DYNAMIC INVERSE PROBLEM FOR JACOBI MATRICES

ALEXANDR MIKHAYLOV* AND VICTOR MIKHAYLOV

St. Petersburg Department of V.A. Steklov Institute of Mathematics of the Russian Academy of Sciences, 27, Fontanka, 191023 St. Petersburg, Russia and Saint Petersburg State University, St.Petersburg State University 7/9 Universitetskaya nab., St. Petersburg, 199034, Russia

(Communicated by Matti Lassas)

Abstract. We consider the inverse dynamic problem for a dynamical system with discrete time associated with a semi-infinite Jacobi matrix. We derive discrete analogs of Krein equations and answer a question on the characterization of dynamic inverse data. As a consequence we obtain a necessary and sufficient condition for a measure on a real line to be a spectral measure of a semi-infinite discrete Schrödinger operator.

1. Introduction. For a given sequence of positive numbers \(\{a_0, a_1, \ldots\}\) and real numbers \(\{b_1, b_2, \ldots\}\), we consider a dynamical system with discrete time associated with a Jacobi matrix:

\[
\begin{aligned}
& u_{n,t+1} + u_{n,t-1} - a_n u_{n+1,t} - a_{n-1} u_{n-1,t} - b_n u_{n,t} = 0, \quad n, t \in \mathbb{N}, \\
& u_{n,-1} = u_{n,0} = 0, \quad n \in \mathbb{N}, \\
& u_{0,t} = f_t, \quad t \in \mathbb{N} \cup \{0\},
\end{aligned}
\]

which is a natural analog of a dynamical systems governed by a wave equation on a semi-axis. By an analogy with continuous problems [6], we treat the real sequence \(f = (f_0, f_1, \ldots)\) as a boundary control. The solution to (1) is denoted by \(u_{n,t}^f\). Having fixed \(\tau \in \mathbb{N}\), we associate the response operators with (1), which maps the control \(f = (f_0, \ldots, f_{\tau-1})\) to \(u_{1,t}^f\):

\[
(R^\tau f)_t := u_{1,t}^f, \quad t = 1, \ldots, \tau.
\]

The inverse problem we will be dealing with consists in recovering the sequences \(\{b_1, b_2, \ldots, b_n\}\), \(\{a_0, a_1, \ldots, a_n\}\) for some \(n\) from \(R^n\). This problems is a natural discrete analog of an inverse problem for a wave equation on a half-line, in which the dynamic Dirichlet-to-Neumann map is used as inverse data [8, 4, 10].

The area of dynamic inverse problems is wide and well-developed, to mention reviews and books [6, 7, 16, 15, 14] and literature cited therein, at the same time the authors are not familiar with any literature on inverse dynamic problems for discrete systems with discrete time. In our approach we use the Boundary Control (BC) method [6, 7]. For the first time the BC method was applied to discrete dynamical systems in [1, 3] in connection with the spectral estimation problem.

\[2010\] Mathematics Subject Classification. Primary: 35R30, 93B30; Secondary: 39A12, 35C15, 35P99.  

Key words and phrases. Inverse problem, Jacobi matrices, discrete Schrödinger operator, boundary control method, characterization of inverse data.
For the same sequences \( \{a_1, a_2, \ldots\} \), \( \{b_1, b_2, \ldots\} \) (we note that \( a_0 \) is an additional parameter used in system (1)) we consider the operator \( H \) corresponding to a semi-infinite Jacobi matrix, defined on \( l^2 \ni \phi = (\phi_1, \phi_2, \ldots) \), given by

\[
(H\phi)_n = a_n\phi_{n+1} + a_{n-1}\phi_{n-1} + b_n\phi_n, \quad n \geq 2,
\]

\[
(H\phi)_1 = b_1\phi_1 + a_1\phi_2, \quad n = 1.
\]

Let us fix \( N \in \mathbb{N} \) and \( h \in \mathbb{R} \), the operator corresponding to a finite Jacobi matrix is denoted by \( H^N \), it is given by (2), (3) and condition at the “right end”:

\[
\phi_{N+1} + h\phi_N = 0.
\]

When all \( a_n = 1 \), \( n = 0, 1, \ldots \) the operator \( H \) is called a discrete Schrödinger operator. In [13] the authors posed a question of the characterization of a spectral measure for a discrete Schrödinger operator. We give the answer on this question as a consequence to a theorem on characterization of dynamic inverse data.

In the second section we study the forward problem: for (1) we prove the analog of d’Alembert integral representation formula, we also introduce and prove representation formulae for the main operators of the BC method: response operator, control and connecting operators. The continuous analogs of dynamic inverse problems for one-dimensional systems were considered in [4, 10, 8], where the authors deal with wave equations on a half-line, in [11] the Dirac system is considered and in [9, 12] the two-velocity system is studied. In the third section we derive Krein equations of the inverse problem and answer a question on the characterization of dynamic inverse data in the case of Jacobi matrices and in the case of discrete Schrödinger operators. More on dynamic approach to Krein equations one can find in [10, 18]. The application of the BC-method to the problem of characterization of dynamic inverse data (different from one in the present paper) is considered in [9, 12, 11]. In the last section we derive spectral representations for response and connecting operators, that gives a possibility to use methods developed in this paper for solving inverse spectral problems for Jacobi matrices in finite and semi-infinite cases. Then we use the results obtained to prove the theorem on the characterization of a spectral measure for a discrete Schrödinger operator. In [5, 19, 20] relationships between dynamic and spectral inverse data were established for some one-dimensional dynamical systems. In [18] the authors established links between BC-method and the method of de Branges on the example of several continuous and discrete one-dimensional systems.

2. Forward problem, operators of the Boundary Control method. We fix some positive integer \( T \) and denote by \( \mathcal{F}^T \) the outer space of the system (1), the space of controls (inputs): \( \mathcal{F}^T := \mathbb{R}^T, f \in \mathcal{F}^T, f = (f_0, \ldots, f_{T-1}) \), we use the notation \( \mathcal{F}^\infty = \mathbb{R}^\infty \) when control acts for all \( t \geq 1 \). The representation formula in the following lemma can be considered as a discrete analog of a d’Alembert integral representation formula for a solution of an initial-boundary value problem for a wave equation with a potential on a half-line. [4].

**Lemma 2.1.** A solution to (1) admits the representation

\[
u^f_{n,t} = \prod_{k=0}^{n-1} a_k f_{t-n} + \sum_{s=n}^{t-1} w_{n,s} f_{t-s-1}, \quad n, t \in \mathbb{N},
\]
where $w_{n,s}$ satisfies the Goursat problem

\begin{equation}
\begin{aligned}
w_{n,s+1} + w_{n,s-1} - a_n w_{n+1,s} - a_{n-1} w_{n-1,s} - b_n w_{n,s} = \\
= -\delta_{s,n}(1 - a_n^2) \prod_{k=0}^{n-1} a_k, \ n, s \in \mathbb{N}, \ s > n, \\
w_{n,n} - b_n \prod_{k=0}^{n-1} a_k - a_{n-1} w_{n-1,n-1} = 0, \ n \in \mathbb{N}, \\
w_{0,t} = 0, \ t \in \mathbb{N}_0.
\end{aligned}
\end{equation}

Proof. We assume that $u_{n,t}^f$ has a form (5) with unknown $w_{n,s}$ and plug it into (1):

\begin{equation}
0 = \prod_{k=0}^{n-1} a_k f_{t+1-n} + \prod_{k=0}^{n-1} a_k f_{t-1-n} - a_n \prod_{k=0}^{n} a_k f_{t-n-1} - a_{n-1} \prod_{k=0}^{n-2} a_k f_{t-n+1}
\end{equation}

\begin{equation}
- b_n \prod_{k=0}^{n-1} a_k f_{t-n} + \sum_{s=n}^{t-1} w_{n,s} f_{t-s} + \sum_{s=n}^{t-2} w_{n,s} f_{t-s-2}
\end{equation}

\begin{equation}
- a_n \sum_{s=n+1}^{t-1} w_{n+1,s} f_{t-s-1} + a_{n-1} \sum_{s=n-1}^{t-1} w_{n-1,s} f_{t-s-1} - b_n w_{n,s} f_{t-s-1}.
\end{equation}

Changing the order of summation and evaluating we get

\begin{equation}
0 = (1 - a_n^2) \prod_{k=0}^{n-1} a_k f_{t-n-1} - b_n \prod_{k=0}^{n-1} a_k f_{t-n}
\end{equation}

\begin{equation}
- \sum_{s=n}^{t-1} f_{t-s-1} (b_n w_{n,s} + a_n w_{n+1,s} + a_{n-1} w_{n-1,s}) + a_n w_{n+1,n} f_{t-n} - a_{n-1} w_{n-1,n} f_{t-n-1}
\end{equation}

\begin{equation}
- a_{n-1} w_{n-1,n-1} f_{t-n} + \sum_{s=n-1}^{t-1} w_{n,s} f_{t-s-1} + \sum_{s=n}^{t-1} w_{n,s} f_{t-s-1}
\end{equation}

\begin{equation}
= \sum_{s=n}^{t-1} f_{t-s-1} (w_{n,s+1} + w_{n,s-1} - a_n w_{n+1,s} - a_{n-1} w_{n-1,s} - b_n w_{n,s})
\end{equation}

\begin{equation}
- b_n \prod_{k=0}^{n-1} a_k f_{t-n} + (1 - a_n^2) \prod_{k=0}^{n-1} a_k f_{t-n-1} + a_n w_{n+1,n} f_{t-n-1}
\end{equation}

\begin{equation}
- a_{n-1} w_{n-1,n-1} f_{t-n} + w_{n,n} f_{t-n} - w_{n,n-1} f_{t-n-1} + b_n \prod_{k=0}^{n-1} a_k
\end{equation}

Finally we arrive at

\begin{equation}
\sum_{s=n}^{t-1} f_{t-s-1} (w_{n,s+1} + w_{n,s-1} - a_n w_{n+1,s} - a_{n-1} w_{n-1,s} - b_n w_{n,s})
\end{equation}

\begin{equation}
+ (1 - a_n^2) \prod_{k=0}^{n-1} a_k \delta_{sn} + f_{t-n} \left( w_{n,n} - a_{n-1} w_{n-1,n-1} - b_n \prod_{k=0}^{n-1} a_k \right) = 0.
\end{equation}

Counting that $w_{n,s} = 0$ when $n > s$ and the arbitrariness of $f$, we arrive at (6). \qed

**Definition 2.2.** For $f, g \in \mathcal{F}^\infty$ we define the convolution $c = f \ast g \in \mathcal{F}^\infty$ by the formula

\begin{equation}
c_t = \sum_{s=0}^{t} f_s g_{t-s}, \quad t \in \mathbb{N} \cup \{0\}.
\end{equation}

The input $\mapsto$ output correspondence in the system (1) is realized by a **response operator**.
Definition 2.3. For (1) the response operator \( R^T : \mathcal{F}^T \rightarrow \mathbb{R}^T \) is defined by the rule
\[
(R^T f)_t = u^f_{1,t}, \quad t = 1, \ldots, T.
\]

This operator plays the role of inverse data. The quantity \( u_{1,t} - u_{0,t} \) is an analog of \( u_x(0,t) \) (derivative in \( x \) direction at \( x = 0 \)) in a continuous case, so such introduced response \( R^T f \) can be considered as a discrete dynamic Dirichlet-to-Neumann map. On the relationship between \( R^T \) and the Weyl function for corresponding Jacobi matrix see [20].

The following statement is equivalent to a boundary controllability of (1).

\[\text{Lemma 2.4.} \quad \text{The operator \( W^T : \mathcal{F}^T \rightarrow \mathcal{H}^T \) is an isomorphism between \( \mathcal{F}^T \) and \( \mathcal{H}^T \).}\]

Proof. We fix some \( a \in \mathcal{H}^T \) and look for a control \( f \in \mathcal{F}^T \) such that \( W^T f = a \). We write down the action of the operator \( W^T \) as
\[
W^T f = \begin{pmatrix}
u_{1,T} \\ u_{2,T} \\ \vdots \\ u_{k,T} \\ u_{T,T}
\end{pmatrix} = \begin{pmatrix}
0 & w_{1,1} & w_{1,2} & \cdots & \cdots & w_{1,T-1} \\
0 & a_0 a_1 & w_{2,2} & \cdots & \cdots & w_{2,T-1} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \Pi_{j=0}^{k-1} a_j & w_{k,k} & \cdots & w_{k,T-1} \\
0 & 0 & 0 & 0 & \cdots & \Pi_{k=0}^{T-1} a_{T-1}
\end{pmatrix} \begin{pmatrix}
0 \\ \vdots \\ f_{T-1} \\ f_{T-2} \\ \vdots \\ f_{T-k-1} \\
f_0
\end{pmatrix}.
\]
On introducing the notations

\[ J_T : F^T \mapsto F^T, \quad (J_T f)_n = f_{T-n}, \quad n = 0, \ldots, T - 1, \]

\[ A \in \mathbb{R}^{T \times T}, \quad a_{ii} = \prod_{k=0}^{i-1} a_k, \quad a_{ij} = 0, \quad i \neq j, \]

\[ K \in \mathbb{R}^{T \times T}, \quad k_{ij} = 0, \quad i \geq j, \quad k_{ij} = w_{ij-1}, \quad i < j, \]

we have that

\[ W^T = (A + K) J_T. \]

Obviously, this operator is invertible, which proves the statement of the lemma. \( \square \)

For the system (1) we introduce the connecting operator \( C^T : F^T \mapsto F^T \) by the quadratic form: for arbitrary \( f, g \in F^T \) we define

\[ (C^T f, g)_{F^T} = \left( u^f_{-T}, u^g_{-T} \right)_{H^T} = (W^T f, W^T g)_{H^T}. \]

That is \( C^T = (W^T)^* W^T \), and by (9), \( C^T \) connects the metrics of the outer and inner spaces.

The fact that the connecting operator can be represented in terms of inverse data is crucial in the BC method.

**Theorem 2.5.** The connecting operator \( C^T \) is an isomorphism in \( F^T \), it admits the representation in terms of inverse data:

\[ C^T = a_0 C^T_{ij}, \quad C^T_{ij} = \sum_{k=0}^{T-\max(i,j)} r_{[i-j]+2k}, \quad r_0 = a_0. \]

\[ C^T = \left( \begin{array}{cccccc}
    r_0 + r_2 + \ldots + r_{2T-2} & r_1 + \ldots + r_{2T-3} & \ldots & r_T + r_{T-2} & r_{T-1} \\
    r_1 + r_3 + \ldots + r_{2T-3} & r_0 + \ldots + r_{2T-4} & \ldots & \ldots & r_{T-2} \\
    \cdot & \cdot & \cdot & \cdot & \cdot \\
    r_T - 3 + r_{T-1} + r_{T+1} & \ldots & r_0 + r_2 + r_4 & r_1 + r_3 & r_2 \\
    r_{T-1} + r_{T-2} & \ldots & r_1 + r_3 & r_0 + r_2 & r_1 \\
    \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array} \right) \]

**Proof.** We observe that \( C^T = (W^T)^* W^T \), so due to Lemma 2.4, \( C^T \) is an isomorphism in \( F^T \). For fixed \( f, g \in F^T \) we introduce the Blagoveshchenskii function by the rule

\[ \psi_{n, t} := \left( u^f_{-n}, u^g_{-t} \right)_{H^T} = \sum_{k=1}^{T} u^f_{k,n} u^g_{k,t}. \]

We show that \( \psi_{n, t} \) satisfies certain difference equation. Indeed, we can evaluate:

\[ \psi_{n, t+1} + \psi_{n, t-1} - \psi_{n+1, t} - \psi_{n-1, t} = \sum_{k=1}^{T} u^f_{k,n} \left( u^g_{k, t+1} + u^g_{k, t-1} \right) - \sum_{k=1}^{T} \left( u^f_{k, n+1} + u^f_{k, n-1} \right) u^g_{k, t} = \sum_{k=1}^{T} u^f_{k,n} \left( a_k u^g_{k+1, t} + a_{k-1} u^g_{k-1, t} + b_k u^g_{k, t} \right) \]
Thus we arrived at the following difference equation on $\psi$

\[
\psi_{n,t+1} + \psi_{n,t-1} - \psi_{n+1,t} - \psi_{n-1,t} = h_{n,t}, \quad n, t \in \mathbb{N}_0,
\]

\[
h_{n,t} = a_0 \left[ g_k(Rf)_n - f_n(Rg)_t \right].
\]

We introduce the set

\[
K(n, t) := \{(n, t) \cup \{(n - 1, t - 1), (n + 1, t - 1)\} \cup \{(n - 2, t - 2), (n, t - 2), (n + 2, t - 2)\} \cup \ldots \cup \{(n - t, 0), (n - t + 2, 0), \ldots, (n - t + 2, 0), (n + t, 0)\}\}
\]

\[
= \bigcup_{\tau = 0}^{t} \bigcup_{k = 0}^{\tau} (n - \tau + 2k, t - \tau).
\]

The solution to (11) is given by (see [17])

\[
\psi_{n,t} = \sum_{(k, \tau) \in K(n, t-1)} h(k, \tau).
\]

We observe that $\psi_{T, T} = (CT f, g)$, so

\[
(C^T f, g) = \sum_{(k, \tau) \in K(T, T-1)} h(k, \tau).
\]

We note that in the right hand side of (12) the argument $k$ runs from 1 to $2T - 1$. Extending $f \in F^T$, $f = (f_0, \ldots, f_{T-1})$ to $f \in F^{2T}$ by:

\[
T - 1, \quad k = 1, 2, \ldots, T - 1,
\]

we deduce that $\sum_{k, \tau \in K(T, T-1)} f_k(R^T g)_\tau = 0$, so (12) leads to

\[
(C^T f, g) = \sum_{k, \tau \in K(T, T-1)} g_\tau \left[ (R^T f)_k \right] = g_0 \left[ (R^T f)_1 + (R^T f)_2 + \ldots + (R^T f)_{2T-2} + \ldots + (R^T f)_{2T-1} \right] + \ldots + g_{T-1} \left[ (R^T f)_{2T-1} \right].
\]

The latter relation yields

\[
C^T f = \left( (R^T f)_1 + \ldots + (R^T f)_{2T-1} \right), \quad C^T f = \left( (R^T f)_2 + \ldots + (R^T f)_{2T-1} \right),
\]

from where the statement of the theorem follows.

3. **Inverse problem.** The speed of a wave propagation in (1) is finite, which implies the following dependence of inverse data on coefficients $\{a_n, b_n\}$: for $M \in \mathbb{N}$ the element $u_{1,2M-1}^f$ depends on $\{a_0, a_1, \ldots, a_{M-1}\}$, $\{b_1, \ldots, b_{M-1}\}$, on observing this we can make the following

**Remark 1.** The response operator $R^{2T}$ (or, what is equivalent, the response vector $(r_0, r_1, \ldots, r_{2T-2})$) depends on $\{a_0, \ldots, a_{T-1}\}$, $\{b_1, \ldots, b_{T-1}\}$. 

**Inverse Problems and Imaging** Volume 13, No. 3 (2019), 431–447
This observation leads to the natural set up of the dynamic inverse problem for the system (1): by the given operator $R^T$ to recover $\{a_0, \ldots, a_{T-1}\}$ and $\{b_1, \ldots, b_{T-1}\}$.

3.1. **Krein equations.** Let $\alpha, \beta \in \mathbb{R}$ and $y$ be a solution to

$$
\begin{aligned}
&\begin{cases}
  a_k y_{k+1} + a_{k-1} y_{k-1} + b_k y_k = 0, \\
y_0 = \alpha, \ y_1 = \beta.
\end{cases}
\end{aligned}
$$

We set up the following special control problem: to find a control $f^T \in F^T$ that drives the system (1) to prescribed state $\tilde{y}^T := (y_1, \ldots, y_T) \in \mathcal{H}^T$:

$$
W^T f^T = \tilde{y}^T, \quad (W^T f^T)_k = y_k, \quad k = 1, \ldots, T.
$$

Due to Lemma 2.4, this control problem has a unique solution $f^T = (W^T)^{-1} \tilde{y}^T$. Let $x^T$ be a solution to

$$
\begin{aligned}
&\begin{cases}
  x^T_{t+1} + x^T_{t-1} = 0, & \quad t = 0, \ldots, T, \\
x^T_T = 0, x^T_{T-1} = 1.
\end{cases}
\end{aligned}
$$

It is important fact that the control $f^T$ can be found as a solution to the Krein equation:

**Theorem 3.1.** The control $f^T = (W^T)^{-1} \tilde{y}^T$ solving the special control problem (14), satisfies the following equation in $F^T$:

$$
C^T f^T = a_0 \left[ \beta x^T - \alpha (R^T)^* x^T \right].
$$

**Proof.** Let $f^T$ be a solution to (14). We observe that for any fixed $g \in F^T$:

$$
\begin{aligned}
u^g_{k, T} = \sum_{t=1}^{T-1} \left( u^g_{k, t+1} + u^g_{k, t-1} \right) x^T_t.
\end{aligned}
$$

Indeed, changing the order of a summation in the right hand side of (17) yields

$$
\begin{aligned}
\sum_{t=1}^{T-1} \left( u^g_{k, t+1} + u^g_{k, t-1} \right) x^T_t = \sum_{t=1}^{T-1} \left( x^T_{t+1} + x^T_{t-1} \right) u^g_{k, t} + u^g_{k, 0} x^T_T - u^g_{k, T} x^T_{T-1},
\end{aligned}
$$

which gives (17) due to (15). Using this observation, we can evaluate

$$
\begin{aligned}
(C^T f^T, g) &= \sum_{k=1}^{T} y_k v^g_{k, T} = \sum_{k=1}^{T} \sum_{t=0}^{T-1} \left( u^g_{k, t+1} + u^g_{k, t-1} \right) x^T_t y_k \\
&= \sum_{t=0}^{T-1} x^T_t \left( \sum_{k=1}^{T} \left( a_k u^g_{k+1, t} y_k + a_{k-1} u^g_{k-1, t} y_k + b_k u^g_{k, t} y_k \right) \right) \\
&= \sum_{t=0}^{T-1} x^T_t \left( \sum_{k=1}^{T} \left( u^g_{k, t} (a_k y_{k+1} + a_{k-1} y_{k-1} + b_k y_k) + a_0 u^g_{0, t} y_0 \\
+ a_T u^g_{T+1, t} y_T - a_0 u^g_{T, 0} y_T - a_T u^g_{T+1, T} y_{T+1} \right) \right) = \sum_{t=0}^{T-1} x^T_t \left( a_0 \beta g_t - a_0 \alpha (R^T g)_t \right),
\end{aligned}
$$

$$
= (x^T, a_0 \left[ \beta g - \alpha (R^T g) \right]) = (a_0 \left[ \beta x^T - \alpha (R^T)^* x^T \right], g).
$$

From where (16) follows. \hfill \square
Now we describe the procedure of the recovering $a_0, a_n, b_n, n = 1, \ldots, T - 1$ from the solutions of Krein (16) equations $f^r \in \mathcal{F}^r$ for $r = 1, \ldots, T$. From (5) and (6) we infer that

$$u_{T,T}^f = \prod_{k=0}^{T-1} a_k f_0^T,$$

$$u_{T-1,T}^f = \prod_{k=0}^{T-2} a_k f_1^T + \prod_{k=0}^{T-2} (b_1 + b_2 + \ldots + b_{T-1}).$$

Notice that we know $a_0 = r_0$. Let $T = 2$, then we have:

$$y_2 = u_{2,2}^f = a_0 a_1 f_0^2,$$

$$y_1 = u_{1,2}^f = a_0 f_1^2 + a_1 f_0^2,$$

In (18) we know $y_1 = \beta, a_0, f_1^2, f_0^2$, so we can recover $b_1$. On the other hand, using (13), we have a system

$$\begin{cases}
y_2 = a_0 a_1 f_0^2, \\
2a_1 y_2 + a_0 \alpha + b_1 \beta = 0
\end{cases}$$

Since $a_1 > 0$, we can recover $y_2$ and $a_1$. We proceed by the induction: assuming that we have already recovered $y_{k-1}, b_{k-2}, a_{k-2}$ for $k \leq n$, we recover $y_n, a_{n-1}, b_{n-1}$. Bearing in mind that

$$y_n = u_{n,n}^f = \prod_{k=0}^{n-2} a_k a_{n-1} f_0^2,$$

(19) $y_{n-1} = u_{n-1,n}^f = \prod_{k=0}^{n-2} a_k f_0^n + \prod_{k=0}^{n-2} a_k (b_1 + \ldots + b_{n-2} + b_{n-1}) f_0^n,$

and that we know $y_{n-1}, f_0^n, f_1^n$, and $a_k, b_k, k \leq n - 2$, we can recover $b_{n-1}$ from (20). Using (13) and (19) leads to the following relations

$$\begin{cases}
y_n = \prod_{k=0}^{n-2} a_k a_{n-1} f_0^2, \\
a_{n-1} y_n + a_{n-2} y_{n-2} + b_{n-1} y_{n-1} = 0,
\end{cases}$$

from which we can recover $a_{n-1}$ and $y_n$.

3.2. Factorization method. We make use of the fact that the matrix $C^T$ has a special structure: it is a product of a triangular matrix and its conjugate. We rewrite the operator $W^T$ as $W^T = \overline{W^T} J$ where

$$W^T f = \begin{pmatrix}
a_0 & w_{1,1} & w_{1,2} & \cdots & w_{1,T-1} \\
0 & a_0 a_1 & w_{2,2} & \cdots & w_{2,T-1} \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \frac{1}{a_j} & w_{k-1,1} & w_{k,T-1} \\
0 & \cdots & \cdots & 1 & 0 & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1 \\
0 & \cdots & \cdots & 0 & 0 & \cdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots \\
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix} \begin{pmatrix}
f_0 \\
f_2 \\
f_{T-k-1} \\
f_{T-1} \\
\end{pmatrix}$$

Using the definition (9) and the invertibility of $W^T$ (cf. Lemma 2.4) gives that

$$C^T = (W^T)^* W^T, \quad \text{or} \quad (W^T)^{-1} W^T = I.$$
The latter equation can be rewritten as
\[(21) \quad \left( (W^T)^{-1} \right)^* \mathcal{C}^T (W^T)^{-1} = I, \quad \mathcal{C}^T = JC^T J, \]
where the matrix \( \mathcal{C}^T \) is defined by
\[(22) \quad \mathcal{C}_{ij} = C_{T+1-j,T+1-i}, \quad \mathcal{C}^T = a_0 \begin{pmatrix} r_0 & r_1 & r_2 & \cdots & r_{T-1} \\ r_1 & r_0 + r_2 & r_1 + r_3 & \cdots & \vdots \\ r_2 & r_1 + r_3 & r_0 + r_2 + r_4 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & 0 \end{pmatrix}, \]
and \((W^T)^{-1}\) has a form
\[(23) \quad (W^T)^{-1} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,T} \\ 0 & a_{2,2} & a_{2,3} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & a_{T,T} \end{pmatrix}. \]

We multiply the \(k\)-th row of \(W^T\) by \(k\)-th column of \((W^T)^{-1}\) to get \(a_{k,k}a_0a_1 \cdots a_{k-1} = 1\), so diagonal elements of \((23)\) satisfy the relation:
\[(24) \quad a_{k,k} = \left( \prod_{j=0}^{k-1} a_j \right)^{-1}. \]

Multiplying the \(k\)-th row of \(W^T\) by \(k+1\)-th column of \((W^T)^{-1}\) leads to the relation
\[a_{k,k+1}a_0a_1 \cdots a_{k-1} + a_{k+1,k+1}w_{k,k} = 0, \]
from where we deduce that
\[(25) \quad a_{k,k+1} = -\left( \prod_{j=0}^{k} a_j \right)^{-2} a_kw_{k,k}. \]

All aforesaid leads to the equivalent form of \((21)\):
\[(26) \quad \begin{pmatrix} a_{1,1} & 0 & 0 \\ a_{1,2} & a_{2,2} & 0 \\ \vdots & \vdots & \vdots \\ a_{1,T} & \cdots & a_{T,T} \end{pmatrix} \begin{pmatrix} \xi_{1,1} & \cdots & \xi_{1,T} \\ \xi_{2,1} & \cdots & \xi_{2,T} \\ \vdots & \ddots & \vdots \\ \xi_{T,1} & \cdots & \xi_{T,T} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,T} \\ 0 & a_{2,2} & \cdots & a_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{T,T} \end{pmatrix} = I. \]

We note that in the case of a discrete Schrödinger operator, \((26)\) is equivalent to Gelfand-Levitan type equations (see [17]). In the above equality \(\xi_{ij}\) are given (see \((22)\)), the entries \(a_{ij}\) are unknown. A direct consequence of \((26)\) is an equality for determinants:
\[\det \left( (W^T)^{-1} \right)^* \det \mathcal{C}^T \det (W^T)^{-1} = 1, \]
which yields
\[a_{1,1} \ast \cdots \ast a_{T,T} = \left( \det \mathcal{C}^T \right)^{-\frac{1}{2}}. \]
From the above equality we derive that

\[ a_{1,1} = \left( \frac{\det C^1}{\det C} \right)^{-\frac{1}{2}}, \quad a_{2,2} = \left( \frac{\det C^2}{\det C} \right)^{-\frac{1}{2}}, \quad a_{k,k} = \left( \frac{\det C^k}{\det C^{k-1}} \right)^{-\frac{1}{2}}. \]

Combining latter relations with (24), we deduce that

\[ \prod_{i=0}^{k-1} a_i = \left( \frac{\det C^k}{\det C^{k-1}} \right)^{\frac{1}{2}}, \]

similarly, for \( k + 1 \):

\[ \prod_{i=0}^{k} a_i = \left( \frac{\det C^{k+1}}{\det C^k} \right)^{\frac{1}{2}}. \]

These two relations lead to

\[ a_k = \frac{\left( \frac{\det C^{k+1}}{\det C^k} \right)^{\frac{1}{2}} \left( \frac{\det C^{k-1}}{\det C^k} \right)^{\frac{1}{2}}}{\det C^k}, \quad k = 1, \ldots, T - 1. \] (27)

Here we set \( \det C^0 = 1, \det C^{-1} = 1 \).

Now using (26) we write down the equation on the last column of \( (\mathbf{W}^T)^{-1} \):

\[
\begin{pmatrix}
  a_{1,1} & 0 & \cdots & 0 \\
  a_{1,2} & a_{2,2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{1,T-1} & \cdots & a_{T-1,T-1} & 0
\end{pmatrix}
\begin{pmatrix}
  \bar{c}_{1,1} & \cdots & \cdots & \bar{c}_{1,T} \\
  \vdots & \cdots & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  \bar{c}_{T-1,1} & \cdots & \cdots & \bar{c}_{T-1,T-1}
\end{pmatrix}
\begin{pmatrix}
  a_{1,T} \\
  a_{2,T} \\
  \vdots \\
  a_{T,T}
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}.
\]

Note that we know \( a_{T,T} \), so we rewrite the above equality in the form of equation on \( (a_{1,T}, \ldots, a_{T-1,T})^T \):

\[
\begin{pmatrix}
  a_{1,1} & 0 & \cdots & 0 \\
  a_{1,2} & a_{2,2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{1,T-1} & \cdots & a_{T-1,T-1} & 0
\end{pmatrix}
\begin{pmatrix}
  \bar{c}_{1,1} & \cdots & \cdots & \bar{c}_{1,T} \\
  \vdots & \cdots & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  \bar{c}_{T-1,1} & \cdots & \cdots & \bar{c}_{T-1,T-1}
\end{pmatrix}
\begin{pmatrix}
  a_{1,T} \\
  a_{2,T} \\
  \vdots \\
  a_{T,T}
\end{pmatrix}
+ a_{T,T} \begin{pmatrix}
  a_{1,1} & 0 & \cdots & 0 \\
  a_{1,2} & a_{2,2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{1,T-1} & \cdots & a_{T-1,T-1} & 0
\end{pmatrix}
\begin{pmatrix}
  a_{1,T} \\
  a_{2,T} \\
  \vdots \\
  a_{T,T}
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}.
\]

which is equivalent to the equation

\[ (28) \]

\[
\begin{pmatrix}
  \bar{c}_{1,1} & \cdots & \cdots & \bar{c}_{1,T} \\
  \vdots & \cdots & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  \bar{c}_{T-1,1} & \cdots & \cdots & \bar{c}_{T-1,T-1}
\end{pmatrix}
\begin{pmatrix}
  a_{1,T} \\
  a_{2,T} \\
  \vdots \\
  a_{T,T}
\end{pmatrix}
= -a_{T,T} \begin{pmatrix}
  a_{1,T} \\
  a_{2,T} \\
  \vdots \\
  a_{T,T}
\end{pmatrix}.
\]

Introduce the notation:

\[
\frac{C^{k-1}}{C_k} := \begin{pmatrix}
  \bar{c}_{1,1} & \cdots & \cdots & \bar{c}_{1,k-2} & \bar{c}_{1,k} \\
  \vdots & \cdots & \cdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \bar{c}_{k-1,1} & \cdots & \cdots & \bar{c}_{k-1,k-2} & \bar{c}_{k-1,k}
\end{pmatrix}.
\]
that is \( C_k^{k-1} \) is constructed from the matrix \( C^{k-1} \) by substituting the last column by \((c_{1,k}, \ldots, c_{k-1,k})\). Then from (28) we deduce that:

\[
a_{T-1,T} = - a_T, \quad \frac{\det C_{T-1}^{T-1}}{\det C^{T-1}},
\]

where we assumed that \( \det C_0^{-1} = 0 \). On the other hand, from (24), (25) we see that

\[
a_{T-1,T} = \left( \prod_{j=0}^{T-1} a_j \right)^{-1} \sum_{k=1}^{T-1} b_k.
\]

Equating (29) and (30) gives that

\[
\sum_{k=1}^{T-1} b_k = - \frac{\det C_{T-1}}{\det C^{T-1}}, \quad \sum_{k=1}^{T} b_k = - \frac{\det C_{T+1}}{\det C^{T}}
\]

from where

\[
b_k = - \frac{\det C_{k+1}}{\det C^{k+1}} + \frac{\det C_k^{k-1}}{\det C^{k-1}}, \quad k = 1, \ldots, T - 1.
\]

### 3.3. Characterization of inverse data.

In the second section the forward problem for (1) was considered: for \((a_0, \ldots, a_{T-1}), (b_1, \ldots, b_{T-1})\) we constructed the matrix \( W^T \) (5), (6), the response vector \((r_0, r_1, \ldots, r_{2T-2})\) (see (7)) and the connecting operator \( \overline{C}^T \) defined in (10), (22). From Lemma 2.4 we know that \( C^T \) (and \( \overline{C}^T \)) is positively definite. We have also shown that if the vector \((r_0, r_1, \ldots, r_{2T-2})\) is a response vector corresponding to (1) with the coefficients \(a_0, \ldots, a_{T-1}, b_1, \ldots, b_{T-1}\), then one can recover those \(a'\)s and \(b'\)s by \(a_0 = r_0\) and formulas (27) and (31).

Now we set up the following question: can one determine whether a vector \((r_0, r_1, r_2, \ldots, r_{2T-2})\) is a response vector for dynamical system (1) with some coefficients \((a_0, \ldots, a_{T-1})\) \((b_1, \ldots, b_{T-1})\)? The answer is the following theorem.

**Theorem 3.2.** The vector \((r_0, r_1, r_2, \ldots, r_{2T-2})\) is a response vector for the dynamical system (1) if and only if the matrix \( C^T \) defined by (10) is positive definite.

**Proof.** Since the necessary part of the theorem is proved in the preceding sections we are left to prove the sufficiency part. We also observe that in the conditions of the theorem we can substitute \( C^T \) by \( \overline{C}^T \) (22).

Let a vector \((r_0, r_1, \ldots, r_{2T-2})\) be such that the matrix \( \overline{C}^T \) constructed from it with the help of (22) is positive definite. Then we can construct the sequences \((a_0, \ldots, a_{T-1}), (b_1, \ldots, b_{T-1})\) using \(a_0 = r_0\) and formulas (27) and (31) and consider dynamical system (1) with this coefficients. For this system we construct the response \((r_{0}^{\text{new}}, r_{1}^{\text{new}}, \ldots, r_{2T-2}^{\text{new}})\), the connecting operator \( C^T \) and corresponding \( \overline{C}^T \) using (10) and (22). We show that the response vector \((r_{0}^{\text{new}}, r_{1}^{\text{new}}, \ldots, r_{2T-2}^{\text{new}})\) coincides with the given one \((r_0, r_1, \ldots, r_{2T-2})\).

We have two matrices constructed by (22), one comes from the response vector \((r_0, r_1, \ldots, r_{2T-2})\) and the other comes from \((r_{0}^{\text{new}}, r_{1}^{\text{new}}, \ldots, r_{2T-2}^{\text{new}})\). Both of them are positive definite \( C^T > 0, \overline{C}^T > 0 \) (\( C^T \) by the conditions of a theorem, and \( \overline{C}^T \) since it comes from a connecting operator \( C^T \)). We note that if we calculate the elements of sequences \((a_1, \ldots, a_{T-1}), (b_1, \ldots, b_{T-1})\) using (27) and (31) from
any of $C^T$ and $\overline{C}^T$ matrices, we get the same answer. If so, we obtain that for $k = 1, \ldots, T - 1$ the following relations hold:

$$\frac{\det C_{k}^{k-1}}{\det C_{k}^{k-1}} - \frac{\det C_{k+1}^{k}}{\det C_{k}^{k-1}} = \frac{\det \overline{C}_{k}^{k-1}}{\det C_{k}^{k-1}} - \frac{\det \overline{C}_{k+1}^{k}}{\det \overline{C}_{k}^{k}},$$

$$\left(\frac{\det C_{k}^{k+1}}{\det C_{k}^{k}}\right)^{\frac{1}{2}} \left(\frac{\det \overline{C}_{k}^{k-1}}{\det \overline{C}_{k}^{k}}\right)^{\frac{1}{2}} = \left(\frac{\det \overline{C}_{k}^{k+1}}{\det \overline{C}_{k}^{k}}\right)^{\frac{1}{2}} \left(\frac{\det \overline{C}_{k}^{k-1}}{\det \overline{C}_{k}^{k}}\right)^{\frac{1}{2}},$$

$$\det C_{0}^{0} = \det \overline{C}_{0}^{0} = 1, \quad \det C_{-1}^{-1} = \det \overline{C}_{-1}^{-1} = 1, \quad \det C_{0}^{-1} = \det \overline{C}_{0}^{-1} = 0.$$

From these equalities we deduce that

$$\det C_{k}^{k} = \det \overline{C}_{k}^{k},$$

$$\det C_{k+1}^{k} = \det \overline{C}_{k+1}^{k}.$$

The above relations immediately yield that

$$r_{k} = r_{k}^{new}, \quad k = 1, \ldots, 2T - 2,$$

which finishes the proof.

3.4. **Discrete Schrödinger operator.** We consider the special case of dynamical system (1): we assume that Jacobi matrix corresponds to a discrete Schrödinger operator, i.e. $a_{k} = 1$, $k \in \mathbb{N} \cup \{0\}$, see also [17]. In this particular case the control operator (8) is given by $W^{T} = (I + K)J^{T}$, so all the diagonal elements of the matrix in (8) are equal to one. The latter immediately implies that $\det W^{T} = 1$. Due to this fact, the connecting operator (9), (10), has a remarkable property that $\det C_{l}^{l} = 1$, $l = 1, \ldots, T$. This implies that not all elements in the response vector are independent: $r_{2m}$ depends on $r_{2l+1}$, $l = 0, \ldots, m - 1$, moreover, this property characterize the dynamic data of discrete Schrödinger operators:

**Theorem 3.3.** The vector $(1, r_{1}, r_{2}, \ldots, r_{2T-2})$ is a response vector for the dynamical system (1) with $a_{k} = 1$ if and only if the matrix $C^{T}$ (10) is positive definite and $\det C_{l}^{l} = 1$, $l = 1, \ldots, T$.

**Proof.** As in Theorem 3.2 we use $\overline{C}^{T}$ instead of $C^{T}$. The necessity of the conditions was explained. We are left with the sufficiency part.

Note that $r_{0} = a_{0} = 1$. Let a vector $(1, r_{1}, \ldots, r_{2T-2})$ be such that the matrix $\overline{C}^{T}$ constructed from it using (22) satisfies conditions of the theorem. We calculate $b_{1}, \ldots, b_{T-1}$ using (31) and consider the dynamical system (1) with these $b_{k}$ and $a_{k} = 1$, $k = 0, \ldots, T - 1$. For this system we construct the response $(1, r_{1}^{new}, \ldots, r_{2T-2}^{new})$ and the connecting operator $\overline{C}^{T}$ using (7), (10) and (22). We will show that the response $(1, r_{1}^{new}, \ldots, r_{2T-2}^{new})$ coincide with the given one.

We note that if we calculate $b_{1}, \ldots, b_{T-1}$ using (31) with any of $\overline{C}^{T}$ or $\overline{C}^{T}$ matrices, we get the same answer. The latter implies (we count that $\det \overline{C}_{k}^{k} = \det \overline{C}_{k}^{k} = 1$)

$$\det \overline{C}_{k}^{k-1} - \det \overline{C}_{k+1}^{k} = \det \overline{C}_{k}^{k-1} - \det \overline{C}_{k+1}^{k},$$

$$\det \overline{C}_{0}^{0} = \det \overline{C}_{0}^{0} = 0.$$
By the induction arguments we obtain that
\[
\det C_{k+1}^k = \det C_{k+1}^k,
\]
\[
\det C^k = \det C^k = 1,
\]
which yields \( r_k = r_k^{new}, k = 1, \ldots, 2T - 2 \). That finishes the proof. \( \square \)

4. Spectral representation of \( C^T \) and \( r_t \). We fix \( N \in \mathbb{N} \) and \( h \in \mathbb{R} \). Along with (1) we consider the dynamical system with discrete time which is an analog of the wave equation with a potential on an interval: imposing a boundary condition at \( n = N + 1 \) we come to

\[
\begin{aligned}
\begin{cases}
v_{n,t+1} + v_{n,t-1} = a_n v_{n+1,t} - a_{n-1} v_{n-1,t} - b_n v_{n,t} = 0, & t \in \mathbb{N} \cup \{0\}, \ n \in 1, \ldots, N, \\
v_{n,-1} = v_{n,0} = 0, & n = 1, 2, \ldots, N + 1, \\
v_{0,t} = f_t, \ v_{N+1,t} + h v_{N,t} = 0, & t \in \mathbb{N}_0, 
\end{cases}
\end{aligned}
\]

where \( f = (f_0, f_1, \ldots) \) is a boundary control. The solution to (32) is denoted by \( v^f \).

Let \( \phi_n(\lambda) \) be a solution to
\[
\begin{aligned}
\begin{cases}
a_n \phi_{n+1} + a_{n-1} \phi_{n-1} + b_n \phi_n = \lambda \phi_n, \\
\phi_0 = 0, \ \phi_1 = 1.
\end{cases}
\end{aligned}
\]
Denote by \( \{\lambda_k\}^{N}_{k=1} \) the roots of the equation \( \phi_{N+1}(\lambda) + h \phi_N(\lambda) = 0 \), it is known [2, 21] that they are real and distinct. We introduce the vectors \( \phi^n \in \mathbb{R}^N \) by the rule \( \phi^n := \phi_i(\lambda_n), n, i = 1, \ldots, N, \) and define the numbers \( \rho_k \) by

\[
(\phi^k, \phi^l) = \delta_{kl} \frac{\rho_k}{a_0},
\]

where \( (\cdot, \cdot) \) is a scalar product in \( \mathbb{R}^N \).

**Definition 4.1.** The set of pairs

\[
\{\lambda_k, \rho_k\}^{N}_{k=1}
\]

is called spectral data of operator \( H^N \) (see (2)–(4)).

Taking arbitrary \( y = (y_1, \ldots, y_N) \in \mathbb{R}^N \), for each \( n \) we multiply the first equation in (32) by \( y_n \), sum up from \( n = 1 \) to \( n = N \) and evaluate, changing the order of summation:

\[
0 = \sum_{n=1}^{N} \left( v_{n,t+1} y_n + v_{n,t-1} y_n - a_n v_{n+1,t} y_n - a_{n-1} v_{n-1,t} y_n - b_n v_{n,t} y_n \right)
\]
\[
= \sum_{n=1}^{N} \left( v_{n,t+1} y_n + v_{n,t-1} y_n - v_{n,t} (a_n y_{n+1} - a_{n-1} y_{n-1} - b_n y_n) \right)
\]
\[
- a_N v_{N+1,t} y_N - a_0 v_0,t y_1 + a_0 v_1,t y_0 + a_N v_N,t y_{N+1}.
\]

Now we choose \( y = \phi^l, l = 1, \ldots, N \). On counting that \( \phi_0^l = 0, \ \phi_{N+1}^l = -h \phi_N^l, \ \phi_1^l = 1, \ \phi_0^l = f_t, \ v_{N+1,t} + h v_{N,t} = 0 \) we evaluate (34) arriving at

\[
\sum_{n=1}^{N} \left( v_{n,t+1} \phi_n^l + v_{n,t-1} \phi_n^l - v_{n,t} (a_n \phi_{n+1}^l + a_{n-1} \phi_{n-1}^l + b_n \phi_{n+1}^l) \right) - a_0 f_t = 0.
\]

Let us look for a solution to (32) in a form

\[
v^f_{n,t} = \begin{cases}
\sum_{k=1}^{N} c_k^k \phi^k_n, & n = 1, \ldots, N, \\
f_t, & n = 0.
\end{cases}
\]
Proposition 1. Coefficients $c^k$ admit the representation:

\[ (37) \quad c^k = \frac{a_0}{\rho_k} T(\lambda_k) * f, \]

where $T(2\lambda) = (T_1(2\lambda), T_2(2\lambda), T_3(2\lambda), \ldots)$ are Chebyshev polynomials of the second kind.

Proof. We plug (36) into (35) and evaluate, counting that

\[
\begin{align*}
\sum_{n=1}^{N} (v_{n,t+1} + v_{n,t-1} - \lambda_l v_{n,t}) \phi^l_n &= a_0 f_t, \\
\sum_{n=1}^{N} \sum_{k=1}^{N} (c^k_{t+1} \phi^k_n + c^k_{t-1} \phi^k_n - \lambda_l c^k_n \phi^l_n) \phi^l_n &= a_0 f_t.
\end{align*}
\]

Changing the order of summation and using (33) yields that $c^k_t$ for $k = 1, \ldots, N$ satisfy the following problem:

\[ (38) \quad \begin{cases} c^k_{t+1} + c^k_{t-1} - \lambda_k c^k_t = \frac{a_0}{\rho_k} f_t, \\
\end{cases} \]

We look for the solution to (38) in a form $c^k_t = \frac{a_0}{\rho_k} T^* f$:

\[ (39) \quad c^k_t = \frac{a_0}{\rho_k} \sum_{l=0}^{t} T_l f_{t-l}. \]

Plugging this expression into (38) yields:

\[
\frac{a_0}{\rho_k} \left( \sum_{l=0}^{t+1} f_l T_{t+1-l} + \sum_{l=0}^{t-1} f_l T_{t-1-l} - \lambda_k \sum_{l=0}^{t} f_l T_{t-l} \right) = \frac{a_0}{\rho_k} f_t,
\]

\[
\sum_{l=0}^{t} f_l (T_{t+1-l} + T_{t-1-l} - \lambda_k T_{t-l}) + f_t T_1 - f_{t-1} T_0 = f_t.
\]

The above equality implies that (39) holds in the case if $T$ solves

\[
\begin{cases} T_{t+1} + T_{t-1} - \lambda_k T_t = 0, \\
T_0 = 0, \quad T_1 = 1.
\end{cases}
\]

Thus $T_k(2\lambda)$ are Chebyshev polynomials of the second kind and the formula (37) is proved.

For the system (32) the control operator $W^F_{N,h} : \mathcal{F}^T \to \mathcal{H}^N$ is defined by the rule

\[ W^F_{N,h} f := v^f_{n,T}, \quad n = 1, \ldots, N. \]

The representation for this operator immediately follows from (36), (37). Due to the dependence of the solution on the coefficients, which was discussed in the third section (see Remark 1), we see that $v^f_{N,N}$ does not “feel” the boundary condition at $n = N$, that is why

\[ (40) \quad u^f_{n,t} = v^f_{n,t}, \quad n \leq t \leq N, \quad \text{and} \quad W^N = W^N_{N,h}. \]

For the system (32) the response operator $R^F_{N,h} : \mathcal{F}^T \to \mathbb{R}^T$ is introduced by the rule

\[ (41) \quad (R^F_{N,h} f)_t = v^f_{1,t}, \quad t = 1, \ldots, T. \]
Taking in (45) on the limit-circle case at infinity). Due to (42), we have that from (43), (44) we deduce that So on introducing the spectral function

\[ R_{2M} = R_{2M,h} \]

eq \{ (46) \}

implies that for \( M \in \mathbb{N} \),

\[ r_{t-1}^{N,h} = \{ R_{N,h}^{T} \} = \{ v_{\delta}^{t} \} \quad t = 1, \ldots, T. \]

on the other hand, we can use (36), (37) to obtain:

\[ v_{\delta}^{t} = \sum_{k=1}^{N} a_{0} T_{t}(\lambda_{k}). \]

So on introducing the spectral function

\[ \rho_{N,h}^{N}(\lambda) = \sum_{\{ k \mid \lambda_{k} < \lambda \}} \frac{a_{0}}{\rho_{k}}, \]

from (43), (44) we deduce that

\[ r_{t-1}^{N,h} = \int_{-\infty}^{\infty} T_{t}(\lambda) \ d\rho_{N,h}^{N}(\lambda), \quad t \in \mathbb{N}. \]

Due to (42), we have that

\[ r_{t-1} = r_{t-1}^{N,h} = \int_{-\infty}^{\infty} T_{t}(\lambda) \ d\rho_{N,h}^{N}(\lambda), \quad t \in 1, \ldots, 2N. \]

Taking in (45) \( N \) to infinity, we come to the spectral representation of the response vector:

\[ r_{t-1} = \int_{-\infty}^{\infty} T_{t}(\lambda) \ d\rho(\lambda), \quad t \in \mathbb{N}, \]

where \( d\rho \) is a spectral measure of the operator \( H \) (2), (3) (non-unique when \( H \) is in the limit-circle case at infinity).

For \( f, g \in F^{T} \) we can evaluate, using the expansion (36):

\[ (C_{N,h}^{T} f, g) = \sum_{n=1}^{N} v_{f}^{n,T} v_{g}^{n,T} = \sum_{n=1}^{N} \sum_{k=1}^{N} \frac{a_{0}}{\rho_{k}} T_{n}(\lambda_{k}) * f \varphi_{n}^{k} \sum_{l=1}^{N} \frac{a_{0}}{\rho_{l}} T_{l}(\lambda_{l}) * g \varphi_{l}^{l} \]

\[ = \sum_{k=1}^{N} \frac{a_{0}}{\rho_{k}} T_{n}(\lambda_{k}) * f T_{n}(\lambda_{k}) * g = \int_{-\infty}^{\infty} \sum_{l=0}^{T-1} \sum_{m=0}^{T-1} T_{T-l}(\lambda) f_{l} g_{m} T_{T-m}(\lambda) \ d\rho_{N,h}^{N}(\lambda) \]

from the above equality the spectral representation of \( C^{T} \) (cf. (10)) follows:

\[ \{ C_{N,h}^{T} \}_{l+1, m+1} = \int_{-\infty}^{\infty} T_{T-l}(\lambda) T_{T-m}(\lambda) \ d\rho_{N,h}^{N}(\lambda), \quad l, m = 0, \ldots, T - 1. \]

Taking into account (40), we obtain that \( C^{T} = C_{N,h}^{T} \) with \( N \geq T \), so (46) yields for \( N \geq T \):

\[ \{ C^{T} \}_{l+1, m+1} = \int_{-\infty}^{\infty} T_{T-l}(\lambda) T_{T-m}(\lambda) \ d\rho_{N,h}^{N}(\lambda), \quad l, m = 0, \ldots, T - 1, \]
and passing to the limit as \( N \to \infty \) yields

\[
\{ C^T \}_{l+1, m+1} = \int_{-\infty}^{\infty} T_{T-l} \left( \lambda \right) T_{T-m} \left( \lambda \right) d\rho \left( \lambda \right), \quad l, m = 0, \ldots, T - 1,
\]

where \( d\rho \) is a spectral measure of \( H \).

It is known \cite{13} that any probability measure on \( \mathbb{R} \) with finite moments give rise to the Jacobi operator, i.e. is a spectral measure of this operator. In \cite{13} the authors posed the question on the characterization of a spectral measure for semi-infinite discrete Schrödinger operator. The following theorem gives an answer to this question.

**Theorem 4.2.** The measure \( d\rho \) is a spectral measure of a semi-infinite discrete Schrödinger operator if and only if for every \( T \) the matrix \( C^T \) is positive definite and \( \det C^T = 1 \), where

\[
(47) \quad C^T_{i,j} = \int_{-\infty}^{\infty} T_{T-i+1} \left( \lambda \right) T_{T-j+1} \left( \lambda \right) d\rho \left( \lambda \right), \quad i, j = 1, \ldots, T.
\]

**Proof.** We consider the system (1) with \( a_k = 1 \). Let \( d\rho \) be a spectral measure of \( H \) (note that this measure is unique since \( H \) is in limit point at infinity case). For every \( T \in \mathbb{N} \) we construct the connecting operator \( C^T \) (see (9)) using the representation (47). According to Theorem 3.3, such \( C^T \) is positive definite and \( \det C^T = 1 \).

On the other hand, if given measure \( d\rho \) satisfies conditions of the theorem, for every \( T \) we can construct \( C^T \) by (47) and by Theorem 3.3 recover coefficients \( b_n \) using (31).

**Acknowledgments.** The research of Victor Mikhaylov was supported by RFBR 17-01-00529 and by the Ministry of Education and Science of Republic of Kazakhstan under grant AP05136197. Alexandr Mikhaylov was supported by RFBR 17-01-00099; A. S. Mikhaylov and V. S. Mikhaylov were partly supported by and by VW Foundation program “Modeling, Analysis, and Approximation Theory toward application in tomography and inverse problems.”

**REFERENCES**

[1] S. Avdonin and A. Bulanova, Boundary control approach to the spectral estimation problem. The case of multiple poles, *Math. Contr. Sign. Syst.*, 22 (2011), 245–265.

[2] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Acad. Press, 1964.

[3] S. Avdonin, A. Bulanova and D. Nibolsky, Boundary control approach to the spectral estimation problem. The case of simple poles, *Sampling Theory in Signal and Image Processing*, 8 (2009), 225–248.

[4] S. Avdonin and V. S. Mikhaylov, The boundary control approach to inverse spectral theory, *Inverse Problems*, 26 (2010), 045009, 19 pp.

[5] S. Avdonin, V. Mikhaylov and A. Rybkin, The boundary control approach to the Titchmarsh-Weyl m–function, *Comm. Math. Phys.*, 275 (2007), 791–803.

[6] M. Belishev, Recent progress in the boundary control method, *Inverse Problems*, 23 (2007), R1–R67.

[7] M. Belishev, Boundary control and tomography of Riemannian manifolds (the BC-method), *Uspekhi Matem. Nauk.*, 72 (2017), 3–66, (in Russian).

[8] M. Belishev, Boundary control and inverse problems: A one-dimensional version of the boundary control method, *J. Math. Sci. (N.Y.*)*, 155 (2008), 343–378.

[9] M. Belishev and S. Ivanov, Characterization of data of dynamical inverse problem for two-velocity system, *J. Math. Sci. (N.Y.*)*, 109 (2002), 1814–1834.

[10] M. Belishev and V. Mikhaylov, Unified approach to classical equations of inverse problem theory, *Journal of Inverse and Ill-Posed Problems*, 20 (2012), 461–488.
[11] M. Belishev and V. Mikhaylov, Inverse problem for one-dimensional dynamical Dirac system (BC-method), *Inverse Problems*, 30 (2014), 125013, 26 pp.

[12] M. Belishev and A. Pestov, Characterization of the inverse problem data for one-dimensional two-velocity dynamical system, *St. Petersburg Mathematical Journal*, 26 (2015), 411–440.

[13] F. Gesztesy and B. Simon, \(m\)-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices, *J. d’Analyse Math.*, 73 (1997), 267–297.

[14] V. Isakov, *Inverse Problems for Partial Differential Equations*, Appl. Math. Studies, Springer, v. 127, 1998.

[15] S. I. Kabanikhin, *Inverse and Ill-Posed Problems. Theory and Applications*, Inverse and Ill-posed Problems Series, 55. Walter de Gruyter GmbH & Co. KG, Berlin, 2012.

[16] A. Kachalov, Y. Kurylev and M. Lassas, *Inverse Boundary Spectral Problems*, Chapman&Hall, 2001.

[17] A. Mikhaylov and V. Mikhaylov, Dynamical inverse problem for the discrete Schrödinger operator, *Nanosystems: Physics, Chemistry, Mathematics*, 7 (2016), 842–854.

[18] A. Mikhaylov and V. Mikhaylov, Boundary Control method and de Branges spaces. Schrödinger operator, Dirac system, discrete Schrödinger operator, *Journal of Mathematical Analysis and Applications*, 460 (2018), 927–953.

[19] A. Mikhaylov and V. Mikhaylov, Relationship between different types of inverse data for the one-dimensional Schrödinger operator on a half-line, *J. Math. Sci. (N.Y.)*, 226 (2017), 779–794.

[20] A. Mikhaylov, V. Mikhaylov and S. Simonov, On the relationship between Weyl functions of Jacobi matrices and response vectors for special dynamical systems with discrete time, *Mathematical Methods in the Applied Sciences*, 41 (2018), 6401–6408.

[21] B. Simon, The classical moment problem as a self-adjoint finite difference operator, *Advances in Math.*, 137 (1998), 82–203.

Received October 2017; revised October 2018.

E-mail address: mikhaylov@pdmi.ras.ru
E-mail address: ftvsm78@gmail.com