Existence and asymptotic stability of quasi-periodic solution of discrete NLS with potential in \( \mathbb{Z} \)

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Abstract

We prove the existence of a 2-parameter family of small quasi-periodic in time solutions of discrete nonlinear Schrödinger equation (DNLS). We further show that all small solutions of DNLS decouples to this quasi-periodic solution and dispersive wave.

1 DNLS

In this paper, we consider the following discrete nonlinear Schrödinger equation (DNLS) on \( \mathbb{Z} \).

\[
i \partial_t u = Hu + |u|^6 u, \quad u : \mathbb{R} \times \mathbb{Z} \to \mathbb{C},
\]

where \( H = -\Delta + V \), and

- \((\Delta u)(n) := u(n + 1) - 2u(n) + u(n - 1)\).
- \((Vu)(n) := V(n)u(n)\).
- \(\sigma_p(H) = \{e_0, e_1\}\) with
  \[
e_0 + n(e_1 - e_0) \notin [0, 4], \quad \forall n \in \mathbb{Z},
  \]

where \(\sigma_p(H)\) is the set of point spectrum of \(H\). Further, set \(\phi_0, \phi_1\) to be the normalized real valued eigenfunctions associated to \(e_0, e_1\) respectively.

Remark 1.1. \([0, 4]\) is the essential spectrum of \(H\). If \(e_0 < 0\) and \(e_1 > 4\), \(\{e_0, e_1\}\) satisfies (1.2).

Remark 1.2. The eigenfunctions \(\phi_0, \phi_1\) decay exponentially.

Definition 1.3. For \(n \in \mathbb{Z}\), let \(\langle n \rangle := (1 + n^2)^{1/2}\). We set

\[
\ell^{p,\alpha}(\mathbb{Z}) := \left\{ u = \{u(n)\}_{n \in \mathbb{Z}} \mid \|u\|_{\ell^{p,\alpha}} := \sum_{n \in \mathbb{Z}} \langle n \rangle^{p\alpha} |u(n)|^p < \infty \right\},
\]

and \(\ell^p(\mathbb{Z}) := \ell^{p,0}(\mathbb{Z})\) for \(p \in [1, \infty)\). Further, for \(a \in \mathbb{R}\), we set

\[
\ell^a_\mathbb{Z} := \left\{ u = \{u(n)\}_{n \in \mathbb{Z}} \mid \|u\|_{\ell^a_\mathbb{Z}} := \sum_{n \in \mathbb{Z}} e^{2a|n|} |u(n)|^2 < \infty \right\}.
\]

We define the inner-product of \(\ell^2(\mathbb{Z})\) by

\[
\langle u, v \rangle := \text{Re} \sum_{n \in \mathbb{Z}} u(n)\overline{v(n)}.
\]
In the following, we use the following notations.

- We often write $a \lesssim b$ by meaning that there exists a constant $C$ s.t. $a \leq Cb$. If we have $a \lesssim b$ and $b \lesssim a$, we write $a \sim b$.
- For a Banach space $X$ equipped with the norm $\| \cdot \|_X$, we set
  \[ B_X(a, \delta) := \{ u \in X \ | \ |u - a|_X < \delta \}. \]

It is well known that (1.1) possesses small periodic solutions which bifurcate from $\phi_j$. For the convenience of the readers, we will give the proof in the appendix of this paper.

**Proposition 1.4.** Fix $j \in \{0, 1\}$. There exist $a_0 > 0$ and $\delta_0 > 0$ s.t. for all $z \in B_{\mathbb{C}}(0, \delta_0)$, there exists $E_j \in C^{\omega}(B_{\mathbb{R}}(0, \delta_0^2); \mathbb{R})$ and $q_j \in C^{\omega}(B_{\mathbb{R}}(0, \delta_0^2); l^{\infty}(\mathbb{Z}; \mathbb{R}))$ s.t. $\langle \phi_j, q_j \rangle = 0$ and
  \[ \phi_j(z) := z \tilde{\phi}_j(|z|^2) = z \left( \phi_j + \tilde{q}_j(|z|^2) \right), \tag{1.3} \]
satisfies
  \[ (H - E_j(|z|^2)) \phi_j(z) + |\phi_j(z)|^6 \phi_j(z) = 0. \tag{1.4} \]
Further, we have $|E_j(|z|^2) - e_j| \lesssim |z|^6$, $||\tilde{q}_j(|z|^2)||_{l^\infty} \lesssim |z|^6$.

**Remark 1.5.** If $\phi$ satisfies (1.4), then $e^{-tE_j}\phi$ is the solution of (1.1).

The first result of this paper is the existence of quasi-periodic solution of (1.1).

**Theorem 1.6.** There exists $a_1 > 0$, $\delta_1 \in (0, \delta_0)$ s.t. there exists
  \[ \psi \in C^{\omega}(B_{\mathbb{C}}(0, \delta_1) \times B_{\mathbb{C}}(0, \delta_1); l^{5\times}(\mathbb{Z}; \mathbb{C})), \quad \text{and} \quad \varepsilon_j \in C^{\omega}(B_{\mathbb{R}}(0, \delta_1^2) \times B_{\mathbb{R}}(0, \delta_1^2); \mathbb{R}), \]
s.t.
  \[ \Psi(z_0, z_1) := \phi_0(z_0) + \phi_1(z_1) + \psi(z_0, z_1) \]
is a solution of (1.1) if $z_j$ ($j = 0, 1$) satisfies
  \[ i\dot{z}_j = (E_j(|z_j|^2) + \varepsilon_j(|z_0|^2, |z_1|^2)) z_j, \]
Further, for arbitrary $\theta \in \mathbb{R}$, we have
  \[ e^{i\theta}\psi(z_0, z_1) = \psi(e^{i\theta}z_0, e^{i\theta}z_1), \tag{1.5} \]
and
  \[ \|\psi(z_0, z_1)\|_{l^{5\times}} \lesssim |z_0||z_1||(|z_0|^5 + |z_1|^5), \tag{1.6} \]
  \[ |\varepsilon_j(|z_0|^2, |z_1|^2)| \lesssim |z_1-j|^2 \left( |z_0|^4 + |z_1|^4 \right). \tag{1.7} \]

**Remark 1.7.** By $\psi \in C^{\omega}(B_{\mathbb{C}}(0, \delta_1) \times B_{\mathbb{C}}(0, \delta_1); H^{a_1})$, we mean $\tilde{\psi}(z_{0,R}, z_{0,I}, z_{1,R}, z_{1,I}) := \psi(z_{0,R} + iz_{0,I}, z_{1,R} + iz_{1,I})$ is real analytic on $B_{\mathbb{R}^2}(0, \delta_1) \times B_{\mathbb{R}^2}(0, \delta_1)$, and so for $\varepsilon_j$.

The second result of this paper is about the asymptotic behavior of small solution of (1.1). In particular, we show that all solutions of (1.1) with $\|u(0)\|_a$ small, decomposes to the quasi-periodic solution $\Psi$ obtained in Theorem 1.6 and free linear solution of $iu_t = -\Delta u$ as $t \to \infty$.  

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Theorem 1.8. Assume $H$ is generic (for the definition see Lemma 5.3 of [22]). Then, there exists $\delta_2 \in (0, \delta_1)$ s.t. if $\|u_0\|_2 < \delta_2$, then the solution of (1.1) with $u(0) = u_0$ exists globally in time and there exist $z_j(t) : [0, \infty) \to \mathbb{C}$, $\rho_{j,+} \in \mathbb{R}_{\geq 0}$ for $j = 0, 1$ and $v_+ \in l^2$ s.t.

$$
\lim_{t \to \infty} \|u(t) - \Psi(z_0(t), z_1(t)) - e^{it\Delta} v_+\|_2 = 0, \quad \lim_{t \to \infty} |z_j(t)| = \rho_{j,+}, \quad (j = 0, 1).
$$

Further, we have $\|v_+\|_2 + \rho_{0,+} + \rho_{1,+} \lesssim \|u_0\|_2$.

Remark 1.9. The assumption that $H$ is generic is used for the linear estimates of $H$. See section 5.

We now recall the known results related to our results on continuous and discrete NLS. There is a long list of papers on asymptotic stability of both large and small standing waves of continuous NLS [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 19, 21, 23, 25, 29, 30, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 45, 46] where standing waves are solutions with the form $e^{it\phi}(x)$ (which corresponds to the periodic solution given in Proposition 1.4).

The asymptotic stability result for small standing wave for continuous NLS was first proved by Soffer-Weinstein [40] assuming the (continuous) Schrödinger operator $H = -\Delta + V$ has exactly one eigenvalue and the initial data is small in some weighted space. Later, Gustafson-Nakanishi-Tsai [25] removed the assumption that the initial data is small in the weighted space and proved the asymptotic stability in the energy space $H^1$ for the 3 dimensional case. One of the main tool of [25] was the endpoint Strichartz estimate [24, 27] which collapse in the 1 and 2 dimensional cases. For 1 and 2 dimensional cases, Mizumachi [34, 35] prove the asymptotic stability result in the energy space by replacing the endpoint Strichartz estimates to Kato type smoothing estimates. The results [25, 34, 35] tells us that the dynamics of small solutions of NLS is similar to the linear Schrödinger equation under the assumption that $H$ has only one eigenvalue. This is because the solution of linear Schrödinger equation also decomposes to a periodic solution associated to the eigenvalue and dispersive solution associated to the absolutely continuous spectrum.

For the case that $H$ has more than two eigenvalues, the asymptotic dynamics are no more similar to the linear Schrödinger equation. Indeed, linear Schrödinger equation posse quasi-periodic solutions associated to the two eigenvalues of $H$. However, [39] proved that there is no quasi-periodic solution for NLS. Further, [43], [45], [46], [44] proved that if $H$ has two eigenvalues with $e_0 < 2\epsilon_1$, all small solution in some weighted space decomposes to a standing wave and a free solution. Recently, [16] extended these result to the case $H$ has more than two eigenvalues and the solution is in the energy space. See also related results for nonlinear Klein-Gordon equation (NLKG) [3, 42] and nonlinear Dirac equation [20]. The mechanism which destroys the quasi-periodic solution is the nonlinear interaction between the eigenvalue and the absolutely continuous spectrum. The non-degeneracy condition for such interaction is called Fermi Golden Rule (FGR) which all the above papers assume.

We now turn to the known results of DNLS. For the case that the discrete Schrödinger operator $H$ has only one eigenvalue, [22, 28] proved the asymptotic stability result in the energy space $l^2$. This result corresponds to the continuous case. However, for the case $H$ has two eigenvalues, [13] proved that the ground state (which is the standing wave corresponding to the smallest eigenvalue) is orbitally stable but not asymptotically stable. For the continuous case, ground state is asymptotically stable, so this result shows that in this case the small solution of continuous and discrete NLS has different asymptotic dynamics. As mentioned in [13], the situation that the standing wave is orbitally stable but not asymptotically stable suggests that there may exists a quasi-periodic solutions. Indeed, Theorem 1.6 shows that there exists a 2-parameter family of quasi-periodic solution which bifurcates from the two eigenvalues of $H$. Note that the fact that the standing wave is not asymptotically stable is a direct consequence of the existence of quasi-periodic solution near standing waves.
We note that there are several results concerning the existence and asymptotic stability of quasi-periodic solution of DNLS and discrete NLKG in the "anticontinuous limit" situation (see for example [2, 26, 33]). This is the case $H$ is replaced by $\varepsilon H$ in (1.1) with $\varepsilon$ small, which is a different type of solution which we are concerning.

We prove the existence of the quasi-periodic solution by implicit function theorem starting from the sum of two standing waves. In particular, we assume that the quasi-periodic solution can be written as $\sum_{n \in \mathbb{Z}} e^{-i(\xi_0 + n(\xi_1 - \xi_0)) t} v_n$, where $\xi_j \sim c_j$ and solve (1.1) for each frequency. Notice that the frequencies $\{\xi_0 + n(\xi_1 - \xi_0)\}_{n \in \mathbb{Z}}$ are generated from the two standing waves and the nonlinearity. Further, these frequencies do not intersect with the continuous spectrum of $H$ because of (1.2). The fact that $\{\xi_0 + n(\xi_1 - \xi_0)\}_{n \in \mathbb{Z}}$ do not intersect with the continuous spectrum is crucial for the existence of quasi-periodic solution. Indeed, for the continuous NLS case, condition (1.2) always fails because the continuous spectrum is $[0, \infty)$. Then, by the nonlinear interaction, we have a damping from the point spectrum to the continuous spectrum which prevents the existence of the quasi-periodic solutions. By the same reason, we conjecture that for the case $H$ has more than 3 eigenvalues there will be no quasi-periodic solution like

$$
\Psi(z_0, z_1, z_2) = z_0 \phi_0 + z_1 \phi_1 + z_2 \phi_2 + \cdots.
$$

This is because the nonlinear interaction between the point spectrum and absolutely continuous spectrum arises again and there will be a damping.

For the asymptotic stability result Theorem 1.8, we start from a standard modulation argument and adapt the nonlinear coordinate given in [25]. However, since our quasi-periodic solution is not a standing wave, it seems to be difficult to get a simple equations for the modulation parameters in this coordinate. To overcome this difficulty, we use the Darboux theorem which was introduced in [14] and used in [2, 16, 17, 20]. In fact, after changing the coordinates by the Darboux theorem, we get a well decoupled equations (see (4.35), (4.36)) which are easy to analyze. We note that although we have made the change of coordinate with a real analytic regularity, we actually need only $C^3$. The real analyticity comes from the real analyticity of the nonlinearity. Therefore, for the asymptotic stability, we do not need real analyticity. However, for the existence of the quasi-periodic solution, we can only handle a polynomial nonlinearity because we have expanded the solution as $\sum_{n \in \mathbb{Z}} e^{-i(\xi_0 + n(\xi_1 - \xi_0)) t} v_n$. Further, real analyticity reduces the amount of some computation for the estimate of the derivatives of the coordinate change (see Lemma 4.10). These are the reasons why we have adapted the real analytic framework for the change of coordinate.

The difference between the proof of [2] and the proof of Theorem 1.8 is that [2] uses the normal form argument infinite times (the Birkhoff normal form). For this method, it is necessary to have the analyticity of the nonlinear term for the convergence of the normal form steps. On the other hand, we only use the normal form argument (the Darboux theorem) once. As mentioned before, our argument only requires $C^3$ regularity for the coordinate change so it is not necessary to have a analytic nonlinearity for the proof of asymptotic stability. However, we need the nonlinearity to be polynomial for the proof of the existence of the quasi-periodic solution.

The paper is organized as follows: In section 2, we prove Theorem 1.6. In section 3, we set up the nonlinear coordinate. In section 4, we prove the Darboux theorem and show that the energy is sufficiently decomposed. In section 5, we introduce the linear estimates which were originally given in [22] and in section 6, we prove Theorem 1.8. In the appendix we give the proof of Proposition 1.4, Lemma 2.4 and a explicit computation of some terms needed for the proof of Theorem 1.6.

We end the introduction with some notation. For Banach spaces $X, Y$, we set $\mathcal{L}(X; Y)$ to be the Banach space of all bounded operators from $X$ to $Y$. We set $\mathcal{L}^n(X; Y)$ inductively by $\mathcal{L}^0(X; Y) = \mathcal{L}(X; \mathcal{L}^{n-1}(X; Y))$ and $\mathcal{L}^n(X; Y) = Y$. Further, we write $\mathcal{L}(X) := \mathcal{L}(X; X)$. We set $C^n(B_X(0, \delta); Y)$ to be all real analytic functions from $B_X(0, \delta)$ to $Y$. By real analytic functions, we...
mean that \( f : B_X(0, \delta) \to Y \) can be written as \( f(x) = \sum_{n \geq 0} a_n x^n \) with \( \sum_{n \geq 0} \|a_n\|_{L^n(X;Y)} r^n < \infty \) for all \( r < \delta \), where \( a_n \in L^n(X;Y) \) and \( a_n x^n := a_n(x, x, \cdots, x) \).

## 2 Proof of Theorem 1.6

We construct solutions of (1.1) which have the form as follows:

\[
\Psi(z_0, z_1) = \phi_0(z_0) + \phi_1(z_1) + \psi(z_0, z_1),
\]
\[
\psi(z_0, z_1) = \sum_{m \geq 0} z_0^{m+1} \overline{z}_1^m v_m(|z_0|^2, |z_1|^2) + \overline{z}_0^m z_1^{m+1} w_m(|z_0|^2, |z_1|^2),
\]

(2.1)

where, \( v_m, w_m \) are real valued and \( \langle v_0, \phi_1 \rangle = \langle w_0, \phi_0 \rangle = 0 \). Fix \( \varepsilon_0, \varepsilon_1 \in \mathbb{R} \) which will be chosen later, and suppose \( i \zeta_j = \mathcal{E}_j \zeta_j \), where \( \mathcal{E}_j = E_j(|z_j|^2) + \varepsilon_j \in \mathbb{R} \) (\( j = 0, 1 \)). Then, we have

\[
i \partial_t \Psi(z_0, z_1) = \mathcal{E}_0 \bar{z}_0 \left( \bar{\phi}_0(|z_0|^2) + v_0 \right) + \mathcal{E}_1 z_1 \left( \bar{\phi}_1(|z_1|^2) + w_0 \right)
\]
\[
+ \sum_{m \geq 1} z_0^{m+1} \overline{z}_1^m ((m+1)\mathcal{E}_0 - \mathcal{E}_1) v_m + \overline{z}_0^m z_1^{m+1} ((m+1)\mathcal{E}_1 - m\mathcal{E}_0) w_m.
\]

\[
H \Psi(z_0, z_1) = \bar{z}_0 \left( H \bar{\phi}_0(|z_0|^2) + Hv_0 \right) + z_1 \left( H \bar{\phi}_1(|z_1|^2) + Hw_0 \right)
\]
\[
+ \sum_{m \geq 1} z_0^{m+1} \overline{z}_1^m Hv_m + \overline{z}_0^m z_1^{m+1} Hw_m,
\]

where \( \bar{\phi}_j(|z_j|^2) \) is given in (1.3). Further, we have

\[
|\Psi(z_0, z_1)|^2 \psi(z_0, z_1) = |\phi_0(z_0)|^2 \phi_0(z_0) + |\phi_1(z_1)|^2 \phi_1(z_1) + N(|z_0|^2, |z_1|^2, \{v_m\}_{m \geq 0}, \{w_m\}_{m \geq 0}),
\]

where

\[
N(|z_0|^2, |z_1|^2, \{v_m\}_{m \geq 0}, \{w_m\}_{m \geq 0}) = \sum_{m \geq 0} z_0^{m+1} \overline{z}_1^m N_m(|z_0|^2, |z_1|^2, \{v_m\}_{m \geq 0}, \{w_m\}_{m \geq 0}),
\]

(2.2)

for some \( \{N_m\}_{m \geq 0}, \{M_m\}_{m \geq 0} \), see Lemma 2.4 below. Therefore, to construct a solution of (1.1) in the form (2.1), it suffices to solve each equations of the coefficients of \( z_0^{m+1} \overline{z}_1^m \) and \( \overline{z}_0^m z_1^{m+1} \). In particular, we solve the system of equations

\[
\begin{aligned}
\varepsilon_0 \bar{\phi}_0(|z_0|^2) + \mathcal{E}_0 v_0 &= Hv_0 + N_0, \\
\varepsilon_1 \bar{\phi}_1(|z_1|^2) + \mathcal{E}_1 w_0 &= Hw_0 + M_0, \\
((m+1)\mathcal{E}_0 - \mathcal{E}_1) v_m &= Hv_m + N_m, (m \geq 1), \\
((m+1)\mathcal{E}_1 - m\mathcal{E}_0) w_m &= Hw_m + M_m, (m \geq 1).
\end{aligned}
\]

(2.3)

To solve (2.3), we use the following function space.

**Definition 2.1.** Let \( r, a > 0 \). We define a space of pairs of sequence \( X_{r,a} \) by

\[
X_{r,a} := \{ \{v_m\}_{m \geq 0}, \{w_m\}_{m \geq 0} \mid v_m, w_m \in L^a, \sum_{m \geq 0} r^{2m+1} (\|v_m\|_2^2 + \|w_m\|_2^2) < \infty \}.
\]
For $V = \{\{v_m\}_{m \geq 0}, \{w_m\}_{m \geq 0}\} \in X_{r,a}$, we set
\[
\|V\|_{r,a} = \sum_{m \geq 0} r^{2m+1} \left(\|v_m\|_{l^r_a} + \|w_m\|_{l^r_a}\right),
\]
and
\[
\mathcal{T}(z_0, z_1)V := \sum_{m \geq 0} z_0^m \hat{z}_1 v_m + \sum_{m \geq 0} \bar{z}_1^m z_1^m w_m.
\]

We first show that if $V \in X_{r,a}$, then $\mathcal{T}(z_0, z_1)V \in l^a_c$ if $|z_0|, |z_1| \leq r$.

**Lemma 2.2.** Let $|z_0|, |z_1| \leq r$ and $V \in X_{r,a}$, then
\[
\|\mathcal{T}(z_0, z_1)V\|_{l^a_c} \leq \|V\|_{r,a}. \tag{2.4}
\]

**Proof.** Set $V = \{\{v_m\}_{m \geq 0}, \{w_m\}_{m \geq 0}\}$. Then, we have
\[
\|\mathcal{T}(z_0, z_1)V\|_{l^a_c} \leq \sum_{m \geq 0} |z_0|^m |z_1|^m \|v_m\|_{l^r_a} + |z_0|^m |z_1|^m \|w_m\|_{l^r_a} \leq \|V\|_{r,a}.
\]
Therefore, we have the conclusion. \(\square\)

**Remark 2.3.** By the above lemma, we see that if $V(\cdot, \cdot) \in C^\omega(B_{R^2}(0, \delta); X_{r,a})$ for $\delta < r^2$, then $\Psi(z_0, z_1) := \mathcal{T}(z_0, z_1)V(|z_0|^2, |z_1|^2) \in l^a_c$ is real analytic with respect to $z_0, z_1, t, z_1, t, z_1, t$.

We next show that if $\psi_1, \psi_2, \psi_3$ has the form of (2.1). Then also $\psi_1 \psi_2 \psi_3$ has the form of (2.1). Repeatedly using this fact, we can verify that $N(|z_0|^2, |z_1|^2, \{\{v_m\}_{m \geq 0}, \{w_m\}_{m \geq 0}\})$ has the form given in (2.2).

**Lemma 2.4.** Assume $|z_0|, |z_1| \leq r$. Let $V_j \in X_{r,a}$ for $j = 1, 2, 3$. Then, there exists $V(|z_0|^2, |z_1|^2) \in X_{r,a}$ s.t.
\[
\mathcal{T}(z_0, z_1)V_1 \mathcal{T}(z_0, z_1)V_2 \mathcal{T}(z_0, z_1)V_3 = \mathcal{T}(z_0, z_1)V(|z_0|^2, |z_1|^2). \tag{2.5}
\]
Further, set $M(|z_0|^2, |z_1|^2, V_1, V_2, V_3) = V(|z_0|^2, |z_1|^2)$, where $V$ is given by (2.5). Then, we have
\[
\|M(|z_0|^2, |z_1|^2, V_1, V_2, V_3)\|_{r,a} \leq \|V_1\|_{r,a} \|V_2\|_{r,a} \|V_3\|_{r,a}, \tag{2.6}
\]
and $M \in C^\omega(B_{R^2}(0, r^2) \times X_{r,a} \times X_{r,a} \times X_{r,a}; X_{r,a})$.

We give the proof of Lemma 2.4 in the appendix of this paper. By Lemm 2.4, we immediately have that $N$ is real analytic with respect to $|z_0|^2$, $|z_1|^2$ and $V$.

**Corollary 2.5.** Let $r, a > 0$ sufficiently small. Then, we have $N \in C^\omega(B_{R^2}(0, r^2) \times X_{r,a}; X_{r,a})$.

**Proof.** Set
\[
\Phi_0(|z_0|^2) = \{\{\delta_{0m} \tilde{\phi}_0(|z_0|^2)\}_{m \geq 0}, \{0\}_{m \geq 0}\} \in X_{r,a}, \tag{2.7}
\]
\[
\Phi_1(|z_0|^2) = \{\{0\}_{m \geq 0}, \{\delta_{0m} \tilde{\phi}_1(|z_1|^2)\}_{m \geq 0}\} \in X_{r,a}. \tag{2.8}
\]
where $\delta_m = 1$ if $m = 0$ and $\delta_m = 0$ if $m \geq 1$ and $a \leq a_0$. Then, the term $N(|z_0|^2, |z_1|^2, V)$ given in (2.2) can be rewritten as

$$N(|z_0|^2, |z_1|^2, V) = M(|z_0|^2, |z_1|^2, V + \Phi_0 + \Phi_1, V + \Phi_0 + \Phi_1, M_2)$$

$$- \sum_{j=0,1} M(|z_0|^2, |z_1|^2, V + \Phi_0 + \Phi_1, M(|z_0|^2, |z_1|^2, V + \Phi_0 + \Phi_1, M_2)))$$

where

$$M_2 := M(|z_0|^2, |z_1|^2, V + \Phi_0 + \Phi_1, V + \Phi_0 + \Phi_1, M_1),$$

$$M_1 := M(|z_0|^2, |z_1|^2, V + \Phi_0 + \Phi_1, V + \Phi_0 + \Phi_1, M_1).$$

Therefore, since the composition of real analytic functions is real analytic, we have the conclusion. □

We next define the operator $A(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1)$ on $X_{r,a}$ by

$$A(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1)V = \{((H - (m + 1)E_0 + mE_1)v_m)_{m \geq 0}, ((H - (m + 1)E_0 + mE_0)w_m)_{m \geq 0}\},$$

where $V = \{(v_m)_{m \geq 0}, (w_m)_{m \geq 0}\}$. We next show that the inverse of $A(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1)$ exists in $X^{c}_{r,a}$ where

$$X^{c}_{r,a} = \{(v_m)_{m \geq 0}, (w_m)_{m \geq 0} \} \in X_{r,a} | \langle v_0, \phi_0 \rangle = \langle w_0, \phi_1 \rangle = 0 \}.$$

**Lemma 2.6.** Let $|z_0|, |z_1|, \varepsilon_0, \varepsilon_1$ sufficiently small. Then, $A(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1)$ restricted on $X^{c}_{r,a}$ is invertible. Further, set $B(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1) := \left( A(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1) \right)_{X^{c}_{r,a}}^{-1}$. Then, there exists $\delta > 0$ s.t. $B \in \mathcal{C}^\omega(B_R, (0, \delta); \mathcal{L}(X^{c}_{r,a}))$.

**Proof.** For each $m \geq 0$, we have

$$H - (m + 1)E_j + mE_{j-1} = (H - (m + 1)E_j + mE_{j-1})^{-1} A_{j,m} + (E_{j-1} - E_{j}) B_{j,m},$$

where $A_{j,m} = (m + 1)(H - (m + 1)E_j + mE_{j-1})^{-1}$ and $B_{j,m} = m(H - (m + 1)E_j + mE_{j-1})^{-1}$. Since $|((H - (m + 1)E_j + mE_{j-1})^{-1} \mathcal{L}(X^{c}) \lesssim (m)^{-1}$, for all $m \geq 0$ and $j = 0, 1$, we see that $H - (m + 1)E_j + mE_{j-1}$ is invertible (on $\{\phi_j\}$ in the case $m = 0$) if $|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1$ is sufficiently small. The fact that $B$ is real analytic with respect to $(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1)$ follows from the fact $E_j$ is real analytic with respect to $|z_j|^2$ and (2.10).

We now set $C(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1) \in X_{r,a}$ for $a \leq a_0$ by

$$C(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1) = \varepsilon_0 \Phi_0(|z_0|^2) + \varepsilon_1 \Phi_1(|z_1|^2),$$

where $\Phi_0$, $\Phi_1$ are given in (2.7) and (2.8). Notice that $C \in \mathcal{C}^\omega(B_R, (0, \delta); X_{r,a})$ for sufficiently small $\delta > 0$. Using $A$ and $C$, we can rewrite the system of equations (2.3) as

$$-A(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1)V + C(|z_0|^2, |z_1|^2, \varepsilon_0, \varepsilon_1) = N(|z_0|^2, |z_1|^2, V).$$

Now, set

$$\varepsilon_0 = \langle N_0(|z_0|^2, |z_1|^2, V), \phi_0 \rangle, \quad \varepsilon_1 = \langle M_0(|z_0|^2, |z_1|^2, V), \phi_1 \rangle.$$
Then, we see that \( \varepsilon \in C^\omega(B_{R^2 \times X_{r,a}^0}(0, \delta); \mathbb{R}) \) for \( \delta > 0 \) sufficiently small. Further, setting

\[
\begin{align*}
\bar{B}(|z_0|^2, |z_1|^2, V) &:= B(|z_0|^2, |z_1|^2, \varepsilon_0(|z_0|^2, |z_1|^2, V), \varepsilon_1(|z_0|^2, |z_1|^2, V)), \\
\bar{C}(|z_0|^2, |z_1|^2, V) &:= C(|z_0|^2, |z_1|^2, \varepsilon_0(|z_0|^2, |z_1|^2, V), \varepsilon_1(|z_0|^2, |z_1|^2, V)),
\end{align*}
\]

we have \( \bar{B} \in C^\omega(B_{R^2 \times X_{r,a}^0}(0, \delta); \mathcal{L}(X_{r,a}^0)), \bar{C} \in C^\omega(B_{R^2 \times X_{r,a}^0}(0, \delta); X_{r,a}) \). Thus, it suffices to solve

\[
\mathcal{F}(|z_0|^2, |z_1|^2, V) := V + \bar{B}(|z_0|^2, |z_1|^2, V) \left( -\bar{C}(|z_0|^2, |z_1|^2, V) + N(|z_0|^2, |z_1|^2, V) \right) = 0. \tag{2.13}
\]

Notice that, \( \mathcal{F} \in C^\omega(B_{R^2 \times X_{r,a}^0}(0, \delta); X_{r,a}) \).

**Proof of Theorem 1.6.** We set \( N^0(|z_0|^2, |z_1|^2), N^1(|z_0|^2, |z_1|^2, V), N^2(|z_0|^2, |z_1|^2, V) \) to satisfy \( N = N^0 + N^1 + N^2 \), where \( N \) is given by (2.9). \( N^0 \) has no \( V \), \( N^1 \) has one \( V \) and \( N^2 \) has more than two \( V \).

See section C in the appendix of this paper for the explicit form of \( N_0(|z_0|^2, |z_1|^2), N_1(|z_0|^2, |z_1|^2, V) \).

We first solve (2.13) by implicit function theorem. Notice that \( \mathcal{F}(0,0,0) = 0 \). Further, since \( \bar{C}(0,0,0) = 0 \) and \( N(0,0,0) = 0 \), we have

\[
D_V \mathcal{F}(0,0,0) = D_V \bar{C}(0,0,0) + D_V N(0,0,0),
\]

where \( D_V \) is the Frechet derivative with respect to \( V \). Now, for \( \tilde{V} = \{ \{\tilde{m}\}_m, \{\tilde{w}_m\}_m \} \in X_{\delta_0, \delta_1, a} \), we have

\[
D_V \bar{C}(0,0,0) = \left< D_V N_0(0,0,0) \tilde{V}, \phi_0 \right> \partial_{z_0} C(0,0,0,0) + \left< D_V M_0(0,0,0) \tilde{V}, \phi_1 \right> \partial_{z_1} C(0,0,0,0).
\]

By the explicit form of \( N_0^1 \) and \( M_0^1 \) given in section C in the appendix of this paper, we obtain \( D_V N_0(0,0,0) = D_V N_0^1(0,0,0) = 0 \) and \( D_V M_0(0,0,0) = D_V M_0^1(0,0,0) = 0 \). This gives us \( D_V \bar{C}(0,0,0) = 0 \). Next, again by the explicit form of \( N^1 \), we have

\[
D_V N(0,0,0) \tilde{V} = D_V N^1(0,0,0) \tilde{V}
\]

\[
= \{ \{ 4\phi_0^2 \phi_1^2 \tilde{v}_{m-3} + 3 \phi_0^2 \phi_1 \tilde{w}_{m-3} \}_m \geq 0, \{ 3 \phi_0^2 \phi_1 \tilde{v}_{m-3} + 4 \phi_0^2 \phi_1 \tilde{w}_{m-3} \}_m \geq 0 \},
\]

where \( \tilde{v}_m = \tilde{w}_m = 0 \) if \( m < 0 \). Therefore, we have

\[
\|D_V N(0,0,0)\|_{L(X_{r,a})} \lesssim \varepsilon^6.
\]

This implies that for sufficiently small \( r \), we have \( D_V \mathcal{F}(0,0,0) \) invertible. Therefore, by implicit function theorem, there exists \( V \in C^\omega(B_{R^2}(0, \delta); X_{r,a}) \) s.t. \( \mathcal{F}(|z_0|^2, |z_1|^2, V(|z_0|^2, |z_1|^2)) = 0 \).

Now, \( \psi(z_0, z_1) := T(z_0, z_1) V(|z_0|^2, |z_1|^2) \) is the desired solution. The final task is to obtain the bounds (1.6), (1.7). However, this is easy because the main part of \( V \), is

\[
\bar{B}(|z_0|^2, |z_1|^2, 0) \left( \bar{C}(|z_0|^2, |z_1|^2, 0) - N(|z_0|^2, |z_1|^2, 0) \right),
\]

and the main part of \( \varepsilon_0 \) and \( \varepsilon_1 \) are \( \langle N_0(|z_0|^2, |z_1|^2, 0), \phi_0 \rangle \) and \( \langle M_0(|z_0|^2, |z_1|^2, 0), \phi_1 \rangle \). Therefore, by the explicit form of \( N^0 \) and \( M^0 \), we have (1.6) and (1.7).
3 Coordinate

In this section, we prepare the modulation arguments for the proof of Theorem 1.8. First, since for fixed $z_0, z_1$, $\Psi(e^{-it\xi_z}z_0, e^{-it\xi_z}z_1)$ is a solution of (1.1), computing $\frac{\partial}{\partial t} |_{t=0} \Psi(e^{-it\xi_z}z_0, e^{-it\xi_z}z_1)$, we have

$$H\Psi + |\Psi|^6\Psi = i \sum_{j=0,1} (E_jz_j, D_j, R\Psi(z_0, z_1) - E_jz_j, R\Psi(z_0, z_1)),$$

(3.1)

where $D_j, A := \partial_{z_j, A}$ for $j = 0, 1$, $A = R, I$.

Now, (1.1) conserves the energy $E$ and the $l^2$ norm, where

$$E(u) = \frac{1}{2} \langle Hu, u \rangle + \frac{1}{8} \langle |u|^6 u, u \rangle. \tag{3.2}$$

Substituting, $\Psi(z_0, z_1) + v$, we have

$$E(\Psi + v) = E(\Psi(z_0, z_1)) + E(v) + \langle H\Psi(z_0, z_1) + |\Psi|^6\Psi, v \rangle + N(z_0, z_1, v)$$

$$= E(\Psi(z_0, z_1)) + E(v) + \sum_{j=0,1} (E_jz_j, iD_j, R\Psi(v) - E_jz_j, R(iD_j, R\Psi(v)) + N(z_0, z_1, v), \tag{3.3}$$

where we have used (3.1) in the second equality and

$$N(z_0, z_1, v) = \sum_{k=2, i+j=k, i \geq j} \langle G_{k,i,j}(z_0, z_1), v^j\bar{v}^i \rangle, \tag{3.4}$$

$$G_{k,i,j}(z_0, z_1) = \sum_{l+r=8-k} C_{k,i,j,l,r} \Psi(z_0, z_1)\bar{\Psi(z_0, z_1)}, \tag{3.5}$$

for some $C_{k,i,j,l,r} \in \mathbb{R}$. We take the orthogonality condition for $v$ to eliminate the first order term of $v$ in (3.3). Therefore, we set

$$\mathcal{H}_{\epsilon}[z_0, z_1] := \{v \in l^2 \mid \langle iv, D_j, A\Psi \rangle = 0, \ j = 0, 1, \ A = R, I\}.$$

We show that choosing appropriate $z_0, z_1$, we can make the remainder $v$ to be in $\mathcal{H}_{\epsilon}[z_0, z_1]$. In the following, we use the notation $\phi_j, R := \phi_j$ and $\phi_{j, l} = i\phi_j$ for $j = 0, 1$.

Lemma 3.1. There exists $\delta > 0$ s.t. there exists $(z_1(\cdot), z_2(\cdot)) \in C^\omega(B_{\epsilon}(0, \delta); \mathbb{C} \times \mathbb{C})$, s.t.

$$v(u) := u - \Psi(z_0(u), z_1(u)) \in \mathcal{H}_{\epsilon}[z_0(u), z_1(u)]. \tag{3.6}$$

Proof. Set

$$\mathcal{F}(u, z_0, z_1) := \left\{ \begin{array}{l}
\langle i(u - \Psi(z_0, z_1)), D_{0, R}\Psi(z_0, z_1) \rangle \\
\langle i(u - \Psi(z_0, z_1)), D_{0, I}\Psi(z_0, z_1) \rangle \\
\langle i(u - \Psi(z_0, z_1)), D_{1, R}\Psi(z_0, z_1) \rangle \\
\langle i(u - \Psi(z_0, z_1)), D_{1, I}\Psi(z_0, z_1) \rangle \\
\end{array} \right\}$$

By implicit function theorem, to obtain $z_0(u), z_1(u)$ which satisfy $\mathcal{F}(u, z_0(u), z_1(u)) = 0$, it suffices to show $\frac{\partial \mathcal{F}}{\partial (z_0, R; z_1, R; z_1, I)} |_{(u, z_0, z_1) = 0}$ is invertible if $\|u\|_2 \ll 1$. Since for $j, k = 0, 1, A, B = R, I$. 

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\[ \Psi = o(1), D_{j,A} \Psi = \phi_{j,A} + o(1) \] and \[ D_{j,A} D_{k,B} \Psi = o(1) \] as \[ |u|, |z_0|, |z_1| \to 0, \] we have

\[ \frac{\partial F}{\partial (z_0, R, z_0, R, z_1, R, z_1, I)}
   \bigg|_{(u, z_0, z_1) = (0, 0, 0)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \]

Therefore, there exists \((z_1(\cdot), z_2(\cdot)) \in C^\omega(B_{\mathbb{C}}(0, \delta); \mathbb{C} \times \mathbb{C})\) s.t. \(F(u, z_1(u)), z_2(u) = 0\) which is equivalent to (3.6).

Next, set

\[ P_d := \sum_{j=0, A=R, I} \langle \cdot, \phi_{j,A} \rangle \phi_{j,A}, \quad P_c := 1 - P_d. \]

Notice that \(H_c[0, 0] = P_c l^2 = l^2\).

We now define the inverse of the map \(P_c|_{H_c[z_0, z_1]}\) which was used in [25].

**Lemma 3.2.** There exists \(\delta > 0\) s.t. there exists \(\alpha_{j,A} \in C^\omega(B_{\mathbb{C}}(0, \delta); l^2_{c}(\mathbb{Z}; \mathbb{C}))\) \((j = 0, 1, A = R, I)\), where, \(a_1\) is the constant given in Theorem 1.6, s.t.

\[ \|\alpha_{j,A}(z_0, z_1)\|_{l^2_{c}} \lesssim |z_0|^6 + |z_1|^6, \]

Further,

\[ R[z_0, z_1] \eta = \eta + \sum_{j=0, A=R, I} \langle \alpha_{j,A}(z_0, z_1), \eta \rangle \phi_{j,A}. \]  

(3.7)

satisfies \(R[z_0, z_1] : l^2_{c} \to H_c[z_0, z_1]\) and \(P_c|_{H_c[z_0, z_1]} = R[z_0, z_1]^{-1}\).

**Proof.** We define \(\beta_{j,A}[z_0, z_1] \eta \in \mathbb{R}\) \((j = 0, 1, A = R, I)\) for \(\eta \in l^2_{c}\) to be the unique solution of

\[ \begin{pmatrix} i \left( \eta + \sum_{j=0, A=R, I} \langle \beta_{j,A}(z_0, z_1) \eta \rangle \phi_{j,A} \right), D_{k,B} \Psi(z_0, z_1) \end{pmatrix} = 0, \]

(3.8)

for \(k = 0, 1, B = R, I\) and set

\[ R[z_0, z_1] \eta = \eta + \sum_{j=0, A=R, I} \langle \beta_{j,A}(z_0, z_1) \eta \rangle \phi_{j,A} \]

By the form of \(R[z_0, z_1]\), it is obvious that \(P_c R[z_0, z_1] = \text{id}_{l^2_{c}}\). On the other hand for \(\eta \in H_c[z_0, z_1]\), we have

\[ R[z_0, z_1] P_c \eta = P_c \eta + \sum_{j=0, A=R, I} \langle \beta_{j,A}(z_0, z_1) P_c \eta \rangle \phi_{j,A} \]

\[ = \eta + \sum_{j=0, A=R, I} \langle \beta_{j,A}(z_0, z_1) P_c \eta \rangle \phi_{j,A} \]

Since \(P_c \eta + \sum_{j=0, A=R, I} \langle \eta, \phi_{j,A} \rangle \phi_{j,A} \in H_c[z_0, z_1]\), by the uniqueness of the solution of (3.8), we have

\[ \beta_{j,A}(z_0, z_1) P_c \eta = \langle \eta, \phi_{j,A} \rangle, \quad j = 0, 1, A = R, I. \]
Therefore, we have $R[z_0, z_1]P_c \eta = \eta$.

We finally prove (3.8) has a unique solution. (3.8) can be written as

$$
\sum_{j=0,1; A=R,I} (\beta_{j,A}(z_0, z_1)\eta) = -\langle \eta, D_{k,B}(q_0 + q_1 + \psi) \rangle,
$$

(3.9)

where $k = 0, 1$ and $B = R, I$. Writing (3.9) in the matrix form, one can see the coefficient matrix becomes invertible. Therefore, we have a unique solution of (3.8) and the solution $\beta_{j,A}(z_0, z_1)\eta$ can be expressed as $\langle \alpha_{j,A}(z_0, z_1), \eta \rangle$ where $\alpha_{j,A}(z_0, z_1)$ are linear combinations of $D_{k,B}(q_0 + q_1 + \psi)$ for $k = 0, 1$ and $B = R, I$. This expression combined with Theorem 1.6 gives us the desired estimates for $\alpha_{j,A}$ for $j = 0, 1$ and $A = R, I$.

Combining Lemmas 3.1, 3.2, we obtain a system of coordinates near the origin of $l^2$.

**Lemma 3.3.** Let $\delta > 0$ sufficiently small. Then there exits a $C^\omega$ diffeomorphism

$$
B_{C^2 \times l^2}(0, \delta) \ni (z_0, z_1, \eta) \mapsto u = \Psi(z_0, z_1) + R[z_0, z_1]\eta \in l^2.
$$

(3.10)

Further, we have

$$
|z_0| + |z_1| + \|\eta\|_2 \sim \|u\|_2.
$$

(3.11)

In the following, we set $(z_0(u), z_1(u), \eta(u)) \in C \times C \times l^2_c$ to be the inverse of the map (3.10).

## 4 Darboux theorem

The coordinate system (3.10) given in Lemma 3.3 is essentially the same coordinate given in [25]. This coordinate system is suitable for the nonlinear analysis. However, in our situation, it seems difficult to control $z_0, z_1$ in this coordinate. Therefore, following [16], we make a change of coordinate to have a “canonical” coordinate system.

**Definition 4.1** (Exterior derivative). Let $F \in C^\infty(l^2; X)$, where $X$ be a Banach space (in particular we are considering the case $X = \mathbb{R}, \mathbb{C}, l^2$). We think $F$ as a 0-form and define its exterior derivative $dF(u)$ (which is a 1-form) by $dF(u) = DF(u)$, where $DF(u)$ is the Fréchet derivative of $F$. Next, let $\omega(u)$ be 1-form. Then, we define its exterior derivative $d\omega(u)$ (which is a 2-form) by

$$
d\omega(u)(X,Y) = \mathcal{L}_X \omega(u)(Y) - \mathcal{L}_Y \omega(u)(X),
$$

(4.1)

where $\mathcal{L}_X$ is the Lie derivative.

**Remark 4.2.** In general, for the definition of the exterior derivative, we have the add another term $-\omega(\mathcal{L}_X(Y))$ to (4.1). However, our space $l^2$ is flat and we only have to consider constant vector fields for the definition, we can define $d\omega$ as (4.1). See section 6.4 of [1].

We set the symplectic form $\Omega$ associated to (1.1) by

$$
\Omega(X,Y) := \langle iX, Y \rangle,
$$

and

$$
B(u)X := \frac{1}{2} \Omega(u, X).
$$

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Then,
\[ dB(u)(X,Y) = \mathcal{L}_X B(u)Y - \mathcal{L}_Y B(u)Y = \frac{1}{2} \Omega(X,Y) - \frac{1}{2} \Omega(Y,X) = \Omega(X,Y), \]
where \( \mathcal{L}_X, \mathcal{L}_Y \) are Lie derivatives. Therefore, we have \( dB(u) = \Omega \).

Next, we introduce a new symplectic form \( \Omega_0 \).

**Definition 4.3.** We define the 1-form \( B_0 \) and 2-form \( \Omega_0 \) by the following.
\[
B_0(u)X := \frac{1}{2} \Omega(\Psi(z_0, z_1), d\Psi(z_0, z_1)(X)) + \frac{1}{2} \Omega(\eta, d\eta(X)),
\]
\[
\Omega_0(X,Y) = \Omega(d\Psi(z_0, z_1)(X), d\Psi(z_0, z_1)(Y)) + \Omega(d\eta(X), d\eta(Y)).
\]

**Remark 4.4.** Computing as (4.2), we see
\[
dB_0(u) = \Omega_0.
\]

**Remark 4.5.** The original symplectic form \( \Omega \) do not depend on \( u \). However, the new symplectic form \( \Omega_0 \) depends on \( u \). So, \( \Omega_0(X,Y) \) should be written as \( \Omega_0(u)(X,Y) \). However, we omit \( u \) since there should be no confusion.

Our aim is to change the coordinate system \((z_0, z_1, \eta)\) to have the new symplectic form \( \Omega_0 \). To do so, we use the Moser’s argument. Let \( \Gamma \) s.t. \( \Omega - \Omega_0 = d\Gamma \) and \( \mathcal{X}^s \) satisfies \( i_{\mathcal{X}^s}(\Omega_0 + s(\Omega - \Omega_0)) = -\Gamma \), where \( i_X \omega(Y) = \omega(X, Y) \). Then, if we set \( \mathcal{Y}_s \) to be the solution map of \( \frac{d}{ds} \mathcal{Y}_s = \mathcal{X}^s(\mathcal{Y}_s) \), we have
\[
\frac{d}{ds} (\mathcal{Y}^*_s \Omega_s) = \mathcal{Y}^*_s (\mathcal{L}_{\mathcal{X}^s} \Omega_s + \partial \Omega_s) = \mathcal{Y}^*_s (d\mathcal{X}^s \Omega_s + d\Gamma) = 0,
\]
where \( \Omega_s = \Omega_0 + s(\Omega - \Omega_0) \). Thus, we have the desired change of coordinate \( \mathcal{Y} = \mathcal{Y}_1 \) which satisfies \( \mathcal{Y}^* \Omega = \Omega_0 \). By this argument, it may look like we have already have the change of the coordinate. However, for the application to the asymptotic stability of the quasi-periodic solution, we need have an estimate of \( \mathcal{Y} \) in some weighted space.

In the following we construct \( \Gamma \) and \( \mathcal{X}^s \) directly.

**Lemma 4.6.** Let \( \delta > 0 \) sufficiently small. Then, there exists \( F_\eta \in C^\omega(B_{C^2 \times P_{l_s^{a_1}}} (0, \delta); l_{c_1}^{a_1}) \) and \( F_{j,A} \in C^\omega(B_{C^2 \times P_{l_s^{a_1}}} (0, \delta); l_{c_1}^{a_1}) \) \( (j = 0, 1, A = R, I) \) s.t. there exists \( C \) s.t.
\[
B(u) - B_0(u) - dC = \sum_{j=0,1, A=R,I} \langle F_{j,A}, \eta \rangle \ dz_{j,A} + \langle F_\eta, d\eta \rangle =: \Gamma.
\]

Further, for \( j = 0, 1, A = R, I \), we have
\[
\| F_\eta \|_{C^0} \lesssim (|z_0|^6 + |z_1|^6) \| \eta \|_{L_{c_1}^{a_1}}, \quad (4.4)
\]
\[
\| F_{j,A} \|_{C^1} \lesssim |z_0|^6 + |z_1|^6. \quad (4.5)
\]

**Proof.** In the following, we write \( \Sigma_{j=0,1, A=R,I} \) as \( \Sigma_{j,A} \). Further \( \Sigma_{k,B} \) and \( \Sigma_{l,C} \) will have the same meaning. First, since
\[
2B(u) = \Omega(u, du)
\]
\[
= \Omega(\Psi + \eta + \sum_{j,A} \langle \alpha_{j,A}, \eta \rangle \phi_{j,A}, d(\Psi + \eta + \sum_{k,B} \langle \alpha_{k,B}, \eta \rangle \phi_{k,B})),
\]
\[
= \Omega(\Psi, d\Psi) + \Omega(\eta, d\eta) + \Omega(\Psi, d\eta) + \Omega(\eta, d\Psi)
\]
\[
+ \sum_{k,B} \Omega(\Psi, \phi_{k,B}) d(\langle \alpha_{k,B}, \eta \rangle) + \sum_{j,A} \sum_{k,B} \Omega(\phi_{j,A}, \phi_{k,B}) (\alpha_{j,A}, \eta) \ d(\langle \alpha_{k,B}, \eta \rangle).
\]
So, we have
\begin{equation}
2(B(u) - B_0(u)) = \Omega(\Psi, d\eta) + \Omega(\eta, d\Psi) + \sum_{k,B} \Omega(\Psi, \phi_k,B) d(\langle \alpha_k,B, \eta \rangle) \tag{4.6}
\end{equation}
\begin{equation}
+ \sum_{j,A} \sum_{k,B} \Omega(\phi_j,A, \phi_k,B) \langle \alpha_{a,A}, \eta \rangle d(\langle \alpha_k,B, \eta \rangle).
\end{equation}

The first and second term of r.h.s. of (4.6) can be rewritten as
\begin{equation}
\Omega(\Psi, d\eta) + \Omega(\eta, d\Psi) = d\Omega(\Psi, \eta) + 2\Omega(\eta, d\Psi). \tag{4.7}
\end{equation}

The third term of r.h.s. of (4.6) can be rewritten as
\begin{equation}
\sum_{k,B} \Omega(\Psi, \phi_k,B) d(\langle \alpha_k,B, \eta \rangle) = d \left( \sum_{k,B} \Omega(\Psi, \phi_k,B) \langle \alpha_k,B, \eta \rangle \right) + \sum_{k,B} \langle \alpha_k,B, \eta \rangle \Omega(\phi_k,B, d\Psi). \tag{4.8}
\end{equation}

The last term of (4.6) can be rewritten as
\begin{equation}
\sum_{j,A} \sum_{k,B} \Omega(\phi_j,A, \phi_k,B) \langle \alpha_{a,A}, \eta \rangle d(\langle \alpha_k,B, \eta \rangle) = \sum_{j,A} \sum_{k,B} \Omega(\phi_j,A, \phi_k,B) \langle \alpha_{a,A}, \eta \rangle (\langle \eta, d\alpha_k,B \rangle + \langle \alpha_k,B, d\eta \rangle). \tag{4.9}
\end{equation}

Combining (4.7), (4.8) and (4.9), we have
\begin{equation}
2(B(u) - B_0(u)) = d \left( \Omega(\Psi, \eta) + \sum_{k,B} \Omega(\Psi, \phi_k,B) \langle \alpha_k,B, \eta \rangle \right) \tag{4.10}
\end{equation}
\begin{equation}
+ \Omega \left( 2\eta + \sum_{k,B} \langle \alpha_k,B, \eta \rangle \phi_k,B, d\Psi \right) + \sum_{l,C} \sum_{k,B} \Omega(\phi_l,C, \phi_k,B) \langle \alpha_{l,C}, \eta \rangle (\langle \eta, d\alpha_k,B \rangle + \langle \alpha_k,B, d\eta \rangle).
\end{equation}

Since \(d\Psi = \sum_{j,A} D_{j,A} \Psi dz_{j,A}, d\alpha_k,B = \sum_{j,A} D_{j,A} \alpha_k,B dz_{j,A}\) and \(\Omega(\eta, D_{j,A} \Psi) = \Omega(\eta, D_{j,A}(q_0 + q_1 + \psi))\) because \(P_{c,\eta} = \eta\), from (4.10), we have
\begin{equation}
B(u) - B_0(u) = dC + \langle F_{j,A}, \eta \rangle dz_{j,A} + \langle F, d\eta \rangle, \tag{4.11}
\end{equation}
where
\begin{equation}
C = \frac{1}{2} \left( \Omega(\Psi, \eta) + \sum_{k,B} \Omega(\Psi, \phi_k,B) \langle \alpha_k,B, \eta \rangle \right),
\end{equation}
\begin{equation}
F_{j,A} = -iD_{j,A}(q_0 + q_1 + \psi) + \frac{1}{2} \sum_{k,B} \Omega(\phi_k,B, D_{j,A} \Psi) \alpha_k,B \tag{4.12}
\end{equation}
\begin{equation}
+ \frac{1}{2} \sum_{k,B, l,C} \Omega(\phi_k,B, \phi_{l,C}) \langle \eta, D_{j,A} \alpha_{l,C} \rangle \alpha_k,B,
\end{equation}
\begin{equation}
F_{\eta} = \sum_{k,B, l,C} \Omega(\phi_k,B, \phi_{l,C}) \langle \alpha_k,B, \eta \rangle \alpha_{l,C}. \tag{4.13}
\end{equation}

The estimates (4.4)–(4.5) follows from (4.12) and (4.13) and Lemma 3.2.
By lemma 3.11, we have

\[ \Omega - \Omega_0 = d(B(u) - B_0(u)) = d(dC + \Gamma) = d\Gamma. \]

We set

\[ \Omega_s = \Omega_0 + s(\Omega - \Omega_0), \]

and try to find a solution $X^s$ of the equation $i_{\Sigma_s}\Omega_s = -\Gamma$.

**Lemma 4.7.** Let $\delta > 0$ sufficiently small. Then, there exist $X^s_0 \in C^\omega(B_{C^2 \times P_e^{-1}}(0, \delta); \mathbb{B})$ and $\lambda^s_{j,A} \in C^\omega(B_{C^2 \times P_e^{-1}}(0, \delta); \mathbb{R})$ for $j = 0, 1$, $A = R, I$ s.t. $X^s := \sum_{j,A} \lambda^s_{j,A} \partial_{\omega_{j,A}} + X^s \nabla \eta$ satisfies $i_{\Sigma_s}\Omega_s = -\Gamma$. Further, we have

\[ \|X^s_0\|_{\mathcal{E}^s} + \sum_{j,A} |\lambda^s_{j,A}| \lesssim (|z_0|^6 + |z_1|^6)\|\eta\|_{\mathcal{E}^{-1}}. \]

**Proof.** We directly solve

\[ \Omega_0(X^s, \cdot) + s(\Omega(X^s, \cdot) - \Omega_0(X^s, \cdot)) = -\Gamma. \quad (4.14) \]

In the following, we omit the summation over $j = 0, 1$, $A = R, I$, etc. and $j, k, l, r$ will always be 0, 1 and $A, B, C, D$ will be $R, I$. First, we have

\[ \Omega_0(X^s, \cdot) = \Omega(d\Psi(X^s), d\Psi) + \Omega(X^s_0, d\eta) \]

\[ = \Omega(D_{k,B}\Psi, D_{j,A}\Psi)X^s_{k,B} + \Omega(X^s_0, d\eta), \]

and

\[ \Omega(X^s, \cdot) = \Omega(d(\Psi + \eta + (\alpha_{i,C}, \eta) \phi_{i,C})(X^s), d(\Psi + \eta + (\alpha_{r,D}, \eta) \phi_{r,D})) \]

\[ = \Omega(D_{k,B}\Psi X^s_{k,B} + X^s_0 + (D_{k,B}(\alpha_{i,C}, \eta))\phi_{i,C} X^s_{k,B} + (\alpha_{i,C}, X^s_0) \phi_{i,C}, \]

\[ D_{j,A}\Psi \partial_{\omega_{j,A}} + d\eta + (D_{j,A}(\alpha_{r,D}, \eta) \phi_{r,D} \partial_{\omega_{j,A}} + (\alpha_{r,D}, d\eta) \phi_{r,D}) \]

\[ = \Omega_0(X^s, \cdot) + (G_{j,A,k,B} X^s_{k,B} + \langle G_{j,A,n}, X^s_n \rangle) d\omega_{j,A} - X^s_{k,B} \langle G_{k,B,n}, d\eta \rangle \]

\[ + \Omega(\phi_{i,C}, \phi_{r,D}) (\alpha_{i,C}, X^s_0) \langle \alpha_{r,D}, d\eta \rangle \]

where

\[ G_{j,A,k,B} = \Omega(D_{k,B}\Psi, \phi_{r,D}) (D_{j,A}(\alpha_{i,C}, \eta) + \langle D_{k,B}(\alpha_{i,C}, \eta) \Omega(\phi_{i,C}, D_{j,A}\Psi) \rangle \]

\[ + \langle D_{k,B}(\alpha_{i,C}, \eta) \Omega(\phi_{i,C}, \phi_{r,D}) \rangle (D_{j,A}(\alpha_{r,D}, \eta), \]

\[ G_{j,A,n} = -iD_{j,A}(q_0 + q_1 + \psi) + \Omega(\phi_{i,C}, D_{j,A}\Psi) \alpha_{i,C} + \Omega(\phi_{i,C}, \phi_{r,D}) (D_{j,A}(\alpha_{r,D}, \eta) \alpha_{r,D}. \]

Therefore, (4.14) can be written as

\[ (\Omega(D_{k,B}\Psi, D_{j,A}\Psi) + sG_{j,A,k,B} X^s_{k,B} + \langle G_{j,A,n}, X^s_n \rangle = -\langle F_{j,A}, \eta \rangle \]

\[ i\lambda^s_0 + s(-\lambda^s_{k,B} G_{k,B,n} + \langle \alpha_{i,C}, X^s_0 \rangle \alpha_{r,D} = -F_\eta, \]
where
\[
\mathcal{A}_\xi = i\Omega(\phi_{l,C}, \phi_{r,D}) \langle \alpha_{l,C}, \xi \rangle \alpha_{r,D}.
\]

Since \( \|A\|_{L^2_\eta} \leq |z_0|^6 + |z_1|^6 \) by Lemma 3.2, \( \|(1 - \mathcal{A})^{-1}\|_{L^2_\eta} \leq 1 \). Therefore, we have
\[
\mathcal{X}_\eta^\ast = (1 - \mathcal{A})^{-1} \left( -i\mathcal{A}_{k,B}G_{k,B,\eta} + iF_\eta \right). \tag{4.17}
\]

Substituting, (4.17) into (4.15), we have
\[
(\Omega(D_{k,B}\Psi, D_{j,A}\Psi) + sG_{j,A,k,B} - \langle G_{j,A,\eta}, is(1 - \mathcal{A})^{-1}G_{k,B,\eta} \rangle) \mathcal{X}_\eta^*_{k,B}
= - \langle F_{j,A,\eta} - \langle G_{j,A,\eta}, i(1 - \mathcal{A})^{-1}F_\eta \rangle. \tag{4.18}
\]

Considering \( \Omega(D_{k,B}\Psi, D_{j,A}\Psi) + sG_{j,A,k,B} + \langle G_{j,A,\eta}, is(1 - \mathcal{A})^{-1}G_{\eta,k,B} \rangle \) as a \( 4 \times 4 \) matrix, this matrix has the form
\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} + o(1).
\]

Since this matrix is invertible, we have the solution of (4.18). Therefore, we have the solution of (4.14). Further, we have
\[
|\mathcal{X}_\eta^*_{k,B}| \lesssim \|F_{j,A}\|_{L^2_\eta} \|\eta\|_{L^{-1}_\eta} + \|G_{j,A,\eta}\|_{L^2} \|F\|_{L^2} \lesssim (|z_0|^6 + |z_1|^6)\|\eta\|_{L^{-1}_\eta},
\]
and
\[
\|\mathcal{X}_\eta^*\|_{L^2_\eta} \lesssim \|\mathcal{X}_{k,B}\|_{L^2_\eta} + \|F\|_{L^2_\eta} \lesssim (|z_0|^6 + |z_1|^6)\|\eta\|_{L^{-1}_\eta}.
\]

We now construct the desired change of coordinate \( \mathcal{Y} \) by the flow of \( \mathcal{X}^\ast \). We consider the following system
\[
\frac{d}{ds} r_j(z_0, z_1, \eta; s) = \mathcal{X}_{k,A}^\ast (z_0 + r_0(z_0, z_1, \eta; s), z_1 + r_1(z_0, z_1, \eta; s), \eta + r_\eta(z_0, z_1, \eta; s)), \tag{4.19}
\]
\[
\frac{d}{ds} r_\eta(z_0, z_1, \eta; s) = \mathcal{X}_{\eta}^\ast (z_0 + r_0(z_0, z_1, \eta; s), z_1 + r_1(z_0, z_1, \eta; s), \eta + r_\eta(z_0, z_1, \eta; s)), \tag{4.20}
\]
with \( j = 0, 1 \) and the initial condition \( r_0(0) = 0, r_1(0) = 0 \) and \( r_\eta(0) = 0 \).

**Lemma 4.8.** Let \( \delta > 0 \) sufficiently small. Then, there exists
\[
(r_0(0), r_1(0), r_\eta(0)) \in C^\omega(B_{C^2 \times P_{L^{-1}_\eta}}(0, \delta); C([0, 1]; C^2 \times L^2_\eta)),
\]

s.t. \( (r_0(z_0, z_1, \eta; \cdot), r_1(z_0, z_1, \eta; \cdot), r_\eta(z_0, z_1, \eta; \cdot)) \) is the solution of system (4.19)–(4.20) and
\[
\sum_{j=0,1} |r_j(z_0, z_1, \eta; 1)| + \|r_\eta(z_0, z_1, \eta; 1)\|_{L^2_\eta} \lesssim (|z_0|^6 + |z_1|^6)\|\eta\|_{L^{-1}_\eta}. \tag{4.21}
\]
Proof. We solve the system (4.19)–(4.20) by implicit function theorem. Let $\delta > 0$ sufficiently small. Let $(x_0, x_1, \xi) \in C([0,1]; B^2_{C^2 \times l_2^2}(0, \delta))$ and set

\[
\Phi(z_0, z_1, \eta, x_0, x_1, \xi)(s) = (\Phi_0(z_0, z_1, \eta, x_0, x_1, \xi)(s), \Phi_1(z_0, z_1, \eta, x_0, x_1, \xi)(s), \Phi_2(z_0, z_1, \eta, x_0, x_1, \xi)(s)),
\]

where

\[
\Phi_j(z_0, z_1, \eta, x_0, x_1, \xi)(s) = x_j(s) - \int_0^s \mathcal{X}_j(z_0 + x_0(\tau), z_1 + x_1(\tau), \eta + \xi(\tau)) \, d\tau, \quad j = 0, 1,
\]

\[
\Phi_2(z_0, z_1, \eta, x_0, x_1, \xi)(s) = \xi(s) - \int_0^s \mathcal{X}_2(z_0 + x_0(\tau), z_1 + x_1(\tau), \eta + \xi(\tau)) \, d\tau.
\]

Notice that $\Phi \in C^\omega(B_{C^2 \times l_2^{\epsilon=1}}(0, \delta) \times B_{C((0,1]; C^2 \times l_2^{\epsilon=1})}(0, \delta); C([0,1]; C^2 \times l_2^2))$. By the estimate of lemma 4.7 and the analyticity of $\mathcal{X}_j$, we have

\[
D_\xi \Phi_j(0, 0, 0, 0, 0, 0) = id_{C((0,1]; l_2^2)}.
\]

Therefore, there exists $\tilde{r}_\eta(z_0, z_1, \eta, x_0, x_1)$ s.t. $\Phi_\eta(z_0, z_1, \eta, x_0, x_1, r_\eta(z_0, z_1, \eta, x_0, x_1)) = 0$. Repeatedly, we will have $\tilde{r}_1(z_0, z_1, \eta, x_0)$ and $r_\eta(z_0, z_1, \eta)$ with desired property. Therefore, setting $r_1 = \tilde{r}_1(z_0, z_1, \eta, x_0(z_0, z_1, \eta))$ and $r_\eta(z_0, z_1, \eta) = \tilde{r}_\eta(z_0, z_1, \eta, x_0(z_0, z_1, \eta), x_1(z_0, z_1, \eta))$, we have the solution of (4.19)–(4.20). We now prove (4.21). From

\[
\begin{align*}
\frac{d}{ds} \mathcal{Y}_s &= \mathcal{X}(\mathcal{Y}_s), \\
\mathcal{Y}_s &= \mathcal{Y}_s(0, 0, 0, 0, 0, 0).
\end{align*}
\]

Combining (4.24) with the fact $A(0) = 0$, we have (4.21). □

Now, define $\mathcal{Y}_s$ by

\[
\mathcal{Y}_s^* z_j = z_j + r_j(z_0, z_1, \eta; s), \quad j = 0, 1, \quad \mathcal{Y}_s^* \eta = \eta + r_\eta(z_0, z_1, \eta; s).
\]

Then, $\mathcal{Y}_s$ satisfies

\[
\frac{d}{ds} \mathcal{Y}_s = \mathcal{X}^*(\mathcal{Y}_s),
\]

which gives us the desired coordinate change by (4.3). Therefore, setting $\mathcal{Y} := \mathcal{Y}_1$, we have

\[
\mathcal{Y}^* \Omega = \Omega_0.
\]

We set $r_j(z_0, z_1, \eta) := r_j(z_0, z_1, \eta; 1)$ for $j = 0, 1$ and $r_\eta(z_0, z_1, \eta) := r_\eta(z_0, z_1, \eta; 1)$. We will say $(z_0', z_1', \eta') := (\mathcal{Y}^* z_0, \mathcal{Y}^* z_1, \mathcal{Y}^* \eta) = (z_0 + r_0, z_1 + r_1, \eta + r_\eta)$ is the “old” coordinate and $(z_0, z_1, \eta)$ is the “new” coordinate. We set the pull-back of the energy by $K$. That is, we set

\[
K(z_0, z_1, \eta) = E \circ \mathcal{Y}(z_0, z_1, \eta) = E(z_0 + r_0, z_1 + r_1, \eta + r_\eta).
\]
Now, we define the Hamiltonian vector field associated to $F$ with respect to the symplectic form $\Omega_0$. We define $X_F$ by

$$\Omega_0(X_F, Y) = \langle \nabla F, Y \rangle.$$ 

Further, set $(X_F)_{j,A} := dz_{j,A} X_F$ for $j = 0, 1$ and $A = R, I$ and $(X_F)_\eta := d\eta (X_F)$. Then, if $u$ is a solution of (1.1), $z_{j,A}$ and $\eta$ satisfies

$$\dot{z}_{j,A} = (X_K)_{j,A}, \quad j = 0, 1, A = R, I, \quad \dot{\eta} = (X_K)_\eta.$$ 

We now directly compute $(X_K)_{j,A}$. By the definition of $X_K$, we have

$$\Omega_0(X_K, Y) = \sum_{j,A} \sum_{k,B} \Omega(D_{k,B} \Psi, D_{j,A} \Psi)(X_K)_{k,B} dz_{j,A} Y + \Omega((X_K)_\eta, d\eta Y),$$

and

$$\langle \nabla K, Y \rangle = \sum_{j,A} D_{j,A} F dz_{j,A} Y + \langle \nabla \eta, d\eta Y \rangle.$$ 

Therefore, we have

$$\dot{\eta} = (X_K)_\eta = -i \nabla \eta K. \tag{4.25}$$

We will postpone the computation of $(X_K)_{j,A}$.

Our next task is to compute the pull-back of the energy $K$. Before computing, we make one observation. The following lemma is corresponds to Lemma 4.11 (Cancellation Lemma) of [16].

**Lemma 4.9.** Let $\delta > 0$ sufficiently small. Then, for any $(z_0, z_1) \in B_{C^2}(0, \delta)$, we have

$$\nabla \eta K(z_0, z_1, 0) = 0.$$ 

**Proof.** First, notice that if $\eta = 0$, from Lemma 4.8, we have $r_0 = r_1 = 0$ and $r_\eta = 0$. Therefore, the new and old coordinate becomes the same.

Next, recall that if we have the initial condition $u(z'_0, z'_1, 0) = \Psi(z'_0, z'_1)$, since $\Psi$ is the quasi-periodic solution, $\eta'$ will always be 0. Therefore, the new and old coordinate will correspond for all time and further, we will have $\eta = 0$ for all time.

Now, suppose $\nabla \eta K(z_0, z_1, 0) \neq 0$. Then, from (4.25), we have

$$i \frac{d}{dt} \bigg|_{t=0} \eta = \nabla \eta K(z_0, z_1, 0) \neq 0. \tag{4.26}$$

However, l.h.s. of (4.26) is 0 because $\eta(t) \equiv 0$. Therefore, we have the conclusion. \qed

We prepare another lemma before computing $K$.

**Lemma 4.10.** Set

$$\delta \Psi(z_0, z_1, \eta) := \Psi(z_0 + r_0(z_0, z_1, \eta), z_1 + r_0(z_0, z_1, \eta)) - \Psi(z_0, z_1),$$

$$\delta \eta(z_0, z_1, \eta) := R[\eta + r_0(z_0, z_1, \eta), z_1 + r_1(z_0, z_1, \eta)](\eta + r_\eta(z_0, z_1, \eta)) - \eta.$$
Then, we have
\[
\|D^m \delta \Psi(z_0, z_1, \eta)\|_{L^2(\mathcal{C})} + \|D^m \delta \eta(z_0, z_1, \eta)\|_{L^2(\mathcal{C})} \lesssim \left( |z_0|^{\max(6-|m|)} + |z_1|^{\max(6-|m|)} \right) \|\eta\|_{L^2(\mathcal{C})}^{\max(|m|)}, \tag{4.27}
\]
\[
\|D^m D_\eta \delta \Psi(z_0, z_1, \eta)\|_{L^2(\mathcal{C})} + \|D^m D_\eta \delta \eta(z_0, z_1, \eta)\|_{L^2(\mathcal{C})} \lesssim |z_0|^{\max(6-|m|)} + |z_1|^{\max(6-|m|)},
\]
\[
\|D^m D_\eta^2 \delta \Psi(z_0, z_1, \eta)(\xi, \eta)\|_{L^2(\mathcal{C})} \lesssim \left( |z_0|^{\max(6-|m|)} + |z_1|^{\max(6-|m|)} \right) \|\xi\|_{L^2(\mathcal{C})}^{\max(|m|)},
\]
where \( m = (m_1, \cdots, m_{|m|}) \) with \( m_j \in \{(k, B) \mid k = 0, 1, B = R, I\} \) and \( D^m = D_{m_1} \cdots D_{m_{|m|}} \).

**Proof.** By Taylor expansion, we have
\[
\delta \Psi(z_0, z_1, \eta) = \sum_{j,A} \int_0^1 D_{j,A} \Psi(z_0 + sr_0, z_1 + sr_1) \, ds r_j, A,
\]
\[
\delta \eta(z_0, z_1, \eta) = r_\eta + \sum_{j,A} \langle \alpha_{j,A}(z_0 + r_0, z_1 + r_1), \eta + r_\eta \rangle \phi_{j,A}.
\]

Combining the above with Lemma 4.8, we have (4.27) with \( |m| = 0 \). The estimates for the derivative respect to \( D_{j,A} \) and \( D_\eta \) also follows from Lemma 4.8 because of the analyticity.

We now compute the expansion of \( K \).

**Lemma 4.11.** We have
\[
K(z_0, z_1, \eta) = E(\Psi(z_0, z_1)) + E(\eta) + N(z_0, z_1, \eta), \tag{4.28}
\]
where \( N \) satisfies
\[
|N(z_0, z_1, \eta)| \lesssim (|z_0|^6 + |z_1|^6 + \|\eta\|_{L^2(\mathcal{C})}) \|\eta\|_{L^2(\mathcal{C})}^2, \tag{4.29}
\]
\[
|D_{j,A}N(z_0, z_1, \eta)| \lesssim (|z_0|^5 + |z_1|^5 + \|\eta\|_{L^2(\mathcal{C})}) \|\eta\|_{L^2(\mathcal{C})}^2, \tag{4.30}
\]
\[
|\nabla_\eta N(z_0, z_1, \eta)|_{L^2(\mathcal{C})} \lesssim (|z_0|^6 + |z_1|^6 + \|\eta\|_{L^2(\mathcal{C})}) \|\eta\|_{L^2(\mathcal{C})}^2, \tag{4.31}
\]
for \( a_2 = a_1/3 \).

**Proof.** By Taylor expansion, we have
\[
E(\Psi(z_0', z_1')) = E(\Psi(z_0, z_1)) + \int_0^1 \langle \nabla E(\Psi(z_0, z_1)) + s \delta \Psi(z_0, z_1), \delta \Psi(z_0, z_1) \rangle \, ds,
\]
\[
E(R[z_0', z_1']) = E(\eta) + \int_0^1 \langle \nabla E(\eta + s \delta \eta(z_0, z_1, \eta)), \delta \eta(z_0, z_1, \eta) \rangle \, ds.
\]

Therefore, by (3.3), we have (4.28), with
\[
N(z_0, z_1, \eta) = \int_0^1 \langle \nabla E(\Psi + s \delta \Psi), \delta \Psi \rangle \, ds + \int_0^1 \langle \nabla E(\eta + s \delta \eta), \delta \eta \rangle \, ds \tag{4.32}
\]
\[
+ \sum_{k=2}^7 \sum_{i+j+k \equiv 0 \pmod{2}} \sum_{\mathbf{i} \geq \mathbf{j}} C_{k,i,j,r} \langle (\Psi + \delta \Psi)^r(\overline{\Psi} + \overline{\delta \Psi})^r, (\eta + \delta \eta)^r(\overline{\eta} + \overline{\delta \eta})^r \rangle.
\]
Therefore, we have

\[ \left| \int_0^1 \langle \nabla E(\eta + s\delta\eta), \delta\eta \rangle \, ds \right| \leq \int_0^1 \| \nabla E(\eta + s\delta\eta) \|_{L^{\infty}} \, ds \| \delta\eta \|_{L^1} \lesssim \| \eta \|_{L^1} \| \delta\eta \|_{L^1} \]

\[ \lesssim (|z_0|^6 + |z_1|^6)\| \delta\eta \|_{L^1}^2. \]

One can also estimate the \( D_{j,A} \) derivative of this term in similar manner. We next compute the \( \nabla \eta \) derivative.

\[ \left\langle \nabla \eta \int_0^1 \langle \nabla E(\eta + s\delta\eta), \delta\eta \rangle \, ds, \xi \right\rangle = \left\langle \nabla \eta \int_0^1 \langle H\eta + sH\delta\eta + (\eta + s\delta\eta)^4(\bar{\eta} + s\delta\eta)^3, \delta\eta \rangle \, ds, \xi \right\rangle \]

\[ = (H\xi, \delta\eta) + \frac{1}{2} \langle HD_{j,}\delta\eta(\xi), \delta\eta \rangle + 4 \int_0^1 \langle (\xi + s(D_{j,}\delta\eta(\xi))) |\eta + s\delta\eta|^6, \delta\eta \rangle \, ds \]

\[ + 3 \int_0^1 \langle \bar{\xi} + sD_{j,}\delta\eta(\xi) \rangle |\eta + s\delta\eta|^4(\eta + s\delta\eta)^2, \delta\eta \rangle \, ds + \int_0^1 \langle \nabla E(\eta + s\delta\eta), D_{j,}\delta\eta(\xi) \rangle \, ds. \]

Therefore, we have

\[ \| \nabla \eta \int_0^1 \langle \nabla E(\eta + s\delta\eta), \delta\eta \rangle \, ds \|_{L^1} \lesssim \| \delta\eta \|_{L^2} + \| D_{j,}\delta\eta \|_{L(1^2, 1^2)} \| \delta\eta \|_{L^2} \]

\[ + (1 + \| D_{j,}\delta\eta \|_{L(1^2, 1^2)})(\| \eta \|_{L^\infty}^6 + \| \delta\eta \|_{L^\infty}^6) \| \delta\eta \|_{L^2} + \| D_{j,}\delta\eta \|_{L(1^2, 1^2)} \int_0^1 \| \nabla E(\eta + s\delta\eta) \|_{L^{\infty}} \, ds \]

\[ \lesssim (|z_0|^6 + |z_1|^6) \| \eta \|_{L^1}^2. \]

The third term of (4.32) can be bounded in similar manner. However, for example, the estimate of \( \langle \Psi, |\eta|^7 \eta \rangle \), we have

\[ \| \langle \Psi, |\eta|^7 \eta \rangle \| \leq (|z_0| + |z_1|)\| \eta \|_{L^\infty}^5 \| \eta \|_{L^1}^2. \]

This is why we have to make \( a_1 \) smaller and replace \( |z_0|^j + |z_1|^j \) to \( |z_0|^j + |z_1|^j + \| \eta \|_{L^2}^j. \)

We finally estimate the first term of (4.32). Expanding \( \nabla E(\Psi + s\delta\Psi) \), we have

\[ \int_0^1 \langle \nabla E(\Psi + s\delta\Psi), \delta\Psi \rangle \, ds = \langle \nabla E(\Psi), \delta\Psi \rangle + \frac{1}{2} \langle H\delta\Psi, \delta\Psi \rangle \]

\[ + \int_0^1 \| \Psi + s\delta\Psi \|_{L^\infty}^6(\Psi + \delta\Psi) - |\Psi|^{l_2} \Psi, \delta\Psi \rangle \, ds. \]

The last two terms, which has at least two \( \delta\Psi \) can be estimated as before. Now, notice that the only possible source of the first order term of \( \eta \) is \( \langle \nabla E(\Psi), \delta\Psi \rangle \). However, by Lemma 4.9, for arbitrary \( \xi \in l^2 \), we have

\[ 0 = \langle \nabla \eta K(z_0, z_1, 0), \xi \rangle = \langle \nabla \eta \langle \nabla E(\Psi), \delta\Psi(z_0, z_1, 0) \rangle, \xi \rangle = \langle \nabla E(\Psi), D_{\eta}\delta\Psi(z_0, z_1, 0)(\xi) \rangle \]

Therefore, by Taylor expansion, we have

\[ \langle E(\Psi(z_0, z_1)), \delta\Psi(z_0, z_1, \eta) \rangle = \int_0^1 (1 - s) \langle E(\Psi), D_{\eta}^2\delta\Psi(z_0, z_1, s\eta)(\eta, \eta) \rangle \, ds, \]

\[ \langle \nabla \eta \langle E(\Psi(z_0, z_1)), \delta\Psi(z_0, z_1, \eta) \rangle, \xi \rangle = \int_0^1 \langle E(\Psi), D_{\eta}^2\delta\Psi(z_0, z_1, s\eta)(\eta, \xi) \rangle \, ds. \]
Thus, by Lemma 4.10, we have
\[
| (E(\Psi), \delta \Psi) | \lesssim \sup_{s \in [0,1]} \| D_{\theta}^{2} \delta \Psi (z_{0}, z_{1}, s\eta) (\eta, \eta) \|_{L^{2}} \lesssim (|z_{0}|^{6} + |z_{1}|^{6}) \| \eta \|^{2}_{L^{2}(-1)} ,
\]
\[
\| \nabla_{\eta} (E(\Psi), \delta \Psi) \|_{L^{2}} \lesssim \sup_{s \in [0,1]} \| D_{\theta} \delta \Psi (z_{0}, z_{1}, s\eta) (\eta, \cdot) \|_{L^{2}(-1)} \lesssim (|z_{0}|^{6} + |z_{1}|^{6}) \| \eta \|_{L^{2}(-1)} .
\]

The estimate for $D_{j,A} (E(\Psi), \delta \Psi)$ can be obtained by similar manner. Therefore, we have the conclusion.

We now try to obtain the equations which $z_{j}$ satisfies. Set
\[
\{ F, G \} := dF(u)X_{G}(u) = \langle \nabla F(u), X_{G}(u) \rangle = \Omega(X_{F}(u), X_{G}(u))
\]
for $F : l^{2} \rightarrow \mathbb{C}$. Then, if $u$ is a solution of (1.1), we have
\[
\frac{d}{dt} F(u) = \{ F, K \} ,
\]
Therefore, setting
\[
K(z_{0}, z_{1}, \eta) = K_{0}(z_{0}, z_{1}) + K_{1}(z_{0}, z_{1}, \eta) ,
\]
where $K_{0}(z_{0}, z_{1}) = E(\Psi(z_{0}, z_{1}))$, we have
\[
\dot{z}_{j} = \{ z_{j}, K_{0} \} + \{ z_{j}, K_{1} \} .
\]

Now, since
\[
\Omega_{0}(X_{K_{n}}, Y) = dK_{n}(u)Y = \sum_{j=0,1; A=R, I} \partial_{z_{j,A}} K_{n} Y_{j,A} + \langle \nabla_{\eta} K_{n}, Y_{\eta} \rangle ,
\]
and
\[
\Omega_{0}(X_{K_{n}}, Y) = \sum_{j,k=0,1; A,B=R, I} \Omega(D_{k,B} \Psi(z_{0}, z_{1}), D_{j,A} \Psi(z_{0}, z_{1}))(X_{K_{n}})_{k,B}(Y)_{j,A} + \Omega((X_{K_{n}})_{\eta}, Y_{\eta}) ,
\]
we have
\[
(X_{F})_{z} = A(z_{0}, z_{1})^{-1} \partial_{z} F ,
\]
where,
\[
(X_{F})_{z} = \left( \begin{array}{c}
(X_{F})_{0,R} \\
(X_{F})_{0,I} \\
(X_{F})_{1,R} \\
(X_{F})_{1,I}
\end{array} \right) ,
\quad \partial_{z} F = \left( \begin{array}{c}
D_{0,R} F \\
D_{0,I} F \\
D_{1,R} F \\
D_{1,I} F
\end{array} \right) ,
\]
and
\[
A(z_{0}, z_{1}) = \left( \begin{array}{cccc}
(a_{0,0,0,R}(z_{0}, z_{1}) & a_{0,1,0,R}(z_{0}, z_{1}) & a_{1,0,0,R}(z_{0}, z_{1}) & a_{1,1,0,R}(z_{0}, z_{1}) \\
a_{0,0,0,I}(z_{0}, z_{1}) & a_{0,1,0,I}(z_{0}, z_{1}) & a_{1,0,0,I}(z_{0}, z_{1}) & a_{1,1,0,I}(z_{0}, z_{1}) \\
a_{0,0,1,R}(z_{0}, z_{1}) & a_{0,1,1,R}(z_{0}, z_{1}) & a_{1,0,1,R}(z_{0}, z_{1}) & a_{1,1,1,R}(z_{0}, z_{1}) \\
a_{0,0,1,I}(z_{0}, z_{1}) & a_{0,1,1,I}(z_{0}, z_{1}) & a_{1,0,1,I}(z_{0}, z_{1}) & a_{1,1,1,I}(z_{0}, z_{1})
\end{array} \right) ,
\]
\[20\]
where \( a_j,A,k,B(z_0, z_1) = \Omega(D_{j,A} \Psi(z_0, z_1), D_{k,B}\Psi(z_0, z_1)) \). Notice that since

\[
A(z_0, z_1) = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix} + o(1),
\]

\( A(z_0, z_1) \) is invertible.

We will not compute \( \{z_j, K_0\} \) directly but use the fact that \( \Psi(z_0, z_1) \) is the solution of (1.1).

**Lemma 4.12.** We have

\[
\{z_j, K_0(z_0, z_1)\} = -i \mathcal{E}(|z_0|^2, |z_1|^2) z_j
\]

**Proof.** First, if \( \eta = 0 \), we have \( z_0' = z_0 \) and \( z_1' = z_1 \) (the old and new coordinate becomes the same). Therefore, since \( \Psi(z_0, z_1) \) is the solution (1.1) if \( i \dot{z}_j = \mathcal{E}(|z_0|^2, |z_1|^2) z_j \), we have

\[
-i \mathcal{E}(|z_0|^2, |z_1|^2) z_j = \dot{z}_j = \{z_j, K(z_0, z_1, \eta)\}_{\eta=0} = \{z_j, K_0(z_0, z_1)\}_{\eta=0} + \{z_j, K_1(z_0, z_1, \eta)\}_{\eta=0}.
\]

On the other hand, from (4.33), we see that \( \{z_j, K_1(z_0, z_1, \eta)\}_{\eta=0} = 0 \) because it consists from the derivative of \( K_1 \). Therefore, we have

\[
-i \mathcal{E}(|z_0|^2, |z_1|^2) z_j = \{z_j, K_0(z_0, z_1)\}_{\eta=0}.
\]

Finally, since the symplectic form \( \Omega_0 \) do not depend on \( \eta \) (although it depends on \( z_j \)), we have the conclusion. \( \square \)

We set \( R_j = \{z_j, K_1(z_0, z_1, \eta)\} \). Then, by (4.33), we have \( R_j = \{z_j, \mathcal{N}(z_0, z_1, \eta)\} \). Further, combining (4.33) with (4.30), we have

\[
|R_j| \lesssim (|z_0|^5 + |z_1|^5 + \|\eta\|_c \|\eta\|_2). \tag{4.34}
\]

As a conclusion of this section, we have the equations of \( z_j \) and \( \eta \).

\[
i \dot{\eta} = H \eta + P_x (|\eta|^6 \eta + \nabla_\eta N), \tag{4.35}
\]

\[
\dot{z}_j = -i \mathcal{E}(|z_0|^2, |z_1|^2) z_j + R_j. \tag{4.36}
\]

## 5 Linear estimates

In this section, we introduce the linear estimates for the proof of Theorem 1.8. Lemmas 5.2–5.5 can be found in [22]. See also [36] and [28]. In the following we always assume \( H \) is generic in the sense of Lemma 5.3 of [22].

**Definition 5.1.** We say the pair of numbers \( (r, p) \) is admissible if

\[
\frac{2}{r} + \frac{1}{p} = \frac{1}{2}, \quad (r, p) \in [4, \infty] \times [2, \infty].
\]

We set

\[
X_{r,p} := l^{\frac{1}{r'}}(Z, L_1^{\infty}([n, n+1], L^p)), \quad X'_{r,p} := l^{\frac{1}{r'}}(Z, L_1^{\frac{1}{n}}([n, n+1], L^{p})),
\]

where \( p' \) is the Hölder conjugate of \( p \) (i.e. \( \frac{1}{p} + \frac{1}{p'} = 1 \)).
**Lemma 5.2** (Dispersive estimate). We have
\[ \| e^{-itPc} \|_{L(t^1; \ell^\infty)} \lesssim (t)^{-1/3}. \]

**Lemma 5.3** (Strichartz estimate). Let \((p, r), (p_1, r_1)\) and \((p_2, r_2)\) admissible. Then, we have
\[ \| e^{-itPc} f \|_{X_{r,p}} \lesssim \| f \|_{l^2}, \]
and
\[ \left\| \int_0^t e^{-i(t-s)H} Pcg(s) \, ds \right\|_{X_{r_1,p_1}} \lesssim \| g \|_{X_{r_2,p_2}}. \]

**Lemma 5.4** (Kato Smoothing). Let \(\sigma > 1\). Then, we have
\[ \| e^{-itPc} f \|_{L(t^1l^2,-\sigma)} \lesssim \| f \|_{l^2}, \]
and
\[ \left\| \int_0^t e^{-i(t-s)H} Pcg(s) \, ds \right\|_{L^2(t^1l^2,-\sigma)} \lesssim \| g \|_{L^2(t^2,\sigma)}. \]

**Lemma 5.5.** Let \(\sigma > 1\) and \((r, p)\) admissible. Then, we have
\[ \left\| \int_0^t e^{-i(t-s)H} Pcg(s, \cdot) \, ds \right\|_{X_{r,p}} \lesssim \| g \|_{L^2(t^2,\sigma)}. \]

**6 Proof of Theorem 1.8**

We are now in the position to prove Theorem 1.8. Fix \(\sigma > 1\) and set \(X = X_{4,\infty} \cap X_{\infty,2} \cap L^2(t^2,\sigma).\)

**Proposition 6.1.** Under the hypothesis of Theorem 1.8, there exists \(\epsilon_0 > 0\) s.t. if \(\| u_0 \|_{l^2} = \epsilon < \epsilon_0\), we have
\[ \| \eta \|_X \lesssim \epsilon, \]
\[ \left\| \frac{d}{dt} |z_j|^2 \right\|_{l^1} \lesssim \epsilon^8. \]

**Proof.** First, by the \(l^2\) conservation of (1.1) we have
\[ |z_0| + |z_1| + \| \eta \|_{l^2} \lesssim \sum_{j=0,1} |z_j'| + |r_j(z_0, z_1, \eta)| + \| \eta \|_{l^2} + \| r_\eta(z_0, z_1, \eta) \|_{l^2} \]
\[ \lesssim \epsilon + (|z_0|^6 + |z_1|^6) \| \eta \|_{l^2}. \]
Therefore, we have
\[ |z_0| + |z_1| + \| \eta \|_{l^2} \lesssim \epsilon, \]
for all time $t$. By (4.35), for any admissible pair $(r,p)$, we have

$$
\|\eta\|_{X_{r,p}} \leq \|e^{-iHt} \eta(0)\|_{X_{r,p}} + \| \int_0^t e^{-i(t-s)H} P_c \nabla_\eta N ds \|_{X_{r,p}} + \| \int_0^t e^{-i(t-s)H} P_c |\eta|^6 \eta ds \|_{X_{r,p}}
$$

where we have used Lemma 5.3 and 5.5 in the first inequality and (4.31) in the second inequality. Again by (4.35) and Lemma 5.3, 5.4, we have

$$
\|\eta\|_{L^{2i_2-\sigma}} \lesssim \|\eta(0)\|_{i_2} + \|\nabla_\eta N\|_{L^{2i_2-\sigma}} + \|\|\eta\|_{L^{2i_2-\sigma}}
$$

where we have used Lemma 5.3 and 5.5 in the first inequality and (4.31) in the second inequality. Therefore, we have

$$
\|\eta\|_{X} \lesssim \|\eta(0)\|_{i_2} + \epsilon^6 \|\eta\|_{X} + \|\eta\|_{X}^2.
$$

By continuity argument, we have (6.1).

Next, multiplying $\tilde{\eta}_j$ to (4.36) and taking the real part, we have

$$
\frac{d}{dt} |\tilde{z}_j|^2 = R_j \tilde{z}_j + \bar{R}_j \tilde{z}_j.
$$

Therefore, by (4.34), we have

$$
\|\frac{d}{dt} |\tilde{z}_j|^2\|_{L^1} \leq (\|z_0\|_{L^\infty} + \|z_1\|_{L^\infty} + \|\eta\|_{L^{2i_2}}\|\eta\|_{L^{2i_2-\sigma}}^2) \lesssim \epsilon^6 \|\eta\|_{L^{2i_2-\sigma}}^2 \lesssim \epsilon^8.
$$

This gives us the conclusion. \hfill \Box

We now prove Theorem 1.8.

Proof of Theorem 1.8. It suffices to prove $|z_j'|$ converges and $R[z_j', z_j'] \eta'$ scatters. By Proposition 6.1, we have the result for the new coordinates. That is, there exists $\rho_{j,+}$ and $v_+$ s.t.

$$|z_j(t)| \to \rho_{j,+}, \quad \text{and} \quad \|\eta(t) - e^{it\Delta} \eta_+\|_{i_2} \to 0,
$$

with $\rho_{0,+} + \rho_{1,+} + \|v_+\|_{i_2} \lesssim \epsilon$. Then, since $\|\eta(t)\|_{i_2} \to 0$ for any $a > 0$ because of Lemma 5.2, we see $\|\eta'(t) - \eta(t)\|_{i_2} \to 0$ and $|z_j(t) - z_j'(t)| \to 0$ as $t \to \infty$. Therefore, we have the conclusion. \hfill \Box

A  Proof of Proposition 1.4

In this section, we prove Proposition 1.4. Before proving Proposition 1.4, we prepare another elementary estimate.

Lemma A.1. Let $\delta > 0$. Then there exists $a(\delta) > 0$ s.t. for $a \in (0, a(\delta))$ and for $\lambda \notin (-\delta, 4 + \delta) \cup (\epsilon_0 - \delta, \epsilon_0 + \delta) \cup (\epsilon_1 - \delta, \epsilon_1 + \delta)$, we have

$$
\| (H - \lambda)^{-1} \|_{L^2} \lesssim \delta \langle \lambda \rangle^{-1}. \quad (A.1)
$$

Further, let $j = 0, 1$. Then, for sufficiently small $a > 0$, we have

$$
\left\| \left( (H - e_j)^{-1} \right)^{-1} \right\|_{L^2} \lesssim 1. \quad (A.2)
$$
Proof. Set $T_{a,N}$ by

$$(T_{a,N}v)(n) = e^{a \min(|n|,N)} v(n).$$

We first claim there exists $B_{a,N} : l^2 \to l^2$ s.t. $\|B_{a,N}\|_{l^2 \to l^2} \lesssim a$ (the implicit constant do not depend on $N$) and

$$T_{a,N}(H - \lambda)T_{a,N}^{-1} = H - \lambda + B_{a,N}.$$

Indeed, setting $B_{a,N} = T_{a,N}(-\Delta)T_{a,N}^{-1} + \Delta$, we have

$$(B_{a}u)(n) = \left(1 - e^{a(\min(|n|,N) - \min(|n+1|,N))}\right) u(n + 1) + \left(1 - e^{a(\min(|n|,N) - \min(|n-1|,N))}\right) u(n - 1).$$

Since $|1 - e^{a(\min(|n|,N) - \min(|n+1|,N))}| \lesssim a$ and $|1 - e^{a(\min(|n|,N) - \min(|n-1|,N))}| \lesssim a$, we have the desired bound for $B_{a,N}$. Now, since

$$T_{a,N}(H - \lambda)^{-1}T_{a,N}^{-1} = (T_{a,N}(H - \lambda)^{-1}T_{a,N})^{-1} = (H - \lambda + B_{a,N})^{-1}$$

$$= (H - \lambda)^{-1}(1 + (H - \lambda)^{-1}B_{a,N}).$$

Therefore, by Neumann expansion and since $\|(H - \lambda)^{-1}\|_{l^2 \to l^2} \lesssim \delta^{-1}$, if we take $a > 0$ sufficiently small s.t. $a\delta^{-1} \ll 1$, we have

$$\|T_{a,N}(H - \lambda)^{-1}T_{a,N}^{-1}\|_{L(l^2)} \lesssim \|(H - \lambda)^{-1}\|_{l^2} \lesssim \lambda^{-1} \delta^{-1}$$

This implies that for $u \in l^2$, $\|T_{a,N}(H - \lambda)^{-1}u\|_{l^2} \lesssim \|T_{a,N}u\|_{l^2} \leq \lambda^{-1} \|u\|_{l^2}$.

Taking $N \to \infty$, we obtain (A.1).

Next, we prove (A.2). Suppose $u, f \perp \phi_j$ and $(H - e_j)u = f$, $u \in l^2$, $f \in l^\alpha_c$. Set $P := \langle \cdot, \phi_j \rangle \phi_j$ and $Q = 1 - P$. Now, we have

$$T_{a,N}f = (H - e_j + B_{a,N})T_{a,N}u = (H - e_j + B_{a,N})(QT_{a,N}u + \langle u, T_{a,N}\phi \rangle \phi).$$

Therefore, we have

$$(H - e_j)QT_{a,N}u = T_{a,N}f - B_{a,N}QT_{a,N}u - \langle u, T_{a,N}\phi \rangle B_{a,N}\phi.$$

Now, by $f, \phi \in l^\alpha_c$, where $a > 0$ is sufficiently small so that $\phi_j \in l^\alpha_c$, we have

$$\|QT_{a,N}u\|_{l^2} \lesssim \|f\|_{l^\alpha_c} + a\|QT_{a,N}u\|_{l^2} + a\|u\|_{l^2}.$$

Thus, for $a$ sufficiently small,

$$\|QT_{a,N}u\|_{l^2} \lesssim \|f\|_{l^\alpha_c} + a\|u\|_{l^2},$$

and

$$\|T_{a,N}u\|_{l^2} \leq \|QT_{a,N}u\|_{l^2} + \|PT_{a,N}u\|_{l^2} \lesssim \|f\|_{l^\alpha_c} + \|u\|_{l^2}.$$

Finally, taking $N \to \infty$, we have

$$\|u\|_{l^2} \lesssim \|f\|_{l^\alpha_c},$$

where we have used the fact that $\|u\|_{l^2} \lesssim \|f\|_{l^2} \leq \|f\|_{l^\alpha_c}$.  

$\Box$
We now prove Proposition 1.4.

Proof of Proposition 1.4. For simplicity, we write \( \phi_j \) as \( \phi \), \( e_j \) as \( e \) and \( E_j \) as \( E \). Consider a solution in the form \( z(\phi + \tilde{q}(|z|^2)) \) with real valued \( \tilde{q} \) with \( \langle \phi, \tilde{q} \rangle = 0 \). Now, substitute it in the equation and we have

\[
Hq + |z|^6|\phi + \tilde{q}|^6(\phi + \tilde{q}) = e\tilde{q} + (E - e)(\phi + \tilde{q}).
\]

Then, we have

\[
(|z|^6|\phi + \tilde{q}|^6(\phi + \tilde{q}), \phi) = E - e,
\]

\[
H\tilde{q} + Q(|z|^6|\phi + \tilde{q}|^6(\phi + \tilde{q})) = E\tilde{q}.
\]

Therefore, we set

\[
E(|z|^2, \tilde{q}) := e + |z|^6 \langle |\phi + \tilde{q}|^6(\phi + \tilde{q}), \phi \rangle,
\]

and we have

\[
(H - e)\tilde{q} = (E(z, \tilde{q}) - e)\tilde{q} - Q(|z|^6|\phi + \tilde{q}|^6(\phi + \tilde{q}))
\]

\[
= |z|^6(f(\tilde{q}), \phi)\tilde{q} - |z|^6Qf(\tilde{q}),
\]

where \( f(\tilde{q}) = |\phi + \tilde{q}|^6(\phi + \tilde{q}) \). We set \( F : Ql^n_c \times \mathbb{R} \to Ql^n_c \) by

\[
F(\tilde{q}, s) := (H - e)\tilde{q} - s^3(f(\tilde{q}), \phi)\tilde{q} + s^3Qf(\tilde{q}).
\]

Then, \( F \) is real analytic with respect to \( \tilde{q} \) and \( s \). Further, since

\[
D_qF(\tilde{q}, s)|_{(\tilde{q}, s) = (0, 0)} = H - q
\]

is invertible in \( Ql^n_c \) for sufficiently small \( a > 0 \), by implicit function theorem, for sufficiently small \( s \), there exists \( \tilde{q}(s) \) s.t. \( \tilde{q}(s) \) is real analytic with respect to \( s \) and \( F(\tilde{q}(s), s) = 0 \). Further, comparing the Taylor series of

\[
\tilde{q}(s) = s^3(H - e)^{-1}((f(\tilde{q}), \phi) - Qf(\tilde{q})),
\]

we see \( \|\tilde{q}(s)\|_z \lesssim s^3 \). Therefore, \( \tilde{q}(|z|^2) \) is the desired solution. Finally, set \( E(s) = E(s, \tilde{q}(s)) \), where the r.h.s. is given in \((A.3)\). Then, since \( E(s, \tilde{q}) \) and \( \tilde{q} \) is both real analytic, \( E(s) \) also becomes real analytic. The estimate \( |E(|z|^2) - e| \lesssim |z|^6 \) also follows from \((A.3)\).

\[\square\]

B Proof of Lemma 2.4

Proof of Lemma 2.4. Set

\[
V_j = \{v_{j,m}\}_{m \geq 0}, \{w_{j,m}\}_{m \geq 0}, \quad j = 1, 2, 3,
\]

and

\[
\psi_j = T(z_0, z_1) V_j = \sum_{m_j \geq 0} z_0^{m_j + 1} z_1^{m_j} v_{j,m_j} + \sum_{m_j \geq 0} z_1^{m_j + 1} w_{j,m_j}, \quad j = 1, 2, 3.
\]
Then, we have

$$\psi_1 \bar{\psi}_2 \psi_3 = \sum_{m \geq 0} z_0^{m+1} z_1^m v_m(|z_0|^2, |z_1|^2) + \sum_{m \geq 1} w_m(|z_0|^2, |z_1|^2),$$

where

$$v_m = \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} |z_0|^{2l} |z_1|^{2l} v_{1,m_1} v_{2,l} v_{3,l-m-m_1} + \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} v_{1,m_2-m_3-1} w_{2,m_2} v_{3,m_3} \tag{B.1}$$

and

$$w_m = \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} |z_0|^{2l+4} |z_1|^{2l} v_{1,m_1} v_{2,m+1} v_{3,l-m_2} + \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} |z_0|^{2l+2} |z_1|^{2l} v_{1,m_1} v_{2,l} v_{3,m_1} v_{3,l}$$

$$+ \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} |z_0|^{2l+4} |z_1|^{2l} v_{1,m_1} v_{2,l} v_{3,l-m_2} w_{3,l-m_1} + \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} w_{1,m_2-m_3-1} v_{2,m_2} w_{3,m_3}$$

Therefore, we see that $\psi_1 \bar{\psi}_2 \psi_3$ formally has the desired form and putting $V = \{v_n\}, \{w_n\}$, we see

$$T(z_0, z_1) V = \psi_1 \bar{\psi}_2 \psi_3.$$

We now prove (1.4). The contribution of the first term in (B.1) can be bounded as

$$\sum_{m \geq 0} r^{2m+1} \left( \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} |z_0|^{2l} |z_1|^{2l} v_{1,m_1} v_{2,l} v_{3,l-m-m_1} \right)$$

$$\leq \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} r^{2l+1} v_{1,m_1} v_{2,l} v_{3,l-m-m_1} + \sum_{l \geq 0 \atop l+m \geq m_1 \geq 0} v_{3,l-m-m_1}$$

provided $|z_0|, |z_1| \leq r$. All the other terms can also be bounded in the same manner. Further, the analyticity also follows from this expression. Therefore, we have the conclusion. \[\square\]
C  Explicit form of $N^0(|z_0|^2, |z_1|^2)$, $N^1(|z_0|^2, |z_1|^2)$

In this section, we give the explicit form of $N^0(|z_0|^2, |z_1|^2)$, $N^1(|z_0|^2, |z_1|^2)$. We set $N^0(|z_0|^2, |z_1|^2) = \{ \{N_m^0\}_{m \geq 0}, \{M_m^0\}_{m \geq 0}\}$. Then, by direct computation, we have

\[
N_0^0(|z_0|^2, |z_1|^2) = 12|z_0|^4|z_1|^2\phi_0\phi_1^2 + 18|z_0|^2|z_1|^4\phi_0^2\phi_1 + 4|z_1|^6\phi_0\phi_1^3,
\]

\[
N_1^0(|z_0|^2, |z_1|^2) = 3|z_0|^4\phi_0^6\phi_1 + 12|z_0|^2|z_1|^2\phi_0^5\phi_1^2 + 6|z_1|^4\phi_0^3\phi_1^3,
\]

\[
N_2^0(|z_0|^2, |z_1|^2) = 3|z_0|^2\phi_0^6\phi_1^2 + 4|z_1|^2\phi_0^5\phi_1,
\]

\[
N_3^0(|z_0|^2, |z_1|^2) = \phi_0^3\phi_1^3,
\]

\[
N_m^0(|z_0|^2, |z_1|^2) = 0, \quad (m \geq 4),
\]

and

\[
M_0^0(|z_0|^2, |z_1|^2) = 4|z_0|^6\phi_0^6\phi_1 + 18|z_0|^4|z_1|^2\phi_0^5\phi_1^2 + 12|z_0|^2|z_1|^4\phi_0^4\phi_1^3 + 6|z_0|^2|z_1|^4\phi_0^3\phi_1^4 \phi_0^3\phi_1^4
\]

\[
M_1^0(|z_0|^2, |z_1|^2) = 6|z_0|^4\phi_0^6\phi_1^2 + 12|z_0|^2|z_1|^2\phi_0^5\phi_1^4 + 3|z_1|^4\phi_0^3\phi_1^6,
\]

\[
M_2^0(|z_0|^2, |z_1|^2) = 4|z_0|^2\phi_0^6\phi_1^5 + 3|z_1|^2\phi_0^5\phi_1^7,
\]

\[
M_3^0(|z_0|^2, |z_1|^2) = \phi_0^3\phi_1^9,
\]

\[
M_k^0(|z_0|^2, |z_1|^2) = 0, \quad (k \geq 4).
\]

Similarly, by direct computation, we have

\[
N_0^1(|z_0|^2, |z_1|^2, V) = \tilde{N}_0^1(|z_0|^2, |z_1|^2, V) + 3(|z_0|^6\phi_0^6 + 8|z_0|^4|z_1|^2\phi_0^5\phi_1^2 + 6|z_0|^2|z_1|^4\phi_0^3\phi_1^4 + 6|z_0|^2|z_1|^4\phi_0^3\phi_1^4)\nu_0
\]

\[
+ 6(|z_0|^6|z_1|^2\phi_0^6\phi_1 + 2|z_0|^4|z_1|^4\phi_0^4\phi_1^2 + 3|z_0|^2|z_1|^6\phi_0^2\phi_1^3 + 3|z_0|^2|z_1|^6\phi_0^2\phi_1^3 + 4|z_0|^4|z_1|^6\phi_0^2\phi_1^3\phi_1^3)
u_0
\]

\[
+ 12(|z_0|^4|z_1|^2\phi_0^4\phi_1^4 + 3|z_0|^2|z_1|^4\phi_0^2\phi_1^6 + |z_1|^6\phi_0^2\phi_1^6)\nu_0
\]

\[
+ 12(|z_0|^4|z_1|^2\phi_0^4\phi_1^4 + |z_0|^2|z_1|^6\phi_0^2\phi_1^3\phi_1^3)\phi_0^3\phi_1^6\nu_0 + 4|z_0|^4|z_1|^6\phi_0^2\phi_1^3\phi_1^3\nu_0
\]

\[
N_1^1(|z_0|^2, |z_1|^2, V) = \tilde{N}_1^1(|z_0|^2, |z_1|^2, V) + 6(|z_0|^4\phi_0^6\phi_1 + |z_0|^2|z_1|^2\phi_0^5\phi_1^2)\nu_0 + 3|z_0|^4|z_1|^4\phi_0^4\phi_1^2 + 12(|z_0|^2|z_1|^2\phi_0^4\phi_1^4 + 3|z_0|^2|z_1|^4\phi_0^2\phi_1^6 + 4|z_0|^4|z_1|^4\phi_0^2\phi_1^6\phi_1^4)
u_0
\]

\[
N_2^1(|z_0|^2, |z_1|^2, V) = \tilde{N}_2^1(|z_0|^2, |z_1|^2, V) + 3|z_0|^2\phi_0^6\phi_1^4 + 4|z_1|^2\phi_0^5\phi_1^6 + 4|z_0|^4|z_1|^4\phi_0^2\phi_1^6 + 4|z_0|^2|z_1|^4\phi_0^2\phi_1^6\phi_1^4\nu_0
\]

\[
N_m^1(|z_0|^2, |z_1|^2, V) = \tilde{N}_m^1(|z_0|^2, |z_1|^2, V), \quad (m \geq 3),
\]

and

\[
M_0^1(|z_0|^2, |z_1|^2, V) = \tilde{M}_0^1(|z_0|^2, |z_1|^2, V) + 12(|z_0|^6\phi_0^6\phi_1 + 3|z_0|^4|z_1|^2\phi_0^5\phi_1^2 + 3|z_0|^2|z_1|^4\phi_0^3\phi_1^4 + |z_0|^2|z_1|^4\phi_0^3\phi_1^4)\nu_0
\]

\[
+ 12(|z_0|^6|z_1|^2\phi_0^6\phi_1 + |z_0|^4|z_1|^4\phi_0^4\phi_1^2 + 4|z_0|^2|z_1|^6\phi_0^2\phi_1^3 + 4|z_0|^2|z_1|^6\phi_0^2\phi_1^3 + 4|z_0|^4|z_1|^6\phi_0^2\phi_1^3\phi_1^3\nu_0
\]

\[
+ 3(6|z_0|^4|z_1|^2\phi_0^4\phi_1^4 + 8|z_0|^2|z_1|^4\phi_0^2\phi_1^6 + |z_1|^6\phi_0^2\phi_1^6)\nu_0
\]

\[
+ 6(2|z_0|^4|z_1|^2\phi_0^4\phi_1^4 + |z_0|^2|z_1|^6\phi_0^2\phi_1^3\phi_1^3)\phi_0^3\phi_1^6\nu_0 + 3|z_0|^4|z_1|^6\phi_0^2\phi_1^3\phi_1^3\nu_0
\]

\[
M_1^1(|z_0|^2, |z_1|^2, V) = \tilde{M}_1^1(|z_0|^2, |z_1|^2, V) + 12(|z_0|^4\phi_0^4\phi_1^2 + |z_0|^2|z_1|^2\phi_0^2\phi_1^4)\nu_0 + 4|z_0|^4|z_1|^4\phi_0^2\phi_1^6 + 6(2|z_0|^2|z_1|^2\phi_0^2\phi_1^4 + |z_1|^4\phi_0^2\phi_1^4)\nu_0
\]

\[
+ 3|z_0|^2|z_1|^4\phi_0^2\phi_1^6 + 3|z_1|^4\phi_0^2\phi_1^6\phi_1^4\nu_0
\]

\[
M_k^1(|z_0|^2, |z_1|^2, V) = \tilde{M}_k^1(|z_0|^2, |z_1|^2, V), \quad (m \geq 3),
\]
where
\[
\tilde{N}_m^1 = \sum_{j=-3}^{3} \tilde{N}_{m,v,j}^1 v_{m+j} + \tilde{N}_{m,w,j}^1 w_{m+j}, \quad \tilde{M}_m^1 = \sum_{j=-3}^{3} \tilde{M}_{m,v,j}^1 v_{m+j} + \tilde{M}_{m,w,j}^1 w_{m+j},
\]
and the explicit form of \(\tilde{N}_{m,x,j}^1\) and \(\tilde{N}_{m,x,j}^1\) for \(x = v, w\) are given by
\[
\begin{align*}
\tilde{N}_{m,v,-3}^1 &= 4 \tilde{\phi}_0^2 \tilde{\phi}_1^3, \\
\tilde{N}_{m,v,-2}^1 &= 12(\vert z_0 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^2 + \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^2), \\
\tilde{N}_{m,v,-1}^1 &= 12(\vert z_0 \vert^4 \tilde{\phi}_0^6 \tilde{\phi}_1 + 3 \vert z_0 \vert^2 \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{N}_{m,v,0}^1 &= 4(\vert z_0 \vert^6 \tilde{\phi}_0^9 + 9 \vert z_0 \vert^4 \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + 9 \vert z_0 \vert^2 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{N}_{m,v,1}^1 &= 12(\vert z_0 \vert^6 \vert z_1 \vert^2 \tilde{\phi}_0^9 \tilde{\phi}_1 + 3 \vert z_0 \vert^4 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_0 \vert^2 \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{N}_{m,v,2}^1 &= 12(\vert z_0 \vert^6 \vert z_1 \vert^4 \tilde{\phi}_0^9 \tilde{\phi}_1^2 + \vert z_0 \vert^4 \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{N}_{m,v,3}^1 &= 4 \vert z_0 \vert^6 \vert z_1 \vert^6 \tilde{\phi}_0^9 \tilde{\phi}_1^4, \\
\tilde{N}_{m,w,-3}^1 &= 3 \tilde{\phi}_0^2 \tilde{\phi}_1^3, \\
\tilde{N}_{m,w,-2}^1 &= 6(\vert z_0 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^1 + \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^1), \\
\tilde{N}_{m,w,-1}^1 &= 9(\vert z_0 \vert^4 \tilde{\phi}_0^6 \tilde{\phi}_1^2 + 8 \vert z_0 \vert^2 \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + 6 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{N}_{m,w,0}^1 &= 12(\vert z_0 \vert^4 \vert z_1 \vert^2 \tilde{\phi}_0^6 \tilde{\phi}_1^2 + 3 \vert z_0 \vert^2 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{N}_{m,w,1}^1 &= 3(6 \vert z_0 \vert^2 \vert z_1 \vert^4 \tilde{\phi}_0^6 \tilde{\phi}_1^4 + 8 \vert z_0 \vert^2 \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^8 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{N}_{m,w,2}^1 &= 3(6 \vert z_0 \vert^4 \vert z_1 \vert^4 \tilde{\phi}_0^6 \tilde{\phi}_1^4 + 8 \vert z_0 \vert^4 \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^8 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{N}_{m,w,3}^1 &= 3 \vert z_0 \vert^4 \vert z_1 \vert^8 \tilde{\phi}_0^3 \tilde{\phi}_1^4, \\
\tilde{M}_{m,v,-3}^1 &= 3 \tilde{\phi}_0^2 \tilde{\phi}_1^3, \\
\tilde{M}_{m,v,-2}^1 &= 6(2 \vert z_0 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^1 + \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^1), \\
\tilde{M}_{m,v,-1}^1 &= 9(\vert z_0 \vert^4 \tilde{\phi}_0^6 \tilde{\phi}_1^2 + 8 \vert z_0 \vert^2 \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{M}_{m,v,0}^1 &= 12(\vert z_0 \vert^6 \tilde{\phi}_0^9 + 9 \vert z_0 \vert^4 \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + 9 \vert z_0 \vert^2 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{M}_{m,v,1}^1 &= 3(6 \vert z_0 \vert^2 \vert z_1 \vert^4 \tilde{\phi}_0^6 \tilde{\phi}_1^4 + 8 \vert z_0 \vert^2 \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^8 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{M}_{m,v,2}^1 &= 6(\vert z_0 \vert^6 \vert z_1 \vert^2 \tilde{\phi}_0^9 \tilde{\phi}_1^2 + 2 \vert z_0 \vert^6 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{M}_{m,v,3}^1 &= 3 \vert z_0 \vert^8 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4, \\
\tilde{M}_{m,w,-3}^1 &= 4 \tilde{\phi}_0^2 \tilde{\phi}_1^3, \\
\tilde{M}_{m,w,-2}^1 &= 12(\vert z_0 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^1 + \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^1), \\
\tilde{M}_{m,w,-1}^1 &= 12(\vert z_0 \vert^4 \tilde{\phi}_0^6 \tilde{\phi}_1^2 + 3 \vert z_0 \vert^2 \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{M}_{m,w,0}^1 &= 4(\vert z_0 \vert^6 \tilde{\phi}_0^9 + 9 \vert z_0 \vert^4 \vert z_1 \vert^2 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + 9 \vert z_0 \vert^2 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{M}_{m,w,1}^1 &= 12(\vert z_0 \vert^6 \vert z_1 \vert^2 \tilde{\phi}_0^9 \tilde{\phi}_1 + 3 \vert z_0 \vert^4 \vert z_1 \vert^4 \tilde{\phi}_0^3 \tilde{\phi}_1^4 + \vert z_0 \vert^2 \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{M}_{m,w,2}^1 &= 12(\vert z_0 \vert^6 \vert z_1 \vert^4 \tilde{\phi}_0^9 \tilde{\phi}_1^2 + \vert z_0 \vert^4 \vert z_1 \vert^6 \tilde{\phi}_0^3 \tilde{\phi}_1^4), \\
\tilde{M}_{m,w,3}^1 &= 4 \vert z_0 \vert^6 \vert z_1 \vert^6 \tilde{\phi}_0^9 \tilde{\phi}_1^4.
\]
where $v_m$, $w_m$ with $m < 0$ is 0.

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References

[1] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*, second ed., Applied Mathematical Sciences, vol. 75, Springer-Verlag, New York, 1988. MR 960687 (89f:58001)

[2] Dario Bambusi, *Asymptotic stability of breathers in some Hamiltonian networks of weakly coupled oscillators*, Comm. Math. Phys. 324 (2013), no. 2, 515–547. MR 3117519

[3] Dario Bambusi and Scipio Cuccagna, *On dispersion of small energy solutions to the nonlinear Klein Gordon equation with a potential*, Amer. J. Math. 133 (2011), no. 5, 1421–1468. MR 2843104

[4] V. S. Buslaev and G. S. Perel’man, *Nonlinear scattering: states that are close to a soliton*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 200 (1992), no. Kraev. Zadachi Mat. Fiz. Smezh. Voprosy Teor. Funktsii. 24, 38–50, 70, 187. MR 1192111 (93k:35040)

[5] , *On nonlinear scattering of states which are close to a soliton*, Astérisque (1992), no. 210, 6, 49–63, Méthodes semi-classiques, Vol. 2 (Nantes, 1991). MR 1221351 (94g:35199)

[6] , *Scattering for the nonlinear Schrödinger equation: states that are close to a soliton*, Algebra i Analiz 4 (1992), no. 6, 63–102. MR 1199635 (94b:35256)

[7] , *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations, Amer. Math. Soc. Transl. Ser. 2, vol. 164, Amer. Math. Soc., Providence, RI, 1995, pp. 75–98, MR 1334139 (96e:35157)

[8] Vladimir S. Buslaev and Catherine Sulem, *On asymptotic stability of solitary waves for nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 20 (2003), no. 3, 419–475. MR 1972870 (2004d:35229)

[9] Scipio Cuccagna, *Stabilization of solutions to nonlinear Schrödinger equations*, Comm. Pure Appl. Math. 54 (2001), no. 9, 1110–1145. MR 1835384 (2002g:35193)

[10] , *On asymptotic stability of ground states of NLS*, Rev. Math. Phys. 15 (2003), no. 8, 877–903. MR 2027616 (2004k:35348)

[11] , *Stability of standing waves for NLS with perturbed Lamé potential*, J. Differential Equations 223 (2006), no. 1, 112–160. MR 2210141 (2007e:35256)

[12] , *On asymptotic stability in energy space of ground states of NLS in 1D*, J. Differential Equations 245 (2008), no. 3, 653–691. MR 2422523 (2009c:35435)
[13] __________.  Orbitally but not asymptotically stable ground states for the discrete NLS, Discrete Contin. Dyn. Syst. 26 (2010), no. 1, 105–134. MR 2552781 (2010j:39024)

[14] __________.  The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states, Comm. Math. Phys. 305 (2011), no. 2, 279–331. MR 2805462 (2012d:37176)

[15] Scipio Cuccagna and Masaya Maeda, On small energy stabilization in the NLS with trapping potential, preprint arXiv:1309.0655.

[16] Scipio Cuccagna and Masaya Maeda, On weak interaction between a ground state and a non-trapping potential, J. Differential Equations 256 (2014), no. 4, 1395–1466. MR 3145762

[17] Scipio Cuccagna and Tetsu Mizumachi, On asymptotic stability in energy space of ground states for nonlinear Schrödinger equations, Comm. Math. Phys. 284 (2008), no. 1, 51–77. MR 2443298 (2009k:35294)

[18] Scipio Cuccagna and Dmitry E. Pelinovsky, The asymptotic stability of solitons in the cubic NLS equation on the line, Appl. Anal. 93 (2014), no. 4, 791–822. MR 3180019

[19] Scipio Cuccagna and Mirko Tarulli, On stabilization of small solutions in the nonlinear Dirac equation with a trapping potential, preprint arXiv:1408.5779.

[20] Scipio Cuccagna and Mirko Tarulli, On asymptotic stability in energy space of ground states of NLS in 2D, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 4, 1361–1386. MR 2542729 (2010i:35366)

[21] Zhou Gang and I. M. Sigal, Asymptotic stability of nonlinear Schrödinger equations with potential, Rev. Math. Phys. 17 (2005), no. 10, 1143–1207. MR 2187292 (2006j:35220)

[22] M. Goldberg and W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, Comm. Math. Phys. 251 (2004), no. 1, 157–178. MR 2096737 (2005g:81339)

[23] Magnus Johansson and Serge Aubry, Existence and stability of quasiperiodic breathers in the discrete nonlinear Schrödinger equation, Nonlinearity 10 (1997), no. 5, 1151–1178. MR 1473378 (99c:39019)

[24] Stephen Gustafson, Kenji Nakanishi, and Tai-Peng Tsai, Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equations with small solitary waves, Int. Math. Res. Not. (2004), no. 66, 3559–3584. MR 2101699 (2005g:35268)

[25] Robert G. I. Moore, Endpoint Strichartz estimates, Amer. J. Math. 120 (1998), no. 5, 955–980. MR 1646048 (2000d:35018)

[26] P. G. Kevrekidis, D. E. Pelinovsky, and A. Stefanov, Asymptotic stability of small bound states in the discrete nonlinear Schrödinger equation, SIAM J. Math. Anal. 41 (2009), no. 5, 2010–2030. MR 2578796 (2011a:37153)
[29] E. Kirr and Ö. Mızrak, *Asymptotic stability of ground states in 3D nonlinear Schrödinger equation including subcritical cases*, J. Funct. Anal. 257 (2009), no. 12, 3691–3747. MR 2557723 (2010k:35460)

[30] E. Kirr and A. Zarnescu, *On the asymptotic stability of bound states in 2D cubic Schrödinger equation*, Comm. Math. Phys. 272 (2007), no. 2, 443–468. MR 2300249 (2008a:35266)

[31] , *Asymptotic stability of ground states in 2D nonlinear Schrödinger equation including subcritical cases*, J. Differential Equations 247 (2009), no. 3, 710–735. MR 2528489 (2010g:35300)

[32] Eva Koo, *Asymptotic stability of small solitary waves for nonlinear Schrödinger equations with electromagnetic potential in $\mathbb{R}^3$*, J. Differential Equations 250 (2011), no. 8, 3473–3503. MR 2772399 (2012f:35508)

[33] R. S. MacKay and S. Aubry, *Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators*, Nonlinearity 7 (1994), no. 6, 1623–1643. MR 1304442 (96a:34085)

[34] Tetsu Mizumachi, *Asymptotic stability of small solitons for 2D nonlinear Schrödinger equations with potential*, J. Math. Kyoto Univ. 47 (2007), no. 3, 599–620. MR 2402517 (2010a:35247)

[35] , *Asymptotic stability of small solitary waves to 1D nonlinear Schrödinger equations with potential*, J. Math. Kyoto Univ. 48 (2008), no. 3, 471–497. MR 2511047 (2010h:35377)

[36] D. E. Pelinovsky and A. Stefanov, *On the spectral theory and dispersive estimates for a discrete Schrödinger equation in one dimension*, J. Math. Phys. 49 (2008), no. 11, 113501, 17. MR 2468536 (2009m:81059)

[37] Galina Perelman, *Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations*, Comm. Partial Differential Equations 29 (2004), no. 7-8, 1051–1095. MR 2097576 (2005g:35277)

[38] Claude-Alain Pillet and C. Eugene Wayne, *Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations*, J. Differential Equations 141 (1997), no. 2, 310–326. MR 1488355 (99b:35193)

[39] I. M. Sigal, *Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasiperiodic solutions*, Comm. Math. Phys. 153 (1993), no. 2, 297–320. MR 1218303 (94d:35012)

[40] A. Soffer and M. I. Weinstein, *Multichannel nonlinear scattering for nonintegrable equations*, Comm. Math. Phys. 133 (1990), no. 1, 119–146. MR 1071238 (91h:35303)

[41] , *Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data*, J. Differential Equations 98 (1992), no. 2, 376–390. MR 1170476 (93i:35137)

[42] , *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations*, Invent. Math. 136 (1999), no. 1, 9–74. MR 1681113 (2000k:37119)

[43] , *Selection of the ground state for nonlinear Schrödinger equations*, Rev. Math. Phys. 16 (2004), no. 8, 977–1071. MR 2101776 (2005g:81095)
[44] Tai-Peng Tsai and Horng-Tzer Yau, *Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data*, Adv. Theor. Math. Phys. 6 (2002), no. 1, 107–139. MR 1992875 (2004m:35254)

[45] ______, *Relaxation of excited states in nonlinear Schrödinger equations*, Int. Math. Res. Not. (2002), no. 31, 1629–1673. MR 1916427 (2004i:35292)

[46] ______, *Stable directions for excited states of nonlinear Schrödinger equations*, Comm. Partial Differential Equations 27 (2002), no. 11-12, 2363–2402. MR 1944033 (2004k:35359)

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