Exact Vacuum Solutions of Jordan, Brans-Dicke Field Equations

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Abstract

We present the static spherically symmetric vacuum solutions of the Jordan, Brans-Dicke field equations. The new solutions are obtained by considering a polar Gaussian, isothermal and radial hyperbolic metrics.

Keywords: scalar-tensor theory, spherical symmetry, exact solutions

1 Introduction

The scalar-tensor theory first time was invented by P. Jordan [1] in the 1950’s, and then taken over by C. Brans and R.H. Dicke [2] some years later. In this theory the scalar field acts as the source of the (local) gravitational coupling with $G \sim \chi = \phi^{-1}$ and consequently the gravitational constant is not in fact a constant but is determined by the total matter in the universe through an auxiliary scalar field equation. In the Jordan conformal frame, the Jordan, Brans, Dicke (JBD) action takes the form [1] (we use geometrized units such that $G = c = 1$ and we follow the signature +,−,−,−).

$$\delta \int \chi^n \left( R - \xi \frac{\chi_i \chi^i}{\chi^2} + L_m \right) \sqrt{-g} d^4x. \quad (1)$$

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where \( \eta \) and \( \xi \) coupling constants, \( L_m \) is the Lagrangian density of ordinary matter. Variation of (1) with respect to \( g_{ik} \) and \( \chi \) gives, respectively, the field equations

\[
\frac{\ddot{x}^i}{x} + \left( \eta - 1 \right) \frac{\dot{x}^i \dot{x}^j}{x^2} = \frac{\eta T}{3\eta^2 - 2\xi},
\]

\[
R_{ik} + \eta \frac{x^{i,k}}{x} - \left[ \xi - \eta (\eta - 1) \right] \frac{\dot{x}^i \dot{x}^k}{x^2} = \chi \left( T_{ik} + g_{ik} \frac{\xi - \eta^2}{3\eta^2 - 2\xi} T \right)
\]  

(2)

where \( R \) is the Ricci scalar, and \( T = T_{kk} \) is the trace of the matter energy momentum tensor.

One of the largest concentrations of literature within the area of exact solution of field equations relativistic gravity theories is static, spherically symmetric solutions. Some of them are discovered at the early stage of development of relativistic theories, but up to now they are often considered as equivalent representations of some ”unique” solution. However, even for standard general relativistic spherical compact objects there are infinitely many exterior solutions of field equations with spherical symmetry, which describe physically and geometrically different space-times [3]. Since Birkhoffs theorem does not hold in the presence of scalar field, several static solution of the JBD theory is possible in spherically symmetric vacuum situation [4]. Many spacetimes satisfying Einstein’s equations, but these are commonly regarded as non-physical, or unrepresentative of the spacetime we live in. On the other hand, what makes a solution realistic is somewhat subjective, and will depend from author to author. Turning now to specific coordinate system, various interpretations of spherical symmetry have favoured certain coordinate systems, and hence particular choice for the radial coordinate. To start with, note that by using the coordinate freedom inherent in general relativity any static spherically symmetric geometry can be put into a form where there are only two independent metric components. The most common coordinate system so common that it has sometimes been referred to as ”canonical” coordinates - is what we shall term Schwarzschild coordinates.

\[
ds^2 = -e^\lambda dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + e^\nu dt^2.
\]

(3)

These are defined by taken \( r \) to be the radial coordinate (the radius of the 2-sphere with proper area \( 4\pi r^2 \)). Schwarzschild coordinates account for roughly 55% of the work in general relativity fluid spheres [5]. The sec-
ond most popular coordinate system used in fluid spheres work, accounting approximately 35% of papers, is isotropic coordinates. In this coordinate system, the spacelike part of metric is Galilean. Of other coordinate choices used, accounting for the remaining 10% of papers, the only one to have been frequently used is Polar Gaussian coordinates [5].

Space structure in Jordan, Brans, Dicke theory is very different from that in ordinary general relativity. Static, spherically symmetric solutions of this theory can be better understood by consider in different coordinate systems. Further we present a few specific examples where we demonstrate some exact solutions. We do not claim this list is exhaustive.

2 Static spherically symmetric vacuum solutions of Jordan, Brans-Dicke theory.

The first exact solution of JBD field equations were obtained in parametric form by Heckmann [6], soon after Jordan proposed scalar tensor theory. The first solution describes the geometry of the space-time exterior to a prefect fluid sphere in hydrostatic equilibrium. One can chose the static spherically symmetric metric in the form (1) Schwarzschild or curvature coordinates. Then the solutions of the gravitational field equations (2) in the vacuum for \( \eta = 1 \) take the form [6]

\[
\begin{align*}
\tau &= \frac{r_0}{\sqrt{\tau^2 (\tau^2 - h^2)}}, \\
e^\lambda &= \frac{4h^2}{\left[\left(\frac{3}{2} + h\right) \tau^2 - \left(\frac{1}{2} - h\right) \tau^2 - h^2\right]^2}, \\
e^\nu &= \tau^\frac{h^2}{2}, \\
\chi &= \chi_0 \tau^\frac{\beta_0}{2},
\end{align*}
\]

(4)

where \( \tau \) parameter, \( r_0, \beta_0 \) arbitrary constant and

\[
h^2 = \frac{1}{4} - \frac{A}{B^2}; A = \frac{\beta_0}{2} (1 + \beta_0 \xi); B = 1 + 2\beta_0.
\]

From this expression we observe that the solution goes to Schwarzshild solution as \( \beta_0 \to 0 \):
\[ r = r_0 \frac{r_0}{1 - \tau}, \quad e^\lambda = \frac{1}{\tau}, \quad e^{\nu} = \tau, \quad \chi = \chi_0 \]

\[ r_0 = 2m. \]  

On the other hand for \( A=0 \) (\( \xi \neq 0, \xi \neq 2 \))

\[ e^\lambda = \frac{1}{1 - r_0/r}, \quad e^{\nu} = (1 - r_0/r)^{\frac{\xi}{\tau - 2}}, \quad \chi = (1 - r_0/r)^{\frac{1}{\tau - 2}}. \]  

\( (\xi=0) \)

\[ e^\lambda = \frac{1}{1 - r_0/r}, \quad e^{\nu} = 1, \quad \chi = \sqrt{1 - r_0/r}. \]  

From (4) it follows that for \( h=0 \) and \( B\neq0 \)

\[ r = \frac{r_0}{\sqrt{\tau} \ln \tau}, \quad e^\lambda = \frac{1}{(1 + \frac{1}{2} \ln \tau)^2}, \quad e^{\nu} = \tau^{\frac{1}{\tau}}, \quad \chi = \chi_0 \tau^{\frac{2B}{2\pi}}, \]

for \( h=0 \) and \( B=0 \) (\( \xi<2 \))

\[ e^\lambda = \frac{4r^2}{r_0^2 + 4r^2}, \quad e^{\nu} = \left( \pm \sqrt{\frac{r_0^2 + 4r^2 - r_0}{2r}} \right)^{\frac{3}{\sqrt{-A}}}, \quad \chi = \chi_0 e^{-\frac{r}{\tau}}. \]  

\( (\xi>2) \)

\[ e^\lambda = \frac{4r^2}{4r^2 - r_0^2}, \quad e^{\nu} = \frac{1}{\sqrt{-A}} \arcsin \frac{r_0}{2r}, \quad \chi = \chi_0 e^{-\frac{r}{\tau}}. \]  

\( (\xi=2) \)

\[ e^\lambda = 1, \quad e^{\nu} = e^{-\frac{2\chi_0}{\tau}}, \quad \chi = \chi_0 e^{-\frac{r}{\tau}}. \]  

Moreover, there are some special cases. If the \( h^2 = -b^2 < 0 \) then:
\[ r = \frac{r_0}{2 \sqrt{\tau} \sin(b \ln \tau)}, \]
\[ e^\lambda = \frac{4b^2}{[2b \cos(b \ln \tau) + \sin(b \ln \tau)]^2}, \]
\[ e^\nu = \tau^{\frac{b}{2}}, \]
\[ \chi = \chi_0 \tau^{\frac{\eta}{2}}, \]

(12)

Note that from solution for \( \eta = 1 \) we can easy compute \([1]\) the solution for every value of \( \eta \). Usually the most appropriate choice of variables and coupling constants in which to study this problem is \( \phi = 1/\chi, \eta = -1, \omega = -\xi \). As mentioned before one can obtain the solution for \( \eta = -1 \); under a mere redefinition of scalar field \( \chi \to 1/\chi \).

For the purpose of solving JBD field equations (2) new and convenient technique is proposed. This technique will provide significant tool to handle the sophisticated and intractable problems of JBD fields. For the further simplification of the problem we will replace variable \( r \) by \( r(\nu) \) \([7]\).

\[ r \to r(\nu). \]

(13)

Hence the solution of the field equations will be assumed to have the form:

\[ r = e^{\frac{(1+2a)\nu}{2}} \sec \gamma, \]
\[ e^\lambda = \kappa^2 \left[ (1+2a) \cos \gamma - \kappa \sin \gamma \right]^2, \]
\[ \phi = \phi_0 e^{a\nu}, \]

(14)

where \( a, b, \phi_0 \) are arbitrary constants and the parameters are connected through the constraint

\[ \kappa = \sqrt{-1 - 2a (1 + a (2 + \omega))}, \gamma = \frac{\kappa (2b + \nu)}{2}. \]

But there is other spherically symmetric vacuum solution to the JBD field equations
\[ r = c e^{(1+2a)\frac{\nu}{2}} \sec \gamma, \]
\[ e^\lambda = -\frac{\nu^4}{-\kappa^4 + \sigma \cos \gamma^2 + k^4 (1+2a) \sin 2\gamma}, \tag{15} \]
\[ \phi = \phi_0 e^{\alpha\nu}, \]

with \( \sigma = 2 + 10a + (28 + 6\omega) a^2 + 8 (5 + 2\omega) a^3 + 4 (8 + 6\omega + \omega^2) a^4. \)

While the other, known as exterior Brans solutions, corresponds to the geometry in isotropic coordinates. Four forms of static spherically symmetric vacuum solution of the JBD theory are constructed by Brans himself \([8]\). However, as it has been shown in \([9]\) among the four different forms of the static spherically symmetric solution of the vacuum JBD theory of gravity only two classes, Brans class I and class IV solutions, are really independent; the remaining solutions are their variant.

The Brans class I solution in isotropic coordinates

\[ ds^2 = -e^\lambda \left( d\rho^2 - \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 \right) + e^\nu dt^2. \tag{16} \]
is given by

\[ ds^2 = -e^{\lambda_0} \left( 1 + \frac{B}{\rho} \right) \left( \frac{1-B}{1+B} \right)^\frac{2(B-C+1)}{2} \left( d\rho^2 - \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 \right) + \]
\[ + e^{\nu_0} \left( \frac{1-B}{1+B} \right)^\frac{\beta}{2} dt^2, \tag{17} \]
\[ \phi = \phi_0 \left( \frac{1-B}{1+B} \right)^\frac{\beta}{2}, \]

with constant condition

\[ \beta^2 = (C+1)^2 - C \left( 1 - \frac{\omega C}{2} \right) \]

were \( \nu_0, \lambda_0, B, C \) are arbitrary constants.

Brans class IV solution can be written as
\[ ds^2 = -e^{\frac{\lambda_0 + 2(C + 1)}{2\nu}} \left( d\rho^2 - \rho^2 d \theta^2 + \rho^2 \sin^2 \theta d \varphi^2 \right) + e^{\frac{2\nu - 2}{2\nu}} dt^2, \]

\[ \phi = \phi_0 e^{\frac{C}{2\nu}}, \]

with

\[ C = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2}. \]

However, the new coordinate representation (13) simplifies field’s equations. Therefore the coordinate \( \nu \) will be used to give a new solution in the form:

\[ ds^2 = -e^{\nu} \left( \frac{(\rho^2 - B^2)^2}{4\nu^2} \right) \left( d\rho^2 - \rho^2 d \theta^2 + \rho^2 \sin^2 \theta d \varphi^2 \right) + \]

\[ +e^{\nu_0} \left( -\frac{\rho+B}{\rho-B} \right)^2 \sqrt{\frac{2}{2+\omega}} dt^2, \]

\[ \phi = \phi_0 \left( -\frac{\rho+B}{\rho-B} \right)^{-\sqrt{\frac{2}{2+\omega}}}, \]

Other examples of exact JBD solution studied in the literature a power generalization of the Schwarzschild metric [10], [11]

\[ ds^2 = -A^{n-1} dr^2 - A^n r^2 \left( d \theta^2 + \sin^2 \theta d \varphi^2 \right) + A^{m+1} dt^2, \]

\[ \phi = \phi_0 A^{-\frac{m+n}{n}}, \]

\[ A = 1 - \frac{2\rho_0}{r}, \]

where \( m, n \) and \( n_0 \) are arbitrary constants. The coupling constant is found from:

\[ \omega = -2\frac{m^2 + n^2 mn + m - n}{(m + n)^2}. \]

The novelty in the current article lies in the fact that we use other coordinate choices. Polar Gaussian coordinate was first used for general relativity by Synge [12]:

7
\[ ds^2 = -dr^2 - A^2 \left( d\theta^2 + \sin^2 \theta \right) d\varphi^2 + e^\nu dt^2. \]  

(21)

A brief computation leads to

\[ \nu = \frac{1}{\sqrt{2(2+\omega)}} \arctan \left( a\sqrt{\frac{2}{2+\omega}} \left( r + C \right) \right) + b, \]

\[ A = \sqrt{(r + C)^2 - \frac{2+\omega}{2a^2}}, \]

\[ \phi = \phi_0 e^{-\sqrt{2+\omega} \arctan \left( a\sqrt{\frac{2}{2+\omega}} (r + C) \right) - b}, \]

(22)

where \( a, b, C \) and \( \phi_0 \) are arbitrary constants.

On the other hand for \( \omega = -2 \)

\[ \nu = b - \frac{2a}{r+C}, \]

\[ A = r + C, \]

\[ \phi = \phi_0 e^{\frac{2}{2+\omega} r} - 2b, \]

(23)

Isothermal coordinates defined according to the metric [12]

\[ ds^2 = e^\nu \left( dt^2 - dr^2 \right) - A^2 \left( d\theta^2 + \sin^2 \theta \right) d\varphi^2. \]

(24)

A brief computation yields

\[ \nu = \frac{1}{\sqrt{3+2\omega}} \ln \left( \frac{-2r+2a\sqrt{3+2\omega} + C}{2r+2a\sqrt{3+2\omega} - C} \right) + b, \]

\[ A = -\frac{1}{2} \sqrt{-4 \left( 2 + 2\omega \right) a^2 + \left( -2r + C \right)^2 \left( \frac{-2r+2a\sqrt{3+2\omega} + C}{2r+2a\sqrt{3+2\omega} - C} \right)^2}, \]

\[ \phi = \phi_0 \frac{2r+2a\sqrt{3+2\omega} - C}{-2r+2a\sqrt{3+2\omega} + C} \sqrt{\frac{1}{3+2\omega}}, \]

(25)

Nikolai Ivanovich Lobacbevsky for the first time in the mathematical literature, a geometric theory was presented based on all Euclidean postulates except for the fifth one which is referred to as the Euclidean postulate of parallels [13]. In the usual way we get for the three dimensional space the metric form
\[ ds^2 = d\rho^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right). \]  

(26)

The difference between the two geometries consists in the dependence of \( r \) on \( \rho \): in the Euclidean geometry \( r = \rho \) while in the Lobachevskyan one \[ r = \kappa \sinh \left( \frac{\rho}{\kappa} \right) \]  

(14)

On the other hand, a radial coordinate of the spatial hyperbolic geometry is necessary if perfect fluids with cosmological constant are considered [15], this was noted by Weyl already in 1919 [16]. Therefore it is interesting to analyse solutions to the JBD field equations with hyperbolic metric:

\[ ds^2 = -e^\lambda dr^2 - \kappa \sinh \left( \frac{r}{\kappa} \right)^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + e^\nu dt^2. \]  

(27)

If the energy-momentum tensor of matter \( T_{ik} \), vanishes, the system of equations (2) have the flat vacuum solution. For the case of hyperbolic metric:

\[ e^\lambda = \cosh \left( \frac{r}{\kappa} \right)^2, \ e^\nu = 1, \ \phi = \phi_0. \]  

(28)

It should be noted, however, there is vacuum solution that identical to solution of general relativity:

\[ e^\lambda = \frac{b \cosh \left( \frac{r}{\kappa} \right)^2 \sinh \left( \frac{r}{\kappa} \right)}{-\kappa a + b \sinh \left( \frac{r}{\kappa} \right)}, \]

\[ e^\nu = \left( b - \kappa \csc h \left( \frac{r}{\kappa} \right) \right), \]  

(29)

\[ \phi = \phi_0. \]

The different possible vacuum solution of equations (2) with hyperbolic metric yield:
\[ e^\lambda = \frac{b \cosh\left(\frac{r}{\kappa}\right)^2 \sinh\left(\frac{r}{\kappa}\right)}{-\kappa a + b \sinh\left(\frac{r}{\kappa}\right)}, \]
\[ e^\nu = \left(\frac{(2+\omega)(b-\kappa \text{csc} h\left(\frac{r}{\kappa}\right))}{\omega}\right)^{\frac{1}{2+\omega}}, \tag{30} \]
\[ \phi = \phi_0 \left(\frac{(2+\omega)(\kappa \text{csc} h\left(\frac{r}{\kappa}\right)-b)}{\omega}\right)^{\frac{1}{2+\omega}}, \]

were \( a, b, \phi_0 \) are arbitrary constants.

3 Conclusion

When the energy-momentum tensor \( T=T_{kk} \) vanishes one can use \( \phi = \text{const} \) for scalar field outside the matter and the solutions of the JBD theory become the similar as the solutions of the Einstein theory. Then in empty space there is not a difference between scalar-tensor theories (as well as vector-metric theories [17]) and Einstein theory. In this case in empty space celestial-mechanical experiments to reveal a difference between scalar-tensor theories and Einstein theory is not possible.

On the other hand due to highly non-linear character of gravitational theories, a desirable pre-requisite for studying strong field condition is to have knowledge of exact solutions of the field equations. Many of those solutions may be useful for understanding the inherent character of gravitational theories. In this work we present different classes of the static spherically symmetric vacuum solution of JBD theory in different coordinate systems. We found new exact solution in Polar Gaussian, isothermal and hyperbolic coordinate for the JBD field equations. In closing we reiterate that while a tremendous amount is already known concerning static spherically symmetric spacetimes discuss the solutions with appropriate choice of coordinate systems fruitful for understanding its physical relevance.

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