On the Derivation of Weights from Incomplete Pairwise Comparisons Matrices via Spanning Trees with Crisp and Fuzzy Confidence Levels

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Abstract

In this paper, we propose a new method for the derivation of a priority vector from an incomplete pairwise comparisons (PC) matrix. We assume that each entry of a PC matrix provided by an expert is also evaluated in terms of the expert’s confidence in a particular judgment. Then, from corresponding graph representations of a given PC matrix, all spanning trees are found. For each spanning tree, a unique priority vector is obtained with the weight corresponding to the confidence levels of entries that constitute this tree. At the end, the final priority vector is obtained through an aggregation of priority vectors achieved from all spanning trees. Confidence levels are modeled by real (crisp) numbers and triangular fuzzy numbers. Numerical examples and comparisons with other methods are also provided. Last, but not least, we introduce a new formula for an upper bound of the number of spanning trees, so that a decision maker gains knowledge (in advance) on how computationally demanding the proposed method is for a given PC matrix.

Keywords: pairwise comparisons, fuzzy numbers, priority vector, spanning tree, multiple-criteria decision making

1. Introduction

Pairwise comparisons (PCs) constitute a fundamental part of many multiple-criteria decision making (MCDM) methods, such as the AHP/ANP, BWM, ELECTRE, MACBETH, PAPRIKA and PROMETHEE etc. [31, 9, 23, 38, 45, 48, 55].

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Usually, it is assumed that all pairwise comparisons by experts are performed and available to a decision maker. However, due to a lack of knowledge, time pressure, uncertainty or other factors, pairwise comparisons might not be complete. The problem of the derivation of a priority vector from incomplete PC matrices has been studied for almost four decades. Probably the first study on incomplete PC matrices came in 1987 from Weiss and Rao [49] in the context of large-scale systems. Soon afterwards, the ’standard’ method for obtaining a priority vector from an incomplete PC matrix was proposed by Harker in [25]. Harker’s completion method is based upon the graph-theoretic structure of the PC matrix and the gradient of the right Perron vector. It can be used with the geometric mean leading to the geometric mean method (GMM) for incomplete PC matrices proposed by Kulakowski [32]. This approach is equivalent to the logarithmic least-square method (LLSM) for incomplete PC matrices proposed and developed earlier by Tone and Bozóki et al. [47, 4]. Ramík [43] proposed estimation of missing matrix elements via spanning trees of an induced graph of an extended matrix while maximally preserving consistency. The method of determining missing values based on inconsistency was also proposed by Alonso et al. [1]. Several recent studies were aimed at theoretical considerations regarding the geometric mean method. Lundy et al. [37] showing that the ranking (a priority vector) based on spanning trees of a PC matrix is equivalent to the ranking obtained by the geometric mean method. Later on, Bozóki and Tsyganok [6] proved that the spanning trees method and the logarithmic least squares method for incomplete PC matrices yield the same result as well.

By far the most popular methods for the derivation of a priority vector are the aforementioned eigenvalue method and geometric mean method (the least squares method). In 2012, Siraj et al. [46] proposed the spanning trees method, originally for complete PC matrices. The method enumerates all spanning trees of a graph representation of a given PC matrix to find priority vectors corresponding to each spanning tree, and, after aggregation, the final priority vector is achieved. The virtue of this approach lies in the fact that, from each spanning tree, a unique priority vector is obtained and that pairwise comparisons included in one spanning tree are necessarily consistent. However, the obvious drawback of this method is the large number (given by Cayley’s theorem) of spanning trees when the order of a PC matrix grows. Nevertheless, for incomplete matrices the number of spanning trees might be significantly reduced and the method can be easily applicable.

Another problem in pairwise comparisons relates to the problem of uncertainty associated with experts’ judgments. Uncertainty in the framework of PCs has been studied by many authors who proposed various sophisticated methods and approaches based on interval numbers, fuzzy sets, intuitionistic fuzzy sets, fuzzy hesitant sets, rough sets, linguistic variables, Z-numbers, and so on [3, 51, 26, 5, 17, 18, 21, 43, 52, 54], to name a few, however, a comprehensive review on uncertainty in the pairwise comparisons method is beyond the scope of this paper.

Therefore, this paper aims to propose a novel procedure for the derivation of a priority vector (a vector of weights of compared objects) from an incomplete
PC matrix which is based on the graph representation of a PC matrix and its set of spanning trees, which allows modeling of uncertainty using confidence levels. The method is based on elaboration and synthesis of the aforementioned studies, where the application of spanning trees was inspired by [46] and the use of confidence levels by [54]. More specifically, we introduce two similar mathematical models with different levels of uncertainty. In both models, confidence levels for each pairwise comparison provided by an expert are provided, creating a confidence matrix. In the first model, the confidence levels are expressed (coded) by crisp numbers corresponding to a given linguistic scale. For instance, an expert can have "strong confidence" or "absolute confidence" in his/her judgments. Hence, the input of the first model consists of a real reciprocal PC matrix and a corresponding real symmetric confidence matrix, and the output is a real priority vector. In our second model, we incorporate triangular fuzzy numbers both for the preferences and confidences. Preferences form a reciprocal fuzzy PC matrix and the confidence values are arranged into a symmetric fuzzy confidence matrix. Both matrices constitute the input of the model, while the output is a fuzzy priority vector. In both models, confidence values are applied to assess the reliability of a priority vector inferred from each spanning tree, where reliability plays the role of weights in the aggregation process by which a final priority vector is obtained. In the presented model, we use triangular fuzzy numbers due to their easy and intuitive interpretation, simple arithmetics, and a large number of their previous applications in the literature. However, our approach can be easily extended to trapezoidal or other types of fuzzy numbers as well.

It has several advantages. Firstly, it does not require a decision maker to complete a PC matrix, which is always associated with a distortion of original preferences. Secondly, it enables an expert to assign confidence levels to his/her judgments, so the more confident preferences (in a particular spanning tree) translate into the higher weight of a corresponding priority vector in the final aggregation. Additionally, incorporation of fuzzy numbers allows an expert to add another level of uncertainty of his/her judgments dealing with pairwise comparisons. Since the number of all pairwise comparisons grows quadratically with the number of compared objects \( n \), PC methods such as AHP are often limited to small-scale problems, typically with \( n \) not greater than 10. However, our approach based on incomplete pairwise comparisons enables to solve problems with relatively high numbers of compared objects as long as the amount of spanning trees induced by an incomplete PC matrix remains manageable. That’s why we provide an estimate for the upper bound of the number of spanning trees as well. Finally, the method is also applicable for complete PC matrices, however, due to the fact that the number of spanning trees grows very quickly with the size of a PC matrix, we recommend its use predominantly in the former case.

The paper is organized as follows: Section 2 provides preliminaries on applied concepts and notation, in Sections 3-4 methods for the derivation of priority vectors are proposed with numerical examples, and the Discussion with Conclusions close the article.
2. Preliminaries

2.1. Multiplicative pairwise comparisons

A pairwise comparison is a binary relation over a finite and discrete set of objects \( V = \{V_1, \ldots, V_n\} \) (usually formed by alternatives, criteria, sub-criteria, etc.) such that \( V \times V \to \mathbb{R}_+ \cup \{0\} \). The value 0 means the comparison is undefined. Further on, every pairwise comparison \( a_{ij} \in \mathbb{R}_+ \) describes the degree of preference (importance, etc.) of an object \( V_i \) over an object \( V_j \).

Pairwise comparisons are usually arranged into a square and positive \( n \times n \) matrix \( A = (a_{ij}) \) called a pairwise comparisons matrix. If matrix contains zeros it is called incomplete. In our paper, we limit ourselves to the multiplicative framework, however, multiplicative pairwise comparisons can be easily transformed into additive or fuzzy systems, and vice versa, see e.g. [11, 12, 10] or [43].

Definition 1. A matrix \( A \) is said to be (multiplicatively) reciprocal if:

\[
\forall i, j \in \{1, \ldots, n\} : a_{ij} \neq 0 \Rightarrow a_{ij} = \frac{1}{a_{ji}}
\]  

(1)

and \( A \) is said to be (multiplicatively) consistent if:

\[
\forall i, j, k \in \{1, \ldots, n\} : a_{ij}, a_{jk}, a_{ki} \neq 0 \Rightarrow a_{ij} \cdot a_{jk} \cdot a_{ki} = 1
\]  

(2)

The result of the pairwise comparisons method is a priority vector (vector of weights) \( w = (w_1, \ldots, w_n) \) that assigns positive values \( w_i \) to each of the \( n \) compared objects. According to the EVM (the eigenvalue method) proposed by Saaty for complete PC matrices, see [44, 45], the vector \( w \) is determined as the rescaled principal eigenvector of a PC matrix \( A \). Thus, assuming that \( Aw = \lambda_{\text{max}}w \) the priority vector \( w \) is given as:

\[
w = [\gamma w_1, \ldots, \gamma w_n]^T,
\]

where \( \gamma \) is a scaling factor. Usually, it is assumed that \( \gamma = [\sum_{i=1}^n w_i]^{-1} \). Extension of EVM for incomplete PC matrices has been defined by Harker [25].

In the geometric mean method (GMM) introduced by Crawford and Williams [16], the weight of the \( i \)-th alternative is estimated as the geometric mean of the \( i \)-th row of \( A \). Thus, the priority vector is given as:

\[
w = \left[ \gamma \left( \prod_{r=1}^n a_{1r} \right)^{\frac{1}{n}}, \ldots, \gamma \left( \prod_{r=1}^n a_{nr} \right)^{\frac{1}{n}} \right]^T
\]

where \( \gamma \) is a scaling factor again. In the case of incomplete matrices, one can use the extension of GMM defined by Kulakowski [32] or equivalent Logarithmic Least-squares Method (LLSM) [47, 7].

Other, less frequently applied prioritization methods include direct least squares, weighted least squares, row and/or column sums, etc., see e.g. [31].
Recently, an interesting method was proposed by Siraj et al. [46, 31]. This method is based on spanning trees of a graph representation of a given PC matrix, and the final priority vector is the arithmetic mean of all spanning trees’ priority vectors:

\[ w = \frac{1}{\eta} \sum_{s=1}^{\eta} w(\tau_s), \]  

(4)

where \( \eta = n^{n-2} \) denotes the number of spanning trees for a complete PC matrix, see e.g. [13], and \( w(\tau_s) \) denotes priority vectors of individual spanning trees.

### 2.2. Graph representation of a PC matrix and spanning trees

Judgments contained in a PC matrix can be represented in a form of a graph in which vertices correspond to the alternatives while edges denote individual pairwise comparisons, see e.g. [31, 19, 46, 30].

**Definition 2.** The undirected graph \( P_A = (V, E) \) is said to be a graph of the PC matrix \( A = (a_{ij}) \) if \( V = \{V_1, \ldots, V_n \} \) is the set of vertices and \( E \subseteq \{ \{V_i, V_j\} \) such that \( V_i, V_j \in V, \ i \neq j \) and \( a_{ij} \) exists.\(^1\) is the set of edges.

Similarly, we can formally define the concept of a path between two vertices.

**Definition 3.** An ordered sequence of distinct vertices \( p = V_{i_1}, V_{i_2}, \ldots, V_{i_m} \) such that \( \{V_{i_1}, V_{i_2}, \ldots, V_{i_m}\} \subseteq V \) is said to be a path between \( V_{i_1} \) and \( V_{i_m} \) with the length \( m-1 \) in \( P_A = (A, E) \) if \( \{V_{i_1}, V_{i_2}\}, \{V_{i_2}, V_{i_3}\}, \ldots, \{V_{i_{m-1}}, V_{i_m}\} \in E. \)

In a connected graph, there will be at least one path for each pair of vertices.

**Definition 4.** The irreducible matrix is one that cannot be transformed by the permutation of rows and columns to form:

\[
\begin{pmatrix}
Q_1 & 0 \\
Q_2 & Q_3
\end{pmatrix},
\]

where \( Q_1, Q_3 \) are square matrices, and 0 denotes the entries filled with zeros.

In the context of a PC matrix, zero means that the given comparison is undefined, and a matrix having such comparisons will be called incomplete. The connectivity of \( P_A \) is equivalent to the irreducibility of the matrix \( A \) [42, p. 185, 186]. It can be shown that, in the case of incomplete PC matrices, the ranking (a priority vector, vector of weights) can only be calculated if \( P_A \) is connected, i.e. \( A \) is irreducible.\(^2\)

\(^1\)For incomplete PC matrices, some comparisons may not be defined [31].

\(^2\)Note that when the graph \( P_A \) is undirected its matrix \( A \) must be symmetric. Thus, the reducibility means that there exists permutation of rows and columns such that \( A \) takes the
2.3 Pairwise comparisons’ confidence and priority vectors’ reliability

**Definition 5.** The spanning tree \( ST = (A, E') \) of \( PA \) is any connected subgraph of \( P_A \) such that \( E' \subset E \) where the number of edges \( |E'| = n - 1 \).

In the case of a complete PC matrix (the case of a complete graph), the number of all spanning trees is given by the well-known Cayley formula \( n^{n-2} \), where \( n \) is the order of the PC matrix. However, this number is significantly reduced for incomplete matrices (see e.g. Examples 1 and 3).

**Definition 6.** The Laplacian matrix \( L(G) = (l_{ij}) \) of a graph \( G = (V, E) \) is a matrix such that

\[
l_{ij} = \begin{cases} 
  s_i & \text{if } i = j \\
  0 & \text{if } i \neq j \text{ and } \{v_i, v_j\} \in E \\
  -1 & \text{if } i \neq j \text{ and } \{v_i, v_j\} \notin E
\end{cases}
\]

where \( V \) is a set of vertices \( V = \{v_1, \ldots, v_n\} \), \( E \) is a set of edges in the form of pairs \( \{v_i, v_j\} \), where \( v_i, v_j \in V \), and \( s_i \) is the number of edges adjacent to \( v_i \) i.e. \( \{v_i, v_j\} \in E \).

The following theorems answer the question about the number of spanning trees for incomplete graphs.

**Theorem 7.** Kirchhoff’s matrix tree theorem \([29]\). Let \( G \) be a graph, and let \( L(G) \) be the Laplacian matrix of \( G \) and let \( t(G) \) denote the number of spanning trees of a graph \( G \). Then \( t(G) \) is equal to all cofactors of \( L(G) \).

**Theorem 8.** Kelmans and Chelnokov \([28]\). Let \( G \) be a graph of \( n \) points, let \( L(G) \) be the Laplacian matrix of \( G \) and let \( 0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) denote the eigenvalues of \( L(G) \). Then:

\[
t(G) = \frac{1}{n} \prod_{k=2}^{n} \lambda_k.
\]

In particular, if \( G \) is a tree itself, then the number of spanning trees \( t(G) = 1 \); when \( G \) is a cycle graph of \( n \) points, then \( t(G) = n \).

A spanning tree corresponding to a \( n \times n \) PC matrix \( A = (a_{ij}) \) induces a unique ranking of compared objects since \( (n - 1) \) edges of a spanning tree corresponds to \( (n - 1) \) equations for \( n \) weights of \( n \) objects and along with the normality condition – the sum of weights being equal to 1 – a system of \( n \) equations is formed. Hence, a unique (normalized) priority vector \( w \) is assigned to each spanning tree.

2.3. Pairwise comparisons’ confidence and priority vectors’ reliability

According to \([41]\) or \([50]\), the word ‘confidence’ is used to describe a person’s strength of belief about the accuracy or quality of a prediction, judgment, or form \( A = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_3 \end{pmatrix} \), where \( Q_1 \) and \( Q_3 \) are the square sub-matrices with the dimensions \( s \times s \) and \( t \times t \) respectively. Assuming that \( A \) corresponds to the comparisons of \( s+t \) alternatives \( V_1, \ldots, V_{s+t} \) it is clear that for every \( i, j \) such that \( 1 \leq i \leq s < j \leq s+t \) the direct comparison \( a_{ij} \) is 0. This implies that non of \( V_1, \ldots, V_s \) is compared to any of \( V_{s+1}, \ldots, V_{s+t} \). Hence, based on \( A \), no common ranking for \( V_1, \ldots, V_{s+t} \) can be calculated.
choice, and confidence can thus be described on a continuum ranging from total certainty to complete doubt, though usually confidence is expressed verbally on a discrete scale.

For each \( n \times n \) PC matrix \( A = (a_{ij}) \), let us define a corresponding \( n \times n \) "confidence" matrix \( C = (c_{ij}) \), where \( c_{ij} \) might be real (crisp) or fuzzy numbers, or possibly other numbers, such as Z-numbers, such that \( c_{ij} \) expresses the confidence of an expert of a value of the pairwise comparison \( a_{ij} \), and it is assumed that \( c_{ij} = c_{ji}, \forall i, j \).

Further on, we assume that an expert can express his/her confidence in PC judgments on a) a finite, discrete and strictly increasing cardinal scale \( S_d = \{ s_1^{(d)}, ..., s_k^{(d)} \} \), where \( s_{i+1}^{(d)} \succ s_i^{(d)}, \forall i \), b) a finite ordinal (linguistic) scale \( S_o = \{ s_1^{(o)}, ..., s_k^{(o)} \} \), where \( s_{i+1}^{(o)} \succ s_i^{(o)}, \forall i \), or c) on a real interval scale \( S_i = [0, 1] \).

Then we define the absolute reliability \( r \) of a priority vector \( w \) derived from a given spanning tree as follows:

**Definition 9.** Let \( A = (a_{ij}) \) be a preference PC matrix, let \( C = (c_{ij}) \) be a corresponding confidence matrix and let \( E^{(ST)} = \{ [ij] \} \) be a set of all edges of a spanning tree \( ST \). Then the absolute reliability \( r \) of a priority vector \( w \) obtained from a spanning tree \( ST \) is given as follows:

\[
r(ST) = \prod c_{ij} | [ij] \in E^{(ST)} \]

Simply put, the value of the absolute reliability \( r \) of a priority vector obtained from a given spanning tree is a product of all \( c_{ij} \) corresponding to the PCs’ elements \( a_{ij} \) (graph edges) of that spanning tree. Next, we define the relative reliability \( R \) of a priority vector obtained from a given spanning tree.

**Definition 10.** Let \( r(ST) \) be the absolute reliability of a priority vector \( w \) obtained from a spanning tree \( ST \). Then the relative reliability \( R \) of the priority vector \( w \) obtained from a spanning tree \( ST \) is given as follows:

\[
R(ST) = \frac{r(ST)}{\sum_{i=1}^{N} r(ST_i)}, \quad (6)
\]

where \( N \) is the number of all spanning trees corresponding to the graph representation of a given PC matrix.

It should be noted that other aggregation functions, see Grabisch [22], such as the minimum, can be applied for the calculation of the absolute reliability (5), though we consider the product to be the most natural.\(^3\)

\(^3\)Let’s consider confidence values from Table 3 and a spanning tree with only two edges/preferences, with corresponding values of confidence 1 (low confidence) and 4 (absolute confidence). Then the absolute reliability of a priority vector acquired from this spanning tree is \( r = 1 \cdot 4 = 4 \). Now consider another spanning tree with corresponding values of confidence 1 and 1. Then the absolute reliability \( r = 1 \cdot 1 = 1 \). If we used, for example, the minimum function, both absolute reliabilities would be 1, neglecting the fact that an expert was more confident in the case of preferences for the first spanning tree, hence the corresponding priority vector should obtain the higher weight.
2.4 Fuzzy sets and triangular fuzzy numbers

As for the most suitable scale for the confidence values $c_{ij}$, we suggest using the 5-item integer scale from 0 (zero confidence in the judgment of $a_{ij}$) to 4 (absolute confidence about $a_{ij}$), see Table 3. Similar scales have been applied, for instance, in [35], or [36].

2.4. Fuzzy sets and triangular fuzzy numbers

Fuzzy sets introduced by L. A. Zadeh [53] provide a convenient theoretical framework for uncertainty modeling. Here, we provide its basic properties.

Definition 11. A fuzzy set $\tilde{U}$ is a pair $(U, \mu_U(x))$ where $U$ denotes the universe of discourse, $x \in U$ and $\mu_U(x)$ is a membership function such that $\mu_U(x) \in [0, 1]$.

Hence, the membership function $\mu_U(x)$ assigns to every member $x$ of the set $U$ its grade of membership in $U$.

Support of a fuzzy set $\tilde{U}$ is a subset of $U$ for which the value of the membership function is greater than 0: $\text{Support}(\tilde{U}) = \{x \in U | \mu_U(x) > 0\}$.

The $\alpha$-cut of a fuzzy set $\tilde{U}$ consists of all elements $x$ in $U$ for which a value of the membership function is equal to or greater than $\alpha$: $\tilde{U}_\alpha = \{x \in U | \mu_U(x) \geq \alpha\}$

Definition 12. A fuzzy number $\tilde{U} \equiv (U, \mu_U(x))$ is a fuzzy set that is piecewise continuous, achieves the membership value 1 at least once and all its alpha cuts are convex sets [20].

Triangular fuzzy number (TFN) constitutes a special case of a fuzzy number such that the membership function has the shape of a triangle and achieves the value of 1 at exactly one point, see Figure 1.

Formally, a TFN can be defined as a triplet of real numbers $(l, m, u)$, $l \leq m \leq u$, where $l$ is the lower bound of the TFN, $m$ is the value of $x$ for which $\mu_U(x) = 1$ and $u$ is the upper bound of the TFN. The membership function of a TFN is given as follows:

$$
\mu_U(x) = \begin{cases} 
0, & \text{if } x < l \\
\frac{x - l}{m - l}, & \text{if } l \leq x \leq m \\
\frac{u - x}{u - m}, & \text{if } m \leq x \leq u \\
0, & \text{if } x > u 
\end{cases} \tag{7}
$$

Obviously, the support of a triangular fuzzy number is a closed interval $[l, u]$.

Arithmetic operations of addition and subtraction with TFNs given by the Extension principle are closed, which means the result of these operations is a TFN again. However, multiplication and division of TFNs do not preserve the triangular shape of a membership function in general [8]. In practice, piece-wise linear approximations are used for the two latter operations, see Table 1.

Fuzzy weights need to be normalized in order to generate a unique priority vector (vector of weights). Otherwise, there is an infinite number of fuzzy weights that can be derived from a fuzzy pairwise comparisons matrix, thus making the weights incomparable and impossible to aggregate in a hierarchical structure (if there is any).
Definition 13. Let \{\tilde{w}_i : i = 1, ..., n\} be the set of fuzzy weights. Let \([l_i^*(\alpha), u_i^*(\alpha)], \alpha \in [0, 1]\) represent the \(\alpha\)-cut of \(\tilde{w}_i\). Then the set \{\tilde{w}_i\} is said to be fuzzy normalized if the following condition holds for every \(\alpha\):

\[
\left[\sum_{i=1}^{n} l_i^*(\alpha)\right] \cdot \left[\sum_{i=1}^{n} u_i^*(\alpha)\right] = 1
\]  

(8)

Further on, if the condition above is satisfied only for \(\alpha = 0\) and \(\alpha = 1\), then it is called relaxed normalization [15]. In this particular case, we obtain:

\[
\begin{cases}
\sum_{i=1}^{n} l_i \cdot \sum_{i=1}^{n} u_i = 1(\alpha = 0) \\
\sum_{i=1}^{n} m_i = 1(\alpha = 1)
\end{cases}
\]  

(9)

Definition 14. A fuzziness \(k(\tilde{U})\) of a fuzzy number \(\tilde{U}\) is defined as the area under the membership function \(\mu_U(x)\) [27]:

\[
k(\tilde{U}) = \int_{\text{support}(\tilde{U})} \mu_U(x)dx
\]  

(10)

In the case of a triangular fuzzy number \(\tilde{U} = (l, m, u)\), the formula above simplifies to

\[
k(\tilde{U}) = \frac{u - l}{2}
\]  

(11)

In a sense, fuzziness determines the 'specificity' contained in a given TFN. For example, \(\tilde{a}_{ij} = (2, 3, 4)\) expresses that an object \(i\) is probably two to four times more preferred than an object \(j\), while \(\tilde{b}_{ij} = (2, 10, 90)\) is less specific, since it says an object \(i\) is probably two to ninety times more preferred than an object \(j\). Hence, too much fuzziness (uncertainty) might be counterproductive, and this notion appears in the formulation of the mathematical model in Section 4.

If a decision maker wishes to obtain a crisp (real) number from a triangular fuzzy number, defuzzification must be performed. In this paper we use the following formula called the center of gravity:

\[
c(\tilde{U}) = \frac{l + m + u}{3},
\]  

(12)

where \(c(\tilde{U})\) denotes a crisp value of a triangular fuzzy number \(\tilde{U} = (l, m, u)\).

3. Derivation of a priority vector via spanning trees with crisp preferences and confidence levels

The input for the proposed method consists of an \(n \times n\) (incomplete) preference PC matrix \(A = (a_{ij})\), where \(a_{ij} \in \mathbb{R}^+\), and an \(n \times n\) confidence matrix...
Figure 1: A typical membership function of a triangular fuzzy number.

Table 1: Standard approximations of arithmetic operations on positive triangular fuzzy numbers.

| Arithmetic operation | Definition |
|----------------------|------------|
| $\tilde{A} \oplus \tilde{B}$ | $\langle l_A + l_B, m_A + m_B, u_A + u_B \rangle$ |
| $\tilde{A} \ominus \tilde{B}$ | $\langle l_A - u_B, m_A - m_B, u_A - l_B \rangle$ |
| $\tilde{A} \otimes \tilde{B}$ | $\langle l_A \cdot l_B, m_A \cdot m_B, u_A \cdot u_B \rangle$ |
| $\frac{\tilde{A}}{\tilde{B}}$ | $\langle \frac{u_A}{n_A}, \frac{m_A}{n_A}, \frac{l_A}{n_A} \rangle$ |
| $\tilde{A}^{-1}$ | $\langle \frac{l_A}{u_A}, \frac{m_A}{u_A}, \frac{1}{u_A} \rangle$ |

$C = (c_{ij}), c_{ij} \in R_0^+$, where the values of $c_{ij}$ express the confidence of corresponding values $a_{ij}$. It is assumed that both $a_{ij}$ and $c_{ij}$ take values from some suitable comparison scale set in advance (for instance, one can apply Saaty’s scale from 1 to 9 for $a_{ij}$ values and a 5-item ordinal linguistic scale for the values of $c_{ij}$).

The method starts with the enumeration of all spanning trees from a graph representation of the matrix $A$ by a suitable algorithm, see e.g. [14]. For each spanning tree, a unique (normalized) priority vector $w$ is calculated from $(n - 1)$ graph edges ($a_{ij}$ values) and the normality condition $\sum_{i=1}^{n} w_i = 1$. Subsequently, this priority vector is assigned its absolute and relative reliability according to relations (5–6). At last, the final priority vector $w^{(f)}$ is obtained as a weighted arithmetic mean of the priority vectors $w^{(STi)}$ obtained from $N$ individual spanning trees, where the relative reliability $R^{(i)}$ of the $i$-th spanning tree plays the role of weights:

$$w^{(f)} = \sum_{i=1}^{N} R^{(i)} \cdot w^{(STi)}$$

(13)
Based on the values of $w^{(f)}_i$ all compared objects can be ranked (ordered from the best to the worst).

To summarize the whole process, each spanning tree of the graph representation of the preference matrix leads to a corresponding (individual) priority vector, and the final priority vector of all compared objects is obtained as an aggregation of priority vectors over all spanning trees. In addition, a confidence value is assigned to each pairwise comparison. Consequently, each priority vector from a given spanning tree has a reliability value, which is the product of the confidence levels of individual pairwise comparisons forming the tree. In this way, each priority vector acquires its weight equal to the reliability in the final aggregation process, where the greater reliability means the greater weight.

The method with the fuzzy extension is analogous; only arithmetic operations over triangular fuzzy numbers are performed differently than for crisp numbers.

The method is illustrated on the following numerical example.

**Example 15.** Consider the incomplete multiplicative PC matrix $A$ and the confidence matrix $C$ shown below. While $a_{ij}$, elements of $A$, express a degree of preference of an object $i$ over an object $j$, the elements $c_{ij}$ of a matrix $C$ express the confidence of an expert of the correct value of $a_{ij}$. The sign "**" denotes a missing element. Notice that while $a_{ij} = 1/a_{ji}, \forall i, j$ (the reciprocity condition), we have $c_{ij} = c_{ji}, \forall i, j$ (the symmetry condition).

\[
A = \begin{bmatrix}
1 & 4 & 1/2 & * \\
1/4 & 1 & 2 & * \\
2 & 1/2 & 1 & 5 \\
* & * & 1/5 & 1 \\
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
-4 & 3 & * \\
4 & -1 & * \\
3 & 1 & -2 \\
* & 2 & - \\
\end{bmatrix},
\]

The graph representation of the matrix $A$ is shown in Figure 2, and all three spanning trees are shown in Figure 3. Each spanning tree provides a unique vector of weights $w = (w_1, ..., w_4)$, $\|w\| = 1$, derived from ratios of all $a_{ij}$ included in a given spanning tree.

In the case of the spanning tree number 1 (see Figure 3), we obtain the following set of equations:

\[
\begin{aligned}
w_1 &= 4w_2, \\
w_2 &= 2w_3, \\
w_4 &= 5w_3, \\
\sum_{i=1}^{4} w_i &= 1
\end{aligned}
\tag{14}
\]

The solution of the system of equations (14) is: $w^{(ST1)} = (0.5, 0.125, 0.0625, 0.3125)$, see also Table 2, second row.
The absolute reliability of this priority vector is estimated as follows:

\[ r^{(1)} = c_{12} \cdot c_{23} \cdot c_{34} = 4 \cdot 1 \cdot 2 = 8, \]

And the relative reliability is given as:

\[ R^{(1)} = 8/(8 + 24 + 6) = 8/38 = 0.211. \]

Further on, \( R^{(2)} = 24/38 = 0.632 \) and \( R^{(3)} = 6/38 = 0.158 \).

All priority vectors and their relative reliability are shown in Table 2. As can be seen, more reliable priority vectors (vectors from spanning trees with higher reliability) achieve higher weights in the final aggregation.

Table 2: Priority vectors and their relative reliability for all three spanning trees

| Spanning tree | Priority vector \( w^{(ST_i)} \) | \( (R) \) |
|---------------|----------------------------------|--------|
| 1             | \( w^{(ST_1)} = (0.5, 0.125, 0.0625, 0.3125) \) | 0.211  |
| 2             | \( w^{(ST_2)} = (0.274, 0.068, 0.548, 0.110) \) | 0.632  |
| 3             | \( w^{(ST_3)} = (0.135, 0.541, 0.270, 0.054) \) | 0.158  |

The final priority vector is estimated as a weighted arithmetic mean (13), where the reliability plays the role of weights of each spanning tree priority vector:

\[ w^{(f)} = \sum_{i=1}^{3} R^{(i)} \cdot w^{(ST_i)} = 0.211 \cdot (0.5, 0.125, 0.0625, 0.3125) + 0.632 \cdot (0.274, 0.068, 0.548, 0.110) + 0.158 \cdot (0.135, 0.541, 0.270, 0.054) = (0.300, 0.155, 0.402, 0.144). \]

According to the final priority vector, the ranking of compared objects is as follows: \((3,1,2,4)\).

Figure 2: Graph representation of a PC matrix \( A \).

A comparison of the proposed method with Harker’s method \[25\] and the geometric mean method modified for incomplete PC matrices \[32\] is provided below, but without confidence levels, of course.

Example 16. Consider the same PC matrix as in Example 15. Harker’s method starts with the construction of an auxiliary matrix \( B = [b_{ij}] \) associated with the PC matrix. We assume that the edge labels correspond to the increasing order of vertices. In other words, the label between \( A_i \) and \( A_j \) is set to \( a_{ij} \) if \( i \leq j \), to \( a_{ji} \) otherwise.

\[ a_{ij} = \begin{cases} 2 & i \leq j \\ 3 & i > j \end{cases} \]
with $A$, such that

$$b_{ij} = \begin{cases} 0 & \text{if } a_{ij} = ? \text{ and } i \neq j \\ c_{ij} & \text{if } a_{ij} \neq ? \text{ and } i \neq j \\ s_i + 1 & \text{if } i = j \end{cases}$$

where $s_i$ denotes the number of missing entries in the $i$-th row of $A$, given as:

$$B = \begin{pmatrix} 2 & 4 & \frac{1}{2} & 0 \\ \frac{1}{3} & 2 & 2 & 0 \\ 2 & \frac{1}{3} & 1 & 5 \\ 0 & 0 & \frac{1}{3} & 3 \end{pmatrix}.$$ 

By solving the equation $Bw = \lambda_{max} w$

we obtain the ranking vector $w_{hm}$, which after the appropriate rescaling equals

$$w_{hm} = \begin{pmatrix} 0.416 \\ 0.252 \\ 0.298 \\ 0.0335 \end{pmatrix}.$$ 

Similarly, using the geometric mean method for incomplete PC matrices [32] we obtain
\[ w_{gm} = \begin{pmatrix} \frac{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} + \frac{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} \\
\frac{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} + \frac{1}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} \\
\frac{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} + \frac{1}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} \\
\frac{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} + \frac{1}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} \\
\frac{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} + \frac{1}{\sqrt[5]{5} + \frac{1}{5} \sqrt[5]{3 \ln(5) - 4 \ln(2)}} \end{pmatrix}, \]

i.e.

\[ w_{gm} = \begin{pmatrix} \frac{5^{2/3}}{5 + 6 \sqrt[3]{5} \cdot 5^{2/3}} \\
\frac{5^{2/3}}{5 + 6 \sqrt[3]{5} \cdot 5^{2/3}} \\
\frac{5^{2/3}}{5 + 6 \sqrt[3]{5} \cdot 5^{2/3}} \\
\frac{5^{2/3}}{5 + 6 \sqrt[3]{5} \cdot 5^{2/3}} \end{pmatrix}, \]

and finally

\[ w_{gm} = \begin{pmatrix} 0.387 \\
0.2439 \\
0.307 \\
0.061 \end{pmatrix}. \]

As can be seen, both Harker’s method and the modified geometric mean method yield the same ranking of the compared objects: \((1,3,2,4)\), which slightly differs from the ranking \((3,1,2,4)\) obtained from the model with confidence levels. This difference can be attributed to the application of additional information contained in the confidence matrix of the proposed model.

4. Derivation of a priority vector via spanning trees with fuzzy preferences and confidence levels

The input for the proposed method consists of a (incomplete) \(n \times n\) preference PC matrix \(\tilde{A} = (\tilde{a}_{ij})\), where \(\tilde{a}_{ij}\) has the form of triangular fuzzy numbers, and an \(n \times n\) confidence matrix \(\tilde{C} = (\tilde{c}_{ij})\), where \(\tilde{c}_{ij}\) are triangular fuzzy numbers as well. In addition, it is assumed that (fuzzy) scales for both pairwise comparisons and confidence levels were selected in advance.

The method begins with the enumeration of all spanning trees from a graphical representation of preferences provided in a PC matrix in the same way as in the previous section.

Let \(\tilde{w}_i \equiv (l_i, m_i, u_i), i \in \{1, \ldots, n\}\) denote weights (coordinates of a priority vector) in the form of TFNs of all compared objects with respect to a given spanning tree that is formed by \((n - 1)\) edges denoted as \([k,l], k,l \in \{1, \ldots, n\}\).

Then, for each spanning tree the following mathematical model for weights derivation comprised of four parts (I-IV) is constructed:

\[(I) : \tilde{w}_i = \tilde{a}_{ij} \cdot \tilde{w}_j, j = i + 1, i \in \{1, \ldots, n\} \quad (15)\]
\((\text{II})\) : \(0 < l_i \leq m_i \leq u_i, \; i \in \{1, \ldots, n\}\) \hfill (16)

\[
\begin{align*}
\begin{cases}
(\text{III}) : \\
(n \sum_{i=1}^{n} l_i) \cdot (n \sum_{i=1}^{n} u_i) = 1 \\
\sum_{i=1}^{n} m_i = 1 \\
(\text{IV}) : \\
\min \sum_{i=1}^{n} (u_i - l_i)
\end{cases}
\end{align*}
\] \hfill (17)

Part (I) is the ‘Preference part’ of the model. It consists of \((n - 1)\) fuzzy equations that can be decomposed into \(3(n - 1)\) crisp equations for the \(3n\) unknown values of \((l_i, m_i, u_i)\). Since there are \(3n\) unknown variables and only \(3(n - 1)\) equations, there are 3 degrees of freedom.

Part (II) is the ‘Structural part’ of the model. It places constraints on the values of lower bounds, middle values and upper bound of all triangular fuzzy numbers. If this part was missing, a solution that contradicts \(0 < l_i \leq m_i \leq u_i\) inequalities could be obtained (this was noted as early as the Chang and Lee paper from 1995,\(^{[15]}\)).

Part (III) is the ‘Normalization part’ of the model since it is usually desired that weights (a priority vector) are normalized. Equations (17) are identical to Equations (11) expressing the condition of the relaxed normalization. These two additional equations reduce the degrees of freedom of the model to 1.

Finally, Part (IV) has the following purpose: the model has one degree of freedom, hence one can expect an infinite number of solutions (if the set of inequalities in Part (II), which provides other constraints for the solution, is satisfied) depending on one parameter (it can be \(u_1\), for example). If this happens, then condition (IV) provides a unique solution with minimal ‘fuzziness’ (a solution in the form of a ‘narrow’ triangular fuzzy number provides more specific information than a ‘wide’ triangular fuzzy number).

After a priority vector \(\tilde{w}\) from a spanning tree \(ST\) with the set of edges \(E^{(ST)}\) is acquired, its absolute reliability \(\tilde{r}\) is given as follows:

\[
\tilde{r}(ST) = \prod \hat{c}_{ij} |[ij] \in E^{(ST)}\}
\] \hfill (19)

The corresponding relative reliability \(\tilde{R}\) of a priority vector \(\tilde{w}\) from a spanning tree \(ST\) is defined as:

\[
\tilde{R}(ST) = \frac{\tilde{r}(ST)}{\sum_{i=1}^{N} \tilde{r}(ST^i)},
\] \hfill (20)

where \(N\) is the number of spanning trees.
Note that operations of multiplication and division are fuzzy operations defined in Table 1.

At last, the final (fuzzy) priority vector \( \tilde{w}(f) \) is obtained as a weighted arithmetic mean of priority vectors \( \tilde{w}(ST_i) \) obtained from \( N \) individual spanning trees \( (ST_i) \), where the relative reliability \( \tilde{R}^{(i)} \) of the i-th spanning tree plays the role of weights:

\[
\tilde{w}(f) = \sum_{i=1}^{N} \tilde{R}^{(i)} \otimes \tilde{w}(ST_i),
\]

(21)

where \( \otimes \) denotes fuzzy multiplication.

The use of the method is illustrated in the following example.

**Example 17.** Consider the incomplete multiplicative matrix PC matrix \( \tilde{A} = (\tilde{a}_{ij}) \), as shown below, and its corresponding confidence matrix \( \tilde{C} = (\tilde{c}_{ij}) \), where the elements of both \( \tilde{A} \) and \( \tilde{C} \) are in the form of triangular fuzzy numbers. In order to enable a comparison of methods applied in Examples 15 and 16, we only transformed the crisp values from Example 15 to the corresponding triangular fuzzy numbers. We applied Saaty’s 9-point scale from [2], see Fig. 4, and the 5-item fuzzy DEMATEL linguistic scale from [34], see Table 3.

From the the first spanning tree (see Figure 3), we obtain the following mathematical model for the priority vector \( \tilde{w} = (\tilde{w}_1, ..., \tilde{w}_n) \):

\[
\begin{align*}
\tilde{w}_1 &= \tilde{a}_{12} \otimes \tilde{w}_2 = (3, 4, 5) \otimes \tilde{w}_2 \\
\tilde{w}_2 &= \tilde{a}_{23} \otimes \tilde{w}_3 = (1, 2, 3) \otimes \tilde{w}_3 \\
\tilde{w}_3 &= \tilde{a}_{34} \otimes \tilde{w}_4 = (4, 5, 6) \otimes \tilde{w}_4 \\
\end{align*}
\]

(22)

\[
\begin{align*}
0 < l_1 &\leq m_1 \leq u_1 \\
0 < l_2 &\leq m_2 \leq u_2 \\
0 < l_3 &\leq m_3 \leq u_3 \\
0 < l_4 &\leq m_4 \leq u_4 \\
\end{align*}
\]

(23)

\[
\begin{align*}
(l_1 + l_2 + l_3 + l_4) \cdot (u_1 + u_2 + u_3 + u_4) &= 1 \\
m_1 + m_2 + m_3 + m_4 &= 1 \\
\end{align*}
\]

(24)

\[
\min[(u_1 + u_2 + u_3 + u_4) - (l_1 + l_2 + l_3 + l_4)]
\]

(25)

The solution of the model (obtained by Mathematica Solver) is as follows:

\[
\begin{align*}
\tilde{w}_1 &= (0.214, 0.714, 2.09) \\
\tilde{w}_2 &= (0.071, 0.179, 0.417) \\
\tilde{w}_3 &= (0.071, 0.089, 0.089) \\
\tilde{w}_4 &= (0.018, 0.018, 0.023) \\
\end{align*}
\]

(26)

The absolute and relative reliabilities of this priority vector are:
\( \tilde{r}(1) = \tilde{c}_{12} \otimes \tilde{c}_{23} \otimes \tilde{c}_{34} = (0.75, 1, 1) \otimes (0, 0.25, 0.5) \otimes (0.25, 0.5, 0.75) = (0, 0.125, 0.375), \) and \( \tilde{R}(1) = (0, 0.211, 4). \)

The results for all three spanning trees are summarized in Table 4 along with the final priority vector \( \tilde{w}^{(f)} \) obtained from relation (21). Here, only the calculation of the first coordinate of \( \tilde{w}^{(f)} \) is shown (the rest is analogous):

\[
\tilde{w}_1^{(f)} = (0, 0.211, 4) \otimes (0.214, 0.714, 2.09) \oplus (0.063, 0.632, 8) \otimes (0.206, 0.274, 0.401) \\
\oplus (0, 0.158, 4) \otimes (0.135, 0.135, 0.169) \\
= (0.013, 0.345, 12.23)
\]

Once the final fuzzy weights (final priority vector) \( \tilde{w}^{(f)} \) are obtained, see the lower part of Table 4 the defuzzification by relation \( (12) \) and subsequent normalization is performed, see the bottom of Table 4 for results. Now, crisp and normalized weights of each object can be easily compared and objects can be ranked for the best to the worst. It should be noted that the defuzzification is not necessary since a comparison of fuzzy numbers is also possible, but it is far from being straightforward and thus less suitable for practical applications.

After the defuzzification by relation \( (12) \), the final (crisp) priority vector is: \( w = (0.353, 0.240, 0.348, 0.059) \). The ranking of objects (from the best to the worst) is as follows: (1, 3, 2, 4), which is the same as the ranking obtained from Harker’s method and the GMM method in Example 16, but different from the result obtained via the spanning trees method with crisp confidence levels in Example 15.

Notice that from the table of arithmetic operations, it follows that \( \tilde{A} \otimes \frac{1}{A} \neq \tilde{1} \). Therefore, in the system of equations (15) and (22), respectively, one should consistently multiply TFNs so that all \( \tilde{c}_{ij} \) have the lower bound \( l_{ij} \geq 1 \) (or vice versa) to obtain replicable results.

\[
\tilde{A} = \begin{bmatrix}
(1,1,1) & (3,4,5) & (1/3,1/2,1) & * \\
(1/5,1/4,1/3) & (1,1,1) & (1,2,3) & * \\
(1,2,3) & (1/3,1/2,1) & (1,1,1) & (4,5,6) \\
* & * & (1/6,1/5,1/4) & (1,1,1)
\end{bmatrix},
\]

\[
\tilde{C} = \begin{bmatrix}
(0,0,0) & (0.75,1,1) & (0.5,0.75,1) & * \\
(0.75,1,1) & (0,0,0) & (0,0.25,0.5) & * \\
(0.5,0.75,1) & (0,0.25,0.5) & (0,0,0) & (0.25,0.5,0.75) \\
* & * & (0.25,0.5,0.75) & (1,1,1)
\end{bmatrix},
\]

5. Discussion

In the proposed approach, we use the term 'confidence' to estimate the belief in the accuracy of individual pairwise comparisons provided by an expert, which
Figure 4: Saaty’s fuzzy 1-9 scale. Source: [2].

| Linguistic term | Crisp value | TFN       |
|-----------------|-------------|-----------|
| zero confidence | 0           | (0, 0.25) |
| low confidence  | 1           | (0.25, 0.5)|
| moderate confidence | 2 | (0.25, 0.5, 0.75) |
| strong confidence | 3       | (0.5, 0.75, 1) |
| absolute confidence | 4 | (0.75, 1, 1) |

Table 3: The fuzzy linguistic scale. Source: [34]

allows an expert to assess the quality of their judgments. This feature remains neglected in the standard pairwise comparisons methods such as the analytic hierarchy/network process, where it is assumed all pairwise comparisons have the same confidence. However, in real-world problems involving (not only) brand new complex phenomena or emerging technologies, it is natural that even experts with the best knowledge might be, in some situations, less confident about their judgments. Moreover, our approach allows straightforward extensions to a multiple-criteria single or group decision-making, where experts’ assessments concerning different criteria may be performed with individually selected confidence.

Apart from our method’s already mentioned advantages (there is no need to fill the incomplete PC matrix and confidence levels enable modeling uncertainty), the method has some limitations, specifically it requires that all spanning trees are found (there are a lot of algorithms [24] that can do the task) and a priority vector and its reliability estimated for each spanning tree. However, the number of spanning trees, especially for complete or almost complete PC matrices (graphs) could be very high, rendering the method excessively time-consuming and computationally demanding to use. That’s why we propose using the method primarily for incomplete matrices.

Before a decision maker applies our method, we suggest approximately estimating the number of spanning trees that would be necessary to process. For this, we postulate the following simple combinatorial theorem which gives an upper bound for the number of spanning trees \(t(A)\) associated with a PC matrix
| Spanning tree 1 | l   | m   | u     |
|----------------|-----|-----|-------|
| \( w_1 \)     | 0.2143 | 0.7143 | 2.087 |
| \( w_2 \)     | 0.0714 | 0.1786 | 0.4174 |
| \( w_3 \)     | 0.0714 | 0.0893 | 0.0893 |
| \( w_4 \)     | 0.0179 | 0.0179 | 0.0232 |
| \( \tilde{w} \) | 0   | 0.2105 | 4     |

| Spanning tree 2 | l   | m   | u     |
|----------------|-----|-----|-------|
| \( w_1 \)     | 0.2055 | 0.2740 | 0.4008 |
| \( w_2 \)     | 0.0685 | 0.0685 | 0.0802 |
| \( w_3 \)     | 0.2055 | 0.5479 | 0.2025 |
| \( w_4 \)     | 0.0514 | 0.1096 | 0.2004 |
| \( \tilde{w} \) | 0.0625 | 0.6315 | 8     |

| Spanning tree 3 | l   | m   | u     |
|----------------|-----|-----|-------|
| \( w_1 \)     | 0.1351 | 0.1351 | 0.1687 |
| \( w_2 \)     | 0.1351 | 0.5405 | 1.518  |
| \( w_3 \)     | 0.1351 | 0.2703 | 0.5060 |
| \( w_4 \)     | 0.0338 | 0.0541 | 0.0843 |
| \( \tilde{w} \) | 0   | 0.1578 | 4     |

| Final priority vector | l   | m   | u     |
|-----------------------|-----|-----|-------|
| \( \tilde{w}^{(f)}_1 \) | 0.0128 | 0.3448 | 12.2292 |
| \( \tilde{w}^{(f)}_2 \) | 0.0043 | 0.1662 | 8.3832  |
| \( \tilde{w}^{(f)}_3 \) | 0.0128 | 0.4075 | 12.0012 |
| \( \tilde{w}^{(f)}_4 \) | 0.0032 | 0.0815 | 2.0332  |

| Final crisp priority vector | after defuzzification | after normalization | rank |
|-----------------------------|----------------------|---------------------|------|
| \( w_1 \)                  | 4.195                | 0.353               | 1    |
| \( w_2 \)                  | 2.851                | 0.240               | 3    |
| \( w_3 \)                  | 4.141                | 0.348               | 2    |
| \( w_4 \)                  | 0.706                | 0.059               | 4    |

A which is based only on the number of vertices, the number of edges and the number of vertices with the degree equal to 1 and does not require formation of the Laplacian matrix and calculation of its cofactors or eigenvalues.

**Theorem 18.** Let \( A \) be a PC matrix, let \( G \) be a graph representation of \( A \) and let \( t(G) \) be the number of spanning trees of \( G \). Further on, let \( n, e, k \) denote the number of vertices, the number of edges and the number of vertices with the degree equal to 1 (called leaf vertex or end vertex, since it is connected to the rest of a graph with only one edge), respectively, of the graph \( G \). Then:

\[
t(G) \leq \binom{e - k}{n - k - 1} \tag{27}
\]

**Proof.** A graph with \( n \) vertices and \( e \) edges requires each spanning tree to have \( n - 1 \) edges. There are \( \binom{e}{n-1} \) possible ways to perform the selection. Of course,
the number of spanning trees would be the same or lower than \( \binom{e-1}{n-1} \) since some choices of \( n-1 \) edges may form circles, leaving the rest of the graph unconnected. Therefore, \( t(G) \leq \binom{e}{n-1} \). Further on, if there is a vertex with the degree equal to 1, the edges from this vertex have to be included in each spanning tree. Thus, the combinatorial number \( \binom{e}{n-1} \) describing a selection of \( n-1 \) edges from \( e \) changes to \( \binom{e-k}{n-k-1} \), since both the number of edges available and the number of edges necessary to form a spanning tree both decrease by 1. Apparently, when there are \( k \) vertices with the degree of one, the combinatorial number becomes \( \binom{e-k}{n-k-1} \).

For a cyclic graph (\( e = n, k = 0 \)) we get \( t(G) \leq \binom{e-k}{n-k-1} = \binom{n}{n-1} = n \), hence the equality holds. The same relationship holds if a graph is a spanning tree itself (\( e = n-1, k = 2 \)): \( t(G) \leq \binom{e-k}{n-k-1} = \binom{n-1-2}{n-2-1} = 1 \).

Using Theorem 18, a decision maker can decide beforehand if the proposed method would be feasible. The following Example 4 illustrates the use of Theorem 18.

**Example 19.** Consider the following incomplete real PC matrix \( A \) of the order \( n = 7 \) with a graph representation depicted in Figure 5. Estimate the upper bound for the number of spanning trees necessary to perform the proposed method.

\[
A = \begin{bmatrix}
1 & 5 & * & * & * & * & 0.25 \\
0.2 & 1 & 4 & 0.25 & 4 & * & * \\
* & 0.25 & 1 & 0.5 & * & * & * \\
* & 8 & 2 & 1 & 2 & * & 6 \\
* & 0.25 & * & 0.5 & 1 & * & * \\
* & * & * & * & * & 1 & 3 \\
4 & * & * & 0.167 & * & 0.333 & 1
\end{bmatrix},
\]

From the matrix \( A \) it is clear that \( n = 7, e = 9 \) and \( k = 1 \) (the object number 6 is connected to the object number 7 only). From Theorem 18 it follows that \( t(G) \leq \binom{e-k}{n-k-1} = \binom{8}{5} = 56 \).

Hence, there are only 56 spanning trees, or less. The Kelmans and Chelnokov formula \(^{28}\) for the exact number of spanning trees gives the value of 28.
6. Conclusion

The aim of the paper was to introduce a novel approach for the derivation of a priority vector from an incomplete PC matrix when additional information on the confidence of the pairwise comparisons is provided. The method has several advantages, namely it does not require the completion of a PC matrix, thus reflecting original preferences without the distortion. Secondly, it enables sophisticated modeling of uncertainty of pairwise comparisons using embedded confidence levels expressed by crisp (real) numbers or triangular fuzzy numbers.

Our approach opens several interesting research directions that can be pursued in the future. Firstly, numerical simulations can be carried out for the comparison of the proposed method and other methods for the derivation of a priority vector. Secondly, the problem of the condition of the order preservation, see e.g. [1, 33, 39], or [40] can be investigated in the context of the proposed method, with the interplay between preferences and confidences on one hand and inconsistency on the other hand. Further on, future research may focus on the extension of the proposed method, for instance towards Z-numbers, and also to the design of a user-friendly and free online software tool for facilitating the method’s computations. In addition, we will also want to investigate the relationship between the level of confrontation and the decision security of the model. We hope that the introduced extension can effectively detect manipulation attempts in the pairwise comparison method.
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