OPERAD GROUPS AS A UNIFIED FRAMEWORK FOR THOMPSON-LIKE GROUPS

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Abstract. We propose a new unified framework for Thompson-like groups using a well-known device called operads and category theory as language. We discuss examples of operad groups which have appeared in the literature before. As a first application, we give sufficient conditions for plain operads to yield groups of type \( F_\infty \). This unifies and extends existing proofs for certain Thompson-like groups in a conceptual manner.

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1. INTRODUCTION

Since the introduction of R. Thompson’s famous group \( F \), a lot of generalizations have appeared in the literature which, in some sense, have “Thompson-esque” feeling to them. Among them are the so-called diagram or picture groups [21], various groups of piecewise linear homeomorphisms of the unit interval [32], groups acting on ultrametric spaces via local similarities [22], higher dimensional Thompson groups \( nV \) [5] and the braided Thompson group \( BV \) [6]. A recurrent theme in the study of these groups are topological finiteness properties, most notably property \( F_\infty \) which means that the group admits a classifying space with compact skeleta. The proof of this property is very similar in each case, going back to a method of Brown, the Brown criterion [7], and a technique of Bestvina and Brady, the discrete Morse Lemma for affine complexes [2]. This program has been conducted in all the above mentioned classes of groups: for diagram or picture groups in [11,12], for the piecewise linear homeomorphisms in [32], for local similarity groups in [13], for the higher dimensional Thompson groups in [17] and for the braided Thompson group in [8].

The motivation to define the class of operad groups was to find a framework in which a lot of the Thompson-like groups in the existing literature could be recovered and in which the established techniques could be performed to show property \( F_\infty \), thus unifying existing proofs in the literature. The main device to define these groups are operads in the category of sets. Operads are well established objects.

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whose importance in mathematics and physics has steadily increased during the last decades. Representations of operads constitute algebras of various types and consequently find applications in such diverse areas as Lie-Theory, Noncommutative Geometry, Algebraic Topology, Differential Geometry and Field Theories.

This article shall serve as an introduction to the concept of seeing Thompson-like groups as fundamental groups of operads. The language will be in large parts category theory flavoured. We therefore review some definitions and techniques from category theory in Section 2 which are needed later in the text. In Section 3 we introduce operads and operad groups. Section 4 deals with the connection of operad groups to existing Thompson-like groups in the literature: We give explicit examples of operads and identify the corresponding operad groups. In Section 5 we review the discrete Morse Lemma of Bestvina and Brady and discuss a version for categories. This is used in the final Section 6 where we prove a $F_\infty$ theorem for a subclass of operads, so-called plain operads.

A more general theorem, applicable to a larger class of operads, will appear in [35]. This theorem will subsume the one in Section 6 but we have decided to publish it in another article because the plain case allows for nice simplifications which make the proof easier to read. The main ideas are already present in the plain case and therefore, it will serve as a guide for the proof of the more general theorem.

In [36], we will present a homological result for operad groups which generalizes the results previously obtained for the subclass of local similarity groups in [30].

1.1. Notation and Conventions. When $f : A \to B$ and $g : B \to C$ are two composable arrows, we strongly prefer the notation $f \circ g$ or $fg$ for the composite $A \to C$ instead of the usual notation $g \circ f$. Consequently, it is often better to plug in arguments from the left. When we do this, we use the notation $x \circ f$ for the evaluation of $f$ at $x$. However, we won’t entirely drop the usual notation $f(x)$ and use both notations side by side. Otherwise, for example, we would have to write $(X, x_0) \triangleright \pi_1$ for the fundamental group $\pi_1(X, x_0)$.

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2. Preliminaries on categories

In this section we review some aspects of category theory which we need for later considerations. In particular, we want to emphasize the concept of seeing categories as topological objects. We refer to the first chapter of Quillen’s seminal paper [28] for another exposition in that direction. Also note that nothing of this section is new and we make no claim of originality.

In the following, we will frequently use the notion of comma categories which we recall now. Let $A \xrightarrow{\gamma} C \xleftarrow{\beta} B$ be two functors. Then the comma category $f \downarrow g$ has as objects all the triples $(A, B, \gamma)$ where $\gamma : f(A) \to g(B)$ is an arrow in $C$. An arrow from $(A, B, \gamma)$ to $(A', B', \gamma')$ is a pair $(\alpha, \beta)$ of arrows $\alpha : A \to A'$ and $\beta : B \to B'$ such that the diagram

$$
\begin{array}{ccc}
  f(A) & \xrightarrow{\gamma} & g(B) \\
  f(\alpha) \downarrow & & \downarrow g(\beta) \\
  f(A') & \xrightarrow{\gamma'} & g(B')
\end{array}
$$

In [36], we will present a homological result for operad groups which generalizes the results previously obtained for the subclass of local similarity groups in [30].
commutes. Composition is given by composing the components.

If $f$ is the inclusion of a subcategory, we write $A ↓ g$ for the comma category $f ↓ g$. Furthermore, if $A$ is just a subcategory with one object $A$ and its identity arrow, we write $A ↓ g$. In this case, the objects of the comma category are pairs $(B, \gamma)$ where $B$ is an object in $B$ and $\gamma: A \rightarrow g(B)$ is an arrow. An arrow from $(B, \gamma)$ to $(B', \gamma')$ is an arrow $\beta: B \rightarrow B'$ such that the triangle

\[ \gamma \]
\[ \downarrow \]
\[ A \]
\[ \gamma' \]
\[ \downarrow \]
\[ g(\beta) \]
\[ \downarrow \]
\[ g(B') \]

commutes. Of course, there are analogous abbreviations for the right factor.

2.1. The classifying space of a category. We assume that the reader is familiar with the basics of simplicial sets (see e.g. [20]). The nerve $N(C)$ of a category $C$ is a simplicial set defined as follows: A $k$-simplex is a sequence $A_0 \xrightarrow{\alpha_0} A_2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{k-1}} A_k$ of composable arrows. The $i$'th face map $d_i: N(C)_k \rightarrow N(C)_{k-1}$ is given by composing the arrows at the object $A_i$. When $i$ is 0 or $k$, then the object $A_i$ is removed from the sequence instead. The $i$'th degeneracy map $s_i: N(C)_k \rightarrow N(C)_{k+1}$ is given by inserting the identity at the object $A_i$.

The geometric realization $|N(C)|$ of $N(C)$ is a CW-complex which we call the classifying space $B(C)$ of $C$. See [37] for the reason why this is called a classifying space. If the category $C$ is a group, then $B(C)$ is the usual classifying space of the group which is defined as the unique space (up to homotopy equivalence) with fundamental group the given group and with higher homotopy groups vanishing.

Since we can view any category as a space via the above construction, any topological notion or concept can be transported to the world of categories. For example, if we say that the category $C$ is connected, then we mean that $B(C)$ is connected. Of course, one can easily think of an intrinsic definition of connectedness for categories and we will give some for other topological concepts below. But there are also concepts for which a combinatorial description is at least unknown, for example higher homotopy groups.

Transporting topological concepts to the category $\text{CAT}$ of (small) categories via the nerve functor can be made precise: The Thomason model structure on $\text{CAT}$ is a model structure Quillen equivalent to the usual model structure on $\text{SSET}$, the category of simplicial sets.

Every simplicial complex is homeomorphic to the classifying space of some category: A simplicial complex can be seen as a partially ordered set of simplices with the order relation given by the face relation. Moreover, a partially ordered set (poset) is just a category with at most one arrow between any two objects. The classifying space of such a poset is exactly the barycentric subdivision of the original simplicial complex.

Even more is true: McDuff showed in [26] that for each connected simplicial complex there is a monoid (i.e. a category with only one object) with classifying space homotopy equivalent to the given complex. Thus, every path-connected space has the weak homotopy type as the classifying space of some monoid. For example,
observe the monoid consisting of the identity element and elements $x_{ij}$ with multiplication rules $x_{ij}x_{kl} = x_{il}$. It’s classifying space has the homotopy type of the 2-sphere (see [14]).

2.2. The fundamental groupoid of a category. Following the philosophy of transporting topological concepts to categories via the nerve functor, we define the fundamental groupoid $\pi_1(C)$ of a category $C$ to be the fundamental groupoid of its nerve. There is also an intrinsic description of the fundamental groupoid of $C$ in terms of the category itself which we will describe now (see e.g. [20], Chapter III, Corollary 1.2) that these two notions are indeed the same). The objects of $\pi_1(C)$ are the objects of $C$ and the arrows of $\pi_1(C)$ are paths modulo homotopy. Here, a path in $C$ from an object $A$ to an object $B$ is a zig-zag of morphisms from $A$ to $B$, i.e. starting from $A$, one travels from object to object over the arrows of $C$, regardless of the direction of the arrows. For example, the following zig-zag is a path in $C$

$$A \leftarrow C_1 \rightarrow C_2 \leftarrow C_3 \leftarrow C_4 \rightarrow C_5 \rightarrow B$$

Paths can be concatenated in the obvious way. The homotopy relation on paths is the smallest equivalence relation respecting the operation of concatenation of paths generated by the following elementary relations:

$$\begin{align*}
   A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C & \sim A \overset{\alpha \beta}{\rightarrow} C \\
   A \overset{\alpha}{\rightarrow} B \overset{\beta}{\leftarrow} C & \sim A \overset{\beta \alpha}{\leftarrow} C \\
   A \overset{\alpha}{\rightarrow} B \overset{\alpha}{\leftarrow} A & \sim A \\
   A \overset{\alpha}{\leftarrow} B \overset{\alpha}{\rightarrow} A & \sim A \\
   A \overset{id}{\leftarrow} A \overset{id}{\rightarrow} A & \sim A
\end{align*}$$

where the $A$’s on the right represent the empty path at $A$. Composition in $\pi_1(C)$ is given by concatenating representatives. The identities are represented by the empty paths. If $A$ is an object of $C$ then we denote by $\pi_1(C, A)$ the automorphism group of $\pi_1(C)$ at $A$ and call it the fundamental group of $C$ at $A$.

The fundamental groupoid of $C$ has two further descriptions: First, denote by $G$ the left adjoint functor to the inclusion functor from groupoids to categories. Then we have $\pi_1(C) = G(C)$. Second, it is the localization $C[C^{-1}]$ of $C$ (at all its morphisms) since it comes with a canonical functor $\varphi: C \rightarrow \pi_1(C)$ satisfying the following universal property: Having any other functor $\eta: C \rightarrow A$ with the property that $\eta(f)$ is an isomorphism in $A$ for every arrow $f$ in $C$, then there is a unique functor $\epsilon: \pi_1(C) \rightarrow A$ such that $\varphi \epsilon = \eta$.

$$\begin{array}{ccc}
   C & \xrightarrow{\varphi} & \pi_1(C) \\
   \downarrow{\eta} & & \downarrow{\epsilon} \\
   A & &
\end{array}$$

2.3. Coverings of categories. Let $P: D \rightarrow C$ be a functor. We say that $P$ is a covering if for every arrow $a$ in $C$ and every object $X$ in $D$ which projects via $P$ onto the domain or the codomain of $a$, there exists exactly one arrow $b$ in $D$ with domain resp. codomain $X$ and projecting onto $a$ via $P$. In other words, arrows can be lifted uniquely provided that the lift of the domain or codomain is given. Of course, $P$ yields a map on the classifying spaces. To justify the definition of covering functor, we have the following.

**Proposition 2.1.** Let $P: D \rightarrow C$ be a functor. Then $P$ is a covering functor if and only if $BP: BD \rightarrow BC$ is a covering map of spaces.

Proof. By [13, Appendix I, 3.2], $BP = |NP|: |ND| \rightarrow |NC|$ is a covering map if and only if $NP: ND \rightarrow NC$ is a covering of simplicial sets as defined in [13] Appendix I, 2.1. This means that every n-simplex in $NC$ uniquely lifts to $ND$ provided that the lift of a vertex of the simplex is given. The lifting property for $P$ as defined above says that this is true for 1-simplices. So it is clear that $P$ is a covering functor provided that $BP$ is a covering map of spaces. For the reverse implication, one exploits special properties of nerves of categories. Not every simplicial set arises as the nerve of a category. The Segal condition gives a necessary and sufficient condition for a simplicial set to come from a category: Every horn $\Lambda^n_i$ as the nerve of a category. The Segal condition gives a necessary and sufficient conditions for these four claims:

- Contractibility and homotopy equivalences. We say that a category is contractible if its classifying space is contractible and we say that a functor $F: C \rightarrow D$ is a homotopy equivalence if $BF: BC \rightarrow BD$ is one. There are some standard conditions which assure that a category is contractible or a functor is a homotopy equivalence.

  A non-empty category $C$ is contractible if
  
  i) $C$ has an initial object.
  
  ii) $C$ has binary products.
  
  iii) $C$ is a poset, i.e. there is at most one arrow between any two objects, and there is an object $X_0$ together with a functor $F: C \rightarrow C$ such that for each object $X$ there exist arrows $X \rightarrow F(X) \leftarrow X_0$ (compare with [29, Subsection 1.5]).
  
  iv) $C$ is filtered which means that for every two objects $X, Y$ there is an object $Z$ with arrows $X \rightarrow Z, Y \rightarrow Z$ and for every two arrows $f, g: A \rightarrow B$ there is an arrow $h: B \rightarrow C$ such that $fh = gh$.

Of course, the dual statements are also true. It is instructive to sketch the arguments for these four claims:

i) Let $I$ be the category with two objects and one non-identity arrow from the first to the second object. The classifying space of $I$ is the unit interval $I$. A natural transformation of two functors $f, g: C \rightarrow D$ can be interpreted as a functor $C \times I \rightarrow D$. On the level of spaces this gives a homotopy $BC \times I \rightarrow BD$. If $C$ is a category with initial object $X_0$, then there is a unique natural transformation from the functor const$_{X_0}$ (sending every arrow of $C$ to id$_{X_0}$) to the identity functor id$_C$. On the level of spaces, this
yields a homotopy between $\text{id}_{BC}$ and the constant map $BC \to BC$ with value the point $X_0$.

ii) Choose an object $X_0$ in $\mathcal{C}$. Let $F: \mathcal{C} \to \mathcal{C}$ be the functor $Y \mapsto X_0 \times Y$. Let $F_0: \mathcal{C} \to \mathcal{C}$ be the constant functor $Y \mapsto X_0$. Projection onto the first factor yields a natural transformation $F \to F_0$ and projection onto the second factor yields a natural transformation $F \to \text{id}_\mathcal{C}$. This gives two homotopies which together give the desired contraction of $BC$.

iii) First note that, if $F, G: \mathcal{C} \to \mathcal{C}$ are two functors with the property that there is an arrow $F(X) \to G(X)$ for each object $X$, then this already defines a natural transformation $F \to G$ by uniqueness of arrows in the poset. Now the conditions on $X_0$ and $F$ yield that there are natural transformations $\text{id}_\mathcal{C} \to F$ and $\text{const}_{X_0} \to F$. On the level of spaces this gives the desired contraction of $BC$.

iv) First, let $\mathcal{D}$ be a finite subcategory of $\mathcal{C}$. We claim that there exists a cocone over $\mathcal{D}$ in $\mathcal{C}$, i.e. there is an object $Z$ in $\mathcal{C}$ and for each object $Y$ in $\mathcal{D}$ an arrow $Y \to Z$ which commute with the arrows in $\mathcal{D}$. The subcategory $\mathcal{D}$ together with $Z$ and the arrows to $Z$ then forms a subcategory which is contractable since $Z$ is a terminal object. A cocone can be constructed as follows: First pick two objects $Y_1, Y_2$ in $\mathcal{D}$ and find an object $Z'$ with arrows $Y_1 \to Z'$ and $Y_2 \to Z'$. Pick another object $Y_3$ and find an object $Z''$ with arrows $Y_3 \to Z''$ and $Z' \to Z''$. Repeating this with all objects of $\mathcal{D}$, we obtain an object $Q$ together with arrows $f_Y: Y \to Q$ for every object $Y$ in $\mathcal{D}$. The $f_Y$ probably won't commute with the arrows in $\mathcal{D}$ yet, but we can repair this by repeatedly applying the second property of filteredness. Pick an arrow $d: Y \to Y'$ in $\mathcal{D}$ and observe the parallel arrows $df_Y$ and $f_Y$. Apply the second property to find an arrow $\omega: Q \to Q'$ with $df_Y \omega = f_Y \omega$. Replace $Q$ by $Q'$ and all the arrows $f_D$ for objects $D$ of $\mathcal{D}$ by $f_D \omega$. Repeat this with all the other arrows in $\mathcal{D}$.

Now to finish the proof of this item, take a map $S^n \to BC$. Since $S^n$ is compact, it can be homotoped to a map such that the image is covered by the geometric realization of a finite subcategory. The cocone over this subcategory then gives the desired null-homotopy. To obtain the first homotopy, one could use Kan’s fibrant replacement functor $\text{Ex}^\infty$ and argue as follows: Since $\text{Ex}^\infty(\mathcal{C})$ is fibrant and weakly homotopy equivalent to $\mathcal{C}$, an element in $\pi_n(BC) = \pi_n(\mathcal{C})$ can be represented by a morphism of simplicial sets $\partial \Delta_n \to \text{Ex}^\infty(\mathcal{C})$. Since $\partial \Delta_n$ has only finitely many non-degenerate simplices, this map factors through $\text{Ex}^k(\mathcal{C})$ for some $k$ and we obtain a map $\partial \Delta_n \to \text{Ex}^k(\mathcal{C})$. By adjointness, this gives a morphism $\text{Sd}^k(\partial \Delta_n) \to \mathcal{C}$ where $\text{Sd}$ is the barycentric subdivision functor. This representative can now be null-homotoped in a cocone as above.

We recall Quillen’s famous Theorem A from [28] which gives a sufficient but in general not necessary condition for a functor to be a homotopy equivalence.

**Theorem 2.2.** Let $f: \mathcal{C} \to \mathcal{D}$ be a functor. If for each object $Y$ in $\mathcal{D}$ the category $Y \downarrow f$ is contractible, then the functor $f$ is a homotopy equivalence. Similarly, if the category $f \downarrow Y$ is contractible for each object $Y$ in $\mathcal{D}$, then $f$ is a homotopy equivalence.

**Remark 2.3.** When applying this theorem to an inclusion $f: \mathcal{A} \to \mathcal{B}$ of a full subcategory, it suffices to check $Y \downarrow f = Y \downarrow \mathcal{A}$ for objects $Y$ not in $\mathcal{A}$. If $Y$ is an object in $\mathcal{A}$, the comma category $Y \downarrow \mathcal{A}$ has the object $(Y, \text{id}_Y)$ as initial object and thus is automatically contractible. Similar remarks apply to the comma categories $f \downarrow Y = \mathcal{A} \downarrow Y$.
Remark 2.4. If \( D \) is a groupoid, then for \( Y, Y' \in D \) the comma categories \( Y \downarrow f \) and \( Y' \downarrow f \) are isomorphic. Thus one has to check contractibility only for one \( Y \). The same remarks apply to the comma categories \( f \downarrow Y \).

2.5. Calculus of fractions and cancellation properties. The next definition is very classical and due to Gabriel and Zisman [18].

Definition 2.5. Let \( C \) be a category. It satisfies the calculus of fractions if the following two conditions are satisfied:

- (Square filling) For every pair of arrows \( f : B \to A \) and \( g : C \to A \) there are arrows \( a : D \to B \) and \( b : D \to C \) such that \( af = bg \).

\[
\begin{array}{ccc}
D & \xrightarrow{a} & B \\
\downarrow b & & \downarrow f \\
C & \xrightarrow{g} & A \\
\end{array}
\]

- (Equalization) Whenever we have arrows \( f, g : A \to B \) and \( a : B \to C \) such that \( fa = ga \) then there exists an arrow \( b : D \to A \) with \( bf = bg \).

\[
\begin{array}{ccc}
D & \xrightarrow{b} & A \\
\downarrow f & & \downarrow g \\
B & \xrightarrow{a} & C \\
\end{array}
\]

More precisely, this is called the right calculus of fractions. There is also a dual left calculus of fractions. Since we are mainly interested in the right calculus of fractions, we omit the word “right”.

Remark 2.6. The existence of binary pullbacks in \( C \) trivially implies the square filling property but it also implies the equalization property [1 Lemma 1.2]. So a category with binary pullbacks satisfies the calculus of fractions.

The calculus of fractions has positive effects on the complexity of the fundamental groupoid \( \pi_1(C) \): One can show (see e.g. [18] or [1]) that each class in \( \pi_1(C) \) can be represented by a span, which is a zig-zag of the form

and two spans

are homotopic if and only if the diagram can be filled in the following way:

In other words, the elements in the localization can be described as fractions and this explains the name of the calculus of fractions. We will frequently write \((\alpha, \beta)\) for a span consisting of arrows \(\alpha\) and \(\beta\) where the first arrow \(\alpha\) points to the left (i.e. is the denominator) and the second arrow \(\beta\) points to the right (i.e. is the numerator).
nominator). Two spans are composed by concatenating representatives to a zig-zag and then transforming the zig-zag to a span by choosing a square filling of the middle cospan.

\[
\begin{array}{c}
\bullet \\
\rightarrow \Downarrow \rightarrow
\end{array}
\]

The canonical functor \( \varphi : C \to \pi_1(C) \) is given by sending an arrow \( \alpha \) to the class represented by the span

\[
\begin{array}{c}
\bullet \\
\Downarrow \rightarrow
\end{array}
\]

Using the special form of the homotopy relation from above, we see that two arrows \( \alpha, \beta : X \to Y \) are homotopic (i.e. the images under the canonical functor are homotopic) if and only if there is an arrow \( \omega : A \to X \) such that \( \omega \alpha = \omega \beta \).

**Theorem 2.7.** If \( C \) is a category satisfying the calculus of fractions, then the canonical functor \( \varphi : C \to \pi_1(C) \) is a homotopy equivalence.

**Proof.** A proof can be found in [10, Section 7]. \( \square \)

Note that a connected groupoid is equivalent, as a category, to any of its automorphism groups. Since groups are aspherical, so are all connected groupoids. In particular, \( \pi_1(C) \) is aspherical if \( C \) is connected. We thus get the following corollary from Theorem 2.7.

**Corollary 2.8.** If \( C \) is a connected category satisfying the calculus of fractions, then \( C \) is aspherical.

We now turn to cancellation properties in categories.

**Definition 2.9.** Let \( C \) be a category. It is called right cancellative if \( fa = ga \) for arrows \( f, g, a \) implies \( f = g \). It is called left cancellative if \( af = ag \) implies \( f = g \). It is called cancellative if it is left and right cancellative.

**Remark 2.10.** Note that we have the following implications:

right cancellation \( \implies \) equalization

equalization + left cancellation \( \implies \) right cancellation

In the presence of cancellation properties, we can give a much simpler proof of Theorem 2.7.

**Proposition 2.11.** Let \( C \) be a category satisfying the cancellative calculus of fractions. Then the canonical functor \( \varphi : C \to \pi_1(C) \) is faithful and a homotopy equivalence. As a consequence of the latter statement, if \( C \) is assumed to be connected, we obtain that \( C \) is aspherical.

**Proof.** Injectivity is easy: Let \( f, g \) be arrows in \( C \) which are mapped to the same arrow in \( \pi_1(C) \). This means that \( f, g \) are homotopic. But as pointed out above, this implies that there is an arrow \( \omega \) with \( \omega f = \omega g \). By the cancellation property, it follows that \( f = g \).

For showing that the functor is a homotopy equivalence, we apply Quillen’s Theorem A (Theorem 2.2) to the functor \( \varphi : C \to \pi_1(C) \). Let \( X \) be an object in \( \pi_1(C) \), i.e. an object in \( C \). We have to check that the comma category \( X_{\downarrow} \phi \) is contractible. Note that this is the universal covering category of the component of \( C \) containing \( X \). First we claim that this category is a poset: Let \( (Z, a) \) and \( (Z', a') \) be objects in \( X_{\downarrow} \phi \) and \( \gamma_1, \gamma_2 \) be two arrows from \( (Z, a) \) to \( (Z', a') \). This means that \( Z, Z' \) are objects in \( C \), \( a : X \to Z \circ \phi \) and \( a' : X \to Z' \circ \phi \) are arrows in \( \pi_1(C) \).
and \( \gamma_i : Z \to Z' \) are arrows in \( \mathcal{C} \) such that \( a \circ (\gamma_1 \circ \varphi) = a' = a \circ (\gamma_2 \circ \varphi) \) in \( \pi_1(\mathcal{C}) \). It follows \( \gamma_1 \circ \varphi = \gamma_2 \circ \varphi \) and therefore \( \gamma_1 = \gamma_2 \) by injectivity.

Now we want to show that this poset is cofiltered. Then we can apply item iv) of Subsection 2.4. So we have to show that for each two objects \( A, A' \) in \( \pi_1(\mathcal{C}) \) there is another object \( B \) and arrows \( B \to A \) and \( B \to A' \). Let \( A = (Z, a) \) and \( A' = (Z', a') \) with arrows \( a : X \to Z \varphi \) and \( a' : X \to Z' \varphi \) which can be represented by spans \((a, \beta)\) and \((a', \beta')\) respectively. Choose a square filling \((\gamma, \gamma')\) of the cospan \((\alpha, \alpha')\).

Then the arrow \( \omega := \gamma \alpha = \gamma' \alpha' \) can be interpreted as the denominator of a span representing an arrow in \( \pi_1(\mathcal{C}) \) which we denote by \( \omega^{-1} \). Furthermore, since \( \varphi : \mathcal{C} \to \pi_1(\mathcal{C}) \) is the identity on objects, we can write \( Y = Y \varphi \). Thus we can define the object \( B := (Y, \omega^{-1}) \) in \( X \downarrow \varphi \). Finally, the arrows \( \gamma \beta \) and \( \gamma' \beta' \) give arrows \( B \to A \) and \( B \to A' \) respectively. \( \square \)

2.6. Monoidal categories and string diagrams. We assume that the reader is acquainted with the definition of monoidal categories, symmetric monoidal categories and braided monoidal categories (see e.g. [25]). In the following, we will always assume the strict versions, i.e. the associator, right and left unitor are identities. Moreover, we will sometimes call a monoidal category planar in order to stress that it’s neither symmetric nor braided.

Joyal and Street introduced the notion of braided monoidal categories in [23]. It is designed such that the braided monoidal category freely generated by a single object is the groupoid with components the braid groups \( B_n \). More precisely, we have an object for each natural number \( n \), there are no morphisms \( n \to m \) with \( n \neq m \) and \( \text{Hom}(n, n) = B_n \). More generally, they introduced the braided monoidal category \( \mathcal{B}\text{ri} \mathcal{M} \mathcal{O} (\mathcal{C}) \) freely generated by another category \( \mathcal{C} \) [23] p. 37]: The objects are free words in the objects of \( \mathcal{C} \), i.e. finite sequences of objects of \( \mathcal{C} \). A morphism consists of a braid \( \beta \in B_n \) where the strands are labelled by morphisms \( \alpha_i : A_i \to B_i \) of \( \mathcal{C} \) with \( i \in \{1, ..., n\} \), yielding an arrow

\[
(\beta, \alpha_1, ..., \alpha_n) : (A_1, ..., A_n) \to (B_{1 \circ \beta^{-1}}, ..., B_{n \circ \beta^{-1}})
\]

in \( \mathcal{B}\text{ri} \mathcal{M} \mathcal{O}(\mathcal{C}) \). Composition is performed by composing the braids and applying composition in \( \mathcal{C} \) to every strand. The tensor product is given by juxtaposition. A set \( \mathcal{C} \) can be viewed as a discrete category, so we also obtain the notion of a braided monoidal category \( \mathcal{B}\text{ri} \mathcal{M} \mathcal{O}(\mathcal{C}) \) freely generated by a set. The arrows are just braids with strands labelled by the elements of \( \mathcal{C} \), i.e. are colored.

The same remarks apply to the symmetric version. In particular, a category \( \mathcal{C} \) freely generates a symmetric monoidal category \( \mathfrak{S}\text{ym}(\mathcal{C}) \).
Even simpler, we can form the free monoidal category \( \mathbf{Mon}(C) \) generated by a category \( C \). The strands are decorated by arrows in \( C \) but they are not allowed to braid or cross each other.

If \( C \) is a (symmetric/braided) monoidal category, then there is exactly one tensor structure on \( \pi_1(C) \) making it into a (symmetric/braided) monoidal category and such that the canonical functor \( \varphi : C \to \pi_1(C) \) (strictly) respects that structure, i.e.

\[
\begin{align*}
\varphi(X \otimes Y) &= \varphi(X) \otimes \varphi(Y) \\
\varphi(\alpha \otimes \beta) &= \varphi(\alpha) \otimes \varphi(\beta) \\
\varphi(I) &= I \\
\varphi(\gamma_{X,Y}) &= \gamma_{\varphi(X),\varphi(Y)}
\end{align*}
\]

for objects \( X, Y \) and arrows \( \alpha, \beta \). Here, \( I \) denotes the unit object and \( \gamma_{X,Y} : X \otimes Y \to Y \otimes X \) the symmetry resp. the braiding. The tensor product on the level of arrows can be constructed as follows: Let one arrow be represented by the zig-zag

\[
A_1 \xleftarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} A_k
\]

and the other arrow by the zig-zag

\[
B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} B_l
\]

Then the tensor product may be represented by the zig-zag

\[
\begin{array}{cccccccccccc}
A_1 & \xleftarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_k} & A_k \\
\otimes & \otimes & \otimes & \cdots & \otimes & \otimes & \otimes \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_l} & B_l
\end{array}
\]

We now review (in an informal manner) the notion of string diagrams for (symmetric/braided) monoidal categories. They were first used by Roger Penrose and later formally introduced by Joyal and Street [24] (see also [31] for a survey on string diagrams for monoidal categories of various sorts).

In a category \( C \), objects \( X \) can be visualized as labelled strings and arrows \( f : X \to Y \) can be visualized as labelled nodes or boxes between strings. Composition \( f \circ g \) with \( f : X \to Y \) and \( g : Y \to Z \) is visualized by connecting the strings of two boxes.

\[
\begin{array}{c}
X \\
\xrightarrow{f} \\
Y
\end{array}
\]

\[
\begin{array}{c}
X \\
\xrightarrow{f} \\
Y \\
\xleftarrow{g} \\
Z
\end{array}
\]

In order to model composition correctly, we have to identify the following string diagrams:

\[
\begin{array}{c}
X \\
\xrightarrow{f} \\
Y \\
\xrightarrow{g} \\
Z = \begin{array}{c}
X \\
\xrightarrow{fg} \\
Z
\end{array}
\end{array}
\]
A box labelled with the identity morphism of an object $X$ is identified with a string labelled $X$.

If $C$ is monoidal, the tensor functor $\otimes$ is visualized by writing strings and boxes side by side. For example, the following diagrams represent the tensor product $X \otimes Y$ of objects $X, Y$ and the tensor product $f \otimes g : X \otimes A \to Y \otimes B$ of morphisms $f : X \to Y$ and $g : A \to B$.

A morphism of the form $f : X_1 \otimes ... \otimes X_k \to Y_1 \otimes ... \otimes Y_l$ is visualized by a box with $k$ entering strings labelled $X_1, ..., X_k$ and $l$ outgoing strings labelled $Y_1, ..., Y_l$. For example, the following diagram represents an arrow $f : X \otimes A \to Y \otimes B$.

Just as in the case of composition, we have to make identifications in order to model the tensor operation correctly:

Strings labelled with the unit object $I$ are identified with empty diagrams.

If $C$ is braided, we have the natural braiding isomorphisms $\gamma_{X,Y} : X \otimes Y \to Y \otimes X$. These are visualized by the diagrams

If $C$ is symmetric, $\gamma_{X,Y}$ takes the form

The crucial observation for this graphical language is the coherence between it and the axioms of monoidal categories (compare with \cite[Theorems 3.1, 3.7 and 3.12]{...}).
Theorem 2.12. A well-formed equation between morphism terms in the language of planar resp. braided resp. symmetric monoidal categories follows from the axioms of planar resp. braided resp. symmetric monoidal categories if and only if it holds, up to isotopy in 2 resp. 3 resp. 4 dimensions, in the graphical language.

These dimensions also reflect the position of planar resp. braided resp. symmetric monoidal categories in the periodic table of higher categories: Planar monoidal categories are 2-categories with one object, braided monoidal categories are 3-categories with one object and one arrow, symmetric monoidal categories are 4-categories with one object, one arrow and one 2-cell.

Above, we have seen that the fundamental groupoid $\pi_1(C)$ of a monoidal category $C$ of some type inherits the structure of a monoidal category of the same type. To model this fundamental groupoid via string diagrams, we also allow to connect outgoing strings of boxes to outgoing strings of other boxes and also ingoing strings to ingoing strings.

\[
\begin{array}{c}
X & \xymatrix{ & f \ar@{<-}[r] & \ar@{<->}[l] Y & g \ar@{<-}[l] & Z} \\
& X & \xymatrix{ & f \ar@{<->}[r] & \ar@{<-}[l] Y & g \ar@{<-}[r] & \ar@{<->}[l] Z} \\
\end{array}
\]

The homotopy relations

\[
A \xrightarrow{\alpha} B \xrightarrow{\beta} C \sim A \xrightarrow{\alpha\beta} C
\]
\[
A \xleftarrow{\alpha} B \xleftarrow{\beta} C \sim A \xleftarrow{\beta\alpha} C
\]
\[
A \xleftarrow{id} A \sim A
\]
\[
A \xrightarrow{id} A \sim A
\]

already become identities in the graphical language. To correctly model the homotopy relations

\[
A \xrightarrow{\alpha} B \xleftarrow{\alpha} A \sim A
\]
\[
A \xleftarrow{\alpha} B \xrightarrow{\alpha} A \sim A
\]

we introduce identifications for string diagrams as follows:

\[
\begin{array}{c}
\begin{array}{c}
A & \xrightarrow{\alpha} B & \xleftarrow{\alpha} A \\
\end{array}
\end{array}
= A
\]
\[
\begin{array}{c}
\begin{array}{c}
A & \xleftarrow{\alpha} B & \xrightarrow{\alpha} A \\
\end{array}
\end{array}
= A
\]

We call the left hand sides dipoles which cancel each other.

With these modifications, we obtain the same coherence Theorem 2.12 for the fundamental groupoids of (symmetric/braided) monoidal categories. This will help us later to handle paths up to homotopy more easily and will establish a connection of operad groups with diagram groups.

2.7. Cones and joins. Let $C, D$ be two categories. We define the join $C \ast D$. The set of objects of $C \ast D$ is the disjoint union of the objects of $C$ and $D$. The set of arrows is the disjoint union of the arrows of $C$ and $D$ together with exactly one arrow $C \to D$ for each pair $(C, D)$ of objects $C$ of $C$ and $D$ of $D$. The composition rules are
the unique ones extending the compositions in $\mathcal{C}$ and $\mathcal{D}$. The classifying space of the join is homotopy equivalent to the join of the classifying spaces $B(\mathcal{C} \star \mathcal{D}) \simeq B\mathcal{C} \star B\mathcal{D}$.

Now we define the cone over a category. The objects of $\text{Cone}(\mathcal{C})$ are the objects of $\mathcal{C}$ plus another object called $\text{tip}$. The arrows are the arrows of $\mathcal{C}$ together with exactly one arrow from $\text{tip}$ to every object in $\mathcal{C}$. Dually, there is a $\text{Cocone}(\mathcal{C})$ over $\mathcal{C}$. In the cocone, the extra arrows go from the objects of $\mathcal{C}$ to the extra object $\text{tip}$. Last but not least, when we have a join $\mathcal{C} \star \mathcal{D}$ of two categories, there is a mixed version $\text{Cone}(\mathcal{C} \star \mathcal{D})$ which we call the cocone over the join. Again, there is one extra object $\text{tip}$ and for every object in $\mathcal{C} \star \mathcal{D}$ we have an extra arrow. When we have an object in $\mathcal{C}$, the extra arrow goes to $\text{tip}$. When we have an arrow in $\mathcal{D}$, the extra arrow comes from $\text{tip}$. The composition of an arrow $C \rightarrow \text{tip}$ with an arrow $\text{tip} \rightarrow D$ is the unique arrow $C \rightarrow D$ from the definition of the join. All three coning versions give the usual coning on the topological level:

$$
B(\text{Cone}(\mathcal{C})) \cong \text{Cone}(B(\mathcal{C}))
$$

$$
B(\text{Cocone}(\mathcal{C})) \cong \text{Cone}(B(\mathcal{C}))
$$

$$
B(\text{Cone}(\mathcal{C} \star \mathcal{D})) \cong \text{Cone}(B(\mathcal{C} \star \mathcal{D}))
$$

The join of two spaces $X$ and $Y$ is defined to be the homotopy pushout of the two projections $X \leftarrow X \times Y \rightarrow Y$. Thus, it is defined only up to homotopy and there is some freedom to choose models of a join. Indeed, there is another construction giving the join of two categories. For this, we need to recall the Grothendieck construction: Let $\mathcal{J}$ be some indexing category and $F : \mathcal{J} \rightarrow \text{CAT}$ a diagram in $\text{CAT}$. The objects of the Grothendieck construction $\int F$ are pairs $(j, X)$ of objects $j$ in $\mathcal{J}$ and $X$ in $\text{Pr}_\mathcal{J} F$. An arrow from $(j, X)$ to $(j', X')$ is a pair $(f, \alpha)$ consisting of an arrow $f : j \rightarrow j'$ and an arrow $\alpha : X \triangleright (f \triangleright X)$ of the underlying category $\text{Pr}_\mathcal{J} F$. Composition is given by $(f, \alpha) \circ (f', \alpha') := (f \circ f', \alpha \circ (f \triangleright \alpha')$. In [33] it is shown that there is a model structure on $\text{CAT}$ Quillen equivalent to $\text{SSET}$, nowadays called the Thomason model structure, and in [34] it is shown that the nerve of the Grothendieck construction $\int F$ is homotopy equivalent to the homotopy pushout of the diagram $F \star N$ which is obtained from the diagram $F$ by applying the nerve functor. In fact, $\int F$ realizes the homotopy pushout of $F$ with respect to the Thomason model structure on $\text{CAT}$ [16, Section 3].

Now let $\mathcal{C}, \mathcal{D}$ be categories. We call the Grothendieck construction of the diagram

$$
\mathcal{C} \xleftarrow{pr_\mathcal{C}} \mathcal{C} \times \mathcal{D} \xrightarrow{pr_\mathcal{D}} \mathcal{D}
$$

the Grothendieck join of $\mathcal{C}$ and $\mathcal{D}$ and denote it by $\mathcal{C} \circ \mathcal{D}$. From [34] we know that $B(\mathcal{C} \circ \mathcal{D})$ is homotopy equivalent to the homotopy pushout of the diagram

$$
B\mathcal{C} \xleftarrow{pr_{B\mathcal{C}}} B\mathcal{C} \times B\mathcal{D} \xrightarrow{pr_{B\mathcal{D}}} B\mathcal{D}
$$

But the latter is the join $B\mathcal{C} \star B\mathcal{D}$ by definition. So we have $B(\mathcal{C} \circ \mathcal{D}) \simeq B\mathcal{C} \star B\mathcal{D}$.

One can show that the Grothendieck join is associative and thus we can write $\mathcal{C}_1 \circ \ldots \circ \mathcal{C}_k$ for a finite collection $\mathcal{C}_i$ of categories. The objects of such an iterated Grothendieck join are elements of the set

$$
\text{obj}(\mathcal{C}_1 \circ \ldots \circ \mathcal{C}_k) = \prod_{S \subseteq \{1, \ldots, k\}} \prod_{s \in S} \text{obj}(\mathcal{C}_s)
$$

Whenever we have $S \subseteq T \subseteq \{1, \ldots, k\}$, objects $(Y_t)_{t \in T}$ and $(X_s)_{s \in S}$ and arrows $\alpha_s : Y_s \rightarrow X_s$ in $\mathcal{C}_s$ for each $s \in S$, then there is an arrow

$$
(\alpha_s)_{s \in S} : (Y_t)_{t \in T} \rightarrow (X_s)_{s \in S}
$$

For $R \subseteq S \subseteq T$ the composition is given by

$$
(\alpha_s)_{s \in S} \circ (\beta_r)_{r \in R} = (\alpha_r \circ \beta_r)_{r \in R}
$$
There is also a dual notion of the Grothendieck join which we define as
\[ C \bullet D := (C^{op} \circ D^{op})^{op} \]
Since \( B(A^{op}) = B(A) \) for any category, we still have \( B(C \bullet D) \simeq BC \circ BD \). Furthermore, it is still associative, so that we can write \( C_1 \bullet \ldots \bullet C_k \) for a finite collection \( C_i \) of categories. The objects of such an iterated dual Grothendieck join are elements of the set
\[ \text{obj}(C_1 \bullet \ldots \bullet C_k) = \prod_{S \subset \{1, \ldots, k\}} \prod_{s \in S} \text{obj}(C_s) \]
Whenever we have \( S \subset T \subset \{1, \ldots, k\} \), objects \( (X_s)_{s \in S} \) and \( (Y_t)_{t \in T} \) and arrows \( \alpha_s: X_s \to Y_s \) in \( C_s \) for each \( s \in S \), then there is an arrow
\[ (\alpha_s)_{s \in S}: (X_s)_{s \in S} \to (Y_t)_{t \in T} \]
For \( R \subset S \subset T \) the composition is given by
\[ (\beta_r)_{r \in R} \ast (\alpha_s)_{s \in S} = (\beta_r \ast \alpha_r)_{r \in R} \]

3. OPERAD GROUPS

We now come to the main objects of this article.

**Definition 3.1.** An operad \( O \) consists of a set of colors \( C \) and sets of operations \( O(a_1, \ldots, a_n; b) \) for each finite ordered sequence \( a_1, \ldots, a_n, b \) of colors in \( C \) (the \( a_i \) are the input colors and \( b \) is the output color) with \( n \geq 1 \) (allowing operations with no inputs is possible, but we won’t consider such operads). See Figure 1 for a visualization of operations. There are composition maps (Figure 1)
\[ O(c_{11}, \ldots, c_{1k_1}; a_1) \times \ldots \times O(c_{nk_n}; a_n) \times O(a_1, \ldots, a_n; b) \]
denoted by \( (\phi_1, \ldots, \phi_n, \theta) \mapsto (\phi_1, \ldots, \phi_n) \ast \theta \). Composition is associative (Figure 2):
\[ (\psi_{11}, \ldots, \psi_{1k_1}) \ast (\phi_1, \ldots, \phi_n) \ast \theta \]

\[ (\psi_{11}, \ldots, \psi_{1k_1}, \psi_{21}, \ldots, \psi_{nk_n}) \ast ((\phi_1, \ldots, \phi_n) \ast \theta) \]
For each color \( a \) there are distinguished unit elements \( 1_a \in O(a; a) \) such that
\[ (1_{a_1}, \ldots, 1_{a_n}) \ast \theta = \theta = \theta \ast 1_b \]
for each operation \( \theta \). Sometimes we call such an operad planar in order to distinguish it from the symmetric or braided versions below.

A symmetric/braided operad comes with additional maps (Figure 3)
\[ x \cdot_\nu: O(a_1, \ldots, a_n; b) \to O(a_{1d}, \ldots, a_{nd}; b) \]
for each \( x \in S_n \) in the symmetric group \( S_n \) or in the braid group \( B_n \) respectively. Here, \( \nu \cdot x \) for \( x \in S_n \) means plugging the element \( i \) into the permutation \( x \) which is considered as a bijection of the set \( \{1, \ldots, n\} \). There is a canonical projection \( B_n \to S_n \), so this makes sense also in the braided case. These maps are assumed to be actions:
\[ x \cdot (y \cdot \theta) = (xy) \cdot \theta \quad 1 \cdot \theta = \theta \]
They also have to be equivariant with respect to composition (Figures 4 and 5):
\[ (\phi_{1d}, \ldots, \phi_{nd}) \ast (x \cdot \theta) = \bar{x} \cdot ((\phi_1, \ldots, \phi_n) \ast \theta) \]
\[ (y_1 \cdot \phi_1, \ldots, y_n \cdot \phi_n) \ast \theta = (y_1, \ldots, y_n) \cdot ((\phi_1, \ldots, \phi_n) \ast \theta) \]
Here, $\bar{x}$ is obtained from $x$ by replacing the $i$'th strand of $x$ by $n_i$ strands and $n_i$ is the number of inputs of $\phi_{i\triangleright x}$. Furthermore, $(y_1, ..., y_n)$ is the concatenation of the permutations resp. braidings $y_i$.

**Figure 1.** Visualization of an operation and composition of operations.

**Figure 2.** Associativity.

**Figure 3.** Action of the braid groups on the operations.

**Remark 3.2.** There is an equivalent way of writing the composition, namely with so-called partial compositions. The $i$'th partial compositions

$$s_i : \mathcal{O}(c_1, ..., c_k; a_i) \times \mathcal{O}(a_1, ..., a_n; b) \to \mathcal{O}(a_1, ..., a_{i-1}, c_1, ..., c_k, a_{i+1}, ..., a_n; b)$$
are defined as
\[ \phi \ast_i \theta := (\text{id}_{a_1}, \ldots, \text{id}_{a_{i-1}}, \phi, \text{id}_{a_{i+1}}, \ldots, \text{id}_{a_n}) \ast \theta \]
Conversely, one could define operads via partial compositions and reobtain the usual composition by a product of several partial compositions.

The planar operads, symmetric operads and braided operads can be organized into categories \( \textsc{op}, \textsc{sym.\,op} \) and \( \textsc{bra.\,op} \) respectively. Denote by \( \textsc{mon}, \textsc{sym.\,mon} \) and \( \textsc{bra.\,mon} \) the categories of monoidal categories, symmetric monoidal categories and
braided monoidal categories respectively. There are functors

\[
\begin{align*}
\text{End}: \text{MON} & \longrightarrow \text{OP} \\
\text{End}: \text{SYM.MON} & \longrightarrow \text{SYM.OP} \\
\text{End}: \text{BRA.MON} & \longrightarrow \text{BRA.OP}
\end{align*}
\]

assigning to each (symmetric/braided) monoidal category \( \mathcal{C} \) an operad \( \text{End}(\mathcal{C}) \), called the endomorphism operad. The colors of \( \text{End}(\mathcal{C}) \) are the objects of \( \mathcal{C} \) and the sets of operations are given by

\[
\text{End}(\mathcal{C})(a_1, \ldots, a_n; b) = \text{Hom}_\mathcal{C}(a_1 \otimes \ldots \otimes a_n, b)
\]

Composition in \( \text{End}(\mathcal{C}) \) is induced by the composition in \( \mathcal{C} \) in the obvious way. The unit element in \( \text{End}(\mathcal{C})(a; a) \) is the identity \( \text{id}_a: a \rightarrow a \) in \( \mathcal{C} \). In the symmetric or braided case, \( \mathcal{C} \) comes with additional natural isomorphisms \( \gamma_{X,Y}: X \otimes Y \rightarrow Y \otimes X \).

These can be used to define the action of the symmetric resp. braid groups on the sets of operations. In the theory of operads, these endomorphism operads play an important role since morphisms of operads

\[
\mathcal{O} \rightarrow \text{End}(\mathcal{C})
\]

are representations of or algebras over the operad \( \mathcal{O} \).

The functors \( \text{End} \) have left adjoints

\[
\begin{align*}
\mathcal{S}: \text{OP} & \longrightarrow \text{MON} \\
\mathcal{S}: \text{SYM.OP} & \longrightarrow \text{SYM.MON} \\
\mathcal{S}: \text{BRA.OP} & \longrightarrow \text{BRA.MON}
\end{align*}
\]

The (symmetric/braided) monoidal category \( \mathcal{S}(\mathcal{O}) \) is called the category of operators. We will define these categories explicitly. We start with the planar case and then use it to define the braided case. The symmetric case is then similar to the braided case.

So let \( \mathcal{O} \) be a planar operad with a set of colors \( \mathcal{C} \). The objects of \( \mathcal{S}(\mathcal{O}) \) are free words in the colors, i.e. finite sequences of colors in \( \mathcal{C} \). An arrow in \( \mathcal{S}(\mathcal{O}) \) is a finite sequence of operations in \( \mathcal{O} \). If \( X_1, \ldots, X_n \) are operations in \( \mathcal{O} \), the (ordered) input colors of \( X_i \) are \( (c^1_i, \ldots, c^k_i) \) and the output color of \( X_i \) is \( d_i \), then the \( X_i \) give an arrow

\[
(X_1, \ldots, X_n): (c^1_1, \ldots, c^k_1, c^1_2, \ldots, c^k_n) \rightarrow (d_1, \ldots, d_n)
\]

of \( \mathcal{S}(\mathcal{O}) \). Composition is induced by the composition in the operad \( \mathcal{O} \) and the identities are given by the identity operations in \( \mathcal{O} \). The tensor product is given by juxtaposition.
Now let $O$ be a braided operad with set of colors $C$. By forgetting the action of the braid groups, we get a planar operad $O_{pl}$. The braided monoidal category $S(O)$ is a certain product $\mathbf{Braid}(C) \boxtimes S(O_{pl})$. The objects of $S(O)$ are once more finite sequences of colors in $C$. Arrows in $S(O)$ are equivalence classes of pairs $(\beta, X) \in \mathbf{Braid}(C) \times S(O_{pl})$ consisting of a $C$-colored braid $\beta$ and a sequence $X = (X_1, \ldots, X_n)$ of operations of $O$ where the codomain of $\beta$ equals the domain of $X$ (Figure 6). The equivalence relation on such pairs is the following: Let $(\beta, X)$ be such a pair with $X = (X_1, \ldots, X_n)$. For each $i = 1, \ldots, n$ let $\sigma_i$ be a $C$-colored braid such that $\sigma_i \cdot X_i$ is defined. Let $\sigma := \sigma_1 \otimes \ldots \otimes \sigma_n$ be the concatenation of the $\sigma_i$ and for such a $\sigma$ define

$$\sigma \cdot (\beta, X) := (\beta \ast \sigma^{-1}, (\sigma_1 \cdot X_1, \ldots, \sigma_n \cdot X_n))$$
We require \((\beta, X)\) and \((\beta', X')\) to be equivalent if there exists a \(\sigma\) as above such that \((\beta', X') = \sigma \cdot (\beta, X)\). Roughly speaking, parts of the braid \(\beta\) can be used to act on the operations \(X_i\) or, conversely, braids acting on the operations \(X_i\) can be merged into \(\beta\). This is visualized in Figure 7.

Composition in \(S(O)\) is defined on representatives \((\beta, X)\) and \((\delta, Y)\). Loosely speaking, we push the colored braid \(\delta\) past the sequence of operations \(X\) just as in the definition of equivariance for operads, obtain another colored braid \(\bar{\delta}\) which is obtained from \(\delta\) by multiplying the strands according to \(X\) and another sequence of operations \(\bar{X}\) which is obtained from \(X\) by permuting the operations according to \(\delta\), and finally compose the left and right side in \(\text{Braid}(C)\) and \(S(O_{\text{pl}})\) respectively, i.e.

\[
(\beta, X) \ast (\delta, Y) := (\bar{\beta} \ast \bar{\delta}, \bar{X} \ast Y)
\]
See Figure 8 for a visualization of this procedure. That this definition is independent of the chosen representatives follows from the equivariance properties of operads.

Last but not least, the tensor product is defined on representatives \((\beta, X)\) and \((\delta, Y)\) via juxtaposition, i.e. \((\beta, X) \otimes (\delta, Y) := (\beta \otimes \delta, X \otimes Y)\). The identity arrows are those represented by a pair of identities.

**Definition 3.3.** The degree of an operation is its number of inputs. The degree of an object in \(S(O)\) is the length of the corresponding color word. The degree of an arrow in \(S(O)\) is the degree of its domain.

**Definition 3.4.** Let \(O\) be a planar, symmetric or braided operad and let \(X\) be an object in \(S(O)\). Then the group

\[
\pi_1(O, X) := \pi_1(S(O), X)
\]

is called the operad group associated to \(O\) based at \(X\).

### 3.1. Normal forms

In case \(O\) is a planar operad, arrows in \(S(O)\) are just tensor products of operations. In the symmetric and braided case, however, arrows are equivalence classes of pairs \((\beta, X)\). In this section we want to give a normal form of such arrows, i.e. canonical representatives \((\beta, X)\). We will treat the braided case, the symmetric case is similar and simpler.

Consider a colored braid \(\beta\) with \(n\) strands. The \(i\)th strand is the strand starting from the node with index \(i \in \{1, \ldots, n\}\). Let \(S\) be a subset of the index set \(\{1, \ldots, n\}\). Deleting all strands in \(\beta\) other than those with an index in \(S\) yields another colored braid \(\beta|_S\). We say that \(\beta\) is unbraided on \(S\) if \(\beta|_S\) is trivial.

Let \((n_1, \ldots, n_k)\) be a sequence of natural numbers with \(1 = n_1 < n_2 < \ldots < n_k = n + 1\). A sequence like this is called a partition of \(n\), denoted by \([n_1, \ldots, n_k]\), because the sets \(S_i := \{n_i, \ldots, n_i+1-1\}\) form a partition of the set \(\{1, \ldots, n\}\). We say \(\beta\) is unbraided with respect to the partition \([n_1, \ldots, n_k]\) if it is unbraided on the sets \(S_i\).

**Lemma 3.5.** Let \([n_1, \ldots, n_k]\) be a partition of \(n\) and \(\beta\) a colored braid with \(n\) strands. Then there is a unique decomposition \(\beta = \beta_p \ast \beta_u\) into colored braids \(\beta_p\) and \(\beta_u\) such that \(\beta_p = \beta_p^1 \otimes \ldots \otimes \beta_p^{k-1}\) is a tensor product of colored braids \(\beta_p^i\) with \(|S_i|\) strands and \(\beta_u\) is unbraided with respect to \([n_1, \ldots, n_k]\).

**Proof.** Define \(\beta_p^i := \beta|_{S_i}\) and

\[
\beta_u := \left(\beta|_{S_1}^{-1} \otimes \ldots \otimes \beta|_{S_{k-1}}^{-1}\right) \ast \beta
\]

Then we have \(\beta = \beta_p \ast \beta_u\) and \(\beta_u\) is unbraided with respect to \([n_1, \ldots, n_k]\). The uniqueness statement is not hard to see. \(\square\)

Now let \([\beta, X]\) be an arrow in \(S(O)\) with \(X = (X_1, \ldots, X_k)\). Assume \(\text{deg}(X_i) = d_i\) and \(d_1 + \ldots + d_k = n\). Define \(n_i = 1 + \sum_{j=1}^{i-1} d_j\) for \(i = 1, \ldots, k + 1\) and observe the partition \([n_1, \ldots, n_{k+1}]\). Decompose the colored braid \(\beta^{-1}\) as in the previous lemma to obtain \(\beta = \tau \ast \rho\) where \(\tau^{-1}\) is unbraided with respect to \([n_1, \ldots, n_{k+1}]\) and \(\rho = \rho_1 \otimes \ldots \otimes \rho_k\) is a tensor product of colored braids \(\rho_i\) with \(d_i\) strands. Define \(Y_i = X_i\). Then from the definition of arrows in \(S(O)\) it follows that

\[
[\beta, X] = [\tau, Y]
\]

with \(Y = (Y_1, \ldots, Y_k)\). So each arrow has a representative \((\tau, Y)\) such that \(\tau^{-1}\) is unbraided in the ranges defined by the domains of the operations in the second component. It is easy to see that there is at most one such pair.

Similarly, in the symmetric case, for each arrow in \(S(O)\), there is a unique representative \((\tau, Y)\) such that the colored permutation \(\tau^{-1}\) is unpermuted on the domains of the operations in the second component.
**Definition 3.6.** The unique representative \((\tau, Y)\) of an arrow in \(S(O)\) with \(\tau^{-1}\) unpermutated resp. unbraided is called the normal form of that arrow.

### 3.2. Calculus of fractions and cancellation properties

In the following, we write \(\theta \approx \psi\) if the two operations \(\theta, \psi\) in an operad are equivalent modulo the action of the symmetric resp. braid groups, i.e. there exists a permutation resp. braid \(\gamma\) such that \(\theta = \gamma \cdot \psi\). Of course, in the planar case, this just means equality of operations.

**Definition 3.7.** Let \(O\) be a (symmetric/braided) operad. We say that \(O\) satisfies the calculus of fractions if the following two conditions are satisfied:

- (Square filling) For every pair of operations \(\theta_1\) and \(\theta_2\) with the same output color, there are sequences of operations \(\Psi_1 = (\psi_1^1, \ldots, \psi_1^k)\) and \(\Psi_2 = (\psi_2^1, \ldots, \psi_2^k)\) such that \(\Psi_1 \ast \theta_1\) is defined for \(i = 1, 2\) and such that \(\Psi_1 \ast \theta_1 \approx \Psi_2 \ast \theta_2\).
- (Equalization) Assume we have an operation \(\theta\) and sequences of operations \(\Psi_1 = (\psi_1^1, \ldots, \psi_1^k)\) and \(\Psi_2 = (\psi_2^1, \ldots, \psi_2^k)\) such that \(\Psi_1 \ast \theta \approx \Psi_2 \ast \theta\), i.e. there is \(\gamma\) with \(\Psi_1 \ast \theta = \gamma \cdot (\Psi_2 \ast \theta)\). Then \(\gamma\) is already of the form \(\gamma = \gamma_1 \otimes \ldots \otimes \gamma_k\) such that \(\gamma_j \cdot \psi_2^j\) is defined for each \(j = 1, \ldots, k\) and there is a sequence of operations \(\Xi_j\) for each \(j = 1, \ldots, k\) such that \(\Xi_j \ast \psi_1^j = \Xi_j \ast (\gamma_j \cdot \psi_2^j)\).

**Proposition 3.8.** \(O\) satisfies the calculus of fractions if and only if \(S(O)\) does.

*Proof.* It is easy to see that the square filling property for \(S(O)\) implies the square filling property for \(O\).

Conversely, assume \([\beta, X]\) and \([\sigma, Y]\) are arrows in \(S(O)\) with common codomain. Let \(X = (X_1, \ldots, X_k)\) and \(Y = (Y_1, \ldots, Y_k)\). For showing the square filling property for this pair of arrows, it suffices to consider the case \(\beta = \text{id}\) and \(\sigma = \text{id}\). But under this assumption, we can use the square filling property for \(O\) for each pair of operations \(X_i\) and \(Y_j\).

Now assume equalization for \(O\). Let \([\beta, X], [\sigma, Y], [\delta, Z]\) be arrows in \(S(O)\) such that \([\beta, X] \ast [\delta, Z] = [\sigma, Y] \ast [\epsilon, Z]\). For showing the equalization property for this situation, it suffices to consider the case \(\delta = \text{id}\). Moreover, we can assume without loss of generality \(\beta = \text{id}\). Then, by assumption, we have that the pairs \((\text{id}, X \ast Z)\) and \((\sigma, Y \ast Z)\) are equivalent. When writing \(X \ast Z = (R_1, \ldots, R_l)\) and \(Y \ast Z = (S_1, \ldots, S_l)\), this implies that \(\sigma\) has to split as a tensor product \(\sigma = \sigma_1 \otimes \ldots \otimes \sigma_l\) such that \(\sigma_1 \cdot S_1 = R_1\). We therefore can assume without loss of generality that \(l = 1\). Setting \(\Psi_1 := X\) and \(\Psi_2 := Y\), we see that the hypothesis of the equalization property for \(O\) is satisfied with \(\gamma := \sigma\). The resulting sequences of operations \(\Xi_j\) can be juxtaposed to yield and arrow which equalizes the arrows \([\text{id}, X]\) and \([\sigma, Y]\).

Conversely, assume equalization for \(S(O)\) and assume we have \(\theta, \Psi_1, \Psi_2, \gamma\) as in the hypothesis of the equalization property for \(O\). The arrow \([\text{id}, \theta]\) then coequalizes the arrows \([\text{id}, \Psi_1]\) and \([\gamma, \Psi_2]\). We therefore get an arrow \([\delta, Z]\) which equalizes the last two arrows. Also \([\text{id}, Z]\) does the same job. But then it follows that \(\gamma\) has to split as required and the \(\Xi_j\) can be obtained by splitting \(Z\).

**Definition 3.9.** Let \(O\) be a (symmetric/braided) operad. We define right cancellativity and left cancellativity for \(O\) as follows:

- (Right cancellativity) Assume we have an operation \(\theta\) and sequences of operations \(\Psi_1 = (\psi_1^1, \ldots, \psi_1^k)\) and \(\Psi_2 = (\psi_2^1, \ldots, \psi_2^k)\) such that \(\Psi_1 \ast \theta \approx \Psi_2 \ast \theta\), i.e. there is \(\gamma\) with \(\Psi_1 \ast \theta = \gamma \cdot (\Psi_2 \ast \theta)\). Then \(\gamma\) is already of the form \(\gamma = \gamma_1 \otimes \ldots \otimes \gamma_k\) such that \(\gamma_j \cdot \psi_2^j\) is defined and equal to \(\psi_1^j\) for each \(j = 1, \ldots, k\).
- (Left cancellativity) Assume we have operations \(\theta_1\) and \(\theta_2\) and a sequence of operations \(\Psi\) such that \(\Psi \ast \theta_1 = \Psi \ast \theta_2\). Then \(\theta_1 = \theta_2\).
We say that $\mathcal{O}$ is cancellative if it is both left and right cancellative.

**Remark 3.10.** Just as in the case of categories, we see the following implications:

right cancellation $\implies$ equalization

equalization + left cancellation $\implies$ right cancellation

**Proposition 3.11.** $\mathcal{O}$ satisfies the left resp. right cancellation property if and only if $S(\mathcal{O})$ does.

**Proof.** The equivalence of right cancellativity of $\mathcal{O}$ with that of $S(\mathcal{O})$ is proven similarly as in the case of the equalization property in Proposition 3.8.

It is easy to prove the left cancellativity of $\mathcal{O}$ using the one of $S(\mathcal{O})$.

Conversely, assume we have left cancellativity for $\mathcal{O}$ and let $[\delta, Z] * [\beta, X]$ and $[\delta, Z]$ be arrows in $S(\mathcal{O})$ with $[\delta, Z] * [\beta, X] = [\delta, Z] * [\sigma, Y]$. For showing left cancellation for $S(\mathcal{O})$ we can assume without loss of generality that $\beta = \text{id}$. Moreover, we can then assume without loss of generality that also $\delta = \text{id}$. Then, by assumption, we have that the pairs $(\text{id}, Z * X)$ and $(\sigma, Z * Y)$ are equivalent where $\sigma$ and $Z$ are obtained from $\sigma$ and $Z$ according to the definition of composition. When writing $Z * X = (R_1, ..., R_l)$ and $Z * Y = (S_1, ..., S_l)$, this implies that $\sigma$ has to split as a tensor product $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_l$ such that $\sigma_i \cdot S_i = R_i$. When writing $X = (K_1, ..., K_l)$ and $Y = (T_1, ..., T_l)$, it follows that also $\sigma$ has to split as a tensor product $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_l$ such that $\sigma_i \cdot T_i$ is defined and, using left cancellativity of $\mathcal{O}$, we have $\sigma_i \cdot T_i = K_i$. This implies $[\text{id}, X] = [\sigma, Y]$. $\square$

4. Examples of operad groups

4.1. Free operads. When specifying a set of colors $C$ and for each $(n+1)$-tuple $(a_1, ..., a_n; b)$ of colors a set of operations with inputs labelled by the colors $a_1, ..., a_n$ and with output labelled by the color $b$, then we may form the free planar operad resp. the free braided resp. the free symmetric operad generated by this data. This construction can be described as the left adjoint of a functor forgetting all the data of an operad except the colors and the sets of operations.

Consider (planar) operads freely generated by a set of colors and operations of degree at least 2. Then the operad groups associated to these operads are the so-called diagram groups defined in [21]. When considering free symmetric operads, we get symmetric versions of diagram groups which are called “braided” in [21] Definition 16.2. The truly braided diagram groups are the ones arising from free braided operads. This correspondence is not hard to see if we use string diagrams (see Subsection 2.6) for representing elements in the fundamental groupoid of $S(\mathcal{O})$ and compare them with so-called pictures in [21] Definitions 4.1 and 16.2. Each generating operation is visualized as a transistor shaped object with several colored ingoing wires and one colored outgoing wire. However, transistors in the definition of pictures can have more than one outgoing wire. To cope with this, we can introduce another color and split the relation into two using the new color. For example, consider the semigroup presentation $(a, b \mid aa = ab)$ and a word $w$ in the alphabet $\{a, b\}$. Let $\Gamma$ be the diagram group based at $w$ associated to this semigroup presentation. Consider instead the presentation $(a, b, c \mid aa = c, c = ab)$ which gives the same diagram group based at $w$. Now the operad $\mathcal{O}$ with $\pi_1(\mathcal{O}, w) \cong \Gamma$ is the free one generated by the colors $\{a, b, c\}$ and two operations, one with inputs labelled $(a, a)$ and output labelled $c$ and the other one with inputs labelled $(a, b)$ and output labelled $c$.

This correspondence also yields that the classical Thompson groups $F$ and $V$ and more generally the Higman–Thompson groups $F_{n,r}$ and $V_{n,r}$ are the operad groups based at $r$ of the free planar operads resp. free symmetric operads generated
by one color and one operation of degree \( n \). In the case of \( F \), this has first been observed in [15]. We can also form the operad group based at \( 1 \) of the free braided operad generated by one color and one binary relation. This yields the braided Thompson group \( BV \) defined in [6].

In [35] we will deal with so-called operads with transformations. These are (symmetric/braided) operads such that each degree 1 operation is invertible. In this context, we can freely generate (symmetric/braided) operads with transformations out of the following data: A set of colors \( C \) together with sets of \( C \)-colored operations as above and, additionally, a groupoid with object set \( C \) which specifies the degree 1 operations in the associated free (symmetric/braided) operad. Again, this can be described formally as the left adjoint of a forgetful functor forgetting all the data of a (symmetric/braided) operad except the data just under discussion.

### 4.2. Cube cutting operads

Fix some dimension \( d \geq 1 \) and consider a brick

\[
[a_1, b_1] \times \ldots \times [a_d, b_d] \quad a_i < b_i
\]

in \( \mathbb{R}^d \). Let \( n \geq 2 \). A \( n \)-cut in direction \( j \) of that brick is the set of bricks

\[
[a_1, b_1] \times \ldots \times \left[ a_j + k \frac{(b_j - a_j)}{n}, a_j + (k + 1) \frac{(b_j - a_j)}{n} \right] \times \ldots \times [a_d, b_d]
\]

for \( k = 0, \ldots, n-1 \). For each \( j = 1, \ldots, d \) let \( N_j \) be a set of finitely many coprime natural numbers greater or equal to 2. A \( (N_j)_j \)-subdivision of the unit \( d \)-dimensional cube \( I^d \) is a finite sequence of successive \( n \)-cuts in direction \( j \) for some \( n \in N_j \) and \( j \in \{1, \ldots, d\} \), starting with the unit cube \( I^d \). That is, first we \( n_1 \)-cut \( I^d \) in direction \( j_1 \) with \( n_1 \in N_{j_1} \). Then we take a resulting brick and \( n_2 \)-cut it in direction \( j_2 \) with \( n_2 \in N_{j_2} \) and so on.

Now we define a symmetric operad out of the data \( (N_j)_j \). The set of colors consists of a single element. An operation of degree \( l \) is a sequence \((f_1, \ldots, f_l)\) of embeddings \( f_i: I^d \to I^d \) which are affine on each coordinate and such that the images of the \( f_i \) form a \( (N_j)_j \)-subdivision of the unit cube. The \( i \)'th partial composition (see Remark 3.2) of \((f_1, \ldots, f_l)\) with \((g_1, \ldots, g_k)\) is given by

\[
(f_1, \ldots, f_l) *_i (g_1, \ldots, g_k) = (g_1, \ldots, g_{i-1}, f_ig_i, \ldots, f_ig_i, g_{i+1}, \ldots, g_k)
\]

The unit operation is given by \((\text{id}_{I^d})\). The symmetric groups act on the operations by permuting the embeddings in the sequence. The following picture visualizes composition in the case \( d = 2 \) with \( N_1 = \{2\} \) and \( N_2 = \{2\} \).

\[
\begin{pmatrix}
2 & 1 \\
1 & \end{pmatrix}, \quad \begin{pmatrix}
1 & 1 \\
2 & \end{pmatrix} \times \begin{pmatrix}
1 & 3 \\
2 & \end{pmatrix} = \begin{pmatrix}
2 & 1 & 4 \\
3 & 5 & \end{pmatrix}
\]

In the case \( d = 1 \), we can also define a planar operad out of the data of \( N \), a set with finitely many coprime natural numbers greater or equal to 2, by requiring that the embeddings in the sequence \((f_1, \ldots, f_l)\) are ordered by their images via the natural ordering on \( I \) and by forgetting the action of the symmetric groups.

**Proposition 4.1.** The cube cutting operads satisfy the cancellative calculus of fractions.

**Proof.** Both the left and right cancellation properties are not hard to verify.

For the square filling property, first consider the case \( d = 1 \) with \( N_1 = N = \{n_1, \ldots, n_k\} \). Let \( \theta_1, \theta_2 \) be two operations. The length of a subbrick in the subdivision of \( I \) corresponding to an operation is of the form \( n_1^{-d_1} \cdots n_k^{-d_k} \) with exponents \( d_i \in \mathbb{N}_0 \). Let \( \delta_i \) be the highest exponent appearing among all the subbricks of \( \theta_i \).
Set \( \delta := \max\{\delta_1, \delta_2\} \). Then we can subdivide \( \theta_1 \) further such that each subbrick has length \((n_1 \cdots n_k)^{-\delta}\). The same holds for \( \theta_2 \). Thus, we obtain the same subdivision. In the case \( d > 1 \), one applies this argument to each coordinate and also deduces this conclusion. Now we permute the embeddings of \( \theta_1 \) (or of \( \theta_2 \)) so that we obtain the same operation. \( \square \)

With this proposition and the fact that the fundamental groupoid of a category satisfying the calculus of fractions can be represented by spans, it is easy to identify the following operad groups:

- The Higman-Thompson groups \( F_{n,r} \) resp. \( V_{n,r} \) arise as the operad groups based at \( r \) associated to the planar resp. symmetric cube cutting operads with \( d = 1 \) and \( N = \{n\} \). These are also the free (symmetric) operads generated by one \( n \)-cut of the unit interval \( I \) (see the previous subsection).
- The groups of piecewise linear homeomorphisms \( F(r, Z[\mathbb{N}, m_1], \langle n_1, ..., n_k \rangle) \) resp. \( G(r, Z[\mathbb{N}, m_2], \langle n_1, ..., n_k \rangle) \) of the (Cantor) unit interval considered in [22] arise as the operad groups based at \( r \) associated to the planar resp. symmetric cube cutting operads with \( d = 1 \) and \( N = \{n_1, ..., n_k\} \).
- The higher dimensional Thompson groups \( nV \) (see [5]) arise as the operad groups associated to the symmetric cube cutting operads with \( d = n \) and \( N_1 = ... = N_d = \{2\} \).

We can modify the cube cutting operads by allowing not only coordinatewise affine embeddings \( f_i \) but also allow such embeddings with an isometry of \( I^d \) precomposed (for example a reflection or a rotation). This gives operads with invertible degree 1 operations other than the identity.

4.3. Local similarity operads of ultrametric spaces. In [22] groups were defined which act in a certain way on compact ultrametric spaces. Recall the definition of a finite similarity structure:

**Definition 4.2.** Let \( X \) be a compact ultrametric space. A finite similarity structure \( \text{Sim}_X \) on \( X \) consists of a finite set \( \text{Sim}_X(B_1, B_2) \) of similarities \( B_1 \rightarrow B_2 \) for every ordered pair of balls \( (B_1, B_2) \) such that the following axioms are satisfied:

- **(Identities)** Each \( \text{Sim}_X(B, B) \) contains the identity.
- **(Inverses)** If \( \gamma \in \text{Sim}_X(B_1, B_2) \), then also \( \gamma^{-1} \in \text{Sim}_X(B_2, B_1) \).
- **(Compositions)** If \( \gamma_1 \in \text{Sim}_X(B_1, B_2) \) and \( \gamma_2 \in \text{Sim}_X(B_2, B_3) \), then also \( \gamma_1 \gamma_2 \in \text{Sim}_X(B_1, B_3) \).
- **(Restrictions)** If \( \gamma \in \text{Sim}_X(B_1, B_2) \) and \( B_3 \subset B_1 \) is a subball, then also \( \gamma|_{B_3} \in \text{Sim}_X(B_3, \gamma(B_3)) \).

Here, a similarity \( \gamma : X \rightarrow Y \) of metric spaces is a homeomorphism such that there is a \( \lambda > 0 \) with \( d(\gamma(x_1), \gamma(x_2)) = \lambda d(x_1, x_2) \) for all \( x_1, x_2 \in X \). Let \( \text{Sim}_X \) be a finite similarity structure on the compact ultrametric space \( X \). A homeomorphism \( \gamma : X \rightarrow X \) is said to be locally determined by \( \text{Sim}_X \) if for every \( x \in X \) there is a ball \( x \in B \subset X \) such that \( \gamma(B) \) is a ball and \( \gamma|_B \in \text{Sim}_X(B, \gamma(B)) \). The set of all such homeomorphisms forms a group which we denote by \( \Gamma(\text{Sim}_X) \).

To a finite similarity structure \( \text{Sim}_X \), we can associate a symmetric operad \( \mathcal{O}(\text{Sim}_X) \) and reobtain the groups \( \Gamma(\text{Sim}_X) \) as operad groups. Two balls \( B_1, B_2 \) in \( X \) are called \( \text{Sim}_X \)-equivalent if \( \text{Sim}_X(B_1, B_2) \neq \emptyset \). The colors of the operad are the \( \text{Sim}_X \)-equivalence classes of balls. In each \( \text{Sim}_X \)-equivalence class of balls, choose one representing ball. An operation with input colors the chosen balls \( B_1, ..., B_l \) and output color the chosen ball \( B \) is a sequence \((f_1, ..., f_l) \) of \( \text{Sim}_X \)-embeddings \( f_i : B_i \rightarrow B \) (i.e. elements in \( \text{Sim}_X \) when the codomain is restricted to the image) such that the images of the \( f_i \) are pairwise disjoint and their union is \( B \). Composition, units and the symmetric group actions are defined just as in the case of
cube cutting operads. Note that when choosing other balls in each class, we get an isomorphic operad. Note also that the operad is multi-colored and that it has invertible degree 1 operations other than the identities, namely exactly the isometries of the chosen balls contained in the similarity structure.

Proposition 4.3. The operads $O(\text{Sim}_X)$ satisfy the cancellative calculus of fractions.

Proof. Similarly as in the case of cube cutting operads, right and left cancellativity is not hard to see.

For the square filling property, let $\theta$ and $\psi$ be two operations with the same codomain color represented by the chosen ball $B$. Write $\theta = (f_i: C_i \to B)_{i=1,\ldots,k}$ and $\psi = (g_j: D_j \to B)_{j=1,\ldots,l}$. We can first precompose with sequences of operations such that we obtain $k = l$ and such that the sets

\[ \{\text{im}(f_i) \mid i = 1, \ldots, k\} = \{\text{im}(g_i) \mid i = 1, \ldots, k\} \]

are equal. Then we can precompose with sequences of degree 1 operations such that the sets

\[ \{f_i \mid i = 1, \ldots, k\} = \{g_i \mid i = 1, \ldots, k\} \]

are equal. Thus, after permuting the embeddings, we get equal operations. □

Using this proposition and the fact that the fundamental groupoid of a category satisfying the calculus of fractions can be represented by spans, it is not hard to establish an isomorphism $\pi_1(O(\text{Sim}_X), [X]) \cong \Gamma(\text{Sim}_X)$.

5. The Morse method

We review a well-known technique to study connectivity of spaces. We first describe the method in the case of simplicial complexes which has been used in [2] to prove finiteness properties of certain groups. We then explain the same method in the context of categories.

Let $C$ be a simplicial complex. Let $v$ be a vertex in $C$. Denote by $C^-v$ the full subcomplex spanned by the vertices of $C$ except $v$. Define the descending link

\[ \text{lk}_\downarrow(v) := \text{lk}_C(v) \cap C^-v \]

to be the intersection of the ordinary link of $v$ in $C$ with the subcomplex $C^-v$. We then have a canonical pushout diagram

\[
\begin{array}{ccc}
\text{lk}_\downarrow(v) & \rightarrow & \text{Cone} \left( \text{lk}_\downarrow(v) \right) \\
\downarrow & & \downarrow \\
C^-v & \rightarrow & C
\end{array}
\]

where $\text{Cone} \left( \text{lk}_\downarrow(v) \right)$ denotes the simplicial cone over $\text{lk}_\downarrow(v)$. The following proposition expresses the connectivity of the pair $(C, C^-v)$ in terms of the connectivity of $\text{lk}_\downarrow(v)$.

Lemma 5.1. Let $X$ and $L$ be two spaces and $L \rightarrow C$ a cofibration into a contractible space $C$. Let

\[
\begin{array}{ccc}
L & \rightarrow & C \\
\downarrow & & \downarrow \\
X & \rightarrow & Z
\end{array}
\]

be a pushout of spaces. If $L$ is $(n-1)$-connected, then the pair $(Z, X)$ is $n$-connected.
Proof. Choose a base point $a \in L$ and denote by $a$ also the corresponding points in the spaces $C, X, Z$ of the pushout. We prove the claim by induction over $n$. First we treat the cases $n = 0$ and $n = 1$. For $n = 0$ the assumption is that $L$ is $(-1)$-connected, i.e. we only know that $L$ is non-empty. It is easy to see in this case that $\pi_0(X,a) \rightarrow \pi_0(Z,a)$ is surjective which means that $\pi_0(Z,a) = 0$, i.e. $(Z,X)$ is $0$-connected. For $n = 1$ the assumption is that $L$ is $0$-connected, i.e. path connected. Seifert–van Kampen applied to the pushout of spaces yields isomorphisms

$$H_k(C,L) \cong H_k(Z,X)$$

for each $k$. So we obtain $H_k(Z,X) = 0$ for $2 \leq k \leq n + 1$. By induction hypothesis we have that $(Z,X)$ is $n$-connected. So by the relative Hurewicz Theorem, we have an isomorphism

$$H_{n+1}(Z,X) \cong \pi_{n+1}(Z,X)$$

Consequently, we also have $\pi_{n+1}(Z,X) = 0$ and $(Z,X)$ is $(n+1)$-connected. \hfill $\square$

More generally, let $\mathcal{X}_0$ be the full subcomplex of the simplicial complex $\mathcal{X}$ spanned by a subset of vertices. Then $\mathcal{X}$ can be built up from $\mathcal{X}_0$ by successively adding vertices. If all the descending links appearing this way are highly connected, then also the pair $(\mathcal{X}, \mathcal{X}_0)$ will be highly connected and, using the long exact homotopy sequence, we obtain the following:

**Proposition 5.2.** Let $x_0 \in \mathcal{X}_0$ be a point. Assume that each descending link is $n$-connected. Then, we have

$$\pi_k(\mathcal{X}_0, x_0) \cong \pi_k(\mathcal{X}, x_0)$$

for $k = 0, \ldots, n$.

Note that, in general, the descending links depend on the order in which the vertices are added. We call two vertices $v_1, v_2$ in $\mathcal{X} \setminus \mathcal{X}_0$ independent if they are not joined by an edge in $\mathcal{X}$. Assume now that we want to add $v_1$ and then $v_2$ at some step of the process. The independence condition ensures that the descending links of $v_1$ and $v_2$ do not depend on the order in which $v_1$ and $v_2$ are added.

The adding order is often encoded in a Morse function. This is a function $f$ assigning to each vertex in $\mathcal{X} \setminus \mathcal{X}_0$ an element in a totally ordered set, e.g. $\mathbb{N}$. We require that vertices with the same $f$-value are pairwise independent. We then add vertices in order of ascending $f$-values. Because of the independence property, the adding order of vertices with the same $f$-value can be chosen arbitrarily. Alternatively, we can add vertices with the same $f$-value all at once.
We now give a version of this concept for categories. Let \( C \) be a category and \( X \) be an object in \( C \) with \( \text{Hom}_C(X,X) = \{ \text{id}_X \} \). Define \( C^{-X} \) to be the full subcategory of \( C \) spanned by the objects of \( C \) except \( X \). We define
\[
\overline{\text{lk}}_i(X) := C^{-X} \downarrow X
\]
to be the descending up link of \( X \) and
\[
\underline{\text{lk}}_i(X) := X \downarrow C^{-X}
\]
to be the descending down link of \( X \). Furthermore, define
\[
\text{lk}_i(X) := \overline{\text{lk}}_i(X) \ast \underline{\text{lk}}_i(X)
\]
to be the descending link of \( X \).

We have a commutative diagram \( \mathcal{D} \) as follows:

\[
\begin{array}{ccc}
\text{lk}_i(X) & \rightarrow & \text{Coone} (\overline{\text{lk}}_i(X) \ast \underline{\text{lk}}_i(X)) \\
\downarrow & & \downarrow \\
C^{-X} & \rightarrow & C
\end{array}
\]

The horizontal arrows are the obvious inclusions. We explain the vertical arrows, starting with
\[
\overline{\text{lk}}_i(X) \ast \underline{\text{lk}}_i(X) \rightarrow C^{-X}
\]
An object either comes from \( \overline{\text{lk}}_i(X) \) and thus is a pair \((Y, Y \rightarrow X)\) with \( Y \) an object in \( C^{-X} \) or comes from \( \underline{\text{lk}}_i(X) \) and thus is a pair \((Y, X \rightarrow Y)\) with \( Y \) an object in \( C^{-X} \). In both cases, the object will be sent to \( Y \). Similarly, on the level of arrows, it is also the canonical projection from \( \overline{\text{lk}}_i(X) \) or \( \underline{\text{lk}}_i(X) \) to \( C^{-X} \). But for each object \((Y, Y \rightarrow X)\) in \( \overline{\text{lk}}_i(X) \) and each object \((Y', X \rightarrow Y')\) in \( \underline{\text{lk}}_i(X) \), there is another unique arrow in the join. Send this arrow to the composition \( Y \rightarrow X \rightarrow Y' \). Next, we will define the arrow
\[
\text{Coone} (\overline{\text{lk}}_i(X) \ast \underline{\text{lk}}_i(X)) \rightarrow C
\]
Of course, in order to make the diagram commutative, this function restricted to the base \( \text{lk}_i(X) \) of the coone is the one already defined above. So we have to define the images of the extra object \( \text{tip} \) and the extra arrows. Send \( \text{tip} \) to \( X \). Let \((Y, Y \rightarrow X)\) be an object of \( \overline{\text{lk}}_i(X) \). The arrow from this object to \( \text{tip} \) is sent to the arrow \( Y \rightarrow X \). Similarly, let \((Y, X \rightarrow Y)\) be an object of \( \underline{\text{lk}}_i(X) \). Then the arrow from \( \text{tip} \) to this object is sent to the arrow \( X \rightarrow Y \).

Our goal is to show that the diagram \( \mathcal{D} \) becomes a pushout on the level of classifying spaces. Unfortunately, this is not always the case. Consider for example the groupoid \( \bullet \rightleftarrows \circ \) with two objects and two non-identity arrows which are inverse to each other. In all these cases, however, the situation is even better:

**Lemma 5.3.** Assume that there is an object \( A \neq X \) and arrows \( \alpha \colon X \rightarrow A \) and \( \beta \colon A \rightarrow X \). Then the inclusion \( C^{-X} \rightarrow C \) is a homotopy equivalence.

**Proof.** We show that \( C^{-X} \downarrow X \) is filtered and thus contractible. The lemma then follows from Theorem 2.2 and Remark 2.3. Let \((Y, \gamma)\) be an object in \( C^{-X} \downarrow X \), i.e., \( \gamma \colon Y \rightarrow X \) is an arrow in \( C \) with \( Y \) an object in \( C^{-X} \). Set \( \epsilon := \gamma \circ \alpha \). Because of the assumption \( \text{Hom}_C(X,X) = \{ \text{id}_X \} \), the arrow \( \alpha \beta \colon X \rightarrow X \) must be the identity. Then we calculate
\[
\gamma = \gamma (\alpha \beta) = (\gamma \circ \alpha) \beta = \epsilon \beta
\]
This shows that \( \epsilon \) represents an arrow \((Y, \gamma) \rightarrow (A, \beta)\) in \( C^{-X} \downarrow X \). In particular, for every two objects in the comma category, there are arrows to the object \((A, \beta)\). This shows the first property of a filtered category.
For the second property, we have to show that any two parallel arrows are co-equalized by another arrow. So let \((Z, \nu)\) and \((Y, \gamma)\) be two objects and \(\epsilon, \epsilon': (Z, \nu) \to (Y, \gamma)\) be two arrows, i.e. \(\epsilon, \epsilon': Z \to Y\) are arrows in \(\mathcal{C}^-X\) and we have \(\epsilon\gamma = \nu = \epsilon'\gamma\). Set \(\mu = \gamma\alpha\) which is an arrow \((Y, \gamma) \to (A, \beta)\) as already pointed out. Then we calculate
\[
\epsilon\mu = \epsilon\gamma\alpha = \nu\alpha = \epsilon'\gamma\alpha = \epsilon'\mu
\]
and we are done. \(\square\)

In all other cases, diagram \(\mathcal{D}\) is indeed a pushout on the level of classifying spaces:

**Lemma 5.4.** Assume that for any object \(A \neq X\) either there are only arrows from \(X\) to \(A\) or there are only arrows from \(A\) to \(X\), but never both. Then the diagram
\[
B(\mathcal{D})
\]

\[
\begin{array}{ccc}
B(lk_\downarrow X) & \longrightarrow & \text{Cone}(B(lk_\downarrow X)) \\
\downarrow & & \downarrow \\
B(\mathcal{C}^-X) & \longrightarrow & B(\mathcal{C})
\end{array}
\]

is a pushout of spaces.

**Proof.** We claim that the nerve functor applied to the diagram \(\mathcal{D}\)
\[
\begin{array}{ccc}
N(lk_\downarrow X) & \longrightarrow & N(\text{Coone}(lk_\downarrow X * lk_\downarrow X)) \\
\downarrow & & \downarrow \\
N(\mathcal{C}^-X) & \longrightarrow & N(\mathcal{C})
\end{array}
\]

yields a pushout in \(\text{SSET}\). Since the geometric realization functor \(|\cdot|: \text{SSET} \to \text{TOP}\) is left adjoint to the singular simplex functor, it preserves all colimits and in particular all pushouts. Therefore, applying the geometric realization functor \(|\cdot|\) to the diagram \(N(\mathcal{D})\), we obtain a pushout in \(\text{TOP}\), as claimed in the lemma.

A simplex in \(N(\mathcal{C})\) is just a string of composable arrows \(A_0 \to \ldots \to A_k\). One can easily deduce from the assumption that whenever there are two occurrences of \(X\) in such a string of composable arrows, then there cannot be objects different from \(X\) in between. In other words, if \(X\) occurs at all, then all the \(X\) in the string are contained in a maximal substring of the form \(X \to X \ldots \to X\) where all the arrows are (necessarily) \(\text{id}_X\). Assume now that we have a commutative diagram as follows
\[
\begin{array}{ccc}
N(lk_\downarrow X) & \longrightarrow & N(\text{Coone}(lk_\downarrow X * lk_\downarrow X)) \\
\downarrow & & \downarrow \\
N(\mathcal{C}^-X) & \longrightarrow & N(\mathcal{C})
\end{array}
\]

We will show that \(h\) is uniquely determined by \(f\) and \(g\). Assume \(\sigma\) is a simplex in \(N(\mathcal{C})\) given by a string of composable arrows \(A_0 \to \ldots \to A_k\). If all the \(A_i\) are contained in the full subcategory \(\mathcal{C}^-X\) then \(\sigma\) is a simplex in the simplicial subset \(N(\mathcal{C}^-X)\) and then necessarily \(h(\sigma) = g(\sigma)\). On the other hand, assume that not all the \(A_i\) are objects of \(\mathcal{C}^-X\), i.e. at least one \(A_i = X\). As pointed out above, \(\sigma\) must be of the form
\[
B_0 \to \ldots \to B_r \to X \to \ldots \to X \to C_0 \to \ldots \to C_s
\]
where $B_i \neq X$ for all $i = 0, \ldots, r$ and $C_j \neq X$ for all $j = 0, \ldots, s$. All the $B_i$ are in the image of $\bar{\text{lk}}_1(X)$ because, after composing, we get an arrow $B_i \to X$. Analogously, all the $C_j$ are in the image of $\bar{\text{lk}}_n(X)$. Such a simplex $\sigma$ always lifts to unique simplex $\tilde{\sigma}$ along the map

$$N(\text{Coone}(\bar{\text{lk}}_1 X \ast \bar{\text{lk}}_n X)) \to N(\mathcal{C})$$

Thus we have $h(\sigma) = f(\tilde{\sigma})$. This proves uniqueness of $h$. Showing that $h$ actually exists is left to the reader.

We combine Lemmas 5.1, 5.3 and 5.4 to get:

**Proposition 5.5.** If the descending link $\text{lk}_1(X)$ is $(n-1)$-connected, then the pair $(\mathcal{C}, \mathcal{C}^\perp)$ is $n$-connected.

More generally, let $\mathcal{X}_0$ be the full subcategory of the category $\mathcal{X}$ spanned by a collection of objects in $\mathcal{X}$. Assume that $\text{Hom}_\mathcal{X}(X, X) = \{\text{id}_X\}$ for all objects $X$ in $\mathcal{X} \setminus \mathcal{X}_0$. Then $\mathcal{X}$ can be built up from $\mathcal{X}_0$ by successively adding objects. If all the descending links appearing this way are highly connected, then also the pair $(\mathcal{X}, \mathcal{X}_0)$ will be highly connected and, using the long exact homotopy sequence, we obtain the following:

**Theorem 5.6.** Let $x_0 \in \mathcal{X}_0$ be an object. Assume that each descending link is $n$-connected. Then, we have

$$\pi_k(\mathcal{X}_0, x_0) \cong \pi_k(\mathcal{X}, x_0)$$

for $k = 0, \ldots, n$.

We say that two objects $x_1$ and $x_2$ in $\mathcal{X} \setminus \mathcal{X}_0$ are independent if there are no arrows $x_1 \to x_2$ or $x_2 \to x_1$ in $\mathcal{X}$. This guarantees independence of $\text{lk}_i(x_1)$ and $\text{lk}_i(x_2)$ from the adding order of $x_1$ and $x_2$.

Again, we can encode the adding order with the help of a Morse function $f$ which assigns to each object in $\mathcal{X} \setminus \mathcal{X}_0$ an element in a totally ordered set, e.g. $\mathbb{N}$. We require that objects with the same $f$-value are pairwise independent and we add objects in order of increasing $f$-values.

6. **A Finiteness Result for Plain Operad Groups**

**Definition 6.1.** A plain operad is a planar operad with no degree 1 operations other than identities.

**Definition 6.2.** Let $\mathcal{O}$ be a plain operad. A non-identity operation is called very elementary if it cannot be obtained from other non-identity operations using partial compositions. In other words, $\theta$ is very elementary if it is indecomposable with respect to partial composition. Denote the set of very elementary operations by the symbol $\text{VE}$.

It is easy to see that every non-identity operation in a plain operad can be written as a partial composition of very elementary operations.

Let $\mathcal{O}$ be a plain operad. We introduce a partial order on the set of operations of $\mathcal{O}$. We say $O_1 \leq O_2$ if $O_2$ can be obtained from $O_1$ by precomposing with a sequence of operations, i.e. if $O_2 = (\theta_1, \ldots, \theta_n) \circ O_1$ for operations $\theta_i$.

**Definition 6.3.** Let $\mathcal{O}$ be plain. We define a set $E$ of operations which we call elementary operations of $\mathcal{O}$. First we define sets of operations $E_i$ for $i \in \{0, 1, 2, \ldots\}$ inductively. Set $E_0 := \text{VE}$. Now assume $E_i$ has been constructed. If $|E_i| \leq 1$, set $E_{i+1} = \emptyset$. Else, for each pair $\theta_1, \theta_2$ of operations in $E_i$ with $\theta_1 \neq \theta_2$, define the set $M_{i+1}(\theta_1, \theta_2)$ as follows: If $\theta_1 < \theta_2$ or $\theta_1 > \theta_2$, let $M_{i+1}(\theta_1, \theta_2)$ be the one element...
set consisting of the element \( \max\{\theta_1, \theta_2\} \). Else, define it to be the set of all the minimal operations in the set
\[
\{ \theta \in O \mid \theta_i < \theta \text{ for } i = 1, 2 \}
\]
Now set
\[
E_{i+1} = \bigcup_{\theta_1 \neq \theta_2 \in E_i} M_{i+1}(\theta_1, \theta_2)
\]
and then \( E = \bigcup_{i=1}^{\infty} E_i \).

Note that each very elementary operation is also elementary. We will occasionally call elementary operations, which are not very elementary, strictly elementary.

**Definition 6.4.** \( O \) is finitely generated if \( VE \) is finite. It is of finite type if \( E \) is finite.

**Definition 6.5.** Let \( O \) be a plain operad. We call a word in the colors of \( O \) reduced if no subword is the domain of a non-identity operation in \( O \). We say that \( O \) is color-tame if the set of colors is finite and the length of reduced color words is bounded from above.

Note that if \( O \) monochromatic (i.e. there is only one color) and non-trivial (i.e. there are non-identity operations), then it is automatically color-tame.

We are now ready to state the main theorem of this section.

**Theorem 6.6.** Let \( O \) be a color-tame plain operad. Assume that at least one of the following holds:

A) \( O \) is free and finitely generated.

B) \( O \) satisfies the right cancellative calculus of fractions and is of finite type.

Then the operad group \( \pi_1(O, X) \) for any object \( X \) in \( S(O) \) is of type \( F_{\infty} \).

Here, free means that \( O \) is isomorphic to a free planar operad generated by its set of colors and operations of degree greater or equal to 2.

**Remark 6.7.** In [35] we will prove a generalization of this theorem which also treats symmetric and braided operads and even operads with invertible degree 1 operations other than identities. This more general theorem will render Theorem 6.6 obsolete. However, the main idea of the proof is the same and the case of plain operads allows a lot of simplifications in the definitions and arguments which make the proof easier to follow. The proof here may then be taken as a guide for the proof of the more general theorem.

**6.1. Outline of the proof.** We assume that the category \( S := S(O) \) is connected. Otherwise, we pass to the connected component of \( X \).

The main ingredient of the proof will be the Morse method for categories applied to the category \( S \). The degree function on the objects of \( S \) gives a Morse function on \( S \). Denote by \( S_k \) the full subcategory of \( S \) spanned by the objects of degree at most \( k \). In Subsection 6.3 below, we will show that the connectivity of the descending links \( lk_j(K) \) for objects \( K \) in \( S \) tends to infinity for \( \deg(K) \to \infty \). Let \( m \in \N \). We then know that there is \( k \) large enough such that \( lk_j(K) \) is \( m \)-connected whenever \( K \) is an object with \( \deg(K) > k \). From Theorem 6.6 it follows
\[
\pi_j(S_k, X) \cong \pi_j(S, X)
\]
for \( j = 0, ..., m \). In case B) of Theorem 6.6 we already know that \( S \) is aspherical by Corollary 2.8. In case A) of the theorem, we use Theorem 6.8 below. In any case, \( S_k \) is a category with the same fundamental group as \( S \) and with higher homotopy groups vanishing up to dimension \( m \).
Since the set of colors of $O$ is finite, there are only finitely many objects in $S_k$. Since the set $VE$ of very elementary operations is finite, there are only finitely many arrows in $S_k$. Thus, $S_k$ is a finite category. It follows that the classifying space $B(S_k)$ of $S_k$ is of finite type, i.e. it is a CW-complex with only finitely many cells in each dimension. In fact, one can even prove that the classifying space is compact since there are only finitely many ways of forming strings of composable non-identity arrows.

We now can take that $B(S_k)$ and attach cells in dimension $m + 2$ and higher to kill all the homotopy groups above dimension $m$. Call this space $B(S_k)^+$. Thus, $B(S_k)^+$ is a classifying space for $\pi_1(S, X)$ with finitely many cells up to dimension $m + 1$. Consequently, $B(S_k)^+$ is a witness that $\pi_1(S, X) = \pi_1(O, X)$ is of type $F_m$ (see e.g. [19, Proposition 7.2.2]).

6.2. Asphericity. In this subsection, we want to prove the following theorem (see [11, Theorem 3.10] for a related result):

**Theorem 6.8.** Let $O$ be a free plain operad. Then each component of $S(O)$ is aspherical.

Assume without loss of generality that $S := S(O)$ is connected. Let $X$ be an object in $S$. Observe the canonical functor $\varphi: S \to \pi_1(S)$ and the comma category $X \downarrow \varphi$ which is the universal covering category $U := U_X(S)$. We show that $U$ is contractible and the result follows from Quillen’s Theorem A or by elementary covering theory.

We do so by introducing a Morse function on $U$ and apply the Morse method for categories. An object in $U$ is a pair $(Y, \alpha: X \to Y)$ where $Y$ is another object in $S$ and $\alpha: X \to Y$ is an arrow in $\pi_1(S)$, i.e. a path in $S$ from $X$ to $Y$ modulo homotopy. We can think of such paths modulo homotopy as string diagrams (see Subsection 2.6) with boxes or transistors representing the operations of $O$. The very elementary operations are exactly the operations which generate $O$ freely. Thus, each operation can be uniquely written as a composition of very elementary operations. It follows that each arrow in $\pi_1(S)$ can be uniquely represented by a reduced (i.e. there are no dipoles) string diagram with very elementary transistors.

For such a diagram $D$, denote by $\#_i(D)$ the number of very elementary operations pointing to the left and by $\#_r(D)$ the number of very elementary operations pointing to the right. Now if $A := (Y, \alpha)$ is an object of $U$ as above and $D$ is the reduced very elementary string diagram representing $\alpha$, then define

$$f(A) := (\#_l(D), \#_r(D))$$

Order the pairs on the right lexicographically. Then this defines a Morse function on the objects of $U$. For example, the string diagram above has Morse height $(5, 1)$. The only object with Morse height $(0, 0)$ is $(X, \text{id}_X)$. Thus, we are building up $U$ starting with a point. If we prove that each descending link $lk_i(A)$ is contractible, it follows that $U$ is contractible by Theorem 5.6.
So let $A = (Y, \alpha) \neq (X, \text{id}_X)$ be an object in $\mathcal{U}$, represented by a reduced very elementary diagram as above. We call a very elementary operation in $A$ exposed if there is no other very elementary operation connected to that operation on the right. For example, the thick bordered operations are the exposed operations in the following diagram.

\[ \begin{array}{c}
\theta \\
\psi \\
\theta
\end{array} \]

Assume that there is at least one exposed operation pointing to the left (in the above diagram we have two). In this case, we will show that $\text{lk} \downarrow (A)$ is contractible. In case there are only right pointing exposed operations, we observe that $\text{lk} \downarrow (A)$ is contractible using similar arguments.

Consider an arrow $\beta : Y \to Z$ which is a tensor product of identities and very elementary operations that form dipoles with the left pointing exposed arrows when $\beta$ is concatenated to the diagram of $A$.

\[ \begin{array}{c}
\theta \\
\psi \\
\theta
\end{array} \]

When reducing the dipoles, the number of left pointing operations and thus the Morse height decreases. So $\beta$ is indeed an object in $\text{lk}_1(A)$. The full subcategory $\mathcal{Z}$ in $\text{lk}_1(A)$ spanned by these objects is a poset realizing a $(k - 1)$-simplex where $k$ is the number of left pointing exposed operations. The top face is represented by the arrow $\beta$ which contains all the left pointing exposed operations.

There are more objects in $\text{lk}_1(A)$ and we want to show that the inclusion $\mathcal{Z} \to \text{lk}_1(A)$ is a homotopy equivalence using Quillen’s Theorem A. Write arrows $Y \to Z$ in $\mathcal{S}$ as reduced very elementary string diagrams with all transistors pointing to the right. An object in $\text{lk}_1(A)$ is an arrow $\beta : Y \to Z$ such that at least one left pointing exposed operation in the diagram of $A$ forms a dipole with one of the very elementary operations in the diagram of $\beta$. Let $\gamma : Y \to Z'$ be the object in $\mathcal{Z}$ containing all the very elementary operations in the diagram of $\beta$ which form dipoles with left pointing exposed operations in the diagram of $A$.

\[ \begin{array}{c}
\theta \\
\psi \\
\theta
\end{array} \]
In \( \underline{lk}_\gamma(A) \), there is exactly one arrow \( \gamma \rightarrow \beta \) and that arrow represents an object in the comma category \( \mathcal{Z}\downarrow \beta \). This object is the terminal object in \( \mathcal{Z}\downarrow \beta \) and thus the inclusion \( \mathcal{Z} \rightarrow \underline{lk}_\gamma(A) \) is a homotopy equivalence by Quillen’s Theorem A.

6.3. Connectivity of the descending links. In the subsection, we prove that the connectivity of the descending links \( \underline{lk}_\gamma(K) \) with respect to the degree Morse function on \( \mathcal{S} \) tends to infinity as the degree of the object \( K \) tends to infinity.

**Definition 6.9.** An arrow in \( \mathcal{S} \) is called (very) elementary if it is not an identity arrow and every non-identity operation in that arrow is (very) elementary. An arrow is called atomic if there is exactly one non-identity operation in that arrow and that operation is very elementary.

Note that the descending up link \( \overline{lk}_\gamma(K) \) is always empty, so we have \( \underline{lk}_\gamma(K) = \overline{lk}_\gamma(K) \). Its objects are pairs \( (Y, \alpha: K \rightarrow Y) \) where \( Y \) is an object of strictly smaller degree than \( K \) and \( \alpha: K \rightarrow Y \) is an arrow in \( \mathcal{S} \). Denote by \( \text{Core}(K) \) the full subcategory of \( \underline{lk}_\gamma(K) \) spanned by the objects \( (Y, \alpha) \) where \( \alpha \) is a very elementary arrow. Denote by \( \text{Corona}(K) \) the full subcategory of \( \underline{lk}_\gamma(K) \) spanned by the objects \( (Y, \alpha) \) where \( \alpha \) is an elementary arrow. So we have

\[
\text{Core}(K) \subset \text{Corona}(K) \subset \underline{lk}_\gamma(K)
\]

and we will study the connectivity of these spaces successively.

6.3.1. The core. First we define a simplicial complex \( \mathcal{C}(K) \) as follows: Let the vertex set be the set of objects \( (Y, \alpha) \) of \( \text{Core}(K) \) with \( \alpha: K \rightarrow Y \) an atomic arrow. Connect two such vertices \( (Y, \alpha), (Y', \alpha') \) by an edge if the domains of the very elementary operations in \( \alpha, \alpha' \) are disjoint in \( K \). Let \( \mathcal{C}(K) \) be the flag complex with 1-skeleton this graph.

**Lemma 6.10.** \( \mathcal{C}(K) \) is homeomorphic to \( B(\text{Core}(K)) \).

**Proof.** First note that if very elementary operations are (partially) composed with another very elementary operation, then the result is definitely not very elementary any more. From this it follows easily that \( \text{Core}(K) \) is a poset.

Now if we view a simplicial complex as a poset of simplices, then this poset satisfies the following two conditions:

i) If \( x, y \) are two elements, then either the set of elements \( z \) with \( z \leq x \) and \( z \leq y \) is empty or has a greatest element.

ii) For each element \( x \) there exists a \( n \geq 0 \) such that the subposet of all elements \( z \) with \( z \leq x \) is isomorphic to the poset of non-empty subsets of the set \( \{0, \ldots, n\} \).

In fact, these two properties characterize the simplicial complexes among the posets. These properties are not hard to verify for \( \text{Core}(K) \), see pictures below.
Theorem 6.11
Subsection 4.2]. For this, we need:

Let \( K \) be a simplicial complex and \( \{D_i\}_{i=1}^n \) a family of subcomplexes which covers \( C \). Let \( k \geq -1 \). Assume that each non-empty intersection \( D_{i_1} \cap \ldots \cap D_{i_t} \) for \( t \geq 1 \) is \((k-t+1)\)-connected. Then \( C \) is \( k \)-connected if and only if \( \mathcal{N} \) is \( k \)-connected where \( \mathcal{N} \) is the nerve of the cover \( \{D_i\}_{i=1}^n \).

Our operad \( \mathcal{O} \) is assumed to be of finite type, in particular finitely generated, i.e. \( V \mathcal{E} \) is finite. Denote by \( \max_{\mathcal{V} \mathcal{E}} \) the maximal degree of all operations in \( \mathcal{V} \mathcal{E} \). The number

\[
\max_C := \max\{\deg(X) \mid X \text{ is reduced}\}
\]

is finite because \( \mathcal{O} \) is assumed to be color-tame. Define

\[
\kappa(n) := \left\lfloor \frac{n - 2\max_C - \max_{\mathcal{V} \mathcal{E}} - 1}{\max_C + \max_{\mathcal{V} \mathcal{E}}} \right\rfloor
\]

Proposition 6.12. \( \mathcal{C}(K) \) and therefore \( \text{Core}(K) \) is \( \kappa(\deg K) \)-connected.

Proof. We will prove: If

\[
\deg(K) \geq \xi_k := \max_C + 1 + (k + 1)(\max_C + \max_{\mathcal{V} \mathcal{E}})
\]

then \( \mathcal{C}(K) \) is \( k \)-connected. We prove this by induction over \( k \). The case \( k = -1 \) means that \( \deg(K) > \max_C \) and therefore \( K \) is not reduced. It follows that \( \mathcal{C}(K) \) is not empty, i.e. \((-1)\)-connected.

Assume now the statement is true for \( k - 1 \) with \( k \geq 0 \) and we want to prove it for \( k \). Let \( \mathcal{V} = \{v_i\}_{i=1}^l \) be the vertices \( v_i = (Y_i, \alpha_i) \) of \( \mathcal{C}(K) \) such that \( \alpha_i = \text{id}_{R_i} \otimes \mathcal{O}_i \otimes \text{id}_{S_i} \) and each \( R_i \) is a reduced object (in particular, it could be the unit object, i.e. the empty word of colors). We want to apply the Nerve Theorem to the cover \( \mathcal{D}_i := \text{st}(v_i) \) where \( \text{st}(v_i) \) denotes the star of \( v_i \) in \( \mathcal{C}(K) \). To see that this is indeed a cover of \( \mathcal{C}(K) \), let \( w_1, \ldots, w_l \) be vertices representing a simplex in \( \mathcal{C}(K) \). If one of these vertices is contained in \( \mathcal{V} \), the simplex is already contained in the star of that vertex. If, on the other hand, none of the \( w_i \) is contained in \( \mathcal{V} \), then there must be a \( v \in \mathcal{V} \) such that each \( w_i \) is connected to \( v \) by an edge and consequently, the simplex represented by the \( w_i \) is contained in \( \text{st}(v) \). Now, for each \( i = 1, \ldots, l \) we have

\[
\deg(S_i) = \deg(K) - \deg(O_i) - \deg(R_i) \\
\geq \xi_k - \max_{\mathcal{V} \mathcal{E}} - \max_C \\
= \max_C + 1 + k(\max_C + \max_{\mathcal{V} \mathcal{E}})
\]

Therefore, we have \( \deg(S_i) \geq \xi_{k-1} > \max_C \). From this we see that the intersection \( \bigcap_{i=1}^l \text{st}(v_i) \) is non-empty. Thus, the nerve \( \mathcal{N} \) of that cover is a simplex and therefore contractible. It remains to show that each non-empty intersection \( \text{st}(v_{i_1}) \cap \ldots \cap \text{st}(v_{i_t}) \) for \( t \geq 2 \) is \((k-t+1)\)-connected. Let \( S \) be the object among the \( S_i \) which is of minimal degree. By definition of the \( v_i \), no two distinct \( v_i \) are joined by an edge. This implies

\[
\text{st}(v_{i_1}) \cap \ldots \cap \text{st}(v_{i_t}) = \text{lk}(v_{i_1}) \cap \ldots \cap \text{lk}(v_{i_t}) \cong \mathcal{C}(S)
\]

where \( \text{lk}(v_i) \subset \text{st}(v_i) \) denotes the link of \( v_i \) in \( \mathcal{C}(K) \). Because of \( \deg(S) \geq \xi_{k-1} \) and by the induction hypothesis, we get that \( \mathcal{C}(S) \) is \((k-1)\)-connected and in particular \((k-t+1)\)-connected. \( \square \)
6.3.2. The corona. First note that $\text{VE} = E$ and therefore $\text{Core}(K) = \text{Corona}(K)$ in the free case A) of Theorem 6.10. So this subsubsection only applies to case B).

We build up $\text{Corona}(K)$ from $\text{Core}(K)$ using again the Morse method for categories. We then get a connectivity result for the corona from the connectivity result for the core. The idea is attributed to [17].

We assumed $\mathcal{O}$ to be of finite type, i.e. the set of elementary operations $E$ is finite. Let $\max_E$ be the maximal degree of all operations in $E$. An object in $\text{Corona}(K)$ is a pair $(Y, \alpha: K \to Y)$ where $\alpha$ is an elementary arrow. For $2 \leq k \leq \max_E$, denote by $\#_{se}(\alpha)$ the number of strictly elementary operations of degree $k$ in the arrow $\alpha$. Note that $\deg(Y)$ is the number of operations in $\alpha$, including the identity operations. Now define

$$f((Y, \alpha)) := \left(\#_{se}^\maxE(\alpha), \#_{se}^\maxE-1(\alpha), \ldots, \#_{se}^2(\alpha), \deg(Y)\right)$$

Order the values of $f$ lexicographically. Then $f$ becomes a Morse function on $\text{Corona}(K)$ with base space $\text{Core}(K)$. Define

$$\chi(n) := \left\lfloor \frac{n - 2 \maxC + \maxVE - 1}{2 \maxC + \maxVE + \maxE + 1} - 2 \right\rfloor$$

**Proposition 6.13.** For each object $(Y, \alpha)$ in $\text{Corona}(K)$ but not in $\text{Core}(K)$, the descending link $\text{lk}_1(Y, \alpha)$ is $\chi(\deg K)$-connected.

From Theorem 5.9 we get that $\text{Core}(K)$ and $\text{Corona}(K)$ share the same homotopy groups up to dimension $\deg(K)$. We already know that $\text{Core}(K)$ is $\kappa(\deg K)$-connected.

**Corollary 6.14.** $\text{Corona}(K)$ is $\min\{\kappa(\deg K), \chi(\deg K)\}$-connected. In particular, its connectivity tends to infinity when $\deg(K) \to \infty$.

In the rest of this subsubsection, we give a proof of 6.13. We distinguish between two sorts of objects $(Y, \alpha)$ in $\text{Corona}(K)$ which are not objects in $\text{Core}(K)$ (i.e. all non-identity operations in $\alpha$ are elementary and there exists at least one which is not very elementary). Such an object is called mixed if there is at least one very elementary operation in $\alpha$. It is called pure if there is no very elementary operation in $\alpha$.

**Lemma 6.15.** Let $(Y, \alpha)$ be mixed. Then $\overline{\text{lk}}_1(Y, \alpha)$ and therefore $\text{lk}_2(Y, \alpha)$ is contractible. In particular, Proposition 6.13 is true for mixed objects.

**Proof.** The data of an object in $\overline{\text{lk}}_1(Y, \alpha)$ is $\Omega = ((L, \beta_1), \beta_2)$ where $L$ is an object in $\mathcal{S}$, $\beta_1$ is an elementary arrow in $\mathcal{S}$ and $\beta_2$ is an arrow in $\mathcal{S}$ such that $\beta_1 \beta_2 = \alpha$. Furthermore, $(L, \beta_1)$ forms an object in $\text{Corona}(K)$ of strictly smaller Morse height than $(Y, \alpha)$, i.e. $\beta_1$ arises from $\alpha$ by splitting at least one operation that is not very elementary. Let $\Omega' = ((L', \beta'_1), \beta'_2)$ be another such object. An arrow $\Omega \to \Omega'$ is represented by an arrow $\delta: L \to L'$ such that $\beta_1 \delta = \beta'_1$ and $\delta \beta_2 = \beta'_2$.

It is easy to see that the right cancellation property of $\mathcal{S}$ implies that $\overline{\text{lk}}_1(Y, \alpha)$ is a poset.

Let $\alpha^v: K \to Y^v$ be the arrow obtained from $\alpha$ by replacing all elementary but not very elementary operations $O$ with $\deg(O)$ identity operations. Then $(Y^v, \alpha^v)$ is an object in $\text{Core}(K)$. Let $\alpha^{ve}$ be the arrow obtained from $\alpha$ by replacing all
very elementary operations by one identity operation each. We have $\alpha^v \alpha^{se} = \alpha$ by definition. Thus, $\alpha^{se}$ represents an arrow $(Y^v, \alpha^v) \to (Y, \alpha)$ in $\text{Corona}(K)$. Moreover, the pair $\Xi := ((Y^v, \alpha^v), \alpha^{se})$ is an object in $\overline{\text{lk}}_K(Y, \alpha)$. An example of $\alpha, \alpha^v, \alpha^{se}$ is pictured below. There, a white triangle represents a strictly elementary operation. A black triangle represents a very elementary operation. A straight horizontal line represents an identity operation.

Let $\Omega = ((L, \beta_1), \beta_2)$ be an object in $\overline{\text{lk}}_K(Y, \alpha)$. We define another object $F(\Omega) = ((M, \gamma_1), \gamma_2)$ of $\overline{\text{lk}}_K(Y, \alpha)$ as follows: The arrows $\gamma_1, \gamma_2$ form the same factorization of $\alpha$ as the arrows $\beta_1, \beta_2$, with the only difference that no very elementary operations of $\alpha$ are splitted. In other words, we pull all the very elementary operations of $\beta_2$, which compose with $\beta_1$ to very elementary operations in $\alpha$, back into $\beta_1$, see the picture below. There, a gray triangle is an elementary operation or an identity operation and a blue triangle can be any operation. Furthermore, $(M, \gamma_1)$ is of strictly lower Morse height than $(Y, \alpha)$, so $F(\Omega)$ is indeed an object in $\overline{\text{lk}}_K(Y, \alpha)$.

We claim that $\Omega \mapsto F(\Omega)$ extends to a functor $\overline{\text{lk}}_K(Y, \alpha) \to \overline{\text{lk}}_K(Y, \alpha)$. To see this, we must show that whenever we have an arrow $\delta: \Omega \to \Omega'$, then there is an arrow $\Delta: F(\Omega) \to F(\Omega')$.

This can easily be verified using the definitions: $\Delta$ is obtained from $\delta$ by replacing all the very elementary operations coming from $\alpha$ (which are splitted in $\beta_1$ but not in $\beta_1'$) with identities.
We also have arrows $\xi_\Omega: \Xi \to F(\Omega)$ and $\iota_\Omega: \Omega \to F(\Omega)$.

The arrow $\xi_\Omega$ is given by an arrow $\gamma_1$ which is obtained from $\alpha$ by replacing all the very elementary operations coming from $\alpha$ with identities. For this arrow to give an arrow $\Xi \to F(\Omega)$ in $\mathcal{K}_i(Y, \alpha)$, the obvious two triangles have to commute, i.e. $\alpha^r \xi_\Omega = \gamma_1$ and $\xi_\Omega \gamma_2 = \alpha^s$. But this is checked easily. The arrow $\iota_\Omega$ is given by an arrow $L \to M$ which is obtained from $\gamma_1$ by replacing all operations with identities except the very elementary ones coming from $\alpha$ which are not present in $\beta_1$. We have $\beta_1 \iota_\Omega = \gamma_1$ and $\iota_\Omega \gamma_2 = \beta_2$, so this is indeed an arrow $\Omega \to F(\Omega)$ in $\mathcal{K}_i(Y, \alpha)$.

The claim of the proposition now follows from item iii) in Subsection 2.4 applied to the functor $F$ and the object $\Xi$. □

**Lemma 6.16.** Let $(Y, \alpha)$ be pure. Then $\mathcal{K}_i(Y, \alpha)$ is $\chi(\deg K)$-connected and Proposition 6.13 is true for pure objects.

**Proof.** First observe the descending up link $\mathcal{K}_i(Y, \alpha)$. Just as in the proof of the previous lemma, an object in $\mathcal{K}_i(Y, \alpha)$ is a pair $((L, \beta_1), \beta_2)$ where $\beta_1$ is obtained from $\alpha$ by splitting at least one operation and $\beta_2$ is an arrow which merges the splittings back to the original operations, i.e. $\beta_1 \beta_2 = \alpha$. Denote by $A_i$ the full subcategory of $\mathcal{K}_i(Y, \alpha)$ spanned by the objects which only split the $i$'th (non-identity) operation in $\alpha$. Denote by $n$ the number of non-identity operations in $\alpha$. Observe now that when splitting operations in $\alpha$ one by one, then we can also split all those operations at once. This observation reveals that $\mathcal{K}_i(Y, \alpha) = A_1 \circ \ldots \circ A_n$ is the Grothendieck join of the $A_i$ introduced in Subsection 2.4. Note that the categories $A_i$ are all non-empty since all the non-identity operations in $\alpha$ are elementary but not very elementary.

Now look at the descending down link $\mathcal{K}_i(Y, \alpha)$. Objects are pairs $((L, \beta_1), \beta_2)$ just as in the up link with the only exception that now $\alpha \beta_2 = \beta_1$. Still, $(L, \beta_1)$ must be of strictly lower Morse height. Looking at the Morse function $f$ for the corona, one sees that $\beta_2$ must be a very elementary arrow and the very elementary operations in $\beta_2$ only compose with identity operations in $\alpha$. Roughly speaking, only very elementary operations are added into identity components of $\alpha$. An identity
component in $\alpha$ is a maximal subsequence of operations which only consists of identity operations. Let $m$ be the number of identity components and denote by $l_i$ for $i = 1, \ldots, m$ the lengths of the identity components. Denote by $l$ the total number of identity operations in $\alpha$, i.e., the sum of the $l_i$. Define $B_i$ to be the full subcategory of $lk_i(Y, \alpha)$ spanned by the objects which only add very elementary operations into the $i$'th identity component. Observe now that when adding very elementary operations into different identity components one by one, then we can also add all those operations at once. This reveals, as above, that $lk_i(Y, \alpha)$ is the Grothendieck join of the $B_i$. Note, though, when inspecting the direction of the arrows in $lk_i(Y, \alpha)$, one sees that it is in fact the dual Grothendieck join. So we have

$$lk_i(Y, \alpha) = B_1 \cdot \ldots \cdot B_m$$

Note that $B_i$ is canonically isomorphic to Core($B_i$) where $B_i$ is the subobject of $K$ corresponding to the $i$'th identity component of $\alpha$. The degree of $B_i$ is therefore $l_i$.

Summarizing we have

$$lk_1(Y, \alpha) = (A_1 \circ \ldots \circ A_n) \ast (B_1 \cdot \ldots \cdot B_m)$$

Recall that a join of a $k_1$-connected space with a $k_2$-connected space gives a space which is $(k_1 + 1) + (k_2 + 1)$-connected. As noted above, each $A_i$ is $(-1)$-connected. Therefore, $lk_i(Y, \alpha)$ is at least $(n - 2)$-connected. Furthermore, we already know that $B_i \cong \text{Core}(B_i)$ is $\kappa(l_i)$-connected. Therefore, $lk_i(Y, \alpha)$ has connectivity at least

$$2m - 2 + \sum_{j=1}^{m} \kappa(l_j)$$

All in all, $lk_1(Y, \alpha)$ has connectivity at least

$$n + 2m - 2 + \sum_{j=1}^{m} \kappa(l_j)$$

Now since $E$ is finite, we have the maximal degree $\max_{E}$ of operations in $E$. Then either we have a lot of non-identity operations in $\alpha$ (which are all elementary) or we have large identity components. In the first case we get high connectivity from $lk_1(Y, \alpha)$ and in the second case we get high connectivity from $lk_k(Y, \alpha)$. We make this explicit by the following calculation:

$$n + 2m - 2 + \sum_{j=1}^{m} \kappa(l_j) = n + 2m - 2 + \sum_{j=1}^{m} \left[ \frac{l_j - 2 \max_{C} - \max_{VE} - 1}{\max_{C} + \max_{VE}} \right]$$

$$\geq n + m - 2 + \sum_{j=1}^{m} \left[ \frac{l_j - 2 \max_{C} - \max_{VE} - 1}{\max_{C} + \max_{VE}} \right]$$

$$\geq n - 2 + \frac{1 + m(-2 \max_{C} - \max_{VE} - 1)}{\max_{C} + \max_{VE}}$$

$$\geq n - 2 + \frac{1 + (n + 1)(-2 \max_{C} - \max_{VE} - 1)}{\max_{C} + \max_{VE}}$$

$$\geq n - 2 + \frac{1 + (n + 1)(-2 \max_{C} - \max_{VE} - 1)}{2 \max_{C} + \max_{VE} + \max_{E} + 1}$$

$$\geq \frac{\deg(K) + (-2 \max_{C} - \max_{VE} - 1)}{2 \max_{C} + \max_{VE} + \max_{E} + 1} - 2$$

$$\geq \chi(\deg K)$$
In the fourth step we have used that $m \leq n + 1$ and in the seventh step we have used that $1 + n \max_E \geq \deg(K)$. 

6.3.3. The whole link. In this last step, we show that the inclusion $\text{Corona}(K) \subset \text{lk}_2(K)$ is a homotopy equivalence. It then follows from Corollary 6.14 that the connectivity of $\text{lk}_i(K)$ tends to infinity as $\deg(K) \to \infty$. This step is analogous to the reduction to the Stein space of elementary intervals in [22]. We again apply the Morse method for categories to build $\text{lk}_2(K)$ up from $\text{Corona}(K)$.

The Morse function on objects of $\text{lk}_2(K)$ which do not lie in $\text{Corona}(K)$ is given by

$$f((Y, \alpha)) := -\deg(Y)$$

Thus, if we look at such an object $(Y, \alpha)$ in $\text{lk}_2(K)$, the only possibility to obtain an object of strictly lower Morse height is to split at least one operation in $\alpha$. In other words, $(\text{lk}_2(K), Y, \alpha) = \emptyset$ and $\text{lk}_2(K, Y, \alpha) = \overline{\text{lk}}_1(Y, \alpha)$. Let $A_i$ be the full subcategory of $\overline{\text{lk}}_1(Y, \alpha)$ spanned by the objects which only split the $i$'th non-identity operation in $\alpha$. Just as in the proof of 6.16, we see that

$$\overline{\text{lk}}_1(Y, \alpha) = A_1 \circ \ldots \circ A_n$$

where $n$ is the number of non-identity operations in $\alpha$. At least one of these non-identity operations must be non-elementary since $(Y, \alpha)$ is not an object in $\text{Corona}(K)$. Without loss of generality, assume that $A_1$ corresponds to that non-elementary operation. If we show that $A_1$ is contractible, it follows that $\text{lk}_1(K, Y, \alpha)$ is contractible. Thus, we are building $\text{lk}_1(K)$ up from $\text{Corona}(K)$ along contractible descending links and it follows from Theorem 5.6 that the inclusion $\text{Corona}(K) \subset \text{lk}_1(K)$ is a homotopy equivalence. That $A_1$ is contractible follows from Proposition 6.18 below. But first we need a general lemma concerning elementary operations:

**Lemma 6.17.** Let $O$ be plain. For each non-elementary operation $O \neq \text{id}$, the set $\{\theta \in E \mid \theta < O\}$ has a greatest element which we call the greatest elementary off-splitting of the non-elementary operation $O$.

**Proof.** First observe that we have $\deg(\theta_1) < \deg(\theta_2)$ for operations $\theta_1 < \theta_2$. This property will be crucial to the proof.

Let $S$ be any set of operations. Assume that $\psi$ is an operation with $s < \psi$ for all $s \in S$. We claim that there is a minimal operation $\eta$ in the set $\{\theta \in O \mid \forall_s \in S \ s < \theta\}$ such that $\eta \leq \psi$. If $\psi$ is minimal, then we can set $\eta = \psi$. If it is not minimal, then there must be another operation $\psi'$ with $s < \psi' < \psi$ for all $s \in S$. Then $\psi'$ has strictly smaller degree than $\psi$. If we repeat this argument with $\psi'$, we have to end up with a minimal element $\eta$ at some time, because the degree function is bounded below. This $\eta$ surely satisfies $\eta \leq \psi$.

Now we find the greatest element in $\Omega := \{\theta \in E \mid \theta < O\}$. Recall the notation in Definition 6.3. For each $i$, set $E_i^O = E_i \cap \Omega$. We claim: There exists exactly one $i_0$ such that $|E_j^O| > 1$ for $j < i_0$, $|E_i^O| = 1$ and $E_j^O = \emptyset$ for $j > i_0$ and the unique operation in $E_i^O$ is the greatest element in $\Omega$. First, it is clear that $E_0^O \neq \emptyset$. Note that either all but finitely many of the $E_i$ are empty or the sequence of numbers $d_i := \min\{\deg(\theta) \mid \theta \in E_i\}$ tends to infinity. But the degree of all the operations in all the $E_i^O$ is bounded by $\deg(O)$. It follows that in any case there must be a $i_0$ such that $E_j^O = \emptyset$ for all $j > i_0$. Choose the $i_0$ which is minimal with respect to this property, i.e. $E_{i_0}^O \neq \emptyset$. Assume $|E_{i_0}^O| > 1$ and let $\theta_1 \neq \theta_2$ be two operations in this set. If these two operations are comparable, e.g. $\theta_1 < \theta_2$, then $\theta_2 \in E_{i_0+1}^O$, a contradiction. Else, write $S = \{\theta_1, \theta_2\}$. Recall that $\theta_1, \theta_2 < O$. Thus, by the observations in the second paragraph, we find that there must be a minimal $\eta \in M_{i_0+1}(\theta_1, \theta_2)$ with $\eta \leq O$. Since $O$ is non-elementary, we have indeed $\eta < O$. By definition, this means $\eta \in E_{i_0+1}^O$, a contradiction again. So we have
indeed $|E_{i_0}^\Omega| = 1$. Next, observe that for any $j$, if $E_j \neq \emptyset$, then $E_{j-1}$ consists of at least two elements. This follows directly from the definitions. Consequently, the same holds for the $E_j^{\Omega}$. From this, it easily follows $|E_{i_0}^{\Omega}| > 1$ for $j < i_0$.

We now use this to prove that the unique operation $P$ in $E_{i_0}$ is the greatest element in $\Omega$, i.e. $\theta \leq P$ whenever $\theta \in \Omega$ with $\theta < \eta$. Let $\eta$ be such an element. If $\theta \neq P$, then there must be some $j < i_0$ such that $\theta \in E_j^{\Omega}$. There is another element $\theta'$ in this $E_j^{\Omega}$. If $\theta$ and $\theta'$ are comparable, then set $\eta := \max\{\theta, \theta'\}$. We then have $\eta \in E_{j+1}^{\Omega}$ and $\theta \leq \eta$. Else, the argument in the second paragraph applied to $S = \{\theta, \theta'\}$ and $\psi = O$ shows that there is $\eta \in E_{j+1}^{\Omega}$ with $\theta < \eta$. If $j + 1 = i_0$, then $\eta$ must be $P$ and we are done. Else, we repeat this process with $\eta$ in place of $\theta$ until we reach level $i_0$. This completes the proof.

In the proof of the announced proposition below, we use the right cancellation property for $S$ several times. This is an assumption in case B) of Theorem 5.6 and a consequence of freeness in case A).

**Proposition 6.18.** Let $O \neq \id$ be a non-elementary operation regarded as an arrow $N \rightarrow A$ in $S$ with $\deg(N) = n$ and $\deg(A) = 1$. Let $\mathcal{M}$ be the full subcategory of $N \downarrow S_{n-1}$ spanned by the objects $(Y, \alpha : N \rightarrow Y)$ with $\deg(Y) > 1$ and

$$\mathcal{L} := \mathcal{M} \downarrow O$$

the descending up link of $O$ with respect to the Morse function $f$ above. Then $\mathcal{L}$ is contractible.

**Proof.** Note that the data of an object of $\mathcal{L}$ is a non-trivial factorization of $O$, i.e. a pair of arrows $(\alpha_1, \alpha_2)$ such that $\alpha_1 \neq \id \neq \alpha_2$ and $\alpha_1 \alpha_2 = O$. An arrow from $(\alpha_1, \alpha_2)$ to $(\beta_1, \beta_2)$ is an arrow $\gamma$ such that $\alpha_1 \gamma = \beta_1$ and $\gamma \beta_2 = \alpha_2$. From the right cancellation property of $S$ it follows that $\mathcal{L}$ is a poset.

Let $O_{ge}$ be the greatest elementary off splitting operation of $O$. Then there is an arrow $\omega$ such that $\omega O_{ge} = O$ and by the right cancellation property it is unique. The pair $(\omega, O_{ge})$ is an object in $\mathcal{L}$. More generally, for an object $(\beta, \theta)$ in $\mathcal{L}$, define $\theta_{ge}$ to be the greatest elementary off splitting of $\theta$ if $\theta$ is non-elementary and let $\theta_{ge} = \omega$ otherwise. In any case, there is an arrow $\alpha_\theta$ such that $\alpha_\theta \theta_{ge} = \theta$ and this arrow is unique because of the right cancellation property. Note that $(\beta \alpha_\theta, \theta_{ge})$ is an object in $\mathcal{L}$. Set $\text{ge}(\beta, \theta) := (\beta \alpha_\theta, \theta_{ge})$.

We will show that $\text{ge} : \mathcal{L} \rightarrow \mathcal{L}$ is a functor. So let $(\beta, \theta)$ and $(\beta', \theta')$ be two objects of $\mathcal{L}$ and $\gamma : (\beta, \theta) \rightarrow (\beta', \theta')$ an arrow. We have $\text{ge}(\beta, \theta) = (\beta \alpha_\theta, \theta_{ge})$ and $\text{ge}(\beta', \theta') = (\beta' \alpha_{\theta'}, \theta'_{ge})$.

\[
\begin{array}{c}
\bullet \\
\downarrow^n_{\beta} \\
\gamma \\
\downarrow^n_{\beta'} \\
\bullet
\end{array}
\]

\[
\begin{array}{c}
\alpha_\theta \\
\downarrow^n_{\theta_{ge}} \\
\theta \\
\downarrow^n_{\beta} \\
\bullet
\end{array}
\]

\[
\begin{array}{c}
\alpha_{\theta'} \\
\downarrow^n_{\theta'_{ge}} \\
\theta' \\
\downarrow^n_{\beta'} \\
\bullet
\end{array}
\]

Observe that if $\theta_{ge} = \theta$ then $\alpha_\theta = \id$ and so $\gamma_{\alpha_\theta}$ is an arrow $\text{ge}(\beta, \theta) \rightarrow \text{ge}(\beta', \theta')$ as required. So assume that $\theta_{ge}$ is indeed the greatest elementary off splitting of $\theta$. Then observe that also $\theta'_{ge}$ is an elementary off splitting of $\theta$ because $(\gamma_{\alpha_\theta})\theta'_{ge} = \theta$. We therefore get an arrow $\varphi$ with $\varphi \theta'_{ge} = \theta_{ge}$. Furthermore, we have

$$(\beta \alpha_\theta \varphi)\theta'_{ge} = O = (\beta' \alpha_{\theta'})\theta'_{ge}$$
By right cancellation we get $(\beta \alpha_\theta) \varphi = \beta' \alpha_\theta$. Consequently, $\varphi$ gives an arrow $ge(\beta, \theta) \to ge(\beta', \theta')$ and we are done.

Last but not least, observe that there is an arrow $(\omega, O_{ge}) \to ge(\beta, \theta)$ because of the following: Since $\theta_{ge} < O$ is elementary and $O_{ge}$ is the greatest elementary operation satisfying that inequality, we get an arrow $\delta$ in $S$ such that $\delta \theta_{ge} = O_{ge}$.

We have

$$(\omega \delta) \theta_{ge} = O = (\beta \alpha_\theta) \theta_{ge}$$

and thus $\omega \delta = \beta \alpha_\theta$ by the right cancellation property. So $\delta$ indeed gives an arrow $(\omega, O_{ge}) \to ge(\beta, \theta)$. Furthermore, $\alpha_\theta$ clearly gives an arrow $(\beta, \theta) \to ge(\beta, \theta)$. The claim now follows from item iii) in Subsection 2.4 applied to the functor $ge$ and the object $(\omega, O_{ge})$. □

6.4. Applications. We now want to apply Theorem 6.6 to our examples of Section 4. The plain operads among them are the

A) free planar operads generated by operations of degree at least 2 and
B) the 1-dimensional planar cube cutting operads.

These two cases correspond to the cases A) and B) of Theorem 6.6.

In case A), the very elementary operations are exactly the free generators of the operad. So it is finitely generated exactly when there are only finitely many free generators. Furthermore, to apply the theorem, the operad should be color-tame, i.e., there are only finitely many colors and the degree of reduced objects is bounded from above.

In [11, Theorem 4.4] it is shown that the diagram groups associated to a finite complete presentation of a finite semigroup are of type $F_\infty$. Assume without loss of generality that one side of each relation in this presentation consists of a single generator only (see the discussion in Subsection 4.1). The finiteness of the presentation then corresponds to finitely many colors and finitely many free generators in our language. Furthermore, if we have a finite complete presentation of a finite semigroup, then there are only finitely many reduced words (see [11, Subsection 4.2]). This implies color-tameness. So our result in the free case is not new. However, note that the proof covers a much bigger class than just the free operads. On the other hand, the length of the proof stays the same regardless of whether we include the free case or not (apart from the asphericity result in Theorem 6.8).

In case B), we only have one color which implies color-tameness. Furthermore, we have already shown in Proposition 4.1 that these 1-dimensional planar cube cutting operads satisfy the right cancellative calculus of fractions. It remains to check the finite type condition. So let the cuttings of the unit interval $I$ be given by the finite set $N$ of coprime integers greater or equal to 2. To each subset $\{s_1, ..., s_k\} = S \subset N$ corresponds an operation $\theta_S$ with the defining property that the length of each subinterval in the subdivision of $I$ equals $(s_1 \cdots s_k)^{-1}$. These are exactly the elementary operations. The very elementary operations are the operations $\theta(n)$ with $n \in N$. Given $\theta_{S_1}$ and $\theta_{S_2}$ with $S_1 \neq S_2$, then $\theta_{S_1 \cup S_2}$ is the smallest operation $\theta$ with $\theta_{S_1} < \theta \not> \theta_{S_2}$.

So we get from the theorem that all the 1-dimensional planar cube cutting operad groups are of type $F_\infty$. This has been shown before in [32]. We believe that there are many more operads satisfying the conditions of Theorem 6.6.

The non-plain operads among the examples of Section 4 will be covered by a more general theorem in [35] (see Remark 6.7).


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