Global integrability and boundary estimates for uniformly elliptic PDE in divergence form

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Abstract. We show that two classically known properties of positive supersolutions of uniformly elliptic PDEs, the boundary point principle (Hopf lemma) and global integrability, can be quantified with respect to each other. We obtain an extension to the boundary of the de Giorgi-Moser weak Harnack inequality, optimal with respect to the norms involved, for equations in divergence form.

1 Introduction and Main Results

This paper is devoted to global estimates for nonnegative supersolutions of divergence-form uniformly elliptic PDE in a given domain. We study bounds in terms of the distance to the boundary, as well as integrability and $L^p$-estimates up to the boundary, of supersolutions and their gradients.

A fundamental property of superharmonic functions is that positivity entails a quantitative version of itself. Specifically, if $u > 0$ is superharmonic in a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^n$, then

$$u \geq c_0 d \quad \text{in} \quad \Omega, \quad \text{where} \quad d(x) := \text{dist}(x, \partial \Omega), \quad c_0 = c_0(u, \Omega) > 0.$$  \hfill (1)

In other words, if $u$ attains a minimum at a boundary point, then the normal derivative of $u$ at this point does not vanish. This is the famous Zaremba-Hopf-Oleinik lemma, also called boundary point lemma or boundary point principle. Because of the importance of this principle, a lot of work has been dedicated to understanding its ramifications and getting optimal conditions for its validity, in terms of the regularity or the geometry of the domain, or of the nature of the coefficients of the elliptic operator. We refer to \cite{21}, \cite{3}, \cite{4}, \cite{19}, \cite{2}, \cite{21}, \cite{22}, \cite{16}, \cite{23} where such conditions, as well as a lot more references and history of this “bedrock” (to quote page 1 of \cite{21}) result in the theory of elliptic PDE can be found.

Another striking property of superharmonic functions, to which a lot of attention has been given, is that positivity implies global integrability. A

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classical result by Armitage [5], [6], states that if $u > 0$ is superharmonic in $\Omega$ then $u \in L^r(\Omega)$ for each $r < n/(n-1)$, and that bound is sharp. Extensions of Armitage’s result to superharmonic functions in more general domains can be found in [1], [18], [27].

The essence of the results below is that global integrability and the boundary point principle quantify each other. Furthermore, we study and quantify how the loss of superharmonicity influences these properties - the integrability is preserved, with the boundary point estimate being corrected with a $L^q$-norm of the “loss”, for $q > n$. We consider general linear operators in divergence form.

Our main result, Theorem 1.1 below, provides a sharp global integrability estimate for the quantity $u/d$, and can also be seen as an optimal global weak Harnack inequality for this quantity. A simple consequence of Theorem 1.1 is an extension to the boundary of the classical de Giorgi-Moser weak Harnack inequality, see Corollary 1.1.

Another consequence of Theorem 1.1 is a novel and surprising global integrability result for the gradient of supersolutions, also quantified in terms of the boundary point property.

We now give the precise statements. We consider weak solutions of inequalities in the form

$$-\text{div}(A(x)\nabla u) + b(x)|\nabla u| \geq -f(x), \quad u \geq 0,$$

where $A$ is a symmetric matrix, for some $\lambda > 0$ and $q > n$

$$A \geq \lambda I, \quad A \in W^{1,q}(\Omega), \quad b, f \in L^q_+(\Omega),$$

$\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded $C^{1,1}$-domain. We set $\Lambda = \|A\|_{W^{1,q}(\Omega)}$. In the sequel, all constants denoted by $C$ will be allowed to depend on $n$, $\lambda$, $\Lambda$, $q$, $\|b\|_{L^q(\Omega)}$, the diameter of $\Omega$, the $C^{1,1}$-norm of $\partial \Omega$, as well as on the positive exponents $r, s, t$ in each of the theorems below.

**Definition 1.1** We say that $u : \Omega \to \mathbb{R}$ is a supersolution of (2) if for each $l \in \mathbb{N}$ the function $u_l := \min\{u, l\}$ belongs to $H^1_{\text{loc}}(\Omega)$, and

$$\int_{\Omega} A\nabla u_l \cdot \nabla \varphi + \int_{\Omega} \varphi b |\nabla u_l| \geq -\int_{\Omega} f \varphi, \quad \text{for each } \varphi \in C^\infty_0(\Omega), \varphi \geq 0. \quad (4)$$

In the literature there are at least four frequently used notions of supersolutions which we briefly recall: in the weak Sobolev sense ($u \in H^1_{\text{loc}}(\Omega)$ and (3) holds for $u$ instead of $u_l$), in the $L^q$-viscosity sense ($u$ is continuous and (2) holds in the essliminf sense for $W^{2,q}$-functions at points where they...
touch \(u\) from below), in the \(C\)-viscosity sense (if \(A, b, f\) are continuous, \(u\) is lower semi-continuous and (2) holds for smooth functions at points where they touch \(u\) from below), and in the potential theory sense (\(u\) is above the solution of the Dirichlet problem in any ball, with \(u\) as boundary value). All these four definitions are included in Definition 1.1. See the Appendix. We note that it is important to consider not just weak Sobolev supersolutions, in order to accommodate supersolutions which are not in the energy space \(H^1_{\text{loc}}\), such as the fundamental solution with pole inside the domain.

Notice that in Definition 1.1 we only ask that \(u_i\) be in \(H^1_{\text{loc}}(\Omega)\), there is no a priori assumption on integrability or behaviour close to the boundary, let alone any boundary condition. The integrability is a consequence of the supersolution property only, as the following shows.

Here is our main result.

**Theorem 1.1** Let \(u\) be a nonnegative supersolution of (2). Then

\[
\left( \int_{\Omega} \left( \frac{u}{d} \right)^s \right)^{1/s} \leq C \left( \inf_{\Omega} \frac{u}{d} + \|f\|_{L^q(\Omega)} \right),
\]

for each

\[
s < 1.
\]

This theorem is an extension of [25, Theorem 1.2] where we showed that (5) holds for some small \(s > 0\), for viscosity supersolutions of more general equations in non-divergence form. As a consequence, in [25, Theorem 1.4] we obtained a full boundary Harnack inequality for nonnegative solutions of inhomogenous equations. Here we upgrade the estimate from [25] to the optimal range \(s < 1\), thanks to the additional variational structure we have.

To our knowledge, even the finiteness of the integral in the left-hand side of (5) is proved here for the first time, for values of \(s\) away from zero.

As a simple consequence of the proof of Theorem 1.1 we get the following quantitative global integrability result.

**Corollary 1.1** Let \(u\) be a nonnegative supersolution of (2). Then

\[
\left( \int_{\Omega} u^r \right)^{1/r} \leq C \left( \inf_{\Omega} \frac{u}{d} + \|f\|_{L^q(\Omega)} \right),
\]

for each

\[
r < \frac{n}{n-1}.
\]
The interior version of Corollary 1.1, when $\Omega$ in the integral and the infimum in (6) is replaced by a compactly included subdomain, is the famous and fundamental de Giorgi-Moser weak Harnack inequality, which is known to hold for $r < n/(n-2)$ (see for instance [11, Theorem 8.18]). It is worth noticing that the interior result does not require any regularity of the leading coefficients of the operator. For results similar to Corollary 1.1 with a small exponent $r$ but for supersolutions of more general $p$-homogeneous equations like the $p$-Laplacian, we refer to [17], [24], [7].

In combination with standard inequalities for supersolutions, Theorem 1.1 implies optimal gradient integrability and a gradient bound for nonnegative supersolutions.

**Theorem 1.2** Let $u$ be a nonnegative supersolution of (2). Then

$$
\left( \int_\Omega |\nabla u|^t \right)^{1/t} \leq C \left( \inf_{\Omega} \frac{u}{d} + \|f\|_{L^q(\Omega)} \right), \tag{7}
$$

for each $t < 1$.

That the integral in (7) is finite for bounded solutions (as opposed to supersolutions) and $t < 1$, and for supersolutions and some $t$ close to zero was proved in [15] for $p$-homogeneous equations ($f = 0$), without an upper bound.

We stress that (5)-(7) are valid for supersolutions, there is no need for $u$ to satisfy an equation or even a two-sided inequality. Also, reading (5)-(7) “from right to left”, that is,

$$u(x) \geq \left( c \max \{ \|u/d\|_{L^s}, \|u\|_{L^p}, \|\nabla u\|_{L^1} \} - C\|f\|_{L^q} \right) d(x) \quad x \in \Omega,$$

is a quantification of the boundary point lemma (i.e. of $c_0(u, \Omega)$ in (1)), as well as an extension of this lemma to inhomogeneous inequations.

As far as optimality is concerned, taking $\Omega$ to have a flat part of its boundary in $\{x_n = 0\}$ containing the origin, and $u = x_n/|x|^n$, we easily see that the ranges for $r$, $s$, and $t$ in the above theorems cannot be improved, even for harmonic functions. Thus, the theorems above show that general supersolutions (with arbitrarily bad behaviour in $\Omega$) are as integrable as the fundamental solution with an isolated singularity on the boundary. At a first sight, this property may not look natural, since supersolutions are supposed to be larger than solutions. That a one-sided elliptic inequality should imply gradient control is even less intuitive.
Given that our results are new even for supersolutions of the Poisson equation in a smooth domain, we have not striven for maximum generality with respect to the regularity of the domain or the coefficients of the elliptic operator. The latter are however quite reasonable – it is known that even the Hopf lemma may fail for some \( A \in C(\overline{\Omega}) \cap W^{1,n}(\Omega) \) and \( b = 0 \), or for \( A = I \) and some \( b \in L^n(\Omega) \) (see for instance [19], [22], [3]). We expect our results to be valid for \( A \in C^\alpha(\Omega) \), \( \alpha > 0 \) (or even for \( A \) Dini continuous in \( \Omega \)), as well as for domains whose boundary is in \( C^{1,Dini} \). It can also be expected that the linearity of the principal part of the operator is not essential, and the theorems above are valid for operators with some sort of “linear-like” structure, like those in [11, Chapter 8].

It is only a matter of technicalities to replace the hypothesis \( f \in L^q(\Omega) \) by \( f \in L^{q/2}(\Omega) \cap L^q(\Omega') \), where \( \Omega' \) is some (small) neighborhood of the boundary in \( \Omega \). Note this intersection is \( L'(\Omega) \) from [25]. In the next section we also establish local versions around a given boundary point of the above theorems.

The main idea of the proof of Theorem 1.1 is to use a Moser-type iteration in order to upgrade the result from [25, Theorem 1.2] to the optimal range \( s < 1 \). We rely on recent embedding results and estimates for weighted Sobolev spaces from [10].

It turns out that implementing an iteration procedure is considerably more delicate at the boundary than in the interior of the domain. The test function we use contains a product of different powers of \( u \) and the distance function, and these powers are varied independently at some points, and together at others. Surprisingly, it is indispensable to track carefully the dependence in these powers of the constants in front of the integrals in order to realize even one step of the iteration (no such necessity appears in the proof of the interior estimate). At several moments this dependence suffices just barely to absorb bad terms into good terms, see for instance (21)–(23) – in this sense the estimate (5) feels very “exact”. Also a sequence of cut-offs that get closer to the boundary is necessary, together with a careful evaluation of their contribution. At the end of the proof we obtain a somewhat unusual recursively defined sequence of Lebesgue exponents, which converges to one.

2 Proofs

It is sufficient to prove (5)–(7) with \( u \) replaced by \( u_l \) and a constant \( C \) independent of \( l \). Indeed, then the monotone convergence theorem implies that \( u \) satisfies the same inequalities. In particular, \( u \) can be assumed bounded, i.e. a usual weak Sobolev supersolution, provided \( C \) is shown to be independent of \( u \). Further, we observe that \( u \) is lower semi-continuous (see the appendix),
in particular $u$ attains its minimum on compacts. We also recall that the minimum of supersolutions is a supersolution.

Thanks to the boundary weak Harnack inequality in [25] we know that (5)-(6) are true if $r, s$ are small positive constants which depend on the right quantities. We note that [25] Theorem 1.2 is stated for $L^q$-viscosity supersolutions but it also applies to bounded supersolutions as in Definition 1.1. This can be seen for instance by repeating almost verbatim the proof of [25], substituting the elliptic theory for viscosity solutions of extremal equations used there by the theory of weak Sobolev supersolutions as developed for instance in [11, Chapter 8] and [28]. Specifically, we need to replace the use of the ABP inequality by a scaled (with respect to the width of a domain) version of Theorem 8.16 in [GT], and notice that weak solutions of equations involving the divergence form operator $L$ have global $C^{1,\alpha}$ regularity and estimates (Corollary 8.36 and Remark following it in [11]). Alternatively, [25] Theorem 1.2 applies to bounded weak Sobolev supersolutions by the equivalence between the notions of supersolutions, see the Appendix.

So our job will be to improve [25] Theorem 1.2 to $\varepsilon_0 < 1$.

2.1 Proof of Theorem 1.1

By a standard argument involving local straightening and covering of $\partial \Omega$, Theorem 1.1 is a consequence of the following local result. We denote with $B_R^+ = \{ x \in \mathbb{R}^n : |x| < R, x_n > 0 \}$ a half-ball whose boundary’s flat portion is included in $\{ x_n = 0 \}$.

**Theorem 2.1** Assume that $u_l \in H^1_{\text{loc}}(B_{2R}^+)$ is a bounded weak Sobolev supersolution of (2) in $B_{2R}^+$, for each $l \in \mathbb{N}$. Then there exists $C > 0$ depending on $n, \lambda, \Lambda, q, s,$ and $R^{1-n/q} \| b \|_{L^q(B_{2R}^+)}$, such that

$$R^{-n/s} \left( \int_{B_R^+} \frac{u}{x_n} \right)^{1/s} \leq C \left( \inf_{B^+_R} \frac{u}{x_n} + R^{1-n/q} \| f^- \|_{L^q(B_{2R}^+)} \right), \quad (8)$$

for each $s < 1$.

This is a consequence of the following particular case of Theorem 1.1.

**Proposition 2.1** Let $\Omega$ be a bounded convex $C^2$-domain, and $u \in H^1(\Omega)$ be a bounded weak Sobolev supersolution of (2) in $\Omega$. Then

$$\left( \int_{\Omega} \frac{u}{d} \right)^{1/s} \leq C \left( \inf_{\Omega} \frac{u}{d} + \| f \|_{L^q(\Omega)} \right), \quad (9)$$
for each $s < 1$.

**Proof of Theorem 2.1** By scaling (change $x \rightarrow x/R$), it is enough to prove (2.1) for $R = 1$. Fix a smooth convex domain $\Omega$, such that $B_{3/2} \subset \Omega \subset B_2$, with the $C^2$-norm of $\partial \Omega$ being a universal constant, and take a monotone sequence of smooth convex domains $\omega_m$ which converges to $\Omega$ in $C^2$ and $\omega_m \subset \Omega \cap \{x_n > 1/m\}$. Apply \[9\] to $\omega_m$ and pass to the limit $m \rightarrow \infty$ with the help of the monotone convergence theorem.

The rest of this section will be devoted to the proof of Proposition 2.1. We are going to use the following weighted Sobolev inequality, which follows from a result due to Filippas, Maz’ya and Tertikas, [10].

**Theorem 2.2** Let $\Omega$ be a bounded convex $C^2$ domain of $\mathbb{R}^n$, $n > 2$. If $\phi \in H^1_0(\Omega)$ then for all

$$a \in \left(\frac{1}{2}, 1\right), \quad t \in \left(2, \frac{2n}{n-2}\right)$$

we have

$$\|\psi^b \phi\|_{L^t(\Omega)} \leq C\|\psi^a \nabla \phi\|_{L^2(\Omega)} + C\|\phi\|_{L^2(\Omega)}$$

where $C = C(n, t, \Omega)$, and we have set

$$b = a - 1 + \frac{t - 2}{2t} n.$$

More generally, if $n > \alpha > 1$,

$$\|\psi^b \phi\|_{L^t(\Omega)} \leq C\|\psi^a \nabla \phi\|_{L^\alpha(\Omega)} + C\|\phi\|_{L^\alpha(\Omega)}, \quad b = a - 1 + \frac{t - \alpha}{\alpha t} n.$$  

**Proof.** This is a particular case of Theorem 4.5 and inequality (4.40) in [10].

Alternatively, Theorem 2.2 is a consequence of Proposition 2 of Souplet [26], combined with the Hölder inequality and the interpolation Lemma 3 in that paper. To accomodate the reader we note that $a$ in [26] is our $2a$, while $b$ in [26, Lemma 3] is our $2 - 2a$.

**Proof of Proposition 2.1** We know there is a uniform neighborhood of size $\delta > 0$ of the boundary $\partial \Omega$ in which the distance function to the boundary is $C^2$-smooth. By scaling we can assume that $\delta = 2$ (translate so that $0 \in \Omega$, and dilate $x \rightarrow R_0 x$, for some $R_0$ which depends on $\delta$, $\text{diam}(\Omega)$, \footnote{note we apply these results with $k = 1$; and that there is a misprint in (4.40), the $L^q$-norm in the right-hand side lacks the power $p$ there.}
min_{x \in \partial \Omega} \cos(x, \nu(x)))$. Set $\Omega' = \Omega'_1 = \{x \in \Omega : \text{dist}(x, \partial \Omega) < 1\}$, $\Omega'_{m} = \{x \in \Omega : \text{dist}(x, \partial \Omega) < 1/m\}$, and $\Omega'' = \{x \in \Omega : \text{dist}(x, \partial \Omega) < 2\}$.

We fix a $C^2$-smooth unitary vector field $\nu(x)$ in $\Omega$, such that for each $x \in \Omega'$ with $u(x) = 1/m$, $\nu(x)$ is the interior normal to the boundary of $\Omega'_{m}$ at $x$, and for $x \in \partial \Omega$, $\nu(x)$ is the interior normal to the boundary of $\Omega$. Let $\psi$ be a smooth function in $\Omega$, such that $0 \leq \psi \leq 2$ in $\Omega$, $\psi = 2$ in $\Omega \setminus \Omega''$, $\psi(x) = d(x)$ in $\Omega'$, $\frac{\partial \psi}{\partial \nu} \geq 0$ in $\Omega$, $\|\psi\|_{C^2(\Omega)} \leq C(\Omega)$.

Note we choose these just to ease some technicalities later, $\psi$ is equivalent to $d$ but is smooth, and we will consider $u/\psi$.

Set $k = \|f\|_{L^q(\Omega)}$ if $f \not\equiv 0$, and let $k > 0$ be arbitrary if $f = 0$. Replace $u$ by $\tilde{u} = u + k$, which solves the same equation. We are going to show that, given $\varepsilon_0 \in (0, 1)$,

$$
\left( \int_{\Omega} \left( \frac{\tilde{u}}{\psi} \right)^s \right)^{1/s} \leq C \left( \int_{\Omega} \left( \frac{\tilde{u}}{\psi} \right)^{\varepsilon_0} \right)^{1/\varepsilon_0}
$$

(10)

for each $s < 1$. Here $C$ depends also on $\varepsilon_0$. Then the theorems follow, by the standard inequalities: for $u, k, \alpha > 0$,

$$
\min\{1, 2^{\alpha - 1}\}(u^\alpha + k^\alpha) \leq (u + k)^\alpha \leq \max\{1, 2^{\alpha - 1}\}(u^\alpha + k^\alpha),
$$

(noting also that $\psi^{-s}$ is integrable in $\Omega$ for $s < 1$, $\|\psi^{-s}\|_{L^1(\Omega)} = C(s, \Omega)$).

Assume first $n > 2$. Fix $s_0 < 1$ and let us prove (10) for $s = s_0$. Also fix some numbers $\sigma \in [0, 1)$, $\gamma \in [\varepsilon_0, (1 + s_0)/2]$, $r \in (\sigma \gamma, 1)$. Later we will specify (and vary) these constants.

In all that follows $C$ will denote a constant which may vary from line to line and depends on the usual quantities, as well as on positive lower bounds on $1 - s$, $1 - \sigma$, $1 - r$, $r - \sigma \gamma$.

We set, for $m \in \mathbb{N}$,

$$
\eta_m(x) = \begin{cases} md(x) & \text{if } d(x) \leq 1/m \\ 1 & \text{if } d(x) \geq 1/m. \end{cases}
$$

We will often omit the subscript $m$, and write $\eta = \eta_m$. Further denote

$$
v = v_{\sigma} = \psi^{-\sigma} \tilde{u}, \quad \text{and} \quad w = v^{\gamma/2}.
$$

We recall that $0 < k \leq \tilde{u} \leq M$ in $\Omega$, with $M = \|u\|_{L^\infty(\Omega)} + k$. 8
It is easy to check that the function $\eta w \in H^1_0(\Omega)$ (near the boundary $\partial \Omega$ we have $\eta w \sim d^{1-\frac{n}{2}}$, and $\sigma \gamma < 1$, so $\nabla (\eta w)$ is square-integrable up to the boundary). Our goal is an estimate of the type

$$\int_{\Omega} \psi^{1+\gamma} |\nabla (\eta w)|^2 \leq C \int_{\Omega} \psi^\gamma (\eta w)^2 + \text{negligible},$$

which together with Theorem 2.2 leads to a reverse Hölder inequality.

We will use the test function

$$\varphi = \psi^{1+r-\sigma \gamma} \tilde{u}^{\gamma-1} \eta^2 = \psi^{1-\sigma + r\gamma-1} \eta^2$$

in the weak formulation of (2)

$$\int_{\Omega} (A \nabla u, \nabla \varphi) + \int_{\Omega} (b. \nabla u) \varphi \geq \int_{\Omega} f \varphi,$$

valid for each $\varphi \in H^1_0(\Omega), \varphi \geq 0$ (by density), and rewrite the resulting inequality in terms of $w$.

We compute, setting $\tilde{f} = f/k, \|\tilde{f}\|_{L^q} \leq 1$, that for each $\epsilon > 0$ we can find $C\epsilon > 0$ such that

$$\left| \int_{\Omega} f \varphi \right| = \left| \int_{\Omega} \psi^{1-\sigma + r} \tilde{f} \psi^\gamma \eta^2 \right|$$

$$\leq \int_{\Omega} \psi^{1+r} \tilde{f}(\eta w)^2 \leq \|\tilde{f}\|_{L^{q/2}} \|\psi^{1+r}(\eta w)^2\|_{L^{(q/2)'}},$$

$$\leq \epsilon \|\psi^{1+r}(\eta w)^2\|_{L^{q/(n-2)}} + C\epsilon \|\psi^{1+r}(\eta w)^2\|_{L^1},$$

$$\leq C\epsilon \int_{\Omega} \psi^{1+r} |\nabla (\eta w)|^2 + C\epsilon \int_{\Omega} \psi^\gamma (\eta w)^2, \quad (12)$$

In the third inequality we used that $q > n$, so $(q/2)' < (n/2)' = n/(n-2)$, Hölder and Young inequalities; to get the last inequality in (12), we applied Theorem 2.2 with

$$a = b = \frac{1 + r}{2}, \quad t = \frac{2n}{n - 2}.$$  

Remark. For this computation we only need $f \in L^{q/2}, q > n$.

Further, since

$$\nabla u = \nabla \tilde{u} = \sigma \psi^{-1} v \nabla \psi + \psi^\gamma \nabla v, \quad v^{\gamma-1} \nabla v = \frac{1}{\gamma} \nabla w^2 = \frac{2}{\gamma} w \nabla w, \quad (13)$$
we have, setting \( \tilde{b} = |b.\nabla\psi| \in L^q(\Omega) \),

\[
\left| \int_{\Omega} (b.\nabla u) \varphi \right| \leq \int_{\Omega} \tilde{b} \psi^r \eta^2 + \frac{C}{\gamma} \int_{\Omega} \psi^{1+r} \eta^2 w b.\nabla w \right|
\leq \int_{\Omega} \tilde{b} \psi^r (\eta w)^2 + \frac{C}{\varepsilon_0} \int_{\Omega} \psi^{1+r} \eta w b. (\nabla (\eta w) - w \nabla \eta) \right|
= \ J_1 + C/\varepsilon_0|J_2 - J_3| \leq J_1 + C|J_2| + C|J_3|.
\]

We evaluate, by \( q' < n' = n/(n - 1) \), for every \( \epsilon > 0 \),

\[
J_1 \leq ||\tilde{b}||_{L^q} ||(\psi^{r/2} \eta w)^2||_{L^{q'}} \leq C \epsilon ||\psi^{r/2} \eta w||^2_{L^{2n/(n-1)}} + C \epsilon ||(\psi^{r/2} \eta w)^2||_{L^1} \leq C \epsilon \int \psi^{1+r} |\nabla (\eta w)|^2 + C \epsilon \int \psi^r (\eta w)^2 \]
where in the last inequality we used Theorem 2.2 with

\[
a = \frac{1 + r}{2}, \quad b = \frac{r}{2}, \quad t = \frac{2n}{n - 1}.
\]

We observe that

\[
\psi |\nabla \eta| \leq \eta,
\]
so

\[
|J_3| \leq C \int |b| \psi^r (\eta w)^2,
\]
and \( J_3 \) can be evaluated exactly like \( J_1 \). On the other hand,

\[
|J_2| \leq \epsilon \int \psi^{1+r} |\nabla (\eta w)|^2 + C \epsilon \int \psi^{1+r} b^2 (\eta w)^2,
\]
and the last integral can be evaluated exactly like in (12), replacing \( \tilde{f} \) there by \( b^2 \in L^{q/2} \).

In the following we denote with \( I_{\text{OK}} \) any integral such that for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) for which

\[
|I_{\text{OK}}| \leq \epsilon \int \psi^{1+r} |\nabla (\eta w)|^2 + C_\epsilon \int \psi^r (\eta w)^2.
\]
We have just shown the lower-order terms in (11) have this property.

We now turn to the highest order integral in (11) of \( (A\nabla u, \nabla \varphi) \), where most care will be needed. We have

\[
\nabla \varphi = (\gamma - 1) \psi^{1-\sigma+r} \eta^2 v^{r-2} \nabla v + (1 - \sigma + r) \psi^{-\sigma+r} \eta^2 v^{\gamma-1} \nabla \psi + \psi^{1-\sigma+r} v^{\gamma-1} \nabla \eta^2,
\]
\[
v^{-2}(A \nabla v, \nabla v) = \frac{4}{\gamma^2}(A \nabla w, \nabla w),
\]
so, recalling (13),
\[
\int_\Omega (A \nabla u, \nabla \varphi) = -\frac{4(1-\gamma)}{\gamma^2} \int_\Omega \psi^{1+r} \eta^2 (A \nabla w, \nabla w) + \frac{1}{\gamma} \int_\Omega \psi^{1+r} (A \nabla w^2, \nabla \eta^2) + \frac{1+r+\sigma(\gamma-2)}{\gamma} \int_\Omega \psi^r \eta^2 (A \nabla w^2, \nabla \psi) + \sigma \int_\Omega \psi^r w^2 (A \nabla \psi, \nabla \eta^2) + \sigma(1-\sigma+r) \int_\Omega \psi^{r-1} w^2 \eta^2 (A \nabla \psi, \nabla \psi) =: I_1 + I_2 + I_3 + I_4 + I_5. \tag{14}
\]

We have
\[
\int_\Omega \psi^r \eta^2 (A \nabla w^2, \nabla \psi) = \int_\Omega \psi^r (A \nabla (\eta w)^2, \nabla \psi) - \int_\Omega \psi^r w^2 (A \nabla \psi, \nabla \eta^2),
\]
and, by the divergence theorem
\[
\int_\Omega \psi^r (A \nabla (\eta w)^2, \nabla \psi) = -r \int_\Omega \psi^{r-1} (\eta w)^2 (A \nabla \psi, \nabla \psi) + \int_\Omega (A \nabla (\psi^r \eta^2 w^2), \nabla \psi) + \int_\partial \Omega \psi^r (\eta w)^2 (A \nabla \psi, -\nu) - r \int_\Omega \psi^{r-1} (\eta w)^2 (A \nabla \psi, \nabla \psi) - \int_\Omega \text{div}(A \nabla \psi) \psi^r (\eta w)^2 + \int_\partial \Omega \psi^r (\eta w)^2 (A \nabla \psi, -\nu).
\]
The integral on the boundary vanishes (recall the definition of \(\psi\), as well as \(w^2 \leq C \psi^{-\sigma} u^3 \leq C M^2 \psi^{-\sigma} \gamma\) and \(\sigma \gamma < r < 1\)), while the penultimate integral can be evaluated exactly like \(J_1\), since \(\|\text{div}(A \nabla \psi)\|_{L^q} \leq C \Lambda\).

We deduce that
\[
I_3 + I_4 + I_5 = \left( \sigma(1-\sigma) - \frac{r}{\gamma}(1+r-2\sigma) \right) \int_\Omega \psi^{r-1} (\eta w)^2 (A \nabla \psi, \nabla \psi) + \frac{1+r-2\sigma}{\gamma} \int_\Omega \psi^r w^2 (A \nabla \psi, \nabla \eta^2) + I_{\text{OK}}.
\]

Next, to evaluate \(I_2\) we observe that
\[
\int_\Omega \psi^{1+r} (A \nabla w^2, \nabla \eta^2) = -(1+r) \int_\Omega \psi^r w^2 (A \nabla \psi, \nabla \eta^2) + \int_\Omega (A \nabla (\psi^{1+r} w^2), \nabla \eta^2),
\]
while by the divergence theorem
\[ \int_{\Omega} (A \nabla (\psi^{1+r} w^2), \nabla \eta^2) = \int_{\Omega} (A \nabla \eta^2, \nabla (\psi^{1+r} w^2)) \]
\[ = \int_{\partial \Omega} \psi^{1+r} w^2 (A \nabla \eta^2, -\nu) - \int_{\Omega} \text{div} (A \nabla \eta^2) \psi^{1+r} w^2 \]
\[ = - \int_{\partial \Omega} A' \nabla \eta^2 \psi^{1+r} w^2 - \int_{\Omega} \text{tr} (A D^2 \eta^2) \psi^{1+r} w^2, \]

where \( A' \) is a matrix containing derivatives of the entries of \( A \), so \( A' \) is bounded in \( L^q \). Since \( |\nabla \eta^2| \leq C \psi^{-1} \eta^2 \), the integral on \( \partial \Omega \) vanishes, and the first integral in the right-hand side of the last equality can again be evaluated like \( J_1 \). Next we deal with the last integral.

We fix a smooth orthonormal basis \((\tau, \nu)\), where \( \nu(x) \) is the vector field we defined above, and let \( T(x) \) be an orthogonal change-of-basis matrix between \((\tau(x), \nu(x)) \) and \( x \) (the \( C^2 \)-norm of \( T \) is bounded in terms of \( \Omega \)), so that \( \nabla = \nabla_x = T \nabla_{\tau, \nu} \). By the definition of \( \eta = \eta_m \), \( \partial_{\alpha \beta} (\eta^2) = 0 \) if \((\alpha, \beta) \in (\tau, \nu)^2 \setminus (\nu, \nu) \), and
\[ \partial_{\nu \nu} (\eta^2) = 2 m^2 \chi(\Omega_m) - 2 m \delta(\partial \Omega_m), \]
where \( \Omega_m = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < 1/m \} \), \( \Omega_m = \Omega \setminus \Omega_m \), \( \chi \) denotes the characteristic function, and \( \delta \) is the Dirac mass concentrated at \( \partial \Omega_m \).

Therefore
\[ \int_{\Omega} (A \nabla (\psi^{1+r} w^2), \nabla \eta^2) = \left( -2 m^2 \int_{\Omega_m} \tilde{a}_{nn} \psi^{1+r} w^2 + I \text{OK} \right) \]
where \( \tilde{a}_{nn} \) denotes the last entry of the matrix \( \tilde{A} = T^{-1} A T \).

We compute that
\[ 0 \leq \lim_{m \to \infty} 2 m \int_{\partial \Omega_m} \tilde{a}_{nn} \psi^{1+r} w^2 \leq \lim_{m \to \infty} 2 C m^{1+r} m^{\sigma \gamma} (\max_{\Omega} \tilde{a}_{nn}) = 0, \]
since \( \psi = 1/m \) on \( \partial \Omega_m \), \( w^2 \leq C M^{\gamma} \psi^{-\sigma \gamma} \) (as above), and \( r > \sigma \gamma \).

Finally,
\[ I_2 + \ldots + I_5 = \frac{-2 (1 + r) + 2 \sigma}{\gamma} \int_{\Omega} \psi^{r} w^2 (A \nabla \psi, \nabla \eta^2) - \frac{m^2}{\gamma} \int_{\Omega_m} \tilde{a}_{nn} \psi^{1+r} w^2 \]
\[ + \left( \sigma (1 - \sigma) - \frac{r}{\gamma} (1 + r - 2 \sigma) \right) \int_{\Omega} \psi^{r-1} (\eta w)^2 (A \nabla \psi, \nabla \psi) + I \text{OK} + o(1), \]

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where $o(1)$ is a quantity which goes to zero as $m \to \infty$. Therefore, by (14) and (11),

$$\frac{4(1 - \gamma)}{\gamma^2} \int_{\Omega} \psi^{1+r} \eta^2 \nabla w, \nabla w \leq \frac{-2(1 + r) + 2\sigma}{\gamma} \int_{\Omega} \psi^r w^2 (A\nabla \psi, \nabla \eta^2) + \left(\sigma(1 - \sigma) - \frac{r}{\gamma}(1 + r - 2\sigma)\right) \int_{\Omega} \psi^{r-1} (\eta w)^2 (A\nabla \psi, \nabla \psi) - \frac{2m^2}{\gamma} \int_{\Omega_m} \tilde{a}_{nn} \psi^{1+r} w^2 + I_{OK} + o(1),$$

For a moment we set in this inequality

$$\sigma = 0, \quad \text{i.e.} \quad v = u, \ w = w_0 = u^{\gamma/2}.$$

All terms in the right-hand side are negative, and $(1 - \gamma)/\gamma^2$ is between two positive constants (by the assumption we made on $\gamma$), so by the uniform positivity of $A$

$$\int_{\Omega} \psi^{1+r} \eta^2 |\nabla w_0|^2 \leq I_{OK} + o(1), \quad \text{for all} \ r \in (0, 1). \quad (15)$$

The constant in $I_{OK}$ in (15) is bounded by what we need, as long as $r$ is bounded away from 0 and 1.

We go back to the general case $\sigma > 0$. By using the Young inequality on the terms involving $\tilde{a}_{\tau \nu}$ in the quadratic form $(\tilde{A} \xi, \xi)$, for any $\delta > 0$ we can find $C_\delta > 0$ such that

$$\tilde{a}_{\tau \nu} \xi_\tau \xi_\nu \geq -\delta/(\lambda n^2)\xi_\nu^2 - C_\delta \xi_\tau^2 \geq -\delta/(n^2)\tilde{a}_{m \eta} \xi_\nu^2 - C_\delta \xi_\tau^2$$

$$(\tilde{A} \xi, \xi) = (\tilde{A}_\tau \xi_\tau, \xi_\tau) + \tilde{a}_{mn} \xi_\nu^2 + \sum_{i=1}^{n-1} \tilde{a}_{\tau \nu} \xi_\tau \xi_\nu$$

$$(1 - \delta)\tilde{a}_{mn} \xi_\nu^2 \leq (\tilde{A} \xi, \xi) + C_\delta |\xi_\tau|^2$$

From now on we set

$$r = \gamma.$$

Hence

$$(1 - \delta) \int_{\Omega} \psi^{1+\gamma} \eta^2 \tilde{a}_{m n} (\partial_\nu w)^2 \leq C_\delta \int_{\Omega} \psi^{1+\gamma} \eta^2 |\nabla_\tau w|^2 + D_1 \int_{\Omega} \psi^\gamma w^2 (A\nabla \psi, \nabla \eta^2) + D_2 \int_{\Omega} \psi^{\gamma-1} (\eta w)^2 (A\nabla \psi, \nabla \psi) - D_3 m^2 \int_{\Omega_m} \tilde{a}_{mn} \psi^{1+\gamma} w^2 + I_{OK} + o(1), \quad (16)$$

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where

\[ D_1 = -\frac{\gamma(1 + \gamma - \sigma)}{2(1 - \gamma)}, \quad D_2 = \frac{\gamma^2 (-\sigma^2 + 3\sigma - 1 - \gamma)}{4(1 - \gamma)}, \quad D_3 = \frac{\gamma}{2(1 - \gamma)}. \]

However, by using \( w^2 = u^\gamma / \psi^{\sigma\gamma} \) and \( \nabla_r \psi = 0 \) in \( \Omega' \), \( \psi \geq 1 \) in \( \Omega \setminus \Omega' \),

\[
\int_\Omega \psi^{1+\gamma} \eta^2 |\nabla_r w|^2 \leq C \int_\Omega \psi^{1+\gamma-\sigma\gamma} \eta^2 |\nabla w_0|^2 \leq I_{\text{OK}} + o(1)
\]

where the last inequality follows from (15) with \( r \) replaced by \((1 - \sigma)\gamma\).

We have \( \psi^{-1} \leq 1 \) in \( \Omega \setminus \Omega' \) and \((A\nabla \psi, \nabla \psi) = a_{nn} \) in \( \Omega' \), so

\[
\int_\Omega \psi\gamma^{-1}(\eta w)^2 (A\nabla \psi, \nabla \psi) \leq \int_\Omega \psi\gamma(\eta w)^2 (A\nabla \psi, \nabla \psi) + \int_\Omega \psi\gamma^{-1}(\eta w)^2 a_{nn} \leq I_{\text{OK}} + \int_{\Omega'} \psi\gamma^{-1}(\eta w)^2 a_{nn}, \quad (17)
\]

where the integral is evaluated exactly like \( J_1 \) above. Note \(|(A\nabla \psi, \nabla \psi)| \leq C\).

We further compute, writing \( \partial \vartheta = \partial_n \),

\[
\gamma \int_\Omega \psi\gamma^{-1} w^2 \eta^2 \partial_n a_{nn} \partial \vartheta + \int_\Omega \psi\gamma \partial(w^2) \eta^2 \partial_n a_{nn} = \int_\Omega \partial(\psi\gamma w^2) \eta^2 \partial_n a_{nn} = -\int_\Omega \psi\gamma w^2 \partial(\eta^2) a_{nn} - \int_\Omega \psi\gamma w^2 \eta^2 \partial a_{nn} + \int_{\partial \Omega} + I_{\text{OK}}
\]

(\( \text{the boundary term again vanishes} \)).

Hence (recall \( \partial \vartheta \geq 0 \) in \( \Omega \) and \( \partial \vartheta = 1 \) in \( \Omega' \))

\[
\int_{\Omega'} \psi\gamma^{-1}(\eta w)^2 a_{nn} \leq \int_\Omega \psi\gamma^{-1} w^2 \eta^2 \partial_n a_{nn} \partial \vartheta \leq -\frac{1}{\gamma} \int_\Omega \psi\gamma \partial(w^2) \eta^2 \partial_n a_{nn} - \frac{1}{\gamma} \int_\Omega \psi\gamma w^2 \partial(\eta^2) a_{nn} + I_{\text{OK}} \quad (18)
\]

Further, we have

\[
\left| \int_\Omega \psi\gamma \partial(w^2) \eta^2 \partial_n a_{nn} \right| \leq 2 \int_\Omega (\psi(\gamma+1)/2 |\partial w| \eta^{-1/2} a_{nn}^{1/2})(\psi(\gamma-1)/2 w \eta a_{nn}^{1/2}) \]

\[
\leq \frac{2}{\gamma} \int_\Omega \psi^{\gamma+1}(\partial w)^2 \eta^2 a_{nn} + \frac{\gamma}{2} \int_\Omega \psi^{-1}(\partial w)^2 \eta^2 a_{nn} \quad (19)
\]

\[
= \frac{2}{\gamma} \int_\Omega \psi^{\gamma+1}(\partial w)^2 \eta^2 a_{nn} + \frac{\gamma}{2} \int_\Omega \psi^{-1}(\partial w)^2 \eta^2 a_{nn} + I_{\text{OK}}
\]
since again $\psi^{-1} \leq 1$ in $\Omega \setminus \Omega'$.

Combining (18) with (19), we get from (17)

$$
\int_{\Omega} \psi^{\gamma-1}(\eta w)^2 (A \nabla \psi, \nabla \psi) \leq \frac{4}{\gamma^2} \int_{\Omega} \psi^{\gamma+1}(\partial w)^2 \eta^2 \tilde{a}_{mn} - \frac{2}{\gamma} \int_{\Omega} \psi^2 \partial(\eta^2) \tilde{a}_{nn} + I_{\text{OK}}.
$$

Hence we get from (16), noticing that $(A \nabla \psi, \nabla \eta^2) = \partial(\eta^2) \tilde{a}_{nn}$,

$$(1 - \delta) \int_{\Omega} \psi^{1+\gamma} \eta^2 \tilde{a}_{nn} (\partial w)^2 \leq (D_1 - \frac{2D_2}{\gamma}) \int_{\Omega} \psi^2 \partial(\eta^2) \tilde{a}_{nn}$$

$$+ \frac{4D_2}{\gamma^2} \int_{\Omega} \psi^{1+\gamma} (\partial w)^2 \eta^2 \tilde{a}_{nn} - D_3 m^2 \int_{\Omega_m} \tilde{a}_{nn} \psi^{1+\gamma} w^2 + I_{\text{OK}} + o(1),$$

Now $\sigma \in (0,1)$ implies

$$\sigma^2 - 3\sigma + 2 > 0$$

which in turn guarantees precisely that

$$\frac{4D_2}{\gamma^2} < 1.$$ 

Hence, by choosing $\delta > 0$ sufficiently small, the second term in the right-hand side of (20) can be absorbed in the left-hand side.

It is easy to check that

$$-d_0 := D_1 - \frac{2D_2}{\gamma} = -\gamma \frac{\sigma^2 - 2\sigma + 1}{2(1 - \gamma)} < 0.$$ 

We have thus shown that for some uniformly positive constants $\lambda_0, d_0$,

$$\lambda_0 \int_{\Omega} \psi^{1+\gamma} \eta^2 |\nabla w|^2 \leq -d_0 m^2 \int_{\Omega_m} \psi^{1+\gamma} w^2 + I_{\text{OK}} + o(1).$$

We can assume $\lambda_0 < d_0$, by further diminishing $\lambda_0$ if necessary.

We have $|\nabla \eta|^2 = m^2 \chi(\Omega'_m)$. Hence

$$\frac{\lambda_0}{2} \int_{\Omega} \psi^{1+\gamma} |\nabla(\eta \psi)|^2 \leq \lambda_0 \int_{\Omega} \psi^{1+\gamma} |\nabla \eta|^2 w^2 + \lambda_0 \int_{\Omega} \psi^{1+\gamma} \eta^2 |\nabla w|^2 + I_{\text{OK}} + o(1)$$

$$\leq (\lambda_0 - d_0)m^2 \int_{\Omega_m} \psi^{1+\gamma} w^2 + I_{\text{OK}} + o(1)$$

$$\leq I_{\text{OK}} + o(1).$$
Taking $\epsilon = \lambda_0/4$ in the definition of $I_{OK}$, we get

$$\int_{\Omega} \psi^{1+\gamma} |\nabla (\eta w)|^2 \leq C \int_{\Omega} \psi^\gamma (\eta w)^2 + o(1),$$

so, by Theorem 2.2 applied with $a = (1 + \gamma)/2$, $b = 0$, we obtain

$$\| (\eta w)^2 \|_{L^\rho(\Omega)} \leq C \| (\eta w)^2 \|_{L^1(\Omega)} + o(1),$$

where

$$\rho = \frac{t}{2} = \frac{n}{n - 1 + \gamma}.$$

Letting $m \to \infty$, by the definition of $w$, $\eta \rightarrow 1$ and the monotone convergence theorem, we get

$$\frac{\tilde{u}}{\psi^\sigma} \in L^\rho(\Omega), \quad \frac{\tilde{u}}{\psi^\sigma} \leq C \frac{\tilde{u}}{\psi^\sigma} \in L^\gamma(\Omega),$$

as long as the right-hand side is finite. This is so in particular if $\gamma = \epsilon_0$ (recall $\sigma \leq 1$), since we already know that $\frac{\tilde{u}}{\psi} \in L^{\epsilon_0}$, by [25, Theorem 1.2].

Set $a_1 := \epsilon_0$. Hence $\frac{u}{\psi^\sigma} \in L^{a_2}$, $a_2 = \frac{n}{n-1+a_1} a_1$. Taking in the above argument $\gamma = \frac{n-1+a_k}{n-1+a_k} a_k$ results in the iteration

$$\frac{\tilde{u}}{\psi^\sigma} \in L^{a_k}, \quad \frac{\tilde{u}}{\psi^\sigma} \leq C^{k-1} \frac{\tilde{u}}{\psi^\sigma} \in L^{a_1}(\Omega),$$

$$a_1 = \epsilon_0, \quad a_{k+1} = \frac{n}{n-1+a_k} a_k, \quad k \in \mathbb{N}, \quad (24)$$

The recursively defined sequence $\{a_k\}$ is increasing as long as $\epsilon_0 < 1$, and then

$$\lim_{k \to \infty} a_k = 1.$$

We claim that the proof is finished after a finite number $k_0$ of iterations, where $k_0$ is the first index such that

$$a_{k_0} \geq \frac{1 + s_0}{2} = \frac{s_0}{\sigma},$$

where the latter equality is how we make our overall choice of $\sigma$,

$$\sigma = \frac{2s_0}{1 + s_0}. $$

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Indeed, by Hölder inequality and $s = s_0 < 1$

$$
\int_{\Omega} \left( \frac{\tilde{u}}{\psi} \right)^s \, \frac{1}{\psi^{(1-\sigma)s}} = \int_{\Omega} \frac{\tilde{u}^s}{\psi^{s\sigma}} \psi^{(1-s)\sigma} \, \frac{1}{\psi^{(1-\sigma)s}} \\
\leq \left( \int_{\Omega} \frac{\tilde{u}^{s/\sigma}}{\psi^{s}} \right)^{\sigma} \left( \int_{\Omega} \frac{1}{\psi^{s}} \right)^{1-\sigma} \\
\leq C \left\| \frac{\tilde{u}}{\psi^\sigma} \right\|_{L^{s/\sigma}(\Omega)} \leq C \left\| \frac{\tilde{u}}{\psi^\sigma} \right\|_{L^{s_0}(\Omega)} \leq C \left\| \frac{\tilde{u}}{\psi} \right\|_{L^{s_0}(\Omega)} ,
$$

and (11) is proved. \[ \square \]

Finally, if $n = 2$, we use the last part of Theorem 2.2. Repeating the above, with trivial modifications, we can show that

$$
\int_{\Omega} \psi^{1+\gamma} |\nabla (\eta w)|^\alpha \leq C_{\alpha} \int_{\Omega} (\eta w)^2 + o(1),
$$

for each $\alpha < 2$, where $C_{\alpha}$ is bounded in terms of a lower bound on $2 - \alpha$. As above we can set up an iteration process which goes up arbitrarily close to a number $\tilde{s}(\alpha) < 1$, which will be the limit of the sequence $\{a_k\}$ above. We readily see that $\tilde{s}(\alpha) \to 1$ as $\alpha \to 2$. So for each initially fixed $s_0 < 1$ we can choose $\alpha < 2$ such that $\tilde{s}(\alpha) > s_0$, and the iteration gives a bound for $u/\psi$ in $L^{s_0}$. The technical details are left to the interested reader.

Proposition 2.1 and Theorem 1.1 are proved.

### 2.2 Proofs of Proposition 1.1 and Theorem 1.2

As in the previous section, Proposition 1.1 and Theorem 1.2 have local versions around any point on the boundary of $\Omega$, so it is sufficient to prove them under the hypothesis of Proposition 2.1.

Proposition 1.1 is simpler than Theorem 1.1 and follows only from (15). This inequality together with Theorem 2.2 applied with $a = (1 + r)/2, b = 0$, implies that for each $\gamma < 1, r < 1$,

$$
|\tilde{u}|_{L^{\rho^\gamma}(\Omega)} \leq C |\tilde{u}|_{L^{\gamma}(\Omega)} , \quad \rho = \frac{n}{n-1+r}
$$

as long as the right-hand side is finite.

We can assume $p \geq 1$. Fix $r < 1$ so small that

$$
\frac{n}{n-1+r} = \frac{1}{2} \left( p + \frac{n}{n-1} \right)
$$
Fix $k$ such that
\[
\left( \frac{n}{n-1+r} \right)^k \varepsilon_0 > 1
\]
and then $\gamma_0 < \varepsilon_0$ such that
\[
\left( \frac{n}{n-1+r} \right)^k \gamma_0 = \delta_0,
\]
where $\delta_0 < 1$ is so close to 1 that
\[
\frac{n}{n-1+r} \delta_0 = \frac{\delta_0}{2} \left( p + \frac{n}{n-1} \right) > p.
\]
With these choices, $k+1$ iterations starting from $\gamma_0$ give the result.

We now prove Theorem 1.2. Fix $t \in (0,1)$. We use the following well-known inequality, valid for bounded weak Sobolev supersolutions of (2). For each $\eta \in C_0^1(\Omega)$, by testing (2) with $\eta^2 \hat{u}^{t-1}$,
\[
\int_\Omega \eta^2 \hat{u}^{t-2} |\nabla u|^2 \leq C \int_\Omega (\eta^2 + |\nabla \eta|^2) \hat{u}^{t},
\]
where $C$ is bounded in terms of positive lower and upper bounds for $1 - t$. This follows from the computation on pages 195-196 in [11], in particular inequalities (8.52)-(8.53) with $\beta = t - 1$ there.

By density, the same inequality is valid for any $\eta \in H_0^1(\Omega)$. Since $d^r$ has square-integrable gradient close to the boundary for each $r > 1/2$, we can apply (25) with
\[
\eta = \psi^{1-t} \leq C d^{1-t/2}.
\]
Then
\[
\eta^2 + |\nabla \eta|^2 \leq C d^{-t},
\]
so (25) becomes
\[
\int_\Omega \eta^2 \hat{u}^{t-2} |\nabla u|^2 \leq C \int_\Omega \left( \frac{\hat{u}}{d} \right)^t,
\]
and we obtain by applying the Hölder inequality
\[
\int_\Omega |\nabla u|^t = \int_\Omega \left( \eta^t \hat{u}^{(2-t)/2} |\nabla u|^t \right) \left( \eta^{-t} \hat{u}^{(2-t)/2} \right)
\leq \left( \int_\Omega \eta^2 \hat{u}^{t-2} |\nabla u|^2 \right)^{t/2} \left( \int_\Omega \eta^{-(2-t)/2} \hat{u}^{(2-t)/2} \right)
\leq C \int_\Omega \left( \frac{\hat{u}}{d} \right)^t \leq C \left( \int_\Omega \left( \frac{u}{d} \right)^t + \|f\|_{L^q(\Omega)} \right).
\]
We conclude by Theorem 1.1.

In the end we recall, for completeness, that in case $u$ is a solution, rather than just a supersolution, the gradient of $u$ is bounded pointwise by the quantity $u/d$. By standard elliptic regularity we know that $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$, with the gradient estimate

$$\sup_K |\nabla u| \leq C \left( \sup_{K'} u + \|f\|_{L^q(K')} \right),$$

for each $K \subset K' \subset \Omega$, with $C$ depending of course on $K$, $K'$. By the Harnack inequality

$$\sup_K |\nabla u| \leq C \left( \inf_{K'} u + \|f\|_{L^q(\Omega)} \right).$$

(26)

Fix $x_0 \in \Omega'$ and $d = d(x_0)$. We apply (26) to the function $\tilde{u}(x) = u(x_0 + dx)$, which satisfies the same equation with $b$ replaced by $db$ and $f$ replaced by $d^2f$ (but $d \leq 1$), and with $K = B_{1/2}(0)$, $K' = B_{3/4}(0)$, $\Omega = B_1(0)$. We deduce

$$d(x_0)|\nabla u(x_0)| = |\nabla \tilde{u}(0)| \leq C \left( \tilde{u}(0) + \|f\|_{L^q(\Omega)} \right) = C \left( u(x_0) + \|f\|_{L^q(\Omega)} \right).$$

By (26) with $K = \Omega \setminus \Omega'$, the same inequality is valid for any $x_0 \in \Omega$ with $d(x_0) \geq 1$. Thus

$$|\nabla u|^\varepsilon \leq C \left( \frac{|u|^\varepsilon}{d} + \frac{\|f\|_{L^q(\Omega)}^\varepsilon}{d^\varepsilon} \right) \quad \text{in } \Omega,$$

which is another way to infer Theorem 1.2 from Theorem 1.1, if we have a solution.

The last argument, combined with [25, Theorem 1.2], also shows that a nonnegative supersolution $u$ of a general non-divergence form inequality as in [25] is such that $|\nabla u|\varepsilon \in L^1(\Omega)$, for some $\varepsilon > 0$. This observation complements the results in [25].

3 Appendix

Here we record some essentially known facts about weak supersolutions.

First, we show that bounded weak Sobolev supersolutions are lower semi-continuous functions (after redefinition on a set of measure zero) which satisfy the definition of a viscosity supersolution. Let $A(x)$ be a bounded uniformly
positive matrix in $\Omega$ (for this we do not need any regularity for $A$) and let $b \in L^q_{\text{loc}}(\Omega)$, $f \in L^{q/2}_{\text{loc}}(\Omega)$, $q > n$, and $v \in H^1_{\text{loc}}(\Omega)$ be a bounded weak Sobolev solution of

$$-Lv = -\text{div}(A(x)Dv) + b(x)\nabla v \geq f(x), \quad (27)$$

Given $B_{2r_0} = B_{2r_0}(x_0) \subset \Omega$ and $r < r_0$ we define

$$m(r) = \inf_{B_r} v$$

(we recall that $\inf$ stands for essential infimum). By the interior weak Harnack inequality (Theorem 8.18 in GT) we have

$$0 \leq \frac{1}{|B_{2r}|} \int_{B_{2r}} (v(x) - m(2r)) \, dx$$

$$\leq C(m(r) - m(2r)) + Cr^{2(1-n/q)} \|f\|_{L^{q/2}(B_{r_0})}.$$

Since $m(r)$ is bounded and monotone, and $q > n$, the latter quantity tends to zero as $r \to 0$. Hence

$$\liminf_{x \to x_0} v(x) = \lim_{r \to 0} m(2r) = \lim_{r \to 0} \frac{1}{|B_{2r}|} \int_{B_{2r}(x_0)} v(x) \, dx.$$

The latter limit is $v(x_0)$ for almost every $x_0$, by the Lebesgue differentiation theorem. But the quantity $\liminf_{x \to x_0} v(x)$ is always lower semi-continuous in $x_0$, for any $v$.

Assume now that $v$ does not satisfy the definition of a $C$-viscosity or $L^{q/2}$-viscosity supersolution of (27) in $\Omega$ (for more details on the first of these two notions we refer to [9], for the second to [8, Definition 2.1]). This means there exist a ball $B = B_{2r_0}(x_0) \subset \Omega$ and a function $\psi \in W^{2,q/2}(B)$ (note $W^{2,q/2}(B) \subset H^1(B) \cap C(\overline{B})$) such that $\psi$ touches $v$ from below at $x_0$, but for some $\delta > 0$

$$-L\psi \leq f - \delta \quad \text{a.e. in } B.$$

Let $\theta \in H^1_0(B)$ be the unique solution (see [11, Theorem 8.3], and [28] where equations with unbounded coefficients are treated) of

$$-L\theta = \delta \quad \text{in } B, \quad \theta = 0 \quad \text{on } \partial B.$$

By de Giorgi’s classical result ([11, Theorem 8.24]) or by regularity we know that $\theta$ is continuous in $B$. By the weak maximum principle ([11, Theorem 8.1]) and the interior Harnack inequality ([11, Theorem 8.18]) we have $\theta > 0$ in $B = B_{2r_0}$. Hence $\theta \geq \theta_0 > 0$ in $B_{r_0}$ for some positive constant $\theta_0$. 20
Thus, the function \( w = v - \psi - \theta \) is such that \( w(x_0) \leq -\theta_0 \) and \( w \) satisfies in the weak Sobolev sense

\[-Lw \geq 0 \text{ in } B, \quad w \geq 0 \text{ on } \partial B.\]

By the maximum principle ([11, Theorem 8.1]) \( w \geq 0 \) in \( B \), a contradiction.

The notions of \( C \)-viscosity and \( L^p \)-viscosity solutions are coherent with each other, see [8]. That potential theory supersolutions coincide with viscosity solutions is a simple exercise – if a function is not one, then it is not the other, as we easily prove with the help of a function like \( \theta \) above.

For possible further reference we note that in the definition of a viscosity solution the minimum at \( x_0 \) of \( v - \psi \) can be assumed to be strict. For a full proof of this fact for equations with unbounded ingredients we refer to [20, Lemma 2.10].

Finally, it is also known that any locally bounded function which satisfies the comparison principle with respect to regular subsolutions (viscosity supersolutions have this property) belongs to \( H^1_{\text{loc}}(\Omega) \) and is a weak Sobolev supersolution. See for instance [14, Theorem 2], and [14, Proposition 14]. For more general \( p \)-laplacian like operators we refer to the book [12].

References

[1] H. Aikawa, Integrability of superharmonic functions in a John domain, Proc. Amer. Math. Soc., 128 (2000), no. 1, 195-201.

[2] R. Alvarado, D. Brigham, V. Maz’ya, M. Mitrea, E. Ziad, On the regularity of domains satisfying a uniform hour-glass condition and a sharp version of the Hopf-Oleinik boundary point principle. Problems in mathematical analysis. No. 57. J. Math. Sci. (N.Y.) 176 (2011), no. 3, 281-360.

[3] D. E. Apushkinskaya, A. I. Nazarov A counterexample to the Hopf-Oleinik lemma (elliptic case) Anal. PDE 9 (2016) 439-458

[4] D. E. Apushkinskaya, A. I. Nazarov On the Boundary Point Principle for divergence-type equations, preprint, arXiv:1802.09636

[5] D. H. Armitage, On the global integrability of superharmonic functions in balls, J. London Math. Soc. (2) 4 (1971), 365-373.

[6] D. H. Armitage, Further result on in the global integrability of superharmonic functions, J. London Math. Soc. (2) 6 (1972), 109-121.
[7] J. Braga, D. Moreira, Inhomogeneous Hopf-Oleinik Lemma and regularity of semiconvex supersolutions via new barriers for the Pucci extremal operators, Adv. Math. 334 (2018), 184-242.

[8] L. Caffarelli, M.G. Crandall, M. Kocan, A. Wiech, On viscosity solutions of fully nonlinear equations with measurable ingredients. Comm. Pure Appl. Math. 49 (1996), 365-97.

[9] M.G. Crandall, H. Ishii, P.-L. Lions, Users Guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) 27 (1992), 1-67.

[10] S. Filippas, V. Mazya, A. Tertikas, Critical Hardy-Sobolev inequalities. J. Math. Pures Appl. (9) 87 (2007), no. 1, 37-56.

[11] D. Gilbarg, N. Trudinger, Elliptic partial differential equation of second order, 2nd ed., Springer-verlag 1983.

[12] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[13] Q. Han, F. Lin, Elliptic partial differential equations. Second edition. Courant Lecture Notes in Mathematics, New York; American Mathematical Society, Providence, RI, 2011.

[14] R.-M. Hervé, M. Hervé, Les fonctions surharmoniques associes un opéra- teur elliptique du second ordre coefficients discontinus. (French. English summary) Ann. Inst. Fourier (Grenoble) 19 (1969) fasc. 1, 305-359.

[15] T. Kilpeläinen, P. Koskela, Global integrability of the gradients of solutions to partial differential equations. Nonlinear Anal. 23 (1994), no. 7, 899-909.

[16] Ü. Kuran, On positive superharmonic functions in $\alpha$-admissible domains. J. London Math. Soc. (2) 29 (1984), no. 2, 269-275.

[17] P. Lindqvist, Global integrability and degenerate quasilinear elliptic equations. J. Anal. Math. 61 (1993), 283-292.

[18] F.-Y. Maeda, N. Suzuki, The integrability of superharmonic functions on Lipschitz domains, Bull. London Math. Soc., 21 (1989), no. 3, 270-278.

[19] A. Nazarov, A centennial of the Zaremba-Hopf-Oleinik lemma. SIAM J. Math. Anal. 44 (2012), no. 1, 437-453.
[20] G. Nornberg, Methods of the regularity theory in the study of partial differential equations with natural growth in the gradient, Ph.D. thesis, PUC-Rio (2018).

[21] P. Pucci, J. Serrin, The maximum principle. Progress in Nonlinear Differential Equations and their Applications, 73. Birkhuser Verlag, Basel, 2007. x+235 pp

[22] M. V. Safonov Non-divergence elliptic equations of second order with unbounded drift. Trans. of the AMS 2010, 29(2):211-232.

[23] M. V. Safonov On the boundary estimates for second-order elliptic equations, Complex Variables and Elliptic Equations, 63:7-8 (2018), 1123-1141.

[24] Saari, Olli Parabolic BMO and global integrability of supersolutions to doubly nonlinear parabolic equations. Rev. Mat. Iberoam. 32 (2016), no. 3, 1001-1018.

[25] B. Sirakov, Boundary Harnack Estimates and Quantitative Strong Maximum Principles for Uniformly Elliptic PDE, International Mathematics Research Notices, rnx107, https://doi.org/10.1093/imrn/rnx107

[26] Ph. Souplet, A priori estimates and bifurcation of solutions for an elliptic equation with semidefinite critical growth in the gradient. Nonlinear Anal. 121 (2015), 412-423.

[27] D.A. Stegenga and D.C.Ullrich, Superharmonic functions on Holder domains, Rocky Mountain Journal Math., 25 (1995), 1539-1556.

[28] N. Trudinger, Linear elliptic operators with measurable coefficients. Ann. Scuola Norm. Sup. Pisa (3) 27 (1973), 265-308.