Research Article

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A necessary and sufficient condition for the continuity of local minima of parabolic variational integrals with linear growth

Abstract: For proper minimizers of parabolic variational integrals with linear growth with respect to $|Du|$, we establish a necessary and sufficient condition for $u$ to be continuous at a point $(x_0, t_0)$, in terms of a sufficient fast decay of the total variation of $u$ about $(x_0, t_0)$. These minimizers arise also as proper solutions to the parabolic 1-Laplacian equation. Hence, the continuity condition continues to hold for such solutions.

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1 Introduction

Let $E$ be an open subset of $\mathbb{R}^N$, and denote by $\text{BV}(E)$ the space of functions $v \in L^1(E)$ with finite total variation [8], i.e.,

$$
\|Dv\|(E) := \sup_{\varphi \in [C^1_0(E)]^N, |\varphi| \leq 1} \left\{ \left \langle Dv, \varphi \right \rangle = - \int_E v \text{ div } \varphi \, dx \right\} < \infty.
$$

Here $Dv = (D_1v, \ldots, D_Nv)$ is the vector valued Radon measure, representing the distributional gradient of $v$. A function $v \in \text{BV}_{\text{loc}}(E)$ if $v \in \text{BV}(E')$ for all open sets $E' \subseteq E$. For $T > 0$, let $E_T = E \times (0, T)$, and denote by $L^1(0, T; \text{BV}(E))$ the collection of all maps $v: [0, T] \to \text{BV}(E)$ such that

$$
v \in L^1(E_T), \quad \|Dv(t)\|(E) \in L^1(0, T),$$

and the maps

$$(0, T) \ni t \mapsto \langle Dv(t), \varphi \rangle$$

are measurable with respect to the Lebesgue measure in $\mathbb{R}$ for all $\varphi \in [C^1_0(E)]^N$.

A function $u \in L^1_{\text{loc}}(0, T; \text{BV}_{\text{loc}}(E))$ is a local parabolic minimizer of the total variation flow in $E_T$ if

$$
\int_0^T \left( \int_E -u \varphi_t \, dx + \|Du(t)\|(E) \right) \, dt \leq \int_0^T \|D(u - \varphi)(t)\|(E) \, dt \tag{1.1}
$$

for all $\varphi \in C^0_0(E_T)$. This notion has been introduced in [3] and modeled in [11]. It is a parabolic version of the elliptic local minima of total variation flow as introduced in [9].

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1.1 The main result

Let $B_ρ(x_o)$ denote the ball of radius $ρ$ about $x_o$. If $x_o = 0$, we write $B_ρ(x_o) = B_ρ$. We introduce the cylinders $Q_ρ(θ) = B_ρ × (−θ, 0)$, where $θ$ is a positive parameter to be chosen as needed. If $θ = 1$, we write $Q_ρ(1) = Q_ρ$.

For a point $(x_o, t_o) ∈ R^{N+1}$ we let $[(x_o, t_o) + Q_ρ(θ)]$ be the cylinder of “vertex” at $(x_o, t_o)$ and congruent to $Q_ρ(θ)$, i.e.,

$$[(x_o, t_o) + Q_ρ(θ)] = B_ρ(x_o) × (t_o − θρ, t_o),$$

and we let $ρ > 0$ be so small that $[(x_o, t_o) + Q_ρ(θ)] ⊂ E_T$.

Theorem 1.1. Let $u ∈ L^1_{loc}(0, T; BV_{loc}(E))$ be a local parabolic minimizer of the total variation flow in $E_T$, satisfying in addition

$$u ∈ L^∞_{loc}(E_T) \quad \text{and} \quad u_t ∈ L^1_{loc}(E_T).$$

Then, $u$ is continuous at some $(x_o, t_o) ∈ E_T$ if and only if

$$\limsup_{ρ ↘ 0} \frac{ρ}{|Q_ρ|} \int_{t_o - ρ}^{t_o} \|Du(·, t)\|(B_ρ(x_o)) \, dt = 0.$$  \hspace{1cm} (1.3)

For stationary, elliptic minimizers, condition (1.3) has been introduced in [9]. The stationary version of (1.3) implies that $u$ is quasi-continuous at $x_o$. For time-dependent minimizers, however, (1.3) gives no information on the possible quasi-continuity of $u$ at $(x_o, t_o)$. Condition (1.3) is only a measure-theoretical restriction on the speed at which a possible discontinuity may develop at $(x_o, t_o)$. For this reason our proof is entirely different than [9], being based instead on a DeGiorgi-type iteration technique that exploits precisely such a measure-theoretical information.

2 Comments on boundedness and continuity

The theorem requires that $u$ is locally bounded and that $u_t ∈ L^1_{loc}(E_T)$. In the elliptic case, local minimizers of the total gradient flow in $E$, are locally bounded ([9, §2]). This is not the case, in general, for parabolic minimizers in $E_T$, even if $u_t ∈ C^∞_{loc}(0, T; L^1_{loc}(E))$. Consider the function

$$B_1 × (−∞, 1) ∋ (x, t) ↦ F(|x|, t) = (1 − t)\frac{N − 1}{|x|} \quad \text{for} \quad N ≥ 3.$$  

Denote by $D_aF$ that component of the measure $DF$ which is absolutely continuous with respect to the Lebesgue measure in $R^N$. One verifies that $DF = D_aF$ and $\|DF(t)\|(B_1) = \|D_aF(t)\|_{1, B_1}$. By direct computation, we have

$$\int_0^T \int_{B_1} \left(−Fφ_t + \frac{D_aF}{|D_aF|} \cdot Dφ \right) \, dx \, dt = 0$$

for all $φ ∈ C^∞_0(B_1 × (0, T)), 0 < T < 1$. It follows that

$$\int_0^T \int_{B_1} \left(−Fφ_t + \frac{D_aF}{|D_aF|} \cdot D_aF \right) \, dx \, dt = \int_0^T \int_{B_1} \frac{D_aF}{|D_aF|} \cdot D_a(F − φ) \, dx \, dt,$$

which yields

$$\int_0^T \int_{B_1} (−Fφ_t + |D_aF|) \, dx \, dt ≤ \int_0^T \int_{B_1} |D_a(F − φ)| \, dx \, dt.$$  

Thus, $F$ is a local, unbounded, parabolic minimizer of the total variation flow. The requirement $u ∈ L^∞_{loc}(E_T)$ could be replaced by $u ∈ L^{r}_{loc}(E_T)$ for some $r > N$. A discussion on this issue is provided in Appendix B.
2.1 On the modulus of continuity

While Theorem 1.1 gives a necessary and sufficient condition for continuity at a given point, it provides no information on the modulus of continuity of \(u\) at \((x_0, t_0)\). Consider the following two time-independent functions in \(B_\rho \times (0, \infty)\) for some \(\rho < 1:\)

\[
\begin{aligned}
    u_1(x_1, x_2) &= \begin{cases} 
        \frac{1}{\ln x_1} & \text{for } x_1 > 0, \\ 
        0 & \text{for } x_1 = 0, \\ 
        -\frac{1}{\ln(x_1)} & \text{for } x_1 < 0,
    \end{cases} \\
    u_2(x_1, x_2) &= \begin{cases} 
        \sqrt{x_1} & \text{for } x_1 > 0, \\ 
        -\sqrt{-x_1} & \text{for } x_1 \leq 0.
    \end{cases}
\end{aligned}
\]

Both of them are stationary parabolic minimizers of the total variation flow in the sense of (1.1)–(1.2) over \(B_\frac{1}{2} \times (0, \infty)\). We establish this for \(u_1\), and note that the statement for \(u_2\) can be verified analogously. Since \(u_1 \in W^{1,1}(B_\rho)\), and is time-independent, one also has \(u \in L^1(0, T; BV(B_\rho))\). To verify (1.1), one needs to show that

\[
\|Du_1\|(B_\rho) \leq \int_0^T \|D(u_1 + \varphi)(\cdot, t)\|(B_\rho) \, dt
\]

for all \(T > 0\) and all \(\varphi \in C^\infty_0(B_\rho \times (0, T))\). Let \(\mathcal{H}^k(A)\) denote the \(k\)-dimensional Hausdorff measure of a Borel set \(A \subset \mathbb{R}^N\). One checks that \(\mathcal{H}^k(\{Du_1 = 0\}) = 0\), and there exists a closed set \(K \subset B_\rho\) such that \(\mathcal{H}^{N-1}(K) = 0\) and

\[
\int_{B_\rho - K} \frac{Du_1}{|Du_1|} \cdot D\varphi \, dx = 0 \quad \text{for all } \varphi \in C^\infty_0(B_\rho - K).
\]

From this and by [5, Lemma 4 in §8], for all \(\psi \in C^\infty_0(B_\rho)\), one has

\[
\|Du_1\|(B_\rho) \leq \|D(u_1 + \psi)\|(B_\rho),
\]

which, in turn, yields (2.1). The two functions \(u_1\) and \(u_2\) can be regarded as equibounded near the origin. They both satisfy (1.3), and exhibit quite different moduli of continuity at the origin. This occurrence is in line with a remark of Evans [7]. A sufficiently smooth minimizer of the elliptic functional \(\|Du\|(E)\) is a function whose level sets are surfaces of zero mean curvature. Thus, if \(u\) is a minimizer, so is \(\varphi(u)\) for all continuous monotone functions \(\varphi(\cdot)\). This implies that a modulus of continuity cannot be identified solely in terms of an upper bound of \(u\).

3 Singular parabolic DeGiorgi classes

Let \(\mathcal{C}(Q_\rho(\theta))\) denote the class of all non-negative, piecewise smooth, cutoff functions \(\zeta\) defined in \(Q_\rho(\theta)\), vanishing outside \(B_\rho\) such that \(\zeta_t \geq 0\) and satisfying

\[
|D\zeta| + \zeta_t \in L^\infty(Q_\rho(\theta)).
\]

For a measurable function \(u: E_T \rightarrow \mathbb{R}\) and \(k \in \mathbb{R}\), set

\[
(u - k)_+ = \{\max(u - k) \wedge 0\}.
\]

The singular, parabolic DeGiorgi class \([DG]^{+}(E_T; \gamma)\) is the collection of all measurable maps

\[
u \in C^0_{\text{loc}}((0, T); L^2_{\text{loc}}(E)) \cap L^1_{\text{loc}}(0, T; BV_{\text{loc}}(E)),
\]

(3.1)
satisfying
\[
\sup_{t_0 - \theta \leq t \leq t_0} \int_{B_\rho(x_0)} (u - k)^2 \xi(x, t) \, dx + \int_{t_0 - \theta \rho}^{t_0} \|D((u - k)_\xi(c)) (B_{\rho}(x_0))\| \, dt \\
\leq \gamma \int_{(x_0, t_0) + Q_{\rho}(\theta)} [(u - k)^2_\xi + (u - k)^2_\xi(\zeta_\xi)] \, dx \, dt + \int_{B_\rho(x_0)} (u - k)^2 \xi(x, t_0 - \theta \rho) \, dx
\]
for all \((x_0, t_0) + Q_{\rho}(\theta) \subset E_T\), all \(k \in \mathbb{R}\), all \(\xi \in \mathcal{C}(E_T)\), and for a given positive constant \(\gamma\). The singular DeGiorgi classes \([DG](E_T; \gamma)\) are defined as \([DG](E_T; \gamma) = [DG]^+(E_T; \gamma) \cap [DG]^{-}(E_T; \gamma)\).

### 3.1 The main result

The main result of this note is that the necessary and sufficient condition of Theorem 1.1 holds for functions \(u \in DG(E_T; \gamma) \cap L^\infty_{\text{loc}}(E_T)\). Indeed, the proof of Theorem 1.1, only uses the local integral inequalities (3.2). In particular, the second of (1.2) is not needed.

**Proposition 3.1.** Let \(u\) in the functional classes (3.1) be a parabolic minimizer of the total variation flow in \(E_T\), in the sense of (1.1), satisfying in addition (1.2). Then, \(u \in DG(E_T; 2)\).

The proof will be given in Appendix A.

**Remark 3.2.** Note that in the context of \(DG(E_T)\) classes, the characteristic condition (1.3) holds with no further requirement that \(u_t \in L^1_{\text{loc}}(E_T)\). The latter, however, is needed to cast a parabolic minimizer of the total variation flow into a \(DG(E_T)\)-class as stated in Proposition 3.1.

### 4 A singular diffusion equation

Consider (formally) the parabolic 1-Laplacian equation
\[
u_t - \text{div} \left( \frac{Du}{|Du|} \right) = 0 \quad \text{formally in } E_T.
\]

Let \(\mathcal{P}\) be the class of all Lipschitz continuous non-decreasing functions \(p(\cdot)\), defined in \(\mathbb{R}\) with \(p'\) compactly supported. Denote by \(\mathcal{C}(E_T)\) the class of all non-negative functions \(\xi\) defined in \(E_T\) such that \(\xi(\cdot, t) \in C^1_{\text{loc}}(E)\) for all \(t \in (0, T)\), and \(0 \leq \xi \leq \zeta < \infty\) in \(E_T\). A function \(u \in C^1_{\text{loc}}(0, T; L^1(E))\) is a local solution to (4.1) if the following hold:

(a) \(p(u) \in L^1_{\text{loc}}(0, T; BV(E))\) for all \(p \in \mathcal{P}\).

(b) There exists a vector valued function \(z \in [L^\infty(E_T)]^N\) with \(\|z\|_{\text{loc}, E} \leq 1\), such that \(u_t = \text{div} z\) in \(\mathcal{P}(E_T)\).

(c) Denoting by \(d\|Dp(u - \ell)\|\) the measure in \(E\) generated by the total variation \(\|Dp(u - \ell)\|(E)\), we have
\[
\int_E \left( \int_0^{t_2} \int_0^{t_1} p(s) \, d\zeta dx \, ds + \int_{t_2}^{t_1} \int_0^{T_2} \zeta d\|Dp(u - \ell)\| \right) \, dt \\
\leq \int_E \left( \int_0^{t_2} \int_0^{t_1} p(s) \, d\zeta dx \, ds + \int_{t_2}^{t_1} \int_0^{T_2} p(s) \, d\zeta dx \, ds \right) \zeta dx \, dt - \int_E z \cdot D\zeta(p(u - \ell) \, dx \, dt
\]
for all \(\ell \in \mathbb{R}\), all \(p \in \mathcal{P}\), all \(\zeta \in \mathcal{C}(E_T)\), and all \([t_1, t_2] \subset (0, T)\).

This notion is a local version of a global one introduced in [1, Chapter 3]. Similar notions are found in [1, 3, 4, 10], associated with issues of existence for the Cauchy problem and boundary value problems associated with (4.1). The notion of solution in [3], called *variational*, is different and closely related to the variational integrals (1.1).
Our results are local in nature and disengaged from any initial or boundary conditions. Let $u$ be a local solution to (4.1) in the indicated sense, which in addition is locally bounded in $E_T$. In (4.2), take $\ell = 0$ and $p_s(u) = \pm (u - k)_s$. Since $u \in L^{10}(E_T)$, one verifies that $p_s \in \mathcal{P}$. Standard calculations then yield that $u$ is in the DeGiorgi classes $[DG]^+(E; \gamma)$ for some fixed $\gamma > 0$. As a consequence, we have the following corollary.

**Corollary 4.1.** Let $u \in L^{10}(E_T)$ be a local solution to (4.1) in $E_T$, in the sense (a)–(c) above. Then, $u$ is continuous at some $(x_0, t_0) \in E_T$ if and only if (1.3) holds true.

## 5 Proof of the necessary condition

Let $u \in [DG](E_T; \gamma)$ be continuous at $(x_0, t_0) \in E_T$, which we may take as the origin of $\mathbb{R}^{N+1}$, and we may assume $u(0, 0) = 0$. In (3.2) for $(u - k)_s$, take $\theta = 1$ and $k = 0$. Let also $\zeta \in \mathcal{C}(Q_{2\rho})$ be such that $\zeta(\cdot, -2\rho) = 0$, $\zeta = 1$ on $Q_{2\rho}$, and

$$|D\zeta| + \zeta_t \leq \frac{3}{\rho}.$$ 

Repeat the same choices in (3.2) for $(u - k)_s$. Adding the resulting inequalities gives

$$\frac{\rho}{|Q_{\rho}|} \int_0^\rho \|D(u\zeta)(\cdot, t)||B_{2\rho}|| dt \leq 2^{N+1} \gamma \iint_{Q_{2\rho}} (u + u^2) \ dx \ dt. \quad (5.1)$$

Since the total variation $\| Dw \|$ of a function $w \in BV$ can be seen as a measure (see, for example, [12, Chapter 1, §1]), we have

$$\frac{\rho}{|Q_{\rho}|} \int_0^\rho \|D(u\zeta)(\cdot, t)||B_{2\rho}|| dt \leq \frac{\rho}{|Q_{\rho}|} \int_0^\rho \|D(u\zeta)(\cdot, t)||B_{2\rho}|| dt.$$ 

On the other hand, $u\zeta \equiv u$ in $Q_{\rho} \supset Q_{\rho}$, and therefore we conclude

$$\frac{\rho}{|Q_{\rho}|} \int_\rho^\rho \|Du(\cdot, t)||B_{\rho}|| dt \leq 2^{N+1} \gamma \iint_{Q_{2\rho}} (u + u^2) \ dx \ dt.$$ 

The right-hand side tends to zero as $\rho \to 0$, thereby implying the necessary condition of Theorem 1.1. \hfill $\square$

## 6 A DeGiorgi-type lemma

For a fixed cylinder $[(y, s) + Q_{\rho}(\theta)] \subset E_T$, denote by $\mu_\pm$ and $\omega$ non-negative numbers such that

$$\mu_+ \geq \text{ess sup}_{[(y, s) + Q_{\rho}(\theta)]} u, \quad \mu_- \leq \text{ess inf}_{[(y, s) + Q_{\rho}(\theta)]} u, \quad \omega \geq \mu_+ - \mu_-.$$ \hfill (6.1)

Let $\zeta \in (0, \tfrac{1}{2})$ be fixed and let $\theta = 2\zeta \omega$. This is an intrinsic cylinder in that its length $\theta \rho$ depends on the oscillation of $u$ within it. We assume momentarily that the indicated choice of parameters can be effected.

**Lemma 6.1.** Let $u$ belong to $[DG]^+(E_T, \gamma)$. There exists a number $\nu_-$ depending on $N$ and $\gamma$ only, such that if

$$[u \leq \mu_- + \zeta \omega] \cap [(y, s) + Q_{\rho}(\theta)] \leq \nu_- |Q_{\rho}(\theta)|,$$ \hfill (6.2)

then

$$u \geq \mu_- + \frac{1}{2} \zeta \omega \quad \text{a.e. in } [(y, s) + Q_{\rho}(\theta)].$$ \hfill (6.3)

Likewise, if $u$ belongs to $[DG]^+(E_T, \gamma)$, there exists a number $\nu_+$ depending on $N$ and $\gamma$ only, such that if

$$[u \leq \mu_+ - \zeta \omega] \cap [(y, s) + Q_{\rho}(\theta)] \leq \nu_+ |Q_{\rho}(\theta)|,$$ \hfill (6.4)

then

$$u \leq \mu_+ - \frac{1}{2} \zeta \omega \quad \text{a.e. in } [(y, s) + Q_{\rho}(\theta)].$$ \hfill (6.5)
Proof. We prove (6.2)–(6.3); the proof of (6.4)–(6.5) is similar. We may assume \((y, s) = (0, 0)\) and for \(n = 0, 1, \ldots\), set
\[
\rho_n = \rho + \frac{\rho}{2^n}, \quad B_n = B_{\rho_n}, \quad Q_n = B_n \times (-\theta \rho_n, 0).
\]
Apply (3.2) over \(B_n\) and \(Q_n\) to \((u - k_n)_-\), for the levels
\[
k_n = \mu_- + \xi_n \alpha, \quad \text{where} \quad \xi_n = \frac{1}{2} \xi + \frac{1}{2^{n+1}} \xi.
\]
The cutoff function \(\zeta\) is taken of the form \(\zeta(x, t) = \zeta_1(x) \zeta_2(t)\), where
\[
\zeta_1 = \begin{cases} 1 & \text{in } B_{n+1}, \\ 0 & \text{in } \mathbb{R}^N - B_n, \end{cases} \quad \text{and} \quad \zeta_2 = \begin{cases} 0 & \text{for } t < -\theta \rho_n, \\ 1 & \text{for } t \geq -\theta \rho_{n+1}, \end{cases}
\]
where
\[
\zeta_1 \leq \frac{1}{\rho} \left( \int_{B_n} (u - k_n)^2 dx + \int_{-\theta \rho_n}^0 |D(u - k_n)_- \zeta_1| |B_n| dt \right) \leq \gamma \frac{2^n}{\rho} \left( \int_{Q_n} (u - k_n)_- dx dt + \frac{1}{\theta} \int_{Q_n} (u - k_n)^2 dx dt \right) \leq \gamma \frac{2^n (\xi \omega)}{\rho} |[u < k_n] \cap Q_n|.
\]

By [6, the embedding Proposition 4.1 in Preliminaries],
\[
\int_{Q_n} [(u - k_n)_- \zeta_1]^{\frac{N+2}{N}} dx dt \leq \frac{1}{\theta \rho_n} \|D[(u - k_n)_- \zeta_1] \| |B_n| dt \left( \int_{-\theta \rho_n}^0 \left( \int_{B_n} (u - k_n)_- \zeta_1 \right)^2 dx \right)^{\frac{1}{2}} \leq \gamma \left( \frac{2^n}{\rho} \xi \omega \right)^{\frac{N+2}{N}} |[u < k_n] \cap Q_n|^{\frac{N+2}{N}}.
\]
Estimate below
\[
\int_{Q_n} [(u - k_n)_- \zeta_1]^{\frac{N+2}{N}} dx dt \geq \left( \frac{\xi \omega}{2^{n+2}} \right)^{\frac{N+2}{N}} |[u < k_n+1] \cap Q_{n+1}|,
\]
and set
\[
Y_n = |[u < k_n] \cap Q_n| \left| Q_n \right|.
\]
Then,
\[
Y_{n+1} \leq \gamma b^n Y_{n}^{1+\frac{1}{b}},
\]
where
\[
b = 2^{\frac{1}{b}} |3N+1|.
\]
By [6, Lemma 5.1 in Preliminaries], \([Y_n] \to 0\) as \(n \to \infty\), provided
\[
Y_0 \leq \gamma^{-N} b^{-N} =: \nu_-
\]
The proof of (6.4)–(6.5) is almost identical. One starts from inequalities (3.2) written for the truncated functions
\[
(u - k_n)_+ \quad \text{with} \quad k_n = \mu_- - \xi_n \alpha
\]
for the same choice of \(\xi_n\).
7 A time expansion of positivity

For a fixed cylinder
\[(y, s) + Q^+_2(\theta) = B_{2\rho}(y) \times (s, s + \theta \rho) \subset E_T,\]
denote by \(\mu_+\) and \(\omega\) the non-negative numbers satisfying the analogue of (6.1). Let also \(\xi \in (0, 1)\) be a fixed parameter. The value of \(\theta\) will be determined by the proof; we momentarily assume that such a choice can be made.

**Lemma 7.1.** Let \(u \in [DG]^-(E_T, \gamma)\) and assume that for some \((y, s) \in E_T\) and some \(\rho > 0\),
\[|u(\cdot, s) + \mu_+ + \xi \omega| \cap B_\rho(y) \geq \frac{1}{2}|B_\rho(y)|.\]
Then, there exist \(\delta\) and \(\varepsilon\) in \((0, 1)\), depending only on \(N, \gamma\) and independent of \(\xi\), such that
\[|u(\cdot, t) - \mu_+ + \varepsilon \xi \omega| \cap B_\rho(y) \geq \frac{1}{4}|B_\rho|\]
for all \((s, s + \delta(\xi \omega) \rho)\).

**Proof.** Assume \((y, s) = (0, 0)\), and for \(k > 0\) and \(t > 0\) set
\[A_{k, \rho}(t) = \{u(\cdot, t) < k\} \cap B_\rho.\]
The assumption implies
\[|A_{\mu_+ + \xi \omega, \rho}(0)| \leq \frac{1}{2}|B_\rho|.\]
Write down inequalities (3.2) for the truncated functions \((u - \mu_+ + \xi \omega)_{\cdot, s} \cdot,\) over the cylinder \(B_{\rho} \times (0, \theta \rho)\), where \(\theta > 0\) is to be chosen. The cutoff function \(\xi\) is taken independent of \(t\), non-negative, and such that
\[\xi = 1 \quad \text{on } B_{(1 - \sigma)\rho} \quad \text{and} \quad |D\xi| \leq \frac{1}{\sigma \rho},\]
where \(\sigma \in (0, 1)\) is to be chosen. Discarding the non-negative term containing \(D(u - \mu_+ + \xi \omega)_{\cdot, s}\) on the left-hand side, these inequalities yield
\[
\int_{B_{(1 - \sigma)\rho}} \frac{(u - \mu_+ + \xi \omega)^2(x, t)}{B_\rho} \, dx \leq \int_{B_\rho} (u - \mu_+ + \xi \omega)^2(x, 0) \, dx + \frac{\sigma \rho}{\sigma \rho} \left( \int_{B_\rho} (u - \mu_+ + \xi \omega)^2 \, dx \right) \, dt
\]
\[\leq (\xi \omega)^2 \left[ \frac{1}{2} + \frac{\theta}{\sigma(\xi \omega)} \right]|B_\rho|
\]
for all \(t \in (0, \theta \rho)\), where we have enforced (7.1). The left-hand side is estimated below by
\[
\int_{B_{(1 - \sigma)\rho}} (u - \mu_+ + \xi \omega)^2(x, t) \, dx \geq \int_{B_{(1 - \sigma)\rho} \cap u < \mu_+ + \varepsilon \xi \omega} (u - \mu_+ + \xi \omega)^2(x, t) \, dx \geq (\xi \omega)^2 (1 - \varepsilon)^2 |A_{\mu_+ + \varepsilon \xi \omega, (1 - \sigma)\rho}(t)|,
\]
where \(\varepsilon \in (0, 1)\) is to be chosen. Next, estimate
\[|A_{\mu_+ + \varepsilon \xi \omega, \rho}(t)| = |A_{\mu_+ + \varepsilon \xi \omega, (1 - \sigma)\rho}(t) \cup (A_{\mu_+ + \varepsilon \xi \omega, \rho}(t) - A_{\mu_+ + \varepsilon \xi \omega, (1 - \sigma)\rho}(t))| \leq |A_{\mu_+ + \varepsilon \xi \omega, (1 - \sigma)\rho}(t)| + |B_\rho - B_{(1 - \sigma)\rho}| \leq |A_{\mu_+ + \varepsilon \xi \omega, (1 - \sigma)\rho}(t)| + N\sigma |B_\rho|.
\]
Combining these estimates gives
\[|A_{\mu_+ + \varepsilon \xi \omega, \rho}(t)| \leq \frac{1}{(\xi \omega)^2 (1 - \varepsilon)^2} \int_{B_{(1 - \sigma)\rho}} (u - \mu_+ + \xi \omega)^2(x, t) \, dx + N\sigma |B_\rho|
\]
\[\leq \frac{1}{(1 - \varepsilon)^2} \left[ \frac{1}{2} + \frac{\gamma \theta}{\sigma(\xi \omega)} + N\sigma \right]|B_\rho|.
\]
Choose \(\theta = \delta(\xi \omega)\) and then set
\[\sigma = \frac{1}{16N}, \quad \varepsilon = \frac{1}{32}, \quad \delta = \frac{1}{2^3 \gamma N}.
\]
This proves the lemma. \(\square\)
8 Proof of the sufficient part of Theorem 1.1

Having fixed \((x_0, t_0) \in E_T\), assume it coincides with the origin of \(\mathbb{R}^{N+1}\) and let \(\rho > 0\) be so small that \(Q_\rho \subset E_T\). Set

\[
\mu_+ = \text{ess sup}_{Q_\rho} u, \quad \mu_- = \text{ess inf}_{Q_\rho} u, \quad \omega = \mu_+ - \mu_- = \text{ess osc}_{Q_\rho} u.
\]

Without loss of generality, we may assume that \(\omega \leq 1\), so that

\[
Q_\rho(\omega) = B_\rho \times (-\omega \rho, 0) \subset Q_\rho \subset E_T
\]

and

\[
\text{ess osc}_u \leq \omega.
\]

If \(u\) were not continuous at \((x_0, t_0)\), there would exist \(\rho_0 > 0\) and \(\omega_0 > 0\) such that

\[
\omega_\rho = \text{ess osc}_u \geq \omega_0 > 0 \quad \text{for all } \rho \leq \rho_0. \quad (8.1)
\]

Let \(\delta\) be determined from the last equality of (7.2). At the time level \(t = -\delta \omega_\rho\), either

\[
\left| \left\{ u(\cdot, t) > \mu_+ + \frac{1}{2} \omega \right\} \cap B_\rho \right| \geq \frac{1}{2} |B_\rho|
\]

or

\[
\left| \left\{ u(\cdot, t) < \mu_+ - \frac{1}{2} \omega \right\} \cap B_\rho \right| \geq \frac{1}{2} |B_\rho|.
\]

Assuming the former holds, by Lemma 7.1,

\[
\left| \left\{ u(\cdot, t) > \mu_+ + \frac{\delta}{\rho} \omega \right\} \cap B_\rho \right| \geq \frac{1}{2} |B_\rho| \quad \text{for all } t \in (-\delta \omega_\rho, 0].
\]

Let \(2\xi = \frac{1}{\rho_0} \delta\). Then,

\[
\left| \left\{ u(\cdot, t) > \mu_+ + 2\xi \omega \right\} \cap B_\rho \right| \geq \frac{1}{2} |B_\rho| \quad \text{for all } t \in (-\xi \omega_\rho, 0]. \quad (8.2)
\]

Next, apply the discrete isoperimetric inequality of [6, Lemma 2.2 in Preliminaries] to the function \(u(\cdot, t)\), for \(t\) in the range \((-\xi \omega_\rho, 0]\), over the ball \(B_\rho\), for the levels

\[
k = \mu_- + \xi \omega \quad \text{and} \quad \ell = \mu_- + 2\xi \omega, \quad \text{so that} \quad \ell - k = \xi \omega.
\]

This inequality is stated and proved in [6] for functions in \(W^{1,1}_{\text{loc}}(E)\). It continues to hold for \(u \in \text{BV}_{\text{loc}}(E)\), by virtue of the approximation procedure of [8, Theorem 1.17]. Taking also into account (8.2), this gives

\[
\xi \omega |\left\{ u(\cdot, t) < \mu_- + \xi \omega \right\} \cap B_\rho| \leq \gamma \rho \int (\rho^0 \text{Du})([u(\cdot, t) > k] \cap B_\rho).
\]

Integrating in \(dt\) over the time interval \((-\xi \omega_\rho, 0]\) gives

\[
\frac{|\left\{ u < \mu_- + \xi \omega \right\} \cap Q_\rho(\xi \omega)|}{|Q_\rho(\xi \omega)|} \leq \frac{\gamma}{(\xi \omega_\rho)^2 |Q_\rho|} \int_{-\rho}^0 \|\text{Du}(\cdot, t)\|(B_\rho) \, dt.
\]

By the assumption, the right-hand side tends to zero as \(\rho \searrow 0\). Hence, there exists \(\rho\) so small that

\[
\frac{|\left\{ u < \mu_- + \xi \omega \right\} \cap Q_\rho(\xi \omega)|}{|Q_\rho(\xi \omega)|} \leq \nu_-,
\]

where \(\nu_-\) is the number claimed by Lemma 6.1 for such choice of parameters. The Lemma then implies

\[
\text{ess inf}_{Q_{\frac{1}{2}}} u \geq \mu_- + \frac{1}{2} \xi \omega,
\]

and hence

\[
\text{ess osc}_{Q_{\frac{1}{2}}(\xi \omega)} u \leq \eta \omega, \quad \text{where} \quad \eta = 1 - \frac{1}{2} \xi \in (0, 1).
\]
Setting $\rho_1 = \frac{1}{2} \xi \omega \rho$ gives
\[
\omega_{\rho_1} = \text{ess osc}_{Q_{\rho_1}} u \leq \eta \omega.
\]
Repeat now the same argument starting from the cylinder $Q_{\rho_1}$, and proceed recursively to generate a decreasing sequence of radii $\{\rho_n\} \to 0$ such that
\[
\omega_0 \leq \text{ess osc}_{Q_{\rho_n}} u \leq \eta^n \omega \quad \text{for all } n \in \mathbb{N}.
\]

**A Proof of Proposition 3.1**

The proof uses an approximation procedure of [2]. Observe first that the assumption $u_t \in L^1_{\text{loc}}(E_T)$ permits to cast (1.1) in the form
\[
\|Du(t)(E)\| \leq \|D(u + \varphi(t))(E)\| + \int_E u_t \varphi \, dx \tag{A.1}
\]
for a.e. $t \in (0, T)$ and for all
\[
\varphi \in BV_{\text{loc}}(E) \cap L^\infty_{\text{loc}}(E) \quad \text{with } \text{supp}\{\varphi\} \subset E.
\]
We only prove the estimate for $(u - k)_+$, the one for $(u - k)_-$ is similar. Fix a cylinder
\[
[(x_o, t_o) + Q_{\rho}(\Theta)] \subset E_T.
\]
Up to a translation, assume that $(x_o, t_o) = (0, 0)$ and fix a time $t \in (-\theta \rho, 0)$ for which
\[
\int_{B_\rho} |u_t(x, t)| \, dx < \infty \quad \text{and} \quad u(\cdot, t) \in BV(E) \cap L^\infty(B_\rho).
\]
The next approximation procedure is carried out for such $t$ fixed and we write $u(\cdot, t) = u$. By [8, Theorem 1.17], there exists $\{u_j\} \subset \mathcal{C}^\infty_0(B_\rho)$ such that
\[
\lim_{j \to \infty} \int_{B_\rho} |u_j - u| \, dx = 0 \quad \text{and} \quad \|Du\|(E) = \lim_{j \to \infty} \int_E |Du_j| \, dx. \tag{A.3}
\]
Test (A.1) with $\varphi = -\zeta(u - k)_+$, where $\zeta \in \mathcal{C}(Q_{\rho}(\Theta))$. This is an admissible choice, since $u \in BV(E) \cap L^\infty(B_\rho)$. Set $\varphi_j = -\zeta(u_j - k)_+$ for $j \in \mathbb{N}$. For a given $\epsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that
\[
\int_E |Du_j| \, dx < \|Du(\cdot, t)\|(E) + \frac{1}{2} \epsilon \quad \text{for all } j \geq j_0.
\]
Here we have used the second equality of (A.3). By the first, $\{(u_j + \varphi_j)\} \to (u + \varphi)$ in $L^1(E)$. Therefore, for any $\psi \in \mathcal{C}^1_0(E)^N$ with $\|\psi\| \leq 1$,
\[
\int_E (u + \varphi) \, \text{div} \, \psi \, dx = \lim_{j \to \infty} \int_E (u_j + \varphi_j) \, \text{div} \, \psi \, dx \leq \liminf_{j \to \infty} \int_E |D(u_j + \varphi_j)| \, dx.
\]
Taking the supremum over all such $\psi$ gives
\[
\|D(u + \varphi)(t)(E)\| \leq \liminf_{j \to \infty} \int_E |D(u_j + \varphi_j)| \, dx.
\]
Therefore, up to redefining \( j_0 \) we may also assume that
\[
\int_E |D(u_j + \varphi)| \, dx \geq \|D(u + \varphi)(E) - \frac{1}{2} \varepsilon \| \text{ for all } j \geq j_0.
\]
Combining the preceding inequalities gives that
\[
\int_E |Du| \, dx < \|D(\cdot, \cdot)(E) + \frac{1}{2} \varepsilon
\]
\[
\leq \|D(u + \varphi)(\cdot, \cdot)(E) + \int_E u_t(\cdot, t) \varphi \, dx + \frac{1}{2} \varepsilon
\]
\[
\leq \int_E |Du_j + \varphi| \, dx + \int_E u_t(\cdot, t) \varphi \, dx + \varepsilon
\]
for all \( j \geq j_0 \). Next, we estimate the first integral on the right-hand side as follows:
\[
\int_E |Du_j + \varphi| \, dx = \int_E |Du_j - \zeta(u_j - k)_+| \, dx
\]
\[
\leq \int_E |Du_j - \zeta(u_j - k)_+| \, dx + \int_E |D\zeta(u_j - k)_+| \, dx
\]
\[
\leq \int_E (1 - \zeta)|Du_j| + \zeta|Du_j - D(u_j - k)_+| \, dx + \int_E |D\zeta(u_j - k)_+| \, dx.
\]
Put this in (A.4), and absorb the first integral on the right-hand side into the left-hand side to obtain
\[
\int_E \zeta|Du_j - k)_+| \, dx \leq \int_E \zeta|Du_j| - |Du_j - D(u_j - k)_+| \, dx
\]
\[
\leq \int_E |D\zeta(u_j - k)_+| \, dx + \int_E u_t(\cdot, t) \varphi \, dx + \varepsilon.
\]
From this, we have
\[
\int_E |D(\zeta(u_j - k)_+)| \, dx \leq 2 \int_E |D\zeta(u_j - k)_+| \, dx + \int_E u_t(\cdot, t) \varphi \, dx + \varepsilon.
\]
Next let \( j \to \infty \), using the lower semicontinuity of the total variation with respect to \( L^1 \)-convergence. This gives
\[
\|D(\zeta(u - k)_+)(B_\rho)\| \leq \liminf_{j \to \infty} \int_E |D(\zeta(u_j - k)_+)| \, dx
\]
\[
\leq \lim_{j \to \infty} 2 \int_E |D\zeta(u_j - k)_+| \, dx + \int_E u_t(\cdot, \cdot) \varphi \, dx + \varepsilon
\]
\[
= 2 \int_E |D\zeta(u - k)_+| \, dx + \int_E u_t(\cdot, \cdot) \varphi \, dx + \varepsilon.
\]
Finally, let \( \varepsilon \to 0 \) and use the definition of \( \varphi \) to get
\[
\|D(\zeta(u - k)_+)(B_\rho)\| \leq 2 \int_{B_\rho} |D\zeta(u - k)_+| \, dx - \int_{B_\rho} \zeta_t(u - k)_+ \, dx.
\]
To conclude the proof, integrate in \( dt \) over \((-\theta \rho, 0)\). \( \square \)
B Boundedness of minimizers

Proposition B.1. Let $u: E_T \to \mathbb{R}$ be a parabolic minimizer of the total variation flow in the sense of (1.1). Furthermore, assume that $u \in L^r_{\text{loc}}(E_T)$ for some $r > N$, and that it can be constructed as the limit in $L^r_{\text{loc}}(E_T)$ of a sequence of parabolic minimizers satisfying (1.2). Then, there exists a positive constant $\gamma$ depending only upon $N, \gamma, r$ such that

$$
\sup_{B_r(y) \times [s, t]} u_s \leq \gamma \left( \frac{\rho_0}{t-s} \right)^{\frac{1}{2n}} \left( \frac{1}{\rho_N(t-s)} \int_{2s-t}^{t} \int_{B_{q_0} (y)} u^r \, dx \, d\tau \right)^{-\frac{1}{2n}} + \frac{\gamma (t-s)}{\rho} \tag{B.1}
$$

for all cylinders $B_{q_0} (y) \times [s - (t-s), s + (t-s)] \subset E_T$.

The constant $\gamma(N, \gamma, r) \to \infty$ as either $r \to N$ or $r \to \infty$.

Remark B.2. The approximations to $u$ do not require to satisfy (1.2) uniformly. The latter is only needed to cast a function satisfying (1.1) into a DeGiorgi class. The proof of the proposition only uses such a membership, and turns such a qualitative, non-uniform information into the quantitative information (B.1).

Proof of Proposition B.1. Let $\{u_j\}$ be a sequence of approximating functions to $u$. Since $u_j$ satisfy (1.2), they belong to the classes $[DG](E_T; 2)$, by Proposition 3.1. It will suffice to establish (B.1) for $u_j$ for a constant $\gamma$ independent of $j$. Thus, in the calculations below we drop the suffix $j$ from $u_j$. The proof will be given for non-negative $u \in [DG]^+(E_T; 2)$, the proof for the remaining case is identical; it is very similar to the proof of [6, Proposition A.2.1]. Assume $(y, s) = (0, 0)$ and for fixed $\sigma \in (0, 1)$ and $n = 0, 1, 2, \ldots$, set

$$
\rho_n = \sigma \rho + \frac{1-\sigma}{2^n} \rho, \quad t_n = -\sigma t - \frac{1-\sigma}{2^n} t,
$$

$$
B_n = B_{\rho_n}, \quad Q_n = B_n \times (t_n, t).
$$

This is a family of nested and shrinking cylinders with common “vertex” at $(0, t)$, and by construction

$$
Q_0 = B_\rho \times (-t, t) \quad \text{and} \quad Q_\infty = B_{\rho_0} \times (-\sigma t, t).
$$

We have assumed that $u$ can be constructed as the limit in $L^r_{\text{loc}}(E_T)$ of a sequence of bounded parabolic minimizers. By working with such approximations, we may assume that $u$ is qualitatively locally bounded. Therefore, set

$$
M = \text{ess sup}_{Q_n} \max_{Q_n} u, \quad M_\infty = \text{ess sup}_{Q_\infty} \max_{Q_\infty} u.
$$

We first find a relationship between $M$ and $M_\infty$. Denote by $\zeta$ a non-negative, piecewise smooth cutoff function in $Q_n$ that equals one on $Q_{n+1}$ and has the form $\zeta(x, t) = \zeta_1(x) \zeta_2(t)$, where

$$
\zeta_1 = \begin{cases} 
1 & \text{in } B_{n+1}, \\
0 & \text{in } \mathbb{R}^N - B_{n},
\end{cases} \quad |D\zeta_1| \leq \frac{2^{n+1}}{(1-\sigma)\rho},
$$

$$
\zeta_2 = \begin{cases} 
0 & \text{for } t \leq t_n, \\
1 & \text{for } t \geq t_{n+1},
\end{cases} \quad 0 \leq \zeta_2 \leq \frac{2^n}{(1-\sigma)t}.
$$

We introduce the increasing sequence of levels $k_n = k - 2^{-n}k$, where $k > 0$ is to be chosen, and in (3.2), take such a test function, to get

$$
\sup_{t_n \leq \tau \leq t} \int_{B_n} \left| (u - k_{n+1})_+ \right|^2 \, dx + \frac{1}{t_n} \int_{t_n}^{t} \|D[(u - k_{n+1})_+] \zeta(\cdot, \tau)\| (B_n) \, d\tau 
\leq \frac{\gamma 2^n}{(1-\sigma)\rho} \int_{Q_n} \left| (u - k_{n+1})_+ \right|^2 \, dx + \frac{\gamma 2^n}{(1-\sigma)t} \int_{Q_n} (u - k_{n+1})_+^2 \, dx \, d\tau.
$$
Estimate
\[
\int_{Q_n}(u - k_n)^+ \, dx \, dt \leq \gamma \frac{n+1}{k^r-1} \int_{Q_n}(u - k_n)^+ \, dx \, dt,
\]
\[
\int_{Q_n}(u - k_n)^2 \, dx \, dt \leq \gamma \frac{n+1}{k^r-2} \int_{Q_n}(u - k_n)^+ \, dx \, dt.
\]
Taking these estimates into account yields
\[
\sup_{t_n \leq t \leq T} \left( \int_{Q_n}(u - k_n)^+, \xi \right)^2 \, dx \, dt + \int_{t_n}^T \|D((u - k_n)^+, \xi)(\cdot, \tau)\|_{(B_n)}^2 \, d\tau 
\leq \gamma \frac{n+1}{k^r-1} \int_{Q_n}(u - k_n)^+ \, dx \, dt.
\]
Assuming that \( k > \frac{t}{\rho} \), this implies
\[
\sup_{t_n \leq t \leq T} \left( \int_{Q_n}(u - k_n)^+, \xi \right)^2 \, dx \, dt + \int_{t_n}^T \|D((u - k_n)^+, \xi)(\cdot, \tau)\|_{(B_n)}^2 \, d\tau 
\leq \gamma \frac{n+1}{k^r-1} \int_{Q_n}(u - k_n)^+ \, dx \, dt.
\]
Set
\[
Y_n = \frac{1}{|Q_n|} \int_{Q_n}(u - k_n)^+ \, dx \, dt,
\]
and estimate
\[
Y_n+1 \leq \|u\|_{L^q(\Omega, \partial \Omega)} \left( \frac{1}{|Q_n|} \int_{Q_n}(u - k_n)^+ \, dx \, dt \right),
\]
where \( q := \frac{n+2}{N} \). Applying [6, the embedding Proposition 4.1 in Preliminaries], the previous inequality can be rewritten as
\[
Y_n+1 \leq \|u\|_{L^q(\Omega, \partial \Omega)} \left( \frac{b^n}{(1 - \alpha)^{2(N+1)}} \frac{1}{k^{(r-2)\frac{n+2}{N}}} \int_{Q_n}(u - k_n)^+/k \right),
\]
where \( b = 2^{\frac{n+1}{2}} \). Apply [6, Lemma 5.1 in Preliminaries], and conclude that \( Y_n \to 0 \) as \( n \to +\infty \), provided \( k \) is chosen to satisfy
\[
Y_\alpha = \int_{Q_n} u^+ \, dx \, dt = \frac{N+1}{(1 - \alpha)^{N+1}} \|u\|_{L^q(\Omega, \partial \Omega)} \left( \frac{t}{\rho} \right)^N \int_{Q_n}(u - k_n)^+ \, dx \, dt \leq \gamma \nu \frac{n+2}{N} \left( \frac{t}{\rho} \right)^N.
\]
which yields
\[
M_\alpha \leq \gamma \frac{n+2}{(1 - \alpha)^{N+1}} \left( \frac{t}{\rho} \right)^{N+1} \int_{Q_n}(u - k_n)^+ \, dx \, dt.
\]
The proof is concluded by [6, the interpolation Lemma 5.2 in Preliminaries].

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