A FUNCTION OBSTRUCTION TO THE EXISTENCE OF COMPLEX STRUCTURES

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ABSTRACT. We construct a function for almost-complex Riemannian manifolds. Non-vanishing of the function for the almost-complex structure implies the almost-complex structure is not integrable. Therefore the constructed function is an obstruction for the existence of complex structures from the almost-complex structure. It is a function, not a tensor, so it is easier to work with.

1. INTRODUCTION

Given $M^n$ a Riemannian manifold, an interesting question is that does there exist any complex manifold structure on $M^n$ that makes $M^n$ a complex manifold? For a given complex manifold, the underline complex structure gives a canonical almost-complex structure. In the study of existence or nonexistence of complex manifold structure for a manifold, one naturally looks at existence of almost-complex structure first, and should any exists, and check whether or not it can be ”integrated” to a complex structure.

An almost-complex structure $J$ on Riemannian manifold $M^n$ is an endomorphism of the tangent bundle $TM$ with $J^2 = -1$. It is known that if $M$ has an almost-complex structure, then $M$ has even dimension $n$ and $M$ is orientable. Nijenhuis tensor $N_J$ for the almost complex structure $J$ is given by the following equation

$$N_J(X,Y) = [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y],$$

for all smooth vector fields $X,Y$. The celebrated Newlander-Nirenberg theorem $[2]$ implies that $N_J = 0$ if and only if $J$ is a canonical almost-complex structure of a complex manifold. An almost-complex structure $J$ is called integrable if $N_J$ vanishes. So in studying existence or nonexistence of complex manifold structure on a manifold, one often studies existence or nonexistence of integrable almost-complex structures, or equivalently studies almost-complex structures and the related Nijenhuis tensor vanishing or non-vanishing property.

On the other hand, though it is great that vanishing or non-vanishing Nijenhuis tensor determines whether the almost-complex structure is integrable or not, and for special manifolds like spheres it is often easy to check the vanishing or non-vanishing property of Nijenhuis tensor, like the author did for spheres $S^n$ ($n$ even, $n > 2$) $[1]$, it is hard to check the vanishing status for the manifolds other than spheres, due the nature of tensor. It would be nice to give some sufficient condition for existence or to give some obstruction for non-existence, of complex structures. In this paper we give an obstruction that is a function $L_J$, not a tensor, which should be much easier.
to work with for general manifold other than spheres. Non-vanishing function $L_J$ implies non-vanishing Nijenhuis tensor $N_J$. We construct the obstruction function in Section 2.

2. AN FUNCTION OBSTRUCTION FOR THE EXISTENCE OF COMPLEX STRUCTURES

We take the convention that we sum on duplicated index in this paper, unless otherwise stated. Our first result is

**Theorem 2.1.** If $J$ is an almost-complex structure on a Riemannian manifold with metric $g$, and if

\[
L_J := -\partial_j(J^i_j J^k_i)\partial_k J^j_i.
\]

is not zero at some point of $M$, Then $J$ is not integrable.

**Proof.** We define a $(4,0)$ tensor $N(X, Z, Y, W)$ by

\[
N(X, Z, Y, W) := \frac{1}{4} \left\{ JN_J \left( N_J(X, Z), Y \right), W \right\}_g + \left\{ JN_J \left( N_J(Y, Z), X \right), W \right\}_g + \left\{ JN_J \left( N_J(X, W), Y \right), Z \right\}_g + \left\{ JN_J \left( N_J(Y, W), X \right), Z \right\}_g \right\},
\]

where $N_J$ is the one in (1.1) and $\langle \cdot, \cdot \rangle_g = g(\cdot, \cdot)$.

Take the trace of the first argument and the third argument and then take the trace of the second and the fourth argument at a point to get a number for that point

\[
g^{ij} g^{kl} N(\varepsilon_i, \varepsilon_k, \varepsilon_i, \varepsilon_k),
\]

where $\{ \varepsilon_i = \frac{\partial}{\partial x^i} \}_{i=1}^n$ is a local frame of local coordinate system $\{ x^i \}_{i=1}^n$, $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, matrix \((g^{ij}) = (g_{ij})^{-1}\). We write $\partial_i = \frac{\partial}{\partial x^i}$, $\nabla_i = \nabla \frac{\partial}{\partial x^i}$ for convenience.

If we can show

\[
g^{ij} g^{kl} N(\varepsilon_i, \varepsilon_k, \varepsilon_i, \varepsilon_k) = L_J
\]

then $L_J$ is constructed from Nijenhuis tensor $N_J$ by contractions. Non-vanishing function $L_J$ implies non-vanishing tensor $N_J$ since $N_J = 0$ implies $L_J = 0$. Therefore the theorem follows.

We now prove (2.2).

We calculate at a point with normal coordinates. So at the point $g_{ij} = \delta_{ij}$, $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$.

Note that $N(X, Z, X, Z) = \left\{ JN_J \left( N_J(X, Z), X \right), Z \right\}_g$. therefore

\[
L_j := N(e_i, e_k, e_i, e_k) = \left\{ JN \left( N(e_i, e_k), e_i \right), e_k \right\}_g = N^r_{ik} N^s_{rs} J^k_s,
\]

where $N^k_{ij}$ are given by $N_J(\partial_i, \partial_j) = N^k_{ij} \partial_k$. It is easy to see

\[
N^k_{ij} = J^k_i (\partial_p J^r_j - \partial_j J^r_p) - J^k_j (\partial_p J^r_i - \partial_i J^r_p).
\]

We write $J_i := J^i_p \partial_j$ for convenience.

\[
N^r_{ik} N^s_{rs} J^k_s = \{ J^p_i (\partial_p J^r_k - \partial_k J^r_p) - J^k_i (\partial_p J^r_j - \partial_j J^r_p) \} \{ J^q_j (\partial_q J^r_s - \partial_s J^r_q) - J^q_s (\partial_q J^r_i - \partial_i J^r_q) \} J^k_s
\]
\[
\{(J^k_p J^p (\partial_p J^p_\pi - \partial_\pi J^p_\beta) - J^p_\pi J^p_\beta (\partial_\beta J^p_\pi - \partial_\pi J^p_\beta))\{J^p_\beta (\partial_p J^p_\pi - \partial_\pi J^p_\beta) - J^p_\pi (\partial_\beta J^p_\pi - \partial_\pi J^p_\beta)\} \\
= \{(J^k_p J^p (\partial_p J^p_\pi - \partial_\pi J^p_\beta)) - J^p_\pi (\partial_\beta J^p_\pi - \partial_\pi J^p_\beta)\} \{\partial_p J^p_\pi - \partial_\pi J^p_\beta\} - J^p_\beta (\partial_\beta J^p_\pi - \partial_\pi J^p_\beta) \\
= \{J^k_p \cdot J^k_\pi \cdot \partial_\pi J^k_\beta - J^k_p \cdot J^k_\beta \cdot \partial_\pi J^k_\beta - J^k_\pi \cdot J^k_\beta \cdot \partial_\pi J^k_\beta\} - \{J^k_p \cdot J^k_\pi \cdot \partial_\pi J^k_\beta - J^k_\pi \cdot J^k_\beta \cdot \partial_\pi J^k_\beta\} - \{J^k_p \cdot J^k_\pi \cdot \partial_\pi J^k_\beta - J^k_\pi \cdot J^k_\beta \cdot \partial_\pi J^k_\beta\} \\
= -J^k_p \cdot J^k_\pi \cdot J^k_\beta + J^k_\pi \cdot J^k_\beta \cdot J^k_\pi + J^k_\pi \cdot J^k_\beta \cdot J^k_\pi + J^k_\pi \cdot J^k_\beta \cdot J^k_\pi - J^k_\beta \cdot J^k_\pi \cdot J^k_\pi \\
- J^k_\beta \cdot J^k_\pi \cdot J^k_\beta + J^k_\beta \cdot J^k_\pi \cdot J^k_\beta + J^k_\beta \cdot J^k_\pi \cdot J^k_\beta \\
= -J^k_p \cdot J^k_\pi \cdot J^k_\beta + J^k_\pi \cdot J^k_\beta \cdot J^k_\pi + J^k_\pi \cdot J^k_\beta \cdot J^k_\pi - J^k_\beta \cdot J^k_\pi \cdot J^k_\beta \\
- J^k_\beta \cdot J^k_\pi \cdot J^k_\beta + J^k_\beta \cdot J^k_\pi \cdot J^k_\beta + J^k_\beta \cdot J^k_\pi \cdot J^k_\beta. \\
\]

IV3, meaning the third term of fourth line in the above four lines after = 

\[-J^3_\beta \partial_\beta J^3 \cdot \partial_\beta J^3 = -\partial_\beta J^3 \cdot \partial_\beta J^3 = J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3 = J^3_\beta \cdot \partial_\beta J^3 = J^3_\beta \cdot \partial_\beta J^3.\]

IV2, meaning the third term of fourth line in the four lines after =. Others are similar.

\[-J^2_\beta \partial_\beta J^2 \cdot \partial_\beta J^2 = -J^2_\beta \partial_\beta J^2 \cdot \partial_\beta J^2 = 0,\]

\[\{\text{II2} + \text{IV3}\} = 0,\]

\[N^r_{\pi k} N^s_{\pi r} N^k_{\pi i} J^k = 0,\]

\[-J^k_\beta \cdot J^k_\pi \cdot J^k_\beta + J^k_\pi \cdot J^k_\beta \cdot J^k_\pi + J^k_\pi \cdot J^k_\beta \cdot J^k_\pi - J^k_\beta \cdot J^k_\pi \cdot J^k_\beta \\
- J^k_\beta \cdot J^k_\pi \cdot J^k_\beta + J^k_\beta \cdot J^k_\pi \cdot J^k_\beta + J^k_\beta \cdot J^k_\pi \cdot J^k_\beta.\]

IV2

\[-J^2_\beta \partial_\beta J^2 \cdot \partial_\beta J^2 = J^2_\beta \cdot \partial_\beta J^2 \cdot \partial_\beta J^2 = J^2_\beta \cdot \partial_\beta J^2 \cdot \partial_\beta J^2 = -J^2_\beta \cdot \partial_\beta J^2 \cdot \partial_\beta J^2 \\
- J^2_\beta \cdot \partial_\beta J^2 \cdot \partial_\beta J^2 + J^2_\beta \cdot \partial_\beta J^2 \cdot \partial_\beta J^2 + J^2_\beta \cdot \partial_\beta J^2 \cdot \partial_\beta J^2 \\
= J^2_\beta \cdot \partial_\beta J^2 \cdot \partial_\beta J^2 = J^2_\beta \cdot \partial_\beta J^2 \cdot \partial_\beta J^2.\]

II5

\[-J^3_\beta \partial_\beta J^3 \cdot \partial_\beta J^3 = J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3 = J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3 = -J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3 \\
- J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3 + J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3 + J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3 \\
= J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3 = J^3_\beta \cdot \partial_\beta J^3 \cdot \partial_\beta J^3.\]
$$+ J_i^q J_s^k \cdot J_i J_k^r \cdot \partial_r J_q^s = - J_q^s J_s^k \cdot J_i J_k^r \cdot \partial_r J_i^s = J_i J_k^r \cdot \partial_r J_i^s = J_i J_k^r \cdot \partial_r J_i^s.$$ 

$\text{III2}$ 

$$- J_k^k \cdot \partial_r J_i^s = - J_k^k \cdot \partial_r J_i^s, $$

$$II2 + III2 = 0.$$ 

$$III5 + IV2 = 0.$$ 

$$N_{ik}^s N_{r_i}^s J_s^k$$ 

$$- J_k^k \cdot J_i J_k^r \cdot J_k J_i^r + J_k^k \cdot J_i J_k^r \cdot J_k J_i^r + J_p^p \cdot J_i J_k^r \cdot J_k J_i^r - J_p^p \cdot J_k J_i^r \cdot J_k J_i^r - J_p^p \cdot J_k J_i^r \cdot J_k J_i^r$$ 

$$- J_q^q J_s^k \cdot J_i J_k^r \cdot \partial_s J_q^s + II2 + III + J_i J_k^r \cdot J_k J_i^r + II5$$ 

$$- J_q^q J_p^p \cdot J_i J_k^r \cdot \partial_s J_q^s + II2 + J_i J_k^r \cdot \partial_s J_i^s$$ 

$$+ J_q^q \partial_s J_q^s \cdot \partial_s J_i^s + IV2 + IV3 + J_i J_k^r \cdot J_k J_i^r \cdot \partial_s J_i^r.$$ 

$\text{III1}$ 

$$- J_q^q J_s^k \cdot J_i J_k^r \cdot J_k J_i^r - J_q^q J_s^k \cdot J_i J_k^r \cdot J_k J_i^r = J_i J_k^r \cdot \partial_s J_i^s.$$ 

$\text{III4}$ 

$$+ J_i J_k^r \cdot \partial_s J_i^s = + J_i J_k^r \cdot \partial_s J_i^s.$$ 

$$II1 + III4 = 0.$$ 

$$N_{ik}^s N_{r_i}^s J_s^k$$ 

$$- J_k^k \cdot J_i J_k^r \cdot J_k J_i^r + J_k^k \cdot J_i J_k^r \cdot J_k J_i^r + J_p^p \cdot J_i J_k^r \cdot J_k J_i^r - J_p^p \cdot J_k J_i^r \cdot J_k J_i^r - J_p^p \cdot J_k J_i^r \cdot J_k J_i^r$$ 

$$II1 + II2 + III + II4 + II5$$ 

$$J_i J_k^r \cdot \partial_s J_i^s + II2 + J_i J_k^r \cdot \partial_s J_i^s$$ 

$$+ J_q^q \partial_s J_q^s \cdot \partial_s J_i^s + IV2 + IV3 + J_i J_k^r \cdot J_k J_i^r \cdot \partial_s J_i^s.$$ 

$\text{IV4}$ 

$$+ J_i J_k^r \cdot J_k J_i^r \cdot J_k J_i^r - J_q^q J_i J_k^r \cdot \partial_s J_i^s = - J_i J_k^r \cdot \partial_s J_i^s = - J_i J_k^r \cdot \partial_s J_i^s.$$ 

$\text{III3}$ 

$$+ J_i J_i^s \cdot \partial_s J_i^s = + J_i J_i^s \cdot \partial_s J_i^s$$ 

$$II1 = II3 = 2 J_i J_k^r \cdot \partial_s J_i^s.$$ 

$$N_{ik}^s N_{r_i}^s J_s^k$$ 

$$- J_i^l \cdot J_i J_k^r \cdot J_k J_i^r + J_i^l \cdot J_i J_k^r \cdot J_k J_i^r + J_i^l \cdot J_i J_k^r \cdot J_k J_i^r - J_i^l \cdot J_i J_k^r \cdot J_k J_i^r$$ 

$$II1 + II2 + III + II4 + II5$$ 

$$J_i J_k^r \cdot \partial_s J_i^s + II2 + J_i J_k^r \cdot \partial_s J_i^s$$ 

$$+ J_i^l \partial_s J_i^s \cdot \partial_s J_i^s + IV2 + IV3 - J_i J_k^r \cdot \partial_s J_i^s.$$ 

$\text{I3}$ 

$$J_i^l \cdot J_i J_k^r \cdot J_k J_i^r = J_i^l \cdot J_i J_k^r \cdot J_k J_i^r \partial_p J_i^l = J_i^l \cdot J_i J_k^r \cdot J_k J_i^r \partial_p J_i^l.$$
\[ \begin{align*}
&= -J_k^i J_j^i \cdot J_i J_k^i \cdot \partial_p J_i^i \cdot J_k J_i^i \cdot \partial_p J_i^i = -J_k^i J_j^i J_k J_i^i \cdot J_k J_i^i \cdot J_k J_i^i .
\end{align*} \]

\[ I2 + I3 = 0. \]

\[ I4 + III1 = 0, \quad III1 = III3. \]

\[ \begin{align*}
N^k_i N^p_s J^k_i
&= -J_j^i J_i J_k^i \cdot J_i J_k^i + I2 + I3 + I4 \\
&= -J_j^i J_i J_k^i + J_i J_k^i \cdot \partial_i J_k^i + J_i J_k^i \cdot \partial_j J_k^i - J_i J_k^i \cdot \partial_k J_i^i \\
&= J_i^j (J_i J_k^i - J_i J_k^i) \cdot \partial_j J_k^i. \\
\end{align*} \]

\[ \begin{align*}
N^k_i N^p_s J^k_i
&= -J_j^i J_i J_k^i \cdot J_i J_k^i + J_i J_k^i \cdot \partial_i J_k^i + J_i J_k^i \cdot \partial_j J_k^i - J_i J_k^i \cdot \partial_k J_i^i \\
&= -J_i^j J_j^i J_k^i + J_i J_k^i \cdot \partial_i J_k^i - J_i J_k^i \cdot \partial_k J_i^i \\
&= -J_i^j J_j^i J_k^i + J_i J_k^i \cdot \partial_i J_k^i - J_i J_k^i \cdot \partial_k J_i^i \\
&= -J_i^j J_j^i J_k^i + J_i J_k^i \cdot \partial_i J_k^i - J_i J_k^i \cdot \partial_k J_i^i .
\end{align*} \]

The first term above is zero. In fact,

\[ \begin{align*}
&-J_i^j J_j^i J_k^i \cdot \partial_i J_k^i - J_i J_k^i \cdot \partial_i J_k^i \\
&= -\partial_i (J_i J_k^i J_i^i J_k^i) \\
&+\partial_i J_i^j J_j^i J_k^i \cdot \partial_j J_k^i \\
&+J_i^j \partial_i J_j^i J_k^i \cdot J_k^i \cdot \partial_j J_k^i \\
&+J_i^j J_k^i \partial_i J_j^i J_k^i \cdot \partial_j J_k^i \\
&+J_i^j J_k^i J_j^i J_k^i \cdot \partial_j J_k^i \\
&= +\partial_i (J_i J_j^i J_k^i) \\
&+J_i^j J_j^i \cdot \partial_i J_k^i J_k^i \cdot \partial_j J_k^i .
\end{align*} \]
\[- J^i_p \cdot \partial_i J^i_p \cdot \partial_j J^j_l \]

\[- J^i_p \cdot \partial_i J^i_p \cdot \partial_j J^j_l \]

\[- J^i_p \cdot \partial_i J^i_p \cdot \partial_j J^j_l \]

\[= J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l \]

\[= J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l \]

\[= J^h_p J^i_p J^i_p \cdot \partial_i J^i_h \cdot \partial_j J^j_l \]

\[= J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l \]

\[= J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l \]

\[= J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l \]

\[J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l \]

Therefore

\[- J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l = J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l \]

\[- J^k_p J^i_p J^i_p \cdot \partial_i J^i_k \cdot \partial_j J^j_l = 0. \]

Now

\[N^i_{jk} N^k_{ir} J^i_r \]

\[= J^i_l \cdot \partial_i J^i_k \cdot \partial_j J^j_l \]

\[= J^i_l \cdot \partial_i J^i_k \cdot \partial_j (J^i_k - J^i_l) \]

\[= J^i_l \cdot \partial_i (J^i_k - J^i_l) \cdot \partial_j J^j_l \]

\[= \partial_j (J^i_k - J^i_l) \cdot \partial_i J^i_l \cdot \partial_j J^j_l \]

\[= \partial_j (-\delta^j_k - J^j_k) \cdot \partial_i J^i_l \cdot \partial_j J^j_l \]

The sum of the last two terms is zero, for

\[\partial_i J^i_k \cdot J^i_k \cdot \partial_j J^j_l = \partial_i J^i_k \cdot J^i_k \cdot \partial_j J^j_l \]

\[= \partial_a J^a_k \cdot J^a_k \cdot \partial_b J^b_a = \partial_a J^a_k \cdot J^a_k \cdot \partial_b J^b_a. \]

Therefore

\[N^i_{jk} N^k_{ir} J^i_r = -\partial_j (J^i_k J^i_k) \cdot \partial_i J^i_l = L_J \]

\[\square\]

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