A QUANTUM CHARACTERIZATION OF NP

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Abstract. In this article, we introduce a new complexity class called $\text{PQMA}_{\log(2)}$. Informally, this is the class of languages for which membership has a logarithmic-size quantum proof with perfect completeness and soundness, which is polynomially close to 1 in a context where the verifier is provided a proof with two unentangled parts. We then show that $\text{PQMA}_{\log(2)} = \text{NP}$. For this to be possible, it is important, when defining the class, not to give too much power to the verifier. This result, when compared to the fact that $\text{QMA}_{\log} = \text{BQP}$, gives us new insight into the power of quantum information and the impact of entanglement.

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1. Introduction

In classical complexity, the concept of proof is extensively used to define very interesting complexity classes such as $\text{NP}$, $\text{MA}$, and $\text{IP}$. When allowing the verifier (and the prover) to be quantum mechanical, we obtain complexity classes such as $\text{QMA}$ and $\text{QIP}$. Quantum complexity classes can sometimes turn out to have surprising properties. For example, in contrast to the classical case, we know that quantum interactive proofs can be restricted to three messages; that is, $\text{QIP} = \text{QIP (3)}$ (Kitaev & Watrous 2000).

Because of the probabilistic nature of quantum computation, the most natural quantum generalization of $\text{NP}$ is $\text{QMA}$. This is the class of languages having polynomial-size quantum proofs. A quantum proof obviously requires a quantum verifier, but
behaves similar to a classical proof with regard to completeness and soundness. Since group non-membership is in $QMA$ (Watrous 2000), but is not known to be in $MA$ (and therefore $NP$), we have an example of a statement having polynomial-size quantum proofs but no known polynomial-size classical proof.

In this paper, we are interested in logarithmic-size quantum proofs. Classically, when considering a polynomial-time verifier, the concept of logarithmic-size classical proofs is not interesting. Any language having logarithmic-size classical proofs would also be in $P$, since one can go through every possible logarithmic-size proof in polynomial time.

In the quantum case, very short quantum proofs could still be interesting. Any reasonable classical description of a quantum proof requires a polynomial number of bits, and thus, one cannot try all quantum proofs using a classical simulator. That being said, if the verifier is simple enough, the optimization problem of finding a proof that makes the verifier accept with high enough probability can be turned into a semidefinite programming problem (Alizadeh 1995; Vandenberghe & Boyd 1996) of polynomial size. Thus, if the verifier is simple enough, then the language is in $P$. Also, if the verifier is in $BQP$, then one still only obtains $BQP$ (Marriott & Watrous 2004).

Although we just argued that logarithmic-size classical and quantum proofs seem uninteresting, by slightly changing the rules of the game, we get an interesting complexity class. In preliminary work (Blier & Tapp 2008), we showed that $NP \subseteq QMA_{\log}(2)$. This class is also defined with the promise that two logarithmic-size unentangled registers are given to the verifier. This promise gives the verifier more leeway to check the proof and limits the prover’s ability to cheat. Therefore, this gives a new perspective on the properties of entanglement.

In parallel with our work, and independently of us, Aaronson et al. (2008) have shown that $3SAT$ is in $QMA_{\log}(\sqrt{n}\text{polylog}(n))$ (i.e., with $\sqrt{n}\text{polylog}(n)$ unentangled registers) with constant completeness and soundness. It seems that only two unentangled registers for the certificate are not enough to check the proof with a constant gap between completeness and soundness; we
achieve perfect completeness but soundness polynomially close to one. They commented on our previous note (Blier & Tapp 2007) emphasizing that this soundness cannot be improved to a constant unless \( \text{QMA}(2) = \text{NEXP} \). Note also that since the length of the proof is logarithmic and the number of registers is constant, showing that \( 3\text{COL} \) (the language of graphs colorable with three colors) is in \( \text{QMA}_{\log}(2) \) would implies that \( \text{NP} \subseteq \text{QMA}_{\log}(2) \). Therefore, constant soundness is achieved in Aaronson et al. (2008) at the cost that their result cannot be generalized to \( \text{NP} \subseteq \text{QMA}_{\log}(\sqrt{n \text{polylog}(n)}) \) because the polynomial reduction would cause the length of the proof to increase polynomially. In a recent article (Beigi 2008), it has been shown how to obtain a soundness of \( 1 - \frac{1}{n^{3+\epsilon}} \) for the language \( 3\text{SAT} \) with two registers.

The class \( \text{QMA}_{\log}(2) \) is small (included in \( \text{QMA} \)) but still contains both \( \text{NP} \) and \( \text{BPP} \). This is an interesting property since no relation is known between \( \text{NP} \) and \( \text{BPP} \). Showing that \( \text{QMA}_{\log}(2) \subseteq \text{NP} \) would somehow imply that classical non-determinism allows us to simulate a polynomial-size quantum circuit. In this paper, we show that by slightly changing the class definition, the paradigm of unentangled logarithmic size registers leads to a characterization of the class \( \text{NP} \). We thus introduce a new complexity class \( \text{PQMA}_{\log}(2) \) and show that \( \text{NP} = \text{PQMA}_{\log}(2) \). Once again, this class is defined to have a polynomially small gap between completeness and soundness. Compared to our preliminary works, where we defined \( \text{QMA}_{\log}(2) \), we do not consider the verifier to work in quantum polynomial time but only to be able to generate a quantum circuit of polynomial size that acts on a logarithmic number of qubits. This still allows the verifier to do the protocol such as in Blier & Tapp (2008) but also to define a classical polynomial-size certificate for the class \( \text{PQMA}_{\log}(2) \) which implies \( \text{PQMA}_{\log}(2) \subseteq \text{NP} \).

2. Definitions and theorem

A formal definition of the class \( \text{PQMA}_{\log}(2) \) (Classical polynomial-time quantum Merlin Arthur with two unentangled logarithmic-size certificates) will follow. Informally, it can be seen as the
class of languages for which there exists a logarithmic quantum proof with the promise that it is separated into two unentangled parts. The verifier works in classical polynomial time. It is allowed to produce quantum circuits of polynomial size acting on a logarithmic number of qubits.

The following definition is simply a formal statement of what is usually referred to as a set of gate one can efficiently approximate.

**Definition 2.1.** A natural gate set $U$ is a finite set of unitary transformations acting on a finite number of qubits such that for all $U \in U$ there exists a classical algorithm that can approximate every element of the matrix $U$ up to $n$ bits in time polynomial in $n$. Furthermore, $C(U)$ is the set of circuit composed of gates from $U$.

With this definition in hand, we can define formally $\text{PQMA}_{\log}(2)$.

**Definition 2.2.** A language $L$ is in $\text{PQMA}_{\log}(2)$ if there exists a natural gate set $U$, polynomials $p$ and $q$, a constant $c$ and a classical algorithm $V$ running in polynomial time that is allowed, for a word $x$ where $|x| = n$, to produce a quantum circuit $Q = V(x) \in C(U)$ of polynomial size $q(n)$ acting on $O(\log(n))$ qubits such that:

1) (Completeness) if $x \in L$, there exists a state $|w\rangle \in \left(\mathcal{H}^{2\log(n)}_2 \right)^2$ s.t.

$$\Pr[Q(|w\rangle) = \text{accept}] = 1, \quad \text{where } |w\rangle = |w_1\rangle \otimes |w_2\rangle;$$

2) (Soundness) if $x \notin L$, with $|x| = n$, then for all states $|w\rangle \in \left(\mathcal{H}^{2\log(n)}_2 \right)^2$,

$$\Pr[Q(|w\rangle) = \text{accept}] < 1 - \frac{1}{p(n)}, \quad \text{where } |w\rangle = |w_1\rangle \otimes |w_2\rangle.$$
Theorem 2.4. \( \text{NP} = \text{PQMA}_{\log}(2) \).

This will be proven in the two following sections. In the next section, we will describe an algorithm showing that \( 3\text{COL} \) is in \( \text{PQMA}_{\log}(2) \). It follows that \( \text{NP} \subseteq \text{PQMA}_{\log}(2) \). Then, it will be proved that \( \text{PQMA}_{\log}(2) \subseteq \text{NP} \).

3. Logarithmic-size quantum proof for \( 3\text{COL} \)

In this section, we will prove the following statement:

Lemma 3.1. \( \text{NP} \subseteq \text{PQMA}_{\log}(2) \).

To prove the statement, we will show that the language \( 3\text{COL} \) is in class \( \text{PQMA}_{\log}(2) \). We will address the completeness in Lemma 3.2 and the soundness in Lemma 3.8. The proof is conclusive since \( 3\text{COL} \) is \( \text{NP} \)-complete over polynomial time reduction. On the one hand, the soundness in \( \text{PQMA}_{\log}(2) \) only has to be polynomially close to one. On the other hand, the proof is of logarithmic size and the number of registers is constant. Therefore, any decision problem in \( \text{NP} \) reduced to \( 3\text{COL} \) will still have a protocol with a logarithmic-size proof and a satisfying soundness.

3.1. Protocol and completeness. We describe the verifier for the language \( 3\text{COL} \). The registers of the proof \( |\Psi\rangle \) and \( |\Phi\rangle \) are both regarded as vectors in \( \mathcal{H}_n \otimes \mathcal{H}_3 \) respectively the node and color part of the register. The verifier performs one of the following three tests with equal probability. If the test succeeds, he accepts, otherwise he rejects.

- **Test 1:** (Equality of the two registers) Perform the swap-test (Buhrman et al. 2001) on \( |\Psi\rangle \) and \( |\Phi\rangle \) and reject if the test fails.

- **Test 2:** (Consistency with the graph) \( |\Psi\rangle \) and \( |\Phi\rangle \) are measured in the computational basis, yielding \( (i, C(i)), (i', C'(i)) \),
  a) if \( i = i' \), verify that \( C(i) = C'(i) \).
  b) otherwise if \( (i, i') \in E \) verify that \( C(i) \neq C'(i) \).

- **Test 3:** (All nodes are present) For both \( |\Psi\rangle \) and \( |\Phi\rangle \), measure the index part of the register and the color part separately in the
Fourier basis. If the outcome of the measurement of the color part is $F_3|0\rangle$ and the outcome of the index part is not $F_n|0\rangle$, then reject.

The following lemma states that the protocol has completeness 1.

**Lemma 3.2.** If $x \in 3\text{COL}$, then there exists a proof that the verifier described above will accept with probability 1.

**Proof.** Let the quantum proof be $|\Psi\rangle = |\Phi\rangle = \frac{1}{\sqrt{n}} \sum_i |i\rangle |C(i)\rangle$ where $C$ is a valid coloring of the graph $G$. The probability that the swap-test outputs equal is $\frac{1}{2} + \frac{|\langle\Psi|\Phi\rangle|^2}{2}$ (Buhrman et al. 2001). Since $|\Psi\rangle = |\Phi\rangle$, the probability that Test 1 succeeds is 1. Because $C$ is a valid coloring of $G$, we have that Test 2 succeeds with probability 1. To see that Test 3 will also succeed with certainty, it is sufficient to see that:

$$\langle I \otimes F_3 | \frac{1}{\sqrt{n}} \sum_j |j\rangle |c_j\rangle = \frac{1}{\sqrt{n}} \sum_j |j\rangle \frac{1}{\sqrt{3}} \sum_k e^{2\pi i c_j k} |k\rangle$$

and therefore, if the color register is measured to be in the state $|0\rangle$, the resulting state will be $\frac{1}{\sqrt{n}} \sum_i |i\rangle = F_n|0\rangle$.

**3.2. Soundness.** Let us now consider the case where $G \not\in 3\text{COL}$. Lemma 3.8 at the end of this section states that if $G$ is not 3-colorable, then there is a non-negligible probability that one of the three tests will fail. To prove this, we will require the following five simple lemmas.

Because we know that the two registers given by the prover are not entangled, they can be written separately as

$$|\Psi\rangle = \sum_i \alpha_i |i\rangle \sum_j \beta_{i,j} |j\rangle \quad |\Phi\rangle = \sum_i \alpha'_i |i\rangle \sum_j \beta'_{i,j} |j\rangle$$

where $\sum_i |\alpha_i|^2 = 1$ and $\forall i, \sum_j |\beta_{i,j}|^2 = 1$ and likewise for $|\Phi\rangle$. It is not difficult to see that the use of unentangled mixed states would not help the prover.

The following lemmas will give us some useful facts on the behavior of the state when measured in the computational basis.
The next lemma says that if Test 1 succeeds with high enough probability, then the distribution of outcomes will be similar for the two states.

**Lemma 3.3.** Let $|\Psi\rangle$ and $|\Phi\rangle$ be as defined earlier. If there exists a $k$ and an $l$ such that $||\alpha_k\beta_{k,l}|-|\alpha'_k\beta'_{k,l}|| \geq 1/n^3$ then Test 1 will fail with probability at least $1/8n^6$.

**Proof.** Let $P_{i,j} = |\alpha_i\beta_{i,j}|^2$ and $Q_{i,j} = |\alpha'_i\beta'_{i,j}|^2$ be the probability distributions when $|\Phi\rangle$ and $|\Psi\rangle$ are measured in the computational basis. We will use the fact that, for any von Neumann measurement, the distances defined below are such that $D(|\Psi\rangle, |\Phi\rangle) \geq D(P, Q)$, where $P$ and $Q$ are the classical outcomes distributions of the measurement. Then,

$$\sqrt{1 - |\langle \Psi | \Phi \rangle|^2} \overset{\text{def}}{=} D(|\Psi\rangle, |\Phi\rangle) \geq D(P, Q) \overset{\text{def}}{=} \frac{1}{2} \sum_{ij} ||\alpha_i\beta_{i,j}||^2 - ||\alpha'_i\beta'_{i,j}||^2 \geq \frac{1}{2} \left| |\alpha_k\beta_{k,l}|^2 - |\alpha'_k\beta'_{k,l}|^2 \right| \geq \frac{1}{2} \cdot \frac{1}{n^3}.$$ 

This means that $|\langle \Psi | \Phi \rangle|^2 \leq 1 - \frac{1}{4n^6}$ and that Test 1 will fail with probability at least $1/8n^6$. $\square$

The next lemma states that nodes with a high enough probability of being observed have a well-defined color.

**Lemma 3.4.** Given that the quantum proof would pass both Test 1 and part a) of Test 2 with probability of failure no larger than $1/8n^6$, it must be that $\forall i$ for which $|\alpha_i| \geq \frac{1}{n^2}$, $\exists! j$ such that $|\beta_{i,j}|^2 \geq \frac{99}{100}$.

**Proof.** Suppose for the sake of contradiction that there exists an $i$ such that $|\alpha_i|^2 \geq \frac{1}{n^2}$ for which two of the $\beta_{i,j}$ have a squared norm larger than 1/200. Hence, w.l.o.g we can assume that $|\beta_{i,0}|^2 > 1/200$ and $|\beta_{i,1}|^2 > 1/200$. Because of Lemma 3.3, we have that $|\alpha'_i|^2 |\beta'_{i,1}|^2 \geq |\alpha_i|^2 |\beta_{i,1}|^2 - \frac{1}{n^3} \geq \frac{200n^2}{400n^2} - \frac{1}{n^3}$. Therefore, the
probability of obtaining \((i, 0)\) when measuring \(|\Psi\rangle\) and \((i, 1)\) when measuring \(|\Phi\rangle\) is at least

\[
\left( \frac{1}{200n^2} \right) \left( \frac{1}{200n^2} - \frac{1}{n^3} \right) \geq \frac{1}{8n^6}
\]

when \(n\) is large enough. This is in contradiction with the hypothesis. Therefore, the norm squared of the amplitude for two of the three colors must be less than \(\frac{1}{200}\), concluding the proof. \(\Box\)

The next three lemmas tell us what Test 3 actually implies.

**Lemma 3.5.** Given that the quantum proof would pass both Test 1 and part a) of Test 2 with probability of failure less than \(\frac{1}{8n^6}\), then the probability of measuring \(|\bar{0}\rangle = F_3|0\rangle\) in the Fourier basis on the color register is greater than \(1/5\) when \(n\) is large enough.

**Proof.** Assume that the node register is measured. If the outcome is \(i\), then the probability of obtaining \(|\bar{0}\rangle\) in the Fourier basis on the color register is given by

\[
\frac{1}{3} |\beta_{i,0} + \beta_{i,1} + \beta_{i,2}|^2.
\]

For all \(i\) with probability larger than \(1/n^2\) Lemma 3.4 applies, in which case we can assume w.l.o.g that \(|\beta_{i,0}|^2 > 99/100\) and \(|\beta_{i,1}|^2 + |\beta_{i,2}|^2 \leq 1/100\). Using the Cauchy–Schwarz inequality, we obtain

\[
\frac{1}{3} |\beta_{i,0} + \beta_{i,1} + \beta_{i,2}|^2 \geq \frac{1}{3} |\beta_{i,0}| - |\beta_{i,1} + \beta_{i,2}|^2
\]

\[
\geq \frac{1}{3} |\beta_{i,0}| - \sqrt{2(|\beta_{i,1}|^2 + |\beta_{i,2}|^2)}
\]

\[
\geq \frac{1}{4}
\]

Now, note that only \(n - 1\) of the nodes can have a probability smaller than \(1/n^2\), and therefore, the probability of obtaining 0 on the color register is at least \((1 - (n - 1)\frac{1}{n^2})\frac{1}{4} \geq \frac{1}{5}\) for large enough \(n\). \(\Box\)
LEMMA 3.6. Given a state \( |X\rangle = \sum_i \gamma_i |i\rangle \) such that there exist an \( l \) with \( |\gamma_l|^2 < \frac{1}{2^n} \), then the probability of not getting \( |\overline{0}\rangle = F_n |0\rangle \) when we measure \( |X\rangle \) in the Fourier basis is at least \( \frac{1}{16n^2} \).

PROOF. Let \( P \) and \( Q \) be the probability distributions when measuring \( |X\rangle \) and \( F_n |0\rangle \) respectively in the computational basis. Using the same techniques as in Lemma 3.3, we get:

\[
\sqrt{1 - |\langle X|\overline{0}\rangle|^2} \overset{\text{def}}{=} D(|X\rangle, |\overline{0}\rangle) \geq D(P, Q) = \frac{1}{2} \sum_i \left| |\gamma_i|^2 - \frac{1}{n} \right| \geq \frac{1}{2} \left| |\gamma_l|^2 - \frac{1}{n} \right| \geq \frac{1}{4n}.
\]

This implies that the probability of failing the test is greater than \( \frac{1}{16n^2} \).

LEMMA 3.7. Assuming Test 1, Test 3 and part a) of Test 2 would succeed with probability larger than \( \frac{1}{8n^6} \), then it must be that for all \( i \), \( |\alpha_i|^2 \geq \frac{1}{10n} \).

PROOF. Because of Lemma 3.5, the probability of measuring 0 on the color register while performing Test 3 is at least 1/5. Let \( |X\rangle = \sum_i \gamma_i |i\rangle \) be the state after measuring 0 on the color register of \( |\Psi\rangle \). Suppose that there was an \( i \) in the original \( |\Psi\rangle \) such that \( |\alpha_i|^2 < 1/(10n) \). Again, because of Lemma 3.5 we must have \( |\gamma_i|^2 < 1/(2n) \). Now, from this fact and Lemma 3.6, we conclude that \( |\Psi\rangle \) would fail Test 3 with too large a probability. Therefore, for all \( i \), \( |\alpha_i|^2 \geq \frac{1}{10n} \).

Now, using Lemma 3.3, Lemma 3.4, Lemma 3.5 and Lemma 3.7, it will be possible to prove the soundness of our verifier.

LEMMA 3.8. If \( x \not\in 3\text{COL} \), then all quantum proofs will fail the test with probability at least \( \frac{1}{24n^6} \).

PROOF. Assume that the graph \( G \) is not 3 colorable and that it would fail Test 1, Test 3 and part a) of Test 2 with probability smaller than \( \frac{1}{8n^6} \). Let \( C(i) = \max_j |\beta_{ij}| \) be a coloring. Because of Lemma 3.3 and Lemma 3.4, this maximum is well defined. Since
the graph is not 3-colorable, there exists two adjacent vertices $v_1$ and $v_2$ in $G$ such that $C(v_1) = C(v_2)$. Because of Lemma 3.7, when performing Test 2, we have a probability of at least $1/(10n^2)$ of measuring $C(v_1)$ in the first register, and because of Lemma 3.3 and Lemma 3.7, we have a probability $1/(10n^2) - 1/n^3$ of measuring $C(v_2)$ in the second register. Combining these results, we have a probability larger than $1/8n^6$ of failing condition b) of Test 2. □

4. Polynomial-size classical proof for languages in $\text{PQMA}_{\log}(2)$

In this section, we will prove the following statement.

**Lemma 4.1.** $\text{PQMA}_{\log}(2) \subseteq \text{NP}$

**Proof.** We will show that for all languages $L$ in $\text{PQMA}_{\log}(2)$, there is a totally classical polynomial-time verifier $V'$ for the language. The verifier $V'$ will receive a classical description (density matrix) of the quantum proof and compute the acceptance probability of the quantum circuit $Q = V(x)$ with enough precision.

Let $g$ be the gap between the soundness and completeness for the verifier associated with the language $L$. Following the notation introduced in Definition 2.2, $g = 1/p(n)$.

More precisely, the (totally) classical verifier $V'$ will do the following in order to accept the language $L$. Let us call the classical witness $\rho$. The verifier $V'$ first simulates $V$ (the verifier as defined for the class $\text{PQMA}_{\log}(2)$) to compute a description of the quantum circuit $Q = V(x) \in C(U)$. Let $U$ be the unitary transformation corresponding to $Q$ and $\Pi_{\text{accept}}$ be the projector corresponding to measuring 1 on the accepting qubit. The verifier $V'$ approximates the value $Tr(\Pi_{\text{accept}}U(\rho \otimes \rho)U^\dagger)$ with precision $g/3$. The verifier $V'$ accepts if the acceptance probability is larger than $1 - g/2$. This value can be evaluated in polynomial time for the number of qubits $Q$ act on is logarithmic, and the number of gates is polynomial. □
5. Conclusion

In this paper, we proved that $\text{NP} = \text{PQMA}_{\log}(2)$. This gives a quantum characterization of one of the most important complexity classes. Moreover, this characterization is interesting in the sense that it is done within the paradigm of interactive proofs. This result gives us insight into the power of quantum information and the subtlety of entanglement.

For future work, on the one hand, a very natural way to improve on our result would be to investigate whether $\text{NP} = \text{PQMA}_{\log}(2)$ even when the gap between soundness and completeness is constant rather than only non-negligible. As stated before, it would be surprising since it could lead to the result $\text{QMA}(2) = \text{NEXP}$. As an intermediate result, one could try to show a better gap between completeness and soundness using a constant number of unentangled parts in the proof. This is related to the question of whether $\text{QMA}_{\log}(k) = \text{QMA}_{\log}(2)$ and also reminiscent of the $\text{QMA}(k)$ vs $\text{QMA}(2)$ problem.

On the other hand, Aaronson et al. (2008) achieved a constant gap at the cost of requiring $\sqrt{n \text{polylog}(n)}$ registers. Is it possible to use fewer unentangled registers and still achieve a constant gap? The ultimate goal would indeed be to show that all languages in NP have logarithmic-size quantum proofs with constant completeness and soundness. If this it is not the case, what is the minimal length of such a proof in order to obtain a constant gap for an NP-complete problem?

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