DECOHERENCE CONTROL IN QUANTUM INFORMATION PROCESSING: SIMPLE MODELS

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We explore a strategy for protecting the evolution of a qubit against the effects of environmental noise based on the application of controlled time-dependent perturbations. In the case of a purely decohering coupling, an explicit sequence of control operations is designed, able to average out the decoherence of the qubit with high efficiency. We argue that, in principle, the effects of arbitrary qubit-environment interactions can be removed through suitable decoupling perturbations acting on the system dynamics over time scales comparable to the correlation time of the environment.

I. INTRODUCTION

Decoherence remains one of the most serious obstacles to the exploitation of the speed-up promised by quantum computation [1]. Broadly speaking, two different philosophies are being investigated to overcome the decoherence problem. On one hand, passive error-prevention schemes have been proposed, based on the idea of encoding logical quantum bits (qubits) within subspaces which do not decohere owing to symmetry properties [2,3]. On the other hand, active error-correction approaches have been formalized within a sophisticated theory of quantum error-correcting codes (QECC), where a logical qubit is encoded in the larger Hilbert space of several physical qubits and suitable feedback operations are conditionally carried out [4,5].

Although a purposeful manipulation is implied in the latter case, quantum error-correcting codes can be properly interpreted in terms of a clever redundancy in the software architecture rather than a physical way to operate on decoherence. In this work, we explore the possibility of using control techniques to modify and eliminate decoherence. Unlike recent proposals for feedback (or closed-loop) control schemes of decoherence in quantum optical systems [6], we apply control in the simpler open-loop configuration to general models of quantum information processing systems [7].

The underlying idea is suggested by high-resolution pulsed Nuclear Magnetic Resonance (NMR), where astonishingly versatile refocusing and decoupling techniques are nowadays available to remove the effects of interactions among the spins that are considered unwanted or uninteresting [8]. In our analysis, we outline the conditions under which analogous procedures can be extended from eliminating interactions internal to the system to suppressing interactions of the system with an external quantized environment. In particular, the role of the environment correlation time will be pointed out as a further parameter to be engineered in the struggle for preserving quantum coherence.

II. QUANTUM BANG-BANG CONTROL OF QUBIT DECOHERENCE

We start by investigating a prototype situation that conveys the basic idea in the simplest form. We will focus on the dynamics of a single memory cell of quantum information (qubit) undergoing decoherence due to the coupling to a thermal reservoir. The physical qubit can be associated either to a fictitious or to a real spin-1/2 system, the latter case allowing for a direct reference to the language of Nuclear Magnetic Resonance and NMR quantum computation [8]. We assume that the fastest relaxation process originated by the interaction with the quantized environment is a purely dephasing process i.e., in NMR terminology, no dissipative $T_1$-type of decay takes place. A general model for the dynamics of the overall qubit + bath system is then provided by the purely decohering spin-boson Hamiltonian ($\hbar = 1$):

$$H_0 = H_S + H_B + H_{SB} = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sigma_z \sum_k (g_k b_k^\dagger + g_k^* b_k).$$

(1)

Here, $\sigma_z$ is the standard diagonal Pauli matrix, with qubit basis states denoted as $|i\rangle$, $i = 0, 1$, while $b_k^\dagger, b_k, g_k$ are bosonic operators and coupling parameters for the $k$-th field mode respectively. Hamiltonian (1) is widely used in

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the quantum computation literature to investigate the effect of phase errors, representing the most nonclassical and dangerous source of errors for quantum qubits [4,9]. Since $[\sigma_z, H_0] = 0$, spin populations are not affected by time evolution and decoherence dynamics is characterized completely by the qubit coherence with respect to the computational basis:

$$\rho_{01}(t) = \langle 0 | \rho_B(t) | 1 \rangle = \langle 0 | \text{Tr}_B \left\{ \rho_{\text{tot}}(t) \right\} | 1 \rangle = \text{Tr}_B, S \left\{ \rho_{\text{tot}}(t_0) \sigma_+(t) \right\} = \langle \sigma_+(t) \rangle,$$

where, starting from the left and using standard notations, the relevant reduced density matrix element $\rho_{01}(t) = \rho^{(\text{tot})}_{01}(t)$ in the Schrödinger picture is linked to the expectation value of the time-evolved ladder operator in the Heisenberg representation

$$\sigma_+(t) = \frac{1}{2} \left( \sigma_x(t) + i \sigma_y(t) \right).$$

The evolution of the qubit coherence (2) can be calculated exactly under the customary assumptions about the initial state of the overall system, i.e., qubit and environment are initially uncorrelated and the environment is in thermal equilibrium at a temperature $T$. In the Heisenberg representation, one may write formally

$$\langle \sigma_+(t) \rangle = \langle G_{\text{tot}}(t_0, t) \sigma_+(t_0) \rangle,$$

with the propagator $G_{\text{tot}}(t_0, t)$ determined by the solution of the Heisenberg equations for the coupled spin + bath motion:

$$\begin{cases}
\hat{\sigma}_x(t) = 0, \\
\hat{\sigma}_+(t) = i \omega_0 \sigma_+(t) + 2i \sum_k (g_k b_k^\dagger(t) + g^*_k b_k(t)) \sigma_+(t), \\
\hat{b}_k(t) = -i \omega_k b_k(t) - ig_k \sigma_z(t), \\
\hat{b}_k^\dagger(t) = +i \omega_k b_k^\dagger(t) + ig^*_k \sigma_z(t).
\end{cases}$$

The result can be written in the following form [3]:

$$\rho_{01}(t) = e^{i\omega_0(t-t_0)-\Gamma_0(t-t_0)} \rho_{01}(t_0),$$

the loss of phase information being characterized by the damping function

$$\Gamma_0(t-t_0) \equiv \sum_k \Gamma_0(k; t-t_0) = \sum_k 4 |g_k|^2 \coth \left( \frac{\omega_k}{2T} \right) \frac{1 - \cos \omega_k(t-t_0)}{\omega_k^2},$$

in units where the Boltzmann constant $k_B = 1$. In the limit of a truly macroscopic environment, a description in terms of a continuum of modes is appropriate and the dependence of the decoherence function [3] on reservoir properties can be cast in a compact form after introducing the spectral density function $I(\omega)$,

$$\Gamma_0(t-t_0) \equiv \int_0^\infty d\omega \Gamma_0(\omega; t-t_0) = \int_0^\infty d\omega I(\omega) 4 \coth \left( \frac{\omega}{2T} \right) \frac{1 - \cos \omega(t-t_0)}{\omega^2}.$$}

Depending on the temperature $T$ and the spectral density $I(\omega)$, qualitatively different open-system evolutions arise in general, with a different interplay between quantum fluctuation and dissipation phenomena. Regardless the details of the spectral density function, however, the existence of a certain ultraviolet cut-off frequency $\omega_c$ is always demanded on physical grounds, leading to

$$I(\omega) \rightarrow 0 \quad \text{for } \omega > \omega_c.$$}

Although the specific value of $\omega_c$ depends on a natural cut-off frequency varying from system to system, $\omega_c$ can be generally associated to a characteristic time $\tau_c \sim \omega_c^{-1}$ setting the fastest (finite !) time scale of the irreversible dynamics. $\tau_c$ is known as the correlation time of the environment. The dynamics of the decoherence process arising from [8] for various choices of $I(\omega)$ has been investigated in detail elsewhere [8,9]. A pictorial representation for the important class of Ohmic reservoirs, $I(\omega) \propto \omega e^{-\omega/\omega_c}$, is shown in Fig. 1.

We introduce now a procedure aimed at improving the coherence properties of the qubit by adding a controllable time-dependent interaction to the original Hamiltonian:

$$H(t) = H_0 + H_1(t) = H_S + H_B + H_{SB} + H_1(t).$$
In the same spirit underlying multiple-pulse techniques in the manipulation of nuclear spin Hamiltonians [8], we try to average out the unwanted effects of the qubit-reservoir coupling $H_{SB}$ by applying a sequence of coherent $\pi$-pulses that repetitively flip the state of the system. Under the assumptions that the duration of the pulses is short enough compared to the typical decoherence time and the strength of the control field is sufficient to override the $H_{SB}$ coupling, the pulsed-mode operation allows one to separate the actions of the bath and the external field, by neglecting $H_{SB}$ while $H_1(t)$ is on. Specifically, $H_1(t)$ is assumed to schematize a train of $n_P$ identical $\pi$-pulses along the $\hat{x}$-axis applied on resonance at instants $t = t_P^{(n)}$, $n = 1, \ldots, n_P$, with pulse separation $t_P^{(n+1)} - t_P^{(n)} = \Delta t$. By invoking, as usual, the rotating-wave approximation, we have

$$H_1(t) = \Xi(t) \left[ \cos(\omega_0 t)\sigma_x + \sin(\omega_0 t)\sigma_y \right],$$

with envelope function

$$\Xi(t) = \sum_{n=1}^{n_P} V \left[ \partial(t - t_P^{(n)}) - \partial(t - t_P^{(n)} - \tau_P) \right], \quad t_P^{(n)} = t_0 + n\Delta t, \ n = 1, \ldots, n_P .$$

In Eq. (12), $\partial(\cdot)$ denotes the Heaviside step function, and the height $V$ and the width $\tau_P$ of each pulse satisfy $2V\tau_P = \pi$. To simplify things, we work henceforth in the limit of infinitely narrow pulses $\tau_P \rightarrow 0$, assuming the kicks of radiofrequency control field large enough to produce instantaneous spin rotations. By analogy with the classical technique of bang-bang (or on-off) controls, whereby piecewise controls with extremal values are exploited [13], one may look at this strategy as an implementation of quantum bang-bang control.

In order to depict the evolution associated to a given pulse sequence, it is convenient to think the latter as formed by applying a sequence of coherent $\pi$-pulses along the $\hat{x}$-axis applied on resonance at instants $t = t_P^{(n)}$, $n = 1, \ldots, n_P$, with pulse separation $t_P^{(n+1)} - t_P^{(n)} = \Delta t$. By invoking, as usual, the rotating-wave approximation, we have

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In order to depict the evolution associated to a given pulse sequence, it is convenient to think the latter as formed by repeated elementary cycles of spin-flips, a complete cycle being able to return the spin back to the starting configuration. For definiteness, let us analyze the first cycle, made of the following steps: evolution under $H_0$ during $t_0 \leq t \leq t_P^{(1)}$; $\pi$-pulse $P_1$ at time $t_P^{(1)}$; evolution under $H_0$ during $t_P^{(1)} \leq t \leq t_P^{(2)}$; $\pi$-pulse $P_2$ at time $t_P^{(2)}$. After a total time $t_1 = t_0 + 2\Delta t$, the first cycle is complete. The description of $\pi$-pulses turns out to be extremely simple in the Heisenberg representation. Nothing happens to the bath operators $b_k, b_k^\dagger$ in the limit of instantaneous pulses, while, by denoting with $t_P^{(+)}$ the times immediately before (after) a pulse respectively, spin operators are transformed as follows:

$$\left\{ \begin{array}{l}
\sigma_x(t_P^{+}) = -\sigma_x(t_P^{-}) , \\
\sigma_z(t_P^{+}) = [\sigma_z(t_P^{-})]^\dagger .
\end{array} \right.$$

(13)

In terms of the free propagator $G_{\text{tot}}(t_i, t_j)$ introduced in [4] to evolve coherence from $t_i$ to $t_j$, the time development during the cycle can be represented as

$$\langle \sigma_+(t_0 + 2\Delta t) \rangle = \langle G_{\text{tot}}(t_0, t_0 + \Delta t) \sigma_+(t_0)G_{\text{tot}}^\dagger(t_0 + \Delta t, t_0 + 2\Delta t) \rangle ,$$

(14)

to be compared with

$$\langle \sigma_+(t_0 + 2\Delta t) \rangle = \langle G_{\text{tot}}(t_0 + \Delta t, t_0 + 2\Delta t)G_{\text{tot}}(t_0, t_0 + \Delta t) \sigma_+(t_0) \rangle$$

(15)

in the absence of pulses. Since instantaneous rotations introduce discontinuous changes in operators [13], care must be taken in evaluating the two propagators $G_{\text{tot}}(t_0, t_0 + \Delta t)$, $G_{\text{tot}}(t_0 + \Delta t, t_0 + 2\Delta t)$ separately, by solving Heisenberg equations of motion [8] with initial conditions at $t = t_0$, $t = t_P^{+} = t_0 + \Delta t$ respectively. Only at the end of the calculation everything can be expressed with respect to the initial time of the cycle. Omitting the details, the result for the coherence evolution over the first complete cycle is [4]

$$\rho_{01}(t_0 + 2\Delta t) = e^{-\Gamma_{\rho}(N=1, \Delta t)} \rho_{01}(t_0) ,$$

(16)

where a new decoherence function for $N = 1$ spin cycles has been introduced:

$$\Gamma_{\rho}(N=1, \Delta t) = \sum_k \Gamma_0(k; 2\Delta t) \left| 1 - \frac{1 - e^{i\omega_k \Delta t}}{1 - e^{2i\omega_k \Delta t}} \right|^2 .$$

(17)

Since, for each mode, the additional factor arising from the pulses is of order $O(\omega_k^2 \Delta t^2) \ll 1$ for small $\Delta t$, we may guess that something interesting is happening in a regime where the state of the qubit is tipped very rapidly. This is made clear by generalizing the description to an arbitrary number $N$ of spin-flip cycles, involving a total number
of \( n_P = 2N \) \( \pi \)-pulses. After straightforward calculations along the same line outlined above, the expression for the qubit coherence at the final time \( t_N = t_0 + 2N\Delta t \) is the following:

\[
\rho_{01}(t_0 + 2N\Delta t) = e^{-\Gamma_P(N,\Delta t)} \rho_{01}(t_0),
\]

with

\[
\Gamma_P(N,\Delta t) = \sum_k \Gamma_0(k;2N\Delta t) \left| 1 - f_k(N,\Delta t) \right|^2,
\]

\[
f_k(N,\Delta t) = \frac{1 - e^{i\omega_k\Delta t}}{1 - e^{2i\omega_k\Delta t}} \sum_{n=1}^{N} e^{2i(n-1)\omega_k\Delta t}.
\]

From inspection of Eq. (19), the contribution due to the pulse sequence turns out to manifest in the typical form of an interference factor. The implications for the decoherence properties are easily stated by considering the mathematical limit where \( \Delta t \to 0, N \to \infty \), subjected to the constraint \( 2N\Delta t = t_N - t_0 \). Under these conditions, one can prove that

\[
\lim_{\Delta t \to 0} f_k(N,\Delta t) = 1 \quad \forall k \quad \Rightarrow \quad \lim_{\Delta t \to 0} \Gamma_P(N,\Delta t) = 0.
\]

Thus, in the limit of continuous flipping, decoherence is completely washed out for any temperature and any spectral density function.

Obviously, a continuous limit of this kind is scarcely meaningful from a physical point of view. However, this ideal situation should be approached if \( \Delta t \to 0 \), \( N \to \infty \), subjected to the constraint \( 2N\Delta t = t_N - t_0 \). Under these conditions, one can prove that

\[
\delta t \ll \tau_c.
\]

More explicitly, this result implies that, given an arbitrary time \( t \), one can always recover the initial state and the coherence of the qubit by making \( t \) the end time of a pulse sequence and by adjusting the parameters to satisfy \( t = t_N = 2N\Delta t \) and \( \delta t \ll \tau_c \). Then, at time \( t \), a coherence echo is formed. Alternatively, by keeping the qubit flipped and restricting the observation to cycle times \( t_N, N = 1, 2, \ldots, \), the system is found to evolve ideally as it would do in the absence of the coupling \( H_{SB} \) responsible for decoherence. A typical behavior originated by the pulsing procedure for the prototype high-temperature Ohmic environment of Fig. 1 is displayed in Fig. 2.

So far, the suppression of decoherence has been derived in a rather formal way. Actually, a simple physical explanation can be provided as well. Similarly to the original well-known spin-echo phenomenon [14], and to the more sophisticated solid-echoes or magic echo experiments [9], the basic argument here is a time-reversal argument. The examination of a single elementary spin-flip cycle suffices to capture the underlying mechanism. Roughly speaking, and looking back at the representation (1), it is the presence of the transformed propagator \( G_{tot}^{-1}(t_0 + \Delta t, t_0 + 2\Delta t) \), generated by the couple of \( \pi \)-pulses, that simulates the effect of a time-reversal. Would the evolution during the second half of the cycle be identical to the one in the first \( \Delta t \) interval, then it would be

\[
G_{tot}^{-1}(t_0 + \Delta t, t_0 + 2\Delta t) = G_{tot}^{-1}(t_0, t_0 + \Delta t),
\]

and, therefore, \( \langle \sigma_+ (t_0 + 2\Delta t) \rangle = \langle \sigma_+ (t_0) \rangle \) as a consequence of the cyclic property in the trace. Instead, this reversal is only approximate in general since the two propagations differ by a dephasing factor \( e^{i\omega_k\Delta t} \) in the evolution of each reservoir mode [9]. However, if the condition (22) is met, then the cycle is effectively equivalent to an exact time-reversal and, by iteration on every cycle, the elimination of decoherence [21] is achieved.

### III. DYNAMICAL DECOUPLING OF QUBIT-ENVIRONMENT INTERACTIONS

In this section, we rederive the result established above in a form that opens up the way to further generalization. The first step is to formally reinterpret the method of reducing environment-induced decoherence by successive application of \( \pi \)-pulses within the general framework of decoupling techniques based on controlled averaging. In NMR, sophisticated decoupling schemes are routinely used to simplify complex spectra by manipulating the underlying spin Hamiltonian to an extent allowing for a successful analysis [13]. In particular, a relevant class of decoupling procedures, including spin-decoupling and multiple-pulse experiments, involves selective averaging in the internal spin
space. The idea is to introduce controlled motions into the system, with the time-dependence designed in such a way that undesired terms in the Hamiltonian are averaged out. In extending similar techniques to the decoupling of interactions between a system and its environment, the major difference stems from the fact that the decoupling action can be easily exerted only on the system variables, the bath degrees of freedom being generally uncontrollable.

Looking back at the Hamiltonian $\mathcal{H}$, we start by seeking a perturbation $H_1(t)$ to be added as a suitable decoupling interaction in order to remove $H_{SB}$, Eq. (14). We restrict to a situation where $H_1(t)$ is cyclic, i.e. satisfying the following conditions ($t_0 = 0$ henceforth):

\[
\begin{align*}
(i) & \quad H_1(t) = H_1(t + \Delta t) \text{ for some } \Delta t; \\
(ii) & \quad U_1(t) \equiv T \exp \left\{ -i \int_0^t ds H_1(s) \right\} = U_1(t + T_c) \text{ for some } T_c.
\end{align*}
\]

From (ii), $U_1(T_c) = 1$ and $T_c$ is called the cycle time. An elegant description of the dynamics arising in the presence of a cyclic perturbation is provided by the so-called average Hamiltonian theory [8,15]. We only recall here the basic ingredients in a language that is appropriate to quantum information processing. In particular, we schematize the following conditions (25) where the unwanted coupling $H_{SB}$ no longer appears up to a certain order $\mathcal{H}^{(r)}$ and higher-order contributions $\mathcal{H}^{(r+1)}, \ldots$ are made negligible.

\[
\begin{align*}
\mathbf{1} & \quad \mathcal{H}_1(t) = H_1(t + \Delta t) \text{ for some } \Delta t; \\
\mathbf{2} & \quad U_1(t) \equiv T \exp \left\{ -i \int_0^t ds H_1(s) \right\} = U_1(t + T_c) \text{ for some } T_c.
\end{align*}
\]

In (26), $H_0$ is given by (1) and, according to the standard NMR literature, the transformed Hamiltonian $\tilde{H}(t)$ is also known as the toggling frame Hamiltonian. If $U_{tot}(t)$ and $\hat{U}_{tot}(t)$ denote the time evolution operators in the Schrödinger and interaction picture respectively, due to (25) one gets

\[
\begin{align*}
U_{tot}(nT_c) = U_1(nT_c)\hat{U}_{tot}(nT_c) = \hat{U}_{tot}(nT_c) = \left[ \hat{U}_{tot}(T_c) \right]^n \equiv e^{-i\mathcal{H}t},
\end{align*}
\]

i.e. provided the observation of the dynamics is restricted to stroboscopic and synchronized sampling, $t_n = nT_c$, it is sufficient to know the evolution over a single cycle as described by the transformed Hamiltonian $\tilde{H}(t)$. In (26), the last equality defines the average Hamiltonian and contains the main result of average Hamiltonian theory: for suitable repeated observations, the motion of the system under the influence of the time-dependent field $H_1(t)$ can be represented by a constant average Hamiltonian $\overline{\mathcal{H}}$. The calculation of $\overline{\mathcal{H}}$ is usually performed on the basis of a standard Magnus expansion of the time-ordered exponential defining $\hat{U}_{tot}(T_c)$ [13], i.e.

\[
\begin{align*}
\hat{U}_{tot}(T_c) = T \exp \left\{ -i \int_0^{T_c} ds \tilde{H}(s) \right\} \equiv e^{-i\mathcal{H}} = e^{-i(\overline{\mathcal{H}}^{(0)} + \overline{\mathcal{H}}^{(1)} + \ldots)}.
\end{align*}
\]

In our case, the evaluation of (26) is simplified since the toggling frame Hamiltonian $\tilde{H}(t)$ is piecewise constant during the intervals $\Delta t_k$ separating consecutive rotations and one can introduce stepwise transformed Hamiltonians:

\[
\hat{U}_{tot}(T_c) = e^{-i\mathcal{H}_{SP}} \ldots e^{-i\mathcal{H}_{SP}} e^{-i\mathcal{H}_{SP}}, \quad \tilde{H}_k = (P_k \ldots P_1)^{-1} H_0 (P_k \ldots P_1),
\]

where, obviously, $\sum_k \Delta t_k = T_c$. Thus, the lowest-order approximation $\overline{\mathcal{H}}^{(0)}$ to the average Hamiltonian in (26) has a particularly simple form:

\[
\overline{\mathcal{H}}^{(0)} = \frac{1}{T_c} \left\{ \tilde{H}_0 \Delta t_0 + \ldots + \tilde{H}_{n_p} \Delta t_{n_p} \right\} = \frac{1}{T_c} \sum_{k=0}^{n_p} \Delta t_k P_k^{-1} \ldots P_1^{-1} H_0 P_k \ldots P_1.
\]
We are now in a position to apply this formalism to the evolution of the decohering qubit, whose Hamiltonian we rewrite for convenience as follows:

$$H_0 = H_S + H_B + H_{SB} = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sigma_z B_z , \quad B_z = \sum_k (g_k b_k^\dagger + g_k^* b_k) . \quad (32)$$

Using \( (27) \), the transformation to the toggling frame associated to \( H_1(t) \) leads to

$$\tilde{H}(t) = H_B + U_1^\dagger(t) H_S U_1(t) + [U_1^\dagger(t) \sigma_z U_1(t)] B_z , \quad (33)$$

and to a zero-th order average Hamiltonian given by \( (34) \):

$$\overline{H}^{(0)} = H_B + \frac{1}{T_c} \sum_{k=0}^{n_p} \Delta t_k \mathcal{P}_k^{-1} H_S \mathcal{P}_k + \left[ \frac{1}{T_c} \sum_{k=0}^{n_p} \Delta t_k \mathcal{P}_k^{-1} \sigma_z \mathcal{P}_k \right] B_z , \quad (34)$$

where, in \( (33) \) and \( (34) \), the fact that reservoir operators are unaffected by the control field has been evidenced and the short notation \( \mathcal{P}_k = P_k \ldots P_1 \) has been introduced. The second and third terms in \( (34) \) correspond, in general, to a transformed qubit Hamiltonian and a transformed qubit-bath interaction. It is immediate to realize that the effect of the \( \pi \)-pulse sequence of Sec. II viewed in this frame is to cause the latter interaction term to vanish. To make the identification explicit, we rearrange the elementary spin-flip cycle as follows:

$$- \Delta t \mathcal{T} = - \frac{\Delta t}{2} - P_x^{180} - \frac{\Delta t}{2} - P_x^{180} - \frac{\Delta t}{2} - \frac{\Delta t}{2} \ldots \quad (35)$$

where the rotation axis of the pulses has been indicated and, to compare with \( (34) \), \( \Delta t_0 = \Delta t_2 = \Delta t/2, \Delta t_1 = \Delta t, T_c = 2\Delta t \). Written in the form \( (35) \), the decoupling sequence for \( (32) \) is nothing but a variant of the famous Carr-Purcell (CP) sequence, that is ordinarily exploited to get rid of static applied-field inhomogeneities \[15\]. At variance with this standard usage of the CP-sequence, however, where the size of \( \Delta t \) is of no importance, the averaging of \( H_{SB} \) at zeroth-order does not guarantee by itself the elimination of decoherence. Obviously, higher-order terms in \( (29) \) have to be quenched. The question under which circumstances the Magnus series can be truncated after the leading term or the few lowest-order corrections is nontrivial. It is possible to show \[14\] that, since the \( \pi \)-th order contribution \( \overline{H}^{(r)} = O(\Delta t^r) \), a sufficient condition is \( \Delta t \omega_c \ll 1 \), which is identical to \( (22) \). Then, from \( (28) \), one ideally gets the stroboscopic equality

$$\rho_S(nT_c) = \text{Tr}_B \left\{ e^{-i \overline{H}^{(0)} nT_c} \rho_S(0) e^{i \overline{H}^{(0)} nT_c} \right\} = \rho_S(0) . \quad (36)$$

The generalization of the above scheme to decouple arbitrary qubit-environment interactions is, in principle, straightforward since the most general bilinear coupling can be expressed as a mixed sum of error-generators \[12\]:

$$H_{SB} = \sigma_x B_x + \sigma_y B_y + \sigma_z B_z , \quad (37)$$

for suitable reservoir operators. Then, provided there is no constraint on the rate of control so that condition \( (22) \) can be assumed, one has to ensure the existence of a gate sequence generating the required temporal average:

$$\sum_{k=0}^{n_p} \Delta t_k \mathcal{P}_k \sigma_\alpha \mathcal{P}_k = 0 , \quad \alpha \in \{ x, y, z \} . \quad (38)$$

The purely decohering coupling corresponds to \( B_x = B_y = 0 \). Actually, binary sequences of the Carr-Purcell type also suffice to decouple any form of interaction \( (27) \) involving at most two error generators. A specially relevant case is a Jaynes-Cummings-like dissipative coupling with \( B_z = 0 \) and \[2\]

$$H_{SB} = \sum_k (g_k b_k^\dagger \sigma_- + \text{h.c.}) \Rightarrow B_z = \sum_k (g_k b_k^\dagger + \text{h.c.}) , \quad B_y = \sum_k (-i g_k b_k^\dagger + \text{h.c.}) , \quad (39)$$

which can be eliminated, in principle, by a sequence of \( \pi \)-pulses along the \( \hat{z} \)-axis. A slightly more elaborated sequence is necessary to decouple the qubit from the simultaneous action of the three error generators. It turns out that it can be derived on the basis of a simple group-theoretic argument. A more detailed and general formulation of the method is presented elsewhere \[14\].
IV. CONCLUSIONS

Our work demonstrates the possibility to modify the evolution of a quantum open system by applying external controllable interactions. From the perspective of quantum information, the analysis suggests a different promising direction compared to conventional quantum error-correction techniques. The practical usefulness of the proposed approach strongly depends, in its present status, on the time scale of the motional processes causing relaxation. The effectiveness of analogous schemes under less idealized assumptions and in the presence of a finite bound on the control rate deserves further investigation, together with the possibility of examining decoherence properties within a fully quantum mechanical control configuration as recently proposed in [18].

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FIG. 1. Qubit decoherence as a function of time for an Ohmic environment. Time is in units of $T^{-1}$ and $\omega_c = 100$. High- and low-temperature behaviors are depicted, (H) $\omega_c/T = 10^{-2}$ and (L) $\omega_c/T = 10^4$ respectively.

FIG. 2. Qubit pulsed decoherence as a function of time for the Ohmic high-temperature environment of Fig. 1. A pulse separation $\Delta t = \tau_c/10$ has been used and coherence stroboscopically evaluated using Eqs. (18)-(19). Long-time deviations from unit value arise from cumulation of errors in the presence of a small but finite $\Delta t$. 

7
\[ e^{-\Gamma(t)} \]

\[ \omega_c t \]

(H)

(L)
\[ e^{-\Gamma(t)} \]