Geometrical Equivalence and Action Type Geometrical Equivalence of Group Representations

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Abstract

The universal algebraic geometry of group representations was considered in [10]. Thereat the concepts of geometrical equivalence and action type geometrical equivalence of group representations were defined. It was proved [10, Corollary 2 from Proposition 4.2.] that if two representations are geometrically equivalent then they are action type geometrically equivalent. Also it was remarked [10, Remark 5.1] that if two representations \((V_1, G_1)\) and \((V_2, G_2)\) are action type geometrically equivalent and groups \(G_1\) and \(G_2\) are geometrically equivalent, the representations \((V_1, G_1)\) and \((V_2, G_2)\) are not necessarily geometrically equivalent. But some specific counterexample was not presented. In this paper we construct the example of two representations \((V_1, G_1)\) and \((V_2, G_2)\) which are action type geometrically equivalent and groups \(G_1\) and \(G_2\) are geometrically equivalent, but the representations \((V_1, G_1)\) and \((V_2, G_2)\) are not geometrically equivalent.

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1 Introduction

All definitions of the basic notions of the universal algebraic geometry can be found, for example, in [5], [6], [7] and [8]. Also, there are fundamental papers [1], [4] and [2], [3].

1
Some problems of the universal algebraic geometry for many-sorted universal algebras were considered also in [10], [12], [13] and [14].

Let $K$ be an arbitrary but fixed field. We consider representations of groups over $K$-vectorspaces. In this article a representation of group is a pair $(V, G)$, where $V$ is a vector space over field $K$ and $G$ is a group. The signature of this algebraic object includes all operations of the vector space $V$ (multiplication by a scalar $\lambda \in K$ we consider as unary operation: $\lambda : V \rightarrow V$, where $\lambda (v) = \lambda v$, $v \in V$), all operations of the group $G$ and the operation of action of the $G$ on vector space $V$. We denote this operation by $\circ$:

$$V \times G \ni (v, g) \rightarrow v \circ g \in V.$$  

From now on we will write briefly "representation" instead of "representation of group". The homomorphism $(\alpha, \beta) : (V, G) \rightarrow (W, H)$ from representation $(V, G)$ to representation $(W, H)$ is a pair, where $\alpha : V \rightarrow W$ is a linear mapping and $\beta : G \rightarrow H$ is a homomorphism of groups, such that for every $v \in V$ and every $g \in G$ the equality

$$\alpha (v \circ g) = \alpha (v) \circ \beta (g)$$

holds. The reader can see that we consider representations as 2-sorted universal algebras: the first sort is a sort of vectors from a vector space and the second sort is a sort of elements from a group. This approach to representations can be found in [11], [15], [10], [13].

The variety of all representations of groups over a fixed field $K$ will be denoted by $\text{REP} - K$.

We say that

**Definition 1.1** The representation $W(X, Y) = (U(X, Y), H(X, Y))$ is called the free representation generated by the sets $X$ and $Y$ if, for every $(V, G) \in \text{REP} - K$ and every mappings $f_1 : X \rightarrow V$ and $f_2 : Y \rightarrow G$, there is only one homomorphism of representations $(\alpha, \beta) : (U(X, Y), H(X, Y)) \rightarrow (V, G)$, such that $\alpha|_X = f_1$, $\beta|_Y = f_2$.

It was proved in [11], that $(U(X, Y), H(X, Y)) = (XKF(Y), F(Y))$, where $F(Y)$ is the free group with the free set of generators $Y$, $KF(Y)$ is the group ring over $F(Y)$ and $XKF(Y) = \bigoplus_{x \in X} xKF(Y)$ is the free $KF(Y)$-module with free basis $X$.

2 Basic notions of the algebraic geometry of representations

From now on we suppose that the sets $X$ and $Y$ of generators of the free representations $(XKF(Y), F(Y))$ are finite. Equations in the algebraic geometry of representations have the form $v_1 = v_2$, where $v_1, v_2 \in XKF(Y)$, or
the form \( f_1 = f_2 \) where \( f_1, f_2 \in F(Y) \). But the equation of the first form is equivalent to the equation \( v_1 - v_2 = 0 \), and the equation of the second form is equivalent to the equation \( f_1f_2^{-1} = 1 \). So, in algebraic geometry of representations we can consider the system of equations \( T = (T_1, T_2) \), where \( T_1 \subseteq XKF(Y), T_2 \subseteq F(Y) \), or, briefly, \( (T_1, T_2) \subseteq (XKF(Y), F(Y)) \). If we solve this system of equations in the representation \( (V, G) \in REP - K \), then the set \( \text{Hom}((XKF(Y), F(Y)), (V, G)) \) has for us the role of the affine space. The solution of the system \( (T_1, T_2) \) in \( (V, G) \) is the set

\[
(T_1, T_2)_{(V,G)} = \{(\alpha, \beta) \in \text{Hom}((XKF(Y), F(Y)), (V, G)) | T_1 \subseteq \ker(\alpha), T_2 \subseteq \ker(\beta)\}.
\]

The algebraic \( (V, G) \)-closure of the system \( (T_1, T_2) \) will be the set

\[
(T_1, T_2)_{(V,G)}'' = \bigcap_{(\alpha, \beta) \in (T_1, T_2)_{(V,G)}} \ker(\alpha), \bigcap_{(\alpha, \beta) \in (T_1, T_2)_{(V,G)}} \ker(\beta) \subseteq (XKF(Y), F(Y)).
\]

This is the maximal system of equations, which has the same solutions as the system \( (T_1, T_2) \).

**Definition 2.1** Let \( (V_1, G_1), (V_2, G_2) \in REP - K \). We say that \( (V_1, G_1) \) and \( (V_2, G_2) \) are **geometrically equivalent** if \( (T_1, T_2)_{(V_1, G_1)} = (T_1, T_2)_{(V_2, G_2)} \), for every \( (T_1, T_2) \subseteq (XKF(Y), F(Y)) \) and every \( X \) and \( Y \). We use the notation \( (V_1, G_1) \sim (V_2, G_2) \).

The notion of geometric equivalence can be defined in an arbitrary variety of universal algebras. For more details, see [5].

In our considerations

**Definition 2.2** The logic formula of the form:

\[
(\bigwedge_{i=1}^{n} w_i) \Rightarrow w_0,
\]

is called **quasi-identity**, where \( w_i \) can be either \( (v_i = 0) \) or \( (f_i = 1) \), for \( v_i \in XKF(Y), f_i \in F(Y) \), \( 0 \leq i \leq n \), \( n \in \mathbb{N} \).

**Definition 2.3** Let \( (V, G) \in REP - K \). We say that \( (V, G) \) **fulfills** the quasi-identity (2.1) if, for every \( (\alpha, \beta) \in \text{Hom}((XKF(Y), F(Y)), (V, G)) \) which fulfills these conditions: for \( 1 \leq i \leq n \), if \( w_i \) is \( (v_i = 0) \) then \( \alpha(v_i) = 0 \), if \( w_i \) is \( (f_i = 1) \) then \( \beta(f_i) = 1 \) - we have that if \( w_0 \) is \( (v_0 = 0) \) then \( \alpha(v_0) = 0 \), or if \( w_0 \) is \( (f_0 = 1) \) then \( \beta(f_0) = 1 \), when \( w_0 \) is \( (f_0 = 1) \).
We will write

$$(V, G) \vdash ((\bigwedge_{i=1}^{n} w_i) \Rightarrow w_0),$$

if $(V, G)$ fulfills the quasi-identity (2.1).

By [9, Theorem 2] we have

**Proposition 2.1** Let $(V_1, G_1), (V_2, G_2) \in REP - K$ and $(V_1, G_1) \sim (V_2, G_2)$

then $(V_1, G_1)$ and $(V_2, G_2)$ fulfill same quasi-identities.

By [5, Proposition 13] we also have

**Proposition 2.2** Let $\Theta$ be some variety of universal algebras, $H_1, H_2$ be
dinitely generated algebras from $\Theta$. Then $H_1 \sim H_2$ if and only if exist injections
$H_1 \hookrightarrow H_2^{I_2}$ and $H_2 \hookrightarrow H_1^{I_1}$, where $I_1, I_2$ are some sets of indexes and $H_1^{I_1}, H_2^{I_2}$
corresponding Cartesian powers of algebras $H_1$ and $H_2$.

In [10] the action type algebraic geometry of representations was consid-
ered. This geometry was studied in order to avoid the influence of the algebraic
geometry of the acting group on the algebraic geometry of representation.

In this geometry we consider only system of action type equations, i.e.,
system of equations which have form $T \subseteq XKF(Y)$. The set of solutions of this
system in the representation $(V, G)$ is the set

$$T'_{(V, G)} = \{ (\alpha, \beta) \in \text{Hom}((XKF(Y), F(Y)), (V, G)) \mid T \subseteq \ker(\alpha) \}.$$

The action type $(V, G)$-closure of the system of equations $T$ is a set

$$T''_{(V, G)} = \bigcap_{(\alpha, \beta) \in T'_{(V, G)}} \ker(\alpha) \subseteq XKF(Y). \quad (2.2)$$

This is the maximal system of action type equations, which has the same solu-
tions as the system $T$.

**Definition 2.4** Let $(V_1, G_1), (V_2, G_2) \in REP - K$. We say that $(V_1, G_1)$ and
$(V_2, G_2)$ are action type geometrically equivalent if $T''_{(V_1, G_1)} = T''_{(V_2, G_2)}$, for
every $T \subseteq XKF(Y)$ and every $X$ and $Y$. We use the notation $(V_1, G_1) \sim at$
$(V_2, G_2)$.

By [10, Corollary 2 from Proposition 4.2], if two representations $(V_1, G_1)$ and
$(V_2, G_2)$ are geometrically equivalent then they are action type geometrically
equivalent.

If $(V, G) \in REP - K$, for every $v \in V$ we can consider the stabilizer of $v$, defined as follows

$$\text{stab}(v) = \{ g \in G \mid v \circ g = v \},$$

and $\ker(V, G)$, defined as follows
\[ \ker(V,G) = \bigcap_{v \in V} \text{stab}(v). \]

One can prove that \( \ker(V,G) \) is a normal subgroup of \( G \). We denote by \( \tilde{G} \) the quotient group \( G/\ker(V,G) \) and by \( \sigma \) the natural epimorphism \( \sigma : G \rightarrow G/\ker(V,G) \). It also is easy to check that we obtain the representation \((V, \tilde{G})\) over the vector spaces \( V \) if we define the action of the group \( \tilde{G} \) over the vector space \( V \) as follows
\[ v \circ \sigma(g) = v \circ g, \]
where \( v \in V, g \in G \).

**Definition 2.5** The representation \((V, \tilde{G})\) is called the **faithful image** of the representation \((V,G)\).

By [10, Corollary 4 from Theorem 5.1], we have

**Proposition 2.3** Every representation \((V,G) \in \text{REP} - K\) is action type geometrically equivalent to its faithful image \((V,\tilde{G})\).

## 3 The relation between geometrical equivalence and action type geometrical equivalence of group representations

In this section we will discuss the following question:

**Problem 1** Let \((V_1, G_1), (V_2, G_2) \in \text{REP} - K\). Can we conclude the geometrical equivalence \((V_1, G_1) \sim (V_2, G_2)\) from the action type geometrical equivalence \((V_1, G_1) \sim_{at} (V_2, G_2)\) and the geometrical equivalence of groups \(G_1 \sim G_2\)?

The negative answer to this question was mentioned in [10, Remark 5.1], but no counterexample was presented. This question makes sense, because in the action type algebraic geometry of representations we consider only specific systems of equations which have form \( T \subseteq XKF(Y) \) and the specific form of the algebraic closure (2.2). By this restriction we avoid the influence of the algebraic geometry of acting groups of representations. But one question which comes naturally is whether we are losing some important information about representations by this restriction. The negative answer to the Problem 1 shows that we indeed lose some information.

**Theorem 3.1** There exists \((V_1, G_1), (V_2, G_2) \in \text{REP} - K\), such that \((V_1, G_1) \sim_{at} (V_2, G_2)\), \(G_1 \sim G_2\), but \((V_1, G_1) \not\sim (V_2, G_2)\).
Proof. We consider a vector space $V$ over arbitrary field $K$, such that $\dim(V) = 2$. Let $\{e_1, e_2\}$ be a basis of $V$. We consider the groups $G_1 = \langle a \rangle \cong \mathbb{Z}_2$ and $G_2 = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We define the action of $G_1$ and $G_2$ over $V$ as follows:

$$e_1 \circ a = e_2, \quad e_2 \circ a = e_1,$$

$$e_1 \circ b = e_1, \quad e_2 \circ b = e_2.$$

Hence, we obtain the representations $(V_1, G_1)$ and $(V_2, G_2)$. Note that $\ker(V, G_2) = \langle b \rangle \cong \mathbb{Z}_2$ and $G_2/\ker(V, G_2) \cong G_1 = \langle a \rangle \cong \mathbb{Z}_2$. Therefore, the faithful image $(V, \tilde{G}_2)$ of the representation $(V, G_2)$ is isomorphic to the representation $(V_1, G_1)$. So, by Proposition 2.3, $(V, G_1) \sim_{at} (V, G_2)$.

The injections $G_1 \hookrightarrow G_2$ and $G_2 \hookrightarrow G_1 \times G_1$ exist. Therefore, by Proposition 2.2, $G_1 \sim G_2$.

Now, we consider the quasi-identity

$$(x \circ y - x = 0) \Rightarrow (y = 1).$$

We have that

$$(V, G_1) \models ((x \circ y - x = 0) \Rightarrow (y = 1)),$$

because $\ker(V, G_1) = \{1\}$, and

$$(V, G_2) \not\models ((x \circ y - x = 0) \Rightarrow (y = 1)),$$

because $\ker(V, G_2) \neq \{1\}$. By Proposition 2.1 this means that $(V_1, G_1) \not\sim (V_2, G_2)$.

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