On the Existence of 4-regular Matchstick Graphs

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Abstract

A matchstick graph is a planar unit-distance graph. That is a graph drawn with straight edges in the plane such that the edges have unit length, and non-adjacent edges do not intersect. We call a matchstick graph 4-regular if every vertex has only degree 4. Examples of 4-regular matchstick graphs with less than 63 vertices are only known for 52, 54, 57, and 60 vertices. It is shown that for all number of vertices \( \geq 63 \) at least one example of a 4-regular matchstick graph exists.

1 Introduction

A matchstick graph is a planar unit-distance graph. That is a graph drawn with straight edges in the plane such that the edges have unit length, and non-adjacent edges do not intersect. We call a matchstick graph 4-regular if every vertex has only degree 4, and \((2; 4)\)-regular if every vertex has only degree 2 or 4.

Examples of 4-regular matchstick graphs are currently known for all number of vertices \( \geq 52 \) except for 53, 55, 56, 58, 59, 61, and 62. The smallest known example with 52 vertices was first presented by Heiko
Harborth in 1986 [2] (see Figure 1a). The examples with 54, 57, 65, 67, 73, 74, 77, and 85 vertices were first presented by the authors in 2016 [1][3]. It is still an open problem if there exists at least a 4-regular matchstick graph with less than 52 vertices or a different example with 52 vertices.

In this article we prove that for all number of vertices \( \geq 63 \) at least one example of a 4-regular matchstick graph exists. This main theorem is given by Theorem 6, which we shall prove at the end of Chapter 5.

Our prove is based on eleven \((2; 4)\)-regular matchstick graphs with a number of vertices between 5 and 49, and six 4-regular matchstick graphs with a number of vertices between 64 and 74. We also show the four known examples with less than 63 vertices. If the uniqueness of a graph is not explicitly specified, there exists at least one more known example with the same number of vertices. Many of those examples and other known matchstick graphs can be found in [6] (cf. Table 2–3), respectively in the authors thread in a graph theory internet forum [9]. A first and slightly different proof for the existence of 4-regular matchstick graphs for all number of vertices \( \geq 63 \) except for 65, 67, 73, 74, 77, and 85 was given by Harborth in 2002 [1].

The geometry, rigidity or flexibility of the graphs in this article has been verified with a computer algebra system named MATCHSTICK GRAPHS CALCULATOR (MGC). This remarkable software created by Stefan Vogel runs directly in web browsers. A special version of the MGC contains all graphs from this article and is available under the weblink given in [5]. The method Vogel used for the calculations he describes in a German article [4].

**Remark 1. Proofs for the existence of the graphs shown in this article.**

The MGC contains a constructive proof for each graph. We are using this online reference, because these proofs are too extensive to reproduce here.

The MGC contains also an animation function, which can be used to show the flexibility of a graph, for example Figure 1d. If a rigid graph is suitable, the animation can also be used for proving the existence of the graph, for example Figure 2b.
2 Examples of 4-regular matchstick graphs with less than 63 vertices

Examples of 4-regular matchstick graphs with less than 63 vertices are only known for 52, 54, 57, and 60 vertices.

**Theorem 2.** There exists at least one example of a 4-regular matchstick graph with 52, 54, 57, and 60 vertices.

*Proof.* Figure 1 shows the only known examples for these number of vertices [5]. A proof for the graph 1a is also given by Harborth [2].

![Figure 1: 4-regular matchstick graphs with 52, 54, 57, and 60 vertices.](image-url)
The graphs 1a and 1b are rigid and have a vertical and a horizontal symmetry. The geometry of the graph 1b is based on the graph 1a, whereby the angles $\neq 60$ degrees are different in both graphs. The graph 1c is rigid and has a rotational symmetry of order 3. The graph 1d is flexible and has a rotational symmetry of order 12.

3 (2; 4)-regular matchstick graphs

Examples of (2; 4)-regular matchstick graphs with less than 42 vertices which contain only two vertices of degree 2 are only known for 22, 30, 31, and 34–41 vertices [6]. For our proof we only need the examples in Figure 2 and 3. All these graphs are rigid and have a vertical symmetry, except the graph 3b which is asymmetric. The number of edges of each graph is twice the number of its vertices less two.

**Theorem 3.** There exists at least one example of a (2; 4)-regular matchstick graph which contains only two vertices of degree 2 with 22, 30, 31, 34, 35, 36, 40, and 41 vertices.

**Proof.** Figure 2 and 3 show examples for these number of vertices [5].

![Figure 2: (2; 4)-regular matchstick graphs with two vertices of degree 2.](image)
4 Examples of 4-regular matchstick graphs with more than 62 and less than 94 vertices

Theorem 4.1. There exists at least one example of a 4-regular matchstick graph for all number of vertices > 62 and < 121 except 64, 65, 67, 69, 73, and 74 vertices.

Proof. According to Theorem 3, by connecting three of the eight (2; 4)-regular graphs we can build $8^2 + \binom{8}{3} = 120$ different rigid 4-regular matchstick graphs with all number of vertices > 62 and < 121 except 64–70, 73, 74, 78, and 116. Table 1 shows the number of these possible graphs $g$ with $63 \leq v \leq 120$ vertices. The graphs 2d and 3a–3d can also be mirrored on the line through their vertices of degree 2 to form a rigid 4-regular graph with 66, 68, 70, 78, and 80 vertices, whereby the graph 3b must be also rotated by 180 degrees. A flexible 4-regular graph with 116 vertices can be constructed from four subgraphs 2b.
Table 1: 120 different possible 4-regular matchstick graphs.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 63 | 1 | 72 | 1 | 81 | 1 | 90 | 4 |
| 64 | 0 | 73 | 0 | 82 | 1 | 91 | 3 |
| 65 | 0 | 74 | 0 | 83 | 1 | 92 | 2 |
| 66 | 0 | 75 | 1 | 84 | 2 | 93 | 4 |
| 67 | 0 | 76 | 1 | 85 | 2 | 94 | 4 |
| 68 | 0 | 77 | 1 | 86 | 1 | 95 | 4 |
| 69 | 0 | 78 | 0 | 87 | 2 | 96 | 3 |
| 70 | 0 | 79 | 1 | 88 | 2 | 97 | 4 |
| 71 | 1 | 80 | 1 | 89 | 4 | 98 | 5 |

Theorem 4.2. There exists at least one example of a 4-regular matchstick graph with 64, 65, 67, 69, 73, and 74 vertices.

Proof. Figure 4 and 5 show examples for these number of vertices [5].

Figure 4: 4-regular matchstick graphs with 64 and 65 vertices.
The graph 4a is rigid, has a vertical and a horizontal symmetry and is constructed from the graph 1a. This is the only known example with 64 vertices. The graphs 4b and 5a are flexible, have a vertical and a horizontal symmetry and their geometries are based on the graph 1d. The graph 5b is rigid, has a rotational symmetry of order 3 and is constructed from the graph 1c. The graphs 5c and 5d are rigid, have a vertical symmetry and their geometries are based on the graph 1c. The only difference between the graphs 5c and 5d is one vertex and six edges in the center of the graphs.
5 Examples of 4-regular matchstick graphs with more than 93 vertices

By using the graph 2a as a subgraph we can construct $(2; 4)$-regular matchstick graphs with 48 and 49 vertices which contain only two vertices of degree 2 as shown in Figure 6. The graphs 6a, 6b, and 6c are flexible. This flexibility allows the distance between its vertices of degree 2 to be set equally [5].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{(2; 4)-regular matchstick graphs with 5, 48, and 49 vertices.}
\end{figure}

**Theorem 5.1.** There exists at least one example of a 4-regular matchstick graph with 94, 95, and 96 vertices.

**Proof.** The graphs 6a and 6c can be mirrored on the line through their vertices of degree 2 to construct 4-regular matchstick graphs with 94 and 96 vertices. Because of the flexibility the graphs 6a and 6c can also be connected together at their vertices of degree 2 to construct a 4-regular matchstick graph with 95 vertices. These graphs with 94, 95, and 96 vertices are flexible [5].

**Theorem 5.2.** There exists at least one example of a 4-regular matchstick graph for all number of vertices \( \geq 97 \).

**Proof.** Because of the flexibility the graph 6b can used as a subgraph to enlarge each of the graphs with 94, 95, and 96 vertices of Theorem 5.1 up to infinity. Thereby each subgraph 6b expands a graph by three vertices. This construction method allows us to build 4-regular matchstick
graphs with $94 + 3n$, $95 + 3n$, and $96 + 3n$ vertices for each $n \in \mathbb{N}$. All these graphs are flexible [5].

Figure 7 shows the graph with $95 + 3n$ vertices as an example for these three types of graphs with up to an infinite number of vertices [5].

![Figure 7: 4-regular matchstick graph with $95 + 3n$ vertices.](image)

**Theorem 6.** There exists at least one example of a 4-regular matchstick graph for all number of vertices $\geq 63$.

**Proof.** From the Theorems 4.1, 4.2, 5.1, and 5.2 it follows directly that Theorem 6 holds. □

### 6 Supplementary notes

**On Chapter 3:** Every $(2; 4)$-regular matchstick graph with only two vertices of degree 2 is composed of smaller $(2; 4)$-regular subgraphs with three or more vertices of degree 2. Therefore different examples can exist for the same number of vertices. The graphs in Figure 2 and 3 show the most symmetrical examples. Except the graph 2c, each of these graphs can be transformed into at least one different example by mirroring and rotating the subgraphs by 180 degrees.

Table 2 shows the number of the currently smallest known examples of $(2; 4)$-regular matchstick graphs with a minimum number of vertices $\leq 41$ which contain only two vertices of degree 2.
On Chapter 4: The selection of the graphs in Figure 4 and 5 for which more than one example is known was made according to the simplicity of the geometry, the grade of symmetry and the symmetrical relation to the graphs in Figure 1.

Table 3 shows the number of the currently known examples of 4-regular matchstick graphs with more than 62 and less than 71 vertices.

| vertices | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
|----------|----|----|----|----|----|----|----|----|
| examples | 3  | 1  | 3  | 9  | 11 | 5  | 3  | 4  |

Table 3

For Table 2 and 3 the rotated and mirrored versions of the graphs have not been considered. Flexible graphs are counted as single example only. The graphs corresponding to Table 2 and 3 can be found in [6].

On the graphs: All graphs in this article are shown in their original size relationship. Therefore the edges in the figures have exactly the same length.

The graphs 1a, 1d, 2a, 2c, 2d, 3c, 4a, and the graphs in Figure 6 and 7 are simply constructed or probably known since 1986 or 2002 [1]–[3]. The list below shows the date of the first presentation of the graphs in this article and its name if available. This information refer to [9], the published articles and websites on this subject.

1a: The HARBOORTH GRAPH. 1986 by Harborth. 1b: The VOGEL-DINKELACKER-WINKLER GRAPH. July 3, 2016 by Vogel, Dinkelacker, and Winkler. 1c: The WINKLER GRAPH. April 15, 2016 by Winkler. 2b: April 23, 2016 by Winkler. 3a: October 23, 2016 by Winkler. 3b: March 12, 2017 by Dinkelacker. 3d: June 28, 2016 by Winkler. 4b: April 6, 2016 by Winkler. 5a: March 29, 2017 by Winkler. 5b: March 7, 2017 by Winkler. 5c and 5d: March 8, 2017 by Winkler.
References

[1] H. Harborth, *Complete Vertex-to-vertex Packings of Congruent Equilateral Triangles*, Geombinatorics Volume XI, Issue 4, April 2002, pp. 115–118.

[2] H. Harborth, *Match Sticks in the Plane*, The Lighter Side of Mathematics. Proceedings of the Eugène Strens Memorial Conference of Recreational Mathematics & its History, Calgary, Canada, July 27 – August 2, 1986 (Washington) (Richard K. Guy and Robert E. Woodrow, eds.), Spectrum Series, The Mathematical Association of America, 1994, pp. 281–288.

[3] H. Harborth, *Private communication*, November 2016 and May 2017.

[4] S. Vogel, *Beweglichkeit eines Streichholzgraphen bestimmen*, July 2016, https://tinyurl.com/yc8at6r7

[5] S. Vogel, *Matchstick Graphs Calculator* (MGC), a software for the construction and calculation of unit distance graphs and matchstick graphs, 2016–today. http://mikematics.de/matchstick-graphs-calculator.htm

[6] M. Winkler, *A catalogue of 4-regular matchstick graphs with 63 – 70 vertices and (2; 4)-regular matchstick graphs with less than 42 vertices which contain only two vertices of degree 2*, May 2017, arXiv:1705.04715[math.CO].

[7] M. Winkler, *Ein neuer 4-regulärer Streichholzgraph*, Mitteilungen der Deutschen Mathematiker Vereinigung (DMV), Band 24, Heft 2, Seiten 74–75, DOI: 10.1515/dmvm-2016-0031, July 2016.

[8] M. Winkler, P. Dinkelacker, and S. Vogel, *New minimal (4; n)-regular matchstick graphs*, Geombinatorics Volume XXVII, Issue 1, July 2017, pp. 26–44, arXiv:1604.07134[math.MG]

[9] M. Winkler, P. Dinkelacker, and S. Vogel, *Streichholzgraphen 4-regulär und 4/h-regulär (n>4) und 2/5*, thread in a graph theory internet forum, used nicknames: P. Dinkelacker (haribo), M. Winkler (Slash), https://tinyurl.com/ydovm2ou