On the Substitution Rule for Lebesgue–Stieltjes Integrals

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INTEGRALS

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Abstract. We establish a generalization for the substitution rule which holds
for arbitrary Lebesgue–Stieltjes integrals.

1. Introduction

The present note is concerned with the classical integration by substitution rule
\[ \int_a^b g(\mu(x))\mu'(x) \, dx = \int_{\mu(a)}^{\mu(b)} g(y) \, dy \] (1.1)
and to what extent it can be extended to Lebesgue–Stieltjes integrals. Of course,
if \( \mu \) is continuous and increasing the natural generalization of the above formula
would be
\[ \int_a^b g(\mu(x)) \, d\mu(x) = \int_{\mu(a)}^{\mu(b)} g(y) \, dy. \] (1.2)

In fact, if \( \mu \) is absolutely continuous it reduces to the first form (1.1) and can be
found in virtually any textbook which covers Lebesgue–Stieltjes integrals. See for
example p323 in [7] which also contains a historic account of this case. However,
it remains valid for continuous \( \mu \), see for example [3, Theorem 6.2.1] or [2, Ex-
ample 3.6.2] where this case is attributed to A.N. Kolmogorov. An extension to
Kurzweil–Henstock–Stieltjes integrals is given in [6].

So this naturally raises the question about the case when \( \mu \) is allowed to have
jumps. However, by letting \( \mu \) be the Heaviside step function, 
\( \mu(x) = 0 \) for \( x < 0 \)
and \( \mu(x) = 1 \) else, such that the corresponding measure \( d\mu \) is a single Dirac measure
sitting at \( x = 0 \), one easily sees that the claim breaks down. So it seems that the
continuous case is best possible.

On the other hand, in a recent paper [5] we needed the general case without the
continuity assumption. In fact, a similar problem arose in [1, Lemma 1.2] where,
for the special case of monomials \( g(x) = x^n \), the following inequality is shown
\[ \int_{(a,b)} \mu(x)^n \, d\mu(x) \leq \frac{\mu(b)^{n+1}}{n+1}, \quad \mu(a) = 0, \] (1.3)
for a nondecreasing left continuous \( \mu \). This inequality is the crucial ingredient in
establishing a Gronwall lemma for Lebesgue–Stieltjes integrals.

By considering again a Dirac measure with a single point mass one can verify
that (1.3) fails in general if \( \mu \) is not left continuous.

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The aim of this paper is not only to prove an inequality but to generalize the integration by substitution rule such that we always have equality. It will reduce to (1.2) for continuous \( \mu \) and it will also produce generalizations of (1.3). Roughly speaking, the idea is as follows: If \( \mu \) has a jump, the corresponding jump will be missing in the range of \( \mu \). Since the length of the missing interval is precisely the weight of the point mass whose contribution is missed by the substitution rule, we can make up for it by including it in the range of integration on the right-hand side if we modify \( g \) such that it is constant on this interval, with the value chosen to be the value at the jump. If we do not modify \( g \), but if \( g \) is monotone along this interval, we can at least get inequalities such as (1.3).

2. The generalized inverse of a monotone function

As a preparation we will need a generalization of the usual inverse which works for arbitrary nondecreasing functions. Such a generalized inverse arises for example as quantile functions in probability theory and we refer to [4] for a nice discussion and further properties. We recall some basic facts and prove a simple lemma which we could not find in its present form in the literature.

So we look at nondecreasing functions \( f : \mathbb{R} \to \mathbb{R} \). By monotonicity the limits from left and right exists at every point and we will denote them by

\[
(2.1) \quad f(x \pm) = \lim_{\varepsilon \to 0} f(x \pm \varepsilon).
\]

Clearly we have \( f(x-) \leq f(x+) \) and a strict inequality can occur only at a countable number of points. By monotonicity the value of \( f \) has to lie between these two values \( f(x-) \leq f(x) \leq f(x+) \). It will also be convenient to extend \( f \) to a function on the extended reals \( \mathbb{R} \cup \{-\infty, +\infty\} \). Again by monotonicity the limits \( f(\pm \infty \pm) = \lim_{x \to \pm \infty} f(x) \) exist and we will set \( f(\pm \infty \pm) = \pm \infty \).

If we want to define an inverse, problems will occur at points where \( f \) jumps and on intervals where \( f \) is constant. Formally speaking, if \( f \) jumps, then the corresponding jump will be missing in the domain of the inverse and if \( f \) is constant, the inverse will be multivalued. For the first case there is a natural fix by choosing the inverse to be constant along the missing interval. In particular, observe that this natural choice is independent of the actual value of \( f \) at the jump and hence the inverse loses this information. The second case will result in a jump for the inverse function and here there is no natural choice for the value at the jump (except that it must be between the left and right limits such that the inverse is again a nondecreasing function).

To give a precise definition it will be convenient to look at relations instead of functions. Recall that a (binary) relation \( R \) on \( \mathbb{R} \) is a subset of \( \mathbb{R}^2 \).

To every nondecreasing function \( f \) associate the relation

\[
(2.2) \quad \Gamma(f) = \{(x, y) | y \in [f(x-), f(x+)]\}.
\]

Note that \( \Gamma(f) \) does not depend on the values of \( f \) at a discontinuity and \( f \) can be partially recovered from \( \Gamma(f) \) using \( f(x-) = \inf \Gamma(f)(x) \) and \( f(x+) = \sup \Gamma(f)(x) \), where \( \Gamma(f)(x) = \{y | (x, y) \in \Gamma(f)\} = [f(x-), f(x+)] \). Moreover, the relation is nondecreasing in the sense that if \( x_1 < x_2 \) implies \( y_1 \leq y_2 \) for \( (x_1, y_1), (x_2, y_2) \in \Gamma(f) \). It is uniquely defined as the largest relation containing the graph of \( f \) with this property.
The graph of any reasonable inverse should be a subset of the inverse relation
\begin{equation}
\Gamma(f)^{-1} = \{(y, x) | (x, y) \in \Gamma(f)\}
\end{equation}
and we will call any function \(f^{-1}\) whose graph is a subset of \(\Gamma(f)^{-1}\) a generalized inverse of \(f\). Note that any generalized inverse is again nondecreasing since a pair of points \((y_1, x_1), (y_2, x_2) \in \Gamma(f)^{-1}\) with \(y_1 < y_2\) and \(x_1 > x_2\) would contradict the fact that \(\Gamma(f)\) is nondecreasing. Moreover, since \(\Gamma(f)^{-1}\) and \(\Gamma(f^{-1})\) are two nondecreasing relations containing the graph of \(f^{-1}\), we conclude
\begin{equation}
\Gamma(f^{-1}) = \Gamma(f)^{-1}
\end{equation}
since both are maximal. In particular, it follows that if \(f^{-1}\) a generalized inverse of \(f\) then \(f\) is a generalized inverse of \(f^{-1}\).

There are two particular choices, namely the left continuous version \(f^-\cdot1(y) = \inf \Gamma(f)^{-1}(y)\) and the right continuous version \(f^+-\cdot1(y) = \sup \Gamma(f)^{-1}(y)\). It is straightforward to verify that they can be equivalently defined via
\begin{equation}
f^-\cdot1(y) = \inf f^{-1}([y, \infty)) = \sup f^{-1}((-\infty, y]),
f^+-\cdot1(y) = \inf f^{-1}((y, \infty)) = \sup f^{-1}((-\infty, y]).
\end{equation}
For example, \(\inf f^{-1}([y, \infty)) = \inf \{x | (x, \bar{y}) \in \Gamma(f), \bar{y} \geq y\} = \inf \Gamma(f)^{-1}(y)\). The first one is typically used in probability theory, where it corresponds to the quantile function of a distribution.

If \(f\) is strictly increasing the generalized inverse coincides with the usual inverse and we have \(f(f^-\cdot1(y)) = y\) for \(y\) in the range of \(f\). The purpose of the next lemma is to investigate to what extend this remains valid for a generalized inverse.

Note that for every \(y\) there is some \(x\) with \(y \in [f(x^-), f(x^+)]\). Moreover, if we can find two values, say \(x_1\) and \(x_2\), with this property, then \(f(x) = y\) is constant for \(x \in (x_1, x_2)\). Hence, the set of all such \(x\) is an interval which is closed since at the left, right boundary point the left, right limit equals \(y\), respectively.

**Lemma 2.1.** Let \(f\) be nondecreasing and \(f^{-1}\) a generalized inverse for \(f\). Then
\begin{itemize}
  \item \(f(f^-\cdot1(y)) \leq y\) if \(f(x) = f(x^-)\) for the largest \(x\) with \(y \in [f(x^-), f(x^+)]\) and
  \item \(f(f^+-\cdot1(y)) \geq y\) if \(f(x) = f(x^+)\) for the smallest \(x\) with \(y \in [f(x^-), f(x^+)]\).
\end{itemize}
In particular, \(f(f^-\cdot1(y)) = y\) if \(f(x) = f(x^-)\) for every \(x\) with \(y \in [f(x^-), f(x^+)]\).

**Proof.** We begin by observing that the graph of the composition \(f \circ f^{-1}\) must be a subset of the composition of the associated relations
\begin{align*}
\Gamma(f) \circ \Gamma(f)^{-1} &= \{(y, z) | (y, x) \in \Gamma(f^{-1}) \text{ and } (x, z) \in \Gamma(f) \text{ for some } x\} \\
&= \{(y, z) | y, z \in [f(x^-), f(x^+)] \text{ for some } x\}.
\end{align*}
Now fix some \(y\) and set \(z = f(f^-\cdot1(y))\). Consider the set of all \(x\) with \(y \in [f(x^-), f(x^+)]\). If there is only one such \(x\), then \(f^-\cdot1(y) = x\) implying \(z = f(x)\) and the claim follows. Otherwise let \(x_0\) be the smallest such \(x\) and \(x_1\) the largest. Then \(f(x_0^-) = y = f(x_1^-)\) and by the above considerations \(z \in [f(x_0^-), f(x_1^+)]\). Now \(f(x_0^-)\) can only be in the range of \(f\) if \(f(x_0) = f(x_0^-)\) by our choice of \(x_0\). Similarly for \(f(x_1^+)\) and since \(z\) must be in the range of \(f\) we get \(z \in [f(x_0), f(x_1)]\). Hence the claim again follows. \(\square\)

Of course an analog result for \(f^{-1}(f(x)) = x\) follows by reversing the roles of \(f\) and \(f^{-1}\).
3. Borel measures on \( \mathbb{R} \)

We are interested in measures defined on the Borel \( \sigma \)-algebra \( \mathcal{B} \) of \( \mathbb{R} \) which assign finite measure to every compact interval. We will simply call such measures Borel measures. To every Borel measure \( \mu : \mathcal{B} \to [0, \infty] \) we can assign its distribution function

\[
(3.1) \quad \mu(x) = \begin{cases} 
-\mu((x,0]), & x < 0, \\
0, & x = 0, \\
\mu((0,x]), & x > 0,
\end{cases}
\]

which is right continuous and nondecreasing.

Conversely, to obtain a measure from a nondecreasing function \( \mu : \mathbb{R} \to \mathbb{R} \) we proceed as follows: Recall that an interval is a subset of the real line of the form

\[
(3.2) \quad I = (a,b), \quad I = [a,b), \quad I = (a,b), \quad \text{or} \quad I = [a,b),
\]

with \( a \leq b, \ a, b \in \mathbb{R} \cup \{-\infty, \infty\} \). Note that \((a,a), \ [a,a), \ \text{and} \ (a,a] \) denote the empty set, whereas \([a,a] \) denotes the singleton \{a\}. For any proper interval with different endpoints we can define its measure to be

\[
(3.3) \quad \mu(I) = \begin{cases} 
\mu(b+) - \mu(a+), & I = (a,b), \\
\mu(b+) - \mu(a-), & I = [a,b), \\
\mu(b-) - \mu(a+), & I = (a,b), \\
\mu(b-) - \mu(a-), & I = [a,b],
\end{cases}
\]

where \( \mu(a\pm) = \lim_{x\to a} \mu(a\pm x) \) (which exist by monotonicity). If one of the endpoints is infinite we agree to use \( \mu(\pm\infty) = \lim_{x\to \pm\infty} \mu(x) \). For the empty set we of course set \( \mu(\emptyset) = 0 \) and for the singletons we set

\[
(3.4) \quad \mu(\{a\}) = \mu(a+) - \mu(a-)
\]

(which agrees with \(3.3\) except for the case \( I = (a,a) \) which would give a negative value for the empty set if \( \mu \) jumps at \( a \)). Note that \( \mu(\{a\}) = 0 \) if and only if \( \mu(x) \) is continuous at \( a \) and that there can be only countably many points with \( \mu(\{a\}) > 0 \) since a nondecreasing function can have at most countably many jumps.

Now we can consider the algebra of finite unions of intervals and extend \(3.3\) to finite unions of disjoint intervals by summing over all intervals. Thus \( \mu \) gives rise to a premeasure which can be uniquely extended to a measure using the Carathéodory procedure. This can be readily found in most textbooks on real analysis and we simply state the result (e.g. \cite{2} Theorem 1.8.1).

**Theorem 3.1.** For every nondecreasing function \( \mu : \mathbb{R} \to \mathbb{R} \) there exists a unique Borel measure \( \mu \) which extends \(3.3\). Two different functions generate the same measure if and only if the difference is a constant away from the discontinuities.

4. Transformation of measures

Finally we want to transform measures. Let \( f : X \to Y \) be a measurable function. Given a measure \( \mu \) on \( X \) we can introduce a measure \( f_* \mu \) on \( Y \) via

\[
(4.1) \quad (f_* \mu)(A) = \mu(f^{-1}(A)).
\]

It is straightforward to check that \( f_* \mu \) is indeed a measure. Moreover, note that \( f_* \mu \) is supported on the range of \( f \). The central result is the following theorem, which can be found in any textbook on measure theory (e.g. \cite{2} Theorem 3.6.1):
Theorem 4.1. Let $f : X \to Y$ be measurable and let $g : Y \to \mathbb{R}$ be a Borel function. Then the Borel function $g \circ f : X \to \mathbb{R}$ is a.e. nonnegative or integrable if and only if $g$ is and in both cases

$$
\int_Y g \, d(f_*\mu) = \int_X g \circ f \, d\mu.
$$

In fact, note that it holds for characteristic functions of measurable sets by definition since $\chi_A \circ f = \chi_{f^{-1}(A)}$. Hence it holds for simple functions (linear combinations of characteristic functions) by linearity of the integral and hence for general functions since they can be approximated by simple functions.

5. Transformation of Borel measures

Now we have all the ingredients to apply this to Borel measures $\mu$. For any nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ we will define its (right continuous) generalized inverse as

$$
f^{-1}(y) = \inf f^{-1}((y, \infty)).
$$

We will need the set

$$
L(f) = \{y | f^{-1}((y, \infty)) = (f^{-1}(y), \infty)\}.
$$

Note that $y \notin L(f)$ if and only if there is some $x$ such that $y \in [f(x-), f(x))$.

Lemma 5.1. Let $\mu$ be a Borel measure on $\mathbb{R}$ and $f$ a nondecreasing function on $\mathbb{R}$ such that $\mu((0, \infty)) < \infty$ if $f$ is bounded above and $\mu((-\infty, 0)) < \infty$ if $f$ is bounded below. Then $f_*\mu$ is a Borel measure whose distribution function coincides with $\mu(f^{-1}(y)) - c$ at every point $y$ which is in $L(f)$ or satisfies $\mu\{f^{-1}(y)\} = 0$. If $y \in [f(x-), f(x))$ and $\mu(f^{-1}(y)) > 0$, then $\mu(f^{-1}(y))$ jumps at $f(x-)$ and $(f_*\mu)(y)$ jumps at $f(x)$.

Proof. First of all note that the assumptions in case $f$ is bounded from above or below ensure that $(f_*\mu)(K) < \infty$ for any compact interval. Moreover, note that we have $f^{-1}((y, \infty)) = (f^{-1}(y), \infty)$ for $y \in L(f)$ and $f^{-1}((y, \infty)) = [f^{-1}(y), \infty)$ else. Hence

$$
(f_*\mu)((y_0, y_1]) = \mu(f^{-1}((y_0, y_1])) = \mu((f^{-1}(y_0), f^{-1}(y_1]))
$$

$$
= \mu(f^{-1}(y_1)) - \mu(f^{-1}(y_0)) = (\mu \circ f^{-1})(y_1) - (\mu \circ f^{-1})(y_0)
$$

if $y_1$ is either in $L(f)$ or satisfies $\mu(f^{-1}(y_1)) = 0$. For the last claim observe that $f^{-1}((y, \infty))$ will jump from $(f^{-1}(y), \infty)$ to $[f^{-1}(y), \infty)$ at $y = f(x)$. \qed

Example. For example, consider $f(x) = \chi_{[0, \infty)}(x)$ and $\mu = \Theta$, the Dirac measure centered at 0 (note that $\Theta(x) = f(x)$). Then

$$
f^{-1}(y) = \begin{cases} +\infty, & 1 \leq y, \\ 0, & 0 \leq y < 1, \\ -\infty, & y < 0, \end{cases}
$$

and $L(f) = (-\infty, 0) \cup [1, \infty)$. Moreover, $\mu(f^{-1}(y)) = \chi_{(0, \infty)}(y)$ and $(f_*\mu)(y) = \chi_{[1, \infty)}(y)$. If we choose $g(x) = \chi_{(0, \infty)}(x)$, then $g^{-1}(y) = f^{-1}(y)$ and $L(g) = \mathbb{R}$. Hence $\mu(g^{-1}(y)) = \chi_{[0, \infty)}(y) = (g_*\mu)(y)$.
If we choose $f$ to be the distribution function of $\mu$ we get the following generalization of the integration by substitution rule. To formulate it we introduce

\begin{equation}
  i_\mu(y) = \mu(\mu^{-1}(y-)).
\end{equation}

By $\text{ran}(\mu)$ we will denote the range of a function and by $\text{hull}(\text{ran}(\mu))$ its convex hull.

**Corollary 5.2.** Suppose $\mu$, $\nu$ are Borel measures on $\mathbb{R}$. Then we have

\begin{equation}
  \int_\mathbb{R} (g \circ \mu) \, d(\nu \circ \mu) = \int_{\text{hull}(\text{ran}(\mu))} (g \circ i_\mu) \, d\nu
\end{equation}

for any Borel function $g$ which is either nonnegative or for which one of the two integrals is finite. In particular, note that

\begin{equation}
  \int_\mathbb{R} (g \circ \mu) \, d(\nu \circ \mu) = \int_{\text{ran}(\mu)} g \, d\nu, \quad \mu(\{x\}) = 0 \forall x.
\end{equation}

**Proof.** First of all we can assume that $\nu$ is supported on $\text{hull}(\text{ran}(\mu))$ without loss of generality. Moreover, abbreviate $\sigma(y) = \mu^{-1}(y-)$. Then

\begin{equation*}
  \int (g \circ \mu) \, d(\nu \circ \mu) = \int (g \circ \mu) \, d\sigma, \nu = \int (g \circ \mu \circ \sigma) \, d\nu.
\end{equation*}

Hence the usual $\int_\mathbb{R} (g \circ \mu) \, d(\nu \circ \mu) = \int_{\text{ran}(\mu)} g \, d\nu$ only holds if $\mu$ is continuous. In fact, the right-hand side loses all point masses of $\mu$. The above formula fixes this problem by rendering $g$ constant along a gap in the range of $\mu$ and includes the gap in the range of integration such that it makes up for the lost point mass. It should be compared with the previous example!

If one does not want to bother with $i_\mu$ one can at least get inequalities for monotone $g$.

**Corollary 5.3.** Suppose $\mu$, $\nu$ are Borel measures on $\mathbb{R}$ and $g$ is monotone. Then we have

\begin{equation}
  \int_\mathbb{R} (g \circ \mu) \, d(\nu \circ \mu) \leq \int_{\text{hull}(\text{ran}(\mu))} g \, d\nu
\end{equation}

if $\mu$ is right continuous and $g$ nonincreasing or $\mu$ left continuous and $g$ nondecreasing. If $\mu$ is right continuous and $g$ nondecreasing or $\mu$ left continuous and $g$ nonincreasing the inequality has to be reversed.

**Proof.** Immediate from the previous corollary together with $i_\mu(y) \leq y$ if $y = f(x) = f(x^+)$ and $i_\mu(y) \geq y$ if $y = f(x) = f(x^-)$ as pointed out before.

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