Condensation phenomena in preferential attachment trees with neighbourhood influence

N. Fountoulakis*,†, T. Iyer*

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Abstract

We introduce a model of evolving preferential attachment trees where vertices are assigned weights, and the evolution of a vertex depends not only on its own weight, but also on the weights of its neighbours. We study the distribution of edges with endpoints having certain weights, and the distribution of degrees of vertices having a given weight. We show that the former exhibits a condensation phenomenon under a certain critical condition, whereas the latter converges almost surely to a distribution that resembles a power law distribution. Moreover, in the absence of condensation, we prove almost-sure setwise convergence of the related quantities. This generalises existing results on the Bianconi-Barabási tree as well as on an evolving tree model introduced by the second author.

Keywords: Preferential attachment trees, random recursive trees, Pólya processes, scale-free.

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1 Introduction

1.1 Background

Complex networks appearing in areas as diverse as the internet, social networks and telecommunications are well known for their ubiquitous, non-trivial properties; in particular, they often have a scale free (power law) degree distribution, and display a small or ultra-small world phenomenon (having diameter of logarithmic or double logarithmic order with respect to the size of the network). In their seminal paper, Albert and Barabási in [1] (later studied rigorously in [2, 3]) observed that these properties emerged naturally in a model where vertices arrive one at a time, and display a “preference” to popular vertices - more precisely, connect to existing vertices with probability proportional to their degree. In the case where the newly arriving vertex connects to a single existing vertex, this gives rise to a well-known model of random trees that has been studied under various names: first under the name ordered recursive tree by Prodinger and Urbanek in [4], nonuniform recursive trees by Szymański in [5], random plane oriented recursive trees in [6, 7], random heap ordered recursive trees

*School of Mathematics, University of Birmingham, Birmingham, UK.
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and scale-free trees [2, 9, 10]. Various other modifications of this model have also been studied, including the case that vertices are chosen according to a super-linear function of their degree in [11], or indeed any positive function of the degree [12] (assuming a certain condition is satisfied). In [13], the latter model is generalised to arbitrary non-negative functions of the degree and is referred to as generalised preferential attachment.

Whilst the preferential attachment model is successful in reproducing the properties of complex networks, it is generally the earlier arriving vertices that are more likely to have higher degrees, since (informally) they have more time to acquire new neighbours, which in turn reinforces the growth of their degree. (Indeed, a result of [14] shows that, from a certain time point onward, the vertex with maximal degree remains fixed in this model.) In contrast, in real world models it is often newly arriving nodes that quickly acquire a large number of links (for example, in the world wide web). Motivated by this, in [15], Bianconi and Barabási introduced their well-known model (also called preferential attachment with multiplicative fitness). There, vertices arrive one at a time, and, upon arrival, each vertex is equipped with a random weight sampled independently from a fixed distribution. At each time-step, the newly arriving vertex \( u \) connects to an existing vertex \( v \) with probability proportional to the product of the weight of \( v \) and its degree. Thus, the random weight may be interpreted as a measure of the intrinsic “attractiveness” of a vertex. Bianconi and Barabási postulated the emergence of an interesting dichotomy in this model which they called Bose-Einstein condensation (motivated by similar phenomena in statistical physics): under a certain critical condition on the weight distribution, a positive proportion of all the edges in tree accumulate around vertices of maximum weight. This dichotomy was first proved rigorously by Borgs et al. in [16] in the case that the weight distribution is supported on an interval, and absolutely continuous with respect to Lebesgue measure (however, they note that other classes of weight distribution are possible). They also showed that in this model, the degree distribution of vertices with a given weight follows a power law, with exponent depending on the weight of the vertex. A similar condensation phenomenon was observed in a variant of this model by Dereich in [17], and later, in a more general, robust setting (in the sense that the results apply to wide variety of model specifications) in [18].

Two other similar models are the preferential attachment with additive fitness introduced by Ergün and Rodgers in [19], where newly arriving vertices now connect to existing vertices with probability proportional to the sum of their weight and degree, and the weighted recursive tree introduced in [20]. In [21], Sénizergues showed that the preferential attachment with additive fitness (with deterministic weights) is equal in distribution to a particular weighted random recursive tree with random weights. In addition, Lodewijks and Ortgiese in [22, 23] uncovered an interesting dichotomy in the maximal degrees of these models, in a robust, evolving graph setting. In [24], the second author studied a model incorporating the weighted recursive tree as well as preferential attachment trees with both additive and multiplicative fitness: here at each time-step vertex with weight \( w \) and degree \( k \) is chosen with probability proportional to \( g(w)(k - 1) + h(w) \), where \( g, h \) are non-negative, measurable functions. In this case, the dynamics of the model depend on \( h \) in a non-trivial way: under a certain critical condition on the weight distribution, \( g \) and \( h \) condensation occurs, but does not occur if \( h \) takes large enough values on certain parts of its domain.

In the case of evolving trees, many of the above models describe the family tree of associated continuous time branching processes (often Crump-Mode-Jagers or multitype branching processes), and this perspective has offered some interesting insights into the evolution of these models. For example, the preferential attachment tree of Albert and Barabási was actually first described in the context of evolution by Yule in [25] and in the context of language by Simon in [26]. In addition, the condensation phenomenon observed by Bianconi and Barabási was first studied in a similar, yet simpler manner, in
the context of evolution by Kingman in [27]. Later, the results of [11, 12, 13, 28] have all exploited the connection to branching processes to derive results related to more general preferential attachment models, and in [24, 29] in relation to inhomogeneous models with a ‘fitness’ component. Often, the associated branching process with the discrete time model is known as the continuous time embedding, or Arthreya-Karlin embedding, based on pioneering work by Arthreya and Karlin in [30] who applied this approach in the context of Pólya urns. As shown in [13, 29], when studying ‘local’ properties such as degrees of vertices, one can observe that the continuous time embedding is a Crump-Mode-Jagers branching process, and apply the results of [31], whilst when studying properties such as the height (which is the same order of magnitude as the diameter), one can apply the results of [32] and an argument of Pittel [33].

In [34], the authors studied condensation in models of reinforced branching processes that generalise the continuous time embedding of the Bianconi-Barabási model, showing that the condensation is non-extensive: whilst a positive proportion of edges in the family tree of the process accumulate around vertices of maximal weight, the maximal degree of the tree remains sub-linear. In addition, in [35], the authors studied another generalisation of the continuous time embedding of the Bianconi-Barabási model, incorporating ‘aging’ effects, and applying this to the study of citation networks; they demonstrated a dichotomy between degree distributions having power law and exponential tails based on the aging parameter.

There are a number of other interesting variations of inhomogeneous preferential attachment models. In [36], Jordan studies a model of preferential attachment where vertices belong to two types, and new vertices connect to one according to an additive fitness mechanism, and the other via a multiplicative fitness. Geometric models have also been considered in [37]: here, new vertices are equipped with a location in a metric space, and connect to existing vertices with probability proportional to the product of their degree, and a positive function (called an attractiveness function) of the distance between them. In [37], the authors demonstrate a dichotomy (depending on the attractiveness function) between behaviour according to the model of Albert and Barabási, and a well known geometric model known as the on line nearest neighbour model.

Inhomogeneous models have also been studied in the context of models with choice in [38, 39], with the appearance of more fascinating condensation phenomena. In this model vertices are equipped with weights, at each time step $r$ vertices are chosen with probability proportional to their degree, and out of these $r$ vertices, a random vertex is chosen as the neighbour of the new-coming vertex (where the probability distribution may depend on the weights of the vertices). In [38], the authors showed that, in the case that the maximal weight vertex is chosen, extensive condensation may occur, that is, under a critical condition on the weight distribution, a positive proportion of edges accumulate around the vertex of maximal degree. In addition, in [39], the authors showed that in certain cases, with random choice rules, the distribution of edges with endpoint having certain weight converges weakly to a random measure where multiple condensation can occur with positive probability (that is, positive proportions of edges accumulate around vertices of multiple weights). In addition, they showed that multiple condensation cannot occur when deterministic choice rules are used, and there exist phase transitions for condensation occurring with probability 0 or 1.

### 1.2 Preferential Attachment Trees with Neighbourhood Influence

As we discussed above, a number of preferential attachment mechanisms which incorporate inhomogeneity have been considered. However, models where the attachment mechanism depends on the weights of the neighbours of a vertex have received far less attention. In this direction, the authors
in [40] recently incorporated higher-dimensional interactions into this notion of preferential attachment, studying a model of evolving simplicial complexes. They proved convergence in probability of the limiting degree distribution to a limiting value, depending on a companion Markov process that tracks the evolution of the neighbourhood of a given vertex. In this paper, we study a simplified version of that model, which involves evolving trees; as a result, we are able to derive stronger statements.

More precisely, we consider a model of weighted directed trees \((\mathcal{T}_n)_{n\in\mathbb{N}_0}\); these are labelled directed trees, where vertices have real valued weights associated to them. Let \(\mathbb{T}\) denote the set of all such weighted trees, and given a tree \(T \in \mathbb{T}\) and a vertex \(j \in T\), let \(N^+(j, T)\) be the weighted tree consisting of \(j\) and all of its out-neighbours. In order to define the model, we will require a probability measure \(\mu\), which, without loss of generality is supported on a subset of an interval \([0, w^*]\), for some \(w^* > 0\) and a fitness function \(f: \mathbb{T} \to \mathbb{R}_+\).

In the model we consider, we start with an initial tree \(T_0\) consisting of a single vertex with random weight \(W_0\) sampled from \(\mu\). Then, given \(T_i\), the model proceeds recursively as follows:

(i) Sample a vertex \(j\) from \(T_i\) with probability \(\frac{f(N^+(j, T)_i)}{Z_i}\), where \(Z_i := \sum_{k=0}^{i} f(N^+(k, T)_i)\) is the partition function associated with the process.

(ii) Form \(T_{i+1}\) by adding the edge \((j, i + 1)\), and assigning vertex \(i + 1\) weight \(W_{i+1}\) sampled independently from \(\mu\).

In this paper, we define \(f\) so that

\[
f(N^+(v, T)) = h(W_v) + \sum_{(v,u) \in E(T)} g(W_v, W_u), \tag{1}
\]

where \(h: [0, w^*] \to [0, \infty)\) and \(g: [0, w^*] \times [0, w^*] \to [0, \infty)\) are bounded and measurable. To ensure that the evolution of the model is well-defined, in all of our results we condition on \(W_0\) satisfying \(h(W_0) > 0\) (which we assume is an event that has positive probability).

Figure 1: A sample transition from \(T_1\) to \(T_2\). In \(T_1\), 0 is chosen with probability proportional to \(f(N^+(0, T)_1) = h(W_0) + g(W_0, W_1)\), while 1 is chosen with probability proportional to \(f(N^+(1, T)_1) = h(W_1)\). In this evolution, 1 is chosen, so the newcomer 2 arrives as an out-neighbour of 1.

Remark 1.1. The form of the fitness function in Equation (1) is sufficiently general to encompass some existing models. In the case where \(g\) and \(h\) are a single constant, we obtain the classic preferential attachment tree of Albert and Barabási. The case \(g(x, y) = h(x) = x\) is the Bianconi-Barabási model, whilst the case \(g(x, y) = 1, h(x) = x\) is the preferential attachment tree with additive fitness. Finally, the case \(g(x, y) = g'(x)\), for some bounded measurable function of a single variable is the generalised preferential attachment with fitness model studied by the second author in [24].
Remark 1.2. One may interpret $(\mathcal{T}_n)_{n \in \mathbb{N}_0}$ in the context of reinforced branching processes as follows: we begin with an individual 0 belonging to its own family that reproduces after an exponentially distributed amount of time, with parameter $h(W_0)$. We say that the ancestral weight of the family is $W_0$. Then, recursively, when a birth event occurs in the $i$th family, with ancestral weight $W_i$, a new individual with random weight $W$ joins the $i$th family, reproducing after an $\text{Exp}(g(W_i,W))$ amount of time; and simultaneously, an individual of weight $W$ begins its own family, with ancestral weight $W$. The out-neighborhood of a vertex $i$ in the tree $\mathcal{T}_n$ (including the vertex $i$ itself) then represents individuals in the $i$th family in the branching process, at the time of the $n$th birth event.

Remark 1.3. One can extend the model from the previous remark further by supplanting it with constants $0 \leq \beta, \gamma \leq 1$, so that when a birth event occurs, independently with probability $\beta$, an individual with random weight $W$ joins the $i$th family, and with probability $\gamma$, an individual with random weight $W'$ (also sampled from $\mu$) initiates its own family with ancestral weight $W'$. While not immediately clear from the way we have defined the model, our methods also extend to this case - this link becomes clearer when viewing individuals as “loops” and “edges” in a Pólya urn similar to Urn E (see Figure 2 below). In this extended model, the case $g(x,y) = h(x) = x$ (and this terminology) was introduced in [34], as a stochastic analogue of the model of Kingman [27].

Let $\mathcal{B}$ denote the Borel $\sigma$-algebra on $[0, w^*]$, and $\mathcal{B} \otimes \mathcal{B}$ the product $\sigma$-algebra on $[0, w^*] \times [0, w^*]$. In this paper, we will generally be concerned with studying following main quantities:

1. Given $A \in \mathcal{B} \otimes \mathcal{B}$, the quantity $\Xi^{(2)}(A,n)$ denotes the number of edges $(v, v')$ in the tree $\mathcal{T}_n$ such that $(W_v, W_{v'}) \in A$, that is,

$$\Xi^{(2)}(A,n) := \sum_{(v,v') \in \mathcal{T}_n} 1_A(W_v, W_{v'}). \quad (2)$$

2. Given $B \in \mathcal{B}$, the quantity $N_{\geq k}(B,n)$ denotes the number of vertices $v$ in the tree $\mathcal{T}_n$ with out-degree at least $k$ and weight $W_v \in B$, that is,

$$N_{\geq k}(B,n) := \sum_{v \in \mathcal{T}_n; \deg^+(v; \mathcal{T}_n) \geq k} 1_B(W_v).$$

3. For $B \in \mathcal{B}$, we also define $\Xi(B, n)$, so that

$$\Xi(B,n) := \sum_{(v,v') \in \mathcal{T}_n} 1_B(W_v) = \Xi^{(2)}(B \times [0, w^*], n) \quad \text{(3)}$$

(where the latter equality is in the almost sure sense).

1.2.1 Notation

We denote by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ - i.e. the natural numbers including 0. Also, in general in this paper, $W$ refers to a generic $\mu$ distributed random variable on a probability space $(\Omega, \mathcal{F}, \Pi)$ taking values in the measure space $([0, w^*], \mathcal{B})$ and $\mathbb{E}[]$ will denote expectation with respect to this random variable. In addition, we will require a probability space with an infinite sequence $W_0, W_1, W_2, \ldots$ of random variables, which are independent and identically distributed; we view these (abusing notation slightly) as belonging to the product space $(\Omega, \mathcal{F}, \Pi) := (\prod_{i \in \mathbb{N}_0} (\Omega_i, \mathcal{F}_i, \Pi_i))$. For brevity, $\mathbb{E}[]$ will also denote expectations with respect to random variables on this product space.

In addition, for $s \in \mathbb{N}$, we denote by $[s]$ the set $\{1, \ldots, s\}$. In addition, for $\ell \in \mathbb{N}$, we denote by $[s]^\ell$ the $\ell$-fold Cartesian product $[s] \times \cdots \times [s]$. Given a set $S \subset \mathcal{S}$, we denote by $S^c$ the complement of this set, and (if $\mathcal{S}$ has a topology made clear from context), we denote by $\overline{S}$ the topological closure of $S$. We also denote the indicator function associated with $S$ by $1_S$. Finally, we will introduce some extra notation related to Section 2 in 2.1.2.
1.3 Main Results

The results in this paper depend on two sets of conditions; intuitively one set of conditions describes the ‘non-condensation’ regime, whilst the other describes the ‘condensation’ regime.

1.3.1 The Non-Condensation Regime

The first main conditions are the following: recalling $g$ and $h$ as defined in (1), assume

**C1** There exists some $\lambda^* > \bar{g}^*$ such that
\[
\mathbb{E} \left[ \frac{h(W)}{\lambda^* - \bar{g}(W)} \right] = 1,
\]
where $\bar{g}(x) := \mathbb{E} [g(x, W)]$ and $\bar{g}^* := \mathbb{E} \left[ \sup_{x \in [0,w^*]} g(x, W) \right]$. We call $\lambda^*$ the Malthusian parameter of the process.

**C2** For some $J > 0, N \in \mathbb{N}$, there exist measurable functions $\phi_j^{(i)} : [0, w^*] \to [0, J], j = 1, 2, i \in [N]$, and a bounded continuous function $\kappa : [0, J]^2N \to \mathbb{R}_+$ such that $g(x, y) = \kappa \left( \phi_1^{(1)}(x), \ldots, \phi_1^{(N)}(x), \phi_2^{(1)}(y), \ldots, \phi_2^{(N)}(y) \right)$.

**Remark 1.4.** We expect similar results under the weaker hypothesis that $g$ and $h$ are measurable and bounded rather than Condition C2. However, this condition still allows many “reasonable” choices of bounded measurable functions $g$. This includes the models mentioned in Remark 1.1, the case where $g$ is continuous, as well as functions of the form $g(x, y) = \alpha \phi_1(x) + \beta \phi_2(y)$ or $g(x, y) = \phi_1(x) \phi_2(y)$, where $\phi_1, \phi_2$ are bounded and measurable and $\alpha, \beta \geq 0$.

Our first theorem concerns the partition function of the process

**Theorem 1.1.** Assume Conditions C1 and C2. Then we have
\[
\lim_{n \to \infty} \frac{Z_n}{n} \to \lambda^*
\]
almost surely, where $Z_n$ and $\lambda^*$ respectively denote the partition function and Malthusian parameter of the process.

Define $\psi(x) = h(x)/(\lambda^* - \bar{g}(x))$, denote by $\psi_* \mu$ the pushforward measure of $\mu$ under $\psi$ - i.e. the measure such that for $A \in \mathcal{B}$
\[
(\psi_* \mu)(A) = \mathbb{E} \left[ \frac{h(W)}{\lambda^* - \bar{g}(W)} 1_A(W) \right].
\]

**Theorem 1.2.** Assume Conditions C1 and C2. Then, with $\Xi^{(2)}(\cdot, n)$ as defined in Equation (2), we have
\[
\frac{\Xi^{(2)}(\cdot, n)}{n} \to (\psi_* \mu \times \mu)(\cdot),
\]
almost surely, in the sense of weak convergence. (Here $\psi_* \mu \times \mu$ denotes the product measure of $\psi_* \mu$ and $\mu$ on $([0, w^*]^2, \mathcal{B} \otimes \mathcal{B})$.)
We include the proofs of Theorems 1.1 and 1.2 in Section 2 in 2.2.4 and 2.2.5. We also prove theorems related to the degree distribution. In order to describe this result, we first describe a companion process \((S_i(w))_{i \geq 0}\) that describes the evolution of the fitness of a vertex with weight \(w\) as its neighbourhood changes. First, let \(W_1, W_2, \ldots\) be independent \(\mu\)-distributed random variables and let \(w \in [0, w^*]\). We then define the random process \((S_i(w))_{i \geq 0}\) inductively so that

\[
S_0(w) := h(w); \quad S_{i+1}(w) := S_i(w) + g(w, W_{i+1}), \quad i \geq 0.
\]

(6)

Recall from Section 1.2.1, \(E[\cdot]\) denotes expectation with respect to the path of \(S_i(W_0)\) (i.e., expectations with respect to the product measure involving the terms \(W_0, W_1, W_2, \ldots\)). We then have the following theorem:

**Theorem 1.3.** Assume Conditions C1 and C2. Then, for any \(B \in \mathcal{B}\), we have

\[
\lim_{n \to \infty} N_{\geq k}(B, n) \cdot \frac{1}{n} = \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i(W_0)}{S_i(W_0) + \lambda^*} \right) 1_B(W_0) \right],
\]

(7)

almost surely.

We prove Theorem 1.3 in Subsection 2.3.3.

**Remark 1.5.** One may interpret the right hand side of Equation (7) as the probability of a sequence of at least \(k\) consecutive heads before a first tail when, sampling \(W_0\) at random, and flipping the \(i\)th coin heads with probability proportional to \(S_i(W_0)\) and having weight \((\Xi(\cdot, n)\) as defined in Equation (3)). First we require the following lemma, which may be of independent interest:

**Lemma 1.4.** Let \((S_i(w))_{i \geq 0}\) denote the process defined in (6) in terms of bounded, measurable functions \(g, h\), suppose \(\tilde{g}(x) := \mathbb{E}[g(x, W)]\) and \(\tilde{g}_+ := \sup_{x \in [0, w^*]} \tilde{g}(x)\). Then, for any \(w \in [0, w^*]\), and \(\lambda \geq \tilde{g}_+\) we have

\[
\sum_{k=1}^{\infty} \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i(w)}{S_i(w) + \lambda} \right) 1_B(W_0) \right] = \mathbb{E} \left[ \frac{h(w)}{\lambda - \tilde{g}(w)} 1_B(W_0) \right],
\]

(8)

where the right hand side is infinite if \(g(w) = \tilde{g}_+\) and \(\lambda = \tilde{g}_+ = \tilde{g}(w)\). In particular,

\[
\sum_{k=1}^{\infty} \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i(w)}{S_i(w) + \lambda} \right) 1_B(W_0) \right] = \mathbb{E} \left[ \frac{h(W)}{\lambda - \tilde{g}(W)} 1_B(W_0) \right].
\]

As the proof of this lemma detracts from the main techniques used in this paper, we delay its proof to the appendix, in Subsection 4.1.

**Remark 1.6.** One may interpret Equation (8) as a generalisation of the classic geometric series formula: if we set \(g(x, y) \equiv 0\), and \(q := h(w)/(h(w) + \lambda)\), the left hand side of (8) is \(\sum_{i=1}^{\infty} q^i = \frac{h(w)}{h(w) + \lambda} = \frac{q}{1-q}\). Indeed, as Remark 1.5 shows, one may interpret the left hand side as the expected value of a generalised geometrically distributed random variable.

Lemma 1.4 allows us to strengthen the weak convergence result of Theorem 1.2 to setwise convergence.
Theorem 1.5. Assume Condition C1. Then, for any set \( A \in \mathcal{B} \) we have

\[
\frac{\Xi(A,n)}{n} \to (\psi_*\mu)(A),
\]

almost surely. In other words, the random probability measure \( \Xi(\cdot,n)/n \) converges almost surely setwise to the limiting measure \( \psi_*\mu \).

Remark 1.7. The setwise convergence in Theorem 1.5 is stronger than the usual weak convergence results appearing in the literature (for example in [16, 34]). However, as the limiting measure is absolutely continuous with respect to \( \mu \), and hence almost surely with respect to the measures \( \Xi(p,\cdot,n) \), one might expect to improve this to total variation convergence. Indeed, this is the result obtained in the simplified model first analysed by Kingman in [27] (Kingman describes the non-condensation regime as the “democratic” regime).

1.3.2 The Condensation Regime

In this paper, we are able to describe a “condensation” result; we first make precise what “condensation” means.

Definition. Suppose we are given a \( \mu \)-null set \( S \subseteq [0,w^*] \) and let \( \Xi(\cdot,n) \) be as in (2). We say that condensation occurs around the set \( S \), if for some nested collection of sets \( (S_\varepsilon)_{\varepsilon \geq 0} \), with \( S_\varepsilon \downarrow S \) as \( \varepsilon \to 0 \) we have

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\Xi(S_\varepsilon,n)}{n} > 0,
\]

with positive probability.

Remark 1.8. Informally, condensation means that, in the limit of the random measure \( \Xi(\cdot,n)/n \), the set \( S \) acquires more mass than one ‘would expect’. Indeed, if we swap limits,

\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{\Xi(S_\varepsilon,n)}{n} = \lim_{n \to \infty} \frac{\Xi(S,n)}{n} = 0,
\]

almost surely, since \( \mu(S) = 0 \).

Our main assumptions are now as follows:

D1 We have

\[
\mathbb{E} \left[ \frac{h(W)}{g^* - \bar{g}(W)} \right] < 1. \tag{9}
\]

D2 The function \( g \) satisfies Condition C2.

D3 There exists a (maximal) set of points \( \mathcal{M} \subseteq \text{Supp}(\mu) \), such that, for any \( x^* \in \mathcal{M} \),

\[
\max_{p \in [0,w^*]} g(p,W) = g(x^*,W) \quad \mathbb{P} - \text{a.s.}
\]

We denote by \( x^* \) a generic point in \( \mathcal{M} \).

D4 For all \( \varepsilon > 0 \) sufficiently small, and a measurable function \( u_\varepsilon : [0,w^*] \to \mathbb{R}_+ \) with \( \lim_{\varepsilon \to 0} u_\varepsilon = 0 \) pointwise, we have

\[
\mathcal{M}_\varepsilon := \{ x : \mathbb{P} (g(x^*,W) - g(x,W) < u_\varepsilon(W)) = 1 \} = \{ x : \mathbb{P} (g(x^*,W) - g(x,W) < u_\varepsilon(W)) > 0 \}.
\tag{10}
\]

Under this assumption, we have \( \mu(\mathcal{M}_\varepsilon) > 0 \).

\(^1\)That is, a collection of sets such that if \( \varepsilon_1 < \varepsilon_2 \), \( S_{\varepsilon_1} \subseteq S_{\varepsilon_2} \).
Remark 1.9. Note that, by the measurability of $g(\cdot, q)$ for any $q \in [0, w^*]$, the function

$$p \mapsto \text{ess sup}_{q \in [0, w^*]} \{g(x^*, q) - g(p, q) - u_\varepsilon(q)\}$$

is also measurable (see, e.g. Theorem 4.7.1., [41]). This ensures that the set $M_\varepsilon \in \mathcal{B}$.

Example 1.10. In the case that $g(x, y) = \phi_1(x)\phi_2(y)$ for bounded, measurable $\phi_1, \phi_2$, if $\phi_1(x)$ is maximised on a set $M$ and $\phi_2(y) > 0 \mu$-a.e., for $\varepsilon > 0$ and $x^* \in M$ we may take $u_\varepsilon = \varepsilon \cdot \phi_2$ and

$$M_\varepsilon := \{x : \phi_1(x^*)\phi_2(W) - \phi_1(x)\phi_2(W) < \varepsilon\phi_2(W)\} = \{x : \phi_1(x^*) - \phi_1(x) < \varepsilon\}.$$

A condition that guarantees that this set has positive measure is assuming continuity of $\phi_1$ at some point $x^* \in M$, as this implies that $M_\varepsilon$ is a neighbourhood of $x^*$.

Remark 1.11. Conditions D1 and D2 may be interpreted as analogues of Conditions C1 and C2 in the condensation regime. One may regard $M$ from D3 as a “dominating set”, in the sense that $P$-a.s., upon arrival of a new vertex into its neighbourhood, the change of the fitness of any vertex is at most the change of the fitness of a vertex with weight with weight in $M$. Condition D4 ensures that this “dominating property” is captured by sets $M_\varepsilon$ of positive measure. Indeed the right hand side of (10) implies that the change of the fitness of any vertex with weight in $M_\varepsilon$ is at most the change of the fitness of a vertex having weight in $M_\varepsilon$. Note that $M_\varepsilon \downarrow M$ as $\varepsilon \to 0$. This accounts for the formation of the condensate in Theorem 1.7, since $\tilde{g}$ is maximised on $M$, by D1 it must be the case that $\mu(M) = 0$.

Theorem 1.6. Assume Conditions D1-D4. Then we have

$$\lim_{n \to \infty} \frac{Z_n}{n} \to \bar{g}^* = g(x^*)$$

almost surely.

Theorem 1.7. Assume Conditions D1-D4. Then, for any $A \in \mathcal{B}$ such that, for $\varepsilon > 0$ sufficiently small $A \cap M_\varepsilon = \emptyset$, we have

$$\frac{\Xi(A, n)}{n} \to (\psi_\varepsilon \mu)(A),$$

almost surely. In addition,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\Xi(M_\varepsilon, n)}{n} = 1 - (\psi_\varepsilon \mu)([0, w^*]) > 0,$$

so that condensation occurs around $M$.

Remark 1.12. As the condensation occurs around the “dominating set” $M$, in the context of reinforced branching processes (see Remarks 1.2 and 1.3), one may interpret this is families with maximum reinforced ‘fitness’ (in this context meaning the ability to produce offspring quickly) acquiring a positive proportion of individuals in the population in the limit. This has an interesting interpretation in the context of evolution.

We have the following corollary:

Corollary 1.8. Assume Conditions D1-D4, and the sets $M_\varepsilon$ in D4 are such that $\overline{M_\varepsilon} \downarrow M$ as $\varepsilon \to 0$ (recalling $\overline{M_\varepsilon}$ denotes the topological closure of $M_\varepsilon$). Also, suppose that $M = \{x^*\}$, and define the measure $\Pi(\cdot)$ such that, for $B \in \mathcal{B}$

$$\Pi(B) = (\psi_\varepsilon \mu)(B) + (1 - (\psi_\varepsilon \mu)([0, w^*])) \delta_{x^*}(B).$$
Then,
\[
\frac{\Xi(\cdot, n)}{n} \to \Pi(\cdot) \text{ almost surely,}
\]
in the sense of weak convergence.

**Example 1.13.** In the case that \(g(x, y) = \phi_1(x)\phi_2(y)\) for a bounded, continuous function \(\phi_1\) and bounded measurable function \(\phi_2\), if \(\phi_1(x)\) is maximised at a unique point \(x^*\) and \(\phi_2(y) > 0\) \(\mu\)-a.e., we may take \(u_\varepsilon\) and \(M_\varepsilon\) as defined in Example 1.10. Indeed, in this case
\[
M_\varepsilon = \{x : \phi_1(x^*) - \phi_1(x) \leq \varepsilon\},
\]
so that \(M_\varepsilon \downarrow \{x^*\}\) as \(\varepsilon \to 0\).

Finally, we have the following extension of Theorem 1.3:

**Theorem 1.9.** Assume Conditions D1-D4. Then, for any \(B \in \mathcal{B}\), we have
\[
\lim_{n \to \infty} \frac{N_{\geq k}(B, n)}{n} = \mathbb{E}\left[\prod_{i=0}^{k-1} \left(\frac{S_i(W)}{S_k(W) + \lambda^*} \right) 1_B(W)\right],
\]
almost surely.

### 1.4 Discussion

In this subsection, we provide an informal discussion of some of the implications of our main results.

#### 1.4.1 Power-Law Degrees

First note that by Theorem 1.3, if \(N_k(B, n)\) denotes the number of vertices with degree \(k\) and weight belonging to \(B\) at time \(n\), then almost surely
\[
\lim_{n \to \infty} \frac{N_k(B, n)}{n} = \lim_{n \to \infty} \left(\frac{N_{\geq k}(B, n)}{n} - \frac{N_{\geq k+1}(B, n)}{n}\right) = \mathbb{E}\left[\frac{\lambda^*}{S_k(W) + \lambda^*} \prod_{i=0}^{k-1} \left(\frac{S_i(W)}{S_k(W) + \lambda^*} \right) 1_B(W)\right].
\]

Now by the strong law of large numbers, one would expect (at least asymptotically), \(S_i(W) \sim h(W) + i\tilde{g}(W)\), and thus it is natural to expect
\[
\lim_{n \to \infty} \frac{N_k(B, n)}{n} \sim \mathbb{E}\left[\frac{\lambda^*}{k\tilde{g}(W) + \lambda^*} \prod_{i=0}^{k-1} \left(\frac{h(W) + i\tilde{g}(W)}{h(W) + i\tilde{g}(W) + \lambda^*}\right)\right].
\]

Now, if we approximate the product on the right hand side as a ratio of gamma functions, and noting that by Stirling’s approximation (as \(k \to \infty\))
\[
\frac{\Gamma(k)}{\Gamma(k + a)} = (1 + O(1/k))k^{-a},
\]
we thus expect that
\[
\lim_{n \to \infty} \frac{N_k(B, n)}{n} \sim \mathbb{E}\left[k^{-1+\lambda^*/\tilde{g}(W)} 1_B(W)\right].
\]

Thus, informally, this model displays a degree distribution of vertices with a given weight satisfies a power law that depends on the weights of the vertices. Noting also that \(\lambda^*/\tilde{g}(W) > 1\), the exponent of this power law is larger than 2. A similar analysis can be applied to the condensation regime by applying Theorem 1.9. Finally, note that these arguments can be made rigorous if the function \(g\) is independent of its second argument - i.e., if \(g(x, y) = g'(x)\) for some function \(g' : [0, w^*] \to [0, \infty)\) - see Section 2 of [24].
1.4.2 The Growth of the Neighbourhood of Fixed Vertex

In the following proposition, we let $f_v(n) = f(N^+(v, T_n))$ denote the fitness (as defined in Equation (1)) of a vertex labelled $v \in \mathbb{N}_0$, with weight $w_v$ in the tree at time $n$. In addition, let $(R_i)_{i \geq v}$ denote the filtration generated by the tree process $(T_i)_{i \geq v}$. Next, set

$$M_v(n) := \frac{f_v(n)}{\prod_{s=v}^{n-1} \left( \frac{Z_s + \tilde{g}(w_v)}{Z_s} \right)}.$$

**Proposition 1.10.** For any vertex $v$, $(M_v(n))_{n \geq v}$ is a martingale with respect to the filtration $(R_i)_{i \geq v}$.

**Proof.** Using the definition of the process, for $n \geq v$ we compute

$$\mathbb{E} \left[ f_v(n+1) | R_n \right] = \frac{f_v(n)}{Z_n} \left( f_v(n) + \tilde{g}(w_v) \right) + \left( 1 - \frac{f_v(n)}{Z_n} \right) f_v(n)$$

$$= f_v(n) \left( \frac{Z_n + \tilde{g}(w_v)}{Z_n} \right).$$

The result follows from the definition of $(M_v(n))_{n \geq v}$. □

Now, here we note two things: first, if $\deg^+_v(t)$ denotes the out-degree of vertex $v$ at time $t$, then we expect $f_v(t) \sim \deg^+_v(t)$ (in fact, by applying Wald’s lemma, one can show $\mathbb{E} \left[ f_v(t) \right] = h(w_v) + \mathbb{E} \left[ \deg^+_v(t) \right] \tilde{g}(w_v)$). Second, by Theorems 1.1 and 1.6, we expect $Z_i \sim \lambda^* i$ and $\tilde{g}^* i$ in the non-condensation and condensation regimes respectively. Thus, we expect

$$\deg^+_v(t) \sim \prod_{s=v}^{t-1} \left( \frac{Z_s + \tilde{g}(w_v)}{Z_s} \right) \sim \begin{cases} t \tilde{g}(w_v)/\lambda^*, & \text{under Conditions C1 and C2;} \\ t \tilde{g}(w_v)/\tilde{g}^*, & \text{under Conditions D1-D4.} \end{cases}$$

Therefore, in the non-condensation regime, we expect each individual vertex to grow like $t \tilde{g}(w_v)/\lambda^* \leq t \tilde{g}^*/\tilde{g}^* < t$, whereas, in the condensation regime, vertices with weight $w_v$ such that $g(w_v)$ is closer and closer to $\tilde{g}^*$ grow at a rate closer and closer to linearity with respect to the size of the network. Note that to turn this argument into a rigorous result in terms of $\mathbb{E} \left[ \deg^+_v(t) \right]$, one requires L1 convergence of the martingale in Proposition 1.10.

1.5 Overview and Techniques

1.5.1 Overview

In Section 2 we prove results about the model related to the non-condensation regime. We first review some background theory about Pólya urns in Subsection 2.1, and then, the result of Subsection 2.2 are used in order to prove Theorems 1.1 and 1.2 in Subsections 2.2.4 and 2.2.5 respectively. Next, the results of Subsection 2.3 are used to prove Theorems 1.3 and 1.5 in 2.3.3 and 2.3.4. In Section 3 we extend the previous results to the condensation regime, proving Theorems 1.6 and 1.7, Corollary 1.8 and Theorem 1.9 in 3.1, 3.2, 3.3 and 3.4 respectively. We prove Lemma 1.4 in the Appendix, in Subsection 4.1.

1.5.2 Techniques

This paper generalises the techniques used in [16] for the study of the Bianconi-Barabási model - using a Pólya urn approximation. However, the generalisation of this model to bounded measurable functions $h$, functions $g$ satisfying Condition C2, and the possibility of arbitrary weight distributions lead
to technical challenges, somewhat analogous to those arising from using a measure-theoretic approach to integration as opposed to the Riemann integral. Applying this approach to studying the degree distribution in the case of uncountably supported weight distributions also appears to be novel. In extending the results to the condensation regime we apply a similar coupling to that used in [24].

One might imagine that many of the results here may follow easily from an application of the theory of Crump-Mode-Jagers branching processes (for example as in [34]). However, the dependence between offspring distributions of a parent and its offspring means that the classic theory is not immediately applicable. This in turn raises the question of whether one can develop a theory of C-M-J branching processes with dependencies.

2 The Non-Condensation Regime

2.1 Generalised Pólya urns

Generalised Pólya urns are a well studied family of stochastic processes representing the composition of an urn containing balls with certain types. If \( \mathcal{T} \) denotes the set of possible types, associated to a ball of type \( t \in \mathcal{T} \) is a non-negative activity \( a(t) \), which depends on the type. The process then evolves in discrete time so that, at each time-step, a ball of type \( t \) is sampled at random from the urn with probability proportional to its activity \( a(t) \), and replaced with a number of different coloured balls according to a (possibly random) replacement rule.

In the case that \( \mathcal{T} \) is finite, the configuration of the urn after \( n \) replacements may be represented as a composition vector \( (X_n)_{n \in \mathbb{N}} \) with entries labelled by type, and the activities encoded in an activity vector \( \mathbf{a} \). In this vector, the \( i \)th entry corresponds to the number of balls of type \( i \in \mathcal{T} \). Let \( (\xi_{ij})_{i,j \in \mathcal{T}} \) be the matrix whose \( ij \)th component denotes the random number of balls of colour \( j \) added, if a ball of colour \( i \) is drawn, and (following the notation of Janson in [42]) define the matrix \( A \) such that

\[
A_{ij} := a_j \mathbb{E}[\xi_{ji}].
\]

The (expected) evolution of the urn in the \((n+1)\)st step, may therefore be obtained by applying the matrix \( A \) to the composition vector \( X_n \). A type \( i \in \mathcal{T} \) is said to be dominating if, for any \( j \in \mathcal{T} \), it is possible to obtain a ball of type \( j \) starting with a ball of type \( i \). If we write \( i \sim j \) for the equivalence relation where \( i \sim j \) if it is possible to obtain \( j \) starting from a ball of type \( i \), and vice versa. This partitions the types into equivalence classes. A class \( \mathcal{C} \subseteq \mathcal{T} \) is dominating if, for every \( i \in \mathcal{C} \), \( i \) is dominating. Moreover, the eigenvalues of \( A \) may be obtained by the restriction of \( A \) to its classes; we say an eigenvalue belongs to a dominating class if it is an eigenvalue of the restriction of \( A \) to this class. Finally, we say that the urn, or the matrix \( A \), is irreducible if there is only one dominating class (note the difference when compared to irreducible matrices in the context of Markov chains: here it is possible for diagonal entries to be negative). Now, assume the following conditions are satisfied:

(A1) For all \( i,j \in \mathcal{T} \), \( \xi_{ij} \geq 0 \) if \( i \neq j \) and \( \xi_{ii} \geq -1 \).

(A2) For all \( i,j \in \mathcal{T} \), \( \mathbb{E}[\xi_{ij}^2] < \infty \).

(A3) The largest real eigenvalue \( \lambda_1 \) of \( A \) is positive.

(A4) The largest real eigenvalue \( \lambda_1 \) is simple.

(A5) We start with at least one ball of a dominating type.

(A6) \( \lambda_1 \) belongs to the dominating class.

The following is a well known result of Janson from 2004 (building on previous work by by Athreya and Karlin, see, for example, Proposition 2 in [30] and Theorem 5 of [43]):
Theorem 2.1 ([42], Theorem 3.16). Assume Conditions (A1)-(A6), and suppose that $v_1$ denotes the right eigenvector, corresponding to the leading eigenvalue $\lambda_1$ of $A$, normalised so that $a^Tv_1 = 1$. Then, we have
\[
\frac{X_n}{n} \xrightarrow{n \to \infty} \lambda_1 v_1,
\]
almost surely, conditional on essential non-extinction (i.e. non-extinction of balls of dominating type).

In addition, the following lemma by Janson provides convenient criteria for satisfying (A1)-(A6):

Lemma 2.2 ([42], Lemma 2.1). If $A$ is irreducible, (A1) and (A2) hold, $\sum_{i \in \mathcal{T}} E[\xi_i] \geq 0$ for all $i \in \mathcal{T}$, with the inequality being strict for some $i \in \mathcal{T}$, then (A1) - (A6) are satisfied and essential extinction does not occur.

2.1.1 Analysing the Tree using Pólya Urns

The idea behind analysing the distribution of edges with a given weight, and the degree distribution in this model, is to consider two different types of Pólya urns, which we call Urn E and Urn D respectively. We illustrate the evolution of both these urns below. Recall, Figure 1 illustrates a possible evolution of a step of the process $(\mathcal{T}_i)_{i \in \mathbb{N}_0}$; Figures 2 and 3 illustrate the corresponding steps in Urn E and Urn D.

In Urn E, we consider a generalised Pólya urn with balls of two types: singletons $x$, and tuples $(x, y)$, corresponding to ‘edges’ and ‘loops’. A ball of type $(x, y)$ has activity $g(x, y)$ and a ball of type $x$ has activity $h(x)$. At each step, if a ball of activity $x$ or $(x, y)$ is selected, we introduce a new ball of random type $W$, and a ball of type $(x, W)$. In relation to the evolving tree, this corresponds to the event that a vertex of weight $x$ has been sampled in the subsequent step.

![Figure 2: The evolution of the tree from $\mathcal{T}_1$ to $\mathcal{T}_2$ from Figure 1 viewed as a transition in Urn E. The event vertex 1 is selected may be interpreted as the event that the ‘loop’ $W_1$ is selected in the Pólya urn - and thus the arrival of the vertex 2 corresponds to the arrival of the ‘loop’ $W_2$ and the ‘edge’ $(W_1, W_2)$ in the Pólya urn.]

In Urn D, we consider a generalised Pólya urn with balls of types corresponding to tuples of varying lengths. A ball of type $(x_0, \ldots, x_k)$ has activity $h(x_0) + \sum_{i=1}^{k} g(x_0, x_i)$, and at each step, if a ball this type is selected, we remove it and introduce new balls of random type $W$, and a ball of type $(x_0, \ldots, x_k, W)$. In relation to the evolving tree, this corresponds to the event that a vertex of weight $x_0$, with neighbours (listed in order of arrival) having weights $x_1, \ldots, x_k$, has been sampled when proceeding to the subsequent step.
Then, there exists a refined good partition of \( A \) that also forms a good partition of \( A \).

### Proof.

Refined good partitions, similar to the above, can be applied to Urn D. The event vertex 1 is selected, and thus the arrival of the vertex 2 corresponds to the addition of the balls \( W_2 \) and \( (W_1, W_2) \) (the latter representing the addition of vertex 2 into the neighbourhood of vertex 1).

Note that, in the manner we have described Urns E and D, the set of possible types may be infinite: the measure \( \mu \) may have infinite support so that \( W \) may take on infinite values, and the neighbourhoods of vertices (in Urn D) may be infinite. Whilst there is some theory related to infinite type Pólya urns within the framework of measure-valued Pólya processes (see, for example, [44]), these results are often non-trivial to apply in practice - see, for example, pages 14-21 of [40]. As a result, we instead approximate these infinite urns with urns of finitely many types - enough to approximate the sigma algebras generated by \( W, g(W, W') \) and \( h(W) \), where \( W, W' \) are i.i.d random variables sampled according to \( \mu \). In Subsection 2.2 we apply this analysis to Urn E, and in Subsection 2.3 we apply it to Urn D. We first introduce some extra notation specific to this section.

#### 2.1.2 Some More Notation and Terminology

In order to apply the finite Pólya urn theory, given a set of types \( \mathcal{F} \), we denote by \( \mathbb{V}_{\mathcal{F}} \) the free vector space over the field \( \mathbb{R} \) generated by \( \mathcal{F} \) (i.e. the vector space where vectors are indexed by the elements of \( \mathcal{F} \)). We will generally view an urn with types \( \mathcal{F} \) as a stochastic process taking values in \( \mathbb{V}_{\mathcal{F}} \). In addition we will generally identify vectors \( v \in \mathbb{V}_{\mathcal{F}} \) interchangeably with functions \( v: \mathcal{F} \to \mathbb{R} \). Thus, for \( x \in \mathcal{F} \), \( v(x) \) denotes the entry of the vector corresponding to \( x \), and for \( v_1, v_2 \in \mathbb{V}_{\mathcal{F}} \), we have \( (v_1 v_2)(x) = v_1(x) v_2(x) \). For \( x \in \mathcal{F} \), we define \( \delta_x \in \mathbb{V}_{\mathcal{F}} \) such that \( \delta_x(y) = 1 \) if \( y = x \) and 0 otherwise.

For a Borel measurable set \( S \subseteq \mathbb{R} \), and a finite set \( \mathcal{A} \) of Borel measurable subsets of \( S \), we say that \( \mathcal{A} \) forms a good partition of \( S \) if, given any two nonempty sets \( A_i, A_j \in \mathcal{A} \), \( A_i \cap A_j \neq \emptyset \iff A_i = A_j \), and \( \bigcup_{i=1}^n A_i = S \). Note that, given two good partitions \( \mathcal{A}_1, \mathcal{A}_2 \) of \( S \), the set

\[
\{ A_1 \cap A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2 \}
\]

also forms a good partition of \( S \). In addition, if \( \mathcal{A} \) is a good partition of \( S \), we say that \( \mathcal{A}' \) forms a refined good partition (often we will just write refined partition) of \( \mathcal{A} \), if, for any \( A' \in \mathcal{A}' \) there exists \( A \in \mathcal{A} \) such that \( A' \subseteq A \). The following lemma (which is well-known) justifies the use of the word ‘refined’.

**Lemma 2.3.** Suppose \( \mathcal{A} \) is a good partition of a set \( S \), and \( \mathcal{A}' \) is a refined partition of \( \mathcal{A} \). Then, for any set \( A \in \mathcal{A} \), there exist sets \( X_1, \ldots, X_n \in \mathcal{A}' \) such that \( A = \bigcup_{i=1}^n X_i \). In particular, \( \{X_i\}_{i \in [n]} \) forms a good partition of \( A \).

**Proof.** For \( A \in \mathcal{A} \), define the sub-family \( \mathcal{X} := \{ A' \in \mathcal{A}' : A' \subseteq A \} \). Suppose \( U := (\bigcup_{X \in \mathcal{X}} X) \neq A \). Then, there exists \( x \in A \setminus U \), and since \( \mathcal{A}' \) partitions \( S \), \( x \in V' \), for some set \( V' \in \mathcal{A}' \) with \( V' \not\subseteq A \). But

Figure 3: The evolution of the tree from \( T_1 \) to \( T_2 \) from Figure 1 viewed as a transition in Urn D. The event vertex 1 is selected may be interpreted as the event that the ball \( W_1 \) is selected in the Pólya urn - and thus the arrival of the vertex 2 corresponds to the addition of the balls \( W_2 \) and \( (W_1, W_2) \) (the latter representing the addition of vertex 2 into the neighbourhood of vertex 1).
then, since \( A' \) is a refined partition of \( A \), \( V' \subseteq V \) for some \( V \in A \). But then, this implies that either \( V \cap A \neq \emptyset \), contradicting the fact that \( A \) is a good partition of \( S \), or \( V = A \), contradicting the fact that \( V' \subseteq A \).

### 2.2 Urn E

In this subsection we will refer to Conditions C1 and C2. We will analyse the process under these conditions by coupling the tree process \( \{T_n\}_{n \in \mathbb{N}_0} \) with Pólya urn processes, parametrised by \( m \in \mathbb{N} \). These may be interpreted as finite approximations of Urn E. Now, for each \( x \in \mathbb{R} \) and \( m \in \mathbb{N} \) we define a good partition of interval \([0,x]\) into into \( 2^m \) intervals (a dyadic partition): set

\[
\mathcal{D}_i^m(x) := [0,2^{-m}x], \quad \text{and} \quad \mathcal{D}_i^m(x) := ((i-1) \cdot 2^{-m}x, i \cdot 2^{-m}x], \quad i \in [2^m]\{1\}.
\]

For \( i \in [2^m] \), we also denote the closure of \( \mathcal{D}_i^m(x) \) by \( \overline{\mathcal{D}_i^m(x)} \), so that

\[
\overline{\mathcal{D}_i^m(x)} = [(i-1) \cdot 2^{-m}x, i \cdot 2^{-m}x].
\]

Supposing \( h : [0,w^*] \to \mathbb{R}_+ \) takes values in \([0,h_{\text{max}}]\) and recalling the functions \( \phi_1^{(j)}, \phi_2^{(j)}, j \in [N] \) from Condition C2, for each \( i \in [2^m], j \in [N] \) and \( k \in [2] \), we set

\[
\mathcal{H}_i^m := h^{-1}(\mathcal{D}_i^m(h_{\text{max}})) \quad \text{and} \quad \Phi_k^m(i,j) := \left( \phi_k^{(j)} \right)^{-1}(\mathcal{D}_i^m(J)).
\]

By the measurability assumptions on the functions \( \phi_k^{(j)} \) and \( h \), for each \( i \in [2^m] \) we have \( \mathcal{H}_i^m, \Phi_k^m(j,k) \in \mathcal{B} \) and thus, the collections of sets \( \{\mathcal{H}_i^m\}_{i \in [2^m]} \) and \( \{\Phi_k^m(i,j)\}_{i \in [2^m]} \) form good partitions of \([0,w^*]\). Now, given \( \nu = (\nu_1, \ldots, \nu_N) \in [2^m]^N \), we define

\[
\mathcal{D}^m_{\nu}(J) := \mathcal{D}_{\nu_1}^m(J) \times \mathcal{D}_{\nu_2}^m(J) \times \cdots \times \mathcal{D}_{\nu_N}^m(J),
\]

and observe that, given \( i, j \in [2^m]^N \), the construction of the sets in Equation (14) are such that \((x,y) \in \Phi_1^m(i) \times \Phi_2^m(j)\) implies that

\[
\left( \phi_1^{(1)}(x), \ldots, \phi_1^{(N)}(x), \phi_2^{(1)}(y), \ldots, \phi_2^{(N)}(y) \right) \in \overline{\mathcal{D}_i^m(J)} \times \overline{\mathcal{D}_j^m(J)}.
\]

Now, recalling the function \( \kappa : [0,J]^{2N} \to [0,9_{\text{max}}] \) from Condition C2, for each \( i, j \in [2^m]^N \), by continuity on the compact set \( \overline{\mathcal{D}_i^m(J)} \times \overline{\mathcal{D}_j^m(J)} \), for \((x,y) \in \Phi_1^m(i) \times \Phi_2^m(j)\) we have

\[
\kappa \left( \phi_1^{(1)}(x), \ldots, \phi_1^{(N)}(x), \phi_2^{(1)}(y), \ldots, \phi_2^{(N)}(y) \right) \geq \inf_{u,v \in \overline{\mathcal{D}_i^m(J)} \times \overline{\mathcal{D}_j^m(J)}} \{\kappa(u,v)\} \leq \max_{u,v \in \overline{\mathcal{D}_i^m(J)} \times \overline{\mathcal{D}_j^m(J)}} \{\kappa(u,v)\} =: \kappa^-(i, j),
\]

and likewise,

\[
\kappa \left( \phi_1^{(1)}(x), \ldots, \phi_1^{(N)}(x), \phi_2^{(1)}(y), \ldots, \phi_2^{(N)}(y) \right) \leq \sup_{u,v \in \overline{\mathcal{D}_i^m(J)} \times \overline{\mathcal{D}_j^m(J)}} \{\kappa(u,v)\} \geq \min_{u,v \in \overline{\mathcal{D}_i^m(J)} \times \overline{\mathcal{D}_j^m(J)}} \{\kappa(u,v)\} =: \kappa^+(i, j).
\]
Now, set
\[ g^-(x,y) := \sum_{i,j \in [2^m]^N} \kappa^-(i,j) 1_{\Phi_1^m(i) \times \Phi_2^m(j)}(x,y), \quad \text{and} \quad g^+(x,y) := \sum_{i,j \in [2^m]^N} \kappa^+(i,j) 1_{\Phi_1^m(i) \times \Phi_2^m(j)}(x,y); \]
and
\[ h^-(x) := \sum_{i=1}^{2^m} (i - 1) \cdot 2^{-m} h_{\max} 1_{\mathcal{H}_i}(x), \quad h^+(x) := \sum_{i=1}^{2^m} i \cdot 2^{-m} h_{\max} 1_{\mathcal{H}_i}(x). \]

One should interpret these functions as lower and upper approximations to \( g \) and \( h \), indeed, by construction, we now have the following lemma:

**Lemma 2.4.** We have \( g^- \uparrow g, h^- \uparrow h, g^+ \downarrow g \) and \( h^+ \downarrow h \) uniformly, as \( m \to \infty \).

**Proof.** We prove the statements regarding \( h^- \) and \( g^- \); the others follow analogously (in the case of \( g^+ \) using Equation (16) instead of (15)). Since the sets \( \{ \mathcal{H}_i^m \}_{i \in [2^m]} \) form a good partition of \([0, w^*]\), for each \( m \in \mathbb{N} \), given \( x \in [0, w^*] \), we have \( x \in \mathcal{H}_j^m \) for some \( j \in [2^m] \), and thus
\[ h^-(x) = (j - 1) \cdot 2^{-m} h_{\max} \leq h(x) \leq h^-(x) + 2^{-m} h_{\max}. \]
The convergence result for \( h^- \) follows. Now, note that by uniform continuity of \( \kappa \) on the compact set \([0, J]^{2N}\), for \( \varepsilon > 0 \), let \( M \) be sufficiently large so that for all \( u, v \in [0, J]^{2N} \)
\[ \|u - v\| < \sqrt{2N} \cdot 2^{-M} J \quad \implies \quad |\kappa(u) - \kappa(v)| < \varepsilon. \]
Now, for any \( m > M \), given \((x,y) \in [0, w^*] \times [0, w^*] \), there exists a unique set \( \Phi_1^m(i) \times \Phi_2^m(j) \) containing \((x,y)\), which implies that
\[ \left( \phi_1^{(1)}(x), \ldots, \phi_1^{(N)}(x), \phi_2^{(1)}(y), \ldots, \phi_2^{(N)}(y) \right) \in \mathcal{D}_1^m(J) \times \mathcal{D}_2^m(J). \]
Thus, for each \( j \in [N] \), combining this equation with the definition of \( \kappa^-(i,j) \) from (15), we have
\[ \kappa^-(i,j) \leq \kappa \left( \phi_1^{(1)}(x), \ldots, \phi_1^{(N)}(x), \phi_2^{(1)}(y), \ldots, \phi_2^{(N)}(y) \right) \leq \kappa^-(i,j) + \varepsilon, \]
and thus
\[ g^-(x,y) \leq g(x,y) \leq g^-(x,y) + \varepsilon. \]
The result now follows. \( \Box \)

Now, using the good partitions \( \{ \mathcal{H}_i^m \}_{i \in [2^m]} \), \( \{ \Phi_1^m(i) \}_{i \in [2^m]^N} \), \( \{ \Phi_2^m(j) \}_{j \in [2^m]} \) and \( \{ \mathcal{D}_i^m(w^*) \}_{i \in [2^m]} \), we will form an even more refined partition, which we will use as the “building blocks” of the evolution of the Pólya urn approximations. For each \( m \), define the good partition \( \mathcal{F}^m \) such that
\[ \mathcal{F}^m := \left\{ i \in \mathcal{B} : I = \mathcal{H}_i^m \cap \mathcal{D}_q^m(w^*) \cap \Phi_1^m(i) \cap \Phi_2^m(j), \ p, q \in [2^m], i, j \in [2^m]^N \right\}. \quad (17) \]
Intuitively, this family of sets is such that the finite \( \sigma \)-algebra \( \sigma(\mathcal{F}^m) \), is “fine enough” to approximate \( \mathcal{B} \), and also capture the behaviour of \( g \) and \( h \). Observe that, for \( m_1 < m_2 \), \( \mathcal{F}^{m_2} \) is a refined partition of \( \mathcal{F}^{m_1} \).

Suppose \( |\mathcal{F}^m| = D_m \); then we label the sets in \( \mathcal{F}^m \) arbitrarily as \( \{ T_i^m \}_{i \in [D_m]} \). Now, for each \((x,y) \in T_i^m \times T_j^m \), \( g^-(x,y) \) and \( g^+(x,y) \) are constant, depending only on \((i,j)\), and likewise, for each
We are now ready to define the urn process \((\mathcal{U}_n)_{n \in \mathbb{N}_0}\). For \(i \in \mathbb{N}\), set

\[ [D_m]^i := [D_m] \times [D_m] \times \cdots \times [D_m] = \{(u_0, \ldots, u_{i-1}) : u_0, \ldots, u_{i-1} \in [D_m]\}, \]

and

\[ \mathcal{B} := [D_m] \cup [D_m]^2 \cup ([D_m + 1] \times [D_m]); \]

this will represent the set of types in Urn E. We now define parameters \(\gamma\) such that, for \(x \in [D_m] \cup [D_m] \times [D_m]\),

\[
\gamma(x) = \begin{cases} 
\frac{\min(i,j)}{\max(i,j)}, & x = (i, j) \in [D_m]^2, g_{\max}(i, j) > 0; \\
\frac{\min(i)}{\max(i)}, & x = i \in [D_m], h_{\max}(i) > 0; \\
0, & \text{otherwise.}
\end{cases}
\] (19)

Then, we define the urn process \((\mathcal{U}^m_n)_{n \in \mathbb{N}_0}\) as the urn process with activities \(a\) such that

\[
a(x) = \begin{cases} 
g_{\max}(i, j) & \text{if } x = (i, j), i, j \in [D_m] \\
\gamma^*(j) & \text{if } x = (i, j), i = D_m + 1, j \in [D_m] \\
h_{\max}(i) & \text{if } x = i \in [D_m].
\end{cases}
\] (20)
and a replacement matrix \( M \) such that, for \( x, x' \in \mathcal{B} \),

\[
M_{x',x} = \begin{cases} 
(\gamma a)(x)p^m_{\ell}, & \text{if } x' = (i, \ell), x \in (\{i\} \times [D_m]) \cup \{i\}, i, \ell \in [D_m]; \\
(a - \gamma a)(x)p^m_{\ell}, & \text{if } x' = (D_m + 1, \ell), x \in \mathcal{B}; \\
a(x)p^m_{\ell}, & \text{if } x' = \ell, x \in \mathcal{B}; \\
0 & \text{otherwise.}
\end{cases}
\]

Note that it is not necessarily the case that \( M \) is irreducible: it may be the case that \( a \cdot p^m_{\ell} = 0 \) for certain \( x \in \mathcal{B} \) (this is possible if \( h_{\max}(i) = 0 \) or \( g_{\max}(i, j) = 0 \)), or it may be the case that \( p^m_{\ell} = 0 \) for certain choices of \( \ell \). We therefore define the following subsets of \( \mathcal{B} \):

\[ U_1 := \{ x \in \mathcal{B} : M_{x,x} = 0 \ \forall x' \in \mathcal{B} \} = \{ x \in \mathcal{B} : a(x) = 0 \}, \]

and

\[ \mathcal{U}_2 := \{ x' \in \mathcal{B} : M_{x',x} = 0 \ \forall x' \in \mathcal{B} \}. \]

Also assume that \( \mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset \); if not, we replace \( \mathcal{U}_1 \) by \( \mathcal{U}_1 \setminus \mathcal{U}_2 \). We then set \( R = \mathcal{B} \setminus (\mathcal{U}_1 \cup \mathcal{U}_2) \), and let \( M_R \) be the restriction of \( M \) to \( R \). It is easy to check that \( M_R \) is irreducible, and thus, by Lemma 2.2, has a unique largest positive eigenvalue \( \lambda_m \) with corresponding eigenvector \( u_R \). But then, writing \( M \) in block form (with columns and rows labelled by \( R, \mathcal{U}_1, \mathcal{U}_2 \)) for suitable matrices \( A, B, C \), we have

\[
M = \begin{pmatrix} R & U_1 & U_2 \\ M_R & 0 & B \\ A & 0 & C \\ 0 & 0 & 0 \end{pmatrix}.
\]

Thus, \( M \) has the same largest positive eigenvalue, with corresponding right eigenvector given (in block form) by

\[
u_m = \begin{bmatrix} u_R \\ \lambda_m^{-1} Au_R \\ 0 \end{bmatrix}.
\]

Here, we assume \( u_m \) is normalised so that \( a \cdot u_m = 1 \). In addition, (assuming we begin with a single ball \( x \in R \)), one readily verifies that the restriction of \( M \) to \( R \) and \( \mathcal{U}_1 \) satisfies conditions (A1)-(A6) of Subsection 2.1. Note also, that at each time-step the probability of adding a ball of type \( x \in \mathcal{U}_2 \) is 0 and thus, for each \( n \in \mathbb{N}_0 \), \( U_n(x) = 0 \) almost surely. Therefore, combining this fact with Theorem 2.1, we have the following corollary.

**Corollary 2.5.** With \( u_m, \lambda_m \) and \( R \) as defined above, assuming we begin with a ball \( x \in R \), we have

\[
U_n^m \xrightarrow{n \to \infty} \lambda_m u_m
\]

almost surely. In particular, almost surely

\[
a \cdot U_n^m \xrightarrow{n \to \infty} \lambda_m.
\]

In the coupling below, the assumption of a ball \( x \in R \) is met by the tree process being initiated by a vertex 0 with weight \( W_0 \) sampled at random from \( \mu \) and satisfying \( h(W_0) > 0 \).
2.2.2 Coupling Urn E with the Tree Process

For $A \in \mathcal{R} \otimes \mathcal{R}$, recall the definition of $\Xi^{(2)}(A, n)$ from Equation (2): this is the number of directed edges $(v, v')$ of $T_n$ where $(W_v, W_{v'}) \in A$.

**Proposition 2.6.** There exists a coupling $(\hat{U}^m_n)_{m \in \mathbb{N}^0}$ of the Pólya urn processes $\{(U^m_n)_{m \in \mathbb{N}}, m \in \mathbb{N}\}$ and the tree process $(T_n)_{n \in \mathbb{N}^0}$ such that, for each $m \in \mathbb{N}$, almost surely (on the coupling space), $\hat{U}^m_n$ consists of a single ball $\ell \in R$ and, in addition, for $(i, j) \in [D_m]^2$, we have

$$\hat{U}^m_n((i, j)) \leq \Xi^{(2)}(T^m_i \times T^m_j, n), \quad (23)$$

$$\sum_{(i, j) \in [D_m]^2} \left( \Xi^{(2)}(T^m_i \times T^m_j, n) - \hat{U}^m_n((i, j)) \right) = \sum_{j=1}^{D_m} \hat{U}^m_n((D_m + 1, j)), \quad (24)$$

and

$$(\gamma a) \cdot \hat{U}^m_n \leq Z_n \leq a \cdot \hat{U}^m_n. \quad (25)$$

for all $n \in \mathbb{N}^0$.

**Proof.** First sample the entire tree process $(\hat{T}_n)_{n \in \mathbb{N}^0}$; we will use this to define the evolution of the urn processes. Moreover, for $i \in [D_m]$ let

$$\eta_n(i) := \sum_{v \in T_n : r'(v) = i} f(N^+(v, T_n)),$$

i.e., the sum of fitnesses of vertices with weight belonging to $T^m_i$. Also, for $i \in [D_m]$ define

$$\theta_n(i) := (\gamma a \hat{U}^m_n)(i) + \sum_{j=1}^{D_m} (\gamma a \hat{U}^m_n)((i, j)).$$

Finally, recall that $Z_n$ denotes the partition function associated with the tree at time $n$. Assume that at time 0 the tree consists of a single vertex 0 such that $r(W_0) = \ell \in [D_m]$. Then, set $\hat{U}^m_0 = \delta_\ell$. Using the definition of $r$, since $W_0 \in T^m_\ell$

$$0 < Z_0 = h(W_0) \leq h_{\max}(\ell) = a \cdot \hat{U}^m_0,$$

and by the choice of $\gamma$, we have

$$\eta_0(\ell) = h(W_0) \geq h_{\min}(\ell) = (\gamma a \hat{U}^m_0)(\ell) = \theta_0(\ell).$$

In this case, (23) and (24) are trivially satisfied since both sides of both equations are 0. Now, assume inductively that after $n$ steps in the urn process, Equations (23) and (24) are satisfied, we have

$$\eta_n(k) \geq \theta_n(k) \quad \text{for each} \quad k \in [D_m], \quad (26)$$

and moreover, $Z_n \leq a \cdot \hat{U}^m_n$. Note that (26) implies the left hand side of (25), since

$$(\gamma a) \cdot \hat{U}^m_n = \sum_{k=1}^{D_m} \eta_n(k) \leq \sum_{k=1}^{D_m} \theta_n(k) = Z_n.$$

Let $s$ be the vertex sampled from $T_n$ in the $(n + 1)$st step, and assume that $r(W_s) = \ell'$, $r(W_{n+1}) = k$. Then, for the $(n + 1)$th step in the urn: sample an independent random variable $U_{n+1}$ uniformly distributed on $[0, 1]$. Then:
• If $U_{n+1} \leq \frac{\theta_n(\ell')\Xi_n}{\eta_n(\ell')a \cdot \hat{U}_n}$, add balls of type $(\ell', k)$ and $k$ to the urn (i.e., set $\hat{U}_{n+1}^m = \hat{U}_n^m + \delta(\ell', k) + \delta_k$).

• Otherwise, add balls of type $(D_m+1, k)$.

Note that, in the first case, we have

$$\Xi(\mathcal{T}_n^m \times \mathcal{T}_n^m, n+1) = \Xi(\mathcal{T}_n^m \times \mathcal{T}_n^m, n) + 1 \geq \hat{U}_n^m((\ell', k)) + 1 = \hat{U}_{n+1}^m((\ell', k))$$

and for $i \neq \ell'$ or $j \neq k$

$$\Xi(\mathcal{T}_n^m \times \mathcal{T}_n^m, n+1) = \Xi(\mathcal{T}_n^m \times \mathcal{T}_n^m, n) \geq \hat{U}_n^m((i, j)) = \hat{U}_{n+1}^m((i, j)).$$

Also, in this case

$$\eta_{n+1}(\ell') = \eta_n(\ell') + g(W_s, W_{n+1}) \geq \theta_n(\ell') + g_{\min}(\ell', k) = \theta_{n+1}(\ell'),$$

and similarly,

$$\eta_{n+1}(k) = \eta_n(k) + h(W_{n+1}) \geq \theta_n(k) + h_{\min}(k) = \theta_{n+1}(k),$$

so that Equation (26) is satisfied. Finally, in this case,

$$Z_{n+1} = Z_n + g(W_s, W_{n+1}) + h(W_{n+1}) \leq a \cdot \hat{U}_n^m + g_{\max}(\ell', k) + h_{\max}(k) = a \cdot \hat{U}_{n+1}^m.$$  

Meanwhile, in the second case $\Xi(\mathcal{T}_n^m \times \mathcal{T}_n^m, n)$ and $\eta_n(\ell')$ increase, while $\sum_{j=1}^{D_m} \hat{U}_n^m((\ell', j))$ and $\theta_n(\ell')$ remain the same, and thus (23) is satisfied and $\eta_{n+1}(\ell') \geq \eta_{n+1}(\ell')$. As this is the only case when $\Xi(\mathcal{T}_n^m \times \mathcal{T}_n^m, n) - \hat{U}_n^m((\ell', k))$ increases, and we add a ball of type $(D_m+1, k)$, (24) also follows. Both $\eta_n(k)$ and $\theta_n(k)$ increase as in the first case. Next,

$$Z_{n+1} = Z_n + g(W_s, W_{n+1}) + h(W_{n+1}) \leq a \cdot \hat{U}_n^m + g_{\max}(k) + h_{\max}(k) = a \cdot \hat{U}_{n+1}^m.$$  

As all other quantities remain the same, Equation (26) is satisfied, and moreover, $Z_{n+1} \leq a \cdot \hat{U}_{n+1}^m$.

To complete the proof, it remains to prove the following claim.

**Claim 2.6.1.** For each $m \in \mathbb{N}$, almost surely (on the coupling space), the urn process $\hat{U}_n^m = (\hat{U}_n^m)_{n \in \mathbb{N}_0}$ is distributed like the Pólya urn process $(U_n^m)_{n \in \mathbb{N}_0}$ with $U_0^m$ consisting of an initial ball $\ell \in R$.

**Proof.** First note that, since $W_0$ is sampled from $\mu$, conditionally on the positive probability event $\{h(W_0) > 0\}$, we have

$$\mathbb{P}(W_0 \in \mathcal{T}_n^m, h(W_0) > 0) \leq \mathbb{P}(W_0 \in \mathcal{T}_n^m) = p_n^m,$$

and thus, $\mathbb{P}$-a.s., we have $W_0 \in \mathcal{T}_n^m$ with $p_n^m > 0$. This, combined with the fact that $0 < h(W_0) \leq h_{\max}(\ell)$, implies that $\mathbb{P}$-a.s., the initial ball $\ell \in R$.

Now, note that in every step in $(\hat{U}_n^m)_{n \in \mathbb{N}_0}$, we add a ball of type $k$ for $k \in [D_m]$ with probability $p_k^m$, which is the same as in $(U_n^m)_{n \in \mathbb{N}_0}$. Moreover, given $\hat{U}_n^m$, the probability of adding balls of type $(k, \ell)$ is

$$p_k^m \left( \frac{\eta_n(k) \Xi_n}{\eta_n(k) a \cdot \hat{U}_n} \right) = p_k^m \left( \frac{\theta_n(k)}{a \cdot \hat{U}_n} \right),$$

which also agrees with the Pólya urn scheme. Finally, the probability of adding a ball of type $(D_m+1, \ell)$ is

$$p_k^m \left( \frac{\sum_{j=1}^{D_m} \left( 1 - \frac{\theta_n(j) \Xi_n}{\eta_n(j) a \cdot \hat{U}_n} \right) \eta_n(j) \Xi_n}{\sum_{j=1}^{D_m} \eta_n(j) a \cdot \hat{U}_n} \right) = p_k^m \left( 1 - \sum_{j=1}^{D_m} \frac{\theta_n(j)}{a \cdot \hat{U}_n} \right),$$

as required. \qed
Note also, that, since the functions $h^+, g^+$ are non-increasing pointwise in $m$, on the coupling we have that for any fixed $n$, $a \cdot U^m_n$ is non-increasing in $m$. Combining this result with Corollary 2.5, we have the following corollary.

**Corollary 2.7.** The sequence $(\lambda_m)_{m \in \mathbb{N}}$ is non-increasing in $m$. In particular, there exists a limit $\lambda_\infty \geq 0$ such that

$$
\lambda_m \downarrow \lambda_\infty
$$

as $m \to \infty$.

### 2.2.3 The Limiting Vectors of Urn Schemes associated with Urn E

We now calculate the limiting vector $u_m$ and the limiting eigenvalue $\lambda_m$. First note that by the definition of the urn process, for each $n \in \mathbb{N}_0$, $\ell \in [D_m]$ we have that $U^m_n(\ell) - U^m_n(\ell)$ is Bernoulli distributed with parameter $p^m_\ell$. Thus, by the strong law of large numbers and Corollary 2.5, we have, for each $\ell \in [D_m]$,

$$
u_m(\ell) = \frac{p^m_\ell}{\lambda_m}.

(27)

Next, for any $i, j \in [D_m]$ using the definitions of $\gamma$ and $a$ ((19) and (20)) we have

$$
\lambda_m u_m((i, j)) = p^m_j \sum_{\ell=1}^{D_m} (\gamma a u_m)((i, \ell)) + p^m_j (\gamma a u_m)(i)

= p^m_j \sum_{\ell=1}^{D_m} g_{\min}(i, \ell) u_m((i, \ell)) + p^m_j h_{\min}(i) u_m(i)

= p^m_j \sum_{\ell=1}^{D_m} g_{\min}(i, \ell) u_m((i, \ell)) + \frac{p^m_j p^m_i h_{\min}(i)}{\lambda_m}.

(28)

We now define

$$
A_i := \sum_{\ell=1}^{D_m} g_{\min}(i, \ell) u_m((i, \ell))

$$

Multiplying both sides of Equation (28) by $g_{\min}(i, j)$ and taking the sum over $j \in [D_m]$ (and recalling the definition of $\tilde{g}_-(i)$ in (18)), we get

$$
\lambda_m A_i = \left(A_i + \frac{p^m_i h_{\min}(i)}{\lambda_m} \sum_{j=1}^{D_m} p^m_j g_{\min}(i, j)\right)

= \left(A_i + \frac{p^m_i h_{\min}(i)}{\lambda_m} \tilde{g}_(i)\right).

$$

Thus, solving for $A_i$

$$
A_i = \frac{p^m_i h_{\min}(i) \tilde{g}_(i)}{\lambda_m (\lambda_m - \tilde{g}_(i))}.

(29)

$$
Substituting Equation (29) into Equation (28), we have
\[
\lambda_m u_m(i, j) = p_j^m \left( \frac{p_i^m h_{\min}(i) \bar{g}^-(i)}{\lambda_m (\lambda_m - \bar{g}^-(i))} + \frac{p_i^m h_{\min}(i)}{\lambda_m} \right) \\
= p_j^m \frac{p_i^m h_{\min}(i)}{\lambda_m - \bar{g}^-(i)}.
\]

Meanwhile, for each \( j \in [D_m] \) we have
\[
\lambda_m u_m(D_m + 1, j) = p_j^m \left( \sum_{\ell=1}^{D_m} (a u_m)((D_m + 1, \ell)) + \sum_{i=1}^{D_m} \sum_{\ell=1}^{D_m} (a - \gamma a)(i, \ell) + \sum_{i=1}^{D_m} (a - \gamma a)(i) \right) \\
= p_j^m \left( \sum_{\ell=1}^{D_m} g^*(\ell) u_m((D_m + 1, \ell)) + \sum_{i=1}^{D_m} \sum_{\ell=1}^{D_m} (g_{\max}(i, \ell) - g_{\min}(i, \ell)) u_m((i, \ell)) + \sum_{i=1}^{D_m} (h_{\max}(i) - h_{\min}(i)) u_m(i) \right) \\
=: p_j^m (\mathcal{B}_m + \mathcal{E}_m);
\]
where, in the last equation we set
\[
\mathcal{B}_m := \sum_{\ell=1}^{D_m} g^*(\ell) u_m((D_m + 1, \ell))
\]
and
\[
\mathcal{E}_m := \sum_{i=1}^{D_m} \sum_{\ell=1}^{D_m} (g_{\max}(i, \ell) - g_{\min}(i, \ell)) u_m((i, \ell)) + \sum_{i=1}^{D_m} (h_{\max}(i) - h_{\min}(i)) u_m(i).
\]
Multiplying both sides of (31) by \( g^*(j) \) and taking the sum over \( j \), we have
\[
\lambda_m \mathcal{B}_m = \left( \sum_{j=1}^{D_m} p_j^m g^*(j) \right) \mathcal{B}_m + \mathcal{E}_m = \bar{g}^+ (\mathcal{B}_m + \mathcal{E}_m)
\]
and thus
\[
\mathcal{B}_m = \frac{\bar{g}^+}{\lambda_m - \bar{g}^+} \mathcal{E}_m.
\]

We now apply Condition \( \text{C1} \) in the following lemma (all of the previous analysis implicitly applied to \( \text{C2} \)):

**Lemma 2.8.** Assume Conditions \( \text{C1} \) and \( \text{C2} \). Then, we have \( \lambda_\infty := \lim_{m \to \infty} \lambda_m > \bar{g}^+ \).

**Proof.** Note that, since we add two balls to the urn at each time-step, we have
\[
\|U_{m+1} - U_m\|_1 = 2.
\]
Thus, by Equation (21), we have \( \|\lambda_m u_m\|_1 = 2 \). Now, by Equation (27), we have \( \lambda_m \sum_{\ell=1}^{D_m} u_m(\ell) = 1 \), and thus, by Equation (30), we have
\[
\sum_{j=1}^{D_m} \sum_{i=1}^{D_m} \lambda_m u_m((i, j)) = \mathbb{E} \left[ \frac{h_{\min}(r(W))}{\lambda_m - \bar{g}^-(r(W))} \right] \leq 1.
\]
Note that as $m \to \infty$, $h_{\min}(r(W)) \uparrow h(W)$ and $\tilde{g}_-(r(W)) \uparrow \tilde{g}(W)$. Thus, by the monotone convergence theorem, we have

$$
E \left[ \frac{h(W)}{\lambda_{\infty} - \tilde{g}(W)} \right] = \lim_{m \to \infty} E \left[ \frac{h_{\min}(r(W))}{\lambda_m - \tilde{g}_-(r(W))} \right] \leq 1.
$$

Now, since the eigenvectors $u_m$ are non-negative, by Equation (32), we have

$$
\lambda_m \gtrsim \tilde{g}^*_+,
$$

and thus, $\lambda_{\infty} = \lim_{m \to \infty} \lambda_m \gtrsim \lim_{m \to \infty} \tilde{g}^*_+ = \tilde{g}^*$. But, if $\lambda_{\infty} = \tilde{g}^*$, since the expression in (4) is decreasing in $\lambda^*$, we would have a contradiction to Condition C1. The result follows. \hfill \Box

**Lemma 2.9.** Assume Conditions C1 and C2. Then, we have $B_m \downarrow 0$ and $E_m \downarrow 0$ as $m \to \infty$. In particular,

$$
E \left[ \frac{h(W)}{\lambda_{\infty} - \tilde{g}(W)} \right] = 1,
$$

so that $\lambda_{\infty} = \lambda^*$.

**Proof.** First, note that by Corollary 2.7 and Lemma 2.8, for each $m \in \mathbb{N}$, we have $\lambda_m \gtrsim \lambda_{\infty} > \tilde{g}^*$. Combining this fact with the boundedness of $g$ and $h$ we observe that

$$
\sup_{x,y \in [0, w^*]} \left\{ \frac{h(x)}{\lambda_m (\lambda_m - \tilde{g}(x))}, \frac{1}{\lambda_m} \right\} < \sup_{x,y \in [0, w^*]} \left\{ \frac{h(x)}{\tilde{g}^* (\lambda_{\infty} - \tilde{g}(x))}, \frac{1}{\lambda_{\infty}} \right\} =: C < \infty,
$$

where the bound on the right is independent of $m$. Now, given $\varepsilon > 0$, by applying Lemma 2.4, let $m$ be sufficiently large that

$$
\sup_{(x,y) \in [0, w^*] \times [0, w^*]} (g^+(x,y) - g^-(x,y)) < \frac{\varepsilon}{2C} \quad \text{and} \quad \sup_{x \in [0, w^*]} \left( h^+(x) - h^-(x) \right) < \frac{\varepsilon}{2C}.
$$

Then we have

$$
E_m = \sum_{i=1}^{D_m} \sum_{j=1}^{D_m} \left( g_{\max}(i,j) - g_{\min}(i,j) \right) u_m((i,j)) + \sum_{\ell=1}^{D_m} \left( h_{\max}(\ell) - h_{\min}(\ell) \right) u_m(\ell)
$$

$$
\overset{(27),(30)}{=} \sum_{i=1}^{D_m} \sum_{j=1}^{D_m} \left( g_{\max}(i,j) - g_{\min}(i,j) \right) \frac{h_{\min}(i) p_i^m p_j^m}{\lambda_m (\lambda_m - \tilde{g}_-(i))} + \sum_{\ell=1}^{D_m} \left( h_{\max}(\ell) - h_{\min}(\ell) \right) \frac{p_i^m}{\lambda_m}
$$

$$
< \frac{\varepsilon}{2C} \cdot C \left( \sum_{i=1}^{D_m} \sum_{j=1}^{D_m} p_i^m p_j^m \right) + \frac{\varepsilon}{2C} \cdot C \left( \sum_{\ell=1}^{D_m} p_i^m \right) = \varepsilon.
$$

The result for $B_m$ then follows from the fact that $\tilde{g}^*_+ \downarrow \tilde{g}^*$, and Lemma 2.8. \hfill \Box

We are now ready to prove our main results of this subsection.

**2.2.4 Proof of Theorem 1.1**

**Proof.** Note that, by Equation (25) from Proposition 2.6, we have

$$
0 \leq a \cdot U_n^m - Z_n \leq (a - \gamma a) \cdot U_n^m.
$$

Dividing by $n$ and taking limits as $n \to \infty$, by Equation (22) we have

$$
0 \leq \lambda_m - \limsup_{n \to \infty} \frac{Z_n}{n} \leq \lambda_m - \liminf_{n \to \infty} \frac{Z_n}{n} \leq \limsup_{n \to \infty} \left( (a - \gamma a) \cdot \frac{U_n^m}{n} \right) = B_m + E_m.
$$

The result follows by applying Lemma 2.9. \hfill \Box
In addition, recalling the definition of $\mathcal{F}^m$ from Equation (17), note that

$$\sigma(\mathcal{F}^m) = \left\{ S \subseteq [0, w^*] : S = \bigcup_{i \in I} I^m_i, I \subseteq [D_m] \right\};$$

(i.e. the $\sigma$-algebra generated by $\mathcal{F}^m$ is the set of finite unions of sets in $\mathcal{F}^m$). Recalling that $\mathcal{F}^{m_2}$ is a refined partition of $\mathcal{F}^{m_1}$ for $m_1 < m_2$, by Lemma 2.3 we have

$$\sigma(\mathcal{F}^{m_1}) \subseteq \sigma(\mathcal{F}^{m_2}).$$

We now prove Theorem 1.2.

### 2.2.5 Proof of Theorem 1.2

**Proof.** We begin by proving the result for Cartesian products of the form $S \times S'$ with $S, S' \in \sigma(\mathcal{F}^{m'})$, for $m' \in \mathbb{N}$. Note that, by the definition of $\Xi^{(2)}(\cdot, n)$, we clearly have finite additivity, that is, for any $S_1, S_2, S_3 \in \mathcal{B}$ if $S_1 \cap S_2 = \emptyset$, we have

$$\Xi^{(2)}((S_1 \cup S_2) \times S_3, n) = \Xi^{(2)}(S_1 \times S_3, n) + \Xi^{(2)}(S_2 \times S_3, n),$$

and similarly,

$$\Xi^{(2)}(S_3 \times (S_1 \cup S_2), n) = \Xi^{(2)}(S_3 \times S_1, n) + \Xi^{(2)}(S_3 \times S_2, n).$$

Combining these facts with Proposition 2.6, Corollary 2.5 and Equation (30), for sets $S \times S'$ with $S, S' \in \sigma(\mathcal{F}^{m'})$ we have, for each $m > m'$,

$$\mathbb{E} \left[ \frac{h^{-}(W)}{\lambda_m - \tilde{g}_{-} (r(W))} 1_S \right] \mu(S') \leq \liminf_{n \to \infty} \frac{\Xi^{(2)}(S \times S', n)}{n} \leq \limsup_{n \to \infty} \frac{\Xi^{(2)}(S \times S', n)}{n} \leq \mathbb{E} \left[ \frac{h^{-}(W)}{\lambda_m - \tilde{g}_{-} (r(W))} 1_S \right] \mu(S') + \mathcal{B}_m + \mathcal{E}_m.$$

Taking limits as $m \to \infty$ and applying Lemma 2.9, this proves the result for this family of sets.

Now, by the Portmanteau Theorem, we need only prove that for all sets $U \in \mathcal{O}$ (where $\mathcal{O}$ denotes the class of open subsets of $[0, w^*] \times [0, w^*]$),

$$\liminf_{n \to \infty} \frac{\Xi^{(2)}(U, n)}{n} \geq (\psi_* \mu \times \mu)(U).$$

Now, let

$$I^m(U) := \bigcup_{i,j \in [D_m]: I^m_i \times I^m_j \subseteq U} I^m_i \times I^m_j.$$

Note that, since $U$ is open, and $\mathcal{F}^m$ is fine enough that the set of dyadic intervals $\{I^m_i(w^*)\}_{i \in [2^m]} \subseteq \sigma(\mathcal{F}^m)$, we have

$$1_{I^m(U)} \uparrow 1_U \quad \text{pointwise as } m \to \infty.$$

In addition, since $I^m(U) \subseteq U$, for each $m \in \mathbb{N}$,

$$(\psi_* \mu \times \mu)(I^m(U)) = \liminf_{n \to \infty} \frac{\Xi^{(2)}(I^m(U), n)}{n} \leq \liminf_{n \to \infty} \frac{\Xi^{(2)}(U, n)}{n}.$$

Equation (35) then follows by taking limits as $m \to \infty$. \qed
2.3 Urn D

In order to analyse the degree distribution in this model under Conditions C1 and C2, we introduce another collection of Pólya urns \((\mathcal{V}_n^{K'})_{n \in \mathbb{N}_0}\), which not only depend on \(m\), but also depends on a parameter \(K' \in \mathbb{N}\). These may be regarded as finite approximations of Urn D. For brevity of notation, wherever possible in this subsection we will omit the dependence of these parameters on \(m\). For \(i \in \mathbb{N}\), define \([D_m]^i\) so that

\[
[D_m]^i := \{(u_0, \ldots, u_{i-1}) : u_0, \ldots, u_{i-1} \in [D_m]\}.
\]

Now, we set

\[
B' := \left( \bigcup_{i=1}^{K'+1} [D_m]^i \right) \cup ([D_m] + 1 \times [D_m]).
\]

The urn process \((\mathcal{V}_n^{K'})_{n \geq 0}\) is then a vector-valued stochastic process taking values in \(\mathbb{V}_{B'}\). We now define the vectors \(\mathbf{a}', \gamma'\) associated with the urn process such that

\[
\mathbf{a}'(x) = \begin{cases} 
    h_{\text{max}}(u_0) + \sum_{j=1}^{k} g_{\text{max}}(u_0, u_j) & \text{if } x = (u_0, \ldots, u_k) \in [D_m]^{k+1} \\
    g_{\text{max}}(\ell) & \text{if } x = (D_m + 1, \ell);
\end{cases}
\]

and

\[
\gamma'(x) = \begin{cases} 
    \frac{h_{\text{min}}(u_0) + \sum_{j=1}^{k} g_{\text{min}}(u_0, u_j)}{h_{\text{max}}(u_0) + \sum_{j=1}^{k} g_{\text{max}}(u_0, u_j)} & \text{if } x = (u_0, \ldots, u_k) \in [D_m]^{k+1}, k < K', \mathbf{a}'(x) > 0; \\
    0 & \text{otherwise}.
\end{cases}
\]

Now, given \(u = (u_0, \ldots, u_k) \in [D_m]^{k+1}, k < K', \) and \(\ell \in [D_m]\), we define their concatenation \((u, \ell) \in [D_m]^{k+2}\) such that

\[
(u, \ell) := (u_0, \ldots, u_k, \ell).
\]

Then, we define the replacement matrix \(M'\) of the urn \((\mathcal{V}_n^{K'})_{n \in \mathbb{N}_0}\) such that, given \(x, x' \in B'\),

\[
M'_{x',x} = \begin{cases} 
    -\langle \gamma' \mathbf{a}' \rangle(x) & \text{if } x' = x, x \in [D_m]^k, k \leq K'; \\
    \langle \gamma' \mathbf{a}' \rangle(x)p_\ell^m & \text{if } x' = (x, \ell), \ell \in [D_m], x \in B'; \\
    (\mathbf{a}' - \gamma' \mathbf{a}')'(x)p_\ell^m & \text{if } x' = (D_m + 1, \ell), \ell \in [D_m], x \in B'; \\
    \mathbf{a}'(x)p_\ell^m & \text{if } x' = \ell, x \in B'; \\
    0 & \text{otherwise}.
\end{cases}
\]

Again, note that it may be the case that \(M'\) is not irreducible, if either \(\mathbf{a}'(x) = 0\) for certain \(x \in B'\) or \(p_\ell^m = 0\) for certain choices of \(\ell\). Nevertheless, we define the sets

\[
\mathcal{U}_1 := \{ x \in B' : M'_{x',x} = 0 \ \forall x' \in B' \} = \{ x \in B' : \mathbf{a}'(x) = 0 \},
\]

and

\[
\mathcal{U}_2 := \{ x' \in B' : M'_{x',x} = 0 \ \forall x \in B' \setminus \{x'\} \}.
\]

Again, we assume that \(\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset\); if not, we replace \(\mathcal{U}_1\) by \(\mathcal{U}_1 \setminus \mathcal{U}_2\). We then set \(R' = B' \setminus (\mathcal{U}_1 \cup \mathcal{U}_2)\), and let \(M'_{R'}\) be the restriction of \(M'\) to \(R'\). As before, \(M'_{R'}\) satisfies the conditions of Lemma 2.2, and thus has a unique largest positive eigenvalue \(\lambda'_{R'}\) with corresponding eigenvector \(\mathbf{v}_{R'}\). But then,
writing $M'$ in block form in a manner analogous to the previous subsection, $M$ has the same largest positive eigenvalue, with corresponding right eigenvector given (in block form) by

$$
\mathbf{V}_K' = \begin{bmatrix}
\mathbf{V}_{R'}' \\
(\lambda_{R'}')^{-1} A'\mathbf{V}_{R'}'
\end{bmatrix}.
$$

Here, we assume $\mathbf{V}_K'$ is normalised so that $a' \cdot \mathbf{V}_K' = 1$. Also in a manner similar to the previous subsection, assuming we begin with a ball of type $x \in R'$, one readily verifies that the restriction of $M'$ to $R'$ and $R_1'$ satisfies conditions (A1)-(A6) of Subsection 2.1, and also, that for each $x \in R_2'$ and $n \in \mathbb{N}_0$, $\mathcal{U}_n(x) = 0$ almost surely. Therefore, applying Theorem 2.1 again, we have the following corollary:

**Corollary 2.10.** With $\mathbf{V}_K', \lambda_{K}'$, and $R'$ as defined above, assuming we begin with a ball $x \in R'$, we have

$$
\frac{\mathbf{V}_K'}{n} \xrightarrow{n \to \infty} \lambda_{K}' \mathbf{V}_K'.
$$

almost surely. In particular, we have

$$
\frac{\mathbf{a} \cdot \mathbf{V}_K'}{n} \xrightarrow{n \to \infty} \lambda_{K}'.
$$

(37)

As in the previous subsection, in the coupling below, the assumption of a ball $x \in R'$ is met by the tree process being initiated by a vertex $0$ with weight $W_0$ sampled at random from $\mu$ and satisfying $h(W_0) > 0$.

### 2.3.1 Coupling Urn D with the Tree Process

Recall that we denote by $N_{\geq k}(B, n)$ the number of vertices of out-degree at least $k$ having weight belonging to $B \in \mathcal{B}$. We also define the analogue $\mathcal{D}_{\geq k}(j, \mathbf{V}_n)$ for $n \in \mathbb{N}_0$ and $j \in [D_m]$ such that

$$
\mathcal{D}_{\geq k}(j, n) := \sum_{j=k}^{K'+1} \sum_{\mathbf{u}_j \in [D_m]} \mathbf{V}^j_n(\mathbf{u}_j) \mathbf{1}_{\{j_1(u_0)\}}.
$$

(38)

This represents the number of balls in the urn $\mathcal{V}^j_n$ with type $\mathbf{u} = (u_0, \ldots)$ having dimension at least $k + 1$, with $u_0 = j$. We then have the following analogue of Proposition 2.6:

**Proposition 2.11.** There exists a coupling $(\hat{\mathbf{V}}^j_n, \hat{T}_n)_{n \in \mathbb{N}_0}$ of the Pólya urn process $(\mathbf{V}_n)$ and the tree process $(T_n)_{n \in \mathbb{N}_0}$ such that, almost surely (on the coupling space), $\mathbf{V}_0$ consists of a single ball $\ell \in R'$ and for all $n \in \mathbb{N}_0$, $k \in \{0\} \cup [K']$, we have

$$
\mathcal{D}_{\geq k}(j, n) \leq N_{\geq k}(T_j^m, n) \quad \text{and} 
$$

$$
\sum_{j=1}^{D_m} (N_{\geq k}(T_j^m, n) - \mathcal{D}_{\geq k}(j, n)) \leq \sum_{j=1}^{D_m} \hat{\mathbf{V}}^j_n((D_m + 1, j)).
$$

(39)

(40)

In addition, we have

$$
(\gamma' a') \cdot \hat{\mathbf{V}}^j_n \leq Z_n \leq a' \cdot \hat{\mathbf{V}}^j_n.
$$

(41)
Proof. We proceed in a somewhat similar manner to Proposition 2.6, however, in this case, we first introduce a “labelled” Pólya urn \((\mathcal{L}_n)_{n \geq 0}\) where balls carry integer labels from \(-D_m, \ldots, 0, \ldots, n\). In addition, for \(j \in \{0\} \cup [n]\), the label is independent of the type of the ball; we denote by \(b_j(n)\) the type of a ball with label \(j\) at time \(n\). One may interpret the ball with label \(j\) as representing the evolution of vertex \(j\) in the tree process - in this sense, the label may be interpreted as a “time-stamp”.

Balls of type \((D_m + 1, j)\), \(j \in [D_m]\), however, are labelled \(-j\) - we denote by \(d_j = d_j(n)\) the number of balls with this label (since here, multiple balls may share the same label). We describe the labelled urn process \(\mathcal{L}_n\) as an evolving vector in \(\mathbb{B}^* \times \mathbb{Z}\), so that \(\mathcal{L}_n = \sum_{j=1}^{D_m} d_j \cdot \delta(b_j(n), j) + \sum_{i=0}^{n} \delta(b_i(n), i)\). We set

\[
a'(\mathcal{L}_n) = \sum_{j=-D_m}^{-1} d_j \cdot a'(b_j(n)) + \sum_{i=0}^{n} a'(b_i(n)), \quad \text{and} \quad (\gamma a')(\mathcal{L}_n) = \sum_{i=0}^{n} (\gamma a')(b_i(n)).
\]

Now, we use \(\mathcal{L}_{n+1}\) to define \(\hat{\gamma}^{K'}_{n+1}\) by “forgetting” labels, so that,

\[
\text{if } \mathcal{L}_{n+1} = \sum_{j=-D_m}^{-1} d_j \cdot \delta(b_{j(n+1), j}) + \sum_{i=0}^{n+1} \delta(b_{i(n+1), i}), \text{ we set } \hat{\gamma}^{K'}_{n+1} = \sum_{j=-D_m}^{-1} d_j \cdot \delta(b_{j(n+1)}) + \sum_{i=0}^{n+1} \delta(b_{i(n+1)}).
\]

Sample the entire tree process \((\hat{T}_n)_{n \in \mathbb{N}_0}\). If, at time 0, the tree consists of a single vertex 0 with weight \(W_0 \in I^m\), then, we set \(\mathcal{L}_0 = \delta(\ell, 0)\), and note that we have

\[
(\gamma a')(\mathcal{L}_0) = h_{\min}(\ell) \leq h(W_0) = \mathcal{Z}_0 \leq a'(\mathcal{L}_0) = h_{\max}(\ell),
\]

and

\[
f(N^+(0, \hat{T}_0)) = h(W_0) \geq (\gamma a')(b_0(0)) = h_{\min}(\ell).
\]

Now, assume inductively that after \(n\) steps in the process, for each \(i \in \{0\} \cup [n]\) we have

\[
f(N^+(i, \hat{T}_n)) \geq (\gamma a')(b_i(n)), \quad \deg^+(i, T_n) \geq \dim(b_i(n)) - 1, \quad (42)
\]

\[
\sum_{i=0}^{n} \left(\deg^+(i, T_n) - \dim(b_i(n)) + 1\right) = \sum_{j=1}^{D_m} \hat{\gamma}^{K'}_{n}((D_m + 1, j), \quad (43)
\]

and Equation (41) is satisfied.

Let \(s\) be the vertex sampled in the tree in the \((n+1)\)st step, assume that \(r(s) = \ell'\) and that \(r(s+1) = k\). Then, for the \((n+1)\)th step in the urn: sample an independent random variable \(U_{n+1}\) uniformly distributed on \([0, 1]\). Then:

- If \(\dim(b_s(n)) \leq K'\) and \(U_{n+1} \leq \frac{(\gamma a')(b_s(n), s)\mathcal{Z}_n}{f(N^+(s, \hat{T}_n))a'(\mathcal{L}_n)}\), remove the ball \((b_s(n), s)\) from the urn, and add balls \(((b_s(n), k), s)\) and \((k, n+1)\) to the urn (i.e. set \(\mathcal{L}_{n+1} = \mathcal{L}_n + \delta((b_s(n), \ell), s) + \delta(k, n+1) - \delta((b_s(n), s)\)). We call this step Case 1.
- Otherwise, add balls of type \(((D_m + 1, k), -k), (k, n+1)\) - we call this Case 2.

First note that

\[
(\gamma a')(b_s(n+1)) - (\gamma a')(b_s(n)) = \begin{cases} g_{\min}(\ell', k), & \text{in Case 1} \\ 0, & \text{in Case 2} \end{cases}
\]

\[
\leq g(W_s, W_{n+1}) = f(N^+(s, \hat{T}_{n+1})) - f(N^+(s, \hat{T}_n)),
\]

and likewise

\[
(\gamma a')(b_{n+1}(n+1)) = h_{\min}(\ell) \leq h(W_{n+1}) = f(N^+(n+1, \hat{T}_{n+1})).
\]
Additionally, in Case 1 the dimension of \( b_s(n) \) and the degree of \( s \) in \( \hat{T}_n \) both increase, whilst in Case 2 only the degree of \( s \) increases whilst the dimension of \( b_s(n) \) remains the same. This proves Equation (42) at time \( n + 1 \). In addition, Case 2 coincides with the addition of a ball of type \((D_m + 1, \ell)\), which yields Equation (43). Finally, \( (\gamma' a') \cdot (\hat{\nu}_{n+1}^{K'} - \hat{\nu}_n^{K'}) = \begin{cases} h_{\min}(k) + g_{\min}(\ell', k), & \text{in Case 1} \\ h_{\min}(k), & \text{in Case 2} \end{cases} \leq h(W_{n+1}) + g(W_s, W_{n+1}) = \mathcal{Z}_{n+1} - \mathcal{Z}_n \leq \begin{cases} h_{\max}(k) + g_{\max}(\ell', k), & \text{in Case 1} \\ h_{\max}(k) + g_{\max}^*(k), & \text{in Case 2} \end{cases} \leq (a') \cdot (\hat{\nu}_{n+1}^{K'} - \hat{\nu}_n^{K'}); \] which shows that Equation (41) is also satisfied at time \( n + 1 \).

**Claim 2.11.1.** Almost surely (on the coupling space), the urn process \( \hat{\nu}_n^{K'} = (\hat{\nu}_n^{K'})_{n \in \mathbb{N}_0} \) is distributed like the Pólya urn \( (\nu_n^{K'})_{n \in \mathbb{N}_0} \) with \( \nu_0^{K'} \) consisting of an initial ball \( \ell \in R' \).

**Proof.** The fact that, \( \mathbb{P} \)-a.s., the initial ball \( \ell \in R' \) follows immediately from the fact that the initial weight \( W_0 \) is sampled from \( \mu \) conditionally on the event \( \{h(W_0) > 0\} \) (analogous to Claim 2.6.1). Moreover, in every step in \( \hat{\nu}_n^{K'} \), we add a ball of type \( k \) for \( k \in [D_m] \) with probability \( p_k^m \), which is the same as in \( \nu_n^{K'} \). Furthermore, given \( \hat{\nu}_n^{K'} \) the probability of removing a ball of type \( u \) with \( \text{dim } u \leq K' \) and adding a ball of type \( (u, \ell) \) is

\[
 p_k^m \sum_{s \in \mathcal{L} : b_s(n) = u} \frac{(\gamma' a')(b_s(n))Z_n}{f(N^+(s, \hat{T}_n))a'(L_n)} \times \frac{f(N^+(s, \hat{T}_n))}{Z_n} = p_k^m \sum_{s \in \mathcal{L} : b_s(n) = u} \frac{(\gamma' a')(b_s(n))}{a'(L_n)}
 = p_k^m \frac{\hat{\nu}_n^{K'}(u)}{Z_n},
\]

which also agrees with the transition law of the Pólya urn scheme \( \mathcal{V} \). Finally, the probability of adding a ball of type \( (D_m + 1, \ell) \) is

\[
 p_k^m \sum_{s \in \mathcal{L} : \text{dim } b_s(n) > K'} \frac{f(N^+(s, \hat{T}_n))}{Z_n} + p_k^m \sum_{s \in \mathcal{L} : \text{dim } b_s(n) \leq K'} \left(1 - \frac{(\gamma' a')(b_s(n))Z_n}{f(N^+(s, \hat{T}_n))a'(L_n)}\right) \times \frac{f(N^+(s, \hat{T}_n))}{Z_n}
 = p_k^m \sum_{s \in \mathcal{L} \backslash b_s(n) \leq K'} \left(1 - \frac{(\gamma' a')(\hat{\nu}_n^{K'}(u))}{a'(\hat{\nu}_n^{K'})}\right),
\]

which agrees with transition rule of \( \nu_n^{K'} \).

Finally, to complete the proof, we verify the following claim.

**Claim 2.11.2.** For all \( n \in \mathbb{N}_0 \), Equations (39) and (40) are satisfied for all \( k \in \{0\} \cup [K'] \).
Proof. If we define \( b_i(n) \mid_0 \) such that \( b_i(n) \mid_0 = x_0 \) if \( b_i(n) = (x_0, \ldots, x_k) \), then, by construction of the labelled urn process \((\mathcal{L}_n)_{n \in \mathbb{N}_0}, b_i(n) \mid_0 = x_0 \implies r(W_i) = x_0 \), so that \( W_i \in \mathcal{I}_m^\ell \). Therefore, for each \( k \in \{0\} \cup [K'], j \in [D_m], \)
\[
\mathcal{D}_{\geq k}(j, n) = \sum_{b_i(n): \dim(b_i(n)) \geq k} 1_{\{j\}}(b_i(n) \mid_0)^{(42)} \sum_{i: \deg(i, \hat{T}_n) \geq k} 1_{\mathcal{I}_j^m}(W_i) = N_{\geq k}(\mathcal{I}_j^m, n).
\]
Moreover, by (43),
\[
\sum_{j=1}^{D_m} \hat{V}_n^{K'}((D_m + 1, j)) = \sum_{i=0}^{n} \left( \deg^+(i, \hat{T}_n) - \dim(b_i(n)) + 1 \right)
= \sum_{k=0}^{n} \sum_{j=1}^{D_m} \left( (N_{\geq k}(\mathcal{I}_j^m, n) - \mathcal{D}_{\geq k}(j, n)) \right),
\]
which implies (40). \( \square \)

2.3.2 Analysis of the Pólya urn \( \mathcal{V}^{K'} \)

We now calculate the limiting vector \( \mathbf{V}_K \) and limiting eigenvalue \( \lambda'_K \) of the Pólya urn scheme \( \mathcal{V}^{K'}_{n \geq 0} \).

We first introduce some more notation: for any vector \( \mathbf{u} = (u_0, \ldots, u_{k-1}) \in [D_m]^k \), and \( i \in [k] \), denote by \( \mathbf{u} \mid_i := (u_0, \ldots, u_{i-1}) \in [D_m]^i \). We also define the following quantities:

\[
\mathcal{R}^{K'} := \sum_{\ell=1}^{D_m} \mathbf{a}'((D_m + 1, \ell)) \mathbf{V}_{K'}((D_m + 1, \ell)), \quad \mathcal{E}^{K'} := \sum_{\mathbf{u} : \dim \mathbf{u} \leq K'} (\mathbf{a}' - \gamma \mathbf{a}') (\mathbf{u}) \mathbf{V}_{K'}(\mathbf{u}),
\]

and

\[
\mathcal{F}^{K'} := \sum_{\mathbf{v} : \dim \mathbf{v} = K'+1} \mathbf{a}'(\mathbf{v}) \mathbf{V}_{K'}(\mathbf{v}).
\]

Proposition 2.12. Let \( \lambda'_K \) and \( \mathbf{V}_{K'} \) denote the limiting leading eigenvalue and corresponding right eigenvector of \( M' \), respectively. Then, \( \mathbf{V}_{K'} \) satisfies

\[
\lambda'_K \mathbf{V}_{K'}(x) = \begin{cases} 
  p^m_{\ell^m}, & x = \ell \in [D_m]; \\
  p^m_{\ell^m} \lambda'_K \mathbf{V}_{K'}(\mathbf{u}) \prod_{i=1}^{k} \left[ p^m_{u_i} \left( \frac{\gamma / u_i}{\mathbf{V}_{K'}}(\mathbf{u}) \right) \right], & x = \mathbf{u} = (u_0, \ldots, u_k) \in [D_m]^{k+1}, k < K'; \\
  p^m_{\ell^m} \prod_{i=1}^{k} \left[ p^m_{u_i} \left( \frac{\gamma / u_i}{\mathbf{V}_{K'}}(\mathbf{u}) \right) \right], & x = \mathbf{u} = (u_0, \ldots, u_K) \in [D_m]^{K'+1}.
\end{cases}
\]

In addition, we have

\[
\mathcal{R}^{K'} = \frac{\mathcal{E}^{K'} + \mathcal{F}^{K'}}{\lambda'_K - g^+_x}.
\]

Proof. First note that, as before, for each \( \ell \in [D_m] \), since we add a ball of type \( \ell \) with probability \( p^m_{\ell^m} \) at each time-step, we have

\[
\lambda'_K \mathbf{V}_{K'}(\ell) = p^m_{\ell^m},
\]

29
Proof. Note that by Equation (46), with Lemma 2.13.

\[
\lambda_{K'} V_{K'}(u) = \begin{cases} 
  p_{u_k}^m (\gamma' a') (u_k | u) V_{K'} (u_k), & u \in [D_m]^{k+1}, k \in [K' - 1]; \\
  p_{u_k}^m (\gamma' a') (u) V_{K'} (u), & u \in [D_m]^{K' + 1},
\end{cases}
\]

so that, if \( u \in [D_m]^{k+1}, k \in [K' - 1], \)

\[
V_{K'}(u) = \frac{p_{u_k}^m (\gamma' a') (u_k | u)}{\lambda_{K'} + (\gamma' a')(u)}.
\]

Applying Equations (48) and (49), recursing backwards, and using the fact that \( V_{K'}(u_0) = \frac{p_{u_0}^m}{\lambda_{K'}}, \) completes the proof of Equation (46). Finally, for each \( j \in [D_m], \) we have

\[
\lambda_{K'} V_{K'}((D_m + 1, j)) = p_j^m \left( \sum_{\ell=1}^{D_m} a'((D_m + 1, \ell)) V_{K'}((D_m + 1, \ell)) + \sum_{u : \dim u \leq K'} (a' - \gamma' a')(u) V_{K'}((u)) \right) \\
+ \sum_{v : \dim v = K' + 1} a'(v) V_{K'}(v)
\]

where, in the last equation we recall the definitions in Equations (44) and (45). Now, multiplying both sides of Equation (50) by \( a'((D_m + 1, j)) = g^*(j) \) and taking the sum over \( j, \) we have

\[
\lambda_{K'} R_{K'} = \left( \sum_{j=1}^{D_m} p_j^m g^*(j) \right) (R_{K'} + \mathcal{E}_{K'} + F_{K'}) = \bar{g}^* (R_{K'} + \mathcal{E}_{K'} + F_{K'}). 
\]

Rearranging this proves Equation (47), thus completing the proof of the proposition. \( \Box \)

Now, we recall the definition of the companion process \((S_i(w))_{i \geq 0}\) from Subsection 1.3: Recall that \( W_1, W_2, \ldots \) were defined to be independent \( \mu \)-distributed random variables and let \( w \in [0, w^*]. \) We then define the random process \((S_i(w))_{i \geq 0}\) inductively so that

\[
S_0(w) := h(w); \quad S_{i+1}(w) := S_i(w) + g(w, W_{i+1}), i \geq 0.
\]

Now, we also define the lower companion process \((S_{i}^-(w))_{i \geq 0}\) in a similar way, but with functions \( h^-; g^- \) instead.

**Lemma 2.13.** Assume Conditions C1 and C2. Then we have

\[
\lim_{K' \to \infty} \lim_{m \to \infty} \mathcal{F}_{K'} = 0.
\]

**Proof.** Note that by Equation (46), with \( J' \) being an upper bound on \( \max \{ h, g \}, \) we have

\[
\mathcal{F}_{K'} = \sum_{u : \dim u = K' + 1} a'(u) V_{K'}(u) = \sum_{u : \dim u = K' + 1} a'(u) p_{uo}^m \prod_{i=1}^{K'} \left[ p_{ui}^m \left( \frac{(\gamma' a')(u_i)}{(\gamma' a')(u_i) + \lambda_{K'}} \right) \right]
\]

\[
\leq J'(K' + 1) \cdot \sum_{u : \dim u = K' + 1} p_{uo}^m \prod_{i=1}^{K'} \left[ p_{ui}^m \left( \frac{(\gamma' a')(u_i)}{(\gamma' a')(u_i) + \lambda_{K'}} \right) \right]
\]

\[
= J'(K' + 1) \cdot \mathbb{E} \left[ \prod_{i=0}^{K'-1} \left( \frac{S_{i}^-(W)}{S_{i}^-(W) + \lambda_{K'}} \right) \right].
\]

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Now, note that for all \( m \in \mathbb{N} \), \( S^{-}(W) \) is stochastically bounded above by \( S(W) \), and by Theorem 1.1 and Equations (37) and (41) \( \lambda'_{K'} \) is bounded below by \( \lambda^* \) uniformly in \( m \) and \( K' \). Therefore, since the function \( x \mapsto \frac{x}{x+\lambda} \) is increasing in \( x \) and decreasing in \( \lambda \), we may bound the previous display above so that

\[
J'(K' + 1) \cdot \mathbb{E} \left[ \prod_{i=0}^{K'-1} \left( \frac{S^{-}_i(W)}{S^+_i(W) + \lambda'_{K'}} \right) \right] \leq J'(K' + 1) \cdot \mathbb{E} \left[ \prod_{i=0}^{K'-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] \]

We complete the proof by proving the following claim.

**Claim 2.13.1.** We have

\[
\lim_{k \to \infty} k \cdot \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] = 0
\]

**Proof.** First observe that

\[
\mathbb{E} \left[ \prod_{i=0}^{\infty} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] \leq \prod_{i=1}^{\infty} \left( \frac{J'_i}{J'_i + \lambda^*} \right) = \prod_{i=0}^{\infty} \left( 1 - \frac{\lambda^*}{J'_i + \lambda^*} \right) \leq e^{-\sum_{i=1}^{\infty} \frac{\lambda^*}{J'_i + \lambda^*} } = 0.
\]

Therefore, we have

\[
k \cdot \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] = k \cdot \sum_{j=k}^{\infty} \mathbb{E} \left[ \left( 1 - \frac{S_j(W)}{S_j(W) + \lambda^*} \right) \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right]
\]

\[
= k \cdot \sum_{j=k}^{\infty} \mathbb{E} \left[ \frac{\lambda^*}{S_j(W) + \lambda^*} \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right]
\]

\[
\leq \sum_{j=k}^{\infty} j \cdot \mathbb{E} \left[ \frac{\lambda^*}{S_j(W) + \lambda^*} \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right].
\]

The series on the right of the previous display consists of non-negative terms, and for each \( N \in \mathbb{N} \), we have

\[
\sum_{j=1}^{N} j \cdot \mathbb{E} \left[ \frac{\lambda^*}{S_j(W) + \lambda^*} \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right]
\]

\[
= \sum_{j=1}^{N} \left[ j \cdot \mathbb{E} \left[ \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] - j \cdot \mathbb{E} \left[ \prod_{i=0}^{j} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] \right]
\]

\[
= \sum_{j=1}^{N} \mathbb{E} \left[ \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] - N \cdot \mathbb{E} \left[ \prod_{i=0}^{N} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right]
\]

\[
\leq \sum_{j=1}^{N} \mathbb{E} \left[ \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right].
\]

Now, note that by Lemma 1.4, we have

\[
\sum_{j=1}^{\infty} \mathbb{E} \left[ \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] < \infty,
\]
and thus by (51) and the monotone convergence theorem, we also have
\[
\sum_{j=1}^{\infty} j \cdot \mathbb{E} \left[ \frac{\lambda^*}{S_j(W) + \lambda^*} \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] < \infty.
\]
Therefore,
\[
\lim_{k \to \infty} k \cdot \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] \leq \lim_{k \to \infty} \sum_{j=k}^{\infty} j \cdot \mathbb{E} \left[ \frac{\lambda^*}{S_j(W) + \lambda^*} \prod_{i=0}^{j-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*} \right) \right] = 0.
\]

\[\square\]

**Lemma 2.14.** Assume Conditions **C1** and **C2**. Then we have
\[
\lim_{K' \to \infty} \lim_{m \to \infty} \mathcal{E}_{K'} = 0, \quad \text{and} \quad \lim_{K' \to \infty} \lim_{m \to \infty} \mathcal{R}_{K'} = 0.
\]  
(52)

In addition,
\[
\lim_{K' \to \infty} \lim_{m \to \infty} \lambda_{K'} = \lambda^*.
\]
(53)

**Proof.** The proof is similar to that of Lemma 2.9. First, let \(\varepsilon > 0\) be given, and, by Lemma 2.4, let \(m\) be sufficiently large that
\[
\sup_{(x,y) \in [0,w^*]^2} (g^+(x,y) - g^-(x,y)) < \frac{\varepsilon \lambda^*}{2(K')^2} \quad \text{and} \quad \sup_{x \in [0,w^*]} (h^+(x) - h^-(x)) < \frac{\varepsilon \lambda^*}{2(K')^2}.
\]
(54)

Now, we have
\[
\mathcal{E}_{K'} = \sum_{\ell \in [D_m]} (h_{\max} (\ell) - h_{\min} (\ell)) V_{K'}(\ell) + \sum_{u \in [D_m]} \sum_{i=2}^{K'} ((a' - \gamma a')(u_i)) V_{K'}(u_i)
\]
\[
= \sum_{\ell \in [D_m]} (h_{\max} (\ell) - h_{\min} (\ell)) \frac{p^m_\ell}{\lambda K'} + \sum_{u \in [D_m]} \sum_{i=2}^{K'} ((a' - \gamma a')(u_i)) V_{K'}(u_i).
\]

By applying Equation (46) again we may write the previous equation as
\[
= \sum_{\ell \in [D_m]} (h_{\max} (\ell) - h_{\min} (\ell)) \frac{p^m_\ell}{\lambda K'} + \sum_{u \in [D_m]} \sum_{i=2}^{K'} ((a' - \gamma a')(u_i)) \frac{p^m u_{i}}{\lambda K'} \prod_{j=1}^{i} p^m_{u_j} \left( \frac{(\gamma a')(u_j)}{(\gamma a')(u_j) + \lambda K'} \right)
\]
\[
< \frac{\varepsilon \lambda^*}{2} \left( \sum_{\ell \in [D_m]} \frac{p^m_\ell}{\lambda K'} \right) + \sum_{u \in [D_m]} \sum_{i=2}^{K'} \frac{((a' - \gamma a')(u_i))}{\lambda K'} \prod_{j=1}^{i} p^m_{u_j},
\]
(55)

Also note that, by Equation (54), for any \(u = (u_0, \ldots, u_{K'-1}) \in [D_m]^{K'}\), and each \(i \in \{2, \ldots, K'\}\) we have
\[
(a' - \gamma a')(u_i) = h_{\max} (u_0) - h_{\min} (u_0) + \sum_{j=1}^{i-1} (g_{\max} (u_0, u_j) - g_{\min} (u_0, u_j)) < \frac{\varepsilon \lambda^*}{2(K')^2} \cdot K' = \frac{\varepsilon \lambda^*}{2K'}.
\]
In addition, noting that uniformly in \(K'\) and \(m\) we have \(\lambda'_{K'} \geq \lambda^*\) (as in the proof of Lemma 2.13) and thus, we may bound (55) by
\[
\frac{\varepsilon}{2} + \frac{\varepsilon}{2K'} \left( \sum_{i=2}^{K'} \sum_{u \in [D_m]} p_{u_0}^m \prod_{j=1}^{i} p_{u_j}^m \right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2K'} \cdot K' = \varepsilon.
\]
First sending \(m \to \infty, \varepsilon \to 0,\) and \(K' \to \infty\) implies the first equation in (52). Next, Equation (47), Lemma 2.13, and the facts that \(\lambda'_{K'} \geq \lambda^*\) and \(\lim_{m \to \infty} \tilde{g}^* = \tilde{g}^* < \lambda^*\) together imply the second limit in (52). Finally, by Equation (41) and Proposition 2.12 we have
\[
\lambda'_{K'} - \lambda^* \leq \mathcal{E}_{K'} + \sum_{\dim u = K' + 1} (a' - \gamma' a')(u) + \mathcal{R}_{K'}
\]
\[
\leq \mathcal{E}_{K'} + \mathcal{F}_{K'} + \mathcal{R}_{K'}.
\]
Equation (53) then follows by taking limits as \(m \to \infty\) and \(K' \to \infty\).

2.3.3 Proof of Theorem 1.3

Proof. First (recalling the definition of \(\mathcal{D}_{\geq k}(\cdot, n)\) from (38)), by Proposition 2.12 for any \(\ell \in [D_m]\) we have
\[
\lim_{n \to \infty} \frac{\mathcal{D}_{\geq k}(\ell, n)}{n} = \sum_{j=k+1}^{K' \kappa+1} \sum_{u \in [D_m]}^{K' \kappa+1} \mathbb{V}_{K'}(u_{ij}) \mathbf{1}_{(\ell)}(u_0)
\]
\[
= \sum_{u \in [D_m]}^{K' \kappa+1} \left( p_{u_0}^m p_{u_{j}}^m \prod_{i=1}^{j} p_{u_i}^m \left( \frac{(\gamma' a')(u_{ij})}{(\gamma' a')(u_{ij}) + \lambda'_{K'}} \right) \right)
\]
\[
+ \sum_{j=k+1}^{K' \kappa+1} p_{u_0}^m \lambda'_{K'} + (\gamma' a')(u_{ij}) \prod_{i=1}^{j-1} p_{u_i}^m \left( \frac{(\gamma' a')(u_{ij})}{(\gamma' a')(u_{ij}) + \lambda'_{K'}} \right) \mathbf{1}_{(\ell)}(u_0).
\]
Now, by the definitions of the functions \(g^-, h^-\) and the definition of expectation, we may write the last equation as
\[
= \mathbb{E} \left[ \prod_{i=0}^{K' \kappa-1} \left( \frac{S_i^-(W)}{S_i^-(W) + \lambda'_{K'}} \right) \mathbf{1}_{T_i} \right] + \sum_{j=k+1}^{K' \kappa+1} \mathbb{E} \left[ \frac{\lambda'_{K'}}{S_{j-1}(W) + \lambda'_{K'}} \prod_{i=0}^{j-2} \left( \frac{S_i^-(W)}{S_i^-(W) + \lambda'_{K'}} \right) \mathbf{1}_{T_i} \right]
\]
\[
= \mathbb{E} \left[ \prod_{i=0}^{K' \kappa-1} \left( \frac{S_i^-(W)}{S_i^-(W) + \lambda'_{K'}} \right) \mathbf{1}_{T_i} \right] + \sum_{j=k+1}^{K' \kappa+1} \mathbb{E} \left[ \left( 1 - \frac{S_{j-1}(W)}{S_{j-1}(W) + \lambda'_{K'}} \right) \prod_{i=0}^{j-2} \left( \frac{S_i^-(W)}{S_i^-(W) + \lambda'_{K'}} \right) \mathbf{1}_{T_i} \right]
\]
\[
= \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i^-(W)}{S_i^-(W) + \lambda'_{K'}} \right) \mathbf{1}_{T_i} \right].
\]
(56)
For \(m' \in \mathbb{N}\), (56) allows us to prove the result for sets \(S \in \sigma(\mathcal{C}^m)\) (where we recall the definition of \(\mathcal{C}^m\) in Equation (17), and the facts that \(\sigma(\mathcal{C}^m)\) consists of finite unions, and is increasing in \(m'\) - Equations (33) and (34)). Since \(N(\cdot, n)\) is finitely additive, if \(S \in \sigma(\mathcal{C}^m)\), by Equation (39) and Equation (56) we have
\[
\mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i^-(W)}{S_i^-(W) + \lambda'_{K'}} \right) 1_S(W) \right] \leq \liminf_{n \to \infty} \frac{N_{\geq k}(S, n)}{n} \leq \limsup_{n \to \infty} \frac{N_{\geq k}(S, n)}{n} \leq \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i^-(W)}{S_i^-(W) + \lambda'_{K'}} \right) 1_S(W) \right] + \mathcal{R}_{K'} + \mathcal{E}_{K'} + \mathcal{F}_{K'}.
\]
Taking limits as $m \to \infty$ and then as $K' \to \infty$, and applying Lemma 2.13 and Lemma 2.14 now proves the result for sets in $\sigma(\mathcal{F}^m)$. Now, note that for each $k \in \mathbb{N}_0$, and measurable sets $S' \in \mathcal{B}$, we have

$$\limsup_{n \to \infty} \frac{N_{\geq k}(S')}n \leq \limsup_{n \to \infty} \frac{N_{\geq 0}(S')}n = \mu(S'),$$

(57)

where the last equality applies the strong law of large numbers.

We now prove the result for sets $U \in \mathcal{O}$ where $\mathcal{O}$ denotes the class of all open subsets of $[0, w^*)$. For a fixed open set $U \in \mathcal{O}$, and $m \in \mathbb{N}$, recall that $\mathcal{I}^m(U) := \bigcup_{j \in [D_m]} \mathcal{I}^m_{jU}$. Also recall Equation (36), which states that $1_{\mathcal{I}^m(U)} \uparrow 1_U$ pointwise as $m \to \infty$. Now, since each $\mathcal{I}^m(U) \in \sigma(\mathcal{F}^m)$, by applying Equation (57) for each $k \leq K'$ we have

$$\mathbb{E}\left[ \prod_{i=0}^{k-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^{K'}} \right) 1_{\mathcal{I}^m(U)} \right] \leq \liminf_{n \to \infty} \frac{N_{\geq k}(U, n)}n \leq \limsup_{n \to \infty} \frac{N_{\geq k}(U, n)}n \leq \mathbb{E}\left[ \prod_{i=0}^{k-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^{K'}} \right) 1_{\mathcal{I}^m(U)} \right] + \mu(U \setminus \mathcal{I}^m(U)).$$

Taking limits as $m \to \infty$ and then $K' \to \infty$ now proves the result for sets belonging to $\mathcal{O}$.

Finally, note that since $\mu$ is a regular measure, for any $A \in \mathcal{B}$ we have

$$\mu(A) = \inf_{U \in \mathcal{O} : A \subseteq U} \{\mu(U)\}.$$

Thus, for a given measurable set $A$, and any $\varepsilon > 0$, there exists an open set $U_\varepsilon$ such that

$$\mu(U_\varepsilon \setminus A) \leq \varepsilon.$$

Therefore by finite additivity and Equation (57)

$$\lim_{n \to \infty} \frac{N_{\geq k}(U_\varepsilon, n)}n - \varepsilon \leq \liminf_{n \to \infty} \frac{N_{\geq k}(A, n)}n \leq \limsup_{n \to \infty} \frac{N_{\geq k}(A, n)}n \leq \lim_{n \to \infty} \frac{N_{\geq k}(U_\varepsilon, n)}n.$$

The proof for the general case now follows by applying the result for the class $\mathcal{O}$, and sending $\varepsilon \to 0$. 

Theorem 1.3 now allows us to prove Theorem 1.5.

### 2.3.4 Proof of Theorem 1.5

Note that, if $N_k(A, n)$ denotes the number of vertices of out-degree $k$ in the tree at time $n$ having weight in $A$, by counting the edges in the tree in two ways we have

$$\Xi(A, n) = \sum_{k=1}^n kN_k(A, n) = \sum_{k=1}^n N_{\geq k}(A, n).$$

But now, Lemma 1.4 and using Fatou's Lemma in the last inequality, we have,

$$(\psi^*_w)\mu(A) = \mathbb{E}\left[ \frac{h(W)}{\lambda^*_w - \bar{g}(W)} 1_A \right] = \sum_{k=1}^\infty \mathbb{E}\left[ \prod_{i=0}^{k-1} \left( \frac{S_i(W)}{S_i(W) + \lambda^*_w} \right) 1_A \right] = \sum_{k=1}^\infty \liminf_{n \to \infty} \frac{N_{\geq k}(A, n)}n \leq \liminf_{n \to \infty} \frac{\Xi(A, n)}n;$$

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and likewise, \( \liminf_{n \to \infty} \frac{\Xi(A, n)}{n} \geq (\psi \ast \mu)(A^c) \). Now, since we add one edge at each time-step, it follows that \( \Xi([0, w^*], n) = n \). Thus, by finite additivity,

\[
1 = \liminf_{n \to \infty} \left( \frac{\Xi(A, n)}{n} + \frac{\Xi(A^c, n)}{n} \right) \leq \limsup_{n \to \infty} \frac{\Xi(A, n)}{n} + \liminf_{n \to \infty} \frac{\Xi(A^c, n)}{n} \leq \limsup_{n \to \infty} \left( \frac{\Xi(A, n)}{n} + \frac{\Xi(A^c, n)}{n} \right) = 1.
\]

But, since Equation (4) implies that \( (\psi \ast \mu)(\cdot) \) is a probability measure, this is only possible if

\[
\limsup_{n \to \infty} \frac{\Xi(A, n)}{n} = (\psi \ast \mu)(A) \quad \text{and} \quad \liminf_{n \to \infty} \frac{\Xi(A^c, n)}{n} = (\psi \ast \mu)(A^c) \quad \text{almost surely}.
\]

The result follows.

### 3 The Condensation Regime

Here, we extend the results of the previous section to the condensation regime. The techniques applied in this section are closely related to those of [24].

The results of this subsection will depend on a sequences of auxiliary trees \( T^{(\varepsilon)}, T^{(-\varepsilon)}, \varepsilon > 0 \). Given \( \varepsilon > 0 \), and \( M_\varepsilon \) as defined in Equation (10), define the functions \( g_\varepsilon, g_{-\varepsilon} \) such that

\[
g_\varepsilon(p, q) := 1_{M_\varepsilon}(p)g(p, q) + 1_{M_\varepsilon}(p)g(x^*, q)
\]

and

\[
g_{-\varepsilon}(p, q) := 1_{M_\varepsilon}(p)g(p, q) + 1_{M_\varepsilon}(p)(g(x^*, q) - u_\varepsilon(q));
\]

and let \( T^{(\varepsilon)}, T^{(-\varepsilon)} \) be the evolving trees with measure \( \mu \), and associated functions \( g_\varepsilon, h \) and \( g_{-\varepsilon}, h \) respectively. We also denote by \((Z_n^{(\varepsilon)})_{n \geq 0} \) and \((Z_n^{(-\varepsilon)})_{n \geq 0} \) the partition functions associated with \( T^{(\varepsilon)}, T^{(-\varepsilon)} \), respectively.

**Lemma 3.1.** Assume Conditions **D1-D4**. Then, for each \( \varepsilon > 0 \) sufficiently small, \( T^{(\varepsilon)} \) and \( T^{(-\varepsilon)} \) satisfy Conditions **C1** and **C2**. In addition, if \( \lambda_\varepsilon, \lambda_{-\varepsilon} \) denote the Malthusian parameters associated with \( T^{(\varepsilon)}, T^{(-\varepsilon)} \), then \( \lambda_\varepsilon \downarrow \bar{g}^* \) and \( \lambda_{-\varepsilon} \uparrow \bar{g}^* \) as \( \varepsilon \downarrow 0 \).

**Proof.** First, since by **D2** \( g \) satisfies Condition **C2**, we have

\[
g(x, y) = \kappa \left( \phi_1^{(1)}(x), \ldots, \phi_1^{(N)}(x), \phi_2^{(1)}(y), \ldots, \phi_2^{(N)}(y) \right),
\]

for measurable functions \( \phi_j^i : [0, w^*] \to [0, J], \; j = 1, 2, \; i \in [N] \) and a bounded continuous function \( \kappa : [0, J]^{2N} \to \mathbb{R}_+ \). Now, if we set \( \phi_1^{(N+1)}(x) := 1_{M_\varepsilon}(x), \phi_1^{(N+2)}(x) := 1_{M_\varepsilon}(x), \phi_2^{(N+1)}(y) := g(x^*, y) - u_\varepsilon(y) \) and define \( \kappa' \) such that

\[
\kappa'(c_1, \ldots, c_{N+2}, d_1, \ldots, d_{N+1}) := c_{N+2} \kappa(c_1, \ldots, c_N, d_1, \ldots, d_N) + c_{N+1} d_{N+1},
\]

we clearly have that \( \phi_1^{(N+1)}, \phi_1^{(N+2)}, \phi_2^{(N+1)} \) are bounded, non-negative measurable functions, and \( \kappa' \) is bounded and continuous, taking values in \( \mathbb{R}_+ \). Noting that

\[
g_{-\varepsilon}(x, y) = \kappa' \left( \phi_1^{(1)}(x), \ldots, \phi_1^{(N+2)}(x), \phi_2^{(1)}(y), \ldots, \phi_2^{(N+1)}(y) \right),
\]

...
it follows that \( g_{-\varepsilon} \) satisfies Condition C2. The proof of C2 for \( g_\varepsilon \) is similar.

For C1, since \( h \) is bounded, for sufficiently large \( \lambda > \hat{g}^* \), we have
\[
\mathbb{E} \left[ \frac{h(W)}{\lambda - \hat{g}_\varepsilon(W)} \right] < 1.
\]

Meanwhile, since, by Condition D4, \( \mu(\mathcal{M}_\varepsilon) > 0 \) and \( \hat{g}_\varepsilon(x) = \hat{g}^* \) for any \( x \in \mathcal{M}_\varepsilon \), by monotone convergence
\[
\lim_{\lambda \downarrow \hat{g}^*} \mathbb{E} \left[ \frac{h(W)}{\lambda - \hat{g}_\varepsilon(W)} \right] = \mathbb{E} \left[ \frac{h(W)}{\hat{g}^* - \hat{g}_\varepsilon(W)} \right] = \infty.
\]

Thus, by continuity in \( \lambda \), Condition C1 is satisfied for \( \mathcal{T}^{(\varepsilon)} \). A similar argument also works for \( \mathcal{T}^{(-\varepsilon)} \): if \( \hat{g}_{-\varepsilon}^* \) denotes the maximum value of \( \hat{g}_{-\varepsilon}(x) \), then this value is also attained on \( \mathcal{M}_\varepsilon \) which has positive measure. If \( \lambda_{\varepsilon}, \lambda_{-\varepsilon} \) denote the associated Malthusian parameters associated with the trees, then, for each \( \varepsilon > 0 \), \( \lambda_\varepsilon > \hat{g}^* \) and \( \lambda_{-\varepsilon} > \hat{g}_{-\varepsilon}^* \). Moreover, since \( g_\varepsilon \) is non-increasing pointwise as \( \varepsilon \) decreases, \( \lambda_\varepsilon \) is non-increasing in \( \varepsilon \); likewise, \( \lambda_{-\varepsilon} \) is non-decreasing in \( \varepsilon \). Now, suppose \( \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon = \lambda_+ > \hat{g}^* \). Then we may apply dominated convergence, and
\[
1 = \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[ \frac{h(W)}{\lambda_\varepsilon - \hat{g}_\varepsilon(W)} \right] = \mathbb{E} \left[ \lim_{\varepsilon \downarrow 0} \frac{h(W)}{\lambda_\varepsilon - \hat{g}_\varepsilon(W)} \right] = \mathbb{E} \left[ \frac{h(W)}{\lambda_+ - \hat{g}(W)} \right],
\]
contradicting Equation (9). The case for \( \lambda_{-\varepsilon} \) follows identically. \( \square \)

**Lemma 3.2.** There exists a coupling \((\hat{T}^{(-\varepsilon)}, \hat{T}, \hat{T}^{(\varepsilon)})\) of these processes such that, almost surely (on the coupling space), for all \( n \in \mathbb{N}_0 \),
\[
Z^{(-\varepsilon)}_n \leq Z_n \leq Z^{(\varepsilon)}_n, \tag{58}
\]
and, for each vertex \( v \) with \( W_v \in \mathcal{M}_\varepsilon \), we have
\[
f(N^+(v, \hat{T}^{(\varepsilon)}_n)) \leq f(N^+(v, \hat{T}_n)) \leq f(N^+(v, \hat{T}^{(-\varepsilon)}_n)) \tag{59}
\]
and
\[
\deg(v, \hat{T}^{(\varepsilon)}_n) \leq \deg(v, \hat{T}_n) \leq \deg(v, \hat{T}^{(-\varepsilon)}_n). \tag{60}
\]

**Proof.** We initialise the trees with a single vertex 0 having weight \( W_0 \) sampled independently from \( \mu \), conditioned on \( \{h(W_0) > 0\} \) and will construct copies of these three tree processes on the same vertex set, which is identified with \( \mathbb{N}_0 \). Now, assume that at the \( n \)th time-step,

\[(\hat{T}^{(-\varepsilon)}_j)_{0 \leq j \leq n} \sim (\hat{T}^{(-\varepsilon)}_j)_{0 \leq j \leq n}, \quad (\hat{T}^{(\varepsilon)}_j)_{0 \leq j \leq n} \sim (\hat{T}^{(\varepsilon)}_j)_{0 \leq j \leq n}, \quad \text{and} \quad (\hat{T}^{(-\varepsilon)}_j)_{0 \leq j \leq n} \sim (\hat{T}^{(-\varepsilon)}_j)_{0 \leq j \leq n}.\]

In addition, assume that Equations (58) and (59) are satisfied up to time \( n \). Now, for the \((n+1)\)st step:

- Introduce vertex \( n+1 \) with weight \( W_{n+1} \) sampled independently from \( \mu \) in \( \hat{T}^{(-\varepsilon)}_n, \hat{T}_n \) and \( \hat{T}^{(\varepsilon)}_n \).
- Form \( \hat{T}^{(-\varepsilon)}_{n+1} \) by sampling the parent \( v \) of \( n+1 \) independently according to the law of \( \mathcal{T}^{(-\varepsilon)} \) (i.e. with probability proportional to \( f(N^+(v, \hat{T}^{(-\varepsilon)}_n)) \)). Then, in order to form \( \hat{T}^{(\varepsilon)}_{n+1} \) sample an independent uniformly distributed random variables \( U_1 \) on \([0,1]\).
  - If \( U_1 \leq \frac{Z^{(-\varepsilon)}_n f(N^+(v, \hat{T}^{(-\varepsilon)}_n))}{Z_n f(N^+(v, \hat{T}^{(-\varepsilon)}_n))} \) and \( W_v \in \mathcal{M}_\varepsilon \), select \( v \) as the parent of \( n+1 \) in \( \hat{T}^{(-\varepsilon)}_{n+1} \) as well.
  - Otherwise, form \( \hat{T}^{(\varepsilon)}_{n+1} \) by selecting the parent \( v' \) of \( n+1 \) with probability proportional to \( f(N^+(v', \hat{T}_n)) \) out of all the vertices with weight \( W_{v'} \in \mathcal{M}_\varepsilon \).
• Then form \( \hat{T}_{n+1}^{(\varepsilon)} \) in a similar manner. Sample an independent uniform random variable \( U_2 \) on \([0, 1]\).

  - If vertex \( v \) (with weight \( W_v \in \mathcal{M}_\varepsilon \)) was chosen as the parent of \( n + 1 \) in \( \hat{T}_{n+1} \) and \( U_2 \leq \frac{Z_n f(N^+(v, \hat{T}_n^{(\varepsilon)}))}{Z_n f(N^+(v, \hat{T}_n))} \), also select \( v \) as the parent of \( n + 1 \) in \( \hat{T}_{n+1}^{(\varepsilon)} \).

  - Otherwise, form \( \hat{T}_{n+1}^{(\varepsilon)} \) by selecting the parent \( v'' \) of \( n + 1 \) with probability proportional to \( f(N^+(v'', \hat{T}_n^{(\varepsilon)})) \) out of all the vertices with weight \( W_w \in \mathcal{M}_\varepsilon \).

Clearly \( \hat{T}_{n+1}^{(\varepsilon)} \sim \hat{T}_{n+1}^{(\varepsilon)} \). On the other hand, in \( \hat{T}_{n+1} \) the probability of choosing a certain parent \( v \) of \( n + 1 \) with weight \( W_v \in \mathcal{M}_\varepsilon \) is

\[
\frac{Z_n^{(-\varepsilon)} f(N^+(v, \hat{T}_n))}{Z_n f(N^+(v, \hat{T}_n^{(-\varepsilon)}))} \frac{f(N^+(v, \hat{T}_n^{(-\varepsilon)}))}{Z_n^{(-\varepsilon)}} = \frac{f(N^+(v, \hat{T}_n))}{Z_n},
\]

whilst the probability of choosing a parent \( v' \) with weight \( W_{v'} \in \mathcal{M}_\varepsilon \) is

\[
\frac{f(N^+(v', \hat{T}_n))}{\sum_{v':W_{v'} \in \mathcal{M}_\varepsilon} f(N^+(v', \hat{T}_n))} \left( \sum_{v:W_v \in \mathcal{M}_\varepsilon} \left( 1 - \frac{Z_n^{(-\varepsilon)} f(N^+(v, \hat{T}_n))}{Z_n f(N^+(v, \hat{T}_n^{(-\varepsilon)}))} \right) \frac{f(N^+(v, \hat{T}_n^{(-\varepsilon)}))}{Z_n^{(-\varepsilon)}} \right) + \sum_{v':W_{v'} \in \mathcal{M}_\varepsilon} \frac{f(N^+(v', \hat{T}_n))}{f(N^+(v', \hat{T}_n))} \left( \sum_{v:W_v \in \mathcal{M}_\varepsilon} \frac{f(N^+(v, \hat{T}_n^{(-\varepsilon)}))}{Z_n^{(-\varepsilon)}} \right) - \sum_{v:W_v \in \mathcal{M}_\varepsilon} \frac{f(N^+(v, \hat{T}_n))}{Z_n} = \frac{f(N^+(v', \hat{T}_n))}{\sum_{v':W_{v'} \in \mathcal{M}_\varepsilon} f(N^+(v', \hat{T}_n))} \frac{Z_n}{Z_n^{(-\varepsilon)}} = \frac{f(N^+(v', \hat{T}_n))}{Z_n},
\]

where we use the fact that \( \sum_v f(N^+(v, \hat{T}_n)) = Z_n \). Thus, we have \( \hat{T}_{n+1}^{(\varepsilon)} \sim \hat{T}_{n+1} \). Now, note that if the parent \( v \) of \( n + 1 \) in \( \hat{T}_{n+1}^{(-\varepsilon)} \) is such that \( W_v \in \mathcal{M}_\varepsilon \), the same parent is chosen in \( \hat{T}_{n+1}^{(-\varepsilon)} \). Since \( W_v \in \mathcal{M}_\varepsilon \), we have

\[
f(N^+(v, \hat{T}_{n+1}^{(-\varepsilon)})) - f(N^+(v, \hat{T}_n^{(-\varepsilon)})) = g_{-\varepsilon}(W_v, W_{n+1}) = g(W_v, W_{n+1}) = f(N^+(v, \hat{T}_{n+1})) - f(N^+(v, \hat{T}_n)).
\]

Otherwise, the parent of \( n + 1 \) in \( \hat{T}_{n+1} \) has weight which belongs to \( \mathcal{M}_\varepsilon \), and thus \( f(N^+(v, \hat{T}_{n+1}^{(-\varepsilon)})) \) increases whilst \( f(N^+(v, \hat{T}_n)) \) stays the same. An increase in \( f(N^+(v, \hat{T}_{n+1}^{(-\varepsilon)})) \) coincides with the increase of \( \text{deg}(v, \hat{T}_{n+1}^{(-\varepsilon)}) \), and thus the right hand sides of Equations (59) and (60) are satisfied for time \( n + 1 \).

Now, note that

\[
Z_{n+1}^{(-\varepsilon)} - Z_n^{(-\varepsilon)} = h(W_{n+1}) + g_{-\varepsilon}(W_v, W_{n+1}), \quad \text{and} \quad Z_{n+1} - Z_n = h(W_{n+1}) + g(W_{v'}, W_{n+1})
\]

where \( v, v' \) denote the parent of \( n + 1 \) in \( \hat{T}_n \) and \( \hat{T}_n^{(\varepsilon)} \) respectively. Then we either have:

- \( v = v' \) (so that \( g_{-\varepsilon}(W_v, W_{n+1}) = g(W_{v'}, W_{n+1}) \)),
- \( v \in \mathcal{M}_\varepsilon \) and \( v' \in \mathcal{M}_\varepsilon \), in which case, \( \mathbb{P} \)-a.s. using D4

\[
g_{-\varepsilon}(W_v, W_{n+1}) = g(W_v, W_{n+1}) \leq g(x^*, W_{n+1}) - u_{\varepsilon}(W_{n+1}) < g(W_{v'}, W_{n+1}),
\]

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• Both \(v, v' \in \mathcal{M}_\varepsilon\), in which case, \(\mathbb{P}\text{-a.s.,}
\)

\[
g_{\varepsilon}(W_v, W_{n+1}) = g(x^n, W_{n+1}) - u_{\varepsilon}(W_{n+1}) < g(W_{v'}, W_{n+1}).
\]

In every case we have \(Z_{n+1}^{(-\varepsilon)} - Z_n^{(-\varepsilon)} \leq Z_{n+1} - Z_n\), and thus Equation (58) is also satisfied at time \(n + 1\).

Each of the statements concerning \(\tilde{\mathcal{T}}^{(\varepsilon)}\) follow in an analogous manner, applying Condition \(\text{D3}\). \(\square\)

### 3.1 Proof of Theorem 1.6

**Proof.** First note that by Equation (58) in Lemma 3.2 (and Theorem 1.1), for each \(\varepsilon > 0\) we have, \(\mathbb{P}\text{-a.s.,}
\)

\[
\lambda_\varepsilon = \lim_{n \to \infty} \frac{Z_n^{(-\varepsilon)}}{n} \leq \liminf_{n \to \infty} \frac{Z_n}{n} \leq \limsup_{n \to \infty} \frac{Z_n}{n} = \lim_{n \to \infty} \frac{Z_n^{(\varepsilon)}}{n} = \lambda_\varepsilon.
\]

The result follows by sending \(\varepsilon \to 0\), using Lemma 3.1. \(\square\)

In the following theorem, recall the definition of the measure \(\psi_\varepsilon \mu\) in Equation (5).

### 3.2 Proof of Theorem 1.7

**Proof.** By assumption, for each \(\varepsilon > 0\) sufficiently small, we have \(A \subseteq \mathcal{M}_\varepsilon^c\). Next, applying Equation (60), if \(\Xi^{(\varepsilon)}\) and \(\Xi^{(-\varepsilon)}\) denote the edge distributions in the coupled trees \(\tilde{\mathcal{T}}^{(\varepsilon)}, \tilde{\mathcal{T}}^{(-\varepsilon)}\), respectively, then for each \(n \in \mathbb{N}_0\)

\[
\Xi^{(\varepsilon)}(A, n) \leq \Xi(A, n) \leq \Xi^{(-\varepsilon)}(A, n),
\]

and thus, by Theorem 1.5, we have

\[
\mathbb{E} \left[ \frac{h(W)}{\lambda_\varepsilon - \bar{g}_\varepsilon(W)} 1_A \right] \leq \liminf_{n \to \infty} \frac{\Xi(A, n)}{n} \leq \limsup_{n \to \infty} \frac{\Xi(A, n)}{n} \leq \mathbb{E} \left[ \frac{h(W)}{\lambda_\varepsilon - \bar{g}_\varepsilon(W)} 1_A \right]. \tag{61}
\]

Now, noting that \(\bar{g}_{-\varepsilon} = \bar{g} = \bar{g}_\varepsilon\) on \(A\), and \(\lambda_{-\varepsilon} > \bar{g}_{-\varepsilon} = \sup_{x \in A} \bar{g}(x)\) and is non-decreasing in \(\varepsilon\), by applying Lemma 3.1 and dominated convergence we have

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \frac{h(W)}{\lambda_\varepsilon - \bar{g}_\varepsilon(W)} 1_A \right] = \lim_{\varepsilon \to 0} \mathbb{E} \left[ \frac{h(W)}{\lambda_{-\varepsilon} - \bar{g}_{-\varepsilon}(W)} 1_A \right] = \mathbb{E} \left[ \frac{h(W)}{\bar{g}_1 - \bar{g}(W)} 1_A \right]. \tag{62}
\]

Equation (11) follows by combining Equations (61) and (62). Moreover, for each \(\varepsilon' > 0\), by setting \(A = \mathcal{M}_{\varepsilon'}\),

\[
\lim_{n \to \infty} \frac{\Xi(\mathcal{M}_{\varepsilon'}, n)}{n} = \lim_{n \to \infty} \left[ 1 - \frac{\Xi(\mathcal{M}_{\varepsilon'}, n)}{n} \right] = 1 - \mathbb{E} \left[ \frac{h(W)}{\bar{g}_1 - \bar{g}(W)} 1_{\mathcal{M}_{\varepsilon'}} \right].
\]

But then, again by dominated convergence,

\[
\lim_{\varepsilon' \to 0} \mathbb{E} \left[ \frac{h(W)}{\bar{g}_1 - \bar{g}(W)} 1_{\mathcal{M}_{\varepsilon'}} \right] = \mathbb{E} \left[ \frac{h(W)}{\bar{g}_1 - \bar{g}(W)} \right],
\]

and Equation (12) follows. \(\square\)
3.3 Proof of Corollary 1.8

Proof. By the Portmanteau theorem, it suffices to show that, \( \mathbb{P} \)-a.s.

\[
\lim_{n \to \infty} \frac{\Xi(A, n)}{n} = \Pi(A),
\]

for any set \( A \in \mathcal{B} \) with \( \mu(\partial A) = 0 \). Now, since \( \mu(\mathcal{M}) = 0 \), it suffices to prove this equation for sets \( A \in \mathcal{B} \) with \( A \cap \mathcal{M} = \emptyset \). In view of Theorem 1.7, we need only show that for all \( \varepsilon > 0 \) sufficiently small, we have \( A \cap \mathcal{M}_\varepsilon = \emptyset \). Indeed, if this were not the case, then, since \( (A \cap \mathcal{M}_{1/n})_{n \in \mathbb{N}} \) is a nested sequence of closed sets, by Cantor's intersection theorem,

\[
\emptyset \neq \bigcap_{n \in \mathbb{N}} (A \cap \mathcal{M}_{1/n}) = A \cap \bigcap_{n \in \mathbb{N}} \mathcal{M}_{1/n} = A \cap \mathcal{M},
\]

a contradiction.

\( \square \)

The coupling also allows us to derive a result for the degree distribution. Recall the definition of the companion process \((S_i)_{i \geq 0}\) in Equation (6), and that, for \( B \in \mathcal{B} \), \( N_{\geq k}(B, n) \) denotes the number of vertices of out-degree at least \( k \) with weight belonging to \( B \) at time \( n \).

3.4 Proof of Theorem 1.9

Proof. Let \( B \in \mathcal{B} \) be given. For \( \varepsilon > 0 \), note that

\[
\frac{N_{\geq k}(B \cap \mathcal{M}_\varepsilon, n)}{n} \leq \frac{N_{\geq k}(B, n)}{n} \leq \frac{N_{\geq k}(B \cap \mathcal{M}_\varepsilon, n)}{n} + \frac{N_{\geq 0}(\mathcal{M}_\varepsilon)}{n}.
\]

Now, by the strong law of large numbers, in the limit as \( n \to \infty \) (as in Equation (57)), the second quantity tends to \( \mu(\mathcal{M}_\varepsilon) \), and thus,

\[
\liminf_{n \to \infty} \frac{N_{\geq k}(B \cap \mathcal{M}_\varepsilon, n)}{n} \leq \limsup_{n \to \infty} \frac{N_{\geq k}(B, n)}{n} \leq \limsup_{n \to \infty} \frac{N_{\geq k}(B \cap \mathcal{M}_\varepsilon, n)}{n} + \mu(\mathcal{M}_\varepsilon). \tag{63}
\]

Now, let \( N_{\geq k}^{(\varepsilon)}(\cdot, n) \), \( N_{\geq k}^{(\varepsilon)}(\cdot, n) \) denote the associated quantities in the trees \( T^{(-\varepsilon)}, T^{(\varepsilon)} \), and denote by \((S_i^{(\varepsilon)})_{i \geq 0}\) and \((S_i^{(\varepsilon)})_{i \geq 0}\) the companion processes defined in terms of the functions \( h, g_{-\varepsilon} \) and \( h, g_{+\varepsilon} \) respectively. Then, by Equation (60), on the coupling in Lemma 3.2, we have

\[
N_{\geq k}^{(\varepsilon)}(B \cap \mathcal{M}_\varepsilon, n) \leq N_{\geq k}(B \cap \mathcal{M}_\varepsilon, n) \leq N_{\geq k}^{(-\varepsilon)}(B \cap \mathcal{M}_\varepsilon, n).
\]

Therefore, by Theorem 1.3, (recalling the definitions of \( \lambda_\varepsilon, \lambda_{-\varepsilon} \) in Lemma 3.1)

\[
\mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i^{(\varepsilon)}(W)}{S_i^{(\varepsilon)}(W) + \lambda_\varepsilon} \right) 1_{B \cap \mathcal{M}_\varepsilon} \right] \leq \liminf_{n \to \infty} \frac{N_{\geq k}(B \cap \mathcal{M}_\varepsilon, n)}{n} \leq \limsup_{n \to \infty} \frac{N_{\geq k}(B \cap \mathcal{M}_\varepsilon, n)}{n} \leq \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i^{(-\varepsilon)}(W)}{S_i^{(-\varepsilon)}(W) + \lambda_{-\varepsilon}} \right) 1_{B \cap \mathcal{M}_\varepsilon} \right],
\]

and thus, by Equation (63), we have

\[
\mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i^{(\varepsilon)}(W)}{S_i^{(\varepsilon)}(W) + \lambda_\varepsilon} \right) 1_{B \cap \mathcal{M}_\varepsilon} \right] \leq \liminf_{n \to \infty} \frac{N_{\geq k}(B, n)}{n} \leq \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i^{(-\varepsilon)}(W)}{S_i^{(-\varepsilon)}(W) + \lambda_{-\varepsilon}} \right) 1_{B \cap \mathcal{M}_\varepsilon} \right] + \mu(\mathcal{M}_\varepsilon). \tag{64}
\]
Now, by dominated convergence, as \( \varepsilon \to 0 \)
\[
\mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i^{(\varepsilon)}(W)}{S_i^{(\varepsilon)}(W) + \lambda_\varepsilon} \right) \mathbf{1}_{B \cap \mathcal{M}_\varepsilon} \right] \to \mathbb{E} \left[ \prod_{i=0}^{k-1} \left( \frac{S_i(W)}{S_i(W) + \tilde{\gamma}} \right) \mathbf{1}_B \right],
\]
and, since \( \mathcal{M} \) is a \( \mu \)-null set (by Equation (9)), \( \mu(\mathcal{M}_\varepsilon) \to 0 \). Combining these statements with (64) completes the proof. \( \square \)

4 Appendix

4.1 Proof of Lemma 1.4

In order to prove Lemma 1.4 we first introduce an auxiliary, piecewise constant continuous time Markov process \((Y_w(t), r_w(t))_{t \geq 0}\) taking values in \(\mathbb{N} \times [0, \infty)\). Let \((W_i)_{i \geq 0}\) be independent \(\mu\)-distributed random variables, and define \((S_i(w))_{i \geq 0}\) according to (6), that is,
\[
S_0(w) := h(w); \quad S_{i+1}(w) := S_i(w) + g(w, W_{i+1}), \ i \geq 0.
\]
In addition, set \(\tau_0 = 0\), and define \((\tau_i)_{i \geq 1}\) recursively so that
\[
\tau_i - \tau_{i-1} = \text{Exp}(r_w(\tau_{i-1})); \quad (65)
\]
where \(\text{Exp}(r_w(\tau_{i-1}))\) denotes an exponentially distributed random variable with parameter \(r_w(\tau_{i-1})\). Then, we set
\[
Y_w(t) := \sum_{n=1}^{\infty} 1_{[\tau_n, \infty)}(t), \quad \text{and} \quad r_w(t) := \sum_{n=0}^{\infty} S_n(w) 1_{[\tau_n, \tau_{n+1})}(t).
\]
Now, let \((\mathcal{F}_t)_{t \geq 0}\) denote the filtration generated by the process \((Y_w(t), r_w(t))_{t \geq 0}\).

Claim 4.0.1. The process \(Y_w(t) - \int_0^t r_w(s) \, ds\) is a martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\).

Proof. This follows from the fact that the difference between jump times is exponentially distributed (Equation (65)), and by applying, for example, [Theorem 1.33, [45]] (page 149). \( \square \)

In addition,

Claim 4.0.2. For all \( t \in [0, \infty) \), we have \( \mathbb{E} [Y_w(t)] < \infty \) almost surely. In particular, for each \( t \in [0, \infty) \),
\[
\mathbb{E} [Y_w(t)] = \int_0^t \mathbb{E} [r_w(s)] \, ds. \quad (66)
\]

Proof. Let \(\alpha\) be an independent exponentially distributed random variable with parameter \(a > 0\), and set \(Y_w(\alpha) := \inf_{t \geq \alpha} Y_w(t)\). Then,
\[
\mathbb{E} \left[ 1_{Y_w(\alpha) > k} | S_{k-1}(w), 1_{Y_w(\alpha) > k-1} \right] = \mathbb{E} \left[ \alpha \geq \tau_k | S_{k-1}(w), 1_{Y_w(\alpha) > k-1} \right] \quad (67)
\]
\[
\mathbb{E} \left[ \min (\alpha - \tau_{k-1}, \tau_k - \tau_{k-1}) = \tau_k - \tau_{k-1} \right] 1_{Y_w(\alpha) > k-1}
\]
\[
\frac{S_{k-1}(w)}{a + S_{k-1}(w)} 1_{Y_w(\alpha) > k-1},
\]

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where in the last equality we have used (65) and the memory-less property of the exponential distribution. Note also, that for any \( j \leq k - 1 \), the random variables \( (S_j(w), \ldots, S_{k-1}(w)) \) and \( 1_{\{Y_w(\alpha) \geq j\}} \) are conditionally independent given the random variables \( S_{j-1}(w), 1_{\{Y_w(\alpha) \geq j-1\}} \). Indeed, for each \( \ell \in \{j, \ldots, k-1\} \),

\[
S_\ell(w) = S_{j-1}(w) + \sum_{i=j}^\ell g(w, W_i),
\]

where \( W_j, \ldots, W_{k-1} \) are independent random variables sampled from \( \mu \), while

\[
1_{\{Y_w(\alpha) \geq j\}} = 1_{\{Y_w(\alpha) \geq j-1\}} \times 1_{\{\min(S_{j-1}, \alpha) = S_{j-1}\}}
\]

where \( S_{j-1} \) is an independent exponentially distributed random variable with parameter \( S_{j-1}(w) \). As a result, we have

\[
\mathbb{E} \left[ \left( \prod_{i=j}^{k-1} \frac{S_i(w)}{S_{j-1}(w) + a} \right) 1_{\{Y_w(\alpha) \geq j\}} \bigg| S_{j-1}(w), 1_{\{Y_w(\alpha) \geq j-1\}} \right] = \mathbb{E} \left[ \left( \prod_{i=j}^{k-1} \frac{S_i(w)}{S_{j-1}(w) + a} \right) 1_{\{Y_w(\alpha) \geq j\}} \bigg| S_{j-1}(w), 1_{\{Y_w(\alpha) \geq j-1\}} \right] \mathbb{E} \left[ 1_{\{Y_w(\alpha) \geq j\}} \bigg| S_{j-1}(w), 1_{\{Y_w(\alpha) \geq j-1\}} \right].
\]

(68)

Therefore, we have

\[
\mathbb{P} (Y_w(\alpha) \geq k) = \mathbb{E} \left[ 1_{\{Y_w(\alpha) \geq k\}} \right] = \mathbb{E} \left[ \mathbb{E} \left[ 1_{\{Y_w(\alpha) \geq k\}} \bigg| S_{k-1}(w), 1_{\{Y_w(\alpha) \geq k-1\}} \right] \right].
\]

(67)

Iterating in this manner and noting that \( Y_w(\alpha) \geq 0 \) almost surely, we deduce that the previous expression is \( \mathbb{E} \left[ \prod_{i=0}^{k-1} \frac{S_i(w)}{a + S_i(w)} \right] \). This now implies that

\[
\mathbb{E} [Y_w(\alpha)] = \sum_{k=1}^\infty \mathbb{E} \left[ \prod_{i=0}^{k-1} \frac{S_i(w)}{a + S_i(w)} \right].
\]

(69)

Now, the display on the right is increasing in \( S_i(w) \), and using the fact that \( g \) and \( h \) are bounded by \( J' \), we may bound this above by

\[
\sum_{k=1}^\infty \prod_{i=1}^k \frac{J'i}{J'i + a} < \infty \quad \text{for all } a > J' \ (\text{by applying, for example, Stirling’s approximation}).
\]

Thus, for a suitable choice of \( a \), \( \mathbb{E} [Y_w(\alpha)] \) is finite, so that, in particular, for each \( t \in [0, \infty) \), since the random variable \( Y_w(t) \) is independent of the event \( \{\alpha \geq t\} \) which occurs with positive probability,

\[
\mathbb{E} [Y_w(t)] \leq \frac{\mathbb{E} [Y_w(\alpha)1_{\{\alpha \geq t\}}]}{\mathbb{P} (\alpha \geq t)} < \infty.
\]
Now (66) follows from Claim 4.0.1.

We require an additional claim:

**Claim 4.0.3.** We have

\[
\mathbb{E} [r_w(t)] = h(w) + \mathbb{E} [g(w, W)] \mathbb{E} [\mathcal{Y}_w(t)] = h(w) + \tilde{g}(w)\mathbb{E} [\mathcal{Y}_w(t)].
\]  

(70)

**Proof.** First note that, since \(r_w(t)\) jumps by \(g(w, W)\) whenever \(\mathcal{Y}_w(t)\) jumps, we have

\[
\mathbb{E} [r_w(t)] - h(w) = \mathbb{E} \left[ \sum_{i=1}^{\mathcal{Y}_w(t)} g(w, W_i) \right].
\]

Assume that \(g(w, W_i)\) are bounded by \(J'\). In addition, for each \(n \in \mathbb{N}\),

\[
\mathbb{E} [g(w, W_n) \mathbb{1}_{\{\mathcal{Y}_w(t) \geq n\}}] = \mathbb{E} [g(w, W_n)] - \mathbb{E} [g(w, W_n) \mathbb{1}_{\{\mathcal{Y}_w(t) < n\}}] \\
= \mathbb{E} [g(w, W_n)] (1 - \mathbb{P} (\mathcal{Y}_w(t) < n)) = \mathbb{E} [g(w, W_n)] \mathbb{P} (\mathcal{Y}_w(t) \geq n),
\]

where the second to last equality follows from the fact that the event \(\{\mathcal{Y}_w(t) < n\}\) depends only on \((S_i(w))_{i=0, \ldots, n-1}\), and is thus independent of \(W_n\). Finally, by Claim 4.0.2, \(\mathbb{E} [\mathcal{Y}_w(t)] < \infty\), and thus the result follows by applying Wald’s Lemma.

**Proof of Lemma 1.4.** First note that by Equations (66) and (70), we have

\[
\frac{d}{dt} \mathbb{E} [\mathcal{Y}_w(t)] = \tilde{g}(w)\mathbb{E} [\mathcal{Y}_w(t)],
\]

and solving this differential equation, with initial condition \(\mathbb{E} [\mathcal{Y}_w(0)] = h(w)\), we have

\[
\mathbb{E} [\mathcal{Y}_w(t)] = h(w) e^{\tilde{g}(w)t}.
\]

(71)

Now, let \(\Lambda\) be an exponentially distributed random variable with parameter \(\lambda\). Then, on the one hand, by Equation (69)

\[
\mathbb{E} [\mathcal{Y}_w(\Lambda)] = \sum_{k=1}^{\infty} \mathbb{E} \left[ \prod_{i=0}^{k-1} \frac{S_i(w)}{S_i(w) + \lambda} \right].
\]

On the other hand,

\[
\mathbb{E} [\mathcal{Y}_w(\Lambda)] = \mathbb{E} [\mathbb{E} [\mathcal{Y}_w(u)|\Lambda = u]] = \int_0^t \lambda e^{-\lambda u} \mathbb{E} [\mathcal{Y}_w(u)] du \overset{(71)}{=} \int_0^t \lambda h(w) e^{-(\tilde{g}(w) - \lambda)u} du = \frac{h(w)}{\lambda - \tilde{g}(w)},
\]

where, in the last equality we have used the fact that \(\lambda > \tilde{g}_+\). The result follows.

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