AVOIDING PATTERNS AND MAKING THE BEST CHOICE

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ABSTRACT. We study a variation of the game of best choice (also known as the secretary problem or game of googol) under an additional assumption that the ranks of interview candidates are restricted using permutation pattern-avoidance. We develop some general machinery for investigating interview orderings with a non-uniform rank distribution, and give a complete description of the optimal strategies for the pattern-avoiding games under each of the size three permutations. The optimal strategy for the “disappointment-free” (i.e. 321-avoiding) interviews has a form that seems to be new, involving thresholds based on value-saturated left-to-right maxima in the permutation.

1. INTRODUCTION

The game of best choice has been considered under many different names by researchers with a wide variety of perspectives. In the classical story, a player conducts interviews with a fixed number $N$ of candidates. After each interview, the player ranks the current candidate against all of the candidates that have previously been considered (without ties). The interviewer must then decide whether to accept the current candidate and end the game or, alternatively, whether to reject the current candidate forever and continue playing in the hope of obtaining a better candidate in the future. (These rules model a “tight” market in which each candidate has many options for employment and will not be available to be recalled for a second interview later in the game.)

Various facets and extensions of this model can be investigated but the most developed line of inquiry is to describe a strategy that maximizes the player’s chance of hiring the candidate ranked best among the $N$ candidates. Notably, the classical analysis of this game assumes that all $N!$ orderings of rankings into interviews are equally likely. It turns out that the form of the optimal strategy is to reject an initial set of candidates and use them as a training set by hiring the next candidate who is better than all of them (or the last candidate if no subsequent candidate is better). The question then becomes when to make the transition from rejection to hiring: If the training set is small, it is likely that our standards will be set too low to capture the best candidate; if it is large, it is likely that it already contains the best candidate who will be interviewed and rejected. After some analysis, it turns out that the asymptotically optimal transition point is after we have rejected the first $1/e \approx 37\%$ of candidates; the probability of success with this strategy is also $1/e$ as $N$ tends to infinity.

Although this analysis is mathematically pithy, involving the constant $e$ in a surprising way, we believe the assumption that all $N!$ interview orderings are equally likely is ultimately unrealistic. Over the period that the player is conducting the $N$ interviews, there can be both extrinsic trends in the candidate pool as well as intrinsic learning on the part of the interviewer. As the interviewer ranks the current candidates at each step, they acquire information about the domain that should allow them to hone the pool to include more relevant candidates at future time steps. Overall, this results in candidate ranks that are improving over time (rather than uniform). We establish in this paper two new models that produce interesting behavior in this direction.

To describe them, we employ pattern-avoidance in order to restrict the interview orderings. This is a natural technique from the viewpoint of algebraic combinatorics although its application for the best
choice game seems to be new. Throughout this paper, we model interview orderings as permutations. The permutation \( \pi \) of \( N \) is expressed in one-line notation as \([\pi_1 \pi_2 \cdots \pi_N]\) where the \( \pi_i \) consist of the elements \( 1, 2, \ldots, N \) (so each element appears exactly once). In the best choice game, \( \pi_i \) is the rank of the \( i \)th candidate interviewed in reality, where rank \( N \) is best and 1 is worst. What the player sees at each step, however, are relative rankings. For example, corresponding to the interview order \( \pi = [2516374] \), the player sees the sequence of permutations

\[
1, 12, 231, 2314, 24153, 241536, 2516374
\]

and must use only this information to determine at which step to transition. The left-to-right maxima of \( \pi \) consist of the elements \( \pi_i \) that are larger in value than all elements \( \pi_j \) lying to the left. For example, in the interview order \( \pi = [2516374] \) the best candidate occurs in the sixth position and it is not difficult to see that we will successfully hire them if and only if we transition from rejection to hiring between the last two left-to-right maxima (i.e. after the fourth or fifth interview).

![Figure 1. \( \pi = [574239618] \)](image)

Given a permutation \( \pi \) of \( N \) and a permutation \( \rho \) of \( M \leq N \), we say that \( \pi \) contains \( \rho \) as a pattern if there exists a subsequence \( \pi_{i_1}\pi_{i_2}\cdots\pi_{i_M} \) of entries from \( \pi \) in the same relative order as the entries \( \rho_1\rho_2\cdots\rho_M \) of \( \rho \). Otherwise, we say \( \pi \) avoids \( \rho \). This has a simple geometric interpretation when we plot our permutations graphically by placing a point \((i, j)\) in the Cartesian plane to represent \( \pi_i = j \). In Figure 1, we show the plot of \([574239618]\) and have highlighted a 321-instance in positions 2, 3, and 8. As the example illustrates, neither the positions nor the values in a pattern instance need to be consecutive. This permutation avoids 54321.

We call the best choice game restricted if we play on some subset of the \( N! \) interview orderings to obtain a distribution with, for simplicity, probability zero on interview orderings that are not in the subset and uniform probability on the orderings that are in the subset. The choice of restriction criteria represents the modeler’s beliefs about the overall effect of the player learning process on the interview orderings. However, the player does not directly impose these restrictions from within the game.

In our first model, we avoid the permutation 231 which requires a sequence of interviews to be status-seeking in the sense that each time we have an interview that is an improvement over some previous interview, a floor is set for all future candidate rankings. Less drastically, one could imagine avoiding \([23 \cdots (k-1)k]1\) for various values of \( k \) (tending towards the classical game as \( k \to \infty \)), but the case we consider is the strongest nontrivial model in this direction. In a variation, avoiding 321 requires interviews to be disappointment-free in the sense that each time we have an interview that is worse than some previous interview, the new interview becomes a floor for all future candidate rankings. These are illustrated in Figure 2. As above, one could extend this to consider a family of patterns of the form \([k(k-1) \cdots 21]\) for various \( k \).

It is a surprising consequence of our analysis that although these patterns have symmetric descriptions in terms of floor setting, they turn out to have very different optimal strategies and asymptotic success probabilities! Namely, the status-seeking interviews have simple and robust optimal strategies that essentially allow the player to transition from rejection to hiring at any time. This results in a probability

\[
F \text{ U R I G E} 1. \pi = [574239618] \]
of success that is a ratio of consecutive Catalan numbers which approaches 25% as $N \to \infty$ (see Theorem 3.6). Observe that this model allows a version of the game in which no candidate prefers another interview position to their own! (By contrast, it is a recent criticism of the classical model that no candidate would prefer to be among the first $1/e$ interview positions as they will be rejected automatically.)

The disappointment-free interviews have a more subtle set of optimal strategies involving thresholds based on value-saturated left-to-right maxima (see Corollary 4.9) but the probability of success approaches a limit that is more than 50% (see Corollary 4.12). Note that the optimal strategies for these models are not mutually exclusive so one may wager à la Pascal and implement both at the same time; the optimal disappointment-free strategy is also optimal for the status-seeking model.

In Robbins’ problem (see [Bru05]) and other variants of the game where the player seeks to maximize expected value, researchers develop ad hoc strategies for analysis (because it is not clear what form an optimal strategy should take). As far as we know, thresholds based on saturated left-to-right maxima have not appeared previously in the literature, but perhaps bear further investigation in light of the fact that they yield the optimal strategy for our disappointment-free interview orders.

We now mention some further ties to earlier work. Martin Gardner’s 1960 Scientific American column popularized what he called “the game of googol,” although the problem has roots which predate this. His article has been reprinted in [Gar95]. One of the first papers to systematically study the game of best choice in detail is [GM66]. Many other variations and some history have been given in [Fer89] and [Fre83]. Recently, researchers (e.g. [BIKR08]) have begun applying the best-choice framework to online auctions where the “candidate rankings” are bids (that may arrive and expire at different times) and the player must choose which bid to accept, ending the auction.

Although there is an established “Cayley” or “full-information” version of the game in which the player observes values from a given distribution, it seems that only a few papers have considered alternative rank distributions directly. The paper [RF88] considers an explicit continuous probability distribution that allows for dependencies between nearby arrival ranks via a single parameter. Inspired by approximation theory, the paper [KKN15] also studies some general properties of non-uniform rank distributions for the secretary problem.

Recent work of Buchbinder et al. [BJS14] uses linear programming to find algorithms for solving best choice problems motivated by applications to online auctions. They give an “incentive-compatible” mechanism for which the probability of selecting a candidate is independent of their position, a property that is closely related to the main finding for our status-seeking game. Under the rule that candidates may not be recalled once rejected, their strategy is shown to succeed with probability $1/4$, exactly the same as our asymptotic result for the status-seeking interviews! At the moment, this seems to be a curious coincidence.

From the algebraic combinatorics perspective, Wilf has collected some results on distributions of left-to-right maxima in [Wil95] and Prodinger [Pro02] has studied these under a geometric random model. Although we phrase our results in terms of the game of best choice, they may also be viewed in some cases as an extension of the literature on distributions of left-to-right maxima to subsets of pattern-avoiding permutations. More recently, several authors have investigated the distribution of various permutation statistics for a random model in which a pattern-avoiding permutation is chosen uniformly at random. For example, [MP14] finds the positions of smallest and largest elements as well as the number
of fixed points in a random permutation avoiding a single pattern of size 3; [MP16] finds the probability that one or two specified points occur in a random permutation avoiding 312; and the work of several authors [DHW03, FMW07] determines the lengths of the longest monotone and alternating subsequences in a random permutation avoiding a single pattern of size 3. We also consider uniformly random 321-avoiding and 231-avoiding permutations in our work, but the statistics we are concerned with arise from the game of best choice. In some sense, our results refine the question of where a uniformly random pattern-avoiding permutation achieves its maximum because in our problem we want to transition so as to capture the maximum value.

We now outline the rest of this paper. In Section 2, we introduce the notion of strike sets that express optimal strategies when we restrict our interview orderings to some subset of permutations, and recall basic properties of the Catalan and ballot numbers that serve as denominators for our probabilities. The rest of the paper characterizes the optimal strategies that arise when we avoid a single pattern of size 3. In Sections 3 and 4 we analyze the status-seeking and disappointment-free models, respectively. We consider the remaining patterns of size 3 in Section 5.

2. STRIKES AND STRATEGIES

Fix a subset \( \mathcal{I}_N \) of permutations of size \( N \). Such a subset defines a restricted game of best choice as follows.

**Definition 2.1.** Given a sequence of \( i \) distinct integers, we define its **flattening** to be the unique permutation of \( \{1, 2, \ldots, i\} \) having the same relative order as the elements of the sequence. Given a permutation \( \pi \), define the \( i \)th **prefix flattening**, denoted \( \pi|_i \), to be the permutation obtained by flattening the sequence \( \pi_1, \pi_2, \ldots, \pi_i \).

In the restricted game of best choice, some \( \pi \in \mathcal{I}_N \) is chosen (uniformly randomly, with probability \( 1/|\mathcal{I}_N| \)) and each prefix flattening \( \pi|_1, \pi|_2, \ldots, \pi|_i \) is presented sequentially to the player. If the player stops at value \( N \), they win; otherwise, they lose.

To describe the form of an optimal strategy for such games, form the **prefix tree** consisting of all possible prefixes partially ordered by the prefix flattenings they contain. For example, the complete tree for \( N = 4 \) is shown in Figure 3. The \( N \)th level always includes all of the actual interview orders from \( \mathcal{I}_N \) that may be encountered.

![Figure 3. Unrestricted prefix tree for \( N = 4 \) with strike probabilities](image)

A **strike strategy** for a restricted game of best choice is defined by a collection of prefixes we call the **strike set**. To play the strategy on a particular interview ordering \( \pi \), compare prefix flattenings to the strike set at each step. As soon as the \( i \)th prefix flattening occurs in the strike set, accept the candidate at position \( i \) and end the game. Otherwise, the strike strategy rejects the candidate at position \( i \) to continue playing.
It follows directly from the definitions that any strategy (including the optimal strategy) for a game of best choice can be represented as a strike strategy because the player has only the relative ranking information captured in the prefix flattenings to guide them as they play. It suffices to restrict our attention to strike sets that are antichains, meaning that no pair of elements are related in the prefix tree. If the descendants of a strike set eventually include every permutation of size $N$, we say the strategy is complete. Given a subset $S$ of prefixes, the completion of $S$ is the strike set consisting of the prefixes from $S$ together with all the permutations from $\mathcal{L}_N$ that are not descendants of any prefix in $S$.

For each prefix $p$, let the strike probability $S_N(p)$ represent the probability of winning the game if $p$ is included in the strike set for the subset of interview orderings containing $p$ as a prefix. Explicitly for $p = p_1 p_2 \cdots p_k$, say that $\pi$ is $p$-winnable if $\pi$ has $p$ as a prefix flattening and value $N$ in the $k$th position. Then $S_N(p)$ is the number of $p$-winnable permutations divided by the total number of permutations from $\mathcal{L}_N$ having $p$ as a prefix flattening. Note this implies that $S_N(p)$ is 0 unless $p$ ends in a left-to-right maximum. For this reason, we refer to a prefix as eligible if it ends in a left-to-right maximum. We have indicated the $S_N(p)$ values for $N = 4$ in Figure 3 with ineligible prefixes shown in gray.

Given a complete strike set $A$, the probability of winning using the corresponding strike strategy is then

$$P(A) = \bigoplus_{p \in A} S_N(p)$$

where

(a) We view the probabilities $S_N(p)$ as pairs of integers given by the numerator (number of $p$-winnable permutations) and denominator (total number of permutations having $p$ as a prefix flattening), not as rational numbers.

(b) We sum $\frac{a}{b}$ and $\frac{c}{d}$ as the probability of the union of their underlying (independent) events, so $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$.

We say that a complete strike set $A$ is optimal if $P(A)$ is maximal among all complete strike sets $A$. Observe that the following “backwards induction” algorithm will always find an optimal strike set $A$.

**Algorithm 2.2.** Begin with $A$ equal to $\mathcal{L}_N$. Choose an eligible prefix $p$ of maximal size that has not yet been considered as part of this algorithm. If $S(p)$ is larger than the probability of winning on the interviews restricted to the subforest strictly below the $p$ in the prefix tree, playing under the best strategy that has been obtained so far, then replace the elements of $A$ having $p$ as a prefix with $p$. Otherwise, continue on to the next unconsidered eligible prefix. Eventually, we will consider the prefix $[1]$ at which point $A$ will be a globally optimal strike set.

**Example 2.3.** By inspection, the optimal strike set for the unrestricted set of interview orders in $N = 4$ is the completion of $\{12, 213, 3124, 3214\}$, contributing 11 winners. (This strategy coincides with the strategy that rejects the first candidate and selects the next left-to-right maximum thereafter.)

In the unrestricted best choice game, the $S_N(p)$ probability for an eligible prefix $p$ is equal to the $S_N(p')$ probability for any other eligible prefix $p'$ of the same size. This follows because we can perform an automorphism of the symmetric group that rearranges all of the permutations with prefix flattening $p$ to have prefix flattening $p'$. Using arguments from [Kad94], the classical results can then be stated in terms of a positional strategy in which the player rejects the first $k$ candidates and accepts the next left-to-right maximum thereafter.

It is possible to generalize this to a trigger strategy by defining a set of prefixes to be triggers in the sense that when a prefix flattening of $\pi$ matches an element of the trigger set, the player rejects the current interview candidate but accepts the next left-to-right maximum thereafter. In the classical case, optimal trigger sets consist of all prefixes having the critical $1/e$ size.

More generally still, we may consider some statistic from the set of prefixes to nonnegative integers (say) together with a threshold function for that statistic that can be used to define a threshold trigger.
or threshold strike strategy. That is, once the statistic is larger than the threshold, accept the current interview candidate (strike) or transition to accept the next left-to-right maximum (trigger). For example, the size statistic together with the $1/e$ trigger threshold defines the optimal strategy in the classical case, and we will see in Section 4 that the number of value-saturated left-to-right maxima together with a quadratic strike threshold function defines the optimal strategy in the disappointment-free (i.e. 321-avoiding) case. It is an interesting problem to characterize or determine properties of $I_N$ that restrict the optimal strategy to one of these classes.

Given two restriction criteria, we say that the resulting prefix trees are isomorphic if there exists a bijection from the nodes of one tree to the other that preserves the tree structure as well as the $S_N(p)$ values. In this situation, we then obtain isomorphic strike sets to describe optimal strategies for the two games and the probabilities of success under optimal play will be equal. In this work, our restriction criteria come from permutation pattern avoidance and we call two patterns best-choice Wilf equivalent if they induce the same optimal probability of success in each restricted game of best choice for all $N$. Clearly, when two patterns induce isomorphic prefix trees they are best-choice Wilf equivalent (and we know of no other examples). In the subsequent sections of this paper, we will find that there are four of these generalized Wilf equivalence classes in $S_3$:

$$231 \cong 132, 321 \cong 312, 123, 213.$$  

Define the integer sequence of Catalan numbers $C_N$ by

$$C_N = \sum_{i+j=N-1} C_i C_j = C_0 C_{N-1} + C_1 C_{N-2} + \cdots + C_{N-1} C_0$$

where $C_0 = 1$ and $C_1 = 1$. The first few terms are $1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$, and we have the explicit formula

$$C_N = \frac{1}{N+1} \binom{2N}{N}.$$ 

One of the earliest enumerative results in permutation pattern avoidance is that the number of permutations avoiding any pattern of size 3 is counted by the Catalan sequence, so these will form denominators in our probability calculations. We refer the reader to the textbook [B12] for details and references.

The ballot numbers (sequence A009766 in [Slo])

$$C(N, k) = \frac{k+1}{N+1} \binom{2N-k}{N}$$

are a refinement of the Catalan numbers (where $C(N, 0) = C(N, 1) = C_N$). Some data is shown in Figure 4. They are defined by (either of) the recurrences

$$C(N, k) = \sum_{i=0}^{k} C(N-i, k+1-i) = C(N-1, k-1) + C(N, k+1)$$

with the initial conditions that $C(N, N) = 1$ for all $N$.

These ballot numbers arise as denominators of $S_N(p)$ probabilities.

3. 231-avoiding (isomorphic with 132-avoiding)

In this section, we derive optimal strategies for the 231-avoiding best choice game, and show that this game is isomorphic to the 132-avoiding best choice game.

**Lemma 3.1.** Every 231-avoiding permutation $\pi$ decomposes as $[\pi_1 \pi_2 \cdots \pi_{i-1} N \pi_{i+1} \cdots \pi_N]$ where each entry of $[\pi_1 \cdots \pi_{i-1}]$ is smaller in value than each entry of $[\pi_{i+1} \cdots \pi_N]$, and both of these are 231-avoiding.
\begin{align*}
N & \quad k = 0 \quad k = 1 \quad k = 2 \quad k = 3 \quad k = 4 \\
1 & \quad 1 \quad 1 \\
2 & \quad 2 \quad 2 \quad 1 \\
3 & \quad 5 \quad 5 \quad 3 \quad 1 \\
4 & \quad 14 \quad 14 \quad 9 \quad 4 \quad 1 \\
5 & \quad 42 \quad 42 \quad 28 \quad 14 \quad 5 \quad 1 \\
6 & \quad 132 \quad 132 \quad 90 \quad 48 \quad 20 \quad 6 \quad 1 \\
7 & \quad 429 \quad 429 \quad 297 \quad 165 \quad 75 \quad 27 \quad 7 \quad 1 \\
8 & \quad 1430 \quad 1430 \quad 1001 \quad 572 \quad 275 \quad 110 \quad 35 \quad 8 \quad 1 \\
9 & \quad 4862 \quad 4862 \quad 3432 \quad 2002 \quad 1001 \quad 429 \quad 154 \quad 44 \quad 9 \quad 1
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure4.png}
\caption{Ballot numbers}
\end{figure}

\textbf{Proof.} Graphically, we are claiming that the diagram of any 231-avoiding permutation must have the form

\begin{center}
\text{\includegraphics[width=0.2\textwidth]{figure5.png}}
\end{center}

where each block is itself 231-avoiding. Observe that this decomposition also realizes the Catalan recurrence. This is straightforward (because \( N \) must play the role of 3 in any 231 instance) and well-known; see [BÍ2] for example.

The complete 231-avoiding prefix tree for \( N = 4 \) is shown in Figure 5 with the strike probabilities \( S_N(p) \) given in parentheses. We mention that these prefix trees (up to some position/value conventions) coincide with the generating trees used by West to give uniform proofs of some early enumerative results on permutation patterns; see [Wes96] for an introduction.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure5.png}
\caption{231-avoiding prefix tree for \( N = 4 \)}
\end{figure}

\textbf{Definition 3.2.} Given a 231-avoiding permutation \( \pi \) of size \( N \), define \( \varphi(\pi) \) as follows. Move the value \( N \) (at least once) one position at a time to the right, keeping all other values in the same relative order. Stop as soon as the permutation avoids 231.

In practice, this means that \( \varphi \) moves \( N \) past any and all entries that form an inversion with the entry immediately to the right of \( N \). Also note that placing \( N \) in the last position is always valid, so \( \varphi \) is defined for all \( \pi \) not having \( N \) in the last position.

For our main result, we define the \textbf{successors} of an eligible prefix \( p \) to be the set of prefixes \( q \) (eligible or not) whose longest proper eligible prefix is \( p \).

\textbf{Theorem 3.3.} For each eligible prefix \( p \) in the 231-avoiding prefix tree, we have

\[ S_N(p) = \bigoplus_{\text{successors } q \text{ of } p} S_N(q). \]
Proof. From the definitions, it is clear that the denominator on the left side is equal to the sum of the denominators on the right side. Suppose \( \pi \) is a 231-avoiding permutation of \( N \) and suppose that \( p \) is the prefix flattening \( \pi|_{[i]} \), which ends in a left-to-right maximum. If including \( p \) in the strike set wins \( \pi \) then \( \pi_i = N \). In this case, \( \varphi(\pi) \) will be winnable with \( q \) in the strike set for precisely one successor \( q \), namely \( q = \varphi(\pi)|_{[j]} \) where \( j \) is the position of \( N \) in \( \varphi(\pi) \). Conversely, if \( q \) is an eligible successor of \( p \) and \( \pi \) is winnable with \( q \) in the strike set, then \( \pi_j = N \) where \( j \) is the size of \( q \). Sliding \( N \) back to position \( i \), where \( i \) is the size of \( p \), and keeping the other values of \( \pi \) in the same relative order yields a permutation that is winnable with \( p \) in the strike set. Since \( \pi_i \) will be the left-to-right maximum immediately prior to \( \pi_j = N \) in \( \pi \), the resulting permutation will remain 231-avoiding. Thus, for a fixed eligible prefix \( p \), we have that \( \varphi \) is a bijection from the set of \( p \)-winnable permutations to the union of the set of \( q \)-winnable permutations where \( q \) is any eligible successor of \( p \). This proves the numerator on the left side is equal to the sum of the numerators on the right side. \( \square \)

Corollary 3.4. For the 231-avoiding interviews, the completion of any antichain of eligible prefixes forms an optimal strike set.

Proof. Consider the set of all 231-avoiding prefixes of size \( N \). Any other antichain in the prefix tree consisting of some eligible prefixes together with some prefixes of size \( N \) can be obtained from this by a sequence of moves in which we replace a collection of successors with their predecessor, as in Algorithm 2.2. By Theorem 3.3, we do not change the probability of success. \( \square \)

Theorem 3.5. Any complete trigger set is optimal. In particular, any positional strategy is optimal.

Proof. We claim that

\[
\mathcal{T}_N(p) = \bigoplus_{\text{children } q \text{ of } p} \mathcal{T}_N(q),
\]

where \( \mathcal{T}_N(p) \) are the trigger probabilities defined as the number of \( \pi \) that are won if \( p \) is included as a trigger (explicitly, the number of \( \pi \) where the last entry of \( p \) lies between the last two left-to-right maxima in \( \pi \)) divided by the total number of \( \pi \) having \( p \) as a prefix flattening. We also allow a null prefix \( p = \emptyset \); as a trigger, this simply selects the first interview candidate. The trigger probabilities are illustrated for \( N = 4 \) in Figure 6.

To verify the equation, let \( \pi \) be a 231-avoiding permutation of \( N \) and suppose that \( p \) is the prefix flattening \( \pi|_{[i-1]} \) and \( q \) is the prefix flattening \( \pi|_{[i]} \). Then, there are several cases.

1. Suppose \( \pi|_{[i]} \) does not end in a left-to-right maximum. Then \( \pi \) is \( p \)-winnable if and only if it is \( q \)-winnable.
2. Suppose \( \pi|_{[i]} \) does end in a left-to-right maximum. Observe that \( \varphi(\pi)|_{[i]} \) also ends in a left-to-right maximum since all entries to the right of \( N \) are larger than entries to the left of \( N \) by Lemma 3.1.
   
   Also note that in \( \pi \), value \( N \) must lie in some position \( j \geq i \) (for otherwise we would be in Case (1) above because \( N \) is always the last left-to-right maximum).

   Now consider the following subcases:
   
   (a) \( \pi \) is \( q \)-winnable. Then \( j > i \).
   (b) \( \pi \) is not \( q \)-winnable but \( \varphi(\pi) \) is \( q \)-winnable. Then we must have \( j = i \).
   (c) Neither \( \pi \) nor \( \varphi(\pi) \) are \( q \)-winnable. Then neither element can be \( p \)-winnable because any \( p \)-winnable element in Case (2) must have \( N \) in position \( i \) and then it is straightforward to see that \( \varphi(\pi) \) would be \( q \)-winnable.

Observe that applying \( \varphi \) is a bijection from the elements in subcase (b) to the elements in subcase (a). Moreover, the elements in subcase (b) are \( p \)-winnable but not \( q \)-winnable, while elements in subcase (a) are \( q \)-winnable but not \( p \)-winnable. The elements in subcase (c) are neither \( p \)-winnable nor \( q \)-winnable.
Thus, the probabilities are preserved as claimed in each case. Since any trigger set can be obtained, starting from all of the size \( N - 1 \) prefixes as our initial trigger set (which agrees with the corresponding initial strike strategy) by a sequence of moves in which we replace a collection of children with their parent, we find that any complete trigger strategy (and hence any positional strategy) is optimal. \( \square \)

**Theorem 3.6.** For the 231-avoiding interviews of size \( N \), the optimal probability of success is the ratio of Catalan numbers \( \frac{C_{N-1}}{C_N} \), which approaches \( \frac{1}{4} \) as \( N \rightarrow \infty \).

**Proof.** Consider the strike set consisting of the \( C_N \) 231-avoiding prefixes of size \( N \). This is optimal by Corollary 3.4. Exactly \( C_{N-1} \) of them are winnable because \( N \) must lie in the last position. Hence, the numerator in the success probability for this strike set is \( C_{N-1} \). Therefore, the probability of success is \( \frac{C_{N-1}}{C_N} \), and the asymptotics for this ratio of Catalan numbers are straightforward from the explicit formula given in Section 2. \( \square \)

Finally, we show that the 231-avoiding prefix tree and \( S_N(p) \) probabilities are isomorphic to those for the 132-avoiding interview orders.

**Definition 3.7.** Let \( \pi = [\pi_L \ N \ \pi_R] \) be a 231-avoiding permutation with the decomposition from Lemma 3.1, so the block \( \pi_L \) has values \( \{1, 2, \ldots, k\} \) and \( \pi_R \) has values \( \{k+1, \ldots, N-1\} \).

Then we define \( \Upsilon \) recursively where \( \Upsilon \) is the identity on permutations of size 2 or less and in general,

\[
\Upsilon(\pi) = [\Upsilon(\pi_L) \uparrow N \ \Upsilon(\pi_R) \downarrow]
\]

where the \( \uparrow \) and \( \downarrow \) operators reverse the blocks of values; i.e. \( \Upsilon(\pi_L) \uparrow \) is a block with values \( \{N-k, \ldots, N-1\} \) and relative order given by \( \Upsilon(\pi_L) \), and \( \Upsilon(\pi_R) \downarrow \) is a block with values \( \{1, 2, \ldots, N-k-1\} \) and relative order given by \( \Upsilon(\pi_R) \).

It is straightforward to check that \( \Upsilon \) is a bijection from the set of 231-avoiding permutations of size \( N \) to the set of 132-avoiding permutations of size \( N \).

**Theorem 3.8.** The prefix tree for 231-avoiding permutations is isomorphic to the prefix tree for 132-avoiding permutations.

**Proof.** Apply the bijection \( \Upsilon \) to each 231-avoiding prefix \( p \). We first claim that the tree structure is preserved. Namely, if \( p \) is a prefix of \( q \) in the 231-avoiding tree then \( \Upsilon(p) \) is a prefix of \( \Upsilon(q) \) in the 132-avoiding tree. We refer to this as the prefix property. Note that it suffices to check this when the size of \( p \) is one less than the size of \( q \) by transitivity.

Our strategy is to apply induction on the number of recursive steps used in the application of \( \Upsilon \). So suppose that \( p \) and \( q \) are prefixes with \( q = [q_1 \cdots q_{i-1} q_i] \) where \( q_{i-1} \) and \( p \) have the same relative order. If we apply \( \Upsilon \) for a single recursive step, we will split \( p \) and \( q \) at their maximum elements, say \( p_{MAX} \) and
If these elements are equal, then the left factors of \( p \) and \( q \) have the same relative order, while the right factors differ by the extra element \( q_i \) at the end of \( q \). Since applying \( \Upsilon \) shifts the values as a block, we see that all of the elements in the right factors of \( q \) (not including \( q_i \)) and \( p \) will continue to have the same relative order. So the prefix property holds by induction in this case. Otherwise, we must have \( q_{\text{MAX}} = q_i \). Then this step has an empty right block in the decomposition of \( q \). So, in the application of \( \Upsilon \) at the next recursive step for \( q \) we find that \( q_{\text{MAX}} \) will agree with \( p_{\text{MAX}} \) in the application at this step for \( p \), so the prefix property continues to hold by induction in this case also.

Since applying \( \Upsilon \) does not change the position of \( N \), we have that \( \pi \) is \( \pi|_{[j]} \)-winnable if and only if \( \Upsilon(\pi) \) is \( \Upsilon(\pi)|_{[j]} \)-winnable. So the \( S_N(p) \) probabilities are preserved as well.

\[ \square \]

4. 321 AVOIDING (ISOMORPHIC WITH 312-AVOIDING)

4.1. Introduction. The best choice game restricted to the 321-avoiding interview orders is a bit more complicated. The prefix trees for \( N = 4 \) and \( N = 5 \) (the latter omits the last level) with the \( S_N(p) \) probabilities are shown in Figure 7.

![Figure 7. 321-avoiding prefix trees for \( N = 4 \) and \( N = 5 \)](image)

The following result describes the \( S_N(p) \) probabilities for arbitrary \( N \).

**Theorem 4.1.** For the 321-avoiding interview orders, the strike probabilities are

\[
S_N(p) = \begin{cases} 
S_{N-1}(\hat{p}) & \text{if } p \text{ is eligible and contains at least one inversion} \\
\left(\frac{N-1}{N}\right) & \text{if } p = [12 \cdots k] 
\end{cases}
\]

where \( \hat{p} \) is the result of removing the value 1 from \( p \) (and flattening).

**Proof.** In any permutation \( \pi \) with prefix \( p \) that contains an inversion, the position of the entry 1 in \( \pi \) is fixed: it must occur in the prefix or else it would create a 321 instance when paired with the inversion in the prefix. Hence, removing the 1 from this position is a bijection to the corresponding subtree of \( N - 1 \) prefixes.

Next, suppose \( p \) has the form \( [12 \cdots k] \) for some \( k \). We prove the formulas for the denominator and numerator separately. We claim the denominators are ballot numbers. To prove this, we show that...
they satisfy the same recurrence. Notice that there two possibilities for any \( \pi \) of size \( N \) having prefix \([12 \cdots k]\):

- If the element in the next position after the prefix is larger than the maximum value in the prefix, then \( \pi \) is counted by the prefix \([12 \cdots (k + 1)]\) in \( N \).
- Otherwise, the next element is smaller and so \( \pi \) is counted by the prefix \([12 \cdots (k - 1)]\) in \( N - 1 \) after applying the \( \bar{\rho} \) bijection.

The initial conditions are that we only have one element with prefix \([12 \cdots N]\) in \( N \). Hence, the denominators agree with the ballot numbers we defined in Section 2.

For the numerator, let \( \pi \) be a \( p \)-winnable permutation with \( p \) still equal to \([12 \cdots k]\). If \( \pi \) has \( N \) in position \( k \) then everything after \( N \) must be increasing (to avoid 321) so we just have to choose the subset of values to place in the prefix. These are counted by the binomial \( \binom{N-1}{k-1} \).

In the prefix tree of rank \( N \), let \( \mathcal{B}^\circ(N, k) \) be the “open” subforest lying under (but not including) \([12 \cdots k]\) and let \( \mathcal{B}(N, k) \) be the “closed” subtree lying under (and including) \([12 \cdots k]\). Then we may interpret the first part of Theorem 4.1 as an isomorphism of prefix “forests” (i.e. disjoint unions of trees).

**Corollary 4.2.** We have a bijection

\[
\mathcal{B}^\circ(N, k) \cong \mathcal{B}(N, k + 1) \cup \bigcup_{i=1}^{k} \mathcal{B}^\circ(N-i, k+1-i)
\]

where each nonzero strike probability on the left side occurs for the isomorphic image on the right side. In particular, we may obtain an optimal strike set for the forest on the left side as the union of (the isomorphic images of) optimal strike sets for each tree in the forest on the right side.

**Proof.** Note that the children of \([12 \cdots k]\) in the prefix tree consist of the permutations of size \( k + 1 \) that are increasing for the first \( k \) entries and end with some value \( i = 1, 2, \ldots, k + 1 \). Only the \([12 \cdots (k + 1)]\) child ends with a left-to-right maximum so we apply the \( \bar{\rho} \) bijection \( i \) times to identify each of the other children with an increasing prefix from some smaller rank. Since none of these children are eligible in rank \( N \), though, we only transfer open subforests, not the node itself. (Hence, we are committing an abuse of notation by including the nonincreasing children of \([12 \cdots k]\) themselves on the left side of the bijection; however, this does not affect any of the nonzero strike probabilities or winning strategies.)

As explained in the proof of Theorem 4.1, the inverse of \( \bar{\rho} \) simply inserts a new lowest entry into the position where 1 appears in \( p \). In particular, the inverse image of a subforest \( \mathcal{B}^\circ(N-i, k+1-i) \) from the right side consists of the prefixes on the left side with their \( i \) smallest entries in fixed position. \( \square \)

**Example 4.3.** Consider \( N = 5 \). Iterating the bijection gives

\[
\mathcal{B}^\circ(5, 1) \cong \mathcal{B}(5, 2) \cup \mathcal{B}^\circ(4, 1) \cong \mathcal{B}(5, 3) \cup \mathcal{B}^\circ(4, 2) \cup \mathcal{B}^\circ(3, 1) \cup \mathcal{B}^\circ(4, 1)
\]

\[
\cong \mathcal{B}(5, 4) \cup \mathcal{B}^\circ(4, 3) \cup \mathcal{B}^\circ(3, 2) \cup \mathcal{B}^\circ(2, 1) \cup \mathcal{B}^\circ(4, 2) \cup \mathcal{B}^\circ(3, 1) \cup \mathcal{B}^\circ(4, 1)
\]

For example, \( \mathcal{B}^\circ(3, 2) \) here corresponds to the subforest under the prefix \([1342]\) in \( N = 5 \). (The bijection inserts value 1 into the first position and 2 into the fourth position of each of the three prefixes under \([12]\) in \( N = 3 \).) We find that \([1234]\) is the optimal strike in \( \mathcal{B}(5, 4) \) so we can stop here. We may obtain the other strikes from smaller ranks using the bijection.
4.2. Optimal strategy. Let \( B_N^o(p) \) denote the probability of success using an optimal strategy for the interview orderings restricted to the 321-avoiding open subforest under (but not including) the prefix \( p \).

The following corollary permits \( B^o \) to be computed recursively.

**Corollary 4.4.** We have

\[
B_N^o(12 \cdots k) = \max\{B_N^o(12 \cdots (k+1)), S_N(12 \cdots (k+1))\} + \bigoplus_{i=1}^{k} B_{N-i}^o(12 \cdots (k+1-i)).
\]

**Equivalently,**

\[
B_N^o(12 \cdots k) = B_{N-1}^o(12 \cdots (k-1)) + \left( B_{N-1}^o(12 \cdots k) - \max(B_{N-1}^o(12 \cdots k), S_{N-1}(12 \cdots (k+1))) \right)
\]

\[
+ \max(B_{N-1}^o(12 \cdots (k+1)), S_N(12 \cdots (k+1))).
\]

**Proof.** This follows directly from the definitions and Corollary 4.2. \(\square\)

In particular, \( B_N^o(1) \) gives the probability of success under the optimal strategy for the entire collection of 321-avoiding interview orders. The \( B_N^o(12 \cdots k) \) numerators for \( N \leq 16 \) are shown in Figure 8; the denominators are all ballot numbers (not shown).

| \(N\) \(\backslash\) k | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2              | 1 * |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 3              |     | 3   | 1   |     |     |     |     |     |     |     |     |     |     |     |     |
| 4              |     | 8   | 5   | 1   |     |     |     |     |     |     |     |     |     |     |     |
| 5              |     | 23  | 15  | 7   | 1   |     |     |     |     |     |     |     |     |     |     |
| 6              |     | 71  | 48  | 25  | 9 * | 1   |     |     |     |     |     |     |     |     |     |
| 7              |     | 729 | 158 | 87  | 39  | 11  | 1   |     |     |     |     |     |     |     |     |
| 8              |     | 759 | 530 | 301 | 143 | 56  | 13  | 1   |     |     |     |     |     |     |     |
| 9              |     | 2568| 1809| 1050| 520 | 219 | 76  | 15  | 1   |     |     |     |     |     |     |
| 10             |     | 8833| 6265| 3697| 1888| 838 | 318 | 99  | 17  | 1   |     |     |     |     |     |
| 11             |     | 30797| 21964| 13131| 6866| 3169| 1281| 443 | 125 | 19  | 1   |     |     |     |     |
| 12             |     | 108613| 77816| 47019| 25055| 11924| 5058| 1889| 608 | 154 | 21  | 1   |     |     |     |
| 13             |     | 386804| 278191| 169578| 91762| 44743| 19688| 7764| 2706| 817 | 186| 23  | 1   |     |     |
| 14             |     | 1389109| 1002305| 615501| 337310| 167732| 75970| 31227| 11539| 3775| 1069| 221| 25  | 1   |     |
| 15             |     | 5024945| 3635836| 2246727| 1244422| 628921| 291611| 123879| 47909| 16682| 5143| 1368| 259 | 27  | 1   |
| 16             |     | 18292738| 13267793| 8242848| 4607012| 2360285| 1115863| 486942| 195331| 71452| 23543| 6861| 1718| 300 | 29  | 1   |

**FIGURE 8.** The first few rows of the \( B^o \) triangle

Say that an entry \((N,k)\) is **optimal** if \( S_N(12 \cdots k) > B_N^o(12 \cdots k) \). Such an entry represents the root of a subtree for which the optimal strike set is simply the prefix itself, so such a prefix is “locally” optimal. The **optimal boundary** in the \( B^o \) triangle is the collection consisting of the leftmost optimal entry from each row and we have highlighted these in bold in Figure 8. Such entries represent increasing prefixes that are globally optimal, so appear in the optimal strike set for \( N \).

We now prove an important structural result about these entries, namely, that the optimal boundary proceeds by diagonal and vertical steps as \( N \) increases.

**Theorem 4.5.** If \((N,k)\) is optimal then so is \((N+1,k+1)\). Also, if \((N,k)\) is not optimal then neither is \((N+1,k)\).

**Proof.** For the first statement in the theorem, note that it is enough to prove that the ratios \( \frac{B_N^o(12 \cdots k)}{S_N(12 \cdots k)} \) are decreasing as we move down diagonals; once \( \frac{B_N^o}{S_N} < 1 \), it then remains so for all subsequent entries along the same diagonal. Equivalently, we show that the ratios \( \frac{B_{N-1}^o(12 \cdots (k-1))}{B_N(12 \cdots k)} \) are bounded below by \( \frac{S_{N-1}(12 \cdots (k-1))}{S_N(12 \cdots k)} = \frac{k-1}{N-1} \).

For the remainder of this proof, we abuse notation to refer to the numerators only; the denominators in each probability are ballot numbers which appear on both sides of the inequality so can be canceled.
Following the ballot number conventions, we also extend the $B^o$ triangle to include an extra column on the left, defining $B^o_N(0) = B^o_N(1)$.

To avoid a profusion of closely related subscripts, we will let letters denote positions in the $B^o$ and $S$ triangles so each letter corresponds to a particular $(N,k)$ pair as defined in Figure 9. If the letter is unadorned, it represents the $B^o$ value at that position. If it has a bar, it represents the complementary term in a recurrence for $B^o$, namely the max$(B^o,S)$ value at that position plus the $B^o - \text{max}(B^o,S)$ value at the position just northwest. Observe that this enables us to write
\[ x = b + \bar{y} \]
by Corollary 4.4. Finally, we represent the $S$ values by capital letters (to emphasize that these are simply binomial coefficients by Theorem 4.1).

**Figure 9.** Letter coordinates and key ratios for the induction argument

Applying the defining recurrence (twice), we have $x = b + \bar{y}$ and $b = c + \bar{a}$, so
\[
(4.1) \quad \frac{b}{x} = \frac{b}{b + \bar{y}} = \frac{1}{1 + \frac{\bar{y}}{b}} = \frac{1}{1 + \frac{\bar{y}}{c+\bar{a}}} = \frac{1}{1 + \frac{\frac{\bar{b}}{\bar{x}}}{\frac{\bar{y}}{\bar{a} + 1}}}.
\]

Now, if we produce lower bounds for $\frac{b}{\bar{x}}$ and $\frac{\bar{b}}{\bar{y}}$ then this expression gives a lower bound for $\frac{b}{x}$. So, we claim the following lower bounds, to be proved by induction:
\[
(4.2) \quad \frac{b}{x} \geq \frac{B}{X}, \quad \frac{b}{\bar{y}} \geq \frac{B}{X-B}, \quad \text{and} \quad \frac{\bar{b}}{\bar{x}} \geq \frac{F - G}{E - F}.
\]

If valid, we say that an inequality holds at $b$, the entry appearing in the numerator on the left side. Recall that the first bound is our primary goal in this proof (but we evidently require the others to facilitate it).

For the base case, we solve the recurrences to find explicit formulas along the rightmost diagonals: we find that $B^o_N(N-1) = 1$ and $B^o_N(N-2) = 2N - 3$ for all $N \geq 3$. The bar value $\bar{y}$ is 2 at position $(N,N-1)$ for all $N \geq 3$ and $\bar{x}$ is $3N - 7$ at position $(N,N-2)$ for all $N \geq 7$.

The binomial formula for $S$ from Theorem 4.1 yields
\[
\frac{B}{X} = \frac{k - 1}{N - 1}, \quad \frac{B}{X - \bar{b}} = \frac{N - k}{N - k}, \quad \frac{F - G}{E - F} = \frac{(k - 2)}{(N - 2)} = \frac{C}{B},
\]

Then we can verify each of the claimed inequalities from (4.2) directly for $k = N - 2$, $N \geq 7$:
\[
\frac{b}{x} = \frac{2N - 5}{2N - 3} \geq \frac{N - 3}{N - 1} = \frac{B}{X}, \quad \frac{b}{\bar{y}} = \frac{2N - 5}{2} \geq \frac{N - 3}{2} = \frac{B}{X - \bar{b}}, \quad \text{and} \quad \frac{\bar{b}}{\bar{x}} = \frac{3N - 10}{3N - 7} \geq \frac{N - 4}{N - 2} = \frac{F - G}{E - F}.
\]

Now assume that we have all three of the inequalities from (4.2) at entries along the current diagonal including the entry $c$ and at all entries on diagonals to the right including the entry $a$. We now derive each of the bounds at $b$. 

By the induction hypothesis,
\[
\frac{\bar{a}}{\bar{y}} \geq \frac{B - C}{X - B} \quad \text{and} \quad \frac{c}{\bar{a}} \geq \frac{C}{B - C}.
\]
Substituting these into our expression from Equation (4.1), we obtain
\[
\frac{b}{x} = \frac{1}{1 + \frac{1}{\bar{y}(\bar{a} + 1)}} \geq \frac{1}{1 + \frac{1}{\frac{B - C}{X - B}(\frac{C}{B - C} + 1)}} = \frac{1}{1 + \frac{X - B}{X - B}} = \frac{1}{X}.
\]
When we simplify this expression, we find that this is precisely equal to \(\frac{b}{x}\), as required. Moreover, this calculation already verifies the second bound as
\[
\frac{b}{x} = \frac{\bar{a}}{\bar{y}} \left( \frac{c}{\bar{a}} + 1 \right) \geq \frac{B - C}{X - B} \left( \frac{C}{B - C} + 1 \right) = \frac{B}{X - B}.
\]
Finally, we evaluate \(\frac{\bar{b}}{\bar{x}}\) in cases. Recall,
\[
\bar{b} = \max(B^c, S)(b) + B^c(c) - \max(B^c, S)(c)
\]
The argument up to here already shows if \(\max(B^c, S) = B^c\) for entry \(c, b, or x\), then it remains so for all entries above to the northwest. So the cases for
\[
\frac{\bar{b}}{\bar{x}} = \frac{\max(B^c, S)(b) + B^c(c) - \max(B^c, S)(c)}{\max(B^c, S)(x) + B^c(b) - \max(B^c, S)(b)}
\]
are: none of \(c, b, or x\) have \(\max = B^c\); \(c\) alone does; \(b\) and \(c\) do; or all three do.
For the first case, we claim that
\[
\frac{\bar{b}}{\bar{x}} = \frac{B + c - C}{X + b - B} \geq \frac{C}{B} = \frac{F - G}{E - F}
\]
because \(\frac{\bar{b}}{\bar{x}} \geq \frac{C}{B}\) and \(\frac{X}{b} \geq \frac{C}{B}\) so \(B(B + c) \geq C(X + b)\) whence
\[
B(B + c - C) \geq C(X + b - B)
\]
yielding the desired inequality.
For the other three cases, we have \(\frac{B + c - C}{X + b - B} = \frac{B}{X + b - B} \geq \frac{B}{X + b - b} = \frac{b}{x}\), and \(\frac{b + c - c}{x + b - b} = \frac{b}{x}\). For the last case, note that \(\frac{b}{x} \geq \frac{B}{X}\) by the induction argument to this point. So in each case we have \(\frac{\bar{b}}{\bar{x}} \geq \frac{B}{X}\) and since \((k - 1)(N - 2) \geq (N - 1)(k - 2)\) for \(k \leq N\), we obtain \(\frac{\bar{b}}{\bar{x}} \geq \frac{E - G}{F - F}\) as desired.

The induction proceeds along each row from right to left. Since \(B^c_N(\emptyset) = B^c_N(1)\), once we prove that \(\frac{b}{x} \geq \frac{B}{X}\) for \(b = (N - 1, 1)\), we automatically get that \(\frac{\bar{b}}{\bar{x}} = \frac{b}{x} \geq \frac{B}{X} = \frac{\bar{F}}{\bar{F}}\). Thus, the induction may proceed to the next row.

The proof of the second statement in the theorem now follows directly. Continuing our notation, we have
\[
1 = \frac{b + \bar{y}}{x} \geq \frac{b + a}{x} \geq \frac{B}{X} + \frac{a}{x}
\]
because \(\bar{y} = a + (\max(B^c, S)(y) - \max(B^c, S)(a))\) and by the inequalities we already proved. Hence,
\[
\frac{a}{x} \leq 1 - \frac{B}{X} = \frac{X - B}{X} = \frac{1}{\frac{1}{X} - \frac{B}{X}} = \frac{1}{\frac{N - k}{N - 1}} = \frac{A}{X}.
\]
Thus, if \(a\) is not optimal then \(1 \leq \frac{a}{A} \leq \frac{A}{X}\) and so \(x\) is not optimal either. \(\Box\)

**Definition 4.6.** We say that the value-saturated left-to-right maxima of a prefix \(p = p_1 \cdots p_k\) are the largest subset of left-to-right maxima in \(p\) whose values form an interval \(\{k-i+1, k-i+2, \ldots, k-1, k\}\) for some \(i\).
Given a node \( p = [p_1p_2\cdots p_k] \) from the prefix tree of 321-avoiding permutations of size \( N \), we refer to \( N \) as the rank of \( p \) and \( k \) as the size of \( p \). Let \( \sigma(i) \) be the minimal \( k \) such that \( (N, k) \) is optimal among the entries where \( N - k = i \). That is, \( \sigma(i) \) is the column containing the \( i \)th vertical step along the optimal boundary of the \( B^\circ \) table. We know that \( \sigma(i) \) is a well-defined weakly increasing function by Theorem 4.5.

We say \( p \) is selected by the threshold if

\[
\# \text{ value-saturated left-to-right maxima in } p \geq \sigma(\text{rank}(p) - \text{size}(p)).
\]

Lemma 4.7. We have that \( p \) is selected by the threshold if and only if \( \tilde{p} \) is selected by the threshold.

Proof. Applying the \( \tilde{p} \) bijection from Theorem 4.1 removes the lowest entry from \( p \) and reduces the rank by 1. This does not change the right side of the threshold inequality (4.3). The only way that removing the lowest entry could change the number of value-saturated left-to-right maxima in \( p \) is if we removed the first entry of a prefix with the form \([12\cdots i] \). But this is not possible because we require at least one inversion in \( p \) in order to apply the bijection. Hence, we do not change the left side of the threshold inequality either.

Theorem 4.8. For any \( N \) and \( k \), we have that \( p \) is included in the optimal strike set for \( B^\circ(N, k) \) if and only if

- \( p \) is eligible,
- \( p \) is selected by the threshold, and
- no proper prefix of \( p \in B^\circ(N, k) \) is selected by the threshold.

Proof. We argue by strong induction on \( N \). For \( N = 1 \) and \( N = 2 \), we have that the prefix \([1]\) is an optimal strike and it is selected by the threshold in each case. So suppose the result holds for all ranks \( M < N \).

Recalling Corollary 4.2, we may view \( B^\circ(N, k) \) as a disjoint union. Any prefix from \( \bigcup_{i=1}^{k} B^\circ(N - i, k + 1 - i) \) will be selected by the threshold if and only if it is optimal, by our induction hypothesis and Lemma 4.7.

So it suffices to show that the prefixes from \( B(N, k + 1) \) conform to the threshold. Let \( j \) be the leftmost optimal entry in row \( N \) of the \( B^\circ \) table. If \( k + 1 \leq j \) then \([12\cdots j]\) is in the optimal strike set by definition. Otherwise, we claim that \([12\cdots (k + 1)]\) is in the optimal strike set. This follows from the fact that the strike numerators for increasing prefixes are binomial coefficients so this prefix must have the largest strike numerator in the subtree under \([12\cdots (k + 1)]\) (or else we contradict \( k + 1 > j \) because the set of optimal entries on each row of the \( B^\circ \) table is an interval by Theorem 4.5).

Now, since the number of value-saturated left-to-right maxima in an increasing prefix is equal to its size, we find that an increasing prefix is selected by the threshold precisely when it corresponds to an optimal entry \((N, k)\) in the \( B^\circ \) table. Thus, the prefixes from \( B(N, k + 1) \) conform to the threshold.

Corollary 4.9. For 321-avoiding interview orders, the optimal strategy is a threshold strike strategy using the number of value-saturated left-to-right maxima as the statistic. Moreover, a single threshold function works simultaneously for all \( N \).

Proof. This follows by applying Theorem 4.8 to \( B^\circ(N, 1) \).

To play the optimal strategy on a given interview ordering \( \pi \), reject candidates until we are at some eligible prefix flattening, \( \pi|_{[k]} \), where we have seen \( \sigma(N - k) \) value-saturated left-to-right maxima. Then select the \( k \)th candidate. From small values of \( N \), we can compute the beginning of the optimal threshold function precisely. The first few values are:

\[
\sigma(N) = \sigma(N - 1) = 1, \sigma(N - 2) = 4, \sigma(N - 3) = 9, \sigma(N - 4) = 16, \ldots
\]
Definition 4.10. Precisely, which gives the total probability of success.

Theorem 4.11. The linear combination of ballot numbers that agrees with $B_{(1,4,9)}^o(N,k)$.

These rules say that by the time you get to interview $k = N$ or $k = N-1$, you should always accept the $k$th interview candidate. However, when $N$ is large enough to permit it and you have seen an “unusually high” number of value-saturated left-to-right maxima, it can also be optimal to accept an earlier candidate.

The complete list of rules (compiled for all $N \leq 10000$) indicate that the threshold function has the form $\sigma(N-i) = i^2$ for all $i \geq 1$. We do know that the rules we computed for these “small” values of $N$ remain in force for all $N$ by Theorem 4.5. It would be interesting to obtain the asymptotic threshold function and probability of success precisely.

4.3. An asymptotic lower bound. In this section we show how to compute the probability of success for the particular threshold strategy defined by using the first four rules, $\sigma(N) = \sigma(N-1) = 1$, $\sigma(N-2) = 4$, and $\sigma(N-3) = 9$, from the optimal strategy. This threshold strategy has an associated triangle of “interior” probabilities that we denote $B_{(1,4,9)}^o(N,k)$. The only difference between this triangle and the $B_{N}(12\cdots k)$ triangle from the last subsection is that the optimal boundary (i.e. the leftmost entries where $S > B_{(1,4,9)}^o(k)$) follows the $k = N-3$ diagonal forever and so all entries on or left of the $k = N-5$ diagonal are obtained from the recurrence

$$B_{(1,4,9)}^o(N,k) = B_{(1,4,9)}^o(N-1,k-1) + B_{(1,4,9)}^o(N,k+1).$$

This is the same recurrence that is satisfied by the ballot numbers, and it turns out we can write $B_{(1,4,9)}^o$ as a linear combination of “shifted” ballot numbers. This allows us to compute the first entry in each row precisely, which gives the total probability of success.

Definition 4.10. Define the $i$-shifted ballot triangle $C_i(N,k)$ to be the result of replacing $N$ by $N-i$ in the ballot number formula from Section 2 (where we interpret any binomial coefficients with negative indices as zero).

Theorem 4.11. We have that $B_{(1,4,9)}^o(N,k)$ agrees with the triangle

$$4C_1(N,k) - 9C_2(N,k) + 2C_4(N,k) + 105C_5(N,k) - 206C_6(N,k) + 95C_7(N,k) - 5C_8(N,k).$$

for all $N \geq 11$ and $1 \leq k \leq N-5$.

Proof. As we have observed, all of the triangles under discussion satisfy the recurrence

$$X(N,k) = X(N-1,k-1) + X(N,k+1)$$

at least in the region of $(N,k)$ values lying on or to the left of the $k = (N-5)$th diagonal. But this can be translated to recurrences for the entries along each particular diagonal.
Let \( x_N = X(N, N-k) \) be the sequence of entries along a particular diagonal. When \( k = 2 \), we have \( x_{N+1} = x_N + c \). Successively replacing each \( x_N \) with its difference \( x_N - x_{N-1} \) we obtain a recurrence for the next diagonal to the left. For \( k = 3 \), for example, we obtain \( x_{N+1} = (x_N - x_{N-1}) + c \) which reduces to \( x_{N+1} = 2x_N - x_{N-1} + c \). In general, the recurrence for entries along the \( k \)th diagonal will have degree \( k \) (and its coefficients will be alternating binomial coefficients; see [GKP94]).

Hence, once we have agreement between \( B_{(1,4,9)}^{N-1} \) and the linear combination of shifted ballot numbers for six terms along the \( k = N - 5 \) diagonal, we must have agreement forever along this diagonal and therefore for all entries along diagonals to the left. A finite computation illustrated in Figure 10 shows that this indeed occurs for \( N \geq 11 \) and \( k \leq N - 5 \).

**Corollary 4.12.** The asymptotic probability of success under the optimal strategy for the 321-avoiding interview orders is at least 32983/65536 which is about 0.5032806396484.

**Proof.** In the linear combination of shifted ballot numbers from Theorem 4.11, set \( k = 1 \), divide by the \( N \)th Catalan number, and take the limit as \( N \to \infty \). We obtain

\[
4 \left( \frac{1}{4} \right)^1 - 9 \left( \frac{1}{4} \right)^2 + 2 \left( \frac{1}{4} \right)^4 + 105 \left( \frac{1}{4} \right)^5 - 206 \left( \frac{1}{4} \right)^6 + 95 \left( \frac{1}{4} \right)^7 - 5 \left( \frac{1}{4} \right)^8.
\]

\( \Box \)

### 4.4. Other strategies

We have also investigated positional and trigger strategies for 321-avoiding interview orders. Although they are not generally optimal, it may be convenient to compare them more directly with the classical best choice game analysis. Positional strategies are also simpler to implement and require less memory. In this subsection, we briefly outline the main results.

The 321-avoiding permutations are completely determined by the values and positions of their left-to-right maxima (as the complementary entries must be increasing to avoid 321). These left-to-right maxima may be encoded by Dyck paths on an \( N \times N \) grid lying above the diagonal. The interviews that are \( k \)-winnable (where \( k \) represents a position to transition from rejection to hiring) then correspond to Dyck paths whose next to last horizontal segment passes through column \( k \). These can be counted recursively to obtain a success probability in the form of a linear combination of ratios of Catalan numbers. For example, the positional strategy of transitioning after \( k = N - 3 \) has a success probability of

\[
\frac{3C_{N-1} - 4C_{N-2} - C_{N-3}}{C_N}.
\]

The main result in this direction is that for all \( N > 8 \), transitioning from rejection to hiring after \( k = N - 3 \) turns out to be the optimal positional strategy and has limiting value \( \frac{34}{64} = 0.484375 \). Further details are given in [FJ18].

More generally, we considered trigger strategies for the 321-avoiding interviews. Replacing Theorem 4.1, we have

**Theorem 4.13.** For the 321-avoiding interview orders, we have

\[
\mathcal{T}_N(p) = \begin{cases} 
\mathcal{T}_{N-1}(\tilde{p}) & \text{if } p \text{ contains at least one inversion} \\
\frac{k(N-1)+\binom{N-1}{k}}{N+1\binom{2N-k}{N}} & \text{if } p = [12\cdots k] (\text{with } k = 0 \text{ corresponding to } p = \emptyset)
\end{cases}
\]

where \( \mathcal{T}_N(p) \) is the probability of winning (restricted to the subtree under \( p \)) if \( p \) is included in the trigger set.

One can define an analogous \( B_N^{(2)} \) triangle with slightly simpler recurrence

\[
B_N^{(2)}(12\cdots k) = B_{N-1}^{(2)}(12\cdots (k-1)) \oplus \max(B_N^{(2)}(12\cdots (k+1)), \mathcal{T}_N(12\cdots (k+1))).
\]
The optimal trigger sets can be translated to a statistical trigger strategy based on value-saturated left-to-right maxima, proved along the same lines as we have done for our strike strategy. The optimal trigger threshold function grows similarly to a quadratic but does not seem to satisfy a simple formula. The first few values are

$$\sigma(N-2) = 1, \sigma(N-3) = 3, \sigma(N-4) = 8, \sigma(N-5) = 15, \sigma(N-6) = 25, \sigma(N-7) = 36, \ldots.$$  

For $N \geq 12$, it turns out that the optimal trigger strategy is not optimal overall. However, a lower bound (proved similarly to Theorem 4.11) for the asymptotic probability of success using the optimal trigger strategy is $8239/16384$, which is about 0.50286865.

4.5. **Bijection for 312-avoiding.** In this subsection, we explain why the 321-avoiding prefix tree is isomorphic to the 312-avoiding prefix tree. This is essentially “West’s bijection” of generating trees described in [Wes95] and [CK09] although some of the position/value conventions are different; we give a self-contained account here for completeness.

Given a prefix permutation of size $N - 1$, there are potentially $N$ children in the prefix tree, each corresponding to a value in the last position which we refer to as index of the child. The other values and positions in the child are then completely determined by the parent prefix.

These indices are determined by inversions in the parent permutation. We will refer to an inversion by its values which we denote by $(b > a)$.

**Lemma 4.14.** Fix a prefix permutation $\pi$ of size $N - 1$ in the 321-avoiding prefix tree. Then, the indices for the children of $\pi$ are

$$\{1, 2, \ldots, N\} \setminus \{j : j \leq a \text{ for an inversion } (b > a) \text{ in } \pi\}.$$  

(So it suffices to consider the inversion with the largest minimal value.)

Fix a prefix permutation $\pi$ of size $N - 1$ in the 312-avoiding prefix tree. Then, the indices for the children of $\pi$ are

$$\{1, 2, \ldots, N\} \setminus \{j : a < j \leq b \text{ for an inversion } (b > a) \text{ in } \pi\}.$$  

**Proof.** It is straightforward to verify that these conditions directly encapsulate the pattern avoidance criteria. □

**Corollary 4.15.** The child indices form a nested decreasing sequence of subsets along any path in the prefix tree.

**Theorem 4.16.** The prefix tree for 321-avoiding permutations is isomorphic to the prefix tree for 312-avoiding permutations.

**Proof.** We will actually claim a little more in order to establish the isomorphism. First, we claim that the number of children for a permutation $\pi$ in the 321-avoiding prefix tree is the same as the number of children in the isomorphic node of the 312-avoiding prefix tree. This means the denominators of the strike probabilities remain the same. Second, we claim that the positions (but not necessarily the values!) of the left-to-right maxima of $\pi$ in the 321-avoiding prefix tree are equal to the positions of the left-to-right maxima of the isomorphic node of the 312-avoiding prefix tree. This means the position of $N$, and hence winnability, is preserved so the numerators of the strike probabilities remain the same.

We work by induction on permutation size. The claims are true for permutations of size $< 3$, establishing a base case.

Suppose that $\pi$ is a 321-avoiding permutation of size $N - 1$ with child indices

$$c_1, c_2, \ldots, c_{m-1} = N - 1, c_m = N$$
Figure 11. Correspondence for $N = 4$

(arranged increasingly). By induction, there exists a corresponding 312-avoiding permutation $\tilde{\pi}$ with the same number of children, say

$$\tilde{\pi}_1 = 1, \tilde{\pi}_2, \ldots, \tilde{\pi}_m = N$$

(also arranged increasingly), and having the same positions for left-to-right maxima as $\pi$.

We now show that each of the children of $\pi$ also has a corresponding element in the 312-avoiding prefix tree. The child $c_m = N$ corresponds to the child $\tilde{c}_m = N$ and each child has the same indices as its parent since no new inversions are created. Hence, the first claim is satisfied by induction. Also, $N$ will be a new left-to-right maximum, so the second claim is satisfied by induction.

Otherwise, we claim that the child $c_i$ will correspond to the child $\tilde{c}_{m-i}$. To see this, note that the value $c_i$ will form the minimal entry of a new inversion ($N > c_i$) in the child permutation. Hence, by Lemma 4.14, the child $c_i$ will need to remove $c_1, c_2, \ldots, c_i$ from its set of indices.

Similarly for the 312-avoiding tree, the value $\tilde{c}_{m-i}$ forms the minimal entry of a new inversion ($N > \tilde{c}_{m-i}$) so the child $\tilde{c}_{m-i}$ will need to remove $\tilde{c}_{m-i+1}, \tilde{c}_{m-i+1}, \ldots, \tilde{c}_m$ by Lemma 4.14, which is also a total of $i$ indices removed. Also, one new index $N + 1$ will be added for each child.

Hence, the first claim is satisfied by induction. Also, neither of the entries in position $N$ will be a left-to-right maximum, so the second claim is satisfied by induction.

□

Example 4.17. The correspondence begins as shown in Figure 11.

5. THE REMAINING SIZE THREE PATTERNS

The 123-avoiding and 213-avoiding interview orders give distinct best choice problems but have distributions that are essentially decreasing so the optimal strategy is to always to choose the next left-to-right maximum after the first entry. Although these are not so interesting from the game perspective, they complete our analysis of all the size 3 patterns. For the following results, we continue to let $C(N, k)$ denote the ballot numbers.

Proposition 5.1. For the 123-avoiding interviews, we have

$$S_N(p) = \begin{cases} 
\frac{C(N - 1, k - 1)}{C(N - 1, k - 1)} & \text{if } p = (k - 1) (k - 2) \cdots 2 1 \text{ and } 2 \leq k \leq N \\
\frac{C(N - 1)}{C(N)} & \text{if } p = 1 \\
0 & \text{otherwise.}
\end{cases}$$
Hence, the optimal strategy is to accept the second left-to-right maximum. This succeeds with asymptotic probability $3/4$.

**Proof.** Observe that a 123-avoiding permutation can have at most two left-to-right maxima, and $S(p)$ is 0 unless $p$ has the form of a decreasing sequence followed by a left-to-right maximum. That is, $p = (k - 1)(k - 2) \cdots 1k$ with one prefix for each $1 \leq k \leq N$.

By rejecting the first interview candidate, we can capture all of the other eligible strikes. When we add these, we obtain total probability of success

$$\frac{C(N - 1, N - 1) + C(N - 1, N - 2) + \cdots + C(N - 1, 1)}{C(N)} = \frac{C(N, 2)}{C(N)} = \frac{3N(N - 1)}{2N(2N - 1)}$$

which is more than the $\frac{C(N-1)}{C(N)}$ as we would obtain by accepting $p = 1$. \hfill $\square$

**Proposition 5.2.** For the 213-avoiding interviews, we have

$$S_N(p) = \begin{cases} \frac{C(N-1,k-1)}{C(N,k)} & \text{if } p = 12 \cdots k \text{ and } 1 \leq k \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the optimal strategy is accept the first or second left-to-right maximum. This succeeds with asymptotic probability $1/4$.

**Proof.** Observe that any 213-avoiding permutation that ends in a left-to-right maximum must be increasing. So, $S_N(p) = 0$ if $p$ is not increasing. Otherwise, $p = 12 \cdots k$, so $S_N(p) = \frac{C(N-1,k-1)}{C(N,k)}$ (as the ballot numbers count the total number of 213-avoiding permutations under a given increasing prefix, so removing the value $N$ gives a bijection to smaller rank). These prefixes all contain each other so we should pick the one with largest number of wins. As the numerators are decreasing, it is optimal to choose the first or second one. \hfill $\square$

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