Automorphism Groups of Comparability Graphs

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Abstract. Comparability graphs are graphs which have transitive orientations. The dimension of a poset is the least number of linear orders whose intersection gives this poset. The dimension \( \text{dim}(X) \) of a comparability graph \( X \) is the dimension of any transitive orientation of \( X \), and by \( k\text{-DIM} \) we denote the class of comparability graphs \( X \) with \( \text{dim}(X) \leq k \).

It is known that the complements of comparability graphs are exactly function graphs and permutation graphs equal \( 2\text{-DIM} \).

In this paper, we characterize the automorphism groups of permutation graphs similarly to Jordan’s characterization for trees (1869). For permutation graphs, there is an extra operation, so there are some extra groups not realized by trees. For \( k \geq 4 \), we show that every finite group can be realized as the automorphism group of some graph in \( k\text{-DIM} \), and testing graph isomorphism for \( k\text{-DIM} \) is \GI-complete.

1 Introduction

Comparability Graphs. A comparability graph is created from a poset by removing the orientation of the edges. Alternatively, every comparability graph \( X \) can be transitively oriented: if \( x \to y \) and \( y \to z \), then \( xz \in E(X) \) and \( x \to z \); see Fig. 1a. This class was first studied by Gallai [10] and we denote it by COMP.

An important parameter of a poset \( P \) is its Dushnik-Miller dimension [5]. It is the least number of linear orderings \( L_1, \ldots, L_k \) such that \( P = L_1 \cap \cdots \cap L_k \). (For a finite poset \( P \), its dimension is always finite since \( P \) is the intersection of all its linear extensions.) Similarly, we define the dimension of a comparability graph \( X \), denoted by \( \text{dim}(X) \), as the dimension of any transitive orientation of \( X \). (It can be shown that every transitive orientation has the same dimension.) By \( k\text{-DIM} \), we denote the subclass consisting of all comparability graphs \( X \) with \( \text{dim}(X) \leq k \). We get the following infinite hierarchy of graph classes:

\[
1\text{-DIM} \subsetneq 2\text{-DIM} \subsetneq 3\text{-DIM} \subsetneq 4\text{-DIM} \subsetneq \cdots \subsetneq \text{COMP}.
\]

Function and Permutation Graphs. An intersection representation of a graph \( X \) is a collection of sets \( \{R_u : u \in V(X)\} \) such that \( R_u \cap R_v \neq \emptyset \) if and only if \( uv \in E(X) \); i.e., it encodes the vertices by sets and the edges by intersections of these sets. To get nice graph classes, one typically restricts the sets \( R_u \) to particular classes of geometrical objects.

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We study the class of function graphs (FUN) which are intersection graphs of continuous functions $R_u : [0, 1] \rightarrow \mathbb{R}$ and its subclass permutation graphs (PERM) which can be represented by linear functions [2]; see Fig. 2.

Surprisingly, these classes are related to comparability graphs. Golombic [11] proved that function graphs are the complements of comparability graphs: $\text{FUN} = \text{co-COMP}$. If two functions do not intersect, we can orient the non-edge from the bottom function to the top one which gives a transitive orientation of the complement. On the other hand, a comparability graph has some dimension $k$, so one of its transitive orientations can be written as $L_1 \cap \cdots \cap L_k$. We place the vertices in these orderings on $k$ vertical lines between $[0, 1]$. Then we represent each vertex by the polyline function which connects this vertex in each of the $k$ vertical lines; see Fig. 1b. We get a function representation of the complement. The second relation $\text{PERM} = \text{COMP} \cap \text{co-COMP} = 2\text{-DIM}$ was shown by Even [7].

**Automorphism Groups of Graphs.** The automorphism group $\text{Aut}(X)$ of a graph $X$ describes its symmetries. Every automorphism is a permutation of the vertices which preserves adjacencies and non-adjacencies. Frucht [9] proved that every finite group is isomorphic to $\text{Aut}(X)$ of some graph $X$. Most graphs are asymmetric, i.e., have only the trivial automorphism [6]. However, many combinatorial and graph theory results rely on highly symmetrical graphs.

**Definition 1.1.** For a class $C$ of graphs, let $\text{Aut}(C)$ be the set $\{\text{Aut}(X) : X \in C\}$ of abstract groups. The class $C$ is called universal if every abstract finite group is contained in $\text{Aut}(C)$, and non-universal otherwise.

In 1869, Jordan [14] characterized the automorphism groups of trees (TREE). The automorphism groups of planar graphs were characterized by Babai [1]; see also [8]. Several results for the automorphism groups of intersection-defined classes of graphs were shown recently by Klavík and Zeman [15]: the automorphism groups of interval graphs (INT) are the same as of trees, the automorphism

**Fig. 2.** (a) A function graph which is not a permutation graph and one of its representations. (b) A permutation graph and one of its representations.
groups of unit interval graphs are the same as of disjoint unions of caterpillars and the automorphism groups of circle graphs are the same as of pseudoforests; see [15] for definitions of these classes. Most superclasses are already universal, e.g., chordal graphs, function graphs, claw-free graphs.

**Graph Isomorphism Problem.** This famous problem asks whether two input graphs \( X \) and \( Y \) are the same up to a relabeling. It obviously belongs to \( \text{NP} \), and it is not known to be polynomially-solvable or \( \text{NP} \)-complete. This is a prime candidate for an intermediate problem with the complexity between \( \text{P} \) and \( \text{NP} \)-complete. It belongs to the low hierarchy of \( \text{NP} \) [18], which implies that it is unlikely \( \text{NP} \)-complete. (Unless the polynomial-time hierarchy collapses to its second level.) It is closely related to computing generators of an automorphism group: \( X \) and \( Y \) are isomorphic if and only if there exists an automorphism swapping them in \( X \cup Y \), and generators of \( \text{Aut}(X) \) can be computed using \( \mathcal{O}(n^4) \) instances of graph isomorphism [17]. By \( \text{GI} \), we denote the complexity class of all problems that can be reduced to graph isomorphism in polynomial time.

For many graph classes, the graph isomorphism problem was shown to be polynomial-time solvable. For classes like interval graphs and planar graphs [4], circle graphs [13] and permutation graphs [6], using known structural results their isomorphism can be reduced to isomorphism of trees. As evidenced by [11,15] and this paper, their automorphism groups also have nice structures. When a class of graphs has very restrictive automorphism groups, it seems that graph isomorphism problem should be relatively easy to solve. Actually, the complexity of graph isomorphism testing of asymmetric graphs is unknown. There are also very complicated polynomial-time algorithms solving graph isomorphism for universal graph classes: graphs of bounded degree [10] and with excluded topological subgraphs [12].

There are graph classes for which testing graph isomorphism is \( \text{GI} \)-complete. For instance, it is \( \text{GI} \)-complete for bipartite graphs: For a graph \( X \), we subdivide its edges which makes it bipartite, alternatively it is the incidence-graph of \( V(X) \) and \( E(X) \). Notice that \( X \cong Y \) if and only if their subdivisions are. Similar constructions are known for chordal graphs [4] and other graph classes. We are not aware of any \( \text{GI} \)-completeness results for classes with very restricted automorphism groups. When the graph isomorphism problem is \( \text{GI} \)-complete, it seems that its automorphism groups have to be rich enough to encode most graphs (not necessary universal).

**Our Results.** Since 1-DIM consists of all complete graphs, \( \text{Aut}(1\text{-DIM}) = \{S_n\} \). Concerning 2-DIM = PERM, it was observed in [15] that it is non-universal since its superclass circle graphs is non-universal. In this paper, we answer the question of [15] and describe their automorphism groups precisely:

**Theorem 1.2.** The class \( \text{Aut}(\text{PERM}) \) is the class of groups closed under

(a) \( \{1\} \in \text{Aut}(\text{PERM}) \),

(b) If \( G_1, G_2 \in \text{Aut}(\text{PERM}) \), then \( G_1 \times G_2 \in \text{Aut}(\text{PERM}) \).

(c) If \( G \in \text{Aut}(\text{PERM}) \), then \( G \wr S_n \in \text{Aut}(\text{PERM}) \).
(d) If \(G_1, G_2, G_3 \in \text{Aut}(\text{PERM})\), then \((G_1 \times G_2 \times G_3) \rtimes \mathbb{Z}_2 \in \text{Aut}(\text{PERM})\).

In (d), \(\mathbb{Z}_2\) acts on \(G_1\) as on the vertices of a rectangle, on \(G_2\) as on centers of two opposite edges, and on \(G_3\) as on centers of the other two opposite edges. Our characterization is similar to Jordan’s characterization [14] of the automorphism groups of trees which consists of (a) to (c). Therefore, \(\text{Aut}(\text{TREE}) \subseteq \text{Aut}(\text{PERM})\).

Inspired by the technique described in [15], we study the induced action of \(\text{Aut}(X)\) on the set of all transitive orientations. In the case of permutation graphs, we study the action on pairs of orientations of the graph and its complement, and show that it is semiregular. They are efficiently captured by the modular decomposition which we encode into the modular tree.

We are not aware of any algorithmic result for computing automorphism groups of permutation graphs. From our description, a polynomial-time algorithm can be constructed. Further, it can give \(\text{Aut}(X)\) in terms of group products of Theorem 1.2 which gives more insight into the structure of \(\text{Aut}(X)\).

Comparability graphs are universal since they contain bipartite graphs; we can orient all edges from one part to the other. Since the automorphism group is preserved by complementation and \(\text{FUN} = \text{co-COMP}\), we have \(\text{Aut}(<\text{FUN}) = \text{Aut}(<\text{COMP})\) and function graphs are also universal. We explain this in more detail using the induced action on all transitive orientations.

It is well-known that general bipartite graphs have arbitrary large dimensions: the crown graph, which is \(K_{n,n}\) without a matching, has dimension \(n\). We give a different construction which encodes any graph \(X\) into a comparability graph \(Y\) with \(\dim(Y) \leq 4\).

**Theorem 1.3.** For every \(k \geq 4\), the class \(k\text{-DIM}\) is universal and its graph isomorphism problem is \(\text{GI}\)-complete.

Yannakakis [19] proved that recognizing 3-DIM is \(\text{NP}\)-complete by a reduction from 3-coloring. For each graph \(X\), a comparability graph \(Y\) with several vertices representing each element of \(V(X) \cup E(X)\) is constructed. It is shown that \(\dim(Y) = 3\) if and only if \(X\) is 3-colorable. Unfortunately, the automorphisms of \(X\) are lost in \(Y\) since it depends on the labels of \(V(X)\) and \(E(X)\) and \(Y\) contains some additional edges according to these labels. We describe a simple completely different construction which achieves only dimension 4, but preserves the automorphism group: for a given graph \(X\), we create \(Y\) by replacing each edge with a path of length eight. However, it is non-trivial to show that \(Y \in 4\text{-DIM}\), and the constructed four linear orderings are inspired by [19].

**Outline.** In Section 2 we describe the modular decomposition and modular trees. In Section 3 we discuss the action of \(\text{Aut}(X)\) on the set of all transitive orientations of a comparability graph \(X\). In Section 4 we describe automorphism groups of permutation graphs. In Section 5 we encode arbitrary graphs into four-dimensional comparability graphs. We conclude this paper with open problems.

**Definitions.** We use \(X\) and \(Y\) for graphs, \(M\) for modules, \(T\) for modular trees and \(G, H\) for groups. The vertices and edges of \(X\) are \(V(X)\) and \(E(X)\). The complement of \(X\) is \(\overline{X}\). A permutation \(\pi\) of \(V(X)\) is an automorphism if \(uv \in E(X) \iff \pi(u)\pi(v) \in E(X)\). \(S_n\) and \(\mathbb{Z}_n\) are the symmetric and cyclic groups.
Given two groups $N$ and $H$, and a group homomorphism $\varphi : H \to \text{Aut}(N)$, we can construct a new group $N \rtimes_{\varphi} H$ as the Cartesian product $N \times H$ with the operation defined as $(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot \varphi(h_1)(n_2), h_1 \cdot h_2)$. The group $N \rtimes_{\varphi} H$ is called the semidirect product of $N$ and $H$ with respect to the homomorphism $\varphi$. The wreath product $G \wr S_n$ is a shorthand for $G^n \ltimes S_n$ where $\psi$ is defined naturally by $\psi(\pi) = (g_1, \ldots, g_n) \mapsto (g_{\pi(1)}, \ldots, g_{\pi(n)})$.

2 Modular Decomposition

In this section, we introduce the modular decomposition of a graph $X$ and show that it can be encoded by a modular tree. We further show that the automorphism group of this modular tree is isomorphic to $\text{Aut}(X)$.

Modules. A module $M$ of a graph $X$ is a set of vertices such that each $x \in V(X) \setminus M$ is either adjacent to all vertices in $M$, or to none of them. Modules generalize connected components, but unlike connected components, one module can be a proper subset of another one. Therefore, modules lead to a recursive decomposition of a graph, instead of just a partition. See Fig. 3a for examples. A module $M$ is called trivial if $M = V(X)$ or $|M| = 1$, and non-trivial otherwise.

If $M$ and $M'$ are two disjoint modules, then either the edges between $M$ and $M'$ form the complete bipartite graph, or there are no edges at all; see Fig. 3a. In the former case, $M$ and $M'$ are called adjacent, otherwise they are non-adjacent.

Quotient Graphs. Let $\mathcal{P} = \{M_1, \ldots, M_k\}$ be a modular partition of $V(X)$, i.e., each $M_i$ is a module of $X$, $M_i \cap M_j = \emptyset$ for every $i \neq j$, and $M_1 \cup \cdots \cup M_k = V(X)$. We define the quotient graph $X/\mathcal{P}$ with the vertices $m_1, \ldots, m_k$ (which correspond to the modules $M_1, \ldots, M_k$) where $m_i m_j \in E(X/\mathcal{P})$ if and only if $M_i$ and $M_j$ are adjacent. In other words, the quotient graph is obtained by contracting each module $M_i$ into a single vertex $m_i$; see Fig. 3b.

Modular Decomposition. We decompose a graph $X$ by finding some modular partition $\mathcal{P} = \{M_1, \ldots, M_k\}$, computing $X/\mathcal{P}$ and recursively decomposing $X/\mathcal{P}$ and each $X[M_i]$. The recursive process stops on prime graphs which are graphs containing only trivial modules. There might be many such decompositions, depending on the choice of $\mathcal{P}$ in each step. In 1960s, Gallai [10] described the modular decomposition in which some special modular partitions are chosen. This modular decomposition encodes all possible decompositions.

The key is the following observation. Let $M$ be a module of $X$ and let $M' \subseteq M$. Then $M'$ is a module of $X$ if and only if it is a module of $X[M]$. We construct the modular decomposition $\mathfrak{MD}$ of a graph $X$ in the following way:
- A graph $X$ is called degenerate if it is $K_n$ or $\overline{K}_n$. If $X$ is a prime or a degenerate graph, then we add $X$ to $\mathfrak{MD}$ and stop. We stop on degenerate graphs.

Fig. 3. (a) A graph $X$ with a modular partition $\mathcal{P}$ formed by its inclusion maximal non-trivial modules. (b) The quotient graph $X/\mathcal{P}$ is prime.
graphs to make the modular decomposition unique; there are many modular partitions for them but they are not very interesting.

Let $X$ and $\overline{X}$ be connected graphs. Gallai [10] shows that the inclusion maximal non-trivial modules of $X$ form a modular partition $P$ of $V(X)$, and the quotient graph $X/P$ is a prime graph; see Fig. 3. We add $X/P$ to $\mathfrak{MD}$ and recursively decompose $X[M]$ for each $M \in P$.

If $X$ is disconnected and $\overline{X}$ is connected, then every union of several connected components is a module. All other modules are subsets of a single connected component. Therefore the connected components form a modular partition $P$ of $V(X)$, and the quotient graph $X/P$ is an independent set. We add $X/P$ to $\mathfrak{MD}$ and recursively decompose $X[M]$ for each $M \in P$.

If $\overline{X}$ is disconnected and $X$ is connected, then the modular decomposition is defined in the same way on the connected components of $\overline{X}$. They form a modular partition $P$ and the quotient graph $X/P$ is a complete graph. We add $X/P$ to $\mathfrak{MD}$ and recursively decompose $X[M]$ for each $M \in P$.

Gallai [10] shows that the modular decomposition of a graph is unique. It is easy to see that it captures all modules of $X$.

**Modular Tree.** Let $\mathfrak{MD}$ be the modular decomposition of $X$. We encode it by the modular tree $T$ which is a graph with two types of vertices (normal and marker vertices) and two types of edges (normal and directed tree edges). The tree edges connect the prime and degenerate graphs obtained in $\mathfrak{MD}$ into a tree. Further every modular tree has an induced subgraph called root node.

If $X$ is a prime or a degenerate graph, we define $T = X$ and its root node is equal $T$. Otherwise, let $P = \{M_1, \ldots, M_k\}$ be the modular partition of $X$ used in $\mathfrak{MD}$ and let $T_1, \ldots, T_k$ be the corresponding modular trees for $X[M_1], \ldots, X[M_k]$ according $\mathfrak{MD}$. The modular tree $T$ is constructed by taking disjoint union of $T_1, \ldots, T_k$ and the quotient $X/P$ with the marker vertices $m_1, \ldots, m_k$. To every graph $T_i$, we add a new marker vertex $m'_i$ such that $m'_i$ is adjacent exactly to the vertices of the root node of $T_i$. We further add a tree edge from $m_i$ to $m'_i$.

For an example, see Fig. 4.

Since the modular decomposition of $X$ is unique, also the modular tree of $X$ is unique. The graphs obtained in $\mathfrak{MD}$ are called nodes of $T$, or alternatively root nodes of some modular tree in the construction of $T$. For a node $N$, its subtree is the modular tree which has $N$ as the root node. Every node either has all vertices as marker vertices, or contains no marker vertices. In the former case, it is called an inner node, otherwise a leaf node.
An automorphism of the modular tree $T$ has to preserve the types of vertices and edges. We denote the automorphism group of $T$ by $\text{Aut}(T)$. For the proof of the following lemma see Appendix A.

**Lemma 2.1.** If $T$ is the modular tree representing a graph $X$, then

$$\text{Aut}(X) \cong \text{Aut}(T).$$

**Recursive Construction.** We can build $\text{Aut}(T)$ from simple groups recursively, similarly to Jordan [14]. Suppose that we know automorphism groups $\text{Aut}(T_1), \ldots, \text{Aut}(T_k)$ of all children $T_1, \ldots, T_k$ of $T$. Let $R$ be the root node of $T$. We further color the marker vertices in $R$ by the colors coding isomorphism classes of the subtrees $T_1, \ldots, T_k$, and let $\text{Aut}(R)$ be the color preserving automorphism group of $R$. Then we get:

**Lemma 2.2.** We have

$$\text{Aut}(T) \cong (\text{Aut}(T_1) \times \cdots \times \text{Aut}(T_k)) \rtimes \text{Aut}(R).$$

*Proof (Sketch).* We isomorphically label the vertices of isomorphic subtrees $T_i$. Each automorphism $\pi \in \text{Aut}(T)$ is a composition of two automorphisms $\sigma \cdot \tau$ where $\sigma$ maps each subtree $T_i$ to itself, and $\tau$ permutes the subtrees as in $\pi$ while preserving the labeling. Therefore, the automorphisms $\sigma$ can be bijectively identified with the elements of the direct product $\text{Aut}(T_1) \times \cdots \times \text{Aut}(T_k)$ and the automorphisms $\tau$ with some element of $\text{Aut}(R)$. The rest of the proof follows from a standard argument from permutation group theory; see Appendix A.  

With no further assumptions on $X$, if $R$ is a prime graph, then $\text{Aut}(R)$ can be isomorphic to an arbitrary group. If $R$ is a degenerate graph, then $\text{Aut}(R)$ is a direct product of symmetric groups.

We note that this procedure does not lead to a polynomial-time algorithm for computing $\text{Aut}(T)$. The reason is that the automorphism groups of prime graphs can be very complicated. To color the marker vertices, we have to be able to solve graph isomorphism of subtrees $T_i$, and then we have to find the subgroup of $\text{Aut}(R)$ which preserves the colors.

### 3 Automorphism Groups of Comparability Graphs

In this section, we give a structural understanding of the automorphism groups of comparability graphs, in terms of actions on sets of transitive orientations.

**Structure of Transitive Orientations.** Let $\to$ be a transitive orientation of $X$ and let $T$ be the modular tree representing $X$. For modules $M_1$ and $M_2$, we write $M_1 \to M_2$ if $x_1 \to x_2$ for all $x_1 \in M_1$ and $x_2 \in M_2$. Gallai [10] shows:

- If two modules $M_1$ and $M_2$ are adjacent, then either $M_1 \to M_2$, or $M_2 \leftarrow M_1$.
- The graph $X$ is a comparability graph if and only if each node of $T$ is a comparability graph.
- Every prime comparability graph has exactly two transitive orientations, one being the reversal of the other.
Fig. 5. Two automorphism reflect $X$ and change the transitive orientation. On the right, their action on the modular tree $T$.

The modular tree $T$ encodes all transitive orientations as follows. For each prime node of $T$, we choose one of the two possible orientations. For each degenerate node, we choose some orientation. (If it is a complete graph $K_n$, it has $n!$ possible orientations, if it is an independent graph $K_n$, it has the unique orientation). A transitive orientation of $X$ is then constructed as follows. We orient the vertices of leaf nodes as above. For every subtree with children $M_1, \ldots, M_k$, we orient $X[M_i] \to X[M_j]$ if and only if $m_i \to m_j$ in the root node. It is easy to check that this gives a valid transitive orientation, and every transitive orientation can be constructed in this way.

The Induced Action. Let $\sigma(X)$ be the set of all transitive orientations of $X$. Let $\pi \in \text{Aut}(X)$ and $O \in \sigma(X)$. We define the orientation $\pi(O)$:

$$xOy \iff \pi(x)\pi(O)\pi(y), \quad \forall x, y \in V(X).$$

We can observe that $\pi(O)$ is a transitive orientation of $X$, so $\pi(O) \in \sigma(X)$; see Fig. 5. Therefore $\text{Aut}(X)$ defines an action on $\sigma(X)$.

Let $S$ be the stabilizer of some orientation $O$. It consists of all automorphisms which preserve this orientation, so they permute only the vertices that are incomparable in $O$. In other words, $S$ is the automorphism group of the poset created by the transitive orientation $O$ of $X$. We want to understand it in terms of $\text{Aut}(T)$ for the modular tree $T$ representing $X$. Each automorphism $\text{Aut}(T)$ somehow acts inside each node, and somehow permutes the nodes, as characterized in Lemma 2.2.

Consider some subtree of $T$ with the children $T_1, \ldots, T_k$. Suppose that $\sigma \in S$ maps $T_i$ to $\sigma(T_i) = T_j$. Then the marker vertices $m_i$ and $m_j$ have to be incomparable in the root node of this subtree. If the root node is an independent set, the isomorphic subtree can be arbitrarily permuted in $S$. If it is a complete graph, all subtrees are preserved in $S$. If it is a prime graph, then isomorphic subtrees of incomparable marker vertices can be permuted.

4 Automorphism Groups of Permutation Graphs

In this section, we prove the characterization of the automorphism groups of permutation graphs of Theorem 1.2.

The Induced Action. Let $X$ be a permutation graph. The main difference is that both $X$ and $\overline{X}$ are comparability graphs. By the results of Section 3, we know that $\text{Aut}(X)$ induces an action on both $\sigma(X)$ and $\sigma(\overline{X})$. We work with these two actions as with one action on the pair $(\sigma(X), \sigma(\overline{X}))$, in other words on pairs $(O, \overline{O})$ such that $O \in \sigma(X)$ and $\overline{O} \in \sigma(\overline{X})$. Figure 6 shows an example.
Fig. 6. The action of $\text{Aut}(X)$ on four pairs of transitive orientations $X$. The black generator flips the orientation of $X$, the gray automorphism of both $X$ and $\overline{X}$.

An action is called **semiregular** if only the identity has a fixed point. In other words, all stabilizers of a semiregular action are trivial.

**Lemma 4.1.** The action of $\text{Aut}(X)$ on $(\sigma(X), \sigma(\overline{X}))$ is semiregular.

**Proof.** We know that a permutation belonging to a stabilizer can only permute incomparable elements. Since incomparable elements in $O$ are exactly the comparable elements in $\overline{O}$, the stabilizer is trivial. \boxed{}

**Lemma 4.2.** For a prime permutation graph $X$, $\text{Aut}(X)$ is a subgroup of $\mathbb{Z}_2^2$.

**Proof.** There are at most four pairs of orientations in $(\sigma(X), \sigma(\overline{X}))$, so by Lemma 4.1 the order of $\text{Aut}(X)$ is at most four. If $\pi \in \text{Aut}(X)$, then $\pi^2$ fixes the orientation of both $X$ and $\overline{X}$. Therefore $\pi^2$ is an identity, $\pi$ an involution and $\text{Aut}(X)$ is a subgroup of $\mathbb{Z}_2^2$. \boxed{}

Now, we are ready to characterize $\text{Aut}(\text{PERM})$:

**Proof (Theorem 1.2, sketch).** To show that $\text{Aut}(\text{PERM})$ is closed under (b) to (d), we use construction as in Fig. 7; see Appendix B.

We apply the recursive procedure of Lemma 2.2. We build $\text{Aut}(T)$ recursively from the leaves to the root of $T$. If the root node $R$ is degenerate, then we can arbitrarily permute the isomorphic subtrees. Therefore, $\text{Aut}(T)$ can be constructed using (b) and (c). If the root node $R$ is a prime graph, we know that $\text{Aut}(R)$ is by Lemma 4.2 a subgroup of $\mathbb{Z}_2^2$. Then $\text{Aut}(T)$ can be constructed using (d). See Appendix B for details. \boxed{}

**Geometry of Permutation Representations.** We explain the result $\text{PERM} = 2\text{-DIM of Even}$ [7]. Let $O \in \sigma(X)$ and $\overline{O} \in \sigma(\overline{X})$, and let $\overline{O}_R$ be the reversal of $\overline{O}$. We construct two linear orderings $L_1 = O \cup \overline{O}$ and $L_2 = O \cup \overline{O}_R$. It follows that the comparable pairs in $L_1 \cap L_2$ are precisely the edges $E(X)$.

Fig. 7. The construction of the operations (b) to (d). It is easy to check that they are permutation graphs with correct automorphism groups.
Consider a permutation representation of a symmetric prime permutation graph. The horizontal reflection corresponds to exchanging \( L_1 \) and \( L_2 \), which is equivalent to reversing \( O \). The vertical reflection corresponds to reversing both \( L_1 \) and \( L_2 \), which is equivalent to reversing both \( O \) and \( \overline{O} \). The central rotation by 180° is the combination of both, which is equivalent to reversing \( O \). See Fig. 8.

5 \( k \)-Dimensional Comparability Graphs

We prove that \( \text{Aut}(4\text{-DIM}) \) contains all finite groups, i.e., each finite group can be realised as an automorphism group of some 4-dimensional comparability graph. Our construction also shows that graph isomorphism testing of 4-DIM is \( \text{GI} \)-complete. Both results easily translate to \( k\text{-DIM} \) for \( k > 4 \) since \( 4\text{-DIM} \subseteq k\text{-DIM} \).

The Construction. Let \( X \) be a graph with \( V(X) = x_1, \ldots, x_n \) and \( E(X) = \{e_1, \ldots, e_m\} \). We define

\[
\begin{align*}
P &= \{p_i : x_i \in V(X)\}, & Q &= \{q_{ik} : x_i \in e_k\}, & R &= \{r_k : e_k \in E(X)\},
\end{align*}
\]

where \( P \) represents the vertices, \( R \) represents the edges and \( Q \) represents the incidences between the vertices and the edges.

The constructed comparability graph \( C_X \) is defined as follows, see Fig. 9:

\[
\begin{align*}
V(C_X) &= P \cup Q \cup R, & E(C_X) &= \{p_i q_{ik} r_k : x_i \in e_k\}.
\end{align*}
\]

Proof of Dimension 4. The harder part is to prove that the constructed graph \( C_X \) has dimension four, which we can do when \( X \) is bipartite.

Lemma 5.1. If \( X \) is a connected bipartite graph, then \( \dim(C_X) \leq 4 \).

Fig. 9. The construction \( C_X \) for the graph \( X = K_{2,3} \).
that $X$ be a connected graph with some automorphism group $\text{Aut}(X)$. It is sufficient to prove the statement for $4$-DIM. Let $X$ be a connected graph with some automorphism group $\text{Aut}(X)$, and we assume that $X \not\cong C_n$. First, we take the bipartite incidence graph $Y$ between $V(X)$ and $E(X)$, and it easily follows that $\text{Aut}(Y) \cong \text{Aut}(X)$. Then we construct $C_Y$. In Appendix B, we have $\text{Aut}(C_Y) \cong \text{Aut}(Y) \cong \text{Aut}(X)$ and by Lemma 10.1, we have that $C_Y \in 4$-DIM. Similarly, if two graphs $X_1$ and $X_2$ are given, we construct $C_{Y_1}$ and $C_{Y_2}$ such that $X_1 \cong X_2$ if and only if $C_{Y_1} \cong C_{Y_2}$; this gives the reduction which shows GI-completeness of graph isomorphism testing.

6 Open Problems

We conclude with the following open problems:

Fig. 10. On the left, the forced edges in $L_1 \cap L_2$, on the right in $L_3 \cap L_4$. Proof. We construct four chains such that $L_1 \cap L_2 \cap L_3 \cap L_4$ have two vertices comparable if and only if they are adjacent in $C_X$. We describe linear chains as words containing each vertex of $V(C_X)$ exactly once. If $S_1, \ldots, S_s$ is a sequence of strings, the symbol $\langle S_i \uparrow t \rangle$ is the concatenation $S_1 S_2 \ldots S_s$ and $\langle S_i \downarrow t \rangle$ is the concatenation $S_s S_{s-1} \ldots S_1$. When the arrows are omitted as in $\langle S_i \rangle$, we concatenate in an arbitrary order.

First, we define the incidence string $I_i$ which codes $p_i$ and its neighbors $q_{ik}$:

$$I_i = p_i\langle q_{ik} : p_i q_{ik} \in E(C_X) \rangle.$$

Notice that the concatenation $I_i I_{j}$ contains the right edges but it further contains edges going from $p_i$ and $q_{ik}$ to $p_j$ and $q_{j\ell}$. We remove these edges by concatenation $I_i I_{j}$ in some other chain.

Since $X$ is bipartite, let $(A, B)$ be partition of its vertices. We define

$$P_A = \{p_i : x_i \in A\}, \quad Q_A = \{q_{ik} : x_i \in A\},$$
$$P_B = \{p_j : x_j \in B\}, \quad Q_B = \{q_{jk} : x_j \in B\}.$$

Each vertex $r_k$ has exactly one neighbor in $Q_A$ and exactly one in $Q_B$.

We construct the four chains as follows:

$$L_1 = \langle p_i : p_i \in P_A \rangle \langle r_k q_{ik} : q_{ik} \in Q_A, \uparrow k \rangle \langle I_i : p_i \in P_B, \uparrow i \rangle,$$
$$L_2 = \langle p_i : p_i \in P_A \rangle \langle r_k q_{ik} : q_{ik} \in Q_A, \downarrow k \rangle \langle I_i : p_i \in P_B, \downarrow i \rangle,$$
$$L_3 = \langle p_j : p_j \in P_B \rangle \langle r_k q_{jk} : q_{jk} \in Q_B, \uparrow k \rangle \langle I_i : p_i \in P_A, \uparrow i \rangle,$$
$$L_4 = \langle p_j : p_j \in P_B \rangle \langle r_k q_{jk} : q_{jk} \in Q_B, \downarrow k \rangle \langle I_i : p_i \in P_A, \downarrow i \rangle.$$

See Fig. 10 for properties of $L_1, \ldots, L_4$. It is routine to verify that the intersection $L_1 \cap L_2 \cap L_3 \cap L_4$ is correct; see Appendix C.

Proof (Theorem 1.3). It is sufficient to prove the statement for 4-DIM. Let $X$ be a connected graph with some automorphism group $\text{Aut}(X)$, and we assume that $X \not\cong C_n$. First, we take the bipartite incidence graph $Y$ between $V(X)$ and $E(X)$, and it easily follows that $\text{Aut}(Y) \cong \text{Aut}(X)$. Then we construct $C_Y$. In Appendix B, we have $\text{Aut}(C_Y) \cong \text{Aut}(Y) \cong \text{Aut}(X)$ and by Lemma 10.1, we have that $C_Y \in 4$-DIM. Similarly, if two graphs $X_1$ and $X_2$ are given, we construct $C_{Y_1}$ and $C_{Y_2}$ such that $X_1 \cong X_2$ if and only if $C_{Y_1} \cong C_{Y_2}$; this gives the reduction which shows GI-completeness of graph isomorphism testing.

6 Open Problems

We conclude with the following open problems:
Problem 6.1. What is $\text{Aut}(3\text{-DIM})$?

In Lemma 4.2 we show that the automorphism group of a prime permutation graph is always a subgroup of $\mathbb{Z}_2^2$. Our proof does not give much structural insight into prime permutation graphs.

Problem 6.2. What is the structure of prime permutation graphs? Can they be characterized?

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A Modular Trees

The following lemma explains that $T$ encodes adjacencies in $X$:

**Lemma A.1.** We have $xy \in E(X)$ if and only if there exists an alternating path $xm_1m_2\ldots m_ky$ in the modular tree $T$ such that each $m_i$ is a marker vertex and precisely the edges $m_{2i-1}m_{2i}$ are tree edges.

**Proof.** Suppose that $xy \in E(X)$. If $xy \in E(T)$, then we are done. We assume that $xy \notin E(T)$. The modular decomposition was constructed by a sequence of quotient operations. At some step of the construction we get the last graph $X_0$ such that $xy \in E(X_0)$. Let $P$ be the modular partition of $X_0$ chosen by the modular decomposition. As in the construction of the modular tree, we denote the marker vertices obtained from the contraction of the modules by $m_1, \ldots, m_k$, and the marker vertices attached to those by tree edges by $m'_1, \ldots, m'_k$.

We consider the next step of the modular decomposition. Suppose that $x \in M_i$ and $y \in M_j$. We have that $x \in V(X_0[M_i])$ and $y \in V(X_0[M_j])$. From the construction of $T$, it follows that $xm'_i$ and $ym'_j$ are normal edges and since $xy \notin E(X_0)$, we also have that $m_im_j \in E(X_0/P)$. The vertices $xm'_im_im_jm'_jy$ form an alternating path.

Now, we recursively construct an alternating path in $T$. From the construction of $T$, we have that the vertices $x$ and $m'_i$ are connected by a normal edge. Since the vertices $x$ and $m'_i$ are adjacent in the graph $X_0[M_i] \cup m'_i$, there exists an alternating path $P_i$ connecting $x$ and $m'_i$ in the subtree of $T$ representing $X_0[M_i] \cup m'_i$. Similarly, we have an alternating path $P_j$ connecting $y$ and $m'_j$ in some subtree of $T$ representing $X_0[M_j] \cup m'_j$. The vertices $xP_im'_im_im_jm'_jP_jy$ form a correct alternating path in $T$.

The converse implication can be easily derived by reversing the process described above.

**Proof (Lemma A.1).** First, we show that each automorphism $\sigma \in \text{Aut}(T)$ induces a unique automorphism of $X$. We define $\alpha = \sigma |_A$. By Lemma A.1 two vertices $x, y \in V(X)$ are adjacent if and only if there exists an alternating path in $T$ connecting them. Since $\sigma$ is an automorphism, we also have an alternating path between $\sigma(x)$ and $\sigma(y)$. Therefore, $xy \in E(X) \iff \alpha(x)\alpha(y) \in E(X)$.

To obtain the converse implication, we prove that $\alpha \in \text{Aut}(X)$ induces a unique automorphism $\sigma \in \text{Aut}(T)$. We define $\sigma(x) = \alpha(x)$ for a non-marker vertex $x$. On the marker vertices, we define $\sigma$ recursively as follows. Let $\mathcal{P} = \{M_1, \ldots, M_k\}$ be a modular partition of $X$ from the construction of the modular decomposition. It is easy to see that the group Aut($X$) induces an action on the partition $\mathcal{P}$. If $\alpha(M_i) = M_j$, then clearly $X[M_i]$ and $X[M_j]$ are isomorphic. We define $\sigma(m_i) = m_j$ and $\sigma(m'_i) = m'_j$, and finish the rest recursively. Since $\sigma$ is an automorphism at each step of the construction, it follows that $\sigma \in \text{Aut}(T)$.

**Proof (Lemma A.2).** We isomorphically label the vertices of isomorphic subtrees $T_i$. Each automorphism $\pi \in \text{Aut}(T)$ is a composition of two automorphisms $\sigma \cdot \tau$ where $\sigma$ maps each subtree $T_i$ to itself, and $\tau$ permutes the subtrees as in $\pi$ while preserving the labeling. Therefore, the automorphisms $\sigma$ can be bijectively
identified with the elements of the direct product \( \text{Aut}(T_1) \times \cdots \times \text{Aut}(T_k) \) and the automorphisms \( \tau \) with some element of \( \text{Aut}(R) \).

Let \( \pi, \pi' \in \text{Aut}(T) \). Consider the composition \( \sigma \cdot \tau \cdot \sigma' \cdot \tau' \), we want to swap \( \tau \) with \( \sigma' \) and rewrite this as a composition \( \sigma \cdot \hat{\sigma} \cdot \hat{\tau} \cdot \tau' \). Clearly the subtrees are permuted in \( \pi \cdot \pi' \) exactly as in \( \tau \cdot \tau' \), so \( \hat{\tau} = \tau \). On the other hand, \( \hat{\sigma} \) is not necessarily equal \( \sigma' \). Let \( \sigma' \) be identified with the vector

\[
(\sigma'_1, \ldots, \sigma'_k) \in \text{Aut}(T_1) \times \cdots \times \text{Aut}(T_k).
\]

Since \( \sigma' \) is applied after \( \tau \), it acts on the subtrees permuted according to \( \tau \). Thus, \( \hat{\sigma} \) is constructed from \( \sigma \) by permuting the coordinates of its vector by \( \tau \):

\[
\hat{\sigma} = (\sigma'_{\tau(1)}, \ldots, \sigma'_{\tau(k)}).
\]

This is precisely the definition of the semidirect product. \( \square \)

**B Permutation Graphs**

*Proof (Theorem 1.2).* Since \( \{1\} \in \text{Aut}(\text{PERM}) \), we need prove that \( \text{Aut}(\text{PERM}) \) is closed under (b)–(d).

- Let \( G_1, G_2 \in \text{Aut}(\text{PERM}) \), and let \( X_1 \) and \( X_2 \) be two permutation graphs such that \( \text{Aut}(X_1) \cong G_1 \) and \( \text{Aut}(X_2) \cong G_2 \). We construct a permutation graph \( X \) by attaching \( X_1 \) and \( X_2 \) to an asymmetric permutation graph; see Figure 7. Clearly, we get \( \text{Aut}(X) \cong G_1 \times G_2 \).

- Let \( G \in \text{Aut}(\text{PERM}) \), and let \( Y \) be connected a permutation graph such that \( \text{Aut}(Y) \cong G \). We construct a graph \( X \) by taking the disjoint union of \( n \) copies of \( Y \); see Figure 7. Clearly, we get \( \text{Aut}(X) \cong G \wr S_n \).

- Let \( G_1, G_2, G_3 \in \text{Aut}(\text{PERM}) \), and let \( X_1, X_2, \) and \( X_3 \) be permutation graphs such that \( \text{Aut}(X_i) \cong G_i \), for \( i = 1, 2, 3 \). We construct a graph \( X \) as shown in Figure 7. Clearly, we get \( \text{Aut}(X) \cong (G_1 \times G_2 \times G_3) \times \mathbb{Z}_2 \).

To show that for a given permutation graph \( X \) the group \( \text{Aut}(X) \in \text{Aut}(\text{PERM}) \) we use Lemma 2.2. Let \( T \) be the modular tree representing \( X \), let \( R \) be its root, and let \( T_1, \ldots, T_k \) be the subtrees of \( R \). By induction, we assume that \( \text{Aut}(T_i) \in \text{Aut}(\text{PERM}) \), and we show that also \( \text{Aut}(T) \in \text{Aut}(\text{PERM}) \). We distinguish two cases.

- If \( R \) is a degenerate node (an independent set or a complete graph), then \( \text{Aut}(R) \) is a direct product of symmetric groups. By Lemma 2.2 we get

\[
\text{Aut}(T) \cong (\text{Aut}(T_1) \times \cdots \times \text{Aut}(T_k)) \rtimes (S_{\ell_1} \times \cdots \times S_{\ell_m}),
\]

where \( \ell_1, \ldots, \ell_m \) are the sizes of the isomorphism classes of \( T_1, \ldots, T_k \). Let \( G_i \) be the direct product of all \( \text{Aut}(T_j) \) such that \( T_j \) belong to the same isomorphism class \( i \). We have

\[
\text{Aut}(T) \cong G_1 \wr S_{\ell_1} \times \cdots \times G_m \wr S_{\ell_m}.
\]

Therefore \( \text{Aut}(X) \cong \text{Aut}(T) \) can be constructed using (b) and (c) and it belongs to \( \text{Aut}(\text{PERM}) \).
If $R$ is a prime node, then by Lemma 4.2, $\text{Aut}(R)$ is a subgroup of $\mathbb{Z}_2^2$. The only interesting case is when $\text{Aut}(R) \cong \mathbb{Z}_2^2$. From the orbit-stabilizer theorem, the action of $\mathbb{Z}_2^2$ on $V(R)$ can have orbits of sizes 4, 2, and 1. Moreover, each orbit of size 2 corresponds to some stabilizer of size 2. Since there are three subgroups of $\mathbb{Z}_2^2$ of size 2, there can be possibly three types of orbits of size 2. By a geometric argument, we show that if $R$ is a prime permutation graph, then one of the three subgroups of size 2 can not be a stabilizer of any orbit of size 2, and therefore there are at most two types of orbits of size 2.

The non-identity elements $(1, 0)$, $(0, 1)$, and $(1, 1)$ of $\mathbb{Z}_2^2$ correspond to the reflection $f$ of the permutation representation along the vertical axis, reflection $f'$ along the horizontal axis, and rotation $r$ around the center by $180^\circ$, respectively; see Figure 8. The reflection $f$ stabilizes only segments that coincide with the vertical axis. Note that there can be at most one such segment, since otherwise $R$ would not be prime. Therefore, the reflection $f$ does not stabilize any orbit of size 2.

Let $G_1$ be the direct product of all $\text{Aut}(T)$ such that $T$ is attached to a vertex of $R$ that belongs to an orbit of size four. The groups $G_2$ and $G_3$ are defined similarly for the two types of orbits of size two, and $G_4$ for the orbits of size one. We have

$$\text{Aut}(T) \cong (G_1^4 \times G_2^2 \times G_3^2 \times G_4) \rtimes \mathbb{Z}_2^2 \cong (G_1^4 \times G_2^2 \times G_3^2) \rtimes \mathbb{Z}_2^2 \times G_4,$$

where $\varphi: \mathbb{Z}_2^2 \to \text{Aut}(G_1^4 \times G_2^2 \times G_3^2)$ is the homomorphism defined as follows. The automorphism $\varphi(1,0)$ swaps the first two components of $G_1^4$, swaps the components of $G_2^2$, fixes the components of $G_3^2$, and fixes $G_4$. The automorphism $\varphi(0,1)$ swaps the second two components of $G_1^4$, fixes the components of $G_2^2$, swaps the components of $G_3^2$, and fixes $G_4$. We get that $\text{Aut}(X) \cong \text{Aut}(T)$ can be constructed using (b) and (d) and it belongs to $\text{Aut}(\text{PERM})$.

### C $k$-dimensional Comparability Graphs

**Lemma C.1.** Let $X$ be a connected graph such that $X \not\cong C_n$. Then

$$\text{Aut}(C_X) \cong \text{Aut}(X).$$

**Proof.** All vertices of $Q$ and $R$ have degree two, and by our assumption at least one vertex $p_i$ in $P$ has a different degree. Therefore, we obtain $P$ as the set of the vertices in $C_X$ whose distance from $p_i$ is divisible by four, $Q$ as the set of their neighbors and $R$ as the remaining vertices. Every automorphism of $C_X$ has to preserve this partition, therefore it induces an automorphism of $X$. Since this construction does not depend on the labeling, every automorphism of $X$ is an automorphism of $C_X$. Therefore, $\text{Aut}(C_X) \cong \text{Aut}(X)$. $\square$
Proof. We conclude the proof of Lemma 5.1 by verify the construction:

\[ L_1 = \langle p_i : p_i \in P_A \rangle \langle r_k q_{ik} : q_{ik} \in Q_A, \uparrow k \rangle \langle I_i : p_i \in P_B, \uparrow i \rangle, \]
\[ L_2 = \langle p_i : p_i \in P_A \rangle \langle r_k q_{ik} : q_{ik} \in Q_A, \downarrow k \rangle \langle I_i : p_i \in P_B, \downarrow i \rangle, \]
\[ L_3 = \langle p_j : p_j \in P_B \rangle \langle r_k q_{jk} : q_{jk} \in Q_B, \uparrow k \rangle \langle I_i : p_i \in P_A, \uparrow i \rangle, \]
\[ L_4 = \langle p_j : p_j \in P_B \rangle \langle r_k q_{jk} : q_{jk} \in Q_B, \downarrow k \rangle \langle I_i : p_i \in P_A, \downarrow i \rangle. \]

The four defined chains have the following properties, see Figure 10:

- The intersection \( L_1 \cap L_2 \) forces the correct edges between \( Q_A \) and \( R \) and between \( P_B \) and \( Q_B \). It poses no restrictions between \( Q_B \) and \( R \) and between \( P_A \) and the rest of the graph.
- Similarly the intersection \( L_3 \cap L_4 \) forces the correct edges between \( Q_B \) and \( R \) and between \( P_A \) and \( Q_A \). It poses no restrictions between \( Q_A \) and \( R \) and between \( P_B \) and the rest of the graph.

Claim 1: The edges in \( Q \cup R \) are correct. For every \( k \), we get \( r_k \) adjacent to both \( q_{ik} \) and \( q_{jk} \) since it appear on the left in \( L_1, \ldots, L_4 \). On the other hand, \( q_{ik} q_{jk} \notin E(C_X) \) since they are ordered differently in \( L_1 \) and \( L_3 \).

For every \( k < \ell \), there are no edges between \( N[r_k] = \{r_k, q_{ik}, q_{jk}\} \) and \( N[r_\ell] = \{r_\ell, q_{\ell i}, q_{\ell j}\} \). This can be shown by checking the four ordering of these six elements:

in \( L_1 \):
\[ r_k q_{ik} \quad r_{\ell q_{\ell i}} \quad q_{jk} \quad q_{\ell t} \]
in \( L_2 \):
\[ r_{\ell q_{\ell i}} \quad r_k q_{ik} q_{jk} \quad q_{\ell t} \]
in \( L_3 \):
\[ r_k q_{jk} \quad r_{\ell q_{\ell t}} \quad q_{ik} \quad q_{\ell i} \]
in \( L_4 \):
\[ r_{\ell q_{\ell t}} \quad r_k q_{jk} q_{ik} \quad q_{\ell i} \]
where the elements of \( N[r_\ell] \) are boxed.

Claim 2: The edges in \( P \) are correct. We show that there are no edges between \( p_i \) and \( p_j \) for \( i \neq j \) as follows. If both belong to \( P_A \) (respectively \( P_B \)), then they are ordered differently in \( L_3 \) and \( L_4 \) (respectively \( L_1 \) and \( L_2 \)). If one belongs to \( P_A \) and the other one to \( P_B \), then they are ordered differently in \( L_1 \) and \( L_3 \).

Claim 3: The edges between \( P \) and \( Q \cup R \) are correct. For every \( p_i \in P \) and \( r_k \in R \), we have \( p_i r_k \notin E(C_X) \) because they are ordered differently in \( L_1 \) and \( L_3 \). On the other hand, \( p_i q_{ik} \in E(C_X) \), because \( p_i \) is before \( q_{ik} \) in \( I_i \), and for \( p_i \in P_A \) in \( L_1 \) and \( L_2 \), and for \( p_i \in P_B \) in \( L_3 \) and \( L_4 \).

It remains to show that \( p_i q_{jk} \notin E(C_X) \) for \( i \neq j \). If both \( p_i \) and \( p_j \) belong to \( P_A \) (respectively \( P_B \)), then \( p_i \) and \( q_{jk} \) are ordered differently in \( L_3 \) and \( L_4 \) (respectively \( L_1 \) and \( L_2 \)). And if one belongs to \( P_A \) and the other one to \( P_B \), then \( p_i \) and \( q_{jk} \) are ordered differently in \( L_1 \) and \( L_3 \).

These three claims show that comparable pairs in the intersection \( L_1 \cap L_2 \cap L_3 \cap L_4 \) are exactly the edges of \( C_X \), so \( C_X \) is a comparability graph with the dimension at most four.