The Form Factors in the Sinh-Gordon Model

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Abstract
The most general solution to the form factor problem in the sinh-Gordon model is presented in an explicit way. The linearly independent classes of solutions correspond to powers of the elementary field. We show how the form factors of exponential operators can be obtained from the general solution by adjusting free parameters. The general formula obtained in the sinh-Gordon case reproduces the form factors of the scaling Lee-Yang model in a certain limit of the coupling constant.
1 Introduction

The famous reconstruction theorem [1] states that once all Wightman functions, i.e. vacuum expectation values of local operators, in a physical model are known one can reconstruct the entire Hilbert space of the theory and the explicit action of the local algebra on it. In a generic physical theory the complete knowledge of the Wightman functions is almost elusive.

However, within the last years considerable progress has been made in the quantum field theoretical treatment of massive integrable models in (1+1) dimensions. In particular there are indications that it might be possible, using the form factor bootstrap approach [2, 3], to compute Wightman functions exactly within certain models in the aforementioned class.

Form factors are matrix elements of a local operator \( O(x) \) between multiparticle states and the vacuum. In the case of the Sinh-Gordon (sinhG) model, where only one species of particles is present, and the multiparticle states can be labeled by the rapidities \( \theta_i \), the object of study is defined to be

\[
F_n(\theta_1, \theta_2, \ldots, \theta_n) = \langle 0 | O(0) | \theta_1, \theta_2, \ldots, \theta_n \rangle. \tag{1}
\]

It has been shown in [2, 3] that due to the physical structure of integrable massive models in (1+1) dimensions the functions \( F_n \) are subject to a set of conditions, which are sometimes referred to as axioms [2]. These conditions, which will be described below in the sinhG case provide equations for the matrix elements (1).

It has been shown for various models that it is feasible to find solutions to these equations; see e.g. [2-17]. Due to the quite complicated structure of the form factor equations the solutions are sometimes known only up to a certain number of incoming particles. The form factor equations do not refer to any particular local operator in the model. Hence, after having found all solutions the next problem is to identify them with particular local operators.

The form factors can then at least in principle be used to make statements about the operator content of the theory and to compute correlation functions of local operators in the quantum field theory by:

\[
\langle 0 | O_1(x) O_2(0) | 0 \rangle = \sum_n \frac{1}{n!} \int F_n^{O_1}(\theta_1, \theta_2, \ldots, \theta_n) F_n^{O_2}(\theta_1, \theta_2, \ldots, \theta_n)^* e^{-m|x|} \sum_{i=1}^n \cosh \theta_i \prod_{i=1}^n d\theta_i \frac{d}{2\pi}. \tag{2}
\]
There is, in fact, some hope that upon a convenient parametrization of the form factors the sum in \( (3) \) can be evaluated. This is one of the motivations for the present work. For recent progress in this direction see \([18]\).

In this work will give an explicit formula for the most general solution to the form factor bootstrap equations in the sinhG model. Our method is motivated by \([14]\) where form factors of the \(O(3)\) nonlinear \(\sigma\) model were calculated. One might therefore be tempted to think that there exists a general method to compute form factors of models within the class of two dimensional integrable field theories. We are convinced that the formula proven in this paper should work with some modifications in other affine Toda field theories as well \([19]\). This is due to the fact that the equations to be solved for the matrix element \( (4) \) in the sinhG case can be reduced to a polynomial recursion equation. This polynomial equation is in turn a particular specialization of the polynomial recursion equations stated in \([19]\) for all ADE affine Toda field theories.

Let us comment on the status of the form factors in the sinhG model. A few low lying solutions corresponding to the elementary field and the trace of the energy momentum tensor have been computed in \([9]\). A closed and complete solution was found in \([12]\) for local operators satisfying the cluster property. These local operators can be identified with exponential fields in the theory and in certain limiting cases with the elementary field and the trace of the energy momentum tensor respectively. However, it turns out that due to free parameters which occur in the process of finding solutions to the form factor equations there are linearly independent solutions which are not covered (at least directly) by the solution found in \([12]\). In this paper we complete the study of form factors in the sinhG model in that we find all possible form factor solutions in a closed form for the sinhG model. We will show that these additional solutions correspond to (renormalized) powers of the elementary field.

This paper is organized as follows. In section 2 we will review some facts about the form factor bootstrap for the sinhG model. In section 3 we will prove the main result of this work which is the general solution to the polynomial recursion equations which follow from the bootstrap equations. For operators satisfying the so called cluster property in the sinhG model a solution of the form factor equations is known \([12]\). We will show in section 4 how this particular solution can be obtained from the general result by adjusting the free parameters in the general solution in a simple way. Since this solution corresponds to exponential operators in the sinhG model we can identify the linearly independent families of the general solutions with renormalized powers of the elementary field. This will be done in section 5.

If the effective coupling in the sinhG model is set to a particular value one can build a
bridge to the scaling Lee-Yang model. In section 6 we will show that even though the form factor bootstrap equations are structurally different in this case our general result can be used to uniquely compute the form factors in this model as well. The quantum equivalence of the two possible primary fields in this model is then just a consequence of the general solution without making any assumptions on the nature of the local operators or the structure of the polynomials. Since these results have already been obtained using different approaches in [7, 9, 11] our presentation serves to establish the usefulness of our approach to compute form factors in integrable massive models in (1+1) dimensions. The last section is left to conclusions.

2 The form factor bootstrap in the sinhG model

In this section some necessary facts about the method to calculate form factors in the sinhG model will be recalled. A detailed exposition of this topic can be found in [9].

The sinhG model is the $A_1^{(1)}$ affine Toda field theory defined by the relativistically invariant Lagrangian in (1+1) dimensions

$$L(x) = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{m^2}{\beta^2} \cosh(\beta \phi(x)).$$

$m^2$ sets the mass scale of the theory and the coupling $\beta$ is taken to be a real number. The bootstrap S-matrix as a function of the rapidity $\theta$ is given by [20]

$$S(\theta) = \frac{i \sinh(\theta) + \sin(\pi B/2)}{i \sinh(\theta) - \sin(\pi B/2)},$$

where the “effective coupling” is given by $B(\beta) = \frac{\beta^2/2\pi}{1+\beta^2/4\pi}$.

According to the form factor bootstrap method of [2, 3] the matrix element $F_n(\theta_1, \ldots, \theta_n)$, which is of course nothing else than a meromorphic function with nontrivial monodromies, is subject to the following conditions. The first two of them are known as Watson’s equations.

$$F_n(\theta_1, \ldots, \theta_l, \theta_{l+1}, \ldots, \theta_n) = S(\theta_l - \theta_{l+1}) \ F_n(\theta_1, \ldots, \theta_{l+1}, \theta_l, \ldots, \theta_n)$$

$$F_n(\theta_1 + 2\pi i, \theta_2, \ldots, \theta_n) = \prod_{l=2}^n S(\theta_l - \theta_1) \times F_n(\theta_1, \theta_2, \ldots, \theta_n).$$

We would like to remark that the second of these equations has recently been derived from quantum field theoretical principles [21].
There is only a single elementary field in the Lagrangian (3), which has the property to be self-conjugate. From this property one can derive the following kinematical residue equation
\[ -i \text{ res}_{\theta = \theta + i\pi} F_{n+2}(\theta', \theta, \theta_1, \ldots, \theta_n) = \left( 1 - \prod_{l=1}^{n} S(\theta - \theta_l) \right) F_n(\theta_1, \ldots, \theta_n). \] (6)

Since there are no fusings in the sinhG model there is – in contrast to other affine Toda field theories – no bound state residue equation.

In the physical strip we now parametrize the matrix element (1) in full generality, using \( x_l = e^{\theta_l} \) in the following way.
\[ F_n(\theta_1, \ldots, \theta_n) = H_n Q_n(x_1, \ldots, x_n) \prod_{i<j} F_{\text{min}}(\theta_i - \theta_j) \frac{x_i + x_j}{x_i + x_j}. \] (7)

\( H_n \) is a constant, and the other symbols in this definition will be explained in what follows. It can now be shown that the procedure of solving the equations (5) and (6) can be split into two steps. From (5) we obtain, after requiring \( Q_n(x_1, \ldots, x_n) \) to be a symmetric function, two equations in the minimal form factors \( F_{\text{min}} \) only.
\[ F_{\text{min}}(\theta) = S(\theta) F_{\text{min}}(-\theta), \quad F_{\text{min}}(2\pi i - \theta) = F_{\text{min}}(\theta). \] (8)

This means that we have restored the non-trivial monodromies arising from (3) in the minimal form factor. These equations can be solved using an integral representation of the S-matrix. Up to normalization the solution is
\[ F_{\text{min}}(\theta) = \exp \left( 8 \int_0^\infty \frac{dt}{t} \frac{\sinh(tB/2) \sinh(t(2 - B)/2) \sinh(t)}{\sinh^2(2t)} \sin^2((i\pi - \theta)t/\pi) \right). \] (9)

This expression can also be expanded into an infinite product of \( \Gamma \) functions, see [3, 15].

We are now left to solve equation (3) which, using the ansatz (4), leads to a polynomial equation.
\[ Q_{n+2}(-x, x; x_1, \ldots, x_n) = (-1)^n D_n(x|x_1, \ldots, x_n) Q_n(x_1, \ldots, x_n), \] (10)

where, using the abbreviation \( \omega = \exp(i\pi B/2) \), the recursion coefficient is given by
\[ D_n(x|x_1, \ldots, x_n) = \frac{x}{2(\omega - \omega^{-1})} \left( \prod_{l=1}^{n} (x + \omega x_l)(x - \omega^{-1} x_l) - \prod_{l=1}^{n} (x - \omega x_l)(x + \omega^{-1} x_l) \right). \] (11)
Let us comment on the structure of $Q_n(x_1, \ldots, x_n)$. Since (10) is a purely polynomial recursion equation, its solutions $Q_n$ obviously have to be polynomials. Upon the requirement mentioned above, $Q_n$ has then to be a symmetric polynomial.

Just by considering (10) it is not obvious that its solutions are symmetric, instead there exists a family of non-symmetric polynomial solutions to this equation. But, as it has been outlined, we are not interested in these non-symmetric solutions.

As it stands, (10) does not connect polynomials of even $n$ and of odd $n$. The reason for this is the $\mathbb{Z}_2$ symmetry of the sinhG model (8). However, we will see that the general symmetric solution for $Q_{n+2}$ to (10) to be given in the next section will explicitly depend on $Q_n$ and also inherit the structure of $Q_{n+1}$. The way in which $Q_{n+2}$ depends on $Q_{n+1}$ is consistent with the $\mathbb{Z}_2$ symmetry of the sinhG model.

Obviously (10) has a one dimensional kernel. This means that at each stage of the recursion process we will have to take one free parameter into account. We therefore expect the polynomials to be of the following structure.

\[
Q_n = A_n Q_{n,n} + A_{n-2} Q_{n,n-2} + \ldots + A_1 Q_{n,1}, \quad n \text{ odd},
\]

\[
Q_n = A_n Q_{n,n} + A_{n-2} Q_{n,n-2} + \ldots + A_2 Q_{n,2}, \quad n \text{ even}.
\]

(12)

The components $Q_{n,l}$ satisfy (10) independently. The coefficients $A_l$ are to be determined by the structure of the local operator which is under consideration in (9). This means that the operator content of the quantum field theory is linked to these parameters. We will come to this point in more detail in sections 4, 5, and 6.

To end this introductory section let us briefly recall some facts about symmetric polynomials [23] which will be needed later. The elementary symmetric polynomials in $n$ variables are denoted by $e^{(n)}_l = e^{(n)}_l(x_1, \ldots, x_n)$ and defined by the relation $\prod_{r=1}^n (1 + tx_r) = \sum_{l=0}^n e^{(n)}_l t^l$. Note that $e^{(n)}_l = 0$ for $l < 0$ and for $l > n$. If $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition, we define $E^{(n)}_\lambda = e^{(n)}_{\lambda_1} \cdots e^{(n)}_{\lambda_m}$.

The monomial symmetric functions $m^{(n)}_\lambda = m^{(n)}_\lambda(x_1, \ldots, x_n)$ are indexed by a partition $\lambda$ and defined by $m^{(n)}_\lambda = \sum_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where the sum runs over all permutations $\alpha$ of the partition $\lambda$.

A special partition which will be needed later is $\delta_n = (n-1, n-2, \ldots, 1, 0)$. Taking the monomial symmetric functions, we can show that

\[
E^{(n)}_{\delta_n} = m^{(n)}_{\delta_n} + \sum_{\mu < \delta} c_\mu m^{(n)}_\mu.
\]

(13)
The sum goes over partitions $\mu$, satisfying a “smaller than” relation \cite{23} with $\lambda$, $c_\mu$ is a combinatorial coefficient, which is not important for the present purposes.

As a last remark we would like to add that the kernel polynomial of (10) can be easily written using a Schur function $s^{(n)}_\lambda = s^{(n)}_\lambda(x_1, \ldots, x_n)$.

$$\prod_{i<j}^n (x_i + x_j) = s^{(n)}_{\delta_n} + \sum_{\mu<\delta} d_\mu E^{(n)}_{\mu} + \sum_{\mu<\delta} d'_\mu E^{(n)}_{\mu}. \quad (14)$$

The coefficients $d_\mu$ and $d'_\mu$ are again of combinatorial nature \cite{23}.

## 3 The general solution

We have shown in the last section that the problem of calculating the matrix element (11) reduces to the problem of finding symmetric polynomial solutions to the equation (10).

Let us add two remarks on the structure of the solutions $Q_n$. If we require the matrix element (11) to be Lorentz invariant it can be shown by standard arguments that the total degree of $Q_n$ is $n(n-1)/2$. Its partial degree is $n-1$. However, if the operator under consideration is of spin $s$ the degree of $Q_n$ is $n(n-1)/2 + s$.

We are now going to present the most general solution to the recursion equation (10) in the spinless case and comment later on the case of nonvanishing spin.

In contrast to other studies, see e.g. \cite{24}, our approach of calculating the polynomials might seem less appealing from the conceptual point of view. However, our method of directly calculating the polynomials seems to be quite effective and does not suffer from any ambiguities. The reader is referred to \cite{25} on this point.

Before presenting the main result of this paper note the following convention. In (12) we have been indicating the general structure of the polynomials $Q_n$. We will denote by $Q'_n$ the polynomial in (12) with all indices of the coefficients $A_l$ shifted by +1. Consequently $Q''_n$ will be the polynomial $Q_n$ with all indices at the $A_l$’s shifted by +2 and so on. For example, if $Q_5 = A_5 Q_{5,5} + A_3 Q_{5,3} + A_1 Q_{5,1}$, then $Q'_5 = A_6 Q_{5,5} + A_4 Q_{5,3} + A_2 Q_{5,1}$. Moreover, we indicate by $(x_1, \ldots, \hat{x}_k, \ldots, x_n)$ that the coordinate $x_k$ does not appear inside the brackets.

We now state the formula for the general symmetric solution of (10) and will then outline a proof.
\[ Q_n(x_1, x_2, \ldots, x_n) = \sum_{k=2}^{n} D_{n-2}(x_k|x_2, \ldots, \hat{x}_k, \ldots, x_n) Q_{n-2}(x_2, \ldots, \hat{x}_k, \ldots, x_n) \prod_{l \neq k}^{n} \frac{x_1 + x_l}{x_k - x_l} \]

\[ + \prod_{l=2}^{n} (x_1 + x_l) \times Q'_{n-1}(x_2, \ldots, x_n). \]

(15)

The first term on the right hand side of this equation is motivated by a result obtained in [14] in the context of the \(O(3)\) nonlinear \(\sigma\) model. A term of this structure cannot be enough for the sinhG model for the following reason. While the total degree of the form factor polynomials in the \(O(3)\) model is the same as in the sinhG model for the spinless case, in the former case the partial degree is \(n - 2\). Considering the product factor in the first term it is obvious that the highest power of \(x_1\) is just \(n - 2\). Therefore something has to be added in the sinhG model. We would, however, like to remark that for solutions where we set \(A_i = 0\), for all \(i > 1\), the first term is sufficient. This particular solution corresponds to the form factors of the elementary field according to [9].

Let us show how to prove the formula. First of all it might seem that due to the product in the denominator in the first term that (15) is not a polynomial. However, if we pull out a factor off the first term, which is just a Vandermonde determinant, \(a_\delta(x_2, \ldots, x_n) = \prod_{i<j}^{n} (x_i - x_j)\), we are left with a polynomial which can be shown to be completely antisymmetric in the variables \(\{x_2, \ldots, x_n\}\). Hence, this part is divisible by \(a_\delta(x_2, \ldots, x_n)\) using standard arguments [23]. This shows that (15) is actually a polynomial which is symmetric in \(\{x_2, \ldots, x_n\}\). The second term is by definition a symmetric polynomial in these variables.

Obviously, if we set \(x_1 = -x_k\) in (15) we will obtain (10). The first term would already be enough to obtain this result, but then, as mentioned above, \(Q_n\) would in general not be symmetric in \(x_1\).

To show that \(Q_n\) as defined in (15) is symmetric in \(x_1\) as well, it is almost sufficient to prove that if we set \(x_k = -x_l\) for some \(k, l > 1\) will also lead to the recursion equation (11).

To prove this one could start making an ansatz of the form (15) where we replace \(Q'_{n-1}\) by any symmetric polynomial \(f^{(n)}(x_2, \ldots, x_n)\) of appropriate degree.

The following identity is crucial in the proof. If we write

\[ H_{n-2}^{(k)}(x_2, \ldots, x_n) = D_{n-2}(x_k|x_2, \ldots, \hat{x}_k, \ldots, x_n) Q_{n-2}(x_2, \ldots, \hat{x}_k, \ldots, x_n), \]

(16)

we can show using the properties of the recursion coefficient (11) that
\[ H^{(k)}_{n-2}(x_2, \ldots, x_n)|_{x_i = -x_r} = H^{(r)}_{n-2}(x_2, \ldots, x_n)|_{x_i = -x_k}, \quad l \neq r \neq k. \] (17)

If we then use the modified ansatz and compute for example \( Q_n(x_1, x_2, x_3, \ldots, x_n)|_{x_2 \to -x_3} \) the function \( f^{(n)}(x_2, x_3, x_4, \ldots, x_n) \) has to obey

\[ f^{(n)}(-x_3, x_3, x_4, \ldots, x_n) = (-1)^{n-3} D_{n-3}(x_3|x_4, \ldots, x_n) f^{(n-2)}(x_4, \ldots, x_n). \] (18)

Equivalent equations hold if we choose any other two variables and take the limit. Obviously (18) is nothing other than the recursion equation (10) for \( n-1 \) variables. This shows that \( f^{(n)} \propto Q_{n-1} \).

It can now be shown by applying induction in \( n \) to (13) that the constants of the polynomial \( Q_{n-1} \) in the expansion (12) have to be chosen according to the rule mentioned in the beginning of this section in order to render the solution polynomial symmetric. Note that the term in \( Q_{n-1} \) does not merely produce a kernel solution to (10). Our proof shows that it is really needed in order to render all linearly independent components (12) in \( Q_n \) symmetric! We will show this explicitly in appendix A.

It is clear that (15), as it stands, provides due to our analysis of the general structure of the solution spaces in (12) the most general symmetric solution to (10). This completes the proof of relation (15).

Let us make a few remarks on the solution we have found.

1. Since the degree of \( Q_1 \) is zero, we set \( Q_1(x_1) = A_1 \). This is the only initial condition we have to choose. \( Q_2 \) already follows uniquely from (15) upon the observation that \( D_0 = 0 \). We will show in appendix A how (15) works and give explicit expressions for the first few polynomials.

2. It turns out naturally from (13) that the highest component \( Q_{n,n} \) (12) of \( Q_n \) is always the kernel of (10), i.e.

\[ Q_{n,n}(x_1, \ldots, x_n) = s^{(n)}_{b_n}(x_1, \ldots, x_n), \] (19)

according to (14).

3. The other components of \( Q_n \) admit a description in terms of symmetric skew polynomials over the ring \( \mathbb{Z}[\omega + \omega^{-1}] \); therefore the solution polynomials can be thought of as relatives of skew Macdonald polynomials [23]. It seems that a similar structure is present in the polynomial solution spaces of the form factors of other affine Toda field theories as well [24], we are, however, not going to elaborate this issue here, but rather intend to address this interesting mathematical problem in a more general context in the future.
4. It is, of course, possible to give straightforwardly an explicit expression for any $Q_n$ just in terms of the $D_r$'s, with $r \leq n - 2$, $Q_2^{(k)}$, and $Q_1^{(q)}$ by using (13) recursively. Since the resulting expression is a bit lengthy it will not be written out here explicitly.

5. In order to find a nice expression for any component $Q_{n,l}$ in (15) it is useful to pull out the Vandermonde determinant $\prod_{i<j}^n (x_i - x_j)$ in (13). Then one can easily read off $Q_{n,l}$ in terms of Schur polynomials, due to the fact that a Schur polynomial can be written as

$$s^{(n)}_{\lambda} = a_{\lambda+\delta}/a_\delta, \quad a_{\lambda+\delta} = \det(x_i^{\lambda_j+n-j})_{1 \leq i,j \leq n}, \quad a_\delta = \prod_{i<j}^n (x_i - x_j). \quad (20)$$

6. According to [6, 4, 9], solutions for higher spin operators can be easily obtained from (15). Since we would like to stress the importance of symmetric skew polynomials in the context of form factors we are going to write the solution found in [6] in terms of just one skew Schur polynomial (for details see appendix B and [23]).

We define the partition $\rho_n = (n^2, n-1, \ldots, 3, 2)$. Note that for $n = 1$ we have a degenerate case, so we set $\rho_1 = (1)$. The second partition we need is $\delta_n$, defined at the end of the previous section. Note that both partitions have the property of being self-conjugate [23].

In order to describe a local operator of spin $2s - 1$ we can simply multiply the matrix element (1) with the function $s^{(n)}_{\rho_n/\delta_n}$. The reason for this is that

$$I_{n+2}^{2s-1} = (-1)^s s^{(n)}_{\rho_n/\delta_n}. \quad (21)$$

4 Relation to the cluster property solution

A remarkable solution to the recursive equations (10) has been found in [12]. We are going to review this solution briefly and point out to which local operators it corresponds. After that it will be shown how these form factors can be found in our general solution (13).

A local operator $\mathcal{O}(x)$ is said to satisfy the cluster property if, after shifting a certain set of variables by a rapidity $\Delta$ and then taking the limit $\Delta \to \infty$, the form factors of $\mathcal{O}(x)$ decompose in the following way.
\[
\lim_{\Delta \to \infty} F_n^O(\theta_1 + \Delta, \ldots, \theta_m + \Delta, \theta_{m+1}, \ldots, \theta_n) \propto F_m^O(\theta_1, \ldots, \theta_m) F_{n-m}^O(\theta_{m+1}, \ldots, \theta_n).
\]  

(22)

Note that in general the form factors appearing on the right hand side of this equation may correspond to local operators different from \(O\).

A family of solutions of (10) corresponding to operators satisfying the property (22) was found in [12].

Introduce the \(q\)-number symbol \([k]_\omega = (\omega^k - \omega^{-k})/(\omega - \omega^{-1})\) and define the following matrix elements, which depend on a parameter \(k\) by

\[
M_{ij}(k) = e^{2j-i} [j - i + k]_\omega.
\]  

(23)

Solutions of (10) are then given by the determinant of this matrix

\[
Q_n(k) = \det(M_{ij}(k)), \quad i, j = 1, \ldots, n - 1.
\]  

(24)

As it was mentioned above, the general solution (15) comes naturally in the form (12). In order to relate the cluster property solution (24) to (15) we have to adjust the parameters \(A_l\).

We will be using a combinatorial argument to derive the desired relation. First notice that for any \(n\) the cluster solution (24) has exactly one term of the form \(E^{(n)}_n \delta_n\), as defined in (13). This term has the coefficient \([k]^{n-1}_\omega\). Since according to (14) \(\delta_n\) is the leading partition in the expansion of the kernel of equation (14), it is clear that the general solution \(Q_n\) will have a term of the form \(E^{(n)}_n \delta_n\) with coefficient 1 in its \(A_n\) component.

We now turn to the recursion coefficient \(D_n\) which was introduced in (11). We expand \(D_n\) in powers of \(x\) and in terms of monomial symmetric functions.

In accordance with other affine Toda field theories [26] it turns out that in this expansion \(D_n\) satisfies a kind of duality in the space of partitions.

This duality is characterized by the fact that in the expansion of \(D_n\) the coefficients of \(x^k m_\lambda^{(n)}\) and \(x^{2n-k} m_\mu^{(n)}\) with \(\lambda + \bar{\mu} = (2^n)\) are identical. In particular it turns out that the highest power of \(x\) comes with a symmetric polynomial \(m_1^{(n)}(100\ldots0)\), while the lowest power of \(x\) has \(m_{(22\ldots21)}^{(n)}\). According to the duality observation, both terms have the same coefficient, which is just 1.

It can now be observed that in the recursion relation \(Q_n \to D_{n-2} Q_{n-2}\), explicitly stated in (14), the structure of the highest partition in any component of \(Q_{n-2}\) and the highest and lowest powers of \(x\) in the expansion of \(D_n\) will uniquely induce the highest partition in the corresponding component of \(Q_n\).
It can be proven using some simple combinatorics of partitions that if \( \delta_{n-2} \) is the highest partition of a component of \( Q_{n-2} \) then the corresponding components of \( Q_n \) will have \( \delta_n \) as its leading partition, apart from \( n = 5 \).

Moreover, it can be proven that a partition \( \delta_n \) can never be generated in the recursion process \((11)\) by a partition \( \mu \), which satisfies \( \mu < \delta_{n-2} \).

Using these properties it is possible to show inductively which components of \( Q_n \) in the form \((12)\) do have \( m_{\delta_n} \), and hence \( E_{\delta_n} \), as leading term. Note that if \( m_{\delta_n} \) or \( E_{\delta_n} \) occur in any component of \( Q_n \) they will come with coefficient 1 due to the aforementioned structure of \( D_n \).

By direct comparison of the solutions \((24)\) and \((15)\) we find for the first few coefficients

\[
A_1 = 1, \quad A_2 = [k]_\omega, \quad A_3 = [k]^2_\omega, \quad A_4 = [k]^3_\omega - [k]_\omega, \quad A_5 = [k]^4_\omega. \tag{25}
\]

Using our combinatorial argument we can now proceed inductively and obtain

\[
A_l = [k]^{l-1}_\omega - [k]^l_\omega, \quad l \geq 6. \tag{26}
\]

By \((25)\) and \((26)\) we have shown that the cluster property solution \((24)\) is just a special, but physically very important, specialization of our general solution \((15)\). We will elaborate this issue in more detail in the next section.

5 On the operator content of the sinhG model

We are now in a position to identify the local operators in the sinhG model with the linearly independent solutions of \((15)\). Again note that \((12)\) we encounter one free parameter at each stage of the recursion process \((10)\). Moreover we state that solutions \( Q_n \) (once given by \((15)\)) for \( n \) odd or even need to be dealt with seperately due to the \( Z_2 \) symmetry.

From \((25)\) and \((26)\) one realizes that there are two special values for the parameter \( k \).

Setting \( k = 0 \) will leave \( A_1 = 1 \) while \( A_l = 0 \) for all \( l \geq 2 \). This particular solution corresponds to the form factors of the elementary field as has been shown by a LSZ analysis in \([9, 12]\).

Moreover, putting \( k = 1 \) will lead to \( A_2 = 1 \) and all the other free parameters in the \( Z_2 \) even sector vanish. This solution corresponds to the form factors of the trace of the energy momentum tensor in the sinhG model \([9, 12]\).
In addition it turns out from (25) and (26) that for generic values of the effective coupling $B$ one cannot project out any other of the linearly independent solutions of (15) by adjusting $k$ in a straightforward way. It was argued in [12] that the solution $Q_n(k)$ given in (24) corresponds to the exponential operators of the form $\exp(kg\phi(x))$, see also [27]. We can then take the form factors of this exponential operator and expand it in the following way (cf. [17])

$$\langle 0 | : e^{kg\phi(0)} : | \theta_1, \ldots, \theta_n \rangle = \sum_{r=0}^{\infty} \frac{k^r g^r}{r!} \langle 0 | : \phi^r(0) : | \theta_1, \ldots, \theta_n \rangle,$$

(27)

where $: \phi^r(0) :$ indicates a normal ordered (possibly renormalized) power of the elementary field in the sinhG model (3).

On obvious grounds we require the matrix element $\langle 0 | : \phi^r(0) : | \theta_1, \ldots, \theta_n \rangle$ to vanish if $r = 0$ or $r > n$. In addition the $\mathbb{Z}_2$ invariance of the model dictates that the matrix element is nonvanishing only if both $r$ and $n$ are either simultaneously even or simultaneously odd. After adjusting a normalization in order to comply with $A_1 = 1$ in (25), by comparing the powers of $k$ in (27) and in (25), (26) we find the following correspondence between the linearly independent solutions characterized by the parameter $A_r$ (cf. (12)) and the local operator $: \phi^r(x) :$

$$A_r \leftrightarrow : \phi^r(x) : .$$

(28)

Note, however, that this procedure does not fix the constants $A_r$ uniquely.

The correspondence (28) was already conjectured in [12]. It would be interesting to retrieve this correspondence from the UV limit of the sinhG model in the manner it was done for the minimal models in [4, 11].

### 6 Relation to the scaling Lee-Yang model

In this section we show that the general formula for the polynomial part of the form factors in the sinhG model can be applied to the scaling Lee-Yang model (SLY) as well. For details on this model we refer to [28, 7, 4]. It is known that this model is integrable, and well studied within the context of perturbed conformal field theories. However, it is known that this model has some problems with unitarity on the quantum field theory level.
We note that the result established in this section is not a new one in the sense that the form factor problem for the SLY model was already solved in [7]. However, we think it is interesting to outline how the general formula (15) in the sinhG case can be applied to the SLY model. Another approach to recover the SLY form factors from the sinhG ones was given in [9].

The S-matrix of the scaling Lee-Yang model can be obtained from the S-matrix of the sinhG model (4) by specializing the effective coupling in the sinhG model [9] to the value $B = -2/3$, yielding [28]

$$S_{SLY}(\theta) = \frac{\sinh(\theta) + i\sin(\pi/3)}{\sinh(\theta) - i\sin(\pi/3)}.$$  (29)

Even though the SLY model can be obtained by specialization from the sinhG model it has one essentially different feature. As in the sinhG case we have just one self-conjugate particle of mass $m$. In contrast to the sinhG case the SLY model has a nonvanishing three point coupling [28]. This fact is reflected in the presence of a pole at the rapidity $\theta = 2i\pi/3$ in the S-matrix (29).

Owing to this additional structure we have one more condition on the matrix elements (1) in addition to equations (5) and (6). We denote the vertex of the fusing by $\Gamma = i\sqrt{23}^{1/4}$, and use the notation $\omega_{SLY} = \exp(i\pi/3)$. This quantity is of course obtained from $\omega$ in the sinhG model by using the aforementioned specialization of $B$.

The so called bound state residue equation [2] which arises due to the presence of a three-point coupling is given by

$$-i\text{res}_{\theta = \theta'} F_{n+1}(\theta' + i\pi/3, \theta - i\pi/3, \theta_1, \ldots, \theta_{n-1}) = \Gamma F_n(\theta, \theta_1, \ldots, \theta_{n-1}).$$  (30)

It was shown in [7] that in the case of the SLY model an ansatz similar to (11) is possible.

$$F_{n_{SLY}}(\theta_1, \ldots, \theta_n) = H_n Q_{n_{SLY}}(x_1, \ldots, x_n) \prod_{i<j}^{n} \frac{f(\theta_i - \theta_j)}{x_i + x_j}.$$  (31)

Again we introduce a constant $H_n$ which was given explicitly in [7]. As in the sinhG case we can determine the degree and the partial degree of the polynomials $Q_{SLY}$ by this ansatz. It turns out that in the case of spinless operators we have that $\text{deg}Q_{SLY} = n(n-1)/2$, while the partial degree of the variables in this polynomial is $n-1$. Hence, these criteria are the same as in the sinhG case.

The functions $f(\theta)$ are the analogues of the minimal form factors in the context of the SLY model and determined as solutions to (8). They are explicitly given by [7]
\[ f(\theta) = \frac{\cosh(\theta) - 1}{\cosh(\theta) + 1/2} v(\theta), \quad (32) \]

with
\[ v(\theta) = \exp \left( 4 \int_0^\infty \frac{\text{dt}}{t} \frac{\sinh(t/2) \sinh(t/3) \sinh(t/6)}{\sinh^2(t)} \cosh(t + it\theta/\pi) \right). \quad (33) \]

The kinematical residue equation for the SLY model leads to an equation for \( Q_{\text{SLY}} \) which is just (10) with \( \omega = \exp(i\pi B/2) \) replaced by \( \omega_{\text{SLY}} = \exp(i\pi/3) \). From now on we assume, unless mentioned explicitly, that we are working with \( \omega_{\text{SLY}} \).

In contrast to the sinhG model we get one more condition on the polynomials \( Q_{\text{SLY}} \) due to the presence of the bound state residue equation (30).

\[ Q_{\text{SLY}}^{n+2}(\omega_{\text{SLY}} x, \omega_{\text{SLY}}^{-1} x, x_1, \ldots, x_n) = x^n \prod_{l=1}^n (x + x_l) \times Q_{\text{SLY}}^{n+2}(x, x_1, \ldots, x_n). \quad (34) \]

The degree of \( Q_{\text{SLY}}^1 \) is zero. In accordance with [7] we just set this polynomial equal to one. Using (34) we can then easily compute \( Q_{\text{SLY}}^2 \).

\[ Q_{\text{SLY}}^1(x_1) = 1, \quad Q_{\text{SLY}}^2(x_1, x_2) = e_1^{(2)}. \quad (35) \]

Comparing this result with the general solution in the sinhG case it is clear that (13) will give the correct solution for \( Q_{\text{SLY}}^2 \) provided that in contrast to using \( Q_1' \) in the second term on the right hand side we just take the polynomial \( Q_1 \) itself.

Since the kinematical residue equations for the sinhG and the SLY model are structurally equivalent, it is clear that we only have to check which constraints (34) puts on the solution (15) with \( \omega \) adjusted to the SLY case.

In order to do that, we have to show under which conditions (13) is consistent with (34), and a few identities are necessary. First we notice that

\[ D_{n-2}(x_2|x_3, \ldots, x_{n-2}, \omega_{\text{SLY}} x, \omega_{\text{SLY}}^{-1} x) = (x_2 - x)(x_2 + x) D_{n-3}(x_2|x_3, \ldots, x_{n-2}, x). \quad (36) \]

Another useful identity for the recursion coefficient of the kinematical residue equation in the SLY case is

\[ D_{n-2}(\omega_{\text{SLY}} x|x_2, x_3, \ldots, x_{n-2}, \omega_{\text{SLY}}^{-1} x) = \omega_{\text{SLY}} x^n \prod_{l=2}^{n-2} (x + x_l)(\omega_{\text{SLY}}^2 x - x_l) \quad (37) \]
It is then tedious but straightforward to show by induction that the following formula, which is very similar to (15), generates the unique symmetric polynomial solution to the equations (10) and (34) in the SLY model.

\[
Q_{\text{SLY}}^n(x_1, x_2, \ldots, x_n) = \sum_{k=2}^n D_{n-2}(x_k| x_2, \ldots, \hat{x}_k, \ldots, x_n) Q_{n-2}(x_2, \ldots, \hat{x}_k, \ldots, x_n) \prod_{l=2, l \neq k}^{n} \frac{x_1 + x_l}{x_k - x_l} + \prod_{l=2}^{n} (x_1 + x_l) \times Q_{n-1}^\text{SLY}(x_2, \ldots, x_n).
\]  

(38)

Having established this formula, it is sufficient to choose only one starting condition, which is as mentioned above \(Q_{\text{SLY}}^1 = 1\). The polynomials for higher particle numbers then follow uniquely using (38). This is analogous to what has been outlined for the sinhG case in section 3, but surprising because in the SLY case we have two conditions to determine the polynomials \(Q_{\text{SLY}}^n\).

It is useful to remark that (38) could be obtained directly from (15) by making the replacement \(\omega \rightarrow \omega_{\text{SLY}}\) and adjusting the parameters \(A_l\) in the general form of the solutions (12). Arguments similar to those used in the previous section allow us to state that the specializations

\[
\omega \rightarrow \omega_{\text{SLY}}, \quad A_1 = A_2 = A_3 = A_5 = 1, \quad A_4 = 0, \quad \text{and} \quad A_l = 0, \quad l \geq 6,
\]

(39)

will provide the unique (symmetric polynomial) solution to (10) and (34) in the SLY case from the solution (15) in the sinhG case.

The solution polynomials in the SLY case can be given in terms of just one skew Schur polynomial (see appendix B for details). If we want to provide a physical interpretation of this result we can split off two elementary symmetric polynomials.

Let us define the following partitions.

\[
\lambda^{(n)} = ((n-1)^{n-1}, n-2, n-3, \ldots, 2, 1), \quad \mu^{(n)} = ((n-2)^2, (n-3)^2, \ldots, 2^2, 1^2),
\]

\[
\rho^{(n)} = ((n-3)^2, (n-4)^2, \ldots, 2^2, 1^2), \quad \nu^{(n)} = ((n-4)^2, (n-5)^2, \ldots, 2^2, 1^2, 0^2).
\]

(40)

We can then write the solution, which was already found in [7], in a compact form

\[
Q_n^{\text{SLY}}(x_1, \ldots, x_n) = s^{(n)}_{\lambda^{(n)}/\mu^{(n)}} = e_1^{(n)} e_{n-1}^{(n)} s^{(n)}_{\rho^{(n)}/\nu^{(n)}}.
\]

(41)

This is consistent with the result obtained in [7] and with an analysis of the form factors of the SLY model in the context of cluster property operators in [11]. We would like to
emphasize that in both approaches certain assumptions on the structure of the polynomials $Q_{n}^{SLY}$ have been made. While in the former, a factorization of the polynomial solution like the last expression in (41) had been assumed, the latter required the local operators in question to be explicitly of cluster type. In our approach no assumption was made. We have just been using the main result (15) and obtained a unique solution to the form factor problem for spinless operators in the SLY model.

Let us comment on the uniqueness of the solution (41). It is believed that the form factor approach is suited to the classification of the operator content of a theory. In general this classification is not an easy task. The way to do this is to adjust the free parameters, which appear when solving the recursion relations for the polynomials $Q_{n}$ to the operator under consideration. As in [9] one could use LSZ techniques to get some information on the factorization properties of the polynomials.

In the case of the SLY model it turns out by (38) that the recursion relations (10) and (34) do not allow for any free parameters. In a standard way of thinking one could interpret this result that there is only one consistent (non-descendent) local quantum operator of zero spin in the SLY model.

According to a LSZ analysis [9, 7] the factorization property of the solution (41) indicates that the local operator to which our solutions belong should be the trace of the energy momentum tensor. The question then is, why does the elementary field not appear as an independent local operator in our analysis?

The SLY model can be understood in the context of perturbed conformal models [7, 28], from which it is clear that the trace of the energy-momentum tensor is equal to the elementary field up to a factor. Hence, our result gives another confirmation of this fact on the quantum field theoretical level.

7 Conclusion and Outlook

In this paper we have been studying the form factor bootstrap of the sinhG model. Having chosen a convenient ansatz (7) for the matrix elements (1) we arrived at a recursion equation (10) for a polynomial $Q_{n}(x_{1}, \ldots, x_{n})$. The appearance of such polynomial equations is a general feature of the form factor bootstrap in the case of affine Toda field theories [14, 19]. By employing a trick [14] we have obtained the most general symmetric polynomial solution (15) to these recursion equations. From the mathematical point of view it turns out that these solutions admit a description in terms of symmetric skew polynomials over the ring $\mathbb{Z}[\omega + \omega^{-1}]$. 
We showed how the remarkable cluster property solution [12] could be obtained by the general formula (15) by adjusting the free parameters in a simple way. In turn this gave us an identification of the linearly independent solutions of (15) with local operators, which have been shown to be the powers of the elementary field.

In the last section it was pointed out that the general formula can as well be applied to the SLY model, yielding a unique solution for the polynomials \( Q_{SLY} \) with no free parameters left. We have given an interpretation of this result.

It should be possible to employ the method to compute form factors outlined in the present work to find solutions to the polynomial recursion equations of other affine Toda field theories [19]. However, in these cases a major difficulty is the fact that more than one species of particles is present and that higher order poles do occur. From the viewpoint of the theory of symmetric polynomials [23] this seems to be an interesting application, since from other affine Toda models we will get symmetric polynomials in several distinguished classes of variables.

However, a straightforward application of our method should be to treat the Bullough-Dodd model [8, 17]. This model is nothing other than the non-minimal version of the SLY model and shares the feature with the theories which were under consideration in this paper of having only one massive field in the Lagrangian.

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Appendix A
In this appendix it is shown how the formula (15) works in practice. We also give explicit results for the polynomials up to \( Q_5 \).
As it was mentioned in section 3 the polynomial \( Q_1 \) is constant and we have to give a name to this constant

\[
Q_1(x_1) = A_1. \tag{42}
\]
The higher solutions will then follow uniquely as will be explained now. We take (15) and use the fact that $D_0 = 0$. Hence only the second term in (15) will contribute.

$$Q_2(x_1, x_2) = (x_1 + x_2)Q_1'(x_2) = A_2e_1^{(2)}$$ (43)

The next step is straightforward, because the second term in (15) will produce merely the kernel solution. We have using (14)

$$Q_3(x_1, x_2, x_3) = D_1(x_2|x_3)Q_1(x_3)\frac{x_1 + x_2}{x_2 - x_3} + D_1(x_3|x_2)Q_1(x_2)\frac{x_1 + x_3}{x_3 - x_2}$$
$$+ \ (x_1 + x_2)(x_1 + x_3)Q_2(x_2, x_3)$$
$$= A_1e_3^{(3)} + A_3(e_2^{(3)}e_1^{(3)} - e_3^{(3)}) = A_3 s^{(3)}_{(210)} + A_1 e_3^{(3)}.$$ (44)

In extracting $Q_4$ from (15) we find that the second term will contribute, besides the kernel, a nontrivial term which is necessary to supply the terms in $A_2$ arising from the first term in (15) in order to make the full $A_2$ component of $Q_4$ symmetric. Using the definition (16) we find

$$Q_4(x_1, \ldots, x_4) = H_2^{(2)}(x_2, x_3, x_4)\frac{(x_1 + x_2)(x_1 + x_4)}{(x_2 - x_3)(x_2 - x_4)} + H_2^{(3)}(x_3, x_2, x_4)\frac{(x_1 + x_2)(x_1 + x_4)}{(x_3 - x_2)(x_3 - x_4)}$$
$$+ H_4^{(4)}(x_1, x_2, x_3, x_4)\frac{(x_1 + x_2)(x_1 + x_3)}{(x_2 - x_3)(x_2 - x_4)}$$
$$+ (x_1 + x_2)(x_1 + x_3)(x_1 + x_4)\left(A_4 s^{(3)}_{(210)}(x_2, x_3, x_4) + A_2 e_3^{(3)}(x_2, x_3, x_4)\right)$$ (45)

We can now rewrite the result in terms of symmetric functions in four arguments

$$Q_4(x_1, \ldots, x_4) = A_4 s^{(4)}_{(3210)} + A_2 e_3^{(4)}e_1^{(4)}.$$ (46)

Computing $Q_5$ goes along similar lines. Here we find that the contribution in $A_3$ arising from the first term in (15) is not symmetric, the symmetry is only achieved if we add the $A_3$ component of the second term arising from $Q_4$. This is a general feature of the recursion equation (16). The kernel solution in $Q_n$ will evolve nontrivially, which means that in going from $Q_r$ to $Q_s$ with $r < s$ the kernel part of $Q_r$ will (apart from $Q_n$ with $n \leq 3$) not be mapped one to one onto the kernel of $Q_r$.

For $Q_5$ we get the following result

$$Q_5(x_1, \ldots, x_5) = A_5 s^{(5)}_{(43210)} + A_3 Q_{5,3}(x_1, \ldots, x_5) + A_1 Q_{5,1}(x_1, \ldots, x_5),$$ (47)

with
\[ Q_{5,3}(x_1, \ldots, x_5) = e_4^{(5)} e_3^{(5)} + 2e_4^{(5)} e_1^{(5)} e_1^{(5)} + e_5^{(5)} e_2^{(5)} e_1^{(5)} - 2e_5^{(5)} e_3^{(5)} e_2^{(5)} - (2 + (\omega + \omega^{-1})^2) e_5^{(5)} e_4^{(5)} e_1^{(5)} + (1 + (\omega + \omega^{-1})^2) e_5^{(5)} e_5^{(5)}, \]  

and

\[ Q_{5,3}(x_1, \ldots, x_5) = e_5^{(5)} e_3^{(5)} e_2^{(5)} - (\omega + \omega^{-1})^2 e_5^{(5)} e_5^{(5)}. \]  

Note that the solution (48) does correspond to the local operator : \( \phi^3(x) \) : and is not covered by the results in [3].

According to what has been said in section 3 it is now straightforward to give explicit expressions of the polynomials \( Q_n \) for \( n > 5 \).

Appendix B

In this appendix we give one (out of many) definition of a skew Schur polynomial [23]. In doing that we do not need to refer to the number of variables of the symmetric functions involved. Note that skew Schur polynomials arise when one introduces a scalar product in the space of symmetric polynomials.

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) be a partition. The conjugate partition \( \lambda' \) is then a partition obtained from \( \lambda \) with entries \( \lambda'_i = \text{Card}\{j : \lambda_j \geq i\} \).

Let \( \mu \) be another partition. Then the skew Schur polynomial can be expressed as a determinant of elementary symmetric polynomials in the following way.

\[ s_{\lambda/\mu} = \det(e_{\lambda'_i - \mu'_j - i + j}). \]  

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