Weyl’s Law and Connes’ Trace Theorem for Noncommutative Two Tori

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Abstract

We prove the analogue of Weyl’s law for a noncommutative Riemannian manifold, namely the noncommutative two torus $\mathbb{T}^2_\theta$ equipped with a general metric, by studying the asymptotic growth of the eigenvalues of its Laplacian. We also prove the analogue of Connes’ trace theorem for pseudodifferential operators of order $-2$ on $\mathbb{T}^2_\theta$ by showing that the Dixmier trace and a noncommutative residue coincide on these operators.

1 Introduction

A celebrated theorem of Weyl states that one can compute the volume of a compact Riemannian manifold from the asymptotic growth of the eigenvalues of its Laplacian. This result, known as Weyl’s law, has been the starting point of numerous developments and conjectures in mathematics and physics, some of which are now presented in the context of spectral geometry, inverse problems, and quantum mechanics. In [19], it is pointed out that the only intrinsic link between a classical observable represented by a real-valued function $h$ on a symplectic phase space and its quantum mechanical description represented by a self-adjoint operator $H$ on a Hilbert space, seems to be the asymptotic correspondence between the eigenvalue counting function for $H$, $N(\Lambda) = \#\{\lambda \leq \Lambda\}$, and the phase space volume $\text{Vol}(h \leq \Lambda)$, in the sense that the ratio of these two numbers approaches a constant as $\Lambda \to \infty$.

Weyl’s law, and similar results in spectral geometry, encoding geometric information like volume and curvature in terms of spectrum of natural differential operators on manifolds, has played a very important role in the development of metric aspects of noncommutative geometry [6, 7, 8, 9, 11, 3, 10]. In noncommutative geometry, a geometric space is encoded by a spectral triple $(A, \mathcal{H}, D)$, consisting of a noncommutative algebra $A$ acting by bounded operators on a Hilbert space $\mathcal{H}$, and an unbounded selfadjoint operator $D$ acting in $\mathcal{H}$ with compact resolvent such that for any $a \in A$, the commutator $[D, a]$ is a bounded operator [8, 11, 9, 10]. Then, under suitable summability and regularity conditions, the terms of the asymptotic expansion for the heat kernel of the Dirac
Laplacian $\triangle = D^2$, 

$$\text{Trace} \left( a e^{-t\triangle} \right) \sim \sum_{n=0}^{\infty} a_n(a, \triangle) t^{\frac{d-2}{2}} \quad (t \to 0),$$

for $a \in \mathcal{A}$, encode information like volume and curvature of a noncommutative space with a geometry of metric dimension $d$. There is an equivalent formulation in terms of spectral zeta functions.

In this paper we prove the analogue of Weyl’s law and Connes’ trace theorem [7] for a noncommutative Riemannian manifold, namely for the noncommutative two torus $\mathbb{T}_\theta^2$ equipped with a translation invariant complex structure and a Weyl factor. In Section 2 we recall the notion of a general metric on $\mathbb{T}_\theta^2$ and the associated Laplacian, defined via a complex number $\tau$ in the upper half-plane representing the conformal class of a metric, and a positive invertible element $k \in C^\infty(\mathbb{T}_\theta^2)$ playing the role of a Weyl factor [13, 15]. In Section 3 we recall Connes’ pseudodifferential calculus as developed in [5], for the canonical dynamical system defining the noncommutative two torus. In Section 4 using this pseudodifferential calculus and applying heat kernel techniques [13, 13, 15], we find the first coefficient in the short time asymptotic of the trace of the heat kernel of the perturbed Laplacian on $\mathbb{T}_\theta^2$. Here, the volume of $\mathbb{T}_\theta^2$ with respect to the perturbed volume form manifests itself, and we use a Tauberian theorem to find its relation with the asymptotic growth of the eigenvalues as the analogue of Weyl’s law for this $C^*$-algebra.

In Section 5 we use the results on the asymptotic growth of eigenvalues, to show that any pseudodifferential operator of order $-2$ on $\mathbb{T}_\theta^2$ is in the domain of the Dixmier trace [14]. Then using this trace we define a positive linear functional and show that it can be represented by integration of the trace of principal symbols against the Lebesgue measure, and this will prove our desired result.

Recently, there has been much progress in understanding the local differential geometry of the noncommutative two torus equipped with a general metric. In [13], the Gauss-Bonnet theorem of [13] is extended to full generality following the pioneering work [4] in the subject. The much more delicate question of computing the scalar curvature of this noncommutative manifold is now also fully settled in [12, 16]. In [12], among other results, one can also find a closed formula for the Ray-Singer analytic torsion in terms of the Dirichlet quadratic form, and the corresponding evolution equation for the metric is shown to give the appropriate analogue of Ricci curvature. A different approach to the question of Ricci curvature is proposed in [2]. Also, a noncommutative residue for pseudodifferential operators on the noncommutative two torus is defined and studied in [17] which we use in this paper.

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2 Preliminaries

Let \( \Sigma \) be a closed, oriented, 2-dimensional smooth manifold equipped with a Riemannian metric \( g \). The spectrum of the Laplacian \( \triangle_g = d^*d \), where \( d \) is the de Rham differential operator acting on smooth functions on \( \Sigma \), encodes important geometric and topological information about \( (\Sigma, g) \). In fact there is an asymptotic expansion for the trace of the heat kernel \( e^{-t\triangle_g} \), which is of the form

\[
\text{Trace} \left( e^{-t\triangle_g} \right) \sim t^{-1} \sum_{n=0}^{\infty} a_{2n}(\triangle_g)t^n \quad (t \to 0),
\]

with \( a_0(\triangle_g) = \frac{1}{16\pi} \text{Vol}(\Sigma) \) and \( a_2(\triangle_g) = \frac{1}{72\pi} K(\Sigma) \), where \( \text{Vol}(\Sigma) \) and \( K(\Sigma) \) denote the volume and total curvature of the Riemannian manifold respectively.

On the other hand \( \text{Trace} \left( e^{-t\triangle_g} \right) \) depends only on the spectrum of the Laplacian:

\[
\text{Trace} \left( e^{-t\triangle_g} \right) = \sum e^{-t\lambda_j},
\]

where the summation is over all eigenvalues \( \lambda_j \) of \( \triangle_g \), counted with multiplicity.

Using Karamata’s Tauberian theorem [1], the term \( a_0 = \frac{1}{16\pi} \text{Vol}(\Sigma) \) in the above asymptotic expansion is seen to depend only on the asymptotic growth of the eigenvalues \( \lambda_j \). More precisely, one obtains

\[
\lim_{j \to \infty} \frac{\lambda_j}{j} = \frac{4\pi}{\text{Vol}(\Sigma)}.
\]

This provides a proof for the celebrated Weyl’s law which states that one can hear the volume of a Riemannian manifold from the asymptotic behavior of the eigenvalues of its Laplacian.

As a first step towards formulating Weyl’s law for the noncommutative two torus, we recall the notion of the perturbed Laplacian \( \triangle' \) attached to \((T^2_\theta, \tau, k)\), where \( \tau \in \mathbb{C} \setminus \mathbb{R} \) represents the conformal class of a metric on the noncommutative two torus \( T^2_\theta \), and \( k \in C^\infty(T^2_\theta) \) is the Weyl factor by the aid of which one can vary inside the conformal class of the metric [13, 15].

2.1 The irrational rotation algebra.

Let \( \theta \) be an irrational number. Recall that the irrational rotation \( C^* \)-algebra \( A_\theta \) is, by definition, the universal unital \( C^* \)-algebra generated by two unitaries \( U, V \) satisfying

\[
VV = e^{2\pi i \theta} UV.
\]

One usually thinks of \( A_\theta \) as the algebra of continuous functions on the noncommutative 2-torus \( T^2_\theta \). There is a continuous action of \( \mathbb{T}^2 \), \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \), on \( A_\theta \) by \( C^* \)-algebra automorphisms \( \{ \alpha_s \} \), \( s \in \mathbb{R}^2 \), defined by

\[
\alpha_s(U^mV^n) = e^{is.(m,n)}U^mV^n.
\]

The space of smooth elements for this action, that is those elements \( a \in A_\theta \) for which the map \( s \mapsto \alpha_s(a) \) is \( C^\infty \) will be denoted by \( A_\theta^\infty \). It is a dense subalgebra
of $A_\theta$ which can be alternatively described as the algebra of elements in $A_\theta$ whose (noncommutative) Fourier expansion has rapidly decreasing coefficients:

$$A_\theta^\infty = \left\{ \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n : \sup_{m,n \in \mathbb{Z}} (|m|^k |n|^q |a_{m,n}|) < \infty, \ k, q \in \mathbb{Z} \right\}.$$

There is a unique normalized trace $t$ on $A_\theta$ whose restriction on smooth elements is given by

$$t \left( \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n \right) = a_{0,0}.$$

The infinitesimal generators of the above action of $\mathbb{T}^2$ on $A_\theta$ are the derivations $\delta_1, \delta_2 : A_\theta^\infty \rightarrow A_\theta^\infty$ defined by

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V.$$

In fact, $\delta_1, \delta_2$ are analogues of the differential operators $\frac{1}{i} \partial / \partial x, \frac{1}{i} \partial / \partial y$ acting on the smooth functions on the ordinary two torus. We have $\delta_j(a^*) = -\delta_j(a)^*$ for $j = 1, 2$ and all $a \in A_\theta^\infty$. Moreover, since $t \circ \delta_j = 0$, for $j = 1, 2$, we have the analogue of integration by parts:

$$t(a \delta_j(b)) = -t(\delta_j(a)b), \quad \forall a, b \in A_\theta^\infty.$$

We define an inner product on $A_\theta$ by

$$\langle a, b \rangle = t(b^* a), \quad a, b \in A_\theta,$$

and complete $A_\theta$ with respect to this inner product to obtain a Hilbert space denoted by $H_0$. The derivations $\delta_1, \delta_2$, as unbounded operators on $H_0$, are formally selfadjoint and have unique extensions to selfadjoint operators.

### 2.2 Conformal structure on $T^2_\theta$.

To any complex number $\tau = \tau_1 + i \tau_2, \tau_1, \tau_2 \in \mathbb{R}$, in the upper half-plane, we can associate a complex structure on the noncommutative two torus by defining

$$\partial = \delta_1 + \tau \delta_2, \quad \partial^* = \delta_1 + \tau \delta_2.$$

To the conformal structure defined by $\tau$, corresponds a positive Hochschild two cocycle on $A_\theta^\infty$ given by (cf. [8])

$$\psi(a, b, c) = -t(a \partial b \partial^* c).$$

We note that $\partial$ is an unbounded operator on $H_0$ and $\partial^*$ is its formal adjoint. The analogue of the space of $(1, 0)$--forms on the ordinary two torus is defined to be the Hilbert space completion of the space of finite sums $\sum a \partial b, a, b \in A_\theta^\infty$, with respect to the inner product defined above, and it is denoted by $H^{(1,0)}_0$.

Now we rescale the metric by choosing a smooth selfadjoint element $h = h^* \in A_\theta^\infty$ (cf. [13]), and define a linear functional $\varphi$ on $A_\theta$ by

$$\varphi(a) = t(a e^{-h}), \quad a \in A_\theta.$$
In fact, $\varphi$ is a positive linear functional which is not a trace, however, it is a twisted trace, and satisfies the KMS condition at $\beta = 1$ for the 1-parameter group $\{\sigma_t\}, \ t \in \mathbb{R},$ of inner automorphisms $\sigma_t = \Delta^{-it},$ where the modular operator for $\varphi$ is defined on $A_\theta$ by (cf. [13])

$$\Delta(x) = e^{-hxe^h}.$$

We define an inner product $\langle , \rangle_\varphi$ on $A_\theta$ by

$$\langle a, b \rangle_\varphi = \varphi(b^*a), \quad a, b \in A_\theta.$$

The Hilbert space obtained from completing $A_\theta$ with respect to this inner product will be denoted by $\mathcal{H}_\varphi$.

### 2.3 Laplacian on $T^2_\theta$.

Using the unbounded operator $\partial : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$ and its formal adjoint $\partial^*$, we can define the Laplacian $\triangle$ on $T^2_\theta$ by

$$\triangle := \partial^* \partial = \delta_1^2 + 2\tau_1 \delta_1 \delta_2 + |\tau|^2 \delta_2^2.$$

Now let $\partial_1, \partial_2$ be the same operator as $\partial$, but viewed as an unbounded operator from $\mathcal{H}_\varphi$ to $\mathcal{H}^{(1,0)}$, and define the modified Laplacian $\triangle'$ by

$$\triangle' := \partial_\varphi^* \partial_\varphi.$$

Obviously, $\triangle'$ is a positive unbounded operator acting in $\mathcal{H}_\varphi$. It is shown in [15] that $\triangle'$ is anti-unitarily equivalent to

$$k \triangle k : \mathcal{H}_0 \to \mathcal{H}_0,$$

where $k := e^{h/2}$ acts by left multiplication.

### 3 Pseudodifferential Operators on $T^2_\theta$

In the case of closed Riemannian manifolds, one can approximate the resolvent of an elliptic differential operator by pseudodifferential operators. This, combined with the Cauchy integral formula, provides an asymptotic expansion for the trace of the heat kernel of the elliptic differential operator [18].

A differential operator of order $n \in \mathbb{Z}_{\geq 0}$ on the noncommutative two torus, is of the form

$$\sum_{j_1,j_2 \geq 0, j_1+j_2 \leq n} a_{j_1,j_2} \partial_1^{j_1} \partial_2^{j_2} : A^\infty_\theta \to A^\infty_\theta,$$

where $a_{j_1,j_2} \in A^\infty_\theta$, and they act by left multiplication. This notion, using Connes’ pseudodifferential calculus for $C^*$-dynamical systems, can be generalized to the notion of pseudodifferential operators [5]. This is achieved by using operator-valued symbols which we recall briefly. We shall use the notion $\partial_1 = \frac{\partial}{\partial x_1}, \partial_2 = \frac{\partial}{\partial x_2}$ for partial differentiation with respect to the coordinates of $\mathbb{R}^2$. 

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Definition 3.1. For an integer \( n \), a smooth map \( \rho : \mathbb{R}^2 \to A_\theta^\infty \) is said to be a symbol of order \( n \), if for all non-negative integers \( i_1, i_2, j_1, j_2 \),
\[
||\delta_{i_1}^1 \delta_{j_1}^1 \delta_{i_2}^2 \delta_{j_2}^2 \rho(\xi)|| \leq c(1 + |\xi|)^{n - j_1 - j_2},
\]
where \( c \) is a constant, and if there exists a smooth map \( k : \mathbb{R}^2 \to A_\theta^\infty \) such that
\[
\lim_{\lambda \to \infty} \lambda^{-n} \rho(\lambda \xi_1, \lambda \xi_2) = k(\xi_1, \xi_2).
\]
The space of symbols of order \( n \) is denoted by \( S_n \).

To a symbol \( \rho \) of order \( n \), one can associate an operator on \( A_\theta^\infty \), denoted by \( P_\rho \), and given by
\[
P_\rho(a) = (2\pi)^{-2} \int \int e^{-is \cdot \xi} \rho(\xi) a_s(a) \, ds \, d\xi. \tag{1}
\]
The operator \( P_\rho \) is said to be a pseudodifferential operator of order \( n \). For example, the differential operator \( \sum_{j_1 + j_2 \leq n} a_{j_1, j_2} \delta_{i_1}^1 \delta_{i_2}^2 \) is associated with the symbol \( \sum_{j_1 + j_2 \leq n} a_{j_1, j_2} \xi_1^{j_1} \xi_2^{j_2} \) via the above formula.

Definition 3.2. Two symbols \( \rho, \rho' \in S_k \) are said to be equivalent if and only if \( \rho - \rho' \) is in \( S_n \) for all integers \( n \). The equivalence of the symbols will be denoted by \( \rho \sim \rho' \).

The following lemma shows that the space of pseudodifferential operators on \( T_\theta \) is an algebra and one can find the symbol of the product of these pseudodifferential operators up to the above equivalence relation. Also, the adjoint of a pseudodifferential operator, with respect to the inner product defined on \( H_0 \) by \( \| \cdot \| \), is a pseudodifferential operator with the symbol given in the following proposition up to the above equivalence relation (c.f. [5]).

Proposition 3.3. Let \( P \) and \( Q \) be pseudodifferential operators with the symbols \( \rho \) and \( \rho' \) respectively. Then the adjoint \( P^* \) and the product \( PQ \) are pseudodifferential operators with the following symbols
\[
\sigma(P^*) \sim \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1! \ell_2!} \delta_{i_1}^1 \delta_{i_2}^2 \delta_{j_1}^1 \delta_{j_2}^2 (\rho(\xi))^*,
\]
\[
\sigma(PQ) \sim \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1! \ell_2!} \delta_{i_1}^1 \delta_{j_1}^1 (\rho(\xi)) \delta_{i_2}^2 \delta_{j_2}^2 (\rho'(\xi)).
\]

We also recall the notion of ellipticity for these pseudodifferential operators:

Definition 3.4. Let \( \rho \) be a symbol of order \( n \). It is said to be elliptic if \( \rho(\xi) \) is invertible for \( \xi \neq 0 \), and if there exists a constant \( c \) such that
\[
||\rho(\xi)^{-1}|| \leq c(1 + |\xi|)^{-n}
\]
for sufficiently large \( |\xi| \).

The Laplacian \( \Delta = \delta_2^2 + 2\tau_1 \delta_1 + |\tau|^2 \delta_2^2 \) is an example of an elliptic operator on the noncommutative two torus.
4 Weyl’s Law for $\mathbb{T}^2_\theta$

Using the Cauchy integral formula, one has

$$e^{-t\triangle'} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(\triangle' - \lambda)^{-1} d\lambda,$$

where $C$ is a curve in the complex plane that goes around the non-negative real axis in such a way that

$$e^{-ts} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(s - \lambda)^{-1} d\lambda.$$

Appealing to this formula and employing similar arguments to those in [18], one can derive an asymptotic expansion for the trace of the heat kernel of the perturbed Laplacian:

$$\text{Trace } (e^{-t\triangle'}) \sim t^{-1} \sum_{n=0}^\infty B_{2n}(\triangle') t^n \quad (t \to 0).$$

Note that the latter trace depends only on the eigenvalues of the perturbed Laplacian. Hence for the purpose of finding the coefficients in the above asymptotic expansion, one can work with the anti-unitarily equivalent operator $k\triangle k$ whose symbol as a pseudodifferential operator [15] is equal to $a_2(\xi) + a_1(\xi) + a_0(\xi)$, where

- $a_2(\xi) = \xi^2 k^2 + |\tau|^2 \xi^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2$,
- $a_1(\xi) = 2\xi_1 k\delta_1(k) + 2|\tau|^2 \xi_1 \xi_2 \delta_2(k) + 2\tau_1 \xi_1 k\delta_2(k) + 2\tau_1 \xi_1 \delta_1(\xi)$,
- $a_0(\xi) = k\delta_1^2(k) + |\tau|^2 \xi_2 \delta_2^2(k) + 2\tau_1 k\delta_1 \delta_2(k)$.

In order to find the first coefficient, one may approximate the inverse of the operator $(\triangle' - \lambda)$ by a pseudodifferential operator $B_\lambda$ whose symbol has an expansion of the form

$$b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \cdots,$$

where $b_j(\xi, \lambda)$ is a symbol of order $-2 - j$, and

$$\sigma(B_\lambda(\triangle' - \lambda)) \sim 1.$$

Then one can see that:

$$B_0(\triangle') = \frac{1}{2\pi i} \int_C \int e^{-\lambda t} (b_0(\xi, \lambda)) d\lambda d\xi.$$  (2)

In order to find the symbols $b_j$ we need to solve the following equation:

$$ (b_0 + b_1 + b_2 + \cdots) \sigma(\triangle' - \lambda) = (b_0 + b_1 + b_2 + \cdots)((a_2 - \lambda) + a_1 + a_0) \sim 1.$$
In fact \( \lambda \) will be treated as a symbol of order 2, and we let \( a'_2 = a_2 - \lambda, a'_1 = a_1, a'_0 = a_0 \). Then the above equation yields

\[
\sum_{j, \ell_1, \ell_2 \geq 0, k=0,1,2} \frac{1}{\ell_1! \ell_2!} \delta_1^{\ell_1} \delta_2^{\ell_2} (b_j) \delta_1^{\ell_1} \delta_2^{\ell_2} (a'_k) \sim 1.
\]

By comparing the symbols of order 0 on both sides, we have \( b_0 a'_2 = 1 \), therefore

\[
b_0 = a'^{-1}_2 = (a_2 - \lambda)^{-1} = (\xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2 \tau_1 \xi_1^2 k^2 - \lambda)^{-1}.
\]

In fact, as it is shown in [15], one can find all terms \( b_j \) inductively.

The following theorem is the analogue of Weyl’s law for the noncommutative two torus equipped with a general metric. In fact we show that the first term in the above asymptotic expansion is intimately related to a natural definition for the volume of \((T^2_\theta, \tau, k]\), and by means of a Tauberian theorem, the relation between the volume and asymptotic growth of the eigenvalue counting function of the perturbed Laplacian is derived.

**Theorem 4.1.** For any positive real number \( \lambda \), let \( N(\lambda) \) denote the number of eigenvalues of the perturbed Laplacian \( \triangle' \) that are less than \( \lambda \). Then as \( \lambda \to \infty \), we have

\[
N(\lambda) \sim \frac{\pi}{\tau_2} \varphi(1) \lambda.
\]

**Proof.** Using (2) and (3), and passing to the coordinates

\[
\xi_1 = r \cos \theta - \frac{\tau_1}{\tau_2} r \sin \theta, \quad \xi_2 = \frac{r}{\tau_2} \sin \theta,
\]

where \( \theta \) ranges from 0 to \( 2\pi \) and \( r \) ranges from 0 to \( \infty \), we can compute the first coefficient in the asymptotic expansion for the trace of the heat kernel of \( \triangle' \) directly:

\[
B_0(\triangle') = \frac{1}{2\pi i} \int \int_C e^{-\lambda t} \{ (b_0(\xi, \lambda)) d\lambda d\xi = \frac{\pi}{\tau_2} t (k^{-2}).
\]

Now the asymptotic behavior of the eigenvalue counting function \( N \) can be determined as follows. In fact, it follows from (2) that

\[
\lim_{t \to 0^+} t \sum_{i} e^{-t \lambda_i} = B_0(\triangle') = \frac{\pi}{\tau_2} t (k^{-2}),
\]

where the summation is over all eigenvalues \( \lambda_j \) of \( \triangle' \). Then it follows immediately from Karamata’s Tauberian theorem that the eigenvalue counting function \( N \) satisfies

\[
N(\lambda) \sim \frac{\pi}{\tau_2 \Gamma(2)} t (k^{-2}) \lambda = \frac{\pi}{\tau_2} t (k^{-2}) \lambda,
\]

as \( \lambda \to \infty \) (cf. [1]).
The Weyl asymptotic behavior of the eigenvalue counting function proved in the above theorem readily shows the Dixmier traceability of $(1 + \Delta')^{-1}$.

**Corollary 4.2.** The operator $(1 + \Delta')^{-1}$ belongs to $\mathcal{L}^{1,\infty}(\mathcal{H}_\phi)$, and its Dixmier trace $\text{Tr}_\omega$ is independent of the choice of the state $\omega$.

**Proof.** Assuming that the eigenvalues $\lambda_j$ of the perturbed Laplacian $\Delta'$ are written in the increasing order $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, the asymptotic estimate (4) can be reformulated to find the asymptotic growth of the eigenvalues: In fact,

$$\lambda_j \sim \frac{\tau_2}{\pi \varphi(1)} j,$$

and consequently

$$(1 + \lambda_j)^{-1} \sim \frac{\pi \varphi(1)}{\tau_2} \left( \frac{\pi \varphi(1)}{\tau_2} + j \right)^{-1},$$

as $j \to \infty$. Hence:

$$\text{Tr}_\omega((1 + \Delta')^{-1}) = \frac{\pi \varphi(1)}{\tau_2}.$$ 

\[\Box\]

Considering the classical case explained at the beginning of Section 2, the above theorem suggests the following formula for the volume of the noncommutative two torus $T^2_{\theta}$ equipped with the metric associated to the conformal structure represented by $\tau = \tau_1 + i \tau_2$, and the Weyl factor $k$:

$$\text{Vol} (T^2_{\theta}) := 4\pi^2 \frac{\tau_2}{\varphi(1)} = 4\pi^2 \frac{\tau_2}{\varphi(1)} t(k^{-2}).$$

### 5 Connes’ Trace Theorem for $T^2_{\theta}$

In [17], a noncommutative residue on the algebra of classical pseudodifferential operators on the noncommutative two torus is defined and it is proved that up to multiplication by a constant, it is the unique continuous trace on this algebra.

A symbol $\rho$ of order $n$ is said to be classical if there is an asymptotic expansion of the form

$$\rho(\xi) \sim \sum_{j=0}^{\infty} \rho_{n-j}(\xi)$$

as $|\xi| \to \infty$, where each $\rho_{n-j} : \mathbb{R}^2 \setminus \{0\} \to A^\infty_{\theta}$ is smooth and positively homogeneous of order $n - j$. It is shown that the homogeneous terms in such an asymptotic expansion are uniquely determined by $\rho$ and the noncommutative residue of $P_{\rho}$ is defined by

$$\text{Res} (P_{\rho}) := \int_{\mathbb{S}^1} t(\rho_{-2}(\xi)) \, d\Omega,$$
where $d\Omega$ is the Lebesgue measure on the unit circle. This is the analogue of Wodzicki’s noncommutative residue [20].

In the following theorem we prove the analogue of Connes’ trace theorem [7] for classical pseudodifferential operators on the noncommutative two torus. Namely, we show that the Dixmier trace and the noncommutative residue defined above coincide on classical pseudodifferential operators of order $-2$ on $T^2_\theta$.

**Theorem 5.1.** Let $\rho$ be a classical pseudodifferential symbol of order $-2$ on the noncommutative two torus. Then $P_\rho \in L^{1,\infty}(\mathcal{H}_0)$ and

$$\text{Tr}_\omega(P_\rho) = \frac{1}{2} \text{Res}(P_\rho).$$

**Proof.** We have $P_\rho = A(1 + \Delta_0)^{-1}$ where $\Delta_0 = \delta_1^2 + \delta_2^2$ is the Laplacian of the metric associated with $\tau = i$ and $k = 1$, and $A = P_\rho(1 + \Delta_0)$. Since $A$ is a pseudodifferential operator of order 0, it is a bounded operator on $\mathcal{H}_0$. Therefore, using Corollary 4.2, $P_\rho$ belongs to the ideal $L^{1,\infty}(\mathcal{H}_0)$.

Similarly one can see that any pseudodifferential operator of order $-3$ is a trace-class operator. Since the Dixmier trace vanishes on trace-class operators, if $\rho(\xi) \sim \sum_{j=0}^{\infty} \rho_{-2-j}(\xi)$, as $\xi \to \infty$, with positively homogeneous terms $\rho_{-2-j}(\xi)$ of order $-2 - j$, then we have:

$$\text{Tr}_\omega(P_\rho - \rho_{-2}P_\rho - P_\rho P_{\rho_{-2}}) = 0.$$  \hspace{1cm} (5)

The symbol of $P_{\rho_{-2}}P_\rho - P_\rho P_{\rho_{-2}}$ is equivalent to

$$\rho_{-2}U - U\rho_{-2} + (\partial_1 \rho_{-2})U + \frac{1}{2!}(\partial_2^2 \rho_{-2})U + \frac{1}{3!}(\partial_1^3 \rho_{-2})U + \cdots.$$  

Since $(\partial_1 \rho_{-2})U + \frac{1}{2!}(\partial_2^2 \rho_{-2})U + \frac{1}{3!}(\partial_1^3 \rho_{-2})U + \cdots$ is equivalent to a symbol of order $-3$, the Dixmier trace vanishes on the corresponding operator. Hence, we have

$$\text{Tr}_\omega(P_{\rho_{-2}}U - P_\rho P_{\rho_{-2}}) = 0.$$  

Note that the last equality follows from the fact that the Dixmier trace is a hyper-trace. Using a similar argument for $V$ instead of $U$, we will have:

$$\text{Tr}_\omega(P_{\rho_{-2}}V) = \text{Tr}_\omega(P_{\rho_{-2}}P_\rho - P_\rho P_{\rho_{-2}}).$$

From this observation, it follows that only $t(\rho_{-2})$ contributes to $\text{Tr}_\omega(P_{\rho_{-2}})$, i.e.

$$\text{Tr}_\omega(P_{\rho_{-2}}) = \text{Tr}_\omega(P_{t(\rho_{-2})}).$$  \hspace{1cm} (6)

In order to analyze the latter we define a linear functional $\mu$ on the space of smooth complex-valued functions on $S^1$ as follows. Given a smooth map
$f : S^1 \to \mathbb{C} \subset A^\infty_{\theta}$, first, we extend it to a homogeneous map of order $-2$ from $\mathbb{R}^2 \setminus \{0\}$ to $A^\infty_{\theta}$, which we denote by $\tilde{f}$. Then we define $\mu(f)$ to be the Dixmier trace of the pseudodifferential operator associated with $\tilde{f}$:

$$
\mu(f) := \text{Tr}_\omega(P_{\tilde{f}}).
$$

Considering the positivity of the Dixmier trace, the fact that the principal symbol of product of two pseudodifferential operators is the product of the principal symbols, and the formula for the symbol of the adjoint of a pseudodifferential operator explained in Proposition 3.3, it follows that $\mu$ is a positive linear functional. So from the Riesz representation theorem it follows that $\mu$ is given by integration against a Borel measure on $S^1$.

Now we show that $\mu$ is rotation invariant. Since for any smooth map $f : S^1 \to \mathbb{C} \subset A^\infty_{\theta}$, its range is in the center of $A^\infty_{\theta}$, in the definition of $P_{\tilde{f}}$ given in (1), the noncommutativity of the algebra does not play a role, and we can think of $P_{\tilde{f}}$ as a pseudodifferential operator on the ordinary two torus. By a straightforward computation, one can see that if $\sigma(x, \xi)$ is an ordinary pseudodifferential symbol on $\mathbb{R}^2$, and if $T$ is a rotation of the plane, then

$$
\mathcal{U}_T P_{\sigma(x, T\xi)} = P_{\sigma(x, \xi)} \mathcal{U}_T,
$$

where $\mathcal{U}_T$ is a unitary operator given by

$$
\mathcal{U}_T(g) := g \circ T^{-1}, \quad \forall g \in C^\infty_c(\mathbb{R}^2).
$$

Using this, one can see that the linear functional $\mu$ is invariant under rotations of the circle, hence $\mu$ is given by integration against a constant multiple of the Lebesgue measure on $S^1$. Denoting this constant by $c$, and using (5) and (6), we can write:

$$
\text{Tr}_\omega(P_{\rho}) = \text{Tr}_\omega(P_{\rho-2}) = \text{Tr}_\omega(P_t(\rho-2)) = \frac{c}{\pi} \int_{S^1} t(\rho-2) d\Omega = c \text{Res}(P_{\rho}).
$$

Finally the constant $c$ is fixed by $(1 + \Delta_0)^{-1}$ as follows. Using Corollary 4.2, we have

$$
\text{Tr}_\omega((1 + \Delta_0)^{-1}) = \pi.
$$

One the other hand one can easily see that

$$
\text{Res}((1 + \Delta_0)^{-1}) = 2\pi,
$$

as the asymptotic expansion with homogeneous terms for its symbol, starts with $1/(\xi_1^2 + \xi_2^2)$. Hence:

$$
c = \frac{1}{2}.
$$
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