ON RELATIVISTIC QUANTUM INFORMATION PROPERTIES OF ENTANGLLED WAVE VECTORS OF MASSIVE FERMIONS

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We study special relativistic effects on the entanglement between either spins or momenta of composite quantum systems of two spin-\(\frac{1}{2}\) massive particles, either indistinguishable or distinguishable, in inertial reference frames in relative motion. For the case of indistinguishable particles, we consider a balanced scenario where the momenta of the pair are well-defined but not maximally entangled in the rest frame while the spins of the pair are described by a one-parameter (\(\eta\)) family of entangled bipartite states. For the case of distinguishable particles, we consider an unbalanced scenario where the momenta of the pair are well-defined and maximally entangled in the rest frame while the spins of the pair are described by a one-parameter (\(\xi\)) family of non-maximally entangled bipartite states. In both cases, we show that neither the spin-spin (\(ss\)) nor the momentum-momentum (\(mm\)) entanglements quantified by means of Wootters’ concurrence are Lorentz invariant quantities: the total amount of entanglement regarded as the sum of these entanglements is not the same in different inertial moving frames. In particular, for any value of the entangling parameters, both \(ss\) and \(mm\)-entanglements are attenuated by Lorentz transformations and their parametric rates of change with respect to the entanglements observed in a rest frame have the same monotonic behavior. However, for indistinguishable (distinguishable) particles, the change in entanglement for the momenta is (is not) the same as the change in entanglement for spins. As a consequence, in both cases, no entanglement compensation between spin and momentum degrees of freedom occurs.

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I. INTRODUCTION

It is known in relativistic thermodynamics that the concept of temperature is observer-dependent because radiation that is perfectly black-body in a given inertial reference frame is not thermal when viewed from a different moving reference frame [1][2]. Therefore, since probability distributions can depend on the frames, many other information theoretic quantities such as the Shannon entropy can exhibit such dependence. Quantum information theory usually involves only a nonrelativistic quantum mechanics. However, relativistic effects are of great importance also in quantum information theory [3]. From a practical point of view, relativistic quantum information is important since it may provide a useful theoretical platform for several possible applications such as quantum-enhanced global positioning [4], quantum clock synchronization [5] and quantum teleportation [6]. From a foundational viewpoint, relativistic extensions of quantum information theory come from quantum cosmology [3]. Specifically, quantum field theory in curved spacetime (black hole physics, in particular) present challenges that everybody who upholds the principle that “information is physical” [7] should respond to. A first consequence of relativity on quantum theory is the existence of an upper bound, the velocity of light, on the speed of propagation of physical effects. A more important consequence of relativity is that there is a hierarchy of dynamical variables [8]; primary variables have relativistic transformation laws that depend only on the Lorentz transformation matrix that acts on the spacetime coordinates. For example, momentum components are primary variables. On the other hand, secondary variables such as spin and polarization have transformation laws that depend not only on the Lorentz transformation matrix, but also on the momentum of the particle. As a consequence, the reduced density matrix for secondary variables, which may be well defined in any coordinate system, has no transformation law relating its components in different Lorentz frames. In relativistic quantum information theory, the notion “spin state of a particle” is meaningless if we don’t specify its complete state, including the momentum variables [9]. It is possible to formally define spin in any Lorentz frame, but there is no relationship between the observable expectation values in different Lorentz frames. Stated otherwise, the answers to such questions, asked in different Lorentz frames, are not related by any transformation group. Under a Lorentz boost, the spin undergoes a Wigner rotation whose direction and magnitude depend on the momentum of the particle. Even if the initial state is a direct product of a function of momentum and a function of spin, the transformed state is not a direct product. Spin and momentum appear to be entangled. This implies that the spin entropy is not a Lorentz scalar and has no invariant meaning in special relativity [9].

Entanglement is one of the key features of quantum mechanics and a deep understanding of its properties and
implementations is essential not only to fundamentally advance our understanding of how Nature works, but also to design more powerful technologies. Entanglement is a property unique to quantum systems. Two systems (microscopic particles or even macroscopic bodies) are said to be quantum entangled if they are described by a joint wave function that cannot be written as a product of wave functions of each of the subsystems (or, for mixed states, if a density matrix cannot be written as a weighted sum of product density matrices). The subsystems can be said not to have a state of their own, even though they may be arbitrarily far apart. The entanglement produces correlations between the subsystems that go beyond what is classically possible. From a relativistic point of view, entanglement is an observer-dependent concept [10]. In particular since Lorentz boosts entangle the spin and momentum degrees of freedom, entanglement may be transferred between them. This may be true for single particles [10] (where entanglement is being considered between degrees of freedom belonging to the same particle), and for pairs [11,12], where the Lorentz boost affects the entanglement between spins. Within such quantum relativistic framework, qubits are realized as discrete degrees of freedom of particles: a qubit can be either a spin of a massive particle or a polarization of a photon. Once the relativistic transformations of the states of massive particles and photons are given, it can be deduced what happens to them when described by observers in relative motion.

In [12], it was stated that the increase in spin entanglement comes at the expense of a loss of momentum entanglement, since the entanglement between all degrees of freedom (spin and momentum) is constant under Lorentz transformations. The misinterpretation of this statement has lead to several misleading remarks in the literature as pointed out in [13]. Actually in [15] it was found that the change in entanglement for the momenta is indeed the same as the change in entanglement for spins and no entanglement transfer occurs. However, there, it was considered a pair of spin-$\frac{1}{2}$ indistinguishable particles with maximally entangled momenta and spins in the rest frame. Hence, the possibility of entanglement transfer between different degrees of freedom is jeopardized by this assumption (it automatically implies that moving from the rest frame, both $ss$ and $mm$-entanglements will diminish). This led us to consider more general scenarios as explained in the following paragraph.

In this article, we investigate special relativistic effects on the entanglement between either spins or momenta of composite quantum system of two spin-$\frac{1}{2}$ massive particles, either indistinguishable or distinguishable, in inertial reference frames in relative motion. Specifically, for the case of indistinguishable particles, we consider a balanced scenario where the momenta of the pair are well-defined but not maximally entangled in the rest frame while the spins of the pair are described by a one-parameter ($\eta$) family of non-maximally entangled bipartite states. In particular, when $\eta = 0$ (symmetry in the spin-wave function and antisymmetry in the momentum-wave function) or $\eta = 1$ (antisymmetry in the spin-wave function and symmetry in the momentum-wave function), we recover the main result appeared in [15]. For the case of distinguishable particles, we take into consideration an unbalanced scenario where the momenta of the pair are well-defined and maximally entangled in the rest frame while the spins of the pair are described by a one-parameter ($\xi$) family of non-maximally entangled bipartite states. Furthermore, in both cases, we show that neither the spin-spin ($ss$) nor the momentum-momentum ($mm$) entanglements quantified by means of Wooters’ concurrence are Lorentz invariant quantities: the total amount of entanglement regarded as the sum of these entanglements is not the same in different inertial moving frames. In particular, for any value of the entangling parameters, both $ss$ and $mm$-entanglements are attenuated by Lorentz transformations and their parametric rates of change with respect to the entanglements observed in a rest frame have the same monotonic behavior. However, for indistinguishable (distinguishable) particles, the change in entanglement for the momenta is (is not) the same as the change in entanglement for spins. We conclude that in both cases no entanglement compensation between spin and momentum degrees of freedom occurs.

The layout of the article is as follows. In Section II, we present preliminary material on relativistic Lorentz transformations for quantum wave-vectors. In Section III, we define the reduced density matrices of the considered composite quantum systems of two spin-$\frac{1}{2}$ particles (either indistinguishable or distinguishable) in a rest frame and analyze the action of relativistic Lorentz boosts on them. In Section IV, we present the relativistic effects on the entanglement of a composite system of two indistinguishable particles (two electrons with wave-vector $|\Psi_{ee}\rangle$) with a balanced (but not maximal) amount of entanglement in the rest frame between momentum and spin. In Section V, we analyze the relativistic effects on the entanglement of a composite system of two distinguishable particles (an electron and a muon with wave-vector $|\Psi_{em}\rangle$) in which we consider an unbalanced scenario. Our concluding remarks appear in Section VI.

**II. LORENTZ TRANSFORMATIONS FOR QUANTUM WAVE-VECTORS**

In [14], Wigner showed that quantum states of relativistic particles are given by unitary irreducible representations of the Poincaré group, the group of translations and Lorentz transformations (boosts) in the Minkowski space. From [14] it is also clear that finite dimensional representations of Lorentz boosts are non-unitary. However, special relativity requires that the physics of quantum states of relativistic particles should not depend on the arbitrary inertial reference
frame from which the states are observed. Thus, we should expect the states to transform unitarily from one inertial reference frame to another. The solution to this paradox resides in the fact that in relativistic quantum mechanics, the creation and annihilation operators, as well as the associated mode functions for the quantum field that creates a given state, transform under Lorentz transformations by local unitary spin-$j$ representations of the three-dimensional rotation group. The key ingredient of such relativistic transformations is the Wigner rotation, a rotation in the rest frame of the particle that leaves the rest momentum invariant, which restores unitarity in the transformations between relativistic single and multi-particle quantum states. Given the explicit expression of the Wigner rotation, the transformation properties of entangled quantum states observed from two inertial reference frames moving with constant relative velocity can be described.

An arbitrary transformation $L (\Lambda, a)$ of the Poincaré group (or, inhomogeneous Lorentz group), where $\Lambda$ denotes a Lorentz transformation and $a$ a constant vector, relates the coordinates $x^\mu$ and $x'^\mu$ of two inertial reference frames $S$ and $S'$ [17].

$$x^\mu \rightarrow x'^\mu \equiv L (\Lambda, a) x^\mu \defeq \Lambda^\mu_\nu x^\nu + a^\mu. \quad (1)$$

The transformation $L (\Lambda, a)$ satisfy the composition law,

$$L (\bar{\Lambda}, \bar{a}) L (\Lambda, a) = L (\bar{\Lambda} \Lambda, \bar{\Lambda} a + \bar{a}), \quad (2)$$

where the bar in (2) is used just to distinguish one Lorentz transformation from the other. Moreover, $L (\Lambda, a)$ induces a linear unitary transformation $U (\Lambda, a)$ on quantum mechanical wave-vectors $|\Psi\rangle$,

$$|\Psi\rangle \rightarrow |\Psi'\rangle \equiv U (\Lambda, a) |\Psi\rangle, \quad (3)$$

where $U (\Lambda, a)$ satisfies the following composition rule,

$$U (\bar{\Lambda}, \bar{a}) U (\Lambda, a) = U (\bar{\Lambda} \Lambda, \bar{\Lambda} a + \bar{a}). \quad (4)$$

For the sake of clarity, let us focus on single particle states and for a more general approach we refer to [17]. Assuming to consider only transformations with $a^\mu = 0$ in the homogeneous Lorentz group, $|\Psi'\rangle$ in (3) for a single particle state reads,

$$|\Psi\rangle \rightarrow |\Psi'\rangle \equiv U (\Lambda) |\Psi\rangle \defeq \sqrt{\frac{(Ap)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}^{(j)} (W (\Lambda, p)) |\Psi_{\Lambda p, \sigma'}\rangle, \quad (5)$$

where $p$ and $\sigma$ labels the momentum and the spin (or helicity for massless particles) degrees of freedom, respectively. The coefficients $D_{\sigma'\sigma}^{(j)} (W (\Lambda, p))$ provide a representation of the Wigner rotation $W (\Lambda, p)$ belonging to the so-called Wigner’s little group for angular-$j$ particles (for instance, $j = \frac{1}{2}$ for spin-$\frac{1}{2}$ particles). The Wigner’s little group element $W (\Lambda, p)$ is defined as,

$$W (\Lambda, p) \defeq L^{-1} (Ap) \Lambda L (p), \quad (6)$$

where $p^\mu \defeq (\vec{p}, p^0)$ with,

$$\vec{p} = \frac{m \vec{\nu}}{\sqrt{1 - (\frac{\nu}{c})^2}} \quad \text{and} \quad p^0 = E = \frac{mc}{\sqrt{1 - (\frac{\nu}{c})^2}}. \quad (7)$$

and where $(Ap)^\mu \defeq (\vec{p}_\Lambda, (Ap)^0)$ with $\mu = 0, 1, 2, 3$. The quantity $L (p)$ is the standard Lorentz transformation such that $p^\mu = L^\mu_k k^\nu$ where $k^\nu = (0, 0, 0, m)$ is the four-momentum in the rest frame of the particle being considered. Stated otherwise, $L (p)$ takes a massive particle from rest to a 4-momentum $p$.

For massive ($m > 0$) spin-$\frac{1}{2}$ particle states it turns out that $D^{(1/2)} (W (\Lambda, p))$ in (3) reads [18],

$$D^{(1/2)} (W (\Lambda, p)) = D^{(1-1/2)} (L (Ap)) D^{(1/2)} (\Lambda) D^{(1/2)} (L (p)). \quad (8)$$

Consider an arbitrary boost characterized by the velocity $\vec{\nu} = v \hat{e}$ with $\hat{e}$ the normal vector in the boost direction. Then, omitting tedious technical details that can be found in [13] [18], it turns out that the explicit expression of $D^{(1/2)} (W (\Lambda, p))$ reads,

$$D^{(1/2)} (W (\Lambda, p)) = \cos \frac{\Omega \vec{p}}{2} + i (\vec{\sigma} \cdot \hat{u}) \sin \frac{\Omega \vec{p}}{2}, \quad (9)$$
where,

\[
\cos \frac{\Omega \delta}{2} = \frac{\cosh \frac{\alpha}{2} \cosh \frac{\delta}{2} + \sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} (\hat{e} \cdot \hat{p})}{\left[ \frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta + \frac{1}{2} (\hat{e} \cdot \hat{p}) \sinh \alpha \sinh \delta \right]^{\frac{1}{2}},}
\]

and,

\[
\hat{n} \sin \frac{\Omega \delta}{2} = \frac{\sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} (\hat{e} \times \hat{p})}{\left[ \frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta + \frac{1}{2} (\hat{e} \cdot \hat{p}) \sinh \alpha \sinh \delta \right]^{\frac{1}{2}}.}
\]

The quantities \( \alpha \) and \( \delta \) in (10) and (11) are such that,

\[
\cosh \alpha = \frac{1}{\sqrt{1 - \beta^2}} \text{ and } \cosh \delta = \frac{p^0}{m},
\]

respectively, where \( \beta \equiv \frac{p}{c} \) with \( c \) the speed of light. The non-relativistic limit is recovered for \( \beta = 0 \). A more detailed description of relativistic transformations of either massive or massless arbitrary multi-particle quantum states can be found in [11] [17].

### III. General Framework: Two-Particles Composite Quantum Systems

In this Section, we study the relativistic action of Lorentz boosts on the density operators of the composite quantum systems being considered in a rest frame \( S \).

We consider two spin-\( \frac{1}{2} \) particles \( A \) and \( B \), either indistinguishable or distinguishable, with positive mass. We focus on two-particle wave-vectors that in a given inertial reference frame at rest \( S \) may be written as,

\[
|\Psi\rangle_S \equiv |\Psi_{pA, \sigma A; pB, \sigma B}\rangle = |\vec{p}_A, \sigma_A; \vec{p}_B, \sigma_B\rangle \overset{\text{def}}{=} |\vec{p}_A, \vec{p}_B\rangle \otimes |\sigma_A, \sigma_B\rangle,
\]

where \( p \) and \( \sigma \) denote the momentum and spin degrees of freedom of each particle. Following [15], we assume that the momentum of particle \( A \) is concentrated around \( \vec{p}_A \) and that of particle \( B \) around \( \vec{p}_B \). Thus, the state for the momenta of the composite system \( A + B \) is a product state that reads,

\[
|\vec{p}_A, \vec{p}_B\rangle = |\vec{p}_A\rangle_A \otimes |\vec{p}_B\rangle_B.
\]

Furthermore, \( |\sigma_A, \sigma_B\rangle \) represents a state for the spins of the two particles. In a different reference frame \( S' \) in relative motion with respect to the rest frame \( S \), the composite wave-vector \( |\Psi\rangle_S \) becomes

\[
|\Psi\rangle_S \overset{S \rightarrow S'}{\rightarrow} |\Psi\rangle_{S'} = \sum_{\sigma_A', \sigma_B'} \mathcal{D}_{\sigma_A' \sigma_B; \sigma A, \sigma B}^{(1/2)} (W (\Lambda, pA)) \mathcal{D}_{\sigma_B' \sigma B; \sigma A, \sigma B}^{(1/2)} (W (\Lambda, pB)) |\Lambda \vec{p}_A, \sigma_A'; \Lambda \vec{p}_B, \sigma_B';\rangle
\]

\[
= |\vec{p}_A, \vec{p}_B\rangle^\Lambda \otimes \mathcal{D}_{\sigma_A' \sigma_B'}^{(1/2)} (W (\Lambda, pA)) \mathcal{D}_{\sigma_B' \sigma B}^{(1/2)} (W (\Lambda, pB)) |\sigma_A, \sigma_B\rangle,
\]

where \( \Lambda \) is the Lorentz boost between the two reference frames. The transformed state for the momenta of the two particles is given by,

\[
|\vec{p}_A, \vec{p}_B\rangle \overset{S \rightarrow S'}{\rightarrow} |\vec{p}_A, \vec{p}_B\rangle^\Lambda = (|\vec{p}_A\rangle_A \otimes |\vec{p}_B\rangle_B)^\Lambda = |\vec{p}_A\rangle_A^\Lambda \otimes |\vec{p}_B\rangle_B^\Lambda.
\]

Moreover, the transformed state for the spins reads,

\[
|\sigma_A, \sigma_B\rangle \overset{S \rightarrow S'}{\rightarrow} \mathcal{D}_{\sigma_A' \sigma_B; \sigma A, \sigma B}^{(1/2)} (W (\Lambda, pA)) \mathcal{D}_{\sigma_B' \sigma B; \sigma A, \sigma B}^{(1/2)} (W (\Lambda, pB)) |\sigma_A, \sigma_B\rangle,
\]

where \( W (\Lambda, p) \) is the Wigner rotation for the Lorentz transformation \( \Lambda \) and \( p^\mu = (\vec{p}, m) \) with \( m > 0 \). In the case under investigation, \( \mathcal{D}_{\sigma_A' \sigma_B; \sigma A, \sigma B}^{(1/2)} (W (\Lambda, p)) \) is a \( 2 \times 2 \) unitary (rotation) matrix.

In the rest of the manuscript, we focus our attention on the quantum Lorentz transformation properties of the following two-particle quantum mechanical state vector \( |\Psi\rangle \) that, when viewed from the reference rest frame \( S \), reads

\[
|\Psi\rangle_S \equiv |\Psi\rangle = \frac{1}{\sqrt{2}} [ |\vec{p}_{A1},\vec{p}_{B1}\rangle \otimes |0\rangle_S + |\vec{p}_{A2},\vec{p}_{B2}\rangle \otimes |0\rangle_S ],
\]
where the state vectors $|0\rangle_S$ and $|0\rangle_S'$ (with $|0\rangle_S \neq |0\rangle_S'$, in general) describe the state of spins of the massive fermions.

We stress that one of the main consequences of the spin-statistics theorem [19] is that the wave-function (both spin and space parts) of a system of identical (indistinguishable) half-integer spin particles changes sign when two particles are swapped. This can be achieved in two ways: either the space wave-function is symmetric and the spin wave-function is antisymmetric, or the space wave-function antisymmetric and the spin wave-function symmetric. Fermions are particles whose wave-function is antisymmetric under exchange. Observe that no measurement can distinguish between identical particles so we must always have this exchange symmetry in the wave-function of identical particles. For distinguishable particles that exhibit different intrinsic physical properties, no exchange symmetry needs to hold.

From [18] we imply that just two pairs of momentum values ($\vec{p}_{A1}, \vec{p}_{B1}$) and ($\vec{p}_{A2}, \vec{p}_{B2}$) are needed to fully characterize the momentum degrees of freedom of the composite system under investigation. As pointed out earlier, we also consider the working hypothesis that such momenta are concentrated closely enough around their distinct values so that this allows us to use orthogonal state vectors for distinguishable concentrations. Within such approximated scenario, we can employ a single Wigner rotation for each concentration [13].

In a reference frame $S'$ in relative motion with respect to the rest frame $S$, the quantum Lorentz transformation of the wave-vector $|\Psi\rangle$ reads

$$|\Psi\rangle \xrightarrow{S \rightarrow S'} |\Psi\rangle_{S'} \equiv |\Psi'\rangle = \frac{1}{\sqrt{2}} \left[ |\vec{p}_{A1}, \vec{p}_{B1}\rangle^\Lambda \otimes |1\rangle_{S'} + |\vec{p}_{A2}, \vec{p}_{B2}\rangle^\Lambda \otimes |2\rangle_{S'} \right],$$

(19)

where the transformed momenta state vector reads,

$$|\vec{p}_{A1}, \vec{p}_{B1}\rangle^\Lambda \equiv |\Lambda \vec{p}_{A1}, \Lambda \vec{p}_{B1}\rangle,$$

(20)

and the transformed spin states $|1\rangle_{S'}$ and $|2\rangle_{S'}$ are given by,

$$|1\rangle_{S'} \equiv |1\rangle \overset{D_{A}^{(1/2)}}{\rightarrow} (W (\Lambda, p_{A1})) D_{B}^{(1/2)} (W (\Lambda, p_{B1})) |0\rangle_{S} = D_{A} (p_{A1}) D_{B} (p_{B1}) |0\rangle_{S},$$

(21)

and,

$$|2\rangle_{S'} \equiv |2\rangle \overset{D_{A}^{(1/2)}}{\rightarrow} (W (\Lambda, p_{A2})) D_{B}^{(1/2)} (W (\Lambda, p_{B2})) |0\rangle_{S} = D_{A} (p_{A2}) D_{B} (p_{B2}) |0\rangle_{S},$$

(22)

respectively. Substituting (20), (21) and (22) into (19), the Lorentz-transformed density operator becomes

$$\hat{\rho}' \equiv |\Psi\rangle \langle \Psi| \xrightarrow{S \rightarrow S'} \hat{\rho}' \equiv |\Psi'\rangle \langle \Psi'| = U (\Lambda) \hat{\rho} U^\dagger (\Lambda),$$

(23)

that is,

$$\hat{\rho}' = \frac{1}{2} \left[ |\Lambda \vec{p}_{A1}, \Lambda \vec{p}_{B1}\rangle \langle \Lambda \vec{p}_{A1}, \Lambda \vec{p}_{B1}| \otimes |1\rangle \langle 1| + |\Lambda \vec{p}_{A1}, \Lambda \vec{p}_{B1}\rangle \langle \Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B2}| \otimes |1\rangle \langle 2| + |\Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B2}\rangle \langle \Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B2}| \otimes |2\rangle \langle 1| + |\Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B2}\rangle \langle \Lambda \vec{p}_{A1}, \Lambda \vec{p}_{B1}| \otimes |2\rangle \langle 2| \right],$$

(24)

where $|1\rangle \equiv |1\rangle_{S}$ and $|2\rangle \equiv |2\rangle_{S}$. The reduced spin density operator is obtained from $\hat{\rho}'$ by tracing over the momentum degrees of freedom and it reads

$$\hat{\rho}'_{\text{spin}} \overset{\text{def}}{=} \text{Tr}_{\text{momentum}} (\hat{\rho}') = \frac{1}{2} \left[ \langle \Lambda \vec{p}_{A1}, \Lambda \vec{p}_{B1}| \langle \Lambda \vec{p}_{A1}, \Lambda \vec{p}_{B1}| \langle 1\rangle \langle 1| + \langle \Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B1}| \langle \Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B1}| \langle 1\rangle \langle 2| + \langle \Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B2}| \langle \Lambda \vec{p}_{A1}, \Lambda \vec{p}_{B1}| \langle 2\rangle \langle 1| + \langle \Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B2}| \langle \Lambda \vec{p}_{A2}, \Lambda \vec{p}_{B2}| \langle 2\rangle \langle 2| \right]$$

$$= \frac{1}{2} \left[ |1\rangle \langle 1| + |2\rangle \langle 2| \right],$$

(25)

with $|1\rangle \equiv |1\rangle_{S'}$ and $|2\rangle \equiv |2\rangle_{S'}$. Similarly, the reduced momentum density operator is obtained from $\hat{\rho}'$ by tracing over the spin degrees of freedom and it reads,

$$\hat{\rho}'_{\text{momentum}} \overset{\text{def}}{=} \text{Tr}_{\text{spin}} (\hat{\rho}') = \frac{1}{2} \left[ \langle 1\rangle \langle 1| + \langle 1\rangle \langle 2| + \langle 2\rangle \langle 1| + \langle 2\rangle \langle 2| \right]$$

(26)

The reduced density operators $\hat{\rho}'_{\text{spin}}$ and $\hat{\rho}'_{\text{momentum}}$ in the moving reference frame $S'$ are the analogues of $\hat{\rho}_{\text{spin}}$ and $\hat{\rho}_{\text{momentum}}$ in the rest reference frame $S$, respectively.
\[ \mathcal{C}_{\text{spin}}(\eta) \equiv \mathcal{C}_{\text{momentum}}(\eta) \text{ vs. } \eta, \ x \equiv \eta : \varphi = 0 \text{ (dash)}, \ \varphi = \frac{\pi}{10} \text{ (thin solid)}, \ \varphi = \frac{\pi}{8} \text{ (thick solid)}. \]

IV. INDISTINGUISHABLE PARTICLES

In this Section, we study the relativistic effects on the entanglement of a composite system of two spin-\(\frac{1}{2}\) indistinguishable massive particles (two electrons in the state \(|\Psi_{ee}\rangle\) in the rest frame \(S\)) with a balanced (but not maximal) amount of entanglement in the rest frame between momentum and spin. We assume that the wave-vector \(|\Psi_S\rangle\) in the rest frame \(S\) is given by,

\[ |\Psi_{ee}\rangle = |\Psi_S\rangle \text{ def } = \frac{1}{\sqrt{2}} |\vec{p}_{A1}, \vec{p}_{B1}\rangle \left[ \sqrt{\eta} |\phi_-\rangle + \sqrt{1-\eta} |\phi_+\rangle \right] + \frac{1}{\sqrt{2}} |\vec{p}_{A2}, \vec{p}_{B2}\rangle \left[ \sqrt{\eta} |\phi_-\rangle - \sqrt{1-\eta} |\phi_+\rangle \right], \tag{27} \]

where \(|\phi_\pm\rangle\) are the maximally entangled Bell-states,

\[ |\phi_\pm\rangle \text{ def } = \frac{|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm 1 \\ 0 \end{pmatrix}, \tag{28} \]

and \(\eta\) is a real parameter such that \(0 \leq \eta \leq 1\). Furthermore, we assume that

\[ \vec{p}_{A1} = -\vec{p}_{A2}, \ \vec{p}_{B1} = -\vec{p}_{B2} \text{ and, } \vec{p}_{A1} = \vec{p}_{B2}, \tag{29} \]

where \(|\vec{p}_{A1}\rangle\) and \(|\vec{p}_{A2}\rangle\) are the eigenvectors of \(\tilde{\sigma}_z^{(A)}\) with eigenvalues +1 and −1, respectively, while \(|\vec{p}_{B1}\rangle\) and \(|\vec{p}_{B2}\rangle\) are the eigenvectors of \(\tilde{\sigma}_z^{(B)}\) with eigenvalues −1 and +1, respectively. Thus, we have

\[ |\vec{p}_{A1}\rangle = |0\rangle_A \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |\vec{p}_{A2}\rangle = |1\rangle_A \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ |\vec{p}_{B1}\rangle = |1\rangle_B \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and, } |\vec{p}_{B2}\rangle = |0\rangle_B \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{30} \]

Observe that from (27) and (30), when \(\eta = 0\) and \(\eta = 1\), \(|\Psi_S\rangle\) in (27) reduces to

\[ |\Psi_+\rangle \text{ def } = \frac{1}{\sqrt{2}} |10\rangle |\phi_+\rangle - \frac{1}{\sqrt{2}} |01\rangle |\phi_+\rangle \text{ and, } |\Psi_-\rangle \text{ def } = \frac{1}{\sqrt{2}} |10\rangle |\phi_-\rangle + \frac{1}{\sqrt{2}} |01\rangle |\phi_-\rangle, \tag{31} \]

respectively. Using (27) and (30), it follows that the reduced spin density operator in the rest frame \(S\) becomes (for further details, see Appendix A),

\[ \rho^{(S)}_{\text{spin}} = \frac{1}{4} \left[ I_{2\times2}^{(A)} \otimes I_{2\times2}^{(B)} + (1 - 2\eta) \sigma_x^{(A)} \otimes \sigma_x^{(B)} + (1 - 2\eta) \sigma_y^{(A)} \otimes \sigma_y^{(B)} - \sigma_z^{(A)} \otimes \sigma_z^{(B)} \right], \tag{32} \]
that is,
\[ \rho_{\text{spin}}^{(S)} \overset{\text{def}}{=} \text{Tr}_{\text{momentum}} (|\Psi\rangle \langle \Psi|) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 - 2\eta & 0 \\ 0 & 1 - 2\eta & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (33)

We quantify the entanglement between the spin degrees of freedom of particle \( A \) and \( B \) by means of Wootter’s concurrence [20],
\[ C\left( \rho_{\text{spin}}^{(S)} \right) \overset{\text{def}}{=} \max \{ 0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \}. \] (34)

Denoting \( \rho_{\text{spin}}^{(S)} \equiv \rho_{\text{spin}} \), the non-negative real numbers \( \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \} \) denote the square roots of the eigenvalues of the non-Hermitian matrix \( \rho_{\text{spin}} \rho_{\text{spin}}^{*} \) where \( \rho_{\text{spin}}^{*} \) is the “time-reversed” matrix defined as,
\[ \rho_{\text{spin}}^{*} \equiv \left( \sigma_y^{(A)} \otimes \sigma_y^{(B)} \right) \rho_{\text{spin}} \left( \sigma_y^{(A)} \otimes \sigma_y^{(B)} \right)^{\dagger}, \] (35)
and \( \rho_{\text{spin}}^{*} \) is the complex conjugate of \( \rho_{\text{spin}} \). It turns out that the concurrence \( C\left( \rho_{\text{spin}}^{(S)} \right) \equiv C_{\text{spin}}^{(S)} (\eta, \varphi) \) reads,
\[ C_{\text{spin}}^{(S)} (\eta, \varphi) = |(1 - 2\eta)|. \] (36)

Following the same line of reasoning, it can also be shown that the reduced momentum density operator is given by,
\[ \rho_{\text{momentum}}^{(S)} \overset{\text{def}}{=} \frac{1}{4} \left[ I_{2 \times 2} \otimes I_{2 \times 2} + (2\eta - 1) \tilde{\sigma}_x^{(A)} \otimes \tilde{\sigma}_x^{(B)} + (2\eta - 1) \tilde{\sigma}_y^{(A)} \otimes \tilde{\sigma}_y^{(B)} - \tilde{\sigma}_z^{(A)} \otimes \tilde{\sigma}_z^{(B)} \right], \] (37)
that is,
\[ \rho_{\text{momentum}}^{(S)} = \text{Tr}_{\text{spin}} (|\Psi\rangle \langle \Psi|) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\eta - 1 & 0 \\ 0 & 2\eta - 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (38)

The concurrence of \( \rho_{\text{momentum}}^{(S)} \) in \( \text{38} \) becomes,
\[ C_{\text{momentum}}^{(S)} (\eta, \varphi) = |(1 - 2\eta)|. \] (39)

From \( \text{36} \) and \( \text{39} \), we conclude that we are considering a balanced but not maximal initial scenario as far as the entanglement in spins and momenta concern.

In a moving reference frame \( S' \) where \( \vec{p}_{A_1}, \vec{p}_{A_2}, \vec{p}_{B_1}, \) and \( \vec{p}_{B_2} \) are along the \( \hat{x} \)-axis and the quantum Lorentz transformations are along the \( \hat{y} \)-direction with velocity \( \vec{v} \), it turns out that the Lorentz-transformed spin density operator in \( \text{25} \) reads,
\[ \rho_{\text{spin}}^{(S')} = \frac{1}{4} \left[ I_{2 \times 2} \otimes I_{2 \times 2} + (2\eta - 1) \cos 2\varphi \tilde{\sigma}_x^{(A)} \otimes \tilde{\sigma}_x^{(B)} + (1 - 2\eta) \cos 2\varphi \tilde{\sigma}_y^{(A)} \otimes \tilde{\sigma}_y^{(B)} - \tilde{\sigma}_z^{(A)} \otimes \tilde{\sigma}_z^{(B)} \right], \] (40)
that is,
\[ \rho_{\text{spin}}^{(S')} (\eta, \varphi) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (1 - 2\eta) \cos 2\varphi & 0 \\ 0 & (1 - 2\eta) \cos 2\varphi & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (41)

The concurrence \( C_{\text{spin}}^{(S')} (\eta, \varphi) \) becomes,
\[ C_{\text{spin}}^{(S')} (\eta, \varphi) = |(1 - 2\eta) \cos 2\varphi|. \] (42)

Similarly, the Lorentz-transformed momentum density operator in \( \text{26} \) is given by,
\[ \rho_{\text{momentum}}^{(S')} = \frac{1}{4} \left[ I_{2 \times 2} \otimes I_{2 \times 2} + (2\eta - 1) \cos 2\varphi \tilde{\sigma}_x^{(A)} \otimes \tilde{\sigma}_x^{(B)} + (2\eta - 1) \cos 2\varphi \tilde{\sigma}_y^{(A)} \otimes \tilde{\sigma}_y^{(B)} - \tilde{\sigma}_z^{(A)} \otimes \tilde{\sigma}_z^{(B)} \right], \] (43)
that is,

\[ \rho_{\text{momentum}}^{(s')} (\eta, \varphi) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & (2\eta - 1) \cos 2\varphi & 0 \\ 0 & (2\eta - 1) \cos 2\varphi & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (44)

The concurrence \( C_{\text{momentum}}^{(s')} (\eta, \varphi) \) is given by,

\[ C_{\text{momentum}}^{(s')} (\eta, \varphi) = |(1 - 2\eta) \cos 2\varphi|. \] (45)

Thus, from (42) and (45), we uncover that in the case of identical spin-\( \frac{1}{2} \) particles,

\[ C_{\text{spin}}^{(s')} (\eta, \varphi) = C_{\text{momentum}}^{(s')} (\eta, \varphi). \] (46)

We stress that when \( \eta = 0 \) (symmetry in the spin-wave function and antisymmetry in the momentum-wave function; first Eq. in [31]) or \( \eta = 1 \) (antisymmetry in the spin-wave function and symmetry in the momentum-wave function; second Eq. in [31]), we recover the main result appeared in [15]. Finally, we emphasize that for indistinguishable particles, the change in entanglement for the momenta is the same as the change in entanglement for spins (see Eq. [46], Figure 1 and Figure 2).

V. DISTINGUISHABLE PARTICLES

In this Section, we present a detailed study of relativistic effects on the entanglement of distinguishable particles (an electron and a muon in the state \( |\Psi_{e\mu}\rangle \) in the rest frame \( S \)) where we consider an unbalanced scenario where the momenta of the pair are well-defined and maximally entangled in the rest frame while the spins of the pair are described by a one-parameter (\( \xi \)) family of non-maximally entangled bipartite states.
A. Spin density operators in inertial reference frames in relative motion

We assume that the state vector $|0\rangle_{S} = |0\rangle_{S} \equiv |0\rangle$ in (18) that describes the state of spins of the massive fermions is given by,

$$|0\rangle_{S} \equiv |0\rangle = |\psi_{\pm} (\xi)\rangle \overset{\text{def}}{=} \sqrt{1 - \xi} \left| \uparrow \downarrow \right\rangle \pm \sqrt{\xi} \left| \downarrow \uparrow \right\rangle,$$

with $0 < \xi < 1$ and where in the limiting case of $\xi = \frac{1}{2}$ we recover standard maximally entangled pure Bell states. As a side remark, notice that the matrix representation of the density operator to $\left| \xi \right\rangle < \left| \xi \right\rangle$ with $0 \leq \xi < 1$ is given by,

$$\hat{\rho}_{\psi_{\pm}} (\xi) \overset{\text{def}}{=} |\psi_{\pm} (\xi)\rangle \langle \psi_{\pm} (\xi)| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - \xi & \pm \sqrt{\xi (1 - \xi)} & 0 \\ 0 & \pm \sqrt{\xi (1 - \xi)} & \xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In order to make as clear as possible the effect of quantum Lorentz transformation on quantum state vectors, it is convenient to re-express $\hat{\rho}_{\psi_{\pm}} (\xi)$ in terms of Pauli matrices and study the effect of Wigner rotations used to construct the Lorentz transformations rotate Pauli’s operators. It turns out that in terms of Pauli matrices, $\hat{\rho}_{\psi_{\pm}} (\xi)$ in (48) may be rewritten as (see Appendix A),

$$\hat{\rho}_{\psi_{\pm}} (\xi) = \frac{1}{4} \left[ I_{2 \times 2} \otimes I_{2 \times 2} + (1 - 2\xi) \sigma_{z} (A) \otimes I_{2 \times 2} - (1 - 2\xi) I_{2 \times 2} \otimes \sigma_{z} (B) \pm 2 \sqrt{\xi (1 - \xi)} \sigma_{y} (A) \otimes \sigma_{y} (B) \right],$$

where $(\sigma_{x} (A), \sigma_{y} (A), \sigma_{z} (A))$ and $(\sigma_{x} (B), \sigma_{y} (B), \sigma_{z} (B))$ denote the Pauli matrices for the spin of particle A and B, respectively.

As a working hypothesis, we assume that,

$$\vec{p}_{A_{1}} = -\vec{p}_{A_{2}} \text{ and } \vec{p}_{B_{1}} = -\vec{p}_{B_{2}} \text{ with } \vec{p}_{A_{1}} = -a \vec{p}_{B_{2}},$$

(50)

where $\vec{p}_{A_{1}}, \vec{p}_{A_{2}}, \vec{p}_{B_{1}}$ and $\vec{p}_{B_{2}}$ are along the $\hat{x}$-axis and the quantum Lorentz transformations are along the $\hat{y}$-direction with velocity $\vec{v}$. The constant real coefficient $a$ must be such that $\varphi_{\text{electron}} = \varphi_{\text{muon}} \equiv \varphi$. In the case under investigation,

$$\vec{p}_{A_{1}} = -a \vec{p}_{B_{2}} \Rightarrow p_{A_{1}} = a p_{B_{2}} \iff \gamma (v_{A_{1}}) m_{e} v_{A_{1}} = a \gamma (v_{B_{2}}) m_{\mu} v_{B_{2}},$$

(51)

that is (setting the speed of light equal to one, $c = 1$),

$$\frac{m_{e} v_{A_{1}}}{\sqrt{1 - v_{A_{1}}^{2}}} = \frac{a m_{\mu} v_{B_{2}}}{\sqrt{1 - v_{B_{2}}^{2}}}.$$  

(52)

After some algebraic manipulations, we obtain

$$v_{A_{1}} = \frac{a m_{\mu} v_{B_{2}}}{\sqrt{m_{e}^{2} + (a^{2} m_{\mu}^{2} - m_{e}^{2}) v_{B_{2}}^{2}}} = \frac{a m_{\mu}}{m_{e}} \frac{1}{\sqrt{1 + \frac{a^{2} m_{\mu}^{2} - m_{e}^{2}}{m_{e}^{2}} v_{B_{2}}^{2}}}.$$  

(53)

In the case being considered, following the line of reasoning presented in [13], the Wigner rotation angle $\varphi$ reads,

$$\tan \varphi = \frac{\sinh \alpha \sinh \delta}{\cosh \alpha + \cosh \delta},$$

(54)

where,

$$\cosh \alpha = \frac{1}{\sqrt{1 - (\frac{v_{\text{boost}}}{c})^{2}}} = \frac{1}{\sqrt{1 - v_{\text{boost}}^{2}}} \text{ and, } \cosh \delta = \frac{p_{0}}{m} = \frac{\gamma (v_{\text{particle}}) m_{e}^{2}}{m} = \frac{1}{\sqrt{1 - v_{\text{particle}}^{2}}}.$$  

(55)

Therefore,

$$\tan \varphi_{1} = \tan \varphi_{2} \iff \frac{\sinh \alpha \sinh \delta_{1}}{\cosh \alpha + \cosh \delta_{1}} = \frac{\sinh \alpha \sinh \delta_{2}}{\cosh \alpha + \cosh \delta_{2}} \iff \frac{\sinh \delta_{1}}{\cosh \alpha + \cosh \delta_{1}} = \frac{\sinh \delta_{2}}{\cosh \alpha + \cosh \delta_{2}},$$

(56)
that is,
\[ \sinh \delta_1 = \sinh \delta_2 \iff \frac{v_{\text{electron}}}{\sqrt{1 - v_{\text{electron}}^2}} = \frac{v_{\text{muon}}}{\sqrt{1 - v_{\text{muon}}^2}} \iff v_{\text{electron}} = v_{\text{muon}}, \] (57)

where \( v_{A_1} \equiv v_{\text{electron}} \) and \( v_{B_2} \equiv v_{\text{muon}} \). Thus, from \( 53 \) and \( 57 \), internal consistency requires
\[ v_{A_1} = a \frac{m_\mu}{m_e} \frac{1}{\sqrt{1 + \frac{\alpha^2 m_\mu^2 - m_e^2}{m_e^2} v_{B_2}^2}} v_{B_2} = v_{B_2}, \] (58)

which is true provided that,
\[ a \frac{m_\mu}{m_e} \frac{1}{\sqrt{1 + \frac{\alpha^2 m_\mu^2 - m_e^2}{m_e^2} v_{B_2}^2}} = 1 \iff a = \frac{m_e}{m_\mu}. \] (59)

In conclusion, we require
\[ \vec{\rho}_{A_1} = -\vec{\rho}_{A_2} \text{ and } \vec{\rho}_{B_1} = -\vec{\rho}_{B_2} \text{ with } \vec{\rho}_{A_1} = -\frac{m_e}{m_\mu} \vec{\rho}_{B_2}. \] (60)

It finally turns out that the Lorentz transformations are defined in terms of the following \( D(p) \) operators,
\[ D_A(p_{A_1}) = \cos \frac{\varphi}{2} - i \sin \frac{\varphi}{2} \sigma_z^{(A)}; \quad D_B(p_{B_1}) = \cos \frac{\varphi}{2} + i \sin \frac{\varphi}{2} \sigma_z^{(B)}, \]
\[ D_A(p_{A_2}) = \cos \frac{\varphi}{2} + i \sin \frac{\varphi}{2} \sigma_z^{(A)}; \quad D_B(p_{B_2}) = \cos \frac{\varphi}{2} - i \sin \frac{\varphi}{2} \sigma_z^{(B)}, \] (61)

where \( \varphi \) is the angle that characterizes the Wigner rotation. From \( 61 \), it turns out that the Wigner rotations \( W(p_{A_1}) \) and \( W(p_{B_1}) \) are rotations by \( \varphi \) around the \( z \)-axis and \( W(p_{A_2}) \) and \( W(p_{B_2}) \) are Wigner rotations by \( -\varphi \) around the \( z \)-axis. Thus, substituting \( 61 \) into \( 21 \) and \( 22 \) and using \( 47 \), the Lorentz transformed spin density operator \( \hat{\rho}_{\text{spin}}^\prime \equiv \hat{\rho}_{\psi_\pm}^\Lambda (\xi, \varphi) \) in \( 25 \) becomes (for further details, see Appendix B),
\[ \hat{\rho}_{\psi_\pm}^\Lambda (\xi, \varphi) = \frac{1}{4} \left[ I^{(A)}_{2 \times 2} \otimes I^{(B)}_{2 \times 2} + (1 - 2\xi) \sigma_z^{(A)} \otimes I^{(B)}_{2 \times 2} - (1 - 2\xi) I^{(A)}_{2 \times 2} \otimes \sigma_z^{(B)} \pm 2 \sqrt{\xi (1 - \xi)} \cos 2\varphi \sigma_x^{(A)} \otimes \sigma_x^{(B)} + \right. \]
\[ \left. \pm 2 \sqrt{\xi (1 - \xi)} \cos 2\varphi \sigma_y^{(A)} \otimes \sigma_y^{(B)} - \sigma_z^{(A)} \otimes \sigma_z^{(B)} \right]. \] (62)

or, after some algebra,
\[ \hat{\rho}_{\psi_\pm}^\Lambda (\xi, \varphi) = \hat{\rho}_{\psi_\pm} (\xi) \cos^2 \varphi + \hat{\rho}_{\psi_\mp} (\xi) \sin^2 \varphi, \] (63)

with \( \hat{\rho}_{\psi_\pm} (\xi) \) given in \( 49 \).

**B. Momentum density operators in inertial reference frames in relative motion**

In agreement with \( 15 \), we regard the momentum states \( |\vec{p}_{A_1}, \vec{p}_{B_2}\rangle \) for the composite quantum system under investigation as two-qubits momentum states. We use \( \vec{\Sigma}^{(A)} = (\vec{\sigma}_x^{(A)}, \vec{\sigma}_y^{(A)}, \vec{\sigma}_z^{(A)}) \) and \( \vec{\Sigma}^{(B)} = (\vec{\sigma}_x^{(B)}, \vec{\sigma}_y^{(B)}, \vec{\sigma}_z^{(B)}) \) to denote the vectors of Pauli matrices used to describe the momentum qubit for particles \( A \) and \( B \), respectively. Specifically, we assume that \( |\vec{p}_{A_1}\rangle \) and \( |\vec{p}_{A_2}\rangle \) are the eigenvectors of \( \vec{\sigma}_z^{(A)} \) with eigenvalues +1 and −1, respectively. Thus, we get
\[ |\vec{p}_{A_1}\rangle = |0\rangle_A \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and, } |\vec{p}_{A_2}\rangle = |1\rangle_A \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (64)

Similarly, we assume that \( |\vec{p}_{B_1}\rangle \) and \( |\vec{p}_{B_2}\rangle \) are the eigenvectors of \( \vec{\sigma}_z^{(B)} \) with eigenvalues +1 and −1, respectively. Thus, we obtain
\[ |\vec{p}_{B_1}\rangle = |0\rangle_B \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and, } |\vec{p}_{B_2}\rangle = |1\rangle_B \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (65)
Combining Eqs. (18), (47), (64) and (65), the quantum mechanical wave-vector in the rest frame $S$ becomes,

$$|\Psi_{\psi}\rangle = \frac{1}{\sqrt{2}} |\vec{p}_{A_1}, \vec{p}_{B_2} \rangle \left[ \sqrt{1 - \xi} |\uparrow\downarrow\rangle \pm \sqrt{\xi} |\downarrow\uparrow\rangle \right] + \frac{1}{\sqrt{2}} |\vec{p}_{A_2}, \vec{p}_{B_2} \rangle \left[ \sqrt{1 - \xi} |\uparrow\downarrow\rangle \pm \sqrt{\xi} |\downarrow\uparrow\rangle \right].$$  \hspace{1cm} (66)

The Lorentz transformed quantum states corresponding to $|\vec{p}_{A_1}\rangle$, $|\vec{p}_{A_2}\rangle$, $|\vec{p}_{B_1}\rangle$ and $|\vec{p}_{B_2}\rangle$ are given by,

$$|\vec{p}_{A_1}\rangle_{S \rightarrow S'} = (\cos \frac{\varphi}{2} - i \sin \frac{\varphi}{2} \tilde{\sigma}(A)) |0\rangle_A, \quad |\vec{p}_{A_2}\rangle_{S \rightarrow S'} = (\cos \frac{\varphi}{2} + i \sin \frac{\varphi}{2} \tilde{\sigma}(A)) |1\rangle_A,$$

$$|\vec{p}_{B_1}\rangle_{S \rightarrow S'} = (\cos \frac{\varphi}{2} + i \sin \frac{\varphi}{2} \tilde{\sigma}(B)) |0\rangle_B, \quad |\vec{p}_{B_2}\rangle_{S \rightarrow S'} = (\cos \frac{\varphi}{2} - i \sin \frac{\varphi}{2} \tilde{\sigma}(B)) |1\rangle_B.$$  \hspace{1cm} (67)

Upon computation of the inner products $\langle 1 | 2 \rangle$ and $\langle 2 | 1 \rangle$ and using Eqs. (64), (65) and (67), it can be shown that $\hat{\rho}'_{\text{momentum}}$ in (26) reads,

$$\hat{\rho}'_{\text{momentum}} (\xi, \varphi) = \frac{1}{4} \left[ I_{2 \times 2} \otimes I_{2 \times 2} + \cos 2\varphi \left( \tilde{\sigma}(A) \otimes \tilde{\sigma}(B) - \tilde{\sigma}(A) \otimes \tilde{\sigma}(B) \right) + \tilde{\sigma}(A) \otimes \tilde{\sigma}(B) + \right]$$

$$+ \left( 1 - 2\xi \right) \sin 2\varphi \left( \tilde{\sigma}(A) \otimes \tilde{\sigma}(B) + \tilde{\sigma}(A) \otimes \tilde{\sigma}(B) \right).$$  \hspace{1cm} (68)

For further details on the derivation of (68), we refer to Appendix C.

### C. Entanglement in inertial reference frames in relative motion

In what follows, we compute the concurrence of the reduced (mixed) density matrices $\hat{\rho}'_{\text{spin}} (\xi, \varphi)$ in (62) and $\hat{\rho}'_{\text{momentum}} (\xi, \varphi)$ in (68).

#### 1. Concurrence of the Lorentz-transformed spin density matrix

We quantify the entanglement between the spin degrees of freedom of particle $A$ and $B$ by means of Wootters’ concurrence [20]. Using (34) and (35), it follows that the concurrence of $\hat{\rho}_{\psi \pm} (\xi)$ in (48) reads

$$C_{\text{spin}}^S (\xi) = \sqrt{4\xi(1 - \xi)},$$  \hspace{1cm} (69)
where \( C_{\text{spin}}(\xi) \) is the concurrence of the spin density matrix as viewed by an inertial observer located in the reference rest frame \( S \). In the moving reference frame \( S' \), it turns out that the matrix representation of \( \hat{\rho}_{\psi \pm}^L(\xi, \varphi) \) in (62) reads

\[
\hat{\rho}_{\psi \pm}^L(\xi, \varphi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - \xi & \pm \sqrt{\xi(1 - \xi)} \cos 2\varphi & 0 \\ 0 & \pm \sqrt{\xi(1 - \xi)} \cos 2\varphi & \xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

(70)

and \( \rho_{\text{spin}} \hat{\rho}_{\text{spin}} \) with \( \rho_{\text{spin}} = \hat{\rho}_{\psi \pm}^L(\xi, \varphi) \) in (70) becomes,

\[
\rho_{\text{spin}} \hat{\rho}_{\text{spin}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \xi (1 - \xi) (1 + \cos^2 2\varphi) & \pm 2 (1 - \xi) \sqrt{\xi(1 - \xi)} \cos 2\varphi & 0 \\ 0 & \pm 2\xi \sqrt{\xi(1 - \xi)} \cos 2\varphi & \xi (1 - \xi) (1 + \cos^2 2\varphi) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

(71)

The eigenvalues of \( \rho_{\text{spin}} \hat{\rho}_{\text{spin}} \) in (71) are,

\[
\lambda_1 = \xi (1 - \xi) (1 + \cos 2\varphi)^2, \quad \lambda_2 = \xi (1 - \xi) (1 - \cos 2\varphi)^2, \quad \lambda_3 = \lambda_4 = 0.
\]

(72)

Finally, the concurrence of the Lorentz-transformed spin density matrix reads,

\[
C_{\text{spin}}^{(S')}(\xi, \varphi) = \sqrt{4\xi(1 - \xi) \cos^2 2\varphi} = \sqrt{4\xi(1 - \xi) |\cos 2\varphi|}.
\]

(73)

### 2. Concurrence of the Lorentz-transformed momentum density matrix

Following our analysis in the previous Subsection, we quantify the entanglement between the momentum degrees of freedom of particle \( A \) and \( B \) by means of Wootter’s concurrence \( C(\rho_{\text{momentum}}) \) where \( \rho_{\text{momentum}} \) denotes the reduced density matrix obtained by tracing over the spin degrees of freedom.

In the reference frame \( S' \), it turns out that the matrix representation of \( \hat{\rho}_{\text{momentum}}(\xi, \varphi) \) in (68) reads

\[
\hat{\rho}_{\text{momentum}}(\xi, \varphi) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \cos 2\varphi - i (1 - 2\xi) \sin 2\varphi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos 2\varphi + i (1 - 2\xi) \sin 2\varphi & 0 & 0 & 1 \end{pmatrix},
\]

(74)
FIG. 5: $C_{\text{spin}}^{(S')}(\varphi)$ vs. $\varphi$ for $x \equiv \varphi \in [0, 1]$: $\xi = \frac{1}{2}$ (dash), $\xi = \frac{1}{4}$ (thin solid), $\xi = \frac{1}{8}$ (thick solid).

and $\rho_{\text{momentum}}\hat{\rho}_{\text{momentum}}$ with $\rho_{\text{momentum}} = \rho'_{\text{momentum}}(\xi, \varphi)$ becomes,

$$
\rho_{\text{momentum}}\hat{\rho}_{\text{momentum}} = \frac{1}{2} \begin{pmatrix}
\xi (\xi - 1) (1 - \cos 2\varphi) + 1 & 0 & 0 & \cos 2\varphi - i (1 - 2\xi) \sin 2\varphi \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\cos 2\varphi + i (1 - 2\xi) \sin 2\varphi & 0 & 0 & \xi (\xi - 1) (1 - \cos 2\varphi) + 1
\end{pmatrix}.
$$

(75)

It turns out that the eigenvalues of $\rho_{\text{momentum}}\hat{\rho}_{\text{momentum}}$ in (75) are given by,

$$
\lambda_1 = \left[ \frac{1}{2} \left( 1 + \sqrt{1 - 4\xi (1 - \xi) \sin^2 2\varphi} \right) \right]^2, \quad \lambda_2 = \left[ \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi (1 - \xi) \sin^2 2\varphi} \right) \right]^2, \quad \lambda_3 = \lambda_4 = 0.
$$

(76)

Finally, the concurrence of the Lorentz-transformed momentum density matrix reads,

$$
C_{\text{momentum}}^{(S')} (\xi, \varphi) = \sqrt{1 - 4\xi (1 - \xi) \sin^2 2\varphi}.
$$

(77)

Let us discuss the main consequences that originate from Eqs. (73) and (77). Observe that both the $ss$ and $mm$-entanglements in the moving inertial reference frame $S'$ are functions of either the entangling parameter $\xi$ or the angle of rotation $\varphi$. This implies that the analysis of the monotonic behavior of the entanglements-changes in different reference frames is tricky since it involves changes to both $\xi$ and $\varphi$. From (73) it follows that for a given value of the rotation angle $\varphi$, the $ss$-entanglement is an increasing function of $\xi$ for $0 < \xi < \frac{1}{2}$ while it is a decreasing function of $\xi$ for $\frac{1}{2} < \xi < 1$ (see Figure 3). Furthermore, for fixed values of $\xi$, Lorentz boosts with greater angles of rotation exhibit an attenuating power on the $ss$-entanglement that is stronger than that related to boosts with smaller angles (see Figure 3). From (77) it follows that, unlike the $ss$-entanglement, for a given value of the rotation angle $\varphi$, the $mm$-entanglement is a decreasing function of $\xi$ for $0 < \xi < \frac{1}{2}$ while it is an increasing function of $\xi$ for $\frac{1}{2} < \xi < 1$ (see Figure 4). However, like the $ss$-entanglement, for fixed values of $\xi$, Lorentz boosts with greater angles of rotation exhibit an attenuating power on the $mm$-entanglement that is stronger than that related to boosts with smaller angles (see Figure 4). For the sake of clarity and without loss of relevant information, we focus our attention in the rest of the manuscript on values of the rotation angle $\varphi$ in the interval $[0, 1]$. From (73) it also turns out that for a given value of the entangling parameter $\xi$, the $ss$-entanglement is a decreasing function of $\varphi$ for $0 < \varphi < \frac{\pi}{4}$ while it is an increasing function of $\varphi$ for $\frac{\pi}{4} < \varphi < 1$ (see Figure 5). In particular, unlike the analysis presented in [15], we are able to show from (73) that the absolute value of the rate of change in $\varphi$ of $C_{\text{spin}}^{(S')} (\varphi)$ (that is, its "speed") depends...
FIG. 6: $C_{\text{momentum}}^{(S')} (\varphi)$ vs. $\varphi$ for $x \equiv \varphi \in [0, 1]$: $\xi = \frac{1}{2}$ (dash), $\xi = \frac{1}{4}$ (thin solid), $\xi = \frac{1}{8}$ (thick solid).

on the degree of entanglement of the initial state in the rest frame: the spin-entanglement degradation occurs at a faster rate for states with a higher degree of entanglement,

$$\left| \left( \frac{dC_{\text{spin}}^{(S')} (\varphi)}{d\varphi} \right)_{\xi=\xi_M} \right| \geq \left| \left( \frac{dC_{\text{spin}}^{(S')} (\varphi)}{d\varphi} \right)_{\xi=\xi_m} \right|,$$

with $\xi_M \geq \xi_m$. Furthermore, for fixed values of $\varphi$, to lower values of $\xi$ correspond lower values of the $ss$-entanglement (see Figure 5). From (77) it follows that, like for the $ss$-entanglement, for a given value of the entangling parameter $\xi$, the $mm$-entanglement is a decreasing function of $\varphi$ for $0 < \varphi < \frac{\pi}{4}$ while it is an increasing function of $\varphi$ for $\frac{\pi}{4} < \varphi < 1$ (see Figure 6). However, unlike the $ss$-entanglement, for fixed values of $\varphi$, to lower values of $\xi$ correspond higher values of the $mm$-entanglement (see Figure 6). In particular, unlike the study appeared in [15], we can state from (77) that the absolute value of the rate of change in $\varphi$ of $C_{\text{momentum}}^{(S')} (\varphi)$ depends on the degree of entanglement of the initial state in the rest frame: the momentum-entanglement degradation occurs at a slower rate for states with higher degree of entanglement,

$$\left| \left( \frac{dC_{\text{momentum}}^{(S')} (\varphi)}{d\varphi} \right)_{\xi=\xi_M} \right| \leq \left| \left( \frac{dC_{\text{momentum}}^{(S')} (\varphi)}{d\varphi} \right)_{\xi=\xi_m} \right|,$$

with $\xi_M \geq \xi_m$. We also stress that our analysis shows that no sum of $ss$ and $mm$-entanglements is conserved in any pair of inertial reference frames in relative motion for any value of the entangling parameter,

$$C_{\text{spin}}^{(S)} (\xi) + C_{\text{momentum}}^{(S)} (\xi) \neq C_{\text{spin}}^{(S)} (\xi, \varphi) + C_{\text{momentum}}^{(S)} (\xi, \varphi).$$

Finally, it turns out that both $\xi$-changes and $\varphi$-changes of variations of either $ss$ or $mm$-entanglements, $\Delta C_{\text{spin}}$ and $\Delta C_{\text{momentum}}$, respectively, exhibit the same monotonic behavior on the permitted range of values for each parameter. More explicitly, it follows that

$$\text{sign} \left( \left( \frac{\partial \Delta C_{\text{spin}}}{\partial \xi} \right)_{\varphi} \right) = \text{sign} \left( \left( \frac{\partial \Delta C_{\text{momentum}}}{\partial \xi} \right)_{\varphi} \right),$$

and,

$$\text{sign} \left( \left( \frac{\partial \Delta C_{\text{spin}}}{\partial \varphi} \right)_{\xi} \right) = \text{sign} \left( \left( \frac{\partial \Delta C_{\text{momentum}}}{\partial \varphi} \right)_{\xi} \right),$$
where the variations $\Delta C_{\text{spin}}$ and $\Delta C_{\text{momentum}}$ are defined as,

$$
\Delta C_{\text{spin}}(\xi, \varphi) \overset{\text{def}}{=} C_{\text{spin}}^{(S)}(\xi) - C_{\text{spin}}^{(S')}(\xi, \varphi),
$$

and,

$$
\Delta C_{\text{momentum}}(\xi, \varphi) \overset{\text{def}}{=} C_{\text{momentum}}^{(S)} - C_{\text{momentum}}^{(S')(\xi, \varphi)},
$$

respectively. For an illustrative justification of Eqs. (81) and (82), see Figures 7 and 8, respectively. This last finding implies that no entanglement compensation between spins and momenta occurs and neither it is required by the Lorentz invariance of the joint entanglement of the entire wave-function since no analytical constraint exists between this quantity and the $ss$ and $mm$-entanglements.

VI. FINAL REMARKS

In this article, we have analyzed special relativistic effects on the entanglement between either spins or momenta of composite quantum systems of two spin-$\frac{1}{2}$ massive particles, either distinguishable or indistinguishable, in inertial reference frames in relative motion.

*Indistinguishable particles with balanced (but not maximal) entanglement configuration.* For the case of indistinguishable particles, we consider a single balanced scenario where the momenta of the pair are well-defined but not maximally entangled in the rest frame while the spins of the pair are described by a one-parameter ($\eta$) family of entangled bipartite states (see Eq. (27)). We find out that in any scenario neither the $ss$ nor the $mm$ entanglements are Lorentz invariant quantities (see Eqs. (42) and (45)). In particular, for any value of the entangling parameters, both $ss$ and $mm$-entanglements are attenuated by Lorentz transformations. Moreover, the change in entanglement for the momenta is the same as the change in entanglement for spins (see Eq. (46) and Figures 1 and 2). In particular (see Eq. (31)), when $\eta = 0$ or $\eta = 1$, we recover the main result appeared in [15].

*Distinguishable particles with unbalanced entanglement configuration.* For the case of distinguishable particles, we consider an unbalanced scenario where the momenta of the pair are well-defined and maximally entangled in the rest frame while the spins of the pair are described by a one-parameter ($\xi$) family of non-maximally entangled bipartite states (see (66)). We present an extensive investigation of the behavior of two-parameters (the angle of rotation $\varphi$ and the entangling parameter $\xi$) $ss$ and $mm$-entanglements for a two-particle quantum composite system of massive fermions in inertial reference frames in relative motion. We show that in any case neither the $ss$ nor the
entanglements quantified by means of Wootters’ concurrence were Lorentz invariant quantities (see Eqs. (73), (77)): the total amount of entanglement regarded as the sum of these entanglements is not the same in all inertial frames. Furthermore, for any value of the entangling parameter, both \(ss\) and \(mm\)-entanglements are attenuated by Lorentz transformations (see Figures 3, 4, 5, 6) and their parametric rates of change with respect to the entanglements observed in a rest frame have the same monotonic behavior (see Figures 7 and 8). In particular, in Appendix D, we also consider an additional scenario of distinguishable particles (see Eq. (D3)) where in the limiting case of \(\xi = \frac{1}{2}\) and \(m_e = m_p \equiv m\) (indistinguishable particles, two identical electrons), the main result of [15] is reproduced. However, unlike the main finding appeared in [15] and our generalized result in (46) which holds for indistinguishable particles, we show that in general the change in entanglement for the momenta is not the same as the change in entanglement for the spins when considering distinguishable particles. Furthermore, unlike the main result presented in [12], we provide clear evidences that the rate of entanglement changes in both parameters \(\xi\) and \(\varphi\) are in the same direction (not in the opposite direction). Surprisingly, even allowing for non-maximal entanglement in the spin degrees in the rest frame \(S\) (while keeping the momentum entanglement maximal in \(S\)), no entanglement compensation from the momentum to spin entanglement occurs for any possible pair of inertial reference frames one of which at rest.

In conclusion, although a thorough characterization of relativistic properties of quantum entanglement is far from being achieved, our study provides general enough results towards such direction.

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Appendix A: Decomposition of density operators in terms of Pauli matrices

In this Appendix, we derive Eq. (49). Consider a two-qubit system with Hilbert space \(\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2\) and computational basis \(\mathcal{B}_\mathcal{H} = \{|00\}, |01\}, |10\}, |11\}\). It can be shown that a general two-qubit state \(\hat{\rho}\) can always be written up to local unitary equivalence to a state of the following form [21],

\[
\hat{\rho} = \frac{1}{4} \left[ I_{4 \times 4} \otimes \sigma + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i \right],
\]

where \(\sigma = \sigma_x, \sigma_y, \sigma_z\) are Pauli matrices.
that is, after some algebra,

\[
\hat{\rho} = \frac{1}{4} \begin{pmatrix}
  a_3 + b_3 + c_3 + 1 & b_1 - ib_2 & a_1 - ia_2 & c_1 - c_2 \\
  b_1 + ib_2 & a_3 - b_3 - c_3 + 1 & c_1 + c_2 & a_1 - ia_2 \\
  a_1 + ia_2 & c_1 + c_2 & b_3 - a_3 - c_3 + 1 & b_1 - ib_2 \\
  c_1 - c_2 & a_1 + ia_2 & b_1 + ib_2 & c_3 - b_3 - a_3 + 1
\end{pmatrix},
\]

(A2)

where \( \vec{a} \) and \( \vec{b} \in \mathbb{R}^3 \) are given by \( \vec{a} = (a_1, a_2, a_3) \), \( \vec{b} = (b_1, b_2, b_3) \) and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is the operator vector of Pauli matrices. Equating \( \hat{\rho}_{\psi_+}(\xi) \) in [A4] to [A2], we obtain

\[
a_1 = a_2 = b_1 = b_2 = 0, \quad c_1 = c_2 = 2\sqrt{\xi} (1 - \xi), \quad a_3 = 1 - 2\xi, \quad b_3 = -(1 - 2\xi), \quad c_3 = -1.
\]

(A3)

Therefore, in terms of Pauli matrices, \( \hat{\rho}_{\psi_+}(\xi) \) becomes

\[
\hat{\rho}_{\psi_+}(\xi) = \frac{1}{4} \left[ I^{(1)}_{2x2} \otimes I^{(2)}_{2x2} + (1 - 2\xi) \sigma_z^{(1)} \otimes I^{(2)}_{2x2} - (1 - 2\xi) I^{(1)}_{2x2} \otimes \sigma_z^{(2)} + 2\sqrt{\xi} (1 - \xi) \sigma_x^{(1)} \otimes \sigma_z^{(2)} + 2\sqrt{\xi} (1 - \xi) \sigma_y^{(1)} \otimes \sigma_z^{(2)} - \sigma_z^{(1)} \otimes \sigma_z^{(2)} \right].
\]

(A4)

Observe that for \( \xi = \frac{1}{2} \) we obtain,

\[
\hat{\rho}_{\psi_+} \left( \frac{1}{2} \right) = \frac{1}{4} \left[ I^{(1)}_{2x2} \otimes I^{(2)}_{2x2} + \sigma_x^{(1)} \otimes \sigma_x^{(2)} + \sigma_y^{(1)} \otimes \sigma_y^{(2)} - \sigma_z^{(1)} \otimes \sigma_z^{(2)} \right].
\]

(A5)

Similarly, equating \( \hat{\rho}_{\psi_-}(\xi) \) in [A4] to [A2], we get

\[
a_1 = a_2 = b_1 = b_2 = 0, \quad c_1 = c_2 = -2\sqrt{\xi} (1 - \xi), \quad a_3 = 1 - 2\xi, \quad b_3 = -(1 - 2\xi), \quad c_3 = -1.
\]

(A6)

In terms of Pauli matrices, \( \hat{\rho}_{\psi_-}(\xi) \) reads,

\[
\hat{\rho}_{\psi_-}(\xi) = \frac{1}{4} \left[ I^{(1)}_{2x2} \otimes I^{(2)}_{2x2} + (1 - 2\xi) \sigma_z^{(1)} \otimes \sigma_z^{(2)} - (1 - 2\xi) I^{(1)}_{2x2} \otimes \sigma_z^{(2)} - 2\sqrt{\xi} (1 - \xi) \sigma_x^{(1)} \otimes \sigma_x^{(2)} - 2\sqrt{\xi} (1 - \xi) \sigma_y^{(1)} \otimes \sigma_y^{(2)} - \sigma_x^{(1)} \otimes \sigma_x^{(2)} \right],
\]

(A7)

and for \( \xi = \frac{1}{2} \) it becomes,

\[
\hat{\rho}_{\psi_-} \left( \frac{1}{2} \right) = \frac{1}{4} \left[ I^{(1)}_{2x2} \otimes I^{(2)}_{2x2} - \sigma_x^{(1)} \otimes \sigma_x^{(2)} - \sigma_y^{(1)} \otimes \sigma_y^{(2)} - \sigma_z^{(1)} \otimes \sigma_z^{(2)} \right].
\]

(A8)

In summary, we have shown that

\[
\hat{\rho}_{\psi}(\xi) = \frac{1}{4} \left[ I^{(1)}_{2x2} \otimes I^{(2)}_{2x2} + (1 - 2\xi) \sigma_z^{(1)} \otimes \sigma_z^{(2)} - (1 - 2\xi) I^{(1)}_{2x2} \otimes \sigma_z^{(2)} \pm 2\sqrt{\xi} (1 - \xi) \sigma_x^{(1)} \otimes \sigma_x^{(2)} \right],
\]

(A9)

and in the limiting case of \( \xi = \frac{1}{2} \) we recover the case of maximally entangled pure Bell states.

**Appendix B: Lorentz transformation of the spin density operator**

In this Appendix, we derive Eq. (B2). Substituting (21) and (22) into (25), we get

\[
\hat{\rho}_{\text{spin}}' = \frac{1}{2} D_A (p_{A_1}) D_B (p_{B_1}) \hat{\rho}_{\text{spin}} D_A^\dagger (p_{A_1}) D_B^\dagger (p_{B_1}) + \frac{1}{2} D_A (p_{A_2}) D_B (p_{B_2}) \hat{\rho}_{\text{spin}} D_A^\dagger (p_{A_2}) D_B^\dagger (p_{B_2}),
\]

(B1)
Therefore, using (B2) and (B3), after some tedious algebra it follows that

\[ \hat{\rho} \]

that is, \( \hat{\rho} \) equals \( \hat{\rho} \) in (C1) becomes

\[ \hat{\rho}'_{\text{spin}} = \frac{1}{4} \left[ I_{2 \times 2} \otimes I_{2 \times 2} + (1 - \xi) I_{2 \times 2} \otimes I_{2 \times 2} - (1 - \xi) I_{2 \times 2} \otimes I_{2 \times 2} \pm 2 \sqrt{(1 - \xi) \cos 2 \phi \sigma_x^A \otimes \sigma_z^b + 2 \xi (1 - \xi) \cos 2 \phi \sigma_x^A \otimes \sigma_z^a + 2 \xi (1 - \xi) \cos 2 \phi \sigma_x^A \otimes \sigma_z^a + 2 \xi (1 - \xi) \cos 2 \phi \sigma_x^A \otimes \sigma_z^a} \right], \]  

that is, \( \hat{\rho}'_{\text{spin}} \) equals \( \hat{\rho}_{\psi++}^A (\xi, \phi) \) in (B1).

**Appendix C: Lorentz transformation of the momentum density operator**

In this Appendix, we derive Eq. (68). Using Eqs. (64), (65) and (67), it follows that

\[ |\Lambda p_{A_1} \rangle \langle \Lambda p_{A_1}| = \frac{I_{2 \times 2} + \hat{\sigma}_z^A}{2}, \quad |\Lambda p_{B_1} \rangle \langle \Lambda p_{B_1}| = \frac{I_{2 \times 2} + \hat{\sigma}_z^B}{2}, \]

\[ |\Lambda p_{B_1} \rangle \langle \Lambda p_{A_2}| = \frac{I_{2 \times 2} - \hat{\sigma}_z^A}{2}, \quad |\Lambda p_{B_2} \rangle \langle \Lambda p_{B_2}| = \frac{I_{2 \times 2} - \hat{\sigma}_z^B}{2}, \]

and,

\[ |\Lambda p_{A_1} \rangle \langle \Lambda p_{A_2}| = \frac{\hat{\sigma}_x^A + i \hat{\sigma}_y^A}{2}, \quad |\Lambda p_{A_1} \rangle \langle \Lambda p_{B_1}| = \frac{\hat{\sigma}_x^A - i \hat{\sigma}_y^A}{2}, \]

\[ |\Lambda p_{B_1} \rangle \langle \Lambda p_{B_2}| = \frac{\hat{\sigma}_x^B + i \hat{\sigma}_y^B}{2}, \quad |\Lambda p_{B_2} \rangle \langle \Lambda p_{B_1}| = \frac{\hat{\sigma}_x^B - i \hat{\sigma}_y^B}{2}. \]
As a final remark, we stress that the density matrices \((\text{D9})\) and \((\text{C7})\) cannot be parametrized by means of the one-parameter \((\xi)\) family of non-maximally entangled bipartite states. Unlike the distinguishable case considered in \((\text{A1})\), for them it is needed the most general parametrization for a two-qubit system with Hilbert space \(\mathbb{C}^2 \otimes \mathbb{C}^2\). It can be shown that any state \(\hat{\rho}\) for such a system may be parametrized as \((\text{22})\),

\[
\hat{\rho}_{\text{momentum}} = \frac{1}{2} \left[ \begin{array}{c|c} 1 \frac{\phi (A) + \phi (B)}{2} & 1 \frac{\phi (A) + i \phi (B)}{2} \\ \hline 1 \frac{\phi (A) + i \phi (B)}{2} & 1 \frac{\phi (A) + \phi (B)}{2} \end{array} \right] + [\cos 2\varphi + i (2\xi - 1) \sin 2\varphi] \left[ \begin{array}{c|c} 1 \frac{\phi (A) + \phi (B)}{2} & 1 \frac{\phi (A) + i \phi (B)}{2} \\ \hline 1 \frac{\phi (A) + i \phi (B)}{2} & 1 \frac{\phi (A) + \phi (B)}{2} \end{array} \right].
\]

After some algebra, \(\hat{\rho}_{\text{momentum}}\) may be finally rewritten as \(\hat{\rho}_{\text{momentum}} (\xi, \varphi)\) in \((\text{68})\),

\[
\hat{\rho}_{\text{momentum}} = \hat{\rho}_{\text{momentum}} (\xi, \varphi) = \frac{1}{4} \left[ \begin{array}{c} I_{2 \times 2} \otimes I_{2 \times 2} + \cos 2\varphi \left( \hat{\sigma}_x \otimes \hat{\sigma}_x - \hat{\sigma}_y \otimes \hat{\sigma}_y \right) + \hat{\sigma}_z \otimes \hat{\sigma}_z + (1 - 2\xi) \sin 2\varphi \left( \hat{\sigma}_x \otimes \hat{\sigma}_y + \hat{\sigma}_y \otimes \hat{\sigma}_x \right) \end{array} \right].
\]

As a final remark, we stress that the density matrices \((\text{D9})\) and \((\text{C7})\) cannot be parametrized by means of the decompositions in \((\text{A1})\). For them it is needed the most general parametrization for a two-qubit system with Hilbert space \(\mathbb{C}^2 \otimes \mathbb{C}^2\). It can be shown that any state \(\hat{\rho}\) for such a system may be parametrized as \((\text{22})\),

\[
\hat{\rho} = \frac{1}{4} \left[ I_{4 \times 4} + \bar{c} \cdot \hat{\sigma} \otimes I_{2 \times 2} + I_{2 \times 2} \otimes \bar{d} \cdot \hat{\sigma} + \sum_{j, k=1}^{3} \gamma_{jk} \sigma_j \otimes \sigma_k \right],
\]

where \(\bar{c}, \bar{d} \in \mathbb{R}^3\) and \(\gamma_{jk}\) are real numbers.

**Appendix D: Additional example for distinguishable particles**

In this Appendix, we consider the case of distinguishable particles with an unbalanced scenario where the momenta of the pair are well-defined and maximally entangled in the rest frame while the spins of the pair are described by a one-parameter \((\xi)\) family of non-maximally entangled bipartite states. Unlike the distinguishable case considered in Section IV, we consider a composite quantum system described in the rest frame \(S\) by a total wave-vector that differs from \((\text{A1})\) and characterized by a set of momentum constraints that differs from that in \((\text{50})\).

Consider the following set of working hypotheses on momentum states,

\[
\vec{p}_{A_1} = -\vec{p}_{A_2}, \quad \vec{p}_{B_1} = -\vec{p}_{B_2} \quad \text{and} \quad \vec{p}_{A_1} = a \vec{p}_{B_2},
\]

where \(a \in \mathbb{R}\) and \(\vec{p}_{A_2}, \vec{p}_{B_1}, \vec{p}_{B_2} \in \mathbb{R}^3\).
where \( \alpha = \frac{m_v}{m_p} \). This expression for \( \alpha \) is obtained by imposing that the two angles of Wigner’s rotations \( \varphi_A \) and \( \varphi_B \) be equal (for an explicit derivation of this specific expression for \( \alpha \), we refer to Eq. (59) in Section V). The momentum states \( |\vec{p}_{A_1}\rangle \) and \( |\vec{p}_{A_2}\rangle \) are the eigenvectors of \( \tilde{\sigma}_z^{(A)} \) with eigenvalues +1 and −1, respectively, while \( |\vec{p}_{B_1}\rangle \) and \( |\vec{p}_{B_2}\rangle \) are the eigenvectors of \( \tilde{\sigma}_z^{(B)} \) with eigenvalues −1 and +1, respectively. Thus, we consider

\[
|\vec{p}_{A_1}\rangle = |0\rangle_A \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\vec{p}_{A_2}\rangle = |1\rangle_A \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |\vec{p}_{B_1}\rangle = |1\rangle_B \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad |\vec{p}_{B_2}\rangle = |0\rangle_B \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

(D2)

In such case, the wave-vector \( |\Psi\rangle_S \) in the rest frame \( S \) reads,

\[
|\Psi_{e\mu}^{(\text{new})}\rangle = |\Psi\rangle_S \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left[ |\vec{p}_{A_1}, \vec{p}_{B_1}\rangle \otimes |0\rangle_S + |\vec{p}_{A_2}, \vec{p}_{B_2}\rangle \otimes |0\rangle_S \right]
\]

\[
= \frac{1}{\sqrt{2}} |10\rangle \left[ \sqrt{1-\xi} |\uparrow\downarrow\rangle \pm \sqrt{\xi} |\downarrow\uparrow\rangle \right] \mp \frac{1}{\sqrt{2}} |01\rangle \left[ \sqrt{1-\xi} |\uparrow\downarrow\rangle \pm \sqrt{\xi} |\downarrow\uparrow\rangle \right].
\]

(D3)

Note that for \( \xi = \frac{1}{2} \), \( |\Psi\rangle_S \) in (D3) becomes the wave-vector studied in [15],

\[
|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}} |10\rangle |\phi_{\pm}\rangle \mp \frac{1}{\sqrt{2}} |01\rangle |\phi_{\pm}\rangle,
\]

(D4)

with |\( \phi_{\pm}\rangle \) defined in [28]. Focusing on \( |\Psi_{+}\rangle \) (the same results are obtained using \( |\Psi_{-}\rangle \) and following the same line of reasoning presented in Section V, we obtain that the spin and momentum reduced density operators in the rest frame \( S \) become,

\[
\rho^{(S)}_{\text{spin}}(\xi) = \frac{1}{4} \left[ I_{2\times2}^{(A)} \otimes I_{2\times2}^{(B)} + (1 - 2\xi) \sigma_x^{(A)} \otimes \sigma_x^{(B)} - (1 - 2\xi) I_{2\times2}^{(A)} \otimes \sigma_z^{(B)} + 2\sqrt{\xi (1-\xi)} \sigma_y^{(A)} \otimes \sigma_x^{(B)} + \right.
\]

\[
+ \left. 2 \sqrt{\xi (1-\xi)} \sigma_y^{(A)} \otimes \sigma_y^{(B)} - \sigma_z^{(A)} \otimes \sigma_z^{(B)} \right]
\]

\[
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 - \xi & \sqrt{\xi (1-\xi)} & 0 \\
0 & \sqrt{\xi (1-\xi)} & \xi & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(D5)

and,

\[
\rho^{(S)}_{\text{momentum}} = \frac{1}{4} \left[ I_{2\times2}^{(A)} \otimes I_{2\times2}^{(B)} - \tilde{\sigma}_z^{(A)} \otimes \tilde{\sigma}_z^{(B)} - \tilde{\sigma}_y^{(A)} \otimes \tilde{\sigma}_y^{(B)} - \tilde{\sigma}_z^{(A)} \otimes \tilde{\sigma}_z^{(B)} \right] = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(D6)

respectively. The concurrence of \( \rho^{(\text{rest-frame})}_{\text{spin}}(\xi) \) and \( \rho^{(\text{rest-frame})}_{\text{momentum}} \) are,

\[
C^{(S)}_{\text{spin}}(\xi) = \sqrt{4\xi (1-\xi)} \quad \text{and} \quad C^{(S)}_{\text{momentum}} = 1,
\]

(D7)

respectively. The Lorentz-transformed spin density operator in [25] reads,

\[
\rho^{(S')}_{\text{spin}}(\xi) = \frac{1}{4} \left[ I_{2\times2}^{(A)} \otimes I_{2\times2}^{(B)} + (1 - 2\xi) \sigma_z^{(A)} \otimes I_{2\times2}^{(B)} - (1 - 2\xi) I_{2\times2}^{(A)} \otimes \sigma_z^{(B)} + 2\sqrt{\xi (1-\xi)} \cos 2\varphi \sigma_x^{(A)} \otimes \sigma_x^{(B)} + \right]
\]

\[
+ 2 \sqrt{\xi (1-\xi)} \cos 2\varphi \sigma_y^{(A)} \otimes \sigma_y^{(B)} - \sigma_z^{(A)} \otimes \sigma_z^{(B)} \right]
\]

\[
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 - \xi & 0 & \sqrt{\xi (1-\xi)} \cos 2\varphi \\
0 & 0 & \xi & 0 \\
0 & \sqrt{\xi (1-\xi)} \cos 2\varphi & 0 & 0
\end{pmatrix},
\]

(D8)
Similarly, the Lorentz-transformed momentum density operator in (26) is given by
\[
\rho_{\text{momentum}}^{(S')} = \frac{1}{4} \left[ I_{2x2}^{(A)} \otimes I_{2x2}^{(B)} - \cos 2\varphi \tilde{\sigma}_x^{(A)} \otimes \tilde{\sigma}_x^{(B)} - \cos 2\varphi \tilde{\sigma}_y^{(A)} \otimes \tilde{\sigma}_y^{(B)} + (1 - 2\xi) \sin 2\varphi \left( \tilde{\sigma}_y^{(A)} \otimes \tilde{\sigma}_x^{(B)} - \tilde{\sigma}_x^{(A)} \otimes \tilde{\sigma}_y^{(B)} \right) - \tilde{\sigma}_x^{(A)} \otimes \tilde{\sigma}_x^{(B)} \right]
\]
\[
= \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -[\cos 2\varphi + i (1 - 2\xi) \sin 2\varphi] \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} .
\] (D9)

Furthermore, omitting technical details, it can be shown that the concurrences of the Lorentz-transformed spin and momentum density matrices in the moving reference frame $S'$ are given by,
\[
C_{\text{spin}}^{(S')} (\xi, \varphi) = \sqrt{4\xi (1 - \xi)} |\cos 2\varphi| ,
\] (D10)
and,
\[
C_{\text{momentum}}^{(S')} (\xi, \varphi) = \sqrt{1 - 4\xi (1 - \xi) \sin^2 2\varphi} ,
\] (D11)
respectively.

We remark that although Eqs. (D3) and (D11) differ from Eqs. (66) and (50), respectively, we have obtained the same final conclusions in both distinguishable cases considered: Eqs. (D10) and (D11) are equal to Eqs. (75) and (77), respectively. We also point out that when $\xi = \frac{1}{2}$ and $m_e = m_\mu$ (two electrons), we recover the main result appeared in [15]. Finally, we emphasize that for distinguishable particles, the change in entanglement for the momenta is not the same as the change in entanglement for spins (see Eqs. (D10) and (D11) in addition to Figures 3, 4, 5, 6).

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