GENERALIZED HILBERT OPERATORS ON WEIGHTED BERGMAN SPACES

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Abstract. The main purpose of this paper is to study the generalized Hilbert operator

$$H_{g}(f)(z) = \int_{0}^{1} f(t)g'(tz) dt$$

acting on the weighted Bergman space $A^{p}_{\omega}$, where the weight function $\omega$ belongs to the class $\mathcal{R}$ of regular radial weights and satisfies the Muckenhoupt type condition

$$\sup_{0 \leq r < 1} \left( \int_{r}^{1} \left( \int_{t}^{1} \omega(s)ds \right)^{\frac{p'}{p}} dt \right)^{\frac{p}{p'}} \int_{0}^{r} (1-t)^{-p} \left( \int_{t}^{1} \omega(s)ds \right) dt < \infty.$$

If $q = p$, the condition on $g$ that characterizes the boundedness (or the compactness) of $H_{g} : A^{p}_{\omega} \to A^{p}_{\omega}$ depends on $p$ only, but the situation is completely different in the case $q \neq p$ in which the inducing weight $\omega$ plays a crucial role. The results obtained also reveal a natural connection to the Muckenhoupt type condition (†). Indeed, it is shown that the classical Hilbert operator (the case $g(z) = \log \frac{1}{1-z}$ of $H_{g}$) is bounded from $L^{p}_{\omega}((0,1))$ (the natural restriction of $A^{p}_{\omega}$ to functions defined on $(0,1)$) to $A^{p}_{\omega}$ if and only if $\omega$ satisfies the condition (†). On the way to these results decomposition norms for the weighted Bergman space $A^{p}_{\omega}$ are established.

1. Introduction

Let $\mathcal{H}(D)$ denote the space of all analytic functions in the unit disc $D$ of the complex plane $\mathbb{C}$. A function $\omega : D \to (0, \infty)$, integrable over $D$, is called a weight function or simply a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in D$. For $0 < p < \infty$ and a weight $\omega$, the weighted Bergman space $A^{p}_{\omega}$ consists of those $f \in \mathcal{H}(D)$ for which

$$\|f\|_{A^{p}_{\omega}} = \int_{D} |f(z)|^{p} \omega(z) dA(z) < \infty,$$

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where \( dA(z) = \frac{dx
dy}{\pi} \) is the normalized Lebesgue area measure on \( \mathbb{D} \). As usual, we write \( A_p^\alpha \) for the classical weighted Bergman space induced by the standard radial weight \( \omega(z) = (1 - |z|^2)^\alpha \) with \(-1 < \alpha < \infty\).

Every \( g \in \mathcal{H}(\mathbb{D}) \) induces the \textit{generalized Hilbert operator} \( \mathcal{H}_g \), defined by

\[
\mathcal{H}_g(f)(z) = \int_0^1 f(t)g(tz) \, dt, \quad f \in \mathcal{H}(\mathbb{D}).
\]

The sharp condition

\[
\int_0^1 \left( \int_0^1 \omega(s)ds \right)^{-\frac{1}{p'-1}} dt < \infty, \quad p > 1,
\]

ensures that the integral in (1) defines an analytic function on \( \mathbb{D} \) for each \( f \in A_p^\omega \).

The choice \( g(z) = \log \frac{1}{1 - z} \) in (1) gives an integral representation of the \textit{classical Hilbert operator} \( \mathcal{H} \). The Hilbert operator \( \mathcal{H} \) is a prototype of a Hankel operator which has attracted a considerable amount of attention during the last years in operator theory on spaces of analytic functions. Questions related to the boundedness, the operator norm and the spectrum of \( \mathcal{H} \) have been studied in [1, 4, 5, 6]. These studies reveal a natural connection from \( \mathcal{H} \) to the weighted composition operators, the Szegö projection and the Legendre functions of the first kind. For further information on \( \mathcal{H} \), the reader is invited to see the recent monograph [7, Chapter 13].

The primary purpose of this paper is to study the operator \( \mathcal{H}_g \) acting on the weighted Bergman space \( A_p^\omega \) induced by a radial weight \( \omega \). We are particularly interested in basic properties such as the question of when \( \mathcal{H}_g : A_p^\omega \to A_q^\omega \), \( 1 < p, q < \infty \), is bounded or compact.

As far as we know, the generalized Hilbert operator \( \mathcal{H}_g \) has not been extensively studied in the existing literature. The operator was introduced recently in [9], where it was shown, among other things, that the membership of the analytic symbol \( g \) to the mean Lipschitz space \( \Lambda \left( p, \frac{1}{p'} \right) \) characterizes the boundedness of \( \mathcal{H}_g \) on the Bergman space \( A_p^\omega \) \((-1 < p - 2 < \alpha < \infty)\), on the Hardy space \( H^p \) \((1 < p \leq 2)\) and also on certain Dirichlet type spaces. The proofs of these results are based on the identity \( (\mathcal{H}_g)'(f) = \mathcal{H}_g'(zf) \) together with properties of the maximum modulus and the smoothness of the sequence of moments \( \left\{ \int_0^1 t^k |f(t)| \, dt \right\}_{k=0}^\infty \) of functions in these spaces.

The approach we take to the study of \( \mathcal{H}_g \) allows us to determine those analytic symbols \( g \) for which \( \mathcal{H}_g : A_p^\omega \to A_q^\omega \), \( 1 < p, q < \infty \), is bounded or compact, provided the regular weight \( \omega \) (see Section 2 for the definition) satisfies the Muckenhoupt type condition

\[
\sup_{0 \leq r < 1} \left( \int_r^1 \left( \int_t^1 \omega(s)ds \right)^{-\frac{1}{p'}} dt \right)^{\frac{p'}{p}} \int_0^r (1 - t)^{-p} \left( \int_t^1 \omega(s)ds \right) dt < \infty.
\]

By the classical results of Muckenhoupt [15], the condition (3) characterizes those real functions \( v \in [0, 1) \rightarrow (0, \infty) \) for which the Hardy type operator \( \int_t^1 \frac{h(s)}{1-ts} \, ds \)
is bounded on $L^p$ \int_0^1 |v(s)| ds$. We mention that each standard radial weight $\omega(r) = (1 - r^2)^\alpha$, $-1 < p - 2 < \alpha < \infty$, is regular and satisfies (3). Our results show the interesting phenomenon that when the inducing powers of the domain and the target spaces are equal, i.e. $q = p$, then the weight function $\omega$ does not play any role in the condition on $g$ that characterizes the boundedness (or the compactness) of $Hg : A^p_\omega \to A^q_\omega$, although the description depends on $p$. However, the situation is completely different in the case $q \neq p$ in which the inducing weight $\omega$ plays a crucial role.

2. Preliminaries and main results

For $0 < p \leq \infty$, the Hardy space $H^p$ consists of those $f \in H(D)$ for which

$$\|f\|_{H^p} = \lim_{r \to 1^-} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

Throughout the paper, the letter $C = C(\cdot)$ will denote a constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

The distortion function of a radial weight $\omega : [0, 1) \to (0, \infty)$ is defined by

$$\psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(s) \, ds, \quad 0 \leq r < 1.$$

It was introduced in [22] on the way to the Littlewood-Paley formulas for weighted Bergman spaces. A radial weight $\omega$ is called regular, if it is continuous and its distortion function satisfies

$$\psi_\omega(r) \asymp (1 - r), \quad 0 \leq r < 1.$$  (4)

The class of all regular weights is denoted by $\mathcal{R}$. For basic properties and concrete examples of regular weights, see [20, Chapter 1] and [22], and references therein. At this point we settle to mention that each standard weight $\omega(r) = (1 - r^2)^\alpha$ with $-1 < \alpha < \infty$ is regular. From now on we will use the notation $\tilde{\omega}(r) = \int_r^1 \omega(s) \, ds$ so that (4) ensures $\tilde{\omega}(r) \asymp \omega(r)(1 - r)$ for $\omega \in \mathcal{R}$. Moreover, for each radial weight $\omega$ we will write $\omega_\gamma(r) = (1 - r^2)^\gamma \omega(r)$, $-\infty < \gamma < \infty$, and $\omega \in \mathcal{M}_p$ if $\omega$ satisfies the Muckenhoupt type condition (3). We will write $\|T\|_{(X,Y)}$ for the norm of an operator $T : X \to Y$, and if no confusion arises with regards to $X$ and $Y$, we will simply write $\|T\|$.
Our study of $\mathcal{H}_g$ on weighted Bergman spaces leads us to consider other classes of weighted spaces. For $0 < p \leq \infty$, $0 < q < \infty$, $0 \leq \gamma < \infty$ and a radial weight $\omega$, the mixed norm space $H(p,q,\omega^{\gamma})$ consists of those $g \in \mathcal{H}(\mathbb{D})$ such that

$$
\|g\|_{H(p,q,\omega^{\gamma})}^q = \int_0^1 M_p^q(r,g)(1-r)^{\gamma}\omega(r)\,dr < \infty.
$$

Moreover, if in addition $-\infty < \beta < \infty$, we will denote $g \in H(p,\infty, (\omega^{\beta})_\gamma)$, whenever

$$
\|g\|_{H(p,\infty,(\omega^{\beta})_\gamma)}^q = \sup_{0 < r < 1} M_p(r,g)(1-r)^{\gamma}\omega(r)^{\beta} < \infty.
$$

We will simply write $H(p,q,\omega)$ and $H(p,\infty, \omega^{\beta})$ if $\gamma = 0$. It is clear that $H(p,p,\omega) = A^p_\omega$. The mixed norm spaces play an essential role in the closely related question of studying the coefficient multipliers on Hardy and weighted Bergman spaces [7].

For given $1 \leq p < \infty$, $0 < \alpha \leq 1$ and $0 \leq \beta < \infty$, we say that $g \in \mathcal{H}(\mathbb{D})$ belongs to $\Lambda(p,\alpha,\omega^{\beta})$, if $g' \in H(p,\infty, (\omega^{\beta})_{1-\alpha})$, that is,

$$
\|g\|_{\Lambda(p,\alpha,\omega^{\beta})} = \sup_{0 < r < 1} \frac{M_p(r,g')(1-r)^{1-\alpha}}{\omega(r)^{\beta}} + |g(0)| < \infty.
$$

Since $0 < \alpha \leq 1$ and $0 \leq \beta < \infty$, we have $\Lambda(p,\alpha,\omega^{\beta}) \subset H^p$, and therefore each function $g \in \Lambda(p,\alpha,\omega^{\beta})$ has a non-tangential limit $g(e^{i\theta})$ almost everywhere on the unit circle $\mathbb{T}$. Indeed, if $\beta = 0$, then $\Lambda(p,\alpha,\omega^{\beta})$ is nothing else but the mean Lipschitz space $\Lambda(p,\alpha)$ that consists of those $g \in \mathcal{H}(\mathbb{D})$ having non-tangential limits $g(e^{i\theta})$ almost everywhere and for which

$$
\sup_{0 < \theta \leq \frac{2\pi}{2\pi}} \left( \int_0^{\frac{2\pi}{2\pi}} |g(e^{i(\theta+\theta)}) - g(e^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} = O(t^{\alpha}), \quad t \to 0,
$$

see a classical result of Hardy and Littlewood [8, Theorem 5.4].

We will see that if $1 < p < \infty$ and $\omega \in \mathcal{R} \cap \mathcal{M}_p$, then $\mathcal{H}_g : A^p_\omega \to A^p_\omega$ is bounded if and only if $g \in \Lambda(p,\frac{1}{p})$. The spaces $\Lambda(p,\frac{1}{p})$ form a nested scale contained in BMOA [3]:

$$
\Lambda \left( q,\frac{1}{q} \right) \subset \Lambda \left( p,\frac{1}{p} \right) \subset \text{BMOA}, \quad 1 \leq q < p < \infty.
$$

The absence of $\omega$ in the condition on $g$ that characterizes the boundedness does not come as a surprise in view of [9, Theorem 3]. However, the situation is completely different in the case $q \neq p$ in which the inducing weight $\omega$ plays a crucial role. In particular, if $q > p$, then the space $\Lambda \left( q,\frac{1}{p},\omega^{\frac{p}{q}-1} \right)$, that can be described also by a growth condition on the modulus of continuity of order $q$ of $g(e^{i\theta})$ by Proposition 24 below, comes naturally to the picture.

Our main result on $\mathcal{H}_g$ reads as follows.

**Theorem 1.** Let $1 < p, q < \infty$, $\omega \in \mathcal{R} \cap \mathcal{M}_p$ and $g \in \mathcal{H}(\mathbb{D})$. 
\(\text{(i) If } 1 < p \leq q < \infty, \text{ then } \mathcal{H}_g : A^p_\omega \to A^q_\omega \text{ is bounded if and only if } g \in \Lambda \left( q, \frac{1}{p}, \frac{1}{q} - \frac{1}{q} \right). \text{ Moreover, if } g \in \Lambda \left( q, \frac{1}{p}, \frac{1}{q} - \frac{1}{q} \right), \text{ then} \\
\|\mathcal{H}_g\|_{(A^p_\omega, A^q_\omega)} \asymp \|g - g(0)\|_{\Lambda \left( q, \frac{1}{p}, \frac{1}{q} - \frac{1}{q} \right)}.\)

\(\text{(ii) If } 1 < q < p < \infty, \text{ then } \mathcal{H}_g : A^p_\omega \to A^q_\omega \text{ is bounded if and only if } g' \in \tilde{H} \left( q, s, \omega_s \left( 1 - \frac{q}{r} \right) \right), \text{ where } \frac{1}{q} - \frac{1}{p} = \frac{1}{s}. \text{ Moreover, if } g' \in \tilde{H} \left( q, s, \omega_s \left( 1 - \frac{q}{r} \right) \right), \text{ then} \\
\|\mathcal{H}_g\|_{(A^p_\omega, A^q_\omega)} \asymp \|g'\|_{\tilde{H} \left( q, s, \omega_s \left( 1 - \frac{q}{r} \right) \right)}.\)

It is easy to see, by using the auxiliary result on \(\mathcal{M}_p\), stated as Lemma 7 below, that the space \(\Lambda \left( q, \frac{1}{p}, \frac{1}{q} - \frac{1}{q} \right)\) is not trivial if \(\omega \in \mathcal{R} \cap \mathcal{M}_p\) no matter how large \(q\) is.

Our approach to the study of the boundedness of \(\mathcal{H}_g\) on weighted Bergman spaces arises the Muckenhoupt type condition (3) in a natural way. In order to explain this phenomenon better, we recall that the sublinear Hilbert operator is defined by

\[\tilde{H}(f)(z) = \int_0^1 \frac{|f(t)|}{1 - tz} dt.\]

We shall see that it behaves like a kind of maximal function for all generalized Hilbert operators under the assumptions of Theorem 1. Indeed, we will show that

\[\|\mathcal{H}_g(f)\|_{A^q_\omega} \lesssim \|f\|_{A^p_\omega} + \|f\|^s_1 \|\tilde{H}(f)\|^s_2, \quad s_1 + s_2 = 1.\]

This together with the sharp inequality

\[\int_0^1 M^p_\omega(r, f) \omega_r(r) \, dr \leq \frac{\pi}{2} \|f\|^p_{A^p_\omega},\]

which can be easily obtained by integrating the known inequality \(\int_0^s M^p_\omega(r, f) \, dr \leq \pi s M^p_\omega(s, f) [21]\), lead us to consider the following result of interest of its own.

**Theorem 2.** Let \(1 < p < \infty\) and \(\omega \in \mathcal{R}\) such that (2) is satisfied. Then the following assertions are equivalent:

(i) \(\mathcal{H} : L^p_\omega \to A^p_\omega\) is bounded;

(ii) \(\tilde{H} : L^p_\omega \to A^p_\omega\) is bounded;

(iii) \(\omega\) satisfies the Muckenhoupt type condition

\[\mathcal{M}_p(\omega) = \sup_{0 < r < 1} \left( \int_r^1 \omega(t)^{1 - \frac{1}{p}} \, dt \right)^{\frac{1}{1 - \frac{1}{p}}} \left( \int_0^r (1 - t)^{-p} \omega(t) \, dt \right)^{\frac{1}{p}} < \infty.\]

Moreover, if \(\omega \in \mathcal{M}_p\), then

\[\|\mathcal{H}\|_{(L^p_\omega, A^p_\omega)} \asymp \|\tilde{H}\|_{(L^p_\omega, A^p_\omega)} \asymp \mathcal{M}_p(\omega).\]

Theorem 2 together with (6) extends [5, Theorem 1] and [9, Theorem 5 (ii)].
Corollary 3. Let $1 < p < \infty$ and $\omega \in \mathcal{R} \cap \mathcal{M}_p$. Then, both the Hilbert operator $\mathcal{H}$ and the sublinear Hilbert operator $\tilde{\mathcal{H}}$ are bounded on $A^p_\omega$.

We also work partially with radial weights $\omega$ for which the quotient $\frac{\psi_\omega(r)}{1-r}$ is not bounded. More precisely, we say that a radial weight $\omega$ is rapidly increasing, if it is continuous and

$$\lim_{r \to 1^-} \frac{\psi_\omega(r)}{1-r} = \infty.$$ 

The class of rapidly increasing weights is denoted by $\mathcal{I}$. It is easy to see that $A^p_\omega \subset A^p_\beta$ for each $\omega \in \mathcal{I}$ and for any $\beta > -1$, see [20, Section 1.4]. Typical examples of rapidly increasing weights are

$$\omega(r) = \left( (1-r) \prod_{n=1}^{N} \log_n \frac{\exp_n 0}{1-r} \left( \frac{\exp_{N+1} 0}{1-r} \right)^{\alpha} \right)^{-1}$$

for all $1 < \alpha < \infty$ and $N \in \mathbb{N} = \{1, 2, \ldots\}$. Here, as usual, $\log_n x = \log(\log_{n-1} x)$, $\log_1 x = \log x$, $\exp_n x = \exp(\exp_{n-1} x)$ and $\exp_1 x = e^x$.

The right choice of the norm used is in many cases a key tool for a good understanding of how a concrete operator acts in a given space. Here, an $l^p$-type norm of the Hardy norms of blocks of the Maclaurin series, whose size depend on the weight $\omega$, provides us an effective skill to study the boundedness and compactness of $\mathcal{H}_p$ on weighted Bergman space $A^p_\omega$. The size of these blocks reflects the growth of the inducing weight $\omega$. We remind the reader that decomposition results have been an important tool for the study of a good number of questions on spaces of analytic function on $D$. They have been applied, for example, when studying coefficient multipliers [7], Carleson measures [10] and the generalized Hilbert operator [9]. The results proved by M. Mateljević and M. Pavlović in [14] (see also [18]) offer such a decomposition result on $A^p_\omega$ when $\omega \in \mathcal{R}$, see also [16, 17] for further results. This because a calculation based on [20, Lemma 1.1] says that [14, Theorem 2.1 (b)] works for $\omega \in \mathcal{R}_\omega$. However, to the best of our knowledge, results in the existing literature do not cover the less understood case of the class $\mathcal{I}$ of rapidly increasing weights. Indeed, only some special cases have been considered in [10, Theorem 6.1].

We will develop a technique that allows us to give a unified treatment for both classes $\mathcal{R}$ and $\mathcal{I}$. Theorem 4 below is our main result in that direction. To give the precise statement, we need to introduce some notation. To do this, let $\omega \in \mathcal{I} \cup \mathcal{R}$ such that $\int_0^1 \omega(r) \, dr = 1$. For each $\alpha > 0$ and $n \in \mathbb{N} \cup \{0\}$, let $r_n = r_n(\omega, \alpha) \in [0, 1)$ be defined by

$$\hat{\omega}(r_n) = \int_{r_n}^{1} \omega(r) \, dr = \frac{1}{2n\alpha}.$$ 

Clearly, $\{r_n\}_{n=0}^{\infty}$ is an increasing sequence of distinct points on $[0,1)$ such that $r_0 = 0$ and $r_n \to 1^-$, as $n \to \infty$. For $x \in [0, \infty)$, let $E(x)$ denote the integer such
that $E(x) \leq x < E(x) + 1$, and set $M_n = E\left(\frac{1}{1-r_n}\right)$ for short. Write

$$I(0) = I_{\omega, \alpha}(0) = \{k \in \mathbb{N} \cup \{0\} : k < M_1\}$$

and

$$I(n) = I_{\omega, \alpha}(n) = \{k \in \mathbb{N} : M_n \leq k < M_{n+1}\}$$

for all $n \in \mathbb{N}$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $D$, define the polynomials $\Delta_{n, \alpha} f$ by

$$\Delta_{n, \alpha} f(z) = \sum_{k \in I_{\omega, \alpha}(n)} a_k z^k, \quad n \in \mathbb{N} \cup \{0\}.$$

If $\alpha = 1$, we will simply write $\Delta_n$ instead of $\Delta_{n, 1}$.

**Theorem 4.** Let $1 < p < \infty$, $0 < \alpha < \infty$ and $\omega \in I \cup R$ such that $\int_1^0 \omega(r) \, dr = 1$, and let $f \in H(D)$.

(i) If $0 < q < \infty$, then $f \in H(p, q, \omega)$ if and only if

$$\sum_{n=0}^{\infty} 2^{-n\alpha} \|\Delta_{n, \alpha} f\|_{H^p}^q < \infty.$$

Moreover,

$$\|f\|_{H(p, q, \omega)} \asymp \left(\sum_{n=0}^{\infty} 2^{-n\alpha} \|\Delta_{n, \alpha} f\|_{H^p}^q\right)^{1/q}.$$

(ii) If $0 < \beta < \infty$, then $f \in H(p, \infty, \hat{\omega}^\beta)$ if and only if

$$\sup_n 2^{-n\alpha\beta} \|\Delta_{n, \alpha} f\|_{H^p} < \infty.$$

Moreover,

$$\|f\|_{H(p, \infty, \hat{\omega}^\beta)} \asymp \sup_n 2^{-n\alpha\beta} \|\Delta_{n, \alpha} f\|_{H^p}.$$

The method of proof that we use to establish Theorem 4 can be employed to characterize certain functions in $A_{\omega}^p$ in terms of the coefficients in their Maclaurin series. In fact, we will see that, whenever $\omega \in R$, a standard lacunary series $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ with $\frac{n_{k+1}}{n_k} \geq c > 1$, belongs to $A_{\omega}^p$ if and only if

$$\sum_{k=0}^{\infty} |a_k|^q \int_0^1 r^{2n_k+1} \omega(r) \, dr < \infty.$$

The same is not true in general if $\omega$ is rapidly increasing. However, the assertion is valid for $\omega \in I$ if the Maclaurin series expansion of $f$ has sufficiently large gaps depending on $\omega$. To give the precise statement, let $\omega$ be a radial weight. We say that $f \in H(D)$ is an $\omega$-lacunary series in $D$ if its Maclaurin series $\sum_{k=0}^{\infty} a_k z^{n_k}$ satisfies

$$\frac{\hat{\omega}\left(1 - \frac{1}{n_k}\right)}{\hat{\omega}\left(1 - \frac{1}{n_{k+1}}\right)} = \frac{\int_1^{1 - \frac{1}{n_k}} \omega(r) \, dr}{\int_1^{1 - \frac{1}{n_{k+1}}} \omega(r) \, dr} \geq \lambda > 1, \quad k \in \mathbb{N} \cup \{0\}.$$
This is a natural generalization of the classical concept of power series with Hadamard gaps, in the sense that, for \( \omega \in \mathbb{R} \), the class of \( \omega \)-lacunary series is nothing else but the set of Hadamard gap series.

**Theorem 5.** Let \( 0 < q < \infty, 0 < p \leq \infty \) and \( \omega \in I \cup \mathbb{R} \) such that \( \int_1^1 \omega(r) \, dr = 1 \), and let \( f \) be an \( \omega \)-lacunary series in \( \mathbb{D} \). Then the following conditions are equivalent:

(i) \( f \in H(p, q, \omega) \);

(ii) \( \sum_{k=0}^{\infty} |a_k|^q \int_0^1 r^{2n_k+1} \omega(r) \, dr < \infty \).

Moreover,

\[
\|f\|_{H(p, q, \omega)}^q \leq \sum_{k=0}^{\infty} |a_k|^q \int_0^1 r^{2n_k+1} \omega(r) \, dr.
\]

The remaining part of the paper is organized as follows. In Section 3 we state and prove some preliminary results on weights and technical results on series with positive coefficients, and prove Theorems 4 and 5. Theorem 2 will be proved in Section 4. In Section 5 we will deal with technical background on Hadamard products which will be used in the proof of Theorem 1, that is given in Section 6. Section 7 is devoted to proving the expected results on the compactness of \( H_g : A^p_\omega \to A^q_\omega \). Finally, in Section 8 we will offer natural alternative descriptions of the spaces appearing in the statement of Theorem 1 and analyze the Muckenhoupt type condition (3) in detail. In particular, we will see that (3) is closely related to the value of \( \lim_{r \to 1^-} \psi_\omega(r)/(1-r) \), if it exists.

### 3. Decomposition theorems

This section is instrumental for the rest of the paper. Here we will discuss basic properties of the radial weights considered and \( L^p_\omega \)-behavior of power series with positive coefficients, and then prove Theorem 4 and other related decomposition theorems. We will also prove Theorem 5 and further discuss the \( \omega \)-lacunary series.

#### 3.1. Preliminaries on weights.

We begin with collecting some necessary definitions and results on weights in \( I \cup \mathbb{R} \). The **Carleson square** \( S(I) \) associated with an interval \( I \subset \mathbb{T} \) is the set \( S(I) = \{ re^{it} \in \mathbb{D} : e^{it} \in I, 1 - |I| \leq r < 1 \} \), where \( |E| \) denotes the Lebesgue measure of the set \( E \subset \mathbb{T} \). For \( 1 < p < \infty \), the letter \( p' \) will denote its conjugate, that is, the number for which \( \frac{1}{p} + \frac{1}{p'} = 1 \).

Let \( 1 < p_0 < \infty \) and \( \eta > -1 \). A weight \( u \) (not necessarily radial) satisfies the **Bekollé-Bonami** \( B_{p_0}(\eta) \)-condition, denoted by \( u \in B_{p_0}(\eta) \), if there exists a constant \( C = C(p_0, \eta, \omega) > 0 \) such that

\[
\left( \int_{S(I)} u(z) (1 - |z|)^{\eta} \, dA(z) \right)^{\frac{1}{p_0}} \leq C |I|^{(2+\eta)p_0}
\]

for every interval \( I \subset \mathbb{T} \). For the proof of the next result, see [20, Lemmas 1.2-1.4].
Lemma A. \hspace{2em} (i) Let \( \omega \in \mathcal{R} \). Then there exist constants \( \alpha = \alpha(\omega) > 0 \) and
\[ \beta = \beta(\omega) \geq \alpha \] such that
\[ \left( \frac{1-r}{1-t} \right)^{\alpha} \tilde{\omega}(t) \leq \tilde{\omega}(r) \leq \left( \frac{1-r}{1-t} \right)^{\beta} \tilde{\omega}(t), \quad 0 \leq r \leq t < 1. \]

(ii) Let \( \omega \in \mathcal{I} \). Then for each \( \beta > 0 \) there exists a constant \( C = C(\beta, \omega) > 0 \) such that
\[ \tilde{\omega}(r) \leq C \left( \frac{1-r}{1-t} \right)^{\beta} \tilde{\omega}(t), \quad 0 \leq r \leq t < 1. \]

(iii) For each radial weight \( \omega \) and \( 0 < \alpha < 1 \), \( \tilde{\omega}(r) = \tilde{\omega}(r) - \omega(r) \) is also a weight and \( \psi(\omega)(r) = \frac{1-\alpha}{\alpha} \psi(r) \) for all \( 0 < r < 1 \).

(iv) If \( \omega \in \mathcal{I} \cap \mathcal{R} \), then
\[ \int_{0}^{1} s^\omega \omega(s) ds \lesssim \tilde{\omega} \left( 1 - \frac{1}{x} \right), \quad x \in [1, \infty). \]

(v) If \( \omega \in \mathcal{R} \), then for each \( p_0 > 1 \) there exists \( \eta_0 = \eta(p_0, \omega) > -1 \) such that for all \( \eta \geq \eta_0 \), \( \frac{\omega(z)}{(1-|z|)^\eta} \) belongs to \( B_{p_0}(\eta) \).

The next lemma is a restatement of [20, Lemma 2.3].

Lemma B. \hspace{2em} (i) If \( \omega \in \mathcal{R} \), then there exists \( \gamma_0 = \gamma_0(\omega) \) such that
\[ \int_{\mathbb{D}} \frac{\omega(z)}{|1-az|^{\gamma+1}} dA(z) \lesssim \frac{\omega(S(a))}{(1-|a|)^{\gamma+1}}, \quad a \in \mathbb{D}, \]
for all \( \gamma > \gamma_0 \).

(ii) If \( \omega \in \mathcal{I} \), then
\[ \int_{\mathbb{D}} \frac{\omega(z)}{|1-az|^{\gamma+1}} dA(z) \lesssim \frac{\omega(S(a))}{(1-|a|)^{\gamma+1}}, \quad a \in \mathbb{D}, \]
for all \( \gamma > 0 \).

Lemma 6. Let \( \omega \in \mathcal{R} \) such that \( \int_{0}^{1} \omega(r) dr = 1 \), and let \( \{r_n\}_{n=0}^{\infty} \) be the sequence defined by (8) with \( \alpha = 1 \). Then there exist constants \( \gamma_2 = \gamma_2(\omega) > \gamma_1 = \gamma_1(\omega) > 0 \) such that
\[ 2^{\gamma_1} M_n \leq M_{n+1} \leq 2^{\gamma_2} M_n, \quad r_n \geq \max \left\{ \frac{1}{2^{\gamma_1}}, \frac{1}{2^{\gamma_2}} \right\}. \]

Proof. Using Lemma A(i) and (8), we obtain
\[ \frac{M_{n+1}}{M_n} \geq \frac{r_{n+1}(1-r_n)}{1-r_{n+1}} \geq 2^{-\gamma_1} \left( \frac{\tilde{\omega}(r_n)}{\tilde{\omega}(r_{n+1})} \right)^{1/\beta} 2^{\frac{\beta}{2} - \gamma_1}, \]
where \( \beta = \beta(\omega) \) is from Lemma A(i). The left hand inequality of the assertion follows by choosing \( \gamma_1 = \frac{1}{2^{\gamma_0}} \). The right hand inequality can be proved in an analogous manner.

Several useful reformulations of the Muckenhoupt type condition (7) are gathered to the following lemma.
Lemma 7. Let $1 < p < \infty$ and let $\omega$ be a radial weight, and denote $u_p(r) = (\tilde{\omega}(r)(1-r))^{-\frac{1}{p}}$. Then the following conditions are equivalent:

(i) $\omega \in \mathcal{M}_p$;
(ii) $\tilde{\omega}^{-\frac{1}{p-1}} \in \mathcal{R}$;
(iii) $u_p \in \mathcal{R}$;
(iv) $\frac{(1-r)^p}{\omega(r)} \int_0^r \frac{\tilde{\omega}(t)}{(1-t)^p} \, dt \asymp 1-r, \quad 0 \leq r < 1.$

Proof. (i)$\iff$(ii). Observe first that

\[
\left( \int_1^r \tilde{\omega}(t)^{-\frac{1}{p-1}} \, dt \right)^{p-1} \int_0^r \frac{\tilde{\omega}(t)}{(1-t)^p} \, dt = \left( \tilde{\omega}(r) \int_1^r \tilde{\omega}(t)^{\frac{1}{p-1}} \, dt \right)^{p-1} \cdot \frac{(1-r)^p}{\omega(r)} \int_0^r \frac{\tilde{\omega}(t)}{(1-t)^p} \, dt \geq 1^{p-1} \cdot \left( \frac{1-(1-r)^{p-1}}{p-1} \right), \quad 0 \leq r < 1,
\]

and hence $\omega \in \mathcal{M}_p$ if and only if (ii) and (iv) are satisfied. Therefore, to see that (i) and (ii) are equivalent, it suffices to show that (ii) implies (iv). To prove this, note that

\[
(9) \quad \tilde{\omega}(r) \asymp \left( \int_1^r \tilde{\omega}(t)^{-\frac{1}{p-1}} \, dt \right)^{p-1}, \quad 0 \leq r < 1,
\]

whenever $\tilde{\omega}^{-\frac{1}{p-1}} \in \mathcal{R}$. Therefore, under the assumption (ii), the condition (iv) is equivalent to

\[
(10) \quad \left( \int_1^r \tilde{\omega}(s)^{-\frac{1}{p-1}} \, ds \right)^{p-1} \int_0^r \frac{dt}{(1-t)^{\frac{1}{p-1}}} \left( \int_t^1 \tilde{\omega}(s)^{-\frac{1}{p-1}} \, ds \right)^{p-1} \asymp 1.
\]

But since $\tilde{\omega}^{-\frac{1}{p-1}} \in \mathcal{R}$, Lemma A(i) shows that there exist $\alpha = \alpha(p, \omega) > 0$ and $\beta = \beta(p, \omega) > 0$ such that

\[
(11) \quad \left( \frac{1-r}{1-t} \right)^{\beta} \leq \left( \int_t^1 \tilde{\omega}(s)^{-\frac{1}{p-1}} \, ds \right) \leq \left( \frac{1-r}{1-t} \right)^{\alpha}, \quad 0 \leq t \leq r < 1.
\]

Hence the left-hand side of (10) is dominated by

\[
(1-r)^{\alpha(p-1)} \int_0^r \frac{dt}{(1-t)^{1+\alpha(p-1)}} \lesssim 1,
\]

and (i)$\iff$(ii) follows. Note that the beginning of this part of the proof also establishes the implication (i)$\Rightarrow$(iv).
(ii)⇔(iii). If \( \hat{\omega}^{-\frac{1}{p-1}} \in \mathcal{R} \), then (9) and (11) yield

\[
\frac{1}{u_p(r)} \int_{r}^{1} u_p(t) \, dt \propto (1-r) \int_{r}^{1} \left( \int_{r}^{1} \hat{\omega}(s)^{-\frac{1}{p-1}} \, ds \right)^{\frac{p-1}{p}} \, dt \propto 1-r,
\]

that is, \( u_p \in \mathcal{R} \). The opposite implication (iii)⇒(ii) can be proved in a similar manner.

(iv)⇒(iii). A calculation based on the assumption (iv) shows that \( F(r) = (1-r) \frac{1}{p} \int_{0}^{r} \hat{\omega}(t) \, dt \) is increasing on \([0, 1]\) for \( K > 0 \) large enough. By using this and (iv) we deduce

\[
1-r \leq \frac{\int_{0}^{1} u_p(s) \, ds}{u_p(r)} \propto (1-r) \int_{r}^{1} \left( \int_{0}^{r} \hat{\omega}(t) (1-t)^{\frac{1}{p}} \, dt \right)^{\frac{1}{p}} \, ds \propto (1-r),
\]

and thus \( u_p \in \mathcal{R} \).

Since (i)⇒(iv) by the first part of the proof, the lemma is now proved. \( \square \)

3.2. \( L_{\omega}^p \) behavior of power series with positive coefficients. We begin with a technical but useful result. Recall that a function \( h \) is called essentially decreasing if there exists a positive constant \( C = C(h) \) such that \( h(x) \leq Ch(y) \) whenever \( y \leq x \). Essentially increasing functions are defined in an analogous manner.

**Lemma 8.** Let \( \omega \in \mathcal{I} \cup \mathcal{R} \) such that \( \int_{0}^{1} \omega(r) \, dr = 1 \). For each \( \alpha > 0 \) and \( n \in \mathbb{N} \cup \{0\} \), let \( r_n = r_n(\omega, \alpha) \in [0, 1) \) be defined by (8). Then the following assertions hold:

(i) For each \( \gamma > 0 \), there exists \( C = C(\alpha, \gamma, \omega) > 0 \) such that

\[
\eta_\gamma(r) = \sum_{n=0}^{\infty} 2^{n\gamma} r^n M_n \leq C \hat{\omega}(r)^{-\frac{\gamma}{\pi}}, \quad 0 \leq r < 1.
\]

(ii) For each \( 0 < \beta < 1 \), there exists \( C = C(\alpha, \beta, \omega) > 0 \) such that

\[
2^{-n\alpha \beta} \int_{0}^{1} r^n M_n \omega(r) \, dr \leq C \int_{0}^{1} r^n M_n \omega(r) \, dr.
\]

(iii) If \( \alpha = 1 \) in (8), \( 1 < p < \infty \), \( p\eta < 1 \) and \( \omega \in \mathcal{R} \cap \mathcal{M}_p \), then there exists \( C = C(\eta, p, \omega) > 0 \) such that

\[
\sum_{n=0}^{\infty} M_n^{p-1} 2^{-n\gamma} r^n M_n \leq C \frac{\hat{\omega}(r)^{\eta}}{(1-r)^{1-p}}, \quad 0 \leq r < 1.
\]

**Proof.** (i). We will begin with proving (12) for \( r = r_N \), where \( N \in \mathbb{N} \). To do this, note first that

\[
\sum_{n=0}^{N} 2^{n\gamma} r_N^n M_n \leq \frac{2^\gamma}{2^\gamma-1} \hat{\omega}(r_N)^{-\frac{\gamma}{\pi}}
\]
by (8). To deal with the remainder of the sum, we apply Lemma A(i)(ii) and (8) to find $\beta = \beta(\omega) > 0$ and $C = C(\beta, \omega) > 0$ such that

$$
\frac{1 - r_n}{1 - r_{n+j}} \geq C \left( \frac{\tilde{\omega}(r_n)}{\tilde{\omega}(r_{n+j})} \right)^{1/\beta} = C 2^{jn}, \quad n, j \in \mathbb{N} \cup \{0\}.
$$

This, the inequality $\log \frac{1}{x} \geq 1 - x$, $0 < x \leq 1$, and (8) give

$$
\sum_{n=N+1}^{\infty} 2^{n\gamma} r_N M_n \leq 2^{N\gamma} \sum_{j=1}^{\infty} 2^j e^{\frac{-r_{N+j}}{1-r_{N+j}}} \leq 2^{N\gamma} \sum_{j=1}^{\infty} 2^j e^{\frac{-r_{N+j}}{1-r_{N+j}}} = C(\beta, \alpha, \omega) \tilde{\omega}(r_N)^{-\frac{1}{\beta}}.
$$

Since $\beta = \beta(\omega)$, this together with (15) gives (12) for $r = r_N$, where $N \in \mathbb{N}$. Now, using standard arguments, it implies (12) for any $r \in (0, 1)$.

(ii). Clearly,

$$
2^{-na\beta} \int_0^{r_n} r^M \tilde{\omega}(r) \, dr \leq 2^{-na\beta} \int_0^{r_n} r^M \omega(r) \, dr \leq \int_0^{r_n} r^M \omega(r) \, dr.
$$

Moreover, Lemma A(iii) yields

$$
2^{-na\beta} \int_{r_n}^{1} r^M \tilde{\omega}(r) \, dr \leq 2^{-na\beta} \int_{r_n}^{1} r^M \omega(r) \, dr \leq \int_{r_n}^{1} r^M \omega(r) \, dr.
$$

By combining (16) and (17) we obtain (ii).

(iii). The proof is similar to that of (i). We will begin with proving (14) for $r = r_N$, where $N \in \mathbb{N}$. Since $\omega \in M_p$, Lemma 7 yields $\tilde{\omega}^{-\frac{1}{p-1}} \in \mathcal{R}$, that is,

$$
\left( \int_{r_n}^{1} \tilde{\omega}^{-\frac{1}{p-1}}(s) \, ds \right)^{\frac{p-1}{p}} \asymp (1 - r)^{\frac{p-1}{p}} \tilde{\omega}(r) = \tilde{\omega}(r_n) \psi(r_n) = \frac{1}{1 - \beta} \tilde{\omega}(r_n) \psi(r_n),
$$

so taking $r = r_n$ and bearing in mind Lemma 6 we deduce that the sequence

$$
\left\{ \frac{2^{\frac{n}{p}}}{M_n} \right\}
$$

is essentially decreasing. Therefore

$$
\sum_{n=0}^{N} M_n^{1 - \frac{n}{p}} 2^{-n\eta} r_N M_n \preceq M_0^{1 - \frac{n}{p}} 2^{-n\eta} M_n^{1 - \frac{n}{p}} 2^{-n\eta} \sum_{n=0}^{N} 2^{n(\frac{1}{p} - \eta)} \times \tilde{\omega}(r_N)^{\eta} \sim (1 - r_n)^{\frac{n}{p}} \tilde{\omega}(r_n) \frac{\eta}{\eta}.
$$

Moreover, bearing in mind Lemma 6, the inequality $\log \frac{1}{x} \geq 1 - x$, $0 < x \leq 1$, and the boundedness of the function $x^s e^{-tx}$, $s, t > 0$, on $[0, \infty)$, we obtain

$$
\sum_{n=N+1}^{\infty} M_n^{1 - \frac{n}{p}} 2^{-n\eta} r_N M_n \preceq 2^{-\eta} \sum_{j=0}^{\infty} M_j^{1 - \frac{n}{p}} 2^{-n\eta} e^{\frac{-C j \eta N^{j+1}}{M_N^{j+1}}} \preceq 2^{-\eta} M_N^{1 - \frac{n}{p}} \times \tilde{\omega}(r_N)^{\eta} \left( 1 - r_N \right)^{\frac{n}{p}},
$$
which together with (18) gives (iii) for \( r = r_N \). Finally, by using Lemma 6, (8) and the fact that \( (1 - r) \frac{ \omega(r) }{ \alpha } \) is essentially decreasing, we obtain (iii) for any \( r \in (0,1) \).

We now present a result on power series with positive coefficients. This result will play a crucial role in the proof of Theorem 4.

**Proposition 9.** Let \( 0 < p, \alpha < \infty \) and \( \omega \in I \cup R \) such that \( \int_0^1 \omega(r) \, dr = 1 \). Let \( f(r) = \sum_{k=0}^{\infty} a_k r^k \), where \( a_k \geq 0 \) for all \( k \in \mathbb{N} \cup \{0\} \), and denote \( t_n = \sum_{k \in I_\omega(n)} a_k \). Then there exists a constant \( C = C(p, \alpha, \omega) > 0 \) such that

\[
\frac{1}{C} \sum_{n=0}^{\infty} 2^{-n \alpha} t_n^p \leq \int_0^1 f(r)^p \omega(r) \, dr \leq C \sum_{n=0}^{\infty} 2^{-n \alpha} t_n^p.
\]

**Proof.** We will use ideas from the proof of [13, Theorem 6]. The definition (8) yields

\[
\int_0^1 f(r)^p \omega(r) \, dr \geq \sum_{n=0}^{\infty} \int_{r_{n+1}}^{r_{n+2}} \left( \sum_{k=0}^{\infty} t_k r_{n+1}^{M_k} \right)^p \omega(r) \, dr
\]

\[
\geq \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} t_k r_{n+1}^{M_k} \right)^p \int_{r_{n+1}}^{r_{n+2}} \omega(r) \, dr
\]

\[
\geq \left( 1 - \frac{1}{2^\alpha} \right) \sum_{n=0}^{\infty} t_n^p r_{n+1}^{pM_n+1} 2^{-(n+1)\alpha} \geq C \sum_{n=0}^{\infty} t_n^p 2^{-n \alpha},
\]

where \( C = C(p, \alpha, \omega) > 0 \) is a constant. This gives the first inequality in (19).

To prove the second inequality in (19), let first \( p > 1 \) and take \( 0 < \gamma < \frac{\alpha}{p-1} \). Then Hölder’s inequality gives

\[
f(r)^p \leq \left( \sum_{n=0}^{\infty} t_n r_{n}^{M_n} \right)^p \leq \eta_\gamma(r)^{p-1} \sum_{n=0}^{\infty} 2^{-n \alpha} t_n^p r_{n}^{M_n}.
\]

Therefore, by (12) and (13) in Lemma 8 and Lemma A(iv) there exist constants \( C_1 = C_1(\alpha, \gamma, p, \omega) > 0 \), \( C_2 = C_2(\alpha, \gamma, p, \omega) > 0 \) and \( C_3 = C_3(\alpha, \gamma, p, \omega) > 0 \) such that

\[
\int_0^1 f(r)^p \omega(r) \, dr \leq \sum_{n=0}^{\infty} 2^{-n \alpha} t_n^p \int_0^1 r_{n}^{M_n} \eta_\gamma(r)^{p-1} \omega(r) \, dr
\]

\[
\leq C_1 \sum_{n=0}^{\infty} 2^{-n \alpha} t_n^p \int_0^1 \frac{r_{n}^{M_n} \omega(r)}{\hat{\omega}(r)^{2(p-1)}} \, dr
\]

\[
\leq C_2 \sum_{n=0}^{\infty} t_n^p \int_0^1 r_{n}^{M_n} \omega(r) \, dr
\]

\[
\leq C_3 \sum_{n=0}^{\infty} t_n^p \hat{\omega}(r_n) \, dr = C_3 \sum_{n=0}^{\infty} t_n^p 2^{-n \alpha}.
\]
Since \( \gamma = \gamma(\alpha, p) \), this gives the assertion for \( 1 < p < \infty \). The proof of the case \( 0 < p \leq 1 \) is similar but easier. \( \square \)

### 3.3. Decomposition theorems.

In this section we will prove Theorem 4 and related results as well as discuss their consequences. For \( g(z) = \sum_{k=0}^{\infty} b_k z^k \in \mathcal{H}(\mathbb{D}) \) and \( n_1, n_2 \in \mathbb{N} \cup \{0\} \), we set

\[
S_{n_1, n_2} g(z) = \sum_{k=n_1}^{n_2-1} b_k z^k, \quad n_1 < n_2.
\]

We will use repeatedly the following auxiliary result.

**Lemma 10.** Let \( 0 < p \leq \infty \) and \( n_1, n_2 \in \mathbb{N} \) with \( n_1 < n_2 \). If \( g(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{H}(\mathbb{D}) \), then

\[
\|S_{n_1, n_2} g\|_{H^p} \asymp M_p \left( 1 - \frac{1}{n_2}, S_{n_1, n_2} g \right).
\]

Lemma 10 can be proved, for example, by using the inequality

\[
r^{n_2} \|S_{n_1, n_2} g\|_{H^p} \leq M_p(r, S_{n_1, n_2} g) \leq r^{n_1} \|S_{n_1, n_2} g\|_{H^p}, \quad 0 < r < 1,
\]

which follows by \([14, \text{Lemma 3.1}].\)

**Proof of Theorem 4.** (i). By the M. Riesz projection theorem and (21),

\[
\|f\|_{H^{(p,q,\omega)}} \geq \sum_{n=0}^{\infty} \|\Delta_n^{\omega,\alpha} f\|_{H^p} \int_{r_{n+1}}^{r_{n+2}} r^{qM_{n+1}} \omega(r) \, dr
\]

\[
\asymp \sum_{n=0}^{\infty} \|\Delta_n^{\omega,\alpha} f\|_{H^p} \int_{r_{n+1}}^{r_{n+2}} \omega(r) \, dr \asymp \sum_{n=0}^{\infty} 2^{-n\alpha} \|\Delta_n^{\omega,\alpha} f\|_{H^p}.
\]

On the other hand, Minkowski’s inequality and (21) give

\[
M_p(r, f) \leq \sum_{n=0}^{\infty} M_p(r, \Delta_n^{\omega,\alpha} f) \leq \sum_{n=0}^{\infty} r^{M_n} \|\Delta_n^{\omega,\alpha} f\|_{H^p},
\]

and hence Proposition 9 yields

\[
\|f\|_{H^{(p,q,\omega)}} \leq \int_0^1 \left( \sum_{n=0}^{\infty} r^{M_n} \|\Delta_n^{\omega,\alpha} f\|_{H^p} \right)^q \omega(r) \, dr \asymp \sum_{n=0}^{\infty} 2^{-n\alpha} \|\Delta_n^{\omega,\alpha} f\|_{H^p}^q.
\]

(ii). Using again the M. Riesz projection theorem and (21) we deduce

\[
\sup_{0 < r < 1} M_p(r, f) \hat{\omega}(r)^\beta \geq r^{M_{n+1}} \|\Delta_n^{\omega,\alpha} f\|_{H^p} 2^{-n\alpha\beta}, \quad n \in \mathbb{N} \cup \{0\},
\]

and hence

\[
\|f\|_{H^{(p,\infty,\hat{\omega}\beta)}} \geq \sup_n 2^{-n\alpha\beta} \|\Delta_n^{\omega,\alpha} f\|_{H^p}.
\]

Conversely, assume that \( M = \sup_n 2^{-n\alpha\beta} \|\Delta_n^{\omega,\alpha} f\|_{H^p} < \infty \). Then (22) and Lemma 8(i) yield

\[
M_p(r, f) \leq \sum_{n=0}^{\infty} r^{M_n} \|\Delta_n^{\omega,\alpha} f\|_{H^p} \leq M \sum_{n=0}^{\infty} 2^{n\alpha\beta} r^{M_n} \lesssim M \hat{\omega}(r)^{-\beta}.
\]
This finishes the proof.

Now we will present a couple of results which will be strongly used in the proof of Theorem 1. We saw in Theorem 4 that the $H(p, q, \omega)$-norm of $f \in H(\mathbb{D})$ can be written in terms of $H^p$-norms of the polynomials $\Delta n g$ if $\omega \in I \cup R$. The next result shows that the same polynomials work also in the case of $H(p, q, \omega)$ whenever $\gamma \geq 0$ and $\omega \in R$. In both, Corollary 11 and Corollary 12, $M_n = E \left( \frac{1}{1 - r_n} \right)$, where $r_n$ is defined by (8) with $\alpha = 1$.

**Corollary 11.** Let $1 < q < \infty$, $0 < p < \infty$, $0 \leq \gamma < \infty$, $\omega \in R$ such that $\int_0^1 \omega(r) \, dr = 1$ and $g \in H(\mathbb{D})$. Then

$$\|g\|_{H(\mathbb{D})}^p = \int_0^1 M_p(r, g)(1 - r)\omega(r) \, dr \leq \sum_{n=0}^{\infty} 2^{-n} \|\Delta n g\|_{H^q}^p.$$  

**Proof.** The inequality

$$\int_0^1 M_p(r, g)(1 - r)\omega(r) \, dr \geq \sum_{n=0}^{\infty} 2^{-n} \|\Delta n g\|_{H^q}^p$$

follows by the M. Riesz projection theorem, Lemma 10 and Lemma 6.

On the other hand, by Lemma A(iii) the weight $\tilde{\omega}_\beta(r) = \omega(r)/\hat{\omega}(r)$ is regular for each $\beta \in (0, 1)$ and then $(1 - r)\tilde{\omega}_\beta(r)$ is also regular. Therefore Lemma A(iv) yields

$$\int_0^1 r^n(1 - r)\tilde{\omega}_\beta(r) \, dr \leq \int_0^1 r^n(1 - r)\tilde{\omega}_\beta(r) \, dr \leq \frac{1}{n!} \int_0^1 r^n \tilde{\omega}_\beta(r) \, dr.$$  

By (22), (20), Lemma 8(i), (23) with $\beta = \eta(p - 1) \in (0, 1)$, (17) and Lemma 6,

$$\int_0^1 M_p(r, g)(1 - r)\omega(r) \, dr \leq \int_0^1 \left( \sum_{n=0}^{\infty} r^M_n \|\Delta n g\|_{H^q}^p \right) (1 - r)\omega(r) \, dr$$

$$\leq \sum_{n=0}^{\infty} 2^{-n\eta(p-1)} \|\Delta n g\|_{H^q}^p \int_0^1 \frac{\omega(r)}{\hat{\omega}(r)}^{\eta(p-1)} \frac{1}{r^M_n} (1 - r)\omega(r) \, dr$$

$$\leq \sum_{n=0}^{\infty} 2^{-n\eta(p-1)} \|\Delta n g\|_{H^q}^p \frac{1}{M_n} \int_{1 - \frac{1}{M_n}}^1 \omega(r)r^{M_n} \, dr$$

$$\leq \sum_{n=0}^{\infty} \|\Delta n g\|_{H^q}^p \frac{1}{M_n} \int_{1 - \frac{1}{M_n}}^1 r^{M_n} \omega(r) \, dr \times \sum_{n=0}^{\infty} 2^{-n} \|\Delta n g\|_{H^q}^p \frac{1}{M_n},$$

and the proof is complete.

The second result generalizes a known characterization of the mean Lipschitz space $\Lambda(p, \alpha)$, where $1 < p < \infty$ and $0 < \alpha \leq 1$, see [14, Theorem 2.1-3.1]. We say that $g \in \lambda(p, \alpha, \hat{\omega}^\beta)$, if

$$\lim_{r \to 1^-} \frac{M_p(r, g')(1 - r)^{1-\alpha}}{\hat{\omega}(r)^\beta} = 0.$$
Corollary 12. Let $1 < q, p < \infty$, $\eta \in \left[0, \frac{1}{p}\right)$, $\omega \in \mathcal{R} \cap \mathcal{M}_p$ such that $\int_0^1 \omega(r) \, dr = 1$ and $g \in \mathcal{H}(\mathbb{D})$.

(i) $g \in \Lambda \left(q, \frac{1}{p}, \omega^\eta\right)$ if and only if $\|\Delta_n g\|_{H^q} \lesssim M_n^{1 - \frac{1}{p}} 2^{-n\eta}$ for all $n \in \mathbb{N}$.
Moreover,

$$
\|g\|_{\Lambda \left(q, \frac{1}{p}, \omega^\eta\right)} \asymp |g(0)| + \sup_n \frac{\|\Delta_n g\|_{H^q 2^{n\eta}}}{M_n^{1 - \frac{1}{p}}}.
$$

(ii) $g \in \lambda(q, \frac{1}{p}, \omega^\eta)$ if and only if

$$
\|\Delta_n g\|_{H^q} = o\left(M_n^{1 - \frac{1}{p}} 2^{-n\eta}\right), \quad n \to \infty.
$$

Corollary 12 can be obtained by following the lines of the proof of Theorem 4 (ii) together with Lemma 8 (iii). We omit the details.

Finally, we will give simple proofs of several known results as a by-product of Theorem 4.

Corollary C. Let $1 < p < \infty$, $0 < \alpha < \infty$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$.

(i) If $0 < \gamma < \infty$, then

$$
\int_0^1 M_p^q(r, f)(1 - r)^{q\gamma - 1} \, dr \asymp |a_0|^q + \sum_{n=0}^{\infty} 2^{-nq\gamma} \left\| \sum_{k=2^n}^{2^{n+1}-1} a_k z^k \right\|_{H^p}^q.
$$

(ii) If $1/q < \beta < \infty$, then

$$
\int_0^1 M_p^q(r, f) \left(\log \frac{2}{1 - r}\right)^{-q\beta} (1 - r)^{-1} \, dr
\asymp \left\| \sum_{k=0}^{3} a_k z^k \right\|_{H^p}^q + \sum_{n=1}^{\infty} 2^{-n(q\beta - 1)} \left\| \sum_{k=2^n}^{2^{n+1}-1} a_k z^k \right\|_{H^p}^q.
$$

Proof. (i) Consider the regular and normalized weight $\omega(r) = q\gamma(1 - r)^{q\gamma - 1}$. Then, by choosing $\alpha = q\gamma$ in (8) and Theorem 4, the result follows.

(ii) Take the normalized rapidly increasing weight

$$
\omega(r) = \frac{q\beta - 1}{\log 2} \frac{1}{(1 - r) \left(\frac{1}{\log 2 \frac{1}{1 - r}}\right)^{q\beta}}, \quad q\beta > 1.
$$

Then, by choosing $\alpha = q\beta - 1$ in (8) and Theorem 4, the result follows. \qed

Corollary C(i) is obtained in [14] as a consequence of a more general result. Corollary C(ii) is nothing else but [10, Theorem 6.1], the original proof of which is more involved and uses the Riesz-Thorin interpolation theorem.
3.4. \( \omega \)-Lacunary series. The main purpose of this section is to stress how different is a weighted Bergman space \( A^p_\omega \), induced by a rapidly increasing weight \( \omega \), from another \( A^p_\omega \), induced by a regular one. This will be done by using strongly the results on power series with positive coefficients obtained in Section 3.2. The reader is invited to see [20] for more information on this topic.

Recall that, for a given radial weight \( \omega \), \( f \in H(\mathbb{D}) \) is said to be an \( \omega \)-lacunary series in \( \mathbb{D} \) if its Maclaurin series

\[
\sum_{k=0}^{\infty} a_k z^k
\]

satisfies

\[
\frac{\hat{\omega}(1 - \frac{1}{n_k})}{\hat{\omega}(1 - \frac{n_k}{n_{k+1}})} = \frac{\int_{1/n_k}^{1} \omega(r) \, dr}{\int_{1/n_{k+1}}^{1} \omega(r) \, dr} \geq \lambda > 1, \quad k \in \mathbb{N} \cup \{0\}. \tag{24}
\]

We begin with proving an extension of Theorem 5 that describes the \( \omega \)-lacunary series in the mixed norm space \( H(p,q,\omega) \) in terms of the coefficients in their Maclaurin series.

**Theorem 13.** Let \( 0 < q, \alpha < \infty, 0 < p \leq \infty \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \) such that \( \int_{0}^{1} \omega(r) \, dr = 1 \), and let \( f \) be an \( \omega \)-lacunary series in \( \mathbb{D} \). Then the following conditions are equivalent:

(i) \( f \in H(p,q,\omega) \);

(ii) \( \sum_{n=0}^{\infty} 2^{-n\alpha} \left( \sum_{n_k \in I_{\omega,\alpha}(n)} |a_k|^2 \right)^{q/2} < \infty \);

(iii) \( \sum_{n=0}^{\infty} 2^{-n\alpha} \left( \sum_{n_k \in I_{\omega,\alpha}(n)} |a_k|^q \right) < \infty \);

(iv) \( \sum_{n=0}^{\infty} 2^{-n\alpha} \left( \sum_{n_k \in I_{\omega,\alpha}(n)} |a_k| \right)^q < \infty \);

(v) \( \sum_{k=0}^{\infty} |a_k|^q \int_{0}^{1} r^{2n_k+1} \omega(r) \, dr < \infty \).

Moreover, each of the sums in (ii)-(v) is comparable to \( \|f\|_{H(p,q,\omega)}^q \).

**Proof.** Let \( f \) be an \( \omega \)-lacunary series in \( \mathbb{D} \). First, we observe that the chain of inequalities

\[
\frac{1}{1 - r_n} \leq n_k < n_{k+s} < \frac{1}{1 - r_{n+1}}
\]

is equivalent to

\[
\frac{1}{2^{n\alpha}} = \hat{\omega}(r_n) \geq \hat{\omega} \left( 1 - \frac{1}{n_k} \right) > \hat{\omega} \left( 1 - \frac{1}{n_{k+s}} \right) > \hat{\omega}(r_{n+1}) = \frac{1}{2^{(n+1)\alpha}} \tag{25}
\]
by (8). This together with (24) shows that there are at most \( \log_\lambda \alpha + 2 \) integers \( n_k \) in each set \( I_{\omega,\alpha}(n) \). Therefore H"older's inequality and standard estimates give

\[
\sum_{n=0}^{\infty} 2^{-n\alpha} \left( \sum_{k \in I_{\omega,\alpha}(n)} |a_k|^2 \right)^{q/2} \leq \sum_{n=0}^{\infty} 2^{-n\alpha} \left( \sum_{k \in I_{\omega,\alpha}(n)} |a_k|^q \right) \leq \sum_{n=0}^{\infty} 2^{-n\alpha} \left( \sum_{n_k \in I_{\omega,\alpha}(n)} |a_k|^q \right)^{q/2},
\]

and thus (ii) \( \iff \) (iii) \( \iff \) (iv). Moreover, by Lemma A(i) (ii) (iv),

\[
\hat{\omega} \left( 1 - \frac{1}{n_k} \right) \leq \hat{\omega} \left( 1 - \frac{1}{2n_k + 1} \right) \times \int_0^1 r^{2n_k+1} \omega(r) \, dr,
\]

and it follows by (25) that (iii) \( \iff \) (v).

By the proof of [20, Lemma 1.2], there exist \( \beta = \beta(\omega) > 0 \) and \( N \in \mathbb{N} \) such that

\[
\hat{\omega} \left( 1 - \frac{1}{n_k} \right) \leq \left( \frac{n_k+1}{n_k} \right)^\beta
\]

for all \( k \geq N \). Therefore \( f \) is a standard lacunary series by (24). In fact, Lemma A(i) shows that an \( \omega \)-lacunary series for \( \omega \in \mathcal{R} \) is just a standard lacunary series. Consequently, if \( 0 < p < \infty \), Zygmund's theorem [23, p. 215] gives

\[
\|f\|_{H(p,q,\omega)}^q \leq \int_0^1 \left( \sum_{k=0}^{\infty} |a_k|^2 r^{2n_k} \right)^{q/2} \omega(r) r \, dr.
\]

Therefore Proposition 9 implies (i) \( \iff \) (ii) and

\[
\|f\|_{H(p,q,\omega)}^q \leq \sum_{n=0}^{\infty} 2^{-n\alpha} \left( \sum_{n_k \in I_{\omega,\alpha}(n)} |a_k|^2 \right)^{q/2}.
\]

This completes the proof for \( 0 < p < \infty \).

Finally, if \( f \in H(\infty,q,\omega) \), then \( f \in H(p,q,\omega) \) for any \( 0 < p < \infty \), so by the previous argument (i) \( \Rightarrow \) (ii). Reciprocally, assume that (iv) holds. Then, by using Proposition 9, we deduce

\[
\|f\|_{H(\infty,q,\omega)}^q \leq \int_0^1 \left( \sum_{k=0}^{\infty} |a_k|^2 r^{n_k} \right)^q \omega(r) r \, dr \leq \sum_{n=0}^{\infty} 2^{-n\alpha} \left( \sum_{n_k \in I_{\omega,\alpha}(n)} |a_k|^q \right)^{q/2} < \infty.
\]

This finishes the proof. \( \Box \)

Theorem 13 gives an easy way to construct functions in \( A^p_{\alpha} \). For example, if \( 0 < p < q < \infty \) and \( \omega \in \mathcal{I} \cup \mathcal{R} \), then Theorem 13, with \( \alpha = 1 \), shows that

\[
f(z) = \sum_{n=0}^{\infty} r^{n/q} M_n, \quad M_n = E \left( \frac{1}{1 - r_n} \right),
\]
where \( r_n \) is given by (8) with \( \alpha = 1 \), belongs to \( A_\omega^p \setminus A_\omega^q \).

It is worth noticing that the equivalence (i) \( \Leftrightarrow \) (ii) in Theorem 13 is valid for standard lacunary series and \( \omega \in I \cup R \). However, (i) \( \Leftrightarrow \) (iii) is no longer true for standard lacunary series if \( \omega \in I \) and \( q \neq 2 \). Namely, let us consider the rapidly increasing weight

\[
v_\beta(r) = (1 - r)^{-1} \left( \log \frac{e}{1 - r} \right)^{-\beta}, \quad \beta > 1.
\]

If (i) and (iii) were equivalent, then the choice \( \alpha = \beta - 1 \) would imply that a standard lacunary series

\[
f(z) = \sum_{n=0}^{\infty} a_n z^{2^n}
\]

belongs to \( A_\omega^q v_\beta \) if and only if

\[
\sum_{n=0}^{\infty} 2^{-n(\beta-1)} \sum_{k=2^n}^{2^{n+1}-1} |a_k|^q \gtrsim \sum_{k=4}^{\infty} |a_k|^q (\log k)^{-\beta+1} < \infty.
\]

But this is impossible. Namely, if \( \beta > 2 \) and \( a_k = k^{-1/p} \), then we would have

\( f \in A_\omega^p \) for \( q \geq p \), but \( f \notin A_\omega^q \) for \( q < p \). A similar reasoning also works for \( 1 < \beta < 2 \). An analogous argument can be used to show that the condition (v) does not characterize standard lacunary series in \( A_\omega^p \) when \( \omega \in I \) and \( q \neq 2 \).

If \( \omega \in I \), then (24) says, roughly speaking, that the smaller the space \( A_\omega^p \) is, the larger the gaps of an \( \omega \)-lacunary series are. Namely, the condition (iii) in Theorem 13 is equivalent to (i) and (ii) when the series \( \sum a_k z^{n_k} \) has very large gaps depending on \( \omega \).

The next result offers a description of \( \omega \)-lacunary series in the mixed norm space \( H(p, \infty, \hat{\omega}_\beta) \).

**Theorem 14.** Let \( 0 < \beta < \infty \) and \( \omega \in I \cup R \) such that \( \int_0^1 \omega(r) \, dr = 1 \). Let \( f(z) = \sum_{n=0}^{\infty} a_n z^{n_k} \) be an \( \omega \)-lacunary series in \( \mathbb{D} \). Then the following assertions are equivalent:

(i) \( f \in H(\infty, \infty, \hat{\omega}_\beta) \);

(ii) \( f \in H(p, \infty, \hat{\omega}_\beta) \) for some \( 0 < p \leq \infty \);

(iii) The coefficients \( \{a_k\} \) of the Maclaurin series of \( f \) satisfy

\[
|a_k| \lesssim \left( \int_0^1 r^{n_k} \omega(s) \, ds \right)^{-\beta}, \quad k \in \mathbb{N} \cup \{0\}.
\]

**Proof.** The implication (i) \( \Rightarrow \) (ii) is trivial. Moreover, as each \( \omega \)-lacunary series is a standard lacunary series, \( f \in H(p, \infty, \hat{\omega}_\beta) \) if and only if \( f \in H(2, \infty, \hat{\omega}_\beta) \). Therefore Cauchy integral formula and Lemma A(iv) easily give (ii) \( \Rightarrow \) (iii). To complete the proof, we will establish (iii) \( \Rightarrow \) (i). If we choose \( \alpha = \frac{1}{\beta} \) in (8), then Lemma 8(i) gives

\[
\sum_{n=1}^{\infty} 2^n |z|^{M_n} \lesssim \hat{\omega}(|z|)^{-\beta}, \quad z \in \mathbb{D},
\]
so it suffices to prove \( \sum_{k=1}^{\infty} \frac{r^{n_k}}{\omega\left(1 - \frac{1}{n_k}\right)^\beta} \lesssim \sum_{n=1}^{\infty} 2^n r^{M_n}. \) Bearing in mind Lemma A(i)-
(ii) and arguing as in the proof of Theorem 13, we deduce

\[
\sum_{k=1}^{\infty} \frac{r^{n_k}}{\omega\left(1 - \frac{1}{n_k}\right)^\beta} = \sum_{n=1}^{\infty} \sum_{n_k \in I_{\omega, 1}(n)} \frac{r^{n_k}}{\omega\left(1 - \frac{1}{n_k}\right)^\beta} \\
\leq \sum_{n=1}^{\infty} r^{M_n} \sum_{n_k \in I_{\omega, 1}(n)} \frac{1}{\omega\left(1 - \frac{1}{M_{n+1}-1}\right)^\beta} \\
\leq (\log_\lambda 2^{1/\beta} + 2) \sum_{n=1}^{\infty} \frac{r^{M_n}}{\omega\left(r_{n+1}\right)^\beta},
\]

which together with (8) finishes the proof. \( \square \)

Theorem 14 generalizes and improves known results in the existing literature. In particular, by taking the regular weight \( \phi_\gamma(r) = \log \gamma e^{1-r} \) and choosing \( \omega \) such that \( \phi_\gamma([0, 1]) \cdot \omega(r) = \phi_\gamma(r) \), we deduce that the lacunary series \( f(z) = \sum_{n=0}^{\infty} a_k z^{n_k} \), where \( \frac{n_k+1}{n_k} \geq \lambda > 1 \), satisfies the Bloch-type condition

\[
M_{\infty}(r, f') = O \left( \frac{1}{(1-r) \log^{\gamma} \frac{1}{1-r}} \right), \quad 0 < \gamma < \infty,
\]

if and only if

\[
|a_k| = O \left( \left( \log n_k \right)^{-\gamma} \right), \quad k \in \mathbb{N}.
\]

4. The role of the sublinear Hilbert operator

The generalized Hilbert operator

\[
\mathcal{H}_g(f)(z) = \int_0^1 f(t) g(tz) \, dt
\]

is well defined whenever

\[
(27) \quad \int_0^1 |f(t)| \, dt < \infty.
\]

Further, if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}) \) satisfies (27), then \( \mathcal{H}_g(f) \) can be written in terms of the coefficients of the Maclaurin series of \( f \) and \( g \). Namely, if \( g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}(\mathbb{D}) \), then

\[
\mathcal{H}_g(f)(z) = \sum_{k=0}^{\infty} \left( (k+1)b_{k+1} \int_0^1 t^k f(t) \, dt \right) z^k \\
= \sum_{k=0}^{\infty} \left( (k+1)b_{k+1} \sum_{n=0}^{\infty} \frac{a_n}{n+k+1} \right) z^k.
\]
We begin with noting that condition (2) implies (27) for any $f \in A_p^\omega$. In fact, by using Hőlder’s inequality and (6), we deduce
\[
\int_0^1 |f(t)| \, dt \leq \left( \int_0^1 |f(t)|^p \, \omega(t) \, dt \right)^{\frac{1}{p}} \left( \int_0^1 \omega(t)^{1 - \frac{1}{p}} \, dt \right)^{\frac{1}{1 - \frac{1}{p}}} \lesssim \|f\|_{A_p^\omega}^p.
\]
The standard radial weight $(1 - |z|^2)^\alpha$ satisfies (2) if and only if $-1 < \alpha < p-2$. Moreover, the function $h(z) = (1 - z)^{-1} \left( \log \frac{1}{1 - z} \right)^{-1}$ belongs to $A_p^\omega$ for all $1 < p < \infty$, but $\int_0^1 |h(t)| \, dt = \infty$. Therefore (2) is a natural sharp condition for both, the generalized Hilbert operator $\mathcal{H}_g$ and the sublinear Hilbert operator
\[
\mathcal{H}(f)(z) = \int_0^1 \frac{|f(t)|}{1 - tz} \, dt
\]
to be well defined. As mentioned in (5), the operator $\hat{\mathcal{H}}$ behaves like a maximal operator with respect to $\mathcal{H}_g$ under appropriate hypotheses on $\omega$ and $g$. Consequently, in view of (6), it is natural to study the boundedness of $\hat{\mathcal{H}}$ on both $L_p^\omega$ and $A_p^\omega$. This is the main aim of this section.

**Proof of Theorem 2.** (i)$\Rightarrow$(iii). This part of the proof uses ideas from [15]. For $r \in [0,1]$, set $\phi_r(t) = \hat{\omega}(t)^{-1} \chi_{[r,1]}(t)$, so that $\phi_r \in L_p^\omega$ for all $r \in [0,1]$ by (2). Here, as usual, $\chi_E$ stands for the characteristic function of the set $E$. Then, bearing in mind (6), we deduce
\[
\|\mathcal{H}(\phi_r)\|_{L_p^\omega} \lesssim \|\mathcal{H}(\phi_r)\|_{A_p^\omega} \lesssim \|\mathcal{H}\|_{L_p^\omega} \|\phi_r\|_{L_p^\omega},
\]
and hence
\[
\int_0^1 \hat{\omega}(s) \left( \int_r^1 \hat{\omega}(s)^{-1} \frac{1}{1 - ts} \, dt \right)^p \, ds \lesssim \int_0^1 \hat{\omega}(t)^{-1} \frac{1}{1 - ts} \, dt.
\]
Therefore
\[
\int_0^r \hat{\omega}(s) \left( \int_r^1 \hat{\omega}(s)^{-1} \frac{1}{1 - ts} \, dt \right)^p \, ds \gtrsim \frac{1}{2p} \left( \int_0^r \hat{\omega}(s) \, ds \right) \left( \int_r^1 \hat{\omega}(t)^{-1} \frac{1}{1 - ts} \, dt \right)^p,
\]
and this together with (28) implies $\omega \in M_p$ and
\[
M_p(\omega) \lesssim \|\mathcal{H}\|_{L_p^\omega}.
\]
This argument also proves (ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(i). Since $\omega \in \mathcal{R}$ by the assumption, $\omega$ is comparable to the differentiable weight $\frac{\int_0^1 \omega(s) \, ds}{(1-s)^p}$, so, by using [19, Theorem 1.1], we deduce
\[
\|f\|_{A_p^\omega}^p \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^p \omega(z) \, dA(z), \quad f \in H(\mathbb{D})
\]
Now, for any $\phi \in L_p^\omega$,
\[
(\mathcal{H}(\phi))^\prime(z) = \int_0^1 \frac{t \phi(t)}{(1 - tz)^2} \, dt.
\]
and so Minkowski’s inequality in continuous form yields
\[
M_p(r, (H(\phi))') = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 \frac{\phi(t) t}{(1 - tre^{i\theta})^2} dt \right|^p d\theta \right)^{\frac{1}{p}} \\
\leq \int_0^1 \phi(t) \left( \int_0^{2\pi} \frac{d\theta}{|1 - tre^{i\theta}|^{2p}} \right)^{\frac{1}{p}} dt \asymp \int_0^1 \frac{\phi(t)}{(1 - tr)^{2 - \frac{1}{p}}} dt,
\]

and hence
\[
(29) \quad \|H(\phi)\|_{A^p} \lesssim I_1(r) + I_2(r) + |H(\phi)(0)|^p
\]
where
\[
I_1(r) = \int_0^1 \left( \int_r^1 \frac{\phi(t)}{(1 - t)^{2 - \frac{1}{p}}} dt \right)^p (1 - r)^p \omega(r) \, dr
\]
and
\[
I_2(r) = \int_0^1 \left( \int_1^r \frac{\phi(t)}{(1 - tr)^{2 - \frac{1}{p}}} dt \right)^p (1 - r)^p \omega(r) \, dr.
\]

We observe that
\[
(30) \quad I_1(r) \lesssim \|\phi\|_{L^p_{\omega}}
\]
can be written as
\[
\int_0^1 \left( \int_0^r \Phi(t) \, dt \right) U^p(r) \, dr \leq \int_0^1 \Phi^p(r) V^p(r) \, dr,
\]
where
\[
U^p(x) = \begin{cases} (1 - x)^{p-1} \tilde{\omega}(x), & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases},
\]
\[
V^p(x) = \begin{cases} (1 - x)^{2p-1} \tilde{\omega}(x), & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases},
\]
and \( \Phi(t) = \frac{\phi(t)}{(1-t)^{2 - \frac{1}{p}}} \). Since \( \tilde{\omega} \) is decreasing,
\[
\left( \int_r^1 U^p(s) \, ds \right)^{\frac{1}{p'}} \left( \int_0^r V^{-p'}(s) \, ds \right)^{\frac{1}{p'}}
\]
\[
= \left( \int_r^1 (1 - s)^{p-1} \tilde{\omega}(s) \, ds \right)^{\frac{1}{p'}} \left( \int_0^r \frac{1}{(1 - s)^{(2 - \frac{1}{p})p'} \tilde{\omega}(s)} \, ds \right)^{\frac{1}{p'}}
\]
\[
\leq \tilde{\omega}(r)^{\frac{1}{p'}} (1 - r) \tilde{\omega}(r)^{-\frac{1}{p'}} \left( \int_0^r \frac{1}{(1 - s)^{(2 - \frac{1}{p})p'}} \, ds \right)^{\frac{1}{p'}} \leq C,
\]
for all \( r \in [0,1) \). Now, \([15, \text{Theorem 1}]\) shows that (30) holds. Moreover, since \( \omega \in M_p \), by applying \([15, \text{Theorem 2}]\) with
\[
U^p(x) = \begin{cases} \frac{\hat{\omega}(x)}{(1-x)^p}, & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases},
\]
and
\[
V^p(x) = \begin{cases} \hat{\omega}(x), & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases},
\]
we deduce
\[
I_2(r) \lesssim \int_0^1 \left( \int_r^1 \phi(t) \frac{dt}{r} \right)^p \frac{\hat{\omega}(r)}{(1-r)^p} dr \lesssim M_p(\omega) \|\phi\|_{L^p_\omega},
\]
which together with (29) and (30) gives (iii) \( \Rightarrow \) (i) and
\[
\|H\|_{(L^p_\omega, A^p_\omega)} \lesssim M_p(\omega).
\]
It is clear that the same argument proves (iii) \( \Rightarrow \) (ii). \( \square \)

It is worth noticing that the implication (iii) \( \Rightarrow \) (i) (as well as (iii) \( \Rightarrow \) (ii)) can also be proved by using the theory of Bekollé-Bonami weights. We will only give an outline of this argument. It is strongly based on the following essentially known result, which follows from Lemma A(v) and \([12, \text{Theorem 2.1}]\).

**Lemma 15.** Let \( 1 < p < \infty \) and \( \omega \in \mathcal{R} \). Then there exists \( \eta_0 = \eta_0(p, \omega) > -1 \) such that for all \( \eta \geq \eta_0 \), the dual of \( A^p_{\omega} \) can be identified with \( A^{p'}_{\omega} \) under the pairing
\[
\langle f, g \rangle_\eta = \int f(z) \overline{g(z)} (1 - |z|)^\eta \, dA(z).
\]
(31)

Reciprocally, the dual of \( A^p_{\omega} \) can be identified with \( A^{p'}_{\omega} \) under the same pairing.

An alternative proof of (iii) \( \Rightarrow \) (i). Let \( \eta_0 = \eta_0(p, \omega) > -1 \) be that of Lemma 15 and fix \( \eta \geq \eta_0 \). For simplicity, we write \( v^p_\omega(z) = \omega(z)^{-\frac{1}{p'}} (1 - |z|)^{p'\eta} \). By Lemma 15, the dual of \( A^p_{v^p_\omega} \) can be identified with \( A^{p'}_{\omega} \) under the pairing defined by (31). Therefore \( \mathcal{H} : L^p_{\omega} \to A^p_{\omega} \) is bounded if and only if
\[
|\langle \mathcal{H}(\phi), h \rangle_\eta| \lesssim \|\phi\|_{L^p_{\omega}} \|h\|_{v^p_\omega}, \quad \phi \in L^p_{\omega}, \ h \in A^p_{v^p_\omega}.
\]
To prove this, let \( \phi \in L^p_{\omega} \) and \( h \in A^p_{v^p_\omega} \). By Fubini’s theorem, the Cauchy integral formula and Hölder’s inequality, we deduce
\[
|\langle \mathcal{H}(\phi), h \rangle_\eta| = 2 \int_0^1 \phi(t) \left( \int_0^1 \frac{dr}{r^2 t} r (1 - r)^\eta \right) dt \lesssim \|\phi\|_{L^p_{\omega}} I(h),
\]
(32)
where
\[ I(h) = \left( \int_0^1 \left( \int_0^1 |h(r^2 t)| r(1-r)^{\eta} \, dr \right)^{p'} \, \hat{\omega}^{-\frac{\nu'}{p'}} (t) \, dt \right)^\frac{1}{p'} . \]

A change of variable, the hypotheses \( \omega \in \mathcal{M}_p, [15, \text{Theorem 1}] \) and (4) give
\begin{align*}
&\int_0^1 \left( \int_0^1 |h(r^2 t)| r(1-r)^{\eta} \, dr \right)^{p'} \, \hat{\omega}^{-\frac{\nu'}{p'}} (t) \, dt \\
&\lesssim \int_0^1 \left( \int_0^t M_{\infty}(s, h)(1-s)^{\eta} \, ds \right)^{p'} \, \hat{\omega}^{-\frac{\nu'}{p'}} (t) \, dt \\
&\lesssim \mathcal{M}_p'(\omega) \int_0^1 M_{\infty}'(t, h)(1-t)^{p'\eta+1} \, \hat{\omega}^{-\frac{\nu'}{p'}} (t) \, dt .
\end{align*}

Now, it follows from [20, p. 9 (i)] that
\[ \int_t^1 v_{p'}(s) \, ds \geq \int_t^{1+t} (1-s)^{p'\eta} \omega^{-\frac{\nu'}{p'}} (s) \, ds \asymp (1-t)^{p'\eta+1} \omega^{-\frac{\nu'}{p'}} (t) . \]

Consequently, this together with (33) and (6) yield
\begin{align*}
&\int_0^1 \left( \int_0^1 |h(r^2 t)| r(1-r)^{\eta} \, dr \right)^{p'} \, \hat{\omega}^{-\frac{\nu'}{p'}} (t) \, dt \\
&\lesssim \mathcal{M}_p'(\omega) \int_0^1 M_{\infty}'(t, h) \left( \int_t^1 v_{p'}(s) \, ds \right) \, dt \lesssim \mathcal{M}_p'(\omega) \|h\|v_{p'} .
\end{align*}

By combining this and (32), the proof is finished. \( \square \)

5. Background on smooth Hadamard products

If \( W(z) = \sum_{k \in J} b_k z^k \) is a polynomial and \( f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(D) \), then the Hadamard product
\[ (W * f)(z) = \sum_{k \in J} b_k a_k z^k \]
is well defined. Further, if \( f \in H^1 \), then
\[ (W * f)(e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} W(e^{i(t-\theta)}) f(e^{i\theta}) \, d\theta \]
is the usual convolution.

If \( \Phi : \mathbb{R} \to \mathbb{C} \) is a \( C^\infty \)-function such that its support \( \text{supp}(\Phi) \) is a compact subset of \( (0, \infty) \), we set
\[ A_{\Phi} = \max_{s \in \mathbb{R}} |\Phi(s)| + \max_{s \in \mathbb{R}} |\Phi''(s)| , \]
and we consider the polynomials
\[ W_N^\Phi(z) = \sum_{k \in \mathbb{N}} \Phi \left( \frac{k}{N} \right) z^k, \quad N \in \mathbb{N} . \]

With this notation we can state the next result on smooth partial sums.
Lemma 16. Let \( \Phi : \mathbb{R} \rightarrow \mathbb{C} \) be a \( C^\infty \)-function such that \( \text{supp}(\Phi) \subset (0, \infty) \) is compact. Then the following assertions hold:

(i) There exists a constant \( C > 0 \) such that
\[
|W_N^\phi(e^{i\theta})| \leq C \min \left\{ N \max_{s \in \mathbb{R}} |\Phi(s)|, N^{1-m} |\theta|^{-m} \max_{s \in \mathbb{R}} |\Phi^{(m)}(s)| \right\},
\]
for all \( m \in \mathbb{N} \cup \{0\} \), \( N \in \mathbb{N} \) and \( 0 < |\theta| < \pi \).

(ii) There exists a constant \( C > 0 \) such that
\[
\left| (W_N^\phi * f)(e^{i\theta}) \right| \leq CA_\Phi M(|f|)(e^{i\theta})
\]
for all \( f \in H^1 \). Here \( M \) denotes the Hardy-Littlewood maximal-operator
\[
M(|f|)(e^{i\theta}) = \sup_{0 < h < \pi} \frac{1}{2h} \int_{\theta-h}^{\theta+h} |f(e^{it})| \, dt.
\]
(iii) For each \( p \in (1, \infty) \) there exists a constant \( C = C(p) > 0 \) such that
\[
\|W_N^\phi * f\|_{H^p} \leq CA_\Phi \|f\|_{H^p}
\]
for all \( f \in H^p \).

(iv) For each \( p \in (1, \infty) \) and a radial weight \( \omega \), there exists a constant \( C = C(p, \omega) > 0 \) such that
\[
\|W_N^\phi * f\|_{A^p_\omega} \leq CA_\Phi \|f\|_{A^p_\omega}
\]
for all \( f \in A^p_\omega \).

Theorem D follows from the results and proofs in [18, p. 111-113]. We will also need the following lemma whose proof follows from (21) and Lemma A.

Lemma 16. Let \( 0 < p < \infty, n_1, n_2 \in \mathbb{N} \) with \( n_1 \leq n_2 \leq Cn_1, \omega \in I \cup R \) and \( g \in \mathcal{H}(\mathbb{D}) \). Then
\[
\|S_{n_1, n_2}g\|_{A^p_\omega} \asymp \left( \int_{1-\frac{1}{n_1}}^{1-\frac{1}{n_2}} \omega(s) \, ds \right)^{1/p} \|S_{n_1, n_2}g\|_{H^p}
\]
\[
\asymp \left( \int_{1-\frac{1}{n_1}}^{1-\frac{1}{n_2}} \omega(s) \, ds \right)^{1/p} \|S_{n_1, n_2}g\|_{H^p}.
\]

The next auxiliary result allows us to prove the maximality of the sublinear Hilbert operator \( \mathcal{H} \) in the study of the boundedness of \( \mathcal{H}_g \) on weighted Bergman spaces. The proof of Lemma 17 is analogous to that of [9, Lemma 7] and is therefore omitted.

Lemma 17. Let \( 1 < p < \infty, \omega \) be a radial weight satisfying (2) and \( n_1, n_2 \in \mathbb{N} \) with \( n_1 < n_2 \). Let \( f \in A^p_\omega, g(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{H}(\mathbb{D}) \) and \( h(z) = \sum_{k=0}^{\infty} c_k \left( \int_0^t t^k f(t) \, dt \right) z^k \). Then there exists a constant \( C = C(p) > 0 \) such that
\[
\|S_{n_1, n_2}h\|_{H^p} \leq C \left( \int_0^1 t^{\frac{n_1}{2}} |f(t)| \, dt \right) \|S_{n_1, n_2}g\|_{H^p}.
\]
The next known result can be proved by summing by parts and using the M. Riesz projection theorem [9, 11].

**Lemma E.** Let $1 < p < \infty$ and $\lambda = \{\lambda_k\}_{k=0}^\infty$ be a monotone sequence of positive numbers. Let $(\lambda g)(z) = \sum_{k=0}^\infty \lambda_k b_k z^k$, where $g(z) = \sum_{k=0}^\infty b_k z^k$.

(i) If $\{\lambda_k\}_{n=0}^\infty$ is nondecreasing, then there exists a constant $C > 0$ such that
\[
C^{-1} \lambda_{n_1} \|S_{n_1, n_2} g\|_{H^p} \leq \|S_{n_1, n_2} \lambda g\|_{H^p} \leq C \lambda_{n_2} \|S_{n_1, n_2} g\|_{H^p}.
\]
(ii) If $\{\lambda_n\}_{n=0}^\infty$ is nonincreasing, then there exists a constant $C > 0$ such that
\[
C^{-1} \lambda_{n_2} \|S_{n_1, n_2} g\|_{H^p} \leq \|S_{n_1, n_2} \lambda g\|_{H^p} \leq C \lambda_{n_1} \|S_{n_1, n_2} g\|_{H^p}.
\]

6. Proof of Theorem 1.

We may assume without loss of generality that $\int_0^1 \omega(r) \, dr = 1$. Throughout the proof $\{r_n\}_{n=0}^\infty$ is the sequence defined by (8) with $\alpha = 1$.

6.1. **Sufficiency.** Theorem 4, with $\alpha = 1$, shows that
\[
\|H_g(f)\|_{A^\varphi^p}^q \geq \sum_{n=0}^\infty 2^{-n} \|\Delta_n^\omega H_g(f)\|_{H^q}^q
\]
for all $f \in H(D)$. Now Lemma 17, Hölder’s inequality and (6) yield
\[
\|\Delta_0^\omega H_g(f)\|_{H^q} \lesssim |g'(0)| \int_0^1 |f(t)| \, dt + \|S_{t, t} H_g(f)\|_{H^q}
\]
\[
\lesssim |g'(0)| \int_0^1 |f(t)| \, dt + \|S_{t, t} g\|_{H^q} \int_0^1 t^{1/4} |f(t)| \, dt
\]
\[
\lesssim (|g'(0)| + \|S_{t, t} g\|_{H^q}) \int_0^1 |f(t)| \, dt
\]
\[
\lesssim \left( \int_0^1 M_{\omega}^q(t, f) \tilde{\omega}(t) \, dt \right)^{1/p} \approx \|f\|_{A^\varphi^p},
\]
where the constants of comparison depend on $p$, $q$, $\omega$ and $g$.

Let first $1 < p \leq q < \infty$ and assume that $g \in A \left( q, \frac{1}{p}, \tilde{\omega}^{\frac{1}{p'}} \right)$, that is,
\[
M_q(r, g') \leq \|g - g(0)\|_{A \left( q, \frac{1}{p}, \tilde{\omega}^{\frac{1}{p'}} \right)} \left( \frac{1}{1 - r} \right)^{1 - \frac{1}{p}}, \quad 0 \leq r < 1.
\]

Lemma 17, Lemma 10, the M. Riesz projection theorem and the assumption give
\[
\|\Delta_n^\omega H_g(f)\|_{H^q}^q \lesssim \left( \int_0^1 t^{-\frac{n}{4}} |f(t)| \, dt \right)^q \|\Delta_n^\omega g\|_{H^q}^q
\]
\[
\lesssim \left( \int_0^1 t^{-\frac{n}{4}} |f(t)| \, dt \right)^q M_q^q \left( \frac{1}{1 - \frac{1}{M_{n+1}}}, g' \right)
\]
\[
\lesssim \left( \int_0^1 t^{-\frac{n}{4}} |f(t)| \, dt \right)^q \tilde{\omega} \left( 1 - \frac{1}{M_{n+1}} \right)^{q \left( \frac{1}{p} - \frac{1}{p'} \right)} M_{n+1}^{q \left( 1 - \frac{1}{p} \right)}.
\]
where in the last inequality the constant of comparison depend on \(||g-g(0)||^q_{\Lambda(a^{1/p}_p, \omega^{1/q}_q)}||.

Let \(f \in A^p_{\omega}||. Then (6) yields \(M_r(f) \lesssim u_p(r) = ((1-r)\omega(r))^{-\frac{1}{p}}||. This together the fact that \(u_p \in \mathcal{R}|| by Lemma 7, and Lemma A(iv) yields

\[
\int_0^1 t^{\frac{M_n}{q}} |f(t)| dt \lesssim \int_0^1 t^{\frac{M_n}{q}} u_p(t) dt \asymp u_p \left(1 - \frac{1}{M_{n+1}}\right) \frac{1 - \frac{1}{M_{n+1}}}{M_{n+1}},
\]

and thus

\[
M_n^{-\frac{1}{p}} \omega \left(1 - \frac{1}{M_{n+1}}\right)^{\frac{1}{p}} \int_0^1 t^{\frac{M_n}{q}} |f(t)| dt \lesssim ||f||_{A^p_{\omega}}.
\]

Let now \(k_0 \in \mathbb{N}|| to be fixed later. Since \(q \geq p||, by using (36), (37), Hölder’s inequality and (6), we deduce

\[
\sum_{n=1}^{k_0} 2^{-n} ||\Delta_{\omega}^n \mathcal{H}_g(f)||_{H^q}^q \lesssim \sum_{n=1}^{k_0} 2^{-n} \left(\int_0^1 t^{\frac{M_n}{q}} |f(t)| dt\right)^q \omega \left(1 - \frac{1}{M_{n+1}}\right)^{q(\frac{1}{p} - \frac{1}{q})} M_{n+1}^{q\left(1 - \frac{1}{p}\right)}
\]

\[
\lesssim ||f||_{A^p_{\omega}}^q \sum_{n=1}^{k_0} 2^{-n} \left(\int_0^1 t^{\frac{M_n}{q}} |f(t)| dt\right)^p M_{n+1}^{p\left(1 - \frac{1}{p}\right)}
\]

\[
\lesssim ||f||_{A^p_{\omega}}^q \int_0^1 |f(t)| dt \omega(t) dt \lesssim ||f||_{A^p_{\omega}}^q,
\]

where the constants of comparison depend on \(p, q, \omega|| and \(k_0||.

Let now \(\gamma_1|| be the constant appearing in Lemma 6, and choose \(k_0|| to be the smallest natural number such that \(r_{k_0} \geq \max \left\{\frac{1}{2\gamma_1}, \frac{1}{2}\right\}|| and \(2^{k_0\gamma_{\omega}} \geq 4||. Then, by
(36), (37) and Lemma 6, we have
\[
\sum_{n=k_0+1}^{\infty} 2^{-n} \| \Delta_n^\omega \mathcal{H}_g(f) \|_{A_p^q}^p \\
\lesssim \sum_{n=k_0+1}^{\infty} 2^{-n} \left( \int_0^1 t^{M_n} |f(t)| \, dt \right)^q \\
\cdot \hat{\omega} \left( 1 - \frac{1}{M_{n+1}} \right)^{q \left( 1 - \frac{1}{p} \right)} M_{n+1}^{q \left( 1 - \frac{1}{p} \right)} \\
\lesssim \| f \|_{A_p^q}^{q-p} \sum_{n=k_0+1}^{\infty} 2^{-n} \left( \int_0^1 t^{M_n} |f(t)| \, dt \right)^p M_{n+1}^{p-1} \\
\leq C \| f \|_{A_p^q}^{q-p} \left( \gamma_2 (p-1)^{-1} \right) \sum_{j=0}^{\infty} 2^{-j} \left( \int_0^1 t^{M_{j+1}} |f(t)| \, dt \right)^p M_{j+1}^{p-1} \\
\lesssim \| f \|_{A_p^q}^{q-p} \sum_{j=0}^{\infty} 2^{-j} \left( \int_0^1 t^{M_{j+1}} |f(t)| \, dt \right)^p M_{j+1}^{p-1}.
\]

On the other hand, the M. Riesz projection theorem and Lemma 10 give
\[
\left\| \Delta_n^\omega \left( \frac{1}{1-z} \right) \right\|_{A_p^p}^p \approx M_{n+1}^{p-1}.
\]

Now, by using Theorem 4 together with Lemma E(ii), we get
\[
\left\| \tilde{\mathcal{H}}(f) \right\|_{A_p^p}^p \approx \sum_{n=0}^{\infty} 2^{-n} \left\| \Delta_n^\omega \tilde{\mathcal{H}}(f) \right\|_{A_p^p}^p \\
\geq \sum_{n=0}^{\infty} 2^{-n} \left( \int_0^1 t^{M_{n+1}} |f(t)| \, dt \right)^p \left\| \Delta_n^\omega \left( \frac{1}{1-z} \right) \right\|_{A_p^p}^p \\
\geq \sum_{n=0}^{\infty} 2^{-n} \left( \int_0^1 t^{M_{n+1}} |f(t)| \, dt \right)^p M_{n+1}^{p-1}.
\]

So, by combining (34), (35), (38), (39) and (40), we finally deduce
\[
\| \mathcal{H}_g(f) \|_{A_p^q}^q \lesssim \| g - g(0) \|_{A_p^q}^q \left( \left\| f \right\|_{A_p^q}^q + \left\| f \right\|_{A_p^q}^{q-p} \right) \left\| \tilde{\mathcal{H}}(f) \right\|_{A_p^q}^p,
\]

which together with Corollary 3 gives
\[
\| \mathcal{H}_g(f) \|_{A_p^q}^q \lesssim \| g - g(0) \|_{A_p^q}^q \left( \left\| f \right\|_{A_p^q}^q + \left\| f \right\|_{A_p^q}^{q-p} \right) \| f \|_{A_p^q}^q.
\]

This finishes the proof of the sufficiency in the case $1 < p \leq q < \infty$. 

Let now $1 < q < p < \infty$ and assume that $g' \in H\left(q, s, \hat{\omega}_{s}\left(1 - \frac{1}{q}\right)\right)$. By (34), (35), Lemma 17 and Hölder’s inequality, we deduce

\begin{equation}
\|H_g(f)\|_{A^q_p, \hat{\omega}} \lesssim \sum_{n=0}^{\infty} 2^{-n} \|\Delta_n^H g(f)\|_{H^q_p} \lesssim \sum_{n=1}^{\infty} 2^{-n} \|\Delta_n^H g(f)\|_{H^q_p} \lesssim \sum_{n=1}^{\infty} 2^{-n} \left(\int_0^1 \frac{M_n}{t^{\frac{1}{q}}} |f(t)| \, dt\right)^{\frac{q}{p}} \|\Delta_n^H g\|_{H^q_p} \lesssim \sum_{n=1}^{\infty} 2^{-n} \frac{\|\Delta_n^H g\|_{H^q_p}}{M_n^{\left(\frac{1}{q} - \frac{1}{p}\right)/s}}. \tag{41}
\end{equation}

Corollary 11 and the assumption $\omega \in \mathcal{R}$ yield

\begin{equation}
\left[\sum_{n=1}^{\infty} 2^{-n} \frac{\|\Delta_n^H g\|_{H^q_p}}{M_n^{\left(\frac{1}{q} - \frac{1}{p}\right)/s}}\right]^{\frac{1}{q}} \lesssim \left[\int_0^1 M_s^q(r, g') (1 - r)^{\left(\frac{1}{s} - \frac{1}{q}\right)/s} \omega(r) \, dr\right]^{\frac{1}{q}} \lesssim \|g'|_{H^q_p}\left(q, s, \hat{\omega}_{s}\left(1 - \frac{1}{q}\right)\right). \tag{42}
\end{equation}

Moreover, by arguing as in the previous case we obtain

\begin{equation}
\left[\sum_{n=1}^{\infty} 2^{-n} \frac{\|\Delta_n^H g\|_{H^q_p}}{M_n^{\left(\frac{1}{q} - \frac{1}{p}\right)/s}}\right]^{\frac{1}{q}} \lesssim \|f\|_{A^p_q, \hat{\omega}} + \|\tilde{H}(f)\|_{A^p_q, \hat{\omega}} \lesssim \|f\|_{A^p_q},
\end{equation}

which together with (41) and (42) gives

\begin{equation}
\|H_g(f)\|_{A^q_p, \hat{\omega}} \lesssim \|g'|_{H^q_p}\left(q, s, \hat{\omega}_{s}\left(1 - \frac{1}{q}\right)\right) \|f\|_{A^p_q, \hat{\omega}} + \|f\|_{A^p_q}. \tag{43}
\end{equation}

The proof of the sufficiency is now complete.

6.2. Test functions. Before passing to the proof of the necessity part of Theorem 1, we will construct appropriate test functions. If $q < p$ we set up a family of functions $Q_\rho \in A^p_q$, depending on $g$, such that

\begin{equation}
\lim_{\rho \to 1^-} \|Q_\rho\|_{A^p_q} = \|g'|_{H^q_p}\left(q, s, \hat{\omega}_{s}\left(1 - \frac{1}{q}\right)\right).
\end{equation}

In the case $q \geq p$ we will use the next result which can be proved by using ideas from [9, Lemma 1].
Lemma 18. Let $0 < p - 1 < \gamma < \infty$ and $\omega \in I \cup R$. Let $E \subset (0, \infty)$ be a bounded set such that $\text{dist}(E, 0) > 0$. For $N \in \mathbb{N}$, let $a_N = 1 - \frac{1}{N}$, and consider the functions

$$
\psi_{N,\omega}(s) = \left[ N^{\gamma+1} \omega(S(a_N)) \right]^{-\frac{1}{p}} \int_0^1 \frac{t^{sN}}{(1 - a_N t)^{\frac{\gamma+1}{p}}} \, dt, \quad s > 0,
$$

and

$$
\varphi_{N,\omega}(s) = \frac{1}{\psi_{N,\omega}(s)}, \quad s > 0.
$$

Then the following assertions hold:

(i) $\psi_{N,\omega}, \varphi_{N,\omega} \in C^\infty((0, \infty))$.

(ii) There exists a constant $C = C(E) > 0$ such that

$$
C^{-1} N^{\gamma+1} \omega(S(a_N))^{-\frac{1}{p}} \leq |\psi_{N,\omega}(s)| \leq C N^{\gamma+1} \omega(S(a_N))^{-\frac{1}{p}}, \quad s \in E, \quad N \to \infty.
$$

(iii) For each $m \in \mathbb{N}$, there exists a constant $C = C(m, E) > 0$ such that

$$
|\psi_{N,\omega}^{(m)}(s)| \leq C N^{\gamma+1} \omega(S(a_N))^{-\frac{1}{p}}, \quad s \in E, \quad N \in \mathbb{N}.
$$

(iv) For each $m \in \mathbb{N}$, there exists a constant $C = C(m, E) > 0$ such that

$$
|\varphi_{N,\omega}^{(m)}(s)| \leq C N^{\gamma+1} \omega(S(a_N))^{-\frac{1}{p}}, \quad s \in E, \quad N \in \mathbb{N}.
$$

Next, we will construct the test functions which will be used in the proof of the case $q < p$. As usual, we write $f_\rho(z) = f(\rho z)$ for each $0 \leq \rho < 1$.

Lemma 19. Let $1 < q < p < \infty$, $\omega \in R \cap M_p$ and $g \in H(D)$ such that $g' \in H\left(q, s, \tilde{\omega}_{s(1-\frac{1}{q})}\right)$. Then the functions

$$
\phi_\rho(r) = \left( M_q(r, g'_\rho)(1 - r)^{-\frac{1}{q}} \right)^{\frac{1}{p-q}}, \quad 0 < \rho < 1,
$$

and

$$
Q_\rho(z) = \int_0^1 \frac{\phi_\rho(t)}{1 - tz} \, dt, \quad z \in D,
$$

satisfy

$$
Q_\rho(t) \gtrsim \phi_\rho(t), \quad 0 \leq t < 1,
$$

and

$$
\|Q_\rho\|_{A_p}^p \asymp \int_0^1 \phi_\rho^p(t) \tilde{\omega}(t) \, dt < \infty.
$$

Proof. Clearly, $Q_\rho \in \mathcal{H}(D)$ for all $0 < \rho < 1$. Moreover,

$$
Q_\rho(r) \geq \int_r^1 \frac{\phi_\rho(t)}{1 - tr} \, dt \geq \frac{M_q^{\frac{q}{p-q}}(r, g'_\rho)}{1 - r^2} \int_r^1 (1 - t)^{\frac{q-1}{p-q}} \, dt \asymp \phi_\rho(r), \quad 0 \leq r < 1.
$$
This and (6) give

\[(47) \quad \|Q_\rho\|_{A^p_\omega}^p \gtrsim \int_0^1 \phi_\rho^p(t) \hat{\omega}(t) \, dt.\]

Since \(\omega \in \mathcal{R}\) by the assumption, \(\omega\) is comparable to the differentiable weight \(\hat{\omega}(r)\), and hence \(n\) consecutive applications of [19, Theorem 1.1] give

\[(48) \quad \|f\|_{A^p_\omega}^p \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)|^p + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|)^{np} \omega(z) \, dA(z), \quad f \in \mathcal{H}(\mathbb{D}).\]

Now

\[Q_\rho^{(n)}(z) = n! \int_0^1 \frac{t^n \phi_\rho(t)}{(1 - tz)^{n+1}} \, dt,\]

and so Minkowski’s inequality in continuous form yields

\[M_p(r, Q_\rho^{(n)}) = \left( \frac{n!}{2\pi} \int_0^{2\pi} \left| \int_0^1 \frac{\phi_\rho(t)}{(1 - tre^{it})^{n+1}} \, dt \right|^p \, d\theta \right)^{\frac{1}{p}} \leq \int_0^1 \phi_\rho(t) \left( \int_0^{2\pi} \left| \frac{d\theta}{1 - tre^{it}} \right|^{np+1} \right)^{\frac{1}{p}} \, dt \times \int_0^1 \frac{\phi_\rho(t)}{(1 - tr)^{n-\frac{1}{p}+1}} \, dt.\]

Choose now \(n \in \mathbb{N}\) such that \(n - \frac{1}{p} - \frac{q-1}{p+q} > 0\). Then

\[\int_0^r \frac{\phi_\rho(t)}{(1 - tr)^{n-\frac{1}{p}+1}} \, dt \leq M_q(r, g'\rho)^{\frac{q-1}{p-q}} \int_0^r \frac{(1 - t)^{\frac{q-1}{p-q}}}{(1 - tr)^{n-\frac{1}{p}+1}} \, dt \leq M_q(r, g'\rho)^{\frac{q}{p-q}} \int_0^r \frac{dt}{(1 - tr)^{n-1-\frac{q-1}{p+q}}} \times \frac{\phi_\rho(r)}{(1 - r)^{n-\frac{1}{p}}}.\]

Moreover, since \(\omega \in \mathcal{M}_\rho\), by applying [15, Theorem 2], with

\[U^p(x) = \begin{cases} \frac{\hat{\omega}(x)}{(1-x)^p}, & 0 \leq x < 1, \\ 0, & x \geq 1 \end{cases},\]

and

\[V^p(x) = \begin{cases} \hat{\omega}(x), & 0 \leq x < 1, \\ 0, & x \geq 1 \end{cases},\]
we deduce
\[
\int_0^1 M_p(r, Q(1-r)^p \omega(r) dr \\
\lesssim \int_0^1 \left( \int_0^r + \int_r^1 \right) \frac{\phi_p(t)}{(1-tr)^{n-s+1}} dt \right)^p (1-r)^p \omega(r) dr \\
\lesssim \int_0^1 \phi_p(r)(1-r)\omega(r) dr + \int_0^1 \left( \int_0^r \phi_p(t) dt \right)^p \frac{\omega(r)}{(1-r)^{p-1}} dr \\
\lesssim \int_0^1 \phi_p(r)\hat{\omega}(r) dr + \int_0^1 \left( \int_0^r \phi_p(t) dt \right)^p \hat{\omega}(r) \frac{\omega(r)}{(1-r)^{p-1}} dr \\
\lesssim \int_0^1 \phi_p(r)\hat{\omega}(r) dr < \infty,
\]
which together with (48) and (47) gives (46). This finishes the proof. \(\square\)

6.3. Necessity. First we deal with the case \(1 < p \leq q < \infty\). Let \(g(z) = \sum_{k=0}^{\infty} b_k z^k\) be the Maclaurin series of \(g\). By Lemma 6 there exists a positive constant \(B_2 = B_2(\omega)\) such that
\[
1 \leq M_n^{1+1} \leq B_2, \quad n \in \mathbb{N}.
\]
Let us consider the functions \(\psi_{M_n,\omega}\) and \(\varphi_{M_n,\omega} = \frac{1}{\psi_{M_n,\omega}}\) defined in Lemma 18. For each \(n \in \mathbb{N}\), we can find a \(C^\infty\)-function \(\Phi_{M_n} : \mathbb{R} \to \mathbb{C}\) with \(\text{supp}(\Phi_{M_n}) \subset (\frac{1}{2}, 2B_2)\), satisfying
\[
\Phi_{M_n}(s) = \varphi_{M_n,\omega}(s), \quad 1 \leq s \leq B_2,
\]
and such that, by using part Lemma 18(iv), for each \(m \in \mathbb{N}\) there exists a constant \(C = C(m) > 0\) for which
\[
|\Phi_{M_n}^{(m)}(s)| \leq C M_n \omega(S(a_{M_n}))^{1/p}, \quad s \in \mathbb{R}, \quad n \in \mathbb{N}.
\]
In particular, by (51) and Lemma 18(ii), we have
\[
A\Phi_{M_n} = \max_{s \in \mathbb{R}} |\Phi_{M_n}(s)| + \max_{s \in \mathbb{R}} |\Phi_{M_n}^{(m)}(s)| \lesssim M_n \omega(S(a_{M_n}))^{1/p}.
\]
Let us now consider the functions
\[
f_{M_n}(z) = \frac{1}{M_n^{1+1} \omega(S(a_{M_n}))^{1/p}} \frac{1}{(1-a_{M_n} z)^{\gamma_{1+1}/p}}, \quad z \in \mathbb{D}, \quad n \in \mathbb{N},
\]
where \(\gamma = \max\{\gamma_0, p-1\}\) and \(\gamma_0 = \gamma_0(\omega) > 0\) is from Lemma B. The \(A_p^\omega\)-norms of the functions \(f_{M_n}\) are uniformly bounded by Lemma B. Therefore
\[
\sup_{n \in \mathbb{N}} \|\mathcal{H}_g(f_{M_n})\|_{A_p^\omega} \lesssim \|\mathcal{H}_g\|_{(A_p^\omega, A_p^\omega)} < \infty
\]
by the hypothesis. This together with Theorem D(iv) and (52) implies
\begin{equation}
\|W_{M_n}^{\Phi} \mathcal{H}_g(f M_n)\|_{A_\omega^q} \lesssim A_{\Phi M_n} \|\mathcal{H}_g(f M_n)\|_{A_\omega^q} \\
\lesssim \|\mathcal{H}_g\|_{(A_\omega^p, A_\omega^q)} M_n \omega \left(S(a_{M_n})\right)^{1/p}.
\end{equation}

On the other hand, bearing in mind the M. Riesz projection theorem, (49), (50), (53) and (43), we deduce
\begin{align*}
\|W_{M_n}^{\Phi M_n} \mathcal{H}_g(f M_n)\|_{A_\omega^q} &= \left\| \sum_{M_n \leq k \leq M_{n+1}-1} (k+1) b_{k+1} \left( \int_0^1 t^k f M_n(t) \, dt \right) \Phi_{M_n} \left( \frac{k}{M_n} \right) z^k \right\|_{A_\omega^q} \\
&= \left\| \sum_{M_n \leq k \leq M_{n+1}-1} (k+1) b_{k+1} \left( \int_0^1 t^k f M_n(t) \, dt \right) \varphi_{M_n, \omega} \left( \frac{k}{M_n} \right) z^k \right\|_{A_\omega^q} \\
&= \|\Delta_n^\omega g\|_{A_\omega^q},
\end{align*}

which together with (54), Lemma 16, Lemma A and (8) gives
\begin{align*}
\|\Delta_n^\omega g\|_{H^q} &\lesssim \|\mathcal{H}_g\|_{(A_\omega^p, A_\omega^q)} M_n^{1-\frac{1}{p}} \left( \tilde{\omega} \left( 1 - \frac{1}{M_n} \right) \right)^{1-p-\frac{1}{q}} \\
&\lesssim \|\mathcal{H}_g\|_{(A_\omega^p, A_\omega^q)} M_n^{1-\frac{1}{p}} 2^{-n \left( \frac{1}{p} - \frac{1}{q} \right)}.
\end{align*}

Finally, Corollary 12(i) implies $g \in \Lambda \left( q, \frac{1}{p}, \tilde{\omega}^{\frac{1}{p} - \frac{1}{q}} \right)$ and
\begin{equation}
\|g - g(0)\|_{\Lambda \left( q, \frac{1}{p}, \tilde{\omega}^{\frac{1}{p} - \frac{1}{q}} \right)} \lesssim \|\mathcal{H}_g\|_{(A_\omega^p, A_\omega^q)}.
\end{equation}

Let now $1 < q < p < \infty$ and assume that $\mathcal{H}_g : A_\omega^p \to A_\omega^q$ is bounded. Let $\{\phi_\rho\}$ and $\{Q_\rho\}$ be the families of functions considered in Lemma 19. Since each $Q_\rho$ is increasing on $[0, 1)$, Lemma 6 gives
\begin{equation}
\int_0^1 t^{M_n} Q_\rho(t) \, dt \asymp \int_0^1 t^{M_{n+1}} Q_\rho(t) \, dt, \quad n \in \mathbb{N}.
\end{equation}

So, Theorem 4, Lemma E and Lemma 10 imply
\begin{align*}
\|\mathcal{H}_g(\rho)\|_{A_\omega^q}^q &\asymp \sum_{n=0}^\infty 2^{-n} \|\Delta_n^\omega \mathcal{H}_g(\rho)\|_{H^q}^q \\
&\asymp \sum_{n=0}^\infty 2^{-n} \left( \int_0^1 t^{M_n} Q_\rho(t) \, dt \right)^q \|\Delta_n^\omega g\|_{H^q}^q \\
&\asymp \sum_{n=0}^\infty 2^{-n} \left( \int_0^1 t^{M_n} Q_\rho(t) \, dt \right)^q M_q^q \left( 1 - \frac{1}{M_{n+1}}, \Delta_n^\omega g\right).
\end{align*}
Since $Q_\rho$ is increasing on $[0,1)$, (45), the M. Riesz projection theorem and Lemma 10 yield

$$\int_0^1 t^{M_n} Q_\rho(t) \, dt \gtrsim \frac{Q_\rho \left( 1 - \frac{1}{M_{n+1}} \right)}{M_{n+1}}$$

(56)

$$\gtrsim \frac{M_{\rho}^{-\frac{q}{1-q}} \left( 1 - \frac{1}{M_{n+1}}, g_\rho' \right)}{M_{n+1}} \gtrsim \frac{\| \Delta_n g_\rho' \|^q_{L^q}}{M_{n+1}^{1+q}}.$$

So, by combining (55), (56), Corollary 11 and Lemma 19, we obtain

$$\| H_{g_\rho} (Q_\rho) \|^q_{A_p^q} \gtrsim \sum_{n=0}^{\infty} 2^{-n} \frac{1}{M_{n+1}} \| \Delta_n g_\rho' \|^q_{L^q}$$

(57)

$$\times \int_0^1 M_{q}^{s} (r, g_\rho' (1 - r)^{\left( 1 - \frac{1}{p} \right)} \omega(r) \, dr$$

$$\times \int_0^1 \phi_p^s (r) \widehat{\omega}(r) \, dr \asymp \| Q_\rho \|^p_{A_p^q}.$$

Further, (55) yields

$$\| H_{g_\rho} (Q_\rho) \|^q_{A_p^q} \leq C \| H_{g_\rho} (Q_\rho) \|^q_{A_p^q}, \quad 0 < \rho < 1,$$

where $C$ does not depend on $\rho$. This together with (57) gives

$$\infty > \| H_{g_\rho} \|^q_{(A_p^q, A_p^q)} \geq \frac{\| H_{g_\rho} (Q_\rho) \|^q_{A_p^q}}{\| Q_\rho \|^q_{A_p^q}} \gtrsim \frac{\| H_{g_\rho} (Q_\rho) \|^q_{A_p^q}}{\| Q_\rho \|^q_{A_p^q}}$$

$$\gtrsim \| Q_\rho \|^q_{A_p^q} \asymp \| g_\rho' \|^q_H \left( q, s, \omega \left( 1 - \frac{1}{q} \right) \right),$$

so, by letting $\rho \to 1^-$, we deduce

$$\| H_{g_\rho} \|^q_{(A_p^q, A_p^q)} \gtrsim \| g_\rho' \|^q_H \left( q, s, \omega \left( 1 - \frac{1}{q} \right) \right).$$

This finishes the proof.

7. Compact and Hilbert-Schmidt operators

7.1. Compactness. The main objective of this section is to prove the following result.

**Theorem 20.** Let $1 < p, q < \infty$, $\omega \in R \cap M_p$ and $g \in H(D)$.

(i) If $1 < p \leq q < \infty$, then $H_g : A_p^q \to A_p^q$ is compact if and only if $g \in \lambda \left( q, \frac{1}{p}, \frac{1}{q}, \frac{1}{p} - \frac{1}{q} \right).$

(ii) If $1 < q < p < \infty$, then $H_g : A_q^p \to A_p^q$ is compact if and only if it is bounded.
We will need the following lemma, which can be easily proved by using (2), Hölder’s inequality and (6).

**Lemma 21.** Let \(1 < p < \infty\) and let \(\omega\) be a radial weight such that (2) is satisfied. Let \(\{f_j\}_{j=1}^\infty\) be a sequence in \(A^p_\omega\) such that \(\sup_j \|f_j\|_{A^p_\omega} = K < \infty\) and \(f_j \to 0\), as \(j \to \infty\), uniformly on compact subsets of \(\mathbb{D}\). Then the following assertions hold:

(i) \(\lim_{j \to \infty} \int_0^1 |f_j(t)| \, dt = 0\);
(ii) \(\mathcal{H}_g(f_j) \to 0\), as \(j \to \infty\), uniformly on compact subsets of \(\mathbb{D}\) for each \(g \in \mathcal{H}(\mathbb{D})\).

Next, we remind the reader that for \(\omega \in \mathcal{I} \cup \mathcal{R}\), the norm convergence in \(A^p_\omega\) implies the uniform convergence on compact subsets of \(\mathbb{D}\) by \([20, \text{Lemma 2.5}]\).

This fact and Lemma 21 are the key tools in the proof of the following result whose proof will be omitted.

**Lemma 22.** Let \(1 < p < \infty\), \(0 < q < \infty\) and \(\omega \in \mathcal{I} \cup \mathcal{R}\) such that (2) is satisfied, and let \(g \in \mathcal{H}(\mathbb{D})\). Then the following conditions are equivalent:

(i) \(\mathcal{H}_g : A^p_\omega \to A^q_\omega\) is compact;
(ii) For each sequence \(\{f_j\}_{j=1}^\infty\) in \(A^p_\omega\) for which

\[
\sup_k \|f_j\|_{A^p_\omega} = K < \infty
\]

and

\[
f_j \to 0, \text{ as } j \to \infty, \text{ uniformly on compact subsets of } \mathbb{D},
\]

we have \(\lim_{j \to \infty} \|\mathcal{H}_g(f_j)\|_{A^q_\omega} = 0\).

**Proof of Theorem 20.** (i). Assume first that \(\mathcal{H}_g : A^p_\omega \to A^q_\omega\) is compact. Let \(\{f_{M_n}\}_{n=0}^\infty\) be the family of test functions

\[
f_{M_n}(z) = \frac{1}{\left( M_n^{\gamma+1} \omega(S(a_{M_n}))^{1/p} \right)^{1/p}} \frac{1}{(1 - a_{M_n} z)^{\frac{\gamma+1}{p}}}, \quad z \in \mathbb{D}, \quad n \in \mathbb{N},
\]

considered in (53). If \(\gamma\) is large enough, Lemma B ensures that \(\{f_{M_n}\}_{n=0}^\infty\) satisfies (58). Now the proof of Lemma [20, Lemma 1.1] shows that \(\lim_{|a| \to 1^{-}} \frac{1 - |a|}{\omega(|a|)} = 0\), if \(\gamma > 0\) is again large enough. So, if \(\gamma\) is fixed appropriately, then

\[
\lim_{n \to \infty} f_{M_n}(z) = \lim_{n \to \infty} \frac{1}{\left( M_n^{\gamma+1} \omega(1 - a_{M_n} z)^{\frac{\gamma+1}{p}} \right)^{1/p}} = 0
\]

uniformly on compact subsets of \(\mathbb{D}\). Thus \(\{f_{M_n}\}_{n=0}^\infty\) satisfies (59). Therefore Lemma 22 implies

\[
\lim_{n \to \infty} \|\mathcal{H}_g(f_{M_n})\|_{A^q_\omega} = 0.
\]
Next, a careful inspection of the proof of the necessity part of Theorem 1 reveals the inequalities
\[
\|\Delta_n^w g'\|_{H^q} \lesssim \|H_g(f_{M_n})\|_{A^p_n} M_n^{1-\frac{2}{|\nu|}} (1 - \frac{1}{M_n})^{\frac{1}{p} - \frac{q}{q}} \\
\lesssim \|H_g(f_{M_n})\|_{A^p_n} M_n^{1-\frac{2}{|\nu|}} 2^{-n(\frac{1}{p} - \frac{q}{q})}, \quad n \in \mathbb{N}.
\]
Finally, (60) and Corollary 12(ii) imply \( \varepsilon > 0 \) and \( g \in \lambda \left( q, \frac{1}{p}, \frac{1}{q} \right) \).

Conversely, let \( \varepsilon > 0 \) and \( g \in \lambda \left( q, \frac{1}{p}, \frac{1}{q} \right) \). Then there exists \( r_0 = r_0(\varepsilon) \in [0, 1) \) such that
\[
M_q^n(r, g') \leq \varepsilon \frac{\tilde{\omega}^{\frac{1}{p}-1}(r)}{(1 - r)^{q(1-\frac{q}{p})}}, \quad r \geq r_0.
\]
Let now \( k_0 \) be the integer which appears in the proof of the sufficiency part of Theorem 1, and choose \( n_0 \geq k_0 \), such that
\[
1 - \frac{1}{M_{n+1}} \geq r_0, \quad n \geq n_0.
\]
Let \( \{f_j\} \) be a sequence of analytic functions in \( \mathbb{D} \) satisfying (58) and (59). By arguing as in the proof of Theorem 1 and bearing in mind that \( \lambda \left( q, \frac{1}{p}, \frac{1}{q} \right) \) \( \Lambda \left( q, \frac{1}{p}, \frac{1}{q} \right) \), we deduce
\[
\sum_{n=0}^{n_0} 2^{-n} \|\Delta_n^w H_g(f_j)\|_{H^q}^q \lesssim \|f_j\|_{A^p_n}^p \sum_{n=0}^{n_0} 2^{-n} \left( \int_0^1 \frac{t^{M_n}}{t^{\frac{q}{p}}} |f_j(t)| dt \right)^p M_{n+1}^{p-1}
\]
\[
\lesssim \left( \int_0^1 |f_j(t)| dt \right)^p,
\]
where the constants of comparison depend on \( g, p, \omega, K \) and \( n_0 \).

On the other hand, an analogous reasoning to that in (36), (61), (62), (37) and (58) give
\[
\|\Delta_n^w H_g(f_j)\|_{H^q}^q \lesssim \left( \int_0^1 \frac{t^{M_n}}{t^{\frac{q}{p}}} |f_j(t)| dt \right)^p \|\Delta_n^w g'\|_{H^q}^q
\]
\[
\lesssim \varepsilon \|f_j\|_{A^p_n}^q \left( \int_0^1 \frac{t^{M_n}}{t^{\frac{q}{p}}} |f_j(t)| dt \right)^p M_{n+1}^{p-1}
\]
\[
\leq \varepsilon K^{q-p} \left( \int_0^1 \frac{M_n}{t^{\frac{q}{p}}} |f_j(t)| dt \right)^p M_{n+1}^{p-1},
\]
so bearing in mind that \( n_0 \geq k_0 \), Corollary 3, (58) and following the proof of Theorem 1, we get
\[
\sum_{n=n_0+1}^{\infty} 2^{-n} \|\Delta_n^w H_g(f_j)\|_{H^q}^q \lesssim \varepsilon K^{q-p} \left\|\widetilde{H}(f_j)\right\|_{A^p_n}^p \lesssim \varepsilon K^{q-p} |f_j|_{A^p_n}^p \lesssim \varepsilon K^q.
\]
This together with (34), (63) and Lemma 21, imply
\[
\lim_{j \to \infty} \|H_\omega(f_j)\|_{A^p_\omega}^p \leq \lim_{j \to \infty} \left( \left( \int_0^1 |f_j(t)|^p \, dt \right) + \epsilon K^q \right) \lesssim \epsilon K^q.
\]

Since \(\epsilon > 0\) is arbitrary, Lemma 22 shows that \(H_\omega : A^p_\omega \to A^q_\omega\) is compact.

(ii). By Theorem 1(ii) it is enough to show that \(H_\omega : A^p_\omega \to A^q_\omega\) is compact if \(g' \in H(q, s, \omega^{(1-q)})\). Let \(\{f_j\}\) be a sequence of analytic functions in \(D\) satisfying (58) and (59). Let \(\epsilon > 0\) be given. By the proof of Corollary 11, there exists \(n_0 \in \mathbb{N}\) such that
\[
\left[ \sum_{n=n_0}^{\infty} 2^{-n} \left( \int_0^1 t^\frac{M_0}{p} |f_j(t)|^p \, dt \right) \right]^{\frac{q}{p}} \lesssim \epsilon^{q/p}
\]
for all \(n \geq n_0\). By Hölder’s inequality, (58) and a reasoning similar to that in the proof of Theorem 1, we obtain
\[
\sum_{n=n_0}^{\infty} 2^{-n} \left( \int_0^1 t^\frac{M_0}{p} |f_j(t)|^p \, dt \right) \lesssim \left( \sum_{n=n_0}^{\infty} 2^{-n} \left( \int_0^1 t^\frac{M_0}{p} |f_j(t)|^p \, dt \right) \right) \lesssim \epsilon^{q/p}
\]
and hence by (34) and Lemma 17
\[
\|H_\omega(f_j)\|_{A^q_\omega}^q \lesssim \left( \int_0^1 |f_j(t)|^q \, dt \right)^{\frac{q}{p}} \sum_{n=n_0}^{\infty} \|\Delta^\omega_n g'\|_{H^q}^q
\]
Finally, since \(\epsilon > 0\) is arbitrary and \(n_0 \in \mathbb{N}\) fixed, Lemma 21 gives
\[
\lim_{j \to \infty} \|H_\omega(f_j)\|_{A^q_\omega} = 0,
\]
which together with Lemma 22 finishes the proof.
7.2. Hilbert-Schmidt operators. In this section we offer a characterization of those symbols $g$ for which the operator $H_g$ is Hilbert-Schmidt on $A^2_\omega$, where $\omega \in \mathcal{R} \cap \mathcal{M}_2$. Recall that the classical Dirichlet space consists of those functions $g \in \mathcal{H}(\mathbb{D})$ for which 

$$
\|g\|_D^2 = |g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 \, dA(z) < \infty.
$$

**Theorem 23.** Let $\omega \in \mathcal{R} \cap \mathcal{M}_2$ and $g \in \mathcal{H}(\mathbb{D})$. Then $H_g$ is Hilbert-Schmidt on $A^2_\omega$ if and only if $g \in D$.

**Proof.** Denote 

$$
\omega_n = \int_0^1 r^{2n+1} \omega(r) \, dr, \quad e_n(z) = \frac{z^n}{\sqrt{\omega_n}}, \quad n \in \mathbb{N},
$$

and consider the basis $\{e_n\}$ of $A^2_\omega$. If $g(z) = \sum_0^\infty b_k z^k \in \mathcal{H}(\mathbb{D})$, then 

$$
\|H_g(e_n)\|_{A^2_\omega}^2 = \frac{1}{2\omega_n} \sum_0^\infty \frac{(k+1)^2 |b_{k+1}|^2 \omega_k}{(n+k+1)^2}.
$$

We claim that 

$$
(64) \quad \sum_0^\infty \frac{1}{(n+k+1)^2 \omega_n} \asymp \frac{1}{(k+1)\omega_k}, \quad k \in \mathbb{N}.
$$

So, assuming this for a moment, we deduce 

$$
\sum_0^\infty \|H_g(e_n)\|_{A^2_\omega}^2 = \frac{1}{2\omega_n} \sum_0^\infty \frac{(k+1)^2 |b_{k+1}|^2 \omega_k}{n+k+1} 
= \frac{1}{2} \sum_0^\infty (k+1)^2 |b_{k+1}|^2 \omega_k \sum_0^\infty \frac{1}{(n+k+1)^2 \omega_n} 
\asymp \sum_0^\infty (k+1) |b_{k+1}|^2 \asymp \|g-g(0)\|_D^2,
$$

which proves the assertion. It remains to prove (64). Clearly,

$$
(65) \quad \sum_0^k \frac{1}{(n+k+1)^2 \omega_n} \geq \frac{1}{\omega_k} \sum_0^\infty \frac{1}{(n+k+1)^2} \asymp \frac{1}{(k+1)\omega_k}, \quad k \in \mathbb{N}.
$$

On the other hand,

$$
(66) \quad \sum_0^k \frac{1}{(n+k+1)^2 \omega_n} \leq \frac{1}{\omega_k} \sum_0^k \frac{1}{(n+k+1)^2} \asymp \frac{1}{(k+1)\omega_k}, \quad k \in \mathbb{N}.
$$

Moreover, since $\omega \in \mathcal{M}_2$, Lemma 7 yields

$$
\int_0^1 \frac{dt}{\omega(t)} \asymp \frac{(1-r)}{\omega(r)},
$$
which together with Lemma A(i) and (iv) gives
\[
\sum_{n=k+1}^{\infty} \frac{1}{(n+k+1)^2 \omega_n} \leq \sum_{n=k+1}^{\infty} \frac{1}{(n+1)^2 \omega_n} \times \sum_{n=k+1}^{\infty} \frac{1}{n} \int_{1-\frac{1}{n}}^{1} \frac{dt}{\omega(t) \left(1 - \frac{1}{k+1}\right)} \times \frac{1}{(k+1) \omega_k}.
\]
This combined with (66) and (65) yields (64) and finishes the proof. □

8. Further results

8.1. Descriptions of weighted spaces. In view of [8, Theorem 5.4] it is natural to expect that, under appropriate assumptions, the space \( \Lambda \left(q, \alpha, \hat{\omega}^\eta \right) \) could be characterized by a weighted \( q \)-mean Lipschitz condition. We show that this is indeed the case for those spaces to which the containment of the symbol \( g \) characterizes the boundedness of \( H_g : A_p^\omega \to A_q^\omega \) when \( 1 < p \leq q < \infty \) and \( \omega \in \mathcal{R} \cap \mathcal{M}_p \).

**Proposition 24.** Let \( 1 < q, p < \infty, \eta \in \left[0, \frac{1}{p}\right) \), \( \omega \in \mathcal{R} \cap \mathcal{M}_p \) and \( g \in \mathcal{H}(\mathbb{D}) \). The following assertions hold:

(i) \( g \in \Lambda \left(q, \frac{1}{p}, \hat{\omega}^\eta \right) \) if and only if \( g \in H^q \) and
\[
\sup_{0 < h \leq t} \left( \int_{0}^{2\pi} |g(e^{i(\theta+h)}) - g(e^{i\theta})|^q \frac{d\theta}{2\pi} \right)^{1/q} = O(t \hat{\omega}^\eta(1-t)), \quad t \to 0.
\]
Moreover,
\[
\|g\|_{\Lambda \left(q, \frac{1}{p}, \hat{\omega}^\eta \right)} \asymp |g(0)| + \sup_{0 < h \leq t} \left( \int_{0}^{2\pi} |g(e^{i(\theta+h)}) - g(e^{i\theta})|^q \frac{d\theta}{2\pi} \right)^{1/q}.
\]

(ii) \( g \in \lambda \left(q, \frac{1}{p}, \hat{\omega}^\eta \right) \) if and only if \( g \in H^q \) and
\[
\sup_{0 < h \leq t} \left( \int_{0}^{2\pi} |g(e^{i(\theta+h)}) - g(e^{i\theta})|^q \frac{d\theta}{2\pi} \right)^{1/q} = o(t \hat{\omega}^\eta(1-t)), \quad t \to 0.
\]

**Proof.** The proof of (i) consists of a direct application of [2, Theorem 2.1(i)]. First, observe that if \( g \in \Lambda \left(q, \frac{1}{p}, \hat{\omega}^\eta \right) \), then \( g \in H^q \). Now, if we choose \( \varrho(t) = t \hat{\omega}^\eta(1-t) \), \( 0 \leq t < 1 \), it suffices to show that \( \varrho \) satisfies both, the Dini condition
\[
(67) \quad \int_{0}^{t} \frac{\varrho(s)}{s} \, ds \lesssim \varrho(t), \quad 0 < t < 1,
\]
and the \( b_1 \)-condition
\[
(68) \quad \int_{t}^{1} \frac{\varrho(s)}{s^2} \, ds \lesssim \frac{\varrho(t)}{t}, \quad 0 < t < 1.
\]
By Lemma A(i), we deduce the inequality
\[
\int_0^t s^{\frac{1}{p}-1} \left( \frac{\tilde{\omega}(1-s)}{\tilde{\omega}(1-t)} \right)^{\eta} ds \leq \frac{1}{t^\eta} \int_0^t s^{\frac{1}{p}+\alpha\eta-1} ds = \frac{t^{\frac{1}{p}}}{t^{\frac{1}{p}+\alpha\eta}},
\]
which is equivalent to (67). Moreover, by using the fact that \( \tilde{\omega}(r)^{\frac{1}{r}} \) is essentially increasing (see the proof of Lemma 8(iii)) and again Lemma A(i), we obtain
\[
\int_t^1 \frac{g(s)}{s^2} ds = \int_0^{1-t} \frac{\tilde{\omega}(r)^{\eta}}{(1-r)^{2-\frac{1}{p}}} dr \leq \frac{\tilde{\omega}(1-t)^{\frac{1}{p}}}{t^{1-\frac{1}{p}}} \int_0^{1-t} \frac{dr}{(1-r)^{\frac{1}{p}-\eta}(1-r)}
\]
\[
\leq \frac{\tilde{\omega}^\eta(1-t)t^{\alpha\left(\frac{1}{p}-\eta\right)}}{t^{1-\frac{1}{p}}} \int_0^{1-t} \frac{dr}{(1-r)^{1+\alpha\left(\frac{1}{p}-\eta\right)}}
\]
\[
\leq \frac{\tilde{\omega}^\eta(1-t)}{t^{1-\frac{1}{p}}} = \frac{g(t)}{t}, \quad 0 < t < 1,
\]
which is (68).

With the proof of [2, Theorem 2.1] in hand, the second assertion (ii) can be proved in an analogous manner with minor modifications. \(\square\)

If we choose \( \omega(r) = (1-r^2)^\alpha \), where \(-1 < p-2 < \alpha < \infty\), then Theorem 1(i) and Proposition 24 show that \( \mathcal{H}_g : A_p^\alpha \to A_q^\alpha, q \geq p \), is bounded if and only if \( g \) belongs to the mean Lipschitz space \( \Lambda \left( q, \frac{2+\alpha}{p} - \frac{1+\alpha}{q} \right) \).

With respect to the condition that characterizes the bounded operators \( \mathcal{H}_g : A_p^\alpha \to A_q^\alpha \), when \( 1 < q < p < \infty \) and \( \omega \in \mathcal{R} \cap \mathcal{M}_p \), it is worth noticing that

\[
g' \in H \left( q, s, \tilde{\omega}_{s(1-\frac{1}{q})} \right) \iff g \in H \left( q, s, \tilde{\omega}_{-\frac{2}{q}} \right)
\]

by [19, Theorem 1.1], provided that \( (1-r)^{-\frac{2}{q}} \tilde{\omega}(r) \in \mathcal{R} \). This last requirement may happen only if \( q < p-1 \), because

\[
(1-r)^{-\frac{2}{q}} \tilde{\omega}(r) = \frac{\tilde{\omega}(r)}{(1-r)^{p-1}} \frac{1}{(1-r)^{1+\frac{p-q}{p-q}-p}}
\]
and \( \frac{\tilde{\omega}(r)}{(1-r)^{p-1}} \) is essentially increasing. In particular, the previous argument says that (69) does not hold if \( 1 < p \leq 2 \). It is also worth noticing that for the standard weight \( \omega(r) = (1-r)^\alpha \) that belongs to \( \mathcal{M}_p \), the equivalence (69) is satisfied when \( p-2 > \alpha > \frac{p}{p-q} - 2 \).
8.2. Analysis on the Muckenhoupt type condition. We saw in Theorem 2 that the Muckenhoupt type condition (3) characterizes the boundedness of both the Hilbert operator $H$ and the sublinear Hilbert operator $\tilde{H}$ from $L^p_\omega$ to $A^p_\omega$ whenever $\omega$ is regular and satisfies the integral condition (2), and further, that the quantity $M_p(\omega)$ is comparable to the operator norm in both cases. Both integral conditions (2) and (3) restrict the behavior of the inducing weight $\omega$ in their own way and thus also affect to the nature of the spaces $L^p_\omega$ and $A^p_\omega$ as well. To understand these conditions better, we compare them to the pointwise behavior of the quotient $\psi_\omega(r)/(1-r)$ appearing in the definitions of the regular and rapidly increasing weights.

Lemma 25. Let $\omega$ a continuous radial weight and $1 < p < \infty$.

(i) If (2) holds and
\[ \liminf_{r \to 1} \frac{\psi_\omega(r)}{1-r} > \frac{1}{p-1}, \]
then $\omega \in M_p$.

(ii) If (2) holds and
\[ \lim_{r \to 1} \frac{\psi_\omega(r)}{1-r} = \frac{1}{p-1}, \]
then $\omega \notin M_p$.

(iii) If there exists $r^* \in (0,1)$ such that
\[ \frac{\psi_\omega(r)}{1-r} \leq \frac{1}{p-1}, \quad r^* \leq r < 1, \]
then (2) does not hold.

Proof. (i). By the assumption, there exist $d > \frac{1}{p-1}$ and $r_0 \in (0,1)$ such that
\[ \frac{\psi_\omega(r)}{1-r} \geq d \quad \text{on } [r_0,1). \]
Therefore the differentiable function $h_d(r) = \frac{\hat{\omega}(r)}{(1-r)^{p-1}}$ is increasing on $[r_0,1)$, and hence
\[ \hat{\omega}(r) \lesssim \left( \frac{1-r}{1-t} \right)^{\frac{p-1}{2}} \hat{\omega}(t), \quad 0 \leq r \leq t < 1. \]

It follows that
\[ \int_r^1 \hat{\omega}(t)^{-\frac{1}{p-1}} dt \lesssim \hat{\omega}(r)^{-\frac{1}{p-1}} (1-r)^{\frac{1}{p-1}} \int_r^1 (1-t)^{-\frac{1}{p-1}} dt \asymp (1-r)\hat{\omega}(r)^{-\frac{1}{p-1}}. \]
Since trivially,
\[ \int_r^1 \hat{\omega}(t)^{-\frac{1}{p-1}} dt \geq (1-r)\hat{\omega}(r)^{-\frac{1}{p-1}}, \]
we deduce $\hat{\omega}^{-\frac{1}{1-1}} \in \mathcal{R}$, and thus $\omega \in M_p$ by Lemma 7.

(ii). The assertion follows by the Bernouilli-l’Hôpital theorem and Lemma 7(i).

(iii). The assumption yields
\[ \hat{\omega}(r) \gtrsim \left( \frac{1-r}{1-t} \right)^{p-1} \hat{\omega}(t), \quad 0 \leq r \leq t < 1, \]
and hence
\[ \int_r^1 \hat{\omega}(t)^{-\frac{1}{p-1}} dt \gtrsim \hat{\omega}(r)^{-\frac{1}{p-1}} (1-r) \int_r^1 \frac{dt}{1-t} = \infty. \]

\[ \Box \]

It is worth noticing that there exists regular weights \( \omega \) such that \( \lim_{r \to 1^{-}} \frac{\psi_r(\omega)}{1-r} \) does not exist. The weight \( \omega \), defined by the identity
\[ \int_r^1 \omega(s) \, ds = 2(1-r) \cos \left( \frac{1}{(1-r)^{1/2}} \right) + 16(1-r)^{1/2}, \]
gives is a concrete example.

The bigger the limit \( \lim_{r \to 1^{-}} \frac{\psi_r(\omega)}{1-r} \) is (if it exists), the smaller the space \( A^p_{\omega} \) is. Therefore, in view of Lemma 25, Theorem 1(i) says, roughly speaking, that the Hilbert operator \( \mathcal{H} \) is well defined and bounded on \( A^p_{\omega} \) for \( \omega \in \mathcal{R} \) whenever the space is small enough. It is known that the weighted Bergman space \( A^p_{\omega} \) induced by a rapidly increasing weight \( \omega \) lies closer to the Hardy space \( H^p \) than any classical weighted Bergman space \( A^p_{\omega} \). Therefore, by the observation above and results in [9], it is natural to expect that if \( \omega \in \mathcal{I} \) (and satisfies some local regularity requirement), then \( \mathcal{H}_g \) is bounded on \( A^p_{\omega} \) if and only if \( g \) belongs to the mean Lipschitz space \( \Lambda \left( p, \frac{1}{p} \right) \). The proof of Theorem 1 with minor modifications show that \( g \in \Lambda \left( p, \frac{1}{p} \right) \) is indeed a necessary condition for \( \mathcal{H}_g : A^p_{\omega} \to A^p_{\omega} \) to be bounded when \( \omega \in \mathcal{I} \). It is also appropriate to mention that the question of characterizing the bounded operators \( \mathcal{H}_g \) on \( A^p_{\omega} \) with \( \omega \in \mathcal{I} \) is more likely related to the open problem of describing those \( g \in \mathcal{H}(\mathbb{D}) \) such that \( \mathcal{H}_g : H^p \to H^p \) is bounded in the case \( 2 < p < \infty \) [9].

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