontal arrow $\Lambda \to \Gamma$ can be treated either as a canonical (embedding or ideal-quotient) functor $\text{proj} \, \Lambda \to \text{proj} \, \Gamma$, or as an adjoint triple $\text{mod} \, \Gamma \to \text{mod} \, \Lambda \to \text{mod} \, \Gamma$ in a standard recollement. Moreover, the vertical arrows in the middle and on the right represent graded derived equivalence.

Analogous to the previous theorem, we have the following result for standard representation-finite self-injective algebras.

**Theorem 0.8** (Theorem 4.16, Corollary 4.19). Let $B$ be a standard representation-finite self-injective algebra of tree class $Q$ with at least 3 vertices, and $h_Q$ the Coxeter number of $Q$.

1. The Koszul dual $E$ of the Auslander algebra of $B$ is a $(2h_Q - 3)$-Auslander algebra.

2. The base algebra $C$ of $E$ is a self-injective $(2h_Q - 3)$-representation-finite algebra.

3. $C$ is a $(2, h_Q - 3)$-Koszul algebra and its quadratic dual is the stable Auslander algebra of $B$.

The $(2h_Q - 3)$-cluster tilting $C$-module corresponding to $E$ in the sense of Theorem 0.2 is given by $C \oplus S$ for an explicit semisimple $C$-module $S$, see Theorem 4.24 for the details. In fact, we have the following remarkable bijection

$$\left\{ \begin{array}{c} \text{1-simple-minded systems} \\ \text{of } \text{mod} \, D \\ \text{with } S \text{ being semisimple} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{basic } (2h_Q - 3)\text{-cluster tilting } C\text{-modules } C \oplus S \end{array} \right\} ,$$

see Theorem 4.28 for details.

Note that by Theorem 0.5 results in Theorem 0.7(1) and 0.8(1) requires a stiff Auslander algebra; the stiffness condition fails for general representation-finite standard (even for 1-Iwanaga-Gorenstein) algebras; see Example 4.22. It is then natural to ask when an Auslander algebra $A$ of a representation-finite algebra is $\ell$-stiff for some $\ell$, see Question 4.21.

1. Preliminaries

We assume that all algebras are finite dimensional algebras over a field $K$ and modules are finitely generated right modules unless stated otherwise. We assume that the reader is familiar with the basics of representation theory of algebras and refer for example to the textbooks [ARS] and [SY]. For background on homological dimensions such as the dominant dimension and related properties we refer for example to [Tac] and [Yam]. For background on Gorenstein homological algebra and maximal Cohen-Macaulay modules we refer to [Che].

Throughout, $D$ denotes the natural duality of the module category of an algebra $A$, $\text{proj} \, A$ denotes the full subcategory of projective $A$-modules and $\text{inj} \, A$ the full subcategory of injective $A$-modules. A module $M$ is called basic if it has no non-zero direct summand of the form $L \oplus_2$ for some indecomposable module $L$. An algebra is called basic if the regular module is a basic module. The Loewy length of a module $M$ will be denoted by $LL(M)$. Recall that the dominant dimension $\text{domdim} \, M$ of a module $M$ with a minimal injective coresolution

$$0 \to M \to I^0 \to I^1 \to \cdots$$

is defined as the smallest $n$ such that $I^n \neq 0$. The codominant dimension $\text{codomdim} \, M$ of a module $M$ is defined as the dominant dimension of the module $D(M)$. $\text{Dom}_n(A)$ denotes the
full subcategory of \( \text{mod} \, A \) consisting of modules having dominant dimension at least \( n \). The dominant dimension of an algebra is defined as the dominant dimension of the regular module. The dominant dimension of an algebra \( A \) is equal to the dominant dimension of its opposite algebra and thus equal to the codominant dimension of \( D(A) \).

Take idempotents \( e, f \in A \) such that \( eA \) is a minimal faithful projective-injective \( A \)-module and \( Af \) is a minimal faithful projective-injective \( A^{\text{op}} \)-module, then we have \( fAf \cong eAe \) as algebras. An algebra has dominant dimension at least two if and only if \( A \cong \text{End}_{A^{\text{op}}}(Af) \) by the Morita-Tachikawa correspondence. Following \([KY1]\) algebras with dominant dimension at least two and self-injective base algebra are called \textit{Morita algebras} and following \([FK]\) such algebras are called \textit{gendo-symmetric} if the base algebra is even a symmetric algebra.

The following proposition is a special case of results from \([Apt]\):

**Proposition 1.1.** Let \( A \) be an algebra with minimal faithful projective-injective \( A^{\text{op}} \)-module \( Af \). Let \( B := fAf \).

1. The adjoint pair \( (- \otimes_A Af, \text{Hom}_B(Af, -)) \) yields equivalences \( \text{Dom}_n(A) \cong \text{mod} \, B \), \( \text{proj} \, A \cong \text{add} \, Af \text{ and proj} \, A \cap \text{inj} \, A \cong \text{inj} \, B \).
2. If \( M \in \text{Dom}_{k+1}(A) \), then \( \text{Ext}^i_A(X, M) \cong \text{Ext}^i_B(X f, M f) \) for all \( i \in [1, k-1] \) and any \( X \in \text{mod} \, A \).
3. \( \text{domdim}(\text{Hom}_B(Af, X)) = \inf \{ i \geq 1 | \text{Ext}^i_B(Af, X) \neq 0 \} + 1 \).

**Proof.** (1) is \([Apt]\), lemma 3.1, (2) is \([Apt]\), theorem 3.2, and (3) is \([Apt]\), proposition 3.7. \( \square \)

An algebra \( A \) is said to be \textit{Iwanaga-Gorenstein} if \( \text{idim} \, A + \text{pdim} \, D(A) < \infty \); in such a case, we have \( \text{idim} \, A = \text{pdim} \, D(A) \) and \( \text{idim} \, A \) is called the \textit{self-injective dimension} of \( A \). A module \( M \) with \( \text{Ext}^i_A(M, A) = 0 \) for all \( i > 0 \) over an Iwanaga-Gorenstein algebra \( A \) is called \textit{maximal Cohen-Macaulay} (also known as Gorenstein projective in the literature). \( \text{CM} \) denotes the full subcategory of maximal Cohen-Macaulay modules of \( A \) and \( \Omega^n(\text{mod} \, A) \) denotes the full subcategory of modules that are isomorphic to \( n \)-th syzygy modules. One can show that an algebra \( A \) is Iwanaga-Gorenstein with self-injective dimension \( n \) if and only if \( \Omega^n(\text{mod} \, A) = \text{CM} \), see for example \([Che]\) theorem 2.3.3. Note that an Iwanaga-Gorenstein algebra \( A \) has finite global dimension if and only if \( \text{CM} = \text{proj} \, A \), in which case the global dimension coincides with the self-injective dimension. We will need the following lemma:

**Lemma 1.2.** \([MV1]\) Proposition 4] Let \( A \) be an algebra of dominant dimension \( n \) equal to \( n \geq 1 \). Then \( \text{Dom}_n(A) = \Omega^n(\text{mod} \, A) \) for all \( i = 0, 1, ..., d \).

In \([Iya3]\), algebras \( A \) having \( \text{gldim} \, A \leq n \leq \text{domdim} \, A \) for an \( n \geq 2 \) are called higher Auslander algebras. We extend this definition in this article and define \textit{higher Auslander algebras} as algebras \( A \) with \( \text{gldim} \, A \leq n \leq \text{domdim} \, A \) for some \( n \geq 1 \). Note that this essentially adds the algebras with \( \text{gldim} \, A = \text{domdim} \, A = 1 \), which are classified as direct products of upper triangular matrix algebras over division rings, see \([Iya5]\).

Recently, in \([JS]\), the authors generalised higher Auslander algebras to the concept of minimal Auslander-Gorenstein algebras which they defined by the condition that they are Iwanaga-Gorenstein algebras with \( \text{idim} \, A \leq n \leq \text{domdim} \, A \) for some \( n \geq 2 \). In this article we extend the definition and define \textit{minimal Auslander-Gorenstein} algebras as algebras \( A \) with \( \text{idim} \, A \leq n \leq \text{domdim} \, A \) for some \( n \geq 1 \).

We refer also to \([CK]\) for an alternative approach to minimal Auslander-Gorenstein algebras and \([ClM]\) for the construction and many example of such algebras. The generator-cogenerators \( M \) such that \( \text{End}_B(M) \cong A \) for a minimal Auslander-Gorenstein algebra \( A \) are called \textit{precluster tilting modules}. For a subcategory \( \mathcal{C} \) of \( \text{mod} \, A \) we define for an integer \( n \)

\[
\mathcal{C}^1 := \{ X \in \text{mod} \, A \mid \text{Ext}_A(X, Y) = 0 \text{ for all } Y \in \mathcal{C} \text{ and } 1 \leq i \leq n \},
\]

\[
\mathcal{C}^1 := \{ X \in \text{mod} \, A \mid \text{Ext}_A(Y, X) = 0 \text{ for all } Y \in \mathcal{C} \text{ and } 1 \leq i \leq n \}.
\]

For example, \( \mathcal{C}^1 = \mathcal{C}^0 \) denotes \( \mathcal{C}^1 = \mathcal{C}^1 = \mathcal{C}^1 = \mathcal{C}^d \) as \( \mathcal{C} \). Note that when \( \mathcal{C} = \text{add} \, M \) for some module \( M \in \text{mod} \, A \) we often write \( M^1 \) or \( M^1 \) instead of \( M^{1} \) or \( M^1 \). For \( d \geq 1 \), we call \( M \in \text{mod} \, A \) \textit{d-cluster tilting} if

\[
\text{add} \, M = M^{1} \oplus M^{1} = \cdots = M^{1} \oplus M^{1}.
\]

The \textit{grade} of a module \( M \) is defined as \( \text{grade} \, M := \inf \{ i \geq 0 \mid \text{Ext}_A^i(M, A) \neq 0 \} \) and dually the \textit{cograde} is defined as \( \text{cograde} \, M := \inf \{ i \geq 0 \mid \text{Ext}_A^i(D(A), M) \neq 0 \} \). The \textit{strong grade} \( s \, \text{grade} \, M \) of a module \( M \) is the infimum of \( \text{grade} \, N \) for all submodules \( N \) of \( M \).
Definition 1.3. [FGR] Let $A$ be an algebra and

\[(1.3.1) \quad 0 \to A \to I^0 \to I^1 \to \cdots \]

a minimal injective coresolution of the $A$-module $A$. We call $A$ Auslander-Gorenstein when $A$ is Iwanaga-Gorenstein and additionally satisfies the following equivalent conditions.

1. $\text{pdim} I^i \leq i$ for all $i \geq 0$.
2. For each $i \geq 1$ and $X \in \text{mod} A$, we have $s.\text{grade Ext}^i_A(X, A) \geq i$.

When an Auslander-Gorenstein algebra $A$ has finite global dimension, $A$ is called Auslander regular. We refer for example to the survey article [C] for more on Auslander-Gorenstein algebras. An algebra $A$ is Auslander-Gorenstein if and only if its opposite algebra is Auslander-Gorenstein. Given an algebra $A$ with minimal injective coresolution (1.3.1), we define the set of dominant numbers of $A$ to be

$$\text{DN}(A) := \{ n \geq 0 \mid \text{pdim} I^n < \text{pdim} I^n \text{ for all } 0 \leq i < n \}.$$ 

For $l \geq 0$ we define the subcategories

$$\text{Gr}_l(A) := \{ M \in \text{mod} A \mid \text{grade} M \geq l \},$$

$$\text{SGr}_l(A) := \{ M \in \text{mod} A \mid s.\text{grade} M \geq l \}.$$

It is elementary that $\text{SGr}_l(A)$ is a Serre subcategory of $\text{mod} A$, that is, closed under subfactor modules and extensions. We will need the following result.

Proposition 1.4. Suppose $A$ is an Auslander-Gorenstein algebra. Then the following hold.

1. $\text{DN}(A) = \text{DN}(A^{\text{op}}) = \{ \text{grade} X \mid X \in \text{mod} A \}$.
2. $\text{Gr}_l(A) = \text{SGr}_l(A)$ and this subcategory is a Serre subcategory of $\text{mod} A$ for all $0 \leq l \leq \text{idim} A$.

Proof. (1) is [Iya1] Theorem 1.1], (2) is a special case of [Iya1] Proposition 2.4] due to the discussion after [Iya1] Definition 2.2].

2. DOMINANT AUSLANDER-GORENSTEIN ALGEBRAS AND BASIC PROPERTIES

2.1. The definition and the first properties.

Definition 2.1. An algebra $A$ a dominant Auslander-Gorenstein algebra if it is Iwanaga-Gorenstein and $\text{domdim} P \geq \text{idim} P$ for every indecomposable projective module $P$. If, furthermore, $\text{gldim} A < \infty$, then we call it a dominant Auslander-regular algebra.

Note that for a dominant Auslander-Gorenstein algebra $\text{domdim} P > \text{idim} P$ holds if and only if $P$ is also injective; otherwise, we have $\text{domdim} P = \text{idim} P$. In particular, a dominant Auslander-Gorenstein algebra is self-injective if and only if we have strict inequality for all $P$.

Proposition 2.2. Suppose $A$ is a dominant Auslander-Gorenstein algebra. Then the following hold.

1. For every indecomposable projective $A$-module $P$ with $d := \text{idim} P \geq 0$, the minimal injective coresolution

\[0 \to P \to T^0 \to T^1 \to \cdots \to T^{d-1} \to I \to 0\]

has indecomposable injective $I$ and projective-injective $T^k$ for all $0 \leq k < d$. In particular, this is also the minimal injective resolution of $I$.
2. The map $P \mapsto I = \Omega^{-\text{idim} P}(P)$ defines a bijection $\pi : \text{ind(proj} A) \to \text{ind(inj} A)$ with inverse $I \mapsto \Omega^{\text{idim} I}(I)$.
3. $A^{\text{op}}$ is also dominant Auslander-Gorenstein.
4. $\text{DN}(A) = \{ \text{idim} P \mid P \text{ indecomposable projective} \}$.
5. $A$ is Auslander-Gorenstein.

Proof. By assumption $A$ is Iwanaga-Gorenstein. Let $n := \text{idim} A$ denote the self-injective dimension of $A$. If $n = 0$, then $A$ is self-injective and the claim is trivial; hence, we will assume $n > 0$ from now on. Label the indecomposable projective (resp. injective) $A$-modules by $P_1, \ldots, P_m$ (resp. $I_1, \ldots, I_m$) so that $\text{top} P_i = \text{soc} I_i$ for all $i$. 

(1) When \( P_i \) is injective, the claim is trivial. Consider a non-injective \( P_i \) with \( d_i := \text{domdim} P_i = \text{idim} P_i \), so there is the following minimal injective coresolution:

\[
0 \to P_i \to T^0_i \to T^1_i \to \cdots \to T^{d_i}_i \to 0.
\]

Since \( P_i \) is non-injective, we have \( \text{domdim} P_i = \text{idim} P_i \), which means that the module \( T^k_i \) is projective-injective for \( 0 \leq k < d_i \). Now \( T^{d_i}_i \) is injective and has no projective direct summands or else the resolution would split off this summand and would not be minimal. If \( T^{d_i}_i = X_1 \oplus X_2 \) for two non-zero direct summands \( X_1 \) and \( X_2 \) then those summands would have both codominant dimension at least \( d_i \) by the above minimal coresolution of \( P_i \). But then \( \Omega^{d_i}(X_1) \) and \( \Omega^{d_i}(X_2) \) would both be non-zero, contradicting that \( P_i \) is indecomposable. Thus \( T^{d_i}_i \) must be indecomposable.

(2) This follows from (1) immediately.

(3) By (1), we have \( \text{codomdim} I_{(\pi(i))} \leq \text{pdim} I_{(\pi(i))} \) with strict inequality if and only if \( I_{(\pi(i))} \nsubseteq P_i \in \text{proj} A \).

By \( K \)-linear duality this means that \( A^{\text{op}} \) is dominant Auslander-Gorenstein.

(4) As \( \text{pdim} T^k_i = 0 \) for all \( 0 \leq k < d_i \), and \( \text{pdim} T^{d_i}_i = \text{pdim} I_{(\pi(i))} = d_i \), the claim on \( \text{DN}(A) \) now follows.

(5) Since the minimal injective coresolution \( (I^k)_k \geq 0 \) of \( A \) is the direct sum of those for \( P_i \) over all \( i \)'s, we have \( \text{pdim} I^k = \text{pdim} \bigoplus I^k_i \), so it immediately follows that \( A \) is Auslander-Gorenstein.

By Theorem 1.4, let \( A \) be a dominant Auslander-Gorenstein algebra with \( \text{DN}(A) = \{d_1 < d_2 < \cdots < d_r\} \), then for each \( 1 \leq i < r \), we have \( \text{Gr}_{d_i+1}(A) = \text{Gr}_{d_i+2}(A) = \cdots = \text{Gr}_{d_i+q}(A) \). In other words, knowledge of the subcategories \( \text{Gr}_{\text{idim} P}(A) \) over all indecomposable projective \( P \) gives complete information on the grades of modules.

**Proposition 2.3.** Let \( A \) be a dominant Auslander-Gorenstein algebra, \( 0 \to A \to I^0 \to I^1 \to \cdots \) a minimal injective coresolution of the regular module \( A \), and \( \text{DN}(A) = \{0 = d_0 < d_1 < \cdots < d_r\} \). Define the idempotents \( f_i \) so that \( D(Af_i) \) is the basic additive generator of \( \text{add}\{\bigoplus_{0 \leq j \leq i} I^j\} \) for all \( 0 \leq i \leq k \). Then \( \text{Gr}_{d_i}(A) \cong \text{mod} A/Af_{i-1}A \) for all \( i \) with \( f_{i-1} = 0 \).

**Proof.** By Theorem 1.4 (b), for any \( 0 \leq t \leq \text{idim} A \), the subcategory \( \text{Gr}_t(A) \) is a Serre subcategory and so there is an idempotent \( e_t \in A \) so that \( \text{Gr}_t(A) \cong \text{mod} A/Ae_tA \).

For \( d = 0 \in \text{DN}(A) \), it is clear that \( \text{Gr}_0(A) = \text{mod} A \). Suppose \( d = d_i \in \text{DN}(A) \) is a dominant number for some \( i > 0 \). A simple module \( S \) is in \( \text{Gr}_d(A) \) if and only if \( \text{Ext}^k_A(S, A) = 0 \) for all \( 0 \leq k < d \), which is equivalent to \( \text{Hom}_A(S, I^k) = 0 \) for all \( 0 \leq k < d \). Thus \( S \in \text{Gr}_d(A) \) if and only if \( S \) is not a submodule of \( I^{<d} := \bigoplus_{k<d} I^k \). For \( k \notin \{d_1, \ldots, d_i = d\} \), as \( I^k \) is projective we have \( \text{soc}(I^k) \in \text{Gr}_0(A) \setminus \text{Gr}_d(A) \). Hence, \( S \) is not a submodule of \( I^{<d} \) if and only if it is not a submodule of \( \bigoplus_{j<i} I^{d_j} = D(Af_{i-1}) \), which is equivalent to \( S \in \text{mod} A/Af_{i-1}A \). \( \square \)

**2.2. Relation with the Gorenstein condition.** The \textit{Gorenstein condition} is a non-commutative modification of a property of commutative Gorenstein rings and was introduced in the study of Artin-Schelter regular algebras. The following condition is an analogue of the Gorenstein condition, and was studied under the additional assumption that \( S \) has finite projective dimension in [H MRS] where it is called \( k \)-regular.

**Definition 2.4.** A simple module \( S \) over an algebra \( A \) is called \( k \)-\textit{Gorenstein} if for \( i \geq 0 \), \( \text{Ext}^i_A(S, A) \neq 0 \) if and only if \( i = k \) and \( \text{Ext}^k_A(S, A) \) is a simple \( A^{\text{op}} \)-module. We will consider the following condition:

(G) \( A \) is Iwanaga-Gorenstein and a simple \( A \)-module \( S \) is \( k \)-Gorenstein for some \( k \geq 1 \) if the injective envelope \( I(S) \) of \( S \) is not projective.

We give the following example, which shows that \( 0 \)-Gorenstein simple modules correspond to simple algebra direct summands for algebras of finite global dimension.

**Example 2.5.** Let \( A \) be a finite dimensional algebra. Any simple \( A \)-module corresponding to a simple algebra direct summand of \( A \) is \( 0 \)-Gorenstein. Conversely, if \( A \) has finite global dimension, then any \( 0 \)-Gorenstein simple \( A \)-module is obtained in this way.

**Proof.** The first assertion is clear. We prove the second one. Without loss of generality, we can assume that \( A \) is not simple. It is basic that \( \text{pdim} M = \sup \{i \geq 0 \mid \text{Ext}^i_A(M, A) \neq 0\} \) holds for each \( M \in \text{mod} A \) with \( \text{pdim} M < \infty \). In particular, each simple module \( S \) that is \( 0 \)-Gorenstein has to be projective since \( A \)
By Proposition 2.2(1), we immediately have $\text{Ext}$. Let $A$ be a simple $A$-module such that $I(S) \notin \text{proj} A \to P := \text{ind}(\text{proj} A) \setminus \text{ind}(\text{inj} A)$.

**Lemma 2.6.** Assume that $A$ satisfies the condition (G). For every indecomposable projective non-injective $A$-module $P$, there exists a unique simple $A$-module $S$ with injective envelope $I(S)$ non-projective and $\text{Ext}_A^k(S, P) \neq 0$ for some $k \geq 1$. In particular, the minimal injective coresolution of $P$ is of the form

$$0 \to P \to I^0 \to I^1 \to \cdots \to I^{k-1} \to I(S) \to 0$$

with $I^0, I^1, \ldots, I^{k-1}$ projective-injective and this defines a bijection $\pi : \text{ind}(\text{proj} A) \to \text{ind}(\text{inj} A)$ given by $P \mapsto \Omega^{-\text{idim} P}(P)$.

**Proof.** We consider a map

$$\alpha : S := \{S \mid \text{simple } A\text{-module such that } I(S) \notin \text{proj} A\} \to P := \text{ind}(\text{proj} A) \setminus \text{ind}(\text{inj} A)$$

defined as follows: By the condition (G), each $S \in S$ is $k$-Gorenstein for some $k \geq 1$. Then $\text{Ext}_A^k(S, A)$ is a simple $A^{op}$-module, so there exists a unique $P \in \text{ind}(\text{proj} A)$ such that $\text{Ext}_A^k(S, P) \neq 0$. Clearly $P \in P$. Now we define $\alpha(S) := P$.

We prove that the map is bijective. Since $\#S = \#P$, it suffices to show that the map $\alpha$ is surjective. For each $P \in P$, let

$$0 \to P \to I^0 \to \cdots \to I^k \to 0$$

be a minimal injective coresolution. Clearly $P \in P$ means that $k \geq 1$ holds. Take a simple submodule $S$ of $I^k$. Since $I^k$ has no indecomposable projective direct summands, we have $S \in S$. By condition (G), $S$ is $k$-Gorenstein, and we have $\alpha(S) = P$. Thus $\alpha$ is surjective. As a consequence of $\alpha$ being bijective, we get that $I^k = I(S)$.

Finally, since there is a canonical bijection $\beta : S \to \text{ind}(\text{inj} A) \setminus \text{ind}(\text{proj} A)$ given by $S \mapsto I(S)$. The restriction of the map $\pi$ to $\text{ind}(\text{proj} A) \setminus \text{ind}(\text{inj} A)$ coincides with $\beta\alpha^{-1}$.

**Theorem 2.7.** Let $A$ be a finite dimensional algebra. Then $A$ satisfies the condition (G) if and only if $A$ is dominant Auslander-Gorenstein.

**Proof.** We prove the “if” part. Assume $A$ is dominant Auslander-Gorenstein.

Let $S$ be simple $A$-module such that its injective envelope $I$ is non-projective and let $d := \text{pdim} I(S)$. By Proposition 2.2(1), we immediately have $\text{Ext}_A^d(S, P) = 0$ whenever $(i, P) \neq (d, \Omega^d I))$ and that the $A^{op}$-module $T := \text{Ext}_A^d(S, A)$ has a unique composition factor. Since $T \simeq \text{RHom}_A(S, A)[d]$ and $\text{RHom}_A(-, A)$ gives a duality $D(A) \to D(A^{op})$, the ring $\text{End}_{A^{op}}(T) \simeq \text{End}_A(S)$ is a division ring. Thus $T$ is a simple $A^{op}$-module, and hence $S$ is $d$-Gorenstein. Therefore, $A$ satisfies condition (G).

We prove the “only if” part. Assume $A$ satisfies condition (G). For $P \in \text{ind}(\text{proj} A)$, then Lemma 2.6 says that $\text{idim} P \leq \text{domdim} P$.

The bijection $\pi$ in Proposition 2.2(2) and in Lemma 2.6 is clearly the same. This is related to the following classical bijection for Auslander-Gorenstein algebras. Let $\text{sim} A$ be the set of (isoclasses of) simple $A$-modules. Recall that an Auslander-Gorenstein algebra $A$ admits a grade bijection $\text{sim} A \to \text{sim} A$ given by $S \mapsto \text{top } D \text{Ext}_A^{\text{grade } S}(S, A)$.

If $A$ is dominant Auslander-regular, then each simple $A$-module $S$ satisfies $\text{Ext}_A^i(S, A) = 0$ for all $i \neq \text{grade } S$, and hence $D \text{Ext}_A^{\text{grade } S}(S, A)$ coincides with the higher Auslander-Reiten translation $\tau_{\text{grade } S} S$.

**Corollary 2.8.** Let $A$ be a dominant Auslander-Gorenstein algebra. For $S \in \text{sim} A$, let $I(S)$ be the injective envelope of $S$. Then the grade bijection coincides with the map $S \mapsto \text{top } \pi^{-1}(I(S))$.

**Proof.** By Theorem 2.7, $A$ satisfies condition (G). The claim then follows from the construction of $\pi = \beta\alpha^{-1}$ in Lemma 2.6.
2.3. The first examples. Before developing more theory of minimal Auslander-Gorenstein algebras, we discuss the construction of examples. The next proposition gives an easy construction of dominant Auslander-Gorenstein algebras from known examples by tensoring with self-injective algebras.

**Proposition 2.9.** Let $A$ be a dominant Auslander-Gorenstein algebra and $B$ be a self-injective algebra over a field $K$. Assume $A$ and $B$ are split algebras over the field $K$. Then the algebra $C := A \otimes_K B$ is again a dominant Auslander-Gorenstein algebra having the same self-injective and dominant dimension.

**Proof.** Since $A$ and $B$ are split over $K$, every indecomposable projective non-injective $C$-module is isomorphic to a module of the form $P \otimes_K Q$ where $P$ is an indecomposable projective non-injective $A$-module and $Q$ is an indecomposable projective $B$-module. By [E17 Proposition 10] we have in this case $\text{idim}(P \otimes_K Q) = \text{idim} P + \text{idim} Q = \text{idim} P$, because $B$ is self-injective and thus $Q$ projective-injective.

By an argument as in [M1] Lemma 6 we have in general $\text{domdim} P \otimes_K Q = \min\{\text{domdim} P, \text{domdim} Q\}$. Since $B$ is self-injective, domdim $Q = \infty$ and thus domdim $P \otimes_K Q = \text{domdim} P = \text{idim} P = \text{idim} P \otimes_K Q$, which shows that $C$ is dominant Auslander-Gorenstein. \hfill \Box

We remark that in general the tensor product for two dominant Auslander-Gorenstein algebras is not dominant Auslander-Gorenstein as easy examples such as two hereditary Nakayama algebras with two simple modules show.

Recall that a quiver algebra is called quadratic if its relations are quadratic and homogeneous.

**Definition 2.10.** Let $n$ be a natural number.

(i) An integer composition of $n$ is a list of positive integers $a = [a_1, ..., a_r]$ with $\sum_{i=1}^r a_i = n$.

Denote by $a_{\leq k} := \sum_{i=1}^k a_i$, then the descent set of $a$ is defined as the subset $D_a = \{a_{\leq 1} = a_1, a_{\leq 2}, ..., a_{\leq r-1}\}$ of $\{1, 2, ..., n-1\}$. The complement $a^{\perp}$ of $a$ is defined as the unique integer composition with sum $n$ such that $D_a^{\perp} = \{1, 2, ..., n-1\} \setminus D_a$.

(ii) For a composition $a = [a_1, ..., a_r]$ of $n$, denote by $N_a := KA_{n+1}/I_a$ the quadratic linear Nakayama algebra given by linear $A_{n+1}$ quiver with $I_a$ generated by paths $a_{\leq k} \rightarrow a_{\leq k} + 1 \rightarrow a_{\leq k} + 2$ for all $1 \leq k < r$:

$$1 \rightarrow 2 \rightarrow \cdots \rightarrow a_{\leq k} \rightarrow a_{\leq k} + 1 \rightarrow a_{\leq k} + 2 \rightarrow \cdots \rightarrow n + 1.$$

All quadratic linear Nakayama algebras are of this form; it is clear that there are $2^{n-1}$ of them (enumerated by the choice of the relations appear). Recall that $\text{LL}(M)$ denotes the Loewy length of a module $M$. The Kupisch series $[\text{LL}(P_1), ..., \text{LL}(P_{n+1})]$ of $N_a$ is given by $[a_1 + 1, a_1, ..., 2, a_2 + 1, a_2, 2, ..., a_r + 1, a_r, ..., 2, 1]$.

**Proposition 2.11.** Let $a$ be a composition of $n+1 \geq 3$.

1. $N_a$ is dominant Auslander-regular with codimdim $I_i = \text{pdim} I_i = \text{LL}(P_i) - 1$ for all non-projective injective $I_i$, where $\text{LL}(P_i)$ is the Loewy length of the indecomposable projective $N_a$-module $P_i$ corresponding the vertex $i$. In particular, $\text{gldim} N_a = \max(a^+) = \max(a^+)$.

2. $N_a$ is higher Auslander if and only if $\min(a^+) = \max(a^+)$. In particular, the number of higher Auslander quadratic linear Nakayama algebra is equal to the number of divisors of the natural number $n - 1$.

**Remark 2.12.** We remark that $N_a^{\perp}$ is the opposite ring of the quadratic dual $(N_a)^{\perp} = KA_{n+1}^{op}/I_a^{\perp}$ (hence, Koszul dual) of $N_a$, where $I_a^{\perp}$ is generated by all length 2 paths $x + 1 \rightarrow x + 1 \rightarrow x$ such that $x \rightarrow x + 1 \rightarrow x + 2$ is not a relation in $N_a$. The ‘duality’ between homological dimension and Loewy length is not a coincidence for Nakayama algebras, but a more general phenomenon that we will detail in Section 4.

**Example 2.13.** Consider $a = [2, 3, 1]$ so that $N_a$ is given by:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7.$$ This has $D_a = \{2, 5\}$ which means that the indecomposable projective-injective modules are $P_i$ for $i \in 1 + (D_a \cup \{0\}) = \{1, 3, 6\}$. More precisely, we have $P_1 = I_3, P_3 = I_6, P_6 = I_7$,.
and they have Loewy lengths 3, 4, 2 respectively. Now we have \( D_{a\perp} = \{1, 2, \ldots, 6\} \setminus D_a = \{1, 3, 4, 6\} \), and this yields \( a^\perp = [1, 2, 1, 2] \). Add 1 to each element of \( D_{a\perp} \) then the resulting set enumerates the indecomposable non-injective projective modules, whose injective coresolutions are shown in the left-hand side as follows; the right-hand side show the corresponding Gorenstein condition.

\[
\begin{align*}
0 & \longrightarrow P_2 \longrightarrow I_3 \longrightarrow I_1 \longrightarrow 0 & \text{Ext}^1_S(S_1, A) = S_{i1}^{\text{depth}} \\
0 & \longrightarrow P_4 \longrightarrow I_6 \longrightarrow I_3 \longrightarrow I_2 \longrightarrow 0 & \text{Ext}^2_S(S_2, A) = S_{21}^{\text{depth}} \\
0 & \longrightarrow P_5 \longrightarrow I_6 \longrightarrow I_4 \longrightarrow 0 & \text{Ext}^2_S(S_4, A) = S_{11}^{\text{depth}} \\
0 & \longrightarrow P_7 \longrightarrow I_7 \longrightarrow I_6 \longrightarrow I_5 \longrightarrow 0 & \text{Ext}^2_S(S_5, A) = S_{11}^{\text{depth}}
\end{align*}
\]

Notice how entries of \( a^\perp \) corresponds to \( \text{idim} P_i \) for \( i \in D_{a\perp} \). Also, the grade bijection \( S \mapsto D\text{Ext}^{\text{grade}}_A(S, A) \) is given by \((1, 2, 4, 5, 7, 6, 3)\), where \((\ldots, i_j, i_{j+1}, \ldots)\) is the cyclic permutation that maps \( i_j \) to \( i_{j+1} \), etc.

The proof of Proposition 2.14 follows from the next result. We give a construction via gluing that illustrates that in general one can expect that there are much more dominant Auslander-Gorenstein algebras than minimal Auslander-Gorenstein algebras.

Consider two algebras \( B := KQ_B/IB \) and \( C = KQ_C/IC \). Choose \( n \) sinks in \( Q_B \) and \( n \) sources in \( Q_C \), label them \( v_1, \ldots, v_n \), so that we obtain a new quiver \( Q \) where the \( v_i \)'s in the two quivers are identified. Define a new algebra

\[ A := KQ/I = KQ/(IB, IC, bC | b \in B, c \in C). \]

Note that \( I \) is an admissible ideal and so \( A \) is finite-dimensional. For \( \Lambda \in \{B, C\} \), we denote by \( P^A_x, I^A_x \) the indecomposable projective, indecomposable injective \( \Lambda \)-module corresponding to vertex \( x \) in the quiver of \( \Lambda \), respectively.

**Proposition 2.14.** Let \( A, B, C \) be algebras as above.

1. If \( B \) and \( C \) are dominant Auslander-Gorenstein algebras, then so is \( A \).
2. In this case, we have

\[
\text{idim} P^A_x = \begin{cases} 
\text{idim} P^B_x, & \text{if } x \in Q_B; \\
\text{idim} P^C_x, & \text{if } x \in Q_C \text{ and } \pi(x) \notin \{v_i\}; \\
\text{idim} P^C_x + \text{idim} P^B_{v_i}, & \text{if } x \in Q_C \text{ and } \pi(x) = v_i, \text{ some } i,
\end{cases}
\]

where \( \pi \) denotes the canonical permutation associated to \( C \). The same formula holds if we replace \( \text{idim} \) by \( \text{domdim} \).

3. Let \( D \) be the union of \( \{\text{idim} C, \text{idim} P^B_x | x \in Q_B \setminus \{v_i\}\} \) and \( \{\text{idim} P^B_{v_i} + \text{idim} I^C_{v_i} | 1 \leq i \leq n\} \), then we have \( \text{idim} A = \max D \) and \( \text{domdim} A = \min D \).

**Proof.** For simplicity, we omit the superscript \( A \) for \( A \)-modules. Let \( I_x \) be an indecomposable projective \( A \)-module corresponding to the vertex \( x \in Q \). Note that if \( P^A_{v_i} \in \text{inj} A \) (resp. \( I^A_{v_i} \in \text{proj} A \)) for any \( \Lambda \in \{B, C\} \), then \( P_y \in \text{inj} A \) (resp. \( I_y \in \text{proj} A \)).

If \( x \in Q_C \), then the minimal projective resolution for \( I^C_x \) lifts to that of \( I_x \) in mod \( A \) naturally (i.e. we can just remove the superscript \( C \) everywhere).

Suppose that \( x \in Q_B \) and let \( p_x := \text{pd} I^B_x \). Since \( B \) is dominant Auslander-Gorenstein, we have \( \Omega^B_y(I^B_x) \sim P^B_y \) for some \( y \in Q_B \). If \( y \notin \{v_i\} \), then the full minimal projective resolution of \( I^B_y \) in mod \( B \) lifts to that of \( I_x \) in mod \( A \). Otherwise, splitting with the (lift of the) minimal projective resolution of \( I^B_y \) with the that of \( I^B_x \) yields the full minimal projective resolution of \( I_x \). Since \( B \) and \( C \) are dominant Auslander-Gorenstein, all terms in these projective resolutions are projective-injective except the last one. The claimed formulae are now clear from these resolutions.

Quadratic linear Nakayama algebras are all obtained from iteratively gluing hereditary Nakayama algebras and in this way one obtains a proof of Proposition 2.14 as a special case of 2.14. \qed
Remark 2.15. Let
\[(2.15.1) \quad \tilde{B} := \left( \begin{array}{cc} B & BM_A \\ 0 & \Lambda \end{array} \right), \quad \tilde{C} := \left( \begin{array}{cc} \Lambda & \Lambda N_C \\ 0 & C \end{array} \right), \quad A := \left( \begin{array}{ccc} B & M & 0 \\ 0 & \Lambda & N \\ 0 & 0 & C \end{array} \right). \]

Proposition 2.14 can be generalized as follows: Suppose we have two dominant Auslander-Gorenstein algebras $\tilde{B}$ and $\tilde{C}$ of the form \[(2.15.1)\]. If $\Lambda$ is self-injective and both $M, N$ are sincere as $\Lambda$-modules, then the algebra $A$ given by \[(2.15.1)\] is also dominant Auslander-Gorenstein.

3. Mixed precluster tilting modules and Dominant Auslander-Gorenstein algebras

3.1. Dominant Auslander-Solberg correspondence. Recall that the higher Auslander-Reiten translates and their inverses are defined by $\tau_n := \tau \Omega^{n-1}$ and $\tau_n^- := \tau^{-1} \Omega^{-(n-1)}$.

Definition-Proposition 3.1. Let $B$ be a finite dimensional algebra, and $C$ a full subcategory of $\text{mod} \ B$ which is functorially finite and satisfies $\text{add} \ C = C$. We say that $C$ is mixed precluster tilting if $B \oplus DB \in C$ and the following equivalent conditions are satisfied.

(i) For each $X \in \text{ind} \ C \setminus \text{ind} \ (\text{proj} \ B)$, there exists $c_X \geq 1$ such that $\text{Ext}^i_B(X, C) = 0$ for all $1 \leq i < c_X$ and $\tau_X(X) \in C$.

(ii) For each $X \in \text{ind} \ C \setminus \text{ind} \ (\text{inj} \ B)$, there exists $d_X \geq 1$ such that $\text{Ext}^i_B(X, C) = 0$ for all $1 \leq i < d_X$ and $\tau_X^-(X) \in C$.

We call $M \in \text{mod} \ B$ is mixed precluster tilting if so is $\text{add} M$.

Now we prove the following generalisation of the Auslander-Solberg correspondence [AS, IS].

Theorem 3.2. There exists a bijection between the following objects.

1. The Morita equivalence classes of dominant Auslander-Gorenstein algebras $A$ with $\text{domdim} \ A \geq 2$.
2. The Morita equivalence classes of pairs $(B, M)$ of finite dimensional algebras $B$ and mixed precluster tilting $B$-modules $M$.

The correspondence from $(2)$ to $(1)$ is given by $(B, M) \mapsto A := \text{End}_B(M)$.

The proof is divided into two parts: Propositions 3.3 and 3.4.

Proposition 3.3. Let $B$ be a finite dimensional $k$-algebra. If $M$ is a mixed precluster tilting $B$-module, then $A = \text{End}_B(M)$ is a dominant Auslander-Gorenstein algebra with $\text{domdim} \ A \geq 2$.

Proof. Let $A = \text{End}_B(M)$. Fix an indecomposable projective $A$-module $P$. Then $P = \text{Hom}_B(M, X)$ holds for some indecomposable $X \in \text{add} M$.

(i) Assume $X \in \text{inj} \ B$. Then $P = \text{Hom}_B(M, X) \simeq D \text{Hom}_B(\nu^{-1}(X), M)$ is an injective $A$-module. Thus $P$ satisfies the desired condition.

(ii) Assume $X \notin \text{inj} \ B$. We take an exact sequence
\[(3.3.1) \quad 0 \rightarrow X \rightarrow I^0 \rightarrow \cdots \rightarrow I^{d_X-1} \rightarrow I^{d_X} \]
of $B$-modules such that each $I^i$ is injective. Applying $\text{Hom}_B(M, -)$ and using $\text{Ext}_B^i(M, X) = 0$ for all $1 \leq i < d_X$, we obtain an exact sequence
\[(3.3.2) \quad 0 \rightarrow P \rightarrow \text{Hom}_B(M, I^0) \rightarrow \cdots \rightarrow \text{Hom}_B(M, I^{d_X-1}) \rightarrow \text{Hom}_B(M, I^{d_X}). \]

Here we use the notation $(M, a)$ to denote the application of the functor $\text{Hom}_B(M, -)$ to $a$. Similarly, we will use the dual notation for functors of the form $\text{Hom}_B(-, M)$. Each $\text{Hom}_B(M, I^i)$ is a projective-injective $A$-module by the same reason as in (i). It remains to show that $\text{Cok}(M, a)$ is an injective $A$-module. By the following commutative diagram, we have $\text{Cok}(M, a) = \text{Cok} D(\nu^{-1}(a), M)$.

\[
\begin{array}{ccc}
\text{Hom}_B(M, I^{d_X-1}) & \xrightarrow{(M,a)} & \text{Hom}_B(M, I^{d_X}) \\
\| & & \| \\
D \text{Hom}_B(\nu^{-1}(I^{d_X-1}), M) & \xrightarrow{D(\nu^{-1}(a), M)} & D \text{Hom}_B(\nu^{-1}(I^{d_X}), M)
\end{array}
\]
From (3.3.1), we have an exact sequence $\nu^{-1}(I^{d+1}) \xrightarrow{\nu^{-1}(a)} \nu^{-1}(I^d) \to \tau^{-1}_{dX}(X) \to 0$. Applying $D\text{Hom}_B(\cdot, M)$, we have an exact sequence

$$D\text{Hom}_B(\nu^{-1}(I^{d+1}), M) \xrightarrow{D(\nu^{-1}(a), M)} D\text{Hom}_B(\nu^{-1}(I^d), M) \rightarrow D\text{Hom}_B(\tau^{-1}_{dX}(X), M) \to 0.$$  

Thus $\text{Cok}(M, a) = D\text{Hom}_B(\tau^{-1}_{dX}(X), M)$ is an injective $A$-module since $\tau^{-1}_{dX}(X) \in \text{add} M$ by our assumption. □

The converse map of Theorem 3.2 is given by the following observation.

**Proposition 3.4.** Let $A$ be a dominant Auslander-Gorenstein algebra with $\text{domdim} A \geq 2$. Take idempotents $e, f \in A$ satisfying $\text{add} eA = \text{proj} A \cap \text{inj} A = \text{add} D(Af)$. Let $B = fAf$. Then $M := Af$ is a mixed precluster tilting $B$-module and satisfies $\text{End}_B(M) = A$.

**Proof.** We have $\text{add} eAf = \text{add} D(Af) = \text{add} DB = \text{inj} B$.

(i) We show $\text{End}_B(M) = A$.

This is routine. Take an injective resolution $0 \to A \to J^0 \to J^1$ with $J^i \in \text{add} D(Af)$. Multiplying $f$ from the right, we have an exact sequence $0 \to M \to J^0 f \to J^1 f$ of $B$-modules. Applying $\text{Hom}_B(M, \cdot) = \text{Hom}_B(Af, \cdot)$ to the second sequence and comparing with the first sequence, we have a commutative diagram of exact sequences

$$0 \xrightarrow{} \text{End}_B(M) \xrightarrow{} \text{Hom}_B(Af, J^0 f) \xrightarrow{} \text{Hom}_B(Af, J^1 f)$$

where we used $\text{Hom}_B(Af, D(Af)f) = D(Af)$. Thus $\text{End}_B(M) = A$.

(ii) We show that $M$ is a mixed precluster tilting $B$-module.

Fix an indecomposable object $X \in \text{add} M \setminus \text{inj} B$. Since $\text{End}_B(M) = A$, there exists a primitive idempotent $g \in A$ such that $X = gAf$ and $gA \notin \text{add} eA$. Since $\text{domdim} A \geq 2$, there exists a minimal injective resolution of the $A$-module $gA$

$$(3.4.1) \quad 0 \to gA \to I^0 \to \cdots \to I^{d-1} \to I^d \to I^{d+1} \to 0,$$

such that $d \geq 1$ and $I^i \in \text{add} eA$ for all $0 \leq i \leq d$. Since $I^{d+1}$ is indecomposable, we can write $I^{d+1} = D(Ag')$ for a primitive idempotent $g' \in A$.

Multiplying $f$ to (3.4.1) from the right, we have an exact sequence

$$(3.4.2) \quad 0 \to X \to I^0 f \to \cdots \to I^{d-1} f \to I^d f \to D(fAg') \to 0$$

with $I^i f \in \text{add} eAf$ for all $0 \leq i \leq d$. Applying $\text{Hom}_B(M, \cdot) = \text{Hom}_B(Af, \cdot)$ to (3.4.2) and comparing with (3.4.1), we have a commutative diagram

$$\text{Hom}_B(Af, I^0 f) \xrightarrow{} \cdots \xrightarrow{} \text{Hom}_B(Af, I^{d-1} f) \xrightarrow{} \text{Hom}_B(Af, I^d f)$$

where we used $\text{Hom}_B(Af, D(Af)f) = D(Af)$. Thus the upper sequence is exact, and we have $\text{Ext}_B^1(M, X) = 0$ for all $1 \leq i \leq d$.

In the rest, we prove $\tau^{-1}_{d}(X) = g' Af$, which completes the proof. Applying $\text{Hom}_B(DB, \cdot)$ to (3.4.2) and $\text{Hom}_A(D(Af), \cdot)$ to (3.4.1) and comparing them, we have a commutative diagram

$$\text{Hom}_B(DB, I^{d-1} f) \xrightarrow{} \text{Hom}_B(DB, I^d f) \xrightarrow{} \tau^{-1}_{d}(X) \xrightarrow{} 0,$$

$$\text{Hom}_A(D(Af), I^{d-1}) \xrightarrow{} \text{Hom}_A(D(Af), I^d) \xrightarrow{} \text{Hom}_A(D(Af), D(Ag')) \xrightarrow{} 0,$$

where we used $\text{Hom}_B(DB, D(Af)f) = B = fAf = \text{Hom}_A(D(Af), D(Af))$. The upper sequence is exact by definition of $\tau^{-1}_d$, and the lower sequence is exact by $D(Af) \in \text{proj} A$. Since the right term of the lower sequence is $\text{Hom}_A(D(Af), D(Ag')) \simeq \text{Hom}_{A^o}(Ag', Af) = g' Af$, we obtain $\tau^{-1}_{d}(X) = g' Af$. □
3.2. Examples of mixed precluster tilting modules. For the next proposition call a subset \( S \) of \( \mathbb{Z}/n\mathbb{Z} \) isolated if it has the property that with \( s \in S \) we have \( s \pm 1 \not\in S \).

Proposition 3.5. Let \( A \) be a symmetric algebra and \( M \) and indecomposable non-projective \( A \)-module with \( \Omega^n(M) \cong M \) for some \( n \geq 1 \) and \( \text{Ext}^i_A(M,M) = 0 \) for each \( 1 \leq i \leq n - 2 \). Assume \( S \) is a isolated subset of \( \mathbb{Z}/n\mathbb{Z} \) with representatives in \( \{0, \ldots, n - 1\} \) and let \( N := A \oplus \bigoplus_{s \in S} \Omega(M) \). Then \( N \) is a mixed precluster tilting \( A \)-module.

Proof. When \( S = \emptyset \), then \( N = A \) is clearly mixed precluster tilting and thus assume \( S \neq \emptyset \) in the following. We recall that in a symmetric algebra we have \( \tau \cong \Omega^2 \) and hence \( \text{mod} \, A \) is a \((-1)\)-Calabi-Yau triangulated category.

Let \( X = \Omega^i(M) \) with \( i \in S \) and \( c_X = \min \{ i \geq 1 \mid i + r + 1 \in S \} \). Using \( \tau = \Omega^2 \), we obtain
\[
\tau_{c_X}(X) = \Omega^{c_X+i+r}(M) \in \text{add} \, N.
\]
It remains to prove \( \text{Ext}^i_A(X,N) = 0 \) for each \( 1 \leq i \leq c_X - 1 \). For each \( i \in \mathbb{Z} \), let \( \text{Ext}^i_A(U,V) := \text{Hom}_A(\Omega^i(U),V) \). Then
\[
\text{Ext}^n_A(M,M) \cong \text{Hom}_A(\Omega^n(M),M) \cong \text{Hom}_A(M,M) \neq 0,
\]
\[
\text{Ext}^{n-1}_A(M,M) \cong \text{Ext}^{-1}_A(M,M) \cong \text{DHom}_A(M,M) \neq 0,
\]
Thus \( \text{Ext}^i_A(M,M) \neq 0 \) if and only if \( i \equiv -1 \) or \( i \equiv 0 \) mod \( n \). Now we have
\[
\text{Ext}^i_A(X,N) = \text{Ext}^i_A(\Omega^i(M),\bigoplus_{s \in S} \Omega^s(M)) = \bigoplus_{s \in S} \text{Ext}^i_A(\Omega^i(M),\Omega^s(M)) = \bigoplus_{s \in S} \text{Ext}^{i+r-s}_A(M,M).
\]
Since \( S \) is isolated, \( i + r - s \not\equiv -1,0 \) mod \( n \) for each \( 1 \leq i \leq c_X - 1 \). Hence this equals 0. \( \square \)

Example 3.6. Let \( A \) be a connected symmetric Nakayama algebra with \( n \) simple modules. Then any simple \( A \)-module \( S \) satisfies \( \Omega^{2n}(S) \cong S \) and \( \text{Ext}^i_A(S,S) = 0 \) for \( 1 \leq i \leq 2n - 2 \). Thus the previous proposition can be applied in this case and leads to a large class of dominant Auslnder-Gorenstein algebras. More generally, a similar process works for any Brauer tree algebra and leads to the recently introduced gendo Brauer tree algebras in [CIM]. This class of algebras were one of our motivating examples for the study of mixed precluster tilting modules.

Following [CIM], the SGC-extension (smallest generator-cogenerator extension) of a basic algebra \( A \) is the algebra \( \text{End}_A(A \oplus D(A)) \). Our next results shows that the SGC-extension of a dominant Auslnder-Gorenstein algebra that is gendo-symmetric, is still dominant Auslnder-Gorenstein.

Proposition 3.7. Let \( A \) be a gendo-symmetric algebra and \( M = D(A) \oplus A \) with \( B := \text{End}_A(M) \) the SGC-extension of \( A \). If \( A \) is dominant (respectively, minimal) Auslnder-Gorenstein, then \( B \) is dominant (respectively, minimal) Auslnder-Gorenstein.

Proof. Up to Morita equivalence, we can assume \( B = \text{End}_A(M) \) is basic for \( M = D(A) \oplus P_1 \oplus \ldots \oplus P_r \) with indecomposable projective \( A \)-modules \( P_i \). We will use two results on gendo-symmetric algebras. The first is that a module \( M \) over a gendo-symmetric algebra \( A \) with dominant dimension at least two has dominant dimension \( \text{dim} \, M = \inf \{ i \geq 1 \mid \text{Ext}^i_A(D(A),M) \not= 0 \} + 1 \), see for example [FK] at the end of section 2. The second needed result is that for a module \( M \) with dominant dimension at least two, we have \( \tau^{-1}(M) \cong \Omega^{-2}(M) \), see [Mar] proposition 4.3. Let \( U_i := \text{Hom}_A(M,P_i) \) be the indecomposable projective non-injective \( B \)-modules corresponding to the indecomposable projective non-injective \( A \)-module \( P_i \) and let \( n_i + 2 \) denote the dominant dimension of the \( P_i \). We have \( n_i + 2 = \text{domdim} \, P_i = \text{domdim} \, U_i \) by the theorem of Fang-Koenig for the dominant dimensions of modules in gendo-symmetric algebras and (3) of [I]. Now note that \( \tau^{-1}(\Omega^{-n_i}(P_i)) = \Omega^{-(n_i+2)}(P_i) \) because the module \( \Omega^{-n_i}(P_i) \) has dominant dimension at least two. But since \( \text{domdim} \, P_i = n_i + 2 \), we can assume \( \text{domdim} \, P_i = \text{idim} \, P_i \), we have \( \Omega^{-(n_i+2)}(P_i) \in \text{add} \, M \) and thus \( M \) is a mixed precluster tilting module and \( B \) is dominant Auslnder-Gorenstein. \( \square \)

The previous proposition can be used to obtain Iwanaga-Gorenstein algebras of infinite global dimension from algebras with finite global dimension. We give one example.
Example 3.8. A representation-finite block of a Schur algebra $A$ is a gendo-symmetric higher Auslander algebra, see for example section 6.1 in [CM]. Applying the previous proposition to this algebra we obtain that the SGC-extension $B$ of $A$ is a minimal Auslander-Gorenstein algebra of infinite global dimension.

3.3. Mixed cluster tilting modules and Dominant Auslander correspondence. In this section we give a description of $\mathcal{Z}(M)$ when $A$ is a dominant Auslander-Gorenstein algebra with dominant dimension at least two that generalises the result from [IS] where such a description was given for minimal Auslander-Gorenstein algebras. We apply this description to give characterisation when dominant Auslander-Gorenstein algebras with dominant dimension at least two have finite global dimension and use this to generalise the classical higher Auslander correspondence.

**Definition-Proposition 3.9.** Let $B$ be an algebra, and $M$ a mixed precluster tilting $B$-module. We consider a full subcategory

$$\mathcal{Z}(M) := \bigcap_{X \in \text{ind} \text{(add } M)} X^{\perp_{\text{CM}} - 1} = \bigcap_{X \in \text{ind} \text{(add } M)} ^{\perp_{\text{CM}} - 1}X.$$

We call $M$ a mixed cluster tilting $B$-module if $\mathcal{Z}(M) = \text{add } M$.

As in the case of precluster tilting modules, we have the following equivalence.

**Theorem 3.10.** Let $A$ be a dominant Auslander-Gorenstein algebra with dominant dimension at least two corresponding to $(B,M)$. Then we have a commutative diagram of equivalences of categories:

$$\begin{array}{ccc}
\mathcal{Z}(M) & \xleftarrow{\text{Hom}_{B}(M,-)} & \text{CM} A \\
\downarrow & & \downarrow \\
\mathcal{Z}(M) & \xrightarrow{\text{Hom}_{A}(\cdot,-)} & (\text{CM} A^{op})^{op}. \\
\end{array}$$

**Proof.** Let $A$ be an Iwanaga-Gorenstein algebra of dominant dimension at least two and self-injective dimension $g$. Note that $\text{CM} A = \Omega^g(\text{mod } A) \subseteq \Omega^2(\text{mod } A) = \text{Dom}_2(A)$ and thus every maximal Cohen-Macaulay module has dominant dimension at least two. By restricting the equivalence between $\text{Dom}_2(A)$ and $\text{mod } fAf$ (see Proposition 1.1), this gives us that there is an equivalence of categories $\text{CM} A \cong C$, where $C := \{X \mid X \in \text{CM} A\}$. We will in fact show that $C = \mathcal{Z}(M)$.

The functor $F := - \otimes_A fAf$ gives an equivalence between $\text{Dom}_2(A)$ and $\text{mod } fAf$ and as explained above $\text{CM} A \subseteq \text{Dom}_2(A)$ shows that $F$ restricts to an equivalence between $\text{CM} A$ and $F(\text{CM} A)$ with inverse $\text{Hom}_{B}(M,-)$. We just have to determine $F(\text{CM} A)$. Let $N$ be a maximal Cohen-Macaulay $A$-module. Then $N$ is characterised by $\text{Ext}^l_A(N,A) = 0$ for all $l \geq 2$. Now $\text{Ext}^l_A(N,A) = 0$ for $l = 1, 2, ..., g$ if and only if $\text{Ext}^l_A(N,P_i) = 0$ for all indecomposable projective non-injective modules $P_i$ with injective dimension $d_i + 1$ and $r = 1, 2, ..., d_i + 1$. By Proposition 1.1 (2) we have $\text{Ext}^l_A(N,P_i) \cong \text{Ext}^l_{fAf}(Nf,P_i) \cong \text{Ext}^l_{fAf}(Nf,M_i)$, where $P_i = \text{Hom}_B(M,M_i)$ and $r = 1, 2, ..., d_i - 1$. Thus $N \in \text{CM} A$ implies that $Nf \in \bigcap_{i=1}^{d_i - 1}M_i = \mathcal{Z}(M)$.

Now let $X \in \bigcap_{i=1}^{d_i - 1}M_i$. We have to show that $N := \text{Hom}_B(M,X) \in \text{CM} A$, which is equivalent to $\text{Ext}^l_A(N,P_i)$ for all indecomposable projective non-injective modules $P_i$ with injective dimension $d_i + 1$ and $r = 1, 2, ..., d_i + 1$. Setting $P_i = \text{Hom}_B(M,M_i)$ and again using Proposition 1.1 (2) we get $\text{Ext}^l_A(N,P_i) = \text{Ext}^l_{fAf}(X,M_i) = 0$ by assumption for $r = 1, ..., d_i - 1$. Since $N \in \text{Dom}_2(A) = \Omega^2(\text{mod } A)$ by Lemma 1.2, we can write $N = \Omega^2(N')$ for some $A$-module $N'$. Then for $t \geq d_i$ we have $\text{Ext}^l_A(N,P_i) = \text{Ext}^l_A(\Omega^2(N'),P_i) = \text{Ext}^{l+2}(N',P_i) = 0$ since $d_i + 1 \leq t + 2$. Altogether we proved $\text{Ext}^l_A(N,A) = 0$ for all $l \geq 1$ and thus $N \in \text{CM} A$.

To see the commutativity of the diagram, first note that $\text{Hom}_{A}(\cdot,-) : \text{CM} A \rightarrow \mathcal{Z}(M)$ is an equivalence. Then its right adjoint functor $\text{Hom}_{B}(M,-) : \mathcal{Z}(M) \rightarrow \text{CM} A$ gives its quasi-inverse. Since $M$ is a projective $A^{op}$-module, we have an isomorphism of functors

$$\text{Hom}_{A^{op}}(\cdot,-) \circ \text{Hom}_{A}(\cdot,-) \cong \text{Hom}_{A^{op}}(\cdot,-).$$
Since $M$ is a generator, $\text{Hom}_A(M,-) : Z(M) \to \text{CM}_A$ is fully faithful and we have an isomorphism of functors

$$\text{Hom}_B(-,M) \simeq \text{Hom}_A(\text{Hom}_B(M,-),\text{Hom}_B(M,M)) = \text{Hom}_A(-,A) \circ \text{Hom}_B(M,-).$$

Thus the commutativity of the diagram follows. In particular, the lower horizontal functors are equivalences, and all claims follow. □

We obtain the next result immediately. Notice that the global dimension of $A$ equals its self-injective dimension in this case.

**Corollary 3.11.** Let $A$ be a dominant Auslander-Gorenstein algebra with dominant dimension at least two associated to $(B,M)$. Then the following are equivalent:

1. $\text{gldim} A < \infty$.
2. $\text{CM}_A = \text{proj} A$.
3. $M$ is a mixed cluster tilting $B$-module.

**Proof.** The equivalence of (1) and (2) follows immediately from the fact that an Iwanghai-Gorenstein algebra has finite global dimension if and only $\text{CM}_A = \text{proj} A$ and in this case the global dimension equals the self-injective dimension of $A$. The equivalence of (2) and (3) follows directly from 3.10. □

As a consequence of Theorem 3.2 and Corollary 3.11, we obtain a generalisation of the classical higher Auslander correspondence. Due to its importance we formulate it as a theorem.

**Theorem 3.12.** There exists a bijection between the following objects.

1. The Morita equivalence classes of dominant Auslander-regular algebras $A$ with $\text{domdim} A \geq 2$.
2. The Morita equivalence classes of pairs $(B,M)$ of finite dimensional algebras $B$ and mixed cluster tilting $B$-modules $M$.

The correspondence from (2) to (1) is given by $(B,M) \mapsto A := \text{End}_B(M)$.

3.4. **Examples of mixed cluster tilting modules.** First we give an example of a mixed cluster tilting module in a Nakayama algebra.

**Example 3.13.** Let $A$ be the linear Nakayama algebra with Kupisch series $[2,3,3,2,1]$. Then the basic version $M$ of $A \oplus D(A)$ is a mixed cluster tilting module. In fact the endomorphism algebra of $M$ is the Nakayama algebra with Kupisch series $[3,3,3,3,2,2,1]$, which is a dominant Auslander-regular algebra with global dimension 4 and dominant dimension 3.

We give an example of a mixed cluster tilting module of a self-injective algebra that is not cluster tilting and in fact has no non-trivial cluster tilting module, see [DK] for a recent classification of non-trivial cluster tilting modules in trivial extension algebras of Dynkin type. Here non-trivial means that the cluster tilting module $M$ in $\text{mod} \; B$ does not have the property $\text{add} \; M = \text{mod} \; B$.

**Example 3.14.** Let $Q$ be the following quiver of Dynkin type $D_5$:

$$
\begin{array}{c}
2 \\
\downarrow
\end{array}
\quad
\begin{array}{c}
1 \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
3 \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
4 \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
5
\end{array}
$$

Let $B$ be the trivial extension algebra of $KQ$ and view the indecomposable projective $KQ$-module $P_1$ as a $B$-module. Then $M = B \oplus P_1 \oplus \Omega^4(P_1) \oplus \Omega^9(P_1)$ is a mixed cluster tilting $B$-module whose endomorphism ring has global dimension 5 and dominant dimension 4. We leave the verification to the reader. By the main result of [DK], there is in fact no non-trivial cluster tilting $B$-module. In forthcoming work we will show that in fact every trivial extension of a Dynkin type path algebra has a non-trivial mixed cluster tilting module while there is no non-trivial cluster tilting module in general.

The next example shows that in contrast to the (pre)cluster tilting situation [Iya4, Theorem 2.3(1)] [IS, Proposition 3.8(b)], the conditions $\tau_{\leq X}^-(X) \in \text{add} \; M$ and $\tau_{\geq X}^-(X) \in \text{add} \; M$ do not follow from the rest of the conditions in the definition of a mixed cluster tilting module.
Example 3.15. Let $A$ be the connected symmetric Nakayama algebra of Loewy length 3 with two simple modules $S$ and $T$. Let $X$ be indecomposable $A$-module of length two with simple top $S$ and simple socle $T$. Take $M := A \oplus S \oplus X$, $d_S = 1 = c_X$ and $d_X = 2 = c_S$. It is routine to check that $S \in M^{ \perp_{c_S-1}}$ and $X \in M^{ \perp_{d_X-1}}$, and $M^{ \perp_1}X = \mathrm{add} M = S^{\perp_1}$. However, $M$ is not mixed cluster tilting; indeed, $B := \mathrm{End}_A(M)$ is given by the

\[
\begin{array}{c}
C & \xleftarrow{a} & D \xleftarrow{b} & \xleftarrow{c} & E \\
\xleftarrow{d} & S \xleftarrow{e} P \\
X & \xleftarrow{f} & T
\end{array}
\]

with relations $ab - cd, dea, ecc$. The indecomposable projective $B$-module corresponding to the point $P_S$ in the quiver has dominant dimension 2 and injective dimension 3. Hence, $B$ is not a dominant Auslander regular algebra and thus $M$ cannot be a mixed cluster tilting. Note that $\tau_2(X) = T \notin \mathrm{add} M$ and $\tau_2^{-1}(S) = \mathrm{rad} P_S \notin \mathrm{add} M$.

4. Koszul dual of a dominant Auslander-regular algebra

In subsection 2.3 we showed that any quadratic Nakayama algebra $A$ (with a linear quiver) is dominant Auslander-regular. Readers familiar with the subject will know that this class of algebras is Koszul and closed under taking Koszul dual, that is, the Yoneda algebra $E(A) := \mathrm{Ext}^*_A(A_0, A_0) = \bigoplus_{k \geq 0} \mathrm{Ext}^k_A(A_0, A_0)$ where $A_0 := A/\mathrm{rad} A$. Additionally, there is some interesting swapping between Loewy lengths and homological dimensions on the two sides. We will prove this phenomenon more generally.

We will assume the reader is familiar with the basics of Koszul algebras and Koszul duality; see [BGS, MV2, PP] for example. However, key results we need will be quoted with appropriate reference.

4.1. Our main results. In the following, for a Koszul algebra $A$, we denote by $P_x, I_x, S_x$ the indecomposable projective, indecomposable injective, simple $A$-module, respectively, indexed by $x \in \mathbb{Z}$ for some indexing set $\mathcal{I}$; and denote by $P_x^{\tau}, I_x^{\tau}, S_x^{\tau}$ those for $E(A)$-modules. We will distinguish $E(A)$ from the quadratic dual $A^!$ of $A$ depending on context. Note that for a Koszul algebra $A$, $E(E(A)) \cong A$ (see, for example, [BGS, Thm 2.10.2]). Recall also that $LL(M)$ denotes the Loewy length of $M$.

Theorem 4.1. Let $A$ be the classes of finite-dimensional Koszul dominant Auslander-regular algebras. Then taking the Koszul dual $A \mapsto E := E(A)$ is an involution on $A$. Moreover, let $\pi : \mathcal{I} \rightarrow \mathcal{I}$ be the grade bijection associated to $A \in \mathfrak{A}$ so that $P_{\pi(x)} = \Omega^\pdim I_x(I_x)$, then the following hold:

1. For every $I_x$ non-projective, we have $I_x \cong P_{\pi(x)}$ and $\mathrm{LL}(I_x) - 1 = \pdim I_x$.
2. For every $I_x \cong P_{\pi(x)}$ projective, we have $d_x := \pdim I_x = \mathrm{LL}(I_x) - 1$ and $S_x$ is $d_x$-Gorenstein with $D \mathrm{Ext}^d_{E}(S_x^{\tau}, E) \cong S_x^{\tau}$.
3. The grade bijections associated to $A$ and $E$ coincide.
4. The bijection $\mathrm{ind}(\mathrm{inj} A) \leftrightarrow \mathrm{ind}(\mathrm{inj} E)$ given by $I_x \leftrightarrow I_x^\tau$ restricts to bijections

\[
\begin{array}{c}
\mathrm{ind}(\mathrm{inj} A) \setminus \mathrm{ind}(\mathrm{proj} A) \leftrightarrow \mathrm{ind}(\mathrm{inj} E \cap \mathrm{proj} E), \\
\mathrm{ind}(\mathrm{inj} A \cap \mathrm{proj} A) \leftrightarrow \mathrm{ind}(\mathrm{inj} E) \setminus \mathrm{ind}(\mathrm{proj} E).
\end{array}
\]

The proof of this theorem will be given in section 4.2.

Example 4.2. Let $A := N_a = N_{[2, 3, 1]}$ as in Example 2.13. We tabulate the information relevant to Theorem 4.1

\[
\begin{array}{c|cccccccc}
i & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\mathrm{LL}(I_i) & 3 & 4 & 2 \\
\pdim I_i & 1 & 2 & 1
\end{array}
\]

As mentioned in Remark 2.12 the Koszul dual of $A$ is given by $N^{\text{op}}_a = N^{\text{op}}_{[1, 2, 1, 2]}$:

\[
1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 6 \leftarrow 7.
\]
Now the projective-injectives are \( P_2 = I_1, P_4 = I_2, P_5 = I_4, P_7 = I_5 \) with Loewy length 2, 3, 2, 3 (these are one plus the entries of \( a \)). The rest of the relevant information are as follows.

\[
\begin{array}{c|ccccccc}
\pi^! & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
LL(I_i) & 2 & 3 & 3 & 2 & 3 & 1 \\
pdim I_i & 2 & & & & & \\
\end{array}
\]

At the end of section 5 in [Gre], Green posed the following question about the characterisation of the Koszul dual of an Auslander algebra:

**Question 4.3** (Green). Suppose \( A \) is a Koszul algebra satisfying the following properties:

(G1) The Loewy length of \( A \) is 3.

(G2) Each indecomposable projective \( A \)-module of Loewy length 3 is injective.

(G3) \( A \) is 1-Gorenstein.

What extra conditions imply that the Yoneda algebra \( E(A) \) is an Auslander algebra?

A consequence of Theorem 4.1 is to give an answer to Green’s question. Note that the algebras satisfying the conditions (G1) and (G2) are called translation algebras in [Guo1], whose study is also originated from the above question of Green.

In [Guo2], Guo introduced \((\ell+1)\)-translation algebras as a generalization of translation algebras. These algebras satisfy a generalization of (G1) and (G2) where the Loewy length 3 is replaced by \( \ell + 1 \) for \( \ell \geq 2 \). The definition in [Guo2] relies on some combinatorial datum on the quiver of the algebra; in contrast, we will use instead the following algebraic formulation that generalises (G1) directly and strengthens the generalized (G2) condition.

**Definition 4.4.** We say that an algebra is \( \ell \)-stiff if

1. it is of Loewy length \( \ell \), and
2. an indecomposable injective module is projective if, and only if, its Loewy length is \( \ell \).

Koszul duality on the class of dominant Auslander-regular algebras restricts to the following.

**Theorem 4.5.** For each \( n \geq 2 \), Koszul duality \( E(E(A)) \cong A \) induces the following correspondence between ring-indecomposable Koszul algebras:

\[
\{(n+1)\text{-stiff dominant Auslander-regular algebra}\} \leftrightarrow \{(n-1)\text{-Auslander algebra}\}
\]

Moreover, this restricts to a correspondence

\[
\{(n+1)\text{-stiff (d-1)-Auslander algebra}\} \leftrightarrow \{(d+1)\text{-stiff (n-1)-Auslander algebra}\}
\]

Note that the case \( n = 2 \) in the first part of Theorem 4.5 answers the question of Green.

We prove Theorem 4.5 as an application of Theorem 4.1.

**Proof.** An \((n-1)\)-Auslander algebra \( A \) is dominant Auslander-regular with \( \text{pdim} I_x = n \) for all indecomposable injective non-projective \( I_x \). It follows from Theorem 4.1 that every projective-injective \( E(A) \)-module has the same Loewy length \( n + 1 \), which then in turn implies that every \( P \in \text{ind} (\text{proj} E(A)) \setminus \text{ind} (\text{inj} E(A)) \) must have \( \text{LL}(P) < n + 1 \), i.e. \( E(A) \) is \((n+1)\)-stiff.

Suppose \( A \) is furthermore \((d+1)\)-stiff. As \( A \) is finite-dimensional, we have \( \text{gldim} E(A) < \infty \) (see, for example, Proposition 4.8 (b)). Applying Theorem 4.1 again we get that \( \text{pdim} I_x = d \) for all \( I_x \in \text{ind} (\text{inj} E(A)) \setminus \text{ind} (\text{proj} E(A)) \), which means that \( \text{gldim} E(A) = d \). Thus, \( E(A) \) is \((d-1)\)-Auslander.

As an immediate corollary, we can answer the above question by Green as follows.

**Corollary 4.6.** Let \( A \) be a ring-indecomposable non-semisimple Koszul algebra. Then \( E(A) \) is an Auslander algebra if and only if the following conditions are satisfied.

(G1) The Loewy length of \( A \) is 3.

(G2) An indecomposable projective module is injective if and only if it has Loewy length 3.

(G3) \( A \) is dominant Auslander-regular.
4.2. Reminder on Koszul algebras and the proof of the main result. We keep the notations $P_x, I_x, S_x, P^+_x, I^+_x, S^+_x$ from before. By graded algebra we always mean $\mathbb{Z}$-graded algebra $\Lambda = \bigoplus_{n \in \mathbb{Z}} \Lambda_n$ with $\Lambda_{<0} = 0$. We assume also that $\Lambda_0$ is semisimple unless otherwise stated.

**Definition 4.7.** Let $A$ be a graded algebra. An $A$-module $M$ is **linear** if its minimal graded projective resolution

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to M \to 0$$

is linear, meaning that the $i$-th term $P^{-i}$ is generated in degree $i$, i.e. $P^{-i} = P^i A$. Dually, a **colinear** module is one whose minimal graded injective coresolution is linear, meaning that the $i$-th term is cogenerated in degree $-i$. We say that $A$ is a **Koszul algebra** if $A_0$ is linear.

We will need the following results on Koszul algebras:

**Proposition 4.8.** Let $A$ be a Koszul algebra. For each $x \in \mathcal{I}$, the following hold.

1. $\operatorname{Ext}^1_A(A_0, I_x) = S^+_x$.
2. $\operatorname{pdim} S^+_x = \operatorname{LL}(I_x) - 1$.
3. If $\operatorname{gldim} A < \infty$, then the injective $E(A)$-module $I^*_x = D \operatorname{Ext}^1_A(S_x, A_0)$ is projective if and only if, $S_x$ is $k$-Gorenstein for some $k \geq 0$. In such a case, we have $\operatorname{LL}(I^*_x) = k + 1$.

**Proof.** (1) This follows from $\operatorname{Ext}^1_A(A_0, M) = \mathcal{H}^*({\operatorname{Hom}}_A(A_0, I^*_M))$ where $I^*_M$ is an injective coresolution of $M$; see also [BGS] Thm 2.12.5(iii).

(2) [GM1] Thm 10.4(4), (5)] says that a colinear $A$-module $M$ satisfies $\operatorname{pdim}_{E(A)} \operatorname{Ext}^1_A(A_0, M) = \operatorname{LL}(M) - 1$. So we can apply this to $M = I_x$ which is clearly colinear and use (1).

(3) This is [MV3] Prop 4.1. Note that the part $\operatorname{LL}(I^*_x) = k + 1$ is implicit in the proof of [MV3] Prop 4.1; alternatively, as $\operatorname{gldim} A < \infty$ the Gorenstein condition implies $\operatorname{pdim} S_x = k$ and the formula follows from (2).

We note that the statements in the cited literature are slightly different from our (2) and (3). This is because the setting they work in deals with left modules with maps composing from right to left, whereas our setting uses right modules with maps composing from left to right. In particular, our $E(A)$ is their $E(A)^\text{op}$. Let us explain more precisely: the essential key to obtain (2) is, *in their setup*, the functor $\operatorname{Ext}^1_A(\_, A_0)$ being an equivalence from linear $A$-modules to linear $E(A)^\text{op}$-modules [GM1] Thm 10.4. Here, the domains takes linear $A$-modules because $\operatorname{Ext}^1(\_, A_0)$ is about the homology of $\operatorname{Hom}_A(\_, A_0)$ on the projective resolution. Remembering this and transferring back to our setup, we then get that $\operatorname{Ext}^1_A(A_0, \_) = \operatorname{Ext}^1_A(S_x, A_0)$ corresponding to the right $A$-simple $S_x$ being injective; so the correct statement in our setting is a characterisation of whether $D \operatorname{Ext}^1_A(S_x, A_0) \cong I^*_x$ is projective.

**Theorem 4.9.** For a graded algebra $A$, denote by $D^2(A)$ the derived category of graded $A$-modules with suspension [1], and by (1) the grading shift given by $M(1)_n = M_{n+1}$. If $A$ is finite-dimensional, has finite global dimension, and is Koszul with Koszul dual $E = E(A)$, then there is an equivalence $F : D^2(A) \simeq D^2(E)$ of triangulated categories such that $F(M) \cong \operatorname{Ext}^1_A(A_0, M)$ for all colinear modules $M$ and $F(1) \simeq [1](-1)F$.

**Proof.** This is the unbounded version of [BGS] Theorem 1.2.6; see, for example, [Mad] Prop 5.3. \hfill \Box

**Proof of Theorem 4.7.** We can assume that $A$ is ring-indecomposable and non-simple. Let $A$ be a Koszul dominant Auslander-regular algebra, and $E := E(A)$ be its Koszul dual. Note that by Proposition 4.8 (2), $\operatorname{LL}(A) < \infty$ implies that $\operatorname{gldim} E < \infty$. Since $A$ is ring-indecomposable non-simple, the same applies for $E$. In particular, by Lemma 2.5 $E$ has no 0-Gorenstein simple module.

Define $I_0 := \{ x \mid I_x \in \operatorname{proj} A \} \subseteq \mathcal{I}$. We first show that for $x \in I_0$, we have $I^*_x$ is non-projective with $d_x$-Gorenstein $S^+_x$ for $d_x := \operatorname{LL}(I_x) - 1$. Indeed, by Proposition 4.8 (3) and the fact that $E(E(A)) \cong A$, $I_x$ being projective is equivalent to $S^+_x$ being $k$-Gorenstein, where $k = \operatorname{LL}(I_x) - 1 = d_x$. As $E$ has no 0-Gorenstein simple, we have $d_x \geq 1$, which means that $\operatorname{Hom}_E(S^+_x, E) = 0$, which in turn is equivalent to saying that $I^*_x$ is non-projective.

Next we show that for $x \in \mathcal{I} \setminus I_0$, we have $I^*_x \in \operatorname{proj} E$. Indeed, by Theorem 2.7 we have $S_x$ is $k$-Gorenstein for some $k \geq 1$ for any $x \in \mathcal{I} \setminus I_0$. Hence, Proposition 4.8 (3) says that $I^*_x \in \operatorname{proj} A$.\hfill \Box
The previous two paragraphs combine to yield bijections
\[
\{ x \mid I_x \in \text{proj } A \} \leftrightarrow \{ x \mid I'_x \notin \text{proj } E \} \leftrightarrow \{ x \mid S'_x \text{ is } k\text{-Gorenstein for some } k \geq 1 \}
\]
and \( \{ x \mid S_x \text{ is } k\text{-Gorenstein for some } k \geq 1 \} \leftrightarrow \{ x \mid I_x \notin \text{proj } A \} \leftrightarrow \{ x \mid I'_x \in \text{proj } E \} \).

This means that \( E \) satisfies the condition (G), hence, by Theorem 2.7, \( E \) is dominant Auslander-regular.

Most of the claim (2) is already proved in the second paragraph of this proof. It remains to show that \( d_x = \text{LL}(I_x) - 1 = \text{pdim } I'_x \) and that \( D \text{Ext}^d_E(S'_x, E) \cong S'_x \). Indeed, recall from Lemma 2.6 that \( S'_x \) being \( k\)-Gorenstein for some \( k \geq 1 \) implies that \( k = \text{pdim } S'_x = \text{pdim } I'_x \) and \( \text{Ext}^d_E(S'_x, E) = \text{Ext}^d_E(E, E) \neq 0 \). This immediately implies that \( d_x = \text{pdim } I'_x \) as required. Now, if we denote by \( \pi' \) the grade bijection associated to \( E \), then the description in Corollary 2.8 tells us that \( S'_x = D \text{Ext}^d_E(S'_x, E) \). By the (derived) Koszul duality Theorem 4.9, we have
\[
0 \neq \text{Ext}^d_E(S'_x, E) = \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{D^b(E)}(S'_x, E[d_x](\ell)) = \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{D^b(A)}(I_x, A_0[d_x + \ell](-\ell)).
\]

Since \( I_x \in \text{proj } A \), we have an isomorphism \( \theta : I_x \cong P_{\pi(x)} \), which induces a map
\[
I_x \rightarrow S_{\pi(x)}(\text{LL}(I_x) - 1) \cong A_0(\text{LL}(I_x) - 1)
\]
that belongs to the subspace \( \text{Hom}_{D^b(A)}(I_x, A_0(\text{LL}(I_x) - 1)) = \text{Hom}_{D^b(A)}(I_x, A_0(d_x)) \) in the right-hand side of the above formula. Hence, \( \text{D}(S'_{\pi(x)}) \in \text{mod } E^{op} \) is generated by \( \theta \), and so \( \pi(x) = \pi'(x) \).

The claim (1) now follows from (2), as we can swap the role of \( A \) with \( E \) by Koszul duality and both of them are dominant Auslander-regular.

Finally, claim (3) and (4) follows immediately from combining (1) and (2). \( \square \)

4.3. The base algebra of the Koszul dual of an Auslander algebra. Let \( (A, E) \) be a Koszul dual pair of algebras and throughout we assume \( A \) is ring-indecomposable and basic. Assume also that \( A \) is an \( d \)-Auslander algebra for some \( d \geq 1 \) and that \( E \) has dominant dimension at least two. Then there is some idempotent \( e = e^2 \in A \) so that \( A \cong \text{End}_{c Ac}(M) \) for some \( d \)-cluster tilting \( c Ac \)-module \( M = Ac \) and \( \text{add}(D(\text{Ac}))) = \text{proj } A \cap \text{inj } A. \) By Theorem 4.1 and Proposition 3.3 \( E \) is a dominant Auslander-regular algebra with an idempotent \( f = f^2 \in E \) so that \( E \cong \text{End}_{f Ef}(N) \) for some mixed cluster tilting \( f Ef \)-module \( N = Ef \) and \( \text{add}(D(\text{Ef}))) = \text{proj } E \cap \text{inj } E. \) In fact, as the degree 0 component of both \( A, E \) coincide, we can regard \( e, f \) to be living in the same space \( S = A_0 = E_0. \) Then Theorem 4.1 furthermore tells us that \( f = 1 - e. \)

In the following, we will consider the case when \( A \) is the Auslander algebra of a representation-finite algebra \( B = c Ac. \) Note that in this case, \( A_0 := A/acA \) is the stable Auslander algebra of \( B, i.e. A = \text{End}_B(L) \) where \( L \) is the direct sum of all indecomposable (one for each isoclass) non-projective \( B \)-modules and \( \text{Hom}_B(X, Y) \) is the stable homomorphism space given by \( \text{Hom}_B(X, Y) \) quotienting out the space of homomorphisms that factor through a projective \( B \)-module. Recall that if \( 0 \rightarrow \tau X \xrightarrow{a} E \rightarrow X \xrightarrow{b} 0 \) is an almost split sequence ending at indecomposable \( B \)-module \( X \), then we have a minimal projective resolution of the simple \( A \)-module \( S_X \) corresponding to \( X \) given by
\[
\xi_X: \quad 0 \rightarrow (M, \tau X) \xrightarrow{a} (M, E) \xrightarrow{b} (M, X) \rightarrow S_X \rightarrow 0,
\]
where \( (M, Y) \) denotes the projective \( A \)-module given by \( \text{Hom}_B(Y, M) \) for any \( B \)-module \( Y \).

For simplicity, we will use underline (—) to denote the canonical functor \( - \otimes_A A : \text{mod } A \rightarrow \text{mod } A. \) Note that \( (M, Y) := \text{Hom}_B(M, Y) \) is a projective \( A \)-module.

Lemma 4.10. [Iya3, 1.4(2)] Let \( X \) be an indecomposable \( B \)-modules with an almost split sequence \( 0 \rightarrow \tau X \rightarrow U \rightarrow X \rightarrow 0. \) If the following condition
\[
(4.10.1) \quad U \in \text{proj } B \Rightarrow \tau X \in \text{proj } B
\]
holds, then applying \( - \otimes_A A \) to the minimal projective resolution \( \xi_X \) of \( S_X \) yields the first three terms of the minimal projective resolution
\[
\xi_X: \quad (M, \tau X) \xrightarrow{a} (M, U) \xrightarrow{b} (M, X) \rightarrow S_X \rightarrow 0
\]
of the simple \( A \)-module \( S_X. \)
Remark 4.11. In the case when $A$ is Koszul, meaning that $\xi_X$ can be lifted to a linear complex, then the $A$-module complex $\xi_X$ can also be lifted to a linear complex, as it only kills direct summands of terms in $\xi_X$.

The following is somewhat well-known to expert. We give an explanation for completeness.

Lemma 4.12. Let $Q$ be a connected Dynkin quiver with $|Q_0| > 1$ and $h_Q$ be its Coxeter number. Let $B$ be a ring-indecomposable representation-finite algebra over a field $K$.

1. If $B$ is standard self-injective of tree class $Q$, then $A$ is $(2h_Q - 1)$-stiff and $\text{LL}(P) = h_Q - 1$ for all projective $A$-modules $P$.

2. If $B = KQ$, then $A$ is $(h_Q - 1)$-stiff and $\text{LL}(A) = h_Q - 2$.

Proof. Denote by $P_X := (M, X) := \text{Hom}_B(M, X)$ the projective $A$-module corresponding to $X \in \text{add} M = \text{mod} B$. Note that projective-injective $A$-modules are given by $P_X$ for $X \in \text{inj} B$. So for the claim on stiffness, it suffices to show that $\text{LL}(P_X)$ is constant over $X \in \text{inj} B$ as the inclusion $X \hookrightarrow I$ of $X$ into its injective envelope induces a mono $P_X = (M, X) \hookrightarrow (M, I) = P_I$ that is a radical map for any non-injective $M$.

1. The claim on $\text{LL}(P)$ is [BLR] 1.1. Notice that $K$ is assumed to be algebraically closed in [BLR], but it is not necessary due to standardness of $B$. We just need to explain the claim on stiffness. Let $I \in \text{ind}(\text{inj} B)$. Since $B$ is self-injective, for every indecomposable injective $I \in \text{ind}(\text{inj} B)$ we have an almost split sequence

$$0 \rightarrow \text{rad} I \xrightarrow{(\iota_I, \iota_I^1)^T} I \oplus \text{rad} I / \text{soc} I \xrightarrow{(\pi_I, \pi_I^1)} I / \text{soc} I \rightarrow 0.$$ 

So $\iota_I, \iota_I^1, \pi_I, \pi_I^1$ are degree 1 elements of $A$.

Consider the simple $B$-module $S := \text{soc} I$. Then $\nu^{-1} \Omega^{-}(S) = P / \text{soc} P$ and $\nu \Omega(S) = \text{rad} I$, where $P$ is the projective cover of $S$. The projection $\theta : P / \text{soc} P \rightarrow S$ satisfies $f \theta = 0 = \theta f$ for any $f$ and does not factors through projective. In other words, $\text{soc} \text{Hom}(M, S) = \text{soc} P_S$ is spanned by $\theta p$, where $p : M \rightarrow P / \text{soc} P$ is the canonical projection to the direct summand $P / \text{soc} P$. This means that $\text{deg}(\theta) = \text{LL}(\theta) = h_Q - 1$, and so $\text{deg}(\theta \pi) = h_Q - 1$.

Dually, consider the inclusion morphism $\phi : S \hookrightarrow \text{rad} I$, then we have $\text{deg}(\iota_I \phi) = h_Q - 1$. Composing the two maps yield $\alpha := (P \xrightarrow{\theta \pi} S \xrightarrow{\iota_I^1} I)$, and so precomposing with the canonical projection yields $M \rightarrow P \xrightarrow{\alpha} I$ that spans $\text{soc} \text{Hom}(M, I)$. Thus, we have $\text{LL}(\text{Hom}(M, I)) + 1 = \text{deg} \alpha = (h_Q - 1) + (h_Q - 1) = 2h_Q - 2$. This finishes the proof.

2. Recall that the trivial extension algebra $T(B) := B \times DB$ is a symmetric algebra with quotient $B$. Moreover, in our case, $T(B)$ is representation-finite symmetric and the restriction functor induces an embedding of mod $B$ into the stable category $\text{mod}(T(B))$ which makes $A$ an idempotent subalgebra of the stable Auslander algebra $\Lambda$ of $T(B)$. Now take $X \in \text{ind}(\text{inj} B)$, since the socle of $P_X$ is the simple $\Lambda$-module corresponding to $\Omega_X^1(B)(X) = \nu_B(X) \in \text{proj} B \subset \text{mod}(T(B))$ and $\text{LL}(P_X) = h_Q - 1$ from (1), we have $\text{LL}(P_X) = \text{LL}_{\Lambda}(P_X) = h_Q - 1$ and $\text{LL}_{\Lambda}(P_X) = \text{LL}_{\Lambda}(P_X) - 1 = h_Q - 2$. The latter equation shows the claim on $\text{LL}(\Lambda)$.

Definition 4.13. [BBK] Let $A = \bigoplus_{n \geq 0} A_n$ be a (finite-dimensional) $\mathbb{Z}_{\geq 0}$-graded algebra with $A_0$ semisimple. It is (right) $(p, q)$-Koszul for some positive integers $p, q$ if

1. $A_{>p} = 0$, and

2. the minimal graded projective resolution $P_\bullet$ of $A_p$ has the following linear truncation

$$0 \rightarrow P_q \xrightarrow{d_1} P_{q-1} \rightarrow \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow 0$$

with $\text{Ker}(d_q : P_q \rightarrow P_{q-1}) \subset A_0(q)$. 

Note that a (classical) Koszul algebra is $(p, q)$-Koszul for all $p \geq \text{LL}(A) - 1$ and $q \geq \text{gldim} A$.

Lemma 4.14. For an indecomposable non-projective $B$-module $X$, condition (4.10.1) holds in the following cases.

1. $B = KQ$ with Dynkin quiver $Q$. In this case, the kernel of the last map in $\xi_X$ is zero, i.e. $\xi_X$ is the minimal projective resolution of the simple $\Lambda$-module $S_X$. In particular, $\Lambda$ is Koszul.
(2) $B$ is self-injective with $\text{LL}(B) > 2$. In this case, the kernel of the last map in $\xi_X$ is a simple $A$-module $\xi_{\Omega(X)}$ associated to the first syzygy $\Omega(X)$ of $X$. Moreover, if $B$ is standard of Dynkin type tree class $Q$, then $\xi_{\Omega(X)}$ is generated in degree $h_Q + 1$. In particular, $A$ is $(h_Q - 2, 2)$-Koszul.

**Proof.** (1) Condition (4.10.1) follows by the fact that $B$ is hereditary. For the second part, by [ARI, Prop 10.2], we have $\text{gldim}_A \leq 2$ and so $\xi_X$ is all we get in the minimal projective resolution of $\xi_X$. Koszulity now follows from the fact that linearity is preserved in $\xi_X$, as discussed in Remark 4.11.

(2) Every indecomposable projective-(injective) $B$-module $P$ appears in the almost split sequence $0 \to \text{rad} P \to P \oplus \text{soc} P \to P/\text{soc} P \to 0$. So $\text{LL}(B) > 2$ guarantees $\text{rad} P/\text{soc} P \neq 0$, meaning that no middle term of the almost split sequence is projective, and so (4.10.1) is a null condition.

To see the kernel $K$ of the last map in $\xi_X$, we recall that the stable module category $\text{mod} B$ has a canonical triangulated structure where the shift is the inverse syzygy $\Omega^{-1}$. The almost split sequence ending at $X$ induces an almost split triangle $\tau X \to E \to X \to \Omega^{-1}(\tau X)$ in $\text{mod} B$, and so we obtain the following long exact sequence from applying $\text{Hom}_B(M, -)$:

$$\cdots \to (M, \Omega(E)) \to (M, \Omega(X)) \to (M, \tau X) \to (M, E) \to \cdots.$$  

Since applying $\Omega$ to the same almost split triangle yields another almost split triangle, the cokernel of the map $(M, \Omega(E)) \to (M, \Omega(X))$ above is the simple $A$-module $\xi_{\Omega(X)}$.

Now suppose that $B$ is standard. Then $A$ is self-injective and $\xi_{\Omega(X)}$ is the socle of the indecomposable projective $A$ module $(M, \tau X)$, which has Lowey height $h_Q - 1$ by Lemma 4.12 (1). So the generating degree of $\xi_{\Omega(X)}$ as the kernel of the last map of $\xi_X$ is just $2 + h_Q - 1 = h_Q + 1$. \hfill \Box

We need one more result before proving the first main aim of this section.

**Proposition 4.15.** [Guo2 Prop 2.6] (see also [BBK Prop 3.2, 3.11]) Let $\Lambda$ be a $(p, q)$-Koszul algebra with $p, q \geq 2$. Then it is quadratic. Moreover, the quadratic dual $A^\perp$ is a $(q, p)$-Koszul algebra isomorphic to the subalgebra $\bigoplus_{k=0}^q \text{Ext}^k_{\Lambda}(A_0, A_0)$ of $E(A)$.

**Theorem 4.16.** Let $Q$ be a connected Dynkin quiver with $|Q_0| > 1$. Consider $B$ a ring-indecomposable representation-finite algebra over a field $k$ that is

(i) either $B = KQ$;

(ii) or standard self-injective of tree class $Q$.

Let $A$ be the Auslander algebra of $B$ and $E = E(A)$ be its Koszul dual. Then the following hold.

(1) $A$ is $(h_Q - 1)$-stiff in case (i) and $(2h_Q - 1)$-stiff in case (ii).

(2) $E$ is 3-stiff.

(3) $E$ is $(h_Q - 3)$-Auslander in case (i) and $(2h_Q - 3)$-Auslander in case (ii).

(4) If $|Q_0| > 2$, then the base algebra $C$ of the higher Auslander algebra $E$ is isomorphic to the quadratic dual $A^\perp$ of the stable Auslander algebra $A = A/AeA$.

(5) $A$ is Koszul in case (i), and $(h_Q - 2, 2)$-Koszul in case (ii). Moreover, $C$ is Koszul in case (i), and $(2, h_Q - 2)$-Koszul in case (ii).

**Proof.** (1): This is Lemma 4.12.

(2), (3): follows from applying (1) to Theorem 4.5.

(4): First note that the conditions on $B$ imply that $A$ is $(h_Q - 2, 2)$-Koszul. Indeed, for case (ii) this is as stated explicitly in Lemma 4.14 (2), whereas for case (i) we have $\text{LL}(A) = h_Q - 3$, $\text{gldim}(A) = 2$ and $A$ is (classical) Koszul by Lemma 4.14 (1), so it is also $(h_Q - 2, 2)$-Koszul. In particular, as $h_Q \geq 2$, Proposition 4.15 says that $A$ is quadratic with a $(2, h_Q - 2)$-Koszul quadratic dual $A^\perp$ that is isomorphic to $\bigoplus_{k=0}^2 \text{Ext}^k_{\Lambda}(A_0, A_0) \subset E(A)$.

We claim that there is the following chain of isomorphism of algebras

$$A^\perp \cong \bigoplus_{k=0}^2 \text{Ext}^k_{\Lambda}(A_0, A_0) \cong \bigoplus_{k=0}^2 \text{Ext}^k_{\Lambda}(A_0, A_0) = \text{Ext}^*_{\Lambda}(A_0, A_0) \cong (1 - e)E(1 - e) = C.$$  

Indeed, by Lemma 4.10 we have $\text{Tor}^0_{\Lambda}(A_0, A_0) = 0$ for $k \in \{1, 2\}$, which implies the second isomorphism. Since $\text{gldim} A \leq 2$, we have $\text{pdim}_{\Lambda} A_0 \leq 2$, and so the third isomorphism follows. The third equality comes from the definition of $E = E(A) = \text{Ext}^*_{\Lambda}(A_0, A_0)$ and the fact that $A_0 = (1 - e)A_0$. The final equality follows from Theorem 4.1 (4). \hfill \Box
Remark 4.17. We give to remark in the case when $|Q_0| = 2$. Firstly, Theorem 4.16 (3) for case (i) says that $E$ is a 0-Auslander algebra, which is by definition an algebra $A$ satisfying $\domdim A \leq 1 \leq \gldim A$. This is equivalent to that $A$ is a hereditary Nakayama algebra.

The second remark is that, while the proof of Theorem 4.16 (4) fails, we can still describe $C$ by manual calculation, namely:

(i) If $B = KQ$, then $B = K(1 \to 2)$ and $A \cong K \cong C$. Hence, Theorem 4.16 (4) still holds.

(ii) If $B$ is self-injective, then $B \cong N_{n,3}$ the self-injective Nakayama algebra with $n \geq 1$ simples and Loewy length 3. In this case, $A \cong N_{2n,2}$ (which means that $A^1 \cong N_{2n,\infty}$) and $C \cong N_{2n,3} \neq A^1$.

For ease of further exposition, we will use the following terminology; note that these algebras are examples of stable translation algebras in [Guo1].

Definition 4.18. We call the base algebra $C$ of $E$ appearing in Theorem 4.16 the Auslander-Koszul complement of $B$.

Corollary 4.19. Let $B$ be a standard representation-finite algebra over a field $K$, and $C$ be its Auslander-Koszul complement.

1. If $B = KQ$ for some Dynkin quiver $Q$ with $|Q_0| > 2$, then $C$ is a $(h_Q - 3)$-representation-finite $(h_Q - 3)$-hereditary algebra, i.e., it is of global dimension $h_Q - 3$ and has a $(h_Q - 3)$-cluster tilting module.

2. If $B$ is self-injective of tree class $Q$ with $|Q_0| > 2$, then $C$ is a $(2h_Q - 3)$-representation-finite self-injective algebra.

Proof. (1): The higher Auslander correspondence and Theorem 4.16 imply the existence of $(h_Q - 3)$-cluster tilting module. It remains the verify the claimed global dimension. By Lemma 4.12 (2), $A$ is Koszul and $LL(A) = h_Q - 2$, so we have $\gldim C = \pdim C_0 = h_Q - 3$ by Proposition 4.8 (2) and left-right symmetry of Koszul algebras (or alternatively, use Proposition 4.15 directly).

(2): Apart from the global dimension part, this uses the same argument as (1) with $h_Q - 3$ replaced by $2h_Q - 3$. Self-injectivity of $C$ follows from Lemma 4.23 below.

It is straightforward to see that this fails for $|Q_0| = 2$ as $\Delta \cong K \cong C$.

Lemma 4.20. Suppose $B = KQ$ for a Dynkin quiver $Q$ with $|Q_0| > 2$. Let $e = e^2$ be the idempotent of $A$ so that $B = eAe$. Then $E/E(1-e)E \cong KQ^{op}/rad^2 KQ^{op} = B^1 \cong E(B)$, and it has Loewy length 2 and global dimension being the length of the longest path on $Q$.

Proof. Recall that $f := 1 - e$ is the idempotent corresponding to the direct sum of all indecomposable non-projective $B$-modules. So the quiver of $E/E(1-e)E$ is $Q^{op}$. Since $E := E(A) \cong A^1$ by Koszulness of $A$, for any $\alpha\beta \neq 0$ on $Q$, we have $\beta^{op}\alpha^{op} = 0$ in $A^1$, hence on $E/AE$. Thus, $E/E(1-e)E = KQ^{op}/rad^2(KQ^{op})$, which is by definition $B^1$. Note that $B$ is always Koszul and so $B^1 \equiv E(B)$.

We pose the following question:

Question 4.21. Let $B$ be a representation-finite algebra with Auslander algebra $A$. When is $A$ $\ell$-stiff for some $\ell$?

By Theorem 4.5 the above question for standard $B$ is equivalent to asking when is $E(A)$ a higher Auslander algebra. We have seen that this is the case when $B$ is a path algebra or self-injective, but one can find simple counter-examples for non-Iwanaga-Gorenstein $B$ - such as the case when $B$ is Nakayama algebra with Kupisch series [3,4]. This answer to this question is still unclear even if we restrict $B$ to 1-Iwanaga-Gorenstein algebras; we demonstrate this in the following example.

Example 4.22. Let $Q$ be the quiver:

$$
\begin{align*}
1 & \overset{a_1}{\longrightarrow} \overset{a_2}{\longleftarrow} 2 \\
& \overset{a_3}{\longleftarrow} \overset{a_4}{\longrightarrow} 3
\end{align*}
$$

and let $I = (a_1a_2, a_2a_1, a_3a_4, a_4a_3)$. Then the algebra $B := KQ/I$ is the quotient $\Pi/\soc(P_2)$ of the (representation-finite) preprojective algebra $\Pi$ of type $A_3$, where we denote by $P_I$ the indecomposable projective associated to $i \in Q_0$. By [BIRS], Prop III.2.2, $B$ is 1-Iwanaga-Gorenstein; $B$ is also representation-finite by Drozd’s rejection lemma. The Auslander algebra $A$ of $B$ is not $\ell$-stiff for any $\ell$. Indeed, let $M$ be
the additive generator of \text{mod} \, B, then the projective-injective \text{A}-module \text{Hom}(M, P_3) = D \text{Hom}(P_1, M) has Loewy length \((= \text{length of longest path from } P_2 \text{ to } I_2 \text{ on the AR-quiv})\) 7 while the projective-injective \text{A}-module \text{Hom}(M, \nu_A(P_2)) = D \text{Hom}(P_2, M) has Loewy length 5.

4.4. Cluster tilting modules over Auslander-Koszul complements. As before, \text{C} is the base algebra \(f \text{Ef} \) of \(E := E(A)\), where \text{A} is the Auslander algebra of \(B \) and \(B \) is a representation-finite path algebra or standard self-injective algebra of tree class \(Q \) with \(|Q_0| > 1\). Our aim in this subsection is to give a concrete description of the higher cluster tilting \(C\)-module \(E f \).

In the following, for an indecomposable \(B\)-module \(X\), we will use the notation \(\underline{P}_X, \underline{I}_X, \underline{S}_X\) to denote the associated indecomposable projective, indecomposable injective, simple \(C\)-modules, respectively. Note that the canonical grading on these modules have \(\underline{P}_X\) concentrated in non-negative degrees with degree 0 being the simple top, \(\underline{I}_X\) concentrated in non-positive degrees with degree 0 being the simple socle, and \(\underline{S}_X\) is concentrated in degree 0.

Lemma 4.23. Suppose that we have \(|Q_0| > 2\).

\((1)\) If \(X\) is an indecomposable non-projective \(B\)-module with an almost split sequence \(0 \to \tau X \to U \to X \to 0\), Then we have

\[
\begin{align*}
(\underline{I}_X)_{-2} &= \underline{I}_X / \text{rad} \underline{I}_X = \underline{S}^{I}_{-X}, \\
(\underline{I}_X)_{-1} &= \text{rad} \underline{I}_X / \text{soc} \underline{I}_X = \underline{S}^{I}_{U_1} \oplus \cdots \oplus \underline{S}^{I}_{U_r},
\end{align*}
\]

where is \(U_1 \oplus \cdots \oplus U_r\), the indecomposable decomposition of the quotient \(U/P\) of \(U\) by the maximal projective direct summand \(P\) of \(U\).

\((2)\) If \(Y\) is an indecomposable non-projective \(B\)-module with an almost split sequence \(0 \to Y \to V \to \tau^{-1}Y \to 0\). Then we have

\[
\begin{align*}
(\underline{P}_Y)_1 &= \text{rad} \underline{P}_Y / \text{soc} \underline{P}_Y = \underline{S}^{P}_{V_1} \oplus \cdots \oplus \underline{S}^{P}_{V_r}, \\
(\underline{P}_Y)_2 &= \text{soc} \underline{P}_Y = \underline{S}^{I}_{Y},
\end{align*}
\]

where is \(V_1 \oplus \cdots \oplus V_r\), the indecomposable decomposition of the quotient \(V/P\) of \(V\) by the maximal projective direct summand \(P\) of \(V\).

\((3)\) There is a bijection \(\text{ind} B \setminus (\text{ind}(\text{proj} B) \cup \text{ind}(\text{inj} B)) \to \text{proj} C \cap \text{inj} C\) given by \(X \mapsto \underline{P}_X\). In particular, if \(B\) is self-injective, then so is \(C\).

Proof. (1): From the proof of Theorem 4.16(d), we see that \(C \cong A^1\) has \(k\)-th graded piece given by \(\text{Ext}_A^k(A_0, A_0)\). Now the claim follows from the form of \(\underline{L}_X\) (see Lemma 4.10) and the fact that \(\text{Hom}_B(M, U) = \text{Hom}_B(M, U/P)\).

(2): This is dual to (1).

(3): Combining (1) and (2) yields \(\underline{P}_X \in \text{inj} \, C\) if and only if \(X\) is non-projective non-injective, and the claim follows. \(\square\)

Theorem 4.24. Let \(Q\) be a connected Dynkin quiver. Consider \(B\) a ring-indecomposable representation-finite algebra over a field \(K\) that is

\((i)\) either \(B = K\mathbf{Q}\) with \(|Q_0| > 2\);

\((ii)\) or standard self-injective of tree class \(Q\) with \(|Q_0| > 1\).

Let \(E\) be the Koszul dual of the Auslander algebra of \(B\), and \(C\) the base algebra of \(E\). Then the higher cluster tilting \(C\)-module \(M\) so that \(E \cong \text{End}_C(M)\) is given by the basic part of

\[
C \oplus DC \oplus \bigoplus_P \underline{S}_P/\text{soc}(P)
\]

where \(P\) varies over all projective-injective \(B\)-modules and \(\underline{S}_X\) is the simple \(C\)-module associated to \(X\).

Proof. \(M\) is a generator-cogenerator, and so \(\text{add} (C \oplus DC) \subset \text{add} M\). By Lemma 4.23(3), we have \(|C \oplus DC| = |\text{proj} B| - |B| - |DB| + |\text{add} B \cap \text{add} DB|\), where \(|\text{add} X|\) and \(|X|\) both denote the number of isoclasses of indecomposable direct summand of \(X\). Therefore, each of the remaining indecomposable direct summand of \(M\) is given by \(e_P M = e_X Ef\) for an indecomposable projective-injective \(B\)-module \(P\) and \(f \in E\) is the idempotent so that \(\text{add}(D(Ef)) = \text{proj} E \cap \text{inj} E\) (i.e. \(M = Ef\) and \(C \cong f Ef\)).
For the case (ii) with $|Q_0| = 2$, the claim can be checked manually, c.f. Remark 4.17. From now on, we assume $|Q_0| > 2$.

As $P$ is a projective $B$-module, $D(Ae_P)$ is a projective-injective $A$-module for $A$ the Auslander algebra of $B$. By Theorem 4.1, the projective $E$-module $e_P E$ is non-injective. The assumption on $|Q_0|$ implies that $E$ is ring-indecomposable, and so we have $\text{LL}(e_P E) \neq 1$. Hence, by Theorem 4.16 (2) we must have $\text{LL}(e_P E) = 2$.

As $P$ is projective-injective, we have an almost split sequence $0 \rightarrow \text{rad} P \rightarrow P \oplus \text{rad} P/\text{soc} P \rightarrow P/\text{soc} P \rightarrow 0$, and $P$ has only one incoming arrow on the quiver of the Auslander algebra of $B$, which is then equivalent to $P$ having only one out-going arrow on the quiver of $E$. Combining with $\text{LL}(e_P E) = 2$ we get that $\text{rad} e_P E$ is just the simple module corresponding to $P/\text{soc} P$. Thus, we have $e_P E f = S'_P/\text{soc} P$ as claimed.

**Example 4.25.** Let $B := KQ$ be the path algebra of the following Dynkin quiver of type $D_4$

$$Q :$$

Then the Koszul dual $E$ of the Auslander algebra of $B$ has quiver given (by the solid arrows) as follows:

The relations are generated by the quadratic monomials $ab$ for each dashed line that spans across arrows $a, b$, and commutation relations $(x \rightarrow x + 1 \rightarrow x + 4) - (x \rightarrow x + j \rightarrow x + 4)$ for $x \in \{3, 7\}, y \in \{2, 3\}$.

The Loewy structure of $E = P_1 \oplus \cdots \oplus P_{12}$ is of the form

$$\frac{1}{5} \oplus \frac{2}{6} \oplus \left( \frac{5}{4} \oplus \frac{6}{7} \oplus \frac{8}{9} \oplus \frac{9}{10} \oplus \left( \frac{7}{8} \oplus \frac{11}{12} \oplus \frac{10}{11} \oplus \frac{12}{13} \oplus \frac{11}{12} \right) \right) \oplus \frac{8}{12} \oplus \frac{9}{11} \oplus \frac{10}{11} \oplus \frac{11}{12} \oplus \frac{12}{13} \oplus \frac{11}{12} \oplus \frac{12}{13}$$

This is a 3-stiff algebra and so the indecomposable projective-injective modules are $P_i \cong I_{i+4}$ for $i \leq 8$.

The Loewy length of the Auslander algebra of $KQ$ is 5 and so $E$ has global dimension 4; hence, the Auslander-Koszul complement $C = f Ef$ (for $f = e_5 + \cdots + e_{12}$) has a 3-cluster tilting module $E f$ of the form

$$\frac{5}{\text{ind(inj) } C} \oplus \frac{6}{\text{ind(proj) } C} \oplus \left( \frac{5}{6} \oplus \frac{6}{7} \oplus \frac{7}{8} \oplus \frac{8}{9} \oplus \left( \frac{7}{6} \oplus \frac{9}{10} \oplus \frac{10}{11} \oplus \frac{11}{12} \oplus \frac{12}{13} \right) \right) \oplus \frac{8}{12} \oplus \frac{9}{11} \oplus \frac{10}{11} \oplus \frac{11}{12} \oplus \frac{12}{13} \oplus \frac{11}{12} \oplus \frac{12}{13} \frac{11}{12} \oplus \frac{12}{13}$$

4.5. **Simple-minded system versus higher cluster tilting.** One consequence of Theorem 4.24 is that it gives a new connection between higher cluster tilting modules and simple-minded systems, which are analogue of projective (tilting) and simple modules respectively, in certain triangulated (singularity) categories. Let us recall the definition of simple-minded systems now.

**Definition 4.26.** Let $S$ be a class of indecomposable objects in a $K$-linear triangulated category $C$ with $K$ algebraically closed. We call $S$ a semibrick if $\text{Hom}_C(S, T)$ is a division $K$-algebra when $S = T$; otherwise, zero. A brick $S \in C$ is an indecomposable object such that $\{S\}$ is a semibrick. We call $S$ a (1-)simple-minded system if it is a semibrick and the filtration closure of $S$ is $C$.

In the following, we will consider the case when $B$ is representation-finite standard self-injective of tree class $Q$ with $|Q_0| > 1$. Note that by standardness, the division $K$-algebra in the definition of simple-minded system of $\text{mod} B$ is just $K$ itself. In the rest, we denote by $\text{ind} B$ the set of isoclasses of indecomposable non-projective $B$-modules.

**Lemma 4.27.** Suppose that we have $|Q_0| > 2$. Suppose $X, Y, Z \in \text{ind} B$. The following hold.

1. $\text{Ext}_B^k(S_X, S_Y) = 0$ for $1 \leq k \leq h_Q - 2$ if and only if $\text{Hom}_B(X, Y) = 0$ or $Y = X$ is a brick in $\text{mod} B$.
2. $\text{Ext}_B^k(S_X, S_Y) = 0$ for $h_Q - 1 \leq k \leq 2h_Q - 4$ if and only if $\text{Hom}_B(Z, \nu X) = 0$ or $Z = \nu X$ is a brick in $\text{mod} B$. 

Proof. Let $\Lambda$ be a basic $(p,q)$-Koszul algebra with $p,q \geq 2$, and $\Gamma := \Lambda^!$ be its quadratic dual. By Proposition 4.15 $\Gamma$ is $(q,p)$-Koszul and $\Lambda \cong \bigoplus_{k=0}^{\infty} \text{Ext}^k_{\Lambda}(\Gamma_0, \Gamma_0)$, which means that the minimal graded projective resolution

$$\cdots \to P_p \to P_{p-1} \to \cdots \to P_0 \to S' \to 0$$

of a simple $\Gamma$-module $S'$ is linear in the first $p+1$ terms, and the composition factors of the $(k+1)$-st socle layer $(D(\Lambda e_k))_k$ of the indecomposable injective $\Lambda$-module $I_k$ correspond to the direct summands of $P_k$, i.e. $P_k = \bigoplus_{y \in \mathcal{P}} (P_y)^{\oplus c_{y_k}}$ where $c_{y,k} = [(D(\Lambda e_k))_k : S_y] = \dim K e_y \Lambda e_y = \dim K e_y \Lambda e_x$. Hence, we have

$$\text{Ext}^k(S', S_y) = 0 \text{ for } 1 \leq k \leq p \Leftrightarrow \begin{cases} c_y \Lambda e_x = 0, & \text{if } x \neq y; \\ c_x \Lambda e_x = K e_x, & \text{if } x = y. \end{cases}$$

For simplicity, we denote by $h := h_{\Omega}$ in the following.

(1): By Lemma 4.14 and Theorem 4.16 we can apply the preceding discussion with $\Lambda = \Lambda$ and $\Gamma = C$. Then the claim follows immediately; note that the condition $c_x \Lambda e_x = e_x$ is satisfied because of the brick assumption.

(2): We first claim that $\Omega^{h-1}(S'') = S''(X)$. Indeed, by the discussion preceding the proof of (1), the $h-1$-st term in the minimal projective resolution of $S''(X)$ is given by $P_{h-2}$. Since $A_{h-2} = 0$, the simple $A$-module $S''(X)$ is the socle of $P_{h-2}$, i.e. $Z = \nu(\Omega(X))$. Hence, $\Omega^{h-1}(S'') = \text{soc} P_{h-2}(X)$, which is $S''(X)$ as $\tau = \nu = \Omega$. For $h-1 \leq k \leq 2h-4$, we have $\text{Ext}^k_{\Omega}(S'', S''(X)) = \text{Ext}^k_{\Omega}(S''(X), S''(Y))$. Moreover, if $T_{k-2} \to \cdots \to T_0 \to S''(X) \to 0$ is the first $h-1$-terms of the minimal injective coresolution of $S''(X)$, then $T_{k-2} \to \cdots \to T_0$ is also the first $h-1$ terms of the minimal injective coresolution of $S''(X)$. Hence, the required Ext-orthogonality is equivalent to $P_{h-2} = P_{h-2}(X)$ being not a direct summand of $T_k$ for $1 \leq k \leq h-2$. Since applying $\bigoplus_{k=0}^{h-2} \text{Ext}^k_{\Omega}(S'') \to \text{Ext}^k_{\Omega}(S''(X), S''(Y))$ yields the projective $A$-module $P_{h-2}(X)$, the multiplicity of $P_{h-2}$ in $T_k$ for $0 \leq k \leq h-2$ is the same as $[P_{h-2}(X) : S''(X)] = \dim K e_{\text{soc} P_{h-2}(X)} e_x = \dim K e_{\nu X} e_X$.

Summarising the previous paragraph, we have that $\text{Ext}^k_{\Omega}(S', S'') = 0$ for all $h-1 \leq k \leq 2h-4$ and $\bigoplus_{k=0}^{h-2} e_{\nu X} A_k e_X = 0$. Now we can proceed in the same manner as (1), namely, if $Z \neq \nu X$, then $\bigoplus_{k=0}^{h-2} e_{\nu X} A_{k-1} e_X = \text{Hom}_B(Z,\nu X)$ and the claim follows; whereas in the case when $Z = \nu X$, $\bigoplus_{k=0}^{h-2} e_{\nu X} A_{k-1} e_X = 0$ is equivalent to $e_{\nu X} A e_X = K e_{\nu X}$, i.e. $\nu X$ is a brick.

\textbf{Theorem 4.28.} Let $B$ be a representation-finite standard self-injective algebra of tree class $Q$ with $|Q_0| > 1$, $C$ be the Auslander-Koszul complement of $B$. Then $S \mapsto M(S) := C \oplus \bigoplus_{X \in S} S_X$ defines a bijection

$$\left\{ \begin{array}{c} 1\text{-simple-minded systems} \\ \text{of } \text{mod} B \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{basic } (2h_Q-3)\text{-cluster tilting } C\text{-modules } M \\ \text{with } M/C \text{ being semisimple} \end{array} \right\}.$$ 

\textbf{Proof.} For the case $|Q_0| = 2$, one can easily check by hand using the description of $C$ in Remark 4.17. We show the remaining case in the following.

For $X \in \text{ind} B$, denote by $P_X$ and $I_X$ the associated indecomposable projective and injective $C$-module respectively. Let $A$ be the stable Auslander algebra of $B$. We will also suppress the subscript $Q$ in the Coxeter number $h_{\Omega}$ in the following.

First note that the Auslander-Koszul complement of $B$ is invariant under replacing $B$ by a stable equivalent algebra as this is the case for $A$. Let $S$ be a 1-simple-minded system of $\text{mod} B$. If $S \neq \{ P/\text{soc}(P) | \text{ } P \in \text{ind proj } B \}$, then we replace $B$ by a stable equivalent algebra such that this holds; this is always possible by [CKL] Thm 4.14 and the set of simple $B$-module corresponds to the set of $\{ P/\text{soc}(P) \}$ by the stable equivalence $\Omega^{-1}$. Now the well-definedness of $S \mapsto M(S)$ follows from Theorem 4.24 and injectivity is clear.

It remains to show surjectivity. Suppose $M = C \oplus N$ is a basic $(2h - 3)$-cluster tilting $C$-module with $N$ semisimple. Let $S$ be the set of $X \in \text{ind} B$ such that $S_X$ is a direct summand of $N$.

By Lemma 4.27 self-Ext-orthogonality of $S_X$ implies that $X$ is a brick, and so is $\nu X$ for any $k \in \mathbb{Z}$. Maximality of $N$ implies that $\nu^k X$ are all direct summands of $N$, which means that $S$ is $\nu$-stable. On the other hand, Lemma 4.27 also says that for $Y \neq X$, $S_Y$ is a direct summand of $N$ implies that $\text{Hom}_B(X,Y) = 0$ and $\text{Hom}_B(\nu Y, X) = 0$. Hence, $S$ is a $\nu$-stable semibrick.
We have shown that $S^1_Y$ is a direct summand of $N$ if and only if $Y$ is a brick and $\text{Hom}_B(X,Y) = 0 = \text{Hom}_B(Y,X)$ for all $X \in \mathcal{S} \setminus \{Y\}$. In other words, for every $Y \in \text{ind}B$, there must be some $X, X' \in \mathcal{S}$ such that $\text{Hom}_B(X,Y) \neq 0$ and $\text{Hom}_B(Y,X') = 0$. This means that the semibrick $S$ in $\text{mod}B$, by definition, a Riedtmann configuration. By [CKL] Thm 3.6, for representation-finite self-injective $B$, a Riedtmann configuration is equivalent to a 1-simple-minded system in $\text{mod}B$. This completes the proof. □

**Example 4.29.** Let $B = K[x]/(x^4)$ be the truncated polynomial ring of (Loewy) length 4, which is of tree class $A_3$. The Auslander-Koszul complement of $B$ is the representation-finite zigzag algebra $Z := Z(A_3)$ of type $A_3$ [HK] given by quiver

$$
\begin{array}{c}
1 & \overset{a_1}{\leftarrow} & 2 & \overset{a_2}{\leftarrow} & 3 \\
\end{array}
$$

with relation $1 \rightarrow 2 \rightarrow 3$, and commutation relation $a_2 a_1 = a_3 a_4$. There are only 2 simple-minded systems, namely $\{S := \text{top} B\}$ and $\{\text{rad} B = \langle x \rangle\}$. They correspond to the 5-cluster tilting modules $Z \oplus S_1$ and $Z \oplus S_3$; see [CM] where we already showed these two modules are 5-cluster tilting.

For reader familiar with Auslander-Reiten theory of representation-finite self-injective algebras of tree class $Q$, we recall that the stable Auslander algebra $A$ of such an algebra $B$ is given by the smash product of the preprojective algebra of type $Q$ by some finite group $G$ such that $ZQ/G$ is the stable Auslander-Reiten quiver of $B$. Dually, the associated Auslander-Koszul complement $C$ can also be described by the smash product of the zigzag algebra of type $Q$ by $G$, and use this to see that $C$ is in fact representation-finite; indeed, this is a special case of [Guo1] Thm 7.7. As a consequence, the Auslander-Koszul complement $C$ of a representation-finite $KQ$ is also representation-finite. Indeed, one can check that $C$ is an idempotent truncation of the Auslander-Koszul complement $C'$ associated to the (representation-finite standard symmetric) trivial extension of $KQ$.

**Example 4.30.** Let $B = Z(A_3)$ be the zigzag algebra of type $A_3$. Then the AR-quiver of $\text{mod}B$ is given as follows.

![Diagram of the AR-quiver of mod B](image)

The boxed vertices $P_a, P_b, P_c$ are the indecomposable projective $B$-modules (corresponding to vertices 1,2,3 respectively in the quiver of the previous example). The set $S = \{P_i/\text{soc} P_i\}_{i=a,b,c}$ corresponding the circled vertices 4,9,5 in the above picture is a 1-simple-minded system of $\text{mod} B$. The corresponding 5-cluster tilting $C$-module over the Auslander-Koszul complement $C$ under Theorem 4.28 is given as follows.

$$M = M(S) = 4 \oplus 9 \oplus 5 \oplus \frac{1}{2} \oplus \frac{3}{5} \oplus \left(\frac{4}{5}\right) \oplus \frac{4}{7} \oplus \frac{5}{8} \oplus \left(\frac{6}{8}\right) \oplus \frac{7}{9} \oplus \frac{8}{1} \oplus \left(\frac{9}{4}\right).$$

The endomorphism ring $\text{End}_C(M) \cong E$ is then given by

$$\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \oplus \frac{3}{5} \oplus \left(\frac{1}{2}\right) \oplus \left(\frac{3}{4}\right) \oplus \frac{4}{5} \oplus \frac{5}{6} \oplus \left(\frac{6}{7}\right) \oplus \frac{7}{9} \oplus \frac{8}{1} \oplus \left(\frac{9}{4}\right)
\end{array}
\end{array}
$$

As we have mentioned in the preceding discussion, the self-injective algebra $C$ in this example is Morita equivalent to the smash product of $Z(A_3)$ with $Z/3Z$, and is representation-finite.

We remark also that if one relaxes $B$ to arbitrary representation-finite standard algebra, then the arising Auslander-Koszul complement may not be representation-finite; indeed, the algebra

$$B = K(\begin{array}{c}
\begin{array}{c}
\frac{1}{a} \rightarrow \frac{1}{b} \rightarrow \frac{1}{c} \rightarrow \frac{1}{a} \rightarrow \frac{1}{b} \rightarrow \frac{1}{c} \rightarrow \frac{1}{a} \\
\end{array}
\end{array})/(ac - b^2a, c^2, b^3)$$

is representation-finite with Auslander-Reiten quiver given as in [BG] p.370, case (5), and one can find the extended Dynkin quiver of type $E_7$ with alternating orientation embedded in the stable part. Let $C := C/\text{soc}(P)$ where $P$ is the direct sum of all projective $C$-module of Loewy length 3. Note that every
indecomposable direct summand of $P$ has simple socle, and so by Drozd rejection lemma we have an embedding $\mod C \rightarrow \mod C$. Since $C$ has radical-squared zero, it is representation-finite if and only if its separated quiver has only Dynkin components by [ARS] Theorem X.2.6. Since the alternating-oriented quiver $E_7$ is a full subquiver of that of $C$, it is also one of $C$, and so the separated quiver contains a non-Dynkin component.

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