A NONSMOOTH VAN DER POL-DUFFING OSCILLATOR (II):
THE SUM OF INDICES OF EQUILIBRIA IS 1

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Abstract. We continue to study the nonsmooth van der Pol-Duffing oscillator
\[ \dot{x} = y, \quad \dot{y} = a_1 x + a_2 x^3 + b_1 y + b_2 |x| y, \]
where \(a_i, b_i\) are real and \(a_2 b_2 \neq 0\), \(i = 1, 2\). Notice that the sum of indices of equilibria is \(-1\) for \(a_2 > 0\) and \(1\) for \(a_2 < 0\). When \(a_2 > 0\), the nonsmooth van der Pol-Duffing oscillator has been studied completely in the companion paper. Attention goes to the bifurcation diagram and all global phase portraits in the Poincaré disc of the nonsmooth van der Pol-Duffing oscillator for \(a_2 < 0\) in this paper. The bifurcation diagram is more complex, which includes two Hopf bifurcation surfaces, one pitchfork bifurcation surface, one homoclinic bifurcation surface, one double limit cycle bifurcation surface and one bifurcation surface for equilibria at infinity. When \(b_2 > 0\) is fixed, this nonsmooth van der Pol-Duffing oscillator cannot be changed into a near-Hamiltonian system for small \(a_1, b_1\). Moreover, the global dynamics of the nonsmooth van der Pol-Duffing oscillator and the van der Pol-Duffing oscillator are different.

1. Introduction and main results. By the scaling \((x, y) \rightarrow ([a_2]^{-1/2} x, [a_2]^{-1/2} y)\), the nonsmooth van der Pol-Duffing oscillator
\[ \dot{x} = y, \quad \dot{y} = a_1 x + a_2 x^3 + b_1 y + b_2 |x| y \]
with \(a_2 b_2 \neq 0\) can be changed into

the sum of indices of equilibria is 1:
\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= \mu_1 x + x^3 + \mu_2 y - \mu_3 |x| y,
\end{align*} \tag{1.1a} \]

the sum of indices of equilibria is \(-1\):
\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= \mu_1 x - x^3 + \mu_2 y - \mu_3 |x| y,
\end{align*} \tag{1.1b} \]
where \(\mu_1 := a_1, \mu_2 := b_1\) are real and \(\mu_3 := -b_2 [a_2]^{-1/2}\) is non-zero (see the companion paper [12]). Moreover, the oscillator can appear in ship motions and rolling motions, also see [1, 6, 9] and the references therein. It is to note that the bifurcation
diagram and all global phase portraits in the Poincaré disc of system (1.1a) have been studied completely in [12]. However, the bifurcation diagram and all global phase portraits in the Poincaré disc of system (1.1b) are still unknown. Similar to system (1.1a), we only need to discuss \( \mu_3 > 0 \) since system (1.1b) is also invariant under the transformation \((x, y, t, \mu_1, \mu_2, \mu_3) \rightarrow (x, -y, -t, \mu_1, -\mu_2, -\mu_3)\). Furthermore, we will show that the global dynamics of system (1.1b) is more complex than the global dynamics of system (1.1a). In other words, they are motivations of this paper.

The following three theorems are our main results.

**Theorem 1.1.** The global bifurcation diagram of system (1.1b) consists of the following bifurcation surfaces:

(a): pitchfork bifurcation surface
\[ P := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 = 0\}; \]

(b): Hopf bifurcation surfaces
\[ H_1 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 > 0, \mu_2 = \mu_3 \sqrt{\mu_1}\}; \]
\[ H_2 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 \leq 0, \mu_2 = 0\}; \]

(c): homoclinic bifurcation surface
\[ H_L := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 > 0, \mu_2 = \varphi_1(\mu_3) \sqrt{\mu_1}\}; \]

(d): double large limit cycle bifurcation surface
\[ DL := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 > 0, \mu_2 = \varphi_2(\mu_3) \sqrt{\mu_1}\}; \]

(e): equilibrium at infinity bifurcation surface
\[ EB := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 = 2\sqrt{\mu_2}\}, \]

where \( \varphi_1(\mu_3), \varphi_2(\mu_3) \) for \( \mu_3 > 0 \) are increasing functions and \( \mu_3/2 < \varphi_2(\mu_3) < \varphi_1(\mu_3) < \mu_3 \).

Notice that the pitchfork bifurcation surface \( P \) intersects the Hopf bifurcation surface \( H_2 \) on the \( \mu_3 \)-axis and \( P \) is divided into two sub-surfaces
\[ P_1 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 = 0, \mu_2 > 0\}, \]
\[ P_2 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 = 0, \mu_2 \leq 0\}. \]

From the following two theorems, the global phase portraits in \( P_1 \) and \( P_2 \) are different. On the one hand, since \((\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+ \) is three-dimensional, the bifurcation diagram is not straightforward. On the other hand, by Theorem 1.1, there are only two kinds of dynamics of orbits near infinity in the slices perpendicular to the \( \mu_3 \)-axis. Therefore, for simplicity, we only give the slices of fixed \( \mu_3 \) of the global bifurcation diagram and they can be divided into two subcases \( 0 < \mu_3 < 2\sqrt{2} \) and \( \mu_3 \geq 2\sqrt{2} \).

**Theorem 1.2.** When \( 0 < \mu_3 < 2\sqrt{2} \), the slice \( \mu_3 = \mu_3^{(1)} \) of the global bifurcation diagram and global phase portraits in the Poincaré disc of (1.1b) are given in Figure 1, where \( 0 < \mu_3^{(1)} < 2\sqrt{2} \) is any fixed value,

\[ G_1 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 < 0, \mu_2 > 0, \mu_3 < 2\sqrt{2}\}, \]
\[ G_2 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 < 0, \mu_2 < 0, \mu_3 < 2\sqrt{2}\}, \]
\[ G_3 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 > 0, \mu_2 < \varphi_2(\mu_3) \sqrt{\mu_1}, \mu_3 < 2\sqrt{2}\}, \]
\[ G_4 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 > 0, \varphi_2(\mu_3) \sqrt{\mu_1} < \mu_2 < \varphi_1(\mu_3) \sqrt{\mu_1}, \mu_3 < 2\sqrt{2}\}, \]
\[ G_5 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 > 0, \varphi_1(\mu_3) \sqrt{\mu_1} < \mu_2 < \varphi_3(\mu_3) \sqrt{\mu_1}, \mu_3 < 2\sqrt{2}\}, \]
\[ G_6 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+: \mu_1 > 0, \mu_2 > \varphi_3(\mu_3) \sqrt{\mu_1}, \mu_3 < 2\sqrt{2}\}. \]
Theorem 1.3. When $\mu_3 \geq 2\sqrt{2}$, the slice $\mu_3 = \mu_3^{(2)}$ of the global bifurcation diagram and global phase portraits in the Poincaré disc of (1.1b) are given in Figure 2, where $\mu_3^{(2)} \geq 2\sqrt{2}$ is any fixed value,

- $S_1 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : \mu_1 < 0, \mu_2 > 0, \mu_3 \geq 2\sqrt{2}\}$,
- $S_2 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : \mu_1 > 0, \mu_2 < \mu_3 \sqrt{\mu_1}, \mu_3 \geq 2\sqrt{2}\}$,
- $S_3 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : \mu_1 > 0, \mu_2 < g_2(\mu_3) \sqrt{\mu_1}, \mu_3 \geq 2\sqrt{2}\}$,
- $S_4 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : \mu_1 > 0, g_2(\mu_3) \sqrt{\mu_1} < \mu_2 < g_1(\mu_3) \sqrt{\mu_1}, \mu_3 \geq 2\sqrt{2}\}$,
- $S_5 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : \mu_1 > 0, g_1(\mu_3) \sqrt{\mu_1} < \mu_2 < \mu_3 \sqrt{\mu_1}, \mu_3 \geq 2\sqrt{2}\}$,
- $S_6 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : \mu_1 > 0, \mu_2 > \mu_3 \sqrt{\mu_1}, \mu_3 \geq 2\sqrt{2}\}$.

Remark: It is to note that the global phase portrait in $S_3$ is incomplete because there are two possible connections between the stable manifolds of the origin and the equilibria at infinity: the $\alpha$-limit set of the stable manifold on the right-hand side of the saddle at the origin is equilibrium at infinity on the positive $y$-axis or the negative $y$-axis, as shown in Figure 3. However, we cannot obtain the number of the bifurcation surfaces when these parameters cross such bifurcation surfaces such that the global phase portrait in the Poincaré disk (see Figure 3(a)) is changed into the global phase portrait in the Poincaré disk (see Figure 3(b)).

The paper is organized as follows. In section 2, the qualitative properties and the bifurcations of equilibria (including equilibria at infinity) of system (1.1b) are obtained. In section 3, we study the number of limit cycles, homoclinic bifurcations.
Figure 2. The slice $\mu_3 = \mu_3^{(2)} \geq 2\sqrt{2}$ of the bifurcation diagram and corresponding global phase portraits.

Figure 3. Two possibilities of connections in $S_3$.

and double limit cycle bifurcations of system (1.1b). We give the proofs of Theorems 1.1-1.3 in section 4 and numerical examples to illustrate our theoretical results in section 5. Some concluding remarks of this paper are presented in section 6.

2. Local dynamics of system (1.1b). In this section, we investigate the qualitative properties and bifurcations for equilibria of system (1.1b). First, the qualitative properties of all possible finite equilibria of system (1.1b) are given in the following two lemmas.

Lemma 2.1. When $\mu_1 > 0$, system (1.1b) exhibits three equilibria $E_0 : (0, 0)$, $E_l : (-\sqrt{\mu_1}, 0)$, $E_r : (\sqrt{\mu_1}, 0)$ and the properties of them are shown in Table 1.
### Table 1. Properties of $E_0$, $E_i$ and $E_r$.  

| possibilities of $(\mu_1, \mu_2)$ | types and stabilities |
|-----------------------------------|-----------------------|
| $\mu_1 > 0$, $\mu_2 < (\mu_3 - 2\sqrt{2})\sqrt{\mu_1}$ | $E_0$ saddle; $E_i$, $E_r$ stable bidirectional nodes |
| $\mu_1 > 0$, $\mu_2 = (\mu_3 - 2\sqrt{2})\sqrt{\mu_1}$ | $E_0$ saddle; $E_i$, $E_r$ stable unidirectional nodes |
| $\mu_1 > 0$, $(\mu_3 - 2\sqrt{2})\sqrt{\mu_1} < \mu_2 < \mu_3\sqrt{\mu_1}$ | $E_0$ saddle; $E_i$, $E_r$ stable rough foci |
| $\mu_1 > 0$, $\mu_2 = \mu_3\sqrt{\mu_1}$ | $E_0$ saddle; $E_i$, $E_r$ unstable weak foci |
| $\mu_1 > 0$, $\mu_3\sqrt{\mu_1} < \mu_2 < (\mu_3 + 2\sqrt{2})\sqrt{\mu_1}$ | $E_0$ saddle; $E_i$, $E_r$ unstable rough foci |
| $\mu_1 > 0$, $\mu_2 = (\mu_3 + 2\sqrt{2})\sqrt{\mu_1}$ | $E_0$ saddle; $E_i$, $E_r$ unstable unidirectional nodes |
| $\mu_1 > 0$, $\mu_2 > (\mu_3 + 2\sqrt{2})\sqrt{\mu_1}$ | $E_0$ saddle; $E_i$, $E_r$ unstable bidirectional nodes |

*Proof.* Solving $\dot{x} = \dot{y} = 0$ for system (1.1b), it is easy to see that it has three equilibria $E_0, E_i, E_r$ when $\mu_1 > 0$. Notice that the Jacobian matrix $J_0$ for system (1.1b) at $E_0$ is

$$J_0 = \begin{pmatrix} 0 & 1 \\ \mu_1 & \mu_2 \end{pmatrix} \quad (2.1)$$

and $\det J_0 = -\mu_1 < 0$. Although the right hand side of the second equation in system (1.1b) is not continuously differentiable at $E_0$, [13, Theorem 4.4 of Chapter 4] is also available, which implies that $E_0$ is a saddle. The reason is that the exceptional directions of $E_0$ are not along the $y$-axis, where the continuous differentiability is lost.

The Jacobian matrices for system (1.1b) at $E_i$ and $E_r$ are both

$$J_i = J_r = \begin{pmatrix} 0 & 1 \\ -2\mu_1 & \mu_2 - \mu_3\sqrt{\mu_1} \end{pmatrix}.\quad (2.1)$$

It follows that $\det J_i = \det J_r = 2\mu_1 > 0$ and $\text{tr} J_i = \text{tr} J_r = \mu_2 - \mu_3\sqrt{\mu_1}$. Then

$$\Delta = (\mu_2 - (\mu_3 + 2\sqrt{2})\sqrt{\mu_1})(\mu_2 - (\mu_3 - 2\sqrt{2})\sqrt{\mu_1}).$$

Thus, $E_i$ and $E_r$ are stable (resp. unstable) bidirectional nodes if $\mu_2 < (\mu_3 - 2\sqrt{2})\sqrt{\mu_1}$ (resp. $\mu_2 > (\mu_3 + 2\sqrt{2})\sqrt{\mu_1}$), stable (resp. unstable) unidirectional nodes if $\mu_2 = (\mu_3 - 2\sqrt{2})\sqrt{\mu_1}$ (resp. $\mu_2 = (\mu_3 + 2\sqrt{2})\sqrt{\mu_1}$), stable (resp. unstable) rough foci if $\mu_2 > (\mu_3 - 2\sqrt{2})\sqrt{\mu_1} < (\mu_3 + 2\sqrt{2})\sqrt{\mu_1}$ (resp. $\mu_3\sqrt{\mu_1} < \mu_2 < (\mu_3 + 2\sqrt{2})\sqrt{\mu_1}$).

Consider the case $\mu_2 = \mu_3\sqrt{\mu_1}$. Since system (1.1b) is invariant under the transformation $(x, y) \rightarrow (-x, -y)$, it is symmetric about $E_0$. So it is enough to consider qualitative properties of $E_r$. By the transformation $(x, y) \rightarrow (x + \sqrt{\mu_1}, y)$, which translates the equilibrium $E_r$ to the origin, system (1.1b) can be reduced into

$$\dot{x} = y, \quad \dot{y} = -2\mu_1 x + \mu_3(\sqrt{\mu_1} - |x + \sqrt{\mu_1}|)y - 3\sqrt{\mu_1}x^2 - x^3. \quad (2.2)$$

Notice that $|x + \sqrt{\mu_1}| = x + \sqrt{\mu_1}$ for small $|x|$. Then in any small neighbourhood of the origin of system (2.2), system (2.2) can be simplified into

$$\dot{x} = y, \quad \dot{y} = -2\mu_1 x - 3\sqrt{\mu_1}x^2 - \mu_3 xy - x^3. \quad (2.3)$$

Applying the scaling

$$(x, y, t) \rightarrow (x/(2\mu_1), y/\sqrt{2\mu_1}, -t/\sqrt{2\mu_1}), \quad (2.4)$$
system (2.3) can be written as
\[ \dot{x} = -y, \quad \dot{y} = x + \frac{3x^2}{4\mu_1\sqrt{\mu_1}} + \frac{\sqrt{2}\mu_3 y}{4\mu_1\sqrt{\mu_1}} + \frac{x^3}{8\mu_2^3}. \]  
(2.5)

One can compute the first focal value \( \mu = -3\sqrt{2}\mu_3\pi/(64\mu_1^2) \) < 0. Therefore, when \( \mu_2 = \mu_3\sqrt{\mu_1} \) the origin of (2.5) is a stable weak focus of order one. Since the orbits of system (1.1b) are reversed over time during the transformation (2.4), \( E_r \) is an unstable weak focus when \( \mu_2 = \mu_3\sqrt{\mu_1} \). So is \( E_l \).

Lemma 2.2. When \( \mu_1 \leq 0 \), there is a unique equilibrium \( E_0 : (0, 0) \) for system (1.1b) and the properties of \( E_0 \) are shown in Table 2.

| possibilities of \((\mu_1, \mu_2)\) | types and stabilities |
|----------------|-----------------------|
| \( \mu_1 = 0 \) | \( \mu_2 < 0 \) \( E_0 \) stable degenerate node |
| \( \mu_1 = 0 \) | \( \mu_2 = 0 \) \( E_0 \) stable nilpotent focus |
| \( \mu_1 = 0 \) | \( \mu_2 > 0 \) \( E_0 \) unstable degenerate node |
| \( \mu_1 < 0 \) | \( \mu_2 = -2\sqrt{-\mu_1} \) \( E_0 \) stable bidirectional node |
| \( \mu_1 < 0 \) | \( \mu_2 = -2\sqrt{-\mu_1} \) \( E_0 \) stable unidirectional node |
| \( \mu_1 < 0 \) | \( -2\sqrt{-\mu_1} < \mu_2 < 0 \) \( E_0 \) stable rough focus |
| \( \mu_1 < 0 \) | \( \mu_2 = 0 \) \( E_0 \) stable weak focus |
| \( \mu_1 < 0 \) | \( 0 < \mu_2 < 2\sqrt{-\mu_1} \) \( E_0 \) unstable rough focus |
| \( \mu_1 < 0 \) | \( \mu_2 = 2\sqrt{-\mu_1} \) \( E_0 \) unstable unidirectional node |
| \( \mu_1 < 0 \) | \( \mu_2 > 2\sqrt{-\mu_1} \) \( E_0 \) unstable bidirectional node |

Table 2. Properties of \( E_0 \).

Proof. Solving \( \dot{x} = \dot{y} = 0 \) for system (1.1b), it is easy to see that \( E_0 \) is the unique equilibrium when \( \mu_1 \leq 0 \). The Jacobian matrix for system (1.1b) at \( E_0 \) is still (2.1), which implies that \( \det J_0 = -\mu_1 \) and \( \tr J_0 = \mu_2 \). Moreover, \( \Delta = \mu_2^2 + 4\mu_1 \).

When \( \mu_1 < 0 \) and \( \mu_2 \neq 0 \), \( E_0 \) is a stable (resp. unstable) bidirectional node if \( \mu_2 < -2\sqrt{-\mu_1} \) (resp. \( \mu_2 > 2\sqrt{-\mu_1} \)), a stable (resp. unstable) unidirectional node if \( \mu_2 = -2\sqrt{-\mu_1} \) (resp. \( \mu_2 = 2\sqrt{-\mu_1} \)), a stable (resp. unstable) rough focus if \( -2\sqrt{-\mu_1} \leq \mu_2 < 0 \) (resp. \( 0 < \mu_2 < 2\sqrt{-\mu_1} \)).

When \( \mu_1 < 0 \) and \( \mu_2 = 0 \), by the transformation
\[ (x, y, t) \rightarrow (-x/\mu_1, y/\sqrt{-\mu_1}, t/\sqrt{-\mu_1}), \]  
(2.6)
system (1.1b) can be reduced into
\[ \dot{x} = -y, \quad \dot{y} = x - \frac{\mu_3|x|y}{\mu_1\sqrt{-\mu_1}} - \frac{x^3}{\mu_2^3}. \]  
(2.7)

We can not calculate focal values of system (2.7) directly because it is only \( C^1 \) at any point on the y-axis. Taking polar coordinates \( x = r \cos \theta \) and \( y = r \sin \theta \), we have
\[ \frac{dr}{d\theta} = \frac{-\mu_3 \sin^2 \theta |\cos \theta| r^2 / (\mu_1 \sqrt{-\mu_1}) - \sin \theta \cos^3 \theta r^3 / \mu_2^3}{1 - \mu_3 \sin \theta \cos \theta |\cos \theta| r / (\mu_1 \sqrt{-\mu_1}) - \cos^4 \theta r^2 / \mu_2^4}. \]  
(2.8)

Therefore,
\[ \frac{dr}{d\theta} = \frac{\mu_3 \sin^2 \theta \cos \theta r^2 / (\mu_1 \sqrt{-\mu_1}) - \sin \theta \cos^3 \theta r^3 / \mu_2^3}{1 + \mu_3 \sin \theta \cos \theta |\cos \theta| r / (\mu_1 \sqrt{-\mu_1}) - \cos^4 \theta r^2 / \mu_2^4}, \]  
if \( \pi/2 \leq \theta \leq 3\pi/2 \)  
(2.9)
and
\[
\frac{dr}{d\theta} = \frac{-\mu_3 \sin^2 \theta \cos \theta r^2 / (\mu_1 \sqrt{-\mu_1}) - \sin \theta \cos^3 \theta r^3 / \mu_1^3}{1 - \mu_3 \sin \theta \cos^2 \theta r / (\mu_1 \sqrt{-\mu_1}) - \cos^4 \theta r^2 / \mu_1^3}, \text{ if } -\pi/2 \leq \theta \leq \pi/2. \quad (2.10)
\]

Consider the solution \( r(\theta, r_0) \) of (2.8) with the initial condition \( r(\pi/2, r_0) = r_0 \), where \( r_0 > 0 \) is sufficiently small. Obviously, \( r(\theta, r_0) \) can be written as \( r^+(\theta, r_0) = r_0 + \sum_{i=2}^{\infty} r_i^+(\theta) r_0^i \) when \( \pi/2 \leq \theta \leq 3\pi/2 \). Substituting it into (2.9), one can obtain that
\[
r^+(3\pi/2, r_0) - r_0 = -\frac{2\mu_3}{3\mu_1 \sqrt{-\mu_1}} r_0^2 + O(r_0^3).
\]

Similarly, \( r(\theta, r_0) \) can be written as \( r^-(\theta, r_0) = r_0 + \sum_{i=2}^{\infty} r_i^-(\theta) r_0^i \) when \( -\pi/2 \leq \theta \leq \pi/2 \). By substituting it into (2.10), one can obtain that
\[
r^-(\pi/2, r_0) - r_0 = \frac{2\mu_3}{3\mu_1 \sqrt{-\mu_1}} r_0^2 + O(r_0^3).
\]

Then
\[
r^+(3\pi/2, r_0) - r^-(\pi/2, r_0) = -\frac{4\mu_3}{3\mu_1 \sqrt{-\mu_1}} r_0^2 + O(r_0^3),
\]

which implies that the origin is an unstable weak focus of system (2.7). Since the orbits of system (1.1b) are reversed over time during the transformation (2.6), \( E_0 \) is a stable weak focus when \( \mu_1 < 0 \) and \( \mu_2 = 0 \).

When \( \mu_1 = 0 \), the origin of system (1.1b) is degenerate. Since system (1.1b) is symmetric about \( E_0 \), it is enough to consider
\[
\dot{x} = y, \quad \dot{y} = \mu_2 y - \mu_3 x y - x^3. \quad (2.11)
\]

When \( \mu_1 = 0 \) and \( \mu_2 \neq 0 \), by the transformation \((x, y, t) \rightarrow (x + y/\mu_2, y, t/\mu_2)\), system (2.11) can be rewritten as
\[
\begin{align*}
\dot{x} &= \frac{\mu_3 x y}{\mu_2^2} + \frac{\mu_3 y^2}{\mu_2^2} + \frac{x^3}{\mu_2^3} + \frac{3x^2 y}{\mu_2^3} + \frac{3xy^2}{\mu_2^3} + \frac{y^3}{\mu_2^3} := M(x, y), \\
\dot{y} &= y - \frac{\mu_3 x y}{\mu_2} - \frac{\mu_3 y^2}{\mu_2^2} - \frac{x^3}{\mu_2} - \frac{3x^2 y}{\mu_2} - \frac{3xy^2}{\mu_2} - \frac{y^3}{\mu_2} := N(x, y). \quad (2.12)
\end{align*}
\]

By the implicit function theorem, there exists a unique function \( \phi(x) \) such that \( N(x, \phi(x)) = 0 \). Then one can calculate \( \phi(x) = x^3/\mu_2 + O(x^4) \). Substituting \( y = \phi(x) \) into \( M(x, y) \), we get \( M(x, \phi(x)) = x^3/\mu_2^3 + O(x^4) \). Thus, the origin of system (2.12) is an unstable degenerate node by \cite{13}. Theorem 7.1 of Chapter 2]. Notice that the orbits of system (2.11) are reversed over time during the transformation \((x, y, t) \rightarrow (x + y/\mu_2, y, t/\mu_2)\) when \( \mu_2 < 0 \). Thus, \( E_0 \) is a degenerate node of system (1.1b) when \( \mu_1 = 0 \) and \( \mu_2 \neq 0 \). Moreover, it is unstable when \( \mu_2 > 0 \) and stable when \( \mu_2 < 0 \).

When \( \mu_1 = 0 \) and \( \mu_2 = 0 \), the origin of system (2.11) is nilpotent. By \cite{13}, Theorem 7.4 of Chapter 2], the origin of system (2.11) is a center or focus. Then \( E_0 \) is a center or focus of system (1.1b). Assume that \( AB \) is the right half part of an orbit near \( E_0 \), and \( A \) (resp. \( B \)) is its intersection point with the positive (resp. negative) \( y \)-axis, as shown in Figure 4. Let \( E_0(x, y) = x^4/4 + y^2/2 \). Then
\[
\frac{dE_0}{dt} \big|_{(1.1b)} = -\mu_3 |x| y^2 \leq 0.
\]

One can get
\[
E_0(B) - E_0(A) = \int_{AB} \frac{dE_0}{dt} dt = \int_{AB} -\mu_3 |x| y^2 dt < 0,
\]
implying that $E_0$ is a stable nilpotent focus when $\mu_1 = 0$ and $\mu_2 = 0$. 

Based on Lemmas 2.1 and 2.2, we investigate bifurcations from finite equilibria in the following three propositions.

**Proposition 2.1.** Consider $\mu_1 > 0$. There is a unique limit cycle occurring in a small neighborhood of $E_l$ and a unique limit cycle occurring in a small neighborhood of $E_r$ if $\mu_2$ varies from $\mu_2 = \mu_3\sqrt{\mu_1}$ to $\mu_2 = \mu_3\sqrt{\mu_1} - \epsilon$ and no limit cycles in any small neighborhood of $E_l$ or $E_r$ if $\mu_3\sqrt{\mu_1} \leq \mu_2 < \mu_3\sqrt{\mu_1} + \epsilon$, where $\epsilon > 0$ is sufficiently small. Moreover, the limit cycles are unstable.

**Proof.** From Lemma 2.1, we know that $E_r$ is an unstable weak focus of order 1 when $\mu_1 > 0$ and $\mu_2 = \mu_3\sqrt{\mu_1}$, and $E_r$ is a stable rough focus when $\mu_1 > 0$ and $(\mu_3 - 2\sqrt{2})\sqrt{\mu_1} < \mu_2 < \mu_3\sqrt{\mu_1}$. By the classical Hopf bifurcation Theorem (see [5, Chapter 3]) we obtain that there is a unique limit cycle, which is unstable, bifurcated from $E_r$ when $\mu_2$ varies from $\mu_2 = \mu_3\sqrt{\mu_1}$ to $\mu_2 = \mu_3\sqrt{\mu_1} - \epsilon$. Moreover, since system (1.1b) is symmetric about $E_r$, we know that a limit cycle bifurcated from $E_l$ appears simultaneously as the one bifurcated from $E_r$. 

By Lemma 2.2, $E_0$ becomes an unstable rough focus from a stable weak focus when $\mu_1 < 0$ and $\mu_2$ varies to $\mu_2 = \epsilon$ from $\mu_2 = 0$, where $\epsilon > 0$ is sufficiently small. Then Hopf bifurcations may occur. But since the vector field of system (1.1b) is only $C^1$, we cannot apply the classical Hopf bifurcation Theorem (see [5, Chapter 3]). When $\mu_1 = \mu_2 = 0$, $E_0$ is a stable nilpotent focus, from which limit cycles may also be bifurcated. Proposition 2.2 gives us generalized Hopf bifurcations from $E_0$, which is available in the aforementioned cases.

Before we show Proposition 2.2, we restate an equivalent form of system (1.1b). By the transformation $(x, y) \rightarrow (x, y - F(x))$, system (1.1b) can be rewritten as

$$
\dot{x} = y - F(x), \quad \dot{y} = -g(x),
$$

where $f(x) = -\mu_2 + \mu_3|x|$, $F(x) = \int_0^x f(t)dt = -\mu_2 x + \mu_3 x|x|/2$, and $g(x) = -\mu_1 x + x^3$. Obviously, system (2.13) has the same topological structure with system (1.1b).

**Proposition 2.2.** Consider $\mu_1 \leq 0$. There is a unique limit cycle occurring in a small neighborhood of $E_0$ if $\mu_2$ varies from $\mu_2 = 0$ to $\mu_2 = \epsilon$ and no limit cycles in any small neighborhood of $E_0$ if $-\epsilon \leq \mu_2 < 0$, where $\epsilon > 0$ is sufficiently small. Moreover, the limit cycle is stable.
Proof. When \( \mu_1 \leq 0 \) and \( \mu_2 > 0 \), consider the existence and uniqueness of limit cycles for equivalent system (2.13). It is easy to see that \( f(x), F(x) \) and \( g(x) \) satisfy:

(i): \( g(x) \) is odd and \( xg(x) > 0 \) when \( x \neq 0 \).
(ii): \( F(x) \) is odd and there exists \( x_0 = 2\mu_2/\mu_3 > 0 \) such that \( F(x) < 0 \) when \( 0 < x < x_0 \), \( F(x) \) is increasing and \( \geq 0 \) when \( x \geq x_0 \),
(iii): \( \int_{-\infty}^{0} f(x)dx = \int_{0}^{\infty} g(x)dx = \infty \),
(iv): \( f(x) \) and \( g(x) \) are Lipschitz functions on any bounded interval.

By [13, Theorem 4.1 of Chapter 4], when \( \mu_1 \leq 0 \) and \( \mu_2 > 0 \) system (2.13) exhibits a unique limit cycle, which is stable. So does system (1.1b).

When \( \mu_1 \leq 0 \) and \( \mu_2 \leq 0 \), the divergence of system (1.1b)
\[
\text{div}(y, \mu_1 x - x^3 + \mu_2 y - \mu_3 |x|y) = \mu_2 - \mu_3 |x| \leq 0.
\]
By Bendixson-Dulac Criterion, system (1.1b) exhibits no limit cycles.

Therefore, when \( \mu_1 \leq 0 \) system (1.1b) exhibits a unique limit cycle if \( \mu_2 \) varies from \( 0 \) to \( \mu_2 = \epsilon \) and no limit cycles if \( -\epsilon \leq \mu_2 < 0 \), where \( \epsilon > 0 \) is sufficiently small. Since \( E_0 \) is the unique finite equilibrium of system (1.1b), the limit cycle must surround \( E_0 \) if it exists. Now we prove that the limit cycle is in a small neighborhood of \( E_0 \) by considering subcases \( \mu_1 < 0 \) and \( \mu_1 = 0 \) respectively.

First, consider \( \mu_1 < 0 \). When \( \mu_2 = \epsilon \), we claim that the unique limit cycle of system (1.1b) in the strip \( x \in (-3\epsilon/\mu_3, 3\epsilon/\mu_3) \), where \( \epsilon > 0 \) is sufficiently small. Since the transformation between (1.1b) and system (2.13) does not change abscissa ordinates, we can prove the assertion by verifying the conditions of [4, Theorem 2.1] in the strip \( x \in (-3\epsilon/\mu_3, 3\epsilon/\mu_3) \) for system (2.13).

Notice that \( f(x) = F'(x) \) is even and continuous on \( (-3\epsilon/\mu_3, 3\epsilon/\mu_3) \) and \( F(0) = 0 \). Then the condition (i) of [4, Theorem 2.1] holds. It is easy to check that the odd function \( g(x) \) is continuous on \( (-3\epsilon/\mu_3, 3\epsilon/\mu_3) \) and \( xg(x) > 0 \) for \( x \in (-3\epsilon/\mu_3, 0) \cup (0, 3\epsilon/\mu_3) \). Then the condition (ii) of [4, Theorem 2.1] holds. Choose \( p_1 = \epsilon/\mu_3 \), which is on the interval \( (0, 3\epsilon/\mu_3) \). Then \( f(x)(x-\epsilon/\mu_3) = \mu_3(x-\epsilon/\mu_3)^2 \geq 0 \) when \( x \in (0, \epsilon/\mu_3) \cup (\epsilon/\mu_3, 3\epsilon/\mu_3) \), implying that the condition (iii) of [4, Theorem 2.1] holds. By the straight calculation,
\[
\frac{d(f(x)/g(x))}{dx} = \frac{-2\mu_3x^3 + 3x^2\epsilon - \mu_1 \epsilon}{x^2(x^2 - \mu_1)^2} > 0
\]
when \( \epsilon/\mu_3 < x < 3\epsilon/\mu_3 \) because \( d(-2\mu_3x^3 + 3x^2\epsilon - \mu_1 \epsilon)/dx = -6x(\mu_3x - \epsilon) < 0 \) and \( -2\mu_3x^3 + 3x^2\epsilon - \mu_1 \epsilon) |_{x=3\epsilon/\mu_3} = -\epsilon(\mu_1\mu_3^3 + 27\epsilon^2)/\mu_3^2 > 0 \) for sufficiently small \( \epsilon > 0 \). Then \( f(x)/g(x) \) is increasing on \( (\epsilon/\mu_3, 3\epsilon/\mu_3) \), which implies that the condition (iv) of [4, Theorem 2.1] holds. One can compute
\[
\int_{0}^{x} F(s)g(s)ds = \frac{1}{120}x^3(10\mu_3x^3 - 24x^2 - 15\mu_1\mu_3x + 40\mu_1\epsilon) := F(x).
\]
Then
\[
F\left(\frac{2\epsilon}{\mu_3}\right) = \frac{2\epsilon^4(5\mu_1\mu_3^2 - 8\epsilon^2)}{15\mu_3^2} < 0
\]
and
\[
F\left(\frac{3\epsilon}{\mu_3}\right) = \frac{-9\epsilon^4(5\mu_1\mu_3^2 - 54\epsilon^2)}{40\mu_3^2} > 0
\]
when \( \mu_1 < 0 \) and \( \mu_2 = \epsilon > 0 \). Thus, \( \rho_2 = 2\epsilon/\mu_3 > 0 \) in the condition (v) of [4, Theorem 2.1] and there is a positive number \( \rho_3 \) with \( \rho_2 < \rho_3 < 3\epsilon/\mu_3 \) such that \( F(\rho_3) = 0 \), yielding that (v) holds. By [4, Theorem 2.1], when \( \mu_1 < 0 \) and \( \mu_2 = \epsilon \) system (2.13) has exactly one limit cycle in the strip \( (-3\epsilon/\mu_3, 3\epsilon/\mu_3) \), where \( \epsilon > 0 \).
is sufficiently small. So does system (1.1b). Thus, the unique limit cycle lies in a small neighborhood of \( E_0 \) when \( \mu_1 < 0 \) and \( \mu_2 = \epsilon > 0 \).

Second, consider \( \mu_1 = 0 \). Notice that the vector field of system (1.1b) is rotated by \([10, 13]\) and system (1.1b) exhibits no limit cycles for \( \mu_2 = 0 \). Assume that the unique limit cycle of system (1.1b) does not lie in a small neighborhood of \( E_0 \) when \( \mu_2 = \epsilon \), where \( \epsilon > 0 \) is sufficiently small. By the rotated property, the unique stable limit cycle still persists when \( \mu_2 = 0 \). This is a contradiction. The proof is finished. \( \square \)

**Remark:** In fact, for the case \( \mu_1 < 0 \), we can also adopt the proof of the case \( \mu_1 = 0 \). However, in order to show the unique limit cycle of system (1.1b) in the strip \( x \in (-3\epsilon/\mu_3, 3\epsilon/\mu_3) \), we use [4, Theorem 2.1] to obtain the conclusion.

**Proposition 2.3.** The bifurcation diagram of system (1.1b) consists the following bifurcation surfaces:

(a): pitchfork bifurcation surface
\[ P := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+ : \mu_1 = 0\}; \]

(b): Hopf bifurcation surface
\[ H_1 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+ : \mu_1 > 0, \mu_2 = \mu_3\sqrt{\mu_1}\}; \]

(c): generalized Hopf bifurcation surface
\[ H_2 := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^2 \times \mathbb{R}^+ : \mu_1 \leq 0, \mu_2 = 0\}. \]

**Proof.** By Lemmas 2.1 and 2.2, the number of equilibria at finity varies from 3 to 1 when \( \mu_1 \) varies from a positive value to a non-positive one. Then \( \mu_1 = 0 \) is a pitchfork bifurcation surface. Thus, (a) is proven.

When \( \mu_1 > 0 \), by Proposition 2.1 the unstable weak foci \( E_l \) and \( E_r \) become stable rough foci and two limit cycles occur at the same time, one is in a small neighborhood of \( E_l \) and the other is in a small neighborhood of \( E_r \), as \( \mu_2 \) changes from \( \mu_3\sqrt{\mu_1} \) to a smaller value. Then \( H_1 \) is a Hopf bifurcation surface and (b) is proven.

When \( \mu_1 < 0 \) (resp. \( \mu_1 = 0 \)), by Proposition 2.2, \( E_0 \) becomes an unstable rough focus (resp. unstable degenerate node) from the stable weak focus (resp. stable nilpotent focus) and one stable limit cycle occurs in a small neighborhood of \( E_0 \) as \( \mu_2 \) changes to a small positive value from 0. Then \( H_2 \) is a generalized Hopf bifurcation surface and (c) is proven. \( \square \)

To investigate the behavior of orbits when either \(|x|\) or \(|y|\) is large, we turn to discuss the possible equilibria at infinity. By a Poincaré transformation \( x = 1/z \) and \( y = u/z \), system (1.1b) can be rewritten as

\[
\frac{du}{d\tau} = -1 - \mu_3u|z| + \mu_1z^2 + \mu_2u^2z^2 - u^2z^2, \quad \frac{dz}{d\tau} = -uz^3, \tag{2.14}
\]

where \( d\tau = z^2dt \). Obviously, system (2.14) has no equilibria on the u-axis. By another transformation \( x = v/z, y = 1/z \), system (1.1b) becomes

\[
\frac{dv}{d\tau} = z^2 + \mu_3v|z| - \mu_2vz^2 + v^4 - \mu_1v^2z^2, \quad \frac{dz}{d\tau} = \mu_3z|v| - \mu_2z^3 + v^3z - \mu_1vz^3, \tag{2.15}
\]

where \( d\tau = z^2dt \). Then \( D : (0,0) \) is a equilibrium of system (2.15) and \( D \) corresponds to the pair of equilibria \( I_y^+ \) and \( I_y^- \) at infinity of system (1.1b), which lie on the positive y-axis and negative y-axis respectively. The following lemma tells us the qualitative properties of \( I_y^+ \) and \( I_y^- \).
Lemma 2.3. When $0 < \mu_3 < 2\sqrt{2}$, $I_y^+$ and $I_y^-$ are cusps. When $\mu_3 \geq 2\sqrt{2}$, $I_y^+$ and $I_y^-$ are degenerate saddle-nodes. Moreover, the properties of $I_y^+$ and $I_y^-$ for system (1.1b) are shown in Figure 5.

**Proof.** Notice that $D : (0, 0)$ is a degenerate equilibrium of system (2.15) and $z = 0$ is an orbit of system (2.15). Since system (2.15) is invariant under the transformation $(v, z) \rightarrow (v, -z)$, it is symmetric about the $v$-axis. So it is enough to consider qualitative properties of $D$ in the upper half $vz$-plane, i.e., $z \geq 0$. Taking polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$, system (2.15) becomes

$$
\frac{1}{r} \frac{dr}{d\theta} = \frac{H(\theta) + O(r)}{G(\theta) + O(r)}, \quad \text{as } r \rightarrow 0,
$$

where $G(\theta) = -\sin^3 \theta$ and $H(\theta) = \cos \theta \sin^2 \theta$. The equation $G(\theta) = 0$ has exactly two real roots $0$ and $\pi$ on the interval $[0, \pi]$. Moreover, $G(0) = H(0) = G(\pi) = H(\pi) = 0$. Let

$$
\mathcal{L}^+ := \{(v, z) \in \mathbb{R}^2 : z = v \tan \alpha, \ 0 < v < \ell\},
$$

where $\ell > 0$ is sufficiently small and $0 < \alpha < \pi/2$ is arbitrarily close to 0. From equations in (2.15), one can obtain a unique horizontal isocline $\mathcal{H}_1$ which is in the upper half $vz$-plane and tangent to $\theta = 0$, where

$$
\mathcal{H}_1 := \{(v, z) \in \mathbb{R}^2 : z = 0, \ 0 < v < \ell\}
$$

and there are no unique vertical isoclines in the open region $\Delta \mathcal{L}^+ \mathcal{H}_1$, as shown in Figure 6 (A). In fact, $\dot{v} > 0$ and $\dot{z} > 0$ in $\Delta \mathcal{L}^+ \mathcal{H}_1$. One can check that

$$
\frac{dz}{dv} = \frac{z(\mu_3 vz - \mu_2 z^2 + v^3 - \mu_1 vz^2)}{z^2 + v(\mu_3 vz - \mu_2 z^2 + v^3 - \mu_1 vz^2)} = \frac{z}{v + \frac{z^2}{(\mu_3 vz - \mu_2 z^2 + v^3 - \mu_1 vz^2)}},
$$

because $\mu_3 vz - \mu_2 z^2 + v^3 - \mu_1 vz^2 > 0$. Since any orbit connecting $D$ along $\theta = 0$ in $\Delta \mathcal{L}^+ \mathcal{H}_1$ is in the form of $z = cv \theta + o(\theta^0)$, where $c > 0$ and $\theta > 1$, it is easy to compute $dz/dv = c\theta v^{\theta-1} + o(v^{\theta-1}) > z/v = cv^{\theta-1} + o(v^{\theta-1})$ in $\Delta \mathcal{L}^+ \mathcal{H}_1$. Then there are no orbits connecting $D$ in the open region $\Delta \mathcal{L}^+ \mathcal{H}_1$.

Let

$$
\mathcal{L}^- := \{(v, z) \in \mathbb{R}^2 : z = -v \tan \alpha, \ -\ell < v < 0\},
$$
where \( \ell > 0 \) is sufficiently small and \( 0 < \alpha < \pi/2 \) is arbitrarily close to 0. From equations in (2.15), one can obtain two horizontal isoclines \( \mathcal{H}_2 \) and \( \mathcal{H}_3 \) which are in the upper half \( vz \)-plane and tangent to \( \theta = \pi \), where
\[
\mathcal{H}_2 := \left\{ (v, z) \in \mathbb{R}^2 : z = v^2/\mu_3 + O(v^3), \ -\ell < v < 0 \right\},
\]
\[
\mathcal{H}_3 := \left\{ (v, z) \in \mathbb{R}^2 : z = 0, \ -\ell < v < 0 \right\}.
\]
But the existence of vertical isoclines depends on the value of \( \mu_3 \). When \( 0 < \mu_3 < 2 \) there are no vertical isoclines in the open region \( \Delta \mathcal{L}^-\mathcal{D}\mathcal{H}_3 \), as shown in Figure 6 (A). One can get that \( \dot{v} > 0 \) and \( \dot{z} < 0 \) in the open region \( \Delta \mathcal{H}_2\mathcal{D}\mathcal{H}_3 \). Then
\[
\frac{dz}{dv} = \frac{z(-\mu_3 vz - \mu_2 z^2 + v^3 - \mu_1 vz^2)}{z^2 + v(-\mu_3 vz - \mu_2 z^2 + v^3 - \mu_1 vz^2)} = \frac{z}{v + z^2/(-\mu_3 vz - \mu_2 z^2 + v^3 - \mu_1 vz^2)} \geq \frac{z}{v},
\]
because \( \mu_3 vz - \mu_2 z^2 + v^3 - \mu_1 vz^2 < 0 \). Since any orbit connecting \( D \) along \( \theta = \pi \) in \( \Delta \mathcal{H}_2\mathcal{D}\mathcal{H}_3 \) can be expressed as \( z = cv^q + o(v^q) \), where \( q > 1 \) is a positive integer and \( c > 0 \) (resp. \( < 0 \)) if \( q \) is even (resp. odd), it is easy to compute \( \frac{dz}{dv} = cq^{q-1} + o(v^{q-1}) < z/v = cv^{q-1} + o(v^{q-1}) \) in \( \Delta \mathcal{H}_2\mathcal{D}\mathcal{H}_3 \). Then there are no orbits connecting \( D \) in the open region \( \Delta \mathcal{H}_2\mathcal{D}\mathcal{H}_3 \) when \( 0 < \mu_3 < 2 \). Moreover, \( \dot{v} > 0 \) and \( \dot{z} > 0 \) in the open region \( \Delta \mathcal{L}^-\mathcal{D}\mathcal{H}_2 \). There are also no orbits connecting \( D \) in \( \Delta \mathcal{L}^-\mathcal{D}\mathcal{H}_2 \) when \( 0 < \mu_3 < 2 \), because any orbit connecting \( D \) in second quadrant has a negative slope.

When \( \mu_3 = 2 \) there is a unique vertical isocline \( \mathcal{V} \) in the open region \( \Delta \mathcal{L}^-\mathcal{D}\mathcal{H}_3 \), where
\[
\mathcal{V} := \left\{ (v, z) \in \mathbb{R}^2 : z = \frac{\mu_3}{2} v^2 + O(v^3), \ -\ell < v < 0 \right\}.
\]
Obviously, $V$ is above $H_2$, as shown in Figure 6 (B). One can check that $\dot{v} > 0$ and $\dot{z} < 0$ in $\Delta H_2^2 D H_3$. As in the case when $0 < \mu_3 < 2$, there are no orbits connecting $D$ in $\Delta H_2^2 D H_3$ when $\mu_3 = 2$. Notice that $\dot{v} > 0$ and $\dot{z} > 0$ in both $\Delta L^2 D V$ and $\Delta V D H_2$. There are also no orbits connecting $D$ in either $\Delta L^2 D V$ or $\Delta V D H_2$ because any orbit connecting $D$ in the second quadrant has a negative slope.

When $\mu_3 > 2$ there are two vertical isoclines $V_1$ and $V_2$ in the open region $\Delta L^2 D H_3$, where

$$V_1 := \{ (v, z) \in \mathbb{R}^2 : z = \left( \frac{\mu_3}{2} - \frac{\sqrt{\mu_3^2 - 4}}{2} \right) v^2 + O(v^3), \quad -\ell < v < 0 \}$$

and

$$V_2 := \{ (v, z) \in \mathbb{R}^2 : z = \left( \frac{\mu_3}{2} + \frac{\sqrt{\mu_3^2 - 4}}{2} \right) v^2 + O(v^3), \quad -\ell < v < 0 \}.$$

One can check that the positions of $V_1$, $V_2$ and $H_2$ are shown in Figure 6 (C) or (D). Similar to the case $\mu_3 = 2$, there are no orbits connecting $D$ in $\Delta H_2^2 D H_3$, $\Delta V_1 D H_2$ or $\Delta L^2 D V_2$.

In the open region $\Delta V_2 D V_1$, we see that $\dot{v} < 0$ and $\dot{z} > 0$. To get the tendency of orbits, we need consider the subcases $2 < \mu_3 < 2\sqrt{2}$ and $\mu_3 \geq 2\sqrt{2}$. When $2 < \mu_3 < 2\sqrt{2}$, we claim that no orbits connect $D$ in $\Delta V_2 D V_1$. In fact, any orbit connecting $D$ along $\theta = \pi$ in $\Delta V_2 D V_1$ can be written as $z = cv^2 + o(v^2)$, where $(\mu_3 - \sqrt{\mu_3^2 - 4})/2 \leq c \leq (\mu_3 + \sqrt{\mu_3^2 - 4})/2$. Then

$$\frac{dz}{dv} \big|_{z=cv^2+o(v^2)} = \frac{c(1 - c \mu_3)}{c^2 - c \mu_3 + 1} v + o(v),$$

which implies

$$\frac{dz}{dv} \big|_{z=cv^2+o(v^2)} - (2cv + o(v)) = -\frac{c(2c^2 - c \mu_3 + 1)}{c^2 - c \mu_3 + 1} v + o(v).$$

Since $2c^2 - c \mu_3 + 1 > 0$ when $2 < \mu_3 < 2\sqrt{2}$, we have $(dz/dv) |_{z=cv^2+o(v^2)} \neq 2cv + o(v)$, implying that no orbits connect $D$ in $\Delta V_2 D V_1$. When $\mu_3 \geq 2\sqrt{2}$, let

$$C := \{ (v, z) \in \mathbb{R}^2 : z = \frac{\mu_3}{4} v^2 - (1 + \frac{\sqrt{2}}{2} \mu_2) v^3, \quad -\ell < v < 0 \},$$

which is between $V_1$ and $V_2$. Since

$$\frac{dz}{dv} \big|_{C - \left( \frac{\mu_3}{2} v - 3(1 + \frac{\sqrt{2}}{2} \mu_2) v^2 \right)} = -\frac{\mu_3(\mu_3^2 - 8)}{2(3\mu_3^2 - 16)} v + O(v^2) > 0$$

when $\mu_3 > 2\sqrt{2}$ and

$$\frac{dz}{dv} \big|_{C - \left( \frac{\mu_3}{2} v - 3(1 + \frac{\sqrt{2}}{2} \mu_2) v^2 \right)} = v^2 + O(v^3) > 0$$

when $\mu_3 = 2\sqrt{2}$, the open region $\Delta V_2 D C$ is a generalized normal sector of Class I [11], as shown in Figure 6 (D). Then there are infinitely many orbits leaving $D$ in $\Delta V_2 D V_1$ as $\tau \to \infty$ when $\mu_3 \geq 2\sqrt{2}$.

Above all, $z = 0$ is an orbit of system (2.15). Since $\dot{v}|_{z=0} > 0$, there is an orbit leaving $D$ along $\theta = 0$, which is the positive $v$-axis, and an orbit approaching $D$ along $\theta = \pi$, which is the negative $v$-axis, as $\tau \to \infty$. Moreover, in the half plane
z > 0, there are no orbits connecting $D$ when $0 < \mu_3 < 2\sqrt{2}$ and there are infinitely many orbits leaving $D$ along $\theta = \pi$ as $\tau \to \infty$ when $\mu_3 \geq 2\sqrt{2}$. Combining with the symmetry of system (2.15) about the $z$-axis, we also can get the qualitative properties in the half plane $z < 0$. As a result, for system (1.1b) the direction of the orbit on the boundary of the Poincaré disc is clockwise. Except the boundary orbits, when $0 < \mu_3 < 2\sqrt{2}$ there are no other orbits connecting either $I^+_y$ or $I^-_y$, as shown in Figure 5 (A), and when $\mu_3 \geq 2\sqrt{2}$ there are infinitely many orbits leaving $I^+_y$ in the direction of the negative $x$-axis and infinitely many orbits leaving $I^-_y$ in the direction of the positive $x$-axis as $t \to \infty$, as shown in Figure 5 (B).

By Lemma 2.3, it is easy to obtain that all the orbits of system (1.1b) are positively bounded when $\mu_3 \geq 2\sqrt{2}$. In the following proposition, we need to judge further the positive boundedness of all the orbits of system (1.1b) when $0 < \mu_3 < 2\sqrt{2}$.

**Proposition 2.4.** All the orbits are positively bounded for system (1.1b). Specially, when $0 < \mu_3 < 2\sqrt{2}$ the dynamical behaviors of system (1.1b) near infinity are shown in Figure 7.

**Proof.** It is sufficient to prove that all the orbits are positively bounded for equivalent system (2.13) with $0 < \mu_3 < 2\sqrt{2}$. The same as system (1.1b), equilibria at infinity of system (2.13) are cusps and orbits nearby are rotating clockwise when $0 < \mu_3 < 2\sqrt{2}$. Let $\Upsilon$ denote the orbit passing through $(x_*,y_*)$, where

$$x_* \begin{cases} = 1, & \text{if } \mu_1 \leq 0, \mu_2 \leq 0, \\ > 0 \text{ satisfying } F(x_*) = 1, & \text{if } \mu_1 \leq 0, \mu_2 > 0, \\ = 2\sqrt{\mu_1}, & \text{if } \mu_1 > 0, \mu_2 \leq 0, \\ > 0 \text{ satisfying } F(x_*) = 1, & \text{if } \mu_1, \mu_2 > 0, \sqrt{\mu_1} \leq 2\mu_2/\mu_3, \\ = 2(\sqrt{\mu_1} - \mu_2/\mu_3), & \text{if } \mu_1, \mu_2 > 0, \sqrt{\mu_1} > 2\mu_2/\mu_3 \end{cases} \tag{2.16}$$

and $y_* < 0$ is sufficiently small. Then $\Upsilon$ intersects the $y$-axis and the vertical line $x = x_*$ successively, denoted the intersection points by $G$, $H$, $I$ and $J$, as shown in Figure 8 (A). Let $y_G$, $y_H$, $y_I$ and $y_J$ be ordinates of $G$, $H$, $I$ and $J$. Obviously, $y_I = y_*$. 

![Figure 7](image_url)
Let
\[ E(x, y) = -\frac{\mu_1 x^2}{2} + \frac{y^2}{2} + \frac{x^4}{4}. \] (2.17)

It is easy to see that
\[ \frac{1}{2}(y_J^2 - y_G^2) = \int_{\mathcal{Y}} dE = \int_{\mathcal{Y}} \frac{1}{2} x^2 (\mu_1 - x^2)(\mu_3|x| - 2\mu_2) dt = \int_{\mathcal{Y}} F(x) dy. \] (2.18)

When \( \mu_1 \leq 0 \) and \( \mu_2 \leq 0 \), it is easy to see that
\[ \frac{1}{2}(y_J^2 - y_G^2) = \int_{\mathcal{Y}} F(x) dy < 0, \] (2.19)

implying that \( y_G + y_J > 0 \).

When \( \mu_1 > 0 \) or \( \mu_2 > 0 \), consider
\[ \int_{\mathcal{Y}} F(x) dy = \int_{\hat{G}H \cup \hat{H}I \cup \hat{I}J} F(x) dy. \] (2.20)

Notice that
\[ \int_{\hat{H}I} F(x) dy < \int_{\hat{H}I} F(x_\ast) dy < \int_{y_H} F(x_\ast) dy < F(x_\ast)(y_H - y_H). \] (2.21)

Then
\[ \frac{1}{2}(y_J^2 - y_H^2) < F(x_\ast)(y_H - y_H) < 0, \]
yielding that \( y_H > -y_\ast \). It follows from (2.21) that
\[ \int_{\hat{H}I} F(x) dy < 2F(x_\ast)y_\ast. \] (2.22)

When \( \mu_1 \leq 0 \) and \( \mu_2 > 0 \), we get that \( g(x) \geq 0 \) for any \( x \) and the equal sign holds only when \( x = 0 \). Then the ordinate of any point on \( GH \) is greater than \( y_H \) and
\[ \int_{\hat{G}H} F(x) dy = -\int_{0}^{2\mu_2/\mu_3} \frac{F(x)g(x)}{y - F(x)} dx - \int_{2\mu_2/\mu_3}^{x_\ast} \frac{F(x)g(x)}{y - F(x)} dx < -\int_{0}^{2\mu_2/\mu_3} \frac{F(x)g(x)}{y - F(x)} dx \leq \frac{M_1}{y_H - m_1} < \frac{M_1}{-y_\ast - m_1}, \] (2.23)
where \( m_1 \) and \( M_1 \) are constants.

When \( \mu_1 > 0 \) and \( \mu_2 \leq 0 \), we see that \( g(x) < 0 \) for \( 0 < x < \sqrt{\mu_1} \) and \( g(x) > 0 \) for \( \sqrt{\mu_1} < x < x_* \). Note that \( g(\sqrt{\mu_1} + \varepsilon) > -g(\sqrt{\mu_1} - \varepsilon) > 0 \) and \( 0 < y - F(\sqrt{\mu_1} + \varepsilon) < y - F(\sqrt{\mu_1} - \varepsilon) < 0 \) when \( 0 < \varepsilon < \sqrt{\mu_1} \) and \( 0 < x < x_* \). Then \( y_H \) is less than the ordinate of any point on \( \partial \Gamma \) and

\[
\int_{\partial \Gamma} F(x)dy = -\int_0^{\sqrt{\mu_1}} \frac{F(x)g(x)}{y - F(x)} dx - \int_{\sqrt{\mu_1}}^{x_*} \frac{F(x)g(x)}{y - F(x)} dx < -\int_{\sqrt{\mu_1}}^{x_*} \frac{F(x)g(x)}{y - F(x)} dx = \frac{m_2}{y_H - m_2} < \frac{M_2}{-y_* - m_2},
\]

(2.24)

where \( m_2 \) and \( M_2 \) are constants.

When \( \mu_1, \mu_2 > 0 \) and \( \sqrt{\mu_1} < 2\mu_2/\mu_3 \), we have \( g(x) > 0 \) for \( x > \sqrt{\mu_1} \). Then the ordinate of any point on \( \partial \Gamma \) with \( x > \sqrt{\mu_1} \) is greater than \( y_H \) and

\[
\int_{\partial \Gamma} F(x)dy = -\int_0^{2\mu_2/\mu_3} \frac{F(x)g(x)}{y - F(x)} dx - \int_{2\mu_2/\mu_3}^{x_*} \frac{F(x)g(x)}{y - F(x)} dx - \int_{\sqrt{\mu_1}}^{x_*} \frac{F(x)g(x)}{y - F(x)} dx < -\int_{2\mu_2/\mu_3}^{x_*} \frac{F(x)g(x)}{y - F(x)} dx = \frac{m_3}{y_H - m_3} < \frac{M_3}{-y_* - m_3},
\]

(2.25)

where \( m_3 \) and \( M_3 \) are constants.

When \( \mu_1, \mu_2 > 0 \) and \( \sqrt{\mu_1} < 2\mu_2/\mu_3 \), we obtain that \( g(x) < 0 \) for \( 0 < x < \sqrt{\mu_1} \) and \( g(x) > 0 \) for \( \sqrt{\mu_1} < x < x_* \). Similar to the case when \( \mu_1 > 0 \) and \( \mu_2 \leq 0 \), \( y_H \) is less than the ordinate of any point on \( \partial \Gamma \) with \( 2\mu_2/\mu_3 < x < \sqrt{\mu_1} \) and

\[
\int_{\partial \Gamma} F(x)dy = -\int_0^{2\mu_2/\mu_3} \frac{F(x)g(x)}{y - F(x)} dx - \int_{2\mu_2/\mu_3}^{x_*} \frac{F(x)g(x)}{y - F(x)} dx - \int_{\sqrt{\mu_1}}^{x_*} \frac{F(x)g(x)}{y - F(x)} dx < -\int_{2\mu_2/\mu_3}^{x_*} \frac{F(x)g(x)}{y - F(x)} dx = \frac{m_4}{y_H - m_4} < \frac{M_4}{-y_* - m_4},
\]

(2.26)

where \( m_4 \) and \( M_4 \) are constants.

Similarly,

\[
\int_{\hat{\Gamma}} F(x)dy = -\int_0^{x_*} \frac{F(x)g(x)}{y - F(x)} dx < \begin{cases} 
M_1/(n_1 - y_*), & \text{if } \mu_1 \leq 0, \mu_2 > 0, \\
M_2/(n_2 - y_*), & \text{if } \mu_1 > 0, \mu_2 \leq 0, \\
M_3/(n_3 - y_*), & \text{if } \mu_1, \mu_2 > 0, \sqrt{\mu_1} \leq 2\mu_2/\mu_3, \\
M_4/(n_4 - y_*), & \text{if } \mu_1, \mu_2 > 0, \sqrt{\mu_1} > 2\mu_2/\mu_3,
\end{cases}
\]

(2.27)

where \( n_1, n_2, n_3 \) and \( n_4 \) are constants. Since \( F(x_*) > 0 \) and \( y_* < 0 \) is sufficiently small, it follows from (2.20) and (2.22-2.27) that

\[
\int_{\hat{\Gamma}} d\xi = \int_{\partial \Gamma \cup \hat{\Gamma}} F(x)dy < 0
\]

even when \( \mu_1 > 0 \) or \( \mu_2 > 0 \). Combined with (2.18) and (2.19),

\[
\frac{1}{2}(y_G^2 - y_J^2) = \int_T d\xi < 0
\]

for any \( \mu_1 \) and \( \mu_2 \), implying that \( y_G + y_J > 0 \).

By the symmetry about the origin of system (2.13), the orbit \( \gamma \), its symmetrical orbit and two straight line segments form a closed curve, and as \( t \to \infty \) all the
orbits cannot leave the region surrounded by it, as shown in Figure 8 (B). That means all the orbits are positively bounded.

3. Limit cycles and homoclinic loops. In this section, sufficient and necessary conditions for the existence of limit cycles and sufficient and necessary conditions for the existence of homoclinic loops are given for (1.1b). For simplicity, the whole parameter space is divided into the following five subsets:

\begin{itemize}
\item[(c1)] \(\mu_1 \in \mathbb{R}, \mu_2 \leq 0, \mu_3 > 0\),
\item[(c2)] \(\mu_1 \leq 0, \mu_2 > 0, \mu_3 > 0\),
\item[(c3)] \(\mu_1 > 0, 0 < \mu_2 \leq \mu_3 \sqrt{\mu_1}/2, \mu_3 > 0\),
\item[(c4)] \(\mu_1 > 0, \mu_3 \sqrt{\mu_1}/2 < \mu_2 < \mu_3 \sqrt{\mu_1}, \mu_3 > 0\),
\item[(c5)] \(\mu_1 > 0, \mu_2 \geq \mu_3 \sqrt{\mu_1}, \mu_3 > 0\).
\end{itemize}

**Lemma 3.1.** When (c1) holds, system (1.1b) exhibits neither limit cycles nor homoclinic loops.

**Proof.** When (c1) holds, one can calculate the divergence of system (1.1b),

\[ \text{div}(y, \mu_1 x - x^3 + \mu_2 y - \mu_3 |x|y) = \mu_2 - \mu_3 |x| \leq 0. \]

By Bendixson-Dulac Criterion, system (1.1b) has no closed orbits, implying the nonexistence of limit cycles and homoclinic loops.

**Lemma 3.2.** When (c2) holds, there is a unique limit cycle for system (1.1b). Moreover, the limit cycle is stable and hyperbolic.

**Proof.** Proceeding as in the proof of Proposition 2.2, all the conditions of [13, Theorem 4.1 of Chapter 4] are still satisfied when (c2) holds. Then system (2.13) exhibits a unique limit cycle, which is stable. Now we study the hyperbolicity of the limit cycle. Let \(G(x) = \int_0^x g(t)dt\). It is easy to check that under the condition \(\mu_2 > 0\),

\[ \frac{d(F(x)/G^{3/2}(x))}{dx} = \frac{2(\mu_2|x| - \mu_1 \mu_3)}{(x^2 - 2\mu_1)^{3/2}} \]

is positive for all \(x\) when \(\mu_1 < 0\) and positive for all \(x \neq 0\) when \(\mu_1 = 0\). By [13, Corollary 2 of Chapter 4], the limit cycle of system (2.13) is hyperbolic.

For the remaining three cases, by Lemma 2.1 system (1.1b) has three equilibria and \(E_0\) is a saddle. Since the index of a saddle is \(-1\) and the sum of the indices of all equilibria surrounded by a limit cycle is 1, there are no limit cycles in a small neighborhood of \(E_0\). Moreover, system (1.1b) is symmetric about \(E_0\). Then limit cycles cannot surround only two equilibria of \(E_0, E_l\) and \(E_r\). Thus, limit cycles may surround all three equilibria \(E_0, E_l\) and \(E_r\) or surround one of equilibria \(E_l\) and \(E_r\). In what follows, let large limit cycles be the ones surrounding three equilibria and small limit cycles be the ones surrounding a single equilibrium for simplicity.

The following lemma gives the nonexistence of closed orbits for system (1.1b) when (c3) holds.

**Lemma 3.3.** When (c3) holds, there are no limit cycles and no homoclinic loops for system (1.1b).
Proof. It suffices to prove the nonexistence of closed orbits (limit cycles or homoclinic loops) for system (2.13). Clearly, when $\mu_1 > 0$ the equivalent system (2.13) has three equilibria $O : (0, 0)$, $K_1 : (-\sqrt{\mu_1}, F(-\sqrt{\mu_1}))$ and $K_2 : (\sqrt{\mu_1}, F(\sqrt{\mu_1}))$, which correspond to equilibria $E_0$, $E_l$ and $E_r$ of system (1.1b). If there exists a closed orbit $\Gamma$ of system (2.13), we get

$$0 = \oint_{\Gamma} dE = \oint_{\Gamma} -g(x)F(x)dt = \oint_{\Gamma} F(x)dy;$$

(3.1)

where energy function $E(x,y)$ is defined in (2.17).

Firstly, assume that there is a large limit cycle $\Gamma_1$ of system (2.13). When $\mu_1 > 0$ and $0 < \mu_2 < \mu_3\sqrt{\mu_1}/2$, the closed orbit $\Gamma_1$ will intersect with the $y$-axis, the lines $x = 2\mu_2/\mu_3$ and $x = \sqrt{\mu_1}$ successively. Denote the intersection points by $A, B, C, D, E$ and $F$ respectively (see Figure 9 (A)). Moreover, since for system (2.13)

$$\dot{y} = \mu_1 x - x^3 > 0 \text{ (resp. } < 0\text{)}$$

the horizontal lines passing $B$ or $E$ also intersect $\Gamma_1$ at $B'$ and $E'$ on the right hand side of $x = \sqrt{\mu_1}$. Notice that $F(x) \leq 0$ and $g(x) \leq 0$ along the arcs $\hat{AB}$ and $\hat{EF}$, and the equal signs hold only at isolated points. Then

$$\int_{AB} F(x)dy + \int_{EF} F(x)dy = \int_{\hat{AB}\cup\hat{EF}} -g(x)F(x)dt < 0.$$  

(3.3)

From (3.2), the arcs $BC$ and $CB'$ can be regarded as graphs of functions $x = x_1(y)$ and $x = x_2(y)$ respectively, where $y_B \leq y \leq y_C$. Then

$$\int_{BC} F(x)dy + \int_{CB'} F(x)dy = \int_{y_B}^{y_C} F(x_1(y))dy + \int_{y_B}^{y_C} F(x_2(y))dy$$

$$= \int_{y_B}^{y_C} (F(x_1(y)) - F(x_2(y)))dy < 0,$$

(3.4)

because $F(x)$ is increasing and $x_1(y) < x_2(y)$ on the interval $(y_B,y_C)$. Similarly, we also have

$$\int_{ED} F(x)dy + \int_{DE} F(x)dy < 0.$$  

(3.5)
From (3.2), the arc $\hat{B}'E'$ also can be regarded as the graph of $x = x_3(y)$, where $y_{E'} \leq y \leq y_{B'}$. Then
\[
\int_{\hat{B}'E'} F(x)dy = -\int_{y_{E'}}^{y_{B'}} F(x_3(y))dy < 0, \tag{3.6}
\]
because $x_3(y) > \sqrt{\mu_1}$ and $F(x_3(y)) > 0$ on the interval $(y_{E'}, y_{B'})$. Notice that system (2.13) is symmetric about its origin. It follows from (3.3-3.6) that
\[
\int_{\Gamma_{1}} F(x)dy = 2 \int_{\hat{A}B \cup \hat{B}C \cup \hat{C}D \cup \hat{D}E \cup \hat{E}F} F(x)dy < 0,
\]
which contradicts (3.1). Therefore, system (2.13) exhibits no large limit cycles when $\mu_1 > 0$ and $0 < \mu_2 < \mu_3\sqrt{\mu_1}/2$. Furthermore, when $\mu_1 > 0$ and $\mu_2 = \mu_3\sqrt{\mu_1}/2$, $B$ coincides with $B'$ and $E$ coincides with $E'$. From (3.3) and (3.6)
\[
f_{\Gamma_{1}} F(x)dy = 2 \int_{\hat{A}B \cup \hat{B}E \cup \hat{E}F} F(x)dy < 0,
\]
implies that system (2.13) has no large limit cycles when $\mu_1 > 0$ and $\mu_2 = \mu_3\sqrt{\mu_1}/2$.

Secondly, assume that there is a small limit cycle $\Gamma_2$ of system (2.13), which surrounds equilibrium $K_2$. Obviously, the closed orbit $\Gamma_2$ will intersect with the line $x = \sqrt{\mu_1}$ and the curve $y = F(x)$ successively. Denote the intersection points by $G$, $H$, $I$ and $J$ respectively (see Figure 9 (B)). From (3.2), $G$ is the highest point and $I$ is the lowest point on $\Gamma_2$. Then the arcs $\hat{G}H \hat{I}$ and $I \hat{J}G$ can be regarded as graphs of functions $x = x_4(y)$ and $x = x_5(y)$ respectively, where $y_H \leq y \leq y_G$. Since $x_4(y) > \sqrt{\mu_1}$ and $0 < x_5(y) < \sqrt{\mu_1}$, one can check that $F(x_5(y)) < F(\sqrt{\mu_1}) < F(x_4(y))$ on the interval $(y_H, y_G)$. Thus,
\[
f_{\Gamma_{2}} F(x)dy = \int_{GH} F(x)dy + \int_{IJG} F(x)dy = \int_{y_H}^{y_I} F(x_4(y))dy + \int_{y_I}^{y_G} F(x_5(y))dy
\]
\[
= \int_{y_I}^{y_G} (F(x_5(y)) - F(x_4(y)))dy < 0, \tag{3.7}
\]
which conflicts with (3.1). Therefore, system (2.13) has no limit cycles surrounding equilibrium $K_2$. By the symmetry of (2.13) about the origin and the origin is a saddle, if it has small limit cycles, one of it must surround $K_2$. Thus, system (2.13) exhibits no small limit cycles.

Thirdly, assume that there is a homoclinic loop $\Gamma_3$ of system (2.13), which surrounds $K_2$. Similar to (3.7), we can calculate that $\int_{\Gamma_{3}} F(x)dy < 0$. Combining the symmetry of (2.13), system (2.13) has no homoclinic loops, which completes the proof.

When (c4) holds, the number and stability of small limit cycles for system (1.1b) are given by Lemma 3.4. Moreover, Lemma 3.5 tells us the divergency along an outer large limit cycle is smaller than the divergency along an inner one if there are at least two large limit cycles for equivalent system (2.13).

**Lemma 3.4.** Consider (c4). There are at most two small limit cycles for system (1.1b). Moreover, the two small limit cycles are unstable and hyperbolic if they exist.

**Proof.** By the explanations above Lemma 3.3, any small limit cycle of system (1.1b) must surround one of $E_1$ and $E_r$. Since system (1.1b) is invariant under the transformation $(x, y) \rightarrow (-x, -y)$, it is symmetric about $E_0$. Then small limit cycles
must appear in pairs. So we only need to prove that system (1.1b) exhibits at most one small limit cycle surrounding $E_r$, which is unstable and hyperbolic. Moreover, the equilibrium $K_2 : (\sqrt{\mu_1}, F(\sqrt{\mu_1}))$ of system (2.13) corresponds to $E_r$. Applying the transformation

$$(x, y, t) \rightarrow (x + \sqrt{\mu_1}, y + F(\sqrt{\mu_1}), -t),$$  

which translates $K_2$ to the origin and reverses the orbits over time, we can change system (2.13) into the form

$$\begin{cases} 
\dot{x} = -y - \mu_2 x + \mu_3 \sqrt{x} + \mu_3 \sqrt{x} / 2 - \mu_1 \mu_3 / 2, \\
\dot{y} = x(\sqrt{\mu_1} + x + 2\sqrt{\mu_1}). 
\end{cases}$$

(3.8)

To investigate the limit cycles only surrounding the origin of system (3.9), which means that limit cycles are in the region $x \in (-\sqrt{\mu_1}, \infty)$, we take $|x + \sqrt{\mu_1}| = x + \sqrt{\mu_1}$. In the strip $x > -\sqrt{\mu_1}$, system (3.9) can be rewritten as

$$\begin{cases} 
\dot{x} = -y + \hat{F}(x), \\
\dot{y} = \hat{g}(x), 
\end{cases}$$

(3.10)

where $\hat{F}(x) = \mu_3 x + \mu_3 \sqrt{\mu_1} - \mu_2$. It is easy to check that $\hat{F}(x)$ has a unique zero $x_0 = (\mu_2 - \mu_3 \sqrt{\mu_1}) / \mu_3$. Since $\mu_3 \sqrt{\mu_1} / 2 < \mu_2 < \mu_3 \sqrt{\mu_1}$, we get $-\sqrt{\mu_1} / 2 < x_0 < 0$. Then $\hat{F}(x) < 0$ as $-\sqrt{\mu_1} / 2 < x < x_0$ and $\hat{F}(x) > 0$ as $x > x_0$. Thus, the condition (i) of [7, Theorem 2.1] holds. Notice that $\hat{F}(0) = \hat{F}(\xi_0) = 0$, where $\xi_0 = 2(\mu_2 - \mu_3 \sqrt{\mu_1}) / \mu_3$. It follows from $\mu_3 \sqrt{\mu_1} / 2 < \mu_2 < \mu_3 \sqrt{\mu_1}$ that $-\sqrt{\mu_1} < \xi_0 < x_0$. Then the condition (ii) of [7, Theorem 2.1] holds. Obviously, $x \hat{g}(x) = x^2 (x + \sqrt{\mu_1})(x + 2\sqrt{\mu_1}) > 0$ for $x \in (-\sqrt{\mu_1}, 0) \cup (0, \infty)$. Then the condition (iii) of [7, Theorem 2.1] holds. The simultaneous equations

$$\hat{F}(x_1) = \hat{F}(x_2), \quad \hat{g}(x_1) / \hat{f}(x_1) = \hat{g}(x_2) / \hat{f}(x_2)$$

with $x_1 < x_2$ have a unique solution

$$x_1 = \frac{\mu_2 - \mu_3 \sqrt{\mu_1}}{\mu_3} - \frac{\sqrt{\mu_1 \mu_3} - \mu_2}{\sqrt{3\mu_3}}, \quad x_2 = \frac{\mu_2 - \mu_3 \sqrt{\mu_1}}{\mu_3} + \frac{\sqrt{\mu_1 \mu_3} - \mu_2}{\sqrt{3\mu_3}}.$$

Since $\mu_3 \sqrt{\mu_1} / 2 < \mu_2 < \mu_3 \sqrt{\mu_1}$, one can calculate

$$x_2 = \frac{-2(\mu_3 \sqrt{\mu_1} - \mu_2)(\mu_3 \sqrt{\mu_1} - \mu_2)}{\sqrt{3\mu_3} \mu_3 \mu_3 - \mu_2^2 + 3\mu_3 (\mu_3 \sqrt{\mu_1} - \mu_2)} > 0 \Rightarrow x_0 > x_1.$$

Then the condition (iv) of [7, Theorem 2.1] holds. We can compute that

$$d \left( \hat{F}(x) \hat{f}(x) / \hat{g}(x) \right) / dx = \frac{\mu_2 \kappa(x)}{2(x + \sqrt{\mu_1})^2(x + 2\sqrt{\mu_1})^2},$$

where

$$\kappa(x) = 3\mu_3 \left( x + \frac{2(\mu_3 \sqrt{\mu_1} - \mu_2)}{3\mu_3} \right)^2 + \frac{2(\mu_3 \sqrt{\mu_1} - \mu_2)(\mu_3 \sqrt{\mu_1} + \mu_2)}{3\mu_3}.$$

It is easy to check that

$$-\sqrt{\mu_1} < -\frac{2(\mu_3 \sqrt{\mu_1} - \mu_2)}{3\mu_3} < \xi_0.$$
because $\mu_2 > \mu_3 \sqrt{\mu_1}/2$. Then on the interval $(-\sqrt{\mu_1}, \xi_0)$,

$$\kappa(x) \leq \max\{\kappa(\xi_0), \kappa(-\sqrt{\mu_1})\} = \max\left\{\frac{2(\mu_3 \sqrt{\mu_1} - 2\mu_2)(\mu_3 \sqrt{\mu_1} - \mu_2)}{\mu_3}, \sqrt{\mu_1}(\mu_3 \sqrt{\mu_1} - 2\mu_2)\right\} < 0,$$

implying that $\dot{F}(x) \dot{f}(x)/\dot{g}(x)$ is decreasing on $(-\sqrt{\mu_1}, \xi_0)$. Thus, the condition (v) of [7, Theorem 2.1] holds.

By [7, Theorem 2.1], system (3.10) exhibits at most one limit cycle in the region $x \in (-\sqrt{\mu_1}, \infty)$. Moreover, the limit cycle is stable and hyperbolic if it exists. Since the transformation (3.8) reverses the orbits of system (2.13) over time, there is at most one limit cycle only surrounding $K_2$ of system (2.13), which is unstable and hyperbolic. The same is true for $E_r$, which completes the proof.

**Lemma 3.5.** When (c4) holds, if there are at least two large limit cycles for system (2.13),

$$\oint_{L_1} \text{div}(y-F(x),-g(x))dt > \oint_{L_2} \text{div}(y-F(x),-g(x))dt,$$

where $L_1, L_2$ are large limit cycles and $L_1$ lies in the region enclosed by $L_2$.

**Proof.** When (c4) holds, it is clear that $L_1$ and $L_2$ will intersect with the $y$-axis and the lines $x = \sqrt{\mu_1}$ successively. Let the intersection points of inner ones by $A_1, B_1, C_1, D_1$ and outer ones be represented respectively by $A_2, B_2, C_2, D_2$ (see Figure 10). By the symmetry of system (2.13) about the origin,

$$\oint_{L_i} \text{div}(y-F(x),-g(x))dt = \oint_{L_i} -f(x)dt = -2 \int_{A_iD_i} f(x)dt, \quad (3.11)$$

where $i = 1, 2$. The orbit segments $\overline{A_1B_1}$ and $\overline{A_2B_2}$ can be regarded respectively as

![Figure 10. Two large limit cycles.](image-url)

the graphs of $y = y_1(x)$ and $y = y_2(x)$, where $0 \leq x \leq \sqrt{\mu_1}$. Then for each $i = 1, 2$,
we have

\[
\int_{A_1B_1} f(x)\,dt = \int_0^{\sqrt{mt}} \frac{f(x)}{y(x) - F(x)} \, dx \\
= \int_0^{\sqrt{mt}} \frac{y'(x)}{y(x) - F(x)} \, dx - \int_0^{\sqrt{mt}} \frac{F'(x)}{y(x) - F(x)} \, dx \\
= \int_0^{\sqrt{mt}} \frac{y'(x)}{y(x) - F(x)} \, dx - \ln \left| \frac{y(\sqrt{mt}) - F(0)}{y(0) - F(0)} \right| \\
= \int_0^{\sqrt{mt}} -\frac{g(x)}{(y(x) - F(x))^2} \, dx - \ln \left| \frac{y(\sqrt{mt}) - F(0)}{y(0) - F(0)} \right| \\
= \int_0^{\sqrt{mt}} -\frac{g(x)}{(y(x) - F(x))^2} \, dx + \int_0^{\sqrt{mt}} \frac{g(x)}{y(x)}(y(x) - F(x)) \, dx \\
- \ln \left| \frac{y(\sqrt{mt}) - F(\sqrt{mt})}{y(\sqrt{mt})} \right| \\
= \int_0^{\sqrt{mt}} -\frac{g(x)F(x)}{y(x)(y(x) - F(x))^2} \, dx - \ln \left| \frac{y(\sqrt{mt}) - F(\sqrt{mt})}{y(\sqrt{mt})} \right|
\]

Thus,

\[
\int_{A_2B_2} f(x)\,dt - \int_{A_1B_1} f(x)\,dt = \int_0^{\sqrt{mt}} \left( \frac{g(x)F(x)}{y(x)(y(x) - F(x))^2} - \frac{g(x)F(x)}{y_2(x)(y_2(x) - F(x))^2} \right) \, dx \\
+ \ln \left| 1 - \frac{F(\sqrt{mt})}{y(\sqrt{mt})} \right| - \ln \left| 1 - \frac{F(\sqrt{mt})}{y(\sqrt{mt})} \right| > 0, \tag{3.12}
\]

because \( y_2(x) > y_1(x) > 0 > F(x) \) and \( g(x) < 0 \) for \( x \in (0, \sqrt{mt}) \). Similarly,

\[
\int_{C_2D_2} f(x)\,dt - \int_{C_1D_1} f(x)\,dt > 0. \tag{3.13}
\]

By straight computation, when \( \mu_2, \sqrt{\mu_1}/2 < \mu_2 < \mu_3/\sqrt{\mu_1} \)

\[
\frac{\partial(F(x)f(x)/g(x))}{\partial x} = 3\mu_2\mu_3(x - (\mu_1\mu_3^2 + 2\mu_2^3)/(3\mu_2\mu_3))^2 + (4\mu_3^2 - 2\mu_1\mu_3^2/3\mu_2\mu_3)/2(x^2 - \mu_1)^2
\]

and

\[
\lim_{x \to -\infty} F(x) < \lim_{x \to \infty} F(x).
\]

Due to the proof in \( [7, (V) \text{ of Theorem 2.1}] \),

\[
\int_{B_2C_2} f(x)\,dt - \int_{B_1C_1} f(x)\,dt > 0. \tag{3.14}
\]

It follows from (3.11-3.14) that

\[
\oint_{L_1} \text{div}(y - F(x), -g(x))\,dt = -2\int_{A_1D_1} f(x)\,dt \\
> -2\int_{A_2D_2} f(x)\,dt \\
= \oint_{L_2} \text{div}(y - F(x), -g(x))\,dt,
\]

which completes the proof. \( \square \)
In the following two lemmas, we give the number and stability of closed orbits for system (1.1b) when (c5) holds.

**Lemma 3.6.** When (c5) holds, there are neither small limit cycles nor homoclinic loops for system (1.1b).

**Proof.** To prove the nonexistence of small limit cycles for system (1.1b), it suffices to prove that there are no limit cycles surrounding $K_2 : (\sqrt{\mu_1}, F(\sqrt{\mu_1}))$ of equivalent system (2.13) in the region $x \in (0, \infty)$.

Firstly, we prove the nonexistence of small limit cycles for system (2.13) when $\mu_2 = \mu_3\sqrt{\mu_1}$. Assume that system (2.13) exhibits small limit cycles surrounding $K_2$ and $\gamma$ is the innermost one. Then

$$\oint_{\gamma} \text{div}(y - F(x), -g(x))dt = \int_{\gamma} -f(x)dt = \int_{\gamma} \frac{\mu_3}{x(\sqrt{\mu_1} + x)}dy.$$  

Let $D$ be the region enclosed by $\gamma$. Notice that $f(x)/g(x) \in C^1$ on the interval $(0, \infty)$ when $\mu_2 = \mu_3\sqrt{\mu_1}$. By Green formula, we get

$$\oint_{\gamma} \text{div}(y - F(x), -g(x))dt = \iint_{D} \mu_3(\sqrt{\mu_1} + 2x)dx dy > 0.$$  

Due to [13, Theorem 2.3 of Chapter 4], the limit cycle $\gamma$ is unstable. By Lemma 2.1, $E_r$ is unstable when $\mu_1 > 0$ and $\mu_2 = \mu_3\sqrt{\mu_1}$. So is $K_2$ of system (2.13), which implies that $\gamma$ is internally stable. This is a contradiction. Therefore, when $\mu_1 > 0$ and $\mu_2 = \mu_3\sqrt{\mu_1}$ system (2.13) exhibits no limit cycles surrounding $K_2$ on the interval $x \in (0, \infty)$.

Secondly, we prove the nonexistence of small limit cycles for system (2.13) when $\mu_2 > \mu_3\sqrt{\mu_1}$. By the transformation $(x, y) \to (-x + \sqrt{\mu_1}, y + F(\sqrt{\mu_1}))$, which translates $K_2$ to the origin and reflects the vector filed across the $y$-axis, system (2.13) can be reduced into

$$\dot{x} = -y + \tilde{F}(x), \quad \dot{y} = \tilde{g}(x), \quad (3.15)$$

where

$$\tilde{F}(x) = F(-x + \sqrt{\mu_1}) - F(\sqrt{\mu_1}) = (\mu_2 - \mu_3\sqrt{\mu_1} + \frac{\mu_3}{2}x^2)/2, \quad (3.16)$$

$$\tilde{g}(x) = -g(-x + \sqrt{\mu_1}) = x(x - \sqrt{\mu_1})(x - 2\sqrt{\mu_1}). \quad (3.17)$$

It is worth noting that we have taken $|x - \sqrt{\mu_1}| = -x + \sqrt{\mu_1}$ in the process of reduction because here we only investigate the existence of limit cycles in the region $x \in (-\infty, \sqrt{\mu_1})$. Let $\tilde{f}(x) := \tilde{F}'(x) = \mu_3x + \mu_2 - \mu_3\sqrt{\mu_1}$. Similar to the proof of Lemma 3.3, one can check that conditions (i), (ii) and (iii) of [7, Theorem 2.1] hold for system (3.15). Moreover, the simultaneous equations

$$\begin{align*}
\tilde{F}(x_1) &= \tilde{F}(x_2), \\
\frac{\tilde{g}(x_1)}{\tilde{f}(x_1)} &= \frac{\tilde{g}(x_2)}{\tilde{f}(x_2)}
\end{align*} \quad (3.18)$$

with $x_1 < x_2$ have a unique solution

$$\begin{align*}
x_1 &= \frac{\mu_3\sqrt{\mu_1} - \mu_2}{\mu_3} - \frac{\sqrt{\mu_1\mu_3^2 - \mu_2^2}}{\sqrt{3}\mu_3}, \\
x_2 &= \frac{\mu_3\sqrt{\mu_1} - \mu_2}{\mu_3} + \frac{\sqrt{\mu_1\mu_3^2 - \mu_2^2}}{\sqrt{3}\mu_3}.
\end{align*}$$

But it follows from $\mu_2 > \mu_3\sqrt{\mu_1}$ that

$$x_2 = \frac{-2(\mu_3\sqrt{\mu_1} - \mu_2)(\mu_3\sqrt{\mu_1} - \mu_2)}{\sqrt{3}\mu_3\sqrt{\mu_1\mu_3^2 - \mu_2^2} - 3\mu_3(\mu_3\sqrt{\mu_1} - \mu_2)} < 0.$$
Then the condition (iv) of [7, Theorem 2.1] does not hold. By [7, Corollary 2.2], system (3.15) exhibits no limit cycles in the region $x \in (-\infty, \sqrt{\mu_1})$ when $\mu_1 > 0$ and $\mu_2 > \mu_3 \sqrt{\mu_1}$. Therefore, when $\mu_1 > 0$ and $\mu_2 > \mu_3 \sqrt{\mu_1}$ system (2.13) has no small limit cycles surrounding $K_2$.

Thirdly, we prove the nonexistence of homoclinic loops for system (1.1b) when $\mu_1 > 0$ and $\mu_2 \geq \mu_3 \sqrt{\mu_1}$. On the one hand, the Jacobian matrix $J_0$ for system (1.1b) at the saddle $E_0$ is shown in (2.1) and $\text{tr} J_0 = \mu_2 > 0$. Notice that the homoclinic loops of system (1.1b) do not traverse the $y$-axis, only on which system (1.1b) is nonsmooth. By [5, Theorem 3.3], the homoclinic loops of $E_0$ is asymptotically unstable if they exist. On the other hand, by Lemma 2.1 $E_r$ is unstable when $\mu_1 > 0$ and $\mu_2 \geq \mu_3 \sqrt{\mu_1}$. That means that there is at least one small limit cycle surrounding $E_r$ if system (1.1b) exhibits homoclinic loops of $E_0$, which contradicts the first two parts of this proof. The proof is completed.

**Lemma 3.7.** When (c5) holds, system (1.1b) exhibits a unique large limit cycle, which is stable and hyperbolic.

**Proof.** We first prove the existence of large limit cycles for equivalent system (2.13). The origin of system (2.13) is a saddle, which corresponds to $E_0$ of system (1.1b). By Lemma 3.6, there are no homoclinic loops at the origin of system (2.13) when (c5) holds. Since all orbits are positive bounded, the unstable manifold of the saddle at the origin cannot connect equilibria at infinity. Therefore, the manifold is shown in Figure 11.

![Figure 11](image)

We claim that the manifold is only as shown in Figure 11(B). Assume that the manifold is as shown in Figure 11(A). In fact, when $\mu_1 > 0$ and $\mu_2 \geq \mu_3 \sqrt{\mu_1}$, the equilibrium $K_2 : (\sqrt{\mu_1}, F(\sqrt{\mu_1}))$ of system (2.13) is unstable because $E_r$ is unstable by Lemma 2.1. By Poincaré-Bendixson Theorem [8], there is at least one small limit cycle for system (1.1b), which contradicts Lemma 3.6. Thus, the assertion is proven. By the symmetry about the origin of system (2.13), the unstable manifolds of the origin and two straight line segments form a closed curve, and the orbits cannot enter the region surrounded by it, as shown in Figure 12(A).

By Proposition 2.4, all orbits of system (1.1b) are positively bounded. Then an orbit near infinity, its symmetrical orbit and two straight line segments form a closed curve, and all the orbits cannot leave the region surrounded by it. As a result, an annular region (see Figure 12(B)), whose $\omega$-limit set lies in itself, can
be constructed. By Poincaré-Bendixson Theorem (see [8]), we get the existence of large limit cycles of system (1.1b).

\[(a) \quad \text{The inner boundary of the annular region}
\]

\[(b) \quad \text{The annular region}
\]

\[\text{Figure 12. Existence of the large limit cycle.}\]

In what follows, we prove the uniqueness, stability and hyperbolicity of large limit cycles of system (1.1b). Assume that there exists a large limit cycle \(\Gamma\) for system (2.13). Obviously, \(\Gamma\) will intersect with the \(y\)-axis, the line \(x = \frac{\mu_2}{\mu_3}\) and the vertical isocline \(y = F(x)\) successively. Denote the intersection points by \(A, B, C, D\) and \(E\) respectively (see Figure 13(A)). Letting \(w = F(x) = -\frac{\mu_2 x + \mu_3 x^3}{2}\), we have

\[x = \begin{cases} \frac{\mu_2 - \sqrt{\mu_2^2 + 2 \mu_3 w}}{\mu_3} =: F_1(w), & \text{if } 0 \leq x < \frac{\mu_2}{\mu_3}, \\ \frac{\mu_2 + \sqrt{\mu_2^2 + 2 \mu_3 w}}{\mu_3} =: F_2(w), & \text{if } x \geq \frac{\mu_2}{\mu_3} \end{cases}\]

For \(x > 0\), system (2.13) can be rewritten as

\[\frac{dy}{dw} = \frac{\lambda_i(w)}{-y + w}, \quad \text{where } \lambda_i(w) = \frac{g(F_i(w))}{f(F_i(w))}, \quad i = 1, 2. \quad (3.19)\]

Thus, the orbits in the \(xy\)-plane with \(x \geq 0\) will be changed into the ones in the \(wy\)-plane. For convenience, in the \(wy\)-plane the corresponding intersection points are also marked as \(A, B, C, D\) and \(E\) (see Figure 13(B)).

We claim that the point \(P : (2\mu_2/\mu_3, 0)\) of the \(xy\)-plane lies in the interior of \(\Gamma\). Otherwise, since

\[\dot{x}|_{x=2\mu_2/\mu_3} \begin{cases} > 0, & \text{if } y > 0, \\ < 0, & \text{if } y < 0, \end{cases}, \quad \dot{y}|_{x=2\mu_2/\mu_3} < 0,
\]

the orbit section \(\overline{BCD}\) of the \(xy\)-plane lies in the region \(\mu_2/\mu_3 \leq x \leq 2\mu_2/\mu_3\). Then the orbit section \(\overline{BCD}\) of the \(wy\)-plane lies in the region \(w \leq 0\), as shown in Figure 13(B). It is easy to see that

\[\frac{dy}{dw}|_{\overline{BA}, w < F(\sqrt{\mu_1})} > 0, \quad \frac{dy}{dw}|_{\overline{BC}, w < F(\sqrt{\mu_1})} < 0. \quad (3.20)\]

Notice that simultaneous equations (3.18) have no solutions satisfying \(x_1 < 0 < x_2\). From (3.16) and (3.17), we get that

\[F(x_1) = F(x_2), \quad \frac{g(x_1)}{f(x_1)} = \frac{g(x_2)}{f(x_2)} \]
also exhibit no solutions satisfying $0 < x_1 < \sqrt{\mu_1} < x_2$. That means $\lambda_1(w) = \lambda_2(w)$ which has no solutions on the interval $(F(\sqrt{\mu_1}), 0)$. One can check that $\lambda_1(w) > 0$ and $\lambda_2(w) > 0$ on $(F(\sqrt{\mu_1}), 0)$. Since $\lambda_2(F(\sqrt{\mu_1})) = \lambda_1(F(\sqrt{\mu_1}))$, we have $\lambda_2(w) > \lambda_1(w)$ on the interval $(F(\sqrt{\mu_1}), 0)$. Then

\[
\frac{dy}{dw}|_{\hat{BC}, w > F(\sqrt{\mu_1})} < \frac{dy}{dw}|_{\hat{BA}, w > F(\sqrt{\mu_1})} < 0.
\]  

(3.21)

From (3.20) and (3.21), $\hat{BA}$ lies above $\hat{BC}$ in the $wy$-plane. Similarly, $\hat{DE}$ lies below $\hat{DC}$. By the symmetry about origin of the system (2.13), it follows that $y_A + y_E = 0$, where $y_A$ and $y_E$ are respectively ordinates of points $A$ and $E$. It is clear that

\[
\int_{E\hat{A}} -g(x)dx + (y - F(x))dy = \int_{y_E}^{y_A} ydy = \frac{1}{2}(y_A^2 - y_E^2) = 0.
\]

Thus,

\[
\int_{\Gamma} -g(x)dx + (y - F(x))dy = 2 \int_{\hat{AB}\cup\hat{BCD}\cup\hat{DE}} -g(x)dx + (y - F(x))dy
\]

\[
= 2 \int_{\hat{AB}\cup\hat{BCD}\cup\hat{DE}\cup\hat{EA}} -g(x)dx + (y - F(x))dy
\]

\[
= -2 \int_{\Omega_{xy}} f(x)dx dy
\]

\[
= -2 \int_{\Omega_{wy}} dw dy < 0,
\]

where $\Omega_{xy}$ (resp. $\Omega_{wy}$) is the interior region by $\hat{AB}$, $\hat{BCD}$, $\hat{DE}$ and $\hat{EA}$ in the $xy$-plane (resp. $wy$-plane). However, it is easy to obtain

\[
\int_{\Gamma} -g(x)dx + (y - F(x))dy = 0.
\]

This is a contradiction.

Since $P : (2\mu_2/\mu_3, 0)$ of the $xy$-plane lies in the interior region of $\Gamma$, the limit cycle $\Gamma$ intersects with line $x = 2\mu_2/\mu_3$ at two points, denoted by $\hat{B}$ and $\hat{D}$, as shown in Figure 14. To prove the uniqueness, stability and hyperbolicity of $\Gamma$, it
A NONSMOOTH VAN DER POL-DUFFING OSCILLATOR (II)

Figure 14. $P$ is in the region enclosed by $\Gamma$.

suffices to show

$$\oint_{\Gamma} \text{div}(y - F(x), -g(x))\,dt = \oint_{\Gamma} f(x)\,dt < 0$$

by [13, Theorem 2.2 of Chapter 4].

Let $y = y_1(w)$ and $y = y_2(w)$ denote the functions of orbit segments $\widehat{BA}$ and $\widehat{BB}$, respectively. Similar to (3.20) and (3.21), we also have

$$\frac{dy}{dw}|_{\widehat{BA}, w < F(\sqrt{\mu_1})} > 0, \quad \frac{dy}{dw}|_{\widehat{BB}, w < F(\sqrt{\mu_1})} < 0$$

and

$$\frac{dy}{dw}|_{\widehat{BB}, w > F(\sqrt{\mu_1})} < 0, \quad \frac{dy}{dw}|_{\widehat{BA}, w > F(\sqrt{\mu_1})} < 0.$$  

Then $y_1(w) > y_2(w)$ for $-\mu_2^2/(2\mu_3) < w < 0$. Thus, from (2.13) and (3.19),

$$\int_{\widehat{AB}} -f(x)\,dt + \int_{\widehat{BB}} -f(x)\,dt = \int_{\widehat{AB}} -f(x)\,dx + \int_{\widehat{BB}} -f(x)\,dx = \int_{-\mu_2^2/(2\mu_3)}^0 \left( -\frac{1}{w - y_1} + \frac{1}{w - y_2} \right) \,dw = \int_{-\mu_2^2/(2\mu_3)}^0 \frac{y_2 - y_1}{(w - y_1)(w - y_2)} \,dw < 0. \quad (3.22)$$

Similarly,

$$\int_{\widehat{DE}} -f(x)\,dt + \int_{\widehat{DD}} -f(x)\,dt < 0. \quad (3.23)$$

Since $f(x) > 0$ along $\widehat{BCD}$,

$$\int_{\widehat{BCD}} -f(x)\,dt < 0. \quad (3.24)$$

From (3.22), (3.23) and (3.24), we get

$$\oint_{\Gamma} -f(t)\,dt = 2\int_{\widehat{ACD}} -f(x)\,dx = 2\int_{\widehat{ABD\cup\widehat{BCD}\cup\widehat{DD\cup\widehat{DE}}}} -f(x)\,dx < 0.$$

Therefore, $\Gamma$ is stable and hyperbolic. Furthermore, the two limit cycles are stable if system (1.1b) exhibits two limit cycles. By the Poincaré-Bendixson Theorem, there
is an unstable limit cycle which lies in the annulus of the two stable limit cycles. This is a contradiction. The uniqueness of limit cycles is proven. \qed

Until now, only the homoclinic loops and large limits cycles in (c4) are still unclear. In what follows we will investigate the changes in the number of homoclinic loops and large limits cycles when (c4) holds. Moreover, the homoclinic bifurcation surface and double limit cycle bifurcation surface will be obtained. Before we do this, we consider another equivalent form of system (1.1b) when \( \mu_1 > 0 \).

By the scaling transformation \((x, y, t) \to (\sqrt{\mu_1 x}, \mu_1 y, t/\sqrt{\mu_1})\), system (1.1b) can be reduced into

\[
\dot{x} = y, \quad \dot{y} = x - x^3 + \mu y - \mu_3 |x|y, \tag{3.25}
\]

where \( \mu := \mu_2/\sqrt{\mu_1} \). It is easy to obtain that system (3.25) is topologically equivalent to system (1.1b) when \( \mu_1 > 0 \). Then all aforementioned lemmas which hold for (1.1b) with \( \mu_1 > 0 \) are also available for (3.25), except that the equilibria of system (3.25) are \((0,0), (-1,0)\) and \((1,0)\), corresponding to \( E_0, E_\ell \) and \( E_r \) of system (1.1b).

Let \( \gamma_{\mu, \mu_3}^+ (c) \) (resp. \( \gamma_{\mu, \mu_3}^- (c) \)) be the positive (resp. negative) orbit of system (3.25) that starts at the point \((0,c)\) (resp. \((0,-c)\)). Then the orbits \( \gamma_{\mu, \mu_3}^+(0) \) and \( \gamma_{\mu, \mu_3}^-(0) \) are respectively exactly the unstable manifold on the right-hand side \( W_{\mu, \mu_3}^+ \) and the stable manifold on the right-hand side \( W_{\mu, \mu_3}^- \) of system (3.25) at the origin. Obviously, \( \gamma_{\mu, \mu_3}^+(c) \) lies in the first quadrant and \( \gamma_{\mu, \mu_3}^-(c) \) lies in the fourth quadrant at the beginning. Notice that

\[
\dot{x}|_{x=0} = y, \quad \dot{y}|_{y=0} = x - x^3 \begin{cases} > 0, & \text{if } 0 < x < 1, \\ < 0, & \text{if } x > 1. \end{cases}
\]

Combining the dynamic structure of equilibria at infinity of system (3.25), which corresponds to \( I_y^+ \) and \( I_y^- \), we know that \( \gamma_{\mu, \mu_3}^+(c) \) intersects the positive x-axis and \( \gamma_{\mu, \mu_3}^-(c) \) intersects the positive x-axis or leaves from \( I_y^- \) as \( t \to \infty \). When \( \gamma_{\mu, \mu_3}^-(c) \) is one of the orbits leaving from \( I_y^- \), we can think that it crosses the x-axis at positive infinity, because the infinity on the positive x-axis is connected with \( I_y^- \) by a unique orbit. Denote the first intersection points of \( \gamma_{\mu, \mu_3}^+(0) \) and \( \gamma_{\mu, \mu_3}^-(0) \) with the positive x-axis by \( A : (x_A^+(\mu, \mu_3), 0) \) and \( B : (x_B^-(\mu, \mu_3), 0) \) respectively. The following lemma tells us how \( x_A^+(\mu, \mu_3) \) and \( x_B^-(\mu, \mu_3) \) continuously depend on \( \mu \) and \( \mu_3 \).

**Lemma 3.8.** For a fixed \( \mu_3 \), \( x_A^+(\mu, \mu_3) \) increases continuously and \( x_B^-(\mu, \mu_3) \) decreases continuously as \( \mu \) increases. For a fixed \( \mu \), \( x_A^+(\mu, \mu_3) \) decreases continuously and \( x_B^-(\mu, \mu_3) \) increases continuously as \( \mu_3 \) increases.

**Proof.** Let \((x, y_{\mu, \mu_3}^+(x))\) and \((x, y_{\mu, \mu_3}^-(x))\) denote the points on \( \gamma_{\mu, \mu_3}^+(c) \) and \( \gamma_{\mu, \mu_3}^-(c) \) respectively. Then

\[
y_{\mu, \mu_3}^+(0) = c, \quad y_{\mu, \mu_3}^-(0) = -c, \quad y_{\mu, \mu_3}^+(x_A^+(\mu, \mu_3)) = 0 \quad \text{and} \quad y_{\mu, \mu_3}^-(x_B^-(\mu, \mu_3)) = 0.
\]

Let \( z_{\mu, \mu_3}^+(x) = y_{\mu, \mu_3}^+(x) - y_{\mu, \mu_3}^+(x) \), \( 0 \leq x \leq \min\{x_A^+(\mu, \mu_3), x_A^+(\mu + \epsilon, \mu_3)\} \) be the vertical distance between \( \gamma_{\mu, \mu_3}^+(c) \) and \( \gamma_{\mu, \mu_3}^+(c) \), where \((x_A^+(\mu + \epsilon, \mu_3), 0) \) is the first intersection point of \( \gamma_{\mu, \mu_3}^+(c) \) and the positive x-axis and \( |\epsilon| \) is sufficiently small. One can check that

\[
z_{\mu, \mu_3}^+(x) = \{y_{\mu, \mu_3}^+(s) - y_{\mu, \mu_3}^+(s)\} = \int_0^x \left( \frac{s - s^3}{y_{\mu, \mu_3}^+(s)} + (\mu + \epsilon) - \mu_3 s \right) ds = H_1(x) + H_2(x), \tag{3.26}
\]
where

\[ H_1(x) = \epsilon x, \quad H_2(x) = \int_0^x z_{\mu_3}(s)H_3(s)ds \quad \text{and} \quad H_3(x) = \frac{-x + x^3}{y_{\mu_3}(x) y_{\mu_3}(x)}. \]

From (3.26), we get

\[ z_{\mu_3}(x)H_3(x) = H_1(x)H_3(x) + H_2(x)H_3(x). \]  \hspace{1cm} (3.27)

It follows from (3.27) that

\[ \frac{dH_2(x)}{dx} - H_2(x)H_3(x) = H_1(x)H_3(x), \]

which is a first order linear differential equation. With the initial condition \( H_2(0) = 0 \), we get

\[ H_2(x) = \int_0^x H_1(\tau)H_3(\tau) \exp \left\{ \int_\tau^x H_3(\eta)d\eta \right\} d\tau. \]  \hspace{1cm} (3.28)

From (3.26) and (3.28),

\[ z_{\mu_3}(x) = H_1(x) + \int_0^x H_1(\tau)H_3(\tau) \exp \left\{ \int_\tau^x H_3(\eta)d\eta \right\} d\tau \]

\[ = H_1(x) - \left[ H_1(\tau) \exp \left\{ \int_\tau^x H_3(\eta)d\eta \right\} \right]_0^x + \int_0^x H_1(\tau) \exp \left\{ \int_\tau^x H_3(\eta)d\eta \right\} d\tau \]

\[ = H_1(0) \exp \left\{ \int_0^x H_3(\eta)d\eta \right\} + \int_0^x H_1(\tau) \exp \left\{ \int_\tau^x H_3(\eta)d\eta \right\} d\tau \]

\[ = \epsilon \int_0^x \exp \left\{ \int_\tau^x H_3(\eta)d\eta \right\} d\tau > 0 \quad \text{(resp. < 0), if } \epsilon > 0 \quad \text{(resp. < 0)}. \]  \hspace{1cm} (3.29)

Then \( y_{\mu_3}(x_A(\mu, \mu_3)) = y_{\mu_3}(x_A(\mu, \mu_3)) = 0 \) when \( \epsilon > 0 \), implying that \( x_A(\mu + \epsilon, \mu_3) > x_A(\mu, \mu_3) \). Moreover, it follows from (3.29) that \( \lim_{\epsilon \to 0} z_{\mu_3}(x) = 0 \). That means \( \lim_{\epsilon \to 0} x_A(\mu + \epsilon, \mu_3) = x_A(\mu, \mu_3) \). Thus \( x_A(\mu, \mu_3) \) increases continuously as \( \mu \) increases.

Let \( z_{\mu, \mu_3}(x) = y_{\mu_3}(x) - y_{\mu_3}(x), 0 \leq x \leq \min\{x_B(\mu, \mu_3), x_B(\mu + \epsilon, \mu_3)\} \), where \( (x_B(\mu, \mu_3), 0) \) is the first intersection point of \( \gamma(\mu + \epsilon, \mu_3) \) and the positive \( x \)-axis and \( |x| \) is sufficiently small. Similar to the calculation of \( z_{\mu_3}(x) \), we get

\[ z_{\mu, \mu_3}(x) = \epsilon \int_0^x \exp \left\{ \int_\tau^x H_3(\eta)d\eta \right\} d\tau > 0 \quad \text{(resp. < 0), if } \epsilon > 0 \quad \text{(resp. < 0)}. \]  \hspace{1cm} (3.30)

where \( \hat{H}_3(x) = (-x + x^3)/(y_{\mu_3}(x)y_{\mu_3}(x)) \). Then \( \hat{H}_3(x) = y_{\mu_3}(x) (x_B(\mu + \epsilon, \mu_3)) > y_{\mu_3}(x_B(\mu + \epsilon, \mu_3)) \) when \( \epsilon > 0 \), implying that \( x_B(\mu + \epsilon, \mu_3) < x_B(\mu, \mu_3) \). Moreover, it follows from (3.30) that \( \lim_{\epsilon \to 0} z_{\mu, \mu_3}(x) = 0 \). That implies \( \lim_{\epsilon \to 0} x_B(\mu + \epsilon, \mu_3) = x_B(\mu, \mu_3) \). Hence, \( x_B(\mu, \mu_3) \) decreases continuously as \( \mu \) increases.

To know how \( x_A(\mu, \mu_3) \) and \( x_B(\mu, \mu_3) \) continuously depend on \( \mu_3 \), let \( z_{\mu, \mu_3}(x) = y_{\mu_3}(x) - y_{\mu_3}(x), 0 \leq x \leq \min\{x_A(\mu, \mu_3), x_B(\mu, \mu_3 + \delta)\} \). One can check that

\[ z_{\mu, \mu_3}(x) = \int_0^x \left\{ \left( \frac{s - s^3}{y_{\mu_3}(s) + \mu - (\mu_3 + \delta) s} \right) - \left( \frac{s - s^3}{y_{\mu_3}(s) + \mu - \mu_3 s} \right) \right\} ds \]

\[ = \hat{H}_1(x) + \hat{H}_2(x). \]
where
\[ H_1(x) = -\frac{1}{2} \delta x^2, \quad H_2(x) = \int_0^x \tilde{z}^{+}_{\mu, \mu_3}(s) H_3(s) ds, \quad \tilde{H}_3(x) = \frac{-x + x^3}{y^{+}_{\mu, \mu_3 + \delta}(x) y^{+}_{\mu, \mu_3}(x)}. \]

Thus,
\[ \tilde{z}^{+}_{\mu, \mu_3}(x) = -\delta \int_0^x \tau \exp \left\{ \int_0^\tau \tilde{H}_3(\eta) d\eta \right\} d\tau < 0 \quad (\text{resp.} > 0), \; \text{if} \; \delta > 0 \; (\text{resp.} < 0). \] (3.31)

Then \( 0 = y^{+}_{\mu, \mu_3 + \delta}(x^{c}_{A}(\mu, \mu_3 + \delta)) < y^{+}_{\mu, \mu_3}(x^{c}_{A}(\mu, \mu_3)) \) when \( \delta > 0 \), implying that \( x^{c}_{A}(\mu, \mu_3 + \delta) < x^{c}_{A}(\mu, \mu_3) \). Moreover, it follows from (3.32) that \( \lim_{\delta \to 0} \tilde{z}^{+}_{\mu, \mu_3}(x) = 0 \).

That implies \( \lim_{\delta \to 0} x^{c}_{A}(\mu, \mu_3 + \delta) = x^{c}_{A}(\mu, \mu_3) \). Thus \( x^{c}_{A}(\mu, \mu_3) \) decreases continuously as \( \mu_3 \) increases.

Let \( \tilde{z}^{-}_{\mu, \mu_3}(x) = y^{-}_{\mu, \mu_3 + \delta}(x) - y^{-}_{\mu, \mu_3}(x) \), where \( 0 \leq x \leq \min\{ x^{c}_{B}(\mu, \mu_3), x^{c}_{B}(\mu, \mu_3 + \delta) \} \). Similarly,
\[ \tilde{z}^{-}_{\mu, \mu_3}(x) = -\delta \int_0^x \tau \exp \left\{ \int_0^\tau \tilde{H}_3(\eta) d\eta \right\} d\tau < 0 \quad (\text{resp.} > 0), \; \text{if} \; \delta > 0 \; (\text{resp.} < 0), \] (3.32)

where \( \tilde{H}_3(x) = (-x + x^3)/y^{-}_{\mu, \mu_3 + \delta}(x) y^{-}_{\mu, \mu_3}(x). \) Then \( 0 = y^{-}_{\mu_1, \mu_2}(x^{c}_{B}(\mu, \mu_3)) > y^{-}_{\mu_1, \mu_2}(x^{c}_{B}(\mu, \mu_3)) \) when \( \delta > 0 \), implying that \( x^{c}_{B}(\mu, \mu_3 + \delta) > x^{c}_{B}(\mu, \mu_3) \). Moreover, it follows from (3.32) that \( \lim_{\delta \to 0} \tilde{z}^{-}_{\mu, \mu_3}(0) = 0 \). That implies \( \lim_{\delta \to 0} x^{c}_{B}(\mu, \mu_3 + \delta) = x^{c}_{B}(\mu, \mu_3) \). Hence, \( x^{c}_{B}(\mu, \mu_3) \) increases continuously as \( \mu_3 \) increases.

\[ \square \]

Lemma 3.8 will be used to prove the existence of homoclinic loops and get a homoclinic bifurcation surface in the following proposition.

**Proposition 3.1.** There is an increasing \( C^\infty \) function \( \varrho_1(\mu_3) \) such that \( \mu_3/2 < \varrho_1(\mu_3) < \mu_3 \) and

(a): system (1.1b) exhibits one figure-eight loop if and only if \( \mu_2 = \varrho_1(\mu_3) \sqrt{\mu_1}; \)

(b): when \( \varrho_1(\mu_3) \sqrt{\mu_1} < \mu_2 < \mu_3 \sqrt{\mu_1}, \) system (1.1b) exhibits three limit cycles, where two limit cycles are unstable and small, another one is stable and large;

(c): when \( \mu_2 = \varrho_1(\mu_3) \sqrt{\mu_1}, \) system (1.1b) has a unique limit cycle, which is stable and large.

**Proof.** For simplicity, we first prove the statements (a), (b) and (c) for equivalent system (3.25). Then consider the monotonicity and smoothness for \( \varrho_1(\mu_3). \) By the symmetry, system (3.25) exhibits one figure-eight loop if and only if \( x^{c}_{A}(\mu, \mu_3) - x^{c}_{B}(\mu, \mu_3) = 0. \)

\[ \text{Figure 15. Unstable and stable manifolds in the right half plane.} \]
For the statement \((a)\), since \(x_A^0(\mu, \mu_3)\) increases and \(x_B^0(\mu, \mu_3)\) decreases continuously with respect to \(\mu\), we can get the existence and uniqueness of homoclinic loops by showing that \(x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3)\) has different signs for \(\mu = \mu_3/2\) and \(\mu = \mu_3\). When \(\mu = \mu_3/2\), we get that \(\mu_1 > 0\) and \(\mu_2 = \mu_3 = \sqrt{\mu_1}/2\). By Lemmas 3.3 there are no homoclinic loops for system (3.25), which implies \(x_A^0(\mu_3/2, \mu_3) - x_B^0(\mu_3/2, \mu_3) \neq 0\). From Lemma 2.1 we know that \(E_r\) is stable when \(\mu_1 > 0\) and \(\mu_2 = \mu_3 = \sqrt{\mu_1}/2\). So is (1, 0) of system (3.25). If \(x_A^0(\mu_3/2, \mu_3) - x_B^0(\mu_3/2, \mu_3) > 0\), due to the sign of \(y\) on the \(x\)-axis, an annular region (see Figure 15 (A)), whose \(\alpha\)-limit set lies in itself, can be constructed. By Poincaré-Bendixson Theorem [8], there is at least one small limit cycle, which contradicts the conclusion of Lemma 3.3. Then \(x_A^0(\mu_3/2, \mu_3) - x_B^0(\mu_3/2, \mu_3) < 0\).

When \(\mu = \mu_3\), we get that \(\mu_1 > 0\) and \(\mu_2 = \mu_3 = \sqrt{\mu_1}\). It follows from Lemma 3.6 that there are no homoclinic loops for system (3.25), which implies \(x_A^0(\mu_3, \mu_3) - x_B^0(\mu_3, \mu_3) \neq 0\). By Lemma 2.1, \(E_r\) is an unstable focus when \(\mu_1 > 0\) and \(\mu_2 = \mu_3 = \sqrt{\mu_1}\). So is (1, 0) of system 3.25. If \(x_A^0(\mu_3, \mu_3) - x_B^0(\mu_3, \mu_3) < 0\), an annular region (see Figure 15 (B)), whose \(\omega\)-limit set lies in itself, can be constructed. By Poincaré-Bendixson Theorem [8], at least one small limit cycle exists, which contradicts the conclusion of Lemma 3.6. Then \(x_A^0(\mu_3, \mu_3) - x_B^0(\mu_3, \mu_3) > 0\). Thus, there exists a unique function \(\mu = \ell_1(\mu_3) \in (\mu_3/2, \mu_3)\) such that \(x_A^0(\ell_1(\mu_3), \mu_3) - x_B^0(\ell_1(\mu_3), \mu_3) = 0\), implying that system (3.25) has a unique homoclinic loop in the right half plane. Moreover, for system (1.1b) there is a unique function \(\mu_3/2 < \ell_1(\mu_3) < \mu_3\), such that it has a figure-eight homoclinic loop when \(\mu_2 = \ell_1(\mu_3) = \sqrt{\mu_1}\).

For the statement \((b)\), \(\ell_1(\mu_3) = \sqrt{\mu_1} < \mu_2 < \mu_3 = \sqrt{\mu_1}\) implies \(\mu > \ell_1(\mu_3)\). Since \(x_A^0(\mu, \mu_3)\) increases and \(x_B^0(\mu, \mu_3)\) decreases continuously as \(\mu\) increases, it follows from the statement \((a)\) that \(x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3) > 0\). From Lemma 2.1, \(E_r\) is stable when \(\mu_2 < \mu_3\). So is (1, 0) of system (3.25). Then one can construct an annular region (see Figure 15 (A)), whose \(\alpha\)-limit set lies in itself. By Poincaré-Bendixson Theorem, at least one limit cycle exists in the strip \(x \in (0, x_B^0(\mu, \mu_3))\). Meanwhile, the uniqueness and instability of this small limit cycle can be obtained by Lemma 3.4. By symmetry, system (1.1b) exhibits two small limit cycles when \(\ell_1(\mu_3) = \sqrt{\mu_1} < \mu_2 < \mu_3 = \sqrt{\mu_1}\), which are unstable and surrounding \(E_r\), \(E_l\) respectively.

The existence of large limit cycles are obtained by Poincaré-Bendixson Theorem. Similar to the proof of Lemma 3.7, the outer boundary of an annular region is given by the fact that all the orbits are positively bounded. Since \(x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3) > 0\), the unstable manifolds and stable manifolds of system (3.25) at the origin and two straight segments on the \(x\)-axis form the inner boundary. Then an annular region whose \(\omega\)-limit set lies in itself, can be constructed.

Assume that there are at least two large limit cycles for (3.25) when \(\ell_1(\mu_3) = \sqrt{\mu_1} < \mu_2 < \mu_3 = \sqrt{\mu_1}\). It follows from \(x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3) > 0\) that the innermost one is internally stable. Consider the equivalent system (2.13), which has the same topological structure with (3.25) when \(\mu_1 > 0\). Then system (2.13) exhibits at least two large limit cycles. Denote the innermost large limit cycles for (2.13) by \(L_1\) and second inner one by \(L_2\). Thus, \(L_1\) is internally stable, implying that \(\int_{L_1} \text{div}(y - F(x), g(x)) \leq 0\). By Lemma 3.5,

\[
\int_{L_2} \text{div}(y - F(x), g(x)) < \int_{L_1} \text{div}(y - F(x), g(x)) \leq 0.
\]

That means \(L_2\) is stable and \(L_1\) is externally unstable. Moreover, the innermost limit cycle of system (3.25) is internally stable and externally unstable. Assume
that the innermost limit cycle of system (3.25) intersects the positive y-axis at (0, c) and denote it by $\Gamma_c$. Then $x_A^c(\mu, \mu_3) - x_B^c(\mu, \mu_3) = 0$. By Lemma 3.8, $x_A^c(\mu - \epsilon, \mu_3) - x_B^c(\mu - \epsilon, \mu_3) < 0$, where $\epsilon > 0$. Due to [13, Theorem 3.4 of Chapter 3.4], at least two limit cycles are bifurcated from $\Gamma_c$, including a stable inner limit cycle and an unstable outer one, when $\mu$ changes to a smaller value. That conflicts with the conclusion of Lemma 3.5, which tells us the divergency of inner limit cycle is greater. Thus, system (3.25) exhibits a unique large limit cycle, which is stable. As a result, there are exactly three limit cycles for system (1.1b) when $g_1(\mu_3)\sqrt{\mu_1} < \mu_2 < \mu_3\sqrt{\mu_1}$, two of them are small and unstable and the remaining one is large and stable.

For the statement (c), it follows from the statement (a) that system (1.1b) exhibits a figure-eight homoclinic loop. Notice that $\text{tr}J_0 = \mu_2 > 0$, where $J_0$ is the Jacobian matrix of system (1.1b) at the saddle $E_0$. By [5, Theorem 3.3], the homoclinic loop of $E_0$ is asymptotically unstable. Then if there are small limit cycles for system (1.1b), the one closest to the homoclinic loop must be externally stable. Thus, the nonexistence of small limit cycles can be obtained from Lemma 3.4. Similar to the statement (b) we can get the existence, uniqueness and stability of large limit cycles.

Now we consider the monotonicity and smoothness of $g_1(\mu_3)$. From the statement (a), there exists a figure-eight homoclinic loop surrounding $(-1, 0)$ and $(1, 0)$ when $\mu = g_1(\mu_3)$, implying that $x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3) = 0$. As $\mu_3$ changes to $\mu_3 + \delta$, the increment of $\mu$ must be $\epsilon = g(\mu_3 + \delta) - g(\mu_3)$ to keep the existence of a figure-eight homoclinic loop, i.e., $x_A^0(\mu + \epsilon, \mu_3 + \delta) - x_B^0(\mu + \epsilon, \mu_3 + \delta) = 0$. Then

$$
x_A^0(\mu, \mu_3 + \delta) - x_A^0(\mu, \mu_3) + x_B^0(\mu + \epsilon, \mu_3 + \delta) - x_B^0(\mu, \mu_3 + \delta) = x_A^0(\mu, \mu_3 + \delta) - x_B^0(\mu, \mu_3 + \delta).
$$

One can calculate that when $\delta > 0$

$$
x_A^0(\mu, \mu_3 + \delta) - x_A^0(\mu, \mu_3) = \int_{x_A^0(\mu, \mu_3)}^{x_A^0(\mu, \mu_3 + \delta)} \frac{dz}{x_A^0(\mu, \mu_3 + \delta)}
$$

$$
= \int_0^{x_A^0(\mu, \mu_3 + \delta)} \frac{dx}{x_A^0(\mu, \mu_3 + \delta) - x_A^0(\mu, \mu_3)}
$$

$$
= \int_0^{x_A^0(\mu, \mu_3 + \delta)} \frac{dy}{y - x^3 - \mu x + \mu_3 x y} dy
$$

$$
= \int_0^{x_A^0(\mu, \mu_3 + \delta)} \frac{dy}{y - x^3 - \mu x + \mu_3 x y} + O(y^3)
$$

$$
= \left[ \frac{y^2}{2(x_A^0(\mu, \mu_3) - (x_A^0(\mu, \mu_3))^3) + O(y^3)} - \frac{\delta}{x_A^0(\mu, \mu_3)} \right]_0^{x_A^0(\mu, \mu_3 + \delta)}
$$

$$
= \delta^2 \left( \int_0^{x_A^0(\mu, \mu_3 + \delta)} \tau \exp \left\{ \int_{\tau}^{x_A^0(\mu, \mu_3 + \delta)} H_3(\eta) d\eta \right\} d\tau \right)^2 + O(\delta^3) \tag{3.34}
$$

and

$$
x_B^0(\mu, \mu_3 + \delta) - x_B^0(\mu, \mu_3) = \int_{x_B^0(\mu, \mu_3)}^{x_B^0(\mu, \mu_3 + \delta)} \frac{dx}{x_B^0(\mu, \mu_3 + \delta) - x_B^0(\mu, \mu_3)}
$$

$$
= \int_0^{x_B^0(\mu, \mu_3)} \frac{dy}{y - x_B^0(\mu, \mu_3) + \mu_3 x y} dy
$$

$$
= \int_0^{x_B^0(\mu, \mu_3)} \frac{dy}{y - x_B^0(\mu, \mu_3) + \mu_3 x y} + O(y^3)
$$

$$
= \delta^2 \left( \int_0^{x_B^0(\mu, \mu_3)} \tau \exp \left\{ \int_{\tau}^{x_B^0(\mu, \mu_3)} H_3(\eta) d\eta \right\} d\tau \right)^2 + O(\delta^3). \tag{3.35}
$$
By Lemma 3.8, \( x_A^0(\mu, \mu_3 + \delta) - x_B^0(\mu, \mu_3) < 0 \) and \( x_B^0(\mu, \mu_3 + \delta) - x_B^0(\mu, \mu_3) > 0 \) when \( \delta > 0 \). It follows from (3.33) that \( x_A^0(\mu + \epsilon, \mu_3 + \delta) - x_A^0(\mu, \mu_3 + \delta) > 0 \) and \( x_B^0(\mu + \epsilon, \mu_3 + \delta) - x_B^0(\mu, \mu_3 + \delta) < 0 \), implying \( \epsilon > 0 \). Thus, \( x_A^0(\mu, \mu_3 + \delta) - x_A^0(\mu, \mu_3 + \delta) \)

\[
\epsilon^2 \left( \int_{\eta_0}^{\eta_1(x)} \exp \left\{ \int_{\eta}^{\eta_1(x)} \tilde{H}_3(\eta) d\eta \right\} d\eta \right)^2 \rightarrow 0 + O(\epsilon^3) \quad (3.36)
\]
and

\[
x_B^0(\mu + \epsilon, \mu_3 + \delta) - x_B^0(\mu, \mu_3 + \delta) \]

\[
\epsilon^2 \left( \int_{\eta_0}^{\eta_1(x)} \exp \left\{ \int_{\eta}^{\eta_1(x)} \tilde{H}_3(\eta) d\eta \right\} d\eta \right)^2 \rightarrow 0 + O(\epsilon^3), \quad (3.37)
\]

where

\[
\tilde{H}_3(x) = \frac{\langle y_{\mu+\epsilon}\rangle(x) \cdot y_{\mu+\epsilon}(x)}{y_{\mu+\epsilon}(x) \cdot y_{\mu+\epsilon}(x)}.
\]

Substituting (3.34-3.37) into (3.33) and taking limits for the both sides as \( \delta \rightarrow 0 \), it is easy to obtain that \( \lim_{\delta \rightarrow 0} \epsilon = 0 \). Moreover, we get

\[
\varrho'(\mu_3) = \lim_{\delta \rightarrow 0} \frac{\epsilon}{\delta} = \frac{\left( \int_{\eta_0}^{\eta_1(x)} \exp \left\{ \int_{\eta}^{\eta_1(x)} \tilde{H}_3(\eta) d\eta \right\} d\eta \right)^2}{2(\varrho^0_{\mu} - \varrho^0_{\mu_3})^2} + O(\epsilon) + O(\delta)
\]

\[
\left( \frac{\int_{\eta_0}^{\eta_1(x)} \exp \left\{ \int_{\eta}^{\eta_1(x)} \tilde{H}_3(\eta) d\eta \right\} d\eta \right)^2 + 2(\varrho^0_{\mu} - \varrho^0_{\mu_3})^2 \right) > 0, \quad (3.38)
\]

where \( x_A^0 = x_A^0(\mu, \mu_3) \), \( x_B^0 = x_B^0(\mu, \mu_3) \), \( H^*_3(x) = (-x + x^3)/y_{\mu+\epsilon}(x) \) and \( H^*_3(x) = (\varrho^0_{\mu} - \varrho^0_{\mu_3})^2 \). When \( \delta < 0 \), we can similarly prove that \( \epsilon < 0 \) and (3.38) still holds. It follows from (3.38) that \( \varrho'(\mu_3) > 0 \). The expression (3.38) shows that \( \varrho_1(\mu_3) \) is \( C^1 \). Furthermore, \( \varrho'(\mu_3) \) has the same smoothness as \( \varrho_1(\mu_3) \), implying the \( C^\infty \) of \( \varrho_1(\mu_3) \).

**Proposition 3.2.** There exists an increasing \( C^0 \) function \( \varrho_2(\mu_3) \) such that \( \mu_3/2 > \varrho_2(\mu_3) > \varrho_1(\mu_3) \) and

(a): when \( \mu_2 = \varrho_2(\mu_3) \), system (1.1b) exhibits a unique limit cycle, which is internally unstable, externally stable and large;

(b): when \( \varrho_2(\mu_3) \) is between \( \varrho_1(\mu_3) \) and \( \mu_2 \), system (1.1b) exhibits two limit cycles, which are large. The inner one is unstable and the outer one is stable;

(c): when \( \mu_3 \) is between \( \mu_2 \) and \( \varrho_2(\mu_3) \), system (1.1b) exhibits no limit cycles.

**Proof.** Firstly, we prove that there are no small limit cycles when \( \mu_3 \) is below \( \varrho_2(\mu_3) \). If system (1.1b) exhibits small limit cycles, there is at least one small limit cycle surrounding equilibrium (1, 0) for system (3.25). Obviously, \( \mu_2 < \varrho_1(\mu_3) \) implies \( \mu < \varrho_1(\mu_3) \). By Proposition 3.1 \( x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3) = 0 \) when \( \mu = \varrho_1(\mu_3) \). It follows from Lemma 3.8 that \( x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3) < 0 \) when
\( \mu < g_1(\mu_3) \). Then the outmost small limit cycle surrounding equilibrium \((1, 0)\) must be externally stable. It contradicts the conclusion of Lemma 3.4.

Secondly, we prove that there are at most two large limit cycles. Moreover, the inner one is unstable and the outer one is stable if they exist. Assume that there are at least three large limit cycles for \((3.25)\). Then for the equivalent system \((2.13)\) there are also at least three large limit cycles. Denote the innermost large limit cycles, seconde inner one and third inner one for \((2.13)\) by \(L_1, L_2\) and \(L_3\) respectively. Thus, \(\int_{L_1} \text{div}(y - F(x), g(x)) \geq 0\). By Lemma 3.5,

\[
\int_{L_1} \text{div}(y - F(x), g(x)) > \int_{L_2} \text{div}(y - F(x), g(x)) > \int_{L_3} \text{div}(y - F(x), g(x)).
\]

Since any two closed orbits with the same stability cannot be adjacent to each other,

\[
\int_{L_1} \text{div}(y - F(x), g(x)) > 0, \quad \int_{L_2} \text{div}(y - F(x), g(x)) = 0, \quad \int_{L_3} \text{div}(y - F(x), g(x)) < 0,
\]

implying that \(L_1\) is unstable, \(L_3\) is stable and \(L_2\) is internally stable and externally unstable. Moreover, the second inner limit cycle \(\Gamma\) of system \((3.25)\) is also internally stable and externally unstable. Similar to the prove of the statement (b) of Proposition 3.1, at least two limit cycles are bifurcated from \(\Gamma\), including a stable inner limit cycle and an unstable outer one, when \(\mu\) changes to a smaller value. That contradicts the conclusion of Lemma 3.5.

Assume that system \((3.25)\) exhibits two large limit cycles, denoted by \(\Gamma_1\) and \(\Gamma_2\) from insider to outsider. Then \(\Gamma_1\) is internally unstable because \(x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3) < 0\). By Proposition 2.4, all the orbits are positively bounded, yielding that \(\Gamma_2\) is externally stable. If \(\Gamma_1\) is internally unstable and externally stable, so is \(\Gamma_2\). Then the divergencies of of equivalent system \((2.13)\) along corresponding large limit cycles are both zero, which contradicts the conclusion of Lemma 3.5. Thus, \(\Gamma_1\) is unstable and \(\Gamma_2\) is stable.

Thirdly, we give the existence interval of large limit cycles on the \(y\)-axis. By Proposition 3.2, there is a unique figure-eight homoclinic loop and a unique large limit cycle \(\Gamma_{\mu_3}\) for system \((3.25)\) when \(\mu_2 = g_1(\mu_3)\sqrt{\mu_1}\). Assume that \(\Gamma_{\mu_3}\) intersects the positive \(y\)-axis at \((0, c_{\mu_3})\). We can prove that any large limit cycle must cross the \(y\)-axis at a point between 0 and \(c_{\mu_3}\) when \(\mu_3/2 < \mu < g_1(\mu_3)\). In fact, since the large limit cycle is stable,

\[
x_A^{c_{\mu_3}+\epsilon}(g_1(\mu_3), \mu_3) - x_B^{c_{\mu_3}+\epsilon}(g_1(\mu_3), \mu_3) < 0,
\]

where \(\epsilon > 0\) is sufficiently small. Obviously, \(x_A^c(\mu, \mu_3)\) and \(x_B^c(\mu, \mu_3)\) depend continuously on \(c\) by the continuous dependence of the solution on parameters and initial values. Then

\[
x_A^c(g_1(\mu_3), \mu_3) - x_B^c(g_1(\mu_3), \mu_3) < 0,
\]

for any \(c > c_{\mu_3}\). If not, there exists another large limit cycle distinct from \(\Gamma_{\mu_3}\). It follows from Lemma 3.8 that when \(\mu < g_1(\mu_3)\)

\[
x_A^c(\mu, \mu_3) - x_B^c(\mu, \mu_3) < 0,
\]

for any \(c \geq c_{\mu_3}\).

Fourthly, we give the proof for statements (a), (b) and (c). We can prove the statement (a) by considering the maximum value of \(x_A^c(\mu, \mu_3) - x_B^c(\mu, \mu_3)\) over the interval \([0, c_{\mu_3}]\). Since \(x_A^c(\mu, \mu_3)\) and \(x_B^c(\mu, \mu_3)\) depend continuously on \(c\), the maximum value of \(x_A^c(\mu, \mu_3) - x_B^c(\mu, \mu_3)\) over the interval \([0, c_{\mu_3}]\) exits, denoted by \(M(\mu, \mu_3)\). Notice that the homoclinic loop is unstable and the large limit cycle is
stable when $\mu = g_1(\mu_3)$. It is easy to check that $x_A^c(\mu_3, \mu_3) - x_B^c(\mu_1(\mu_3), \mu_3) > 0$ for $c \in (0, c_{\mu_3})$. Then $M(\mu_1(\mu_3), \mu_3) > 0$.

By Lemma 3.8, we have that $x_A^0(\mu_3/2, \mu_3) - x_B^0(\mu_3/2, \mu_3) < 0$ and $x_A^{c_{\mu_3}}(\mu_3/2, \mu_3) - x_B^{c_{\mu_3}}(\mu_3/2, \mu_3) < 0$ because $\mu = \mu_3/2 < g_1(\mu_3)$. Thus, $x_A^c(\mu_3/2, \mu_3) - x_B(\mu_3/2, \mu_3) < 0$ for all $0 < c \leq c_{\mu_3}$. If not, by the continuity of $x_A^c(\mu_3/2, \mu_3) - x_B(\mu_3/2, \mu_3)$ with respect to $c$ there exists a $c^*$ such that $x_A^c(\mu_3/2, \mu_3) - x_B^c(\mu_3/2, \mu_3) = 0$, which conflicts with the conclusion of Lemma 3.3. Thus, $M(\mu_3/2, \mu_3) < 0$.

We claim that $M(\mu, \mu_3)$ continuously increases as $\mu$ increases. In fact, it is easy to check that $M(\mu, \mu_3)$ is continuous with respect to $\mu$ because both $x_A^c(\mu, \mu_3)$ and $x_B^c(\mu, \mu_3)$ depend continuously on $\mu$ for any $c \in (0, c_{\mu_3})$. For any $\epsilon > 0$,

$$M(\mu + \epsilon, \mu_3) - M(\mu, \mu_3) = M(\mu + \epsilon, \mu_3) - (x_A^c(\mu, \mu_3) - x_B^c(\mu, \mu_3))$$

$$\geq M(\mu + \epsilon, \mu_3) - (x_A^c(\mu + \epsilon, \mu_3) - x_B^c(\mu + \epsilon, \mu_3))$$

where $c = c^*$ is the point such that $x_A^c(\mu, \mu_3) - x_B^c(\mu, \mu_3)$ takes the maximum value. It follows from $M(\mu_1(\mu_3), \mu_3) > 0$ and $M(\mu_3/2, \mu_3) < 0$ that there exists a unique $g_2(\mu_3) \in (\mu_3/2, g_1(\mu_3))$ such that $M(g_2(\mu_3), \mu_3) = 0$. That implies when $\mu_2 = g_2(\mu_3)\sqrt{\mu_1}$ system (1.1b) exhibits a unique limit cycle. Assume that this limit cycle intersects the $y$-axis at $(0, c^*)$. Then $x_A^c(g_2(\mu_3), \mu_3) - x_B^c(g_2(\mu_3), \mu_3) < 0$ for $c \in [0, c^*]$, implying that the limit cycle is internally unstable, externally stable. Thus, the conclusion (a) holds.

For the statement (b), $g_2(\mu_3)\sqrt{\mu_1} < \mu_2 < g_1(\mu_3)\sqrt{\mu_1}$ implies that $g_2(\mu_3) < \mu < g_1(\mu_3)$. By the monotonicity of $M(\mu, \mu_3)$, we get $M(\mu, \mu_3) > 0$. Then there exists a $c_1^* \in (0, c_{\mu_3})$ such that

$$x_A^{c_1^*}(\mu, \mu_3) - x_B^{c_1^*}(\mu, \mu_3) > 0.$$  

It follows from Lemma 3.8 that

$$x_A^0(\mu, \mu_3) - x_B^0(\mu, \mu_3) < x_A^0(g_1(\mu_3), \mu_3) - x_B^0(g_1(\mu_3), \mu_3) = 0$$

and

$$x_A^{c_{\mu_3}}(\mu, \mu_3) - x_B^{c_{\mu_3}}(\mu, \mu_3) < x_A^{c_{\mu_3}}(g_1(\mu_3), \mu_3) - x_B^{c_{\mu_3}}(g_1(\mu_3), \mu_3) = 0$$

when $\mu < g_1(\mu_3)$. Thus, there exist $c_1 \in (0, c_1^*)$ and $c_2 \in (c_1^*, c_{\mu_3})$ such that

$$x_A^{c_1}(\mu, \mu_3) - x_B^{c_1}(\mu, \mu_3) = 0, \quad x_A^{c_2}(\mu, \mu_3) - x_B^{c_2}(\mu, \mu_3) = 0.$$  

That means there are two large limit cycles for system (1.1b). From the first and second steps, there are no other limit cycles and the inner limit cycle is unstable and the outer one is stable. Thus, the conclusion (b) is proven.

For the statement (c), $\mu_2 < g_2(\mu_3)\sqrt{\mu_1}$ implies that $\mu < g_2(\mu_3)$. By the monotonicity of $M(\mu, \mu_3)$, we get $M(\mu, \mu_3) < 0$. Then $x_A^c(\mu, \mu_3) - x_B^c(\mu, \mu_3) < 0$ for all $c \in (0, c_{\mu_3})$, implying that there are no limit cycles across the $y$-axis at the point between 0 and $c_{\mu_3}$. From the first and third step, there are no limit cycles for system (1.1b) when $\mu_3\sqrt{\mu_1}/2 < \mu_2 < g_2(\mu_3)\sqrt{\mu_1}$. Thus, the conclusion (c) is proven.

Finally, we show that $g_2(\mu_3)$ is increasing and $C^0$. From the third step, we know

$$x_A^c(\mu_3, \mu_3) - x_B^c(\mu_3, \mu_3) < 0, \quad \forall c \in (c_{\mu_3}, \infty)$$

when $\mu_3/2 < \mu < g_1(\mu_3)\sqrt{\mu_1}$. Then for $0 \leq c \leq c_{\mu_3} + \delta$,

$$x_A^c(g_2(\mu_3), \mu_3 + \delta) - x_B^c(g_2(\mu_3), \mu_3 + \delta) < x_A^c(g_2(\mu_3), \mu_3) - x_B^c(g_2(\mu_3), \mu_3) \leq 0,$$
where $\delta > 0$ is sufficiently small. That means $M(\varphi_2(\mu_3), \mu_3 + \delta) < 0$. From the monotonicity of $M(\mu, \mu_3)$ with respect to $\mu$ and $M(\varphi_2(\mu_3 + \delta), \mu_3 + \delta) = 0$, we get that $\varphi_2(\mu_3 + \delta) - \varphi_2(\mu_3) > 0$ and $\varphi_2(\mu_3)$ is increasing. Moreover,

$$\lim_{\delta \to 0} (x'_a(\varphi_2(\mu_3), \mu_3 + \delta) - x'_b(\varphi_2(\mu_3), \mu_3 + \delta)) = \lim_{\delta \to 0} (x'_a(\varphi_2(\mu_3), \mu_3) - x'_b(\varphi_2(\mu_3), \mu_3))$$

for any $c \in [0, 0_{\varphi_2}]$, implying that

$$\lim_{\delta \to 0} (M(\varphi_2(\mu_3), \mu_3 + \delta)) = M(\varphi_2(\mu_3), \mu_3) = 0.$$

By the uniqueness of $\varphi_2(\mu_3)$, we get that $\lim_{\delta \to 0} \varphi_2(\mu_3 + \delta) - \varphi_2(\mu_3) = 0$, which completes the proof. □

4. Proofs of Theorems 1.1-1.3. Proof of Theorem 1.1 The statements (a) and (b) can be obtained from Proposition 2.3. Furthermore, the statements (c) and (d) are directly from Propositions 3.1 and 3.2. By Lemma 2.3, equilibria at infinity become degenerate saddle-nodes from cusps as $\mu_3$ increases from a value less than $2\sqrt{2}$ to a value greater than or equal to $2\sqrt{2}$. Then the statement (e) is proven.

Proof of Theorem 1.2 When $0 < \mu_3 < 2\sqrt{2}$, by Lemma 2.3 system (1.1b) exhibits two infinite equilibrium $I_y^-$ and $I_y^+$, which are cusps. Then there are no orbits connecting $I_y^-$ or $I_y^+$ from points at infinity. Moreover, from Proposition 2.4 all the orbits are positive bounded, as shown in Figure 7.

Consider $(\mu_1, \mu_2, \mu_3) \in G_1$ or $(\mu_1, \mu_2, \mu_3) \in P_1$. By Lemma 2.2, system (1.1b) exhibits a unique finite equilibrium $E_0$, which is unstable. It follows from Lemma 3.2 that there is a unique limit cycle for system (1.1b), which is stable. Thus, the limit cycle is the $\omega$-limit set of all the orbits except $E_0$.

Consider $(\mu_1, \mu_2, \mu_3) \in G_2$ or $(\mu_1, \mu_2, \mu_3) \in P_2$ or $(\mu_1, \mu_2, \mu_3) \in H_2$. By Lemma 2.2 the unique finite equilibrium $E_0$ of system (1.1b) is stable. From Lemma 3.1 there are no closed orbits for system (1.1b). Thus, $E_0$ is the $\omega$-limit set of all the orbits.

Consider $(\mu_1, \mu_2, \mu_3) \in G_3$ or $(\mu_1, \mu_2, \mu_3) \in DL$ or $(\mu_1, \mu_2, \mu_3) \in G_4$. By Lemma 2.1 there are three finite equilibria $E_0$, $E_l$ and $E_r$. Obviously, $E_l$ and $E_r$ are stable because $\varphi_2(\mu_3) < g_3(\mu_3) < \mu_3$, and $E_0$ is a saddle. It follows from Lemmas 3.1, 3.3 and Proposition 3.2 that there are no small limit cycles for system (1.1b). According to the aforementioned explanation of Lemma 3.8, the unstable manifold on the right-hand side of $E_0$ will cross the positive $x$-axis. Notice that

$$\dot{x}|_{x=0} = \dot{y}, \quad \dot{y}|_{y=0} = \mu_1 x - x^3 \begin{cases} > 0, & \text{if } 0 < x < \sqrt{\mu_1}, \\ < 0, & \text{if } x > \sqrt{\mu_1}. \end{cases}$$

Then the stable manifold on the right-hand side cannot intersect the positive $x$-axis at the point which is on the left of the intersection of the unstable manifold on the right-hand side and the positive $x$-axis. If not, an annular region can be construct whose $\alpha$-limit set lies in itself (see Figure 15 (A)), which contradicts the nonexistence of small limit cycles. Thus, the $\omega$-limit set of the unstable manifold on the right-hand side must be $E_r$. By the symmetry of system (1.1b), the $\omega$-limit set of the unstable manifold on the left-hand side must be $E_l$. Due to Lemmas 3.1, 3.3 and Proposition 3.2, system (1.1b) exhibits no large limit cycles, a unique large limit cycle and two large limit cycles when $(\mu_1, \mu_2, \mu_3) \in G_3$, $(\mu_1, \mu_2, \mu_3) \in DL$ and $(\mu_1, \mu_2, \mu_3) \in G_4$ respectively. Combining the stability of limit cycles, we can get the global phase portraits in the Poincaré disk.
Consider \((\mu_1, \mu_2, \mu_3) \in HL\). Similar to the case when \((\mu_1, \mu_2, \mu_3) \in G_4, E_0\) is a saddle and \(E_l, E_r\) are stable. By Proposition 3.1, there is a figure-eight homoclinic loop and a unique large limit cycle for system (1.1b).

Consider \((\mu_1, \mu_2, \mu_3) \in G_5\). Similar to the case when \((\mu_1, \mu_2, \mu_3) \in G_4, E_0\) is a saddle and \(E_l, E_r\) are stable. But by Proposition 3.1 there is a pair of small limit cycles for system (1.1b), which are unstable. Then the stable manifold on the right-hand side of \(E_0\) must intersect the positive \(x\)-axis at the point which is on the left of the intersection of the unstable manifold on the right-hand side and the positive \(x\)-axis. If not, it conflicts with the stability of small limit cycle surrounding \(E_r\). Thus, the \(\alpha\)-limit set of the stable manifold on the right-hand side must be the small limit cycle surrounding \(E_r\). By the symmetry of system (1.1b), the \(\alpha\)-limit set of the stable manifold on the left-hand side must be the small limit cycle surrounding \(E_l\). Moreover, the \(\omega\)-limit set of unstable manifolds of \(E_0\) is a large limit cycle, which is unique and stable by Proposition 3.1.

Consider \((\mu_1, \mu_2, \mu_3) \in G_6 \) or \((\mu_1, \mu_2, \mu_3) \in H_1\). By Lemma 2.1 \(E_0\) is a saddle and \(E_l, E_r\) are unstable. Due to Lemma 3.6, there are no small limit cycles for system (1.1b). Then the stable manifold on the right-hand side of \(E_0\) must intersect the positive \(x\)-axis at the point which is on the left of the intersection of the unstable manifold on the right-hand side and the positive \(x\)-axis. If not, an annular region can be construct whose \(\omega\)-limit set lies in itself (see Figure 15 (B)), which contradicts the nonexistence of small limit cycles. Moreover, Lemma 3.7 gives the existence, uniqueness and stability of large limit cycles. □

**Proof of Theorem 1.2** When \(\mu_3 \geq 2\sqrt{2}\), by Lemma 2.3 the infinite equilibrium \(I_y^+\) and \(I_y^-\) are degenerate saddle-nodes, as shown in Figure 5 (B). No matter what positive \(\mu_3\) is, all the orbits are positively bounded for system (1.1b) by Proposition 2.4. Then except at infinity, the qualitative properties of system (1.1b) are the same as those of \(0 < \mu_3 < 2\sqrt{2}\). Moreover, the orbits near infinity connect \(I_y^+\) or \(I_y^-\) when \(t \to -\infty\). The proof is completed. □

5. **Numerical examples and discussions.** In this section, we will show phase portraits and display bifurcations of (1.1b) by numerical simulations. For simplicity, in the following numerical phase portraits, denote “unstable manifold” and “stable manifold” by “UM” and “SM”, respectively.

In the numerical simulation, the qualitative properties of the system (1.1b) at infinity can not be reflected. Except the concavity factor of bifurcation curves, the slice \(u_3 = u_3^*\) of bifurcation diagram are same for different \(u_3^*\). So we fix \(\mu_3 = 1\) in following numerical simulations.

We first consider the case \(\mu_1 \leq 0\). By Lemma 2.2, system (1.1b) exhibits a unique equilibrium \(E_0\).

**Example 1.** When \(\mu_1 = -4, \mu_2 = 1\) and \(\mu_3 = 1\), \(E_0\) is an unstable focus. Moreover, system (1.1b) exhibits a stable limit cycle, as shown in Figure 16 (A). When \(\mu_1 = -4, \mu_2 = -1\) and \(\mu_3 = 1\), \(E_0\) is a stable focus, as shown in Figure 16 (B).

**Example 2.** When \(\mu_1 = 0, \mu_2 = 1\) and \(\mu_3 = 1\), \(E_0\) is an unstable node. Moreover, system (1.1b) exhibits a stable limit cycle, as shown in Figure 17 (A). When \(\mu_1 = 0, \mu_2 = 0\) and \(\mu_3 = 1\), \(E_0\) is a stable focus, as shown in Figure 17 (B).

Then we consider the case \(\mu_1 > 0\). By Lemma 2.1 system (1.1b) exhibits three equilibria \(E_0, E_l\) and \(E_r\). We fix \(\mu_1 = 4\) and \(\mu_3 = 1\) and see how the phase diagram changes as \(\mu_2\) increases.
Example 3. When $\mu_1 = 4$, $\mu_2 = 0$ and $\mu_3 = 1$, $E_0$ is a saddle, $E_l$ and $E_r$ are stable foci, as shown in Figure 18 (A). When $\mu_1 = 4$, $\mu_2 = 1.55779$ and $\mu_3 = 1$, $E_0$ is a saddle, $E_l$ and $E_r$ are stable foci. System (1.1b) has a limit cycle, which is internally unstable, externally stable, as shown in Figure 18 (B). When $\mu_1 = 4$, $\mu_2 = 1.6$ and $\mu_3 = 1$, $E_0$ is a saddle, $E_l$ and $E_r$ are stable foci. System (1.1b) has two limit cycles, the inner one is unstable and the outer one is stable, as shown in Figure 18 (C). When $\mu_1 = 4$, $\mu_2 = 1.67285$ and $\mu_3 = 1$, $E_0$ is a saddle, $E_l$ and $E_r$ are stable foci. System (1.1b) has a stable limit cycle and an unstable figure-eight homoclinic loop, as shown in Figure 18 (D). When $\mu_1 = 4$, $\mu_2 = 1.8$ and $\mu_3 = 1$, $E_0$ is a saddle, $E_l$ and $E_r$ are stable foci. System (1.1b) has three limit cycles, one is stable and large, the other two are unstable and small, as shown in Figure 18 (E). When $\mu_1 = 4$, $\mu_2 = 2$ and $\mu_3 = 1$, $E_0$ is a saddle, $E_l$ and $E_r$ are unstable foci. System (1.1b) has a stable limit cycle, as shown in Figure 18 (F).

Moreover, Example 3 gives us estimations of $\varrho_1$ and $\varrho_2$ at $\mu_3 = 1$. Furthermore, $\varrho_1(1) \approx 1.67285$ and $\varrho_2(1) \approx 1.55779$. 
6. Concluding remarks. In this section we compare the global dynamics of the nonsmooth van der Pol-Duffing oscillator (1.1b) and the van der Pol-Duffing oscillator
\[ \dot{x} = y - (bx + x^3), \quad \dot{y} = ax - x^3, \tag{6.1} \]
where \(a, b\) are real. The bifurcation diagram and global phase portraits in the Poincaré disc of system (6.1) were given by [2]. It is easy to see that system (1.1b) has a similar form with system (6.1), but system (1.1b) cannot be simplified into a
two-parameter system except that the sign of $\mu_1$ or $\mu_2$ is fixed. In addition, the difference between systems (1.1b) and (6.1) is that the highest terms in $\dot{x}$ are $\mu_3 x|x|/2$ and $x^3$ respectively. And the difference does not change the symmetry of system (1.1b) about the origin. However, the bifurcation for equilibria at infinity cannot occur in system (6.1). In conclusion, both the dimension of the bifurcation diagrams of the two systems and all global phase portraits in the Poincaré disc are different. Furthermore, many classic results on smooth vector fields cannot be applied in system (1.1b) since the vector field of system (1.1b) is only $C^1$. Interestingly, when $a$ and $b$ are small, system (6.1) can be changed into a near Hamiltonian system by a scaling transformation so that local bifurcation diagram and local phase portraits of system (6.1) can be obtained by Abelian integral, see [5, Section 2 of Chapter 4]. However, when $\mu_3$ is fixed, $\mu_1$ and $\mu_2$ are small, notice that system (1.1b) cannot be changed into a near Hamiltonian system.

When the sum of indices of equilibria is $-1$, the global dynamics of the van der Pol-Duffing oscillator and the nonsmooth van der Pol-Duffing oscillator were studied respectively in [3, 12]. Moreover, the difference were discussed in [12]. In conclusions, the global dynamics of the van der Pol-Duffing oscillator and the nonsmooth van der Pol-Duffing oscillator have been studied completely by these four papers.

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