Compatibility complexes of overdetermined PDEs of finite type, with applications to the Killing equation

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Abstract

In linearized gravity, two linearized metrics are considered gauge-equivalent, \( h_{ab} \sim h_{ab} + K_{ab}[v] \), when they differ by the image of the Killing operator, \( K_{ab}[v] = \nabla_a v_b + \nabla_b v_a \). A universal (or complete) compatibility operator for \( K \) is a differential operator \( K_1 \) such that \( K_1 \circ K = 0 \) and any other operator annihilating \( K \) must factor through \( K_1 \). The components of \( K_1 \) can be interpreted as a complete (or generating) set of local gauge-invariant observables in linearized gravity. By appealing to known results in the formal theory of overdetermined PDEs and basic notions from homological algebra, we solve the problem of constructing the Killing compatibility operator \( K_1 \) on an arbitrary background geometry, as well as of extending it to a full compatibility complex \( K_i \) (\( i \geq 1 \)), meaning that for each \( K_i \) the operator \( K_{i+1} \) is its universal compatibility operator. Our solution is practical enough that we apply it explicitly in two examples, giving the first construction of full compatibility complexes for the Killing operator on these geometries. The first example consists of the cosmological FLRW spacetimes, in any dimension. The second consists of a generalization of the Schwarzschild–Tangherlini black hole spacetimes, also in any dimension. The generalization allows an arbitrary cosmological constant and the replacement of spherical symmetry by planar or pseudo-spherical symmetry.

Keywords: linearized gravity, gauge-invariant observables, FLRW, schwarzschild
1. Introduction

An important aspect of General Relativity is its invariance under diffeomorphisms, also called gauge transformations of this theory. Of course, this invariance survives linearization about some fixed background metric $g$ and the linearized diffeomorphisms (or linearized gauge transformations) change the linearized metric as $h_{ab} \mapsto h_{ab} + K_{ab} [v]$, where $K_{ab} [v] = \nabla_a v_b + \nabla_b v_a$ is the Killing operator with respect to the background metric $g$. Solutions of the Killing equation $K[v] = 0$ are Killing vectors $v_a$. Because two linearized metric configurations are considered physically equivalent if they differ only by a linearized gauge transformation, an inescapable part of the study of linearized gravity (linearized General Relativity) is the need to separate gauge and physical degrees of freedom; the latter essentially parametrize equivalence classes of linearized metrics under linearized gauge transformations.

A local gauge-invariant observable is a differential operator $O[h]$ such that $O[K[v]] = 0$ for an arbitrary argument $v_a$. Clearly, such differential operators have many potential applications in linearized gravity and, not surprisingly, their study has a long history [34]. While not all useful gauge-invariant observables $O[h]$ are local (where $O$ is local if it is a differential, rather than an integral, operator), the local ones are distinguished by the property that they preserve supports, $\text{supp} O[h] \subseteq \text{supp} h$, which helps to disentangle the gauge-invariant information contained in $h$ from infrared or asymptotic properties of $h$. Further discussion of these issues, with brief surveys of previous work, can be found [14], in the context of cosmological perturbations, and in [1, 21, 31], in the context of black hole perturbations.

In this work, we are interested in the problem of explicitly constructing complete (or generating) sets of local gauge-invariant observables on spacetime backgrounds of physical interest. Completeness refers to the ability to express any local gauge-invariant observable in terms of linear combinations of derivatives of a given set. For technical reasons [22, 23], it also becomes important to identify complete sets of differential relations between them, complete sets of differential relations between these differential relations, and so on. For instance, once a complete set of local gauge-invariant observables is known, the complete set of differential relations between them may allow us to reduce the number of independent invariants at the price of integrating some differential equations (if any sufficiently simple ones can be identified among the differential relations). This is the way in which Teukolsky–Starobinsky identities are used to relate the Teukolsky scalars to other invariants on the Kerr background [1, 39]. Also, the differential relations among invariants may play a role in the construction of a wave equation satisfied by the invariants, just as the Bianchi identities $\nabla_a F_{bc} = 0$ play a role in obtaining the wave equation for the Maxwell tensor $F_{bc} = \nabla [A_b]$ (whose components constitute a complete set of invariants in electrodynamics). For linearized gravity on maximally symmetric spaces, this idea was used in [23] to identify wave equations satisfied by the invariants, the differential relations between them, as well as all higher differential relations. Finally, knowledge of all the higher differential relations has interesting applications to the analysis of the symplectic and Poisson structures on the space of solutions of linearized gravity [22] [23, section 5]. Thus, phrased in mathematical terms, given a background metric $g$, we are interested in constructing a (full) compatibility complex for the corresponding Killing operator $K[v]$, where full refers to the continuation of the sequence of differential relations until it terminates (becomes identically zero), a property that is usually required implicitly.

An unfortunate aspect of the study of local gauge-invariant observables $O[h]$ is that their structure depends strongly on the background metric $g$, since the Killing operator $K[v]$, which determines the structure of gauge-equivalence classes, depends on $g$ in an essential way. Thus, in principle, this problem needs to be attacked anew for each background metric of interest. Unfortunately, a full solution (a complete set of gauge-invariants, relations between them, etc)
can be found in the literature only in very few cases, even if we restrict ourselves only to the
to the construction of complete sets of gauge-invariants (and not relations between them, etc). To our
knowledge, the full Killing compatibility complex is known only for flat (Minkowski) and con-
tant curvature (de Sitter or anti-de Sitter) spacetimes [23]. In principle, the methods of [15, 16]
could have been used to generate the compatibility complex on locally symmetric spacetimes
(those with a covariantly constant Riemann tensor), but to our knowledge they have never been
explicitly elaborated in the Lorentzian setting [7]. In addition, complete sets of local gauge-
invariant observables are known only for cosmological (inflationary FLRW) spacetimes in any
dimension, due to the recent construction in [9, 13, 14], and for the 4-dimensional Kerr black
hole, as recently highlighted in [1]. Full proofs of the results announced in [1] will appear in [2]
and will be based on the methods to be presented in this work.

The major obstacle to solving the problem that we have posed (the construction of a com-
patibility complex for the Killing operator) has so far been proving completeness (of a set of
gauge-invariants, of a set of relations between them, etc). In the flat and constant curvature
cases, the proof was basically due to Calabi [8, 23], and was specific to those geometries.
In the cosmological case, the proof is due to [13], but is somewhat ad-hoc and without clear
generalizations.

The main innovation in this work is the application of methods from the formal theory of
PDEs [18, 30, 32, 35] and homological algebra [38] to the problem of constructing Killing
compatibility complexes. In fact, a method for systematically constructing a complete com-
patibility operator for any overdetermined linear differential operator (under mild regularity
conditions) has been known for a long time [18] (it was this method that was applied in [15, 16]).
Unfortunately, it is rather cumbersome to apply directly. There do exist computer algebra
implementations of this method [6], but they suffer from the problem that the input and output
of this computer algebra construction must be matrices of scalar differential operators written
in some explicit coordinates, which often destroys any manifest symmetry or other structure
that the original linear differential operator had. This is certainly an undesirable feature when
dealing with the Killing operator on a spacetime with some symmetry, product or warped
product structure. However, there is a significant simplification of the general systematic
construction when we restrict our attention to differential operators of regular finite type, of
which the Killing operator is often an example. We will take full advantage of this simplifica-
tion, together with some basic notions from homological algebra, to give a practical sufficient
condition (lemma 4) for the completeness of a given set of local gauge-invariant observa-
tives in linearized gravity (or more generally, the completeness of a compatibility operator for
any operator). In practice, this criterion also leads to a way to construct (theorem 9) the full
Killing compatibility complex (or more generally, the compatibility complex for any operator
of regular finite type), which can preserve various structural properties of a given background
spacetime geometry.

In section 2, we introduce some ideas from homological algebra, applied to linear differ-
ential operators, and use it to show how to explicitly construct a compatibility complex for a
PDE of regular finite type (the appropriateness of our definition of finite type operator is dis-
cussed at length in appendix A). This technique is applied to the Killing equation in section 3,
with some examples. In particular, we treat in detail the examples of spacetimes of constant
curvature (section 3.1), cosmological FLRW spacetimes (section 3.2) and Schwarzschild–
Tangherlini black holes (section 3.3). In each case we make some remarks about the relation
of our results with the literature. appendix B gives a helpful reference for the notation used in
different subsections of section 3. In all examples, we keep the spacetime dimension $n$ general
(that is, we allow at least $n \geq 4$). The results of sections 3.2 and 3.3 are new. Finally, we con-
clude with a discussion of further work in section 4.
Whenever speaking of differential operators, we will specifically mean a linear differential operator with smooth coefficients acting on smooth functions. More precisely, we will consider differential operators that map between sections of vector bundles, say $V_1 \to M$ and $V_2 \to M$, on some fixed manifold $M$. $K: \Gamma(V_1) \to \Gamma(V_2)$. The source and target bundle of a differential operator, $V_1 \to M$ and $V_2 \to M$ respectively in the last example, will be considered as part of its definition and will most often be omitted from the notation. We will denote the composition of two differential operators $K$ and $L$ by $K \circ L$, or simply by $KL$, if no confusion is possible. A local section of a vector bundle $V \to M$ is a section of the restriction bundle $V|_U \to U$ for some open $U \subset M$. A local section $\nu$ that solves the differential equation $K[\nu] = 0$ on its domain of definition is a local solution.

2. Compatibility operators

We start by introducing some basic notions from homological algebra [38].

**Definition 1.** A (possibly infinite) composable sequence $K_l$ of linear maps, $l = l_{\text{min}}, \ldots, l_{\text{max}}$, such that $K_{l+1} \circ K_l = 0$ when possible, is called a (cochain) complex. Given complexes $K_l$ and $K'_l$ a sequence $C_l$ of linear maps, as in the diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_{l-1} & \longrightarrow & K_l & \longrightarrow & K_{l+1} & \longrightarrow & \cdots \\
\cdots \downarrow C_{l-1} & & \downarrow C_l & & \downarrow C_{l+1} & & \downarrow C_{l+2} & & \\
\cdots & \longrightarrow & K'_{l-1} & \longrightarrow & K'_l & \longrightarrow & K'_{l+1} & \longrightarrow & \cdots
\end{array}
$$

(1)

such that its squares commute, that is $K'_l \circ C_l = C_{l+1} \circ K_l$ when possible, is called a cochain map or a morphism between complexes. A homotopy between complexes $K_l$ and $K'_l$ (which could also be the same complex, $K_l = K'_l$) is a sequence of morphism, as the dashed arrows in the diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_{l-1} & \longrightarrow & K_l & \longrightarrow & K_{l+1} & \longrightarrow & \cdots \\
\cdots \downarrow H_{l-1} & & \downarrow H_l & & \downarrow H_{l+1} & & \downarrow H_{l+2} & & \\
\cdots & \longrightarrow & K'_{l-1} & \longrightarrow & K'_l & \longrightarrow & K'_{l+1} & \longrightarrow & \cdots
\end{array}
$$

(2)

and the sequence of maps $C_l = K'_{l+1} \circ H_{l-1} + H_l \circ K_l$ is said to be a morphism induced by the homotopy $H_l$. An equivalence up to homotopy between complexes $K_l$ and $K'_l$ is a pair of morphisms $C_l$ and $D_l$ between them, as in the diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_{l-1} & \longrightarrow & K_l & \longrightarrow & K_{l+1} & \longrightarrow & \cdots \\
\cdots \downarrow H_{l-1} & & \downarrow H_l & & \downarrow H_{l+1} & & \downarrow H_{l+2} & & \\
\cdots & \longrightarrow & K'_{l-1} & \longrightarrow & K'_l & \longrightarrow & K'_{l+1} & \longrightarrow & \cdots
\end{array}
$$

(3)

such that $C_l$ and $D_l$ are mutual inverses up to homotopy ($H_l$ and $H'_l$), that is

$$D_l \circ C_l = \text{id} - K_{l-1} \circ H_{l-1} - H_l \circ K_l,$$

(4)
\[ C_l \circ D_l = \text{id} - K_{l-1}' \circ H_{l-1}' - H_l' \circ K_l', \quad (5) \]

with the special end cases
\[
\begin{align*}
D_{l_{\min}} \circ C_{l_{\min}} &= \text{id} - \tilde{H}_{l_{\min}-1} - H_{l_{\min}} \circ K_{l_{\min}}, & K_{l_{\min}} \circ \tilde{H}_{l_{\min}-1} &= 0, \quad (6) \\
C_{l_{\min}} \circ D_{l_{\min}} &= \text{id} - \tilde{H}_{l_{\min}-1}' - H_{l_{\min}}' \circ K_{l_{\min}}', & K_{l_{\min}}' \circ \tilde{H}_{l_{\min}-1}' &= 0, \quad (7) \\
D_{l_{\max}+1} \circ C_{l_{\max}+1} &= \text{id} - H_{l_{\max}} \circ K_{l_{\max}} - \tilde{H}_{l_{\max}+1}, & \tilde{H}_{l_{\max}+1} \circ K_{l_{\max}} &= 0, \quad (8) \\
C_{l_{\max}+1} \circ D_{l_{\max}+1} &= \text{id} - H_{l_{\max}}' \circ K_{l_{\max}}' - \tilde{H}_{l_{\max}+1}', & \tilde{H}_{l_{\max}+1}' \circ K_{l_{\max}}' &= 0, \quad (9) \\
\end{align*}
\]

where the \( \tilde{H} \) maps are allowed to be arbitrary, as long as they satisfy the given identities.

Note that our definition of equivalence up to homotopy between complexes of finite length is set up in a way that allows an equivalence between longer complexes to be truncated and still remain an equivalence.

Next, we restrict our attention to the case where all maps are given by differential operators.

**Definition 2** (See [30, definition 10.5.4], [35, definition 1.2.2]). Given a differential operator \( K \), any composable differential operator \( L \) such that \( L \circ K = 0 \) is a compatibility operator for \( K \). If \( K_1 \) is a compatibility operator for \( K \), it is called complete or universal when any other compatibility operator \( L \) can be factored through \( L = L' \circ K_1 \) for some differential operator \( L' \). A complex of differential operators \( K_0, l = 0, 1, \ldots \) is called a compatibility complex for \( K \) when \( K_0 = K \) and, for each \( l \geq 1 \), \( K_l \) is a complete compatibility operator for \( K_{l-1} \).

**Definition 3.** Given a (possibly infinite) complex of differential operators \( K_l, l = l_{\min}, l_{\min} + 1, \ldots, l_{\max} \), we say that it is locally exact at a point \( x \) when, for every \( l_{\min} < l < l_{\max} \), for every smooth function \( f_l \) defined on an open neighborhood \( U \ni x \) such that \( K_l[f_l] = 0 \), there exists a smooth function \( g_{l-1} \) defined on a possibly smaller open neighborhood \( V \ni x \) such that \( f_l = K_{l-1}[g_{l-1}] \). Locally exact (without specifying a point \( x \)) means locally exact at every \( x \).

Note that a complete compatibility operator, say \( K_1 \), need not be unique. But, by its universal factorization property, any two compatibility operators, say \( K_1 \) and \( K_1' \), must factor through each other, \( K_1 = L_1 \circ K_1' \) and \( K_1' = L_1' \circ K_1 \) for some differential operators \( L_1 \) and \( L_1' \).

Given two composable operators, \( K \) and \( K_1 \), the compatibility condition \( K_1 \circ K = 0 \) is very easy to check. On the other hand, it may be quite challenging to check completeness/universality. One way to do it is to compare \( K \) and \( K_1 \) with another pair of operators which are already known to satisfy the universality condition.

**Lemma 4.** Consider two complexes of differential operators \( K_l \) and \( K_l' \), for \( l = 0, 1 \). If these complexes are equivalent up to homotopy, as in the diagram

\[
\begin{array}{c}
\bullet \quad \bullet \\
\downarrow \quad \downarrow \\
K_0 \quad K_1 \\
\downarrow \quad \downarrow \\
\bullet \quad \bullet \\
D_0 \quad D_1 \\
\downarrow \quad \downarrow \\
H_0 \quad H_1 \\
\downarrow \quad \downarrow \\
C_0 \quad C_1 \\
\downarrow \quad \downarrow \\
K_0' \quad K_1' \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \bullet \\
\downarrow \quad \downarrow \\
K_0 \quad K_1 \\
\downarrow \quad \downarrow \\
\bullet \quad \bullet \\
D_0 \quad D_1 \\
\downarrow \quad \downarrow \\
H_0 \quad H_1 \\
\downarrow \quad \downarrow \\
C_0 \quad C_1 \\
\downarrow \quad \downarrow \\
K_0' \quad K_1' \\
\end{array}
\]
where we really only require all squares to be commutative and the identities $D_1 \circ C_1 = \text{id} - K_0 \circ H_0 - H_1 \circ K_1$ and $C_1 \circ D_1 = \text{id} - K_0' \circ H_0' - H_1' \circ K_1'$ to hold, then $K_1$ is universal iff $K_1'$ is universal.

Furthermore, the complex $K_l$, $l = 0, 1$, is locally exact iff the complex $K'_l$, $l = 0, 1$, is locally exact.

**Proof.** Without loss of generality, assume that $K'_1$ is universal. Let $L \circ K_0 = 0$. Then $(L \circ D_1) \circ K_0' = L \circ K_0 \circ D_0 = 0$. By universality of $K'_1$, there exists a differential operator $L'$ such that $L \circ D_1 = L' \circ K'_1$. Recall that from our hypotheses that $D_1 \circ C_1 = \text{id} - K_0 \circ H_0 - H_1 \circ K_1$. But then

$$
L = L \circ (D_1 \circ C_1 + K_0 \circ H_0 + H_1 \circ K_1)
= L' \circ (K'_1 \circ C_1) + (L \circ H_1) \circ K_1
= (L' \circ C_2) \circ K_1 + (L \circ H_1) \circ K_1 = L'' \circ K_1,
$$

where $L'' = L' \circ C_2 + L \circ H_1$. This demonstrates the universality of $K_1$.

Next, without loss of generality, assume that $K'_0$ is locally exact. Pick a point $x$, an open neighborhood $U \ni x$, and a smooth function $f$ such that $K_1[f] = 0$. Then $K'_1[C_1[f]] = C_2[K_1[f]] = 0$. Hence, by local exactness, there exists a smooth $g'$ defined on a possibly smaller open neighborhood $V \ni x$ such that $K'_0[g'] = C_1[f]$. Setting $g = D_0[g'] + H_0[f]$ on $V \ni x$, direct calculation shows that

$$
K_0[g] = K_0[D_0[g']] + K_0[H_0[f]] = D_1[K'_0[g']] + K_0[H_0[f]]
= D_1 \circ C_1[f] + K_0[H_0[f]] = f - H_1[K_1[f]],
$$

which shows that the $K_1$ complex is also locally exact. 

Next, we will show how to construct a universal compatibility operator for a differential operator $K$ if it is equivalent, in the sense of a complex consisting of one operator, to some operator with a known universal compatibility operator. This construction is not unknown in homological algebra, but we include a proof for completeness.

**Lemma 5 (see [35, proposition 1.2.7]).** Consider differential operators $K_0$ and $K'_0$. Suppose that $K_0$ and $K'_0$ are equivalent up to homotopy, in the sense of the diagram

\[
\begin{array}{ccc}
\tilde{H} & \xrightarrow{K_0} & H_0 \\
C_0 \downarrow & & \downarrow C_1 \\
\tilde{H}' & \xrightarrow{K'_0} & H'_0
\end{array}
\]

where we require all squares to be commutative and the identities $D_0 \circ C_0 = \text{id} - \tilde{H} - H_0 \circ K_0$, $C_0 \circ D_0 = \text{id} - \tilde{H}' - H'_0 \circ K'_0$ to hold, with $K_0 \circ \tilde{H} = 0$ and $K'_0 \circ \tilde{H}' = 0$. Then, if a universal compatibility operator $K'_1$ for $K'_0$ is known, we can complete the above diagram to the following equivalence up to homotopy
with some differential operators $H'_1, D'_2$.

**Proof.** From our hypotheses, $C_0$ and $D_0$ are mutual inverses, up to a homotopy correction. Our first observation is that the same property then holds for $C_1$ and $D_1$. Namely,

\[
(id - K_0 \circ H_0 - D_1 \circ C_1) \circ K_0 = K_0 - K_0 \circ H_0 \circ K_0 - K_0 \circ (D_0 \circ C_0)
= K_0 \circ (id - H_0 \circ K_0 - D_0 \circ C_0)
= K_0 \circ \hat{H} = 0,
\]

(14)

\[
(id - K'_0 \circ H'_0 - C_1 \circ D_1) \circ K'_0 = 0,
\]

(15)

where the second identity is completely analogous to the first one. Then we also have

\[
(id - K_0 \circ H_0 - D_1 \circ C_1) \circ D_1 \circ K'_0 = (id - K_0 \circ H_0 - D_1 \circ C_1) \circ K_0 \circ D_0 = 0.
\]

(16)

Since we know that $K'_1$ is a universal compatibility operator for $K'_0$, there must exist differential operators $H'_1$ and $D'_2$ such that

\[
(id - K'_0 \circ H'_0 - C_1 \circ D_1) \circ K'_1 = H'_1 \circ K'_1,
\]

(17)

\[
(id - K'_0 \circ H'_0 - C_1 \circ D_1) \circ D_2 \circ K'_1 = D'_2 \circ K'_1.
\]

(18)

Next, defining the operators $K_1, H_1, C_2$ and $D_2$ as in the diagram (13), the remaining identities needed to show that this diagram is a homotopy equivalence are

\[
D_1 \circ C_1 \equiv id - K_0 \circ H_0 - H_1 \circ K_1,
\]

(19a)

\[
C_2 \circ K_1 = K'_1 \circ C_1,
\]

(19b)

\[
D_2 \circ K'_1 = K_1 \circ D_1,
\]

(19c)

\[
(id - K_1 \circ H_1 - D_2 \circ C_2) \circ K_1 = 0,
\]

(19d)

\[
(id - K'_1 \circ H'_1 - C_2 \circ D_2) \circ K'_1 = 0.
\]

(19e)

The last two, (d) and (e), follow from the same argument as in the first paragraph of this proof. The first two, (a) and (b), follow from direct calculation and the identities that we have already established earlier in the proof. To get (c), it remains to check the following identity
\[(K'_1 \circ C_1) \circ D_1 = K'_1 \circ (\text{id} - K'_0 \circ H'_0 - H'_1 \circ K'_1) = (\text{id} - K'_1 \circ H'_1) \circ K'_1. \tag{20}\]

This completes the proof. \hfill \Box

**Definition 6.** A (linear) connection \(\mathbb{D}\) on a vector bundle \(V \to M\) is a linear differential operator \(\mathbb{D} : \Gamma(V) \to \Gamma(T^*M \otimes_M V)\) that in local coordinates \((x^a)\) has the form \(\mathbb{D}_a \nu(x) = \frac{\partial}{\partial x^a} \nu(x) + \gamma_a(x) \nu(x)\), with the \(\gamma_a(x)\) being smooth local sections of the endomorphism bundle of \(V \to M\). The connection is flat when locally its components commute, \([\mathbb{D}_a, \mathbb{D}_b] = 0\). The equation \(\mathbb{D}f = 0\) is called the \((\mathbb{D})\)-flat section equation.

A flat connection \(\mathbb{D}\) gives rise to a complex of differential operators \(d^\mathbb{D} : \Gamma(\Lambda^l T^*M \otimes_M V) \to \Gamma(\Lambda^{l+1} T^*M \otimes_M V), \ l = 0, 1, \ldots, n, \ldots,\) with \(d^\mathbb{D} = \mathbb{D}\), locally \((d^\mathbb{D} w)_{a_1 \ldots a_{l+1}} = (l + 1)\mathbb{D}[a_1 w_{a_2 \ldots a_{l+1}}]\) for \(0 < l < n\), and \(d^\mathbb{D}_l = 0\) for \(l \geq n\), the \((\mathbb{D})\)-twisted de Rham complex.

**Definition 7.** A differential operator \(K\) defines a PDE of regular finite type \(K[f] = 0\), when \(K_0 = K\) is equivalent to a flat connection \(K'_0 = \mathbb{D}\), in the sense of one operator complexes (definition 1), with the extra requirement that \(H'_1 = 0\) and \(H'_n = 0\).

**Remark 1.** One can find in the literature different definitions for PDEs of finite type. Our definition 7, with the extra regular modifier, is most convenient for the purposes of this work and is well-known to be equivalent to other common definitions when the regularity conditions are precisely specified. The relation between the different definitions is discussed in the appendix and an elementary proof of the equivalence is given in proposition A.2.

Once we know that we are faced with a PDE of regular finite type, we can exploit its equivalence to the equation for \(\mathbb{D}\)-flat sections, together with our preceding results and the following well known proposition.

**Proposition 8.** Given a flat connection \(\mathbb{D}\), the corresponding twisted de Rham complex \(d^\mathbb{D}_l\), \(l = 0, 1, \ldots, n, \ldots\) is locally exact and is also a compatibility complex for \(\mathbb{D} = d^\mathbb{D}_0\).

**Proof.** A well-known property of flat connections is that they are locally equivalent to \(\text{rk}_M V\) copies of exterior derivative operator (we can set the \(\gamma_a(x)\) matrices to zero by locally choosing the fiber coordinates on \(V \to M\) adapted to the foliating by flat sections). Thus, locally, the twisted de Rham complex turns into \(\text{rk}_M V\) copies of the ordinary de Rham complex, which is known to satisfy all the desired properties (it suffices to combine the Poincaré lemma [35, proposition 1.2.41] with [35, propositions 1.2.13, 1.2.39]). \hfill \Box

**Theorem 9.** Let \(K[f] = 0\) be PDE of regular finite type. Then, starting from an equivalence of \(K\) with a flat connection \(\mathbb{D}\), we can explicitly construct a locally exact compatibility complex \(K_l\) for \(K, l = 0, 1, \ldots\).

**Proof.** The proof is by induction. Setting \(K_0 = K\) and \(K'_0 = \mathbb{D} = d^\mathbb{D}_0\), the regular finite type hypothesis implies that we can satisfy the hypotheses of lemma 5 and extend the equivalence of \(K_0\) and \(K'_0\) to an equivalence of \(K_l\) and \(K'_l, l = 0, 1, \ldots, K_l\) explicitly constructed. Suppose, inductively, that we have an equivalence up to homotopy between the complexes \(K_l\) and \(K'_l, l = 0, 1, \ldots, m\), for some \(m > 0\). Iterating the previous argument, we can extend it to an equivalence up to homotopy between the complexes \(K_l\) and \(K'_l, l = 0, 1, \ldots, m + 1,\)
where $K_{m+1}$ is an explicitly constructed as in lemma 5.

Since the above construction of the complex $K_{l}, l = 0, 1, \ldots$, comes with an equivalence up to homotopy with the twisted de Rham complex $\mathcal{K}_{j}^{l} = d_{j}^{l}, l = 0, 1, \ldots$, lemma 4 and proposition 8 allow us to conclude that $K_{l}$ is both locally exact and is a compatibility complex for $K$. That is, $K_{1}$ is a universal compatibility operator for $K_{0} = K$ and $K_{l+1}$ is a universal compatibility operator for $K_{l}$, for $l > 1$.

Note that, even though the twisted de Rham complex $d_{j}^{l}$ terminates after $l = n$, or rather becomes trivial $d_{j}^{l} = 0$ for $l \geq n$, the result of theorem 9 is not guaranteed to produce a complex that eventually terminates in the same way. For instance, the simple example $K = \text{id}$, produces the complex $K_{l}, l = 0, 1, \ldots$, where $K_{2k} = \text{id}$ and $K_{2k+1} = 0$, with the source and target of every operator $K_{l}$ being the same as for $K$. Of course, in this simple example, we can force this complex to terminate by setting the target of $K_{l}$ to be zero dimensional, and setting $K_{l} = 0$ as a map between zero dimensional spaces, for each $l > 1$.

Since our goal is not a fully algorithmic construction of compatibility complexes, but rather one where human intervention is allowed along the way, we can apply similar simplifications at each step of the iterative construction given in the proof of theorem 9. At each step, before proceeding to the next one, having obtained the operator $K_{l+1}$, we can replace it by a potentially simpler operator $\tilde{K}_{l+1}$ without breaking its universality property. Here, a trivial, but helpful, observation is that an operator $\tilde{K}_{l+1}$ such that $K_{l+1} = \tilde{K}_{l+1} \circ K_{l+1}$ is a universal compatibility operator for $K_{l}$ whenever $K_{l+1}$ is. This way, on general principles, we expect to be able to produce a compatibility complex $K_{l}, l = 0, 1, \ldots$, that becomes trivial $K_{l} = 0$ for $l > n$.

We will not try to give a rigorous proof of the fact that we can always produce a compatibility complex $K_{l}$ that trivializes to $K_{l} = 0$ for $l > n$. Instead, in the next section, we will present examples of PDEs of regular finite type with compatibility complexes of finite length. In each case, we will give an explicit equivalence up to homotopy of a given complex to a twisted de Rham complex, which together with lemma 4 and proposition 8 serves as a witness to the fact that it is a compatibility complex. However, the reader should understand that this compatibility complex was produced by the construction given in the proof of theorem 9, with intermediate simplifications as described above.

### 3. Killing equation

Consider an $n$-dimensional (pseudo-)Riemannian manifold $(M, g)$, with Levi-Civita connection $\nabla$. In Lorentzian signature, we refer to $(M, g)$ as a spacetime. The Killing equation is an equation on sections $v \in \Gamma(TM)$, namely

$$K_{ab}[v] = \nabla_{a}v_{b} + \nabla_{b}v_{a} = 0.$$  \hspace{1cm} (21)

The Lie derivative identity $K[v] = L_{v}g$ implies that solutions of the Killing equations are infinitesimal isometries of $(M, g)$. In the context of linearized gravity (that is, the theory of linearized Einstein equations), metric perturbations $h \in \Gamma(S^{2}T^{*}M)$ are grouped into gauge equivalence classes, $h \sim h + K[v]$ for $v \in \Gamma(TM)$. A differential operator $L[h]$ such that $L \circ K = 0$, a compatibility operator for $K$ in the terminology of section 2, is interpreted as a (local) gauge-invariant observable or gauge-invariant field combination [34]. The components of a complete compatibility operator $K_{1}$ for the Killing operator $K$ can be interpreted, by the universality property, as a generating set for all gauge-invariants, also known as a complete set of gauge-invariant observables.
It is well known that the Killing equation is of regular finite type (definition 7), provided a precise regularity condition holds. The quickest way to see that is to put it into the so-called tractor form [11] or the form of the Killing transport equation [17, appendix B]. Namely, we have the equivalence up to homotopy

\[
\begin{align*}
T_{a_1}^a & \quad \begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} \\
\quad \begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} h_{a_1a} \\ w_{[bc];a_1} \end{bmatrix} \\
\quad \begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} 0 \\ -v_{[bc]} \end{bmatrix}
\end{align*}
\]
(22)

where the connection operator\(^1\) is

\[
T_{a_1}^a \begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} = \begin{bmatrix} \nabla_{a_1} v_a - w_{[a_1a]} \\ \nabla_{a_1} w_{[bc]} + R_{a_1d b c} v^d \end{bmatrix},
\]
(23)

which uses the Riemann tensor\(^2\)

\[
R_{abcd} v^d = 2 \nabla_{[a} v_{b]} - \frac{1}{2} K_{bc} [v].
\]
(24)

Then, the antisymmetry \(w_{cb} = -w_{bc}\) and the algebraic Bianchi identity \(R_{ab[cd]} = 0\) allow us to write

\[
2(\nabla_{a} w_{bc} + R_{abdc} v^d) = (2\nabla_{[a} w_{bc]} - R_{abcd} v^d) - (2\nabla_{[c} w_{ab]} - R_{bcda} v^d) + (2\nabla_{[e} w_{a]} - R_{eabc} v^d)
\]
\[
= 2\nabla_{[a} K_{bc]} [v].
\]
(25)

In general, the connection \(\alpha\) is not a flat. In fact, it is flat iff \((M, g)\) is of constant curvature, \(R_{abcd} = \alpha(g_{ac} g_{bd} - g_{ad} g_{bc})\) for some constant \(\alpha\). In general, the dimension of the space of local solutions of \(T_{a_1}^a \begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} = 0\) need not be constant over \(M\). However, when it is (this is the needed regularity condition), the local solutions span a vector bundle and the restriction of \(T\) to this sub-bundle is flat and the corresponding flat section equation is equivalent to the original Killing equation (lemma A.3), hence implying its regular finite type. Thus, the required regularity condition is that the local solutions of the Killing equation, in tractor form, span

\(^{1}\) The form of this connection was already derived in [17, equation (B.2)], though there it has a typo. The sign of \(R_{abdc} v^d\) is opposite compared to ours.

\(^{2}\) We follow the curvature conventions of [37].
a sub-bundle. Or, equivalently, the pointwise dimension of the span of these local solutions is constant. However, in special cases, this particular way of reducing the Killing operator to a flat connection may not be the preferred one, and a different reduction might be more convenient.

Consider a tensor $T[g]$ built covariantly out of the metric $g$, the Riemann tensor $R$, and the covariant derivatives $\nabla R$, $\nabla^2 R$, $\ldots$. Define its linearization $\dot{T}$ about $g$ by the identity $T[g + \varepsilon h] = \dot{T}[g] + \varepsilon T[h] + O(\varepsilon^2)$. Recall the standard identity between the Lie derivative, $T$ and $\dot{T}$:

$$\mathcal{L}_v T[g] = \dot{T}[\mathcal{L}_v [g]] = \dot{T}[K[v]],$$

(26) which guarantees that $\dot{T} \circ K = 0$ for the linearization $g \mapsto g + \varepsilon h$ whenever $T[g] = 0$ or some expression involving only constants and Kronecker $\delta$'s. This result is sometimes known as the Stewart–Walker lemma [34, lemma 2.2]. Alternatively, when $T[g] \neq 0$, this identity can be used to extract some components of $v_a$ or $\nabla_a v_b$ by applying $\dot{T}$ to $K[v]$.

Sections 3.2 and 3.3 below will involve some algebraic constructions with tensors for which it is convenient to introduce the following notation. For $A_a$ and $B_{bd}$ symmetric tensors, we denote the Kulkarni–Nomizu product by

$$(A \circ B)_{abcd} = A_{ac}B_{bd} - A_{bc}B_{ad} - A_{ad}B_{bc} + A_{bd}B_{ac}. \quad (28)$$

Clearly $A \circ B = B \circ A$ and the result has the symmetry type of the Riemann tensor. For tensors with two or four indices, we define the contractions

$$(A \cdot B)_{ab} = A^a_B, \quad \text{and} \quad (R \cdot S)_{abcd} = R_{ab}^\delta S_{\delta f c d}. \quad (29)$$

With these definitions, when $A, B, C$ and $D$ are symmetric, we have the useful identities

$$[(A \circ B) \cdot (C \circ D)]_{abcd} = 2[(A \cdot C) \circ (B \cdot D) + (A \cdot D) \circ (B \cdot C)]_{abcd}, \quad (30)$$

$$(A \circ B)_{ab}^b_d = [A \cdot B - (\text{tr}A)B - A(\text{tr}B) + B \cdot A]_{ad}. \quad (31)$$

### 3.1. Constant curvature spacetime

An $n$-dimensional constant curvature spacetime $(M, g)$ of sectional curvature $\alpha$ is defined by a Riemann curvature tensor of the form $R_{abcd} = \alpha (g_{ac}g_{bd} - g_{ad}g_{bc})$, with $\alpha$ a constant. It is well-known that in this case the Killing transport (or tractor) connection defined in (23) is actually flat. Thus, we could use the methods of section 2, in particular theorem 9, to construct a compatibility complex for the Killing operator $K$ on $(M, g)$. However, in this particular situation, the compatibility complex for $K$ is already known independently. It is sometimes called the Calabi complex [23]. We will denote it by $C_i$, $i = 0, 1, \ldots$, with $C_0 = K$ and $C_i = 0$ for $l \geq n$. The operator $C_1$ is essentially the linearized Riemann tensor $R_{ab}^\delta \epsilon [g]$, while $C_2$ is the linearized differential Bianchi identity. The remaining operators $C_i$, for $i > 2$, are essentially higher rank Bianchi identities. These operators have the following explicit formulas (see [23, section 2.2]):

$$C_0[v]_{ab} = \nabla_a v_b + \nabla_b v_a, \quad (32a)$$

$$C_1[h]_{abcd} = (\nabla \nabla \circ h)_{abcd} + \alpha (g \circ h)_{abcd}, \quad (32b)$$
\[ C_2[r]_{abcde} = 3\nabla_{[abcde]}, \quad \text{(32c)} \]
\[ C_3[b]_{abcd|ef} = 4\nabla_{[ab|bcd|ef]}, \quad \text{(32d)} \]
\[ \vdots \quad \text{(32e)} \]
\[ C_i[b]_{a_i\ldots a_i|bc} = (i + 1)\nabla_{[a_i|b]a_i\ldots a_i|bc} \quad (i \geq 2). \quad \text{(32f)} \]

The : notation only serves to visually separate groups of indices that are independently anti-symmetric. \( C_0 \) has the symmetric index pair \( a:b \), while \( C_1 \) has the index group \( abcd \) satisfying the algebraic symmetries of the Riemann tensor. More generally, the tensor symmetry type of the target of each \( C_i \) operator is best described using Young symmetrizers (see [23, section 2.1] for complete details). Ignoring corresponding algebraic symmetry conditions on the tensors entering into the Calabi complex may violate its property of being a compatibility complex. Below we list the Young symmetry types and ranks of the tensor bundles serving as domains and codomains for the operators of the Calabi complex:

| Young type | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) | \( n \geq 2 \) |
|------------|------------|------------|------------|------------|
| \( C_0 \)  | (1) 2      | 3          | 4          | \( n \)    |
| \( C_1 \)  | (2) 3      | 6          | 10         | \( \frac{n(n+1)}{2} \) |
| \( C_2 \)  | (2,2) 1    | 6          | 20         | \( \frac{n}{3} \left( \frac{n+1}{3} \right) \) |
| \( C_i \)  | (2,2,1^{i-2}) 3 | 20 | \( \frac{n-1}{2} \left( \frac{n+1}{i+1} \right) \) | \( \vdots \) |
| \( C_{n-1} \) | (2,2,1^{n-2}) 1      | 3          | 6          | \( \frac{n(n-1)}{2} \) |

In the diagram (22) for the equivalence up to homotopy between \( K \) and \( T \), the top and bottom lines can be extended to their compatibility complexes, the Calabi \( C_i \) complex for \( K \) and the twisted de Rham complex \( d_i^T \) (definition 6) for \( T \). Then, using the same argument as at the beginning of the proof of lemma 5, the vertical equivalence maps can be propagated to the rest of the complexes, thus giving a full equivalence up to homotopy between them.

\begin{align*}
C_0 = K & \quad \leftrightarrow \quad C_1 & \quad \cdots & \quad \leftrightarrow \quad C_{n-1} & \quad \leftrightarrow \quad 0 \quad \leftrightarrow \quad \cdots \\
\cdots & \quad \leftrightarrow \quad d_0^T = T & \quad \leftrightarrow \quad d_1^T \quad \cdots & \quad \leftrightarrow \quad d_{n-1}^T \quad \leftrightarrow \quad 0 \quad \leftrightarrow \quad \cdots
\end{align*}

\[ \text{(33)} \]
We will not discuss the explicit formulas for the vertical equivalence differential operators. For our purposes it is sufficient to know that they exist. However, if these operators were to be given explicitly, then according to lemma 4 the equivalence diagram (33) would constitute an independent proof of the fact that the operators $C_i$ constitute a compatibility complex for the Killing operator $K$ on the constant curvature spacetime $(M, g)$.

### 3.2. FLRW spacetimes

Consider an FLRW spacetime $(M, g)$, where $M = I \times F$, with $I \subset \mathbb{R}$ an open interval with coordinate $t$ and $\dim F = m$, and $g = -dt^2 + f^2 g^F$, where the scale factor $f = f(t)$ is a positive scalar function and $g^F_{ab} = (\pi^*g^F)_{ab}$ is the pullback of a constant curvature Riemannian metric (with sectional curvature $\alpha$) on $F$ along the standard projection $\pi: I \times F \to F$. Let us denote $U_a = -(\partial t)_a$, and note that $U_a U^a = -1$ and $f^2 g^F_{ab} = g_{ab} + U_a U_b$.

Below, it will be convenient to extensively rely on the product structure $M = I \times F$ and naturally decompose all tensors on $M$ with respect to it. For instance, $v_a^I = v_a^I U^b + v_b^I U^b + f^2 \tilde{v}_a^b + \tilde{v}_b^a$, where $v_a^I$ is a scalar and $\tilde{v}_a^I$, $\tilde{v}_b^a$ and $\tilde{v}_b^a$ are respectively sections of the pullback bundles $\pi^*T^IF$, $\pi^*T^F$ and $\pi^*(T \otimes T^F)$ over $M$, or equivalently simply sections $TM$, $T^*M$ and $(T \otimes T^F)M$ that are annihilated by $g$-contraction with $U_a$ on any index. It will also be convenient to insert extra factors of $f$ into some tensor decompositions of this type, with the sole purpose of simplifying some forthcoming formulas (which we have unfortunately managed to do only in an ad-hoc way). If $X_a \in \Gamma(T^*F)$ then we will denote its pullback by $\tilde{X}_a = \pi^*X_a \in \Gamma(T^*M)$ and note that it satisfies $U^a \tilde{X}_a = 0$, where the metric $g$ is used for contractions. On the other hand, if no such $X_a$ was introduced previously (which will be true in most cases), we used $\tilde{X}_a \in \Gamma(T^*M)$ to denote any section that satisfies $U^a \tilde{X}_a = 0$, even if it has non-trivial dependence on $t$. This should not generate any confusion for the reader. The same convention is extended to all purely covariant tensors.

Each of the factors has an auxiliary pseudo-Riemannian structure, $(J, -dt^2)$ and $(F, g^F)$, with corresponding Levi-Civita connections, $\nabla^I$ and $\nabla^F$, which we can extend to all covariant tensor fields on the product manifold $M = I \times F$. Let us denote the extensions of $\nabla^I$ and $\nabla^F$ respectively by $-U_a \partial_t$ and $\nabla$, which incidentally defines the convenient differential operator $\partial_t = U^a (-U_a \partial_t)$. The defining properties of these operators are the usual Leibniz rule, together with $\partial_t t = 1$, $\partial_t U_a = 0$, and $\partial_t \tilde{X}_a = \partial_t (\pi^*X_a) = 0$ for any covariant tensor field $X$ on $F$, and also $\nabla t = 0$, $\nabla U_a = 0$ and $\nabla \tilde{X}_a = \nabla^F \tilde{X}_a = \pi^* (\nabla^F X_a)$ for any covariant tensor field $X$ on $F$. Since any covariant tensor on $M$ is locally a limit of sums of products of covariant tensors pulled back from $I$ and $M$, these properties are sufficient to uniquely define $\partial_t$ and $\nabla$ as linear differential operators. Note that we will also frequently use the abbreviation $(-)^{'} = \partial_t (-)$.

By the above remarks, any covector field $v_a$ on $M$ can be parametrized as

$$v_a = A f U_b + f^2 \tilde{X}_a$$

(34)

where $A$ is a scalar function on $M$ and $\tilde{X} \in \Gamma(\pi^* T^* F) \subset \Gamma(T^*M)$, so that $U^a \tilde{X}_a = 0$. Now note that with our conventions, for any scalar $A$, its exterior derivative is given by $(dA)_a = -A' U_a + \nabla_a A$, while the Levi-Civita connection on $(M, g)$ is given in the above parametrization by

$$\nabla_a (Af U_b + f^2 \tilde{X}_b) = (dA)_a U_b + Af' g_{ab} - 2f' f U_a \tilde{X}_b - f^2 U_a \tilde{X}'_b + f^2 \nabla_a \tilde{X}_b.$$  

(35)

Parametrizing symmetric 2-tensors $h_{ab}$ on $M$ as

$$h_{ab} = p U_a U_b - 2f^2 U_{(a} \tilde{Y}_{b)} + f^2 \tilde{Z}_{ab},$$  

(36)
the Killing operator on covectors \( h_{ab} = K_{ab}[v] = 2\nabla_{(a} v_{b)} \) becomes
\[
\begin{bmatrix}
\frac{p}{\tilde{Y}} \\
\frac{\dot{Z}}{\tilde{Z}}
\end{bmatrix} = K \begin{bmatrix}
A \\
\tilde{X}
\end{bmatrix} = \begin{bmatrix}
-2(Af)' \\
K[X] + 2Af'gF
\end{bmatrix} = \begin{bmatrix}
-2\partial_t f \\
-\frac{\tilde{X}}{K}
\end{bmatrix} \begin{bmatrix}
A \\
\tilde{X}
\end{bmatrix}
\]
(37)

in block matrix operator notation, where \( \tilde{K}_{bc}[\tilde{X}] = 2\tilde{\nabla}_{(b} \tilde{X}_{c)} \), which is simply our extension of the Killing operator on covectors from \((F, gF)\) to \((M, g)\).

**Remark 2.** Each entry in our block operator matrices is a linear differential operator between some covariant tensor bundles over \( M \). We use vertical and horizontal lines to further partition block operator matrices with respect to some special direct sum decomposition of their domain or codomain. We will use \( \text{id} \) to denote the identity endomorphism on any vector bundle and \( 0 \) to denote the zero morphism between any two vector bundles. In each case, the domains and codomains of these operators can be deduced from the context.

It is worth noting that setting the \( A \) component of \( v_a \) to zero simplifies the Killing operator to
\[
K \begin{bmatrix}
0 \\
\text{id}
\end{bmatrix} = \begin{bmatrix}
0 \\
\partial_t \tilde{K}
\end{bmatrix}
\]
(38)

For a generic FLRW spacetime (see [9, definition 2.1] for a breakdown of FLRW geometries into special and generic classes, based on the properties of the scale factor \( f \)), it is well-known that the only Killing vectors are those that reduce to the Killing vectors of the spatial slices \((F, gF)\), appropriately propagated in time. We will see shortly that, equivalently, each Killing vector on \((M, g)\) has the form \( v_a = 0 + f^2 \tilde{X}_a \), where \( \tilde{K}_{ab} = 0 \) and \( \partial_t \tilde{X}_a = 0 \). Now, since the spatial slices \((F, gF)\) are of constant curvature, the spatial Killing operator \( \tilde{K} \) is of the type discussed in section 3.1. This means that \( \tilde{K} \) is equivalent up to homotopy to the flat spatial Killing transport connection \( \tilde{T} \), so that the following two operators are also equivalent up to homotopy:
\[
\begin{bmatrix}
\partial_t \\
\tilde{K}
\end{bmatrix} \quad \text{and} \quad \tilde{T} = \begin{bmatrix}
\partial_t \\
\tilde{T}
\end{bmatrix}
\]
(39)

Since both \([\partial_t, \tilde{K}] = 0 \) and \([\partial_t, \tilde{T}] = 0 \), it is easy to see that \( \tilde{T} \) itself defines a flat connection. Let the operators \( \tilde{C}_j \) be the extensions of the Calabi complex (32) from the constant curvature geometry \((F, gF)\) to \( M \), where we have simply replaced \( \nabla^F \) by \( \tilde{\nabla} \) and \( gF \) by \( \tilde{g}F \) whenever necessary. Since such an extension preserves all operator identities, including the suitably extended equivalence up to homotopy in (33). Now, it is a straightforward exercise to check that the twisted de Rham complex \( d^T \) can be represented as the bottom line in the following diagram, and also to use the mentioned identities to construct the corresponding the vertical differential operators that complete this diagram to an equivalence up to homotopy:
\[
\begin{array}{ccc}
\begin{bmatrix}
\partial_t \\
\tilde{C}_1
\end{bmatrix} & \begin{bmatrix}
-\tilde{C}_0 \\
0
\end{bmatrix} & \begin{bmatrix}
\partial_t \\
\tilde{C}_1
\end{bmatrix} \\
\begin{bmatrix}
\partial_t \\
\tilde{C}_2
\end{bmatrix} & \begin{bmatrix}
-\tilde{C}_1 \\
0
\end{bmatrix} & \begin{bmatrix}
\partial_t \\
\tilde{C}_2
\end{bmatrix} \\
\vdots & \vdots & \vdots
\end{array}
\]
\[
\begin{array}{ccc}
0 & \begin{bmatrix}
\partial_t \\
\tilde{C}_{m-1}
\end{bmatrix} & \begin{bmatrix}
\partial_t \\
\tilde{C}_m
\end{bmatrix} \\
0 & \begin{bmatrix}
\partial_t \\
\tilde{C}_{m-1}
\end{bmatrix} & \begin{bmatrix}
\partial_t \\
\tilde{C}_m
\end{bmatrix}
\end{array}
\]
(40)
Hence, by lemma 4, the top complex in (40) is also a compatibility complex.

Now, we need to examine the integrability conditions that will help us establish an explicit equivalence of the full Killing equation $K_{ab}[v] = 0$ with the system of equations $\partial_t \tilde{X}_a = 0$, $\tilde{K}_{ab}[\tilde{X}] = 0$, whose compatibility complex is given by the top line of (40). All of that crucially depends on the structure of the curvature of $(M, g)$.

The Riemann curvature tensor, the Ricci tensor and the Ricci scalar of $(M, g)$ are (recalling the notation from (28)) given by

$$R_{abcd} = \left( g \odot \left[ \frac{1}{2} \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) g - \left( \left( \frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right) UU \right] \right)_{abcd},$$

$$R_{ab} = -(m-1) \left[ \left( \frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right] U_a U_b$$

$$+ \left[ \left( \left( \frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right) + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) \right] g_{ab},$$

$$\mathcal{R} = \left( \left( \frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right) + (m + 1) \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right).$$

We suppose that the FLRW spacetime is non-degenerate, that is both that $f'/f \neq 0$ and that the scalar curvature $\mathcal{R}$ is not constant,

$$\mathcal{R}' = \left[ \left( \left( \frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right) + (m + 1) \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) \right]' \neq 0.$$

To make use of identity (26), we compute the Lie derivative

$$\mathcal{L}_v \mathcal{R} = v^a \nabla_a \mathcal{R} = Af \mathcal{R}'.$$

Thus, defining the operator

$$J[h] = \frac{1}{\mathcal{R} \mathcal{R}'} \mathcal{R} \mathcal{R}'[h],$$

we have the identities

$$J \circ K \left[ \frac{A}{X} \right] = A \quad \text{or} \quad J \circ K = [\text{id} \ | \ 0].$$

The last equation also implies that

$$J \circ \left( K \left[ \frac{0}{X} \right] \right) = [\text{id} \ | \ 0] \left[ \frac{0}{X} \right] = 0 \quad \text{and} \quad J \circ \left( K \left[ \frac{\text{id}}{0} \right] \right) = [\text{id} \ | \ 0] \left[ \frac{\text{id}}{0} \right] = \text{id}.$$

Let us denote the block matrix components of $J$ as $J = \begin{bmatrix} J^p & J^y \\ J^z & J^v \end{bmatrix}$ and also $\tilde{J} = \begin{bmatrix} J^v & J^z \\ J^y & J^p \end{bmatrix}$. Combining the above identities with formula (38), we get

$$\tilde{J} \circ \left( K \left[ \frac{0}{X} \right] \right) = \begin{bmatrix} J^y & J^z \\ J^v & J^p \end{bmatrix} \left[ \frac{\partial}{K} \right] = 0.$$
Hence, knowing the first compatibility operator from the top line of (40), it must be possible to factor
\[
J = \tilde{H}_J \begin{bmatrix} -\tilde{C}_0 & \partial_t \\ 0 & \tilde{C}_1 \end{bmatrix}.
\] (48)

Of course, the full operator \( J \) can be then factored as
\[
J = H_J \begin{bmatrix} \text{id} & 0 \\ 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{bmatrix},
\] (49)

where \( H_J = [J^\dagger | \tilde{H}_J] \). For much of what follows, we will only need the fact that \( \tilde{H}_J \) or \( H_J \) exists. However, a direct calculation shows that its explicit form can be deduced from the identity
\[
\begin{align*}
\hat{R} \begin{bmatrix} p \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} &= f^{-2} \tilde{\Delta} p + m(f^{2}/f')[(f^2/f') p]' + m(m + 1)(f^2/f') p
\end{align*}

\begin{align*}
&= f^{-m-1}[\hat{J}^{m+1}(2\text{div} \tilde{Y} - \tilde{t} \tilde{Z})]' + f^{-2}[\tilde{\Delta} \tilde{t} \tilde{Z} + \text{div} \text{div} \tilde{Z}_{ab} - (m - 1) \alpha \tilde{t} \tilde{Z}]
\end{align*}

\begin{align*}
&= f^{-2} \tilde{\Delta} p + \frac{m}{f^{2}f'}[\hat{J}^{m+1}(f^2/f') p]' + f^{-m-1}[\hat{J}^{m+1} \tilde{t} \tilde{Z} - C_0[Y]]' - \frac{1}{2} f^{-2} \tilde{t} \tilde{t} \tilde{t} \tilde{C}_1 [\tilde{Z}],
\end{align*}
\] (50)

meaning that
\[
J = \frac{1}{fR^2} \begin{bmatrix} f^{-2} \tilde{\Delta} + \frac{m}{f^{2}f'} \partial_t f^{m+1} \tilde{C}_0 \\ \frac{1}{f^{2}f'} \tilde{t} \partial_t f^{m+1} \tilde{C}_0 - \frac{1}{2} f^{-2} \tilde{t} \tilde{t} \tilde{C}_1 \end{bmatrix},
\] (51)

and hence
\[
H_J = \frac{1}{fR^2} \begin{bmatrix} f^{-2} \tilde{\Delta} + \frac{m}{f^{2}f'} \partial_t f^{m+1} \tilde{C}_0 \\ \frac{1}{f^{2}f'} \partial_t f^{m+1} \tilde{r} - \frac{1}{2} f^{-2} \tilde{r} \tilde{r} \tilde{r} \end{bmatrix},
\] (52)

where of course we have defined \( \tilde{r}, \text{div} \) and \( \tilde{\Delta} \) such that
\[
\tilde{\Delta} X = (g^f)^{ab} \nabla_a \nabla_b X, \quad \text{div} \tilde{Y} = \nabla^a Y_a, \quad (\tilde{t} \tilde{r} X)_{ab} = (g^f)^{cd} X_{abcd}, \quad \text{and} \quad \tilde{r} \tilde{Z} = (g^f)^{ab} Z_{ab}.
\] (53)

Now we are ready to follow the proof of theorem 9 to construct a compatibility complex for the Killing operator \( K \) by lifting the compatibility complex from (40). The results of these calculations will be presented below directly in diagrammatic form, where the arrows in the diagrams satisfy the identities introduced in section 2. All the relevant identities are easily checked by direct calculation, relying on the key identity (45), the basic commutation relations \([\partial_t, \hat{\nabla}] = [\partial_t, \tilde{C}_1] = 0\), the compatibility identities \( \tilde{C}_{i+1} \circ \tilde{C}_i = 0 \) of the operators of the Calabi complex, which were introduced in section 3.1.

We start by applying the information obtained above to give an explicit reduction of the Killing equation to the first operator from the top line of (40):
Next, we proceed by iterating the construction from lemma 5, while simultaneously applying the simplifications discussed after the proof of theorem 9. The following diagram should be appended on the right to (54):

\[
\begin{bmatrix}
0 & \text{id} \\
\text{id} & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & \text{id} & 0 \\
\text{id} & 0 & \text{id}
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
\text{id} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
-2\partial_t f & 0 \\
-f^{-1} \nabla & \partial_t \\
2 f' \nabla^F & \tilde{K}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\partial_t \\
\tilde{C}_0 = \tilde{K}
\end{bmatrix}
\]

Note that we do not repeat the labels on the left-most vertical arrows, which can be read off as the right-most vertical arrows in (54).

Two more iterations of lemma 5 (with simultaneous simplifications) gives the following diagram, to be appended on the right to (55):
From this point on, the compatibility complex for $K$ and the top line of (40) become identical.

**Theorem 10.** Consider a non-degenerate FLRW spacetime $(M, g)$, $M = I \times F$, as introduced at the top of section 3.2, which spatially has the structure of an $m$-dimensional constant curvature space $(F, g^F)$, with sectional curvature $\alpha$. The full compatibility complex $K_i$ for the Killing operator $K_0 = K (37)$ is given by

$$
K_0 = \begin{bmatrix}
-2\partial_t f & 0 \\
-f^{-1} \vec{\nabla} & \partial_t \\
2f' \vec{g}^F & \vec{K}
\end{bmatrix},
$$

(57a)

$$
K_1 = \begin{bmatrix}
\text{id} & 0 & 0 \\
0 & -\vec{C}_0 & \partial_t \\
0 & 0 & \vec{C}_1
\end{bmatrix}
\begin{bmatrix}
\text{id} - K
\end{bmatrix},
$$

(57b)

$$
K_2 = \begin{bmatrix}
H_J & 0 \\
0 & -\vec{C}_1 & \partial_t \\
0 & 0 & \vec{C}_2
\end{bmatrix},
$$

(57c)

$$
K_3 = \begin{bmatrix}
0 & -\vec{C}_2 & \partial_t \\
0 & 0 & \vec{C}_3
\end{bmatrix},
$$

(57d)

$$
K_i = \begin{bmatrix}
-\vec{C}_{i-1} & \partial_t \\
0 & \vec{C}_i
\end{bmatrix} (3 < i < m),
$$

(57e)

$$
K_m = \begin{bmatrix}
-\vec{C}_{m-1} & \partial_t \\
0 & \vec{C}_m
\end{bmatrix},
$$

(57f)

$$
K_i = 0 (m < i),
$$

(57g)
where the operator $H_J$ is defined in (49), $\partial_t$ and $\tilde{\nabla}$ are the covariant derivatives pulled back along the product structure $t: I \times F \to I$ and $I \times F \to F$, while $\tilde{C}_i$ are the operators from the Calabi complex associated to the constant curvature space $(M, g^3)$, as introduced in section 3.1. (See appendix B.2 for a more complete summary of the notation.)

**Proof.** The argument given around diagram (40) shows that its top line constitutes a full compatibility complex, which coincides with the bottom line of the diagram obtained by gluing (from left to right) the diagrams (54)–(56), which are continued by identifying the top and bottom rows. From the preceding discussion in the current section, it is clear that each pair of consecutive squares in this glued diagram satisfies the hypotheses of lemma 4. Thus, the top line of this glued diagram is itself a full compatibility complex, but that complex consists precisely of the operators $K_i$ in (57).

\[ \square \]

The non-vanishing ranks of the vector bundles in the $K_i$ complex have the following pattern, which can be compared to a similar table for the constant curvature case at the end of section 3.1 (where $n = m + 1$, for comparison):

|       | $m = 2$ | $m = 3$ | $m = 4$ | $m \geq 2$ |
|-------|---------|---------|---------|------------|
| $K_0$ | 3       | 4       | 5       | $m + 1$    |
| $K_1$ | 6       | 10      | 15      | $\frac{m(m+1)}{2} + m + 1$ |
| $K_2$ | 5       | 13      | 31      | $m\left(\frac{m+1}{3}\right) + \frac{m(m+1)}{2} + 1$ |
| $K_3$ | 2       | 10      | 41      | $m\left(\frac{m+1}{4}\right) + \frac{m(m+1)}{2} + 1$ |
| $\vdots$ | $\frac{3m}{2}$ | $\frac{m(m+1)}{3}$ | $\frac{m(m+1)}{4}$ | $\vdots$ |
| $K_i$  ($3 < i$) | $\frac{m(i-1)}{2} + \frac{m(i-2)}{3} + \frac{m+1}{i+1}$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $K_m$  | 2       | 3       | 6       | $\frac{m(m-1)}{2}$ |

**Remark 3.** In $n = 4$ ($m = 3$) dimensions, the common choice for gauge-invariant variables on cosmological FLRW spacetimes are the so-called Bardeen potentials [4]. They include two scalars components, $\Phi$ and $\Psi$, a divergence-free spatial vector field, $\Phi_a$, with 3 independent components, and a divergence-free trace-free spatial symmetric 2-tensor, $\hat{E}_{ab}$, with 5 independent components. Hence, the total number of independent components (not counting the differential relations coming from the divergence-free conditions) is $1 + 1 + 3 + 5 = 10$, which is less than the 13 components of $K_1$ that we have counted above. The difference of course between our $K_1$ and the Bardeen potentials is that our expressions are all local (given only in terms of differential operators), while the Bardeen potentials are non-local (their definition involves inverted spatial Laplacians; see the Introduction to [14] for details). While we cannot claim that our construction gives the minimal number of local gauge-invariant quantities, it is not surprising that we get a larger number than a construction that allows non-local expressions.
Remark 4. It is worth noting that the $K_i$ complex presented above is not continuously deformable through the class of generic FLRW spacetimes to the constant or zero curvature cases, which correspond to special choices of the scale factor $f(t)$. The main reason is that the operator $J$ introduced in equation (44) is proportional to $1/R'$, which diverges when the background scalar curvature becomes constant. Since the operator $J$ and the related operator $H_j$ appear in several places in the formulas (57) for the operators $K_i$ (for $i \geq 1$), it seems difficult to directly compare them with the corresponding operators $C_j$ on constant curvature backgrounds in (32). In particular, an explicit expression for the linearized Riemann tensor $C_1$ on $(M, g)$ would be rather long and unenlightening in our notation, as can already be glimpsed from the formula for the linearized scalar curvature in (50). A more fruitful comparison would be to try to express the components of our $K_1$ in terms of the linearized IDEAL characterization that were recently constructed for the FLRW geometries [9].

However, we might gain some qualitative insight into the components of $K_1$, which can be interpreted as a complete set of local gauge-invariant observables, from the alternative formula that was indicated in diagram (55):

$$\begin{bmatrix} q \\ \tilde{Y}_{a:b} \end{bmatrix} = K_1 \begin{bmatrix} p \\ \tilde{Z}_{a:b,cd} \end{bmatrix} = \left( id - \begin{bmatrix} \text{id} & 0 & 0 \\ 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{bmatrix} \right) \begin{bmatrix} H_j \\ 0 \end{bmatrix} \begin{bmatrix} \text{id} & 0 & 0 \\ 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{bmatrix} \begin{bmatrix} p \\ \tilde{Y}_a \\ \tilde{Z}_{ab} \end{bmatrix}. \tag{58}$$

The $\tilde{Z}_{a:b,cd}$ invariants are roughly coming from the linearized Riemann operator $\tilde{C}_1$ on the spatial slice $(F, g^F)$. While these components can be obtained from a purely spatial projection of the linearized Riemann operator on the FLRW spacetime $(M, g)$, since the components of the linearized spacetime Riemann tensor are not by themselves invariant, they need to be deformed through the $H_j$ and subsequent operators in the above formula to be truly invariant. The $q$ scalar invariant comes from the difference between the $p$ component of $h_{ab}$ and the $p$ component of $K[h]_{ab}$. When precomposed with $h_{ab} = K[v]_{ab}$, both terms in the difference depend only on the $A$ component of $v_a$ and in exactly the same way, hence cancelling to give an invariant quantity. The $\tilde{Y}_{a:b}$ components are harder to interpret in familiar terms.

3.3. Schwarzschild–Tangherlini spacetimes

Consider an $n$-dimensional spacetime $(\mathcal{M}, \mathcal{g})$ where $\mathcal{M} = \mathcal{M} \times S$, where $\dim \mathcal{M} = 2$ and $\dim S = n - 2$ [20, 25, 26]. Let Greek indices $\mu, \nu, \ldots$ correspond to tensors on $\mathcal{M}$. We take the second factor $(S, \Omega)$ to be a maximally symmetric space, hence a constant curvature Riemannian space with sectional curvature $\alpha$, where $\alpha = 1$ for a unit sphere, $\alpha = 0$ for Euclidean space, and $\alpha = -1$ for hyperbolic space (or pseudo-sphere). Let upper case Latin indices $A, B, \ldots$ correspond to tensors on $\mathcal{S}$. Let us denote local coordinates on $\mathcal{S}$ by $\theta^a$ and the Levi-Civita connection on $(S, \Omega)$ by $D_\mu$; its curvature tensor is then

$$R_{ABCD} = \alpha(\Omega_{AC}\Omega_{BD} - \Omega_{AD}\Omega_{BC}) = \frac{\alpha}{2}(\Omega \odot \Omega)_{ABCD}, \tag{59}$$

where we have used the Kulkarni–Nomizu product (28). The other factor $(\mathcal{M}, g)$ has signature $(- +)$. Let lower case Latin indices $a, b, \ldots$ correspond to tensors on $\mathcal{M}$, which we will presume has a timelike Killing vector $r^\mu$. Let us denote local coordinates on $\mathcal{M}$ by $y^a$ and the Levi-Civita connection on $(\mathcal{M}, g)$ by $\nabla_a$. Because $\dim \mathcal{M} = 2$, its curvature is given by
\[ R_{abcd} = \frac{\mathcal{R}}{2} (g_{ac} g_{bd} - g_{ad} g_{bc}) = \frac{\mathcal{R}}{4} (g \otimes g)_{abcd}, \]  
\[ (60) \]
where \( \mathcal{R} = R_{ab}^{ab} \) is the corresponding Ricci scalar. The above notation and index conventions are chosen to match those of [28] as closely as possible.

We are interested in warped product [29, Chapter 7] metrics of the form [20, 25, 26]
\[ \bar{g} = g_{ab} dy^a dy^b + r^2 \Omega_{ab} d\theta^a d\theta^b, \]  
\[ (61) \]
where \( r = r(y) \) and \( g_{ab} \) is static in the (Schwarzschild) coordinates \((y^a) = (t, r)\),
\[ g_{ab} = -f dt_a dt_b + \frac{1}{f} dr_a dr_b, \]  
\[ (62) \]
with \( f = f(r) \). In these coordinates, the timelike Killing vector has the form \( \tau^a = (\partial_t)^a \). For convenience, we also introduce the notation \( \tau^a = g_{ab} \tau^b \) and \( r_a = dr_a \). They are related as \( \tau^a = -\varepsilon^{ab} r_b \), where \( \varepsilon^{ab} = (d\tau \wedge d\sigma)_{ab} \). Then, of course, \( r_a \tau^a = f \) and \( \tau_a \tau^a = -f \).

As we will see shortly, under our assumptions, the Einstein equations with a cosmological constant \( \Lambda \),
\[ \bar{R}^{ab} - \frac{1}{2} \bar{g}^{ab} \bar{R} + \Lambda \bar{g}_{ab} = 0, \]  
\[ (63) \]
are solved by [25, equation (2.15)]
\[ f(r) = \alpha - \frac{2M}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)} r^2, \]  
\[ (64) \]
where \( M \) is a constant. When \( \alpha = 1, \Lambda = 0 \) and \( n \geq 4 \), this metric describes the higher dimensional spherically symmetric static black holes, the so-called Schwarzschild–Tangherlini solutions, specializing to the Schwarzschild solution when \( n = 4 \). When \( n = 3 \), we are forced to have \( \alpha = 0 \) and the spacetime is actually of constant curvature. With \( n = 3 \) and \( \Lambda < 0 \), we get the BTZ metric [3]. In terms of the parameter \( M \), the black hole mass is given by
\[ \frac{(n-2)A_{n-2} M}{8\pi G} = \frac{(n-2) A_{n-2} M}{2 A_2 G}, \]  
\[ (65) \]
where \( G \) is the \( n \)-dimensional Newton’s constant and
\[ A_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma[(n-1)/2]} \]  
\[ (66) \]
is the area of the unit \((n-2)\)-sphere. When \( \alpha = 0 \), we get the higher dimensional version of Taub’s plane-symmetric spacetime [36], [33, equation (15.29), [5, equation (2.2)]. When \( \alpha = -1 \), we get the higher dimensional version of a pseudo-Schwarzschild wormhole spacetime [27].

In what follows, we restrict our attention to \( n \geq 4 \), which is physically reasonable, but is also forced upon us by some of our formulas, which have poles at \( n = 1, 2 \) or 3.

For convenience, let us introduce the notations
\[ f_1(r) = rf'(r) = (n-3) \frac{2M}{r^{n-3}} - \frac{4\Lambda}{(n-1)(n-2)} r^2, \]  
\[ (67) \]
\[ f_2(r) = rf'_1(r) = -(n-3) \frac{2M}{r^{n-3}} - \frac{8\Lambda}{(n-1)(n-2)} r^2, \]  
\[ (68) \]
as well as note that the formula (64) for $f$ parametrized by the constants $M$ and $\Lambda$, with $\alpha$ fixed, gives the general solution to the differential equation

$$f_3 + (n - 5) f_1 - 2(n - 3) f = 0.$$  \hspace{1cm} (69)

Any tensor on $M$ decomposes as $T_a = T_d d_a + T_r d_r$ or $T_a \rightarrow (T, T_r)$, with obvious extension to higher rank tensors. With respect to this decomposition and the coordinates $(t, r)$, the Levi-Civita connection on $(M, g)$ is then \[25, \text{equation (2.18)}\]

$$\nabla_a v_b \rightarrow \frac{\partial v_t}{\partial x^a} \frac{\partial v_r}{\partial x^b} + \left[ 0 \quad 0 \quad -\frac{f}{2} \right] v_t + \left[ -f \frac{f}{2} \quad 0 \quad \frac{f}{2} \right] v_r.$$  \hspace{1cm} (70)

Equivalently, we can summarize this information by giving the covariant derivatives of the frame $(t_a, r_a)$,

$$\nabla_a b = \frac{f_1}{2r} \varepsilon_{ab} \quad \text{and} \quad \nabla_a r_b = \frac{f_1}{2r} g_{ab}. \hspace{1cm} (71)$$

A direct calculation gives the Ricci scalar on $(M, g)$ as

$$R = \frac{f_1 - f_2}{r^2}.$$  \hspace{1cm} (72)

And finally, symmetrizing the covariant derivative as written in (70) or (71), the explicit form of the Killing operator on $(M, g)$ is

$$2\nabla_a (\nabla b) = -2 t(a \nabla b) \frac{v_t}{f} - 2 t_a b \frac{f_1}{2r} v_r - 2 t_a r b \frac{1}{f} \frac{v_t}{r} + r_a b \frac{1}{f} (2 f \partial_r v_r + f \frac{f}{2} v_r)$$

$$\rightarrow \left[ f (2 \partial_r \frac{f}{2} v_r - \frac{f}{2} v_r) \quad \delta v_r + f \partial_r \frac{f}{2} v_r \right] \frac{\partial v_r}{\partial x^a} + f \partial_r \frac{f}{2} v_r + \frac{f}{2} \left(2 f \partial_r v_r + \frac{f}{2} v_r\right). \hspace{1cm} (73)$$

Greek indices $\mu, \nu, \ldots$ on $M$-tensors are raised and lowered by $\tilde{g}_{\mu\nu}$. Lower case Latin indices $a, b, c, \ldots$ on $M$-tensors are raised and lowered by $g_{ab}$. And upper case Latin indices $A, B, C, \ldots$ on $S$-tensors are raised and lowered by $\Omega_{AB}$. Any $M$-tensor decomposes into sectors\(^3\), according to $T_\mu = T_d (d^\mu)_d + r T_A (d^\mu)_A \rightarrow (T_d, r T_A)$ and $T^\mu = T^\mu (\partial_d)^\mu + \frac{1}{2} T^\mu (\partial_A)^\mu \rightarrow (T^d, \frac{1}{2} T^A)$, with obvious extension to higher rank tensors. With a slight departure from this convention, let us define some $M$-tensors by their sector decomposition

$$g_{\mu\nu} \rightarrow \begin{bmatrix} g_{ab} & 0 \\ 0 & 0 \end{bmatrix}, \quad g^{\mu\nu} \rightarrow \begin{bmatrix} g^{ab} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \Omega_{\mu\nu} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & \Omega_{AB} \end{bmatrix}, \quad \Omega^{\mu\nu} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & \Omega^{AB} \end{bmatrix}. \hspace{1cm} (74)$$

so that $\tilde{g}_{\mu\nu} = g_{\mu\nu} + r^2 \Omega_{\mu\nu}$ and $\tilde{g}^{\mu\nu} = g^{\mu\nu} + r^{-2} \Omega^{\mu\nu}$.

The pair $(\nabla_a, D_A)$ defines a connection on $\mathcal{M} = M \times S$, which differs \[26, \text{appendix A}\] from the Levi-Civita connection $\nabla_\mu$ as follows

$$\nabla_\mu v_\nu \rightarrow \frac{\nabla_a v_b}{r D_A \frac{v_a}{T}} \left[ r^2 \partial_a v_B \frac{v_b}{T} \right] + \frac{0}{0} \frac{2}{r^2 \Omega_{AB} \frac{v_c}{T}} v_c + \frac{0}{0} \frac{-r \delta_A^c}{0} v_c. \hspace{1cm} (75)$$

**Remark 5.** In giving the formula for $\nabla_\mu$, we have essentially extended the action of $\nabla_a$ and $D_A$ as linear differential operators to tensors defined on $\mathcal{M} = M \times S$. The extension is done in exact analogy with the procedure described at the start of section 3.2. Recall that the
covariant derivatives $\nabla_a$ and $D_A$ simply act as coordinate derivatives on scalars on $\mathcal{M}$ and $\mathcal{S}$. Suitably extending these coordinate derivatives to $\bar{\mathcal{M}}$, the same can be said for $(\mathcal{M}, \mathcal{S})$-mixed tensors on $\bar{\mathcal{M}}$. But for the sake of uniformity in notation, we continue to use the notation $\nabla_a$ and $D_A$ even when they act on $\mathcal{M}$- and $\mathcal{S}$-scalars respectively.

Next, we need to carefully study the structure of the curvature tensor. The spacetime Riemann curvature tensor on $(\bar{\mathcal{M}}, \bar{g})$ is [26, appendix A]

$$R_{\mu\nu\lambda\kappa} = \frac{R}{2} (\bar{g} \circ \bar{g})_{\mu\nu\lambda\kappa} + \frac{1}{2r^2} (\alpha - ra^\mu)(r^2 \Omega \circ r^2 \Omega)_{\mu\nu\lambda\kappa} - \left( \frac{\nabla \nabla r}{r} \circ r^2 \Omega \right)_{\mu\nu\lambda\kappa},$$

with the corresponding Ricci tensor

$$\bar{R}_{\mu\lambda} = \frac{f_1 - f_2}{4r^2} (\bar{g} \circ \bar{g})_{\mu\nu\lambda\kappa} + \frac{(\alpha - f)}{2r^2} (r^2 \Omega \circ r^2 \Omega)_{\mu\nu\lambda\kappa} - \frac{f_1}{2r^2} (\bar{g} \circ r^2 \Omega)_{\mu\nu\lambda\kappa},$$

(76)

To satisfy Einstein’s equations in the presence of a cosmological constant (63), we must have

$$\bar{R}_{\mu\lambda} = \frac{2\Lambda}{(n-2)} g_{\mu\lambda} = \frac{2\Lambda}{(n-2)} (g + r^2 \Omega)_{\mu\lambda},$$

(77)

which implies

$$f_1 = -(n-3)(f - \alpha) - \frac{2\Lambda}{(n-2)} r^2, \quad f_2 = -(n-3)f_1 - \frac{4\Lambda}{(n-2)} r^2.$$

(79)

Eliminating the explicit dependence on $\Lambda$, we obtain precisely the second order ODE (69) whose general solution is given by $f(r)$ in (64).

Recalling the definition of the Kulkarni–Nomizu and contraction products (28) and (29), we get the following useful identities, where we used $\bar{g}_{\mu\nu}$ for contractions:

$$(g \circ g) \cdot (g \circ g) = 4(g \circ g),$$

(81)

$$\quad (r^2 \Omega \circ r^2 \Omega) \cdot (r^2 \Omega \circ r^2 \Omega) = 4(r^2 \Omega \circ r^2 \Omega),$$

(82)

$$\quad (g \circ r^2 \Omega) \cdot (g \circ r^2 \Omega) = 2(g \circ r^2 \Omega),$$

(83)

$$\quad (g \circ r^2 \Omega) \cdot (g \circ g) = 0,$$

(84)

$$\quad (g \circ r^2 \Omega) \cdot (r^2 \Omega \circ r^2 \Omega) = 0,$$

(85)

$$\quad (g \circ g) \cdot (r^2 \Omega \circ r^2 \Omega) = 0,$$

(86)

$$\quad (g \circ g)_{\lambda\nu}^{\nu} \kappa = -2g_{\lambda\kappa}.$$  

(87)
\[
(r^2 \Omega \odot r^2 \Omega)_{\lambda \mu \nu \kappa} = -2(n - 3)(r^2 \Omega)_{\lambda \kappa},
\]
(88)
\[
(g \odot r^2 \Omega)_{\lambda \mu \nu \kappa} = -2(r^2 \Omega)_{\lambda \kappa} - (n - 2)g_{\lambda \kappa}.
\]
(89)
\[
(g \odot g)_{\mu \lambda \nu \kappa} r^{\lambda} r^{\kappa} = 2(r^4 r_{\lambda}) g_{\mu \nu} - 2r_{\mu} r_{\nu},
\]
(90)
\[
(g \odot r^2 \Omega)_{\mu \lambda \nu \kappa} r^{\lambda} r^{\kappa} = (r^4 r_{\lambda})(r^2 \Omega)_{\mu \nu},
\]
(91)
\[
(r^2 \Omega \odot r^2 \Omega)_{\mu \lambda \nu \kappa} r^{\lambda} r^{\kappa} = 0.
\]
(92)

Defining
\[
\bar{T}_{\mu \nu \lambda \kappa} = \bar{R}_{\mu \nu \lambda \kappa} - \frac{\Lambda}{(n - 1)(n - 2)} (\bar{g} \odot \bar{g})_{\mu \nu \lambda \kappa}
\]
\[
= \frac{M}{r^{n-1}} \left[ (n - 2)(n - 3) + \frac{2}{2}(g \odot g)_{\mu \nu \lambda \kappa} + (r^2 \Omega \odot r^2 \Omega)_{\mu \nu \lambda \kappa} - (n - 3)(g \odot r^2 \Omega)_{\mu \nu \lambda \kappa} \right],
\]
(93)
we also get the identities
\[
\bar{T} \cdot \bar{T} = \left( \frac{M}{r^{n-1}} \right)^2 \left[ (n - 2)^2(n - 3)^2 + 4(r^2 \Omega \odot r^2 \Omega) + 2(n - 3)^2(g \odot r^2 \Omega) \right]
\]
\[
\bar{T} \cdot T = \frac{M}{r^{n-1}} \left[ (n - 2)^2(n - 3)^2 + 4(r^2 \Omega \odot r^2 \Omega) + 8(n - 3)^2(g \odot r^2 \Omega) \right]
\]
\[
(\bar{T} \cdot T)_{\mu \nu \kappa} = -\frac{M}{r^{n-1}} \left[ 2(n - 2)^2(n - 3)^2 g + 8(n - 3)r^2 \Omega + 2(n - 3)^2(2r^2 \Omega + (n - 2)g) \right]_{\mu \kappa}
\]
\[
= -\frac{M}{r^{n-1}} \left[ 2(n - 1)(n - 2)(n - 3)^2 g + 4(n - 1)(n - 3)r^2 \Omega \right]_{\mu \kappa}
\]
\[
(\bar{T} \cdot T)_{\mu \nu} = \frac{M}{r^{n-1}} \left[ 4(n - 1)(n - 2)(n - 3)^2 + 4(n - 1)(n - 2)(n - 3) \right]
\]
\[
= 4(n - 1)(n - 2)^2(n - 3) \left( \frac{M}{r^{n-1}} \right)^2
\]
\[
\nabla_\lambda (\bar{T} \cdot T)_{\mu \nu} = -2(n - 1) \frac{r_{\lambda}}{r}
\]

We would like to use these identities to write \((r^2 \Omega)_{\lambda \kappa}\) and a simple \(r\)-dependent scalar as covariant expression in the curvature. For the latter, the simplest choice seems to be
\[
T^{(1)} [g] := \frac{(\bar{T} \cdot T)_{\mu \nu}}{4(n - 1)(n - 2)^2(n - 3)} = \left( \frac{M}{r^{n-1}} \right)^2.
\]
(94)
Next, we encounter a slight dimension dependence in the expression for \((r^2 \Omega)_{\lambda \kappa}\). When \(n > 4\), we can use
\[
(r^2 \Omega)_{\lambda \kappa} = \frac{2(n - 2)^2}{(n - 1)(n - 4)} \frac{(\bar{T} \cdot T)_{\lambda \mu \nu \kappa}}{(\bar{T} \cdot T)_{\mu \nu}} + \frac{(n - 2)(n - 3)}{(n - 1)(n - 4)} \Omega_{\lambda \kappa} =: \bar{T}_{\lambda \kappa}^{(2)} [g],
\]
(95)
while for $n = 4$ the simplest expression we could find is
\[
(r^2 \Omega)_{\lambda \kappa} = \frac{2}{(n - 3)^3} [\nabla (T \cdot T)]_{\mu \nu} \eta^{\mu \nu} \cdot
\left(T \cdot T \cdot T - \frac{(n - 3)}{(n - 1)} (T \cdot T)_{\mu \nu} \cdot T \cdot T\right)_{\lambda \mu \nu \sigma}
\]
\[
= \nabla^\rho (T \cdot T)_{\mu \nu} \eta^{\mu \nu} \cdot \nabla^\rho (T \cdot T)_{\mu \nu} \eta^{\mu \nu}
\]
\[
= : T^{(3)}_{\lambda \kappa}[g].
\]
(96)

Although, since it also works for $n > 4$, if desired, the more complicated expression $T^{(3)}_{\lambda \kappa}[g]$ could actually be used in higher dimensions too.

To make use of identity (26), we compute the Lie derivatives
\[
\mathcal{L}_v \left( \frac{M}{r^{n-1}} \right)^2 = -2(n - 1) \frac{r^2 v^e}{f} \left( \frac{M}{r^{n-1}} \right)^2,
\]
(97)
\[
\mathcal{L}_v (r^2 \Omega)_{\mu \nu} \rightarrow \left[ \frac{0}{r(r \nabla_\nu \frac{r \nabla_\nu}{r})} \right].
\]
(98)

Hence, defining the linear operators
\[
J_1 [h] := \frac{1}{2(n - 1)} r \cdot \left( \frac{M}{r^{n-1}} \right)^{-2} \hat{T}^{(1)}[h],
\]
(99)
\[
J_2 [h]_{ab} := \frac{1}{r^2} \left[ \hat{T}^{(2)}_{ab}[h] \right] (n > 4)
\]
\[
J_2 [h]_{ab} := \frac{1}{r^2} \left[ \hat{T}^{(3)}_{ab}[h] \right] (n = 4),
\]
(100)

the Lie derivative formula (26) implies the compositional identities
\[
J_1 \circ K[v] = \frac{r^2 v^e}{f},
\]
(101)
\[
J_2 \circ K[v]_{ab} = \frac{\nabla_a \omega_b}{r}.
\]
(102)

Based on equation (75), the explicit expression for the Killing operator is
\[
\hat{K}_{\mu \nu}[v] = 2 \nabla_{(\mu} v_{\nu)} \rightarrow \left[ \frac{2 \nabla_{(a} v_{b)}}{r} \frac{2 \nabla_{(a} v_{b)} + r D_{ab} \frac{r}{r}}{2 r^2 D_{(a} v_{b)} + 2 r^2 \frac{r}{r}} \right].
\]
(103)

For further convenience, we parametrize
\[
v_{\mu} \rightarrow \left[ \frac{u_\mu f d u_\lambda + u_\mu d r_{\lambda}}{r(r X_\lambda)} \right]
\]
and
\[
h_{\mu \nu} \rightarrow \left[ \frac{p r a r b - 2 t (a w b)}{r(r X_\lambda)} \right].
\]
(104)

The Killing equation $h = \hat{K}[v]$ then becomes
\[
\begin{bmatrix}
\frac{p}{w} \\
Y \\
Z
\end{bmatrix} = \hat{K}
\begin{bmatrix}
u_r \\
u_t \\
u_X
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{r} \left( 2 f \sigma_t + \frac{D}{r} \right) \\
\frac{d r}{f} \frac{1}{r} \partial_{\sigma} - d t \frac{D}{r^2} \\
\frac{d r}{r} D \frac{1}{f} + d t \frac{D}{r^2} \nabla \frac{D_0}{f} \left[ 0 \\
\nabla \frac{D_0}{f} \\
0 \nabla \frac{D_0}{f} \nabla \frac{D_0}{f}
\end{bmatrix}
\begin{bmatrix}
u_r \\
u_t \\
u_X
\end{bmatrix}.
\]
(105)
where \( C_0[X]_{AB} = D_A X_B + D_B X_A \) is the Killing operator on the constant curvature factor \((S, \Omega)\), and hence the first operator of the Calabi complex \( C_i, i \geq 0 \), which constitutes a compatibility complex for \( C_0 \) (section 3.1).

**Remark 6.** Note that we are continuing here to use the block matrix notation for differential operators, as discussed previously in remark 2.

With the above parametrizations for \( v \) and \( h \), the compositional identities for the operators \( J_1 \) (99) and \( J_2 \) (100) simplify to

\[
J_1 \circ \bar{K} = \begin{bmatrix}
J_1^p & J_1^w & J_1^Y & J_1^Z
\end{bmatrix} \circ \bar{K} = \begin{bmatrix}
\text{id} & 0 & 0
\end{bmatrix},
\]

(106a)

\[
J_2 \circ \bar{K} = \begin{bmatrix}
J_2^p & J_2^w & J_2^Y & J_2^Z
\end{bmatrix} \circ \bar{K} = \begin{bmatrix}
0 & 0 & \nabla
\end{bmatrix}.
\]

(106b)

Now we have all the information that we need to use the methods of section 2 to construct a compatibility complex for the Killing operator \( \bar{K} \). We will follow roughly the same outline as we did in the section 3.2 on cosmological FLRW geometries.

From now on, our strategy will be to show that our Killing operator \( \bar{K}_0 = \bar{K} \) is equivalent to each of the operators \( \tilde{K}_0 \) first to \( \tilde{K}_0 \) and finally to \( \bar{K}_0 \). This known compatibility complex has the form

\[
\tilde{K}_0 \begin{bmatrix}
\bar{d}_0 = \nabla \\
\bar{d}_1 \\
\bar{d}_2 \\
\bar{d}_0 = \nabla
\end{bmatrix} \begin{bmatrix}
\bar{X}
\end{bmatrix} \begin{bmatrix}
\bar{u}_t
\end{bmatrix},
\]

(107)

where we have introduced the notation \( \bar{d}_i \) and \( \bar{d}_0 \), for the usual exterior derivatives acting on \( i \)-forms on \( \bar{M} \) and \( M \) respectively (hence the corresponding de Rham complexes). In the sequel, we will use the notations \( \bar{d}_0 \) and \( \nabla \) completely interchangeably. Then, we will lift the known compatibility complex for \( K_0 \) first to \( \tilde{K}_0 \) and finally to \( \bar{K}_0 \). This known compatibility complex has the form

\[
K_0 = \begin{bmatrix}
\bar{d}_0 & 0 & 0 \\
0 & \bar{d}_0 & 0 \\
0 & 0 & C_0
\end{bmatrix},
\]

(108a)

\[
K_1 = \begin{bmatrix}
\bar{d}_1 & 0 & 0 \\
0 & \bar{d}_1 & 0 \\
0 & -C_0 & \bar{d}_0 \\
0 & 0 & C_1
\end{bmatrix},
\]

(108b)

\[
K_2 = \begin{bmatrix}
\bar{d}_2 & 0 & 0 & 0 \\
0 & \bar{C}_0 & \bar{d}_1 & 0 \\
0 & 0 & -C_1 & \bar{d}_0 \\
0 & 0 & 0 & C_2
\end{bmatrix},
\]

(108c)
It is straightforward to construct an equivalence between this complex and a twisted de Rham complex, similar to how it was done in (40), thus showing that each of the above compatibility operators is complete.

We start with the explicit reduction of $\bar{K}_i$ to $\tilde{K}_i$ and then to $K_i$. Here and in each subsequent step, we give pairs of diagrams, which could be concatenated vertically, illustrating the passage from the $\bar{K}_i$ to the $\tilde{K}_i$ and to the $K_i$ sequences. All the diagrams below illustrate equivalences up to homotopy, as discussed in section 2. All the required identities can be checked by direct calculation, making careful use of the known identities $d_i + 1 \circ d_i = 0$, $\bar{d}_i + 1 \circ \bar{d}_i = 0$, $C_i + 1 \circ C_i = 0$, as well as the compositional identities (106).
Next, as we did previously in section 3.2, we iterate the construction from lemma 5, while applying the simplifications discussed after the proof of theorem 9. As before, the $\tilde{K}_i$ and $\tilde{K}_0$ complexes are built up by appending the following diagrams to the right of the diagrams in (109). Also as before, we do not repeat the labels on the vertical arrows if they can be read off a preceding diagram.

The resulting operators $\tilde{K}_1$ and $\tilde{K}_0$ will be, respectively, compatibility operators for $\tilde{K}_0$ and $\tilde{K}_0$. Some of the auxiliary arrows in these diagrams use the operators $\tilde{H}J_1$ and $HJ_1$, which are defined as follows. Noting that

$$J_1 \bar{K}_0 = \begin{bmatrix} \nabla & 0 \\ dt \frac{1}{\rho} D & \nabla \\ 0 & C_0 \end{bmatrix}$$

we must be able to factor

$$[J_1^w J_1^0 J_1^2] = HJ_1 \tilde{K}_1,$$

through some operators $\tilde{H}J_1$ and $HJ_1$. A bit more precisely, $HJ_1 = [J_1^p \mid \tilde{H}J_1]$.
Above, we have used the notations $dt \cdot (-) = \text{data}(-)$ and $dr \cdot (-) = \text{dra}(-)$. Also, the operator $\tilde{H}_J$ is defined as follows. Noting that
we must be able to factor

\[
\begin{bmatrix}
J_2 & J_2^Y \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
J_2^e & J_2^Y \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\overline{d}_0 = \nabla \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 & \nabla \\
0 & 0 \\
\end{bmatrix}
= \nabla = [id & 0] \begin{bmatrix}
\nabla \\
\nabla \\
\end{bmatrix},
\]

through some operator \( \tilde{H}_{J_2} \).

For convenience, we note that

\[
\begin{bmatrix}
id & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{K}_2
\end{bmatrix}
= \begin{bmatrix}
\overline{d}_1 & 0 & 0 \\
0 & d_1 & 0 \\
0 & -C_0 & d_0 \\
0 & 0 & C_1
\end{bmatrix}
\]

while on the other hand

\[
\begin{bmatrix}
0 & \tilde{H}_{J_2} \\
\end{bmatrix}
\begin{bmatrix}
\tilde{K}_1
\end{bmatrix}
= \begin{bmatrix}
\overline{d}_1 & 0 & 0 \\
0 & d_1 & 0 \\
0 & -C_0 & d_0 \\
0 & 0 & C_1
\end{bmatrix}
\]

Then, defining

\[
H_{J_2} = \begin{bmatrix}
-J_2^e & \tilde{H}_{J_2}
\end{bmatrix}
\]

we have \( H_{J_2} \tilde{K}_1 = 0 \).

With the next iteration of lemma 5, we construct the compatibility operators \( \tilde{K}_2 \) and \( \tilde{K}_2 \).
With two more iterations of lemma (5), we construct the compatibility operators $\tilde{K}_3$, $\tilde{K}_4$ and $\tilde{K}_3$, $\tilde{K}_4$.

With two more iterations of lemma (5), we construct the compatibility operators $\tilde{K}_3$, $\tilde{K}_4$ and $\tilde{K}_3$, $\tilde{K}_4$.
From this point on, the complexes \( \mathcal{K}_i \), \( \mathcal{K}_4 \), become identical with \( K_i \) from (108).

**Theorem 11.** Consider the family of \( n \)-dimensional \((n \geq 4)\) spacetimes \((\mathcal{M}, \tilde{g})\) introduced at the top of section 3.3, warped products of a static 2-dimensional factor \((\mathcal{M}, g)\) and a constant curvature factor \((\mathcal{S}, \Omega)\) with sectional curvature \( \alpha \), which includes the higher dimensional Schwarzschild (Schwarzschild–Tangherlini), Taub and pseudo-Schwarzschild solutions, possibly with a nonzero cosmological constant. The full compatibility complex \( K_i \) for the Killing operator \( \mathcal{K}_0 = \mathcal{K} (105) \) is given by

\[
\mathcal{K}_0 = \begin{bmatrix}
\frac{1}{2} (2 f \partial_r + \frac{f_r}{r^2} ) & 0 & 0 \\
\frac{1}{r^2} D_r \frac{f_r}{r^2} D_r & dt \frac{f_r}{r^2} D_r \\
\frac{1}{2} \Omega_r & 0 & C_0
\end{bmatrix},
\]

(116a)

\[
\mathcal{K}_1 = \begin{bmatrix}
\text{id} & 0 & 0 \\
0 & [0 \ - dr \cdot (-)] & dr \cdot (-) \\
0 & \bar{d}_1 & \bar{d}_1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & r^2 dt \cdot (-) & 0 \\
0 & d_1 & - C_0 & d_0 \\
0 & 0 & 0 & C_1
\end{bmatrix},
\]

(116b)
\[
\tilde{K}_2 = \begin{bmatrix}
& H_{J_1} \\
0 & & H_{J_2} \\
0 & 0 & d_2 & 0 & 0 & 0 \\
0 & 0 & 0 & C_0 & d_1 & 0 \\
0 & 0 & 0 & 0 & -C_1 & d_0 \\
0 & 0 & 0 & 0 & 0 & C_2
\end{bmatrix},
\tag{116c}
\]

\[
\tilde{K}_3 = \begin{bmatrix}
& d_3 & 0 & 0 & 0 \\
0 & 0 & 0 & C_1 & d_1 & 0 \\
0 & 0 & 0 & 0 & -C_2 & d_0 \\
0 & 0 & 0 & 0 & 0 & C_3
\end{bmatrix},
\tag{116d}
\]

\[
\bar{K}_i = \begin{bmatrix}
\bar{d}_i & 0 & 0 & 0 \\
0 & C_{i-2} & d_i & 0 \\
0 & 0 & -C_{i-1} & d_0 \\
0 & 0 & 0 & C_i
\end{bmatrix},
\tag{3 < i < n - 2)}
\]

\[
\tilde{K}_{n-2} = \begin{bmatrix}
\bar{d}_{n-2} & 0 & 0 & 0 \\
0 & C_{n-4} & d_1 & 0 \\
0 & 0 & -C_{n-3} & d_0
\end{bmatrix},
\tag{116f}
\]

\[
\tilde{K}_{n-1} = \begin{bmatrix}
\bar{d}_{n-1} & 0 & 0 \\
0 & C_{n-3} & d_1
\end{bmatrix},
\tag{116g}
\]

\[
\tilde{K}_i = 0 \quad (n \leq i).
\tag{116h}
\]

where \( f(r) \) is defined in (64) and \( f_1 = rf'(r) \). \( \bar{d}_i \) and \( d_i \) denote the exterior derivatives on \( i \)-forms, on \( \bar{M} \) and \( M \) respectively, while \( D \) and \( C_i \) are the covariant derivative and the Calabi complex operators (32) on \( (S, \Omega) \), and we have also used the operators \( J_1 \) (99), \( J_2 \) (100), \( H_{J_1} \) (110), \( H_{J_2} \) (113). (See appendix B.3 for a more complete summary of the notation.)

While we have unambiguously defined the operators \( J_1, J_2, H_{J_1}, \) and \( H_{J_2} \), we have not computed them explicitly. For our purposes here, it is sufficient that they exist and satisfy a few defining properties. Of course, in individual cases, they could be easily computed using computer algebra.

**Proof.** The proof is very much parallel to the proof of theorem 10. We start with the knowledge that the complex (108) is a full compatibility complex. Then, gluing together (from left to right) the diagrams (109), (111), (114) and (115), we observe that the glued diagrams satisfy the hypotheses of lemma 4. This implies, that \( \tilde{K}_i \) is a full compatibility complex as well, which in turn implies that so is \( \bar{K}_i \), whose operators we have explicitly listed in (116).

The non-vanishing ranks of the vector bundles in the \( \bar{K}_i \) complex have the following pattern, which can be compared to similar table for the constant curvature (section 3.1) and FLRW cases (section 3.2, where \( m = n - 1 \), for comparison):
| $n$ | $\bar{K}_0$ | $\bar{K}_1$ | $\bar{K}_2$ | $\bar{K}_3$ | $\bar{K}_{n-1}$ |
|-----|-------------|-------------|-------------|-------------|---------------|
| 4   | 4           | 10          | 18          | 12          | 2             |
| 5   | 5           | 15          | 35          | 35          | 17            |
| 6   | 6           | 21          | 64          | 95          | 81            |
| $n \geq 4$ | $(n-2)+1+1$ | $\frac{(n-1)(n-2)}{2}+2(n-2)+3$ | $\frac{n^2}{2}(\frac{n-1}{3})+(n-1)(n-2)+(n-2)+\binom{n}{2}+(n-2)+1$ | $(n-2)\left(\frac{n-1}{4}\right)+(n-2)\left(\frac{n-4}{3}\right)+\frac{(n-1)(n-2)}{2}+\binom{n}{2}+(n-2)+1$ | $\frac{(n-2)(n-5)}{2}+1$ |

**Remark 7.** In $n = 4$ dimensions, it is well-known [21, 31] that, for practical purposes, taking the linearized Einstein equations into account, the gauge invariant degrees of freedom for linear perturbations on the Schwarzschild background reduce to the Regge–Wheeler (axial) and Zerilli (polar) scalars, or equivalently the complex Teukolsky scalar. It is even possible to give the Regge–Wheeler and Zerilli scalars local and manifestly gauge-invariant definitions, based on the linearization of curvature tensors vanishing on the Schwarzschild background [10]. However, it is also known that there exist so-called algebraically special modes that are not pure gauge but lie in the kernel of these gauge-invariants [39]. Hence, this small set of invariants cannot be considered complete in our sense. In our construction, $K_1$ has 18 independent components (without taking the linearized Einstein equations into account, though). But our construction proves that they form a complete set of local gauge-invariants.

**Remark 8.** In analogy with remark 4 about FLRW geometries, it is worth noting that the $K_i$ complex presented above is not continuously deformable through the class of gST spacetimes to the $M = 0$ case, which corresponds to the constant curvature limit. The main reason, again, is that the operators $J_1$ and $J_2$, introduced in equations (99) and (100), are proportional to $1/M$ and hence diverge in that limit. These operators, together with their factorizations $H_{J_i}$ and $H_{J_2}$, appear in several places in the formulas (116) for the operators $K_i$ (for $i \geq 1$). Thus, also in this case, it would be difficult to compare the local gauge-invariant components of our $K_i$ operator to the components of the linearized Riemann operator, which would also be given by long and unenlightening expressions. A more fruitful comparison would be to try to express the components of our $K_1$ in terms of the linearized IDEAL characterization tensors that were recently constructed for the gST geometries [24]. However, the intuition proposed in the second paragraph of remark 4 still largely applies to the components of our $K_1$. In particular, our $J_1$ operator is directly analogous to the $J$ operator
4. Discussion

In this work, we have studied the construction of the compatibility complex (definition 2) $K_l$, $l = 0, 1, 2, \ldots$, for a linear differential operator $K_0$ of regular finite type (definition 7). The construction proceeds by putting the operator $K_0$ into a canonical form of a flat connection and then lifting the resulting twisted de Rham complex to a compatibility complex for $K_0$ (theorem 9). Our primary and motivating example of an operator of regular finite type is the Killing operator $K_{ab}[v] = \nabla_a v_b + \nabla_b v_a$ on a Lorentzian (or even pseudo-Riemannian) manifold $(M, g)$. Once known, the components of the first compatibility operator $K_1$ can be interpreted (as discussed in the Introduction) as a complete set of local gauge-invariant observables in linearized gravity on $(M, g)$.

We have applied the abstract construction of section 2 to several physically motivated examples: flat (Minkowski) and constant curvature (de Sitter or anti-de Sitter) spacetimes in section 3.1, cosmological (FLRW) spacetimes in section 3.2, (Schwarzschild–Tangherlini) spherically symmetric black hole spacetimes$^4$ in section 3.3. In each case, we have kept the dimension $n = \dim M$ general, allowing at least $n \geq 4$. While the contents of section 3.1 are well-known (they were previously reviewed in more detail in [23]), the Killing compatibility complexes constructed in sections 3.2 and 3.3 are new.

One may wish to compare the main result for FLRW geometries, theorem 10, with the recent works [9, 13, 14], which were the first to (a) construct, (b) give a geometric interpretation to and (c) prove completeness for the first compatibility operator $K_1$ in a context very similar the one considered in section 3.2 (the difference is that here we do not include the presence of a dynamical scalar inflaton field on an cosmological FLRW geometry). The systematic approach developed in this work can also be easily applied in the presence of an inflaton field. Then, the systematically constructed compatibility operator $K_1$ would be necessarily equivalent to what was obtained in [9, 13, 14]. The difference is that our systematic construction automatically comes with a proof of completeness, while the previous proof of completeness given in [13] relied very heavily on parallels with known results for the flat and constant curvature cases [19, 23], without an obvious way to generalize it. On the other hand, our systematic construction does not give a Stewart-Walker-like (see the introduction to section 3) geometric interpretation to $K_1$ as a linear local gauge-invariant observable. On the other hand, the approach put forward in [9, 14], of constructing a candidate $K_1$ by linearizing an IDEAL characterization of the background geometry, automatically gives $K_1$ a Stewart-Walker-like geometric interpretation, but does not automatically prove completeness$^5$. Thus, we see great potential in joining the methods of the current work with those of [9, 14] to construct universal Killing compatibility operators (equivalently, complete sets of linear local gauge-invariant observables) on a variety of backgrounds, while getting the benefits of straightforward geometric interpretation and of a systematic way to prove completeness.

---

$^4$The family of spacetimes considered section 3.3 is actually richer than just asymptotically flat spherically symmetric black holes (the Schwarzschild–Tangherlini ones). More generally, it allows for a non-zero cosmological constant and also allows to substitute spherical symmetry for planar or pseudo-spherical symmetry, which respectively give rise Taub’s plane symmetric spacetimes or to pseudo-Schwarzschild solutions.

$^5$Although, the only possibility we know in which completeness might fail is when the IDEAL characterization tensors vanish at quadratic or higher order when approaching the isometry class of the characterized geometry in the space of metrics. Then their linearization might fail to capture all of the linear invariants.
For the Schwarzschild black hole (and its higher dimensional generalizations), the Regge–Wheeler and Zerilli local gauge-invariants have been known for a long time [25]. Other local gauge-invariants have also been proposed (see [1, 21, 31] for a brief review). However, to our knowledge, no claim of completeness has ever been made for an explicit set of local gauge-invariants on Schwarzschild. Thus, even our construction of the first compatibility $K_1$ operator in section 3.3 appears to be new. On the other hand, the 4-dimensional Schwarzschild black hole does have a known IDEAL characterization [12], recently extended to higher dimensions [24], so as was argued in the previous paragraph its linearization would have provided a good candidate for $K_1$. To our knowledge, this has not been done explicitly in the literature. Again, comparing that heuristic construction with our systematic approach would be very interesting.

The next logical step is to apply our methods to the Kerr black hole and higher dimensional (Myers–Perry) generalizations. As a first step, we intend to construct a Killing compatibility complex for the Kerr geometry [2], thus providing a proof of completeness for the list of local gauge-invariants recently proposed in [1].

Once the Killing compatibility complex is known on a given geometry, this information has interesting applications to the symplectic and Poisson structures on the space of solutions of linearized gravity [23, section 5].

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Appendix A. Flat connection form for PDEs of regular finite type

We have chosen to express our definition of a PDE of (regular) finite type (definition 7) directly in terms of a flat connection (definition 6). Elsewhere in the literature, the definition is given in different terms, but the equivalence with the form of a flat connection is well-known, even for nonlinear equations, once the notion of regularity is made precise. For instance, the contents of remark 2.3.3, remark 2.3.6, and example 2.3.17 of [30] concern precisely the equivalence between these two possible definitions.

For the convenience of the reader, we give an elementary proof of this equivalence for linear equations, which is all that we will need. When we speak of equivalence below, we mean in the sense of one operator complexes of definition 1. The length of our proof mostly reflects the amount of notation that we have needed to introduce along the way to make the argument as explicit as possible.

For the following definition, we need to quickly introduce the notion of jets and jet bundles [30, sections 2.1–2]. Given a vector bundle $V \rightarrow M$, the $N$-jet $j^N_x v$ at $x \in M$ of a section $v \in \Gamma(V)$ is the equivalence class of sections that have the same Taylor expansion about $x$ up to order $N$ in local adapted coordinates on $V \rightarrow M$. The definition clearly does not depend on the choice of adapted coordinates. Denote by $J^N_x V$ the vector space of all $N$-jets at $x$ and let $J^N V = \bigsqcup_{x \in M} J^N_x V$ be the $N$-jet bundle of $V$, which can naturally be given the structure of a smooth vector bundle $J^N V \rightarrow M$, with $J^0 V = V$. By throwing away higher terms of Taylor series expansions, we can define natural projections $\pi^N_{N'} : J^N V \rightarrow J^{N'} V$ when $N \geq N'$. By assigning to a section $v \in \Gamma(V)$ its own $N$-jet at each point of $M$, we can define the natural
\textit{N-jet extension} differential operator $J^k : \Gamma(V) \to \Gamma(J^N V)$, which is universal in the sense that for any differential operator $D : \Gamma(V) \to \Gamma(W)$ of order at most $N$, there exists a unique vector bundle map $p^k(D) : J^N V \to W$ such that $D[v] = d(J^k v)$. Extending this notation, we denote by $p^k(D) = p^k(J^k \circ D)$ the $l$-th prolongation of $D$.

\textbf{Definition A.1.} Let $V \to M$ and $W \to M$ be vector bundles and $K : \Gamma(V) \to \Gamma(W)$ a linear differential operator. The PDE $K[v] = 0$ is said to be of \textit{finite type} when (a) locally there exists an integer $N < \infty$, a vector bundle morphism $\kappa : J^k V \to J^{k+1} V$ and a differential operator $\lambda : \Gamma(W) \to \Gamma(J^{N+1} V)$ such that $J^{N+1} v - \kappa(J^k v) = \lambda(K[v])$ for any $v \in \Gamma(V)$, and moreover regular when (b) locally the dimension of the solution space is finite and constant.

\textbf{Proposition A.2.} If a differential operator $K : \Gamma(V) \to \Gamma(W)$ between vector bundles $V \to M$, $W \to M$ defines a PDE of regular finite type $K[v] = 0$, then $K$ is equivalent to a flat connection operator $\nabla$ on some vector bundle $U \to M$.

\textbf{Proof.} Our proof will consist of three steps: (a) equivalence of the original differential operator $K$ to a connection $\nabla$ on $J^k V$ together with a non-differential constraint $E : J^k V \to W'$, (b) equivalence to the restriction $\nabla|_U$ of $\nabla$ to the sub-bundle $U \to J^k V \to M$ satisfying the non-differential constraint $E(\nabla|_U) = 0$, and (c) equivalence to the restriction $\nabla|_U$ of $\nabla$ to the sub-bundle $U \to \nabla|_U M$ spanned by flat sections.

Since all of our definitions and claims are local, we might as well work in local adapted coordinates on $V$, $W$ and any other vector bundles. For instance, we will use coordinates $(x^a)$ on $M$, $(x^a, v^\alpha)$ on $V$ and $(x^a, v^\alpha, v^{\alpha_1}, \ldots, v^{\alpha_N})$ on $J^k V$, such that $v_{\alpha_1}(J^k \phi(x)) = \partial^k v^\alpha(\phi(x))$, where $\partial^k$ stands for all possible independent partial derivatives of order $k$ with respect to the $(v^\alpha)$ coordinates, with similar notations used for other bundles. Also, when there is no confusion, we will denote a general section of a vector bundle $V$ by $v$, a general section of $W$ by $w$, and so on. We will denote a general section of $J^k V$ by $v^{(N)} = (v, v^1, \ldots, v^N)$.

Before proceeding, let us establish some notation. Namely, suppose that $K$ is a differential operator of order $k$, there is a unique non-differential bundle map that factors $K$ through $k$-jets, which we denote by $p^k(K)(J^k v) = K[v]$ and similarly $p^l(K)(J^{k+l} v) = J^l K[v]$. Also, we will need the algebraic operators $\tau_1$ defined by the identity\footnote{The non-triviality of the identity stems from the fact that $\partial^k(\partial^l v) \neq \partial^{k+l} v$, if we do not symmetrize the partial derivatives between the $\partial^k$ and $\partial^l$ operators.} $\partial^k(\partial^l v) = \tau_1(\partial^l v)$, as well as the differential operators $\Delta^k_1$ defined by the identity

\begin{equation}
\Delta^k_1[\partial v - \tau_1(v^1), \ldots, \partial v^{N-1} - \tau_1(v^N), \partial v^N - \tau_1(v^{N+1})] = \partial^k v - \tau_1(v^k), \tag{A.1}\end{equation}

for $k \leq N$. These operators basically encode the identities $\partial^k v = \tau_1(v^1)$, $\partial^k v - \partial^2 v = \partial(\partial v - \tau_1(v^1)) + (\partial v^1 - \tau_1(v^2))$, ....

(a) Essentially, all the information that we will need to establish the first equivalence is contained in the bundle map $\kappa$ and the differential operator $\lambda$ from definition 7. The non-trivial information is contained in the highest order components, $\partial^{N+1} v - \kappa^{N+1}(v, \partial v, \ldots, \partial^N v) = \lambda^{N+1}[K[v]]$, which implies the identity

\begin{equation}
\partial(\partial^{N+1} v - \tau_1(\kappa^{N+1}(v, \partial v, \ldots, \partial^N v))) = \tau_1(\lambda^{N+1}[K[v]]). \tag{A.2}\end{equation}
That last identity will be a crucial piece of our definition of a connection on $J^N V$.
To complete the necessary definitions, we will need the differential operator
\[ P_1 : \Gamma(W) \to \Gamma(T^* M \otimes M J^N V) \]
given by
\[ P_1[w] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \iota_1(\lambda^{N+1}[w]) \end{bmatrix}. \]  
(A.3)

Next, the bundle $W'$ and the bundle map $E : J^N \to W'$ are chosen so that 
\[ \ker E = \pi_0(J^N) \ker p^N(K) \] for some $N'$. Since we are allowed to specify $N$ and $N'$ as we like, we can pick them so that $N' > 0, N > k$. The meaning is that $E$ takes into account all integrability conditions of order $N$ that can be obtained by prolonging the equation $K[v] = 0$ by $N'$ differentiations. By construction, there must exist a differential operator $P_0 : \Gamma(W) \to \Gamma(W')$ such that $P_0 K[v] = E(J^N v)$. Also, since we have presumed that $N > k$, we have $\pi_k(\ker E) \subset \ker p^0(K)$ and there must exist a bundle map $P_0 : W' \to W$ such that $P_0 E = p^0(K) \pi_k$. It remains to define the connection on $J^N V$, which we give by the formula
\[ \begin{bmatrix} v \\ \vdots \\ v^{N-1} \\ v^N \end{bmatrix} \mapsto \begin{bmatrix} \partial v - \iota_1(v^1) \\ \partial v^1 - \iota_1(v^2) \\ \vdots \\ \partial v^{N-1} - \iota_1(v^N) \\ \partial v^N - \iota_1(e^{N+1}(v, v^1, \ldots, v^N)) \end{bmatrix}. \]  
(A.4)

Having introduced all the necessary notation. The desired equivalence is explicitly exhibited by the diagram

where $v^N$ denotes a general section of $T^* M \otimes M J^N V \to M$. To prove that we have an equivalence, we must verify all the conditions required by definition 1. The commutativity of the squares formed by solid arrows follows from direct computations. One is involves only the defining properties of $P_0$ and $P_1$, while the other uses the definitions of $\Delta^N_k$ and $P_0$. 

The remaining checks involve the homotopy corrections:
\[\pi_0^N f^N = \text{id} - 0,\]
\[j^N \pi_0^N = \text{id} + (j^N \pi_0^N - \text{id}) = \text{id} + \Delta_N^N \mathbb{D},\]
\[
\left(\begin{array}{cc}
\text{id} - p^0(K) \Delta_N^N
\end{array}\right) \tilde{P}_0
\left[
\begin{array}{c}
[0, 0] - [P_1, P_0]
\end{array}\right] K =
K = p^0(K) \Delta_N^N (P_1 K) - \tilde{P}_0 (P_0 K)
= K - p^0(K) \Delta_N^N - (\tilde{P}_0 E) f^N
= K - p^0(K) (j^N \pi_0^N - \pi_0^N) f^N - p^0(K) f^N = 0.
\]
(b) The next step is to eliminate the non-differential \(E[v, v', \ldots, v^N] = 0\) constraint. By introducing a sub-bundle \(\tilde{U} \hookrightarrow J^{N'} V \to M\) such that \(\tilde{U} = \ker E\). In general \(\ker E\) need not be a vector bundle (the fiber ranks may be non-constant over \(M\)), hence \(\tilde{U}\) might not exist as a bundle. However, from requirement in definition A.1(b), we know that the dimension of the solution space of \(K[v] = 0\) is finite and locally constant, which by part (a) of our proof also applies to the solution space of \(\mathbb{D}[v^{(N)}] = 0\) under the \(E\)-constraint. In fact, the dimension of the solution space on a neighborhood of \(x \in M\) is bounded from above by \(\dim \ker_{E}\), simply because a \(\mathbb{D}\)-flat section is uniquely determined by its value at any one point. The only reason that an element \(\tilde{u}_e \in \ker_{E}\) might not correspond to a local solution is that there might exist some higher order differential consequence of \(\mathbb{D}[v^{(N)}] = 0\) (or equivalently of \(K[v] = 0\)) that imposes further integrability conditions on \(j^N v\), which \(\tilde{u}_e\) may not satisfy. However, from the theory of formal integrability of PDEs (Cartan–Kuranishi theorem [30, section7.4]), it is well known that past a certain finite differential order \(N'\), no further constraints on \(j^N v\) will appear from considering \(\partial^{N'} K[v] = 0\) or higher order differential consequences. Let us use this order \(N'\) (or any higher one) to influence the definition of the \(E\)-constraint that we introduced in part (a) of the proof. That is, we are free to assume that \(E\) has been chosen such that every element \(\tilde{u}_e \in \ker_{E}\) defines a unique local solution of \(K[v] = 0\) with \(j^N v(x) = \tilde{u}_e\). In other words, \(\dim \ker_{E}\) is equal to the local dimension of the solution space about \(x \in M\). But then, by the regular finite type hypothesis on \(K[v] = 0\), we know that \(\dim \ker_{E}\) is locally constant, meaning that \(\ker E\) is indeed a vector bundle, which we can denote by \(\iota: \tilde{U} \to J^{N'} V \to M\).

\[\text{The existence of such a solution is guaranteed by applying } \mathbb{D}\text{-parallel transport.}\]
Since we are working locally, we are free to presume that there also exists a projection bundle map \( q : J^N V \to \tilde{U} \) such that \( q_\iota = \text{id} \). In the other direction, we have the identity \((\text{id} - \iota q)\iota = 0\), which means that there must exist a bundle map \( h : W' \to J^N V \) such that \( \iota q = \text{id} - h E \). Further, we can define a connection operator \( \tilde{\nabla} \) on \( \tilde{U} \) by the formula

\[
\tilde{\nabla} \tilde{u} = q_1 \nabla_{\iota}(\tilde{u}),
\]

(A.6)

where we have introduced the convenient notation \( q_1 = \text{id} \otimes q : T^* M \otimes_M J^N V \to T^* M \otimes_M \tilde{U} \).

We will use the same convention also for \( E_1 = \text{id} \otimes E \) and \( t_1 = \text{id} \otimes \iota \). We finally have all the ingredients to exhibit the next equivalence

\[
\begin{bmatrix}
\tilde{D} \\
\tilde{E}
\end{bmatrix}
= \begin{bmatrix}
D \\
E
\end{bmatrix}
\begin{bmatrix}
q_1 \\
- q_1 \nabla h
\end{bmatrix}
\begin{bmatrix}
\iota \\
0
\end{bmatrix}
\begin{bmatrix}
\iota_1 \\
0
\end{bmatrix}.
\]

(A.7)

For the commutativity of the solid arrow squares we first need one more identity. Let us define

\[
E' = h_1 E_1 \nabla - \nabla h E
\]

and note that it is a non-differential operator (as follows from the basic properties of the connection operator). From its definition, \( E'(v^{(N)}) = 0 \) is also an integrability condition. But by the discussion from the preceding paragraph, \( E(V^{(N)}) = 0 \) already takes into account all possible integrability conditions. Hence, we must be able to factor \( E' = h'/E \) for some bundle map \( h' : W' \to T^* M \otimes_M W' \), from which follows the identity

\[
h_1 E_1 \nabla = \nabla h E + h'E = (\nabla h + h') E.
\]

(A.9)

Hence,

\[
\begin{bmatrix}
D \\
E
\end{bmatrix} - \begin{bmatrix}
\iota_1 \\
0
\end{bmatrix} \nabla = \begin{bmatrix}
D\iota - (\iota q_1)\nabla \iota \\
E\iota
\end{bmatrix} = \begin{bmatrix}
(h_1 E_1 \nabla)\iota \\
0
\end{bmatrix} = \begin{bmatrix}
(\nabla h + h') (E\iota) \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

\[
\tilde{\nabla} q - [q_1 - q_1 \nabla h] \begin{bmatrix}
D \\
E
\end{bmatrix} = q_1 \nabla (\iota q) - q_1 \nabla (\text{id} - h E) = 0.
\]

The first set of identities involving the homotopy corrections also easily follows from the definition of the \( \iota \) and \( q \) bundle maps. We check the remaining ones by direct computation:
\[
\text{id} - [q_1 - q_1 h] \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \text{id} - q_1 \iota_1 = 0,
\]

\[
\begin{pmatrix}
\text{id} & 0 \\
0 & \text{id}
\end{pmatrix}
\begin{bmatrix}
\iota_1 \\
0
\end{bmatrix}
\begin{pmatrix}
|q_1 - q_1 h| & \frac{\partial}{\partial E} \\
0 & 0
\end{pmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
\text{id} - \iota_1 q_1 & \iota_1 q_1 h - \frac{\partial}{\partial E} \\
0 & \text{id} - \frac{\partial}{\partial E}
\end{bmatrix}
= \begin{bmatrix}
h_1 E_1 & -(h_1 E_1 h) \\
0 & \text{id} - \frac{\partial}{\partial E}
\end{bmatrix}
= \begin{bmatrix}
h_1 E_1 & -(\frac{\partial}{\partial E} + h') \\
0 & \text{id} - \frac{\partial}{\partial E}
\end{bmatrix}
= \begin{bmatrix}
\text{id} & (\frac{\partial}{\partial E} + h') \\
0 & \text{id} - \frac{\partial}{\partial E}
\end{bmatrix}
\begin{bmatrix}
h_1 E_1 & -(\frac{\partial}{\partial E} + h') \\
0 & \text{id} - \frac{\partial}{\partial E}
\end{bmatrix},
\]

where the last factor has the property
\[
\begin{bmatrix}
h_1 E_1 & -(\frac{\partial}{\partial E} + h') \\
0 & \text{id} - \frac{\partial}{\partial E}
\end{bmatrix}
\begin{pmatrix}
\frac{\partial}{\partial E} \\
E
\end{pmatrix}
= \begin{bmatrix}
h_1 E_1 \frac{\partial}{\partial E} - E \frac{\partial}{\partial E} - h' E \\
E \text{id} - h E
\end{bmatrix}
= \begin{bmatrix}
0 \\
\text{id} - (E_1 q_1)
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

(c) At this point, we know that the local solutions of \( K[v] = 0 \) are in bijection with the \( \tilde{D} \)-flat local sections of \( \tilde{U} \to M \). In principle, it is now sufficient to check that \( D \) is flat (if it were not flat, then the rank of \( \tilde{U} \) could not coincide with the dimension of the local solution space of \( K[v] = 0 \), though the two do coincide by construction from part (b) of our proof). However, we will take a slightly indirect route and show a more general result, that will also be referred to in our discussion of the Killing equation in section 3. Namely, provided the local solutions of \( \tilde{D} \tilde{u} = 0 \) span a sub-bundle \( \iota: \tilde{U} \to U \to M \), we will show that the restriction of \( \tilde{D} \to U \to M \) is flat and the original \( \tilde{D} \tilde{u} = 0 \) equation is equivalent to the new \( \tilde{D} \tilde{u} = 0 \) equation.

In our case, from the regular finite type assumption on \( K[v] = 0 \), we know that the local solution space has locally constant (finite) dimension, which is easily seen to be equivalent to the local solutions of \( \tilde{D} \tilde{u} = 0 \) spanning a sub-bundle.

Now, under our hypotheses and since we are working locally, we can choose a frame on \( U \to M \) which corresponds to flat sections \( \tilde{u}_\beta \) on \( \tilde{U} \). Namely, we define \( \iota(\tilde{u}) = \tilde{u}_\beta \tilde{u}_\beta \). Locally, there also exists the projection bundle map \( \tilde{q}: \tilde{U} \to \tilde{U} \to M \), which satisfies \( \tilde{q} \tilde{u} = \text{id} \) and hence acts as \( \tilde{q}(\tilde{u} \tilde{u}_\beta) = \tilde{u} \). Hence, using again the notation \( \iota_1 = 1 \otimes \iota \), we get the identity
\[
\tilde{D} [\iota(\tilde{u})] = d \tilde{u} \tilde{u}_\beta \otimes \tilde{u}_\beta + \tilde{u}_\beta (\tilde{D} \tilde{u}_\beta) = d \tilde{u} \tilde{u}_\beta \otimes \tilde{u}_\beta = \iota_1 (\tilde{D} \tilde{u}),
\]

where \( d \) is simply the exterior derivative acting on the scalars \( \tilde{u}_\beta \) and we have defined \( \tilde{D} \) to act on the frame components of \( \tilde{u} \) as \( (\tilde{D} \tilde{u})^\beta = d \tilde{u} \tilde{u}_\beta \). The operator \( \tilde{D} \) is clearly a flat connection on the bundle \( \tilde{U} \to M \). It remains only to exhibit the equivalence between the equations \( \tilde{D} \tilde{u} = 0 \) and \( D \tilde{u} = 0 \).

As we already discussed in part (b) of our proof, when \( \iota \) is not surjective, there must be some integrability conditions that follow from the differential consequences of \( \tilde{D} \tilde{u} = 0 \). In other words, there exists a differential operator \( \lambda \) such that \( \lambda \tilde{D} \) is a non-differential operator, satisfying \( (\lambda \tilde{D}) \iota_1 = 0 \), with \( \iota: \tilde{U} \to \ker \lambda \tilde{D} \) actually being an isomorphism. Again, as before, this means that there exists an operator \( h \) such that \( \tilde{q} \tilde{q} = h(\lambda \tilde{D}) \). Recalling again the notation, \( \tilde{U}_1 = T^* M \otimes_M \tilde{U} \) and \( U_1 = T^* M \otimes_M U \), as well as \( \tilde{q}_1 = \text{id} \otimes \tilde{q} \) and \( \iota_1 = \text{id} \otimes \iota \). With that in mind, the desired equivalence is explicitly given by the diagram
The arguments to check all the required identities are similar to those in part (b). We check that the solid arrows form commutative squares by direct computation:

\[
\tilde{D}q = q_1 \tilde{D}(i\tilde{q}) = q_1 \tilde{D} - q_1 \tilde{D}h\tilde{\lambda} = q_1 (id - \tilde{D}h\tilde{\lambda}) \tilde{D}, \\
\tilde{D}i = i_1 \tilde{D}.
\]

To check some of the identities with the homotopy corrections, we need one more identity. For ease of notation, define \(\tilde{E} = \tilde{\lambda} \tilde{D} \), which by assumption is a non-differential operator which incorporates all the integrability conditions of the equation \(\tilde{D}u = 0\). Just as in part (b), since all integrability conditions must factor through \(\tilde{E}\), there must exist an operator \(\tilde{h}^1\) such that \(\tilde{h}^1 \tilde{E} - \tilde{D}h \tilde{E} = \tilde{h}' \tilde{E}\), which implies the identity

\[
[h_1 \tilde{E} - (\tilde{D}h + \tilde{h}^1) \tilde{\lambda}] \tilde{D} = 0.
\]  

This concludes the proof. □

Since according to definition A.1 the flat section equation \(\tilde{D}v = 0\) for a flat connection \(\tilde{D}\) is itself of regular finite type (with \(N = 0\)), proposition A.2 shows that definitions 7 and A.1 are clearly equivalent and can be used interchangeably.
The reader might notice that the structure of parts (b) and (c) in the proof of proposition A.2 is rather similar. The reason that we have included both of them in detail is that part (c) can basically be read independently and establishes the following (of course also well-know) more specific result:

**Lemma A.3.** Let $\mathbb{D}$ be a connection on a vector bundle $V$. If the local solutions of the flat section equation $\mathbb{D}v = 0$ span a sub-bundle $W \hookrightarrow V$, then the restriction $\mathbb{D} = \mathbb{D}|_W$ of $\mathbb{D}$ to $W$ is a flat connection on $W$. Moreover, $\mathbb{D}$ is equivalent to $\mathbb{D}$ in the sense of one operator complexes (definition 1).

### Appendix B. Notation Reference

#### B.1. Constant curvature spacetime

| Notation | Description | Reference |
|----------|-------------|-----------|
| $\alpha$ | Curvature constant | Section 3.1 |
| $C_i, C_0 = K$ | Calabi compatibility complex for Killing operator $K$ | (32) |
| $S \otimes T$ | Kulkarni–Nomizu product | (28) |

#### B.2. FLRW spacetimes

| Notation | Description | Reference |
|----------|-------------|-----------|
| $(M, g) = (I \times F, -dt^2 + f^2 g^F)$ | FLRW geometry, with scale factor $f$ | Section 3.2 |
| $\alpha$ | Curvature constant | Section 3.2 |
| $U_a$ | Unit covector normal to $F$ factor | Section 3.2 |
| $R_{abcd}, R_{ab}, R$ | Background Riemann, Ricci and scalar curvatures on $(M, g)$ | (41) |
| $\partial, \tilde{\nabla}$ | Derivative operators extended from $I$ and $F$ to $M$ | Section 3.2 |
| $\tilde{\Delta}, \tilde{\text{div}}, \tilde{\text{tr}}$ | Laplacian, divergence and trace extended from $F$ to $M$ | (53) |
| $v_a = A f U_b + f^2 \tilde{X}_a$ | Covector parametrization, $U^a \tilde{X}_a = 0$ | (34) |
| $h_{ab} = p U_a U_b - 2 f^2 U_a Y_b + f^2 Z_{ab}$ | Symmetric 2-tensor parametrization, $U^a Y_a = 0 = U^a Z_{ab}$ | (36) |
| $K$ | Killing operator on $(M, g)$ | (37) |
| $C_i, C_0 = \tilde{K}$ | Extension of Calabi and Killing operators from $(F, g^F)$ to $M$ | (40) |
| $J, \tilde{J}$ | Operator to extract $A = J \circ K[v]$ | (44), (47), (51) |
| $H, \tilde{H}$ | Factorization of $J, J$ | (49), (48), (52) |
B.3. Schwarzschild–Tangherlini spacetimes

\[(\bar{\mathcal{M}}, \bar{g}) = (\mathcal{M} \times S, g + r^2 \Omega), \bar{\nabla}_\mu\]

\(R_{abcd}, R_{ab}, \bar{\mathcal{R}}\)

Background Riemann, Ricci and scalar curvatures on \((\bar{\mathcal{M}}, \bar{g})\)

\(\nabla_{\bar{\mathcal{M}}}\)

A-shifted background Riemann curvature on \((\bar{\mathcal{M}}, \bar{g})\)

\((\mathcal{S}, \Omega), D_\Lambda\)

Constant curvature factor, covariant derivative extended to \(\mathcal{M}\)

\((\mathcal{M}, g), \nabla_\alpha\)

Radio-temporal factor, covariant derivative extended to \(\mathcal{M}\)

\(R_{abcd}, R_{ab}, \mathcal{R}\)

Background Riemann, Ricci and scalar curvatures on \((\mathcal{M}, g)\)

\(-f(r)\)

The \(\bar{g}_{\alpha\beta}\) metric component

\(f_1(r) = r f'(r), f_2(r) = r' f_1(r)\)

Derivatives of \(f(r)\)

\(M, \Lambda, \alpha\)

Mass, cosmological, curvature constants

\(t_a = -f dt_a\)

Timelike Killing covector on \((\mathcal{M}, g)\)

\(r, \rho_a = dr_a\)

Radial coordinate and covector on \((\mathcal{M}, g)\)

\(v_\mu \to \left[ \frac{m f d t_a + u d r_a}{r (r \Lambda_a)} \right]\)

Covector parametrization on \(\bar{\mathcal{M}}\)

\(h_{\mu\nu} \to \left[ \frac{p r_a r_b - 2 i (\omega w_b) r (r Y_{ab})}{r (r Y_{ab})} \right]\)

Symmetric 2-tensor parametrization on \(\mathcal{M}\)

\(\bar{\mathcal{K}}\)

Killing operator on \((\bar{\mathcal{M}}, \bar{g})\)

\(C_1, C_0 = K\)

Extension of Calabi and Killing operators from \((\bar{S}, \Omega)\) to \(\mathcal{M}\)

\(J_1\)

Operator to extract \(u = J_1 \circ \bar{\mathcal{K}}[v]\)

\(J_2\)

Operator to extract \(\nabla_\alpha \bar{\mathcal{K}}[v] = J_2 \circ \bar{\mathcal{K}}[v]_{ab}\)

\(H_{\alpha\beta}, \tilde{H}_{\alpha\beta}\)

Factorization of \(J_1\)

\(H_{\alpha\beta}, \tilde{H}_{\alpha\beta}\)

Factorization of parts of \(J_2\)

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