Radius theorems for subregularity in infinite dimensions

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Abstract
The paper continues our previous work (Dontchev et al. in Set-Valued Var Anal 28:451–473, 2020) on the radius of subregularity that was initiated by Asen Dontchev. We extend the results of (Dontchev et al. in Set-Valued Var Anal 28:451–473, 2020) to general Banach/Asplund spaces and to other classes of perturbations, and sharpen the coderivative tools used in the analysis of the robustness of well-posedness of mathematical problems and related regularity properties of mappings involved in the statements. We also expand the selection of classes of perturbations, for which the formula for the radius of strong subregularity is valid.

Keywords  Subregularity · Generalized differentiation · Radius theorems

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1 Introduction

The paper continues the line of “radius of good behaviour” theorems initiated by Dontchev, Lewis & Rockafellar [10] in 2003, and aiming at quantifying the “distance” from a given well-posed problem to the set of ill-posed problems of the same kind. Radius theorems go further than just establishing stability of a problem: they provide quantitative estimates of how far the problem can be perturbed before well-posedness is lost. This is of significant importance, e.g., for computational methods.
Precursors of radius theorems can be traced back to the Eckart–Young theorem [14] dated 1936, which says that, for any nonsingular \( n \times n \) matrix \( A \), it holds

\[
\inf_{B \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)} \{ \|B\| \mid A + B \text{ singular} \} = \frac{1}{\|A^{-1}\|},
\]

where \( \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n) \) denotes the set of all \( n \times n \) matrices, and \( \| \cdot \| \) is the usual operator norm. It gives the exact value for the distance from a given nonsingular square matrix to the set of all singular square matrices over the class of perturbations by arbitrary square matrices, with the distance being measured by the norm of a perturbation matrix. This theorem is connected with the conditioning of a matrix. We refer the readers to the monograph [3] for a broad coverage of the mathematics around condition numbers and conditioning, and their role in numerical algorithms.

Far reaching generalizations of the Eckart–Young theorem were established in Dontchev, Lewis & Rockafellar [10] for the fundamental property of metric regularity, and then in Dontchev & Rockafellar [11] for the important properties of strong metric regularity and strong metric subregularity of set-valued mappings; see also [12, Section 6A] and [24, Section 5.4.1]. The authors of [10, 11] considered perturbations over the classes of affine, Lipschitz continuous and calm single-valued functions, used Lipschitz and calmness moduli of the perturbation functions to measure the distance, and in finite dimensions established exact formulas for the radii in terms of the moduli of the respective regularity properties. In general Banach spaces, lower bounds (providing sufficient conditions for the stability of the properties) were obtained [23, 42] in terms of the respective regularity moduli as well as exact formulas in some particular cases [5, 10, 38]. The definitions of the mentioned regularity properties and the radius theorems from [10, 11] are collated in Definition 2.1 and Theorem 2.3, respectively.

There have been several attempts to study stability of metric regularity with respect to set-valued perturbations under certain assumptions either of sum-stability or on the diameter of the image of perturbation mappings [1, 22, 43]. In line with the general pattern of radius theorems in [10, 11], certain radius theorems for monotone mappings have been established recently in Dontchev, Eberhard & Rockafellar [8].

One thing strikes the eye of the reader of [10, 11]: the absence of any radius estimates for another fundamental regularity property of set-valued mappings, that of (not strong) metric subregularity (see Definition 2.1(iii)). On the contrary, it was shown in [11] that the radius estimates similar to those in Theorem 2.3 do not hold for metric subregularity. In particular, the class of affine functions is not appropriate for estimating the radius of subregularity, neither is the conventional subregularity modulus. The main reason of the metric subregularity property being quite different and difficult to study lies in the well-known fact that, unlike its better studied siblings, it is not always stable.

The properties of metric subregularity and strong metric subregularity are key tools in analyzing convergence rates of numerical algorithms. From the enormous number of papers on this subject we mention here two very recent ones [35, 47], where it is shown that metric subregularity at the solution yields linear convergence rates of several algorithms for solving generalized equations. On the other hand, it is
demonstrated in [34] that metric subregularity is even necessary for the linear convergence of a certain class of algorithms. Further, it is shown in [6] that strong metric subregularity ensures superlinear convergence of the Josephy–Newton method [25] and some of its variants.

For a long time, studies of stability of metric subregularity have been limited to that of error bounds of (mainly convex or almost convex) extended-real-valued functions. (The latter property is equivalent to subregularity of the corresponding epigraphical multifunctions.) Some sufficient and necessary conditions for the stability of error bounds have been obtained in [33, 36, 40, 49, 51, 53]. We mention the radius of error bounds formulas and estimates for several classes of perturbations in [33]. Stability of metric subregularity for general set-valued mappings under smooth perturbations in finite dimensions has been studied in [18]. In the very recent paper by Zheng & Ng [52], stability properties of metric subregularity is studied, mainly under the assumption that the perturbations are normally regular at the point under consideration. In particular, the authors show that, for a set-valued mapping between Asplund spaces, which is either metrically regular or strongly subregular at a reference point, metric subregularity is stable with respect to small calm subsmooth perturbations. In some cases, they provide radius-type estimates. Some radius estimates for a special mapping defined by a system of linear inequalities have been established in [4].

The fundamental results from [10, 11] motivated Dontchev, Gfrerer et al. in their recent paper [9], restricted to finite dimensions, to pursue another approach and employ other tools when studying stability of metric subregularity. Instead of the conventional subregularity modulus, several new “primal-dual” subregularity constants are used in [9] for estimating radii of subregularity. Besides the standard class of Lipschitz continuous functions, semismooth and continuously differentiable perturbations are examined. In the case of Lipschitz continuous perturbations, lower and upper bounds for the radius of subregularity are established, which differ by a factor of at most two. The radii of subregularity over the classes of semismooth and continuously differentiable functions are shown to coincide, and the exact formula is obtained; see Theorem 2.6.

In this paper, we extend the results of [9] to general Banach/Asplund spaces. We consider the standard class of Lipschitz continuous perturbations as well as three new important for applications classes of functions: firmly calm (see Definitions 2.2), Lipschitz semismooth* (see Definitions 4.1 and 4.4), and firmly calm semismooth*. We also sharpen the primal-dual tools used in the analysis.

The motivation to study semismooth* perturbations stems from the successful application of the recently introduced semismooth* Newton method [19, 20] to generalized equations in finite dimensions; see [17, 21]. In a first attempt to generalize the semismooth* Newton method, we carry over the notion of semismooth* sets and mappings to infinite dimensions and state some basic properties.

In our main Theorem 5.5, we establish upper bounds for the radii of metric subregularity over all classes of perturbations, except Lipschitz semismooth*, in general Banach spaces, as well as lower bounds over all four classes in Asplund spaces.

In the case of Lipschitz continuous perturbations, the bounds differ by a factor of at most two. As a byproduct, this gives a characterization of stability of
subregularity under small Lipschitz continuous perturbations. In finite dimensions, the bounds reduce to those in [9, Theorem 3.2]. For firmly calm perturbations, in Asplund spaces we obtain an exact formula for the radius, which however gives a positive radius only in exceptional cases. This has motivated us to consider a more narrow class of firmly calm semismooth* perturbations, imposing additionally a new semismooth* property. For this class, we again have lower and upper bounds for the radius.

The proofs of the lower bounds for the radii of metric subregularity are based on the application of the conventional quantitative sufficient conditions for subregularity from [15, 32] (see Theorem 5.3(ii)) coupled with a sum rule for set-valued mappings in Theorem 3.4(ii). The proofs of the upper bounds employ a rather non-conventional result in [15, Theorem 3.2(2)] (see Theorem 5.3(i)), and are much more involved.

Using similar techniques, we briefly consider the property of strong metric subregularity, and establish in Theorem 5.11 an exact formula for the radius over the classes of calm, firmly calm and firmly calm semismooth* perturbations in general Banach spaces, thus, strengthening the correspondent assertion in [11, Theorem 4.6].

As mentioned above, we consider in this paper two new properties arising naturally in radius of subregularity considerations: firm calmness and semismoothness*. The first property is defined for single-valued functions and requires a function to be calm at a reference point and Lipschitz continuous around every other point nearby; see Definition 2.2. This property has the potential to be used outside the scope of the current topic to substitute the stronger local Lipschitz continuity property in some studies. The semismoothness* property is defined for sets and set-valued mappings (see Definitions 4.1 and 4.4) as an extension of the corresponding definitions in finite dimensions introduced recently in [19]. It is motivated by the formula for the radius of subregularity in [9], where the conventional semismooth perturbations were considered. The semismoothness* property is weaker than the conventional semismoothness. It plays an important role in [19] when constructing Newton-type methods for generalized equations. Some characterizations of semismoothness* of sets and mappings as well as sufficient conditions ensuring the property are established. In particular, it is shown that a positively homogenous function is semismooth*; see Corollary 4.7.

The structure of the paper is as follows. The next Sect. 2 provides some preliminary material used throughout the paper. This includes basic notation and general conventions, the definition of the new firm calmness property, definitions of the classes of perturbations typically used in stability analysis and corresponding radii, and certain primal-dual subregularity constants used in the radius estimates. In Sect. 3, we establish certain sum rules for mappings between normed (in most cases Asplund) spaces that are used in the sequel. Section 4 is dedicated to new concepts of semismooth* sets and mappings, being infinite-dimensional extensions of the corresponding properties introduced recently in [19]. In Sect. 5, we formulate several new estimates and formulas for the radii of subregularity and strong subregularity, and introduce new primal-dual tools for quantitative characterization of
subregularity of mappings and its stability, and potentially other related properties. The proofs of the main results are in the separate Sect. 6.

2 Preliminaries

Notation and basic conventions Our basic notation is standard; see [12, 39, 45]. Throughout the paper, if not explicitly stated otherwise, we assume that $X$ and $Y$ are normed or, more specifically, Banach or Asplund spaces. Their topological duals are denoted by $X^*$ and $Y^*$, respectively, while $(\cdot, \cdot)$ denotes the bilinear form defining the pairing between the spaces. Recall that a Banach space is Asplund if every continuous convex function on an open convex set is Fréchet differentiable on a dense subset, or equivalently, if the dual of each separable subspace is separable [44]. All reflexive, particularly, all finite dimensional Banach spaces are Asplund.

We normally use the letters $x$ and $u$, often with subscripts, for elements of $X$, and the letters $y$ and $v$ for elements of $Y$. Elements belonging to the corresponding dual spaces are marked with $*$ (i.e., $x^*$, $y^*$, etc.). The open unit balls in a normed space and its dual are denoted by $B$ and $B^*$, respectively, while $S$ and $S^*$ stand for the unit spheres (possibly with a subscript denoting the space). $B_{\delta}(x)$ denotes the open ball with radius $\delta > 0$ and centre $x$. Norms and distances in all spaces are denoted by the same symbols $\|\cdot\|$ and $d(\cdot, \cdot)$, respectively. $d(x, \Omega) := \inf_{\omega \in \Omega} \|x - \omega\|$ is the point-to-set distance from $x$ to $\Omega$.

Symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of all real numbers, all nonnegative real numbers and all positive integers, respectively. We use the following conventions: $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$, where $\emptyset$ (possibly with a subscript) denotes the empty subset (of a given set).

If not specified otherwise, products of primal and dual normed spaces are assumed to be equipped with the sum and maximum norms, respectively:

$$
\|(x, y)\| = \|x\| + \|y\|, \quad (x, y) \in X \times Y,
\|(x^*, y^*)\| = \max\{\|x^*\|, \|y^*\|\}, \quad (x^*, y^*) \in X^* \times Y^*.
$$

We denote by $F : X \rightrightarrows Y$ a set-valued mapping acting from $X$ to subsets of $Y$. We write $f : X \to Y$ to denote a single-valued function. The graph and domain of $F$ are defined as $\text{gph } F := \{(x, y) \in X \times Y \mid y \in F(x)\}$ and $\text{dom } F := \{x \in X \mid F(x) \neq \emptyset\}$, respectively. The inverse of $F$ is the mapping $y \mapsto F^{-1}(y) := \{x \in X \mid y \in F(x)\}$. $L(X, Y)$ denotes the space of all linear continuous maps $X \to Y$ equipped with the conventional operator norm.

Regularity, subregularity, Aubin property, and calmness The regularity and Lipschitz-like continuity properties of set-valued mappings in the definition below are defined in metric terms. They have been widely used in variational analysis and optimization, and well studied; cf. [6, 9–13, 15, 16, 24, 27, 32, 37, 39, 41, 45, 46, 48, 54]. For brevity, we drop the word “metric” from the names of the regularity properties. For the definition of a single-valued localization of a set-valued mapping we refer the readers to [12].
Definition 2.1 Let \( F : X \rightrightarrows Y \) be a mapping between metric spaces, and \((\bar{x}, \bar{y}) \in \text{gph } F\).

(i) The mapping \( F \) is regular at \((\bar{x}, \bar{y})\) if there exists an \( \alpha > 0 \) such that
\[
\text{ad}(x, F^{-1}(y)) \leq d(y, F(x)) \text{ for all } (x, y) \text{ near } (\bar{x}, \bar{y}).
\] (2)

If, additionally, \( F^{-1} \) has a single-valued localization around \((\bar{y}, \bar{x})\), then \( F \) is strongly regular at \((\bar{x}, \bar{y})\).

(ii) The mapping \( F \) is subregular at \((\bar{x}, \bar{y})\) if there exists an \( \alpha > 0 \) such that
\[
\text{ad}(x, F^{-1}(y)) \leq d(\bar{y}, F(x)) \text{ for all } x \text{ near } \bar{x}.
\] (3)

If, additionally, \( \bar{x} \) is an isolated point of \( F^{-1}(\bar{y}) \), then \( F \) is strongly subregular at \((\bar{x}, \bar{y})\).

(iii) The mapping \( F \) has the Aubin property at \((\bar{x}, \bar{y})\) if there exists an \( \alpha > 0 \) such that
\[
d(y, F(x)) \leq \alpha d(x, x') \text{ for all } x, x' \text{ near } \bar{x} \text{ and } y \in F(x') \text{ near } \bar{y}.
\] (4)

(iv) The mapping \( F \) is calm at \((\bar{x}, \bar{y})\) if there exists an \( \alpha > 0 \) such that
\[
d(y, F(\bar{x})) \leq \alpha d(x, \bar{x}) \text{ for all } x \text{ near } \bar{x} \text{ and } y \in F(x) \text{ near } \bar{y}.
\] (5)

We denote the (possibly infinite) supremum of all \( \alpha \) satisfying (2) (resp., (3)) by \( \text{rg } F(\bar{x}, \bar{y}) \) (resp., \( \text{srg } F(\bar{x}, \bar{y}) \)), and call it the regularity (resp., subregularity) modulus of \( F \) at \((\bar{x}, \bar{y})\). Thus,
\[
\text{rg } F(\bar{x}, \bar{y}) = \liminf_{(x, y) \to (\bar{x}, \bar{y}), x \notin F^{-1}(\bar{y})} \frac{d(y, F(x))}{d(x, F^{-1}(y))},
\] (6)

\[
\text{srg } F(\bar{x}, \bar{y}) = \liminf_{F^{-1}(\bar{y}) \ni x \to \bar{x}, x \neq \bar{x}} \frac{d(\bar{y}, F(x))}{d(x, F^{-1}(\bar{y}))}.
\] (7)

The case \( \text{rg } F(\bar{x}, \bar{y}) = 0 \) (resp., \( \text{srg } F(\bar{x}, \bar{y}) = 0 \)) indicates the absence of regularity (resp., subregularity).

We denote the (possibly infinite) infimum of all \( \alpha \) satisfying (4) (resp., (5)) by \( \text{lip } F(\bar{x}, \bar{y}) \) (resp., \( \text{clm } F(\bar{x}, \bar{y}) \)), and call it the Lipschitz (resp., calmness) modulus of \( F \) at \((\bar{x}, \bar{y})\). Thus,
\[
\text{lip } F(\bar{x}, \bar{y}) = \limsup_{x, x' \to \bar{x}, x \neq x'} \frac{d(y, F(x))}{d(x, x')}, \quad \text{clm } F(\bar{x}, \bar{y}) = \limsup_{\bar{x} \neq \bar{x}, F(x) \ni y \to \bar{y}} \frac{d(y, F(\bar{x}))}{d(x, \bar{x})}.
\]

It is easy to see that
\[
(\text{rg } F(\bar{x}, \bar{y}))^{-1} = \text{lip } F^{-1}(\bar{y}, \bar{x}), \quad (\text{srg } F(\bar{x}, \bar{y}))^{-1} = \text{clm } F^{-1}(\bar{y}, \bar{x}),
\]
and $F$ is regular (resp., subregular) at $(\bar{x}, \bar{y})$ if and only if $F^{-1}$ has the Aubin property (resp., is calm) at $(\bar{y}, \bar{x})$.

If $\bar{x}$ is an isolated point of $F^{-1}(\bar{y})$, formula (7) admits a simplification:

$$srg F(\bar{x}, \bar{y}) = \liminf_{\bar{y} \neq \bar{x} \rightarrow \bar{x}} \frac{d(\bar{y}, F(\bar{x}))}{d(\bar{y}, \bar{x})} = \liminf_{\bar{y} \neq \bar{x} \rightarrow \bar{x}, y \in F(\bar{x})} \frac{d(y, \bar{y})}{d(y, \bar{x})}. \quad (8)$$

If the lower limit in (8) is zero, this indicates the absence of strong subregularity. Note that in this case the lower limit in (7) can still be nonzero (when $\bar{x}$ is not an isolated point of $F^{-1}(\bar{y})$), i.e. $F$ can be (not strongly) subregular at $(\bar{x}, \bar{y})$. This is the only case when $srg F(\bar{x}, \bar{y})$ defined by (7) is not applicable for characterizing strong subregularity.

In the case of a single-valued function $f : X \rightarrow Y$, we write simply $\text{lip } f(\bar{x})$ and $\text{clm } f(\bar{x})$ to denote its Lipschitz and calmness moduli, i.e.,

$$\text{lip } f(\bar{x}) = \limsup_{x', x \neq x' \rightarrow \bar{x}, x \neq \bar{x}} \frac{d(f(x), f(\bar{x}'))}{d(x, x')}, \quad \text{clm } f(\bar{x}) = \limsup_{x \rightarrow \bar{x}, x \neq \bar{x}} \frac{d(f(x), f(\bar{x}))}{d(x, \bar{x})}.$$  

If $\text{lip } f(\bar{x}) < +\infty$ (resp., $\text{clm } f(\bar{x}) < +\infty$), this indicates that $f$ is Lipschitz continuous near $\bar{x}$ (resp., calm at $\bar{x}$). It obviously holds $0 \leq \text{clm } f(\bar{x}) \leq \text{lip } f(\bar{x})$. Hence, if $f$ is Lipschitz continuous near $\bar{x}$, it is automatically calm at $\bar{x}$. Simple examples show that the converse implication does not hold in general. If $f$ is affine, then obviously $\text{clm } f(\bar{x}) = \text{lip } f(\bar{x})$.

We are going to use a new continuity property lying strictly between the calmness and Lipschitz continuity. It arises naturally when dealing with radius of subregularity estimates, and is likely to be of importance in other areas of analysis.

**Definition 2.2** A function $f : X \rightarrow Y$ is firmly calm at $\bar{x} \in X$ if it is calm at $\bar{x}$ and Lipschitz continuous around every $x \neq \bar{x}$ near $\bar{x}$.

Note that a function which is firmly calm at a point is not necessarily Lipschitz continuous near this point.

**Normal cones and coderivatives** Dual estimates of the regularity radii require certain dual tools – normal cones and coderivatives; cf. [28, 29, 31, 39].

Given a subset $\Omega \subset X$, a point $\bar{x} \in \Omega$, and a number $\varepsilon \geq 0$, the sets

$$N_{\Omega, \varepsilon}(\bar{x}) := \left\{ x^* \in X^* \left| \limsup_{\Omega \ni x \rightarrow \bar{x}, x \neq \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right. \right\}, \quad (9)$$

$$\bar{N}_{\Omega}(\bar{x}) := \limsup_{\Omega \ni x \rightarrow \bar{x}, \varepsilon \downarrow 0} N_{\Omega, \varepsilon}(x) \quad (10)$$

are called, respectively, the set of Fréchet $\varepsilon$-normals and the limiting normal cone to $\Omega$ at $\bar{x}$. If $\varepsilon = 0$, the set (9) reduces to the Fréchet normal cone $N_{\Omega}(\bar{x})$. The lim sup in (10) is the sequential upper limit (in the sense of Painlevé–Kuratowski) with respect to the strong topology in $X$ and the weak* topology in $X^*$. If $\Omega$ is convex, both $N_{\Omega}(\bar{x})$
and \( \overline{N}_\Omega(\bar{x}) \) coincide with the conventional normal cone in the sense of convex analysis. If \( X \) is Asplund, \( \varepsilon \) in (10) can be dropped. If \( \overline{N}_\Omega(\bar{x}) = N_\Omega(\bar{x}) \), then \( \Omega \) is said to be normally regular at \( \bar{x} \).

Given a mapping \( F : X \rightrightarrows Y \), a point \((\bar{x}, \bar{y}) \in \text{gph } F\), and a number \( \varepsilon \geq 0 \), the mappings \( Y^* \rightrightarrows X^* \) defined for all \( y^* \in Y^* \) by

\[
D^\varepsilon F(\bar{x}, \bar{y})(y^*) := \{ x^* \in X^* \mid (x^*, -y^*) \in N_{\text{gph } F, \varepsilon}(\bar{x}, \bar{y}) \},
\]

\[
\overline{D}^\varepsilon F(\bar{x}, \bar{y})(y^*) := \{ x^* \in X^* \mid (x^*, -y^*) \in \overline{N}_{\text{gph } F}(\bar{x}, \bar{y}) \}
\]

are called, respectively, the Fréchet \( \varepsilon \)-coderivative and the limiting coderivative of \( F \) at \((\bar{x}, \bar{y})\). If \( \varepsilon = 0 \), the first one reduces to the Fréchet coderivative \( D^* F(\bar{x}, \bar{y}) \). In the case of a (single-valued) function \( f : X \to Y \), we simply write \( D^\varepsilon f(\bar{x})(y^*) \) and \( D^* f(\bar{x})(y^*) \) for all \( y^* \in Y^* \).

Certain “directional” versions of (10) and (12) have been introduced in \([9, 15, 16]\): for all \( u \in X \) and \( y^* \in Y^* \),

\[
\tilde{N}_\Omega(\bar{x}; u) := \limsup_{\Omega \ni (\bar{x}, t(x-\bar{x})-u), t \to 0} N_\Omega(u),
\]

\[
\tilde{D}F(\bar{x}, \bar{y})(u, y^*) := \{ (x^*, v) \in X^* \times Y \mid (x^*, -y^*) \in \tilde{N}_{\text{gph } F}((\bar{x}, \bar{y}); (u, v)) \}.
\]

They are called, respectively, the directional limiting normal cone to \( \Omega \) at \( \bar{x} \) in the direction \( u \in X \), and the primal-dual derivative of \( F \) at \((\bar{x}, \bar{y})\). Observe that \( \tilde{D}F(\bar{x}, \bar{y}) \) acts from \( X \times Y^* \) to \( X^* \times Y \). This explains the name.

It is well known \([30, 39]\) that in Asplund spaces, when \( \text{gph } F \) is closed near \((\bar{x}, \bar{y})\), the regularity modulus (6) admits a dual representation:

\[
\text{rg } F(\bar{x}, \bar{y}) = \liminf_{\text{gph } F \ni (\bar{x}, \bar{y}) \to (\bar{x}, \bar{y})} \| x^* \|.
\]

In finite dimensions, representation (14) can be simplified:

\[
\text{rg } F(\bar{x}, \bar{y}) = \inf_{x^* \in \overline{D}^* F(\bar{x}, \bar{y})(S_{y^*})} \| x^* \|.
\]

Unlike the regularity modulus (6), its subregularity counterpart (7) does not in general possess dual representations. This is another reflection of the fact that the property lacks robustness. Several dual (and primal-dual) subregularity constants have been used in \([9, 15, 32]\) to provide estimates of the subregularity modulus (7) as well as sufficient (and in some cases also necessary) conditions for subregularity. None of them is in general equal to (7).

**Radius theorems** Below we formulate the key radius theorems from \([9–11]\) which form the foundation for the results in this paper. We consider a mapping \( F : X \rightrightarrows Y \) between normed spaces and the following classes of perturbations of \( F \) near a point \( \bar{x} \in X \):
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We refer the readers to [12] for the definition of semismooth functions (in finite dimensions). We obviously have
\[ F_{\text{lin}} \subseteq F_{C^1} \subseteq F_{ss} \subseteq F_{\text{lip}} \subseteq F_{\text{clm}}. \]
More classes are introduced in Sect. 5. Without loss of generality, we will assume that perturbation functions \( f \) in the above definitions satisfy \( f(\bar{x}) = 0 \). (Thus, the functions in \( F_{\text{lin}} \) are actually linear.)

The radii at a point \((\bar{x}, \bar{y}) \in \text{gph } F\) are defined as follows:
\[
\text{rad}[\text{Property}]_F(\bar{x}, \bar{y}) := \inf_{f \in F_{\text{ Property}}} \{ \text{mod } f(\bar{x}) \mid F + f \text{ fails the ‘Property’ at } (\bar{x}, \bar{y}) \}.
\]

Here, ‘Property’ stands for ‘regularity’, ‘subregularity’, ‘strong regularity’ or ‘strong subregularity’. For brevity, we will write ‘R’, ‘SR’, ‘sR’ and ‘sSR’, respectively, in the notation of the radius. \( \mathcal{P} \) indicates the class of perturbations: \( \text{clm}, \text{lip}, C^1, \text{ss} \) or \( \text{lin} \); more classes will be considered in Sect. 5. \( \text{mod } f(\bar{x}) \) identifies the modulus of \( f \) at \( \bar{x} \) used in the computation of a particular radius: it can be either \( \text{lip } f(\bar{x}) \) or \( \text{clm } f(\bar{x}) \). The first one is used when considering perturbations from \( F_{\text{lin}} \) or any its subclass, and the latter for \( F_{\text{clm}} \) and potentially other classes containing non-Lipschitz functions. For instance, the definition of the radius of regularity over the class of Lipschitz continuous perturbations looks like this:
\[
\text{rad}[R]_{\text{lip}} F(\bar{x}, \bar{y}) := \inf_{f \in F_{\text{lip}}} \{ \text{lip } f(\bar{x}) \mid F + f \text{ is not regular at } (\bar{x}, \bar{y}) \}.
\]

The next theorem combines [10, Theorem 1.5] and [11, Theorems 4.6 and 5.12].

**Theorem 2.3** Let \( X \) and \( Y \) be Banach spaces, \( F : X \rightrightarrows Y \) a mapping with closed graph, and \((\bar{x}, \bar{y}) \in \text{gph } F\). Then
\[
\text{rad}[R]_{\text{lin}} F(\bar{x}, \bar{y}) \geq \text{rad}[R]_{\text{lip}} F(\bar{x}, \bar{y}) \geq \text{rg } F(\bar{x}, \bar{y}),
\]
\[
\text{rad}[sSR]_{\text{lin}} F(\bar{x}, \bar{y}) \geq \text{rad}[sSR]_{\text{clm}} F(\bar{x}, \bar{y}) \geq \text{srg } F(\bar{x}, \bar{y}).
\]

If \( \dim X < \infty \) and \( \dim Y < \infty \), then
\[
\text{rad}[R]_{\text{lin}} F(\bar{x}, \bar{y}) = \text{rad}[R]_{\text{lip}} F(\bar{x}, \bar{y}) = \text{rg } F(\bar{x}, \bar{y}),
\]
Moreover, the equalities remain valid if $\mathcal{F}_{lin}$ is restricted to affine functions of rank 1.

If $F$ is strongly regular at $(\bar{x}, \bar{y})$, then conditions (22) and (24) remain valid with $\text{rad}[R]_{lip}F(\bar{x}, \bar{y})$ in place of $\text{rad}[R]_{lip}F(\bar{x}, \bar{y})$.

It is important to observe that the radii of regularity and strong regularity are considered in Theorem 2.3 with respect to Lipschitz continuous perturbations, and the regularity modulus is used in the estimates, while in the case of strong subregularity, calm perturbations and the subregularity modulus are employed. As observed in [52, p. 2434], it follows from [12, Example 1E.5] that strong metric regularity is not stable with respect to small calm (even “0-calm”) perturbations. In fact, it was shown in [52, p. 2435] that the perturbed mapping may fail to be even metrically regular.

In Theorem 5.11, we show that, when $F$ is strongly subregular at $(\bar{x}, \bar{y})$, the second equality in (25) holds in general Banach spaces. At the same time, in infinite dimensions, the inequality $\text{rad}[R]_{lin}F(\bar{x}, \bar{y}) \geq \text{rg} F(\bar{x}, \bar{y})$ in (22) can be strict; cf. [24, Theorem 5.61].

Note that Theorem 2.3 says nothing about the fundamental property of (not strong) subregularity, which turns out to be quite different. The next two examples show that it does not fit into the pattern of the conditions in Theorem 2.3.

**Example 2.4** [Subregularity: perturbations from $\mathcal{F}_{lin}$] Let a function $F : \mathbb{R} \rightarrow \mathbb{R}$ be given by $F(x) = x$ ($x \in X$). By Definition 2.1(i), $F$ is (strongly) regular, hence also (strongly) subregular everywhere, and by (6) and (7) (or (8)), $\text{rg} F(0, 0) = \text{srg} F(0, 0) = 1$. Any function $f \in \mathcal{F}_{lin}$ is of the form $f(x) = \lambda x$, where $\lambda \in \mathbb{R}$. Hence, $(F + f)(x) = (1 + \lambda)x$. Thus, $F + f \in \mathcal{F}_{lin}$, and this function is (strongly) regular everywhere for any $\lambda \neq -1$, while with $\lambda = -1$, it is the zero function that is neither regular nor strongly subregular. It follows that $\text{rad}[R]_{lin}F(0, 0) = \text{rad}[sR]_{lin}F(0, 0) = 1$, which agrees with Theorem 2.3. At the same time, the zero function is trivially subregular (with subregularity modulus equal $+\infty$); hence, $\text{rad}[SR]_{lin}F(0, 0) = +\infty$. Thus, for subregularity an analogue of (24) and (25) fails.

**Example 2.5** [Subregularity: perturbations from $\mathcal{F}_{lip}$ and $\mathcal{F}_{clm}$] Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be the zero function, i.e., $F(\bar{x}) = 0$ for all $\bar{x} \in \mathbb{R}$. As observed in Example 2.4, $F$ is subregular everywhere, and $\text{rg} F(0, 0) = +\infty$. Consider the function $f \in \mathcal{F}_{lip}$ given by $f(x) = x^2$ ($x \in X$), and observe that it is not subregular at $(0, 0)$, while $\text{lip} f(0) = \text{clm} f(0) = 0$. Hence, $\text{rad}[SR]_{lip}F(0, 0) = \text{rad}[SR]_{clm}F(0, 0) = 0$. Thus, for subregularity an analogue of (22) and (23) fails.

In view of Example 2.4, the class of perturbations $\mathcal{F}_{lin}$ does not seem appropriate for estimating the radius of subregularity. Example 2.5 shows that, if we want to use the natural classes of perturbations $\mathcal{F}_{lip}$ and $\mathcal{F}_{clm}$, it seems unlikely that the
subregularity modulus \( \text{srg} F(\bar{x}, \bar{y}) \) can be used for estimating the radii of subregularity or, at least, the troublesome zero function (or, more generally, constant functions) should somehow be excluded.

It has been shown in our recent paper [9] that in finite dimensions certain “primal-dual” subregularity constants:

\[
\begin{align*}
\text{srg}^+ F(\bar{x}, \bar{y}) & := \inf_{(x^*, v) \in \hat{DF}(\bar{x}, \bar{y})(\mathbb{S}_{X^*})} \max \{ \| v \| + \| x^* \| \}, \\
\text{srg}^* F(\bar{x}, \bar{y}) & := \inf_{(x^*, v) \in \hat{DF}(\bar{x}, \bar{y})(\mathbb{S}_{X^*})} \| B \|,
\end{align*}
\]

employing the primal-dual derivative \( \hat{DF}(\bar{x}, \bar{y}) \) defined by (13), can be used for estimating radii of subregularity in finite dimensions, and one can consider additionally \( C^1 \) and semismooth perturbations.

The next statement is [9, Theorem 3.2]. It gives lower and upper estimates for the radius of subregularity with respect to Lipschitz continuous perturbations, and the exact formula for the radii with respect to semismooth and \( C^1 \) perturbations.

**Theorem 2.6** Let \( \text{dim } X < \infty, \text{dim } Y < \infty, F : X \rightrightarrows Y \) a mapping with closed graph, and \( (\bar{x}, \bar{y}) \in \text{gph } F \). Then

\[
\begin{align*}
\widehat{\text{srg}} F(\bar{x}, \bar{y}) & \leq \text{rad}[\text{SR}]_{\text{lip}} F(\bar{x}, \bar{y}) \leq \text{srg}^+ F(\bar{x}, \bar{y}), \\
\text{rad}[\text{SR}]_{\text{ss}} F(\bar{x}, \bar{y}) & = \text{rad}[\text{SR}]_{C^1} F(\bar{x}, \bar{y}) = \text{srg}^* F(\bar{x}, \bar{y}).
\end{align*}
\]

Observe that in the case of a constant function \( X \ni x \mapsto F(x) := c \in Y \) we have \( \widehat{\text{srg}} F(\bar{x}, c) = \text{srg}^+ F(\bar{x}, c) = \text{srg}^* F(\bar{x}, c) = 0 \) for any \( \bar{x} \in X \). Thus, each of the conditions \( \text{srg} F(\bar{x}, \bar{y}) > 0 \) or \( \text{srg}^* F(\bar{x}, \bar{y}) > 0 \) eliminates the troublesome constant functions, something that condition \( \text{srg} F(\bar{x}, \bar{y}) > 0 \) fails to do. In fact, for the zero function in Example 2.5, Theorem 2.6 gives \( \text{rad}[\text{SR}]_{\text{lip}} F(0, 0) = \text{rad}[\text{SR}]_{\text{ss}} F(0, 0) = \text{rad}[\text{SR}]_{C^1} F(0, 0) = 0 \).

Further observe that \( \widehat{\text{srg}} F(\bar{x}, \bar{y}) \leq \text{srg}^+ F(\bar{x}, \bar{y}) \leq 2\text{srg} F(\bar{x}, \bar{y}) \). Thus, Theorem 2.6 gives reasonably tight upper and lower bounds for the radius of subregularity under Lipschitz continuous perturbations. The property is stable if and only if \( \text{srg}^+ F(\bar{x}, \bar{y}) \) (or \( \text{srg}^* F(\bar{x}, \bar{y}) \)) is positive.

### 3 Sum rules for set-valued mappings

In this section, we establish certain sum rules for mappings between normed (in most cases Asplund) spaces.

The next statement is a **fuzzy intersection rule** [39, Lemma 3.1].
Lemma 3.1 Let $X$ be an Asplund space, $\Omega_1, \Omega_2$ be closed subsets of $X$, $\bar{x} \in \Omega_1 \cap \Omega_2$, and $x^* \in N_{\Omega_1 \cap \Omega_2}(\bar{x})$. Then, for any $\varepsilon > 0$, there exist $x_1 \in \Omega_1 \cap B_\varepsilon(\bar{x})$, $x_2 \in \Omega_2 \cap B_\varepsilon(\bar{x})$, $x_1^* \in N_{\Omega_1}(x_1)$, $x_2^* \in N_{\Omega_2}(x_2)$, and $\lambda \geq 0$ such that

$$\lambda x^* = x_1^* + x_2^* \quad \text{and} \quad \max\{\lambda, \|x_1^*\|, \|x_2^*\|\} = 1.$$ (29)

The above lemma is instrumental in proving the intersection rule in the next theorem. It is needed to prove the sum rule in Theorem 3.4.

Theorem 3.2 Let $X$ be an Asplund space, $\Omega_1, \Omega_2$ be closed subsets of $X$, $\bar{x} \in \Omega_1 \cap \Omega_2$, and $x^* \in N_{\Omega_1 \cap \Omega_2}(\bar{x})$. The following assertions hold true.

(i) For any $\varepsilon > 0$, there exist $x_1 \in \Omega_1 \cap B_\varepsilon(\bar{x})$, $x_2 \in \Omega_2 \cap B_\varepsilon(\bar{x})$, $x_1^* \in N_{\Omega_1}(x_1)$, $x_2^* \in N_{\Omega_2}(x_2)$, and $\lambda \geq 0$ such that

$$\|\lambda x^* - x_1^* - x_2^*\| < \varepsilon \quad \text{and} \quad \max\{\lambda, \|x_1^*\|, \|x_2^*\|\} = 1.$$ (30)

(ii) Suppose that $\Omega_1, \Omega_2$ satisfy at $\bar{x}$ the following normal qualification condition: there exist $\tau > 0$ and $\delta > 0$ such that, for all $x_1 \in \Omega_1 \cap B_\delta(\bar{x})$, $x_2 \in \Omega_2 \cap B_\delta(\bar{x})$, $x_1^* \in N_{\Omega_1}(x_1)$ and $x_2^* \in N_{\Omega_2}(x_2)$, it holds

$$\|x_1^* + x_2^*\| \geq \tau \max\{|x_1^*|, |x_2^*|\}.$$ (31)

Proof

(i) Let $\varepsilon > 0$. Choose a positive number $\varepsilon' < \min\{\varepsilon/3, 1\}$. By Lemma 3.1, there exist $x_1 \in \Omega_1 \cap B_\varepsilon(\bar{x})$, $x_2 \in \Omega_2 \cap B_\varepsilon(\bar{x})$, $u_1^* \in N_{\Omega_1}(x_1) + \varepsilon' B^*$, $u_2^* \in N_{\Omega_2}(x_2) + \varepsilon' B^*$, and $\lambda' \geq 0$ such that $\lambda' x^* = u_1^* + u_2^*$ and $\max\{\lambda', \|u_1^*\|, \|u_2^*\|\} = 1$. Without loss of generality, we suppose that $\max\{\lambda', \|u_1^*\|, \|u_2^*\|\} = 1$. Indeed, if $\|u_2^*\| > 1$, we can replace $\lambda'$, $u_1^*$ and $u_2^*$ with $\lambda'/\|u_2^*\|$, $u_1^*/\|u_2^*\|$ and $u_2^*/\|u_2^*\|$, respectively. There exist $\hat{u}_1^* \in N_{\Omega_1}(x_1)$ and $\hat{u}_2^* \in N_{\Omega_2}(x_2)$ such that $\|\hat{u}_1^* - u_1^*\| < \varepsilon'$ and $\|\hat{u}_2^* - u_2^*\| < \varepsilon'$. Hence, $\|\lambda' x^* - \hat{u}_1^* - \hat{u}_2^*\| < 2\varepsilon'$. Set $\alpha := \max\{\lambda', \|\hat{u}_1^*\|, \|\hat{u}_2^*\|\}$. If $\alpha \geq 1$, we set $\lambda := \lambda'/\alpha$, $x_1^* := \hat{u}_1^*/\alpha$ and $x_2^* := \hat{u}_2^*/\alpha$, and obtain $\max\{\lambda, \|x_1^*\|, \|x_2^*\|\} = 1$ and $\|\lambda x^* - x_1^* - x_2^*\| < 2\varepsilon' < \varepsilon$. Let $\alpha < 1$. Then $\lambda' < 1$ and $\max\{\|u_1^*\|, \|u_2^*\|\} = 1$. Without loss of generality, $\|u_1^*\| \leq \|u_2^*\| = 1$. Then $\|\hat{u}_2^*\| > 1 - \varepsilon' > 0$. We set $\lambda := \lambda'/\alpha$, $x_1^* := \hat{u}_1^*/\alpha$ and $x_2^* := \hat{u}_2^*/\|\hat{u}_2^*\|$. Thus, $\max\{\lambda, \|x_1^*\|, \|x_2^*\|\} = 1$, $\|x_1^* - \hat{u}_1^*\| = 1 - \|\hat{u}_2^*\| < \varepsilon'$, and consequently, $\|\lambda x^* - x_1^* - x_2^*\| < 3\varepsilon' < \varepsilon$.

(ii) Let $\varepsilon > 0$. If $x^* = 0$, the conclusion holds true trivially. Let $x^* \neq 0$. Set $\gamma := \max\{1, |x^*|/\tau\}$, and choose a positive number $\varepsilon' < \min\{\varepsilon/2\gamma, \tau/2, \delta\}$.

By (i), there exist $x_1 \in \Omega_1 \cap B_\varepsilon(\bar{x})$, $x_2 \in \Omega_2 \cap B_\varepsilon(\bar{x})$, $x_1^* \in N_{\Omega_1}(x_1)$,
\(x_2^* \in N_{\Omega_1}(x_2),\) and \(\lambda \geq 0\) such that conditions (29) hold with \(\epsilon'\) in place of \(\epsilon.\) Then \(\epsilon' < \epsilon/(2\gamma) \leq \epsilon/2 < \epsilon,\) and consequently, \(x_1 \in B_\epsilon(\bar{x})\) and \(x_2 \in B_\epsilon(\bar{x}).\) Moreover, using conditions (29) and (30), we obtain

\[
\lambda' = \max \{\lambda, \lambda'_{\|x^*\|/\tau}\} \geq \max \{\lambda, (\lambda'_{\|x^*\| + \epsilon'/\tau} - \epsilon'/\tau\} \\
\geq \max \{\lambda, \|x_1^* + x_2^*\|/\tau\} - 1/2 \geq \max \{\lambda, \|x_1^*\|, \|x_2^*\|\} - 1/2 = 1/2.
\]

Hence, \(\|x^* - x_1^*/\lambda - x_2^*/\lambda'\| < \epsilon'/\lambda \leq 2\gamma\epsilon' < \epsilon,\) i.e. condition (31) is satisfied.

\[\square\]

**Remark 3.3**

(i) It is easy to show that the conclusions of Lemma 3.1 and Theorem 3.2(i) are equivalent.

(ii) The normal qualification condition in Theorem 3.2(ii) can be rewritten equivalently in the limiting form: for any sequences \(\{x_{1k}\} \subset \Omega_1\) and \(\{x_{2k}\} \subset \Omega_2\) converging to \(\bar{x},\) and \(\{x_{1k}^*\}, \{x_{2k}^*\} \subset B^*\) with \(x_{1k}^* \in N_{\Omega_1}(x_i)\) for \(i = 1, 2\) and all \(k \in \mathbb{N},\) it holds

\[
\|x_{1k}^* + x_{2k}^*\| \to 0 \implies x_{1k}^* \to 0 \quad \text{and} \quad x_{2k}^* \to 0.
\]

In the case of closed sets in an Asplund space, it is equivalent to the limiting qualification condition in \([39, \text{Definition 3.2(ii)}].\)

The next theorem is a key tool for establishing radius of subregularity estimates.

**Theorem 3.4** Let \(X\) and \(Y\) be Asplund spaces, \(F_1, F_2 : X \rightrightarrows Y\) be set-valued mappings with closed graphs, \(\bar{x} \in X,\) \(\bar{y}_1 \in F_1(\bar{x}),\) \(\bar{y}_2 \in F_2(\bar{x}),\) \(y^* \in Y^*\) and \(x^* \in D^*(F_1 + F_2)(\bar{x}, \bar{y}_1 + \bar{y}_2)(y^*).\) The following assertions hold true.

(i) For any \(\epsilon > 0,\) there exist \((x_1, y_1) \in \text{gph} F_1 \cap B_\epsilon(\bar{x}, \bar{y}_1), (x_2, y_2) \in \text{gph} F_2 \cap B_\epsilon(\bar{x}, \bar{y}_2), (x_1^*, y_1^*) \in Y^*, (x_2^*, y_2^*) \in Y^*.\) \(x_1^* \in D^*F_1(x_1, y_1)(y_1^*),\) \(x_2^* \in D^*F_2(x_2, y_2)(y_2^*),\) and \(\lambda \geq 0\) such that

\[
\|(\lambda x^* - x_1^* - x_2^*, \lambda y^* - y_1^*, \lambda y^* - y_2^*)\| < \epsilon, \\
\max \{\lambda, \|x_1^*\|, \|y_1^*\|\} = 1.
\]

(ii) Suppose that \(F_2\) satisfies the Aubin property near \((\bar{x}, \bar{y}).\) Then, for any \(\epsilon > 0,\) there exist \((x_1, y_1) \in \text{gph} F_1 \cap B_\epsilon(\bar{x}, \bar{y}_1), (x_2, y_2) \in \text{gph} F_2 \cap B_\epsilon(\bar{x}, \bar{y}_2),\) and \(y_1^*, y_2^* \in B_\epsilon(y^*)\) such that

\[
x^* \in D^*F_1(x_1, y_1)(y_1^*) + D^*F_2(x_2, y_2)(y_2^*) + \epsilon B^*.
\]  

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Proof Observe that $X \times Y$ and $X \times Y \times Y$ are Asplund spaces, and define the closed sets

$$\Omega_i := \{(x, y_1, y_2) \in X \times Y \times Y \mid (x, y_i) \in \text{gph} F_i\}, \quad i = 1, 2.$$ 

Observe also that

$$\Omega_1 \cap \Omega_2 = \{(x, y_1, y_2) \in X \times Y \times Y \mid (x, y_1) \in \text{gph} F_1, (x, y_2) \in \text{gph} F_2\}$$

and, for all $(x, y_1, y_2) \in X \times Y \times Y$,

$$N_{\Omega_1}(x, y_1, y_2) = \{(u^*, v^*) \in X^* \times Y^* \times Y^* \mid (u^*, v^*) \in N_{\text{gph} F_1}(x, y_1)\},$$

$$N_{\Omega_2}(x, y_1, y_2) = \{(u^*, 0, v^*) \in X^* \times Y^* \times Y^* \mid (u^*, v^*) \in N_{\text{gph} F_2}(x, y_2)\},$$

$$N_{\text{gph}(F_1 + F_2)}(x, y_1 + y_2) \subseteq \{(u^*, v^*) \in X^* \times Y^* \mid (u^*, v^*, v^*) \in N_{\Omega_1 \cap \Omega_2}(x, y_1, y_2)\}.$$

Assertion (i) is now a direct consequence of Theorem 3.2(ii).

To prove assertion (ii), we first show that, thanks to the Aubin property of $F_2$, the sets $\Omega_1, \Omega_2$ satisfy at $(\bar{x}, \bar{y}, \bar{y})$ the normal qualification condition in Theorem 3.2(ii). Indeed, if $F_2$ satisfies the Aubin property, then, by Definition 2.1(iii) and the definition of the Fréchet normal cone, there exist numbers $\alpha > 0$ and $\delta > 0$ such that, for all $(x, y) \in \text{gph} F_2 \cap B_\delta(\bar{x}, \bar{y})$ and all $(u^*, v^*) \in N_{\text{gph} F_2}(x, y)$, it holds $\|u^*\| \leq \alpha\|v^*\|$. Hence, if $(x_1, y_1, v_1) \in \Omega_1 \cap B_\delta(\bar{x}, \bar{y}, \bar{y})$, $(x_2, v_2, y_2) \in \Omega_2 \cap B_\delta(\bar{x}, \bar{y}, \bar{y})$, $(x_1^*, y_1^*, 0) \in N_{\Omega_1}(x_1, y_1, v_1)$, $(x_2^*, 0, y_2^*) \in N_{\Omega_2}(x_2, v_2, y_2)$ with $(x_1^*, y_1^*) \in N_{\text{gph} F_1}(x_1, y_1)$ and $(x_2^*, y_2^*) \in N_{\text{gph} F_2}(x_2, y_2)$, then

$$\max\{\|(x_1^*, y_1^*, 0)\|, \|(x_2^*, 0, y_2^*)\|\} = \max\{\|x_1^*\|, \|x_2^*\|, \|y_1^*\|, \|y_2^*\|\} \leq \max\{\|x_1^* + x_2^*\|, \|y_1^*\|, \|y_2^*\|\} = \max\{\|x_1^* + x_2^*\|, \|y_1^*\|, \|y_2^*\|\} \leq \max\{\|x_1^* + x_2^*\|, \|y_1^*\|, \|y_2^*\|\} \leq (\alpha + 1)\|x_1^* + x_2^*\|, \|y_1^*\|, \|y_2^*\| = (\alpha + 1)\|x_1^*, y_1^*, 0\| + (x_2^*, 0, y_2^*),$$

i.e. $\Omega_1, \Omega_2$ satisfy at $(\bar{x}, \bar{y}, \bar{y})$ the normal qualification condition with $\tau := (\alpha + 1)^{-1}$. By Theorem 3.2(ii), there exist $(x_1, y_1) \in \text{gph} F_1 \cap B_\varepsilon(\bar{x}, \bar{y})$, $(x_2, y_2) \in \text{gph} F_2 \cap B_\varepsilon(\bar{x}, \bar{y})$, $y_1^*, y_2^* \in Y^*$, $x_1^* \in D^* F_1(x_1, y_1)(y_1^*)$, and $x_2^* \in D^* F_2(x_2, y_2)(y_2^*)$ such that

$$\|(x^* - x_1^*, y^* - y_1^*, y^* - y_2^*)\| < \varepsilon,$$

i.e. $y_1^*, y_2^* \in B_\varepsilon(y^*)$ and condition (32) is satisfied. 

The next statement complements Theorem 3.4(ii). It extends [39, Theorem 1.62(i)] which addresses the case $\varepsilon = 0$. 

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Theorem 3.5 Let $F : X \rightrightarrows Y$, $f : X \to Y$, $(\bar{x}, \bar{y}) \in \text{gph } F$, and $\varepsilon \geq 0$. Suppose that $f$ is Fréchet differentiable at $\bar{x}$. Set $\varepsilon_1 := (\|\nabla f(\bar{x})\| + 1)^{-1}\varepsilon$ and $\varepsilon_2 := (\|\nabla f(\bar{x})\| + 1)\varepsilon$. Then, for all $y^* \in Y^*$, it holds

$$D^*_{\varepsilon_1} F(\bar{x}, \bar{y})(y^*) \subset D^*_{\varepsilon}(F + f)(\bar{x}, \bar{y} + f(\bar{x}))(y^*) - \nabla f(\bar{x})^*y^* \subset D^*_{\varepsilon_2} F(\bar{x}, \bar{y})(y^*).$$

(33)

Proof Let $y^* \in Y^*$ and $x^* \in D^*_{\varepsilon_1} F(\bar{x}, \bar{y})(y^*)$. By the definitions of the Fréchet derivative, Fréchet $\varepsilon$-coderivative (11), and the set of $\varepsilon$-normals (9),

$$\lim_{\bar{x} \neq x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0,$$

$$\limsup_{\text{gph } F \ni (x, y) \to (\bar{x}, \bar{y})} \frac{\langle x^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} \rangle}{\| (x, y) - (\bar{x}, \bar{y}) \|} \leq \varepsilon_1.$$ 

Hence,

$$\limsup_{\text{gph } F \ni (x, y) \to (\bar{x}, \bar{y})} \frac{\langle x^* + \nabla f(\bar{x})^*y^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} - f(\bar{x}) \rangle}{\| (x, y) - (\bar{x}, \bar{y} + f(\bar{x})) \|}$$

$$= \limsup_{\text{gph } F \ni (x, y) \to (\bar{x}, \bar{y})} \frac{\langle x^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} \rangle}{\|x - \bar{x}\| + \|y - \bar{y} + f(x) - f(\bar{x})\|}$$

$$\leq (\|\nabla f(\bar{x})\| + 1) \limsup_{\text{gph } F \ni (x, y) \to (\bar{x}, \bar{y})} \frac{\langle x^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} \rangle}{\|x - \bar{x}\| + \|y - \bar{y} + f(x) - f(\bar{x})\|}$$

$$\leq (\|\nabla f(\bar{x})\| + 1) \limsup_{\text{gph } F \ni (x, y) \to (\bar{x}, \bar{y})} \frac{\langle x^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} \rangle}{\|x - \bar{x}\| + \|y - \bar{y}\|} \leq (\|\nabla f(\bar{x})\| + 1)\varepsilon_1 = \varepsilon.$$ 

This proves the first inclusion in (33). The second inclusion is a consequence of the first one applied with $F + f$, $-f$ and $\varepsilon_2$ in place of $F$, $f$ and $\varepsilon$, respectively. \qed

Lemma 3.6 Let $F : X \rightrightarrows Y$, $f : X \to Y$, $(\bar{x}, \bar{y}) \in \text{gph } F$, and $\varepsilon \geq 0$. Suppose that $f$ is calm at $\bar{x}$ with $c := \text{clm } f(\bar{x}) < 1$. Then, for all $y^* \in Y^*$ and $x^* \in D^*_{\varepsilon_1} F(\bar{x}, \bar{y})(y^*)$, it holds $x^* \in D^*_{\delta}(F + f)(\bar{x}, \bar{y} + f(\bar{x}))(y^*)$ with $\delta := (\varepsilon + c\|y^*\|)/(1 - c)$.

Proof Choose a number $c' \in (c, 1)$. For all $x$ near $\bar{x}$ and all $y \in Y$, we have
\[
\| (x - \bar{x}, (y + f(x)) - (\bar{y} + f(\bar{x}))) \| = \| x - \bar{x} \| + \| (y + f(x)) - (\bar{y} + f(\bar{x})) \| \\
\geq \| x - \bar{x} \| + \| y - \bar{y} \| - \| f(x) - f(\bar{x}) \| \\
\geq (1 - c') \| x - \bar{x} \| + \| y - \bar{y} \| \\
\geq (1 - c') \| (x, y - \bar{y}) \|.
\]

Hence,

\[
\limsup_{gph (F + f) \ni (x, y) \to (\bar{x}, \bar{y})} \frac{\langle x^*, x - \bar{x} \rangle - \langle y^*, y - (\bar{y} + f(\bar{x})) \rangle}{\| (x - \bar{x}, y - (\bar{y} + f(\bar{x})) \|}
= \limsup_{gph F \ni (x, y) \to (\bar{x}, \bar{y})} \frac{\langle x^*, x - \bar{x} \rangle - \langle y^*, (y + f(x)) - (\bar{y} + f(\bar{x})) \rangle}{\| (x - \bar{x}, (y + f(x)) - (\bar{y} + f(\bar{x})) \|}
\leq \limsup_{(x, y) \to (\bar{x}, \bar{y})} \frac{\| y^* \| \| f(x) - f(\bar{x}) \|}{(1 - c') \| x - \bar{x} \|}
\leq \frac{\varepsilon + c' \| y^* \|}{1 - c'}.
\]

Taking infimum over all \( c' \in (0, 1) \), we arrive at the assertion. \( \square \)

### 4 Semismooth\(^*\) sets and mappings

We are going to employ an infinite-dimensional extension of the semismooth\(^*\) property introduced recently in [19]. In finite dimensions, this property is weaker than the conventional semismoothness, and is important when constructing Newton-type methods for generalized equations.

**Definition 4.1** A set \( \Omega \subset X \) is semismooth\(^*\) at \( \bar{x} \in \Omega \) if, for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that

\[
| \langle x^*, x - \bar{x} \rangle | \leq \varepsilon \| x - \bar{x} \| \tag{34}
\]

for all \( x \in \Omega \cap B_\delta(\bar{x}) \) and \( x^* \in N_{\Omega,\delta}(x) \cap S_X^* \).

**Theorem 4.2** Let \( X \) be an Asplund space, and \( \Omega \subset X \) be closed around \( \bar{x} \in \Omega \). The set \( \Omega \) is semismooth\(^*\) at \( \bar{x} \) if and only if, for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that condition (34) holds for all \( x \in \Omega \cap B_\delta(\bar{x}) \) and \( x^* \in N_{\Omega}(x) \cap S_X^* \).

**Proof** Since for all \( x \in \Omega \) and \( \delta \geq 0 \) we have \( N_{\Omega}(x) \subset N_{\Omega,\delta}(x) \), the “only if” part is true trivially. We prove the “if” part by contradiction. Suppose that \( \Omega \) is not semismooth\(^*\) at \( \bar{x} \), i.e. there are a number \( \varepsilon > 0 \) and sequences \( x_k \in \Omega \) with \( x_k \to \bar{x}, \delta_k \downarrow 0 \), and \( x^*_k \in N_{\Omega,\delta_k}(x_k) \cap S_X^* \), such that

\[
| \langle x^*_k, x_k - \bar{x} \rangle | > \varepsilon \| x_k - \bar{x} \|.
\]
Then, for every \( k \in \mathbb{N} \), there exist points \( x'_k \in \Omega \) and \( x^*_k \in N_{\bar{\Omega}}(x'_k) \) satisfying \( \|x'_k - x_k\| \leq \frac{1}{k} \|x_k - \bar{x}\| \) and \( \|x^*_k - x'_k\| \leq \delta_k + \frac{1}{k} \|x_k - \bar{x}\| \), and \( \|x^*_k - x'_k\| + \frac{1}{k} \|x^*_k\| < \varepsilon \); cf. [39, formula (2.51)]. Hence,

\[
\|x'_k - \bar{x}\| \leq \|x_k - \bar{x}\| + \|x'_k - x_k\| \leq \left( 1 + \frac{1}{k} \right) \|x_k - \bar{x}\|
\]

and consequently,

\[
\langle x^*_k, x'_k - \bar{x} \rangle \geq \|x^*_k - x'_k\| \|x_k - \bar{x}\| - \|x^*_k\| \|x_k - x'_k\| > \left( \varepsilon - \|x^*_k - x'_k\| - \frac{1}{k} \|x^*_k\| \right) \|x_k - \bar{x}\| \\
\geq \left( \varepsilon - \|x^*_k - x'_k\| - \frac{1}{k} \|x^*_k\| \right) \frac{k}{k+1} \|x'_k - \bar{x}\|.
\]

Since \( \|x^*_k\| = 1 \) and \( \|x^*_k - x'_k\| \to 0 \) as \( k \to \infty \), we can set \( v^*_k := \frac{x^*_k}{\|x^*_k\|} \) and conclude that

\[
\left| \langle v^*_k, x'_k - \bar{x} \rangle \right| > \frac{\varepsilon}{2} \|x'_k - \bar{x}\|
\]

for all sufficiently large \( k \in \mathbb{N} \). This completes the proof. \( \square \)

**Remark 4.3**

(i) Condition (34) in Definition 4.1 and Theorem 4.2 is obviously equivalent to the following one:

\[
|\langle x^*, x - \bar{x} \rangle| \leq \varepsilon \|x^*\| \|x - \bar{x}\|.
\]

Moreover, with the latter condition, the restriction \( x^* \in \mathbb{S}_X \) in Definition 4.1 and Theorem 4.2 can be dropped.

(ii) Definition 4.1 differs from the corresponding finite dimensional definition in [19, Definition 3.1]. Thanks to [19, Proposition 3.2], the two definitions are equivalent (in finite dimensions).

The class of semismooth* sets is rather broad. E.g., it follows from [26, Theorem 2] that in finite dimensions every closed subanalytic set is semismooth* at any of its points.

**Definition 4.4** A mapping \( F : X \rightrightarrows Y \) is semismooth* at \( (\bar{x}, \bar{y}) \in \text{gph } F \) if \( \text{gph } F \) is semismooth* at \( (\bar{x}, \bar{y}) \).

In view of Remark 4.3(i), we have the following explicit reformulation of the semismooth* property of a set-valued mapping.

**Proposition 4.5** A mapping \( F : X \rightrightarrows Y \) is semismooth* at \( (\bar{x}, \bar{y}) \in \text{gph } F \) if and only if, for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that

\[
\langle x^*, x - \bar{x} \rangle \leq \varepsilon \|x^*\| \|x - \bar{x}\|.
\]
\[
|\langle x^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} \rangle| \leq \varepsilon \|\langle x^*, y^* \rangle\|\|x - \bar{x}, y - \bar{y}\| \tag{35}
\]
for all \((x, y) \in \text{gph} F \cap B_0(\bar{x}, \bar{y}), y^* \in Y^*, \text{ and } x^* \in D_0^x F(x, y)(y^*)\).

**Proposition 4.6** Let \(f : X \to Y\) and \(\bar{x} \in X\). Suppose that the directional derivatives \(f'(x; x - \bar{x})\) and \(f'(x; \bar{x} - x)\) exist for all \(x \neq \bar{x}\) near \(\bar{x}\), and

\[
\lim_{x, \bar{x} \to \bar{x}} \max \left\{ \|f(x) - f(\bar{x}) - f'(x; x - \bar{x})\|, \|f(x) - f(\bar{x}) + f'(x; \bar{x} - x)\| \right\} / \|f(x) - f(\bar{x}), x - \bar{x}\| = 0.
\]

Then \(f\) is semismooth* at \(\bar{x}\).

**Proof** Let a number \(\varepsilon > 0\) be fixed. Choose a number \(\delta > 0\) such that \(\delta(1 + \varepsilon/2) \leq \varepsilon/2\), and

\[
\max \left\{ \|f(x) - f(\bar{x}) - f'(x; x - \bar{x})\|, \|f(x) - f(\bar{x}) + f'(x; \bar{x} - x)\| \right\} < \frac{\varepsilon}{2} \|\langle x - \bar{x}, f(x) - f(\bar{x}) \rangle\| \text{ for all } x \in B_0(\bar{x}) \setminus \{\bar{x}\}.
\]

Let \(x \in B_0(\bar{x}) \setminus \{\bar{x}\}\) and \((x^*, y^*) \in N_{\text{gph} f, \delta}(x, f(x)) \cap \mathcal{S}_{\text{gph} f, \delta}(x, y)^*\). Then

\[
\frac{\langle x^*, x - \bar{x} \rangle + \langle y^*, f'(x; x - \bar{x}) \rangle}{\|x - \bar{x}, f'(x; x - \bar{x})\|} = \lim_{\varepsilon \to 0} \frac{\langle x^*, x + \varepsilon(x - \bar{x}) - x \rangle + \langle y^*, f(x + \varepsilon(x - \bar{x})) - f(x) \rangle}{\|x + \varepsilon(x - \bar{x}) - x, f(x + \varepsilon(x - \bar{x})) - f(x)\|} \leq \delta.
\]

Hence,

\[
\langle x^*, x - \bar{x} \rangle + \langle y^*, f(x) - f(\bar{x}) \rangle = \langle x^*, x - \bar{x} \rangle + \langle y^*, f'(x; x - \bar{x}) \rangle + \langle y^*, f(x) - f(\bar{x}) - f'(x; x - \bar{x}) \rangle \\
\leq \delta \|\langle x - \bar{x}, f'(x; x - \bar{x}) \rangle\| + \|f(x) - f(\bar{x}) - f'(x; x - \bar{x})\| \\
\leq \delta \|\langle x - \bar{x}, f(x) - f(\bar{x}) \rangle\| + (\delta + 1)\|\langle x - \bar{x}, f(x) - f(\bar{x}) \rangle\| - f'(x; x - \bar{x})\| \\
\leq (\delta + (\delta + 1)\varepsilon/2)\|\langle x - \bar{x}, f(x) - f(\bar{x}) \rangle\| \leq \varepsilon \|\langle x - \bar{x}, f(x) - f(\bar{x}) \rangle\|.
\]

Similarly, \(\langle x^*, \bar{x} - x \rangle + \langle y^*, f(\bar{x}) - f(x) \rangle \leq \varepsilon \|\langle x - \bar{x}, f(x) - f(\bar{x}) \rangle\|, \text{ and consequently,} \]

\[
|\langle x^*, x - \bar{x} \rangle + \langle y^*, f(x) - f(\bar{x}) \rangle| \leq \varepsilon \|\langle x - \bar{x}, f(x) - f(\bar{x}) \rangle\|. \tag*{□}
\]

**Corollary 4.7** If a function \(f : X \to Y\) is positively homogenous at \(\bar{x} \in X\), i.e.,

\[
f(\bar{x} + \lambda(x - \bar{x})) = f(\bar{x}) + \lambda(f(x) - f(\bar{x})) \text{ for all } x \in X, \lambda > 0,
\]

then it is semismooth* at \(\bar{x}\).

Let us now briefly compare the semismoothness* properties in Definitions 4.1 and 4.4 with the corresponding extensions of smoothness and convexity due to Aussel, Daniilidis & Thibault \[2\], and Zheng & Ng \[50, 52\] known as subsmoothness and \(w\)-subsmoothness. The definitions below employ the Clarke normal cones \(N^C_\Omega(x)\). We do not use this type of cones in the current paper and refer the readers to \[7\] for the respective definition and properties of such objects.
Definition 4.8  Let $\Omega \subset X$ and $\bar{x} \in \Omega$.

(i) $\Omega$ is subsmooth at $\bar{x}$ if for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$
\langle x^*, u - x \rangle \leq \epsilon \|u - x\|
$$

for all $x, u \in \Omega \cap B_\delta(\bar{x})$ and $x^* \in N^C_\Omega(x) \cap \mathbb{B}_{X^*}$.

(ii) $\Omega$ is $w$-subsmooth at $\bar{x}$ if for any $\epsilon > 0$ there is a $\delta > 0$ such that condition (36) holds with $x = \bar{x}$ for all $u \in \Omega \cap B_\delta(\bar{x})$ and $x^* \in N^C_\Omega(\bar{x}) \cap \mathbb{B}_{X^*}$.

Observe the differences between the definitions of semismooth* in Definition 4.1 and $w$-subsmoothness in Definition 4.8(ii): we fix $(u =) \bar{x}$ in the first one, and $x = \bar{x}$ in the latter. In addition, condition (34) involves taking the absolute value in the left-hand side. Besides, in Definition 4.8(ii) (Clarke) normals are computed at $\bar{x}$, while Definition 4.1 requires computing (Fréchet $\delta$-)normals at all $x$ near $\bar{x}$.

Definition 4.9  A mapping $F : X \rightrightarrows Y$ is subsmooth ($w$-subsmooth) at $(\bar{x}, \bar{y}) \in \text{gph} F$ if $\text{gph} F$ is subsmooth ($w$-subsmooth) at $(\bar{x}, \bar{y})$.

Recall [39] that in a Banach space it holds $N^C_\Omega(x) \subset c^1 \text{co} \overline{\Omega}(x)$ for all $x \in \Omega$, and the inclusion holds as equality if the space is Asplund, and $\Omega$ is locally closed around $x$. It follows immediately that subsmoothness of a set $\Omega$ at $\bar{x} \in \Omega$ implies that

$$
\overline{\Omega}(x) \cap \mathbb{B}_{X^*} \subset N^C_\Omega(x) \cap \mathbb{B}_{X^*} \subset N_{\Omega, \epsilon}(x)
$$

for any $\epsilon > 0$ and all $x \in \Omega$ sufficiently close to $\bar{x}$. Furthermore, whenever a set $\Omega$ is subsmooth or $w$-subsmooth at $\bar{x}$, it is normally regular at $\bar{x}$. On the other hand, a semismooth* set needs not to be normally regular at the point under consideration. As an easy example consider the complementarity angle $\Omega = \{ (x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 x_2 = 0 \}$. It is obviously semismooth* at $(0, 0)$ but not normally regular, hence, not ($w$-) subsmooth.

Since ($w$-)subsmoothness and the semismooth* properties for a set-valued mapping are defined via the respective properties of its graph, similar considerations also apply to mappings.

5 Radii of subregularity

In this section, $F : X \rightrightarrows Y$ is a mapping between normed spaces, and $(\bar{x}, \bar{y}) \in \text{gph} F$.

Several quantities employing ($\epsilon$-)coderivatives can be used for quantitative characterization of subregularity of mappings and its stability, and potentially other related properties. Given an $\epsilon \geq 0$, we are going to use the set

$$
\Delta_\epsilon F := \{ (x, y, x^*) \in X \times Y \times X^* \mid (x, y) \in \text{gph} F, \ x^* \in D_\epsilon^y F(x, y)(\mathbb{S}_Y) \}
$$

of the primal and dual (in $X^*$ corresponding to unit vectors in $Y^*$) components of the $\epsilon$-coderivative of $F$, as well as its projection on $X \times Y$.
\[ \Delta^\epsilon F := \{(x, y) \in \text{gph } F \mid D^\epsilon_F(x, y)(\mathbb{S}_Y) \neq \emptyset\}. \]  

(38)

When \( \epsilon = 0 \), we will drop the subscript in (37) and (38). Given parameters \( \epsilon > 0 \) and \( \delta \geq 0 \), we define local analogues of (37) and (38), involving the additional restriction (35) which arises when dealing with semismooth* perturbations; cf. Proposition 4.5:

\[ Y_{\epsilon, \delta}^\omega F(\bar{x}, \bar{y}) := \{(x, y, x^*)|(x, y) \in \text{gph } F, x^* \in D^\omega_F(x, y)(y^*), y^* \in \mathbb{S}_Y, \] 
\[ \langle x^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} \rangle \leq \epsilon \|x^*\|\|y^*\|\|x - \bar{x}, y - \bar{y}\| \} \].

(39)

\[ Y_{\epsilon, \delta}^\omega F(\bar{x}, \bar{y}) := \{(x, y) \mid (x, y, x^*) \in Y_{\epsilon, \delta}^\omega F(\bar{x}, \bar{y}) \text{ for some } x^* \in X^* \}. \]  

(40)

When \( \delta = 0 \), we will write \( Y_{\epsilon}^\omega F(\bar{x}, \bar{y}) \) and \( Y_{\epsilon}^\omega F(\bar{x}, \bar{y}) \).

Here is the list of regularity constants to be used in the sequel when estimating the radii of subregularity with respect to different classes of perturbations. They employ notations (37)–(40).

\[ \text{srg}_1 F(\bar{x}, \bar{y}) := \sup_{\epsilon > 0} \inf_{(x, y, x^*) \in \Delta F, 0 < \|x - \bar{x}\| < \epsilon} \max \left\{ \frac{\|y - \bar{y}\|}{\|x - \bar{x}\|}, \|x^*\| \right\}, \]  

(41)

(42)

\[ \text{srg}_2 F(\bar{x}, \bar{y}) := \sup_{\epsilon > 0} \inf_{(x, y) \in \Delta^\epsilon F, 0 < \|x - \bar{x}\| < \epsilon} \frac{\|y - \bar{y}\|}{\|x - \bar{x}\|}, \]  

\[ \text{srg}_3 F(\bar{x}, \bar{y}) := \sup_{\epsilon > 0} \inf_{(x, y, x^*) \in \Delta F, 0 < \|x - \bar{x}\| < \epsilon} \max \left\{ \frac{\|y - \bar{y}\|}{\|x - \bar{x}\|}, \|x^*\| \right\}, \]  

(43)

\[ \text{srg}_4 F(\bar{x}, \bar{y}) := \sup_{\epsilon > 0} \inf_{(x, y) \in Y_{\epsilon} F(\bar{x}, \bar{y}), 0 < \|x - \bar{x}\| < \epsilon} \frac{\|y - \bar{y}\|}{\|x - \bar{x}\|}, \]  

(44)

\[ \text{srg}_1^+ F(\bar{x}, \bar{y}) := \sup_{\epsilon > 0} \inf_{(x, y, x^*) \in \Delta^\epsilon F, 0 < \|x - \bar{x}\| < \epsilon} \left( \frac{\|y - \bar{y}\|}{\|x - \bar{x}\|} + \|x^*\| \right), \]  

\[ \text{srg}_2^+ F(\bar{x}, \bar{y}) := \sup_{\epsilon > 0} \inf_{(x, y, x^*) \in \Delta^\epsilon F, 0 < \|x - \bar{x}\| < \epsilon} \frac{\|y - \bar{y}\|}{\|x - \bar{x}\|}, \]  

\[ \text{srg}_4^+ F(\bar{x}, \bar{y}) := \sup_{\epsilon > 0} \inf_{(x, y) \in Y_{\epsilon} F(\bar{x}, \bar{y}), 0 < \|x - \bar{x}\| < \epsilon} \frac{\|y - \bar{y}\|}{\|x - \bar{x}\|}. \]  

(45)

(46)

(47)

Remark 5.1
The sup inf constructions in (41)–(47) can be rewritten as lim inf. We have chosen this explicit form to avoid confusion with the common usage of lim inf in set-valued analysis.

Thanks to the term \( \|y - \tilde{y}\| / \|x - \tilde{x}\| \) present in all the definitions (41)–(47), the inequality \( \|x - \tilde{x}\| < \epsilon \) in these definitions can be replaced with the stronger one: \( \|(x, y) - (\tilde{x}, \tilde{y})\| < \epsilon \).

In view of the definition (1) of the dual norm on the product space, \( \| (x^*, y^*) \| \) in (39) equals \( \max \{ \|x^*\|, 1 \} \).

In (43), the set \( Y_4(x, \tilde{y}) \) can be replaced with its slightly simplified version with the inequality in the definition (39) substituted by the next one:

\[
\langle x^*, x - \tilde{x} \rangle - \langle y^*, y - \tilde{y} \rangle \leq \epsilon \| (x - \tilde{x}, y - \tilde{y}) \|.
\]

**Proposition 5.2** Let \( F : X \rightrightarrows Y \) and \( (\tilde{x}, \tilde{y}) \in \text{gph} F \).

**Proof** Assertions (i)–(iii) follow immediately from the definitions of the respective constants. To justify (48), first note that the Fréchet \( \varepsilon \)-coderivative is defined by Fréchet \( \varepsilon \)-normals (see (11)). Second, given a closed subset \( \Omega \) of an Asplund space, and any \( \tilde{x} \in \Omega \), \( \varepsilon > 0 \) and \( x^*_\varepsilon \in N_{\Omega, \varepsilon}(\tilde{x}) \), there are \( x \in \Omega \cap \text{B}_\varepsilon(\tilde{x}) \) and \( \tilde{x} \in N_{\Omega}(x) \)
satisfying \( \|x^* - x^*\| < 2\varepsilon \); cf. [39, formula (2.51)]. Hence, \( \Delta \varepsilon F \) in (45) can be replaced with \( \Delta F \), yielding (48). The same arguments show that, in the Asplund space setting, one can set \( \delta = 0 \) in (46) and (47), yielding (49) and (50). Note that in the case of (49), conditions \((x, y, x^*) \in \Delta_0 F \) and \( 0 \cdot \|x^*\| < \varepsilon \) are equivalent to \((x, y) \in \Delta^0 F \). Having established (48)–(50), the estimates (iv)–(vi) easily follow. Finally, to verify (vii) and (viii), consider suitable sequences \((x_k, y_k) \to (\bar{x}, \bar{y}) \) and \((x_k^*, y_k^*) \) with \( x_k^* \in D^* F(\bar{x}_k, y_k)(y_k^*) \) and \( y_k^* \in \mathbb{S}_Y \), and

\[
\text{srg}_1 F(\bar{x}, \bar{y}) = \lim_{k \to \infty} \max \left\{ \frac{\|y_k - \bar{y}\|}{\|x_k - \bar{x}\|}, \frac{\|x_k^*\|}{\|x_k - \bar{x}\|}, \frac{\|y_k - \bar{y}\|}{\|x_k - \bar{x}\|} + \|x_k^*\| \right\} \quad \text{or} \quad \text{srg}_1^+ F(\bar{x}, \bar{y}) = \lim_{k \to \infty} \frac{\|y_k - \bar{y}\|}{\|x_k - \bar{x}\|} + \|x_k^*\|.
\]

Since both \( X \) and \( Y \) are finite dimensional, by passing to subsequences without relabelling, we can assume that

\[
\left( \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|}, \frac{y_k - \bar{y}}{\|x_k - \bar{x}\|}, x_k^*, y_k^* \right) \to (u, v, x^*, y^*).
\]

(51)

It follows that \((u, v, x^*) \in \mathbb{S}_{X \times Y} \) and \((x^*, v) \in \hat{D}^* F(\bar{x}, \bar{y})(u, y^*) \), and consequently,

\[
\text{srg} F(\bar{x}, \bar{y}) \leq \max\{\|u\|, \|x^*\|\} = \text{srg}_1 F(\bar{x}, \bar{y}) \quad \text{or} \quad \text{srg}_1^+ F(\bar{x}, \bar{y}) \leq \|u\| + \|x^*\| = \text{srg}_1^+ F(\bar{x}, \bar{y}) .
\]

The opposite inequalities are valid in arbitrary normed spaces and rely on the fact that, by the definition of \( \hat{D} F \), for any \((u, y^*) \in \mathbb{S}_{X \times Y} \) and \((x^*, v) \in \hat{D}^* F(\bar{x}, \bar{y})(u, y^*) \), there are sequences \((x_k, y_k) \to (\bar{x}, \bar{y}) \) and \((x_k^*, y_k^*) \) with \( x_k^* \in D^* F(x_k, y_k)(y_k^*) \) and \( y_k^* \in \mathbb{S}_Y \), verifying (51).

The next theorem contains characterizations of subregularity from [15], which play a key role when establishing radii estimates. Note that condition \( \text{srg}_1^+ F(\bar{x}, \bar{y}) = 0 \) in part (i) does not in general imply the absence of subregularity; consider the zero function in Example 2.5.

**Theorem 5.3** Let \( X \) and \( Y \) be Banach spaces, \( F : X \rightrightarrows Y \), and \((\bar{x}, \bar{y}) \in \text{gph} F \).

(i) If \( \text{srg}_1^+ F(\bar{x}, \bar{y}) = 0 \), then there exists a \( C^1 \) function \( f : X \to Y \) with \( f(\bar{x}) = 0 \) and \( V f(\bar{x}) = 0 \) such that \( F + f \) is not subregular at \((\bar{x}, \bar{y})\).

(ii) Suppose that \( X \) and \( Y \) are Asplund spaces, and \( \text{gph} F \) is closed. If \( \text{srg}_1 F(\bar{x}, \bar{y}) > 0 \), then \( F \) is subregular at \((\bar{x}, \bar{y})\).

**Proof** Assertion (i) is a consequence of [15, Theorem 3.2(2)]. Assertion (ii) follows from the Asplund space part of [15, Theorem 3.2(1)] after observing that condition \( \text{srg}_1 F(\bar{x}, \bar{y}) > 0 \) is equivalent to \((0, 0) \notin \text{Cr}_0(\bar{x}, \bar{y}) \), where

\[
\text{Cr}_0(\bar{x}, \bar{y}) := \left\{ (v, x^*) \in Y \times X^* \left| \exists t_k \downarrow 0, (u_k^*, x_k^*) \to (v, x^*), (u_k, y_k^*) \in \mathbb{S}_X \times \mathbb{S}_Y^* \right. \right. \text{ with } x_k^* \in D^* F(\bar{x} + t_k u_k, \bar{y} + t_k y_k)(y_k^*) \right\}
\]
is the limit set critical for subregularity [15]. □

**Remark 5.4** Using [32, Corollary 5.8] with condition (g), one can strengthen Theorem 5.3(ii) and show that $\text{srg}_1 F(\bar{x}, \bar{y}) \leq \text{srg} F(\bar{x}, \bar{y})$.

In addition to $\mathcal{F}_{\text{clm}}$ and $\mathcal{F}_{\text{lip}}$ defined by (16) and (17), respectively, we are going to consider the following three classes of perturbations of $F$ near $\bar{x}$:

\[
\mathcal{F}_{\text{clm}} := \{ f : X \to Y \mid f \text{ is firmly calm at } \bar{x} \},
\]

\[
\mathcal{F}_{\text{lip}+\text{ss}^*} := \{ f : X \to Y \mid f \text{ is Lipschitz continuous around } \bar{x} \text{ and semismooth}^* \text{ at } \bar{x} \},
\]

\[
\mathcal{F}_{\text{clm}+\text{ss}^*} := \{ f : X \to Y \mid f \text{ is firmly calm and semismooth}^* \text{ at } \bar{x} \}.
\]

The next relations are immediate from the definitions:

\[
\mathcal{F}_{\text{lip}+\text{ss}^*} = \mathcal{F}_{\text{lip}} \cap \mathcal{F}_{\text{clm}+\text{ss}^*}, \quad \mathcal{F}_{\text{lip}} \cup \mathcal{F}_{\text{clm}+\text{ss}^*} \subset \mathcal{F}_{\text{clm}}.
\]

We now formulate stability estimates for the property of subregularity with respect to these classes of perturbations. Without loss of generality, we will assume that perturbation functions $f$ in the above definitions satisfy $f(\bar{x}) = 0$. In accordance with (21), the corresponding radii are defined as follows:

\[
\text{rad}[\text{SR}]_{\text{clm}} F(\bar{x}, \bar{y}) := \inf_{f \in \mathcal{F}_{\text{clm}}} \{ \text{clm} f(\bar{x}) \mid F + f \text{ is not subregular at } (\bar{x}, \bar{y}) \},
\]

\[
\text{rad}[\text{SR}]_{\text{lip}+\text{ss}^*} F(\bar{x}, \bar{y}) := \inf_{f \in \mathcal{F}_{\text{lip}+\text{ss}^*}} \{ \text{lip} f(\bar{x}) \mid F + f \text{ is not subregular at } (\bar{x}, \bar{y}) \},
\]

\[
\text{rad}[\text{SR}]_{\text{clm}+\text{ss}^*} F(\bar{x}, \bar{y}) := \inf_{f \in \mathcal{F}_{\text{clm}+\text{ss}^*}} \{ \text{clm} f(\bar{x}) \mid F + f \text{ is not subregular at } (\bar{x}, \bar{y}) \}.
\]

It is easy to check that

\[
\text{rad}[\text{SR}]_{\text{clm}} F(\bar{x}, \bar{y}) \leq \min \{ \text{rad}[\text{SR}]_{\text{lip}} F(\bar{x}, \bar{y}), \text{rad}[\text{SR}]_{\text{clm}+\text{ss}^*} F(\bar{x}, \bar{y}) \},
\]

\[
\max \{ \text{rad}[\text{SR}]_{\text{lip}} F(\bar{x}, \bar{y}), \text{rad}[\text{SR}]_{\text{clm}+\text{ss}^*} F(\bar{x}, \bar{y}) \} \leq \text{rad}[\text{SR}]_{\text{lip}+\text{ss}^*} F(\bar{x}, \bar{y}).
\]

**Theorem 5.5** Let $X$ and $Y$ be Banach spaces, $F : X \rightharpoonup Y$, and $(\bar{x}, \bar{y}) \in \text{gph } F$. Then

\[
\text{rad}[\text{SR}]_{\text{lip}} F(\bar{x}, \bar{y}) \leq \text{srg}^+_1 F(\bar{x}, \bar{y}),
\]

\[
\text{rad}[\text{SR}]_{\text{clm}} F(\bar{x}, \bar{y}) \leq \text{srg}^+_2 F(\bar{x}, \bar{y}),
\]

\[
\text{rad}[\text{SR}]_{\text{clm}+\text{ss}^*} F(\bar{x}, \bar{y}) \leq \text{srg}^+_4 F(\bar{x}, \bar{y}).
\]

Suppose that $X$ and $Y$ are Asplund spaces and $\text{gph } F$ is closed. Then

\[
\text{srg}^+_1 F(\bar{x}, \bar{y}) \leq \text{rad}[\text{SR}]_{\text{lip}} F(\bar{x}, \bar{y}) \leq \text{srg}^+_1 F(\bar{x}, \bar{y}),
\]

\[
\text{srg}^+_1 F(\bar{x}, \bar{y}) \leq \text{rad}[\text{SR}]_{\text{clm}} F(\bar{x}, \bar{y}) \leq \text{srg}^+_1 F(\bar{x}, \bar{y})
\]

\[
\text{srg}^+_1 F(\bar{x}, \bar{y}) \leq \text{rad}[\text{SR}]_{\text{clm}+\text{ss}^*} F(\bar{x}, \bar{y}) \leq \text{srg}^+_1 F(\bar{x}, \bar{y}).
\]
Remark 5.6

(i) In view of Proposition 5.2(vii) and (viii), in finite dimensions, estimates (55) in Theorem 5.5 recapture (28) in Theorem 2.6.

(ii) In view of Proposition 5.2(iv), the upper bound \( srg^+_1 F(\bar{x}, \bar{y}) \) for \( \text{rad}[\text{SR}]_{\text{lip} \ast} F(\bar{x}, \bar{y}) \) in (55) differs from the lower bound \( srg_1 F(\bar{x}, \bar{y}) \) by a factor of at most two. As a consequence, in the Asplund space setting, the property of subregularity is stable under small Lipschitz continuous perturbations if and only if \( srg_1 F(\bar{x}, \bar{y}) > 0 \).

(iii) Formula (56) gives the exact value for the radius of subregularity under firmly calm perturbations in Asplund spaces.

(iv) We do not know an upper bound for the radius of subregularity under semismooth* and Lipschitz continuous perturbations.

(v) For firmly calm and semismooth* perturbations, we do not know how much the upper bound \( srg^+_4 F(\bar{x}, \bar{y}) \) differs from the lower bound \( srg_4 F(\bar{x}, \bar{y}) \).

The next lemma is a key ingredient (together with Theorem 5.3(i)) of the proof of the first part of Theorem 5.5. The proofs of the lemma and the second part of Theorem 5.5 are given in Sect. 6.

Lemma 5.7 Let \( X \) and \( Y \) be Banach spaces, \( F : X \rightrightarrows Y, (\bar{x}, \bar{y}) \in \text{gph } F, \) and \( \gamma > 0 \).

(i) If \( srg^+_1 F(\bar{x}, \bar{y}) < \gamma \), then there exists a function \( f \in \mathcal{F}_{\text{lip}} \) such that \( \text{lip } f(\bar{x}) < \gamma \) and \( srg^+_1 (F + f) (\bar{x}, \bar{y}) = 0 \).

(ii) If \( srg^+_2 F(\bar{x}, \bar{y}) < \gamma \), then there exists a function \( f \in \mathcal{F}_{\text{fclm}} \) such that \( \text{clm } f(\bar{x}) < \gamma \) and \( srg^+_1 (F + f) (\bar{x}, \bar{y}) = 0 \).

(iii) If \( srg^+_4 F(\bar{x}, \bar{y}) < \gamma \), then there exists a function \( f \in \mathcal{F}_{\text{fclm} + \ast} \) such that \( \text{clm } f(\bar{x}) < \gamma \) and \( srg^+_1 (F + f) (\bar{x}, \bar{y}) = 0 \).

The next statement is a consequence of Lemma 5.7 and Theorem 5.3(i). It complements Theorem 5.3(i) and immediately implies the estimates in the first part of Theorem 5.5.

Corollary 5.8 Let \( X \) and \( Y \) be Banach spaces, \( F : X \rightrightarrows Y, (\bar{x}, \bar{y}) \in \text{gph } F, \) and \( \gamma > 0 \).
(i) If \( \text{srg}_1^+ F(\bar{x}, \bar{y}) < \gamma \), then there exists a function \( f \in \mathcal{F}_{lip} \) such that \( \text{lip} f(\bar{x}) < \gamma \), and \( F + f \) is not subregular at \((\bar{x}, \bar{y})\).

(ii) If \( \text{srg}_1^+ F(\bar{x}, \bar{y}) < \gamma \), then there exists a function \( f \in \mathcal{F}_{fclm} \) such that \( \text{clm} f(\bar{x}) < \gamma \), and \( F + f \) is not subregular at \((\bar{x}, \bar{y})\).

(iii) If \( \text{srg}_1^+ F(\bar{x}, \bar{y}) < \gamma \), then there exists a function \( f \in \mathcal{F}_{fclm+ss} \) such that \( \text{clm} f(\bar{x}) < \gamma \), and \( F + f \) is not subregular at \((\bar{x}, \bar{y})\).

Example 5.9 Let \( F : \mathbb{R} \to \mathbb{R} \) be given by

\[
F(x) := \begin{cases} 
    x \sin \frac{1}{x} & \text{if } x \neq 0, \\
    0 & \text{if } x = 0.
\end{cases}
\]

and \( \bar{x} = \bar{y} = 0 \). The function \( F \) is obviously subregular at \((0, 0)\). Next we show that the property is stable with respect to perturbations from \( \mathcal{F}_{fclm+ss} \). For every \( x \neq 0 \), we have \( y := F(x) = x \sin \frac{1}{x} \) and \( y' = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \). If \( |y'| = 1 \) and \( x^* = y^* y^* \), then

\[
\alpha := \frac{|\langle x^*, x - \bar{x} \rangle - \langle y^*, y - \bar{y} \rangle|}{\|\langle x^*, y^* \rangle\|\|\langle x, x^* \rangle\|} = \frac{\left| \left( \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \right) y^* x - y^* x \sin \frac{1}{x} \right|}{\max\{|x^*|, 1\} \left( |x| + |x \sin \frac{1}{x}| \right)}.
\]

If \( |x^*| \to +\infty \), then \( \frac{1}{x} \cos \frac{1}{x} \to +\infty \), in which case \( \frac{1}{x} \cos \frac{1}{x} \to 1 \). Hence, \( \alpha \to 0 \) if and only if \( \frac{1}{x} \cos \frac{1}{x} \to 0 \) as \( x \to 0 \). The latter condition obviously implies \( \sin \frac{1}{x} \to 1 \), and consequently, \( |x^*| \to 1 \) and \( |y|/|x| \to 1 \). By (39), (40), (44) and (47), \( \text{srg}_1^+ F(\bar{x}, \bar{y}) = \text{srg}_1^+ F(\bar{x}, \bar{y}) = 1 \), and it follows from Theorem 5.5 that \( \text{rad}[\text{SR}]_{fclm+ss} F(\bar{x}, \bar{y}) = 1 \).

At the same time, the property is not stable with respect to perturbations from \( \mathcal{F}_{fclm} \). Set \( x_k := (k \pi)^{-1} \) for all \( k \in \mathbb{N} \). Then \( F(x_k) = 0 \), \( F \) is differentiable at \( x_k \), and \( x_k \to \bar{x} \). By (37), (38) and (42), \( \text{srg}_2^+ F(\bar{x}, \bar{y}) = 0 \), and it follows from Theorem 5.5 that \( \text{rad}[\text{SR}]_{fclm} F(\bar{x}, \bar{y}) = 0 \).

The formulas for \( \text{rad}[\text{SR}]_{fclm} F(\bar{x}, \bar{y}) \) in Theorem 5.5 together with definitions (37), (38), (42) and (46) suggest that the property of subregularity is not very likely to be stable with respect to perturbations from \( \mathcal{F}_{fclm} \) whenever the limit in (8) equals 0, i.e., \( F \) is not strongly subregular at \((\bar{x}, \bar{y})\). The next example illustrates such an unlikely situation.

Example 5.10 Let \( F : \mathbb{R} \to \mathbb{R} \) be given by

\[
F(x) := \begin{cases} 
    x \sin \frac{1}{x} & \text{if } x \neq 0, \\
    0 & \text{if } x = 0.
\end{cases}
\]
\[ F(x) := \begin{cases} 
-x, x & \text{if } x = 1/k \text{ for some } k \in \mathbb{N}, \\
x & \text{otherwise}, 
\end{cases} \]

and \( \bar{x} = \bar{y} = 0 \). With \( x_k := 1/k \), we obviously have \( 0 \in F(x_k) \), and \( \bar{x} \neq x_k \to \bar{x} \); hence, the limit in (8) equals 0. At the same time, for any \( y \in (-x_k, x_k) \) and \((x^*, y^*) \in N_{\text{gph} F(x_k, y)} \), it holds \( y^* = 0 \), i.e., \( D^* F(x_k, y)(S_{Y^*}) = \emptyset \). At all other points \((x, y) \in \text{gph} F \) with \( x \neq 0 \), i.e., when either \( x = x_k \) and \( y = \pm x_k \) for some \( k \in \mathbb{N} \) or \( y = x \), we have \( ||y - \bar{y}||/||x - \bar{x}|| = 1 \), and \( D^* F(x, y)(S_{Y^*}) \neq \emptyset \). By (37), (38), (42) and (46), \( srg_2 F(\bar{x}, \bar{y}) = srg_2^* F(\bar{x}, \bar{y}) = 1 \), and it follows from Theorem 5.5 that \( \text{rad}[SR]_{\text{fclm}} F(\bar{x}, \bar{y}) = 1 \).

In the case when the lower limit in (8) is strictly positive, i.e., \( F \) is strongly subregular at \((\bar{x}, \bar{y})\), the property of subregularity is stable with respect to perturbations from \( \mathcal{F}_{\text{fclm}} \) by Theorem 5.5. We now present the exact formula for the radii of strong subregularity \( \text{rad}[sSR]_{\text{clm}} F(\bar{x}, \bar{y}) \) defined in accordance with (21) under calm, firmly calm, or firmly calm semismooth* perturbations, i.e., with \( \mathcal{P} \) equal to \text{clm}, \text{fclm}, or \text{fclm + ss}*. 

**Theorem 5.11** Let \( X \) and \( Y \) be Banach spaces, and \( F : X \rightrightarrows Y \) be strongly subregular at \((\bar{x}, \bar{y}) \in \text{gph} F \). Then

\[
\text{rad}[sSR]_{\text{fclm + ss}^*} F(\bar{x}, \bar{y}) = \text{rad}[sSR]_{\text{clm}} F(\bar{x}, \bar{y}) = \text{rad}[sSR]_{\text{fclm}} F(\bar{x}, \bar{y}) = srg F(\bar{x}, \bar{y}).
\]

**Remark 5.12** The last equality in (59) strengthens the corresponding assertion in Theorem 2.3 (taken from [11, Theorem 4.6]) which only guarantees this equality in finite dimensions.

The next lemma is a key ingredient in the proof of Theorem 5.11. The proof of the lemma is given in Sect. 6.

**Lemma 5.13** Let \( X \) and \( Y \) be Banach spaces, and \( F : X \rightrightarrows Y \) be strongly subregular at \((\bar{x}, \bar{y}) \in \text{gph} F \). Then

\[
\text{rad}[sSR]_{\text{fclm + ss}^*} F(\bar{x}, \bar{y}) \leq srg F(\bar{x}, \bar{y}).
\]

**Proof** of Theorem 5.11 The statement is a consequence of Lemma 5.13 and the next chain of inequalities:

\[
srg F(\bar{x}, \bar{y}) \leq \text{rad}[sSR]_{\text{clm}} F(\bar{x}, \bar{y}) \leq \text{rad}[sSR]_{\text{fclm}} F(\bar{x}, \bar{y}) \leq \text{rad}[sSR]_{\text{fclm + ss}^*} F(\bar{x}, \bar{y}).
\]

For the first inequality we refer to [6, Theorem 2.1], while the other two are trivial since \( \mathcal{F}_{\text{fclm + ss}^*} \subset \mathcal{F}_{\text{fclm}} \subset \mathcal{F}_{\text{clm}} \). \(\square\)
6 Proofs of the estimates for the radii of subregularity and strong subregularity

6.1 Lipschitz perturbations: the upper estimate

We prove Lemma 5.7(i) which provides a key ingredient for the proof of the estimate (52) in Theorem 5.5.

Suppose that \( srg^+ F(\bar{x}, \bar{y}) < \gamma < +\infty \). By definitions (37) and (45), there exist sequences \( x_k \to \bar{x} \) with \( x_k \neq \bar{x} \), \( \varepsilon_k \downarrow 0 \), \( y_k \in F(x_k) \), \( y_k^* \in \mathbb{S}_Y \), and \( x_k^* \in D^* \varepsilon_k F(x_k, y_k)(y_k^*) \) such that

\[
\gamma' := \sup_{k \in \mathbb{N}} \left( \left\| x_k^* \right\| + \frac{\left\| y_k - \bar{y} \right\|}{\left\| x_k - \bar{x} \right\|} \right) < \gamma. \tag{61}
\]

Denote \( t_k := \left\| x_k - \bar{x} \right\| \). By passing to subsequences, we can ensure that

\[
t_{k+1} < \frac{t_k}{2(k+1)} \quad (k \in \mathbb{N}). \tag{62}
\]

Set

\[
\rho_k := \frac{k}{k+1}, \quad \alpha_k := t_k + \rho_k \quad \text{and} \quad \beta_k := t_k - \rho_k. \tag{63}
\]

From (62) and (63) we obtain

\[
\alpha_{k+1} = \frac{2k+3}{k+2} t_{k+1} < \frac{2k+3}{2(k+1)(k+2)} t_k < \frac{1}{k+1} t_k = \beta_k. \tag{64}
\]

Observe that \( \beta_k < \left\| x - \bar{x} \right\| < \alpha_k \) for all \( x \in B_{\rho_k}(x_k) \). Hence,

\[
B_{\rho_k}(x_k) \cap B_{\rho_i}(x_i) = \emptyset \quad \text{for all} \quad k \neq i. \tag{65}
\]

For each \( k \in \mathbb{N} \), choose a point \( v_k \in Y \) such that

\[
\left\langle y_k^*, v_k \right\rangle = 1 \quad \text{and} \quad \left\| v_k \right\| < 1 + 1/k. \tag{66a}
\]

For all \( k \in \mathbb{N} \) and \( x \in X \), set

\[
s_k(x) := \max \left\{ 1 - \left( \left\| x - x_k \right\| / \rho_k \right)^{1 + \frac{1}{r}}, 0 \right\}, \tag{66b}
\]

\[
g_k(x) := y_k - \bar{y} + \left\langle x_k^*, x - x_k \right\rangle v_k, \tag{66c}
\]

\[
f_k(x) := s_k(x) g_k(x). \tag{66d}
\]

Observe that \( s_k(x) = 0 \) and \( f_k(x) = 0 \) for all \( x \notin B_{\rho_k}(x_k) \). In view of (64), the function
\[ f(x) := -\sum_{k=1}^{\infty} f_k(x), \quad x \in X, \] (67)

is well defined, \( f(x) = -f_k(x) \) for all \( x \in B_{\rho_k}(x_k) \) and all \( k \in \mathbb{N} \), and \( f(x) = 0 \) for all \( x \not\in \bigcup_{k=1}^{\infty} B_{\rho_k}(x_k) \). In particular, \( f(\bar{x}) = 0 \). Observing that \( s_k(x_k) = 1 \), and the function \( s_k \) is differentiable at \( x_k \) with \( \nabla s_k(x_k) = 0 \), we have

\[
  f(x_k) = \bar{y} - y_k \quad \text{and} \quad \nabla f(x_k) = -\langle x_k^*, \cdot \rangle v_k \quad \text{for all} \quad k \in \mathbb{N}. \quad (68)
\]

Given any \( x, x' \in \overline{B}_{\rho_k}(x_k) \), by the mean-value theorem applied to the function \( t \mapsto t^{1+\frac{1}{k}} \) on \( \mathbb{R}_+ \), there is a number \( \theta \in [0, 1] \) such that

\[
s_k(x) - s_k(x') = -\left(1 + \frac{1}{k}\right)\rho_k^{-1+\frac{1}{k}}(\theta\|x-x_k\| + (1-\theta)\|x'-x_k\|)\frac{1}{1+\frac{1}{k}}\|x-x_k\|\|x-x'\|
\]

and consequently,

\[
|s_k(x) - s_k(x')| \leq \left(1 + \frac{1}{k}\right)\rho_k^{-1+\frac{1}{k}}\max\{\|x-x_k\|, \|x'-x_k\|\}\frac{1}{1+\frac{1}{k}}\|x-x'\|.
\]

If \( x \neq x' \) and \( \|x-x_k\| \leq \|x'-x_k\| \), we obtain:

\[
\frac{|s_k(x) - s_k(x')|}{\|x-x'\|} \leq \left(1 + \frac{1}{k}\right)\rho_k^{-1+\frac{1}{k}}\left(\frac{\|y_k - \bar{y}\|}{\rho_k} + \frac{\|x'_k\|\|v_k\|\|x-x_k\|}{\rho_k}\right)
\]

\[
\leq \left(1 + \frac{1}{k}\right)\left(\frac{\|y_k - \bar{y}\|}{\rho_k} + \left(\frac{\|x'-x_k\|}{\rho_k}\right)^{1+\frac{1}{k}}\|x'_k\|\|v_k\|ight).
\]

\[
s_k(x')\frac{\|g_k(x) - g_k(x')\|}{\|x-x'\|} \leq \left(1 - \left(\frac{\|x'-x_k\|}{\rho_k}\right)^{1+\frac{1}{k}}\right)\|x'_k\|\|v_k\|.
\]

Combining the above estimates, we get

\[
\frac{|f_k(x) - f_k(x')|}{\|x-x'\|} \leq \frac{|s_k(x) - s_k(x')|}{\|x-x'\|} \frac{\|g_k(x)\| + s_k(x')\frac{\|g_k(x) - g_k(x')\|}{\|x-x'\|}}{\|x-x'\|}
\]

\[
\leq \left(1 + \frac{1}{k}\right)\frac{\|y_k - \bar{y}\|}{\rho_k} + \frac{1}{k}\left(\frac{\|x'-x_k\|}{\rho_k}\right)^{1+\frac{1}{k}}\|x'_k\|\|v_k\| + \|x'_k\|\|v_k\|
\]

\[
\leq \left(1 + \frac{1}{k}\right)\left(\frac{\|y_k - \bar{y}\|}{\rho_k} + \|x'_k\|\|v_k\|\right),
\]

(69)

and it follows from (61), (63), (65) and (69) that

\[
\frac{|f_k(x) - f_k(x')|}{\|x-x'\|} \leq \left(1 + \frac{1}{k}\right)^2 \left(\frac{\|y_k - \bar{y}\|}{t_k} + \|x'_k\|\right) \leq \left(1 + \frac{1}{k}\right)^2 \gamma'.
\]
Choose numbers \( \hat{k} \in \mathbb{N} \) and \( \gamma'' \) such that \((1 + 1/\hat{k})^2 \gamma' < \gamma'' < \gamma\). Thus, for any \( k > \hat{k} \), the function \( f_k \) is Lipschitz continuous on \( B_{\rho_k}(x_k) \) with modulus \( \gamma'' \). In particular, given any \( x \in B_{\rho_k}(x_k) \) with \( x \neq x_k \), and taking \( x' := x_k + \rho_k \frac{x-x_k}{\|x-x_k\|} \in B_{\rho_k}(x_k) \), we have \( f_k(x') = 0 \), and consequently, \( \|f_k(x)\| \leq \gamma'' \|x - x'\| \). Thus,

\[
\|f_k(x)\| \leq \gamma''(\rho_k - \|x - x_k\|) \quad \text{for all} \quad x \in B_{\rho_k}(x_k). \tag{70}
\]

Next, we show that the function \( f \) is Lipschitz continuous with modulus \( \gamma'' \) on \( B_{\rho_k}(\bar{x}) \) for any \( k > \hat{k} \). Indeed, let \( k \leq \hat{k}, x, x' \in B_{\rho_k}(\bar{x}) \) and \( x \neq x' \).

1) If \( x, x' \in B_{\rho_k}(x_i) \) for some \( i \geq k \), then, as shown above, \( \|f(x) - f(x')\| < \gamma'' \|x - x'\| \).

2) If \( x \in B_{\rho_k}(x_i) \) and \( x' \in B_{\rho_j}(x_j) \) for some \( i, j \geq k \) with \( i \neq j \), then, thanks to (64), we have \( \|x_i - x_j\| \geq \rho_i + \rho_j \), and using (70), we obtain

\[
\|f(x) - f(x')\| \leq \|f(x)\| + \|f(x')\| \leq \gamma''(\rho_i + \rho_j - \|x - x_i\| - \|x' - x_j\|) \
\leq \gamma''(\|x_i - x_j\| - \|x - x_i\| - \|x' - x_j\|) \leq \gamma'' \|x' - x'\|.
\]

3) If \( x \in B_{\rho_k}(x_i) \) for some \( i \geq k \), and \( x' \notin \bigcup_{j=k}^{\infty} B_{\rho_j}(x_j) \), then \( f(x') = 0 \), \( \|x' - x_i\| \geq \rho_i \), and using (70), we obtain

\[
\|f(x) - f(x')\| \leq \gamma''(\|x' - x_i\| - \|x - x_i\|) \leq \gamma'' \|x' - x'\|.
\]

4) If \( x, x' \notin \bigcup_{j=k}^{\infty} B_{\rho_j}(x_j) \), then \( f(x) = f(x') = 0 \).

Thus, in all cases, \( \|f(x) - f(x')\| \leq \gamma'' \|x' - x'\| \). Hence, \( f \in \mathcal{F}_{lip} \) and \( \text{lip}_{f}(\bar{x}) \leq \gamma'' < \gamma \). In view of (68) and (65), for all \( k \in \mathbb{N} \), we have \( \bar{y} = y_k + f(x_k) \) and \( D^*f(x_k)(y_k^*) = -\langle y_k^*, v_k \rangle x_k^* = -x_k^* \); hence \( \bar{y} \in (F + f)(x_k) \) and, by Theorem 3.5, \( 0 \in D^*_{\epsilon_k}(F + f)(x_k, \bar{y})(\gamma_k^*) \), where \( \epsilon_k^* := (\|\nabla f(x_k)\| + 1)\epsilon_k \). Thanks to (61), (65) and (68),

\[
\epsilon_k^* = (\|x_k^*\| \|v_k\| + 1)\epsilon_k \leq (2\|x_k^*\| + 1)\epsilon_k < (2\gamma + 1)\epsilon_k.
\]

Thus, \( \text{sgn}_{\text{sgn}}^*(F + f)(\bar{x}, \bar{y}) = 0 \).

### 6.2 Firmly calm perturbations: the upper estimate

We prove Lemma 5.7(ii) which provides a key ingredient for the proof of the estimate (53) in Theorem 5.5.

Suppose that \( \text{sgn}_{\text{sgn}}^2 F(\bar{x}, \bar{y}) < \gamma < +\infty \). By definitions (37) and (46), there exist sequences \( x_k \to \bar{x} \) with \( x_k \neq \bar{x}, \epsilon_k \downarrow 0, y_k \in F(x_k), y_k^* \in S_{y^*}, \) and \( x_k^* \in D^*_{\epsilon_k}(F(x_k, y_k)(y_k^*) \) such that

\[
\lim_{k \to \infty} \epsilon_k \|x_k^*\| = 0, \tag{71}
\]

\[
\gamma' := \sup_{k \in \mathbb{N}} \frac{\|y_k - \bar{y}\|}{\|x_k - \bar{x}\|} < \gamma. \tag{72}
\]

Denote \( t_k := \|x_k - \bar{x}\| \). By passing to subsequences, we can ensure estimate (62). Set
\[ \rho_k := \min \left\{ \frac{1}{k+1}, \frac{\gamma^\prime}{(k+1)(1 + \|x_k\|^2)} \right\} t_k, \tag{73} \]

\[ \alpha_k := t_k + \rho_k \text{ and } \beta_k := t_k - \rho_k, \text{ and observe that, thanks to (62),} \]

\[ \beta_k - \alpha_{k+1} \geq \frac{kt_k}{k+1} - \frac{k+3}{k+2} t_{k+1} > \left( \frac{k+3}{2(k+2)} \right) t_k + \frac{(k-1)t_k}{k+1} \geq 0. \]

As a consequence, condition (64) holds. For each \( k \in \mathbb{N} \), we choose a point \( v_k \in Y \) satisfying (65), and then define functions \( s_k, g_k, f_k \), and \( f \) by (66) and (67). Thanks to (64), function \( f \) is well defined. Moreover, it is differentiable at \( x_k \), and satisfies (68).

Hence \( \tilde{y} \in (F + f)(x_k) \) and, by Theorem 3.5, \( 0 \in D^*_k(F + f)(x_k, \tilde{y})(y_k^*) \), where \( \varepsilon_k^0 := (\|\nabla f(x_k)\| + 1) \varepsilon_k. \) Thanks to (65), (68), and (71), \( \varepsilon_k^0 = (\|x_k^*\|\|v_k\| + 1) \varepsilon_k \leq (2\|x_k^*\| + 1) \varepsilon_k \to 0 \) as \( k \to \infty. \) Thus, \( \text{sr}_0^+(F + f)(\tilde{x}, \tilde{y}) = 0. \)

As shown in Sect. 6.1, estimates (69) hold true for all \( x, x^* \in B_{\rho_k}(x_k) \), i.e., \( f \) is Lipschitz continuous on \( B_{\rho_k}(x_k) \) with modulus \( l_k := (1 + 1/k)(\|y_k - \tilde{y}\|/\rho_k + \|x_k^*\|\|v_k\|). \)

Since, \( f(x) = 0 \) for all \( x \notin \cup_{k=1}^{\infty} B_{\rho_k}(x_k) \), the function \( f \) is Lipschitz continuous around every \( x \neq \tilde{x} \) near \( \tilde{x}. \) We now show that \( \text{clm} f(\tilde{x}) < \gamma. \) Choose numbers \( k \in \mathbb{N} \) and \( \gamma'' \) such that \( (1 + 1/k)^2 \gamma' < \gamma'' < \gamma. \) Consider a point \( x \in X \) with \( 0 < \|x - \tilde{x}\| < t_k \) and \( f(x) \neq 0. \) Then there is a unique index \( k = k(x) \geq k \) such that \( x \in B_{\rho_k}(x_k) \) and \( f(x) = -f_k(x_k). \) Thus, by (65) and (73), we have

\[ \|x - \tilde{x}\| > t_k - \rho_k \geq kt_k/(k+1), \]

\[ \|g_k(x)\| \leq \|y_k - \tilde{y}\| + \rho_k \|x_k^*\| \|v_k\| < \|y_k - \tilde{y}\| + \gamma' t_k/2. \]

Since \( s_k(x) \leq 1 \) and \( s_k(\tilde{x}) = 0, \) we obtain from (66) and (72):

\[
\frac{\|f(x) - f(\tilde{x})\|}{\|x - \tilde{x}\|} \leq \frac{\|g_k(x)\|}{\|x - \tilde{x}\|} < \left( 1 + \frac{1}{k} \right) \left( \frac{\|y_k - \tilde{y}\|}{\|x_k^* - \tilde{x}\|} + \frac{\gamma'}{k} \right) \leq \left( 1 + \frac{1}{k} \right)^2 \gamma' < \gamma''.
\]

Hence, \( \text{clm} f(\tilde{x}) \leq \gamma'' < \gamma. \)

### 6.3 Firmly calm semismooth* perturbations: the upper estimate

We prove Lemma 5.7(iii) which provides a key ingredient for the proof of the estimate (54) in Theorem 5.5.

Suppose that \( \text{sr}_0^+(F, \tilde{y}) < \gamma < +\infty. \) By definitions (39) and (47), there exist sequences \( x_k \to \tilde{x} \) with \( x_k \neq \tilde{x}, \varepsilon_k \downarrow 0, y_k \in F(x_k), y_k^* \in S_{y^*}, \) and \( x_k^* \in D_k^*(F(x_k, y_k)(y_k^*) \) such that conditions (71) and (72) are satisfied, and

\[ \frac{\|x_k^* - x_k - \tilde{x}\|}{\|x_k^* - \tilde{x}\|} \leq \varepsilon_k, \tag{74} \]

For every \( k \in \mathbb{N}, \) set \( t_k := \|x_k - \tilde{x}\| \) and \( u_k := (x_k - \tilde{x})/t_k \in S_x, \) and choose points \( u_k^* \in S_{x^*} \) and \( v_k \in Y \) such that

\[ \varepsilon \ Springer
\[ \langle u_k^*, u_k \rangle = 1, \quad \langle y_k^*, v_k \rangle = 1, \quad \|v_k\| < 2. \]  

(75)

Set
\[ \alpha_k(x) := \langle u_k^*, x \rangle, \quad r_k(x) := x - \alpha_k(x)u_k, \quad \hat{x}_k^* := x_k^* - \langle x_k^*, u_k \rangle u_k^*. \]  

(76)

Observe that
\[ \alpha_k(u_k) = 1, \quad r_k(u_k) = 0, \quad \langle \hat{x}_k^*, u_k \rangle = 0, \quad \langle \hat{x}_k^*, r_k(x) \rangle = \langle \hat{x}_k^*, x \rangle. \]  

(77)

Case 1: no element of the sequence \((u_k)\) appears infinitely many times. Without changing the notation, we now construct inductively a subsequence of \((u_k)\) (and the corresponding to it other involved sequences). For each \(k = 1, \ldots, \) we do the following. If \(u_k = \lim_{j \to \infty} u_j,\) remove \(u_k\) and all its finitely many copies from the sequence. Then \(s_k := \lim_{j \to \infty} \|u_j - u_k\| > 0.\) Choose a subsequence of \((u_j)_{j=k+1}^\infty\) such that \(\|u_j - u_k\| > s_k/2\) for all \(j > k.\)

Defining
\[ \rho_k := \inf\{\|u_j - u_k\| \mid j \neq k\}/2, \]

we have \(\rho_k \geq \min_{j \leq k} s_j > 0\) for all \(k \in \mathbb{N},\) and \(\|u_k - u_i\| \geq 2 \max\{\rho_k, \rho_i\}\) for all \(k \neq i,\)

and consequently,
\[ B_{\rho_k}(u_k) \cap B_{\rho_i}(u_i) = \emptyset \quad \text{for all} \quad k \neq i. \]  

(78)

Let \(\gamma'' \in (\gamma', \gamma).\) For each \(k \in \mathbb{N},\) choose a number \(\tau_k \in (0, \rho_k/2)\) such that \(\tau_k \|x_k^*\| < (\gamma'' - \gamma')/4,\) and define
\[ s_k(x) := \begin{cases} \max \left\{ 1 - \left( \frac{\|r_k(x)\|}{\tau_k \alpha_k(x)} \right)^2, 0 \right\} & \text{if} \ \alpha_k(x) > 0, \\ 0 & \text{otherwise}, \end{cases} \]  

(79)

\[ A_k(x) := \alpha_k(x)(y_k - \bar{y})/t_k + \langle \hat{x}_k^*, x \rangle v_k, \]  

(80)

\[ \hat{A}_k(x) := s_k(x)A_k(x). \]  

(81)

Observe that \(s_k(x) = 0\) and \(\hat{A}_k(x) = 0\) for all \(x \in X\) with \(\|r_k(x)\| \geq \tau_k \alpha_k(x).\) If \(\|r_k(x)\| < \tau_k \alpha_k(x),\) then by (76), \(x \neq 0\) and
\[
\|x - ||x||u_k\| = \|r_k(x) + (\alpha_k(x) - ||x||)u_k\| \leq \|r_k(x)|| + \|x\| - \alpha_k(x) \\
\leq 2\|r_k(x)|| < 2\tau_k \alpha_k(x) < \rho_k ||x||;
\]

hence, \(x/||x|| \in B_{\rho_k}(u_k).\) It follows from (78) that for every \(x \in X\) there is at most one \(k \in \mathbb{N}\) with \(\hat{A}_k(x) \neq 0.\) Thus, the function
is well defined. The function \( \hat{A}_k(x) \) is Lipschitz continuous around any point \( x \neq 0 \); hence, \( f \) is Lipschitz continuous around any point \( x \neq \bar{x} \). Further, \( s_k \) is positively homogenous at 0, while \( A_k \) is linear. Therefore \( \hat{A}_k \) is positively homogenous at 0, and \( f \) is positively homogenous at \( \bar{x} \) and, by Corollary 4.7, semismooth* at \( \bar{x} \). If \( \hat{A}_k(x) \neq 0 \), then \( \| r_k(x) \| < \tau_k \alpha_k(x) \) and, thanks to (72) and (75)-(77),

\[
\| \hat{A}_k(x) \| \leq \| A_k(x) \| \leq \alpha_k(x) \| y_k - \bar{y} \| / t_k + 2 \| \langle \hat{x}_k^*, x \rangle \| \leq \alpha_k(x) \gamma' + 2 \| \hat{x}_k^*, r_k(x) \| \leq (\gamma' + 2 \| \hat{x}_k^* \| \alpha_k(x)) \leq (\gamma' + 4 \| x_k^* \| \tau_k) \| x \| \leq (\gamma' + (\gamma'' - \gamma')) \| x \| = \gamma'' \| x \| ,
\]

and consequently, \( clm \hat{A}_k(0) \leq \gamma'' \). Hence, \( clm f(\bar{x}) \leq \gamma'' < \gamma \). Thus, \( f \in \mathcal{F}_{fclm+ss^*} \).

In view of (79) and (77), \( s_k(x_k - \bar{x}) = 1 \). Further, \( s_k \) is Fréchet differentiable at \( x_k - \bar{x} \) with derivative \( \nabla s_k(x_k - \bar{x}) = 0 \). Hence, using (77) again, we obtain:

\[
f(x_k) = -\hat{A}_k(x_k - \bar{x}) = \hat{y} - y_k , \tag{83}
\]

\[
\nabla f(x_k) = -\nabla \hat{A}_k(x_k - \bar{x}) = -\nabla A_k(x_k - \bar{x}) = -\langle u_k^*, \cdot \rangle (y_k - \bar{y}) / t_k - \langle \hat{x}_k^*, \cdot \rangle v_k . \tag{84}
\]

Thanks to (75) and (76), the latter equality yields

\[
D^* f(x_k)(v_k) = -\langle y_k^*, y_k - \bar{y} \rangle u_k^* / t_k - \hat{x}_k^* \langle y_k^*, v_k \rangle = -x_k^* - (\langle y_k^*, y_k - \bar{y} \rangle - \langle x_k^*, y_k - \bar{y} \rangle) u_k^* / t_k.
\]

By (83), \( \bar{y} \in (F + f)(x_k) \), and, by Theorem 3.5,

\[
\hat{u}_k^* = -\langle x_k^*, x_k - \bar{x} \rangle - \langle y_k^*, y_k - \bar{y} \rangle u_k^* / t_k \in D_{\varepsilon_k'}^*(F + f)(x_k, \bar{y})(y_k^*),
\]

where \( \varepsilon_k' := (\| \nabla f(x_k) \| + 1) \varepsilon_k \). In view of (72), (74), (76) and (84), we have

\[
\| \hat{u}_k^* \| = \| \langle x_k^*, x_k - \bar{x} \rangle - \langle y_k^*, y_k - \bar{y} \rangle \| / t_k \\
= \| \langle x_k^*, x_k - \bar{x} \rangle - \langle y_k^*, y_k - \bar{y} \rangle \| \| x_k - \bar{x} \| + \| y_k - \bar{y} \| / t_k \| x_k^* \|, \max \{ \| x_k^* \|, 1 \}
\leq \varepsilon_k (1 + \gamma) \max \{ \| x_k^* \|, 1 \},
\]

\[
\varepsilon_k' < (\gamma + 4 \| x_k^* \| + 1) \varepsilon_k. \tag{86}
\]

It follows from (71) that \( \hat{u}_k^* \to 0 \) and \( \varepsilon_k' \to 0 \) as \( k \to \infty \). Thus, \( srg_1^+(F + f)(\bar{x}, \bar{y}) = 0 \).

Case 2: There is an element \( u \) of the sequence \( (u_k) \) which appears infinitely many times. From now on, we consider the stationary subsequence \( (u) \) of the sequence \( (u_k) \), i.e., without changing the notation, we assume that \( (x_k - \bar{x}) / t_k = u \in S_X \) for all \( k \in \mathbb{N} \). Thus, subscript \( k \) can be dropped in many of the formulas in (75), (76) and (77):
\[ \langle u^*, u \rangle = \| u^* \| = 1, \quad \alpha(x) := \langle u^*, x \rangle, \quad r(x) := x - \alpha(x)u, \quad \hat{x}^*_k := x^*_k - \langle x^*_k, u \rangle u^*, \]  
\[
(87) \quad \alpha(u) = 1, \quad r(u) = 0, \quad \langle \hat{x}^*_k, u \rangle = 0, \quad \langle \hat{x}^*_k, r(x) \rangle = \langle \hat{x}^*_k, x \rangle. \tag{88}
\]

By passing to a subsequence again, we can also assume that the sequence \((t_k)\) satisfies
\[ t_k < e^{-x}t_{k-1}, \quad \text{or equivalently,} \quad \ln(t_{k-1}/t_k) > k \quad (k \geq 2). \tag{89} \]

Let \(\gamma'' \in (\gamma', \gamma)\). For each \(k \in \mathbb{N}\), choose a number \(r_k \in (0, 1)\) such that \(r_k \| x^*_k \| < (\gamma'' - \gamma')/4\), and consider mappings \(s_k, A_k\) and \(\hat{A}_k : X \to Y \) \((k \geq 1)\) defined by \((79)-(81)\). Note that in the considered case \(\alpha_k\) and \(r_k\) do not depend on \(k\); hence, the first two mappings take a slightly simplified form:

\[
s_k(x) := \begin{cases} 
\max \left\{ 1 - \left( \frac{\| r(x) \|}{\tau_k \alpha(x)} \right)^2, 0 \right\} & \text{if } \alpha(x) > 0, \\
0 & \text{otherwise,} 
\end{cases} \tag{90} \]

\[
A_k(x) := \alpha(x)(y_k - \bar{y})/t_k + \langle \hat{x}^*_k, x \rangle v_k. \tag{91} \]

Next we define mappings \(T_k : \{ x \in X \mid \alpha(x) > 0 \} \to Y \) \((k \geq 2)\) and \(g, f : X \to Y:\)

\[
T_k(x) := \hat{A}_k(x) + \frac{\ln(\alpha(x)/t_k)}{\ln(t_{k-1}/t_k)} (\hat{A}_{k-1}(x) - \hat{A}_k(x)), \tag{92} \]

\[
g(x) := \begin{cases} 
\hat{A}_1(x) & \text{if } \alpha(x) \geq t_1, \\
T_k(x) & \text{if } \alpha(x) \in [t_k, t_{k-1}) \text{ for some } k \geq 2, \\
0 & \text{if } \alpha(x) \leq 0, 
\end{cases} \tag{93} \]

\[
f(x) := - g(x - \bar{x}). \tag{94} \]

We will now show that \(g\) is Lipschitz continuous around every point in \(U := \{ x \in X \mid 0 < \| x \| < t_1 \}\), and semismooth* and calm at 0 with \(\text{clm} g(0) \leq \gamma''\).

The mappings \(T_k\) are Lipschitz continuous around every \(x \in X\) with \(\alpha(x) > 0\) as compositions of mappings having this property, and satisfy

\[
T_k(x) = T_{k+1}(x) = \hat{A}_k(x) \quad \text{whenever} \quad \alpha(x) = t_k. \tag{95} \]

Hence, \(g\) is Lipschitz continuous around every point \(x \in U\) with \(\alpha(x) > 0\). By \((90)\), the functions \(s_k\) vanish on the open set \(U_0 := \{ x \mid \alpha(x) < \| r(x) \| \}\). By \((81), (92)\) and \((93)\), so does \(g\). If \(\alpha(x) = \| r(x) \| = 0\), then, by \((87)\), \(x = 0\). Hence, \(\alpha(x) > 0\) for all \(x \in U \setminus U_0\), and consequently, \(g\) is Lipschitz continuous around every point in \(U\).

As shown in the preceding case, \(\hat{A}_k\) is positively homogeneous at 0, and
\[ \|\hat{A}_k(x)\| < \gamma''\|x\| \quad \text{for all } x \neq 0. \quad (96) \]

Let \(x \in U \setminus U_0\). Then \(0 < \alpha(x) < t_1\), and we can find a unique integer \(\hat{k} \geq 2\) such that \(\alpha(x) \in [t_{\hat{k}}, t_{\hat{k}-1})\). Thus, \(\xi := \ln(\alpha(x)/t_{\hat{k}}) / \ln(t_{\hat{k}-1}/t_{\hat{k}}) \in [0, 1)\), and from (92), (93) and (96) we obtain:

\[ \|g(x)\| = \|(1 - \xi)\hat{A}_{\hat{k}}(x) + \xi\hat{A}_{\hat{k}-1}(x)\| < (1 - \xi)\gamma''\|x\| + \xi\gamma''\|x\| = \gamma''\|x\|, \]

showing \(\text{clm } g(0) \leq \gamma''\).

The mapping \(A_k\) is linear and continuous, while the function \(s_k\), in view of (87) and (90), satisfies \(s_k(x \pm tx) = s_k(x)\) for all \(t \in (0, 1)\). Hence, \(A_k'(x; \pm x) = \pm \hat{A}_k(x)\). It is easy to check from (87) that \(\left(\ln(\alpha(\cdot)/t_k)\right)'(x; \pm x) = \pm 1\). Now it follows from (92) that

\[ T_k'(x; \pm x) = \pm \left( T_k(x) + \frac{\hat{A}_{k-1}(x) - \hat{A}_k(x)}{\ln(t_{k-1}/t_k)} \right). \quad (97) \]

If \(\alpha(x) \in (t_{\hat{k}}, t_{\hat{k}-1})\), then, by (93), \(g\) coincides with \(T_k\) around \(x\), yielding \(g'(x; \pm x) = T_k'(x; \pm x)\). Taking into account (89), (96) and (97), we obtain:

\[ \|g(x) - g'(x; x)\| = \left\| \frac{\hat{A}_{k-1}(x) - \hat{A}_k(x)}{\ln(t_{k-1}/t_k)} \right\| < \frac{2 \gamma\|x\|}{\hat{k}}, \quad (98) \]

and similarly, \(\|g(x) + g'(x; -x)\| < 2\gamma\|x\|/\hat{k}\). If \(\alpha(x) = t_{\hat{k}} (> 0)\), then we have \(g'(x, x) = T_k(x, x)\), yielding (98), and \(g'(x; -x) = T_k'(x; -x)\), yielding

\[ \|g(x) + g'(x; -x)\| = \left\| \frac{\hat{A}_k(x) - \hat{A}_{k+1}(x)}{\ln(t_k/t_{k+1})} \right\| < \frac{2 \gamma\|x\|}{k + 1}. \]

Since \(\alpha(x) \to 0\) and \(k \to \infty\) as \(U \setminus U_0 \ni x \to 0\), it follows from Proposition 4.6 that \(g\) is semismooth* at 0. We conclude that the function \(f\) defined by (94) belongs to \(\mathcal{F}_{\text{fclm+ss}}\), and \(\text{clm } f(\bar{x}) \leq \gamma'' < \gamma\).

Now, consider the functions \(h_k := g - \hat{A}_k\) \((k \geq 2)\). We claim that \(\text{clm } h_k(t_k u) < c_k := 2\gamma/k\). By (88), (93) and (95), \(a(t_k u) = t_k\), \(g(t_k u) = T_k(t_k u) = \hat{A}_k(t_k u)\), and consequently, \(h_k(t_k u) = 0\). In view of (92) and (93), the function \(h_k\) admits the following representation near \(t_k u\):

\[ h_k(x) = \begin{cases} h_k^1(x) & \text{if } \alpha(x) \in [t_k, t_{k-1}), \\ h_k^2(x) & \text{if } \alpha(x) \in [t_{k+1}, t_k), \end{cases} \]

where

\[ h_k^1(x) := \frac{\ln(\alpha(x)/t_k)}{\ln(t_{k-1}/t_k)} (\hat{A}_{k-1}(x) - \hat{A}_k(x)), \quad h_k^2(x) := \frac{\ln(\alpha(x)/t_k)}{\ln(t_{k+1}/t_k)} (\hat{A}_{k+1}(x) - \hat{A}_k(x)). \]
By (88) and (90), \( s_k(t_k u) = 1 \), and \( s_k \) is Fréchet differentiable at \( t_k u \) with \( \nabla s_k(t_k u) = 0 \). Moreover, \( \ln(\alpha(t_j u) / t_k) = 0 \). Thus, in view of (81), (87) and (88), \( h_k^1 \) is differentiable at \( t_k u \), and its derivative amounts to

\[
\nabla h_k^1(t_k u) = \frac{\alpha(\cdot)}{\ln(t_{k-1} / t_k)} (A_{k-1}(u) - A_k(u)) = \frac{\alpha(\cdot)}{\ln(t_{k-1} / t_k)} \left( \frac{y_{k-1} - \tilde{y}}{t_{k-1}} - \frac{y_k - \tilde{y}}{t_k} \right).
\]

Hence, in view of (72) and (89), we have \( \| \nabla h_k^1(t_k u) \| < 2\gamma / k = c_k \). The same arguments yield \( \| \nabla h_k^2(t_k u) \| < 2\gamma / (k + 1) = c_k \), and consequently,

\[
\text{clm} h_k(t_k u) = \max \{ \| \nabla h_k^1(t_k u) \|, \| \nabla h_k^2(t_k u) \| \} < c_k.
\]  

(99)

Next, for all \( x \in X \) and \( k \in \mathbb{N} \), set \( C_k(x) := \tilde{A}_k(x - \bar{x}) \). Then

\[
C_k(x_k) = s(t_k u) A_k(t_k u) = t_k A_k(u) = y_k - \bar{y}.
\]  

(100)

and \( C_k \) is Fréchet differentiable at \( x_k \) with

\[
\nabla C_k(x_k) = \nabla A_k(t_k u) = \langle u^*, \cdot \rangle (y_k - \bar{y}) / t_k + \left\langle \hat{x}_k^*, \cdot \right\rangle v_k.
\]

(101)

Thus, in view of (87),

\[
D^* C_k(x_k)(y_k^*) = \left\langle y_k^*, y_k - \bar{y} \right\rangle u^*/t_k + \hat{x}_k^* = x_k^* - \hat{u}_k^*.
\]

where \( \hat{u}_k^* := \left( \left\langle x_k^*, u \right\rangle - \left\langle y_k^*, y_k - \bar{y} \right\rangle \right) u^*/t_k \). By (100), \( \bar{y} \in (F - C_k)(x_k) \) and, by Theorem 3.5,

\[
\hat{u}_k^* \in D^*_{\epsilon_k'} (F - C_k)(x_k, \bar{y})(y_k^*),
\]

(102)

where \( \epsilon_k' := (\| \nabla C_k(x_k) \| + 1) \epsilon_k \). In view of (72), (74), (87) and (101), estimates (85) and (86) hold true. It follows from (71) that \( \hat{u}_k^* \to 0 \) and \( \epsilon_k' \to 0 \) as \( k \to \infty \). Observe that \( f(x) = -C_k(x) - h_k(x - \bar{x}) \) for all \( x \in X \). By (99) and (100), we have

\[
f(x_k) = -C_k(x_k) = \bar{y} - y_k \quad \text{and} \quad \text{clm} (f + C_k)(x_k) = \text{clm} h_k(t_k u) < c_k.
\]  

(103)

In view of (102) and (103), it follows from Lemma 3.6 that \( \hat{u}_k^* \in D^*_{\delta_k}(F + f)(x_k, \bar{y})(y_k^*) \), where \( \delta_k := (\epsilon_k' + c_k) / (1 - c_k) \to 0 \) as \( k \to \infty \). Hence, \( \text{srg}_1(F + f)(\bar{x}, \bar{y}) = 0 \).

### 6.4 The lower estimates

We prove the lower estimates in (55)–(58). Combined with the first part of (5.5), they prove the second part of the theorem.

Suppose that \( F + f \) is not subregular at \( (\bar{x}, \bar{y}) \) for some function \( f \in \mathcal{F}_{\text{clm}} \). It follows from Theorem 5.3(ii) that \( \text{srg}_1(F + f)(\bar{x}, \bar{y}) = 0 \). By definitions (37) and (41), there exist sequences \( x_k \neq \bar{x}, \ y_k \in F(x_k), \ y_k^* \in S_{Y^*} \) and \( x_k^* \in D^*(F + f)(x_k, y_k + f(x_k))(y_k^*) \) such that \( f \) is Lipschitz continuous near \( x_k \), and
\[ \alpha_k := \max \left\{ \| x_k - \bar{x} \|, \| x_k^* \|, \frac{\| y_k + f(x_k) - \bar{y} \|}{\| x_k - \bar{x} \|} \right\} \to 0. \quad (104) \]

For each \( k \in \mathbb{N} \), set \( t_k := \| x_k - \bar{x} \| \), and choose a positive number \( \epsilon_k < \min \left\{ \frac{\alpha_k t_k}{\alpha_k + 1}, \frac{1}{2} \right\} \). \( \quad (105) \)

Then \( 0 < \epsilon_k < t_k \leq \alpha_k \). By Theorem 3.4(ii), there exist \( x_{1k}, x_{2k} \in B_{\epsilon_k}(x_k), \)
\( y_{1k} \in F(x_{1k}) \cap B_{\epsilon_k}(y_k), \quad y_{1k}^*, y_{2k}^* \in B_{\epsilon_k}(y_k^*), \quad x_{1k}^* \in D^*F(x_{1k}, y_{1k}^*) \)
and \( x_{2k}^* \in D^*f(x_{2k}, y_{2k}^*) \) such that
\[ \| x_{1k}^* + x_{2k}^* - x_k^* \| < \epsilon_k. \quad (106) \]

Observe that
\[ \min\{\| y_{1k}^* \|, \| y_{2k}^* \| \} > 1 - \epsilon_k > \frac{1}{2}, \]
\[ \min\{\| x_{1k} - \bar{x} \|, \| x_{2k} - \bar{x} \| \} > t_k - \epsilon_k > \frac{t_k}{\alpha_k + 1} > 0. \quad (108) \]

Let \( \text{clm} f(\bar{x}) < \gamma < +\infty \), and choose a number \( \gamma' \in (\text{clm} f(\bar{x}), \gamma) \). In view of (104), (105) and (108), we have for all sufficiently large \( k \in \mathbb{N} \):
\[ \frac{\| y_{1k} - \bar{y} \|}{\| x_{1k} - \bar{x} \|} < \frac{\| y_k - \bar{y} \| + \epsilon_k}{(\alpha_k + 1)^{-1}t_k} \leq \frac{\| f(x_k) \| + \| y_k + f(x_k) - \bar{y} \| + \epsilon_k}{(\alpha_k + 1)^{-1}t_k} \]
\[ < (\alpha_k + 1)(\gamma' + \alpha_k) + \alpha_k = \gamma' + \alpha_k(\gamma' + \alpha_k + 1) < \gamma. \quad (109) \]

In particular, \( y_{1k} \to \bar{y} \) as \( k \to \infty \). Letting \( \gamma \downarrow \text{clm} f(\bar{x}) \) and observing that the function \( f \in \mathcal{P}_{\text{clm}} \) is arbitrary, inequality (109) implies \( \text{rad}[\mathcal{S}R]_{\text{clm}} F(\bar{x}, \bar{y}) \geq \text{srg}_2 F(\bar{x}, \bar{y}) \). Combined with the established above inequality (53) and Proposition 5.2(v), this proves (56).

We now start imposing additional assumptions on the function \( f \) and the other parameters introduced above. Recall that \( \mathcal{P}_{\text{lip}} \cup \mathcal{P}_{\text{clm} + \text{ss}^*} \subset \mathcal{P}_{\text{clm}} \). The presentation splits into considering two cases.

**Case 1:** \( f \in \mathcal{P}_{\text{lip}} \). Let \( \text{lip} f(\bar{x}) < \gamma < +\infty \) and \( \gamma' \in (\text{lip} f(\bar{x}), \gamma) \). In view of (104) and (106), we have for all sufficiently large \( k \in \mathbb{N} \):
\[ \| x_{1k}^* \| \leq \| x_{2k}^* \| + \| x_{1k}^* + x_{2k}^* - x_k^* \| < \gamma' \| y_{2k}^* \| + \alpha_k + \epsilon_k \]
\[ < \gamma'(1 + \epsilon_k) + \alpha_k + \epsilon_k < \gamma' + (\gamma' + 2\alpha_k) < \gamma. \quad (110) \]

Letting \( \gamma \downarrow \text{lip} f(\bar{x}) \) and observing that the function \( f \in \mathcal{P}_{\text{lip}} \) is arbitrary, inequalities (109) and (110) imply \( \text{rad}[\mathcal{S}R]_{\text{lip}} F(\bar{x}, \bar{y}) \geq \text{srg}_1 F(\bar{x}, \bar{y}) \). Combined with the established above inequality (52), this proves (55).

**Case 2:** \( f \in \mathcal{P}_{\text{clm} + \text{ss}^*} \). Without loss of generality, for each \( k \in \mathbb{N} \), there is a \( \delta_k > 0 \) such that \( f \) is Lipschitz continuous on \( B_{\delta_k}(x_k) \) with some constant \( \eta_k > 0 \).
and observe that $\epsilon_k$ satisfies (105). Thus, we can assume that the sequences defined above satisfy conditions (106)–(109). Since $f$ is semismooth* at $\bar{x}$, we have

$$\lim_{k \to \infty} \frac{\langle x^{*}_{2k} \cdot x_{2k} - \bar{x} \rangle - \langle y^{*}_{2k} \cdot f(x_{2k}) \rangle}{\|x^{*}_{2k} \cdot y^{*}_{2k}\| \|x_{2k} - \bar{x}, f(x_{2k})\|} = 0,$$

where the denominator is nonzero for all $k \in \mathbb{N}$. We are going to show that

$$\lim_{k \to \infty} \frac{\langle x^{*}_{1k} \cdot x_{1k} - \bar{x} \rangle - \langle y^{*}_{1k} \cdot y_{1k} - \bar{y} \rangle}{\|x^{*}_{1k} \cdot y^{*}_{1k}\| \|(x_{1k} - \bar{x}, y_{1k} - \bar{y})\|} = 0.$$

To transform (112) into (113), we next compare the corresponding components of the two expressions one by one. In view of (106), (107) and (104), we have for all $k \in \mathbb{N}$:

$$\frac{\|x^{*}_{1k} \cdot y^{*}_{1k}\|}{\|x^{*}_{2k} \cdot y^{*}_{2k}\|} - 1 = \frac{\|x^{*}_{1k} \cdot y^{*}_{1k}\| - \|x^{*}_{1k} \cdot y^{*}_{1k}\|}{\|x^{*}_{2k} \cdot y^{*}_{2k}\|} \leq \frac{\|x^{*}_{2k} + x^{*}_{1k} \cdot y^{*}_{2k} - y^{*}_{1k}\|}{\|y^{*}_{2k}\|} < 2 \max\{\|x^{*}_{1k}\| + \epsilon_k, 2\epsilon_k\} < 4\alpha_k.$$

Thanks to the Lipschitz continuity of $f$ near $x_k$ and its calmness at $\bar{x}$, and in view of (104) and (105), we have for all sufficiently large $k \in \mathbb{N}$:

$$\|f(x_{2k}) + y_{1k} - \bar{y}\| \leq \|f(x_{2k}) - f(x_{1k})\| + \|f(x_{1k}) + y_{1k} - \bar{y}\| + \|y_{1k} - y_{k}\| < \eta_k \epsilon_k + \alpha_k \epsilon_k + \epsilon_k = (\eta_k + 1)\epsilon_k + \alpha_k \epsilon_k,$$

$$\|f(x_{2k})\| \leq \gamma \|x_{2k} - \bar{x}\| < \gamma (t_k + \epsilon_k) < \gamma t_k (1 + \alpha_k (\alpha_k + 1)^{-1}) < 2\gamma t_k,$$

$$\|x_{2k}\| \leq \eta_k \|y_{2k}\| < \eta_k (1 + \epsilon_k) < 2\eta_k.$$

As a consequence, employing (105), (108), (111) and (115), we obtain:

$$\frac{\|x_{1k} - \bar{x}, y_{1k} - \bar{y}\|}{\|x_{2k} - \bar{x}, f(x_{2k})\|} - 1 = \frac{\|x_{2k} - \bar{x}, f(x_{2k})\| - \|x_{1k} - \bar{x}, y_{1k} - \bar{y}\|}{\|x_{2k} - \bar{x}, f(x_{2k})\|} \leq \frac{\|x_{2k} - x_{1k}, f(x_{2k}) + y_{1k} - \bar{y}\|}{\|x_{2k} - \bar{x}\|} < \frac{(\eta_k + 1)\epsilon_k + \alpha_k \epsilon_k + \epsilon_k}{(\alpha_k + 1)^{-1} t_k} < \alpha_k (\alpha_k + 3).$$

Similarly, employing (106)–(108) and (115)–(117), we obtain:
\[
\frac{\langle x_{2k}^*, x_{2k} - \bar{x} \rangle + \langle x_{1k}^*, x_{1k} - \bar{x} \rangle}{\| (x_{1k}^*, y_{1k}^*) \| \| (x_{1k} - \bar{x}, y_{1k} - \bar{y}) \|} \leq \frac{\| x_{1k}^* + x_{2k}^* \| \| x_{1k} - \bar{x} \| + \| x_{2k}^* \| \| x_{2k} - x_{1k} \|}{\| y_{1k}^* \| \| x_{1k} - \bar{x} \|} \\
< 2 \left( \| x_{k}^* \| + \varepsilon_k + \frac{4\eta_k \varepsilon_k}{(\alpha_k + 1)^{-1} t_k} \right) < 12\alpha_k,
\]

(119)

\[
\frac{\langle y_{2k}^*, f(x_{2k}) \rangle + \langle y_{1k}^*, y_{1k} - \bar{y} \rangle}{\| (x_{1k}^*, y_{1k}^*) \| \| (x_{1k} - \bar{x}, y_{1k} - \bar{y}) \|} \leq \frac{\| y_{2k}^* - y_{1k}^* \| \| f(x_{2k}) \| + \| y_{1k}^* \| \| f(x_{2k}) + y_{1k} - \bar{y} \|}{\| y_{1k}^* \| \| x_{1k} - \bar{x} \|} \\
< \frac{\alpha_k + 1}{t_k} \left( 8\gamma \varepsilon_k t_k + (\eta_k + 1)\varepsilon_k + \alpha_k t_k \right) \\
= (\alpha_k + 1)(8\gamma \varepsilon_k + (\eta_k + 1)\varepsilon_k / t_k + \alpha_k) \\
< \alpha_k((\alpha_k + 1)(8\gamma + 1) + 1).
\]

(120)

Conditions (104), (114) and (118)–(120) yield

\[
\lim_{k \to \infty} \frac{\| (x_{1k}^*, y_{1k}^*) \|}{\| (x_{2k}^*, y_{2k}^*) \|} = \lim_{k \to \infty} \frac{\| (x_{1k} - \bar{x}, y_{1k} - \bar{y}) \|}{\| (x_{2k} - \bar{x}, f(x_{2k})) \|} = 1,
\]

(121)

\[
\lim_{k \to \infty} \frac{\langle x_{2k}^*, x_{2k} - \bar{x} \rangle + \langle x_{1k}^*, x_{1k} - \bar{x} \rangle}{\| (x_{1k}^*, y_{1k}^*) \| \| (x_{1k} - \bar{x}, y_{1k} - \bar{y}) \|} = \lim_{k \to \infty} \frac{\langle y_{2k}^*, f(x_{2k}) \rangle + \langle y_{1k}^*, y_{1k} - \bar{y} \rangle}{\| (x_{1k}^*, y_{1k}^*) \| \| (x_{1k} - \bar{x}, y_{1k} - \bar{y}) \|} = 0.
\]

(122)

Combining (121) and (122) with (112), we show (113). Letting \( \gamma \downarrow \text{clm} f(\bar{x}) \) and observing that the function \( f \in \mathcal{F}_{\text{fclm} + \text{ss}} \) is arbitrary, conditions (109) and (113), and definition (44) imply \( \text{rad}[\text{SR}]_{\text{fclm} + \text{ss}} F(\bar{x}, \bar{y}) \geq \text{sr}_{\text{F}} F(\bar{x}, \bar{y}) \). Combined with the established above inequality (54), this proves (58).

If \( f \in \mathcal{F}_{\text{lip} + \text{ss}} ^* \) and \( \gamma > \text{lip} f(\bar{x}) \), then, in addition to (109) and (113), we also have condition (110). Letting \( \gamma \downarrow \text{lip} f(\bar{x}) \), conditions (109), (110) and (113), and definition (43) imply (57).

### 6.5 The upper estimate for the radius of strong subregularity

We prove Lemma 5.13 which provides a key ingredient for the proof of Theorem 5.11. The proof below is an adaptation of the one in Sect. 6.3 for Lemma 5.7(iii). As in Sect. 6.3, we construct a function \( f \) such that \( F + f \) is not strongly regular, but we make this conclusion by the definition of strong regularity without appealing to Theorem 3.5 and Lemma 3.6. Hence, we skip all the estimates for coderivatives involved in Sect. 6.3. This also explains why \( \text{gph} F \) in the statement of Lemma 5.13 is not required to be closed.

Suppose that \( \text{sr}_{\text{F}} F(\bar{x}, \bar{y}) < \gamma < +\infty \). By (8), there exist sequences \( x_k \to \bar{x} \) with \( x_k \neq \bar{x} \), and \( y_k \in F(x_k) \) such that condition (72) is satisfied. For every \( k \in \mathbb{N} \), set \( t_k := \| x_k - \bar{x} \| \), \( u_k := (x_k - \bar{x}) / t_k \in \mathbb{S}_X \) and \( v_k := (y_k - \bar{y}) / t_k \in \gamma' \mathbb{B}_Y \), and choose a vector \( u_k \in X^* \) satisfying
\[ \langle u_k^*, u_k \rangle = \|u_k^*\| = 1. \] (123)

Set
\[ \alpha_k(x) := \langle u_k^*, x \rangle, \quad r_k(x) := x - \alpha_k(x)u_k, \quad A_k(x) := \alpha_k(x)v_k. \] (124)

Observe that
\[ \alpha_k(u_k) = 1, \quad r_k(u_k) = 0, \quad A_k(u_k) = v_k. \] (125)

**Case 1:** no element of the sequence \((u_k)\) appears infinitely many times. As shown in Sect. 6.3, there exists a subsequence of \((u_k)\) (we keep the original notation) such that \(\rho_k := \inf\{\|u_j - u_k\| : j \neq k\} > 0\) for all \(k \in \mathbb{N}\), and condition (78) is satisfied. For each \(k \in \mathbb{N}\), set \(\tau_k := \rho_k/2\) and consider mappings \(s_k : X \to \mathbb{R}, \hat{A}_k : X \to Y\) (\(k \geq 1\)) and \(f : X \to \mathbb{R}\) defined by (79), (81) and (82), respectively. Observe that \(s_k(x) = 0\) and \(\hat{A}_k(x) = 0\) for all \(x \in X\) with \(\|r_k(x)\| \geq \tau_k\alpha_k(x)\). If \(\|r_k(x)\| < \tau_k\alpha_k(x)\), then, by (123) and (124), \(x \neq 0\) and
\[
\|x - \|x\|u_k\| = \|r_k(x) + (\alpha_k(x) - \|x\|)u_k\| \leq \|r_k(x)\| + \|x\| - \alpha_k(x)
\leq 2\|r_k(x)\| < 2\tau_k\alpha_k(x) \leq \rho_k\|x\|;
\]
hence, \(x/\|x\| \in B_{\rho_k}(u_k)\), and it follows from (78) that for every \(x \in X\) there is at most one \(k \in \mathbb{N}\) with \(\hat{A}_k(x) \neq 0\). Thus, the function \(f\) is well defined by (82). The function \(\hat{A}_k(x)\) is Lipschitz continuous around any point \(x \neq 0\); hence, \(f\) is Lipschitz continuous around any point \(x \neq \bar{x}\). Further, \(s_k\) is positively homogenous at 0, while \(A_k\) is linear. Therefore \(\hat{A}_k\) is positively homogenous at 0, and \(f\) is positively homogenous at \(\bar{x}\) and, by Corollary 4.7, semismooth* at \(\bar{x}\). If \(\hat{A}_k(x) \neq 0\), then, by (72) and (79), (81), (123) and (124), \(\|r_k(x)\| < \tau_k\alpha_k(x)\), and
\[
\|\hat{A}_k(x)\| \leq \alpha_k(x)\|v_k\| = \gamma'\|x\|,
\] (126)
and consequently, \(\text{clm} \hat{A}_k(0) \leq \gamma'. \) Hence, \(\text{clm} f(\bar{x}) \leq \gamma' < \gamma. \) Thus, \(f \in \mathcal{F}_{fclm+ss}^*\).

In view of (79) and (125), \(s_k(x_k - \bar{x}) = 1\). This yields (83). By (83), \(\bar{y} \in (F + f)(x_k)\), and in view of (8), \(\text{sr}(F + f)(\bar{x}, \bar{y}) = 0\), i.e., \(F + f\) is not strongly subregular at \((\bar{x}, \bar{y})\). Hence, (60) holds true.

**Case 2:** There is an element \(u\) of the sequence \((u_k)\) which appears infinitely many times. From now on, we consider the stationary subsequence \((u)\) of the sequence \((u_k)\), i.e. without changing the notation, we assume that \((x_k - \bar{x})/t_k = u \in \mathcal{S}_x\) for all \(k \in \mathbb{N}\). Thus, subscript \(k\) can be dropped in many of the formulas in (123)–(125):
\[
\langle u^*, u \rangle = \|u^*\| = 1, \quad \alpha(x) := \langle u^*, x \rangle, \quad r(x) := x - \alpha(x)u, \quad A_k(x) := \alpha(x)v_k,
\] (127)
\[
\alpha(u) = 1, \quad r(u) = 0.
\] (128)

By passing to a subsequence again, we can also assume that the sequence \((t_k)\) satisfies conditions (89).
Next we define mappings $T_k : \{x \in X | \alpha(x) > 0\} \to Y$ ($k \geq 2$) and $g, f : X \to Y$ by (92)–(94).

As shown in Section 6.3, $f \in \mathcal{F}_{clm+ss^*}$, and $clm f(\bar{x}) < \gamma$. By (92)–(94) and (127)–(130), $\alpha(t_k u) = t_k$, and

$$f(x_k) = -g(t_k u) = -T_k(t_k u) = -\hat{A}_k(t_k u) = -s(t_k u)A_k(t_k u) = -t_k v_k = \bar{y} - y_k.$$  

Thus, $\bar{y} \in (F + f)(x_k)$, and in view of (8), $sgn(F + f)(\bar{x}, \bar{y}) = 0$, i.e., $F + f$ is not strongly subregular at $(\bar{x}, \bar{y})$. Hence, (60) holds true.

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**Declarations**

**Conflict of interest** The authors have no competing interests to declare that are relevant to the content of this article.

**References**

1. Adly, S., Cibulka, R., Ngai, H.V.: Newton’s method for solving inclusions using set-valued approximations. SIAM J. Optim. **25**(1), 159–184 (2015). https://doi.org/10.1137/130926730
2. Aussel, D., Daniilidis, A., Thibault, L.: Subsmooth sets: functional characterizations and related concepts. Trans. Am. Math. Soc. **357**(4), 1275–1301 (2005). https://doi.org/10.1090/S0002-9947-04-03718-3
3. Bürgisser, P., Cucker, F.: Condition. The Geometry of Numerical Algorithms, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 349. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-38896-5
4. Camacho, J., Cánovas, M.J., López, M.A., Parra, J.: Robust and continuous metric subregularity for linear inequality systems. Comput. Optim. Appl. To appear
5. Cánovas, M.J., Dontchev, A.L., López, M.A., Parra, J.: Metric regularity of semi-infinite constraint systems. Math. Program. Ser. B **104**(2–3), 329–346 (2005). https://doi.org/10.1007/s10107-005-0618-z
6. Cibulka, R., Dontchev, A.L., Kruger, A.Y.: Strong metric subregularity of mappings in variational analysis and optimization. J. Math. Anal. Appl. **457**(2), 1247–1282 (2018). https://doi.org/10.1016/j.jmaa.2016.11.045
7. Clarke, F.H.: Optimization and Nonsmooth Analysis. John Wiley & Sons Inc., New York (1983)
8. Dontchev, A.L., Eberhard, A., Rockafellar, R.T.: Radius theorems for monotone mappings. Set-Valued Var. Anal. **27**(3), 605–621 (2019). https://doi.org/10.1007/s11228-018-0469-4
9. Dontchev, A.L., Gfrerer, H., Kruger, A.Y., Outrata, J.: The radius of metric subregularity. Set-Valued Var. Anal. 28(3), 451–473 (2020). https://doi.org/10.1007/s11228-019-00523-2
10. Dontchev, A.L., Lewis, A.S., Rockafellar, R.T.: The radius of metric regularity. Trans. Am. Math. Soc. 355(2), 493–517 (2003)
11. Dontchev, A.L., Rockafellar, R.T.: Regularity and conditioning of solution mappings in variational analysis. Set-Valued Anal. 12(1–2), 79–109 (2004)
12. Dontchev, A.L., Rockafellar, R.T.: Implicit Functions and Solution Mappings. A View from Variational Analysis, 2 edn. Springer Series in Operations Research and Financial Engineering. Springer, New York (2014). https://doi.org/10.1007/978-1-4939-1037-3
13. Durea, M., Strugariu, R.: Metric subregularity of composition set-valued mappings with applications to fixed point theory. Set-Valued Var. Anal. 24(2), 231–251 (2016). https://doi.org/10.1007/s11228-015-0327-6
14. Eckart, C., Young, G.: The approximation of one matrix by another of lower rank. Psychometrica 1, 211–218 (1936)
15. Gfrerer, H.: First order and second order characterizations of metric subregularity and calmness of constraint set mappings. SIAM J. Optim. 21(4), 1439–1474 (2011)
16. Gfrerer, H.: On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs. Set-Valued Var. Anal. 21(2), 151–176 (2013). https://doi.org/10.1007/s11228-012-0220-5
17. Gfrerer, H., Mandimayr, M., Outrata, J.V., Valdman, J.: On the SCD semismooth* Newton method for generalized equations with application to a class of static contact problems with Coulomb friction. Comput. Optim. Appl. (2022). https://doi.org/10.1007/s10589-022-00429-0
18. Gfrerer, H., Outrata, J.V.: On Lipschitzian properties of implicit multifunctions. SIAM J. Optim. 26(4), 2160–2189 (2016). https://doi.org/10.1137/15M1052299
19. Gfrerer, H., Outrata, J.V.: On a semismooth* Newton method for solving generalized equations. SIAM J. Optim. 31(1), 489–517 (2021). https://doi.org/10.1137/19M1257408
20. Gfrerer, H., Outrata, J.V.: On (local) analysis of multifunctions via subspaces contained in graphs of generalized derivatives. J. Math. Anal. Appl. 508(2), 125895 (2022). https://doi.org/10.1016/j.jmaa.2021.125895
21. Gfrerer, H., Outrata, J.V., Valdman, J.: On the application of the SCD semismooth* Newton method to variational inequalities of the second kind. Set-Valued Var. Anal. (2022). https://doi.org/10.1007/s11228-022-00651-2
22. He, Y., Xu, W.: An improved stability result on the metric regularity under Lipschitz set-valued perturbations. J. Math. Anal. Appl. 514(1), 126253 (2022). https://doi.org/10.1016/j.jmaa.2022.126253
23. Ioffe, A.D.: On perturbation stability of metric regularity. Set-Valued Anal. 9(1–2), 101–109 (2001)
24. Ioffe, A.D.: Variational Analysis of Regular Mappings. Theory and Applications. Springer Monographs in Mathematics. Springer (2017). https://doi.org/10.1007/978-3-319-64277-2
25. Jourani, A.: Radiality and semismoothness. Control Cybernet. 36(3), 669–680 (2007)
26. Klatt, D., Kummer, B.: Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications, Nonconvex Optimization and its Applications, vol. 60. Kluwer Academic Publishers, Dordrecht (2002)
27. Jourani, A.: Radiality and semismoothness. Control Cybernet. 36(3), 669–680 (2007)
28. Kruger, A.Y.: Generalized differentials of nonsmooth functions. VINITI no. 1332-81. Minsk (1981). 67 pp. In Russian. Available from: https://asterius.federation.edu.au/akruger/research/publications.html
29. Kruger, A.Y.: ε-semidifferentials and ε-normal elements. VINITI no. 1331–81. Minsk (1981). 76 pp. In Russian. Available from: https://asterius.federation.edu.au/akruger/research/publications.html
30. Kruger, A.Y.: A covering theorem for set-valued mappings. Optimization 19(6), 763–780 (1988). https://doi.org/10.1080/02331938808843391
31. Kruger, A.Y.: On Fréchet subdifferentials. J. Math. Sci. (N.Y.) 116(3), 3325–3358 (2003). https://doi.org/10.1023/A:1023673105317
32. Kruger, A.Y.: Error bounds and metric subregularity. Optimization 64(1), 49–79 (2015). https://doi.org/10.1080/02331934.2014.938074
33. Kruger, A.Y., López, M.A., Théra, M.A.: Perturbation of error bounds. Math. Program. Ser. B 168(1–2), 533–554 (2018). https://doi.org/10.1007/s10107-017-1129-4
34. Luke, D.R., Teboulle, M., Thao, N.H.: Necessary conditions for linear convergence of iterated expansive, set-valued mappings. Math. Program. 180(1), 1–31 (2020). https://doi.org/10.1007/s10107-018-1343-8
35. Luke, D.R., Thao, N.H., Tam, M.K.: Quantitative convergence analysis of iterated expansive, set-valued mappings. Math. Operations Res. 43(4), 1143–1176 (2018). https://doi.org/10.1287/moor.2017.0898
36. Luo, Z.Q., Tseng, P.: Perturbation analysis of a condition number for linear systems. SIAM J. Matrix Anal. Appl. 15(2), 636–660 (1994)
37. Maréchal, M.: Metric subregularity in generalized equations. J. Optim. Theory Appl. 176(3), 527–540 (2018). https://doi.org/10.1007/s10957-018-1246-0
38. Mordukhovich, B.S.: Coderivative analysis of variational systems. J. Global Optim. 28(3–4), 347–362 (2004). https://doi.org/10.1023/B:JOGO.0000026454.56343.b9
39. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. I: Basic Theory, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 330. Springer, Berlin (2006)
40. Ngai, H.V., Kruger, A.Y., Théra, M.: Stability of error bounds for semi-infinite convex constraint systems. SIAM J. Optim. 20(4), 2080–2096 (2010). https://doi.org/10.1137/090767819
41. Ngai, H.V., Phan, N.T.: Metric subregularity of multifunctions: first and second order infinitesimal characterizations. Math. Oper. Res. 40(3), 703–724 (2015). https://doi.org/10.1287/moor.2014.0691
42. Ngai, H.V., Théra, M.: Error bounds in metric spaces and application to the perturbation stability of metric regularity. SIAM J. Optim. 19(1), 1–20 (2008). https://doi.org/10.1137/060675721
43. Ngai, H.V., Tron, N.H., Théra, M.: Metric regularity of the sum of multifunctions and applications. J. Optim. Theory Appl. 160(2), 355–390 (2014). https://doi.org/10.1007/s10957-013-0385-6
44. Phelps, R.R.: Convex Functions, Monotone Operators and Differentiability, Lecture Notes in Mathematics, vol. 1364, 2nd edn. Springer-Verlag, Berlin (1993)
45. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Springer, Berlin (1998)
46. Uderzo, A.: A strong metric subregularity analysis of nonsmooth mappings via steepest displacement rate. J. Optim. Theory Appl. 171(2), 573–599 (2016). https://doi.org/10.1007/s10957-016-0952-8
47. Ye, J.J., Yuan, X., Zeng, S., Zhang, J.: Variational analysis perspective on linear convergence of some first order methods for nonsmooth convex optimization problems. Set-Valued Var. Anal. 29(4), 803–837 (2021). https://doi.org/10.1007/s11228-021-00591-3
48. Zheng, X.Y.: Metric subregularity for a multifunction. J. Math. Study 49(4), 379–392 (2016). https://doi.org/10.4208/jms.v49n4.16.03
49. Zheng, X.Y., Ng, K.F.: Perturbation analysis of error bounds for systems of conic linear inequalities in Banach spaces. SIAM J. Optim. 15(4), 1026–1041 (2005)
50. Zheng, X.Y., Ng, K.F.: Calmness for L-subsmooth multifunctions in Banach spaces. SIAM J. Optim. 19(4), 1648–1673 (2008). https://doi.org/10.1137/080714129
51. Zheng, X.Y., Ng, K.F.: Stability of error bounds for conic subsmooth inequalities. ESAIM: Control Optim. Calc. Var. 25, 55 (2019). https://doi.org/10.1051/cocv/2018047
52. Zheng, X.Y., Ng, K.F.: Perturbation analysis of metric subregularity for multifunctions. SIAM J. Optim. 31(3), 2429–2454 (2021). https://doi.org/10.1137/19M1309171
53. Zheng, X.Y., Wei, Z.: Perturbation analysis of error bounds for quasi-subsmooth inequalities and semi-infinite constraint systems. SIAM J. Optim. 22(1), 41–65 (2012). https://doi.org/10.1137/100806199
54. Zheng, X.Y., Zhu, J.: Generalized metric subregularity and regularity with respect to an admissible function. SIAM J. Optim. 26(1), 535–563 (2016). https://doi.org/10.1137/15M1016345

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