The Differential Entropy of Mixtures: New Bounds and Applications

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Abstract

Mixture distributions are extensively used as a modeling tool in diverse areas from machine learning to communications engineering to physics, and obtaining bounds on the entropy of probability distributions is of fundamental importance in many of these applications. This article provides sharp bounds on the entropy concavity deficit, which is the difference between the entropy of the mixture and the weighted sum of entropies of constituent components. Toward establishing lower and upper bounds on the concavity deficit, results that are of importance in their own right are obtained. In order to obtain nontrivial upper bounds, properties of the skew-divergence are developed and notions of “skew” $f$-divergences are introduced; a reverse Pinsker inequality and a bound on Jensen-Shannon divergence are obtained along the way. Complementary lower bounds are derived with special attention paid to the case that corresponds to independent summation of a continuous and a discrete random variable. Several applications of the bounds are delineated, including to mutual information of additive noise channels, thermodynamics of computation, and functional inequalities.

I. INTRODUCTION

Mixture models are extensively employed in diverse disciplines including genetics, biology, medicine, economics, speech recognition, as the distribution of a signal at the receiver of a communication channel when the transmitter sends a random variable taking values in the form

$$p_Z(z) = \sum_i p_i f_i(z),$$

where $f_i$ is a probability density function and each $p_i$ is a nonnegative weight with $\sum_i p_i = 1$. Such mixture densities have a natural probabilistic meaning as outcomes of a two stage random process with the first stage being a random draw, $i$, from the probability mass function $p$, followed by choosing a real-valued vector $x$ following a distribution $f_i(\cdot)$; equivalently, it is the density of $X + Z$, where $X$ is a discrete random variable taking values $x_i$ with probabilities $p_i$, and $Z$ is a dependent variable such that $P(Z \in A | X = x_i) = \int_A f_i(z - x_i) dz$. The differential entropy of this mixture is of significant interest.

Our original motivation for this paper came from the fundamental study of thermodynamics of computation, in which memory models are well approximated by mixture models, while the erasure of a bit of information is akin to the state described by a single unimodal density. Of fundamental importance here is the entropy of the mixture model which is used to estimate the thermodynamic change in entropy in an erasure process and for other computations \cite{31, 57}. It is not possible, analytically, to determine the differential entropy of the mixture model $\sum_i p_i f_i$, even in the simplest case where $f_i$ are normal distributions, and hence one is interested in refined bounds on the same. While this was our original motivation, the results of this paper are more broadly applicable and we strive to give general statements so as not to limit the applicability.

For a random vector $Z$ taking values in $\mathbb{R}^d$ with probability density density $f$, the differential entropy is defined as $h(Z) = h(f) = -\int_{\mathbb{R}^d} f(z) \ln f(z) dz$, where the integral is taken with respect to Lebesgue measure. We will frequently omit the qualifier “differential” when this is obvious from context and simply call it the entropy. It is to be noted that, unlike $h(\sum_i p_i f_i)$, the quantity $\sum_i p_i h(f_i)$ is more readily determinable and thus the concavity deficit $h(\sum_i p_i f_i) - \sum_i p_i h(f_i)$ is of interest. This quantity can also be interpreted as a generalization of the Jensen-Shannon divergence \cite{5}, and its quantum analog (with density functions replaced by density matrices and Shannon entropy replaced by von Neumann entropy) is the Holevo information, which plays a key role in Holevo’s theorem bounding the amount of accessible (classical) information in a quantum state \cite{26}.

It is a classical fact going back to the origins of information theory that the entropy $h$ is a concave function, which means that the concavity deficit is always nonnegative:

$$h(f) - \sum_i p_i h(f_i) \geq 0. \quad (1)$$

Let $X$ be a random variable that takes values in a countable set where $X = x_i$ with probability $p_i$. Intimately related to the entropy $h$ of a mixture distribution $f = \sum_i p_i f_i$ and the concavity deficit are the quantities $H(p) := -\sum_i p_i \log p_i$, and the conditional entropies $h(Z | X)$ and $H(X | Z)$. Indeed, it is easy to show an upper bound on the concavity deficit (see, e.g., \cite{64}) in the form

$$h(f) - \sum_i p_i h(f_i) \leq H(p), \quad (2)$$
which relates the entropy of continuous variable Z with density \( f \) to the entropy of a random variable that lies in a countable set. A main thrust of this article will be to provide refined upper and lower bounds on the concavity deficit that improve upon the basic bounds (1) and (2).

The main upper bound we establish is inspired by bounds in the quantum setting developed by Audenaert [3] and utilizes the total variation distance. Given two probability densities \( f_1 \) and \( f_2 \) with respect to a common measure \( \mu \), the total variation distance between them is defined as \( \|f_1 - f_2\|_{TV} = \frac{1}{2} \int |f_1 - f_2| d\mu \).

We will state the following theorem in terms of the usual differential entropy, on Euclidean space with respect to the Lebesgue measure. Within the article, the statements and proofs will be given for a general Polish measure space \((E, \gamma)\) from which the result below can be recovered as a special case.

**Theorem I.1.** Suppose \( f = \sum_i p_i f_i \), where \( f_i \) are probability density functions on \( \mathbb{R}^d \), \( p_i \geq 0 \), \( \sum_i p_i = 1 \). Define the mixture complement of \( f \) by \( \tilde{f}(z) = \sum_{i \notin j} \frac{p_i}{1-p_i} f_i \). Then

\[
\frac{1}{2} \|f_1 - \tilde{f}_1\|_{TV} \leq \mathcal{T}_f \, \text{H}(p)
\]

where

\[
\mathcal{T}_f := \sup_i \|f_i - \tilde{f}_i\|_{TV}.
\]

Theorem I.1 shows that as distributions cluster in total variation distance, the concavity deficit vanishes. The above result thus considerably reduces the conservativeness of the upper bound on the concavity deficit given by (2). Indeed consider the following example with \( f_1(z) = e^{-(z-a)^2/2}/\sqrt{2\pi} \) and \( f_2(z) = e^{-(z+a)^2/2}/\sqrt{2\pi} \). By (2), for \( p \in (0,1) \),

\[
\frac{1}{2} \|f_1 - f_2\|_{TV} \leq \text{H}(p) + \frac{1}{2} \log 2\pi e.
\]

However, noting that, \( \tilde{f}_1 = f_2 \) and \( \tilde{f}_2 = f_1 \) implies,

\[
\mathcal{T}_f = \|f_1 - f_2\|_{TV} = 2 \int_{-\infty}^{\infty} \left( e^{-(z-a)^2/2}/\sqrt{2\pi} - e^{-(z+a)^2/2}/\sqrt{2\pi} \right) dz = \Phi(a) - \Phi(-a),
\]

where \( \Phi \) is the standard normal distribution function \( \Phi(t) := \int_{-\infty}^{t} e^{x^2/2}/\sqrt{2\pi} dx \). Since \( \Phi(a) - \Phi(-a) \leq a \sqrt{2/\pi} \), Theorem I.1 gives

\[
\frac{1}{2} \|f_1 - f_2\|_{TV} \leq a \sqrt{2/\pi} \text{H}(p) + \frac{1}{2} \log 2\pi e.
\]

Another interpretation of Theorem I.1 is as a generalization of the classical bounds on the Jensen-Shannon divergence by the total variation distance [35], [59], which is recovered by taking \( p_1 = p_2 = \frac{1}{2} \), see Corollary III.11. Let us point out that \( \Phi(a) - \Phi(-a) = 1 - \mathbb{P}(|Z| > a) \) where \( Z \) is a standard normal variable, thus an alternative representation of these bounds in this special case is as mutual information bounds

\[
I(X; Z) \leq \text{H}(X) - \mathbb{P}(|Z| > a) \text{H}(X)
\]

where \( X \) denotes an independent Bernoulli taking the values \( \pm a \) with probability \( p \) and \( 1-p \).

The methods and technical development toward establishing Theorem I.1 are of independent interest. We develop a notion of skew divergence for general f-divergences generalizing the skew divergence (or skew relative entropy) introduced by Lee [32], and in Theorem III.1 show that the class of f-divergences is stable under the skew operation. After proving elementary properties of the skew entropy in Proposition III.3 and an introduced skew chi-squared divergence in Proposition III.8, we adapt arguments due to Audenaert [3] from the quantum setting to prove the two f-divergences to be intertwined through a differential equality and that the classical upper bound of the relative entropy by the chi-square divergence can be generalized to the skew setting (see Theorem III.2). After further bounding the skew chi-square divergence by the total variation distance, we integrate the differential equality and obtain a bound of the skew divergence by the total variation in Theorem III.3. As a corollary we obtain a reverse Pinsker inequality due to Verdu [62]. With these tools in hand, Theorem I.1 is proven and we demonstrate that the bound of the Jensen-Shannon divergence by total variation distance [35], [59] is an immediate special case.

In the converse direction that provides lower bounds on the concavity deficit, our main result applies to the case where all the component densities come from perturbations of a random vector W in \( \mathbb{R}^d \) that has a log-concave and spherically symmetric distribution. We say a random vector W has a log-concave distribution when it possesses a density \( \varphi \) satisfying \( \varphi((1-t)z + ty) \geq \varphi^1(z) \varphi^t(y) \). We say W has a spherically symmetric distribution when there exists \( \psi : \mathbb{R} \rightarrow [0, \infty) \) such that the density \( \varphi(z) = \psi(|z|) \) for every \( z \in \mathbb{R}^d \), where \( |z| := \sqrt{z_1^2 + z_2^2 + \ldots + z_d^2} \). We employ the notation \( B_\lambda = \{x \in \mathbb{R}^d : \|x\| \leq \lambda\} \).
$|x| \leq \lambda$ for the centered closed ball of radius $\lambda$ in $\mathbb{R}^d$, $\mathcal{T}(t) := \mathcal{T}_W(t) := \mathbb{P}(|W| > t)$ for the tail probability of $W$, and $|A| := \sup_{w \in A} \|Aw\|_2$ for the operator norm of a matrix $A : \mathbb{R}^d \to \mathbb{R}^d$.

**Theorem I.2.** Suppose that there exists $\tau \geq 1$ such that for each $x \in X$, $Z|X = x$ has distribution given by $T_x(W)$ where $W$ has density $\varphi$, spherically symmetric and log-concave and $T_x$ is a $\sqrt{\tau}$ bi-Lipschitz function. For $i, k \in X$, take $T_{ij} := T_i^{-1} \circ T_j$ and further assume there exists $\lambda > 0$ such that for any $k \neq i$ $T_{ij}(B_\lambda) \cap T_{kj}(B_\lambda) = \emptyset$. Then

$$h(Z) - h(Z|X) \geq H(X) - \tilde{C}(W),$$

where $\tilde{C}$ is the following function dependent heavily on the tail behavior of $|W|$, 

$$\tilde{C}(W) = \mathcal{T}(\lambda)(1 + h(W)) + \mathcal{T}^{1/2}(\lambda)(\sqrt{d} + K(\varphi))$$

with

$$K(\varphi) := \log \left[ \tau^d \left( \|\varphi\|_\infty + \left( \frac{3}{\lambda} \right) \omega_d^{-1} \right) \right] \mathcal{T}^{1/2}(\lambda) \log \left[ 1 + \tau + \frac{\tau^2|w|}{\lambda} \right] dw$$

where $\omega_d$ denoting the volume of the $d$-dimensional unit ball, $B_\lambda^\circ$ denotes the complement of $B_\lambda \in \mathbb{R}^d$.

We note the quantity $H(X|Z)$ connotes the uncertainty in the discrete variable $X$ conditioned on the continuous variable $Z$; such a quantity needs to be defined/determined from the knowledge of probabilities, $p_i$, that the discrete variable $X = x_i$ and the description of the conditional probability density function $p(z|x_i) = f_i(z)$. These notions are made precise in Section II. Here, it is also established that (2) can be equivalently formulated as $H(X|Z) \leq H(X)$ for a particular coupling of a discrete variable $X$ taking values with probabilities $\{p_i\}$, and a variable $Z$ with density $f_i$ when conditioned on $X = x_i$. From this perspective the super-concavity bound of Theorem IV.1 gives $H(X|Z) \leq \tilde{C}(W)$. One should also note that when $\{p_i\}_{i=1}^n$ is a finite sequence, the classical bounds on $H(X|Z)$ are provided by Fano’s inequality: for a Markov triple of random variables $X \to Z \to \hat{X}$, and $e = \{X \neq \hat{X}\}$,

$$H(X|Z) \leq H(e) + \mathbb{P}(e) \log(\#\mathcal{X} - 1),$$

where $\#\mathcal{X}$ denotes the cardinality of the set $\mathcal{X}$, gives a strengthening of concavity in many situations, where we have employed the notation for a measurable set $A$, $H(A) = -\mathbb{P}(A) \log \mathbb{P}(A) - (1 - \mathbb{P}(A)) \log(1 - \mathbb{P}(A))$. It yields

$$h(f) \geq \sum_{i=1}^n p_i h(f_i) + H(p) - (H(e) + \mathbb{P}(e) \log(\#\mathcal{X} - 1)).$$

To compare the strength of the bounds derived in Theorem IV.1 to Fano’s we compare $H(e) + \mathbb{P}(e) \log(\#\mathcal{X} - 1)$ and $\tilde{C}(W)$; as is established in Section V even in simple cases, $\tilde{C}(W)$ can be arbitrarily small even while $\min_{x \in \mathcal{X}} H(e) + \mathbb{P}(e) \log(\#\mathcal{X} - 1)$ is arbitrarily large.

The study of entropy of mixtures has a long history and is scattered in a variety of papers that often have other primary emphases. Consequently it is difficult to exhaustively review all related work. Nonetheless, the references that we were able to find that attempt to obtain refined bounds under various circumstances are [1], [8], [7], [9], [42], [46]. In all of these papers, however, either the bounds deal with specialized situations, or with a general setup but employing different (and typically far more) information than we require. We emphasize that our bounds deal with general multidimensional situations (including in particular multivariate Gaussian mixtures, which are historically and practically of high interest) and in that sense go beyond the previous literature.

The article is organized as follows. In Section II we will give notation and preliminaries, where we delineate definitions and relationship for entropies, conditional entropies emphasizing a mix of continuous and discrete variables. Section III is devoted to the proof of Theorem I.1 (a preliminary version has appeared in the conference paper [40]). In Section IV we prove Theorem IV.1. These results give a considerable generalization of earlier work of authors in [39], [50]. The result will hinge on a Lemma IV.2 bounding the sum $\sum \varphi(x_i)$ for $x_i$ well spaced and $\varphi$ log-concave and spherically symmetric, and a concentration result from convex geometry [22]. We close discussing bounds of $\mathbb{P}(|W| \geq t)$ in the case that $W$ is log-concave and strongly log-concave, see Corollary IV.9. Section V we demonstrate applications of the theorems to a diverse group of problems; hypothesis testing, capacity estimation, nanoscale energetics, and functional inequalities.

**II. Notation and Preliminaries**

In this part of the article, we will elucidate definitions and results for conditional entropy and mutual information when a mix of discrete valued and continuous random variables are involved. We will assume that

1) $X$ takes values in a discrete countable set $\mathcal{X}$, and that $\mathbb{P}(X = x) = p_x > 0$. 
2) \( Z \) is a random variable which takes values in a Polish space \( E \). The conditional distribution is described by \( \mathbb{P}(Z \in A | X = x) = \int_A f_x(z) d\gamma(x) \) where \( f_x(z) \) is a density function with respect to a Radon reference measure \( \gamma \).

We will denote by \( m \) the counting measure on \( \mathcal{X} \), so that for \( A \subseteq \mathcal{X} \), the measure of \( A \) is its cardinality,

\[
m(A) = \#(A). \tag{10}\]

Integration with respect to the counting measure, corresponding to summation; for \( g : \mathcal{X} \to \mathbb{R} \) such that \( \sum_{x \in \mathcal{X}} |g(x)| < \infty \),

\[
\int_{\mathcal{X}} g(x) dm := \sum_{x \in \mathcal{X}} g(x). \tag{11}\]

We will denote by \( dm \, d\gamma \) the product measure on \( \mathcal{X} \times E \) where, for \( A \subseteq \mathcal{X} \) and measurable \( B \subseteq \mathbb{R}^d \),

\[
\int_{\mathcal{X} \times E} \mathbb{1}_{A \times B}(x, z) dm(x) \, d\gamma(z) := m(A) \gamma(B), \tag{12}\]

when \( \gamma \) denotes the \( d \)-dimensional Lebesgue measure, we will use \( |B|_d \) or \( |B| \) when there is no risk of confusion to denote the Lebesgue volume of a measurable set \( B \). For measures \( \mathbb{P} \) and \( \mathbb{Q} \) on a shared measure space such that \( \mathbb{P} \) has a density \( \varphi \), with respect to \( \mathbb{Q} \), when any measurable set \( A \) satisfies,

\[
\mathbb{P}(A) = \int_A \varphi \, d\mathbb{Q}. \tag{13}\]

Such a \( \varphi \) will also be written as \( \frac{d\mathbb{P}}{d\mathbb{Q}} \).

For random variables \( U \) and \( V \) whose induced probability measures admit densities with respect to a reference measure \( \gamma \), in the sense that \( \mu(A) := \mathbb{P}(U \in A) = \int_A u \, d\gamma \) and \( \nu(A) := \mathbb{P}(V \in A) = \int_A v \, d\gamma \) where \( u \) and \( v \) are density functions with respect to \( \gamma \), the relative entropy (or KL divergence) is defined as

\[
D(U||V) := D(\mu||\nu) := D(u||v) := \int u \log \frac{u}{v} \, d\gamma. \tag{14}\]

When \( U \) has an \( E \) valued random variable density \( u \) with respect to a reference measure \( \gamma \), denote the entropy

\[
h\gamma(U) := h\gamma(u) = -\int u(z) \log u(z) d\gamma(z), \tag{15}\]

whenever the above integral is well defined. When \( \gamma \) is the Lebesgue measure we denote the usual differential entropy,

\[
h(U) := h(u) := -\int u(x) \log u(x) \, dx, \tag{16}\]

When \( U \) is discrete, taking values \( x \subseteq \mathcal{X} \) with probability \( p_x \), define

\[
H(U) := H(p) := -\sum_{x \in \mathcal{X}} p_x \log p_x. \tag{17}\]

When \( t \in [0,1] \), we define \( H(t) \) to the the entropy of a Bernoulli random variable with parameter \( t \), \( H(t) := -(1-t) \log(1-t) - t \log t \). When \( A \) is an event, \( H(A) := H(\mathbb{P}(A)) \).

The following proposition elucidates notions of conditional entropy and joint entropy of a mix of discrete and continuous random variables.

**Proposition II.1.** Suppose \( X \) is a discrete random variable with values in a countable set \( \mathcal{X} \) and \( Z \) is a Borel measurable random variable taking values in \( E \). Suppose, for all \( x \in \mathcal{X} \), \( \mathbb{P}(Z \in A | X = x) = \int_A f_x(z) d\gamma(z) \) for density \( f_x(z) \) with respect to a common reference measure \( \gamma \). Then the following hold.

- The joint distribution of \((X,Z)\) on \( \mathcal{X} \times E \), has a density

  \[
  F(x,z) = p_x f_x(z) \tag{18}\]

  with respect to \( dm \, d\gamma \).

- \( Z \) has a density

  \[
  f(z) = \sum_{x \in \mathcal{X}} p_x f_x(z) \tag{19}\]

  with respect to \( \gamma \) on \( E \).

- The conditional density of \( X \) with respect to \( Z = z \) defined as

  \[
  p(x|z) = \begin{cases} \frac{p_x f_x(z)}{f(z)} & \text{for } f(z) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{20}\]
satisfies $\mathbb{P}(X = x) = \int_E p(x|z)f(z)dz = p_x$.

Proof. Note that since a set $A \subseteq X \times E$ can be decomposed into a countable union of disjoint sets $\{x\} \times A_x$ where $A_x = \{z \in E: (x, z) \in A\}$, to prove $F(x, z)$ is the joint density function of $(X, Z)$, it suffices to prove

$$\mathbb{P}_{XZ}(\{x\} \times A) = \int_{\{x\} \times A} F(x, z)dm(x) \, d\gamma(z). \quad (21)$$

Indeed,

$$\mathbb{P}_{XZ}(A) = \mathbb{P}_{XZ}(\cup_x \{x\} \times A_x) = \sum_x \mathbb{P}_{XZ}(\{x\} \times A_x), \quad (22)$$

while

$$\int_A F(x, z)dm(x) \, d\gamma(z) = \int_{\cup_x \{x\} \times A_x} F(x, z)dm(x) \, d\gamma(z) = \sum_x \int_{\{x\} \times A_x} F(x, z)dm(x) \, d\gamma(z). \quad (24)$$

Since $(21)$ would give equality of the summands of $(23)$ and $(25)$ the result would follow. We compute directly.

$$\mathbb{P}_{XZ}(\{x\} \times A) = \mathbb{P}(X = x, Z \in A) \quad (26)$$

$$= \mathbb{P}(X = x)\mathbb{P}(Z \in A|X = x) \quad (27)$$

$$= p_x \int_A f_x(z)d\gamma(z) \quad (28)$$

$$= \int_{\{x\}} p_x \int_A f_x(z)d\gamma(z)dm(x) \quad (29)$$

$$= \int_{\{x\} \times A} F(x, z)dm(x) \, d\gamma(z). \quad (30)$$

This gives the first claim. For the second,

$$\mathbb{P}(Z \in A) = \sum_{x \in X} \mathbb{P}(X = x, Z \in A) \quad (31)$$

$$= \sum_{x \in X} p_x \int_A f_x(z)d\gamma(z) \quad (32)$$

$$= \int_A f(z)d\gamma(z). \quad (33)$$

The last assertion is immediate,

$$\int_E p(z|x)f(z)d\gamma(z) = \int_E \frac{p_x f_x(z)}{f(z)}f(z)d\gamma(z) = p_x. \quad (34)$$

Proposition II.1 allows the following definitions of conditional entropies.

1. $h_\gamma(Z|X = x) = -\int f(Z|X = x) \log f(Z|X = x) = -\int f_x \log f_x d\gamma(z)$ and thus

$$h(Z|X) := E_x[h(Z|X = x)] = -\sum_{x \in X} p_x \int f_x(z) \log f_x(z)d\gamma(z) = \sum_{x \in X} p_x h_\gamma(f_x). \quad (35)$$

2. $H(X|Z) = -\sum_{x \in X} p(X = x|Z = z) \log p(X = x|Z = z) = -\sum_{x \in X} p(x|z) \log p(x|z) = H(p|\cdot|z)$ and thus

$$H(X|Z) = E_Z[H(X|Z = z)] := -\int_E \left(\sum_{x \in X} p(x|z) \log p(x|z)\right) f(z)d\gamma(z). \quad (36)$$

Let us note how the entropy of a mixture can be related to its relative entropy with respect to a dominating distribution $g$. The entropy concavity deficit of a convex combination of densities $f_i$, is the convexity deficit of the relative entropy with respect to a reference measure in the following sense.

**Proposition II.2.** For a density $g$ such that $\sum_x p_x D(f_x||g) < \infty$,

$$h_\gamma(f) - \sum x p_x h_\gamma(f_x) = \sum x p_x D(f_x||g) - D(f||g). \quad (37)$$
Proof.

\[
\sum_x p_x D(f_x||g) - D(f||g) = \sum_x p_x \left( \int f_x \log \frac{f_x}{g} - f_x \log \frac{f}{g} d\gamma \right) \\
= \sum_x p_x \left( \int f_x \log f_x - f_x \log f d\gamma \right) \\
= h_\gamma(f) - \sum_x p_x h_\gamma(f_x).
\]

\[
\sum_x p_x D(f_x||g_1) - D(f||g_1) = \sum_x p_x D(f_x||g_2) - D(f||g_2).
\]

Note that the left hand side of Proposition II.2 is invariant with respect to \(g\). Thus for \(g_1, g_2\) such that \(\sum_x p_x D(f_x||g_j) < \infty\),

\[
\sum_x p_x D(f_x||g_1) - D(f||g_1) = \sum_x p_x D(f_x||g_2) - D(f||g_2).
\]

Taking \(g_1 = g\) and \(g_2 = f = \sum_x p_x f_x\) yields the compensation identity,

\[
\sum_x p_x D(f_x||g) = \sum_x p_x D(f_x||f) + D(f||g),
\]

which is often used to obtain its immediate corollary

\[
\min_g \sum_x p_x D(f_x||g) = \sum_x p_x D(f_x||f).
\]

We define the mutual information between probability measures \(P_U\) and \(P_V\) with joint distribution \(P_{UV}\) and their product distribution \(P_{U/P}\), as the relative entropy of the product distribution from the joint distribution,

\[
I(P_U; P_V) = D(P_{UV}||P_{U/P}).
\]

For the random variables \(U\) and \(V\) inducing probability measures \(P_U\) and \(P_V\), we will write \(I(U; V) = I(P_U; P_V)\).

Proposition II.3. For \(X\) discrete with \(P(X = x) = p_x\) and \(Z\) satisfying \(P(Z \in B|X = x) = \int_B f_x(z)dz\),

\[
I(X; Z) = h_\gamma(Z) - h_\gamma(Z|X) = H(X) - H(X|Z) = \sum_{x \in X} p_x D(f_x||f).
\]

Proof. By Proposition II.1, \(P_{XZ}\) has density \(F(x, z) = p_x f_x(z)\) with respect to \(dm(x) d\gamma(z)\) the product of the counting measure \(m\) and \(\gamma\). The product measure \(P_{XZ}\) has density \(G(x, z) = p_x f(z)\) with respect to \(dm d\gamma\) and it follows that

\[
\frac{dP_{XZ}}{dP_X P_Z}(x, z) = \frac{dP_{XZ}}{dm d\gamma}(x, z) = \frac{F(x, z)}{G(x, z)} = \frac{f_x(z)}{f(z)}.
\]

By equation (44),

\[
D(P_{XZ}||P_X P_Z) = \int_{X \times R^d} F(x, z) \log \frac{f_x(z)}{f(z)} dm d\gamma
\]

Recalling \(p(z|x)\) from Proposition II.1 using the algebra of logarithms and Fubini-Tonelli,

\[
\int_E \sum_{x \in X} p_x f_x(z) \log \frac{f_x(z)}{f(z)} d\gamma(z)
\]

\[
= \int_E \sum_{x \in X} p_x f_x(z) \log \frac{p(z|x)}{p_x} d\gamma(z)
\]

\[
= - \sum_{x \in X} p_x \log p_x \int_E f_x(z) d\gamma(z) + \int_E f(z) \sum_{x \in X} p(x|z) \log p(x|z) d\gamma(z)
\]

\[
= H(p) - \int_{x \in X} f(z) H(p(x|z)) d\gamma(z)
\]

\[
= H(X) - H(X|Z),
\]
we have expression (43). By Proposition II.2,

\[ \int_E \sum_{x \in X} p_x f_x(z) \log \frac{f_x(z)}{f(z)} \, d\gamma(z) = \sum_{x \in X} \int_E f_x(z) \log \frac{f_x(z)}{f(z)} \, d\gamma(z) \]

(52)

\[ = \sum_{x \in X} \rho_x D(f_x||f), \]

(53)

we have expression (43). By Proposition II.2,

\[ \sum_{x \in X} p_x D(f_x||f) = h_\gamma(f) - \sum_{x \in X} p_x h(f_x) \]

(54)

\[ = h(Z) - h(Z|X), \]

(55)

(41) follows.

Using Proposition II.3 we can give a simple information theoretic proof of a result proved analytically in [9], [64].

**Corollary II.4.** When \( X \subseteq E \), and \( \gamma \) is a Haar measure, then \( X \) and \( Z \) satisfy,

\[ h_\gamma(X + Z) \leq H(X) + h_\gamma(Z|X) \]

(56)

which reduces to

\[ h_\gamma(X + Z) \leq H(X) + h_\gamma(Z) \]

(57)

in the case that \( X \) and \( Z \) are independent.

**Proof.** Applying Proposition II.3 to \( X \) and \( Z = X + Z \) we have

\[ h_\gamma(X + Z) = H(X) + h_\gamma(X + Z|X) - H(X|X + Z) \]

(58)

\[ = H(X) + h_\gamma(Z|X) - H(X|X + Z), \]

(59)

where the second equality follows from the assumption that \( \gamma \) is a Haar measure. Since \( H(X|X + Z) \geq 0 \), (56) follows, while (57) follows from \( h_\gamma(X + Z|X) = h_\gamma(Z) \) under the assumption of independence. \( \Box \)

Incidentally, the main use of Corollary II.4 in [64] is to give a rearrangement-based proof of the entropy power inequality (see [37] for much more in this vein).

### III. Lower Bounds

In this section we will provide lower bounds to the concavity deficit and provide a proof of Theorem [41]. We will first introduce the notion of \( f \)-divergences.

**Definition III.1.** For a convex function \( f \) satisfying \( f(1) = 0 \), and probability measures \( \mu \) and \( \nu \), with densities \( u = \frac{d\mu}{d\gamma} \) and \( v = \frac{d\nu}{d\gamma} \) with respect to a common reference measure \( \gamma \), the \( f \)-divergence from \( \mu \) to \( \nu \) is

\[ D_f(\mu||\nu) := D_f(u||v) := \int f \left( \frac{u}{v} \right) \, dv \, d\gamma. \]

(60)

Note that a common reference measure for measures \( \mu \) and \( \nu \) always exists, take \( \frac{1}{2}(\mu + \nu) \) for instance, and the value of \( D_f(\mu||\nu) \) is independent of the choice of reference measure as can be seen by comparing a reference measures to one it has a density with respect to. When the value of a functional of a pair of probability distributions \( \mu, \nu \) is given by (60) we will call the functional an \( f \)-divergence. An \( f \)-divergence satisfies the following. (i) Non-negativity, \( D_f(\mu||\nu) \geq 0 \) and (ii) The map \((\mu, \nu) \mapsto D_f(\mu||\nu)\) is convex. We direct the reader to [34], [50], [51] for further background on \( f \)-divergences and their properties. When \( f(x) = x \log x \), the divergence induced is the relative entropy.

This section is organized as follows. We first introduce the concept of skewing which is the \( f \)-divergence from a convex combination \((1 - t)\mu + tv \) to \( \nu \). Skewing provides a more regular version of the original divergence measure, for example the Radon-Nikodym derivative of \( \mu \) with respect to \((1 - t)\mu + tv \) always exists even if Radon-Nikodym derivative of \( \mu \) with respect to \( \nu \) may not, whereby skew divergence is well defined unlike divergence. Skew divergence as we will develop preserves important features of the original divergence. We first state elementary properties of the skew relative information, corresponding to skewing the relative entropy with proofs given in an appendix, and then introduce a skew \( \chi^2 \)-divergence which interpolates between the well known Neyman \( \chi^2 \) divergence and the Pearson \( \chi^2 \) divergence.

We will pause to demonstrate that the class of \( f \)-divergences is stable under skewing and recover as a special case; a recent result of Nielsen [47], that the generalized Jensen-Shannon divergence is an \( f \)-divergence. Then we establish several inequalities between the skew relative information and the introduced skew \( \chi^2 \)-divergence. We will show in Theorem III.2 that the skew relative information can be controlled by the skew \( \chi^2 \)-square divergence extending the classical bound of relative
entropy by Pearson $\chi^2$ divergence, and using an argument due to Audenart in the quantum setting [3], we show that the rate of decrease of the skew relative information with respect to the skewing parameter can be described exactly as a multiple of the skew $\chi^2$ divergence.

Theorem III.3 also appropriates a quantum argument [3] to show that though neither the Neyman or Pearson divergences can be controlled by total variation, their skewed counterparts can be. We harness this bound along side the differential relationship between the two skew divergences to bound the skew relative entropy by the total variation as well. As a brief aside we demonstrate that the this bound is equivalent to a reverse Pinsker type inequality derived by Verdu [62], before using Theorem III.3 to give our proof of Theorem I.1. Finally to conclude the section, we demonstrate that one may obtain the classical result of Lin [35] bounding the Jensen-Shannon divergence by total variation as a special case of Theorem I.1.

A. Skew Relative Information

We will consider the following generalization of the relative entropy due to Lee.

Definition III.2. [32] For probability measures $\mu$ and $\nu$ on a common set $\mathcal{X}$ and $t \in [0,1]$ define their Skew relative information

$$S_t(\mu||\nu) = \int \log \frac{d\mu}{d(\nu(t\mu + (1-t)\nu))} d\mu$$

In the case that $d\mu = ud\gamma$, and $d\nu = v\gamma$ we will also write

$$S_t(u||v) = S_t(\mu||\nu).$$

We state some important properties of skew relative information with the proofs provided in the Appendix.

Proposition III.3. For probability measures $\mu$ and $\nu$ on a common set and $t \in [0,1]$ the Skew Relative information satisfies the following properties.

1) $S_t(\mu||\nu) = D(\mu||\nu(t\mu + (1-t)\nu))$. In particular, $S_0(\mu||\nu) = D(\mu||\nu)$.
2) $S_t(\mu||\nu) = 0$ iff $t = 1$ or $\mu = \nu$.
3) For $0 < t < 1$ the Radon-Nikodym derivative of $\mu$ with respect to $t\mu + (1-t)\nu$ does exist, and $S_t(\mu||\nu) \leq -\log t$.
4) $S_t(\mu||\nu)$ is convex, non-negative, and decreasing in $t$.
5) $S_t$ is an $f$-divergence with $f(x) = x \log(x/(tx + (1-t)))$.

Motivated by the fact that the act of skewing the relative entropy preserves its status as an $f$-divergence we introduce the act of skewing of an $f$-divergence

Definition III.4. Given a convex function $f : [0, \infty) \to \mathbb{R}$ with $f(1) = 0$ and its associated divergence $D_f(\cdot||\cdot)$, define the $r,t$-skew of $D_f$ by

$$S_{f,r,t}(\mu||\nu) := D_f(r\mu + (1-r)\nu||t\mu + (1-t)\nu).$$

It can be shown that for $t \in (0,1)$, $S_{f,r,t}(\mu||\nu) < \infty$.

Theorem III.1. The class of $f$-divergences is stable under skewing. That is, if $f$ is convex, satisfying $f(1) = 0$, then

$$\hat{f}(x) := (tx + (1-t))f \left( \frac{rx + (1-r)}{tx + (1-t)} \right)$$

is convex with $\hat{f}(1) = 0$ as well, so that the $r,t$ skew of $D_f$ defined in (61) is an $f$-divergence as well.

Proof. If $\mu$ and $\nu$ have respective densities $u$ and $v$ with respect to a reference measure $\gamma$, then $r\mu + (1-r)\nu$ and $t\mu + (1-t)\nu$ have densities $ru + (1-r)v$ and $tu + (1-t)v$

$$S_{f,r,t}(\mu||\nu) = \int f \left( \frac{ru + (1-r)v}{tu + (1-t)v} \right) (tu + (1-t)v) d\gamma$$

(63)

$$= \int f \left( \frac{ru + (1-r)v}{tu + (1-t)v} \right) \left( \frac{u}{v} + (1-t) \right) v d\gamma$$

(64)

$$= \int \hat{f} \left( \frac{u}{v} \right) v d\gamma.$$  (65)

Since $\hat{f}(1) = f(1) = 0$, we need only prove $\hat{f}$ convex. For this, recall that the conic transform $g$ of a convex function $f$ defined by $g(x,y) = yf(x/y)$ for $y > 0$ is convex, since

$$\frac{y_1 + y_2}{2} f \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \frac{y_1 + y_2}{2} f \left( \frac{\frac{y_1}{y_1 + y_2} x_1 + \frac{y_2}{y_1 + y_2} x_2}{y_1 + y_2} \right)$$

(66)

$$\leq \frac{y_1}{2} f(x_1/y_1) + \frac{y_2}{2} f(x_2/y_2).$$

(67)
Our result follows since $\hat{f}$ is the composition of the affine function $A(x) = (rx + (1-r), tx + (1-t))$ with the conic transform of $f$,

$$\hat{f}(x) = g(A(x)). \quad (68)$$

Let us note that in the special case that $D_f$ corresponds to relative entropy, Theorem [III.1] demonstrates that the “Generalized Jensen-Shannon divergence” developed recently by Nielsen see [47, Definition 1] is in fact an $f$-divergence, as it is defined as the weighted sum of $r_i, t$-skew divergences associated to the relative entropy.

**Corollary III.5.** For a vector $\alpha \in [0, 1]^k$ and $w_i > 0$ such that $\sum_i w_i = 1$, the $(\alpha, w)$-Jensen-Shannon divergence between two densities $p, q$ defined by:

$$JS^{\alpha,w}(p : q) := \sum_{i=1}^k w_i D((1 - \alpha_i)p + \alpha_i q || (1 - \bar{\alpha})p + \bar{\alpha} q) \quad (69)$$

with $\bar{\alpha} = \sum_i w_i \alpha_i$, is an $f$-divergence.

**Proof.** By Theorem III.1 the mapping $(p, q) \mapsto D((1 - \alpha_i)p + \alpha_i q || (1 - \bar{\alpha})p + \bar{\alpha} q)$ is an $f$-divergence, and the result follows since the class of $f$-divergences is stable under non-negative linear combinations. \hfill \Box

We will only further pursue the case that $r = 1$, and write $S_{f,1}(\mu || \nu) := S_{f,1}(\mu || \nu)$. We now skew, Pearson’s $\chi^2$ divergence which we recall below.

**Definition III.6.** [49] For measures $\mu$ and $\nu$ absolutely continuous with respect to a common reference measure $d\gamma$ so that $d\mu = ud\gamma$ and $d\nu = vd\gamma$, define

$$\chi^2(\mu; \nu) = \int \left(1 - \frac{d\mu}{d\nu}\right)^2 d\nu = \int \frac{(u - v)^2}{v} d\gamma,$$

and $\chi^2(\mu; \nu) = \infty$ when $\frac{d\mu}{d\nu}$ does not exist.

**Definition III.7.** For $t \in [0, 1]$ and measures $\mu$ and $\nu$, define the skew $\chi^2_t$ via:

$$\chi^2_t(\mu; \nu) = \int \left(1 - \frac{d\mu}{d\nu}\right)^2 t \frac{d\mu}{d\nu} + (1 - t) d\nu.$$

Formally, the $\chi^2$ divergence of Neyman [44] differs only by a notational convention $\chi^2_N(\nu; \mu) = \chi^2(\mu; \nu)$, see [34] for a more modern treatment and [17] for background on the distances significance in statistics. Now let us present a skew $\chi^2$ divergence, which interpolates the Pearson and Neyman $\chi^2$ divergences.

**Proposition III.8.** The skew $\chi^2_t$ divergence satisfies the following:

1) When $d\mu = ud\gamma$ and $d\nu = vd\gamma$ with respect to some reference measure $\gamma$, then

$$\chi^2_t(\mu; \nu) = \int \frac{(u - v)^2}{tu + (1 - t)v} d\gamma.$$

2) For $t = 0$, $(1 - t)^2 \chi^2_t(\mu; \nu) = \chi^2(\mu; t\mu + (1 - t)\nu).

3) $\chi^2_t(\mu; \nu) = \chi^2_{1-t}(\nu; \mu)$.

4) $\chi^2_t$ is an $f$-divergence with $f(x) = (x - 1)^2 / (1 + tx - 1)$.

5) The skew $\chi^2_t$ interpolates the divergences of Neyman and Pearson, $\chi^2_N(\mu; \nu) = \chi^2(\mu; \nu)$ and $\chi^2_0(\mu; \nu) = \chi^2_N(\mu; \nu)$.

**Proof.** For (1), the formula follows in the case $\mu \ll \nu$, from the fact that on the support of $\nu$,

$$\frac{u}{v} = \frac{d\mu}{d\nu},$$

so that

$$\chi^2_t(\mu; \nu) = \int \frac{(1 - \frac{u}{v})^2}{t \frac{u}{v} + (1 - t)} v d\gamma = \int \frac{(u - v)^2}{tu + (1 - t) v} d\gamma.$$
To prove (2), we use (1). Note that $d\mu = ud\gamma$ and $d\nu = v d\gamma$ implies that $d(t\mu + (1-t)\nu) = (tu + (1-t)v)d\gamma$ so that

$$\chi^2_t(\mu; t\mu + (1-t)\nu) = \int \frac{(u - (tu+(1-t)v))^2}{tu + (1-t)v} d\gamma$$

$$= (1-t)^2 \int \frac{(u - v)^2}{tu + (1-t)v} d\gamma$$

$$= (1-t)^2 \chi^2_t(\mu; \nu).$$

It is immediate from (1) that (3) holds. That $\chi^2_0(\mu; \nu) = \chi^2(\mu; \nu)$ is immediate from the definition. Applying this and symmetry from (3) we have $\chi^2_t(\mu; \nu) = \chi^2(t\mu; \nu) = \chi^2(\nu; \mu) = \chi^2_N(\mu; \nu).$ □

The skew divergence and skew $\chi^2$ inherit bounds from $t = 0$ case, and enjoy an interrelation unique to the skew setting as described below.

**Theorem III.2.** For probability measures $\mu$ and $\nu$ and $t \in (0,1)$

$$S_t(\mu||\nu) \leq (1-t)^2 \chi^2_t(\mu; \nu)$$

and

$$\frac{d}{dt} S_t(\mu||\nu) = (t-1)\chi^2_t(\mu; \nu).$$

**Proof.** Recall that when $t = 0$, the concavity of logarithm bounds $\log x$ by its tangent line $x - 1$ so that,

$$\int \log \left( \frac{d\mu}{d\nu} \right) d\mu \leq \int \left( \frac{d\mu}{d\nu} - 1 \right) d\mu$$

$$= \int \left( \frac{d\mu}{d\nu} - 1 \right)^2 d\nu,$$

giving the classical bound,

$$D(\mu || \nu) \leq \chi^2(\mu; \nu).$$

Applying (74) to the identities Proposition III.3(1) and Proposition III.8(2) gives

$$S_t(\mu||\nu) = D(\mu||t\mu + (1-t)\nu)$$

$$\leq \chi^2(\mu; t\mu + (1-t)\nu)$$

$$= (1-t)^2 \chi^2(t\mu; \nu);$$

Applying the identity $(1-t)(y - 1) = y - (ty + (1-t))$ we have

$$(1-t)\chi^2_t(\mu; \nu) = \int \frac{(\frac{d\mu}{d\nu} - 1)(\frac{d\mu}{d\nu} - (t \frac{d\mu}{d\nu} + (1-t)))}{t \frac{d\mu}{d\nu} + (1-t)} d\nu$$

$$= \int \frac{\frac{d\mu}{d\nu} - 1}{t \frac{d\mu}{d\nu} + (1-t)} d\mu - \int (\frac{d\mu}{d\nu} - 1) d\nu$$

$$= \int \frac{\frac{d\mu}{d\nu} - 1}{t \frac{d\mu}{d\nu} + (1-t)} d\mu.$$

Observing the expression

$$S_t(\mu||\nu) = \int \log \frac{d\mu}{d\nu} - \log \left( t \frac{d\mu}{d\nu} + (1-t) \right) d\mu,$$

we compute directly,

$$\frac{d}{dt} S_t(\mu||\nu) = - \int \frac{\frac{d\mu}{d\nu} - 1}{t \frac{d\mu}{d\nu} + (1-t)} d\mu.$$
Theorem III.3. For μ and ν, and t ∈ (0, 1),
\[\chi_t^2(\mu; \nu) \leq \frac{\|\mu - \nu\|_{TV}}{t(1-t)}\]  
(82)
\[S_t(\mu||\nu) \leq -\log t\|\mu - \nu\|_{TV}.\]  
(83)

Proof. From the identity in (78) we have
\[\chi_t^2(\mu; \nu) = \frac{1}{1-t} \int \frac{d\mu}{d\nu} \left( \frac{d\mu}{d\nu} - 1 \right) d\nu \leq \frac{1}{t(1-t)} \int \left( \frac{d\mu}{d\nu} - 1 \right) d\nu = \frac{\|\mu - \nu\|_{TV}}{t(1-t)}.\]

Define the function
\[\varphi(\lambda) := S_{e^{-\lambda}}(\mu||\nu),\]
for \(\lambda \in [0, \infty)\) and note that \(\varphi(0) = D(\mu||\mu) = 0\). Thus we can write
\[\varphi(\lambda) = \int_0^\lambda \frac{d}{ds} S_{e^{-s}}(\mu||\nu) ds = \int_0^\lambda e^{-s}(1 - e^{-s})\chi_{e^{-s}}^2(\mu; \nu) ds.\]

Applying (82) gives
\[S_{e^{-\lambda}}(\mu||\nu) = \varphi(\lambda) \leq \int_0^\lambda \|\mu - \nu\|_{TV} ds = \lambda\|\mu - \nu\|_{TV}.\]
The substitution \(t = e^{-\lambda}\) gives (83). □

Observe that (83) of Theorem III.3 recovers a reverse Pinsker inequality due to Verdu [62].

Corollary III.9 (62 Theorem 7). For probability measures \(\mu\) and \(\gamma\) such that \(\frac{d\mu}{d\gamma} \leq \frac{1}{\beta} \) with \(\beta \in (0, 1)\)
\[\|\mu - \gamma\|_{TV} \geq \frac{1 - \beta}{\log \frac{1}{\beta}} D(\mu||\gamma).\]

Proof. The hypothesis implies that \(\nu = \frac{\gamma - \beta\mu}{1 - \beta}\) is a probability measure satisfying \(\gamma = \beta\mu + (1 - \beta)\nu\). Applying (83)
\[D(\mu||\nu) = S_{\beta}(\mu||\nu) \leq -\log \beta\|\mu - \nu\|_{TV} = \frac{-\log \beta}{1 - \beta}\|\mu - \gamma\|_{TV}.\]

It is easily seen that the two results, (83) and Theorem 7 of [62] are actually equivalent. In contrast the proof of (83) hinges on foundational properties of the divergence metrics, while Verdu leverages the monotonicity of \(x \ln x/(x - 1)\) for \(x > 1\).

Proof of Theorem I.7 From Proposition II.2
\[h(\sum p_i f_i) = \sum p_i h(f_i) + \sum p_i D(f_i||f)\]  
(84)
\[= \sum p_i h(f_i) + \sum p_i S_{p_i}(f_i||\tilde{f}_i).\]  
(85)

By Theorem III.3 \(S_{p_i}(f_i||\tilde{f}_i) \leq \log \frac{1}{p_i}\|f_i - \tilde{f}_i\|_{TV}\). Applying Hölder’s inequality completes the proof,
\[\sum p_i S_{p_i}(f_i||\tilde{f}_i) \leq \sum p_i \log \frac{1}{p_i} \|f_i - \tilde{f}_i\|_{TV}\]  
(86)
\[\leq \mathcal{T} \sum p_i \log \frac{1}{p_i},\]  
(87)
where we recall \(\mathcal{T} := \sup_i \|f_i - \tilde{f}_i\|_{TV}\). □
Since the total variation of any two measures is bounded above by 1 this is indeed a sharpening of (2). Expressed in random variables it is

\[ h_\gamma(Z) \leq TH(X) + h_\gamma(Z|X). \] (88)

which when \( \gamma \) is a Haar measure and we apply to \( \tilde{Z} = X + Z \) gives

\[ h_\gamma(X + Z) \leq TH(X) + h_\gamma(Z|X), \] (89)

while the right hand side of (89) reduces further to

\[ h_\gamma(X + Z) \leq TH(X) + h_\gamma(Z) \] (90)

in the case that \( X \) and \( Z \) are independent.

Note that the quantity \( h_\gamma(\sum_i p_i f_i) - \sum_i p_i h_\gamma(f_i) = \sum_i p_i D(f_i || f) \) can be considered a generalized Jensen-Shannon divergence, as the case that \( n = 2 \) and \( p_1 = p_2 = \frac{1}{2} \) this is exactly the Jensen-Shannon Divergence.

**Definition III.10.** For probability measures \( \mu \) and \( \nu \) define the Jensen-Shannon divergence,

\[ \text{JSD}(\mu || \nu) = \frac{1}{2} \left( D(\mu || 2^{-1}(\mu + \nu)) + D(\nu || 2^{-1}(\mu + \nu)) \right). \] (91)

Theorem [1] recovers the classical bound of the Jensen-Shannon divergence by the total variation, due to Lin, see also [58]. [59] for other proofs.

**Corollary III.11.** [35] For \( \mu \) and \( \nu \) probability measures,

\[ \text{JSD}(\mu || \nu) \leq \|\mu - \nu\|_{TV} \log 2. \]

**Proof.** Apply Theorem [1] to the Jensen-Shannon divergence, and observe that \( T = \|\mu - \nu\|_{TV} \) in the case of two summands. \( \square \)

**IV. UPPER BOUNDS**

Let us state our assumptions and notations for this section.

1) \( X \) is a random variable taking values in countable space \( \mathcal{X} \), such that for \( i \in \mathcal{X} \), \( \mathbb{P}(X = i) = p_i. \)
2) \( Z \) is an \( \mathbb{R}^d \) valued random variable, with conditional densities, \( f_i \) satisfying,

\[ \mathbb{P}(Z \in A | X = i) = \int_A f_i(z)dz = \mathbb{P}(T_i(W) \in A). \] (92)

for \( T_i \), a \( \sqrt{T} \) bi-Lipschitz function, and \( W \) is a spherically symmetric log-concave random vector with density \( \varphi \).
3) There exists \( \lambda, M > 0 \) such that for any \( i, j \),

\[ \# \{ k : T_{kj}(B_\lambda) \cap T_{ij}(B_\lambda) \neq \emptyset \} \leq M, \] (93)

with \# denoting cardinality and \( T_{ij} := T_{ij}^{-1} \circ T_j \).

Our assumption that \( W \) is log-concave and spherically symmetric is equivalent to \( W \) possessing a density \( \varphi \) that is spherically symmetric in the sense that \( \varphi(x) = \varphi(y) \) for \( |x| = |y| \) and log-concave in the sense that \( \varphi((1-t)x + ty) \geq \varphi^{1-t}(x)\varphi^t(y) \) holds for \( t \in [0, 1] \) and \( x, y \in \mathbb{R}^d \). By the spherical symmetry of \( \varphi \), there exists \( \psi: [0, \infty) \rightarrow [0, \infty) \) such that \( \varphi(x) = \psi(|x|) \). Note that by Radamacher’s theorem, Lipschitz continuous functions are almost everywhere differentiable, and since bi-Lipschitz functions are necessarily invertible. Thus, using this and [14] it follows that \( Z \) has a density given by the following expression

\[ f(z) = \sum_i p_i f_i(z) = \sum_i p_i \varphi(T_i^{-1}(z)) \det((T_i^{-1})'(z)). \] (94)

Note that \( T_i \) being \( \sqrt{T} \) bi-Lipschitz implies \( T_i^{-1} \) is \( \sqrt{T} \) bi-Lipschitz as well, thus \( T_{ij} \) is \( \tau \)-bi-Lipschitz, thus after potentially adjusting \( T_{ij} \) on set of measure zero, we have \( \frac{1}{\tau} \leq \|T_{ij}(z)\| \leq \tau \). Under these assumptions we will prove the following generalization of \( [1, 2] \)

**Theorem IV.1.** For \( X \) and \( Z \) satisfying the assumptions of Section [IV]

\[ h(Z) - h(Z|X) \geq H(X) - \bar{C}(W), \] (95)

where \( \bar{C} \) is the following function dependent heavily on the tail behavior of \( |W| \),

\[ \bar{C}(W) = (M - 1)(1 - \mathcal{F}(\lambda \tau)) + \mathcal{F}(\lambda)(M + h(W)) + \mathcal{F}^2(\lambda)(\sqrt{d} + K(\varphi)) \] (96)
with
\[
K(\varphi) := \log \left[ \tau^d M \left( \|\varphi\|_\infty + \left( \frac{3}{\varepsilon} \right) \omega_d^{-1} \right) \right] \mathbb{P}^\perp (|W| > \lambda) + d \left( \int_{B_x^\perp} \varphi(w) \log^2 \left[ 1 + \varepsilon \tau^2 |w| \lambda \right] dw \right)^{\frac{1}{2}}
\]
where \( \omega_d \) denoting the volume of the \( d \)-dimensional unit ball, \( B_x^\perp \) denotes the complement of \( B_x \in \mathbb{R}^d \).

Note that when \( M = 1 \), Theorem IV.1 reduces to Theorem I.2. Additionally observe that \( \log^2(x) \) is a concave function for \( x \geq e \). If one writes \( m := \max\{e, 1 + \tau \} \) then by Jensen’s inequality,
\[
\left( \int_{B_x^\perp} \varphi(w) \log^2 \left[ 1 + \tau^2 |w| \lambda \right] dw \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^d} \varphi(w) \log^2 \left[ m + \tau^2 |w| \lambda \right] dw \right)^{\frac{1}{2}} \leq \log \left( m + \tau^2 \int \varphi(w) |w| dw \right).
\]

Thus we can further bound
\[
K(\varphi) \leq \log \left[ \tau^d M \left( \|\varphi\|_\infty + \left( \frac{3}{\varepsilon} \right) \omega_d^{-1} \right) \right] \mathbb{P}^\perp (|W| > \lambda) + d \log \left( m + \tau^2 \int \varphi(w) |w| dw \right).
\]

We now derive some implications of our assumptions on \( T_{ij} \), a partitioning result on \( T_{ij}(B_x) \) based on the axiom of choice and for the reader’s convenience we prove some elementary consequences of the boundedness of the derivatives of \( T_{ij} \).

Proposition IV.1. For \( T_{ij} = T_i^{-1} \circ T_j \)
\[
T_{ij}(0) + B_{\lambda/\tau} \subseteq T_{ij}(B_{\lambda}) \subseteq T_{ij}(0) + B_{\lambda \tau}
\]

That
\[
\# \{ k : T_{ji}(B_{\lambda}) \cap T_{jk}(B_{\lambda}) \} \leq M
\]
implies that any collection \( \mathcal{X} \subseteq \mathbb{N} \) has a partition \( \mathcal{X}_1, \ldots, \mathcal{X}_n \), with \( n \leq M \) such that \( x_1, x_2 \in \mathcal{X}_k \) implies \( T_{x_1}(B_{\lambda}) \cap T_{x_2}(B_{\lambda}) = \emptyset \).

Proof. To prove (101), observe that \( T_{ij} \tau\)-bi-Lipschitz implies, that \( T_{ij}^{-1} \) exists and is \( \tau\)-bi-Lipschitz as well. Observing that \( \tau\)-Lipschitz implies,
\[
|T_{ij}(x) - T_{ij}(0)| \leq \tau |x|.
\]
If we take \( |x| < \lambda \), this inequality shows \( T_{ij}(B_{\lambda}) \subseteq T_{ij}(0) + B_{\lambda \tau} \). For the other inclusion, observe that since \( T_{ij}^{-1} \) is \( \tau\)-Lipschitz as well,
\[
|T_{ij}^{-1}(T(0) + x)| = |T_{ij}^{-1}(T_{ij}(0) + x) - T_{ij}^{-1}(T_{ij}(0))| \leq \tau |x|.
\]
Taking \( |x| < \lambda \), this shows that \( T_{ij}^{-1}(T_{ij}(0) + B_{\lambda/\tau}) \subseteq B_{\lambda} \), which the desired inclusion follows from.

Now we prove the existence of the partitioning. If \( M = 1 \), the result is obvious, and we proceed by induction. Choose \( \mathcal{X}_1 \) to be a maximal subset of \( \mathcal{X} \) such that \( \{ T_{x} \}_{x \in \mathcal{X}_1} \) are disjoint. For \( x_0 \in \mathcal{X} - \mathcal{X}_1 \),
\[
\# \{ k : \mathcal{X} - \mathcal{X}_1 : T_{ji}(B_{\lambda}) \cap T_{jk}(B_{\lambda}) \neq \emptyset \} \leq M - 1.
\]
Indeed for every \( k \in \mathcal{X} \), \( T_{jk}(B_{\lambda}) \) intersects at most \( M \) others, and since \( \mathcal{X}_1 \) is maximal and \( k \notin \mathcal{X}_1 \) \( T_{jk}(B_{\lambda}) \) must intersect one of the \( T_{x}(B_{\lambda}) \) for \( x \in \mathcal{X}_1 \) which leaves \( T_{jk}(B_{\lambda}) \) to intersect at most \( M - 1 \) of \( \{ T_{x}(B_{\lambda}) \}_{x \in \mathcal{X} - \mathcal{X}_1} \). By induction the result follows.

We will need the following concentration result for the information content of a log-concave vector \([22, 45, 63]\).

Theorem IV.2. For a log-concave density function \( \varphi \) on \( \mathbb{R}^d \),
\[
\int \left( \log \frac{1}{\varphi(x)} - h(\varphi) \right)^2 \varphi(x) dx \leq d.
\]
where \( h(\varphi) \) is the entropy of the density \( \varphi \).

See [21] for a generalization to convex measures, which can be heavy-tailed.

The following upper bounds the sum of a sequence whose values are obtained by evaluating a spherically symmetric density at well spaced points.
Lemma IV.2. If $\phi$ is a density on $\mathbb{R}^d$, not necessarily log-concave, given by $\phi(x) = \psi(|x|)$ for $\psi : [0, \infty) \to [0, \infty)$ decreasing, $\lambda > 0$, and a discrete set $\mathcal{X} \subseteq \mathbb{R}^d$ admitting a partition $\mathcal{X}_1, \ldots, \mathcal{X}_M$ such that distinct $x, y \in \mathcal{X}_k$ satisfy $|x - y| \geq 2\lambda$, then there exists an absolute constant $c \leq 3$ such that
\[
\sum_{x \in \mathcal{X}} \phi(x) \leq M \left( \|\phi\|_{\infty} + \left( \frac{c}{\lambda} \right)^d \omega_d^{-1} \right),
\] (108)
where $\omega_d = |\{x : |x| \leq 1\}|_d$, where we recall $|\cdot|_d$ as the $d$-dimensional Lebesgue volume. In particular, if for all $x_0 \in \mathcal{X}$
\[
\#\{x \in \mathcal{X} : |x - x_0| < 2\lambda\} \leq M,
\] (109)
then (108) holds.

Note, that when $M = 1$ and $\phi$ is the uniform distribution on a $d$-dimensional ball, this reduces to a sphere packing bound,
\[
\#\{\text{disjoint } \lambda\text{-balls contained in } B_{R+\lambda}\} \leq 1 + \left( \frac{Rc}{\lambda} \right)^d.
\] (110)
From which it follows, due to classical bounds of Minkowski, that $c \geq \frac{1}{2}$.

For the proof below we use the notations $B_\lambda(x) = \{w \in \mathbb{R}^d \mid |w - x| \leq \lambda\}$ and we identify $B_\lambda \equiv B_\lambda(0)$.

Proof. Let us first see that it is enough to prove the result when $M = 1$.
\[
\sum_{x \in \mathcal{X}} \phi(x) = \sum_{k=1}^{M} \sum_{x \in \mathcal{X}_k} \phi(x)
\] (111)
\[
\leq \sum_{k=1}^{M} \left( \|\phi\|_{\infty} + \left( \frac{c}{\lambda} \right)^d \omega_d^{-1} \right)
\] (112)
\[
= M \left( \|\phi\|_{\infty} + \left( \frac{c}{\lambda} \right)^d \omega_d^{-1} \right)
\] (113)
where (112) follows if we have the result when $M = 1$. We proceed in the case that $M = 1$ and observe that $\psi$ non-increasing enables the following Riemann sum bound,
\[
1 = \int_{\mathcal{X}} \phi(x)dx
\] (114)
\[
\geq \sum_{k=0}^{\infty} \psi(k\lambda)\omega_d\lambda^d ((k + 1)^d - k^d),
\] (115)
where $\omega_d\lambda^d ((k + 1)^d - k^d)$ is the volume of the annulus $B_{(k+1)\lambda} - B_{k\lambda}$. Define
\[
\Lambda_k := \{x \in \mathcal{X} : |x| \in [k\lambda, (k + 1)\lambda)\},
\] (116)
then
\[
\sum_{x \in \mathcal{X}} \phi(x) = \sum_{k=0}^{\infty} \sum_{x \in \Lambda_k} \phi(x)
\] (117)
\[
\leq \sum_{k=0}^{\infty} \#\Lambda_k \psi(k\lambda),
\] (118)
as $\psi$ is non-increasing. Let us now bound $\#\Lambda_k$. Using the assumption that any two elements $x$ and $y$ in $\mathcal{X}$ satisfy $|x - y| \geq 2\lambda$,
\[
| \bigcup_{x \in \Lambda_k} \{x + B_\lambda\}| = \#\Lambda_k |B_\lambda|
\] (119)
\[
= \#\Lambda_k \omega_d\lambda^d,
\] (120)
so that it suffices to bound $| \bigcup_{x \in \Lambda_k} \{x + B_\lambda\}|$. Observe that we also have $\bigcup_{x \in \Lambda_k} \{x + B_\lambda\}$ contained in an annulus,
\[
\bigcup_{x \in \Lambda_k} B_\lambda(x) \subseteq \{x : |x| \in [(k - 1)\lambda, (k + 2)\lambda)\},
\] (121)
which combined with (120) gives
\[
\#\Lambda_k \omega_d\lambda^d \leq |\{x : |x| \in [(k - 1)\lambda, (k + 2)\lambda)\}|_d
\] (122)
\[
= \omega_d\lambda^d ((k + 2)^d - (k - 1)^d),
\] (123)
so that
\[ \#\Lambda_k \leq (k + 2)^d - (k - 1)^d. \]  
(124)

Note the following bound, for \( k \geq 1 \)
\[ (k + 2)^d - (k - 1)^d \leq 3^d \left( (k + 1)^d - k^d \right). \]  
(125)

Indeed, by the mean value theorem, there exists \( x_0 \in [k - 1, k + 2] \)
\[ (k + 2)^d - (k - 1)^d = 3dx_0^{d-1} \]
\[ \leq 3d(k + 2)^{d-1}. \]  
(126)
(127)

and there exists \( y_0 \in [k, k + 1] \) such that,
\[ (k + 1)^d - k^d = dy_0^{d-1} \]
\[ \geq dk^{d-1}. \]  
(128)
(129)

Thus our bound follows from the obvious fact that for \( k \geq 1 \)
\[ (k + 2)^d - (k - 1)^d \leq 3^{d-1}k^{d-1}. \]  
(130)

Compiling the above,
\[ (k + 2)^d - (k - 1)^d \leq 3d(k + 2)^{d-1} \]
\[ \leq 3d3^{d-1}k^{d-1} \]
\[ \leq 3^d \left( (k + 1)^d - k^d \right). \]  
(131)
(132)
(133)

Thus for \( k \geq 1 \), (124) and (125) give,
\[ \#\Lambda_k \leq 3^d \left( (k + 1)^d - k^d \right). \]  
(134)

Applying this inequality to (117) gives
\[ \sum_{x \in \mathcal{X}} \phi(x) \leq \sum_{k=0}^{\infty} \sum_{x \in \Lambda_k} \psi(\lambda k) \]
\[ \leq ||\phi||_{\mathcal{X}} \sum_{k=1}^{\infty} \psi(\lambda k)\#\Lambda_k \]
\[ \leq ||\phi||_{\mathcal{X}} \sum_{k=1}^{\infty} \psi(\lambda k)3^d \left( (k + 1)^d - k^d \right) \]
\[ \leq ||\phi||_{\mathcal{X}} + \omega_b^{-1} \left( \frac{3}{\lambda} \right)^d, \]  
(135)
(136)
(137)
where (137) follows from the fact that \( \#\Lambda_0 \leq 1 \) (any \( x \in \Lambda_0 \) has 0 in \( \{x + B_{\lambda}\} \)), and the last inequality follows from the Riemann sum bound (115).

Let us now show that any \( \mathcal{X} \) satisfying (109) necessarily satisfy (108), by show that \( \mathcal{X} \) satisfying (109) can be partitioned into \( M \) subsets \( \{\mathcal{X}_1, \ldots, \mathcal{X}_M\} \) such \( \cup_{k=1}^{M} \mathcal{X}_k = \mathcal{X} \) and \( x, y \in \mathcal{X}_k \) implies \( |x - y| \geq 2\lambda \) for \( x \neq y \). Choose \( \mathcal{X}_1 \) to be a subset of \( \mathcal{X} \) maximal with respect to the property \( x, y \in \mathcal{X}_1 \) implies \( |x - y| \geq 2\lambda \). We note that such a maximal subset necessarily exists by Zorn’s Lemma with the partial order given by set theoretic inclusion. Now suppose that \( x_0 \in \mathcal{X}' = \mathcal{X} - \mathcal{X}_1 \). Then if follows that
\[ \#\{x \in \mathcal{X}' : |x - x_0| < 2\lambda \} \leq M - 1. \]  
(139)

Indeed, as \( x_0 \notin \mathcal{X}_1 \) we have from the maximality of \( \mathcal{X}_1 \), that there exists a \( x_0' \in \mathcal{X}_1 \) such that \( |x_0 - x_0'| \leq 2\lambda \). Note that the cardinality of the set \( \{x \in \mathcal{X} : |x - x_0| \leq 2\lambda\} \) is at most \( M \) with \( x_0' \notin \mathcal{X}' := \mathcal{X} - \mathcal{X}_1 \) in the set. It follows that \( \#\{x \in \mathcal{X}' : |x - x_0| \leq 2\lambda\} \leq M - 1 \). Applying the result inductively on \( \mathcal{X}' \) our claim follows. \( \square \)

Lemma IV.3. For \( T_{i} \) \( \sqrt{\tau} \)-bi-Lipshitz satisfying \( \#\{l : |T_{ij}(0) - T_{ij}(0)| < 2\lambda\} \leq M \) for all \( x \), then for \( \epsilon > 0 \),
\[ \#\{l : |T_{ij}(w) - T_{ij}(w)| < \epsilon \} \leq M \left( \frac{\lambda + \epsilon + 2\tau|w|}{\lambda} \right)^{d}. \]  
(140)

In particular,
\[ \#\{l : |T_{ij}(0) - T_{ij}(0)| < \epsilon \} \leq M \left( \frac{\epsilon + \lambda}{\lambda} \right)^{d}. \]  
(141)
Proof. We will first prove and then leverage (141). Note that there is nothing to prove when \( \varepsilon \leq 2\lambda \) as (141) is weaker than the assumption. When \( 2\lambda < \varepsilon \) choose (by Zorn’s lemma for instance) \( \Lambda \) to be a maximal subset of \( \mathbb{N} \) such that for \( k \in \Lambda \), \( |T_{kj}(0) - T_{ij}(0)| < \varepsilon \) and \( |T_{kj}(0) - T_{kj}(0)| \geq 2\lambda \) for \( k, k' \in \Lambda \), with \( k \neq k' \). By construction, for a fixed \( j \), \( T_{kj}(0) + B_{\lambda} \) are disjoint over \( k \in \Lambda \) contained in \( T_{ij}(0) + B_{\lambda + \varepsilon} \). Thus
\[
\lambda \omega_d \# \Lambda = |\cup_{k \in \Lambda} \{T_{kj}(0) + B_{\lambda}\}|
\leq |\{T_{ij}(0) + B_{\lambda + \varepsilon}\}|
= (\lambda + \varepsilon)^d \omega_d,
\]
and we have the following bound on the cardinality of \( \Lambda \),
\[
\# \Lambda \leq \left( \frac{\lambda + \varepsilon}{\lambda} \right)^d.
\]
Applying (145), the assumed cardinality bounds, and the maximality of \( \Lambda \), which implies \( \cup_{k \in \Lambda} \{T_{kj}(0) + B_{\lambda}\} \) contains every \( T_{ij}(0) \) such that \( |T_{ij}(0) - T_{ij}(0)| < \varepsilon \), we have
\[
\# \{l : |T_{ij}(0) - T_{ij}(0)| < \varepsilon\} \leq \sum_{k \in \Lambda} \# \{m : |T_{mj}(0) - T_{kj}(0)| < \lambda\}
\leq M \left( \frac{\lambda + \varepsilon}{\lambda} \right)^d.
\]
Towards (140), by the mean value theorem, there exists \( t \in [0, 1] \) such that
\[
T_{ij}(w) - T_{ij}(0) = T_{ij}(0) - T_{ij}(0) + (T'_{ij}(tw) - T'_{ij}(0))w.
\]
Note that if \( |T_{ij}(w) - T_{ij}(0)| < \varepsilon \), then
\[
|T_{ij}(0) - T_{ij}(0)| = |T_{ij}(w) - T_{ij}(0) - (T'_{ij}(tw) - T'_{ij}(0))w| 
\leq |T_{ij}(w) - T_{ij}(0)| + |(T'_{ij}(tw) - T'_{ij}(0))w| 
\leq \varepsilon + 2\tau |w|.
\]
Thus
\[
\# \{l : |T_{ij}(w) - T_{ij}(0)| < \varepsilon\} \leq \# \{l : |T_{ij}(0) - T_{ij}(0)| < \varepsilon + 2\tau |w|\}.
\]
Applying (141),
\[
\# \{l : |T_{ij}(w) - T_{ij}(0)| < \varepsilon\} \leq M \left( \frac{\lambda + \varepsilon + 2\tau |w|}{\lambda} \right)^d.
\]
\( \square \)

Corollary IV.4. For a density \( \phi(w) = \psi(|w|) \), with \( \psi \) decreasing, \( \varepsilon > 0 \), \( T_i \sqrt{\tau} \)-bi-Lipschitz, \( T_{ij} = T_i^{-1} \circ T_j \), such that there exists \( M \geq 1 \) such that for any \( i \)
\[
||\{k : T_{ij}(B_{\lambda}) \cap T_{kj}(B_{\lambda}) \neq \emptyset\}|| \leq M,
\]
then
\[
\sum_i \phi(T_{ij}(w)) \det(T'_{ij}(w)) \leq \tau^d M \left( 1 + \frac{\varepsilon \tau + \tau^2 |w|}{\lambda} \right)^d \left( \|\phi\|_\infty + \left( \frac{3}{\varepsilon} \right)^d \omega_d^{-1} \right).
\]

Proof. For any \( k \) suppose \( |T_{ij}(0) - T_{kj}(0)| < 2\lambda/\tau \), then \( \{T_{ij}(0) + B_{\lambda/\tau}\} \cap \{T_{kj}(0) + B_{\lambda/\tau}\} \neq \emptyset \) which by Proposition IV.1 implies \( T_{ij}(B_{\lambda}) \cap T_{kj}(B_{\lambda}) \neq \emptyset \). Thus, \( \# \{k : |T_{ij}(0) - T_{kj}(0)| < 2\lambda/\tau\} \leq M \). By Lemma IV.3
\[
\# \{l : |T_{ij}(w) - T_{ij}(0)| < 2\varepsilon\} \leq M \left( \frac{2\lambda + 2\varepsilon + 2\tau |w|}{2\lambda} \right)^d
= M \left( 1 + \frac{\varepsilon \tau + \tau^2 |w|}{\lambda} \right)^d.
\]
This shows that (109) holds for \( x_i = T_{ij}(w) \), \( \lambda = \varepsilon \) and \( M \) in (109) identified with \( M \left( 1 + \frac{\varepsilon \tau + \tau^2 |w|}{\lambda} \right)^d \). It follows from Lemma IV.2 that
\[
\sum_i \phi(T_{ij}(w)) \leq M \left( 1 + \frac{\varepsilon \tau + \tau^2 |w|}{\lambda} \right)^d \left( \|\phi\|_\infty + \left( \frac{3}{\varepsilon} \right)^d \omega_d^{-1} \right).
\]
This, combined with $\|T_{ij}(x)\| \leq \tau$ giving the determinant bounds $\det(T_{ij}(w)) \leq \tau^d$ yields,
\begin{equation}
\sum_i \phi(T_{ij}(w))\det(T_{ij}(w)) \leq \tau^d \sum_i \phi(T_{ij}(w)) \leq \tau^d M \left(1 + \frac{\tau + \tau^2|w|}{\lambda}\right)^d \left(\|\phi\|_{\infty} + \left(\frac{3}{\varepsilon}\right)^d \omega_d^{-1}\right).
\end{equation}

\[\text{Proof.}\]
Note that, non-increasing except to ensure that the relevant statistics are finite.

\[\text{Proof.}\]

**Corollary IV.5.** Consider $T_i$, $\sqrt{\tau}$-bi-Lipschitz and suppose there exists $M, \lambda > 0$ such that $\#\{k : T_{kj}(B_{\lambda}) \cap T_{ij}(B_{\lambda}) \neq \emptyset\} \leq M$ holds for any $i, j$. Then for a spherically symmetric log-concave density $\varphi(x) = \psi(|x|)$,
\begin{equation}
\int_{B_{\lambda}^c} \varphi(w) \log \left(\sum_i p_i \sum_j \phi(T_{ij}(w))\det(T_{ij}(w))\right) \leq K(\varphi) \mathbb{P}^\frac{1}{2}(|W| > \lambda),
\end{equation}

where
\begin{equation}
K(\varphi) := \log \left[\tau^d M \left(\|\varphi\|_{\infty} + \left(\frac{3}{\lambda}\right)^d \omega_d^{-1}\right)\right] \mathbb{P}^\frac{1}{2}(|W| > \lambda) + d \left(\int_{B_{\lambda}^c} \varphi(w) \log^2 \left[1 + \tau + \frac{\tau^2|w|}{\lambda}\right] dw\right)^\frac{1}{2}.
\end{equation}

Note that $K(\varphi)$ depends on only on the statistics of $\varphi$, its maximum and its a logarithmic scaling of its norm, which can be easily further bounded by concavity results. The proof does not leverage log-concavity, a stronger assumption than $\psi$ non-increasing except to ensure that the relevant statistics are finite.

**Proof.** Note that,
\begin{equation}
\int_{B_{\lambda}^c} \varphi(w) \log \left(\sum_i p_i \sum_j \phi(T_{ij}(w))\det(T_{ij}(w))\right) dw \leq \int \varphi(w) I_{B_{\lambda}^c} \log \left(\tau^d M \left(1 + \tau + \frac{\tau^2|w|}{\lambda}\right)^d \left(\|\varphi\|_{\infty} + \left(\frac{3}{\lambda}\right)^d \omega_d^{-1}\right)\right) dw \leq K(\varphi) \mathbb{P}^\frac{1}{2}(|W| > \lambda)
\end{equation}

where the first inequality is an application of Corollary [IV.4] with $\varepsilon = \lambda$ and the second from Cauchy-Schwartz.

**Proof of Theorem [IV.4]** Using $f_i(x) = \varphi(T_{i}^{-1}(x))\det((T_{i}^{-1})'(x))$ and applying the substitution $x = T_{i}(w)$ we can write
\begin{equation}
\int f_i(x) \log \left(1 + \frac{\sum_{j \neq i} p_j f_j(x)}{p_i f_i(x)}\right) dx \leq \int \varphi(T_{i}^{-1}(x))\det((T_{i}^{-1})'(x)) \log \left(1 + \frac{\sum_{j \neq i} p_j \varphi(T_{j}^{-1}(x))\det((T_{j}^{-1})'(x))}{p_i \varphi(T_{i}^{-1}(x))\det((T_{i}^{-1})'(x))}\right) dx,
\end{equation}

\begin{equation}
= \int \varphi(w) \log \left(1 + \frac{\sum_{j \neq i} p_j \varphi(T_{j}(w))\det((T_{i}^{-1})'(T_{i}(w)))\det(T_{j}(w))}{p_i \varphi(w)}\right) dw,
\end{equation}

\begin{equation}
= \int \varphi(w) \log \left(1 + \frac{\sum_{j \neq i} p_j \varphi(T_{j}(w))\det(T_{j}(w))}{p_i \varphi(w)}\right) dw.
\end{equation}

Thus, applying Jensen’s inequality
\begin{equation}
\sum_i p_i \int f_i(x) \log \left(1 + \frac{\sum_{j \neq i} p_j f_j(x)}{p_i f_i(x)}\right) dx \leq \sum_i p_i \int \varphi(w) \log \left(1 + \frac{\sum_{j \neq i} p_j \varphi(T_{j}(w))\det(T_{j}(w))}{p_i \varphi(w)}\right) dw \leq \int \varphi(w) \log \left(1 + \frac{\sum_i \sum_{j \neq i} p_j \varphi(T_{j}(w))\det(T_{j}(w))}{\varphi(w)}\right) dw.
\end{equation}

Under spherical symmetry and log-concavity $\|\varphi\|_{\infty} = \varphi(0)$. Indeed, $\varphi(0) = \varphi((-\varepsilon,x)) \geq \sqrt{\varepsilon} \varphi(-x) \varphi(x) = \varphi(x)$. Using log-concavity again for $t \in (0, 1)$, $\varphi((1-t)0 + tx) \geq \varphi(x)$, taking $t = (1-t)$, $\varphi'(x) \geq \varphi(x) = \psi(|x|)$. Thus it follows that $\psi$ is non-increasing.
We will split the integral in to two pieces. Using \( \log(1 + x) \leq x \) on \( B_\lambda \)

\[
\int_{B_\lambda} \varphi(w) \log \left( 1 + \sum_j p_j \sum_{i \neq j} \varphi(T_{ji}(w)) \det(T'_{ji}(w)) \frac{1}{\varphi(w)} \right) \, dw
\]

\[
\leq \int_{B_\lambda} \sum_j p_j \sum_{i \neq j} \varphi(T_{ji}(w)) \det(T'_{ji}(w)) \, dw
\]

\[
= \sum_j p_j \sum_{i \neq j} \int_{B_\lambda} \varphi(T_{ji}(w)) \det(T'_{ji}(w)) \, dw
\]

\[
= \sum_j p_j \left( \sum_{i \neq j} \int_{B_\lambda} \varphi(x) \, dx \right)
\]

\[
= \sum_j p_j \left( \sum_{i \neq j \in \mathcal{T}_j(B_\lambda) \cap B_\lambda \neq \emptyset} \int_{T_j(B_\lambda)} \varphi(x) \, dx + \sum_{i \in \mathcal{T}_j(B_\lambda) \cap B_\lambda = \emptyset} \int_{T_j(B_\lambda)} \varphi(x) \, dx \right)
\]

\[
\leq \sum_j p_j \left( \int_{\mathcal{T}_j(B_\lambda) \cap B_\lambda \neq \emptyset} \varphi(x) \, dx \right) + M \int_{B_\lambda} \varphi(x) \, dx
\]

\[
\leq \sum_j p_j \left( M - 1 \right) \int_{B_\lambda} \varphi(x) \, dx + M \int_{B_\lambda} \varphi(x) \, dx
\]

\[
= (M - 1) \mathbb{P}(|W| \leq \lambda \tau) + M \mathbb{P}(|W| > \lambda)
\]

where inequality (179) follows from Proposition [IV.1] and inequality (180) follows another application of Proposition [IV.1] and the fact that the map \( s(x) = \int_{x + B_\lambda} \varphi(z) \, dz \) is maximized at 0. To see this, observe that \( s \) can be realized as the convolution of two spherically symmetric unimodal functions, explicitly \( s(x) = \varphi * \mathbb{1}_{B_\lambda}(x) \). Since the class of such functions is stable under convolution, see for instance [33 Proposition 8], \( s \) is unimodal and spherically symmetric which obviously implies \( s(0) \geq s(x) \) for all \( x \).

Using the fact that \( T_{ji}(w) = w \),

\[
\int_{B_\lambda^c} \varphi(w) \log \left( 1 + \sum_j p_j \sum_{i \neq j} \varphi(T_{ji}(w)) \det(T'_{ji}(w)) \frac{1}{\varphi(w)} \right) \, dw
\]

\[
= \int_{B_\lambda^c} \varphi(w) \log \left( \sum_j p_j \sum_i \varphi(T_{ji}(w)) \det(T'_{ji}(w)) \right) \, dw - \int_{B_\lambda^c} \varphi(w) \log \varphi(w) \, dw
\]

\[
\leq K(\varphi) \mathbb{P}^\frac{1}{2}(|W| \leq \lambda) - \int_{B_\lambda^c} \varphi(w) \log \varphi(w) \, dw,
\]

where the bound \( K(\varphi) \) is defined from Corollary [IV.5] By Cauchy-Schwartz, followed by Theorem [IV.2]

\[
- \int_{B_\lambda^c} \varphi(w) \log \varphi(w) \, dw
\]

\[
= h(W) \mathbb{P}(|W| \geq \lambda) + \int_{B_\lambda^c} \varphi(w) \left( \log \frac{1}{\varphi(w)} - h(W) \right) \, dw
\]

\[
\leq h(W) \mathbb{P}(|W| \geq \lambda) + \sqrt{\int \left( \log \frac{1}{\varphi(w)} - h(\varphi) \right)^2 \varphi(w) \, dw} \int_{B_\lambda^c} \varphi(w) \, dw
\]

\[
\leq h(W) \mathbb{P}(|W| \geq \lambda) + \sqrt{d} \mathbb{P}^\frac{1}{2}(|W| \geq \lambda).
\]

\[\square\]

A. Commentary on Theorem [IV.1]

Let us comment on the nature of \( \mathbb{P}(|W| \geq \lambda) \) for \( W \) log-concave. It is well known that in broad generality (see [10]–[12], [24], [36]), log-concave random variables satisfy “sub-exponential” large deviation inequalities. The following is enough to suit our needs.

Lemma IV.6. [36 Theorem 2.8] When \( W \) is a log-concave and \( t, r > 0 \), then

\[
\mathbb{P}(|W| > rt) \leq \mathbb{P}(|W| > t)^{\frac{r^2}{t^2}}.
\]
Proof. By Caffarelli’s celebrated result [13],

\[ \mathbb{P}(|W| > c) \leq C e^{-ct}, \]

where \( C = 2^{-1/2}, c = \frac{\log 2}{2\sqrt{d\sigma^2}} \).

Proof. By Chebyshev’s inequality \( \mathbb{P}(|W| > r\sqrt{2d\sigma^2}) \leq \frac{1}{r^2} \). Hence for \( r > 1 \), by Lemma IV.6

\[ \mathbb{P}(|W| > r\sqrt{2d\sigma^2}) \leq \mathbb{P}(|W| > \sqrt{2d\sigma^2}) \frac{1}{r^2} \leq 2^{-\frac{r^2}{r^2 - 1}}. \]

Taking \( t = r\sqrt{2d\sigma^2} \) gives the result.

Lemma IV.8. Suppose that the spherically symmetric log-concave \( W \) is strongly log-concave in the sense that its density function \( \varphi \) satisfies \( \varphi((1-t)x + ty) \geq e^{t(1-t)|x-y|^2/2} \varphi^{1-t}(x) \varphi^t(y) \), for \( x, y \in \mathbb{R}^d \) and \( t \in [0, 1] \), then

\[ \mathbb{P}(|W| > t) \leq \mathbb{P}(|Z| > t), \quad (189) \]

where \( Z \) is a standard normal vector.

Proof. By the celebrated result of Caffarelli [13], \( W \) can be expressed as \( T(Z) \) where \( T \) is the necessarily 1-Lipschitz Brenier transportation map. Moreover it follows from the assumed spherically symmetry of \( W \), the radial symmetry of the Gaussian, and [13] Lemma 1 that \( T \) is spherically symmetric. In particular \( T(0) = 0 \), so that \( |Z| \leq t \) implies \( |T(Z)| = |T(Z) - T(0)| \leq |Z - 0| \leq t \), and our result follows.

Corollary IV.9. Suppose \( X \) and \( Y \) are variables satisfying the conditions of Section IV for \( \tau, M = 1, \lambda \), and \( W \) possessing spherically symmetric log-concave density \( \varphi \). Then

\[ H(X|Y) \leq \tilde{C} 2^{-\frac{1 + \lambda \sqrt{2d} + \lambda}{2}}. \quad (190) \]

Furthermore, if \( W \) is strongly log-concave then

\[ H(X|Y) \leq \tilde{C} \mathbb{P}^Z(|Z| > t), \quad (191) \]

where \( Z \) is a standard Gaussian vector and \( \tilde{C} = \left(1 + \sqrt{d} + h(W) + K(\varphi)\right) \), with \( K(\varphi) \) defined as in (97).

Proof. The proof is an immediate application of Corollary IV.7 and Lemma IV.8 to Theorem IV.1.

V. Applications

Mixture distributions are ubiquitous, and their entropy is a fundamental quantity. We now give special attention to how the ideas of Section IV can be sharpened in the case that \( W \) is a Gaussian.

Proposition V.1. When \( X \) and \( Y \) satisfy the assumptions of Section IV for \( \tau, M, \lambda, T_{ij} \), and \( W \sim \varphi(w) = e^{-|w|^2/(2\pi)^d/2} \), then

\[ H(X|Y) \leq (M - 1)\mathbb{P}(|W| \leq \lambda\tau) + J_d(\varphi)\mathbb{P}(|W| > \lambda) \quad (192) \]

with

\[ J_d(\varphi) = \log \left[ e^{(\lambda/\sigma)^2 + M(\tau e)^d} \left(1 + \tau^2 + \frac{\tau^2 \sigma^2}{\lambda^2}\right)^d \left(1 + \sqrt{\frac{3\pi}{2}} \frac{\sigma}{\lambda}\right)^d \right], \quad (193) \]

for \( d \geq 2 \) and

\[ J_1(\varphi) = \log \left[ e^{(\lambda/\sigma)^2 + M + \tau M} \left(1 + \tau^2 + \frac{\tau^2 \sigma^2}{\lambda^2}\right)^d \left(1 + \sqrt{\frac{3\pi}{2}} \frac{\sigma}{\lambda}\right)^d \right], \quad (194) \]

when \( d = 1 \).

The proof of the case \( d \geq 2 \) is given below, a similar argument in the \( d = 1 \) case is given in the appendix as Proposition A.4.
Proof. As in the proof of Theorem [IV.4],

\[
H(X|Y) = \sum_i p_i \int \varphi(w) \log \left(1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)}\right) dw
\]

(195)

\[
\leq \int_{B_\lambda^c} \varphi(w) \log \left(1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)}\right) dw + \int_{B_\lambda} \varphi(w) \log \left(1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)}\right) dw.
\]

(196)

(197)

We use the general bound from the proof of Theorem [IV.1] for

\[
\int_{B_\lambda^c} \varphi(w) \log \left(1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)}\right) dw \leq (M - 1) \mathbb{P}(|W| \leq \lambda \tau) + M \mathbb{P}(|W| > \lambda).
\]

(198)

(199)

Splitting the integral,

\[
\int_{B_\lambda} \varphi(w) \log \left(1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)}\right) dw
\]

(200)

\[
= \int_{B_\lambda^c} \varphi(w) \log \left(\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))\right) dw - \int_{B_\lambda} \varphi(w) \log \varphi(w) dw.
\]

(201)

Using Corollary [IV.4], Jensen’s inequality, and then Proposition [A.2]

\[
\int_{B_\lambda} \varphi(w) \log \left(\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))\right) dw
\]

(202)

\[
\leq \int_{B_\lambda} \varphi(w) \log \left[\tau^d M \left(1 + \tau + \frac{\tau^2 |w|}{\lambda}\right)^d \left(\|\varphi\|_\infty + \left(\frac{3}{\lambda}\right)^d \omega_d^{-1}\right)\right]
\]

(203)

\[
\leq \mathbb{P}(|W| > \lambda) \log \left[\tau^d M \left(1 + \tau + \frac{\tau^2 \int \mathbb{1}_{|w| > \lambda} \varphi(w)|w| dw}{\mathbb{P}(|W| > \lambda)} \right)^d \left(\|\varphi\|_\infty + \left(\frac{3}{\lambda}\right)^d \omega_d^{-1}\right)\right]
\]

(204)

\[
\leq \mathbb{P}(|W| > \lambda) \log \left[\tau^d M \left(1 + \tau + \frac{\tau^2 (\lambda + \sigma d)}{\lambda}\right)^d \left(\|\varphi\|_\infty + \left(\frac{3}{\lambda}\right)^d \omega_d^{-1}\right)\right].
\]

(205)

Then applying Proposition [A.1]

\[- \int_{B_\lambda^c} \varphi(w) \log \varphi(w) dw \leq \left(\frac{d}{2} \log 2\pi e^2 \sigma^2 + \lambda^2 / \sigma^2\right) \mathbb{P}(|W| > \lambda)
\]

(206)

Combining (199), (205), and (206) we have

\[
H(X|Y) \leq (M - 1) \mathbb{P}(|W| \leq \lambda \tau) + J_d(\varphi) \mathbb{P}(|W| > \lambda)
\]

(207)

with

\[
J_d(\varphi) = \log \left[e^{(\lambda/\sigma)^2 + M (\tau e \sqrt{2\pi})^d M} \left(1 + \tau + \frac{\tau^2 \sigma d}{\lambda}\right)^d \left(2\pi \sigma^2\right)^{-\frac{d}{2}} + \left(\frac{3}{\lambda}\right)^d \omega_d^{-1}\right]
\]

(208)

\[
= \log \left[e^{(\lambda/\sigma)^2 + M (\tau e \sqrt{2\pi})^d M} \left(1 + \tau + \frac{\tau^2 \sigma d}{\lambda}\right)^d \left(2\pi\right)^{-\frac{d}{2}} + \left(\frac{3\sigma}{\lambda}\right)^d \omega_d^{-1}\right].
\]

(209)

For example, when $X$ takes values $\{x_i\} \in \mathbb{R}$ such that $|x_i - x_j| \geq 2\lambda$ and $Y$ is given by $X + W$ where $W$ is independent Gaussian noise with variance $\sigma^2$, then $Y$ has density $\sum p_j f_i(y)$ with $f_i(y) = \varphi_\sigma(y - x_i)$. Let $T_i(y) = y - x_i$. Thus $T_{ij}(B_\lambda) = B_\lambda + x_j - x_i$, so that $\{T_{kj}(B_\lambda)\}_{k}$ are disjoint and we can take $M = 1$ and $\tau = 1$. Applying Proposition [IV.1] we have

\[
H(X|Y) = H(X|X + W) \leq J_1(\varphi) \mathbb{P}(|W| > \lambda) = J_1(\varphi) \mathbb{P}(|Z| > \lambda / \sigma)
\]

(210)
with
\[
J_1(\varphi) = \log \left[ e^{(\lambda/\sigma)^2 + 3} \left( 3 + \frac{\sigma}{\lambda^2} \right) \left( 1 + \sqrt{\frac{9\pi \sigma}{2 \lambda}} \right) \right].
\] (211)

Notice that for \( t > 0 \), \( tX \) and \( tX + tW \) satisfy the same conditions with \( \tilde{\lambda} = t\lambda \) and \( \tilde{\sigma} = t\sigma \), and since \( H(tX|tX + tW) = H(X|X + W) \), after applying the result to \( tX, tX + tW \) we have
\[
H(X|X + W) = H(tX|tX + tW)
\] (212)
\[
\leq \log \left[ e^{(\lambda/\sigma)^2 + 3} \left( 3 + \frac{\sigma}{\tilde{\lambda}^2} \right) \left( 1 + \sqrt{\frac{9\pi \sigma}{2 \tilde{\lambda}}} \right) \right] \mathbb{P}(|Z| > \lambda/\sigma).
\] (213)

Taking the limit with \( t \to \infty \), we obtain,
\[
H(X|X + W) \leq \log \left[ 3e^{(\lambda/\sigma)^2 + 3} \left( 1 + \sqrt{\frac{9\pi \sigma}{2 \lambda}} \right) \right] \mathbb{P}(|Z| > \lambda/\sigma).
\] (214)

We collect these observations in the following Corollary.

**Corollary V.2.** When \( X \) is a discrete \( \mathbb{R} \) valued random variable taking values \( \{x_i\} \) such that \( |x_i - x_j| \geq 2\lambda \) for \( i \neq j \) and \( W \) is an independent Gaussian variable with variance \( \sigma^2 \), then
\[
H(X|X + W) \leq \log \left[ 3e^{(\lambda/\sigma)^2 + 3} \left( 1 + \sqrt{\frac{9\pi \sigma}{2 \lambda}} \right) \right] \mathbb{P}(|Z| > \lambda/\sigma).
\] (215)

**A. Fano’s inequality**

A multiple hypothesis testing problem is described in the following, with \( X \) an index \( i \in \mathcal{X} \) is drawn and then samples are drawn from the distribution \( f_i \), with a goal of determining the value \( i \). If \( Z \) denotes a random variable with \( P(Z \in A|X = i) = \int_A f_i(z)dz \), then by the commutativity of mutual information proven in Proposition 1.3, \( H(X|Z) = h(Z|X) - h(Z) + H(X) \). Thus bounds on the mixture distribution are equivalent to bounds \( H(X|Z) \). For \( X = g(Z) \), Fano’s inequality provides the following bound
\[
H(X|Z) \leq H(e) + \mathbb{P}(e) \log(2^{\#\mathcal{X}} - 1)
\] (216)
where \( e = \{ \hat{X} \neq X \} \) is the occurrence of an error. Fano and Fano-like inequalities are important in multiple hypothesis testing, as they can be leveraged to deliver bounds on the Bayes risk (and hence min/max risk); we direct the reader to [7], [25], and [68] for more background. Fano’s inequality gives a lower bound on the entropy of a mixture distribution, that can also give a non-trivial improvement on the concavity of entropy through the equality \( H(X|Z) = H(X) + h(Z|X) - h(Z) \). Combined with (216),
\[
h(\sum_i p_i f_i) - \sum_i p_i h(f_i) \geq H(p) - (H(e) + \mathbb{P}(e) \log(2^{\#\mathcal{X}} - 1)).
\] (217)

In concert with with Theorem 1.1, we have the following corollary.

**Corollary V.3.** For \( X \) distributed on indices \( i \in \mathcal{X} \), and \( Z \) such that \( Z|\{X = i\} \) is distributed according to \( f_i \), then given an estimator \( \hat{X} = f(Z) \), with \( e = \{ X \neq \hat{X} \} \)
\[
(1 - T_f)H(X) \leq H(e) + \mathbb{P}(e) \log(|\mathcal{X}| - 1).
\] (218)

**Proof.** By Fano’s inequality \( H(e) + \mathbb{P}(e) \log(N - 1) \geq H(X|Z) \). Recalling that \( H(X|Z) = H(X) - (h(\sum_i p_i f_i) - \sum_i p_i h(f_i)) \) and by Theorem 1.1
\[
H(X) - (h(\sum_i p_i f_i) - \sum_i p_i h(f_i)) \geq H(X) - T_f H(X),
\] (219)
gives our result.

Heuristically, this demonstrates that “good estimators” are only possible for hypothesis distributions discernible in total variation distance. For example in the simplest case of binary hypothesis testing where \( n = 2 \), the inequality is \( (1 - \|f_1 - f_2\|_{TV})H(X) \leq H(e) \), demonstrating that existence of an estimator improving on the trivial lower bound \( H(X) \), is limited explicitly by the total variation distance of the two densities.

We note that the pursuit of good estimators \( \hat{X} \) is a non-trivial problem in most interesting cases, so much so that Fano’s inequality is often used to provide a lower bound on the potential performance of a general estimator by the ostensibly simpler
quantity $H(X|Z)$, as determining an optimal value for $\mathbb{P}(\epsilon)$ is often intractable. A virtue of Theorem IV.1 is that it provides upper bounds on $H(X|Z)$, in terms of tail bounds of a single log-concave variable $W$. Thus, Theorem IV.1 asserts that for a large class of models, $H(X|Z)$ can be controlled by a single easily computable quantity, which in the case that $M = 1$, decays sub-exponentially in $\lambda$ to 0. However, the example delineated below, demonstrates that even in simple cases where an optimal estimator of $X$ admits explicit computation, the bounds of Theorem IV.1 may outperform the best possible bounds based on Fano’s inequality.

Suppose that $X$ is uniformly distributed on $\{1, 2, \ldots, N\}$ and that $W$ is an independent, symmetric log-concave variable with density $\varphi$, and $Z = X + W$, then $Z$ has density $f(z) = \frac{1}{N} \sum_{i=1}^{N} f_i(z)$ with $f_i(z) = \varphi(z - i)$. The optimal (Bayes) estimator of $X$ is given by $\Theta(z) = \arg\max_i \{f_i(z) : i \in \{1, 2, \ldots, N\}\}$, which by the assumption of symmetric log-concavity can be expressed explicitly as:

$$\Theta(z) = \mathbb{1}_{(-\infty, \frac{1}{2})} + \sum_{i=2}^{N-1} i\mathbb{1}_{(i-\frac{1}{2}, i+\frac{1}{2})} + N\mathbb{1}_{(N-\frac{1}{2}, \infty)}.$$  \hfill (220)

Thus, $\mathbb{P}(\Theta \neq X)$ can be written explicitly as well. Indeed,

$$\mathbb{P}(X = \Theta(Z)) = \sum_i \mathbb{P}(X = i, i = \Theta(i + W))$$

$$= \frac{1}{N} \mathbb{P}(W \leq \frac{1}{2}) + \frac{1}{N} \mathbb{P}(W \geq -\frac{1}{2}) + \sum_{i=2}^{N-1} \frac{\mathbb{P}(W \in (-\frac{1}{2}, \frac{1}{2}))}{N}$$

$$= \mathbb{P}(W \in (-\frac{1}{2}, \frac{1}{2})) + \frac{2}{N}\mathbb{P}(W \geq \frac{1}{2}).$$  \hfill (223)

Thus writing $P(\epsilon) = \mathbb{P}(X \neq \Theta)$, we have

$$P(\epsilon) = 2\mathbb{P}(W \geq \frac{1}{2})\left(1 - \frac{1}{N}\right)$$

$$= \mathbb{P}(|W| \geq 1/2)\left(1 - \frac{1}{N}\right)$$

$$= \mathbb{P}(|X| \geq 1/2\sigma)\left(1 - \frac{1}{N}\right),$$  \hfill (226)

where $Z$ is a standard normal variable. Thus the optimal bounds achievable through Fano’s inequality are described by,

$$H(X|Z) \leq H(\epsilon) + P(\epsilon)\log(N - 1)$$ \hfill (227)

with $P(\epsilon) = \mathbb{P}(|Z| \geq 1/2\sigma)\left(1 - \frac{1}{N}\right)$. Note that with $N \to \infty$, the bounds attainable through Fano’s inequality become meaningless since $\lim_{N \to \infty} H(\epsilon) + P(\epsilon)\log(N - 1) = \infty$ independent of $\sigma$. In contrast, since $Z = X + W$, Corollary V.2 gives the following bound:

$$H(X|Z) \leq \log \left[3e^{(\lambda/\sigma)^2 + 3} \left(1 + \frac{\sqrt{\frac{9\pi\sigma}{2}}}{\lambda}\right)\right] \mathbb{P}(|Z| > \lambda/\sigma),$$  \hfill (228)

independent of $N$.

B. Channel Capacity

In the case of a channel that admits discrete inputs (and possibly continuous inputs as well) with output density $f_i(z) = p(z|i)$ when conditioned on an input $i$. Suppose the input $X$ takes value $i$ with probability $p_i$ then the output $Z$ distribution will have a density function $\sum_i p_i f_i$. Thus

$$I(Z; X) = h(Z) - h(Z|X) = H(X) - H(X|Z) = h(\sum_i p_i f_i) - \sum_i p_i h(f_i).$$ \hfill (229)

Thus, any choice of input $X$ gives a lower bound on the capacity of the channel. In the context of additive white Gaussian noise channel [48] gave rigorous bounds to the findings of [60], that finite input can nearly achieve capacity.
Theorem V.1 (Ozarow-Wyner [48]). Suppose $X$ is uniformly distributed on $N$ evenly spaced points, $\{2\lambda, 4\lambda, \ldots, 2N\lambda\}$ and its variance $E X^2 - E^2 X = \lambda^2 \left( \frac{3^2 - 1}{4} \right)$. If $Z = X + W$, where $W$ is Gaussian with variance one and independent of $X$. Then

1) \[ I(Z; X) \geq (1 - (\pi K)^{-1/2} e^{-K}) H(X) - h((\pi K)^{-1/2} e^{-K}) \] (232)

where

\[ K = \frac{3}{2}\alpha^2 (1 - 2^{-2C}) \] (233)

\[ \alpha = N 2^{-C} \] (234)

\[ C = \frac{1}{2} \log \left( 1 + \lambda^2 N^2 - \frac{1}{3} \right) \] (235)

2) \[ I(Z; X) \geq C - \frac{1}{2} \log \frac{\pi e}{6} \left( 1 + \frac{\alpha^2}{2} \right) \] (236)

In the notation of this paper $K = \frac{\lambda^2}{2} (1 - \frac{1}{\sqrt{\pi}})$. Defining,

\[ p_o := \frac{e^{-\frac{\lambda^2}{2} (1 - \frac{1}{\sqrt{\pi}})}}{\sqrt{\pi \lambda^2} (1 - \frac{1}{\sqrt{\pi}})} \] (237)

we can re-write (232) as

\[ I(X|Z) \geq H(X) - (p_o H(X) + h(p_o)) \] (238)

Note that $N 2^{-C} = \left( \frac{N}{\lambda + 1/2} \right)^{-1} = \sqrt{\frac{1 + 3(\sigma/\lambda)^2}{1 + \sigma^2}}$, so that $\alpha \approx \sqrt{3}/\lambda$ for $\sigma$ large. Thus (232) gives a bound with sub-Gaussian-like convergence in $\lambda$ to $H(X)$ for fixed $N$, but gives worse than trivial bounds for fixed $\lambda$ and $N \to \infty$. In contrast, by Corollary V.2 gives

\[ I(X|Z) \geq H(X) - \log \left[ 3 e^{\lambda^2 + \frac{3}{2}} \left( 1 + \sqrt{\frac{9\pi \sigma}{2 \lambda}} \right) \right] P(|Z| > \lambda) \] (239)

Comparing the bound on the gap between $I(X|Z)$ and $H(X)$ provided by (238) and Corollary V.2 we see that Corollary V.2 outperforms Theorem V.1 for large $\lambda$. Indeed, one can easily find an explicit rational function $q$ such that

\[ \frac{p_o H(X) + h(p_o)}{\log \left[ 3 e^{\lambda^2 + \frac{3}{2}} \left( 1 + \sqrt{\frac{9\pi \sigma}{2 \lambda}} \right) \right]} \geq q(\lambda) e^{\frac{\lambda^2}{2\pi}} \] (240)

Additionally, (239) gives a universal bound, independent of $N$.

These results have been of recent interest, see for example [18], [19], where the results improving and generalizing [2] have been studied in a form

\[ H(X) - \text{gap}^* \leq I(Z; X) \leq H(X), \] (241)

with an emphasis on achieving $\text{gap}^*$ bounds that are independent of $N$, and viable for more general noise models. The significance of the results of Theorem [IV.1] in this context is that the $\text{gap}^*$ bounds provided converge exponentially fast to zero in $\lambda$, independent of $H(X)$, while for example in [19], the $\text{gap}^*$ satisfies

\[ \text{gap}^* \geq \frac{1}{2} \log \frac{2\pi e}{12}. \] (242)

Additionally, the tools developed can be extended to perturbations of the $Y$ and signal dependent noise through Theorem [IV.1].

A related investigation of recent interest is the relationship between finite input approximations of capacity achieving distributions, particularly the number of “constellations” needed to approach capacity. For example [65], [66] the rate of convergence in $n$ of the capacity of an $n$ input power constrained additive white Gaussian noise channel to the usual additive white noise Gaussian channel is obtained. In many practical situations, although a Gaussian input is capacity achieving, discrete inputs are used. We direct the reader to [67] for background on the role this practice plays in Multiple Input- Multiple Output channels pivotal in the development of 5G technology.

Additionally in the amplitude constrained discrete time additive white Gaussian noise channel, the capacity achieving distribution is itself discrete [55]. In fact, many important channels achieve capacity for discrete distributions, see for example [2], [14], [28], [24], [61]. Thus in the case that the noise model is independent of input, the capacity achieving output will be a mixture distribution, and the capacity of the channel is given by calculating the entropy of said mixture.
Theorem [V.1] shows that for sparse input, relative to the strength of the noise, the mutual information of the input and output distributions is sub-exponentially close to the entropy of the the input in the case of log-concave noise, and sub-Gaussian from the the entropy of the input in the case of strongly log-concave noise, which includes Gaussian noise as a special case. In contrast Theorem [I.1] gives a reverse inequality, demonstrating that when the mixture distributions are close to one another in the sense that their total variation distance from the mixture “with themselves removed” is small, then the mutual information is quantifiably lessened.

C. Energetics of Non-equilibrium thermodynamics

For a process \( x_t \) that satisfies an overdamped Langevin stochastic differential equation, 
\[
\begin{align*}
\dot{x}_t &= -\nabla U(x_t, t) + \sqrt{2\gamma} \zeta_t, \\
\zeta_t &\sim \mathcal{N}(0, \gamma dt),
\end{align*}
\]
with time varying potential \( U : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) and \( \zeta_t \) a Brownian motion with \( D = k_B T / \gamma \) where \( \gamma \) is the viscosity constant, \( k_B \) is Boltzman’s constant and \( T \) is temperature, one can define natural thermodynamic quantities, in particular trajectory dependent notions of work done \( W \) on the system (see [29]) and heat dissipated \( Q \), respectively,
\[
W := \int_0^{t_f} \partial_t U(x_t, t) dt
\]
and
\[
Q := -\int_0^{t_f} \nabla_x U(x_t, t) \circ dx_t,
\]
where the above is a Stratonovich stochastic integral. Recall that Stratonovich integrals satisfy a chain rule
\[
\int_0^{t_f} \left( \partial_t U(x_t, t) + \nabla_x U(x_t, t) \circ dx_t \right) dt = \int_0^{t_f} \partial_t U(x_t, t) dt + \int_0^{t_f} \nabla_x U(x_t, t) \circ dx_t
\]
where
\[
\Delta U := U(t_f, x(t_f)) - U(0, x(0))
\]
Further, if \( \rho_0 \) denotes the distribution of \( x_t \) at time \( t \), satisfying the Fokker-Planck equation then it can be shown [4] (see also [15], [38], [53]).
\[
\mathbb{E} Q = k_B T \left( h(\rho_0) - h(\rho_{t_f}) \right) + \int_0^{t_f} \mathbb{E} |v(t, x_t)|^2 dt,
\]
where \( v \) is mean local velocity (see [4] or as the current velocity in [43]). In the quasistatic limit where the non-negative term \( \int_0^{t_f} \mathbb{E} |v(t, x_t)|^2 dt \) goes to 0, one has a fundamental lower bound on the efficiency of a process’s evolution, the average heat dissipated in a transfer from configuration \( \rho_0 \) to \( \rho_{t_f} \) is bounded below by the change in entropy.
\[
\mathbb{E} Q = k_B T \left( h(\rho_0) - h(\rho_{t_f}) \right).
\]
A celebrated example of this inequality is Landauer’s principle [31], which proposes fundamental thermodynamic limits to the efficiency of a computer utilizing logically irreversible computations (see also [6]). More explicitly (249) suggests that the average heat dissipated in the erasure of a bit, that is, the act of transforming a random bit to a deterministic 0 is at least \( k_B T \log 2 \). This can be reasoned to in the above, by presuming the entropy of a random bit should satisfy \( h(\rho_0) = \log 2 \) and that the reset bit should satisfy \( h(\rho_{t_f}) = 0 \).

In the context of nanoscale investigations, (like protein pulling or the intracellular transport of cargo by molecular motors) it is often the case that phenomena take one of finitely many configurations with an empirically derived probability. However at this scale, thermal fluctuations can make discrete modeling of the phenomena unreasonable, and hence the distributions \( \rho_0 \) and \( \rho_{t_f} \) in such problems are more accurately modeled as a discrete distribution disrupted by thermal noise, and are thus, mixture distributions. Consequently bounds on the entropy of mixture distributions translate directly to bounds on the energetics of nanoscale phenomena [41], [56]. For example, in the context of Landauer’s bound, the distribution of the position of a physical bit is typically modeled by a Gaussian bistable well, explicitly by the density
\[
f_p(z) = p e^{-(x-a)^2/2\sigma^2} / \sqrt{2\pi\sigma^2} + (1-p) e^{-(x+a)^2/2\sigma^2} / \sqrt{2\pi\sigma^2}.
\]
The variable \( p \) denotes the probability that the bit takes the value 1, and \( (1-p) \) the probability the bit takes the value 0. This can be modeled by \( X_p \) a Bernoulli variable taking values \( \pm a \) and \( Z_p = X_p + \sigma Z \) where \( Z \) is a standard normal, so that \( Z_p \) has distribution \( f_p \).

Corollary V.4. The average heat dissipated \( Q_0 \) in an optimal erasure protocol, resetting a random bit to zero in the framework of (250) can be bounded above and below,
\[
\hat{C}_L \mathbb{P}(|Z| > a/\sigma) \leq \mathbb{E} Q_0 - k_B T \log 2 \leq \hat{C}_U \mathbb{P}(|Z| > a/\sigma)
\]
where \( \bar{C}_L = -k_B T \left( \log \left[ 3e^{(a/\sigma)^2/3} \left( 1 + \sqrt{\frac{9\pi}{2} \frac{\sigma}{a}} \right) \right] \right) \) and \( \bar{C}_U = k_B T \left( \log \left[ 3e^{(a/\sigma)^2/3} \left( 1 + \sqrt{\frac{9\pi}{2} \frac{\sigma}{a}} \right) \right] \right). \\

More generally, in the case that the erasure is imperfect, so that the probability of failure is non-negligible we have the following bound,

\[
C_L \mathbb{P}(|Z| > a/\sigma) \leq \mathbb{E}Q_0 - k_B T (H(p_0) - H(p_1)) \leq C_U \mathbb{P}(|Z| > a/\sigma) 
\]

(252)

where

\[
C_L = -k_B T \left( \log \left[ 3e^{(a/\sigma)^2/3} \left( 1 + \sqrt{\frac{9\pi}{2} \frac{\sigma}{a}} \right) \right] - H(X_{p_1}) \right) 
\]

(253)

\[
C_U = k_B T \left( \log \left[ 3e^{(a/\sigma)^2/3} \left( 1 + \sqrt{\frac{9\pi}{2} \frac{\sigma}{a}} \right) \right] - H(X_{p_0}) \right). 
\]

(254)

**Proof.** First let us note that we understand a random bit to be the case that \( p_0 = \frac{1}{2} \), while an erased bit is to be understood as a deterministic \( X \) with \( p_1 = 0 \). Thus, (251) follows immediately from (252) If we let \( p_0 = f_{p_0} \) and \( p_1 = f_{p_1} \), and let \( Z_{p_i} = X_{p_i} + \sigma Z \) denote a variable then (249) gives

\[
\mathbb{E}Q_0 = k_B T (h(f_{p_0}) - h(f_{p_1})) = k_B T (I(Z_{p_0}; X_{p_0}) - I(Z_{p_1}; X_{p_1})), 
\]

(255)

(256)

where the second equality follows from the fact that both \( Z_{p_i} \) variables are conditionally Gaussian of the same variance. Using the corollary of Theorem 1.3 obtained in (228),

\[
I(Z_{p_1}; X_{p_1}) \geq H(X_{p_1}) - \log \left[ 3e^{(a/\sigma)^2/3} \left( 1 + \sqrt{\frac{9\pi}{2} \frac{\sigma}{a}} \right) \right] \mathbb{P}(|Z| > a/\sigma), 
\]

(257)

while Theorem 1.4 applied as in (4) as

\[
I(Z_{p_1}; X_{p_1}) \leq T_f H(X_{p_1}) = H(X_{p_1}) - \mathbb{P}(|Z| > a/\sigma)H(X_{p_1}), 
\]

(258)

(259)

since \( T_f = 1 - \mathbb{P}(|Z| > a/\sigma) \). Combining these results gives

\[
I(Z_{p_0}; X_{p_0}) - I(Z_{p_1}; X_{p_1}) \leq H(X_{p_0}) - H(X_{p_1}) + \mathbb{P}(|Z| > a/\sigma) \left( \log \left[ 3e^{(a/\sigma)^2/3} \left( 1 + \sqrt{\frac{9\pi}{2} \frac{\sigma}{a}} \right) \right] - H(X_{p_0}) \right) 
\]

(260)

\[
I(Z_{p_0}; X_{p_0}) - I(Z_{p_1}; X_{p_1}) \geq H(X_{p_0}) - H(X_{p_1}) - \mathbb{P}(|Z| > a/\sigma) \left( \log \left[ 3e^{(a/\sigma)^2/3} \left( 1 + \sqrt{\frac{9\pi}{2} \frac{\sigma}{a}} \right) \right] - H(X_{p_1}) \right). 
\]

(261)

Inserting these equations into (256) completes the proof. \( \square \)

**D. Functional inequalities**

Mixture distributions arise naturally in mathematical contexts as well. For example in [9] Bobkov and Marsiglietti found interesting application of \( h(X + Z) \leq H(X) + h(Z) \) for \( X \) discrete and \( Z \) independent and continuous in the investigation of entropic Central Limit Theorem for discrete random variables under smoothing.

In the study of stability in the Gaussian log-Sobolev inequalities, Eldan, Lehec, and Shenfeld [20], it is proven as Proposition 5 that the deficit in the Gaussian log-Sobolev inequality, defined as

\[
\delta(\mu) = \frac{I(\mu||\gamma)}{2} - D(\mu||\gamma) 
\]

(262)

for a measure \( \mu \) and \( \gamma \) the standard \( d \)-dimensional Gaussian measure, and \( I \) is relative Fisher information,

\[
I(\mu||\gamma) := \int_{\mathbb{R}^d} \log \left( \frac{d\mu}{d\gamma} \right) d\gamma, 
\]

(263)

is small for Gaussian mixtures. More explicitly for \( p_i \) non-negative numbers summing to 1,

\[
\delta \left( \sum_i p_i \gamma_i \right) \leq H(p). 
\]

(264)

In the language of Theorem 1.1 a sharper bound can be achieved.
Corollary V.5. When $\gamma_i$ are translates of the standard Gaussian measure then

$$\delta \left( \sum_i p_i \gamma_i \right) \leq T \mathcal{H}(p),$$

where $T$ is defined as in Theorem I.1.

Proof. By the convexity of the relative Fisher information, the equality $D(\sum_i p_i \gamma_i \| \gamma) = \sum_i p_i D(\gamma_i \| \gamma) + h(\sum_i p_i \gamma_i) - \sum_i p_i h(\gamma_i), \frac{I(\gamma_i \| \gamma)}{2} - D(\gamma_i \| \gamma) = 0$, and the application of Theorem I.1 we have

$$\delta \left( \sum_i p_i \gamma_i \right) = \frac{I(\gamma_i \| \gamma)}{2} - D(\gamma_i \| \gamma)$$

$$\leq \sum_i p_i \left( \frac{I(\gamma_i \| \gamma)}{2} - D(\gamma_i \| \gamma) \right) + \left( h(\sum_i p_i \gamma_i) - h(\sum_i p_i h(\gamma_i)) \right)$$

$$= h(\sum_i p_i \gamma_i) - \sum_i p_i h(\gamma_i)$$

$$\leq T \mathcal{H}(p).$$

\[ \square \]

VI. Conclusions

In this article, the entropy of mixture distributions is estimated by providing tight upper and lower bounds. The efficacy of the bounds is demonstrated, for example, by demonstrating that existing bounds on the conditional entropy, $H(X|Z)$ of a random variable, $X$ taking values in a countable set, $X$ conditioned on a continuous random variable, $Z$, become meaningless as the cardinality of the set $X$ increases while the bounds obtained here remain relevant. Significantly enhanced upper bounds on mutual information of channels that admit discrete input with continuous output are obtained based on the bounds on the entropy of mixture distributions. The technical methodology developed is of interest in its own right whereby connections to existing results either can be derived as corollaries of more general theorems in the article or are improved upon by the results in the article. These include the reverse Pinsker inequality, and bounds on Jensen-Shannon divergence, and bounds that are obtainable via Fano’s inequality.

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APPENDIX

Proof of Proposition III.3. There is nothing to prove in (1), this is exactly the definition of the usual relative entropy from $\mu$ to $t\mu + (1-t)\nu$. For (2), by (1) $S_t(\mu \| \nu) = 0$ iff $D(\mu \| t\mu + (1-t)\nu) = 0$ which is true iff $\mu = t\mu + (1-t)\nu$ which happens iff $t = 1$ or $\mu = \nu$. To prove (3), observe that for a Borel set $A$

$$\mu(A) \leq \frac{1}{t} (t\mu + (1-t)\nu)(A).$$

This gives the following inequality, from which absolute continuity, and the existence of $\frac{d\mu}{d(t\mu + (1-t)\nu)}$ follow immediately.

$$\frac{d\mu}{d(t\mu + (1-t)\nu)} \leq \frac{1}{t}. $$

(270)

Integrating (270) gives

$$S_t(\mu \| \nu) \leq - \log t.$$

(271)

To prove (4), notice that for fixed $\mu$ and $\nu$, the map $\Phi_t = t\mu + (1-t)\nu$ is affine, and since the relative entropy is jointly convex, convexity in $t$ follows from the computation below.

$$S_{(1-\lambda)t_1 + \lambda t_2}(\mu \| \nu) = D(\mu \| \Phi_{(1-\lambda)t_1 + \lambda t_2})$$

$$= D(\mu \| (1-\lambda)\Phi_{t_1} + \lambda \Phi_{t_2})$$

$$\leq (1-\lambda)D(\mu \| \Phi_{t_1}) + \lambda D(\mu \| \Phi_{t_2})$$

$$= (1-\lambda)S_{t_1}(\mu \| \nu) + \lambda S_{t_2}(\mu \| \nu).$$
Since \( t \mapsto S_t(\mu||\nu) \) is a non-negative convex function on \((0, 1]\) with \( S_1(\mu||\nu) = 0 \) it is necessarily non-increasing. When \( \mu \neq \nu, \mu \neq t\mu + (1-t)\nu \) so that \( S_t(\mu||\nu) > 0 \) for \( t < 1 \), so that as a function of \( t \) the skew divergence is strictly decreasing. To prove that \( S_t \) is an \( f \)-divergence recall Definition \[ 3.1 \] It is straightforward that \( S_t \) can be expressed in form \((60)\) with \( f(x) = x \log(x/(tx + (1-t))) \). Convexity of \( f \) follows from the second derivative computation,

\[
f''(x) = \frac{(t-1)^2}{x(t(1-t))^2} > 0.
\]

Since \( f(1) = 0 \) the proof is complete. \( \Box \)

In this section we consider \( W \sim \varphi \sigma \) with \( \varphi(\omega) = e^{-|\omega|^2/2\sigma^2}/(2\pi\sigma^2)^{\frac{d}{2}} \), and use \( \varphi \) to denote \( \varphi_1 \) and use \( Z \) in place of \( W \) in this case.

**Proposition A.1.** For \( d \geq 2 \)

\[
-\int_{B_\lambda^d} \varphi(\omega) \log \varphi(\omega) d\omega \leq \left( \frac{d}{2} \log 2\pi e^2 \sigma^2 + \frac{\lambda^2}{\sigma^2} \right) \mathbb{P}(|W| > \lambda)
\]  

(272)

**Proof.** We first show that the result for general \( \sigma \) follows from the case that \( \sigma = 1 \). Indeed, assuming (272), the substitution \( u = \varphi / \sigma \) gives

\[
\int_{B_{\lambda/\sigma}^d} \varphi(\omega) \log \varphi(\omega) d\omega = \mathbb{P}(|Z| > \lambda/\sigma) d\log \sigma - \int_{B_{\lambda/\sigma}^d} \varphi(u) \log \varphi(u) du
\]

\[
\leq \left[ \frac{d}{2} \log 2\pi e^2 \sigma^2 + \frac{\lambda^2}{\sigma^2} \right] \mathbb{P}(|Z| > \lambda/\sigma),
\]

(274)

where we have applied (272) to achieve the inequality. Since \( W \) has the same distribution as \( \sigma Z \), the reduction holds. By direct computation,

\[
-\int_{B_\lambda^d} \varphi(\omega) \log \varphi(\omega) d\omega = \int_{B_\lambda^d} \varphi(\omega) \frac{d}{2} \log 2\pi d\omega + \int_{B_\lambda^d} \frac{|\omega|^2}{2} \varphi(\omega)
\]

\[
= \mathbb{P}(|Z| > \lambda) \left( \frac{d}{2} \log 2\pi + \frac{(2\pi)^{-d/2} \omega d \int_\lambda^\infty \rho^{d+1} e^{-r^2/2} dr}{(2\pi)^{-d/2} \omega d \int_\lambda^\infty \rho^{d-1} e^{-r^2/2} dr} \right)
\]

\[
= \mathbb{P}(|Z| > \lambda) \left( \frac{d}{2} \log 2\pi e^2 + \frac{\lambda^2 e^{-\lambda^2/2}}{\int_\lambda^\infty \rho^{d-1} e^{-r^2/2} dr} \right)
\]

(277)

Using \( \rho^{d-1} \geq r \lambda^{d-2} \) for \( r \geq \lambda \) when \( d \geq 2 \),

\[
\int_\lambda^\infty \rho^{d-1} e^{-r^2/2} dr \geq \lambda^{d-2} \int_\lambda^\infty r e^{-r^2/2} dr = \lambda^{d-2} e^{-r^2/2}.
\]

(279)

Thus,

\[
-\int_{B_\lambda^d} \varphi(\omega) \log \varphi(\omega) d\omega \leq \mathbb{P}(|Z| > \lambda) \left( \frac{d}{2} \log 2\pi e^2 + \frac{\lambda^2 e^{-\lambda^2/2}}{\lambda^{d-2} e^{-r^2/2} dr} \right)
\]

(280)

**Proposition A.2.** For \( d \geq 2 \),

\[
\int_{B_\lambda^d} \varphi(w) w |dw| \leq (\lambda + d\sigma) \mathbb{P}(|W| > \lambda)
\]

(281)

**Proof.** Again we reduce to the case that \( \sigma = 1 \). Substituting \( u = w/\sigma \) gives

\[
\int_{B_{\lambda/\sigma}^d} \varphi_\sigma(\omega) w |dw| = \sigma \int_{B_{\lambda/\sigma}^d} \varphi(u) |u| du
\]

\[
\leq \left( \frac{\lambda}{\sigma} + d \right) \sigma \mathbb{P}(|Z| > \lambda/\sigma)
\]

(283)

\[
= (\lambda + d\sigma) \mathbb{P}(|W| > \lambda),
\]

(284)
where we have used (281) for the inequality and \( \sigma Z \) being equidistributed with \( W \) for the last equality. We now proceed in the reduced case. By change of coordinates and integration by parts,

$$
\int_{B^c} \varphi(w)|w|dw = \left[ \int_{B^c} r^d e^{-r^2/2}dr \right]_{\lambda}^{r_0} - \int_{\lambda}^{r_0} r^{d-1} e^{-r^2/2} dr + d \int_{\lambda}^{r_0} r^{d-1} e^{-r^2/2} dr
$$

(285)

$$
= \frac{\lambda^{d-1} e^{-\lambda^2/2}}{\int_{\lambda}^{r_0} r^{d-1} e^{-r^2/2} dr} + d
$$

(286)

$$
= \frac{\lambda^{d-1} e^{-\lambda^2/2}}{\int_{\lambda}^{r_0} r^{d-1} e^{-r^2/2} dr} + d
$$

(287)

Using \( r \geq \lambda \), for \( d \geq 2 \),

$$
\int_{\lambda}^{r_0} r^{d-1} e^{-r^2/2} dr \geq \lambda^{d-2} \int_{\lambda}^{r_0} r e^{-r^2/2} dr = \lambda^{d-2} e^{-\lambda^2/2}.
$$

(288)

Thus,

$$
\frac{\int_{B^c} \varphi(w)|w|dw}{\Pr(|Z| > \lambda)} \leq \lambda + d.
$$

(289)

\( \square \)

**Proposition A.3.** When \( d = 1 \), so that \( W \sim \varphi_\sigma(w) = e^{-w^2/2\sigma^2}/\sqrt{2\pi\sigma^2} \), we have the following bounds for \( \lambda > 0 \),

$$
\sigma^2 \varphi_\sigma(\lambda) = \int_\lambda^{\infty} w \varphi_\sigma(w) dw \leq \int_\lambda^{\infty} \varphi_\sigma(w) dw \left( \lambda + \frac{\sigma^2}{\lambda} \right)
$$

(290)

$$
- \int_{\lambda}^{\infty} \varphi_\sigma(w) \log \varphi_\sigma(w) dw \leq \left( \lambda^2/\sigma^2 + 2 + \log(\sqrt{2\pi}\sigma) \right) \int_{\lambda}^{\infty} \varphi_\sigma(w) dw
$$

(291)

*Proof.* The inequality (290) is standard. The inequality can be reduced to the \( \sigma = 1 \) by applying (290) after change of variables \( u = w/\sigma \). The proof then follows from the \( \sigma = 1 \) case. Recall \( \varphi'(w) = w\varphi(w) \) and observe that the function

$$
g(\lambda) = \int_{\lambda}^{\infty} \varphi(w) dw - \frac{\lambda^2}{\lambda^2 + 1} \varphi(\lambda)
$$

(292)

satisfies \( g(0) > 0 \), \( \lim_{\lambda \to \infty} g(\lambda) = 0 \), and has derivative

$$
- \frac{2\varphi(\lambda)}{(\lambda^2 + 1)^2} < 0,
$$

(293)

so that \( g(\lambda) > 0 \) which is equivalent to (290). To prove (291), we again reduce to the \( \sigma = 1 \) case by the substitution \( u = w/\sigma \). Then compute directly using integration by parts,

$$
- \int_{\lambda}^{\infty} \varphi(w) \log \varphi(w) dw = \int_{\lambda}^{\infty} \log \sqrt{2\pi}\varphi(w) dw + \int_{\lambda}^{\infty} w^2\varphi(w) dw
$$

(294)

$$
= \log \sqrt{2\pi} \int_{\lambda}^{\infty} \varphi(w) dw + \lambda \varphi(\lambda) + \int_{\lambda}^{\infty} \varphi(w) dw
$$

(295)

$$
\leq \left( \lambda^2 + 2 + \log(\sqrt{2\pi}) \right) \int_{\lambda}^{\infty} \varphi(w) dw.
$$

(296)

The inequality is an application of (290). \( \square \)

**Proposition A.4.** When \( X \) and \( Z \) satisfy the conditions of section IV for the one dimensional Gaussian \( W \sim \varphi_\sigma(w) = e^{-w^2/2\sigma^2}/\sqrt{2\pi\sigma^2} \),

$$
H(X|Z) \leq (M - 1)\Pr(|W| \leq \tau \lambda) + J(\varphi)\Pr(|W| \geq \lambda)
$$

(297)

with

$$
J(\varphi) = \log \left[ e^{(\lambda/\sigma)^2 + M + 2\tau} M \sigma \sqrt{2\pi} \left( 1 + \tau + \tau^2 + \frac{\tau^2 \sigma}{\lambda^2} \right) \left( (2\pi \sigma^2)^{-1} + \frac{3}{2\lambda} \right) \right]
$$

(298)
Proof. As in the proof of Theorem IV.1

\[
H(X|Z) = \sum_i p_i \int \varphi(w) \log \left( 1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{p_i \varphi(w)} \right) dw
\]  

(299)

\[
\leq \int_{B_x} \varphi(w) \log \left( 1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)} \right) dw
\]  

(300)

\[
+ \int_{B_x} \varphi(w) \log \left( 1 + \frac{\sum_j p_j \sum_{i \neq j} \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)} \right) dw.
\]  

(301)

with

\[
\int_{B_x} \varphi(w) \log \left( 1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)} \right) dw
\]  

(302)

\[
\leq (M - 1) \mathbb{P}(|W| \leq \lambda \tau) + M \mathbb{P}(|W| > \lambda).
\]  

(303)

Splitting the integral,

\[
\int_{B_x} \varphi(w) \log \left( 1 + \frac{\sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w))}{\varphi(w)} \right) dw
\]  

(304)

\[
= \int_{B_x} \varphi(w) \log \left( \sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w)) \right) dw - \int_{B_x} \varphi(w) \log \varphi(w) dw.
\]  

(305)

Using Corollary IV.4, Jensen’s inequality, and (290),

\[
\int_{B_x} \varphi(w) \log \left( \sum_j p_j \varphi(T_{ji}(w)) \det(T'_{ji}(w)) \right) dw
\]  

(306)

\[
\leq \int_{B_x} \varphi(w) \log \left[ \tau M \left( 1 + \tau + \frac{\tau^2 |w|}{\lambda} \left( \|\varphi\|_\infty + \left( \frac{3}{\lambda} \right)^{\frac{1}{2}} \right) \right) \right]
\]  

(307)

\[
\leq \mathbb{P}(|W| > \lambda) \log \left[ \tau M \left( 1 + \tau + \frac{\tau^2 \mathbb{1}_{|w| > \lambda} \varphi(w)|w|}{\mathbb{P}(|W| > \lambda) \lambda} \right) \left( \frac{1}{\sqrt{2\pi\sigma}} + \frac{3}{2\lambda} \right) \right]
\]  

(308)

\[
\leq \mathbb{P}(|W| > \lambda) \log \left[ \tau M \left( 1 + \tau + \frac{\tau^2 (\lambda + \frac{2}{\lambda})}{\lambda} \right) \left( \frac{1}{\sqrt{2\pi\sigma}} + \frac{3}{2\lambda} \right) \right].
\]  

(309)

Then applying (291),

\[- \int_{B_x} \varphi(w) \log \varphi(w) dw \leq \left( \frac{\lambda}{\sigma} \right)^2 + 2 + \log \sqrt{2\pi\sigma} \right\} \mathbb{P}(|W| > \lambda)
\]  

(310)

Combining 303, 309 and 310 we have

\[H(X|Z) \leq (M - 1) \mathbb{P}(|W| \leq \lambda \tau) + J(\varphi) \mathbb{P}(|W| > \lambda)
\]  

(311)

with

\[J(\varphi) = \log \left[ e^{\left( \frac{\lambda}{\sigma} \right)^2 + M + 2 \tau M \sigma \sqrt{2\pi} \left( 1 + \tau + \frac{\tau^2 (\lambda + \frac{2}{\lambda})}{\lambda} \right)^{\frac{1}{2}} + \frac{3}{2\lambda} \right] \]  

(312)

\]
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