On threshold amplitudes II: Amplitudes nullification via classical symmetries

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Abstract

The Ward identities for amplitudes at the tree level are derived from symmetries of the corresponding classical dynamical systems. The result are applied to some $2 \to n$ amplitudes

*supported by the Łódź University grant No. 269.
†supported by KBN grant No. 5 P03B 060 21
I Introduction

Threshold amplitudes provide one of the rare examples of problems in quantum field theory which are both interesting and tractable, to some extend at least, by analytical methods. They were subject of numerous studies (see [1] for a review).

An interesting aspect of the problem is amplitude nullification which exhibit some models [2]. In particular, Libanov et al. [3] have shown that tree amplitudes vanish on the threshold in the model with $O(2)$-symmetry broken softly by the mass term. Moreover, they gave a beautiful argument in favour of the idea that nullification is related to the symmetries of the reduced classical mechanical system [4](see also [5]).

The line of reasoning presented in Ref. [4] can be used to construct other theories exhibiting amplitude nullification [6]. Moreover, the conclusions of [4] were confirmed by more traditional approach based on Ward identities following from the symmetry [7].

In the present note we derive the Ward identities for tree-level threshold amplitudes by functional techniques. We assume that the reduced hamiltonian system posses some symmetry and compute the corresponding Ward identities at the tree level. These Ward identities are shown to imply amplitude nullification.

The paper is organized as follows. In Sec. II we remind the description of canonical symmetries on lagrangian level. Then, in Sec. III, the Ward identities for tree-level amplitudes are computed assuming the existence of symmetry for the reduced dynamical system obtained by neglecting space-dependence of the initial system. In Sec.IV these results are applied to specific $2 \rightarrow n$ processes. Finally, Sec.V is devoted to some conclusions.

II Canonical symmetries on lagrangian level

Generically, the symmetries enforcing amplitudes nullification are canonical rather than point transformations. However, it is more convenient to do the pertubation theory on the lagrangian level. Therefore, we shall first remind how the canonical symmetries show up in the lagrangian formalism. As a first step consider a generalized lagrangian formalism; the generalization consists in the assumption that the second time derivatives are also admitted in the lagrangian function

$$L = L(q, \dot{q}, \ddot{q})$$

(1)

The corresponding equations of motion read

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0$$

(2)

The class of point transformations allowed is now wider: new coordinates can depend also on old velocities. The transformation $(q, t) \rightarrow (q', t')$ is a symmetry, if

$$L(q', \frac{dq'}{dt}, \frac{d^2q'}{dt^2})\frac{dt'}{dt} = L(q, \frac{dq}{dt}, \frac{d^2q}{dt^2}) + \frac{df(q, \frac{dq}{dt}, t)}{dt};$$

(3)
note that \( f \) can now also depend on \( \frac{dq}{dt} \). Eq.(3) results in the following Noether identity for infinitesimal transformations \( q'_i = q_1 + \varepsilon y_i, \quad t' = t + \varepsilon \tau, \quad f \to \varepsilon f \),

\[
(y_i - \tau \dot{q}_i) \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{q}_i} \right) \right) + \\
+ \frac{d}{dt} \left( L \tau - f \right) + \frac{\partial L}{\partial q_i} (y_i - \tau \dot{q}_i) \frac{\partial L}{\partial \dot{q}_i} + \\
- (y_i - \tau \dot{q}_i) \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right) = 0 \quad (4)
\]

Eq. (4) implies the relevant conservation law. The advantage of the generalized formalism is that it gives a more general form of Noether symmetries also in the standard case. Assume that \( L \) does not depend on \( \ddot{q}_i \)'s. Then eqs.(3), (4) describe symmetries depending on velocities, even for usual lagrangians.

Assume now that we have a canonical symmetry described by the generator \( F(q, p) \), i.e. infinitesimally

\[
q'_i = q_i + \varepsilon \{ q_i, F \} = q_i + \varepsilon \frac{\partial F}{\partial p_i} \quad (5)\\
p'_i = p_i + \varepsilon \{ p_i, F \} = p_i - \varepsilon \frac{\partial F}{\partial q_i} \quad (6)
\]

The symmetry condition reads

\[
H(q, p) = H'(q', p') = H(q', p') \quad (7)
\]

i.e.

\[
\{ F, H \} = 0 \quad (8)
\]

In order to describe this symmetry on the lagrangian level we express \( p_i \) in terms of \( \dot{q}_i \) through

\[
\dot{q}_i = \frac{\partial H(q, p)}{\partial p_i} \quad (9)
\]

and define the lagrangian

\[
L(q, \dot{q}) = p_i (q, \dot{q}) \dot{q}_i - H(q, p(q, \dot{q})) \quad (10)
\]

Consider now the transformation implied by (5)

\[
q'_i = q_i + \varepsilon \frac{\partial F}{\partial p_i} (q, p(q, \dot{q})) \quad (11)
\]

This is a generalized point transformation in the sense described at the beginning of the present section. Short computation using eqs. (7) \( \div \) (10) gives

\[
L(q', \dot{q'}) = L(q, \dot{q}) + \frac{d}{dt} \left( (p_i \frac{\partial F}{\partial p_i} - F) \big|_{p=p(q,\dot{q})} \right) \quad (12)
\]
which, due to eq.(4), implies $F$ to be a constant of motion.

We conclude that the canonical symmetry can be viewed as the generalized point symmetry once the generalized momenta are expressed in terms of velocities.

Note, in particular, that the approach sketched above allows to derive the energy conservation without addressing to point transformations with time variation.

III Ward identities and amplitudes nullification.

Our starting point is the following lagrangian

$$L = \frac{1}{2} \sum_{i=1}^{k} (\partial_{\mu}\Phi_{i}\partial^{\mu}\Phi_{i} - m_{i}^{2}\Phi_{i}^{2}) - V(\Phi, \Lambda)$$

(13)

where $\Lambda$ is the set of coupling constants and $V(\Phi, \Lambda)$ starts from the terms of third order in $\Phi$’s. The classical field equations read

$$(\Box + m_{i}^{2})\Phi_{i} + \frac{\partial V}{\partial\Phi_{i}} = 0$$

(14)

In order to compute the tree-level amplitudes one can proceed as follows [8], [3], [4], [5]. We look for the solution of the integral equation

$$\Phi_{i}(x) = \Phi_{0i}(x) - \int d^{4}y \Delta_{Fi}(x-y) \frac{\partial V}{\partial\Phi_{j}(y)}$$

(15)

where $\Delta_{Fi}(x) = \Delta_{F}(x; m_{i}^{2})\delta_{ij}$ is the Feynman propagator, while

$$\Phi_{0i}(x) = \sum_{n}(\beta_{i}^{(n)} f_{ni}(x) + \overline{\beta}_{i}^{(n)} \overline{f}_{ni}(x))$$

(16)

is the solution to the free-field equations ( here $\{f_{ni}(x)\}_{n=0}^{\infty}$ is a complete set of normalized positive-energy solutions to the corresponding Klein-Gordon equation). Eq.(15) implies eq.(14) together with the boundary condition

$$\Phi_{i}(x) \mid_{\Lambda=0} = \Phi_{0i}(x)$$

(17)

but the converse is not true.

Once eq.(15) is solved we obtain $\Phi_{i}(x)$ as a function of $\beta_{i}^{(n)}$ and $\overline{\beta}_{i}^{(n)}$. Taking derivatives with respect to $\beta$’s and $\overline{\beta}$’s at $\beta = \overline{\beta} = 0$ one obtains matrix elements of field operator between in - and out-states of on - shell particles in the tree approximation [5]. In order to get the relevant amplitude it remains to reduce, through LZS, the last particle carried by the field $\Phi_{i}(x)$.

Everything simplifies considerably in the case of threshold amplitudes. Then things
become translational invariant and only time dependence remains. The counterparts of (13) ÷ (16) read

\[ L = \frac{1}{2} \sum_{i=1}^{p} (\dot{\Phi}_i^2 - m_i^2\Phi_i^2) - V(\Phi; \Delta) \]  

(18)

\[ \Phi_i(t) = \Phi_{0i}(t) - \int_{-\infty}^{\infty} dt' \Delta_{Fij}(t - t') \frac{\partial V}{\partial \Phi_j(t')} \]  

(19)

\[ \Phi_{0i} = \beta_i e^{-im_i t} + \overline{\beta}_i e^{im_i t} \]  

(20)

Now, consider the threshold amplitudes under the assumption that the initial and final particles are of different kinds. Then (20) can be replaced by

\[ \Phi_{0i}(t) = \begin{cases} \beta_i e^{-im_i t}, & \text{if } "i" \text{ is an initial particle} \\ \overline{\beta}_i e^{im_i t}, & \text{if } "i" \text{ is a final particle} \end{cases} \]  

(21)

The solution to (19) may not exist or may be not expandable in power series in \( \lambda \). To see this note that the iterative solution to (19) produces the tree-graph expansion. However, the latter may be not well-defined due to vanishing denominators of some propagators. This can be cured if we assume eq.(21) to hold. Indeed, assume the masses \( m_i \) are such that no combination \( \sum_{i=1}^{p} n_i m_i, \quad n_i \text{ being integers}, \) vanishes. Then no propagator becomes divergent and the solution to (19), viewed as, at least, formal series in \( \lambda \), exists and is unique; the expansion coefficients are given by the sums of relevant tree-graphs. This series may be divergent due to the problem of ”small denominators” but this fact is irrelevant for us. Note also that our assumption concerning masses \( m_i \) implies that the free counterpart of (18) (\( \lambda = 0 \)) has no global integrals of motion except individual energies of oscillators (is not superintegrable).

Assume now that the theory defined by (18) exhibits some symmetry. We assume the generic situation that the symmetry is the canonical transformation rather than less general point transformation. As explained in Sec.II, this means that at the lagrangian level, we allow for transformations depending on velocities. So, the infinitesimal form of the symmetry reads

\[ \Phi'_i = \Phi_i + \alpha g_i(\Phi, \dot{\Phi}, m, \lambda) \]  

(22)

where \( \alpha \) is an infinitesimal parameter.

Furthermore, we make the following additional assumptions:
(i) eq.(22) is a symmetry for generic values of the masses \( m_i \);
(ii) the \( \lambda \)-independent part of eq.(22) reads

\[ g_i(\Phi, \dot{\Phi}; m, \lambda = 0) = d_i(m) \dot{\Phi}_i \]  

(23)

and contains all linear terms. Actually, (ii) is almost a consequence of (i). Indeed, in the limit of vanishing interaction we are dealing with the set of decoupled harmonic oscillators with generic frequencies. They have no globally defined integrals of motion.
except individual energies of oscillators. Therefore, the generator of symmetry transformations must be a function of them and linearity of the transformation implies that it is a linear combination of energies. Using eq. (21) one can translate (23) into the transformation property of $\beta'$s:

$$\beta'_i = \beta_i - i\alpha m_i d_i(m)\beta_i$$
$$\beta'_j = \beta'_i + i\alpha m_i d_i(m)\beta'_i$$

The solution to (19) and (20) will be denoted below by $\Phi_i(x | \beta, \beta)$. The uniqueness of the solution (viewed as a formal power series) implies the following identity (again, in terms of formal power series) resulting from the symmetry (22) [4].

$$\Phi'_i(x | \beta, \beta) = \Phi_i(x | \beta', \beta)$$

Using (22) and (24) we can rewrite this identity to the first order in $\alpha$; then we get

$$g_i(\Phi(t), \dot{\Phi}(t); m, \Lambda) = -i \sum_{k-\text{initial}} (-\beta_k m_k d_k \frac{\partial \Phi_i(t)}{\partial \beta_k} ) + i \sum_{k-\text{final}} \beta_k m_k d_k \frac{\partial \Phi_i(t)}{\partial \beta_k}$$

This is the basic identity to be used below.

Let us apply to both sides the operator

$$\prod_{k-\text{initial}} \frac{\partial^{n_k}}{\partial \beta_k^{n_k}} \prod_{l-\text{final}} \frac{\partial^{n_l}}{\partial \beta_l^{n_l}},$$

put $\beta = \beta = 0$ and then apply the K-G operator $\frac{\partial^2}{\partial \beta^2} + m_i^2$. The result reads

$$(-i) \prod_{k-\text{initial}} \frac{\partial^{n_k}}{\partial \beta_k^{n_k}} \prod_{l-\text{final}} \frac{\partial^{n_l}}{\partial \beta_l^{n_l}} (\frac{\partial^2}{\partial \beta^2} + m_i^2) g_i(\Phi(t), \dot{\Phi}(t); m, \Lambda) |_{\beta=\beta=0} =$$

$$= (\sum_{p-\text{initial}} (-n_p m_p d_p) + \sum_{r-\text{final}} n_r m_r d_r) \prod_{k-\text{initial}} \frac{\partial^{n_k}}{\partial \beta_k^{n_k}} \prod_{l-\text{final}} \frac{\partial^{n_l}}{\partial \beta_l^{n_l}} (\frac{\partial^2}{\partial \beta^2} + m_i^2) \Phi_i(t) |_{\beta=\beta=0}$$

Let us now vary the masses $m_i$ in such a way that the relation $\sum_i n_i m_i = 0$ is allowed for some integers $n_i$. Consider the set of all integer solutions to

$$\sum_i n_i m_i = 0$$

with not all $n_i$ vanishing. Call $\{\pi_i\}$ the solution to (28) with the following properties:

(a) $\sum | n_i |$ attains minimal value

(b) the amplitudes with $| \pi_i |$ external lines corresponding to $i$-th kind of particles are a priori allowed by the topology of the graphs implied by the lagrangian (18)

Now we can make precise which kinds of particles are viewed as initial and final ones. Namely, if $\pi_i > 0$ ($\pi_i < 0$), the "$i$"-th kind of particles will be called initial (final) (one can forget about those particles for which $\pi_i = 0$).
Let us now consider eq.(27) with \( n_k = | \pi_k |, \ k \neq i \) and \( n_i = | \pi_i | - 1 \). We multiply it by \( \exp(\pm im_i t) \), depending on whether "i"-th particle is initial or final, and integrate over \( t \). It is easy to see that, due to our assumptions, everything here is perfectly finite—the only divergence in propagator, emerging due to the relation \( \sum_k \pi_k m_k = 0 \), is killed by K-G operator. One can say even more. On the right-hand side one gets the amplitude \( A(\pi \to \pi) \) describing the production of final particles out of initial ones, all of multiplicities \( | \pi_k | \), multiplied by the factor \( ( - \sum_p \pi_p m_p d_p \pm m_i d_i ) \) (the sign depends on whether "i"-th particle is initial or final). On the left-hand side only the linear term in \( g_i \) survives. Taking into account energy conservation and the form (23) of linear part of \( g_i \) we conclude that the left-hand side equals \( \pm m_i d_i A(\pi \to \pi) \). Collecting all that we arrive at the following identity

\[
(\sum_p \pi_p m_p d_p) A(\pi \to \pi) = 0
\]

which implies \( A(\pi \to \pi) \neq 0 \) only provided

\[
\sum_p \pi_p m_p d_p = 0
\]

The latter, however, does not hold in general.

Let us reconsider the examples described in Ref.[4]. For the 0(2)-symmetric model broken softly by the mass term one has two kinds of particles with \( d_1 = 0 \), \( d_2 = \frac{1}{2}(4m_1^2 - m_2^2) \). Note that in this case the identity (19) has been also proven diagramatically in [7]. Another model considered in Ref.[4] is the Henon-Heiles system in its integrable version. Again, we are dealing with two kinds of particles with \( d_1 = \frac{1}{2}(4m_1^2 - m_2^2) \), \( d_2 = 0 \).

In general our basic identity allows us to check in each case whether a given symmetry implies amplitude nullification.

IV 2 \( \to n \) in 0(N)-symmetric models.

In order to find more interesting application of our Ward identities let us consider the 0(N) model defined by the Lagrangian

\[
L = \frac{1}{2} \sum_{i=1}^{N} (\partial_\mu \Phi_i \partial^\mu \Phi_1 - m^2 \Phi_i^2) - \frac{\lambda}{4!} (\sum_{i=1}^{N} \Phi_i^2)^2
\]

Consider the production of \( n \) particles at the threshold by two particles with three-momenta \( \vec{p} \) and \( -\vec{p} \), \( \vec{p} \neq 0 \); we assume that both initial particles are carried by \( \Phi_1 \) while the final ones correspond to other indices. The algorithm to calculate such an amplitude at tree level is well known [9]. First, we solve field equations for the fields \( \Phi_i, \ i \neq 1 \), with the boundary conditions:

\[
\Phi_1 \equiv 0 \\
\Phi_i \mid_{\lambda=0} = z_i e^{i \omega_i t}, \ i \neq 1
\]
The solution reads (cf. [8])

$$
\Phi_{icl}(t) = \frac{z_1 e^{int}}{1 - \frac{\lambda}{48 \pi^2} \left( \sum_{k \neq 1} z_k^2 \right) e^{2int}}
$$

(33)

Then the field equation for $\Phi_1$ is written out keeping only linear terms in $\Phi_1$:

$$
(\Box + m^2)\Phi_1 + \frac{\lambda}{3!} \Phi_1 \left( \sum_{k \neq 1} \Phi_k^2 \right) = 0
$$

(34)

Comparing eqs. (33) and (34) with eq. (3.3) of Ref. [9] we conclude that (34) and (3.3) coincide provided the identification $a=2$, $z_0^2 \rightarrow \sum_{k \neq 1} z_k^2$ has been made. Then the results of Ref [9] imply that the scattering amplitude for the process $2 \rightarrow n$ vanishes unless $n=2$ which, however, is excluded by energy conservation ($\vec{p} \neq 0$).

To find a deeper reason for such nullification consider the tree graph contributing to the amplitude under consideration. The nonvanishing three momentum $\vec{p}$ flows only through the continuous line of propagators connecting both initial lines. One can immediately conclude that our amplitude coincides with the one for the process $2 \rightarrow n$ with all, both initial and final, particles at threshold and the mass squared $m^2$ of the first particle replaced by $M^2 = m^2 + \vec{p}^2$; the “effective” Lagrangian reads

$$
L = \frac{1}{2} \sum_{i=2}^{N} \left( \partial_\mu \Phi_i \partial^\mu \Phi_i - m^2 \Phi_i^2 \right) + \frac{1}{2} \left( \partial_\mu \Phi_1 \partial^\mu \Phi_1 - M^2 \Phi_1^2 \right) - \frac{\lambda}{4!} \left( \Phi_1^2 + \sum_{i=2}^{N} \Phi_i^2 \right)^2
$$

(35)

When reduced to one dimension, $\partial_\mu \Phi_i \rightarrow \dot{\Phi}_i$, this becomes the so-called Garnier system which is integrable [10], the additional, apart from energy, integrals of motion being the generators $l_{ij}, i, j \neq 1$ of O(N-1) symmetry together with

$$
F = \frac{\lambda \cdot \sum_{k=2}^{N} \frac{l_{1k}^2}{(M^2 - m^2)}}{4!} + p_1^2 + \frac{\lambda}{4!} \left( \Phi_1^2 \right) \left( \sum_{k=1}^{N} \Phi_k^2 \right)
$$

(36)

Note that the transformation generated by F obeys (i), (ii) (as for as (i) is concerned, it can be checked that F, suitably modified, continues to be a constant of motion even if the masses of other particles are varying as long as they are different from M). In particular,

$$
d_i(m) = 2\delta_{i1}
$$

(37)

Therefore, eq. ( ) gives

$$
4MA(2 \rightarrow n) = 0
$$

(38)

For $\vec{p} \rightarrow 0$, F (multiplied by $M^2 - m^2$) becomes a function of $0(N)$ generators and the above reasoning breaks down. The process $2 \rightarrow 2$ becomes then possible and the relevant amplitude is obviously nonvanishing.
V Conclusions.

We have derived the basic identity (26) which generates all Ward identities on tree level. Under some further assumptions these identities imply vanishing of certain threshold amplitudes. We saw that the important point here is the form of coupling constant(s) independent part of symmetry transformations. Indeed, eq.(30) express the conservation law for the generator of free-field ($\lambda = 0$) counterpart of symmetry transformations. Let us recall the general field-theoretical framework of selection rules for scattering amplitudes. Symmetry implies conservation law for some charge $Q$ built from interpolating fields and their conjugate momenta. The infinitesimal action of symmetry on interpolating fields is given by their commutator with $Q$. If this action is linear in fields we can use LZS asymptotic condition together with constancy of $Q$ to find the transformation properties of asymptotic fields which, in turn, allows to reconstruct $Q_{in}$ and $Q_{out}$; obviously $Q_{in} = Q = Q_{out}$ which implies appropriate selection rule.

For nonlinear transformations situation is more involved; the (weak) LZS asymptotic condition is insufficient to derive the asymptotic form of symmetry in such a straightforward manner. In our case it is reflected by the fact that the conservation law for free-part of generator holds only in the lowest-energy sector allowed by energy conservation. For higher sectors nonlinear parts of symmetry transformations survive asymptotics, i.e. the amputation procedure (more precisely, the relevant contributions are ambiguous due to divergences in some propagators).

Eq.(30) gives an additional (apart from space-time symmetries) constraint which implies amplitude nullification. The reasoning made here strongly resembles the arguments in favour of the well-known Coleman-Mandula \[11\] theorem. There, mixing of internal and space-time symmetries implies the existence of conserved charges equipped with space-time indices. Therefore the relevant conservation laws give rise to the selection rules involving fourmomenta which contradicts analyticity \[11\].

In our case the role of space-time and internal symmetries mixing is played by the fact that the symmetry transformations involve (even in their linear part) time derivatives of fields which results in the appearance of masses in eq. (30). The analogy with Coleman-Mandula theorem is also clearly seen in the case of $2 \rightarrow n$ process. As long as $\vec{p} \neq 0$ ($M^2 \neq m^2$) we have at our disposal an integral of motion with ”space-time indices”, due to appearance of time derivatives in F (through $p_1$). In the limit $\vec{p} \rightarrow 0$ the symmetry is purely internal and the process $2 \rightarrow 2$ becomes possible.

The last example is also interesting because it provides an explanation for nullification phenomena in some $2 \rightarrow n$ processes. It seems that the trick consisting in replacing our process with the one with all particles at rest at the price of modifying the masses of initial particles is more general at least for theories with scalar particles only. One has to embedd the relevant dynamics into the one described by lagrangian with a symmetry which forces some of the amplitudes to vanish.
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