The Flavour Symmetry: $S_3$

A. Mondragón
Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, México 01000 D.F., México
E-mail: mondra@fisica.unam.mx

Abstract. By introducing in the theory three Higgs fields that are $SU(2)$ doublets and a flavour permutational symmetry, $S_3$, we extend the concepts of flavour and generations to the Higgs sector and formulate a Minimal $S_3$–Invariant Extension of the Standard Model. The mass matrices of the neutrinos and charged leptons are re-parameterized in terms of their eigenvalues, then the neutrino mixing matrix, $V_{PMNS}$, is computed and exact, explicit analytical expressions for the neutrino mixing angles as functions of the masses of neutrinos and charged leptons are obtained in excellent agreement with the latest experimental data. We also compute the branching ratios of some selected flavour-changing neutral current (FCNC) processes, as well as the contribution of the exchange of neutral flavour-changing scalars to the anomaly of the magnetic moment of the muon, as functions of the masses of charged leptons and the neutral Higgs bosons. We find that the $S_3 \times Z_2$ flavour symmetry and the strong mass hierarchy of the charged leptons strongly suppress the FCNC processes in the leptonic sector, well below the present experimental bounds by many orders of magnitude. The contribution of FCNC’s to the anomaly of the muon’s magnetic moment is small, but non-negligible.

1. Introduction
The observation of flavour oscillations of solar, atmospheric, reactor, and accelerator neutrinos established that they have non-vanishing masses and mix among themselves, much like the quarks do [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. This discovery brought out very forcibly the need of extending the Standard Model (SM) in order to accomodate in the theory the new data on neutrino physics in a consistent way that would allow a unified and systematic treatment of the observed hierarchy of masses and mixings of all fermions. At the same time, the number of free parameters in the extended form of the SM had to be drastically reduced in order to give predictive power to the theory. These two seemingly contradictory demands are met by a flavour symmetry under which the families transform in a non-trivial fashion.

In the Minimal $S_3$–Invariant Extension of the Standard Model [21, 22, 23, 24, 25, 26, 27], the concept of flavour and generations is extended to the Higgs sector in such a way that all the matter fields –Higgs, quarks, and lepton fields, including the right-handed neutrino fields– have three species and transform under the flavour symmetry group as the three dimensional representation $1 \oplus 2$ of the permutational group $S_3$. A model with more than one Higgs $SU(2)$ doublet has tree level flavour changing neutral currents whose exchange may give rise to lepton flavour violating processes and may also contribute to the anomalous magnetic moment of the muon. An effective test of the phenomenological success of the model is obtained by verifying that all flavour changing neutral current processes and the magnetic anomaly of the muon,
computed in the $S_3$-Invariant extended form of the Standard Model, agree with the experimental values.

2. The Minimal $S_3$-invariant Extension of the Standard Model

In the Standard Model analogous fermions in different generations have identical couplings to all gauge bosons of the strong, weak and electromagnetic interactions. Prior to the introduction of the Higgs boson and mass terms, the Lagrangian is chiral and invariant with respect to permutations of the left and right fermionic fields.

The six possible permutations of three objects $(f_1, f_2, f_3)$ are elements of the permutational group $S_3$. This is the discrete, non-Abelian group with the smallest number of elements. The three-dimensional real representation is not an irreducible representation of $S_3$. It can be decomposed into the direct sum of a doublet $f_D$ and a singlet $f_s$, where

$$f_s = \frac{1}{\sqrt{3}} (f_1 + f_2 + f_3),$$

$$f_D^s = \left( \frac{1}{\sqrt{2}} (f_1 - f_2), \frac{1}{\sqrt{6}} (f_1 + f_2 - 2f_3) \right).$$

The direct product of two doublets $pD^T = (p_D1, p_D2)$ and $qD = (q_D1, q_D2)$ may be decomposed into the direct sum of two singlets $r_s$ and $r_s'$, and one doublet $r_D^T$ where

$$r_s = pD1qD1 + pD2qD2, \quad r_s' = pD1qD2 - pD2qD1,$$

$$r_D^T = (r_D1, r_D2) = (pD1qD2 + pD2qD1, pD1qD1 - pD2qD2).$$

The antisymmetric singlet $r_s'$ is not invariant under $S_3$.

Since the Standard Model has only one Higgs $SU(2)_L$ doublet, which can only be an $S_3$ singlet, it can only give mass to the quark or charged lepton in the $S_3$ singlet representation, one in each family, without breaking the $S_3$ symmetry.

Hence, in order to impose $S_3$ as a fundamental symmetry, unbroken at the Fermi scale, we are led to extend the Higgs sector of the theory. The quark, lepton and Higgs fields are

$$Q^T = (u_L, d_L), \quad u_R, \quad d_R,$$

$$L^T = (\nu_L, e_L), \quad e_R, \quad \nu_R \quad \text{and} \quad H,$$

in an obvious notation. All of these fields have three species, and we assume that each one forms a reducible representation $1_S \oplus 2$. The doublets carry capital indices $I$ and $J$, which run from 1 to 2, and the singlets are denoted by $Q_3, \ u_{3R}, \ d_{3R}, \ L_3, \ e_{3R}, \ \nu_{3R}$ and $H_S$. Note that the subscript 3 denotes the singlet representation and not the third generation. The most general renormalizable Yukawa interactions of this model are given by

$$\mathcal{L}_Y = \mathcal{L}_Y + \mathcal{L}_Y^D + \mathcal{L}_Y^U + \mathcal{L}_Y^E + \mathcal{L}_Y^c,$$

where

$$\mathcal{L}_Y^D = -Y_{11}^D T_1 H_S e_{1R} - Y_{33}^D T_3 H_S e_{3R} - Y_{33}^D [T_1 H_{11} H_{1e_{JR}} + T_3 H_{31} H_{3e_{JR}}] - Y_{44}^D T_4 H_{1e_{1R}} - Y_{44}^D T_4 H_{1e_{3R}} + \text{h.c.},$$

$$\mathcal{L}_Y^U = -Y_{11}^U T_1 (i\sigma_2) H_S^H \nu_{1R} - Y_{33}^U T_3 (i\sigma_2) H_S^H \nu_{3R} - Y_{33}^U [T_1 (i\sigma_2) H_{11} \nu_{1R} + T_3 (i\sigma_2) H_{31} \nu_{3R}] - Y_{44}^U T_4 (i\sigma_2) H_{11} \nu_{1R} - Y_{44}^U T_4 (i\sigma_2) H_{11} \nu_{3R} + \text{h.c.},$$

$$\mathcal{L}_Y^E = -Y_{11}^E T_1 (i\sigma_2) H_S^H \nu_{1R} - Y_{33}^E T_3 (i\sigma_2) H_S^H \nu_{3R} - Y_{33}^E [T_1 (i\sigma_2) H_{11} \nu_{1R} + T_3 (i\sigma_2) H_{31} \nu_{3R}] - Y_{44}^E T_4 (i\sigma_2) H_{11} \nu_{1R} - Y_{44}^E T_4 (i\sigma_2) H_{11} \nu_{3R} + \text{h.c.},$$
Table 1. \( Z_2 \) assignment in the leptonic sector.

| \( H_S, \nu_{3R} \) | \(+\) |
|-----------------|-----|

and

\[ \kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (8)

\( \mathcal{L}_{Y_D} \) and \( \mathcal{L}_{Y_U} \) have similar expressions to \( \mathcal{L}_{Y_E} \) and \( \mathcal{L}_{Y_\nu} \) respectively.

Furthermore, we add to the Lagrangian the Majorana mass terms for the right-handed neutrinos

\[ \mathcal{L}_M = -M_1 \nu_{1R}^T C \nu_{1R} - M_3 \nu_{3R}^T C \nu_{3R}. \] (9)

The extended Higgs sector has three \( SU(2) \) doublets, in a reducible representation \( 1_{S} \circ 2 \) of the flavour group \( S_3 \). The Higgs potential, invariant under \( S_3 \), has an additional reflection symmetry \( R : H_s \rightarrow -H_s \) and an accidental permutational symmetry \( S'_2 : H_1 \leftrightarrow H_2 \). Hence, \( \langle H_1 \rangle = \langle H_2 \rangle \). Then the Yukawa interactions yield mass matrices for all fermions in the theory, of the general form [21]

\[ M = \begin{pmatrix} \mu_1 + \mu_2 & \mu_2 & \mu_5 \\ \mu_2 & \mu_1 - \mu_2 & \mu_5 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}. \] (10)

The Majorana masses for the left handed neutrinos \( \nu_L \) are generated by the see-saw mechanism. The corresponding mass matrix is given by

\[ M_\nu = M_{\nu_D} \check{M}^{-1} (M_{\nu_D})^T, \] (11)

where \( \check{M} = \text{diag}(M_1, M_1, M_3) \).

In principle, all entries in the mass matrices can be complex since there is no restriction coming from the flavour symmetry \( S_3 \). The mass matrices are diagonalized by bi-unitary transformations as

\[ U_{d(u,e)}^\dagger M_{d(u,e)} U_{d(u,e)} = \text{diag}(m_{d(u,e)}, m_{s(c,\mu)}, m_{b(t,\tau)}), \] (12)

\[ U_{\nu}^T M_\nu U_{\nu} = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}). \]

The entries in the diagonal matrices may be complex, so the physical masses are their absolute values.

The mixing matrices are, by definition,

\[ V_{CKM} = U_{uL}^\dagger U_{dL}, \quad V_{PMNS} = U_{eL}^\dagger U_{\nu} K. \] (13)

where \( K \) is the diagonal matrix of the Majorana phase factors.

3. The mass matrices in the leptonic sector and \( Z_2 \) symmetry

A further reduction of the number of parameters in the leptonic sector may be achieved by means of an Abelian \( Z_2 \) symmetry. A possible set of charge assignments of \( Z_2 \), compatible with the experimental data on masses and mixings in the leptonic sector is given in Table 1. These \( Z_2 \) assignments forbid the following Yukawa couplings \( Y_1^1, \ Y_2^2, \ Y_3^3 \) and \( Y_5^5 \). Therefore, the corresponding entries in the mass matrices vanish, \( i.e. \), \( \mu_1^1 = \mu_3^3 = 0 \) and \( \mu_1^5 = \mu_5^5 = 0 \).
3.1. The mass matrix of the charged leptons

The remaining three parameters in the mass matrix of the charged leptons $|\tilde{\mu}_2|$, $|\tilde{\mu}_4|$ and $|\tilde{\mu}_5|$ may readily be expressed in terms of the charged lepton masses [23]. The resulting expression for $M_e$, written to order $(m_\mu m_e/m_\tau^2)^2$ and $x^4 = (m_e/m_\mu)^4$ is

$$M_e \approx m_\tau \begin{pmatrix} \frac{1}{\sqrt{2}} \sqrt{1+x^2} & \frac{1}{\sqrt{2}} \sqrt{1+x^2} & \frac{1}{\sqrt{2}} \sqrt{1+x^2} \\ \frac{1}{\sqrt{2}} \sqrt{1+x^2} & -\frac{1}{\sqrt{2}} \sqrt{1+x^2} & \frac{1}{\sqrt{2}} \sqrt{1+x^2} \\ \frac{m_\mu(1+x^2)}{\sqrt{1+x^2-m_\mu^2}} & \frac{m_\mu(1+x^2)}{\sqrt{1+x^2-m_\mu^2}} & 0 \end{pmatrix}. \quad (14)$$

This approximation is numerically exact up to order $10^{-9}$ in units of the $\tau$ mass. Notice that this matrix has no free parameters other than the Dirac phase $\delta_e$.

The unitary matrix $U_{eL}$ that diagonalizes $M_e M_e^\dagger$ and enters in the definition of the neutrino mixing matrix $V_{PMNS}$ may be written in the polar form as $U_{eL} = P_{e\ell} O_{eL}$ [24], where $P_{e\ell}$ is a diagonal matrix of phases and the orthogonal matrix $O_{eL}$ can be written as $M_e$, as follows

$$O_{eL} \approx \begin{pmatrix} \frac{1}{\sqrt{2}} \frac{x(1+2\delta_2^2+4\delta_5^4+2\delta_4^2)}{\sqrt{1+m_\mu^2+5x^2-m_\mu^2+m_\mu^2+12x^2}} & -\frac{1}{\sqrt{2}} \frac{(1-2\delta_4^2+2\delta_5^2)}{\sqrt{1-4m_\mu^2+x^2+6m_\mu^4-4m_\mu^6-5m_\mu^2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \frac{x(1+2\delta_2^2-4\delta_5^2)}{\sqrt{1+m_\mu^2+5x^2-m_\mu^2-m_\mu^2+12x^2}} & \frac{1}{\sqrt{2}} \frac{(1-2\delta_4^2+4\delta_5^2)}{\sqrt{1-4m_\mu^2+x^2+6m_\mu^4-4m_\mu^6-5m_\mu^2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \frac{x(1+2x^2-m_\mu^2-m_\mu^2)(1+2x^2-2\delta_5^2)}{\sqrt{1+m_\mu^2+5x^2-m_\mu^2-m_\mu^2+12x^2}} & -\frac{1}{\sqrt{2}} \frac{(1+x^2-2\delta_5^2)}{\sqrt{1+2x^2-m_\mu^2}} & \frac{1}{\sqrt{2}} \frac{1+x^2-m_\mu^2}{\sqrt{1+x^2-m_\mu^2}} \end{pmatrix}. \quad (15)$$

where, as before, $m_\mu = m_\mu/m_\tau$, $m_\tau = m_e/m_\tau$ and $x = m_e/m_\mu$.

3.2. The mass matrix of the neutrinos

According to the $\hat{Z}_2$ selection rule, the mass matrix of the Dirac neutrinos takes the form

$$M_{\nu_D} = \begin{pmatrix} \mu_2^\nu & \mu_2^\nu & 0 \\ \mu_2^\nu & -\mu_2^\nu & 0 \\ \mu_3^\nu & \mu_3^\nu & \mu_3^\nu \end{pmatrix}. \quad (16)$$

Then, the mass matrix for the left-handed Majorana neutrinos, $M_\nu$, obtained from the see-saw mechanism, $M_\nu = M_{\nu_D} M^{-1}(M_{\nu_D})^T$, is

$$M_\nu = \begin{pmatrix} 2(\rho_2^\nu)^2 & 0 & 2\rho_2^\nu \rho_3^\nu \\ 0 & 2(\rho_2^\nu)^2 & 0 \\ 2\rho_2^\nu \rho_3^\nu & 0 & 2(\rho_3^\nu)^2 + (\rho_3^\nu)^2 \end{pmatrix}, \quad (17)$$

where $\rho_2^\nu = (\mu_2^\nu)/M_1^{1/2}$, $\rho_3^\nu = (\mu_3^\nu)/M_2^{1/2}$ and $\rho_3^\nu = (\mu_3^\nu)/M_3^{1/2}$, $M_1$ and $M_3$ are the masses of the right handed neutrinos appearing in (9).

The non-Hermitian, complex, symmetric neutrino mass matrix $M_\nu$ may be brought to a diagonal form by a unitary transformation, as

$$U_\nu^T M_\nu U_\nu = \text{diag} \left( |m_{\nu_1}| e^{i\phi_1}, |m_{\nu_2}| e^{i\phi_2}, |m_{\nu_3}| e^{i\phi_3} \right), \quad (18)$$
where $U_\nu$ is the matrix that diagonalizes the matrix $M^\dagger \nu M_\nu$.

As in the case of the charged leptons, the matrices $M_\nu$ and $U_\nu$ can be reparametrized in terms of the complex neutrino masses. Then [23, 24]

$$M_\nu = \begin{pmatrix} m_{\nu_3} & 0 & \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_2} - m_{\nu_3})} e^{-i\delta_\nu} \\ 0 & m_{\nu_3} & 0 \\ \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_2} - m_{\nu_3})} e^{-i\delta_\nu} & 0 & (m_{\nu_1} + m_{\nu_2} - m_{\nu_3}) e^{-2i\delta_\nu} \end{pmatrix}$$

(19)

and

$$U_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_\nu} \end{pmatrix} \begin{pmatrix} \cos \eta & \sin \eta & 0 \\ 0 & 0 & 1 \\ -\sin \eta & \cos \eta & 0 \end{pmatrix},$$

(20)

where

$$\sin^2 \eta = \frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}}, \quad \cos^2 \eta = \frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}}.$$  

(21)

The unitarity of $U_\nu$ constrains $\sin \eta$ to be real and thus $|\sin \eta| \leq 1$, this condition fixes the phases $\phi_1$ and $\phi_2$ as

$$|m_{\nu_1}| \sin \phi_1 = |m_{\nu_2}| \sin \phi_2 = |m_{\nu_3}| \sin \phi_\nu.$$  

(22)

The only free parameters in the matrices $M_\nu$ and $U_\nu$, are the phase $\phi_\nu$, implicit in $m_{\nu_1}$, $m_{\nu_2}$ and $m_{\nu_3}$, and the Dirac phase $\delta_\nu$.

3.3. The neutrino mixing matrix

The neutrino mixing matrix $V_{PMNS}$ is the product $U_L^\dagger U_\nu K$, where $K$ is the diagonal matrix of the Majorana phase factors, defined by

$$\text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) = K^\dagger \text{diag}(|m_{\nu_1}|, |m_{\nu_2}|, |m_{\nu_3}|) K.$$  

(23)

Except for an overall phase factor $e^{i\phi_1}$, which can be ignored, $K$ is

$$K = \text{diag}(1, e^{i\alpha}, e^{i\beta}),$$

(24)

where $\alpha = 1/2(\phi_1 - \phi_2)$ and $\beta = 1/2(\phi_1 - \phi_\nu)$ are the Majorana phases.

Therefore, the theoretical mixing matrix $V_{PMNS}^{th}$, is given by

$$V_{PMNS}^{th} = \begin{pmatrix} O_{11} \cos \eta + O_{31} \sin \eta e^{i\delta} & O_{11} \sin \eta - O_{31} \cos \eta e^{i\delta} & -O_{21} \\ -O_{12} \cos \eta + O_{32} \sin \eta e^{i\delta} & -O_{12} \sin \eta - O_{32} \cos \eta e^{i\delta} & O_{22} \\ O_{13} \cos \eta - O_{33} \sin \eta e^{i\delta} & O_{13} \sin \eta + O_{33} \cos \eta e^{i\delta} & O_{23} \end{pmatrix} \times K,$$

(25)

where $\cos \eta$ and $\sin \eta$ are given eq. (21) $O_{ij}$ are given in (15), and $\delta = \delta_\nu - \delta_e$.

To find how our results are related to the neutrino mixing angles we make use of the equality of the absolute values of the elements of $V_{PMNS}^{th}$ and $V_{PMNS}^{PDG}$ [28], that is

$$|V_{PMNS}^{th}| = |V_{PMNS}^{PDG}|.$$  

(26)

This relation allows us to derive expressions for the mixing angles in terms of the charged lepton and neutrino masses.

The magnitudes of the reactor and atmospheric mixing angles, $\theta_{13}$ and $\theta_{23}$, are determined by the masses of the charged leptons only. Keeping only terms of order $(m_e^2/m_\mu^2)$ and $(m_e/m_\tau)^4$, we get

$$\sin \theta_{13} \approx \frac{1}{\sqrt{2}} \frac{(1+4x^2-5m_e^2)}{\sqrt{1+m_e^2+5x^2-6m_e^2}}, \quad \sin \theta_{23} \approx \frac{1}{\sqrt{2}} \frac{1+\frac{1}{4}x^2-2m_e^2+m_e^4}{\sqrt{1-4m_e^2+x^2+6m_e^2}},$$

(27)
The magnitude of the solar angle depends on charged lepton and neutrino masses, as well as, on the Dirac and Majorana phases,

$$|\tan \theta_{12}|^2 = \frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}} \left(1 - 2 \frac{O_{31}}{O_{33}} \cos \delta \sqrt{\frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}}} + \left(\frac{O_{31}}{O_{33}}\right)^2 \frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}}\right).$$  \hspace{1cm} (28)$$

The dependence of \(\tan \theta_{12}\) on the Dirac phase \(\delta\), see (28), is very weak, since \(O_{31} \sim 1\) but \(O_{11} \sim 1/\sqrt{2}(m_e/m_\mu)\). Hence, we may neglect it when comparing (28) with the data on neutrino mixings.

The dependence of \(\tan \theta_{12}\) on the phase \(\phi_\nu\) and the physical masses of the neutrinos enters through the ratio of the neutrino mass differences, it can be made explicit with the help of the unitarity constraint on \(U_\nu\), eq. (22),

$$\frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}} = \frac{|m_{\nu_2}|^2 - |m_{\nu_1}|^2 |\sin^2 \phi_\nu|^{1/2} - |m_{\nu_3}| |\cos \phi_\nu|}{|m_{\nu_2}|^2 - |m_{\nu_3}|^2 |\sin^2 \phi_\nu|^{1/2} + |m_{\nu_1}| |\cos \phi_\nu|}. \hspace{1cm} (29)$$

4. Neutrino mass spectrum

In the present model, the numerical values of \(\sin \theta_{13}\) and \(\sin^2 \theta_{23}\) are determined by the masses of the charged leptons only, in very good agreement with the experimental values [12, 13, 29, 30, 31, 32],

$$(\sin \theta_{13})^{th} = 3.4 \times 10^{-3}, \hspace{1cm} -0.179 \leq (\sin \theta_{13})^{exp} \leq 0.11, \hspace{1cm} (30)$$

and

$$(\sin^2 \theta_{23})^{th} = 0.5, \hspace{1cm} (\sin^2 \theta_{23})^{exp} = 0.5^{+0.06}_{-0.05}. \hspace{1cm} (31)$$

In this model, the experimental restriction \(|\Delta m^2_{12}| < |\Delta m^2_{13}|\) implies an inverted neutrino mass spectrum, \(|m_{\nu_2}| < |m_{\nu_1}| < |m_{\nu_3}| \ [21].

As can be seen from eqs. (28) and (29), the solar mixing angle is sensitive to the neutrino mass differences and the phase \(\phi_\nu\), but is only very weakly sensitive to the charged lepton masses. If we neglect the small terms proportional to \(O_{11}\) and \(O_{21}^2\) in (28), we get

$$\tan^2 \theta_{12} = \frac{|\Delta m^2_{12} + |m_{\nu_2}|^2 |\cos \phi_\nu|^{1/2} - |m_{\nu_1}| |\cos \phi_\nu|}{|\Delta m^2_{13} + |m_{\nu_3}|^2 |\cos \phi_\nu|^{1/2} + |m_{\nu_1}| |\cos \phi_\nu|}. \hspace{1cm} (32)$$

From this expression, we may readily derive expressions for the neutrino masses in terms of \(\tan \theta_{12}\) and \(\phi_\nu\) and the differences of the squared masses of the neutrinos masses

$$|m_{\nu_3}| = \frac{\sqrt{\Delta m^2_{13} + |m_{\nu_1}|^2} \tan \theta_{12}}{2 \cos \phi_\nu \tan \theta_{12}} \frac{1 - \tan^4 \theta_{12} + r^2}{\sqrt{1 + \tan^2 \theta_{12}} \sqrt{1 + \tan^2 \theta_{12} + r^2}}, \hspace{1cm} (33)$$

where \(r^2 = \Delta m^2_{21}/\Delta m^2_{23} = 3 \times 10^{-2}\).

The other two masses, \(|m_{\nu_1}|\) and \(|m_{\nu_2}|\) are immediately obtained from the knowledge of \(|m_{\nu_3}|\) and \(\Delta m^2_{12}\) and \(\Delta m^2_{13}\).

The cosmological upper bound on the sum of neutrino masses sets a lower bound for \(\cos \phi_\nu\) [18] \[\sum |m_{\nu}| \leq 0.17 \text{ eV} \rightarrow \cos \phi_\nu \geq 0.55. \hspace{1cm} (34)\]

Since, for small values of \(\phi_\nu\), the neutrino masses change very slowly with \(\cos \phi_\nu\), in the absence of any other experimental information, we set \(\phi_\nu = 0\) in our formulas. Hence, we find

$$|m_{\nu_2}| \approx 0.056 \text{eV} \hspace{1cm} |m_{\nu_1}| \approx 0.055 \text{eV} \hspace{1cm} |m_{\nu_3}| \approx 0.022 \text{eV}, \hspace{1cm} (35)$$

where we used the values \(\Delta m^2_{13} = 2.6 \times 10^{-3} \text{eV}^2\), \(\Delta m^2_{21} = 7.9 \times 10^{-5} \text{eV}^2\) and \(\tan \theta_{12} = 0.667\), taken from [14].
5. $V_{PMNS}^{th}$ and the tri-bimaximal form

Once the numerical values of the neutrino masses are determined, we may readily verify that the theoretical mixing matrix, $V_{PMNS}^{th}$, is very close to the tri-bimaximal form of the mixing matrix [33],

$$V_{PMNS}^{th} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} + \delta V_{PMNS}^{tri},$$

where $\delta V_{PMNS}^{tri} = V_{PMNS}^{th} - V_{PMNS}^{tri}$. From eq. (25), the correction term to the tri-bimaximal form of the mixing matrix comes out as

$$\delta V_{PMNS}^{tri} \approx \begin{pmatrix} 1.94 \times 10^{-2} & -2.84 \times 10^{-2} & -3.4 \times 10^{-3} \\ 2.21 \times 10^{-2} & 1.5 \times 10^{-2} & -8.2 \times 10^{-6} \\ 1.8 \times 10^{-2} & 1.24 \times 10^{-2} & 3.1 \times 10^{-10} \end{pmatrix}.$$  

A more complete discussion of the deviation from the tri-bimaximal form in the framework of the minimal $S_3$-invariant extension of the SM can be found in [26].

6. Flavour Changing Neutral Currents (FCNC)

Models with more than one Higgs $SU(2)$ doublet have tree level flavour changing neutral currents. In the Minimal $S_3$-invariant Extension of the Standard Model considered here, there is one Higgs $SU(2)$ doublet per generation coupling to all fermions. The flavour changing Yukawa couplings may be written in a flavour labelled, symmetry adapted weak basis as

$$L^{FCNC} = \begin{pmatrix} E_a L Y^{E1}_{ab} E_b R + U_{aL} Y^{US}_{ab} U_{bR} + \overline{D}_a L Y^{DS}_{ab} D_{bR} \end{pmatrix} H_{S}^0$$

$$+ \begin{pmatrix} E_a L Y^{E2}_{ab} E_b R + U_{aL} Y^{U1}_{ab} U_{bR} + \overline{D}_a L Y^{D1}_{ab} D_{bR} \end{pmatrix} H_{1}^0 +$$

$$+ \begin{pmatrix} E_a L Y^{E2}_{ab} E_b R + U_{aL} Y^{U2}_{ab} U_{bR} + \overline{D}_a L Y^{D2}_{ab} D_{bR} \end{pmatrix} H_{2}^0 + \text{h.c.}$$

The Yukawa couplings of immediate physical interest in the computation of the flavour changing neutral currents are those defined in the mass basis, according to $Y^m_{mL} = U^*_{eL} Y^{E1}_{mL} U_{eR}$, where $U_{eL}$ and $U_{eR}$ are the matrices that diagonalize the charged lepton mass matrix defined in eqs. (12). We obtain [24]

$$\hat{Y}^{E1}_m \approx \frac{m}{v_1} \begin{pmatrix} 2\tilde{m}_e & -\frac{1}{2}\tilde{m}_e & \frac{1}{2}x \\ -\tilde{m}_\mu & \frac{1}{2}\tilde{m}_\mu & -\frac{1}{2} \\ \frac{1}{2}\tilde{m}_\mu x^2 & -\frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \end{pmatrix}_m,$$

and

$$\hat{Y}^{E2}_m \approx \frac{m}{v_2} \begin{pmatrix} -\tilde{m}_e & \frac{1}{2}\tilde{m}_e & -\frac{1}{2}x \\ \tilde{m}_\mu & \frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \\ -\frac{1}{2}\tilde{m}_\mu x^2 & \frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \end{pmatrix}_m,$$

where $\tilde{m}_\mu = 5.94 \times 10^{-2}$, $\tilde{m}_e = 2.876 \times 10^{-4}$ and $x = m_e/m_\mu = 4.84 \times 10^{-3}$. All the non-diagonal elements are responsible for tree-level FCNC processes. If the $S_3'$ symmetry in the Higgs sector is preserved [34], $\langle H_1^0 \rangle = \langle H_2^0 \rangle = v$. 


Table 2. Leptonic FCNC processes, calculated with $M_{H_{1,2}} \sim 120$ GeV.

| FCNC processes | Theoretical BR | Experimental upper bound BR | References |
|----------------|----------------|-----------------------------|------------|
| $\tau \rightarrow 3\mu$ | $8.43 \times 10^{-14}$ | $2 \times 10^{-7}$ | B. Aubert et. al. [37] |
| $\tau \rightarrow \mu e^+ e^-$ | $3.15 \times 10^{-11}$ | $2.7 \times 10^{-7}$ | B. Aubert et. al. [37] |
| $\tau \rightarrow \mu \gamma$ | $9.24 \times 10^{-13}$ | $6.8 \times 10^{-8}$ | B. Aubert et. al. [38] |
| $\tau \rightarrow e \gamma$ | $5.22 \times 10^{-16}$ | $1.1 \times 10^{-11}$ | B. Aubert et. al. [39] |
| $\mu \rightarrow 3e$ | $2.53 \times 10^{-16}$ | $1 \times 10^{-12}$ | U. Bellgardt et. al. [36] |
| $\mu \rightarrow e \gamma$ | $2.42 \times 10^{-20}$ | $1.2 \times 10^{-11}$ | M. L. Brooks et al. [40] |

The amplitude of the flavour violating process $\mu \rightarrow 3e$, is proportional to $\tilde{Y}_{\mu e}^E \tilde{Y}_{ee}^E$ [35]. Then, the leptonic branching ratio,

$$Br(\mu \rightarrow 3e) = \frac{\Gamma(\mu \rightarrow 3e)}{\Gamma(\mu \rightarrow e\nu\bar{\nu})}$$

(41)

and

$$\Gamma(\mu \rightarrow 3e) \approx \frac{m_{\mu}^5}{3 \times 2^{10/3} \pi^3} \frac{Y_{\mu e}^{1.2} Y_{ee}^{1.2}}{M_{H_{1,2}}^4},$$

(42)

which is the dominant term, and the well known expression for $\Gamma(\mu \rightarrow e\nu\bar{\nu})$ [28], give

$$Br(\mu \rightarrow 3e) \approx 2(2 + \tan^2 \beta)^2 \left( \frac{m_e m_\mu}{m_\tau^2} \right)^2 \left( \frac{m_\tau}{M_H} \right)^4,$$

(43)

taking for $M_H \approx 120$ GeV and $\tan \beta = 1$ we obtain $Br(\mu \rightarrow 3e) = 2.53 \times 10^{-16}$, well below the experimental upper bound for this process, which is $1 \times 10^{-12}$ [36].

Similar computations give the numerical estimates of the branching ratios for some others flavour violating processes in the leptonic sector. These results, and the corresponding experimental upper bounds are shown in Table 2. In all cases considered, the theoretical estimations made in the framework of the minimal $S_3$-invariant extension of the SM are well below the experimental upper bounds [24].

7. Muon anomalous magnetic moment

In the minimal $S_3$-invariant extension of the Standard Model we are considering here, we have three Higgs $SU(2)$ doublets, one in the singlet and the other two in the doublet representations of the $S_3$ flavour group. The $Z_2$ symmetry decouples the charged leptons from the Higgs boson in the $S_3$ singlet representation. Therefore, in the leading order of perturbation theory there are two neutral scalars and two neutral pseudoscalars whose exchange will contribute to the anomalous magnetic moment of the muon. Since the heavier generations have larger flavour-changing couplings, the largest contribution comes from the heaviest charged leptons coupled to the lightest of the neutral Higgs bosons.

A straightforward computation gives

$$\delta a_{\mu}^{(H)} = \frac{Y_{\mu e} Y_{\mu \mu} m_\mu m_\tau}{16\pi^2 M_H^2} \left( \log \left( \frac{M_H^2}{m_\tau^2} \right) - \frac{3}{2} \right).$$

(44)

With the help of (39) and (40) we may write $\delta a_{\mu}^{(H)}$ as

$$\delta a_{\mu}^{(H)} = \frac{m_\tau^2}{(246 \text{ GeV})^2} \frac{(2 + \tan^2 \beta) m_\mu^2}{32\pi^2 M_H^2} \left( \log \left( \frac{M_H^2}{m_\tau^2} \right) - \frac{3}{2} \right).$$

(45)
Taking again $M_H = 120 \text{ GeV}$ and the upper bound for $\tan \beta = 14$ gives an estimate of the largest possible contribution of the FCNC to the anomaly of the muon’s magnetic moment $\delta a^{(H)}_\mu \approx 1.7 \times 10^{-10}$. This number has to be compared with the difference between the experimental value and the Standard Model prediction for the anomaly of the muon’s magnetic moment $[41]$

$$\Delta a_\mu = a^{exp}_\mu - a^{SM}_\mu = (28.7 \pm 9.1) \times 10^{-10},$$

which means

$$\frac{\delta a^{(H)}_\mu}{\Delta a_\mu} \approx 0.06. \quad (47)$$

Hence, the contribution of the flavour changing neutral currents to the anomaly of the magnetic moment of the muon is smaller than or of the order of 6% of the discrepancy between the experimental value and the Standard Model prediction.

8. Conclusions

By introducing three Higgs fields that are $SU(2)_L$ doublets in the theory, we extended the concept of flavour and generations to the Higgs sector and formulated a minimal $S_3$-invariant extension of the Standard Model $[21]$. A well defined structure of the Yukawa couplings is obtained, which permits the calculations of mass and mixings matrices for quark and leptons in a unified way. A further reduction of redundant parameters is achieved in the leptonic sector by introducing a $Z_2$ symmetry. The flavour symmetry group $S_3 \times Z_2$ relates the neutrino mass spectrum and mixings. This allowed us to compute the neutrino mixing matrix, $V_{PMNS}$, explicitly in terms of the masses of the charged leptons and neutrinos and two phases $\delta$ and $\phi_\nu$ $[23, 24]$. In this model, the tri-bimaximal mixing structure of $V_{PMNS}$ and the magnitudes of the three mixing angles are determined by the interplay of the flavour $S_3 \times Z_2$, the see-saw mechanism and the lepton mass hierarchy. We also found that $V_{PMNS}$ has three CP-violating phases, namely, one Dirac phase $\delta = \delta_\nu - \delta_e$ and two Majorana phases, $\alpha$ and $\beta$, which are functions of the neutrino masses and the phase $\phi_\nu$, which is independent of the Dirac phase. The numerical values of the reactor, $\theta_{13}$, and the atmospheric, $\theta_{23}$, mixing angles are determined by the masses of the charged leptons only, in very good agreement with the latest analysis of the experimental data on neutrino oscillations and mixings $[30, 32, 31]$. The solar mixing angle, $\theta_{12}$, is almost insensitive to the values of the masses, but its experimental value allowed us to fix the scale and origin of the neutrino mass spectrum, which leads to the prediction of an inverted hierarchy, with the values $|m_{\nu_2}| = 0.056\text{eV}$, $|m_{\nu_1}| = 0.055\text{eV}$ and $|m_{\nu_3}| = 0.022\text{eV}$. We also obtained explicit expressions for the matrices of the Yukawa couplings of the lepton sector parametrized in terms of the charged lepton masses and the VEVs of the neutral Higgs bosons in the $S_3$-doublet representation. These Yukawa matrices are closely related to the fermion mass matrices and have a structure of small and very small entries reflecting the observed charged lepton mass hierarchy. With the help of the Yukawa matrices, we computed the branching ratios of a number of FCNC processes and found that the branching ratios of all FCNC processes considered here are strongly suppressed by powers of the small mass ratios $m_e/m_\tau$ and $m_\mu/m_\tau$, and by the ratio $\left(m_\tau/M_{H_{12}}\right)^4$, where $M_{H_{12}}$ is the mass of the neutral Higgs bosons in the $S_3$-doublet. Taking for $M_{H_{12}}$ a very conservative value ($M_{H_{12}} \approx 120 \text{ GeV}$), we found that the numerical values of the branching ratios of the FCNC in the leptonic sector are well below the corresponding experimental upper bounds by many orders of magnitude. It has already been argued that small FCNC processes mediating non-standard quark-neutrino interactions could be important in the theoretical description of the gravitational core collapse and shock generation in the explosion stage of a supernova $[42, 43, 44]$. Finally, the contribution of the flavour changing neutral currents to the anomalous magnetic moment of the muon is small but
non-negligible, and it is compatible with the best, state of the art measurements and theoretical computations.

9. Acknowledgements

We thank Prof. Jens Erler and Dr. Genaro Toledo-Sánchez for helpful discussions about $g - 2$. This work was partially supported by CONACYT México under contracts No 51554-F and 82291 and by DGAPA-UNAM under contract PAPIIT-IN112709.

References

[1] Jung C K, McGrew C, Kajita T and Mann T 2001 Ann. Rev. Nucl. Part. Sci. 51 451
[2] Mohapatra R N and Smirnov A Y 2006 Ann. Rev. Nucl. Part. Sci. 56 569 [hep-ph/0603118]
[3] GNO, Altmann M ‘et al’. 2005 Phys. Lett. B 616 174 [hep-ex/0504037]
[4] Super-Kamiokande, Smy M B ‘et al’. 2004 Phys. Rev. D 69 011104 [hep-ex/0309011]
[5] SNO, Ahmad Q R ‘et al’. 2002 Phys. Rev. Lett. 89 011302 [nucl-ex/0204009]
[6] SNO, Ahmad Q R ‘et al’. 2002 Phys. Rev. Lett. 89 011301 [nucl-ex/0204008]
[7] SNO, Aharmim B ‘et al’. 2005 Phys. Rev. C 72 055502 [nucl-ex/0502021]
[8] Super-Kamiokande, Fukuda S ‘et al’. 2002 Phys. Lett. B 539 179 [hep-ex/0205075]
[9] Super-Kamiokande, Ashie Y ‘et al’. 2005 Phys. Rev. D 71 112005 [hep-ex/0501064]
[10] Bemporad C, Gratta G and Vogel P 2002 Rev. Mod. Phys. 74 297 [hep-ph/0107277]
[11] KamLAND, Araki T ‘et al’. 2005 Phys. Rev. Lett. 94 081801 [hep-ex/0406035]
[12] Maltoni M, Schwetz T, Tortola M A and Valle J W F 2004 New J. Phys. 6 122 [hep-ph/0405172]
[13] Schwetz T 2005 Acta Phys. Polon. B36 3203 [hep-ph/0510331]
[14] Gonzalez-Garcia M C and Maltoni M 2008 Phys. Rept. 460 1 [0704.1800]
[15] CHOOZ, Apollonio M ‘et al’. 2003 Eur. Phys. J. C 27 331 [hep-ex/0301017]
[16] Eitel K 2005 Nucl. Phys. Proc. Suppl. 143 197
[17] Elliott S R and Engel J 2004 J. Phys. G 30 R183 [hep-ph/0405078]
[18] Seljak U, Slosar A and McDonald P 2006 JCAP 0610 014 [astro-ph/0604335]
[19] Elgaroy O and Lahav O 2005 New J. Phys. 7 61 [hep-ph/0412075]
[20] Lesgourgues J and Pastor S 2006 Phys. Rept. 429 307 [astro-ph/0603494]
[21] Kubo J, Mondragon A, Mondragon M and Rodriguez-Jauregui E 2003 Prog. Theor. Phys. 109 795 [hep-ph/0302196]
[22] Kubo J ‘et al’. 2005 J. Phys. Conf. Series: 18 380
[23] Felix O, Mondragon A, Mondragon M and Peinado E 2007 AIP Conf. Proc. 917 383 [hep-ph/0610061]
[24] Mondragon A, Mondragon M and Peinado P 2007 Phys. Rev. D 76 076003 [0706.0354]
[25] Mondragon A, Mondragon M and Peinado E 2008 J. Phys. A 41 304035 [0712.1799]
[26] Mondragon A, Mondragon M and Peinado E 2008 AIP Conf. Proc. 1026 164 [0712.2488]
[27] Mondragon A, Mondragon M and Peinado E 2008 Rev. Mex. Fis. S 54 3 81 [0805.3507]
[28] Particle Data Group, Yao W M ‘et al’. 2006 J. Phys. G 33 1
[29] Fogli G L, Lisi E, Marrone A and Palazzo A 2006 Prog. Part. Nucl. Phys. 57 742 [hep-ph/0506083]
[30] Fogli G L, Lisi E, Marrone A, Palazzo A and Rotunno A M 2009 SNO, KamLAND and neutrino oscillations: theta(13), talk presented at NEUTEL 2009, XIII International Workshop on Neutrino Telescopes [0905.3549]
[31] Ge H L, Giunti C and Liu Q Y 2009 Phys. Rev. D 80 053009 [0810.5443]
[32] Roa J E, Latimer D C and Ernst D J 2009 Phys. Rev. Lett. 103 061804 [0904.3930]
[33] Harrison P F, Perkins D H and Scott W G 2002 Phys. Lett. B 530 167 [hep-ph/0202074]
[34] Pakvasa S and Sugawara H 1978 Phys. Lett. B 73 61
[35] Sher M and Yuan Y 1991 Phys. Rev. D 44 1461
[36] SINDRUM, Bellgardt U ‘et al’. 1988 Nucl. Phys. B 299 1
[37] BABAR, Aubert B ‘et al’. 2004 Phys. Rev. Lett. 92 121801 [hep-ex/0312027]
[38] BABAR, Aubert B ‘et al’. 2005 Phys. Rev. Lett. 95 041802 [hep-ex/0502032]
[39] BABAR, Aubert B ‘et al’. 2006 Phys. Rev. Lett. 96 041801 [hep-ex/0508012]
[40] MEGA, Brooks M L ‘et al’. 1999 Phys. Rev. Lett. 83 1521 [hep-ph/9905013]
[41] Jegerlehner F 2007 Acta Phys. Polon. B 38 3021 [hep-ph/0703125]
[42] Raffelt G G 2007 Supernova neutrino observations: What can we learn? Proceedings of the 22nd International Conference on Neutrino Physics and Astrophysics (Neutrino 2006) [astro-ph/0701677]
[43] Amanik P S, Fuller G M and Grinstein B 2005 Astropart. Phys. 24 160 [hep-ph/0407130]
[44] Esteban-Pretel A, Tomas R and Valle J W F 2007 Phys. Rev. D D76 053001 [0704.0032]