ON TITCHMARSH-WEYL FUNCTIONS AND EIGENFUNCTION EXPANSIONS OF FIRST-ORDER SYMMETRIC SYSTEMS

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Abstract. We study general (not necessarily Hamiltonian) first-order symmetric systems
\[ Jy'(t) - B(t)y(t) = \Delta(t)f(t) \]
on an interval \( I = [a, b] \) with the regular endpoint \( a \). It is assumed that the deficiency indices \( n_{\pm}(T_{\text{min}}) \) of the minimal relation \( T_{\text{min}} \) in \( L^2(\Delta(I)) \) satisfy \( n_-(T_{\text{min}}) \leq n_+(T_{\text{min}}) \). By using a Nevanlinna boundary parameter \( \tau = \tau(\lambda) \) at the singular endpoint \( b \) we define self-adjoint and \( \lambda \)-depending Nevanlinna boundary conditions which are analogs of separated self-adjoint boundary conditions for Hamiltonian systems. With a boundary value problem involving such conditions we associate the \( m \)-function \( m(\cdot) \), which is an analog of the Titchmarsh-Weyl coefficient for the Hamiltonian system. By using \( m \)-function we obtain the Fourier transform \( V : L^2(\Delta(I)) \rightarrow L^2(\Sigma) \) with the spectral function \( \Sigma(\cdot) \) of the minimally possible dimension. If \( V \) is an isometry, then the (exit space) self-adjoint extension \( \tilde{T} \) of \( T_{\text{min}} \) induced by the boundary problem is unitarily equivalent to the multiplication operator in \( L^2(\Sigma) \); hence the spectrum of \( \tilde{T} \) is defined by the spectral function \( \Sigma(\cdot) \). We show that all the objects of the boundary problem are determined by the parameter \( \tau \), which enables us to parametrize all spectral function \( \Sigma(\cdot) \) immediately in terms of \( \tau \). Similar results for various classes of boundary problems were obtained by Kac and Krein, Fulton, Hinton and Shaw and other authors.

1. Introduction

Let \( H \) and \( \hat{H} \) be finite dimensional Hilbert spaces and let
\[
(1.1) \quad H_0 := H \oplus \hat{H}, \quad \mathbb{H} := H_0 \oplus H = H \oplus \hat{H} \oplus H.
\]
The main object of the paper is first-order symmetric system of differential equations defined on an interval \( I = [a, b] \), \( -\infty < a < b \leq \infty \), with the regular endpoint \( a \) and regular or singular endpoint \( b \). Such a system is of the form [3, 15]
\[
(1.2) \quad Jy'(t) - B(t)y(t) = \Delta(t)f(t), \quad t \in I,
\]
where \( B(t) = B^*(t) \) and \( \Delta(t) \geq 0 \) are the \( [\mathbb{H}] \)-valued functions on \( I \) and
\[
(1.3) \quad J = \begin{pmatrix} 0 & 0 & -I_H \\ 0 & iI_{\hat{H}} & 0 \\ I_H & 0 & 0 \end{pmatrix} : H \oplus \hat{H} \oplus H \rightarrow H \oplus \hat{H} \oplus H.
\]
Throughout the paper we assume that the system (1.2) is definite. The latter means that for any \( \lambda \in \mathbb{C} \) each common solution of the equations
\[
(1.4) \quad Jy'(t) - B(t)y(t) = \lambda \Delta(t)y(t)
\]
and \( \Delta(t)y(t) = 0 \) (a.e. on \( I \) is trivial, i.e., \( y(t) = 0 \), \( t \in I \).

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System (1.2) is called Hamiltonian system if \( \hat{H} = \{0\} \). In this case one has

\[
J = \begin{pmatrix} 0 & -I_H \\ I_H & 0 \end{pmatrix} : H \oplus H \to H \oplus H.
\]

In what follows we denote by \( L^2_\Delta(\mathcal{I}) \) the Hilbert space of \( \mathbb{H} \)-valued Borel measurable functions \( f(\cdot) \) (in fact, equivalence classes) on \( \mathcal{I} \) satisfying \( ||f||^2_\Delta := \int (\Delta(t)f(t), f(t))_\mathbb{H} \, dt < \infty \).

Investigations of symmetric systems is motivated by several reasons. For instance, systems (1.4) form more general objet than formally self-adjoint differential equation of arbitrary order with matrix coefficients. Such equation is reduced to a system of the form (1.4) with \( J \) given by (1.3) (see [28]). Emphasize that presence of the term \( iI_B \) in (1.3) under this reduction characterizes odd order equations, although even order equations are reduced to Hamiltonian systems (with \( J \) given by (1.5)). Moreover, the Krein-Feller string equation is also reduced to Hamiltonian system (1.4) ([15, Chapter 6, §8]).

As it is known, the extension theory of symmetric linear relations gives a natural framework for investigation of the boundary value problems for symmetric systems (see [4, 11, 12, 18, 25, 32, 33, 44] and references therein). According to [25, 33, 44] the system (1.2) generates the minimal linear relation \( T_{\min} \) and the maximal linear relation \( T_{\max} \) for \( \hat{H} = \{0\} \). It turns out that \( T_{\min} \) is a closed symmetric relation with not necessarily equal deficiency indices \( n_\pm(T_{\min}) \). Since system (1.2) is assumed to be definite, \( n_\pm(T_{\min}) \) can be defined as a number of \( L^2_\Delta \)-solutions of (1.4) for \( \lambda \in \mathbb{C}_\pm \). Moreover, \( T_{\max} = T_{\max}^* \) and the equality

\[
[y, z]_b = \lim_{t \to b} (Jy(t), z(t)), \quad y, z \in \text{dom} \, T_{\max},
\]
defines a skew-Hermitian bilinear form on the domain of \( T_{\max} \).

A description of various classes of extensions of \( T_{\min} \) (self-adjoint, \( m \)-dissipative, etc.) in terms of boundary conditions is an important problem in the spectral theory of symmetric systems. Assume that the system (1.2) is Hamiltonian and \( n_+(T_{\min}) = n_-(T_{\min}) \). Let \( y(t) = \{y_0(t), y_1(t)\} \in H \oplus H \) be the representation of a function \( y \in \text{dom} \, T_{\max} \). Then according to [20] the general form of self-adjoint separated boundary conditions is

\[
\cos B_1 y_0(a) + \sin B_1 y_1(a) = 0, \quad [y, \chi_j]_b = 0, \quad j = \{1, \ldots, \nu_b\}, \quad y \in \text{dom} \, T_{\max},
\]

where \( B_1 \) is a self-adjoint operator on \( H \) and \( \{\chi_j\}_1^{\nu_b}, \nu_b = n_\pm(T_{\min}) - \dim H \), is a certain system of functions from \( \text{dom} \, T_{\max} \). The vector \( y_b := \{[y, \chi_j]_b\}_1^{\nu_b} \in \mathbb{C}^{\nu_b} \) is called a singular boundary value of a function \( y \in \text{dom} \, T_{\max} \). Observe that for ordinary differential operators description (1.7) goes back to I.M. Glazman (see [1, Appendix 2, §5]), while the form of the boundary conditions (at regular endpoints) goes back to F.S. Rofe-Beketov [45]. Note also that the notion of a singular boundary value can be found in the book [13, Ch.13.2]).

Boundary conditions (1.7) generate a self-adjoint extension \( \hat{A} \) of \( T_{\min} \) given by \( \tilde{A} = \{[y, f] \in T_{\max} : y \text{ satisfies (1.7)}\} \). The resolvent \( (\hat{A} - \lambda)^{-1} \) of \( \hat{A} \) is defined as follows: for any \( f \in L^2_\Delta(\mathcal{I}) \) vector \( y = (\hat{A} - \lambda)^{-1}f \) is the \( L^2_\Delta \)-solution of the equation

\[
Jy' - B(t)y(t) = \lambda \Delta(t)y + \Delta(t)f(t), \quad f \in L^2_\Delta(\mathcal{I}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

subject to the boundary conditions (1.7). Moreover, according to [20] the Titchmarsh - Weyl coefficient \( M_{TW}(\lambda)(\in [H]) \) of the boundary problem (1.8), (1.7) is defined by the relations

\[
u(t, \lambda) := \varphi(t, \lambda)M_{TW}(\lambda) + \psi(t, \lambda) \in L^2_\Delta[H, \mathbb{H}],
\]

\[
[v(\cdot, \lambda)h, \chi_j]_b = 0, \quad h \in H, \quad j = \{1, \ldots, \nu_b\}.
\]
Here \( \varphi(\cdot, \lambda) \) and \( \psi(\cdot, \lambda) \) are the \([H, \mathbb{H} \oplus \mathbb{H}]\)-valued operator solutions of Eq. (1.4) with the initial data
\[
\varphi(a, \lambda) = (\sin B_1, -\cos B_1)\top \quad \text{and} \quad \psi(a, \lambda) = (-\cos B_1, \sin B_1)\top.
\]
Note also the papers [27, 30], where the Titchmarsh - Weyl coefficient for Hamiltonian systems is defined in another way. By using \( M_{TW}(\cdot) \) one obtains the Fourier transform with the spectral function \( \Sigma(\cdot) \) of the minimally possible dimension \( N_\Sigma = \dim H \) (see [11, 12, 22, 25]).

It turns out that for general (not necessarily Hamiltonian) symmetric systems the situation is more complicated. In particular, it was shown in [42] that non-Hamiltonian system (1.2) does not admit separated self-adjoint boundary conditions. Moreover, the inequality \( n_+(T_{\min}) \neq n_-(T_{\min}) \), and hence absence of self-adjoint boundary conditions is a typical situation for such systems. For instance, in the limit point case at \( b \) one has \( n_+(T_{\min}) = \dim H \) and \( n_-(T_{\min}) = \dim H + \dim \tilde{H} \). Such circumstances make it natural to investigate the following problems:

- To find (might be \( \lambda \)-depending) analogs of self-adjoint separated boundary conditions for general systems (1.2) and describe such type conditions;
- To describe in terms of boundary conditions all spectral matrix functions that have the minimally possible dimension and investigate the corresponding Fourier transforms.

In the paper we solve these problems for symmetric systems (1.2) assuming that \( n_-(T_{\min}) \leq n_+(T_{\min}) \). However to simplify presentation we assume within this section that \( n_+(T_{\min}) = n_-(T_{\min}) \) (the case \( n_+(T_{\min}) < n_-(T_{\min}) \) will be treated elsewhere). We first show that there exists a finite-dimensional Hilbert space \( \mathcal{H}_b \) and a surjective linear mapping
\[
\Gamma_b = (\Gamma_{0b}, \tilde{\Gamma}_b, \Gamma_{1b})\top : \text{dom } T_{\max} \to \mathcal{H}_b \oplus \tilde{H} \oplus \mathcal{H}_b
\]
such that the bilinear form (1.6) admits the representation
\[
[y, z]_b = (\Gamma_{0b}y, \Gamma_{1b}z) - (\Gamma_{1b}y, \Gamma_{0b}z) + i(\tilde{\Gamma}_by, \tilde{\Gamma}_bz), \quad y, z \in \text{dom } T_{\max}.
\]
It turns out that \( \Gamma_{by} \) can be chosen in the form of a singular boundary value of \( y \in \text{dom } T_{\max} \) (see Remark 3.5). Moreover, each proper extension of \( T_{\min} \) can be defined by means of boundary conditions imposed on vectors \( y(a) = \{y_0(a), \tilde{y}(a), y_1(a)\}(\in H \oplus \tilde{H} \oplus H) \) and \( \Gamma_{by} = \{\Gamma_{0b}y, \tilde{\Gamma}_by, \Gamma_{1b}y\}(\in \mathcal{H}_b \oplus \tilde{H} \oplus \mathcal{H}_b) \). In particular, a linear relation \( T \) given by
\[
T := \{(y, f) \in T_{\max} : y_1(a) = 0, \tilde{y}(a) = \tilde{\Gamma}_by, \Gamma_{0b}y = \Gamma_{1b}y = 0\},
\]
is a symmetric extension of \( T_{\min} \) and plays a crucial role in our considerations.

Recall that a generalized resolvent of \( T \) is an operator-valued function given by
\[
R(\lambda) = P_{L^2(\mathcal{I})}(\tilde{T} - \lambda)^{-1} | L^2(\mathcal{I}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
where \( \tilde{T} \) is an exit space self-adjoint extension of \( T \) acting in a wider Hilbert space \( \tilde{\mathcal{H}} \supset L^2(\mathcal{I}) \). Moreover, the spectral function of \( T \) is defined by
\[
F(t) = P_{L^2(\mathcal{I})}E(t) | L^2(\mathcal{I}), \quad t \in \mathbb{R},
\]
where \( E(\cdot) \) is the orthogonal spectral function (resolution of identity) of \( \tilde{T} \). We show that each generalized resolvent \( y = R(\lambda)f, \ f \in L^2(\mathcal{I}) \), is given as the \( L^2(\mathcal{I}) \)-solution of the
following boundary-value problem with \( \lambda \)-depending boundary conditions:
\[
Jy' - B(t)y = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},
\]
(1.10) \[
y_1(a) = 0, \quad \hat{y}(a) = \hat{\Gamma}_b y,
\]
(1.11) \[
C_0(\lambda)\Gamma_{0b} y + C_1(\lambda)\Gamma_{1b} y = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
Here \( C_0(\cdot) \) and \( C_1(\cdot) \) are the components of a Nevanlinna operator pair \( \tau(\cdot) = \{ (C_0(\cdot), C_1(\cdot)) \} \) with values in \( [\mathcal{H}_0] \oplus [\mathcal{H}_0] \), so that formula (1.11) defines a Nevanlinna boundary condition at the singular endpoint \( b \). One may consider a pair \( \tau = \tau(\cdot) \) as a boundary parameter, since \( R(\lambda) \) runs over the set of all generalized resolvents of \( T \) when \( \tau \) runs over the set of all Nevanlinna operator pairs. To indicate this fact explicitly we write \( R(\lambda) = R_{\tau}(\lambda) \) and \( F(t) = F_{\tau}(t) \) for the generalized resolvents and spectral functions of \( T \) respectively. Moreover, we denote by \( \tilde{T} = \tilde{T}_{\tau} \) the exit space self-adjoint extension of \( T \) generating \( R_{\tau}(\cdot) \) and \( F_{\tau}(\cdot) \).

The boundary value problem (1.9)-(1.11) defines a canonical resolvent \( R_{\tau}(\lambda) \) if and only if \( \tau \) is a self-adjoint operator pair \( \tau = \{ (\cos B, \sin B) \} \) with some \( B = B^* \in [\mathcal{H}_0] \). In this case \( R_{\tau}(\lambda) = (\tilde{T}_{\tau} - \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \), where \( \tilde{T}_{\tau} \) is a self-adjoint extension of \( T \) in \( L^2_\Delta(\mathcal{I}) \) defined by the following mixed boundary conditions:
\[
\tilde{T}_{\tau} = \{(y, f) \in T_{\max} : y_1(a) = 0, \quad \hat{y}(a) = \hat{\Gamma}_b y, \quad \cos B \cdot \Gamma_{0b} y + \sin B \cdot \Gamma_{1b} y = 0 \}
\]
(1.12) For Hamiltonian systems the equalities in the right-hand side of (1.12) take the form of self-adjoint separated boundary conditions
(1.13) \[
y_1(a) = 0, \quad \cos B \cdot \Gamma_{0b} y + \sin B \cdot \Gamma_{1b} y = 0.
\]
Formula (1.13) seems to be more convenient than (1.7), because it enables one to parametrize singular self-adjoint boundary conditions (at the endpoint \( b \)) by means of a self-adjoint boundary parameter \( B \).

Next assume that \( \varphi(\cdot, \lambda) \) and \( \psi(\cdot, \lambda) \) are \([\mathcal{H}_0, \mathbb{B}]\)-valued operator solutions of equation (1.4) satisfying the initial conditions
\[
\varphi(a, \lambda) = \begin{pmatrix} I_{\mathcal{H}_0} \\ 0 \end{pmatrix} \in [\mathcal{H}_0, \mathcal{H}_0 \oplus \mathcal{H}], \quad \psi(a, \lambda) = \begin{pmatrix} -iP_B \\ -P_H \end{pmatrix} \in [\mathcal{H}_0, \mathcal{H}_0 \oplus \mathcal{H}].
\]
We show that, for each Nevanlinna boundary parameter \( \tau = \{ (C_0(\lambda), C_1(\lambda)) \} \), there exists a unique operator function \( m_{\tau}(\lambda) \in [\mathcal{H}_0] \) such that the operator solution
\[
v_{\tau}(t, \lambda) := \varphi(t, \lambda)m_{\tau}(\lambda) + \psi(t, \lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
of Eq. (1.4) has the following property: for every \( h_0 \in \mathcal{H}_0 \) the function \( y = v_{\tau}(t, \lambda)h_0 \) belongs to \( L^2_\Delta(\mathcal{I}) \) and satisfies the boundary conditions
\[
i(\hat{y}(a) - \hat{\Gamma}_b y) = P_B h_0, \quad C_0(\lambda)\Gamma_{0b} y + C_1(\lambda)\Gamma_{1b} y = 0.
\]
We call \( m_{\tau}(\cdot) \) the \( m \)-function corresponding to the boundary problem (1.9)-(1.11). It turns out that \( m_{\tau}(\cdot) \) is a Nevanlinna operator function satisfying the inequality
\[
(\operatorname{Im} \lambda)^{-1} \cdot \operatorname{Im} m_{\tau}(\lambda) \geq \int_I v_{\tau}^*(t, \lambda)\Delta(t)v_{\tau}(t, \lambda) dt, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
Moreover, in the case of the Hamiltonian system the \( m \)-function of the "canonical" boundary problem (1.9), (1.13) coincides with the Titchmarsh - Weyl coefficient \( M_{\tau W}(\cdot) \) in the sense of [20, 30, 27]. Note also that a concept of the Titchmarsh - Weyl function for the general system (1.2) with separated \( \lambda \)-depending boundary conditions was proposed in [27]. This
function is no longer a Nevanlinna function, that does not allow one to define the spectral function of the corresponding boundary value problem (cf. (1.16) below).

In the final part of the paper we study eigenfunction expansions of the boundary value problems for symmetric systems. Namely, let \( \tau = \{ (C_0(\cdot), C_1(\cdot)) \} \) be a boundary parameter and let \( F_\tau(\cdot) \) be the spectral function of \( T \) generated by the boundary value problem (1.9)–(1.11). A nondecreasing left-continuous operator-valued function \( \Sigma_\tau(\cdot) : \mathbb{R} \to [H_0] \) is called a spectral function of this problem if, for each function \( f \in L^2_\Delta(\mathcal{I}) \) with compact support, the Fourier transform

\[
\hat{f}(s) = \int_{\mathcal{I}} \varphi^*(t, s) \Delta(t) f(t) \, dt
\]

satisfies

\[
((F_\tau(\beta) - F_\tau(\alpha)) f, f)_{L^2_\Delta(\mathcal{I})} = \int_{[\alpha, \beta]} (d\Sigma_\tau(s) \hat{f}(s), \hat{f}(s))
\]

for any compact interval \([\alpha, \beta] \subset \mathbb{R}\). We show that for each boundary parameter \( \tau \) there exists unique spectral function \( \Sigma_\tau(\cdot) \) and it is recovered from the \( m \)-function \( m_\tau(\cdot) \) by means of the Stieltjes inversion formula

\[
\Sigma_\tau(s) = \lim_{\delta 	o +0} \lim_{\varepsilon 	o +0} \frac{1}{\pi} \int_{-\delta}^{\varepsilon-\delta} \text{Im} m_\tau(\sigma + i\varepsilon) \, d\sigma.
\]

Below (within this section) we assume for simplicity that \( T \) is a (not necessarily densely defined) operator, i.e., \( \text{mul} T = \{0\} \).

It follows from (1.15) that, the mapping \( Vf = \hat{f} \), originally defined by (1.14) for functions with compact supports, admits a continuous extension to a contractive map \( V : L^2_\Delta(\mathcal{I}) \to L^2(\Sigma; H_0) \) (for the strict definition of the Hilbert space \( L^2(\Sigma; H_0) \) see [13, 24, 35] and also Section 6.2). In the following theorem we characterize the most interesting case when the mapping \( V \) is isometric.

**Theorem 1.1.** For each boundary parameter \( \tau \) the following statements are equivalent:

(i) The Fourier transform \( V \) is an isometry from \( L^2_\Delta(\mathcal{I}) \) to \( L^2(\Sigma; H_0) \), or, equivalently, the Parseval equality \( ||f||_{L^2(\Sigma; H_0)} = ||f||_{L^2_\Delta(\mathcal{I})} \) holds for every \( f \in L^2_\Delta(\mathcal{I}) \).

(ii) The exit space self-adjoint extension \( \tilde{T}^\tau \) (in \( \tilde{\mathcal{H}} \)) is the operator, that is \( \text{mul} \tilde{T}^\tau = \{0\} \). If (i) (hence (ii)) is valid, then:

(1) For each \( f \in L^2_\Delta(\mathcal{I}) \) the inverse Fourier transform is given by

\[ f(t) = \int_{\mathbb{R}} \varphi(t, s) d\Sigma_\tau(s) \hat{f}(s) \]

where the integral is understood in an appropriate sense.

(2) There exists a unitary extension \( U \) of the operator \( V \) that maps \( \tilde{\mathcal{H}} \) onto \( L^2(\Sigma; H_0) \) and such that the operator \( \tilde{T}^\tau \) is unitarily equivalent to the multiplication operator \( \Lambda \) on \( L^2(\Sigma; H_0) \), \( \tilde{T}^\tau = U^* \Lambda U \). Hence, the operators \( \tilde{T}^\tau \) and \( \Lambda \) have the same spectral properties; for instance, the multiplicity of spectrum of \( \tilde{T}^\tau \) does not exceed \( \dim H_0 (= \dim H + \dim \tilde{H}) \).

It follows from Theorem 1.1 that \( V \) is a unitary operator from \( L^2_\Delta(\mathcal{I}) \) onto \( L^2(\Sigma; H_0) \) if and only if \( \tau = \{ \cos B, \sin B \} \) is a selfadjoint operator pair and the self-adjoint extension (1.12) of \( T \) is the operator. Observe also that the statements (i) and (ii) hold for any boundary parameter \( \tau \) if and only if \( T \) is a densely defined operator.
Next, we show that all spectral functions $\Sigma_\tau(\cdot)$ can be parametrized immediately in terms of the boundary parameter $\tau$. More precisely the following theorem holds.

**Theorem 1.2.** There exists a Nevanlinna operator function

\begin{equation}
M(\lambda) = \begin{pmatrix} m_0(\lambda) & M_1(\lambda) \\ M_2(\lambda) & M_3(\lambda) \end{pmatrix} : H_0 \oplus H_b \rightarrow H_0 \oplus H_b, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\end{equation}

such that for each Nevanlinna boundary parameter $\tau = \{(C_0(\cdot), C_1(\cdot))\}$ the corresponding $m$-function $m_\tau(\cdot)$ is given by

\begin{equation}
m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda) (C_0(\lambda) - C_1(\lambda) M_4(\lambda))^{-1} C_1(\lambda) M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

Thus, formula (1.18) together with the Stieltjes inversion formula (1.16) defines (unique) spectral function $\Sigma_\tau(\cdot)$ of the boundary problem (1.9)-(1.11). Moreover, the Fourier transform $V$ is an isometry if and only if the following two conditions are fulfilled:

\begin{align}
\lim_{y \to \pm \infty} \frac{1}{y} (C_0(iy) - C_1(iy) M_4(iy))^{-1} C_1(iy) &= 0, \\
\lim_{y \to \pm \infty} \frac{1}{y} M_4(iy) (C_0(iy) - C_1(iy) M_4(iy))^{-1} C_0(iy) &= 0.
\end{align}

Note that a description of spectral functions for various classes of boundary problems in the form close to (1.18), (1.16) can be found in [14, 16, 19, 23, 26, 40].

The above results are obtained in the framework of the new approach to the extension theory of symmetric operators developed during three last decades (see [7, 9, 10, 17, 34, 36, 37, 39] and references therein). This approach is based on concepts of boundary triplets and the corresponding Weyl functions. To apply this method to boundary value problems for system (1.2) we construct an appropriate boundary triplet for the relation $T_{\text{max}}$ (see Proposition 3.6). Moreover, in Proposition 4.4 and Corollary 4.5 we express the corresponding Weyl function $M(\cdot)$ in the sense of [9, 34, 39] in terms of the boundary values of respective matrix solutions of (1.4). It is worth to mention that the operator-valued function (1.17) coincides with the Weyl function $M(\cdot)$ computed in Corollary 4.5. Note also that conditions (1.19), (1.20) are implied by general result on II-admissibility from [7, 8].

We complete the paper by explicit example illustrating the main results.

Some results of the paper have been published as a preprint [2].

## 2. Preliminaries

### 2.1. Notations.

The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on the Hilbert space $\mathcal{H}_1$ with values in the Hilbert space $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $A \mid \mathcal{L}$ is the restriction of an operator $A$ onto the linear manifold $\mathcal{L}$; $P_\mathcal{L}$ is the orthogonal projector in $\mathfrak{H}$ onto the subspace $\mathcal{L} \subset \mathfrak{H}$; $C_+ (C_-)$ is the upper (lower) half-plane of the complex plane.

Recall that a closed linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$ is a closed linear subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (in $\mathcal{H}$) will be denoted by $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ ($\mathcal{C}(\mathcal{H})$). A closed linear operator $T$ from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\text{gr}T \in \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$.

For a linear relation $T \in \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ we denote by $\text{dom}T$, $\text{ran}T$, $\text{ker}T$ and $\text{mul}T$ the domain, range, kernel and the multivalued part of $T$ respectively. Recall also that the inverse and adjoint linear relations of $T$ are the relations $T^{-1} \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_0)$ and $T^* \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_0)$.
defined by
\[ T^{-1} = \{ (h_1, h_0) \in \mathcal{H}_1 \oplus \mathcal{H}_0 : \{h_0, h_1\} \in T \} \]
(2.1) \[ T^* = \{ (k_1, k_0) \in \mathcal{H}_1 \oplus \mathcal{H}_0 : (k_0, h_0) - (k_1, h_1) = 0, \{h_0, h_1\} \in T \}. \]

In the case \( T \in \tilde{C}(\mathcal{H}_0, \mathcal{H}_1) \) we write \( 0 \in \rho(T) \) if \( \ker T = \{0\} \) and \( \text{ran} T = \mathcal{H}_1 \), or equivalently if \( T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0] ; 0 \in \tilde{\rho}(T) \) if \( \ker T = \{0\} \) and \( T \) is a closed subspace in \( \mathcal{H}_1 \). For a linear relation \( T \in \tilde{C}(\mathcal{H}) \) we denote by \( \rho(T) := \{ \lambda \in \mathbb{C} : 0 \in \tilde{\rho}(T - \lambda) \} \) and \( \tilde{\rho}(T) = \{ \lambda \in \mathbb{C} : 0 \in \tilde{\rho}(T - \lambda) \} \) the resolvent set and the set of regular type points of \( T \) respectively.

A linear relation \( T \in \tilde{C}(\mathcal{H}) \) is called symmetric (self-adjoint) if \( T \subset T^* \) (resp. \( T = T^* \)). For each \( T = T^* \in \tilde{C}(\mathcal{H}) \) the following decompositions hold
\[ \mathcal{H} = \mathcal{H}' \oplus \text{mul} T, \quad T = T' \oplus \text{mul} T, \]
where \( \text{mul} T = \{0\} \oplus \text{mul} T \) and \( T' \) is the self-adjoint operator in \( \mathcal{H}' \) (the operator part of \( T \)).

Let \( T = T^* \in \tilde{C}(\mathcal{H}) \), let \( \mathcal{B} \) be the Borel \( \sigma \)-algebra of \( \mathbb{R} \) and let \( E'(: \mathcal{B} \to [\mathcal{H}'] \) be the orthogonal spectral measure of \( T' \). Then the spectral measure \( E(\cdot) \) of \( T \) is defined as \( E(\delta) = E'(\delta) P_{\mathcal{H}}, \delta \in \mathcal{B} \).

Recall also the following definition.

**Definition 2.1.** A holomorphic operator function \( \Phi(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathcal{H}] \) is called a Nevanlinna function if \( \text{Im} \lambda \cdot \text{Im} \Phi(\lambda) \geq 0 \) and \( \Phi^*(\lambda) = \Phi^*(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \).

### 2.2. Holomorphic operator pairs

Let \( \Lambda \) be an open set in \( \mathbb{C} \), let \( \mathcal{K}, \mathcal{H}_0, \mathcal{H}_1 \) be Hilbert spaces and let
\[ (C_0(\lambda), C_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K}, \quad \lambda \in \Lambda, \]
be a pair of holomorphic operator functions \( C_j(\cdot) : \Lambda \to [\mathcal{H}_j, \mathcal{K}], \ j \in \{0, 1\} \) (in short a holomorphic pair). Two such pairs \( C_j(\cdot) : \Lambda \to [\mathcal{H}_j, \mathcal{K}] \) and \( C'_j(\cdot) : \Lambda \to [\mathcal{H}_j, \mathcal{K}'] \) are said to be equivalent if there exists a holomorphic isomorphism \( \varphi(\cdot) : \Lambda \to [\mathcal{K}, \mathcal{K}'] \) such that \( C'_j(\lambda) = \varphi(\lambda) C_j(\lambda), \lambda \in \Lambda, j \in \{0, 1\} \). Clearly, the set of all holomorphic pairs splits into disjoint equivalence classes; moreover, the equality
\[ \tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K} \} := \{(h_0, h_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0(\lambda) h_0 + C_1(\lambda) h_1 = 0 \}
\]
allows us to identify such a class with the \( \tilde{C}(\mathcal{H}_0, \mathcal{H}_1) \)-valued function \( \tau(\lambda), \lambda \in \Lambda \).

In what follows, unless otherwise stated, \( \mathcal{H}_0 \) is a Hilbert space, \( \mathcal{H}_1 \) is a subspace in \( \mathcal{H}_0, \mathcal{H}_2 := \mathcal{H}_0 \oplus \mathcal{H}_1 \) and \( P_j \) is the orthoprojector in \( \mathcal{H}_0 \) onto \( \mathcal{H}_j, j \in \{1, 2\} \).

With each linear relation \( \theta \in \tilde{C}(\mathcal{H}_0, \mathcal{H}_1) \) we associate the \( \times \)-adjoint linear relation \( \theta^\times \in \tilde{C}(\mathcal{H}_0, \mathcal{H}_1) \) given by
\[ \theta^\times = \{ (k_0, k_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1 : (k_1, h_0) - (k_0, h_1) + i(P_k h_0, P_k h_0) = 0, \{h_0, h_1\} \in \theta \}. \]

It follows from (2.1) that in the case \( \mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H} \) one has \( \theta^\times = \theta^* \).

Next assume that
\[ \tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{H}_0 \}, \quad \lambda \in \mathbb{C}_+; \]
\[ \tau_-(\lambda) = \{(D_0(\lambda), D_1(\lambda)); \mathcal{H}_1 \}, \quad \lambda \in \mathbb{C}_- \]
are equivalence classes of the holomorphic pairs
\[ (C_0(\lambda), C_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0, \quad \lambda \in \mathbb{C}_+ \]
(2.5) \[ (D_0(\lambda), D_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_1, \quad \lambda \in \mathbb{C}_- \]
(2.6)
Assume also that
\[
C_0(\lambda) = (C_{01}(\lambda), C_{02}(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_0
\]
\[
D_0(\lambda) = (D_{01}(\lambda), D_{02}(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1
\]
are the block representations of \(C_0(\lambda)\) and \(D_0(\lambda)\).

**Definition 2.2.** A collection \(\tau = \{\tau_+, \tau_–\}\) of two holomorphic pairs (2.4) (more precisely, of the equivalence classes of the corresponding pairs) belongs to the class \(\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)\) if it satisfies the following relations:

\[
2 \Im(C_1(\lambda)C_{01}^\ast(\lambda)) + C_{02}(\lambda)C_{02}^\ast(\lambda) \geq 0,
\]
\[
2 \Im(D_1(\lambda)D_{01}^\ast(\lambda)) + D_{02}(\lambda)D_{02}^\ast(\lambda) \leq 0,
\]
\[
C_1(\lambda)D_{01}^\ast(\lambda) - C_{01}(\lambda)D_1^\ast(\lambda) + iC_{02}(\lambda)D_{02}^\ast(\lambda) = 0, \quad \lambda \in \mathbb{C}_+
\]
\[
0 \in \rho(C_0(\lambda) - iC_1(\lambda)P_1), \quad \lambda \in \mathbb{C}_+ ; \quad 0 \in \rho(D_0(\lambda) + iD_1(\lambda)), \quad \lambda \in \mathbb{C}_-.
\]

A collection \(\tau = \{\tau_+, \tau_–\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)\) belongs to the class \(\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)\) if for some (and hence for any) \(\lambda \in \mathbb{C}_+\) one has
\[
2 \Im(C_1(\lambda)C_{01}^\ast(\lambda)) + C_{02}(\lambda)C_{02}^\ast(\lambda) = 0 \quad \text{and} \quad 0 \in \rho(C_0(\lambda) + iC_1(\lambda)).
\]

The following proposition is immediate from Definition 2.2 and the results of [38].

**Proposition 2.3.** (1) If \(\tau = \{\tau_+, \tau_–\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)\), then \((-\tau_\pm(\lambda))^\ast = -\tau_\pm(\lambda)\), \(\lambda \in \mathbb{C}_\pm\), and the following equality holds
\[
\tau_\pm(\lambda) = \{\{h_1 - iP_2h_0, -P_1h_0\} : h_1, h_0 \in (\tau_\pm(\lambda))^\ast\}.
\]

(2) The set \(\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)\) is not empty if and only if \(\dim \mathcal{H}_0 = \dim \mathcal{H}_1\). This implies that in the case \(\dim \mathcal{H}_1 < \infty\) the set \(\widetilde{R}_0(\mathcal{H}_0, \mathcal{H}_1)\) is not empty if and only if \(\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}\).

(3) Each collection \(\tau = \{\tau_+, \tau_–\} \in \widetilde{R}_0(\mathcal{H}_0, \mathcal{H}_1)\) can be represented as a constant
\[
\tau_\pm(\lambda) = \{(C_0, C_1) : \mathcal{H}_0 \} = \theta(\mathcal{H}(\mathcal{H}_0, \mathcal{H}_1)), \quad \lambda \in \mathbb{C}_\pm,
\]
where \(C_j \in \mathcal{H}_j, \mathcal{H}_0\), \(j \in \{0, 1\}\), and \((-\theta)^\ast = -\theta\).

Moreover, one can easily prove the following proposition.

**Proposition 2.4.** If \(\dim \mathcal{H}_0 < \infty\), then a collection \(\tau = \{\tau_+, \tau_–\}\) of two holomorphic pairs (2.4) belongs to the class \(\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)\) if and only if (2.7)–(2.9) holds and
\[
\text{ran}(C_0(\lambda), C_1(\lambda)) = \mathcal{H}_0, \quad \lambda \in \mathbb{C}_+ ; \quad \text{ran}(D_0(\lambda), D_1(\lambda)) = \mathcal{H}_1, \quad \lambda \in \mathbb{C}_-.
\]

**Remark 2.5.** If \(\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H}\), then the class \(\widetilde{R}(\mathcal{H}) := \widetilde{R}(\mathcal{H}, \mathcal{H})\) coincides with the well-known class of Nevanlinna functions \(\tau(\cdot)\) with values in \(\mathcal{C}(\mathcal{H})\) (see, for instance, [7]). In this case the collection (2.4) turns into the Nevanlinna pair
\[
\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)) : \mathcal{H}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
with \(C_0(\lambda), C_1(\lambda) \in \mathcal{H}\). In view of (2.7)–(2.10) such a pair is characterized by the relations (cf. [7, Definition 2.2])
\[
\text{Im} \lambda \cdot \text{Im}(C_1(\lambda)C_0^\ast(\lambda)) \geq 0, \quad C_1(\lambda)C_0^\ast(\lambda) - C_0(\lambda)C_1^\ast(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
\[
0 \in \rho(C_0(\lambda) - iC_1(\lambda)), \quad \lambda \in \mathbb{C}_+ ; \quad 0 \in \rho(C_0(\lambda) + iC_1(\lambda)), \quad \lambda \in \mathbb{C}_-.
\]
Moreover, the function \( \tau(\cdot) \) belongs to the class \( \tilde{\mathcal{R}}^0(\mathcal{H}) := \tilde{\mathcal{R}}^0(\mathcal{H}, \mathcal{H}) \) if and only if it admits the representation in the form of the constant (cf. (2.12))

\[
\tau(\lambda) \equiv \{(C_0, C_1); \mathcal{H}\} = \theta(\in \tilde{\mathcal{C}}(\mathcal{H})), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]

with the operators \( C_j \in [\mathcal{H}] \) such that \( \text{Im}(C_1^*C_0^*) = 0 \) and \( 0 \in \rho(C_0 \pm iC_1) \) (this means that \( \theta = \theta^* \)). Observe also that according to [45] each \( \tau \in \tilde{\mathcal{R}}^0(\mathcal{H}) \) admits the normalized representation (2.17) with

\[
C_0 = \cos B, \quad C_1 = \sin B, \quad B = B^* \in [\mathcal{H}].
\]

Assume now that \( n := \dim \mathcal{H} < \infty, e = \{e_j\}_1^n \) is an orthonormal basis in \( \mathcal{H} \), \( \tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{H}\} \) is a pair of holomorphic operator-functions \( C_i(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathcal{H}] \) and \( C_i(\lambda) = (c_{kj,l}(\lambda))_{k,j=1}^n \) is the matrix representations of the operator \( C_i(\lambda), l \in \{0, 1\} \), in the basis \( e \). Then by Proposition 2.4 \( \tau \) belongs to the class \( \tilde{\mathcal{R}}(\mathcal{H}) \) if and only if the matrices \( C_0(\lambda) \) and \( C_1(\lambda) \) satisfy (2.15) and the following equality:

\[
\text{rank}(C_0(\lambda), C_1(\lambda)) = n, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Moreover, the operator pair \( \theta = \{(C_0, C_1); \mathcal{H}\} \) belongs to the class \( \tilde{\mathcal{R}}^0(\mathcal{H}) \) if and only if \( \text{Im}(C_1C_0^*) = 0 \) and \( \text{rank}(C_0, C_1) = n \) (here \( C_l = (c_{kj,l}(\lambda))_{k,j=1}^n \) is the matrix representation of the operator \( C_l, l \in \{0, 1\} \), in the basis \( e \)). Note that such a "matrix" definition of the classes \( \tilde{\mathcal{R}}(\mathcal{H}) \) and \( \tilde{\mathcal{R}}^0(\mathcal{H}) \) in the case dim \( \mathcal{H} < \infty \) can be found, e.g. in [12, 29].

### 2.3. Boundary triplets and Weyl functions.
Here we recall definitions of boundary triplets, the corresponding Weyl functions, and \( \gamma \)-fields following [9, 10, 34, 39].

Let \( A \) be a closed symmetric linear relation in the Hilbert space \( \mathfrak{H} \), let \( \mathfrak{H}_A(A) = \ker (A^* - \lambda) \) (\( \lambda \in \tilde{\rho}(A) \)) be a defect subspace of \( A \), let \( \mathfrak{H}_\lambda(A) = \{f, \lambda f : f \in \mathfrak{H}_\lambda(A)\} \) and let \( n_\pm(A) := \dim \mathfrak{H}_\lambda(A) \leq \infty, \lambda \in \mathbb{C}, \) be deficiency indices of \( A \). Denote by \( \text{Ext}_A \) the set of all proper extensions of \( A \), i.e., the set of all relations \( A \in \tilde{\mathfrak{H}}(\mathfrak{H}) \) such that \( A \subset A \subset A^* \).

Next assume that \( \mathcal{H}_0 \) is a Hilbert space, \( \mathcal{H}_1 \) is a subspace in \( \mathcal{H}_0 \) and \( \mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1 \), so that \( \mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2 \). Denote by \( P_j \) the orthoprojector in \( \mathcal{H}_0 \) onto \( \mathcal{H}_j, j \in \{1, 2\} \).

**Definition 2.6.** A collection \( \Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\} \), where \( \Gamma_j : A^* \to \mathcal{H}_j, j \in \{0, 1\} \), are linear mappings, is called a boundary triplet for \( A^* \), if the mapping \( \Gamma : \tilde{f} \to (\Gamma_0 \tilde{f}, \Gamma_1 \tilde{f}), \tilde{f} \in A^* \), from \( A^* \) into \( \mathcal{H}_0 \oplus \mathcal{H}_1 \) is surjective and the following Green’s identity

\[
(f', g) - (f, g') = (\Gamma_1 \tilde{f}, \Gamma_0 \tilde{g})_{\mathcal{H}_0} - (\Gamma_0 \tilde{f}, \Gamma_1 \tilde{g})_{\mathcal{H}_0} + i(P_2 \Gamma_0 \tilde{f}, P_2 \Gamma_0 \tilde{g})_{\mathcal{H}_2}
\]

holds for all \( \tilde{f} = \{f, f'\}, \tilde{g} = \{g, g'\} \in A^* \).

**Proposition 2.7.** Let \( \Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( A^* \). Then:

1. \( \dim \mathcal{H}_1 = n_-(A) \leq n_+(A) = \dim \mathcal{H}_0 \).
2. \( \ker \Gamma_0 \cap \ker \Gamma_1 = A \) and \( \Gamma_j \) is a bounded operator from \( A^* \) into \( \mathcal{H}_j, j \in \{0, 1\} \).
3. The equality

\[
A_0 := \ker \Gamma_0 = \{\tilde{f} \in A^* : \Gamma_0 \tilde{f} = 0\}
\]

defines the maximal symmetric extension \( A_0 \in \text{Ext}_A \) such that \( \mathbb{C}_+ \subset \rho(A_0) \).

**Proposition 2.8.** [39] Let \( \Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( A^* \). Denote also by \( \pi_1 \) the orthoprojector in \( \mathfrak{H} \oplus \mathfrak{H} \) onto \( \mathfrak{H} \oplus \{0\} \). Then the operators \( \Gamma_0 \mid \mathfrak{H}_\lambda(A), \lambda \in \mathbb{C}_+, \) and
Moreover, the equality
\[ (2.22) \]
correctly define the operator functions \( \gamma_+ \) and \( M_+ \) defined in Proposition 2.8 are called the \( \gamma \)-fields and the Weyl functions, respectively, corresponding to the boundary triplet \( \Pi \).

**Definition 2.9.** [39] The operator functions \( \gamma_\pm \) and \( M_\pm \) defined in Proposition 2.8 are the \( \gamma \)-fields and the Weyl functions, respectively. Moreover, let the spaces \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) be decomposed as
\[ (2.24) \]
(see also Remark 2.11 below).

**Proposition 2.10.** Let \( \Pi = \{ [\mathcal{H}_0 \oplus \mathcal{H}_1], \mathcal{H}_0, \mathcal{H}_1 \} \) be a boundary triplet for \( A^* \) and let \( \gamma_\pm \) and \( M_\pm \) be the corresponding \( \gamma \)-fields and Weyl functions, respectively. Moreover, let the spaces \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) be decomposed as
\[ (2.25) \]
If in addition \( n_\pm(A) < \infty \), then the deficiency indices of \( \tilde{A} \) are \( n_\pm(\tilde{A}) = n_\pm(A) - \dim \tilde{\mathcal{H}} \).

(2) The collection \( \tilde{\Pi} = \{ [\mathcal{H}_0 \oplus \mathcal{H}_1], \tilde{\Gamma}_0, \tilde{\Gamma}_1 \} \) is a boundary triplet for \( \tilde{A}^* \).

(3) The \( \gamma \)-fields \( \gamma_\pm \) and the Weyl functions \( M_\pm \) corresponding to \( \tilde{\Pi} \) are given by
\[ (2.26) \]
We omit the proof of Proposition 2.10, since it is similar to that of Proposition 4.1 in [7] (see also Remark 2.11 below).

**Remark 2.11.** If \( \mathcal{H}_0 = \mathcal{H}_1 \), then the boundary triplet in the sense of Definition 2.6 turns into the boundary triplet \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) for \( A^* \) in the sense of [17, 34]. In this case \( n_+(A) = n_-(A) = \dim \mathcal{H} \), \( A_0(= \ker \Gamma_0) \) is a self-adjoint extension of \( A \) and according to [9, 34, 10] the relations
\[ (2.27) \]
define the $\gamma$-field $\gamma(\cdot) : \rho(A_0) \to \mathcal{H}$ and the Weyl function $M(\cdot) : \rho(A_0) \to \mathcal{H}$ corresponding to the triplet $\Pi$. It follows from (2.25) that $\gamma(\cdot)$ and $M(\cdot)$ are associated with the operator functions $\gamma_\pm(\cdot)$ and $M_\pm(\cdot)$ from Definition 2.9 via $\gamma(\lambda) = \gamma_\pm(\lambda)$ and $M(\lambda) = M_\pm(\lambda)$, $\lambda \in \mathbb{C}_\pm$. Moreover, for such a triplet the identity

\begin{equation}
M(\mu) - M^*(\lambda) = (\mu - \lambda)\gamma^*(\lambda)\gamma(\mu), \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

holds, which implies that $M(\cdot)$ is a Nevanlinna operator function. Observe also that for the triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ all the results in this subsection were obtained in [9, 34, 10, 7].

In what follows a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ in the sense of [17, 34] will be sometimes called an ordinary boundary triplet for $A^*$.

### 2.4. Generalized resolvents and spectral functions.

Let $\mathfrak{H}$ be a subspace in a Hilbert space $\mathfrak{F}$, let $\tilde{A} = \tilde{A}^* \in \mathcal{C}(\mathfrak{F})$ and let $E(\cdot)$ be the spectral measure of $\tilde{A}$.

**Definition 2.12.** The relation $\tilde{A}$ is called $\mathfrak{H}$-minimal if it satisfies at least one of the following equivalent conditions:

1. $\text{span}\{\mathfrak{F}, (\tilde{A} - \lambda)^{-1}\mathfrak{F} : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \mathfrak{F}$.
2. There is not a nontrivial subspace $\mathfrak{H}' \subset \mathfrak{F} \ominus \mathfrak{H}$ such that $E([\alpha, \beta])\mathfrak{H}' \subset \mathfrak{H}'$ for each bounded interval $[\alpha, \beta) \subset \mathbb{R}$.

**Definition 2.13.** The relations $T_j \in \mathcal{C}(\mathfrak{F}_j)$, $j \in \{1, 2\}$, are said to be unitarily equivalent (by means of a unitary operator $U \in [\mathfrak{F}_1, \mathfrak{F}_2]$) if $T_2 = UT_1$ with $U = U \oplus U \in [\mathfrak{F}_1^2, \mathfrak{F}_2^2]$.

**Proposition 2.14.** Let $\mathfrak{F}_j$ be a subspace in a Hilbert space $\mathfrak{F}_j$, and let $\tilde{A}_j = \tilde{A}^*_j \in \mathcal{C}(\mathfrak{F}_j)$ be a $\mathfrak{F}_j$-minimal relation, $j \in \{1, 2\}$. Assume also that $V \in [\mathfrak{F}_1, \mathfrak{F}_2]$ is a unitary operator such that

\[ P_{\mathfrak{F}_1}(\tilde{A}_1 - \lambda)^{-1} \mid \mathfrak{F}_1 = V^*(P_{\mathfrak{F}_2}(\tilde{A}_2 - \lambda)^{-1} \mid \mathfrak{F}_2) V. \]

Then there exists a unitary operator $U \in [\mathfrak{F}_1, \mathfrak{F}_2]$ such that $U \mid \mathfrak{F}_1 = V$ and the relations $\tilde{A}_1$ and $\tilde{A}_2$ are unitarily equivalent by means of $U$.

In the case $\mathfrak{F}_1 = \mathfrak{F}_2 =: \mathfrak{F}$ and $V = I_\mathfrak{F}$ the proof of this proposition can be found in [31]. In general case the proof is similar.

Recall further the following definition.

**Definition 2.15.** Let $A$ be a symmetric relation in a Hilbert space $\mathfrak{F}$. The operator functions $R(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathfrak{F}]$ and $F(\cdot) : \mathbb{R} \to [\mathfrak{F}]$ are called the generalized resolvent and the spectral function of $A$ respectively if there exist a Hilbert space $\mathfrak{F} \supset \mathfrak{F}$ and a self-adjoint relation $\tilde{A} \in \mathcal{C}(\mathfrak{F})$ such that $A \subset \tilde{A}$ and the following equalities hold:

\begin{align}
R(\lambda) &= P_{\mathfrak{F}}(\tilde{A} - \lambda)^{-1} \mid \mathfrak{F}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \\
F(t) &= P_{\mathfrak{F}}E((-\infty, t)) \mid \mathfrak{F}, \quad t \in \mathbb{R}
\end{align}

(in formula (2.28) $E(\cdot)$ is the spectral measure of $\tilde{A}$).

The relation $\tilde{A}$ in (2.27) is called an exit space extension of $A$.

It follows from (2.27) and (2.28) that the generalized resolvent $R(\cdot)$ and the spectral function $F(\cdot)$ generated by the same extension $\tilde{A}$ of $A$ are connected by

\begin{equation}
R(\lambda) = \int_{\mathbb{R}} \frac{dF(t)}{t - \lambda}, \quad \lambda \in \mathbb{R}.
\end{equation}
Moreover, (2.28) yields
\begin{equation}
F(\infty):= s - \lim_{t\to+\infty} F(t) = P_{\tilde{A}}P_{\tilde{\mathcal{H}}_0} | \tilde{\mathcal{H}},
\end{equation}
where \( \tilde{\mathcal{H}}_0 = \tilde{\mathcal{H}} \oplus \text{mul} \tilde{\mathcal{A}} \).

According to [31] each generalized resolvent of \( \mathcal{A} \) is generated by some \( \mathcal{A} \)-minimal exit space extension \( \tilde{A} \) of \( \mathcal{A} \). Moreover, if the \( \mathcal{A} \)-minimal exit space extensions \( \tilde{A}_1 \in \mathcal{C}(\tilde{\mathcal{H}}_1) \) and \( \tilde{A}_2 \in \mathcal{C}(\tilde{\mathcal{H}}_2) \) of \( \mathcal{A} \) induce the same generalized resolvent \( R(\lambda) \), then in view of Proposition 2.14 there exists a unitary operator \( V' \in [\tilde{\mathcal{H}}_1 \oplus \tilde{\mathcal{H}}_2 \ominus \tilde{\mathcal{H}}] \) such that \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are unitarily equivalent by means of \( \tilde{U} = I_{\mathcal{H}} \oplus V' \). By using this fact we suppose in the following that the exit space extension \( \tilde{A} \) in (2.27) is \( \mathcal{A} \)-minimal, so that \( \tilde{A} \) is defined by \( R(\cdot) \) uniquely up to the unitary equivalence.

**Definition 2.16.** The generalized resolvent (2.27) and the spectral function (2.28) are called canonical if \( \tilde{\mathcal{H}} = \mathcal{H} \), i.e., if \( R(\lambda) = (\tilde{A} - \lambda)^{-1} \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), is the resolvent of the extension \( A = A^* \in \mathcal{C}(\mathcal{H}) \) of \( \mathcal{A} \) and \( F(t) = E((-\infty, t)), t \in \mathbb{R} \), is the spectral function of \( \tilde{A} \).

Clearly, canonical resolvents and spectral functions exist if and only if \( n_+(\mathcal{A}) = n_-(\mathcal{A}) \).

A description of all generalized resolvents of \( \mathcal{A} \) in terms of boundary triplets for \( A^* \) is given in the following theorem (see [6, 34] for the case \( n_+(\mathcal{A}) = n_-(\mathcal{A}) \) and [39] for the case of arbitrary deficiency indices \( n_{\pm}(\mathcal{A}) \)).

**Theorem 2.17.** Let \( \Pi = \{ \mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( A^* \). If \( \tau = \{ \tau_+, \tau_- \} \in R(\mathcal{H}_0, \mathcal{H}_1) \) is a collection of holomorphic pairs (2.4), then for every \( g \in \mathcal{H} \) and \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the abstract boundary value problem
\begin{align}
&\{ f, \lambda f + g \} \in A^* \tag{2.31} \\
&C_0(\lambda) \Gamma_0 \{ f, \lambda f + g \} - C_1(\lambda) \Gamma_1 \{ f, \lambda f + g \} = 0, \quad \lambda \in \mathbb{C}_+ \tag{2.32} \\
&D_0(\lambda) \Gamma_0 \{ f, \lambda f + g \} - D_1(\lambda) \Gamma_1 \{ f, \lambda f + g \} = 0, \quad \lambda \in \mathbb{C}_- \tag{2.33}
\end{align}
has a unique solution \( f = f(g, \lambda) \) and the equality \( R(\lambda)g := f(g, \lambda) \) defines a generalized resolvent \( \tilde{R}(\lambda) = R_\tau(\lambda) \) of the relation \( \mathcal{A} \). Conversely, for each generalized resolvent \( \tilde{R}(\lambda) \) of \( \mathcal{A} \) there exists a unique \( \tau \in R(\mathcal{H}_0, \mathcal{H}_1) \) such that \( R(\lambda) = R_\tau(\lambda) \). Moreover, \( R_\tau(\lambda) \) is a canonical resolvent if and only if \( \tau \in R(\mathcal{H}_0, \mathcal{H}_1) \).

3. Boundary triplets for symmetric systems

3.1. Notations. Let \( \mathcal{I} = [a, b) \) \((-\infty < a < b \leq \infty)\) be an interval of the real line (the symbol \( \in \) means that the endpoint \( b \) might be included to \( \mathcal{I} \) or not). Further, let \( \mathbb{H} \) be a finite-dimensional Hilbert space, let \( AC(\mathcal{I}; \mathbb{H}) \) be the set of functions \( f(\cdot) : \mathcal{I} \to \mathbb{H} \) which are absolutely continuous on each segment \([a, \beta] \subset \mathcal{I} \) and let \( AC(\mathcal{I}) := AC(\mathcal{I}; \mathbb{C}) \). Denote also by \( L_{1, \text{loc}}(\mathcal{I}; [\mathbb{H}]) \) the set of Borel operator-valued functions \( F(\cdot) \) defined almost everywhere on \( \mathcal{I} \) with values in \( [\mathbb{H}] \) and such that \( \int_{[a, \beta]} \| F(t) \| \, dt < \infty \) for each \( \beta \in \mathcal{I} \).

Next assume that \( \Delta(\cdot) \in L_{1, \text{loc}}(\mathcal{I}; [\mathbb{H}]) \) is an operator function such that \( \Delta(t) \geq 0 \) a.e. on \( \mathcal{I} \). Denote by \( L_{2, \Delta}(\mathcal{I}) \) the linear space of all Borel-measurable vector-functions \( f(\cdot) : \mathcal{I} \to \mathbb{H} \) satisfying
\begin{equation}
\int_{\mathcal{I}} (\Delta(t)f(t), f(t))_{\mathbb{H}} \, dt = \int_{\mathcal{I}} ||\Delta^+(t)f(t)||^2 \, dt < \infty.
\end{equation}
Moreover, for a given finite-dimensional Hilbert space $\mathcal{K}$ denote by $L^2_{\mathcal{A}}[\mathcal{K},\mathbb{H}]$ the set of all Borel operator-functions $F(\cdot) : \mathcal{I} \to [\mathcal{K},\mathbb{H}]$ such that $F(t)h \in L^2_{\mathcal{A}}(\mathcal{I})$ for each $h \in \mathcal{K}$. It is clear that the latter condition is equivalent to $\int_{\mathcal{I}} ||\Delta^2(t)F(t)||^2 \, dt < \infty$.

It is known [24, 13, 35] that $L^2_{\mathcal{A}}(\mathcal{I})$ is a semi-Hilbert space with the semi-definite inner product $(\cdot,\cdot)_\Delta$ and the semi-norm $||\cdot||_\Delta$ given by

\[(3.1) \quad (f,g)_\Delta = \int_{\mathcal{I}} (\Delta(t)f(t),g(t))_\mathbb{H} \, dt, \quad ||f||_\Delta = ((f,f)_\Delta)^{\frac{1}{2}}, \quad f, g \in L^2_{\mathcal{A}}(\mathcal{I}).\]

The semi-Hilbert space $L^2_{\mathcal{A}}(\mathcal{I})$ gives rise to the quotient Hilbert space $L^2_{\mathcal{A}}(\mathcal{I}) = L^2_{\mathcal{A}}(\mathcal{I})/\{f \in L^2(\mathcal{I}) : ||f||_\Delta = 0\}$. The inner product and the norm in $L^2_{\mathcal{A}}(\mathcal{I})$ are defined by

\[(3.2) \quad (\tilde{f},\tilde{g})_\Delta = (f,g)_\Delta, \quad ||\tilde{f}||_\Delta = ||f||_\Delta, \quad \tilde{f}, \tilde{g} \in L^2_{\mathcal{A}}(\mathcal{I}),\]

respectively, where $f \in \tilde{f}$ ($g \in \tilde{g}$) is any representative of the class $\tilde{f}$ (resp. $\tilde{g}$).

In the sequel we systematically use the quotient map $\pi$ from $L^2_{\mathcal{A}}(\mathcal{I})$ onto $\tilde{L}^2_{\mathcal{A}}(\mathcal{I})$ given by $\pi f = \tilde{f}$ ($f \in L^2_{\mathcal{A}}(\mathcal{I})$). Moreover, we let $\tilde{\pi} = \pi \oplus \pi : (L^2_{\mathcal{A}}(\mathcal{I}))^2 \to (L^2_{\mathcal{A}}(\mathcal{I}))^2$, so that $\tilde{\pi}(f,g) = (f,\tilde{g})$, $f, g \in L^2_{\mathcal{A}}(\mathcal{I})$.

### 3.2. Symmetric systems.

In this subsection we provide some known results on symmetric systems of differential equations following [15, 25, 28, 33, 44].

Let as above $\mathcal{I} = [a,b]$ ($-\infty < a < b \leq \infty$) be an interval and let $\mathbb{H}$ be a Hilbert space with $n := \dim \mathbb{H} < \infty$. Moreover, let $B(\cdot), \Delta(\cdot) \in L^1_{\text{loc}}(\mathcal{I};[\mathbb{H}])$ be operator functions such that $B(t) = B^*(t)$ and $\Delta(t) \geq 0$ a.e. on $\mathcal{I}$ and let $J \in [\mathbb{H}]$ be a signature operator (this means that $J^* = J^{-1} = -J$).

A first-order symmetric system on an interval $\mathcal{I}$ (with the regular endpoint $a$) is a system of differential equations of the form

\[(3.2) \quad Jy'(t) - B(t)y(t) = \Delta(t)f(t), \quad t \in \mathcal{I},\]

where $f(\cdot) \in L^2_{\mathcal{A}}(\mathcal{I})$. Together with (3.2) we consider also the homogeneous system

\[(3.3) \quad Jy'(t) - B(t)y(t) = \lambda \Delta(t)y(t), \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}.\]

A function $y \in AC(\mathcal{I};[\mathbb{H}])$ is a solution of (3.2) (resp. (3.3)) if the equality (3.2) (resp. (3.3)) holds a.e. on $\mathcal{I}$. Moreover, a function $Y(\cdot,\lambda) : \mathcal{I} \to [\mathcal{K},\mathbb{H}]$ is an operator solution of the equation (3.3) if $y(t) = Y(t,\lambda)h$ is a (vector) solution of this equation for each $h \in \mathcal{K}$ (here $\mathcal{K}$ is a Hilbert space with dim $\mathcal{K} < \infty$).

In what follows we always assume that system (3.2) is definite in the sense of the following definition.

**Definition 3.1.** [15, 28] The symmetric system (3.2) is called definite if for each $\lambda \in \mathbb{C}$ and each solution $y$ of (3.3) the equality $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) implies $y(t) = 0$, $t \in \mathcal{I}$.

As it is known [44, 25, 33] symmetric system (3.2) gives rise to the maximal linear relations $\mathcal{T}_{\text{max}}$ and $\mathcal{T}^*_{\text{max}}$ in $L^2_{\mathcal{A}}(\mathcal{I})$ and $L^2_{\mathcal{A}}(\mathcal{I})$, respectively. They are given by

\[(3.4) \quad \mathcal{T}_{\text{max}} = \{ (y,f) \in (L^2_{\mathcal{A}}(\mathcal{I}))^2 : y \in AC(\mathcal{I};[\mathbb{H}]) \text{ and } Jy'(t) - B(t)y(t) = \Delta(t)f(t) \text{ a.e. on } \mathcal{I} \}, \]

and $\mathcal{T}^*_{\text{max}} = \tilde{\pi}\mathcal{T}_{\text{max}}$. Moreover the Lagrange’s identity

\[(3.5) \quad (f,z)_\Delta - (y,g)_\Delta = [y,z]_h - (Jy(a), z(a)), \quad \{ y,f \}, \{ z,g \} \in \mathcal{T}_{\text{max}}.\]
holds with
(3.6) \[ [y, z]_b := \lim_{t \to h}(Jy(t), z(t)), \quad y, z \in \text{dom} \ T_{\max}. \]

Formula (3.6) defines the boundary bilinear form \([\cdot, \cdot]_b\) on \text{dom} \ T_{\max}, which plays a crucial role in our considerations. By using this form we define the minimal relations \(T_{\min}\) in \(L^2_\Delta(I)\) and \(T_{\min}^*\) in \(L^2_\Delta(I)\) via
\[ T_{\min} = \{ \{y, f\} \in T_{\max} : y(a) = 0 \text{ and } [y, z]_b = 0 \text{ for each } z \in \text{dom} \ T_{\max} \}. \]
and \(T_{\min}^* = \pi T_{\min}\). According to [44, 33] \(T_{\min}\) is a closed symmetric linear relation in \(L^2_\Delta(I)\) and \(T_{\min}^* = T_{\max}\).

**Remark 3.2.** It is known (see e.g. [33]) that the maximal relation \(T_{\max}\) induced by the definite symmetric system (3.2) possesses the following regularity property: for each \(\{\tilde{y}, \tilde{f}\} \in T_{\max}\) there exists unique function \(y \in AC(I; \mathbb{H}) \cap L^2_\Delta(I)\) such that \(y \equiv \tilde{y}\) and \([y, f]_b \in T_{\max}\) for each \(f \in \tilde{f}\). Below we associate such a function \(y \in AC(I; \mathbb{H}) \cap L^2_\Delta(I)\) with each pair \(\{\tilde{y}, \tilde{f}\} \in T_{\max}\).

For any \(\lambda \in \mathbb{C}\) denote by \(\mathcal{N}_\lambda\) the linear space of solutions of the homogeneous system (3.3) belonging to \(L^2_\Delta(I)\). Definition (3.4) of \(T_{\max}\) implies
\[ \mathcal{N}_\lambda = \ker (T_{\max} - \lambda) = \{y \in L^2_\Delta(I) : \{y, \lambda y\} \in T_{\max}\}, \quad \lambda \in \mathbb{C}, \]
and hence \(\mathcal{N}_\lambda \subset \text{dom} \ T_{\max}\).

As usual, denote by \(n_\pm(T_{\min}) = \dim \mathcal{N}_\lambda(T_{\min}), \quad \lambda \in \mathbb{C}_\pm, \)
the deficiency indices of \(T_{\min}\). Since the system (3.2) is definite, \(\pi \mathcal{N}_\lambda = \mathcal{N}_\lambda(T_{\min})\) and \(\ker (\pi \mid \mathcal{N}_\lambda) = \{0\}, \quad \lambda \in \mathbb{C}. \) This implies that \(\dim \mathcal{N}_\lambda = n_\pm(T_{\min}), \quad \lambda \in \mathbb{C}_\pm. \)

The following lemma will be useful in the sequel.

**Lemma 3.3.** (1) If \(Y(\cdot, \lambda) \in L^2_\Delta(K, \mathbb{H})\) is an operator solution of Eq. (3.3), then the relation
(3.7) \[ K \ni h \to (Y(\lambda)h)(t) = Y(t, \lambda)h \in \mathcal{N}_\lambda, \]
defines the linear mapping \(Y(\lambda) : K \to \mathcal{N}_\lambda\) and, conversely, for each such a mapping \(Y(\lambda)\) there exists unique operator-valued solution \(Y(\cdot, \lambda) \in L^2_\Delta(K, \mathbb{H})\) of equation (3.3) such that (3.7) holds.

(2) Let \(Y(\cdot, \lambda) \in L^2_\Delta(K, \mathbb{H})\) be an operator solution of Eq. (3.3) and let \(F(\lambda) = \pi Y(\lambda)(\in [K, L^2_\Delta(I)])\). Then for each \(f \in L^2_\Delta(I)\)
\[ F^*(\lambda) \bar{f} = \int_I Y^*(t, \lambda) \Delta(t)f(t) dt, \quad f \in \bar{f}. \]

The first statement of this lemma is obvious, while the second one can be proved in the same way as formula (3.70) in [41] (see also formula (2.40) in [33]).

Let \(J \in [\mathbb{H}]\) be the signature operator in (3.2) and let
\[ \nu_+ = \dim \ker (iJ - I) \quad \text{and} \quad \nu_- = \dim \ker (iJ + I). \]
In what follows we suppose that
(3.9) \[ \nu := \nu_- - \nu_+ \geq 0. \]
In this case one can assume without loss of generality that the following statements hold:
(i) the Hilbert space $\mathbb{H}$ is of the form

$$\mathbb{H} = H \oplus \hat{H} \oplus H,$$

where $H$ and $\hat{H}$ are finite dimensional Hilbert spaces with

$$\text{dim } H = \nu_+, \quad \text{dim } \hat{H} = \tilde{\nu};$$

(ii) the operator $J$ is of the form (1.3).

Introducing the Hilbert space

$$H_0 = H \oplus \hat{H}$$

one can represent the equality (3.10) as

$$H = (H \oplus \hat{H}) \oplus H = H_0 \oplus H.$$

Let $\nu_+$ and $\nu_-$ be inertia indices of the skew-Hermitian bilinear form (3.6). Then $\nu_{\pm} < \infty$ and the following equalities hold [4, 42]

$$n_+(T_{\min}) = \nu_+ + \nu_{b+}, \quad n_-(T_{\min}) = \nu_- + \nu_{b-}.$$ 

This yields the equivalence

$$n_+(T_{\min}) = n_-(T_{\min}) \iff \tilde{\nu} = \nu_+ - \nu_-.$$

Next assume that

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix} : H \oplus \hat{H} \oplus H \rightarrow \hat{H} \oplus H$$

is the operator satisfying the relations

$$\text{ran } U = \hat{H} \oplus H,$$

$$iu_2u_2^* - u_1u_3^* + u_3u_1^* = iI_{\hat{H}}, \quad iu_5u_2^* - u_4u_3^* + u_6u_1^* = 0,$$

$$iu_5u_5^* + u_6u_4^* - u_4u_6^* = 0.$$ 

One can prove that the operator (3.16) admits an extension to the $J$-unitary operator

$$\tilde{U} = \begin{pmatrix} u_7 & u_8 & u_9 \\ u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix} : H \oplus \hat{H} \oplus H \rightarrow H \oplus \hat{H} \oplus H,$$

i.e. the operator satisfying $\tilde{U}^*J\tilde{U} = J$. The operator (3.20) induces the linear mapping $\Gamma_a : AC(I; \mathbb{H}) \rightarrow \mathbb{H}$ given by

$$\Gamma_a y = \tilde{U}y(a), \quad y \in AC(I; \mathbb{H}).$$

In accordance with the decomposition (3.10) $\Gamma_a$ admits the block representation

$$\Gamma_a = \begin{pmatrix} \Gamma_{0a}, \widehat{\Gamma}_a, \Gamma_{1a} \end{pmatrix}^\top : AC(I; \mathbb{H}) \rightarrow H \oplus \hat{H} \oplus H.$$

If a function $y \in AC(I; \mathbb{H})$ is decomposed as

$$y(t) = \{y_0(t), \widehat{y}(t), y_1(t)\} \in H \oplus \hat{H} \oplus H, \quad t \in I,$$
then the mappings $\Gamma_{ja} : AC(\mathcal{I}; \mathbb{H}) \to H$, $j \in \{0, 1\}$, and $\tilde{\Gamma}_a : AC(\mathcal{I}; \mathbb{H}) \to \tilde{H}$ in (3.22) can be represented as

$$\Gamma_{0a} y = u_7 y_0(a) + u_8 \tilde{y}(a) + u_9 y_1(a), \quad y \in AC(\mathcal{I}; \mathbb{H})$$

(3.23)

$$\tilde{\Gamma}_a y = u_1 y_0(a) + u_2 \tilde{y}(a) + u_3 y_1(a), \quad \Gamma_{1a} y = u_4 y_0(a) + u_5 \tilde{y}(a) + u_6 y_1(a).$$

(3.24)

This implies that $\tilde{\Gamma}_{a}$ and $\Gamma_{1a}$ are determined by the operator $U$, while $\Gamma_{0a}$ is determined by the extension $\tilde{U}$.

Let $\lambda \in \mathbb{C}$ and $\mathcal{K}$ be a finite-dimensional Hilbert space. By using the operator (3.20) we associate with each operator solution $Y(\cdot, \lambda) : \mathcal{I} \to [\mathcal{K}, \mathbb{H}]$ of equation (3.3) the operator $Y_a(\lambda) \in [\mathcal{K}, \mathbb{H}]$ given by

$$Y_a(\lambda) = \tilde{U} Y(a, \lambda).$$

(3.25)

If in addition $Y(\cdot, \lambda) \in \mathcal{L}_2^2(\mathcal{K}, \mathbb{H})$, then the operator (3.25) admits the representation

$$Y_a(\lambda) = \Gamma_a Y(\lambda),$$

(3.26)

where $Y(\lambda)$ is defined in Lemma 3.3.

In what follows we associate with each operator $U$ (see (3.16)) the operator solution $\varphi(\cdot, \lambda) = \varphi_U(\cdot, \lambda)(\in [H_0, \mathbb{H}])$, $\lambda \in \mathbb{C}$, of Eq. (3.3) with the initial data

$$\varphi_U(a, \lambda) = \left(\begin{array}{c} u_0^* \\ u_2 \\ -u_4^* \\ -iu_3^* \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right) : H \oplus \tilde{H} \to H \oplus \tilde{H} \oplus H.$$  

(3.27)

One can easily verify that for each $J$-unitary extension $\tilde{U}$ of $U$ the following equality holds

$$\varphi_{U,a}(\lambda) = \tilde{U} \varphi_U(a, \lambda) = \left(\begin{array}{c} I_{H_0} \\ 0 \end{array}\right) : H_0 \to H_0 \oplus H.$$  

(3.28)

The particular case of the operator $U$ and its $J$-unitary extension $\tilde{U}$ is (cf. [21])

$$U = \left(\begin{array}{ccc} 0 & I_{\tilde{H}} & 0 \\ \cos B & 0 & \sin B \\ 0 & \sin B & 0 \end{array}\right), \quad \tilde{U} = \left(\begin{array}{ccc} \sin B & 0 & -\cos B \\ 0 & I_{\tilde{H}} & 0 \\ \cos B & 0 & \sin B \end{array}\right),$$

where $B = B^* \in [H]$. For such $U$ the solution $\varphi_U(\cdot, \lambda)$ is defined by the initial data

$$\varphi_U(a, \lambda) = \left(\begin{array}{c} \sin B \\ 0 \\ -\cos B \end{array}\right) : H \oplus \tilde{H} \to H \oplus \tilde{H} \oplus H.$$  

3.3. Decomposing boundary triplets. We start with the following lemma.

**Lemma 3.4.** If $n_-(T_{\min}) \leq n_+(T_{\min})$, then there exist a finite dimensional Hilbert space $\mathcal{H}_b$, a subspace $\mathcal{H}_b \subset \mathcal{H}_b$ and a surjective linear mapping

$$\Gamma_b = \left(\begin{array}{c} \Gamma_{0b} \\ \Gamma_{1b} \end{array}\right) : \text{dom } \mathcal{T}_{\max} \to \mathcal{H}_b \oplus \tilde{H} \oplus \mathcal{H}_b$$

(3.29)

such that for all $y, z \in \text{dom } \mathcal{T}_{\max}$ the following identity is valid

$$[y, z]_b = (\Gamma_{0b} y, \Gamma_{1b} z)_{\mathcal{H}_b} - (\Gamma_{1b} y, \Gamma_{0b} z)_{\mathcal{H}_b} + i(\mathcal{P}_{\mathcal{H}_b} \Gamma_{0b} y, \mathcal{P}_{\mathcal{H}_b} \Gamma_{0b} z)_{\mathcal{H}_b} + i(\tilde{\mathcal{I}}_{b} y, \tilde{\mathcal{I}}_{b} z)_{\tilde{H}}$$

(3.30)
Moreover, for each such a mapping $\Gamma_b$ one has

$$\dim \mathcal{H}_b = \nu_{b-}, \quad \dim \mathcal{H}_b = \nu_{b+} - \tilde{\nu}$$

and the following equivalence holds

$$n_+(T_{\min}) = n_-(T_{\min}) \iff \mathcal{H}_b = \mathcal{H}_b.$$  

Therefore in the case of equal deficiency indices $n_+(T_{\min}) = n_-(T_{\min})$ the identity (3.30) takes the form

$$[y, z]_b = (\Gamma_{0b} y, \Gamma_{1b} z)_{\mathcal{H}_b} - (\Gamma_{1b} y, \Gamma_{0b} z)_{\mathcal{H}_b} + i(\hat{\Gamma}_b y, \hat{\Gamma}_b z)_{\tilde{\mathcal{H}}}.$$  

Proof. In view of (3.14) and (3.9) one has $\nu_{b+} - \nu_{b-} \geq \tilde{\nu}$. Therefore by [42, Lemma 5.1] there exist Hilbert spaces $\mathcal{H}_b$ and $\mathcal{H}_b$ and a surjective linear mapping

$$\Gamma_b = (\Gamma'_{0b}, \hat{\Gamma}_b, \Gamma_{1b})^\top : \text{dom } \mathcal{T}_{\max} \to \mathcal{H}_b \oplus \mathcal{H}_b \oplus \mathcal{H}_b$$

such that

$$[y, z]_b = (\Gamma'_{0b} y, \Gamma_{1b} z)_{\mathcal{H}_b} - (\Gamma_{1b} y, \Gamma'_{0b} z)_{\mathcal{H}_b} + i(\hat{\Gamma}_b y, \hat{\Gamma}_b z)_{\tilde{\mathcal{H}}}, \quad y, z \in \text{dom } \mathcal{T}_{\max}.$$  

Moreover, for such a mapping $\Gamma_b$ one has

$$\dim \mathcal{H}_b = \nu_{b-}, \quad \dim \mathcal{H}_b = \nu_{b+} - \nu_{b-},$$

which in view of (3.11) yields $\dim \mathcal{H}_b \geq \dim \tilde{\mathcal{H}}$. Therefore without loss of generality one may assume that $\tilde{\mathcal{H}} \subset \mathcal{H}_b$ and hence

$$\mathcal{H}_b = \mathcal{H}_b' \oplus \tilde{\mathcal{H}}$$

with $\mathcal{H}_b' = \mathcal{H}_b \oplus \mathcal{H}_b$ (so that $\mathcal{H}_b \subset \mathcal{H}_b$) and let $\Gamma_{0b} : \text{dom } \mathcal{T}_{\max} \to \mathcal{H}_b$ and $\hat{\Gamma}_b : \text{dom } \mathcal{T}_{\max} \to \tilde{\mathcal{H}}$ be the linear mappings given by

$$\Gamma_{0b} = \Gamma'_{0b} + P_{\mathcal{H}_b'} \hat{\Gamma}_b, \quad \hat{\Gamma}_b = P_{\tilde{\mathcal{H}}} \hat{\Gamma}_b.$$  

Then (3.33) can be written in the form (3.29) and the direct calculation gives the identity (3.30). Therefore, for such a mapping $\Gamma_b$ one has

$$\dim \mathcal{H}_b = \dim \mathcal{H}_b + (\dim \mathcal{H}_b - \dim \tilde{\mathcal{H}}),$$

which together with (3.34) and the second equality in (3.11) yields (3.31). Finally, the equivalence (3.32) is implied by (3.15) and (3.31).

Remark 3.5. (1) Since the mapping $\Gamma_b$ is surjective, it follows from (3.30) that $\Gamma_b y = 0$ for each function $y \in \text{dom } \mathcal{T}_{\max}$ such that $y(t) = 0$ on some interval $[\beta, b] \subset \mathcal{I}$. Therefore, if $y_1, y_2 \in \text{dom } \mathcal{T}_{\max}$ and $y_1(t) = y_2(t)$ on some interval $[\beta, b]$, then $\Gamma_b y_1 = \Gamma_b y_2$.

(2) In the case of the regular system (3.2) (i.e., when $\mathcal{I} = [a, b]$ is a compact interval and both integrals $\int_{\mathcal{I}} ||B(t)|| \, dt$ and $\int_{\mathcal{I}} ||\Delta(t)|| \, dt$ are finite) one can put in (3.29) $\mathcal{H}_b = \mathcal{H}_b = H$ and $\Gamma_b y = X_b y(b), \ y \in \text{dom } \mathcal{T}_{\max}$, where $X_b \in \mathbb{H}$ and $X^*_b J X_b = J$.

In general case Remark 5.2 in [42] implies that the mapping (3.29) can be constructed with the aid of the following assertion:

— there exist systems of functions $\{\theta_j^{(1)}\}_{1}^{\nu_{b+}-\tilde{\nu}}, \{\theta_j^{(2)}\}_{1}^{\nu_{b-}}$ and $\{\theta_j^{(3)}\}_{1}^{\nu_{b-}}$ in dom $\mathcal{T}_{\max}$ such that the operators

$$\Gamma_{0b} y = \{[y, \theta_j^{(1)}]_{1}\}_{1}^{\nu_{b+}-\tilde{\nu}}, \quad \hat{\Gamma}_b y = \Sigma_{1}^{\nu_{b-}} [y, \theta_j^{(2)}]_{1} e_j, \quad \Gamma_{1b} y = \{[y, \theta_j^{(3)}]_{1}\}_{1}^{\nu_{b-}}$$

($y \in \text{dom } \mathcal{T}_{\max}$) form the surjective linear mapping $\Gamma_b = (\Gamma_{0b}, \hat{\Gamma}_b, \Gamma_{1b})^\top : \text{dom } \mathcal{T}_{\max} \to \mathbb{C}^{\nu_{b+}-\tilde{\nu}} \oplus \tilde{\mathcal{H}} \oplus \mathbb{C}^{\nu_{b-}}$ satisfying the identity (3.30) (here $\{e_j\}_{1}^{\nu_{b-}}$ is an orthonormal basis in $\tilde{\mathcal{H}}$).
In the case \( \nu_{b-} = 0, \nu_{b+} = \tilde{\nu} \) and \( \dim \tilde{H} = 1 \) one has \( \tilde{H}_b = H_b = \{0\} \). In this case one can put
\[
\Gamma_{by} = |y, \theta|_b e,
\]
where \( e \) is an ort in \( \tilde{H} \) and \( \theta \) is a function in \( \text{dom} \mathcal{T}_{\text{max}} \) such that \( |\theta, \theta|_b = i \).

These assertions show that one may consider \( \Gamma_{by} \) as a singular boundary value of a function \( y \in \text{dom} \mathcal{T}_{\text{max}} \) (cf. [13, Ch. 13.2]).

The following proposition is immediate from [42, Theorem 5.8] and (3.32).

**Proposition 3.6.** Assume that \( n_-(T_{\text{min}}) \leq n_+ (T_{\text{min}}) \), \( \tilde{U} \) is the \( J \)-unitary operator (3.20), \( \Gamma_a \) is the linear mapping (3.21) with the block representation (3.22) and \( \Gamma_b \) is the surjective linear mapping (3.29) satisfying the identity (3.30).

Moreover, let \( H_0 \) and \( H_1(\subset H_0) \) be finite dimensional Hilbert spaces defined by

\[
(3.36) \quad H_0 = H_0 \oplus \tilde{H}_b, \quad H_1 = H_0 \oplus H_b
\]

and let \( \Gamma_j : \mathcal{T}_{\text{max}} \to H_j, j \in \{0, 1\}, \) be the operators given by

\[
(3.37) \quad \Gamma_0 \{\tilde{y}, \tilde{f}\} = \{-\Gamma_{1a} y + i(\tilde{\Gamma}_a - \tilde{\Gamma}_b) y, \Gamma_{0b} y\} \in \{H_0 \oplus \tilde{H}_b\},
\]

\[
(3.38) \quad \Gamma_1 \{\tilde{y}, \tilde{f}\} = \{\Gamma_{0a} y + \frac{1}{i}(\tilde{\Gamma}_a + \tilde{\Gamma}_b) y, -\Gamma_{1b} y\} \in \{H_0 \oplus H_b\}, \quad \{\tilde{y}, \tilde{f}\} \in \mathcal{T}_{\text{max}}.
\]

(here \( y \in \text{dom} \mathcal{T}_{\text{max}} \) is the function corresponding to \( \{\tilde{y}, \tilde{f}\} \in \mathcal{T}_{\text{max}} \) according to Remark 3.2. Then the collection \( \Pi = \{H_0 \oplus H_1, \Gamma_0, \Gamma_1\} \) is a boundary triplet for \( \mathcal{T}_{\text{max}} \). If in addition \( n_+(T_{\text{min}}) = n_-(T_{\text{min}}) \), then \( \Pi \) turns into an ordinary boundary triplet \( \Pi = \{H, \Gamma_0, \Gamma_1\} \) for \( \mathcal{T}_{\text{max}} \), where \( H = H_0 \oplus H_b \) and \( \Gamma_j : \mathcal{T}_{\text{max}} \to H, j \in \{0, 1\}, \) are the operators given by (3.37) and (3.38) with \( \hat{H}_b = H_b \).

**Definition 3.7.** The boundary triplet \( \Pi = \{H_0 \oplus H_1, \Gamma_0, \Gamma_1\} \) constructed in Proposition 3.6 will be called a decomposing boundary triplet for \( T_{\text{max}} \).

**Proposition 3.8.** Let \( n_-(T_{\text{min}}) \leq n_+(T_{\text{min}}) \), let \( U \) be the operator (3.16), let \( \tilde{\Gamma}_a \) and \( \Gamma_{1a} \) be the linear mappings (3.24) and let \( \tilde{\Gamma}_b \) be the linear mapping (3.29). Then:

1. The equalities

\[
(3.39) \quad T = \{\tilde{y}, \tilde{f}\} \in \mathcal{T}_{\text{max}} : \Gamma_{1a} y = 0, \tilde{\Gamma}_a y = \tilde{\Gamma}_b y, \Gamma_{0b} y = \Gamma_{1b} y = 0
\]

\[
(3.40) \quad T^* = \{\tilde{y}, \tilde{f}\} \in \mathcal{T}_{\text{max}} : \Gamma_{1a} y = 0, \tilde{\Gamma}_a y = \tilde{\Gamma}_b y
\]

define a symmetric extension \( T \) of \( T_{\text{min}} \) and its adjoint \( T^* \). Moreover, the deficiency indices of \( T \) are \( n_+(T) = \nu_{b+} - \tilde{\nu} \) and \( n_-(T) = \nu_{b-} \).

2. The collection \( \tilde{\Pi} = \{H_b \oplus H_b, \tilde{\Gamma}_0, \tilde{\Gamma}_1\} \) with the operators

\[
(3.41) \quad \tilde{\Gamma}_0 \{\tilde{y}, \tilde{f}\} = \Gamma_{0b} y, \quad \tilde{\Gamma}_1 \{\tilde{y}, \tilde{f}\} = -\Gamma_{1b} y, \quad \{\tilde{y}, \tilde{f}\} \in T^*
\]

is a boundary triplet for \( T^* \) and the (maximal symmetric) relation \( A_0(= \ker \tilde{\Gamma}_0) \) is of the form

\[
(3.42) \quad A_0 = \{\tilde{y}, \tilde{f}\} \in \mathcal{T}_{\text{max}} : \Gamma_{1a} y = 0, \tilde{\Gamma}_a y = \tilde{\Gamma}_b y, \Gamma_{0b} y = 0
\]

If in addition \( n_+(T_{\text{min}}) = n_-(T_{\text{min}}) \), then \( n_+(T) = n_-(T) = \nu_{b-} \), \( \tilde{\Pi} = \{H_b, \tilde{\Gamma}_0, \tilde{\Gamma}_1\} \) is an ordinary boundary triplet for \( T^* \) and \( A_0 = A_0^2 \).

**Proof.** Let \( \tilde{U} \) be the \( J \)-unitary extension (3.20) of \( U \), let \( \Gamma_{0a} \) be the operator (3.23) and let \( \Pi = \{H_0 \oplus H_1, \Gamma_0, \Gamma_1\} \) be the decomposing boundary triplet (3.37), (3.38) for \( \mathcal{T}_{\text{max}} \). Applying to this triplet Proposition 2.10 one obtains the desired statements. \( \square \)
Remark 3.9. Clearly, \(\text{mul } T = \text{mul } T^*\) if and only if the following condition is fulfilled:

(C1) For each function \(y \in \text{dom } \mathcal{T}_{\max}\) the equalities

\[
\Gamma_{1a}y = 0, \quad \widehat{\Gamma}_a y = \widehat{\Gamma}_b y \quad \text{and} \quad \Delta(t)y(t) = 0 \quad \text{(a.e. on } \mathcal{I})
\]

yield \(\Gamma_{0b}y = \Gamma_{1b}y = 0\).

Moreover, \(\text{mul } T^* = 0\) (i.e., \(T\) is a densely defined operator) if and only if the following condition is satisfied:

(C2) For each \(y \in \text{dom } \mathcal{T}_{\max}\) the equalities (3.43) yield \(y = 0\).

4. \(L^2_\Delta\)-SOLUTIONS OF BOUNDARY VALUE PROBLEMS

In what follows we suppose that the symmetric system (3.2) satisfies the condition \(n_-(T_{\min}) \leq n_+(T_{\min})\). Our considerations will be also based on the following assumptions:

(A1) \(U\) is the operator (3.16) satisfying the relations (3.17) - (3.19) and \(\widehat{\Gamma}_a\) and \(\Gamma_{1a}\) are the linear mappings (3.24).

(A2) \(\hat{H}_b\) and \(H_b(\subset \hat{H}_b)\) are finite dimensional Hilbert spaces and \(\Gamma_b\) is the surjective linear mapping (3.29) such that (3.30) holds.

In addition to (A1)–(A2) we will sometimes use the following assumption:

(A3) \(\bar{U}\) is a \(J\)-unitary extension (3.20) of \(U\) and \(\Gamma_{0a}\) is the mapping (3.23).

Let (A1)–(A2) be satisfied and let \(\tau = \{\tau_+, \tau_-\} \in \hat{R}(\hat{H}_b, H_b)\) be a collection of holomorphic operator pairs (2.4) with \(C_0(\lambda) \in [\hat{H}_b], C_1(\lambda) \in [\hat{H}_b, \hat{H}_b], \lambda \in \mathbb{C}_+, \) and \(D_0(\lambda) \in [\hat{H}_b, H_b], D_1(\lambda) \in [\hat{H}_b], \lambda \in \mathbb{C}_-\). For a given function \(f \in L^2_\Delta(\mathcal{I})\) consider the following boundary value problem:

\[
\begin{align*}
Jy' - B(t)y & = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I}, \\
\Gamma_{1a}y & = 0, \quad \widehat{\Gamma}_a y = \widehat{\Gamma}_b y, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \\
C_0(\lambda)\Gamma_{0b}y + C_1(\lambda)\Gamma_{1b}y & = 0, \quad \lambda \in \mathbb{C}_+ \\
D_0(\lambda)\Gamma_{0b}y + D_1(\lambda)\Gamma_{1b}y & = 0, \quad \lambda \in \mathbb{C}_-.
\end{align*}
\]

A function \(y(\cdot, \cdot) : \mathcal{I} \times (\mathbb{C} \setminus \mathbb{R}) \to \mathbb{H}\) is called a solution of this problem if for each \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) the function \(y(\cdot, \lambda)\) belongs to \(AC(\mathcal{I}; \mathbb{H}) \cap L^2_\Delta(\mathcal{I})\) and satisfies the equation (4.1) a.e. on \(\mathcal{I}\) (so that \(y \in \text{dom } \mathcal{T}_{\max}\)) and the boundary conditions (4.2) – (4.4).

If \(n_+(T_{\min}) = n_-(T_{\min})\), then in view of (3.32) \(\hat{H}_b = H_b\) and the collection \(\tau\) turns into a Nevanlinna operator pair \(\tau \in \hat{R}(H_b)\) defined by (2.14) with \(C_j(\lambda) \in [H_b], \lambda \in \mathbb{C} \setminus \mathbb{R}, j \in \{0, 1\}\). In this case the boundary conditions (4.3)–(4.4) takes the form

\[
C_0(\lambda)\Gamma_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

If in addition \(\tau \in \hat{R}^0(H_b)\) is a selfadjoint operator pair (2.17) with \(C_j \in [H_b], j \in \{0, 1\}\), then (4.5) becomes a self-adjoint boundary condition (at the endpoint \(b\)):

\[
C_0\Gamma_{0b}y + C_1\Gamma_{1b}y = 0.
\]

Theorem 4.1. Let \(T\) be a symmetric relation in \(L^2_\Delta(\mathcal{I})\) defined by (3.39). If \(\tau = \{\tau_+, \tau_-\} \in \hat{R}(\hat{H}_b, H_b)\) is a collection (2.4), then for every \(f \in L^2_\Delta(\mathcal{I})\) the boundary problem (4.1) - (4.4) has a unique solution \(y(t, \lambda) = y_f(t, \lambda)\) and the equality

\[
R(\lambda)\bar{f} = \pi(y_f(\cdot, \lambda)), \quad \bar{f} \in L^2_\Delta(\mathcal{I}), \quad f \in \bar{f}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

is valid.
defines a generalized resolvent \( R(\lambda) =: R_\tau(\lambda) \) of \( T \). Conversely, for each generalized resolvent \( R(\lambda) \) of \( T \) there exists a unique \( \tau \in \hat{R}(\mathcal{H}_b, \mathcal{H}_b) \) such that \( R(\lambda) = R_\tau(\lambda) \).

If in addition \( n_+(T_{\min}) = n_-(T_{\min}) \), then \( n_+(T) = n_-(T) \) and the above statements hold with Nevanlinna operator pairs \( \tau \in \hat{R}(\mathcal{H}_b) \) of the form (2.14) and the boundary condition (4.5) in place of (4.3) and (4.4). Moreover, \( R_\tau(\lambda) \) is a canonical resolvent of \( T \) if and only if \( \tau \in \hat{R}^0(\mathcal{H}_b) \) is a self-adjoint operator pair (2.17), in which case \( R_\tau(\lambda) = (\hat{T}^\tau - \lambda)^{-1} \) with

\[
\hat{T}^\tau = \left\{ \{\tilde{y}, \tilde{f}\} \in T_{\max} : \Gamma_{1a} y = 0, \tilde{\Gamma}_a y = \tilde{\Gamma}_b y, \ C_0 \Gamma_0 b y + C_1 \Gamma_1 b y = 0 \right\}.
\]

Proof. Let \( \hat{\Pi} = \{\mathcal{H}_b \oplus \mathcal{H}_b, \hat{\Gamma}_0, \hat{\Gamma}_1\} \) be the boundary triplet (3.41) for \( T^* \). Applying to this triplet Theorem 2.17 we obtain the required statements. \( \square \)

Remark 4.2. Let in Theorem 4.1 \( \tau_0 = \{\tau_+, \tau_-\} \in \hat{R}(\mathcal{H}_b, \mathcal{H}_b) \) be defined by (2.4) with

\[
C_0(\lambda) \equiv I_{\hat{\mathcal{H}}_b}, \quad C_1(\lambda) \equiv 0, \quad D_0(\lambda) \equiv P_{\mathcal{H}_b}(\in [\hat{\mathcal{H}}_b, \mathcal{H}_b]), \quad D_1(\lambda) \equiv 0
\]

and let \( R_0(\lambda) := R_{\tau_0}(\lambda) \) be the corresponding generalized resolvent of \( T \). Then

\[
R_0(\lambda) = (A_0 - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+ \quad \text{and} \quad R_0(\lambda) = (A_0^* - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_-,
\]

where \( A_0 \) is given by (3.42).

If in addition \( n_+(T_{\min}) = n_-(T_{\min}) \), then \( \tau_0 \) turns into a self-adjoint operator pair \( \tau_0 = \{(I_{\mathcal{H}_b}, 0); \mathcal{H}_b\} \in \hat{R}^0(\mathcal{H}_b) \) and \( R_0(\lambda) = (A_0 - \lambda)^{-1} \) is a canonical resolvent of \( T \).

Proposition 4.3. Let the assumptions (A1) and (A2) be satisfied. Then:

1. For every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) there exists a unique operator solution \( v_0(\cdot, \lambda) \in \mathcal{L}_A^2[H_0, \mathbb{H}] \) of Eq. (3.3) such that

\[
\Gamma_{1a} v_0(\lambda) = -P_H, \quad i(\tilde{\Gamma}_a - \tilde{\Gamma}_b)v_0(\lambda) = P_{\hat{\mathcal{H}}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]

\[
\Gamma_{0b} v_0(\lambda) = 0, \quad \lambda \in \mathbb{C}_+; \quad P_{\mathcal{H}_b} \Gamma_{0b} v_0(\lambda) = 0, \quad \lambda \in \mathbb{C}_-
\]

2. For every \( \lambda \in \mathbb{C}_+ \) (resp. \( \lambda \in \mathbb{C}_- \)) there exists a unique operator solution \( u_+(\cdot, \lambda) \in \mathcal{L}_A^2[H_0, \mathbb{H}] \) (resp. \( u_-(-\cdot, \lambda) \in \mathcal{L}_A^2[H_0, \mathbb{H}] \)) of Eq. (3.3) such that

\[
\Gamma_{1a} u_+(\lambda) = 0, \quad i(\tilde{\Gamma}_a - \tilde{\Gamma}_b)u_+(\lambda) = 0, \quad \lambda \in \mathbb{C}_+,
\]

\[
\Gamma_{0b} u_+(\lambda) = I_{\hat{\mathcal{H}}_b}, \quad \lambda \in \mathbb{C}_+; \quad P_{\mathcal{H}_b} \Gamma_{0b} u_-(\lambda) = I_{\hat{\mathcal{H}}_b}, \quad \lambda \in \mathbb{C}_-.
\]

In formulas (4.10)–(4.13) \( v_0(\lambda) \) and \( u_\pm(\lambda) \) are linear mappings from Lemma 3.3, (1) corresponding to the solutions \( v_0(\cdot, \lambda) \) and \( u_\pm(\cdot, \lambda) \) respectively.

Proof. Let \( \tilde{U} \) be the \( J \)-unitary extension (3.20) of \( U \), let \( \hat{\Gamma}_{0a} \) be the operator (3.23) and let \( \Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \hat{\Gamma}_0, \hat{\Gamma}_1\} \) be the decomposing boundary triplet (3.37), (3.38) for \( T_{\max} \). Assume also that \( \gamma_\pm(\cdot) \) are the \( \gamma \)-fields of \( \Pi \). Since the quotient mapping \( \pi \) isometrically maps \( \mathcal{N}_\lambda \) onto \( \mathcal{N}_\lambda(T_{\min}) \), it follows that for every \( \lambda \in \mathbb{C}_+ \) (resp. \( \lambda \in \mathbb{C}_- \)) there exists an isomorphism \( Z_+(\lambda) : \mathcal{H}_0 \rightarrow \mathcal{N}_\lambda \) (resp. \( Z_-(\lambda) : \mathcal{H}_1 \rightarrow \mathcal{N}_\lambda \)) such that

\[
\gamma_+(\lambda) = \pi Z_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(\lambda) = \pi Z_-(\lambda), \quad \lambda \in \mathbb{C}_-.
\]

Let \( \Gamma_0' \) and \( \Gamma_1' \) be the linear mappings given by

\[
\Gamma_0' = \begin{pmatrix} -\Gamma_{1a} + i(\tilde{\Gamma}_a - \tilde{\Gamma}_b) \\ \Gamma_{0b} \end{pmatrix} : \text{dom} \, T_{\max} \rightarrow H_0 \oplus \hat{\mathcal{H}}_b,
\]

\[
\Gamma_1' = \begin{pmatrix} \Gamma_{0a} + \frac{1}{2}(\tilde{\Gamma}_a + \tilde{\Gamma}_b) \\ -\Gamma_{1b} \end{pmatrix} : \text{dom} \, T_{\max} \rightarrow H_0 \oplus \hat{\mathcal{H}}_b.
\]
Then by (3.37) and (3.38) one has $\Gamma_j\{\pi y, \lambda \pi y\} = \Gamma_j y, \ y \in \mathcal{N}_\lambda, \ j \in \{0, 1\}$. Combining of this equality with (4.14) and (2.24) gives

\begin{equation}
\Gamma_0' Z_+(\lambda) = I_{H_0}, \ \lambda \in \mathbb{C}_+; \quad P_{H_0} \oplus H_1 \Gamma_0' Z_-(\lambda) = I_{H_1}, \ \lambda \in \mathbb{C}_-,
\end{equation}

which in view of (4.15) can be written as

\begin{equation}
\begin{pmatrix} -\Gamma_{1a} + i(\tilde{\Gamma}_a - \tilde{\Gamma}_b) \\ P_{\mathcal{H}_b} \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} I_{H_0} & 0 \\ 0 & I_{H_b} \end{pmatrix}, \ \lambda \in \mathbb{C}_+
\end{equation}

(4.17)

\begin{equation}
\begin{pmatrix} -\Gamma_{1a} + i(\tilde{\Gamma}_a - \tilde{\Gamma}_b) \\ P_{\mathcal{H}_b} \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} I_{H_0} & 0 \\ 0 & I_{H_b} \end{pmatrix}, \ \lambda \in \mathbb{C}_-
\end{equation}

(4.18)

It follows from (4.17) and (4.18) that

\begin{equation}
\Gamma_1 Z_+(\lambda) = (-P_H, 0), \quad \frac{\sqrt{2}}{2}(\tilde{\Gamma}_a - \tilde{\Gamma}_b) Z_+(\lambda) = (-\frac{\sqrt{2}}{2} P_H, 0), \ \lambda \in \mathbb{C}_+
\end{equation}

(4.19)

\begin{equation}
\Gamma_{0b} Z_-(\lambda) = (0, \Gamma_{H_b}), \ \lambda \in \mathbb{C}_+; \quad P_{\mathcal{H}_b} \Gamma_{0b} Z_-(\lambda) = (0, I_{H_b}), \ \lambda \in \mathbb{C}_-
\end{equation}

(4.20)

Assume now that the block representations of $Z_+(\lambda)$ are

\begin{equation}
Z_+(\lambda) = (v_0(\lambda), u_+(\lambda)) : H_0 \oplus \tilde{\mathcal{H}}_b \rightarrow \mathcal{N}_\lambda, \ \lambda \in \mathbb{C}_+
\end{equation}

(4.21)

\begin{equation}
Z_-(\lambda) = (v_0(\lambda), u_-(\lambda)) : H_0 \oplus \mathcal{H}_b \rightarrow \mathcal{N}_\lambda, \ \lambda \in \mathbb{C}_-
\end{equation}

(4.22)

and let $v_0(\cdot, \lambda) \in \mathcal{L}_2^2[H_0, \mathbb{H}], \ u_+(\cdot, \lambda) \in \mathcal{L}_2^2[\tilde{\mathcal{H}}_b, \mathbb{H}]$ and $u_-(\cdot, \lambda) \in \mathcal{L}_2^2[\mathcal{H}_b, \mathbb{H}]$ be the operator solutions of Eq. (3.3) corresponding to $v_0(\lambda)$, $u_+(\lambda)$ and $u_-(\lambda)$ respectively (see Lemma 3.3). Then the representations (4.21) and (4.22) together with (4.19) and (4.20) yield the relations (4.10)-(4.13) for $v_0(\cdot, \lambda)$ and $u_+\cdot, \lambda$).

To prove uniqueness of $v_0(\cdot, \lambda)$ and $u_+\cdot, \lambda$ assume that $\tilde{v}_0(\cdot, \lambda) \in \mathcal{L}_2^2[H_0, \mathbb{H}], \tilde{u}_+\cdot, \lambda) \in \mathcal{L}_2^2[\tilde{\mathcal{H}}_b, \mathbb{H}]$ and $\tilde{u}_-(\cdot, \lambda) \in \mathcal{L}_2^2[\mathcal{H}_b, \mathbb{H}]$ are other solutions of Eq. (3.3) satisfying (4.10)-(4.13). Then for each $h_0 \in \mathcal{H}_0$, $\tilde{h}_b \in \tilde{\mathcal{H}}_b$ and $h_b \in \mathcal{H}_b$ the functions $y_1 = (v_0(t, \lambda) - \tilde{v}_0(t, \lambda)) h_0, \ y_2 = (u_+(t, \lambda) - \tilde{u}_+(t, \lambda)) \tilde{h}_b$ and $y_3 = (u_-(t, \lambda) - \tilde{u}_-(t, \lambda)) h_b$ are solutions of the homogeneous boundary problem (4.1) - (4.4) with $f = 0$ and $C_j(\lambda), D_j(\lambda), \ j \in \{0, 1\}$, defined by (4.9). Since by Theorem 4.1 such a problem has a unique solution $y = 0$, it follows that $y_1 = y_2 = y_3 = 0$ and, consequently, $v_0 = \tilde{v}_0$ and $u_+ = \tilde{u}_+$.

\begin{proposition}
Assume the hypothesis (A1)-(A3). Let $\Pi = \{H_0 \oplus H_1, \Gamma_0, \Gamma_1\}$ be the decomposing boundary triplet for $T_{\text{max}}$ defined in Proposition 3.6, let $\gamma_{\pm}(\cdot)$ and $M_{\pm}(\cdot)$ be the corresponding $\gamma$-field and the Weyl function, respectively. Then $\gamma_{\pm}(\cdot)$ is connected with the solutions $v_0(\cdot, \lambda)$ and $u_+\cdot, \lambda$ from Proposition 4.3 by

\begin{equation}
\gamma_{\pm}(\lambda) \upharpoonright H_0 = \pi v_0(\lambda), \ \lambda \in \mathbb{C}_+;
\end{equation}

(4.23)

\begin{equation}
\gamma_+(\lambda) \upharpoonright \tilde{\mathcal{H}}_b = \pi u_+(\lambda), \ \lambda \in \mathbb{C}_+; \quad \gamma_-(\lambda) \upharpoonright \mathcal{H}_b = \pi u_-(\lambda), \ \lambda \in \mathbb{C}_-
\end{equation}

and the block representations

\begin{equation}
M_+(\lambda) = \begin{pmatrix} m_0(\lambda) & M_2(\lambda) \\ M_1(\lambda) & M_4(\lambda) \end{pmatrix} : H_0 \oplus \tilde{\mathcal{H}}_b \rightarrow H_0 \oplus \mathcal{H}_b, \ \lambda \in \mathbb{C}_+
\end{equation}

(4.25)

\begin{equation}
M_-(\lambda) = \begin{pmatrix} m_0(\lambda) & M_2(-\lambda) \\ M_1(-\lambda) & M_4(-\lambda) \end{pmatrix} : H_0 \oplus \mathcal{H}_b \rightarrow H_0 \oplus \tilde{\mathcal{H}}_b, \ \lambda \in \mathbb{C}_-
\end{equation}

(4.26)
hold with
\begin{align}
(4.27) & \quad m_0(\lambda) = (\Gamma_{0a} + \tilde{\Gamma}_a)v_0(\lambda) + \frac{i}{2}P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \\
(4.28) & \quad M_{2\pm}(\lambda) = (\Gamma_{0a} + \tilde{\Gamma}_a)u_{\pm}(\lambda), \quad \lambda \in \mathbb{C}_\pm \\
(4.29) & \quad M_{3\pm}(\lambda) = -\Gamma_{1b}v_0(\lambda), \quad M_{4\pm}(\lambda) = -\Gamma_{1b}u_+(\lambda), \quad \lambda \in \mathbb{C}_+ \\
(4.30) & \quad M_{3-}(\lambda) = (-\Gamma_{1b} + iP_H\Gamma_0)v_0(\lambda), \\
(4.31) & \quad M_{4-}(\lambda) = (-\Gamma_{1b} + iP_H\Gamma_0)u_-(\lambda), \quad \lambda \in \mathbb{C}_-.
\end{align}

**Proof.** The equalities (4.23) and (4.24) are immediate from (4.14) and (4.21), (4.22).

Next assume that the assumption \((A3)\) and \(\Gamma_1\) are the linear mappings (4.15) and let \(M_\pm(\cdot)\) have the block representations (4.25), (4.26). Then by using (4.14) and (2.22), (2.23) one obtains

\[
\Gamma_1Z_+(\lambda) = M_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad (\Gamma_1' + iP_H\Gamma_0')Z_-(\lambda) = M_-(\lambda), \quad \lambda \in \mathbb{C}_-,
\]

which can be represented as
\begin{align}
(4.32) & \quad \begin{pmatrix}
\Gamma_{0a} + \frac{1}{2}(\tilde{\Gamma}_a + \tilde{\Gamma}_b) \\
-\Gamma_{1b} + iP_H\Gamma_0
\end{pmatrix}
Z_+(\lambda) = \begin{pmatrix}
m_0(\lambda) \\
M_{3+}(\lambda)
\end{pmatrix}, \quad \lambda \in \mathbb{C}_+ \\
(4.33) & \quad \begin{pmatrix}
\Gamma_{0a} + \frac{1}{2}(\tilde{\Gamma}_a + \tilde{\Gamma}_b) \\
-\Gamma_{1b} + iP_H\Gamma_0
\end{pmatrix}
Z_-(\lambda) = \begin{pmatrix}
m_0(\lambda) \\
M_{3-}(\lambda)
\end{pmatrix}, \quad \lambda \in \mathbb{C}_-.
\end{align}

Hence
\begin{align}
(4.34) & \quad \Gamma_{0a}Z_+(\lambda) = (P_Hm_0(\lambda), P_HM_{2\pm}(\lambda)), \quad \lambda \in \mathbb{C}_+ \\
(4.35) & \quad \frac{1}{2}(\tilde{\Gamma}_a + \tilde{\Gamma}_b)Z_+(\lambda) = (P_Hm_0(\lambda), P_HM_{2\pm}(\lambda)), \quad \lambda \in \mathbb{C}_+ \\
(4.36) & \quad \Gamma_{1b}Z_+(\lambda) = (-M_{3+}(\lambda), -M_{4+}(\lambda)), \quad \lambda \in \mathbb{C}_+ \\
(4.37) & \quad (-\Gamma_{1b} + iP_H\Gamma_0)Z_-(\lambda) = (M_{3-}(\lambda), M_{4-}(\lambda)), \quad \lambda \in \mathbb{C}_-.
\end{align}

Summing up the second equality in (4.19) with (4.34) and (4.35) one obtains
\begin{align}
(4.38) & \quad (\Gamma_{0a} + \tilde{\Gamma}_a)Z_\pm(\lambda) = (m_0(\lambda) - \frac{i}{2}P_H, M_{2\pm}(\lambda)), \quad \lambda \in \mathbb{C}_+.
\end{align}

Combining now (4.36)-(4.38) with the block representations (4.21) and (4.22) of \(Z_\pm(\lambda)\) we arrive at the equalities (4.27)-(4.31). \(\square\)

In the case of equal deficiency indices the statements of Propositions 4.3 and 4.4 can be rather simplified. Namely the following corollary is obvious.

**Corollary 4.5.** Let the assumptions \((A1)\) and \((A2)\) be satisfied, \(n_+(T_{\text{min}}) = n_-(T_{\text{min}})\), and let \(A_0\) be the selfadjoint extension of \(T_{\text{min}}\) given by (3.42). Then for every \(\lambda \in \rho(A_0)\) there exists a unique pair of operator-valued solutions \(v_0(\cdot, \lambda) \in L^2_{\Lambda}[H_0, \mathbb{H}]\) and \(u(\cdot, \lambda) \in L^2_{\Lambda}[H_b, \mathbb{H}]\) of Eq. (3.3) satisfying the following boundary conditions:
\[
\begin{align*}
\Gamma_{1a}v_0(\lambda) & = -P_H, \quad i(\tilde{\Gamma}_a - \tilde{\Gamma}_b)v_0(\lambda) = P_H, \quad \Gamma_{0b}v_0(\lambda) = 0, \quad \lambda \in \rho(A_0), \\
\Gamma_{1a}u(\lambda) & = 0, \quad i(\tilde{\Gamma}_a - \tilde{\Gamma}_b)u(\lambda) = 0, \quad \Gamma_{0b}u(\lambda) = I_{H_b}, \quad \lambda \in \rho(A_0).
\end{align*}
\]

Assume, in addition, that the assumption \((A3)\) is fulfilled and \(\Pi = \{H, \Gamma_0, \Gamma_1\}\) is an ordinary decomposing boundary triplet (3.37), (3.38) for \(T_{\text{max}}\). Then the corresponding Weyl function \(M(\cdot)\) admits a block matrix representation
\begin{align}
(4.39) & \quad M(\lambda) = \begin{pmatrix}
m_0(\lambda) & M_2(\lambda) \\
M_3(\lambda) & M_4(\lambda)
\end{pmatrix}: H_0 \oplus H_b \rightarrow H_0 \oplus H_b, \quad \lambda \in \rho(A_0)
\end{align}
with the entries given by

\begin{align}
(4.40) \quad m_0(\lambda) &= (\Gamma_{0a} + \hat{\Gamma}_a)v_0(\lambda) + \frac{\hat{\tau}}{\rho} P_H, \quad M_2(\lambda) = (\Gamma_{0a} + \hat{\Gamma}_a)u(\lambda), \\
(4.41) \quad M_3(\lambda) &= -\Gamma_{1b}v_0(\lambda), \quad M_4(\lambda) = -\Gamma_{1b}u(\lambda), \quad \lambda \in \rho(A_0).
\end{align}

**Theorem 4.6.** Let the assumptions (A1) and (A2) be fulfilled and let \( \tau = \{ \tau_+, \tau_- \} \in \mathcal{R}(\mathcal{H}_b, \mathcal{H}_b) \) be a collection of operator pairs (2.4). Then for each \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) there exists a unique operator solution \( v_\tau(\cdot, \lambda) \in L^2_\Delta[H_0, \mathbb{H}] \) of Eq. (3.3) satisfying the boundary conditions

\begin{align}
(4.42) \quad \Gamma_{1a}v_\tau(\lambda) &= -P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \\
(4.43) \quad i(\hat{\Gamma}_a - \hat{\Gamma}_b)v_\tau(\lambda) &= P_H^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \\
(4.44) \quad C_0(\lambda)\Gamma_{0b}v_\tau(\lambda) + C_1(\lambda)\Gamma_{1b}v_\tau(\lambda) &= 0, \quad \lambda \in \mathbb{C}_+, \\
(4.45) \quad D_0(\lambda)\Gamma_{0b}v_\tau(\lambda) + D_1(\lambda)\Gamma_{1b}v_\tau(\lambda) &= 0, \quad \lambda \in \mathbb{C}_-.
\end{align}

(here \( P_H \) and \( P_H^* \) are the orthoprojectors in \( H_0 \) onto \( H \) and \( \tilde{H} \) respectively and \( v_\tau(\cdot, \lambda) \) is the linear map from Lemma 3.3 corresponding to the solution \( v_\tau(\cdot, \lambda) \)). Moreover, \( v_\tau(\cdot, \lambda) \) is connected with the solutions \( v_0(\cdot, \lambda) \) and \( u_\pm(\cdot, \lambda) \) from Proposition 4.3 via the equalities

\begin{align}
(4.46) \quad v_\tau(t, \lambda) &= v_0(t, \lambda) - u_+(t, \lambda)(\tau_+(\lambda) + M_{4+}(\lambda))^{-1}M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+ \\
(4.47) \quad v_\tau(t, \lambda) &= v_0(t, \lambda) - u_-(t, \lambda)(\tau_-(\lambda)^\ast + M_{4-}(\lambda))^{-1}M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-,
\end{align}

in which \( M_{3\pm}(\cdot) \) and \( M_{4\pm}(\cdot) \) are the operator functions given by (4.29)–(4.31).

If in addition \( n_+(T_{\min}) = n_-(T_{\min}) \), then \( \tau \in \mathcal{R}(\mathcal{H}_b) \) is given by (2.14) and the boundary conditions (4.44) and (4.45) take the form

\begin{align}
C_0(\lambda)\Gamma_{0b}v_\tau(\lambda) + C_1(\lambda)\Gamma_{1b}v_\tau(\lambda) &= 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{align}

**Proof.** To prove the theorem it is sufficient to show that the equalities (4.46) and (4.47) define a unique solution \( v_\tau(\cdot, \lambda) \in L^2_\Delta[H_0, \mathbb{H}] \) of Eq. (3.3) satisfying (4.42)–(4.45).

Let \( \Pi = \{ \mathcal{H}_b \oplus \mathcal{H}_b, \hat{\Gamma}_0, \hat{\Gamma}_1 \} \) be a boundary triplet (3.41) for \( T^* \). Then by Proposition 2.10, (3) the corresponding Weyl function is \( \hat{M}_\tau(\lambda) = M_{4+}(\lambda) \) and according to [39] one has \( 0 \in \rho(\tau_+(\lambda) + M_{4+}(\lambda)), \lambda \in \mathbb{C}_+ \), and \( 0 \in \rho(\tau_-(\lambda)^\ast + M_{4-}(\lambda)), \lambda \in \mathbb{C}_- \). Therefore for each \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the equalities (4.46) and (4.47) correctly define the solution \( v_\tau(\cdot, \lambda) \in L^2_\Delta[H_0, \mathbb{H}] \) of Eq. (3.3).

Combining (4.46) and (4.47) with (4.10) and (4.12) one gets the equalities (4.42) and (4.43). To prove (4.44) and (4.45) we let \( T_+(\lambda) = (\tau_+(\lambda) + M_{4+}(\lambda))^{-1}, \lambda \in \mathbb{C}_+, \) and \( T_-(\lambda) = (\tau_-(\lambda)^\ast + M_{4-}(\lambda))^{-1}, \lambda \in \mathbb{C}_- \). Then

\begin{align}
(4.48) \quad \tau_+(\lambda) &= \{ \{ T_+(\lambda)h, (I - M_{4+}(\lambda)T_+(\lambda))h \} : h \in \mathcal{H}_b \} \\
\text{and} \quad \tau_-(\lambda) &= \{ \{ -T_-(\lambda) - iP_{\mathcal{H}_b} + iP_{\mathcal{H}_b}M_{4-}(\lambda)T_-(\lambda))h, \\
&\quad (P_{\mathcal{H}_b} + P_{\mathcal{H}_b}M_{4-}(\lambda)T_-(\lambda))h \} : h \in \mathcal{H}_b \},
\end{align}

which in view of (2.11) yields

\begin{align}
(4.49) \quad \tau_+(\lambda) &= \{ \{ T_+(\lambda) - iP_{\mathcal{H}_b}M_{3+}(\lambda)T_+(\lambda)h, \} : h \in \mathcal{H}_b \}, \\
\text{and} \quad \tau_-(\lambda) &= \{ \{ -T_-(\lambda) - iP_{\mathcal{H}_b}M_{3-}(\lambda)T_-(\lambda)h, \} : h \in \mathcal{H}_b \}.
\end{align}

It follows from (4.11), (4.13) and (4.30), (4.31) that

\begin{align}
(4.50) \quad \Gamma_{0b}v_0(\lambda) &= -iP_{\mathcal{H}_b}M_{3+}(\lambda), \quad \Gamma_{1b}v_0(\lambda) = -P_{\mathcal{H}_b}M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_- \\
(4.51) \quad \Gamma_{0b}u_- (\lambda) &= I_{\mathcal{H}_b} - iP_{\mathcal{H}_b}M_{4-}(\lambda), \quad \Gamma_{1b}u_- (\lambda) = -P_{\mathcal{H}_b}M_{4-}(\lambda), \quad \lambda \in \mathbb{C}_-
\end{align}
and the relations (4.46) and (4.47) with taking (4.11), (4.13), (4.29) and (4.50), (4.51) into account give
\[
\begin{align*}
\Gamma_{0b} v_\tau(\lambda) &= -T_+(\lambda)M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+,
\Gamma_{1b} v_\tau(\lambda) &= -(I - M_{4+}(\lambda)T_+(\lambda))M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+,
\Gamma_{0b} v_\tau(\lambda) &= (-iP_{H_b^+} - T_-(\lambda) + iP_{H_b^+} M_{4-}(\lambda)T_-(\lambda))M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-,
\Gamma_{1b} v_\tau(\lambda) &= (-P_{H_b} + P_{H_b} M_{4-}(\lambda)T_-(\lambda))M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-.
\end{align*}
\]

Hence by (4.48) and (4.49) one has
\[
(4.52) \quad \{\Gamma_{0b} v_\tau(\lambda)h_0, \Gamma_{1b} v_\tau(\lambda)h_0\} \in \tau_\pm(\lambda), \quad h_0 \in H_0, \quad \lambda \in \mathbb{C}_\pm,
\]
which in view of the equalities (2.4) yields (4.44) and (4.45).

Finally, one proves uniqueness of \(v_\tau(\cdot, \lambda)\) in the same way as uniqueness of \(v_0(\cdot, \lambda)\) in Proposition 4.3. \(\square\)

5. \(m\)-FUNCTIONS

Let the assumptions (A1) and (A2) at the beginning of Section 4 be fulfilled.

**Definition 5.1.** A boundary parameter \(\tau\) (at the endpoint \(b\)) is a collection \(\tau = \{\tau_+, \tau_-\}\) of holomorphic operator pairs (2.4) belonging to the class \(\bar{R}(\mathcal{H}_b, \mathcal{H}_b)\).

In the case of equal deficiency indices \(n_+(T_{\min}) = n_-(T_{\min})\) one has \(\bar{H}_b = \mathcal{H}_b\) and, therefore, a boundary parameter \(\tau \in \bar{R}(\mathcal{H}_b)\) of the form (2.14).

Let in addition to (A1) and (A2) the assumption (A3) be satisfied, let \(\tau\) be a boundary parameter and let \(v_\tau(\cdot, \lambda) \in \mathcal{L}^2_{\mathbb{R}}[H_0, \mathbb{H}]\) be the corresponding operator solution of Eq. (3.3) defined in Theorem 4.6.

**Definition 5.2.** The operator function \(m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [H_0]\) defined by
\[
(5.1) \quad m_\tau(\lambda) = (\Gamma_{0a} + \Gamma_{1a}) v_\tau(\lambda) + \frac{\lambda}{2} P_{\hat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
will be called the \(m\)-function corresponding to the boundary parameter \(\tau\) or, equivalently, to the boundary value problem (4.1)–(4.4).

If \(n_+(T_{\min}) = n_-(T_{\min})\), then \(m_\tau(\cdot)\) corresponds to the boundary value problem (4.1), (4.2) and (4.5). In this case the \(m\)-function \(m_\tau(\cdot)\) will be called canonical if \(\tau \in \bar{R}^0(\mathcal{H}_b)\).

It follows from (4.42) that \(m_\tau(\cdot)\) satisfies the equality
\[
(5.2) \quad v_{\tau, a}(\lambda) = \left(\frac{\Gamma_{0a} + \Gamma_{1a}}{\Gamma_{1a}} v_\tau(\lambda)\right) = \left(m_\tau(\lambda) - \frac{i}{2} P_{\hat{H}}\right) : H_0 \to H_0 \oplus H.
\]

It turns out that for a given operator \(U\) and a boundary parameter \(\tau\) the \(m\)-function \(m_\tau(\cdot)\) is defined up to a self-adjoint constant. More precisely, the following proposition holds.

**Proposition 5.3.** Suppose that under the assumptions (A1) and (A2)
\[
\bar{U}_j = \begin{pmatrix} u_{7}^{(j)} & u_{8}^{(j)} & u_{9}^{(j)} \\ u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix} : H \oplus \hat{H} \oplus H \to H \oplus \hat{H} \oplus H, \quad j \in \{1, 2\}
\]
are two \(J\)-unitary extensions of \(U\) and \(\Gamma_{0a}^{(j)} : AC(\mathcal{I}, \mathbb{H}) \to H, \quad j \in \{1, 2\}\), are the mappings (3.23). Moreover, let \(\tau\) be a boundary parameter and let
\[
m_{\tau}^{(j)}(\lambda) = (\Gamma_{0a}^{(j)} + \Gamma_{1a}) v_\tau(\lambda) + \frac{\lambda}{2} P_{\hat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad j \in \{1, 2\}
\]
be the corresponding \( m \)-functions. Then there exists an operator \( B = B^* \in [H] \) such that the equality
\begin{equation}
(5.3) \quad m^{(2)}_\tau(\lambda) = m^{(1)}_\tau(\lambda) + \tilde{B}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\end{equation}
holds with the operator \( \tilde{B} = B^* \in [H_0] \) given by \( \tilde{B} = BP_H \).

Proof. By using the equality \( \tilde{U}_j^* J \tilde{U}_j = J \), \( j \in \{1, 2\} \), one can easily prove that there exists \( B = B^* \in [H] \) such that \( \tilde{U}_2 = X \tilde{U}_1 \) with
\[
X = \begin{pmatrix} I & 0 & -B \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} : H \oplus \hat{H} \oplus H \to H \oplus \hat{H} \oplus H.
\]

Therefore the mappings \( \Gamma^{(j)}_a \) are connected by \( \Gamma^{(2)}_0 = \Gamma^{(1)}_0 - B \Gamma^{(1)}_{1a} \).

Now by using (4.42) one obtains
\[
m^{(2)}_\tau(\lambda) = (\Gamma^{(2)}_{0a} + \tilde{\Gamma}_a)v_\tau(\lambda) + \frac{i}{2}P_H =
(\Gamma^{(1)}_{0a} + \tilde{\Gamma}_a)v_\tau(\lambda) + \frac{i}{2}P_H - B \Gamma_{1a}v_\tau(\lambda) = m^{(1)}_\tau(\lambda) + BP_H,
\]
which proves (5.3). \( \square \)

In the following proposition we show that the \( m \)-function \( m_\tau(\cdot) \) can be defined in a somewhat different way.

**Proposition 5.4.** Let under the assumptions (A1)–(A3) \( \tau \) be a boundary parameter at the endpoint \( b \), let \( \varphi_U(\cdot, \lambda)(\in [H_0, \mathbb{H}]) \) be the operator solution defined by (3.27) and let \( \psi(\cdot, \lambda)(\in [H_0, \mathbb{H}]) \), \( \lambda \in \mathbb{C} \), be the operator solutions of Eq. (3.3) with the initial data
\begin{equation}
(5.4) \quad \psi_a(\lambda)(= \tilde{U}_j \psi(a, \lambda)) = \left( \begin{array}{c} -\frac{i}{2}P_H \\ -P_H \end{array} \right) : H_0 \to H_0 \oplus H.
\end{equation}

Then there exists a unique operator function \( m(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [H_0] \) such that for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the operator solution \( v(\cdot, \lambda) \) of Eq. (3.3) given by
\begin{equation}
(5.5) \quad v(t, \lambda) := \varphi_U(t, \lambda)m(\lambda) + \psi(t, \lambda)
\end{equation}
belongs to \( \mathcal{L}_2^a[H_0, \mathbb{H}] \) and satisfies the boundary conditions (4.43)–(4.45). Moreover, the equalities \( v(t, \lambda) = v_\tau(t, \lambda) \) and \( m(\lambda) = m_\tau(\lambda) \) are valid.

Proof. Let \( m_\tau(\cdot) \) be the \( m \)-function in the sense of Definition 5.2 and let \( v(\cdot, \lambda) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), be the solution of Eq. (3.3) given by (5.5) with \( m(\lambda) = m_\tau(\lambda) \). Then in view of (3.28),(5.4) and (5.2) one has \( v_a(\lambda) = v_\tau(\lambda) \) and, consequently, \( v(t, \lambda) = v_\tau(t, \lambda) \). Therefore by Theorem 4.6 \( v(\cdot, \lambda) \) belongs to \( \mathcal{L}_2^a[H_0, \mathbb{H}] \) and satisfies the boundary conditions (4.43)–(4.45) . Hence there exists an operator function \( m(\lambda)(= m_\tau(\lambda)) \) with the desired properties.

Assume now that the solution \( v(\cdot, \lambda) \) of Eq. (3.3) given by (5.5) with some \( m(\lambda) \) belongs to \( \mathcal{L}_2^a[H_0, \mathbb{H}] \) and satisfies the boundary conditions (4.43)–(4.45). Then in view of (3.28) and (5.4) \( \Gamma^j_{1a}v(\lambda) = -P_H \) and according to Theorem 4.6 \( v(t, \lambda) = v_\tau(t, \lambda) \). Therefore \( m(\lambda) = m_\tau(\lambda) \), which proves uniqueness of \( m(\lambda) \). \( \square \)

Description of all \( m \)-functions immediately in terms of the boundary parameter \( \tau \) is contained in the following theorem.
Theorem 5.5. Let the assumptions (A1)–(A3) be satisfied and let $M_\pm(\cdot)$ be the operator functions given by (4.35)–(4.38) (that is, $M_\pm(\cdot)$ are the Weyl functions of the decomposing boundary triplet $\Pi = \{H_0 \oplus H_1, \Gamma_0, \Gamma_1\}$). Moreover, let $\tau_0 = \{\tau_+, \tau_-\}$ be a boundary parameter defined by (2.4) and the equality (4.9). Then $m_0(\lambda) = m_{\tau_0}(\lambda)$ and for every boundary parameter $\tau = \{\tau_+, \tau_-\}$ defined by (2.4) the corresponding $m$-function $m_\tau(\cdot)$ is of the form
\begin{equation}
(5.10) \quad m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C}_+
\end{equation}

Proof. One can easily verify that $v_0(t, \lambda) = v_{\tau_0}(t, \lambda)$, where $v_0(\cdot, \lambda) \in L_2^m[H_0, \mathbb{H}]$ is the solution of Eq. (3.3) defined in Proposition 4.3. This and the equality (4.27) imply that $m_0(\lambda) = m_{\tau_0}(\lambda)$. Next, applying the operator $\Gamma_{0a} + \hat{\Gamma}_a$ to the equalities (4.46) and (4.47) with taking (4.27) and (4.28) into account one obtains
\begin{equation}
(5.7) \quad m_\tau(\lambda) = m_0(\lambda) - M_2(\lambda)(\tau_+(\lambda) + M_4(\lambda))^{-1}M_3(\lambda), \quad \lambda \in \mathbb{C}_+,
\end{equation}
\begin{equation}
(5.8) \quad m_\tau(\lambda) = m_0(\lambda) - M_2(\lambda)(\tau_-\lambda + M_4(\lambda))^{-1}M_3(\lambda), \quad \lambda \in \mathbb{C}_-.
\end{equation}

Moreover, according to [36, Lemma 2.1] $0 \in \rho(C_0(\lambda) - C_1(\lambda)M_4(\lambda))$, $\lambda \in \mathbb{C}_+$, and
\[-(\tau_+(\lambda) + M_4(\lambda))^{-1} = (C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda), \quad \lambda \in \mathbb{C}_+,
\]
which together with (5.7) yields (5.6).

The following corollary is immediate from Theorem 5.5.

Corollary 5.6. Let under the assumptions (A1)–(A3) $n_+(T_{\min}) = n_-(T_{\min})$ and let $M(\cdot)$ be the operator function given by (4.39)–(4.41) (so that $M(\cdot)$ is the Weyl function of the ordinary decomposing boundary triplet $\Pi = \{H, \Gamma_0, \Gamma_1\}$ for $T_{\max}$). Moreover, let $\tau_0 = \{\{\Gamma_{0a}, 0\}; \mathcal{H}_a\} \in R^0(\mathcal{H}_a)$. Then $m_0(\lambda) = m_{\tau_0}(\lambda)$ and for every boundary parameter $\tau$ defined by (2.14) the corresponding $m$-function $m_\tau(\cdot)$ is
\begin{equation}
(5.9) \quad m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

Proposition 5.7. The $m$-function $m_\tau(\cdot)$ is a Nevanlinna operator function such that the relation
\begin{equation}
(5.10) \quad (\text{Im } \lambda)^{-1} \cdot \text{Im } m_\tau(\lambda) \geq \int_I v^*_\tau(t, \lambda)\Delta(t)v_\tau(t, \lambda) dt
\end{equation}
holds for all $\lambda \in \mathbb{C}_+$. If in addition $n_+(T_{\min}) = n_-(T_{\min})$, then (5.10) holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. It follows from (5.7) and (5.8) that the operator function $m_\tau(\cdot)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$. Next, the equality $M_+(\lambda) = M_-(\lambda)$ for the Weyl functions (4.25) and (4.26) implies that $m_0(\lambda) = m_0(\lambda)$, $M_2(\lambda) = M_3(\lambda)$, $M_2(\lambda) = M_3(\lambda)$, $M_4(\lambda) = M_4(\lambda)$ and $M_4(\lambda) = M_4(\lambda)$. This and (5.7), (5.8) yield the equality $m_\tau(\lambda) = m_\tau(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now it remains to show that $m_\tau(\cdot)$ satisfies (5.10).

Let $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_a, \mathcal{H}_b)$ be a boundary parameter defined by (2.4). Assume that $\lambda \in \mathbb{C}_+$, $h_0 \in H_0$ and let $y := v_\tau(\cdot)h_0$, so that $y = y(t) = v_\tau(t, \lambda)h_0$, $t \in I$. It follows from (3.21) that
\begin{equation}
(5.11) \quad (Jy(a), y(a)) = (J\Gamma_ay, \Gamma_ay) = -2i \text{Im}(\Gamma_{1a}y, \Gamma_{0a}y) + i ||\tilde{\Gamma}_ay||^2.
\end{equation}

Applying now the Lagrange's identity (3.5) to $\{y, \lambda y\} \in T_{\max}$ and taking the equalities (5.11) and (3.30) into account one obtains
\begin{equation}
(5.12) \quad \text{Im } \lambda \cdot (y, y)_\Delta = \frac{1}{2} (||\tilde{\Gamma}_ay||^2 - ||\tilde{\Gamma}_ay||^2) + \text{Im} (\Gamma_{1a}y, \Gamma_{0a}y) - \text{Im} (\Gamma_{1b}y, \Gamma_{0b}y) - \frac{1}{2} ||P_{\mathcal{H}_a} \Gamma_{0b}y||^2.
\end{equation}
It follows from (4.43) that $\tilde{\Gamma}_{by} = \tilde{\Gamma}_{ay} + iP_H h_0$ and, therefore, 
\begin{equation}
||\tilde{\Gamma}_{by}||^2 - ||\tilde{\Gamma}_{ay}||^2 = ||P_H h_0||^2 + 2\text{Im}(\tilde{\Gamma}_{ay}, P_H h_0).
\end{equation}
According to (5.2) 
\begin{equation}
\Gamma_{0a} y = P_H m_r(\lambda) h_0, \quad \Gamma_{1a} y = -P_H h_0.
\end{equation}
(5.15) 
and substitution of (5.15) to the right hand part of (5.13) yields
\begin{equation}
\label{5.16}
||\tilde{\Gamma}_{by}||^2 - ||\tilde{\Gamma}_{ay}||^2 = 2\text{Im}(P_H m_r(\lambda) h_0, P_H h_0).
\end{equation}
Moreover, by (5.14) one has 
\begin{equation}
\label{5.17}
\text{Im}(\Gamma_{1a} y, \Gamma_{0a} y) = \text{Im}(P_H m_r(\lambda) h_0, P_H h_0).
\end{equation}
Substituting now (5.16) and (5.17) to (5.12) we obtain 
\begin{equation}
\label{5.18}
\text{Im} \lambda \cdot (y, y)_\Delta = \text{Im} (m_r(\lambda) h_0, h_0) - (\text{Im} (\Gamma_{1b} y, \Gamma_{0b} y) - \frac{1}{2}||P_{H_0} \Gamma_{0b} y||^2).
\end{equation}
It follows from (4.44) that \{\Gamma_{0b} y, \Gamma_{1b} y\} \in \tau_+(\lambda). Therefore according to [38, Proposition 4.3] 
\begin{equation}
\label{5.19}
\text{Im} (\Gamma_{1b} y, \Gamma_{0b} y) - \frac{1}{2}||P_{H_0} \Gamma_{0b} y||^2 \geq 0.
\end{equation}
Moreover, in view of (3.1) one has 
\begin{equation}
\label{5.20}
(y, y)_\Delta = ((\int_I v_\tau^*(t, \lambda) \Delta(t) v_r(t, \lambda) dt) h_0, h_0).
\end{equation}
Combining now (5.19) and (5.20) with (5.18) we arrive at the relation (5.10). \hfill \Box

**Corollary 5.8.** For each boundary parameter $\tau$ the following equality holds:
\begin{equation}
\label{5.21}
\phi_U(x, \lambda) v_{\tau}^*(x, \lambda) - v_r(x, \lambda) \phi_U^*(x, \lambda) = J, \quad x \in I, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

**Proof.** Let $\tilde{U}$ be a $J$-unitary extension (3.20) of $U$ and let $Y_0(x, \lambda)(\in \mathbb{H})$ be the operator solution of Eq. (3.3) with $Y_{0,\alpha}(\lambda)(= \tilde{U}Y_0(a, \lambda)) = I_{H_0}$. Then by the Lagrange’s identity (3.5) one has 
\begin{equation}
Y_0^*(x, \overline{\lambda})J Y_0(x, \lambda) = Y_0^*(a, \overline{\lambda})J Y_0(a, \lambda) = \tilde{U}^{-1} J \tilde{U}^{-1} = J
\end{equation}
and, consequently, 
\begin{equation}
Y_0(x, \lambda) J Y_0^*(x, \overline{\lambda}) = J, \quad x \in I, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}
Since by Proposition 5.7 $m_\tau^*(\lambda) = m_r(\lambda)$, it follows from (5.2) that 
\begin{equation}
v_{\tau,\alpha}^*(\lambda) = (m_r(\lambda) + \frac{i}{2}P_H, -I_H) : H_0 \oplus H \to H_0.
\end{equation}
Combining of this equality with (3.28) yields 
\begin{equation}
\phi_{U,\alpha}(\lambda) v_{\tau,\alpha}^*(\lambda) - v_r(\lambda) \phi_{U,\alpha}^*(\lambda) = \begin{pmatrix} I_{H_0} & 0 \\ 0 & 0 \end{pmatrix} (m_r(\lambda) + \frac{i}{2}P_H, -I_H) - 
\begin{pmatrix} m_r(\lambda) - \frac{i}{2}P_H \\ P_H - I_H \end{pmatrix} (I_{H_0}, 0) = \begin{pmatrix} P_H & -I_H \\ P_H & 0 \end{pmatrix} = J.
\end{equation}
Now by using (5.22) one obtains
\[
\varphi_U(x, \lambda)v_\tau^*(x, \bar{\lambda}) - v_\tau(x, \lambda)\varphi_U^*(x, \bar{\lambda}) = \\
(Y_0(x, \lambda)\varphi_{U,a}(\lambda))(Y_0(x, \bar{\lambda})v_{\tau,a}(\bar{\lambda}))^* - (Y_0(x, \lambda)v_{\tau,a}(\lambda))(Y_0(x, \bar{\lambda})\varphi_{U,a}(\bar{\lambda}))^* = \\
Y_0(x, \lambda)(\varphi_{U,a}(\lambda)v_{\tau,a}(\bar{\lambda}))^* - v_{\tau,a}(\lambda)\varphi_{U,a}^*(\bar{\lambda}))Y_0^*(x, \bar{\lambda}) = Y_0(x, \lambda)JY_0^*(x, \bar{\lambda}) = J.
\]
\[
\square
\]

In the following proposition we show that a canonical $m$-function $m_\tau(\cdot)$ is the Weyl function of some symmetric extension of $T_{\min}$ (in the sense of Definition 2.9).

**Proposition 5.9.** Let the assumptions (A1)–(A3) be satisfied and let $n_+(T_{\min}) = n_-(T_{\min})$.
Moreover, let $\tau \in \tilde{R}^0(H_b)$ be a boundary parameter (2.17), let $v_{\tau}(\cdot, \lambda) \in \mathcal{L}_A^2[H_0, \mathbb{H}]$ be the operator solution of Eq. (3.3) defined in Theorem 4.6 and let $m_\tau(\cdot)$ be the corresponding $m$-function. Then:

1. The equalities
   \[
   \tilde{T} = \{\{\bar{y}, \bar{f}\} \in T_{\max} : y(a) = 0, \bar{\Gamma}_b y = 0, C_0\Gamma_0 b y + C_1 \bar{\Gamma}_b y = 0\},
   \]
   define a symmetric extension $\tilde{T}$ of $T_{\min}$ and its adjoint $\tilde{T}^*$;

2. The collection $\tilde{\Pi} = \{H_0, \bar{\Gamma}_0, \bar{\Gamma}_1\}$ with the operators
   \[
   \bar{\Gamma}_0(y, \tilde{f}) = -\bar{\Gamma}_a y + i(\bar{\Gamma}_a - \bar{\Gamma}_b) y, \quad \bar{\Gamma}_1(y, \tilde{f}) = \Gamma_0 a y + \frac{1}{2}(\bar{\Gamma}_a + \bar{\Gamma}_b) y, \quad \{\bar{y}, \bar{f}\} \in \tilde{T}^*,
   \]
   is a boundary triplet for $\tilde{T}^*$. Moreover, the $\gamma$-field $\tilde{\gamma}(\cdot)$ and Weyl function $\tilde{M}(\cdot)$ of $\tilde{\Pi}$ are
   \[
   \tilde{\gamma}(\lambda) = \pi v_{\tau}(\lambda), \quad \tilde{M}(\lambda) = m_\tau(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
   \]

3. The following identity holds
   \[
   m_\tau(\mu) - m_\tau^*(\lambda) = (\mu - \bar{\lambda}) \int_I v_{\tau}^*(t, \lambda)\Delta(t)v_{\tau}(t, \mu) dt, \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R}.
   \]
   This implies that for the canonical $m$-function $m_\tau(\cdot)$ the inequality (5.10) turns into the equality, which holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

**Proof.** Clearly, we may assume that $\tau$ is given in the normalized form (2.18), in which case the operators

\[
\bar{\Gamma}_0(y, \tilde{f}) = \{-\Gamma_a y + i(\Gamma_a - \Gamma_b) y, C_0\Gamma_0 b y + C_1 \Gamma_1 b y\} \in H_0 \oplus H_b,
\]

\[
\bar{\Gamma}_1(y, \tilde{f}) = \{\Gamma_0 a y + \frac{1}{2}(\Gamma_a + \Gamma_b) y, C_1 \Gamma_1 b y - C_0 \Gamma_1 b y\} \in H_0 \oplus H_b
\]

form a decomposing boundary triplet $\tilde{\Pi} = \{\mathcal{H}, \bar{\Gamma}_0, \bar{\Gamma}_1\}$ for $T_{\max}$ with $H = H_0 \oplus H_b$.

Let $\tilde{\gamma}(\lambda)$ be the $\gamma$-field and

\[
\tilde{M}(\lambda) = \begin{pmatrix} \frac{\bar{\pi}_0(\lambda)}{\mathbf{M}_0(\lambda)} & \frac{\bar{\pi}_1(\lambda)}{\mathbf{M}_1(\lambda)} \end{pmatrix} : H_0 \oplus H_b \to H_0 \oplus H_b, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

be the Weyl function of the triplet $\tilde{\Pi}$. Assume also that $\tilde{\pi}_0(\cdot, \lambda) \in \mathcal{L}_A^2[H_0, \mathbb{H}]$ is the operator solution of Eq. (3.3) defined in Proposition 4.3 (for the triplet $\tilde{\Pi}$). Then $\tilde{\pi}_0(t, \lambda) = v_{\tau}(t, \lambda)$ and (4.23) yields $\tilde{\gamma}(\lambda) | H_0 = \pi v_{\tau}(\lambda)$. Moreover, in view of (4.40) one has $\tilde{\pi}_0(\lambda) = m_\tau(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}$. Applying now Proposition 2.10 to the triplet $\tilde{\Pi}$ (with $H_0 = \mathcal{H}_1 = H_0$) we obtain statements (1) and (2). Finally, (5.25) follows from the identity (2.26) for the triplet $\tilde{\Pi}$ and Lemma 3.3, (2) applied to the solution $v_{\tau}(\cdot, \lambda)$. \qed
Remark 5.10. Let under the assumptions (A1)–(A3) $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary decomposing boundary triplet for $T_{\max}$, let $\tau \in \mathbb{R}^3(\mathcal{H}_b)$ be a boundary parameter given in the normalized form (2.17), (2.18) and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be another decomposing boundary triplet for $T_{\max}$ defined by (5.26) and (5.27). The triplets $\Pi$ and $\Pi$ are connected by means of linear fractional transformation,

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix},$$

where $X_j \in [H_0 \oplus \mathcal{H}_b]$ are defined as follows:

$$X_1 = \begin{pmatrix} I & 0 \\ 0 & C_0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 0 & -C_1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 0 & C_1 \end{pmatrix}, \quad X_4 = \begin{pmatrix} I & 0 \\ 0 & C_0 \end{pmatrix}.$$ 

Therefore according to [10] the Weyl functions $M(\cdot)$ and $\overline{M}(\cdot)$ of the triplets $\Pi$ and $\Pi$ respectively are connected by means of linear fractional transformation,

$$(5.29) \quad \overline{M}(\lambda) = (X_3 + X_4 M(\lambda)) (X_1 + X_2 M(\lambda))^{-1}.$$ 

By using the block representation (4.39) of $M(\lambda)$ one obtains

$$(X_1 + X_2 M(\lambda))^{-1} = \begin{pmatrix} I & 0 \\ -C_1 M_3 & C_0 - C_1 M_4 \end{pmatrix}^{-1} = \begin{pmatrix} I \\ (C_0 - C_1 M_4)^{-1} C_1 M_3 \\ (C_0 - C_1 M_4)^{-1} \end{pmatrix}$$

and (5.29), (5.28) imply that $\overline{m}_0(\lambda)$ coincides with the right-hand side of (5.9). This and the equality $m_\tau(\lambda) = \overline{m}_0(\lambda)$ obtained in the proof of Proposition 5.9 yield (5.9). Thus, for canonical $m$-functions $m_\tau(\cdot)$ formula (5.9) is a simple consequence of the relation (5.29) for Weyl functions.

Note that in this proof we follow the reasonings of [10, Remark 86], where the Krein formula for canonical resolvents was proved in a similar way.

6. Spectral functions

6.1. Green’s function. In the sequel we put $\mathfrak{S} := L^2(\mathcal{I})$ and denote by $\mathfrak{S}_b$ the set of all $\tilde{f} \in \mathfrak{S}$ with the following property: there exists $\beta \in \mathcal{I}$ (depending on $\tilde{f}$) such that for some (and hence for all) function $f \in \tilde{f}$ the equality $\Delta(t)f(t) = 0$ holds a.e. on $(\beta, b)$.

Assume hypothesis (A1) and (A2). Let $\varphi_U(\cdot, \lambda)$ be the operator-valued solution (3.27), let $T$ be a boundary parameter and let $v_\tau(\cdot, \lambda) \in L^2(\mathcal{I}; \mathbb{H})$ be the operator-valued solution of Eq. (3.3) defined in Theorem 4.6.

**Definition 6.1.** The operator function $G_\tau(\cdot, \cdot, \lambda) : \mathcal{I} \times \mathcal{I} \to [\mathbb{H}]$ given by

$$(6.1) \quad G_\tau(x, t, \lambda) = \begin{cases} v_\tau(x, \lambda) \varphi_U^*(t, x), & x > t \\ \varphi_U(x, \lambda) v_\tau^*(t, x), & x < t \end{cases}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

will be called the Green’s function corresponding to the boundary parameter $\tau$.

Next we compute the generalized resolvent of $T$ in terms of the Green’s function.

**Theorem 6.2.** Let $\tau$ be a boundary parameter and let $R_\tau(\cdot)$ be the corresponding generalized resolvent of the relation $T$ (see Theorem 4.1). Then

$$(6.2) \quad R_\tau(\lambda) \tilde{f} = \pi \left( \int_{\mathcal{I}} G_\tau(\cdot, t, \lambda) \Delta(t) f(t) dt \right), \quad \tilde{f} \in \mathfrak{S}, \quad f \in \tilde{f}.$$
Proof. Since \( v_\tau(\cdot, \lambda) \in L_\lambda^2[H_0, \mathbb{H}] \), it follows from (6.1) that
\[
\int_I \| G_\tau(x, t, \lambda) \Delta^\frac{1}{2}(t) \|^2 \, dt < \infty, \quad x \in I.
\]
Hence for each \( f \in L_\lambda^2(I) \) and \( x \in I \) one has
\[
\int_I \| G_\tau(x, t, \lambda) \Delta(t) f(t) \| \, dt \leq \int_I \| G_\tau(x, t, \lambda) \Delta^\frac{1}{2}(t) \| : \| \Delta^\frac{1}{2}(t) f(t) \| \, dt < \infty
\]
and, therefore, the equality
\[
y_f = y_f(x, \lambda) := \int_I G_\tau(x, t, \lambda) \Delta(t) f(t) \, dt, \quad f \in L_\lambda^2(I), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]
correctly defines the function \( y_f(\cdot, \lambda) : I \times \mathbb{C} \setminus \mathbb{R} \to \mathbb{H} \). This implies that (6.2) is equivalent to the following statement: for each \( f \in \mathcal{H} \)
\[
y_f(\cdot, \lambda) \in L_\lambda^2(I) \quad \text{and} \quad R_\tau(\lambda) \hat{f} = \pi(y_f(\cdot, \lambda)), \quad f \in \mathcal{F}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
To prove (6.4) first assume that \( \hat{f} \in \mathcal{H}_b \). We show that in this case the function \( y_f(\cdot, \lambda) \) given by (6.3) is a solution of the boundary problem (4.1)–(4.4). It follows from (6.1) that
\[
y_f = y_f(x, \lambda) = \varphi_u(x, \lambda) C_1(x, \lambda) + v_\tau(x, \lambda) C_2(x, \lambda) = Y(x, \lambda) C(x, \lambda),
\]
where
\[
C_1(x, \lambda) = \int_x^b v_\tau^*(t, \lambda) \Delta(t) f(t) \, dt, \quad C_2(x, \lambda) = \int_a^x \phi_{\hat{U}}^*(t, \lambda) \Delta(t) f(t) \, dt,
\]
\[
Y(x, \lambda) = (\varphi_u(x, \lambda), v_\tau(x, \lambda)), \quad C(x, \lambda) = \{C_1(x, \lambda), C_2(x, \lambda)\} \in H_0 \oplus H_0.
\]
Moreover, by (6.5) and the equality \( \Delta(t) f(t) = 0 \) (a.e. on \((\beta, b)\)) one has
\[
y_f(x, \lambda) = v_\tau(x, \lambda) \int_I \phi_{\hat{U}}^*(t, \lambda) \Delta(t) f(t) \, dt, \quad x \in (\beta, b).
\]
This implies that \( y_f \in AC(I; \mathbb{H}) \cap L_\lambda^2(I) \). Next, in view of (5.21) one has
\[
Y(x, \lambda) C'(x, \lambda) = (-\varphi_u(x, \lambda) v_\tau^*(x, \lambda) + v_\tau(x, \lambda) \phi_{\hat{U}}^*(x, \lambda)) \Delta(x) f(x) = -J \Delta(x) f(x).
\]
By using this equality we obtain
\[
J y_f(x, \lambda) - B(x) y_f(x, \lambda) = (J Y'(x, \lambda) - B(x) Y(x, \lambda)) C(x, \lambda) +
J Y(x, \lambda) C'(x, \lambda) = \lambda \Delta(x) Y(x, \lambda) C(x, \lambda) - J^2 \Delta(x) f(x) = \lambda \Delta(x) y_f(x, \lambda) + \Delta(x) f(x).
\]
Thus, for each \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the function \( y_f(\cdot, \lambda) \) satisfies (4.1) a.e. on \( I \).

Next we show that \( y_f(\cdot, \lambda) \) satisfies the boundary conditions (4.2) (4.4). Let \( \hat{U} \) be a \( J \)-unitary extension (3.20) of \( U \) and let \( \Gamma_a \) be the mapping (3.21). Since by (6.5)
\[
\Gamma_a y_f = \varphi_{U,a}(\lambda) \int_I v_\tau^*(t, \lambda) \Delta(t) f(t) \, dt,
\]
it follows from (3.28) that
\[
\hat{\Gamma}_a y_f = P_H \int_I v_\tau^*(t, \lambda) \Delta(t) f(t) \, dt = \int_I (v_\tau^*(t, \lambda) \upharpoonright \hat{H})^* \Delta(t) f(t) \, dt,
\]
(6.7) \( \hat{\Gamma}_a y_f = \Gamma_{1a} y_f = 0 \).
Moreover, according to Remark 3.5, (1) the equality (6.6) yields

$$\hat{\Gamma}_byf = \hat{\Gamma}_b\varphi(t, \lambda) \int_I \varphi_U(t, \lambda) \Delta(t)f(t) dt,$$

(6.10) $$\Gamma_{byf} = \Gamma_{b}\varphi(t, \lambda) \int_I \varphi_U(t, \lambda) \Delta(t)f(t) dt,$$

(6.11) $$\Gamma_{byf} = \Gamma_{by}\varphi(t, \lambda) \int_I \varphi_U(t, \lambda) \Delta(t)f(t) dt.$$  

In view of (6.8) the first condition in (4.2) is fulfilled. Next, by (4.43) and (5.2) one has

(6.12) $$\hat{\Gamma}_byf = \hat{\Gamma}_a\varphi(t, \lambda) + iP_{\hat{H}} = (P_{H}\varphi(t, \lambda) - \frac{i}{2}P_{H}) + iP_{\hat{H}} = P_{H}(\varphi(t, \lambda) + \frac{i}{2}I_{H_0}).$$

Moreover, in view of (5.2)

$$v_{r,a}(\lambda) \mid \hat{H} = \left( (\varphi(t, \lambda) - \frac{i}{2}I_{H_0}) \mid \hat{H} \right) : H_0 \to H_0 \oplus H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and (3.28) gives $$v_r(t, \lambda) \mid \hat{H} = \varphi_U(t, \lambda)(\varphi(t, \lambda) - \frac{i}{2}I_{H_0}) \mid \hat{H}.$$ Combining this equality with (6.9), (6.12) and (6.7) one obtains

$$\hat{\Gamma}_byf = P_{H}(\varphi(t, \lambda) + \frac{i}{2}I_{H_0}) \int_I \varphi_U(t, \lambda) \Delta(t)f(t) dt =$$

$$\int_I (\varphi_U(t, \lambda)(\varphi(t, \lambda) - \frac{i}{2}I_{H_0}) \mid \hat{H})^* \Delta(t)f(t) dt =$$

$$\int_I (v_r(t, \lambda) \mid \hat{H})^* \Delta(t)f(t) dt = \hat{\Gamma}_byf$$

(here we make use of the relation $$m^*_a(\lambda) = m^*_r(\lambda)).$$ Hence the second condition in (4.2) is fulfilled. Finally combining (6.10) and (6.11) with (4.44) and (4.45) one obtains the relations (4.3) and (4.4) for $$y_f.$$ Thus $$y_f \cdot \lambda$$ is a solution of the boundary problem (4.1)–(4.4) and by Theorem 4.1 the relations (6.4) hold.

Now assume that $$\tilde{f} \in \mathcal{B}$$ is arbitrary, $$f \in \tilde{f},$$ and $$y_f = y_f(x, \lambda)$$ is given by (6.3). Assume also that $$f_n = f(x, \lambda)$$ and $$y_{f_n}(x, \lambda)$$ be given by (6.3) with $$f_n(t)$$ in place of $$f(t).$$ Moreover, let a function $$y_R \in \mathcal{L}_\Delta^2(\mathcal{I})$$ be such that $$\pi y_R = R(\lambda)\tilde{f}.$$ Since $$\tilde{f}_n \to \tilde{f}$$ and $$\pi y_{f_n} = R(\lambda)\tilde{f}_n,$$ it follows that $$\|y_{f_n} - y_f\| \to 0.$$ On the other hand, $$y_{f_n}(x, \lambda) \to y_f(x, \lambda), \ x \in \mathcal{I},$$ and, consequently, $$\Delta(x)(y_f(x, \lambda) - y_f(x, \lambda)) = 0$$ a.e. on $$\mathcal{I}.$$ Hence $$y_f \in \mathcal{L}_\Delta^2(\mathcal{I})$$ and $$\pi y_f = \pi y_R = R(\lambda)\tilde{f},$$ which gives the relations (6.4) for $$\tilde{f}.$$  

Remark 6.3. Theorem 6.2 generalizes several results in this direction. More precisely, in the case of Hamiltonian system (3.2) ($$\hat{H} = \{0\}$$) and separated boundary conditions formulas (6.1) and (6.2) for canonical resolvents of $$T_{min},$$ were proved in [20, 30]. Moreover, assuming that the minimal operator $$T_{min}$$ is generated by Hamiltonian system with the minimal deficiency indices $$n_\pm (T_{min}) = \dim H,$$ formulas (6.1) and (6.2) for generalized resolvents of $$T_{min}$$ have been obtained in [11, 12]. Note also that formulas for canonical and generalized resolvents of even order ordinary differential equations subject to separated boundary conditions are known as late as the middle of nineteenth (see e.g. [13, 43, 46]).
6.2. The space $L^2(\Sigma; \mathcal{H})$. Let $\mathcal{H}$ be a finite dimensional Hilbert space.

**Definition 6.4.** A non-decreasing operator function $\Sigma(\cdot) : \mathbb{R} \to [\mathcal{H}]$ is called a distribution function if it is left continuous and satisfies the equality $\Sigma(0) = 0$.

Next recall the definition of the space $L^2(\Sigma; \mathcal{H})$ (see e.g. [43, Section 20.5], [5, Section 7.2.3]). Denote by $C_0(\mathcal{H})$ the set of continuous vector functions $f : \mathbb{R} \to \mathcal{H}$ having compact supports. Introduce the semi-scalar product on $C_0(\mathcal{H})$ by setting

$$
(f,g)_{L^2(\Sigma; \mathcal{H})} = \int_\mathbb{R} (d\Sigma(t)f(t), g(t)d\mu) = \lim_{d(\pi_n) \to 0} \sum_{k=1}^n (\Sigma(\Delta_k)f(\xi_k), g(\xi_k)).
$$

(6.13)

Here $\pi_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$ denotes a partition of a segment $[a, b]$ containing the supports of functions $f$ and $g$, $d(\pi_n)$ is the diameter of the partition $\pi_n$. $\Sigma(\Delta_k) := \Sigma(t_k) - \Sigma(t_{k-1})$, and $\xi_k \in [t_{k-1}, t_k]$. The limit in (6.13) is understood in the same sense as in the definition of the Riemann-Sieltjes integral, i.e., a particular choice of $\pi_n$ with a given diameter and of $\xi_k \in [t_{k-1}, t_k]$ is irrelevant.

The completion of $C_0(\mathcal{H})$ with respect to the semi-norm $p(f) := (f, f)_{L^2(\Sigma; \mathcal{H})}^\frac{1}{2}$ gives rise to a semi-Hilbert space $\tilde{L}^2(\Sigma, \mathcal{H})$ (i.e., to a complete space with a semi-norm in place of norm). Denoting by $\ker p := \{f \in \tilde{L}^2(\Sigma, \mathcal{H}) : p(f) = 0\}$ the kernel of the semi-norm, we introduce the quotient space $L^2(\Sigma; \mathcal{H}) := \tilde{L}^2(\Sigma, \mathcal{H})/\ker p$ which is already Hilbert space.

Let $\Sigma = (\sigma_{ij})_{i,j=1}^n$ be a matrix valued measure generated by a distribution function $\Sigma(\cdot)$ and let $\sigma = \sum_{i,j} \sigma_{ij}$. Clearly, the measure $\Sigma(\cdot)$ is absolutely continuous with respect to $\sigma$ (in fact both measures are equivalent). Therefore, by the Radon-Nykodim theorem, there exists a $\sigma$-measurable matrix density $\Psi(\cdot) = (\psi_{ij}(\cdot))_{i,j=1}^n$ such that

$$
\Sigma(\delta) = \int_\delta \Psi(t)d\sigma(t), \quad \Psi(t) := \psi_{ij}(t)_{i,j=1}^n = (d\sigma_{ij}/d\sigma)_{i,j=1}^n, \quad \delta \in B_0(\mathbb{R}).
$$

Let $\tilde{L}^2_0(\Sigma, \mathbb{C}^n)$ be the set of $\sigma$-measurable vector-valued functions $f : \mathbb{R} \to \mathbb{C}^n$ satisfying

$$
||f||^2_{\tilde{L}^2_0(\Sigma, \mathbb{C}^n)} := \int_{\mathbb{R}} (\Psi(t)f(t), f(t)) d\sigma(t) < \infty.
$$

(6.14)

**Theorem 6.5.** [24] The spaces $\tilde{L}^2(\Sigma, \mathbb{C}^n)$ and $L^2(\Sigma, \mathbb{C}^n)$ are identified isometrically with the spaces $\tilde{L}^2_0(\Sigma, \mathbb{C}^n)$ and $L^2_0(\Sigma, \mathbb{C}^n) := \tilde{L}^2_0(\Sigma, \mathbb{C}^n)/N_0$, respectively, where $N_0 = \{f \in \tilde{L}^2_0(\Sigma, \mathbb{C}^n) : ||f||_{\tilde{L}^2_0(\Sigma, \mathbb{C}^n)} = 0\}$ is the kernel of the semi-norm. Therefore, $f \in \tilde{L}^2(\Sigma, \mathbb{C}^n)$ if and only if $f$ is $\sigma$-measurable and the norm (6.14) is finite.

It was shown in [35] that the spaces $\tilde{L}^2(\Sigma, \mathbb{C}^n)$ and $L^2(\Sigma, \mathbb{C}^n)$ admit the representation in the form of direct integrals

$$
\tilde{L}^2(\Sigma, \mathbb{C}^n) = \int_{\mathbb{R}} \oplus \tilde{G}(t)d\sigma(t), \quad L^2(\Sigma, \mathbb{C}^n) = \int_{\mathbb{R}} \oplus G(t)d\sigma(t),
$$

(6.15)

where $\tilde{G}(t)$ is the $n$-dimensional Euclidian space with the semi-scalar product $\langle f, g \rangle := (\Psi(t)f, g)$ and $G(t) = \tilde{G}(t)/\{f \in \tilde{G}(t) : (\Psi(t)f, f) = 0\}$. In particular, representation (6.15) gives a simple proof of Theorem 6.5 (as distinguished from the known proofs in [24] and [13]).
6.3. **Spectral functions and the Fourier transform.** Let the assumptions (A1) and (A2) at the beginning of Section 4 be satisfied and let \( \phi_U(\cdot, \lambda) \in [H_0, \mathbb{H}] \) be the operator solution of Eq. (3.3) with the initial data (3.27). For each \( \tilde{f} \in \mathcal{S}_0 \) introduce the Fourier transform \( \hat{f}(\cdot) : \mathbb{R} \to H_0 \) by setting

\[
(6.16) \quad \hat{f}(s) = \int \varphi_U^* (t, s) \Delta(t) f(t) \, dt, \quad f \in \tilde{f}.
\]

Note that \( \hat{f}(\cdot) \) is uniquely defined by \( \tilde{f} \), i.e., it does not depend on the choice of \( f \in \tilde{f} \).

Next assume that \( \tau = \{ \tau_+, \tau_- \} \in \tilde{R} (\mathcal{H}_0, \mathcal{H}_b) \) is a boundary parameter given by (2.4) (with \( \mathcal{H}_0 = \mathcal{H}_b \) and \( \mathcal{H}_1 = \mathcal{H}_b \)). Then according to Theorem 4.1 the corresponding boundary problem (4.1)–(4.4) generates the generalized resolvent

\[
R_\tau (\lambda) = P_\mathcal{S}_0 (\tilde{T}^\tau - \lambda)^{-1} |_{\mathcal{S}_0}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

of the symmetric relation \( T \in \tilde{C} (\mathcal{S}_0) \). The equality (6.17) uniquely (up to the unitary equivalence) defines a self-adjoint \( \mathcal{S}_0 \)-minimal relation \( \tilde{T}^\tau \) in \( \mathcal{S}_0 \supset \mathcal{S}_0 \) such that \( T \subset \tilde{T}^\tau \). Denote also by \( F_\tau (\cdot) \) the corresponding spectral function of \( T \), so that in view of (2.29)

\[
(6.18) \quad R_\tau (\lambda) = \int_{\mathbb{R}} \frac{dF_\tau (t)}{t - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

In the following we fix some \( J \)-unitary extension \( \tilde{U} \) of \( U \) (see (3.20)) and denote by \( m_\tau (\cdot) \) the \( m \)-function of the boundary problem (4.1)–(4.4). Note that in view of Proposition 5.3 a choice of \( \tilde{U} \) does not matter in our further considerations.

**Definition 6.6.** A distribution function \( \Sigma (\cdot) = \Sigma_\tau (\cdot) : \mathbb{R} \to [H_0] \) is called a spectral function of the boundary problem (4.1)–(4.4) if, for each \( \tilde{f} \in \mathcal{S}_0 \) and for each finite interval \( [\alpha, \beta] \subset \mathbb{R} \), the Fourier transform (6.16) satisfies the equality

\[
(6.19) \quad (F_\tau (\beta) - F_\tau (\alpha)) \tilde{f}, \tilde{f} \rangle_{\mathcal{S}_0} = \int_{[\alpha, \beta]} \langle d\Sigma (s) \tilde{f}(s), \tilde{f}(s) \rangle_{\mathcal{S}_0}.
\]

Note that the integral on the right-hand side of (6.19) exists, since the function \( \hat{f}(\cdot) \) is continuous (and even holomorphic) on \( \mathbb{R} \); moreover, by (6.19) \( \tilde{f} \in L^2 (\Sigma_\tau; H_0) \).

Let \( \tilde{\mathcal{S}}_0 := \tilde{\mathcal{S}} \oplus \text{mul} \, \tilde{T}^\tau \), so that

\[
(6.20) \quad \tilde{\mathcal{S}} = \tilde{\mathcal{S}}_0 \oplus \text{mul} \, \tilde{T}^\tau
\]

Then by (2.30) and (6.19) one has

\[
(6.21) \quad \|P_{\tilde{\mathcal{S}}_0} \tilde{f}\|_{\tilde{\mathcal{S}}} = \|\tilde{f}\|_{L^2 (\Sigma_\tau; H_0)}, \quad \tilde{f} \in \tilde{\mathcal{S}}_0
\]

and, consequently, \( \|\tilde{f}\| \leq \|\tilde{f}\| \). Therefore for each \( \tilde{f} \in \tilde{\mathcal{S}} \) there exists a function \( \hat{f} \in L^2 (\Sigma_\tau; H_0) \) (the Fourier transform of \( \tilde{f} \)) such that

\[
\lim_{\beta \searrow \alpha} \|\hat{f} - \int_\alpha^\beta \varphi_U (t, \cdot) \Delta(t) f(t) \, dt \|_{L^2 (\Sigma_\tau; H_0)} = 0, \quad f \in \tilde{f},
\]

and the equality \( V \tilde{f} = \hat{f}, \tilde{f} \in \tilde{\mathcal{S}} \), defines the linear operator \( V : \tilde{\mathcal{S}} \to L^2 (\Sigma_\tau; H_0) \) such that

\[
(6.22) \quad \|V \tilde{f}\|_{L^2 (\Sigma_\tau; H_0)} = \|P_{\tilde{\mathcal{S}}_0} \tilde{f}\|_{\tilde{\mathcal{S}}_0}, \quad \tilde{f} \in \tilde{\mathcal{S}}.
\]

This implies that \( V \) is a contraction from \( \tilde{\mathcal{S}} \) to \( L^2 (\Sigma_\tau; H_0) \).
Theorem 6.7. For each boundary parameter $\tau$ there exists a unique spectral function $\Sigma_\tau(\cdot)$ of the boundary problem (4.1)--(4.4). This function is defined by the Stieltjes inversion formula

\begin{equation}
\Sigma_\tau(s) = \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \text{Im} \ m_\tau(\sigma + i\varepsilon) \, d\sigma.
\end{equation}

Proof. It follows from Proposition 5.4 that the Green function (6.1) admits the representation

\[ G_\tau(x,t,\lambda) = \varphi_U(x,\lambda)m_\tau(\lambda)\varphi_U^*(t,\overline{\lambda}) + G_0(x,t,\lambda), \]

where $G_0(x,t,\lambda) =\begin{cases} \psi(x,\lambda)\varphi_U^*(t,\overline{\lambda}), & x > t \\ \varphi_U(x,\lambda)\psi^*(t,\overline{\lambda}), & x < t \end{cases}$.

Now by using (6.2) and the Stieltjes-Livšic inversion formula one proves the theorem in the same way as Theorem 4 in [46].

Next, similarly to [13, 43, 46] one can prove the following theorem.

Theorem 6.8. Let $V : \mathcal{H} \to L^2(\Sigma_\tau;H_0)$ be the Fourier transform corresponding to the spectral function $\Sigma_\tau(\cdot)$ and let $V^*$ be the operator adjoint to $V$. Then for each function $g = g(s) \in L^2(\Sigma_\tau;H_0)$ with the compact support the function

\[ f_g(t) := \int_{\mathbb{R}} \varphi_U(t,s) \, d\Sigma_\tau(s) g(s) \]

belongs to $L^2_\Lambda(I)$ and $V^*g = \pi f_g$. Therefore

\begin{equation}
V^*g = \pi \left( \int_{\mathbb{R}} \varphi_U(\cdot,s) \, d\Sigma_\tau(s) g(s) \right), \quad g = g(s) \in L^2(\Sigma_\tau;H_0),
\end{equation}

where the integral converges in the semi-norm (3.1).

For a spectral function $\Sigma_\tau(\cdot)$ denote by $\Lambda$ the multiplication operator in $L^2(\Sigma_\tau;H_0)$ given by the relations

\begin{equation}
\text{dom } \Lambda = \{ f \in L^2(\Sigma_\tau;H_0) : tf(t) \in L^2(\Sigma_\tau;H_0) \},
\end{equation}

and

\[ (\Lambda f)(t) = tf(t), \quad f \in \text{dom } \Lambda. \]

As is known $\Lambda$ is a self-adjoint operator and the spectral measure $E_\Lambda(\cdot)$ of $\Lambda$ is

\[ (E_\Lambda(\delta)f)(t) = \chi_\delta(t)f(t), \quad f \in L^2(\Sigma_\tau;H_0), \quad \delta \in \mathcal{B}, \]

where $\chi_\delta(\cdot)$ is the indicator of the Borel set $\delta$. Moreover, in view of (6.19) one has

\[ F_\tau(\beta) - F_\tau(\alpha) = V^*E_\Lambda([\alpha,\beta])V, \quad [\alpha,\beta] \subset \mathbb{R}. \]

Proposition 6.9. Let $\Sigma_\tau(\cdot)$ be a spectral function of the boundary problem (4.1)--(4.4), let $V : \mathcal{H} \to L^2(\Sigma_\tau;H_0)$ be the corresponding Fourier transform and let

\begin{equation}
L_0 := \text{clos} \left( V\mathcal{H} \right)
\end{equation}

Then the operator $\Lambda$ is $L_0$-minimal (in the sense of Definition 2.12).
Proof. Let $L_1 := \ker V^* = L^2(\Sigma; H_0 \oplus L_0)$ and let $g \in L_1$ be a vector such that $E_{\Lambda}(\alpha, \beta)g \in L_1$ for each bounded interval $[\alpha, \beta] \subset \mathbb{R}$. Then $\Delta E_{\Lambda}(\alpha, \beta)g \in L_1$ and, consequently, $V^*E_{\Lambda}(\alpha, \beta)g = 0$ and $V^*\Delta E_{\Lambda}(\alpha, \beta)g = 0$. This and (6.24) imply that the functions
\begin{equation}
(6.29)
 y(t) = \int_{[\alpha, \beta]} \varphi_U(t, s) d\Sigma_\tau(s) g(s), \quad f(t) = \int_{[\alpha, \beta]} \varphi_U(t, s) d\Sigma_\tau(s) s g(s)
\end{equation}
satisfy the equalities
\begin{equation}
(6.30)
\Delta(t) y(t) = 0 \quad \text{and} \quad \Delta(t) f(t) = 0 \quad \text{a.e. on} \quad I.
\end{equation}
On the other hand, in view of (6.29) one has
\begin{equation}
Jy'(t) - B(t) y(t) = \int_{[\alpha, \beta]} (J\varphi_U'(t, s) - B(t)\varphi_U(t, s)) d\Sigma_\tau(s) g(s) =
\int_{[\alpha, \beta]} (s\Delta(t)\varphi_U(t, s)) d\Sigma_\tau(s) g(s) = \Delta(t) f(t).
\end{equation}
Combining this equality with (6.30) and taking definiteness of the system (3.2) into account one gets
\begin{equation}
(6.31)
 y(t) = \int_{[\alpha, \beta]} \varphi_U(t, s) d\Sigma_\tau(s) g(s) = 0, \quad t \in I, \quad [\alpha, \beta] \subset \mathbb{R}.
\end{equation}
It follows from (3.27) and (3.28) that the operator $\varphi_U(a, s)$ does not depend on $s$ and $\ker \varphi_u(a, s) = \{0\}$. This and (6.31) yield
\begin{equation}
\int_{[\alpha, \beta]} d\Sigma_\tau(s) g(s) = 0, \quad [\alpha, \beta] \subset \mathbb{R},
\end{equation}
which gives the equality $g = 0$. Thus the condition (2) of Definition 2.12 is satisfied. \hfill \Box

Let $\tilde{\mathcal{H}}$ be decomposed as in (6.20) and let
\begin{equation}
(6.32)
\mathcal{H}_V := \tilde{\mathcal{H}}_0 \cap \mathcal{H}, \quad \mathcal{H}_k := \text{mul} \tilde{T}^* \cap \mathcal{H}, \quad \mathcal{H}_c := \mathcal{H} \ominus (\mathcal{H}_V \oplus \mathcal{H}_k).
\end{equation}
Then
\begin{equation}
(6.33)
\mathcal{H} = \mathcal{H}_V \oplus \mathcal{H}_k \oplus \mathcal{H}_c
\end{equation}
and by (6.21) the operator $V$ (the Fourier transform) is isometric on $\mathcal{H}_V$, strictly contractive on $\mathcal{H}_c$ and has $\mathcal{H}_k$ as a kernel. Observe also that $\text{mul} T \subset \mathcal{H}_k$, so that $V \upharpoonright \text{mul} T = 0$.

Next assume that $\mathcal{H}_0 := \mathcal{H} \ominus \text{mul} T$, so that $\mathcal{H}$ can be represented as
\begin{equation}
(6.34)
\mathcal{H} = \mathcal{H}_0 \oplus \text{mul} T.
\end{equation}
It follows from (6.33) that $\mathcal{H}_0$ is the maximally possible subspace of $\mathcal{H}$ on which the Fourier transform $V$ may be isometric.

**Definition 6.10.** A spectral function $\Sigma_\tau(\cdot)$ of the boundary problem (4.1)–(4.4) will be referred to the class $SF_0$ if the operator
\begin{equation}
(6.35)
V_0 := V \upharpoonright \mathcal{H}_0
\end{equation}
is an isometry from $\mathcal{H}_0$ to $L^2(\Sigma; H_0)$. 

ON TITCHMARSH-WEYL FUNCTIONS
By using $\mathfrak{H}$-minimality of $\tilde{T}^\tau$ one can easily show that
\begin{equation}
\Sigma_\tau(\cdot) \in SF_0 \iff \mul \tilde{T}^\tau = \mul T.
\end{equation}
Therefore all spectral functions belong to $SF_0$ if and only if $\mul T = \mul T^*$. If $\Sigma_\tau(\cdot) \in SF_0$, then by (6.24) for each $f \in \mathfrak{H}$ the inverse Fourier transform is
\begin{equation}
\tilde{f} = \pi \left( \int_\mathbb{R} \varphi_U(\cdot, s) d\Sigma_\tau(s) \tilde{f}(s) \right)
\end{equation}
\begin{theorem}
Let $\tau$ be a boundary parameter, let $\Sigma_\tau(\cdot)$ be the spectral function of the boundary problem (4.1)--(4.4) and let $V$ be the corresponding Fourier transform. Assume also that $\tilde{T}^\tau \in \tilde{\mathcal{C}}(\tilde{\mathfrak{H}})$ is the (exit space) self-adjoint extension of $T$ defined by (6.17), $\tilde{\mathfrak{H}}_0 \subset \tilde{\mathfrak{H}}$ and $\mathfrak{H}_0 \subset \mathfrak{H}$ are the subspaces from decompositions (6.20) and (6.34) respectively and $T^\tau$ is the operator part of $\tilde{T}^\tau$ (so that $T^\tau$ is a self-adjoint operator in $\mathfrak{H}_0$). If $\Sigma_\tau(\cdot) \in SF_0$, then $\mathfrak{H}_0 \subset \mathfrak{H}_0$ and there exists a unitary operator $\tilde{V} \in [\tilde{\mathfrak{H}}_0, L^2(\Sigma_\tau; H_0)]$ such that $\tilde{V} | \mathfrak{H}_0 = V_0 (= V | \mathfrak{H}_0)$ and the operators $T^\tau$ and $\Lambda$ are unitarily equivalent by means of $\tilde{V}$.

Moreover, if $\mul T = \mul T^*$ (that is, the condition (C1) in Remark 3.9 is fulfilled), then the statements of the theorem hold for each spectral function $\Sigma_\tau(\cdot)$.
\end{theorem}
\begin{proof}
Since in view of (6.36) $\mul \tilde{T}^\tau = \mul T$, it follows that $\mathfrak{H}_0 \subset \tilde{\mathfrak{H}}_0$ and the decomposition (6.20) takes the form
\begin{equation}
\tilde{\mathfrak{H}} = \mathfrak{H}_0 \oplus \mul T.
\end{equation}
It follows from (6.27) and (2.28) that for each finite interval $[\alpha, \beta] \subset \mathbb{R}$ the spectral function $E_\tau(\cdot)$ of $T^\tau$ satisfies the equality
\begin{equation}
P_{\mathfrak{H}_0} E_\tau([\alpha, \beta]) | \mathfrak{H}_0 = V_0^* E_\Lambda([\alpha, \beta]) V_0 = V_0^* (P_{L_0} E_\Lambda([\alpha, \beta]) | L_0) V_0,
\end{equation}
where $L_0 = V_0 | \mathfrak{H}_0$. This implies that
\begin{equation}
P_{\mathfrak{H}_0} (T^\tau - \lambda)^{-1} | \mathfrak{H}_0 = V_0^* (P_{L_0} (\Lambda - \lambda)^{-1} | L_0) V_0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}
Since $\tilde{T}^\tau$ is $\mathfrak{H}$-minimal, it follows from (6.34) and (6.38) that the operator $T^\tau$ is $\mathfrak{H}_0$-minimal.

Moreover, according to Proposition 6.9 the operator $\Lambda$ is $L_0$-minimal. Now, applying Proposition 2.14 to operators $T^\tau$ and $\Lambda$ we arrive at the desired statements for $\Sigma_\tau(\cdot) \in SF_0$.

The last statement of the theorem follows from the fact that in the case $\mul T = \mul T^*$ the inclusion $\Sigma_\tau(\cdot) \in SF_0$ holds for each spectral function $\Sigma_\tau(\cdot)$.
\end{proof}

Combining of Theorems 6.11 and 4.1 yields the following corollary.
\begin{corollary}
Let $\tau$ be a boundary parameter and let $\Sigma_\tau(\cdot)$ be the spectral function of the boundary problem (4.1)--(4.4). Then the following statements are equivalent:
\begin{enumerate}
\item $n_+(T_{\min}) = n_-(T_{\min})$, $\tau \in \tilde{\mathcal{C}}(\mathfrak{H}_0)$ and the canonical self-adjoint extension $\tilde{T}^\tau$ of $T$
\item given by (4.8) satisfies the equality $\mul \tilde{T}^\tau = \mul T$
\end{enumerate}
(2) The Fourier transform $V$ isometrically maps $\mathfrak{H}_0$ onto $L^2(\Sigma_\tau; H_0)$ (that is, $V | \mathfrak{H}_0$ is a unitary operator).

If the statement (1) (and hence (2)) is valid, then the operator $T^\tau$ (the self-adjoint part of $\tilde{T}^\tau$) and the multiplication operator $\Lambda$ are unitarily equivalent by means of $V$.
\end{corollary}
\begin{theorem}
Assume that $T$ is a densely defined operator, that is, the condition (C2) in Remark 3.9 is fulfilled. Then for each boundary parameter $\tau$ and the corresponding spectral function $\Sigma_\tau(\cdot)$ the following hold: (i) $T^\tau$ is an operator, that is, $T^\tau = T^*$; (ii) the Fourier
transform $V$ is an isometry; (iii) there exists a unitary operator $\tilde{V} \in [\tilde{\mathcal{H}}, L^2(\Sigma; H_0)]$ such that $V \upharpoonright \tilde{\mathcal{H}} = V$ and the operators $T^\tau$ and $\Lambda$ are unitarily equivalent by means of $\tilde{V}$.

Moreover, the following statements are equivalent:

1. $n_+(T_{\min}) = n_-(T_{\min})$ and $\tau \in \tilde{R}(\mathcal{H}_b)$, so that $\tilde{T}^\tau$ is the canonical self-adjoint extension of $T$ given by the boundary conditions (4.8);

2. $V\tilde{\mathcal{H}} = L^2(\Sigma; H_0)$, that is the fourier transform $V$ is a unitary operator.

If the statement (1) (and hence (2)) is true, then the operators $\tilde{T}^\tau$ and $\Lambda$ are unitarily equivalent by means of $\tilde{V}$.

Proof. Since $\text{null } T = \text{null } T^* = \{0\}$, the required statements are implied by Theorem 6.11 and Corollary 6.12.

It follows from Theorem 6.11 that the operators $T^\tau$ and $\Lambda$ have the same spectral properties. This implies, in particular, the following corollary.

Corollary 6.14. (1) If $\tau$ is a boundary parameter such that $\Sigma_\tau(\cdot) \in SF_0$, then the spectral multiplicity of the operator $T^\tau$ does not exceed $\nu_- (= \dim H_0)$.

(2) If the condition (C1) in Remark 3.9 is fulfilled, then the above statement on the spectral multiplicity of $T^\tau$ holds for each boundary parameter $\tau$.

In the next theorem we give a parametrization of all spectral functions $\Sigma_\tau(\cdot)$ in terms of a boundary parameter $\tau$.

**Theorem 6.15.** Let $n_+(T_{\min}) = n_-(T_{\min})$ and let $M(\cdot)$ be the operator function defined by (4.39)–(4.41). Then, for each boundary parameter $\tau \in \tilde{R}(\mathcal{H}_b)$ given by (2.14) the equality

$$m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

(6.40) together with (6.23) defines a (unique) spectral function $\Sigma_\tau(\cdot)$ of the boundary problem (4.1)–(4.4). Moreover, the following hold:

1. $\Sigma_\tau(\cdot) \in SF_0$ if and only if the following two conditions are satisfied:

$$\lim_{y \to -\infty} \frac{1}{y}(C_0(iy) - C_1(iy)M_4(iy))^{-1}C_1(iy) = 0,$$

(6.41)

$$\lim_{y \to \infty} \frac{1}{y}M_4(iy)(C_0(iy) - C_1(iy)M_4(iy))^{-1}C_0(iy) = 0.$$

(6.42)

2. Each spectral function $\Sigma_\tau(\cdot)$ belongs to the class $SF_0$ if and only if

$$\lim_{y \to \infty} \frac{1}{y}M_4(iy) = 0 \quad \text{and} \quad \lim_{y \to -\infty} y \text{Im}(M_4(iy)h, h) = +\infty, \quad h \in \mathcal{H}_b, \quad h \neq 0.$$

Proof. The main statement of the theorem directly follows from Corollary 5.6 and Theorem 6.7.

Next, consider the boundary triplet $\tilde{\Pi} = \{\mathcal{H}_b, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for $T^*$ defined in Proposition 3.8. Since $M(\cdot)$ is the Weyl function of the decomposing boundary triplet (3.37), (3.38) for $T_{\max}$, it follows from Proposition 2.10, (3) that the Weyl function of the triplet $\tilde{\Pi}$ coincides with $M_4(\lambda)$. Now applying to the boundary triplet $\tilde{\Pi}$ the results of [10, 7] we obtain statements (1) and (2).
6.4. The case of minimal equal deficiency indices. In this subsection we reformulate the above results for the simplest case of minimally possible equal deficiency indices of $T_{\min}$, which in view of (3.14) are

\begin{equation}
(6.43) \quad n_+(T_{\min}) = n_-(T_{\min}) = \nu_-
\end{equation}

In this case $\nu_{-} = 0$, $\nu_{+} = \nu$ and according to Lemma 3.4 there exists a surjective linear mapping $\hat{\Gamma}_b : \text{dom} \ T_{\max} \to \hat{H}$ such that

\begin{equation}
(6.44) \quad [y, z]_b = i(\hat{\Gamma}_b y, \hat{\Gamma}_b z), \quad y, z \in \text{dom} \ T_{\max}.
\end{equation}

Below we suppose that the assumption (A1) at the beginning of Section 4 is fulfilled and that $\hat{\Gamma}_b$ is a surjective operator satisfying (6.44).

It follows from Proposition 3.8 and Theorem 4.1 that the equality

$$T = \{ \{ \bar{y}, \bar{f} \} \in T_{\max} : \Gamma_{1a} y = 0, \ \hat{\Gamma}_a y = \hat{\Gamma}_b y \}$$

defines a self-adjoint relation $T$ in $\mathcal{H}(= L_2^2(I))$ and the (canonical) resolvent of $T$ is given by the boundary value problem

\begin{equation}
(6.45) \quad J y' - B(t)y = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in I,
\end{equation}

\begin{equation}
(6.46) \quad \Gamma_{1a} y = 0, \quad \hat{\Gamma}_a y = \hat{\Gamma}_b y, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

Next, in view of Theorem 4.6 for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique operator solution $v(\cdot, \lambda) \in \mathcal{L}_2^2[H_0, \mathbb{H}]$ of Eq. (3.3) such that

\begin{equation}
(6.47) \quad \Gamma_{1a} v(\lambda) = -P_H, \quad i(\hat{\Gamma}_a - \hat{\Gamma}_b) v(\lambda) = P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

Moreover, if $\tilde{U}$ is a $J$-unitary extension (3.20) of $U$ and $\Gamma_{0a}$ is the mapping (3.23), then the (canonical) $m$-function $m(\cdot)$ of the problem (6.45), (6.46) is given by the equality

\begin{equation}
(6.48) \quad m(\lambda) = (\Gamma_{0a} + \hat{\Gamma}_a) v(\lambda) + \frac{i}{2} P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\end{equation}

or, equivalently, by the relations

$$v(t, \lambda) := \varphi_U(t, \lambda) m(\lambda) + \psi(t, \lambda) \in \mathcal{L}_2^2[H_0, \mathbb{H}], \quad i(\hat{\Gamma}_a - \hat{\Gamma}_b) v(\lambda) = P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

The boundary problem (6.45), (6.46) has a unique spectral function $\Sigma(\cdot)$, which is defined by the Stieltjes formula (6.23) with $m_{\cdot}(\cdot) = m(\cdot)$. Moreover, Corollary 6.12 implies that the corresponding Fourier transform $V$ isometrically maps $\mathcal{H}_0(= \mathcal{H} \ominus \mu T)$ onto $L^2(\Sigma; H_0)$.

6.5. Example. In this subsection we provide an example illustrating the results of the paper.

Let $I = [0, \infty)$ and let $\delta(\cdot)$ be a Borel function on $I$ such that $\delta(t) > 0$ (a.e. in $I$) and

$$C := \int_0^\infty \delta(t) \, dt < \infty.$$ 

Assume also that in formulas (3.10) and (3.12) $H = \hat{H} = \mathbb{C}$, so that $\mathbb{H} = \mathbb{C}^3$ and $H_0 = \mathbb{C}^2$. Consider the symmetric system

\begin{equation}
(6.49) \quad J y' = \Delta(t)f(t), \quad t \in I, \quad f \in \mathcal{L}_2^2(I),
\end{equation}

where $J$ and $\Delta(t)$ are given by the matrices

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & i & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Delta(t) = \begin{pmatrix} \frac{i}{2}(\delta(t) + 1) & 0 & \frac{i}{2}(\delta(t) - 1) \\ 0 & 0 & 1 \\ -\frac{i}{2}(\delta(t) - 1) & 0 & \frac{i}{2}(\delta(t) + 1) \end{pmatrix}.$$
Clearly, $\Delta(t)$ is a nonnegative invertible matrix (a.e. on $I$); therefore the system (6.49) is definite and $T_{\text{min}}$ is a densely defined operator in $L_2^\Delta(I)$. The immediate checking shows that the homogeneous system

$$Jy' = \lambda \Delta(t)y, \quad t \in I, \quad \lambda \in \mathbb{C},$$

has a fundamental solution

$$Y(t, \lambda) = \begin{pmatrix} e^{-i\lambda \Phi(t)} & 0 & e^{i\lambda t} \\ 0 & e^{-i\lambda t} & 0 \\ -ie^{-i\lambda \Phi(t)} & 0 & ie^{i\lambda t} \end{pmatrix},$$

where

$$\Phi(t) := \int_0^t \delta(s) \, ds.$$

Denote by $y^{(1)}(\cdot, \lambda)$, $y^{(2)}(\cdot, \lambda)$ and $y^{(3)}(\cdot, \lambda)$ vector solutions of Eq. (6.50) formed by the first, second and third columns of the matrix (6.51) respectively. It is easily seen that $y^{(1)}(\cdot, \lambda)$, $y^{(2)}(\cdot, \lambda) \in L_2^\Delta(I)$, $y^{(2)}(\cdot, \lambda) \notin L_2^\Delta(I)$ for all $\lambda \in \mathbb{C}_+$ and $y^{(1)}(\cdot, \lambda)$, $y^{(2)}(\cdot, \lambda) \in L_2^\Delta(I)$, $y^{(3)}(\cdot, \lambda) \notin L_2^\Delta(I)$ for all $\lambda \in \mathbb{C}_-$. Therefore the operator $T_{\text{min}}$ has minimally possible equal deficiency indices $n_+(T_{\text{min}}) = n_-(T_{\text{min}}) = 2$.

Let $\theta(\cdot) \in L_2^\Delta(I)$ be the solution of Eq. (6.50) given by

$$\theta(t) = \frac{i}{\sqrt{2}} e^{-C} y^{(1)}(t, i) = \frac{i}{\sqrt{2}} e^{-C} \{ e^{i\Phi(t)}, 0, -ie^{i\Phi(t)} \}.$$

Since $[\theta, \theta]_\infty = i$, it follows from Remark 3.5, (2) that the equality $\hat{\Gamma}_b y = [y, \theta]_\infty$, $y \in \text{dom} \, T_{\text{max}}$, defines the surjective linear mapping $\hat{\Gamma}_b : \text{dom} \, T_{\text{max}} \to \mathcal{C}$ satisfying (6.44).

We assume that $\hat{U} = I$ (see (3.20)). Then for each function $y \in \text{dom} \, T_{\text{max}}$ decomposed as

$$y(t) = \{ y_0(t), \hat{y}(t), y_1(t) \} \in \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, \quad t \in I,$$

one has $\Gamma_{0a} y = y_0(0)$, $\hat{\Gamma}_a y = \hat{y}(0)$, $\Gamma_{1a} y = y_1(0)$ and the boundary problem (6.45), (6.46) can be written as

$$Jy' = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in I,$$

$$y_1(0) = 0, \quad \hat{y}(0) = [y, \theta]_b, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

According to Subsection 6.4 there exists a unique operator solution

$$v(t, \lambda) = \begin{pmatrix} r_0(t, \lambda) & q_0(t, \lambda) \\ \hat{r}(t, \lambda) & \hat{q}(t, \lambda) \\ r_1(t, \lambda) & q_1(t, \lambda) \end{pmatrix} : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

of Eq. (6.50) belonging to $L_2^\Delta[H_0, \mathbb{R}]$ and satisfying the boundary conditions (6.47), which in our case take the form

$$r_1(0, \lambda) = -1, \quad \hat{r}(0, \lambda) - [r, \theta]_\infty = 0, \quad q_1(0, \lambda) = 0, \quad \hat{q}(0, \lambda) - [q, \theta]_\infty = -i.$$

The immediate checking shows that for $\lambda \in \mathbb{C}_+$ such a solution is

$$v(t, \lambda) = \begin{pmatrix} ie^{i\lambda t} & \frac{i}{\sqrt{2}} e^{i\lambda C} (e^{-i\lambda \Phi(t)} + e^{i\lambda t}) \\ 0 & 0 \\ -e^{i\lambda t} & \frac{i}{\sqrt{2}} e^{i\lambda C} (-ie^{-i\lambda \Phi(t)} + ie^{i\lambda t}) \end{pmatrix}.$$

Combining of (6.48) with (6.54) implies that the $m$-function of the problem (6.52), (6.53) is

$$m(\lambda) = \begin{pmatrix} r_0(0, \lambda) & q_0(0, \lambda) \\ \hat{r}(0, \lambda) & \hat{q}(0, \lambda) + \frac{i}{\sqrt{2}} \end{pmatrix}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$
Therefore by (6.55) one has

\[ m(\lambda) = \begin{pmatrix} i & i\sqrt{2}\Phi(t) \\ 0 & \sqrt{2} \end{pmatrix}, \quad \lambda \in \mathbb{C}_+. \]

Applying the Stieltjes formula (6.23) to \( m(\cdot) \) one obtains the spectral function of the boundary problem (6.52), (6.53):

\[
\Sigma(s) = \frac{1}{\pi} \begin{pmatrix} s \quad -\frac{i\sqrt{2}}{s} \\ \frac{i\sqrt{2}}{s} & \frac{1}{s} \end{pmatrix}
\]

Since \( \Sigma(s) \) has the continuous derivative

\[ \Sigma'(s) = \frac{1}{\pi} \begin{pmatrix} 1 & \frac{i\sqrt{2}}{s} \\ \frac{i\sqrt{2}}{s} & \frac{1}{s} \end{pmatrix}, \]

it follows that \( L^2(\Sigma; \mathbb{C}^2) \) is the set of all functions \( g(\cdot) \) such that

\[
\int_{\mathbb{R}} (\Sigma'(s)g(s), g(s)) \, ds < \infty.
\]

To simplify further considerations we pass to the new orthonormal basis \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \) in \( \mathbb{C}^3 \) with \( \hat{e}_1 = \{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\} \), \( \hat{e}_2 = \{0, 1, 0\} \) and \( \hat{e}_3 = \{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\} \). Then the Hilbert space \( L^2_\Delta(I) \) can be identified with the set of all Borel functions \( f(\cdot): I \rightarrow \mathbb{C}^3 \) of the form

\[ f(t) = f_1(t)\hat{e}_1 + f_2(t)\hat{e}_2 + f_3(t)\hat{e}_3 =: \{\hat{f}_1(t), \hat{f}_2(t), \hat{f}_3(t)\}, \]

where \( \delta^{1/2}\hat{f}_1 \in L^2(I) \) and \( \hat{f}_2, \hat{f}_3 \in L^2(I) \).

Next, the equality

\[ \varphi(t, \lambda) = \begin{pmatrix} \frac{1}{2}(e^{-i\lambda} + e^{i\lambda}) & 0 \\ \frac{i}{2}(-e^{-i\lambda} + e^{i\lambda}) & e^{-i\lambda} \end{pmatrix} : \mathbb{C} \oplus \mathbb{C} \rightarrow \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, \quad \lambda \in \mathbb{C}, \]

defines the operator solution of Eq. (6.50) with \( \varphi(0, \lambda) = \begin{pmatrix} I_{\mathbb{H}_0} \\ 0 \end{pmatrix} \). This and formula (6.16) (with \( \varphi_U(t, \lambda) = \varphi(t, \lambda) \)) imply that for each function \( f(\cdot) = \{\hat{f}_1(\cdot), \hat{f}_2(\cdot), \hat{f}_3(\cdot)\} \in L^2_\Delta(I) \) the Fourier transform \( \hat{f}(\cdot) = \{\hat{f}_1(\cdot), \hat{f}_2(\cdot), \hat{f}_3(\cdot)\} \in L^2(\Sigma; \mathbb{C}^2) \) is given by

\[
\hat{f}_1(s) = \frac{1}{\sqrt{2}} \int_0^\infty (e^{ist}\Phi(t)\delta(t)\hat{f}_1(t) + e^{-ist}\hat{f}_3(t)) \, dt, \quad \hat{f}_2(s) = \int_0^\infty e^{ist}\hat{f}_2(t) \, dt.
\]

According to Theorem 6.13 \( Vf = \hat{f} \) is a unitary operator from \( L^2_\Delta(I) \) onto \( L^2(\Sigma; \mathbb{C}^2) \) and by using (6.37) one can easily prove that the inverse Fourier transform is

\[
\hat{f}_1(t) = \frac{1}{\pi\sqrt{2}} \int_{\mathbb{R}} (e^{-is\Phi(t)}\hat{f}_1(s) + \frac{i}{\sqrt{2}} e^{-is(\Phi(t)-C)}\hat{f}_2(s)) \, ds,
\]

\[
\hat{f}_2(t) = \frac{1}{\pi\sqrt{2}} \int_{\mathbb{R}} (e^{-is(t+C)}\hat{f}_1(s) + \frac{i}{\sqrt{2}} e^{-ist}\hat{f}_2(s)) \, ds,
\]

\[
\hat{f}_3(t) = \frac{1}{\pi\sqrt{2}} \int_{\mathbb{R}} (e^{ist}\hat{f}_1(s) + \frac{i}{\sqrt{2}} e^{is(t+C)}\hat{f}_2(s)) \, ds
\]

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