Core-Periphery in Networks: An Axiomatic Approach

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November 11, 2014

Abstract

Recent evidence shows that in many societies worldwide the relative sizes of the economic and social elites are continuously shrinking. Is this a natural social phenomenon? What are the forces that shape this process? We try to address these questions by studying a Core-Periphery social structure composed of a social elite, namely, a relatively small but well-connected and highly influential group of powerful individuals, and the rest of society, the periphery. Herein, we present a novel axiom-based model for the forces governing the mutual influences between the elite and the periphery. Assuming a simple set of axioms, capturing the elite’s dominance, robustness, compactness and density, we are able to draw strong conclusions about the elite-periphery structure. In particular, we show that a balance of powers between elite and periphery and an elite size that is sub-linear in the network size are universal properties of elites in social networks that satisfy our axioms. We note that the latter is in controversy to the common belief that the elite size converges to a linear fraction of society (most recently claimed to be 1%). We accompany these findings with a large scale empirical study on about 100 real-world networks, which supports our results.
1 Introduction

In his book *Mind and Society* [24], Vilfredo Pareto wrote what is by now widely accepted by sociologists: “Every people is governed by an *elite*, by a chosen element of the population”. Indeed, with the exception of some rare examples of utopian or totally egalitarian societies, almost all societies exhibit an (often radically) uneven distribution of power, influence and wealth among their members, and, in particular, between the *elite* and its complement, often called the *periphery* or the *masses*. Typically, the elite is small, powerful and influential and the periphery is larger, less organized and less dominant. This division is usually referred to as a *core-periphery* partition [4]1 and the problem of identifying this partition has recently received increased interest [26, 32, 16]. However, while the core-periphery structure is perhaps the most high-level structure of society, it has so far not received a satisfactory *quantitative* definition. In fact, it seems unlikely that a single formal definition exists that suits all elite types in all social networks. Moreover, it is not clear that the elite must be unique.

To illustrate this last point, let us consider the following mental experiment. Sort the members of society by decreasing order of influence2 and add them to a set $\mathcal{E}$ (representing the intended elite) one by one in this order. After the first step, $\mathcal{E}$ contains only one member of society, albeit the most influential one, hence clearly it cannot yet be thought of as “the elite” - it simply has insufficient power. This holds true also for the next few sets obtained in this way. On the other extreme, if the process is continued to its conclusion, we end up with $\mathcal{E}$ containing the entire society, which is clearly too large to be considered “the elite”. The question is therefore: at which point along this process does $\mathcal{E}$ qualify as an elite?

Intuitively, the “break-point” where the process should be halted is the point where adding new members into $\mathcal{E}$ no longer serves to significantly strengthen the group but rather “dilutes” its power (relative to its size). This point is not well-defined, and depends to some extent on the specific circumstances of the society. In this paper we propose a concrete choice for this break-point, referred to as the *power symmetry point* of the elite. The core and periphery are at their power symmetry point if the overall influence of the elite (nearly) equals that of the periphery.

Our main contribution is a characterization of elites, i.e., a set of properties (formulated as “axioms”) that any definition for an elite must adhere to. While this characterization does not lead to pinpointing a single definition for the elite, it narrows down the range of subsets

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1Hereafter we use the terms *elite* (mostly used for social networks) and *core* (used for any network) interchangeably.

2The exact definition of “influence” is immaterial here, and will be discussed later on.
Let us next describe two of our main conclusions. The first result, stated in Theorem 3.1, is that elites satisfying the axioms of our model are at their power symmetry point. This implies that the power of the periphery is similar to that of the elite, or in other words, that as society evolves, the “natural” core-periphery partition maintains a balance between these two groups.

Our second conclusion applies to the size of the elite. Recent reports show that the gap between the richest people and the masses keeps increasing, and that decreasingly fewer people amass more and more wealth [13, 23]. Claims like “The top 10 percent no longer takes in one-third of our income – it now takes half,” made by President Obama [28] when recently addressing the issue, are interpreted as implying that the economic and political
elites become increasingly more greedy and overbearing. Such claims are often used in order to criticize governments and regulatory financial institutions for neglecting to cope with this disturbing development. The question raised by us is: can society help it, or is this phenomenon an unavoidable by-product of some inherent natural properties of society? We claim that in fact, one can predict the shrinkage of elite size with time (as a fraction of the entire society size) based on the very nature of social elites. In particular, in our model, such shrinkage is the natural result of a combination of three facts: First, society grows. Second, elites are denser than peripheries (informally meaning that they are much better connected). Third, the size of a dense elite at the power symmetry point is sub-linear in the size of the society. Combining these facts implies that the fraction of the total population size comprised by dense elites will decrease as the population grows with time. We prove this formally in Theorem 4.2. The empirical evidence we present in this work lends additional support to the above claim.

A dual question we are interested in concerns the stable size of the elite in a growing society: How small can the elite be while still maintaining its inherent properties? We prove that under our model, an elite cannot be smaller than $\Omega(\sqrt{m})$, where $m$ is the number of network edges.

Consequently, we assert that the elite’s symmetry of powers and sublinear size properties should indeed join the growing list of universal properties of social networks established in recent years, such as short average path lengths (a.k.a. the “small world” phenomenon), high clustering coefficients, heavy-tailed degree distributions (e.g., scale-free networks), navigability, and more recently dynamic properties such as densification and a shrinking diameter.

As a small illustrative example for the meaning of our terms we consider, in Figure 1(a)-(c), the network of top 139 Marvel’s superheroes and the 924 links interconnecting them (where two heroes are connected by a link if they appeared together in a story). We partitioned this network into a core and a periphery as shown by the colors in Figure 1(a). Several striking features can be clearly observed in this figure. First, the core (containing, e.g., Captain America, Spiderman and Thor) is dense and organized while the periphery is much sparser and less structured. Second, despite their considerable size difference, both the elite and the periphery have almost the same number of internal edges ($\approx 250$), thus exhibiting what we refer to as a symmetry of powers. Third, the number of “crossing” edges connecting the core to the periphery is almost twice as large (425), reaching most of the

3A property of a class of networks is universal if it keeps appearing in different types of networks and contexts.

4The partitioning methods used, briefly described later on, are tangential to the current discussion.
vertices in the periphery. Last, the size of the core is “only” 27 vertices (with 112 vertices in the periphery). We argue, and support by evidence, that it makes sense to consider 27 as about $\sqrt{924}$, and more generally, view the core size as roughly the square root of the number of edges in the network. For almost all networks, a core of size about $\sqrt{m}$ (where $m$ is the number of edges) will not grow as a linear fraction of the number of vertices.

The rest of the paper is organized as follows. Section 2 presents our model of influence and axioms. Section 3 discusses the property of power symmetry (Appendix B expands on power symmetry in random graphs.) In Section 4, we analyze the size properties of the elite. Section 5 presents our empirical results. Related work is provided in Section 6, and finally, we conclude with a discussion in Section 7.

2 An Axiomatic Approach

The conceptual approach we adopt towards studying core-periphery properties diverges from the established traditions in the field of social networks. The common approach to explaining empirical results on social networks is based on providing a new concrete (usually random) evolutionary model and comparing its predictions to the observed data. In contrast, we propose an axiomatic approach to the questions at hand. This approach is based on postulating a small set of axioms, capturing certain expectations about the network structure and the basic properties that an elite must exhibit in order to maintain its power in the society. Two main advantages of the axiomatic approach are that a suitable set of axioms attaches an “interpretation” or “semantics” to observed phenomena, and moreover, once agreeing on the axioms, it becomes possible to draw conclusions using logical arguments. For example, it may become possible to infer some information on the asymptotic behavior of a growing network, which is not always clear from empirical findings.

The framework presented here for describing the elite-periphery structure in a social network revolves around the fundamental notion of influence among groups of vertices. The underlying assumption is that the elite has more influence than the rest of the population, allowing it on the one hand to control the rest of the population, and on the other to protect its members from being controlled by others outside the elite. We refer to these two capabilities, respectively, as dominance and robustness.

\footnote{In fact, such growth can only occur in near-complete networks, which are rarely seen in real life.}
2.1 Influence and Core-Periphery Partition

We consider influence to be a measurable quantity between any two (not necessarily distinct) groups of vertices $X$ and $Y$, abstractly denoted by $\mathcal{I}(X,Y)$. The groups $X$ and $Y$ do not necessarily have to be distinct, and in particular, we are also interested in the internal influence exerted by the vertices of a group $X$ on themselves, denoted $\mathcal{I}(X,X)$.

Two central strengths of an elite in a given society are its dominance on the rest of society on the one hand, and its resistance to influence by the rest of society on the other. We will quantify these two aspects shortly. Denote the elite by a set $\mathcal{E}$ and the rest of society (the “periphery”) by $\mathcal{P}$. We call such a pair $(\mathcal{E}, \mathcal{P})$ a core-periphery partition, and will be interested in the four basic influence quantities $\mathcal{I}(\mathcal{E}, \mathcal{E}), \mathcal{I}(\mathcal{P}, \mathcal{P}), \mathcal{I}(\mathcal{E}, \mathcal{P})$ and $\mathcal{I}(\mathcal{P}, \mathcal{E})$.

Let us next lend the abstract notion of influence in social networks a concrete interpretation. A social network is modeled as a graph $G = (V,E)$, with a set $V$ of $n$ vertices representing the members of society, connected by a set $E$ of $m$ edges. In a social network, a network edge represents some social relation between the two connected vertices, such as friendship, citations, following on Twitter, etc. For our purposes, we may abstract and unify the interpretation of edges by simply stating that an edge connecting two vertices represents some kind of a channel of influence between the two vertices. To reflect the self-influence of every individual’s opinion on itself, we assume that each vertex has a self-loop, namely, an edge connecting it to itself.

For every vertex $v$ and set of vertices $X$, denote the set of edges connecting $v$ to vertices in $X$ by $E(v,X)$. Similarly, for vertex sets $X,Y \subseteq V$, let $E(X,Y)$ denote the set of edges connecting vertices in $X$ to vertices in $Y$. Define the degree of $v$ with respect to $X$ as $d_X(v) = |E(v,X)|$.

Given a partition of $V$ to an elite $\mathcal{E}$ and a periphery $\mathcal{P}$ sets, we can now partition the edge set $E$ to four edge sets $E(\mathcal{E}, \mathcal{E}), E(\mathcal{E}, \mathcal{P}), E(\mathcal{P}, \mathcal{E})$ and $E(\mathcal{P}, \mathcal{P})$.

$$
\begin{bmatrix}
E(\mathcal{E}, \mathcal{E}) & E(\mathcal{E}, \mathcal{P}) \\
E(\mathcal{P}, \mathcal{E}) & E(\mathcal{P}, \mathcal{P})
\end{bmatrix}
$$

These sets correspond to the four basic parts of the block-model matrix representation [14] of the adjacency matrix of a core-periphery network [6]. The adjacency matrix of the
network $G$ can be now written as in the figure to the right. See also Figure 1(d) for an example of such representation of the Marvel superhero network.

Now, for $X, Y \subseteq V$, define the influence of $X$ on $Y$ as

$$I(X,Y) = |E(X,Y)|.$$  

Herein we consider only undirected graphs where $I(X,Y) = I(Y,X)$\footnote{A more elaborate model may allow also for the possibility of directed edges, representing uni-directional influence; this extension of our framework is left for future study.}. Also note that due to the existence of self-loops, if an vertex $v$ belongs to both $X$ and $Y$, then $(v,v) \in E(X,Y)$. Hence in particular, $I(X,X) \geq |X|$ for every $X \subseteq V$.

A major question that we study is what a natural $(\mathcal{E}, \mathcal{P})$ partition is and what are the properties of the basic influence quantities. We use the following additional definitions.

Define the total influence of a group $X$ in society as the sum of its internal and external influence on the society:

$$I(X) = I(X,X) + I(X,V \setminus X).$$
Alternatively, \( I(X) = \sum_{v \in X} d_V(v) \).

Note that for undirected graphs \( I(\mathcal{E}, \mathcal{P}) = I(\mathcal{P}, \mathcal{E}) \), so by Eq. (2)

\[ I(\mathcal{E}) = I(\mathcal{P}) \implies I(\mathcal{E}, \mathcal{E}) = I(\mathcal{P}, \mathcal{P}). \]

### 2.2 Core-Periphery Axioms

We now propose and state a set of four simple axioms to capture elite and periphery properties of a \((\mathcal{E}, \mathcal{P})\) partition, illustrated in Figure 2 (A1)-(A4). Intuitively, to dominate the rest of society, the elite \( \mathcal{E} \) aspires to maintain a large amount of external influence on the periphery \( \mathcal{P} \), higher than or at least comparable to the internal influence that the periphery has on itself. Similarly, to maintain its robustness, hold its position and stick to its opinions, the elite must be able to resist “outside” pressure in the form of external influence. To achieve that, the elite \( \mathcal{E} \) must maintain the internal influence that it has on itself higher than (or at least not significantly lower than) the external influence exerted on it by the periphery. Both high dominance and high robustness are essential for the elite to be able to maintain its superior status in society. Moreover, all other things being equal, one may expect the elite size to tend to be as small as possible. In social terms this may be motivated by the idea that if membership in the elite entails benefits, then maintaining the elite size as small as possible will increase the share coming to each of its members. We express these requirements in the form of the following three axioms.

Let \( c_d \) and \( c_r \) be two positive constants.

**(A1) Dominance:** The elite’s influence dominates the periphery, or formally:

\[ I(\mathcal{E}, \mathcal{P}) \geq c_d \cdot I(\mathcal{P}, \mathcal{P}). \]

**(A2) Robustness:** The elite can withstand outside influence from the periphery, or formally:

\[ I(\mathcal{E}, \mathcal{E}) \geq c_r \cdot I(\mathcal{P}, \mathcal{E}). \]

**(A3) Compactness:** The elite is a minimal set satisfying the dominance and robustness axioms (A1) and (A2).

The forth axiom states that the elite members are better connected among themselves than the periphery members. This assertion is justified by some of the classical elite definitions, which state the elite is a “clique” where “everyone knows everyone”. (In fact, having high density is a weaker requirement than being a clique.)
Formally, define the density of a set $X \subseteq V$ as $\delta_X = \log|E(X,X)|/\log|X|$. (Written differently, this says that the number of edges internal to $X$ is $|E(X,X)| = |X|^{\delta_X}$.)

**(A4) Density**: The elite is denser than the whole network, namely, $\delta_V/\delta_E < 1$. 

Next, we discuss the implications of our axioms on power symmetry and elite size in social networks satisfying our axioms.

## 3 Core-Periphery Power Symmetry

A key notion in this paper is the power symmetry point. Consider a social network $G(V,E)$ and assume some ordering $\pi$ (e.g., one reflecting influence via degrees or other centrality measures) on the vertices of $V$. Start with the elite defined as the empty set, and the periphery containing all the vertices of the network, namely, $\mathcal{E} = \emptyset$ and $\mathcal{P} = V$. One by one, move the vertices from the periphery to the elite, according to the given ordering $\pi$. As this transition evolves, the influences of the elite and the periphery undergo a gradual shift, where the internal influence $I(\mathcal{E},\mathcal{E})$ increases, the internal influence $I(\mathcal{P},\mathcal{P})$ decreases, and the cross influence $I(\mathcal{E},\mathcal{P})$ first increases and then decreases. This can be illustrated by an elite influence shift diagram, such as the one in Fig. 1(e). The elite size for which the plots of $I(\mathcal{E},\mathcal{E})$ and $I(\mathcal{P},\mathcal{P})$ intersect is referred to as the power symmetry point of the network and the ordering. More formally, a given partition $(\mathcal{E},\mathcal{P})$ is said to be at a power symmetry point if 

$$I(\mathcal{E}) = I(\mathcal{P}).$$

(Similarly, $(\mathcal{E},\mathcal{P})$ is said to be near its symmetry point if $I(\mathcal{E}) \approx I(\mathcal{P})$, i.e., the two are equal up to a constant factor.)

Note that for a given network there are many partitions at a symmetry point. In fact, for each ordering $\pi'$ of the vertices there is an elite at a symmetry point, obtained by the iterative process described above.

Assuming the first three axioms regarding an elite-periphery partition $(\mathcal{E},\mathcal{P})$ of an undirected social network allows us to draw our first major result about symmetry.

**Theorem 3.1.** Let $(\mathcal{E},\mathcal{P})$ be a core-periphery partition that satisfy the dominance, robustness and compactness axioms (A1), (A2) and (A3). Then the partition is near its symmetry point, i.e., $I(\mathcal{E}) \approx I(\mathcal{P})$. Moreover, $I(\mathcal{E}) = \Theta(m)$ and $I(\mathcal{P}) = \Theta(m)$. 

This means that for any elite that satisfies Axioms (A1)-(A3), the overall influence of the elite, $I(\mathcal{E})$, is (nearly) equal to the overall influence of the periphery, $I(\mathcal{P})$, which makes
the elite-periphery power symmetry a universal property and justifies the symmetry point as a natural “candidate” breakpoint for selecting the elite. We expect this “symmetry of powers” between the elite and the periphery to be recognized as a significant element in understanding the internal balances in social networks.

To prove Theorem 3.1 we need a simple fact and two lemmas. Recall that the number of edges in the graph is $m$. As $(E, P)$ forms a partition of the network vertices, we have

Fact 3.2. $I(E,E) + I(E,P) + I(P,P) = m$.

Lemma 3.3. If $(E, P)$ satisfies the dominance and robustness axioms (A1)-(A2), then for some constants $c_1, c_2 > 0$,

1. $I(E,E) \geq c_1 \cdot I(P,P)$,
2. $I(E,E) \geq c_2 \cdot m$.

Proof. By the two axioms, we have that

$I(E,E) \geq c_r \cdot I(P,E) \geq c_r c_d \cdot I(P,P)$,

implying the first claim with $c_1 = c_r c_d$.

Also, by Fact 3.2 combined with the two axioms,

$m = I(E,E) + I(E,P) + I(P,P) \leq \left( 1 + \frac{1}{c_r} + \frac{1}{c_r c_d} \right) I(E,E)$.

Hence $I(E,E) \geq c_2 m$ for $c_2 = (1 + 1/c_r + 1/(c_r c_d))^{-1}$. The second claim follows. □

Let us next consider the implications of the compactness axiom (A3).

Lemma 3.4. If the elite $E$ satisfies also the compactness axiom (A3), then $I(E,P) = \Omega(m)$.

Proof. We assume $|P| \geq |E|$, or in other words $|P| \geq n/2$. Since each vertex of $P$ has a self loop, $I(P,P)$ contains at least $n/2$ self loops. Combining this with Axiom (A1), we get that $I(E,P) \geq c_d \cdot n/2$. Hence if the network has only a linear number of edges altogether, say, at most $4n/c_2$ edges for the constant $c_2$ of Lemma 3.3, then the lemma holds trivially. Hence herafter we consider networks where

$m > (4/c_2) \cdot n.$ (3)

Consider an elite $E$ that satisfies Axiom (A3), i.e., it is a minimal set of vertices satisfying Axioms (A1) and (A2). This implies that for every vertex $v \in E$, moving $v$ from $E$ to $P$ violates either (A1) or (A2).
Let us first consider the case where there exists a vertex \( v \in \mathcal{E} \) whose movement from \( \mathcal{E} \) to \( \mathcal{P} \) violates the robustness Axiom (A2). In other words,

\[
\mathcal{I}(\mathcal{E}, \mathcal{E}) - d_{\mathcal{E}}(v) < c_r \cdot (\mathcal{I}(\mathcal{P}, \mathcal{E}) + (d_{\mathcal{E}}(v) - 1) - d_{\mathcal{P}}(v)).
\]

Rearranging, we get that

\[
\mathcal{I}(\mathcal{E}, \mathcal{P}) > \frac{\mathcal{I}(\mathcal{E}, \mathcal{E}) - (1 + c_r) d_{\mathcal{E}}(v)}{c_r} > \frac{\mathcal{I}(\mathcal{E}, \mathcal{E}) - 2n}{c_r}.
\]

Applying Lemma 3.3 (2) and Eq. 3 we get that

\[
\mathcal{I}(\mathcal{E}, \mathcal{P}) > \frac{1}{c_r} \cdot (c_2 m - 2n) > \frac{c_2}{2c_r} \cdot m.
\]

Next, let us consider the complementary case, where for every vertex \( v \in \mathcal{E} \), moving \( v \) from \( \mathcal{E} \) to \( \mathcal{P} \) does not violate the robustness Axiom (A2). In this case, for every vertex \( v \in \mathcal{E} \), moving \( v \) from \( \mathcal{E} \) to \( \mathcal{P} \) necessarily violates the dominance Axiom (A1). This means that for every vertex \( v \in \mathcal{E} \),

\[
\mathcal{I}(\mathcal{E}, \mathcal{P}) + (d_{\mathcal{E}}(v) - 1) - d_{\mathcal{P}}(v) < c_d \cdot (\mathcal{I}(\mathcal{P}, \mathcal{P}) + (d_{\mathcal{P}}(v) + 1)).
\]

On the other hand we have by Axiom (A1) that

\[
\mathcal{I}(\mathcal{E}, \mathcal{P}) \geq c_d \cdot \mathcal{I}(\mathcal{P}, \mathcal{P}).
\]

Adding up these two inequalities and simplifying,

\[
d_{\mathcal{E}}(v) < (1 + c_d) \cdot d_{\mathcal{P}}(v) + 2 < 2d_{\mathcal{P}}(v) + 2.
\]

Summing over all \( v \in \mathcal{E} \),

\[
2\mathcal{I}(\mathcal{E}, \mathcal{E}) < 2\mathcal{I}(\mathcal{E}, \mathcal{P}) + 2|\mathcal{E}|,
\]

so

\[
\mathcal{I}(\mathcal{E}, \mathcal{P}) > \mathcal{I}(\mathcal{E}, \mathcal{E}) - |\mathcal{E}| \geq c_2 m - n \geq c_2 \cdot m.
\]

Theorem 3.1 now follows by the above lemmas.

In fact, a slightly stronger observation can be made. For a compact elite \( \mathcal{E} \), we say that \( \mathcal{E} \) is \textit{over-dominant} if for every \( v \in \mathcal{E} \), moving \( v \) from the elite to the periphery will not violate the elites dominance (but will, necessarily, violate robustness).

\textbf{Lemma 3.5.} If a compact elite \( \mathcal{E} \) is not over-dominant, then also \( \mathcal{I}(\mathcal{P}, \mathcal{P}) = \Omega(m) \).
Proof. Suppose $E$ is compact but not over-dominant. Let $c_3$ be the constant implied by Lemma 3.4, namely, such that $I(E, \mathcal{P}) \geq c_3 \cdot m$. As in the proof of Lemma 3.4, we assume that $|\mathcal{P}| \geq n/2$, hence $I(\mathcal{P}, \mathcal{P}) \geq n/2$. Hence if the network has only a linear number of edges altogether, say, at most $4(1 + c_d)n/c_3$ edges, then the lemma holds trivially. Hence hereafter we consider networks where $m > (4(1 + c_d)/c_3) \cdot n$, or equivalently

$$n < \frac{c_3}{4(1 + c_d)} \cdot m. \quad (4)$$

The fact that $E$ is not over-dominant means that there is some $v \in E$ whose movement from the elite to the periphery will violate the dominance axiom (A1). Hence $I(E, \mathcal{P}) + (d_E(v) - 1) - d_P(v) < c_d \cdot (I(\mathcal{P}, \mathcal{P}) + (d_P(v) + 1))$. Rearranging, we get that

$$I(\mathcal{P}, \mathcal{P}) > \frac{1}{c_d} (I(E, \mathcal{P}) + d_E(v) - (1 + c_d)(d_P(v) + 1))$$

$$> \frac{1}{c_d} (I(E, \mathcal{P}) - (1 + c_d)2n).$$

By Lemma 3.4 and Eq. (4) we get

$$I(\mathcal{P}, \mathcal{P}) > \frac{1}{c_d} (c_3m - (1 + c_d)2n)$$

$$> \frac{c_3m}{c_d} - \frac{(1 + c_d)2}{c_d} \cdot \frac{c_3m}{4(1 + c_d)} = \frac{c_3}{2c_d} \cdot m.$$ 

The claim follows. \qed

Moreover, one can draw an additional interesting conclusion concerning a universal property related to the power symmetry point.

**Observation 3.6.** For any ordering of the vertices and corresponding elite influence shift diagram, the crossing influence $I(E, \mathcal{P})$ is maximized at the symmetry point.

This property is clearly observed in our empirical results (see for example Figure 1(e) and Section 5), and we prove it formally for a very general random graph model known as the *configuration model* (see Appendix B).

## 4 The Size of the Elite

The above discussion about the symmetry point leaves open the question of the elite size in networks. In fact, there are networks for which the elite can be at a symmetry point
but with significantly different sizes varying from linear size to $\sqrt{n}$. (see Figure 6). Our first contribution in this direction concerns the question of how small the elite can be while still preserving its properties. We show that once satisfying the axioms, the elite cannot be smaller than $\Omega(\sqrt{m})$.

**Theorem 4.1.** If $(\mathcal{E}, \mathcal{P})$ satisfies the dominance and robustness axioms $(A1)$ and $(A2)$, then $|\mathcal{E}| \geq \Omega(\sqrt{m})$.

**Proof.** Graph-theoretical considerations dictate that $I(\mathcal{E}, \mathcal{E}) \leq \binom{|\mathcal{E}|}{2} \leq |\mathcal{E}|^2$, implying that $|\mathcal{E}| \geq \sqrt{I(\mathcal{E}, \mathcal{E})}$. Combined with Lemma 3.3(2), the theorem follows. □

We complement this result by providing an example of what we call a purely elitistic society, where the elite reaches its minimum possible size of $\Theta(\sqrt{m})$ in Appendix A.

In reality, however, most social networks are not purely elitistic, which leaves the question of an upper bound for the “typical” elite unanswered: does the “universal” size of elites (if exists) converge to a linear, or a sublinear, function of the network size? For illustration, consider the US population of about 314 million people. An elite of 0.1% will consist of 314,000 people, while an elite of size $\sqrt{314M}$ will consist of only about 18,000 people. These numbers differ by an order of magnitude; which of them is more plausible?

Considering also Axiom $(A4)$, we can clarify this important point and prove that the elite size is sublinear.

**Theorem 4.2.** If $(\mathcal{E}, \mathcal{P})$ satisfies the dominance, robustness and density axioms axioms $(A1)$, $(A2)$ and $(A4)$, then the elite size is sub-linear in the size of society, namely, $|\mathcal{E}| \leq n^{\delta_{V}/\delta_{E}} = nc$ for $c < 1$.

We find it remarkable that three simple and intuitive assumptions lead to such a strong implication on the elite size. Note that Theorem 4.2 is in controversy to the common belief that the elite size converges to a linear fraction of the society (most recently claimed to be 1% [25]). This discrepancy may perhaps be attributed to the fact that our axiom-based approach characterizes the elite differently than in previous approaches. In the next section we present evidence that many social networks and complex networks tend to have sublinear elites.

Theorem 4.2 is derived from the following lemma. Recall that the density of a set $X \subseteq V$ is $\delta_X = \log|E(X, X)|/\log|X|$, so in particular

$$I(\mathcal{E}, \mathcal{E}) = |E(X, X)| = |\mathcal{E}|^{\delta_{E}}.$$
We show that the elite density tightly determines its size.

**Lemma 4.3.** If \((\mathcal{E}, \mathcal{P})\) satisfies axioms (A1) and (A2) and \(\mathcal{E}\) has density \(\delta_e\), then \(|\mathcal{E}| = \Theta\left(\frac{1}{m^{\frac{1}{\delta_e}}}\right)\).

**Proof.** By Fact 3.2, \(\mathcal{I}(\mathcal{E}, \mathcal{E}) \leq m\). By Lemma 3.3, \(\mathcal{I}(\mathcal{E}, \mathcal{E}) \geq m/c'\). Hence relying on Eq. 5, we get \(m/c' \leq |\mathcal{E}|^{\delta_e} \leq m\). The lemma follows. \(\square\)

If \(\delta_X = 2\) we say that \(X\) is *dense*, i.e., in graph terms it is very close to a clique. In this case, \(|\mathcal{E}| = \Theta(\sqrt{m})\).

## 5 Empirical Results

### 5.1 Symmetry Point and the Axioms

We investigated both the static properties of the network elites and their dynamics over time. In total we analyzed about 100 social and complex networks. Their observed behavior (w.r.t. elite and periphery relationships) is surprisingly consistent. While it is clear that the issue of identifying the elite members of a given network is of paramount importance, with some recent developments \[26, 32\], this issue is not discussed in the present work. Instead, in order to conduct our experiments on given networks, we construct an approximation of the elite, \(\mathcal{E}_k\), of size \(k\). Once the \(k\) members of the elite are selected, the rest of the vertices are considered as forming the periphery \(\mathcal{P}_k\) (of size \(n - k\)), and the values of \(I(\mathcal{E}_k, \mathcal{E}_k), I(\mathcal{E}_k, \mathcal{P}_k)\) and \(I(\mathcal{P}_k, \mathcal{P}_k)\) can be calculated directly. We use two known methods for approximating an elite \(\mathcal{E}_k\) of size \(k\). The first method is based on the notion of the \(k\)-rich-club, and we denote the size-\(k\) elite it generates for a given network \(G\) by \(\mathcal{E}_k^{\text{rich}}(G)\) (and omit \(G\) and rich when they are clear from context). The second method relies on looking for a \(c\)-core in the network \[12\], and it is a bit more complex. The two methods produce different elites (albeit with some overlap). We present in the main text only the results for the \(k\)-rich-club elites, but similar results were obtained when we used the \(c\)-core method (see Appendix C for details on the results with the \(c\)-core method).

The \(k\)-rich-club \[33\] is perhaps the most intuitive and natural approximation for an elite of size \(k\). To build it for a given network \(G\), we sort the network vertices according to their degree, and choose the \(k\) highest degree vertices as the members of \(\mathcal{E}_k^{\text{rich}}\) (breaking ties arbitrarily). Note that this method allows us to generate an elite for any desirable size \(1 \leq k \leq n\).
Figure 3: (a) The elite influence shift diagrams (plotting the influence ratios $I(\mathcal{E}, \mathcal{E})/m$, $I(\mathcal{P}, \mathcal{P})/m$ and $I(\mathcal{E}, \mathcal{P})/m$) for the $k$-rich-clubs of nine networks. The X axis is on a logarithmic scale, where a point $x$ (in $[0,1]$) represents a $k$-rich club of size $k = n^x$. A point $y$ on the Y axis indicates the fraction $I(\cdot, \cdot)/m$, where $m$ is the total number of edges. The range of $x$ values between $x_0$ such that $k = \sqrt{m} = n^{x_0}$ and $x_1$, the symmetry point, such that $k = k_{sp} = n^{x_1}$ is highlighted. (b) Networks nodes and edges numbers. (c) Median results of the elite influence shift diagram for the $k$-rich-club elites for the 98 networks examined, for increasing $k$. 

| Network     | Nodes | Edges   |
|-------------|-------|---------|
| Slashdot    | 51083 | 116573  |
| Buzznet     | 101163| 2763066 |
| Livemocha   | 104103| 2193083 |
| Delicious   | 536408| 1366136 |
| Digg        | 771229| 5907413 |
| LastFm      | 1191812| 4519340 |
| Pokec       | 1632803| 22301964|
| LiveJournal | 2238731| 12816184 |
| Flixster    | 2523386| 7918801  |
We present results from the 98 networks we evaluated. (Detailed results are shown only for nine example networks, but the median values were calculated based on all the 98 networks studied.) For each network we considered an elite selection method (i.e., \(k\)-rich-club or \(c\)-core) and then examined the influence and density of the elite and periphery and validated the axioms for different elite sizes and networks.

We first consider the elite influence shift diagrams which presents the changes in \(I(E_k, E_k), I(P_k, P_k)\) and \(I(E_k, P_k)\) as the elite (i.e., \(\mathcal{E}_{\text{rich}}(G)\)) grows from its minimum size of 1 to its maximum size of \(n\). Figure 3 shows the elite influence shift diagrams of the \(k\)-rich-club for 9 networks and the median results of the \(k\)-rich-club for all of the 98 networks in our experiments. Note that the X-axis is logarithmic and presented as \(k = n^x\) where \(x\) goes from 0 to 1. The point \(x = 0.5\) therefore indicates a \(\sqrt{n}\)-rich-club. To normalize the Y-axis for different networks, each graph plots \(I(E_k, E_k)/m, I(P_k, P_k)/m,\) and \(I(E_k, P_k)/m\). Observe that the networks exhibit a similar pattern: the number of crossing edges between the elite and the periphery, \(I(E_k, P_k)\), grows with the elite size \(k\) up to some maximum, and then starts decreasing. An interesting and less obvious pattern is that \(I(E_k, P_k)\) is larger than \(I(E_k, E_k)\) right from the beginning and remains larger until the maximum point. The relation between these numbers changes only after \(I(E_k, P_k)\) begins to decrease, while \(I(E_k, E_k)\) continues to grow.

As mentioned before, a particularly interesting point in these graphs is the symmetry point, namely, the elite size, denoted as \(k_{sp}\), where the internal influence of the core and the internal influence of the periphery are equal, i.e., \(I(\mathcal{E}_{k_{sp}}, \mathcal{E}_{k_{sp}}) = I(\mathcal{P}_{k_{sp}}, \mathcal{P}_{k_{sp}}).\) Recall that any elite that satisfies Axioms (A1), (A2) and (A3) must be near its symmetry point. An important observation (stated earlier in Observation 3.6) is that \(I(\mathcal{E}_k, \mathcal{P}_k)\) achieves its maximum at (or very close to) the symmetry point where \(k = k_{sp}\).

One of the main contributions of this paper is the claim that elites in social networks are “small”, that is, of sub-linear size. We address this issue in detail shortly, but for now we note that in Figure 3 for both the example networks and the median of all networks, the symmetry point occurs mostly at elite sizes between \(n^{0.6}\) and \(n^{0.85}\), and its median value is at about \(n^{0.77}\).

In all figures we highlight the range \([x_0, x_1]\) of sizes for which it was shown that an elite can satisfy Axioms (A1) and (A2). More explicitly, this range starts at the lower bound on the elite size, i.e., the point \(x_0\) such that \(k = \sqrt{m} = n^{x_0}\), and ends at the symmetry point, i.e., the point \(x_1\) such that \(k = k_{sp} = n^{x_1}\). Note that since each network has different structure and different average degree, the values \(x_0\) and \(x_1\) are different for each network (hence in Figure 3(c), the leftmost point \(x_0\) of the highlighted range represents a median
Figure 4: (a) Observed dominance ratio, $\text{dom}(\mathcal{E}_k)$, observed robustness ratios, $\text{rob}(\mathcal{E}_k)$ and the observed density increase, $\text{dns}(\mathcal{E}_k) = \delta(\mathcal{E}_k) - \delta$ for nine example networks. The x axis is logarithmic and marks the sizes of the $k$-rich-club elites with $k = n^x$. The highlighted area is $\mathcal{E}_k$'s of sizes between $k = \sqrt{m}$ and the symmetry point $k = k_{sp}$. (b) Median, Min and Max results for 98 networks at $k = \sqrt{m}$ and $k = k_{sp}$. (c) Median results for 98 networks.
We now turn to the claim that small elites, and in particular that elites near the symmetry point \(k_{sp}\) satisfy Axioms (A1), (A2) and (A3), that is, they are dominant, robust and denser than the whole network. Recall that by Thm. 4.2 if an elite satisfies these three axioms, then its size must be sublinear. For a given network \(G\) with elite \(E_k\) (of size \(k\)) and periphery \(P_k\), define the observed dominance ratio of \(E_k\) on \(G\) as \(\text{dom}(E_k) = \frac{I(E_k, P_k)}{I(P_k, P_k)}\) and the observed robustness ratio of \(E\) on \(G\) as \(\text{rob}(E_k) = \frac{I(E_k, E_k)}{I(E_k, P_k)}\). The observed density of the \(E_k\) is denoted as \(\delta(E_k)\). Note that the density of the network is \(\delta(E_n)\) and is denoted simply as \(\delta\). We define the observed density increase as \(\text{dns}(E_k) = (\delta(E_k) - \delta)/\delta\), which indicates for each elite \(E_k\) the ratio by which it is denser than the whole graph.

Figure 4 (a) shows the values of the observed dominance and robustness ratios and the observed density increase, \(\text{dom}(E_k)\), \(\text{rob}(E_k)\) and \(\text{dns}(E_k)\) respectively, for the nine example networks \(G\) with \(k\)-rich club elites of different sizes \(k\). Figure 4 (c) shows the median of these values for all of the 98 networks.

The X-axis for each graph is again on a logarithmic scale, where an \(x\) value represents a \(k\)-rich-club of size \(n^x\). We focus on values of \(\text{dom}(E_k)\) and \(\text{rob}(E_k)\) up to 1, and ignore higher values. As before, we highlight in each figure the range of \(k\) for \(k\)-rich-club of sizes between \(k = \sqrt{m}\) and the symmetry point \(k = k_{sp}\).

Although it is not necessitated by the model or the axioms, it is clear from the figures that as the elite \(E_k\) grows in size, it attains more dominance over the rest of the network (the periphery, \(P_k\)), namely, its observed dominance ratio increases, and its observed robustness ratio grows as well. We therefore focus on the highlighted area of interest. One can see that the social networks under study exhibit high values for both \(\text{dom}(E_k)\) and \(\text{rob}(E_k)\) at the highlighted area. Recall that by our previous analysis, elites that satisfy Axioms (A1) and (A2) are of size \(k \geq \sqrt{m}\). Hence the smallest possible size for an elite is \(k = \sqrt{m}\) and at that size, one might expect the elites to exhibit relatively small values of \(\text{dom}(E_k)\) and \(\text{rob}(E_k)\). Somewhat surprisingly, the calculated values are relatively high and are bounded away from zero. For example, At \(k = \sqrt{m}\), the elite of ‘Buzznet’ has observed dominance ratio \(\text{dom}(E_k)\) way beyond 1, and the elite of ‘Digg’ has observed ratio of about 1, exhibiting high dominance over the rest of the network. ‘Pocek’, which exhibits one of the lowest observed dominance ratios at \(k = \sqrt{m}\), still has \(\text{dom}(E_k)\) of about 0.1, which is also reasonably high. Figure 3 (c) shows the median value of \(\text{dom}(E_k)\) of all 98 networks. At \(k = \sqrt{m}\), this median value is very high, about 0.6, exhibiting very high dominance of elites of this size. The dominance of elites of size \(k_{sp}\) at the symmetry point is even higher. All the example networks have observed dominance ratios \(\text{dom}(E_{sp}) > 1\) so the median \(\text{dom}(E_{sp})\) value is also greater than 1.
at the symmetry point $k_{sp}$. These empirical results show that in real social networks, elites of size greater than $\sqrt{m}$ and smaller than the symmetry point exhibit high dominance and satisfy Axiom (A1). Similar results, albeit somewhat weaker, were obtained for $c$-core elites.

Observed robustness ratios exhibit a similar behavior to observed dominance ratios. For small elites of size $k = \sqrt{m}$, all the example networks have observed robustness ratios well bounded away from zero, and so is the median value for all 98 networks. The lowest value of any of the example networks, observed for 'Flixter', is $\text{rob}(E_{\sqrt{m}}) = 0.02$. The highest value, observed for 'Digg', is 0.13. The median observed robustness ratio at that point is about 0.1. At the symmetry point $k_{sp}$ the observed ratios are higher. For example, 'Pokec' has an observed robustness ratio of $\text{rob}(E_{sp}) = 0.65$, and the smallest value, observed for 'Flixter', is 0.18. The median value at that point is about 0.45. Again, it follows that elites of size greater than $\sqrt{m}$ and smaller than the symmetry point in social networks have a high robustness. Similar results, in fact somewhat stronger, were obtained for the $c$-core elites. In general, elites produced by $c$-core exhibit lower dominance but higher robustness than those based on $k$-rich clubs.

Another interesting observation that can be deduced from Figures 4 is that in almost all the social networks that we tested, viewing elites of increasing size, we note that the elites attain dominance before attaining robustness.

Turning to density, the figures clearly reveal that the density of the elite, $\delta(E_k)$, is significantly higher than the density of the whole graph. The median results for 98 networks exhibit that the elite at the symmetry point is about 20% denser than the whole graph. The density at $k = \sqrt{m}$ is even higher. Interestingly, $E_k$ seems to reach its maximum density around this point, when $k = \sqrt{m}$.

To conclude the discussion of Figure 4 we state that our empirical results provide strong evidence that it is a universal property that the $k$-rich-club elite, at its symmetry point, satisfies Axioms (A1), (A2) and (A4). From Theorem 4.2 we therefore conclude that $k_{sp}$, the elite size at the symmetry point, is a sublinear function of $n$.

### 5.2 The Scaling Law: Elite Size in growing networks

In the networks we examined so far, the sizes of the elites at the symmetry point (in both the $k$-rich-club and the $c$-core methods) appear to be sublinear, and specifically, between $n^{0.6}$ and $n^{0.85}$. Although this result was obtained for networks of different sizes, all of them were static, and therefore the data collection does not allow us to ascertain whether the size of the elite in the network's symmetry point is indeed asymptotically sublinear.
Figure 5: (a) The fraction of the elite at the symmetry point, from the whole population, as the networks evolve over time. 4 examples. (b) Duration of the example networks (c) The median out of 8 networks.
To study this crucial question more carefully, we turned to data collected on dynamic networks, namely, networks for which information is available on their evolution over time. If the elite size at the symmetry point is indeed sublinear, then one should observe a decrease in the relative size of the elite (its fraction of the network size) as the network grows.

Figure 5 presents the data collected for dynamic networks. We evaluated 8 networks for which information was available about the creation time of each edge. Using this information, we simulated the evolution of the network. As data on the appearance time of each network vertex was not available, we made the assumption that each vertex has joined the network at the same time when the first edge incident to it has appeared. We then divided the evolution time of the network into 20 time frames, each corresponding to a time period during which the network size increased by 5% of the total (final) network size. For each time frame $t$, we calculated the elite $E_k(t)$ and the symmetry point $k = k_{sp}(t)$ in the snapshot of the network at time $t$. Figure 5 shows the ratio $r = |E_{k_{sp}}(t)|/n(t) = k_{sp}(t)/n(t)$ of the number of vertices in the elite at the symmetry point, where $k_{sp}(t)$ is the elite size at the symmetry point at time $t$ and $n(t)$ is the number of vertices in the entire network at time $t$. In each figure we added the trend of $\sqrt{m(t)/n(t)}$, where $m(t)$ is the number of edges in the network at time $t$. Clearly, unless the network is dense (i.e., with an order of $n(t)^2$ edges at time $t$), the ratio $\sqrt{m(t)/n(t)}$ converges to zero. Figure 5 (a) shows the results for four example networks out of the 8, and Figure 5 (b) shows the median result of all 8 networks. These figures demonstrate that the elite size at the symmetry point is a relatively small fraction of the entire network (starting at a median of 10% and ending at a median of 5%). Furthermore, it can be seen that as the network evolves and grows, the ratio $r$ decreases, following a pattern similar to the function $\sqrt{m(t)}$, implying that asymptotically indeed the elite size in the symmetry point has a sublinear size.

6 Related Work

As identifying the most influential vertices in a network is crucial to understanding its members’ behaviour, many studies considered a variety of notions related to the elite and core-periphery decompositions (see [8] for a recent survey). Borgatti and Everett [4] measured the similarity between the adjacency matrix of a graph and the block matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This captures the intuition that social networks have a dense, cohesive core and a sparse, disconnected periphery. Core/periphery networks revolve around a set of central vertices that are well-connected with each other as well as with the periphery. In addition to formalizing these intuitions, Borgatti and Everett devised algorithms for detecting core/periphery
structures, along with statistical tests for verifying a-priori hypotheses [5]. Other efforts at identifying such structures and decomposing networks include a coefficient to measure if a network exhibits a clear-cut core-periphery dichotomy [16], a method to extract cores based on a modularity parameter [10], a centrality measure computed as a continuous value along a core-periphery spectrum [26], a coreness value attributed to each node, qualifying its position and role based on random walks [11], a detection method using spectral analysis and geodesic paths [9], and a decomposition method using statistical inference [32]. Mislove et al. [20] defined the core of a network to be any (minimal) set of vertices that satisfies two properties. First, the core must be essential for ensuring the connectivity of the network (i.e., removing it breaks the remainder of the vertices into many small, disconnected clusters). Second, the core must be strongly connected with a relatively small diameter. They observed that for such cores, the path lengths increase with the core size when progressively including vertices ordered inversely by their degree. The graphs studied in [20] have a densely connected core comprising of between 1% and 10% of the highest degree vertices, such that removing this core completely disconnects the graph. A recent article [31] argues that the core/periphery structure is simply the result of several overlapping communities and proposes a community detection method coping with overlap.

One of the first papers to focus on the fact that the highest degree vertices are well-connected [33] coined the term rich-club coefficient for the density of the vertices of degree \( k \) or more. Colizza et al. [7] refined this notion to account for the fact that higher degree vertices have a higher probability of sharing an edge than lower degree vertices, and suggested to use baseline networks to avoid a false identification of a rich-club. Xu et al. [30] shows that the rich-club connectivity has a strong influence on the assortativity and transitivity of a network. Weighted and hierarchical versions of the rich-club coefficient have been studied in [19, 22, 27, 34].

The nestedness of a network represents the likelihood of a vertex to be connected to the neighbors of higher degree vertices. Lee et al [17] defined a nestedness measure capturing the degree to which different groups in networks interact. Yet another perspective is offered in [15] where a network formation game and its equilibria are studied (benefits from connections exhibit decreasing returns and decay with network distance).

7 Discussion

In this article we address the forces responsible for the creation of elites in social networks. We provide axioms modeling the influence relationships between the elite and the periphery.
We prove that at the power symmetry point, the size of the elite is sublinear in the size of the network. In particular that means that an elite is much smaller than a constant fraction of the network, evidence of which is often observed in the widening gap between the very rich and the rest of societies. To understand better what these axioms mean in practice, we studied a multitude of large real-world social networks. We approximated the elites by rich-clubs of various sizes. Our findings indicate that in these networks, rich-clubs near the symmetry point exhibit elite properties such as disproportionate dominance, robustness and density as stated by the axioms.

Our results do not only advance the theoretical understanding of the elite of social structures, but may also help to improve infrastructure and algorithms targeted at online social networks (e.g., [3]), organize institutions better or identify sources of power in social networks in general.

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Appendix

A  Extreme Elite Size

We have seen that a compact elite (satisfying axioms A1, A2, A3) has $\mathcal{I}(\mathcal{E}, \mathcal{E}) = \Omega(m)$ and $\mathcal{I}(\mathcal{E}, \mathcal{P}) = \Omega(m)$. A natural question is whether it is also guaranteed that $\mathcal{I}(\mathcal{P}, \mathcal{P}) = \Omega(m)$ (which may imply a stronger version of a symmetry point than the one defined earlier). We now demonstrate that unfortunately this is not the case. This is done by showing an example of a family of graphs with elites satisfying Axioms (A1), (A2), (A3), where $\mathcal{I}(\mathcal{E}, \mathcal{E}) = \Omega(m)$ and $\mathcal{I}(\mathcal{E}, \mathcal{P}) = \Omega(m)$ but $\mathcal{I}(\mathcal{P}, \mathcal{P}) = O(n)$.

We assume that the axioms are defined with constants $c_r, c_d \leq 1$ and for simplicity we assume also that $c_r = 1/b$ for constant integer $b$ (can easily be modified to a rational $c_r$, i.e., $c_r = a/b$ for constant integers $a, b$).

The construction of the graph uses a parameter $Z$ (to be thought of as roughly $n^{1/4}$), namely, the graph family contains one graph $G_Z$ for any integer $Z > 0$.

The elite $\mathcal{E}$ consists of a complete network of $\varepsilon = |\mathcal{E}| = 4Z^3 - 1$ vertices. The periphery $\mathcal{P}$ consists of $|\mathcal{P}| = 2bZ\varepsilon = 2bZ(4Z^3 - 1)$ vertices with no connections between them (except for a self-loop for each vertex). Hence altogether the graph $G_Z$ contains $n = 8bZ^4 + 4Z^3 - 2bZ - 1$ vertices.

Every vertex in $\mathcal{E}$ is connected to $2bZ^3$ vertices in $\mathcal{P}$, and every vertex in $\mathcal{P}$ is connected to $Z^2$ vertices in $\mathcal{E}$. Hence we have

\begin{align*}
\mathcal{I}(\mathcal{E}, \mathcal{E}) & = \varepsilon(\varepsilon + 1)/2 = 2Z^3(4Z^3 - 1) \\
\mathcal{I}(\mathcal{E}, \mathcal{P}) & = Z^2|\mathcal{P}| = 2bZ^3\varepsilon = 2bZ^3(4Z^3 - 1) \\
\mathcal{I}(\mathcal{P}, \mathcal{P}) & = |\mathcal{P}| = 2bZ(4Z^3 - 1).
\end{align*}

Note that $n = \Theta(Z^4)$ and $m = \Theta(Z^6) = \Theta(n^{3/2})$, and indeed $\mathcal{I}(\mathcal{E}, \mathcal{E}), \mathcal{I}(\mathcal{E}, \mathcal{P}) = \Omega(m)$ and $\mathcal{I}(\mathcal{P}, \mathcal{P}) = O(n)$.

We need to verify that this construction satisfies the three axioms. Axiom (A1) says that $\mathcal{I}(\mathcal{E}, \mathcal{P}) \geq c_d \cdot \mathcal{I}(\mathcal{P}, \mathcal{P})$. This follows readily from Eq. (6) and the assumption that $c_d \leq 1$.

Axiom (A2) says that $\mathcal{I}(\mathcal{E}, \mathcal{E}) \geq \mathcal{I}(\mathcal{E}, \mathcal{P})/b$. In fact, Eq. (6) establishes equality here.

To prove Axiom (A3), we need to show that every subset $\emptyset \subset \mathcal{E'} \subset \mathcal{E}$ violates either axiom (A1) or (A2). Concretely, we show that it violates Axiom (A2), i.e., that

\[ \mathcal{I}(\mathcal{E'}, \mathcal{E'}) < \mathcal{I}(\mathcal{E'}, \mathcal{P'})/b. \]
Consider some $\emptyset \subset \mathcal{E}' \subset \mathcal{E}$ and let $\mathcal{X} = \mathcal{E} \setminus \mathcal{E}'$. Note that
\[
\mathcal{I}(\mathcal{E}', \mathcal{E}') = \mathcal{I}(\mathcal{E}, \mathcal{E}) - \mathcal{I}(\mathcal{X}, \mathcal{E}') - \mathcal{I}(\mathcal{X}, \mathcal{X}) \\
\mathcal{I}(\mathcal{E}', \mathcal{P}') = \mathcal{I}(\mathcal{E}, \mathcal{P}) + \mathcal{I}(\mathcal{X}, \mathcal{E}') - \mathcal{I}(\mathcal{X}, \mathcal{P})
\]
hence we need to prove that
\[
b \cdot (\mathcal{I}(\mathcal{E}, \mathcal{E}) - \mathcal{I}(\mathcal{X}, \mathcal{E}') - \mathcal{I}(\mathcal{X}, \mathcal{X})) < \mathcal{I}(\mathcal{E}, \mathcal{P}) - \mathcal{I}(\mathcal{X}, \mathcal{E}') - \mathcal{I}(\mathcal{X}, \mathcal{P}) .
\]
Let $\chi = |\mathcal{X}|$ and $\epsilon' = |c \mathcal{E}'| = \epsilon - \chi$. Then plugging the quantities of Eq. (6), it remains to prove that
\[
b(2Z^3(4Z^3 - 1) - \chi(4Z^3 - 1 - \chi) - \chi(\chi + 1)/2) < 2bZ^3(4Z^3 - 1) + \chi(4Z^3 - 1 - \chi) - 2bZ^3 \chi .
\]
Simplifying, we get that the above indeed holds for every $\chi < \epsilon$. 

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Figure 6: Extreme examples of elite-periphery partitions that satisfy our axioms and are close to the symmetry point. In both networks we assume each vertex has a self-loop edge that represents self-influence and is not drown for simplicity. Elite vertices are gray. (a) A “purely elitistic society” example with 41 vertices of which 6 are the elite. One can extend this example to a network of $n$ vertices where the number of edges is $m < 2n$ edges. The elite then is of sub-linear size and consists of about $\sqrt{m} \approx \sqrt{n}$ vertices. But it is dense, namely, the elite contains about $m$ edges. The periphery also contains about $m$ edges so the partition is at a symmetry point. (b) A “purely egalitarian society” example of a grid network with 49 vertices from which 11 are the elite. This example can be extended to a network with $n$ vertices and $m < 4n$ edges. It can be shown then that every dominant elite must be of size linear in $n$ (and $m$). It follows that both the elite and the periphery will have a linear number of edges and therefore the partition is close to the symmetry point. Most social networks observed in reality exhibit neither of these extreme behaviors.
Symmetry in Random Networks

Let $d = d_1, d_2, \ldots, d_n$, where $1 \leq d_i \leq n - 1$ be a positive degree sequence. A *random configuration graph* $G(n, d)$ is constructed in the following way over the set of vertices $[n]$. Let $W = [2m]$ be the set of *configuration points* where $m = \sum_1^n d_i / 2$ is the number of edges. Define the ranges $W_j = [1 + \sum_1^{j-1} d_i, \sum_1^j d_i]$ for $j \in [n]$. Given a pairing $F$ (i.e., a partition of $W$ into $m$ pairs) we obtain a (multi) graph $G_F$ with vertex set $[n]$ and for each pair $(i, j) \in F$ we add an edge $(i, j)$ to $G_F$. Choosing $F$ uniformly from all possible pairings of $W$ we obtain a random (multi-)graph $G(n, d)$.

For a given degree sequence $d$, define its *symmetry point* $\kappa(d)$ as follows:

$$\kappa(d) = \arg\min_j \left| \sum_1^j d_i - \sum_{i=j+1}^n d_i \right|.$$ 

Given $d$ and an index $1 \leq i \leq n$, let $\mathcal{E}_i$ denote the core as the vertex set $[i] = \{1, \ldots, i\}$ and the periphery $\mathcal{P}_i$ as the vertex set $\{i + 1, \ldots, n\}$.

**Theorem B.1.** Let $G(n, d)$ be a random configuration graph of given positive degree sequence $d$ and let $\kappa$ be the symmetry point of $d$ i.e., $\kappa(d) = \arg\min_j \left| \sum_1^j d_i - \sum_{i=j+1}^n d_i \right|$. Then,

1. $\mathbb{E}[\mathcal{I}(\mathcal{E}_i, \mathcal{P}_i)] \leq \mathbb{E}[\mathcal{I}(\mathcal{E}_\kappa, \mathcal{P}_\kappa)]$ for every $i$

2. $\mathbb{E}[\mathcal{I}(\mathcal{E}_\kappa, \mathcal{P}_\kappa)] \geq 4m/9$

3. $\mathbb{E}[\mathcal{I}(\mathcal{E}_\kappa, \mathcal{E}_\kappa)] \geq m/9$

4. $\mathbb{E}[\mathcal{I}(\mathcal{P}_\kappa, \mathcal{P}_\kappa)] \geq m/9$

**Proof sketch.** Let $D(k, l) = \sum_{i=k}^l d_i / 2m$. Given a partition index $i$, the expected number of edges of each component is the following:

$$\mathbb{E}[\mathcal{I}(\mathcal{E}_i, \mathcal{P}_i)] = 2D(1, i)D(i + 1, n)m,$$

$$\mathbb{E}[\mathcal{I}(\mathcal{E}_i, \mathcal{E}_i)] = D(1, i)^2m,$$

$$\mathbb{E}[\mathcal{I}(\mathcal{E}_i, \mathcal{E}_i)] = D(i + 1, n)^2m.$$ 

Now, since $D(1, i) + D(i + 1, n) = 2m$, the maximum value of $D(1, i)D(i + 1, n)$ is when $i = \arg\min_j |D(1, i) - D(i + 1, n)|$ which is exactly $\kappa(d)$. What are lower bounds for the three influences? Note that $|D(1, \kappa) - D(\kappa + 1, n)|$ will be maximized when $d_\kappa = n - 1$ and $D(1, \kappa - 1) = D(\kappa + 1, n)$ so the worst case is when $D(1, \kappa - 1) = D(\kappa + 1, n) = n - 1$ and $d_\kappa = n - 1$, yielding the claimed result.
Figure 7: $c$-core elites: $I(\mathcal{E}, \mathcal{E})/m$, $I(\mathcal{P}, \mathcal{P})/m$ and $I(\mathcal{E}, \mathcal{P})/m$ for 9 networks for increasing $k_c$ and their medians for the 98 networks in our measurements. A point $x$ on the $X$ axis shows a $c$-core of size $k_c = n^x$, for $x$ in the range $[0,1]$. The $Y$ axis gives the percentage of the measured value from the total number of edges. The range of $x$ values between $n^x = \sqrt{m}$ and the symmetry point is highlighted.

Section 5 described the results obtained when we examined the changes in $I(\mathcal{E}, \mathcal{E}), I(\mathcal{P}, \mathcal{P})$ and $I(\mathcal{E}, \mathcal{P})$ for various elite sizes when the elites were chosen using the $k$-rich-club method.
In this section we consider the setting when the elites are chosen using the $c$-core method, and show the results obtained in this setting.

As in Figures 3, we show the changes in $\mathcal{I}(\mathcal{E}, \mathcal{E})$, $\mathcal{I}(\mathcal{P}, \mathcal{P})$ and $\mathcal{I}(\mathcal{E}, \mathcal{P})$ as the elite grows from its minimum to its maximum possible size. In the $c$-core case, these sizes are determined by the network structure, and cannot be set by us. Specifically, the minimum size is the smallest possible $k_c > 0$ and the maximum size is $k_1$.

Figure 7 shows our results for nine example networks and the median results for the $c$-cores of all 98 networks included in our experiments. Each graph contains three plots. The first is for $\mathcal{I}(\mathcal{E}, \mathcal{E})/m$, the second is for $\mathcal{I}(\mathcal{P}, \mathcal{P})/m$ and the third is for $\mathcal{I}(\mathcal{E}, \mathcal{P})/m$. The X-axis for each graph is on a logarithmic scale, where an $x$ value represents a $c$-core of size $k_c = n^x$. Although the results in the setting of $c$-core elites are not as consistent and smooth for all networks as in the setting of $k$-rich-club elites, one can observe that the results here are similar to those obtained in the $k$-rich-club setting. Here, too, most of the networks exhibit similar patterns: as the elite size grows, $\mathcal{I}(\mathcal{E}, \mathcal{E})$ grows as well and $\mathcal{I}(\mathcal{P}, \mathcal{P})$ decreases, while the number of crossing edges $\mathcal{I}(\mathcal{E}, \mathcal{P})$ grows with the elite size up to some maximum and then starts decreasing. Here too, $\mathcal{I}(\mathcal{E}, \mathcal{P})$ is larger than $\mathcal{I}(\mathcal{E}, \mathcal{E})$ (almost) right from the beginning and remains larger until the maximum point, and this relation changes only after $\mathcal{I}(\mathcal{E}, \mathcal{P})$ begins to decrease (while $\mathcal{I}(\mathcal{E}, \mathcal{E})$ continues to grow).

Figures 7 also show that in the setting of $c$-core elites, just as with $k$-rich-club elites (as mentioned in Section 3), $\mathcal{I}(\mathcal{E}, \mathcal{P})$ attains its maximum at (or very close to) the symmetry point.

Section 5 presented also the values of the dominance and robustness constants, $\rho_d$ and $\rho_r$ respectively, for various elite sizes in the setting of $k$-rich-club elites. Here, we present the corresponding results for $c$-core elites.

As in Figure 4 we show the changes in $\rho_d$ and $\rho_r$ as the elite grows from its minimum to its maximum possible size. Figure 8 shows results for our nine example networks and the median results for the $c$-core elites of all the 98 networks included in our experiments. The X-axis for each graph is, again, on a logarithmic scale, where an $x$ value represents a $c$-core of size $k_c = n^x$. There are two plots in each graph, one for the dominance constant $\rho_d$ and the other for the robustness constant $\rho_r$. We focus on values of $\rho_d$ and $\rho_r$ up to 1, and ignore higher values. As before, we highlight in each figure the $c$-cores of sizes in the range $\sqrt{m} \leq k_c \leq SP$.

Clearly, elites chosen by the $c$-core method are different from same size elites chosen by the $k$-rich-club method. Indeed, the results presented for the $c$-core setting are not as
Figure 8: Dominance and robustness constants for the nine example networks and their median for the 98 networks. The x axis is logarithmic and represents the sizes of c-cores with $k = n^x$. The highlighted area of $x$ values corresponds to sizes between $k_c = \sqrt{m}$ and the symmetry point.

consistent as in the $k$-rich-club setting. Nevertheless, our results show that the same basic characteristics, namely, high dominance and robustness in elites of sizes in the range of $\sqrt{m} \leq k_c \leq SP$, hold also for c-core elites. It is worth mentioning that in the setting of c-core elites, it does not hold for all networks that dominance is achieved before robustness.

Finally, turning to dynamic data, in Fig. 9 we repeated in the c-core setting the analysis
Figure 9: The fraction of the elite at the symmetry point, from the whole population, as the networks evolve over time. 4 examples and the median out of 8 networks. Elites are chosen by c-core method.

of Fig. 5 from Section 5 for the k-rich-club setting. Specifically, we looked at the percentage of the number of vertices in the elite, \( r = \frac{|\mathcal{E}_{sp}|}{n} \), for elite sizes \( |\mathcal{E}_{sp}| \) at the symmetry point.

We used the same assumptions made in Section 5 about the time information and followed the same procedure as described therein. Fig. 9 depicts the ratio \( r \) as it evolves in time. Figure 9 shows the results for four example networks (this is a different set from the nine networks studied above, and the median result of all 8 networks for which evolutionary data was available to us. Once again, the results in the setting of c-core elites are not as smooth as in the setting of k-rich-club elites. Nevertheless, here too the figures demonstrate that the elite size at the symmetry point is a relatively small portion of the entire network (starting at median of 9% and ending at median of 6%). Furthermore, elites chosen by the c-core
method exhibit similar behavior of evolution and growth as the \( k \)-rich-club setting. Here too the ratio \( r \) decrease, implying that asymptotically, the elite size at the symmetry point is indeed sub-linear.
### D Datasets

| Data      | Repository | $n$  | $m$   | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description                                                                 |
|-----------|------------|------|-------|----------------|------------|----------------------|------------------|-----------------------------------------------------------------------------|
| Academia  | BGU        | 200169 | 1022441 | 5.1            | 10693      | D                    |                  | Platform for academics to share research papers.                           |
| Blog      | ASU        | 88784  | 2093195 | 23.6           | 9444       | U                    |                  | Social blog directory which manages bloggers their blog                   |
| Blog2     | ASU        | 97884  | 1668647 | 17.0           | 27849      | U                    |                  | Social blog directory which manages bloggers their blog                   |
| Blog3     | ASU        | 10312  | 333983  | 32.4           | 3992       | U                    |                  | Social blog directory which manages bloggers their blog                   |
| Buzznet   | ASU        | 101163 | 2763066 | 27.3           | 64289      | U                    |                  | Photo, journal, and video-sharing social media network                     |
| Delicious | ASU        | 536408 | 1366136 | 2.5            | 3216       | U                    |                  | Social bookmarking web service for storing sharing and discovering web bookmarks |
| Digg      | ASU        | 771229 | 5907413 | 7.7            | 17643      | U                    |                  | Social news website                                                        |

Continued on next page
| Data        | Repository | $n$   | $m$    | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description                                                                 |
|-------------|------------|-------|--------|----------------|------------|---------------------|------------------|-----------------------------------------------------------------------------|
| Douban      | ASU        | 154908| 327162 | 2.1            | 287        | U                   |                  | Chinese website providing user review and recommendation services for movies books and music |
| Flickr      | ASU        | 80513 | 5899882| 73.3           | 5706       | U                   |                  | An image hosting and video hosting website web services suite and online community |
| Flixster    | ASU        | 252336| 7918801| 3.1            | 1474       | U                   |                  | Social movie site allowing users to share movie ratings discover new movies and meet others with similar movie taste |
| Foursquare  | ASU        | 639014| 3214986| 5.0            | 106218     | U                   |                  | Location-based social networking website software for mobile devices. This service is available to users with GPS enabled mobile devices such as iPhones and Blackberries |

Continued on next page
| Data     | Repository | \( n \)  | \( m \)  | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description |
|----------|------------|----------|----------|----------------|------------|----------------------|------------------|-------------|
| Friendster | ASU         | 5689498  | 14067887 | 2.5            | 4423       | U                    |                  | Social networking website. The service allows users to contact other members maintain those contacts and share online content and media with those contacts. |
| Hyves    | ASU         | 1402673  | 2777419  | 2.0            | 31883      | U                    |                  | The most popular social networking site in the Netherlands with mainly Dutch visitors and members and competes in this country with sites such as Facebook and MySpace. |
| LastFm   | ASU         | 1191812  | 4519340  | 3.8            | 5150       | U                    |                  | Music website founded in the United Kingdom in 2002. It has claimed over 40 million active users based in more than 190 countries. |

Continued on next page
| Data          | Repository | $n$     | $m$     | Average Degree | Max Degree | Undirected | Duration (months) | Description                                                                 |
|---------------|------------|---------|---------|----------------|------------|-------------|------------------|--------------------------------------------------------------------------------|
| LiveJournal   | ASU        | 2238731 | 12816184 | 5.7            | 5873       | U           |                  | Virtual community where Internet users can keep a blog journal or diary       |
| Livemocha     | ASU        | 104103  | 2193083 | 21.1           | 2980       | U           |                  | The world’s largest online language learning community offering free and paid online language courses in 35 languages to more than 6 million members from over 200 countries around the world. |
| Twitter       | ASU        | 11316811 | 63555749 | 5.6            | 564795     | D           |                  | Social news website. It can be viewed as a hybrid of email instant messaging and sms messaging all rolled into one neat and simple package. It’s a new and easy way to discover the latest news related to subjects you care about. |

Continued on next page
| Data          | Repository | $n$   | $m$   | Average Degree | Max Degree | Undirected | Duration (months) | Description                                                                 |
|---------------|------------|-------|-------|----------------|------------|-------------|-------------------|-----------------------------------------------------------------------------|
| YouTube       | ASU        | 13723 | 76765 | 5.6            | 534        | $U$         |                   | Video-sharing website on which users can upload share and view videos      |
| YouTube-2     | ASU        | 13242 | 1940806 | 146.6 | 3068 | $U$         |                   | Video-sharing website on which users can upload share and view videos      |
| YouTube-3     | ASU        | 11765 | 5574249 | 473.8 | 6745 | $U$         |                   | Video-sharing website on which users can upload share and view videos      |
| YouTube-4     | ASU        | 10455 | 2239440 | 214.2 | 5958 | $U$         |                   | Video-sharing website on which users can upload share and view videos      |
| YouTube-5     | ASU        | 13160 | 3797635 | 288.6 | 5759 | $U$         |                   | Video-sharing website on which users can upload share and view videos      |

Continued on next page
| Data                | Repository | \( n \) | \( m \) | Average Degree | Max Degree | Undirected | Duration (months) | Description                                                                 |
|---------------------|------------|---------|--------|----------------|------------|-------------|------------------|-----------------------------------------------------------------------------|
| AnyBeat             | BGU        | 12645   | 49132  | 3.9            | 4800       | D           |                  | Online community. A public gathering place where you can interact with people from around your neighborhood or across the world |
| GooglePlus          | BGU        | 211186  | 1141650| 5.4            | 1790       | D           |                  | Google+ is a social networking service and website offered by Google        |
| TheMarkerCafe       | BGU        | 69413   | 1644843| 23.7           | 8930       | U           |                  | Israeli online social network site that allows its member to connect and interact |
| arXivHep-thKDDCup   | Konect     | 27769   | 352285 | 12.7           | 2468       | D           |                  | Citation network from arXiv’s section on high energy physics theory (hep-th) as used in the KDD cup 2003. |
| BaiduInternal      | Konect     | 2140198 | 17014946| 8.0            | 97848      | D           |                  | Hyperlinks between the articles of the Chinese online encyclopedia Baidu. |

Continued on next page
| Data             | Repository | $n$   | $m$    | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description                                                                 |
|------------------|------------|-------|--------|----------------|------------|----------------------|------------------|-----------------------------------------------------------------------------|
| BaiduRelated     | Konect     | 415624| 2374044| 5.7            | 127066     | D                    |                  | "Related to" links between articles of the Chinese online encyclopedia Baidu. |
| Catster          | Konect     | 149684| 5448196| 36.4           | 80634      | U                    |                  | This Network contains friendships between users of the website catster.com. |
| CatsterDogster   | Konect     | 623748| 15695166| 25.2          | 80636      | U                    |                  | Family links between cats and dogs as well as dogs and dogs from the social websites catster.com and dogster.com. Also included are cat-cat and dog-dog friendships. |
| CiteSeer         | Konect     | 384054| 1736172| 4.5            | 1739       | D                    |                  | Citation network extracted from the CiteSeer digital library.               |
| CoraCitation     | Konect     | 23166 | 89157  | 3.8            | 377        | D                    |                  | Cora citation network.                                                      |

Continued on next page
| Data     | Repository | $n$   | $m$     | Average Degree | Max Degree | Undirected | Directed | Duration (months) | Description                                                                                                                                 |
|----------|------------|-------|---------|----------------|------------|-------------|----------|-------------------|------------------------------------------------------------------------------------------------------------------------------------------|
| DBLP     | Konect     | 1103412 | 4225686 | 3.8            | 1189       | U           |         | 913               | Collaboration graph of authors of scientific papers from DBLP computer science bibliography                                           |
| Digg     | Konect     | 30360  | 85155   | 2.8            | 283        | D           |         |                   | Reply network of the social news website Digg                                                                                             |
| Dogster  | Konect     | 426816 | 8543548 | 20.0           | 46503      | U           |         |                   | This Network contains friendships between users of the website dogster.com.                                                                |
| Epinions | Konect     | 131580 | 711210  | 5.4            | 3558       | D           |         | 31                | Trust and distrust network of Epinions, an online product rating site                                                                     |
| Facebook | Konect     | 45813  | 183412  | 4.0            | 223        | D           |         | 52                | Directed network of a small subset of posts to other user’s wall on Facebook                                                             |
| Facebook-WOSN | Konect | 63731  | 817090  | 12.8           | 1098       | U           |         | 29                | Friendship data of facebook users                                                                                                      |
| Flickr   | Konect     | 2302925 | 22838276 | 9.9            | 27937      | D           |         | 7                 | Social network of users and their friendship connections                                                                                   |
| Data             | Repository | $n$   | $m$   | Average Degree | Max Degree | Undirected | Duration (months) | Description                                                                 |
|------------------|------------|-------|-------|----------------|------------|-------------|------------------|-----------------------------------------------------------------------------|
| FlickrLinks      | Konect     | 1715255 | 15555041 | 9.1            | 27236      | U           |                  | Social network of Flickr users and their connections.                      |
| HudongInternal   | Konect     | 1974655 | 14428382 | 7.3            | 61440      | D           |                  | Hyperlinks between articles of the Chinese online encyclopedia Hudong.     |
| HudongRelated    | Konect     | 2452673 | 18691099 | 7.6            | 204282     | D           |                  | ”Related to” links between articles of the Chinese online encyclopedia Hudong. |
| LibimsetiCZ      | Konect     | 220970  | 17233144 | 78.0           | 33389      | D           |                  | Czech dating site. This is the network of ratings given by users to other users. |
| PrettyGoodPrivacy| Konect     | 10680  | 24316  | 2.3            | 205        | U           |                  | Interaction network of users of the Pretty Good Privacy (PGP) algorithm.    |
| Slashdot         | Konect     | 51083  | 116573 | 2.3            | 2915       | D           | 33               | Reply network of technology website                                         |

Continued on next page
| Data         | Repository | $n$   | $m$   | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description                                                                 |
|--------------|------------|-------|-------|----------------|------------|----------------------|------------------|---------------------------------------------------------------------------|
| SlashdotZoo  | Konect     | 79120 | 467869| 5.9            | 2537       | D                    |                  | Signed social network of users of the technology news site Slashdot connected by directed "friend" and "foe" relations. |
| TRECWT10g    | Konect     | 1601787| 6679248| 4.2            | 25609      | D                    |                  | Web Research Collections (TREC Web, Terabyte and Blog Tracks)              |
| TwitterICWSM | Konect     | 465017| 833541| 1.8            | 677        | D                    |                  | Who follows whom on Twitter.                                              |
| USpatents    | Konect     | 3774768| 16518947| 4.4           | 793        | D                    |                  | Citation network of patents registered with the United States Patent and Trademark Office. |
| WikipediaChinese | Konect | 1930270| 8956902| 4.6            | 29005      | D                    |                  | Network of hyperlinks between the articles of Wikipedia in Chinese.        |

Continued on next page
| Data                  | Repository | $n$     | $m$       | Average Degree | Max Degree | Undirected | Duration (months) | Description                                                                 |
|----------------------|------------|---------|-----------|----------------|------------|-------------|-------------------|-----------------------------------------------------------------------------|
| WikipediaEnglish     | Konect     | 1870709 | 36532531  | 19.5           | 226073     | D           | 75                | Hyperlink network of the English Wikipedia with edge arrival times.         |
| WikipediaLinksDE     | Konect     | 3783012 | 68714064  | 18.2           | 437732     | D           |                   | Wikilinks inside the German Wikipedia.                                    |
| WikipediaLinksFR     | Konect     | 4905934 | 104591689 | 21.3           | 1274642    | D           |                   | Wikilinks inside the French Wikipedia.                                     |
| WikipediaLinksIT     | Konect     | 2790019 | 86754664  | 31.1           | 825147     | D           |                   | Wikilinks inside the Italian Wikipedia.                                    |
| WikipediaLinksJA     | Konect     | 2140579 | 58200970  | 27.2           | 390239     | D           |                   | Wikilinks inside the Japanese Wikipedia.                                   |
| WikipediaLinksPL     | Konect     | 1646203 | 41216900  | 25.0           | 215361     | D           |                   | Wikilinks inside the Polish Wikipedia.                                    |
| WikipediaLinksPT     | Konect     | 2804569 | 51539953  | 18.4           | 628617     | D           |                   | Wikilinks inside the Portuguese Wikipedia.                                 |
| WikipediaLinksRU     | Konect     | 2819989 | 64066427  | 22.7           | 587438     | D           |                   | Wikilinks inside the Russian Wikipedia.                                   |

Continued on next page
| Data      | Repository | $n$  | $m$    | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description                                                                                                                                                                                                 |
|-----------|------------|------|--------|----------------|------------|----------------------|------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| YouTube   | Konect     | 3223589 | 9376594 | 2.9            | 91751      | U                    | 7                | Video-sharing website on which users can upload share and view videos. Social network of users and their friendship connections                                                                                     |
| Amazon    | SNAP       | 334863 | 925872 | 2.8            | 549        | U                    |                  | Based on ‘Customers Who Bought This Item Also Bought’ feature of the Amazon website. If a product i is frequently co-purchased with product j the graph contains an undirected edge from i to j.                               |
| amazon0302| SNAP       | 262111 | 899792 | 3.4            | 420        | D                    |                  | Amazon product co-purchasing network from March 2 2003                                                                                                                                                    |
| amazon0312| SNAP       | 400727 | 2349869 | 5.9            | 2747       | D                    |                  | Amazon product co-purchasing network from March 12 2003                                                                                                                                                     |

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Table 1 – continued from previous page

| Data               | Repository | $n$  | $m$   | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description                                                                 |
|--------------------|------------|------|-------|----------------|------------|---------------------|------------------|-----------------------------------------------------------------------------|
| as-Caida \[41\] [155] | SNAP       | 26475 | 53381 | 2.0            | 2628       | D                   |                  | The CAIDA AS Relationships Datasets, from January 2004 to November 2007     |
| as-Skitter \[42\] [155] | SNAP       | 1696415 | 11095298 | 6.5            | 35455      | U                   |                  | Internet topology graph, from traceroutes run daily in 2005               |
| ca-AstroPh \[49\] [158] | SNAP       | 18771 | 198050 | 10.6           | 504        | U                   |                  | Arxiv ASTRO-PH (Astro Physics) collaboration network                      |
| ca-CondMat \[50\] [158] | SNAP       | 23133 | 93439 | 4.0            | 279        | U                   |                  | Arxiv COND-MAT (Condense Matter Physics) collaboration network            |
| ca-HepPh \[51\] [158]  | SNAP       | 12006 | 118489 | 9.9            | 491        | U                   |                  | Arxiv HEP-PH (High Energy Physics - Phenomenology) collaboration network   |
| cit-Patents \[56\] [156] | SNAP       | 3774768 | 16518947 | 4.4            | 793        | D                   |                  | Citations made by patents granted between 1975 and 1999                  |

Continued on next page
| Data          | Repository | $n$   | $m$    | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description                                                                 |
|--------------|------------|-------|--------|----------------|------------|---------------------|------------------|-----------------------------------------------------------------------------|
| cit-HepPh    | SNAP       | 34546 | 420877 | 12.2           | 846        | D                   |                  | Arxiv HEP-PH (high energy physics phenomenology) citation graph. If a paper i cites paper j the graph contains a directed edge from i to j |
| cit-HepTh    | SNAP       | 27769 | 352285 | 12.7           | 2468       | D                   |                  | Arxiv HEP-TH (high energy physics theory) citation graph. If a paper i cites paper j the graph contains a directed edge from i to j |
| DBLP         | SNAP       | 317080| 1049866| 3.3            | 343        | U                   |                  | Co-authorship network from DBLP computer science bibliography, where two authors are connected if they publish at least one paper together. |
| ego-Gplus    | SNAP       | 107614| 12238285| 113.7        | 20127      | D                   |                  | Social circles from Google+                                                |
| ego-Twitter  | SNAP       | 81306 | 1342296| 16.5           | 3383       | D                   |                  | Social circles from Twitter                                                |

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| Data                  | Repository | $n$  | $m$     | Average Degree | Max Degree | Undirected | Directed | Duration (months) | Description                                                                 |
|-----------------------|------------|------|---------|----------------|------------|------------|----------|------------------|-----------------------------------------------------------------------------|
| email-Enron           | SNAP       | 36692| 183831  | 5.0            | 1383       | U          |          |                  | Email communication network from Enron                                     |
| email-EuAll           | SNAP       | 26509| 364481  | 1.4            | 7636       | D          |          |                  | Email network from a EU research institution                               |
| epinions              | SNAP       | 75879| 405740  | 5.3            | 3044       | D          |          |                  | General consumer review site. An online social network of Who-trust-whom.   |
| flickr                | SNAP       | 10593| 231692  | 21.9           | 5425       | U          |          |                  | Images sharing common metadata on Flickr                                    |
| higgs-twitter-friendship | SNAP     | 45663| 1250842 | 27.4           | 51386      | D          |          |                  | Spreading processes of the announcement of the discovery of a new particle with the features of the Higgs boson on 4th July 2012. |

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| Data                        | Repository | $n$   | $m$   | Average Degree | Max Degree | Undirected | Duration (months) | Description                                                                                                                                 |
|-----------------------------|------------|-------|-------|----------------|------------|-------------|-------------------|--------------------------------------------------------------------------------------------------------------------------------------------|
| higgs-twitter-mention       | SNAP       | 30253 | 436816| 1.4            | 22790      | D           |                   | Spreading processes of the announcement of the discovery of a new particle with the features of the Higgs boson on 4th July 2012.           |
| higgs-twitter-retweet       | SNAP       | 42508 | 732790| 1.7            | 31556      | D           |                   | Spreading processes of the announcement of the discovery of a new particle with the features of the Higgs boson on 4th July 2012.           |
| LiveJournal                 | SNAP       | 4846609 | 42851237 | 8.8      | 20333      | D           |                   | Virtual community where Internet users can keep a blog journal or diary                                                                      |
| LiveJournalCom             | SNAP       | 3997962 | 34681189 | 8.7       | 14815      | U           |                   | Virtual community where Internet users can keep a blog journal or diary                                                                      |

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| Data               | Repository | \( n \) | \( m \) | Average Degree | Max Degree | Undirected Directed | Duration (months) | Description                                                                 |
|--------------------|------------|----------|----------|----------------|------------|----------------------|-------------------|-----------------------------------------------------------------------------|
| loc-brightkite     | SNAP       | 58228    | 214078   | 3.7            | 1134       | U                    |                   | Location-based social networking service provider where users shared their locations by checking-in. |
| loc-gowalla        | SNAP       | 196591   | 950327   | 4.8            | 14730      | U                    |                   | Location-based social networking website where users share their locations by checking-in.     |
| Oregon-1-1         | SNAP       | 10670    | 22002    | 2.1            | 2312       | U                    |                   | AS peering information inferred from Oregon route-views between March 31 and May 26 2001 |
| Oregon-2-1         | SNAP       | 10900    | 31180    | 2.9            | 2343       | U                    |                   | AS peering information inferred from Oregon route-views between March 31 and May 26 2001 |
| p2p-Gnutella31     | SNAP       | 62586    | 147892   | 2.4            | 95         | D                    |                   | Gnutella peer to peer network from August 31 2002                             |

Continued on next page
| Data          | Repository | $n$     | $m$       | Average Degree | Max Degree | Undirected | Duration (months) | Description                                                                 |
|--------------|------------|---------|-----------|----------------|------------|-------------|--------------------|-----------------------------------------------------------------------------|
| Pokec        | SNAP       | 1632803 | 22301964  | 13.7           | 14854      | D           |                    | Pokec is the most popular on-line social network in Slovakia.               |
| roadNet-CA   | SNAP       | 1965206 | 2766607   | 1.4            | 12         | U           |                    | Road network of California                                                 |
| roadNet-PA   | SNAP       | 1088092 | 1541898   | 1.4            | 9          | U           |                    | Road network of Pennsylvania                                              |
| roadNet-TX   | SNAP       | 1379917 | 1921660   | 1.4            | 12         | U           |                    | Road network of Texas                                                     |
| slashdot1    | SNAP       | 77360   | 469180    | 6.1            | 2539       | D           |                    | Technology-related news website. Friend/foe network. Obtained in November 2008. |
| web-BerStan  | SNAP       | 685230  | 6649470   | 9.7            | 84230      | D           |                    | Nodes represent pages from berkely.edu and stanford.edu domains and directed edges represent hyperlinks between them. The data was collected in 2002. |

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| Data        | Repository | $n$   | $m$     | Average Degree | Max Degree | Undirected | Duration (months) | Description                                                                                                                                                           |
|------------|------------|-------|---------|----------------|------------|-------------|------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| web-Google | SNAP       | 875713| 4322051 | 4.9            | 6332       | D           |                  | Nodes represent web pages and directed edges represent hyperlinks between them. The data was released in 2002 by Google as a part of Google Programming Contest              |
| web-NotreDame | SNAP   | 325729| 1090108 | 3.3            | 10721      | D           |                  | Nodes represent pages from University of Notre Dame (domain nd.edu) and directed edges represent hyperlinks between them. The data was collected in 1999 by Albert Jeong and Barabasi. |
| web-Stanford | SNAP     | 281903| 1992636 | 7.1            | 38625      | D           |                  | Nodes represent pages from Stanford University (stanford.edu) and directed edges represent hyperlinks between them.                                                                 |

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Table 1: List of All Tested Networks

| Data        | Repository | $n$    | $m$    | Average Degree | Max Degree | Undirected | Directed | Duration (months) | Description |
|-------------|------------|--------|--------|----------------|------------|------------|----------|-------------------|-------------|
| wiki-talk   | SNAP       | 2394385| 4659565| 1.9            | 100029     | D          |          |                   | Wikipedia’s registered users talk pages. A directed edge from node i to node j represents that user i at least once once edited a talk page of user j. |
| YouTube     | SNAP       | 1134890| 2987624| 2.6            | 28754      | U          |          |                   | Video-sharing website on which users can upload share and view videos. Social network of users and their friendship connections |
## Dominance and robustness

| Data                        | $\rho_d$ | $\rho_r$ |
|-----------------------------|----------|----------|
|                             | $\sqrt{m}$ | SP      | $\sqrt{m}$ | SP      |
| Blog                        | 2.64     | 3.54     | 0.19       | 0.28    |
| Blog2                       | 6.52     | 6.16     | 0.17       | 0.16    |
| Blog3                       | 3.23     | 3.58     | 0.25       | 0.28    |
| Buzznet                     | 2.47     | 4.30     | 0.10       | 0.23    |
| Delicious                   | 0.35     | 2.62     | 0.04       | 0.38    |
| Digg                        | 0.88     | 1.96     | 0.13       | 0.51    |
| Douban                      | 0.18     | 4.50     | 0.07       | 0.22    |
| Flickr                      | 0.72     | 1.38     | 0.31       | 0.73    |
| Flixster                    | 0.40     | 5.50     | 0.02       | 0.18    |
| Foursquare                  | 1.20     | 2.35     | 0.04       | 0.43    |
| Friendster                  | 0.54     | 13.79    | 0.01       | 0.07    |
| Hyves                       | 0.32     | 6.28     | 0.02       | 0.16    |
| LastFm                      | 0.44     | 4.06     | 0.03       | 0.25    |
| LiveJournal                 | 0.33     | 3.07     | 0.02       | 0.33    |
| Livemocha                   | 0.81     | 2.57     | 0.10       | 0.39    |
| Twitter                     | 3.15     | 8.40     | 0.04       | 0.12    |
| YouTube                     | 0.51     | 1.89     | 0.07       | 0.53    |
| YouTube 2                   | 1.40     | 1.72     | 0.43       | 0.58    |
| YouTube 3                   | 2.66     | 2.16     | 0.60       | 0.46    |
| YouTube 4                   | 1.49     | 1.52     | 0.62       | 0.66    |
| YouTube 5                   | 3.21     | 2.39     | 0.60       | 0.42    |
| Academia                    | 0.27     | 1.65     | 0.06       | 0.61    |
| AnyBeat                     | 2.17     | 3.11     | 0.19       | 0.32    |
| GooglePlus                  | 0.38     | 0.91     | 0.19       | 1.10    |
| TheMarkerCafe               | 1.46     | 2.60     | 0.18       | 0.38    |
| arXivHep-thKDDCup Reference | 0.45     | 1.34     | 0.10       | 0.75    |
| BaiduInternal Reference     | 0.54     | 2.63     | 0.03       | 0.38    |
| BaiduRelated Reference      | 1.02     | 2.86     | 0.19       | 0.35    |
| CatsterDogster Social       | 1.08     | 2.32     | 0.17       | 0.43    |
| Catster Social              | 3.57     | 4.12     | 0.20       | 0.24    |
| CiteSeer Reference          | 0.19     | 2.12     | 0.01       | 0.47    |

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| Data                               | $\rho_d$ | $\sqrt{m}$ | SP | $\rho_r$ | $\sqrt{m}$ | SP |
|------------------------------------|----------|------------|----|----------|------------|----|
| CoraCitation Reference            | 0.30     | 1.83       | 0.05 | 0.55    |             |     |
| DBLP Contact                      | 0.09     | 1.36       | 0.07 | 0.73    |             |     |
| Digg Communication                | 0.34     | 1.99       | 0.09 | 0.50    |             |     |
| Dogster Social                    | 1.32     | 2.93       | 0.11 | 0.34    |             |     |
| Epinions Social                   | 0.88     | 1.85       | 0.18 | 0.54    |             |     |
| Facebook Communication            | 0.18     | 1.22       | 0.10 | 0.82    |             |     |
| Facebook WOSN Social              | 0.25     | 1.13       | 0.16 | 0.89    |             |     |
| FlickrLinks Social                | 0.62     | 1.19       | 0.22 | 0.84    |             |     |
| Flickr Social                     | 0.60     | 1.22       | 0.20 | 0.82    |             |     |
| HudongInternal Reference          | 0.81     | 3.89       | 0.01 | 0.26    |             |     |
| HudongRelated Reference           | 0.55     | 9.61       | 0.01 | 0.10    |             |     |
| LibimsetiCZ Social                | 1.15     | 3.18       | 0.08 | 0.31    |             |     |
| PrettyGoodPrivacy Contact         | 0.26     | 0.85       | 0.36 | 1.18    |             |     |
| SlashdotZoo Social                | 0.63     | 2.14       | 0.11 | 0.47    |             |     |
| Slashdot Communication            | 0.55     | 2.40       | 0.09 | 0.42    |             |     |
| TRECWT10g Reference               | 0.54     | 3.46       | 0.01 | 0.29    |             |     |
| TwitterICWSM Social               | 1.17     | 57.05      | 0.01 | 0.02    |             |     |
| USpatents Reference               | 0.03     | 1.40       | 0.04 | 0.71    |             |     |
| WikipediaChinese Reference        | 0.54     | 2.54       | 0.06 | 0.39    |             |     |
| WikipediaEnglish Reference        | 0.73     | 3.07       | 0.03 | 0.33    |             |     |
| WikipediaLinksDE Reference        | 0.58     | 2.62       | 0.01 | 0.38    |             |     |
| WikipediaLinksFR Reference        | 1.00     | 3.16       | 0.01 | 0.32    |             |     |
| WikipediaLinksIT Reference        | 0.99     | 2.62       | 0.03 | 0.38    |             |     |
| WikipediaLinksJA Reference        | 0.72     | 2.57       | 0.03 | 0.39    |             |     |
| WikipediaLinksPL Reference        | 0.63     | 1.97       | 0.08 | 0.51    |             |     |
| WikipediaLinksPT Reference        | 1.17     | 3.19       | 0.05 | 0.31    |             |     |
| WikipediaLinksRU Reference        | 0.76     | 2.24       | 0.02 | 0.45    |             |     |
| YouTube Social                    | 0.77     | 2.16       | 0.03 | 0.46    |             |     |
| Amazon                            | 0.07     | 3.05       | 0.01 | 0.33    |             |     |
| amazon0302                        | 0.07     | 2.14       | 0.02 | 0.47    |             |     |
| amazon0312                        | 0.11     | 2.42       | 0.01 | 0.41    |             |     |
| as-Caida                          | 2.18     | 6.03       | 0.07 | 0.17    |             |     |

Continued on next page
| Data                                | $\rho_d$ | $\sqrt{m}$ | SP | $\rho_r$ | $\sqrt{m}$ | SP |
|-------------------------------------|----------|------------|----|----------|------------|----|
| as-Skitter                          | 0.65     | 2.82       | 0.03 | 0.35     | 0.03       | 0.35 |
| ca-AstroPh                          | 0.35     | 1.17       | 0.20 | 0.85     | 0.20       | 0.85 |
| ca-CondMat                          | 0.23     | 1.41       | 0.11 | 0.71     | 0.11       | 0.71 |
| ca-HepPh                            | 0.34     | 0.42       | 1.49 | 2.41     | 1.49       | 2.41 |
| cit-HepPh                           | 0.31     | 1.62       | 0.07 | 0.62     | 0.07       | 0.62 |
| cit-HepTh                           | 0.45     | 1.34       | 0.10 | 0.75     | 0.10       | 0.75 |
| cit Patents                         | 0.03     | 1.40       | 0.04 | 0.71     | 0.04       | 0.71 |
| DBLP                                | 0.08     | 1.43       | 0.20 | 0.70     | 0.20       | 0.70 |
| ego-Gplus                           | 1.31     | 1.83       | 0.30 | 0.55     | 0.30       | 0.55 |
| ego-Twitter                         | 0.42     | 1.76       | 0.11 | 0.57     | 0.11       | 0.57 |
| email-Enron                         | 0.90     | 1.76       | 0.16 | 0.57     | 0.16       | 0.57 |
| email-EuAll                         | 4.13     | 10.07      | 0.05 | 0.10     | 0.05       | 0.10 |
| epinions                            | 0.76     | 1.65       | 0.20 | 0.61     | 0.20       | 0.61 |
| flickr                              | 0.35     | 0.93       | 0.55 | 1.08     | 0.55       | 1.08 |
| higgs-twitter-friendship            | 1.06     | 3.45       | 0.02 | 0.29     | 0.02       | 0.29 |
| higgs-twitter-mention               | 1.36     | 5.63       | 0.01 | 0.18     | 0.01       | 0.18 |
| higgs-twitter-retweet               | 1.63     | 6.83       | 0.01 | 0.15     | 0.01       | 0.15 |
| LiveJournal                         | 0.12     | 1.04       | 0.10 | 0.96     | 0.10       | 0.96 |
| LiveJournalCom                      | 0.10     | 1.08       | 0.11 | 0.93     | 0.11       | 0.93 |
| loc-brightkite                      | 0.39     | 1.39       | 0.18 | 0.72     | 0.18       | 0.72 |
| loc-gowalla                         | 0.38     | 1.23       | 0.13 | 0.81     | 0.13       | 0.81 |
| Oregon-1-1                          | 2.81     | 5.86       | 0.08 | 0.17     | 0.08       | 0.17 |
| Oregon-2-1                          | 1.97     | 3.10       | 0.18 | 0.32     | 0.18       | 0.32 |
| p2p-Gnutella31                      | 0.09     | 2.27       | 0.02 | 0.44     | 0.02       | 0.44 |
| Pokec                               | 0.08     | 1.53       | 0.04 | 0.65     | 0.04       | 0.65 |
| roadNet-CA                          | 0.01     | 0.87       | 0.01 | 1.15     | 0.01       | 1.15 |
| roadNet-PA                          | 0.01     | 0.97       | 0.01 | 1.03     | 0.01       | 1.03 |
| roadNet-TX                          | 0.01     | 0.87       | 0.01 | 1.16     | 0.01       | 1.16 |
| slashdot1                           | 0.63     | 2.14       | 0.11 | 0.47     | 0.11       | 0.47 |
| web-BerStan                         | 1.12     | 2.30       | 0.01 | 0.43     | 0.01       | 0.43 |
| web-Google                          | 0.26     | 2.31       | 0.01 | 0.43     | 0.01       | 0.43 |
| web-NotreDame                       | 0.42     | 1.66       | 0.12 | 0.60     | 0.12       | 0.60 |

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Table 2 – continued from previous page

| Data          | $\rho_d$ | $\rho_r$ |
|---------------|----------|----------|
|               | $\sqrt{m}$ | SP  | $\sqrt{m}$ | SP  |
| web-Stanford  | 1.14     | 2.56    | 0.01     | 0.39 |
| wiki-talk     | 3.06     | 9.48    | 0.05     | 0.11 |
| YouTube       | 0.57     | 1.96    | 0.04     | 0.51 |
| Median        | 0.59     | 2.28    | 0.07     | 0.44 |
| Average       | 0.95     | 3.41    | 0.13     | 0.51 |
| STD           | 1.06     | 5.90    | 0.19     | 0.33 |
| Min           | 0.01     | 0.42    | 0.01     | 0.02 |

Table 2: Domination and robustness values at the symmetry point and $\sqrt{m}$ of all Networks.

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