Distributed CONGEST Approximation of Weighted Vertex Covers and Matchings

Salwa Faour, Marc Fuchs, and Fabian Kuhn
University of Freiburg, Germany

Abstract

We provide CONGEST model algorithms for approximating the minimum weighted vertex cover and the maximum weighted matching problem. For bipartite graphs, we show that a \((1 + \varepsilon)\)-approximate weighted vertex cover can be computed deterministically in \(\text{poly}\left(\frac{\log n}{\varepsilon}\right)\) rounds. This generalizes a corresponding result for the unweighted vertex cover problem shown in [Faour, Kuhn; OPODIS ’20]. Moreover, we show that in general weighted graph families that are closed under taking subgraphs and in which we can compute an independent set of weight at least \(\lambda \cdot w(V)\) (where \(w(V)\) denotes the total weight of all nodes) in polylogarithmic time in the CONGEST model, one can compute a \((2 - 2\lambda + \varepsilon)\)-approximate weighted vertex cover in \(\text{poly}\left(\frac{\log n}{\varepsilon}\right)\) rounds in the CONGEST model. Our result in particular implies that in graphs of arboricity \(a\), one can compute a \((2 - 1/a + \varepsilon)\)-approximate weighted vertex cover problem in \(\text{poly}\left(\frac{\log n}{\varepsilon}\right)\) rounds in the CONGEST model.

For maximum weighted matchings, we show that a \((1 - \varepsilon)\)-approximate solution can be computed deterministically in time \(2^{O(1/\varepsilon)} \cdot \text{poly log } n\) in the CONGEST model. We also provide a randomized algorithm that with arbitrarily good constant probability succeeds in computing a \((1 - \varepsilon)\)-approximate weighted matching in time \(2^{O(1/\varepsilon)} \cdot \text{poly log}(\Delta W) \cdot \text{log}^* n\), where \(W\) denotes the ratio between the largest and the smallest edge weight. Our algorithm generalizes results of [Lotker, Patt-Shamir, Pettie; SPAA ’08] and [Bar-Yehuda, Hillel, Ghaffari, Schwartzman; PODC ’17], who gave \(2^{O(1/\varepsilon)} \cdot \text{log } n\) and \(2^{O(1/\varepsilon)} \cdot \frac{\text{log } \Delta}{\text{log log } \Delta}\)-round randomized approximations for the unweighted matching problem.

Finally, we show that even in the LOCAL model and in bipartite graphs of degree \(\leq 3\), if \(\varepsilon < \varepsilon_0\) for some constant \(\varepsilon_0 > 0\), then computing a \((1 + \varepsilon)\)-approximation for the unweighted minimum vertex cover problem requires \(\Omega\left(\frac{\log n}{\varepsilon}\right)\) rounds. This generalizes a result of [Göös, Suomela; DISC ’12], who showed that computing a \((1 + \varepsilon_0)\)-approximation in such graphs requires \(\Omega(\log n)\) rounds.

1 Introduction and Related Work

Maximum matching (MM) and minimum vertex cover (MVC) are two classic optimization problems that have been studied intensively in the context of distributed graph algorithms (e.g., [AKO18, AK20, ABB+19, AFP+09, AS10, BEKS18, BCS16, BCGS17, BCM+20, BEKS19, CH03, CHS04, EMR15, FK20, Fis17, GJN20, GS14, GKP08, KP08, Har20, HKL06, II86, KY11, KMW04, KMW06, LPR09, LPP15, Nie08, WW04]). The problems are closely related to each other: the fractional relaxations

*salwa.faour@cs.uni-freiburg.de
†marc.fuchs@cs.uni-freiburg.de
‡kuhn@cs.uni-freiburg.de
of the unweighted variants of the problem are linear programming (LP) duals of each other. The problems are however also fundamentally different. While a maximum (weighted) matching can be found in polynomial time in all graphs [Edm65a, Edm65b], for the minimum (weighted) vertex cover problem, this is only true for bipartite graphs [Ege31, K31]. In general graphs, even for the unweighted MVC problem, the best polynomial-time approximation algorithms have an approximation ratio of $2 - o(1)$ [Kar09]. The MVC problem is known to be APX-hard [DS05, Hás01], and if the unique games conjecture holds, the current $(2 - o(1))$-approximation algorithms are essentially best possible [KR08].

In the distributed context, most prominently, the problems have been studied in the standard message passing models in graphs, in the LOCAL model and the CONGEST model [Pel00]. In both models, the graph $G = (V, E)$ on which we want to solve some graph problem also represents the network and it is assumed that the nodes $V$ of $G$ can communicate with each other in synchronous rounds by exchanging messages over the edges $E$ of $G$. In the LOCAL model, the size of those messages is not restricted, whereas in the CONGEST model, it is assumed that each message has to consist of at most $O(\log n)$ bits, where $n = |V|$ is the number of nodes of the network graph $G$. In the following discussion of existing work on distributed matching and vertex cover algorithms, we concentrate on polylogarithmic-time distributed algorithms that also work for the weighted variants of the problems.

**Distributed Complexity of Weighted Matchings.** While the unweighted versions of both problems can be approximated within a factor of 2 by computing a maximal matching, a little more work is needed for weighted matchings and vertex covers. The first polylogarithmic-time distributed algorithm for computing a constant approximation for the maximum weighted matching (MWM) problem was presented in [WW04]. This algorithm was then improved in [LPR09] and in [LPP15], where it is shown that a $(1/2 - \varepsilon)$-approximation for MWM can be computed in $O(\log(1/\varepsilon) \log n)$ rounds in the CONGEST model. In [BCGS17], it was further shown that one can compute a $1/2$-approximation for MWM in time $O(\log W \cdot T_{\text{MIS}})$ in the CONGEST model, where $W$ is the ratio between the largest and smallest edge weight and where $T_{\text{MIS}}$ is the time for computing a maximal independent set. The paper also shows that for constant $\varepsilon$, a $(1/2 - \varepsilon)$-approximation can be computed in only $O\left(\frac{\log \Delta}{\log \log \Delta}\right)$ rounds. Note that as shown in [KMW06], this time complexity is best possible for any constant approximation algorithm, even in the LOCAL model. All the above algorithms are randomized. In [Fis17], Fischer gave a deterministic CONGEST algorithm to compute a $(1/2 - \varepsilon)$-approximate weighted matching with a round complexity of $O(\log^2 \Delta \cdot \log 1/\varepsilon + \log^* n)$. This algorithm was refined in [AKO18], where it was shown that in time $O\left(\frac{\log^2 \Delta}{\varepsilon} + \log^* n + \frac{\log \Delta W}{\varepsilon}\right)$, it is even possible to deterministically compute a $(2/3 - \varepsilon)$-approximation for the MWM problem in general graphs and a $(1 - \varepsilon)$-approximation for the MWM problem in bipartite graphs, in the CONGEST model. To the best of our knowledge, this is the only existing polylogarithmic-time CONGEST algorithm to obtain an approximation ratio that is better than 1/2. It has been observed already in [KY11, LPP15, Nie08] that in the LOCAL model, better approximations for maximum weighted matching can be computed efficiently. In particular, [LPP15, Nie08] show that even in general graphs, a $(1 - \varepsilon)$-approximation can be computed in poly $\left(\frac{\log \Delta}{\varepsilon}\right)$ rounds. It has later been shown that this can also be achieved deterministically [GKM18]. The best known LOCAL MWM approximation algorithms are by Harris [Har20], who shows that a $(1 - \varepsilon)$-approximation can be computed in randomized time $\tilde{O}\left(\frac{\log \Delta}{\varepsilon^2}\right) + \text{poly log } \left(\frac{\log \log n}{\varepsilon}\right)$ and in deterministic time $\tilde{O}\left(\frac{\log^2 \Delta}{\varepsilon} + \frac{\log^* n}{\varepsilon}\right)$. Those algorithms are based on computing large matchings in hypergraphs defined by paths of length $O(1/\varepsilon)$ and they unfortunately cannot directly turned into efficient CONGEST algorithms. To the best of our knowledge, even constant $\varepsilon > 0$, the only efficient CONGEST algorithms are
for the unweighted maximum matching problem. Lotker, Patt-Shamir, and Pettie [LPP15] give an algorithm to compute a \((1 - \varepsilon)\)-approximation for the unweighted maximum matching problem in time only \(2^{O(1/\varepsilon)} \cdot \log n\) in the randomized CONGEST model. In [BCGS17] (full version), this was even improved to \(2^{O(1/\varepsilon)} \cdot \frac{\log \Delta}{\log \log \Delta}\). As one of our main contributions, we obtain similar algorithms for the weighted matching problem. Obtaining a \((1 - \varepsilon)\)-approximation in poly \((\log n \varepsilon^{-1})\) CONGEST rounds is one of the key open questions in understanding the distributed complexity of maximum matching. Fischer, Mitrović, and Uitto [FMU21] recently settled a related problem for unweighted matchings in the streaming model and in the latest version of their paper, they even obtain a poly \((\log n \varepsilon^{-1})\)-round CONGEST algorithm for the unweighted matching problem.

**Distributed Complexity of Weighted Vertex Covers.** The first distributed constant-factor approximation algorithm for the minimum weighted vertex cover (MWVC) problem is due to Khuller, Vishkin, and Young [KVV94]. They describe a simple deterministic algorithm to obtain a \((2 + \varepsilon)\)-approximation for MWVC. The algorithm can directly be implemented in \(O(\log(n \cdot \log(1/\varepsilon)))\) rounds in the CONGEST model. The time for computing a \((2 + \varepsilon)\)-approximation has subsequently been improved to \(O(\log(\Delta)/\text{poly}(\varepsilon))\) in [KMW06] and to \(O(\log \Delta/\log \log \Delta)\) in [BCS16, BEKS18, BEKS19] (with a very minor dependency on \(\varepsilon\) in [BEKS19]). Note that as for maximum matching, this dependency on \(\Delta\) is optimal for any constant-factor approximations [KMW04]. The algorithm of [BEKS19] can also be used to compute a 2-approximate weighted vertex cover in time \(O(\log n)\). Other polylogarithmic-time algorithms to compute 2-approximations for MWVC appeared in [GKPS08, KVV94, KY11]. In the LOCAL model, one can use generic techniques from [GKM17, RG20] (or the techniques from this paper) to deterministically compute a \((1 + \varepsilon)\)-approximate minimum weighted vertex cover in time poly \((\log n \varepsilon^{-1})\). Further, in [GS14], it was shown that even on bipartite graphs with maximum degree 3, there exists a constant \(\varepsilon_0 > 0\) such that computing a \((1 + \varepsilon_0)\)-approximate (unweighted) vertex cover requires \(\Omega(\log n)\) rounds, even in the LOCAL model and even when using randomization. We generalize this result and show that for computing a \((1 + \varepsilon)\)-approximation, one requires \(\Omega(\log(n)/\varepsilon)\) rounds. While for maximum matching, there are several CONGEST algorithms that achieve approximation ratios that are better than 1/2, for the minimum vertex cover problem, efficiently achieving an approximation ratio significantly below 2 in general graphs might be a hard problem. In this case, computing an exact solution even has a lower bound of \(\tilde{\Omega}(n^2)\) rounds in the CONGEST model and it is therefore basically as hard as any graph problem can be in this model [CHKP17]. To what extent we can achieve approximation ratios below 2 in the CONGEST model for variants of the minimum vertex cover problem is an interesting open question. There recently has been some progress. In [BbKS19], it is shown that the minimum vertex cover problem (and also the maximum matching problem) can be solved more efficiently if the optimal solution is small. In particular, if the size of an optimal vertex cover is at most \(k\), a minimum vertex cover can be computed deterministically in time \(O(k^2)\) and a \((2 - \varepsilon)\)-approximate solution can be computed deterministically in time \(O(k + (\varepsilon k)^2)\) (and slightly more efficiently with randomization). This was the first efficient CONGEST algorithm that achieves an approximation ratio below 2 for the minimum vertex cover problem for some graphs. In [FK20], it was shown that in bipartite graphs, a \((1 + \varepsilon)\)-approximation can be computed in time poly \((\log n/\varepsilon)\). One of the main results of this paper is a generalization of this result to the weighted vertex cover problem. Further, it has recently been shown that on the square graph \(G^2\), it is possible to compute a \((1 + \varepsilon)\)-approximate (unweighted) vertex cover in time \(O(n/\varepsilon)\) in the CONGEST model (on \(G\)) [BCM+20].
1.1 Our Contributions

We next state our main contributions in detail. We prove new CONGEST upper bounds for approximating minimum weighted vertex cover and maximum weighted matching (MWM). We start by describing our results for the vertex cover problem. In [FK20], it was shown that in bipartite graphs, the unweighted vertex cover problem can be \((1 + \varepsilon)\)-approximated in \(\text{poly} \log (\log n)\) time in the CONGEST model. The following theorem is a generalization of the result of [FK20] to weighted graphs.

**Theorem 1.1.** For every \(\varepsilon \in (0, 1]\), there is a deterministic CONGEST algorithm to compute a \((1 + \varepsilon)\)-approximation for the minimum weighted vertex cover problem in bipartite graphs in time \(\text{poly} (\varepsilon^{-1})\).

The next theorem shows that in graph families that are closed under taking (induced) subgraphs and in which we can efficiently compute large (or heavy) independent sets, we can efficiently approximate minimum (weighted) vertex cover with an approximation ratio that is better than 2.

**Theorem 1.2.** Let \(\mathcal{G}\) be a family of weighted graphs that is closed under taking induced subgraphs and such that for some \(\lambda \in (0, 1]\) and any \(n\)-node graph \(G = (V, E, w)\) of \(\mathcal{G}\), there is a \(T_\lambda(n)\)-round CONGEST algorithm to compute an independent set \(S\) of weight \(w(S) \geq \lambda w(V)\). Then, there is \(T_\lambda(n) + \text{poly} (\log n/\varepsilon)\)-round CONGEST algorithm to compute a \((2 - 2\lambda + \varepsilon)\)-approximate weighted vertex cover for graphs of \(\mathcal{G}\). If the independent set algorithm is deterministic, then also the vertex cover algorithm is deterministic.

Note that the algorithm of Theorem 1.2 uses the bipartite vertex cover algorithm of Theorem 1.1 as a subroutine. Theorem 1.2 in particular implies that for graphs for which we can compute a coloring with a small number of colors, we can efficiently compute a non-trivial vertex cover approximation.

**Corollary 1.3.** Let \(\mathcal{G}\) be a family of weighted graphs such that for some non-negative integer \(C\), for any \(n\)-node graph \(G = (V, E, w)\) of \(\mathcal{G}\), there is a \(T_C(n)\)-round CONGEST algorithm to compute a vertex coloring of \(G\) with \(C\) colors. Then, there is a \(T_C(n) + \text{poly} (\log n/\varepsilon)\)-round CONGEST algorithm to compute a \((2 - 2/C + \varepsilon)\)-approximation of the minimum weighted vertex cover problem for graphs of \(\mathcal{G}\). If the coloring algorithm is deterministic, then also the vertex cover algorithm is deterministic.

In order to efficiently compute an independent set \(S\) of weight \(w(S) \geq w(V)/C\) from a \(C\)-coloring, we need the graph to be of small diameter. However, by using standard clustering techniques (which we anyways need to apply also for our bipartite vertex cover algorithm), one can reduce the minimum (weighted) vertex cover problem on general \(n\)-node graphs to graphs of diameter \(\text{poly} (\log n)\). In particular, in graphs of arboricity \(a\), we can (deterministically) compute a \((2 + \varepsilon)a\)-coloring in time \(O(\log^3 a \cdot \log n)\) [GK21]. As a consequence, we get a deterministic \(\text{poly} (\log n/\varepsilon)\)-round CONGEST algorithm for computing a \((2 - 1/a + \varepsilon)\)-approximation of minimum weighted vertex cover in graphs of arboricity \(a\).

In addition to our CONGEST algorithms for approximating minimum weighted vertex cover, we also provide new CONGEST algorithms for approximating maximum weighted matching. The following theorem can be seen as a generalization of Theorem 3.15 in [LPP15] and of Theorem B.12 in [BCGS17] (full version).

**Theorem 1.4.** For every \(\varepsilon, \delta \in (0, 1]\), there is a randomized CONGEST algorithm that with probability at least \(1 - \delta\) computes a \((1 - \varepsilon)\)-approximation to the maximum weighted matching problem in \(2^{O(1/\varepsilon)} \cdot (\log(W\Delta) + \log^2 \Delta + \log^* n) \cdot \log^3 (1/\delta)\) rounds. Further, there is a deterministic CONGEST
algorithm to compute a $(1 - \varepsilon)$-approximation for the minimum weighted matching problem in time $2^{O(1/\varepsilon)} \cdot \text{poly log } n$.

Note that except for the $\log^* n$ term, for constant error probability $\delta$, the round complexity of our randomized algorithm is independent of the number of nodes $n$. Moreover, for constant $\varepsilon$ and $\delta$, in bounded-degree graphs with bounded weights, the round complexity of the randomized algorithm is only $O(\log^* n)$. For unweighted matchings, a round complexity that is completely independent of $n$ was obtained by [BCGS17]. Interestingly, Göös and Suomela in [GS14] showed that such a result is not possible for the minimum vertex cover problem, even in the LOCAL model. They show that even for bipartite graphs of maximum degree 3, there exists a constant $\varepsilon_0 > 0$ such that any randomized distributed $(1 + \varepsilon_0)$-approximation algorithm for the (unweighted) minimum vertex cover problem requires $\Omega(\log n)$ rounds. As our last contribution, we generalize the result of [GS14] to computing $(1 + \varepsilon)$-approximate solutions for any sufficiently small $\varepsilon > 0$.

**Theorem 1.5.** There exists a constant $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, any randomized LOCAL model algorithm to compute a $(1 + \varepsilon)$-approximation for the (unweighted) minimum vertex cover problem in bipartite graphs of maximum degree 3 requires $\Omega\left(\frac{\log n}{\varepsilon}\right)$ rounds.

Theorem 1.5 is obtained by a relatively simple reduction to the $(1 + \varepsilon_0)$-approximation lower bound proven in [GS14] for bipartite graphs of maximum degree 3. We note that if we only require the approximation factor to hold in expectation, as discussed at the end of Section 3.1, in the LOCAL model the theorem is tight even for general graphs and even for the weighted vertex cover problem.

**Organization of the paper:** The remainder of the paper is organized as follows. In Section 2, we define the communication model and we introduce all the necessary mathematical notations and definitions. In Section 3, we give an overview over all our algorithms and our most important ideas and techniques. In Section 4, we provide the additional technical details needed for the weighted vertex cover algorithms, Section 5 is devoted to the details of the maximum weighted matching algorithms, and in Section 6, we formally prove the lower bound on approximating vertex cover in bipartite graphs in the LOCAL model. Finally, in Appendix A, we describe some basic algorithmic tools that we need for our algorithms and which already appear in the literature in a very similar form.

## 2 Model and Preliminaries

### 2.1 Mathematical Notation

Let $G = (V, E, w)$ be an undirected weighted graph, where $w$ is a non-negative weight function. We will use node and edge weights in the paper and depending on the context, we will use $w$ to assigns weights to nodes and/or edges. Generally for a set $X$ of nodes and/or edges, we use $w(X)$ to denote the sum of the weights of all nodes/edges in $X$. For example, if we have node weights, $w(V)$ denotes the sum of the weights of all the nodes. Throughout the paper, we assume that all weights are integers that are polynomially bounded in the number of node of the graph. However, as long as we can communicate a single weight in a single message, all our algorithms can be adapted to also work at no significant additional asymptotic cost for more general weight assignments. We further use the following notation for graphs. For a node $v \in V$, we use $N(v) \subseteq V$ to denote the set of neighbors of $v$ and we use $E(v) \subseteq E$ to denote the set of edges that are incident to $v$. 

5
For a graph \( G = (V, E) \), the bipartite double cover is defined as the graph \( G_2 := G \times K_2 = (V \times \{0, 1\}, E_2) \), where there is an edge between two nodes \((u, i)\) and \((v, j)\) in \( E_2 \) if and only if \( \{u, v\} \in E \) and \( i \neq j \). Hence, in \( G_2 \), every node \( u \) of \( G \) is replaced by two nodes \((u, 0)\) and \((u, 1)\) and every edge \( \{u, v\} \) of \( G \) is replaced by the two edges \( \{(u, 0), (v, 1)\} \) and \( \{(u, 1), (v, 0)\} \). Moreover, if \( G \) is a weighted graph with weight function \( w \), we assume that the bipartite double cover \( G_2 \) is also weighted and that the corresponding nodes and/or edges have the same weight as in \( G \). That is, in case of node weights, for every \( u \in V \), we define \( w((u, 0)) = w((u, 1)) = w(u) \) and in case of edge weights, for every \( \{u, v\} \in E \), we define \( w(\{u, i\}, \{v, 1 - i\}) = w(\{u, v\}) \) for \( i \in \{0, 1\} \).

2.2 Problem Definitions

In this paper, we consider the minimum weighted vertex cover (MWVC) and the maximum weighted matching (MWM) problems. Formally, in the MWVC problem, we are given a weighted graph \( G = (V, E, w) \) with positive node weights. A vertex cover of \( G \) is a set \( S \subseteq V \) of nodes such that for every edge \( \{u, v\} \in E \), \( S \cap \{u, v\} \neq \emptyset \). The goal of the MWVC problem is to find a vertex cover \( S \) of minimum total weight \( w(S) \). In the MWM problem, we are given a weighted graph \( G = (V, E, w) \) with positive edge weights. A matching of \( G \) is a set \( M \subseteq E \) of edges such that no two edges in \( M \) are adjacent. The goal of the MWM problem is to find a matching \( M \) of maximum total weight \( w(M) \). The unweighted versions of the two problems are closely related to each other as their natural fractional linear programming (LP) relaxations are duals of each other. In the paper, we will also use the fractional relaxation of the MWVC problem and its dual problem. In the fractional MWVC problem on \( G \), every node \( u \in V \) is assigned a value \( x_u \in [0, 1] \) such that for every edge \( \{u, v\} \in E \), \( x_u + x_v \geq 1 \) and such that the sum \( \sum_{u \in V} w(u) \cdot x_u \) is minimized. The dual LP of this problem, which for a given weight function \( w \), we in the following call the fractional \( w \)-matching problem, is defined as follows. Every edge \( e \in E \) is assigned a (fractional) value \( y_e \geq 0 \) such that for every node \( u \in V \), we have \( \sum_{e:u \notin e} y_e \leq w(u) \) and such that the sum \( \sum_{e \in E} y_e \) is maximized. We use the vector \( y \) to refer to a fractional solution that assigns a fractional value \( y_e \) to every edge. Further for convenience, for a set of edges \( F \), we also use the short notation \( y(F) := \sum_{e \in F} y_e \). LP duality directly implies that the value of any fractional \( w \)-matching cannot be larger than the weight of any vertex cover:

**Lemma 2.1.** Let \( G = (V, E, w) \) be a node-weighted graph and let \( y \) be a fractional \( w \)-matching of \( G \). It then holds that \( y(E) \leq w(S) \) for every vertex cover \( S \) of \( G \).

**Proof.** We have

\[
w(S) = \sum_{v \in S} w(v) \geq \sum_{v \in S} \sum_{e: v \in e} y_e \geq \sum_{e \in E} y_e = y(E).
\]

The first inequality holds because the values \( y_e \) form a valid fractional \( w \)-matching and the second inequality holds because \( S \) is a vertex cover.

The approximation ratio of an approximation algorithm for the MWVC or MWM problem is defined as the worst-case ratio between the total weight of a vertex cover or matching computed by the algorithm over the total weight of an optimal vertex cover or matching. That is, we define the approximation ratio such that it is \( \geq 1 \) for minimization and \( \leq 1 \) for maximization problems.

2.3 Low-Diameter Clustering

Many of our algorithms have some components that require global communication in the network. In order to achieve a polylogarithmic round complexity, we therefore need a graph with polylogarithmic diameter. We achieve this by applying standard clustering techniques. Formally, we use
the clusterings as in [FK20] described in the following. Let \( G = (V, E, w) \) be a weighted graph with non-negative node and edge weights. A clustering of \( G \) is a collection \( \{S_1, \ldots, S_k\} \) of disjoint node sets \( S_i \subseteq V \). For \( \lambda \in [0, 1] \), a clustering \( \{S_1, \ldots, S_k\} \) is called \( \lambda \)-dense if the total weight of all nodes and edges in the induced subgraphs \( G[S_i] \) for \( i \in \{1, \ldots, k\} \) is at least \( \lambda (w(V) + w(E)) \). Further, for an integer \( h \geq 1 \), a clustering \( \{S_1, \ldots, S_k\} \) is called \( h \)-hop separated if for any two clusters \( S_i \) and \( S_j \) (\( i \neq j \)) and any pair of nodes \( (u, v) \in S_i \times S_j \), we have \( d_G(u, v) \geq h \), where \( d_G(u, v) \) denotes the hop-distance between \( u \) and \( v \). Further for two integers \( c, d \geq 1 \), a clustering \( \{S_1, \ldots, S_k\} \) is defined to be \((c,d)\)-routable if we are given a collection of \( T_1, \ldots, T_k \) trees in \( G \) such that for each \( i \in \{1, \ldots, k\} \), the nodes \( S_i \) are contained in \( T_i \), every tree \( T_i \) has diameter at most \( d \), and every edge \( e \in E \) of \( G \) is contained in at most \( c \) of the trees \( T_1, \ldots, T_k \). Note that this implies that each cluster of a \((c,d)\)-routable clustering has weak diameter at most \( d \) and if the nodes of \( T_i \) are all contained in \( S_i \), it implies that the strong diameter of cluster \( S_i \) is at most \( d \).

### 2.4 Communication Model

Throughout the paper, we assume a standard synchronous message passing model on graphs. That is, the network is modeled as an undirected \( n \)-node graph \( G = (V, E) \). Each node is equipped with a unique \( O(\log n) \)-bit identifier. The nodes \( V \) communicate in synchronous rounds over the edges \( E \) such that in each round, every node can send an arbitrary message to each of its neighbors. Internal computations at the nodes are free. Initially, the nodes do not know anything about the topology of the network. When computing a vertex cover or a matching, at the end of the algorithm, every node must output if it is in the vertex cover or which of its edges belong to the matching. The time complexity of an algorithm is defined as the number of rounds that are needed until all nodes terminate. If the size of the messages is not restricted, this model is known as the \textsc{LOCAL} model [Pel00]. In the more restrictive \textsc{CONGEST} model, all messages must consist of at most \( O(\log n) \) bits [Pel00]. In several of our algorithms, we will first compute a clustering as defined above in Section 2.3 and we afterwards run \textsc{CONGEST} algorithms on the clusters. If we are given a \((c,d)\)-routable clustering, we are only guaranteed that the diameter of each cluster \( S_i \) is small if we add the nodes and edges of the tree \( T_i \) to the cluster. For running our algorithms on individual clusters, we therefore need an extension of the classic \textsc{CONGEST} model, which has been introduced as the \textsc{Supported CONGEST} model in [FKRS19, SS13]. In the \textsc{Supported CONGEST} model, we are given two graphs, a communication graph \( H = (V_H, E_H) \) and a logical graph \( G = (V, E) \), which is a subgraph of \( H \). When solving a graph problem such as MWVC or MWM in the \textsc{Supported CONGEST} model, we need to solve the graph problem on the logical graph \( G \), we can however use \textsc{CONGEST} algorithms on the underlying communication graph \( H \) to do so. Note that if we are given a \((c,d)\)-routable clustering, we can define \( G_i := G[S_i] \) and \( H_i \) as the union of the graph \( G_i \) and the tree \( T_i \) for each cluster and we can then in parallel run 1 round of a \textsc{Supported CONGEST} algorithm on each cluster in \( c \) \textsc{CONGEST} rounds on \( G \).

### 3 Technical Overview

In this section, we provide an overview of the core ideas and techniques for all our results. While we try to provide intuitive arguments for everything, most of the formal proofs appear in Sections 4 to 6. We start by describing how we can use clustering to reduce the problem of approximating MWVC or MWM on general graphs to the case of approximating the same problems on graphs of small diameter.
3.1 Reducing to Small Diameter

The high-level idea that we use to reduce the diameter is a classic one. We find a disjoint and sufficiently separated collection of low-diameter clusters such that only a small fraction of the graph is outside of the clusters (see, e.g., [AP90, LS93, MPX13, Pel00, RG20] for constructions of such clusterings). We then compute a good approximation for a given problem inside each cluster and we use a coarse approximation to extend the solution to the parts of the graph outside of the clusters. If we want this to work in general graphs, we have to adapt the standard clustering constructions such that the part of the graph that is outside of the clusters contains only a small fraction of the number of nodes and/or edges of the graph. A generic way to achieve this in the LOCAL model has been described in [GKM17] and a method that can also be used in the CONGEST model has recently been described in [FK20] for the unweighted minimum vertex cover problem. The following theorem shows how to extend the approach of [FK20] to work also for weighted vertex cover and matching. The theorem shows that at the cost of a $(1 + \varepsilon)$-factor, the problems of approximating MWVC and MWM can efficiently be reduced to approximating the problems in the SUPPORTED CONGEST model with a small-diameter communication graph.

**Theorem 3.1** (Diameter Reduction). Let $T_{SC}^\alpha(n, D)$ be the time required for computing an $\alpha$-approximation for the MWVC or the MWM problem in the SUPPORTED CONGEST model with a communication graph of diameter $D$. Then, for every $\varepsilon \in (0, 1]$, there is a poly $(\log n/\varepsilon) + O(\log n \cdot T_{SC}^\alpha(n, O((\log^3 n/\varepsilon)))$-round CONGEST algorithm to compute a $(1 - \varepsilon)\alpha$-approximation of MWM or an $(1 + \varepsilon)\alpha$-approximation of MWVC in the CONGEST model. If the given SUPPORTED CONGEST model algorithm is deterministic, then the resulting CONGEST model algorithm is also deterministic. Also, if we want to solve MWVC or MWM in the CONGEST model on a bipartite graph, then it is sufficient to have a SUPPORTED CONGEST model algorithm that works for a bipartite communication (and thus also logical) graph.

**Proof.** For both problems, we first compute node or edge weights that we will use to compute a good clustering of the graph. Assume that we have a graph $G = (V, E, w)$, where for the MWVC problem, $w$ is a function that assigns positive integer weights to the nodes and for the MWM problem, $w$ is a function that assigns positive integer weight to the edges.

For the MWVC problem, we first compute a fractional $w$-matching $y$ as follows.\(^1\) We initialize the value of each edge $e = \{u, v\}$ to $y_e := \min\{w(u), w(v)\}/\Delta$. Note that this is a feasible fractional $w$-matching. Now, we improve the fractional $w$-matching as follows in phases. We call a node $v$ half-tight if $\sum_{e \in E(v)} y_e > w(v)/2$. In each phase, we let $E_f$ be the set of edges $\{u, v\}$ for which both $u$ and $v$ are not half-tight and we double the $y_e$-value of all edges in $E_f$. Like this, $y$ remains a feasible fractional $w$-matching and after $O(\log \Delta)$ phases, we obtain a fractional $w$-matching $y$ such that at least one node of every edge is half-tight. Hence, the set $S$ of half-tight nodes is a vertex cover of weight

$$w(S) \leq \sum_{v \in S} w(v) \leq \sum_{v \in S} 2 \sum_{e \in E(v)} y_e \leq 4w(E) \leq 4w(S^*),$$

where $S^*$ is an optimal weighted vertex cover of $G$. The last inequality follows from **Theorem 2.1**. Further, clearly, one phase requires $O(1)$ rounds and we therefore need $O(\log n)$ rounds for computing the fractional $w$-matching $y$.

\(^1\)A similar parallel approximation algorithm for MWVC to the best of our knowledge first appeared in [KVY94].
For the MWM problem, we proceed as follows. The LP dual of the fractional relaxation of the MWM problem asks for assigning a value \( x_v \geq 0 \) to each node such that \( x_u + x_v \geq w(e) \) for every edge \( e = \{u, v\} \) and such that \( \sum_{v \in V} x_v \) is minimized. This is a fractional covering problem and we can for example compute a \( 4 \)-approximation for it by using the CONGEST algorithm of [KMW06]. For the given covering problem, the round complexity of the algorithm of [KMW06] is \( O(\log(\Delta W)) \) if \( W \) is the largest edge weight. Since we assumed that \( W \) is at most polynomial in \( n \), the round complexity for computing the assignment of \( x_v \) values if \( O(\log n) \).

In both cases, we now compute a \((1 - \varepsilon/4)\)-dense, 3-hop separated clustering of \( G \) by using Theorem A.2. In the case of MWVC, we use the fractional \( w \)-matching \( y \) as edge weights for the clustering and we set all the node weights to 0. In the case of MWM, we use the assignment of \( x_v \) values as node weights and we set all the edge weights to 0. By Theorem A.2, the time for computing such a clustering is \( \text{poly}\left(\frac{\log n}{\varepsilon}\right) \). Let \( S_1, \ldots, S_t \) be the collection of clusters returned by the clustering algorithm. In both cases, we define extended clusters \( S'_1, \ldots, S'_t \), where cluster \( S'_i \) consists of all nodes in \( S_i \) and of all neighbors of nodes in \( S_i \). Note that because of the 3-hop separation of clusters, the clusters \( S'_i \) are still vertex-disjoint. By Theorem A.2, the clustering is \( (O(\log n), O\left(\frac{\log^3 n}{\varepsilon}\right)) \)-routeable. On the induced subgraphs \( G[S'_i] \) of the extended clusters, we can therefore run SUPPORTED CONGEST algorithms with a communication graph of diameter \( O\left(\frac{\log^3 n}{\varepsilon}\right) \). If we run such algorithm on all clusters in parallel, we have a slowdown of \( O(\log n) \). By the assumption of the lemma, we can therefore in parallel for all graph \( G[S'_i] \) compute an \( \alpha \)-approximate solution for our given problem (MWVC or MWM) in time \( O(\log n \cdot T^\alpha_{SC}(O\left(\frac{\log^3 n}{\varepsilon}\right))) \).

For weighted matchings, we are now done. By LP duality, the maximum weight matching for a subgraph induced by a subset \( F \subseteq E \) of the edges of \( G \) is upper bounded by the sum of the \( x_v \) values assigned to all the nodes incident to edges in \( F \). Hence the maximum weight matching of the edges that are outside the graphs \( G[S'_i] \) is upper bounded by the sum of the \( x_v \) values of nodes outside the original clusters \( S_i \) and because the clustering is \((1 - \varepsilon/4)\)-dense, we know that the sum of those \( x_v \) values is at most \( \varepsilon/4 \cdot \sum_{v \in V} x_v \). Because the assignment \( x_v \) constitutes a 4-approximation of the dual of the fractional weighted matching problem, we also know that the weight of an optimal matching is at least \( 1/4 \cdot \sum_{v \in V} x_v \). The claim of the theorem thus follows for the MWM problem.

For the MWVC problem, we know that the collection of the \( \alpha \)-approximate weighted vertex covers of the graphs \( G[S'_i] \) is clearly upper bounded by \( \alpha w(S^*) \), where \( w(S^*) \) is the weight of an optimal weighted vertex cover of the whole graph \( G \). However, the collection of the vertex covers of the clusters is not a valid vertex cover of \( G \). Some of the edges outside clusters \( S'_i \) might not be covered. The edges outside clusters \( S'_i \) can however be covered by the set of half-tight nodes (w.r.t. the fractional \( w \)-matching \( y \)) outside the original clusters \( S_i \). The total weight of those nodes is at most 4 times the fractional values \( y_e \) of all the uncovered edges and thus at most \( \varepsilon \cdot y(E) \leq \varepsilon w(S^*) \) (because of (1) and because the clustering is \((1 - \varepsilon/4)\)-dense).

**Remark.** We note that by using a variant of the randomized clustering algorithm of Miller, Peng, and Xu [MPX13], one can compute a \((1, O\left(\frac{\log n}{\varepsilon}\right))\)-routeable, 3-hop separated clustering with expected density \( 1 - \varepsilon \) in time \( O\left(\frac{\log n}{\varepsilon}\right) \) (also cf. Appendix A.2 and [FK20]). In combination with the argument in the above proof, this in particular implies that it in \( O\left(\frac{\log n}{\varepsilon}\right) \) rounds in the LOCAL model, it is possible to compute weighted matchings and weighted vertex covers with expected approximation ratios \( 1 - \varepsilon \) and \( 1 + \varepsilon \), respectively.
3.2 Basic Bipartite Weighted Vertex Cover Algorithm

It is well-known that in bipartite graphs, the size of a maximum matching is equal to the size of a minimum vertex cover (this is known as König’s theorem [Die05, K31]). The theorem was also independently discovered in [Ege31] by Egerváry, who also more generally proved that on node-weighted bipartite graphs, the total value of an optimal (fractional) $w$-matching is equal to the weight of a minimum weighted vertex cover, where a fractional $w$-matching of a node-weighted graph $G = (V, E, w)$ is an assignment of fractional values $y_e \geq 0$ to all edges such that the edges of each node $v$ sum up to at most $w(v)$. In both cases, the theorem can be proven in a constructive way. Given a maximum matching or more generally a maximum fractional $w$-matching, there is a simple (and efficient) algorithm to compute a vertex cover of the same size or weight.

Moreover as shown in [FMS15], if we are given a good approximate matching or $w$-matching with some additional properties, the constructive proof of [Ege31, K31] can be adapted to obtain a good approximate (weighted) vertex cover. For the unweighted case, this method is at the core of the CONGEST model bipartite vertex cover algorithms of [FK20]. We next describe how to use this technique to approximate MWVC in the CONGEST model.

Let $G = (V, E, w)$ be a node-weighted graph, where $w$ is a non-negative node weight function. For a node $v \in V$ and a given fractional $w$-matching $y_e$ for $e \in E$, we define the slack of $v$ as $s(v) := w(v) - \sum_{e: v \in e} y_e$. Given a fractional $w$-matching $y_e$ for $e \in E$, an augmenting path is an odd length path $P = (v_0, \ldots, v_{2k+1})$ where the end nodes $v_0$ and $v_{2k+1}$ have positive slack $s(v_0), s(v_{2k+1}) > 0$, and for $i \in \{1, 2, \ldots, k\}$, each even edge $e = (v_{2i-1}, v_{2i})$ has a positive fractional value $y_e > 0$. Assume that we are given a bipartite graph $G = (A \cup B, E, w)$ and that we are given a fractional $w$-matching $y$ of $G$ such that there are no augmenting paths of length at most $2k - 1$ for some integer $k \geq 1$. We can then apply the following algorithm to compute a vertex cover $S$ of $G$. The algorithm computes disjoint sets $A_0, A_1, \ldots, A_k \subseteq A$ and disjoint sets $B_1, \ldots, B_k \subseteq B$. In the following for a set of nodes $X$, we use $N_{y > 0}(X)$ to denote the set of nodes that are connected to a node in $X$ through an edge $e$ with $y_e > 0$.

**Basic Approximate Weighted Vertex Cover Algorithm**

1. Define $A_0 := \{v \in A : s(v) > 0\}$ as the nodes in $A$ with positive slack and $B_0 := \emptyset$.
2. For every $i \in \{1, \ldots, k\}$, define $B_i := \{v \in B \setminus \bigcup_{j=0}^{i-1} B_j : v \in N(A_{i-1})\}$.
3. For every $i \in \{1, \ldots, k\}$, define $A_i := \{v \in A \setminus \bigcup_{j=0}^{i-1} A_j : v \in N_{y > 0}(B_i)\}$.
4. Define $i^* := \arg \min_{i \in \{1, \ldots, k\}} w(B_i)$.
5. Output $S := \bigcup_{i=1}^{i^*} B_i \cup (A \setminus \bigcup_{i=0}^{i^* - 1} A_i)$.

That is, the sets $A_0, B_1, A_1, B_2, A_2 \ldots$ are the levels of a BFS traversal of the graph starting at the nodes in $A_0$ and where steps from $B_i$ to $A_i$ have to be over an edge $e$ with positive fractional value $y_e > 0$.

**Lemma 3.2.** Given a weighted bipartite graph $G = (A \cup B, E, w)$, an integer $k \geq 1$, and a fractional $w$-matching of $G$ with no augmenting paths of length at most $2k - 1$, the above algorithm computes a $(1 + 1/k)$-approximate weighted vertex cover of $G$. Further, the above algorithm can be deterministically implemented in $O(D + k)$ rounds in the SUPPORTED CONGEST model if the communication graph is also bipartite and has diameter at most $D$. 

10
Theorem 2.1
Section 3.2
FK20 basically converts a good approximation (i.e., f rom LP duality (i.e., from the unweighted setting, there exists a randomized CONGEST algorithm to compute an integral matching with no short augmenting paths) with a standard pipeline scheme for the root to compute the weights of all the sets $B_i$, determine the index $i^*$ of the smallest weight amongst them, and broadcast it to all nodes in $G$.

Finally, we discuss how the above algorithm can be efficiently implemented in time $O(D + k)$ rounds in the SUPPORTED CONGEST model, where $D$ is the diameter of the communication graph $H$. In $O(D)$ rounds, one can compute a BFS spanning tree of $H$ and use it to compute the bipartition of the nodes of $H$ and thus of $G$ into sets $A$ and $B$. Then in $O(k)$ rounds, the algorithm constructs the sets $A_i$ and $B_i$ for $i \in \{1, 2, \ldots, k\}$ by running the first $2k$ iterations of parallel BFS on $G$ starting from set $A_0$ (where edges from $B_i$ to $A_i$ need to have positive fractional values). Finally, in another $O(D + k)$ rounds, we use the precomputed BFS spanning tree on $H$ and a standard pipelining scheme for the root to compute the weights of all the sets $B_i$, determine the index $i^*$ of the smallest weight amongst them, and broadcast it to all nodes in $G$.

We remark that although the above algorithm only requires a BFS traversal from all nodes in $A_0$ for $k$ levels, the algorithm still requires time $D$ for two reasons. First, the algorithm needs to know the bipartition $A \cup B$ of $G$ and computing the bipartition requires $\Omega(D)$ time. Second, even if the bipartition is given initially, the algorithm still needs $\Omega(D)$ time to determine the optimal level $i^*$.

3.3 Getting Rid of Short Augmenting Paths

The basic MWVC algorithm described in Section 3.2 basically converts a good approximation of fractional $w$-matching into a good MWVC approximation. However the algorithm needs a fractional $w$-matching with the additional property that there are no short augmenting paths. For the unweighted setting, there exists a randomized CONGEST algorithm to compute an integral matching with no short augmenting paths [LPP15]. It is however not clear if the algorithm of [LPP15] can be generalized to the fractional $w$-matching problem. Further, even in the unweighted case, we do not have a deterministic CONGEST algorithm to compute such a matching. As in the deterministic, unweighted MVC algorithm of [FK20], we therefore use a different approach. In the unweighted setting, we first compute a $(1 - \delta)$-approximate matching that can potentially have short augmenting paths. We then get rid of those short augmenting paths by removing at least one unmatched node or both nodes of a matching edge from the graph. The removed nodes are at the end added to the vertex cover to make sure that all edges are covered. The selection of a smallest possible number of unmatched nodes and matching edges that hit all short augmenting paths can be phrased as a minimum set cover problem, which we can approximate efficiently in the CONGEST model. In the weighted case, we use a generalization of this approach. Because of the weights, the process and its analysis however becomes more subtle and we have to be more careful.
Assume that we want to compute a \((1 + O(\varepsilon))\)-approximate weighted vertex cover for a node-weighted bipartite graph \(G = (A \cup B, E, w)\). In a first step, we compute a \((1 - \delta)\)-approximate fractional \(w\)-matching \(y := \{y_e : e \in E\}\) of \(G\) for some parameter \(\delta \ll \varepsilon\). We can do this efficiently by using Theorem A.3. The fractional \(w\)-matching \(y\) however might have short augmenting paths. In a second step, we then convert our graph \(G\) and the fractional \(w\)-matching \(y\) such that we obtain an instance with no short augmenting paths and that we can thus apply Theorem 3.2. More concretely, we decrease some of the weights \(w(v)\) and some of the fractional values \(y_e\) such that for the resulting weights \(w'(v)\) and the resulting \(w'\)-matching \(y'\), the graph \(G\) has no short augmenting paths and such that a \((1 + \varepsilon)\)-approximate weighted vertex cover of \(G\) with the weights \(w'(v)\) is a \((1 + O(\varepsilon))\)-approximate weighted vertex cover of \(G\) for the original weights. We next describe the main ideas of this transformation.

Formally, the conversion can be defined by a set \(X \subseteq \{v \in A \cup B : s(v) > 0\}\) of nodes with positive slack and a set \(F \subseteq \{e \in E : y_e > 0\}\) of edges with positive fractional value. The new fractional values \(y'_e\) and the new weights \(w'(v)\) are defined as follows:

\[
y'_e := \begin{cases} 0 & \text{if } e \in F \\ y_e & \text{if } e \not\in F \end{cases}, \quad w'(v) := \begin{cases} w(v) - s(v) - y(E(v) \cap F) & \text{if } v \in X \\ w(v) - y(E(v) \cap F) & \text{if } v \not\in X \end{cases}
\]

**Lemma 3.3.** Any augmenting path of \(G\) w.r.t. the weight function \(w'\) and the fractional \(w'\)-matching \(y'\) is also an augmenting path w.r.t. the original weight function \(w\) and the original fractional \(w\)-matching \(y\).

**Proof.** First note that whenever we decrease a value \(y_e\) to \(y'_e < y_e\) for some edge \(e = \{u, v\}\), we also decrease the weights of \(u\) and \(v\) by the same amount. Therefore, the slack of a node \(v\) w.r.t. \(w'\) and \(y'\) cannot be larger than the slack of \(v\) w.r.t. \(w\) and \(y\). Therefore any odd-length path \(P\) in \(G\) that starts and ends at a node with positive slack w.r.t. \(w'\) and \(y'\) also starts and ends at a node with positive slack w.r.t. \(w\) and \(y\). Further, if every even edge \(e\) of such a path \(P\) has a positive fractional value \(y'_e > 0\), then it also holds that \(y_e > 0\). \(\square\)

Note that the definition of \(w'\) and \(y'\) in (2) guarantees that all nodes \(v \in X\) have slack 0 w.r.t. the new weights \(w'\) and the new fractional values \(y'\). Consider some augmenting path \(P = (v_0, \ldots, v_{2\ell+1})\) of \(G\) w.r.t. \(w\) and \(y\). If we have \(v_0 \in X\) or \(v_{2\ell+1} \in X\) or if we have \(e \in F\) for one of the even edges \(\{v_{2i-1}, v_{2i}\}\) of \(P\), then \(P\) is not an augmenting path of \(G\) w.r.t. \(w'\) and \(y'\). In order to get rid of all short augmenting paths, we therefore need to choose one of the end nodes or one of the even edges of each such path and add them to \(X\) or \(F\). We can then use Theorem 3.2 to efficiently compute a good vertex cover approximation for \(G\) w.r.t. the new weights \(w'\). The quality of such a vertex cover w.r.t. the original weights \(w\) can be bounded as follows.

**Lemma 3.4.** Let \(S^*\) be an optimal weighted vertex cover of \(G = (V, E)\) w.r.t. the weights \(w\) and assume that for some \(\alpha \geq 1\), \(S\) is an \(\alpha\)-approximate weighted vertex cover of \(G\) w.r.t. the weights \(w'\). It then holds that

\[
w(S) \leq \alpha \cdot w(S^*) + s(X) + y(F), \quad \text{where } s(X) := \sum_{v \in X} s(v).
\]

**Proof.** Let \(S'\) be an optimal weighted vertex cover of \(G\) w.r.t. the weights \(w'\). Because any vertex cover must contain at least one node of every edge in \(F\), we have \(w'(S') \leq w(S^*) - y(F)\). We therefore have

\[
w(S) \leq w'(S) + s(X) + 2y(F) \leq \alpha w'(S') + s(X) + 2y(F) \leq \alpha w(S^*) + s(X) + y(F). \quad \square
\]
In order to optimize the approximation, we thus need to determine the sets $X$ and $F$ such that $s(X) + y(F)$ is as small as possible and such that we ‘cover’ all short augmenting paths. The problem of finding the best possible sets $X$ and $F$ can naturally be phrased as a weighted set cover problem. One can further show that if the parameter $\delta$ that determines the quality of the fractional $w$-matching $y$ is chosen sufficiently small (but still as $\delta = \text{poly}(\varepsilon / \log n)$), even a logarithmic approximation to this weighted set cover instance guarantees that $s(X) + y(F) = O(\varepsilon \cdot w(S^*))$.

Further, by sequentially going over the possible short augmenting path lengths and adapting existing algorithms of [LPP15] and [FK20], a variant of the greedy algorithm for this weighted set cover instance can be implemented efficiently in the CONGEST model on $G$. Given an efficient algorithm to find appropriate sets $X$ and $F$, the claim of Theorem 1.1 then follows almost immediately by combining with Theorem 3.1 and Theorem 3.2. The details appear in Section 4.

### 3.4 Generalization to Non-Bipartite Graphs

For approximating the MWVC problem in general graphs, we employ a standard approach that is for example described in [Hoc97]. We describe the details for completeness. The minimum fractional (weighted) vertex cover problem is the natural LP relaxation of the minimum (weighted) vertex cover problem. That is, a fractional vertex cover of a graph $G = (V, E)$ is an assignment of values $x_v \in [0, 1]$ to all nodes such that for every edge $\{u, v\}$, $x_u + x_v \geq 1$. While the integrality gap of the (weighted and unweighted) vertex cover can be arbitrarily close to 2, it is well-known that there are optimal fractional solutions that are half-integral. That is, there are optimal fractional solutions such that for all nodes $v$, $x_v \in \{0, 1/2, 1\}$. Given such a fractional solution, let $S_x$ be the set of nodes $v$ with $x_v = x$ for $x \in \{0, 1/2, 1\}$. Let $I_{1/2}$ be an independent set of the induced subgraph $G[S_{1/2}]$ of the half-integral nodes. It is not hard to see that $S := S_1 \cup S_{1/2} \setminus I_{1/2}$ is a vertex cover of $G$ and if $w(I_{1/2}) \geq \lambda \cdot w(S_{1/2})$, then the set $S$ is a $(2 - 2\lambda)$-approximate solution for the MWVC problem on $G$ with weights $w$. If the half-integral fractional solution is only an $\alpha$-approximate fractional weighted vertex cover, the resulting approximation is $(2 - 2\lambda) \cdot \alpha$.

The core to proving Theorem 1.2 is therefore to first compute a $(1+\varepsilon)$-approximate half-integral fractional solution for a given weighted graph $G = (V, E, w)$. The following lemma uses a standard approach to achieve this by first computing an approximate solution to the (integral) MWVC problem for the bipartite double cover $G_2$ of $G$ (see definition in Section 2).

**Lemma 3.5.** Let $G = (V, E, w)$ be a weighted graph and let $G_2 = (V_2, E_2)$ be the bipartite double cover of $G$, where each node $(v, i) \in V_2$ for $v \in V$ gets assigned weight $w(v)$. Let $S$ be a vertex cover of $G_2$ and define $x_v := |\{(v, 0), (v, 1)\} \cap S|/2$ for every $v \in V$. If $S$ is an $\alpha$-approximate weighted vertex cover of $G_2$ for some $\alpha \geq 1$, then $x = \{x_v : v \in V\}$ is a half-integral $\alpha$-approximate fractional weighted vertex cover of $G$.

**Proof.** We first show that the weight $\sum_{v \in V} w(v) \cdot z_v$ of an optimal fractional weighted vertex cover $z$ of $G = (V, E, w)$ is exactly half the weight of an optimal fractional weighted vertex cover of $G_2$ with weights assigned as defined by the claim of the lemma. To see this, let $z$ be a fractional vertex cover of $G$. We can then get a valid fractional vertex cover of $G_2$ by setting $z_{(v, 0)} = z_{(v, 1)} = z_v$ for every node $v \in V$ of $G$ and the corresponding nodes $(v, 0)$ and $(v, 1)$ in $G_2$. In the other direction, for a fractional vertex cover $z$ in $G_2$, we obtain a fractional vertex cover of $G$ by setting $z_v := (z_{(u, 0)} + z_{(u, 1)})/2$. Note that because $G_2$ is a bipartite graph, an optimal (integral) weighted vertex cover of $G_2$ is also an optimal fractional weighted vertex cover of $G_2$ and therefore the vertex cover $S$ is an $\alpha$-approximate fractional weighted vertex cover of $G_2$. The fractional vertex cover $x$ of $G$ as given by the lemma statement is of half the weight of $S$ in $G_2$ and it therefore is an $\alpha$-approximate fractional weighted vertex cover of $G$. Clearly, $x$ is half-integral. \[\square\]
The proofs of Theorems 1.2 and 1.3 appear in Section 4.2.

3.5 Weighted Matching Approximation

We provide a randomized and a deterministic CONGEST algorithm to approximate the MWM problem. Both algorithms are based on the following key idea that was developed for the unweighted maximum matching problem by Lotker, Patt-Shamir, and Pettie [LPP15] and that was extended by Bar-Yehuda et al. [BCGS17]. We iteratively adapt an initial matching $M_0$ of a given weighted graph $G = (V, E, w)$ as follows. We repeatedly sample bipartite subgraphs of $G$ and we then find a good matching on this bipartition to improve the matching of $G$. In [LPP15, BCGS17], the matching is improved by finding short augmenting paths in the sampled bipartite subgraph and augmenting the existing matching along those paths. While, as discussed below, also for maximum weighted matching, it is in principle possible to find augmenting paths and cycles, in the weighted case, we do not know how to do this efficiently in the CONGEST model. Instead, we use the bipartite MWM approximation algorithm of [AKO18] to find a good matching in each sampled bipartite graph. Unlike when using augmenting paths, this approach can potentially also lead to a worse matching (if the existing matching is already a very good matching of the sampled bipartite graph). We will however see that when computing a sufficiently good approximation in each sampled bipartition, we can use the approach to improve the matching of $G$ sufficiently often.

Before explaining our algorithms in more detail, we need to introduce the notion of augmenting paths and cycles for weighted matchings. Given a matching $M$, a path or cycle in which the edges alternate between edges $\in M$ and edges $\notin M$ is called an alternating path or cycle w.r.t. matching $M$. An alternating path or cycle is called an augmenting path or cycle w.r.t. $M$ if swapping the matching edges with the non-matching edges increases the weight of the matching. This increase is termed the gain of an augmenting path/cycle w.r.t. $M$ (we will omit the qualification ‘w.r.t.’ if it is clear from the context). By definition, the gain of an augmenting path is positive. Note that alternating cycles (and therefore also augmenting cycles) are always of even length.

Let $M^\ast$ be a maximum weighted matching in $G$. Consider the symmetric difference $F = M^\ast \triangle M$ of $M^\ast$ and some arbitrary matching $M$. The set $F$ consists of (vertex-disjoint) alternating paths and cycles where the edges alternate between $M$ and $M^\ast$. All of those alternating paths and cycles are either augmenting paths/cycles, or their $M$-edges have exactly the same weight as their $M^\ast$-edges. The total gain of all the (vertex-disjoint) augmenting paths and cycles induced by $F$ is therefore exactly $w(M^\ast) - w(M)$.

Dealing with augmenting paths and cycles in the context of edge-weighted graphs is much more challenging than in the unweighted case. Every augmenting path in unweighted graphs improves the matching by exactly one and augmenting cycles do not exist. Further, the classic results of Hopcraft and Karp [HK73] imply that if the shortest augmenting path is of length $\ell$, augmenting over an augmenting path of length $\ell$ cannot create new augmenting paths of length $\leq \ell$ and after augmenting over a maximal set of vertex-disjoint augmenting paths of length $\ell$, one gets a matching with a shortest augmenting path length $\geq \ell + 2$. If in some matching $M$, the shortest augmenting path has length $\ell = 2k - 1 \geq 1$, we further know that $M$ already is a $(1 - \frac{1}{k})$-approximation of the optimal matching. The $(1 - \varepsilon)$-approximation algorithm for unweighted matching in [LPP15] heavily relies on all those properties.

Some of the properties of augmenting paths for unweighted matchings also carry over to the weighted case. We will later show that there exists a set of vertex-disjoint augmenting paths and cycles of length at most $\ell$ for some $\ell = O(1/\varepsilon)$ such that augmenting over all those paths/cycles improves the current matching by at least $\frac{1}{4} \cdot w(M^\ast)$ (see Theorem 5.1). Basically, the existence of this collection of short augmenting paths can be proven by breaking long augmenting paths and
cycles in the symmetric difference $F = M \triangle M^*$ into short augmenting paths. However, while in the unweighted case, a large set of vertex-disjoint short augmenting paths can be computed efficiently in the CONGEST model (see [BCGS17, LPP15]), it is not clear how to efficiently compute such a set of augmenting paths/cycles for weighted matchings in the CONGEST model, even in bipartite graphs (there are efficient algorithms in the LOCAL model and this has also been exploited in the literature, e.g., in [GKMU18, LPP15, Nie08]). Fortunately, there still is an efficient CONGEST algorithm for computing a $(1 - \varepsilon)$-approximate weighted matching in bipartite graphs [AKO18] (see also Lemma 5.2). Unlike the existing CONGEST algorithms that are based on the Hopcroft/Karp framework, the algorithm of [AKO18] is even deterministic. It is however not based on augmenting along short augmenting paths or cycles. Instead, it is based on linear programming and on a deterministic rounding scheme that was introduced in [Fis17]. As a result, the matching computed by the algorithm of [AKO18] does not have the nice structural properties of the matchings computed by algorithms based on the Hopcroft/Karp framework (such as not having any short augmenting paths or cycles).

Our randomized weighted matching algorithm. The general idea of our approach is as simple as the algorithm for the unweighted case in [LPP15].

We first describe the randomized version of our algorithm. We start with an initial matching $M_0$ of the given weighted graph $G = (V, E, w)$ and it then consists of iterations $i = 1, 2, \ldots$. In each iteration, we update the given matching such that at the end of iteration $i$, we have matching $M_i$. In each iteration $i$, we sample a bipartite subgraph $H_i = (\hat{V}_i, \hat{E}_i)$ of $G$ as follows. Every node $v \in V$ colors itself black or white independently with probability $1/2$. An edge is called monochromatic if both of its endpoints have the same color, otherwise, we call the edge bichromatic. To preserve good intermediate results, we keep all the matching edges of the previous matching not occurring in the bipartition, i.e., we keep monochromatic matching edges. We call a node free regarding to matching $M$ if none of its incident edges are $\in M$.

**Construct bipartite subgraph $H_i = (\hat{V}_i, \hat{E}_i)$ of $G$ based on matching $M_{i-1}$:**

1. Color each node black or white.
2. $\hat{V}_i := \{u \in V \mid u$ is free or $\exists\{u, v\} \in M_{i-1}$ s.t. $\{u, v\}$ is bichromatic$\}.
3. $\hat{E}_i := \{\{u, v\} \in E \mid u, v \in \hat{V}_i$ and $\{u, v\}$ is bichromatic$\}$.

After sampling the bipartite subgraph $H_i$, we use the algorithm of [AKO18] to compute a $(1 - \lambda)$-approximate weighted matching of $H_i$ for a sufficiently small parameter $\lambda > 0$ ($\lambda$ will be exponentially small in $1/\varepsilon$). We then update the existing matching $M_{i-1}$ by replacing the bichromatic matching edges with the matching edges of the newly computed matching of $H_i$. Because there is a collection of short augmenting paths and cycles with total gain $\Theta(w(M^*) - w(M_{i-1}))$ (where $M^*$ is an optimal matching of $G$), we have a reasonable chance of sampling such paths and cycles so that the matching on $H_i$ can potentially be improved by a sufficiently large amount. Note however that our algorithm does not guarantee that the quality of the matching improves monotonically during the algorithm. However, because the algorithm of [AKO18] has deterministic guarantees if we choose $\lambda$ sufficiently small, we have the guarantee that $w(M_i)$ cannot be much worse than $w(M_{i-1})$. With the right choice of parameters, it turns out that $2^{O(1/\varepsilon)}$ iterations are sufficient to obtain a $(1 - \varepsilon)$-approximate matching with at least constant probability. The details appear in Section 5.1.

\footnote{Our algorithm is essentially the same one as the one in [LPP15, BCGS17]. We just replace the bipartite matching algorithm used as a subroutine. The analysis is then however different from the analysis in [LPP15, BCGS17].}
Our deterministic weighted matching algorithm. The basic idea of our deterministic algorithm is the same as for the randomized algorithm. However, we now have to compute the bipartition into black and white nodes in each iteration deterministically. For this purpose, we show in Theorem 5.6 that for some $T = 2^{O(1/\varepsilon)} \ln n$, there exist a collection of bipartitions $H_1, \ldots, H_T$ such that every path/cycle of length at most $O(1/\varepsilon)$ (and thus every augmenting path/cycle of this length) of $G$ appears in at least one of these bipartitions. Of course, after changing the matching, also the set of augmenting paths and cycles changes and one can therefore not just iterate over all bipartitions $H_1, \ldots, H_T$, improve the matching for each bipartition by using a generic MWM approximation algorithm, and guarantee that at the end the resulting matching is a sufficiently good approximation. However, the property of $H_1, \ldots, H_T$ guarantees that for a fixed initial matching $M$, when going over all $T$ bipartitions, there exists one bipartition that can improve the weight of $M$ by $\Theta(w(M)/T)$. We therefore proceed as follows. We iterate $O(T)$ times through the sequence $H_1, \ldots, H_T$ of bipartitions. For each bipartition, we use the (deterministic) algorithm of [AKO18] to compute a $(1 - \lambda)$-approximate weighted matching of the current bipartite graph. We however then switch to the new matching if the new matching improves the old one by a sufficiently large amount. Checking if a given bipartition leads to a sufficiently large improvement of the current matching can be done efficiently by first applying the diameter reduction technique given by Theorem 3.1. The details appear in Section 5.2.

3.6 Vertex Cover Lower Bound

Recall that Göös and Suomela in [GS14] showed that there exists a constant $\varepsilon_0 > 0$ such that computing a $(1 + \varepsilon_0)$-approximation of minimum (unweighted) vertex cover in bipartite graphs of maximum degree 3 requires $\Omega(\log n)$ rounds even in the LOCAL model. We next describe the high-level idea of extending this lower bound to show that for $\varepsilon \leq \varepsilon_0$, computing a $(1 + \varepsilon)$-approximate vertex cover in bipartite graphs of maximum degree 3 requires $\Omega(\frac{\log n}{\varepsilon})$ rounds in the LOCAL model. Assume that $G$ is the lower bound graph of the proof of [GS14]. We transform $G$ into a bipartite graph $H$ by replacing each edge of $G$ with a path of length $2k + 1$ for some parameter $k = \Theta(1/\varepsilon)$. One can then show that every vertex cover $S_H$ of $H$ can be transformed (in $O(k)$ rounds on $H$) into a vertex cover $S'_H$ of $H$ of size $|S'_H| \leq |S_H|$ such that $S'_H$ consists of exactly $k$ inner nodes of every path replacing a $G$-edge and a vertex cover of $G$ (composed of the original $G$-nodes in $H$ that are contained in $S'_H$). By choosing $k$ appropriately, one can then show that a $(1 + \varepsilon)$-approximate vertex cover on $H$ induces a $(1 + \varepsilon_0)$-approximate vertex cover on $G$. The lower bound then follows because the nodes of $G$ can locally simulate graph $H$ such that $k = O(1/\varepsilon)$ rounds on $H$ can be simulated in $O(1)$ rounds on $G$. A detailed proof appears in Section 6.

4 Technical Details: Weighted Vertex Cover Algorithms

4.1 Bipartite Weighted Vertex Cover

We are now going to provide the additional technical details that are necessary to prove Theorem 1.1, which states that in bipartite graphs, one can deterministically compute a $(1 + \varepsilon)$-approximation of the MWVC problem in time $\text{poly}\left(\frac{\log n}{\varepsilon}\right)$. As discussed in Section 3.1, we first use Theorem 3.1 to cluster the graph into clusters of small weak diameter. Further, in Section 3.2, we gave a basic distributed MWVC approximation algorithm that can be run efficiently in all clusters. However, to run this algorithm on a weighted bipartite graph, we need a fractional $w$-matching with no short augmenting paths. In Section 3.3, we described the high-level idea that we use to obtain such a fractional $w$-matching. Starting from a $(1 - \delta)$-approximate fractional $w$-matching, where $\delta$ needs to
be chosen sufficiently small, we reduce the weights of some nodes and the fractional values of some edges to obtain an instance with weights \( w' \) and a fractional \( y' \)-matching with no short augmenting paths so that we can then apply the MWVC algorithm of Section 3.2. We will now provide the details of this step.

Let us therefore assume that we are given a bipartite node-weighted graph \( G = (A \cup B, E, w) \) and that for an appropriate \( \delta > 0 \), we are given a \((1 - \delta)\)-approximate fractional \( w \)-matching \( y \) of \( G \). For technical reasons, we further assume that for every \( e \in E \), either \( y_e = 0 \) or \( y_e > 1/n^c \) for some constant \( c > 0 \). Similarly, we assume that for every node \( v \in V \), \( s(v) = 0 \) or \( s(v) > 1/n^c \). We can guarantee this requirement by making sure that every fractional value \( y_e \) is an integer multiple of \( 1/n^c \). As long as \( \delta \geq 1/\text{poly}(n) \) (which is much smaller than what we need), this is straightforward to guarantee. As discussed in Section 3.3, for getting rid of short augmenting paths at small cost, we need to find sets \( X \subseteq \{v \in A \cup B : s(v) > 0\} \) and \( F \subseteq \{e \in E : y_e > 0\} \) such that all augmenting paths of length at most \( 2k - 1 \) (for a given integer \( k \geq 1 \)) are ‘covered’ and \( s(X) + y(F) \) is as small as possible. We now first show that we can find sets \( X \) and \( F \) such that \( s(X) + y(F) \) is within a small factor of \( s(X^*) + y(F^*) \) for best possible sets \( X^* \) and \( F^* \). To compute \( X \) and \( F \), we go through stages \( d = 1, 3, \ldots, 2k - 1 \), where in stage \( d \) we select sets \( X_d \) and \( F_d \) that cover all remaining augmenting paths of length \( d \). In the end, we set \( X = X_1 \cup \cdots \cup X_{2k-1} \) and \( F = F_1 \cup \cdots \cup F_{2k-1} \).

Let us now focus on one stage \( d \) in the following part.

### 4.1.1 Getting rid of augmenting paths of length \( d \).

First note that since we covered all augmenting paths of length shorter than \( d \) before stage \( d \), at the beginning of stage \( d \), there are no augmenting paths of length shorter than \( d \). To simplify notation, we will in the following use \( w \) and \( y \) to denote the node weights and the fractional edge values at the beginning of stage \( d \). We therefore assume that w.r.t. \( w \) and \( y \), \( G \) has no augmenting paths of length shorter than \( d \) and the goal is to find sets \( X_d \) and \( F_d \) to cover all augmenting paths of length \( d \). The problem of finding such sets that minimizes \( s(X_d) \) and \( y(F_d) \) can be phrased as a minimum weighed set cover problem as follows. The ground set \( \mathcal{P}_d \) is the set of all augmenting paths of length \( d \) w.r.t. the fractional \( w \)-matching \( y \) in \( G \). We define

\[
S_d := \{ v \in A \cup B : s(v) > 0 \land v \text{ is an end node of some augmenting path of length } d \},
\]

\[
Y_d := \{ e \in E : y_e > 0 \land e \text{ is an even edge of some augmenting path of length } d \}.
\]

For each \( v \in S_d \), let \( P_v \) be the set of augmenting paths of length \( d \) for which node \( v \) is one of the end nodes. Similarly, for each \( e \in Y_d \), let \( P_e \) be the set of augmenting paths of length \( d \) for which it contains \( e \) as an even edge. The collection of sets \( \mathcal{S} \in 2^{\mathcal{P}_d} \) is then made up of sets \( P_v \) for each node \( v \in S_d \) and \( P_e \) for each edge \( e \in Y_d \). The weight of \( P_v \) is \( s(v) \) and the weight of \( P_e \) is \( y_e \). A solution to this set cover instance is a collection of sets \( P_v \) for \( v \in X_d \) and \( P_e \) for \( e \in F_d \) where \( X_d \subseteq S_d \) and \( F_d \subseteq Y_d \), and such that \( \bigcup_{v \in X_d} P_v \cup \bigcup_{e \in F_d} P_e = \mathcal{P}_d \). The weight of such a set cover is exactly \( s(X_d) + y(F_d) \). Note that since we assumed that all non-zero slack values \( s(v) \) and fractional values \( y_e \) are at least \( 1/\text{poly}(n) \) and because we assumed that all node weights are upper bounded by a polynomial in \( n \), the weights of all sets in our set cover instance are between \( 1/\text{poly}(n) \) and \( \text{poly}(n) \).

In the following, we use \( w_{\text{min}} \) and \( w_{\text{max}} \) to refer to the minimum and the maximum weight of a set in the current set cover instance.

**Layered structure of shortest augmenting paths.** To achieve our goal, we first have a look at the structure of the augmenting paths of length \( d \) in \( G = (A \cup B, E, w) \). For this, we define the following layered graph \( H = (V_0 \cup \cdots \cup V_d, E_H) \). The layer \( V_0 := \{v \in A : s(v) > 0\} \) consists of the
nodes in $A$ with positive slack. For every odd $i$, layer $V_i$ consists of the nodes in $B \setminus \bigcup_{j=0}^{i-1} V_j$ for which there exists an edge from a node in $V_{i-1}$. For every even $i \geq 2$, layer $V_i$ consists of the nodes in $A \setminus \bigcup_{j=0}^{i-1} V_j$ for which there exists an edge $e$ with $y_e > 0$ from a node in $V_{i-1}$. Finally, layer $V_d$ consists of nodes in $B$ with positive slack $s(v)$ such that there exists an edge from a node in $V_{d-1}$. Note that if the bipartition of the nodes of $G$ into $A$ and $B$ is known, the layers $V_0, \ldots, V_d$ can be computed in a simple parallel BFS exploration of $G$ in $d$ rounds. Note also that by definition of the layered structure, every path $p = (v_0, \ldots, v_d)$ such that for every $i \in \{0, \ldots, d\}$, $v_i \in V_i$, is an augmenting path. Further, because the shortest augmenting path length is $d$, every augmenting path of length $d$ must have this structure and we also know that all nodes $v$ in odd layers $V_i$ for $1 < i < d$ have slack $s(v) = 0$.

**Implementation of greedy weighted set cover algorithm.** We use a variant of the greedy weighted set cover algorithm here to find the sets $X_d$ and $F_d$ covering all the shortest augmenting paths in $G$. We initially set $X_d = F_d = \emptyset$ and we call all nodes and edges active. Whenever we add a node $v$ to $X_d$ we deactivate $v$ and if we add an edge $e$ to $F_d$, we deactivate $e$. The set of active augmenting paths of length $d$ is defined as the set of paths $(v_0, \ldots, v_d)$ with $v_i \in V_i$ such that all nodes and edges of the path are active. Note that the sets $X_d$ and $F_d$ cover all the shortest augmenting paths (i.e., they form a solution of the weighted set cover instance we need to solve) if and only if the number of active augmenting paths of length $d$ is zero. The efficiency of a set $P_v \in \mathcal{P}_d$ for $v \in A \cup B$ is defined as $|\{P \in P_v : P \text{ active}\}|/s(v)$ and the efficiency of a set $P_e \in \mathcal{P}_d$ for $e \in E$ is defined as $|\{P \in P_e : P \text{ active}\}|/y_e$. Note that the efficiency of a set $P_v$ or $P_e$ is the number of newly covered paths per weight when adding $P_v$ or $P_e$ to the set cover (and thus $v$ to $X_d$ or $e$ to $F_d$). Since every node or edge can be in at most $\Delta^d$ different paths of length $d$, the minimum possible efficiency is equal to $1/w_{\max}$ and the maximum possible efficiency is equal to $\Delta^d/w_{\min}$. Computing the efficiencies requires to count the number of active augmenting paths that pass through each node and edge. This can be done in $O(d^2)$ rounds in the CONGEST model by slightly adapting an algorithm described in [BCGS17, LPP15]. The details appear in Theorem A.4 in Appendix A.4. The algorithm consists of phases $i = 1, 2, \ldots, \lceil \log (\Delta^d \cdot w_{\max}/w_{\min}) \rceil$, where in phase $i$ we add sets $P_v$ and $P_e$ with efficiency $\geq 2^{-i} \cdot \frac{\Delta^d}{w_{\min}}$ to the set cover:

**Covering Paths of Length $d$: Phase $i \in \{1, \ldots, \lceil \log (\Delta^d \cdot w_{\max}/w_{\min}) \rceil \}$**

First go over level $\ell = 0$ and then sequentially iterate over all odd levels $\ell = 1, 3, \ldots, d$:

1. Count the number of active augmenting paths of length $d$ passing through each node and edge (by using Theorem A.4).

2. If $\ell \in \{0, d\}$, for each active node $v \in V_\ell$ do

   - Compute efficiency of set $P_v$ (note that $s(v) > 0$ because $v \in V_0 \cup V_d$)
   - If efficiency of $P_v$ is $\geq 2^{-i} \cdot \frac{\Delta^d}{w_{\min}}$, add $v$ to $X_d$ and deactivate $v$,

3. If $\ell \in \{1, \ldots, d-2\}$, for each edge $e = \{u, v\}$ with $u \in V_\ell$ and $v \in V_{\ell+1}$ do

   - Compute efficiency of set $P_e$ (note that $y_e > 0$ by construction of layers)
   - If efficiency of $P_e$ is $\geq 2^{-i} \cdot \frac{\Delta^d}{w_{\min}}$, add $e$ to $F_d$ and deactivate $e$.

By iterating over the layers $V_\ell$, for $\ell = 0, \ldots, d$, we can add sets $P_v$ (or $P_e$) for $v$ (or $e$) on the same layer in parallel (note that such set of paths need to be disjoint and they therefore do not
influence each other). At the end of a phase \(i\), it is guaranteed that there is no set of efficiency \(\geq 2^{-i} \cdot \frac{\Delta^d}{w_{\text{min}}}\) left. The algorithm therefore always adds sets with an efficiency that is within a factor of 2 of the largest current efficiency. The details of the algorithm for phase \(i\) is given in the following.

Next, we show that the above algorithm can be implemented efficiently in the CONGEST model and that the total weight \(s(X_d) + y(F_d)\) of the solution output by the algorithm in stage \(d\) is small compared to the weight of an optimal vertex cover.

**Lemma 4.1.** Let \(y^*\) be an optimal fractional \(w\)-matching of the graph \(G = (A \cup B, E, w)\) in the current stage. If the given fractional \(w\)-matching \(y\) satisfies \(y^*(E) - y(E) \leq \delta \cdot w(S^*)\) for an optimal weighted set cover of \(G\), then the deterministic algorithm above finds a set \(X_d \subseteq S_d\) and a set \(F_d \subseteq Y_d\) such that \(s(X_d) + y(F_d) \leq \alpha_d \delta \cdot w(S^*)\), where \(\alpha_d = (d + 3)(1 + d \ln \Delta)\). The time complexity of the algorithm is \(O(d^4 \cdot \log n)\) in the CONGEST model.

**Proof.** First, we look at the time complexity. The algorithm consists of \(O(\log(\Delta^d w_{\text{max}}/w_{\text{min}})) = O(d \log n)\) phases, where in each phase we iterate over the \(O(d)\) levels and for each level the time complexity is upper bounded by computing the best efficiency, which essentially comes down to finding the number of active augmenting paths passing through each node or edge. By Theorem A.4, the resulting time complexity is therefore \(O(d^4 \cdot \log n)\) in the CONGEST model.

Next, we show that our distributed algorithm is equivalent to a variant of the sequential greedy weighted set cover algorithm and compute its approximation ratio. Recall that in the standard sequential greedy weighted set cover algorithm, one starts with an empty set cover and iteratively adds to it a set with the current maximum efficiency. It is also well known that the approximation ratio of this greedy algorithm is at most \(H(q) = 1 + \ln q\), where \(q\) is the maximum set size and \(H(q)\) denotes the \(q\)th harmonic number [Chv79]. Further, it is easy to see that if we relax each greedy step to pick a set of at least half the current maximum efficiency, then the algorithm will have an approximation ratio that is at most \(2H(q)\) (this is the greedy set cover algorithm variant that we are considering here). To see this, notice that a set of at least half the maximum efficiency can be turned into a set of maximum efficiency by reducing its weight by a factor of at most \(2\). The resulting solution is therefore an \(H(q)\)-approximation of a set cover instance in which all weights are divided by a factor between \(1\) and \(2\).

Returning now to the distributed set cover algorithm for this lemma, notice that we sequentially iterate over level 0 and the odd levels \(1, 3, ..., d\) and in each iteration, we only add sets \(P_v\) (or \(P_e\)) on the same level whose efficiency is at least half the current maximum efficiency. Moreover note that sets \(P_v\) for nodes \(v\) in one particular level \(\ell\) and sets \(P_e\) for edges \(e\) in one particular level \(\ell\) cover disjoint sets of paths. Hence, in each parallel step, our algorithm always adds sets covering a disjoint set of paths (and therefore a disjoint set of elements in the set cover instance). Our distributed algorithm is therefore equivalent to an execution of the described variant of the sequential greedy algorithm\(^3\). Therefore our algorithm has an approximation ratio of at most \(2H(\Delta^d) \leq 2(1 + \ln \Delta^d) = 2(1 + d \ln \Delta)\). Note that in our setting, the maximum set size is at most \(\Delta^d\) because \(|P_v| \leq \Delta^d\) and \(|P_e| \leq \Delta^d\). To upper bound the absolute cost \(s(X_d) + y(F_d)\) of the computed solution, we also need to understand the cost of an optimal set cover solution.

**Interpreting as minimum weighted hypergraph vertex cover.** To better understand our set cover instance, we interpret it as an instance of a weighted minimum vertex cover problem on

\(^3\)Note that in each iteration, in step 1, we also recompute the number of paths passing through each node and edge so that we always only add sets that cover a sufficient number of still uncovered paths.
hypergraphs. We define a weighted hypergraph \( H_d = (V_H, E_H, \omega) \) as follows. The vertex set \( V_H \) consists of the sets \( P_v \) with \( v \in S_d \) and the sets \( P_e \) with \( e \in Y_d \). That is, we have a node in \( V_H \) for each \( v \in S_d \) and for each \( e \in Y_d \). The hyperedges \( E_H \) of \( H_d \) are the augmenting paths of length \( d \). A node \( P_v \in V_H \) is incident to the hyperedge defined by an augmenting path \( p \) if \( p \in P_v \), i.e., \( p \) starts at node \( v \). A node \( P_e \in V_H \) is incident to the hyperedge defined by an augmenting path \( p \) if \( p \in P_e \), i.e., if \( e \) is contained in \( p \). The weights of vertices \( P_v \) and \( P_e \) of \( H_d \) are defined as \( \omega(P_v) = s(v) \) and \( \omega(P_e) = y(e) \). Observe that the weighted vertex cover on hypergraph \( H_d \) is exactly equivalent to our weighted set cover instance. The dual problem of the natural LP relaxation of weighted vertex cover on hypergraphs is a natural extension of the fractional \( w \)-matching problem defined for graphs in Section 2.2.

For our hypergraph \( H_d \) with weight function \( \omega \), we can define a fractional \( \omega \)-matching of \( H_d \) as an assignment \( z \) of non-negative fractional values \( z_{ep} > 0 \) for every hyperedge \( e_p \in E_H \) such that for every node \( x \in V_H \), the sum of the fractional values \( z_{ep} \) of its incident hyperedges sums up to at most \( \omega(x) \). A fractional assignment is said to be maximal if for each hyperedge \( e_p \), there exists a vertex \( e \in e_p \) that is saturated, i.e., for which the fractional values of its hyperedges sum up to \( \omega(x) \). Note that if we are given a maximal \( \omega \)-matching \( z \) of a hypergraph with node weights \( \omega \), the set of saturated vertices w.r.t. \( z \) form a vertex cover of the hypergraph. If the rank of the hypergraph is \( R \), every hyperedge can have at most \( R \) saturated vertices and the total weight of the saturated nodes is therefore at most \( R \) times the total sum of all fractional hyperedge values. Because every augmenting path of length \( d \) is in exactly 2 of the sets \( P_i \) and in exactly \( (d - 1)/2 \) of the sets \( P_e \), the rank of our hypergraph \( H_d \) is equal to \( 2 + (d - 1)/2 = (d + 3)/2 \).

The weight of an optimal vertex cover of the hypergraph \( H_d = (V_H, E_H, \omega) \) is therefore at most \( \frac{d+3}{2} \cdot z(E_H) \), where \( z(E_H) = \sum_{e_p \in E_H} z_{ep} \) if \( z \) is a maximal fractional \( \omega \)-matching of \( H_d \).

**Upper bounding the value of a fractional \( \omega \)-matching of \( H_d \).** In order to relate the total value \( z(E_H) \) of an \( \omega \)-matching \( z \) of \( H_d \) to the cost of an optimal weighted vertex cover on our original graph \( G \), we interpret the fractional assignment \( z \) on graph \( G \). Since the hyperedges in \( H_d \) correspond to the augmenting paths of length \( d \) of the graph \( G \), the fractional \( \omega \)-matching \( z \) of \( H_d \) assigns a fractional value \( z_{ep} > 0 \) to each such augmenting path \( p \) of \( G \). Because \( z \) is a valid fractional \( \omega \)-matching of \( H_d \), we know that for every node \( v \in V_0 \cup V_d \) and for every edge \( e \in E \) between two levels \( V_{2i-1} \) and \( V_{2i} \), the sum of the \( z_{ep} \)-values for all augmenting paths \( p \) that include \( v \) is at most \( s(v) \) and the sum of the \( z_{ep} \)-values for all augmenting paths \( p \) that include \( e \) is at most \( y_e \). This however implies that we can in parallel augment our existing fractional \( w \)-matching of \( G \) on all augmenting paths \( p \) as follows For every edge \( e \) of \( G \), let \( P_d(e) \subseteq P_d \) be the set of augmenting paths of length \( d \) that contain edge \( e \). For every edge \( e \) between an even level \( V_{2i} \) and a consecutive odd level \( V_{2i+1} \), we set \( y'_e := y_e + \sum_{p \in P_d(e)} z_p \) and for every edge \( e \) between an odd level \( V_{2i-1} \) and a consecutive even level \( V_{2i} \), we set \( y'_e := y_e - \sum_{p \in P_d(e)} z_p \). By construction, the resulting fractional assignment \( y' \) is still a valid \( w \)-matching of \( G \) and we have \( y'(E) - y(E) = \sum_{p \in P_d} z_p = z(E_H) \). By the assumptions of the lemma, we further know that \( y'(E) - y(E) \leq y^*(E) - y(E) \leq \delta w(S^*) \).

Let \( X_d^* \) and \( F_d^* \) be the node and edge sets induced by an optimal solution of the weighted set cover instance. Combining everything, we therefore obtain

\[
\begin{align*}
\frac{s(X_d) + y(F_d)}{2(1 + d \ln \Delta)} \cdot (s(X_d^*) + y(F_d^*)) & \leq 2(1 + d \ln \Delta) \cdot \frac{d+3}{2} \cdot z(E_H) \\
& \leq (d+3)(1 + d \ln \Delta) \cdot \delta \cdot w(S^*).
\end{align*}
\]

This concludes the proof of the lemma. 

\[\square\]
We now have everything that we need to prove our first main theorem, which shows that a \((1+\varepsilon)\)-approximation of the minimum weighted vertex cover problem can be computed deterministically in \(\text{poly}\left(\frac{\log n}{\varepsilon}\right)\) rounds in the \text{CONGEST} model.

**Proof of Theorem 1.1.** By Theorem 3.1, it is sufficient to show that we can compute a \((1+\varepsilon)\)-approximate weighted vertex cover solution in time \(\text{poly}\left(\frac{\log n}{\varepsilon}\right) \cdot D\) in bipartite graphs in the \text{SUPPORTED CONGEST} model with a bipartite communication graph of diameter \(D\). Let us therefore assume that we are given a node-weighted bipartite graph \(G = (A \cup B, E, w)\) on which we want to compute a \((1+\varepsilon)\)-approximate weighted vertex cover in the \text{SUPPORTED CONGEST} model with a bipartite communication graph \(H\) of diameter \(D\). Note that because \(H\) has diameter \(D\), we can compute a 2-coloring of \(H\) in \(O(D)\) rounds in the \text{CONGEST} model and because \(G\) is a subgraph of \(H\), we can therefore also compute the bipartition of the nodes of \(G\) into \(A\) and \(B\) in time \(O(D)\). In the following, we will therefore assume that the nodes of \(G\) know if they are in \(A\) or in \(B\).

As described in Section 3.3, as a first step for solving \(\text{MWVC}\) in \(G\), we choose a sufficiently small parameter \(\delta > 0\) and we compute a \((1-\delta)\)-approximate fractional \(w\)-matching \(y\) of \(G\). By using Theorem A.3, we can do this in time \(\text{poly}\left(\frac{\log n}{\varepsilon}\right)\) (recall that we assume that all weights are polynomially bounded integers). As discussed in Section 3.3, we now need to transform our instance so that we have no short augmenting paths and can afterwards apply Theorem 3.2 to compute a vertex cover of \(G\). To transform the instance, we need to determine sets \(X \subseteq \bigcup_{d=1}^{2k-1} S_d\) and \(F \subseteq \bigcup_{d=1}^{2k-1} Y_d\) such that for every augmenting path \(P = (v_0, v_1, \ldots, v_d)\) of length \(d \leq 2k-1\) for \(k = \left\lceil \frac{2}{\varepsilon} \right\rceil\), either \(v_0\) or \(v_d\) is in \(X\) or one of the edges \(\{v_{2i-1}, v_{2i}\}\) for \(i \in \{1, \ldots, (d-1)/2\}\) is in \(F\). If we have such sets \(X\) and \(F\), then compute new node weights \(w'\) and a new fractional \(w'\)-matching \(y'\) as given by Equation (2). Theorem 3.3 implies that \(w.r.t.\) weights \(w'\) and the fractional \(w'\)-matching \(y'\), graph \(G\) then has no augmenting paths of length at most \(2k-1\). Further, Theorem 3.4 shows that if we compute an \((1+\varepsilon/2)\)-approximate weighted vertex cover \(S\) of \(G\) for weights \(w'\), then \(w(S) \leq (1+\varepsilon/2) \cdot w(S^*) + s(X) + y(F)\), where \(S^*\) is an optimal weighted vertex cover of \(G\) \(w.r.t.\) the original weights \(w\). By Theorem 3.2, the \((1+\varepsilon/2)\)-approximate vertex cover of \(G\) \(w.r.t.\) weights \(w'\) can be compute in time \(O(D+1/\varepsilon)\) in the \text{SUPPORTED CONGEST} model with a communication graph of diameter \(D\).

We therefore need to show that we can find appropriate sets \(X\) and \(F\) such that \(s(X) + y(F) \leq \frac{\varepsilon}{2} \cdot w(S^*)\). We compute the sets \(X\) and \(F\) in stages \(d = 1, 3, \ldots\) by iterating over the possible augmenting path lengths and by using Theorem 4.1. After each augmenting path length, we will apply the conversion given by (2). For each stage \(d = 1, 3, \ldots, 2k-1\), we define \(w_d\) to be the weight function that is used in stage \(d\) and \(y_d\) to be the fractional \(w_d\)-matching that is used in stage \(d\). For the first stage, we therefore have \(w_1 = w\) and \(y_1 = y\). For each stage \(d\), we further define \(y_d^*\) to be an optimal fractional \(w_d\)-matching. Note that because \(y_1\) is a \((1-\delta)\)-approximate \(w\)-matching of \(G = (V, E, w)\), we have \(y_1^*(E) - y_1(E) \leq \delta y_1^*(E) \leq \delta w(S^*)\), where the last inequality follows from Theorem 2.1. We can therefore apply Theorem 4.1 for \(d = 1\) and we obtain sets \(X_1\) and \(F_1\). We now obtain new weights \(w_2\) and a new fractional \(w_2\)-matching \(y_2\) by applying (2). That is, we set \(w_2(v) = w_1(v) - s(v)\) for every \(v \in X_1\), we set \(y_{2e} = 0\) for every \(e \in F_1\), and we reduce the weights of the incident nodes of such edges accordingly (for each reduced edge value, the corresponding node weight is reduced by the same amount). Note that whenever we reduce the weights of both nodes of an edge \(e\) by the same amount \(x\), then the value of an optimal fractional \(w\)-matching is reduced by at least \(x\). We therefore have \(y_2^*(E) \leq y_1^*(E) - y_1(F_1)\). By construction, we also have \(y_2(E) = y_1(E) - y_1(F_1)\) and we therefore get \(y_2^*(E) \leq \delta \cdot w(S^*)\). We can therefore again apply Theorem 4.1 for \(d = 3\). If we continue like this, we obtain sets \(X_1, \ldots, X_{2k-1}\) and set \(F_1, \ldots, F_{2k-1}\) such that for all \(d\), \(s(X_d) + y(F_d) \leq (d+3)(1+d \ln \Delta) \cdot \delta \cdot w(S^*) \in O(k^2 \log \Delta) \cdot \delta \cdot w(S^*)\).
By construction, the sets $X := X_1 \cup X_3 \cup \cdots \cup X_{2k-1}$ and $F := F_1 \cup F_3 \cup \cdots \cup F_{2k-1}$ cover all augmenting paths of length at most $2k-1$ w.r.t. to the original weight function $w$ and the original fractional $w$-matching $y$. Because there are $k$ stages, we further have

$$s(X) + y(F) \in O(k^3 \log \Delta) \cdot \delta \cdot w(S^*) \in O\left(\frac{\log \Delta}{\varepsilon^3}\right) \cdot \delta \cdot w(S^*).$$

If we choose $\delta \leq c \cdot \varepsilon^4$ for a sufficiently small constant $c > 0$, we therefore have $s(X) + y(F) \leq \frac{\varepsilon}{2} \cdot w(S^*)$. With this choice for $\delta$, both the algorithm to reduce the diameter (cf. Theorem 3.1) and the algorithm for getting rid of short augmenting paths (cf. Theorem 4.1) have a round complexity of $\text{poly}\left(\frac{\log n}{\varepsilon}\right)$, which concludes the proof of the theorem. \[ \square \]

### 4.2 Weighted Vertex Cover in General Graphs

We next discuss the additional technical details needed to prove our upper bound for approximating minimum (weighted) vertex cover in general graphs. As discussed in Section 3.4, the high-level idea is as follows. We first (virtually) construct the bipartite double cover $G_2$ of our graph $G$ and we then apply the $(1 + \varepsilon)$-approximation algorithm for bipartite graph from Theorem 1.1. We can then transform this solution to a half-integral $(1 + \varepsilon)$-approximation of the fractional weighted vertex cover problem on $G$. Removing an independent set from the graph induced by the half-integral nodes gives the desired approximation. The details are given in the following proof of Theorem 1.2.

**Proof of Theorem 1.2.** As discussed, given a weighted graph $G = (V, E, w)$, we first construct the bipartite double cover $G_2 = (V_2, E_2, w)$, where the weight function $w$ is extended to $G_2$ in the obvious way (and as described in Section 3.4). Note that since every node of $G$ only needs to simulate 2 nodes in $G_2$ and every edge of $G$ is replaced by 2 edges in $G_2$, CONGEST algorithms on $G_2$ can be simulated on $G$ with only constant overhead. By using Theorem 1.1, we can therefore compute a $(1 + \varepsilon)$-approximation of MWVC on $G_2$ in time poly $\left(\frac{\log n}{\varepsilon}\right)$. By Theorem 3.5, this can directly be turned into a half-integral $(1 + \varepsilon)$-approximate fractional weighted vertex cover of $G$. Let $S_1$ be the nodes of $G$ that have a fractional vertex cover value of 1 and let $S_{1/2}$ be the nodes of $G$ that have a fractional vertex cover value of $1/2$. Assume further that $I_{1/2}$ is an independent set of the induced subgraph $G[S_{1/2}]$. Clearly, $S := S_1 \cup S_{1/2} \setminus I_{1/2}$ is a vertex cover of $G$. If the weight $w(I_{1/2})$ is $w(I_{1/2}) \geq \lambda \cdot w(S_{1/2})$, then the weight of the vertex cover $S$ can be bounded as

$$w(S) = w(S_1) + w(S_{1/2}) - w(I_{1/2}) \leq w(S_1) + (1 - \lambda) \cdot w(S_{1/2}) \leq (2 - 2\lambda) \cdot \left(\frac{w(S_1)}{1} + \frac{1}{2} \cdot w(S_{1/2})\right) \leq (2 - 2\lambda) \cdot (1 + \varepsilon) \cdot w(S^*),$$

where $S^*$ is an optimal weighted vertex cover of $G$. The claim of the theorem now follows. \[ \square \]

We can now also directly prove Theorem 1.3, which states that in graphs that can efficiently be colored with $C$ colors, we can compute a $(2 - 2/C + \varepsilon)$-approximation of MWVC in polylogarithmic time.

**Proof of Theorem 1.3.** By Theorem 3.1, we can first reduce the diameter of the communication graph in time poly $\left(\frac{\log n}{\varepsilon}\right)$ while only losing a $(1 + \varepsilon/2)$-factor in the approximation. Now assume that we are given a weighted graph $G$ with a $C$-coloring, that we have a communication graph
H of diameter $D$, and that we can use the SUPPORTED CONGEST model. In time $O(D)$ in the SUPPORTED CONGEST model, we can then use the C-coloring to compute an independent set of weight at least $w(V(G))/C$ by just keeping the heaviest color class. The corollary therefore follows directly by combining Theorem 3.1 and Theorem 1.2.

5 Technical Details: Weighted Matching Algorithms

Before we start with analyzing our matching algorithms, we first provide two helpful lemmas, which we will use in the randomized and deterministic approach. The first lemma states that if a matching $M$ is sufficiently far from optimal, then augmenting along a carefully chosen set of vertex-disjoint augmenting paths and cycles up to some maximal length guarantees a gain that is half as large as the maximum possible gain. Similar statements appeared before and were used in parallel algorithms and in the distributed context in the LOCAL model (e.g., [DH03, GKMU18, HV06, LPP15, Nie08]). Throughout this section, we assume that we are given an edge-weighted $n$-node graph $G = (V, E, w)$ with maximum degree $\Delta$ and edge weights $w(e) \in \{1, \ldots, W\}$ (for $W \leq n^{O(1)}$), and we assume that $M^*$ is an optimal weighted matching of $G$.

Lemma 5.1. For every matching $M$ with $w(M) < (1 - \varepsilon/2) \cdot w(M^*)$, there exists a set of vertex-disjoint augmenting paths and cycles of length at most $\ell = O(1/\varepsilon)$ such that augmenting over all those structures increases the weight of $M$ by at least $\frac{\varepsilon}{4} \cdot w(M^*)$.

Proof. Let $F := M \Delta M^*$ be the symmetric difference between $M$ and some optimal matching $M^*$. Clearly, $F$ induces a collection of paths and even cycles. By optimality of $M^*$, each of those paths/cycles either is an augmenting path/cycle or the total weight of all $M$-edges on the path/cycle is equal to the total weight of the total weight of the $M^*$-edges on the path/cycle. The set of augmenting paths and cycles induced by $F$ are vertex-disjoint and they together have a gain of exactly $w(M^*) - w(M) \geq \varepsilon/2 \cdot w(M^*)$. Some of the augmenting paths and cycles might however be long (we could potentially have a single augmenting path or cycle of length $|M| + |M^*|$) and we need to split into short vertex-disjoint augmenting paths and cycles.

To do this, we first (arbitrarily) partition each of the long augmenting paths and cycles into subpaths such that all except one subpath are of length exactly $x := \lceil \ell / 3 \rceil$ and the last subpath is of length between $x$ and $2x - 1$. In all of those subpaths, we define a light edge, where the light edge is defined as an $e \in M$ with the lowest weight among all edges in $M$ in this subpath. We now remove the light edge of every subpath. Because we remove one edge of each of the subpaths, this splits each long augmenting path or cycle into short paths of length at most $(x - 1) + (2x - 1 - 1) = 3(x - 1) \leq \ell$.

Since every light edge $e$ is chosen to be minimal among the edges of $M$ and there are at least $x/2 - 1 \geq \ell/6 - 1$ edges of $M$ in each subpath, the total weight of the light edges is at most a $\frac{6}{\ell - 6}$-fraction of the weight of all edges in $M$. Summing this up over all subpath and choosing $\ell \geq 24/\varepsilon + 6$, we can lower bound the total gain of the remaining augmenting paths and cycles of length at most $\ell$ by

$$w(M^*) - w(M) - \left(\frac{6}{\ell - 6}\right)w(M) \geq w(M^*) - \left(1 - \frac{\varepsilon}{2}\right) \cdot w(M^*) - \frac{\varepsilon}{4} \cdot w(M)$$

$$\geq \frac{\varepsilon}{4} \cdot w(M^*)$$

The second lemma was proven in [AKO18] and it in particular shows the existence of an efficient deterministic distributed approximation scheme for maximum weighted matchings in bipartite graphs. We use this algorithm as a subroutine in our randomized and in our deterministic weighted matching algorithm.
Lemma 5.2. [AKO18] There is a deterministic CONGEST algorithm to compute a \((1 - \varepsilon)\)-approximate MWM in bipartite graphs in time \(O \left( \frac{\log(\Delta W)}{\varepsilon^2} + \frac{\log^2(\Delta/\varepsilon) + \log^*(n)}{\varepsilon} \right)\). Further, in the same asymptotic time a \((2/3 - \varepsilon)\)-approximation can be computed on general graphs.

Proof. Combining Theorem 2 and 3 of [AKO18] directly gives this result.

5.1 Randomized Matching

The randomized version of our matching algorithm works as follows: Initially, a \((1/2)\)-approximation of the MWM problem is computed on the whole graph \(G\). This can be done using Theorem 5.2 (by setting \(\varepsilon = 1/6\)). Let the resulting matching of \(G\) be \(M_0\). Starting from this matching, we perform \(T\) (the value of \(T\) will be determined later) iterations of the following procedure. Consider iteration \(i \in \{1, \ldots, T\}\). In iteration \(i\), we transform an existing matching \(M_{i-1}\) into a new matching \(M_i\).

To achieve this, each node independently colors itself black or white with probability \(1/2\). Using that coloring, we construct a bipartite subgraph \(H_i = (\hat{V}_i, \hat{E}_i)\) by using the construction described in Section 3.5. We then use Theorem 5.2 to find a \((1 - \lambda)\)-approximate weighted matching of the sampled bipartite graph \(H_i\), where the parameter \(\lambda\) is set to \(\varepsilon/(2T)\). The new matching \(M_i\) is then defined as the union of the matching edges of this matching of \(H_i\) and the matching edges of \(M_{i-1}\) that are located outside of the bipartition, i.e., the monochromatic edges from the previous matching.

In the subsequent analysis, we will show that it is sufficient for any \(\delta \in (0, 1/2]\) to choose the number of iterations \(T\) (i.e., the number of bipartitions) as \(T = 2^{O(1/\varepsilon)} \cdot \ln^3(1/\delta)\) to find a \((1 - \varepsilon)\)-approximation with probability at least \(1 - \delta\).

When computing the matching on \(H_i\), the edges of the existing matching \(M_{i-1}\) that are part of the graph \(H_i\) are completely ignored. As a result, it is possible that \(w(M_i) < w(M_{i-1})\), that is, we cannot guarantee that the matching weight is monotonically increasing during our algorithm. To analyze this potential loss in the matching quality, we fix some optimal weighted matching \(M^*\) of \(G\) and we take a look at the weight difference between the following matchings: \(\hat{M}_i^* := \hat{E}_i \cap M^*\) contains the edges of \(M^*\) that are sampled in iteration \(i\) and \(M_i' := \hat{E}_i \cap M_{i-1}\) contains the edges of \(M_{i-1}\) that are sampled in iteration \(i\).

Lemma 5.3. In any iteration \(i > 0\) in which \(w(M_{i-1}) < (1 - \varepsilon/2) \cdot w(M^*)\), the expected value of \(w(\hat{M}_i^*) - w(M'_i)\) is at least \(2^{-O(1/\varepsilon)} \cdot w(M^*)\). Moreover, there is a positive value \(c(\varepsilon) = 2^{-O(1/\varepsilon)}\) such that we have \(w(\hat{M}_i^*) - w(M'_i) > c(\varepsilon) \cdot w(M^*)\) with probability at least \(c(\varepsilon)/8\).

Proof. Let \(P\) be an augmenting path or cycle of length \(k\) w.r.t. \(M_{i-1}\) in \(G\). If \(P\) is an augmenting path, then the probability that \(P\) is contained in graph \(H_i\), i.e., \(\mathbb{P}(E(P) \subseteq \hat{E}_i) = 2^{-k}\) (where \(E(P)\) denotes the set of edges of \(P\)). If \(P\) is an augmenting cycle of (even) length \(k\), then \(\mathbb{P}(P \subseteq \hat{E}_i) = 2^{-k+1}\).

By Theorem 5.1, we know that there exists a set of vertex-disjoint augmenting paths and cycles of length at most \(\ell = O(1/\varepsilon)\) and with total gain at least \(\frac{\varepsilon}{4} \cdot w(M^*)\). By the above observation, each of those augmenting paths or cycles is contained in \(H_i\) with probability at least \(\varepsilon/2 = 2^{-O(1/\varepsilon)}\). The lower bound on the expectation of \(w(\hat{M}_i^*) - w(M'_i)\) then follows by linearity of expectation, i.e., \(\mathbb{E}[w(\hat{M}_i^*) - w(M'_i)] \geq 2^{-O(1/\varepsilon)} \cdot w(M^*)\). From here we can conclude that there exist some value \(c(\varepsilon) = 2^{-O(1/\varepsilon)}\) s.t. \(\mathbb{E}[w(\hat{M}_i^*) - w(M'_i)] \geq 2c(\varepsilon) \cdot w(M^*)\). The lower bound on the probability follows because the vertex-disjoint paths and cycles are sampled independently where each path/cycle has
a gain of at most \(w(M^*)\) and we can therefore apply a Chernoff bound (see Theorem A.1):

\[
\mathbb{P} \left( w(\hat{M}_i^*) - w(M_i^*) \leq c(\varepsilon) \cdot w(M^*) \right) = \mathbb{P} \left( w(\hat{M}_i^*) - w(M_i^*) \leq \left( 1 - \frac{1}{2} \right) \cdot 2c(\varepsilon) \cdot w(M^*) \right) \\
\leq e^{- \frac{1}{5} \cdot 2c(\varepsilon) \cdot \frac{w(M^*)}{w(M_i^*)}} = e^{- \frac{c(\varepsilon)}{4}} \leq 1 - \frac{c(\varepsilon)}{8}
\]

In the last step we use that \(e^{-x} \leq 1 - \frac{x}{2}\) for \(0 \leq x \leq 1\). \(\square\)

**Definition 5.1.** Let \(c(\varepsilon) = 2^{-O(1/\varepsilon)}\) (and \(c(\varepsilon) \leq 1\)) be chosen as required by Theorem 5.3. We call an iteration \(i\) **successful** if \(w(\hat{M}_i^*) - w(M_i^*) > c(\varepsilon) \cdot w(M^*)\) is true.

Note that by Theorem 5.3, as long as \(w(M) < (1 - \varepsilon/2) \cdot w(M^*)\), an iteration is successful with probability at least \(c(\varepsilon)/8\). Let \(T\) be the overall number of iterations and recall that \(\lambda\) states the approximation factor used to compute the matching in every bipartition.

**Lemma 5.4.** Choosing \(\lambda = \frac{\varepsilon}{2T}\) guarantees that if the algorithm ever computes a matching \(M_i\) where \(w(M_i) \geq (1 - \varepsilon/2) \cdot w(M^*)\), there will be no subsequent iteration \(j > i\) (and \(j \leq T\)) such that \(w(M_j) < (1 - \varepsilon) \cdot w(M^*)\).

**Proof.** Since a \((1 - \lambda)\)-approximation is computed on the bipartite subgraph \(H_i\) in every iteration \(i\), it is clear that \(w(M_i) \geq w(M_i) - \lambda w(M^*)\). Even assuming a worst-case scenario, where we lose \(\lambda w(M^*) = \frac{\varepsilon}{2T} w(M^*)\) in all \(T\) rounds, the stated approximation is achieved:

\[
w(M_T) \geq \left( 1 - \frac{\varepsilon}{2} \right) w(M^*) - \frac{\varepsilon}{2T} \cdot T \cdot w(M^*) = (1 - \varepsilon) w(M^*)
\]

As a result of the previous lemma, we can essentially ignore executions in which we reach \(w(M_i) \geq (1 - \varepsilon/2) \cdot w(M^*)\) at some point. In the following, as soon as \(w(M_i) \geq (1 - \varepsilon/2) \cdot w(M^*)\), all subsequent iterations are called successful independently of how the matching weight changes.

We will now show that for \(T = \Theta \left( \frac{1}{c(\varepsilon)^2} \log \left( \frac{1}{\delta} \right) \right)\) iterations, the first part of Theorem 1.4 holds.

**Lemma 5.5.** For every \(\varepsilon, \delta \in (0, \frac{1}{2}]\), our procedure computes a \((1 - \varepsilon)\)-approximation of the maximum weighted matching problem in \(2^{O(1/\varepsilon)} \cdot (\log^2 \Delta + \log(\Delta \cdot W) + \log^* n) \cdot \log^3 (1/\delta)\) rounds with probability at least \(1 - \delta\) in CONGEST.

**Proof.** Let \(S\) be the number of successful iterations among all the \(T := \frac{16}{c(\varepsilon)^2} \ln \left( \frac{1}{\delta} \right)\) iterations. From the definition of a successful iteration and from Theorem 5.3, we know that every iteration is successful with probability at least \(c(\varepsilon)/8\), independently of the what happened in previous iterations. We therefore have \(\mathbb{E}[S] \geq T \cdot \frac{c(\varepsilon)}{8} = 2 \cdot \frac{1}{c(\varepsilon)} \ln \left( \frac{1}{\delta} \right)\). Because the probability for being successful is always at least \(c(\varepsilon)/8\), independently of previous iterations, the random variable \(S\) dominates a random variable that is defined as the sum of \(T\) independent random \((0,1)\)-random variables, which are all equal to 1 with probability exactly \(c(\varepsilon)/8\). Applying a Chernoff bound (Theorem A.1) results in the subsequent statement.

\[
\mathbb{P} \left( S \leq \frac{\ln \left( \frac{1}{\delta} \right)}{c(\varepsilon)} \right) = \mathbb{P} \left( S \leq \frac{\mathbb{E}[S]}{2} \right) \leq e^{\frac{\ln \delta}{2 c(\varepsilon)}} \leq \delta
\]
In the following, assume that \( w(M_i) \geq (1 - \varepsilon/2) \cdot w(M^*) \) holds at no time (otherwise, Theorem 5.4 implies that we obtain at least a \((1 - \varepsilon)\)-approximation at the end). We then now know that with probability at least \( 1 - \delta \) we have more than \( \frac{1}{c(\varepsilon)} \ln \left( \frac{\delta}{\varepsilon} \right) \) successful iterations, where each successful iteration improves the current matching by at least \((1 - \lambda) c(\varepsilon) \cdot w(M^*)\) while in non-successful iterations we lose at most \( \lambda \cdot w(M^*) \). Recall that we choose \( \lambda = \frac{2}{T^2} \) (regarding Theorem 5.4) and \( w(M_0) \geq w(M^*)/2 \). The overall weight after \( T \) iterations can be bounded as follows:
\[
w(M_T) \geq w(M_0) + S \cdot (1 - \lambda)c(\varepsilon) \cdot w(M^*) - (T - S)\lambda \cdot w(M^*) \\
\geq w(M^*)/2 + c(\varepsilon) \cdot S \cdot w(M^*) - \lambda \cdot T \cdot w(M^*).
\]
By the assumption that \( \delta \leq 1/2 \), this term simplifies to \((1 - \varepsilon)w(M^*)\) with probability at least \( 1 - \delta \), because then \( c(\varepsilon) \cdot S > \ln(1/\delta) > 1/2 \). This proves the stated approximation guarantee.

Since by Theorem 5.2, each weighted bipartite matching instance (and therefore each iteration) runs in \( O \left( \frac{\log(\Delta W)}{\lambda^2} + \frac{\log^2(\Delta/\lambda) + \log^2(n)}{\lambda} \right) \) rounds in CONGEST, the runtime for all \( T \) iterations is as stated.

5.2 Deterministic Matching

As stated in Section 3.5, the deterministic version of our weighted matching algorithm differs from the randomized one in a few ways. First, instead of running the algorithm on the whole graph, we use Theorem 3.1 to first decompose the network into clusters of small (weak) diameter, where afterwards each cluster independently computes a good approximation. The clusters allow us to efficiently compute the exact weight of some matching \( M \). In each iteration \( i \), we first compute a new matching \( M_i \) by applying the deterministic weighted matching algorithm of [AKO18]. In each cluster, we then compare \( w(M_i) \) to the weight \( w(M_{i-1}) \) of the previous matching. If \( w(M_i) - w(M_{i-1}) < \frac{\varepsilon}{T^2} \cdot w(M_{i-1}) \), we discard the new matching and we simply set \( M_i \) to be equal to \( M_{i-1} \). Note that the parameter \( \lambda \) actually has the same meaning but a different value here than in the randomized approach. The number of bipartitions used is again denoted by \( T \), whereby we have to work multiple times on the same bipartition here. For that reason the algorithm proceeds in stages. In each stage the algorithm iterates through the bipartitions \( H_1, \ldots, H_T \), each time executing the above procedure. We will later see that in each stage there exists at least one bipartition \( H_i \) where our approximation improves the current matching \( M_{i-1} \) by at least \( \frac{\varepsilon}{T^2} \cdot w(M_{i-1}) \). Proceeding through \( O(T) \) many stages is then sufficient to compute the desired \((1 - \varepsilon)\)-approximation. In the following, we concentrate on the computation in a single cluster of the decomposition. More specifically, assume that we can use the SUPPORTED CONGEST model with a communication graph of diameter \( \log \frac{n}{\varepsilon} \). The result can then be lifted to the CONGEST model in general graphs by applying Theorem 3.1. Working in the SUPPORTED CONGEST model allows us to assume that the nodes are uniquely labeled with IDs from 1 to \( n \) (we can relabel the nodes in such a way in time linear in the diameter of the communication graph).

As discussed in Section 3.5, in our deterministic algorithm, we need to compute the bipartitions deterministically. The following lemma shows that there exists a sequence of bipartitions that is ‘good’ in every graph and for every collection of short augmenting paths and cycles.

**Lemma 5.6.** Let \( N > 0 \) and \( k \leq N \) be two integers. There exists a collection of \( T = O(k \cdot 2^k \cdot \log N) \) functions \( f_1, \ldots, f_T : [N] \to \{0, 1\} \) such that for every vector \( (x_1, \ldots, x_k) \in [N]^k \) with pairwise disjoint entries, there exists a function \( f_i \) in the collection such that \( f_i(x_j) = j \mod 2 \) for every \( 1 \leq j \leq k \).
Theorem 5.6: Theorem 5.6: Theorem 5.6: Theorem 5.6
and as discussed above, we only keep this matching, each stage improves the weight of the matching.

Theorem 3.1). Since Theorem 5.2 sufficiently large.

In each phase, this is to compute a bipartition of our graph as follows. Assume that the graph gives a final $T = k \cdot 2^k$. In $N$ pushes this probability below 1, which shows that there must exist a collection of $T$ functions $f_1, \ldots, f_T$ s.t. for every possible vector at least one of those functions will take care.

We can use Theorem 5.6 to compute a bipartition of our graph as follows. Assume that the nodes are labeled with unique IDs from 1 to $N$. When using a particular function $f_i$ to partition the nodes, each node just applies $f_i$ to its ID and gets colored black or white accordingly. Note that as observed above, we can assume that $N \leq n$. Our algorithm now works in consecutive phases, where in each phase, we iterate through all the $T$ bipartitions given by Theorem 5.6. In each phase, this induces $T$ bipartite graphs $H_1, \ldots, H_T$. We choose $k = O(1/\varepsilon)$ in the constructions large enough so that Theorem 5.1 applies (when setting $\ell$ to $k$). We compute a $(1-\lambda)$-approximate matching on each of these graphs $H_i$ by using Theorem 5.2 and as discussed above, we only keep this matching if it improves the weight of the current matching by at least $\frac{\lambda}{8T} \cdot w(M_i)$. The next lemma shows that if $\lambda$ is chosen sufficiently small, there is an iteration in every stage, in which we improve the matching.

Lemma 5.7. Assume that $\lambda \leq \varepsilon/(8T)$. For every matching $M$ with $w(M) \leq (1-\varepsilon)w(M^*)$, there exists one bipartition $H_i$ from the collection $H_1, \ldots, H_T$, such that computing a $(1-\lambda)$-approximate weighted matching on $H_i$ improves $M$ by at least $\frac{\lambda}{8T} \cdot w(M)$.

Proof. Recall that we choose the parameter $k$ in the construction of Theorem 5.6 sufficiently large so that Theorem 5.1 guarantees that (w.r.t. matching $M$) there exists a collection of vertex-disjoint augmenting paths and cycles of length at most $k$ and with total gain at least $\frac{\lambda}{4} \cdot w(M^*)$. The statement of Theorem 5.6 leads to the fact that each of these augmenting paths or cycles appears in at least one of the bipartite graphs $H_1, \ldots, H_T$. By the pigeonhole principle, there is a bipartite graph $H_i$ in which we can make a total gain of $\frac{\lambda}{8T} \cdot w(M^*)$. If we choose $\lambda \leq \varepsilon/(8T)$, then a $(1-\lambda)$-approximate matching in $H_i$ still guarantees a gain of $\frac{\lambda}{8T} \cdot w(M^*) \geq \frac{\lambda}{8T} \cdot w(M)$.

We can now prove the second part of Theorem 1.4.

Lemma 5.8. There exists a deterministic CONGEST algorithm to compute a $(1-\varepsilon)$-approximation to the weighted matching in time

$$2^{O(1/\varepsilon)} \cdot \text{poly} \log n.$$ 

Proof. We first prove the claim on the approximation ratio. Recall that we start with a $1/2$-approximate weighted matching. Then, by Lemma 5.7, each stage improves the weight of the given matching $M$ by at least $\frac{\lambda}{8T} \cdot w(M) \geq \frac{\lambda}{16T} \cdot w(M^*)$. Therefore, $O(T/\varepsilon)$ stages suffice to achieve a final $(1-\varepsilon)$-approximation for MWM.

We now show that $O(T/\varepsilon)$ stages of length $T$ can be implemented in the claimed round complexity. By Theorem 3.1, it is sufficient to prove the claim of the lemma for algorithms in the SUPPORTED CONGEST model with a communication graph of polylogarithmic diameter. The algorithm for each bipartition then clearly requires at most $\text{poly}(\frac{\log n}{\varepsilon})$ rounds (Theorem 5.2). Since
we have $T$ bipartitions in every stage and $O(T/\varepsilon)$ many stages, the total round complexity is bounded by
\[
\text{poly}\left(\frac{\log n}{\lambda}\right) \cdot T \cdot O\left(\frac{T}{\varepsilon}\right) = T^2 \cdot \text{poly}\left(\frac{\log n}{\lambda \varepsilon}\right).
\]
The claim of the lemma now follows because by Theorem 5.6, we have $T = 2^{O(1/\varepsilon)} \cdot \log n$ and because by Theorem 5.7, we can choose $\lambda = \Theta(\varepsilon/T)$. \hfill \square

Theorem 5.5 and Theorem 5.8 together complete the proof of Theorem 1.4.

6 Bipartite Vertex Cover Lower Bound

We next prove Theorem 1.5, i.e., we prove that even in bipartite graph of maximum degree 3, there exists a constant $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, computing a $(1 + \varepsilon)$-approximate vertex cover requires $\Omega(\frac{\log n}{\varepsilon})$ rounds in the LOCAL model.

**Proof of Theorem 1.5.** In [GS14], Göös and Suomela showed that there exists a bipartite graph $G = (V_G, E_G)$ with maximum degree 3 and a constant $\varepsilon_0 > 0$ such that no randomized distributed algorithm with running time $o(\log n)$ can find a $(1 + \varepsilon_0)$-approximate vertex cover on $G$. To extend this proof to smaller approximation ratios, we proceed as follows. Given a positive integer parameter $k$, we construct a new lower bound graph $H$ as follows. Graph $H$ is obtained from graph $G$ by replacing every edge $e$ of $G$ by a path $P_e$ of length $2k + 1$.

Assume that we are given a vertex cover $S_H$ of graph $H$. We first describe a method to transform the vertex cover $S_H$ into a vertex cover $S'_H$ of $H$ such that $|S'_H| \leq |S_H|$ and such that $S'_H$ has the following form. For each edge $e$ of $G$, $S'_H$ contains exactly $k$ of the inner nodes of the path $P_e$ in $H$ and it contains at least one of the end nodes of $P_e$. The transformation is done independently for each path $P_e$ as follows. Let $P_e = (v_0, v_1, \ldots, v_{2k+1})$ be the path that replaces edge $e = \{v_0, v_{2k+1}\}$ in $G$. If $S_H \cap \{v_0, v_{2k+1}\} \neq \emptyset$, we add the nodes $S_H \cap \{v_0, v_{2k+1}\}$ also to $S'_H$. Otherwise, if $S_H \cap \{v_0, v_{2k+1}\} = \emptyset$, we arbitrarily add either $v_0$ or $v_{2k+1}$ to $S'_H$. Further $S'_H$ contains exactly $k$ of the inner nodes $v_1, \ldots, v_{2k}$ of $P_e$ in such a way that every edge of $P_e$ is covered by some node in $S'_H$. If $v_0 \in S'_H$, we can add all nodes $v_{2i}$ for $i \in \{1, \ldots, k\}$ and otherwise we can add all nodes $v_{2i-1}$ for $i \in \{1, \ldots, k\}$. Clearly $S'_H$ is a vertex cover of $H$. To see that $|S'_H| \leq |S_H|$, observe that in order to cover all $2k + 1$ edges of $P_e$, every vertex cover of $H$ must contain at least $k + 1$ of the nodes of $P_e$ and it also must contain at least $k$ of the inner nodes of $P_e$. If $S_H \cap \{v_0, v_{2k+1}\} \neq \emptyset$, $S'_H$ only differs in terms of the inner nodes of $P_e$ from $S_H$ and we know that also $S_H$ must contain at least $k$ inner nodes of $P_e$. If $S_H \cap \{v_0, v_{2k+1}\} = \emptyset$, we add either $v_0$ or $v_{2k+1}$ to $S'_H$. However in this case, $S_H$ contains at least $k + 1$ inner nodes of $P_e$ and $S'_H$ only contains $k$ inner nodes of $P_e$. The transformation from $S_H$ to $S'_H$ can be done independently for each of the paths $P_e$ of length $2k + 1$ and it can therefore be done in $O(k)$ rounds in the LOCAL model.

Let $e_G$ be the number of edges of $G$ and let $s_G$ be the size of an optimal vertex cover of $G$. Because the maximum degree of $G$ is 3 and we have an optimal vertex cover of $G$, we get that $e_G = c \cdot s_G$ for some constant $c \leq 3$. By the observation above, any vertex cover $S_H$ of $H$ can be transformed into an equally good vertex cover $S'_H$ with a nice structure. Note that $S'_H$ consists of exactly $k$ inner nodes of each path $P_e$ for $e \in E_G$ and it consists of at least one of the end nodes of each such path (i.e., of a vertex cover of $G$). We therefore obtain that there is an optimal vertex cover of $H$ that consists of an optimal vertex cover of $G$ and of $k$ inner nodes of each $(2k + 1)$-hop path $P_e$ replacing an edge $e$ of $G$. The size $s_H$ of an optimal vertex cover of $H$ is therefore exactly $s_H = s_G + k \cdot e_G = (1 + ck) \cdot s_G$. 

28
Assume now that we have a $T$-round algorithm to compute a $(1 + \varepsilon)$-approximate vertex cover $S_H$ on graph $H$ for some $\varepsilon \leq \varepsilon_0/(1+3k) \leq \varepsilon_0/(1+ck)$. By the above observation, in $O(k)$ rounds, we can transform this vertex cover into a vertex cover $S'_H$, which contains $k \cdot e_G = ck \cdot s_G$ inner path nodes and at least one of the end nodes of each $(2k + 1)$-hop path replacing an edge of $G$ in $H$. The vertex cover $S'_H$ of $H$ therefore induces a vertex cover of $G$ of size $|S'_H| - k \cdot e_G = |S'_H| - ck \cdot s_G$. Because we assumed that $\varepsilon \leq \varepsilon_0/(1+ck)$, we have $|S'_H| \leq (1+\varepsilon) \cdot (1+ck) \cdot s_G \leq (1+\varepsilon_0) s_G + ck \cdot s_G$. The vertex cover $S'_H$ of $H$ therefore induces a $(1+\varepsilon_0)$-approximate vertex cover of $G$. Assume that we want to compute a vertex cover of $G$. To do this, the nodes of $G$ can simulate graph $H$ by adding $2k$ virtual nodes on each edge of $G$. An $R$-round algorithm on $H$ can then be run in $O([R/k]) = O(1+R/k)$ rounds on $G$. We can therefore compute the vertex cover $S'_H$ on the virtual graph $H$ in $O((1+(T+k)/k) = O(1+T/k)$ rounds on $G$. Since $k = \Theta(1/\varepsilon)$, the lower bound of $[GS14]$ implies a lower bound of $\Omega(\frac{n}{\varepsilon \log(n)})$ on the time $T$ for computing a $(1+\varepsilon)$-approximate vertex cover on $H$. Finally note that since $G$ is bipartite, then $H$ is also bipartite and clearly if the maximum degree of $G$ is 3, then the maximum degree of $H$ is also 3.

References

[ABB⁺19] S. Assadi, M. Bateni, A. Bernstein, V. Mirrokni, and C. Stein. Coresets meet EDCS: Algorithms for matching and vertex cover on massive graphs. In Proc. 30th ACM-SIAM Symp. on Discrete Algorithms (SODA), pages 1616–1635, 2019.

[ÅFP⁺09] M. Åstrand, P. Flöreen, V. Polishchuk, J. Rybicki, J. Suomela, and J. Uitto. A local 2-approximation algorithm for the vertex cover problem. In Proc. 23rd Symp. on Distributed Computing (DISC), pages 191–205, 2009.

[AK20] M. Ahmadi and F. Kuhn. Distributed maximum matching verification in CONGEST. In Proc. 34th Symp. on Distributed Computing (DISC), pages 37:1–37:18, 2020.

[AKO18] M. Ahmadi, F. Kuhn, and R. Oshman. Distributed approximate maximum matching in the CONGEST model. In Proc. 32nd Symp. on Distributed Computing (DISC), pages 6:1–6:17, 2018.

[AP90] B. Awerbuch and D. Peleg. Sparse partitions. In Proc. 31st IEEE Symp. on Foundations of Computer Science (FOCS), pages 503–513, 1990.

[ÅS10] M. Åstrand and J. Suomela. Fast distributed approximation algorithms for vertex cover and set cover in anonymous networks. In Proc. 22nd ACM Symp. on Parallelism in Algorithms and Architectures (SPAA), pages 294–302, 2010.

[BBiKS19] Ran Ben-Basat, Ken ichi Kawarabayashi, and Gregory Schwartzman. Parameterized distributed algorithms. In Proc. 33rd In. Symp. on Distributed Computing (DISC), pages 6:1–6:16, 2019.

[BCGS17] R. Bar-Yehuda, K. Censor-Hillel, M. Ghaffari, and G. Schwartzman. Distributed approximation of maximum independent set and maximum matching. In Proc. 36th ACM Symp. on Principles of Distributed Computing (PODC), full version: arXiv:1708.00276, pages 165–174, 2017.
[BCM+20] R. Bar-Yehuda, K. Censor-Hillel, Y. Maus, S. Pai, and S. V. Pemmaraju. Distributed approximation on power graphs. In Proc. 39th ACM Symp. on Principles of Distributed Computing (PODC), pages 501–510, 2020.

[BCS16] R. Bar-Yehuda, K. Censor-Hillel, and G. Schwartzman. A distributed $(2+\varepsilon)$-approximation for vertex cover in $O(\log \Delta/\varepsilon \log \log \Delta)$ rounds. In Proc. 35th ACM Symp. on Principles of Distributed Computing (PODC), pages 3–8, 2016.

[BEKS18] R. Ben-Basat, G. Even, Kawarabayashi K, and G. Schwartzman. A deterministic distributed 2-approximation for weighted vertex cover in $O(\log N \log \Delta/\log^2 \log \Delta)$ rounds. In Proc. 25th Coll. on Structural Information and Comm. Complexity (SIROCCO), pages 226–236, 2018.

[BEKS19] R. Ben-Basat, G. Even, K. Kawarabayashi, and G. Schwartzman. Optimal distributed covering algorithms. In Proc. 33rd Symp. on Distributed Computing (DISC), pages 5:1–5:15, 2019.

[CG21] Y.-J. Chang and M. Ghaffari. Strong-diameter network decomposition. CoRR, abs/2102.09820, 2021.

[CH03] A. Czygrinow and M. Hańćkowiak. Distributed algorithm for better approximation of the maximum matching. In 9th Annual Int. Computing and Combinatorics Conf. (COCOON), pages 242–251, 2003.

[CHKP17] K. Censor-Hillel, S. Khoury, and A. Paz. Quadratic and near-quadratic lower bounds for the CONGEST model. In Proc. 31st Symp. on Distr. Computing (DISC), pages 10:1–10:16, 2017.

[CHS04] A. Czygrinow, M. Hańćkowiak, and E. Szymanska. A fast distributed algorithm for approximating the maximum matching. In Proceedings of 12th Annual European Symp. on Algorithms (ESA), pages 252–263, 2004.

[Chv79] V. Chvátal. A greedy heuristic for the set-covering problem. Mathematics of Operations Research, 4(3), 1979.

[DH03] D. E. Drake and S. Hougardy. Improved linear time approximation algorithms for weighted matchings. In Proc. 6th Int. Workshop on Approximation Algorithms for Combinatorial Optimization Problems, pages 14–23, 2003.

[Die05] R. Diestel. Graph Theory, chapter 2.1. Springer, Berlin, 3rd edition, 2005.

[DS05] I. Dinur and S. Safra. On the hardness of approximating vertex cover. Annals of Mathematics, 162(1):439–485, 2005.

[Edm65a] J. Edmonds. Maximum matching and a polyhedron with 0, 1-vertices. J. of Res. the Nat. Bureau of Standards, 69 B:125–130, 1965.

[Edm65b] J. Edmonds. Paths, trees, and flowers. Canadian Journal of Mathematics, 17(3):449–467, 1965.

[Egc31] J. Egerváry. Matrixok kombinatorius tulajdonságairól [On combinatorial properties of matrices]. Matematikai és Fizikai Lapok (in Hungarian), 38(16–28), 1931.
[EMR15] G. Even, M. Medina, and D. Ron. Distributed maximum matching in bounded degree
graphs. In Proceedings of the 2015 Int. Conf. on Distributed Computing and Networking
(ICDCN), pages 18:1–18:10, 2015.

[Fis17] M. Fischer. Improved deterministic distributed matching via rounding. In Proc. 31st
Symp. on Distributed Computing (DISC), pages 17:1–17:15, 2017.

[FK20] S. Faour and F. Kuhn. Approximating bipartite minimum vertex cover in the CON-
GEST model. In Proc. 24th Conf. on Principles of Distr. Systems (OPODIS), pages
29:1–29:16, 2020.

[FKRS19] K.-T. Foerster, J. H. Korhonen, J. Rybicki, and S. Schmid. Brief announcement: Does
preprocessing help under congestion? In Proc. 38th ACM Symp. on Principles of
Distributed Computing (PODC), pages 259–261, 2019.

[FMS15] U. Feige, Y. Mansour, and R. E. Schapire. Learning and inference in the presence of
corrupted inputs. In Proc. 28th Conf. on Learning Theory (COLT), pages 637–657,
2015.

[FMU21] M. Fischer, S. Mitrovic, and J. Uitto. Deterministic (1 + \(\varepsilon\))-approximate maximum
matching with poly(1/\(\varepsilon\)) passes in the semi-streaming model. CoRR, abs/2106.04179,
2021.

[GGR21] M. Ghaffari, C. Grunau, and V. Rozhoň. Improved deterministic network decom-
position. In Proc. 33rd ACM-SIAM Symp. on Discrete Algorithms (SODA), pages
2904–2923, 2021.

[GJN20] M. Ghaffari, C. Jin, and D. Nilis. A massively parallel algorithm for minimum weight
vertex cover. In Proc. 32nd ACM Symp. on Parallelism in Algorithms and Architectures
(SPAA), pages 259–268, 2020.

[GK21] M. Ghaffari and F. Kuhn. Deterministic distributed vertex coloring: Simpler, faster,
and without network decomposition. In Proc. 62nd IEEE Symp. on Foundations of
Computer Science (FOCS), 2021.

[GKM17] M. Ghaffari, F. Kuhn, and Y. Maus. On the complexity of local distributed graph
problems. In Proc. 39th ACM Symp. on Theory of Computing (STOC), pages 784–797,
2017.

[GKMU18] M. Ghaffari, F. Kuhn, Y. Maus, and J. Uitto. Deterministic distributed edge-coloring
with fewer colors. In Proc. 50th ACM Symp. on Theory of Comp. (STOC), pages
418–430, 2018.

[GKP08] F. Grandoni, J. Könemann, and A. Panconesi. Distributed weighted vertex cover via
maximal matchings. ACM Trans. Algorithms, 5(1):6:1–6:12, 2008.

[GKPS08] F. Grandoni, J. Könemann, A. Panconesi, and M. Sozio. A primal-dual bicriteria
distributed algorithm for capacitated vertex cover. SIAM J. Comput., 38(3):825–840,
2008.

[GS14] M. Göös and J. Suomela. No sublogarithmic-time approximation scheme for bipartite
vertex cover. Distributed Computing, 27(6):435–443, 2014.
[Har20] D. G. Harris. Distributed local approximation algorithms for maximum matching in graphs and hypergraphs. *SIAM J. Computing, 49*(4):711–746, 2020.

[Häst01] J. Hästad. Some optimal inapproximability results. *J. of the ACM, 48*(4):798—859, 2001.

[HK73] J. E. Hopcroft and R. M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM J. Comput., 2*(4):225–231, 1973.

[HKL06] J.H. Hoepman, S. Kutten, and Z. Lotker. Efficient distributed weighted matchings on trees. In *Proc 13th Coll. on Structural Inf. and Comm. Complexity (SIROCCO)*, pages 115–129, 2006.

[Hoc97] D. Hochbaum, editor. *Approximation Algorithms for NP-hard Problems*. PWS Publishing Company, 1997.

[HV06] S. Hougardy and D. E. Drake Vinkemeier. Approximating weighted matchings in parallel. *Inf. Process. Lett., 99*(3):119–123, 2006.

[II86] A. Israeli and A. Itai. A fast and simple randomized parallel algorithm for maximal matching. *Inf. Process. Lett., 22*(2):77–80, 1986.

[Kar09] G. Karakostas. A better approximation ratio for the vertex cover problem. *ACM Trans. Algorithms, 5*(4):41:1–41:8, 2009.

[Kön31] D. König. Gráfok és mátrixok. *Matematikai és Fizikai Lapok, 38*:116–119, 1931.

[KMW04] F. Kuhn, T. Moscibroda, and R. Wattenhofer. What cannot be computed locally! In *Proc. 23rd ACM Symp. on Principles of Distributed Computing (PODC)*, pages 300–309, 2004.

[KMW06] F. Kuhn, T. Moscibroda, and R. Wattenhofer. The price of being near-sighted. In *Proceedings of 17th Symp. on Discrete Algorithms (SODA)*, pages 980–989, 2006.

[KR08] S. Khot and O. Regev. Vertex cover might be hard to approximate to within 2-epsilon. *J. Comput. Syst. Sci., 74*(3):335–349, 2008.

[KVY94] S. Khuller, U. Vishkin, and N. E. Young. A primal-dual parallel approximation technique applied to weighted set and vertex covers. *J. Algorithms, 17*(2):280–289, 1994.

[KY11] C. Koufogiannakis and N. E. Young. Distributed algorithms for covering, packing and maximum weighted matching. *Distributed Computing, 24*(1):45–63, 2011.

[LPP15] Z. Lotker, B. Patt-Shamir, and S. Pettie. Improved distributed approximate matching. *J. ACM, 62*(5):38:1–38:17, 2015.

[LPR09] Z. Lotker, B. Patt-Shamir, and A. Rosén. Distributed approximate matching. *SIAM Journal on Computing, 39*(2):445–460, 2009.

[LS93] N. Linial and M. Saks. Low diameter graph decompositions. *Combinatorica, 13*(4):441–454, 1993.

[MPX13] G. L. Miller, R. Peng, and S. C. Xu. Parallel graph decompositions using random shifts. In *Proc. 25th ACM Symp. on Parallelism in Alg. and Arch. (SPAA)*, pages 196–203, 2013.
[Nie08] T. Nieberg. Local, distributed weighted matching on general and wireless topologies. In Proc. Joint Workshop on Foundations of Mobile Computing (DIALM-POMC), pages 87–92, 2008.

[Pel00] D. Peleg. Distributed Computing: A Locality-Sensitive Approach. SIAM, 2000.

[RG20] V. Rozhoň and M. Ghaffari. Polylogarithmic-time deterministic network decomposition and distributed derandomization. In Proc. 52nd ACM Symp. on Theory of Computing (STOC), pages 350–363, 2020.

[SS13] S. Schmid and J. Suomela. Exploiting locality in distributed SDN control. In Proc. 2nd ACM SIGCOMM Works. on Hot Topics in Software Defined Netw. (HotSDN), pages 121–126, 2013.

[WW04] M. Wattenhofer and R. Wattenhofer. Distributed weighted matching. In Proc. 18th Conf. on Distributed Computing (DISC), pages 335–348, 2004.

A Basic Tools

A.1 Chernoff Bound

We use the following Chernoff bound.

**Theorem A.1.** Let $X_1, \ldots, X_k$ be independent random variables taking values in $[0, Q]$ for some $Q > 0$. Let $X := \sum_{i=1}^k X_i$ and let $\mu$ be such that $\mathbb{E}[X] \geq \mu$. Then, for any $\delta \in [0, 1]$, it holds

$$\mathbb{P}(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2Q}}.$$

Note that in the standard form, the above Chernoff bound is stated for independent random variables in $\{0,1\}$. A standard convexity argument allows to obtain the same bound for independent random variables in the interval $[0,1]$. The generalization to random variables from the interval $[0,Q]$ is then obtained by scaling.

A.2 Low-Diameter Clustering

As discussed in Section 3.1, we need an efficient deterministic clustering algorithm so that afterwards, we can concentrate on approximating MWVC and MWM in graphs of polylogarithmic diameter. The following theorem is a slight generalization of Lemma 13 in [FK20], which by itself is a relatively straightforward adaptation of one phase of the network decomposition algorithm of Rozhoň and Ghaffari [RG20].

**Theorem A.2.** Let $G = (V, E, w)$ be a weighted $n$-node graph with node and edge weights $w(v), w(e) \in [1,\text{poly}(n)]$, let $h$ be a positive integer. For every $\eta \in (0,1]$, there is a deterministic CONGEST algorithm to compute a $(1-\eta)$-dense, $h$-hop separated, and $(O(\log n), O(\frac{h \cdot \log^4 n}{\eta}))$-routable clustering of $G$ in poly $(\frac{h \cdot \log n}{\eta})$ rounds.

**Proof.** Lemma 13 in [FK20] is only stated for edge weights $\in \{0,1\}$ and no node weights. However, the proof of Lemma 13 explicitly states that the lemma also holds for weights that are polynomially bounded in $n$. Further, in the proof of Lemma 13, in the first step, the edge weights are transformed into node weights. The statement of the lemma therefore directly also applies to node weights. Finally, in Lemma 13, the parameter $h$ is set to $h = 3$. The proof guarantees 3-hop separation
by applying Theorem 2.12 of [RG20] to $G^k$ for $k = 2$. If the theorem is applied to $G^k$ for $k = \max\{1, h - 1\}$, Theorem 2.12 of [RG20] directly implies that the computed clusters are $(k + 1)$-hop separated.

Remarks: In the above proof, we do not optimize the log $n$-factors and the $1/\eta$-factors in the time complexity. We note that the round complexity of the algorithm of [RG20] has been slightly improved in [GGR21]. Further, in [CG21], Chang and Ghaffari give a CONGEST algorithm to compute a strong diameter network decomposition. We think that it should be possible to adapt their algorithm to obtain a deterministic poly log $n$-time CONGEST algorithm to compute an $h$-hop separated $(1 - \eta)$-dense clustering into clusters of strong diameter $O\left(\frac{h \cdot \log n}{\eta}\right)$. We further note that by using a variant of the randomized clustering algorithm of [MPX13], one gets a randomized $O\left(\frac{h \cdot \log n}{\eta}\right)$-round CONGEST algorithm to compute a clustering into clusters of diameter $O\left(\frac{h \cdot \log n}{\eta}\right)$ such that the clustering is $(1 - \eta)$-dense in expectation.

A.3 Fractional Approximation Algorithm

In all our upper bounds, we need an efficient deterministic distributed approximation scheme for the fractional variants of the MWVC and the MWM problem. In [AKO18], Ahmadi, Kuhn, and Oshman showed that for every instance of the MWM problem with weights in the range $[1, W]$ and for every $\varepsilon \in (0, 1]$, it is possible to deterministically compute a $(1 - \varepsilon)$-approximate fractional solution in time $O((\log(\Delta W))/\varepsilon^2)$ in the CONGEST model. In our algorithm to solve the MWVC problem, we need a variant of this algorithm, which works for the (unweighted) fractional $w$-matching problem. The following theorem shows that this can be done with the same asymptotic cost that we have for the fractional MWM problem.

Theorem A.3. Let $G = (V, E, w)$ be an undirected $n$-node graph with integer node weights $w(v) \in \{1, \ldots, W\}$. Then, for every $\varepsilon \in (0, 1]$, there is a deterministic $O((\log(\Delta W))/\varepsilon^2)$-round CONGEST algorithm to compute a $(1 - \varepsilon)$-approximate solution to the fractional $w$-matching problem in $G$ and a $(1 + \varepsilon)$-approximate solution to the minimum fractional weighted vertex cover problem.

Proof. The theorem could be proven for arbitrary weights in the range $[1, W]$ by adapting the algorithm of [AKO18]. We here give a generic reduction, which works for integer weights and which allows to use the result of [AKO18] (almost) in a blackbox manner.

We define an unweighted graph $G' = (V', E')$ as follows. For every node $v \in V$, $V'$ contains $w(v)$ nodes $(v, 1), \ldots, (v, w(v))$. Further, for every edge $\{u, v\} \in E$, we add a complete bipartite graph between the corresponding nodes to $G'$, that is, $E'$ contains all edges $\{(u, i), (v, j)\}$ for $i \in \{1, \ldots, w(v)\}$ and $j \in \{1, \ldots, w(u)\}$. We then apply the unweighted fractional matching algorithm of [AKO18] to compute a $(1 - \varepsilon)$-approximate fractional matching of $G'$. The maximum degree of any node in $G'$ is at most $\Delta \cdot W$ and when running the algorithm in the CONGEST model on $G'$, the round complexity of the algorithm is therefore $O((\log(\Delta W))/\varepsilon^2)$ as claimed (cf. Theorem 2 in [AKO18]).

For an edge $e \in E'$ of $G'$, assume that $z_e$ is the fractional matching value of $e$ in the computed fractional matching of $G'$. We can transform the fractional matching of $G'$ into a fractional $w$-matching of $G$ as follows. For each edge $\{u, v\} \in E$ of $G$, we define $y_{\{u, v\}} := \sum_{i=1}^{w(u)} \sum_{j=1}^{w(v)} z_{\{(u, i),(v, j)\}}$. Note that we obtain a valid fractional $w$-matching because for every node $u \in V$ of $G$, the sum of the fractional values of its edges is at most equal to number of copies of $u$ in $G'$, which is equal to $w(u)$. In the other direction, given a fractional $w$-matching $y_e$ of $G$, we can compute a fractional matching $z_e$ of $G'$ of the same size in the following way. For each edge $\{u, v\} \in E$ of $G$, we assign $z_{\{(u, i),(v, j)\}} := y_{\{u, v\}}/(w(u) \cdot w(v))$. The size of a maximum fractional $w$-matching on $G$ is therefore
equal to the size of a maximum fractional matching on \( G' \) and given a \((1 - \varepsilon)\)-approximation of maximum fractional matching on \( G' \), we therefore also obtain a \((1 - \varepsilon)\)-approximation of maximum fractional \( w \)-matching on \( G \).

It remains to show that we can efficiently run the fractional matching algorithm of [AKO18] in the CONGEST model on \( G \). Since every node of \( G \) has potentially a large number of copies in \( G' \), it is not true that any CONGEST algorithm on \( G' \) can be run efficiently in the CONGEST model on \( G \). However, the behavior of the algorithm of [AKO18] is independent of the node IDs and since the algorithm is deterministic, all copies of a node \( u \in V \) are symmetric and therefore behave in exactly the same way. Each node of \( G \) can therefore simulate all its copies in \( G' \) at no additional cost.

We note that the same reduction has also been used in [GKP08], where a maximal matching of \( G' \) is used to compute a 2-approximate weighted vertex cover of \( G \).

\[ \square \]

### A.4 Counting Paths in Layered Graphs

As part of our bipartite minimum vertex cover algorithm, we need to count the number of shortest augmenting paths (w.r.t. to a given fractional \( w \)-matching) that pass through each node and edge of a bipartite graph. For this, we slightly generalize an algorithm that has been introduced in [LPP15] and refined in [BCGS17]. The following lemma provides a generic version of this algorithm, which works for general layered bipartite graphs.

**Lemma A.4.** Let \( H = (V, E) \) be a bipartite \( n \)-node graph, where the nodes are partitioned into \( L \) layers \( V_1, \ldots, V_L \) such that \( E \) only contains edges between adjacent layers \( V_i \) and \( V_{i+1} \) for \( i \in \{1, \ldots, L - 1\} \). A top-down path \( P \) is a path of length \( L - 1 \) that consists of exactly 1 node from each layer \( V_i \). If the partition into layers is known, there is a deterministic \( O(L^2) \)-round CONGEST algorithm that for every \( v \in V \) and every \( e \in E \) computes the number of top-down paths passing through \( v \) and \( e \).

**Proof.** For every \( i \in \{1, \ldots, L\} \) and every \( v \in V_i \), we first compute the number of paths \( \alpha(v) \) of length \( i - 1 \) that start at a node in \( V_1 \) and pass through the layers \( V_1, \ldots, V_i \). The number can be computed inductively as follows. For each \( v \in V_1 \), we have \( \alpha(v) = 1 \) and for each \( v \in V_i \) for \( i > 1 \), we have \( \alpha(v) = \sum_{u \in N(v) \cap V_{i-1}} \alpha(u) \). Similarly, for every \( i \in \{1, \ldots, L\} \) and every \( v \in V_i \), we can also compute the number of paths \( \beta(v) \) of length \( L - i \) that start at a node in \( V_i \) and pass through the layers \( V_i, V_{i-1}, \ldots, V_1 \). Note that we have \( \alpha(v), \beta(v) \leq n^L \) for all nodes \( v \) and we can therefore compute all values \( \alpha(v) \) and \( \beta(v) \) by a distributed algorithm in \( L \) rounds with messages of size \( O(L \log n) \). The values can therefore be computed in the CONGEST model in time \( O(L^2) \). The number of top-down path passing through a node \( v \) can now be computed as \( \alpha(v) \cdot \beta(v) \) and for an edge \( \{u, v\} \in E \) with \( u \in V_i \) and \( v \in V_{i+1} \) for some \( i \in \{1, \ldots, L - 1\} \), the number of top-down paths passing through edge \( \{u, v\} \) is equal to \( \alpha(u) \cdot \beta(v) \). \[ \square \]