SYMMETRY, SINGULARITIES AND INTEGRABILITY IN COMPLEX DYNAMICS IV:
PAINLEVÉ INTEGRABILITY OF ISOTROPIC COSMOLOGIES

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We apply the results of singularity analysis to the isotropic cosmological models in general relativity and string theory with a variety of matter terms. For some of these models the standard Painlevé test is sufficient to demonstrate integrability or nonintegrability in the sense of Painlevé. For others of these models it is necessary to use a less algorithmic procedure.

1. Introduction

In the modelling of various phenomena in relativistic cosmology the final result of the modelling process is a system of ordinary differential equations, whose solution is by no means obvious. Generally these systems are of the first order and autonomous, so that, if the system is two-dimensional, one is assured of the existence of solutions under mild conditions on the terms in the equations. For systems of dimension three or more the question of integrability or nonintegrability is extended by the possibility of chaotic behaviour in the general solution (for a recent review of the relevant dynamical systems methods in cosmology see \([33]\)).

One method to determine the integrability of a system is by a performance of the so-called singularity, or Painlevé, analysis in an effort to examine whether or not there exists a Laurent expansion of the solution about a movable pole which contains the number of arbitrary constants necessary for a general solution. Any other singularities are not permitted except in the case of branch point singularities which give rise to what is called the weak Painlevé property. A system which is integrable in the sense of Painlevé has its general solution analytic except at the polelike singularity.

The singularity analysis of ordinary differential equations goes back to the end of the nineteenth century to the pioneering works of Kowalevskaya \([17]\), Painlevé \([24, 25, 23]\) and others \([13, 14, 5]\) and has enjoyed a resurgence of interest in the last thirty years because of the growing attention paid to nonlinear equations, both partial and ordinary. The practical use of the singularity analysis has been greatly enhanced by the development of the ARS algorithm \([1, 2, 3]\), the simplifications proposed by Kruskal \([6, 16]\) and the review by Ramani, Grammaticos and Bountis \([28]\). This practical approach has at times led to misinterpretations at the hands of practitioners. The works of Conte \([20]\) have pointed the way to a rigorous application of the singularity analysis.

In this paper, which is the fourth in a series \([19, 20, 21]\) devoted to the investigation of connections between the three main topics in dynamics namely, symmetry, singularities and integrability, the topics which superficially are unrelated in their mathematics and yet are intimately intertwined, we apply the techniques of singularity analysis to a number of model systems which have been derived for different problems in cosmology. The problems which we consider represent three classes, namely, those of fluid models in general relativity, scalar field models coupled to fluid cosmologies again in general relativity and lastly isotropic string cosmologies. (For more information about the physical characteristics of the models treated below the reader is urged to consult the cited papers.) More specifically, we consider the following systems:

1. The one-fluid FRW model in general relativity \([28]\) (p. 122):

\[
\dot{H} = -(1 + q)H, \\
\dot{\Omega} = -2q(1 - \Omega), \quad q = \frac{1}{3}(3\gamma - 2)\Omega
\]

the latter of which is, of course, equivalent to the scalar equation

\[
\dot{\Omega} = (2 - 3\gamma)(1 - \Omega)\Omega.
\]

Here, \(H\) is the Hubble variable, \(\Omega\) the density parameter, \(q\) the deceleration parameter and derivatives are taken with respect to a dimensionless time parameter \(\tau\) defined so that \(dt/d\tau = 1/H\), \(t\) being the usual time variable, giving the space sections of constant curvature in these models. As usual, the parameter \(\gamma \in [0, 2]\) describes a barotropic fluid. Although this is the simplest cosmological system, it is included here for completeness and comparison with more sophisticated ones introduced below.

2. The two-fluid FRW model in general relativity \([28]\) (p. 126):

\[
\dot{\Omega} = -\frac{1}{2}(b - x) \cos 2\Omega \cos \Omega, \\
\dot{\chi} = (1 - \chi^2) \sin \Omega,
\]

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where the so-called compactified density parameter $\Omega \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and the transition variable $\chi \in [-1,1]$ defined in the above reference are used to describe which fluid is dominant dynamically. Here $b > -1$.

3. The flat one-fluid FRW space-time with a scalar field $\phi$ with an exponential potential in general relativity [30] which reduces to the two-dimensional system

$$\dot{x} = -3x + 3\left(1 - \frac{1}{2}\gamma\right)x^3 - \frac{3}{2}\gamma xy^2 + \frac{1}{2}\gamma \lambda y^2, \quad \dot{y} = 3\left(1 - \frac{1}{2}\gamma\right)x^2y - \frac{3}{2}\gamma y^3 - \frac{3}{2}\gamma \lambda xy,$$  

(1.4)

where differentiation is again with respect to a dimensionless time variable. Here $\lambda$ is a positive constant appearing in the scalar field potential and $\gamma \in (0,2)$, that is, we exclude the “extreme” cases of stiff matter and scalar field coming from the fluid contribution.

4. General-relativistic one-fluid FRW models with $n$ scalar fields $\phi_i$, $i = 1, \ldots, n$ with exponential potentials [32] described in dimensionless variables ($\Omega, \Phi, \Psi$) by the system (this strictly excludes positively curved variables wherein the variables are not defined)

$$\dot{\Psi}_i = \Psi_i(q - 2) - \sqrt{\frac{2}{3}}k_i\Phi_i^2, \quad \dot{\Phi}_i = \Phi_i\left(q + 1 + \frac{2}{3}k_i\Psi_i\right), \quad \dot{\Omega} = \Omega(2q - 3\gamma + 2),$$  

(1.5)

where

$$q = \frac{1}{2}(3\gamma - 2)\Omega + 2\sum_{i=1}^{n}\Psi_i^2 - \sum_{i=1}^{n}\Phi_i^2.$$  

(1.6)

The variables ($\Phi, \Psi$) are defined in terms of the potential and kinetic energies of the scalar fields and $\gamma \in [0,2]$.

5. The four-dimensional flat string FRW model with negative central charge deficit described in the compactifying variables ($\xi, \eta$) by the system [32]

$$\dot{\Psi}_i = \xi^2(1 - \eta^2), \quad \dot{\xi} = \left(\sqrt{3} + \eta\xi\right)(1 - \xi^2).$$  

(1.7)

6. The ten-dimensional flat string FRW model in the RR sector and with a positive cosmological constant described by the system [32]

$$\dot{x} = (x + \sqrt{3})(1 - x^2 - y - z) + \frac{1}{2}z(x - \sqrt{3}), \quad \dot{y} = 2y[(x + \sqrt{3})(1 - x^2 - y - z) + \frac{1}{2}z(x - \sqrt{3})], \quad \dot{z} = 2z[(x + \sqrt{3})(1 - x^2 - y - z) + \frac{1}{2}z(x - \sqrt{3})],$$  

(1.8)

for which the dimensionally reduced (four-dimensional) model is obtained by placing $y = 0$ in [32].

Another approach to determining the integrability of sets of differential equations is the use of Noether’s theorem for Lagrangian systems and the Lie theory of extended groups for differential equations in general. For the systems which we study here the Lie symmetry approach is not generally successful because we need to find generalised or nonlocal symmetries to supplement the point symmetries which are the easiest to obtain.

For the benefit of the reader we describe in Sec. 2 the main points of the Painlevé process (for a more detailed albeit elementary introduction we refer the reader to [27]). This is then used in Sec. 3 to study the singularity features of our cosmological systems. Conclusions are given in Sec. 4.

2. Methodology of Painlevé analysis

The singularity analysis which lies at the basis of the Painlevé test as systematised in the ARS algorithm [1–3] is a specific form of a more general analysis which examines a set of differential equations for leading order behaviour and next to leading order behaviour [18, 12]. The latter analysis does not take into consideration the necessity for a Laurent expansion about a polelike singularity (or rational branch point) which is required for the Painlevé test. The essence of the Painlevé test (for a set of ordinary differential equations which is the only type which we consider here) is that the solution of an $n$-dimensional set of equations

$$\dot{x}_i = f_i(t, x),$$  

(2.1)

where the functions $f_i$ are rational in the dependent variables and algebraic in the independent variable, can be written as either

$$x_i(\tau) = \sum_{j=0}^{\infty}a_j\tau^{-p_j+q_j}$$  

(2.2)

in the case of a Right Painlevé Series or

$$x_i(\tau) = \sum_{j=0}^{\infty}a_j\tau^{-p_j-q_j}$$  

(2.3)

in the case of a Left Painlevé Series [22, 23] where $\tau = t - t_0$ and $t_0$ is arbitrary, the parameter $p_i$ are strictly positive integers (rational numbers in the case of the so-called weak Painlevé test), the parameter $q_i$ is a (positive) integer (respectively rational number) and in the coefficients there are $n - 1$ arbitrary constants which, together with $t_0$, give the required number of arbitrary constants for the general solution of the system [23].

The ARS algorithm provides a mechanistic procedure for the determination of the leading order behaviour and the resonances which are where the arbitrary constants of integration arise. The first step is to determine the leading order behaviour by means of the substitution

$$x_i = a_i\tau^{-p_i}.$$  

(2.4)
into the system (2.1) and to assemble all possible patterns of values of the exponents \( p_i, i = 1, n \) from the dominant terms of the system (2.3). For each of these possible patterns the coefficients \( \alpha_i \) are determined and the substitution
\[
\begin{align*}
  x_i = \alpha_i \tau^{-p_i} + \mu_i \tau^{-p_i}
\end{align*}
\]  
(2.5)
is made to determine the “resonances” \( r \) at which the arbitrary constants, \( \mu_i \), are introduced. The resonances and the arbitrary constants are determined from the leading order behaviour of the system (2.1) by a linearisation process. The final step for each particular pattern of leading order behaviour is to substitute a series, truncated at the highest resonance, to ensure that there is compatibility at the resonances. If, for a particular pattern of leading order behaviour, these conditions are satisfied and the series contains \( n - 1 \) arbitrary constants (the \( n \)th comes from \( t_0 \)), that particular pattern is said to pass the Painlevé test. If this is true for all possible patterns of leading order behaviour, the set of equations is said to possess the Painlevé property and to be integrable. We note that the concept of Painlevé integrability means the possession of a Laurent series, \( i.e., \) the solution is analytic except at the polelike (branch point in the case of the weak property) movable singularities.

The algorithm described in the previous paragraph is not always possible or easy to implement. The first problem arises when not all of the exponents obtained in the leading order analysis are strictly positive. In the case that some or all of the exponents are strictly negative one can perform a homeographic transformation, which preserves the Painlevé property, on the affected variables so that the exponents will now be strictly positive. The test can then be continued algorithmically. There is no such consolation in the case of zero exponents in the leading order behaviour. The algorithm fails. In this case it is necessary to make a series substitution commencing at the leading order behaviour to establish whether or not a sufficient number of arbitrary constants is introduced. If these constants enter at early terms in the series, the task is not too difficult with one of the symbolic manipulation codes. If this is not the case, the task can become impossible for a system of moderate complexity.

Another possibility which can occur and create a problem of interpretation is that the number of arbitrary constants introduced at the resonances is insufficient to produce the general solution for the system. This possibility was already observed by Ince, citing from a paper by Chazy written in 1909, in 1927 [15] (p. 355) and treated in some more detail in [10]. Ince described such a solution as a “singular” solution. This is in accord with the usual usage of the word in the case of first order equations. However, it is a little unfortunate that the word “singular” has to perform two distinct functions in the description of the integrability of one system. In an attempt to separate the two uses of the word, Cotsakis and Leach [10] described this type of solution as a “partial” solution confined to a submanifold of the space of initial conditions, to the other was lacking one arbitrary constant. This demonstration was for a single example and one would want a sounder basis for making a definite claim about integrability in cases for which the requirements of the Painlevé property were only partially satisfied for some patterns of leading order behaviour.

Finally we recall that the possession of the Painlevé Property is representation dependent and is definitely preserved only under a homeographic transformation. However, there are times when we can introduce a transformation with profit such as in the system described by (1.3). The nature of the integrability of the original system in comparison with that of the transformed system will be tempered by the nature of the particular transformation.

3. Applications of Painlevé analysis to models

3.1. One-fluid FRW model, (1.1)

In the case of (1.1), it is useful to introduce the rescaling transformation
\[
\begin{align*}
  H \rightarrow x, \quad \Omega \rightarrow y, \quad t \rightarrow \frac{1}{2}(3\gamma - 2)t
\end{align*}
\]  
(3.1)
to give the system
\[
\begin{align*}
  \dot{x} &= -(\sigma + y)x, \\
  \dot{y} &= -2y(1 - y),
\end{align*}
\]  
(3.2)
where \( \sigma = \frac{1}{2}(3\gamma - 2) \). When we make the usual substitution for the leading order behaviour, we obtain
\[
\begin{align*}
  \alpha \tau^{p-1} &= -\alpha \beta \tau^{p-q} \\
  \beta q \tau^{q-1} &= 2\beta^2 \tau^{2q}.
\end{align*}
\]  
(3.3)
From the second Eq. (3.3) it is evident that \( q = -1 \) and \( \beta = -\frac{1}{2} \). From the exponents of the first Eq. (3.3) the value of \( p \) is arbitrary. However, equality of the coefficients of the leading order powers imposes the requirement that \( p = \frac{1}{4} \). Although the value of \( p \) is not strictly negative, we can proceed with the Painlevé test without a further transformation since it is a nonintegral rational number. We note that the value of \( \alpha \) is unspecified.

To obtain the resonances we make the substitution
\[
x = \alpha \tau^{1/2} + \mu \tau^{i+1/2},
\]
\[
y = \beta \tau^{-1} + \nu \tau^{-1}
\]
to obtain the condition
\[
\begin{vmatrix}
\frac{r}{\tau} + \frac{\alpha}{2} + \beta & 0 \\
0 & r - 1 - 4\beta
\end{vmatrix} = 0
\]
that there be a nontrivial solution. The condition (3.3) gives the resonances \( r = -1, 0 \) and so the system (3.2) passes the Painlevé test. Hence the system (1.1) possesses the Painlevé property. The first few terms of the Laurent expansion are
\[
x(\tau) = a_0 \left\{ \tau^{1/2} - \frac{1}{2}(\sigma + \frac{1}{2})\tau^{-1/2} + \frac{3}{4}(\sigma + \frac{3}{2})\tau^{-3/2} + \ldots \right\},
\]
\[
y(\tau) = -\frac{1}{2}\tau^{-1} + \frac{1}{\tau} - \frac{1}{6}\tau^{2} + \ldots .
\]

**3.2. Two-fluid model, (1.3)**

The system (1.3) is not in a suitable form for applying the Painlevé analysis. We have two choices for the introduction of new variables. Interestingly both changes of variables produce essentially the same results. In the first instance we introduce the new variables
\[
x = \chi \quad \text{and} \quad y = \cosec \Omega,
\]
so that the system (1.3) becomes
\[
\dot{xy} = (1 - x^2)
\]
\[
\dot{yy}^2 = \frac{1}{2}(b - x)(y^2 - 1)(y^2 - 2).
\]
The usual leading term analysis gives the exponents \( p = -1 \) and \( q = 0 \), so that the ARS logarithm is not applicable.

We do not continue with the analysis of the system in (3.5) since, as we noted above, the second change of variables produces essentially the same results.

The second transformation, which we consider in detail here, has the advantage of being one-to-one and continuous over the defined intervals of the original variables. We set
\[
x = \chi \quad \text{and} \quad y = \sin \Omega
\]
to obtain the system
\[
\dot{x} = (1 - x^2)y
\]
\[
\dot{y} = -\frac{1}{2}(b - x)(1 - 2y^2)(1 - y^2).
\]
The same exponents are obtained as for (3.8). We make the Ansatz
\[
x = \sum_{i=0} a_i \tau^{i-1}, \quad y = \sum_{i=0} b_i \tau^{i}
\]
and substitute this into the system (3.10) to obtain the pair of relations
\[
(i - 1)a_i \tau^{i-2} = -a_i a_j b_k \tau^{i+j+k-2} + b_i \tau^{i},
\]
\[
ib_i \tau^{i-1} = -\frac{1}{2}b(1 - 3b_0b_2 \tau^{i+j} + 2b_1b_2b_3 \tau^{i+j+k-1}) + a_i (\tau^{i-1} - 3b_2b_3 \tau^{i+j+k-1}) + 2b_3b_4b_5 \tau^{i+j+k+1} + m-1),
\]
from which we are able to deduce the first few terms of the expansions for \( x \) and \( y \). From the first two terms of (3.12) we obtain
\[
a_0b_0 = 1 \quad \text{and} \quad a_1 = -\frac{1}{4}a_0b_1
\]
and from the first term of (3.13)
\[
a_0 (1 - 3b_0^2 + 2b_1^2) = 0.
\]
Since \( a_0 \neq 0 \), it follows from (3.13) that \( b_0^2 = 1, \frac{1}{4} \). In either case the second term of (3.13) gives \( b_1 = 0 \) and consequently \( a_1 = 0 \). The third term of (3.13) reduces to \( b_2 = b_0^2b_2 \), so that, for the first possibility for the value of \( b_0, b_2 \) is arbitrary and, for the second possibility, zero. In fact for the second possibility all subsequent coefficients \( b_i \) are zero and we obtain the solution
\[
x(\tau) = \frac{1}{b_0} + \frac{b_0\tau}{3} - \frac{(b_0\tau)^3}{45} + \ldots,
\]
\[
y(\tau) = b_0,
\]
where \( b_0^2 = 1/2 \), which is certainly a peculiar solution.

In the case that \( b_0^2 = 1 \) we obtain a more standard solution. We have that \( b_2 \) is arbitrary and this provides us with the second arbitrary constant required for a general solution of the original system. We have
\[
\dot{x} = a_0 \tau^{-1} + \frac{1}{3}(b_0 - 2b_2) \tau + \frac{1}{2}b_0b_2 \tau^2 + \ldots,
\]
\[
y = b_0 + b_2 \tau^2 + \ldots,
\]
where the coefficients \( a_0 \) and \( b_0 \) have been given above.

In the expansions we have obtained, only the second one has the required number of arbitrary constants, and we cannot conclude that the system (1.3) is integrable in the sense of Painlevé. However, we do note that it is possible to obtain a first integral of the original system (1.3) and reduce the solution to a rather complicated quadrature. In fact the system can be written in terms of a Lagrangian and is Hamiltonian, so that the existence of the first integral immediately guarantees integrability in the sense of Liouville.
3.3. Flat FRW with one fluid and an exponential potential, \((1.4)\)

Before we begin the singularity analysis of \((1.4)\), it is appropriate to simplify the system by the rescaling

\[
\begin{align*}
t & \rightarrow \frac{2(2 - \gamma)}{3\lambda^2} t, \\
x & \rightarrow \frac{\lambda\sqrt{3}}{\sqrt{2}(2 - \gamma)} x, \\
y & \rightarrow \beta y \quad \text{with} \quad \beta^2 = \frac{3\lambda^2}{2\gamma(2 - \gamma)},
\end{align*}
\]

in which the sign of \(\beta\) may be taken as positive without loss of generality. The system \((1.4)\) now has the simpler appearance

\[
\begin{align*}
\dot{x} &= -Ax + By^2 + x^3 - xy^2 \\
\dot{y} &= -xy + x^2y - y^3, \\
\end{align*}
\]

where \(A = 8(1 - \frac{1}{2}\gamma) / (3\lambda^2)\) and \(B = \lambda^2 / (\gamma(2 - \gamma))\).

We determine the leading order behaviour of the system \((3.14)\) to be \(x(\tau) = a\tau^{-1/2}\) and \(y(\tau) = \beta\tau^{-1/2}\) with the constraint \(a^2 - \beta^2 = -1/2\). We find that the resonances are at \(r = -1, 0\), where the first resonance is generic and the second indicates that one of the coefficients of the leading order behaviour is arbitrary, which is in accordance with the above constraint. If we make the substitutions

\[
\begin{align*}
x(\tau) &= \sum_{i=0} a_i \tau^{i-1/2}, \\
y(\tau) &= \sum_{i=0} b_i \tau^{i-1/2},
\end{align*}
\]

we find that the first few terms of the expansion are given by

\[
\begin{align*}
a_0 &= \text{arbitrary} \\
a_1 &= B + \frac{2}{3}(2 + 5B)a_0^2 + \frac{8}{3}(1 + B)a_0^4 \\
a_2 &= \frac{1}{6}a_0(3B(1 + 5B) + (1 + 2B(4 + 5B))a_0^2 + 28(1 + B)(2 + 5B)a_0^4 + 80(1 + B)^2a_0^6 - 6A(1 + a_0^2)) \\
b_0^2 &= a_0^2 + \frac{1}{2} \\
b_1 &= \frac{2}{3}a_0(-1 + 2B + 4(1 + B)a_0^2) \\
b_2 &= \frac{1}{6}b_0(3(-1 + B)B + 2(-2 - 3A + 2B + 16B^2)a_0^2 + 4(1 + B)(2 + 23B)a_0^4 + 80(1 + B)^2a_0^6).
\end{align*}
\]

We conclude that the system \((1.4)\) is integrable in the sense of Painlevé.

3.4. Exponential potential with one fluid \((1.5)\)

We consider the case in which \(n = 1\). When we make the substitution for \(q\) in the system \((1.5)\), we obtain

\[
\begin{align*}
\dot{\Psi} &= \frac{1}{3}(3\gamma - 2)\Psi + 2\Psi^3 - \Phi^2\Psi - 2\Psi - \sqrt{\frac{2}{3}K}\Phi^2, \\
\dot{\Phi} &= \frac{1}{3}(3\gamma - 2)\Phi + 2\Phi^3 - \Phi^3 + \Phi + \sqrt{\frac{2}{3}K}\Phi\psi, \\
\dot{\Omega} &= (3\gamma - 2)\Omega^2 + 4\Phi^2\Omega - 2\Phi^2\Omega - (3\gamma - 2)\Omega. \\
\end{align*}
\]

We find that the leading order behaviour is given by

\[
\Psi(\tau) = a\tau^{-1/2}, \quad \Phi(\tau) = b\tau^{-1/2}, \quad \Omega(\tau) = c\tau^{(3,23)},
\]

subject to the constraint that

\[
(3\gamma - 2)c + 4a^2 - 2b^2 = -1.
\]

The analysis of the dominant terms for the resonances is facilitated by the substitutions

\[
\begin{align*}
x &= 4\Psi^2, \\
y &= -2\Phi^2, \\
z &= (3\gamma - 2)\Omega.
\end{align*}
\]

(3.24)

The leading order behaviour of \((3.22)\) is given by

\[
\begin{align*}
\dot{x} &= x(x + y + z) \\
\dot{y} &= y(x + y + z) \\
\dot{z} &= z(x + y + z).
\end{align*}
\]

(3.26)

The leading order behaviour of \((3.26)\) is given by

\[
\begin{align*}
x &= \alpha\tau^{-1}, \\
y &= \beta\tau^{-1}, \\
z &= \gamma\tau^{-1},
\end{align*}
\]

subject to the constraint \(\alpha + \beta + \gamma = -1\). (The constant \(\gamma\) in \((3.27)\) is not to be confused with the physical constant in the original system.) We determine the resonances by substituting into \((3.26)\)

\[
\begin{align*}
x &= \alpha\tau^{-1} + \mu\tau^{-1}, \\
y &= \beta\tau^{-1} + \nu\tau^{-1}, \\
z &= \gamma\tau^{-1} + \rho\tau^{-1}
\end{align*}
\]

(3.28)

to obtain the linearised system

\[
\begin{bmatrix}
s - \alpha & -\alpha & -\alpha \\
-\beta & s - \beta & -\beta \\
-\gamma & -\gamma & s - \gamma
\end{bmatrix}
\begin{bmatrix}
\mu \\
\nu \\
\rho
\end{bmatrix} = 0
\]

(3.29)

which has a nontrivial solution if \(s = -1, 0\). Thus we see that there is a double zero resonance which is consistent with the constraint. These results pass over to the original system and, since two arbitrary constants enter at the leading order terms, the system
In each of these cases it is necessary to make a series substitution since the ARS algorithm for the application of the Painlevé test is no longer appropriate. We summarise the results:

1. We substitute the series
   \[
   \Psi = \sum_{i=0} b_i \tau^{(i-1)/2}, \quad \Phi = \sum_{i=0} a_i \tau^{(i-1)/2},
   \]
   \[
   \Omega = \sum_{i=0} c_i \tau^{i/2}
   \]
   to find that \(a_0\) is arbitrary and \(c_0 = c_1 = c_2 = \ldots = 0\), so that we have a peculiar solution with two arbitrary constants \((a_0\) and \(t_0)\) and one of the functions, \(c(\tau)\), having only the trivial solution.

2. The series substituted are now
   \[
   \Psi = \sum_{i=0} a_i \tau^{i/2}, \quad \Phi = \sum_{i=0} b_i \tau^{(i-1)/2},
   \]
   \[
   \Omega = \sum_{i=0} c_i \tau^{(i-2)/2},
   \]
   We find that \(a_0\) and \(c_0\) are arbitrary and that \(b_0 = \sqrt{(1 + A c_0)}/2\), so that the series solutions do contain three arbitrary constants when \(t_0\) is included.

3. In this case we put
   \[
   \Psi = \sum_{i=0} a_i \tau^{(i-1)/2}, \quad \Phi = \sum_{i=0} b_i \tau^{(i-1)/2},
   \]
   \[
   \Omega = \sum_{i=0} c_i \tau^{(i-2)/2},
   \]
   We distinguish three subcases.
   Subcase (a): the coefficients \(a_0 = a_1 = a_2 = \ldots = 0\) and \(b_0 = b_1 = b_2 = \ldots = 0\) and \(c_0 = -1/A\) indicate that we have trivial solutions for \(\Psi\) and \(\Phi\) and a nontrivial series with arbitrary constant \(t_0\) for \(\Omega\).
   Subcase (b): we find that \(a_0\) is arbitrary and \(b_0 = b_1 = b_2 = \ldots = 0\), so that we have the trivial solution for \(\Phi\) and a two parameter solution \((a_0\) and \(t_0)\) for \(\Psi\) and \(\Omega\).
   Subcase (c): In this subcase \(a_0 = 1/2 \tau, b_0 = b_1 = b_2 = \ldots = 0\) and \(c_0 = c_1 = c_2 = \ldots = 0\), so that we have trivial solutions for \(\Phi\) and \(\Omega\) and a one parameter solution for \(\Psi\).

   All three subcases present us with peculiar solutions in the form of series.

4. With the substitution
   \[
   \Psi = \sum_{i=0} a_i (\tau-1/2), \quad \Phi = \sum_{i=0} b_i \tau^{i/2},
   \]
   \[
   \Omega = \sum_{i=0} c_i \tau^{i/2}
   \]
   we find that \(a_0 = 0\), so that there is no singular behaviour in any of the series representations of the three functions. However, the constants \(a_1, b_0\) and \(c_0\), the leading terms of each of the series, are arbitrary, and so we obtain a three-parameter representation of the solution.
5. When we make the substitution
\[ \Psi = \sum_{i=0} a_i \tau^{(i-1)/2}, \quad \Phi = \sum_{i=0} b_i \tau^{(i-1)/2}, \]
\[ \Omega = \sum_{i=0} c_i \tau^{i/2}, \quad (3.37) \]
we find two subcases.

Subcase (a): the coefficient \( b_0 = 0 \), which removes any possible singular behaviour, and the coefficients \( a_0, b_1 \) and \( c_0 \) are arbitrary, thereby providing a three-parameter series representation for \( \Psi, \Phi \) and \( \Omega \).

Subcase (b): the coefficient \( b_0 = 2^{-1/2} \). All coefficients \( c_i \) are zero and all coefficients \( a_i, b_i, \ i > 0 \) are expressed in terms of the parameter \( A \) of the system. Thus we have a trivial solution for \( \Omega \) and a one-parameter solution (the location of the singularity \( t_0 \) of \( \Phi \)) for the two functions \( \Psi \) and \( \Phi \).

The first subcase presents a solution without a singularity and the second one a peculiar solution with singularity.

6. For the last case we substitute
\[ \Psi = \sum_{i=0} a_i \tau^{i/2}, \quad \Phi = \sum_{i=0} b_i \tau^{i/2}, \]
\[ \Omega = \sum_{i=0} c_i \tau^{(i-2)/2}. \quad (3.38) \]
We find that \( c_0 = c_1 = 0 \), thereby removing any possible singular behaviour, and that \( a_0, b_0 \) and \( c_2 \) are arbitrary. Thus we have a three-parameter series representation of the solutions for \( \Psi, \Phi \) and \( \Omega \).

Altogether the results support the proposition that the system (1.8) is integrable.

3.6. 10-dimensional flat string FRW, (1.8)
For the system (1.8)
\[ \dot{x} = (x + \sqrt{3})(1 - x^2 - y - z) + \frac{1}{2} z(x - \sqrt{3}), \quad (3.42) \]
\[ \dot{y} = 2y \left[ (1 - x^2 - y - z) + \frac{1}{2} z \right], \quad (3.43) \]
\[ \dot{z} = 2z \left[ (1 - x^2 - y - z) - \frac{1}{2}(1 - z - \sqrt{3}x) \right], \quad (3.44) \]
we determine the leading order behaviour of this system to be \( x(\tau) = \alpha \tau^{-1/2}, \ y(\tau) = \beta \tau^{-1} \) and \( z(\tau) = \gamma \tau^{-1} \) with the constraint \( 2\alpha^2 + 2\beta + \gamma = 1 \). We determine the resonances by the substitution of
\[ x(\tau) = \alpha \tau^{-1/2} + \mu \tau^{s-1}, \]
\[ y(\tau) = \beta \tau^{-1} + \nu \tau^{s-1}, \]
\[ z(\tau) = \gamma \tau^{-1} + \rho \tau^{s-1} \quad (3.45) \]
into (1.8) to obtain the linearised system
\[ \begin{pmatrix} s + 2\alpha^2 & \alpha & \alpha/2 \\ 4\alpha\beta & s + 2\beta & \beta \\ 4\alpha\gamma & 2\gamma & s + \gamma \end{pmatrix} \begin{pmatrix} \mu \\ \nu \\ \rho \end{pmatrix} = 0 \quad (3.46) \]
which has a non-trivial solution if \( s = -1, 0, 2 \). There is a double zero resonance which is consistent with the constraint and the system passes the Painlevé test for this pattern of leading order behaviour. The first few terms of the Laurent expansion
\[ x(\tau) = a_0 \tau^{-1/2} + a_1 + a_2 \tau^{1/2} + \ldots \]
\[ y(\tau) = b_0 \tau^{-1} + b_1 \tau^{-1/2} + b_2 + b_3 \tau^{1/2} + \ldots \]
\[ z(\tau) = c_0 \tau^{-1} + c_1 \tau^{-1/2} + c_2 + c_3 \tau^{1/2} + \ldots \quad (3.47) \]
are given by
\[ a_1 = \left( \frac{1}{\sqrt{3}} \right) \left[ 3(-3 + 4b_0) \right. \]
\[ -2a_0^6 [ -11 + 6a_0^2 + 6b_0] \]
\[ a_2 = (a_0/2) \left[ -93 + 180a_0 + 4b_0 (68 - 45b_0) \right. \]
\[ +6a_0^4 \left( -77 + 6b_0 \right. \]
\[ +2a_0^2 \left( 188 + 3b_0 (-107 + 30b_0) \right) \]
\[ b_1 = - \left( \frac{4}{\sqrt{3}} \right) a_0 b_0 (-5 + 6a_0^2 + 6b_0) \]
\[ b_2 = (b_0/3) \left[ -63 + 168b_0 + 2(342a_0^6 - 54b_0) \right. \]
\[ +3a_0^4 \left( -199 + 228b_0 \right. \]
\[ +a_0^2 \left( 290 + 3b_0 (-217 + 114b_0) \right) \]
\[ c_0 = 1 - 2a_0^2 - 2b_0 \]
\[ c_1 = \left( \frac{2}{\sqrt{3}} \right) a_0 \left[ -1 + 2a_0^2 + 2b_0 \right] (-13 \]
\[ +12a_0^2 + 12b_0) \]
\[ c_2 = (-1/3) \left[ -1 + 2a_0^2 + 2b_0 \right] [684a_0^6 \]
\[ +6a_0^4 (-229 + 228b_0) \]
\[ +2a_0^2 (392 + 57b_0 (-15 - 6b_0)) \]
\[ -3 (31 + 4b_0 (-17 + 9b_0)) \].

Note that the constraint is apparent in the first of the relations for the coefficients \( c_i \).
4. Discussion

The examples discussed in this paper may prove useful when one is interested in having a theory to decide the question of what is the general singularity pattern of isotropic cosmologies in different gravity theories. This question, apart from its purely mathematical interest, is believed to be related to recent observations of the possible oscillatory nature of the universe, exemplified in an oscillatory behaviour in the Hubble parameter [1].

Among the models discussed in this paper, the two-fluid FRW model in general relativity presents the most interesting dynamical behaviour. All other models are either strictly integrable in the sense of possessing the strong Painlevé property, or they have branch-point singularities indicating weak integrability in the sense of Painlevé.

The two-fluid model possesses the interesting property of what can be called singular envelopes, first discussed by Ince and rediscovered in a different context in [10]. That is, although the general solution is unknown, any possible nonintegrable or even chaotic behaviour may be confined to a region of phase space enveloped by the peculiar solutions in the sense given at the end of the previous section. It is interesting to ask whether this property is rigid, that is, is it maintained when two-fluid models are considered either in other gravity theories, or in more general Bianchi models in general relativity. The former question is currently under investigation and can be reformulated as a two-fluid plus a scalar field model in the Einstein frame.

Another question is whether integrability in the case of scalar field models is maintained when one considers more general potentials. This is known not to be the case even for a general FRW model with a scalar field that has a simple quadratic potential [1]. One would like to be able to relate the integrability properties of different cosmological spacetimes in the context of different gravity theories and matter fields in an effort to understand the significance of exceptional non-integrable cases as opposed to generic, integrable ones in the simple frame of isotropic models before one moves on to more difficult homogeneous but anisotropic case. Problems in this direction are currently under investigation [11].

Acknowledgement

PGLL thanks the Director of GEODYSYNC, Dr S Cotsakis, and the Department of Mathematics, University of the Aegean, for the use of their facilities during the time when this work was performed and acknowledges the continuing support of the National Research Foundation of South Africa and the University of Natal.

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