Mc2G: An Efficient Algorithm for Matrix Completion with Social and Item Similarity Graphs

Qiaosheng Zhang, Geewon Suh, Changho Suh, Senior Member, IEEE, Vincent Y. F. Tan, Senior Member, IEEE

Abstract—In this paper, we design and analyze Mc2G (Matrix Completion with 2 Graphs), an algorithm that performs matrix completion in the presence of social and item similarity graphs. Mc2G runs in quasilinear time and is parameter free. It is based on spectral clustering and local refinement steps. The expected number of sampled entries required for Mc2G to succeed (i.e., recover the clusters in the graphs and complete the matrix) matches an information-theoretic lower bound up to a constant factor for a wide range of parameters. We show via extensive experiments on both synthetic and real datasets that Mc2G outperforms other state-of-the-art matrix completion algorithms that leverage graph side information.

Index Terms—Matrix completion, Community detection, Stochastic block model, Spectral method, Graph side-information.

I. INTRODUCTION

With the ubiquity of social networks such as Facebook and Twitter, it is increasingly convenient to collect similarity information amongst users. It has been shown that exploiting this similarity information in the form of a social graph can significantly improve the quality of recommender systems [1]–[6] compared to traditional recommendation algorithms (e.g., collaborative filtering [7], [8]) that rely merely on rating information. This improvement is particularly pronounced in the presence of the so-called cold-start problem in which we would like to recommend items to a user who has not rated any items, but we do possess his/her similarity information with other users. Similarly, an item similarity graph is sometimes also available for exploitation—it can be constructed either from the features of items [9], [10], or from users’ behavior history (as has been done by Taobao [11]). Again, this can help in solving the dual cold-start problem, namely the learner has no information about new items that have not been rated by any user.

While there have been numerous studies considering how to exploit graph side information to enhance recommender systems, most of the algorithms developed so far exploit only one graph (either the social or the item similarity graph). As mentioned above, both graphs are often available in many real-life applications, and it has been shown in a prior theoretical study [12] that there are scenarios in which exploiting two graphs yields strictly more benefits than exploiting only one graph. This work builds upon [12] which focuses on fundamental limits, but does not propose computationally efficient algorithms that achieve the limits. Our main contribution is to design and analyze a computationally efficient algorithm—which we name Mc2G—for a matrix completion problem, wherein both the social and item similarity graphs are available. We also provide theoretical guarantees on the expected number of sampled entries for Mc2G to succeed, and further show that it meets an information-theoretic lower bound up to a constant factor. It is worth highlighting that Mc2G is applicable to a more general setting than that considered in [12], thus the theoretical results developed in this work further generalize the theory in [12]. For example, we consider general discrete-valued ratings instead of binary ratings.

We consider a setting in which there are $n$ users and $m$ items. Users are partitioned into $k_1 \geq 2$ clusters, while items are partitioned into $k_2 \geq 2$ clusters. Users’ ratings to items are then assumed to be sampled independently with probability $p$; (ii) a social graph generated according to a celebrated generative model for random graphs—the stochastic block model (SBM) [14]; and (iii) an item similarity graph generated according to another SBM. The task is to exactly recover the clusters of both users and items, as well as to complete the matrix. Our model significantly generalizes the models considered in several related works with theoretical guarantees [4], [5], [12], by relaxing some constraints therein, e.g., (i) users/items are only partitioned into two equal-sized clusters, and (ii) only binary ratings are allowed.

A. Main contributions

Our main contributions are summarized as follows.

1) We develop a computationally efficient algorithm Mc2G that runs in quasilinear time. Mc2G is a multi-stage algorithm that follows the “from global to local” principle [14]; it first adopts a spectral clustering method on graphs to obtain initial estimates of user/item clusters, and then refines each user/item individually based on local maximum likelihood estimation (MLE). Mc2G is also a parameter-free algorithm that does not need the knowledge of the model parameters.

2) The Mc2G algorithm is applicable to matrices with general values, which can be ternary, real-valued, or even real-valued with a small range.

3) Mc2G is also applicable to other problems such as community detection [17]–[19], phase retrieval [20], [21], etc.
Under the symmetric setting wherein both the social and item similarity graphs are generated according to symmetric SBMs [18, Def. 2], we show that Mc2G succeeds in the sense of recovering the missing entries of the sub-sampled matrix and the clusters with high probability as long as the number of samples exceeds a bound presented in Theorem 1. While the theoretical guarantee requires the symmetric assumption, we emphasize that Mc2G is universally applicable to all matrix completion problems with two-sided graph side information.

2) We also provide an information-theoretic lower bound that matches the bound in Theorem 1 up to a constant factor; this demonstrates the order-wise optimality of Mc2G. As a by-product, the aforementioned theoretical results also generalize the theory developed in the prior work [12], which was focused on a simpler setting in which both users and items are partitioned into two clusters.

3) We conduct extensive experiments on synthetic datasets to verify that the results show keen agreement with the derived theoretical guarantee of Mc2G in Theorem 1. We further demonstrate the superior performance of Mc2G by comparing it with several state-of-the-art matrix completion algorithms that leverage graph side information, such as matrix factorization with social regularization (SoReg) [3], and a spectral clustering method with local refinements using only the social graph or only the item graph [4]. Mc2G is often orders of magnitude better than the competing algorithms in terms of the mean absolute error (MAE).

4) Finally we apply Mc2G to datasets with real social and item similarity graphs (i.e., the LastFM social network [22] and political blogs network [23]). Our experimental results show that Mc2G works well when the observed graphs are derived from real-world applications; this further confirms that Mc2G is universal, as the real graphs do not satisfy the symmetry assumptions. Finally, we compare Mc2G with the other aforementioned matrix completion algorithms on the dataset with real graphs. Experimental results show that Mc2G outperforms these existing algorithms.

B. Related works

Due to the wide applicability of matrix completion (such as recommender systems), the past decade has witnessed the developments of many efficient matrix completion algorithms, such as [24]–[28]. In the context of recommender systems, the design of algorithms that exploit graph side information (especially the social graph) has attracted much attention, and some of the works [6], [29], [30] exploit both the social and item similarity graphs. Although these algorithms usually have better empirical performance than traditional ones, most of them neither quantify the gains of exploiting graph side information, nor provide any theoretical guarantees.

We note that another line of works focused on characterizing the fundamental limits of matrix completion in which the matrix to be recovered is generated according to a certain generative model for the clusterings of the users and/or items. Ahn et al. [4] considered a simple setting where ratings are binary and a graph encodes the structure of two clusters, and characterized the expected number of sampled entries required for the matrix completion task. Follow-up works [5], [31] relaxed the assumptions in [4], but are still restricted to exploiting the use of a social graph. The recent work [12] investigated a more general setting in which both the social and item similarity graphs are available, and quantified the gains of exploiting two graphs by establishing information-theoretic lower and upper bounds. However, a computationally efficient algorithm that achieves the limit promised by MLE was not developed in [12]. Given that the MLE is not computationally feasible, there is a pressing need to develop efficient algorithms. This precisely sets the goal of this work. Additionally, this work studies a generalized model that spans multiple user/item clusters and discrete-valued rating matrices. This is in contrast to [12] which focuses on two clusters and binary ratings.

Another field relevant to this work is community detection, which is the problem of partitioning nodes of an undirected graph into different clusters/communities. When the graphs are generated according to SBMs, the information-theoretic limits for exact recovery of clusters [17], [18], [32]–[35] have been established. These limits also play a role in establishing the theoretical guarantee of Mc2G (see the third item in Remark 4 for details), as our algorithm includes the clustering step for users and items in the process of matrix completion. It has also been shown that side information is in general helpful for community detection [36]–[39]. This observation is pertinent and related to our work because our problem can also be viewed as recovering users/items clusters with side information in the form of a rating matrix. Besides, our problem is also related to the labelled or weighted SBM problem, if the two SBMs that govern the social and item similarity graphs are merged to a single unified SBM (interested readers are referred to [12, Remark 4] for details).

C. Outline

This paper is organized as follows. We first introduce the problem setup in Section II, and then describe our efficient algorithm Mc2G in Section III. Section IV presents our main theoretical results: (i) the theoretical guarantee for Mc2G and (ii) the information-theoretical lower bound. These results are proved in Sections V and VI respectively. Experimental results are presented in Section VII.

II. PROBLEM STATEMENT

We consider a recommender system with \( n \) users and \( m \) items. Ratings from users to items are chosen from an arbitrary finite alphabet \( \mathcal{Z} \) (e.g., \( \mathcal{Z} = \{1, 2, 3, 4, 5\} \)). It is assumed that users are partitioned into \( k_1 \geq 2 \) disjoint clusters \( \{U_1, U_2, \ldots, U_{k_1}\} \), and items are partitioned into \( k_2 \geq 2 \) disjoint clusters \( \{I_1, I_2, \ldots, I_{k_2}\} \). We define \( \sigma : [n] \rightarrow [k_1] \) as the label function for users such that \( \sigma(i) = a \) if user \( i \) belongs to cluster \( U_a \). On the contrary, each cluster \( U_a \) can be represented as \( U_a = \{i \in [n] : \sigma(i) = a\} \). Thus, \( \sigma \) can be viewed as an alternative (and more concise) representation of the clusterings of users \( \{U_a\}_{a \in [k_1]} \). Similarly, we define

\[ \text{For any integer } s \geq 1, \text{ let } [s] = \{1, \ldots, s\}. \]
TABLE I: Nominal ratings from users to items

| Cluster $\mathcal{U}_1$ | Cluster $\mathcal{U}_2$ | . . . . . . | Cluster $\mathcal{U}_k_2$ |
|------------------------|------------------------|-------------|------------------------|
| Cluster $\mathcal{U}_1$ | $z_{11}$ | $z_{12}$ | . . . | $z_{1k_2}$ |
| Cluster $\mathcal{U}_2$ | $z_{21}$ | $z_{22}$ | . . . | $z_{2k_2}$ |
| . . . . . .             | . . . . . .             | . . . . . . | . . . . . .             |
| Cluster $\mathcal{U}_k_2$ | $z_{k_11}$ | $z_{k_12}$ | . . . | $z_{k_2k_2}$ |

Fig. 1: An example with 6 users (partitioned into 3 clusters) and 6 items (partitioned into 2 clusters). The nominal ratings are chosen from $\mathcal{Z} = \{1, 2, 3, 4, 5\}$, and are set to be $z_{11} = 5$, $z_{12} = 1$, $z_{21} = 1$, $z_{22} = 4$, $z_{31} = 3$, $z_{32} = 2$.

An example of the personalized rating matrix is illustrated in Fig. 1b.

### A. Observations

The learner observes three pieces of information:

1) A sub-sampled rating matrix $\mathcal{U}$, with each entry $\mathcal{U}_{ij} = V_{ij}$ with probability (w.p.) $p$ and $\mathcal{U}_{ij} = e$ (erasure symbol) w.p. $1 - p$. We refer to $p$ as the sample probability and $mn$p as the expected number of sampled entries.

2) A social graph $G_1 = (\mathcal{V}_1, \mathcal{E}_1)$, where $\mathcal{V}_1$ is the set of $n$ user nodes. Let $\mathcal{B}$ be a $k_1 \times k_1$ symmetric connectivity matrix that represents the probabilities of connecting two nodes in $G_1$. Each pair of nodes $(i, i')$ is connected (i.e. $(i, i') \in \mathcal{E}_1$) independently w.p. $\mathcal{B}_{\sigma(i)\sigma(i')}$.

3) An item graph $G_2 = (\mathcal{V}_2, \mathcal{E}_2)$, where $\mathcal{V}_2$ is the set of $m$ item nodes. Let $\mathcal{B}'$ be a $k_2 \times k_2$ symmetric connectivity matrix that represents the probabilities of connecting two nodes in $G_2$. Each pair of nodes $(j, j')$ is connected (i.e. $(j, j') \in \mathcal{E}_2$) independently w.p. $\mathcal{B}'_{\tau(j)\tau(j')}$. 

### B. Objective

The learner is tasked to design an estimator $\hat{\phi} = \phi(\mathcal{U}, G_1, G_2)$ to exactly recover both the user clusters $\{\mathcal{U}_a\}_{a \in [k_1]}$ and item clusters $\{\mathcal{I}_b\}_{b \in [k_2]}$ (or equivalently, the label functions $\sigma$ and $\tau$), as well as to reconstruct the nominal matrix $\mathcal{N}$. The output of the estimator $\hat{\phi}$ is denoted by $\hat{\sigma}(\hat{\tau}, \hat{\mathcal{N}})$.

To measure the accuracies of the estimated label functions $\hat{\sigma}$ and $\hat{\tau}$, we define the classification proportions as

\[
l_1(\hat{\sigma}, \sigma) := \min_{\pi \in \mathcal{S}_{k_1}} \frac{1}{n} \sum_{i \in [n]} \mathbb{I}\{\hat{\sigma}(i) \neq \pi(\sigma(i))\},
\]

\[
l_2(\hat{\tau}, \tau) := \min_{\pi \in \mathcal{S}_{k_2}} \frac{1}{m} \sum_{j \in [m]} \mathbb{I}\{\hat{\tau}(j) \neq \pi(\tau(j))\},
\]

where $\mathcal{S}_{k_1}$ (resp. $\mathcal{S}_{k_2}$) is the set of all permutations of $[k_1]$ (resp. $[k_2]$). The permutations are introduced because it is only possible to recover the $\textit{partitions}$ of users/items, rather than the actual labels (i.e., the best we can hope for is to ensure $l_1(\hat{\sigma}, \sigma) = 0$ and $l_2(\hat{\tau}, \tau) = 0$).

Furthermore, we also define the concept of weak recovery which plays a role in the intermediate steps of our algorithm.

**Definition 1.** An estimate $\hat{\sigma}$ (resp. $\hat{\tau}$) is said to achieve weak recovery if the misclassification proportion $l_1(\hat{\sigma}, \sigma) \to 0$ as $n$ tends to infinity (resp. $l_2(\hat{\tau}, \tau) \to 0$ as $m$ tends to infinity).

### III. MC2G: A Computationally Efficient, Statistically Optimal Algorithm

In this section, we present a computationally efficient multi-stage algorithm called MC2G for recovering the clusters of users and items, and the nominal matrix $\mathcal{N}$, given the social and item similarity graphs. Knowledge of the model parameters (e.g., connectivity matrices $\mathcal{B}$ and $\mathcal{B}'$ and personalized distribution $Q_{\mathcal{V}|\mathcal{Z}}$) is not needed for MC2G to succeed, as they will be estimated on-the-fly. Roughly speaking, MC2G consists of four stages: Stage 1 achieves weak recovery of the user/item clusters; Stage 2 estimates the model parameters $\mathcal{B}$, $\mathcal{B}'$, the label functions $\hat{\sigma}(\hat{\tau}, \hat{\mathcal{N}})$, and the personalized rating $V_{ij}$.
B', and $Q_{V|Z}$; and Stages 3 and 4 respectively refine these estimates of users and items via local refinements steps. The inputs include the sub-sampled rating matrix $U$ and two graphs $G_1$ and $G_2$.

Before describing our algorithm Mc2G in detail in Subsection III-B, we want to first point out a common issue that often arises in the analysis of multi-stage algorithms. When analyzing the error probability of multi-stage algorithms, one needs to be cognizant of the dependencies between random variables in different stages. For example, a pair of random variables that are initially independent may become dependent conditioned on the success of a preceding stage. We circumvent this issue by using an information splitting method inspired by prior works [17], [32], [41] on community detection. As a concrete example, Fig. 2 illustrates how we split the information of the social graph into two pieces, where the first piece is for Stage 1 and the second piece is for subsequent stages. Information splitting can be viewed as a preliminary step for our algorithm Mc2G, and is formally described in Section III-A.

**Remark 1.** An alternative approach to circumvent the aforementioned issue of dependence is to use the so-called uniform analysis technique, which has been adopted by some other works [4], [5], [42]. However, this requires more rounds of local refinements and thus increases the computational complexity.

**A. Information splitting**

The high-level idea is to split the observations $(U, G_1, G_2)$ into two parts—the first part, denoted as $(G^a_1, G^b_2)$, is used for weak recovery of users and items in Stage 1; while the second part, denoted as $(U, G^b_1, G^a_2)$, is used for estimating the parameters and for local refinements (exact recovery) of each user and item in Stages 2–4. We elaborate on the information splitting method as follows.

1) Let $H_1 = (V_1, \tilde{E}_1)$ be the complete graph with vertex set $V_1 = [n]$ and edge set $\tilde{E}_1$, which contains all the $\binom{n}{2}$ edges on $V_1$. We randomly partition $H_1$ into two sub-graphs $H^a_1 = (V_1, \tilde{E}^a_1)$ and $H^b_1 = (V_1, \tilde{E}^b_1)$ such that $H^a_1$ is an Erdős-Rényi (ER) graph on $V_1$ with edge probability $1/\sqrt{\log n}$. That is, each $e \in \tilde{E}_1$ is sampled (independently) to $\tilde{E}^a_1$ with probability $1/\sqrt{\log n}$, and to $\tilde{E}^b_1$ with probability $1 - 1/\sqrt{\log n}$, where $\tilde{E}^b_1$ is the complement of $\tilde{E}^a_1$. An example is illustrated in Fig. 2.

This partition is done independently of the generation of the SBM $G_1$. For any realizations $H^a_1 \cap G_1$ and $H^b_1 \cap G_1$, let $G^a_1 := h^a_1 \cap G_1$ and $G^b_1 := h^b_1 \cap G_1$. be two sub-SBMs on sub-graphs $h^a_1$ and $h^b_1$, respectively. Similarly, let $H_2 = (V_2, \tilde{E}_2)$ be the complete graph with vertex set $V_2 = [m]$ and edge set $\tilde{E}_2$. $H^a_2$ is an ER graph on $V_2$ with edge probability $1/\sqrt{\log m}$, and $\tilde{E}^b_2$ is the complement of $\tilde{E}^a_2$. For any $H^a_2 = h^a_2$ and $H^b_2 = h^b_2$, we also define $G^a_2 := h^a_2 \cap G_2$ and $G^b_2 := h^b_2 \cap G_2$.

**Algorithm 1: Mc2G**

**Input:** $(U, G_1, G_2) = (G^a_1, G^b_2) \cup (U, G^b_1, G^a_2)$

**Output:** Clusters $\{\hat{U}_a\}_{a \in [k_1]}$ and $\{\hat{I}_b\}_{b \in [k_2]}$ (or label functions $\hat{\sigma}$ and $\hat{\tau}$), nominal matrix $\hat{N}$

**Stage 1 (Weak recovery of communities)**

Apply the spectral clustering method on $G^a_1$ and $G^b_2$ to obtain initial estimates $\{\hat{U}_a^{(0)}\}_{a \in [k_1]}$ and $\{\hat{I}_b^{(0)}\}_{b \in [k_2]}$.

**Stage 2 (Parameters estimation)**

Estimate connectivity matrices $B$ and $B'$ as per (5)-(6); Estimate personalization distribution $\{Q_{ab}\}$ as per (7).

**Stage 3 (Local refinements of users)**

for user $i = 1$ to $n$ do

- Calculate likelihood functions $\{L_a(i)\}_{a \in [k_1]}$;
- Let $a^*_i = \arg \max_{a \in [k_1]} L_a(i)$, and declare $i \in \hat{U}_a^*$;
end

**Stage 4 (Local refinements of items)**

for item $j = 1$ to $m$ do

- Calculate likelihood functions $\{L_b(j)\}_{b \in [k_2]}$;
- Let $b^*_j = \arg \max_{b \in [k_2]} L_b(j)$, and declare $j \in \hat{I}_b^*$;
end

Reconstruct the nominal matrix $\hat{N}$ as per (10).

**B. Algorithm description**

**Stage 1 (Weak recovery of clusters):** We run a spectral clustering method (e.g., Algorithm 2 in [17]) on the social graph $G^a_1$ to obtain an initial estimate of the label function $\sigma$ (denoted by $\sigma^{(0)}$), and also run a spectral clustering method on the item graph $G^b_2$ to obtain an initial estimate of the label function $\tau$ (denoted by $\tau^{(0)}$). The estimated user clusters corresponding to $\sigma^{(0)}$ are denoted by $\{\hat{U}_a^{(0)}\}_{a \in [k_1]}$ (i.e., $U_a^{(0)} = (\sigma^{(0)})^{-1}(a)$), and the estimated item clusters corresponding to $\tau^{(0)}$ are denoted by $\{\hat{I}_b^{(0)}\}_{b \in [k_2]}$. These initial estimates $\sigma^{(0)}$ and $\tau^{(0)}$ are expected to serve as good approximations of the

3With a slight abuse of notations, we use $h^a_1 \cap G_1$ (resp. $h^b_1 \cap G_1$) to represent a graph with edge set being the intersection between the edge sets of $h^a_1$ (resp. $h^b_1$) and $G_1$. More specifically, for the sub-SBM $G^a_1$ (resp. $G^b_1$), any pairs of nodes $(i, i')$ are connected with probability $B_{aa',\sigma'}$, if $(i, i') \in \tilde{E}^a_1$ (resp. $(i, i') \in \tilde{E}^b_1$), and with probability zero otherwise.

4To achieve weak recovery of the clusterings of users and items, one can also apply different variants of spectral clustering methods [17], [19], [41], semidefinite programming-based methods [44], belief propagation-based methods [45], or non-backtracking matrix-based methods [46].
true clusters, such that both $\sigma^{(0)}$ and $\tau^{(0)}$ satisfy the weak recovery criteria defined in Definition 1.

Stage 2 (Parameters estimation): For any two sets of nodes $V$ and $V'$, the number of edges connecting $V$ and $V'$ (in $G_1^b$ or $G_2^b$) is denoted as $e(V, V')$. Based on the initial estimates $\{U^{(0)}_a\}_{a \in [k]}$ and $\{I^{(0)}_b\}_{b \in [K]}$, we then obtain the MLE for the connectivity matrices $B$ and $B'$ of the social and item graphs

$$B_{aa'} = \begin{cases} e(U^{(0)}_a, U^{(0)}_{a'}) / \left(\sum_{i=1}^{k} I^{(0)}_i \right), & \text{if } a = a'; \\ e(U^{(0)}_a, U^{(0)}_{a'}) / \left(\sum_{i=1}^{k} I^{(0)}_i \right), & \text{if } a \neq a'; \end{cases}$$

(5)

$$B_{bb'} = \begin{cases} e(I^{(0)}_b, I^{(0)}_{b'}) / \left(\sum_{i=1}^{k} I^{(0)}_i \right), & \text{if } b = b'; \\ e(I^{(0)}_b, I^{(0)}_{b'}) / \left(\sum_{i=1}^{k} I^{(0)}_i \right), & \text{if } b \neq b'; \end{cases}$$

(6)

where $a, a' \in [k]$ and $b, b' \in [K]$. Moreover, we define sets of $(i, j)$-pairs $Q^a_{ab}$ (where $a \in [k], b \in [k]$, and $z \in \mathbb{Z}$) as

$$Q^a_{ab} := \left\{ (i, j) : U_{ij} = z, i \in U^{(0)}_a, j \in I^{(0)}_b \right\}.$$  

Then, the estimated personalization distribution is given by

$$\hat{Q}_{ab}(z) := \frac{|Q^a_{ab}|}{\sum_{z \in \mathbb{Z}} |Q^a_{ab}|}, \quad \forall a \in [k], b \in [k].$$

(7)

Stage 3 (Local refinements of users): This stage refines the classification of each user locally, based on the ratings in $U$, the social graph $G_1^b$, and the initial estimates $\{U^{(0)}_a\}_{a \in [k]}$ and $\{I^{(0)}_b\}_{b \in [K]}$. For each user $i \in [n]$, we essentially adopt a local MLE to determine which cluster it belongs to. We define the likelihood function that reflects how likely user $i$ belongs to cluster $U_a$ as:

$$L_a(i) := \sum_{a' \in [k]} e\left\{ i, U^{(0)}_{a'} \right\} \cdot \log \left( \hat{B}_{aa'} / (1 - \hat{B}_{aa'}) \right) + \sum_{b \in [k]} \sum_{j \in I^{(0)}_b} \mathbb{I}\{U_{ij} \neq \epsilon\} \cdot \log \hat{Q}_{ab}(U_{ij}).$$

(8)

Let $a^*_i := \arg \max_{a \in [k]} L_a(i)$ be the index of the most likely user cluster for user $i$. Mc2G then declares $i \in U_{a^*_i}$; or equivalently, $\hat{\sigma}(i) = a^*_i$.

Stage 4 (Local refinements of items): This stage refines the classification of each item locally, based on $U$, $G_2^b$, and the initial estimates $\{U^{(0)}_a\}_{a \in [k]}$ and $\{I^{(0)}_b\}_{b \in [K]}$. For each item $j \in [m]$, we define the likelihood function that reflects how likely item $j$ belongs to cluster $I_b$ as:

$$L_b(j) := \sum_{b' \in [K]} e\left\{ j, I^{(0)}_{b'} \right\} \cdot \log \left( \hat{B}_{bb'} / (1 - \hat{B}_{bb'}) \right) + \sum_{a \in [k]} \sum_{\epsilon \in I^{(0)}_a} \mathbb{I}\{U_{ij} \neq \epsilon\} \cdot \log \hat{Q}_{ab}(U_{ij}).$$

(9)

Let $b^*_j := \arg \max_{a \in [k]} L_b(j)$ be the index of the most likely item cluster for item $j$. Mc2G then declares $j \in I_{b^*_j}$; or equivalently, $\hat{\tau}(j) = b^*_j$.

Finally, one can recover the nominal matrix $\hat{N}$ by setting

$$\hat{N}_{ij} = \arg \max_{z \in \mathbb{Z}} \hat{Q}_{ab}(z), \quad \text{for } i \in U_{a}, j \in I_{b}.$$  

(10)

Remark 2. The information splitting method introduced in this section is merely for the purpose of analysis (as discussed in the second paragraph of this section); however, it may not be practical when $n$ and $m$ are not sufficiently large, in which case the first part of the graphs ($G_1^1, G_2^1$) may be too sparse to achieve weak recovery of the true clusters in Stage 1. Thus, in practice, instead of splitting the graphs ($G_1, G_2$) into ($G_1^1, G_2^1$) (on which Stage 1 is applied) and ($G_1^2, G_2^2$) (on which Stages 2–4 are applied), one can skip the information splitting step in Section III-A and simply apply every stage on the fully-observed graphs ($G_1, G_2$) for weak recovery, parameter estimations, and local refinements—this is referred to as the simplified version of Mc2G. In our experiments (Section VII), we adopt this simplified version of Mc2G, and show that it also works well on both synthetic and real datasets.

C. Computational Complexity

Using the iterative power method [48], the spectral clustering method used to obtain initial estimates of $G_1^1$ and $G_2^1$ run in times at most $O(|E_1| \log n)$ and $O(|E_2| \log m)$ respectively, where $|E_1| = O(n \log n)$ and $|E_2| = O(m \log m)$ with high probability. In each of the following steps, Mc2G requires (at most) a single pass of all the sub-sampled entries in the rating matrix $U$ and the edge sets $E_1$ and $E_2$, which amounts to at most $O(n \log n, m \log m)$ time. Therefore, the overall computational complexity is $O(\max\{n(\log n)^2, m(\log m)^2\})$ (i.e., quasilinear in $m \times n$) with high probability.

IV. THEORETICAL GUARANTEES OF MC2G AND INFORMATION-THEORETIC LOWER BOUNDS

This section provides theoretical guarantees for Mc2G. Under the symmetric setting defined in Subsection IV-A, we characterize the expected number of sampled entries required for Mc2G to succeed; the key message there is that this quantity depends critically on (i) the “qualities” of the social and item similarity graphs, and (ii) the squared Hellinger distance between the rating statistics of different user/item clusters. We further establish an information-theoretic lower bound on the expected number of sampled entries. This bound matches the achievability bound up to a constant factor, thus demonstrates the order-wise optimality of Mc2G.

A. The Symmetric Setting

Under the symmetric setting, it is assumed that (i) the user clusters are of equal size (i.e., $|U_a| = n/k_1$ for all $a \in [k_1]$) and the item clusters are of equal size (i.e., $|I_b| = m/k_2$ for all $b \in [k_2]$); and (ii) the connection probability for each pair of nodes depends only on whether they belong to the same cluster, i.e., the connectivity matrices $B$ and $B'$ satisfy

$$B_{aa'} = \begin{cases} \alpha_1, & \text{if } a = a'; \\ \beta_1, & \text{if } a \neq a'; \end{cases} \quad \text{and} \quad B'_{bb'} = \begin{cases} \alpha_2, & \text{if } b = b'; \\ \beta_2, & \text{if } b \neq b'; \end{cases}$$

Similar to the prior work [12], we assume $m = \omega(\log n)$ and $n = \omega(\log m)$ such that $m \to \infty$ as $n \to \infty$.

3We implicitly assume that $n$ is divisible by $k_1$ and $m$ is divisible by $k_2$. In the case that $n$ and $m$ are not multiples of $k_1$ and $k_2$ respectively, rounding operations required to define the set $\Xi$. Such rounding operations, however, do not affect the calculations and results downstream.
We note that Mc2G is not restricted to the symmetric setting; it can be applied more generally to asymmetric scenarios. Indeed, for the experiments in Section VII, we do not make the symmetric assumption. In this section, however, we make this assumption to simplify the presentation of Theorem I and to clearly understand the effect of the parameters of the model on the minimum expected number of sampled entries required for Mc2G to succeed.

In the following, we formally define the notion of exact recovery. Note that the model is governed by the pair of label functions \((\sigma, \tau)\) together with the nominal matrix \(N\), and we define the parameter space that contains all valid \((\sigma, \tau, N)\) under the symmetric setting as

\[
\Xi \triangleq \{ (\sigma, \tau, N) : \sigma : [n] \to [k_1], \{ i \in [n] : \sigma_i = a \} \leq n/k_1, \forall a \in [k_1]; \\
\tau : [m] \to [k_2], \{ j \in [m] : \tau_j = b \} \leq m/k_2, \forall b \in [k_2]; \\
N \in \mathbb{Z}^{n \times m}, N_{ij} = N_{c',j}, \text{ if } \sigma(i) = \sigma(i') \text{ and } \tau(j) = \tau(j') \}.
\]

Let \((\sigma, \tau, N)\) be the ground truth, and \((\hat{\sigma}, \hat{\tau}, \hat{N})\) be the output of the estimator \(\hat{\phi} = \phi(U, G_1, G_2)\). We say the event \(\mathcal{E}(\sigma, \tau, N)\) occurs if the output \((\hat{\sigma}, \hat{\tau}, \hat{N})\) of the estimator \(\phi\) satisfies one of the following three criteria: (i) \(\{I_1(\hat{\sigma}, \hat{\tau}) \neq 0\}\), (ii) \(\{I_2(\hat{\sigma}, \hat{\tau}) \neq 0\}\), and (iii) \(\{N \neq \hat{N}\}\).

**Definition 2** (Exact recovery). For any estimator \(\phi\), its corresponding (maximum) error probability is defined as

\[
P_{\text{err}}(\phi) := \max_{(\sigma, \tau, N) \in \Xi} P(\sigma, \tau, N)(\phi(U, G_1, G_2) \in \mathcal{E}(\sigma, \tau, N)),
\]

where \(P(\sigma, \tau, N)(\cdot)\) is the probability that \((U, G_1, G_2)\) is generated according to the model governed by \((\sigma, \tau, N)\). A sequence of estimators \(\Phi = \{\phi_n\}_{n=1}^{\infty}\) achieves exact recovery if

\[
\lim_{n \to \infty} P_{\text{err}}(\phi_n) = 0.
\]

**Definition 3** (Sample complexity). The sample complexity is defined as the minimum expected number of samples in the matrix \(U\) such that there exists \(\Phi\) for which (11) holds.

### B. Theoretical guarantees of Mc2G

As we shall see, the “qualities” of the social and item graphs play a key role in the performance of Mc2G. Specifically, we define a measure of the quality of the social graph \(G_1\) as

\[
I_1 := n(\alpha_1 - \sqrt{\beta_1})^2/\log n.
\]

A larger value of \(I_1\) implies a better quality of the graph, since the structures of the clusters are more clearly delineated when the difference between the intra-cluster probability \(\alpha_1\) and the inter-cluster probability \(\beta_1\) is larger. Analogously, we define a measure of the quality of the item graph \(G_2\) as

\[
I_2 := m(\sqrt{\alpha_2} - \sqrt{\beta_2})^2/\log m.
\]

The performance of Mc2G also depends on the statistics of the rating matrix. Intuitively, if the rating statistics of two clusters are further apart, it is then easier to distinguish them. It turns out that under the symmetric setting, the distance between the rating statistics of different clusters can be measured by the squared Hellinger distance:

\[
H^2(P, Q) := 1 - \sum_{z \in \mathbb{Z}} \sqrt{P(z)Q(z)},
\]

for probability distributions \(P\) and \(Q\). We then define the discrepancy between user clusters \(U_a\) and \(U_{a'}\) as a measure of the discrepancy between user clusters \(U_a\) and \(U_{a'}\) (where \(a, a' \in [k_1]\)), and

\[
d_{U} := \min_{a \neq a'} d_{U}(U_a, U_{a'})
\]

as the minimal discrepancy over all pairs of user clusters. A larger value of \(d_{U}\) means that it is easier to distinguish all the user clusters. Analogously, we define the discrepancy between item clusters \(I_a\) and \(I_{a'}\) as

\[
d_{I} := \min_{b \neq b'} d_{I}(I_b, I_{b'})
\]

as the minimal discrepancy over all pairs of item clusters.

**Remark 3.** The squared Hellinger distance \(H^2(P, Q)\) satisfies \(H^2(P, Q) \in [0, 1]\) and \(H^2(P, Q) = 0\) if and only if \(P = Q\).

Theorem I below states the expected number of sampled entries needed for Mc2G to succeed under the symmetric setting.

**Theorem 1** (Performance of Mc2G). For any \(\epsilon > 0\), if the expected number of sampled entries \(mnp\) satisfies

\[
\text{max} \left\{ \frac{(1+\epsilon) - \frac{I_1}{k_2}}{d_{U}/k_2}, \frac{(1+\epsilon) - \frac{I_2}{k_1}}{d_{I}/k_1} \right\}
\]

then Mc2G ensures \(P_{\text{err}} \to 0\) as \(n \to \infty\).

**Remark 4.** Some remarks on Theorem I are in order.

1) Roughly speaking, the first term on the RHS of (12) is the threshold for Stage 3 (local refinements of users) to succeed. This is because when \(mnp\) exceeds the first term, the probability that a single user is misclassified to an incorrect cluster (in Stage 3) is at most \(n^{-\ell}\) for some \(\ell > 1\). Thus, taking a union bound over all the \(n\) users still results in a vanishing error probability. Similarly, the second term on the RHS of (12) is the threshold for Stage 4 (local refinements of items) to succeed.

2) Our result in (12) confirms our intuitive belief that increasing \(d_{U}\) and \(d_{I}\) (the minimum discrepancies between user and item clusters) indeed helps to reduce the number of samples required for exact recovery. Similarly, increasing \(I_1\) and \(I_2\) (the qualities of the social and item graphs) also helps to reduce the sample complexity.

3) It is also worth noting that when \(I_1 > k_1\), the first term in (12) becomes non-positive (thus inactive); this means that performing local refinements of users in Stage 3 is no longer needed, which is due to the fact that the spectral clustering method in Stage 1 has already ensured exact recovery of the \(k_1\) user clusters. This observation coincides with the theoretical result of community detection in the symmetric SBM [17], which states that exact recovery of \(k_1\) clusters is possible when \(I_1 > k_1\). Similarly, when \(I_2 > k_2\), performing local refinements of items in Stage 4 is no longer needed, as the spectral clustering method in Stage 1 has already ensured exact recovery of the \(k_2\) item clusters.

4) While the theoretical result in Theorem I is dedicated to this symmetric setting, Mc2G is applicable to a more general matrix completion problem with social and item similarity graphs, where the sizes of user/item clusters may be different. This is confirmed by the experiments in Section VII.
C. Information-theoretic lower bound

Theorem 2 below provides an information-theoretic lower bound on the sample complexity under the symmetric setting. Again, the lower bound is a function of \( I_1, I_2 \) (the quality of the social/item graph), and \( d_{t1} \) and \( d_{t2} \) (the minimum discrepancies measured in terms of the squared Hellinger distances of user/item clusters).

**Theorem 2 (Impossibility result).** For any \( \epsilon > 0 \), if
\[
mnp \leq \max \left\{ \frac{\left(\frac{1}{2} - \frac{1}{k_1}\right) n \log n}{d_{t1}/k_2}, \frac{\left(\frac{1}{2} - \frac{1}{k_2}\right) m \log m}{d_{t2}/k_1} \right\},
\]
(13) then \( \lim_{n \to \infty} P_{\text{err}}(\phi) = 1 \) for any estimator \( \phi \).

Theorem 2 states that any estimator must necessarily fail if the expected number of samples is smaller than the maximal term in (13). Thus, the sample complexity defined in Definition 1 is upper-bounded by the RHS of (12), and lower-bounded by the RHS of (13). In particular, Theorem 2 guarantees that \( P_{\text{err}} \) approaches one as \( n \to \infty \); this is the so-called strong converse [49] in the information theory parlance. Comparing (13) with the achievability bound in (12), we note that they match up to a constant factor, and this further demonstrates the order-wise optimality of the proposed computationally efficient algorithm Mc2G.

V. PROOF OF THEOREM 1

Analysis of Stage 1: Note that the sub-SBM \( G_1^n \) is generated on the sub-graph \( h_1^n \), thus the performance of the spectral clustering method on \( G_1^n \) essentially depends on the realization \( h_1^n \). A similar argument also applies to \( G_2^n \).

To circumvent the difficulties of analyzing fixed \( h_1^n \) and \( h_2^n \), we first consider two artificial SBMs \( G_1^* \) and \( G_2^* \), where \( G_1^* \) is generated on the \( n \) user nodes and has connectivity matrix \( B/\sqrt{\log n} \), and \( G_2^* \) is generated on the \( m \) item nodes and has connectivity matrix \( B'/\sqrt{\log m} \). A priori result in [47] Theorem 6] shows that there exist vanishing sequences \( \epsilon_n, \eta_n, \) and \( \gamma_n \) (depending on \( B \) and \( B' \)) such that with probability at least \( 1 - \epsilon_n \), the spectral clustering method running on \( G_1^* \) and \( G_2^* \) respectively ensure that
\[
l_1(\sigma(0), \sigma) \leq \eta_n \quad \text{and} \quad l_2(\tau(0), \tau) \leq \gamma_n.
\]
(14)

Based on the good performances of spectral clustering methods running on \( G_1^* \) and \( G_2^* \), we next show that spectral clustering methods running on \( G_1^n \) and \( G_2^n \) also provide satisfactory initialization results with high probability.

**Definition 4.** Let \( h = (h_1^n, h_2^n, h_3^n, h_4^n) \) be an aggregation of realizations of the sub-graphs.

1) A sub-graph \( h_1^n \) is said to be good if the probability that “a spectral clustering method running on \( G_1^n \) (which depends on \( h_1^n \)) ensures \( l_1(\sigma(0), \sigma) \leq \eta_n \)” is at least \( 1 - \sqrt{\epsilon_n} \). A sub-graph \( h_2^n \) is said to be good if the degree of any node in \( h_2^n \) is at least \( n/(1 - 2/\sqrt{\log n}) \).

2) A sub-graph \( h_3^n \) is said to be good if the probability that “a spectral clustering method running on \( G_2^n \) ensures \( l_2(\tau(0), \tau) \leq \gamma_n \)” is at least \( 1 - \sqrt{\epsilon_n} \). A sub-graph \( h_4^n \) is said to be good if the degree of any node in \( h_4^n \) is at least \( m/(1 - 2/\sqrt{\log m}) \).

3) Let \( G' \) and \( B' \) be two disjoint sets of \( h \). We say \( h \in G' \) if all the elements in \( h \) are good, and \( h \in B \) otherwise.

**Lemma 1.** The randomly generated sub-graphs \( H_1^n, H_1^b, H_2^n, H_2^b \) are all good with probability at least \( (1 - \sqrt{\epsilon_n})^2 \). Equivalently, we have
\[
\sum_{h \in G} P(h) \geq (1 - \sqrt{\epsilon_n})^2.
\]
(15)

**Proof.** See Appendix A.

We define \( G' \) as the set of label functions that are close to the true label functions \( (\sigma, \tau) \), i.e.,
\[
G' := \left\{ (\sigma', \tau') : l_1(\sigma', \sigma) \leq \eta_n, l_2(\tau', \tau) \leq \gamma_n \right\},
\]
(16)

and \( B' := \left\{ (\sigma', \sigma) : (\sigma', \sigma) \notin G' \right\} \) as the complement of \( G' \). By definition, we know that when the randomly generated sub-graphs \( h \in G \), running spectral clustering methods on \( G_1^n \) and \( G_2^n \) yields \( (\sigma(0), \tau(0)) \in G' \) with high probability, i.e.,
\[
\sum_{(\sigma(0), \tau(0)) \in G'} P((\sigma(0), \tau(0))) \geq (1 - \sqrt{\epsilon_n})^2,
\]
(17)

which is uniform in \( h \in G \) (i.e., the sequence \( \{\epsilon_n\} \) does not depend on \( h \)).

**Remark 5.** Lemma 7 above conveys two important messages: (i) Although the sub-graphs \( H_1^n \) and \( H_2^n \) are much sparser compared to \( H_1^b \) and \( H_2^b \) (or equivalently, the information contained in \( H_1^n \) and \( H_2^n \) is much less), they still guarantee the success of running spectral clustering methods (with high probability). (ii) The densities of sub-graphs \( H_1^b \) and \( H_2^b \) are almost the same as those of \( H_1^n \) and \( H_2^n \), and this property is critical in Stages 2–4 for proving the theoretical guarantees of Mc2G.

Analysis of Stage 2: Note that the estimates \( \hat{B}, \hat{B'} \) depend on both \( h \) and \( (\sigma(0), \tau(0)) \). In Stage 2, we show in Lemmas 2 and 3 below that conditioned on \( h \in G \) and \( (\sigma(0), \tau(0)) \in G' \), the estimates are accurate with high probability.

**Lemma 2.** Suppose \( h \in G \) and \( (\sigma(0), \tau(0)) \in G' \). With probability \( 1 - o(1) \), there exists a sequence \( \varepsilon_n \in \Omega(\max\{\gamma_n, \eta_n, 1/\sqrt{\log n}\}) \) such that for all \( a, a' \in [k_1] \) and \( b, b' \in [k_2] \),
\[
|B_{aa'} - B_{aa'}'| \leq \varepsilon_n, \quad |B_{bb'} - B_{bb'}'| \leq \varepsilon_n.
\]
(18)

**Proof.** See Appendix B.

**Lemma 3.** Suppose \( h \in G \) and \( (\sigma(0), \tau(0)) \in G' \). With probability \( 1 - o(1) \), there exists a sequence \( \varepsilon'_n \in \Omega(\max\{\gamma_n, \eta_n, 1/\sqrt{\log n}\}) \) such that for all \( a \in [k_1] \), \( b \in [k_2] \), and \( z \in \mathbb{Z} \),
\[
|\hat{Q}_{ab}(z) - Q_{ab}(z)| - 1 \leq \varepsilon'_n.
\]
(19)

**Proof.** See Appendix C.

**Remark 6.** In Lemmas 2 and 3 above, we implicitly assume (without loss of generality) that the permutations minimizing
l_1(\sigma(0), \sigma) and l_2(\tau(0), \tau) are both the identity permutation, i.e., l_1(\sigma(0), \sigma) = \sum_{i \in [n]} 1{\sigma(0)(i) \neq \sigma(i)}/n and l_2(\tau(0), \tau) = \sum_{j \in [m]} 1{\tau(0)(j) \neq \tau(j)}/m as Per Eqns. (1) and (2). Without this assumption, one needs to introduce the permutations \pi_1^* and \pi_2^* that respectively minimize l_1(\sigma(0), \sigma) and l_2(\tau(0), \tau)—this unnecessarily complicates the presentations of Lemmas 2 and 3 e.g., (19) will be written as

\[ \left| \left( \frac{Q_{\bar{a}b}(z)}{Q_{\pi_1^*(a)\pi_2^*(b)}(z)} \right) - 1 \right| \leq \epsilon_n. \]

The same assumptions are made in the analysis of Stages 3 and 4 below.

Analysis of Stage 3: Note that the likelihood function defined in (8) depends on the estimated values \(\hat{B}\) and \(\{\hat{Q}_{ab}\}\) of the model parameters. For ease of analysis, we first ignore the imprecisions of these estimates, and define the exact likelihood function \(\hat{L}_a(i)\), which depends on the exact values of \(B\) and \(Q_{ab}\), as

\[ \hat{L}_a(i) := \sum_{a' \in [k_1]} e(i, B_{a'a'}) \cdot \log (B_{a'a'}/(1 - B_{a'a'})) + \sum_{b \in [k_2]} \sum_{j \in Z_{ij}^{(0)}} 1\{U_{ij} \neq e\} \cdot \log Q_{ab}(U_{ij}). \] (20)

We now consider a specific user \(i \in [n]\), which belongs to cluster \(U_a\) for some \(a \in [k_1]\). Lemma 4 below shows that, with probability \(1 - o(1/n)\), \(\hat{L}_a(i)\) is larger than any other likelihood functions \(\hat{L}_b(i)\) by at least \((e/2)\log n\).

**Lemma 4.** Suppose \(h \in G\) and \((\sigma(0), \tau(0)) \in G'\). If

\[ \text{mnp} \geq \frac{[(1 + \epsilon) - (I_1/k_1)] n \log n}{d_q/k_2}, \] (21)

with probability at least \(1 - n^{-(1+\frac{4}{3})}\),

\[ \hat{L}_a(i) > \max_{a' \in [k_1]} \hat{L}_{a'}(i) + (e/2) \log n. \] (22)

**Proof.** Consider a specific \(a \neq \bar{a}\). Under the symmetric setting, the entries in the connectivity matrix \(B\) are either \(a_1^*\) or \(a_2^*\); thus, we define \(\lambda_1 := \log (1 - a_1^* / (1 - a_1^*)^2)\) and one can show that

\[ \hat{L}_a(i) - \hat{L}_{\bar{a}}(i) = \lambda_1 e(i, B_{a_1^*}) - \lambda_1 e(i, B_{\bar{a}2^*}) + \sum_{b \in [k_2]} \sum_{j \in Z_{ij}^{(0)}} 1\{U_{ij} \neq e\} \cdot \log Q_{ab}(U_{ij}). \] (23)

For \(a, \bar{a} \in [k_1]\), let \(S_{\bar{a}a} := U_{\bar{a}} \cap U_{a}^{(0)}\) be the set of users that belong to cluster \(U_{\bar{a}}\) and are classified to \(U_{a}^{(0)}\) after Stage 1. By introducing random variables \(X_k\) \(\overset{i.i.d.}{\sim}\) Bern(\(\alpha_1\)) and \(Y_k\) \(\overset{i.i.d.}{\sim}\) Bern(\(\beta_1\)), one can rewrite \(e(i, B_{a_1^*}) - e(i, B_{\bar{a}2^*})\) as

\[ \sum_{k \in S_{\bar{a}a}} X_k + \sum_{k \in U_{\bar{a}} \setminus U_{a}} Y_k - \sum_{k \in U_{a}^{(0)}} Y_k - \sum_{k \in S_{\bar{a}a}} X_k. \] (24)

For \(b, \bar{b} \in [k_2]\), let \(T_{bb} := T_{b} \cap T_{\bar{b}}^{(0)}\) be the set of items that belong to cluster \(T_b\) and are classified to \(T_{\bar{b}}^{(0)}\) after Stage 1. By introducing random variables \(T_{ij}\) \(\overset{i.i.d.}{\sim}\) Bern(\(\pi_1^*(a)\)) and \(Z_{ij}^{(0)}\) \(\overset{i.i.d.}{\sim}\) \(Q_{ab}\), one can rewrite the second part in (23) as

\[ \sum_{b \neq \bar{b}} \sum_{i \in T_{bb}} \left[ \sum_{j \in T_{bj}} \log \frac{Q_{ab}(Z_{ij}^{(0)})}{Q_{\pi_1^*(a)\pi_2^*(b)}(Z_{ij}^{(0)})} \right] + \sum_{b \neq \bar{b}} \sum_{j \in T_{bj}} \log \frac{Q_{ab}(Z_{ij}^{(0)})}{Q_{\pi_1^*(a)\pi_2^*(b)}(Z_{ij}^{(0)})}. \]

(25)

Representing \(\hat{L}_{aa'}(i)\) in terms of (24) and (25), and applying the Chernoff bound \(P(X > \kappa) \leq \min_{m > 0} e^{-t\kappa} \cdot E(e^{tX})\) with \(t = 1/2\), we then have

\[ P \left( \hat{L}_a(i) - \hat{L}_{\bar{a}}(i) < (e/2) \log n \right) \]

\[ \leq \exp \left[ \frac{(e/2) \log n - (1 - o(1)) I_1 \log n}{k_1} \right] - (1 - o(1)) \sum_{b \neq \bar{b}} m_p H^2(Q_{ab}, Q_{\bar{b}b})/k_2 \] (26)

\[ \leq n^{-(1+\frac{4}{3})}, \] (27)

where (26) follows from the facts that (i) \(E(e^{-\frac{1}{2}A_{ij}^{(s)}}) = 1 - pH^2(Q_{ab}, Q_{\bar{b}b})\), (ii) \(E(e^{-\frac{1}{2}Y_k}) = I_1 \log n/n\), and (iii) the misclassification proportions in \(\sigma(0)\) and \(\tau(0)\) are negligible, i.e., \(|S_{aa'}| \geq (\frac{1}{k_1} - o(1))n\), \(U_{a}^{(0)} \cap U_{b}^{(0)} \geq (\frac{1}{k_2} - o(1))n\), and \(|Tbb| \geq (\frac{1}{k_2} - o(1))m\). Eqn. (27) holds since \(m_p H^2(Q_{ab}, Q_{\bar{b}b})\). Finally, by taking a union bound over all the clusters \(U_b\) such that \(\bar{a} \in [k_1] \setminus \{a\}\), we complete the proof of Lemma 4.

Note that Lemma 4 is for a specific user \(i \in [n]\). Taking a union bound over the \(n\) users yields that with probability \(1 - o(1)\), all the users \(i \in [n]\) satisfy

\[ \hat{L}_a(i) > \max_{a' \in [k_1]} \hat{L}_{a'}(i) + (e/2) \log n. \] (28)

where \(\sigma(i)\) is the user cluster that user \(i\) belongs to.

Finally, it is shown in Lemma 5 below that the difference between the exact likelihood function \(\hat{L}_a(i)\) and the original likelihood function \(\hat{L}_a(i)\) is negligible.

**Lemma 5.** With probability \(1 - o(1)\), there exists a sequence \(\xi_n \in \Omega(\max\{\epsilon_n, \epsilon'_n\}) \cap o(1)\) such that for all \(a \in [k_1]\) and all users \(i \in [n]\), \(|\hat{L}_a(i) - \hat{L}_a(i)| \leq \xi_n \log n\).

Theproof of Lemma 5 can be found in Appendix D. Combining (25) and Lemma 5 via the triangle inequality, we have that all the users satisfy \(L_{\sigma(i)}(i) > \max_{a \in [k_1]} L_{a}(i)\). This ensures the success of Stage 3, i.e., \(\hat{\sigma}(i) = \sigma(i)\) for all \(i \in [n]\).

**Analysis of Stage 4:** The analysis of Stage 4 is similar to that of Stage 3. Lemma 6 below states that all the \(m\) items can be classified into the correct cluster when \(\text{mnp} \) satisfies (29).

**Lemma 6.** Suppose \(h \in G\) and \((\sigma(0), \tau(0)) \in G'\). If

\[ \text{mnp} \geq \frac{[(1 + \epsilon) - (I_2/k_2)] m \log m}{d_q/k_1}, \] (29)

with probability \(1 - o(1)\), all the items \(j \in [m]\) satisfy

\[ L'_{\tau(j)}(j) > \max_{b \in [k_2] \setminus \{\tau(j)\}} L_{b}^{(j)}. \] (30)


where \( \sigma \) is uniform in \( \tau \), and we then set
\[
\hat{N}_{ij} = u_{ab}, \quad \text{if } i \in \hat{U}_a, j \in \hat{I}_b.
\]
(31)

The correctness of (31) follows from the fact that
\[
\sum_{i \in \hat{U}_a} \sum_{j \in \hat{I}_b} \mathbb{1} \{U_{ij} = z\} \approx mnQ_{ab}(z)/(k_1 k_2)
\]
which is due to the Chernoff bound.

A. The Overall Success Probability

Let \( E_{\text{succ}} \) be the event that Mc2G exactly recovers the nominal matrix. From the analyses of Stages 2–4, we know that for any \( h \in G \) and \( (\sigma^{(0)}, \tau^{(0)}) \in G' \),
\[
P(E_{\text{succ}}|h, (\sigma^{(0)}, \tau^{(0)})) \geq 1 - o(1),
\]
(32)
where (32) is uniform in \( h \in G \) and \( (\sigma^{(0)}, \tau^{(0)}) \in G' \). Therefore, the overall success probability is lower bounded as
\[
P(E_{\text{succ}}) = \sum_{h \in G} P(h)P(E_{\text{succ}}|h) + \sum_{h \in \Sigma_h} P(h)P(E_{\text{succ}}|h)
\geq \sum_{h \in G} P(h) \sum_{(\sigma^{(0)}, \tau^{(0)}) \in G'} P((\sigma^{(0)}, \tau^{(0)}))
\times [P(E_{\text{succ}}|h, (\sigma^{(0)}, \tau^{(0)}))]
\geq (1 - o(1)) \sum_{h \in G} P(h) \sum_{(\sigma^{(0)}, \tau^{(0)}) \in G'} [P((\sigma^{(0)}, \tau^{(0)}))] \]
(33)
\geq (1 - o(1))(1 - \sqrt{\epsilon_m})^2
= (1 - o(1)),
(34)
where (33) is due to (32), and (34) follows from (17).

VI. PROOF SKETCH OF THEOREM 2

The proof techniques used for Theorem 2 is a generalization of the techniques used in [12] Sec. IV-B], thus we only provide a proof sketch here. The key idea is to first show that the maximum likelihood (ML) estimator \( \phi_{\text{ML}} \) is the optimal estimator (as proved in [12] Eqn. (33)), and then analyze the error probability with respect to \( \phi_{\text{ML}} \)—the crux of the analysis is to focus on a subset of events that are most likely to induce errors, and to prove the tightness of the Chernoff bound.

To analyze \( \phi_{\text{ML}} \), we first show that under the model parameter \( (\sigma, \tau, N) \) (where a single parameter \( \xi \) is used to be the abbreviation of \( (\sigma, \tau, N) \) in the following), the log-likelihood of observing \( (U, G_1, G_2) \) is
\[
\log P_\xi(U, G_1, G_2) = c_1 \log \frac{\beta_1 (1 - \alpha_1)}{\alpha_1 (1 - \beta_1)} + c_2 \log \frac{\beta_2 (1 - \alpha_2)}{\alpha_2 (1 - \beta_2)}
+ \sum_{a \in [k_1]} \sum_{b \in [k_2]} \sum_{z \in Z} |D_{ab}^z(\xi)| \cdot \log Q_{ab}(z) + C_0,
\]
(35)
where \( c_1 \) is the number of inter-cluster edges in \( G_1 \) with respect to \( \sigma \); \( c_2 \) is the number of inter-cluster edges in \( G_2 \) with respect to \( \tau \); \( D_{ab}^z(\xi) = \{(i, j) \in [n] \times [m] : \sigma(i) = a, \tau(j) = b, U_{ij} = z\} \) is the number of observed ratings \( z \) corresponding to user cluster \( U_a \) and item cluster \( I_b \); and \( C_0 \) is a constant that is independent of \( (\sigma, \tau, N) \).

Suppose \( \xi \) is the ground truth that governs the model from now on, and note that the ML estimator \( \phi_{\text{ML}} \) succeeds if \( \xi \) is the most likely model parameter in \( \Xi \) conditioned on the observation \( (U, G_1, G_2) \), i.e., \( \log P_\xi(U, G_1, G_2) \) is larger than any other \( \log P_\xi(U, G_1, G_2) \) for \( \xi' \in \Xi \setminus \{\xi\} \). In fact, what we show in the converse proof is that when \( mnp \) is less than the bound in (13), with high probability there exists another model parameter \( \xi' \in \Xi \setminus \{\xi\} \) such that the likelihood \( \log P_\xi(U, G_1, G_2) \) achieves the maximum.

Specifically, let \( \xi' \neq \xi \) be a model parameter that is identical to \( \xi \) except that its first component \( \sigma' \) differs from \( \sigma \) by only two labels, i.e., \( \sum_{i \in [n]} 1\{\sigma'(i) \neq \sigma(i)\} = 2 \). As the distinction between \( \xi' \) and \( \xi \) is small, the probability that \( \log P_\xi(U, G_1, G_2) \geq \log P_{\xi'}(U, G_1, G_2) \) turns out to be relatively large, which is at least
\[
\frac{1}{4} \exp \left\{-\frac{2f_1(\log n)}{k_1} - \frac{2mnpd\xi}{k_2}\right\}
\]
(36)
due to the tightness of the Chernoff bound (which can be proved by generalizing [12] Lemma 2). In fact, one can find a subset \( \Xi_0 \subseteq \Xi \) of model parameters such that \( |\Xi_0| = \Theta(n) \) and each element in \( \xi_0 \in \Xi_0 \) satisfies (36) (i.e., the probability that \( \xi_0 \) induces an error is relatively large). This, together with the assumption that \( mnp < k_2 \frac{1}{2\epsilon_m - k_1} n \log n / d_{\xi} \), eventually implies that with probability approaching one, there exists at least one \( \xi_0 \in \Xi_0 \) such that \( \log P_{\xi_0}(U, G_1, G_2) \geq \log P_{\xi'}(U, G_1, G_2) \). Thus, the ML estimator fails.

In a similar and symmetric fashion, one can show that the ML estimator fails with probability approaching one, when \( mnp < k_1 \frac{1}{1 - \frac{1}{d_{\xi}}} m \log m / d_{\xi} \). This completes the proof of the converse part.

VII. EXPERIMENTS

In this section, we apply the simplified version of Mc2G mentioned in Remark 2 (without the information splitting step), as the sizes of the graphs \( m \) and \( n \) cannot be made arbitrarily large in the experiments. That is, the four stages are applied to the fully-observed graphs \( (G_1, G_2) \). While this implementation is slightly different from the original algorithm as described in Algorithm 1, its empirical performance nonetheless demonstrates a keen agreement with the theoretical guarantee for the original Mc2G in Theorem 1 (as shown in Section VII-A below).

A. Verification of Theorem 1 on synthetic data

We verify the theoretical guarantee provided in Theorem 1 on a synthetic dataset generated according to a symmetric setting described as follows. The setting contains \( k_1 = 3 \) user

\footnote{As discussed in Remark 2, the information splitting method is merely for the purpose of analysis, and the first part of the graphs \( (G_1, G_2) \) turns out to be too sparse to achieve weak recovery of clusters when \( m \) and \( n \) are not large enough.}
sample complexity is defined (according to Theorem 1) as
rate is averaged over three different values of as a function of the rate and item graphs) to
We set \( n = 2m \), and both \( I_1 \) and \( I_2 \) (the qualities of social and item graphs) to 2. Fig. 3 shows the empirical success rate as a function of the normalized sample complexity for three different values of \( m \) and \( n \). The empirical success rate is averaged over 400 random trials, and the normalized sample complexity is defined (according to Theorem 1) as \( mnp \) divided by
\[
\max \left\{ \frac{(1 - (I_1/k_1))n \log n}{d_{k_1} / k_1}, \frac{(1 - (I_2/k_2))m \log m}{d_{k_2} / k_1} \right\}. \tag{38}
\]
It can be seen from Fig. 3 that the empirical success rate also increases and becomes close to one when the normalized sample complexity exceeds one (corresponding exactly to the condition for Mc2G to succeed).

B. Comparing Mc2G with other algorithms on synthetic data

Next, we compare Mc2G to several existing recommendation algorithms that leverage graph side information on another synthetic dataset. The competitors include the matrix factorization with social regularization (SoReg) \[3\], and a spectral clustering method with local refinements using only the social graph or only the item graph as side information by Ahn et al. \[4\]. In fact, we have also compared our algorithm to other matrix completion algorithms such as biased matrix factorization (MP) \[50\] and TrustSVD \[51\], but they did not perform as well as the competitors we chose and Mc2G. This synthetic dataset is simpler compared to the one in Section VII-A as we need to choose the ratings \( Z \) to be binary (as other competing algorithms are amenable only to binary ratings). It contains \( n = 3000 \) users partitioned into two user clusters, \( m = 3000 \) items partitioned into three item clusters, and we set the qualities of graphs \( I_1 = 1.5 \) and \( I_2 = 2 \), as well as the nominal ratings to be \( z_{11} = 0, z_{12} = 1, z_{13} = 0, z_{21} = 0, z_{22} = 0, z_{23} = 1 \). The personalization distributions are modelled as additive Bern(0.25) noise, i.e., \( Q_{V|Z}(v|z) \) equals 0.75 if \( v = z \), and equals 0.25 otherwise.

To ensure that the comparisons are fair, we quantize the outputs of the other algorithms to be \( \{0, 1\} \)-valued. We measure the performances using the mean absolute error (MAE)
\[
\text{MAE} := \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} |N_{ij} - \hat{N}_{ij}|. \tag{39}
\]
Fig. 4 shows the MAE (averaged over 100 random trials) of each algorithm when \( p \in [0.001, 0.01] \). It is clear that Mc2G is orders of magnitude better than the competing algorithms in terms of the MAEs for this synthetic dataset.

C. Comparing Mc2G with other algorithms on real graphs

To demonstrate that Mc2G is amenable to datasets with real graphs, we applied it to real social and item similarity graphs.

- We adopt the LastFM social network \[22\] (collected in March 2020) as the social graph. Each node is a LastFM user, while each edge represents mutual follower relationships between users. We sub-sample \( n = 1806 \) users from the LastFM social network. These users are partitioned into four clusters with sizes \((n_1, n_2, n_3, n_4) = (497, 327, 552, 430)\), and the empirical connection probabilities are
\[
B = \begin{pmatrix}
9.07 & 0.10 & 0.05 & 0.14 \\
0.10 & 18.3 & 0.17 & 0.25 \\
0.05 & 0.17 & 9.3 & 0.21 \\
0.14 & 0.25 & 0.21 & 22.0
\end{pmatrix} \times 10^{-3}.
\]

- We adopt the political blogs network \[23\] as the item similarity graph. Each node represents a blog that is either liberal-leaning or conservative-leaning, and each edge represents a link between two blogs. This network contains \( m = 1222 \) blogs which are partitioned into two clusters with
sizes \((m_1, m_2) = (586, 636)\), and the empirical connection probabilities are

\[
B' = \begin{bmatrix} 42.6 & 4.2 \\ 4.2 & 38.8 \end{bmatrix} \times 10^{-3}.
\]

We also choose \(Z = \{0, 1\}\), set the nominal ratings to be

\[
\begin{align*}
z_{11} &= 0, & z_{21} &= 0, & z_{31} &= 1, & z_{41} &= 1, \\
z_{12} &= 0, & z_{22} &= 1, & z_{32} &= 0, & z_{42} &= 1,
\end{align*}
\]

and model the personalization distributions as additive Bern(0.1) noise. The personalized ratings matrix \(V\) is then synthesized based on the user and item clusters, nominal ratings, and personalization distributions described above.

We compare MC2G to the algorithms introduced in Section VII-B on this semi-real dataset (real social and item similarity graphs with synthetic ratings). Fig. 5 shows the MAE (averaged over 100 trials) of each algorithm when \(p \in [0.001, 0.012]\). Clearly, MC2G is superior to the other algorithms, and the advantage is more significant when the sample probability \(p\) is small. In addition, the errorbars above and below each data point (representing one standard deviation) for MC2G are fairly small, demonstrating the statistical robustness of MC2G. The average running time (in seconds) of each algorithm, when \(p = 0.01\), is as follows, showing that the running time of MC2G is commensurate with its prediction abilities.

| Algorithm  | Time (s) |
|------------|----------|
| MC2G       | 5.199s   |
| Ahn et al.  | 4.647s   |
| SoReg      | 0.973s   |
| Ahn et al.  | 0.010s   |

It is worth mentioning that the running times of MC2G and the algorithm in Ahn et al. with either a social or an item graph are dominated by the spectral initialization steps—this is the reason why the running times of MC2G are longer than the algorithm in [4] (as MC2G performs spectral clustering for both social and item graphs). SoReg [3] runs faster since it does not perform spectral clustering; however, its performance is rather poor as can be seen from Figs. 4 and 5.

**Fig. 5:** Comparisons of MAEs of different algorithms under the semi-real setting described in Section VII-C where we adopt the LastFM social network and political blog networks as social and item similarity graphs respectively. The length of each errorbar above and below each data point represents the standard deviations across the 100 independent trials.

**APPENDIX A**

**Proof of Lemma 1**

Consider the process of first generating a sub-graph \(H_1^a\) and then generating a sub-SBM \(G_1^a\) on the sub-graph \(H_1^a\). The probability that an edge \(E_{ii'}\) (connecting nodes \(i\) and \(i'\)) appears in \(G_2^a\) equals \(\frac{1}{\sqrt{\log n}}\) multiplied by \(\alpha_2 \text{ or } \beta_1\) (depending on whether \(i\) and \(i'\) are in the same community). Thus, a key observation is that the aforementioned process is equivalent to generating \(G_1\) directly. By this observation and recalling that a spectral clustering method running on \(G_1\) ensures \(l_1(\sigma(0), \sigma) \leq \eta_n\) with probability at least \(1 - \epsilon_n\) [47], Theorem 6], we have

\[
\sum_{h_1^a} \mathbb{P}(H_1^a = h_1^a) \mathbb{P}_{\text{succ}}(h_1^a) \geq 1 - \epsilon_n,
\]

where \(\mathbb{P}_{\text{succ}}(h_1^a)\) is the probability that a spectral clustering method running on \(G_1\) (which depends on \(h_1^a\)) ensures \(l_1(\sigma(0), \sigma) \leq \eta_n\). Let \(H_1^a_G\) and \(H_1^a_B\) respectively be the sets of good and bad sub-graphs \(h_1^a\). Suppose the probability of generating a good sub-graph (i.e., \(h_1^a \in H_1^a_G\)) is less than \(1 - \sqrt{\epsilon_n}\). Then, by the definition of the good sub-graphs \(h_1^a\),

\[
\sum_{h_1^a} \mathbb{P}(H_1^a = h_1^a) \mathbb{P}_{\text{succ}}(h_1^a) < \sum_{h_1^a \in H_1^a_G} \mathbb{P}(H_1^a = h_1^a) + \sum_{h_1^a \in H_1^a_B} \mathbb{P}(H_1^a = h_1^a)(1 - \sqrt{\epsilon_n})
\]

\[
= \sum_{h_1^a \in H_1^a_G} \mathbb{P}(H_1^a = h_1^a) + (1 - \sqrt{\epsilon_n}) \left( 1 - \sum_{h_1^a \in H_1^a_B} \mathbb{P}(H_1^a = h_1^a) \right)
\]

\[
< 1 - \epsilon_n,
\]

which yields a contradiction to (40). Thus, we conclude that with probability at least \(1 - \sqrt{\epsilon_n}\) over the generation of \(H_1^a\), the randomly generated \(h_1^a\) is a good sub-graph.

For each user node \(i \in [n]\), the expected degree of \(i\) in \(H_1^a\) is \((n - 1)/\sqrt{\log n}\). By applying the multiplicative form of the Chernoff bound, one can show that with probability at least \(1 - \exp(-\Theta(n/\sqrt{\log n}))\), the degree of \(i\) in the randomly generated sub-graph \(h_1^a\) is at most \(2n/\sqrt{\log n}\). A union bound over all user nodes guarantees that, with high probability, the degrees of all the nodes in \(h_1^a\) are at most \(2n/\sqrt{\log n}\), which further implies the sub-graph \(h_1^a\) is good. Finally, applying a union bound implies that with high probability, both \(h_1^a\) and \(h_1^b\) are good sub-graphs simultaneously.

In a similar manner, we can also prove the analogous statements for \(H_2^a\) and \(H_2^b\).

**APPENDIX B**

**Proof of Lemma 2**

First recall the definitions of \(S_{\bar{a}a}\) and \(T_{\bar{a}b}\) in Section V (Analysis of Stage 3). As it is assumed that \(h \in G\) and \((\sigma(0), \tau(0)) \in G'\), we know that (i) \(|S_{\bar{a}a}| \geq (1/k_1 - \eta_n)n\) when \(a = \bar{a}\), and \(|S_{\bar{a}a}| \leq \eta_n n\) when \(a \neq \bar{a}\); and (ii)

\(\sum_{h \in \mathbb{N}} P(h) = 1\)

Specifically, the probability that an edge \(E_{ii'}\) appears in \(G_1^a\) is equal to the probability of \(E_{ii'}\) belonging to \(H_1^a\), multiplied by the probability of generating \(E_{ii'}\) in the sub-SBM \(G_1^a\).
\[ |T_{bb}| \geq (1/k_2 - \gamma_n)m \] when \( b = \hat{b} \), and \( |T_{bb}| \leq \gamma_n m \) when \( b \neq \hat{b} \).

We first analyze the estimates \( \{\hat{B}_{aa}\}_{a \in [k_1]} \) in Eqn. (5).

By letting \( \{X_k\} \overset{i.i.d.}{\sim} \text{Bern}(\alpha_1) \) and \( \{Y_k\} \overset{i.i.d.}{\sim} \text{Bern}(\beta_1) \), we have
\[
eq \sum_{b_1} X_k + \sum_{b_2} Y_k, \text{ where } B_1 := \sum_{a \in [k_1]} (X_a) \text{ and } B_2 := (Y_a) - B_1. \]

Note that
\[
\mu_{B_{aa}} := \mathbb{E}[e(U_{a}^0, U_{a}^0)] \leq \alpha_1 \left( \frac{|U_{a}^0|}{2} \right).
\]

On the other hand, since the degree of any node in \( h_1 \) is at least \( n(1 - 2/\sqrt{\log n}) \) (or equivalently, the number of non-edges of any node is at most \( 2n/\sqrt{\log n} \)), we know that
\[
eq \sum_{k=1}^{B_1 - \frac{\sqrt{2}}{\sqrt{\log n}}} X_k, \text{ and } \mu_{B_{aa}} \geq \left[ B_1 - \frac{n^2}{\sqrt{\log n}} \right] \alpha_1.
\]

Applying the multiplicative form of the Chernoff bound yields that for any \( \delta \in (0, 1) \), with probability at least \( 1 - 2\exp(-\delta^2 \mu_{B_{aa}}/3) \), the numerator \( e(U_{a}^0, U_{a}^0) \) satisfies
\[
(1 - \delta) \left( B_1 - \frac{n^2}{\sqrt{\log n}} \right) \alpha_1 \leq e(U_{a}^0, U_{a}^0) \leq (1 + \delta) \alpha_1 \left( \frac{|U_{a}^0|}{2} \right).
\]

As the estimate \( \hat{B}_{aa} = e(U_{a}^0, U_{a}^0) \left( \frac{|U_{a}^0|}{2} \right) \), we then have
\[
\left( 1 - \delta - c_1 \eta_n - \frac{c_2}{\sqrt{\log n}} \right) \alpha_1 \leq \hat{B}_{aa} \leq (1 + \delta) \alpha_1,
\]
for some constant \( c_1, c_2 > 0 \). By choosing \( \delta = 1/\sqrt{\log n} \), we complete the proof for \( B_{aa} \).

The analyses of other estimates \( \hat{B}_{aa} \) are similar, thus we omit them for brevity (except that we need to replace \( \eta_n \) by \( \gamma_n \) for \( \{\hat{B}_{bb}\}_{b \in [k_2]} \), and \( \{\hat{B}_{bb}\}_{b \neq b} \)). Therefore, one can find a sequence \( \varepsilon_n \in \Omega(\max\{\gamma_n, \eta_n, 1/\sqrt{\log n}\}) \cap o(1) \) such that (18) holds.

**APPENDIX C**

**Proof of Lemma 3**

Let us recall the definition of \( \hat{Q}_{ab}(z) \) in (5), in which the numerator takes the form
\[
|Q_{ab}| = \sum_{i \in (U_{a}^0) \cap (U_{b}^0)} \{U_{ij} = z\}. \tag{41}
\]

Let \( \{T_{ij}\} \overset{i.i.d.}{\sim} \text{Bern}(p) \), \( \{Z_{ij}^{ab}\} \overset{i.i.d.}{\sim} Q_{ab} \) for all \( a \in [k_1] \) and \( b \in [k_2] \). Thus, the RHS of (41) can be rewritten as
\[
\sum_{i \in S_{aa}} \sum_{j \in T_{bb}} T_{ij} (Z_{ij}^{ab} = z) + \sum_{a \neq a'} \sum_{b \neq b'} \sum_{j \in T_{bb}} T_{ij} (Z_{ij}^{ab} = z).
\]

Note that the number of summands in the first term
\[
|i \in S_{aa}| |j \in T_{bb}| \geq [(n/k_1 - \eta_n) (m/k_2 - \gamma_n m)] := L.
\]

Thus, the expectation of \( |Q_{ab}^z| \) satisfies
\[
\mathbb{E}(|Q_{ab}^z|) \geq L \cdot \mathbb{E}(T_{ij} (Z_{ij}^{ab} = z)) \geq L \cdot p \cdot Q_{ab}(z), \text{ and } 
\mathbb{E}(|Q_{ab}^z|) \leq L \cdot \mathbb{E}(T_{ij} (Z_{ij} = z)) \leq L \mathbb{E}(T_{ij})
\]
where the upper bound is due to the fact that \( \mathbb{P}(Z_{ij}^{ab} = z) \leq 1 \).

Applying the Chernoff bound yields that with probability \( 1 - \exp(-\Theta(\delta^2 \mathbb{P}(|Q_{ab}^z|))) \), for all \( z \in \mathbb{Z} \),
\[
|Q_{ab}^z| \geq (1 - \delta) L \cdot p \cdot Q_{ab}(z), \text{ and } 
|Q_{ab}^z| \leq (1 + \delta) (1 + \Theta(\max(\eta_n, \gamma_n))) L \cdot p \cdot Q_{ab}(z), \tag{42}
\]
where \( \delta \in (0, 1) \). Choosing \( \delta = 1/\sqrt{\log n} \), we ensure that with probability \( 1 - o(1) \), for all \( z \in \mathbb{Z} \), \( a \in [k_1] \), and \( b \in [k_2] \),
\[
\hat{Q}_{ab}(z) \leq |Q_{ab}(z) - 1| = O(\max(\eta_n, \gamma_n, 1/\sqrt{\log n})). \tag{44}
\]

**APPENDIX D**

**Proof of Lemma 5**

First note that
\[
|L_a(i) - \bar{L}_a(i)| \leq \sum_{a' \in [k_1]} e(\{i\}, U_{a'}^{0}) \log \frac{B_{aa'}(1 - \hat{B}_{aa'})}{B_{aa'}(1 - B_{aa'})} + \sum_{b' \in [k_2]} \sum_{j \in T_{bb}^{0}} \mathbb{I}(U_{ij} \neq e) \cdot \log \frac{Q_{ab}(U_{ij})}{Q_{ab}(U_{ij})} \tag{45}
\]
As \( \sum_{a' \in [k_1]} e(\{i\}, U_{a'}^{0}) \) represents the degree of user \( i \) in the social graph, and its expectation \( \mu_i \) satisfies \( n \mu_i \leq \mu_i \leq n \alpha_1 \). By applying the Chernoff bound, we have that for any \( \kappa > 1 \),
\[
\mathbb{P} \left( \sum_{a' \in [k_1]} e(\{i\}, U_{a'}^{0}) \geq (1 + \kappa) n \alpha_1 \right) \leq e^{-\frac{\kappa^2}{4} n \alpha_1}. \tag{46}
\]
We choose \( \kappa \) to be a large enough constant that ensures the RHS of (46) to scale as \( o(n^{-1}) \). Then, by applying the union bound over all the \( n \) users, we have that with probability \( 1 - o(1) \), all the users \( i \in [n] \) satisfy
\[
\sum_{a' \in [k_1]} e(\{i\}, U_{a'}^{0}) \leq (1 + \kappa) n \alpha_1 = c_1 \log n, \tag{47}
\]
for some constant \( c_1 > 0 \). Also, note that the term \( \sum_{b' \in [k_2]} \sum_{j \in T_{bb}^{0}} \mathbb{I}(U_{ij} \neq e) \) corresponds to the number of observed ratings for each user. By a similar analysis (based on the Chernoff bound), one can show that with probability \( 1 - o(1) \), all the users \( i \in [n] \) satisfy
\[
\sum_{b' \in [k_2]} \sum_{j \in T_{bb}^{0}} \mathbb{I}(U_{ij} \neq e) \leq c_2 \log n, \tag{48}
\]
for some constant \( c_2 > 0 \).

Recall from Lemmas 2 and 3 that the estimated connection probabilities satisfy \( \hat{B}_{aa'} - B_{aa'} \leq \varepsilon_n \) for all \( a, a' \in [k_1] \), and the estimated personalization distribution \( \hat{Q}_{ab}(z)/Q_{ab}(z) - 1 \leq \varepsilon_n \) for all \( a \in [k_1], b \in [k_2], z \in \mathbb{Z} \).

By applying a Taylor series expansion, we then have
\[
\log \frac{B_{aa'}(1 - \hat{B}_{aa'})}{B_{aa'}(1 - B_{aa'})} \leq 2\varepsilon_n, \quad \log \frac{Q_{ab}(U_{ij})}{Q_{ab}(U_{ij})} \leq 2\varepsilon_n. \tag{49}
\]
Combining (47), (48), and (49), we complete the proof of Lemma 5.
