A hybrid differential game with switching thermostatic-type dynamics and cost

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Abstract

In this paper we consider an infinite horizon zero-sum differential game where the dynamics of each player and the running cost are also depending on the evolution of some discrete (switching) variables. In particular, such switching variables evolve according to the switching law of a so-called thermostatic delayed relay, applied to the players’ states. We first address the problem of the continuity of both lower and upper value function. Then, by a suitable representation of the problem as a coupling of several exit-time differential games, we characterize those value functions as, respectively, the unique solution of a coupling of several Dirichlet problems for Hamilton-Jacobi-Isaacs equations. The concept of viscosity solutions and a suitable definition of boundary conditions in the viscosity sense is used in the paper. Finally, we give some sufficient conditions for the existence of an equilibrium.

Keywords: Differential games, hybrid systems, switching, exit costs, Hamilton-Jacobi-Isaacs equations, viscosity solutions, non-anticipating strategies, delayed thermostat

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1 Introduction

We consider an infinite horizon zero-sum differential game where both players, in their decoupled dynamics, as well as the running cost, are also affected by some switching variable whose switching evolution is described by a so-called thermostatic-type switching rule, subject to the evolution of the players’ continuous state-variables.

More precisely, we consider two decoupled dynamics for the two players with, respectively, state-variables $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$, as

$$
\begin{align*}
X'(t) &= f(X(t), W(t), \alpha(t)), \quad t > 0 \\
W(t) &= h[X, w](t) \\
X(0) &= x, \quad W(0) = w
\end{align*}
$$

and

$$
\begin{align*}
Y'(t) &= g(Y(t), Z(t), \beta(t)), \quad t > 0 \\
Z(t) &= h[Y, z](t) \\
Y(0) &= y, \quad Z(0) = z
\end{align*}
$$

where $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$ are the measurable controls, and $W, Z \in \{-1, 1\}$ are the switching variables, whose state-dependent switching rules are represented by the second lines of the systems \[1\]. In particular, the switch from 1 to $-1$ and the switch from $-1$ to 1 occur at two different thresholds.

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that must be reached by the continuous state variable (see Figure 1 and in general Section 4 for more precise details). Moreover, player $X$ wants to minimize, whereas player $Y$ wants to maximize, a discounted infinite horizon cost of the form

$$J(x, y, w, z, \alpha, \beta) = \int_0^{+\infty} e^{-\lambda t} \ell(X(t), Y(t), W(t), Z(t), \alpha(t), \beta(t)) \, dt$$

where $w, z \in \{-1, +1\}$ are the initial states of the switching variables.

As usual, following Elliot-Kalton [16], we define the lower and the upper value functions respectively as

$$\underline{V}(x, y, w, z) = \inf_{\gamma \in \Gamma} \sup_{\beta \in B} J(x, y, w, z, \gamma, \beta),$$

$$\overline{V}(x, y, w, z) = \sup_{\xi \in \Xi} \inf_{\alpha \in \mathcal{A}} J(x, y, w, z, \alpha, \xi \alpha),$$

where $\Gamma$ and $\Xi$ are the set of non-anticipating strategies for player $X$ and player $Y$, respectively.

The main goal of the paper is to derive two suitable problems for Hamilton-Jacobi-Isaacs (HJI) equations in such a way to characterize $\underline{V}$ and $\overline{V}$ as the unique viscosity solutions of those problems, respectively. As a consequence, we will also get the existence of an equilibrium (i.e. $\underline{V} = \overline{V}$) under the standard Isaacs condition. To achieve the main goal we perform several steps.

The first step is to prove the continuity in the space variables of the value functions, that is, for every $(w, z) \in \{-1, 1\} \times \{-1, 1\}$ fixed, the continuity of $(x, y) \mapsto \overline{V}(x, y, w, z)$ (here and somewhere in the sequel, by $\overline{V}$, we will denote any one of the two value functions (2), regardless whether it is the lower or the upper one). Under the hypothesis of decoupled dynamics and another decoupling hypothesis on the running cost $\ell$, such continuity is proved using a suitable construction of non-anticipating strategies. Indeed, in our switching differential game, we need the existence of some non-anticipating strategies which make the players, when they are on a switching threshold, to be able to switch or not (i.e. to cross the threshold or not) in dependence on its convenience. At the same time, such non-anticipating strategies must not penalize the cost too much. In the simpler case of an optimal control problem (one player only) this can be achieved by the Soner’s construction of the so-called constrained controls [26]. Indeed, the switching problem with state-dependent switching thresholds (as our problem is) is strongly related to state-constraints as well as exit-time problems. However, for the differential game situation, the much stricter requirement that the construction of the Soner-like control must be non-anticipating (i.e. non-dependent on future behaviors of the trajectories and of the controls), is a fundamental issue. Such an issue was addressed in the recent work by Bagagiolo-Maggistro-Zoppello [2]. In that work the authors studied (for the first time) an exit-time/exit-costs differential game in the framework of dynamic programming and viscosity solutions theory for Isaacs equations with boundary conditions in the viscosity sense. In particular, the fundamental issue above is there largely treated and solved under the decoupling hypotheses and further controllability hypotheses. In the present work, we are going to make large use of the results of [2], of which one of the main motivations was, by the way, to arrive to study the thermostatic problem as the one here presented.

The second step is to rewrite our infinite horizon problem as four exit-time/exit-costs problems coupled to each other by the exit costs. More precisely, the state space $\mathbb{R}^n \times \mathbb{R}^m \ni (X, Y)$ is divided in four partially overlapped sectors where the switching variables $(W, Z)$ remain constant. In each one of those sectors, the problem is seen as an exit-time differential game with exit costs
mutually exchanged with the other sectors: the value function $V$ has been evaluated on the new entered sector when the previous one is left (i.e. one or both switching variable are switched), see Figure 4. This step is achieved by dynamic programming techniques, using the decoupled feature of the dynamics, a controllability hypothesis and the already proved continuity of $V$.

The third step is, using the second step above, to write a system of four HJI equations, one for anyone of the four sectors above respectively, and coupled by the boundary conditions (for every sector, the boundary datum is the unknown function on the other sectors). Using the results of [2] on exit-time differential games, the value function $V$ will turn out to be a viscosity solution of such a system, where the boundary conditions are interpreted in a suitable viscosity sense.

The last step is to prove that $V$ is actually the unique solution of the system of HJI equations above. This is achieved by a fixed point procedure, using the uniqueness results of [2]. There, under some hypotheses, it is proved that the value function $V$ for a differential game of exit-time/exit-costs type (when the exit costs are given, not as here, where the exit costs are represented by the unknown solution itself) is the unique solution of a Dirichlet problem for the corresponding HJI equation. In particular, the boundary conditions are interpreted in a suitable viscosity sense that takes account of the min-max feature of the problem and benefits of the decoupling and controllability of the dynamics.

Motivations and literature.
There are different situations that can be interpreted as differential games with dynamics affected by switching. Just think to a pursuit evasion game (see Shinar-Glizer-Turetsky [24, 25]) where the switching dynamics is either the one of pursuer or the one of the evader. We can also imagine a race between two cars where the switching variable(s) may represent the position of an automatic gears or the diesel/electric regime of an hybrid car as in Dextreit-Kolmanovsky [15].

Also the well known shallow lake problem that arises in ecological economics (see e.g., Reddy-Schumacher-Engwerda [23]) can be seen as a differential game with switching dynamics as well as the international pollution problem with evolving environmental costs. Such costs, for less developed countries, change according to their cumulative revenue, see Masoudi-Zaccour [22]. We point out that the above mentioned switching dynamics are also called hybrid dynamics. A recent study of hybrid differential games can be found also in Gromov-Gromova [19], where the authors formulated necessary optimality conditions for determining optimal strategies in both cooperative and non-cooperative cases. A particular class of differential games with changing structure is also considered in Bonneuil-Boucekkine [10], where the transition to renewable energy leads to the change of the system’s dynamics, and in Kort-Wrzaczek [21], where the change of a monopolist firm’s dynamics is due to the entrance in the market of a firm offering the same products. The switching can occurs not only in the dynamics but also in the cost function as for example in Fabra-Garca [17] where a dynamic competition is analysed and the market prices change.

There are also mathematical motivations that suggest the study of differential games with thermostatic dynamics, that are similar to the ones for studying optimal control problems with thermostatic dynamics (see Bagagiolo-Maggistro [3], Ceragioli-De Persis-Frasca [13] and Bagagiolo-Danieli [11]). In [3] optimal controls problems with dynamics inside a network are considered. A delay thermostat is introduced to overcome the discontinuity’s problem arising when passing from an arc to another one due to the different dynamics and running cost on each branch of the network. In [13] the authors make a rigorous treatment of continuous-time average consensus dynamics with uniform quantization in communications. The consensus is reached by quantized
measurement which are transmitted using a delay thermostat. Similarly, in [1] they consider an optimal control problem exhibiting several internal switching variables whose discrete evolutions are governed by some delayed thermostatic laws. A zero-sum differential games involving hybrid controls was also considered in Dharmatti-Ramaswamy [14]. Here, the state of the system is changed discontinuously and the associated lower and upper value functions are characterized as the unique viscosity solutions of the corresponding quasi-variational inequalities. Moreover, they give an Isaacs like condition for the game to have a value.

Up to the knowledge of the present authors, the present work is the first attempt to study a switching/hybrid differential game in the framework of dynamic programming and viscosity solutions of Hamilton-Jacobi-Isaacs equations, and especially in connection with hybrid delayed thermostatic laws and, more in general, state-dependent switching.

In conclusion we refer the reader to Bardi-Capuzzo Dolcetta [4] for a comprehensive account to viscosity solutions theory and applications to optimal control problems and differential games (for differential games see also Buckdahn-Cardaliaguet-Quincampoix [11]). Moreover other studies on constrained trajectories and non-anticipating strategies as well as on possible relations with optimal control problems and differential games can be found in Koike [20], Bardi-Koike-Soravia [5], Cardaliaguet-Quincampoix-Saint Pierre [12], Bettiol-Cardaliaguet-Quincampoix [6] and Frankowska-Marchini-Mazzola [18].

Plan of the paper.

This paper is organized as follows: In Section 2, we introduce the hybrid thermostatic delayed relay, and its connection with ordinary differential equations. In Section 3, we briefly review the results in Bagagiolo-Maggistro-Zoppello [2] about exit-time/exit-costs differential games. In Section 4 we state the main assumptions about the infinite horizon switching differential game under study, and we argue about cost estimates on the switching trajectories. In section 5 we prove the continuity of the value functions and give a dynamic programming-like results, connecting the infinite horizon switching problem with exit-time problems. In Section 6 we characterize the value functions as the unique viscosity solutions of the systems of Isaacs equations.

2 The hybrid thermostatic delayed relay

A hybrid delayed thermostat with thresholds \( \rho = (\rho_{-1}, \rho_1) \), \( \rho_{-1} < \rho_1 \), and initial output \( w \in \{-1, 1\} \), is the operator

\[
h_{\rho}[; w] : C^0(0, +\infty) \to L^\infty(0, +\infty), \quad X \mapsto h_{\rho}[X; w],
\]

whose behavior is described by Figure 1. In particular it maps a time-continuous scalar input \( X \) to a measurable time-dependent output function \( W = h_{\rho}[X; w] \) which can only takes values in \( \{-1, 1\} \) and whose switching law is the following (see Visintin [27] for a more systematic treatment
of the delayed relay)\

\[
\begin{aligned}
    & t \geq 0, \quad X(t) > \rho_1 \implies h_{\rho}(X; w)(t) = 1 \\
    & t \geq 0, \quad X(t) < \rho_{-1} \implies h_{\rho}[X; w](t) = -1 \\
    & \rho_{-1} \leq X(0) \leq \rho_1 \implies h_{\rho}(X; w)(0) = w \\
    & t \geq 0, \quad \rho_{-1} \leq X(t) \leq \rho_1 \implies h_{\rho}(X; w)(t) = \lim_{s \to \tau} h_{\rho}(X; w)(s)
\end{aligned}
\]

if \( \tau = \sup\{0 \leq \tau < t \mid X(\tau) < \rho_{-1} \text{ or } X(\tau) > \rho_1\} > 0 \)

\[ h_{\rho}(X; w)(t) = h_{\rho}(X; w)(0) = w \text{ otherwise} \]

In other words, looking to Figure 1, if at certain time \( \tau \) we have \( h_{\rho}(X; w)(\tau) = -1 \) (which certainly means \( X(\tau) \leq \rho_1 \)), then a switch from \(-1\) to \(1\) can only occur (and must occur) at a possible subsequent time \( t \geq \tau \) if and only if, at that time \( t \), the input \( X \) crosses, strictly increasing, the upper threshold \( \rho_1 \), being \( X(t) = \rho_1 \). In that case, it is \( h_{\rho}(X; w)(t) = -1 \) and \( h_{\rho}(X; w) \equiv 1 \) in, at least, a left-open right neighborhood of \( t, [t, t + \delta] \). Similarly, if at certain time \( \tau \) we have \( h_{\rho}(X; w)(\tau) = 1 \) (which certainly means \( X(\tau) \geq \rho_{-1} \)), then a switch from \( 1 \) to \(-1\) can only occur (and must occur) at a possible subsequent time \( t \geq \tau \) if and only if, at that time \( t \), the input \( X \) crosses, strictly decreasing, the lower threshold \( \rho_{-1} \), being \( X(t) = \rho_{-1} \). In that case, it is \( h_{\rho}(X; w)(t) = 1 \) and \( h_{\rho}(X; w) \equiv -1 \) in, at least, a left-open right neighborhood of \( t, [t, t + \delta] \). Note that, by such a definition, and in particular because of the strict inequality \( \rho_{-1} < \rho_1 \), the output \( h_{\rho}(X; w) \) is left-continuous. In particular, we remark that, at a switching instant \( t \), the value of the output \( h_{\rho}(X; w)(t) \) is still the previous one (i.e. it is not switched yet) and it will be equal to the new switched one at subsequent instants after \( t \) only (if \( X \) has crossed the threshold). Finally note that the given initial output \( w \) plays a role only if \( \rho_{-1} \leq X(0) \leq \rho_1 \). An important property

![Figure 1: Delayed thermostat with thresholds \( \rho = (\rho_{-1}, \rho_1) \).](image)

of the thermostatic delayed relay is the following semigroup property. For every \( X \) and \( w \), and for every \( t, \tau \geq 0 \), it is

\[
h_{\rho}(X; w)(t + \tau) = h_{\rho}[(X(\cdot) + t); h_{\rho}(X; w)(t)](\tau)
\] (3)
The switching evolution of $W = h_ρ[X; w]$ can be also described in the following way. Looking to Figure 1, we define

$$O_{−1} = ]−∞, ρ_1] \times \{-1\}; \quad O_1 = [ρ_{−1}, +∞] \times \{1\}$$

which correspond to the sets where the pair $(X, W)$ can evolve without switching. Given $X \in C^0(0, +∞)$ and $w \in \{-1, +1\}$ such that $(X(0), w) \in O_{−1} \cup O_1$, the output $h_ρ[X; w]$ is characterized as the unique left-continuous function $W$ such that

$$(X(t), w(t)) \in O_{−1} \cup O_1 \forall t ≥ 0,$$

$$\text{Var}_{[0,t]} W = \min \{\text{Var}_{[0,t]} W| (X(τ), W(τ)) \in O_{−1} \cup O_1 \forall τ \in [0, t]\} \quad \forall t ≥ 0$$

where $\text{Var}_{[0,t]}$ is the total variation in the interval $[0, t]$ (which, for the delayed thermostat, corresponds to twice the number of switchings in $[0, t]$ (every switching has variation equal to 2)). Hence, the switching law $t \mapsto W(t)$ is the unique one that satisfies the constraint $(X(t), W(t)) \in O_{−1} \cup O_1$ and, in any time interval, minimizes the number of switchings.

Such interpretation is also useful to define what is a the solution of the switching scalar ordinary differential equation

$$\begin{cases}
X'(t) = f(t, X(t), W(t)) \\
W(t) = h_ρ[X; w](t) \\
X(0) = x, \quad (x, w) \in O_{−1} \cup O_1
\end{cases}$$

where $f : [0, +∞] × \mathbb{R} \times \{-1, 1\} \to \mathbb{R}$, $(t, x, w) \mapsto f(t, x, w)$ is bounded, measurable in $t \in [0, +∞]$ and Lipschitz continuous in $x \in \mathbb{R}$ uniformly with respect to $(t, w) \in [0, +∞] \times \{-1, 1\}$. The solution is the unique function $t \mapsto (X(t), W(t))$ such that: i) $X$ is continuous and $W$ is left continuous; ii) $X(t) = x + \int_0^t f(s, X(s), W(s))ds$ and $(X(t), W(t)) \in O_{−1} \cup O_1$ for all $t ≥ 0$; iii) minimizes in $[0, t]$ for all $t$, the number of switchings of $W$, among all pairs $(X, W)$ satisfying i) and ii). Such a unique solution can be constructed in the following way: consider the evolution of $X_1$, starting from $x$ with dynamics $f(\cdot, \cdot, w)$, and maintain such evolution until the possible time $t_1$ of switching for $W = h_ρ[X_1; w]$. Then in $[t_1, +∞)$ consider the trajectory $X_2$ starting from $X_1(t_1)$ with dynamics $f(\cdot, \cdot, −w)$, and maintain it until the possible time $t_2 > t_1$ of switching for $h_ρ[X_2, −w]$. Then, in $[t_2, +∞)$ consider the trajectory $X_3$ with dynamics $f(\cdot, \cdot, w)$ and starting from $X_2(t_2)$. Since the switching thresholds are different, $ρ_{−1} < ρ_1$, then the number of those changes of dynamics is bounded in any compact set $[0, T]$, and hence, gluing together the pieces of trajectories $X_i$, we get the unique solution of (1) defined above.

When the evolution is in $\mathbb{R}^n$ (as the controlled evolution of the next sections) we are going to suppose that the switching dependence of the dynamics is subject to the evolution of a fixed component of the continuous evolution $X \in \mathbb{R}^n$: $X \cdot ζ$ where $ζ \in \mathbb{R}^n$ is a unit vector. We then consider the $(n + 1)$-dimensional system in the variable $(X, W) \in \mathbb{R}^n × \{-1, 1\}$

$$\begin{cases}
X'(t) = f(t, X(t), W(t)) \\
W(t) = h_ρ[X \cdot ζ; w](t) \\
X(0) = x, \quad (x \cdot ζ, w) \in O_{−1} \cup O_1
\end{cases}$$

for which the same considerations as above hold.
Remark 1. The previous definition of the switching rule $W = h_\nu[X; w]$, as well as the subsequent definition of the solution of (4), is certainly linked to an exit-time feature: for example, a switch from $-1$ to $1$ occurs if and only if the pair $(X, W)$ exits from the closed set $\mathcal{O}_{-1}$. And indeed, in the following, we are going to interpret our infinite horizon switching differential games as a coupling of some exit-time differential games with exit from some suitable closed sets. One can also recast the problem as an exit-time problem with exit from an open set. Concerning the switching rule, it corresponds to the immediate switching when the threshold is touched (on the contrary, our definition is when the threshold is bypassed). This, for example, would correspond to the fact that a switch from $-1$ to $1$ occurs if and only if the pair $(X, W)$ exits from the “open” set $]-\infty, \rho_1[\times\{-1, 1\}$.

We use the interpretation as exit from a closed set because it is more prone to treat the Dirichlet problem from the Isaacs equation, since in such a case the boundary is “physically” part of the problem, and it is even viable. However, assuming some suitable controllability conditions, as we are going to do, the two problems (exit from the open set and exit from the closed set), are in general extremely linked to each other, as the corresponding value functions in general coincide in the interior of the set. This fact is certainly known for optimal control problems but we do not investigate here such argument for the differential games situation.

3 On exit-time/exit-costs differential games

In this section we briefly recall the results of Bagagiolo-Maggistro-Zoppello [2], which will be used in the next sections.

Let us consider two open domains $\Omega_X \subseteq \mathbb{R}^n$, $\Omega_Y \subseteq \mathbb{R}^m$, with $C^2$ boundary and two decoupled controlled dynamics, in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively

$$
\begin{align*}
X' &= f(X(t), \alpha(t)) \quad Y' = g(Y(t), \beta(t)) \\
X(0) &= x_0 \in \Omega_X \quad Y(0) = y_0 \in \Omega_Y
\end{align*}
$$

(5)

where for given compact sets $A, B$

$$
\begin{align*}
\alpha &\in \mathcal{A} = \{\alpha : [0, +\infty[ \to A \text{ measurable} \}, \\
\beta &\in \mathcal{B} = \{\beta : [0, +\infty[ \to B \text{ measurable} \}, \\
f &: \mathbb{R}^n \times A \to \mathbb{R}^n, \quad g : \mathbb{R}^m \times B \to \mathbb{R}^m, \\
f, g &\text{ are bounded and continuous and there exists } L > 0 \\
&\text{ such that for all } x_1, x_2 \in \mathbb{R}^n, \ a \in A, \ y_1, y_2 \in \mathbb{R}^m, \ b \in B \\
&\|f(x_1, a) - f(x_2, a)\| \leq L\|x_1 - x_2\|, \quad \|g(y_1, b) - g(y_2, b)\| \leq L\|y_1 - y_2\|.
\end{align*}
$$

We also consider the following functions

$$
\ell : \mathbb{R}^n \times \mathbb{R}^m \times A \times B \to [0, +\infty[, \quad \ell_1, \ell_2 : \mathbb{R}^n \times \mathbb{R}^m \times A \to [0, +\infty[, \quad \ell_1, \ell_2 : \mathbb{R}^n \times \mathbb{R}^m \times B \to [0, +\infty[, \\
\Psi_X : \Omega_X \to [0, +\infty[, \quad \Psi_Y : \Omega_Y \to [0, +\infty[, \quad \Psi_{XY} : \partial \Omega_X \times \partial \Omega_Y \to [0, +\infty[, 
$$

(7)

with the assumption that $\ell_1, \ell_2, \Psi_X, \Psi_Y, \Psi_{XY}$ are bounded and continuous and that there exists $L > 0$ such that for every $a \in A, b \in B$ fixed, and for all $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in \mathbb{R}^m$,

$$
\|\ell_1(x_1, y_1, a) - \ell_1(x_2, y_2, a)\|, \quad \|\ell_2(x_1, y_1, b) - \ell_2(x_2, y_2, b)\| \leq L\|(x_1, y_1) - (x_2, y_2)\|.
$$

(8)
We consider the differential game given by the cost, for \( x \in \Omega_X, y \in \Omega_Y \),
\[
J(x, y, \alpha, \beta) = \int_0^{\tau_{(x,y)}(\alpha,\beta)} e^{-\lambda t} \ell(X(t), Y(t), \alpha(t), \beta(t)) dt + e^{-\lambda \tau_{(x,y)}(\alpha,\beta)} \Psi(X(\tau_{(x,y)}(\alpha,\beta)), Y(\tau_{(x,y)}(\alpha,\beta))),
\]
where \( \lambda > 0 \) and, for \( \alpha \in A, \beta \in B, X(\cdot), Y(\cdot) \) are the trajectories given by \([5]\), sometimes also denoted as \( X(\cdot; x, \alpha), Y(\cdot; y, \beta) \),
\[
\begin{align*}
\tau_{(x,y)}(\alpha, \beta) &= \min\{\tau_x(\alpha), \tau_y(\beta)\}, \\
\tau_x(\alpha) &= \inf\{t \geq 0 \text{ such that } X(t) \in \Omega_X\}, \\
\tau_y(\beta) &= \inf\{t \geq 0 \text{ such that } Y(t) \in \Omega_Y\},
\end{align*}
\]
and
\[
\Psi(X(\tau_{(x,y)}(\alpha, \beta)), Y(\tau_{(x,y)}(\alpha, \beta))) = \begin{cases} \\
\Psi_X(X(\tau_x(\alpha)), Y(\tau_x(\alpha))) & \text{if } \tau_{(x,y)}(\alpha, \beta) = \tau_x(\alpha) < \tau_y(\beta), \\
\Psi_Y(X(\tau_y(\beta)), Y(\tau_y(\beta))) & \text{if } \tau_{(x,y)}(\alpha, \beta) = \tau_y(\beta) < \tau_x(\alpha), \\
\Psi_{XY}(X(\tau_{(x,y)}(\alpha, \beta)), Y(\tau_{(x,y)}(\alpha, \beta))) & \text{if } \tau_{(x,y)}(\alpha, \beta) = \tau_x(\alpha) = \tau_y(\beta),
\end{cases}
\]
and, of course, \( \inf \emptyset = +\infty \) and \( e^{-\infty} \Psi := 0 \) Roughly speaking, the cost is paid as the integral of the discounted running cost up to the first exit-time of one of the two trajectories \( X \) and \( Y \) from its set of reference, \( \Omega_X \) and \( \Omega_Y \), respectively. And then a discounted exit cost is paid, which is given by three different exit costs, \( \Psi_X, \Psi_Y, \Psi_{XY} \), depending whether player \( X \) only exits from \( \Omega_X \) (i.e. \( \tau_x < \tau_y \)), or player \( Y \) only exits from \( \Omega_Y \) (i.e. \( \tau_y < \tau_x \)), or they both simultaneously exit from their closed reference sets (i.e. \( \tau_x = \tau_y \)).

The exit-time/exit-costs differential game is given by the fact that \( X \) wants to minimize \( J \) and \( Y \) wants to maximize it. We then define the non-anticipating strategies for player \( X \) and for player \( Y \) respectively as:
\[
\begin{align*}
\Gamma &= \left\{ \gamma : B \to \mathcal{A}, \beta \mapsto \gamma[\beta] \text{ such that } \right. \\
\beta_1 &= \beta_2 \text{ a. e. in } [0, t] \implies \gamma[\beta_1] = \gamma[\beta_2] \text{ a. e. in } [0, t], \forall t \geq 0 \left\}; \\
\Xi &= \left\{ \xi : \mathcal{A} \to B, \alpha \mapsto \xi[\alpha] \text{ such that } \right. \\
\alpha_1 &= \alpha_2 \text{ a. e. in } [0, t] \implies \xi[\alpha_1] = \xi[\alpha_2] \text{ a. e. in } [0, t], \forall t \geq 0 \left\};
\end{align*}
\]
The lower and upper value functions are respectively defined as, for \( (x, y) \in \Omega_X \times \Omega_Y \),
\[
\begin{align*}
\underline{v}(x, y) &= \inf_{\gamma \in \Gamma} \sup_{\beta \in B} J(x, y, \gamma[\beta], \beta), \\
\overline{v}(x, y) &= \sup_{\xi \in \Xi} \inf_{\alpha \in \mathcal{A}} J(x, y, \alpha, \xi[\alpha]).
\end{align*}
\]
Similarly to the non-anticipating strategies, we define a non-anticipating tuning for both players: a non-anticipating tuning is any function \( \mathcal{K} \to \mathcal{K}, k \mapsto \hat{k} \), where \( \mathcal{K} \) is either \( \mathcal{A} \) or \( \mathcal{B} \), such that
\[
k_1 = k_2 \text{ a. e. in } [0, t] \implies \hat{k}_1 = \hat{k}_2 \text{ a. e. in } [0, t], \forall t \geq 0.
\]
Note the difference: a non-anticipating strategy is a function from the set of measurable controls for one player to the set of measurable controls for the other player; a non-anticipating tuning is a function from the set of measurable controls for one player to itself.

We then assume the following controllability and compatibility hypotheses

\[ \forall x \in \partial \Omega_X \ni a_1, a_2 \in A \text{ such that } f(x, a_1) \cdot n_X(x) < 0 < f(x, a_2) \cdot n_X(x), \]
\[ \forall y \in \partial \Omega_Y \ni b_1, b_2 \in B \text{ such that } g(y, b_1) \cdot n_Y(y) < 0 < g(y, b_2) \cdot n_Y(y), \]

(12)

\[ \Psi_X(x, y) \leq \Psi_X(x, y) \leq \Psi_X(x, y) \quad \forall (x, y) \in \partial \Omega_X \times \partial \Omega_Y, \]

where \( n_X(x) \) and \( n_Y(y) \) are, respectively, the outer normal unit vector to \( \Omega_X \) in \( x \) and to \( \Omega_Y \) in \( y \).

Finally, we define, respectively, the upper Hamiltonian and the lower Hamiltonian for \((x, y, p, q) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m\), as

\[ UH(x, y, p, q) = \min_{b \in B} \max_{a \in A} \{ -f(x, a) \cdot p - g(y, b) \cdot q - \ell(x, y, a, b) \}, \]
\[ LH(x, y, p, q) = \max_{a \in A} \min_{b \in B} \{ -f(x, a) \cdot p - g(y, b) \cdot q - \ell(x, y, a, b) \}. \]

**Theorem 2** Under hypotheses (6)-(12), the value functions \( v \) and \( \varpi \) are bounded and continuous in \( \Omega_X \times \Omega_Y \). Moreover they are, respectively, the unique bounded and continuous function \( u : \Omega_X \times \Omega_Y \to \mathbb{R} \) which satisfies, in the viscosity sense, the following Dirichlet problems for the Isaacs equations

\[
\begin{cases}
\lambda u(x, y) + UH(x, y, \nabla_x u(x, y), \nabla_y u(x, y)) = 0 & \text{in } \Omega_X \times \Omega_Y, \\
u = \Psi_X & \text{on } \partial \Omega_X \times \Omega_Y \\
u = \Psi_Y & \text{on } \Omega_X \times \partial \Omega_Y \\
u = \Psi_X \text{ or } u = \Psi_Y & \text{on } \partial \Omega_X \times \partial \Omega_Y \\
\end{cases}
\]

(13)

\[
\begin{cases}
\lambda u(x, y) + LH(x, y, \nabla_x u(x, y), \nabla_y (x, y)) = 0 & \text{in } \Omega_X \times \Omega_Y, \\
u = \Psi_X & \text{on } \partial \Omega_X \times \Omega_Y \\
u = \Psi_Y & \text{on } \Omega_X \times \partial \Omega_Y \\
u = \Psi_X \text{ or } u = \Psi_Y & \text{on } \partial \Omega_X \times \partial \Omega_Y \\
\end{cases}
\]

(14)

where \( \nabla_x \) and \( \nabla_y \) stay, respectively, for the gradient with respect to the \( x \in \mathbb{R}^n \) variable, and the gradient with respect to the \( y \in \mathbb{R}^m \) variable.

In [2], an ad-hoc definition of viscosity solution is given and used, especially for what concerns the boundary conditions, and in order to suitably treat the min-max feature of the problem and the separation of the three exist costs. By a solution in the viscosity sense of the problem (13) (and similarly for (14)), we mean the following: let \( \varphi \in C^1(\overline{\Omega_X} \times \overline{\Omega_Y}) \) and \((x_0, y_0) \in \overline{\Omega_X} \times \overline{\Omega_Y} \), then the following facts i) and ii) hold true:

i) if \((x_0, y_0)\) is a point of local maximum for \( u - \varphi \), with respect to \( \overline{\Omega_X} \times \overline{\Omega_Y} \), then we have the following four implications (one per every line)

\[
\begin{cases}
(x_0, y_0) \in \Omega_X \times \Omega_Y, \\
(x_0, y_0) \in \partial \Omega_X \times \partial \Omega_Y, \ u(x_0, y_0) > \psi_X(x_0, y_0), \\
(x_0, y_0) \in \Omega_X \times \partial \Omega_Y, \ u(x_0, y_0) > \psi_Y(x_0, y_0), \\
(x_0, y_0) \in \partial \Omega_X \times \partial \Omega_Y, \ \psi_X(x_0, y_0) \neq u(x_0, y_0) > \psi_Y(x_0, y_0) \\
\end{cases}
\]

\[ \lambda u(x_0, y_0) + UH(x_0, y_0, \varphi(x_0, y_0), \varphi_y(x_0, y_0)) \leq 0; \]

(15)
ii) if \((x_0, y_0)\) is a point of local minimum for \(u - \varphi\), with respect to \(\overline{\Omega}_X \times \overline{\Omega}_Y\), then we have the following four implications (one per every line)

\[
\begin{align*}
(x_0, y_0) \in \Omega_X \times \Omega_Y, \\
(x_0, y_0) \in \partial \Omega_X \times \Omega_Y, \quad u(x_0, y_0) < \psi_X(x_0, y_0), \\
(x_0, y_0) \in \Omega_X \times \partial \Omega_Y, \quad u(x_0, y_0) < \psi_Y(x_0, y_0), \\
(x_0, y_0) \in \partial \Omega_X \times \partial \Omega_Y, \quad \psi_Y(x_0, y_0) \neq u(x_0, y_0) < \psi_X(x_0, y_0)
\end{align*}
\]

\(\implies (16)\)

\[
\lambda u(x_0, y_0) + UH(x_0, y_0, \varphi_x(x_0, y_0), \varphi_y(x_0, y_0)) \geq 0;
\]

The implications given by the second, third and fourth lines of \((15) - (16)\) represent the boundary conditions in the viscosity sense.

**Remark 3** Note that in the definition of the boundary conditions in viscosity sense here above, the exit cost \(\Psi_{XY}\) for the simultaneous exit of both players, actually does not play any role. This is a consequence of the compatibility condition in \((12)\), which is, in some sense, a sort of stability: the exit cost for the minimizing player is larger than the cost of the maximizing one.

For the continuity and uniqueness results of Theorem 2, very important roles are played by the decoupled feature of the dynamics \((5)\) and of the running cost \((7)\), by the controllability and compatibility conditions in \((12)\), and by the regularity of the boundaries.

In particular, for what concerns the continuity, in [2], it is proved that, under hypotheses \((6) - (12)\), the following property holds:

for every \(T > 0\), for every \(K \subset \mathbb{R}^n \times \mathbb{R}^m\) compact, there exist \(\delta > 0\) and a modulus of continuity \(O_{T,K}\), and:

1) for every \((x_1, y_1), (x_2, y_2)\) \(\in K \cap (\overline{\Omega}_X \times \overline{\Omega}_Y)\), with \(\|(x_1, y_1) - (x_2, y_2)\| \leq \delta\) there exists a non-anticipating tuning \(\beta \rightarrow \overline{\beta}\) from \(\mathcal{B}\) to itself, and there exists a way to associate \(\gamma \in \Gamma\) to any \(\gamma \in \Gamma\), such that, for every \(\beta \in \mathcal{B}\), \(\gamma \in \Gamma\), we have

\[
\begin{align*}
i) & \quad 0 \leq \tau_{x_1} (\gamma(\beta)) - \tau_{x_2} (\gamma(\beta)) \leq O_{T,K} (\|x_1 - x_2\|), \\
ii) & \quad 0 \leq \tau_{y_1} (\beta) - \tau_{y_2} (\beta) \leq O_{T,K} (\|y_1 - y_2\|), \\
niiii) & \quad \|X(t; x_1, \gamma(\beta)) - X(t; x_2, \gamma(\beta))\| \leq O_{T,K} (\|x_1 - x_2\|), \quad \forall t \in [0, \tilde{\tau}] \\
niv) & \quad \|Y(t; y_1, \beta) - Y(t; y_2, \overline{\beta})\| \leq O_{T,K} (\|y_1 - y_2\|), \quad \forall t \in [0, \tilde{\tau}] \\
v) & \quad |J_{\beta}(x_1, y_1, \gamma(\beta), \beta) - J_{\beta}(x_2, y_2, \gamma(\overline{\beta}), \overline{\beta})| \\
& \quad \leq O_{T,K} (\|x_1 - x_2\|, \|y_1 - y_2\|)
\end{align*}
\]

where \(\tilde{\tau} = \min\{\tau_{x_2} (\gamma(\overline{\beta})), \tau_{y_1} (\beta), T\}\), and \(J_{\beta}\) is the integral of the discounted running cost up to the time \(\tilde{\tau}\): \(\int_0^{\tilde{\tau}} e^{-\lambda t} dt\).

II) Similarly it holds reversing the roles of \(X\) and \(Y\), \(\gamma \in \Gamma\) and \(\xi \in \Xi\), \(\alpha \in \mathcal{A}\) and \(\beta \in \mathcal{B}\).

We point out that the construction of \(\overline{\beta}\) and of \(\overline{\gamma}\) are made independently on the behavior of the other player. That is, \(\overline{\beta}(t)\) and \(\overline{\gamma}(\beta)\) are dependent only on the behavior, up to the time \(t\), of the trajectories \(Y(\cdot; y_1, \beta)\) and \(X(\cdot; x_2, \gamma(\beta))\), respectively. This is possible essentially due to the decoupling feature in the controls of the running cost \(\ell(7)\) (see [2], section 7, Remark 12).

Note that i) means that the trajectory starting from \(x_1\) with control \(\gamma(\beta)\) does not exit before the trajectory starting from \(x_2\) with control \(\gamma(\overline{\beta})\), and moreover, the difference of the two exit instants are controlled by the initial distance of the points; ii) means that the trajectory starting from \(y_2\) with control \(\beta\) does not exit before the trajectory starting from \(y_1\) with control \(\beta\), and the difference
is controlled by the initial distance; iii), iv), v) mean that the distance of those (and other similar)
trajectories and their costs are controlled by the initial distances. Of course, if \( \tau_X(\gamma[\beta]) = +\infty \),
then both trajectories never exit, and similarly if \( \tau_Y(\beta) = +\infty \).

This property is essential in order to prove the continuity. Under the hypotheses here stated,
its validity is proven in [2] (see Assumption 2, points 3) and 7) of Proposition 3, and (7)–(10)),
suitably adapting the construction in Soner [26] to the non-anticipating framework.

We are going to use (17) in the next sections.

4 The switching infinite horizon differential game

The decoupled controlled dynamics of the players are respectively given by

\[
\begin{align*}
X'(t) &= f(X(t), W(t), \alpha(t)), \quad t > 0 \\
W(t) &= h_\rho[X \cdot \zeta_X; w](t) \\
(X(0), w(0)) &= (x, w)
\end{align*}
\]

\[
\begin{align*}
Y'(t) &= g(Y(t), Z(t), \beta(t)), \quad t > 0 \\
Z(t) &= h_\eta[Y \cdot \zeta_Y; z](t) \\
(Y(0), z(0)) &= (y, z)
\end{align*}
\]

(18)

where each dynamics is affected by a delayed thermostatic switching rule.

Here and in the sequel we will assume the following hypotheses and use the following notations:

**Main Assumptions**

\begin{itemize}
  \item \( X(t) = X(t; x, w, \alpha) \in \mathbb{R}^n \), \( Y(t) = Y(t; y, z, \beta) \in \mathbb{R}^m \) are the states at time \( t \) of the player \( X \) and player \( Y \) whose evolution is given by the trajectories of (18), respectively (here and in
the sequel, the names of the players will be identified with the names of their state variable);
  \item \( h_\rho \) and \( h_\eta \) are delayed switching thermostat with thresholds \( \rho_1 < \rho \) and \( \eta_1 < \eta \), respectively;
  \item \( W(t) = W(t; x, w, \alpha) \in \{-1, 1\} \) and \( Z(t) = Z(t; y, z, \beta) \in \{-1, 1\} \) are the switching variables,
with evolution given by (18), respectively;
  \item \( \zeta_X \in \mathbb{R}^n \), \( \zeta_Y \in \mathbb{R}^m \) are unit vectors; \( X \cdot \zeta_X \), \( Y \cdot \zeta_Y \) are scalar products in \( \mathbb{R}^n \) and \( \mathbb{R}^m \),
respectively, and represent the input functions \( t \mapsto X(t) \cdot \zeta_X, t \mapsto Y(t) \cdot \zeta_Y \) to which, via the
delayed thermostats, the switching laws of the variables \( W \) and \( Z \) are subject;
  \item \( A, B \) (the sets of constants controls), \( A, B \) (the sets of measurable controls), \( \Gamma \) and \( \Xi \) (the
sets of non-anticipating strategies) are defined as in (6) and (10); a non-anticipating tuning
\( k \mapsto \tilde{k} \) is defined as in (11);
  \item \( (x, w) \in \mathbb{R}^n \times \{-1, 1\} \), \( (y, z) \in \mathbb{R}^m \times \{-1, 1\} \) are suitable initial data;
  \item \( f : \mathbb{R}^n \times \{-1, 1\} \times A \rightarrow \mathbb{R}^n \), \( g : \mathbb{R}^m \times \{-1, 1\} \times B \rightarrow \mathbb{R}^m \) are the switching controlled
dynamics of player \( X \) and player \( Y \), respectively. Moreover they are continuous, bounded
and Lipschitz in the state variables, i.e.
\[ \exists M > 0 \text{ such that } \forall (x, y, w, z, a, b) \quad \|f(x, w, a)\|, \|g(y, z, b)\| \leq M, \]
\end{itemize}
and
\[ \exists L > 0 \text{ such that } \forall (x_1, w, a), (x_2, w, a), (y_1, z, b), (y_2, z, b), \]
\[ \| f(x_1, w, a) - f(x_2, w, a) \| \leq L \| x_1 - x_2 \|, \]
\[ | g(y_1, z, b) - g(y_2, z, b) | \leq L \| y_1 - y_2 \|. \]

• \( \ell : \mathbb{R}^n \times \mathbb{R}^m \times \{-1, 1\} \times \{-1, 1\} \times A \times B \rightarrow [0, +\infty] \), \( (x, y, w, z, a, b) \mapsto \ell(x, y, w, z, a, b) = \ell_1(x, y, w, z, a) + \ell_2(x, y, w, z, b) \) is the running cost, decoupled in the controls, where \( \ell_1 \) and \( \ell_2 \) are continuous, bounded and Lipschitz continuous with respect to the state variables; in particular \( \exists M, L > 0 \) such that \( \forall x, y, x_1, x_2, y_1, y_2, w, z, a, b, \)
\[ \| \ell_1(x, y, w, z, a) \|, \| \ell_2(x, y, w, z, b) \| \leq M, \]
\[ \| \ell_1(x_1, y, w, z, a) - \ell_1(x_2, y, w, z, a) \| \leq L \| (x_1, y_1) - (x_2, y_2) \|, \]
\[ \| \ell_2(x_1, y_1, w, z, b) - \ell_2(x_2, y_2, w, z, b) \| \leq L \| (x_1, y_1) - (x_2, y_2) \|. \]

• \( \lambda > 0 \) is the discount factor.

We consider an infinite horizon discounted problem where, as usual, \( X \) wants to minimize and \( Y \) wants to maximize a cost of the form
\[ J(x, y, w, z, \alpha, \beta) = \int_0^{+\infty} e^{-\lambda t} \ell(X(t), Y(t), W(t), Z(t), \alpha(t), \beta(t)) \, dt, \]
Note that the cost \( J \) is also depending on the switching variables \( W \) and \( Z \).

We then define the lower and upper value functions as, respectively
\[ \underline{V}(x, y, w, z) = \inf_{\gamma \in \Gamma} \sup_{\beta \in B} J(x, y, w, z, \gamma[\beta], \beta), \]
\[ \overline{V}(x, y, w, z) = \sup_{\xi \in \Xi} \inf_{\alpha \in A} J(x, y, w, z, \alpha, \xi[\alpha]). \] (19)

In order to simplify notations, we assume that \( \xi_X \) and \( \xi_Y \) are the first unit canonical vectors, so that \( X_1 = X \cdot \xi_X \) and \( Y_1 = Y \cdot \xi_Y \) are the first coordinates of \( X \) and \( Y \), respectively.

Let us consider the evolution of \( X \) given by (18). We can interpret such an evolution as a switching evolution governed by two dynamics-modes, \( f(\cdot, 1, \cdot) \) and \( f(\cdot, -1, \cdot) \), where the switching between the two modes is governed by the delayed thermostat \( h_\rho \) subject to the evolution of \( X_1 \). Similarly, the evolution of the player \( Y \) given by (18), which is affected by the delayed thermostat \( h_\eta \) subject to the evolution of \( Y_1 \), switches between the two dynamics \( g(\cdot, 1, \cdot) \) and \( g(\cdot, -1, \cdot) \). The behavior of the projection on the first coordinates of \( X \) and \( Y \) respectively is described by Figure 2. For example, for given controls \( \alpha \in A \), and \( \beta \in B \), the filled curve is the evolution with dynamics \( (f(\cdot, -1, \alpha), g(\cdot, 1, \beta)) \), the short dashed curve is the evolution with \( (f(\cdot, 1, \alpha), g(\cdot, 1, \beta)) \), the long dashed curve is the evolution with \( (f(\cdot, 1, \alpha), g(\cdot, -1, \beta)) \) and the point-dashed one is the evolution with \( (f(\cdot, -1, \alpha), g(\cdot, -1, \beta)) \).
As it is easily deduced by the trajectories described in Figure 2, the state space $\mathbb{R}^n \times \mathbb{R}^m$ can be divided in 4 (non-disjointed, but overlapped) closed sectors, every one indexed by the corresponding 2-string $(w, z)$ of 1 and $-1$. More precisely

$$\mathbb{R}^n \times \mathbb{R}^m = \bigcup \{B_{(w,z)} : (w,z) \in \{-1, 1\}^2\}$$

where

$$B_{(1,1)} = [\rho_{-1}, +\infty[ \times [\eta_{-1}, +\infty[ \times [\mathbb{R}^{n-1} \times [\mathbb{R}^{m-1}.$$

$$B_{(-1,1)} = (\rho_1, +\infty] \times [\eta_{-1}, +\infty[ \times [\mathbb{R}^{n-1} \times [\mathbb{R}^{m-1}.$$

$$B_{(1,-1)} = (\eta_1, +\infty] \times [\rho_{-1}, +\infty[ \times [\mathbb{R}^{n-1} \times [\mathbb{R}^{m-1}.$$

$$B_{(-1,-1)} = (\eta_1, +\infty] \times [\rho_{-1}, +\infty[ \times [\mathbb{R}^{n-1} \times [\mathbb{R}^{m-1}.$$

When we start to move inside one of the sectors, then we continue to move in the same mode $(f(\cdot, w, \cdot), g(\cdot, z, \cdot))$ until we leave that sector, and after that we move in the new modality (corresponding to the index of the new sector) determined by the delayed thermostatic switching rules $h_\rho$ and $h_\eta$. In the next section we are going to interpret the switching infinite horizon problem as four exit-time/exit-costs problems, one per every sectors, and coupled by mutually exchanged exit-costs. In order to recast such exit-time/exit-costs problems in the framework of Section 3, let us note that every sector is of the form $B_{(w,z)} = \Omega^w_X \times \Omega^z_Y$ where, for example, $\Omega^1_X = [\rho_{-1}, +\infty[ \times [\mathbb{R}^{n-1} \times [\mathbb{R}^{m-1}.$

Note that $\partial \Omega^w_X = \{\rho_{-w}\} \times [\mathbb{R}^{n-1}$ and $\partial \Omega^z_Y = \{\eta_{-z}\} \times [\mathbb{R}^{m-1}$, that is the boundaries are the switching thresholds points (and the threshold is crossed when the first component crosses it). Moreover, we also require the following controllability assumption

$$\forall w \in \{-1, 1\}, \forall x \in \partial \Omega^w_X \exists a_1, a_2 \in A \text{ such that } f_1(x, w, a_1) < 0 < f_1(x, w, a_2),$$

$$\forall z \in \{-1, 1\}, \forall y \in \partial \Omega^z_Y \exists b_1, b_2 \in B \text{ such that } g_1(y, z, b_1) < 0 < g_1(y, z, b_2).$$

(21)
where \( f_1 \) and \( g_1 \) are, respectively, the first component of the dynamics \( f \) and \( g \). Note that \cite{21} means that, when \( X \) or \( Y \) are in a switching threshold, then, due to the decoupled feature of the dynamics, they can freely choose whether to switch or not (remember that a switch occur only when the threshold is bypassed).

Finally, note that whenever a finite time \( T > 0 \) is fixed, then any pair of switching trajectories \((W, Z)\) given by \cite{18} can switch only an a-priori bounded finite number of times in \([0, T]\). This is true because the dynamics are bounded and the thresholds are disjoint: for example, the trajectory \( X_1 \) needs a uniform positive time \( T > 0 \) in order to pass from \( \rho_{-1} \) to \( \rho_1 \) and vice-versa. Let \( N_T > 0 \) be such an a-priori bound for the number of switches in \([0, T]\). For every \((x, y, w, z)\) and every control \( \alpha \in \mathcal{A}, \beta \in \mathcal{B} \), and for every \( T > 0 \) we then have a finite sequence of switching instants (possibly empty, if the trajectories never switch) in \([0, T]\):

\[
\begin{align*}
\tau_1^X(x, w, \alpha) < \tau_2^X(x, w, \alpha) < \cdots &< \tau_N^X(x, w, \alpha) \\
\tau_1^Y(y, z, \beta) < \tau_2^Y(y, z, \beta) < \cdots &< \tau_N^Y(y, z, \beta) \\
\tau_1^{XY}(x, y, w, z, \alpha, \beta) < \tau_2^{XY}(x, y, w, z, \alpha, \beta) < \cdots &< \tau_N^{XY}(x, y, w, z, \alpha, \beta)
\end{align*}
\]

(22)

with \( N = N_X + N_Y + N_{XY} \leq N_T \), and where \( \tau_X \) correspond to switches of \( X \) only, \( \tau_Y \) to switches of \( Y \) only, and \( \tau_{XY} \) to simultaneous switches of \( X \) and \( Y \) (which means \( \tau_X = \tau_Y \)). We can merge such three sequences, in order to get a unique sequence

\[
\tau^0 := 0 < \tau^1 < \tau^2 < \cdots < \tau^N \leq T =: \tau^{N+1}, \quad N \leq N_T.
\]

(23)

Moreover, we denote by \((w, z)^i\) the new values of the switching variables after the \( i \)-th switch, \( i = 1, \ldots, N \), and define \((w, z)^0 = (w, z)\). We have

\[
\begin{align*}
J_T(x, y, w, z, \alpha, \beta) = & \sum_{i=1}^{N+1} \int_{\tau^i}^{\tau^{i+1}} e^{-\lambda t} \ell(X(t), Y(t), (w, z)^{i-1}, \alpha(t), \beta(t)) \, dt = \\
& \sum_{i=1}^{N+1} \int_0^{\tau^{i+1}-\tau^i} e^{-\lambda t} \ell(X^{i-1}(t), Y^{i-1}(t), (w, z)^{i-1}, \alpha^{i-1}, \beta^{i-1}) \, dt =
\end{align*}
\]

(24)

where \((X^{i-1}, Y^{i-1})\) is the trajectory starting from \((X(\tau^{i-1}), Y(\tau^{i-1}), (w, z)^{i-1})\) with controls \((\alpha^{i-1}(\cdot), \beta^{i-1}(\cdot)) = (\alpha(\cdot + \tau^{i-1}), \beta(\cdot + \tau^{i-1}))\).

Using the representation \cite{24}, in the spirit of \cite{17}, we now construct a non-anticipating tuning and a non-anticipating strategy which work for our switching problem. Take \( \mu > 0 \) such that if a trajectory switches at time \( \tau \), then it does not switch in the time interval \([\tau, \tau + \mu]\). Take \( T > 0 \) and \( K \subset \mathbb{R}^n \times \mathbb{R}^m \) compact. Fix \((w, z)\) and \((x_1, y_1), (x_2, y_2) \in K \cap \overline{\mathcal{B}}(w, z)\) such that \( \|(x_1, y_1) - (x_2, y_2)\| \leq \delta \), where \( \delta \) is as in \cite{17}, and moreover such that \( \mathcal{O}_{T, K}(e^{LT} \delta) < \mu/2 \). Take \( \beta \in \mathcal{B} \) and \( \gamma \in \Gamma \). By \cite{17}, with the notations of \cite{22}, we get the non-anticipating tuning \( \beta^0 \) and the non-anticipating strategy \( \tau^0 \) such that
0 \leq \tau^1_t(y_2, z, \beta) - \tau^2_t(y_1, z, \beta) \leq \mathcal{O}_{T,K}(\|y_1 - y_2\|)
0 \leq \tau^1_t(x, w, \gamma[\beta]) - \tau^3_t(x, w, \gamma[\beta]) \leq \mathcal{O}_{T,K}(\|x - x_2\|),

\|Y_1(\tau^1_t(y_1, z, \beta)) - Y_2(\tau^1_t(y_2, z, \beta))\| \leq \mathcal{O}_{T,K}(\|y_1 - y_2\|)
\|X_1(\tau^1_t(x, w, \gamma[\beta])) - X_2(\tau^3_t(x, w, \gamma[\beta]))\| \leq \mathcal{O}_{T,K}(\|x - x_2\|)
\|J_\tau(x_1, y_1, w, z, \gamma[\beta]) - J_\tau(x_2, y_2, w, z, \gamma[\beta])\| \leq \mathcal{O}_{T,K}(\|(x_1, y_1) - (x_2, y_2)\|)

(25)

where \(\tau^1 = \min\{\tau^1_t(x_2, w, \gamma[\beta]), \tau^3_t(y_1, z, \beta), T\}\). Our goal is to estimate the difference of the two \(J_T\) costs. If \(\tau^1 = T\), then we are done. Otherwise, we have some cases. We analyze some of them, being the others similarly treated.

1) Suppose that, using the notation of (23),

\[
\tau^1 = \tau^1_t(x_2, w, \gamma[\beta]) < \tau^2 = \tau^1_t(y_1, z, \beta) < \\
\tau^3 = \tau^1_t(x_1, w, \gamma[\beta]) < \tau^4 = \tau^1_t(y_2, z, \beta)
\]

By (25), this implies that all four switchings occur in a lap of time not greater than \(2\mathcal{O}_{T,K}(\|(x_1, y_1) - (x_2, y_2)\|) < \mu\), which also implies that, in the meanwhile, no trajectory can switch two times. We then have the pairs \((X_1(\tau^1), -w, X_2(\tau^4), -w)\), as well as the pairs \((Y_1(\tau^4), -z, Y_2(\tau^4), -z)\).

At the instant \(\tau^3\), \(X_1\) updates its non-anticipating strategy using \(\tau^3\), as given in the view of (17), referring to the points \((X_1(\tau^3), -w), (X_2(\tau^3), -w)\), to the trajectory \(X_2(\cdot; X_2(\tau^3), -w, \gamma[\beta](\cdot + \tau^3))\), and with respect the exit from \(\bar{\Omega}^{X^z}\), in the time interval \([0, T - \tau^3]\). Similarly, at \(\tau^4\), \(Y_2\) updates its non-anticipating tuning using \(\tau^4\), as given in the view of (17), referring to the points \((Y_1(\tau^4), -z), (Y_2(\tau^4), -z)\), to the trajectory \(Y_1(\cdot; Y_1(\tau^4), -z, \gamma[\beta](\cdot + \tau^4))\), and with respect the exit from \(\bar{\Omega}^{Y^z}\), in the time interval \([0, T - \tau^4]\). Gluing together, we get the non-anticipating tuning and non-anticipating strategy

\[
\beta \mapsto \hat{\beta} : t \mapsto \begin{cases} 
\beta^0(t) & \text{if } 0 \leq t \leq \tau^4 \\
\beta^1(t - \tau^4) & \text{if } t > \tau^4
\end{cases} \forall \beta \in \mathcal{B}
\]

(26)

\[
\beta \mapsto \hat{\gamma}(\beta) : t \mapsto \begin{cases} 
\tau^0(\beta)(t) & \text{if } 0 \leq t \leq \tau^3 \\
\tau^3(\beta)(t - \tau^3) & \text{if } t > \tau^3
\end{cases} \forall \beta \in \mathcal{B}
\]

(27)

Note that such constructions are non-anticipating in the sense of (10), (11), because they use already given non-anticipating constructions and glue them in dependence of the behavior of the trajectories (solutions of (18)), which are non-anticipating (the state-position only depends on the past behavior).

Let \(\tau^5 \geq \tau^4\) be a possible subsequent switching of one of the trajectories when continuing to move with \(\hat{\beta}\) and \(\hat{\gamma}(\beta)\). Hence, looking to (24), we have
Given the Main Assumptions of Section 4 and (21), the value functions $V_\gamma$, $\overline{V}_\gamma$ are bounded and continuous in $(x, y) \in \overline{B}_{(w, z)}$, for all $(w, z) \in \{-1, 1\} \times \{-1, 1\}$.
Proof: We prove the proposition for $V$, being the proof for $\overline{V}$ similar. We are going to use the notations of Section 4.

The boundedness of $\overline{V}$ is easily seen, by the boundedness of $\ell$ and the positivity of the discount factor $\lambda > 0$.

Let us fix $\varepsilon > 0$ and take $T > 0$ such that, for all possible trajectories and controls entering the cost $\ell$, it is $\int_T^{\infty} e^{-\lambda t} \ell dt \leq \varepsilon$. Moreover take a compact $K \subset \mathbb{R}^n \times \mathbb{R}^m$, and, for a fixed pair $(w, z)$ take $(x_1, y_1), (x_2, y_2) \in \overline{B}_{(w, z)} \cap K$ such that $\|(x_1, y_1) - (x_2, y_2)\| \leq \delta$, where $\delta > 0$ is given in Proposition 4 with respect to $T$ and $K$.

Take $\gamma_2 \in \Gamma$ which realizes $V(x_2, y_2, w, z)$ up to an $\varepsilon$-error, and consider $\hat{\gamma}_2 \in \Gamma$ as the one in Proposition 4 with respect to $T, K, (w, z)$, and $(x_1, y_1), (x_2, y_2)$. We get

$$V(x_1, y_1, w, z) - V(x_2, y_2, w, z) \leq \sup_{\beta \in \mathcal{B}} J(x_1, y_1, w, z, \gamma_2[\beta], \beta) - \sup_{\beta \in \mathcal{B}} J(x_2, y_2, w, z, \gamma_2[\beta], \beta) + \varepsilon \leq (30)$$

where, $\beta_1 \in \mathcal{B}$ realizes the supremum in the first addendum of the second line up to an $\varepsilon$-error, and $\hat{\beta}_1$ is as in Proposition 4 as before. Recalling the definition of $T$, using again Proposition 4 and continuing with the inequalities (30), we get

$$V(x_1, y_1, w, z) - V(x_2, y_2, w, z) \leq J_T(x_1, y_1, w, z, \gamma_2[\hat{\beta}_1], \hat{\beta}_1) - J_T(x_2, y_2, w, z, \gamma_2[\hat{\beta}_1], \hat{\beta}_1) + 4\varepsilon \leq \omega_{T,K} \|(x_1, y_1) - (x_2, y_2)\| + 4\varepsilon$$

from which, as usual, by the arbitrariness of $\varepsilon > 0$, of the compact $K$ and of the points, we get the required continuity. \hfill $\square$

**Proposition 6** Given the Main Assumptions of Section 4 and (27), then $V$ and $\overline{V}$ respectively satisfy

$$V(x, y, w, z) = \inf_{\gamma \in \Gamma} \sup_{\beta \in \mathcal{B}} \left( \int_0^T e^{-\lambda s} \ell(X(s), Y(s), w, z, \gamma[\beta](s), \beta(s)) \, ds + e^{-\lambda T} V(X(T), Y(T), (w, z)^+) \right)$$

$$\overline{V}(x, y, w, z) = \sup_{\xi \in \chi} \sup_{\alpha \in \mathcal{A}} \left( \int_0^T e^{-\lambda s} \ell(X(s), Y(s), w, z, \alpha(s), \xi[s](s)) \, ds + e^{-\lambda T} \overline{V}(X(T), Y(T), (w, z)^+) \right)$$

where $X(s) = X(s; x, \gamma[\beta], \beta), Y(s) = Y(s; y, \alpha, \xi[\alpha]), \tau$ is the first switching instant and

$$(w, z)^+ = \begin{cases} (-w, z) & \text{if } \tau = \tau_X < \tau_Y \\ (-w - z) & \text{if } \tau = \tau_X = \tau_Y \\ (w, -z) & \text{if } \tau = \tau_Y < \tau_X \end{cases}$$

is the first “switched” label.

Proof. We only prove the equality for $V$. We recall that, in our definition, the switching occurs when the threshold is by passed. This is the reason for which we consider instants a little bit
larger than the switching time, see \( \tau_n \) here below. We will also use the estimates [34] which will be discussed in the next section.

Let us denote by \( p = (x, y, w, z) \in \mathcal{B}_{(w, z)} \times \{(w, z)\} \) any admissible state and, by \( p_p(\cdot; \gamma[\beta], \beta) \) the corresponding trajectory. Let us denote by \( \omega(p) \) the right-hand side of the equality.

Let us fix \( \varepsilon > 0 \) and, and for any \( p' \) let \( \gamma_{p'} \in \Gamma \) be such that

\[
V(p') \geq \sup_{\beta \in B} J(p', \gamma_{p'}[\beta], \beta) - \varepsilon
\]

Claim: \( V(p) \leq \omega(p) \).

For every natural number \( n > 0 \), let us take \( \gamma_n \in \Gamma \) such that

\[
\omega_n(p) \geq \sup_{\beta \in B} \left( \int_0^{\tau_n} e^{-\lambda s} \ell(X(s), Y(s), w(s), z(s), \gamma_n[\beta](s), \beta(s)) \, ds + e^{-\lambda \tau_n} V(X(\tau_n), Y(\tau_n), w(\tau_n), z(\tau_n)) \right) - \varepsilon
\]

where \( \tau_n = \tau_p[\gamma_n, \beta] + 1/n \) and

\[
\omega_n(p) = \inf_{\gamma \in \Gamma} \sup_{\beta \in B} \left( \int_0^{\tau_n} e^{-\lambda s} \ell(X(s), Y(s), w(s), z(s), \gamma[\beta](s), \beta(s)) \, ds + e^{-\lambda \tau_n} V(X(\tau_n), Y(\tau_n), w(\tau_n), z(\tau_n)) \right)
\]

For \( \beta \in \mathcal{B} \), we defining \( p_n = p_p(\tau_n; \gamma[\beta], \beta) \), and \( \delta_n \in \Gamma \) as

\[
\delta_n[\beta](s) = \begin{cases} 
\gamma_n[\beta](s) & \text{if } 0 \leq s \leq \tau_n \\
\gamma_p[\beta](s - \tau_n) & \text{if } s \geq \tau_n
\end{cases}
\]

Arguing as in Bardi-Capuzzo Dolcetta [4] page 437-438, we eventually get

\[
\omega_n(p) \geq \sup_{\beta \in B} J(p, \delta_n[\beta], \beta) - 2\varepsilon \geq V(p) - 2\varepsilon
\]

By the arbitrariness of \( \varepsilon > 0 \) the claim is proved if we prove that \( \omega_n(p) - \omega(p) \leq \mathcal{O}(1/n) \) as \( n \to +\infty \), where \( \mathcal{O} \) is an infinitesimal function as its argument tends to zero. For \( \varepsilon > 0 \), take \( \gamma_\varepsilon \in \Gamma \) and \( \beta_\varepsilon \in \mathcal{B} \) such that

\[
\omega_n(p) - \omega(p) \leq \int_0^{\tau_\varepsilon} e^{-\lambda s} \ell(X(s), Y(s), W(s), Z(s), \gamma_\varepsilon[\beta_\varepsilon](s), \beta_\varepsilon(s)) \, ds + e^{-\lambda \tau_\varepsilon} V(p_n) - \int_0^{\tau_\varepsilon} e^{-\lambda s} \ell(X(s), Y(s), w, z, \gamma_\varepsilon[\beta_\varepsilon](s), \beta_\varepsilon(s)) - e^{-\lambda \tau_\varepsilon} V(p_{\text{switched}}) + 2\varepsilon
\]

where \( \tau_\varepsilon \) is the first switching instant, depending on \( \gamma_\varepsilon \) and \( \beta_\varepsilon \). In particular,

\[
\omega(p) \geq \sup_{\beta \in B} \left( \int_0^{\tau_\varepsilon} e^{-\lambda s} \ell(X(s), Y(s), w, z, \gamma_\varepsilon[\beta](s), \beta(s)) \, ds + e^{-\lambda \tau_\varepsilon} V(p_{\text{switched}}) \right) - \varepsilon
\]
and $p_{\text{switched}} = (X(\tau^\varepsilon), Y(\tau^\varepsilon), (w, z)^+)$.
In order to estimate the second member in (31), we essentially need to compare the values $V(p_n)$ and $V(p_{\text{switched}})$, which may have different switching variables, if $p_n$ has an immediate switching after the one of $p_{\text{switched}}$ at time $\tau^\varepsilon$.

We denote $p_{\text{switch}} = (X(\tau^\varepsilon), Y(\tau^\varepsilon), w, z)$. It is not restrictive to assume that $\gamma_\varepsilon$ is such that, whenever for some $\beta$ it is $p_{\text{switch}} \in \partial \Omega_X^w \times \partial \Omega_Y^z \times \{(w, z)\}$ (i.e. it is a point of possible double switch), $(w, z)^+ = (w, -z)$ (that is only $Y$ switches), and $V(p_{\text{switched}}) < V(X(\tau^\varepsilon), Y(\tau^\varepsilon), -w, -z)$, then there exists $\zeta > 0$ such that the trajectory does not switch again (that is $X$ does not switch) in $[\tau^\varepsilon, \tau^\varepsilon + \zeta]$ (if $\zeta$ is sufficiently small, then certainly $Y$ does not switch again because it has just switched and hence, to do that, it needs to reach the other threshold). Indeed, we can consider $\gamma'_\varepsilon$ defined as

$$
\gamma'_\varepsilon[\beta](t) = \begin{cases} 
\gamma_\varepsilon[\beta](t) & \text{if } 0 \leq t \leq \tau^\varepsilon \\
\gamma_\varepsilon[\beta](t) & \text{if } t \geq \tau^\varepsilon \text{ and } p_{\text{switched}} \notin \partial \Omega_X^w \times \partial \Omega_Y^z \times \{(w, z)\} \\
\gamma_\varepsilon[\beta](t) & \text{if } t \geq \tau^\varepsilon, \text{ and } V(p_{\text{switched}}) = V(X(\tau^\varepsilon), Y(\tau^\varepsilon), -w, -z) \\
a_0 & \text{if } t \geq \tau^\varepsilon, \text{ and } V(p_{\text{switched}}) < V(X(\tau^\varepsilon), Y(\tau^\varepsilon), -w, -z)
\end{cases}
$$

where $a_0$ is inward-pointing in $(X(\tau^\varepsilon), w)$, and we have that $\gamma'_\varepsilon$ still satisfies (32). We can then always assume that, if $p_{\text{switch}} \in \partial \Omega_X^w \times \partial \Omega_Y^z \times \{(w, z)\}$, then

$$(w, z)^+ = (w, -z) \text{ and } V(p_{\text{switched}}) < V(X(\tau^\varepsilon), Y(\tau^\varepsilon), -w, -z) \implies (w_n, z_n) = (w, -z) \text{ for large } n,
$$

where $(w_n, z_n)$ is the actual switching variable of $p_n$. Since $(w_n, z_n) = (w, z)^+$ for large $n$ whenever $p_{\text{switch}} \notin \partial \Omega_X^w \times \partial \Omega_Y^z \times \{(w, z)\}$, as well as whenever $p_{\text{switch}} \in \partial \Omega_X^w \times \partial \Omega_Y^z \times \{(w, z)\}$ and $(w, z) = (-w, -z)$, in these three cases, by the continuity of $V$, we get the convergence $V(p_n) \to V(p_{\text{switched}})$. Two other cases remain when $p_{\text{switch}} \in \partial \Omega_X^w \times \partial \Omega_Y^z \times \{(w, z)\}$. In both we use (34) and the continuity of $V$.

1) \((w, z)^+ = (-w, z) \implies V(p_{\text{switched}}) \geq V(p_n) + O \left(\frac{1}{n}\right)\);

2) \((w, z)^+ = (w, -z) \text{ and } V(p_{\text{switched}}) = V(X(\tau^\varepsilon), Y(\tau^\varepsilon), -w, -z) \implies V(p_{\text{switched}}) \geq V(p_n) + O \left(\frac{1}{n}\right)\),

where, in 2) we used the fact that, by (34), it cannot be $V(p_{\text{switched}}) > V(X(\tau^\varepsilon), Y(\tau^\varepsilon), -w, -z)$, and that, if $(w, z)^+ = (w, -z)$ it cannot be $(w_n, z_n) = (w, -z)$ because $z$ is already switched at the time $\tau^\varepsilon$.

Hence, in any case we get $V(p_{\text{switched}}) \geq V(p_n) + O \left(\frac{1}{n}\right)$ and, by the obvious convergence of the integrals in (31), we got the desired estimate.

Claim: \(V(x, y, w, z) \geq \omega(x, y, w, z)\).

Arguing as in Bardi-Capuzzo Dolcetta [11] page 437–438 we can prove that (with the same notations as in the previous step), for any $\varepsilon > 0$

$$
\omega_n(p) \leq V(p) + 3\varepsilon
$$
The conclusion then still holds because we also have $\omega(p) - \omega_n(p) \leq O(1/n)$. Indeed, reversing the roles of $\omega$ and $\omega_n$ in (31), (32), we have that if $(w_n, z_n) = (-w, z)$ or $(w_n, z_n) = (w, -z)$ then $(w, z)^+ = (w_n, z_n)$; moreover if $(w_n, z_n) = (-w, -z)$ for all $n$ then also $(w, z)^+ = (-w, -z)$. □

6 The HJI systems and uniqueness

By Proposition 6, for every $(w, z) \in \{-1, 1\}$, on $\overline{B}_{w, z}$ the lower value function $V$ of the infinite horizon problem can be interpreted as the lower value function of the exit-time/exit-costs differential game with dynamics $(x, a) \mapsto f(x, w, a)$, $(y, b) \mapsto g(y, z, b)$, running cost $(x, y, a, b) \mapsto \ell(x, y, w, z, a, b)$ and exit costs (using the same notations as in Section 3)

$$
\Psi_X(\cdot, \cdot) = V(\cdot, \cdot, -w, z), \quad \Psi_{XY}(\cdot, \cdot) = V(\cdot, \cdot, -w, -z), \quad \Psi_Y(\cdot, \cdot) = V(\cdot, \cdot, w, -z),
$$

and similarly for the upper value function $\nabla$. Moreover, under the controllability hypothesis (21) such costs satisfy the compatibility hypothesis in (12), here stated for $V \in \{V, \nabla\}$: for all $(x, y) \in \partial \Omega_X^w \times \partial \Omega_Y^w$

$$
V(x, y, w, -z) \leq V(x, y, -w, -z) \leq V(x, y, -w, z),
$$

(34)

Indeed, consider the double switched state $(x, y, -w, -z)$. Since $X$ minimizes, and since from the $Y$-only switched state $(x, y, w, -z)$ it may freely decide to switch or not, it is

$$
V(x, y, w, -z) \leq V(x, y, -w, -z).
$$

Indeed, from the position $(x, y, w, -z)$ $X$ has at its disposal all the admissible trajectories starting from $(x, y)$ and lying in $\overline{B}_{(w, -z)}$, at least for a while, as well as all the admissible trajectories starting from $(x, y)$ and immediately moving in $\overline{B}_{(w, -z)}$. By minimization, the previous inequality holds. Symmetrically it holds for the maximizing player $Y$, getting $V(x, y, -w, -z) \leq V(x, y, -w, z)$.

For every $(w, z)$ fixed, we define, respectively, the upper Hamiltonian and the lower Hamiltonian for $(x, y, p, q) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, as

$$
UH_{(w, z)}(x, y, p, q) = \min_{b \in B} \max_{a \in A} \{ -f(x, w, a) \cdot p - g(y, z, b) \cdot q - \ell(x, y, w, z, a, b) \},
$$

$$
LH_{(w, z)}(x, y, p, q) = \max_{a \in A} \min_{b \in B} \{ -f(x, w, a) \cdot p - g(y, z, b) \cdot q - \ell(x, y, w, z, a, b) \}
$$

By the interpretation as exit-time/exit-costs on every sector $\overline{B}_{(w, z)}$, by the continuity of the value functions (Proposition 5), by the compatibility (34), and by Theorem 2 for every $(w, z)$ fixed, $V(\cdot, \cdot, w, z)$ and $\nabla(\cdot, \cdot, w, z)$ are the unique bounded and continuous viscosity solutions $u : \overline{B}_{(w, z)} \to \mathbb{R}$ of the following Isaacs Dirichlet problems in $\overline{B}_{(w, z)} (= \overline{\Omega}_X^w \times \overline{\Omega}_Y^z)$, with $H_{(w, z)} = UH_{(w, z)}$ and $H_{(w, z)} = LH_{(w, z)}$ respectively

$$
\begin{cases}
\lambda u(x, y) + H_{(w, z)}(x, y, \nabla_x u(x, y), \nabla_y u(x, y)) = 0 & \text{in } \text{int} \overline{B}_{(w, z)} \\
u(x, y) = V(x, y, -w, z) & \text{on } \partial \Omega_X^w \times \overline{\Omega}_Y^z \\
u(x, y) = V(x, y, -w, -z) & \text{on } \partial \Omega_X^w \times \partial \Omega_Y^z \\
u(x, y) = V(x, y, w, z) & \text{on } \Omega_X^w \times \partial \Omega_Y^z \\
u(x, y) = V(x, y, w, -z) & \text{on } \Omega_X^w \times \partial \Omega_Y^z
\end{cases}
$$

(35)

where the boundary conditions must be also interpreted in the viscosity sense as in (15)–(16).
For a function
\[ u : \bigcup_{(w,z)\in\{-1,1\} \times \{-1,1\}} (\overline{B}_{(w,z)} \times (w,z)) \to \mathbb{R} \]
we consider the following problem (four Isaacs Dirichlet problems coupled by the boundary conditions in the viscosity sense, that are mutually exchanged)
\[
\begin{cases}
\forall (w, z) \in \{-1,1\} \times \{-1,1\}, u \text{ solves} \\
\lambda u(x, y, w, z) + H_{(w,z)}(x, y, \nabla_x u, \nabla_y u) = 0 \quad \text{in } \text{int} \overline{B}_{(w,z)} \\
u(x, y, x, w) = u(x, y, -w, z) \quad \text{on } \partial \Omega^u_X \times \Omega^z_Y \\
u(x, y, w, z) = u(x, y, -w, -z) \quad \text{on } \partial \Omega^u_Y \times \partial \Omega^z_Y \\
u(x, y, w, z) = \alpha(x, y, w, z) \quad \text{on } \Omega^u_X \times \Omega^z_Y 
\end{cases}
\tag{36}
\]

**Theorem 7** Under the Main Assumptions in Section 4 and the controllability (21), we get that \( V \) (respectively \( \overline{V} \)) is the unique bounded and continuous viscosity solution of (36) with \( H_{(w,z)} = UH_{(w,z)} \) (respectively \( H_{(w,z)} = LH_{(w,z)} \)), satisfying, for every \((x, y, w, z) \in \partial \Omega^u_X \times \partial \Omega^z_Y \times \{(w, z)\}\)
\[
u(x, y, w, z) \leq u(x, y, -w, -z) \leq u(x, y, -w, z) \leq u(x, y, -w, z).
\tag{37}
\]

**Proof.** The fact that \(V\) and \(\overline{V}\) are solutions is explained here above. For the uniqueness, we are going to use a fixed point argument applied to a suitable functional defined on the subset \(C\) of the space
\[
X = \left\{ u : \bigcup_{w, z \in \{-1,1\}} (\overline{B}_{(w,z)} \times \{(w, z)\}) \to \mathbb{R} \left| u \text{ is continuous and bounded} \right. \right\}
\]
given by
\[
C = \left\{ u \in X \left| u \text{ satisfies \eqref{37}} \right. \right\}
\]
Endowed with the uniform convergence topology, \(C\) is a complete metric space. Also note that \(V, \overline{V} \in C\). We prove the uniqueness results for the system (36) corresponding to the case \( H_{(w,z)} = UH_{(w,z)} \) (i.e. the case solved by \(V\)). The other case is similar.

The construction of the functional is performed in three steps.

**First step.** We construct a functional \(T_1 : C \to X\) in the following way. Given \(u \in C\), for every \((w, z)\) fixed, the functions \(u(\cdot, \cdot, -w, z), u(\cdot, \cdot, -w, -z)\) and \(u(\cdot, \cdot, w, -z)\) give suitable boundary conditions for the sub-problem in (36) with that \((w, z)\) fixed (also compare with (35)). In particular, they are continuous and satisfy \eqref{37}. Let us denote by \(T_1[u; w, z] : \overline{B}_{(w,z)} \to \mathbb{R}\) such a unique solution. By Theorem 2, \(T_1[u; w, z]\) is the lower value function of the exit-time exit cost differential game with dynamics \((x, a) \mapsto f(x, w, a), \ (y, b) \mapsto g(y, z, b)\), running cost \((x, y, a, b) \mapsto \ell(x, y, w, z, a, b)\) and exit costs given by the values of \(u\) as before. Hence, we define the image of \(u \in C\) via \(T_1\) as
\[
T_1[u] : \bigcup_{(w,z)\in\{-1,1\} \times \{-1,1\}} (\overline{B}_{(w,z)} \times (w,z)) \to \mathbb{R}, \ (x, y, w, z) \mapsto T_1[u; w, z](x, y)
\tag{38}
\]
In general, we cannot guarantee that $T_1[u] \in C$, because it may not satisfy (37), but certainly it belongs to $X$. However, it satisfies similar inequalities as (37), that is, for every $(x, y, w, z) \in \partial \Omega_X \times \partial \Omega_Y \times \{(w, z)\}$

$$T_1[u](x, y, w, -z) \leq u(x, y, w, z) \leq T_1[u](x, y, w, -z) \leq T_1[u](x, y, -w, z)$$ \hspace{1cm} (39)

which can be proved similarly to (34), because, for example, from the point $(x, y, w, -z)$ if $X$ exits (the minimizing player), then the cost paid is $u(x, y, w, -z)$.

Second step. We construct a functional $T_2 : C \to X$ similarly to $T_1$ with the only difference that, the exit costs on the corner points $(x, y, w, z) \in \partial \Omega_X \times \partial \Omega_Y \times \{(w, z)\}$ are given by $T_1[u](x, y, w, z)$ and $T_1[u](y, w, -z)$ $(\Psi_X)$ and $T_1[u](y, w, w)$ $(\Psi_Y)$ if only $X$ or only $Y$ exits, respectively, and by $u(x, y, -w, -z)$ itself $(\Psi_{XY})$ for the case of simultaneous exit. In this way, by (39), the costs $\Psi_X, \Psi_Y, \Psi_{XY}$ satisfies (37). We then construct $T_2[u]$ as in the first step. Again, we have for every $(x, y, w, z) \in \partial \Omega_X \times \partial \Omega_Y \times \{(w, z)\}$

$$T_2[u](x, y, w, -z) \leq T_1[u](x, y, w, -z) \leq T_2[u](x, y, w, -z).$$

Note that, when $(x, y, w, z) \in \partial \Omega_X \times \partial \Omega_Y \times \{(w, z)\}$, then from $(x, y, w, -z)$ only $X$ can exit from $B(w, -z)$, and hence, in that case, the paid cost is $T_1[u]$.

Third step. We construct a functional $T_3 : C \to X$, as in the second step, but using $T_2[u]$ as exit costs for the exit of $X$ and $Y$ only, and $T_1[u]$ for the simultaneous exit.

It is evident that any solution $u \in C$ of (36) is a fixed point of $T_3$ (as well as of $T_1$ and $T_2$), and in particular $V$ is a fixed point of $T_3$. Hence a fixed point of $T_3$ exists: $V$. We now prove that $T_3$ is a contraction and so it admits at most one fixed point, which means that (36) admits at most one solution in $C$, $V$. From which the proof will be concluded.

Let us take two functions $u^1, u^2 \in C$ and a point $(x, y, w, z)$. By construction

$$T_1[u^i](x, y, w, z) = V_{(w, z)}^i(x, y) := \inf_{\gamma \in \Gamma} \sup_{\beta \in B} J_{(w, z)}^i(x, y, \gamma[\beta], \beta) = \inf_{\gamma \in \Gamma} \sup_{\beta \in B} \left( \int_0^{\tau} \ell(X(s), Y(s), \gamma[\beta](s), \beta(s), w, z) + e^{-\lambda \tau} u^i(X(\tau), Y(\tau), (w, z)^+) \right)$$

where $\tau$ is the exit time from $B(w, z)$, and $(w, z)^+$ is $(-w, z), (w, -z), (-w, -z)$ if only $X$ exits, only $Y$ exits, or they simultaneously exit, respectively.

Let us fix $\varepsilon > 0$. For suitable $\tilde{\gamma} \in \Gamma$, $\tilde{\beta} \in B$, by definition of infimum and supremum, we have

$$T_1[u^1](x, y, w, z) - T_1[u^2](x, y, w, z) = V_{(w, z)}^1(x, y) - V_{(w, z)}^2(x, y) = \leq J_{(w, z)}^1(x, y, \gamma[\beta], \beta) - J_{(w, z)}^2(x, y, \gamma[\beta], \beta) + 2\varepsilon$$ \hspace{1cm} (40)

Note that both $J^1$ and $J^2$ are concerning with the same dynamics and running cost, because they are governed by the same controls $\gamma[\beta]$ and $\beta$, starting from the same point $(x, y)$, the possible exit time $\tau$ from $B(w, z)$ is the same, and moreover the possible switched label $(w, z)^+$ is also the same. Hence, they may differ only for the paid exit cost, which is then paid in the same point $(X(\tau), Y(\tau))$ and with the same discount $e^{-\lambda \tau}$. If the trajectory does not exit, then the difference of the $J^i$’s in the second member of (40) is zero. Otherwise it is of the form $e^{-\lambda \tau}(u^1(X(\tau), Y(\tau), (w, z)^+) - u^2(X(\tau), Y(\tau), (w, z)^+))$. Hence we have
\[ T_1[u^1](x, y, w, z) - T_1[u^2](x, y, w, z) \leq e^{-\lambda T} u^1(X(\tau), Y(\tau), (w, z)^+) - u^2(X(\tau), Y(\tau), (w, z)^+) + 2\varepsilon \] (41)

Note that, in case of exit in finite time, if \( w^{+} = -w \) then \( X(\tau) \) has a distance from \( \partial \Omega^{-w}_X \) equal to \( 0 < \rho_1 - \rho_{-1} \), if \( z^{+} = -z \) then \( Y(\tau) \) has a distance from \( \partial \Omega^{-z}_Y \) equal to \( 0 < \eta_1 - \eta_{-1} \).

Applying the same reasoning to \( T_2 \), we obtain:

\[ T_2[u^1](x, y, w, z) - T_2[u^2](x, y, w, z) \leq e^{-\lambda \sigma} (\Psi^1(X(\sigma), Y(\sigma), (w, z)^+) - \Phi^1(X(\sigma), Y(\sigma), (w, z)^+)) + 2\varepsilon \]

where \( \Psi^i \) is the corresponding exit cost as from the construction of \( T_2 \) and \( \sigma \) is the possible exit time (corresponding to the choice of suitable \( \bar{\gamma} \) and \( \bar{\beta} \) as before). In particular, if only \( X \) or only \( Y \) exits then \( \Psi^i = T_1[u^i] \), otherwise, in case of simultaneous exit, it is \( u^i \).

Similarly for \( T_3 \)

\[ T_3[u^1](x, y, w, z) - T_3[u^2](x, y, w, z) \leq e^{-\lambda \nu} (\Phi^1(X(\nu), Y(\nu), (w, z)^+) - \Phi^2(X(\nu), Y(\nu), (w, z)^+)) + 2\varepsilon \]

where \( \Phi^i \) is the corresponding exit cost as from the construction of \( T_3 \) and \( \nu \) is the possible exit time (corresponding to the choice of suitable \( \bar{\gamma} \) and \( \bar{\beta} \) as before). In particular, if only \( X \) or only \( Y \) exits then \( \Phi^i = T_2[u^i] \), otherwise, in case of simultaneous exit, it is \( T_1[u^i] \).

Now, suppose that \( (X(\nu), Y(\nu), (w, z)^+) = (X(\nu), Y(\nu), -w, z) \), then

\[ T_3[u^1](x, y, w, z) - T_3[u^2](x, y, w, z) \leq e^{-\lambda \nu} (T_2[u^1](x, y, -w, z) - T_2[u^2](x, y, -w, z)) + 2\varepsilon. \]

It is

\[ T_2[u^1](x, y, -w, z) - T_2[u^2](x, y, -w, z) \leq e^{-\lambda \sigma} (\Psi^1(X(\sigma), Y(\sigma), (-w, z)^+)) + 2\varepsilon, \]

and suppose that \( (-w, z)^+ = (w, -z) \), then \( \Psi^i = u^i \) and \( \|X(\nu) - X(\sigma)\| \geq \rho_1 - \rho_{-1} \), which implies

\[ \sigma > \frac{\rho_1 - \rho_{-1}}{M} > 0 \]

where \( M \) is a bound for the dynamics. We then get

\[ T_3[u^1](x, y, w, z) - T_3[u^2](x, y, w, z) \leq e^{-\lambda(\nu + \sigma)} (u^1(X(\sigma), Y(\sigma), -w, z) - u^2(X(\sigma), Y(\sigma), -w, z)) + 4\varepsilon \leq e^{-\lambda(\sigma + \nu)} \|u^1 - u^2\|_{\infty} + 4\varepsilon \]

If instead, for example, the sequence of the switching variables is \((w, z) \rightarrow (-w, z) \rightarrow (-w, -z) \rightarrow (-w, z)\), then

\[ T_3[u^1](x, y, w, z) - T_3[u^2](x, y, w, z) \leq e^{-\lambda(\nu + \sigma + \tau)} (u^1(X(\tau), Y(\tau), -w, z) - u^2(X(\tau), Y(\tau), -w, z)) + 6\varepsilon \leq e^{-\lambda(\nu + \sigma + \tau)} \|u^1 - u^2\|_{\infty} + 6\varepsilon \]

23
where
\[\tau > \frac{\eta_1 - \eta_{-1}}{M} > 0\]
because \(\|Y(\sigma) - Y(\tau)\| \geq \eta_1 - \eta_{-1}\). Similarly for the other cases.

By the arbitrariness of \(\varepsilon > 0\) and of the point \((x, y, w, z)\), setting \(h = \min \left\{ \frac{\rho_1 - \rho_{-1}}{M}, \frac{\eta_1 - \eta_{-1}}{M} \right\} > 0\) we get
\[\|T_3[u^1] - T_3[u^2]\|_{\infty} \leq e^{-\lambda h} \|u^1 - u^2\|_{\infty}.\]

\[\square\]

Remark 8 By the uniqueness Theorem 7, whenever the Hamiltonians \(LH\) and \(UH\) are equal, then we get \(V = \overline{V}\); that is the differential game has an equilibrium. As usual, the equality of the Hamiltonians can be assured by some particular structure of the running cost \(\ell\), for example if, besides the already assumed decoupling feature as in the Main Assumptions in Section 4, it is also of the form \(\ell(x, y, w, z, a, b) = \ell_1(x, y, w, z) + \ell_2(a) + \ell_3(b)\).

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