Derivation of a generalized Schrödinger equation from the theory of scale relativity

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Using Nottale’s theory of scale relativity relying on a fractal space-time, we derive a generalized Schrödinger equation taking into account the interaction of the system with the external environment. This equation describes the irreversible evolution of the system towards a static quantum state. We first interpret the scale-covariant equation of dynamics stemming from Nottale’s theory as a hydrodynamic viscous Burgers equation for a potential flow involving a complex velocity field and an imaginary viscosity. We show that the Schrödinger equation can be directly obtained from this equation by performing a Cole-Hopf transformation equivalent to the WKB transformation. We then introduce a friction force proportional and opposite to the complex velocity in the scale-covariant equation of dynamics in a way that preserves the local conservation of the normalization condition. We find that the resulting generalized Schrödinger equation, or the corresponding fluid equations obtained from the Madelung transformation, involve not only a damping term but also an effective thermal term. The friction coefficient and the temperature are related to the real and imaginary parts of the complex friction coefficient in the scale-covariant equation of dynamics. This may be viewed as a form of fluctuation-dissipation theorem. We show that our generalized Schrödinger equation satisfies an $H$-theorem for the quantum Boltzmann free energy. As a result, the probability distribution relaxes towards an equilibrium state which can be viewed as a Boltzmann distribution including a quantum potential. We propose to apply this generalized Schrödinger equation to dark matter halos in the Universe, possibly made of self-gravitating Bose-Einstein condensates.

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I. INTRODUCTION

The Schrödinger equation is the cornerstone of quantum mechanics. Schrödinger introduced this equation in 1926 from an ingenious procedure obtained by combining the de Broglie relations and the standard wave equation (second order in time) with a space dependent phase velocity.\(^1\) The Schrödinger equation can also be obtained from a correspondence principle by writing the energy of the particle as $E = p^2 / 2m + m\Phi$ and replacing the variables $E$ and $p$ by the operators $i\hbar\partial_t$ and $-i\hbar\nabla$ acting on the wave function $\psi$. This is how the Schrödinger equation is usually introduced in standard textbooks of quantum mechanics. The Schrödinger equation has proven to be extremely successful. Its time-independent solution leads to a fundamental eigenvalue problem $\hat{H}\psi = E\psi$ which determines the quantification of the energy.\(^2\) For the Coulomb potential, Schrödinger recovered the energy spectrum of the hydrogen atom heuristically obtained by Bohr.\(^{11,12}\) For the harmonic oscillator and for the rotator, Schrödinger recovered the results previously obtained by Heisenberg from his more abstract matrix mechanics. More generally, Schrödinger showed the equivalence between his wave mechanics and the Heisenberg-Born-Jordan quantum mechanics.\(^{13,14}\) On the other hand, by introducing a relation $\psi = e^{i\mathcal{S}/\hbar}$ between the wave function $\psi$ and the complex action $\mathcal{S}$, Wentzel,\(^{10}\) Brillouin,\(^{17}\) and Kramers\(^{18}\) showed that the Schrödinger equation reduces to the Hamilton-Jacobi equation in the semi-classical limit $\hbar \to 0$. This is the so-called WKB approximation which allows one to study the semi-classical regime of a quantum system. One recovers the classical mechanics from the wave mechanics in the limit $\hbar \to 0$ in the same manner that one recovers geometric optics from the theory of undulatory optics when the wavelength $\lambda \to 0$. Despite all its successes, the Schrödinger equation remains very mysterious (in part because of the complex nature of the wave function) and it is fair to say that this equation has been essentially postulated rather than being derived from first principles.

The physical interpretation of the Schrödinger equation when applied to the hydrogen atom posed itself since the start. Schrödinger\(^2\) introduced a density $\rho = |\psi|^2$ and a current and derived a local conservation equation for the density. This gives an interpretation to the wave function in the sense that $|\psi|^2(r,t)$ characterizes the “presence” of the particle at some point. Schrödinger thought that the wave function represents a particle that is spread out, most of the particle being where the modulus of the wave function $|\psi|^2$ is large. For example, according to Schrödinger’s

\(^1\) A short account of the early development of quantum mechanics is given in the Introduction of \(^2\) Schrödinger initially obtained this eigenvalue equation from a variational principle before introducing his time-dependent equation (see \(^2\) for a short review).
view, the charge of the electron is not concentrated in a point, but is spread out through the whole space, proportional to the quantity $|\psi|^2$. Alternatively, Born \cite{19,21} developed a probabilistic interpretation of the wave function. Born proposed that the magnitude of the wavefunction $\psi(r, t)$ does not tell us how much of the particle is at position $r$ at time $t$, but rather the probability that the particle is at $r$ at time $t$. This gives an interpretation to the wave function in the sense that $|\psi|^2$ represents the probability of presence of the electron at some point. Therefore, quantum mechanics essentially provides a probabilistic description of Nature. However, the origin of this probabilistic description divided the researchers. In the complementarity school led by Bohr, the quantum theory is complete. It must be accepted as it is, and one should not try to understand the origin of probabilities from a more fundamental principle. This interpretation of quantum mechanics was criticized by the statistical school associated with Einstein who argued that the quantum theory is incomplete and that one must understand the origin of probabilities from a causal subdynamics (“God does not play dice”). Therefore, the interpretation of quantum mechanics gave rise to intense debates between the view of Bohr who considers that the wave function defines a physical state completely and the view of Einstein who did not accept this view and preferred a more stochastic and statistical interpretation. In brief, Bohr argues that the physical state of an individual system is completely specified by a wave function that determines only the probabilities of actual results that can be obtained in a statistical ensemble of similar experiments. Alternatively, Einstein believes that, even at the quantum level, there must exist dynamical variables determining (as in classical physics) the actual behavior of each individual system, and not merely its probable behavior. Following this point of view, there have been attempts to explain quantum phenomena within the framework of classical statistical mechanics. Several researchers tried to found quantum mechanics in terms of more familiar stochastic models and probability theory in order to gain a better understanding. Without attempting to be exhaustive, we give below a short account of different attempts to justify the postulates of quantum mechanics, and derive the Schrödinger equation, in terms of classical stochastic theories.

In a very intriguing paper, Madelung \cite{23} showed that the complex, linear, Schrödinger equation could be transformed into a pair of real, nonlinear, hydrodynamic equations similar to an equation of continuity and an Euler equation involving a quantum potential arising from the finite value of $\hbar$. As a result, the Schrödinger equation can be represented by the picture of an irrotational flow of a compressible perfect fluid with a particular quantum potential (also equivalent to an anisotropic pressure tensor). The quantum potential, or quantum force, accounts for the Heisenberg uncertainty principle. Because of this quantum term, the particle’s motion does not exactly follow the laws of classical mechanics. The Madelung transformation suggests the possibility of a semiclassical description of quantum systems through the fluid dynamical viewpoint. Initially, very little attention was paid to this approach and Pauli expressed the opinion that it is not interesting (see the comment in Ref. \cite{23}).

At about the same period, de Broglie \cite{24,28} (see also London \cite{29}) developed a relativistic hydrodynamic representation of the Klein-Gordon (KG) equation (he was apparently not aware of the work of Madelung). He derived a relativistic Euler equation that contains a Lorentz invariant quantum potential. This is the relativistic version of the classical Madelung quantum potential. De Broglie interpreted the quantum force as a force of internal tensions existing around the corpuscles. He also interpreted the continuity equation as a conservation equation for a density transported by a velocity. The aim of de Broglie was to provide a causal and objective interpretation of wave mechanics, in accordance with the wish expressed many times by Einstein, and in contrast to the purely probabilistic interpretation of quantum mechanics put forward by Born, Bohr, and Heisenberg. This is what he called the pilot-wave theory because the particle is guided by the wave $\psi$. The pilot-wave theory of de Broglie was criticized by Pauli during the October 1927 Solvay Physics Conference (see the comment in \cite{31}), and de Broglie abandoned it.

An interpretation of the Schrödinger equation in terms of particle trajectories was proposed by Bohm \cite{31} in 1952. In this interpretation, the evolution of each individual system is determined by definite laws analogous to (but not identical with) the classical equations of motion. Bohm interpreted the deviation from the Newtonian equations of motion as being due to a quantum-mechanical potential associated with the wave function. This quantum potential, which depends on the density and density gradients of the system, suggests an interpretation of the quantum theory in terms of “hidden” variables, in the same sense that in macroscopic physics the coordinates and momenta of individual atoms are hidden variables which in a large scale system manifest themselves only as statistical averages. Actually, Bohm rediscovered the quantum potential previously introduced Madelung and de Broglie in 1926.\footnote{Bohm \cite{31} mentions that the work of de Broglie was drawn to his attention after he completed his paper. On the other hand, the paper} The quantum

\footnote{Madelung's hydrodynamic approach was further developed by Kennard \cite{24} in 1927. However, apart from that work, the paper of Madelung was very little quoted.}

\footnote{As reported by Bhaumik \cite{22}, Born was strongly influenced by Einstein in his interpretation as he stated in his Nobel lecture: “Again an idea of Einstein’s gave me the lead. He had tried to make the duality of particles - light quanta or photons - and waves comprehensible by interpreting the square of the optical wave amplitudes as probability density for the occurrence of photons. This concept could at once be carried over to the wave function: $|\psi|^2$ ought to represent the probability density for electrons (or other particles).”}
potential plays a central role in the deterministic interpretation of quantum mechanics discussed by Bohm. This renewal of interest for a causal interpretation of quantum mechanics stimulated de Broglie to return to the problem again and undertake a fresh examination of his old ideas [32–34].

The interpretation of quantum mechanics in terms of a classical statistical or hydrodynamical picture (based on the Madelung-de Broglie transformation) was further discussed by Takabayasi [35, 36]. This formulation is based on the procedure to transform the equation for a state vector (the wave function) into an ensemble of classical motions. There is a formal equivalence between the picture of waves and that of an ensemble of trajectories. This suggests an alternative formulation of quantum mechanics in terms of a classical picture. The quantum potential, which depends on the probability itself, leads to a statistical blurring of the classical trajectories. Diffusion of wave packets, interferences, tunnel effect etc. may be interpreted in terms of this quantum force as previously emphasized by Bohm [31]. In this approach, the trajectories deviate, due to the quantum force, from purely classical ones, generally showing very complicated fluctuations in non-stationary cases. According to Takabayasi, it might be conceivable to deduce the fluctuations of the particles’ trajectories due to the quantum potential through a mechanism similar to that of Brownian motion; that is, we may regard the fluctuations as produced by the random action from the outside medium.

In this viewpoint, we must postulate a virtual medium (an aether). Therefore, “hidden variables” are introduced not only as the simultaneously defined particle position and velocity as in the original statistical picture, but also as the freedom of the medium. However, despite some resemblances between quantum mechanical motion and diffusion phenomena as the result of the formal similarity between the Schrödinger equation and the diffusion equation, Takabayasi [35, 36] emphasized that the nature of the stochastic process in the two cases is very different. In the quantum theory, the trajectories may have very complicated fluctuations but these fluctuations are not at random since, for each individual trajectory, they are completely determined by the density \( \rho = |\psi|^2 \) appearing in the quantum potential. By contrast, in Brownian theory, the particle experiences a fluctuating force that is uncorrelated at every successive time and independent on the probability distribution \( \rho \).

In a remarkable paper, Nelson [45] proposed a derivation of the Schrödinger equation from Newtonian mechanics using a stochastic theory based on Fokker-Planck equations. The hypothesis is that every particle of mass \( m \) is subject to a Brownian motion with a diffusion coefficient \( D = \hbar/2m \) (where \( \hbar \) is the Planck constant) and no friction. In ordinary Brownian motion, friction plays an important role. In Nelson’s theory, quantum particles experience no friction in order to preserve Galilean invariance. As a result, the quantum Brownian motion is non dissipative and reversible while the usual Brownian motion is dissipative and irreversible (this is in agreement with the fact that the Schrödinger equation conserves the energy while the Fokker-Planck equation dissipates the free energy). In this stochastic formulation of quantum mechanics, the random motion represents quantum fluctuations due to a subquantum medium (a sort of aether). Nelson introduced a mean forward velocity \( \mathbf{u}_+ \) and a mean backward velocity \( \mathbf{u}_- \), associated with forward and backward Fokker-Planck equations, defined a mean acceleration, and derived a pair of coupled hydrodynamic equations for the current velocity \( \mathbf{u} = (\mathbf{u}_+ + \mathbf{u}_-)/2 \) and the osmotic velocity \( \mathbf{u}_Q = (\mathbf{u}_+ - \mathbf{u}_-)/2 \). Using the Madelung transformation, he showed that these equations are equivalent to the Schrödinger equation!

More recently, Nottale [46] developed a theory of scale relativity and managed to derive the Schrödinger equation from Newton’s law by using a principle of scale covariance. Nottale gives up the concept of differentiability of space-time and postulates that space-time is fractal. In a fractal space-time the trajectories of the quantum particles are continuous but nowhere differentiable. The velocity is the sum of a differentiable part and a non-differentiable (fractal) part. The new component is an explicit scale-dependent fractal fluctuation. The fractal dimension of the Brownian motion is \( D_F = 2 \). The theory of scale relativity extends Einstein’s theory of relativity to scale transformations of resolutions. In this approach, the particles have a stochastic motion that is due to the fractal nature of the space-time itself. The fractal (non-differentiable) nature of the trajectories leads to introducing twin velocities \( \mathbf{u}_+ \) and \( \mathbf{u}_- \) whose

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6 Takabayasi also criticized certain aspects of Bohm’s interpretation.

7 Similar interpretations of the Schrödinger equation have been developed by other authors. For example, Bohm and Vigier [37] introduced the notion of random fluctuations arising from the interaction with a subquantum medium. On the other hand, Fürth [38], Fényes [39], Weizel [40], Kershaw [41] and Comisar [42] tried to describe the motion of quantum particles in terms of a Markov process by analogy with Brownian motion. They pointed out the formal analogy between the Schrödinger equation and the Fokker-Planck equation and introduced an imaginary diffusion coefficient \( D = i\hbar/2m \) (this formula first appeared in the paper of Fürth [38]). In the work of Weizel [40], the random aspects of the motion of a quantum particle are due to the interaction with hypothetical particles that he called zerons.

8 It is interesting to note that, in the context of generalized thermodynamics and nonlinear Fokker-Planck (NFP) equations, the random force acting on a particle is allowed to depend on the probability distribution \( \rho(r, t) \) (see, e.g., [33, 43] for a review on NFP equations).

9 Many authors have pointed out the formal equivalence between the Schrödinger and the Fokker-Planck equation with imaginary time (see footnote 7) but the approach of Nelson is different.

10 Nelson [45] does not explicitly refers to Madelung’s paper.
meaning is different from the one given in the stochastic quantum mechanics of Nelson.\textsuperscript{11} From these twin velocities, one can form a classical velocity \( \mathbf{u} = (\mathbf{u}_+ + \mathbf{u}_-) / 2 \) and a quantum velocity \( \mathbf{u}_Q = (\mathbf{u}_+ - \mathbf{u}_-) / 2 \). These real velocities can be combined together into a complex velocity \( \mathbf{U} = \mathbf{u} - i\mathbf{u}_Q \) and this duality is viewed as the fundamental origin of the complex nature of the wave function \( \psi \) in quantum mechanics (the wave function is related to the complex velocity by \( \mathbf{U} = -i(\hbar/m)\nabla \ln \psi \)).

The Schrödinger equation applies to systems that are isolated from the surrounding. As a consequence, the energy is conserved and the evolution is reversible. However, in reality, a system is never totally isolated from the surrounding. It is therefore of interest to try to generalize the Schrödinger equation by taking into account the interaction of the system with the external environment. This gives rise to dissipative effects and irreversibility. How can one introduce dissipative effects into the Schrödinger equation? This apparently cannot be done from the correspondence principle. This shows the limit of this heuristic approach. By contrast, it is possible to generalize the approach of Nottale \cite{46} by including a damping force in the fundamental equation of dynamics (Newton’s law). However, there is a little difficulty. If we naively introduce a friction force \( -\gamma \mathbf{U} \) in the scale-covariant equation of dynamics [see Eq. (71)], we obtain a damped Schrödinger equation [see Eq. (74)] that does not conserve the normalization condition locally. Therefore, we slightly generalize the scale-covariant equation of dynamics by writing the friction force as \( -\text{Re}(\gamma \mathbf{U}) \) [see Eq. (60)] in order to obtain a damped Schrödinger equation [see Eqs. (68) and (74)] that conserves the normalization condition locally. Interestingly, when we perform the Madelung transformation on this equation, we obtain a quantum Euler equation [see Eq. (78)] that includes not only a friction term \( -\xi \mathbf{u} \) (as expected) but also a pressure term associated with an effective isothermal equation of state \( P = \rho k_B T/m \) [see Eq. (79)]. The friction coefficient \( \xi \) and the temperature \( T \) are related to the real and imaginary parts of the complex friction \( \gamma \) [see Eq. (76)]. The emergence of both friction and temperature is reminiscent of the fluctuation-dissipation theorem in statistical mechanics but it takes here a novel form. It arises from the complex nature of quantum mechanics which is intrinsically due to the non-differentiability of space-time. Indeed, \( \xi \) and \( 2k_B T/\hbar \) may be viewed as twin friction coefficients forming a complex friction coefficient \( \gamma = \xi + i2k_B T/\hbar \) similarly to the twin velocities \( \mathbf{u} \) and \( \mathbf{u}_Q \) forming a complex velocity \( \mathbf{U} = \mathbf{u} - i\mathbf{u}_Q \).

The paper is organized as follows. In Sec. \textsuperscript{11} we recall the scale-covariant equation of dynamics [see Eq. (11)] obtained from Nottale’s theory of scale relativity and emphasize its resemblance with a complex viscous Burgers equation for a potential flow in hydrodynamics [see Eq. (12)]. Then, making the complex Cole-Hopf transformation [see Eq. (15)], and noting its resemblance with the WKB transformation [see Eq. (17)], we derive the Schrödinger equation [see Eq. (21)]. This direct derivation considerably simplifies and clarifies the derivation of Nottale \cite{46}. It shows that the Schrödinger equation is formally equivalent to the complex viscous Burgers equation in the same sense that the diffusion equation is equivalent to the real viscous Burgers equation. Having obtained the Schrödinger equation, we recall the Madelung transformation and show that the Schrödinger equation is formally equivalent to real hydrodynamic equations formed by a continuity equation and an Euler equation including a quantum potential [see Eqs. (18)-(20)]. In Sec. \textsuperscript{11} we start from a scale-covariant equation of dynamics including a friction force that preserves the local conservation of the normalization condition [see Eq. (66)] and derive a generalized Schrödinger equation. We obtain a nonlinear Schrödinger equation [see Eq. (71)] exhibiting simultaneously dissipative effects and (effective) thermal effects. Using the Madelung transformation, we show that our generalized Schrödinger equation is equivalent to quantum damped isothermal Euler equations [see Eqs. (76)-(79)]. In the strong friction limit \( \xi \to +\infty \), we obtain a quantum Smoluchowski equation [see Eq. (82)]. In Sec. \textsuperscript{11} we develop an effective thermodynamic formalism. We show that our generalized Schrödinger equation satisfies an \( H \)-theorem for a quantum free energy associated with the Boltzmann entropy. As a result, the probability distribution relaxes towards an equilibrium state which can be viewed as a Boltzmann distribution including a quantum potential. This is at variance with the Schrödinger equation and the quantum Euler equation (without friction) that admit steady states but that do not relax towards such states from a generic initial condition. This is because the Schrödinger equation is reversible while our generalized Schrödinger equation is irreversible. In Sec. \textsuperscript{11} we establish the virial theorem associated with our generalized Schrödinger equation. In Sec. \textsuperscript{11} we discuss the connection between our generalized Schrödinger equation and the nonlinear Schrödinger equations obtained previously by Kostin \cite{48} and Bialynicki-Birula and Mycielski \cite{49} who introduced individually dissipative effects and nonlinear effects equivalent to thermal effects. In a sense, our approach based on the theory of scale relativity unifies these two equations into a single one and shows that dissipative effects and thermal effects arise from a single complex friction coefficient [see Eq. (75)]. In Sec. \textsuperscript{11} we consider another dissipative Schrödinger equation. In the Conclusion, we propose to apply our results to self-gravitating Bose-Einstein condensates (BECs) possibly describing dark matter halos in cosmology (this model is developed in detail in \cite{50}). The Appendices contain additional material that is used throughout the paper. In this paper, we do not go into

\textsuperscript{11} The theory of Nottale differs fundamentally from the one of Nelson even if there exist close analogies (see Appendix \textsuperscript{23}).
the details of the interpretation of the theory of scale relativity that are explained in the book of Nottale [46]. Rather, we use the principle of scale covariance as a mathematical tool to obtain generalized Schrödinger equations. We note that frictional effects cannot be directly obtained from the correspondence principle. Therefore, the formalism of scale-relativity seems to be more general and more constructive from this point of view.

II. DERIVATION OF THE SCHRODINGER EQUATION

In this section, we recall the derivation of the Schrödinger equation from the theory of scale relativity. We propose a more direct derivation than the one given by Nottale [46]. We interpret the scale-covariant equation of dynamics stemming from Nottale’s theory as a hydrodynamic viscous Burgers equation for a potential flow involving a complex velocity field and an imaginary viscosity. We show that the Schrödinger equation can be directly obtained from this equation by performing the complex Cole-Hopf transformation equivalent to the WKB transformation. Therefore, the Schrödinger equation is formally equivalent to the complex viscous Burgers equation in the same sense that the diffusion equation is equivalent to the real viscous Burgers equation (see Appendix A).

A. Basic tools of scale relativity

A fundamental consequence of the non-differentiable nature of space-time, breaking the symmetry $dt \leftrightarrow -dt$, is the two-valuedness character of the velocity which is at the origin of complex numbers in quantum mechanics. Indeed, when the flow $r(t, dt)$ is not differentiable, the derivative $d/dt$ is not defined (contrary to standard mechanics) and one has to introduce two velocities $u_+(r(t), t)$ and $u_-(r(t), t)$ defined from $t - dt$ to $t$ for $u_-$ and from $t$ to $t + dt$ for $u_+$. The elementary displacement $dr_{\pm}$ for both processes is the sum of a differential part $dr_{\pm} = u_{\pm} dt$ and a non-differentiable part which is a scale-dependent fractal fluctuation $d b_{\pm}$. This fractal fluctuation is described by a stochastic variable which, by definition, is of zero mean $\langle d b_{\pm} \rangle = 0$. It can be shown that quantum mechanics has a fractal dimension $D_F = 2$ [47] similar to that of Brownian motion or more generally of Markov processes. Therefore, we can write

$$dr_{\pm} = u_{\pm} dt + d b_{\pm},$$

with

$$\langle d b_{\pm} \rangle = 0,$$

$$\langle d b_{\pm} d b_{\pm} \rangle = \pm 2 D \delta_{ij} dt,$$  

where $D$ characterizes the amplitude of the fractal fluctuations to a sort of “quantum diffusion coefficient” measuring the covariance of the noise. In other words, $D$ characterizes the amplitude of the fractal fluctuations. We can also introduce two classical derivative operators $d_+ / dt$ and $d_- / dt$ which yield the twin classical velocities when they are applied to the position vector $r$, namely

$$\frac{d_+ r}{dt} = u_+, \quad \frac{d_- r}{dt} = u_-.$$  

It proves convenient in the formalism to replace the twin velocities $(u_+, u_-)$ by the couple $(u, u_Q)$ where

$$u = \frac{u_+ + u_-}{2}, \quad u_Q = \frac{u_+ - u_-}{2}.$$  

With these two velocities, we can form a complex velocity

$$U = u - i u_Q.$$  

We note that it does not correspond to the velocity of a point-like particle but as that of the fluid of geodesics itself [46]. In the classical limit, we have $u_+ = u_-$ (differentiable trajectories), so the real part $u$ can be identified with the classical velocity $u$ and the imaginary part $u_Q$ vanishes. More generally, the velocity $u$ is interpreted as the classical velocity and the velocity $u_Q$ as the quantum velocity. The quantum velocity is at the origin of the complex fluctuations.

12 In Eq. (2), we consider that $dt > 0$ for the $(+)$ process and $dt < 0$ for the $(-)$ process so that $\pm 2 D \delta_{ij} dt$ is always positive [46].
number $i$ in the equations of quantum mechanics. We can also define a complex derivative operator from the classical (differential) parts as

$$\frac{D}{Dt} = \frac{d_+ + d_-}{2dt} - i \frac{d_+ - d_-}{2dt}$$

in terms of which

$$\frac{Dr}{Dt} = U.$$  (7)

The total derivative with respect to time of a function $f(r(t), t)$ of fractal dimension $D_F = 2$ writes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f \cdot \frac{dr}{dt} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{dx_i dx_j}{dt}.$$  (8)

Using Eq. (2), we find that the classical (differentiable) part of this expression is

$$\frac{d f}{dt} = \frac{\partial f}{\partial t} + u^\pm \cdot \nabla f \pm D \Delta f.$$  (9)

Substituting Eq. (9) into Eq. (6), we obtain the expression of the complex time derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U \cdot \nabla - i D \Delta.$$  (10)

Now, the fundamental postulate of Nottale’s theory of scale relativity is that the equations of quantum mechanics (non-differentiable trajectories) can be obtained from the equations of classical mechanics (differentiable trajectories) by replacing the standard velocity $u$ by the complex velocity $U$ and the standard time derivative $d/dt$ by the complex time derivative $D/Dt$. In other words, $D/Dt$ plays the role of a “covariant derivative operator” in terms of which the fundamental equations of physics keep the same form in the classical and quantum regimes. This is similar to the principle of covariance in Einstein’s theory of relativity according to which the form of the equations of physics should be conserved under all transformations of coordinates.

### B. The Schrödinger equation

We now use the principle of covariance to derive the Schrödinger equation from Newton’s law. Using the covariance principle, we write the fundamental equation of dynamics as

$$\frac{D}{Dt} U = -\nabla \Phi,$$  (11)

where $\mathbf{F} = -\nabla \Phi$ is the force by unit of mass exerted on the particle. This equation can also be obtained from an action principle based on a complex action $\mathcal{S}$. Indeed, it corresponds to the Euler-Lagrange equation associated with the stationarity of the complex action: $\delta \mathcal{S} = 0$. Using the expression (10) of the complex time derivative operator, the foregoing equation can be rewritten as

$$\frac{\partial U}{\partial t} + (U \cdot \nabla) U = i D \Delta U - \nabla \Phi.$$  (12)

This equation is similar to the viscous Burgers equation of fluid mechanics, except that in the present case the velocity field $U(r, t)$ is complex and the viscosity $\nu = i D$ is imaginary. Therefore, quantum mechanics may be interpreted as a generalized hydrodynamics involving a complex velocity field and an imaginary viscosity. In the Lagrangian formalism, the complex impulse $P = mU$ and the complex energy $\mathcal{E}$ write

$$P = \nabla \mathcal{S}, \quad \mathcal{E} = -\frac{\partial \mathcal{S}}{\partial t}.$$  (13)

where $m$ is the mass of the particle and $\mathcal{S}(r, t)$ is the complex action. The complex velocity field is potential since it can be written as the gradient of a function:

$$U = \frac{\nabla \mathcal{S}}{m}.$$  (14)
As a consequence of Eq. 14, the flow is irrotational: \( \nabla \times \mathbf{U} = 0 \). Using the well-known identities of fluid mechanics 

\[
(U \cdot \nabla) U = \nabla((U^2)/2) - U \times (\nabla \times U) \quad \text{and} \quad \Delta U = \nabla(\nabla \cdot U) - \nabla \times (\nabla \times U)
\]

which reduce to \((U \cdot \nabla) U = \nabla((U^2)/2)\) and \(\Delta U = \nabla(\nabla \cdot U)\) for an irrotational flow, and using the identity \(\nabla \cdot U = \Delta S/m\) resulting from Eq. 14, we find that Eq. 12 is equivalent to

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 - iD\Delta S + m\Phi = 0.
\]  

Equation 15 can be viewed as a quantum Hamilton-Jacobi equation for a complex action, or as a Bernoulli equation for a complex potential. We define the wave function \(\psi(r, t)\) through the complex Cole-Hopf transformation

\[
S = -2imD \ln \psi.
\]

Since macroscopic bodies do not feel the non-differentiable nature of space-time, the quantum diffusion coefficient \(D\) must depend on the mass \(m\) of the particle and must tend to zero when \(m \to +\infty\). We shall assume that \(D\) is inversely proportional to \(m\) and set

\[
D = \frac{\hbar}{2m}.
\]

The constant \(\hbar\) has the dimension of an action. We shall see that it can be identified with the Planck constant. With the relation 17, Eq. 16 can be rewritten as

\[
S = -i\hbar \ln \psi,
\]

where the mass \(m\) of the particle does not appear anymore (this is a strong argument in favor of Eq. 17). Equation 18 can be rewritten in the WKB form

\[
\psi = e^{iS/\hbar},
\]

relating the wavefunction to the complex action. Therefore, the complex Cole-Hopf transformation 16 is equivalent to the WKB transformation 19 provided that the relation 17 holds. In particular, the factor 2 in Eq. 17 has the same origin as the one arising in the Cole-Hopf transformation 16. Substituting Eq. 18 into Eq. 15, and using the identity

\[
\Delta(\ln \psi) = \Delta \psi \frac{1}{\psi^2 (\nabla \psi)^2},
\]

we obtain the Schrödinger equation

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi.
\]

This justifies interpreting \(\hbar\) as the Planck constant. This also justifies Eq. 17 a posteriori. From the Schrödinger equation 21, we can easily derive the identity

\[
\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \mathbf{j} = 0,
\]

where

\[
\mathbf{j} = \frac{\hbar}{2im} (\psi^* \nabla \psi - \psi \nabla \psi^*)
\]

is a current. Since the Schrödinger equation 21 conserves the integral \(\int |\psi|^2(r, t) \, dr\), we can interpret

\[
\rho(r, t) = |\psi|^2(r, t)
\]

13 This relation first appeared in Nelson’s stochastic interpretation of quantum mechanics 32 so we shall call it the Nelson relation.

14 As shown in Appendix B, it is only when the relation 17 is satisfied in Eq. 19 that the integral \(\int |\psi|^2 \, dr\) is conserved, allowing us to identify \(|\psi|^2\) with a probability density (see Eqs. 23 and 25).
as the probability density of finding the particle at position \( \mathbf{r} \) at time \( t \) (Born’s postulate).\(^{15}\) This interpretation yields the normalization condition
\[
\int |\psi|^2(\mathbf{r}, t) \, d\mathbf{r} = 1. \tag{25}
\]
Equation (25) shows that the conservation of the normalization condition is locally satisfied.

Remark: By making the complex Cole-Hopf transformation (16), equivalent to the WKB transformation (19), we have shown that the complex viscous Burgers equation (12) is equivalent to the Schrödinger equation (21) in the same sense that the real viscous Burgers equation is equivalent to the diffusion equation (see Appendix A). It is also possible to proceed the other way round. Starting from the Schrödinger equation (21) and making the WKB transformation (19), we obtain the complex quantum Hamilton-Jacobi equation (15). This equation was known by the founders of quantum mechanics (see the Introduction of [5]). If we define a complex field according to Eq. (14), we obtain the complex viscous Burgers equation (12) with an imaginary viscosity given by \( \nu = i\hbar/2m \). Therefore, the Schrödinger equation is equivalent to a hydrodynamic equation involving a complex velocity and an imaginary viscosity. To our knowledge, this complex hydrodynamic equation was “missed” by the founders of quantum mechanics and first appeared in the work of Nottale [46] at the very start of his theory of scale relativity. It could have been obtained much earlier by directly starting from the Schrödinger equation. Of course, the interest of Nottale’s theory is to justify this equation from first principles and then to derive the Schrödinger equation from it.

Remark: For \( \hbar = 0 \), Eq. (15) reduces to the classical Hamilton-Jacobi equation
\[
[S] \equiv \frac{\partial S}{\partial \ell} + \frac{1}{2m}(\nabla S)^2 + m\Phi = 0. \tag{26}
\]
The complex quantum Hamilton-Jacobi equation (15) can be written as
\[
[S] = i\frac{\hbar}{2m}\Delta S. \tag{27}
\]
This equation connects the classical mechanics to the wave mechanics. It could be used as postulate, similar to the correspondence principle, to derive the Schrödinger equation (21). Actually, this is how De Donder [51] derived the KG equation in the relativistic context (see footnote 7 in [3]).

Remark: In the previous calculations, \( \hbar \) needs not be the Planck constant. Indeed, Nottale’s formalism may apply to situations different from quantum mechanics [46]. In any case, the “constant” \( \hbar \) (that Nottale writes \( S_0 \)) must be related to \( D \) according to Eq. (17), see footnote 14. Therefore, the general Schrödinger equation can be written as
\[
iD \frac{\partial \psi}{\partial \ell} = -D^2\Delta \psi + \frac{1}{2}\Phi \psi, \tag{28}
\]
where \( D \) depends on the system under consideration. It is interesting to note that, under that form, the mass \( m \) of the particle does not appear anymore, as is obvious from the original equation (11).

C. The Madelung transformation

Using the Madelung [23] transformation, the Schrödinger equation (21) can be written in the form of real hydrodynamic equations. We write the wavefunction as
\[
\psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)/\hbar}, \tag{29}
\]
where \( \rho \) is the probability density for the particle position and \( S \) is the real action. They can be expressed in terms of the wave function as
\[
\rho = |\psi|^2, \quad S = -\frac{i\hbar}{2} \ln \left( \frac{\psi}{\psi^*} \right). \tag{30}
\]
Following Madelung, we introduce the real potential velocity field
\[
\mathbf{u} = \frac{\nabla S}{m}. \tag{31}
\]
\(^{15}\) See Sec. [11] for an interpretation of this result in the framework of Nottale’s theory of scale relativity.
The flow defined in this way is irrotational since $\nabla \times u = 0$. We also define the energy

$$E = \frac{\partial S}{\partial t}. \quad (32)$$

Substituting Eq. (29) into Eq. (21), separating real and imaginary parts, and using Eq. (31), we obtain the hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \quad (33)$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + m\Phi + Q = 0, \quad (34)$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla \Phi - \frac{1}{m} \nabla Q, \quad (35)$$

where

$$Q = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] = -\frac{\hbar^2}{8m} (\nabla \ln \rho)^2 - \frac{\hbar^2}{4m} (\nabla \ln \rho) \quad (36)$$

is the quantum potential (we have used the identity (20) to obtain the last equality). Equation (33) is similar to the continuity equation in hydrodynamics. It accounts for the local conservation of the normalization condition. It arises from the imaginary part of the Schrödinger equation in the Madelung transformation. Equation (34) is similar to the Hamilton-Jacobi equation or to the Bernoulli equation with an additional quantum potential $Q$. It arises from the real part of the Schrödinger equation in the Madelung transformation. Equation (35) has a form similar to the Euler equation of hydrodynamics with an additional quantum force $-\nabla Q$. It is obtained by taking the gradient of Eq. (34) and using the well-known identity $(u \cdot \nabla) u = \nabla (u^2/2) - u \times (\nabla \times u)$ which reduces to $(u \cdot \nabla) u = \nabla (u^2/2)$ for an irrotational flow. Equations (33)-(35) are called the quantum Euler equations.

### D. The quantum force

The quantum potential (36) first appeared in the paper of Madelung [23] (see also his less well-known paper [52]) and was rediscovered by Bohm [31]. For that reason, it is sometimes called “the Bohm potential”. The quantum force by unit of mass writes

$$F_Q = -\frac{1}{m} \nabla Q. \quad (37)$$

We note the identity

$$(F_Q)_i = -\frac{1}{m} \partial_i Q = -\frac{1}{\rho} \partial_j P_{ij}, \quad (38)$$

where $P_{ij}$ is the quantum stress (or pressure) tensor defined by

$$P_{ij}^{(1)} = -\frac{\hbar^2}{4m^2} \rho \partial_i \partial_j \ln \rho \quad \text{or} \quad P_{ij}^{(2)} = \frac{\hbar^2}{4m^2} \left( \frac{1}{\rho} \partial_i \rho \partial_j \rho - \delta_{ij} \Delta \rho \right). \quad (39)$$

This tensor is manifestly symmetric: $P_{ij} = P_{ji}$. The identity (38) shows that the quantum force $-\nabla Q$ is equivalent to the force produced by an anisotropic pressure tensor $P_{ij}$. The tensors defined by Eq. (39) are related to each other by

$$P_{ij}^{(1)} = P_{ij}^{(2)} + \frac{\hbar^2}{4m^2} \delta_{ij} \Delta \rho - \partial_i \partial_j \rho. \quad (40)$$

---

16 Taking the gradient of the action in Eq. (30) and comparing its expression with Eq. (22), we find that $j = \rho u$. As a result, the continuity equation (33) is equivalent to Eq. (22).

17 A relativistic version of the quantum potential appeared in the works of de Broglie [26,28] and London [29] who developed a hydrodynamic representation of the KG equation independently of Madelung [23].
They differ by a tensor $\chi_{ij} = \delta_{ij} \Delta \rho - \partial_i \partial_j \rho$ satisfying $\partial_j \chi_{ij} = 0$. Contracting the indices, we obtain

$$P^{(1)}_{ii} = -\frac{\hbar^2}{4m^2} \rho \Delta \ln \rho, \quad P^{(2)}_{ii} = \frac{\hbar^2}{4m^2} \left[ \frac{(\nabla \rho)^2}{\rho} - d \Delta \rho \right],$$

and the relation

$$P^{(1)}_{ii} = P^{(2)}_{ii} + (d - 1) \frac{\hbar^2}{4m^2} \Delta \rho.$$  

According to Eq. (48) we note that

$$\langle F_Q \rangle = \int \rho F_Q \, dr = 0$$

so there is no resultant of the quantum force. This property is at the basis of the Ehrenfest theorem [52] (see Appendix C).

**Remark:** We provide below another derivation of Eq. (43). Using Eqs. (36) and (37), the average value of the quantum force is

$$\langle F_Q \rangle = \int \rho F_Q \, dr = -\frac{1}{m} \int \rho \nabla Q \, dr = \frac{1}{m} \int Q \nabla \rho \, dr = -\frac{\hbar^2}{2m^2} \int \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \nabla \rho \, dr = -\frac{\hbar^2}{m^2} \int \Delta \sqrt{\rho} \nabla \sqrt{\rho} \, dr.$$  

Since

$$\int \Delta \sqrt{\rho} \nabla \sqrt{\rho} \, dr = -\int \nabla \sqrt{\rho} \Delta \sqrt{\rho} \, dr = 0,$$

obtained by integration by parts, we recover Eq. (43).

**E. The conservation of the average energy**

Using Eqs. (31) and (32), the quantum Hamilton-Jacobi equation (34) can be written as

$$E(r, t) = \frac{1}{2} m u^2 + m \Phi + Q.$$  

The energy $E(r, t)$ is the sum of the kinetic energy, the classical potential and the quantum potential. This is the quantum generalization of the classical equation of mechanics $E = m u^2 / 2 + m \Phi$. We note that the Euler equation (35) can be written as

$$\frac{\partial u}{\partial t} = -\frac{\nabla E}{m}.$$  

The average energy is

$$\langle E \rangle = \int \rho E \, dr = \int \rho m \frac{u^2}{2} \, dr + \int \rho m \Phi \, dr + \int \rho Q \, dr.$$  

We can show that it is conserved (see Appendix D):

$$\langle \dot{E} \rangle = 0.$$  

The energies defined by Eqs. (32) and (46) first appeared in the papers of Madelung [23, 52]. He also mentioned the conservation of the average energy [48].

**F. Stationary quantum states**

The time-independent solutions of a quantum system are special solutions $\rho(r)$ and $u(r)$ of the hydrodynamic equations [43-46]. Two types of stationary solutions must be considered.
1. Static states

For static states, $\rho(r)$ is independent on time and $u(r) = 0$. The continuity equation (33) is automatically satisfied and the quantum Euler equation (35) reduces to the condition of quantum hydrostatic equilibrium

$$\nabla \Phi + \frac{1}{m} \nabla Q = 0. \quad (50)$$

Therefore, a static state results from the balance between the external force and the quantum force. On the other hand, according to Eq. (31), $\nabla S = 0$ so that $S$ depends only on time: $S = S(t)$. From Eqs. (32) and (34), we obtain

$$E = -\frac{dS}{dt} = m\Phi + Q. \quad (51)$$

Since the second term depends only on time and the third term depends only on $r$, $E$ is necessarily a constant. We then find that $S = -Et$. Taking the gradient of Eq. (51), we recover the condition of quantum hydrostatic equilibrium (50). On the other hand, since $S = -Et$, the wave function (29) can be written as

$$\psi(r, t) = \phi(r) e^{-iEt/\hbar}, \quad (52)$$

where $\phi(r) = \sqrt{\rho(r)}$ is real. Therefore, we can define a quantum static state by Eq. (52). Substituting Eq. (52) into Eq. (21), we obtain the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \Delta \phi + m\Phi \phi = E\phi. \quad (53)$$

Equation (53) is an eigenvalue equation for the wave function $\phi(r)$ where the eigenvalue $E$ is the energy. It is equivalent to the previous equations expressed in terms of $\rho$. Indeed, dividing Eq. (53) by $\phi$ and using $\rho = \phi^2$, we get

$$-\frac{\hbar^2}{2m} \Delta \sqrt{\rho} + m\Phi \sqrt{\rho} = E \quad \text{i.e.} \quad Q + m\Phi = E. \quad (54)$$

As shown above, this equation can be directly derived from the quantum Hamilton-Jacobi equation (34) by setting $S = -Et$. It can also be obtained from Eq. (51) by taking $E(r, t) = E$ and $u = 0$.

2. Dynamic states

For dynamic states, $\rho(r)$ and $u(r)$ are independent on time and $u(r) \neq 0$. The continuity equation (33) and the quantum Euler equation (35) reduce to

$$\nabla \cdot (\rho u) = 0, \quad (55)$$

$$(u \cdot \nabla)u = -\nabla \Phi - \frac{1}{m} \nabla Q. \quad (56)$$

Therefore, a dynamic state results from the balance between the inertia, the external force and the quantum force. On the other hand, according to Eq. (31), $\nabla S$ does not depend on time. Therefore, $S$ must be of the form $S = S_1(r) + S_2(t)$. From Eqs. (32) and (34), we obtain

$$E = -\frac{dS}{dt} = \frac{1}{2} m u^2 + m\Phi + Q. \quad (57)$$

Since the second term depends only on time and the third term depends only on $r$, $E$ is necessarily a constant. We then find that $S_2 = -Et$ so that $S = S_1(r) - Et$. Taking the gradient of Eq. (57), we obtain

$$\nabla \cdot \left( \frac{u^2}{2} + \Phi + \frac{Q}{m} \right) = 0, \quad (58)$$

which is equivalent to Eq. (56). On the other hand, since $S = S_1(r) - Et$, the wave function (29) can be written as

$$\psi(r, t) = \Psi(r) e^{-iEt/\hbar}, \quad (59)$$
where $\Psi(r) = \phi(r)e^{iS_1(r)/\hbar}$ is complex and $\phi(r) = \sqrt{\rho(r)}$ is real. Substituting Eq. (59) into Eq. (21), we obtain the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \Delta \Psi + m\Phi \Psi = E\Psi, \quad (60)$$

where $\Psi$ is complex. Equation (60) is equivalent to the previous equations expressed in terms of $\rho$ and $u$.

### G. Traditional interpretation of the Madelung transformation

In the hydrodynamic representation of the Schrödinger equation developed by Madelung [23], Bohm [31], and Takabayasi [35, 36], Eqs. (33)-(35) are regarded as representing an ensemble of particle trajectories which belong to a single velocity potential $S$ by the relation of Eq. (31). Introducing the material derivative $d/dt = \partial/\partial t + u \cdot \nabla$, the quantum Euler equation (35) can be rewritten as

$$m\frac{du}{dt} = -m\nabla\Phi - \nabla Q, \quad (61)$$

where $u(t) = u(r(t), t) = dr/dt$ is the Lagrangian velocity of a fluid particle. Equation (61) is similar to the classical equation of motion of a particle. However, in the present case, the particle experiences an extra quantum force $-\nabla Q$. This quantum force is extremely complex since it depends on the state of the system itself via its density $\rho$. This is what leads to the peculiar effects of quantum mechanics such as diffusion of wave packets, interferences, tunnel effect etc. As Bohm [31] states, the quantum potential corresponds to the interaction of the $\psi$ field with its own particle.

If we define the energy of a quantum particle by

$$E(t) = \frac{1}{2}m u^2 + m\Phi + Q, \quad (62)$$

we get

$$\dot{E} = \frac{\partial Q}{\partial t} \quad (63)$$

since the quantum potential $Q(r, t)$ depends explicitly on time. Therefore, the energy of a quantum particle is not conserved. As a result, the trajectory of a quantum particle exhibits a complicated fluctuation. This is responsible for the blurring of the classical trajectory. However, if we consider the ensemble of particle trajectories, the average energy $\langle E \rangle$ given by Eq. (48) is conserved (see Appendix D).

Because of the quantum potential, the particle can reach an equilibrium state in which $u = 0$. Bohm [31] writes that “the absence of motion is possible because the applied force, $-m\nabla\Phi$, is balanced by the quantum-mechanical force, $-\nabla Q$, produced by the Schrödinger $\psi$-field acting on its own particle. There is, however, a statistical ensemble of possible positions of the particle, with a probability density, $P(r) = |\psi(r)|^2$.” In the hydrodynamic language, this corresponds to the condition of quantum hydrostatic equilibrium

$$m\nabla\Phi - \nabla Q = 0, \quad (64)$$

For example, in the case of the hydrogen atom, Eq. (64) describes the balance between the attractive Coulomb force and the repulsion due to the quantum force. This fluid dynamic formulation offers a simple physical explanation of the stability of matter because of quantum mechanics. Without the quantum potential, the electron would fall on the nucleus. In this sense, the hydrodynamic picture leads to a conception completely different from the original Bohr model. The tunnel effect can also be explained in terms of the quantum potential. Bohm [31] mentions that the existence of $Q$ partially compensates for the large value of $\Phi$ and enables the particle to “ride over the barrier”. In the wave interpretation of the Schrödinger equation, the effect of the quantum potential is played by the Heisenberg uncertainty principle.

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18 This is what de Broglie [24–28] called the pilot-wave theory in his relativistic approach based on the KG equation since the velocity is determined by the wave $\psi$.

19 This corresponds to Eq. (46) with $E(t) = E(r(t), t)$ in the Lagrangian description.
Remark: We may note certain analogies between the motion of a quantum particle and the notion of phase mixing and violent relaxation introduced by Lynden-Bell \[54\] in the context of collisionless stellar systems described by the Vlasov-Poisson equations. For example, the energy of an individual star in a protogalaxy is not conserved because of the fluctuations of the gravitational field [compare Eq. (1) in \[54\] with Eq. (63)] but the total energy of the system is conserved. On the other hand, as shown in \[55\], collective effects in a stellar system produce a sort of pressure force [see Eq. (5.11) in \[53\]] that can balance the gravitational force and lead to a quasi-stationary state (QSS) on a coarse-grained scale. This pressure force plays a role similar to the quantum force, which can also be interpreted as a pressure force according to Eq. (63).

H. Interpretation of the Madelung transformation in the theory of scale relativity

Historically, the Madelung transformation is based on the Schrödinger equation (21) which is at the basis of the wave mechanics initiated by de Broglie [4]. In that context, the fundamental object is the wave function \(\psi(r, t)\). According to the Born postulate, the square norm of the wave function \(\rho(r, t) = |\psi|^2\) is interpreted as the probability density of finding the particle at \(r\) at time \(t\). Starting from the wave function, Madelung [23] defined a velocity field by Eq. (31), in a rather ad hoc manner, and showed that the Schrödinger equation is equivalent to the hydrodynamic equations (33)-(35). However, this is essentially a formal analogy. In the framework of conventional quantum mechanics, this analogy with hydrodynamics remains relatively mysterious.\[20\] One important new feature of this hydrodynamic representation is the quantum potential \(U_{\text{q}}\) that accounts for all quantum phenomena.

In the theory of scale relativity, the interpretation is very different\[21\] because the fundamental object is the complex velocity \(\mathbf{U}\) of the fractal geodesics and the complex hydrodynamic equation (12) of these geodesics from which the Schrödinger equation (21) is derived. Now, we expect the fluid of geodesics to be more concentrated at some places and less at others as does a real fluid. To find the probability density of presence of the paths we can remark that Eqs. (12) and (21) are equivalent to the real fluid equations (33)-(35) see Appendix E. In that context, Eq. (31) is not an ad hoc definition but it corresponds to the classical velocity (the real part of \(\mathbf{U}\)). On the other hand, in the theory of scale relativity, the quantum potential is a manifestation of the geometry of spacetime, namely, of its non-differentiability and fractality, in similarity with the Newtonian potential being a manifestation of the curvature of spacetime in Einstein’s theory of general relativity. Then, Eqs. (33)-(35), which are equivalent to Eqs. (12) and (21), describe a fluid of fractal geodesics in a non-differentiable space-time. They have therefore a clear physical interpretation. As a result \(\rho(r, t) = |\psi|^2\) represents the density of the geodesic fluid, and the probability density for the quantum “particle” to be found at a given position must be proportional to \(\rho(r, t)\) (the constant of proportionality can be set equal to unity). As a result, the Born postulate is naturally justified in the theory of scale relativity.

In a sense, Madelung [23] and Nottale [46] proceed in opposite directions. Madelung starts from the wave equation of Schrödinger and shows that it is equivalent to fluid equations. Inversely, Nottale’s theory is directly based on fluid equations [see Eq. (12)] and he shows that they are equivalent to the Schrödinger equation.

It is interesting to come back to the manner how the density \(\rho(r, t)\) arises in the theories of Born, Nelson, and Nottale. In the original interpretation of quantum mechanics by Born [19, 21], the relation \(\rho(r, t) = |\psi|^2\) is introduced at the end as an independent postulate that gives an interpretation to the Schrödinger equation: \(\rho(r, t) = |\psi|^2\) determines the probability of presence of a particle. We note that the Schrödinger equation (21) is obtained without having to specify \(\rho(r, t)\) (see the original derivations of the Schrödinger equation summarized in Appendix F of [5]). In the stochastic interpretation of Nelson [45], \(\rho(r, t)\) is introduced from the start by analogy with a diffusion process. Finally, in Nottale’s theory of scale relativity [46], the relation \(\rho = |\psi|^2\) that represents the density of the geodesic fluid, or the probability of presence of a particle, is a new postulate. It is fixed at the end by Eqs. (83)-(85) that are equivalent to the equation of fractal geodesics (12). We note that the Schrödinger equation (21) is obtained without having to specify \(\rho(r, t)\).

I. Complex and real hydrodynamic equations

As we have seen previously, the Schrödinger equation is formally equivalent to hydrodynamic equations. These hydrodynamic equations can be either complex or real.

\[20\] It can be given a more physical interpretation in the context of BECs as discussed in the Conclusion.

\[21\] We briefly summarize the main points of the discussion of Nottale [46].
By making the complex Cole-Hopf transformation (16), equivalent to the WKB transformation (19), the Schrödinger equation (21) can be transformed into the complex viscous Burgers equation (12). This is a single complex equation for the complex velocity $U$ with an imaginary viscosity $\nu = \frac{i \hbar}{2m}$ and without quantum potential.

By making the Madelung transformation (29)-(31), the Schrödinger equation (21) can be transformed into a set of two real hydrodynamic equations, the continuity equation (33) for the density $\rho$ and the quantum Euler equation (35) for the real velocity $u$. The Euler equation involves a quantum potential $Q$ but no viscosity.

As shown in Appendix E 4, the hydrodynamic equations (33) and (35) correspond to the real and imaginary parts of the complex viscous Burgers equation (12). Furthermore, the viscous term $i D \nabla U$ in the complex viscous Burgers equation (12) can be interpreted as a complex quantum force (see Appendix E 2).

Remark: Nottale [46] does not interpret Eq. (12) as a complex Burgers equation with an imaginary viscosity but rather as an equation of geodesics. Similarly, he does not interpret Eq. (16) as a Cole-Hopf transformation. We believe that our “hydrodynamic interpretation” clarifies the derivation of the Schrödinger equation by making a close analogy with fluid mechanics (see Appendix A).

### III. DERIVATION OF A GENERALIZED SCHRODINGER EQUATION

In this section, we use the formalism of scale relativity to derive a generalized Schrödinger equation that includes dissipative effects. As we shall see, our procedure automatically yields a term accounting for (effective) thermal effects. The two terms are related by a sort of fluctuation-dissipation theorem.

#### A. The generalized Schrödinger equation

In order to incorporate dissipative effects into the Schrödinger equation, it would seem natural to introduce a damping term $-\gamma U$ in the fundamental equation of dynamics (11). However, if we proceed too naively, we obtain a generalized Schrödinger equation that does not satisfy the local conservation of the normalization condition (see Appendix E). In order to conserve the normalization condition locally, while taking dissipative effects into account, we write the fundamental equation of dynamics under the form

$$\frac{D U}{D t} = -\nabla \Phi - \text{Re}(\gamma U),$$  \hspace{1cm} (65)

where $\gamma$ is a complex friction coefficient. As shown in Sec. II C, the continuity equation (33) is obtained by performing the Madelung transformation and taking the imaginary part of the Schrödinger equation. Therefore, by projecting the damping term $-\gamma U$ on the real axis, we are guaranteed to obtain a generalized Schrödinger equation that satisfies the local conservation of the normalization condition since the continuity equation will not be affected by the damping term. Using the expression (10) of the covariant derivative, Eq. (65) can be rewritten as a damped complex viscous Burgers equation of the form

$$\frac{\partial U}{\partial t} + (U \cdot \nabla) U = i D \nabla U - \nabla \Phi - \text{Re}(\gamma U).$$  \hspace{1cm} (66)

Using Eq. (14), and proceeding as in Sec. II B, the corresponding complex Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - i D \nabla S + m \Phi + V(t) + \text{Re}(\gamma S) = 0,$$  \hspace{1cm} (67)

where $V(t)$ is a “constant” of integration depending on time that has a nonzero value when dissipative effects are taken into account (see below). Introducing the wave function from Eq. (18) and repeating the derivation of Sec. II B with the additional damping term, we obtain the generalized Schrödinger equation

$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + V \psi + \hbar \text{Im}(\gamma \ln \psi) \psi.$$  \hspace{1cm} (68)

Writing $\gamma = \gamma_R + i \gamma_I$, where $\gamma_R$ is the classical friction coefficient and $\gamma_I$ is the quantum friction coefficient, and using the identity

$$\text{Im}(\gamma \ln \psi) = \gamma_I \ln |\psi| - \frac{1}{2} i \gamma_R \ln \left( \frac{\psi}{\psi^*} \right),$$  \hspace{1cm} (69)
we can rewrite Eq. (68) in the equivalent form

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + V\psi + h\gamma_I \ln|\psi| \psi - i\frac{\hbar}{2}\gamma_R \ln\left(\frac{\psi}{\psi^*}\right) \psi. \]

(70)

Introducing the notations

\[ \gamma_R = \xi, \quad \gamma_I = \frac{2k_B T}{\hbar}, \]

(71)

the generalized Schrödinger equation (70) takes the form

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + V\psi + 2k_B T \ln|\psi| \psi - i\frac{\hbar}{2}\xi \ln\left(\frac{\psi}{\psi^*}\right) \psi. \]

(72)

Using the hydrodynamic representation of the generalized Schrödinger equation (72), we will see in Sec. III B that \( \xi \) plays the role of an ordinary friction coefficient while \( T \) plays the role of an effective temperature. The friction coefficient \( \xi \) must be positive in order to guarantee that the generalized Schrödinger equation (72) relaxes towards a stable equilibrium state (see Sec. IV C). By contrast, since the temperature is effective, it can be positive or negative. Finally, we choose the function \( V(t) \) so that the average value of the friction term proportional to \( \xi \) is equal to zero. This gives

\[ V(t) = \frac{\hbar}{2}\xi \left\langle \ln\left(\frac{\psi}{\psi^*}\right)\right\rangle, \]

(73)

where \( \langle X \rangle = \int \rho X \, dr \). We finally obtain the generalized Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + 2k_B T \ln|\psi| \psi - i\frac{\hbar}{2}\xi \left[ \ln\left(\frac{\psi}{\psi^*}\right) - \left\langle \ln\left(\frac{\psi}{\psi^*}\right)\right\rangle \right] \psi. \]

(74)

We note that the temperature term in the generalized Schrödinger equation can be written as \( k_B T \ln\rho \psi \) and the dissipative term as \( \xi(S - \langle S \rangle)\psi \). It is interesting to note that the complex nature of the friction coefficient

\[ \gamma = \xi + i\frac{2k_B T}{\hbar} \]

(75)

leads to a generalized Schrödinger equation exhibiting simultaneously a friction term (as expected) and an effective temperature term (unexpected). They correspond to the real and imaginary parts of \( \gamma \). This may be viewed as a new form of fluctuation-dissipation theorem (see Appendix G).

**Remark:** Equation (74) can be interpreted as a dissipative Gross-Pitaevskii (GP) equation with a logarithmic nonlinearity. This is a particular case of the generalized GP equations considered in [50]. General properties of these equations are established in [50]. In the present paper, we give the main properties of the dissipative logarithmic Schrödinger equation (74) and refer to [50] for more details, derivations, and generalizations.

### B. The Madelung transformation

Using the Madelung transformation, and proceeding as in Sec. III C we find that the generalized Schrödinger equation (74) is equivalent to the hydrodynamic equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \]

(76)

\[ \frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + m\Phi + Q + k_B T \ln \rho + \xi(S - \langle S \rangle) = 0, \]

(77)

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla P - \nabla \Phi - \frac{1}{m} \nabla Q - \xi u, \]

(78)
\[ P = \frac{k_B T}{m}, \]  
(79)

where \( Q \) is the quantum potential defined in Eq. (36). The first equation is the continuity equation, the second equation is the quantum Hamilton-Jacobi equation (or the quantum Bernoulli equation), and the third equation is the quantum Euler equation. It involves a damping term and a pressure term with an isothermal equation of state. For that reason, it is called the quantum damped isothermal Euler equation [56]. Using the continuity equation (76), it can be rewritten as

\[ \frac{\partial}{\partial t} (\rho u) + \nabla (\rho u \otimes u) = -\nabla P - \rho \nabla \Phi - \frac{\rho}{m} \nabla Q - \xi \rho u. \]  
(80)

When the quantum potential is neglected, we recover the classical damped isothermal Euler equation. For dissipationless systems (\( \xi = 0 \)), they reduce to the quantum and classical isothermal Euler equations. On the other hand, in the overdamped limit \( \xi \to +\infty \), we can formally neglect the inertial term in Eq. (78) so that

\[ \xi u \simeq -\frac{1}{\rho} \nabla P - \nabla \Phi - \frac{1}{m} \nabla Q. \]  
(81)

Substituting this relation into the continuity equation (76), we obtain the quantum Smoluchowski equation [56]:

\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \nabla P + \rho \nabla \Phi + \frac{\rho}{m} \nabla Q \right). \]  
(82)

When the quantum potential is neglected, we obtain the classical Smoluchowski equation with a diffusion coefficient \( D = k_B T/\xi m \) (see Appendix G). Finally, if we neglect the advection term \( \nabla (\rho u \otimes u) \) in Eq. (80), but retain the term \( \partial (\rho u)/\partial t \), and combine the resulting equation with the continuity equation (76), we obtain the quantum telegraphic equation

\[ \frac{\partial^2 \rho}{\partial t^2} + \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \nabla P + \rho \nabla \Phi + \frac{\rho}{m} \nabla Q \right) \]  
(83)

which can be seen as a generalization of the quantum Smoluchowski equation (82) taking inertial (or memory) effects into account. When the quantum potential is neglected, we obtain the classical telegraphic equation.

\[ \text{C. The energy} \]

If we define the energy by

\[ E = -\left( \frac{\partial S}{\partial t} \right)_{\xi=0} \]  
(84)

and use the Hamilton-Jacobi equation (77), we obtain

\[ E(\mathbf{r}, t) = \frac{1}{2} m \mathbf{u}^2 + m \Phi + Q + k_B T \ln \rho. \]  
(85)

This is the sum of the kinetic energy, the classical potential, the quantum potential, and the enthalpy [51]. We note that the quantum damped isothermal Euler equation (78) can be written as

\[ \frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla E}{m} - \xi \mathbf{u}. \]  
(86)

\[ \text{22 The classical Smoluchowski equation [57] was introduced in the context of the Brownian theory initiated by Einstein [58]. It describes the evolution of the probability density of an overdamped Brownian particle moving in a potential \( \Phi \) and experiencing a random force. In that case, it can be interpreted as a Fokker-Planck equation [59–61]. It is interesting to note that the Smoluchowski equation can also be obtained from the generalized Schrödinger equation [43] in a strong friction limit. However, we emphasize that (besides the presence of the quantum potential) its physical interpretation is completely different. For example, the diffusive term in the Smoluchowski equation for Brownian particles comes from stochastic processes (it is due to a random force) while the diffusive term in the Smoluchowski equation derived from the generalized Schrödinger equation (74) comes from the logarithmic nonlinearity. On the other hand, the quantum damped Euler equation (78) can be rigorously derived from the generalized Schrödinger equation (74) while the classical damped Euler equation is not rigorously justified for Brownian particles except in the strong friction limit \( \xi \to +\infty \) where it reduces to the classical Smoluchowski equation (see the discussion in [62, 63]).} \]
D. Time-independent solutions

If we consider a wave function of the form
\[ \psi(r, t) = \phi(r) e^{-iEt/\hbar}, \] (87)
where \( \phi(r) = \sqrt{\rho(r)} \) is real, and substitute Eq. (87) into Eq. (71), we obtain the generalized time-independent Schrödinger equation
\[ -\frac{\hbar^2}{2m} \Delta \phi + m\Phi \phi + 2k_B T \ln \phi = E\phi. \] (88)

Equation (88) is an eigenvalue equation for the wave function \( \phi(r) \) where the eigenvalue \( E \) is the energy. Dividing Eq. (88) by \( \phi \) and using \( \rho = \phi^2 \), we get
\[ -\frac{\hbar^2}{2m} \Delta \sqrt{\rho} + m\Phi + k_B T \ln \rho = E \quad \text{i.e.} \quad Q + m\Phi + k_B T \ln \rho = E. \] (89)

Using the last equality of Eq. (36), we can rewrite Eq. (89) into the form
\[ -\frac{\hbar^2}{8m} (\nabla \ln \rho)^2 + \frac{\hbar^2}{4m} \Delta \ln \rho + m\Phi + k_B T \ln \rho = E. \] (90)

Equation (88) can also be derived from the damped quantum Hamilton-Jacobi equation (77) by setting \( S = -Et \) or from Eq. (88) by setting \( E(r, t) = E \) and \( u = 0 \). We note that dissipative effects do not alter the time-independent solutions of the generalized Schrödinger equation because \( S = -Et \) is uniform so that \( \xi(S - \langle S \rangle) = 0 \).

Remark: If we take \( V = 0 \) in the generalized Schrödinger equation (72), we get a Hamilton-Jacobi equation of the form
\[ \frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + m\Phi + Q + k_B T \ln \rho + \xi S = 0. \] (91)

We note that Eqs. (70) and (73) are not modified. If we look for a static solution of the generalized Schrödinger equation such that \( S \) depends only on time (this implies \( u = 0 \)) while \( \rho \) depends only on position, the Hamilton-Jacobi equation (91) can be written as
\[ -\frac{dS}{dt} - \xi S = m\Phi + Q + k_B T \ln \rho = E_0. \] (92)

Since the first term depends only on time and the second term depends only on position, they must be individually equal to a constant \( E_0 \). The second part of Eq. (92) returns Eq. (90) where \( E \) is replaced by \( E_0 \). The first part of Eq. (92) can be solved for \( S(t) \) leading to
\[ S(t) = \frac{E_0}{\xi} (e^{-\xi t} - 1), \] (93)
where the constant of integrating has been determined such that Eq. (88) reduces to \( S \sim E_0 t \) when \( \xi \to 0 \) or when \( t \to 0 \). We note that \( S(t) \to -E_0/\xi \) when \( t \to +\infty \). Defining the energy by \( E(t) = -dS/dt \), we obtain
\[ E(t) = E_0 e^{-\xi t}. \] (94)

This equation shows that \( E_0 \) represents the initial energy. Furthermore, it shows that the energy \( E(t) \) is dissipated in the static state and that it decays to zero for \( t \to +\infty \).

E. Hydrostatic equilibrium

The time-independent Schrödinger equation (88) can also be obtained from the damped quantum isothermal Euler equation (78) since it is equivalent to the generalized Schrödinger equation (74). The equilibrium state of the damped quantum isothermal Euler equation (78), obtained by setting \( \partial_t \rho = 0 \) and \( u = 0 \), satisfies
\[ \nabla P + \rho \nabla \Phi + \frac{\rho}{m} \nabla Q = 0 \quad \text{i.e.} \quad \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi - \frac{\hbar^2 \rho}{2m^2} \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 0. \] (95)
This equation generalizes the usual condition of hydrostatic equilibrium by incorporating the contribution of the quantum potential. From the hydrodynamic representation, we clearly understand why frictional effects do not influence the equilibrium state since they vanish when \( u = 0 \). Equation (95) describes the (possible) balance between the repulsion (for \( T > 0 \)) or the attraction (for \( T < 0 \)) due to the pressure \( P \), the repulsion or the attraction due to the potential \( \Phi \), and the repulsion due to the quantum potential \( Q \) arising from the Heisenberg uncertainty principle. This equation is equivalent to Eq. (89). Indeed, dividing Eq. (95) by \( \rho \) and integrating the resulting equation with the aid of Eq. (79), we obtain Eq. (89) where the eigenenergy \( E \) appears as a constant of integration.

IV. EFFECTIVE THERMODYNAMICS

In this section, we show that the generalized Schrödinger equation (74) is consistent with a notion of effective thermodynamics based on the Boltzmann entropy.

A. The free energy

The free energy associated with the generalized Schrödinger equation (74), or equivalently with the damped quantum isothermal Euler equations (76)-(79), can be written as

\[
F = \Theta_c + \Theta_Q + W + U.
\]

The first two terms in Eq. (96) correspond to the total kinetic energy \( \Theta \) [see Eq. (H6)]. Using the Madelung transformation, it can be decomposed into the classical kinetic energy

\[
\Theta_c = \int \rho m \frac{u^2}{2} \, dr
\]

and the quantum kinetic energy\(^{23}\)

\[
\Theta_Q = \frac{\hbar^2}{8m} \int \frac{(\nabla \rho)^2}{\rho} \, dr.
\]

Using Eq. (96), integrating by parts, and assuming that the boundary term can be neglected, we get

\[
\int \rho Q \, dr = -\frac{\hbar^2}{2m} \int \sqrt{\rho} \Delta \sqrt{\rho} \, dr = \frac{\hbar^2}{2m} \int (\nabla \sqrt{\rho})^2 \, dr = \frac{\hbar^2}{8m} \int \frac{(\nabla \rho)^2}{\rho} \, dr.
\]

Therefore, the quantum kinetic energy can be rewritten as

\[
\Theta_Q = \int \rho Q \, dr.
\]

It can be interpreted as a potential energy associated with the quantum potential \( Q \).\(^{24}\) The third term in Eq. (96) is the potential energy

\[
W = \int \rho m \Phi \, dr.
\]

The fourth term in Eq. (96) is the internal energy

\[
U = m \int \rho \int \rho' \frac{P(\rho')}{\rho'^2} \, d\rho' \, dr.
\]

\(^{23}\) This functional was introduced by von Weizsäcker.\(^{64}\) It is related to the Fisher entropy \( S_F = \int (\nabla \rho)^2 / \rho \, dr \) by the relation \( \Theta_Q = (\hbar^2/8m) S_F \). Actually, the functional \( \Theta_Q \) was already introduced by Madelung\(^{23, 52}\) under the equivalent form \( \Theta_Q = -(\hbar^2/2m) \int \sqrt{\rho} \Delta \sqrt{\rho} \, dr \) [see Eq. (99)].

\(^{24}\) This is not obvious since \( Q \) is a function of the density (it is not an external potential).
The density of internal energy $\rho m u$ satisfies the first law of thermodynamic $du = -P d(1/\rho)$. The internal energy takes into account the contribution of the nonlinear potential in the generalized Schrödinger equation. For the logarithmic potential, leading to the effective isothermal equation of state \[ U = k_B T \int \rho (\ln \rho - 1) \, dr. \] (103)

We note that

$$ U = -TS $$

(104)

where

$$ S = -k_B \int \rho (\ln \rho - 1) \, dr $$

(105)

is the Boltzmann entropy. Regrouping all these results, the free energy can be explicitly written as

$$ F = \int \rho E d^2r + \int \rho Q d^2r + \int \rho m\Phi d^2r + k_B T \int \rho (\ln \rho - 1) \, dr. $$

(106)

Introducing the energy

$$ E_\ast = \Theta_c + \Theta_Q + W $$

(107)

that includes the classical kinetic energy $\Theta_c$, the quantum kinetic energy $\Theta_Q$ and the potential energy $W$, the free energy can be written in the standard form

$$ F = E_\ast - TS. $$

(108)

It coincides with the Boltzmann free energy. The free energy associated with the quantum Smoluchowski equation is

$$ F = \Theta_Q + W + U = \Theta_Q + W - TS $$

(109)

since the classical kinetic energy $\Theta_c$, which is of order $O(\xi^{-2})$, can be neglected in the overdamped limit $\xi \to +\infty$.

### B. Difference between the average energy and the free energy

The average value of the energy $E(r,t)$ defined by Eq. \[83\] is

$$ \langle E \rangle = \int \rho E d^2r = \int \rho m \frac{u^2}{2} d^2r + \int \rho m\Phi d^2r + \int \rho Q d^2r + k_B T \int \rho \ln \rho d^2r = \Theta_c + W + \Theta_Q + U + k_B T. $$

(110)

It coincides with the average value of the energy operator (see Appendix E). Comparing Eqs. (106) and (111), we find that

$$ F = \langle E \rangle - k_B T. $$

(111)

Therefore

$$ F \neq \langle E \rangle. $$

(112)

It is only in the case of the linear Schrödinger equation ($T = 0$) that $F = \langle E \rangle$. For the logarithmic Schrödinger equation, $F$ and $\langle E \rangle$ differ by a constant $k_B T$. Since they just differ by a constant, they will satisfy the same properties. For more general nonlinear Schrödinger equations, $F$ and $\langle E \rangle$ differ by a more complicated functional and can behave very differently. \[50\]
C. The $H$-theorem

It is shown in Appendix D of [50] that the time derivative of the free energy (96) satisfies the identity

$$\dot{F} = -\xi \int \rho m u^2 \, dr \quad \text{i.e.} \quad \dot{F} = -2\xi \Theta_c. \quad (113)$$

We have to consider two situations:

(i) For dissipationless systems ($\xi = 0$), Eq. (113) shows that the generalized Schrödinger equation (74), or the quantum isothermal Euler equations (76)-(79), conserve the free energy ($\dot{F} = 0$). In that case, it can be shown from general arguments [66] that a minimum of free energy at fixed mass determines a steady state of the generalized Schrödinger equation, or of the quantum isothermal Euler equations, that is formally nonlinearly dynamically stable.

(ii) For dissipative systems ($\xi > 0$), Eq. (113) shows that the generalized Schrödinger equation (74), or the quantum damped isothermal Euler equations (76)-(79), decrease the free energy ($\dot{F} \leq 0$). When $\dot{F} = 0$, Eq. (113) implies that $u = 0$. From the Euler equation (78), we obtain the condition of hydrostatic equilibrium (95). Therefore, Eq. (113) forms an $H$-theorem for the generalized Schrödinger equation or for the damped quantum isothermal Euler equations: $\dot{F} \leq 0$ and $\dot{F} = 0$ if, and only if, the system is at equilibrium. In that case, $F$ is called a Lyapunov functional. From Lyapunov’s direct method, we can show that the system will relax, for $t \to +\infty$, towards an equilibrium state that is a (local) minimum of free energy at fixed mass. Maxima or saddle points of free energy are unstable. If several local minima of free energy exist, the selection will depend on the initial condition and on a notion of basin of attraction.

The free energy associated with the quantum Smoluchowski equation (82) is given by Eq. (109). Its time derivative satisfies the identity

$$\dot{F} = -\frac{1}{\xi} \int \frac{m}{\rho} \left( \nabla P + \rho \nabla \Phi + \frac{\rho}{m} \nabla Q \right)^2 \, dr. \quad (114)$$

This equation can be obtained from the quantum Smoluchowski equation (see Appendix D of [50]). It can also be directly obtained from Eq. (113) by using Eq. (81) which is valid in the strong friction limit $\xi \to +\infty$. When $\dot{F} = 0$, Eq. (114) implies that the term in parenthesis vanishes, leading to the condition of hydrostatic equilibrium (95). Therefore, Eq. (114) forms an $H$-theorem for the quantum Smoluchowski equation.

Remarks:

Since the dissipative ($\xi \neq 0$) Schrödinger equation (74), the damped quantum isothermal Euler equations (76)-(79), and the quantum Smoluchowski equation (82) are relaxation equations, they can be used as numerical algorithms to compute stable equilibrium states of the conservative ($\xi = 0$) Schrödinger equation, or quantum isothermal Euler equations. This can be very useful on a practical point of view because it is generally not easy to solve the time-independent equations (89) and (95) directly and be sure that the solution is stable.

D. The equilibrium state

According to the previous discussion, a stable equilibrium state of the generalized Schrödinger equation (74), or equivalently of the damped quantum isothermal Euler equations (76)-(79), is the solution of the minimization problem

$$F(M, T) = \min_{\rho, u} \left\{ F[\rho, u] \left| \int \rho \, dr = 1 \right. \right\}. \quad (115)$$

A critical point of free energy satisfying the normalization condition is determined by the variational principle

$$\delta F - \mu \delta \left( \int \rho \, dr \right) = 0, \quad (116)$$

25 For dissipationless systems, $F$ is called the total energy $E_{\text{tot}}$ of the system, not the free energy. However, for convenience, we shall always refer to $F$ as the free energy.

26 For the classical diffusion equation (corresponding to Eq. (82) with $\Phi = Q = E_* = 0$, $P = \rho k_B T / m$, $D = k_B T / \xi m$ and $F_B = -TS_B$ where $S_B$ is the Boltzmann entropy) we find from Eq. (114) that $\dot{S}_B = Dk_B S_F$ where $S_F$ is the Fisher entropy.

27 See Appendix E of [13], and [67], for numerical algorithms in the form of relaxation equations that can be used to construct stable steady states of the Vlasov-Poisson and 2D Euler-Poisson equations.
where $\mu$ is a Lagrange multiplier taking into account the normalization constraint. Using the results of Appendix C of [50], this variational problem gives $u = 0$ (the equilibrium state is static) and the condition

$$Q + m\Phi + k_B T \ln \rho = \mu. \quad (117)$$

Taking the gradient of Eq. (117) and using Eq. (79), we recover the condition of hydrostatic equilibrium (85). Equation (117) is equivalent to the time-independent Schrödinger equation (89) provided that we make the identification

$$\mu = E. \quad (118)$$

This shows that the Lagrange multiplier $\mu$ (chemical potential) in the variational problem associated with Eq. (115) can be identified with the eigenenergy $E$. Inversely, the eigenenergy $E$ may be interpreted as a chemical potential $\mu$.

Equation (117) can be written as

$$\rho = e^{-\beta (m\phi + Q - \mu)}. \quad (119)$$

This equation shows that the equilibrium state of the generalized Schrödinger equation (74), or equivalently of the damped quantum isothermal Euler equations (76)-(79), is given by a generalized Boltzmann distribution including the contribution of the quantum potential (this is actually a differential equation). If we neglect the contribution of the quantum potential ($Q = 0$), we recover the usual Boltzmann distribution. Finally, considering the second order variations of free energy, we find that the equilibrium state is stable if, and only if,

$$\delta^2 F = k_B T \int \frac{(\delta \rho)^2}{2\rho} \, dr + \frac{\hbar^2}{8m} \int \frac{1}{\rho} \left( \frac{\Delta \rho}{\rho} - \frac{(\nabla \rho)^2}{\rho^2} \right)(\delta \rho)^2 + (\nabla \delta \rho)^2 \, dr > 0 \quad (120)$$

for all perturbations that conserve the normalization condition: $\int \delta \rho \, dr = 0$. This inequality can also be written as

$$\delta^2 F = k_B T \int \frac{(\delta \rho)^2}{2\rho} \, dr + \frac{\hbar^2}{8m} \int \left[ \nabla \left( \frac{\delta \rho}{\sqrt{\rho}} \right) \right]^2 \, dr + \frac{\hbar^2}{8m} \int \frac{\Delta \sqrt{\rho}}{\rho^{3/2}} (\delta \rho)^2 \, dr > 0. \quad (121)$$

**E. Equilibrium relations**

At equilibrium ($\Theta_c = 0$), the free energy (108) reduces to

$$F = \Theta Q + W - TS. \quad (122)$$

On the other hand, multiplying Eq. (89) by $\rho$ and integrating over $r$, we obtain

$$E = \Theta Q + W + k_B T - TS. \quad (123)$$

Comparing Eqs. (122) and (123), we get

$$F = E - k_B T. \quad (124)$$

Equations (122)-(124) are the equilibrium forms of Eqs. (108), (110) and (111). We note that, at equilibrium, $\langle E \rangle = E$.

**V. THE VIRIAL THEOREM FOR A HARMONIC POTENTIAL**

In this section, we provide the form of the virial theorem associated with the generalized Schrödinger equation (74) or, equivalently, with the quantum damped isothermal Euler equations (76)-(79). To make the results more explicit, we restrict ourselves to the case of a harmonic potential (see [50] for generalizations). We write the harmonic potential as

$$\Phi = \frac{1}{2} \omega_0^2 r^2. \quad (125)$$

When $\omega_0^2 = -\Omega^2 < 0$, this potential mimics the effect of a solid-body rotation of the system (this analogy is exact in $d = 2$). When $\omega_0^2 > 0$, this potential mimics the effect of a confining trap. For the harmonic potential (125), the force
(by unit of mass) exerted on the particle is \( F = -\nabla \Phi = -\omega_0^2 r \). On the other hand, the potential energy \( W \) can be written as

\[
W = \frac{1}{2} \omega_0^2 I, \tag{126}
\]

where

\[
I = \int m\rho r^2 \, dr \tag{127}
\]

is the moment of inertia. We note that \( I = m\langle r^2 \rangle \), where \( \langle r^2 \rangle \) measures the quantum spread of the position of the particle.

For the harmonic potential \( V \), the time-dependent virial theorem writes (see Appendix G of [50]):

\[
\frac{1}{2} \ddot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2(\Theta_c + \Theta_Q) + \kappa_B T. \tag{128}
\]

In the strong friction limit \( \xi \to +\infty \), corresponding to the quantum Smoluchowski equation \( \xi \), it reduces to

\[
\frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2\Theta_Q + \kappa_B T. \tag{129}
\]

At equilibrium (\( \dot{I} = \ddot{I} = \Theta_c = 0 \)), we obtain

\[
2\Theta_Q + \kappa_B T - \omega_0^2 I = 0. \tag{130}
\]

On the other hand, for the harmonic potential \( V \), the relations \( \omega \) and \( \omega_0 \) become

\[
F = \Theta_Q + \frac{1}{2} \omega_0^2 I - TS \tag{131}
\]

and

\[
E = \Theta_Q + \frac{1}{2} \omega_0^2 I + k_B T - TS. \tag{132}
\]

### VI. COMPARISON WITH OTHER WORKS

The generalized Schrödinger equation \( \xi \) includes two new terms with respect to the Schrödinger equation \( \omega \): a friction term \( \xi \) and a temperature term \( \omega \). These terms correspond to the real and imaginary parts of the complex friction coefficient \( \omega \). These two terms have a common origin and they satisfy a sort of fluctuation-dissipation theorem. The Schrödinger equation \( \omega \) is recovered when \( \xi = \omega = 0 \). In this section, we mention the connection between the generalized Schrödinger equation \( \xi \) and nonlinear Schrödinger equations that have been introduced in the past with different arguments.

#### A. The damped Schrödinger equation

A Schrödinger equation including a damping term similar to the one present in Eq. \( \xi \), namely

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi - i\frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi, \tag{133}
\]

has been introduced by Kostin [48]. He derived it from the Heisenberg-Langevin equation describing a quantum Brownian particle interacting with a thermal bath environment. In Ref. [50], we independently obtained this equation by looking for the nonlinear Schrödinger equation that leads, through the Madelung transformation, to a quantum Euler equation with a linear friction force proportional and opposite to the velocity. Our approach gives another interpretation of the damped Schrödinger equation \( \xi \) from a hydrodynamic representation of the wave equation (see Sec. [113]) associated with an effective thermodynamic formalism (see Sec. [IV]).
A nonlinear Schrödinger equation with a logarithmic potential $-b \ln |\psi|^2$ similar to the one present in Eq. (74), namely

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi - 2b \ln |\psi| \psi,$$

(134)

has been introduced by Bialynicki-Birula and Mycielski [49] as a possible generalization of the Schrödinger equation in quantum mechanics. Their original motivation was the following. If we consider a free particle described by the Schrödinger equation (21) with $\Phi = 0$, the kinetic term in the Schrödinger equation, accounting for the Heisenberg uncertainty principle, leads to the spreading of the wavepacket. As a result, the position of the particle becomes more and more uncertain and the Schrödinger equation does not have localized stationary solutions. When applied to macroscopic objects, this leads to a problem because, according to classical mechanics, a macroscopic object has a well-defined position. This problem can be solved if we assume that the Schrödinger equation is replaced by a nonlinear Schrödinger equation with a logarithmic potential $-b \ln |\psi|^2$. The coefficient $b$ can be positive or negative. When it is positive, the nonlinearity $-b \ln |\psi|^2$ counteracts the spreading of the wave packet, thereby allowing solutions which behave macroscopically as classical particles. The balance between the dispersion of the wave packet and the logarithmic nonlinearity leads to a stationary solution of the wave equation with a Gaussian profile and a finite width called a gausson. The radius of a gausson is $R = \hbar/(2mb)^{1/2}$ (see Appendix I). In the interpretation of Bialynicki-Birula and Mycielski [49], the Schrödinger equation is an approximation of this nonlinear wave equation. Therefore, the logarithmic nonlinearity has a fundamental origin and the coefficient $b < 0$ is interpreted as a fundamental constant of physics. Of course, it must be sufficiently small in order to satisfy the constraints set by laboratory experiments. Bialynicki-Birula and Mycielski [49] obtained $b < 4 \times 10^{-10}$ eV which implies a bound to the electron soliton spatial width of 10 µm. Shull et al. [58] obtained $b < 3.4 \times 10^{-13}$ eV. Finally, an upper limit $b < 3.3 \times 10^{-15}$ eV was obtained by Gühler et al. [71] from precise measurements of Fresnel diffraction with slow neutrons. This implies a bound to the electron soliton spatial width of 3 mm. For $m \to +\infty$, the density probability becomes a delta-function and the particle is localized. It means that there exists a classical motion of a particle with a sufficient big mass.

In Ref. [50], we independently obtained the logarithmic Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + 2k_B T \ln |\psi| \psi,$$

(135)

by looking for the nonlinear Schrödinger equation that leads, through the Madelung transformation, to an Euler equation including a pressure term with an isothermal equation of state (we also obtained more general nonlinear Schrödinger equations associated with arbitrary barotropic equations of state). This leads to a logarithmic potential of the form $k_B T \ln |\psi|^2$ where $T$ is an effective temperature that can be positive or negative. Comparing Eqs. (134) and (135), we find that

$$b = -k_B T$$

(136)

so that a positive coefficient $b$ corresponds to a negative effective temperature. Our approach gives another interpretation of the logarithmic Schrödinger equation (134) from a hydrodynamic representation of the wave equation (see Sec. IV B) associated with an effective thermodynamic formalism (see Sec. IV). In this hydrodynamic representation, the evolution of a free quantum particle described by the Schrödinger equation (21) with $\Phi = 0$ is governed by the quantum Euler equation (35). The evolution of the fluid particle is due to the quantum potential which has a fundamental origin.

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28 Other forms of nonlinearity in the Schrödinger equation can prevent the spreading of the wave packet. However, nonlinear Schrödinger equations have in general the undesirable feature to generate correlation between non-interacting particles unless the nonlinearity is logarithmic. Therefore, the motivation of Bialynicki-Birula and Mycielski [49] to introduce a logarithmic nonlinearity in the Schrödinger equation is that this term still satisfies the additivity property for non-interacting particles while solving the spreading of the wave packet problem. Therefore, the logarithmic term was selected by assuming the factorization of the wave function for the composed system. This is what they called the property of separability of noninteracting subsystems. They suggested that to maintain the lack of correlation between noninteracting particle subsystems (separation property). As we have seen in our effective thermodynamic formalism, this logarithmic potential is associated with the Boltzmann entropy. We note that a power-law nonlinearity $-|\psi|^{2(\gamma-1)}$, which corresponds to a polytropic equation of state (see Ref. [54]), also solves the spreading of the wave packet problem but does not satisfy the additivity property for noninteracting subsystems. A polytropic equation of state is associated with the Tsallis entropy (see Ref. [57]) that has been introduced precisely in order to deal with non-extensive and non-additive systems [65]. Therefore, the Tsallis entropy may find applications in the context of nonlinear Schrödinger equations with a power-law nonlinearity.
Note that the number of bosons is not conserved in the stochastic GP equation \[ \text{(71)}. \] When the effective temperature is negative \( (T < 0) \), the pressure term produces an attractive force corresponding to a negative pressure that can balance the repulsion due to the quantum potential and leads to a stationary solution given by the Boltzmann distribution \[ \text{(111)}, \] resulting from the condition of hydrostatic equilibrium \[ \text{(95)}. \] This is the hydrodynamic representation of the gausson. We note that the radius of the gausson can be written as \( R = \hbar / \sqrt{2m|k_B T|} \) which can be interpreted as a generalized de Broglie relation.

C. The unified description

In this paper, we have derived a generalized Schrödinger equation \[ \text{(74)} \] that unifies the damped Schrödinger equation \[ \text{(153)}, \] and the logarithmic Schrödinger equation \[ \text{(131)} \] or \[ \text{(135)}. \] We have shown that the friction and temperature terms in the generalized Schrödinger equation \[ \text{(74)} \] have a common origin and that they can be obtained from a unified description based on Nottale’s theory of scale relativity. They satisfy a sort of fluctuation-dissipation theorem. In our formalism

\[
\xi = \gamma_R \quad \text{and} \quad b = -k_B T = -\frac{1}{2} \hbar \gamma_I.
\]

(137)

If we assume that \( \gamma \sim 1 \), we find that \( \xi \sim 1 \) and \( b = -k_B T \sim \hbar \). This establishes the fact that \( b = -k_B T \sim \hbar \) has its origin in quantum mechanics (it vanishes in the classical limit \( \hbar \to 0 \)) while \( \xi \sim 1 \) survives in the classical limit. We can also reach this conclusion by noticing that the term \( \text{Re}(\gamma U) \) in Eq. \[ \text{(65)} \] can be written as \( \text{Re}(\gamma U) = \gamma_R u + \gamma_I u_Q \) (see Appendix E). Therefore, the real friction coefficient \( \gamma_R = \xi \) (friction term) acts on the classical velocity \( u \) and the imaginary friction coefficient \( \gamma_I = -2b/\hbar = 2k_B T/\hbar \) (temperature term) acts on the quantum velocity \( u_Q = (\hbar/2m) \ln \rho | \text{see Eqs. (E32) and (E33)}. \) On the other hand, the temperature term in Eq. \[ \text{(74)} \] can be written as \( \frac{1}{2} \hbar \gamma_I \ln \rho \psi \) (disappearing when \( \hbar \to 0 \)) and the friction term can be written as \( \gamma_R (S - \langle S \rangle) \psi \) (surviving when \( \hbar \to 0 \)). The fact that \( b = -k_B T \) is proportional to the Planck constant \( \hbar \) may explain its small value and why it is not detectable in earth experiments. We also note, parenthetically, that the scaling \( T \sim \hbar \) is similar to the one arising in the Hawking temperature \( k_B T = \hbar c^3/8\pi GM \) of a black hole. This may suggest a connection to quantum gravity.

Remark: The gausson is a black hole if its radius \( R = \hbar / \sqrt{2mb} \) is of the order of the Schwarzschild radius \( R = 2Gm/c^2 \). This corresponds to a mass \( m \sim (\hbar^2 c^4 / 8G^2)^{1/3} \). Since \( b = -k_B T \), this relation can be rewritten as \( k_B T \approx \hbar^2 c^4 / 8G^2 m^3 \approx G\hbar^2 / c^2 R^3 \). This relation may be compared with the Hawking formula \( k_B T = \hbar c^3 / 8\pi GM = hc/4\pi R \) for the temperature of a Schwarzschild black hole. If we identify the two, we obtain \( m \sim m_P \) where \( m_P = \left( \frac{\hbar c}{G} \right)^{1/2} \) is the Planck mass, \( R \sim l_P \) where \( l_P = \left( \frac{\hbar G}{c^3} \right)^{1/2} \) is the Planck length, and \( k_B T \sim m_P c^2 \sim k_B T_P \) where \( T_P = \left( \frac{\hbar c^5}{Gk_B^2} \right)^{1/2} \) is the Planck temperature. This corresponds to a Planck black hole. We also note that \( R \sim \lambda_C \) where \( \lambda_C = \hbar / m c \) is the Compton wavelength. These results are of course only qualitative and suggestive. It is not clear whether they have a deeper meaning than is apparent at first sight from, essentially, dimensional analysis.

VII. ANOTHER GENERALIZED SCHröDINGER EQUATION

In this section, we consider another form of dissipative Schrödinger equation. We write it as

\[
\hbar \frac{\partial \psi}{\partial t} = (1 - i\gamma) \left( -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi - \langle E \rangle \psi \right),
\]

(138)

where \( \gamma \) is a dimensionless dissipation coefficient and \( \langle E \rangle(t) \) is the time-dependent average energy defined by Eq. \[ \text{(155)}. \] This equation is inspired by the so-called stochastic GP equation introduced in the context of BECs \[ \text{(11)}. \] However, for the simplicity of the presentation, we have ignored the stochastic term and the nonlinearity (they can be introduced straightforwardly). More importantly, we have replaced the constant chemical potential \( \mu \) by the time-dependent average energy \( \langle E \rangle(t) \). Physically, this means that the generalized Schrödinger equation \[ \text{(158)} \] is written in a frame that rotates with an angular velocity \( \omega(t) = \langle E \rangle(t)/\hbar \) that adapts itself at each time. As will be shown below, this allows us to satisfy the conservation of the normalization condition globally.\[ \text{(30)}. \]

\[ \text{29. The dissipative term} -i\gamma \text{ was introduced by Pitaevskii \[ \text{(72)} \text{ (see also \[ \text{(73), (74)}. \)} \]}

\[ \text{30. Note that the number of bosons is not conserved in the stochastic GP equation \[ \text{(71)}. \]}

Making the Madelung transformation \[29\] and proceeding as in Sec. II C, we obtain the hydrodynamic equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = -\frac{2\gamma}{\hbar} \rho \left( \frac{E}{2m} \mathbf{u}^2 + m \Phi + Q - \langle E \rangle \right),
\]

\(139\)

\[
\frac{\partial S}{\partial t} + (\nabla S)^2 - \frac{m}{2m} - m \Phi + Q - \langle E \rangle - \frac{\hbar}{2m} \left( \nabla \rho \cdot \nabla S + \Delta S \right) = 0,
\]

\(140\)

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \Phi - \frac{\hbar}{2m} \left( \nabla \cdot (\rho \mathbf{u}) \right) + \frac{\gamma \hbar}{2m} \Delta \mathbf{u},
\]

\(141\)

where \(\langle E \rangle(t)\) is given by Eq. \(138\). Equation \(139\) is a generalized continuity equation. We note that the normalization condition is not conserved locally. However, it is conserved globally thanks to the time-dependent chemical potential \(\langle E \rangle(t)\) introduced in Eq. \(138\). On the other hand, we see that the dissipation coefficient \(\gamma\) introduces a viscous term \(\nu \Delta \mathbf{u}\) in the momentum equation \(141\) with a quantum viscosity

\[
\nu = \frac{\gamma \hbar}{2m}.
\]

In this sense, Eq. \(141\) can be interpreted as a quantum Navier-Stokes equation. The hydrodynamic equations \(139\) to \(141\) can be written as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = -\frac{2\gamma}{\hbar} \rho \left( E - \langle E \rangle \right),
\]

\(143\)

\[
\frac{\partial S}{\partial t} + E - \langle E \rangle - \frac{\hbar}{2m} \Delta S + m \Phi = 0,
\]

\(144\)

\[
\frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla E}{m} + \frac{\hbar}{2m} \nabla \left( \frac{\rho \mathbf{u}}{\rho} \right),
\]

\(145\)

where \(E(\mathbf{r}, t)\) is defined by Eq. \(136\). We note that \(E \neq -\partial S/\partial t\) in the present situation.

Making the WKB transformation \(13\) in Eq. \(138\), we obtain

\[
\frac{1}{1 - i\gamma} \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - i \frac{\hbar}{2m} \Delta S + m \Phi - \langle E \rangle = 0.
\]

\(146\)

Defining the complex velocity by Eq. \(14\), we find that the generalized Schrödinger equation \(138\) is equivalent to the complex hydrodynamic equation

\[
\frac{1}{1 - i\gamma} \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = i \frac{\hbar}{2m} \Delta \mathbf{U} - \nabla \Phi.
\]

\(147\)

We can interpret this equation as a viscous Navier-Stokes equation with a complex time \((1 - i\gamma)t\). It would be interesting to see if this equation can be derived from Nottale’s theory of scale relativity. We note that the average energy \(\langle E \rangle(t)\) does not appear in this equation. Therefore, Eq. \(147\) can be viewed as the most basic equation of the theory. If we start from this equation and derive the generalized Schrödinger equation \(138\) from this equation by proceeding backwards (following a presentation similar to the one given in Secs. II and III), we find that the average energy \(\langle E \rangle(t)\) (chemical potential) appears in Eqs. \(138\) and \(146\) as a “constant” of integration possibly depending on time. We are then free to choose it so as to assure the global conservation of the normalization condition as done above.

If we consider a static \((\partial_t \rho = 0, \mathbf{u} = 0)\) solution of Eqs. \(139\) to \(141\), we get

\[
m \Phi + Q = E \quad \text{and} \quad S = 0,
\]

\(148\)

where \(E\) is a constant. We then have \(\langle E \rangle = E\). The wavefunction \(\psi(\mathbf{r}) = \sqrt{\rho(\mathbf{r})} = \phi(\mathbf{r})\) is real and independent on time because we are already in the frame rotating with angular velocity \(\omega = E/\hbar\). It satisfies the eigenvalue problem

\[
-\frac{\hbar^2}{2m} \Delta \phi + m \Phi \phi = E \phi
\]

\(149\)
identical to the time-independent solution \(\langle \dot{E}\rangle = 0\) of the conservative \((\gamma = 0)\) Schrödinger equation.

In the presence of dissipation \((\gamma \neq 0)\), the average energy \(\langle E\rangle\) defined by Eq. (15) is not conserved. Taking its time derivative and proceeding as in Appendix D by using Eqs. (143) and (145), we obtain

\[
\langle \dot{E}\rangle = -\frac{2\gamma}{\hbar}\langle (E^2) - \langle E\rangle^2 \rangle - \frac{\gamma}{2} \int \frac{\nabla (\rho u)^2}{\rho} \, dr \leq 0,
\]

(150)

where \(\langle E\rangle = \int \rho E \, dr\) and \(\langle E^2\rangle = \int \rho E^2 \, dr\). At equilibrium, where \(\langle \dot{E}\rangle = 0\), Eq. (150) implies that \(u(r) = 0\) and \(E(r) = \langle E\rangle = E\), leading to Eq. (148). Therefore, Eq. (150) forms an \(H\)-theorem. We can obtain this \(H\)-theorem directly from the generalized Schrödinger equation (138). Using the results of Appendix H1 this equation can be written as

\[
i\hbar \frac{\partial \psi}{\partial t} = (1 - i\gamma)(\hat{H} - \langle E\rangle)\psi = (1 - i\gamma) \left[ \frac{\delta H}{\delta \psi^*} - \langle E\rangle \psi \right].
\]

(151)

From Eqs. (149) and (151), we obtain

\[
\hat{H} = -\frac{2\gamma}{\hbar} \int (|\dot{\psi}|^2 - \langle E\rangle) \psi^2 \, dr,
\]

(152)

which is equivalent to Eq. (150). Therefore, the generalized Schrödinger equation (138) relaxes towards an equilibrium state that minimizes the energy \(H = \langle E\rangle\) defined by Eqs. (148) and (145) under the normalization condition (25). This leads to the variational principle (114) returning the static state (148). Therefore, the generalized Schrödinger equation (138) is a relaxation equation that can serve as a numerical algorithm to compute time-independent solutions of the conservative \((\gamma = 0)\) Schrödinger equation (see the discussion in Sec. IV C).

Remark: In situations of physical interest \(\gamma \ll 1\). Inversely, if we consider formally the limit \(\gamma \to +\infty\), we find that Eqs. (143) and (145) reduce to

\[
\frac{\partial \rho}{\partial t} = -\frac{2\gamma}{\hbar} \rho (E - \langle E\rangle),
\]

(153)

where \(E\) and \(\langle E\rangle\) are defined by Eqs. (146) and (148) with \(u = 0\). Equation (153) provides an even simpler numerical algorithm relaxing towards the static state (148). Indeed, taking the time derivative of \(\langle E\rangle\) and using Eq. (153), we obtain the \(H\)-theorem

\[
\langle \dot{E}\rangle = -\frac{2\gamma}{\hbar} \langle (E^2) - \langle E\rangle^2 \rangle \leq 0.
\]

(154)

It can also be obtained from Eq. (150) with \(u = 0\).

VIII. CONCLUSION

In this paper, we have derived the generalized Schrödinger equation (24) from Nottale’s theory of scale relativity. This equation is nonlinear and irreversible. It takes into account the interaction of the system with an external environment. We started from the covariant form of the equation of dynamics (65) including a friction force. This equation can be interpreted as a damped viscous Burgers equation (66) for a complex velocity field \(U\) with an imaginary viscosity \(\nu = i\hbar/2m\) and a complex friction coefficient \(\gamma\). The real part of the friction coefficient gives rise to ordinary frictional effects while the imaginary part of the friction coefficient gives rise to thermal effects [see Eq. (75)]. The friction and the temperature are connected to each other by a form of fluctuation-dissipation theorem. Using Madelung’s transformation, we have developed a hydrodynamic representation of the generalized Schrödinger equation (24) and obtained the quantum damped isothermal Euler equations (76)-(79). We have also shown that the generalized Schrödinger equation (24) is consistent with a generalized thermodynamic formalism based on the Boltzmann entropy (105). In particular, we have shown that the generalized Schrödinger equation satisfies an \(H\)-theorem (113) for the Boltzmann free energy (108).

The hydrodynamic representation of Madelung may lack a clear physical interpretation when the Schrödinger equation describes just one particle. However, it takes more sense when the Schrödinger equation describes a Bose-Einstein condensate (BEC) made of many particles in the same quantum state (72). In that case, the BEC can be interpreted as a real fluid described by the quantum Euler equations. For self-interacting BECs, there is an additional pressure force coming from the self-interaction potential. It depends on the scattering length \(a_s\) of the bosons.
BECs found unexpected applications in the context of dark matter (see the Introduction of [76] for a review). Indeed, it has been proposed that dark matter halos are self-gravitating BECs at $T = 0$ described by the Gross-Pitaevskii-Poisson (GPP) equations

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + \frac{1}{2} m \omega_0^2 r^2 \psi + \frac{4 \pi a_s \hbar^2}{m^2} |\psi|^2 \psi,$$  

$$\Delta \Phi = 4 \pi G |\psi|^2.$$  

(155)

(156)

For the sake of generality, we have included a harmonic potential that may mimic a confinement due to tidal effects when $\omega_0^2 > 0$ or a solid rotation when $\omega_0^2 < 0$. The quantum nature of the BECs may solve the problems of the cold dark matter (CDM) model such as the cusp problem and the missing satellite problem. However, the standard BEC model based on the GPP equations at $T = 0$ faces apparent difficulties. First of all, these equations are conservative so their spontaneous relaxation towards an equilibrium state, corresponding to a dark matter halo, is not obvious at first sight. On the other hand, the mass-radius relation obtained by determining the stable steady state of the GPP equations [76] is not consistent with the observations of large dark matter halos. When the bosons are self-interacting, one finds that the radius of the halos should decrease with their mass as $T_{\text{dark matter}} = 0$ described by the Gross-Pitaevskii-Poisson equations only describes the stable steady state of the GPP equations only describes the core of the halos that is completely condensed (ground state). This solitonic/BEC core must be surrounded by an approximately isothermal atmosphere. This isothermal atmosphere is necessary to account for the flat rotation curves of the galaxies. It also explains why the size of dark matter halos increases with their mass. Indeed, for large halos, it is the isothermal atmosphere that fixes their size, not their solitonic core. Therefore, the flat rotation curves of the galaxies and the fact that their size increases with their mass suggest that dark matter halos have an almost isothermal envelope with a relatively large temperature. This seems to be paradoxical because one expects from general considerations that the thermodynamic temperature of dark matter is very low, much lower than the condensation temperature $T_c$, so that taking $T = 0$ is an excellent approximation. Therefore, the origin of an atmosphere with a nonzero temperature $T \neq 0$ is not obvious at first sight. As a result, in order to agree with the observations, we must find a source of dissipation and understand the formation of this approximately isothermal envelope surrounding the core.

In Ref. [50], we have proposed to describe dark matter halos by the generalized GPP equations

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + \frac{1}{2} m \omega_0^2 r^2 \psi + \frac{4 \pi a_s \hbar^2}{m^2} |\psi|^2 \psi + 2k_B T \ln |\psi| \psi - i \frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi,$$  

$$\Delta \Phi = 4 \pi G |\psi|^2,$$  

(157)

(158)

or by the corresponding quantum damped isothermal Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$  

(159)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{k_B T}{m} \nabla \ln \rho - \frac{4 \pi a_s \hbar^2}{m^3} \nabla \rho - \nabla \Phi - \frac{\omega_0^2 r}{m} \nabla Q - \xi \mathbf{u},$$  

$$\Delta \Phi = 4 \pi G \rho,$$  

(160)

(161)

that include a friction term and a temperature term. The original GPP equations [155] and [156] are recovered for $\xi = T = 0$. The friction term accounts for the relaxation of the system towards a steady state. The temperature term accounts for their isothermal atmosphere leading to flat rotation curves and to a mass-radius relation consistent with the observations. The quantum pressure (due to the Heisenberg exclusion principle or to the scattering of the bosons) accounts for the formation of a solitonic core. The system described by the generalized GPP equations [157] and [158] undergoes damped oscillations and finally reaches an equilibrium state with a core-halo structure described by an equation of state

$$P = \rho \frac{k_B T}{m} + \frac{2 \pi a_s \hbar^2 \rho^2}{m^3}.$$  

(162)
We note that the exponent $\alpha = 3$ is the closest exponent to $\alpha = 2$ that yields a halo with a (marginally) finite mass.\[31\]

We have proposed in \[54\] three possible justifications of the generalized GPP equations \[157\] and \[158\]:

(i) In a first interpretation, the generalized GP equation \[157\] may be justified by physical processes. The dissipation could be due to non ideal effects or to the interaction of the system with an external environment and the nonlinear potential $2k_B T \ln |\psi|$ may account for the self-interaction of the bosons. In that case, $T$ would be a formal measure of the collisionless interactions inside the BEC at zero (thermal) temperature.

(ii) In a second interpretation, the generalized GPP equations \[157\] and \[158\] provide a heuristic parametrization of the process of gravitational cooling \[77\]. Gravitational cooling is a relaxation mechanism by which a self-gravitating BEC, initially out-of-equilibrium, ejects some mass and energy in order to reach a stable steady state (ground state) which is a stable stationary solution of the GPP equations \[159\] and \[160\]. As a result of this process, accompanied by damped oscillations (virialization), the system takes a core-halo structure in which the core is a soliton/BEC (ground state) and the halo is made of scalar radiation. The halo behaves approximately as an isothermal atmosphere like in the process of violent collisionless relaxation \[54, 55\]. However, it cannot be exactly isothermal otherwise it would have an infinite mass \[78\]. In reality, the halo has a density profile decreasing at large distances as $r^{-3}$ similarly to the Navarro-Frenk-White (NFW) \[79\] and Burkert \[80\] profiles instead of decreasing as $r^{-2}$ as implied by the self-gravitating isothermal sphere \[72\].\[31\] The difference between these two profiles may be due to complicated physical processes such as incomplete violent relaxation \[54, 55\], tidal effects \[81, 82\], stochastic perturbations... In our model, an additional source of confinement (modeling, e.g., tidal effects) can be taken into account by the harmonic potential $\omega_0^2 r^2/2$. In this interpretation, the fundamental equations of the problem are the GPP equations \[159\] and \[160\] at $T = 0$ but the generalized GPP equations \[157\] and \[158\] with a bar on $\psi$ may provide an effective parametrization of the GPP equations on the coarse-grained scale \[32\] in the same manner that the relaxation equations introduced in \[55\] provide a parametrization of the Vlasov-Poisson equations on the coarse-grained scale. The damping term can heuristically explain how a system of self-gravitating bosons rapidly reaches an equilibrium state due to gravitational cooling. Furthermore, this equilibrium state is characterized by a nonzero effective temperature $T$ (the one that appears in Eq. \[157\]) even if $T = 0$ fundamentally. The equilibrium structure of the dark matter halos result from the balance between the repulsive quantum force, the repulsive pressure due to the scattering of the bosons, the repulsive pressure due to the effective temperature, and the gravitational attraction. The equilibrium state of the generalized GPP equations \[157\] and \[158\] has a core-halo structure. It is made of a compact condensed core (BEC/soliton) which is a stable stationary solution of the GPP equation at $T = 0$ (ground state) surrounded by a halo of scalar radiation similar to an isothermal atmosphere at temperature $T$. Therefore, Eqs. \[157\] and \[158\] could provide a relevant model of dark matter halos experiencing gravitational cooling. In that interpretation, the friction coefficient and the temperature are effective parameters modeling the process of gravitational cooling. Their values may differ from halo to halo depending on the efficiency of the relaxation process.

(iii) Another, more speculative, interpretation of the generalized GP equation \[157\] is possible. We may argue that dark matter halos are fundamentally described by the generalized GP/Schrödinger equation \[157\] for which the standard GP/Schrödinger equation \[155\] is an approximation. In this interpretation, the nonlinear terms of friction and temperature in the generalized GP equation \[157\] have a fundamental origin. They are due to the existence of an hypothetical aether in which the quantum particles move.\[33\] This aether is responsible for the stochastic motion of the particles due to quantum fluctuations. In this interpretation, $\xi$ and $T$ are fundamental constants that represent the friction with the aether and the temperature of the aether. For bosons with an ultra small mass, one can show that the temperature of the halos is so small (see \[50\] for details) that the term $2k_B T \ln |\psi|\psi$ cannot be detected in laboratory experiments. However, this term could manifest itself at large scales, in astrophysics and cosmology, and account for the nonzero effective temperature of the dark matter halos and their flat rotation curves. In that interpretation, the flat rotation curves of the galaxies would have a fundamental origin related to the nonlinear term $2k_B T \ln |\psi|\psi$ in the generalized Schrödinger equation. However, if $T$ is a fundamental constant (the temperature of the aether), all the dark matter halos should have the same temperature while it can be shown that their temperature

\[31\] We note that the exponent $\alpha = 3$ is the closest exponent to $\alpha = 2$ that yields a halo with a (marginally) finite mass.

\[32\] In particular, they are able to account for the damped oscillations of a system experiencing gravitational cooling \[83, 84\]. This damping is apparent on the simplified equation of motion \[153\].

\[33\] This aether may be the manifestation of the fractal nature of the space time in Nottale’s theory.
increases linearly with their size as $T \propto R$ (see [50] for details). At that stage, we do not know how to solve this difficulty. Therefore, the interpretations (i) and (ii) should be privileged over the interpretation (iii).

Appendix A: Connection between the Smoluchowski equation, the Schrödinger equation, and the viscous Burgers equation

1. Connection between the Smoluchowski equation and the Schrödinger equation

We consider a Brownian particle of mass $m$ moving in an external potential $U(r)$. In the strong friction limit $\xi \to +\infty$, where $\xi$ is the friction coefficient, the evolution of the density probability $\rho(r, t)$ of finding the Brownian particle at position $r$ at time $t$ is governed by a Fokker-Planck equation of the form

$$\frac{\partial \rho}{\partial t} = D \nabla \cdot (\nabla \rho + \beta \rho U)$$

(A1)

called the Smoluchowski equation. Here, $D$ is the diffusion coefficient and $\beta = m/k_B T$ is the inverse temperature. The diffusion coefficient, the friction coefficient and the temperature are related to each other by the Einstein relation $D = k_B T/\xi m$. The equilibrium solution of the Smoluchowski equation (A1) is the Boltzmann distribution

$$\rho_{eq}(r) = Ae^{-\beta U(r)}.$$  

(A2)

If we make the change of variables

$$\rho(r, t) = \psi(r, t)e^{-\frac{1}{2}\beta U(r)},$$

(A3)

we can transform the Smoluchowski equation (A1) into a Schrödinger equation in imaginary time

$$\frac{\partial \psi}{\partial t} = D\Delta \psi + \frac{1}{2D}V\psi$$

(A4)

with a potential

$$V(r) = D^2 \beta \Delta U - \frac{1}{2}D^2 \beta^2 (\nabla U)^2.$$  

(A5)

In order to make the analogy with the Schrödinger equation closer, we write the diffusion coefficient as

$$D = \frac{\hbar}{2m},$$  

(A6)

where $\hbar$, which has the dimension of an action, is analogous to the Planck constant (in the present case, we have $\hbar = 2k_B T/\xi$ according to the Einstein relation). We also introduce the potential

$$\Phi(r) = -V(r).$$

(A7)

Then, we can rewrite Eq. (A4) as

$$-\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi.$$  

(A8)

Equation (A8) can be viewed as a Schrödinger equation in imaginary time with a potential $\Phi$ opposite to the potential $V$ appearing in Eq. (A4). Therefore, in all the cases where we can solve the Schrödinger equation (A8) for the potential $\Phi$, we can solve the Smoluchowski equation (A1) for the potential $U$ related to $\Phi$ by Eqs. (A5) and (A7). Inversely, if we known an analytical solution of the Smoluchowski equation (A1) for the potential $U$, we can obtain an analytical solution of the Schrödinger equation (A8) with the corresponding potential $\Phi$.

If we consider a solution of the Smoluchowski equation (A1) of the form (A3) with

$$\psi(r, t) = \phi(r)e^{-\lambda t},$$

(A9)
we obtain from Eq. (A8) the eigenvalue equation

\[ D\Delta \phi + \frac{1}{2D} V\phi = -\lambda \phi, \quad (A10) \]

where the eigenvalue \( \lambda \) gives the growth or damping rate of the eigenmode (A9). In comparison, if we consider a stationary solution of the Schrödinger equation (21) of the form

\[ \psi(r, t) = \phi(r)e^{-iEt/\hbar}, \quad (A11) \]

we obtain the eigenvalue equation

\[ -\frac{\hbar^2}{2m}\Delta \phi + m\Phi \phi = E\phi, \quad (A12) \]

where \( E \) is the eigenenergy so that \( \omega = E/\hbar \) gives the pulsation of the wave. We note that the two eigenvalue equations (A10) and (A12) coincide provided that we make the correspondences of Eqs. (A6), (A7) and set \( E/\hbar = \omega = \lambda \). Therefore, if we know analytically the eigenvalues of the Schrödinger equation for the potential \( \Phi \) we can obtain analytically the eigenvalues of the Smoluchowski equation for the potential \( U \) that is related to \( \Phi \) according to Eqs. (A5) and (A7).

**Remark:** For \( \lambda = 0 \) the solution of Eq. (A10) is given by [see Eq. (A2)]:

\[ \psi_{eq}(r) = Ae^{-\frac{1}{2}B\Phi(r)}, \quad (A13) \]

where \( U \) is related to \( V \) according to Eq. (A5). This is also the solution of the stationary Schrödinger equation (A12) for \( E = 0 \). However, to make the solution explicit, we need to solve Eq. (A5) to express \( U \) as a function of \( \Phi = -V \). We note that Eq. (A5) has the form of a Riccati equation for \( \nabla U \).

2. **Connection between the viscous Burgers equation and the Schrödinger equation: The Cole-Hopf transformation**

The Navier-Stokes equation without pressure

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u - \nabla V \quad (A14) \]

is called the viscous Burgers equation. For the sake of generality, we have introduced an external potential \( V(r) \). For a potential flow, the velocity field can be written as \( u = \nabla \theta \). In that case, the flow is irrotational: \( \nabla \times u = 0 \). Using the identity \( (u \cdot \nabla)u = \nabla(u^2/2) - u \times (\nabla \times u) \) which reduces to \( (u \cdot \nabla)u = \nabla(u^2/2) \) for an irrotational flow, Eq. (A14) can be integrated. This yields the viscous Bernoulli equation

\[ \frac{\partial \theta}{\partial t} + \frac{(\nabla \theta)^2}{2} = \nu \Delta \theta - V, \quad (A15) \]

where the constant of integration \( C(t) \) has been set equal to zero. If we make the change of variables

\[ \theta = -2\nu \ln \psi \quad (A16) \]

in Eq. (A15), we obtain

\[ \frac{\partial \psi}{\partial t} = \nu \Delta \psi + \frac{1}{2\nu} V\psi, \quad (A17) \]

For \( V = 0 \), Eq. (A17) reduces to the classical diffusion equation. Therefore, the viscous Burgers equation (A14) for a potential flow is equivalent to the diffusion equation. The transformation (A10) leading from Eq. (A14) to Eq. (A17) is called the Cole-Hopf transformation [85, 86]. With the presence of the external potential \( V \), Eq. (A17) is similar to the Schrödinger equation in imaginary time. To make the analogy closer, we write the viscosity as

\[ \nu = \frac{\hbar}{2m}, \quad (A18) \]
where $\hbar$, which has the dimension of an action, is analogous to the Planck constant (this is a pure notation here since the mass $m$ has no physical meaning). We also introduce the potential

$$\Phi(\mathbf{r}) = -V(\mathbf{r}).$$  \hfill (A19)

Then, we can rewrite Eq. (A17) as

$$-\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi.$$  \hfill (A20)

Equation (A20) can be viewed as a Schrödinger equation in imaginary time with a potential $\Phi$ opposite to the potential $V$ appearing in the viscous Burgers equation (A14). Therefore, in all the cases where we can solve the Schrödinger equation (A20) for the potential $\Phi$ we can solve the Burgers equation (A14) for the potential $V = -\Phi$, and conversely.

We look for the general stationary solution ($\mathbf{u}(\mathbf{r})$ independent of time) of the viscous Burgers equation (A14). The condition $\partial_t \mathbf{u} = 0$ is equivalent to $\partial_t \nabla \theta = 0$ implying $\theta(\mathbf{r}, t) = \theta_1(\mathbf{r}) + \theta_2(t)$. Substituting this decomposition into Eq. (A15), we obtain

$$\frac{d\theta_2}{dt} = -\frac{(\nabla \theta_1)^2}{2} + \nu \Delta \theta_1 - V \equiv 2\nu \lambda.$$  \hfill (A21)

Since the first term depends only on $t$ and the second term depends only on $\mathbf{r}$, they must be individually equal to a constant that we note $2\nu \lambda$. We then obtain $\theta_2 = 2\nu \lambda t$. On the other hand, making the transformation $\theta_1 = -2\nu \ln \phi$, we find that the general stationary solution of the viscous Burgers equation (A14) is

$$\theta(\mathbf{r}, t) = -2\nu (\ln \phi - \lambda t),$$  \hfill (A22)

$$\mathbf{u}(\mathbf{r}) = \nabla \theta = -2\nu \frac{\nabla \phi}{\phi}.$$  \hfill (A23)

where $\phi$ is the solution of the eigenvalue equation

$$-2\nu^2 \Delta \phi - V \phi = 2\nu \lambda \phi.$$  \hfill (A24)

Using Eqs. (A18) and (A19) and defining $E/\hbar = \omega = \lambda$, we can rewrite Eq. (A24) into the form of a stationary Schrödinger equation

$$-\frac{\hbar^2}{2m} \Delta \phi + m \Phi \phi = E \phi.$$  \hfill (A25)

The stationary solution of the viscous Burgers equation with a potential $V$ corresponds to the stationary solution of the Schrödinger equation with a potential $\Phi = -V$. Therefore, in all the cases where we can solve the stationary Schrödinger equation analytically, we can obtain an analytical solution of the stationary Burgers equation and conversely.

### 3. Connection between the Smoluchowski equation and the viscous Burgers equation

According to the results of Appendix A1 the Smoluchowski equation (A1) can be transformed into the Schrödinger equation in imaginary time (A3) with $D = \hbar/2m$ and $\Phi = -V$, where $V$ is related to $U$ by Eq. (A5). According to the results of Appendix A2 the viscous Burgers equation (A14) can be transformed into the Schrödinger equation in imaginary time (A20) with $\nu = \hbar/2m$ and $\Phi = -V$. As a result, the Smoluchowski equation (A1) can be transformed into the viscous Burgers equation (A14) with $\nu = D$ and $V$ given by Eq. (A5). The velocity field $\mathbf{u}$ in the Burgers equation (A14) is related to the density $\rho$ and to the potential $U$ in the Smoluchowski equation (A1) by means of the Cole-Hopf transformation giving

$$\theta = -2\nu \ln \psi = -2D \ln \psi = -2D \left( \ln \rho + \frac{1}{2} \beta U \right),$$  \hfill (A26)

$$\mathbf{u} = \nabla \theta = -2D \left( \frac{\nabla \rho}{\rho} + \frac{1}{2} \beta \nabla U \right).$$  \hfill (A27)
For some simple potentials $U$, we can solve the Smoluchowski equation \((A1)\) analytically. Using Eqs. \((A26)\) and \((A27)\), we can then obtain an analytical solution of the viscous Burgers equation for the corresponding potential $V$ defined by Eq. \((A5)\).

Let us consider a solution of the Smoluchowski equation of the form of Eq. \((A3)\) with Eq. \((A9)\). The rate $\lambda$ of this eigenmode is the eigenvalue of the stationary Schrödinger equation \((A10)\). Using the Cole-Hopf transformation, we find that

$$\theta(r, t) = -2\nu \ln \psi = -2D \ln \psi = -2(D(\ln \phi - \lambda t)), \quad (A28)$$

$$u(r) = \nabla \theta = -2D \nabla \phi. \quad (A29)$$

This returns the general stationary solution \((A22)-(A24)\) of the viscous Burgers equation with an external potential (see Appendix A2). More specifically, we consider the equilibrium solution of the Smoluchowski equation corresponding to $\lambda = 0$. Using Eq. \((A13)\) and making the Cole-Hopf transformation, we find that

$$\theta(r) = -2\nu \ln \psi = -2D \ln \psi = -2D \left( \ln A - \frac{1}{2} \beta U \right), \quad (A30)$$

$$u(r) = \nabla \theta = D\beta \nabla U. \quad (A31)$$

This provides a particular stationary solution of the viscous Burgers equation with a potential $V$ related to $U$ by Eq. \((A5)\).

**Appendix B: Necessity of the Nelson relation \((17)\)**

If we define the wave function by Eq. \((18)\) without imposing the relation \((17)\), we obtain an equation of the form

$$i\hbar \frac{\partial \psi}{\partial t} = \hbar \left( D - \frac{\hbar}{2m} \right) \left( \frac{\nabla \psi}{\psi} \right)^2 - D\hbar \Delta \psi + m\Phi \psi. \quad (B1)$$

Applying the Madelung transformation to this equation, we obtain the hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = \left( 1 - \frac{2m}{\hbar} D \right) \rho \nabla \cdot u, \quad (B2)$$

$$\frac{\partial S}{\partial t} + \frac{D}{\hbar} (\nabla S)^2 + m\Phi - \frac{1}{2} D\hbar \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2}\rho \left( \nabla \rho \right)^2 \right] + \hbar \left( D - \frac{\hbar}{2m} \right) \left[ \frac{1}{4\rho^2} \left( \nabla \rho \right)^2 - \frac{1}{\hbar^2} (\nabla S)^2 \right] = 0, \quad (B3)$$

$$\frac{\partial u}{\partial t} + \frac{2m}{\hbar} D\nabla \left( \frac{u^2}{2} \right) = -\nabla \Phi + D \frac{\hbar}{2m} \nabla \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2\rho^2} (\nabla \rho)^2 \right] - \frac{\hbar}{m} \left( D - \frac{\hbar}{2m} \right) \nabla \left[ \frac{1}{4\rho^2} (\nabla \rho)^2 - \frac{1}{\hbar^2} (\nabla S)^2 \right]. \quad (B4)$$

We see from Eq. \((B2)\) that Eq. \((B1)\) conserves the integral $\int |\psi|^2 \, dr$ if, and only if, the quantum diffusion coefficient $D$ satisfies the Nelson relation \((17)\). In that case, Eqs. \((B1)-(B4)\) reduce to the Schrödinger equation \((21)\) and to the hydrodynamic equations \((33)-(36)\).

**Appendix C: Generalized Ehrenfest theorem**

In this Appendix, using the fluid representation \((70)-(73)\) of the generalized Schrödinger equation \((74)\), we derive a generalization of the Ehrenfest \((53)\) theorem that takes dissipative effects into account.

In the hydrodynamic representation, the average values of the position and velocity of the quantum particle are given by

$$\langle r \rangle(t) = \int \rho r \, dr, \quad \langle u \rangle(t) = \int \rho u \, dr. \quad (C1)$$
Taking the time derivative of the first relation in Eq. (C1), using the continuity equation (76), integrating by parts, and comparing with the second relation in Eq. (C1), we obtain
\[ \frac{d}{dt} \langle r \rangle = \langle u \rangle. \] (C2)

Taking the time derivative of the second relation in Eq. (C1), using the Euler equation (80), and recalling that the average value of the quantum force vanishes [see Eq. (43)], we obtain
\[ \frac{d}{dt} \langle u \rangle = -\langle \nabla \Phi \rangle - \xi \langle u \rangle. \] (C3)

Therefore, the average values of \( r \) and \( u \) follow ordinary equations of motion as in classical mechanics. This is the meaning of the Ehrenfest theorem. We see that this theorem can be generalized to the case of dissipative systems.\(^{34}\)

We note, however, that the Ehrenfest theorem involves the average value of the force \( \langle \nabla \Phi \rangle(t) = \int \rho \nabla \Phi \, dr \), not the force \( \nabla \Phi(\langle r \rangle(t), t) \) taken at the average position of the particle. In general, they are not the same.

In order to establish that Eqs. (C2) and (C3) are really equivalent to the Ehrenfest equations, we need to discuss the relation between the hydrodynamic variables and the quantum operators. According to the correspondence principle, the impulse operator is defined by \( \hat{p} = -i\hbar \nabla \) [see Eq. (H2)]. The average value of the impulse is
\[ \langle p \rangle = \langle \psi | \hat{p} | \psi \rangle = -i\hbar \int \psi^* \nabla \psi \, dr. \] (C4)

In Nottale’s theory, the complex impulse is defined by \( P = mU = \nabla S = -i\hbar \nabla \ln \psi \) [see Eq. (E2)]. Its average value is
\[ \langle P \rangle = \int P |\psi|^2 \, dr = -i\hbar \int \psi^* \nabla \psi \, dr. \] (C5)

It coincides with the average value of the impulse: \( \langle P \rangle = \langle p \rangle \). On the other hand, we note that \( \langle u_Q \rangle = 0 \) according to Eq. (E9). Therefore, \( \langle U \rangle = \langle u \rangle \). This first shows that the average value of the complex velocity is real and that it satisfies Eq. (C3). Furthermore, \( \langle P \rangle = m\langle U \rangle = m\langle u \rangle \) so that \( \langle p \rangle = m\langle u \rangle \). Therefore, Eqs. (C2) and (C3) are equivalent to the Ehrenfest equations appropriately generalized to the case of dissipative systems.

**Appendix D: Conservation of the average energy in the hydrodynamic representation**

In the hydrodynamic representation, the average energy (18) associated with the Schrödinger equation (21) can be written as
\[ \langle E \rangle = \Theta_c + \Theta_Q + W, \] (D1)

where \( \Theta_c \) is the classical kinetic energy (97), \( \Theta_Q \) is the quantum kinetic energy (100), and \( W \) is the potential energy (101). Their first variations are
\[ \delta \Theta_c = \int m \frac{u^2}{2} \delta \rho \, dr + \int \rho m u \cdot \delta u \, dr, \] (D2)
\[ \delta \Theta_Q = \int Q \delta \rho \, dr, \] (D3)
\[ \delta W = \int m \Phi \delta \rho \, dr. \] (D4)

\(^{34}\) It can also be generalized to the class of nonlinear Schrödinger equations considered in [51] that are equivalent to hydrodynamic equations with an arbitrary barotropic equation of state \( P(\rho) \). Indeed, in the derivation of Eq. (C3), the term \( \int \nabla P \, dr \) vanishes for all functions \( P(\mathbf{r}, t) \) that tend to zero at infinity sufficiently rapidly.
Taking the time derivative of the average energy \( \langle \dot{E} \rangle \), and using Eqs. (D2)-(D4), we get

\[
\langle \dot{E} \rangle = \int \left( \frac{m \mathbf{u}^2}{2} + Q + m\Phi \right) \frac{\partial \rho}{\partial t} \, d\mathbf{r} + \int \rho m \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{r}.
\]  

(D5)

Substituting the continuity equation (33) into Eq. (D5), integrating by parts, and using the Euler equation (35) or (47), we obtain \( \langle \dot{E} \rangle = 0 \) showing the conservation of the average energy.

Remark: The calculations leading to relations (D2)-(D4) are straightforward except, maybe, the ones leading Eq. (D3). We give below three different derivations of this relation:

(i) From Eqs. (36), (99) and (100) we directly obtain

\[
\delta \Theta_Q = \frac{\hbar^2}{2m} \int \nabla \sqrt{\rho} \cdot \nabla \delta \sqrt{\rho} \, d\mathbf{r} = -\frac{\hbar^2}{2m} \int \delta \sqrt{\rho} \Delta \sqrt{\rho} \, d\mathbf{r} = \int Q \delta \rho \, d\mathbf{r}.
\]  

(D6)

(ii) From the first equality of Eq. (36), we find that

\[
\delta Q = -\frac{\hbar^2}{2m} \nabla \cdot \left( \sqrt{\rho} \nabla \delta \sqrt{\rho} - \delta \sqrt{\rho} \nabla \sqrt{\rho} \right) = -\frac{\hbar^2}{4m\rho} \nabla \cdot (\rho \delta \nabla \ln \rho).
\]  

(D7)

This implies the identity

\[
\int \rho \delta Q \, d\mathbf{r} = 0,
\]  

(D8)

from which we get \( \delta \Theta_Q = \int \rho \delta Q \, d\mathbf{r} + \int Q \delta \rho \, d\mathbf{r} = \int Q \delta \rho \, d\mathbf{r} \). We note that Eq. (D7) is the equivalent to the tensorial equation \( \text{H3} \) with Eq. \( \text{H9} \) except that it applies to a perturbation \( \delta \) instead of a space derivative \( \partial_i \).

(iii) Taking the first order variations of Eq. (E12) and using Eq. (E9), we obtain

\[
\rho \delta Q = -\frac{\hbar}{2} \nabla \cdot (\rho \delta \mathbf{u}_Q),
\]  

(D9)

implying Eq. (D8), then Eq. (D3).

Appendix E: Derivation of the generalized Nelson equations

1. The complex impulse

The wave function can be written in the WKB form as

\[
\psi = e^{iS/\hbar},
\]  

(E1)

where \( S \) is the complex action. Since \( S = -i\hbar \ln \psi \), the complex velocity field \( \mathbf{U} = \nabla S/m \) and the complex impulse \( \mathbf{P} = m\mathbf{U} = \nabla S \) are given by

\[
\mathbf{U} = -\frac{i\hbar}{m} \nabla \ln \psi \quad \text{and} \quad \mathbf{P} = -i\hbar \nabla \ln \psi.
\]  

(E2)

We note that

\[
\mathbf{P} \psi = -i\hbar \nabla \psi
\]  

(E3)

which is similar to the correspondance principle \( \text{H2} \). The total kinetic energy is defined by Eq. \( \text{H6} \). Using Eq. \( \text{E2} \) it can be written as

\[
\Theta = \frac{1}{2} \int \rho m |\mathbf{U}|^2 \, d\mathbf{r}.
\]  

(E4)

It has the form of an ordinary kinetic energy for a complex velocity field.

The wave function can also be written in the Madelung form as

\[
\psi = \sqrt{\rho} e^{iS/\hbar},
\]  

(E5)
where $S$ is the real action and $\rho = |\psi|^2$ is the probability density. Comparing the two expressions (E1) and (E5) of the wave function, we find that the complex action is related to the real action and to the probability density by

$$S = S - i\frac{\hbar}{2} \ln \rho. \quad (E6)$$

As a result, the complex velocity field $U = \nabla S/m$ can be rewritten as

$$U = u - iu_Q, \quad (E7)$$

where

$$u = \frac{\nabla S}{m} \quad (E8)$$

is the classical velocity and

$$u_Q = \frac{\hbar}{2m} \nabla \ln \rho \quad (E9)$$

is the quantum velocity.\(^{35}\) Using Eq. (E7), the total kinetic energy (E4) can be written as $\Theta = \Theta_c + \Theta_Q$ where

$$\Theta_c = \frac{1}{2} \int \rho m u^2 dr \quad (E10)$$

is the classical kinetic energy and

$$\Theta_Q = \frac{1}{2} \int \rho m u_Q^2 dr \quad (E11)$$

is the quantum kinetic energy. Using Eq. (E9) we can check that the quantum kinetic energy $\Theta_Q$ is equivalent to Eq. (100) which has the form of a quantum potential energy. Alternatively, Eq. (E11) indicates that the quantum kinetic energy can be written as an ordinary kinetic energy for a quantum velocity field.

*Remark:* Using Eq. (E9), the quantum potential (36) can be expressed in terms of the quantum velocity as

$$Q = -\frac{\hbar}{2} \nabla \cdot u_Q - \frac{1}{2} m u_Q^2. \quad (E12)$$

If we consider a static state of the Schrödinger equation (21), we have $Q + m\Phi = E$ [see Eq. (54)]. Using Eq. (E12), we obtain

$$-\frac{\hbar}{2} \nabla \cdot u_Q - \frac{1}{2} m u_Q^2 = E - m\Phi(r). \quad (E13)$$

This equation can be viewed as a Riccati equation that may be easier to solve than the Schrödinger equation itself.

2. The complex quantum potential

Using Eq. (17), the scale-covariant equation of dynamics (12) can be written as

$$\frac{\partial U}{\partial t} + (U \cdot \nabla) U = -i \frac{\hbar}{2m} \Delta U - \nabla \Phi. \quad (E14)$$

In Sec. III.B we have interpreted the term $i(\hbar/2m)\Delta U$ as a viscous term with an imaginary viscosity $\nu = i\hbar/2m$. Alternatively, using the identity $\Delta U = \nabla(\nabla \cdot U) = \nabla(\Delta S/m)$ valid for a potential flow, the term $i(\hbar/2m)\Delta U$ can be interpreted as a complex quantum force by unit of mass

$$F_Q = -\frac{1}{m} \nabla q \quad (E15)$$

\(^{35}\) The complex velocity (E7) combining the classical velocity (E8) and the quantum velocity (E9) was first introduced by Madelung in a not well-known paper\(^{52}\). It also appears in Refs.\(^ {53} \)\(^{54}\).
deriving from a complex quantum potential

\[ q = -\frac{i}{2}\hbar \nabla \cdot \underline{U} = -i\frac{\hbar}{2m}\Delta S = -\frac{\hbar^2}{2m}\Delta \ln \psi. \quad (E16) \]

Using Eq. (E7), the quantum potential can be written in terms of the classical and quantum velocities as

\[ q = -\frac{\hbar^2}{2m}\nabla \cdot \underline{u}_Q - i\frac{\hbar}{2}\nabla \cdot \underline{u}. \quad (E17) \]

On the other hand, using Eq. (29) and the identity (20), we find that

\[ q = -\frac{\hbar^2}{4m}\Delta (\ln \rho) - i\frac{\hbar}{2m}\Delta S = -\frac{\hbar^2}{4m} \left[ \frac{\Delta \rho}{\rho} - \frac{(\nabla \rho)^2}{\rho^2} \right] - i\frac{\hbar}{2m}\Delta S. \quad (E18) \]

Comparing Eq. (E18) with Eq. (36), or Eq. (E17) with Eq. (E12), we note that the real part of the complex quantum potential \( q \) is not the quantum potential \( Q \).

3. The complex energy

According to Eq. (E1), the complex energy \( E = -\partial S/\partial t \) is given by

\[ E = i\hbar \frac{\partial \ln \psi}{\partial t}. \quad (E19) \]

We note that

\[ E\psi = i\hbar \frac{\partial \psi}{\partial t} \quad (E20) \]

which is similar to the correspondence principle (H2). Using Eq. (E6), we find that \( E = E + iE_Q \), where

\[ E = -\frac{\partial S}{\partial t} \quad (E21) \]

and

\[ E_Q = \frac{\hbar}{2\rho} \frac{\partial \rho}{\partial t}. \quad (E22) \]

The complex Hamilton-Jacobi equation (15) can be written as

\[ E = \frac{1}{2m} (\nabla S)^2 - iD\Delta S + m\Phi. \quad (E23) \]

Using Eq. (14), we obtain

\[ E = \frac{1}{2}m\underline{U}^2 - i\frac{\hbar}{2}\nabla \cdot \underline{U} + m\Phi. \quad (E24) \]

Introducing the complex quantum potential (E16), the complex energy takes the form

\[ E = \frac{1}{2}m\underline{U}^2 + q + m\Phi. \quad (E25) \]

It is the sum of the complex kinetic energy, the complex quantum potential, and the external potential. As noted by Nottale [46], the theory of scale relativity yields a new contribution \( q \) to the energy that comes from the very geometry of space-time. This is similar to the new contribution \( mc^2 \) to the energy in Einstein’s theory of relativity. Using Eqs. (E7) and (E17), we find that \( E = E + iE_Q \) with

\[ E = \frac{1}{2}m\underline{u}^2 - \frac{1}{2}m\underline{u}_Q^2 - \frac{\hbar}{2}\nabla \cdot \underline{u}_Q + m\Phi. \quad (E26) \]
and
\[ E_Q = -mu \cdot u_Q - \frac{\hbar}{2} \nabla \cdot u. \] (E27)

Using Eq. (E12), we can rewrite Eq. (E26) as
\[ E = \frac{1}{2} mu^2 + Q + m \Phi. \] (E28)

Equations (E8), (E21) and (E28) return the quantum Hamilton-Jacobi equations (34) and (46). On the other hand, using Eq. (E9), we can rewrite Eq. (E27) as
\[ E_Q = -\frac{\hbar}{2\rho} \nabla (\rho u). \] (E29)

Equations (E22) and (E29) return the equation of continuity (33). When \( \gamma = 0 \), the average energy \( \langle E \rangle \) is conserved (see Appendix D). On the other hand, from Eq. (E29), we see that \( \langle E_Q \rangle \) is trivially conserved since \( \langle E_Q \rangle = 0 \). As a result, \( \langle E \rangle = \langle E \rangle \) is conserved.

Remark: Using Eqs. (21) and (E20), we find that
\[ E(r,t) = -\frac{\hbar^2}{2m} \Delta \psi \psi + m \Phi. \] (E30)

This equation can be rewritten as
\[ -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi = E(r,t) \psi. \] (E31)

For a wave function of the form (52), we find that \( E(r,t) = E \) where \( E \) is a constant. In that case, Eq. (E31) reduces to the time-independent Schrödinger equation (53).

4. The real and imaginary parts of the generalized complex viscous Burgers equation

Writing the complex velocity field as \( U = u - iu_Q \) and taking the real and imaginary parts of the generalized complex viscous Burgers equation (66), we obtain the two real coupled equations
\[ \frac{\partial u_Q}{\partial t} + (u_Q \cdot \nabla) u + (u \cdot \nabla) u_Q = -\frac{\hbar}{2m} \Delta u, \] (E32)
\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u - (u_Q \cdot \nabla) u_Q = \frac{\hbar}{2m} \Delta u_Q - \nabla \Phi - \gamma_R u - \gamma_I u_Q. \] (E33)

When \( \gamma = 0 \) these equations coincide with those derived by Nelson [45] in his stochastic interpretation of quantum mechanics. In his theory, \( u = \nabla S/m \) is called the “current” velocity and \(-u_Q = D \nabla \ln \rho \) the “osmotic” velocity. According to Einstein’s theory of Brownian motion, the terminal velocity \( u = -(1/\xi) \nabla \Phi \) acquired by a Brownian particle submitted to an external force must balance the process of diffusion produced by the thermal molecular motion. The first process is described by a current \( j = \rho u = -(1/\xi) \rho \nabla \Phi \). The second process is described by a current \( J = -D \nabla \rho \) given by Fick’s law, leading to a diffusion equation \( \partial \rho / \partial t = -\nabla \cdot J = D \Delta \rho \). Therefore, \(-D \nabla \rho - (1/\xi) \rho \nabla \Phi = 0 \). When compared to the condition of equilibrium \( \nabla P + \rho \nabla \Phi = 0 \) where \( P = k_B T/\xi m \) is the osmotic pressure, one gets the Einstein relation \( \gamma = k_B T/\xi m \). Furthermore, the velocity can be written as \( u = D \nabla \ln \rho \), hence the name “osmotic velocity” given by Nelson [45] by analogy with Brownian motion.  

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\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u - (u_Q \cdot \nabla) u_Q = \frac{\hbar}{2m} \Delta u_Q - \nabla \Phi - \gamma_R u - \gamma_I u_Q. \] (E33)

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We first consider Eq. (E32). Using the identities \( \Delta \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) \) and \( \nabla(\mathbf{u} \cdot \mathbf{u}_Q) = (\mathbf{u} \cdot \nabla)\mathbf{u}_Q + (\mathbf{u}_Q \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}_Q) + \mathbf{u}_Q \times (\nabla \times \mathbf{u}) \) which reduce to \( \Delta \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) \) and \( \nabla(\mathbf{u} \cdot \mathbf{u}_Q) = (\mathbf{u} \cdot \nabla)\mathbf{u}_Q + (\mathbf{u}_Q \cdot \nabla)\mathbf{u} \) for an irrotational flow \( (\nabla \times \mathbf{u} = \nabla \times \mathbf{u}_Q = 0) \), Eq. (E32) can be rewritten as

\[
\frac{\partial \mathbf{u}_Q}{\partial t} + \nabla \cdot (\mathbf{u} \cdot \mathbf{u}_Q) = -\frac{\hbar}{2m} \nabla(\nabla \cdot \mathbf{u}).
\]  
(E34)

Using Eq. (E9), we recover the equation of continuity (76).

We now consider Eq. (E33). Using the identities \( \Delta \mathbf{u}_Q = \nabla(\nabla \cdot \mathbf{u}_Q) - \nabla \times (\nabla \times \mathbf{u}_Q) \) and \( (\mathbf{u}_Q \cdot \nabla)\mathbf{u}_Q = \nabla(\mathbf{u}_Q^2)/2 - \mathbf{u}_Q \times (\nabla \times \mathbf{u}_Q) \) which reduce to \( \Delta \mathbf{u}_Q = \nabla(\nabla \cdot \mathbf{u}_Q) \) and \( (\mathbf{u}_Q \cdot \nabla)\mathbf{u}_Q = \nabla(\mathbf{u}_Q^2)/2 \) for an irrotational flow \( (\nabla \times \mathbf{u}_Q = 0) \), Eq. (E33) can be rewritten as

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \Phi - \gamma_R \mathbf{u} - \gamma_I \mathbf{u}_Q + \frac{\hbar}{2m} \nabla(\nabla \cdot \mathbf{u}_Q) + \nabla \left( \frac{\mathbf{u}_Q^2}{2} \right).
\]  
(E35)

Using Eqs. (E9) and (E12), and introducing the notations of Eq. (71), we recover the quantum damped isothermal Euler equation (78). We note that the friction term corresponds to \(-\gamma_R \mathbf{u}\) and the pressure term corresponds to \(-\gamma_I \mathbf{u}_Q\).

**Appendix F: A damped Schrödinger equation that does not satisfy the local conservation of the normalization condition**

If we attempt to take into account dissipative effects in the Schrödinger equation by writing the scale-covariant equation of dynamics as

\[
\frac{DU}{Dt} = -\nabla \Phi - \gamma U.
\]  
(F1)

where \( \gamma \) is a complex friction coefficient, we obtain a generalized complex viscous Burgers equation of the form

\[
\frac{\partial U}{\partial t} + (U \cdot \nabla)U = iD \Delta U - \nabla \Phi - \gamma U.
\]  
(F2)

The corresponding complex Hamilton-Jacobi equation writes

\[
\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - iD \Delta S + m \Phi + V(t) + \gamma S = 0.
\]  
(F3)

Introducing the wave function \( \psi \) and repeating the calculations of Sec. II B we obtain the generalized Schrödinger equation

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + V \psi - i\gamma \hbar (\ln \psi) \psi.
\]  
(F4)

As we shall see, this equation with \( V = 0 \) does not conserve the normalization condition. Therefore, it makes sense to determine \( V(t) \) so that the average value of the friction term is equal to zero. This yields

\[
V(t) = i\gamma \hbar (\ln \psi).
\]  
(F5)

In that case, we obtain a generalized Schrödinger equation of the form

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi - i\gamma \hbar (\ln \psi - (\ln \psi)) \psi.
\]  
(F6)

Introducing the notations of Eq. (71), it can be rewritten as

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi - i\xi \hbar (\ln \psi - (\ln \psi)) \psi + 2k_B T (\ln \psi - (\ln \psi)) \psi.
\]  
(F7)

Making the Madelung transformation, we obtain the hydrodynamic equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = -\xi \rho (\ln \rho - (\ln \rho)) + \frac{4k_B T}{\hbar^2} \rho (S - \langle S \rangle),
\]  
(F8)
\[ \frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + m\Phi + Q + \xi(S - \langle S \rangle) + k_B T (\ln \rho - \langle \ln \rho \rangle) = 0, \quad (F9) \]

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p - \nabla \Phi - \frac{1}{m} \nabla Q - \xi u. \quad (F10) \]

Since Eq. (F8) differs from the continuity equation, we conclude that Eq. (F6) does not conserve the normalization condition locally. We note, however, that it conserves the normalization condition globally. Indeed

\[ \frac{d}{dt} \int \rho \, d\mathbf{r} = 0. \quad (F11) \]

This is not the case of Eq. (F4) with \( V = 0 \) which leads to Eq. (F8) without the terms in bracket. In that case, we get

\[ \frac{d}{dt} \int \rho \, d\mathbf{r} = -\xi \int \rho \ln \rho \, d\mathbf{r} + \frac{4k_B T}{\hbar^2} \int \rho S \, d\mathbf{r}, \quad (F12) \]

which is usually different from zero.\(^{37}\)

Remark: Using the identity

\[ \ln \psi = \frac{1}{2} \ln \left( \frac{\psi}{\psi^*} \right) + \ln |\psi|, \quad (F13) \]

we can rewrite Eq. (F4) as

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + V \psi - i\gamma \hbar \ln |\psi| \psi - i\frac{\gamma}{2} \hbar \ln \left( \frac{\psi}{\psi^*} \right) \psi. \quad (F14) \]

This is similar to Eq. (70) with the crucial difference that the coefficients in front of \( \ln |\psi| \) and \( \ln(\psi/\psi^*) \) are complex while they are real and imaginary in Eq. (70).

### Appendix G: Einstein-like relation

The quantum diffusion coefficient that appears in the theory of scale relativity, in particular in the Fokker-Planck-like equation (9), is related to the Planck constant by the Nelson relation (17). Combining Eqs. (17) and (71), we obtain the identity

\[ D = \frac{k_B T}{m \gamma_T}, \quad (G1) \]

which is similar to the Einstein relation in Brownian theory. This can be interpreted as a sort of fluctuation-dissipation theorem.

On the other hand, in the strong friction limit \( \xi \to +\infty \), the generalized Schrödinger equation (74) is equivalent, through the Madelung transformation, to the quantum Smoluchowski equation

\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi + \frac{\rho}{m} \nabla Q \right). \quad (G2) \]

It involves a classical diffusion coefficient

\[ D = \frac{k_B T}{m \xi} = \frac{k_B T}{m \gamma_R} \quad (G3) \]

which is given by the standard Einstein relation.

\(^{37}\) We note that the source terms in Eqs. (F3) and (F12) involve the Boltzmann entropy \( S_B = -k_B (\ln \rho) \).
The classical and quantum diffusion coefficients satisfy the relation

\[ \frac{D}{\mathcal{D}} = \frac{\gamma_I}{\gamma_R} \]  

(G4)

We note that they coincide \((D = \mathcal{D} \text{ or, equivalently, } \gamma_R = \gamma_I)\) when

\[ \frac{2k_B T}{\xi} = \hbar. \]  

(G5)

Appendix H: The energy operator and the Hamiltonian

1. The Schrödinger equation

The Hamiltonian of a classical particle moving in a potential \(\Phi\) is

\[ H = \frac{\hat{p}^2}{2m} + m\Phi. \]  

(H1)

Using the correspondence principle

\[ \hat{p} = -i\hbar \nabla \text{ and } \hat{H} = i\hbar \partial_t, \]  

(H2)

we obtain the Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \]  

(H3)

with the Hamiltonian operator

\[ \hat{H} = -\frac{\hbar^2}{2m} \Delta + m\Phi. \]  

(H4)

The average value of the Hamiltonian is

\[ H = \langle \psi | \hat{H} | \psi \rangle = -\frac{\hbar^2}{2m} \int \psi^* \Delta \psi d\mathbf{r} + m \int \psi^* \Phi \psi d\mathbf{r} = \frac{\hbar^2}{2m} \int |\nabla \psi|^2 d\mathbf{r} + m \int |\psi|^2 \Phi d\mathbf{r}. \]  

(H5)

This is the sum of the kinetic energy

\[ \Theta = \langle \psi | -\frac{\hbar^2}{2m} \Delta | \psi \rangle = \frac{\hbar^2}{2m} \int |\nabla \psi|^2 d\mathbf{r} \]  

(H6)

and the potential energy

\[ W = \langle \psi | m\Phi | \psi \rangle = m \int |\psi|^2 \Phi d\mathbf{r}. \]  

(H7)

Using the Madelung transformation, the kinetic energy \((\Theta)\) can be decomposed into the classical kinetic energy \((97)\) and the quantum kinetic energy \((98)\). The potential energy can be rewritten as in Eq. \((101)\). Using the results of Sec. \([\text{IV A}]\), we see that the average value of the Hamiltonian \((H1)\) coincides with the average energy given by Eq. \((46)\), namely

\[ H = \langle E \rangle. \]  

(H8)

Taking the first variations of the energy functional \((H5)\), we get

\[ \delta H = \int \left( -\frac{\hbar^2}{2m} \Delta \psi^* + m\Phi \psi^* \right) \delta \psi d\mathbf{r} + \text{c.c.} \]  

(H9)
The term in parenthesis coincides with the Hamiltonian operator applied on $\psi^\ast$. Therefore

$$\frac{\delta H}{\delta \psi^\ast} = \hat{H}\psi.$$  \hspace{1cm} (H10)

As a result, the Schrödinger equation can be rewritten as

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\delta H}{\delta \psi^\ast}.$$  \hspace{1cm} (H11)

This expression shows that $H$ represents the true Hamiltonian of the particle. Indeed, in terms of the wavefunction $\psi(r, t)$ and its canonical momentum $\pi(r, t) = i\hbar \psi^\ast(r, t)$, the Schrödinger equation is exactly reproduced by the Hamilton equations

$$\frac{\partial \psi}{\partial t} = \frac{\delta H}{\delta \pi}, \quad \frac{\partial \pi}{\partial t} = -\frac{\delta H}{\delta \psi}.$$  \hspace{1cm} (H12)

This formulation directly implies the conservation of the total energy $H$ since

$$\dot{H} = \int \frac{\delta H}{\delta \psi} \frac{\partial \psi}{\partial t} \, dr + \int \frac{\delta H}{\delta \pi} \frac{\partial \pi}{\partial t} \, dr = 0.$$  \hspace{1cm} (H13)

From general arguments, a minimum of the energy functional $H$ given by Eq. (H5) under the normalization condition is a stationary solution of the Schrödinger equation that is formally nonlinearly dynamically stable. Writing the variational principle as

$$\delta H - \mu \delta \int |\psi|^2 \, dr = 0,$$  \hspace{1cm} (H14)

where $\mu$ is a Lagrange multiplier (chemical potential), we obtain the time-independent Schrödinger equation with $\mu = E$. This shows that the chemical potential $\mu$ can be identified with the eigenenergy $E$. The variational principle was introduced by Schrödinger in his first paper on wave mechanics. This is actually how he derived the fundamental eigenvalue equation (see Appendix F of [5]).

2. The generalized Schrödinger equation

The generalized Schrödinger equation can be written as

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{E}\psi - i\frac{\hbar}{2\xi} \left[ \ln \left( \frac{\psi}{\psi^\ast} \right) - \left\langle \ln \left( \frac{\psi}{\psi^\ast} \right) \right\rangle \right] \psi,$$  \hspace{1cm} (H15)

where the energy operator is given by

$$\hat{E} = -\frac{\hbar^2}{2m} \Delta + m\Phi + 2k_B T \ln |\psi|.$$  \hspace{1cm} (H16)

Its average value is

$$\langle E \rangle = \frac{\hbar^2}{2m} \int |\nabla \psi|^2 \, dr + m \int |\psi|^2 \Phi \, dr + k_B T \int |\psi|^2 \ln |\psi|^2 \, dr.$$  \hspace{1cm} (H17)

Using the results of Sec. IV A, we see that the average value of the energy operator coincides with the average value of the energy given by Eq. (H10). It differs from the free energy which can be rewritten as

$$F = \frac{\hbar^2}{2m} \int |\nabla \psi|^2 \, dr + m \int |\psi|^2 \Phi \, dr + k_B T \int |\psi|^2 \ln |\psi|^2 - 1 \, dr.$$  \hspace{1cm} (H18)

We have

$$F = \langle E \rangle - k_B T.$$  \hspace{1cm} (H19)
From Eq. (H15), we easily obtain the identity

$$\dot{F} = -\xi \int \frac{\hbar^2}{4m} |\psi|^2 \left| \nabla \ln \left( \frac{\psi}{\psi^*} \right) \right|^2 \, dr,$$

(H20)

which coincides with the $H$-theorem (I13).

Remark: In the context of nonlinear Schrödinger equations [50], it can be shown that the true Hamiltonian $H$ of the system is $F$, not $\langle E \rangle$. This is because the nonlinear Schrödinger equations considered in [50], and the corresponding fluid equations, can be expressed in terms of the functional derivative of the free energy $F$ (see Sec. III.F and Appendix B of [50]). For the standard Schrödinger equation (21), we have $F = H = \langle E \rangle$. For the logarithmic Schrödinger equation (74), since $F$ and $\langle E \rangle$ only differ by a constant, we can also interpret $\langle E \rangle$ as the Hamiltonian of the system. However, this identification is not true anymore for other nonlinear Schrödinger equations [50]. In general, $F = H \neq \langle E \rangle$.

Appendix I: Exact solution of the generalized Schrödinger equations (74) and (138) with a harmonic potential

When the external potential is harmonic [see Eq. (I25)], leading to a force $-\nabla \Phi = -\omega^2 r$ depending linearly on the distance, we can obtain an exact analytical solution of the generalized Schrödinger equations (74) and (138) or of the equivalent hydrodynamic equations (76)-(79) and (139)-(141). We consider a wave function of the form

$$\psi(r,t) = \left[ \frac{2}{S_d \Gamma(d/2)(R(t))^d} \right]^{1/2} e^{-\frac{r^2}{2m\hbar}} e^{i m H(t) r^2 / 2\hbar} e^{i S_0(t) / \hbar},$$

(I1)

where $R(t)$ measures the size of the wave packet. Comparing Eq. (I1) with Eq. (29), we find that the density and action are given by

$$\rho(r,t) = \frac{2}{S_d \Gamma(d/2)(R(t))^d} e^{-\frac{r^2}{2m\hbar}}$$

and

$$S(r,t) = \frac{1}{2} m H(t) r^2 + S_0(t).$$

The velocity defined by Eq. (I1) is then given by

$$u(r,t) = H(t) r.$$  

(I3)

Using Eqs. (I2) and (I3), we easily obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = -\rho \left( \frac{\dot{R}}{R} - H \right) \left( d - 2 \frac{r^2}{R^2} \right),$$

(I4)

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = (\dot{H} + H^2) r,$$

(I5)

$$\frac{\nabla \rho}{\rho} = -\frac{2r}{R^2}, \quad \nabla Q = \frac{\hbar^2}{m} \frac{r}{R^4},$$

(I6)

$$E - \langle E \rangle = \left( \frac{1}{2} m H^2 + \frac{1}{2} m \omega_0^2 - \frac{\hbar^2}{2mR^4} \right) \left( r^2 - \frac{d}{2} R^2 \right),$$

(I7)

$$\nabla \left( \frac{u \cdot \nabla \rho}{\rho} \right) = -4H \frac{r}{R^2}, \quad \Delta u = 0.$$  

(I8)

We also have

$$I = \alpha MR^2, \quad \Theta_e = \frac{1}{2} \alpha M \left( \frac{dR}{dt} \right)^2, \quad \Theta_Q = \sigma \frac{\hbar^2 M}{m^2 R^2},$$

(I9)

$$U = -d \frac{k_B T}{m} M \ln R + C, \quad W = \frac{1}{2} \omega_0^2 \alpha MR^2,$$

(I10)

with the coefficients

$$\alpha = \frac{d}{2}, \quad \sigma = \frac{d}{4}, \quad C = \frac{k_B T}{m} M \left[ \ln \left( \frac{2M}{S_d \Gamma(d/2)} \right) - 1 - \alpha \right].$$

(I11)
1. The generalized Schrödinger equation (74)

Substituting Eq. (I4) into the continuity equation (76), we obtain the relation

$$H = \frac{\dot{R}}{R}. \quad (I12)$$

The function $H$ is similar to the Hubble parameter in cosmology, where $R$ plays the role of the scale factor. Using Eqs. (I3), (I5), (I6) and (I12), the quantum damped isothermal Euler equation (78) becomes

$$\ddot{R} + \dot{\xi} \dot{R} + \omega_0^2 R = \frac{2k_BT}{mR} + \frac{\hbar^2}{m^2R^3}. \quad (I13)$$

In the strong friction limit $\dot{\xi} \to +\infty$, corresponding to the quantum Smoluchowski equation (82), it reduces to

$$\dot{\xi} \dot{R} + \omega_0^2 R = \frac{2k_BT}{mR} + \frac{\hbar^2}{m^2R^3}. \quad (I14)$$

Equations (I13) and (I14) are closed ordinary differential equation determining $R(t)$. They are studied in our companion paper [89]. Together with Eq. (I1), they determine a particular exact solution of the generalized Schrödinger equation (74) with a harmonic potential. This solution exhibits damped oscillations about the equilibrium state because of the friction term $\dot{\xi}$.

The free energy (96) can be written as

$$F = \frac{1}{2} \alpha M \left(\frac{dR}{dt}\right)^2 + V(R), \quad (I15)$$

where the first term is the classical kinetic energy and the second term is an effective potential energy

$$V(R) = \sigma \frac{\hbar^2 M}{2m^2R^2} + \frac{1}{2} \omega_0^2 \alpha MR^2 - d \frac{Mk_BT}{m} \ln R + C \quad (I16)$$

including the quantum kinetic energy, the potential energy, and the internal energy. The virial theorem (128) becomes

$$\alpha M \frac{d^2R}{dt^2} + \xi \alpha M \frac{dR}{dt} = - \frac{dV}{dR}, \quad (I17)$$

which is equivalent to Eq. (I13). The $H$-theorem (113) takes the form

$$\frac{dF}{dt} = -\xi \alpha M \left(\frac{dR}{dt}\right)^2 \leq 0. \quad (I18)$$

It can also be obtained from Eqs. (I15) and (I17). In the strong friction limit $\dot{\xi} \to +\infty$, where $F = V$, Eqs. (I17) and (I18) reduce to

$$\xi \alpha M \frac{dR}{dt} = - \frac{dV}{dR} \quad \text{and} \quad \frac{dF}{dt} = - \frac{1}{\xi \alpha M} \left(\frac{dV}{dR}\right)^2 \leq 0. \quad (I19)$$

They can also be obtained from Eqs. (129) and (114).

Remark: When $T < 0$ and $\omega_0 = 0$, the equilibrium state of the generalized Schrödinger equation (74) is the gausson. Its radius, given by Eq. (I13), is $R = h/\sqrt{2mk_BT} = h/\sqrt{2mb}$.

2. The generalized Schrödinger equation (138)

Substituting Eqs. (I4) and (I7) into the generalized continuity equation (139), we obtain

$$\frac{m\gamma R^2}{2\hbar} \dot{H}^2 - \dot{H} + \frac{\gamma m\omega_0^2 R^2}{2\hbar} - \frac{\gamma h}{2mR^2} + \frac{\dot{R}}{R} = 0. \quad (I20)$$
Substituting Eqs. (15), (16) and (18) into the quantum Navier-Stokes equation (141), we get

\[ \dot{H} + H^2 = -\omega_0^2 + \frac{\hbar^2}{m^2 R^4} - \frac{2\gamma \hbar H}{m R^2}. \]  

Equation (120) is a second degree equation giving \( H \) as a function of \( \dot{R} \) and \( R \). When \( H(R, \dot{R}) \) is inserted into Eq. (121) we obtain a closed differential equation determining \( R(t) \). It is studied in our companion paper [89]. Together with Eq. (11), it determines a particular exact solution of the generalized Schrödinger equation (138) with a harmonic potential.

For \( \gamma \ll 1 \), we can expand Eq. (120) to first order in \( \gamma \), and obtain

\[ H = \frac{\dot{R}}{R} + \frac{\gamma m}{2\hbar} \omega_0^2 R^2 - \frac{\gamma \hbar}{2m R} + \frac{m \gamma \dot{R} R}{2\hbar}. \]  

Taking its time derivative and combining the result with Eq. (121) we obtain

\[ \ddot{R} + \frac{3\gamma \hbar}{m R^2} \dot{R} + \frac{\gamma m}{\hbar} \omega_0^2 R^2 \dot{R} + \frac{m \gamma \dot{R} R^2}{2\hbar} + \frac{m \gamma R \dot{R}^2}{2\hbar} + \omega_0^2 R = \frac{\hbar^2}{m^2 R^3}. \]  

If we neglect the term in the r.h.s. of Eq. (139), accounting for the fact that \( \gamma \ll 1 \), it reduces to the ordinary continuity equation (76) and Eq. (120) reduces to Eq. (112). Substituting this relation into Eq. (121), we obtain the simpler differential equation

\[ \ddot{R} + \frac{2\gamma \hbar}{m R} \dot{R} + \omega_0^2 R = \frac{\hbar^2}{m^2 R^3}. \]  

The solution of this equation exhibits damped oscillations about the equilibrium state because of the friction term \( \gamma \). We stress that the damping is not due to the viscous term \( \gamma \), but to the term \( \gamma \hbar/2m \nabla (u \cdot \nabla \rho) \).

In the case \( \gamma \gg 1 \), we have \( u \approx 0 \) and \( H \approx 0 \) (see the Remark in Sec. VII), and Eq. (120) reduces to

\[ \dot{R} = \frac{\gamma \hbar}{2mR} - \frac{\gamma m \omega_0^2 R^3}{2\hbar}. \]  

We note Eq. (125) does not correspond to the strong friction limit of Eq. (121) which has been established under the assumption \( \gamma \ll 1 \).

Appendix J: Dispersion relation for the generalized Schrödinger equations (74) and (138) without external potential

1. The generalized Schrödinger equation (74)

We consider the generalized Schrödinger equation (74) without external potential (\( \Phi = 0 \)). We use its hydrodynamic representation (76) - (79). We consider a homogeneous equilibrium state with \( \rho(r, t) = \rho \), \( u(r, t) = 0 \) and \( S(r, t) = -Et \) with \( E = k_B T \ln \rho \). Considering a small perturbation about this equilibrium state and writing the linearized hydrodynamic equations, we find after simple calculations that the equation for the density contrast \( \delta(r, t) = \delta \rho(r, t)/\rho \) writes

\[ \frac{\partial^2 \delta}{\partial t^2} + \xi \frac{\partial \delta}{\partial t} = - \frac{\hbar^2}{4m^2} \Delta^2 \delta + c_s^2 \Delta \delta, \]  

where \( c_s^2 = P'(\rho) = k_B T/m \) is the square of the speed of sound.\(^{38}\) Considering plane waves of the form \( \delta \propto e^{i(k \cdot r - \omega t)} \), we obtain the dispersion relation

\[ \omega^2 + i \xi \omega = \frac{\hbar^2 k^4}{4m^2} + c_s^2 k^2 \quad \Rightarrow \quad \omega = -i \frac{\xi}{2} \pm \sqrt{ - \frac{\xi^2}{4} + c_s^2 k^2 + \frac{\hbar^2 k^4}{4m^2} }. \]  

\(^{38}\) We use the notation \( c_s^2 \) instead of \( k_B T/m \) so that our results can be immediately generalized to the class of nonlinear Schrödinger equations considered in [54].
This dispersion relation is studied in detail in our companion paper \cite{89}. When $c_s^2 > 0$, it involves the characteristic wavenumber

$$k_s^2 = \frac{2m^2c_s^2}{\hbar^2} + \sqrt{\frac{4m^4c_s^4}{\hbar^4} + \frac{m^2c_s^2}{\hbar^2}} \quad \text{(J3)}$$

that separates the regime where the perturbations are purely damped ($k < k_s$) from the regime where the perturbations have damped oscillations ($k > k_s$). When $c_s^2 < 0$, there is an additional critical wavenumber

$$k_c^2 = \frac{4m^2|c_s^2|}{\hbar^2} \quad \text{(J4)}$$

When $k < k_s$ the perturbations grow exponentially rapidly, when $k_c < k < k_s$ they are purely damped, and when $k > k_s$ they have damped oscillations. We maximum growth rate and the most unstable wavenumber are

$$\gamma_{\text{max}} = -\frac{\xi}{2} + \sqrt{\frac{\xi^2}{4} + \frac{c_s^4m^2}{\hbar^4}}, \quad k_m^2 = \frac{2m^2|c_s^2|}{\hbar^2} \quad \text{(J5)}$$

Remark: For $\xi = 0$, we recover the results obtained in Sec. V of \cite{74} in the nongravitational case. For the generalized Schrödinger equation \cite{136} for which $c_s^2 = P'(\rho) = k_BT/m$, the critical length $\lambda_c = \pi\hbar/\sqrt{mk_BT} = \pi\hbar/(mb)^{1/2}$ obtained in the linear regime of the dynamics [see Eq. \cite{144}] coincides with the radius $R = \hbar/(2mb)^{1/2}$ of the gaussian obtained in the strongly nonlinear (clustered) regime of the dynamics (see Sec. VI.B and Appendix I).

2. The generalized Schrödinger equation \cite{138}

We consider the generalized Schrödinger equation \cite{138} without external potential ($\Phi = 0$). We use its hydrodynamic representation \cite{139}, \cite{141}. We consider a homogeneous equilibrium state with $\rho(\mathbf{r}, t) = \rho$, $\mathbf{u}(\mathbf{r}, t) = \mathbf{0}$ and $S(\mathbf{r}, t) = 0$. Considering a small perturbation about this equilibrium state and writing the linearized hydrodynamic equations, we find after simple calculations that the equation for the density contrast $\delta(\mathbf{r}, t) = \delta\rho(\mathbf{r}, t)/\rho$ writes

$$\frac{\partial^2 \delta}{\partial t^2} - 2\nu \frac{\partial \delta}{\partial t} = - \left( \frac{\hbar^2}{4m^2} + \nu^2 \right) \Delta^2 \delta \quad \text{(J6)}$$

where $\nu$ is the quantum viscosity defined by Eq. \cite{142}. Considering plane waves of the form $\delta \propto e^{i(k \cdot r - \omega t)}$, we obtain the dispersion relation

$$\omega^2 + 2i\nu k^2 \omega = \left( \frac{\hbar^2}{4m^2} + \nu^2 \right) k^4 \quad \Rightarrow \quad \omega = \pm \frac{\hbar k^2}{2m} - i\nu k^2. \quad \text{(J7)}$$

The perturbation oscillates with a pulsation $\Omega = \hbar k^2/2m$ and is damped with a damping rate $\gamma = -\nu k^2$.

Appendix K: Uncertainty principle

In this Appendix, we present a simple derivation of the Heisenberg uncertainty principle. According to the correspondence principle \cite{142}, the mean square value of the impulse is

$$\langle \mathbf{p}^2 \rangle = \langle \psi | \mathbf{p}^2 | \psi \rangle = -\hbar^2 \int \psi^* \Delta \psi \, d\mathbf{r} = \hbar^2 \int |\nabla \psi|^2 \, d\mathbf{r}. \quad \text{(K1)}$$

Using Eqs. \cite{141}, \cite{127} and \cite{140}, we obtain the identities

$$\langle \mathbf{p}^2 \rangle = 2m\Theta = \langle |\mathbf{P}|^2 \rangle = m^2\langle |\mathbf{U}|^2 \rangle = m^2\langle \mathbf{u}^2 \rangle + m^2\langle \mathbf{u}_Q^2 \rangle. \quad \text{(K2)}$$

Therefore, the mean square value of the impulse is equal to the mean square value of the modulus of the complex impulse, which is the sum of the mean square value of the classical impulse and the mean square value of the quantum impulse. This shows that the mean square value of the impulse is always larger than the mean square value of the
quantum impulse (they become equal in a steady state where \( u = 0 \)). For the quantum harmonic oscillator, using the exact Gaussian solution of Appendix I, we have

\[
\langle u_Q^2 \rangle = \frac{2\Theta Q}{m} = \frac{d}{2m^2 R^2} \quad \text{and} \quad \langle r^2 \rangle = \frac{I}{m} = \frac{d}{2R^2}.
\]  

(K3)

This implies

\[
m^2 \langle u_Q^2 \rangle \langle r^2 \rangle = \frac{d^2}{4}\hbar^2.
\]  

(K4)

From Eqs. (K2) and (K4), we obtain the inequality

\[
\langle (\Delta p_i)^2 \rangle \langle (\Delta x_j)^2 \rangle \geq \frac{1}{4} \hbar^2 \delta_{ij},
\]  

(K5)

corresponding to the Heisenberg uncertainty principle.

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