NONUNIQUENESS OF SOLUTIONS OF THE NAVIER-STOKES EQUATIONS ON NEGATIVELY CURVED RIEMANNIAN MANIFOLDS

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Abstract. In a well-known work, M. Anderson constructed a Hadamard manifold \((M^n, g)\) which carries non-zero \(L^2\) harmonic \(p\)-forms when \(p \neq n/2\), thus disproving the Dodziuk-Singer conjecture. In this paper, we use the manifold \((M^3, g)\) in order to solve another problem in geometric analysis, namely the nonuniqueness of solutions of Leray-Hopf type of the Navier-Stokes equations.

1. Introduction

The aim of this paper is to prove that the Cauchy problem for the Navier-Stokes equations is ill-posed on the Riemannian manifold \((M^3, g)\) constructed by Anderson \cite{1} in his counterexample to the Dodziuk-Singer conjecture (see also \cite{11}, \cite{22}, \cite{10}, \cite{15}). This answers a question raised by Chan and Czubak in \cite{6}, pursued also by Khesin and Misiolek \cite{19}.

In order to provide the appropriate context, we begin with a brief review of some pertinent results connected to the uniqueness problem. Recall that the Cauchy problem for the Navier-Stokes equations on \(\mathbb{R}^n\) is

\[
\begin{cases}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \\
\text{div } u = 0, \\
u > 0, \\
u = u(0, x) = u_0(x),
\end{cases}
\]

(1.1)

where \(u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\) is the velocity vector field, \(\nu > 0\) is a viscosity parameter and \(p : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\) is the pressure. From now on, we consider only the cases \(n = 2, 3\), which are of special interest from the standpoint of classical mechanics and have been studied extensively over the past century.

Among the first fundamental contributions to this problem, going back to 1934, were the works of Leray \cite{21} and Hopf \cite{17}, establishing that \((1.1)\) admits global weak solutions

\[
\begin{align*}
\begin{cases}
\quad u \in L^\infty([0, \infty), L^2(\mathbb{R}^n)) \cap L^2([0, \infty), H^1(\mathbb{R}^n))
\end{cases}
\end{align*}
\]

(1.2)

satisfying the energy inequality

\[
\|u(t, \cdot)\|_{L^2}^2 + 4 \int_0^t \|\text{Def}(u(s, \cdot))\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2.
\]

(1.3)

\textbf{Date:} February 24, 2015.

\textbf{2000 Mathematics Subject Classification.} Primary 76D05; Secondary 32Q05.

\textbf{Key words and phrases.} Navier-Stokes, harmonic vector fields, ill-posedness.
where \( \text{Def}(u) = \frac{1}{2}(\nabla u + (\nabla u)\cdot) \) is the so-called deformation tensor. Solutions of (1.1) satisfying (1.2) and (1.3) are said to be of Leray-Hopf type.

When \( n = 2 \), it is known that Leray-Hopf solutions are, in fact, smooth and unique, but this is not known to hold for \( n = 3 \) (see [20], [7] for these and related results).

In 1976, Heywood [16] considered (1.1) on a domain \( D \subset \mathbb{R}^n \), under suitable boundary conditions, and showed that the uniqueness problem is related to the geometry of \( D \). For instance, on

\[
D = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0, \text{ or } y^2 + z^2 < 1\}
\]

there are nonunique solutions, whereas on bounded domains uniqueness always holds.

In view of these developments, it is natural to consider the more general question of uniqueness of solutions of (1.1) on a general Riemannian manifold \((M, g)\).

In what follows, we are concerned only with the case when \((M, g)\) is not flat. In this context, the system (1.1) takes the form

\[
\begin{aligned}
\partial_t u + \nabla_u u - \nu L u + \nabla p &= 0, \\
d \text{div } u &= 0, \\
u(0, x) &= u_0(x),
\end{aligned}
\]

where \( \nabla \) is the Levi-Civita connection, and \( L \) is an operator generalizing the Euclidean Laplacian (see below). If \( \nu = 0 \) in the first equation of (1.5), we obtain the well-known Euler equation

\[
\partial_t u + \nabla_u u + \nabla p = 0.
\]

Throughout the literature, we encounter two natural choices for the operator \( L \). The first one is the Bochner (or rough) Laplacian, given by \( \Delta_B = \nabla^* \nabla \), where \( \nabla^* \) is the formal adjoint of \( \nabla \). The other possibility is to take \( L \) to be the Hodge (or damped) Laplacian (cf. [2]), given by \( \Delta_H = dd^* + d^*d \), where \( d \) and \( d^* \) are the exterior derivative and its formal adjoint.

The operators \( \Delta_H \) and \( \Delta_B \) are related by the Bochner-Weitzenb"ock formula (see [24]) on divergence-free 1-forms \( \omega \), given by

\[
\Delta_B(\omega) = \Delta_H(\omega) - \text{Ric}(\omega, \cdot),
\]

where \( \text{Ric} \) is the Ricci tensor and we identify 1-forms with vector fields via \( g \).

According to Ebin and Marsden [12], when \((M, g)\) is Einstein (i.e., \( \text{Ric} = \lambda g \), for some constant \( \lambda \)), the appropriate operator is \( L = \Delta_B \). Nevertheless, one can also consider the system (1.5) with \( L = \Delta_H \) (referred to in [6] as the modified Navier-Stokes equations), and this was done in a number of works (e.g., [3], [26], [5], [2], [18], [6]).

The case of compact Riemannian manifolds \((M, g)\), with or without boundary, has been studied by several authors (e.g., [9], [18], [13], [12]). For instance, in [12] it was proved that (1.5) is well-posed on Sobolev spaces, in any dimension.

For non-compact Riemannian manifolds, however, the situation is quite different. In sharp contrast with the classical results in \( \mathbb{R}^2 \) mentioned above, Chan and Czubak [6] proved in 2010 that there are nonunique Leray-Hopf solutions of (1.5) on the two-dimensional hyperbolic space \( \mathbb{H}^2(-a^2) \). In [6], the authors use \( L = \Delta_B \) (since \( \mathbb{H}^2(-a^2) \) is Einstein), and also prove a nonuniqueness result where \( L = \Delta_H \) for more general negatively curved two-dimensional manifolds.
The question was raised in [6] as to whether their methods could be extended to three-dimensional manifolds. In the case of $\mathbb{H}^3(-a^2)$, the above question was answered negatively by Khesin and Misiolek [19] in 2012, who showed that the nonuniqueness phenomenon observed in [6] is essentially due to the Hodge-Kodaira decomposition, and that similar techniques cannot produce nonunique solutions on $\mathbb{H}^n(-a^2)$, for $n > 2$. Some of their findings can be summarized in the following theorem, which incorporates an earlier result of Dodziuk [10].

**Theorem 1.1.** Given a Riemannian manifold $(M,g)$, every $L^2$ harmonic vector field that belongs to the Sobolev space $W^{1,4}(\text{TM})$ defines a time-independent solution of the Euler equations (1.6). Furthermore, there is an infinite-dimensional space of such vector fields on the Hyperbolic plane $\mathbb{H}^2(-a^2)$. On the other hand, for $n \geq 3$ any $L^2$ harmonic vector field on $\mathbb{H}^n(-a^2)$ vanishes identically.

The relationship between Theorem 1.1 and the Cauchy problem for the Navier-Stokes equations (1.5) is as follows. Let $v = \nabla \phi$ be a time-independent solution of the Euler equation (1.6), for some function $\phi : M \to \mathbb{R}$. If $v$ decays as in Theorem 1.1, one obtains infinitely many Leray-Hopf solutions $u = f(t)v$ of (1.5), for carefully chosen real-valued functions $f(t)$. This was the basic technique employed in [6] to obtain a nonuniqueness result in $\mathbb{H}^2(-a^2)$. In particular, one sees from Theorem 1.1 that the methods of [6] do not yield nonuniqueness of (1.5) in the case of $\mathbb{H}^3(-a^2)$.

Nevertheless, we show that the approach introduced in [6] can be implemented on certain three-dimensional manifolds of non-constant negative curvature. As a matter of fact, the manifolds introduced by Anderson [1] in his counterexample to the Dodziuk-Singer conjecture can be shown, after some delicate estimates, to carry nontrivial harmonic vector fields with sufficient decay:

**Theorem 1.2.** The Anderson manifold $(M^3,g)$ has the property that the space of $L^2$ harmonic vector fields belonging to the Sobolev space $W^{1,2}(\text{TM}) \cap W^{1,4}(\text{TM})$ is infinite-dimensional.

It is worth pointing out that, in addition to the requirement that the vector fields we construct lie in $W^{1,4}(\text{TM})$, as stated in Theorem 1.1, we also need them to be of Sobolev class $W^{1,2}(\text{TM})$, if we want to guarantee that the solutions are of Leray-Hopf type, due to the energy inequality they must satisfy (1.3). We will carry out both estimates at once.

As an immediate corollary of Theorems 1.1 and 1.2, and the construction in [6] mentioned above, we obtain the following nonuniqueness result for Navier-Stokes on non-compact negatively curved Riemannian manifolds with large isometry groups.

**Theorem 1.3.** The Cauchy problem (1.5), with $L = \Delta_H$, is ill-posed on $(M^3,g)$ in the sense that there are infinitely many smooth initial conditions $u_0$, such that for each of those $u_0$ one has infinitely many smooth Leray-Hopf solutions of (1.5).

The paper is organized as follows. In Section 2, besides reviewing the construction of the manifold $(M^3,g)$, we produce supersolutions that are more refined than the ones used in [1] to create nontrivial $L^2$ harmonic vector fields. This is necessary in order to ensure that the new harmonic vector fields have the desired decay.

The proof of Theorem 1.2 is completed in Section 3, where we carry out the estimates necessary to prove that the vector fields constructed in Section 2 satisfy the hypothesis of Theorem 1.1.
For other results concerning the Navier-Stokes equations on a Riemannian manifold, see [6] and references therein.

**Acknowledgements.** The author wishes to thank Prof. Gerard Misiolek for suggesting the problem and for many inspiring discussions, and Prof. Frederico Xavier for his interest and helpful comments on this work. This research was supported by the Richard and Peggy Notebaert Fellowship.

2. The construction of \((M^3, g)\)

Throughout this paper, whenever \(E \to M\) is a vector bundle equipped with a Riemannian metric, we denote by \(L^p(E)\) the space of \(p\)-integrable sections of \(E\) and by \(W^{s,p}(E)\) the \(L^p\)-based Sobolev space of sections of \(E\), with smoothness index \(s\). See [24] for more details.

In this section, we recall Anderson’s construction of a complete Riemannian three-dimensional manifold \((M^3, g)\) and perform the modifications needed in our case. The latter are necessary in order to ensure that the constructed harmonic vector fields obey appropriate decay conditions at infinity. Our basic reference for this section is [1].

Consider the set
\[
H = \{(x, y) \in \mathbb{C} : x^2 + y^2 < 1, \ x > 0\}.
\]

(2.1)

Fix a number \(a > 1\) and equip \(H\) with the hyperbolic metric \(ds_a^2\) of sectional curvature \(-a^2\). In cartesian coordinates \(z = x + iy\), it takes the form
\[
ds_a^2 = 4 \frac{dx^2 + dy^2}{a^2(1 - |z|^2)^2}.
\]

(2.2)

Consider a warped product \(H \times_f S^1\), where \(S^1\) is the circle and \(f : H \to \mathbb{R}\) is a smooth function to be defined. Let \(L\) be the line segment from \((0, -1)\) to \((0, 1)\) and let \(\rho_a(z)\) be the distance, with respect to \(ds_a^2\), from the point \(z \in H\) to \(L\). More explicitly, we have (see [4, p. 162])
\[
\rho_a(z) = \frac{1}{a} \text{arcsinh} \left( \frac{2x}{1 - |z|^2} \right).
\]

(2.3)

Define \(f\) by
\[
f(z) = \sinh \circ (\rho_a(z)).
\]

(2.4)

The explicit formula for \(f\) will be used later on, but for now we only remark that from the above expression it follows that \(f\) extends smoothly to \(H \cup L\), and that the corresponding warped metric extends to \(M = H \times_f S^1\), which is topologically just \(\mathbb{R}^3\) (for the metric, see [25] p. 12)]. Using cartesian coordinates \(z = x + iy\) on \(H\) and the standard angular coordinate \(\theta\) on \(S^1\), the metric on \(M\) reads
\[
ds_M^2 = 4 \frac{dx^2 + dy^2}{a^2(1 - |z|^2)^2} + f^2 d\theta^2.
\]

(2.5)

To obtain a large space of harmonic 1-forms on \(M\), we first decompose the Hodge Laplacian \(\Delta_M\) acting on 1-forms on \(M\) as
\[
\Delta_M = \Delta_{\mathbb{R}^2(-a^2)} - d \circ i_F + i_F \circ d,
\]

(2.6)
where $F = \nabla \log(f)$, $i_F$ denotes interior multiplication by $F$ and $\nabla$ is the gradient of the Riemannian metric on $M$. Since $f = 0$ on $L$, the above formula only makes sense on $H \times F S^1$. However, if we require that solutions of the equation

\[(2.7) \quad \Delta_{H^2(a^2)}(\omega) - d(\omega(F)) + (d\omega(F, \cdot)) = 0,\]

satisfy a suitable Neumann condition at $L$, then they can be extended smoothly to $M$, and the extension will be harmonic. First, we require $\omega$ to be invariant under isometric reflection through $L$. Second, we require that $\omega$ be invariant under the isometric action of the $S^1$-factor. This reduces the problem to solving equation $(2.7)$ on $H$.

**Remark.** From this point on, we specialize to the case $a = 2$ for simplicity. Other values of $a > 1$ would work as well, with minor modifications.

Assume now that $\omega = du$. Let $\Delta = \Delta_{H^2(-4)}$. Then $(2.7)$ becomes

\[(2.8) \quad d(\Delta u - du(F)) = 0,\]

so that it is enough to solve

\[(2.9) \quad \Delta u - du(F) = 0\]

on $H$. We would like to rewrite this equation on the strip

\[(2.10) \quad \Omega = \{(r, s) \in \mathbb{C} : 0 < s < \pi/2\},\]

where it takes a simpler form. Thus, we introduce the biholomorphisms $\varphi : \Omega \to H$ and $\varphi^{-1} = \psi : H \to \Omega$ given by

\[(2.11) \quad \varphi(w) = ie^{-w} - 1, \quad \psi(z) = \log \left( \frac{i - z}{i + z} \right).\]

Note that $\psi(L) = \mathbb{R} \times \{0\}$, and that the image of the full unit disk under $\psi$ is

\[(2.12) \quad \tilde{\Omega} = \{(r, s) : -\pi/2 < s < \pi/2\}.\]

Letting $v = u \circ \varphi$, $(2.9)$ can be rewritten on $\Omega$ as

\[(2.13) \quad L_{\Omega}(v) = \frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial s^2} + \frac{\partial}{\partial s} \left( \log(f) \right) = 0,\]

where $\tilde{f} = f \circ \varphi$. This means that $u$ is a solution of $(2.9)$ if and only if $v$ is a solution of $(2.13)$. A long but straightforward computation using $(2.4)$ gives

\[(2.14) \quad \tilde{f}(r, s) = \frac{1}{2} \frac{(1 + \sin(s))^{1/2} - (1 - \sin(s))^{1/2}}{\cos(s)^{1/2}}.\]

Observe that $\tilde{f}$ depends only on one variable. Each solution of $(2.13)$ gives a harmonic 1-form on $H$. Consider the mixed boundary value problem

\[(2.15) \quad \begin{cases} L_{\Omega}(v) = 0, \\ \frac{\partial v}{\partial s}(r, 0) = 0, \\ v(r, \pi/2) = v_0(r), \end{cases}\]
where $v_0 : \mathbb{R} \to \mathbb{R}$ is a smooth function with uniformly bounded derivatives of all orders. It was shown in [1] that (2.15) has a unique solution $v$ which can be extended by reflection to

$$\tilde{\Omega} = \{(r, s) : -\pi/2 < s < \pi/2\}.$$  
(2.16)

This solution is smooth in $\tilde{\Omega}$ and continuous up to the boundary

$$\partial \tilde{\Omega} = \{(r, s) : s = \pi/2\} \cup \{(r, s) : s = -\pi/2\}.$$  
(2.17)

In particular, by construction, $v = v_0$ on the boundary of $\tilde{\Omega}$. By the maximum principle applied to the operator $L_{\Omega}$ (see [13, p. 35]), constructing appropriate supersolutions of (2.15) yields bounds on $v$ and its first and second derivatives. More precisely, if $v^+$ is a positive supersolution of (2.15) and $v$ is chosen such that $v(r, \pi/2) \leq v^+(r, \pi/2)$, then

$$|v| + |v_r| + |v_s| + |v_{rr}| + |v_{rs}| + |v_{ss}| \leq \text{const. } v^+.$$  
(2.18)

holds pointwise. It should be remarked that (2.18) relies on estimates obtained from the method used to prove the existence of solutions to (2.15) in [1], rather than a general argument.

Let

$$v^+(r, s) = e^{-\delta|r|} p(s),$$  
(2.19)

where $p : [0, \pi/2] \to (0, +\infty)$ is a smooth function and $\delta > 0$. We can see directly from (2.13) that $v^+$ is a positive supersolution of (2.15) (with initial condition $v_0^+(r) = e^{-\delta|r|} p(\pi/2)$) if and only if

$$\begin{cases} p''(s) + \left(\frac{\bar{f}}{f}\right) p'(s) + \delta^2 p(s) \leq 0, \\ p'(0) = 0. \end{cases}$$  
(2.20)

In [1], the author uses $p(s) = -c_1 s^2 + c_2$, for carefully chosen $c_1, c_2$ and $\delta > 0$ sufficiently small. In that case, for optimal $c_1$ and $c_2$, the number $\delta$ has to be less than $\sqrt{6}/\pi \approx 0.78$. However, our construction requires that $\delta$ be at least 1. For this, we need the following

**Lemma 2.1.** The function

$$v^+(r, s) = e^{-|r|} (8s^4 - 50s^2 + 75),$$  
(2.21)

is a supersolution of (2.15).

**Proof.** We must check that $p(s) = 8s^4 - 50s^2 + 75$ is positive on $[0, \pi/2]$ and that it satisfies (2.20) with $\delta = 1$. Positivity follows from the fact that the roots of $p(s)$ are $\pm \sqrt{5}/2, \pm \sqrt{15}/2$, and $p(0) > 0$. The inequality in (2.20) becomes

$$\begin{cases} 8s^4 + 46s^2 - 25 + G(s)(32s^3 - 100s) \leq 0, \\ G(s) = \bar{f}/f. \end{cases}$$  
(2.22)

It is clear that $G(s) > 0$ on $(0, \pi/2)$ and $G(s) \to +\infty$ as $s \to 0$ or $s \to \pi/2$, so that $G$ is bounded from below on $(0, \pi/2)$. Also, $p_1(s) = 32s^3 - 100s$ is negative on $(0, \pi/2)$. We break (2.22) into two cases:

$$\begin{cases} 0 < s \leq \sqrt{2}/2 \\ \sqrt{2}/2 < s < \pi/2 \end{cases}$$
In the first case, \( p_2(s) = 8s^4 + 46s^2 - 25 \) is negative, so (2.22) obviously holds. In the second case, we have \( \frac{1}{2} < \sin(s) \leq 1 \), and this gives
\[
\frac{(1 + \sin(s))^{1/2} + (1 - \sin(s))^{1/2}}{(1 + \sin(s))^{1/2} - (1 - \sin(s))^{1/2}} \geq \frac{\sqrt{3}}{\sqrt{2} - \sqrt{2}} \geq 4,
\]
so that
\[
G(s) \geq \frac{2}{\cos(s)}.
\]
It is then easy to check that (2.22) holds on \( (\sqrt{2}/2, \pi/2) \), since \( G(s)(32s^3 - 100s) \to -\infty \) as \( s \to \pi/2 \). □

We will only work with solutions \( v \) of (2.15) obeying (2.18). Lemma 2.1 implies the pointwise estimate
\[
|v| + |v_r| + |v_s| + |v_{rr}| + |v_{rs}| + |v_{ss}| \leq \text{const.} e^{-|r|}.
\]
We finish this section with two Lemmas. The last one is the main ingredient needed in the next section to establish that \( du \in L^2(T^*M) \cap W^{1,4}(T^*M) \).

Lemma 2.2. Let \( z \in H \) and \( \psi(z) = \psi_1(z) + i\psi_2(z) \), where \( \psi_1, \psi_2 \) are real-valued. The functions
\[
f_1(z) = (1 - |z|^2)|\psi'(z)| \quad \text{and} \quad f_2(z) = e^{-|\psi_1(z)|}|\psi'(z)|
\]
are bounded on \( H \).

Proof. That \( f_1 \) is bounded follows directly from the triangle inequality. As for \( f_2 \), note that \( |\psi'(z)| \) has only two singularities, located at \( \pm i \). We analyze the one at \( z = i \), the other one being completely analogous. We have
\[
e^{-2|\psi_1(z)||\psi'(z)|^2} \leq \text{const.} \frac{x^2 + (1 - y)^2}{1 - x^2 + y^2 + 4x^2y^2},
\]
and switching to polar coordinates \( (r, \theta) \),
\[
e^{-2|\psi_1(z)||\psi'(z)|^2} \leq \text{const.} \frac{1 - 2r \sin(\theta) + r^2}{1 + 2r \cos(2\theta) + r^4} \leq \text{const.} \frac{1}{(1 + r)^2},
\]
since \( \theta \approx \pi/2 \). □

Lemma 2.3. For \( u = v \circ \psi \), where \( v \) is a solution of (2.15) satisfying (2.25), we have the pointwise estimates
\[
|u_x| + |u_y| \leq \text{const.}
\]
and
\[
|u_{xx}| + |u_{xy}| + |u_{yy}| \leq \text{const.} |\psi'|.
\]
Proof. From Lemma 2.2 and (2.25), we have
\[
|u_x| + |u_y| \leq \text{const.} e^{-|\psi_1||\psi'|} \leq \text{const.}
\]
For the second derivatives of \(u\), using the Cauchy-Riemann equations and the bound \(|\psi''(z)| \leq |\psi'(z)|^2\),

\[
|u_{xx}| = |v_{rr}(\psi_1)^2 + 2v_{rs}(\psi_1_x(\psi_2)_x + v_{ss}(\psi_2)^2 + v_r(\psi_1)_{xx} + v_s(\psi_2)_{xx}|
\leq \text{const.} \ e^{-|\psi_1|} \left( |\psi_1|^2 + 2|\psi_1\psi_2|^2 + (\psi_2)^2 + |\psi_1| + |\psi_2|^2 \right)
\leq \text{const.} \ e^{-|\psi_1|} \left( |\psi'|^2 + |\psi''| \right)
\leq \text{const.} \ |\psi'|^2.
\]

(2.32)

The same estimate holds for \(|u_{xy}|\) and \(|u_{yy}|\).

\[\square\]

3. Sobolev estimates

In this section, we show that the \(L^2\), \(W^{1,2}\) and \(W^{1,4}\) norms of \(du\) are finite. That the \(W^{1,2}\) norm of \(du\) is finite will become clear after we compute the other two, so we leave that to the end.

Recall that, up to a set of measure zero, \(M = H \times_f S^1\). Since \(S^1\) is compact and the integrands will not depend on \(\theta\), we can reduce all computations to \(H\).

Denote by \(\|\|_g\) the norm with respect to the metric \((g_{ij})\) of \(M\), so that from (2.34) we have

\[
\|dx\|_g = \|dy\|_g = (1 - |z|^2) \quad \text{and} \quad \|d\theta\|_g = 1/f,
\]

where \(z = x + iy\) as before. Then, using (2.29),

\[
(3.1) \quad \int_H \|du\|^2_\gamma f \frac{dx\ dy}{(1 - |z|^2)^2} = \int_H (u_x^2 + u_y^2) f \ dx\ dy \leq \text{const.} \ \int_H f \ dx\ dy.
\]

\(L^2\) estimate. We show explicitly that the last integral in (3.1) is finite. Expanding (2.29) gives

\[
f(x, y) = -\frac{1}{2} \left( \frac{2x}{1 - |z|^2} + \sqrt{1 + \frac{4x^2}{(1 - |z|^2)^2}} \right)^{-1/2}
+ \frac{1}{2} \left( \frac{2x}{1 - |z|^2} + \sqrt{1 + \frac{4x^2}{(1 - |z|^2)^2}} \right)^{1/2}.
\]

The first term above is bounded by 2 in absolute value, since

\[
(3.2) \quad \frac{2x}{1 - |z|^2} + \sqrt{1 + \frac{4x^2}{(1 - |z|^2)^2}} \geq 1.
\]

As for the second term, note that

\[
(3.3) \quad \left( \frac{2x}{1 - |z|^2} + \sqrt{1 + \frac{4x^2}{(1 - |z|^2)^2}} \right)^{1/2} \leq \left( \frac{4x}{1 - |z|^2} + 1 \right)^{1/2}
\leq \sqrt{\frac{4x}{1 - |z|^2} + 1}.
\]
Using polar coordinates,

\[ \int_{\mathcal{H}} f \, dx \, dy \leq \text{const.} + \text{const.} \int_{-\pi/2}^{\pi/2} \int_{0}^{1} \frac{2r}{\sqrt{1-r^2}} \, dr \, d\theta < \infty. \] (3.4)

Therefore, we have \( du \in L^2(T^* M) \).

\( W^{1,4} \) estimate. From (3.4) and the fact that

\[ \| du \|_4^4 = (1 - |z|^2)^4(u_x^2 + u_y^2)^2 \leq \text{const}. \ (1 - |z|^2)^2, \] (3.5)

we see immediately that \( du \in L^4(T^* M) \). It remains to show that \( \nabla du \in L^4(T^* M \otimes T^* M) \), where \( \nabla \) denotes the Levi-Civita connection acting on 1-forms. First, we compute and find an upper bound for

\[ \| \nabla du \|_g^2 = \| u_{xx} dx \otimes dx + u_{xy} dx \otimes dy + u_{xy} dy \otimes dx + u_{yy} dy \otimes dy \|_g^2 \]

\[ + 2 \langle u_{xx} dx \otimes dx + u_{xy} dx \otimes dy + u_{xy} dy \otimes dx + u_{yy} dy \otimes dy, u_x \nabla dx + u_y \nabla dy \rangle \]

\[ + \| u_x \nabla dx + u_y \nabla dy \|_g^2 \]

and then estimate each one separately.

Let us start from the last term containing only the lower order derivatives of \( u \) appearing in \( \nabla du \). Recall that, using Einstein’s summation convention, we have

\[ \nabla dx^k = -\Gamma^k_{ij} dx^i \otimes dx^j, \]

where \( \Gamma^k_{ij} \) are the Christoffel symbols, given in terms of the metric \( (g_{ij}) \) of \( M \) by the well-known formula

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right). \]

From this we compute

\[ \nabla dx = \frac{2x}{1 - |z|^2} \, dx \otimes dx + \frac{2y}{1 - |z|^2} \, dx \otimes dy + \frac{2y}{1 - |z|^2} \, dy \otimes dx \]

\[ + \frac{-2x}{1 - |z|^2} \, dy \otimes dy - (1 - |z|^2)^2 f_x d\theta \otimes d\theta, \]

(3.6)

\[ \nabla dy = \frac{-2y}{1 - |z|^2} \, dx \otimes dx + \frac{2x}{1 - |z|^2} \, dx \otimes dy + \frac{2x}{1 - |z|^2} \, dy \otimes dx \]

\[ + \frac{2y}{1 - |z|^2} \, dy \otimes dy - (1 - |z|^2)^2 f_y d\theta \otimes d\theta. \]

To obtain an upper bound for \( \| u_x \nabla dx + u_y \nabla dy \|_g^2 \), we split

\[ \| u_x \nabla dx + u_y \nabla dy \|_g^2 = \| u_x \nabla dx \|_g^2 + \| u_y \nabla dy \|_g^2 + 2 \langle u_x \nabla dx, u_y \nabla dy \rangle \]
Going back to (3.11) for the term containing only second order derivatives, using Lemma 2

This takes care of the term containing only first order derivatives of coordinates, which is the same bound as before. Finally, for the mixed term that contains first and second derivatives of u in (\nabla du, \nabla du), we again use (2.29), (2.30) and Lemma
\[2\left( u_{xx} dx \otimes dx + u_{xy} dx \otimes dy + u_{xy} dy \otimes dx + u_{yy} dy \otimes dy, u_x \nabla dx + u_y \nabla dy \right) \leq 2(1 - |z|^2)^3 \left[ |u_x| \left( 2|u_{xx}| + 4|u_{xy}| + 2|u_{yy}| \right) + |u_y| \left( 2|u_{xx}| + 4|u_{xy}| + 2|u_{yy}| \right) \right] \]
\[\leq \text{const.} (1 - |z|^2)^3 \left[ |u_x||\psi'(z)| + |u_y||\psi'(z)| \right] \]
\[\leq \text{const.} (1 - |z|^2)^2. \]

Combining all the above estimates and squaring leads to
\[\|\nabla du\|_g^4 \leq \text{const.} (1 - |z|^2)^4, \]
so that
\[\int_H \|\nabla du\|_g^4 f \frac{dx \ dy}{(1 - |z|^2)^2} \leq \text{const.} \int_H f \ dx \ dy \]
and this integral is finite, as in (3.14). This shows that \(du \in L^2(T^*M) \cap W^{1,4}(T^*M)\).
Moreover, one sees directly from the estimate above that
\[\int_H \|\nabla du\|_g^2 f \frac{dx \ dy}{(1 - |z|^2)^2} < \infty, \]
so that finite dissipation, \(du \in W^{1,2}(T^*M)\), is also satisfied.

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