Limits of random trees II

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Abstract

Local convergence of bounded degree graphs was introduced by Benjamini and Schramm. This result was extended further by Lyons to bounded average degree graphs. In this paper we study the convergence of random tree sequences with given degree distributions. Denote by \( D_n \) the set of possible degree sequences of a tree on \( n \) nodes. Let \( D_n \) be a random variable on \( D_n \) and \( T(D_n) \) be a uniform random tree with degree sequence \( D_n \). We show that the sequence \( T(D_n) \) converges in probability if and only if \( D_n \to D = (D(i))_{i=1}^\infty \), where \( D(i) \sim D(j), \mathbb{E}(D(1)) = 2 \) and \( D(1) \) is a random variable on \( \mathbb{N}^+ \).

Keywords: sparse graph limits, random trees

1 Introduction

In recent years the study of the structure and behavior of real world networks has received wide attention. The degree sequence of these networks appear to have special properties (like power law degree distribution). Classical random graph models (like the Erdős-Rényi model) have very different degree sequence. An obvious solution is to study a random graph with given degree sequence. More generally generate a random graph with a degree sequence from a family of degree sequences. In [7] Chatterjee, Diaconis and Sly studied random dense graphs (graphs whose number of edges is comparable to the square of the number of vertices) with a given degree sequence.

It is not always easy to generate a truly random graph with a given degree sequence. There is a fairly large literature on the configuration model (for the exact definition of the model see [5]), where for a given degree sequence for each node \( i \) we consider \( d_i \) stubs and take a random pairing of the stubs and connect the corresponding nodes with an edge. This model creates the required degree distribution, but gives a graph with possible loops and parallel edges.
A notion of convergence for (dense) graph sequences was developed by Borgs, Chayes, Lovász, Sós and Vesztergombi in [6]. The limit objects were described by Lovász and Szegedy in [12]. Using this limit theory, the authors in [7] described the structure of random (dense) graphs from the configuration model. They defined the convergence of degree sequences and for convergent degree sequences they gave a sufficient condition on the degree sequence, which implies the convergence of the random graph sequence (taken from the configuration model).

What can we say if the graphs we want to study are sparse (the number of edges is comparable to the number of vertices) and not dense? Is there a similar characterization for sparse graphs with given degree sequence?

The notion of convergence of bounded degree graphs (that is the degree of each vertex is bounded above by some uniform constant $d$) was first introduced by Benjamini and Schramm [3]. This notion was extended by Lyons [13] to bounded average degree graphs.

In [8] the author described the behavior of a random tree sequence with a given degree distribution. In this paper we extend this result and prove a similar characterization as in [7] for random trees with given degree sequence. We define the convergence of degree sequences and give a necessary and sufficient condition on the degree sequence, which implies the convergence of the tree sequence $T(D_n)$ in the sense of Lyons [13]. In the case of convergence we describe the limit.

This paper is organized as follows: In Section 2 we give the basic definitions and notations. In Section 3 we describe the basic properties and the limit of a sequence of random degree sequences. At the end of the section we state our main theorem. In Section 4 we deal with labeled homomorphisms and in Section 5 we describe the limit object.

2 Basic definitions and notations

2.1 Random weak limit of graph sequences

Let $G = G(V, E)$ be a finite simple graph on $n$ nodes. For $S \subseteq V(G)$ denote by $G[S]$ the subgraph of $G$ spanned by the vertices $v \in S$. For a finite simple graph $G$ on $n$ nodes, let $B_G(v, R)$ be the rooted $R$-ball around the node $v$, that is the subgraph induced by the nodes at distance at most $R$ from $v$:

$$B_G(v, R) = G\left[\{u \in V(G) : \text{dist}_G(u, v) \leq R\}\right].$$

Given a positive integer $R$, a finite rooted graph $F$ and a probability distribution $\rho$ on rooted graphs, let $p(R, F, \rho)$ denote the probability that the graph $F$ is rooted isomorphic to the $R$-ball around the root of a rooted graph chosen with distribution $\rho$. For a finite graph $G$, let $U(G)$ denote the distribution of rooted graphs obtained by choosing a uniform random vertex of $G$ as root of $G$. 


It is easy to see, that for any finite graph $G$ we have

$$p(R, F, U(G)) = \frac{\{v \in V(G) : B_G(v, R) \text{ is rooted isomorphic to } F\}}{|V(G)|}.$$  

**Definition 1.** Let $(G_n)$ be a sequence of finite graphs on $n$ nodes, $\rho$ a probability distribution on infinite rooted graphs. We say that the random weak limit of $G_n$ is $\rho$, if for any positive integer $R$ and finite rooted graph $F$, we have

$$\lim_{n \to \infty} p(R, F, U(G_n)) = p(R, F, \rho).$$  

**Definition 2.** Let $(G_n)$ be a sequence of random finite graphs on $n$ nodes, $\rho$ a probability distribution on infinite rooted graphs. We say that the random weak limit of $G_n$ is $\rho$, if for any $\epsilon > 0$, $R \in \mathbb{N}^+$ and finite rooted graph $F$, we have

$$\lim_{n \to \infty} \mathbb{P}(|p(R, F, U(G_n)) - p(R, F, \rho)| > \epsilon) = 0.$$  

### 2.2 Other notations

We will denote random variables with bold characters. For a probability space $(\Omega, \mathcal{B}, \mu)$ and $A \in \mathcal{B}$ denote by $I(A)$ the indicator variable of the event $A$. Denote by $D_n$ the set of possible degree sequences of a tree on $n$ nodes. Let $D_n$ be a random variable on $D_n$. Denote by $T(D_n)$ the uniform random tree on $n$ nodes, with degree sequence $D_n$. Denote the degree sequence of a tree $T$ by $D_T = (D_T(i))_{i=1}^n$. It is easy to see that

$$\mathbb{P}(T(D_n) = T) = \frac{\mathbb{P}(D_n = D_T)\cdot n - 2}{D_T(1), D_T(2), \ldots, D_T(n)}.$$  

If it does not cause any confusion, we will use $D, T_n$ instead of $D_T, T(D_n)$ respectively.

Let $U^l$ denote the set of all finite $l$-deep rooted tree. Denote by $T^l_x$ an $l$-deep rooted tree on $k$ nodes with root $x$. Let us denote the nodes at the $i$th level with $T_i$, and $|T_i| = t_i$ ($t_0$ is just 1, $t_1$ is the degree of the root and $t_l$ is the number of leafs).

Let $\mathcal{T}$ be the set of all infinite rooted trees. For a $T \in \mathcal{T}$ denote by $T(R)$ the $R$-neighborhood of the root of $T$. For an $R$-deep rooted tree $F$ define the set

$$\mathcal{T}(F) = \{T \in \mathcal{T} : T(R) \text{ is rooted isomorphic to } F\}.$$  

Let $\mathcal{F}$ be the sigma-algebra generated by the sets $(\mathcal{T}(F))_F$, where $F$ is an arbitrary finite rooted tree. $(\mathcal{T}, \mathcal{F})$ is a probability field. We call a probability measure $\mu$ on $\mathcal{T}$ an infinite rooted random tree.

It is easy to see that for a random infinite rooted tree $\mu$ and $F \in U^R$, the following holds:

$$p(R, F, \mu) = \sum_{H \in U^{R+1}, H(R) \approx F} p(R + 1, H, \mu).$$  

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This is a necessary consistency criterion for the neighborhood densities. A version of the above should hold for any distribution $\rho$ on rooted infinite graphs. Note that if we want to prove convergence of random tree sequences, then we need to have (2) the convergence of the neighborhood densities and also (3) the consistency of these densities.

3 Limits of degree sequences

Consider a random degree sequence $D_n = (D_n(i))_{i=1}^n$ and construct a tree $T(D_n)$ with uniform distribution given the degree sequence. We want to describe the limit of $T(D_n)$ as $n \to \infty$. We give a characterization of the degree sequences for which $T(D_n)$ has a random tree limit. To describe the model and the limit, we need to define and understand the limit of a random degree sequence $D_n$. Here we only deal with degree sequences of trees. Also we consider degree sequences for which relabeling does not change the distribution. In other words we consider only exchangeable degree sequences, that is for any $\sigma \in S_n$ we have

$$(D_n(i))_{i=1}^n \sim (D_n(\sigma(i)))_{i=1}^n.$$ 

For more on exchangeable random variables we refer to [2].

**Definition 3.** We say that an exchangeable sequence $D_n$ is convergent and $D_n \to D$, where $D$ is a random infinite sequence, if for every $k \in \mathbb{N}$ we have

$$(D_n(i))_{i=1}^k \xrightarrow{p} (D(i))_{i=1}^k.$$ 

It is easy to see, that if $D_n$ is exchangeable and $D_n \to D$ then $D$ is also an exchangeable sequence. The following theorem (see [11]) describes the limits of exchangeable sequences.

**Theorem 1.** Let $X$ be a random infinite exchangeable array. Then $X$ is a mixture of infinite dimensional iid distributions, that is

$$X = \int_{\text{IID}} \lambda d\Pr(\lambda),$$

where $\Pr$ is a distribution on infinite dimensional IID distributions $\lambda$.

As a result we have that the limit of an exchangeable degree sequence is an infinite exchangeable sequence and so a mixture of IID distributions. Note that if $D_n$ is not exchangeable then we can take a random permutation $\sigma \in S_n$ and define the exchangeable degree sequence $D_n(i) = D_n(\sigma(i))$.

**Lemma 1.** Let $X$ be an infinite exchangeable random sequence. Further assume that we have

$$\Pr(X(1) = i, X(2) = i) = \Pr(X(1) = i)\Pr(X(2) = i).$$

Then $X$ is an infinite IID distribution ($\Pr$ is concentrated on one distribution).
Proof: From Jensen’s inequality we have that
\[ \int_{IID} \lambda(i)^2 dp(\lambda) \geq \left( \int_{IID} \lambda(i) dp(\lambda) \right)^2. \] (4)

Also from Theorem 1 we have
\[ \int_{IID} \lambda(i)^2 dp(\lambda) = \mathbb{P}(X(1) = i, X(2) = i) = \mathbb{P}(X(1) = i) \mathbb{P}(X(2) = i) = \left( \int_{IID} \lambda(i) dp(\lambda) \right)^2. \]

It follows that in (4) equality holds which means that \( p \) is a degenerate distribution and so proves our lemma. \( \square \)

We will see that if \( T(D_n) \) is convergent, then \( D_n \) satisfies the assumptions in Lemma 1. So for a convergent random tree sequence \( T(D_n) \) the limit of the degree sequence \( D_n \) needs to be an infinite IID distribution. As the average degree of a tree is \( 2n - 1n \), one would expect that in the limit distribution the expected degree of a node is 2, that is \( E(D(i)) = 2 \) for every \( i \).

Example 1. Let \( X \) be a uniform random element of \([n] \). Consider the degree sequence
\[ D(i) = \begin{cases} n - 1, & \text{if } i = X \\ 1, & \text{otherwise} \end{cases} \]
\( T(D_n) \) will be the star-graph on \( n \) nodes. The limit degree sequence is just the constant 1 vector \( 1 = (1, 1, \ldots) \). Obviously the expected degree of a node is 1. It is not hard to see, that if \( F \) is not a single edge, then \( u(R, F, T_n) = 0 \) for every \( n > |V(F)| \). Thus there is no limit distribution \( \rho \) on infinite graphs such that \( \mathbb{P}(|p(R, F, U(T_n)) - p(R, F, \rho)| > \epsilon) \to 0 \) for every \( F \).

It turns out that it is enough to have that the degree sequence converges and \( E(D(i)) = 2 \) holds \( \forall i \).

Theorem 2. Let \( D_n \) be a sequence of random degree sequences (\( D_n \in D_n \)). The random tree sequence \( T(D_n) \) is convergent and converges to an infinite random tree if and only if \( D_n \to D \), where \( D = (D_0, D_0, \ldots) \) is an infinite IID sequence and \( E(D_0) = 2 \).

4 Labeled subgraph densities

From now on let \( T \) be a fixed tree on \( k \) nodes. Assign values \( r_i \) to each node \( i \) of \( T \) and call it the remainder degree of node \( i \). Let \( S = \{s_1, s_2, \ldots, s_k\} \) be an ordered subset of \([n] = \{1, 2, \ldots, n\} \) \( (s_i \neq s_j \text{ for } i \neq j) \). Now by \( T_S \) we denote the tree \( T \), with label \( s_i \) at node \( i \).
Similarly let \( F \) be a forest with \( m_F \) nodes and \( c_F \) connected components. Denote these components by \( C_1, \ldots, C_{c_F} \). As above, define the remainder degrees for \( F \), and denote them by \((r_1, \ldots, r_{m_F})\). Denote by \( F_S \) the labeled forest with label \( s_i \) at node \( i \).

For two labeled trees \( T_S, T_{S'} \), with remainder degrees \( r, r' \) we define the operation of gluing in the usual way. We identify nodes \( i \in V(T), j \in V(T') \) if \( s_i = s_j' \) and keep the label \( s_i (= s_j') \). For nodes \( i (j) \), where \( s_i \notin S' (s_j' \notin S) \), we do nothing. If \( S \cap S' = \emptyset \), then the resulting graph is just the disjoint union.

We say that a gluing is valid, if it results in a forest and the remainder degrees are consistent, that is \( s_i = s'_j \Rightarrow r_i = r'_j \forall i, j \). Denote the gluing by \( g(T_S, T_{S'}) \).

We define the gluing of two labeled forests similarly. Let

\[
I_n(T_S) = I (\{ T_n[S] \cong T \text{ and } \forall i, D_n(s_i) = D_T(i) + r_i \})
\]

\[
X_T^n = \sum_{S \subseteq [n]} I_n(T_S) = \text{inj lab}(T, T(D_n)).
\]

Here \( \text{inj lab}(T, G) \) denotes the number of labeled copies of \( T \) in \( G \), that is the number of copies of \( T \) in \( G \) with \( r_i = |N(i) \setminus V(T)|, \forall i \in V(T) \). We define \( I_n(F_S), X_F^n \) similarly for a forest \( F \). If it does not cause any confusion, we use \( m_1, c_1, I_S \) instead of \( m_F, c_F, I_n(T_S) \).

As before \( B_G(v, l) \) is the rooted \( l \)-ball around \( v \) in \( G \) and \( T_n \) is the uniform random tree on \( n \) nodes with degree distribution \( D_n \). Let \( \sigma, \rho \in \text{Aut}(T^n_\ell) \) be two automorphisms of the rooted tree \( T^n_\ell \). We say that \( \sigma \sim \rho \) if and only if there exists \( \tau \in \text{Aut}(T^n_\ell) \), such that \( \tau \) fixes every vertex not in \( T_1 \) and \( \sigma \circ \tau = \rho \). \( \sim \) is an equivalence relation. The equivalence classes have \( \prod_{i \in T_1} (D(i) - 1)! \) elements, hence it follows

\[
|\text{Aut}(T^n_\ell)| = |\text{Aut}(T^n_\ell)/\sim| \prod_{i \in T_1} (D(i) - 1)!
\]

It is easy to see that

\[
p(l, T^n_\ell, T_n) = \frac{1}{n} \frac{X^n_{T'}}{|\text{Aut}(T^n_\ell)/\sim|}, \text{ where}
\]

\[
T' = T^n_\ell \setminus T_1
\]

\[
r_i = \begin{cases} 0 & i \notin T_1 \cup T_{l-1} \\ D_{T^n_\ell}(i) - 1 & i \in T_{l-1} \end{cases}
\]

If \( T_n \) is a convergent random tree sequence then from (2) and (6) we have that for any \( T' \) defined above we have that

\[
D^2 \left( \frac{X^n_{T'}}{n} \right) \to 0
\]

The above condition ensures that the limit object will be a random tree, not a distribution on random tree families. We remark here, that for bounded degree
graphs the convergence of the neighborhood densities means that the sequence is convergent in the sense of Benjamini and Schramm. Example 1 shows that for bounded average degree graphs the convergence of the neighborhood densities alone is not enough. We need also (3) to hold.

The reason is that for a fixed \( k \) the \( k \)-neighborhood of the large degree nodes is large (\( O(n) \)). In Example 1 even if \( k = 1 \), every node "sees" the center node (e.g. every neighborhood with radius 1 contains the center node) and so every 2 radius neighborhood contains \( O(n) \) vertices, which is unbounded.

As \( X_T^F \) is a sum of indicator variables, we want to express the probability \( \mathbb{P}(I_n(T_{S_1}) \cdots I_n(T_{S_h}) = 1) \) and \( \mathbb{P}(I_n(T_{S_1}) \cdots I_n(T_{S_h}) = 1\mid I_n(T_{S'_1}) \cdots I_n(T_{S'_h}) = 1) \). The following two lemmas give us these probabilities in a more general setting. For a random degree sequence \( D_n \) let \( D_S = (D_{n}(s_i))_{i=1}^k \).

**Lemma 2.** Let \( D_n \in \mathcal{D}_n \) be a random degree sequence and \( T_n = T_n(D_n) \). Let \( F \) be an arbitrary forest on \( m \) nodes with remainder degrees \( r = (r_1, \cdots, r_m) \). Let \( R = \sum r_i \) and denote by \( c \) the number of connected components of \( F \). The probability that on an ordered subset \( S = (s_1, \cdots, s_m) \) of the nodes of \( T_n \) we see the forest \( F \) and \( \forall i, D(s_i) = D_F(i) + r_i \), is

\[
\mathbb{P}(I_n(F_S) = 1) = \frac{(n - m + c - 2)!}{(n - 2)!} H(r, F) \mathbb{P}(D_S = D_T),
\]

where \( H(r, F) = \prod_{i=1}^c \left[ \left( \sum_{j \in C_i} r_j \right) \prod_{j \in C_i} \frac{(D_F(i) + r_j - 1)!}{(r_j)!} \right] \) is a constant depending only on \( F \) and the remainder degrees \( r \).

**Proof:** Let \( R_i = \sum_{j \in C_i} r_j \). Fix a degree sequence \( D = (D(i))_{i=1}^n \). The number of trees realizing this degree sequence is just \( \frac{n-2}{D(1)-1, \cdots, D(n)-1} \). The number of trees realizing the degree sequence \( D \) and having \( F \) on the first \( m \) nodes is

\[
\prod_{i=1}^c \left[ \frac{R_i!}{\prod_{j \in C_i} r_j!} \right] \left( R_1 - 1, \cdots, R_c - 1, D(m + 1) - 1, \cdots, D(n) - 1 \right).
\]

From this it follows that

\[
\mathbb{P}(I_n(F_{\{1, \cdots, m\}}) = 1\mid D_n = D) = \frac{n - m + c - 2}{(R_1 - 1, \cdots, R_c - 1, D(m + 1) - 1, \cdots, D(n) - 1)} \prod_{i=1}^c \frac{R_i!}{\prod_{j \in C_i} r_j!}.
\]

(9)

Note that the degree sequence \( D \) should be such that \( D(i) = D_F(i) + r_i, i = 1, \cdots, m \) holds for the first \( m \) degree. We need to sum this probability for every possible degree sequence. In our case we sum over degree sequences for which \( D(i) = D_F(i) + r_i, i = 1, \cdots, m \) holds. As in equation (9) the right hand side
Proof: The proof follows immediately from the definition of the conditional probability.

**Remark 1.** Note that we consider exchangeable degree distributions only. For any k element set S we have

\[ P(I_n(T_S)) = P(I_n(T_{\{1, \ldots, k\}})) \]

If \( T_n \) is convergent then we need that for every \( T' \) defined in (7) we have \( D^2(\frac{X_n^{T'}}{n}) \to 0 \). For any T we have

\[ D^2 \left( \frac{X_n^{T}}{n} \right) = \frac{1}{n^2} \sum_{S_i, S_j \in S} P(I_n(T_{S_i})I_n(T_{S_j})) - P(I_n(T_{S_i}))P(I_n(T_{S_j})) \] (11)

Now if we split the sum by the size of the intersection of \( S_i \) and \( S_j \) and use Remark 1, we have

\[ D^2 \left( \frac{X_n^{T}}{n} \right) = \frac{1}{n^2} \sum_{i=0}^{k} \sum_{|S_i \cap S_j|=i} n(n-1) \cdots (n-2k+i+1) \cdot (P(I_n(T_{S_i})I_n(T_{S_j})) - P(I_n(T_{S_i}))P(I_n(T_{S_j}))) \] (12)
From Lemma 2 and 3 we can easily derive that the order of the terms corresponding to \( i \neq 0 \) is \( O(n) \). Now the condition that \( D^2(X_n^{T'}) \to 0 \) is equivalent to

\[
\frac{(n-1) \ldots (n-2k+1)}{n} \left( \mathbb{P}(I_n(T_{S_1}')I_n(T_{S_2}')) - \mathbb{P}(I_n(T_{S_1}'))\mathbb{P}(I_n(T_{S_2}')) \right) \to 0.
\]

Using again Lemma 2 and 3 we can easily derive the following:

\[
\forall T', D^2 \left( \frac{X_n^{T'}}{n} \right) \to 0 \iff \forall S_1, S_2 \subseteq [n] : |S_1| = |S_2| = k, S_1 \cap S_2 = \emptyset \quad \mathbb{P}(D_{S_1} = D_T', D_{S_2} = D_T') \to \mathbb{P}(D_{S_1} = D_T')\mathbb{P}(D_{S_2} = D_T') \quad (13)
\]

**Remark 2.** Putting together Lemma 1 and (13) we conclude that the labeled subgraph densities of a random tree sequence converge in probability if and only if the corresponding degree sequence converges to an infinite IID sequence.

**Remark 3.** The formula in Lemma 2 yields an easy result on the probability that two vertices \( i, j \) with degrees \( d_i, d_j \) are connected:

\[
\mathbb{P}(ij \in E(T(D_n)) \mid D_n(i) = d_i, D_n(j) = d_j) = \frac{d_i + d_j - 2}{n-2}.
\]

Similarly for a given edge \( ij \in E(T(D_n)) \) the degree distribution of the vertices \( i \) and \( j \) can be expressed:

\[
\mathbb{P}(D_n(i) = d_i, D_n(j) = d_j \mid ij \in E(T(D_n))) = \frac{n}{n-2} \frac{d_i + d_j - 2}{2} \mathbb{P}(D_n(i) = d_i, D_n(j) = d_j)
\]

### 5 The limit of \( T(D_n) \)

As before let \( U^l \) denote the set of all finite \( l \)-deep rooted tree. Consider an \( l \)-deep rooted tree with root \( x \): \( T_x^l \in U^l \), with \( |T_x^l| = k \). Let us denote the nodes at the \( i \)th level with \( T_i \), and \( |T_i| = t_i \) (\( t_0 \) is just 1, \( t_1 \) is the degree of the root and \( t_1 \) is the number of leafs). \( B_G(v,l) \) is the rooted \( l \)-ball around \( v \) in \( G \) and \( T_n = T(D_n) \) is a random tree with degree distribution \( D_n \).

If we assign remainder degrees \( r \) to the rooted tree \( T_x^l \) and forget the root, then using Lemma 2 we get

\[
E(X_n^{T'}) = E \left( \sum_{S \subseteq [n]} I_n(T_S) \right) = \frac{n!}{(n-k)!} \mathbb{P}(I_n(T_S) = 1) = \frac{n}{n-2} \frac{d_i + d_j - 2}{2} \mathbb{P}(D_n(i) = d_i, D_n(j) = d_j)
\]

\[
E(X_n^{T'}) = \mathbb{E} \left( \sum_{S \subseteq [n]} I_n(T_S) \right) = \frac{n!}{(n-k)!} \mathbb{P}(I_n(T_S) = 1) = \frac{n}{n-2} \frac{d_i + d_j - 2}{2} \mathbb{P}(D_n(i) = d_i, D_n(j) = d_j)
\]

\( \propto (n-1) \ldots (n-2k+1) \mathbb{P}(I_n(T_{S_1}')I_n(T_{S_2}')) - \mathbb{P}(I_n(T_{S_1}'))\mathbb{P}(I_n(T_{S_2}')) \to 0. \)

Using again Lemma 2 and 3 we can easily derive the following:

\[
\forall T', D^2 \left( \frac{X_n^{T'}}{n} \right) \to 0 \iff \forall S_1, S_2 \subseteq [n] : |S_1| = |S_2| = k, S_1 \cap S_2 = \emptyset \quad \mathbb{P}(D_{S_1} = D_T', D_{S_2} = D_T') \to \mathbb{P}(D_{S_1} = D_T')\mathbb{P}(D_{S_2} = D_T') \quad (13)
\]

**Remark 2.** Putting together Lemma 1 and (13) we conclude that the labeled subgraph densities of a random tree sequence converge in probability if and only if the corresponding degree sequence converges to an infinite IID sequence.

**Remark 3.** The formula in Lemma 2 yields an easy result on the probability that two vertices \( i, j \) with degrees \( d_i, d_j \) are connected:

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\mathbb{P}(ij \in E(T(D_n)) \mid D_n(i) = d_i, D_n(j) = d_j) = \frac{d_i + d_j - 2}{n-2}.
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Similarly for a given edge \( ij \in E(T(D_n)) \) the degree distribution of the vertices \( i \) and \( j \) can be expressed:

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\mathbb{P}(D_n(i) = d_i, D_n(j) = d_j \mid ij \in E(T(D_n))) = \frac{n}{n-2} \frac{d_i + d_j - 2}{2} \mathbb{P}(D_n(i) = d_i, D_n(j) = d_j)
\]

### 5 The limit of \( T(D_n) \)

As before let \( U^l \) denote the set of all finite \( l \)-deep rooted tree. Consider an \( l \)-deep rooted tree with root \( x \): \( T_x^l \in U^l \), with \( |T_x^l| = k \). Let us denote the nodes at the \( i \)th level with \( T_i \), and \( |T_i| = t_i \) (\( t_0 \) is just 1, \( t_1 \) is the degree of the root and \( t_1 \) is the number of leafs). \( B_G(v,l) \) is the rooted \( l \)-ball around \( v \) in \( G \) and \( T_n = T(D_n) \) is a random tree with degree distribution \( D_n \).

If we assign remainder degrees \( r \) to the rooted tree \( T_x^l \) and forget the root, then using Lemma 2 we get

\[
E(X_n^{T'}) = E \left( \sum_{S \subseteq [n]} I_n(T_S) \right) = \frac{n!}{(n-k)!} \mathbb{P}(I_n(T_S) = 1) = \frac{n}{n-2} \frac{d_i + d_j - 2}{2} \mathbb{P}(D_n(i) = d_i, D_n(j) = d_j)
\]

\[
E(X_n^{T'}) = \mathbb{E} \left( \sum_{S \subseteq [n]} I_n(T_S) \right) = \frac{n!}{(n-k)!} \mathbb{P}(I_n(T_S) = 1) = \frac{n}{n-2} \frac{d_i + d_j - 2}{2} \mathbb{P}(D_n(i) = d_i, D_n(j) = d_j)
\]

\[5\]
From (7) we have that
\[ p(l, T^l_x, T_n) = \frac{1}{n} \frac{X_n^T}{|\text{Aut}(T^l_x)/\sim|}. \]

Let
\[ \mu_n(T^l_x) = \frac{1}{n} \frac{\mathbb{E}(X_n^T)}{|\text{Aut}(T^l_x)/\sim|}. \]

Assume we have a convergent sequence of random trees \( T_n \) with degree sequence \( D_n \). Further assume that \( D_n \rightarrow D = (D_0, D_0, \cdots) \) and let \( \gamma = \mathbb{E}(D_0) - 1 \). Define
\[
p(T^l_x) = \lim_{n \rightarrow \infty} \mu_n(T^l_x) = \frac{1}{|\text{Aut}(T^l_x)/\sim|} \frac{n-1}{n-k} \mathbb{P}(D_n(\{1, 2, \cdots, k\}) = D_{T^l_x}) H(r, T^l_x) = \prod_{i \not\in T^l_x} \mathbb{P}(D_0 = d_i)(d_i - 1)! \frac{1}{|\text{Aut}(T^l_x)|} t_i.
\]

Define \( \mu(T(T)) = p(F) \). As the sets \( T(F) \) generate the \( \sigma \)-algebra, \( \mu \) extends to \( T \). For a convergent sequence \( T_n \) we need \( \mu \) to be a probability measure on \( T \).

**Lemma 4.** Let \( D_n \) be an exchangeable random degree sequence and assume that \( D_n \rightarrow D \), where \( D \) is an infinite IID random sequence of the variable \( D_0 \). Let \( \gamma \) be the associated measure defined above. \( \mu \) extends to a probability measure on \( G \) if and only if \( \mathbb{E}(D_0) = 2 \) (or equivalently \( \gamma = 1 \)).

**Proof:** We only need to show that
\[ p(T^{l-1}_x) = \sum_{T^l_x : B_{T^l_x}(x, l - 1) \cong T^{l-1}_x} p(T^l_x) \Leftrightarrow \gamma = 1 \quad (15) \]

We have
\[ p(T^l_x) = \prod_{i \not\in T^l_x} \mathbb{P}(D_0 = d_i)(d_i - 1)! \frac{1}{|\text{Aut}(T^l_x)|} t_i. \]

Now rearranging the sum by the degrees of the leafs of \( T^{l-1}_x \) in \( T^l_x \) we have
\[
\sum_{T^l_x : B_{T^l_x}(x, l - 1) \cong T^{l-1}_x} p(T^l_x) = \sum_{D_{T^l_x(i)}(\mathbb{1}_{i \not\in T^{l-1}_x})} p(T^{l-1}_x \cup (d_i, i \in T^{l-1}_x)) = \prod_{i \not\in T^{l-1}_x \cup T^l_x} \mathbb{P}(D_0(j) = d_j)(d_j - 1)! \frac{1}{|\text{Aut}(T^l_x)|} t_i = \prod_{j \not\in T^{l-1}_x \cup T^l_x} \mathbb{P}(D_0(j) = d_j)(d_j - 1)! \frac{1}{|\text{Aut}(T^l_x \setminus T^l_x)|} t_{l-1} \gamma,
\]
where the last equation follows from the fact that for fixed \( d_i, \ i \in T_{l-1} \) every \( \sigma \in Aut(T_{l-1}^i) \) has only one extension in \( Aut(T_l)/\sim \). Now (15) will hold only if \( \gamma = 1 \). Now we have that (15) holds if and only if \( E(D_0) = 2 \ (\gamma = 1) \).

From Lemma 4 it follows that \( \mu \) is indeed the limit of the random tree sequence \( T_n \). This completes the proof of Theorem 2.

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