Stress-strain State of Circular Plates with Concentric Inserts under Tension

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Abstract. The article discusses the method of determining stresses in circular plates with concentric absolutely rigid inserts. The plate is stretched by a uniformly distributed load. A design scheme is proposed and an analysis of mathematical expressions describing the deformation process at the interface of the plate and rigid insert is performed. Mathematical dependences are obtained for the calculation of normal and tangential stresses, allowing to determine their distribution at the design stage. It was established that the zone of influence on the stress distribution of a rigid insert does not exceed the diameter of the insert. The use of various welding technologies allows changing the resultant field of normal and tangential stresses and increasing the load capacity of the permanent joint.

1. Introduction

In the context of scientific and technological progress, the development of science, technology and production is especially important. When developing complex metal structures, techniques and algorithms are necessary for calculating the strength characteristics [1–6], including the characteristics of permanent joints.

This type of joints include metal welding, which in many industries is one of the main factors determining the pace of technological progress, and has a significant impact on the efficiency of social production. There is hardly any mechanical engineering industry, which doesn’t use metal welding (Figure 1) [7–13].

Residual stresses are those that exist in the structure or in its individual elements in the absence of external force, thermal and other actions. In engineering to designate residual stresses the names of the technological processes are also used: welding stresses, quenching stresses, deformation stresses, straightening stresses or finishing stresses.

Residual (preliminary) stresses have a significant role in engineering [14–22]. Knowledge of the internal stress state in materials and structures makes it possible to calculate the structure more accurately. There are plenty of examples of damage caused by large technological stresses. One of the most frequent is the destruction of pipelines in which cracks can develop up to tens of kilometers in length.

The welding of different types of metal structures made it possible to most effectively use workpieces obtained by rolling, bending, punching, casting and forging, as well as metals with different physicochemical properties. Welded structures are more light-weighted and less labor intensive. Permanent joint of almost all metals and alloys of various thicknesses are obtained by welding.

To determine the stresses in a circular plate with a rigid insert under external load, installed by welding, one can use the solution of the plane elasticity problem for annular plates in polar coordinates. The polar coordinates of the points are the polar radius $r$ and the polar angle $\theta$, and the stress state is determined by the tangents $\tau$, normal radial $\sigma_r$ and tangential $\sigma_\theta$ stresses [23–25].
2. Determination of stresses and strains

2.1. Stress-strain state in the absence of external load

Let us determine residual stresses in a circular plate of radius $R_1$ with an absolutely circular rigid insert of radius $R$ (Figure 2) in the absence of external forces ($p = 0$).

**Figure 1.** Welded structures with circular welded insert.

**Figure 2.** Circular plate with rigid insert under uniaxial tension with uniform load.

In this case, the displacements of the points of the plate on the boundary with the insert in the Cartesian coordinate system are

$$g_1 = u_1 + iv_1,$$

where $u_1 = \Delta_1 \cos \theta$ – the displacements of points in the longitudinal (along the axis OX) direction; $v_1 = \Delta_2 \sin \theta$ – the displacements of points in the transverse (along the axis OY) direction; $\Delta_1, \Delta_2$ – the longitudinal and transverse shrinkage of the welded seam [23].

Representing radial $u$ and tangential $v$ displacements in polar coordinates as

$$\begin{align*}
&u = u_1 \cos \theta + v_1 \sin \theta; \\
v = -u_1 \sin \theta + v_1 \cos \theta.
\end{align*}$$

we obtain the following

$$\begin{align*}
&u = 0.5(\Delta_1 + \Delta_2) + 0.5(\Delta_1 - \Delta_2) \cos 2\theta; \\
v = -0.5(\Delta_1 - \Delta_2) \sin 2\theta.
\end{align*}$$

Boundary conditions on the contour $r = R$ will be written as

$$\begin{align*}
&u(R, \theta) = 0.5(\Delta_1 + \Delta_2) + (\Delta_1 - \Delta_2) \cos 2\theta; \\
v(R, \theta) = -(\Delta_1 - \Delta_2) \sin 2\theta.
\end{align*}$$

Boundary conditions on the contour $r = R_1$ will be written as

$$\begin{align*}
&\sigma_r(R_1, \theta) = 0; \\
&\tau(R_1, \theta) = 0.
\end{align*}$$

For annular plates the general solution is known in the form of an infinite trigonometric series in $\sin(n\theta)$ and $\cos(n\theta)$ ($n$ – is the series term number).

Since the boundary conditions (4, 5) are homogeneous or contain only expansion terms with numbers $n=0$ and $n=2$, then in general expressions for stresses it is sufficient to leave the terms with the indicated numbers and only those that determine the expansion of displacements $u$ by cosines, and $v$ - by sinuses.

Therefore, displacements can be determined by the following expressions:
where \( E \) – modulus of elasticity (Young's modulus); \( \mu \) – coefficient of transverse deformation (Poisson's ratio); \( a_0, a_1, a_2, a_3, a_4, a_5 \) – unknown coefficients to be determined.

Stresses can be determined by the following expressions:

\[
\begin{align*}
\sigma_r &= \frac{a_1}{r^2} + 2a_0 - \left(a_2 + \frac{a_3}{r^4} + \frac{2a_5}{r^2}\right) \cos 2\theta; \\
\sigma_\theta &= -\frac{a_1}{r^2} + 2a_0 + \left(a_2 + \frac{a_3}{r^4} + \frac{a_4}{r^2}\right) \cos 2\theta; \\
\tau_{r\theta} &= \frac{(R^2 - R_1^2)}{r^2} + \left(a_2 - \frac{a_1}{r^4} + a_4 r^2 - \frac{a_5}{r^2}\right) \sin 2\theta.
\end{align*}
\]  

Substituting displacements (1) and stresses (2) into the boundary conditions at \( r = R \) and \( r = R_1 \), we determine the unknown coefficients, equating the free terms to coefficients at \( \cos 2\theta \), \( \sin 2\theta \).

Substituting the coefficient values into expression (2), we obtain the following expressions for the residual stresses when welding absolutely rigid insert of radius \( R \) into the plate of radius \( R_1 \):

\[
\begin{align*}
\sigma_{r1} &= \frac{E(A_1 + A_2)R}{2(1 - \mu)R^2 + (1 + \mu)R_1^2} \left(1 - \frac{R_1^2}{r^2}\right) - \frac{(A_1 - A_2)ERR_1^4[(1 + \mu)R_0^6 + (3 - \mu)R^6]}{A + B} \\
&\times \left\{ \frac{3R_1^4((1 + \mu)(R_2^2 - R_1^2)) - [(1 + \mu)R_0^6 + (3 - \mu)R^6]}{2R_1^2[(1 + \mu)R_1^6 + (3 - \mu)R^6] - 3R_1^2R^2[(1 + \mu)R_0^4 + (3 - \mu)R^4] + 2\frac{(R^2 - R_1^2)}{r^2}} \right\} \cos 2\theta.
\end{align*}
\]  

\[
\begin{align*}
\sigma_{\theta1} &= \frac{E(A_1 + A_2)R}{2(1 - \mu)R^2 + (1 + \mu)R_1^2} \left(1 - \frac{R_1^2}{r^2}\right) - \frac{(A_1 - A_2)ERR_1^4[(1 + \mu)R_0^6 + (3 - \mu)R^6]}{A + B} \\
&\times \left\{ \frac{3R_1^4((1 + \mu)(R_2^2 - R_1^2)) - [(1 + \mu)R_0^6 + (3 - \mu)R^6]}{2R_1^2[(1 + \mu)R_1^6 + (3 - \mu)R^6] - 3R_1^2R^2[(1 + \mu)R_0^4 + (3 - \mu)R^4] + 6(1 + \mu)(R_2^2 - R_1^2)r^2} \right\} \cos 2\theta.
\end{align*}
\]  

\[
\begin{align*}
\tau_{r\theta1} &= \frac{(A_1 - A_2)ERR_1^4[(1 + \mu)R_0^6 + (3 - \mu)R^6]}{A + B} \\
&\times \left\{ \frac{3R_1^4((1 + \mu)(R_2^2 - R_1^2)) - [(1 + \mu)R_0^6 + (3 - \mu)R^6]}{2R_1^2[(1 + \mu)R_1^6 + (3 - \mu)R^6] - 3R_1^2R^2[(1 + \mu)R_0^4 + (3 - \mu)R^4]} - 3(1 + \mu)(R_2^2 - R_1^2)r^2 - \frac{1}{r^2} \right\} \sin 2\theta.
\end{align*}
\]  

where the following notation is used...
\[ A = 3(1 + \mu)(R^2 - R_1^2)R^4[(1 + \mu)R_1^4 + (3 - \mu)R^4]; \]
\[ B = [(1 + \mu)R_1^6 + (3 - \mu)R^6][(1 + \mu)R_1^6 + (3 - \mu)R^6][(1 + \mu)(R^2 - R_1^2)(R_1^2 - 3R^2) + 4R_1^4]. \]

When radius \( R_1 \) tends to infinity in expressions for \( \sigma_{r1}, \sigma_{\theta1}, \tau_{r\theta1} \), we pass to expressions for residual stresses in an infinite plate with a welded absolutely rigid circular insert.

2.2. Stress-strain state under the action of external load

In the case of tension of the plate along the OX axis at the edge \( r = R_1 \) by the normal load \( p \) (Figure 2), additional stresses are introduced into the plate. It can be assumed that at the edge points \( r = R_1 \), the stress field in the plate is determined in Cartesian coordinates by the components \( \sigma_x = p \), \( \sigma_y = \tau_{xy} = 0 \), and in polar coordinates according to the formulas:

\[
\begin{align*}
\sigma_{r2} &= 0.5p(1 + \cos 2\theta); \\
\tau_{\theta2} &= -0.5p\sin 2\theta.
\end{align*}
\]

The insert is considered to be non-deformable or absolutely rigid and firmly connected to the plate along the line of their connection. So the radial and tangential displacements of the points of the insertion edge \( r = R \) are equal to zero. The expressions for the boundary conditions are as follows: on the contour \( r = R - u_2(R,\theta) = 0 \) and \( v_2(R,\theta) = 0 \); on the contour \( r = R_1 - \sigma_{r2}(R_1,\theta) = 0.5p(1 + \cos 2\theta) \) and \( \tau_{\theta2}(R_1,\theta) = -0.5p\sin 2\theta \).

Boundary conditions on the contour \( r = R \) will be written as

\[
\begin{align*}
(R,\theta) &= 0; \\
v(R,\theta) &= 0.
\end{align*}
\]

Boundary conditions on the contour \( r = R_1 \) will be written as

\[
\begin{align*}
\sigma_r(R_1,\theta) &= 0.5(\Delta_1 + \Delta_2) + (\Delta_1 - \Delta_2)\cos 2\theta; \\
\tau_\theta(R_1,\theta) &= -(\Delta_1 - \Delta_2)\sin 2\theta.
\end{align*}
\]

Expressions for stresses by analogy with the welded insert are taken as (2), and for displacements as (1), given that the boundary conditions contain only terms with numbers \( n=0 \) and \( n=2 \). Substituting these expressions into boundary conditions, we determine the unknown coefficients.

Considering these coefficients in expression (2), we obtain the following formulas for determining stresses under uniaxial tension of the circular plate having an absolutely rigid insert firmly connected to the plate

\[
\sigma_{r3} = \frac{pR_1^2}{2[(1 - \mu)R^2 + (1 + \mu)R_1^2]}
\left( 1 + \mu + \frac{(1 - \mu)R^2}{r^2} \right) +
\left\{ \frac{p}{2} + C \right\}
\left[ \frac{(3 - \mu)R^6 + (1 + \mu)R_1^6}{R_1^2[3(R^2 - R_1^2) - R_1^4]} + \frac{3R^2R_1^2(3 - \mu)R^4 - (1 + \mu)R_1^4}{2r^4[(1 + \mu)R_1^6 + (3 - \mu)R^6]} \right] \cos 2\theta. \]
\begin{equation}
\sigma_{\theta 3} = \frac{pR_i^2}{2[(1-\mu)R^2 + (1+\mu)R_i^2]} \left( 1 + \mu - \frac{(1-\mu)R^2}{r^2} \right)
- \left( \frac{p}{2} + C \left[ \frac{(3-\mu)R^6 + (1+\mu)R_i^4 [3(R^2 - R_i^2) - R_i^4]}{R_i^2 2[(3-\mu)R^6 - (1+\mu)R_i^6]} \right. \right.
+ \left. \left. \frac{3R_i^2 R^2(3-\mu)R^4 - (1+\mu)R_i^4}{2r^4[(1+\mu)R_i^6 + (3-\mu)R^6]} + \frac{6(1+\mu)(R^2 - R_i^2)r^2}{[(1+\mu)R_i^6 + (3-\mu)R^6]} \right] \cos 2\theta \right)
\end{equation}

\begin{equation}
\tau_{r\theta 3} = \left( - \frac{p}{2} - C \left[ \frac{(1+\mu)R_i^4 [3(R^2 - R_i^2) - R_i^4]}{R_i^2 2[(3-\mu)R^6 - (1+\mu)R_i^6]} \right. \right.
- \left. \left. \frac{3R_i^2 R^2(3-\mu)R^4 - (1+\mu)R_i^4}{2r^4[(1+\mu)R_i^6 + (3-\mu)R^6]} + \frac{3(1+\mu)(R^2 - R_i^2)r^2}{[(1+\mu)R_i^6 + (3-\mu)R^6]} + \frac{1}{r^2} \right] \right) \sin 2\theta.
\end{equation}

where

\[ C = pR_i^4 R^2 (1+\mu)[(1+\mu)R_i^6 + (3-\mu)R^6] [R^2 (1+\mu)(R^2 - R_i^2)][R^2 (3-\mu)(R_i^4 - 3R^4) - 2R_i^4 (3(1+\mu)R^2 - 2R_i^2)] - [3(1+\mu)r^2(1+\mu)R_i^6][1+\mu)(R_i^4 - 3R^4) - 2R_i^4 (2R_i^2 - (1+\mu)R_i^2)]^{-1}. \]

When the radius \( R_i \) tends to infinity in the expressions for \( \sigma_{r 3}, \sigma_{\theta 3}, \tau_{r\theta 3} \) we come to the expressions for the tension stresses of the infinite plate with the rigid circular insert \([23]\).

The stresses in the stretched plate with the welded circular rigid insert are determined by the overlapping of the stresses of the two cases considered above, i.e.

\[
\begin{align*}
\sigma_{r 4} &= \sigma_{r 1} + \sigma_{r 3}; \\
\sigma_{\theta 4} &= \sigma_{\theta 1} + \sigma_{\theta 3}; \\
\tau_{r\theta 4} &= \tau_{r\theta 1} + \tau_{r\theta 3}.
\end{align*}
\]

If the circular plate with the non-deformable insert firmly connected to it is under external pressure \( p \) (Figure 2), then the boundary conditions in the problem of determining the stresses in the plate are expressed as follows: on the contour \( r = R - u_3(R, \theta) = 0 \) and \( v_3(R, \theta) = 0 \); on the contour \( r = R_1 - \sigma_{r 5}(R_1, \theta) = -p \) and \( \tau_{r\theta 5}(R_1, \theta) = 0 \).

Substituting the expressions for displacements (1) and stresses (2) into these boundary conditions, we obtain that the coefficients, \( a_2, a_3, a_4, a_5 \) are zero. Having determined the other two coefficients \( a_0 \) and \( a_1 \) and substituting them into formulas (2), we calculate the stresses in the plate with non-deformable insert under external pressure:

\[
\sigma_{r 6} = \frac{p}{(1-\mu)R^2 + (1+\mu)R_i^2} \left( \frac{R_i^2 R_i^2}{r^2} - (1+\mu)R_i^2 - (2-\mu)R^2 \right),
\]

\[
\sigma_{\theta 3} = \frac{p}{(1-\mu)R^2 + (1+\mu)R_i^2} \left( \frac{R_i^2 R_i^2}{r^2} + (1+\mu)R_i^2 + (2-\mu)R^2 \right),
\]

\[
\tau_{r\theta 6} = 0.
\]

When the external pressure acts on a stretched plate with the welded rigid insert, the stresses in the plate are determined by the formulas
\[
\begin{align*}
\sigma_r &= \sigma_{r4} + \sigma_{r6}; \\
\sigma_\theta &= \sigma_{\theta4} + \sigma_{\theta6}; \\
\tau_{r\theta} &= \tau_{r\theta6}.
\end{align*}
\] (20)

3. Calculation results and discussion

Formulas (1)–(20) determine the mathematical basis of the calculation method of the residual stress field in the annular plate with the absolutely rigid concentric insert under tension by external load.

Figure 3 shows the stresses at the boundary of the rigid insert. Figure 4 shows the residual welding stresses in the circular plate with the welded rigid insert. It was assumed that the stresses at the weld boundary reached the yield strength of the plate material. It can be seen that both radial \( \sigma_{r1} \) and tangential stresses \( \sigma_{\theta1} \) decay at a distance equal to two radii from the edge of the hole \( r/R = 2 \).

![Figure 3. Distribution of radial stresses along the boundary of the rigid insert in the circular plate under uniaxial tension by a uniform load.](image)

![Figure 4. Distribution of residual welding stresses in the circular plate with welded rigid insert: 1 – \( \sigma_{r1}/\sigma_T \); 1 – \( \sigma_{\theta1}/\sigma_T \).](image)

4. Conclusion

It is established that the width of the zone of significant influence of rigid insert on the distribution of residual stresses depends on the size of the insert and does not exceed the diameter of the insert. The field of residual stresses can be changed by the variation of the welding parameters. The ability to determine the residual stresses by calculation can improve the strength characteristics of complex metal structures at the design stage.

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