Stepanov-like Weighted Pseudo-Almost Automorphic Solutions on Time Scales for a Novel High-order BAM Neural Network with Mixed Time-varying Delays in the Leakage Terms

Adnène Arbi

Abstract. We first propose the concept of Stepanov-like weighted pseudo almost automorphic on time-space scales and we apply this type of oscillation to high-order BAM neural networks with mixed delays. Then, we study the existence and exponential stability of Sp-weighted pseudo-almost automorphic on time-space scales solutions for the suggested system. Some criteria assuring the convergence are proposed. Our method is mainly based on the Banach fixed point theorem, the theory of calculus on time scales and the Lyapunov-Krasovskii functional method. Moreover, a numerical example is given to show the effectiveness of the main results.

Keywords. Time scales; High-order BAM neural networks; Stepanov-like weighted pseudo-almost automorphic solution; Global exponential stability; Leakage delays.

1 Introduction

The concept of weighted pseudo almost automorphic on time-space scales functions for the nabla and delta derivative is recently introduced (see [1], [2], [3]). It is a natural generalization of almost automorphic on time-space scales functions introduced in [4]. In 2010, the concept of Stepanov-like weighted pseudo almost automorphy which is a natural generalization of the almost automorphy is presented (see [5]). Moreover, there is no definition of the notion of Stepanov-like almost automorphy and Stepanov-like weighted pseudo almost periodic on time-space scales in the previous work.
On the other hand, many researchers have been devoting the dynamics of various class of neural networks due to its wide application in pattern recognition, associative memory, image, and signal processing (see [3], [6], [7], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]). Furthermore, BAM neural networks as an extension of the unidirectional autoassociator of Hopfield neural network (see [8], [10], [11], [12]), was firstly introduced by Kosko [19]. In recent years, many scholars pay much attention to the dynamical behavior of bidirectional associative memory (BAM) neural networks. Considering that time delays are unavoidable because of the finite switching of amplifiers in practical implementation of neural networks, and the time delay may result in oscillation and instability; many authors focus on the dynamical properties of BAM neural networks with time delays (see [14], [20], [21], [22], [23], [24], [25]). In real application, when robot move, the joints are properly described by almost periodic solutions of a dynamic neural network. For this reason, it is very important to study almost periodic solutions of neural networks models. In [20], the authors investigated the almost periodic solution of

\[
\begin{aligned}
\dot{x}_i(t) &= -\alpha_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(y_j(t - \tau)) + I_i(t), \\
\dot{y}_j(t) &= -\alpha_j(t)y_j(t) + \sum_{i=1}^{n} \bar{b}_{ij}(t)f_i(x_i(t - \sigma)) + J_j(t), \\
x_i(t) &= \phi_i(t), \\
y_j(t) &= \psi_j(t), \\
i &= 1, ..., n, \\
j &= 1, ..., m,
\end{aligned}
\]

The generalized high-order BAM neural network with mixed delays, defined as follows, has faster convergence rate, higher fault tolerance, and stronger approximation property. The problem of existence and global exponential stability of periodic solution for high-order discrete-time BAM neural networks has been studied in [31]. In fact, the study of the existence periodic solutions, as well as its numerous generalizations to almost periodic solutions, pseudo almost periodic solutions, weighted pseudo almost periodic solutions, and so forth, is one of the most attracting topics in the qualitative theory of differential equations due both to its mathematical interest as well as to their applications in various areas of applied science. The authors in [32] propose some several sufficient conditions for ensuring existence, global attractivity and global asymptotic stability of the periodic solution for the higher-order bidirectional associative memory neural networks with periodic coefficients and delays by using the continuation theorem of Mawhin’s coincidence degree.
theory, the Lyapunov functional and the non-singular $M$-matrix:

$$
\begin{align*}
\dot{x}_{i}(t) &= -\alpha_{i}x_{i}(t) + \sum_{j=1}^{m} b_{ij}(t)f_{j}(y_{j}(t - \tau)) \\
+ \sum_{j=1}^{m} \sum_{l=1}^{m} c_{ijl}(t)f_{j}(y_{j}(t - \tau))f_{l}(y_{l}(t - \tau)) + I_{i}(t), \\
\dot{y}_{j}(t) &= -c_{ij}y_{j}(t) + \sum_{j=1}^{n} \bar{b}_{ij}(t)f_{j}(x_{j}(t - \tau)) \\
+ \sum_{j=1}^{n} \sum_{l=1}^{n} \bar{c}_{ijl}(t)g_{l}(x_{l}(t - \tau))g_{l}(x_{l}(t - \tau)) + J_{j}(t), \quad t \geq 0.
\end{align*}
$$

With initial condition

$$
\begin{align*}
x_{i}(t) &= \phi_{i}(t), \quad y_{j}(t) = \psi_{j}(t), \quad t \in [-\tau^{*}, 0], \\
i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad n, m \in \mathbb{Z}_{+}.
\end{align*}
$$

Furthermore, it has been reported that if the parameters and time delays are appropriately chosen, the delayed high-order BAM neural network can exhibit complicated behaviors even with strange chaotic attractors. Based on the aforementioned arguments, the study of the high-order BAM neural network with mixed delays and its analogous equations have attracted worldwide interest (see [26, 27, 28, 29, 30, 31, 32, 33, 34, 35]). In fact, it is important that systems contain some information about the derivative of the past state to further describe the dynamics for such complex neural reactions. In real world, the mixed time-varying delays and leakage delay should be taken into account when modeling realistic neural networks (see [36, 13, 14]).

As a continuation of our previous published results, we shall consider a high-order BAM neural network with mixed delays:

$$
\begin{align*}
x_{i}^{\Delta}(t) &= -\alpha_{i}x_{i}(t - \eta_{i}(t)) + \sum_{j=1}^{m} D_{ij}(t)f_{j}(x_{j}(t)) \\
+ \sum_{j=1}^{m} D_{ij}(t)f_{j}(x_{j}(t - \tau_{ij}(t))) + \sum_{j=1}^{m} \overline{D}_{ij}(t)f_{j}(x_{j}(s)) \Delta s \\
+ \sum_{j=1}^{m} \overline{D}_{ij}(t)f_{j}(x_{j}(s)) \Delta s + I_{i}(t) \\
+ \sum_{j=1}^{m} \sum_{k=1}^{m} T_{ijk}(t)f_{k}(x_{k}(t - \chi_{k}(t)))f_{j}(x_{j}(t - \chi_{j}(t))), \quad i = 1, \ldots, n, \\
y_{j}^{\Delta}(t) &= -c_{ij}y_{j}(t - \zeta_{j}(t)) + \sum_{i=1}^{n} E_{ij}(t)f_{j}(x_{j}(t)) \\
+ \sum_{i=1}^{n} E_{ij}(t)f_{j}(x_{j}(t - \tau_{ij}(t))) + \sum_{i=1}^{n} \overline{E}_{ij}(t)f_{j}(x_{j}(s)) \Delta s \\
+ \sum_{i=1}^{n} \overline{E}_{ij}(t)f_{j}(x_{j}(s)) \Delta s + J_{j}(t) \\
+ \sum_{i=1}^{n} \sum_{k=1}^{n} \overline{T}_{ijk}(t)f_{k}(x_{k}(t - \chi_{k}(t)))f_{j}(x_{j}(t - \chi_{j}(t))), \quad t \in \mathbb{T}, \quad j = 1, \ldots, m.
\end{align*}
$$

To the best of our knowledge, the existence of Stepanov-like weighted pseudo-almost automorphic solution on time-space scales to BAM neural networks (BAMs) and high-order BAM neural network (HOBAMs) with variable
coefficients, mixed delays and leakage delays have not been studied yet. It has been reported that if the parameters and time delays are appropriately chosen, the delayed neural networks in leakage term can exhibit complicated behaviors even with strange chaotic attractors (see [3], [13], [14], [33], [34], [35], [36]). In addition, the theory of time scales, which has recently received much attention, was introduced by Hilger in his PhD thesis in 1988 to unify continuous and discrete analysis [37].

Our main purpose of this paper is to introduce the Stepanov-like weighted pseudo almost automorphic functions on time-space scales, study some of their basic properties and establish the existence, uniqueness, stability and convergence of Stepanov-like weighted pseudo almost automorphic solutions of HOBAMs on time-space scales. we prove new composition theorems for Stepanov-like weighted pseudo almost automorphic functions on time-space scales.

The remainder of this paper is organized as follows. In Section 2, we will present the model of HOBAMs. In section 3, we will introduce some necessary notations, definitions and fundamental properties of the weighted pseudo-almost automorphic on time-space scales environment, which will be used in the paper. In Section 4, some sufficient conditions will be derived ensuring the existence of the Stepanov-like weighted pseudo-almost automorphic solution on time-space scales. Section 5 will be devoted to the exponential stability of the Stepanov-like weighted pseudo-almost automorphic solution on time-space scales of a HOBAMs model, and the convergence of all solutions to its unique Stepanov-like weighted pseudo-almost automorphic solution. At last, one illustrative numerical example will be given.

2 Preliminaries and function spaces

In the following, we introduce some definitions and state some preliminary results.

2.1 Time-space scales and delta derivative

**Definition 1** ([3]) Let \( T \) be a nonempty closed subset (time scale) of \( \mathbb{R} \). The forward and backward jump operators \( \sigma, \rho : T \to T \) and the graininess \( \nu : T \to \mathbb{R}_+ \) are defined, respectively, by \( \sigma(t) = \inf \{ s \in T : s > t \} \), \( \rho(t) = \sup \{ s \in T : s < t \} \) and \( \nu(t) = \sigma(t) - t \).

**Lemma 1** ([3], [38]) Considering that \( f, g \) be delta differentiable functions on \( T \), then:

(i) \( (\lambda_1 f + \lambda_2 g)^\Delta = \lambda_1 f^\Delta + \lambda_2 g^\Delta \), for any constants \( \lambda_1, \lambda_2 \);
(ii) \( (fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)) \);
(iii) If \( f \) and \( f^\Delta \) are continuous, then \( \left( \int_a^t f(t,s) \Delta s \right)^\Delta = f(\sigma(t),t) + \int_a^t f(t,s) \Delta s \).
Lemma 2 (38, 39) Assume that $p, q : \mathbb{T} \rightarrow \mathbb{R}$ are two regressive functions, then

(i) $c_0(t, s) \equiv 1$ and $c_p(t, t) \equiv 1$;
(ii) $c_p(t, s) = \frac{1}{t - s} e^{c_p(s, t)}$;
(iii) $c_p(t, s) c_p(s, r) = c_p(t, r)$;
(iv) $c_p(t, s) \Delta = p(t) c_p(t, s)$.

Lemma 3 (38) Assume that $p(t) \geq 0$ for $t \geq s$, then $c_p(t, s) \geq 1$.

Definition 2 (39) A function $p : \mathbb{T} \rightarrow \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$; $p : \mathbb{T} \rightarrow \mathbb{R}$ is called positively regressive provided $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ and the set of all positively regressive functions and rd-continuous functions will be denoted $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R})$.

Lemma 4 (38) Suppose that $p \in \mathcal{R}^+$, then:

(i) $c_p(t, s) > 0$, for all $t, s \in \mathbb{T}$;
(ii) if $p(t) \leq q(t)$ for all $t \geq s$, then $c_p(t, s) \leq c_q(t, s)$ for all $t \geq s$.

Lemma 5 (38, 39) If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then $[c_p(a, .)]^\Delta = -p[c_p(a, .)]^\sigma$, and $\int_a^b p(t) c_p(c, \sigma(t)) \Delta t = c_p(c, a) - c_p(c, b)$.

Lemma 6 (38, 39) Let $a \in \mathbb{T}^k$, $b \in \mathbb{T}$ and assume that $f : \mathbb{T} \times \mathbb{T}^k \rightarrow \mathbb{R}$ is continuous at $(t, t)$, where $t \in \mathbb{T}^k$ with $t > a$. Additionally assume that $f^\Delta(t, .)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\epsilon > 0$, there exists a neighborhood $U$ of $\tau \in [a, \sigma(t)]$ such that

$$|f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|, \ \forall s \in U,$$

where $f^\Delta$ denotes the derivative of $f$ with respect to the first variable. Then

(i) $g(t) := \int_a^t f(t, \tau) \Delta \tau$ implies $g^\Delta(t) := \int_a^t f^\Delta(t, \tau) \Delta \tau + f(\sigma(t), t)$;
(ii) $h(t) := \int_t^b f(t, \tau) \Delta \tau$ implies $h^\Delta(t) := \int_t^b f^\Delta(t, \tau) \Delta \tau - f(\sigma(t), t)$.

For more details of time scales and $\Delta$-measurability, one is referred to read the excellent books (38, 39).

2.2 Stepanov-like weighted pseudo-almost automorphic functions on time-space scales

In the following, we recall some definitions of Stepanov almost automorphic functions and Stepanov-like weighted pseudo-almost automorphic functions on time-space scales.

Definition 3 (11) A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \emptyset.$$
Definition 4 A function $f : \mathbb{T} \to \mathbb{R}$ is Bochner integrable, or integrable for short, if there is a sequence of functions such that $f_n(t) \to f(t)$ pointwise a.e. in $\mathbb{T}$ and

$$\lim_{n \to +\infty} \int_{\mathbb{T}} \|f(s) - f_n(s)\| \Delta s = 0,$$

and the integral of $f$ is defined by

$$\int_{\mathbb{T}} f(s) \Delta s = \lim_{n \to +\infty} f_n(s) \Delta s,$$

where the limit exists strongly in $\mathbb{R}$.

Definition 5 Let $E \subset \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be a strongly $\Delta$-measurable function. If, for a given $p$, $1 \leq p < \infty$, $f$ satisfies

$$\int_K \|f(s)\|^p \Delta s < +\infty,$$

where $K$ is a compact subset of $E$, then $f$ is called $p$-integrable in the Bochner sense. The set of all such functions is denoted by $L^p_{\text{loc}}(\mathbb{T}, \mathbb{R})$.

From now on, for $a, b \in \mathbb{R}$ and $a \leq b$, we denote $a^* = \inf\{s \in \mathbb{T}, s \geq a\}$, $b^* = \inf\{s \in \mathbb{T}, s \geq b\}$ and for integrable function $f$, we denote $\int_a^{b^*} f(s) \Delta s = \int_{a^*}^{b^*} f(s) \Delta s$. Obviously, $a^*, b^* \in \mathbb{T}$. If $a \in \mathbb{T}$, then $a^* = a$; if $b \in \mathbb{T}$, then $b^* = b$.

Throughout the rest of paper we fix $p$, $1 \leq p < \infty$. We say that a function $f \in L^p_{\text{loc}}(\mathbb{T}, \mathbb{R})$ is $p$-Stepanov bounded ($S^p_{\text{S}}$-bounded) if

$$\|f\|_{S^p_{\text{S}}} = \sup_{t \in \mathbb{T}} \left( \frac{1}{l} \int_{t-l}^{t+l} \|f(s)\|^p \Delta s \right)^{\frac{1}{p}} < +\infty,$$

where $l > 0$ is a constant.

We denote by $L^p_{\text{S}}$ the set of all $S^p_{\text{S}}$-bounded functions from $\mathbb{T}$ into $\mathbb{R}$.

Definition 6 Let $\mathbb{T}$ be an almost periodic time scale. A function $f(t) : \mathbb{T} \to \mathbb{R}^n$ is said to be $S^p$-almost automorphic, if for any sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, there is a subsequence $\{\tau_n\}_{n=1}^{\infty} \subset \{s_n\}_{n=1}^{\infty}$ such that

$$\|g(t) - f(t + \tau_n)\|_{S^p_{\text{A}}} \to 0, \text{ as } n \to +\infty,$$

is well defined for each $t \in \Pi$ and

$$\|g(t - \tau_n) - f(t)\|_{S^p_{\text{A}}} \to 0, \text{ as } n \to +\infty,$$

for each $t \in \Pi$. Denote by $S^p_{\text{AA}}(\mathbb{T}, \mathbb{R}^n)$ the set of all such functions.
Let

\[ S^p \text{AA}(\mathbb{T}) = \{ f \in S^p C_{rd}(\mathbb{T}, \mathbb{R}^n) : f \text{ is Stepanov almost automorphic} \} \]

and \( S^p \text{BC}(\mathbb{T}, \mathbb{R}^n) \) denote the space of all bounded continuous functions, in the Stepanov sens, from \( \mathbb{T} \) to \( \mathbb{R}^n \).

Let \( \mathcal{U} \) be the set of all functions \( \nu : \mathbb{T} \to (0, +\infty) \) which are positive and locally \( \Delta \)-integrable over \( \mathbb{T} \). For given \( r \in (0, +\infty) \cap \Pi \), set

\[
m(r, \nu, t_0) := \int_{Q_r} \nu(s) \Delta s, \quad \text{for each } \nu \in \mathcal{U},
\]

where \( Q_r := [t_0 - r, t_0 + r] \mathbb{T} \) (\( t_0 = \min\{0, \infty\} \mathbb{T} \}). If \( \nu(s) = 1 \) for each \( \nu \in \mathbb{T} \), then \( \lim_{r \to +\infty} \nu(Q_r) = \infty \). Consequently, we define the space of weights \( \mathcal{U}_\infty \) by

\[
\mathcal{U}_\infty := \left\{ \nu \in \mathcal{U} : \lim_{r \to +\infty} m(r, \nu, t_0) = +\infty \right\}.
\]

In addition to the aforesaid section, we define the set of weights \( \mathcal{U}_B \) by

\[
\mathcal{U}_B := \left\{ \nu \in \mathcal{U}_\infty : \nu \text{ is bounded in the Stepanov sens and } \inf_{s \in \mathbb{T}} \nu(s) > 0 \right\}.
\]

It is clear that \( \mathcal{U}_B \subset \mathcal{U}_\infty \subset \mathcal{U} \). Let \( S^p \text{BCU}(0)(\mathbb{T}, \mathbb{R}^n) \) denote the space of all bounded uniformly continuous functions from \( \mathbb{T} \) to \( \mathbb{R}^n \),

\[
S^p \text{AA}(0)(\mathbb{T}) = S^p \text{AA}(0)(\mathbb{T}, \mathbb{R}^n) = \{ f \in S^p \text{BCU}(\mathbb{T}, \mathbb{R}^n) : f \text{ is Stepanov almost automorphic} \}
\]

and define for \( t_0 \in \mathbb{T}, r \in \Pi \), the class of functions \( WPAA_0(\mathbb{T}, \nu, t_0) \) as follows:

\[
WPAA_0(\mathbb{T}, \nu, t_0) = \{ f \in S^p \text{BCU}(\mathbb{T}, \mathbb{R}^n) : f \text{ is delta measurable such that } \lim_{r \to +\infty} \frac{1}{m(r, \nu, t_0)} \int_{t_0-r}^{t_0+r} |f(s)| \nu(s) \Delta s = 0 \}.
\]

We are now ready to introduce the sets \( S^p WPAA(\mathbb{T}, \nu) \) of Stepanov-like weighted pseudo-almost automorphic on time-space scales functions:

**Definition 7** A function \( f \in S^p C_{rd}(\mathbb{T}, \mathbb{R}^n) \) is called Stepanov-like weighted pseudo almost automorphic on time-space scales if \( f = g + \phi \), where \( g \in S^p \text{AA}(\mathbb{T}) \) and \( \phi \in WPAP_0(\mathbb{T}, \nu) \). Denote by \( S^p WPAA(\mathbb{T}) \), the set of Stepanov-like weighted pseudo-almost automorphic on time-space scales functions.
2.3 Results on composition theorems

By Definition 7 one can easily show that

\textbf{Lemma 7} Let $\phi \in S^pBC_{rd}(\mathbb{T}, \mathbb{R}^n)$, then $\phi \in WPAA_0(\mathbb{T}, \nu)$, where $\nu \in U_B$ if and only if, for every $\epsilon > 0$,

\[ \lim_{r \to +\infty} \frac{1}{m(r, \nu, t_0)} \mu_\Delta(M_{r, \epsilon, t_0}(\phi)) = 0, \]

where $r \in \Pi$ and $M_{r, \epsilon, t_0}(\phi) := \{ t \in [t_0 - r, t_0 + r]_\mathbb{T} : \| \phi(t) \| \geq \epsilon \}$.

Proof. The demonstration is similar to the proof of Lemma 3.3 in [1].

\textbf{Lemma 8} WPAA_0(\mathbb{T}, \nu)$ is a translation invariant set of $S^pBC_{rd}(\mathbb{T}, \mathbb{R}^n)$ with respect to $\Pi$ if $\nu \in U_B$, i.e. for any $s \in \Pi$, one has

$\phi(t + s) := \theta_s \phi \in WPAA_0(\mathbb{T}, \nu)$ if $\nu \in U_B$.

Proof. Similar to proof of Lemma 3.3 in [1].

\textbf{Lemma 9} Let $\phi \in S^pAA(\mathbb{T}, \mathbb{R}^n)$, then the range of $\phi$, $\phi(\mathbb{T})$ is a relatively compact subset $\mathbb{R}^n$.

Proof. Similar to proof of Lemma 3.3 in [1].

\textbf{Lemma 10} If $f = g + \phi$ with $g \in S^pAA(\mathbb{T}, \mathbb{R}^n)$ and $\phi \in WPAA_0(\mathbb{T}, \nu)$, where $\nu \in U_B$, then $g(\mathbb{T}) \subset \overline{f(\mathbb{T})}$.

Proof. The demonstration is similar to the proof of Lemma 3.5 in [1].

\textbf{Lemma 11} The decomposition of a Stepanov-like weighted pseudo-almost automorphic on time-space scales function according to $S^pAA \oplus WPAA_0$ is unique for any $\nu \in U_B$.

Proof. Assume that $f_1 = g_1 + \phi_1$ and $f_2 = g_2 + \phi_2$. Then $(g_1 - g_2) + (\phi_1 - \phi_2) = 0$. Since $g_1 - g_2 \in S^pAA(\mathbb{T}, \mathbb{R}^n)$, and $\phi_1 - \phi_2 \in WPAA_0$ in view of Lemma 10 we deduce that $g_1 - g_2 = 0$. Consequently $\phi_1 - \phi_2 = 0$, i.e. $\phi_1 = \phi_2$. This completes the proof.

\textbf{Lemma 12} For $\nu \in U_B$, $(S^pWPAA(\mathbb{T}, \nu), \| \cdot \|_{S^p})$ is a Banach space.

Proof. Assume that $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $S^pWPAA(\mathbb{T}, \nu)$. We can write uniquely $f_n = g_n + \phi_n$. Using Lemma 10 we see that $\| g_n - g \| \leq \| f_n - f \|_{\infty}$, from which we deduce that $\{g_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $AA(\mathbb{T}, \mathbb{R}^n)$. Hence, $\phi_n = f_n - g_n$ is a Cauchy sequence in $WPAA_0(\mathbb{T}, \nu)$. We deduce that $g_n \to g \in AA(\mathbb{T}, \mathbb{R}^n)$, $\phi_n \to \phi \in WPAA(\mathbb{T}, \nu)$ and finally $f_n \to g + \phi \in S^pWPAA(\mathbb{T}, \nu)$. This complete the proof.

\textbf{Definition 8} ([1]) Let $\nu_1, \nu_2 \in U_\infty$. One says that $\nu_1$ is equivalent to $\nu_2$, written $\nu_1 \sim \nu_2$ if $\nu_1 / \nu_2 \in U_B$.

\textbf{Lemma 13} Let $\nu_1, \nu_2 \in U_\infty$. If $\nu_1 \sim \nu_2$, then $S^pWPAA(\mathbb{T}, \nu_1) = S^pWPAA(\mathbb{T}, \nu_2)$. 

Proof. The demonstration is similar to the proof of Theorem 3.8 in [1].

Lemma 14 Let \( f = g + \phi \in S^p WPAA(\mathbb{T}, \nu) \), where \( \nu \in \mathcal{U}_B \). Assume that \( f \) and \( g \) are Lipschitzian in \( x \in \mathbb{R}^n \) uniformly in \( t \in \mathbb{T} \), then \( f(., h(.)) \in S^p WPAA(\mathbb{T}, \nu) \) if \( h \in S^p WPAA(\mathbb{T}, \nu) \).

Proof. Similar to proof of Theorem 3.10 in [1].

Lemma 15 If \( f(t) \) is almost automorphic, \( F(.) \) is uniformly continuous on the value field of \( f(t) \), then \( F \circ f \) is almost automorphic.

Lemma 16 If \( f \in S^p C(\mathbb{R}, \mathbb{R}) \) satisfies the Lipschitz condition (with \( L \) is a constant of Lipschitz), \( \varphi \in S^p WPAA(\mathbb{T}, \nu) \), \( \theta \in S^p C_0^1(\mathbb{T}, \Pi) \) and \( \eta := \inf_{t \in \mathbb{T}} (1 - \theta^2(t)) > 0 \), then \( f(\varphi - \theta(t))) \in S^p WPAA(\mathbb{T}, \nu) \).

Proof. From Definition [4], we have \( \varphi = \varphi_1 + \varphi_2 \), where \( \varphi_1 \in AP(\mathbb{T}) \) and \( \varphi_2 \in WPAP(\mathbb{T}, \nu, t_0) \). Set

\[
E(t) = f(\varphi_1(t - \theta(t))) = f(\varphi_2(t - \theta(t))) + [f(\varphi_1(t - \theta(t))) - \varphi_2(t - \theta(t))] - f(t - \varphi_1(t - \theta(t))) = E_1(t) + E_2(t).
\]

Firstly, it follows from Lemma [15] that \( E_1 \in S^p AP(\mathbb{T}) \). Next, we show that \( E_2 \in WPAA_0(\mathbb{T}, \nu, t_0) \). Since

\[
\lim_{r \to +\infty} \frac{1}{m(r, \nu, t_0)} \int_{t_0 - r}^{t_0 + r} |E_2(s)| |\nu(s)| \Delta s = \lim_{r \to +\infty} \frac{1}{m(r, \nu, t_0)} \int_{t_0 - r}^{t_0 + r} |f(\varphi_1(t - \theta(t))) - \varphi_2(t - \theta(t))| \nu(s) \Delta s
\]

and

\[
0 \leq \lim_{r \to +\infty} \frac{L}{m(r, \nu, t_0)} \int_{t_0 - r}^{t_0 + r} |\varphi_2(t - \theta(t)))| |\nu(s)| \Delta s
\]

\[
= \frac{L}{m(r, \nu, t_0)} \int_{t_0 - r - \theta(t_0 - r)}^{t_0 + r - \theta(t_0 - r)} \frac{1}{1 - \theta^2(s)} |\varphi_2(u)| |\nu(u)| \Delta u
\]

\[
\leq \frac{r + \theta^+}{\eta} \frac{L}{m(r, \nu, t_0)} \int_{t_0 - \theta^+}^{t_0 + r + \theta^+} |\varphi_2(u)| |\nu(u)| \Delta u = 0,
\]

\( E_2 \in WPAA_0(\mathbb{T}, \nu, t_0) \). Thus \( E \in S^p WPAA(\mathbb{T}, \nu) \). The proof is achieved.
3 Model description and hypotheses

In this paper, we consider a class of \( n \)-neuron high-order BAM neural networks (HOBAMs) with mixed time-varying delays and leakage delays on time-space scales which are defined in the following lines:

\[
\begin{aligned}
    x_i^t(t) &= -\alpha_i(t)x_i(t - \eta_i(t)) + \sum_{j=1}^{m} D_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{m} D_{ij}^\sigma(t) f_j(x_j(t - \tau_{ij}(t))) \\
    &= + \sum_{j=1}^{m} D_{ij}(t) f_i^{\sigma} \left( y_j(t - \eta_j(t)) \right) \Delta s + \sum_{j=1}^{m} \tilde{D}_{ij}(t) f_i^{\sigma} \left( y_j^\sigma(t) \right) \Delta s \\
    &+ \sum_{j=1}^{m} \sum_{k=1}^{m} T_{ijk}(t) f_k(y_k(t - \chi_k(t))) f_j(x_j(t - \chi_j(t))) + I_i(t), \\
    y_j^\Delta(t) &= c_j(t) y_j(t - \eta_j(t)) + \sum_{i=1}^{n} E_{ij}(t) f_j(x_j(t)) + \sum_{i=1}^{n} E_{ij}^\sigma(t) f_j(x_j(t - \tau_{ij}(t))) \\
    &+ \sum_{i=1}^{n} \tilde{E}_{ij}(t) f_i^{\sigma} \left( x_j(t - \eta_j(t)) \right) \Delta s + \sum_{i=1}^{n} \tilde{E}_{ij}(t) f_i^{\sigma} \left( x_j^\sigma(t) \right) \Delta s \\
    &+ \sum_{i=1}^{n} \sum_{k=1}^{n} \tilde{T}_{ijk}(t) f_k(x_k(t - \chi_k(t))) f_j(x_j(t - \chi_j(t))) + J_j(t), \quad t \in \mathbb{T},
\end{aligned}
\]

(2)

where \( i = 1, \cdots, n \) and \( j = 1, \cdots, m \); \( \mathbb{T} \) is an almost periodic time scale; \( x_i(t) \) and \( y_i(t) \) are the neuron current activity level of ith neuron in the first layer and the jth neuron in the second layer respectively at time \( t \) \((i = 1, \cdots, n, \ j = 1, \cdots, m)\); \( \alpha_i(t), c_j(t) \) are the time variable of the neuron \( i \) in the first layer and the neuron \( j \) in the second neuron respectively; \( f_j(x_j(t)) \) and \( f_j(x_j(t - \tau_{ij}(t))) \) are the output of neurons; \( I_i(t) \) and \( J_j(t) \) denote the external inputs on the ith neuron at time \( t \) for the first layer and the jth neuron at the second layer at time \( t \);

\[
\begin{aligned}
    &t \mapsto D_{ij}(t) \\
    &t \mapsto T_{ijk}(t) \\
    &t \mapsto \tilde{T}_{ijk}(t) \\
    &t \mapsto E_{ij}(t) \\
    &t \mapsto \tilde{E}_{ij}(t) \\
    &t \mapsto \tilde{T}_{ijk}(t)
\end{aligned}
\]

represent the connection weights and the synaptic weights of delayed feedback between the ith neuron and the jth neuron respectively;

for all \( i, k = 1, \cdots, n, \ j = 1, \cdots, m \); \( t \mapsto T_{ijk}(t) \) and \( t \mapsto \tilde{T}_{ijk}(t) \) are the second-order connection weights of delayed feedback;

\( t \mapsto I_i(t), t \mapsto J_i(t) \) denote the external inputs on the ith neuron at time \( t \); \( t \mapsto \eta_i(t) \) and \( t \mapsto \zeta_i(t) \) are leakage delays and satisfy \( t - \eta_i(t) \in \mathbb{T}, t - \zeta_i(t) \in \mathbb{T} \) for \( t \in \mathbb{T} \); \( \tau_{ij}(t), \sigma_{ij}(t), \chi_k(t) \) and \( \xi_{ij}(t) \) are transmission delays and satisfy \( t - \tau_{ij}(t) \in \mathbb{T}, t - \sigma_{ij}(t) \in \mathbb{T}, t - \chi_k(t) \in \mathbb{T} \) and \( t - \xi_{ij}(t) \in \mathbb{T} \) for \( t \in \mathbb{T} \).
For convenience, we introduce the following notations:

\[ \alpha_i^+ = \sup_{t \in T} |\alpha_i(t)|, \quad \alpha_i^- = \inf_{t \in T} |\alpha_i(t)| > 0, \quad c_i^+ = \sup_{t \in T} |c_i(t)|, \quad c_i^- = \inf_{t \in T} |c_i(t)| > 0, \]

\[ \eta_i^+ = \sup_{t \in T} |\eta_i(t)|, \quad \zeta_i^+ = \sup_{t \in T} |\zeta_i(t)|, \quad D_i^+ = \sup_{t \in T} |D_i(t)|, \quad (D_i^t)^+ = \sup_{t \in T} |D_i^t(t)|, \]

\[ D_{ij}^+ = \sup_{t \in T} |D_{ij}(t)|, \quad (D_{ij}^t)^+ = \sup_{t \in T} |D_{ij}^t(t)|, \quad \overline{D}_{ij}^+ = \sup_{t \in T} |\overline{D}_{ij}(t)|, \]

\[ (\tilde{D}_{ij})^+ = \sup_{t \in T} |\tilde{D}_{ij}(t)|, \quad E_{ij}^+ = \sup_{t \in T} |E_{ij}(t)|, \quad (E_{ij}^t)^+ = \sup_{t \in T} |E_{ij}^t(t)|, \quad E_{ij}^+ = \sup_{t \in T} |E_{ij}(t)|, \]

\[ (E_{ij}^t)^+ = \sup_{t \in T} |E_{ij}^t(t)|, \quad \overline{E}_{ij}^+ = \sup_{t \in T} |\overline{E}_{ij}(t)|, \quad (\tilde{E}_{ij})^+ = \sup_{t \in T} |\tilde{E}_{ij}(t)|, \]

\[ T_{ijk}^+ = \sup_{t \in T} |T_{ijk}(t)|, \quad T_{ij}^+ = \sup_{t \in T} |T_{ij}(t)|, \quad \tau_{ij}^+ = \sup_{t \in T} |\tau_{ij}(t)|, \]

\[ \sigma_{ij}^+ = \sup_{t \in T} |\sigma_{ij}(t)|, \quad \xi_{ij}^+ = \sup_{t \in T} |\xi_{ij}(t)|, \quad \chi_{ij}^+ = \sup_{t \in T} |\chi_{ij}(t)|, \quad \nu_{ij}^+ = \sup_{t \in T} |\nu_{ij}(t)|, \quad \phi_{ij}^+ = \sup_{t \in T} |\phi_{ij}(t)|. \]

We denote that \([a, b]_\mathbb{T} = \{t, t \in [a, b] \cap \mathbb{T}\}. The initial conditions associated with system (2), are of the form:

\[ x_i(s) = \varphi_i(s), \quad y_i(s) = \phi_i(s), \quad s \in [-\theta, 0], 1 \leq i \leq n, \quad 1 \leq j \leq m, \]

where \(\varphi_i(.)\) and \(\phi_i(.)\) are the real-valued bounded \(\Delta\)-differentiable functions defined on \([-\theta, 0].\)

\[ \theta = \max\{\eta, \tau, \chi, \sigma, \xi, \zeta\}, \quad \eta = \max_{1 \leq i \leq n} \eta_i^+, \quad \tau = \max_{1 \leq i \leq n, 1 \leq j \leq m} \tau_{ij}^+, \quad \chi = \max_{1 \leq j \leq m} \chi_{ij}^+, \]

\[ \sigma = \max_{1 \leq i \leq n, 1 \leq j \leq m} \sigma_{ij}^+, \quad \xi = \max_{1 \leq i \leq n, 1 \leq j \leq m} \xi_{ij}^+, \quad \zeta = \max_{1 \leq i \leq n, 1 \leq j \leq m} \zeta_{ij}^+. \]

**Remark 1** This is the first time to study the Stepanov-like weighted pseudo-almost automorphic solutions of system (2) for both cases: continuous and discrete. Furthermore, there is no result about the automorphic, Stepanov almost automorphic and Stepanov-like weighted pseudo-almost automorphic solutions of networks (2).

Let us list some assumptions that will be used throughout the rest of this paper.

\((H_1)\) For all \(1 \leq i, j \leq n,\), the functions \(\alpha_i(\cdot), c_i(\cdot) \in \mathcal{R}_+^p\) and \(D_i(\cdot), D_i^t(\cdot) \in \mathcal{R}_+^p\), \(\overline{D}_i(\cdot), \tilde{D}_i(\cdot), T_{ij}(\cdot), E_{ij}(\cdot), E_{ij}^t(\cdot), \overline{E}_{ij}(\cdot), \tilde{E}_{ij}(\cdot), \eta_i(\cdot), \xi_i(\cdot), \tau_i(\cdot), \chi_i(\cdot), \sigma_i(\cdot), \xi_i(\cdot), I_i(\cdot), J_i(\cdot)\) are \(\Delta\)-continuous Stepanov-like weighted pseudo-automorphic functions for \(i = 1, \ldots, n, j = 1, \ldots, m.\)

\((H_2)\) The functions \(f_j(\cdot)\) are \(\Delta\)-differentiable and satisfy the Lipschitz condition, i.e., there are constants \(L_j > 0\) such that for all \(x, y \in \mathbb{R}\), and for all \(1 \leq j \leq \max\{n, m\}\), one has \(|f_j(x) - f_j(y)| \leq L_j |x - y|\).
\[(H_3)\]
\[
\max_{1 \leq i \leq n} \left\{ \frac{M_i}{\alpha_i} \left( 1 + \frac{\alpha^+}{\alpha_i} \right) M_i, \frac{N_i}{\alpha_i} \left( 1 + \frac{\alpha^+}{\alpha_i} \right) N_i \right\} \leq r
\]
and \[
\max_{1 \leq i \leq n} \left\{ \frac{M_i}{\alpha_i} \left( 1 + \frac{\alpha^+}{\alpha_i} \right) M_i, \frac{N_i}{\alpha_i} \left( 1 + \frac{\alpha^+}{\alpha_i} \right) N_i \right\} \leq 1,
\]
where \( r \) is a constant, for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m, \)
\[
M_i = \alpha_i^+ \eta_i^+ r + \sum_{j=1}^{m} \left( D_{ij}^+ + (D_{ij}^+)^+ + \overline{D}_{ij}^+ \sigma_{ij}^+ + \overline{D}_{ij}^+ \xi_{ij}^+ \right) (L_j r + |f_j(0)|)
\]
\[
+ \sum_{j=1}^{m} \sum_{k=1}^{m} \overline{T}_{ijk}^+ (L_k r + |f_k(0)|) (L_j r + |f_j(0)|) + I_j^+,
\]
\[
N_i = c_i^+ \xi_i^+ r + \sum_{j=1}^{n} \left( E_{ij}^+ + (E_{ij}^+)^+ + \overline{E}_{ij}^+ \sigma_{ij}^+ + \overline{E}_{ij}^+ \xi_{ij}^+ \right) (L_i r + |f_i(0)|)
\]
\[
+ \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{T}_{ijk}^+ (L_k r + |f_k(0)|) (L_i r + |f_i(0)|) + J_i^+,
\]
\[
N_j = c_j^+ \xi_j^+ + \sum_{i=1}^{n} \left( E_{ij}^+ + (E_{ij}^+)^+ + \overline{E}_{ij}^+ \sigma_{ij}^+ + \overline{E}_{ij}^+ \xi_{ij}^+ \right) L_i
\]
\[
+ \sum_{i=1}^{n} \sum_{k=1}^{n} \overline{T}_{ijk}^+ (L_k r + |f_k(0)|).
\]
\[(H_4)\] \( \inf_{t \in T} (1 - \sigma_{ij}(t)) > 0, \) \( \inf_{t \in T} (1 - \xi_{ij}(t)) > 0, \) and for all \( s \in \Pi, \)
\[
\limsup_{|t| \to +\infty} \frac{\nu(t + s) - \nu(t)}{\nu(t)} < \infty.
\]

Remark 2 The bidirectional associative memory (BAM) neural networks with mixed time-varying delays and leakage time-varying delays on time-space scales is investigated in [14]. Some sufficient conditions are given for the existence, convergence and the global exponential stability of the weighted pseudo almost-periodic solution. However, Theorem 4.1, Theorem 5.1 and Theorem 6.1 proposed in [14] are not applicable for the HOBAMs with mixed time-varying delays in the leakage terms.
4 The existence of Stepanov-like weighted pseudo-almost automorphic on time-space scales solutions

In this section, based on Banach’s fixed point theorem and the theory of calculus on time-space scales, we will present a new condition for the existence and uniqueness of weighted pseudo-almost automorphic on time-space scales solutions of (2). Additionally, we will show a result about the delta derivative and uniqueness of weighted pseudo-almost automorphic on time-space scales calculus on time-space scales, we will present a new condition for the existence of Stepanov-like weighted pseudo-almost automorphic on time-space scales solutions of system (2).

Let

\[ \mathcal{B} = \{(\varphi_1, \varphi_2, ..., \varphi_n, \phi_1, \phi_2, ..., \phi_m)^T : \varphi_i, \phi_j \in C^1(T, \mathbb{R}), \quad i = 1, ..., n, \quad j = 1, ..., m \} \]

with the norm \( \|\psi\|_{\mathcal{B}} = \sup_{t \in T} \max_{i=1,...,n} \max_{j=1,...,m} \{|\varphi_i(t)|, |\phi_j(t)|, |\varphi_i^\Delta(t)|, |\phi_j^\Delta(t)|\} \), then \((\mathcal{B}, \|\psi\|_{\mathcal{B}})\) is a Banach space.

For every \( \psi = (\varphi_1, ..., \varphi_n, \phi_1, ..., \phi_m) \in \mathcal{B} \), we consider the following system

\[ x_i^\Delta(t) = -\alpha_i(t)x_i(t) + F_i(t, \varphi_i), \quad y_j^\Delta(t) = -c_j(t)y_j(t) + G_j(t, \phi_j), \quad t \in T, \quad (3) \]

where, for \( i = 1, ..., n \) and \( j = 1, ..., m \)

\[ F_i(t, \varphi_i(t)) = \alpha_i(t) \int_{t-h_i(t)}^t \varphi_i^\Delta(s) \Delta s + \sum_{j=1}^m D_{ij}(t) f_j(\phi_j(t)) \]

\[ + \sum_{j=1}^m \tilde{D}_{ij}(t) f_j(\phi_j(t - \tau_j(t))) + \sum_{j=1}^m \tilde{D}_{ij}(t) \int_{t-\sigma_j(t)}^t f_j(\phi_j(s)) \Delta s \]

\[ G_j(t, \phi_j(t)) = c_j(t) \int_{t-s_j(t)}^t \phi_j^\Delta(s) \Delta s + \sum_{i=1}^n E_{ij}(t) f_i(\varphi_i(t)) \]

\[ + \sum_{i=1}^n \tilde{E}_{ij}(t) f_i(\varphi_i(t - \tau_i(t))) + \sum_{i=1}^n \tilde{E}_{ij}(t) \int_{t-\sigma_i(t)}^t f_i(\varphi_i(s)) \Delta s \]

\[ + \sum_{i=1}^n \frac{n}{k=1} T_{ijk}(t)f_k(\varphi_k(t - \chi_k(t)))f_j(\phi_j(t - \chi_j(t)) + J_j(t). \]
Let \( y_\psi(t) = (x_{\varphi_1}(t), ..., x_{\varphi_n}(t), y_{\phi_1}(t), ..., y_{\phi_m}(t))^T \), where:

\[
x_{\varphi_i}(t) = \int_{-\infty}^{t} \hat{e}_{-\alpha_i}(t, \sigma(s)) F_i(t, \varphi_i(s)) \Delta s,
\]

\[
y_{\phi_j}(t) = \int_{-\infty}^{t} \hat{e}_{-\epsilon_j}(t, \sigma(s)) G_j(t, \phi_j(s)) \Delta s,
\]

**Lemma 17** Suppose that assumptions \((H_1) - (H_4)\) hold. Define the nonlinear operator \( \Gamma : \mathbb{F} \rightarrow \mathbb{F} \) by for each \( \psi \in WPAA(\mathbb{T}, \nu) \)

\[
(\Gamma \psi)(t) = y_\psi(t), \quad \psi \in \mathbb{F}.
\]

Then \( \Gamma \) maps \( WPAA(\mathbb{T}, \nu) \) into itself.

**Proof.**

We show that for any \( \psi \in \mathbb{F}, \Gamma \psi \in \mathbb{F} \).

\[
|F_i(s, \varphi_i(s))| \leq \alpha_i^+ \eta_i^+ r + \sum_{j=1}^{n} D_{ij}^+ (L_j |\varphi_j(s)| + |f_j(0)|)
\]

\[
+ \sum_{j=1}^{n} (D_{ij}^+) \left( L_j |\varphi_j(s - \tau_{ij}(s))| + |f_j(0)| \right)
\]

\[
+ \sum_{j=1}^{n} D_{ij}^+ \sigma_{ij}^+ (L_j r + |f_j(0)|) + \sum_{j=1}^{n} \hat{D}_{ij}^+ \xi_{ij}^+ (L_j r + |f_j(0)|)
\]

\[
+ \sum_{j=1}^{n} \sum_{k=1}^{n} T_{ijk}^+ (L_k r + |f_k(0)|) (L_j r + |f_j(0)|) + I_i^+
\]

\[
\leq M_i.
\]

In a similar way, we have

\[
|G_j(s, \phi_j(s))| \leq N_j.
\]

Which leads to, for \( i = 1, ..., n, \ j = 1, ..., m \).

\[
\sup_{t \in \mathbb{T}} |x_{\varphi_i}(t)| = \sup_{t \in \mathbb{T}} \left| \int_{-\infty}^{t} \hat{e}_{-\alpha_i}(t, \sigma(s)) F_i(t, \varphi_i(s)) \Delta s \right|
\]

\[
\leq \sup_{t \in \mathbb{T}} \int_{-\infty}^{t} \hat{e}_{-\alpha_i}(t, \sigma(s)) |F_i(s, \varphi_i(s))| \Delta s
\]

\[
\leq \frac{M_i}{\alpha_i},
\]

and

\[
\sup_{t \in \mathbb{T}} |y_{\phi_j}(t)| = \sup_{t \in \mathbb{T}} \left| \int_{-\infty}^{t} \hat{e}_{-\epsilon_j}(t, \sigma(s)) G_j(t, \phi_j(s)) \Delta s \right|
\]

\[
\leq \frac{N_i}{\epsilon_j}.
\]
Otherwise, for \( i = 1, \ldots, n \), we have

\[
\sup_{t \in \mathbb{T}} |x^2_{\alpha_i}(t)| = \sup_{t \in \mathbb{T}} \left| \left( \int_{-\infty}^{t} \hat{e}_{-\alpha_i}(t, \sigma(s)) F_i(t, \varphi_i(s)) \Delta s \right)^{\Delta} \right|
\]
\[
= \sup_{t \in \mathbb{T}} \left| F_i(t, \varphi_i(t)) - \alpha_i(t) \int_{-\infty}^{t} \hat{e}_{-\alpha_i}(t, \sigma(s)) F_i(t, \varphi_i(s)) \Delta s \right|
\]
\[
\leq \alpha_i^+ \eta_i^+ r + \sum_{j=1}^{m} \left( D_{ij}^+ + (D_{ij}^+)^+ + (\overline{D}_{ij})^+ \sigma_{ij}^+ + (\overline{D}_{ij})^+ \xi_{ij}^+ \right) \left( L_i r + |f_j(0)| \right)
\]
\[
+ \sum_{j=1}^{m} \sum_{k=1}^{n} T_{ijk}^+ (L_k r + |f_k(0)|) (L_i r + |f_j(0)|)
\]
\[
+ \frac{\alpha_i^+}{\alpha_i^+} \left( \alpha_i^+ \eta_i^+ r + \sum_{j=1}^{m} (D_{ij}^+ + (D_{ij}^+)^+ + (\overline{D}_{ij})^+ \sigma_{ij}^+ \right)
\]
\[
+ (\overline{D}_{ij})^+ \xi_{ij}^+ \right) \left( L_i r + |f_j(0)| \right) + \sum_{j=1}^{m} \sum_{k=1}^{n} T_{ijk}^+ (L_k r + |f_k(0)|)
\]
\[
\times (L_i r + |f_j(0)|) + I_i^+
\]
\[
= \left( 1 + \frac{\alpha_i^+}{\alpha_i^+} \right) M_i.
\]

Similarly,

\[
\sup_{t \in \mathbb{T}} \left| y_{\alpha_i}^2(t) \right| \leq \left( 1 + \frac{c_j^+}{c_j^+} \right) N_j.
\]

From hypothesis (H3), we can obtain

\[
\|\Gamma \psi\|_{B} \leq r,
\]

which implies that operator \( \Gamma \) is a self-mapping from \( F \) to \( F \).

**Theorem 1** Let \((H_1) - (H_4)\) hold. The system \( F \) has a unique Stepanov-like weighted pseudo-almost automorphic solution in the region \( F = \{ \psi \in B : \|\psi\|_{B} \leq r \} \).

**Proof.**

First, for \( \psi = (\varphi_1, \ldots, \varphi_n, \phi_1, \ldots, \phi_m)^T \), \( \Omega = (u_1, \ldots, u_n, v_1, \ldots, v_m)^T \in F \), we
have

\[
\sup_{s \in T} \| x_{\phi_i}(s) - x_{u_i}(s) \| \leq \frac{1}{\alpha_i} \left( \alpha_i^+ \eta_i^+ + \sum_{j=1}^m \left( D_{ij}^+ + (D_{ij}^+) + (\tilde{D}_{ij})^+ \sigma_{ij}^+ + (\tilde{D}_{ij})^+ \xi_{ij}^+ \right) L_j 
+ \sum_{j=1}^m \sum_{k=1}^m (T_{ijk}^+ + T_{ikj}^+) (L_{kr} + |f_k(0)|) \right) \| \psi - \Omega \|_B 
= \frac{M_i}{\alpha_i} \| \psi - \Omega \|_B.
\]

Besides,

\[
\sup_{s \in T} \| x_{\phi_j}(s) - x_{u_j}(s) \| \leq \left( \alpha_i^+ \eta_i^+ + \sum_{j=1}^m \left( D_{ij}^+ + (D_{ij}^+) + (\tilde{D}_{ij})^+ \sigma_{ij}^+ + (\tilde{D}_{ij})^+ \xi_{ij}^+ \right) L_j 
+ \sum_{j=1}^m \sum_{k=1}^m (T_{ijk}^+ + T_{ikj}^+) (L_{kr} + |f_k(0)|) \right) \| \psi - \Omega \|_B 
+ \frac{\alpha_i^+}{\alpha_i} \left( \alpha_i^+ \eta_i^+ \| \phi_i^a(s) - u_i^a(s) \| + \sum_{j=1}^m D_{ij}^+ \| \phi_i(t) - u_i(t) \| 
+ \sum_{j=1}^m \sigma_{ij}^+ \| \phi_j^a(s - \tau_j(s)) - u_j(s - \tau_j(s)) \| 
+ \sum_{j=1}^m \xi_{ij}^+ \| \phi_j^a(s) - u_j(s) \| 
+ \sum_{j=1}^m \sum_{k=1}^m (T_{ijk}^+ + T_{ikj}^+) (L_{kr} + |f_k(0)|) \| \phi_j(s) - u_j(s) \| 
+ \sum_{j=1}^m \xi_{ij}^+ \| \phi_j^a(s) - u_j^a(s) \| \right) 
\leq \left( 1 + \alpha_i^+ \right) \frac{M_i}{\alpha_i} \| \psi - \Omega \|_B.
\]

Similarly,

\[
\sup_{s \in T} \| y_{\phi_i}(s) - y_{u_i}(s) \| \leq \left( \frac{N_j}{c_j} \right) \| \psi - \Omega \|_B,
\]

and

\[
\sup_{s \in T} \| y_{\phi_j}(s) - y_{u_j}(s) \| \leq \left( 1 + \frac{c_j^+}{c_j} \right) N_j \| \psi - \Omega \|_B,
\]

therefore,

\[
\| \Gamma \psi - \Gamma \Omega \| \leq \kappa \| \psi - \Omega \|_B, \text{ where } \kappa < 1.
\]
According to the well-known contraction principle there exists a unique fixed point \( h^\ast ( \cdot ) \) such that \( \Gamma h^\ast = h^\ast \). So, \( h^\ast \) is a weighted pseudo-almost automorphic on time-space scales solution of the model (2) in \( F = \{ \psi \in B : \| \psi \|_B \leq r \} \). This completes the proof.

**Remark 3** To the best of our knowledge, there have been no results focused on the automorphic solutions, pseudo-almost automorphic ones and Stepanov-like weighted pseudo-almost automorphic solutions on time-space scales for high-order BAM neural networks with time varying coefficients, mixed delays and leakage until now. Hence, the obtained results are essentially new and the investigation methods used in this paper can also be applied to study the Stepanov-like weighted pseudo-almost automorphic solutions on time-space scales for some other models of dynamical neural networks, such as Cohen-Grossberg neural networks.

**Theorem 2** Let \((H_1) - (H_4)\) hold. The delta derivative of the only Stepanov-like weighted pseudo-almost automorphic on time-space scales solution of system (2) is also Stepanov-like weighted pseudo-almost automorphic on time-space scales (i.e. the unique solution of (2) is delta differentiable Stepanov-like weighted pseudo-almost automorphic on time-space scales).

**Proof.** From system (2), the expression of delta derivative of the only Stepanov-like weighted pseudo-almost automorphic on time-space scales solution is

\[
x^3_i (t) = - \alpha_i (t) x_i (t - \eta_i (t)) + \sum_{j=1}^{m} D_{ij} (t) f_j (y_j (t))
\]

\[
+ \sum_{j=1}^{m} \partial_{\sigma_j_i (t)} \sum_{j=1}^{m} \partial_{\sigma_j_i (t)} \int_{t-\sigma_j_i (t)}^{t} f_j (y_j (s)) \Delta s
\]

\[
+ \sum_{j=1}^{m} T_{ij} (t) \int_{t-\tau_{ij} (t)}^{t} f_j (y_j (t - \tau_{ij} (t)))
\]

\[
+ \sum_{j=1}^{m} \tilde{D}_{ij} (t) \int_{t-\xi_{ij} (t)}^{t} f_j (y_j (s)) \Delta s + I_i (t),
\]

and

\[
y^3_j (t) = - c_j (t) y_j (t - \eta_j (t)) + \sum_{i=1}^{n} E_{ij} (t) f_j (x_j (t))
\]

\[
+ \sum_{i=1}^{n} E_{ij} (t) f_j (x_j (t - \tau_{ij} (t))) \sum_{i=1}^{n} E_{ij} (t) \int_{t-\sigma_{ij} (t)}^{t} f_j (x_j (s)) \Delta s
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} (t) f_j (x_j (t - \chi_j (t))) f_j (x_j (t - \chi_j (t)))
\]

\[
+ \sum_{i=1}^{n} \tilde{D}_{ij} (t) \int_{t-\xi_{ij} (t)}^{t} f_j (x_j (s)) \Delta s + J_j (t), \quad t \in T.
\]
Since all coefficients of the system (2) are Stepanov-like weighted pseudo-almost automorphic on time-space scales, derivative of solution of system (2) is Stepanov-like weighted pseudo-almost automorphic on time-space scales.

**Remark 4** In practice, time delays, leakage delay and parameter perturbations are unavoidably encountered in the implementation of high-order BAM neural networks, and they may destroy the stability of Stepanov-like weighted pseudo-almost automorphic solution of HOBAMs, so it is necessary and vital to study the dynamic behaviors of Stepanov-like weighted pseudo-almost automorphic on time-space scales solution of HOBAMs with time delays, leakage term and parameter perturbations.

5 Global exponential stability and convergence of Stepanov-like weighted pseudo-almost automorphic on time-space scales solution

5.1 Global exponential stability

**Definition 9** Let \( Z^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t), y_1^*(t), y_2^*(t), \ldots, y_m^*(t))^T \) be the Stepanov-like weighted pseudo-almost automorphic solution on time-space scales of system (2) with initial value \( \psi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \ldots, \varphi_n^*(t), \phi_1(t), \phi_2(t), \ldots, \phi_m(t))^T \). \( Z^*(\cdot) \) is said to be globally exponential stable if there exist constants \( \gamma > 0, \psi_0 \in \mathbb{R}_+ \) and \( M > 1 \) such that for every solution

\[
Z(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T
\]

of system (2) with any initial value

\[
\psi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \phi_1(t), \phi_2(t), \ldots, \phi_m(t))^T, \quad \forall t \in (0, +\infty)_T, t \geq t_0,
\]

\[
\|Z(t) - Z^*(t)\|_0 = \max \{ \|x(t) - x^*(t)\|_\infty, \|x^\Delta(t) - x^\Delta^*(t)\|_\infty, \\
\|y(t) - y^*(t)\|_\infty, \|y^\Delta(t) - y^\Delta^*(t)\|_\infty \}
\]

\[
\leq M \psi(\cdot, t_0)\|\psi\|_1
\]

\[
= M\sup_{t \in (-\theta, 0)_T} \max \{ \|\varphi(t) - \varphi^*(t)\|_\infty, \|\varphi^\Delta(t) - \varphi^\Delta^*(t)\|_\infty, \\
\|\phi(t) - \phi^*(t)\|_\infty, \|\phi^\Delta(t) - \phi^\Delta^*(t)\|_\infty \},
\]

where \( t_0 = \max\{(-\theta, 0)_T\} \).

**Theorem 3** Let \((H_1) - (H_4)\) hold. The unique Stepanov-like weighted pseudo-almost automorphic on time-space scales solution of system (2) is globally exponentially stable.

Proof. From Theorem 4 the system (2) has one and only one weighted pseudo-almost automorphic on time-space scales solution on time scales

\[
Z^*(t) = (x_1^*(t), \ldots, x_n^*(t), y_1^*(t), \ldots, y_m^*(t))^T \in \mathbb{R}^{n \times m},
\]
with the initial condition
\[ \psi^*(t) = (\varphi^*_1(t),...,\varphi^*_n(t),\phi^*_1(t),...,\phi^*_m(t))^T. \]

Let \( Z(t) = (x_1(t),...,x_n(t),y_1(t),...,y_m(t)) \) one arbitrary solution of (2) with initial condition \( \psi(t) = (\varphi_1(t),...,\varphi_n(t),\phi_1(t),...,\phi_m(t))^T \).

From system (2), for \( t \in \mathbb{T} \), we obtain:

\[
\begin{cases}
  u_i^\Delta(t) = -\alpha_i(t)u_i(t) + \alpha_i(t) \int_{t-N_i(t)}^t u_i^\Delta(s)\Delta s + \sum_{j=1}^m D_{ij}(t) p_j(v_j(t)) \\
  + \sum_{j=1}^m \tilde{D}_{ij}(t) \int_{t-\xi_{ij}(t)}^t h_j(u_i^\Delta(s))\Delta s + \sum_{j=1}^m \sum_{k=1}^n T_{ijk}(t) q_{j,k}(v_j(t)-\chi_j(t)), v_k(t)-\chi_j(t)), \\
  v_i^\Delta(t) = -c_i(t)v_i(t) + c_i(t) \int_{t-\xi_{i}(t)}^t v_i^\Delta(s)\Delta s + \sum_{i=1}^n E_{ij}(t) p_j(u_j(t)) \\
  + \sum_{i=1}^n \tilde{E}_{ij}(t) \int_{t-\xi_{i}(t)}^t h_j(v_i^\Delta(s))\Delta s + \sum_{i=1}^n \sum_{k=1}^n \overline{T}_{ijk}(t) q_{i,k}(u_j(t)-\chi_j(t)), u_k(t)-\chi_j(t)), \tag{4}
\end{cases}
\]

where

\[
\begin{align*}
  u_i(t) &= x_i(t) - x_i^*(t), \quad v_i(t) = y_i(t) - y_i^*(t), \quad p_j(u_j(t)) = f_j(x_j(t)) - f_j(x_j^*(t)), \\
  h_j(u_i^\Delta(t)) &= f_j(x_j^\Delta(t)) - f_j(x_j^*\Delta(t)), \\
  p_j(v_j(t)) &= f_j(y_j(t)) - f_j(y_j^*(t)), \\
  h_j(v_i^\Delta(t)) &= f_j(y_j^\Delta(t)) - f_j(y_j^*\Delta(t)), \\
  q_{j,k}(u_j(t),u_k(t)) &= f_k(x_k(t))f_j(x_j(t)) - f_k(x_k^*(t))f_j(x_j^*(t)) \\
  q_{j,k}(v_j(t),v_k(t)) &= f_k(y_k(t))f_j(y_j(t)) - f_k(y_k^*(t))f_j(y_j^*(t)).
\end{align*}
\]

For \( i = 1,...,n \) and \( j = 1,...,m \), the initial condition of (4) is

\[
u_i(s) = \varphi_i(s) - \varphi_i^*(s), \quad v_j(s) = \phi_j(s) - \phi_j^*(s), \quad s \in [-\theta,0]_\mathbb{T}.
\]

Multiplying the first equation in system (4) by \( \dot{c}_{-\alpha_i}(t_0,\sigma(s)) \) and the second equation by \( \dot{c}_{-\alpha_j}(t_0,\sigma(s)) \), and integrating over \([t_0,t]_\mathbb{T} \), where \( t_0 \in [-\theta,0]_\mathbb{T} \),
we obtain

\[
\begin{align*}
    u_i(t) &= u_i(t_0) \hat{e}_{-\alpha_i}(t, t_0) + \int_{t_0}^{t} \hat{e}_{-\alpha_i}(t, \sigma(s)) \left( \alpha_i(s) \int_{s-\tau_i(s)}^{t} u_i^\Delta(u) \Delta u \right) + \sum_{j=1}^{m} D_{ij}(s) p_j(v_j(s)) + \sum_{j=1}^{m} D_{ij}^\tau(s) p_j(v_j((s - \tau_{ij}(s))) \\
    &+ \sum_{j=1}^{m} \tilde{D}_{ij}(s) \int_{s-\xi_{ij}(s)}^{t} h_j(v_j^A(u)) \Delta u + \sum_{i=1}^{n} \sum_{k=1}^{m} T_{ijk}(s) q_{jk}(v_j(s - \chi_j(s)), v_k(s - \chi_k(s))) \\
    &+ \sum_{j=1}^{m} \tilde{D}_{ij}(s) \int_{s-\xi_{ij}(s)}^{t} h_j(v_j^A(u)) \Delta u) \Delta s, \\
    v_j(t) &= v_j(t_0) \hat{e}_{-c_j}(t, t_0) + \int_{t_0}^{t} \hat{e}_{-c_j}(t, \sigma(s)) \left( c_j(s) \int_{s-\xi_j(s)}^{t} v_j^A(u) \Delta u \right) + \sum_{i=1}^{n} E_{ij}(s) p_i(u_i(s)) + \sum_{i=1}^{n} E_{ij}^\tau(s) p_i(u_i((s - \tau_{ij}(s))) \\
    &+ \sum_{i=1}^{n} \tilde{E}_{ij}(s) \int_{s-\xi_{ij}(s)}^{t} h_i(u_i^A(u)) \Delta u \right) \Delta s, \quad t \in \mathbb{T},
\end{align*}
\]

Now, we define \( G_i, \mathcal{G}_j, H_i \) and \( \mathcal{H}_j \) as follows:

\[
G_i(w) = \alpha_i^- - w - \left( \exp \left( \sup_{s \in \mathbb{T}} \nu(s) \right) \left( \alpha_i^+ \eta_i^+ \exp \left( w \eta_i^+ \right) + \sum_{j=1}^{m} D_{ij}^+ L_j \right) \\
+ \sum_{j=1}^{m} (D_{ij}^+) L_j \exp \left( w \tau_{ij}^+ \right) + \sum_{j=1}^{m} \mathcal{T}_{ij} L_j \sigma_{ij}^+ \exp \left( w \sigma_{ij}^+ \right) + \sum_{j=1}^{m} \tilde{D}_{ij}^+ L_j \xi_{ij}^+ \exp \left( w \xi_{ij}^+ \right) \\
+ \sum_{j=1}^{m} \sum_{k=1}^{n} T_{ijk}(s) \left( L_k r + |f_k(0)| \right) (L_j r + |f_j(0)|) \right),
\]

\[
\mathcal{G}_j(w) = c_j^- - w - \left( \exp \left( \sup_{s \in \mathbb{T}} \nu(s) \right) \left( c_j^+ \eta_j^+ \exp \left( w \eta_j^+ \right) + \sum_{i=1}^{n} E_{ij}^+ L_i \right) \\
+ \sum_{i=1}^{n} (E_{ij}^+) L_i \exp \left( w \tau_{ij}^+ \right) + \sum_{i=1}^{n} \mathcal{T}_{ij} L_i \sigma_{ij}^+ \exp \left( w \sigma_{ij}^+ \right) + \sum_{i=1}^{n} \tilde{E}_{ij}^+ L_i \xi_{ij}^+ \exp \left( w \xi_{ij}^+ \right) \\
+ \sum_{i=1}^{n} \sum_{k=1}^{m} \mathcal{T}_{ijk}(s) \left( L_k r + |f_k(0)| \right) (L_i r + |f_i(0)|) \right),
\]

\[
H_i(w) = \alpha_i^- - w - \left( \alpha_i^+ \exp \left( \sup_{s \in \mathbb{T}} \nu(s) + \alpha_i^- - \beta \right) \left( \alpha_i^+ \eta_i^+ \exp \left( w \eta_i^+ \right) \\
+ \sum_{j=1}^{m} D_{ij}^+ L_j \exp \left( w \tau_{ij}^+ \right) + \sum_{j=1}^{m} \mathcal{T}_{ij} L_j \sigma_{ij}^+ \exp \left( w \sigma_{ij}^+ \right) + \sum_{j=1}^{m} \tilde{D}_{ij}^+ L_j \xi_{ij}^+ \exp \left( w \xi_{ij}^+ \right) \right)
\]
\[
\sum_{j=1}^{m} \sum_{k=1}^{n} T_{jk}^+ (L_k r + |f_k(0)|)(L_j r + |f_j(0)|),
\]

\[
\bar{H}_j(w) = c_j^{-} - w - \left( c_j^+ \exp \left( w \sup_{s \in T} \nu(s) + c_j^{-} - \beta \right) \right) (c_j^+ \sigma_j^+ \exp (w \xi_j^+))
\]

\[
+ \sum_{i=1}^{n} E_{ij}^+ L_i + \sum_{i=1}^{n} (E_{ij}^+)^+ L_i \exp (w \xi_j^+) + \sum_{i=1}^{n} \bar{E}_{ij}^+ L_i \sigma_j^+ \exp (w \xi_j^+) + \sum_{i=1}^{n} \bar{E}_{ij}^+ L_i \xi_j^+ \exp (w \xi_j^+)
\]

\[
+ \sum_{i=1}^{n} \sum_{k=1}^{m} T_{ijk}^+ (L_k r + |f_k(0)|)(L_j r + |f_j(0)|),
\]

where \( i = 1, \ldots, n, j = 1, \ldots, m, w \in (0, +\infty) \).

From \((H_3)\), we have

\[
G_i(0) = \alpha_i^{-} - \left( \alpha_i^{+} \eta_i^{+} + \sum_{j=1}^{m} D_{ij}^+ L_j + \sum_{j=1}^{m} (D_{ij}^+)^+ L_j + \sum_{j=1}^{m} \bar{D}_{ij}^+ L_j \sigma_j^+ \right.
\]

\[
\left. + \sum_{j=1}^{m} \bar{D}_{ij}^+ L_j \xi_j^+ + \sum_{j=1}^{m} \sum_{k=1}^{n} T_{ijk}^+ (L_k r + |f_k(0)|)(L_j r + |f_j(0)|) \right),
\]

\[
\bar{G}_j(0) = c_j^{-} + \sum_{i=1}^{n} E_{ij}^+ L_i + \sum_{i=1}^{n} (E_{ij}^+)^+ L_i + \sum_{i=1}^{n} \bar{E}_{ij}^+ L_i \sigma_j^+
\]

\[
+ \sum_{i=1}^{n} \bar{E}_{ij}^+ L_i \xi_j^+ + \sum_{i=1}^{n} \sum_{k=1}^{m} \bar{T}_{ijk}^+ (L_k r + |f_k(0)|)(L_j r + |f_j(0)|),
\]

\[
H_i(0) = \alpha_i^{-} - \alpha_i^{+} \exp (\alpha_i^{-} - \beta) \left( \alpha_i^{+} \eta_i^{+} + \sum_{j=1}^{m} D_{ij}^+ L_j + \sum_{j=1}^{m} (D_{ij}^+)^+ L_j + \sum_{j=1}^{m} \bar{D}_{ij}^+ L_j \sigma_j^+ \right.
\]

\[
\left. + \sum_{j=1}^{m} \bar{D}_{ij}^+ L_j \xi_j^+ + \sum_{j=1}^{m} \sum_{k=1}^{n} T_{ijk}^+ (L_k r + |f_k(0)|)(L_j r + |f_j(0)|) \right),
\]

\[
\bar{H}_j(0) = c_j^{-} - c_j^{+} \exp (c_j^{-} - \beta) \left( c_j^{+} \sigma_j^+ + \sum_{i=1}^{n} E_{ij}^+ L_i + \sum_{i=1}^{n} (E_{ij}^+)^+ L_i + \sum_{i=1}^{n} \bar{E}_{ij}^+ L_i \sigma_j^+ \right.
\]

\[
\left. + \sum_{i=1}^{n} \bar{E}_{ij}^+ L_i \xi_j^+ + \sum_{i=1}^{n} \sum_{k=1}^{m} \bar{T}_{ijk}^+ (L_k r + |f_k(0)|)(L_j r + |f_j(0)|) \right),
\]

Since the functions \( G_i(\cdot), \bar{G}_j(\cdot), H_i(\cdot) \) and \( \bar{H}_j(\cdot) \) are continuous on \([0, +\infty)\) and \( G_i(w), \bar{G}_j(w), H_i(w), \bar{H}_j(w) \to -\infty \) when \( w \to +\infty \), it exist \( \eta_i, \bar{\eta}_j, \sigma_i, \bar{\sigma}_j \) >
0 such as
\[ H_i(\eta_i) = H_j(\eta_j) = G_i(e_i) = \overline{G}_j(\epsilon_j) = 0 \]
and
\[ G_i(w) > 0 \text{ for } w \in (0, \eta_i), \quad \overline{G}_j(w) > 0 \text{ for } w \in (0, \eta_j), \]
\[ H_i(w) > 0 \text{ for } w \in (0, \epsilon_i), \quad \overline{H}_j(w) > 0 \text{ for } w \in (0, \epsilon_j). \]

Let \( a = \min_{1 \leq i \leq n} \{ \eta_i, \eta_j, \epsilon_i, \epsilon_j \} \), we obtain
\[ H_i(a) \geq 0, \quad \overline{H}_j(a) \geq 0, \quad G_i(a) \geq 0, \quad \text{and} \quad \overline{G}_j(a) \geq 0, \quad i = 1, \ldots, n, j = 1, \ldots, m. \]

So, we can choose the positive constant \( 0 < \gamma < \min_{1 \leq i \leq n, 1 \leq j \leq m} \{ a_i, c_j \} \),
such that \( H_i(\gamma) > 0, \quad \overline{H}_j(\gamma) > 0, \quad G_i(\gamma) > 0 \quad \text{and} \quad \overline{G}_j(\gamma) > 0, \quad i = 1, \ldots, n, j = 1, \ldots, m. \) which imply that, for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \)
\[
\frac{1}{\alpha_i^+ - \gamma} \sum_{j=1}^{m} \left( \exp(\gamma \sup_{s \in T} \nu(s)) \left( \alpha_i^+ \eta_i^+ + \sum_{j=1}^{m} \left( D_{ij}^+ + (D_{ij}^-)^+ + \overline{D}_{ij} \sigma_{ij}^+ + \overline{D}_{ij} \xi_{ij}^+ \right) L_j \right) \right) < 1,
\]
\[
\frac{1}{c_j^+ - \gamma} \sum_{i=1}^{n} \left( \exp(\gamma \sup_{s \in T} \nu(s)) \left( c_j^+ \xi_j^+ + \sum_{i=1}^{n} \left( E_{ij}^+ + (E_{ij}^-)^+ + \overline{E}_{ij} \sigma_{ij}^+ + \overline{E}_{ij} \xi_{ij}^+ \right) L_i \right) \right) < 1,
\]
\[
\left( 1 + \frac{\alpha_i^+ \exp(\gamma \sup_{s \in T} \nu(s))}{\alpha_i^- - \gamma} \right) \sum_{j=1}^{m} \left( \exp(\gamma \sup_{s \in T} \nu(s)) \left( \alpha_i^+ \eta_i^+ + \sum_{j=1}^{m} \left( D_{ij}^+ + (D_{ij}^-)^+ \right) \right) \right) < 1,
\]
\[
\left( 1 + \frac{c_j^+ \exp(\gamma \sup_{s \in T} \nu(s))}{c_j^- - \gamma} \right) \sum_{i=1}^{n} \left( \exp(\gamma \sup_{s \in T} \nu(s)) \left( c_j^+ \xi_j^+ + \sum_{i=1}^{n} \left( E_{ij}^+ + (E_{ij}^-)^+ \right) \right) \right) < 1,
\]

Let
\[
K = \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ \frac{\alpha_i^+}{K^*}, \frac{c_j^-}{K^*} \right\}, \quad (6)
\]
where

\[
K^* = \alpha^+_i \eta^+_i + \sum_{j=1}^m \left( D_{ij}^+ + (D_{ij}^T)^+ + \mathcal{T}_{ij}^+ r_{ij}^+ + \tilde{D}_{ij}^+ e_{ij}^+ \right) L_j
\]

\[+ \sum_{j=1}^m \sum_{k=1}^m T_{ijk}^+ (L_k r + |f_k(0)|) (L_j r + |f_j(0)|),
\]

and

\[
P^* = c_i^+ \varsigma_i^+ + \sum_{i=1}^n \left( E_{ij}^+ + (E_{ij}^T)^+ + \mathcal{E}_{ij}^+ \sigma_{ij}^+ + \tilde{E}_{ij}^+ e_{ij}^+ \right) L_i
\]

\[+ \sum_{i=1}^n \sum_{k=1}^n \mathcal{T}_{ijk}^+ (L_k r + |f_k(0)|) (L_i r + |f_i(0)|).
\]

By hypothesis \((H_3)\), we have \(K > 1\), therefore,

\[\|Z(t) - Z^*(t)\| \leq K \hat{e}_{\Theta \nu \gamma}(t, t_0) \|\psi\|_0, \quad \forall t \in [t_0, 0], \quad (7)\]

where \(\Theta \nu \gamma \in \mathcal{R}_{ij}^+\). We claim that

\[\|Z(t) - Z^*(t)\| \leq K \hat{e}_{\Theta \nu \gamma}(t, t_0) \|\psi\|_0, \quad \forall t \in [t_0, +\infty]. \quad (8)\]

To prove (8), we show that for any \(\omega > 1\), the following inequality holds:

\[\|Z(t) - Z^*(t)\| \leq \omega K \hat{e}_{\Theta \nu \gamma}(t, t_0) \|\psi\|_0, \quad \forall t \in [t_0, +\infty]. \quad (9)\]

If (9) is not true, then there must be some \(t_1 \in (0, +\infty)\), \(d \geq 1\) such that

\[\|Z(t_1) - Z^*(t_1)\| = d \omega K \hat{e}_{\Theta \nu \gamma}(t_1, t_0) \|\psi\|_0, \quad (10)\]

and

\[\|Z(t) - Z^*(t)\| \leq d \omega K \hat{e}_{\Theta \nu \gamma}(t, t_0) \|\psi\|_0, \quad t \in [t_0, t_1]. \quad (11)\]
By (5), (10), (11) and \((H_1) - (H_3)\), we have for \(i = 1, \ldots, n\)

\[
|u_i(t_1)| \leq \hat{e}_{-\alpha_i}(t_1, t_0) \|\psi\|_0 + d\varpi K \hat{e}_{\Theta - \gamma}(t_1, t_0) \|\psi\|_0 \int_{t_0}^{t_1} \hat{e}_{-\alpha_i}(t_1, \sigma(s)) \hat{e}_{\gamma}(t_1, \sigma(s)) \\
\times \left( \alpha_i^+ \int_{s - \eta_i(s)}^s \hat{e}_{\gamma}(\sigma(u), u) \Delta u + \sum_{j=1}^n D_{ij}^+ L_j \hat{e}_{\gamma}(\sigma(s), s) \\
+ \sum_{j=1}^n \sum_{k=1}^n T_{ijk}^+ (L_k r + |f_k(0)| (L_j r + |f_j(0)|) \hat{e}_{\gamma}(\sigma(s), s - \chi_j(s)) \hat{e}_{\gamma}(\sigma(s), s - \chi_k(s)) \\
+ (D_{ij}^+)^T L_j \hat{e}_{\gamma}(\sigma(s), s - \tau_{ij}(s)) + \sum_{j=1}^n D_{ij}^+ L_j \int_{s - \sigma_{ij}(s)}^s \hat{e}_{\gamma}(\sigma(u), u) \Delta u \\
+ \sum_{j=1}^n D_{ij}^+ L_j \int_{s - \xi_{ij}(s)}^s \hat{e}_{\gamma}(\sigma(u), u) \Delta u \right) \|\psi\|_0 \int_{t_0}^{t_1} \hat{e}_{-\alpha_i}(t_1, \sigma(s)) \hat{e}_{\gamma}(t_1, \sigma(s)) \\
\times \left( \alpha_i^+ \hat{e}_{\gamma}(\sigma(s), s - \eta_i(s)) + \sum_{j=1}^n D_{ij}^+ L_j \hat{e}_{\gamma}(\sigma(s), s) \\
+ \sum_{j=1}^n \sum_{k=1}^n T_{ijk}^+ (L_k r + |f_k(0)| (L_j r + |f_j(0)|) \hat{e}_{\gamma}(\sigma(s), s - \chi_j(s)) \hat{e}_{\gamma}(\sigma(s), s - \chi_k(s)) \\
+ \sum_{j=1}^n (D_{ij}^+)^T L_j \hat{e}_{\gamma}(\sigma(s), s - \tau_{ij}(s)) + \sum_{j=1}^n D_{ij}^+ L_j \int_{s - \sigma_{ij}(s)}^s \hat{e}_{\gamma}(\sigma(u), u) \Delta u \\
+ \sum_{j=1}^n D_{ij}^+ L_j \int_{s - \xi_{ij}(s)}^s \hat{e}_{\gamma}(\sigma(u), u) \Delta u \right) \Delta s \\
\leq \hat{e}_{-\alpha_i}(t_1, t_0) \|\psi\|_0 + d\varpi K \hat{e}_{\Theta - \gamma}(t_1, t_0) \|\psi\|_0 \int_{t_0}^{t_1} \hat{e}_{-\alpha_i}(t_1, \sigma(s)) \hat{e}_{\gamma}(t_1, \sigma(s)) \\
\times \left( \alpha_i^+ \eta_i^+ \exp \left[ \gamma \left( \eta_i^+ \sup_{s \in T} \nu(s) \right) \right] + \sum_{j=1}^n D_{ij}^+ L_j \exp \left[ \gamma \sup_{s \in T} \nu(s) \right] \\
+ (D_{ij}^+)^T L_j \exp \left[ \gamma \left( \tau_{ij}^+ \sup_{s \in T} \nu(s) \right) \right] + \sum_{j=1}^n D_{ij}^+ L_j \exp \left[ \gamma \left( \sigma_{ij}^+ \sup_{s \in T} \nu(s) \right) \right] \\
+ \sum_{j=1}^n D_{ij}^+ L_j \exp \left[ \gamma \left( \xi_{ij}^+ \sup_{s \in T} \nu(s) \right) \right] + \sum_{j=1}^n \sum_{k=1}^n T_{ijk}^+ (L_k r + |f_k(0)| (L_j r + |f_j(0)|) \\
\times \exp \left[ \gamma \left( \chi_{ij}^+ \sup_{s \in T} \nu(s) \right) \right] \exp \left[ \gamma \left( \chi_{ij}^+ \sup_{s \in T} \nu(s) \right) \right] \right) \Delta s
\[ \leq d\pi K\hat{e}_{\hat{\Theta}_i}(t_1, t_0)\| \psi \|_0 \left\{ \frac{1}{K} \hat{e}_{-\alpha_i, \hat{\Theta}_i}(t_1, t_0) + \left[ \exp \left( \gamma \sup_{s \in T} \nu(s) \right) \exp(\gamma \tau_{i,j}^+) \right] \right\} \]
\[ \times \left( \alpha_i^+ \eta_i^+ \exp(\gamma \xi_i^+) + \sum_{j=1}^n D_{ij}^+ L_j + \sum_{j=1}^n (D_{ij}^+)^+ L_j \exp(\gamma \tau_{i,j}^+) + \sum_{j=1}^n \bar{D}_{ij} \sigma_{ij}^+ L_j \exp(\gamma \sigma_{ij}^+) \right) \]
\[ + \sum_{j=1}^n \tilde{D}_{ij}^+ L_j \xi_{ij}^+ \exp(\gamma \xi_{ij}^+) + \sum_{j=1}^n \sum_{k=1}^n T_{ij}^+(L_k r + |f_k(0)|)(L_j r + |f_j(0)|) \]
\[ \times \exp(\gamma \chi_i^+) \exp(\gamma \chi_j^+) \left[ \frac{1 - \hat{e}_{-\alpha_i, \hat{\Theta}_i}(t_1, t_0)}{\alpha_i^+ - \gamma} \right] \]
\[ \leq d\pi K\hat{e}_{\hat{\Theta}_i}(t_1, t_0)\| \psi \|_0 \left\{ \frac{1}{K} - \frac{1}{\alpha_i^+} - \gamma \left[ \exp \left( \gamma \sup_{s \in T} \nu(s) \right) \left( \alpha_i^+ \eta_i^+ \exp(\gamma \xi_i^+) \right) \right] \right\} \]
\[ \times \left( \alpha_i^+ \eta_i^+ \exp(\gamma \xi_i^+) + \sum_{j=1}^n D_{ij}^+ L_j + \sum_{j=1}^n (D_{ij}^+)^+ L_j \exp(\gamma \tau_{i,j}^+) + \sum_{j=1}^n \bar{D}_{ij} \sigma_{ij}^+ L_j \exp(\gamma \sigma_{ij}^+) \right) \]
\[ + \sum_{j=1}^n \tilde{D}_{ij}^+ L_j \xi_{ij}^+ \exp(\gamma \xi_{ij}^+) + \sum_{j=1}^n \sum_{k=1}^n T_{ij}^+(L_k r + |f_k(0)|)(L_j r + |f_j(0)|) \]
\[ \times \exp(\gamma \chi_i^+) \exp(\gamma \chi_j^+) \right\} \}
\[ \leq d\pi K\hat{e}_{\hat{\Theta}_i}(t_1, t_0)\| \psi \|_0. \]
\begin{align*}
\leq & \, d\varpi K \hat{e}_{\oplus,\gamma}(t_1, t_0) \|\psi\|_0 \left\{ \frac{1}{K} \hat{e}_{-c_j \oplus \nu}(t_1, t_0) + \left[ \exp \left( \gamma \sup_{s \in T} \nu(s) \right) \right] \right. \\
& \times \left( c_j^+ \xi_j^+ \exp(\gamma \xi_j^+) + \sum_{i=1}^{n} E_i^{(2)} L_i + \sum_{i=1}^{n} (E_i^{(1)})^+ L_i \exp(\gamma \tau_i^{(1)}) + \sum_{i=1}^{n} \bar{E}_i^{(1)} L_i \exp(\gamma \tau_i^{(1)}) \right) \\
& + \sum_{i=1}^{n} \tilde{E}_i^{(1)} L_i \xi_j^+ \exp(\gamma \xi_j^+) + \sum_{i=1}^{n} \sum_{k=1}^{n} T_{ijk} (L_k r + \|f_k(0)\|) (L_i r + \|f_i(0)\|) \\
& \times \exp(\gamma \chi_k^{+}) \exp(\gamma \chi_k^{+})) \right\} \frac{1}{c_j^+} \exp(\gamma \xi_j^+) \\
& + \sum_{i=1}^{n} E_i^{(1)} L_i + \sum_{i=1}^{n} (E_i^{(2)})^+ L_i \exp(\gamma \tau_i^{(2)}) + \sum_{i=1}^{n} \bar{E}_i^{(2)} L_i \exp(\gamma \tau_i^{(2)}) \right) \\
& + \sum_{i=1}^{n} \tilde{E}_i^{(2)} L_i \xi_j^+ \exp(\gamma \xi_j^+) + \sum_{i=1}^{n} \sum_{k=1}^{n} T_{ijk} (L_k r + \|f_k(0)\|) (L_i r + \|f_i(0)\|) \\
& \times \exp(\gamma \chi_k^{+}) \exp(\gamma \chi_k^{+})) \right\} \frac{1}{c_j^+} \exp(\gamma \xi_j^+) \\
& + \sum_{i=1}^{n} E_i^{(2)} L_i + \sum_{i=1}^{n} (E_i^{(1)})^+ L_i \exp(\gamma \tau_i^{(1)}) + \sum_{i=1}^{n} \bar{E}_i^{(1)} L_i \exp(\gamma \tau_i^{(1)}) \right) \\
& + \sum_{i=1}^{n} \tilde{E}_i^{(1)} L_i \xi_j^+ \exp(\gamma \xi_j^+) + \sum_{i=1}^{n} \sum_{k=1}^{n} T_{ijk} (L_k r + \|f_k(0)\|) (L_i r + \|f_i(0)\|) \\
& \times \exp(\gamma \chi_k^{+}) \exp(\gamma \chi_k^{+})) \right\} \frac{1}{c_j^+} \exp(\gamma \xi_j^+) \\
& \leq d\varpi K \hat{e}_{\oplus,\gamma}(t_1, t_0) \|\psi\|_0.
\end{align*}

We can easily obtain some upper bound of the derivative \(|u_1^A(t_1)|\) and \(|u_2^A(t_1)|\) as follow:

\begin{equation}
|u_1^A(t_1)| \leq d\varpi K \hat{e}_{\oplus,\gamma}(t_1, t_0) \|\psi\|_0,
\end{equation}

and

\begin{equation}
|\sigma_1^A(t_1)| \leq d\varpi K \hat{e}_{\oplus,\gamma}(t_1, t_0) \|\psi\|_0.
\end{equation}

From (12) - (14), we obtain

\begin{equation}
\|Z(t_1) - Z^*(t_1)\| < d\varpi K \hat{e}_{\oplus,\gamma}(t_1, t_0) \|\psi\|_0.
\end{equation}

which contradicts (10), therefore (9) holds. Letting \(\varpi \to 1\), then (8) holds. Which implies that only Stepanov-like weighted pseudo-almost automorphic on time-space scales solution of system (2) is globally exponentially stable.
5.2 Convergence

**Definition 10** ([3]) For each \( t \in \mathbb{T} \), let \( N \) be a neighborhood of \( t \). Then, we define the generalized derivative (or Dini derivative on time-space scales) \( D^+ V^\Delta(t) \), to mean that, given \( \epsilon > 0 \), there exists a right neighborhood \( N_\epsilon \subset N \) of \( t \) such

\[
D^+ V^\Delta(t) = D^+ V^\Delta(t, x(t)) = \frac{V(\sigma(t)), x(\sigma(t)) - V(t, x(t))}{\nu(t)}.
\]

**Theorem 4** Suppose that assumptions (\( H_1 \))-\( (H_4) \) hold. Let \( h^* (\cdot) = (x^*_1 (\cdot), \cdots, x^*_n (\cdot), y^*_1 (\cdot), \cdots, y^*_m (\cdot))^T \) be a Stepanov-like weighted pseudo-almost automorphic on time-space scales solution of system (2). If

\[
\alpha_i^- - \sum_{j=1}^m L_j \left( D^+_i (D^+_j)^+ + (\bar{D}^+_i)^+ \sigma^+_i + (\bar{D}^+_i)^+ \xi^+_i + \sum_{k=1}^m T^+_i (L_k r + |f_k(0)|) \right)
+ (L_j r + |f_j(0)|) \sum_{k=1}^m T^+_i L_k > 0,
\]

and

\[
c_j^- - \sum_{i=1}^n L_i \left( E^+_i (E^+_i)^+ + (\bar{E}^+_i)^+ \sigma^+_i + (\bar{E}^+_i)^+ \xi^+_i + \sum_{k=1}^n T^+_i (L_k r + |f_k(0)|) \right)
+ (L_i r + |f_i(0)|) \sum_{k=1}^n T^+_i L_k > 0,
\]

then all solutions \( \psi = (\varphi_1, \ldots, \varphi_n, \phi_1, \ldots, \phi_n) \) of (2) satisfying

\[
x^*_i (0) = \varphi_i (0), \quad y^*_j (0) = \phi_j (0), \quad 1 \leq i \leq n, \quad 1 \leq j \leq m
\]

converge to its unique Stepanov-like weighted pseudo-almost automorphic on time-space scales solution \( h^* \).

Proof. Let \( h^* (\cdot) = (x^*_1 (\cdot), \cdots, x^*_n (\cdot), y^*_1 (\cdot), \cdots, y^*_m (\cdot)) \) be a solution of (2) and \( \psi (\cdot) = (\varphi_1 (\cdot), \ldots, \varphi_n (\cdot), \phi_1 (\cdot), \ldots, \phi_m (\cdot)) \) be a Stepanov-like weighted pseudo almost automorphic on time-space scales solution of (2). First, one verifies
without difficulty that
\[
\left( x_i^+ (t) - \alpha_i (t) \int_{t-\eta_i (t)}^t x_i (u) \Delta u \right)^\Delta - \left( \varphi_i^+ (t) - \alpha_i (t) \int_{t-\eta_i (t)}^t \varphi_i (u) \Delta u \right)^\Delta \\
= -\alpha_i (t) (x_i^+ (t - \eta_i (t)) - \varphi_i (t - \eta_i (t))) + \sum_{j=1}^n D_{ij} (t) \left[ f_j (x_j^+ (t)) - f_j (\varphi_j (t)) \right]
\]
\[+ \sum_{j=1}^n D_{ij}^+ (t) \left[ f_j (x_j^+ (t - \tau_j (t))) - f_j (\varphi_j (t - \tau_j (t))) \right]
\]
\[+ \sum_{j=1}^n \tilde{D}_{ij} (t) \int_{t-\sigma_{ij} (t)}^t \left[ f_j (x_j^+ (t - u)) - f_j (\varphi_j (t - u)) \right] \Delta u
\]
\[+ \sum_{j=1}^n \tilde{D}_{ij}^+ (t) \int_{t-\xi_{ij} (t)}^t \left( f_j ((x_j^+)^\Delta (u)) - f_j (\varphi_j^\Delta (u)) \right) \Delta u
\]
\[+ \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n T_{ijk} (t) [f_k (x_k (t - \chi_k (t))) f_j (x_j (t - \chi_j (t))) - f_k (\varphi_k (t - \chi_k (t))) f_j (\varphi_j (t - \chi_j (t)))],
\]
and
\[
\left( y_j^+ (t) - c_j (t) \int_{t-\varsigma_j (t)}^t y_j (u) \Delta u \right)^\Delta - \left( \phi_j^+ (t) - c_j (t) \int_{t-\varsigma_j (t)}^t \phi_j (u) \Delta u \right)^\Delta
\]
\[= -c_j (t) (y_j^+ (t - \varsigma_j (t)) - \phi_j (t - \varsigma_j (t))) + \sum_{j=1}^n E_{ij} (t) \left[ f_j (x_j^+ (t)) - f_j (\varphi_j (t)) \right]
\]
\[+ \sum_{j=1}^n E_{ij}^+ (t) \left[ f_j (x_j^+ (t - \tau_j (t))) - f_j (\varphi_j (t - \tau_j (t))) \right]
\]
\[+ \sum_{j=1}^n \tilde{E}_{ij} (t) \int_{t-\sigma_{ij} (t)}^t \left[ f_j (x_j^+ (t - u)) - f_j (\varphi_j (t - u)) \right] \Delta u
\]
\[+ \sum_{j=1}^n \tilde{E}_{ij}^+ (t) \int_{t-\xi_{ij} (t)}^t \left( f_j ((x_j^+)^\Delta (u)) - f_j (\varphi_j^\Delta (u)) \right) \Delta u
\]
\[+ \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n T_{ijk} (t) [f_k (x_k (t - \chi_k (t))) f_j (x_j (t - \chi_j (t))) - f_k (\varphi_k (t - \chi_k (t))) f_j (\varphi_j (t - \chi_j (t)))],
\]
Now, consider the following (ad-hoc) Lyapunov-Krasovskii functional

\[V : \mathbb{R} \rightarrow \mathbb{S} WPAP (T, \mathbb{R}^n)
\]
\[t \mapsto V_1 (t) + V_2 (t) + V_3 (t) + V_4 (t) + V_5 (t),
\]
where

\[ V_1(t) = \sum_{i=1}^{n} \left| \left( x_i^*(t) - \alpha_i(t) \int_{t- \eta_i (t)}^{t} x_i(u) \Delta u \right) - \left( \varphi_i^*(t) - \alpha_i(t) \int_{t- \eta_i (t)}^{t} \varphi_i(u) \Delta u \right) \right|, \]

\[ V_2(t) = \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{t-\tau_j (t)}^{t} L_j \left( D_j^+ + (D_j^+) \right) |x_i^*(s) - \varphi_i(s)| \Delta s, \]

\[ V_3(t) = \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{t-\sigma_j (t)}^{t} L_j \tilde{D}_{ij}^+ |x_i^*(u) - \varphi_i(u)| \Delta u \Delta s, \]

\[ V_4(t) = \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{t-\xi_j (t)}^{t} \int_{s \Delta u \Delta s}, \]

and

\[ V_5(t) = \sum_{j=1}^{n} \left| \left( y_j^*(t) - c_j(t) \int_{t- \gamma_j (t)}^{t} y_j(u) \Delta u \right) - \left( \phi_j^*(t) - c_j(t) \int_{t- \gamma_j (t)}^{t} \phi_j(u) \Delta u \right) \right|. \]

Let us calculate the upper right Dini derivative on time-space scales \( D^+ V^\Delta(t) \) of \( V \) along the trajectory of the solution of the equation above. Then one has

\[ D^+ V^\Delta(t) \leq - \sum_{i=1}^{m} \alpha_i^+ |x_i^*(t) - \varphi_i(t)| + \sum_{i=1}^{m} \sum_{j=1}^{m} D_j^+ L_j |x_j^*(t) - \varphi_j(t)| \]

\[ + \sum_{i=1}^{m} \sum_{j=1}^{m} \left( D_j^+ \right) |x_j^*(t - \tau_j(t)) - \varphi_j(t - \tau_j(t))| \]

\[ + \sum_{i=1}^{m} \sum_{j=1}^{m} \tilde{D}_{ij}^+ \int_{t-\sigma_j (t)}^{t} L_j |x_j^*(t-u) - \varphi_j(t-u)| \Delta u \]

\[ + \sum_{i=1}^{m} \sum_{j=1}^{m} \tilde{D}_{ij}^+ \int_{t-\xi_j (t)}^{t} L_j |(x_j^*)^\Delta(u) - \varphi_j^\Delta(u)| \Delta u \]

\[ + \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} T_{ijk}^+ (L_k r + |f_k(0)|) L_j |x_j(t-\chi_j(t)) - \varphi_j(t-\chi_j(t))| \]

\[ + \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} T_{ijk}^+ (L_j r + |f_j(0)|) L_k |x_k(t-\chi_k(t)) - \varphi_k(t-\chi_k(t))| \]
Obviously,

\[ D^+ V_2^\Delta(t) \leq \sum_{j=1}^{m} \sum_{i=1}^{m} L_j(D_{ij}^+ + (D_{ij}^+)^+) \left[ |x_i^*(t) - \varphi_i(t)| - |x_i^*(t - \tau_j(t)) - \varphi_i(t - \tau_j(t))| \right] \]

\[ \leq \sum_{j=1}^{m} \sum_{i=1}^{m} L_j(D_{ij}^+ + (D_{ij}^+)^+) |x_i^*(t) - \varphi_i(t)| \]

\[ - \sum_{j=1}^{m} \sum_{i=1}^{m} L_j(D_{ij}^+ + (D_{ij}^+)^+) |x_i^*(t - \tau_j(t)) - \varphi_i(t - \tau_j(t))| \]

and

\[ D^+ V_3^\Delta(t) \leq \sum_{j=1}^{m} \sum_{i=1}^{m} L_j D_{ij}^{\Delta} \int_{t-\sigma_j(t)}^{t} \left[ |x_i^*(t) - \varphi_i(t)| - |x_i^*(t - s) - \varphi_i(t - s)| \right] \Delta s \]

\[ \leq \sum_{j=1}^{m} \sum_{i=1}^{m} L_j D_{ij}^{\Delta} \int_{t-\sigma_j(t)}^{t} |x_i^*(t) - \varphi_i(t)| \Delta s \]

\[ - \sum_{j=1}^{m} \sum_{i=1}^{m} L_j D_{ij}^{\Delta} \int_{t-\sigma_j(t)}^{t} |x_i^*(t - s) - \varphi_i(t - s)| \Delta s. \]

Reasoning in a similar way, we can obtain the following estimation

\[ D^+ V_4^\Delta(t) = \sum_{j=1}^{m} \sum_{i=1}^{m} \int_{t-\xi_i(t)}^{t} \int_{s}^{0} L_j \tilde{D}_{ij}^+ |(x_i^*)_\Delta(u) - \varphi_i^\Delta(u)| \Delta u \Delta s \]

\[ \leq - \sum_{j=1}^{m} \sum_{i=1}^{m} L_j \tilde{D}_{ij}^+ \int_{t-\xi_i(t)}^{t} |(x_i^*)_\Delta(s) - \varphi_i^\Delta(s)| \Delta s, \]

and,

\[ D^+ V_5^\Delta(t) \leq - \sum_{j=1}^{n} c_j^\Delta |y_j^*(t) - \phi_j(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} (E_{ij}^+) L_j |y_j^*(t) - \phi_j(t)| \]

\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} (E_{ij}^+) L_j |y_j^*(t - \tau_j(t)) - \phi_j(t - \tau_j(t))| \]

\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{E}_{ij}^+ \int_{t-\sigma_j(t)}^{t} L_j |(y_j^*)_\Delta(u) - \phi_j^\Delta(u)| \Delta u \]

\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{E}_{ij}^+ \int_{t-\xi_i(t)}^{t} L_j |(y_j^*)_\Delta(u) - \phi_j^\Delta(u)| \Delta u + |y_j(t) - \phi_j(t)|. \]
By using the inequality of the Dini derivative on time-space scales
\[ D^+ (F^A_1 + F^A_2) \leq D^+ (F^A_1) + D^+ (F^A_2), \]
we get
\[
D^+ (V^A(t)) \leq D^+ V^A_1(t) + D^+ V^A_2(t) + D^+ V^A_3(t) + D^+ V^A_4(t)
\]
\[
\leq - \sum_{i=1}^m \sum_{j=1}^n \min \left\{ \alpha_i - D^+_{ij} L_j - (D^+_{ij})^+ L_j - D^+_{ij} \sigma_{ij} L_j, \right. \\
- \tilde{D}^+_{ij} \xi_{ij} L_j + L_j \sum_{k=1}^n \tilde{T}^+_{ijk} (L_k r + |f_k(0)|) + (L_j r + |f_j(0)|) \sum_{k=1}^n \tilde{T}^+_{ijk} L_k, \\
\left. e_j - E^+_{ij} L_j - (E^+_{ij})^+ L_j - \tilde{E}^+_{ij} \sigma_{ij} L_j + E^+_{ij} \xi_{ij} L_j + L_j \sum_{k=1}^n \tilde{T}^+_{ijk} (L_k r + |f_k(0)|) \right\} \| h^*(t) - \psi (t) \|
\]
\[
= - \sum_{i=1}^n \sum_{j=1}^m \min \{ \beta_i, \beta_j \} \| h^*(t) - \psi (t) \| < 0.
\]
By integrating the above inequality from \( t_0 \) to \( t \), we get
\[
V(t) + \sum_{i=1}^n \sum_{j=1}^m \min \{ \beta_i, \beta_j \} \int_{t_0}^t \| y^*(t) - \psi (t) \| \Delta s < V(t_0) < +\infty.
\]
Now, we remark that \( V(t) > 0 \). It follows that
\[
\lim_{t \to +\infty} \sup_{t_0} \int_{t_0}^t \min \{ \beta_i, \beta_j \} \| h^*(s) - \psi (s) \| \Delta s < V(t_0) < +\infty.
\]
Note that \( h^*(\cdot) \) is bounded on \( T^+ \). Therefore
\[
\lim_{t \to +\infty} \| h^*(t) - \psi (t) \| = 0.
\]
The proof of this theorem is now completed.

**Remark 5** Theorem 1, Theorem 2, Theorem 3 and Theorem 4 are new even for the both cases of differential equations (\( T = \mathbb{R} \)) and difference equations (\( T = \mathbb{Z} \)).
6 Numerical example

In system (2), let $n = 3$, $m = 2$, and take coefficients as follows:

$$\begin{bmatrix}
D_{11}(t) & D_{12}(t) \\
D_{21}(t) & D_{22}(t) \\
D_{31}(t) & D_{32}(t)
\end{bmatrix} = \begin{bmatrix}
D_{11}^*(t) & D_{12}^*(t) \\
D_{21}^*(t) & D_{22}^*(t) \\
D_{31}^*(t) & D_{32}^*(t)
\end{bmatrix} = \begin{bmatrix}
\hat{D}_{11}(t) & \hat{D}_{12}(t) \\
\hat{D}_{21}(t) & \hat{D}_{22}(t) \\
\hat{D}_{31}(t) & \hat{D}_{32}(t)
\end{bmatrix};$$

$$\begin{bmatrix}
E_{11}(t) & E_{12}(t) \\
E_{21}(t) & E_{22}(t) \\
E_{31}(t) & E_{32}(t)
\end{bmatrix} = \begin{bmatrix}
E_{11}^*(t) & E_{12}^*(t) \\
E_{21}^*(t) & E_{22}^*(t) \\
E_{31}^*(t) & E_{32}^*(t)
\end{bmatrix} = \begin{bmatrix}
\hat{E}_{11}(t) & \hat{E}_{12}(t) \\
\hat{E}_{21}(t) & \hat{E}_{22}(t) \\
\hat{E}_{31}(t) & \hat{E}_{32}(t)
\end{bmatrix};$$

$$\alpha_1(t) = \alpha_2(t) = 0.73 + 0.02 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)};$$

$$f_1(x) = f_2(x) = 0.1 \arctan x; \quad \nu(t) = \hat{c}(t, \sigma(t)) \text{ for all } t \in [0, \infty)^\tau;$$

$$\nu(t) = 1 \text{ for all } t \in (-\infty, 0)^\tau;$$

$$c_1(t) = 0.54 - 0.02 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)};$$

$$c_2(t) = 0.54 + 0.02 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)};$$

$$I_1(t) = 0.01 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)} + 0.02 e^{-t^2 \cos^2(t)};$$

$$I_2(t) = 0.02 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)} + 0.02 e^{-t^2 \cos^2(t)};$$

$$J_1(t) = 0.02 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + 0.02 e^{-t^2 \sin^2(t)};$$

$$J_2(t) = 0.1 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)};$$

$$\eta_1(t) = 0.04 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)} + 0.03 e^{-t^2 \cos^2(t)};$$

$$\eta_2(t) = 0.01 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + 0.01 e^{-t^2 \cos^2(t)};$$

$$\zeta_1(t) = 0.01 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + 0.01 e^{-t^2 \sin^2(t)};$$

$$\zeta_2(t) = 0.01 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + 0.01 e^{-t^2 \sin^2(t)};$$

$$\tau_1(t) = 0.02 \sin(\sqrt{2}t) + e^{-t^2 \sin^2(t)};$$

$$\tau_2(t) = 0.02 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)} + 0.01 e^{-t^2 \sin^2(t)};$$

$$\tau_2(t) = 0.02 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)} + 0.01 e^{-t^2 \cos^2(t)};$$
By a simple calculation, we have

\[
\begin{align*}
\sigma_{11}(t) &= 0.02 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + 0.02e^{-t^2 \sin^4(t)}; \\
\sigma_{12}(t) &= 0.01 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + 0.01e^{-t^2 \sin^4(t)}; \\
\sigma_{21}(t) &= 0.02 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)} + 0.01e^{-t^4 \cos^2(t)}; \\
\sigma_{12}(t) &= 0.02 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + e^{-t^2 \sin^4(t)}; \\
\xi_{11}(t) &= 0.02 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + 0.01e^{-t^4 \cos^2(t)}; \\
\xi_{12}(t) &= 0.03 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)} + 0.02e^{-t^2 \sin^4(t)}; \\
\xi_{21}(t) &= 0.01 \sin \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} + 0.01e^{-t^4 \cos^2(t)}; \\
\xi_{22}(t) &= 0.02 \sin \frac{1}{2 + \sin(t) + \sin(\sqrt{2}t)} + 0.02e^{-t^2 \sin^4(t)}.
\end{align*}
\]

\[
\begin{pmatrix}
D_{11}^+ & D_{12}^+ \\
D_{21}^+ & D_{22}^+
\end{pmatrix}
= 
\begin{pmatrix}
(D_{11}^+)^{+} & (D_{12}^+)^{+} \\
(D_{21}^+)^{+} & (D_{22}^+)^{+}
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{D}_{11}^+ & \tilde{D}_{12}^+ \\
\tilde{D}_{21}^+ & \tilde{D}_{22}^+
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{D}_{11}^+ & \tilde{D}_{12}^+ \\
\tilde{D}_{21}^+ & \tilde{D}_{22}^+
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{37} & \frac{1}{37} \\
\frac{1}{37} & \frac{1}{37}
\end{pmatrix};
\]

\[
\begin{pmatrix}
E_{11}^+ & E_{12}^+ \\
E_{21}^+ & E_{22}^+
\end{pmatrix}
= 
\begin{pmatrix}
(E_{11}^+)^{+} & (E_{12}^+)^{+} \\
(E_{21}^+)^{+} & (E_{22}^+)^{+}
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{E}_{11}^+ & \tilde{E}_{12}^+ \\
\tilde{E}_{21}^+ & \tilde{E}_{22}^+
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{E}_{11}^+ & \tilde{E}_{12}^+ \\
\tilde{E}_{21}^+ & \tilde{E}_{22}^+
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{13} & \frac{1}{13} \\
\frac{1}{13} & \frac{1}{13}
\end{pmatrix};
\]

\[
I_1^+ = 0.02; \quad I_2^+ = 0.04; \quad I_3^+ = 0.05; \quad J_1^+ = 0.01; \quad J_2^+ = -0.1; \\
\eta_1^+ = 0.07; \quad \eta_2^+ = 0.02; \quad \eta_3^+ = 0.02; \quad \nu_2^+ = 0.02;
\]

\[
\begin{pmatrix}
\sigma_{ij}^+ \mid 1 \leq i \leq 3, 1 \leq j \leq 2
\end{pmatrix}
= 
\begin{pmatrix}
1.04 & 1.02 \\
1.03 & 1.02 \\
1.03 & 1.02
\end{pmatrix}; \\
\begin{pmatrix}
\xi_{ij}^+ \mid 1 \leq i \leq 3, 1 \leq j \leq 2
\end{pmatrix}
= 
\begin{pmatrix}
1.03 & 1.05 \\
1.02 & 1.04 \\
1.02 & 1.04
\end{pmatrix};
\]

\[
\begin{pmatrix}
(T_{1jk})_{2 \times 2}
\end{pmatrix}
= 
\begin{pmatrix}
0.01 \sin \sqrt{2}t & 0.01 \cos \sqrt{2}t \\
0.01 \cos \sqrt{2}t & 0.01 \cos \sqrt{2}t
\end{pmatrix}; \\
\begin{pmatrix}
(T_{2jk})_{2 \times 2}
\end{pmatrix}
= 
\begin{pmatrix}
0.01 \sin \sqrt{2}t & 0.01 \cos \sqrt{2}t \\
0.01 \cos \sqrt{2}t & 0.01 \cos \sqrt{2}t
\end{pmatrix};
\]

\[
\begin{pmatrix}
(T_{3jk})_{2 \times 2}
\end{pmatrix}
= 
\begin{pmatrix}
0.01 \sin \sqrt{2}t & 0.01 \cos \sqrt{2}t \\
0.01 \sin \sqrt{2}t & 0.01 \sin \sqrt{2}t
\end{pmatrix}; \\
\begin{pmatrix}
(T_{1jk})_{2 \times 2}
\end{pmatrix}
= 
\begin{pmatrix}
0.01 \sin \sqrt{2}t & 0.01 \cos \sqrt{2}t \\
0.01 \cos \sqrt{2}t & 0.01 \cos \sqrt{2}t
\end{pmatrix};
\]

\[
\begin{pmatrix}
(T_{2jk})_{2 \times 2}
\end{pmatrix}
= 
\begin{pmatrix}
0.01 \sin \sqrt{2}t & 0.01 \cos \sqrt{2}t \\
0.01 \sin \sqrt{2}t & 0.01 \sin \sqrt{2}t
\end{pmatrix}; \\
\begin{pmatrix}
(T_{3jk})_{2 \times 2}
\end{pmatrix}
= 
\begin{pmatrix}
0.01 \sin \sqrt{2}t & 0.01 \cos \sqrt{2}t \\
0.01 \cos \sqrt{2}t & 0.01 \cos \sqrt{2}t
\end{pmatrix};
\]
We can take $L_1 = L_2 = 0.1$, $r = 0.43$ and we have

$$
M_3 = \alpha_1^+ \eta_1^+ r + \sum_{j=1}^{2} \left( D_{ij}^+ + (D_{ij}^+)^* + D_{ij}^+ \sigma_{ij}^+ + \tilde{D}_{ij}^+ \xi_{ij}^+ \right) (L_j r + |f_j(0)|)
\begin{align*}
&+ \sum_{j=1}^{2} \sum_{k=1}^{2} T_{ij}^+ (L_k r + |f_k(0)|)(L_j r + |f_j(0)|) + I_3^+, \\
&= 0.75 \times 0.07 \times 0.43 + \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 1.04 + \frac{1}{15} \times 1.03 \right) (0.1 \times 0.43 + 0.1) \\
&+ \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 1.02 + \frac{1}{15} \times 1.05 \right) (0.1 \times 0.43 + 0.1) \\
&+ 0.04 (0.1 \times 0.43 + 0.1)^2 + 0.02 \\
&= 0.119
\end{align*}
$$

$$
\overline{M}_1 = \alpha_1^+ \eta_1^+ + \sum_{j=1}^{2} \left( D_{ij}^+ + (D_{ij}^+)^* + D_{ij}^+ \sigma_{ij}^+ + \tilde{D}_{ij}^+ \xi_{ij}^+ \right) L_j
\begin{align*}
&+ \sum_{j=1}^{2} \sum_{k=1}^{2} (T_{ij}^+ + T_{ik}^+)(L_k r + |f_k(0)|) \\
&= 0.75 \times 0.07 + \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 1.03 + \frac{1}{15} \times 1.02 \right) (0.1 \times 0.43 + 0.1) \\
&+ \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 1.02 + \frac{1}{15} \times 1.04 \right) (0.1 \times 0.43 + 0.1) + 0.08 (0.1 \times 0.43 + 0.1) \\
&= 0.139
\end{align*}
$$

$$
N_1 = c_1^+ \xi_1^+ r + \sum_{i=1}^{3} \left( E_{i1}^+ + (E_{i1}^+)^* + E_{i1}^+ \sigma_{i1}^+ + \tilde{E}_{i1}^+ \xi_{i1}^+ \right) (L_i r + |f_i(0)|)
\begin{align*}
&+ \sum_{i=1}^{3} \sum_{k=1}^{3} T_{i1k}^+ (L_k r + |f_k(0)|)(L_i r + |f_i(0)|) + J_1^+
\end{align*}
\begin{align*}
&= 0.56 \times 1.02 \times 0.43 + \left( \frac{1}{20} + \frac{1}{20} + \frac{1}{20} \times 1.04 + \frac{1}{20} \times 1.03 \right) (0.1 \times 0.43 + 0.1) \\
&+ \left( \frac{1}{20} + \frac{1}{20} + \frac{1}{20} \times 1.03 + \frac{1}{20} \times 1.02 \right) (0.1 \times 0.43 + 0.1) + 0.04 (0.1 \times 0.43 + 0.1)^2 + 0.04 \\
&= 0.343
\end{align*}
$$

$$
\overline{N}_1 = c_1^+ \xi_1^+ + \sum_{i=1}^{3} \left( E_{i1}^+ + (E_{i1}^+)^* + E_{i1}^+ \sigma_{i1}^+ + \tilde{E}_{i1}^+ \xi_{i1}^+ \right) L_i
\begin{align*}
&+ \sum_{i=1}^{3} \sum_{k=1}^{3} (T_{i1k}^+ + T_{i1k}) (L_k r + |f_k(0)|) \\
&= 0.56 \times 1.02 + \left( \frac{6}{20} + 1.04 \times \frac{1}{20} + 3.09 \times \frac{1}{20} + 2.04 \times \frac{1}{20} \right) \times 0.1 + 0.018 \\
&= 0.65
\end{align*}
$$

$$
M_2 = \alpha_2^+ \eta_2^+ r + \sum_{j=1}^{2} \left( D_{2j}^+ + (D_{2j}^+)^* + D_{2j}^+ \sigma_{2j}^+ + \tilde{D}_{2j}^+ \xi_{2j}^+ \right) (L_j r + |f_j(0)|)
\begin{align*}
&+ \sum_{j=1}^{2} \sum_{k=1}^{2} T_{2j}^+ (L_k r + |f_k(0)|)(L_j r + |f_j(0)|) + I_2^+
\end{align*}
\begin{align*}
&= 0.75 \times 0.02 \times 0.43 + \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 1.03 + \frac{1}{15} \times 1.02 \right) (0.1 \times 0.43 + 0.1) \\
&+ \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 1.02 + \frac{1}{15} \times 1.04 \right) (0.1 \times 0.43 + 0.1) \\
&+ 0.04 + 0.08 (0.1 \times 0.43 + 0.1) \\
&= 0.52
\end{align*}
$$
\[ M_2 = a_3^+ \eta_3^2 + 2 \sum_{j=1}^{2} \left( D_3^+ + (D_3^j)^+ + D_3^j \sigma_3^2 + \bar{D}_3^j \xi_3^j \right) L_j \]

\[ + \sum_{j=1}^{2} \sum_{k=1}^{2} T^+_{2jk}(L_k r + |f_k(0)|)(L_j r + |f_j(0)|) \]

\[ = 0.75 \times 0.02 + \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 0.103 + \frac{1}{15} \times 1.02 \right) \times 0.1 \]

\[ + \left( \frac{1}{15} + \frac{1}{15} \times 1.02 + \frac{1}{15} \times 1.04 \right) \times 0.1 + 0.04 (0.1 \times 0.43 + 0.1)^2 \]

\[ = 0.069 \]

\[ N_2 = c^+_2 \varsigma_2^2 r + \sum_{i=1}^{3} \left( E^{+}_{i2} + (E^{+}_{i2})^+ + \bar{E}^{+}_{i2} \sigma_3^2 + \tilde{E}^{+}_{i2} \xi_3^j \right) (L_i r + |f_i(0)|) \]

\[ + \sum_{i=1}^{3} \sum_{k=1}^{2} T^+_{i2k} (L_k r + |f_k(0)|) (L_i r + |f_i(0)|) + J_2^+ \]

\[ = 0.213 \]

\[ \bar{N}_2 = c^+_2 \varsigma_2^2 + \sum_{i=1}^{3} \left( E^{+}_{i2} + (E^{+}_{i2})^+ + \bar{E}^{+}_{i2} \sigma_3^2 + \tilde{E}^{+}_{i2} \xi_3^j \right) L_i \]

\[ + \sum_{i=1}^{3} \sum_{j=1}^{2} (T^+_{i2k} + T^+_{ik2})(L_k r + |f_k(0)|) \]

\[ = 0.343 \]

\[ M_3 = a_3^+ \eta_3^2 r + 2 \sum_{j=1}^{2} \left( D_3^+ + (D_3^j)^+ + D_3^j \sigma_3^2 + \bar{D}_3^j \xi_3^j \right) (L_j r + |f_j(0)|) \]

\[ + \sum_{j=1}^{2} \sum_{k=1}^{2} T^+_{3jk}(L_k r + |f_k(0)|)(L_j r + |f_j(0)|) + I_3^+ \]

\[ = 0.75 \times 0.07 \times 0.43 + \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 0.04 + \frac{1}{15} \times 0.03 \right) (0.1 \times 0.43 + 0.1) \]

\[ + \left( \frac{1}{15} + \frac{1}{15} \times 0.02 + \frac{1}{15} \times 0.05 \right) (0.1 \times 0.43 + 0.1) \]

\[ + 0.04 (0.1 \times 0.43 + 0.1)^2 + 0.02 \]

\[ = 0.12 \]

\[ \bar{M}_3 = a_3^+ \eta_3^2 + 2 \sum_{j=1}^{2} \left( D_3^+ + (D_3^j)^+ + D_3^j \sigma_3^2 + \bar{D}_3^j \xi_3^j \right) L_j \]

\[ + \sum_{j=1}^{2} \sum_{k=1}^{2} (T^+_{3jk} + T^+_{ikj})(L_k r + |f_k(0)|) \]

\[ = 0.75 \times 0.07 + \left( \frac{1}{15} + \frac{1}{15} + \frac{1}{15} \times 0.03 + \frac{1}{15} \times 0.02 \right) (0.1 \times 0.43 + 0.1) \]

\[ + \left( \frac{1}{15} + \frac{1}{15} \times 0.02 + \frac{1}{15} \times 0.04 \right) (0.1 \times 0.43 + 0.1) + 0.08 (0.1 \times 0.43 + 0.1) \]

\[ = 0.0136 \]
The conditions \((H_1), (H_2)\) and \((H_4)\) are satisfied and it is easy to verify

\[
\max \left\{ \frac{M_1}{\alpha_1}, \frac{M_2}{\alpha_2}, \frac{M_3}{\alpha_3}, \left(1 + \frac{\alpha_1^+}{\alpha_1}\right) M_1, \left(1 + \frac{\alpha_2^+}{\alpha_2}\right) M_2, \left(1 + \frac{\alpha_3^+}{\alpha_3}\right) M_3, \frac{N_1}{c_1}, \left(1 + \frac{c_1^+}{c_1}\right) N_1, \frac{N_2}{c_2}, \left(1 + \frac{c_2^+}{c_2}\right) N_2 \right\} \leq 0.43,
\]

and

\[
\max \left\{ \frac{\overline{M}_1}{\alpha_1}, \frac{\overline{M}_2}{\alpha_2}, \frac{\overline{M}_3}{\alpha_3}, \left(1 + \frac{\alpha_1^+}{\alpha_1}\right) \overline{M}_1, \left(1 + \frac{\alpha_2^+}{\alpha_2}\right) \overline{M}_2, \left(1 + \frac{\alpha_3^+}{\alpha_3}\right) \overline{M}_3, \frac{\overline{N}_1}{c_1}, \left(1 + \frac{c_1^+}{c_1}\right) \overline{N}_1, \frac{\overline{N}_2}{c_2}, \left(1 + \frac{c_2^+}{c_2}\right) \overline{N}_2 \right\} \leq 1.
\]

So, condition \((H_3)\) holds. Therefore, using Theorem 4 and Theorem 5, we conclude that the HOBAMs \((2)\) with the coefficients and parameters defined above has one and only one Stepanov-like weighted pseudo-almost periodic solution. Besides, this unique solution is globally exponentially stable.

Let \(h^* (\cdot) = (x_1^* (\cdot), x_2^* (\cdot), x_3^* (\cdot), y_1^* (\cdot), y_2^* (\cdot))^T\) be a Stepanov-like weighted pseudo-almost periodic on time-space scales solution of system \((2)\). We have clearly for \(i = 1, 2, 3\) and \(j = 1, 2\)

\[
\alpha_i^- - \sum_{j=1}^{2} L_j \left( D_{ij}^+ + (D_{ij})^+ + (D_{ij})^+ \sigma_{ij} + (\bar{D}_{ij})^+ \xi_{ij} + \sum_{k=1}^{2} T_{ijk}^+ (L_k r + |f_k(0)|) \right) \\
+ (L_j r + |f_j(0)|) \sum_{k=1}^{2} T_{ijk}^+ L_k > 0,
\]

and

\[
c_j^- - \sum_{i=1}^{3} L_i \left( E_{ij}^+ + (E_{ij})^+ + (E_{ij})^+ \sigma_{ij} + (\bar{E}_{ij})^+ \xi_{ij} + \sum_{k=1}^{3} T_{ijk}^+ (L_k r + |f_k(0)|) \right) \\
+ (L_i r + |f_i(0)|) \sum_{k=1}^{3} T_{ijk}^+ L_k > 0,
\]

Then from Theorem 4 all solutions \(\psi = (\varphi_1, \varphi_2, \varphi_3, \phi_1, \phi_2)\) of \((2)\) satisfying

\[
x_i^* (0) = \varphi_i (0), \ y_j^* (0) = \phi_j (0), 1 \leq i \leq 3, \ 1 \leq j \leq 2,
\]

converge to its unique Stepanov-like weighted pseudo-almost periodic on time-space scales solution \(h^*\).

Remark 6 The model studied in [14] is without the second-order connection weights of delayed feedback. Moreover, the results published in [14] can not be applicable for our example in this brief. Hence, the results presented here
are more general than that outcomes published in [14], [20] and [35]. Consequently, this analysis of dynamics behavior of the Stepanov weighted pseudo almost automorphic on time-space scales solutions for HOBAMs model with Stepanov-like weighted pseudo almost automorphic (SWPAA) coefficients and mixed delays improve the previous study in [14], [20] and [35].

7 Conclusion and open problem

In this paper, the Stepanov-like weighted pseudo-almost automorphic on time-space scales are concerned for HOBAMs with mixed delays and leakage time-varying delays by using Banachs fixed point theorem, the theory of calculus on time scales and the Lyapunov-Krasovskii functional method method. The results obtained in this paper are completely new and complement the previously known works of (Ref. [14,20,35]). Finally, numerical example was given to demonstrate the effectiveness of our theory. As an interesting problem, other almost automorphic on time scales type of solutions for neural networks were deserved to be studied by applying some suitable methods, such as the study of Stepanov-like weighted pseudo-almost automorphic on time-space scales for Cohen Grossberg BAM neural networks. The corresponding results will appear in the near future.

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