A Fast Noniterative Algorithm for Compressive Sensing Using Binary Measurement Matrices

Mahsa Lotfi and Mathukumalli Vidyasagar *

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Abstract

In this paper we present a new algorithm for compressive sensing that makes use of binary measurement matrices and achieves exact recovery of sparse vectors, in a single pass, without any iterations. Our algorithm is hundreds of times faster than \(\ell_1\)-norm minimization, and methods based on expander graphs (which require multiple iterations). Moreover, our method requires the fewest measurements amongst all methods that use binary measurement matrices. The algorithm can accommodate nearly sparse vectors, in which case it recovers the largest components, and can also accommodate noisy measurements.

Numerical experiments with randomly generated sparse vectors indicate that the sufficient conditions for our algorithm to work are very close to being necessary. In contrast, the best known sufficient condition for \(\ell_1\)-norm minimization to recover a sparse vector, namely the Restricted Isometry Property (RIP), is about thirty times away from being necessary. Therefore it would be worthwhile to explore alternate and improved sufficient conditions for \(\ell_1\)-norm minimization to achieve the recovery of sparse vectors.

1 Introduction

1.1 Summary

Compressive sensing refers to the recovery of sparse (or nearly sparse) high-dimensional vectors using a very small number of measurements, often far fewer than required by the Nyquist sampling theorem. To date there have been two different approaches to compressive sensing. The older, and more pervasive, approach is based on \(\ell_1\)-norm minimization, while a recent approach is based on using expander graphs. In \(\ell_1\)-norm minimization, the measurement matrix \(A\) has to satisfy a property known as the restricted isometry property (RIP). There are two approaches to the construction of measurement matrices satisfying the RIP, namely probabilistic and deterministic. In principle, probabilistic approaches lead to a smaller number of measurements. But in practice, for vectors of dimension a few million or less, deterministic methods require fewer measurements than probabilistic methods. Indeed, for vectors of dimension \(10^4\) or so, the probabilistic approach requires more measurements than the dimension of the vector! A more recent approach is based on expander graphs, and makes use of measurement matrices that are binary. If the expander
graph satisfies appropriate properties, this algorithm recovers the unknown vector \textit{exactly} in a finite number of iterations.

In the present paper, we propose a new algorithm for compressive sensing that makes use of a suitably constructed binary measurement matrix. Our algorithm differs from current algorithms in a number of ways. First, our algorithm is \textit{noniterative}, in that it requires only a single pass through the dimension of the unknown vector. If the unknown vector is sparse, our algorithm recovers the unknown vector \textit{exactly}. When the unknown vector is nearly but not exactly sparse, our algorithm recovers the dominant part of the unknown vector, that is, the components that are the largest in magnitude. When the unknown vector is sparse and there is measurement noise, our algorithm still recovers the unknown vector exactly, provided the support set of the noise has sufficiently small cardinality. Finally, when the unknown vector is nearly sparse and there is measurement noise, our algorithm recovers the support set of the dominant part of the unknown vector, that is, the locations of the largest components of the unknown vector. It is worth mentioning that with \(\ell_1\)-norm minimization, there is no guarantee that the support set of the dominant components would be recovered. Due to its noniterative nature, our method is by far the fastest among all known algorithms for compressive sensing, being hundreds of times faster than the existing algorithm based on expander graphs, and about a hundred times faster than \(\ell_1\)-norm minimization. It is shown that the binary measurement matrix used by us can be justified via both the RIP approach and the expander-graph approach. However, by a careful analysis, we are able to establish that our method requires the fewest number of measurements amongst all methods that use binary measurements.\footnote{There is another method based on chirp matrices that uses fewer measurements, but the associated algorithm runs more slowly, not just compared to ours, but also compared to other methods based on \(\ell_1\)-norm minimization.}

To illustrate the new algorithm and to compare it with existing algorithms, we generated 100 vectors at random with \(n = 20,000\) components, of which only \(k = 6\) are nonzero. Appropriate measurement matrices were generated using known sufficient conditions based on the restricted isometry property (for \(\ell_1\)-norm minimization), the expander-graph approach, and our approach. The performance of all approaches were compared, and as expected, our algorithm outperformed both \(\ell_1\)-norm minimization and the expander-graph based approach. However, an interesting phenomenon was then observed. We retained the measurement matrices designed for \(k = 6\), but increased the \textit{actual} number of nonzero components, to see at what value of \(k\) each of these approaches broke down (in terms of failure to recover all 100 randomly generated \(k\)-sparse \(n\)-vectors). Our algorithm broke down for \(k > 9\), suggesting that the sufficient condition derived here is very close to being necessary. In contrast, \(\ell_1\)-norm minimization continued to recover all 100 \(k\)-sparse \(n\)-vectors for \(k\) up to 208, even though the measurement matrix was guaranteed to work only for \(k = 6\). This suggests that the current approach of using the restricted isometry property to guarantee the recovery of sparse vectors leads to bounds that are enormously conservative. This would be an excellent topic for further research.

### 1.2 Precise Definition of Compressive Sensing

In this subsection we give a very precise formulation of what "compressive sensing" means. The terminology more or less follows that in \cite{1, 2}. Throughout, \(n\) denotes the dimension of the unknown vector, and \([n]\) stands for the set \(\{1, \ldots, n\}\). The symbol \(\text{supp}(x) \subseteq [n]\) denotes the "support" of a vector \(x \in \mathbb{R}^n\); that is

\[
\text{supp}(x) := \{i \in [n] : x_i \neq 0\}.
\]
If \( k < n \) is a specified integer, then \( \Sigma_k \subseteq \mathbb{R}^n \) denotes the set of \( k \)-sparse vectors, that is
\[
\Sigma_k := \{ x \in \mathbb{R}^n : |\text{supp}(x)| \leq k \}.
\]

Suppose \( \| \cdot \| \) is some specified norm on \( \mathbb{R}^n \), and \( k < n \) is a specified integer. Then the sparsity index \( \sigma_k(x, \| \cdot \|) \) is defined by
\[
\sigma_k(x, \| \cdot \|) := \min_{z \in \Sigma_k} \| x - z \|.
\]

For a given \( x \in \mathbb{R}^n \) and an integer \( k < n \), the symbols \( x_d \in \mathbb{R}^n \) and \( x_r \in \mathbb{R}^n \) denote respectively the dominant part and the residual part of \( x \). Thus \( x_d \) is the vector consisting of the \( k \) largest components by magnitude of \( x \) with the remaining components set equal to zero, and \( x_r = x - x_d \).

Following the notation in [2], we view compressive sensing as consisting of two maps: A measurement matrix \( A \in \mathbb{R}^{mn} \) where \( m < n \) is the number of measurements, and a decoding map \( \Delta : \mathbb{R}^m \rightarrow \mathbb{R}^n \).

**Definition 1** A pair \( (A, \Delta) \) is said to achieve exact sparse recovery of order \( k \) if
\[
\Delta(Ax) = x, \quad \forall x \in \Sigma_k.
\]

A pair \( (A, \Delta) \) is said to achieve stable sparse recovery of order \( k \) if there exists a constant \( C \) such that
\[
\| \Delta(Ax) - x \|_2 \leq C \sigma_k(x, \| \cdot \|_1), \quad \forall x \in \mathbb{R}^n.
\]

A pair \( (A, \Delta) \) is said to achieve robust sparse recovery of order \( k \) if there exist constants \( C, D \) such that, whenever \( \eta \in \mathbb{R}^m \) satisfies \( \| \eta \|_2 \leq \epsilon \), we have that
\[
\| \Delta(Ax + \eta) - x \|_2 \leq C \sigma_k(x, \| \cdot \|_1) + D \epsilon, \quad \forall x \in \mathbb{R}^n.
\]

It is easy to check that robust recovery implies stable recovery, which in turn implies exact recovery.

### 1.3 Compressive Sensing via \( \ell_1 \)-Norm Minimization

One of the most popular approaches to compressive sensing is \( \ell_1 \)-norm minimization. This approach is originally introduced in [3, 4] as a heuristic and is called “basis pursuit.” The method consists of defining the decoding map \( \Delta \) as
\[
\Delta(y) = \hat{x} := \arg\min_z \| z \|_1 \quad \text{s.t.} \quad y = Az
\]
in the case of noiseless measurements, and
\[
\Delta(y) = \hat{x} := \arg\min_z \| z \|_1 \quad \text{s.t.} \quad \| y - Az \|_2 \leq \epsilon
\]
in the case of noisy measurements. In several papers over the years, beginning with [5, 6], it is shown that \( \ell_1 \)-norm minimization achieves robust sparse recovery under suitable conditions. At present, sufficient conditions for \( \ell_1 \)-norm minimization to achieve robust sparse recovery are usually stated in terms of the so-called restricted isometry property, introduced in [5], defined next.
Definition 2 A matrix $A \in \mathbb{R}^{m \times n}$ is said to satisfy the restricted isometry property (RIP) of order $k$ with constant $\delta$ if

$$(1 - \delta)\|u\|_2^2 \leq \|Au\|_2^2 \leq (1 + \delta)\|u\|_2^2, \forall u \in \Sigma_k. \quad (7)$$

Available results show that $\ell_1$-norm minimization achieves robust sparse recovery provided the measurement matrix $A$ satisfies the RIP with a sufficiently small constant. The definitive results in this direction are derived in [7].

Theorem 1 (See Theorems 1.1 and 2.1 of [7].) Suppose that, for some number $t > 1$, the matrix $A$ satisfies the RIP of order $\lceil tk \rceil$ with constant $\delta < \sqrt{(t - 1)/t}$. Then $\ell_1$-norm minimization as in (6) achieves robust sparse recovery.

Theorem 2 (See [7, Theorem 2.2].) Suppose $t \geq 4/3$. Then for all $\xi > 0$ and all $k \geq 5/\xi$, there exists a matrix $A$ that satisfies the RIP of order $tk$ with constant $\delta_{tk} < \sqrt{(t - 1)/t} + \xi$, and a vector $x \in \Sigma_k$ such that

1. With the noise-free measurement $y = Ax$, the decoder map $\Delta$ defined in (5) fails to recover $x$.

2. With a noisy measurement $y = Ax + \eta$ where $\|\eta\|_2 \leq \epsilon$, the decoder map $\Delta$ defined in (6) fails to recover $x$.

This naturally raises the question as to how one might generate matrices that satisfy the RIP. There are two distinct approaches, namely probabilistic and deterministic. In [5] and several subsequent papers (of which we cite only [8] as an example), it is shown that if the matrix $A$ consists of suitably scaled i.i.d. samples of a sub-Gaussian random variable with zero mean and unit variance, then with high probability the resulting matrix $A$ satisfies the RIP. Moreover, the number of measurements is $m = O(k \log(n/k))$.

At present there are very few deterministic approaches to constructing matrices that satisfy the RIP. In fact the authors are aware of only three approaches. In [9], the author presents a deterministic method for constructing a binary measurement matrix that is guaranteed to satisfy the RIP. This paper is most relevant to our work. In [10], the authors begin with a discrete Fourier matrix (which is unitary), and give a method for choosing a subset of the rows so as to ensure that the resulting matrix satisfies the RIP. Finally, in [11], the authors interpret an earlier paper [12] to show that a suitably constructed “chirp” matrix satisfies the RIP. The number of measurements in both [9] and [10] is the same, and equals $q^2$ where $q$ is an appropriately chosen prime number. In the chirp matrix method [12], the number of measurements equals $q$ where $q$ is an appropriately chosen prime number.

The main criticism of deterministic methods for constructing measurement matrices is that $m = O(n^{1/2})$, compared to $m = O(k \log(n/k))$ for probabilistic methods. However, as shown in the Appendix, the $O$ symbol hides a huge constant. Table 7 in the Appendix shows the number of measurements required by both the probabilistic as well as deterministic methods. From this table it can be seen that deterministic methods require fewer measurements than probabilistic methods in most realistic situations. For this reason, in our numerical experiments we use only deterministic construction methods.

Note that any algorithm based on $\ell_1$-norm minimization will never recover a sparse vector exactly. This is because any implementation of $\ell_1$-norm minimization will make use of some convex optimization algorithm, which will be iterative in nature, and will stop when some termination
criterion is satisfied. As a consequence, the recovered vector \( \hat{x} \) will be close to but never exactly equal to the true vector \( x \). Moreover, the recovered vector \( \hat{x} \) is not necessarily \( k \)-sparse, nor is it the case that \( (\hat{x})_d \approx x_d \). In other words, \( \ell_1 \)-norm minimization is not guaranteed to recover the dominant part of the unknown vector \( x \). In contrast, the algorithm proposed here does do so, under suitable conditions.

1.4 Compressive Sensing Using Expander Graphs

A recent development is the application of ideas from algebraic coding theory to compressive sensing. In [13], a method called “sudocodes” is proposed, which is based on low density parity check (LDPC) codes, which are well-established in coding theory. The sudocodes method can recover sparse signals with high probability. Motivated by this method, Xu and Hassibi in [14] proposed a method based on expander graphs, which are a special type of bipartite graph. The measurement matrix used in [14] is the biadjacency matrix of an expander graph with an expansion factor of \( 3/4 \) or more, and is therefore binary. Using such a binary measurement matrix, in [14] an iterative algorithm is proposed for updating the estimate \( \hat{x} \) component by component. It is shown in [14] that every \( k \)-sparse \( n \)-dimensional vector can be recovered exactly in \( O(k \log n) \) iterations. In [15], the authors improve upon the Xu-Hassibi algorithm by using expander graphs with an expansion factor of \( 1 - \epsilon \), where \( \epsilon \leq 1/4 \). With this modification, it is shown in [15] that a \( k \)-sparse \( n \)-dimensional vector can be recovered in \( 2k \) or fewer iterations. However, making the “expansion factor” closer and closer to 1 increases the number of required measurements. A key issue in applying these algorithms is the construction of expander graphs with the requisite expansion factor. In both [14] and [15] the authors use Hoeffding’s inequality to show that randomly generated bipartite graphs are indeed expanders, and use randomly generated bipartite graphs in their numerical examples, without ascertaining whether they are actually expanders or not. Therefore the numerical examples given in [14, 15] do not correspond to the theory, strictly speaking.

2 The New Algorithm

We present straightaway our new algorithm, and show that it can exactly recover sparse signals in a single pass, without any iterations. Then we compare its performance with those of existing algorithms in subsequent sections.

2.1 The New Algorithm

Suppose a matrix \( A \in \{0, 1\}^{m \times n} \) has the following properties, referred to as the main assumption:

1. Every column \( a_j \) of \( A \) has precisely \( q \) entries of 1 and \( m - q \) entries of 0.
2. If \( a_j, a_t \) are distinct columns of \( A \), then \( \langle a_j, a_t \rangle \leq r - 1 \).

Suppose \( x \in \Sigma_k \) is a \( k \)-sparse \( n \)-dimensional vector, and define \( y = Ax \) to be the measurement vector. For a given index \( j \in [n] \), let \( \{v_1(j), \ldots, v_q(j)\} \subseteq [m] \) denote the \( q \) rows such that \( a_{ij} = 1 \). For an index \( j \in [n] \), the reduced measurement vector \( \bar{y}_j \in \mathbb{R}^q \) is defined as

\[
\bar{y}_j := [y_{v_1(j)} \ldots y_{v_q(j)}]^T.
\]

The main result is given next. Note that we use the standard notation \( \|v\|_0 \) to denote the number of nonzero components of a vector \( v \).

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2 These terms are defined precisely in subsequent sections.
**Theorem 3** Suppose \( x \in \Sigma_k \), \( y = Ax \). Then:

1. If \( j \not\in \text{supp}(x) \), then \( \|\bar{y}_j\|_0 \leq k(r - 1) \).

2. If \( j \in \text{supp}(x) \), then \( \bar{y}_j \) contains at least \( q - (k - 1)(r - 1) \) components that are all equal to \( x_j \).

**Proof:** For \( t \in [n] \), let \( e_t \in \mathbb{R}^n \) denote the \( t \)-th canonical basis vector, which has a 1 as its \( t \)-th element, and zeros elsewhere, and let \( 1_q \in \mathbb{R}^q \) denote the column vector consisting of all ones. Then we can write:

\[
x = \sum_{t \in \text{supp}(x)} x_t e_t,
\]

\[
y = Ax = \sum_{t \in \text{supp}(x)} x_t A e_t = \sum_{t \in \text{supp}(x)} x_t a_t,
\]

where \( a_t \) denotes the \( t \)-th column of \( A \). Therefore, for a fixed \( j \in [n] \) and \( l \in [q] \), we have that

\[
y_{v_l(j)} = \sum_{t \in \text{supp}(x)} x_t (a_t)_{v_l(j)}.
\]

Letting \( l \) range over \([q]\) shows that

\[
\bar{y}_j = \sum_{t \in \text{supp}(x)} x_t (\bar{a}_t)_j,
\]

where \((\bar{a}_t)_j\) is the reduced vector of \( a_t \) consisting of \((a_t)_{v_l(j)}, \ldots, (a_t)_{v_q(j)}\).

**Proof of (1):** Suppose \( j \not\in \text{supp}(x) \). Then \( j \not= t \) for all \( t \in \text{supp}(x) \). Therefore, according to item (ii) of the main assumption, we have that \( \langle a_j, a_t \rangle \leq r - 1 \). Recall that \( v_1(j), \ldots, v_q(j) \) are the row indices of column \( j \) that contain a 1. Therefore, for a fixed index \( t \not= j \), the number of 1’s in the set \( \{\langle a_t, v_l(j)\rangle, \ldots, \langle a_t, v_q(j)\rangle\} \) equals the inner product \( \langle a_j, a_t \rangle \) and thus cannot exceed \( r - 1 \). Therefore, for a fixed index \( t \in \text{supp}(x) \), the vector \( x_t(\bar{a}_t)_j \) contains no more than \( r - 1 \) nonzero entries. Substituting this fact into (8) shows that \( \bar{y}_j \) is the sum of at most \( k \) vectors (because \( x \) is \( k \)-sparse), each of which has no more than \( r - 1 \) nonzero entries. Therefore \( \|\bar{y}_j\|_0 \leq k(r - 1) \).

**Proof of (2):** Suppose \( j \in \text{supp}(x) \). Then we can write

\[
\bar{y}_j = \sum_{t \in \text{supp}(x)} x_t (\bar{a}_t)_j
\]

\[
= x_j 1_q + \sum_{t \in \text{supp}(x) \setminus \{j\}} x_t (\bar{a}_t)_j,
\]

because the “reduced vector” \((\bar{a}_t)_j\) consists of \( q \) 1’s, as denoted by \( 1_q \). By the same reasoning as in the proof of (1), it follows that

\[
\left\| \sum_{t \in \text{supp}(x) \setminus \{j\}} x_t (\bar{a}_t)_j \right\|_0 \leq (k - 1)(r - 1).
\]

Therefore at least \( q - (k - 1)(r - 1) \) terms in \( \bar{y}_j \) equal \( x_j \). \( \square \)

In view of Theorem 3, we can formulate an algorithm for the recovery of \( k \)-sparse vectors, as follows:

Note that there is no iterative process involved in the recovery – the estimate \( \hat{x} \) is generated after a single pass through all \( n \) indices.
New Recovery Algorithm

1: for \( j \in [n] \) do
2: Construct the reduced measurement vector \( \bar{y}_j \).
3: Find the number of the elements of \( \bar{y}_j \) that are nonzero; call it \( \nu \). \( \triangleright \) (In implementation, we find the number of elements that are greater than some tolerance \( \delta \).)
4: if \( \nu > q/2 \) then
5: Find a group of \( q/2 \) elements in \( \bar{y}_j \) that are equal; call this value \( \theta_j \). \( \triangleright \) (In implementation, we allow some tolerance here.)
6: \( \hat{x}_j = \theta_j \).
7: else
8: \( \hat{x}_j = 0 \)
9: end
10: end

Theorem 4 If \( x \) is \( k \)-sparse, and \( A \) satisfies the main assumption with \( q > 2k(r-1) \), then \( \hat{x} = x \).

Proof: Note \( q > 2k(r-1) \) implies that
\[
k(r-1) < q/2, q - (k-1)(r-1) > q - k(r-1) > q/2.
\]
Therefore, by Statement 1 of Theorem 3, it follows that if \( j \notin \text{supp}(x) \), then \( \|\bar{y}_j\|_0 \leq k(r-1) < q/2 \). Taking the contrapositive shows that if \( \|\bar{y}_j\|_0 \geq q/2 \), then \( j \in \text{supp}(x) \). Therefore, by Statement 2 of Theorem 3, it follows that at least \( q - (k-1)(r-1) > q - k(r-1) > q/2 \) elements of \( \bar{y}_j \) equal \( x_j \).

Next we present the extension of our basic algorithm to the cases of a sparse signal with measurement noise, and a nearly sparse signal.

2.2 Recovery of Sparse Signals with Measurement Noise

In previous work, the model for noisy measurements is that \( y = Ax + \eta \) where there is a prior bound of the form \( \|\eta\|_2 \leq \epsilon \). If \( x \in \Sigma_k \), then \( \sigma_k(x, \cdot \|1\|) = 0 \). Therefore, if robust sparse recovery is achieved, then the bound in (4) becomes \( \|\hat{x} - x\|_2 \leq D\epsilon \). However, our approach draws its inspiration from coding theory, wherein it is possible to recover a transmitted signal correctly provided the transmission is not corrupted in too many places. Therefore our noise model is that \( \|\eta\|_0 \leq M \).

In other words, it is assumed that a maximum of \( M \) components of the “true” measurement \( Ax \) are corrupted by additive noise, but there are no assumptions regarding the magnitude of the error signal \( \eta \). In this case it is shown that, by increasing the number of measurements, it is possible to recover the true sparse vector \( x \) perfectly.

Theorem 5 Suppose \( x \in \Sigma_k \), and that \( y = Ax + \eta \) where \( \|\eta\|_0 \leq M \). Suppose further that the matrix \( A \) satisfies the main assumption. Then

1. If \( j \notin \text{supp}(x) \), then \( \bar{y}_j \) contains no more than \( k(r-1) + M \) nonzero components.
2. If \( j \in \text{supp}(x) \), then \( \bar{y}_j \) contains at least \( q - [(k-1)(r-1) + M] \) components that are all equal to \( x_j \).
3. Suppose the new recovery algorithm is applied with a measurement matrix \( A \) that satisfies the main assumption with \( q > 2[k(r-1) + M] \). Then \( \hat{x} = x \).
Proof: Suppose \( x \in \Sigma_k \) and let \( y = Ax + \eta \) where \( A \) satisfies the main assumption and \( \|\eta\|_0 \leq M \). Let \( u = Ax \) denote the uncorrupted measurement. For a fixed index \( j \in [n] \), let \( \bar{y}_j \in \mathbb{R}^q \) denote the reduced measurement vector, consisting of the components \( y_{v_1(j)} \), \( y_{v_2(j)} \), and define \( \bar{u}_j \in \mathbb{R}^q \) and \( \bar{\eta}_j \in \mathbb{R}^q \) analogously.

First suppose \( j \notin \text{supp}(x) \). Then it follows from Item (1) of Theorem 3 that \( \|\bar{u}_j\|_0 \leq k(r - 1) \). Moreover, because \( \eta \) has no more than \( M \) nonzero components and \( \bar{\eta}_j \) is a subvector of \( \eta \), it follows that \( \|\bar{\eta}_j\|_0 \leq M \). Therefore

\[
\|\bar{\eta}_j\|_0 = \|\bar{u}_j + \bar{\eta}_j\|_0 \leq \|\bar{u}_j\|_0 + \|\bar{\eta}_j\|_0 \leq k(r - 1) + M.
\]

This is Item (1) above. Next, suppose that \( j \in \text{supp}(x) \). Then it follows from Item (1) of Theorem 3 that at least \( q - (k - 1)(r - 1) \) elements of \( \bar{u}_j \) equal \( x_j \). Because \( \|\bar{\eta}_j\|_0 \leq M \), it follows that at least \( q - (k - 1)(r - 1) - M \) components of \( \bar{y}_j \) equal \( x_j \). This is Item (2) above. Finally, if \( q > 2k(r - 1) + 2M \), it follows as in the proof of Theorem 4 that \( \hat{x} = x \). \( \square \)

Note that the assumption on the noise signal \( \eta \) can be modified to \( \|\bar{\eta}_j\|_0 \leq M \) for each \( j \in [n] \). In other words, instead of assuming that \( \eta \) has no more than \( M \) nonzero components, one can assume that every reduced vector \( \bar{\eta}_j \) has no more than \( E \) nonzero components.

### 2.3 Recovery of Nearly Sparse Signals

As before, if \( x \notin \Sigma_k \), then let \( x_d \in \mathbb{R}^n \) denote the projection of \( x \) onto its \( k \) largest components, and let \( x_r = x - x_d \). We refer to \( x_d, x_r \) as the dominant part and the residual respectively. Note that, for any \( p \in [1, \infty] \), we have that the sparsity index \( \sigma_k(x, \|\cdot\|_p) \) equals \( \|x_r\|_p \). To (nearly) recover such a vector, we modify the New Recovery Algorithm slightly. Let \( \delta \) be a specified threshold.

**Modified Recovery Algorithm**

1. for \( j \in [n] \) do
2. Construct the reduced measurement vector \( \bar{y}_j \).
3. Find the number of the elements of \( \bar{y}_j \) that are greater than \( \delta \) in magnitude; call it \( \nu \).
4. if \( \nu > q/2 \) then
5. Find a group of \( q/2 \) elements in \( \bar{y}_j \) such that the difference between the largest and smallest elements is no larger than \( 2\delta \); Let \( \theta_j \) denote the average of these numbers.
6. \( \hat{x}_j = \theta_j \).
7. else
8. \( \hat{x}_j = 0 \)
9. end
10. end

**Theorem 6** Suppose \( x \in \mathbb{R}^n \) and that \( \sigma_k(x, \|\cdot\|_1) \leq \delta \). Write \( x = x_d + x_r \) where \( x_d \) is the dominant part of \( x \) consisting of its \( k \) largest components, and \( x_r = x - x_d \) is the residual. Let \( y = Ax \) where \( A \) satisfies the main assumption with \( q > 2k(r - 1) \), and apply the modified recovery algorithm. Then (i) \( \text{supp}(\hat{x}) = \text{supp}(x_d) \), and (ii) \( \|\hat{x} - x_d\|_\infty \leq \delta \).

**Proof:** Write \( x = x_d + x_r \) where \( x_d \) consists of the dominant part of \( x \) and \( x_r \) consists of the residual part. By assumption, \( \|x_r\|_1 \leq \delta \). Note that the measurement \( y \) equals \( Ax = Ax_d + Ax_r \). Let \( u = Ax_d \) and observe that \( x_d \in \Sigma_k \). Further, observe that, because the matrix \( A \) is binary, we
have that the induced matrix norm
\[ \|A\|_{1\rightarrow\infty} := \sup_{v \neq 0} \frac{\|Av\|_{\infty}}{\|v\|_{1}} = \max_{i,j} |a_{ij}| = 1. \]

Therefore \( \|Ax_r\|_{\infty} \leq \|x_r\|_1 \leq \delta \). Now, by Item (1) of Theorem 3, we know that if \( j \notin \text{supp}(x_d) \), then no more than \( k(r-1) \) components of the reduced vector \( \hat{y}_j \) are nonzero. Therefore then no more than \( k(r-1) \) components of the reduced vector \( \tilde{y}_j \) have magnitude more than \( \delta \). By Item (2) of Theorem 3, we know that if \( j \in \text{supp}(x_d) \), then at least \( q-(k-1)(r-1) \) components of \( \hat{u}_j \) equal \( x_j \). Therefore at least \( q-(k-1)(r-1) \) components of \( \hat{y}_j \) lie in the interval \([x_j-\delta,x_j+\delta]\). Finally, if \( q > 2k(r-1) \), then there is only one collection of \( q-(k-1)(r-1) > q/2 \) components of the reduced vector \( \tilde{y}_j \) that lie in an interval of width \( 2\delta \). The true \( x_j \) lies somewhere within this interval, and we can set \( \hat{x}_j \) equal to the midpoint of the interval containing all of these components. In this case \( |\hat{x}_j - x_j| \leq \delta \). Because this is true for all \( j \in \text{supp}(x_d) \), it follows that (i) \( \text{supp}(\hat{x}) = \text{supp}(x_d) \), and (ii) \( \|\hat{x} - x_d\|_{\infty} \leq \delta \). \( \square \)

Finally, it is easy to combine the two proof techniques and to establish the following theorem for the case where \( x \) is not exactly sparse and the measurements are noisy.

**Theorem 7** Suppose \( x \in \mathbb{R}^n \) and that \( \sigma_1(x,\|\cdot\|_1) \leq \delta \). Write \( x = x_d + x_r \) where \( x_d \) is the dominant part of \( x \) consisting of its \( k \) largest components, and \( x_r = x - x_d \) is the residual. Let \( y = Ax + \eta \) where \( \|\eta\|_0 \leq M \), and \( A \) satisfies the main assumption with \( q > 2k(r-1) + 2M \). Apply the modified recovery algorithm. Then (i) \( \text{supp}(\hat{x}) = \text{supp}(x_d) \), and (ii) \( \|\hat{x} - x_d\|_{\infty} \leq \delta \).

### 2.4 Construction of a Binary Measurement Matrix

The results presented until now show that the key to the procedure is the construction of a binary matrix \( A \) that satisfies the main assumption. In this subsection, it is shown that previous work by DeVore [9] provides a simple recipe for constructing a binary matrix with the desired properties. Note that [9] was the first paper to propose a completely deterministic procedure for constructing a matrix that satisfies the restricted isometry property. Therefore the reinterpretation of this construction to achieve sparse recovery in a highly efficient manner is one of the contributions of the present paper. We now describe the construction in [9].

Suppose \( q \) is a prime number. Then \( \mathbb{F}_q \) denotes the set \( \{0, 1, \ldots, q-1\} \) with arithmetic modulo \( q \), and is a field. If \( q = p^s \) where \( p \) is a prime number and \( s > 1 \), then it is possible to define a finite field \( \mathbb{F}_q \) with \( q \) elements by identifying an irreducible polynomial \( \psi \) of degree \( s \) with coefficients in \( \mathbb{F}_p \), and then doing arithmetic on the polynomial ring \( \mathbb{F}_p[z] \) modulo \( \psi(z) \). Note that for rapid implementation in **Matlab**, we would choose \( q \) to be a prime number, in which case \( \mathbb{F}_q \) denotes the set \( \{0, 1, \ldots, q-1\} \) with arithmetic modulo \( q \). However, the theory itself is applicable even to the case where \( q \) is a power of a prime number but not a prime number.

Suppose \( a \) is a polynomial of degree \( r-1 \) or less with coefficients in \( \mathbb{F}_q \), and define its “graph” as the set of all pairs \( (x, a(x)) \) as \( x \) varies over \( \mathbb{F}_q \). Now construct a vector \( u_a \in \{0, 1\}^{q^2 \times 1} \) by setting the entry in row \( (i, j) \) to 1 if \( j = a(i) \), and to zero otherwise. To illustrate, suppose \( q = 3 \), \( r = 4 \), and that \( a(x) = 1 + 2x + x^2 + x^3 \). Because \( q \) is a prime number, the associated field \( \mathbb{F}_q \) is just \( \{0, 1, 2\} \) with arithmetic modulo 3. Therefore \( a(0) = 1 \), \( a(1) = 2 \), and \( a(2) = 2 \). The corresponding \( 9 \times 1 \) column vector has 1’s in positions \( (0, 1), (1, 2), (2, 2) \) and zeros elsewhere. This construction results in a \( q^2 \times 1 \) column vector \( u_a \) that contains \( q \) elements of 1 and the rest equal to zero. In fact the column vector has even more structure, though we do not make use of it. The vector \( u_a \) consists of \( q \) blocks of size \( q \times 1 \), each of which contains a single 1 and \( q-1 \) zeros.
Now let \( \Pi_{r-1}(\mathbb{F}_q) \) denote the set of all polynomials of degree \( r - 1 \) or less with coefficients in \( \mathbb{F}_q \). In other words,

\[
\Pi_{r-1}(\mathbb{F}_q) := \left\{ a(x) = \sum_{i=0}^{r-1} a_i x^i, a_i \in \mathbb{F}_q \right\}.
\]

Note that \( \Pi_{r-1}(\mathbb{F}_q) \) contains precisely \( q^r \) polynomials, because each of the \( r \) coefficients can assume \( q \) different values.\(^3\) Now define

\[
A := [u_a, a \in \Pi_{r-1}(\mathbb{F}_q)] \in \{0,1\}^{q^2 \times q^r}. \tag{11}
\]

The following theorem shows that the matrix \( A \) constructed as above satisfies the main assumption.

**Theorem 8** (See [9, Theorem 3.1]) For the matrix \( A \in \{0,1\}^{q^2 \times q^r} \) defined in (11), we have that

\[
\langle u_a, u_b \rangle \leq r - 1 \quad (12)
\]

whenever \( a, b \) are distinct polynomials in \( \Pi_{r-1}(\mathbb{F}_q) \).

Theorem 8 suggests the following procedure for constructing a binary matrix that satisfies the conditions of Theorem 4. Given integers \( k \) and \( n \) with \( k \ll n \), choose a prime number \( q \) and an integer \( r \geq 3 \) such that \( q > 2k(r - 1) \) and \( q^r \geq n \). Construct a binary matrix \( A \in \{0,1\}^{q^2 \times q^r} \) as in (11), and then retain any \( n \) columns of this matrix, which we continue to refer to as \( A \). Theorem 4 can be applied using this matrix. Note that the two inequalities \( q > 2k(r - 1) \) and \( q^r \geq n \) provide lower bounds on \( q \). An upper bound is provided by the requirement that \( q^2 \leq n \), so that the number of measurements \( m \) does not exceed \( n \).

## 3 Compressive Sensing Using Expander Graphs

In the previous section it was shown that the construction of a binary matrix satisfying the main assumption can be interpreted in terms of the restricted isometry property. One of the objectives of this section is to show that the main assumption can also be interpreted in terms of expander graphs. This permits us to compare our algorithm with other algorithms in the literature based on expander graphs.

### 3.1 Basics of Expander Graphs

Expander graphs are the subject of much recent investigation in the compressive sensing literature. In this section we give a brief overview that covers only those aspects that are relevant to compressive sensing. We deal with undirected bipartite graphs, consisting of a set \( \mathcal{V}_I \) of input vertices, a set \( \mathcal{V}_O \) of output vertices, and an edge set \( \mathcal{E} \subseteq \mathcal{V}_O \times \mathcal{V}_I \), where \( (i,j) \in \mathcal{E} \) if and only if there is an edge between node \( i \in \mathcal{V}_O \) and node \( j \in \mathcal{V}_I \). The corresponding matrix \( A \in \{0,1\}^{\left|\mathcal{V}_O\right| \times \left|\mathcal{V}_I\right|} \) is called the **biadjacency matrix** of the bipartite graph. The graph is said to be **left-regular** of degree \( D \), or **D-left regular**, if every input node has degree \( D \). This is equivalent to requiring that every column of the biadjacency matrix \( A \) has exactly \( D \) elements equal to 1. Given an input vertex \( j \in \mathcal{V}_I \), let \( \mathcal{N}(i) \subseteq \mathcal{V}_O \) denote the set of its neighbors, defined as

\[
\mathcal{N}(j) := \{ i \in \mathcal{V}_O : (i,j) \in \mathcal{E} \}.
\]

\(^3\)If the leading coefficient of a polynomial is zero, then the degree would be less than \( r \).
Given set of input vertices $S \subseteq V_I$, the set of its neighbors $N(S) \subseteq V_O$ is defined as

$$N(S) := \bigcup_{j \in S} N(j) = \{i \in V_O : \exists j \in S \text{ s.t. } (i, j) \in E\}.$$ 

**Definition 3** A $D$-left regular bipartite graph $(V_I, V_O, E)$ is said to be a $(K, 1 - \beta)$-expander for some integer $K$ and some number $\beta \in (0, 1)$ if, for every $S \subseteq V_I$ with $|S| \leq K$, we have that $|N(S)| \geq (1 - \beta)D|S|$.

In [14], Xu and Hassibi introduce a new signal recovery algorithm in which the bi-adjacency matrix of an expander graph with $\beta \leq 1/4$ is used as the measurement matrix. It is referred to here as the “Expander Recovery Algorithm”. Xu and Hassibi show that their algorithm recovers an unknown $k$-sparse vector $x$ exactly in $O(k \log n)$ iterations. Subsequently, their method was updated in [15] by increasing the expansion factor from $1 - 1/4 = 3/4$ to $1 - \epsilon$ in which $\epsilon < \frac{1}{4}$. With this change, it is shown that the number of recovery iterations required is $O(k)$. However, the number of measurements is more than in the Xu-Hassibi algorithm.

**Expander Recovery Algorithm**

1: Initialize $x - 0_{n \times 1}$
2: if $Y = Ax$ then return output $x$ and exit
3: else
4: find a variable $x_j$ such that at least $(1 - 2\epsilon)D$ of the measurements it participates in have identical gap $g$
5: $x_j \leftarrow x_j + g$ and go to step 2
6: end if

3.2 Construction of Expander Graphs

The Expander Recovery Algorithm depends on using the biadjacency matrix of a left-regular expander graph as the measurement matrix. It is known that randomly generated left-regular graphs are expanders with high probability. However, given a graph with $n$ input nodes, it requires $O(n^K)$ operations to determine whether a particular (possibly randomly generated) left-regular graph is, or is not, a $(K, A)$ expander. Therefore in both [14] and [15] the authors simply use randomly generated graphs without checking whether they are expanders.

However, there are explicit construction methods that are guaranteed to produce an expander graph. One such procedure is given in [16, Section 3]. Since it is germane to our method, it is described here without proof.

Let $q$ be a prime number or a power of a prime number. Let $F_q$ denote the finite field with $q$ elements. For fixed integers $r$ and $s$, the bipartite graph has $F_q^r$ as the set of input vertices $V_I$, and $F_q^{s+1}$ as the set of output vertices. Let $E(Y)$ be any irreducible polynomial of degree $r$ with coefficients in $F_q$. Then we can identify $F_q^r$ with the set of all polynomials of degree $r - 1$ or less in some indeterminate $Y$, with coefficients in the field $F_q$. Addition and subtraction of polynomials is done in the usual manner, while multiplication is modulo $E(Y)$. In particular, if $\phi(Y)$ is a polynomial of degree $r - 1$ or less, and $j$ is any integer, then $\phi^j(Y)$ is interpreted modulo $E(Y)$ and is thus another polynomial of degree $r - 1$ or less, and thus belongs to $F_q^r$.

The construction in [16] is as follows: Let $h \geq 2$ be any integer. Then the map $\Gamma : F_q^r \times F_q \rightarrow F_q^{s+1}$ is defined as

$$\Gamma(f, y) := [y, f(y), f^h(y), f^{2h}(y), \ldots, f^{(s+1)}(y)]. \quad (13)$$
An alternate way to express the function $\Gamma$ is:

$$\Gamma(f, y) = [y, (f^{h_i}(y), i = 0, \ldots, s - 1)].$$

In the definition of the function $\Gamma$, $y$ ranges over $\mathbb{F}_q$ as the “counter,” and the above graph is left-regular with degree $q$. The set of input vertices is $\mathbb{F}_q$, consisting of polynomials in some indeterminate $Y$ with coefficients in $\mathbb{F}_q$ of degree no larger than $r - 1$. The set of input vertices has cardinality $q^r$. The set of output vertices is $\mathbb{F}^{s+1}$ and each output vertex is an $(s + 1)$-tuple consisting of elements from $\mathbb{F}_q$. The set of output vertices has cardinality $q^{s+1}$. Note that the graph is $q$-left regular in that every input vertex has exactly $q$ outgoing edges.

**Theorem 9** (See [16, Theorem 3.3].) For every pair of integers $h, s$, the bipartite graph defined in (13) is a $(h^s, 1 - \beta)$-expander with

$$\beta = \frac{(r - 1)(h - 1)s}{q}.$$

(14)

Now we relate the construction of DeVore with that in [16].

**Theorem 10** The matrix $A$ constructed in [9] is a special case of the graph in Theorem 9 with $s = 1$, and any value for $h$.

**Proof:** Suppose that $s = 1$ and that $h$ is any integer. In this case each polynomial $f$ with coefficients in $\mathbb{F}_q$ of degree $r - 1$ or less gets mapped into the pair $(y, f(y))$ as $y$ ranges over $\mathbb{F}_q$. This is precisely what was called the “graph” of the polynomial $f$ in [9]. $\square$

### 4 Computational Results

Theorems 8 and 10 show that the measurement matrix proposed by us (based on DeVore’s construction) falls within the ambit of both the restricted isometry property as well as expander graphs. Specifically, it follows from (12) that the column-normalized matrix $A' = (1/\sqrt{q})A$ satisfies the RIP of order $k$ with constant $\delta_k = (k - 1)(r - 1)/q$. Hence this matrix can be used together with $\ell_1$-norm minimization to achieve compressive sensing. Moreover, because the matrix $A$ satisfies the main assumption, it can also be used in our new algorithm. In this section, we compare the computational performance of our new algorithm with $\ell_1$-norm minimization as well as the Xu-Hassibi algorithm.

#### 4.1 Number of Measurements Required by Various Methods

In this subsection we compare the number of measurements required by $\ell_1$-norm minimization, expander graphs, and our method.

In $\ell_1$-norm minimization, as shown in Theorem 8, the matrix $A$, after column normalization dividing each column by $\sqrt{q}$, satisfies the RIP with constant $\delta_k = (k - 1)(r - 1)/q$. Combined with Theorem 1, we conclude that $\ell_1$-norm minimization with the DeVore construction achieves robust $k$-sparse recovery whenever

$$\frac{([tk] - 1)(r - 1)}{q} < \sqrt{\frac{t - 1}{t}}.$$  

(15)
To maximize the value of $k$ for which the above inequality holds, we set $r$ to its minimum permissible value, which is $r = 3$. Also, we replace $\lceil tk \rceil - 1$ by its upper bound $tk$, which leads to

\[
\frac{2tk}{q} < \sqrt{\frac{t-1}{t}}, \quad \text{or} \quad \frac{2k}{q} < \sqrt{\frac{t-1}{t^3}}.
\]

Elementary calculus shows that the right side is maximized when $t = 1.5$. So the RIP constant of the measurement matrix must satisfy

\[
\delta_{tk} < \sqrt{(t-1)/t} = 1/\sqrt{3} \approx 0.577.
\]

Let us choose a value of 0.5 for $\delta_{tk}$ to give some “cushion.” Substituting the values $t = 1.5, r = 2$ in (15) and ignoring the rounding operations finally leads to the condition

\[
\frac{3k}{q} < 0.5, \quad \text{or} \quad q > 6k.
\]  

For expander graphs, we can calculate the expansion factor $1 - \epsilon$ from Theorem 9. This gives

\[
\epsilon = \frac{(r - 1)(h - 1)s}{q}.
\]

Since we wish the expansion factor $1 - \epsilon$ to be as close to one as possible, or equivalently, $\epsilon$ to be as small as possible, we choose $s$ to be its minimum value, namely $s = 1$. Now we substitute $r = 3, h = 2k$ (following [14]), and set $1 - \epsilon \geq 3/4$, or equivalently $\epsilon \leq 1/4$. This leads to

\[
\frac{2(2k-1)}{q} \leq 1/4, \quad \text{or} \quad q \geq 8(2k-1).
\]

Finally, for the new algorithm, it has already been shown that $q \geq 2(r-1)k = 4k$. Therefore the required number of measurements for each of the three algorithms are as shown in Table 1. Note that, since the matrix $A$ has $q^3$ columns, we must also have that $n \leq q^3$.

### 4.2 Numerical Examples

In this section we present a numerical example to compare the three methods. We chose $n = 20,000$ to be the dimension of the unknown vector $x$. Since all three methods produce a measurement matrix with $m = q^2$ rows, we must have $q < 141 \approx \sqrt{20000}$, because otherwise the number of measurements would exceed the dimension of the vector! Since the expander graph method requires the most measurements, the sparsity count $k$ must satisfy $8(2k-1) < 141$, which gives $k \leq 9$. However, if we try to recover $k$-sparse vectors with $k = 9$ using the expander graph method,
Table 2: Recovery error and the processing time of the expander graph and our new recovery algorithms. The expander graph algorithm used $89^2 = 7,921$ measurements while the new algorithm used $29^2 = 841$ measurements. The column REC shows the number of correctly recovered vectors.

| Exp. Graph Method | New Algorithm |
|-------------------|---------------|
| $k$   | Error | Time | Rec. | Err. | Time | Rec. |
| 6   | 0     | 76   | 100  | 0    | 0.0951 | 100  |
| 9   | 0     | 122  | 100  | 0    | 0.096  | 100  |
| 10  | 0     | 136  | 100  | 0.0121 | 0.1045 | 98   |
| 12  | 0     | 153  | 100  | 0.0672 | 0.1446 | 80   |
| 15  | 0     | 158  | 100  | 0.5818 | 0.1348 | 27   |
| 16  | 0     | 212  | 100  | 1    | 0.1325 | 11   |
| 17  | 0     | 232  | 100  | 1.0327 | 0.2403 | 0    |
| 18  | 0     | 246  | 100  | 1.3179 | 0.1191 | 0    |
| 20  | 0     | 273  | 100  | 1.6910 | 0.1380 | 0    |
| 32  | 0     | 438  | 100  | 3.1917 | 0.2406 | 0    |
| 33  | $\approx$ 0 | 452 | 80  | 3.2620 | 0.2378 | 0    |
| 70  | $\approx$ 0 | 958 | 0   | 4.8240 | 0.1334 | 0    |
| 79  | $\approx$ 0 | 1072 | 0   | 5.1545 | 0.1320 | 0    |
| 80  | 4.12  | 1577 | 0   | 5.1472 | 0.1445 | 0    |
| 208 | 8.49  | 1733 | 0   | 8.3321 | 0.1533 | 0    |

Table 2: Recovery error and the processing time of the expander graph and our new recovery algorithms. The expander graph algorithm used $89^2 = 7,921$ measurements while the new algorithm used $29^2 = 841$ measurements. The column REC shows the number of correctly recovered vectors.

the number of measurements $m$ would be essentially equal to the dimension of the vector $n$. Hence we chose value of $k = 6$. With this choice, the values of $q$ and the number of measurements are shown in Table 1. Note that $q$ must be chosen as a prime number.

Having chosen the values of $n$ and $k$, we generated 100 different $k$-sparse $n$-dimensional vectors, with both the support set of size $k$ and the nonzero values of $x$ generated at random. As expected, both the expander graph method and the new algorithm recovered the unknown vector $x$ exactly in all 100 cases. The $\ell_1$-norm minimization method recovers $x$ with very small error.

Next, we examined how close the sufficient conditions in Table 1 are to being necessary. So for this purpose, we retained $n$ at 20,000 and retained the binary measurement matrix $A$ that is appropriate for each method, increased $k$ from 6 to ever-larger numbers, and noted how many of the 100 randomly generated vectors were recovered using each method. Note that the number of measurements for each method was as shown in Table 1. Thus the number of measurements for the expander graph method was roughly ten times that for our approach. The results are shown in Tables 2 and 3. It is important to note that the number of measurements with the new algorithm is roughly 10% of those used in the expander graph algorithm, and roughly 60% of those used in $\ell_1$-norm minimization.

Next, to make a “like to like” comparison, we ran the Xu-Hassibi algorithm and the new algorithm with the same number of measurements, namely $m = 89^2 = 7,921$, and compared their performance. Note that $q = 89$ is the smallest value of $q$ to satisfy the constraints for the Xu and Hassibi expander recovery algorithm with $k = 6$. The results are shown in Table 4. From this table, it can be seen that for the same number of measurements, our new algorithm achieves complete recovery for higher values of $k$ compared to Xu and Hassibi expander recovery algorithm ($k = 39$

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**Note:** Matlab codes are available from the authors.
Table 3: Recovery error and the processing time of the $l_1$-norm minimization and our new recovery algorithms. The $l_1$-norm minimization algorithm used $37^2 = 1,369$ measurements while our algorithm used $29^2 = 841$ measurements. The column REC shows the number of correctly recovered vectors.

| $k$ | $\ell_1$-Norm Min. | New Algorithm |
|-----|---------------------|---------------|
|     | $\approx 0$        | $\approx 0$   |
| 6   | 21.09              | 0             | 0.0951  | 100   |
| 9   | 15.89              | 0             | 0.096   | 100   |
| 10  | $\approx 0$        | 17.33         | 0.0121  | 100   |
| 12  | $\approx 0$        | 18.66         | 0.0672  | 80    |
| 15  | $\approx 0$        | 24.60         | 0.5818  | 27    |
| 16  | $\approx 0$        | 23.71         | 1       | 1325  | 11    |
| 17  | $\approx 0$        | 19.40         | 1.0327  | 2403  | 0     |
| 18  | $\approx 0$        | 16.26         | 1.3179  | 0.1191| 0     |
| 20  | $\approx 0$        | 31.80         | 1.6910  | 0.1380| 0     |
| 32  | $\approx 0$        | 16.83         | 3.1917  | 0.2406| 0     |
| 33  | $\approx 0$        | 14.51         | 3.2620  | 0.2378| 0     |
| 70  | $\approx 0$        | 27.25         | 4.8240  | 0.1334| 0     |
| 79  | $\approx 0$        | 26.58         | 5.1545  | 0.1320| 0     |
| 80  | $\approx 0$        | 29.47         | 5.1472  | 0.1445| 0     |
| 208 | $\approx 0$        | 50.94         | 8.3321  | 0.1533| 0     |
| 209 | 0.1342             | 34.75         | 8.3168  | 0.1191| 0     |

versus $k = 33$). Moreover, the computation time is one-thousandth of the time required by the expander recovery algorithm. If we compare the results in Table 2 and Table 4, we see that by increasing the value of $q$ in our algorithm, we are able to achieve exact recovery for higher values of $k$ (38 versus 32) compared to the expander graph algorithm.

Similarly, we ran $l_1$-norm minimization and the new algorithm with the same number of measurements, namely $m = 37^2 = 1,369$, and compared their performance. The results are shown in Table 5. We have already seen from Table 3 that with $q = 37$, $l_1$-norm minimization achieves exact recovery for far higher values of $k$ than our new algorithm with $q = 29$. When $q$ is increased to 37 in our algorithm, the same situation persists as can be seen from Table 5. However, the computational time of our algorithm continues to be far smaller than that for $l_1$-norm minimization, roughly one-hundredth.

As a final example, we introduced measurement noise into the output. As per Theorem 5, if $y = Ax + \eta$ where $\|\eta\|_0 \leq M$, then it is still possible to recover $x$ exactly by increasing the prime number $q$. Note that the only thing that matters here is the number of nonzero components of the noise $\eta$, and not their magnitudes. One would expect that, if the norm of the noise gets larger and larger, our algorithm would continue to recover the unknown sparse vector exactly, while $l_1$-norm minimization would not be able to. In other words, our algorithm is tolerant to “shot” noise whereas $l_1$-norm minimization is not. The computational results bear this out. We choose $n = 20,000$ and $k = 6$ as before, and $M = 6$, so that we perturb the true measurement $Ax$ in six locations. Specifically we chose $\eta = \alpha v$ where each component of $v$ is normally distributed, and then increased the scale factor $\alpha$. Each experiment was repeated with 100 randomly generated
### Table 4: Recovery error and the processing time of the Expander recovery algorithm and our new recovery algorithm using $89^2 = 7,921$ measurements. The column REC shows the number of correctly recovered vectors.

| k   | Error  | Time | Rec. | Error  | Time | Rec. |
|-----|--------|------|------|--------|------|------|
| 6   | 0      | 76.75| 100  | 0      | 0.1360| 100  |
| 9   | 0      | 122.98| 100  | 0      | 0.1304| 100  |
| 20  | 0      | 273.24| 100  | 0      | 0.1418| 100  |
| 32  | 0      | 438.24| 100  | 0      | 0.2004| 100  |
| 33  | 6.0666e-17 | 452.84 | 80  | 0      | 0.1345| 100  |
| 38  | 1.4415e-16 | 487.99 | 45  | 0      | 0.1394| 100  |
| 39  | 1.3761e-16 | 497.89 | 45  | 0.0068 | 0.1484| 99   |

### Table 5: Recovery error and the processing time of the $\ell_1$-norm minimization and our new recovery algorithms using $37^2 = 1,369$ measurements. The column REC shows the number of correctly recovered vectors.

| k   | $\ell_1$-norm minimization | New Method |
|-----|----------------------------|------------|
|     | Error  | Time | Error  | Time | Rec. |
| 6   | 1.044e-09 | 21.09 | 0      | 0.1345| 100  |
| 7   | 5.5505e-08 | 22.92 | 0      | 0.0851| 100  |
| 9   | 1.5985e-09 | 15.89 | 0      | 0.1117| 100  |
| 12  | 2.314e-08  | 18.66 | 0      | 0.1251| 100  |
| 13  | 1.0329e-07 | 26.59 | 0.0046 | 0.1443| 99   |
| 15  | 1.135e-08  | 24.60 | 0.0330 | 0.1155| 91   |

Table 4: Recovery error and the processing time of the Expander recovery algorithm and our new recovery algorithm using $89^2 = 7,921$ measurements. The column REC shows the number of correctly recovered vectors.

Table 5: Recovery error and the processing time of the $\ell_1$-norm minimization and our new recovery algorithms using $37^2 = 1,369$ measurements. The column REC shows the number of correctly recovered vectors.
sparse vectors and shot noise. The results are shown in Table 6.

| Alpha | New Algorithm | $\ell_1$-norm minimization |
|-------|---------------|-----------------------------|
|       | Alpha | Err. | Time | Rec. | Err. | Time | Rec. |
| $10^{-5}$ | 0 | 0.1335 | 100 | 3.2887e-06 | 26.8822 | 0 |
| $10^{-4}$ | 0 | 0.1325 | 100 | 3.2975e-05 | 26.6398 | 0 |
| $10^{-3}$ | 0 | 0.1336 | 100 | 3.3641e-04 | 28.1876 | 0 |
| $10^{-2}$ | 0 | 0.1357 | 100 | 0.0033 | 23.1727 | 0 |
| $10^{-1}$ | 0 | 0.1571 | 100 | 0.033 | 28.9145 | 0 |
| 10 | 0 | 0.1409 | 100 | 1.3742 | 26.6362 | 0 |
| 20 | 0 | 0.1494 | 100 | 1.3967 | 26.5336 | 0 |

Table 6: Performance of new algorithm and $\ell_1$-norm minimization with additive shot noise

5 Discussion and Conclusions

In this paper we have presented a new algorithm for compressive sensing that makes use of binary measurement matrices and achieves exact recovery of sparse vectors, without any iterations. Exact recovery continues to hold even when the measurements are corrupted by a noise vector with a sufficiently small support set; this noise model is reminiscent of the model used in algebraic coding. When the unknown vector is not exactly sparse, but is nearly sparse with a sufficiently small residual, our algorithm exactly recovers the support set of the dominant components, and finds an approximation for the dominant part of the unknown vector. Because our algorithm is noniterative, it executes orders of magnitude faster than algorithms based on $\ell_1$-norm minimization and methods based on expander graphs (both of which require multiple iterations). Moreover, our method requires the fewest measurements amongst all methods that use binary measurement matrices. On test examples of $k$-sparse $n$-dimensional vectors with $k = 6$ and $n = 20,000$, our algorithm executes roughly 1,000 times faster than the Xu-Hassibi algorithm [14] based on expander graphs, and roughly 200 times faster than $\ell_1$-norm minimization.

On the other hand, these two methods do have their own advantages over the algorithm proposed here. The Xu-Hassibi algorithm [14] and its extension in [15] can be used with any expander graph with an expansion factor that is sufficiently close to one. In contrast, our algorithm makes use of a particular family of expander graphs whose biadjacency matrix satisfies the “main assumption.” Similarly, if $\ell_1$-norm minimization is used to reconstruct a vector, then a bound of the form (4) holds no matter what the unknown vector is. In contrast, our error bounds require that the residual part of the unknown vector must be sufficiently small compared to the dominant part of the vector. This might not be a serious drawback however, because the objective of compressed sensing is to recover nearly sparse vectors, and not arbitrary vectors.

Computational results also show that $\ell_1$-norm minimization continues to recover the unknown vector for far larger values of $k$ than our method. Whereas we designed the measurement matrix for $k = 6$, $\ell_1$-norm minimization achieved sparse recovery up to $k = 208$. In contrast, our method worked only for $k$ up to 9. This suggests that our sufficient condition that provides a bound on $k$ is nearly tight, which is good. This also suggests that the known sufficient conditions for $\ell_1$-norm minimization to achieve robust sparse recovery, namely those based on the restricted isometry property (RIP), are overly conservative and can be improved. Therefore an interesting area for
further research is to find alternate sufficient conditions that are not based on the RIP for \( \ell_1 \)-norm minimization to achieve robust sparse recovery.

**Appendix**

In this appendix, we compare the number of measurements used by probabilistic as well as deterministic methods to guarantee that the corresponding measurement matrix \( A \) satisfies the restricted isometry property (RIP), as stated in Theorem 1. Note that the number of measurements is computed from the best available sufficient condition. In principle it is possible that matrices with fewer rows might also satisfy the RIP. But there would not be any theoretical justification for using such matrices.

In probabilistic methods, the number of measurements \( m \) is \( O(k \log(n/k)) \). However, in reality the \( O \) symbol hides a huge constant. It is possible to replace the \( O \) symbol by carefully collating the relevant theorems in [1]. This leads to the following explicit bounds.

**Theorem 11** Suppose \( X \) is a random variable with zero mean, unit variance, and suppose in addition that there exists a constant \( c \) such that

\[
E[\exp(\theta X)] \leq \exp(c\theta^2), \quad \forall \theta \in \mathbb{R}.
\]  

Define

\[
\gamma = 2, \zeta = 1/(4c), \alpha = \gamma e^{-\zeta} + e^\zeta, \beta = \zeta, \quad (18)
\]

\[\tilde{c} := \frac{\beta^2}{2(2\alpha + \beta)}. \quad (19)\]

Suppose an integer \( k \) and real numbers \( \delta, \xi \in (0, 1) \) are specified, and that \( A = (1/\sqrt{m})\Phi \), where \( \Phi \in \mathbb{R}^{m \times n} \) consists of independent samples of \( X \). Then \( A \) satisfies the RIP of order \( k \) with constant \( \delta \) with probability \( \geq 1 - \xi \) provided

\[
m \geq \frac{1}{c\delta^2} \left( \frac{4}{3} k \ln \frac{en}{k} + \frac{14k}{3} + \frac{4}{3} \ln \frac{2}{\xi} \right). \quad (20)
\]

In (20), the number of measurements \( m \) is indeed \( O(k \log(n/k)) \). However for realistic values of \( n \) and \( k \), the number of measurements \( n \) would be comparable to, or even to exceed, \( n \), which would render “compressed” sensing meaningless.\(^6\) For “pure” Gaussian variables it is possible to find improved bounds for \( m \) (see Also, for binary random variables where \( X \) equals \( \pm 1 \) with equal probability, another set of bounds is available [17]. While all of these bounds are \( O(k \log(n/k)) \), in practical situations the bounds are not useful.

This suggests that it is worthwhile to study deterministic methods for generating measurement matrices that satisfy the RIP. There are very few such methods. Indeed, the authors are aware of only three methods. The paper [9] uses a finite field method to construct a binary matrix, and this method is used in the present paper. The paper [10] gives a procedure for choosing rows from a unitary Fourier matrix such that the resulting matrix satisfies the RIP. This method leads to the same values for the number of measurements \( m \) as that in [9]. Constructing partial Fourier matrices is an important part of reconstructing time-domain sparse signals from a limited number of

\(^5\)Such a random variable is said to be sub-Gaussian. A normal random variable satisfies (17) with \( c = 1/2.\)

\(^6\)In many papers on compressed sensing, especially those using Gaussian measurement matrices, the number of measurements \( m \) is not chosen in accordance with any theory, but simply picked out of the air.
frequency measurements (or vice versa). Therefore the results of [10] can be used in this situation. In both of these methods, \( m = q^2 \) where \( q \) is appropriately chosen prime number. Finally, in [11] a method is given based on chirp matrices. In this case \( m \) equals a prime number \( q \). Note that the partial Fourier matrix and the chirp matrix are complex, whereas the method in [9] leads to a binary matrix. In all three methods, \( m = O(n^{1/2}) \), which grows faster than \( O(k \log(n/k)) \). However, the constant under this \( O \) symbol is quite small. Therefore for realistic values of \( k \) and \( n \), the bounds for \( m \) from these methods are much smaller than those derived using probabilistic methods.

Table 7 gives the values of \( m \) for various values of \( n \) and \( k \). Also, while the chirp matrix has fewer measurements than the binary matrix, \( \ell_1 \)-norm minimization with the binary matrix runs much faster than with the chirp matrix, due to the sparsity of the binary matrix. In view of these numbers, in the present paper we used DeVore’s construction as the benchmark for the recovery of sparse vectors.

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| $n$  | $k$ | $m_G$  | $m_{SG}$ | $m_A$  | $m_D$  | $m_C$  |
|------|-----|--------|----------|--------|--------|--------|
| $10^4$ |  5  |  5,333 |  28,973  |  3,492 |  841   |  197   |
| $10^4$ |  6  |  5,785 |  31,780  |  3,830 |  1,369 |  257   |
| $10^4$ |  7  |  6,674 |  37,308  |  4,496 |  1,681 |  401   |
| $10^4$ |  8  |  7,111 |  40,035  |  4,825 |  2,209 |  487   |
| $10^4$ |  9  |  7,972 |  45,424  |  5,474 |  2,809 |  677   |
| $10^4$ | 10  |  8,396 |  48,089  |  5,796 |  3,481 |  787   |
| $10^5$ | 10  | 10,025 |  57,260  |  6,901 |  3,481 |  787   |
| $10^5$ | 12  | 11,620 |  66,988  |  8,073 |  5,041 |  1,163 |
| $10^5$ | 14  | 13,190 |  76,582  |  9,229 |  6,889 |  1,601 |
| $10^5$ | 16  | 14,739 |  86,061  | 10,372 |  9,409 |  2,129 |
| $10^5$ | 18  | 16,268 |  95,441  | 11,502 | 11,449 |  2,707 |
| $10^5$ | 20  | 17,781 | 104,733  | 12,622 | 16,129 |  3,371 |
| $10^6$ |  5  |  7,009 |  38,756  |  4,671 | 10,201 |  1,009 |
| $10^6$ | 10  | 11,639 |  66,431  |  8,006 | 10,201 |  1,009 |
| $10^6$ | 15  | 16,730 |  96,976  | 11,687 | 10,201 |  1,949 |
| $10^6$ | 20  | 21,069 | 123,076  | 14,832 | 16,129 |  3,371 |
| $10^6$ | 25  | 25,931 | 152,373  | 18,363 | 22,201 |  5,477 |
| $10^6$ | 30  | 30,116 | 177,635  | 21,407 | 32,041 |  7,753 |
| $10^6$ | 50  | 47,527 | 283,042  | 34,110 | 94,249 | 21,911 |
| $10^6$ | 60  | 55,993 | 334,440  | 40,304 | 128,881| 31,687 |
| $10^6$ | 70  | 64,335 | 385,171  | 46,417 | 175,561| 43,271 |
| $10^6$ | 80  | 72,573 | 435,331  | 52,462 | 229,441| 56,659 |
| $10^6$ | 90  | 80,718 | 484,992  | 58,447 | 292,681| 71,837 |
| $10^6$ |100  | 88,781 | 534,210  | 64,378 | 358,801| 88,807 |

Table 7: Best available bounds for the number of measurements for various choices of $n$ and $k$ using both probabilistic and deterministic constructions. For probabilistic constructions, the failure probability is $\xi = 10^{-9}$. $m_G, m_{SG}, m_A$ denote respectively the bounds on the number of measurements using a normal Gaussian, a sub-Gaussian with $c = 1/2$, and a bipolar random variable and the bound of Achlioptas. For deterministic methods $m_D$ denotes the number of measurements using DeVore’s construction, while $m_C$ denotes the number of measurements using chirp matrices.
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