Quantum-Classical Correspondence for Isolated Systems of Interacting Particles: Localization and Ergodicity in Energy Space *

F.M. Izrailev

Instituto de Física, Universidad Autónoma de Puebla, Apartado Postal J-48, 72570, Puebla, México

Generic properties of the strength function (local density of states (LDOS)) and chaotic eigenstates are analyzed for isolated systems of interacting particles. Both random matrix models and dynamical systems are considered in the unique approach. Specific attention is paid to the quantum-classical correspondence for the form of the LDOS and eigenstates, and to the localization in the energy shell. New effect of the non-ergodicity of individual eigenstates in a deep semiclassical limit is briefly discussed.

PACS numbers: 05.45.-a, 31.25.-v, 31.50.+w, 32.30.-r

I. INTRODUCTION

In recent years, the growing attention was paid to the so-called Quantum Chaos, (see e.g. [1] and references therein). Nowadays, this term is used in very different situations and often leads to a kind of confusion. To clarify the subject, one needs to remind that the origin of this term relates to quantum systems which, first, have the well-defined classical limit, and second, in this limit the corresponding classical system is assumed to manifest strong chaotic properties. One should stress that systems under consideration are dynamical ones (or, the same, there are no random parameters in their description). Therefore, this term was used in relation to the problem of the quantum-classical correspondence for dynamically chaotic systems.

Later, it was discovered that distinctive properties of Quantum Chaos (properties of quantum systems with the dynamical chaos in the classical limit) are generic for many other physical systems. For this reason, on recent conferences and workshops on Quantum Chaos there were many talks devoted to specific problems in atomic and nuclear physics, molecular and solid state physics, optics and acoustics etc.

In order to classify the subject, we suggest to use global terms Quantum Complexity and Wave Chaos. The first term refers quantum systems with complex behavior (or complex properties of spectrum and eigenstates), both with and without the classical limit (see Fig.1). Therefore, we stress that properties of complex quantum systems may have generic features, although the mechanism can be different. In the case of Quantum Chaos the mechanism is closely related to the deterministic classical chaos, in contrast to quantum systems without the classical limit, where the mechanism of complexity is either due to a disorder (Disordered Chaos) or due to dynamical reasons (quite complex interactions, although dynamical ones). As for the Wave Chaos, it refers to the chaotic properties of classical systems described by wave equations which are similar to the quantum Schrödinger equation. Therefore, many properties of such classical systems have much in common with those of Quantum Chaos and Quantum Disorder. There are many physical situations in electrodynamics, optics, acoustics, etc., where chaotic properties of systems are well described by the methods developed in the study of Quantum Complexity.

Below we discuss generic features of complex quantum systems, both dynamical and random ones, paying the main attention to the structure of strength functions and chaotic eigenstates. In what follows we consider the models of isolated systems of interacting particles which can be represented as a sum of the unperturbed Hamiltonian $H_0$ and the interaction $V$,

$$H = H_0 + V.$$  \hspace{1cm} (1)

Here $H_0$ describes finite number of non-interacting particles or quasi-particles. The term $V$ stands for a two-body interaction between the particles, and it is assumed to be responsible for chaotic properties of the total Hamiltonian $H$. For disordered models the interaction $V$ is random by assumption, however, the two-body nature of an interaction leads to important restrictions for chaos, see below. For dynamical systems the mechanism of chaos is not obvious, however, it is known to be related to classical chaos, if the classical limit exists. As for dynamical quantum systems without classical limit, this...

*presented to the Proceedings of the Nobel Simposia “Quantum Chaos Y2K”
case is less studied, however, the knowledge of properties of Quantum Chaos and Disordered Chaos helps very much. We would like to stress that both the Hamiltonian $H_0$ and interaction $V$ can be integrable (for dynamical systems), however, the total Hamiltonian $H$ may reveal strong chaotic properties.

Such a separation of a total Hamiltonian (1) into two parts is common in the description of complex atoms and nuclei and is known as the "mean field approximation". The core of this approximation is to choose such a basis in which regular part of the total Hamiltonian is absorbed in $H_0$ thus resulting in new single-particle states (quasi-particles). In contrast, the residual interaction $V$ accumulates all other terms which can not be embedded into $H_0$ due to their very complicated structure. We note that the choice of the mean field is not well-defined procedure, however, one can expect that if the most regular features of a system are described by $H_0$, main results are not very sensitive to a specific choice of the mean field.

II. RANDOM MATRIX MODELS

In the limit case of a very strong and "chaotic" interaction $V \gg H_0$, it is naturally to assume that system may be approximately represented by a completely random matrix of a given symmetry, this idea was in the origin of the Random Matrix Theory (see, for example, [2] and references therein). Giving correct predictions for statistical fluctuations of energy levels of complex physical systems, such as heavy nuclei and highly excited many-electron atoms, full random matrices are too general and can not describe global properties of physical systems such as dependencies on the energy, strength of interaction, number of interacting particles etc. It is interesting to note that one of the first random matrix models studied in 1955 by Wigner (3) has more complicated form in comparison with full random matrices. In analogy with (3), it consists of two parts, one of which is diagonal matrix with increasing entries $\epsilon_j$ and another is a band matrix $V_{ij}$ with random elements inside the band,

$$H_{ij} = \epsilon_j \delta_{ij} + V_{ij}. \quad (2)$$

In original papers (3) the "unperturbed" spectrum was taken in the form of "picked fence", $\epsilon_j = jD$, where $D$ is the spacing between two close energies and $j$ is a running integer number, however, more generic case corresponds to random values $\epsilon_j$ with the mean spacing $D$, reordered in an increasing way ($\delta_{ij}$ is the delta-function). As for off-diagonal matrix elements $V_{ij}$, they are assumed to be random and independent variables inside the band $|i - j| \leq b$, with the zero mean and given variance, $< V_{ij} = 0 >$ and $< V_{ij}^2 > = V_0^2$. Outside the band, the matrix elements equal to zero. Thus, the control parameters of this model are the ratio $V_0/D$ of a typical matrix element to the mean spacing, and the band-width $b$. As one can see, the first term in (2) corresponds to the "mean field" $H_0$ and the fact that the interaction has a finite energy range is directly taken into account.

In spite of impressive results of the RMT in many different applications (see e.g. [3]), standard random matrix ensembles suffer from the lack of information about the nature of interparticle interaction. For this reason a new kind of random models was suggested in (3) (see also the review [4]); for a long time they were forgotten and recently they have again attracted much attention in the connection with the onset of chaos due to inter-particle interaction (see, for example, [4] and references therein). Standard model of this kind describes $n$ interacting particles which can occupy $m$ single-particle states (orbitals), for Fermi-particles it has the form

$$H = \sum \epsilon_s a_s^\dagger a_s + \frac{1}{2} \sum V_{s_1s_2s_3s_4} a_{s_1}^\dagger a_{s_2}^\dagger a_{s_3} a_{s_4}. \quad (3)$$

Here matrix elements $V_{s_1s_2s_3s_4}$ of the perturbation $V$ stand for a two-body interaction (indices $s_1, s_2, s_3, s_4$ indicate initial $(s_1, s_3)$ and final $(s_2, s_4)$ single-particle states coupled by this interaction), and $\epsilon_s$ is the energy of a single-particle state $|s\rangle$. Creation-annihilation operators $a_s^\dagger$ and $a_s$ define the many-particle basis $|k\rangle = a_{s_1}^\dagger \ldots a_{s_n}^\dagger |0\rangle$ of non-interacting particles, in which the unperturbed Hamiltonian $H_0 = \sum \epsilon_s a_s^\dagger a_s$ is diagonal. It is convenient to reorder this basis according to an increase of the unperturbed energy $E_k = \sum \epsilon_s n_s^{(k)}$ with an increase of the index $k = 1, \ldots, N$. In the Two-Body Random Interaction (TBRI) model all matrix elements $V_{s_1s_2s_3s_4}$ are assumed to be random independent variables. In realistic applications such as complex atoms and nuclei, matrix elements of the two-body interaction are calculated directly by using a proper mean field approximation, see, for example, (3).

The TBRI model is very useful for the study of the role of a two-body interaction, as well as for establishing generic properties of Quantum Complexity, such as the localization, onset of chaos and thermalization in closed systems. To compare with Wigner Band Random Matrices (WBRM) of the type (3), one should stress the following. Having globally a band-like structure, the matrix $H_{ij}$ defined by (3) has many zero off-diagonal matrix elements inside the band, even in the extreme case when there are no additional integrals of motions and dynamical constraints, therefore, when all single-particle states are coupled by the (two-body) interaction. For $m \gg n$ the ratio of the number of non-zero matrix elements $V_{ij}$ to the total number of matrix elements is very small, therefore, the matrix turns out to be very sparse. Moreover, non-zero off-diagonal matrix elements $H_{ij}$ have been found (3) to be slightly correlated, in spite of the complete randomness of two-body matrix elements. The latter fact results in specific correlations between chaotic compound states and may lead
III. LDOS: BREIT-WIGNER REGION

In order to characterize global properties of ergodic eigenstates, it is convenient to introduce the $F$–function $F_j^{(n)}$ which gives the envelope of eigenstates,

$$F_j^{(n)} \equiv \overline{w_j^{(n)}} = \left| C_j^{(n)} \right|^2 = \left| C_j(E^{(n)}) \right|^2 \quad (4)$$

Here $C_j^{(n)}$ are components of exact eigenstates $|n\rangle$ of the total Hamiltonian $H$ expressed in terms of basis states $|j\rangle$ of $H_0$, $|n\rangle = \sum_j C_j^{(n)} |j\rangle$, and the bar stands for the average inside small windows centered at $j$ or $n$. We use here the notations which refer low indices to the basis states, and upper indices to the exact (compound) eigenstates. Thus, the structure of eigenstates is given by the dependence $w_j^{(n)}$ on $j$ for fixed values of $n$.

On the other hand, if we fix the index $j$ and explore the dependence $w_j^{(n)}$ on $n$, one can find how the unperturbed state $|j\rangle$ is coupled to other basis states due to interaction. The latter quantity is very important since it gives the information about the spread of the energy, initially concentrated in a specific basis state $|j\rangle$, over other basis states, when switching on the interaction. The envelope of this function $w_j^{(n)}$ in the energy representation is related to the strength function or local spectral density of states (LDOS) which can be defined as follows,

$$W(E^{(m)}, j) = \sum_{n} |C_j^{(m)}|^2 \delta(E - E^{(n)}) \quad (5)$$

The sum is taken over a number of eigenstates $|n\rangle$ chosen from a small energy window centered at the energy $E^{(m)}$. One can see that this function $W(E, j)$ is originated from the same matrix $w_j^{(n)}$ which gives the structure of eigenstates, therefore, the shapes of strength functions and eigenstates are related to each other. Normalized to the mean energy level spacing, the strength function $W(E, j)$ determines an effective number $N_{pc}$ of principal components of compound states $|n\rangle$ which are present in the basis state $|j\rangle$.

For the first time the form of the LDOS for WBRM (2) has been analyzed in Ref. (3). It was found that the form of the strength function essentially depends on one parameter $q = \frac{\rho_0 V_0^2}{\Delta^2}$ only. Wigner analytically proved (3) that for relatively strong perturbation, $V_0 \gg D$, in the limit $q \ll 1$ the form of the LDOS is the Lorentzian,

$$W_{BW} (E) = \frac{1}{2\pi} \frac{\Gamma_{BW}}{E^2 + \frac{1}{4} \Gamma_{BW}^2}, \quad E = E - D j \quad (6)$$

which nowadays is known as the Breit-Wigner (BW) dependence. Here the energy $E$ refers to the center of the distribution. The spreading width $\Gamma_{BW}$ (half-width of the distribution (3)) is given by the famous expression,

$$\Gamma_{BW} = 2\pi \rho_0 V_0^2 \quad (7)$$

In other limit $q \gg 1$ the influence of the unperturbed part $H_0$ can be neglected and the shape of the LDOS tends to the shape of density of states of band random matrix $V$, which is known to be the semicircle.

Recently, Wigner’s results have been extended to matrices $H$ with a general form of $V$, when the variance of off-diagonal matrix elements decreases smoothly with the distance $r = |i - j|$ from the principal diagonal. In this case the effective band size $b$ is defined by the second moment of the envelope function $f(r)$. Another important generalization of the WBRM studied in (5), is an additional sparsity of the matrix $V$, which may mimic realistic Hamiltonians.

Random matrix models of the type (3) are very useful for the understanding some important properties of the LDOS. Let us, first, write the condition for the LDOS to be of the BW form (3),

$$D \ll \Gamma_{BW} \ll \Delta E; \quad \Delta E = b D \quad (8)$$

to non-statistical behavior of physical systems with completely random two-body interaction even in the region where conventional statistical description is assumed to be valid, see details in (10).

The TBRI model (3) allows both to study the conditions for the onset of chaos and thermalization in closed systems, and to relate chaotic properties of spectra and eigenstates in many-particle basis to the properties of single-particle operators. One of important results obtained recently in (10) in the frame of this model is the Anderson-like transition which occurs in the Hilbert space determined by many-particle states of $H_0$. The critical value $V_{cr}$ for this transition is determined by the density of states $\rho_f = d_f^{-1}$ of those basis states which are directly coupled be a two-body interaction. The point that the interaction should be compared not with the total density of states $\rho_0$ of $H_0$ but with $\rho_f$ for the first time was noted in (12) when considering specific model of a rotating nuclei (see also discussion in (13)).

When the interaction is very weak, $V_0 \ll d_f$, exact eigenstates are delta-like functions in the unperturbed basis, with a very small admixture of other components which can be found by the standard perturbation theory. With an increase of the interaction, the number of principal components $N_{pc}$ increases and can be very large, $N_{pc} \gg 1$. However, if the interaction is still not too strong, $\pi^{-2} \sqrt{d_f D} \ll V_0 \ll d_f$, (12), the eigenstates are sparse, with extremely large fluctuations of components. In order to have ergodic eigenstates which fill some energy range (see below), one needs to have the perturbation large enough, $V_0 \gg d_f$ (for a large number of particles this transition is sharp and, in fact, one needs the weaker condition, $V_0 \geq d_f$).
The left part of this relation refers to the non-perturbative character of the interaction, according to which many of unperturbed basis states are strongly coupled by an interaction. On the other hand, the interaction should be not very strong, namely, the width $\Gamma_{BW}$ determined by Eq.\((7)\), has to be less than the width $\Delta_E$ of the interaction in energy representation. The latter condition is generic for systems with finite range of the interaction $V$. One should also stress that, strictly speaking, the BW form is not correct for such systems since its second moment diverges. As was shown in \((7)\), outside the energy range $|E| > \Delta_E$ the LDOS in the model \((2)\) decreases faster than pure exponent.

For other models, such as the TBRI model where the band width $b$ of the interaction is not well-defined, instead of $\Delta_E$ it is more convenient to use another important quantity, the variance $\sigma_0^2$ of the LDOS. The latter quantity can be rigorously expressed through off-diagonal matrix elements of the interaction, $\sigma_0^2 = \sum_{i < j} V_{ij}^2$ for $i \neq j$, therefore, $\sigma_0^2 = 2bV^2$ for the model \((2)\). As a result, we have $\Delta_E = \pi\sigma_0^2/\Gamma_{BW}$ and Eq.\((8)\) can be written as

$$D \ll \Gamma_{BW} \ll \sigma_0\sqrt{\pi} \tag{9}$$

Numerical data \((8,13,16)\) for the WBRM and TBRI models show that on the border $\Gamma_{BW} \approx 2\sigma_0$ the form of the LDOS is quite close to the Gaussian, and this transition from the BW dependence to the Gaussian-like turns out to be quite sharp. One should note that the form of the LDOS determines the dynamics of wave packets in the energy representation, see the data for the WBRM in Ref. \((17)\).

**IV. LDOS: TRANSITION TO THE GAUSSIAN**

For a long time it was assumed that in real physical systems the LDOS has the universal BW-dependence \((3)\). However, when studying the structure of the LDOS and eigenfunctions of the Ce atom, it was observed clear deviation from the BW-shape. Moreover, recent numerical investigation of nuclear shell models \((3)\) has shown that the form of the LDOS is much closer to the Gaussian rather that to the BW. This fact is due to a quite strong interaction $V \sim H_0$, since the mean field $H_0$ often includes a large regular part of the interaction, thus leaving a “disordered part” (residual interaction) in $V$.

As one can see, it is of great importance to find an analytical description of the LDOS in dependence of the strength of interaction. Although the extreme limit of a very strong interaction, $q \gg 1$, (or, the same, $W_{BW} \gg \sigma_0$) has been studied by Wigner in the WBRM model \((3)\), the semicircle form of the LDOS seems to be unphysical. Indeed, this form is originated from the semicircle dependence of the total density $\rho_V(E)$, and the latter is known to be an artifact of the standard RMT.

As was shown quite long ago (see the review \((3)\) and reference therein), this result formally corresponds to random $n$-body interaction between Fermi or Bose particles with $n = 1, \ldots, \infty$. On the other hand, physical interaction is typically of a two-body nature, this fact is directly taken into account in TBRI models of the type \((3)\).

It is clear that analytical treatment of the TBRI models is much more difficult in comparison with full and band random matrices. For this reason, numerical data are very important since they may give a hint for rigorous results to be proved analytically (see, for example, \((4)\)).

Using general result of the WBRM model, see Eq.\((7)\), one can expect that if the spreading width $\Gamma_{BW}$, determined by the density of states $\rho_V$, has the gaussian form. This result is expected from the point of view of combinatorics, however, the rigorous proof is non-trivial.

Using general result of the WBRM model, see Eq.\((7)\), one can expect that if the spreading width $\Gamma_{BW}$, determined by the density of states $\rho_V$, is the random two-body interaction $V$. As was shown in \((4)\), for large number of particles and orbitals, $m \gg n \gg 1$, the density of states $\rho_V$ has the gaussian form. This result is expected from the point of view of combinatorics, however, the rigorous proof is non-trivial.

As was shown quite long ago (see the review \((3)\) and reference therein), this result formally corresponds to random $n$-body interaction between Fermi or Bose particles with $n = 1, \ldots, \infty$. On the other hand, physical interaction is typically of a two-body nature, this fact is directly taken into account in TBRI models of the type \((3)\).

It is clear that analytical treatment of the TBRI models is much more difficult in comparison with full and band random matrices. For this reason, numerical data are very important since they may give a hint for rigorous results to be proved analytically (see, for example, \((4)\)).

Using general result of the WBRM model, see Eq.\((7)\), one can expect that if the spreading width $\Gamma_{BW}$, determined by the density of states $\rho_V$, is the random two-body interaction $V$. As was shown in \((4)\), for large number of particles and orbitals, $m \gg n \gg 1$, the density of states $\rho_V$ has the gaussian form. This result is expected from the point of view of combinatorics, however, the rigorous proof is non-trivial.
(energy of the basis state $|k\rangle$ of $H_0$). The method used in [14], is an extension of the approach developed in [21], which takes into account specific structure of the Hamiltonian $H$. The dependence of $W_k(E)$ was found to have the form,

$$W_k(E) = \frac{1}{2\pi} \frac{\Gamma_k(E)}{(E_k - E)^2 + \frac{1}{4} \Gamma_k^2(E)}$$  \hspace{1cm} (12)$$

with some function $\Gamma_k(E)$. In the case of a relatively weak interaction, $1 \ll \lambda \leq 1$, the function $\Gamma_k(E)$ is almost constant on the scale of the energy width $\Delta E$, therefore, the conventional BW-expression is recovered for $W_k(E)$. In contrast, in the case of strong interaction, $\lambda \geq 1$, the dependence of $\Gamma_k(E)$ on the energy $E$ can not be neglected and the nominator in (12) is the leading one. As was shown in [14], the function $\Gamma_k(E)$ is defined by the density of states for one-body and two-body transitions, and for large number of particles and orbitals has the Gaussian form with the variance $\sigma_0$. Detailed numerical study of the form of the LDOS in the region $\Gamma_{BW} \geq 2\sigma_0$ have shown that the LDOS coincides with the Gaussian with a very high accuracy.

It is important to stress that the condition for the existence of a smooth energy dependence $\Gamma_k(E)$ is defined by the condition $V_0 \geq d_f$ of ergodicity for the components $C_k^{(i)}$ of the LDOS (or, the same, for exact eigenstates). Below this transition, for $V_0 \leq d_f$, the smooth solution of the equation for the function $\Gamma_k(E)$ does not exist. In this sense, the result obtained in [14] can be treated as the independent proof of the onset of chaos in the TBRI model when $V_0 \geq d_f$.

Having formal solution for the LDOS, it is, however, convenient to find a simple phenomenological expression which depends on two control parameters only, the width $\Gamma_{BW}$ defined by (10), and the width $\sigma_0$, see Eq. (11). This was recently done in [14] where both the WBRM and TBRI models have been used to compare numerical data with the following expression,

$$W(z; \lambda, \beta) = A \exp\left(-\frac{\gamma^2 z^2}{2}ight)$$  \hspace{1cm} (13)$$

Here $z = (E - E_c)/\sigma_0$ is the normalized energy which refers to the center of the LDOS, with $\sigma_0$ given by (11). Taking the parameter $\beta$ as the fitting parameter, two other parameters $A$ and $\gamma$ are determined from the normalization conditions $\int W(z) \, dz = 1$ and $\int z^2 W(z) \, dz = 1$. As one can see, in the normalized energy units there is only one control parameter $\lambda$. In the region $\lambda \ll 1$ the BW-dependence for $W(z)$ is recovered, and for $\lambda \gg 1$ the Gaussian emerges, see details in [16].

Numerical data show that this dependence gives quite good description of the LDOS for any values of $\lambda$ defined by $\Gamma_{BW}$ and $\sigma_0$. It turns out that the transition from the BW-dependence to the Gaussian is quite sharp and takes place when $\lambda \approx 1.0$ (see also recent numerical data [21]). The above phenomenological expression (13) with the fitting function $\beta(\lambda)$ is quite useful in the applications.

So far, we have discussed the form of the LDOS, provided the eigenstates are chaotic in the sense that the fluctuations of their components $C_j^{(i)}$ around the smooth envelope given by $W(E)$ are gaussian-like. Similar analysis can be performed for the form $w(E)$ of exact compound eigenstates in the unperturbed energy representation. As was noted above, these two functions, $W(E)$ and $w(E)$ are related to each other through the matrix $w_j^{(n)}$, see Eq. (4). Analytical estimates for the TBRI model show that for not very strong interaction, when $\sigma_0 \ll \Delta E$ with $\Delta E$ as the total width of the energy spectrum of $H_0$, the two functions are close to each other. This fact allows to use the results obtained for the LDOS, when considering typical structure of exact eigenstates.

V. DISTRIBUTION OF OCCUPATION NUMBERS

The knowledge of the F-function is very important for the relation of chaotic properties of the Hamiltonian $H$ in the many–body representation, with the properties of single-particle operators. Indeed, exact eigenstates $|i\rangle$ of the total Hamiltonian $H$ are given as

$$|i\rangle = \sum_k C_k^{(i)} |k\rangle, \quad |k\rangle = a^\dagger_{s_1} \cdots a^\dagger_{s_n} |0\rangle$$  \hspace{1cm} (14)$$

where $C_k^{(i)}$ is the $k$–th component of the compound state $|i\rangle$ in the unperturbed basis. These coefficients $C_k^{(i)}$ determine the so-called occupation numbers $n_s$,

$$n_s^{(i)} = \langle i| \hat{n}_s |i\rangle = \sum_k \left| C_k^{(i)} \right|^2 \langle k| \hat{n}_s |k\rangle$$  \hspace{1cm} (15)$$

where $\hat{n}_s = a^\dagger_{s} a_s$ stands for the occupation number operator. This quantity gives the probability that one of $n$ particles occupies an orbital $s$ specified by the one-particle state $|s\rangle$ for the fixed exact state $|i\rangle$. According to this expression, this probability can be found by projecting the state $|i\rangle$ onto the basis of unperturbed states, for which the relation between the positions of all particles in the single-particle basis and the specific many-particle basis state is known by the construction of the latter. One can see that the probability $n_s = n_s^{(E^{(i)})}$ is the sum of probabilities over a number of basis states which construct the exact state. For Fermi-particles the occupation number $n_s^{(k)} = \langle k| \hat{n}_s |k\rangle$ is equal to 0 or 1 depending on whether any of the particles in the basis state $|k\rangle$ occupies or not the single-particle state $|s\rangle$.

For chaotic eigenstates the $n_s$–distribution is a fluctuating function of the total energy $E = E^{(i)}$ of a system,
due to strong (gaussian) fluctuations of the components $C_k^{(s)}$. In order to obtain a smooth dependence, one should make an average over a small energy window centered at $E^{(i)}$, which is in the spirit of the conventional statistical mechanics for systems in the contact with thermostat. In fact, such an average is a kind of microcanonical averaging since it is done for the fixed total energy $E$ of a system. Therefore, one can define the $n_s$—distribution through the $F$—function, see (5) and (15),

$$n_s(E) = \sum_k F(E_k, E^{(i)}) \langle k| \hat{n}_s |k \rangle . \quad (16)$$

The $n_s$—distribution plays essential role in the statistical approach to finite systems of interaction particles, see details in [4,6]. It is clear that non-trivial part in the above expression for $n_s$ is the $F$—function $F(E_k, E^{(i)})$ which absorbs statistical effects of the two-body interaction $V$. The important point is that in order to find the distribution of occupation numbers, one needs to know the envelope of exact eigenstates in the basis (or energy) representation, not the eigenstates themselves. Therefore, the knowledge of the $F$—function and the density of states $\rho_0(E)$ of the unperturbed Hamiltonian $H_0$ (it appears when passing from the summation to the integration in Eq.(16)) gives us the possibility to relate global properties of eigenstates to the distribution of occupation numbers of single–particle states.

The form of $n_s$ in the TBRI model (3) has been studied in detail in connection with the problem of thermalization and of the onset of Fermi-Dirac (FD) distribution. It should be stressed that the occurrence of the FD-distribution in closed systems of finite number of interacting particles is far from being trivial, especially, when the number $n$ of particles is not very large. It has to be noted that in many applications $n$ is the number of particles (quasi-particles) above the Fermi level, therefore, it can be relatively small. For example, in the mean field description of the Ce atom there are $n = 4$ interacting electrons in the outer shell [33], and $n = 12$ in the standard $s – d$ shell model of heavy nuclei [3].

One of the most interesting results obtained in the frame of the TBRI model (3) is that the FD-distribution may occur even for 4 interacting Fermi-particles, provided the (random) interaction $V$ is strong enough. In this case one can obtain an analytical expression for the temperature and chemical potential which stand in the standard Fermi-Dirac form for the occupation number distribution $n_s$. Another important result is that even if the form of the $n_s$—distribution is far from the FD-distribution, it can be described analytically, the simplest case is when the form of the $F$—function is close to the Gaussian. The study of the $n_s$—distribution allows to determine the onset of thermalization in terms of the ergodicity of components of compound states in the energy range defined by the $F$—function. In fact, the ergodicity of eigenstates results in standard (gaussian-like) fluctuations, both for components of eigenstates around the envelope ($F$—function) and for the $n_s$ numbers, when changing slightly the total energy of a system (see details in [33,14,6]). The latter property (existence of a smooth dependence of the $n_s$—distribution with gaussian fluctuations around this dependence) is, in essence, the existence of a statistical equilibrium for interacting particles in a closed system.

VI. QUANTUM-CLASSICAL CORRESPONDENCE

One of the important questions is the quantum-classical correspondence for the shape of eigenstates (SE) and the LDOS. As was pointed out in Ref. [22], there is quite simple approach for finding classical shape of eigenstates and classical LDOS. This approach is generic and can be used for different physical models [23, 27].

Let us start with the classical SE. We consider the total Hamiltonian in the form (1) where the “unperturbed part” $H_0$ can be represented as the sum of single-particles Hamiltonians $H_k^0$ describing the motion of $n$ non-interacting particles, $H_0 = \sum_{k=1}^{n} H_k^0(p_k, q_k)$. The interaction $V$ between the particles is assumed to result in a chaotic behavior of the system $H$. Now let us fix the total energy $E$ of the Hamiltonian $H(t)$ and find (numerically) the trajectory $p_k(t), q_k(t)$ by computing Hamiltonian equations. Since the total Hamiltonian $H$ is chaotic, any choice of initial conditions $p_k(0), q_k(0)$ gives the same result if one computes the trajectory for a sufficiently large time. When time is running, let us collect the values of the unperturbed Hamiltonian $H_0(t)$ at fixed times $t = T, 2T, 3T, \ldots$, and construct the distribution of energies $E_0(t)$ along chaotic trajectory of the total Hamiltonian $H$. In this way one can obtain the distribution $w(E_0; E = \text{const})$. Comparing with the quantum model, one can see that this function $w(E_0; E = \text{const})$ is the classical analog of the average shape of eigenstates in the energy representation. Indeed, any exact eigenstate corresponds to a fixed total energy $E = \text{const}$ and it is represented in the unperturbed basis of $H_0$. Thus, one can expect that for chaotic eigenstates in a deep semi-classical region the two above quantities, classical and quantum ones, correspond to each other.

In the same way one can consider the complimentary situation. Let us fix the unperturbed energy $E_0$ and compute the trajectory $p_k^{(0)}(t), q_k^{(0)}(t)$ which belongs to the unperturbed Hamiltonian $H_0(t)$. Similar to the previous case, let us put this unperturbed trajectory into the total Hamiltonian $H(t)$ and collect the values of the total energy $E(t)$ along the unperturbed trajectory for discrete values of time. It allows us to find the distribution $W(E; E_0 = \text{const})$ which now should be compared with the LDOS in the corresponding quantum model. However, in this case one should perform an average over
many initial conditions $p_k(0), q_k(0)$ with the same energy $E_0$, since the unperturbed Hamiltonian $H_0$ is separable. One can see that the above two classical distributions $w(E_0; E = \text{const})$ and $W_0(E; E_0 = \text{const})$ determine the ergodic measure of two energy shells, the first one, when projecting the phase space surface of $H$ onto $H_0$, and the second one, when projecting the surface of $H_0$ onto $H$ (see discussion in [22]).

VII. TWO INTERACTING SPINS

The first example is the model of two interacting rotors (see [23,24] and references therein),

$$H = L_z + M_z + L_z M_z$$  (17)

with angular momentum $\vec{L}$ and $\vec{M}$. This model may be used to describe the interaction of quasi-spins in nuclear physics or pseudo-spins in solid state systems. Comparing with Eq. (6), here $H_0 = L_z + M_z$ and $V = L_z M_z$. The constants of motion are $H = E, L^2$ and $M^2$, which are connected by the relation $E^2 \leq E_{\text{max}}^2 = (L^2 + 1)(M^2 + 1)$ for $LM > 1$. It is worth to mention that in (17) the dynamical variables $\vec{L}, \vec{M}$ are not canonical, however, keeping $L^2$ and $M^2$ as constants, one can present $H$ in the canonical variables.

It is known (see references in [23,24]) that in the classical limit the phase space consists of both regular and chaotic regions, the size of which depends on the values of $L^2$ and $M^2$. We consider the simplest case $L = M$ for which trajectories are regular when $|E|$ is close to $E_{\text{max}} = L^2 + 1$, and are chaotic at the center of the energy band, $E \approx 0$.

Quantization of this model can be done in the standard way, with angular momenta quantized according to the relations $L^2 = M^2 = h^2(l(l+1))$ where $l$ is an integer number. Therefore, for a given $l$ the Hamiltonian matrix is finite and the semiclassical limit corresponds to the limit $l \to \infty$ and $h \to 0$ while keeping $L^2$ constant. According to the approach discussed above, the Hamiltonian is represented in the two-particles basis $|l_z, m_z\rangle$ and has the dimension $N = 2l^2 + 2l + 1$. Due to the symmetry degeneracy with respect to the exchange of particles, there are symmetric and antisymmetric states which are not coupled by the interaction (only symmetric states are studied in [23,24]).

As was pointed out when discussing the TBRI matrix model [3], it is important to reorder the unperturbed basis in such a way that the energy of many-particles states of $H_0$ increases. This kind of sorting corresponds to the classical model and allows to establish quantum-classical correspondence for the shape of LDOS and eigenstates. After this reordering, the unperturbed Hamiltonian $H_0$ has the shell structure, with the same value on energy inside one shell. Diagonal matrix elements are given by the eigenvalues $-2i\hbar, (-2l + 2)\hbar, \ldots, 2l\hbar$ of the unperturbed Hamiltonian $H_0$, and diagonal elements of the perturbation $V$ vanish due to a particular form of the interaction.

The global structure of the matrix $H_{n,n}$ is quite specific, however, it has some similarity with the TBRI matrices. Apart from the principal diagonal $H_{ii}$ determined by the eigenvalues of $H_0$, the next (to the principal one) diagonals $H_{n,n \pm 1}$ correspond to transitions inside each $H_0$-shell while the “arcs” connecting the two corners represent transitions between neighboring shells having $\Delta H_0 = \pm 2\hbar$. Therefore, the matrix $H_{ij}$ has only four off-diagonal “curves” along which non-zero off-diagonal elements are located. All other matrix elements vanish thus resulting in extremely sparse matrix $H_{ij}$. In spite of a clear band structure, the Hamiltonian matrix $H_{ij}$ can not be compared with Wigner Band Random Matrices of the type [3] since its band size is not much less than the total size of the matrix. Indeed, the band width is $b = 2l + 1$ in the middle of the matrix, therefore, the control parameter $b^2/N$ is of the order 1 and the finiteness of the matrix is crucial (see details in [23,24]). It is also important to stress that non-zero matrix elements can not be treated as random ones since the model (17) is dynamical. All non-zero off-diagonal matrix elements are positive and the mean and variance of the distribution of these matrix elements depend on the classical parameter $L^2 = h^2(l(l+1))$ only.

Numerical data for the LDOS in the center of the energy band, $E \approx 0$, have been obtained in the dynamical approach discussed in the previous Section. More precisely, this distribution was numerically calculated taking a sample of chaotic trajectories having the same fixed values of $E$ and $L^2 = M^2$. Following these trajectories, one can calculate $H_0(t) = L_z(t) + M_z(t)$ taken at equal instants of time and find the distribution of $H_0$. In this case the classical distribution can be also evaluated analytically since the classical unperturbed Hamiltonian $H_0$ is integrable.

The result is presented in Fig.2 in the normal and semi-log scales. The quantum distribution Eq. (6) was obtained by averaging over $l + 1$ values of individual dependencies $w_i^{(k)}$ for the largest shell with $\epsilon_0 = 0$. Here, the normalized energies are used, $\epsilon_0 = E_0/E_{\text{max}}$ and $\epsilon = E/E_{\text{max}}$, with $E_0$ and $E$ as the unperturbed and exact energies. One can see a remarkable correspondence between the quantum LDOS and its classical counterpart. The only difference can be found in long tails of the distribution, which manifests a kind of quantum tunneling beyond the energy region given by the classical Hamiltonian. Also, one can notice some discrepancy at the center of the energy band, where quantum effects are also expected to be relatively strong due to a finite value of the semiclassical parameter $l$.

A very good correspondence has been also found for the shape of exact eigenstates SE, see Fig.3. Numerical
procedure of the averaging was taken in the way similar to that for the LDOS, with the average over \( l + 1 \) exact eigenstates taken from the center of the energy band. Again, one can see some difference in the tails and at the center of the distribution, which is caused by pure quantum effects.

It should be pointed out that at the edges of the energy spectrum where the classical motion is regular, individual eigenstates differ very strongly one from the other, therefore, one needs take a relatively large number of eigenstates for the average, in order to obtain a smooth dependence on the energy. This means that although the quantum-classical correspondence can be achieved in the regular region, it has no sense since the energy interval to be taken for the average can be of the order the whole spectrum.

It was shown \([27,28]\) that the dynamics of a classical particle in this billiard manifests generic properties of the transition to chaos with an increase of the amplitude \( a \) for a fixed \( d \). A possible experimental realization of this model with finite length is a mesoscopic electron wave guide. In \([27]\) classical transport properties of this billiard have been related to its dynamical properties, thus giving a transport signature of chaos. The analysis of the quantum motion in the infinitely long rippled channel with the periodic upper boundary \( y(x + L) = y(x) \) is useful for the understanding of universal features of electronic band structures of real crystals, propagation in periodic waveguides, quantum wires and films, see references in \([27,28]\). Detailed study of the energy band structure of the quantum version of this periodic rippled channel is presented in Ref. \([28]\).

The control parameter that determines the degree of chaos in the classical model is \( K = \frac{2d}{\pi a} \), and strong chaos occurs when \( K \gtrsim 1 \). In the quantum description one should solve the Schrödinger equation \( \hat{H} = \frac{\hbar^2}{2m}(\hat{p}_x^2 + \hat{p}_y^2) \) with the boundary conditions \( \psi(x, y) = 0 \) at \( y = 0 \) and \( y = d + a \cos x \). Instead, in Ref. \([23,25]\) the approach has been used which is based on the transition to the
new coordinates \( u = x \), and \( v = \frac{y}{d \sin(a \cos(x))} \), for which

the boundary conditions become trivial, \( \Psi(u,v) = 0 \) at \( v = 0, d \). On the other hand, in the new variables the specific interaction between two degrees of freedom arises and the Hamiltonian acquires a much more complicated form,

\[
\hat{H} = \frac{\hbar^2}{2} g^{-1/4} \hat{P}_a g^{\alpha\beta} g^{1/4} \hat{P}_b g^{-1/4}, \quad \alpha, \beta = u, v, \quad (18)
\]

which is simply the kinetic energy expressed in the covariant form. The momentum is now given by \( \hat{P}_a = -i\hbar \left[ \partial_a + \frac{1}{2} \partial_a \ln(g) \right] = -i\hbar g^{-1/4} \partial_a g^{1/4} \), where \( g^{\alpha\beta} \) is the metric tensor, and \( g = \det(g_{\alpha\beta}) = [1 + \epsilon \cos u]^2, \epsilon \equiv a/d \), (see [28] for details). As a result, the original model of one particle in the rippled billiard is formally transformed into the model of two interacting particles. In this way, the complexity of the boundary in the original model is incorporated into the “interaction potential”.

The solution for the wave function can be represented by expanding the function \( \Phi_k(u,v) \) in a double Fourier series, \( \Psi^\alpha(u,v,k) = \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} C_{mn}^{\alpha}(k) \phi_{mn}^k(u,v) \), where \( \alpha \) stands for the eigenstate of energy \( E^\alpha(k) \), and \( \phi_{mn}^k(u,v) = \pi^{-1/2} g^{-1/4} \sin(m \pi v) e^{i(k+n)u} \) and the Bloch number was fixed \( k = 0.1 \) in the numerical study. The index \( m \) refers to the mode (or channel) number and \( n \) stands for the Brillouin zone number. Note that \( \phi_{mn}^k \) are the eigenstates of the unperturbed momenta squared (and, therefore, of the unperturbed Hamiltonian): \( \hat{P}_u^2 \phi_{mn}^k \equiv \hbar^2 (m^2 \pi^2) \phi_{mn}^k, \quad \hat{P}_v^2 \phi_{mn}^k = \hbar^2 (k+n)^2 \phi_{mn}^k \).

![FIG. 5. (above) Shape of eigenfuctions (the F-function) in the energy representation (broken line) and its classical counterpart (smooth curve); (below) the LDOS and its classical counterpart. For the F-function the average is taken over exact eigenstates from the interval 1870 < \( \alpha < 1950 \) and for the LDOS the average is over basis states from the interval 1900 < \( j < 1990 \), after [26].](image)

In the classical limit one should perform the coordinate transformation in the same way as it was done for the quantum model. The classical Hamiltonian has the form \( H = \frac{1}{2} g^{\alpha\beta} P_\alpha P_\beta \), with the same boundary conditions in \((u,v)\) coordinates. Expanding this expression, one can obtain,

\[
H = \frac{1}{2} \left[ P_u^2 - \frac{2e\nu \xi u}{1 + \epsilon \xi} P_u P_v + \frac{1 + (e\nu \xi u)^2}{(1 + \epsilon \xi)^2} P_v^2 \right], \quad (19)
\]

where \( \xi \equiv \cos u, \xi_u \equiv d\xi/du, \) and \( \epsilon \equiv a/d \). It should be noted that the transformation from the old to new variables is canonical. In accordance with the quantum model, the separation of the total Hamiltonian into two parts, \( H = H^0 + V \), has to be done as follows,

\[
H^0 \equiv \frac{1}{2} \left[ P_u^2 + P_v^2 \right]; \quad V \equiv H - H^0. \quad (20)
\]

Let us now compare the quantum LDOS and the shape of exact eigenstates with their classical counterparts which are numerically obtained [28] in the described above approach, see Fig.5. One can see, that, in general, there is quite good global correspondence between quantum and classical data (the role of quantum fluctuations around the envelopes will be discuss below). The interesting point is that the forms of the LDOS and the SE are highly non-trivial, if to compare with the Breit-Wigner or Gaussian distribution. Remarkably, the averaging procedure reveals a three-peak structure for both classical and quantum quantities.

The origin of these peaks can be understood by a detailed analysis of the classical trajectories [28]. Specifically, it was found that even for the chaotic motion the particle can be located for a long time in the neighborhood of stable and unstable periodic orbits of period 1. One can find that the unstable (stable) periodic orbit is defined by \( P_u = 0 \) and \( x = 0; \pi \) respectively. The right peak corresponds to the motion perpendicular to the \( x \) axis at \( x = \pi \) (unstable orbit). Similarly, the left one results from the stable orbit at \( x = 0 \). In contrast, the central peak is originated from the trajectories which are nearly parallel to the \( x \)-axis. A similar analysis explains the origin of the structure of the classical LDOS.

As for the comparison between the form of the LDOS and that of the SE, it is easy to detect a difference, especially, in the central region. The reason for this is still not clear, in Ref. [28] it was suggested that it is due to the difference between time averages and phase space averages for finite times.
IX. LOCALIZATION AND NON-ERGODICITY IN THE ENERGY SHELL

Now let us discuss the problem of localization in the energy shell. As was claimed above, with an increase of the interaction in the TBRI model, the Anderson-like transition is expected \([13]\) in the Hilbert space of many-particle states. It occurs when the interaction strength \(V_0\) exceeds the mean level spacing between basis states which are directly coupled by the two-body interaction, \(V_0 \geq d_f\). In terms of the structure of compound states, this means that above this threshold the components \(C_j^{(i)}\) ergodically fill an available energy shell. More specifically, the fluctuations of these components fluctuate around the mean given by the \(F\)–function, in accordance with the conventional statistical mechanics, typically, according to the Gaussian distribution. One should stress that for closed systems the width of the energy shell is always finite due to a finite range of interaction, and it can be much less than the total size of the energy spectrum. However, the number \(N_{pc}\) of principal components can be very large. In this sense, one can rigorously define the onset of chaos (or maximal complexity in eigenstates), as the ergodic filling of the energy shell with the Gaussian fluctuations around the envelope given by the \(F\)–function.

If the system has the classical limit, the \(F\)–function has its classical counterpart which can be easily found from Newton equations of motion. More difficult situation is when there is no classical limit, in this case, although the \(F\)–function exists, it is of the problem to find it without direct diagonalization (and the average over a number of eigenstates) of a (typically huge) Hamiltonian matrix. Using two examples of simple quantum dynamical models with the classical chaos, we have shown here that in a deep semiclassical limit there is a good correspondence for the form of both the LDOS and eigenstates. Strange enough, this quantum-classical correspondence is studied scarcely, there are mainly numerical results discussed above (see also, \([23]\) ). As the data for the rippled channel show, a kind of semiclassical periodic orbit theory can be developed both for the LDOS and SE.

The knowledge of the SE is extremely important for the study of any kind of localization in the energy shell. The non-trivial role of the \(F\)–function is that it gives a reference to which individual eigenstates should be compared in order to see the localization. Let us first consider the TBRI model where there is no influence of classical periodic orbits which result in a specific sort of localization (scars, if to speak about the quantum-classical correspondence in the configuration space). For such kind of models, above the threshold \(V_0 \approx d_f\) the gaussian fluctuations for the coefficients \(C_j^{(i)}\) are expected. This was checked for excited states of the Ce atom which is known to be quite chaotic dynamical system \([24]\). Numerical data have shown, that, indeed, the fluctuations turns out to be very close to the Gaussian. Note, that for the study of these fluctuations, first, one needs to normalize each value \(C_j^{(i)}\) to its mean-square-root value which itself depends on the index \(i\) (or on the energy \(E^{(i)}\), if to analyze the fluctuations in the energy space). Therefore, one should find this mean value (square root of the \(F\)–function) by a kind of the average.

Below the threshold \(V_0 \approx d_f\) the fluctuations of the amplitudes \(C_j^{(i)}\) are extremely large since the components which correspond to those states, which are not coupled directly by a two–body interaction, are very small, thus resulting in a large peak in the distribution of \(P(C)\) around zero. As one can see, the quantum localization results in the sparsity of the \(F\)–function and the LDOS, in contrast to the standard Anderson localization in one and quasi-1D disordered models, where eigenstates are “dense” in the configuration space and occupy a finite number of sites (analog of basis states in our models). Therefore, the fingerprint of the Anderson localization is an exponential decay of the amplitudes \(C_k\) far from the “centers” of eigenstates, unlike the localization in the energy shell for closed systems, where the cut-off of tails always exists due to finiteness of the energy shell. On the other hand, for a very strong localization, when eigenstates consist of few basis states (for \(V_0 \ll d_f\) ), the situation is quite similar to that for the Anderson localization (this typically happens for low energy states due to a relatively large values of \(d_f\) ).

A much more delicate situation occurs for dynamical quantum systems with the classical limit. In this case in addition to the quantum localization, even in a strongly chaotic classical region, the influence of tiny structure of the classical phase space (for example, short unstable periodic orbits) may be quite important. In this sense, one can speak about the localization which is caused by both classical properties (periodic orbits) and quantum ones (scarring effect). For this reason, the mesoscopic region where both classical and quantum effects can be equally strong, is of the most difficult for the analysis.

Coming back to the numerical data for the rippled channel \([24]\), a noticeable fraction of extremely localized eigenstates has been detected in the region of a strong classical chaos where typically eigenstates are ergodic. It was found that these eigenstates are originated from those basis states which are located on the edges of the shells of \(H_0\). It turns out that these basis states are very slightly coupled with the others by the interaction \(V\). These localized states correspond to almost free motion of a particle along the channel. The presence of very localized states has been also observed in the model \([17]\) of two interacting spin \([23,24]\). In this case, their origin is not so clear and deserves an additional study.

Finally, we would like to raise a new problem related to the quantum-classical correspondence for chaotic clas-
sical systems. It refers to the ergodicity of individual eigenstates in a deep semiclassical region. According to the Shnirelman theorem, it is believed that the closer this limit, the less influence of quantum effects. To see this fact, a kind of the average over a small energy range is typically assumed. On the other hand, in a deep semi-classical limit the eigenstates have a very large number of components. This allows one to perform the statistical analysis inside individual eigenstates, without any energy or ensemble average. This is in a spirit of dynamical chaos for which statistical properties can be revealed along a single chaotic trajectory. Specifically, we can ask: how fast these fluctuations approach those ones prescribed by the standard statistical mechanics? Numerical data [26] manifest that individual eigenstates can not be rigorously treated as random ones since non-statistical deviations seem to persist for any finite values of $\bar{\hbar}$. In this sense, one can speak about non-ergodic character of individual eigenstates, in spite of the existence of the classical limit itself. Definitely, this question deserves a detailed study, both numerically and analytically.

X. ACKNOWLEDGMENTS

The author is very grateful to his co-authors with whom the works have been done on the subject discussed in this paper. This work was supported by CONACyT (Mexico) Grant No. 28626-E.

[1] Proceedings of the International School of Physics "Enrico Fermi", Course CXLIII, Eds. G. Casati, I. Guarneri and U. Smilansky (IOS Press, Amsterdam, 2000).
[2] T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep., 200, 189 (1999).
[3] E.P. Wigner, Ann. Math. 62, 548 (1955); Ann. Math. 65, 203 (1957).
[4] J.B. French and S.S. M. Wong, Phys. Lett. B 35, 5 (1970); O. Bohigas and J. Flores, Phys. Lett. B 34, 261 (1971).
[5] T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. 53, 385 (1981).
[6] F.M. Izrailev, "Quantum Chaos and Thermalization for Interacting Particles", cond-mat/991207.
[7] V.V. Flambaum, A.A. Gribakina, G.F. Gribakin, and M.G. Kozlov, Phys.Rev. A 50, 267 (1994).
[8] V.V. Flambaum, A.A. Gribakina, and G.F. Gribakin, Phys. Rev. E 52, 5667 (1995); V.V. Flambaum, A.A. Gribakina, G.F. Gribakin, and I.V. Ponomarev, Phys. Rev. E 57, 4933 (1998).
[9] N. Frazier, B.A. Brown and V. Zelevinsky, Phys. Rev. C 54, 1665 (1996); V. Zelevinsky, B.A. Brown, M. Horoi and N. Frazier, Phys. Rep., 276, 85 (1996).
[10] V.V. Flambaum, G.F. Gribakin and F.M. Izrailev, Phys. Rev. E 53, 5729 (1996).
[11] B.L. Altshuler, Y. Gefen, A. Kamenev and L.S. Levitov, Phys. Rev. Lett., 78, 2803 (1997).
[12] S. Aberg, Phys. Rev. Lett., 26, 3119 (1990).
[13] D.L. Shepelyansky and O.P. Sushkov, Europhys. Lett. 37, 121 (1997).
[14] V.V. Flambaum and F.M. Izrailev, Phys. Rev. E 56, 5144 (1997).
[15] Y.V. Fyodorov, O.A. Chubykalo, F.M. Izrailev and G. Casati, Phys. Rev. Lett., 76, 1603 (1996).
[16] G.Casati, V.V.Flambaum, and F.M.Izrailev, to be published.
[17] D. Cohen, F.M.Izrailev, and T. Kottos, Phys. Rev. Lett., 84 (2000) 2052.
[18] V.V.Flambaum, F.M.Izrailev, and G. Casati, Phys. Rev. E 54, 2136 (1996); V.V. Flambaum and F.M.Izraileva, Phys. Rev. E 55, R13 (1997).
[19] V.V.Flambaum and F.M.Izrailev, Phys. Rev. E., 61, 2539 (2000).
[20] A. Bohr and B. Mottelson, Nuclear structure, Vol. 1 (Benjamin, New York, 1969).
[21] V.K.B. Kota, nucl-th/000607.
[22] G. Casati, B.V. Chirikov, I. Guarneri and F.M. Izrailev, Phys. Lett. A 223, 430 (1996).
[23] F. Borgonovi, I. Guarneri, F.M. Izrailev and G. Casati, Phys. Lett. A. 247, 140 (1998).
[24] F. Borgonovi, I. Guarneri and F.M. Izrailev, Phys. Rev. E 57, 5291 (1998).
[25] W. Wang, F.M. Izrailev and G. Casati, Phys. Rev. E 57, 323 (1998); L. Benet, F.M. Izraileva, T.H. Seligman, A. Suarez-Moreno, chao-dyn/9912033; L. Meza-Montes, F.M. Izrailev, and S. E. Ulloa, Phys. Status Solidi, 2000, to appear.
[26] G.A. Luna-Acosta, J.A. Mendez-Bermudez, and F.M. Izrailev, cond-mat/000607.
[27] G.A. Luna-Acosta, A.A. Krokhin, M.A. Rodriguez, and P.H. Hernandez-Tejeda, Phys. Rev. B 54, 11410 (1996).
[28] G.A. Luna-Acosta, K. Na, L.E. Reichl, and A. Krokhin, Phys. Rev. E 53, 3271 (1996).
[29] B.V. Chirikov, Phys. Lett. A 108, 68 (1985).