On the exceptional generalised Lie derivative for $d \geq 7$

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Abstract

In this work we revisit the $E_8 \times \mathbb{R}^+$ generalised Lie derivative encoding the algebra of diffeomorphisms and gauge transformations of compactifications of M-theory on eight-dimensional manifolds, by extending certain features of the $E_7 \times \mathbb{R}^+$ one. Compared to its $E_d \times \mathbb{R}^+$, $d \leq 7$ counterparts, a new term is needed for closure of the algebra. However, we find that no compensating parameters need to be introduced, but rather that the new term can be written in terms of the ordinary generalised gauge parameters by means of a connection. This implies that no further degrees of freedom beyond those of the $E_8$ or $E_{11}$ spectrum are needed to have a well defined theory. We discuss the implications of the form of the $E_8 \times \mathbb{R}^+$ generalised transformation on the construction of the $d = 8$ generalised geometry. Finally, we suggest how to extend the Lie derivative to eleven dimensions.
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1 Introduction

The most remarkable property of string theory is perhaps that the non linear sigma model formulation of it on different backgrounds may define the same string theory at the quantum level. This property is known as “duality”. The transformations between equivalent backgrounds can be packaged in some groups of gauge symmetry. It is well known, for instance, that $d = 11$ supergravity (or the effective action of type II string theory) compactified on a $d$ torus $T^d (T^{(d-1)})$ has an $E_d(\mathbb{R})$ duality, for $d \leq 10$, see [1] and references therein. For the full compactified theory on $T^{11}$ it is conjectured that $E_{11}$ is a symmetry [2], which actually could be a symmetry of the full M-theory, independent of the compactification [3].

The bosonic $d = 10$ supergravity whose field content is the metric $g_{\mu\nu}$ the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton $\varphi$ has a non manifest $O(10, 10)$ symmetry, this symmetry is known as T-duality. The T-dual covariant description of this supergravity theory is based on two different approaches. On the one hand double field theory (DFT) [4] (earlier versions of DFT can be found in [5], [6], [7], [8] and [9]) which describes string backgrounds in terms of fields on a doubled twenty dimensional space transforming under the $O(10, 10)$ group. On the other hand generalised geometry [10], [11] which unifies the local diffeomorphism and gauge transformations of the 2-form on a generalised tangent space $TM \otimes T^*M$ which has a natural $O(10, 10)$ structure. In DFT all fields and parameters are required to satisfy the section condition or strong constraint. This implies that they depend on only ten coordinates, therefore locally DFT is equivalent to generalised geometry. In DFT the strong constraint can be relaxed [12], [13] however the geometric interpretation of this relaxed theory is still not clear.

In the context of the exceptional groups the generalised geometry approach was first presented in [14] to describe the U-dual invariant $E_d$ theories with $d \leq 7$ and later developed in a series of papers [15], [16] and [17]. The extended field theory, as the counterpart of double field theory, was first presented in [18], [19], for $d = 4, 5$ and for $d = 6, 7$ using the $E_{11}$ non linear formalism in [20]. The geometric counterpart of DFT for the $E_7$ U-duality group in $d = 7$ was developed in [21]. In this last works the relation between four
dimensional gauged maximal supergravity theory \cite{22} and U-duality extended $E_7 \times \mathbb{R}^+$ theory was pointed out. More recent applications of the $E_d \times \mathbb{R}^+$ generalised geometry can be found in \cite{23}, \cite{24}. For other extensions and applications of DFT and the extended field theory see \cite{25}, \cite{26} and \cite{27}.

For $d \geq 8$, some extended-like works have appeared in the literature. In \cite{28}, using the non linear realization of the $E_8$ group, the authors were able to write the supergravity action restricted to eight dimensions including the dual graviton field but not the gauge transformations of this. In \cite{29}, starting from the extended $E_d \times \mathbb{R}^+$ with $d \leq 7$ generalised Lie derivative, it was attempted to complete it to get the eleven dimensional transformation, using the tensor hierarchy mechanism \cite{30}, \cite{31}, \cite{32}. Also, it was shown that even the generalised transformations, beyond seven dimensions, have a gauge structure up to a some given point (at the adjoint representation level of the $E_d$ group) some obstructions of consistency and covariance came out. This is the exactly the reason why for the $E_8$ group the algebra does not work when one extends the $E_d$-series ($d \leq 7$) of generalised Lie derivatives to $E_8$ \cite{33}. For the $E_8$ group the fundamental and the adjoint representation are essentially the same.

In this line, what is called Exceptional field theory has been developed in \cite{34}, \cite{35} and \cite{36}. This theory uses the $E_d \times \mathbb{R}^+$ with $d \leq 8$ gauge transformation but embedded in eleven dimensions. The use of the tensor hierarchy mechanism is not enough to achieve the closure of the algebra, thus a mysterious compensating field has to be added to compensate the failure of it. This new field is beyond the spectrum of all the exceptional groups, hence this field lies in a new direction on the extended space, which is equivalent to say that the $E_d$-generalised tangent space gets bigger. Some issues of this approach are discussed in section (3).

In $d = 8$, the dual gravity and higher dual fields become relevant. The dual graviton, for instance, is described through a field with a mixed symmetry $A_{(1,8)}$ whose gauge parameter is a mixed symmetry tensor $\tau_{(1,7)}$. The conventional gauge field theories seem not to work for this kind of fields. For this reason a consistent generalised geometry or

\footnote{See \cite{33} for a discussion about it for all $E_d$ with $d \leq 7$.}
extended description based on the $E_8 \times \mathbb{R}^+$ group can not be found yet in the literature.

In this paper we present the $E_8 \times \mathbb{R}^+$ generalised transformation, which could be the base for establishing the generalised geometry description of $d = 8$ U-duality theory and perhaps going beyond the $d = 8$. The transformation is given by
\[(\hat{\delta}_\xi V)^M = \xi^P \partial_P V^M - f^M_{\ A \ N} f^{AP} Q \partial_P \xi^Q V^N + \partial_P \xi^P V^M - f^{MP}_{\ Q} \Sigma_P V^Q, \quad (1.1)\]
but the crucial difference with [36] is that $\Sigma$ is not an independent parameter but it is given by
\[\Sigma_P = \frac{1}{60} f^K_{\ J \ L} \hat{\Omega}_{PK} L \xi^J, \quad (1.2)\]
where $f^K_{\ J \ L}$ are the structure constants of the $E_8$ group and the derivative only has components in eight directions, also the field $\hat{\Omega}_{PK} L$ in the index $P$. Actually, this field is a generalised connection on the 8-dimensional manifold.

Gauged maximal supergravity theories in lower dimensions can be constructed making no references to string theory or $d = 10, 11$ supergravity. This kind of theories are consistent upon the tensor hierarchy mechanism. The most general theories of gauged maximal supergravity are those where the trombone symmetry are gauged [37]. In these the gaugings are distributed over some given representations of the exceptional groups, including the fundamental one. Comparing with those supergravity theories found from compactification of the $d = 11$ supergravity, it seems, the first one has much more gaugings than fields that could generate these gaugings from eleven dimensions.

A possible solution to this mismatch is to add more fields than those that $d = 11$ supergravity has. In fact, adding to it an infinite number of fields, starting at the fourth level of the $E_{11}$ algebra, it is possible to reproduce all the gauged maximal supergravity theories [38], [39]. In this paper we will focus on the $d = 3$ maximal supergravity whose duality group is $E_8 \times \mathbb{R}^+$, the $\mathbb{R}^+$ factor is associated with the trombone symmetry. Form the $E_{11}$ approach this is equivalent to break $E_{11} \rightarrow GL(3) \times E_8$. Then we will make the correspondence between the gaugings (fluxes) and the generalised Lie derivative in eight dimensions. Concretely, since the fluxes are defined as the coefficients of the expansion of
this object $\hat{\delta}_{E_A}E_B$ in the base $E_C$, namely

$$\hat{\delta}_{E_A}E_B = F_{AB}^C E_C$$

which holds for all $E_d \times \mathbb{R}^+$ with $d \leq 7$, we will proceed assuming (1.3) also holds for $d = 8$. Finally, we will suggest how the $d = 8$ generalised transformation can be extended to eleven dimensions.

The paper is organised as follows. In section 2 we make a summary of some previous results regarding the $E_7 \times \mathbb{R}^+$ generalised Lie derivative and the fluxes. Then, with the aim of making contact with the generalised geometry approach, we present the $SL(8)$ and $SL(7)$ decomposition of the $E_7$ group and the generalised Lie derivative. Along the presentation we point out some facts that are used after in the next section. In section 3 we review the known approaches regarding $E_8 \times \mathbb{R}^+$ generalised transformation. We introduce a detailed $E_8$ group-theoretic analysis and then perform the $SL(9)$ and $SL(8)$ decomposition of the known $d = 8$ generalised transformation. Based on it and the lessons learnt from the $E_7 \times \mathbb{R}^+$ case we move to the construction of the generalised $E_8 \times \mathbb{R}^+$ transformation. Next, we check the consistency and compatibility of our approach. The section 4 is dedicated to discuss the possible extension of the $d = 8$ generalised transformation to eleven dimension. After that, the conclusions are presented in section 5.

2 $E_7 \times \mathbb{R}^+$

2.1 Summary of previous results

In this section we are interested in exploring the extended $E_7 \times \mathbb{R}^+$ generalised transformation, which is only valid for seven dimensions. We also want to obtain the generalised geometry expressions for the generalised Lie derivative and then explore why it is not possible to extend it to eleven dimensions. As we will see, it is not possible when one starts from $E_7 \times \mathbb{R}^+$. The obstruction for doing this extension comes from some ambiguities in the writing of the dual diffeomorphism in $d = 7$ and the closure of the algebra.
These obstructions can be avoided if instead of starting with the extension from $E_7 \times \mathbb{R}^+$ one starts from the $E_8 \times \mathbb{R}^+$ group in $d = 8$. Discussions on how to build the $E_8 \times \mathbb{R}^+$ extended transformations and the implications for the generalised geometry are made in the next section.

Our starting point will be the generalised $E_7 \times \mathbb{R}^+$ transformation [33], [21], which given a local generalised patch, is written as

\[(\delta \xi V)^M = (L_\xi V)^M = \xi^P \partial_P V^M - A^M_{NP} Q \partial_P \xi^Q V^N = (L_\xi V)^M + Y^M_{NP} Q \partial_P \xi^Q V^N \quad (2.1)\]

where $V$ and $\xi$ are generalised vectors (all $E_7 \times \mathbb{R}^+$ generalised vectors are weighted such that $V = e^{-\Delta} \tilde{V}$, being $\tilde{V}$ a pure $E_7$ vector and $e^{-2\Delta} = det(e)$), $M = 1 \ldots 56$ is an index in the fundamental representation of $E_7$, $L_\xi$ is the usual Lie derivative on the generalised tangent space

\[TM \oplus \Lambda^2 T^* M \oplus \Lambda^5 T^* M \oplus (\Lambda T^* M \otimes \Lambda^7 T^* M), \quad (2.2)\]

\[A^M_{NP} Q = 12 P_{(adj)}^M_{NP} Q - \frac{1}{2} \delta^M_{NP} \delta^P_Q, \quad (2.3)\]

\[Y^M_{NP} Q = -A^M_{NP} Q + \delta^M_Q \delta^P_N \quad (2.4)\]

and

\[P_{(adj)}^M_{NP} Q = K_{ab}(t^a)_{NP} (t^b)_{Q}, \quad a = 1 \ldots 133 \quad (2.5)\]

is a projector $P_{(adj)} : 56 \times 56^* \rightarrow 133 = adj(E_7)$ being $t^a$ the generators of the $E_7$ algebra and $K_{ab}$ the Cartan-Killing metric. In terms of the seven dimensional objects the generalised vector can be identified with

\[V = (v, \omega_2, \sigma_5, \tau_{(1,7)}) \quad (2.6)\]

where $v$ is a vector, $\omega_2$ is a 2-form, $\sigma_5$ is a 5-form and $\tau_{(1,7)}$ is a tensor of a mixed symmetry. The transformation (2.1) satisfies the relation

\[[L_{\xi_1}, L_{\xi_2}] V = L_{[\xi_1, \xi_2]} V, \quad (2.7)\]

provided the section condition holds

\[\Omega^{PQ} \partial_P \otimes \partial_Q = P_{(adj)MN}^{PQ} \partial_P \otimes \partial_Q = 0 \quad (2.8)\]
which implies
\[ Y^M N^P Q \partial_M \otimes \partial_P = 0, \]  
(2.9)
where \( \Omega_{PQ} \) is the symplectic invariant. The relation (2.7) ensures the closure of the algebra and the Leibnitz property. There are at least two known solutions of the section condition [21]. Here, we are interested in making contact with the generalised geometry approach [16], for this reason we only focus on the \( SL(8) \) decomposition of (2.1). In this decomposition the derivative is embedded in the U-duality group as
\[ \partial_\alpha = \frac{1}{2} \partial_{[\alpha 8]} ; \quad \partial_{[\alpha \beta]} = 0 ; \quad \tilde{\partial}^{[\tilde{\alpha} \tilde{\beta}]} = 0 \]  
(2.10)
where the hatted indices run from 1 \ldots 8 while the unhatted ones run from 1 \ldots 7.

Given a generalised paretellisable manifold [24] it is possible to pick a base \( E_A \) on the generalised tangent space and define the so called generalised fluxes
\[ \mathcal{L}_{E_A} E_B = F_{AB}^C E_C. \]  
(2.11)
Using (2.1) it is possible to prove that \( F_{AB}^C \) belongs to the representations dictated by gauge supergravity [22], \( F \in 56 + 912 \).

In the following we display the proof that \( F_{AB}^C \) is in the \( 56 + 912 \) representations of the \( E_7 \) group. The interest for doing it, is because one may take advantage of this result to propose a general principle to build the generalised transformations, which holds not only for \( E_d \times \mathbb{R}^+ \), with \( d \leq 7 \), but also for the \( E_8 \times \mathbb{R}^+ \) group and perhaps for \( d \leq 11 \).

The first we note is that in curved indices the fluxes can be written as [21]
\[ F_{MN}^P = \Omega_{MN}^P - 12 P_{(adj)}^P N^R S^R \Omega_{RM}^S + \frac{1}{2} \Omega_{RM}^R \delta_N^P \]  
(2.12)
where
\[ \Omega_{MN}^P = E_N^A \partial_M E_A^P = \Omega_M^0 (t_0)_N^P + \tilde{\Omega}_M^a (t_a)_N^P \]  
(2.13)
with \( \Omega_M^0 = \partial_M \Delta \) and \( (t_0)_N^P = -\delta_N^P \) is the generator in \( \mathbb{R}^+ \). Using the expression for the projectors \( P_{(R_i)} : R_1 \times R_2 \rightarrow R_i \) where \( R_1 = 56 \), \( R_2 = 133 \) and \( R_3 = 912 \) are the three first irreducible representations of the \( E_7 \) group, explicitly they are [40]

\[ ^2 \text{In general } F_{AB}^C \text{ is not a constant.} \]
\[ P_{(56)M}^a R_b = \frac{8}{19} (t^a)_M^K (t_b)_K^R \]  
\[ P_{(912)M}^a R_b = \frac{1}{7} \delta_M^a \delta_b^a - \frac{12}{7} (t^a)_K^R (t_b)_M^K + \frac{4}{7} (t^a)_M^K (t_b)_K^R, \]

we get

\[ F_{MN}^P = \mathbb{P}_{(56+912)M}^a R_b \tilde{\Omega}_R^b (t_a)_N^P + \lambda \Omega_{RM}^R \delta_N^P \]
\[ + \Omega_M^0 (t_0)_N^P - a P_{(adj)}^P N R S \Omega_R^0 (t_0)_M^S \]

or [37]

\[ F_{MN}^P = \Theta_M^a (t_a)_N^P + (8 P_{(adj)}^P N R S - \delta_N^P \delta_M^R) \theta_R. \]  

We have defined the embedding tensor \( \Theta_M^a \) as

\[ \Theta_M^a = 7 P_{(912)M}^a R_b \tilde{\Omega}_R^b \]

and

\[ \theta_M = -\frac{1}{2} E_M^A e^{2A} \partial_P (e^{-2A} E_A^P) = -\frac{1}{2} E_M^A \nabla_P E_A^P \]

is the gauging associated to the trombone symmetry. For \( E_7 \times \mathbb{R}^+ \)

\[ \mathbb{P}_{(56+912)M}^a R_b = 7 P_{(912)M}^a R_b - \frac{19}{2} P_{(56)M}^a R_b, \]

\( \lambda = \frac{1}{2} \) and \( a = 12 \). In fact, taking the appropriate projection \( \mathbb{P}_{(R_1+R_3)M}^a R_b \), according with gauged supergravity, \([2.15]\) is valid for all \( E_d \times \mathbb{R}^+ \) with \( d \leq 7 \). In the next section we will take \([2.15]\) as a conjecture valid for all \( d \leq 11 \) and as a first test we will use it to build the generalised transformation for the \( E_8 \times \mathbb{R}^+ \) case.

### 2.2 The \( SL(8) \) decomposition

The fundamental representation of \( E_7 \) breaks into the \( SL(8) \) group as \( 56 = 28 + 28 \).

The index \( M \) breaks according to

\[ V^M = (V^{[\alpha \beta]}, V_{[\alpha \beta]}) \]  

(2.20)
where $\hat{\alpha} = 1 \ldots 8$, also the components of the generalised Lie derivative \((2.1)\)

$$(\mathcal{L}_\xi V)^M = ((\mathcal{L}_\xi V)_{[\hat{\alpha}\hat{\beta}]} , (\mathcal{L}_\xi V)_{[\hat{\alpha}\hat{\beta}]}).$$

(2.21)

The adjoint representation breaks into the $SL(8)$ group as $133 = 63 + 70$

$$V^a = (V_{\hat{\alpha}} \hat{\beta} , V_{[\hat{\alpha}_1 \ldots \hat{\alpha}_4]}) ; \quad V_{\hat{\alpha}} \hat{\beta} = 0.$$  

(2.22)

The generators in the $SL(8)$ decomposition take the simple form \[3\]

$$(t_{\hat{\alpha}_1 \hat{\beta}_1})_{\hat{\alpha}_2 \hat{\alpha}_3 \hat{\beta}_2 \hat{\beta}_3} = -\delta_{[\hat{\alpha}_2 \hat{\alpha}_3]}^{\hat{\alpha}_1} \delta_{\hat{\beta}_2 \hat{\beta}_3}^{\hat{\beta}_1} - \frac{1}{8} \delta^{\hat{\beta}_1}_{\hat{\alpha}_1} \delta_{\hat{\beta}_2 \hat{\beta}_3}^{\hat{\beta}_2 \hat{\beta}_3}= - (t_{\hat{\alpha}_1 \hat{\beta}_1})_{\hat{\alpha}_2 \hat{\alpha}_3 \hat{\beta}_2 \hat{\beta}_3}$$

(2.23)

$$(t_{\hat{\alpha}_1 \ldots \hat{\alpha}_4})_{\hat{\beta}_1 \ldots \hat{\alpha}_4} = \frac{1}{24} \epsilon_{\hat{\alpha}_1 \ldots \hat{\alpha}_4 \ldots \hat{\beta}_4}$$

(2.24)

The Cartan-Killing metric is given by

$$K^{ab} = \begin{pmatrix} K_{\hat{\alpha}_1 \hat{\beta}_1 \hat{\alpha}_2 \hat{\beta}_2} & 0 \\ 0 & \frac{1}{12} \epsilon_{\hat{\alpha}_1 \ldots \hat{\alpha}_8} \end{pmatrix}$$

(2.25)

where

$$K_{\hat{\alpha}_1 \hat{\beta}_1 \hat{\alpha}_2 \hat{\beta}_2} = 3(\delta_{\hat{\alpha}_1 \hat{\alpha}_2}^{\hat{\beta}_1 \hat{\beta}_2} - \frac{1}{8} \delta_{\hat{\alpha}_1 \hat{\alpha}_2}^{\hat{\beta}_1 \hat{\beta}_2}).$$

(2.26)

Having the decomposition of the Cartan-Killing metric and the generators one can compute the components of (2.5) involved in (2.1) these are

$$P^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\alpha}_2 \hat{\beta}_2} = (t_{\hat{\rho}_1 \hat{\sigma}_1})_{\hat{\alpha}_1 \hat{\alpha}_2} (K^{-1})_{\hat{\hat{\beta}_1 \hat{\beta}_2}} \hat{\rho}_1 \hat{\sigma}_2 (t_{\hat{\rho}_2 \hat{\sigma}_2})_{\hat{\hat{\beta}_1 \hat{\beta}_2}} \hat{\gamma}_1 \hat{\gamma}_2$$

(2.27)

After a long computation the two components of the generalised Lie derivative in the $28 + \overline{28}$ are given by

$\epsilon_{\hat{\alpha}_1 \ldots \hat{\alpha}_p} \epsilon^{\hat{\beta}_1 \ldots \hat{\beta}_p} = p! \delta^{\hat{\alpha}_1 \ldots \hat{\beta}_p}_{[\hat{\alpha}_1 \ldots \hat{\alpha}_p]}$.
\[ (\mathcal{L}_\xi V)^{\hat{\alpha}_1 \hat{\alpha}_2} = \xi^{\hat{\rho}_1 \hat{\rho}_2} \partial_{\hat{\rho}_1 \hat{\rho}_2} V^{\hat{\alpha}_1 \hat{\alpha}_2} + 2V^{\hat{\rho}_1 \hat{\alpha}_1} \partial_{\hat{\rho}_1 \hat{\sigma}_1} \xi^{\hat{\sigma}_2 \hat{\sigma}_1} - 2V^{\hat{\rho}_1 \hat{\alpha}_2} \partial_{\hat{\rho}_1 \hat{\sigma}_1} \xi^{\hat{\sigma}_2 \hat{\alpha}_2} \]  

(2.28)

\[ -\frac{1}{4} \epsilon^{\hat{\alpha}_1 \hat{\alpha}_2 \hat{\beta}_1 \hat{\beta}_2 \hat{\gamma}_1 \hat{\gamma}_2} V_{\hat{\beta}_1 \hat{\beta}_2} \partial_{\hat{\gamma}_1 \hat{\gamma}_2} \xi^{\hat{\alpha}_1 \hat{\alpha}_2} + \partial_{\hat{\rho}_1 \hat{\rho}_2} \xi^{\hat{\rho}_1 \hat{\rho}_2} V^{\hat{\alpha}_1 \hat{\alpha}_2} \]

\[ (\mathcal{L}_\xi V)_{\hat{\alpha}_1 \hat{\alpha}_2} = \xi^{\hat{\rho}_1 \hat{\rho}_2} \partial_{\hat{\rho}_1 \hat{\rho}_2} V_{\alpha_1 \alpha_2} + 2V^{\hat{\rho}_1 \hat{\alpha}_1} \partial_{\hat{\rho}_1 \hat{\sigma}_1} \xi^{\hat{\sigma}_2 \hat{\sigma}_1} - 2V^{\hat{\rho}_1 \hat{\alpha}_2} \partial_{\hat{\rho}_1 \hat{\sigma}_1} \xi^{\hat{\sigma}_2 \hat{\alpha}_2} \]  

(2.29)

\[ -6V^{\hat{\rho}_1 \hat{\rho}_2} \partial_{\hat{\rho}_1 \hat{\rho}_2} \xi^{\hat{\alpha}_1 \hat{\alpha}_2}. \]

The next step in the construction is to look at only the unhatted indices, or equivalently to take the $SL(7)$ decomposition and then make the identification with the dual components, i.e

\[ V^{\alpha 8} = v^\alpha \quad V_{\alpha \beta} = \omega_{\alpha \beta} \]  

(2.30)

\[ V^{\alpha \beta} = \frac{1}{5!} \epsilon^{\alpha \beta \gamma_1 \ldots \gamma_5} \sigma_{\gamma_1 \ldots \gamma_5} \quad V^{\alpha 8} = \frac{1}{7!} \epsilon^{\beta_1 \ldots \beta_7} \tau_{\alpha, \beta_1 \ldots \beta_7}. \]

Now looking at the unhatted components of (2.28) and (2.29) we get

\[ (\mathcal{L}_V V'')^\alpha = (L_{V'} V'')^\alpha \]  

(2.31)

\[ (\mathcal{L}_V V'')_{\alpha \alpha_1 \alpha_2} = (L_{V'} V'')_{\alpha \alpha_1 \alpha_2} - (\epsilon_{\alpha \gamma} d\omega')_{\alpha_1 \alpha_2} \]

\[ (\mathcal{L}_V V'')_{\alpha \ldots \alpha_5} = (L_{V'} V'')_{\alpha \ldots \alpha_5} - (\epsilon_{\alpha \gamma} d\sigma')_{\alpha_1 \ldots \alpha_5} - (\omega'' \wedge d\omega')_{\alpha_1 \ldots \alpha_5} \]

\[ (\mathcal{L}_V V'')_{\alpha_1 \ldots \alpha_7} = (L_{V'} V'')_{\alpha_1 \ldots \alpha_7} - \frac{7!}{116!} \omega''_{\alpha_1} (d\sigma')_{\alpha_2 \ldots \alpha_7} - \frac{7!}{215!} (d\omega')_{\alpha_1 \alpha_2 \sigma''_{\beta_3 \ldots \beta_7}}. \]

Let us make some remarks about the last two terms of (2.31). In order to write the generalised Lie derivative independent of the coordinates\(^4\) we need to write this two terms in a coordinate-independent way. We note

\[ \frac{7!}{116!} \omega''_{\alpha_1} (d\sigma')_{\alpha_2 \ldots \alpha_7} = e^\alpha_{\alpha} \otimes (\epsilon_{\alpha \gamma} \omega'' \wedge d\sigma')_{\beta_1 \beta_2 \ldots \beta_7} \]  

(2.32)

where $e^\alpha_{\alpha}$ and $e^\alpha$ are some basis on $TM$ and its dual on $TM^*$ respectively. Defining the function $j$ as

\[ j(\cdot, \cdot) : \Lambda^n T^* M \otimes \Lambda^n T^* M \to \Lambda T^* M \otimes \Lambda^{n-1+p} T^* M \]  

(2.33)

\(^4\)Notice that the three first lines of (2.31) can be straightforwardly written in a coordinate-independent way.
where \((\cdot, \cdot)\) has been explicited just to point out the fact that \(j\) is a function with two inputs. This can be written as

\[
j(\cdot, \cdot) = e^\alpha \otimes (\iota_{e^\alpha} \cdot) \wedge \cdot,
\]

(2.34)

note that although the base appears explicitly in the definition it is well defined and is independent of the coordinates. Collecting the information and plugging it in (2.31), the generalised Lie derivative can be written in a coordinate-independent way as [16]

\[
\begin{align*}
(\mathcal{L}_{V'}V'')^1 &= L_{v'} v'' \\
(\mathcal{L}_{V'}V'')^2 &= L_{v'} \omega'' - \iota_{v'} d\omega' \\
(\mathcal{L}_{V'}V'')^3 &= L_{v'} \sigma'' - \iota_{v'} d\sigma' - \omega'' \wedge d\omega' \\
(\mathcal{L}_{V'}V'')^4 &= L_{v'} \tau'' - j(\omega'', d\sigma') - j(d\omega', \sigma'').
\end{align*}
\]

Given (2.35) a natural question that arises is, whether this transformation works beyond seven dimensions. The first we note against a possible extension is that the last two term in (2.35) can be written, in seven dimension, in two equivalent forms. For example,

\[
j(d\omega', \sigma'') = j(\sigma'', d\omega').
\]

(2.38)

We have been able to do that since the first term of the right hand side of (2.37) is identically zero only in seven dimensions. If the manifold had a higher dimension the result would be completely different. Consequently, if an extension were possible a reasonable doubt would exist, since we would not be sure which one of both expression (2.38) is the correct one beyond seven dimensions, the same happens with the other \(j\)-terms. In the next section we will see how the algebra of (2.35) closes only for \(n \leq 7\), giving us another proof that a straightforward extension from seven dimensions is impossible.
2.3 Consistency conditions

Consistency conditions of the transformation (2.35) can be condensed in a single expression like (2.7), the antisymmetric part of this expression is called closure of the algebra while the symmetric part is the Leibnitz property. Computing the consistency conditions for (2.35) we get

$$\left( [L_V, L_{V'}] V'' - L_{L_V V'} V'' \right)^4 = e^a \otimes \iota e^a (\iota_{e_a} d\omega \land d\sigma' - \iota_{e_a} d\omega' \land d\sigma) - \omega'' \land d\omega' \land d\omega \quad (2.39)$$

the three first components are identically zero. Notice that in seven dimensions there are no consistency issues since every term in the right hand side of (2.39) is or comes from an 8-form. What is clear is that beyond seven dimensions the algebra does not close. We want to stress some facts which will be helpful to address the extension beyond seven dimensions. Notice that in (2.35) the $\iota_{e''}$ and $\tau'_{(1,7)}$ terms are missing in the fourth component, also that the right hand side of (2.39) is a $(1,7)$ plus an 8 tensor. These facts are giving us an indication that to achieve the closure of (2.39) for $d > 7$, rather than introducing a new parameter lying in a new direction of the generalised $E_d$ tangent space, one has to complete the transformation (2.35) with the proper $\iota_{e''}$ and $\tau'_{(1,7)}$ terms\footnote{The $\tau'_{(1,7)}$ component for $d > 7$ is a $(1,7) + 8$ tensor, this will be clarified in the next section.}.

To avoid this inconsistency when we try to extend (2.35) to $d = 11$ we will pass to the $E_8 \times \mathbb{R}^+$ group. In this case the algebra does not close from the beginning but, as we will see, there is at least one case where it is possible to move forward achieving the closure of the algebra.

3 $E_8 \times \mathbb{R}^+$

3.1 Summary of previous results

The $E_8 \times \mathbb{R}^+$ is more tricky since from the beginning one of the known transformations \cite{33} does not close and the other \cite{36} needs a new parameter to compensate the failure of the closure of the algebra. The uncomfortable part of the last mentioned approach is
that this new parameter gives rise a new degree of freedom which is not present in the $E_8$ spectrum, neither in the $E_{11}$ one. Let us briefly review these two approaches and then present the SL(9) decomposition of the $E_8 \times \mathbb{R}^+$ transformation.

The proposal of [33] to the generalised Lie derivative for the $E_8 \times \mathbb{R}^+$ case has, essentially, the same form as in (2.1) but with $A^M_{NPQ}$ given by

$$A^M_{NPQ} = 60 P_{(ad)}^M_{NPQ} - \delta^M_N \delta^P_Q.$$  \hfill (3.1)

In the $E_8$ group the fundamental and the adjoint representations can be identified and the generators of the algebra can be written as

$$(t^M)_P = - f^M_{NP}$$  \hfill (3.2)

where $f^M_{NP}$ are the structure constants. The generalised vectors are weighted as follow

$$V^M = e^{-2\Delta} \tilde{V}; \quad e^{-2\Delta} = det(e),$$  \hfill (3.3)

where $\tilde{V}$ is a pure $E_8$ vector, and the generalised transformation reads

$$ (\delta \xi V)^M = \xi^P \partial_P V^M - K_{AB} f^{AM}_{NP} f^{BP}_{Q} \partial_P \xi^Q V^N + \partial_P \xi^P V^M. $$  \hfill (3.4)

This transformation does not close into an algebra, the failure of the closure is given by

$$ ([\delta_{\xi_1}, \delta_{\xi_2}] V - \delta_{\xi_1} \delta_{\xi_2} V)^M = f^{MJ}_{NP} f^P_{IQ} \partial_J \partial_P \xi^Q \xi^I V^N. $$  \hfill (3.5)

The second proposal [36] introduces the parameter $\Sigma$, according to [36], the generalised transformation is given by

$$ (\delta(\xi, \Sigma) V)^M = \xi^P \partial_P V^M - K_{AB} f^{AM}_{NP} f^{BP}_{Q} \partial_P \xi^Q V^N + \partial_P \xi^P V^M - f^{PM}_{NP} \Sigma_P V^N. $$  \hfill (3.6)

The $\Sigma_P$ parameter is called “covariant constrained compensating field” and has to satisfy some constraints. Essentially, these constraints are the same the derivative $\partial_P$ satisfies, or in other words, $\Sigma_P$ is embedded in the U-duality group in the same way as the derivative

\footnote{Notice that we do not use $\mathcal{L}$ to denote the transformation, this distinction will be clarified in section [33].}
is embedded. The transformation of the new parameter is fixed by demanding the closure of the algebra, according to [36],

\[ \Sigma_{12,M} = -2 \Sigma_{[2,M} \partial_N \Lambda_{1]}^N + 2 \Lambda_{[2,M}^N \partial_{\Sigma_{1},M} - 2 \Sigma_{[2,M} \partial_{N} \Lambda_{1]}^N + f^N_{KL} \Lambda_{[2}^K \partial_{M} \partial_{N} \Lambda_{1]}^L. \] (3.7)

Let us comment about some issues of the last transformation. When one defines a transformation, the consistency of this one has to be independent of the choice of the vector components. As an example, one can see that in (2.35) it is possible to turn off whatever of the components of the generalised vector \( V' \) and/or \( V'' \), then compute the closure of the algebra [2,39], and it will still close. In this line of thinking, we could compute the closure of (3.6) with the following choice

\[ \left( [\hat{\delta}(\xi_1,0), \hat{\delta}(\xi_2,0)] - \hat{\delta}(\xi_1,0)(\xi_2,0) \right)(V, 0). \] (3.8)

It is straightforward to see the algebra will not close since

\[ (\hat{\delta}(\xi,0)V)^{M} = (\delta \xi V)^{M} \] (3.9)

and the transformation of the right hand side of the last expression does not close into an algebra. This fact gives us a clue that the new parameter \( \Sigma \) could not be an independent one. If it is the case, (as we will see, it is not an independent parameter) one of the consequences is that this parameter does not induce a new degree of freedom as in [36].

### 3.2 The \( SL(9) \) decomposition

Before preforming the \( SL(9) \) decomposition of the \( E_8 \times \mathbb{R}^+ \) generalised transformation we need to introduce some group-theoretic analysis about the \( E_8 \) group and its representations [42], [28]. The fundamental and the adjoint representations of the \( E_8 \) group, both, have dimension 248, thus one may take the generators as the structure constants (3.2).

A vector \( V^M \), with \( M = 1 \ldots 248 \), decomposes into \( SL(9) \) as

\[ V^M = (V^{\hat{\alpha} \hat{\beta}}, V^{\hat{\alpha} \hat{\beta} \hat{\gamma}}, V_{\hat{\alpha} \hat{\beta} \hat{\gamma}}) \; ; \; \; \; V^{\hat{\alpha}}_{\hat{\beta}} = 0 \] (3.10)

\[ \text{Particularly, we could take } V = (v, \omega, 0, \tau), V' = (v', \omega', 0, \tau') \text{ and } V'' = (v'', \omega'', 0, \tau''), \text{ hence } \left( [\mathcal{L}_V, \mathcal{L}_{V'}] V'' - \mathcal{L}_{\mathcal{L}_V} V'' \right)^4 = -\omega'' \wedge d\omega' \wedge d\omega = 0, \text{ in } d = 7. \]
where the hatted indices run from 1 to 9. The algebra \([t^M, t^N] = f^{MN} P_t P^t\) of the \(E_8\) group in the \(SL(9)\) decomposition is as follow

\[
[t^\dot{\alpha}_\beta, t^\dot{\gamma}_\delta] = \delta^\gamma_\beta t^\dot{\alpha}_\delta - \delta^\alpha_\delta t^\dot{\gamma}_\beta. \tag{3.11}
\]

\[
[t^{\dot{\alpha}_\beta}, t^{\dot{\gamma}_1...\dot{\gamma}_3}] = 3\delta^{[\dot{\gamma}_1}_\beta t^{\dot{\gamma}_2\dot{\gamma}_3]\dot{\alpha}} - \frac{1}{3} \delta^\dot{\gamma}_\beta t^{\dot{\gamma}_1...\dot{\gamma}_3},
\]

\[
[t^{\dot{\alpha}_\beta}, t^{\dot{\gamma}_1...\dot{\gamma}_3}] = -3\delta^{[\dot{\gamma}_1}_\beta t^{\dot{\gamma}_2\dot{\gamma}_3]\dot{\beta}} + \frac{1}{3} \delta^\dot{\gamma}_\beta t^{\dot{\gamma}_1...\dot{\gamma}_3},
\]

\[
[t^{\dot{\alpha}_1...\dot{\alpha}_3}, t^{\dot{\beta}_1...\dot{\beta}_3}] = 18\delta^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3\dot{\gamma}_1\dot{\gamma}_2\dot{\gamma}_3},
\]

\[
[t^{\dot{\alpha}_1...\dot{\alpha}_3}, t^{\dot{\beta}_1...\dot{\beta}_3}] = \frac{1}{3!} e^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3\dot{\gamma}_1\dot{\gamma}_2\dot{\gamma}_3} t^{\dot{\gamma}_1...\dot{\gamma}_3}.
\]

From (3.11) it is possible to read the structure constants, these are

\[
f^{\dot{\alpha}_1\dot{\alpha}_2\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3\dot{\beta}_3} = \delta^\dot{\beta}_2\beta_1\dot{\alpha}_3\delta^\dot{\beta}_3\beta_2 - \delta^\dot{\alpha}_1\dot{\alpha}_2\delta^\dot{\beta}_3\beta_1,
\]

\[
f^{\dot{\alpha}_1\dot{\alpha}_2\dot{\beta}_1\dot{\beta}_2\dot{\beta}_2\dot{\gamma}_3} = 3\delta^{[\dot{\beta}_2\dot{\gamma}_2}\dot{\alpha}_1\dot{\beta}_1\dot{\alpha}_3\dot{\beta}_2\dot{\gamma}_2 - \frac{1}{3} \delta^\dot{\gamma}_2\beta_1\delta^\dot{\gamma}_3\beta_2,
\]

\[
f^{\dot{\alpha}_1\dot{\alpha}_2\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3\dot{\gamma}_3} = -3\delta^{\dot{\beta}_1\dot{\alpha}_2\dot{\beta}_2\dot{\gamma}_3\dot{\alpha}_3\dot{\beta}_3\dot{\gamma}_3} + \frac{1}{3} \delta^\dot{\alpha}_1\dot{\alpha}_2\delta^\dot{\beta}_3\beta_1\dot{\beta}_2\dot{\gamma}_3,
\]

\[
f^{\dot{\alpha}_1\dot{\beta}_1\dot{\gamma}_3\dot{\beta}_1\dot{\beta}_2\dot{\beta}_2\dot{\gamma}_3\dot{\beta}_3\dot{\gamma}_3} = 18\delta^{\dot{\beta}_3\dot{\alpha}_3\dot{\beta}_2\dot{\beta}_1\dot{\gamma}_1\dot{\beta}_1\dot{\beta}_3\dot{\gamma}_3},
\]

\[
f^{\dot{\alpha}_1\dot{\beta}_1\dot{\gamma}_1\dot{\alpha}_2\dot{\beta}_2\dot{\beta}_2\dot{\gamma}_2\dot{\alpha}_2\dot{\beta}_1\dot{\beta}_3\dot{\gamma}_3} = -\frac{1}{3!} e^{\dot{\alpha}_1\dot{\alpha}_2\dot{\beta}_1\dot{\beta}_2\dot{\beta}_2\dot{\beta}_2\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3\dot{\gamma}_3},
\]

\[
f^{\dot{\alpha}_1\dot{\beta}_1\dot{\beta}_1\dot{\gamma}_1\dot{\alpha}_2\dot{\beta}_2\dot{\beta}_2\dot{\beta}_2\dot{\beta}_3\dot{\gamma}_3} = \frac{1}{3!} e^{\dot{\alpha}_1\dot{\alpha}_2\dot{\beta}_1\dot{\beta}_2\dot{\beta}_2\dot{\beta}_2\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3\dot{\gamma}_3}.
\]

Having the structure constants we are ready to compute the Cartan-Killing metric, it is defined as

\[
K^{AB} = \frac{1}{60} f^{A K L} f^{B L K}.
\]

After a very long computation we get

\[
K^{AB} = \begin{pmatrix}
k_{\dot{\alpha}_1\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3\dot{\gamma}_1\dot{\gamma}_2\dot{\gamma}_3} & 0 & 0 \\
0 & 0 & 6\delta^{\dot{\alpha}_1\dot{\beta}_1\dot{\gamma}_1\dot{\beta}_2\dot{\beta}_2\dot{\gamma}_2} \\
0 & 6\delta^{\dot{\alpha}_2\dot{\beta}_2\dot{\gamma}_2} & 0
\end{pmatrix}
\]
where

\[ K_{\hat{\alpha}_1 \hat{\beta}_1 \hat{\beta}_2} = \delta^\hat{\beta}_2 \delta_{\hat{\alpha}_1} - \frac{1}{9} \delta_{\hat{\alpha}_1} \delta^\hat{\beta}_2 \]  

(3.15)

and the identity takes the simple form

\[
\delta^B_A = \begin{pmatrix}
K_{\hat{\alpha}_1 \hat{\beta}_1 \hat{\beta}_2} & 0 & 0 \\
0 & \delta_{\hat{\alpha}_1} \delta_{\hat{\beta}_2} \delta_{\hat{\gamma}_2} & 0 \\
0 & 0 & \delta_{\hat{\alpha}_2} \delta_{\hat{\beta}_2} \delta_{\hat{\gamma}_2}
\end{pmatrix},
\]  

(3.16)

finally notice that \( \delta^A_A = 248 \).

To have a better idea about what is going on with the transformation of the \( E_8 \times \mathbb{R}^+ \) group we follow the discussion performing the \( SL(9) \) split of (3.4), but first we will show how the derivative is embedded in the duality group. The partial derivative breaks according to

\[ \partial_M = (\partial_{\hat{\alpha}}, \partial_{\hat{\alpha}_1 \hat{\gamma}_1}, \partial_{\hat{\alpha}_2 \hat{\gamma}_2}) = (\partial_{\hat{\alpha}}, 0, 0) \]  

(3.17)

and

\[ \partial_{\hat{\alpha}} \rightarrow \partial_{\hat{\alpha}_9} = \partial_{\alpha}, \quad \partial_{\hat{\beta}} = \partial_{\alpha} = 0. \]  

(3.18)

This embedding is a solution of the equations (section condition) \([33], [36]\)

\[ K^{MN} \partial_M \otimes \partial_N = 0 \]  

(3.19)

\[ f^{AMN} \partial_M \otimes \partial_N = 0 \]

\[ (f^{AMP} f_{AN} - 2\delta^{(M} \delta_{Q)} \partial_M \otimes \partial_N = 0 \]

which implies

\[ Y^{MN} {\hat{P}Q} \partial_M \otimes \partial_P = 0, \]  

(3.20)

where the \( Y \) tensor is defined as in (2.4), but now adapted to the \( E_8 \times \mathbb{R}^+ \) case.

A very tedious calculation yields the three components of the generalised transformation of the \( E_8 \times \mathbb{R}^+ \), the first one is given by
\[(\delta_\xi V)^{\hat{\alpha}}_{\hat{\beta}} = \xi^{\alpha_1} \partial_{\alpha_1} V^{\hat{\alpha}}_{\hat{\beta}} - V^{\alpha_1}_{\beta} \partial_{\alpha_1} \xi^{\hat{\alpha}} g + \partial_{\alpha_1} \xi^{\alpha_1} \partial_\xi V^{\hat{\beta}_1}_{\hat{\beta}} \]
\[+ \partial_{\beta} \xi^{\hat{\beta}_1} g V^{\hat{\alpha}_{\beta_1}}_{\hat{\beta}_1} - \partial_{\alpha_1} \xi^{\alpha_1} \partial_\xi V^{\hat{\alpha}_{\beta_1}}_{\hat{\beta}_1} g + \partial_{\alpha_1} \xi^{\hat{\alpha}_1} V^{\hat{\beta}_1}_{\hat{\beta}_1} \]
\[+ \frac{9}{6} \delta^{\hat{\alpha}}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1]}} \hat{\beta}_1 \partial_{\hat{\alpha}_2} [\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]^{\hat{\alpha}_2} - \frac{1}{6} \delta^{\hat{\alpha}}_{\hat{\beta}_1} \partial_{\hat{\alpha}_2} [\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]^{\hat{\alpha}_2} V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{\beta}_1 \hat{\gamma}_1} \]
\[+ \frac{9}{6} \partial_{\hat{\alpha}_1} \hat{\beta}_1 \hat{\alpha}_2 \xi^{\hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1]}} \delta^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1]}} \partial_{\hat{\alpha}_2} \xi^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1]}} V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1]}} \]
and as expected
\[(\delta_\xi V)^{\hat{\alpha}}_{\hat{\alpha}} = 0. \] (3.22)

The other two components are given by
\[(\delta_\xi V)^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1} = \xi^{\alpha} \partial_{\alpha} V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1} - 3 \partial_{\alpha} \xi^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} + \partial_{\alpha} \xi^{\alpha} V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1} \]
\[-9 V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_2 \hat{\beta}_1 \hat{\gamma}_1]}} \partial_{\alpha} \xi^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_2 \hat{\beta}_1 \hat{\gamma}_1]}} \]
\[+ \frac{1}{12} \xi^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} \]
and
\[(\delta_\xi V)^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1} = \xi^{\alpha} \partial_{\alpha} V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1} + 3 V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} \partial_{\alpha} \xi^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} - 3 \partial_{\alpha} \xi^{\alpha} V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1} \]
\[-9 V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} \partial_{\alpha} \xi^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} \]
\[+ \frac{1}{12} \xi^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} V^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1}_{\hat{[\hat{\alpha}_1 \hat{\beta}_1 \hat{\gamma}_1]}} \]
where the underlined indices mean they are fully antisymmetric.

The next step, as in the $E_7 \times \mathbb{R}^+$ case, is to look at only the unhatted components ($SL(9) \rightarrow SL(8)$) and then identify these with its corresponding dual. Dualizing on the $E_8$ group is more complicated than for $E_7$ since the fundamental representation of the first one has dimension 248. From table (11) it is straightforward to see that the generalised tangent space is (see [13] for a discussion about it)
\[
V^M \quad SL(8) \text{ repr}
\]

\[
V^\alpha_9 = v^\alpha \\
V_{\alpha_1 \alpha_2 9} = \omega_{\alpha_1 \alpha_2} \\
V_{\beta_1 \beta_2 \beta_3} = -\frac{1}{5!} \epsilon^{\beta_1 \beta_2 \beta_3 \alpha_1 \ldots \alpha_5} \sigma_{\alpha_1 \ldots \alpha_5} \\
V^{\alpha}_{\beta} = \frac{1}{18} \epsilon^{\alpha \gamma_1 \ldots \gamma_7} (\tau_{\beta, \gamma_1 \ldots \gamma_7} + \Lambda_{\beta \gamma_1 \ldots \gamma_7}) \\
V_{\beta_1 \beta_2 \beta_3} = \frac{1}{5!} \epsilon^{\gamma_1 \ldots \gamma_7} \xi_{\beta_1 \beta_2 \beta_3, \gamma_1 \ldots \gamma_7} \\
V_{\beta_1 \beta_2 9} = \frac{1}{6!} \epsilon^{\beta_1 \beta_2 \alpha_1 \ldots \alpha_6} \epsilon^{\gamma_1 \ldots \gamma_7} \xi_{\alpha_1 \ldots \alpha_6, \gamma_1 \ldots \gamma_7} \\
V^9_{\gamma} = \frac{1}{8! \cdot 8!} \epsilon^{\alpha_1 \ldots \alpha_8} \epsilon^{\beta_1 \ldots \beta_8} \xi_{\gamma, \alpha_1 \ldots \alpha_8, \beta_1 \ldots \beta_8} \\
\]

Table 1: \(SL(8)\) decomposition of \(E_8\)

\[
TM \oplus \Lambda^2 T^* M \oplus \Lambda^5 T^* M \oplus (\Lambda T^* M \otimes \Lambda^7 T^* M) \\
\oplus (\Lambda^3 T^* M \otimes \Lambda^8 T^* M) \oplus (\Lambda^6 T^* M \otimes \Lambda^8 T^* M) \oplus (\Lambda T^* M \otimes (\Lambda^8 T^* M)^2)
\]

and now a generalised vector is represented in components as

\[
V = (v, \omega_2, \sigma_5, \tau_{(1,7)} + \Lambda_{(8)}, \xi_{(3,8)}, \xi_{(6,8)}, \xi_{(1,8,8)}).
\]

Here, we are only interested in finding the \(E_8 \times \mathbb{R}^+\) transformation down to the dual diffeomorphism. This is why we will compute the components of the generalised transformations that correspond with the first line of (3.25), or the four first lines of table (1). A straightforward computations for the components \((\delta V')^\alpha_9\), \((\delta V)^{\alpha_1 \alpha_2 9}\) and \((\delta V)^{\alpha_1 \alpha_2 \alpha_3}\) leads to

\[
(\delta V')^\alpha \quad = \quad (L'_\nu V')^\alpha \\
(\delta V')_{\alpha_1 \alpha_2} \quad = \quad (L'_\nu \omega')_{\alpha_1 \alpha_2} - (\iota_{\nu'} d\omega')_{\alpha_1 \alpha_2} \\
(\delta V''')_{\alpha_1 \ldots \alpha_5} \quad = \quad (L'_\nu \sigma''')_{\alpha_1 \ldots \alpha_5} - (\iota_{\nu'} d\sigma')_{\alpha_1 \ldots \alpha_5} - (\omega'' \wedge d\omega')_{\alpha_1 \ldots \alpha_5}.
\]

As in the \(E_7 \times \mathbb{R}^+\) case these three components can be written in a coordinate-independent way. It can be easily done just erasing the indices in (3.27). The \((\delta V)^\alpha_{\beta}\) component is

\^Notice that \(\tau_{[\beta, \gamma_1 \ldots \gamma_7]} = 0\).
quite hard to compute but at the end we get

\[(\delta V' V'')_{\alpha,\beta} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]
\[\times \delta^\alpha_{\beta,\gamma_1...\gamma_7} \frac{-j(\omega'' , d\sigma')}{-8! v''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) e^{\alpha_1\gamma_1...\gamma_7}} + \frac{1}{24} (d\omega' \wedge \sigma'') \]
\[\delta^\alpha_{\beta,\gamma_1...\gamma_7} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]
\[\times \delta^\alpha_{\beta,\gamma_1...\gamma_7} \frac{-j(\omega'' , d\sigma')}{-8! v''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) e^{\alpha_1\gamma_1...\gamma_7}} + \frac{1}{24} (d\omega' \wedge \sigma'') \]
\[\delta^\alpha_{\beta,\gamma_1...\gamma_7} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]
\[\times \delta^\alpha_{\beta,\gamma_1...\gamma_7} \frac{-j(\omega'' , d\sigma')}{-8! v''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) e^{\alpha_1\gamma_1...\gamma_7}} + \frac{1}{24} (d\omega' \wedge \sigma'') \]
\[\delta^\alpha_{\beta,\gamma_1...\gamma_7} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]
\[\times \delta^\alpha_{\beta,\gamma_1...\gamma_7} \frac{-j(\omega'' , d\sigma')}{-8! v''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) e^{\alpha_1\gamma_1...\gamma_7}} + \frac{1}{24} (d\omega' \wedge \sigma'') \]
\[\delta^\alpha_{\beta,\gamma_1...\gamma_7} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]
\[\times \delta^\alpha_{\beta,\gamma_1...\gamma_7} \frac{-j(\omega'' , d\sigma')}{-8! v''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) e^{\alpha_1\gamma_1...\gamma_7}} + \frac{1}{24} (d\omega' \wedge \sigma'') \]
\[\delta^\alpha_{\beta,\gamma_1...\gamma_7} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]

Let us point out some facts about (3.28). First, we have explicitly separated the reducible representation $63 + 1 \rightarrow \delta \tau + \delta \Lambda$. Notice that if $d = 7$ we recover the expression of the fourth component of the generalised Lie derivative (2.35). Ignoring for the moment the $\Lambda$ variation

\[\delta^\alpha_{\beta,\gamma_1...\gamma_7} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]
\[\times \delta^\alpha_{\beta,\gamma_1...\gamma_7} \frac{-j(\omega'' , d\sigma')}{-8! v''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) e^{\alpha_1\gamma_1...\gamma_7}} + \frac{1}{24} (d\omega' \wedge \sigma'') \]
\[\delta^\alpha_{\beta,\gamma_1...\gamma_7} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]
\[\times \delta^\alpha_{\beta,\gamma_1...\gamma_7} \frac{-j(\omega'' , d\sigma')}{-8! v''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) e^{\alpha_1\gamma_1...\gamma_7}} + \frac{1}{24} (d\omega' \wedge \sigma'') \]
\[\delta^\alpha_{\beta,\gamma_1...\gamma_7} = \frac{1}{7!} \epsilon^{\gamma_1...\gamma_7} (L_\nu \tau''_{\beta,\gamma_1...\gamma_7} - 8 \nu''\rho (\partial_\nu \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]
\[\times \delta^\alpha_{\beta,\gamma_1...\gamma_7} \frac{-j(\omega'' , d\sigma')}{-8! v''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) e^{\alpha_1\gamma_1...\gamma_7}} + \frac{1}{24} (d\omega' \wedge \sigma'') \]

Having this cleaned expression, it is easier to check $(\delta V' V'')_{[\alpha,\gamma_1...\gamma_7]} = 0$. On the other hand one may see that getting a coordinate-independent writing of (3.29) is impossible since the equivalent term to $-(\iota_{\nu'} d)$ in (3.29) is

\[-8 \nu''\rho (\partial_\rho \tau'_\beta \gamma_1...\gamma_7 + \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7 - \partial_\beta \Lambda'_\rho \gamma_1...\gamma_7) \]

and this term has a coordinate-dependent writing, namely, this term is not a tensor. It is one of the reason why the algebra does not close. Notice that the terms in the parenthesis of (3.30) are exactly the transformation predicted by [44], [45] for a tensor with a $(1,8)$ mixed symmetry (on a linearised background).

This is not the end of the story, in the next section we will see that as long as the theory is defined on a generalised parallelisable manifold the transformation can be consistently defined in $d = 8$ and perhaps extended to $d > 8$. 

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3.3 Building the $E_8 \times \mathbb{R}^+$ generalised gauge transformation

To build the $E_8 \times \mathbb{R}^+$ generalised gauge transformation the starting point will be (2.15), but adapted to this case. The more general theory of gauged supergravity in $d = 3$, where the trombone symmetry is gauged [37], needs, for consistency, gaugings living in the $1 + 248 + 3865$ representations of $E_8$. Taking that into account, the proposal is as follows

$$F_{MN}^P = \mathbb{P}_{(1+248+3865)} M^A R_B \tilde{\Omega}_R^B (t_A)_N^P + \lambda \Omega_{RM}^R \delta_N^P$$

$$+ \Omega_M^0 (t_0)_N^P - a P_{(adj)}^P N R S \Omega_R^0 (t_0)_M^S,$$

where, given (3.3),

$$\Omega_M^0 = 2 \partial_M \Delta,$$

note that all the indices are capital ones. To fix $\mathbb{P}_{(1+248+3865)} M^A R_B$ we need to know explicitly the projectors to each representation involved, these are given by [37]

$$P_{(1)}^M N^P Q = \frac{1}{248} \delta_M^N \delta_Q^P$$

$$P_{(248)}^M N^P Q = \frac{1}{60} f^{AN} M f_A^P Q$$

$$P_{(3875)}^M N^P Q = \frac{1}{28} f^{AP} M f_A^N Q - \frac{1}{28} f^{APN} f_{AMQ} + \frac{1}{14} \delta_M^P \delta_N^Q - \frac{1}{56} \delta_M^N \delta_Q^P + \frac{1}{14} K^{PN} K_{QM}.$$

Writing $\mathbb{P}_{(1+248+3865)}$ as a linear combination of these projectors, i.e

$$\mathbb{P}_{(1+248+3865)} = a_1 P_{(1)} + a_2 P_{(248)} + a_3 P_{(3875)}.$$  

(3.34)

The coefficients $a_i$ can be fixed demanding $F_{MN}^P$ (or in the planar $F_{AB}^C$ version) is the generalised transformation with all vectors being basis, $(\delta_{E_A} E_B)^M = F_{AB}^C E_C^M$.

We see that taking $a_1 = 62$, $a_2 = -30$, $a_3 = 14$, $\lambda = 1$ and $a = 60$ the fluxes can be written as

$$F_{AB}^C E_C^M = E_A^P \partial_P E_B^M - f_A^M N f^{AP} \partial_P E_A^Q E_B^N + \partial_P E_A^P E_B^M$$

$$+ K_{IQ} \tilde{\Omega}_P^Q (t^P)_J^M E_A^I E_B^J.$$

---

We know (3.4) is not a well defined transformation, but from the $SL(9)$ decomposition (3.27) and (3.28) we already know (3.4) is close to the right one.
From the right hand side of (3.35) it is possible to read the generalised transformation.

The next step is to extend the transformation to general vectors and not only on basis, namely $E_A \rightarrow \xi$ and $E_B \rightarrow V$. This can be done for all terms except for the last one, getting

$$
(\delta_{\xi} V)^M = \xi^P \partial_P V^M - f^M_{\, N} f^{A P} Q \partial_P \xi^Q V^N + \partial_P \xi^P V^M - f^{M P} Q \Sigma_P V^Q.
$$

(3.36)

Remarkably we have obtained a generalised transformation that has the same form as the one presented in [36] but in our transformation $\Sigma$ is not a parameter, this can be written by means of a connection as

$$
\Sigma_P = \frac{1}{60} f^J_{\, L} \tilde{\Omega}_{PK} \xi^J.
$$

(3.37)

where we have used $\tilde{\Omega}_{PK} = \tilde{\Omega}_{PQ} (t_Q)_K^L$, we recall $\tilde{\Omega}_{PK} = \tilde{E}_K \tilde{E}_A \partial_P \tilde{E}_A^L$. Furthermore, the index $P$ is one that corresponds to the derivative, hence it is straightforward to see $\Sigma_P$ gets embedded in the U-duality group as the derivative does. Also, due to the fact that $f^J_{\, L}$ and $\tilde{\Omega}_{PK}$ have weights $\lambda = 0$ and $\lambda = 1$ respectively, $\Sigma$ does not have weight, namely

$$
\Sigma_P = \frac{1}{60} f^J_{\, L} \tilde{\Omega}_{PK} \xi^J.
$$

(3.38)

Moreover, one can see it is the only transformation that can be built without introducing further parameters or degrees of freedom on the manifold. Also, (3.36) is consistent with gauged supergravity in three dimensions. Regarding this last statement, now we are able to write the generalised fluxes (3.31) as in [37],

$$
F_{MN}^P = \Theta_{M}^A (t_A)_N^P + \frac{1}{2} (t_A)_N^P (t_A)_M^Q - \delta_M^P \delta_{M}^Q \partial_Q
$$

(3.39)

where

$$
\Theta_{M}^A = \mathbb{P}_{(1+3865)} R^A_{B} \tilde{\Omega}_{RB}
$$

(3.40)

is the embedding tensor in the $1 + 3865$ representations of the $E_8$ group, and the gauging associated to the trombone symmetry is written as

$$
\partial_M = -E_M \tilde{A} e^{2\Delta} \partial_P (e^{-2\Delta} E_A^P) = -E_M \tilde{A} \nabla_P E_A^P,
$$

(3.41)
we recall that $E_A^P = e^{-2\Delta \bar{E}_A^P}$. In general, provided $\Theta_M^A$ and $\vartheta_Q$ are not constants in this context, $F_{MN}^P$ is not a constant.

In order to write the generalised transformation we have introduced the generalised Weitsenböck connection which is defined in terms of the generalised base as in \(2.13\). In the generalised base are encoded the degrees of freedom of the theory\(^{10}\), for this reason, no further fields or parameters are needed to get a well defined transformation. However, something interesting happens, since now the parameters and the degrees of freedom will be mixed in the transformation, as happens in closed string field theory (CSFT), see for a discussion on gauge transformation in CSFT \([46]\).

Let us, brief and schematically, discuss this last statement in this context. In CSFT usually the algebra of the gauge parameters is represented as follow

\[
[\delta_{\xi_1}, \delta_{\xi_2}]|\Psi\rangle = \delta_{\xi_{12}(\Psi)}|\Psi\rangle \quad + \quad \text{(on-shell=0 terms)} \tag{3.42}
\]

where $|\Psi\rangle$ is the field string containing all the excitations of string theory and, it is said, $\xi_{12}(\Psi)$ is a field dependent parameter. Translating to our case, the equivalent to the truncated string field should be the generalised base $E_A$, strictly speaking, it should be a combination of certain components of the base. The equivalent to the parameter $\xi_{12}(\Psi)$, in our language, should be the generalised transformation \(3.36\). The analogues $E_8 \times \mathbb{R}^+$ expression to \(3.42\) should be

\[
[\delta_{\xi_1}, \delta_{\xi_2}]E_A = \delta_{(\delta_{\xi_1}\xi_2 + f_{MPQ}\Sigma(E, \xi_1)\xi_2^Q)}E_A, \tag{3.43}
\]

which is the consistency condition of our theory. These facts are showing an intricate and intriguing relation between CSFT and the exceptional approach, beyond DFT \([47], [4]\), which is worth exploring and maybe relate it with the full $E_{11}$ theory.

To know the full form of the fourth component of the $E_8 \times \mathbb{R}^+$ generalised transformation, we need to know the $SL(8)$ decomposition of $f^{MPQ}\Sigma_PV_Q$. Taking into account

\[
\Sigma_P = (\Sigma_{\hat{\alpha}}, \Sigma_{\hat{\alpha}\hat{\beta}\hat{\gamma}}, \Sigma^{\hat{\alpha}\hat{\beta}\hat{\gamma}}) = (\Sigma_{\hat{\alpha}}, 0, 0), \tag{3.44}
\]

\(^{10}\)In general the base is the degree of freedom of the theory but there is one special base which can be written directly in terms of $e^{\hat{a}}, A_3, A_6, A_{(1,8)}, A_{(3,9)}, A_{(6,9)}, A_{(1,8,9)}$.\}
where
\[ \Sigma^\beta_{\alpha} \rightarrow \Sigma^0_{\beta} = \Sigma_{\beta} , \quad \Sigma^\alpha_\beta = \Sigma^\beta_\alpha = 0. \] (3.45)
and (3.12) we get
\[ f^{MP}_Q \Sigma_P V^Q = \left( (\Sigma_\beta v^\alpha - \frac{1}{8} \delta^\beta_\alpha \Sigma_\rho v^\rho) + \frac{1}{8} \delta^\alpha_\beta \Sigma_\rho v^\rho , 0 , 0 \right). \] (3.46)
The final expression for the 63 + 1 component can be read from (3.28) plus the terms (3.46), with \( \Sigma_\beta = \frac{1}{60} f^{K}_{J K L} \delta_{\beta K L} \).

3.4 Consistency conditions and compatibility

Closure of the algebra

Now we proceed to compute the closure of the algebra of the transformation (3.36). We write the generalised transformation as
\[ (\hat{\delta}_V)^M = (\delta_V)^M - f^{MP}_Q \Sigma_P V^Q \] (3.47)
where \((\delta_V)^M\) is (3.4) whose algebra does not close and the failure of it is given by (3.5).
Computing
\[ ([\hat{\delta}_{\xi_1}, \hat{\delta}_{\xi_2}] V - \hat{\delta}_{[\xi_1, \xi_2]} V)^M = f^{MP}_Q \left( \hat{\delta}_{\xi_1} \Sigma_{2}^P - L_{\xi_1} \Sigma_{2}^P - \partial_{\xi_1} \Sigma_{1}^P \right) \]
\[ + f^{P}_{J} \partial_{P} \partial_{\xi_1} \Sigma_{1}^P \] (3.48)
From the above expression we can see that the closure of the algebra implies
\[ \hat{\delta}_{\xi_1} \Sigma_{2}^P = L_{\xi_1} \Sigma_{2}^P + \partial_{\xi_1} \Sigma_{1}^P - f^{P}_{J} \partial_{P} \partial_{\xi_1} \Sigma_{1}^P + \partial_{P} \Sigma_{1} \Sigma_{2}^P. \] (3.49)
The antisymmetrization has been removed including no terms symmetric in \( \xi_1, \xi_2 \). As we will see, from the compatibility with the transformation \( \hat{\delta}_{\xi_1} \Sigma_{2}^P \) computed using directly that \( \Sigma \) is given by (3.37), (3.49) is the right expression.

In the computation of the closure of the algebra we have used the identity
\[ f^P_M f_{P N}^L C_K \otimes C_L^\prime = C_M \otimes C_N^\prime + C_N \otimes C_M^\prime \] (3.50)
where $C$ and $C'$ are embedding in the U-duality group as

$$C_M \to (C^\alpha_\beta, 0, 0) \to (C^g_\beta, 0, 0), \quad (3.51)$$

this has been proved in [36], also one can see that the identity is a consequence of the last line of (3.19) which can proved directly from (3.12). Another useful identity that follows from the last one is

$$\delta f^N \rho \Sigma N P V^M = f^M_J (\partial_I \Sigma_P P + \partial_P \Sigma_I P) V^J \quad (3.52)$$

where

$$\Sigma^P N = (\Sigma^\alpha_\beta P, 0, 0) \to (\Sigma^g_\beta P, 0, 0). \quad (3.53)$$

**Compatibility**

If $\Sigma$ were an independent parameter we should check the closure of the algebra to this new component of the generalised vector, i.e we should compute

$$(\hat{\delta}_{\xi_1} \hat{\delta}_{\xi_2} \Sigma - \hat{\delta}_{\xi_1 [\xi_2]} \Sigma) M, \quad (3.54)$$

as usually this computation is performed in the tensor hierarchy [29]. However as $\Sigma$ actually is not an independent parameter the consistency check becomes a compatibility check.

Having a notion of generalised transformation it is possible to define the generalised covariant derivative $\nabla$. In general for some generalised connection $\Gamma$ it is written in an local generalised patch as

$$\nabla_M V^N = \partial_M V^N + \Gamma_{MN}^K V^K \quad (3.55)$$

Demanding that $\nabla_M V^N$ transforms as a tensor with respect to the generalised transformation (3.36) we get

$$\hat{\delta}_{\xi_1} \Gamma_{MN} = \hat{\nabla}_{\xi_1} \Gamma_{MN} + f_A^N K f^P Q \partial_M \partial_P \xi_1^Q - f^NP Q \partial_M \Sigma_{1P} + \partial_M \partial_P \xi_1^P \delta_N \quad (3.56)$$

where $\hat{\nabla}_{\xi_1} \Gamma$ is written just to denote the tensorial part of the $\Gamma$ transformation. Using

$$\nabla_M E^N_A = W_{MA}^B E^N_B = \partial_M E^N_A + \Gamma_{MN}^K E^K_A. \quad (3.57)$$
where $W_{MA}^B$ is the generalised spin connection, it is easy to prove

$$\left(\hat{\delta}_{\xi_1} - \mathcal{L}_{\xi_1}\right)\Gamma_{MK}^N = -\left(\hat{\delta}_{\xi_1} - \mathcal{L}_{\xi_1}\right)\Omega_{MK}^N. \tag{3.58}$$

Given the last expression, for the Weitsenböck connection (3.56) becomes

$$\hat{\delta}_{\xi_1}\Omega_{MK}^N = \mathcal{L}_{\xi_1}\Omega_{MK}^N - f_A^N K f^{AP} Q \partial_M \partial_P \xi_Q^I + f^{NP} Q \partial_M \Sigma_{1P} - \partial_M \partial_P \xi^P \delta_K^N. \tag{3.59}$$

In particular (3.59) also holds for $\tilde{\Omega}_{MK}^N$. Now, contracting with $f^K_I N \xi_2^I$, using (3.13) and $f^K_I K = 0$ we get

$$\hat{\delta}_{\xi_1} \left(\frac{1}{60} f^K_I N \tilde{\Omega}_{MK}^N \xi_2^I\right) = \mathcal{L}_{\xi_1} \left(\frac{1}{60} f^K_I N \tilde{\Omega}_{MK}^N \xi_2^I\right) - f^K_I P Q \partial_M \partial_P \xi_Q^I \xi_2^I + \partial_M \Sigma_{1P} \xi_2^I, \tag{3.60}$$

or, from (3.37)

$$\hat{\delta}_{\xi_1} \Sigma_{2M} = \mathcal{L}_{\xi_1} \Sigma_{2M} - f^K_I P \partial_M \partial_P \xi_Q^I \xi_2^I + \partial_M \Sigma_{1P} \xi_2^I, \tag{3.61}$$

where, due to the fact that $\Sigma$ does not have weight,

$$\hat{\mathcal{L}}_{\xi_1} \Sigma_{2M} = L_{\xi_1} \Sigma_{2M} + \partial_P \xi_Q^P \Sigma_{2M}. \tag{3.62}$$

Notice that (3.61) is exactly the same expression (3.49) we computed through the closure of the algebra, therefore our check of consistency and compatibility is completed.

## 4 Eleven dimensions

### 4.1 From $E_8 \times \mathbb{R}^+$ to eleven dimensions

Going further from eight dimensions should be quit straightforward, at least for the three first levels of $E_{11}$. Before we discuss this statement we will expose a few facts about the generalised Lie derivative on the groups $E_d \times \mathbb{R}^+$ with $d \leq 7$. On the one hand, one interesting thing that happens with the three first lines of the generalised Lie derivative (2.35) is that it is not only valid for $d \leq 6$ but also for all $d$. When the algebra is extended with the fourth line of (2.35), the validity of the generalised Lie derivative gets reduced to $d = 7$. 

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On the other hand, the closure of the algebra of the $E_8 \times \mathbb{R}^+$ generalised transformation does not depend on the dimension of the manifold, hence this transformation close into an algebra for all $d$. However, we have to be very careful, since the transformation depends on having a well defined generalised parallelisable manifold and for $d > 8$ we can not claim its existence since these kind of manifolds have not been studied yet.

For $d > 8$ the U-duality groups are infinite dimensional, thus a generalised base $E_A^P$ on these groups would be a set of infinite vectors with infinite components. To write explicitly the full Weitsenböck connection $\Omega_{MN}^P = E_N^A \partial_M E_A^P$ in terms of the degrees of freedom would be more complicated, or impossible, since an infinite sum is involved in its definition. Regarding last fact, recently appeared two interesting papers [48], [49] which based on them could be worth to explore how going beyond eight dimension within the approach displayed in this work.

Finally, we are confident that under the $SL(d)$ decomposition and correct truncation (in particular the $SL(11)$ decomposition and correct truncation of $E_{11}$) an extension of the $E_8 \times \mathbb{R}^+$ generalised transformation to $d > 8$ could be possible and straightforwardly performed.

5 Conclusions

In this work we have constructed the $E_8 \times \mathbb{R}^+$ generalised transformation which is conceptually different to the one presented in [36]. Remarkably, its consistency is not subject to any compensating parameter, thus only the parameter and the degrees of freedom of the $E_8 \times \mathbb{R}^+$ spectrum are involved in the transformation. Although no compensating fields are needed, the generalised transformation seems not to have a covariant coordinate-independent writing. This could be a problem for the covariance of the theory. However, when the theory is defined on a generalised parallelisable manifold a consistent transformation is achieved upon the introduction of the generalised Weitsenböck connection.

The extended $E_7 \times \mathbb{R}^+$ generalised approach was used as a laboratory. In particular, we present the $SL(8)$ and $SL(7)$ decomposition of the extended generalised $E_7 \times \mathbb{R}^+$
transformation, obtaining perfect agreement with [17]. A very precise definition of the
\( j \)-function, presented in [17], was given here. From the \( SL(7) \) perspective we computed
the consistency conditions, which indeed are the closure of the algebra and the Leibnitz
property, and we analysed under what conditions the transformation is consistent. As
expected, in seven dimensions there is no problem with the generalised Lie derivative.
However, beyond seven dimensions the algebra does not close, hence a straightforward
extension from \( d = 7 \) is impossible.

Working out explicitly the \( E_7 \times \mathbb{R}^+ \) flux definition, it was possible to show they can be
written as a combination of projectors to the \( 56 + 912 \) irreducible representations of the
\( E_7 \) group acting on the Weitsenböck connection, plus terms associated with the conformal
factor. In fact, it is valid for all \( E_d \times \mathbb{R}^+ \) with \( d \leq 7 \), where now the projection is on the
correspondent \( \mathbf{R}_1 + \mathbf{R}_3 \) irreducible representations of the \( E_d \) group. Interestingly enough,
the same expression can be written for \( d = 8 \), but the difference with the other exceptional
groups is that \( \mathbf{R}_3 \) is a reducible representation of \( E_8 \).

Using the lessons learnt from \( E_7 \times \mathbb{R}^+ \) case, we move forward to the \( E_8 \times \mathbb{R}^+ \) one. We
presented the full \( d = 8 \) generalised transformation, written in terms of the fundamental
indices of the \( E_8 \) group and splitted in indices of the \( SL(8) \) one. Consistency and com-
patibility were checked, showing that the transformation of \( \Sigma \) can be computed through
the closure of the algebra or using its own definition in terms of the vector parameter
and the Weitsenböck connection. Actually, it is a non trivial statement whose checking
strengthens everything presented here.

There are several unanswered questions. Maybe, the two ones who need to be impe-
ratively answered are,

- Is it possible to go beyond the parallelisable manifold?

- Is it possible to get a coordinate-independent writing of the fourth component of
  the \( E_8 \times \mathbb{R}^+ \) generalised transformation?

Probably the answer to the first question gives us a clue to answer the second one.
The only place where we used the information, apart from the flux definition, that the
manifold has to be a generalised parallelisable one is in the compatibility check. Note that (3.61) only needs (3.59) which indeed is (3.56). Thus given a general connection on a general manifold, transforming as minus (3.56), seems to be sufficient for the algebra to close. The minus sign is not a problem in the definition of the transformation of $\Gamma$. The point is, having a general connection with the proper transformation is just a necessary condition. One requirement for the algebra to close is that $\Sigma_P$ has to satisfies (3.44). In terms of a generalised connection $\Sigma$ should be

$$
\Sigma_P = \frac{1}{60} f_J^K \Gamma_{PK} L \xi^J.
$$

(5.1)

The above expression implies that to give an $E_8 \times \mathbb{R}^+$ generalised transformation on a general manifold, this manifold has to be equipped with a non zero generalised connection which has to be embedded, in its first index, in the U-duality group as the derivative is embedded in this one. Concretely,

$$
\nabla_M = (\nabla_{\alpha^\hat{\beta}}, 0, 0) = (\nabla_{\alpha^9}, 0, 0)
$$

(5.2)

or

$$
Y^M_{\ N} P \nabla_M \otimes \nabla_P = 0.
$$

(5.3)

This means that the section condition should be extended to the covariant derivatives. The relation (5.3) would have strong implications on the definition of the generalised torsion, curvature and Ricci tensors.

The answer to the second question is more elusive. One possibility could be that on a general manifold the generalised transformations takes the following form

$$
(\hat{\delta}_\xi V)^M = \xi^P \nabla_P V^M - f_A^M_N f_A^P Q \nabla_P \xi^Q \nabla_N + \nabla_P \xi^P V^M,
$$

(5.4)

for some general generalised connection. Then by some mechanism or imposing certain conditions like the torsion free connection one, (5.3) etc; (5.1) reduces to (3.36). However, the meaning of this constrains as well as the definition of torsion tensor, in this context, in not clear for us yet.
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