Einstein and Planck on mass–energy equivalence in 1905–1906: a modern perspective

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Abstract

Einstein’s theoretical analysis of mass–energy equivalence, already at the time, experimentally evident in radioactive decays, in two papers published in 1905, as well as Planck’s introduction, in 1906, of the concepts of relativistic momentum, and, by invoking Hamilton’s principle, relativistic energy, are reviewed and discussed. Claims in the literature that Einstein’s analysis was flawed, lacked generality, or was not rigorous, are rebutted.

Keywords: special relativity, mass–energy equivalence, relativistic kinematics

1. Introduction

Laws of physics are expressed as mathematical equations, so in order to discover a new law the corresponding equation must first be written down. However, the meaning of the law is only made manifest when the precise connection between the symbols in the equation and measurable physical phenomena is made clear. The discoverer of the equation is therefore not necessarily the same person who elucidates the meaning of the corresponding physical law. Once the latter is established it can be expressed either as an equation, where the operational meaning of all the symbols is quite clear, or alternatively, in words, in which case it can be better understood by less mathematically literate readers. It will be shown below that in Einstein’s seminal work on mass–energy equivalence, published in 1905, both of the above possibilities for the formulation of a new physical law were realized. In fact it was already manifest in an equation in the original special relativity paper of June 1905 [1] when this equation is correctly interpreted physically, but this interpretation was not given by Einstein. In contrast, in the September 1905 paper [2], where the equivalence of mass and energy was clearly stated verbally, the corresponding equation was not written down.
In the present paper, Einstein’s September 1905 paper ‘Does the inertia of a body depend on its energy content’ [2] and Planck’s 1906 paper on relativistic kinematics [3] will be reviewed from three different perspectives: (i) Einstein’s and Planck’s original arguments; (ii) arguments Einstein or Planck could have used, given the state of knowledge in 1905–06; (iii) a review of the related literature critical of [2] which claimed that Einstein’s arguments were flawed or lacked generality or rigour. This paper is organized as follows. In the two following sections the arguments of [2] and [3], respectively, are reviewed. In section 4 equations given in Einstein’s June 1905 paper [1] are discussed in relation to the conclusions of his September 1905 paper [2] and the related work of Planck. Section 5 deals with criticisms of [2] by Planck [4], Ives [5], Ohanian [6] and Hecht [7]. Conclusions are given in section 6.

Throughout this paper the concept of ‘mass’ (denoted by \( m \)) refers only to the Lorentz-invariant quantity ‘rest mass’ that is proportional to the energy in the static limit. For a critical discussion of the concept of velocity-dependent mass see [8] and references therein. Discussion of Einstein’s work on mass–energy equivalence after 1905 is beyond the scope of the present paper. See [7, 9, 10] for references to this later work and the related literature.

2. Einstein’s September 1905 paper on mass–energy equivalence

As Einstein’s original notation is somewhat cumbersome, a more modern one, similar to that of [7], will be used for the analysis of Einstein’s gedanken experiment. A process in which a body radiates ‘plane waves of light’ of equal energy content, in opposite directions, is considered both in the frame in which the body is at rest and one in which it moves with constant speed \( v \). Since, in the rest frame of the body, no momentum is carried away by the light, then by Newton’s second law no force acts on the body in this frame. The velocity of the body, \( f \), produced by the radiation process from the body \( i \), is then the same as that of \( i \), in any frame. Einstein assumes energy conservation in both frames:

\[
E_i = E_f + E(L) = E_f + \Delta E_i, \tag{2.1}
\]

\[
E_i' = E_f' + E'(L) = E_f' + \Delta E_i', \tag{2.2}
\]

where primed quantities refer to the frame in which the objects are in motion and \( \Delta E_i, \Delta E_i' \) are the absolute values of the changes in the energy of \( i \) due to the radiation, which are equal to the energies \( E(L), E'(L) \), respectively, of the radiated light. Note that, already at this stage, Einstein has tacitly introduced in (2.1) and (2.2) the concept of the ‘rest energies’ \( E_i \) and \( E_f \) of the objects \( i \) and \( f \) as well as the ‘total energies’ \( E_i' \) and \( E_f' \) of the same objects in motion. Using the formula from §8 of Einstein’s earlier 1905 relativity paper [1] to transform the energy of a ‘light complex’ (identified with the total energy of the two ‘plane waves’ of light) between the two inertial frames gives

\[
E'(L) = \frac{E(L)}{\sqrt{1 - \beta^2}}, \tag{2.3}
\]

where \( \beta \equiv v/c \). Subtracting (2.1) from (2.2) and using (2.3) gives

\[
E_i' - E_i = (E_f' - E_f) = E(L) \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right]. \tag{2.4}
\]
The kinetic energies of the bodies $K_i, K_f$ are introduced as

$$K_i \equiv E_i - E_i - C,$$  

(2.5)

$$K_f \equiv E_f - E_f - C,$$  

(2.6)

where $C$ is a constant, assumed to be the same for the bodies $i$ and $f$. Combining (2.4), (2.5) and (2.6) and retaining only $O(\beta^2)$ terms on the right side of (2.4) gives

$$K_i - K_f = \frac{1}{2} \frac{E(L)v^2}{c^2} + O(\beta^4).$$  

(2.7)

This is the last equation in [2]. Einstein then states, à propos of this equation:1

‘From this equation it follows directly that:—

*If a body gives off the energy $E(L)$ in the form of radiation, its mass diminishes by $E(L)c^2$. The fact that the energy withdrawn from the body becomes energy of radiation evidently makes no difference so that we are lead to the more general conclusion that*

The mass of a body is a measure of its energy content; if the energy changes by $\Delta E$, the mass changes in the same sense by $\Delta E/9 	imes 10^{20}$, the energy being measured in ergs, and the mass in grammes.

It is not impossible that with bodies whose energy-content is variable to a high degree (e.g. with radium salts) the theory may be successfully put to the test.

If the theory corresponds to the facts, radiation conveys inertia between the emitting and absorbing bodies.’ (Einstein’s italics.)

The content of Einstein’s italicized statement and the following sentence, written as an equation is

$$m_i - m_f = \frac{E(L)}{c^2} \Delta E_i = \frac{\Delta E_i}{c^2}. $$  

(2.8)

The last three paragraphs of [2], quoted above, show that Einstein had not only discovered equation (2.8) but also understood, very clearly, the physical significance of the equation.

In order to derive (2.8) from (2.7) it is necessary to assume the Newtonian formula for kinetic energy: $K(v) = (1/2)mv^2$. It must be assumed that Einstein did this tacitly. It may appear, because of this assumption, that the relation (2.8) is only an approximate one, valid when $v \ll c$. However, as pointed out by Stachel and Torreti [11], by defining the rest mass of an object as

$$m \equiv [\text{Lim } v \to 0] \frac{K(v)}{v^2/2} $$  

(2.9) 

and assuming that Newtonian mechanics is the correct $v \to 0$ limit of relativistic kinematics, enables (2.7) to yield the relation

$$[\text{Lim } v \to 0] \left\{ \frac{K_i}{v^2/2} - \frac{K_f}{v^2/2} \right\} = [\text{Lim } v \to 0] \left\{ \frac{\Delta E_i}{c^2} + O(\beta^2) \right\}. $$  

(2.10)

from which, on using (2.9), (2.8) immediately follows. Einstein’s final (verbally presented) equation (2.8) is therefore an exact (velocity independent) relation. Setting $m_i = 0$, $\Delta E_i = E_i$

1 The symbols used to denote energies in [2] are replaced by the corresponding symbols used in the present paper.
in (2.8) gives

\[ E_i(\beta = 0) = m_i c^2. \]  

(2.11)

This is indeed \( E_0 = mc^2 \). Einstein’s final verbal statement of the equivalence of mass and energy, expressed by equation (2.8), is a necessary consequence of his premises which are:

(A) conservation of energy, together with the tacit introduction in equations (2.1) and (2.2) of the concept of ‘total energies’ of objects either at rest or in motion, (B) the transformation law for the energy of radiation, and (C) Newtonian mechanics as the low velocity limit of relativistic mechanics. Whether a philosopher would consider this derivation ‘rigorous’ is perhaps an open question but I submit, in agreement with Stachel and Torretti [11] and Fadner [9], and contrary to Planck [3], Ives [5] and recent assertions of Hecht [7] and Ohanian [6], that most physicists would. Further discussion of this controversial point is found in section 6 below.

3. Planck’s 1906 derivation of relativistic energy and momentum

In [3] Planck unconventionally denoted the Lagrangian function by \( H \) and the Hamiltonian function by \( L \). In the present discussion, to be in accordance with modern convention, this nomenclature will be inverted, and the scaled velocity in three-vector notation \( \vec{\beta} = (1/c) \overrightarrow{d \beta / dt} \) where \( \overrightarrow{d} \) is the spatial displacement of an object, of rest mass \( m \), in motion in free space, employed throughout. The essential initial ansatz\(^2\) of [3] is essentially a relativistic statement of Newton’s second law of mechanics containing a definition of relativistic momentum:

\[ \frac{d\vec{\beta}}{dr} \equiv \frac{d}{dr} \left( \frac{mc\dot{\beta}}{\sqrt{1 - \beta^2}} \right) = \vec{F}, \]  

(3.1)

where \( \vec{F} \) is a force that Planck considered to be produced by electric and magnetic fields, although this is inessential in the subsequent derivation of the formulas for relativistic energy. From the Lagrange equation, a necessary consequence of Hamilton’s Principle [12],

\[ p = \frac{1}{c} \frac{\partial L}{\partial \dot{\beta}} = \frac{mc\dot{\beta}}{\sqrt{1 - \beta^2}}, \]  

(3.2)

and (3.1) Planck derived the Lagrangian, \( L \), for a free particle:

\[ L = -mc^2 \sqrt{1 - \beta^2} + C, \]  

(3.3)

where \( C \) is an arbitrary constant. The Hamiltonian, \( H \), is constructed from the momentum \( \vec{p} \) and the Lagrangian according to the relation [13]

\(^2\) In [3] the formula (3.1) was actually obtained by manipulation of a modified Lorentz force equation, using reasoning that is unclear to the present author.
The quantity $E$ is called by Planck ‘lebendige Kraft’. If this is interpreted as ‘kinetic energy’ then the constant $C$ in (3.4) takes the value $mc^2$ and, writing the conventional symbol $T$ for kinetic energy, (3.4) reduces to the formula

$$T = mc^2 \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right],$$

as derived in Einstein’s first special relativity paper [1]. Alternatively, choosing $C = 0$ in (3.4) gives the total relativistic energy:

$$E = \frac{mc^2}{\sqrt{1 - \beta^2}}.$$

Equations (3.5) and (3.6) show that the total and kinetic relativistic energies are related according to:

$$E = T + mc^2.$$

The famous formula $E_0 \equiv E(\beta = 0) = mc^2$ is an immediate consequence of (3.7). Planck gave at first however only the formula (3.4) and there was no discussion of the value or physical significance of the constant $C$, or of any distinction between total, kinetic and rest relativistic energies. However (3.6) can be written as

$$E = mc^2 \left[ \frac{1}{1 - \beta^2} \right] = mc^2 \left[ 1 + \frac{\beta^2}{1 - \beta^2} \right] = mc^2 \left[ 1 + \frac{p^2}{m^2c^4} \right],$$

where in the last member the definition in (3.1) of relativistic momentum, $p$, has been used. It also follows from (3.7) and (3.8) that

$$T = mc^2 \left[ 1 + \frac{p^2}{m^2c^4} \right] - mc^2.$$

Planck did give equation (3.8) (with an additional additive constant, $C_1$, on the right side) in [3]. The meaning of Planck’s quantity $L$ ($H$ in the notation of the present paper) then depends on the value of this constant $C_1$: $L \equiv E$, $C_1 = 0$ or $L \equiv T$, $C_1 = -mc^2$. Equation (3.8), Planck’s formula with $C_1 = 0$, when transposed, is nothing else than the Lorentz invariant relation between the relativistic energy, relativistic momentum and the rest mass of a moving ponderable object in free space:

$$E^2 - p^2c^2 = m^2c^4.$$
Now for point (ii), the consideration of further arguments, concerning mass–energy equivalence, that Einstein could have used, given his knowledge of the contents of his first 1905 relativity paper [1]. In this paper Einstein derived the kinetic energy, \( K \), of a ponderable object in terms of its rest mass, \( m \), and velocity, \( v = \beta c \), as

\[
K(m, \beta) = mc^2 \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right]. \tag{4.1}
\]

Clearly \( K(m, \beta = 0) = 0 \). Writing explicitly the velocity dependence of the energies in (2.5) and (2.6):

\[
K_{i,f}(m, \beta) = E_{i,f}(\beta) - E_{i,f}(0) - C. \tag{4.2}
\]

Since from (4.1) \( K_{i,f}(m, 0) = 0 \) and \( E_{i,f}(0) = E_{i,f}(0) \) Einstein’s constant \( C \) in (2.5) and (2.6) must vanish. Combining (2.5) and (2.6) (with or without \( C = 0 \)) with (2.4) gives

\[
K_i - K_f = E(L) \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right], \tag{4.3}
\]

which is the exact relation, of which (2.7) is the \( O(\beta^2) \) approximation. Combining (4.1) and (4.3) gives

\[
K_i - K_f = (m_i - m_f)c^2 \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right] = E(L) \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right]. \tag{4.4}
\]

from which equation (2.8) follows as an exact (to all orders in \( \beta \)) relation, without any necessity to consider the \( \beta \to 0 \) limit, as in the derivation of equation (2.8) in section 2 above.

It is now interesting to recall that the factor \( 1/\sqrt{1 - \beta^2} - 1 \) that cancels from both sides of the last member in (4.4) is derived in different and independent ways in equations (4.1) and (4.3). In equation (4.3) it follows from the transformation of the energy of 'plane waves of light'. The kinetic energy in (4.1) is instead found by calculating the work, \( W \), done on an electron, initially at rest, acted on by a constant electric field. For this the relativistic generalization of Newton’s second law given in §10 of [1] is used:

\[
my^3 \frac{d^2x}{dt^2} = my^2 \frac{dv}{dt} = eE_x = F = \text{const}, \tag{4.5}
\]

where \( \gamma \equiv 1/\sqrt{1 - \beta^2} \) to give

\[
K = W = \int dW = \int Fdx = m \int_0^\nu \gamma^2 \frac{dx}{dt} dv = m \int_0^\nu \gamma^2 dv = mc^2(\gamma - 1). \tag{4.6}
\]

Although an electrostatic force is considered in (4.5) it is clear that the relation (4.6) is of complete generality, independent of the nature of the constant force \( F \). Thus this relation holds, not only for electrons, but for any ponderable object of rest mass \( m \) subjected to a constant force. Transposing (4.6) gives

\[
K(m, \beta) + mc^2 \equiv E(m, \beta) = \gamma mc^2. \tag{4.7}
\]

The relation \( E(m, 0) = mc^2 \) is evidently a direct consequence of (4.7), so that a special thought experiment such as that discussed by Einstein in [2] was not necessary to derive it.
However, this thought experiment is of great importance as the first example of a theoretical analysis of the transformation of part of the mass of an object into the energies of different physical objects—Einstein’s ‘plane waves of light’ but what we would today understand, in a realistic realization of the thought experiment, as a pair of photons of equal energy and opposite momenta. It can be argued that the fact that this phenomenon—modification or destruction and creation of particles—occurs in nature, is the single most important physical discovery of the 20th Century both for its conceptual importance and for its practical ramifications. It realized, in the hands of Rutherford, the alchemist’s dream of the transmutation of elements and gave birth to the new disciplines of nuclear physics and elementary particle physics, which in turn revolutionized the understanding of astrophysics. Indeed, already in 1903 in a paper on radioactivity by Rutherford and Soddy [14] can be found the prophetic statement

‘All these considerations point to the conclusion that the energy latent in the atom must be enormous compared with that rendered free in ordinary chemical change. Now the radio elements differ in no way from the other elements in their chemical and physical behaviour. On the one hand they resemble chemically their inactive prototypes in the periodic table very closely and on the other they possess no common chemical characteristics that could be associated with their radioactivity. Hence there is no reason to assume that this enormous store of energy is possessed by the radio elements alone. It seems probable that atomic energy in general is of a similar high order of magnitude, although the absence of change prevents its existence being manifested.’

In this passage Rutherford and Soddy conjecture, on the basis of experimental evidence, the universal nature of mass/energy equivalence, as later derived by Einstein in the first 1905 special relativity paper [1], as well as the necessity of transmutation, as considered in the second one [2], in order demonstrate its existence. This was just what was suggested by Einstein in the passage from [2] quoted above. Indeed, as pointed out by Fadner [9], the equivalence of mass and energy in radioactive decays, as suggested by Einstein, to test equation (2.8) in [2] had previously been conjectured, in an experimental context, by Soddy in a book published the previous year [16]:

‘...it is not to be expected that the law of conservation of mass will hold true for radioactive phenomena. The work of Kaufmann may be taken as an experimental proof of the increase of apparent mass of the electron when its speed approaches that of light. Since during disintegration electrons are expelled at speeds very near to that of light, which, after expulsion, experience resistance and suffer diminution of velocity, the total mass must be less after disintegration than before. On this view, atomic mass must be regarded as a function of the internal energy, and the dissipation of the latter occurs at the expense, to some extent at least, of the mass of the system.’

Einstein might also have noticed the similarity between (4.7) and (2.3) in order to write the relation

\[ E'(L, \beta) \equiv \gamma m(L)c^2, \] (4.8)

This passage was quoted by Pais [15] who remarked that, in it, the modern concept of atomic energy was first introduced.
where \(m(L)\) is the ‘effective mass’ of the radiated light. In this case the energy conservation equation (2.1) becomes, in virtue of (4.7) and (4.8)

\[
m_i c^2 = m_f c^2 + m(L) c^2,
\]

which is the conservation law of mass in Newtonian mechanics, also valid for static systems in relativistic mechanics. Consideration of (4.9) and the corresponding equation, (2.2), in the moving frame:

\[
gam m_i c^2 = 
\]

shows that equation (2.8) may be derived without any necessity to consider the transformation formula for the energy of ‘light waves’ since the energies of the objects \(i\) and \(f\) and the radiation \(L\) all have the same transformation law (4.7) that was already implicit in the general relation (4.1) first given in [1]. As clearly and correctly stated by Einstein, the relation (2.8) is indeed of complete generality, the specialization to radiated energy in the thought experiment being of no importance.

Of particular practical interest are equations (4.8) and (4.9) for the special case \(m_f = 0\) so that the entire energy content of the object \(i\) is constituted by the two ‘plane waves’ that are created—photons in modern parlance. Measurement of the total energy of the photons in the rest frame of \(i\) then enables the mass of the object to be determined. It was by such measurements that the neutral pi-meson was discovered in 1950 by observation of the decay mode: \(\pi^0 \rightarrow \gamma\gamma\) [17]. More recently, observation of the decay: \(H \rightarrow \gamma\gamma\) of the Higgs boson (H) made an important contribution to the discovery of this particle [18–20].

The formula for the transformation law of a ‘light complex’ or ‘plane wave’ derived in [1] and assumed in [2]:

\[
E'(\pm) = \gamma E (1 \pm \beta \cos \phi),
\]

where \(\phi\) is the angle between the direction of one of the ‘plane waves’ and the direction of motion in the rest frame of the object \(i\), was used to derive equation (2.3) above as \(E'(L) = E'(+) + E'(-)\). It is the same, as Einstein remarked in [1], to the transformation law of the frequency, \(\nu\), of the light wave. Perhaps surprisingly, he did not notice—or if he noticed, chose not to say—that the identity of the transformations is a necessary consequence of the Planck relation \(E = \hbar \nu\), and although he had published, earlier in the same year, the paper [21] in which the light quantum concept was introduced, did not then identify the ‘light complex’ of [1] with a light quantum or a group of light quanta in parallel motion. In fact, identical transformation laws for energy and frequency require that the ratio \(\nu E/\hbar\) is the same in all inertial frames, and gives an alternative way to introduce Planck’s constant, \(\hbar\), into physics [22]. For further discussion of the possible role of Einstein’s light quantum concept in the genesis of [1] see [23].

It is interesting to note that if Einstein had considered another application of Newton’s second law, where force is equated to the time derivative of momentum, a straightforward variation of the kinetic energy calculation of equation (4.6) from [1] leads directly to the formula for relativistic momentum given later by Planck [3]:

\[
p = \int dp = \int F dt = m \int_0^\nu \gamma \frac{dv}{dt} dt = mc \int_0^\beta \frac{d\beta}{1 - \beta^2} = \frac{mv}{\sqrt{1 - \beta^2}}.
\]

Combining the relativistic energy: \(E = \gamma mc^2\), momentum: \(p = \gamma mv\) and equation (3.10) relating \(E, p\) and \(m\) gives \(v = pc^2/E = pc^2 \sqrt{m^2c^4 + p^2c^2}\) so that for a massless particle \(v = c\) and \(E = pc\). Einstein could, in this way, at any time after 1906, have derived the
second postulate of special relativity—the constancy of the speed of light—by assuming that the light quanta that he proposed [21] in 1905 were massless particles [24].

5. Discussion

Einstein’s derivation of the equivalence of mass and energy in [2], expressed mathematically in equation (2.8) above, was questioned by Planck in 1907 [4], Ives in 1952 [5] and more recently by Ohanian [6] and Hecht [7].

Planck’s objection, discussed by Fadner [9], was that, as a consequence of a possible contribution of thermal radiation to the rest energy of a system, the latter would be frame-invariant only to first order in \( \frac{v}{c} \). However the total momentum of thermal radiation in the rest system vanishes, so that just as in the case of the two-photon system considered in section 5 above, the thermal radiation component, \( E_0^{\text{rad}} \), of the rest energy has the same transformation law: \( E^{\text{rad}} = \gamma E_0^{\text{rad}} \) as a ponderable object of mass \( E_0^{\text{rad}} c^2 \), an exact formula that is valid at all orders in \( \frac{v}{c} \).

In the paper ‘Derivation of the Mass–Energy Relation’ [5] Ives reproduced Einstein’s thought experiment of [2] replacing the energy conservation postulate with relativistic momentum conservation. He then accused Einstein of the logical error of petitio principii (introducing a premise logically equivalent to the claimed conclusion). The last sentence of the paper is the bald statement:

‘The relation \( E = m_M c^2 \) was not derived by Einstein.’

Here ‘\( m_M \)’ is the mass of ‘matter’ as opposed to the effective mass of electromagnetic radiation ‘\( m_R \)’ also considered in the paper. This conclusion was subsequently quoted in an uncritical manner in several textbooks [25–27].

Ives’s thought experiment was identical to that considered by Einstein, except that the equal pulses of radiation of energy \( E/2 \) in the rest frame of the object \( i \), were assumed to be emitted parallel and anti-parallel to the direction of motion of the objects \( i \) and \( f \). The energy of the pulses, in the frame where the objects are in motion with speed \( \beta c \), using the same transformation formula, (4.11), for the energy of a ‘light complex’ as that derived by Einstein in [1] and used in [2] are

\[
(E/2)\gamma (1 + \beta), \quad (E/2)\gamma (1 - \beta),
\]

as also given by Ives in [5]. Ives then invoked a relation given by Poincaré in 1900 [28] stating that the momentum of electromagnetic radiation in free space is \( S/c^2 \) where \( S \) is the energy flux. Further writing \( S = E c \), Ives obtained for the (oppositely directed) momenta of the radiation pulses

\[
[E/(2c)]\gamma (1 + \beta), \quad [E/(2c)]\gamma (1 - \beta),
\]

so that the net momentum of the radiation in the frame in which \( i \) and \( f \) are in motion is \( E \beta c \). Ives then invokes the formula (unknown to Einstein at the time of writing [2]) for the relativistic momentum of a ponderable object of rest mass \( m \) and velocity \( \beta c \): \( p = \gamma \beta m c \). Imposing conservation of relativistic momentum

\[
\gamma \beta m_i c = \gamma \beta m_f c + E \beta /c
\]

immediately yields equation (2.8) above: \( m_i - m_f = E/c^2 \), the mathematical expression of Einstein’s verbal conclusion, concerning mass/energy equivalence, in [2].

Ives therefore obtains exactly the same result as Einstein in [2] but assumes in addition to the transformation law of radiant energy, also the definition of the relativistic momentum of a ponderable object as well as Poincaré’s relation between the energy flux and momentum of
electromagnetic radiation. Instead, like Einstein in [2], of imposing energy conservation and the validity of Newtonian kinematics in the $\beta \to 0$ limit (both very weak postulates) Ives’ derivation thus requires two additional, strong, postulates of relativistic physics.

In his criticism of Einstein’s analysis of the thought experiment Ives, unlike Einstein, invokes the relativistic formula for the kinetic energy of a ponderable object from [1]

$$K_i = m_i \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right].$$  \hspace{1cm} (5.1)

$$K_f = m_f \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right].$$  \hspace{1cm} (5.2)

As pointed out above, combining these equations with Einstein’s exact relation (2.4), from [2], enables derivation of (4.4) from which follows in all generality—without consideration of the $\beta \to 0$ limit—the relation $m_i - m_f = E(L)c^2$. By false logic, Ives construed the general derivation of equation (2.8) just presented as a petitio principii. To do this he combined (5.1), (5.2) and (2.4) to obtain

$$E_i - E_f = (E_f - E_f) = \frac{E(L)}{(m_i - m_f)c^2} (K_i - K_f),$$  \hspace{1cm} (5.3)

which, Ives stated, follows necessarily from the equations

$$E_i - E_i = \frac{E(L)}{(m_i - m_f)c^2} (K_i + C),$$  \hspace{1cm} (5.4)

$$E_f - E_f = \frac{E(L)}{(m_i - m_f)c^2} (K_f + C).$$  \hspace{1cm} (5.5)

These equations are consistent with Einstein’s definitions of kinetic energy if, and only if, $E(L)[(m_i - m_f)c^2] = 1$. Ives then concluded that, in writing (2.5) and (2.6), as the definitions of kinetic energy, Einstein must have (implicitly) assumed that $E(L)[(m_i - m_f)c^2] = 1$ which is the result he claimed to prove! But, as pointed out in section 4 above, the relation $E(m, \beta = 0) = mc^2$ (which Ives claims Einstein did not derive) is a consequence of (5.1) or (5.2) alone. It is also a consequence of (2.8) alone since on setting $m_f = 0$ in this equation it is found that

$$m_ic^2 = \Delta E_i = E(m, \beta = 0) = E(L).$$  \hspace{1cm} (5.6)

It is also possible to derive, in a similar manner, equation (4.1) from equations (2.4)–(2.6) and (2.8) from [2]. Combining these equations gives

$$K_i = K_f = (m_i - m_f)c^2 \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right].$$  \hspace{1cm} (5.7)

equation (4.1) follows on setting $K_f = 0, m_f = 0$.

Unlike Ives, Einstein did not invoke the relativistic kinetic energy formula (4.1) in [2]. The relation $E = mc^2$ mentioned in the last sentence of [5], quoted above, which, in the notation of the present paper, is $E(m, \beta = 0) = mc^2$, can be derived either from (2.8) (without invoking (4.1)) which is what is done in [2] or, from (4.1) in [1] (without invoking (2.8))—which can be done by simply physically interpreting this equation—although Einstein
did not do this in [1]. In neither case did Einstein introduce a premise logically equivalent to a result he claimed to derive. For further critical discussion of Ives’ assertion that Einstein was guilty of *petitio principii* in [2], see [9] and [11].

In a recent paper [6] Ohanian has claimed that Einstein’s derivation of \( E_0 = mc^2 \) in [2] is flawed by the assumption that the kinetic energy, \( K \), of an extended body is given by the Newtonian formula \( K = mv^2/2 \) in the low-velocity limit. It is conjectured that the Newtonian separation of internal energy, contributing to the rest mass \( m \), and translational kinetic energy \( K \), may break down in the case that internal constituents of the body are in motion with relativistic velocities. To be specific, the approximate formula for the kinetic energy as (tacitly) assumed by Einstein,

\[
K (v) = E - E_0 = \frac{1}{2} mv^2, \tag{5.8}
\]

is conjectured by Ohanian to be modified to

\[
\tilde{K} (v) = \tilde{E} - E_0 = \frac{1}{2} \tilde{m}(v)v^2, \tag{5.9}
\]

where \( \tilde{m}(v) \neq m \) when \( v \neq 0 \) in the case that constituents of a system with rest mass \( m \) have relativistic velocities.

A Taylor expansion of \( \tilde{m}(v) \) gives

\[
\tilde{m}(v) = m + \tilde{m}(v)v + \frac{\tilde{m}(v)^v}{2!}v^2 + \ldots \tag{5.10}
\]

since, by the definition of rest mass, \( \tilde{m}(0) = m \). Defining rest masses, following Stachel and Torreti [11] as

\[
m \equiv \lim_{v \to 0} \frac{K (v)}{v^2/2}, \tag{5.11}
\]

\[
\tilde{m}_0 \equiv \lim_{v \to 0} \frac{\tilde{K} (v)}{v^2/2}, \tag{5.12}
\]

it follows from (5.9) and (5.10) that \( \tilde{m}_0 = m \) so that modifications of the Newtonian kinetic energy (5.8) as suggested by Ohanian cannot invalidate the derivation of \( E_0 = mc^2 \) given in [2]. The reason for this is that both the rest mass \( m \) and \( E_0 \) in the formula \( E_0 = mc^2 \) are, by definition, *static properties* of a ponderable body, for which there is no difference between Newtonian and special relativistic kinematics.

In [6], Ohanian recalls and demonstrates a proof due to Klein [29] of the four-vector character of the quantities: \( \int T^{00} dx \) where \( T^{00} \) are elements of the energy-momentum tensor of an arbitrary extended and closed physical system. It is claimed in [6] that a corollary of this proof is that

‘For any closed system, with a timelike energy momentum four-vector the energy \( E_0 \) in the zero momentum frame ‘rest frame’ is related to the mass by \( E_0 = mc^2 \).’

To show this Ohanian then considers a Lorentz transformation of the energy momentum four-vector: \( (E_0/c, 0, 0, 0) \) to give the energy and momentum in an arbitrary inertial frame

\[
E = \gamma E_0, \tag{5.13}
\]
Equation (5.13) gives, for the relativistic kinetic energy $K = \gamma E_0 - E_0$. Comparing this with the relativistic kinetic energy formula $K = mc^2\gamma - mc^2$ given by Einstein in [1] Ohanian claims to have then derived the formula $E_0 = mc^2$ as a rigorous consequence of the four-vector character of certain elements of the energy momentum tensor, as proved by Klein. However, since Ohanian also assumes the correctness of (4.6)

$$K = mc^2\gamma - mc^2$$

or, defining verbally the terms in this equation

kinetic energy($K$) = energy in motion($E$) − rest energy($E_0$),

the relation $E_0 = mc^2$, as already pointed out in section 4 above, is a consequence of equation (4.6) alone and the definitions of kinetic energy, energy in motion and rest energy. Klein’s proof is therefore irrelevant for the derivation of $E_0 = mc^2$—it follows directly from equation (4.6) which Ohanian introduces as a separate premise in his proof. Unlike Einstein in [2], as claimed by Ives, Ohanian is indeed guilty of petitio principii in [6]!

The abstract of a recent paper ‘How Einstein confirmed $E_0 = mc^2$’ [7] by Hecht concludes with the assertion

‘Although he repeatedly confirmed the efficacy of $E_0 = mc^2$, he never constructed a general proof. Leaving aside that it continues to be affirmed experimentally, a rigorous proof of the mass–energy equivalence is probably beyond the purview of the special theory.’

A detailed reading of [7] did not reveal to the present author what type of ‘proof’ of mass–energy equivalence Hecht would consider to be ‘rigorous’. Presumably not, in view if the above quotation, the experimental proof that it is indeed observed to be an important property of the real world. In mathematics, a ‘proof’ is the demonstration that a certain assertion is a logical consequence of certain stated premises. Whether the premises describe, exactly or approximately, some measurable feature of the real world is irrelevant to the correctness or level of rigour of such a proof. On the other hand, physics is only concerned with premises that do describe, exactly or approximately, some measurable feature of the real world. Einstein’s proof of mass–energy equivalence in [2] is based on just three premises: conservation of energy, the transformation law of electromagnetic energy, and the assumption that in the low velocity limit ($v \ll c$), the laws of Newtonian mechanics are valid. It should be noticed that all of these premises do correctly describe observed features of the real world and so are physically valid ones. Unless there is some logical mistake in the mathematical reasoning (essentially only algebra) leading to Einstein’s conclusion (2.8)—the exact mathematical expression of his verbal statement of mass–energy equivalence—the proof certainly follows logically from the premises. No convincing argument is given by Hecht in support of his assertion that it is ‘not rigorous’.

One objection raised by Hecht is that

‘At this time he could imagine that there might be some kind of quiescent matter that possessed residual inert mass even if all its energy was somehow removed. After all, light was an entity with energy and no mass; perhaps there was matter with mass and no energy.’
Since setting $m_i = 0$ in equation (2.8) gives $m_i = E_i(v = 0)c^2$ not $m_i + \mu_i = E_i(v = 0)c^2$ where the symbol $\mu_i$ represents Hecht’s ‘matter with mass and no energy’ the existence of such ‘energyless mass’ with $\mu_i \neq 0$ is then forbidden if equation (2.8) is correct. Such mass can therefore exist only if one of the three premises on which equation (2.8) is based is false. Hecht, following Ohanian, suggests that it may be the last premise (made tacitly by Einstein) that is false, i.e. that there is a breakdown of Newtonian mechanics at low velocities. As pointed out above, this objection is rebutted since the equation $E_0 = mc^2$ describes a static system so that the precise definition of ‘kinetic energy’ used to derive equation (2.8) is irrelevant. Indeed, from dimensional analysis alone, the relation between rest mass $m$ and the corresponding ‘rest energy’ $E_0$ must be of the form $E_0 = \kappa m$ where $\kappa$ is a universal constant with the dimensions of velocity squared. Then, necessarily, $m = 0$ when $E_0 = 0$ and vice versa. The existence of matter in motion with ‘energy and no mass’ is a known special feature of relativistic kinematics. Taking the simultaneous limits: $m \to 0, \beta \to 1$ in the formula (3.6) for relativistic energy gives, formally $0 \times \infty = E = pc$, as is clear from inspection of the formula (3.10) given by squaring both sides of Planck’s formula (3.8) relating the rest mass of an object to its relativistic energy and momentum.

As discussed in section 4 above, Hecht’s assertion in [7] that ‘Einstein said nothing about’ ($E_0 = mc^2$) ‘in his June 1905 paper’ [1], while perhaps literally true, is highly misleading as to the actual scientific content of this paper. If ‘said’ is interpreted only as verbal expression, the statement is correct. However, the relation $E_0 = mc^2$ follows trivially from equation (4.7) which is simply a transposition of Einstein’s equation (4.1) from [1], in association with the definition of relativistic energy: $E \equiv \gamma mc^2$. The manner of stating mass–energy equivalence is inverted in the June 1905 paper [1], where equation (4.1) is given without verbal explanation, as compared to the September 1905 paper [2] where an exact verbal description of this equivalence is given, but not the corresponding equation (2.8).

Hecht also states, à propos of the September 1905 paper that

‘This derivation came very early in the development of relativity and the formal concept of ‘rest energy’ had not yet evolved, nor had $E_0$ been introduced to symbolize it.’

This statement is untrue. The quantities $E_i$ and $E_f$ in equation (2.1) (called by Einstein $E_0$ and $E_1$ in [2]) are, by definition, the rest energies of the objects $i$ and $f$! Thus Einstein did use the symbol $E_0$ to denote the energy of the object $i$ when it is at rest. The formula in [2] corresponding to equation (2.1) is not mentioned in [7]. Also not mentioned is Einstein’s verbal statement of the exact conversion factor between mass and energy given in the concluding passage of [2] quoted above. Hecht states instead that

‘Nowhere did he write that mass and energy are ‘equivalent’, that would come later.’

However Einstein did state that one gramme of matter corresponds to $9 \times 10^{20}$ ergs of energy, and to say that one gramme is ‘equivalent’ to $9 \times 10^{20}$ ergs does not have a different meaning. Indeed, in [7] Hecht does not consider several written passages or equations in [2] that actually contradict some assertions made in the [7].

The conclusion of [7] also contains the statement (not directly related to the meaning of Einstein’s work in [1, 2]):

‘For several compelling reasons many physicists have come to accept that mass is invariant, even though there is no proof— theoretical or experimental— that it is.’

13
The general ‘theoretical proof’ that rest mass is invariant is provided by the straightforward generalization of equation (4.8) above, where the ‘effective mass’ of a two-photon system is introduced, to an arbitrary final state of \( N (N \geq 2) \) particles. One ‘experimental proof’ is provided by the observation of the Higgs boson, that is produced in the laboratory system at the LHC with a wide energy spectrum \([19, 20]\), via its two-photon decay mode, as mentioned in section 4 above. Another is the identification of 149 distinct decay modes of the \( J/\psi \) (a bound state of a charm quark and a charm anti-quark) \([30]\) by calculation of the Lorentz-invariant effective mass of the different decay products and observing that they are all equal to the mass of the \( J/\psi \). The evidence, both theoretical and experimental, is indeed ‘compelling’. What kind of further ‘proof’, that would convince Hecht that rest mass is indeed invariant and equivalent to energy, is not revealed in \([7]\).

6. Conclusions

The relation \( E_0 = mc^2 \) and the concept of total relativistic energy \( E = \gamma mc^2 \) were already implicit in equation (4.1) given in Einstein’s June 1905 special relativity paper \([1]\). The important new contribution to physics brought by the September 1905 paper \([2]\) was therefore not only an alternative derivation of \( E_0 = mc^2 \), but the first correct theoretical analysis of a process in which mass was transformed into energy and (in modern language) new particles were created from this energy. Experimental studies of radioactive decays had already shown by 1905 that such energy transformation processes must occur in nature, and Einstein suggested that such radioactive decays might be used to test his theory. Planck wrote down in \([3]\) the formula for relativistic momentum, \( p = \gamma mv \), and by invoking Hamilton’s principle, rederived Einstein’s relativistic energy equation (4.1), thus completing the theory of the relativistic kinematics of objects (massive or massless) in motion in free space. The formula for relativistic momentum can alternatively be derived by time integration, as in equation (4.11), of the relativistic generalization of Newton’s second law: equation (4.5), as given in \([1]\).

In agreement with previous work of Stachel and Torreti \([11]\) and Fadner \([9]\), claims in the literature that Einstein’s discovery of mass–energy equivalence, as presented in \([2]\), was flawed, incomplete, or lacked generality or rigour \([4–7]\) are shown to have no foundation.

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