The Inverse first passage time method for a two compartment model as a tool to relate Inverse Gaussian and Gamma spike distributions

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Abstract
In a previous paper [7] we related the stochastic leaky integrate and fire (LIF) and the Gamma models. The question investigated in that paper was the possibility of getting a Gamma distributed output from a LIF model in the presence of coherent choices of the parameters. The used LIF model was the classical one-dimensional Ornstein-Uhlenbeck process and we applied the Inverse first passage time method to verify the possibility to get a Gamma distributed output from this model.

The two compartment model proposed in [8] is a LIF model that distinguish the dynamics between the trigger and the synaptic zones. In some instances, the observed neuronal output adapts to the Gamma distribution while in others it exhibits heavy tails like the Inverse Gaussian distribution.

Here we adapt the Inverse first passage time method to a two dimensional process and we apply it when its output is Inverse Gaussian or Gamma distributed. Here we discuss when the two compartment model is compatible with such outputs.

Keywords: Inverse First-passage-time problem, two-compartment leaky integrate and fire model, Gamma, Inverse Gaussian.

1 Introduction
Inter spike intervals (ISIs) are the typical quantities used to describe neuronal output. Often observed data fit one of the two classes of distributions:

- the Inverse Gaussian (IG);
- the Gamma.
The first distribution was first proposed in the pioneering work by Gerstein and Mandelbrot [5] who observed data characterized by stable distribution and heavy tails. Starting from these observations, they proposed their celebrated model, i.e. a one compartment model in which the membrane potential evolves according to a Brownian motion and the ISIs correspond to First Passage Times (FPTs) of the Brownian motion through a constant boundary. Next, variants and improvements of this model appeared in the literature to improve the realism of the description of the membrane potential dynamics. These new models were characterized by different FPTs distributions (see [10]). However it is still interesting to investigate the existence of models generating ISIs that have IG distribution. In Subsection 2.1 we review the main features of this distribution.

Beside data characterized by heavy tails, other neurons have ISIs Gamma distributed. In Subsection 2.2 we list the main features of this distribution.

In a previous work [7], we investigated the Gamma distribution as an output of a neuron modeled through a Leaky integrate and fire model [9]. There, we studied possible thresholds corresponding to Gamma distributed ISIs. For this aim we used the Inverse first passage time method [11, 14] applied to the OU model.

Here we investigate the possibility to have an unique model generating Gamma or IG distributed ISIs. To this aim we resort again to the Inverse first passage time method and we extended it to the case of the two-dimensional process corresponding to the two-compartment neuronal model proposed in [8] and studied in [4]. In Section 3 we briefly review this model adding some properties that we use to deal with the Inverse first passage time problem. In Section 4 we briefly introduce the Inverse first passage time method for a two dimensional process while we postpone to a future work the mathematical details related to its convergence. Finally in Section 5 we discuss the results and we compare the thresholds corresponding to IG or Gamma ISIs distributions.

2 Two ISIs distributions

Two statistical models describing the randomness of the ISIs are the IG and the Gamma random variables. We remind here their main features that we will use for our analysis.

2.1 Inverse Gaussian model of interspike intervals

The IG random variable $T$ has probability density function (pdf)

$$f_T(t) = \left[ \frac{\lambda}{2\pi t^3} \right]^{1/2} \exp \left[ -\frac{\lambda(t - \rho)^2}{2\rho^2} \right], \quad t \geq 0,$$

(1)
where $\rho > 0$ is the mean and $\lambda > 0$ is the shape parameter. Mean, variance and coefficient of variation are given by

$$
\begin{align*}
\mathbb{E}(T) &= \rho, \\
\text{Var}(T) &= \frac{\rho^3}{\lambda}, \\
\text{CV} &= \text{CV}(T) = \frac{\sqrt{\text{Var}(T)}}{\mathbb{E}(T)} = \sqrt{\frac{\rho}{\lambda}}.
\end{align*}
$$

Different shapes of the IG distribution are shown in Figure 1 panel a for a fixed mean and different values of CV. Panel b will be discussed in Section 5. When $\rho$ is finite, the IG distribution has light tails but if $\rho \to \infty$ the IG becomes

$$
f_T(t) = \left[\frac{\lambda}{2\pi t^3}\right]^{1/2} \exp\left[-\frac{\lambda}{2t}\right], \quad t \geq 0,
$$

and it catches the heavy tails feature observed in some data [5, 6, 12]. The density (3) is related to the density of the FPT of a Brownian motion with zero drift and diffusion coefficient $\nu$ through a constant boundary $b$ with the relation

$$
\lambda = \frac{b^2}{\nu^2}.
$$

Since $\mathbb{E}(T) = \infty$, it makes no sense to compute the CV but, in order to compare light and heavy tails distributions we use the same values of $\lambda$. In Figure 2 panel a, different shapes of the pdfs are shown. Panel b will be discussed in Section 5.
Figure 2: Probability densities (a) and evaluated boundaries (b) in the case of Inverse Gaussian-distributed ISIs with heavy tails. Different lines correspond to different values of the parameter: $\lambda = 1$ (red), $\lambda = 1.5$ (magenta), $\lambda = 2$ (black). The parameters of the two compartment model are $\alpha = 0.33$, $\beta = 0.2$, $\mu = 0$ and $\sigma = 1$. Probability mass=[0.37 0.55 0.65] for $\lambda = [1 \ 1.5 \ 2]$.

### 2.2 Gamma model of interspike intervals

A random variable $T$ is Gamma distributed if its pdf is

$$f_T(t) = \frac{\gamma^\kappa}{\Gamma(\kappa)} t^{\kappa-1} e^{-\gamma t}, \quad t \geq 0. \quad (5)$$

Here $\gamma > 0$ is the rate parameter and $\kappa > 0$ is the shape parameter. Such a random variable is characterized by the following mean, variance and coefficient of variation

$$E(T) = \frac{\kappa}{\gamma} \quad (6)$$

$$Var(T) = \frac{\kappa}{\gamma^2}$$

$$CV = CV(T) = \frac{1}{\sqrt{\kappa}}.$$

The shapes of Gamma pdf for different values of the parameters $\gamma$ and $\kappa$ can be seen in Figure 3 panel a. Panel b will be discussed in Section 5. The shapes of the Gamma pdf strongly change with the value of $CV$. We recall that $CV = 1$ corresponds to ISIs exponentially distributed. Tails of the Gamma distribution are light decaying to zero as an exponential. Note that, when $T$ is used to model ISIs, the case $CV > 1$ allows to describe bursting activity while, for decreasing $CV$ the neuronal activity tends to regularity.
Figure 3: Probability densities (a) and evaluated boundaries (b) in the case of Gamma-distributed ISIs with mean ISI equal to 4. Different lines correspond to different shapes of the Gamma densities, $CV = 0.5$ (red), $CV = 0.75$ (magenta), $CV = 1$ (black), $CV = 1.5$ (blue), $CV = 2$ (green). The parameters of the two compartment model are $\alpha = 0.33$, $\beta = 0.2$, $\mu = 0$ and $\sigma = 1$.

3 The model

Simplest neuronal models resort to one dimensional processes to describe the membrane potential evolution. This choice implies a strong simplification of the neuronal structure that is identified by a single point. More complex models introduce bivariate stochastic processes to discriminate the membrane potential dynamics in the dendritic or in the trigger zone [8]. This type of model assumes an interaction between the membrane potential dynamics in the two zones but the release of the spike is a phenomenon involving only the trigger component. Furthermore, a noisy term is added only to the dendritic component. The reset after the spike can include both the components or only the trigger zone. Here we will consider only the case of the total resetting of the two components.

In [8] the stochastic process $X = \{(X_1(t), X_2(t)), t \geq 0\}$ describes the depolarization of the trigger zone and the dendritic one, respectively. The model assumes that external inputs, with intensity $\mu$ and variability $\sigma$, influence only the second compartment and takes into account the interconnection between the parts of the neuron through a weight $\beta$. According to these hypothesis, the process $X$ is solution of the following stochastic differential system

$$
\begin{align*}
    dX_1(t) &= \{-\alpha X_1(t) + \beta [X_2(t) - X_1(t)]\} dt \\
    dX_2(t) &= \{-\alpha X_2(t) + \beta [X_1(t) - X_2(t)] + \mu\} dt + \sigma dB_t
\end{align*}
$$

with $X(0) = 0$ and where $B$ is a one-dimensional standard Brownian motion. Here, $\alpha$ is a constant related to the spontaneous membrane potential decay (cf. Figure 4).
Figure 4: Schematic representation of the two-compartment model

The stochastic differential system (7) is linear and can be written in matrix form

\[ dX(t) = [AX(t) + M(t)]dt + GdB(t) \]  

(8)

where

\[ A = \begin{pmatrix} -\alpha - \beta & \beta \\ \beta & -\alpha - \beta \end{pmatrix}, \quad M(t) = \begin{pmatrix} 0 \\ \mu \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}. \]

It is an autonomous linear stochastic differential equation, in particular it is a two-dimensional Ornstein Uhlenbeck process, special case of a Gauss-Markov process [2]. The solution of (8) is

\[
\begin{cases}
X_1(t) = \frac{\mu}{2} \left( \frac{1 - e^{-\alpha t}}{\alpha} - \frac{1 - e^{-\alpha(\alpha+2\beta)t}}{\alpha+2\beta} \right) - \frac{\sigma}{2} \int_0^t \left( e^{-\alpha(t-s)} - e^{-\alpha(\alpha+2\beta)(t-s)} \right) dB(s) \\
X_2(t) = \frac{\mu}{2} \left( \frac{1 - e^{-\alpha t}}{\alpha} + \frac{1 - e^{-\alpha(\alpha+2\beta)t}}{\alpha+2\beta} \right) + \frac{\sigma}{2} \int_0^t \left( e^{-\alpha(t-s)} + e^{-\alpha(\alpha+2\beta)(t-s)} \right) dB(s).
\end{cases}
\]

(9)

It is a Gaussian vector with mean

\[
m(t) = \mathbb{E}(X(t)) = \begin{pmatrix} \frac{\mu}{2} \left( \frac{1 - e^{-\alpha t}}{\alpha} - \frac{1 - e^{-\alpha(\alpha+2\beta)t}}{\alpha+2\beta} \right) \\ \frac{\mu}{2} \left( \frac{1 - e^{-\alpha t}}{\alpha} + \frac{1 - e^{-\alpha(\alpha+2\beta)t}}{\alpha+2\beta} \right) \end{pmatrix}
\]

(10)

and variance-covariance matrix \( Q(t-s) \) where,

\[
Q(t) = \begin{pmatrix} Q^{(11)}(t) & Q^{(12)}(t) \\ Q^{(12)}(t) & Q^{(22)}(t) \end{pmatrix}
\]

(11)

and

\[
\begin{align*}
Q^{(11)}(t) &= \frac{1}{2} \left( \frac{1}{\alpha} - \frac{2}{\alpha + \beta} + \frac{1}{\alpha + 2\beta} - e^{-2\alpha t} \left( \frac{1}{\alpha} - \frac{2e^{-2\beta t}}{\alpha + 2\beta} + \frac{e^{-4\beta t}}{\alpha + 2\beta} \right) \right) \\
Q^{(12)}(t) &= \frac{1}{2} \left( 1 - e^{-2\alpha t} - \frac{1}{\alpha} + \frac{2}{\alpha + 2\beta} \right) \\
Q^{(22)}(t) &= \frac{1}{2} \left( \frac{1}{\alpha} + \frac{2}{\alpha + \beta} + \frac{1}{\alpha + 2\beta} - e^{-2\alpha t} \left( \frac{1}{\alpha} + \frac{2e^{-2\beta t}}{\alpha + \beta} + \frac{e^{-4\beta t}}{\alpha + 2\beta} \right) \right).
\end{align*}
\]

Trajectories of the process are plotted in Figure 5. The different behavior of the two components is evident: the noisy behavior is prevalent in \( X_2 \), while the
first component $X_1$ is more smooth since it is more affected by the integrated part.

The neuron releases a spike at time

$$T = \inf\{ t > 0 : X_1(t) > S(t) \}$$

where $S(t)$ is a continuous function representing the threshold for the membrane potential. When $X_1(t)$ attains $S(t)$ the ionic channels open suddenly determining the spike generation.

Note that it is possible to rewrite (9) in iterative form. This version is useful for simulation purposes, in order to generate the trajectories in an exact way. Discretizing the time interval $[0, T]$ with the partition $\pi : 0 = t_0 < t_1 < \cdots < t_N = T$ in $N$ subintervals of constant length $h = \frac{T}{N}$, we can express the position of the process at time $t_{k+1}$ in terms of the position of the process at time $t_k$

$$X(t_{k+1}) = \frac{1}{2} \left[ \begin{array}{cc} e^{-\alpha h} + e^{-(\alpha+2\beta)h} & e^{-\alpha h} - e^{-(\alpha+2\beta)h} \\ e^{-\alpha h} - e^{-(\alpha+2\beta)h} & e^{-\alpha h} + e^{-(\alpha+2\beta)h} \end{array} \right] X(t_k) + \frac{\mu}{2} \left[ \begin{array}{c} 1 - e^{-\alpha h} \\ 1 - e^{-\alpha h} \end{array} \right] + \frac{\sigma}{2} I_k$$

where the term

$$I_k = \left[ \int_{t_k}^{t_{k+1}} (e^{-\theta(t_{k+1} - s)} - e^{-(\alpha+2\beta)(t_{k+1} - s)}) dB(s) \right]$$

known as innovation, is a Gaussian vector with zero mean and variance-covariance matrix $Q(h)$.

For algorithmic purposes we also need the conditioned mean of the first component

$$m^{(1)}(t|(X_1(\theta), X_2(\theta)), \theta) = \mathbb{E}(X_1(t)|X(\theta) = (X_1(\theta), X_2(\theta)))$$

$$= \frac{\mu}{2} \left( \frac{2\beta - \alpha e^{-\alpha(t-\theta)} - 2\beta e^{-\alpha(t-\theta)} + \alpha e^{-(\alpha+2\beta)(t-\theta)}}{\alpha(\alpha + 2\beta)} \right)$$

$$- \frac{X_1(\theta)e^{-\alpha(t-\theta)}}{2}(1 + e^{-2\beta(t-\theta)}) - \frac{X_2(\theta)e^{-\alpha(t-\theta)}}{2}(1 - e^{-2\beta(t-\theta)})$$

and the conditioned variance of the first component

$$Q^{(1)}(t|(X_1(\theta), X_2(\theta)), \theta) = \text{Var}(X_1(t)|X(\theta) = (X_1(\theta), X_2(\theta)))$$

$$= \frac{\sigma^2 e^{-2\alpha(t-\theta)}}{8} \times$$

$$\frac{2\alpha e^{-2\beta(t-\theta)}(\alpha + 2\beta) - \alpha e^{-4\beta(t-\theta)}(\alpha + \beta) + 2\beta^2 e^{2\alpha(t-\theta)} - \alpha^2 - 3\alpha\beta - 2\beta^2}{\alpha(\alpha + \beta)(\alpha + 2\beta)}$$

Assuming that after each spike the system is reset to its initial value, the time between two spikes, i.e. the time when the membrane potential changes
its dynamics with a sudden hyper-polarization, is described by the FPT of the first component through the boundary $S(t)$.

In some instances can be useful to transfer the time dependency from the boundary shape $S(t)$ to an input $M(t)$. Mathematically it is possible to relate these two situations with a simple space transformation. Indeed, the space transformation

$$Y(t) = X(t) - S(t) + \Sigma.$$  

changes our process $X$ given by (4), originated in $X(0) = x_0$ in presence of a time dependent boundary $S(t)$

$$
\begin{aligned}
  dX_1(t) &= \{-\alpha X_1(t) + \beta [X_2(t) - X_1(t)]\} \, dt \\
  dX_2(t) &= \{-\alpha X_2(t) + \beta [X_1(t) - X_2(t)] + \mu\} \, dt + \sigma dB_t \\
  X(0) &= x_0 \\
  S(t) &= \Sigma 
\end{aligned}
$$

(18)

into a two dimensional process characterized by time dependent input $M(t)$ and constant threshold $\Sigma$

$$
\begin{aligned}
  dY_1(t) &= \{-\alpha Y_1(t) + \beta [X_2(t) - Y_1(t)] + \mu_1(t)\} \, dt \\
  dX_2(t) &= \{-\alpha X_2(t) + \beta [Y_1(t) - X_2(t)] + \mu_2(t)\} \, dt + \sigma dB_t \\
  X(0) &= x_0 - S(0) + \Sigma \\
  \Sigma &= \Sigma
\end{aligned}
$$

(19)
Here, the term
\[ M(t) = \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \end{bmatrix} = \begin{bmatrix} -(\alpha + \beta)(S(t) - \Sigma) + S'(t) \\ \mu + \beta(S(t) - \Sigma) \end{bmatrix} \] (20)
can be interpreted as an external input acting with different weights on the two compartments.

4 Inverse first passage time method

The inverse FPT problem consists in searching the unknown boundary \( S(t) \) given that the FPT density \( g(t) \) is known. We work under the assumption that the boundary \( S(t) \) exists, it is unique and sufficiently regular.

Let us consider a diffusion process \( X = \{(X_1(t), X_2(t)), t \geq 0\} \), solution of the stochastic differential equation (8). The proposed method is based on the numerical approximation of the following Volterra integral equation [4]

\[ 1 - \text{Erf} \left( \frac{S(t) - m^{(1)}(t)}{\sqrt{2Q^{(11)}(t)}} \right) = \int_0^t d\theta g(\theta) \cdot E_{Z(\theta)} \left[ 1 - \text{Erf} \left( \frac{S(t) - m^{(1)}(t)(S(\theta), X_2(\theta)), \theta}{\sqrt{2Q^{(11)}(t)(S(\theta), X_2(\theta)), \theta}} \right) \right] \] (21)

where \( Z(t) \) is a random variable that represents the position of the second component \( X_2 \) of the process when the first component \( X_1 \) hits the boundary at time \( t \), i.e.

\[ P(Z(t) < z) = P(X_2(T) < z|T = t, X(t_0) = y). \] (22)

Note that the range of the random variable \( X_2(T) \), i.e. when the \( X_1 \) hits the boundary, is \([kS, \infty)\), where

\[ k = \frac{\alpha + \beta}{\beta}. \]

Let us fix a time interval \([0, T]\) and a partition \( \pi : 0 = t_0 < t_1 < \cdots < t_N = T \) in \( N \) subintervals of constant length \( h = \frac{T}{N} \). Using Euler formula for integrals [1], equation (21) can be approximated as

\[ 1 - \text{Erf} \left( \frac{S^*(t_i) - m^{(1)}(t_i)}{\sqrt{2Q^{(11)}(t_i)}} \right) = h \cdot \sum_{j=1}^i g(t_j) \cdot E_{Z(t_i)} \left[ 1 - \text{Erf} \left( \frac{S^*(t_i) - m^{(1)}(t_i)(S^*(t_j), X_2(t_j)), t_j)}{\sqrt{2Q^{(11)}(t_i)(S^*(t_j), X_2(t_j)), t_j}} \right) \right] \] (23)

\( \forall i = 1 \ldots N, \)
Equation (23) represents a non linear system of $N$ equations in $N$ unknowns $S^*(t_1), \ldots, S^*(t_N)$ that can be solved by means of root finding iterative algorithms [3]. Its solution gives an approximation $S^*(t)$ of the boundary $S(t)$ in the partition points $\tau$. Note that in step $i$ the only unknown quantity is $S(t_i)$ and it is estimated using the boundary approximations $S^*(t_1), \ldots, S^*(t_{i-1})$, computed in the previous steps.

The quantity

$$\theta_{i,k} = \mathbb{E}_{Z(t_k)} \left[ 1 - \text{erf} \left( \frac{S^*(t_i) - m^{(1)}(t_i|S^*(t_k), X_2(t_k)), t_k)}{\sqrt{2Q^{(1)}(t_i|S^*(t_k), X_2(t_k)), t_k)}} \right) \right]$$

(24)

is not easily handled because it depends on the unknown time dependent boundary. In general, the computation of $\theta_{k,i}$ is not trivial but, performing a suitable limit on the considered process we can show that $\theta_{k,i} = 2$ for each value of $k$. To compute (24) when $k \neq i$, we use a Monte Carlo method: we simulate the process $X$ until the first component exceeds the threshold and we save the corresponding value of $Z$. At step $i$, we need to compute $\theta_{i,k}$ for $k = 1, \ldots, i-1$. The presence of an expectation with respect to $Z(t_k)$ determines a difficulty for the estimation of (24) through Monte Carlo because we need the value of $Z(t_k) = X_2(t_k)$ at time $T = t_k$. To circumvent this problem we introduce an approximate approach as follows. At step $i$ we approximate the threshold with a piecewise linear curve with knots in the already computed boundary values. Hence, for $\tau \in [t_{j-1}, t_j]$, $j = 1, \ldots, i-1$ we substitute the exact boundary with

$$\hat{S}(\tau) = \frac{S^*(t_j) - S^*(t_{j-1})}{t_j - t_{j-1}} \tau + \frac{t_j S^*(t_{j-1}) - S^*(t_j)t_{j-1}}{t_j - t_{j-1}}$$

(25)

and we simulate the process up to $t_{i-1}$ or until it reaches the threshold. To compute $\theta_{i,k}$, $k = 1, \ldots, i-1$ we use only the trajectories that crossed the approximated boundary (25) in a neighbourhood of $t_k$. Then, in correspondence to each of these sample paths we identify with $\{Z_k, k = 1 \ldots M\}$ the sequence of values of the second component of the process $X$ (when the first component has exceeded the threshold). In this way, the Monte Carlo estimate for $\theta_{i,j}$ is

$$\hat{\theta}_{i,j} = 1 - \frac{\sum_{k=1}^{M} \text{erf} \left( \frac{S(t_j) - m^{(1)}(t_j|S(t_j), Z_k), t_j)}{\sqrt{2Q^{(1)}(t_j|S(t_j), Z_k), t_j)}} \right)}{M}.$$ 

It is possible to prove that this further approximation does not seriously influence the reliability of the algorithm.

5 Results

In this Section we illustrate the use of the Inverse first passage time method through a set of examples. Our aim is to investigate the role of the $CV$ of ISIs distributions on the features of the two compartment model [7] as well as to
enlighten differences between Inverse Gaussian and Gamma ISIs distributions when (7) is the underlying model. Hence, although we choose the parameter values of the two compartment model according to the literature we do not perform a sensitivity analysis on their values, limiting ourselves to change the CV of the ISI distribution while $E(T)$ is fixed.

Following the approach presented in [7], we first study when Inverse Gaussian and Gamma distributions describe the firing activity modeled by the two-compartment model as their CV varies. Applying the algorithm illustrated in Section 4 we compute the boundary $S(t)$ corresponding to the two-compartment diffusion model (7) whose FPT density function is Inverse Gaussian (1) or Gamma (5) distributed. We fix the values of the two compartment model as follows: $\alpha = 0.33 \text{ msec}^{-1}, \beta = 0.2 \text{ msec} / \sigma = 1 \text{ mV/msec}$. In Section 2 we introduced Figure 1 (panel (a)) and Figure 3 (panel (a)) to illustrate the shapes of the IG and of the Gamma distributions, respectively, corresponding to $E[T] = 4$ and varying CV. Here, we determine the corresponding shapes of the time varying thresholds illustrated in panels (b) of Figure 1 and 3, for the IG and the Gamma ISIs distributions, respectively.

As far as the IG firing distribution is concerned, from (2) we see that changes of CV imply changes of the shape parameter $\lambda$. Moreover, as CV increases, the density becomes more peaked. The corresponding shape of the boundary presents a maximum that tends to disappear as CV grows to higher values. As the intuition suggests, a low amount of short ISIs corresponds to the presence of an higher threshold value that makes the spike activity exceptional. When the size of ISIs increases, their probability mass becomes relevant and the corresponding values of the time dependent threshold decreases, to facilitate the boundary crossing according to model (7). The same analysis is performed in panel (b) of Figure 3 on the ISIs as an output of a Gamma renewal process. Unlike the case of IG ISIs, here as CV increases, the maximum of the boundary disappears and the threshold time varying threshold becomes flat or, eventually when $CV = 2$, increasing. This phenomenon captures the lighter tails of the ISIs distribution for high values of the CV, it indicates that very long ISIs are exceptional.

As a further example, Figure 2 considers ISIs distributed as an IG (3) with heavy tails (panel (a)) and the corresponding boundaries (panel(b)). Since the mean ISI is infinite, we cannot use the CV to determine the different shapes. Hence, we draw the boundary for different choices of the parameter $\lambda$. The heavy tails of this distribution determine a new shape for the threshold that has a decreasing maximum as $\lambda$ increases, followed by a minimum and by an increasing shape of the boundary. The maximum tends to disappear for large values of $\lambda$ and the values of the boundary are essentially positive. It is possible that for larger values of $t$ the growth of the boundary stops to allow the crossing of the samples determining the tail of the distribution. In Figure 2 we stop at time $t = 20$ that correspond to a low probabilistic mass. A check for longer times does not change the results and reaches a higher probabilistic mass (figure not shown).

Note that only the IG distributions with heavy tails always determine posi-
tive boundaries; all the other cases correspond to boundaries becoming negative. This result is biologically difficult to interpret, but it represents no formal problem. The reason is that numerous trajectories of the two compartment process lives below zero and thus below the threshold. Therefore, a negative threshold gradually absorbs these trajectories (hyper-polarized below zero), and it creates the tail of the FPT density. Furthermore, reinterpreting this result in terms of time varying input clarifies the biological reading.

In Figure 6 we show the two components $\mu(t)$ of $M(t)$, when the boundary is transformed into a constant through (17). Here we illustrate the cases of FPT distributed as IG (a-b), IG with heavy tails (c-d) and Gamma (e-f), respectively. Reinterpreting the time dependent boundary in terms of a modification of the drift allows to interpret our results in terms of excitatory and inhibitory inputs. A positive excitation on the first component is always necessary to get the prescribed firing distribution. When $CV \leq 1$ the excitatory input necessary in cases of ISIs distributed as Gamma or IG are similar. On the contrary, when tails of the IG are heavy (panel (c)) or the CV is large enough (panels (a) or (e)), the tuning of the excitatory input on the first component strongly changes becoming decreasing. Interestingly the behavior of the second component (panels (b),(d) and (f)) is the opposite of that of $\mu_1(t)$. This phenomenon reveals the presence of a finer tuning in the two component model with respect to simpler one dimensional models. Indeed, while the first component is tuned through excitatory input, the second component is influenced by inhibition.

To complete the discussion we compare the boundaries corresponding to IG and to Gamma distributions to enlighten differences between these different outputs. Figure 7 shows a comparison between the time varying boundaries corresponding to the IG (dashed) and the Gamma-distributed (solid) ISIs with the same mean value and the same $CV$. In this example we fixed the mean ISIs equal to 10 and we vary the $CV$. Densities and the corresponding boundaries become more and more different as we increase the $CV$ value. The different spreading of probability mass of the two classes of distributions is reflected in different shapes of the corresponding boundaries. Since the IG density has heavier tails, the probability mass should not be consumed for short times. For this reason the boundary increases (or the inhibition acts on the input) allowing crossings for large times.

In the next step we fix the values of the two compartment model as follows: $\alpha = 0.02$ msec$^{-1}$, $\beta = 0.02$ msec and $\sigma = 0.4$ mV/msec. We consider examples of boundaries corresponding the IG or Gamma spiking densities for $CV = 0.5$ (Figure 8) or $CV = 1$ (Figure 9). In one dimensional instances the spiking dynamics changes in sub or supra-threshold regimes, depending on the relative ordering of boundary and mean membrane potential. Here the presence of two components complicates the framework but the firing depends only from the first component. We compare the boundaries with the mean potential of the first component. Here, the concept of supra-threshold behavior changes with respect to the one dimensional case. From the Figures we note that boundary and mean potentials curves always have an intersection. Interestingly, if we fix the firing distribution and its $CV$, the intersection value is the same for different
values of the parameter $\mu$. This fact can be easily understood by noting that a change in $\mu$ determines the same shift both on the mean potential and on the boundary shape. However, this value changes considering different CVs.

6 Conclusions

The extension of the Inverse FPT method to two-dimensional diffusion processes allows to study the connection between a two compartment model and given FPT pdf. We proved that a two compartment model can generate ISIs whose distribution has heavy or light tails at time dependent boundaries. The same behavior had already been found for a Gamma ISI distribution in the case of an Orstein Uhlenbeck process, i.e. a one dimensional LIF model [7]. However, the interpretation in terms of the input shows that the two-compartment model allows a more refined tuning using the two components. We discover that the first component is tuned through excitatory input while the second component is influenced by inhibition.

Some of the identified boundaries have little biological significance. However this is partly related to the choice of parameters and different choices can make these cases with greater biological adhesion.

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Figure 7: Comparison between the time varying boundaries corresponding to the Inverse Gaussian (dashed) and the Gamma-distributed (solid) ISIs with mean ISI equal to 10 and with varying CV, the probability densities are on the subplots. The parameters of the two compartment model are $\alpha = 0.02$, $\beta = 0.02$, $\mu = 0$ and $\sigma = 0.4$.

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Figure 8: Inverse-Gaussian (a) and Gamma (c) distributed ISIs with mean ISI equal to 10 and $CV = 0.5$. Corresponding boundaries and mean membrane potential (b,d). Different lines correspond to different values of the mean input: $\mu = 0$ (black), $\mu = 0.3$ (blue), $\mu = 0.6$ (cyan). Thicker lines correspond to the time varying boundaries. The vertical dotted lines give the ISI distribution quantiles. The parameters of the two compartment model are $\alpha = 0.02$, $\beta = 0.02$, $\mu = 0$ and $\sigma = 0.4$. 
Figure 9: Inverse-Gaussian (a) and Gamma (c) distributed ISIs with mean ISI equal to 10 and $CV = 1$. Corresponding boundaries and mean membrane potential (b,d). Different lines correspond to different values of the mean input: $\mu = 0$ (black), $\mu = 0.3$ (blue), $\mu = 0.6$ (cyan). Thicker lines correspond to the time varying boundaries. The vertical dotted lines give the ISI distribution quantiles. The parameters of the two compartment model are $\alpha = 0.02$, $\beta = 0.02$, $\mu = 0$ and $\sigma = 0.4$. 