The Stoner-Wohlfarth model of Ferromagnetism: Static properties

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Recent advances in high-density magnetic storage and spin electronics are based on the combined use of magnetic materials with conventional microelectronic materials (metals, insulators and semiconductors). The unit of information (bit) is stored as a magnetization state in some ferromagnetic material (FM) and controlled with an external field altering the magnetization state. As device size is shrinking steadily toward the nanometer and the need to increase the processing bandwidth prevails, racing toward higher frequencies is getting even more challenging. In magnetic systems, denser storage leads to finer magnetic grains and smaller size leads to single magnetic domain physics. The Stoner-Wohlfarth model is the simplest model that describes adequately the physics of fine magnetic grains containing single domains and where magnetization state changes by rotation or switching (abrupt reversal). The SW model is reviewed and discussed with its consequences and potential applications in the physics of magnetism and spin electronics.

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I. INTRODUCTION

Moore’s law of Microelectronics (currently still valid since more than 40 years) states that one should expect a doubling of CPU performance every 18 to 24 months. While high-density magnetic storage is advancing (over the long term) at a rate similar to Moore’s law, rates surpassing Moore’s law were recently observed. When bit density gets larger, bit length that scales as $\sim 1/\sqrt{d}$ gets smaller at a rate that recently reached almost the double value of the microelectronics feature size rate. In the latter case, a bit is stored as an electronic charge as in Flash memory, flip-flops, registers and cache memory (the charge can also be periodically refreshed as in capacitor based random access memories D-RAM), while in a mass storage magnetic medium (Disk, floppy, tape...) a bit corresponds to a well defined orientation of magnetization and is usually stored within many ferromagnetic grains possessing an average orientation. When the typical bit size shrinks, we get closer to the limit where one magnetic grain is able of holding a single bit of information.

This is the realm of nanometer electronics (nanoelectronics) and spin electronics (spintronics). Spintronics is a new field of electronics with the coexistence of classical materials used in standard electronics (metals, semiconductors and insulators) and magnetic materials (ferromagnetic, anti-ferromagnetic, ferrimagnetic, paramagnetic, diamagnetic etc...). The goal of spintronics is to make novel devices that are controlled by the simultaneous actions of electric and magnetic fields. A spintronic device has an additional degree of freedom with respect to a standard device controlled by charge only; it has also a polarisation (or a magnetic state: up or down in the vertical technology and right left in the horizontal case).

For instance in an ordinary p-n junction diode, the total current is obtained from electronic $n$ and hole $p$ charge densities, whereas in a spin polarised diode, the current is composed of $n_\uparrow$, $n_\downarrow$ and $p_\uparrow$, $p_\downarrow$, where spin degrees of freedom $\uparrow$, $\downarrow$ play a role in addition to charge.

A magnetic memory such as an MRAM retains any information stored (being of magnetic type) even when the system is powered down (or hangs up) in contrast to electronic RAM’s and registers (since information is of electronic type). That means, in a PC containing MRAM’s the operating system is loaded once for all (at first boot) and if the system hangs or is powered down, all temporary information is preserved. Additionally the magnetic polarization degree of freedom can lead to new devices of interest in quantum computing, communication and storage devices, so called quantum information devices.

The Stoner-Wohlfarth (SW) model of Ferromagnetism is the simplest model that is adequate to describe the physics of tiny magnetic grains containing single magnetic domains. It can be considered as a sort of Hydrogen model of Ferromagnetism. The physics of the SW model is built on a series of assumptions that ought to be placed into perspective in order to highlight and understand the progress and insight in magnetism and magnetic materials.

This review is made of two parts: the first tackles the static properties and the second the dynamic and statistical properties (dynamics of the magnetization reversal under either temperature or a time dependent magnetic field).

This part is organised as follows: section 2 describes the basics of the Stoner-Wohlfarth model. Section 3 describes the hysteresis curves associated with the model. Section 4 details the energetics of the SW model (nature of the energy...
barrier to cross when there is a change in magnetization state). Finally, section 5 discusses some of the limitations of the model.

II. THE STONER-WOHLFARTH MODEL

A FM used typically in recording media (hard disks, floppies, tapes . . . ) can be considered as made of a large number of interacting magnetic moments (on the order of \( N_A \) the Avogadro number for a mole of material). When a magnetic field is applied to a FM, a magnetization change takes place. One way to understand the underlying phenomena is to plot the value of the magnetization \( M \) projected along the direction of the applied field \( H \). The locus of the magnetization \( M \) measured along the direction of the applied magnetic field \( M(H) \) versus the field and depicted in the \( M - H \) plane is the hysteresis loop (see fig. 1). The term hysteresis (delay in Greek) means that when the material is field cycled (i.e. the field \( H \) is increased then decreased) two different non-overlapping curves (“ascending and descending branches”) \( M(H) \) are obtained.

The main characteristics of the hysteresis loop are the saturation magnetization \( M_s \) (saturation is attained when all the magnetic moments are aligned along some common direction resulting in the largest value of the magnetization), the remanent magnetization \( M_r \) (leftover magnetization, that is magnetization when the field \( H = 0 \) and the coercive field \( H_c \) (at which \( M = 0 \)) and the anisotropy field \( H_K \). The hysteresis loop might be viewed as a sort of magnetic \( I - V \) characteristic (\( I \) corresponds to \( M \) and \( V \) to \( H \)). The characteristic is non-linear and the output \( M \) is delayed with respect to input \( H \). The input-output delay is proportional to the width of the loop. The ratio \( M_r/M_s \) called squareness is close to 1 when the applied magnetic field is close to some orientation defined as the easy axis (EA) and the hysteresis loop is closest to a square shape. Once the EA is determined, the angle the magnetic field makes with the EA (say \( \phi \)) is varied and the hysteresis loop is graphed for different angles (see fig. 2). When the angle \( \phi \) is increased the opening of the hysteresis loop is reduced; it is largest when the magnetic field is most parallel to the EA and smallest when the magnetic field is most parallel to the so called hard axis (in simple systems the hard axis is perpendicular to the EA). As shown in fig. 1 most characteristics of the hysteresis loop are depicted and for a given temperature and frequency of the applied field \( H \), quantities such as the remanent magnetization \( M_r \), and the coercive field \( H_c \) depend on the angle \( \phi \).

When temperature or field frequency are varied, the hysteresis loop shape changes and may even be seriously altered by the frequency of the field. The hysteresis loop branches might collapse altogether over a single curve above a given temperature (Curie temperature), the material becoming paramagnetic.

The SW model (also called coherent rotation [8]) considers a FM as represented by a single magnetic moment (thus the name coherent as in optics for fixed phase relationship or as in a superconductor where a single wavefunction represents all electrons in the material). The material is therefore considered as a single magnetic domain, thus all domain related effects or inhomogeneities are not considered. A single domain occurs when the size of the grain is smaller than the exchange length (given by the size of a domain wall, see subsection II-A and Appendix) and contains about \( 10^{12}-10^{18} \) atoms. At \( T = 0K \) a grain carrying a single moment \( M \), is an ellipsoid-shaped object (see fig. 2) since a material with uniform magnetization ought to have an ellipsoid form [1].

The grain possesses a uniaxial anisotropy (meaning an axis along which the magnetization prefers to lie in order to minimize the energy) and is subjected to an externally applied static magnetic field \( H \). \( M \) can evolve strictly in a two dimensional space (see fig. 2), therefore it is characterized by a single angle \( \theta \), the angle \( M \) makes with the anisotropy axis (also the EA, in this case [8]). \( \phi \) is the angle the external applied magnetic field (taken as the \( z \)-axis) makes with the EA.

Thus, the moment \( M \) is subjected to two competing alignment forces: one is due to a uniaxial anisotropy characterised by \( K \) favoring some direction (see fig. 2) and the other is due to an external magnetic field \( H \). Therefore, the total energy is the anisotropy energy \( E_A \) and the Zeeman energy \( E_Z = -M \cdot H \). At \( T = 0K \), the energy is then:

\[
E = E_A + E_Z = K \sin^2 \theta - HM_s \cos(\theta - \phi)
\]

The anisotropy energy \( K \sin^2 \theta \) is minimum \((=0)\) when \( \theta = 0 \) for \( K > 0 \) (see note [6]). The moment will select a direction such that the total energy \( E_A + E_Z \) is minimized. The orientation change might occur smoothly (rotation) or suddenly (switching) implying that the magnetization is discontinuous at some value of the magnetic field \( H \).

A FM of a finite size (like the ellipsoidal grain) magnetized uniformly (the magnetization \( M \) is represented by its components \( M_{ij} \)) contains a magnetic energy (called also magnetostatic energy) given by \( 2\pi N_{ij} M_i M_j \) (Einstein summation is considered). The \( N_{ij} \) coefficients are the demagnetization coefficients of the body determined by its shape.
The origin of the terminology is due to the resemblance to the familiar anisotropy energy of the form $K_{ij}M_iM_j/M_j^2$. The coefficients depend on the geometry of the material. For simple symmetric geometries, one has three positive coefficients along three directions $N_{xx}, N_{yy}$ and $N_{zz}$ (the off-diagonal terms are all 0). This is the case of wires, disks, thin films and spheres. All three coefficients are positive, smaller than 1 and their sum is equal to 1. For a sphere, all three coefficients are equal to $\frac{1}{3}$. For a disk they are given by 0, 0, 1 if the z axis is perpendicular the disk lying in the $xy$ plane. For an infinite length cylindrical wire with its axis lying along the $z$ direction, the values are $\frac{1}{3}, \frac{1}{3}, 0$.

A FM (assumed to possess a single uniform magnetization $M$ even though it does not have an ellipsoid shape) cut in the form of a cylinder, disk or a thin film possessing a uniaxial anisotropy can be considered as having the anisotropy energy $K \sin^2 \theta$ with $K > 0$ (see [3]). If we add to anisotropy the demagnetization energy $2\pi N_{\alpha\beta}M_{\alpha}M_{\beta}$ (due to the body finite size along one or several directions), we get a competition between the two energies:

$$K \sin^2 \theta + 2\pi (N_{\perp} M_{\perp}^2 + N_{\perp} M_{\perp}^2 + N_{\parallel} M_{\parallel}^2) = [K + 2\pi M_{\parallel}^2 (N_{\perp} - N_{\parallel})] \sin^2 \theta + \text{ const.}$$

The indices $||, \perp$ of the demagnetization coefficients denote respectively parallel or perpendicular to the $z$ axis. Thus we define an effective anisotropy $K_{\text{eff}}$ as:

$$K_{\text{eff}} = [K + 2\pi M_{\parallel}^2 (N_{\perp} - N_{\parallel})] \sin^2 \theta$$

The total energy of the SW model (called from now on Stoner particle) can then be written as: $E = K_{\text{eff}} \sin^2(\theta) - M_s H \cos(\theta - \phi)$.

At equilibrium, the magnetization points along a direction defined by an angle $\theta^*$ that minimizes the energy. The behaviour of the energy as a function of $\theta$ for a fixed angle $\phi = 30^\circ$ and for various fields $h = H/H_K$ (the anisotropy field $H_K = 2K_{\text{eff}}/M_s$) is depicted in fig. 3. One observes that despite the wide variability of the energy landscape versus $\theta$, a couple of points are not affected by $h$. In addition, the landscape becomes quite flat for some value of $h$. The minimum condition at $\theta^*$ is:

$$\left(\frac{\partial E}{\partial \theta}\right)_{\theta = \theta^*} = 0 \quad \text{and} \quad \left(\frac{\partial^2 E}{\partial \theta^2}\right)_{\theta = \theta^*} > 0 \quad (1)$$

Normalising the magnetisation by its saturation value, such that $m = M/M_s$, yields:

$$[\sin(\theta) \cos(\theta) + h \sin(\theta - \phi)]_{\theta = \theta^*} = 0 \quad \text{and} \quad [\cos(2\theta) + h \cos(\theta - \phi)]_{\theta = \theta^*} \geq 0 \quad (2)$$

For general $\phi$, the above equations cannot be solved analytically, except for $\phi = 0, \pi/4, \pi/2$. In order to present the possible solutions, we define two components of the magnetization:

1. The longitudinal magnetization i.e. the projection of $M$ along $H$, $m_{||} = \cos(\theta - \phi)$.
2. The transverse magnetization i.e. the projection of $M$ perpendicularly to $H$, $m_{\perp} = \sin(\theta - \phi)$.

Let us find the minimum analytically for the cases: $\phi = 0$ and $\pi/2$.

1. $\phi = 0$: eqs. 2 give the solution:

$$\theta^* = \cos^{-1}(-h) \quad \text{when} \quad h \leq 1 \quad \text{otherwise} \quad \theta^* = 0, \pi \quad \text{yielding the square hysteresis loop in fig. 4 and the line} \quad m_{\perp} = 0$$

in fig. 5.

2. $\phi = \pi/2$: eqs. 2 give:

$$\theta^* = \sin^{-1}(h) \quad \text{when} \quad h \leq 1 \quad \text{otherwise} \quad \theta^* = \pi/2 \quad \text{yielding the main diagonal} \quad (m_{||} = h) \quad \text{hysteresis loop in fig. 4 and the circle} \quad m_{||} = \pm \sqrt{1 - h^2} \quad \text{in fig. 5}$$

Remarkably, both types of hysteresis curves (depicted in fig. 4 and fig. 5) exist and are encountered in many physical systems.

The questions that arise are then:

- What are the conditions for expecting a Stoner particle behaviour?
- Is there some characteristic size below which a single domain is expected?

Attempt at answering the above are discussed next.
A. Stoner particle and critical radius

When a magnetic medium is made of non-interacting grains, there is possibility for observing Stoner particle behaviour (single domain) when the typical size of the grain is below some length scale termed "critical radius $R_c$" of the grain. A simple argument given in Landau-Lifshitz Electrodynamics of Continuous Media [1] is based on the following: When the demagnetization energy $2\pi N c M_i M_j \sim 2\pi N c M^2$ (where $N c$ is the demagnetization coefficient along some preferred axis, usually the long one in an ellipsoid-shaped grain) is equal to the exchange energy $A_{ij} \frac{\partial M_i}{\partial x_j} \sim A_{ij}^2$ ($A_{ij}, i, j, k = 1, 2, 3$ is the exchange stiffness constant along $i, j$ directions).

Considering that $A_{ij} \sim A$ a typical exchange stiffness constant (regardless of $i, j$) results in $R_c \sim \sqrt{\frac{\pi N c}{2 \pi N c}}$. Exchange energy is the largest contribution to non-uniformity energy due to spatial variation of the magnetization $M$. When the change in the direction of $M$ occurs over distances that are large compared to interatomic distances, non-uniformity energy can be expressed with derivatives of $M$ with respect to spatial coordinates (see Landau-Lifshitz [1] and Brown [2]). Exchange stiffness constant $A_{ij}$ is on the order of Heisenberg exchange energy per unit length $J/a_1$ ($a_1$ is the average nearest neighbour distance within the grain). Typically $J \sim 10$ meV and $a \sim 1\AA$, hence we get $A_{ij} \sim 10^{-6}$ erg/cm (see fig. 3). In a more elaborate fashion, Frei et al. [12] define $R_c$ from a minimization of the energy using Euler variational equations obtaining the equation:

$$R_c = \sqrt{\frac{3A}{2\pi N c M^2} [\ln(\frac{4R_c}{a_1}) - 1]} = 0$$

(3)

Solving eq. 3 for $R_c$, in the case of standard ferromagnetic transition metals (Fe, Ni and Co), fig. 6 gives the variation of $R_c$ with $N c$ (the long axis demagnetization coefficient). From the figure, we infer that for elongated grains made of Iron, Nickel or Cobalt, $R_c$ is within a few 100 nm range.

III. HYSTERESIS IN THE STONER-WOHLFARTH MODEL

Applying a magnetic field $H$ to a FM and measuring a resulting magnetization $M$ as a response can be considered as a classical signal input-output problem. The input-output characteristic $M(H)$ is that of a peculiar non-linear filter except at very low fields where $M$ is simply proportional to $H$. A simple illustration of non-linearity is to observe the output as a square signal whereas the input is a sinusoidal excitation (see ref. [9]). In addition, the material imposes a propagation delay to the signal proportional to the width of the hysteresis loop (twice the coercive field).

Hysteretic behaviour is generally exploited in control systems, for instance, because different values of the output are required as the input excitation is varied in an increasing or decreasing fashion.

We consider two types of hysteresis curves, as mentioned in the previous section:

1. A longitudinal hysteresis curve with the reduced magnetization $m_{||}$ taken along the direction of the applied external magnetic field (see fig. 2).

2. A transverse hysteresis curve with the reduced magnetization $m_{\perp}$ taken along the direction perpendicular to the applied external field (see fig. 3).

As an illustration, the hysteresis loop depicted in fig. 4 is found by calculating the component of $M$ along $H$ from the set of $\theta$ angles at a given angle $\phi$, that minimize the energy $E$ (conditions given in eq. 2). The anisotropy field $H_K$ is obtained from the slope break of the hysteresis loop when $\phi = \pi/2$, that is when $H$ is along the hard axis. In this simple model, the coercive field $H_c$ (at $\phi=0$) (for which $M = 0$) is found as $H_c = H_K$. The loop is found to be broadest when $\phi=0$, and it gets thinner as $\phi$ is increased to collapse into a simple line (for $\phi = \pi/2$). That line breaks its slope in order to reach saturation behaviour ($M = \pm M_s$) for $H = \pm H_K$.

The question of the occurrence of hysteresis is addressed next.

A. Magnetization reversal in the Stoner-Wohlfarth model

Hysteresis boundaries versus applied field are determined from the simultaneous nulling of the first and second derivative of the energy (that refers to the observed flatness of the energy landscape versus $\theta$ as discussed previously). Thus one obtains the astroid equation (see fig. 1):
The fields $H_{\perp}$ and $H_{\parallel}$ are the components of the field $H$ along the hard and easy axes. The critical field (equal in this case to the coercive field at $\phi=0$) for which $M$ jumps (at a given orientation of the field) is obtained from the conditions:

\[
\left( \frac{\partial E}{\partial \theta} \right)_{\phi} = 0 \quad \text{and} \quad \left( \frac{\partial^2 E}{\partial \theta^2} \right)_{\phi} = 0 \tag{4}
\]

as:

\[
H_s(\phi) = \frac{H_K}{\left[ \sin^{2/3} \phi + \cos^{2/3} \phi \right]^{3/2}}
\]

In spite of the tremendous simplifying assumptions of the SW model and the fact several derived quantities appear to be equal (e.g. the coercive field at $\phi=0$ and the anisotropy field $H_K$), it is extremely helpful since it captures, in many cases, the essential physics of the problem; in addition, many quantities of interest can be derived analytically (see ref. [4]).

The critical field for which the magnetization value jumps from one energy minimum to another equivalent to the first is denoted $H_s$ (the $s$ index refers to switching emphasising the need to distinguish $H_s$ from the coercive field $H_c$).

Depending on the shape of the hysteresis loop, there are two cases to consider:

1. Large slope case: $H_s$ may be considered as an external magnetic field for which the absolute derivative $|dM/dH|$ diverges or is very large, meaning that we have $|dM/dH|_{H=H_s} \to \infty$.

2. Small slope case: $H_s$ is found below from an energy equality condition: $E(M_1)|_{H=H_s} = E(M_2)|_{H=H_s}$ where $E(M_1)$ (resp. $E(M_2)$) is the energy with $M_{1,2}$ the magnetization in the first minimum state (resp. in the second state). The above conditions insure the magnetization jumps from one minimum to another.

One might use the standard energy derivative (first and second) conditions as given above in eq. 4. In fig. 8 the critical field $H_s$ versus angle $\phi$ is displayed showing the minimum field to reverse the magnetization, being half the anisotropy field, must be applied with an angle of 135° ($90^\circ + 45^\circ$) with respect to the EA.

The critical magnetization is the magnetization at $H_s$ and can be evaluated from the critical angle:

\[
\theta_c = \phi + \tan^{-1} [\tan(\phi)]^{1/3}.
\]

We use this value to follow the variation of the longitudinal $m_{\parallel,c} = \cos(\theta_c - \phi)$ and transverse magnetization $m_{\perp,c} = \sin(\theta_c - \phi)$ as functions of the critical field as displayed in fig. 9.

IV. BARRIER HEIGHT AND ITS DEPENDENCE ON APPLIED FIELD

It is interesting to examine the behaviour of the energy barrier separating two magnetization states with an applied field. A deep understanding of the nature of the barrier, its characteristics and how it is altered by material composition, field or temperature will help us control and finely tune the behaviour of magnetic states inside a FM. Some of the questions one might ask are the following:

1. What are the exact characteristics of the energy barrier $\Delta E$?

2. Does $\Delta E$ vary with the nature and shape of the magnetic grain?

3. How does $\Delta E$ vary with an external applied field?

4. If there is some interaction between grains, how does it affect the barrier height?

First of all, the energy barrier is defined as the minimum energy separating two neighbouring energy minima (one of them being a local minimum). The applied field modifies the shape of the energy barrier as depicted in fig. 8 leading to consider at least two field choices ($H_K$ or $H_s$), i.e. $\Delta E = (1 - H/H_K)^\beta$ or $\Delta E = (1 - H/H_s)^\beta$, where $\Delta E$ is normalised by the effective anisotropy constant. The behaviour of the barriers are depicted in fig. 10 and fig. 11. The often quoted exponent $\beta = 2$ is valid only when $\phi = 0$, or $\pi/2$. Moreover we observe that when we normalise the
applied field with respect to $H_K$ (fig. [10]) the curvature is opposite to what is observed in the $H_s$ normalisation case (fig. [11]). Again the often cited exponent 2 is valid only when $\phi = 0$, or $\pi/2$ exactly as in the $H_K$ normalisation case. A popular analytic approximation for the barrier is the Pfeiffer approximation given by the formula: $\Delta E = (1 - H/H_s(\phi))^{[0.86+1.14H_s(\phi)]}$ yielding an exponent $\beta_0 = 0.86 + 1.14H_s(\phi)$. Fig. [11] indicates that this approximation is not so bad when compared to our exact numerical calculation.

When an assembly of interacting grains are considered, it is possible to recast the barrier formula in a form similar to the non-interacting Stoner particle case, however, the nature of the interaction between the grains and the way it is accounted for will determine the value of the exponent.

V. LIMITATIONS OF THE STONER-WOHLFARTH MODEL

The SW model is a macrospin approach to magnetic systems (like "coarse-graining" in Statistical physics, or Ehrenfest approach in Quantum systems) because of the complexity of a direct microscopic (nanoscopic being more appropriate) description.

Its main concern is single domain physics despite the fact magnetic materials, in general, possess a multi-domain structure. Regardless of this consideration, several limitations are already built into the SW model. One limitation is the crossover problem indicating that hysteresis branches may cross for certain values of the angle $\phi$.

The minimum energy equation may be cast in the form: $\sin(\theta) \cos(\theta) + h \sin(\theta - \phi) = 0$ through the replacement: $m = \cos(\theta - \phi)$ obtaining the expressions for the upper and lower branches:

$$h_1 = -m \cos(2\phi) + \frac{(2m^2-1)}{2\sqrt{1-m^2}} \sin(2\phi)$$

$$h_1 = -m \cos(2\phi) - \frac{(2m^2-1)}{2\sqrt{1-m^2}} \sin(2\phi)$$

A crossing (crossover) between the two branches occurs at $h_x$ when $h_1 = h_1$. Solving this equation, we obtain $m = \pm 1/\sqrt{2}$ yielding $h_x = -\cos(2\phi)/\sqrt{2}$. The crossing angle is defined as the angle for which the field $h_x$ is equal to the critical field, that is for an angle $\phi_x$ satisfying:

$$\cos 2\phi_x = -\frac{\sqrt{2}}{\left[\sin^{2/3}(\phi_x) + \cos^{2/3}(\phi_x)\right]^{3/2}}$$

Writing $u = \cos^2 \phi_x$ we transform this equation into: $(1 - u)^{1/3} + u^{1/3} = \frac{2^{1/3}}{(1-2u)^{1/3}}$. Using the transformation, $u = \frac{1}{4}(1 - \frac{2}{\sqrt{v}})$ we finally obtain the equation in $v$ as:

$$[1 + \frac{2}{\sqrt{v}^2}]^{1/3} + [1 - \frac{2}{\sqrt{v}^2}]^{1/3} = \frac{1}{\sqrt{v}}, \ v > 0$$

Remarkably, this transcendental equation has a unique integer solution $v = 5$.

Hence the crossover angle $\phi_x = \frac{1}{2} \cos^{-1}(\frac{-2}{\sqrt{5}}) \sim 76.72^\circ$.

For large $\phi$ angles (typically $> 76^\circ$) the hysteresis branches cross as seen in fig. [12] for the particular case $\phi = 85^\circ$.

The branch crossing problem has been observed in the SW work but no remedy was offered. One may slightly modify the expression of the branches for large $\phi$ angles as done in the work of Stancu and Chiorescu [13].

Another limitation is the ellipsoidal form of the grain used to simplify demagnetization fields.

The uniaxial anisotropy of the grain appeals to materials such as Cobalt, whereas other transition metal FM’s such as Iron or Nickel possess cubic anisotropy. Other types of anisotropy occur as discussed in the second part of this work.

Several other limitations such as overestimation of the coercive field, linear dependence with respect of the energy with respect to grain volume and quantum effects (see for instance the review by Awschalom and DiVincenzo [14]).

Despite all these limitations, the SW provide a correct overall picture in uniformly magnetised materials even if some inadequacies exist in explaining certain experimental details.

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[1] L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, Pergamon, Oxford, p.195 (1975).
[2] W.F. Brown Jr., Micromagnetics, Wiley Interscience Publishers, New-York (1963).
[3] M. J. Pechan, and H. Jaeger, Am. J. Phys. 71, 825 August 2003.
[4] E.C. Stoner and E.P. Wohlfarth, Phil. Tran. Roy. Soc. Lond., A240, 599 (1948).
[5] Anisotropy energy can be considered a contribution to energy given by the term $K \sin^2 \theta$ where $\theta$ is the angle, the magnetization makes with the $z$ axis. If $K > 0$ the energy minimum ($=0$) occurs when $\theta = 0$ that is alignment with the $z$ axis (EA case). Taking the $z$ axis perpendicular to a thin film or a disk plane ($xy$ plane), and considering $K < 0$, the energy minimum ($-|K|$) is obtained for $\theta = \pi/2$ favoring an in-plane magnetization (easy plane case).

TABLE I: Correspondance between magnetic Units in the MKS and CGS unit systems. Note that induction $B$ has same units as $\mu_0 M_s$ in MKS and $4\pi M_s$ in CGS. In addition, the saturation magnetization $M_s$, in CGS, is in Gauss, whereas the corresponding molar moment $N_A n_B \mu_B$ is in emu, where $N_A$ is Avogadro constant and $n_B$ is the number of Bohr magnetons implying that each atom or molecule carries a moment given by $n_B \mu_B$. It is rarely emphasized in the litterature that $M_s$ should be called saturation magnetization per unit volume. In vacuum or non-magnetic materials in CGS values in Oe and Gauss are same. Typical exchange fields are a thousand fold stronger than anisotropy's.
Associated quantities such as coercive field $H_c$, anisotropy field $H_K$, and remanent magnetization $M_r$ are shown. The thick line is the hysteresis loop when the field is along the hard axis ($\phi=90$ degrees, in this case) and $H_K$ is the field value at the slope break. Quantities such as $H_c$ and $M_r$ depend on $\phi$ whereas the saturation magnetization $M_s$ does not.

FIG. 2: Single domain grain with the magnetization $\mathbf{M}$ and the external applied magnetic field. The anisotropy of strength $K$ competes with the magnetic field $\mathbf{H}$ along the $z$ axis in the alignment of the magnetic moment that is restricted to the 2D $xOz$ plane. Note that the anisotropy energy $K_{ij}M_i M_j / M_s^2$ reduces in the uniaxial case to $K \sin^2 \theta$. It is minimum ($=0$) when $\theta=0$ and $K > 0$ (see note [6].)
FIG. 3: Variation of the energy landscape with angle θ for different values of the normalised magnetic field $h = H/H_k$. The field orientation is held fixed at the value $\phi = 30^\circ$ with the anisotropy axis.

FIG. 4: VSM (or longitudinal) hysteresis loop for various angles $\phi$ of the field $H$ with the EA. Note the square loop for $\phi = 0$ and the diagonal line $m_\parallel = h$ for $\phi = \pi/2$. 
FIG. 5: VSM-RTM (or transverse) hysteresis loop for various angles $\phi$ of the field $H$ with the EA. Note the horizontal line $m_{\perp} = 0$ for $\phi = 0$ and the circle for $\phi = \pi/2$.

FIG. 6: Critical radius (nm) for Iron, Nickel and Cobalt for a prolate ellipsoid (elongated) as a function of the demagnetization coefficient along its axis $N_c$. The room-temperature data for these curves (taken from Kittel [7]) are $M_s = 1710$ (Fe), 485 (Ni) and 1440 (Co) (in Gauss); $a_1 = 2.48$ (Fe), 2.49 (Ni) and 2.50 (Co) (in Å). The exchange stiffness constant $A$ in all cases is taken as $10^{-6}$ erg/cm.
FIG. 7: The inside the astroid domain is made of the field values for which a reversal of the magnetization is possible. Outside the astroid domain, no reversal is possible.

FIG. 8: Normalised critical field $H_s/H_K$ versus angle $\phi$ showing that the minimum field to reverse the magnetization is half the anisotropy field. In fact, this is valid in the static case only. In the dynamic case where the applied field is time dependent the minimum field can be quite smaller as discussed in the next part of the paper.
Critical magnetization and field of the Stoner–Wohlfarth model

FIG. 9: Critical longitudinal \( m_\parallel,c = \cos(\theta_c - \phi) \) and transverse magnetization \( m_\perp,c = \sin(\theta_c - \phi) \) versus critical magnetic field.

Energy barrier exponent \( \beta \) in \( \Delta E = (1 - H/H_K)^\beta \)

FIG. 10: Variation of the barrier exponent \( \beta \) with field normalised with the anisotropy field. The barrier is given by \( \Delta E = (1 - H/H_K)^\beta \).
FIG. 11: Variation of the barrier exponent $\beta_s$ with field normalised with the critical field. The barrier is given by $\Delta E = (1 - H/H_s)^{\beta_s}$. Displayed is also a comparison with the Pfeiffer approximation whose barrier expression is given by $\Delta E = (1 - H/H_s(\phi))^{0.86 + 1.14H_s(\phi)}$.

FIG. 12: Hysteresis loop showing the crossover effect for an angle $\phi=85^\circ$. 

Barrier exponent in $\Delta E = (1-H/H_s)^{\beta_s}$ and Pfeiffer model