The Resolution of Euclidean Massless Field Operators of Higher Spins on $\mathbb{R}^6$ and the $L^2$ Method

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Abstract

The resolution of 4 dimensional massless field operators of higher spins was constructed by Eastwood–Penrose–Wells by using the twistor method. Recently physicists are interested in 6 dimensional physics including the massless field operators of higher spins on Lorentzian space $\mathbb{R}^{5,1}$. Its Euclidean version $\mathcal{D}_0$ and their function theory are discussed in (Complex Anal Oper Theory 12:1219–1235, 2018). In this paper, we construct an exact sequence of weighted $L^2$ spaces resolving $\mathcal{D}_0$:

$$L^2_\phi(\mathbb{R}^6, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} L^2_\phi(\mathbb{R}^6, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} L^2_\phi(\mathbb{R}^6, \mathcal{V}_2) \xrightarrow{\mathcal{D}_2} L^2_\phi(\mathbb{R}^6, \mathcal{V}_3) \rightarrow 0,$$

with suitable operators $\mathcal{D}_l$ and vector spaces $\mathcal{V}_l$. Namely, we can solve $\mathcal{D}_l u = f$ in $L^2_\phi(\mathbb{R}^6, \mathcal{V}_l)$ when $\mathcal{D}_{l+1} f = 0$ for $f \in L^2_\phi(\mathbb{R}^6, \mathcal{V}_{l+1})$. This is proved by using the $L^2$...
method in the theory of several complex variables, which is a general framework to solve overdetermined PDEs under the compatibility condition. To apply this method here, it is necessary to consider weighted $L^2$ spaces, an advantage of which is that any polynomial is $L^2$ integrable. As a corollary, we prove that

$$
P(\mathbb{R}^6, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} P(\mathbb{R}^6, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} P(\mathbb{R}^6, \mathcal{V}_2) \xrightarrow{\mathcal{D}_2} P(\mathbb{R}^6, \mathcal{V}_3) \rightarrow 0$$

is a resolution, where $P(\mathbb{R}^6, \mathcal{V}_i)$ is the space of all $\mathcal{V}_i$-valued polynomials. This provides an analytic way to construct a resolution of a differential operator acting on vector valued polynomials.

**Keywords** Resolution · Euclidean massless field operator of high spins · The $L^2$ method · Overdetermined PDEs · The compatibility condition · Differential complexes

### 1 Introduction

The resolution of massless field operators of higher spins over the complexified Minkowski space $\mathbb{C}^4$ was constructed by Eastwood–Penrose–Wells [7] by using twistor method. The Euclidean version of massless field operator of spin $k/2$ is also called $k$-Cauchy–Fueter operator in quaternionic analysis (cf. [4, 24] and references therein). Recently physicists are interested in 6 dimensional physics including the massless field operators of higher spins on Lorentzian space $\mathbb{R}^{5,1}$ (cf. [16, 19] and references there in). The Euclidean version of these operators are

$$\mathcal{D}_0 : C^\infty(\mathbb{R}^6, \odot^k \mathbb{C}^4) \longrightarrow C^\infty(\mathbb{R}^6, \mathbb{C}^4 \otimes \odot^{k-1} \mathbb{C}^4),$$

for $k = 1, 2, \ldots$, where $\odot^p \mathbb{C}^4$ is $p$-th symmetric power of $\mathbb{C}^4$. A $\odot^k \mathbb{C}^4$-valued distribution $f$ is called $k$-monogenic if it satisfies $\mathcal{D}_0 f = 0$. In [14], we proved various properties for $k$-monogenic functions, e.g. the existence of infinite number of $k$-monogenic polynomials. In order to study $k$-monogenic functions, we need to solve the nonhomogeneous equation $\mathcal{D}_0 u = f$, which is overdetermined for $k > 1$. So we need to find the compatibility condition for solvability, and more generally a resolution of $\mathcal{D}_0$.

Motivated by 4 dimensional and the quaternionic cases (c.f. [1–3, 22] and references therein), a natural candidate of the resolution is

$$\begin{align*}
C^\infty(\mathbb{R}^6, V_0) &\xrightarrow{\mathcal{D}_0} C^\infty(\mathbb{R}^6, V_1) \xrightarrow{\mathcal{D}_1} C^\infty(\mathbb{R}^6, V_2) \xrightarrow{\mathcal{D}_2} C^\infty(\mathbb{R}^6, V_3) \\
&\xrightarrow{\mathcal{D}_3} C^\infty(\mathbb{R}^6, V_4) \rightarrow 0
\end{align*}$$

(1.1)

with

$$V_i := \odot^{k-i} \mathbb{C}^4 \otimes \wedge^i \mathbb{C}^4,$$

(1.2)
when \( k \geq 4 \). But it is already known [14, Section 1] that the image of \( D_0 \) consists of functions only valued in a subspace of \( V_1 \), the kernel of the contraction

\[
\mathcal{C} : \mathbb{C}^4 \otimes \wedge^p \mathbb{C}^4 \to \mathbb{C}^{p-1} \otimes \wedge^q \mathbb{C}^4
\]  

(1.3)

with \( p = k - 1, q = 1 \). Denote

\[
\mathcal{V}_1 := \ker \mathcal{C} |_{V_1}.
\]  

(1.4)

Then \( \mathcal{V}_4 = \{0\} \) automatically. Denote by \( \mathcal{D}_l \) the restriction of \( D_l \) in (1.1) to \( C^\infty(\mathbb{R}^6, \mathcal{V}_l) \). We construct the following differential complex:

\[
0 \longrightarrow C^\infty(\mathbb{R}^6, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} C^\infty(\mathbb{R}^6, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} C^\infty(\mathbb{R}^6, \mathcal{V}_2) \xrightarrow{\mathcal{D}_2} C^\infty(\mathbb{R}^6, \mathcal{V}_3) \longrightarrow 0,
\]  

(1.5)

and call it \( k\)-monogenic complex, \( k = 4, 5, \ldots \). Note that \( \mathbb{C}^4 \) is the spin representation of \( \mathfrak{so}(6, \mathbb{C}) \) and \( \mathcal{V}_l \) as the contraction of \( V_l \) is an irreducible representation of \( \mathfrak{so}(6, \mathbb{C}) \) (cf. [8]).

The \( L^2 \) method is a powerful method to solve the \( \overline{\partial} \)-equation in the theory of several complex variables (cf. e.g. [5, 10, 11]). In fact, it is a general framework to solve overdetermined PDEs under the compatibility condition, which is also given by a system of PDEs. The main difficulty to use this method is to prove the corresponding \( L^2 \) estimate. It was applied to the \( k\)-Cauchy–Fueter complex over \( \mathbb{R}^{4n} \) in [23] and also the Neumann problem associated to the \( k\)-Cauchy–Fueter complex on \( k\)-pseudoconvex domains in \( \mathbb{R}^4 \) [24]. The latter case is restricted to dimension 4 because only over \( \mathbb{R}^4 \) the corresponding \( L^2 \) estimates was proved.

In this paper we consider the weighted \( L^2 \) estimate of the \( k\)-monogenic complex as in [23]. Define an inner product \( \langle \cdot, \cdot \rangle \) on \( V_l \) and \( \mathcal{V}_l \) induced from \( \mathfrak{so}(4, \mathbb{C}) \). Let \( L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_l) \) be the Hilbert space of \( \mathcal{V}_l \)-valued \( L^2_{\varphi} \)-integrable functions on \( \mathbb{R}^6 \) with the weighted inner product

\[
\langle f, h \rangle_{\varphi} := \int_{\mathbb{R}^6} \langle f(x), h(x) \rangle e^{-\varphi} \, dx,
\]  

(1.6)

with weight \( \varphi = |x|^2 \). Denote weighted norm \( \| f \|_{\varphi} := \langle f, f \rangle_{\varphi}^{\frac{1}{2}} \).

Recall that \( \text{Dom}(\mathcal{D}_l) \) consists of \( f \in L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_l) \) such that \( \mathcal{D}_l f = u \) for some \( u \in L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_{l+1}) \) in the weak sense, i.e.

\[
\langle u, g \rangle_{\varphi} = \langle f, \Theta_l g \rangle_{\varphi}
\]  

(1.7)

for any \( g \in C^\infty_0(\mathbb{R}^6, \mathcal{V}_{l+1}) \), where \( \Theta_l \) is the formal adjoint of \( \mathcal{D}_l \). The differential operator \( \mathcal{D}_l \) defines a linear, closed, densely defined operator from \( L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_l) \) to \( L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_{l+1}) \), which we also denote by \( \mathcal{D}_l \).

The sequence

\[
L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_2) \xrightarrow{\mathcal{D}_2} L^2_{\varphi}(\mathbb{R}^6, \mathcal{V}_3) \longrightarrow 0,
\]  

(1.8)
is a complex of Hilbert spaces, i.e., for any \( u \in \text{Dom}(\mathcal{D}_l) \) and \( \mathcal{D}_l u \in \text{Dom}(\mathcal{D}_{l+1}) \), we have \( \mathcal{D}_{l+1} \mathcal{D}_l u = 0 \). To find solution to the equation

\[
\mathcal{D}_l u = f, \tag{1.9}
\]

for \( f \in L^2(\mathbb{R}^6, \mathcal{V}_{l+1}) \) satisfying

\[
\mathcal{D}_{l+1} f = 0, \tag{1.10}
\]

we consider the associated Hodge Laplacian operator \( \Box_l : L^2_\varphi(\mathbb{R}^6, \mathcal{V}_l) \rightarrow L^2_\varphi(\mathbb{R}^6, \mathcal{V}_l) \) given by

\[
\Box_l := \mathcal{D}_{l-1} \mathcal{D}_{l-1}^* + \mathcal{D}_l^* \mathcal{D}_l, \quad l = 1, 2, \text{ and } \Box_3 := \mathcal{D}_2 \mathcal{D}_2^*, \tag{1.11}
\]

where \( \mathcal{D}_l^* : L^2_\varphi(\mathbb{R}^6, \mathcal{V}_{l+1}) \rightarrow L^2_\varphi(\mathbb{R}^6, \mathcal{V}_l) \) is the adjoint operator of \( \mathcal{D}_l \).

**Theorem 1.1** Suppose that \( \varphi(x) = |x|^2 \) and \( k = 6, 7, \ldots \). There exists a constant \( C > 0 \) only depending on \( k \) such that (1) \( \Box_l \) has a bounded self-adjoint inverse \( N_l \) such that

\[
\| N_l f \varphi \| \leq C \| f \varphi \| \quad \text{for any } f \in L^2_\varphi(\mathbb{R}^6, \mathcal{V}_l). \quad (1.12)
\]

(2) For \( f \in L^2_\varphi(\mathbb{R}^6, \mathcal{V}_{l+1}) \), \( \mathcal{D}_l^* N_{l+1} f \) is the canonical solution to the nonhomogeneous equation (1.9)–(1.10), i.e.,

\[
\mathcal{D}_l (\mathcal{D}_l^* N_{l+1} f) = f,
\]

if \( \mathcal{D}_{l+1} f = 0 \), and \( \mathcal{D}_l^* N_{l+1} f \) is orthogonal to \( \ker \mathcal{D}_l \). Moreover,

\[
\| \mathcal{D}_l^* N_{l+1} f \varphi \| \leq C \| f \varphi \|, \quad \| \mathcal{D}_{l+1} N_{l+1} f \varphi \| \leq C \| f \varphi \|. \quad (1.13)
\]

**Corollary 1.1** When \( k = 6, 7, \ldots \), the sequence (1.8) is exact.

The key step to prove Theorem 1.1 is to establish the following weighted \( L^2 \) estimate.

**Theorem 1.2** Assume as in Theorem 1.1. There exists a constant \( C > 0 \) only depending on \( k \) such that

\[
\| f \varphi \|_2^2 \leq C \left( \| \mathcal{D}_l f \varphi \|_2^2 + \| \mathcal{D}_{l-1}^* f \varphi \|_2^2 \right), \quad (1.14)
\]

for any \( f \in \text{Dom}(\mathcal{D}_l) \cap \text{Dom}(\mathcal{D}_{l-1}^*), l = 1, 2, 3. \)

The massless field operators of higher spins on any dimensional Euclidean space was introduced by Souček earlier [20, 21], as generalizations of the Dirac operator [6]. We only consider 6 dimensional case here because we can use spin indices based on \( \mathfrak{s}(6, \mathbb{C}) \cong \mathfrak{s}(4, \mathbb{C}) \oplus \mathfrak{s}(2, \mathbb{C}) \oplus \mathfrak{s}(2, \mathbb{C}) \), as two-component notation in dimension 4 based on \( \mathfrak{s}(4, \mathbb{C}) \cong \mathfrak{s}(2, \mathbb{C}) \oplus \mathfrak{s}(2, \mathbb{C}) \).

An advantage to consider weighted \( L^2 \) space is that any polynomial on \( \mathbb{R}^6 \) is \( L^2_\varphi \) integrable. This allows us to deduce a resolution of the operator \( \mathcal{D}_0 \) on \( P(\mathbb{R}^6, \mathcal{V}_0) \), the module of \( \mathcal{V}_0 \)-valued polynomials over \( \mathbb{R}^6 \).
Theorem 1.3 When \( k = 6, 7, \ldots \), the sequence

\[
P(\mathbb{R}^6, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} P(\mathbb{R}^6, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} P(\mathbb{R}^6, \mathcal{V}_2) \xrightarrow{\mathcal{D}_2} P(\mathbb{R}^6, \mathcal{V}_3) \longrightarrow 0,
\]

is exact.

To prove this theorem, we need the following proposition.

Proposition 1.1 When \( k = 4, 5, \ldots \), the complex (1.5) is an elliptic complex.

Since for \( k \leq 3 \), some operator in the sequence may be of second order, we restrict \( k \) to be bigger than 3 in Proposition 1.1. The reason of the further restriction \( k \geq 6 \) in Theorem 1.2 is that we only prove the estimate (1.14) in this case (cf. (3.32)).

Compared to the quaternionic case \([23]\), the main difficulty comes from the algebraic complexity when passing from vector space \( \mathcal{V}_l = \bigotimes^{k-l} \mathbb{C}^4 \otimes \bigwedge^l \mathbb{C}^4 \) to its contracted subspace \( \mathcal{V}_l \), although only linear algebra is used to overcome it. In Sect. 2, we give some basic propositions on symmetrization and antisymmetrization to handle functions valued in the subspace \( \mathcal{V}_l \). To write down the formal adjoint operator of \( \mathcal{D}_l \) explicitly, we introduce the orthogonal projection \( P_l \) from \( \mathcal{V}_l \) to \( \mathcal{V}_l^\perp \). In Sect. 3, we give the expression of operator \( \mathcal{D}_l \) and its formal adjoint operators. Then we prove the \( L^2 \) estimate (1.14). In Sect. 4, we give the canonical solution to the nonhomogeneous equations (1.9)–(1.10) by the general framework to solve nonhomogeneous overdetermined PDEs. As a corollary, we show the sequence (1.8) is exact. Since the \( \mathcal{V}_l \)-valued polynomials over \( \mathbb{R}^6 \) are \( L^2 \) integrable, we show that (1.15) is also exact. In Sect. 5, we establish the ellipticity of the differential complex (1.5) by showing the exactness of its symbol sequence, based on which we show \( \Box_l \) is an elliptic differential operator and then prove Theorem 1.3.

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2 Linear Algebra for Symmetric and Exterior Forms

2.1 Symmetrization and Antisymmetrization

An element \( \xi \in \bigotimes^t \mathbb{C}^4 \) is a tuple \( (\xi_{A_1 \ldots A_t}) \) with \( \xi_{A_1 \ldots A_t} \in \mathbb{C} \), where \( A_1, \ldots, A_t = 1, 2, 3, 4 \). The symmetric power \( \bigotimes^t \mathbb{C}^4 \) is a subspace of \( \bigotimes^t \mathbb{C}^4 \), whose element is a \( 4^t \)-tuple \( (\xi_{A_1 \ldots A_t}) \) such that \( \xi_{B_1 \ldots B_t} \) is invariant under permutations of subscripts. An element of \( \bigotimes^{k-l} \mathbb{C}^4 \otimes \bigwedge^l \mathbb{C}^4 \) is denoted by \( (\eta_{B_1 \ldots B_{k-l}}) \), where \( \eta_{A_1 \ldots A_t} \) is symmetric in \( B_1, \ldots, B_{k-l} \) and antisymmetric in \( A_1, \ldots, A_t \). The norm \( \| \xi \| \) for \( \xi \in \bigotimes^{k-l} \mathbb{C}^4 \otimes \bigwedge^l \mathbb{C}^4 \) is the norm of \( \xi \) as an element of \( \bigotimes^k \mathbb{C}^4 \), i.e., \( \| \xi \|^2 = \sum_{A_1, \ldots, A_t} |\xi_{A_1 \ldots A_t}|^2 \).

Symmetrization and antisymmetrization of an element \( \xi \in \bigotimes^t \mathbb{C}^4 \) is given by

\[
\xi(A_1 \ldots A_t) := \frac{1}{t!} \sum_{\sigma \in S_t} \xi_{A_{\sigma_1} \ldots A_{\sigma_t}}, \quad \xi[A_1 \ldots A_t] := \frac{1}{t!} \sum_{\sigma \in S_t} \epsilon_{A_{\sigma_1} \ldots A_{\sigma_t}} \xi_{A_{\sigma_1} \ldots A_{\sigma_t}},
\]
respectively, where $S_t$ denotes the permutation group of $t$ elements and $\epsilon_{\sigma_1 \ldots \sigma_t}$ is the sign of the permutation from $(1, \ldots, t)$ to $(\sigma_1, \ldots, \sigma_t)$. The symmetrization or antisymmetrization of $\xi \in \otimes^t \mathbb{C}^4$ is an element of $\otimes^t \mathbb{C}^4$ or $\wedge^t \mathbb{C}^4$.

**Lemma 2.1** (cf. [23]) (1) For any $\xi \in \otimes^t \mathbb{C}^4$ and $\zeta \in \otimes^t \mathbb{C}^4$, we have

$$\sum_{B_1, \ldots, B_t} \xi_{B_1 \ldots B_t} \xi_{B_1 \ldots B_t} = \sum_{B_1, \ldots, B_t} \xi_{B_1 \ldots B_t} \xi_{(B_1 \ldots B_t)}.$$  

(2) For any $\xi \in \wedge^t \mathbb{C}^4$ and $\zeta \in \otimes^t \mathbb{C}^4$, we have

$$\sum_{B_1, \ldots, B_t} \xi_{B_1 \ldots B_t} \xi_{B_1 \ldots B_t} = \sum_{B_1, \ldots, B_t} \xi_{B_1 \ldots B_t} \xi_{[B_1 \ldots B_t]}.$$  

**Proof** By definition (2.1), we have

$$\sum_{B_1, \ldots, B_t} \xi_{B_1 \ldots B_t} \xi_{(B_1 \ldots B_t)} = \frac{1}{t!} \sum_{B_1, \ldots, B_t} \sum_{\sigma \in S_t} \xi_{B_1 \ldots B_t} \xi_{B_{\sigma_1} \ldots B_{\sigma_t}}$$

$$= \frac{1}{t!} \sum_{\sigma \in S_t} \sum_{B_1, \ldots, B_t} \xi_{B_{\sigma_1} \ldots B_{\sigma_t}} \xi_{B_{\sigma_1} \ldots B_{\sigma_t}} = \sum_{B_1, \ldots, B_t} \xi_{B_1 \ldots B_t} \xi_{B_1 \ldots B_t},$$

by $\xi \in \otimes^t \mathbb{C}^4$ and relabeling indices. The proof of (2) is analogous to (1).

**Lemma 2.2** (1) For $(\xi_{B_1 \ldots B_t}) \in \otimes^t \mathbb{C}^4$ symmetric in $B_2 \ldots B_t$, we have

$$\hat{\xi}(B_1 \ldots B_t) = \frac{1}{t} \sum_{s=1}^{t} \xi_{B_2 \ldots \hat{B}_s \ldots B_t},$$

where $\hat{B}_s$ means omitting $B_s$.

(2) For $(\xi_{B_1 \ldots B_t}) \in \otimes^t \mathbb{C}^4$ antisymmetric in $B_2 \ldots B_t$, we have

$$\hat{\xi}[B_1 \ldots B_t] = \frac{1}{t} \left( \xi_{B_1 \ldots B_t} - \sum_{s=2}^{t} \xi_{B_2 \ldots B_1 \ldots B_t} \right),$$

(3) For $\xi \in \otimes^t \mathbb{C}^4$, we have

$$\hat{\xi}[B_1 \ldots B_t] = \hat{\xi}[(B_1 \ldots B_t)]B_{t+1} \ldots B_t].$$

**Proof** (1) Its proof is similar to (2).

(2) By the definition of antisymmetrization (2.1), we have

$$\hat{\xi}[B_1 \ldots B_t] = \frac{1}{t!} \sum_{\sigma \in S_t} \epsilon_{\sigma_1 \ldots \sigma_t} \xi_{B_{\sigma_1} \ldots B_{\sigma_t}} = \frac{1}{t!} \sum_{s=1}^{t} \sum_{\sigma_1 \ldots \sigma_t} \epsilon_{\sigma_1 \ldots \sigma_t} \xi_{B_s \ldots B_t},$$

(2.2)
Since $f$ is antisymmetric in the last $t - 1$ indices, we get
\[
\xi_{[B_1...B_t]} = \frac{(t - 1)!}{t!} \xi_{B_1...B_t} + \frac{1}{t!} \sum_{s=2}^{t} \sum_{\sigma \in S_t, \sigma_1 = s} \epsilon_{1...t}^{\sigma_2...\sigma_t} \epsilon_{s}^{2...t} \xi_{B_s B_{s+1}...B_t}.
\]
\[
= \frac{1}{t} \xi_{B_1...B_t} - \frac{1}{t} \sum_{s=2}^{t} \xi_{B_s B_{s+1}...B_t}.
\]

This completes the proof of (2).

(3) Denote $\Xi_{[B_1...B_t]} := \xi_{[B_1...B_t]} |_{B_t+1...B_l}$. By definition of antisymmetrization, we have
\[
\Xi_{[B_1...B_t]} = \frac{1}{t!} \sum_{\sigma \in S_t} \epsilon_{1...t}^{\sigma_1...\sigma_t} \Xi_{B_{\sigma_1}...B_{\sigma_t}} = \frac{1}{t! t_1!} \sum_{\tau \in S_{t_1}} \sum_{\sigma \in S_{t_2}} \epsilon_{1...t}^{\sigma_1...\sigma_{t_1}} \epsilon_{t_1+1...t}^{\tau_1...\tau_{t_2}} \xi_{B_{\sigma_1}...B_{\tau_1}...B_{\sigma_t}}.
\]
\[
= \frac{1}{t! t_1!} \sum_{\tau \in S_{t_1}} \sum_{\sigma \in S_{t_2}} \epsilon_{1...t}^{\tau_1...\tau_{t_2}} \xi_{B_{\tau_1}...B_{\tau_{t_2}}...B_{\sigma_t}} = \xi_{[B_1...B_t]},
\]
by relabeling indices and permutations in the third identity. □

2.2 The Orthogonal Projection $\mathcal{P}_l : V_l \to \mathcal{V}_l^\perp$

The contraction $\mathcal{C}$ (1.3) given by
\[
\mathcal{C} (f)^{A_1...A_{q-1}}_{B_1...B_{p-1}} = \sum_{C=1}^{4} f^{C A_1...A_{q-1}}_{B_1...B_{p-1} C},
\]
(2.4)
satisfies
\[
\mathcal{C} \circ \mathcal{C} f = 0.
\]
(2.5)

This is because for any fixed $A_1, \ldots, A_{q-2}, B_1, \ldots, B_{p-2},$
\[
(\mathcal{C} \circ \mathcal{C} f)^{A_1...A_{q-2}}_{B_1...B_{p-2}} = \sum_{A} \mathcal{C} (f)^{A A_1...A_{q-2}}_{B_1...B_{p-2} A} = \sum_{A, C} f^{A A_1...A_{q-2}}_{B_1...B_{p-2} A C} = 0,
\]
by $f$ symmetric in subscripts and antisymmetric in superscripts.

Let $\mathcal{V}_l^\perp$ be the orthogonal complement of $\mathcal{V}_l$ in $V_l$. Now we construct a linear transformation $\mathcal{P}_l$ from $V_l$ to $\mathcal{V}_l^\perp$ ($l = 1, 2$) as follows:
\[
\mathcal{P}_1 (f)^{A}_{B_1 B_2 ... B_{k-1}} := \frac{k - 1}{k + 2} \mathcal{C} (f)^{A}_{B_1 B_2 ... B_{k-1}}, \quad f \in V_1,
\]
\[
\mathcal{P}_2 (f)^{A_1 A_2}_{B_1 B_2 ... B_{k-2}} := \frac{2(k - 2)}{k} \mathcal{C} (f)^{A_1 A_2}_{B_1 B_2 ... B_{k-2}}, \quad f \in V_2.
\]

Proposition 2.1 $\mathcal{P}_l$ is an orthogonal projection from $V_l$ to $\mathcal{V}_l^\perp$, $l = 1, 2$. 
Proof We only prove the case \( l = 2 \) since it is similar for the case \( l = 1 \). Note that

\[
\mathcal{C}(\mathcal{P}_2 f) = \mathcal{C}(f)
\]

(2.7)

for any \( f \in V_2 = \bigodot^{k-2} C^4 \otimes \Lambda^2 C^4 \), since

\[
\mathcal{C}(\mathcal{P}_2 f) A_{2}^{A_{2}}_{B_{1}\ldots B_{k-3}} = \sum_{A_{1}} \mathcal{P}_2(f) A_{1}^{A_{2}}_{B_{1}\ldots B_{k-3} A_{1}}
\]

\[
= \frac{1}{k} \sum_{A_{1}} \left( \delta_{B_{1}} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{2}\ldots B_{k-3} A_{1}} + \cdots + \delta_{B_{k-3}} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots B_{k-4} A_{1}} + \delta_{A_{1}} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots B_{k-3}}
\right.
\]

\[
- \delta_{B_{1}} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{2}\ldots B_{k-3} A_{1}} - \cdots - \delta_{B_{k-3}} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots B_{k-4} A_{1}} - \delta_{A_{1}} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots B_{k-3}}
\left.\right)
\]

= \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots B_{k-3}},
\]

by (2.4), (2.5), definition (2.6) and Lemma 2.2 (1). Then \( \mathcal{P}_2 \) is a projection since

\[
\mathcal{P}_2(\mathcal{P}_2 f) A_{1}^{A_{2}}_{B_{1}\ldots B_{k-2}} = \frac{2(k-2)}{k} \delta_{(B_{1})} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{2}\ldots B_{k-2}} = \frac{2(k-2)}{k} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{2}\ldots B_{k-2}}
\]

= \mathcal{P}_2(f) A_{1}^{A_{2}}_{B_{1}\ldots B_{k-2}},
\]

by (2.7). For any \( h \in V_2 \), we have

\[
k(\mathcal{P}_2 f, h)
\]

\[
= \sum_{A_{1}, A_{2}, B_{1}, \ldots, B_{k-2}} \left( \sum_{s=1}^{k-2} \left( \delta_{B_{s}} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots \hat{B}_{s}\ldots B_{k-2}} - \delta_{B_{s}} A_{1} \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots \hat{B}_{s}\ldots B_{k-2}} \right), h A_{1}^{A_{2}} A_{2}^{A_{2}} B_{1}\ldots B_{k-2} \right)
\]

\[
= \sum_{s=1}^{k-2} \left\{ \sum_{A_{1}, A_{2}, \ldots, \hat{B}_{s}, \ldots} \left( \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots \hat{B}_{s}\ldots B_{k-2}} \right), h B_{1}\ldots B_{k-2} \right\}
\]

\[
- \sum_{A_{1}, A_{2}, \ldots, \hat{B}_{s}, \ldots} \left( \mathcal{C}(f) A_{2}^{A_{2}}_{B_{1}\ldots \hat{B}_{s}\ldots B_{k-2}} \right), h B_{1}\ldots B_{k-2} \right\}
\]

= 0,
\]

by definition (2.6) and \( \mathcal{C}h = 0 \). Hence, \( \mathcal{P}_2 f \notin V_2^\perp \).

For any \( f \in \ker \mathcal{P}_2 \), we know \( f \notin V_2 \) since we have \( \mathcal{C}(f) = \mathcal{C}(\mathcal{P}_2 f) = 0 \). On the other hand, for any \( f \in V_2 \), we know \( \mathcal{P}_2 f = 0 \) by definition (2.6). Then \( f \in \ker \mathcal{P}_2 \) if and only if \( f \notin V_2 \). Hence \( \mathcal{P}_2 \) is an orthogonal projection from \( V_2 \) to \( V_2^\perp \).

We also need to know the norm of \( \mathcal{P}_2(\xi) \) for \( \xi \in V_1 \) in the proof of the \( L^2 \) estimate.
Proposition 2.2 We have

\[
\| \mathcal{P}_1(\xi) \|^2 = \frac{k - 1}{k + 2} \| \mathcal{C}(\xi) \|^2, \quad \text{for } \xi \in V_1,
\]

\[
\| \mathcal{P}_2(\xi) \|^2 = \frac{2(k - 2)}{k} \| \mathcal{C}(\xi) \|^2, \quad \text{for } \xi \in V_2.
\] (2.8)

Proof For \( \xi \in V_1 \), we have

\[
(k + 2)^2 \| \mathcal{P}_1(\xi) \|^2 = (k + 2)^2 \sum_{A, B_1, \ldots, B_{k-1}} \| \mathcal{P}_1(\xi)^A_{B_1 \ldots B_{k-1}} \|^2
\]

\[
= \sum_{A, B_1, \ldots, B_{k-1}} \left( \sum_{j, l = 1}^{k-1} \left( \delta^A_{B_j} (\mathcal{C}(\xi)^{B_1 \ldots B_{k-1}}_{B_j A}) , \delta^A_{B_l} (\mathcal{C}(\xi)^{B_1 \ldots B_{k-1}}_{B_l A}) \right) \right)
\]

\[
= \sum_{A, B_1, \ldots, B_{k-1}} \left( \sum_{j = 1}^{k-1} \left( \delta^A_{B_j} (\mathcal{C}(\xi)^{B_1 \ldots B_{k-1}}_{B_j A}) , \delta^A_{B_j} (\mathcal{C}(\xi)^{B_1 \ldots B_{k-1}}_{B_j A}) \right) \right)
\]

\[
+ \sum_{j \neq l} \left( \delta^A_{B_j} (\mathcal{C}(\xi)^{B_1 \ldots B_{k-1}}_{B_j A}) , \delta^A_{B_j} (\mathcal{C}(\xi)^{B_1 \ldots B_{k-1}}_{B_j A}) \right)
\]

\[
:= S_1 + S_2,
\] (2.9)

by (2.6). But we have

\[
S_1 = 4(k - 1) \sum_{B_1, \ldots, B_{k-2}} \| (\mathcal{C}(\xi)^{B_1 \ldots B_{k-2}}) \|^2 = 4(k - 1) \| \mathcal{C}(\xi) \|^2,
\] (2.10)

by relabeling indices and

\[
S_2 = \sum_{j \neq l} \sum_{B_1, \ldots, B_{k-2}} \sum_A \left( (\mathcal{C}(\xi)^{B_1 \ldots B_{k-2}}_{B_j A}) , (\mathcal{C}(\xi)^{B_1 \ldots B_{k-2}}_{B_l A}) \right)
\]

\[
= (k - 1)(k - 2) \sum_{B_1, \ldots, B_{k-2}} \| (\mathcal{C}(\xi)^{B_1 \ldots B_{k-2}}) \|^2 = (k - 1)(k - 2) \| \mathcal{C}(\xi) \|^2,
\] (2.11)

by relabeling indices. Then the sum of (2.10) and (2.11) gives us the first identity of (2.8).
For $\xi \in V_2$, we rewrite $k^2 \| \mathcal{P}_2(\xi) \|$ as in (2.9) as

$$k^2 \| \mathcal{P}_2(\xi) \|^2 = 4 \sum_{A_1, A_2, B_1, \ldots, B_{k-2}} \left( \sum_j \delta_{B_j}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right)^2$$

$$+ 4 \sum_j \delta_{B_j}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \sum_l \delta_{B_l}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}}$$

$$+ 4 \sum_{j \neq l} \left( \delta_{B_j}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \delta_{B_l}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right)$$

$$:= S_1 + S_2,$$ (2.12)

by (2.6). Since $S_1 = S_2 = 0$ if $A_1 = A_2$, we only need to consider the summation over $A_1 \neq A_2$. We have

$$S_1 = \sum_j \sum_{B_1, \ldots, B_{k-2}} \sum_{A_1 \neq A_2} \left| \delta_{B_j}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} - \delta_{B_j}^{A_2} (\xi)^{A_1}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right|^2$$

$$= \sum_j \sum_{B_1, \ldots, B_{k-2}} \sum_{A_1 \neq A_2} \left( \left| (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right|^2 + \left| (\xi)^{A_1}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right|^2 \right)$$

$$= 6(k-2) \sum_{A, B_1, \ldots, B_{k-3}} \left| (\xi)^{A}_{B_1 \ldots B_{k-3}} \right|^2 = 6(k-2) \| \xi \|^2.$$ (2.13)

By taking summation over $B_j$ at first and then $B_l$, we see that

$$S_2 = \sum_{j \neq l} \sum_{A_1 \neq A_2, B_1, \ldots, B_{k-2}} \left( \delta_{B_j}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} - \delta_{B_j}^{A_2} (\xi)^{A_1}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right)$$

$$+ \sum_{j \neq l} \left( \delta_{B_j}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} + \delta_{B_j}^{A_2} (\xi)^{A_1}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right)$$

$$- \sum_{j \neq l} \left( \delta_{B_j}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} - \delta_{B_j}^{A_2} (\xi)^{A_1}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right)$$

$$- \sum_{j \neq l} \left( \delta_{B_j}^{A_1} (\xi)^{A_2}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} - \delta_{B_j}^{A_2} (\xi)^{A_1}_{B_1 \ldots \hat{B}_j \ldots B_{k-2}} \right)$$

...
\[= 2 \sum_{j \neq l} \sum_{A_1 \neq A_2, \ldots, \hat{B}_j, \ldots, \hat{B}_l} \left( \langle \xi \rangle_{A_2} B_{1 \ldots} \hat{B}_j \ldots A_1 \ldots \hat{B}_l \ldots \right) \]

\[= 2 \sum_{j \neq l} \sum_{A_2, B_1, \ldots, B_{k-3}} \left\| \langle \xi \rangle_{A_2} B_{1 \ldots} B_{k-3} \right\|^2 = 2(k - 3)(k - 2) \| \langle \xi \rangle \|^2. \quad (2.14)\]

Here last two terms in the right hand side of the first identity vanish by \( \mathcal{C} \circ \mathcal{C} f = 0 \) in (2.5). We relabel indices in the third identity. Apply (2.13)–(2.14) to (2.12) to get the second identity of (2.8).

\[\square\]

3 The \( L^2 \) Estimate

3.1 The Euclidean Massless Field Operator

Sümann, Wolf [19] and Mason et al. [16] used the embedding \( \mathbb{R}^{5,1} \leftrightarrow \mathbb{M} \subseteq \mathbb{C}^{4 \times 4} \):

\[(x^0, x^1, \ldots, x^5) \mapsto \begin{pmatrix}
0 & x^0 + x^5 & -x^3 - ix^4 & -x^1 + ix^2 \\
-x^0 - x^5 & 0 & -x^1 - ix^2 & x^3 - ix^4 \\
x^3 + ix^4 & x^1 + ix^2 & 0 & -x^0 + x^5 \\
x^1 - ix^2 & -x^3 + ix^4 & x^0 - x^5 & 0
\end{pmatrix}, \quad (3.1)\]

to study massless field equation, where \( \mathbb{CM} = \wedge \mathbb{C}^4 \) is the 6 dimensional space of complex antisymmetric \( 4 \times 4 \) matrices. This embedding is the generalization of the embedding of the Minkowski space into \( 2 \times 2 \)-Hermitian matrix space: \( \mathbb{R}^{3,1} \leftrightarrow \mathbb{C}^{2 \times 2} \),

\[(x^0, x^1, x^2, x^3) \mapsto \begin{pmatrix}
x^0 + x^1 & x^2 + ix^3 \\
x^2 - ix^3 & x^0 - x^1
\end{pmatrix}.\]

The advantage of this embedding is that one can use two-component notation generalizing Penrose’s two-spinor notation [17, 18] and apply the twistor method to study these operators. On the other hand, we can embed 4 dimensional Euclidean space, the quaternionic space \( \mathbb{H} \), into a real subspace of \( \mathbb{C}^4 \) by \( \mathbb{H} \leftrightarrow \mathbb{C}^{2 \times 2} \),

\[x^0 + ix^1 + jx^2 + kx^3 \mapsto \begin{pmatrix}
x^0 + ix^1 & -x^2 - ix^3 \\
x^2 - ix^3 & x^0 - x^1
\end{pmatrix},\]

to obtain the elliptic version of the differential operators corresponding to massless field equations of higher spins on \( \mathbb{R}^4 \), which are called \( k \)-Cauchy–Fueter operators in [22]. For the higher dimensional case, we use the embedding \( \mathbb{H}^{n} \leftrightarrow \mathbb{C}^{2n \times 2} \), and also apply the twistor method to study \( k \)-Cauchy–Fueter equations, e.g. to find series expansion of \( k \)-regular functions on \( \mathbb{H}^{n} \) by Penrose integral formula (cf. [12, 13]). Motivated by the quaternionic case, we introduce the embedding of 6 dimensional Euclidean space into \( \mathbb{C}^{4 \times 4} \) in [14] by \( \iota : \mathbb{R}^6 \leftrightarrow \wedge^2 \mathbb{C}^4 \subseteq \mathbb{C}^{4 \times 4} \) given by
\[ \mathbb{R}^6 \ni x = (x^0, x^1, \ldots, x^5) \mapsto \iota(x) \]
\[ = \begin{pmatrix}
0 & ix^0 + x^5 & x^3 + ix^4 & x^1 + ix^2 \\
-ix^0 - x^5 & 0 & x^1 - ix^2 & -x^3 + ix^4 \\
-x^3 - ix^4 & -x^1 + ix^2 & 0 & -ix^0 + x^5 \\
x^1 - ix^2 & x^3 - ix^4 & ix^0 - x^5 & 0
\end{pmatrix}. \tag{3.2} \]

This is essentially the embedding (3.1) with \( x^0 \) replaced by \( ix^0 \), up to conjugate and sign of some terms. The Euclidean version \( D_0 \) of these massless field operators are

\[ D_0 : C^\infty(\mathbb{R}^6, \mathbb{C}^4) \rightarrow C^\infty(\mathbb{R}^6, \mathbb{C}^4 \otimes \mathbb{C}^{k-1}) \]

with

\[ D_0(f)_{B_2\ldots B_k}^A := \sum_{B_1} \nabla^{B_1 A} f_{B_1 B_2\ldots B_k}, \tag{3.3} \]

where \( \nabla^{AB} \) are complex vector fields and the matrix \( \nabla^{AB} \) is just the embedding matrix (3.2) with the coordinate function \( x_j \) replaced by \( \partial_{x^j} \), i.e.,

\[ \left( \nabla^{AB} \right) := \begin{pmatrix}
0 & i\partial_{x^0} - \partial_{x^5} & \partial_{x^3} + i\partial_{x^4} & \partial_{x^1} + i\partial_{x^2} \\
-i\partial_{x^0} - \partial_{x^5} & 0 & \partial_{x^1} - i\partial_{x^2} & -\partial_{x^3} + i\partial_{x^4} \\
-\partial_{x^3} - i\partial_{x^4} & -\partial_{x^1} + i\partial_{x^2} & 0 & -i\partial_{x^0} + \partial_{x^5} \\
-\partial_{x^1} - i\partial_{x^2} & \partial_{x^3} - i\partial_{x^4} & i\partial_{x^0} - \partial_{x^5} & 0
\end{pmatrix}. \tag{3.4} \]

Let \( \epsilon_{ABCD} \equiv \epsilon^{ABCD} \) be the sign of the permutation from \((1, 2, 3, 4)\) to \((A, B, C, D)\). Then \( \epsilon_{ABCD} \) vanishes if \( \{A, B, C, D\} \neq \{1, 2, 3, 4\} \). We use \( \epsilon^{ABCD} \) and \( \epsilon_{ABCD} \) to raise and low indices respectively. For example,

\[ \nabla_{AB} := \frac{1}{2} \sum_{C,D=1}^4 \epsilon_{ABCD} \nabla^{CD} \text{and} \quad \nabla^{AB} = \frac{1}{2} \sum_{C,D=1}^4 \epsilon^{ABCD} \nabla_{CD}, \]

since \( \sum_{C,D} \epsilon_{ABCD} \epsilon^{CDEF} = 2(\delta^E_A \delta^F_B - \delta^F_A \delta^E_B) \) by definition (cf. [16, P. 6]). By (3.4) we have

\[ \nabla^{AB} = -\nabla^{BA}, \quad \nabla^{AB} = \frac{1}{2} \nabla_{CD} \epsilon_{ABCD}. \]

Define the differential operator \( D_l : C^\infty(\mathbb{R}^6, V_l) \rightarrow C^\infty(\mathbb{R}^6, V_{l+1}) \) by

\[ (D_l f)_{A_1\ldots A_{l+1}}^{B_2\ldots B_{k-l}} := \sum_{B_1} \nabla_{B_1[A_1} f_{B_1 B_2\ldots B_{k-l}]. \tag{3.5} \]

**Proposition 3.1** *The sequence*

\[ 0 \rightarrow C^\infty(\mathbb{R}^6, V_0) \overset{D_0}{\rightarrow} C^\infty(\mathbb{R}^6, V_1) \overset{D_1}{\rightarrow} C^\infty(\mathbb{R}^6, V_2) \overset{D_2}{\rightarrow} C^\infty(\mathbb{R}^6, V_3) \overset{D_3}{\rightarrow} C^\infty(\mathbb{R}^6, V_4) \rightarrow 0 \tag{3.6} \]
is a differential complex, i.e., \( D_{l+1} D_l = 0 \).

**Proof** By definition, we have

\[
(D_{l+1} D_l f)_{B_3 \ldots B_{k-l}}^{A_1 \ldots A_{l+2}} = \sum_{B_2} \nabla B_2 [A_1 (D_l f)]_{B_2 \ldots B_{k-l}}^{A_2 \ldots A_{l+2}} = \sum_{B_1, B_2} \nabla B_2 [A_1 \nabla B_1 [A_2 f]_{B_1 \ldots B_{k-l}}^{A_3 \ldots A_{l+2}}]
\]

\[
= \sum_{B_1, B_2} \nabla B_2 [[A_1 \nabla B_1 [A_2 f]_{B_1 \ldots B_{k-l}}^{A_3 \ldots A_{l+2}}]]_{B_2} \quad \quad (3.7)
\]

by using Lemma 2.2 (3). Here \([\ldots | \ldots | \ldots] \) means we do not antisymmetrize indices inside \(| \cdots | \). Note that \( \nabla^{BC} \) commutes with \( \nabla^{DA} \) since they are differential operators with constant coefficients. So

\[
2 \sum_{B, D} \nabla B [A \nabla D f]_{B D}^{\cdots} = \sum_{B, D} \left( \nabla^{BA} \nabla^{DC} - \nabla^{BC} \nabla^{DA} \right) f_{\cdots B D}^{\cdots}
\]

\[
= \sum_{B, D} \nabla^{BA} \nabla^{DC} f_{\cdots B D}^{\cdots} - \sum_{B, D} \nabla^{BA} \nabla^{DC} f_{\cdots D B}^{\cdots} = 0,
\]

(3.8)

by relabeling \( B \) and \( D \) in the second identity and \( f \) symmetric in \( B \) and \( D \). (3.7) vanishes by (3.8).

Recall that \( \mathcal{C}(D_0 f) = 0 \) for any \( f \in C^1(\mathbb{R}^6, \mathcal{V}_0) \) (cf. [14, Introduction]). This fact is true in general.

**Proposition 3.2** (1) For \( f \in C^\infty(\mathbb{R}^6, \mathcal{V}_l) \), we have \( D_l f \in C^\infty(\mathbb{R}^6, \mathcal{V}_{l+1}) \), \( l = 0, 1, 2 \);

(2) \( \mathcal{V}_4 = \{0\} \).

**Proof** (1) This is because

\[
\mathcal{C}(D_l f)_{B_1 \ldots B_{k-l-2}}^{A_1 \ldots A_l} = \sum_C D_l (f)_{B_1 \ldots B_{k-l-2} C}^{C A_1 \ldots A_l} = \sum_{C, B} \nabla B [C f]_{B_1 \ldots B_{k-l-2} C}^{A_1 \ldots A_l}
\]

\[
= \frac{1}{l+1} \sum_{C, B} \nabla^{BC} [f]_{B_1 \ldots B_{k-l-2} C}^{A_1 \ldots A_l} - \sum_{s=1}^{l} \nabla^{BA_s} [f]_{B_1 \ldots B_{k-l-2} C}^{A_1 \ldots A_s \ldots A_l} = 0,
\]

(3.9)

by using (2.4), Lemma 2.2 (2), \( \mathcal{C} f = 0 \) and \( \nabla^{BC} \) antisymmetric in \( B \) and \( C \) while \( f \) symmetric in \( B \) and \( C \).

(2) For \( f = (f_{B_1 \ldots B_{k-l-4}}^{A_1 A_2 A_3 A_4}) \in \mathcal{V}_4 \), it is obvious that \( f_{B_1 \ldots B_{k-l-4}}^{A_1 A_2 A_3 A_4} \neq 0 \) only if \( \{A_1, A_2, A_3, A_4\} = \{1, 2, 3, 4\} \) and so \( B_j \) must equal to one of \( A_1, \ldots, A_4 \). Without loss of generality, we assume \( A_1 = B_{k-4} \). It follows from \( \mathcal{C}(f) = 0 \) that

\[
f_{B_1 \ldots B_{k-5} A_1}^{A_1 A_2 A_3 A_4} = \sum_C f_{B_1 \ldots B_{k-5} C}^{C A_2 A_3 A_4} = 0,
\]

for any fixed \( A_2, \ldots, A_4, B_1, \ldots, B_{k-5} \). So \( f = 0 \).
Since $D_l$ is the restriction of $D_l$ on $C^\infty(\mathbb{R}^6, \gamma_l)$, we have the following corollary.

**Corollary 3.1** \((1.5)\) is a differential complex.

### 3.2 The Formal Adjoint Operators

Recall that [14, Proposition 2.1] the operators $\nabla^{AB}$ and $\nabla_{AB}$ defined above satisfy

$$\nabla_{AB} = \nabla^{AB} \quad \text{and} \quad \sum_A \nabla_{AB_1} \nabla^{AB_2} = \delta^{B_2}_{B_1} \Delta. \quad (3.10)$$

Let $\Theta_{AB}$ be a scalar differential operator defined by

$$\Theta_{AB} f := -e^\varphi \nabla_{AB}(e^{-\varphi} f) = -\nabla_{AB} f + (\nabla_{AB}\varphi) f. \quad (3.11)$$

**Lemma 3.1** The formal adjoint of the scalar differential operator $\nabla^{AB}$ is $\Theta_{AB}$.

**Proof** For any $u, v \in C^1_0(\mathbb{R}^6, \mathbb{C})$, we have

$$\langle \nabla^{AB} u, v \rangle_\varphi = \int_{\mathbb{R}^6} (\nabla^{AB} u) \bar{v} e^{-\varphi} dV = \int_{\mathbb{R}^6} \bigg\{ \nabla^{AB} (u \bar{v} e^{-\varphi}) - u \cdot \nabla_{AB}(v e^{-\varphi}) \bigg\} dV$$

$$= \int_{\mathbb{R}^6} u \cdot \Theta_{AB} \bar{v} e^{-\varphi} dV = \langle u, \Theta_{AB} v \rangle_\varphi,$$

by using Stokes’ formula. \qed

**Proposition 3.3** The formal adjoint $D^*_l$ of $D_l$ is given by

$$(D^*_l f)^{A_1...A_l}_{B_1...B_{k-l}} = -\sum_E \Theta_{E(B_1} f^{E A_1...A_l}_{B_2...B_{k-l})}, \quad (3.12)$$

for $f \in C^\infty_0(\mathbb{R}^6, V_{l+1})$, $l = 0, 1, 2$.

**Proof** By definition of the formal adjoint operator, for any $h \in C^\infty_0(\mathbb{R}^6, V_{l+1})$, we have

$$\langle f, D^*_l h \rangle_\varphi = \langle D_l f, h \rangle_\varphi = \sum_{A_1,...,A_{l+1}, B_2,...,B_{k-l}} \left( \sum_{B_1} \nabla_{B_1}[A_1 f^{A_2...A_{l+1}}_{B_1...B_{k-l}}, h^{A_1...A_{l+1}}_{B_2...B_{k-l}}] \right)_\varphi$$

$$= \sum_{A_1,...,B_1,...} \left( \nabla_{B_1} f^{A_2...A_{l+1}}_{B_1...B_{k-l}}, h^{A_1...A_{l+1}}_{B_2...B_{k-l}} \right)_\varphi$$

$$= \sum_{A_1,...,B_1,...} \left( f^{A_2...A_{l+1}}_{B_1...B_{k-l}}, B_1 h^{A_1...A_{l+1}}_{B_2...B_{k-l}} \right)_\varphi$$

$$= \sum_{A_1,...,B_1,...} \left( f^{A_2...A_{l+1}}_{B_1...B_{k-l}}, -\sum_{A_1} \Theta_{A_1(B_1} h^{A_1...A_{l+1}}_{B_2...B_{k-l})} \right)_\varphi.$$
by using Lemma 2.1 twice and Lemma 3.1. The result follows. □

**Lemma 3.2** For \( \varphi = |x|^2 \), we have

\[
[\nabla^{AB}, \Theta_{CD}] = 8 \delta^A_C \delta^B_D.
\]

**Proof** Since \( \nabla^{AB}, \nabla_{CD} \) are scalar differential operators with constant coefficients, we have

\[
[\nabla^{AB}, \Theta_{CD}] = [\nabla^{AB}, -\nabla_{CD} + \nabla_{CD} \varphi] = [\nabla^{AB}, \nabla_{CD} \varphi] = \nabla^{AB} \nabla_{CD} \varphi,
\]

by (3.11). Denote

\[
z^{AB} := \begin{pmatrix}
0 & ix_0 + x_5 & x_3 + ix_4 & x_1 + ix_2 \\
-ix_0 - x_5 & 0 & x_1 - ix_2 & -x_3 + ix_4 \\
-x_3 - ix_4 & -x_1 + ix_2 & 0 & -ix_0 + x_5 \\
-x_1 - ix_2 & x_3 - ix_4 & ix_0 - x_5 & 0
\end{pmatrix}.
\]

Note that for any fixed \( D \in \{1, 2, 3, 4\} \), we have \( \varphi = \sum_E E \frac{z^{ED}}{z^{ED}} \), and

\[
\nabla_{CD} z^{ED} = 2 \delta^E_C, \quad \nabla_{CD} \overline{z^{ED}} = 0, \quad \nabla^{AB} z^{CD} = 2(\delta^C_A \delta^D_B - \delta^D_A \delta^C_B),
\]

by [14, Lemma 2.1], from which we see that

\[
\nabla_{CD} \varphi = \sum_E (\nabla_{CD} z^{ED} \cdot \overline{z^{ED}} + \overline{z^{ED}} \cdot \nabla_{CD} z^{ED}) = \sum_E 2 \delta^E_C \cdot \overline{z^{ED}} = 2 z^{CD}.
\]

(3.15)

Apply (3.15) to (3.13) to get

\[
[\nabla^{AB}, \Theta_{CD}] = 2 \nabla^{AB} z^{CD} = 2 \nabla_{AB} z^{CD} = 4(\delta^C_A \delta^D_B - \delta^D_A \delta^C_B).
\]

The lemma is proved. □

By Proposition 3.3, we know \( \Theta_l f \) belongs to \( C_0^\infty(\mathbb{R}^6, V_l) \) for \( f \in C_0^\infty(\mathbb{R}^6, V_{l+1}) \), and so \( \Theta_l f - \mathcal{P}_l(\Theta_l f) \in C_0^\infty(\mathbb{R}^6, V_l) \). Then the formal adjoint \( \Theta_l : C_0^\infty(\mathbb{R}^6, V_{l+1}) \to C_0^\infty(\mathbb{R}^6, V_l) \) of \( \mathcal{D}_l \) satisfies

\[
\Theta_l f = \mathcal{D}_l^* f - \mathcal{P}_l(\mathcal{D}_l^* f).
\]

(3.16)

This is because for \( f \in C_0^\infty(\mathbb{R}^6, V_{l+1}) \),

\[
\langle h, \Theta_l f \rangle_\varphi = \langle \mathcal{D}_l h, f \rangle_\varphi = \langle D_l h, f \rangle_\varphi = \langle h, D_l^* f \rangle_\varphi = \langle h, D_l^* f - \mathcal{P}_l(D_l^* f) \rangle_\varphi,
\]

for any \( h \in C_0^\infty(\mathbb{R}^6, V_l) \), by using Proposition 3.2 (1) and Proposition 2.1.
3.3 Proof of the $L^2$ Estimate

It is a well known fact that differential operator $\mathcal{D}_i : L^2_\psi(\mathbb{R}^6, \mathcal{Y}_i) \to L^2_\psi(\mathbb{R}^6, \mathcal{Y}_{i+1})$ defines a linear, closed, densely defined operator.

**Lemma 3.3** Suppose that $\eta_n \in C_0^\infty(\mathbb{R}^6, \mathbb{R})$ with $\eta_n \equiv 1$ on $B(0, n)$, supp $\eta_n \subset B(0, n + 2)$ and $|\text{grad} \eta_n| \leq 1$. For $f \in \text{Dom}(\mathcal{D}_i) \cap \text{Dom}(\mathcal{D}_i^*)$, we have $\eta_n f \in \text{Dom}(\mathcal{D}_i) \cap \text{Dom}(\mathcal{D}_i^*)$ and

$$\|f - \eta_n f\|_\psi + \|\eta_n \mathcal{D}_i(f) - \mathcal{D}_i(\eta_n f)\|_\psi + \|\eta_n \mathcal{D}_i^*(f) - \mathcal{D}_i^*(\eta_n f)\|_\psi \to 0, \text{ as } n \to +\infty.$$  

**Proof** Let $u = \mathcal{D}_i f$ in the weak sense. In definition (1.7) the operator $\Theta_i$ can be replaced by the formal adjoint operator $\mathcal{D}_i^*$ by (3.16). Then

$$\langle \mathcal{D}_i(\eta_n f), g \rangle_\psi = \langle \eta_n f, \Theta_i g \rangle_\psi = \langle f, \eta_n \mathcal{D}_i^* g \rangle_\psi$$

$$= \langle f, \mathcal{D}_i^*(\eta_n g) \rangle_\psi - \sum_{E, A_1, \ldots, A_l} \int_{B_1} \nabla [E \eta_n \cdot f_{A_1 \ldots A_l}] e^{-\eta_n} dV$$

$$= \langle \eta_n \mathcal{D}_i(f), g \rangle_\psi + \sum_{E, A_1, \ldots, A_l} \int_{B_1} \nabla [E \eta_n \cdot f_{A_1 \ldots A_l}] g_{E A_1 \ldots A_l} e^{-\eta_n} dV$$

for any $g \in C_0^\infty(\mathbb{R}^6, \mathcal{Y}_{i+1})$. Consequently, we have

$$\|\mathcal{D}_i(\eta_n f) - \eta_n \mathcal{D}_i(f)\|_\psi^2 = \sum_{E, A_1, \ldots, B_2, \ldots} \int_{B_1} \left| \nabla B_1 [E \eta_n \cdot f_{B_1 \ldots B_2}] e^{-\eta_n} dV \right|^2$$

$$\leq C \sum_{E, A_1, \ldots, B_2, \ldots} \int_{B(0,n+2) \setminus B(0,n)} \left| f_{A_1 \ldots A_l} \right|^2 e^{-\eta_n} dV \to 0,$$

for some absolute constant $C > 0$. Similarly, we have

$$\left( \mathcal{D}_{i-1}^*(\eta_n f) - \eta_n \mathcal{D}_{i-1}^*(f) \right)_{A_2 \ldots A_l} = \sum_{E} \nabla E \eta_n \cdot f_{E A_2 \ldots A_l},$$

and so $\|\mathcal{D}_{i-1}^*(\eta_n f) - \eta_n \mathcal{D}_{i-1}^*(f)\|_\psi \to 0$. The result follows. \qed

**Proof of Theorem 1.2** We only need to prove the estimate (1.14) for any $f \in C_0^\infty(\mathbb{R}^6, \mathcal{Y}_i)$. This is because we can assume $f \in L^2_\psi(\mathbb{R}^6, \mathcal{Y}_i)$ is compactly supported by Lemma 3.3, and can check by definition that $\delta$ regularization $f_\delta = f * \psi_\delta$, for nonnegative $\psi \in C_0^\infty(\mathbb{R}^6, \mathbb{R})$ with supp $\psi \subset B(0, 1)$ and $\int \psi = 1$, satisfies

$$\|f - f_\delta\|_\psi + \|\mathcal{D}_i(f) - \mathcal{D}_i(f_\delta)\|_\psi + \|\mathcal{D}_{i-1}^*(f) - \mathcal{D}_{i-1}^*(f_\delta)\|_\psi \to 0, \text{ as } \delta \to 0,$$

(cf. [5, Proposition 4.3.2]).
(1) For \( l = 1 \), noting that \( \mathcal{D}_0^* = \mathcal{D}_0^* \) for any \( f \in C_0^\infty(\mathbb{R}^6, \mathcal{V}_1) \), we have

\[
\begin{align*}
\kappa \| \mathcal{D}_0^* f \|_{\varphi}^2 &= k \sum_{B_1, \ldots, B_k} \left\langle \sum_C \Theta C(B_1 f_{B_2 \ldots B_k}), \sum_D \Theta D(B_1 f_{B_2 \ldots B_k}) \right\rangle_{\varphi} \\
&= k \sum_{C, D, B_1, \ldots, B_k} \left\langle \Theta C(B_1 f_{B_2 \ldots B_k}), \Theta D(B_1 f_{B_2 \ldots B_k}) \right\rangle_{\varphi} \\
&= \sum_{C, D, B_1, \ldots, B_k} \left( \Theta_C B_1 f_{B_2 \ldots B_k}, \Theta_D B_1 f_{B_2 \ldots B_k} \right)_{\varphi} \\
&+ \sum_{s=2}^k \sum_{C, D, B_1, \ldots, B_k} \left( \Theta_C B_s f_{B_1 \ldots \hat{B}_s \ldots B_k}, \Theta_D B_1 f_{B_2 \ldots B_k} \right)_{\varphi} : = \Sigma_1 + \Sigma_2,
\end{align*}
\]

\[\text{(3.17)}\]

by using (3.12) and Lemma 2.1 twice. We find that

\[
\Sigma_1 = \sum_{B_1, \ldots, B_k} \left\| \sum_C \Theta C_{B_k} f_{B_1 \ldots B_k - 1} \right\|_{\varphi}^2 \geq 0,
\]

\[\text{(3.18)}\]

To handle \( \Sigma_2 \), take adjoint and use commutators to change the order of operators to get

\[
\begin{align*}
\Sigma_2 &= \sum_{s=2}^k \sum_{C, D, B_1, \ldots, B_k} \left\langle \nabla DB_1 \Theta_C B_s f_{B_1 \ldots \hat{B}_s \ldots B_k}, f_{B_2 \ldots B_k} \right\rangle_{\varphi} \\
&= \sum_{s=2}^k \sum_{C, D, B_1, \ldots, B_k} \left( \Theta_C B_s \nabla DB_1 f_{B_1 \ldots \hat{B}_s \ldots B_k}, f_{B_2 \ldots B_k} \right)_{\varphi} \\
&+ \sum_{s=2}^k \sum_{C, D, B_1, \ldots, B_k} \left[ \nabla(DB_1, \Theta_C B_s) f_{B_1 \ldots \hat{B}_s \ldots B_k}, f_{B_2 \ldots B_k} \right]_{\varphi} : = \Sigma_3 + \Sigma_4,
\end{align*}
\]

and

\[
\begin{align*}
\Sigma_4 &= \sum_{s=2}^k \sum_{C, D, B_1, \ldots, B_k} \left\langle 8 \delta[D_S B_1] f_{B_1 \ldots \hat{B}_s \ldots B_k}, f_{B_2 \ldots B_k} \right\rangle_{\varphi} \\
&= 4 \sum_{s=2}^k \sum_{C, B_2, \ldots, B_k} \left( f_{B_2 \ldots B_k}, f_{B_2 \ldots B_k} \right)_{\varphi} - 4 \sum_{s=2}^k \sum_{B_1, \ldots, B_k} \left( f_{B_1 \ldots \hat{B}_s \ldots B_k}, f_{B_2 \ldots B_k} \right)_{\varphi} \\
&= 4(k - 1) \| f \|_{\varphi}^2,
\end{align*}
\]

\[\text{(3.19)}\]
by using Lemma 3.2 and \( \mathcal{C} f = 0 \). This term is the main term that we need to control. To control \( \Sigma_3 \), note that

\[
\Sigma_3 = \sum_{s=2}^{k} \sum_{C,D,B_1,...,B_k} \left\langle \nabla^{DB_1} f_{B_1...B_s...B_k}, \nabla^{CB_s} f_{B_2...B_k} \right\rangle \varphi
\]

\[
= \sum_{s=2}^{k} \sum_{C,D,B_1,...,B_k} \left\langle \nabla^{DB_1} f_{B_1...B_s...B_k}, -2 \nabla^{B_s}[C f_{B_2...B_k}] \right\rangle \varphi
\]

\[
+ \sum_{s=2}^{k} \sum_{C,D,B_1,...,B_k} \left\langle \nabla^{DB_1} f_{B_1...B_s...B_k}, \nabla^{DB_s} f_{B_2...B_k} \right\rangle \varphi := \Sigma_{31} + \Sigma_{32},
\]

by using Lemma 3.1 and \( \nabla^{AB} \) antisymmetric in \( A, B \). We see that

\[
\Sigma_{32} = \sum_{s=2}^{k} \sum_{C,D,B_1,...,B_{k-2}} \left\| \sum_{E} \nabla^{DE} f_{E D_1...D_{k-2}} \right\|^2 \varphi \geq 0,
\]

by relabeling indices and

\[
\Sigma_{31} = -2 \sum_{s=2}^{k} \sum_{C,D,B_1,...,B_k} \left\langle \nabla^{B_1}[C f_{B_1...B_s...B_k}], \nabla^{B_s}[C f_{B_2...B_k}] \right\rangle \varphi
\]

\[
= -2 \sum_{s=2}^{k} \sum_{C,D,B_2,...,B_{s-1},...B_k} \left\langle \mathcal{D}_1(f)^{CD}_{B_2...B_s...B_k}, \mathcal{D}_1(f)^{CD}_{B_2...B_s...B_k} \right\rangle \varphi
\]

\[
= -2(k - 1) \| \mathcal{D}_1 f \|^2 \varphi,
\]

by using Lemma 2.1. So we get

\[
\Sigma_3 \geq -2(k - 1) \| \mathcal{D}_1 f \|^2 \varphi. \tag{3.20}
\]

By (3.17)–(3.20), we get the \( L^2 \) estimate for \( l = 1 \):

\[
\| f \|^2_{\varphi} \leq \frac{k}{4(k-1)} \| \mathcal{D}_0 f \|^2_{\varphi} + \frac{1}{2} \| \mathcal{D}_1 f \|^2_{\varphi}.
\]

(2) For \( l = 2 \), the proof is similar, but is more complicated, because we have to use projection \( \mathcal{P}_1 \). Note that

\[
\| \mathcal{D}_1^* f \|^2_{\varphi} = \left\langle \mathcal{D}_1^* f, \mathcal{D}_1^* f \right\rangle_{\varphi} = \| \mathcal{D}_1^* f \|^2_{\varphi} + \| \mathcal{P}_1(\mathcal{D}_1^* f) \|^2_{\varphi}, \tag{3.21}
\]

by (3.16). Apply Proposition 2.2 to \( \| \mathcal{P}_1(\mathcal{D}_1^* f) \|^2_{\varphi} \) to get

\[
\| \mathcal{P}_1(\mathcal{D}_1^* f) \|^2_{\varphi} = \frac{k - 1}{k + 2} \sum_{B_1,...,B_{k-2}} \left\| \sum_{E} \mathcal{D}_1^*(f)^{E}_{B_1...B_{k-2}E} \right\|^2_{\varphi}
\]

\[\text{by (3.16).} \]
\[
\begin{align*}
&= \frac{k-1}{k+2} \sum_{B_1, \ldots, B_{k-2}} \left\lVert \sum_{C, E} \Theta_C (B_1 f_{B_2 \ldots B_{k-2} E}) \right\rVert_\varphi^2 \\
&= \frac{1}{(k-1)(k+2)} \sum_{B_1, \ldots, B_{k-2}} \left\lVert \sum_{C, E} \Theta_C f_{B_1 B_2 \ldots B_{k-2}} \right\rVert_\varphi^2 \\
&\leq \frac{4}{(k-1)(k+2)} \sum_{B_1, \ldots, B_{k-2}} \left\lVert \sum_{C, E} \Theta_C f_{B_1 B_2 \ldots B_{k-2}} \right\rVert_\varphi^2 \\
&\leq \frac{4}{(k-1)(k+2)} \sum_{A_1, B_1, \ldots, B_{k-1}} \left\lVert \sum_{C} \Theta_C f_{B_1 B_2 \ldots B_{k-1}} \right\rVert_\varphi^2,
\end{align*}
\]

by (3.12) and \( C f = 0 \). We use the inequality \(|\sum_E a_E|^2 \leq 4 \sum_E |a_E|^2\) in the first inequality and add extra nonnegative terms in the second inequality. Now we have

\[
(k - 1) \left\lVert \mathcal{D}_s^* f \right\rVert_\varphi^2 = (k - 1) \left\lVert \mathcal{D}_1^* f \right\rVert_\varphi^2 - (k - 1) \left\lVert \mathcal{P}_1 \mathcal{D}_s^* (f) \right\rVert_\varphi^2 \\
= \left\{ \sum_{C, D, A_1, B_1, \ldots, B_{k-1}} \left( \Theta_{C B_1 f_{A_1 B_2 \ldots B_{k-1}}} , \Theta_{D B_1 f_{A_1 B_2 \ldots B_{k-1}}} \right) \right\} \\
- (k - 1) \left\lVert \mathcal{P}_1 \mathcal{D}_s^* (f) \right\rVert_\varphi^2 \\
+ \sum_{s=2}^{k-1} \sum_{C, D, A_1, B_1, \ldots, B_{k-1}} \left( \Theta_{C B_1 f_{A_1 B_2 \ldots B_{k-1}}} , \Theta_{D B_1 f_{A_1 B_2 \ldots B_{k-1}}} \right) \varphi
\]

\[
:= \Sigma_1 + \Sigma_2,
\]

by (3.21) and expanding symmetrization as in (3.17). Apply (3.22) to \( \Sigma_1 \) in (3.23) to get

\[
\Sigma_1 \geq \frac{k-2}{k+2} \sum_{A_1, B_1, \ldots, B_{k-1}} \left\lVert \sum_{C} \Theta_{C B_1 f_{A_1 B_2 \ldots B_{k-1}}} \right\rVert_\varphi^2 \geq 0,
\]

if \( k \geq 2 \). To control \( \Sigma_2 \), we use commutator to change order of differential operator again to get

\[
\Sigma_2 = \sum_{s=2}^{k-1} \sum_{C, D, A_1, B_1, \ldots, B_{k-1}} \left( \Theta_{C B_1 f_{A_1 B_2 \ldots B_{k-1}}} , \Theta_{D B_1 f_{A_1 B_2 \ldots B_{k-1}}} \right) \varphi \\
= \sum_{s=2}^{k-1} \sum_{C, D, A_1, B_1, \ldots, B_{k-1}} \left( \Theta_{C B_1 f_{A_1 B_2 \ldots B_{k-1}}} , \Theta_{D B_1 f_{A_1 B_2 \ldots B_{k-1}}} \right) \varphi
\]
\[ + \sum_{s=2}^{k-1} \sum_{C,D,A_1,B_1,...,B_{k-1}} \left\langle \nabla^D f_{B_1...B_{k-1}}, \Theta f_{C A_1} \right\rangle := \Sigma_3 + \Sigma_4, \]  

(3.25)

by using Lemma 3.1. As in the case \( l = 1 \), we have

\[ \Sigma_4 = 4(k - 2) \| f \|_\varphi^2. \]  

(3.26)

Rewrite \( \Sigma_3 \) as

\[ \frac{1}{k - 2} \Sigma_3 = \sum_{s=2}^{k-1} \sum_{C,D,A_1,B_1,...,B_{k-1}} \left\langle \nabla^D f_{B_1...B_{k-1}}, \Theta f_{C A_1} \right\rangle \]

\[ = \sum_{C,D,A_1,E_1,E_2,B_3,...,B_{k-1}} \left\{ -3 \left\langle \nabla^D E_1 f_{B_3...B_{k-1} E_1}, \Theta f_{C A_1} \right\rangle \left\langle \nabla^D E_2 f_{B_3...B_{k-1} E_2}, \Theta f_{C A_1} \right\rangle \right\} \]

\[ := \Sigma_3 + \Sigma_{32} + \Sigma_{33}, \]  

(3.27)

by using Lemma 3.1, Lemma 2.2 (2) and relabeling indices. It is easy to see \( \Sigma_{33} \) is a squared sum, which is nonnegative and

\[ \Sigma_{31} = -3 \sum_{C,D,A_1,E_1,E_2,B_3,...,B_{k-1}} \left\langle \nabla E_1 f_{B_3...B_{k-1} E_1}, \Theta f_{C A_1} \right\rangle \left\langle \nabla E_2 f_{B_3...B_{k-1} E_2}, \Theta f_{C A_1} \right\rangle \]

\[ = -3 \sum_{C,D,A_1,E_1,E_2,B_3,...,B_{k-1}} \left\| \Theta f_{B_3...B_{k-1} E_1} \right\|_\varphi^2 = -3 \| \Theta f \|_\varphi^2. \]

It follows from the expression of \( \frac{1}{k - 2} \Sigma_3 \) in the second identity in (3.27) that

\[ \Sigma_{32} = - \sum_{D,F,G,E_1,E_2,B_3,...,B_{k-1}} \left\langle \nabla^D E_1 f_{B_3...B_{k-1} E_1}, \Theta f_{C A_1} \right\rangle \left\langle \nabla^D E_2 f_{B_3...B_{k-1} E_2}, \Theta f_{C A_1} \right\rangle \]

\[ = - \frac{1}{k - 2} \Sigma_3, \]  

where \( \Theta f \) denotes the adjoint of \( f \) with respect to the inner product on \( L^2(\varphi) \).
by relabeling indices $A_1$ as $G$ and $C$ as $F$ and using $f_{AB}^{AB}$ antisymmetric in $A, B$. Hence, we get

$$\Sigma_3 \geq -\frac{3(k-2)}{2} \| \mathcal{D}_2 f \|_\varphi^2,$$  \hspace{1cm} (3.28)

when $k > 2$. By (3.23)–(3.28), we get

$$\| f \|_\varphi^2 \leq \frac{k-1}{4(k-2)} \| \mathcal{D}_2^* f \|_\varphi^2 + \frac{3}{8} \| \mathcal{D}_2 f \|_\varphi^2.$$

(3) For $l = 3$, since $\mathcal{D}_3 f = 0$, we need to prove $\| f \|_\varphi^2 \leq C \| \mathcal{D}_2^* f \|_\varphi^2$. Similar to (3.21), we have

$$\| \mathcal{D}_2^* f \|_\varphi^2 = \| \mathcal{D}_2 f \|_\varphi^2 + \| \mathcal{P}_2 \mathcal{D}_2^*(f) \|_\varphi^2.$$

Apply Proposition 2.2 to $\| \mathcal{P}_2 \mathcal{D}_2^*(f) \|_\varphi^2$ to get

$$(k-2) \| \mathcal{P}_2 \mathcal{D}_2^* f \|_\varphi^2 = \frac{2(k-2)^2}{k} \sum_{A_1, B_1, \ldots, B_{k-3}} \left| \sum_{C,E} \Theta_{C,E} f^{A_1 E} B_1 \ldots B_{k-3} \right|_\varphi^2$$

$$= \frac{2}{k} \sum_{A_1, B_1, \ldots, B_{k-3}} \left| \sum_{C,E} \Theta_{C,E} f^{A_1 E} B_1 \ldots B_{k-3} \right|_\varphi^2$$

$$\leq \frac{6}{k} \sum_{A_1, B_1, \ldots, B_{k-3}} \sum_{E \neq A_1} \left| \sum_{C} \Theta_{CE} f^{CA_1 E} B_1 \ldots B_{k-3} \right|_\varphi^2$$

$$\leq \frac{6}{k} \sum_{A_1, A_2, \ldots, B_{k-2}} \left| \sum_{C} \Theta_{CB} f^{CA_1 A_2} B_1 \ldots B_{k-3} \right|_\varphi^2,$$  \hspace{1cm} (3.30)

by using $\mathcal{C} f = 0$ again, where we use the inequality $| \sum_{j=1}^3 a_j |^2 \leq 3 \sum_{j=1}^3 |a_j|^2$ in the first inequality and add some nonnegative terms in the second inequality.

As in the case $l = 2$ in (3.23), we have

$$(k-2) \| \mathcal{P}_2 \mathcal{D}_2^* f \|_\varphi^2 = (k-2) \| \mathcal{D}_2^* f \|_\varphi^2 - (k-2) \| \mathcal{P}_2 \mathcal{D}_2^*(f) \|_\varphi^2$$

$$= \left\{ \sum_{C,D,A_1,A_2, B_1, \ldots, B_{k-2}} \left( \Theta_{CB} f^{CA_1 A_2} B_1 \ldots B_{k-2}, \Theta_{DB} f^{DA_1 A_2} B_1 \ldots B_{k-2} \right) \right\} - (k-2) \| \mathcal{P}_2 \mathcal{D}_2^*(f) \|_\varphi^2$$

$$+ \sum_{s=2}^{k-2} \sum_{C,D,A_1,A_2, B_1, \ldots, B_{k-2}} \left( \Theta_{CB} f^{CA_1 A_2} B_1 \ldots B_{k-2}, \Theta_{DB} f^{DA_1 A_2} B_1 \ldots B_{k-2} \right) := \Sigma_1 + \Sigma_2.$$  \hspace{1cm} (3.31)
Apply (3.30) to $\Sigma_1$ in (3.31) to get

$$\Sigma_1 \geq \left(1 - \frac{6}{k}\right) \sum_{D,A_1,A_2,B_1,...} \left\| \sum_C \Theta_{CB_1} f_{B_2...B_{k-2}}^{CA_1A_2} \right\|_\varphi^2 \geq 0,$$  \hspace{1cm} (3.32)

if $k \geq 6$. For $\Sigma_2$, we can rewrite it as

$$\Sigma_2 = \sum_{s=2}^{k-2} \sum_{C,D,A_1,A_2,B_1,...} \left\{ \nabla^{DB_1} \Theta_{CB_1} f_{B_1...B_{k-2}}^{CA_1A_2}, f_{B_2...B_{k-2}}^{DA_1A_2} \right\}_\varphi$$

$$= \sum_{s=2}^{k-2} \sum_{C,D,A_1,A_2,B_1,...} \left\{ \Theta_{CB_1} \nabla^{DB_1} f_{B_1...B_{k-2}}^{CA_1A_2}, f_{B_2...B_{k-2}}^{DA_1A_2} \right\}_\varphi$$

$$+ \sum_{s=2}^{k-2} \sum_{C,D,A_1,A_2,B_1,...} \left\{ \nabla^{DB_1}, \Theta_{CB_1} f_{B_1...B_{k-2}}^{CA_1A_2}, f_{B_2...B_{k-2}}^{DA_1A_2} \right\}_\varphi := \Sigma_3 + \Sigma_4,$$

by Lemma 3.1. Similarly to the case $l = 1$, we have

$$\Sigma_4 = 4(k - 3) \| f \|_\varphi^2.$$  \hspace{1cm} (3.33)

To control $\Sigma_3$, we write

$$\frac{1}{k-3} \Sigma_3 = \frac{1}{k-3} \sum_{s=2}^{k-2} \sum_{C,D,A_1,A_2,B_1,...,B_{k-2}} \left\{ \nabla^{DB_1} f_{B_1...B_{k-2}}^{CA_1A_2}, \Theta_{CB_1} f_{B_2...B_{k-2}}^{DA_1A_2} \right\}_\varphi$$

$$= \sum_{C,D,A_1,A_2,E_1,E_2,B_1,...,B_{k-4}} \left\{ \nabla^{DE_1} f_{B_1...B_{k-4}E_1}^{CA_1A_2}, \nabla^{CE_2} f_{B_1...B_{k-4}E_2}^{DA_1A_2} \right\}_\varphi$$

$$= \sum_{C,D,A_1,A_2,E_1,E_2,B_1,...,B_{k-4}} \left\{ -4 \left\{ \nabla^{DE_1} f_{B_1...B_{k-4}E_1}^{CA_1A_2}, \nabla^{E_2} f_{B_1...B_{k-4}E_2}^{DA_1A_2} \right\}_\varphi \right.$$  

$$+ \left\{ \nabla^{DE_1} f_{B_1...B_{k-4}E_1}^{CA_1A_2}, \nabla^{E_2} f_{B_1...B_{k-4}E_2}^{CA_1A_2} \right\}_\varphi \right.$$  

$$+ \left\{ \nabla^{DE_1} f_{B_1...B_{k-4}E_1}^{CA_1A_2}, \nabla^{E_2} f_{B_1...B_{k-4}E_2}^{DA_1A_2} \right\}_\varphi \right.$$  

$$+ \left\{ \nabla^{DE_1} f_{B_1...B_{k-4}E_1}^{CA_1A_2}, \nabla^{E_2} f_{B_1...B_{k-4}E_2}^{DA_1A_2} \right\}_\varphi \right.$$  

$$:= \Sigma_{31} + \Sigma_{32} + \Sigma_{33} + \Sigma_{34},$$
by Lemma 3.1 and relabeling indices. It is easy to see that $\Sigma_{34}$ is a nonnegative squared norm, and

$$\Sigma_{31} = -4 \sum_{C,D,A_1,A_2,E_1,E_2} \left\{ \nabla E_1 [C f^D A_1 A_2] \nabla E_2 [C f^D A_1 A_2] \right\}_\varphi$$

$$= -4 \| \mathcal{D}_3 f \|_\varphi^2 = 0,$$

by Lemma 2.1, while

$$\Sigma_{32} = \Sigma_{33} = -\frac{1}{k-3} \Sigma_3,$$

by relabeling indices again. Hence,

$$\Sigma_3 \geq 0.$$

(3.34)

Apply (3.32)–(3.34) to (3.31) to get

$$\| f \|_\varphi^2 \leq \frac{k - 2}{4(k - 3)} \| \mathcal{D}_l^* f \|_\varphi^2.$$

The estimate (1.14) is proved. \(\square\)

### 4 Proof of Main Theorems

We use a general machine to deduce the existence of solution from the $L^2$ estimate (cf. e.g. [5]).

**Proposition 4.1** The $\Box_l$ is a densely defined, closed, self-adjoint and non-negative operator with domain

$$\text{Dom}(\Box_l) = \{ f \in L^2_\varphi(\mathbb{R}^6, \mathcal{V}_l) | f \in \text{Dom}(\mathcal{D}_l),$$

$$f \in \text{Dom}(\mathcal{D}_l^*), \mathcal{D}_l^* f \in \text{Dom}(\mathcal{D}_l^*), \mathcal{D}_l f \in \text{Dom}(\mathcal{D}_l^*) \}.$$

This general fact from functional analysis essentially due to Gaffney [9] (See also [5, Proposition 4.2.3], [23, Proposition 3.1]). So we omit its proof here.

**Proof of Theorem 1.1** (1) Theorem 1.2 implies that

$$\frac{1}{C} \| h \|_\varphi^2 \leq \| \mathcal{D}_{l-1}^* h \|_\varphi^2 + \| \mathcal{D}_l h \|_\varphi^2 = \langle \Box_l h, h \rangle_\varphi \leq \| \Box_l \|_\varphi \| h \|_\varphi,$$

for $h \in \text{Dom}(\Box_l)$. Thus $\Box_l$ is bounded from below and injective. Since $\Box_l$ is self-adjoint and closed, $\text{Range} \Box_l$ is a dense subset of $L^2_\varphi(\mathbb{R}^6, \mathcal{V}_l)$ by Proposition 4.1. For fixed $f \in L^2_\varphi(\mathbb{R}^6, \mathcal{V}_l)$, we define the complex anti-linear functional
\[ \lambda_f : \Box_I h \longrightarrow (f, h)_\varphi, \]

which is well defined on the dense subset \( \text{Range } \Box_I \) of \( L^2_\varphi(\mathbb{R}^6, \mathcal{V}) \), since

\[ |\lambda_f(\Box_I h)| = |(f, h)_\varphi| \leq \|f\|_\varphi \|h\|_\varphi \leq C \|f\|_\varphi \|\Box_I h\|_\varphi, \]

for \( h \in \text{Dom}\, \Box_I \). We see that \( \lambda_f \) is bounded on a dense subset and can be uniquely extended to the whole space \( L^2_\varphi(\mathbb{R}^6, \mathcal{V}) \). By the Riesz representation theorem, there exists a unique \( \lambda_f \) such that \( \lambda_f(G) = (f, G)_\varphi \) for any \( G \in L^2_\varphi(\mathbb{R}^6, \mathcal{V}) \) and \( \|F\|_\varphi = |\lambda_f| \leq C \|f\|_\varphi \). So we have \((F, \Box_I h)_\varphi = (f, h)_\varphi \) for any \( h \in \text{Dom}\,(\Box_I) \). This implies \( F \in \text{Dom}(\Box_I^*) \) and \( \Box_I^* F = f \). Since \( \Box_I \) is self-adjoint, \( F \in \text{Dom}(\Box_I) \) and \( \Box_I F = h \). We write \( F = N_I f \). Then \( \|N_I f\|_\varphi \leq C \|f\|_\varphi \).

(2) For \( f \in L^2_\varphi(\mathbb{R}^6, \mathcal{V}_{l+1}) \), since \( N_I f = 0 \) and \( \Box_I N_I f = 0 \) for any \( H \in \text{Dom}(\Box_I) \), the above identity implies \( \Box_I^* N_I f = 0 \) for any \( H \in \text{Dom}(\Box_I) \) and

\[ \Box_I N_I f = 0, \]

by \( \Box_I \) acting on both sides of (4.1). Then

\[ 0 = (\Box_I^* N_I f, \Box_I N_I f)_\varphi = \|\Box_I^* N_I f\|_\varphi^2, \]

i.e., \( \Box_I^* N_I f = 0 \). Hence, by (4.1), we have

\[ \Box_I N_I f = f. \]

Moreover, we have \( \Box_I^* N_I f \perp \ker \Box_I \) since \( (H, \Box_I^* N_I f)_\varphi = (\Box_I H, N_I f)_\varphi = 0 \) for any \( H \in \ker \Box_I \). The estimate (1.13) follows from

\[ \|\Box_I^* N_I f\|_\varphi^2 + \|\Box_I N_I f\|_\varphi^2 = (\Box_I N_I f, N_I f)_\varphi \leq C \|f\|_\varphi^2. \]

The theorem is proved. \( Q.E.D. \)

**Proof of Theorem 1.3** Note that \( \|f\|_\varphi^2 < +\infty \) for \( f \in P(\mathbb{R}^6, \mathcal{V}_{l+1}) \), where \( \varphi = |x|^2 \). So there exists \( u \in L^2(\mathbb{R}^6, \mathcal{V}) \), such that \( \Box_I u = f \) and \( \Box_{l-1} u = 0 \) by Theorem 1.1. Consequently, \( \Theta_{l-1} u = 0 \) in the sense of distributions, and so

\[ \Box_I u = \Theta_I \Box_I u + \Box_{l-1} \Theta_{l-1} u = \Theta_I f, \quad l = 1, 2, 3, \quad (4.2) \]

in the sense of distributions, where \( \Theta_I f \) is a polynomial, and the formal adjoint \( \Theta_I \) of \( \Box_I \) is a differential operator of the first order with coefficients of linear functions by
the expression of \( \Theta_l \) in (3.16) and Proposition 3.3. Hence, \( \Box_l \) is a differential operator of the second order with coefficients of linear functions.

On the other hand, \( \Box_l \) is elliptic. This is because

\[
\langle \sigma (\Box_l) \xi, \xi \rangle = \langle \sigma_l \xi, \sigma_l \xi \rangle + \langle \sigma_{l-1}^* \xi, \sigma_{l-1}^* \xi \rangle,
\]

for \( \xi \in \mathcal{V}_l \), where the inner product is the Euclidean inner product of \( \mathcal{V}_l \), and \( \sigma(\Box_l) \) and \( \sigma_l \) are principal symbols of operators \( \Box_l \) and \( D_l \) (cf. (5.1)), respectively. We see that

\[
\ker \sigma (\Box_l) = \ker \sigma_l \cap \ker \sigma_{l-1}^* = \text{Im} \sigma_{l-1} \cap \ker \sigma_{l-1}^* = \{0\},
\]

by Proposition 1.1. Thus we know the solution \( u \) of (4.2) is real analytic by applying Theorem 6.6.1 in [15] to elliptic differential operator \( \Box_l \) of the second order with real analytic coefficients. We write the Taylor expression of \( u \) as

\[
u = \sum_{m=0}^{\infty} u_m,
\]

where \( u_m \) is a homogeneous polynomial of degree \( m \). Suppose \( f \) is a polynomial of degree \( L \). Since \( D_l \) is a differential operator of the first order with constant coefficients, then \( D_l u_m \) is a homogeneous polynomial of degree \( m - 1 \) or vanishes. Hence, \( D_l f = 0 \) implies that

\[
D_l \left( \sum_{m=0}^{L+1} u_m \right) = f.
\]

So we get a polynomial solution to \( D_l u = f \) if \( D_{l+1} f = 0 \). The result follows. \( \Box \)

5 The Ellipticity of \( k \)-Monogenic-Complex

Recall that the principal symbol of the matrix differential operator \( D = \sum_{|\alpha| \leq m} A_{\alpha_1 \ldots \alpha_N} (x) \partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N} : C^\infty (\Omega, W) \rightarrow C^\infty (\Omega, W') \) at \( (x, v) \in \Omega \times \mathbb{R}^N \) is defined to be

\[
\sigma (D)_{(x,v)} := \sum_{|\alpha| = m} A_{\alpha_1 \ldots \alpha_N} (x) \left( \frac{v_1}{i} \right)^{\alpha_1} \ldots \left( \frac{v_N}{i} \right)^{\alpha_N} : W \rightarrow W', \quad (5.1)
\]

where \( \Omega \) is a domain in \( \mathbb{R}^N \) and \( A_{\alpha_1 \ldots \alpha_N} \) is a linear transformation from vector space \( W \) to \( W' \). A differential complex

\[
C^\infty (\Omega, W_0) \xrightarrow{D_0} \cdots \xrightarrow{D_{n-1}} C^\infty (\Omega, W_n)
\]

is called elliptic if its symbol sequence

\[
W_0 \xrightarrow{\sigma(D_0)_{(x,v)}} \cdots \xrightarrow{\sigma(D_{n-1})_{(x,v)}} W_n
\]

is exact for any \( x \in \Omega, v \in \mathbb{R}^N \setminus \{0\} \), that is, \( \ker (\sigma(D_l)_{(x,v)}) = \text{Im} \sigma(D_{l-1})_{(x,v)} \).
Proof of Proposition 1.1  Let us prove the symbol sequence

$$0 \to \mathcal{V}_0 \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_2} \mathcal{V}_3 \to 0,$$

is exact for fixed \(x \in \mathbb{R}^6\) and \(v \in \mathbb{R}^6 \setminus \{0\}\), where \(\sigma_l := \sigma(\mathcal{D}_l(x,v))\). Note that

$$(\sigma_l f)_{B_1\ldots B_{k-l}}^{A_1\ldots A_{l+1}} = \sum_{B_1=1}^{4} M_{E}^{B_1[A_1} f_{B_1\ldots B_{k-l}}^{A_2\ldots A_{l+1}]}$$

with

$$M_{AB} := \frac{1}{i} \begin{pmatrix} 0 & iv_0 + v_5 & v_3 + iv_4 & v_1 + iv_2 \\ -iv_0 - v_5 & 0 & v_1 - iv_2 - v_3 + iv_4 \\ -v_3 - iv_4 & v_2 + iv_4 & 0 & -iv_0 + v_5 \\ -v_1 - iv_2 & v_3 - iv_4 & iv_0 - v_5 & 0 \end{pmatrix}$$

an antisymmetric matrix. Since \(\sigma_{l+1} \circ \sigma_l = 0\) follows from \(\mathcal{D}_{l+1} \circ \mathcal{D}_l = 0\), we only need to prove \(\sigma_0\) is injective, \(\ker \sigma_l \subseteq \text{Im} \sigma_{l-1}, l = 1, 2\), and \(\sigma_2\) is surjective.

(1) For any \(\xi \in \ker \sigma_0\), we have

$$\sigma_0(\xi)_{B_1\ldots B_k}^{A_1} = \sum_{B_1} M_{E}^{B_1[A_1} \xi_{B_1} B_2\ldots B_k = 0,$$

for any fixed \(A_1, B_2, \ldots, B_k\). It is known that the determinant of \(M\) is nonvanishing for any \(v \neq 0\) since \(MM^T = |v|^2 I_{4 \times 4}\) which can be deduced from [14, (2.5) and Proposition 2.1]. This comes from the fact that \(\mathcal{D}_0\) is essentially the Dirac operator. So we have \(\xi_{B_1\ldots B_k} = 0\) for any \(B_1,\ldots, B_k\). Hence, \(\sigma_0\) is injective.

(2) For any \(\xi \in \ker \sigma_1\), let \(Z \in \mathcal{V}_0\) be given by

$$Z_{B_1\ldots B_k} := \sum_{E} M_{E(B_1\xi B_2\ldots B_k)}^{-1},$$

where \(M^{-1}\) is the inverse of \(M\). Then

$$\sigma_0(Z)_{A_1}^{B_2\ldots B_k} = \sum_{B_1} M_{E}^{B_1[A_1} Z_{B_1} B_2\ldots B_k$$

$$= \frac{1}{k} \sum_{E, B_1} \left[ M_{E}^{-1} M_{E}^{B_1[A_1} \xi_{B_2\ldots B_k} + \sum_{s=2}^{k} M_{E}^{-1} M_{E}^{B_1[A_1} \xi_{B_1\ldots \hat{B}_s\ldots B_k} \right] ,$$

by using Lemma 2.2 (1). Since \(\xi \in \ker \sigma_1\), we have

$$0 = 2\sigma_1(\xi)_{B_2\ldots B_k}^{A_1} = \sum_{B_1} M_{E}^{B_1[A_1} \xi_{B_1\ldots \hat{B}_s\ldots B_k} - \sum_{B_1} M_{E}^{B_1[A_1} \xi_{B_1\ldots \hat{B}_s\ldots B_k} ,$$
by Lemma 2.2 (2). Apply (5.5) to (5.4) to get

$$\sigma_0(\Xi)^{A_1\ldots A_k}_{B_1\ldots B_k} = \frac{1}{k} \left[ \sum_E \delta^{A_1\ldots A_k}_{B_1\ldots B_k} + \sum_{s=2}^k \sum_{B_1} \delta^{B_1\xi}_{B_1\ldots B_s^s B_1\ldots B_k} \right] = \xi^{A_1\ldots A_k}_{B_1\ldots B_k},$$

since $M^{-1}$ is the inverse of $M$. Thus $\sigma_0 \Xi = \xi$ and so $\ker \sigma_1 \subseteq \operatorname{Im} \sigma_0$.

(3) For any $\xi \in \ker \sigma_2$, set

$$\Xi^{A_1\ldots A_k}_{B_1\ldots B_k} = \sum_E M^{-1}_E(\Xi^{A_1\ldots A_k}_{B_1\ldots B_k}).$$

We claim $\Xi \in \mathcal{Y}_1$. Then

$$\sigma_1(\Xi)^{A_1 A_2}_{B_1 \ldots B_k} = \sum_{B_1} M^{B_1[A_1 \Xi^{A_2]}_{B_1 B_2 \ldots B_k}} = \frac{1}{k-1} \sum_{E, B_1} \left[ \sum_{E B_1} M^{-1}_E B_1 M^{B_1[A_1 \xi A_2]}_{E B_2 \ldots B_k} + \sum_{s=2}^{k-1} \sum_{B_1} M^{-1}_E B_1 M^{B_1[A_1 \xi A_2]}_{E B_s B_1 \ldots B_k} \right],$$

by Lemma 2.2 (1). Since $\xi \in \ker \sigma_2$, then for fixed $s \in \{2, \ldots, k-1\}$, we have

$$0 = 3 \sigma_2(\xi)^{A_1 A_2 E}_{B_2 \ldots B_s \ldots B_k} = 3 \sum_{B_1} M^{B_1[A_1 \xi A_2 E]}_{B_2 \ldots B_s \ldots B_k} = 2 \sum_{B_1} M^{B_1[A_1 \xi A_2 E]}_{B_2 \ldots B_s \ldots B_k} + \sum_{B_1} M^{B_1 E \xi A_2 A_2}_{B_2 \ldots B_s \ldots B_k},$$

by (5.3) and Lemma 2.2 (2). Apply (5.7) to (5.6) to get

$$\sigma_1(\Xi)^{A_1 A_2}_{B_2 \ldots B_k} = \frac{1}{k-1} \left( \sum_{E} \delta^{A_1 A_2 E}_{B_2 \ldots B_k} - \frac{1}{2} \sum_{s=2}^{k-1} \sum_{B_1} \delta^{B_1 E A_2}_{B_2 \ldots B_s A_1 \ldots B_k} \right)$$

$$= \frac{-k}{2(k-1)} \xi^{A_1 A_2}_{B_2 \ldots B_k},$$

by $M^{-1}$ inverse to $M$ again. Thus $\sigma_1 \left( \frac{2(k-1)}{-k} \Xi \right) = \xi$.

It remains to show the claim $\mathcal{C}(\Xi) = 0$. Note that for any fixed $B_1, \ldots, B_{k-2}$,

$$(k-1)\mathcal{C}(\Xi)_{B_1 \ldots B_{k-2}} = (k-1) \sum_{A_1, A_2} M^{-1}_A A_2 \xi^{A_1 A_2}_{B_1 A_2 B_2 \ldots B_{k-2} A_1}$$

$$= \sum_{A_1, A_2} \left[ M^{-1}_A A_2 \xi^{A_1 A_2}_{B_2 A_1 B_1 \ldots B_{k-2}} + \sum_{s=1}^{k-2} M^{-1}_A A_2 \xi^{A_1 A_2}_{B_2 \ldots B_s A_1 \ldots B_{k-2}} \right].$$
by \( \mathcal{C} \xi = 0 \). Since \( \text{det} \ M \neq 0 \), \( \mathcal{C}(\Xi) = 0 \) follows from

\[
(k - 1) \sum_{B_1} M^{E B_1} \mathcal{C}(\Xi)_{B_1 B_{k-2}} = \sum_{A_1, A_2, B_1} M^{B_1 E} M^{-1}_{A_2 A_1} \xi^{A_1 A_2}_{B_1 B_{k-2}}
\]

\[
= - \sum_{A_1, A_2} \sum_{B_1} \left( M^{B_1 A_1} \xi^{A_2 E}_{B_1 B_{k-2}} - M^{B_1 A_2} \xi^{A_1 E}_{B_1 B_{k-2}} \right) M^{-1}_{A_2 A_1}
\]

\[
= \sum_{A_1, B_1} \delta^{B_1 A_1} \xi^{A_2 E}_{B_1 B_{k-2}} + \sum_{A_1, B_1} \delta^{B_1 A_2} \xi^{A_1 E}_{B_1 B_{k-2}} = 2 \sum_{B_1} \xi^{B_1 E}_{B_1 B_{k-2}} = 0,
\]

for all indices \( E, B_2, \ldots, B_{k-2} \), by using (5.7), \( M \) antisymmetric and \( \mathcal{C} \xi = 0 \). So \( \Xi \in \mathcal{V}_1 \), \( \ker \sigma_2 \subseteq \text{Im} \sigma_1 \) is proved.

(4) For any \( \xi \in \ker \sigma_3 = \mathcal{V}_3 \), we do not know whether \( \sum_{E} M^{-1}_{E(B_1 \xi B_2 \ldots B_{k-2})} \) belongs to \( \mathcal{V}_2 \) or not. But note that the diagram

\[
\begin{array}{ccc}
\circ^{k-1} \mathbb{C}^4 \otimes \wedge^3 \mathbb{C}^4 & \xrightarrow{4\mathcal{C}} & \circ^{k-2} \mathbb{C}^4 \otimes \wedge^4 \mathbb{C}^4 \\
\downarrow \mathcal{C} & & \downarrow \mathcal{C} \\
\mathcal{V}_2 & \xrightarrow{-3\sigma_2} & \mathcal{V}_3 \\
\end{array}
\]

is commutative, i.e., \( -3\sigma_2 \mathcal{C} = 4\mathcal{C} \tilde{\sigma} \), where \( \tilde{\sigma} : \circ^{k-1} \mathbb{C}^4 \otimes \wedge^3 \mathbb{C}^4 \rightarrow \circ^{k-2} \mathbb{C}^4 \otimes \wedge^4 \mathbb{C}^4 \) is given by

\[
(\tilde{\sigma} \Xi)_{A_1 \ldots A_4 B_2 \ldots B_{k-1}} = \sum_{B_1} M^{B_1 [A_1} \Xi^{A_2 \ldots A_4]}_{B_1 B_2 \ldots B_{k-1}}.
\]

This is because

\[
-3(\sigma_2 \mathcal{C} \Xi)_{B_1 \ldots B_{k-3}} = -3 \sum_{E,F} M^{E[A_1} \Xi^{F A_2 A_3]}_{B_1 B_2 \ldots B_{k-3} EF}
\]

\[
= - \sum_{E,F} \left( M^{E A_1} \Xi^{F A_2 A_3}_{B_1 B_2 \ldots B_{k-3} EF} - M^{E A_2} \Xi^{F A_1 A_3}_{B_1 B_2 \ldots B_{k-3} EF} - M^{E A_3} \Xi^{F A_2 A_1}_{B_1 B_2 \ldots B_{k-3} EF} \right)
\]

\[
= 4 \sum_{E,F} M^{E[F} \Xi^{A_1 A_2 A_3]}_{B_1 B_2 \ldots B_{k-3} EF} = 4(\mathcal{C} \tilde{\sigma}) \Xi_{B_1 B_2 \ldots B_{k-3}}
\]

by \( \sum_{E,F} M^{EF} \Xi^{A_1 A_2 A_3}_{B_1 B_2 \ldots B_{k-3} EF} = 0 \) since \( \Xi \) is symmetric in \( E, F \) while \( M \) is antisymmetric in \( E, F \).

Now we construct an inverse image of \( \sigma_2 \) by an inverse image of \( \tilde{\sigma} \). Suppose that \( A_1, \ldots, A_4 \) are different. There must be at least one of \( A_1, A_2, A_3, A_4 \) equal to one of \( B_1, \ldots, B_{k-2} \). Without loss of generality, we assume \( A_1 = B_{k-2} \). For \( \xi \in \mathcal{V}_3 \), we construct a lifting \( \tilde{\xi} \in \circ^{k-2} \mathbb{C}^4 \otimes \wedge^4 \mathbb{C}^4 \) as follows

\[
\tilde{\xi}^{A_1 A_2 A_3 A_4}_{B_1 B_2 \ldots B_{k-3}} = \xi^{A_2 A_3 A_4}_{B_1 B_{k-2}},
\]

(5.10)
when \( A_1 = B_{k-2} \). \( \tilde{\xi} \) is well defined because if there also exists \( A_2 = B_{k-3} \), we must have \( \tilde{\xi} B_{1...B_{k-2}} = -\xi B_{1...B_{k-2}} \) by \( \xi B_{1...B_{k-2}} = -\xi B_{1...B_{k-2}} \). The latter identity follows from

\[
0 = \sum_{E} \xi E A_3 A_4 = \sum_{E=A_1, A_2} \xi E A_3 A_4 \quad \text{by } \langle \xi \rangle = 0 \quad \text{for } \xi \in \mathcal{V}_2.
\]

We have

\[
\langle \xi \rangle B_{1...B_{k-3}} = \sum_{C} \xi C A_1 A_2 A_3\]

for any fixed \( A_1, \ldots, B_1, \ldots, B_{k-3} \). Now define \( \tilde{\zeta} \in \otimes^{k-1} C^4 \otimes \wedge^3 C^4 \) by

\[
\tilde{\zeta} E_2 A_1 A_2 \quad \text{for some constant } \xi
\]

and \( \Sigma := \langle \xi \rangle \tilde{\zeta} \). Then

\[
\Sigma B_{1...B_{k-2}} = \sum_{E_1, E_2} M^{-1} E_1 B_1 \tilde{\zeta} E_2 A_1 A_2
\]

and \( \Sigma \in \mathcal{V}_2 \), since \( \langle \xi \rangle \circ \langle \xi \rangle \Sigma = 0 \). Now we show \( \sigma_2 \Sigma = C \xi \) for some constant \( C \neq 0 \).

\[
(k - 1)(\sigma_2 \Sigma) B_{1...B_{k-2}} = (k - 1) \sum_{B_1} M^{B_1[A_1]} \Sigma A_2 A_3
\]

by expanding symmetrization and using (5.11). It is easy to see that

\[
\Sigma_2 = - \sum_{E_1} \delta^{A_1 A_2 A_3 E_1} = -\xi B_{2...B_{k-2}}^4.
\]
On the other hand, it follows from $\xi \in \mathcal{V}_3 = \text{ker}\sigma_3$, i.e. $(\sigma_3\xi)_{B_2\cdots B_{k-2}} = 0$, that

$$0 = 4 \sum_{B_1} M^{B_1[A_1\xi E_1A_2A_3]}_{B_1\cdots B_{k-2}} = 3 \sum_{B_1} M^{B_1[A_1\xi A_2A_3]E_1}_{B_1\cdots B_{k-2}} - \sum_{B_1} M^{B_1E_1[A_1A_2A_3]}_{B_1\cdots B_{k-2}}. $$

Apply this identity to $\Sigma_3$ in (5.12) to get

$$\Sigma_3 = -\frac{1}{3} k^{-2} \sum_{s=2}^{k-1} M^{-1}_{E_1B_3} M^{B_1E_1[A_1A_2A_3]}_{B_1\cdots B_{k-2}} = -\frac{1}{3} k^{-2} \sum_{s=2}^{k-1} \delta [B_1\xi A_1A_2A_3]_{B_1\cdots B_{k-2}} \xi$$

by $M^{-1}$ inverse to $M$ again. Note that $M^{B_1[A_1\xi E_2A_3A_2]}_{E_1B_1\cdots B_{k-2}} = 0$ by $\wedge^5 \mathbb{C}^4 = \{0\}$, which implies

$$M^{B_1E_2[A_1A_2A_3]}_{B_1\cdots B_{k-2}} + M^{B_1E_1[A_1A_2A_3]}_{E_2B_1\cdots B_{k-2}} = 3 M^{B_1[A_1\xi E_2A_3A_2]}_{E_1B_1\cdots B_{k-2}}. $$

Then apply (5.15) to $\Sigma_1$ in (5.12) to get

$$\Sigma_1 = \frac{1}{3} \sum_{B_1, E_1, E_2} M^{-1}_{E_1E_2} \left( M^{B_1E_2[A_1A_2A_3]}_{B_1\cdots B_{k-2}} + M^{B_1E_1[A_1A_2A_3]}_{E_2B_1\cdots B_{k-2}} \right)$$

by (5.11). Now apply (5.13), (5.14) and (5.16) to (5.12) to get

$$(k - 1)(\sigma_2 \Xi)_{A_1A_2A_3}^{B_2\cdots B_{k-2}} = \frac{2 - k}{3} \xi A_1A_2A_3.$$

Hence, $\sigma_2$ is surjective. Proposition 1.1 is proved. \hfill $\square$

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