On arrangements of real roots of a real polynomial and its derivatives

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Abstract

We prove that all arrangements (consistent with the Rolle theorem and some other natural restrictions) of the real roots of a real polynomial and of its $s$-th derivative are realizable by real polynomials.

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In the present paper we consider a real polynomial of one real variable $P(x, a) = x^n + a_1x^{n-2} + \ldots + a_{n-1}$. We are interested in the question what arrangements between the real roots of $P$ and $P^{(s)}$ are possible $(1 \leq s \leq n-1)$. To define an arrangement means to write down the roots of $P$ and $P^{(s)}$ in a chain in which every two consecutive roots are connected either by an equality or by an inequality $<$. The arrangement $\alpha$ is said to belong to the closure of the arrangement $\beta$ if it is obtained from $\beta$ by replacing some inequalities by equalities. The results are the first step towards the study of real discriminant sets \{ $a \in \mathbb{R}^{n-1} | \text{Res}(P, P^{(s)}) = 0$ \}.

In an earlier paper [KoSh] it is shown that if $P$ is hyperbolic, i.e. with $n$ real roots, then the standard Rolle restrictions are necessary and sufficient conditions for a root arrangement to be realizable (see Theorems 2 and 4.4 in [KoSh]). Namely, denote by $x_1 \leq \ldots \leq x_n$ the roots of $P$ and by $\xi_1 \leq \ldots \leq \xi_{n-s}$ the ones of $P^{(s)}$ (which is also hyperbolic). Then one has

$$x_l \leq \xi_l \leq x_{l+s}$$

for $l = 1, \ldots, n-s$ and every arrangement of the roots of $P$ and $P^{(s)}$ which is consistent with (1) is realizable. One presumes also that the following conditions hold:

A) If a root of $P$ of multiplicity $d > s$ coincides with a root of $P^{(s)}$ of multiplicity $g$, then $g = d - s$ (self-evident).

B) If a root $\xi$ of $P^{(s)}$ coincides with a root of $P$ of multiplicity $\kappa \leq s$, then $\xi$ is a simple root of $P^{(s)}$ (see [KoSh], Lemma 4.2) and one has $\kappa \leq s - 1$.

C) If $x_l = \xi_l$ or $x_{l+s} = \xi_l$, then $x_l = x_{l+1} = \ldots = x_{l+s} = \xi_l$ (self-evident for $s = 1$ and easy to prove by induction on $s$ for $s > 1$).

Example 1 If $n = 2$, $s = 1$, then there are two possible arrangements (i.e. consistent with (1), A) B) and C)): $x_1 < \xi_1 < x_2$ and $x_1 = \xi_1 = x_2$. They are both realizable by hyperbolic polynomials.

In the present paper we treat the case when $P$ is arbitrary (not necessarily hyperbolic). (Notice that $P^{(s)}$ can be hyperbolic even if $P$ is not.)
Definition 2 Suppose that $P$ has $m$ conjugate couples of complex roots and $n - 2m$ real roots. Then a priori $P^{(s)}$ has at least $n - 2m - s$ real roots counted with the multiplicities. Indeed, a real root of $P^{(i)}$ of multiplicity $l \geq 1$ is a root of $P^{(i+1)}$ of multiplicity $l - 1$ and between every two real roots of $P^{(i)}$ there is a root of $P^{(i+1)}$. Iterating this rule $s$ times one obtains the existence of $n - 2m - s$ real roots of $P^{(s)}$ (we call them Rolle roots) which together with the real roots of $P$ satisfy conditions (1), (A) and (B). A Rolle root is multiple only if it coincides with a root of $P$ of multiplicity $> s$. Eventually, $P^{(s)}$ can have $\leq 2m$ other (non-Rolle) real roots counted with the multiplicities some (or all) of which can coincide with Rolle ones. Which real roots of $P^{(s)}$ should be chosen as Rolle and which as non-Rolle ones is not always uniquely defined and when it is not we assume that a choice is made.

Example 3 The polynomial $x^6 - x^2 = x^2(x^2 - 1)(x^2 + 1)$ has real roots $x_1 = -1, x_2 = x_3 = 0, x_4 = 1$ (and complex roots $\pm i$). One has $P' = 6x^5 - 2x = 2x(\sqrt{3}x^2 - 1)(\sqrt{3}x^2 + 1)$, i.e. $P'$ has three Rolle roots (and no non-Rolle ones) $- 0$ and $\pm 1/3^{1/4}$ where 0 is a common root for $P$ and $P'$, see (A). It has also two complex roots $\pm i/3^{1/4}$. One has $P'' = 30x^4 - 2$, i.e. $P''$ has two Rolle roots $\pm 1/15^{1/4}$, no non-Rolle ones and two complex roots $\pm i/15^{1/4}$. One has $P''' = 120x^3$, i.e. $P'''$ has a triple real root at 0 and no complex roots. One copy of this real root should be considered as a Rolle one and which as non-Rolle ones.

Proposition 4 Suppose that a real root of $P$ of multiplicity $d$ coincides with a real root of $P^{(s)}$ of multiplicity $g$. Then

1) if $d > s$, then one has $g = d - s$; in this case this is a Rolle root of $P^{(s)}$ of multiplicity $d - s$;

2) if $0 \leq d \leq s$, then one has $g \leq 2m + 1$ (and if $g \geq 1$, then $d < s$).

Observe that in the above example one has $m = 1$ and for $s = 3$ the estimation $2m + 1$ is attained by the multiplicity of 0 as a root of $P'''$. The proposition generalizes conditions A) and B) in the case of arbitrary $m$.

Proof:

Part 1) is self-evident. Prove part 2). If the root is non-Rolle and does not coincide with a Rolle one, then its multiplicity is $\leq 2m$. If the root is Rolle and does not coincide with a non-Rolle one, then either it coincides with a root of $P$ of multiplicity $> s$ and we are in case 1) or it is a simple root. Finally, if the root is Rolle and coincides with a non-Rolle one, then the Rolle root must be simple (otherwise there will be a contradiction with part 1)) and the sum of their two multiplicities is $\leq 2m + 1$. □

Definition 5 An arrangement of the real roots of $P$ and $P^{(s)}$ is called a priori admissible if there exist $n - 2m - s$ Rolle roots of $P^{(s)}$ in the sense of Definition 2 and if conditions 1) and 2) of Proposition 4 hold.

Theorem 6 All a priori admissible root arrangements are realizable by real polynomials of degree $n$.

Proof:

1. We explain first in $1^0 - 7^0$ why all a priori admissible arrangements in which the derivative $P^{(s)}$ is hyperbolic and which are the least generic are realizable. "Least generic" means that all non-Rolle roots of $P^{(s)}$ coincide with Rolle ones or with roots of $P$. The general case is treated in $8^0 - 11^0$.

To realize an a priori admissible arrangement with $P^{(s)}$ hyperbolic and with the necessary multiplicities of the real roots of $P$ consider the family of polynomials.
\[ P(x, w, g, t) = \prod_{j=1}^{q} (x - w_j)^m_j \prod_{j=1}^{m} ((x - g_j)^2 + t_j^2) \]

where \( w_j, j = 1, \ldots, q \), are the real roots of \( P \), of multiplicities \( m_j \) \((w_0 = 0 \leq w_1 \leq \ldots \leq w_q \leq 1 = w_{q+1}) \), and \( g_j \pm it_j \) are its complex roots (not necessarily distinct), \( t_j \geq 0, 0 \leq g_j \leq 1 \). We allow here equalities between the roots \( w_j \) for convenience; it will be shown that the necessary arrangement is realizable for roots with strict inequalities between them.

Denote by \( \xi_1 \leq \ldots \leq \xi_{n-s} \) the real parts of the roots of \( P^{(s)} \) \((n - 2m - s \text{ of them are just Rolle roots}) \) and by \( \eta_1 \leq \ldots \leq \eta_m \) the biggest nonnegative imaginary parts of the roots of \( P^{(s)} \) (recall that for a least generic arrangement one has \( \eta_j = 0 \)). Set \( \xi_0 = 0, \xi_{n-s+1} = 1 \). (Notice that \( P^{(s)} \) has not more conjugate couples of complex roots than \( P \), i.e. not more than \( m \).) The functions \( \xi_i, \eta_j \) are continuous in \((w, g, t)\).

2°. Suppose that for the desired arrangement of the real roots of \( P \) and \( P^{(s)} \) the Rolle and non-Rolle roots of \( P^{(s)} \) are fixed. Denote the non-Rolle roots by \( u_1 \leq \ldots \leq u_{2m} \). Impose additional requirements upon the numbers \( g_j \) as follows: if the non-Rolle roots with odd indices \( u_{2p-1}, u_{2p+1}, \ldots, u_{2p+2p'-1} \) belong to the interval \([w_j, w_{j+1}]\), \( j < q \), or to \([w_q, w_{q+1}]\), then we require that \( w_j \leq g_p \leq \ldots \leq g_{p+p'} \leq w_{j+1} \). Define the variables \( h_1 \leq \ldots \leq h_{q+m} \) as the union of the variables \( w_j (j = 1, \ldots, q) \) and \( g_i (i = 1, \ldots, m) \) with the order defined above. Hence, they belong to the unit simplex \( \Sigma_{q+m} \).

3°. In what follows we assume that the variables \( t_j \) belong to some interval \([0, N] \) where \( N > 1 \). We define with the help of the variables \( h_j, t_i \) continuous functions \( \eta_j, \zeta_i \) such that \( (\eta_1, \ldots, \eta_{q+m}), (\zeta_1, \ldots, \zeta_n) \in [0, N] \). The set \( S = \Sigma_{q+m} \times [0, N]^m \) is homeomorphic to \( \Sigma_{q+2m} \). By the Brouwer fixed point theorem (see [Do], p. 57), there exists a fixed point of the mapping \( \tau : S \to S, \tau : (h, t) \to (\eta, \zeta) \), i.e. a point where one has \( \eta_j = h_j, \zeta_i = t_i \). The functions \( \eta_j, \zeta_i \) are defined such that the arrangement of the real roots of \( P \) and \( P^{(s)} \) at the fixed point is the required one.

4°. Define the functions \( \eta_j \) by the following rules:

1) Want to achieve the additional conditions (at the fixed point) \( g_p = u_{2p-1}, \ldots, g_{p+p'} = u_{2p+2p'-1} \) for all appropriate indices, see 2°; therefore we set \( \eta_i = \xi_{i_2} \) whenever \( h_i \) is a variable \( g_{p+p} \) and \( \xi_{i_2} \) is the corresponding function \( u_{2p+2l-1} \);

2) If a variable \( h_j \), which is a root \( w_i \) of multiplicity \( s + 1 \), must coincide with a simple root \( \xi_k \) of \( P^{(s)} \) or, more generally, with the roots \( \xi_k = \xi_{k+1} = \ldots = \xi_{k+l} \), then we set \( \eta_j = \xi_k \);

3) If the variables \( h_r < h_{r+1} < \ldots < h_{r+l} \) (which are all consecutive roots \( w_j \) and among which there might be roots \( w_j \) of multiplicity \( \geq s + 1 \)) lie between the Rolle roots \( \xi_k \) and \( \xi_{k+v} \) of \( P^{(s)} \) and all roots among the roots \( \xi_{k+1}, \ldots, \xi_{k+v-1} \) (if \( v > 1 \)) coincide with roots \( w_j (r \leq j \leq r+l) \) of multiplicity \( \geq s + 1 \), then we set
\[
\eta_{r+j} = \xi_k + (j+1)(\xi_{k+v} - \xi_k)/(l+2), j = 0, 1, \ldots, l.
\]

Remark 7 It follows from rules 1) – 3) that there are \( q + m \) functions \( \eta_j – as many as the variables \( h_j \).

Recall that the arrangement is least generic, i.e. for every non-Rolle root \( \xi_i \) of \( P^{(s)} \) one has either \( \xi_i = \xi_{i_1} \) where \( \xi_{i_1} \) is a Rolle one or \( \xi_i = w_{i_2} = h_j \) for some \( i_2, j \). Denote by \( l_1, \ldots, l_{2m} \) the absolute values \( |\xi_i - \xi_{i_1}| \) and \( |\xi_i - w_{i_2}| \) for all \( i, i_1 \) and \( i_2 \) as above. Set \( \Phi = l_1 + \ldots + l_{2m} \) and
\[
\xi_i = \left| t_i - \frac{1}{3m} \sum_{j=1}^{m} \theta_j - \frac{t_i}{3(N+1)m} |t_1 t_2 \ldots t_m - 1| - \frac{t_i}{12m} \Phi \right| \quad (3)
\]
Denote by $t_{i_0}$ the greatest variable $t_i$ at the fixed point (see 3). Observe first that one can assume that $t_{i_0} > 0$. Indeed, if $t_{i_0} = 0$, then $t_i = 0$ for all $i$, $P$ is hyperbolic and the roots of $P$ and $P^{(s)}$ define an arrangement $\alpha$ from the closure of the desired least generic one $\beta$.

**Lemma 8** If $t_{i_0} = 0$, then there exists a real-analytic deformation of $P$ into a real polynomial which together with its $s$-th derivative defines the arrangement $\beta$.

The lemma is proved after the theorem. It allows one to consider only the case $t_{i_0} > 0$. One has

$$\xi_{i_0} = t_{i_0} - \frac{1}{3m} \sum_{j=1}^{m} \theta_j - \frac{t_{i_0}}{3(N+1)^m} |t_1 t_2 \ldots t_m - 1| - \frac{t_{i_0}}{12m} \Phi.$$  

Indeed, all roots of $P^{(s)}$ lie within the convex hull of all roots of $P$ (see [PoSz], p. 108). Hence, one has $\theta_j \leq t_{i_0}$, $j = 1, \ldots, m$. One has also $|t_1 t_2 \ldots t_m - 1| \leq t_1 t_2 \ldots t_m + 1 < (N+1)^m$ and $\Phi \leq 4m$ (because for each term $l_j$ one has $l_j \leq 2$). Thus

$$\frac{1}{3m} \sum_{j=1}^{m} \theta_j + \frac{t_{i_0}}{3(N+1)^m} |t_1 t_2 \ldots t_m - 1| + \frac{t_{i_0}}{12m} \Phi < mt_{i_0}/3m + t_{i_0}/3 + 4mt_{i_0}/12m = t_{i_0}$$  

and for $i = i_0$ one can delete the absolute value sign in the right hand-side of (3). But then to have $\xi_{i_0} = t_{i_0}$ one must have $\theta_j = 0$ for $j = 1, \ldots, m$, $t_1 t_2 \ldots t_m - 1 = 0$ and $l_1 = \ldots = l_{2m} = 0$.

This means that $t_j \neq 0$, i.e. no root $g_j + i t_j$ of $P$ will be real, that $P^{(s)}$ will indeed be hyperbolic ($\theta_j = 0$) and that all non-Rolle roots of $P^{(s)}$ equal either roots $w_j$ of $P$ or Rolle roots of $P^{(s)}$.

**Remark 9** The condition $N > 1$ makes possible the choice of the values of the variables $t_i$ so that $t_1 t_2 \ldots t_m - 1 = 0$. One can prove by analogy with (4) that $|\xi_i| < N$, i.e. the mapping $\tau$ is indeed from $S$ into itself.

7. A priori the fixed point assures the existence of an arrangement only from the closure of the necessary one. The fact that at the fixed point no inequality between roots of $P$ is replaced by equality is proved by analogy with 6 - 7 of the proof of Theorem 4.4 from [KoSh] where the case of $P$ hyperbolic is considered. The proof there shows that equalities replacing inequalities between roots of $P$ imply that a root of $P$ of multiplicity $m \geq s+1$ is a root of $P^{(s)}$ of multiplicity $\geq m - s + 1$ which contradicts part 1) of Proposition 4. In the general case ($P$ not necessarily hyperbolic) the proof is essentially the same, the presence of eventual non-Rolle roots can only increase the multiplicity of the root as a root of $P^{(s)}$.

Hence, the fixed point provides the necessary arrangement.

8. To obtain (in 8 - 9) all arrangements in which $P^{(s)}$ is hyperbolic but which are not necessarily least generic we use the same construction but with another function $\Phi$. Namely, consider a family of such functions $\Phi$ depending on a parameter $b \in (\mathbb{R}_+, 0)$ defined as follows: if instead of $\xi_i - \xi_{i_1} = 0$, see 4, one must have $\xi_i - \xi_{i_1} > 0$ or $\xi_i - \xi_{i_2} < 0$ (and no root $\xi_j$ or $w_j$ lies between $\xi_i$ and $\xi_{i_1}$), then in $\Phi$ we replace the absolute value $l_\nu = |\xi_i - \xi_{i_1}|$ by $|\xi_i - \xi_{i_1} - b|$ (resp. by $|\xi_i - \xi_{i_2} + b|$); in the same way for $\xi_i - w_{i_2}$, see 4. In a sense, we obtain the not least generic arrangements by deforming least generic ones the deformation parameter being $b$.

9. Denote by $F(b)$ the set of fixed points of the mapping $\tau$ from 3. For $b$ small enough one has $(\eta, \zeta) \in S$. The set $F(0)$ contains all limit points of the family of sets $F(b)$ when $b \to 0$ and there exists at least one such limit point because all sets $F(b)$ (for $b$ small enough) are
non-empty and belong to $S$ which is compact. Hence, one can choose $b > 0$ small enough and a fixed point of $F(b)$ at which there is an inequality between two roots in the arrangement if there is an inequality in the arrangement for $b = 0$, and the equalities $\xi - \xi_1 = 0$ or $\xi - w_i = 0$ where this is necessary are replaced by the desired inequalities.

10$^0$. Obtain all arrangements in which $P^{(s)}$ is non hyperbolic and which are least generic. Suppose that $P^{(s)}$ must have exactly $m'$ conjugate couples of complex roots. In this case we assume that $m'$ of the couples of roots $g_j \pm it_j$ are replaced by a couple $\pm iv$ where $v > 0$ is “large”, i.e. much bigger than $N$. Hence, $P^{(s)}$ also has exactly $m'$ couples of conjugate complex roots with “large” imaginary parts. One has

$$Q := P/v^{2m'} = (1 + x^2/v^2)^{m'} \prod_{j=1}^q (x - w_j)^{m_j} \prod_{j=1}^{m-m'} ((x - g_j)^2 + t_j^2),$$

i.e. the family $Q$ is a one-parameter deformation of a family of polynomials like (2) (the role of the small parameter is played by $1/v^2$) and the existence of the necessary arrangements can be deduced by analogy with $10^0 - 7^0$ (see $9^0$ for the role of the small parameter; however, the function $\Phi$ is the one from $10^0 - 7^0$).

11$^0$. To obtain the existence of all arrangements (which are not necessarily least generic and with $P^{(s)}$ not necessarily hyperbolic) one has to combine $8^0$, $9^0$ and $10^0$. The theorem is proved. \hfill \Box

Proof of Lemma 8:

1$^0$. We assume that $P$ has the same number of distinct real roots as in the desired arrangement $\beta$. If not, then one can first deform $P$ within the class of hyperbolic polynomials (while remaining in the closure of $\beta$) to achieve this condition. See [Ko] for such deformations.

We begin with two observations:

1) for $a > 0$, $\mu \in \mathbb{N} \cup \{0\}$ and $\nu$ even the polynomial $Q = x^\mu(x^\nu + a)$ has a $\mu$-fold root for $x = 0$ and its $s$-th derivative for $s > \mu$ has a $(\mu + \nu - s)$-fold one; $Q$ has also $\nu/2$ couples of conjugate complex roots;

2) with $a$, $\mu$ and $\nu$ as above, the polynomial $Q_1 = x^\mu(x^\nu + a + aQ_2(x, a))$ where $Q_2$ is a polynomial in $x$ of degree $\leq \nu - 1$, $Q_2(0, a) \equiv 0$, has $\nu$ complex zeros for $a$ small enough and a real $\mu$-fold root at 0; to see this set $a = c^\nu$, $x = cy$; one has $Q_1(cy, c^\nu) = c^{\mu+\nu}y^\mu(y^\nu + 1 + Q_2(cy, c^\nu));$ the last polynomial has a $\mu$-fold root at 0 and $\nu$ roots which for $c$ small enough are close to the roots of $y^\nu + 1$, hence, are complex.

2$^0$. Suppose that the polynomial $P$ of degree $n$ realizing with $P^{(s)}$ the arrangement $\alpha$ has a real root of multiplicity $\mu + \nu$ (with $\nu$ even) which (in order to obtain the arrangement $\beta$) must split into $\nu/2$ couples of conjugate complex roots and into a real root of multiplicity $\mu$. (If several roots of $P$ must split, we make them split one by one.) Suppose in addition that in the deformed polynomial (denoted by $R$) the real root of multiplicity $\mu$ must coincide with a root of $R^{(s)}$ of multiplicity $\mu + \nu - s$. Assume that the bifurcating root is at 0 and that

$$P = x^{\mu+\nu}(1 + h(x)) \quad h(0) = 0$$

($P$ is not necessarily monic). Construct the necessary deformation of $P$ in the form

$$R(x, a) = x^\mu(x^\nu + a + b_{s-\mu}x^{s-\mu} + \ldots + b_{\nu-1}x^{\nu-1})(1 + g(x, a))$$

where $a \in (\mathbb{R}, 0)$ and $b_i = b_i(a)$ and $g(x, a)$ ($g(0, a) \equiv 0$) are defined such that all equalities of the form $x_i = \xi_j$ defining the arrangement $\beta$ will be preserved.

3$^0$. Suppose first that in (6) one has $g(x, a) \equiv h(x)$. The condition
Suppose that in (6) one has $g = h(x) + \sum_{j=1}^{l} d_j h_j(x, d)$ where $d = (d_1, \ldots, d_l) \in (\mathbb{R}^l, 0)$ and $h_j$ depend smoothly on $d$. Then condition (A) defines unique functions $b_i(a, d) = b_i^* a + a \sum_{j=1}^{l} d_j h_{i,j}(d)$ where $b_i^* \in \mathbb{R}$ and $h_{i,j}$ are smooth in $d$. This can also be checked directly.

4°. For each root $w_j \neq 0$ of $P$ of multiplicity $< s$ which must be equal to a root $\xi_i$ of $P(s)$ denote by $d_j$ the deviation from its position in a deformation of $P$. Admitting such deviations means that in (5) the function $h$ should be replaced by $h(x) + \sum_{j=1}^{l} d_j h_j(x, d)$.

Denote by (B) the system of all conditions $w_j = \xi_i$ for all such equalities with $w_j \neq 0$ characterizing the arrangement $\beta$.

5°. For any deformation $R^s(x, a, d) = x^\mu (x^\nu + a + b_{s-\mu} x^{s-\mu} + \ldots + b_{\nu-1} x^{\nu-1})(1 + g(x, d))$ of $P$ (where $b_k$ are considered as small parameters) one can find $d$ depending smoothly on $a$ and $b_k$ such that for all $a$ small enough all equalities from (B) hold. This follows from Propositions 11 and 13 from [Ko] where it is shown that the linearizations of the conditions (B) w.r.t. $d$ are linearly independent. (In [Ko] their linear independence is proved only when $P$ is hyperbolic; this independence is an “open” property, so it holds for all nearby polynomials as well.)

6°. The independence of these linearizations implies that for $a$ small enough the system of conditions (B) applied to the deformation

$$\tilde{R}(x, a, d) = x^\mu (x^\nu + a + b_{s-\mu} (a, d) x^{s-\mu} + \ldots + b_{\nu-1} (a, d) x^{\nu-1})(1 + h(x) + \sum_{j=1}^{l} d_j h_j(x, d))$$

(with $b_i(a, d)$ defined as in 3°) defines unique $d_j = d_j(a)$ smooth in $a$. Indeed, the linearizations w.r.t. $d$ of the system of conditions (B) from 6° and from 5° are the same.

On the other hand, $b_i$ were defined such that condition (A) holds. Hence, for $d = d(a)$ and $b_i = b_i(a, d(a))$ (where $a > 0$ is small enough) the $(\mu + \nu)$-fold root of $P$ at 0 splits into a real $\mu$-fold root at 0 and $\nu$ complex roots close to 0 (see observation 2) from 1° and $P(s)$ has a $(\mu + \nu - s)$-fold root at 0. The arrangement of the other real roots of $P$ and $P(s)$ remains the same. □

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