THE PRIME DIVISORS IN EVERY CLASS CONTAIN ARBITRARY LARGE TRUNCATED CLASSES

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Abstract. Let $\mathbb{F}_q$ be the finite field with $q$ elements, and $\hat{C}$ a projective curve over $\mathbb{F}_q$. We show that the prime divisors on $\hat{C}$ in every class contain arbitrary large truncated generalized classes of finite effective divisors.

1. Introduction

Green-Tao [GT1] proved that the primes contains arbitrary long arithmetic progression. Thai [Thai] proved a polynomial analog of the above result. In this paper we shall prove a more general geometric analog.

Let $\mathbb{F}_q$ be the finite field with $q$ elements, $\hat{C}$ a projective curve over $\mathbb{F}_q$, and $K$ the function field of $\hat{C}$. We fix an element $t \in K$ which is transcendental over $\mathbb{F}_q$.

**Definition 1.1.** If $P$ is a pole of $t$, then we call $P$ a point at infinity, and write $P|\infty$.

Let $C$ be the finite part of $\hat{C}$.

**Definition 1.2.** For each nonzero function $f \in K$, the divisor of $f$ on $C$ is defined to be

$$(f) := \sum_{P \in C} \text{ord}_P(f) \cdot P.$$

**Definition 1.3.** For every divisor $D$ on $C$, we write

$L(D) := \{f \in K^\times \mid (f) + D \geq 0\} \cup \{0\}$.

**Definition 1.4.** Let $M, D_1, D_2$ be effective divisors on $C$. If there is a function $f \in K^\times$ such that $f - 1 \in L(D_1 - M)$ and $D_2 = (f) + D_1$, then $D_2$ is said to be equivalent to $D_1$ modulo $M$.

**Definition 1.5.** Let $M$ and $D$ be two divisors on $C$ such that $D \geq M$. Let $a \in L(D)$ and $r > 0$. We call

$$\{f \in L(D) \mid f - a \in L(M), \text{ord}_\infty(f - a) > -r\}$$

a truncated residue class of $L(D)$, where

$$\text{ord}_\infty(f) = \min_{P|\infty}\{\text{ord}_P(f)\}.$$

The truncated class is called principal if $M$ is principal.
Note that the function $\text{ord}_\infty(\cdot)$ extends to

$$K_\infty := \prod_{P|\infty} K_P,$$

where $K_P$ is the completion of $K$ at $P$, and $K$ is embedded into $K_\infty$ canonically.

**Definition 1.6.** Let $D$ be a divisor on $C$, and let $A$ be a truncated residue class of $L(D)$. We call

$$\{(f) + D \mid f \in A\}$$

a truncated generalized class of effective divisors on $C$.

In this paper we shall prove the following generalization of the result of Thai in [Thai].

**Theorem 1.7.** The prime divisors on $C$ in every equivalence class contain arbitrary large truncated equivalence classes of effective divisors.

A positive density version of the above theorem can be proved similarly.

2. Pseudo-random measures on inverse systems

In this section we establish the relationship between two kinds of measures on inverse systems.

Let $k$ be a fixed positive integer, $D$ a fixed nonzero effective divisor on $C$, and $I$ the set of polynomials in $\mathbb{F}_q[t]$ which are prime to every nonzero function in $O_C \cap B_k$, where $O_C$ is the ring of regular functions on $C$, and

$$B_k := \{f \in K_\infty \mid \text{ord}_\infty(f) > -k\}.$$

Then $\{L(D)/(NL(D))\}_{N \in I}$ is an inverse system of finite groups. For each $j \in O_C \cap B_k$, we write $e_j = (O_C \cap B_k) \setminus \{j\}$. Then $(O_C \cap B_k, \{e_j\}_{j \in O_C \cap B_k})$ is a hyper-graph. To each edge $e_j$, we associate the system $\{(L(D)/(NL(D)))^{e_j}\}_{N \in I}$. Thus the system $\{(L(D)/(NL(D)))^{e_j}\}_{N \in I,j \in O_C \cap B_k}$ maybe regarded as an inverse system on the hyper-graph $(O_C \cap B_k, \{e_j\}_{j \in O_C \cap B_k})$. For each $j \in O_C \cap B_k$, and for each $N \in I$, let $\tilde{\nu}_{N,j}$ be a nonnegative function on $(L(D)/(NL(D)))^{e_j}$.

**Definition 2.1.** The system $\{\tilde{\nu}_{N,j}\}_{N \in I,j \in O_C \cap B_k}$ is called a pseudo-random system of measures on the system $\{L(D)/(NL(D))\}_{N \in I,j \in O_C \cap B_k}$ if the following conditions are satisfied.

1. For all $j \in O_C \cap B_k$, and for all $\Omega_j \subseteq \{0, 1\}^{e_j} \setminus \{0\}$, one has

$$\frac{1}{q^{\deg N} \cdot K_F(t) \cdot \deg N} \sum_{x^{(1)} \in (L(D)/(NL(D)))^{e_j}} \prod_{\omega \in \Omega_j} \tilde{\nu}_{N,j}(x^{(\omega)}) = O(1),$$

uniformly for all $x^{(0)} \in (L(D)/(NL(D)))^{e_j}$.
(2) Given any choice $\Omega_j \subseteq \{0, 1\}^{e_j}$ for each $j \in O_C \cap B_k$, one has
\[ \frac{1}{q^{|O_C \cap B_k||K:F_q(t)|}\deg N} \sum_{x^{(0)},x^{(1)} \in (L(D)/NL(D))^{O_C \cap B_k}} \prod_{j \in O_C \cap B_k} \tilde{\nu}_{N,j}(x^{(\omega)}) = 1 + o(1), \]
as $\deg N \to \infty$.
(3) For all $j \in O_C \cap B_k$, for all $i \in e_j$, for all $\Omega_j \subseteq \{0, 1\}^{e_j}$, and for all $M \in \mathbb{N}$, we have
\[ \frac{1}{q^{2(|e_j| - 1)|K:F_q(t)|}\deg N} \sum_{x^{(0)},x^{(1)} \in (L(D)/NL(D))^{e_j}} \text{auto}(x, \tilde{\nu}_{N,j})^M = O(1), \]
where
\[ \text{auto}(x, \tilde{\nu}_{N,j}) := \frac{1}{q^{2|K:F_q(t)|\deg N}} \sum_{x^{(0)},x^{(1)} \in (L(D)/NL(D))} \prod_{j \in O_C \cap B_k} \tilde{\nu}_{N,j}(x^{(\omega)}). \]

For each positive integer $N \in I$, let $\tilde{\nu}_N$ be a nonnegative function on $L(D)/(NL(D))$.

**Definition 2.2.** The system $\{\tilde{\nu}_N\}$ is said to satisfy the $k$-cross-correlation condition if, given any positive integers $s \leq |O_C \cap B_k|2^{|O_C \cap B_k|}$, $m \leq 2|O_C \cap B_k|$, and given any mutually independent linear forms $\psi_1, \cdots, \psi_s$ in $m$ variables whose coefficients are functions in $O_C \cap B_k$, we have
\[ \frac{1}{q^{m|K:F_q(t)|\deg N}} \sum_{x \in (L(D)/(NL(D))} \prod_{j=1}^s \tilde{\nu}_N(\psi_j(x) + b_j) = 1 + o(1), \quad N \to \infty \]
uniformly for all $b_1, \cdots, b_s \in L(D)/(NL(D))$.

**Definition 2.3.** The system $\{\tilde{\nu}_N\}$ is said to satisfy the $k$-auto-correlation condition if, given any positive integers $s \leq |O_C \cap B_k|2^{|O_C \cap B_k|}$, there exists a system $\{\tilde{\tau}_N\}$ of nonnegative functions on $\{L(D)/(NL(D))\}$ which obeys the moment condition
\[ \frac{1}{q^{m|K:F_q(t)|\deg N}} \sum_{x \in (L(D)/(NL(D))} \tilde{\tau}_N(x)^M = O_M, s(1), \forall M \in \mathbb{N} \]
such that
\[ \frac{1}{q^{m|K:F_q(t)|\deg N}} \sum_{x \in (L(D)/(NL(D))} \prod_{j=1}^s \tilde{\nu}_N(x + y_i) \leq \sum_{1 \leq i < j \leq s} \tilde{\nu}_N(y_i - y_j). \]

**Definition 2.4.** The system $\{\tilde{\nu}_N\}$ is called a $k$-pseudo-random system of measure on the inverse system $\{L(D)/(NL(D))\}$ if it satisfies the $k$-cross-correlation condition and the $k$-auto-correlation condition.

From now on we assume that
\[ \tilde{\nu}_{N,j}(x) := \tilde{\nu}_N(\sum_{i \in e_j} (i - j)x_i). \]

**Theorem 2.5.** If $\{\tilde{\nu}_N\}$ is $k$-pseudo-random, then $\{\tilde{\nu}_{N,j}\}$ is pseudo-random.
Proof First, we show that, for all \( j \in O_C \cap B_k \), and for all \( \Omega_j \subseteq \{ 0, 1 \}^{e_j} \setminus \{ 0 \} \),

\[
\frac{1}{q^{|e_j|} |K : F_q(t)| \deg N} \sum_{x^{(1)} \in (L(D)/NL(D))^{e_j}} \prod_{\omega \in \Omega_j} \nu_{N,j}(x^{(\omega)}) = O(1),
\]

uniformly for all \( x^{(0)} \in (L(D)/NL(D))^{e_j} \). For each \( \omega \in \Omega_j \), set

\[
\psi_\omega(x^{(1)}) = \sum_{i \in e_j, \omega_i = 1} (i - j)x_i^{(1)},
\]

and

\[
b_\omega = \sum_{i \in e_j, \omega_i = 0} (i - j)x_i^{(0)}.\]

Then

\[
\frac{1}{q^{|e_j|} |K : F_q(t)| \deg N} \sum_{x^{(1)} \in (L(D)/NL(D))^{e_j}} \prod_{\omega \in \Omega_j} \nu_{N,j}(x^{(\omega)}) = 1 + o(1), \quad \deg N \to \infty.
\]

For each pair \( (j, \omega) \) with \( j \in O_C \cap B_k \) and \( \omega \in \Omega_j \), set

\[
\psi_{(j, \omega)}(x) = \sum_{i \in O_C \cap B_k, \omega_i = \delta} (i - j)x_i^{(\delta)}.
\]

Then

\[
\frac{1}{q^{|e_j|} |K : F_q(t)| \deg N} \sum_{x^{(0)}, x^{(1)} \in (L(D)/NL(D))^{O_C \cap B_k}} \prod_{j \in O_C \cap B_k} \prod_{\omega \in \Omega_j} \nu_{N,j}(x^{(\omega)})
\]

\[
= 1 + o(1).
\]

Finally we show that, for all \( j \in O_C \cap B_k \), for all \( i \in e_j \), for all \( \Omega_j \subseteq \{ 0, 1 \}^{e_j} \), and for all \( M \in \mathbb{N} \),

\[
\frac{1}{q^{|e_j|} |K : F_q(t)| \deg N} \sum_{x^{(0)}, x^{(1)} \in (L(D)/NL(D))^{\Omega_j \setminus \{ 1 \}}} \text{auto}(x, \nu_{N,j})^M = O(1).
\]

By Cauchy-Schwartz it suffices to show that, for \( a = 0, 1 \),

\[
\frac{1}{q^{|e_j|} |K : F_q(t)| \deg N} \sum_{x^{(0)}, x^{(1)} \in (L(D)/NL(D))^{\Omega_j \setminus \{ 1 \}}} \text{auto}(x, \nu_{N,j}, a)^{2M} = O(1),
\]
where
\[ \text{auto}(x, \tilde{\nu}_{N,j}, a) := \frac{1}{q^\deg N[K:F_q(t)]} \sum_{x^{(n)} \in L(D)/NL(D)} \prod_{\omega \in \Omega, \omega_i = a} \tilde{\nu}_{N,j}(x^{(\omega)}). \]

For each \( \omega \in \Omega_j \) with \( \omega_i = a \), set
\[ \psi_\omega(x) = \sum_{l \in \epsilon_j \setminus \{i\}} (l - j)x^{(\omega_l)}. \]

Then
\[ \frac{1}{q^{2(|\epsilon_j| - 1)|K:F_q(t)| \deg N}} \sum_{x^{(0)}, x^{(1)} \in (L(D)/NL(D))^{\epsilon_j \setminus \{i\}}} \text{auto}(x, \tilde{\nu}_{N,j}, a)^{2M} \]
\[ \leq \frac{1}{q^{2(|\epsilon_j| - 1)|K:F_q(t)| \deg N}} \sum_{x^{(0)}, x^{(1)} \in (L(D)/NL(D))^{\epsilon_j \setminus \{i\}}} \sum_{\omega, \omega' \in \Omega_j, \omega_i = \omega_i' = a} \hat{\tau}^{2M}(\psi_\omega(x) - \psi_{\omega'}(x)) \]
\[ = \sum_{\omega, \omega' \in \Omega_j, \omega_i = \omega_i' = a} \frac{1}{q^{K:Q[\deg N]}} \sum_{x \in (L(D)/(NL(D))} \hat{\tau}^{2M}(x) = O(1). \]

\[ \blacksquare \]

3. Pseudo-random measures on \( L(D) \)

In this section we establish the relationship between measures on inverse systems and measures on \( L(D) \).

Let \( A \) be a positive constant. For \( r \in \mathbb{N} \), let \( \nu_r \ll r^A \) be a nonnegative function on \( L(D) \).

**Definition 3.1.** The system \( \{ \nu_r \} \) is said to satisfy the \( k \)-cross-correlation condition if, given any open compact \( \mathbb{F}_q[[1/t]] \)-module \( I \) in \( K_\infty \), given any positive integers \( s \leq |O_C \cap B_k| |O_C \cap B_k|, m \leq 2|O_C \cap B_k| \), and given any mutually independent linear forms \( \psi_1, \cdots, \psi_s \) in \( m \) variables whose coefficients are functions in \( O_C \cap B_k \), we have
\[ \frac{1}{|L(D) \cap (rD)|^m} \sum_{x \in L(D) \cap (rD)} \prod_{j=1}^s \nu_r(\psi_j(x) + b_j) = 1 + o(1), \quad r \to \infty \]
uniformly for all functions \( b_1, \cdots, b_s \in L(D) \).

**Definition 3.2.** The system \( \{ \nu_r \} \) is said to satisfy the \( k \)-auto-correlation condition if given any positive integers \( s \leq |O_C \cap B_k| |O_C \cap B_k| \), there exists a system \( \{ \tau_r \} \) of nonnegative function on \( L(D) \) such that, given any open compact \( \mathbb{F}_q[[1/t]] \)-module \( I \) in \( K_\infty \),
\[ \frac{1}{|(rI) \cap L(D)|} \sum_{x \in (rI) \cap L(D)} \tau_r^M(x) = O_M(1), \quad r \to \infty, \forall M \in \mathbb{N} \]
Let \( \eta, \cdots, \eta_n \) be a \( \mathbb{F}_q[t] \)-basis of \( L(D) \), and set
\[
G = \sum_{j=1}^n \mathbb{F}_q[[1/t]]\eta_j \subseteq K_{\infty}.
\]
From on on we assume that, for each \( N \in I \),
\[
\hat{\nu}_N(x + NL(D)) = \nu_{degN}(x) \text{ if } x \in t^{degNG}.
\]
We now prove the following.

**Theorem 3.4.** If the system \( \{\nu_r\} \) is \( k \)-pseudo-random, then the system \( \{\hat{\nu}_N\}_{N \in I} \) is also \( k \)-pseudo-random.

**Proof** First we show that, given any positive integers \( s \leq |O_C \cap B_k|^{2|O_C \cap B_k|}, m \leq 2|O_C \cap B_k| \), and given any mutually independent linear forms \( \psi_1, \cdots, \psi_s \) in \( m \) variables whose coefficients are functions in \( O_C \cap B_k \),
\[
\frac{1}{q^{m[K:\mathbb{F}_q(t)] degN}} \sum_{x_i \in L(D)/(NL(D))} \prod_{j=1}^s \hat{\nu}_N(\psi_j(x) + b_j) = 1 + o(1), \text{ deg } N \to \infty
\]
uniformly for all \( b_1, \cdots, b_s \in L(D)/(NL(D)) \). It suffices to show that for any \( S' \subset \{1, \cdots, s\} \),
\[
\frac{1}{q^{m[K:\mathbb{F}_q(t)] degN}} \sum_{x_i \in \degNG \cap L(D)} \prod_{j \in S'} \left( \hat{\nu}_N(\psi_j(x) + b_j) - 1 \right) = o(1), \text{ } N \to \infty
\]
uniformly for all \( b_1, \cdots, b_s \in L(D) \). Let \( c \) be the maximal degree of the coefficients of the matrix of \( \psi \) with respect to \( \{\eta_i\} \). Then
\[
\frac{1}{q^{m[K:\mathbb{F}_q(t)] degN}} \sum_{x_i \in \degNG \cap L(D)} \prod_{j \in S'} \left( \hat{\nu}_N(\psi_j(x) + b_j) - 1 \right)
\]
\[
= \frac{1}{q^{m[K:\mathbb{F}_q(t)] degN}} \sum_{y_i \in t^{degNG} \cap L(D)} \sum_{x_i \in \degNG \cap L(D)} \prod_{j \in S'} \left( \hat{\nu}_N(\psi_j(x) + t^{degN-c}\psi_j(y) + b_j) - 1 \right)
\]
\[
= \frac{1}{q^{m[K:\mathbb{F}_q(t)] degN}} \sum_{y_i \in t^{degNG} \cap L(D)} \sum_{x_i \in \degNG \cap L(D)} \prod_{j \in S'} \left( \nu_{degN}(\psi_j(x) + b_j) - 1 \right) = o(1),
\]
where \( b_j' = t^{degN-c}\psi_j(y) + b_j \mod NL(D) \) lies in \( t^{degNG} \).
Secondly we show that, given any positive integers $s \leq |O_C \cap B_k|2^{[O_C \cap B_k]}$, 
\[
\frac{1}{q^{\deg N[K:F_q(t)]}} \sum_{x \in L(D)/(N(D))} \prod_{i=1}^{s} \tilde{\nu}_N(x + y_i) \leq \sum_{1 \leq i < j \leq s} \tilde{\tau}(y_i - y_j),
\]
where 
\[
\tilde{\tau}_N(x + NL(D)) = \tau_{\deg N}(x) \text{ if } x \in t^{\deg N}G.
\]
In fact, we have 
\[
\frac{1}{q^{\deg N[K:F_q(t)]}} \sum_{x \in L(D)/(NL(D))} \prod_{i=1}^{s} \tilde{\nu}_N(x + y_i)
\]
\[
= \frac{1}{q^{\deg N[K:F_q(t)]}} \sum_{x \in t^{\deg N}G} \prod_{i=1}^{s} \mu_{\deg N}(x + y'_i)
\]
\[
\leq \sum_{1 \leq i < j \leq s} \tilde{\tau}_{\deg N}(y'_i - y'_j)
\]
\[
= \sum_{1 \leq i < j \leq s} \tilde{\tau}_N(y_i - y_j),
\]
where $y'_i \equiv y_i (\text{mod} NL(D))$ lies in $t^{\deg N}G$. The theorem is proved. 

\section{The geometric relative Szemerédi theorem}

In this section we prove the geometric relative Szemerédi theorem.

For each $N \in I$, let $\tilde{A}_N$ be a subset of $L(D)/(NL(D))$.

**Definition 4.1.** The upper density of $\{\tilde{A}_N\}$ relative to $\{\tilde{\nu}_N\}$ is defined to be 
\[
\limsup_{I \ni N \to \infty} \frac{\sum_{x \in \tilde{A}_N} \tilde{\nu}_N(x)}{\sum_{x \in L(D)/(NL(D))} \tilde{\nu}_N(x)}.
\]

The following version of the geometric relative Szemerédi theorem follows from a theorem of Tao in [Tao].

**Theorem 4.2.** If the system $\{\tilde{\nu}_{N,j}\}$ is pseudo-random, and $\{\tilde{A}_N\}$ has positive upper density relative to $\{\tilde{\nu}_N\}$, then there is a subset $\tilde{A}_N$ and a truncated principal residue class $A$ of $L(D)$ of size $|O_C \cap B_k|$ such that 
\[
A(\text{mod} NL(D)) \subseteq \tilde{A}_N.
\]

The above theorem, along with Theorem 2.5, implies the following.

**Theorem 4.3.** If the system $\{\tilde{\nu}_N\}$ is $k$-pseudo-random, and $\{\tilde{A}_N\}$ has positive upper density relative to $\{\tilde{\nu}_N\}$, then there is a subset $\tilde{A}_N$ and a truncated residue class $A$ of $L(D)$ of size $|O_C \cap B_k|$ such that 
\[
A(\text{mod} NL(D)) \subseteq \tilde{A}_N.
\]
Definition 4.4. For \( r \in \mathbb{N} \), let \( A_r \) be a subset of \( L(D) \cap B_r \). The upper density of \( \{ A_r \} \) relative to \( \{ \nu_r \} \) is defined to be

\[
\limsup_{r \to \infty} \frac{\sum_{x \in A_r} \nu_r(g)}{\sum_{x \in L(D) \cap B_r} \nu_r(x)}.
\]

We now prove the following.

Theorem 4.5. If \( \{ \nu_r \} \) is \( k \)-pseudo-random, \( c \) is a sufficiently large positive constant depending only on \( k \), \( C \) and \( D \), and \( \{ A_r \cap B_{r-c} \} \) has positive upper density relative to \( \{ \nu_r \} \), then there is a subset \( A_r \) that contains a truncated residue class of \( L(D) \) of size \(| O_C \cap B_k |\).

Proof We have

\[
\frac{\sum_{x \in A_{\deg N \cap B_{\deg N - k}}} \tilde{\nu}_N(x)}{\sum_{x \in L(D)/(NL(D))} \nu_N(x)} = \frac{1}{q[K:F_q(t)] \deg N} \sum_{x \in A_{\deg N \cap B_{\deg N - c}}} \tilde{\nu}_N(x) + o(1) = \frac{1}{q[K:F_q(t)] \deg N} \sum_{x \in A_{\deg N \cap B_{\deg N - c}}} \nu_{\deg N}(x) + o(1) + \sum_{x \in L(D)/(NL(D))} \nu_N(g) = \sum_{x \in L(D)/(NL(D))} \nu_{\deg N}(x) + o(1).
\]

So \( \{ A_{\deg N \cap B_{\deg N - c}}(\text{mod } NL(D)) \} \) has positive upper density relative to \( \{ \tilde{\nu}_N \} \).

By Theorem 4.3, there is a subset \( A_{\deg N \cap B_{\deg N - c}}(\text{mod } NL(D)) \) and a truncated principal residue class \( A \) of \( L(D) \) of size \(| O_C \cap B_k |\) such that

\[
A(\text{mod } NL(D)) \subseteq A_{\deg N \cap B_{\deg N - c}}(\text{mod } NL(D)).
\]

Replace \( A \) by a translation if necessary, we conclude that

\[
A \subset A_{\deg N \cap B_{\deg N - c}}.
\]

The theorem follows.

5. The cross-correlation of the truncated von Mangoldt function

In this section we shall establish the cross-correlation of the truncated von Mangoldt function.

The truncated von Mangoldt function for the rational number field was introduced by Heath-Brown [HB]. The truncated von Mangoldt function for the Gaussian number field was introduced by Tao [Tao]. The cross-correlation of the truncated von Mangoldt function for the rational number field were studied by Goldston-Yıldırım in [GY1, GY2, GY3], and by Green-Tao in [GT1, GT2]. The cross-correlation of the truncated von Mangoldt function for the rational function field were studied by [Thai].

Let \( \varphi : \mathbb{R} \to \mathbb{R}^+ \) be a smooth bump function supported on \([-1, 1]\) which equals 1 at 0, and let \( R > 1 \) be a parameter. We now define the truncated von Mangoldt function for the function field \( K \).
**Definition 5.1.** We define the truncated *von Mangoldt function* $\Lambda_{K,R}$ of $K$ by the formula

$$\Lambda_{K,R}(D) := \sum_{M \leq D} \mu_K(M) \varphi\left(\frac{\deg M}{R}\right), \quad \forall D \geq 0,$$

where $\mu_K$ is the *Möbius function* of $K$ defined by the formula

$$\mu_K(D) = \begin{cases} (-1)^k, & D \text{ is a sum of } k \text{ distinct prime divisors}, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $\Lambda_{K,R}(D) = 1$ if $D$ is a prime divisor of degree $\geq R$.

Let $\zeta_K(z)$ be the zeta function of $K$ defined by the formula

$$\zeta_K(z) = \prod_P \frac{1}{1 - q^{-z} \deg P}, \quad \text{Re } z > 1,$$

where $P$ runs through the set of closed points on $C$. Write

$$\hat{\varphi}(x) = \int_{-\infty}^{\infty} e^{it} \varphi(t) e^{ixt} \, dt,$$

and

$$c_{\varphi} := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(1 + iy)(1 + iy')}{(2 + iy + iy')} \hat{\varphi}(y) \hat{\varphi}(y') \, dy \, dy'.$$

From now on, for each $r \in \mathbb{N}$, let

$$\nu_r(x) = \frac{\phi_K(W) R \cdot \text{Res}_{z=1} \zeta_K(z)}{c_{\varphi} q^{\deg W[K:F]} \Lambda_{K,R}((Wx + \alpha)L(D)^{-1})}.$$ 

Here

$$R = \frac{r}{8|O_C \cap B_k|^2|O_{C} \cap B_k|},$$

$W$ is the product of monic irreducible polynomials of degree $\leq w := \log \log r$,

$$\phi_K(W) := |O_C/(W)|^\times,$$

and $\alpha$ a number prime to $W$.

We now prove the following.

**Theorem 5.2.** The system $\{\nu_r\}$ satisfies the $k$-cross-correlation condition.

**Proof** Given any open compact $\mathbb{F}_q[[1/t]]$-module $I$ in $K_\infty$, given any positive integers $s \leq |O_C \cap B_k|^2|O_{C} \cap B_k|$, $m \leq 2|O_C \cap B_k|$, and given any mutually independent linear forms $\psi_1, \ldots, \psi_s$ in $m$ variables whose coefficients are functions in $O_C \cap B_k$, we show that

$$\frac{1}{|L(D) \cap (t^r I)|^m} \sum_{z \in L(D) \cap (t^r I)} \prod_{j=1}^{s} \nu_r(\psi_j(x) + b_j) = 1 + o(1), \quad r \to \infty$$

uniformly for all functions $b_1, \cdots, b_s \in L(D)$. 

Define $\omega$ and, for $t,t'$, it is easy to see that, for all $d \mid L = \{t \leq 10\}$. We have

$$\omega = \sum_{\varnothing, \varnothing'} \omega((\varnothing_1 \cap \varnothing_1')_{1 \leq i \leq s}) \prod_{i=1}^{s} \mu_K(\varnothing_i) \mu_K(\varnothing_i') \varphi(\frac{\deg \varnothing_i}{R}) \varphi(\frac{\deg \varnothing_i'}{R}),$$

where $\varnothing$ and $\varnothing'$ run over $s$-tuples of effective divisors on $C$, and

$$\omega((\varnothing_1)_{1 \leq i \leq s}) = \frac{|\{x \in (L(D)/(L(D - \varnothing)) \leq (W_0 t + b'_i) + D, \forall i = 1, \ldots, s\}|}{(Nd)^m}$$

with $\varnothing = \text{l.c.m.}(\varnothing_1, \ldots, \varnothing_s)$ and $b'_i = Wb_i + \alpha$.

Define

$$F(t, t') = \sum_{\varnothing, \varnothing'} \omega((\varnothing_1 \cap \varnothing_1')_{1 \leq j \leq s}) \prod_{j=1}^{s} \frac{\mu_K(\varnothing_j) \mu_K(\varnothing_j')}{N(\varnothing_j) \cdot N(\varnothing_j')}, \quad t, t' \in \mathbb{R}^s,$$

where $\varnothing$ and $\varnothing'$ run over $s$-tuples of divisors on $C$.

It is easy to see that, for all $B > 0$,

$$e^x \varphi(x) = \int_{-R}^{R} \varphi(t) e^{-it} dt + O(R^{-B}).$$

It follows that for all $B > 0$,

$$\frac{1}{|L(D) \cap (t' I)|} \left( \frac{q^{[K: \mathbb{Q}]} \deg W}{\varphi_K(W) R \cdot \text{Res}_{z=1} \xi_K(z)} \right)^s \sum_{i \in L(D) \cap (t' I)} \prod_{j=1}^{s} \mu_r(\psi_j(x) + b_j)$$

$$= \int_{-R}^{R} \int_{-R}^{R} F(t, t') \varphi(t) \varphi(t') dt dt' + O(R^{-B}) \cdot \sum_{\varnothing, \varnothing'} \omega((\varnothing_1 \cap \varnothing_1')_{1 \leq j \leq s}) \prod_{i=1}^{s} \frac{\mu_K(\varnothing_j) \mu_K(\varnothing_j')}{N(\varnothing_j) \cdot N(\varnothing_j')},$$

Hence we are reduced to prove the following.

$$\sum_{\varnothing, \varnothing'} \omega((\varnothing_1 \cap \varnothing_1')_{1 \leq j \leq s}) \prod_{j=1}^{s} \frac{\mu_K(\varnothing_j) \mu_K(\varnothing_j')}{N(\varnothing_j) \cdot N(\varnothing_j')} \ll R^{O_s(1)},$$

and, for $t, t' \in [-\sqrt{R}, \sqrt{R}]^s$,

$$F(t, t') = (1 + o(1)) \left( \frac{q^{[K: \mathbb{Q}]} \deg W}{\varphi_K(W) R \cdot \text{Res}_{z=1} \xi_K(z)} \right)^s \prod_{j=1}^{s} \frac{(1 + it_j)(1 + it'_j)}{(2 + it_j + it'_j)}.$$

We prove the equality first. Applying the Chinese remainder theorem, one can show that

$$\omega((\varnothing_1)_{1 \leq j \leq s}) = \prod_{\varnothing} \omega((\varnothing, \varnothing)_{1 \leq j \leq s}).$$
where $\varphi$ runs over prime divisors on $C$. One can also show that

$$\omega((\mathfrak{d}_j, \varphi))_{1 \leq j \leq s} = \begin{cases} 1, & \prod_{j=1}^{s} (\mathfrak{d}_j, \varphi) = 0, \\ 0, & \prod_{j=1}^{s} (\mathfrak{d}_j, \varphi) \neq 0, (W) \geq \varphi. \end{cases}$$

And, if $\varphi \mid W$ and $W$ is sufficiently large, then one can show that

$$\omega((\mathfrak{d}_j, \varphi))_{1 \leq j \leq s} \begin{cases} = 1/|\varphi|, & \prod_{j=1}^{s} (\mathfrak{d}_j, \varphi) = \varphi, \\ \leq 1/|\varphi|^2, & \varphi^2 \mid \prod_{j=1}^{s} (\mathfrak{d}_j, \varphi). \end{cases}$$

It follows that

$$\begin{align*}
F(t, t') &= \prod_{\varphi \mid W} \sum_{d, \varphi' \mid \varphi, \varphi' \neq \varphi} \omega((\mathfrak{d}_j, \varphi))_{1 \leq j \leq s} \prod_{j=1}^{s} \frac{\mu_K(\mathfrak{d}_j) \mu_K(\mathfrak{d}'_j)}{N(\mathfrak{d}_j)^{1/2} N(\mathfrak{d}'_j)^{1/2}} \\
&= \prod_{\varphi \mid W} \left(1 + \sum_{j=1}^{s} -N\varphi^{-1} \frac{1 + it_j}{R} - N\varphi^{-1} \frac{1 + it'_j}{R} + N\varphi^{-1} \frac{2 + it_j + it'_j}{R} + O_s \left(\frac{1}{N\varphi^2}\right)\right) \\
&= \prod_{p \mid W} \left(1 + O_s \left(\frac{1}{q^{2 \deg p}}\right)\right) \prod_{\varphi \mid W} \left(\frac{1 - N\varphi^{-1} \frac{1 + it_j}{R}}{1 - N\varphi^{-1} \frac{2 + it_j + it'_j}{R}}\right) \prod_{\varphi \mid W} \left(1 - N\varphi^{-1} \frac{2 + it_j + it'_j}{R}\right) \\
&= (1 + O(\frac{1}{R})) \prod_{j=1}^{s} \frac{\zeta_K(1 + \frac{2 + it_j + it'_j}{R})}{\zeta_K(1 + \frac{1 + it_j}{R})} \prod_{\varphi \mid W} \left(\frac{1 - N\varphi^{-1} \frac{1 + it_j}{R}}{1 - N\varphi^{-1} \frac{2 + it_j + it'_j}{R}}\right) \prod_{\varphi \mid W} \left(1 - N\varphi^{-1} \frac{2 + it_j + it'_j}{R}\right).
\end{align*}$$

From the estimate

$$\zeta_K(z) = \frac{\text{Res}_{z=1} \zeta_K(z)}{z - 1} + O(1), \ z \to 1,$$

and the estimate

$$e^z = 1 + O(z), \ z \to 0,$$

we infer that

$$\begin{align*}
F(t, t') &= (1 + O(\frac{1}{R})) \prod_{p \mid W} \left(1 + O(\frac{\deg \varphi}{\deg p R^{1/2}})\right) \cdot \\
&\left(\frac{q^{[K:\mathcal{O}(t)] \deg W}}{\varphi_K(W) R \cdot \text{Res}_{z=1} \zeta_K(z)}\right)^s \prod_{j=1}^{s} \frac{(1 + it_j)(1 + it'_j)}{(2 + it_j + it'_j)}.
\end{align*}$$

Applying the estimate

$$\prod_{p \mid W} \left(1 + \frac{\log N\varphi}{N\varphi}\right) = O(e^{\log^2 w}),$$

we arrive at

$$\begin{align*}
F(t, t') &= (1 + o(1)) \left(\frac{q^{[K:\mathcal{O}(t)] \deg W}}{\varphi_K(W) R \cdot \text{Res}_{z=1} \zeta_K(z)}\right)^s \prod_{j=1}^{s} \frac{(1 + it_j)(1 + it'_j)}{(2 + it_j + it'_j)}
\end{align*}$$

as required.

We now turn to prove the estimate

$$\sum_{\mathfrak{d}, \mathfrak{d}'} \omega((\mathfrak{d}_j, \varphi))_{1 \leq j \leq s} \prod_{j=1}^{s} \frac{|\mu_K(\mathfrak{d}_j) \mu_K(\mathfrak{d}'_j)|}{N(\mathfrak{d}_j)^{1/2} N(\mathfrak{d}'_j)^{1/2} \log R} \ll R^{O(1)}.$$

We have
\[ \sum_{\mathfrak{d}, \mathfrak{d}'} \omega((\mathfrak{d}_j \cap \mathfrak{d}'_j)_{1 \leq j \leq s}) \prod_{j=1}^{s} \frac{|\mu_K(\mathfrak{d}_j)\mu_K(\mathfrak{d}'_j)|}{N(\mathfrak{d}_j)^{1/2}N(\mathfrak{d}'_j)^{1/2} \log R} \]
\[ = \prod_{\mathfrak{p} \mid W} \sum_{\mathfrak{d}_j \mid \mathfrak{d}, \mathfrak{d}'_j \mid \mathfrak{d}'} \omega((\mathfrak{d}_j \cap \mathfrak{d}'_j)_{1 \leq j \leq s}) \prod_{j=1}^{s} \frac{1}{N(\mathfrak{d}_j)^{1/2}N(\mathfrak{d}'_j)^{1/2}} \]
\[ = \prod_{\mathfrak{p} \mid W} (1 + N\mathfrak{p}^{-1 - \frac{1}{W}}) O(1) \]
\[ = \prod_{\mathfrak{p} \mid W} (1 + q^{-1 - \frac{1}{W}}) \deg p) O(1) = \zeta_F(q(t)) (1 + \frac{1}{R}) O(1) \ll R O(1). \]
This completes the proof of the theorem. 

6. The auto-correlation of the truncated von Mangoldt function

In this section we shall establish the auto-correlation of the truncated von Mangoldt function.

The auto-correlation of the truncated von Mangoldt function for the rational number field was studied by Goldston-Yıldırım in [GY1, GY2, GY3], and by Green-Tao in [GT1, GT2]. The auto-correlation of the truncated von Mangoldt function for the rational function field were studied by [Thai].

We now prove the following.

**Theorem 6.1.** The system \( \{ \nu_r \} \) satisfies the \( k \)-auto-correlation condition.

The above theorem follows from the following lemma.

**Lemma 6.2.** Let \( I \) be any open compact \( \mathbb{F}_q[[1/t]] \)-module in \( K_\infty \). Then
\[ \frac{1}{|\langle t \rangle \cap L(D)|} \sum_{x \in \langle t \rangle \cap L(D)} \prod_{i=1}^{s} \nu_r(x + y_i) \ll \prod_{1 \leq i < j \leq s} \nu_r(y_i - y_j) (1 + O_s(\frac{1}{N \mathfrak{p}})) \]
uniformly for all \( s \)-tuples \( y \in L(D)^s \) with distinct coordinates.

**Proof** We may assume that
\[ \Delta := \prod_{i \neq j} (y_i - y_j) \neq 0. \]

Define
\[ \omega_2((\mathfrak{d}_i)_{1 \leq i \leq s}) = \left| \left\{ x \in L(D) / L(D - \mathfrak{d}) : \mathfrak{d}_i \leq (Wx + h_i) + D, \forall i = 1, \cdots, s \right\} \right| / N \mathfrak{d} \]
where \( h_i = Wb(y_i) + Wy_i + \alpha \). Then
\[ \frac{1}{|\langle t \rangle \cap L(D)|} \sum_{x \in \langle t \rangle \cap L(D)} q^{[K : \mathbb{F}_q](t) \deg W} \phi_K(W \mathfrak{R}) \prod_{i=1}^{s} \nu_r(x + y_i) \]

\[ \cdots \cdots \]
Hence we are reduced to prove the following.

\[ d \]

where \( d \) and \( d' \) run over \( s \)-tuples of divisors on \( C \). Define

\[ F_2(t, t') = \sum_{d, d'} \omega_2((d_i \cap d'_i)_{1 \leq i \leq s}) \prod_{j=1}^{s} \frac{\mu_K(d_j) \mu_K(d'_j)}{N(d_j)^{1/R} N(d'_j)^{1/R}}, \quad t, t' \in \mathbb{R}^s, \]

where \( d \) and \( d' \) run over \( s \)-tuples of ideals of \( O_C \).

For all \( B > 0 \), we have

\[ \frac{1}{|tr I \cap L(D)|} \sum_{x \in (tr I) \cap L(D)} \left( \frac{q^{[K: \mathbb{Q}(t)]} \deg W}{\phi_K(W) R} \right)^s \prod_{i=1}^{s} \nu_r(x + y_i) \]

\[ = \int_{-\sqrt{R}, \sqrt{R}} \int_{-\sqrt{R}, \sqrt{R}} F_2(t, t') \psi(t) \psi(t') dt dt' + O_B(R^{-B}) \cdot \sum_{d, d'} \omega_2((d_i \cap d'_i)_{1 \leq i \leq s}) \prod_{j=1}^{s} \left| \frac{\mu_K(d_j) \mu_K(d'_j)}{N(d_j)^{1/R} N(d'_j)^{1/R}} \right| \]

Hence we are reduced to prove the following.

\[ \sum_{d, d'} \omega_2((d_i \cap d'_i)_{1 \leq i \leq s}) \prod_{j=1}^{s} \left| \frac{\mu_K(d_j) \mu_K(d'_j)}{N(d_j)^{1/R} N(d'_j)^{1/R}} \right| \ll R^{O_s(1)}, \]

and for \( t, t' \in [-\sqrt{R}, \sqrt{R}]^s \),

\[ F_2(t, t') \ll \left( \frac{q^{[K: \mathbb{Q}(t)]} \deg W}{\phi_K(W) R} \right)^s \prod_{\varphi \mid \Delta, \varphi \mid W} \left( 1 + O_s(1) \right) \prod_{j=1}^{s} \left( 1 + |t_j| (1 + |t'_j|) \right). \]

We prove the second inequality but omit the proof of first one. Applying the Chinese remainder theorem, one can show that

\[ \omega_2((d_i)_{1 \leq i \leq s}) = \prod_{\varphi} \omega_2((d_i, \varphi)_{1 \leq i \leq s}). \]

One can also show that

\[ \omega_2((d_i, \varphi)_{1 \leq i \leq s}) = \begin{cases} 1, & \prod_{i=1}^{s} (d_i, \varphi) = (1), \\ 0, & \prod_{i=1}^{s} (d_i, \varphi) \neq (1), \varphi \mid W. \end{cases} \]

And, if \( \varphi \mid W \) and \( w \) is sufficiently large, then one can show that

\[ \omega_2((d_i, \varphi)_{1 \leq i \leq s}) \leq 1/N \varphi, \quad \varphi^2 \mid \prod_{i=1}^{s} (d_i, \varphi), \varphi \mid \Delta. \]

It follows that

\[ F_2(t, t') = \prod_{\varphi \mid W, \varphi \mid \Delta} \sum_{d, d'} \omega_2((d_i \cap d'_i)_{1 \leq i \leq s}) \prod_{j=1}^{s} \frac{\mu_K(d_j) \mu_K(d'_j)}{N(d_j)^{1/R} N(d'_j)^{1/R}} \prod_{j=1}^{s} \left( 1 + O_s(1) \right) \]

\[ = \prod_{\varphi \mid W, \varphi \mid \Delta} \left( 1 + \sum_{j=1}^{s} -N \varphi^{-1 - \frac{1 + i_j}{R} - \frac{1 + \nu_j'/R}{R} + N \varphi^{-1 - \frac{2 + i_j + \nu_j'}{R}} \right) \prod_{\varphi \mid W, \varphi \mid \Delta} \left( 1 + O_s(1/N \varphi) \right) \]
\[
\ll \prod_{\wp|\Delta, \wp|W} (1 + O_s(\frac{1}{N\wp})) \prod_{j=1}^{s} \prod_{\wp|W} (1 - N\wp^{-1 - \frac{1 + it_j}{n}})(1 - N\wp^{-1 - \frac{1 + it_j'}{n}}) \\
\ll (\frac{q^{|K:F_q(t)| \deg W}}{\phi_K(W)})^s \prod_{\wp|\Delta, \wp|W} (1 + O_s(\frac{1}{N\wp})) \prod_{j=1}^{s} \frac{\zeta_K(1 + \frac{1 + it_j + it_j'}{R})}{\zeta_K(1 + \frac{1 + it_j}{R})} \\
\ll (\frac{q^{|K:F_q(t)| \deg W}}{\phi_K(W)R})^s \prod_{\wp|\Delta, \wp|W} (1 + O_s(\frac{1}{N\wp})) \prod_{j=1}^{s} \frac{1 + |t_j| + |t_j'|}{(2 + |t_j| + |t_j'|)}.
\]

This completes the proof of the lemma.

7. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 1.7. We begin with the following lemma.

**Lemma 7.1.** There is a positive constant \(c_K\) such that every principal divisor on \(C\) is the divisor of a function \(\xi\) satisfying

\[
\text{ord}_P(\xi) \geq -\frac{\deg(\xi)}{|K:F_q(t)|} - c_K, \quad \forall P \mid \infty.
\]

**Proof** Let \(\varepsilon_1, \ldots, \varepsilon_r\) be a system of fundamental units of \(O_K\). For each \(i = 1, \ldots, r\), set

\[
y_i := (\text{ord}_P(\varepsilon_i))_{P | \infty}.
\]

It is known that the vectors \(y_1, \ldots, y_r\) are linearly independent. Let \(\xi\) be a nonzero function in \(K\). We have

\[
\sum_{P | \infty} [K_P : F_q((1/t))] (\text{ord}_P(\xi) + \frac{\deg(\xi)}{|K:F_q(t)|}) = 0.
\]

It follows that the vector

\[
(\text{ord}_P(\xi) + \frac{\deg(\xi)}{|K:F_q(t)|})_{P | \infty}
\]

as well as the vectors \(y_1, \ldots, y_r\) are orthogonal to the vector \([K_P : F_q((1/t))]_{P | \infty}\). So

\[
(\text{ord}_P(\xi) + \frac{\deg(\xi)}{|K:F_q(t)|})_{P | \infty} = a_1 y_1 + \cdots + a_r y_r.
\]

Multiplying \(\xi\) by a function in \(O_K^\times\) if necessary, we may assume that \(0 \leq a_i < 1\). It follows that, for every \(P \mid \infty\),

\[
\text{ord}_P(\xi) \geq -\frac{\deg(\xi)}{|K:F_q(t)|} - c_K.
\]

The lemma now follows.

For each \(r \in \mathbb{N}\), and for each \(\alpha \in L(D)\) with \((\alpha, WL(D)) = L(D)\), set

\[
A_{r, \alpha} = \{x \in L(D) \cap B_r \mid (WX + \alpha) + D \text{ is prime}\}.
\]

By Theorem 4.5, Theorem 1.7 follows from the following theorem.
Theorem 7.2. Let $c$ be a sufficiently large positive constant depending only on $k$, $C$ and $D$. For each $r \in \mathbb{N}$, there is a number $\alpha_r \in (t^{\deg W}G) \cap L(D)$ with $((\alpha_r) + D, W) = 1$ such that the system $\{A_{r, \alpha} \cap B_{r-c}\}$ has positive upper density relative to $\{\nu_r\}$.

Proof By the above lemma, there is a positive constant $c_K$ such that every principal divisor on $C$ is the divisor of a function $\xi$ satisfying 
$$\text{ord}_P(\xi) \geq \frac{-\deg(\xi)}{|K : \mathbb{F}_q(t)|} - c_K, \forall P | \infty.$$ 

It follows that, for any divisor $n \in [D]$, there is an element $x \in L(D)$ such that $n = (x) + D$ and that 
$$\text{ord}_P(x) \geq \frac{\deg D - \deg n}{|K : \mathbb{F}_q(t)|} - c_K, \forall P | \infty.$$ 

In particular, for each $r \in \mathbb{N}$, and for any prime divisor $\wp \in [D]$ satisfying $(\wp, W) = 1$ and $\deg \wp < \deg D + |K : \mathbb{F}_q(t)| (r - c - c_K - \deg W)$, there is a function $x \in A_{r, \alpha} \cap B_{r-c}$, and a function $\alpha \in (t^{\deg W}G) \cap L(D)$ with $((\alpha) + D, W) = 1$ such that $\wp = (Wx + \alpha) + D$. So 
$$\sum_{\alpha \in (t^{\deg W}G) \cap L(D)} \sum_{x \in A_{r, \alpha} \cap B_{r-c}} A_{K, R}^2((Wx + \alpha) + D) \geq \sum_{\wp \in [D], (\wp, W) = 1} A_{K, R}(\wp) \gg \frac{c \cdot q^{\deg W} |K : \mathbb{F}_q(t)|}{R \cdot \text{Res}_{z=1} \zeta_K(z)} \cdot |L(D) \cap B_{r}|.$$ 

The theorem now follows by the pigeonhole principle.

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