Large Squeezing Behavior of Cosmological Entropy Generation

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Abstract

We consider the generation of entropy when particle pairs are created at a cosmological level. Making a reduction via the particle number basis, we compute the classical limit for the entropy generation due to the evolution of the matter field fluctuations (squeeze transformation), obtaining that it is linear in the squeeze parameter for a general class of initial states.

We also discuss the dependence of the generated entropy on the coarse graining criteria.

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1 Introduction

The cosmological particle production that occurs in early universe models, because of the changing space-time metric, could be a relevant source for entropy generation. In this respect, work has been done introducing an entropy measure that increases as the quantum field that represents these particles evolves.

If we approximate the metric by a given time-dependent background, the evolution of the matter field fluctuations can be modelled; while, as noted by L. P. Grishchuk and Y. V. Sidorov [1], the corresponding Bogoliubov transformation that gives the relation between the in and out creation and annihilation operators can be expressed as a squeeze transformation.

The generation of entropy due to particle production can also be studied in the formalism of squeezed states. The problem is that of obtaining in an unambiguous way, a notion of entropy sensitive to the unitary squeeze transformation $\hat{S}$ that the field undergoes.

In Ref. [2], B. L. Hu and D. Pavon associated the entropy generation with the increase of the mean particle number and the loss of coherence of the initial state.

Alternatively, in Ref. [3], T. Prokopec considers the entropy as that obtained by coarse graining the density matrix, that is, by reducing the density matrix with respect to a given basis. In that work the entropy change is considered, when the initial state is the vacuum and the reduction is done via the occupation number basis or the (over-complete) basis of coherent states. In both cases, the classical limit (large squeezing limit) for entropy generation coincides.

Another possibility, recently suggested by M. Gasperini and M. Giovannini [4], is to use the basis of eigenstates of $\hat{x}$, the superfluctuant quadrature of the field. In the $\hat{x}$-basis the calculation can be exactly done, the generated entropy is linear in the squeeze parameter for all values of squeezing (see Refs. [5] and [6]).

In general, we could think in coarse graining a given density matrix $\rho$ with respect to an observable $\hat{A}$, having eigenfunctions $|a\rangle$, and define the entropy as

$$S = -\sum_a \langle a|\rho|a\rangle \ln \langle a|\rho|a\rangle$$

(1)

Then, an important question that arises is to what extent the generation of entropy
due to the squeezing process depends on the initial state and the coarse grain criteria.

In order to study these issues we will take the evolution of the system in the Schrödinger picture. If one considers a pure state $|\psi\rangle$, Eq. (1) reduces to

$$S = - \sum_a |\langle a|\psi\rangle|^2 \ln |\langle a|\psi\rangle|^2$$ (2)

This is indeed the Shannon entropy for the probability distribution, in the state $|\psi\rangle$, associated with the observable $\hat{A}$. The intuitive meaning of this expression is that $e^S$ is the number of basis vectors which are "appreciably involved" in the representation of $|\psi\rangle$ ("its richness"), see Ref. [7].

Therefore, given a coarse graining criteria, we are interested in the way the richness of the initial state of the field changes, due to the parametric squeezing (production of particle pairs).

In Ref. [1] each mode $\vec{k}$ of the field is independently associated with a squeeze transformation. A better description is given in Refs. [3] and [4] where a two mode transformation that mixes the modes $\vec{k}$ and $-\vec{k}$ is considered, conserving the momentum during the production of pairs.

For the sake of simplicity we will first consider the change of the Shannon entropy under a one mode squeeze transformation $|\psi\rangle \rightarrow \hat{S}|\psi\rangle$, $\hat{S} = e^{-i\hat{G}}$, where

$$\hat{G} = \frac{r}{2} i (a^\dagger a^2 - a^2) , \quad r > 0$$ (3)

($a^\dagger$, $a$ are the usual creation and annihilation operators).

In Ref. [5] we studied the interplay between the quantum fluctuations in the superfluctuant (resp. squeezed) quadrature and the corresponding loss (resp. gain) of information.

In this article we will study the classical limit for entropy generation showing that it is the same for a general class of initial states (here the coarse graining is done via $\hat{N}$, the particle number operator). Secondly, we will study the dependence of the classical limit on coarse graining by considering some particular examples.

In section §2 we review the eigenstates of $\hat{G}$ and their properties which lead to a very simple description of the state evolution. In section §3 we compute the leading order behavior (in the squeezing parameter $r$) for $\hat{A} = \hat{N}$ considering a general class of normalizable initial states (which include particle number eigenstates). In section §4 we discuss the dependence of the entropy generation on the coarse graining criteria, and finally, in section §5 we compute the two mode case.

2 The Eigenstates of the Squeeze Operator and the Propagation Kernel

In order to study the physical consequences of the squeezing process it is convenient to use the wave function representation and in particular a set of eigenfunctions of
the squeeze generator \((3)\). By defining the two canonically conjugate quadrature operators:

\[
\hat{x} = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad \hat{p} = \frac{1}{\sqrt{2}}i(a^\dagger - a)
\]

we get a simple form for the squeeze generator \(\hat{G}\), that corresponds to a dilation transformation:

\[
\hat{G} = \frac{r}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}), \quad [\hat{x}, \hat{p}] = i
\]

The eigenfunctions of the dilation operator \((5)\) have been introduced by C. G. Bollini and J. J. Giambiagi (see Ref. \([8]\)). By choosing the following realization for \(\hat{x}\) and \(\hat{p}\)

\[
\hat{x} = x, \quad \hat{p} = \frac{1}{i} \frac{d}{dx}
\]

the eigenvalue equation \(\hat{G}\psi = \mu r\psi\) reduces to

\[
x \frac{d}{dx} \psi = (i\mu - \frac{1}{2})\psi
\]

which leads to the solutions

\[
\psi_+^\mu = \frac{1}{\sqrt{2\pi}} x_+^{\mu - \frac{1}{2}} \quad \text{and} \quad \psi_-^\mu = \frac{1}{\sqrt{2\pi}} x_-^{\mu - \frac{1}{2}}
\]

where

\[
x_+^\lambda = \begin{cases} x^\lambda \quad \text{if } x > 0 \\ 0 \quad \text{if } x < 0 \end{cases} \quad \text{and} \quad x_-^\lambda = \begin{cases} 0 \quad \text{if } x > 0 \\ |x|^\lambda \quad \text{if } x < 0 \end{cases}
\]

Note that the spectrum is continuous and extends from \(-\infty\) to \(+\infty\).

The functions given in \((7)\) form a complete set and satisfy \(\delta\)-function normalization:

\[
\int_{-\infty}^{+\infty} d\mu (\psi_+^\mu(x_1)\overline{\psi}_+^\mu(x_2) + \psi_-^\mu(x_1)\overline{\psi}_-^\mu(x_2)) = \delta(x_1 - x_2)
\]

\[
\langle \psi_+^\mu | \psi_+^{\mu_2} \rangle = \delta(\mu_1 - \mu_2), \quad \langle \psi_-^\mu | \psi_-^{\mu_2} \rangle = 0
\]

In Ref. \([5]\) we obtained the kernel associated with the squeezing operator \(\hat{S} = e^{-i\hat{G}}\), i.e., \(K(x, x') = \langle x | \hat{S} | x' \rangle\), which can be rewritten using the completness relation:

\[
K(x, x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\mu \frac{1}{\sqrt{x'x}} (x_+^{i\mu} x_+^{-i\mu} + x_-^{i\mu} x_-^{-i\mu}) e^{-i\mu r}
\]

Therefore, if \(x'\) and \(x\) are greater than zero we obtain

\[
K(x, x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\mu \frac{1}{\sqrt{x'x}} e^{i\mu(\ln x - \ln x' - r)}
\]

\[
= \frac{1}{\sqrt{x'x}} \delta(\ln x - \ln x' - r)
\]
If \( x' \) and \( x \) are both negative,

\[
K(x, x') = \frac{1}{\sqrt{x'x}} \delta(\ln |x| - \ln |x'| - r)
\]

and if \( x' \) and \( x \) have opposite signs, we see from (11) that \( K(x, x') \) is zero. These properties can be summarized by noting that the kernel \( K(x, x') \) is non zero only for \( x = x'e^r \):

\[
K(x, x') = e^{-\frac{r}{2}} \delta(x' - xe^{-r})
\]

Analogously, we get:

\[
K(p, p') = e^{r} \delta(p' - pe^r)
\]

This can be directly seen from the symmetry of our system under the change \( \hat{x} \rightarrow \hat{p}, \hat{p} \rightarrow -\hat{x}, r \rightarrow -r \) (cf. (5)).

From (13) we see that under squeezing, an initial state given by \( \langle x|\psi \rangle = \psi(x) \) evolves into

\[
\langle x|\hat{S}|\psi \rangle = \int_{-\infty}^{+\infty} dx' K(x|x')\psi(x')
\]

\[
= e^{-\frac{r}{2}} \int_{-\infty}^{+\infty} dx' \delta(x' - xe^{-r})\psi(x')
\]

\[
= e^{-\frac{r}{2}} \psi(xe^{-r})
\]

Similarly,

\[
\langle p|\hat{S}|\psi \rangle = e^{\frac{r}{2}} \varphi(pe^{r})
\]

where \( \varphi(p) = \langle p|\psi \rangle \).

It is straightforward to verify that the normalization is preserved during the evolution.

We can see that, for \( r > 0 \), \( \hat{x} \) (resp. \( \hat{p} \)) is the superfluctuant (resp. squeezed) quadrature: using (15) and (16) we have for the dispersions in \( \hat{x} \) and \( \hat{p} \),

\[
\sigma_r(\hat{x}) = e^r \sigma_0(\hat{x}) \quad , \quad \sigma_r(\hat{p}) = e^{-r} \sigma_0(\hat{p})
\]

Note also that the eigenfunctions (7) are the analog to plane waves but instead of being invariant (up to a phase) under translations, they are invariant under dilations.

### 3 Entropy Generation and the Initial State

We will consider here the entropy generation which is obtained by coarse graining the density matrix. To do so we have to choose an observable \( \hat{A} \) with eigenfunctions \( |a \rangle \) and write the density matrix as

\[
\rho = \sum_a \sum_{a'} |a\rangle\langle a|\rho|a'\rangle\langle a'|
\]
Then, this density matrix is reduced by setting to zero the off-diagonal terms to get:

$$\rho_{\text{red}} = \sum_a |a\rangle\langle a| \rho |a\rangle\langle a|$$  \hspace{1cm} (19)

and the associated expression for the entropy is

$$S = -Tr \rho_{\text{red}} \ln \rho_{\text{red}} = -\sum_a |a\rangle\langle a| \ln |a\rangle\langle a|$$  \hspace{1cm} (20)

In contrast with the entropy computed with the full density matrix, which is unaltered by squeezing, the entropy (20) is sensitive to this evolution and depends on the reduction scheme. A natural scheme is that associated with the occupation number basis, for which we will study the large squeezing regime ($r >> 1$) of the entropy generation.

We will consider an initial pure state $|\psi\rangle$ which is supposed to be normalizable. The corresponding density matrix $\rho = |\psi\rangle\langle \psi|$ leads to a Shannon entropy with respect to the particle number given by

$$S = -\sum_n |\langle n|\psi(r)\rangle|^2 \ln |\langle n|\psi(r)\rangle|^2$$  \hspace{1cm} (21)

where $|\psi(r)\rangle = \hat{S}|\psi\rangle$. Now, in order to obtain the large squeezing behavior of (21), we first note that the amplitudes $\langle n|\psi(r)\rangle$ go to zero in that limit, for in the $\hat{x}$-representation we have (cf. (15)):

$$|\langle n|\psi(r)\rangle| \leq e^{-r^2/2} \int_{-\infty}^{+\infty} dx \psi_n(x) \psi(e^{-r}x)$$

$$\leq e^{-r^2/2} M \int_{-\infty}^{+\infty} dx \frac{|H_n(x)|e^{-x^2/2}}{\sqrt{2^n n! \pi^{1/4}}}$$  \hspace{1cm} (22)

where $M$ is an upper bound on the values of $|\psi(x)|$, $\psi_n(x)$ are the eigenfunctions of the harmonic oscillator and $H_n(x)$ are the Hermite polynomials. As the integral in (22) is convergent, the overlapping $\langle n|\psi(r)\rangle$ goes to zero at least exponentially for large $r$ (and each term in (21) goes to zero at least as $re^{-r}$).

Then, for large squeezing, the leading order of (21) comes from the infinite sum, and we can compute it by summing from a given $n_0$:

$$S \sim -\sum_{n=n_0}^{\infty} |\langle n|\psi(r)\rangle|^2 \ln |\langle n|\psi(r)\rangle|^2, \hspace{1cm} r >> 1$$  \hspace{1cm} (23)

Now, in order to make the computation of the asymptotic behavior easier we will consider the states in the $\hat{p}$-representation, the quadrature which is being squeezed, and we will suppose that $\varphi(p)$ is an even function (a similar reasoning applies for $\varphi(p)$ odd). From Eq. (11) we have

$$\langle 2k|\psi(r)\rangle = \int_{-\infty}^{+\infty} dp \varphi_{2k}(p) e^{i\tilde{z}} \varphi(e^r p)$$

$$= 2e^{i\tilde{z}} \left( \int_{0}^{\delta} + \int_{\delta}^{+\infty} \right) dp \varphi_{2k}(p) \varphi(e^r p)$$  \hspace{1cm} (24)
where \( \delta \) is a fixed number. In the first integral we can use (see [9])

\[
\varphi_{2k}(p) = \frac{1}{\sqrt{\pi k^{1/4}}} \cos(2p\sqrt{k + 1/4})(1 + \mathcal{O}(1/\sqrt{k}))
\] (25)

which is valid for large values of \( k \) (and \( p \) bounded). Therefore,

\[
\langle 2k | \psi(r) \rangle = 2e^{r^2} \int_{0}^{+\infty} dp \cos(2p\sqrt{k + 1/4})(1 + \mathcal{O}(1/\sqrt{k})) \varphi(e^r p)
\]

\[
+2e^{r^2} \int_{\delta}^{+\infty} dp \varphi_{2k}(p) \varphi(e^r p)
\] (26)

In order to set an upper bound on the integrals over the interval \((\delta, \infty)\) we demand that the associated wave function \( \varphi(p) \) satisfy, for large \( p \),

\[
|\varphi(p)| \leq \text{const.} \, p^{-2}
\]

\[
\left| \frac{d\varphi(p)}{dp} \right| \leq \text{const.} \, p^{-3}.
\]

In particular, this condition is fulfilled by the eigenstates of the particle number operator. We end up with an expression of the form

\[
\langle 2k | \psi(r) \rangle = 2e^{r^2} \int_{0}^{+\infty} dp \cos(2p\sqrt{k}) \left( 1 + \mathcal{O}(1/\sqrt{k}) \right) \varphi(e^r p) + \mathcal{O}(e^{-r/\sqrt{k}})
\] (27)

where we have replaced \( k + 1/4 \) by \( k \) within the same order of approximation considered. Finally the change of variable \( p \to e^{-r} p \) leads to

\[
\langle 2k | \psi(r) \rangle = f(k e^{-2r}) e^{-r} + g(k e^{-2r}) \mathcal{O}(e^{-2r})
\] (28)

with

\[
f(k) = 2 \int_{0}^{\infty} dp \frac{\cos(2p\sqrt{k})}{\sqrt{\pi k^{1/4}}} \varphi(p)
\] (29)

and \( g(k) \) being a function which for large \( k \) vanishes faster than \( 1/\sqrt{k} \).

Now, replacing the sum by the integral, we can compute the leading order behavior of (23):

\[
S \sim -\int_{k_0}^{\infty} dk \left| \langle 2k | \psi(r) \rangle \right|^2 \ln \left| \langle 2k | \psi(r) \rangle \right|^2
\] (30)

Choosing \( k_0 \) so as to make valid the assumption (23), and using (28), we see that

\[
S \sim -\int_{k_0}^{\infty} dk e^{-2r} f^2(k e^{-2r})(-2r + \ln f^2(k e^{-2r}))
\]

\[
\sim 2r \int_{k_0 e^{-2r}}^{\infty} dk f^2(k) - \int_{k_0 e^{-2r}}^{\infty} dk f^2(k) \ln f^2(k)
\] (31)

as \( k_0 \) is independent of \( r \) we can integrate the last expression from zero (in the large squeezing regime). From (29) we see that

\[
\int_{0}^{\infty} dk f^2(k) = \int_{-\infty}^{+\infty} dp' \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} dk \frac{1}{k^{1/4}} \cos(2\sqrt{k}p') \cos(2\sqrt{k}p) \tilde{\varphi}(p') \varphi(p)
\]

\[
= \int_{-\infty}^{+\infty} dp \tilde{\varphi}(p) \varphi(p) = 1
\] (32)
The second term in (31) is an approximation to the series for the initial entropy. Although this approximation is not so good, the difference with the exact series is $r$-independent and does not affect the leading order behavior of the generated entropy. Then, using (32), we finally get:

$$\Delta S = S - S_0 \sim 2r \quad , \quad r >> 1$$  (33)

It is straightforward to generalize this result for a general initial mixture of normalizable states $|b\rangle$:

$$\rho = \sum_{b,b'} p_{bb'} |b\rangle \langle b'| \quad . \quad Tr \rho = 1$$  (34)

Following essentially the same steps that led from (21) to (33), for large values of squeezing, we also get

$$S = -\sum_{b,b'} p_{bb'} \sum_k \langle 2k|b(r)\rangle \langle b'(r)|2k\rangle \ln \left( \sum_{b,b'} p_{bb'} \langle 2k|b(r)\rangle \langle b'(r)|2k\rangle \right) \sim 2r$$  (35)

4 Entropy Generation and the Coarse Graining Criteria

In the previous section we obtained that the generated entropy, when coarse graining with respect to $\hat{N}$, goes like $\Delta S \sim 2r$, in the large squeezing regime. In general, if one considers another coarse graining basis, a different asymptotic behavior could be expected. For instance, if we take the singular case of the basis of the dilation operator eigenstates (cf. (7)), the associated entropy does not change under a squeeze transformation. This is precisely because these functions are stationary under squeezing. Now, let us take as an example the set of Laguerre functions

$$\chi_n(x) = \frac{1}{\sqrt{2}} e^{-\frac{|x|^2}{2}} L_n(|x|)$$  (36)

which are similar (in the sense of their localization properties) to the harmonic oscillator case, and let us consider the entropy change under a squeeze transformation of the state

$$\psi(x) = \frac{1}{\sqrt{2}} e^{-\frac{|x|^2}{2}}$$  (37)

Using (see [10])

$$\int_0^{+\infty} dx e^{-ax} L_n(x) = \frac{(a - 1)^n}{a^{n+1}} , \quad \Re(a) > 0$$  (38)

we have

$$\langle \chi_n|\psi(r)\rangle = (-1)^n \frac{\tanh^n(\frac{r}{2})}{\cosh(\frac{r}{2})}$$  (39)
and for the associated entropy we get

\[ S = - \sum_n |\langle \chi_n | \psi(r) \rangle|^2 \ln |\langle \chi_n | \psi(r) \rangle|^2 \]

\[ = \ln \left( \cosh^2 \left( \frac{r}{2} \right) \right) + \sinh^2 \left( \frac{r}{2} \right) \ln \tanh^2 \left( \frac{r}{2} \right) \]

\[ \sim r \]  

(40)

then, the leading order behavior of the generated entropy gives \( \Delta S \sim r \).

There is however a factor of two with respect to the reduction done via the \( \hat{N} \)-basis. This can be qualitatively understood by using the intuitive meaning of the Shannon entropy of a state with respect to a given basis [7]: \( e^S \) is the number of basis vectors which are “appreciably involved” in the representation of \( |\psi \rangle \). As the squeezing develops, the width of the wave function in the \( \hat{x} \)-representation grows as \( e^r \). In the case of the \( \hat{N} \)-basis, in order to represent \( |\psi(r) \rangle \), the \( |n \rangle \) which are significant, can be estimated as those having \( \sqrt{\langle x^2 \rangle_n} \) less than the width of the wave function we are considering. Taking into account that for the harmonic oscillator \( \sqrt{\langle x^2 \rangle_n} \sim \sigma_0 e^r \) (\( \sigma_0 \) is the width of \( \psi(x) \)) which gives \( S \sim \ln n \sim 2r \) (cf. (33)). On the other hand, in the case of the Laguerre functions, we have \( \sqrt{\langle x^2 \rangle_n} \sim \sqrt{6n} \), so applying the previous argument we obtain \( S \sim \ln n \sim r \) (cf. (40)).

We will now consider a coarse graining via the superfluctuant quadrature \( \hat{\chi} \). This is motivated by the approximated eigenfunctions used in Eq. (28) which are plane waves (Dirac \( \delta \)'s in the \( \hat{x} \) representation). This alternative basis was recently suggested by M. Gasperini and M. Giovannini. In this case, the calculation can be exactly done and the generated entropy turns out to be linear for all values of squeezing (see Refs. [4] and [5]). In Ref. [3], this property was shown in the case where the initial density matrix is diagonal in the occupation number basis. From the state transformation given by Eq. (15), regardless the initial density matrix, the reduction done via the \( \hat{x} \) operator leads to a linear generation of entropy under a squeezing transformation. If we make the reduction of (34) using \( \hat{A} = \hat{\chi} \) in equation (19), we obtain:

\[ S = - \sum_{b,b'} p_{bb'} \int dx \langle x | b(r) \rangle \langle b'(r) | x \rangle \ln \left( \sum_{b,b'} p_{bb'} \langle x | b(r) \rangle \langle b'(r) | x \rangle \right) \]  

(41)

where we have made explicit the squeezing of the initial state. Now, using (15) we see that

\[ \langle x | b(r) \rangle \langle b'(r) | x \rangle = e^{-r} \varphi_b(e^{-r} x) \varphi_{b'}(e^{-r} x) \]  

(42)

where \( \varphi_b(x) \) is the wave function \( \langle x | b \rangle \), so the expression for the entropy is:

\[ S = r \int dx \langle x | \hat{S} \rho \hat{S}^\dagger | x \rangle \]

\[ - \int dx e^{-r} \sum_{b,b'} p_{bb'} \varphi_b(e^{-r} x) \varphi_{b'}(e^{-r} x) \ln \sum_{b,b'} p_{bb'} \varphi_b(e^{-r} x) \varphi_{b'}(e^{-r} x) \]  

(43)
The second term is just the initial entropy as one can see by changing variables $e^{-r}x \to x$; while the first term is $r \text{Tr} \hat{S} \rho \hat{S}^\dagger = r \text{Tr} \rho = r$ so we get:

$$\Delta S = r$$

(44)

This coincides with what we would have expected from our qualitative argument: the width of the state grows like $e^r$, so the “number” of δ’s involved grows as $e^r$, and the entropy goes like $r$.

Finally, a reduction via the basis of coherent states can be considered. Starting from the vacuum, the leading order behavior can be calculated, giving $\Delta S \sim r$. We can understand this behavior by noting that the localization properties of the over-complete basis of coherent states are similar to those of the Dirac delta’s: the width is the same for every coherent state and they are uniformly distributed on the real axis.

5 The Two Mode Squeeze Operator

In Ref. [1] each mode $\vec{k}$ of the field is independently associated with a squeeze transformation. A better description is given in Refs. [3] and [4] where it is considered a two mode transformation that mixes the modes $\vec{k}$ and $-\vec{k}$, conserving the momentum during the production of pairs.

The evolution operator can be expressed as (see [12])

$$\hat{U} = \hat{R} \hat{S}$$

(45)

where

$$\hat{R} = e^{i \int_0^t dt' H_0(t')} \quad , \quad H_0 = \sum_{k,k_x>0} \Omega_k(t)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} + 1)$$

(46)

$$\hat{S} = e^{i \hat{G}} \quad , \quad \hat{G} = \sum_{k,k_x>0} ir_k(t) \left( e^{-2i\varphi_k(t)} a_k a_{-k} - e^{2i\varphi_k(t)} a_k^\dagger a_{-k}^\dagger \right)$$

(47)

By defining new operators $b_k$ and $c_k$ ($k_x > 0$) according to:

$$a_k = \frac{1}{\sqrt{2}}(b_k - ic_k)e^{-i\varphi_k} \quad , \quad a_{-k} = \frac{1}{\sqrt{2}}(b_k + ic_k)e^{-i\varphi_k}$$

(48)

which satisfy the canonical commutation relations, $\hat{S}$ can be expressed in terms of two single mode squeeze transformations of the form given in [3]:

$$\hat{S} = \prod_{k,k_x>0} \exp \left( \frac{r_k}{2}(b_k^2 - b_k^2) \right) \exp \left( \frac{r_k}{2}(c_k^2 - c_k^2) \right)$$

(49)

Now, we will consider the generation of entropy under the evolution (45) (in each pair of modes $\vec{k}$ and $-\vec{k}$). For simplicity, we will first consider as coarse graining
basis: $|n_b, n_c\rangle$ (the eigenstates of $\hat{N}_b = b_k^\dagger b_k$ and $\hat{N}_c = c_k^\dagger c_k$) and then the more natural one, $|n_k, n_-k\rangle$ (the eigenstates of $\hat{N}_k = a_k^\dagger a_k$ and $\hat{N}_-k = a_k^\dagger a_-k$).

In the first case we have that the entropy at time $t$ is given by

$$ S = - \sum_{n_b, n_c} \left| \langle n_b, n_c | \hat{U}_k(t) | \psi \rangle \right|^2 \ln \left| \langle n_b, n_c | \hat{U}_k(t) | \psi \rangle \right|^2 $$

(50)

where $\hat{U}_k(t)$ is the part of (45) that contains the modes $\vec{k}$ and $-\vec{k}$. From (46) and (48) we see that the corresponding free part of the hamiltonian takes the form

$$ \Omega_k(t) (b_k^\dagger b_k + c_k^\dagger c_k + 1) $$

which gives an irrelevant phase factor in the amplitudes of (50).

Now, using an approximation similar to (28), but for two single mode squeeze operators, we obtain

$$ \langle n_b, n_c | \hat{S}_k(t) | \psi \rangle \approx e^{-2r} f(k_b e^{-2r}, k_c e^{-2r}) $$

$$ f(k_b, k_c) = \int_{-\infty}^{\infty} dp_b \int_{-\infty}^{\infty} dp_c \cos(2p_b \sqrt{k_b}) \cos(2p_c \sqrt{k_c}) \varphi(p_b, p_c) $$

(51)

where $n_b = 2k_b$, $n_c = 2k_c$, $\hat{S}_k(t)$ is the factor appearing in (49) and we have supposed, as we did in (28), that the initial state is even. From (51), we finally get $\Delta S(t) \sim 4r_k(t)$.

The second case is more interesting because it corresponds to the occupation number basis, which has a direct physical significance. Let us consider for example an initial state having $n_0$ particles with momentum $\vec{k}$ and $n_0$ particles with momentum $-\vec{k}$ (the total momentum is zero). Again, the free part of (45) plays no role and the entropy at time $t$ is

$$ S = - \sum_{n_k, n_-k} \left| \langle n_k, n_-k | \hat{S}_k(t) | n_0, n_0 \rangle \right|^2 \ln \left| \langle n_k, n_-k | \hat{S}_k(t) | n_0, n_0 \rangle \right|^2 $$

(52)

Taking into account that the squeeze operator $\hat{S}_k(t)$ creates and annihilates pairs of particles in the modes $\vec{k}$ and $-\vec{k}$ (cf. (17)), we have that the surviving terms in (52) are those having $n_k = n_-k = n$:

$$ S = - \sum_{n} \left| \langle n, n | \hat{S}_k(t) | n_0, n_0 \rangle \right|^2 \ln \left| \langle n, n | \hat{S}_k(t) | n_0, n_0 \rangle \right|^2 $$

(53)

We can change $|n, n\rangle$ to the basis $|n_a, n_b\rangle$:

$$ |n, n\rangle = \frac{(b_k^\dagger)^n}{\sqrt{n!}} |0, 0\rangle = \frac{1}{2^n} \sum_{j=0}^{n} \frac{\sqrt{(2n-2j)!(2j)!}}{(n-j)!j!} |2n-2j, 2j\rangle' $$

(54)

where the prime in the second member denotes that the ket is given in the basis $|n_b, n_c\rangle$. 

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Using Stirling’s formula we can approximate the factorials in (54) as
\[
\frac{\sqrt{(2n-2j)!(2j)!}}{2^n(n-j)!j!} \approx \frac{1}{\sqrt{\pi(n-j)j}}
\]
moreover, for \( n \) large we shall replace the sum by an integral. Note however that in spite of the approximation we still have the normalization condition:
\[
\int_0^n dj \frac{1}{\pi((n-j)j)^{\frac{3}{2}}} = \frac{1}{\pi} B\left(\frac{1}{2}, \frac{1}{2}\right) = 1, \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}
\]
Now, for large squeezing, we obtain from (51)
\[
\langle n_0, n_0 | \hat{S}_k(t) | n, n \rangle \approx e^{-2r} \int_0^n dj \frac{1}{\sqrt{\pi((n-j)j)^{\frac{3}{2}}}} f((n-j)e^{-2r}, je^{-2r})
\]
\[
= e^{-r} \int_0^{ne^{-2r}} dj \frac{1}{[(ne^{-2r}-j)j]^{\frac{3}{4}}} f(ne^{-2r}-j, j)
\]
\[
= e^{-r} \tilde{f}(ne^{-2r})
\]
Approximating the sum in (53) by the integral and using (57) as well as the result (see Appendix) \( \int_0^\infty dn \tilde{f}(n)^2 = 1 \) (similar to (22)) we obtain the leading order behavior for the generated entropy in the occupation number basis:
\[
\Delta S(t) \sim 2r_k(t)
\]

6 Discussion

In this paper we studied the dependence of the entropy generation on the initial state and the coarse graining criteria, relating these issues to the Shannon entropy; i.e., the change of the richness of a state in a given basis as the squeezing develops. For this purpose we considered the eigenstates of the squeeze operator which enabled us to describe the evolution as a dilation transformation of the wave function.

Making a reduction via the particle number basis, we computed the classical limit for the entropy generation under a one mode squeeze transformation, obtaining that for a general class of initial states it is linear in the squeeze parameter. This is also the case when the reduction is done via the superfluctuant quadrature of the field (here, the linear behavior is valid for any squeezing).

A deep understanding of entropy generation due to the creation of particle pairs will come from the knowledge of the physical mechanism that singles out a coarse graining basis. We note however that for a wide class of basis, the associated entropy will increase as the system evolves. This generation of entropy can be understood if we recall that the Shannon entropy gives the richness of a state with respect to a given basis. In the superfluctuant quadrature representation the wave function of the initial state flattens (cf. (15)); and for a basis sharing the localization properties
of the occupation number basis or that of Dirac’s δ-functions we will have that at each stage more basis vectors are necessary to describe the state.

Then, we studied the entropy generation (in each pair of modes $\vec{k}$ and $-\vec{k}$) when the matter field undergoes a two mode squeeze evolution, that is, when particle pairs are created or annihilated from the initial state, conserving momentum. Here, we considered the reduction via two different basis: the occupation number basis for the modes $b_k$ and $c_k$ that factorize the two mode squeeze transformation, and the occupation number basis for particles with momentum $\vec{k}$ and $-\vec{k}$. In the first case the classical behavior of the generated entropy is $S \sim 4 r_k(t)$ (twice the value for the one mode case) while in the second case, starting from a state with definite particle number and zero momentum, it is $S \sim 2 r_k(t)$. This calculation generalizes the result given in Ref. [3] where the initial state is the vacuum.

Then, in the cases we have considered, we can see that the initial state does not leave its trace, when we look at the classical behavior of the generated entropy. This behavior turns out to be linear in the squeeze parameter with a proportionality factor that depends on the coarse graining criteria.

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We would like to thank D. Harari and E. Calzetta for usefulness discussions on related matters.
Appendix

Here we shall show the result used in (58):

\[ \int_0^\infty dn \tilde{f}(n)^2 = 1 \]  
(59)

In order to obtain an expression for \( \tilde{f}(n) \) we should find the equivalent of equations (25) and (29) for the two mode case.

First, using (55) and (25) we have

\[ \langle p_b p_c | 0, n_j \rangle \approx \frac{1}{\sqrt{\pi}} J_0 \int_0^n dj \frac{1}{[(n - j)j]^\frac{1}{2}} \langle p_b p_c | 2n - 2j, 2j' \rangle' \]
\[ \approx \frac{1}{\pi^\frac{1}{2}} J_0 \int_0^n dj \frac{1}{[(n - j)j]^\frac{1}{2}} \cos(2p_b \sqrt{n - j}) \cos(2p_c \sqrt{j}) \]
\[ = \frac{1}{\sqrt{\pi}} J_0 \left( 2\sqrt{n(p_b^2 + p_c^2)} \right) \]  
(60)

\( J_0 \) is a Bessel function.

Now we can compute \( \tilde{f}(n) \),

\[ \tilde{f}(n) = \int dp_b dp_c \langle n_0 n_0 | p_b p_c \rangle \frac{1}{\sqrt{\pi}} J_0 \left( 2\sqrt{n(p_b^2 + p_c^2)} \right) \]
\[ = \frac{1}{2^{n_0}} \sum_{j=0}^{n_0} \frac{\sqrt{(2n_0 - 2j)!(2j)!}}{\sqrt{\pi(n_0 - j)!}j!} \int dp_b dp_c \langle p_b | 2n_0 - 2j \rangle' \langle p_c | 2j \rangle' J_0 \left( 2\sqrt{n(p_b^2 + p_c^2)} \right) \]
\[ = \frac{1}{2^{2n_0}} \sum_{j=0}^{n_0} \frac{1}{\pi(n_0 - j)!j!} \int dp_b dp_c H_{2n_0 - 2j}(p_b) H_{2j}(p_c) e^{-\frac{p_b^2 + p_c^2}{2}} J_0 \left( 2\sqrt{n(p_b^2 + p_c^2)} \right) \]
\[ = (-1)^{n_0} \int_0^{2\pi} du e^{-u/2} J_0(2\sqrt{n u}) L_{n_0}(u) \]  
(62)

where we changed into polar coordinates \((p, \alpha)\) in the plane \(p_b, p_c\) and applied the identity (see [10])

\[ \int_0^{2\pi} d\alpha H_{2n}(p \cos \alpha) H_{2m}(p \sin \alpha) = 2\pi (-1)^{n+m} \frac{(2n)!(2m)!}{n!m!} L_{n+m}(p^2) \]  
(63)

involving Hermite \((H)\) and Laguerre \((L)\) polynomials. We also used the normalization condition for the state \(|n_0, n_0\rangle\) in the basis \(|n_b, n_c\rangle\) (cf. (54)):

\[ \frac{1}{2^{2n_0}} \sum_{j=0}^{n_0} \frac{(2n_0 - 2j)!(2j)!}{[(n_0 - j)!j!]^2} = 1 \]

With the above expression for \( \tilde{f}(n) \) we finally get:

\[ \int_0^\infty dn \tilde{f}(n)^2 = \int_0^\infty du du' e^{-\frac{u+u'}{2}} \int_0^\infty dn J_0(2\sqrt{n u}) J_0(2\sqrt{n u'}) L_{n_0}(u) L_{n_0}(u') \]
\[ = \int_0^\infty du e^{-u} L_{n_0}^2(u) = 1 \]  
(64)

where we have used the orthogonality of Bessel functions and the normalization of Laguerre polynomials.
References

[1] L. P. Grishchuk and Y. V. Sidorov, Class. Quantum Grav. 6, L161 (1989); Phys. Rev. D42, 3413 (1990)

[2] B. L. Hu and D. Pavon, Phys. Lett. B180, 329 (1986)

[3] T. Prokopec, Class. Quantum Grav. 10, 2295 (1993)

[4] M. Gasperini and M. Giovannini, Phys. Lett. B301, 334 (1993)

[5] C. G. Bollini and L. E. Oxman, Phys. Rev. A47, 2339 (1993)

[6] M. Gasperini and M. Giovannini, Class. Quantum Grav. 10, L133 (1993)

[7] A. Peres in “Proceedings of the Adriatico Research Conference on Quantum Chaos”, Ed. H. A. Cerdeira et al. (World Scientific, 1991) p. 73

[8] C. G. Bollini and J. J. Giambiagi “On Tachyon Quantization” in J. Tiomno Festschrift (World Scientific, 1992)

[9] “Handbook of Mathematical Functions”, Edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards, AMS 55 (1972) p. 508, 510

[10] I.S. Gradshteyn, I.M. Ryshik “Table of Integrals, Series and Products” (New York, Academic Press, 1965) p. 501, 844

[11] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. D48, R439 (1993)

[12] A. Albrecht, P. Ferreira, M. Joyce and T. Prokopec “Inflation and squeezed quantum states”, Preprint Imperial/TP/92-93/21, astro-ph@babbage.sissa.it