Dunkl shift operators and Bannai-Ito polynomials

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Abstract

We consider the most general Dunkl shift operator $L$ with the following properties: (i) $L$ is of first order in the shift operator and involves reflections; (ii) $L$ preserves the space of polynomials of a given degree; (iii) $L$ is potentially self-adjoint. We show that under these conditions, the operator $L$ has eigenfunctions which coincide with the Bannai-Ito polynomials. We construct a polynomial basis which is lower-triangular and two-diagonal with respect to the action of the operator $L$. This allows to express the BI polynomials explicitly. We also present an anti-commutator $AW(3)$ algebra corresponding to this operator. From the representations of this algebra, we derive the structure and recurrence relations of the BI polynomials. We introduce new orthogonal polynomials - referred to as the complementary BI polynomials - as an alternative $q \to -1$ limit of the Askey-Wilson polynomials. These complementary BI polynomials lead to a new explicit expression for the BI polynomials in terms of the ordinary Wilson polynomials.

Keywords: Bannai-Ito polynomials, Dunkl shift operators, Askey-Wilson algebra.

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1. Introduction

Two new families of “classical” polynomials [32], [33] were introduced recently through limits when $q$ goes to -1 of the little and big $q$-Jacobi polynomials. The term ”classical” is taken to mean that the polynomials $P_n(x)$ are eigenfunctions of some differential or difference operator $L$:

$$LP_n(x) = \lambda_n P_n(x). \quad (1.1)$$

In these instances, the operator $L$ is of first order in the derivative operator $\partial_x$ and contains moreover the reflection operator $R$ defined by $Rf(x) = f(-x)$; it can be identified as a first order operator of Dunkl type [10] written as

$$L = F(x)(I - R) + F_1(x)\partial_x + G(x)\partial_x R \quad (1.2)$$

with some real rational functions $F(x), F_1(x), G(x)$.

The eigenvalue equation (1.1) (with $L$ as in (1.2)) has further been investigated generally in [34]. As a rule the polynomial eigenfunctions $P_n(x)$ are not orthogonal. However, if $F_1(x) = 0$ the big and little -1 Jacobi polynomials are seen to be the only classes of orthogonal polynomials consisting in eigenfunctions of such first order Dunkl differential operators.

The main purpose of the present paper is to study a difference analogue of the operator $L$ and the corresponding eigenpolynomials $P_n(x)$. We introduce a natural generalization of the operator $L$ which (apart from the multiplication operator) only contains 2 nontrivial operators: the shift $T^+$ and the reflection $R$. We shall call such operators Dunkl shift operators on the uniform grid. We find the conditions for such operators to transform any polynomial into a polynomial of the same degree. Under these conditions the eigenvalue problem can be posited and we can look for the corresponding eigenpolynomial solutions $P_n(x)$.

The main result of our study is that the polynomial eigenfunctions of such a generic Dunkl shift operator, coincide with the Bannai-Ito (BI) polynomials that depend on 4 parameters.

Recall that the Bannai-Ito polynomials were first proposed in [3] as a $q \rightarrow -1$ limit of the $q$-Racah polynomials. Bannai and Ito showed that these polynomials together with the $q$-Racah polynomials (and their specializations and limiting cases) are the most general orthogonal polynomial systems satisfying the Leonard duality property. Bannai and Ito derived the three-term recurrence relation for their polynomials and also presented explicit expressions in terms of linear combinations of two hypergeometric functions $\text{4F3}(1)$ [3].

Further developments of the theory of BI polynomials and their applications can be found in papers by Terwilliger [29], [30], Curtin [8], [9] and Vidumas [31].

Our approach is completely different: we start with a generic first order Dunkl shift operator that preserves the space of polynomials and then show that its polynomial eigensolutions coincide with the BI polynomials. This approach gives a new type of eigenvalue equation for the BI polynomials - one that involves a combination of the ordinary shift operator with reflections. Moreover, we give an algebraic interpretation of the BI polynomials in terms of a $q = -1$ version of the $\text{AW}(3)$ algebra. This interpretation allows one to derive the structure
and recurrence relations for the BI polynomials. We introduce also a new class of orthogonal polynomials -
the complementary Bannai-Ito polynomials. They can be obtained from the Askey-Wilson polynomials by a
limiting process \( q \to -1 \) which is slightly different from the one used by Bannai and Ito in [3] to obtain their
polynomials. In contrast to the BI polynomials, the complementary BI polynomials do not possess the Leonard
duality property. Nevertheless, the complementary BI polynomials have a very simple expression in terms of
the ordinary Wilson polynomials. Moreover, the complementary BI polynomials are Christoffel transforms of
the BI polynomials. This allows one to present a new explicit expression of the BI polynomials in the form of
a linear combination of two Wilson polynomials.

The present paper is organized as follows.

In the next section we introduce a class of (potentially symmetric) operators \( L \) which has the following
properties:

(i) \( L \) is of first order with respect to \( T^+ \) and contains also the reflection operator \( R \);

(ii) \( L \) preserves the space of polynomials (i.e. it transforms any polynomial into a polynomial of the same
degree).

These properties imply the existence of eigenpolynomials \( P_n(x) \) satisfying (1.1).

In Section 3, we determine these eigenpolynomials \( P_n(x) \) explicitly. Most useful to that end is the observation
that there is a polynomial basis \( (\phi_n(x)) \) in which the operator \( L \) is lower triangular with only 2 diagonals, i.e.

\[
L\phi_n(x) = \lambda_n \phi_n(x) + \nu_n \phi_{n-1}(x).
\]

This property leads to an explicit formula for the polynomials \( P_n(x) \) in terms of a linear combination of two
hypergeometric functions \( {}_4F_3(1) \).

In Section 4, we show that the operator \( L \) together with the operator multiplication by \( x \), forms a \( q = -1 \)
(i.e. an anticommutator) analogue of the AW(3) algebra which we call the Bannai-Ito algebra. Using the
representations of this algebra, we determine the structure relations of the BI polynomials \( P_n(x) \) and show that
these polynomials satisfy expectedly a 3-term recurrence relation

\[
P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x),
\]

where the recurrence coefficients \( u_n, b_n \) are derived explicitly. This allows to identify the eigenpolynomials \( P_n(x) \)
with the Bannai-Ito polynomials first introduced in [3].

In Section 5, we construct the complementary Bannai-Ito polynomials and obtain a new explicit expression
of the BI polynomials in terms of ordinary Wilson polynomials.

In Section 6, we demonstrate that the operator \( L \) is symmetrizable. This means that there exists a function
\( \varphi(x) \) such that the operator \( \varphi(x)L \) is symmetric

\[
(\varphi(x)L)^* = \varphi(x)L,
\]

where \( L^* \) means the formal conjugate of \( L \) (constructed according to natural rules).
In Section 7, we introduce the Bannai-Ito grid, which is a discrete invariant set of the BI operator. The BI operator becomes a 3-diagonal matrix when restricted to this set. Using this observation we derive the weight function for the case when the BI polynomials are orthogonal on a finite set of points of the real line.

In Section 8, it is shown how the BI and complementary BI polynomials can be obtained by limiting processes from the Askey-Wilson polynomials.

In Section 9, it is indicated how the Dunkl shift operator can be derived from the Askey-Wilson difference operator when \( q \to -1 \).

In Section 10, we consider a special symmetric case of the BI polynomials. This leads to the construction of a new family of orthogonal polynomials (explicitly expressed in terms of dual continuous Hahn polynomials) with a purely continuous weight function on the whole real axis. This is the first nontrivial example of an infinite positive family of BI polynomials. (Recall that only finite systems of BI polynomials have been considered in details so far).

In Section 11 we show that a limiting process \( x \to x/h, h \to 0 \) reduces the Dunkl shift operator to a Dunkl differential operator \( L \) having the form (1.2) with \( F_1(x) = 0 \). This means that correspondingly, the Bannai-Ito polynomials tend in this limit to the big -1 Jacobi polynomials introduced in [33].

2. Dunkl shift operators on the uniform grid and their polynomial eigensolutions

In complete analogy with the differential case [34], let us consider the most general linear operator \( L \) of first order with respect to the shift operator \( T^+ \) which also contains the reflection operator \( R \):

\[
L = F_0(x) + F_1(x)R + G_0(x)T^+ + G_1(x)T^+R,
\]

where \( F_0(x), F_1(x), G_0(x), G_1(x) \) are arbitrary functions. The shift operator is defined as usual by:

\[
T^+ = \exp(\partial_x)
\]

so that \( T^+f(x) = f(x + 1) \) for any function \( f(x) \). The operator \( T^+R \) acts on functions \( f(x) \) according to \( T^+Rf(x) = f(-x - 1) \).

We are seeking orthogonal polynomial eigensolutions of the operator \( L \), i.e. for every \( n \) we assume that there exists a monic polynomial \( P_n(x) = x^n + O(x^{n-1}) \) which is an eigenfunction of the operator \( L \) with eigenvalue \( \lambda_n \) (1.1). In what follows we will suppose that

\[
\lambda_n \neq 0 \quad \text{for} \quad n = 1, 2, \ldots, \quad \lambda_n \neq \lambda_m \quad \text{for} \quad n \neq m.
\]

We first establish the necessary conditions for the existence of such eigensolutions.

We will assume the operator \( L \) to be potentially self-adjoint. This leads to the condition

\[
G_0(x) = 0
\]
(see also Sect.5 for further details). On the one hand, the formal adjoint of the operator $T^+$ is the backward-shift operator $(T^+)^* = T^-$, where

$$T^- f(x) = f(x - 1).$$

On the other hand, for the adjoint of the operator $T^+ R$ we have

$$(T^+ R)^* = R^* T^- = R T^- = T^+ R.$$  

Thus, the operator $T^+ R$ is (formally) self-adjoint, while the operator $T^+$ is not. Hence the symmetric (or potentially self-adjoint) operators of type (2.1) cannot contain a term with the operator $T^+$ only.

Moreover, without loss of generality we can assume that $F_0 + F_1 + G_1 = 0$. Indeed, $L\{1\} = constant$. We can assume that $L\{1\} = 0$ (otherwise we can add a constant to $F_0(x)$ to ensure this). This leads to the desired condition.

We thus can present the operator $L$ in the form

$$L = F(x)(I - R) + G(x)(T^+ R - I)$$  (2.4)

with only two unknown functions $F(x), G(x)$. Here $I$ is identity operator.

Equivalently, we can write the action of the operator $L$ on functions $f(x)$ as:

$$Lf(x) = F(x)(f(x) - f(-x)) + G(x)(f(-x - 1) - f(x)).$$  (2.5)

Considering the action of the operator $L$ on $x$ and $x^2$ we arrive at the necessary conditions

$$F(x) = \frac{q_1(x) + q_2(x)}{2x}, \quad G(x) = \frac{q_2(x)}{2x + 1},$$  (2.6)

where by $q_i(x)$, $i = 1, 2$ we mean arbitrary polynomials of degree $i$, i.e.

$$q_1(x) = \xi_1 x + \xi_0, \quad q_2(x) = \eta_2 x^2 + \eta_1 x + \eta_0$$  (2.7)

with arbitrary coefficients $\xi_0, \ldots, \eta_2$.

It is easily seen that these conditions are also sufficient; namely, that the operator $L$ defined by (2.4) with functions $F(x), G(x)$ given by (2.6), (2.7), preserves the linear space of polynomials. More exactly, for any polynomial $Q(x)$ of degree $n$ we have $LQ(x) = \tilde{Q}(x)$, where $\tilde{Q}(x)$ is another polynomial of degree $n$. For monomials $x^n$ we find

$$Lx^n = \lambda_n x^n + O(x^{n-1}),$$  (2.8)

where

$$\lambda_n = \begin{cases} \frac{n \eta_2}{2} & \text{if } n \text{ is even} \\ \xi_1 - \frac{\eta_2(n - 1)}{2} & \text{if } n \text{ is odd} \end{cases}$$  (2.9)

From this expression it is clear that necessarily

$$\eta_2 \neq 0, \quad \xi_1 \neq \eta_2 N, \quad N = 0, 1, 2, \ldots$$  (2.10)
(otherwise the operator $L$ becomes degenerate: it does not preserve the degree of the polynomial).

Assuming that conditions (2.10) are valid, we can construct the eigensolutions

$$LP_n(x) = \lambda_n P_n(x),$$

for every $n = 0, 1, 2, \ldots$, where $P_n(x) = x^n + O(x^{n-1})$ is a monic polynomial of degree $n$.

Since $\eta_2 \neq 0$, we can always assume that $\eta_2 = 1$. We can thus present the functions $F(x)$ and $G(x)$ as

$$G(x) = \frac{(x - r_1 + 1/2)(x - r_2 + 1/2)}{2x + 1}, \quad F(x) = \frac{(x - \rho_1)(x - \rho_2)}{2x}$$

with 4 arbitrary real parameters $r_1, r_2, \rho_1, \rho_2$. We will use this parametrization in what follows.

The eigenvalue $\lambda_n$ has the expression

$$\lambda_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ r_1 + r_2 - \rho_1 - \rho_2 - \frac{(n+1)}{2} & \text{if } n \text{ is odd}. \end{cases}$$

From the properties of the operator $L$ described above, it follows that the monic polynomials $P_n(x)$ exist for all $n = 0, 1, 2, \ldots$ and are the unique polynomial solutions of equation (2.11) for the given eigenvalues $\lambda_n$.

The polynomials $P_n(x) = P_n(x; r_1, r_2, \rho_1, \rho_2)$ depend on 4 parameters $r_1, r_2, \rho_1, \rho_2$. There is however an obvious invariance under the action of the Klein group $Z_2 \times Z_2$. Indeed, the permutations $r_1 \leftrightarrow r_2$ and $\rho_1 \leftrightarrow \rho_2$ leave the operator $L$ unchanged and hence the polynomials $P_n(x; r_1, r_2, \rho_1, \rho_2)$ are invariant as well under these permutations. (With the identity this $Z_2 \times Z_2$ group contains 4 elements.)

3. A 2-diagonal basis and an explicit expression in terms of hypergeometric functions

Choose the following polynomial basis. For even degree, take

$$\phi_{2n}(x) = (x - r_1 + 1/2)_n(-x - r_1 + 1/2)_n$$

and for odd degree, take

$$\phi_{2n+1}(x) = (x - r_1 + 1/2)_{n+1}(-x - r_1 + 1/2)_n = (x - r_1 + 1/2)(x - r_1 + 3/2)_n(-x - r_1 + 1/2)_n.$$  

Here $(x)_n = x(x+1)(x+2)\ldots(x+n-1)$ stands as usual for the shifted factorial (Pochhammer symbol). Obviously, $\phi_n(x)$ is a polynomial of degree $n$:

$$\phi_n(x) = (-1)^{n(n-1)/2}x^n + O(x^{n-1}).$$

It is directly verified that the operator $L$ is two-diagonal in this basis

$$L\phi_n(x) = \lambda_n \phi_n(x) + \nu_n \phi_{n-1}(x),$$
where
\[ \nu_n = \begin{cases} 
\frac{n}{2}(r_1 + r_2 - n^2) & \text{if } n \text{ is even} \\
(r_1 - r_1 + n/2)(\rho_2 - r_1 + n/2) & \text{if } n \text{ is odd}
\end{cases} \] (3.5)

Using this striking observation, we can explicitly construct the polynomial eigensolutions of the operator \( L \) in the same manner as in [33].

Let us expand the polynomials \( P_n(x) \) over the basis \( \phi_n(x) \):
\[ P_n(x) = \sum_{s=0}^{n} A_{ns}\phi_s(x). \]

From the eigenvalue problem (2.11), we have the recurrence relation for the coefficients \( A_{ns} \):
\[ A_{n,s+1} = \frac{A_{ns}(\lambda_n - \lambda_s)}{\nu_{s+1}}. \] (3.6)

Hence the coefficients \( A_{ns} \) can easily be found in terms of \( A_{n0} \):
\[ A_{ns} = A_{n0} \frac{(\lambda_n - \lambda_0)(\lambda_n - \lambda_1) \ldots (\lambda_n - \lambda_{s-1})}{\nu_1\nu_2 \ldots \nu_s}. \] (3.7)

or in terms of the coefficient \( A_{nn} \):
\[ A_{ns} = A_{nn} \frac{\nu_n\nu_{n-1} \ldots \nu_{s+1}}{(\lambda_n - \lambda_{n-1})(\lambda_n - \lambda_{n-2}) \ldots (\lambda_n - \lambda_s)} . \] (3.8)

We thus have the following explicit formula for the polynomials \( P_n(x) \)
\[ P_n(x) = A_{n0} \sum_{s=0}^{n} \frac{(\lambda_n - \lambda_0)(\lambda_n - \lambda_1) \ldots (\lambda_n - \lambda_{s-1})}{\nu_1\nu_2 \ldots \nu_s} \phi_s(x). \] (3.9)

Expression (3.9) resembles Gauss’ hypergeometric function and can be considered as a nontrivial generalization of it.

The expansion coefficients \( A_{ns} \) have different expressions depending on the parity of the numbers \( n \) and \( s \).

When \( n \) is even \( n = 2, 4, 6, \ldots \) we have
\[ A_{n,2s}/A_{n0} = \frac{(-n/2)_s(n/2 + \rho_1 + \rho_2 - r_1 - r_2)_s}{s!(1 - r_2 - r_1)_s(1/2 + \rho_1 - r_1)_s(1/2 + \rho_2 - r_1)_s}, \quad s = 0, 1, \ldots n/2 \] (3.10)
and
\[ A_{n,2s+1}/A_{n0} = \xi_n \frac{(1 - n/2)_s(n/2 + \rho_1 + \rho_2 - r_1 - r_2)_s}{s!(1 - r_2 - r_1)_s(3/2 + \rho_1 - r_1)_s(3/2 + \rho_2 - r_1)_s}, \quad s = 0, 1, \ldots n/2 - 1, \] (3.11)
where
\[ \xi_n = \frac{n}{2(1/2 + \rho_1 - r_1)(1/2 + \rho_2 - r_1)} \]

When \( n \) is odd \( n = 1, 3, 5, \ldots \) then
\[ A_{n,2s}/A_{n0} = \frac{((n - 1)/2)_s((n + 1)/2 + \rho_1 + \rho_2 - r_1 - r_2)_s}{s!(1 - r_2 - r_1)_s(1/2 + \rho_1 - r_1)_s(1/2 + \rho_2 - r_1)_s}, \quad s = 0, 1, \ldots (n - 1)/2 \] (3.12)
and
\[ A_{n,2s+1}/A_{n0} = -\eta_n \frac{((n - 3)/2)_s((n + 3)/2 + \rho_1 + \rho_2 - r_1 - r_2)_s}{s!(1 - r_2 - r_1)_s(3/2 + \rho_1 - r_1)_s(3/2 + \rho_2 - r_1)_s}, \quad s = 0, 1, \ldots (n - 1)/2, \] (3.13)
where
\[ \eta_n = \frac{(n+1)/2 + \rho_1 + \rho_2 - r_1 - r_2}{(1/2 + \rho_1 - r_1)(1/2 + \rho_2 - r_1)}. \]
It is now convenient to separate the even and odd terms in the expression for the polynomials:
\[
P_n(x) = \sum_{s=0} A_{n,2s} \phi_{2s}(x) + \sum_{s=0} A_{n,2s+1} \phi_{2s+1}(x). \quad (3.14)
\]
Taking into account the explicit formulas (3.10)-(3.13) for the expansion coefficients, we finally obtain the following expressions in terms of hypergeometric functions:

(i) If \( n \) is even
\[
\frac{P_n(x)}{A_{n0}} = 4 F_3 \left( \begin{array}{c} -n/2, n/2 + 1 + \rho_1 + \rho_2 - r_1 - r_2, x - r_1 + 1/2, -x + r_1 + 1/2 \\ 1 - r_1 - r_2, 1/2 + \rho_1 - r_1, 1/2 + \rho_2 - r_1 \\ \end{array} ; 1 \right) + \eta_n (x - r_1 + 1/2) 4 F_3 \left( \begin{array}{c} 1 - n/2, n/2 + 1 + \rho_1 + \rho_2 - r_1 - r_2, x - r_1 + 3/2, -x + r_1 + 1/2 \\ 1 - r_1 - r_2, 3/2 + \rho_1 - r_1, 3/2 + \rho_2 - r_1 \\ \end{array} ; 1 \right); \quad (3.15)
\]

(ii) If \( n \) is odd
\[
\frac{P_n(x)}{A_{n0}} = 4 F_3 \left( \begin{array}{c} -(n-1)/2, (n+1)/2 + \rho_1 + \rho_2 - r_1 - r_2, x - r_1 + 1/2, -x + r_1 + 1/2 \\ 1 - r_1 - r_2, 1/2 + \rho_1 - r_1, 1/2 + \rho_2 - r_1 \\ \end{array} ; 1 \right) - \eta_n (x - r_1 + 1/2) 4 F_3 \left( \begin{array}{c} -(n-1)/2, (n+3)/2 + \rho_1 + \rho_2 - r_1 - r_2, x - r_1 + 3/2, -x + r_1 + 1/2 \\ 1 - r_1 - r_2, 3/2 + \rho_1 - r_1, 3/2 + \rho_2 - r_1 \\ \end{array} ; 1 \right). \quad (3.16)
\]

Note that we have terminating hypergeometric functions (i.e. only a finite number of terms in the hypergeometric series appear). Hence, as is easily seen, the expressions in the rhs of (3.15) and (3.16) are polynomials of degree \( n \) in the argument \( x \). The coefficient \( A_{n0} \) (ensuring that the polynomials \( P_n(x) \) are monic) can be found from the coefficient in front of the leading term \( x^n \).

Remark. At first sight one might be tempted to identify the hypergeometric functions \( 4 F_3(1) \) with the Wilson (Racah) polynomials which are known to be expressible in terms of truncated \( 4 F_3(1) \) functions [17]. This is not so, however, because the balance is not right.

Recall that the generalized hypergeometric function
\[ p F_q \left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \\ \end{array} ; 1 \right) \]
is said to be \( k \)-balanced if \( a_1 + a_2 + \ldots + a_p = k + b_1 + b_2 + \ldots + b_q \). The Racah polynomials are expressed in terms of \( -1 \) balanced hypergeometric functions \( 4 F_3(1) \) [17]. In (3.15), (3.16), as is easily seen, all hypergeometric functions \( 4 F_3(1) \) are instead zero-balanced. In what follows we will give another explicit formula for the polynomials \( P_n(x) \) which will relate them to "true" Wilson polynomials.

Remark that the basis \( \phi_n(x) \) is important in Terwilliger’s approach to Leonard duality [29], [30]. Similar bases exist for \( -1 \) Jacobi polynomials as well [33].

Equivalently, the basis \( \phi_n(x) \) can be presented as
\[ \phi_n(x) = (-1)^n(n-1)/2 \prod_{i=1}^n (x - \alpha_i), \]
where
\[ \alpha_n = (-1)^n \left( r_1 + \frac{n}{2} - \frac{1}{4} \right) - \frac{1}{4}. \]

This means that the \( \phi_n(x) \) form a special case of the Newtonian interpolation polynomial basis \( \Omega_n(x) = \prod_{i=1}^{n} (x - \alpha_i) \), where \( \alpha_i \) are fixed interpolation nodes [13]. In the theory of orthogonal polynomials such bases were first proposed by Geronimus [11]. All known explicit examples of orthogonal polynomials (including the Askey-Wilson polynomials at the top level) have simple expansion coefficients with respect to a Newtonian basis \( \Omega_n(x) \) with appropriately chosen nodes \( \alpha_i \) [29], [30]. We thus see that the BI polynomials satisfy this property as well.

4. Bannai-Ito algebra, structure relations and recurrence coefficients for the BI polynomials

In this section we derive the 3-term recurrence relations as well as the structure relations for the BI-polynomials. The main tool will be the Bannai-Ito algebra (BI-algebra for brevity) which is a special case of the AW(3) algebra introduced in [35]. For the Askey-Wilson polynomials the structure relations could be derived in a similar manner using representations of the AW(3) algebra [18], [15]. For a study of the structure relations of the -1 Jacobi polynomials see [28].

Consider the operators
\[ X = 2L + \kappa, \quad Y = 2x + 1/2, \]
where by \( x \) we mean the operator multiplication by \( x \) and where
\[ \kappa = \rho_1 + \rho_2 - r_1 - r_2 + 1/2. \]

We define also the third operator
\[ Z = \frac{r_1 r_2}{x + 1/2} + \frac{\rho_1 \rho_2}{x} - \frac{(x - \rho_1)(x - \rho_2)}{x} R - \frac{(x - r_1 + 1/2)(x - r_2 + 1/2)}{x + 1/2} T + R. \]

It is easily verified that the operator \( Z \) transforms any polynomial of degree \( n \) into a polynomial of degree \( n + 1 \) and that
\[ Z x^n = 2(-1)^{n+1} x^{n+1} + O(x^n). \]

The operators \( X, Y, Z \) are seen to have the following Jordan or anticommutator products which are taken to be the defining relations of the BI algebra:
\[ \{ X, Y \} = Z + \omega_3, \quad \{ Z, Y \} = X + \omega_1, \quad \{ X, Z \} = Y + \omega_2, \]
where
\[ \omega_1 = 4(\rho_1 \rho_2 + r_1 r_2), \quad \omega_2 = 2(\rho_1^2 + \rho_2^2 - r_1^2 - r_2^2), \quad \omega_3 = 4(\rho_1 \rho_2 - r_1 r_2). \]
In the simplest case $\omega_1 = \omega_2 = \omega_3 = 0$, the BI algebra was considered in [1], [12], [25], [26] as an anticommutator analogue of the rotation algebra $so(3)$. A contraction of (4.4) has also been used in [28] to provide an algebraic account of structural properties of the big -1 Jacobi polynomials.

The Casimir operator $Q$ commuting with all generators $X,Y,Z$ of the BI algebra has a simple form:

$$Q = X^2 + Y^2 + Z^2. \quad (4.5)$$

In the given realization of the operators $X,Y,Z$ in terms of difference-reflection operators, this Casimir operator takes the value

$$Q = 2(\rho_1^2 + \rho_2^2 + r_1^2 + r_2^2) - 1/4. \quad (4.6)$$

The BI algebra can be exploited to find explicitly the 3-term recurrence relation of the polynomials $P_n(x)$.

We first introduce two important operators $J_+$ and $J_-$ by the formulas

$$J_+ = (Y + Z)(X - 1/2) - \frac{\omega_2 + \omega_3}{2} \quad (4.7)$$

and

$$J_- = (Y - Z)(X + 1/2) + \frac{\omega_2 - \omega_3}{2}. \quad (4.8)$$

From the commutation relations (4.4), we find that the operators $J_\pm$ satisfy the (anti)commutation relations

$$\{X, J_+\} = J_+, \quad \{X, J_-\} = -J_. \quad (4.9)$$

Moreover, from (4.9) it is seen that both $J_+^2$ and $J_-^2$ commute with the operator $X$:

$$[X, J_+^2] = [X, J_-^2] = 0. \quad (4.10)$$

The operator $J_+$ has the following property: it annihilates any constant $J_+ \{1\} = 0$ and transforms any monomial $x^n$ of even degree $n$ into a polynomial of the same degree $n$ (providing that $r_1 + r_2 \neq n/2$):

$$J_+ x^n = n(n - 2r_1 - 2r_2)x^n + O(x^{n-1}), \quad n = 0, 2, 4, \ldots., \quad (4.11)$$

while any monomial $x^n$ of odd degree is transformed into a polynomial of even degree $n + 1$ (providing that $r_1 + r_2 - \rho_1 - \rho_2 \neq n + 1$)

$$J_+ x^n = 4(r_1 + r_2 - \rho_1 - \rho_2 - n - 1)x^{n+1} + O(x^n), \quad n = 1, 3, 5, \ldots. \quad (4.12)$$

Quite similarly, the operator $J_-$ transforms any monomial $x^n$ of even degree $n$ into a polynomial of odd degree $n + 1$:

$$J_- x^n = 4(\rho_1 + \rho_2 - r_1 - r_2 + n + 1)x^{n+1} + O(x^n), \quad n = 0, 2, 4, \ldots., \quad (4.13)$$

while the monomial $x^n$ of odd degree is transformed to a polynomial of the same degree $n$:

$$J_- x^n = (4(\rho_1 \rho_2 - r_1 r_2) - n^2 + 2n(r_1 + r_2)) x^n + O(x^{n-1}), \quad n = 1, 3, 5, \ldots. \quad (4.14)$$
A careful analysis (where the leading and next to leading monomials are considered), shows that the operator $J_+$ transforms any polynomial of even degree $n$ into a polynomial of degree $n$ or less, while it maps any polynomial of odd degree into a polynomial of exact degree $n + 1$; similarly, it is seen that the operator $J_-$ transforms any polynomial of even degree $n$ into a polynomial of exact degree $n + 1$, while it maps any polynomial of odd degree into a polynomial of degree $n$ or less.

Now, let $\psi_n(x)$ be any eigenfunction of the operator $X$:  

$$X\psi_n(x) = \mu_n \psi_n(x),$$

where  

$$\mu_n = 2\lambda_n + \kappa = (-1)^n(n + \kappa). \quad (4.15)$$

It then follows from relations (4.9) that the function $\tilde{\psi}_n(x) = J_+ \psi_n$ will again be an eigenfunction of the operator $X$ corresponding to the eigenvalue $\tilde{\mu}_n = 1 - \mu_n$. It is seen that $\tilde{\mu}_n = \mu_{n-1}$ if $n$ is even and that $\tilde{\mu}_n = \mu_{n+1}$ if $n$ is odd.

A similar property is found for the operator $J_-$. In this case the function $J_- \psi_n$ will be an eigenfunction of the operator $X$ with eigenvalue $-1 - \mu_n$ which is $\mu_{n+1}$ for even $n$ and $\mu_{n-1}$ for odd $n$.

We have already established that the operators $J_{\pm}$ transform polynomials into polynomials. Hence from the above observations,

$$J_+ P_n(x) = \begin{cases} 
\alpha_n^{(0)} P_{n-1}(x), & \text{if } n \text{ even,} \\
\alpha_n^{(1)} P_{n+1}(x), & \text{if } n \text{ odd}
\end{cases} \quad (4.16)$$

and similarly,

$$J_- P_n(x) = \begin{cases} 
\beta_n^{(0)} P_{n+1}(x), & \text{if } n \text{ even} \\
\beta_n^{(1)} P_{n-1}(x), & \text{if } n \text{ odd}
\end{cases} \quad (4.17)$$

The coefficients in the rhs of these formulas can be found by comparison of highest-order terms:

$$\alpha_n^{(0)} = \frac{2n(r_1 + r_2 + n/2)(r_1 + r_2 - n/2)(r_1 + r_2 - \rho_1 - \rho_2 - n/2)}{r_1 + r_2 - \rho_1 - \rho_2 - n} \quad \text{and} \quad \alpha_n^{(1)} = 4(r_1 + r_2 - \rho_1 - \rho_2 - n - 1)$$

and

$$\beta_n^{(0)} = -4(r_1 + r_2 - \rho_1 - \rho_2 - n - 1) = -\alpha_n^{(1)}, \quad \beta_n^{(1)} = \frac{4(r_1 - n/2)(r_1 - r_2 + n/2)(r_2 - r_1 + n/2)(r_2 - r_2 + n/2)}{\rho_1 + \rho_2 - r_1 - r_2 + n}.$$

Note that formulas (4.16) and (4.17) show that the operators $J_{\pm}$ are block-diagonal in the basis of the polynomials $P_n(x)$ with each block a $2 \times 2$ matrix.

From these formulas it follows that the operators $J_{\pm}^2$ have the polynomials $P_n(x)$ as eigenfunctions

$$J_{\pm}^2 P_n(x) = \begin{cases} 
\alpha_n^{(0)} \alpha_{n-1}^{(0)} P_n(x), & \text{if } n \text{ even} \\
\alpha_n^{(0)} \alpha_{n+1}^{(1)} P_n(x), & \text{if } n \text{ odd}
\end{cases} \quad (4.18)$$

and similarly

$$J_{\pm}^2 P_n(x) = \begin{cases} 
\beta_n^{(0)} \beta_{n+1}^{(0)} P_n(x), & \text{if } n \text{ even} \\
\beta_n^{(0)} \beta_{n-1}^{(1)} P_n(x), & \text{if } n \text{ odd}
\end{cases} \quad (4.19)$$
Now, from the commutation relations (4.4) we find

\[ J_+^2 = (Y + Z)^2(-X^2 + X - 1/4) + (\omega_2 + \omega_3)^2/4. \]  

(4.20)

Using formulas (4.5) and (4.4) we also see that

\[ (Y + Z)^2 = Y^2 + Z^2 + \{Y, Z\} = Q - X^2 + X + \omega_1. \]

We thus have

\[ J_+^2 = (Q - X^2 + X + \omega_1)(-X^2 + X - 1/4) + (\omega_2 + \omega_3)^2/4, \]  

(4.21)

where \( Q \) is the Casimir operator (4.5). Taking into account that in our realization of the BI algebra the Casimir operator takes the value (4.6), we arrive at the expression

\[ J_+^2 = \left( (X + \rho_1 + \rho_2 - 1/2)^2 - (r_1 + r_2)^2 \right) \left( (X - \rho_1 - \rho_2 - 1/2)^2 - (r_1 + r_2)^2 \right). \]  

(4.22)

Quite similarly we obtain

\[ J_-^2 = \left( (X + \rho_2 - \rho_1 + 1/2)^2 - (r_2 - r_1)^2 \right) \left( (X + \rho_2 + 1/2)^2 - (r_2 - r_1)^2 \right). \]  

(4.23)

We see that both operators \( J_+^2 \) and \( J_-^2 \) (which commute with the operator \( X \)) are expressible as 4-th degree polynomials in \( X \).

Starting from relations (4.16) and (4.17) one can now derive many useful relations for the polynomials \( P_n(x) \).

As a first example we present the structure relations of the polynomials \( P_n(x) \).

Define the operator \( U^{(1)}_n \), \( n = 0, 1, 2, \ldots \) as

\[ U^{(1)}_n = \begin{cases} J_+ & \text{if } n \text{ even} \\ J_- & \text{if } n \text{ odd} \end{cases}. \]  

(4.24)

We have in this case

\[ U^{(1)}_n P_n(x) = \epsilon^{(1)}_n P_{n-1}(x), \quad \epsilon^{(1)}_n = \begin{cases} \alpha^{(0)}_n & \text{if } n \text{ even} \\ \beta^{(1)}_n & \text{if } n \text{ odd} \end{cases}. \]  

(4.25)

Similarly, define the operator \( U^{(2)}_n \), \( n = 0, 1, 2, \ldots \) as

\[ U^{(2)}_n = \begin{cases} J_- & \text{if } n \text{ even} \\ J_+ & \text{if } n \text{ odd} \end{cases}. \]  

(4.26)

Then we have

\[ U^{(2)}_n P_n(x) = \epsilon^{(2)}_n P_{n+1}(x), \quad \epsilon^{(2)}_n = \begin{cases} \beta^{(0)}_n & \text{if } n \text{ even} \\ \alpha^{(1)}_n & \text{if } n \text{ odd} \end{cases}. \]  

(4.27)

It is seen that the operator \( U^{(1)}_n \) plays the role of the lowering operator, while the operator \( U^{(2)}_n \) serves as the raising operator for the polynomials \( P_n(x) \).

We are now ready to derive, as a second example, the 3-term recurrence relations of the polynomials \( P_n(x) \).

To do this we consider the operator

\[ V = J_+(X + 1/2) + J_-(X - 1/2) = 2Y(X^2 - 1/4) - \omega_3 X - \omega_2/2. \]  

(4.28)
By construction, this operator is 2-diagonal in the basis of the polynomials \( P_n(x) \):

\[
VP_n(x) = \begin{cases} 
(\mu_n + 1/2)\alpha_n^{(0)}P_{n-1}(x) + (\mu_n - 1/2)\beta_n^{(0)}P_{n+1}(x), & \text{if } n \text{ even} \\
(\mu_n - 1/2)\beta_n^{(1)}P_{n-1}(x) + (\mu_n + 1/2)\alpha_n^{(1)}P_{n+1}(x), & \text{if } n \text{ odd}
\end{cases}
\]  

(4.29)

where \( \mu_n \) is the eigenvalue of the operator \( X \) given in (4.15).

Moreover, from (4.1), the operator \( Y \) coincides (up to an affine transformation) with the multiplication by \( x \), that is, in the polynomial basis \( P_n(x) \), we have \( VP_n(x) = (2x + 1/2)P_n(x) \). Hence, by the second equality in (4.28),

\[
VP_n(x) = ((\mu_n^2 - 1/4)4x + 1) - \omega_3\mu_n - \omega_2/2)P_n(x).
\]

(4.30)

Comparing (4.29) and (4.30), we arrive at the 3-term recurrence relation for the polynomials \( P_n(x) \):

\[
xP_n(x) = P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x),
\]

(4.31)

where the coefficients are

\[
b_n = -1/4 + \frac{\omega_3\mu_n + \omega_2/2}{4\mu_n^2 - 1}
\]

(4.32)

and

\[
u_n = \begin{cases} 
\frac{\alpha_n^{(0)}}{4\mu_n^2 - 2}, & \text{if } n \text{ even} \\
\frac{\beta_n^{(1)}}{4\mu_n^2 - 2}, & \text{if } n \text{ odd}
\end{cases}
\]

(4.33)

We can also present the expressions for the coefficients \( u_n \) in a more detailed form. For even \( n = 0, 2, 4, \ldots \) we have

\[
u_n = -\frac{n(n + 2\tau_1 + 2\tau_2)(n - 2r_1 - 2r_2)(n + 2\rho_1 + 2\rho_2 - 2r_1 - 2r_2)}{16(n + \rho_1 + \rho_2 - r_1 - r_2)^2}.
\]

(4.34)

For odd \( n = 1, 3, 5, \ldots \) we have

\[
u_n = -\frac{(n + 2\rho_1 - 2r_1)(n + 2\rho_1 - 2r_2)(n + 2\rho_2 - 2r_1)(n + 2\rho_2 - 2r_2)}{16(n + \rho_1 + \rho_2 - r_1 - r_2)^2}.
\]

(4.35)

We can now identify these recurrence coefficients with those of the Bannai-Ito polynomials.

Recall that the monic Bannai-Ito polynomials can be defined through the 3-term recurrence relation (4.31), with the recurrence coefficients [3], [33]

\[
u_n = A_{n-1}C_n, \quad b_n = \theta_0 - A_n - C_n,
\]

(4.36)

where

\[
A_n = \begin{cases} 
\frac{2h(n+1+\tau_1)(n+1+\tau_2)}{2n+2-s^*}, & \text{if } n \text{ even} \\
\frac{2h(n+1-s^*)(n+1+\tau_2)}{2n+2-s^*}, & \text{if } n \text{ odd}
\end{cases}
\]

(4.37)

and

\[
C_n = \begin{cases} 
\frac{-2hn(s^*+\tau_2)}{2n-s^*}, & \text{if } n \text{ even} \\
\frac{-2hn(s^*-\tau_2)(s^*-\tau_3)}{2n-s^*}, & \text{if } n \text{ odd}
\end{cases}
\]

(4.38)

In these formulas the parameters \( \tau_1, \tau_2, \tau_3, s^* \) are the 4 essential parameters of the Bannai-Ito polynomials while the parameters \( h \) and \( \theta_0 \) define an affine transformation \( x \to (x + \theta_0)h^{-1} \) of the argument of the polynomials \( P_n(x) \).
An easy analysis shows that the coefficients (4.32) and (4.33) coincide with the recurrence coefficients (4.36) under the identifications

\[ \theta_0 = \rho_1, \quad h = 1/4, \quad \tau_1 = 2(\rho_1 - r_1), \quad \tau_2 = 2(\rho_1 - r_2), \quad \tau_3 = -2(\rho_1 + \rho_2), \quad s^* = 2(r_1 + r_2 - \rho_1 - \rho_2). \]  

(4.39)

Note that apart from the solution (4.39), there exist 3 other solutions which can be obtained from (4.39) by the permutations \( r_1 \leftrightarrow r_2, \rho_1 \leftrightarrow \rho_2 \). This corresponds to the previously mentioned symmetry of the BI polynomials with respect to the \( Z_2 \times Z_2 \) group.

We thus see that the orthogonal polynomials \( P_n(x) \) coincide with the general Bannai-Ito polynomials.

For the reader’s convenience we record the explicit formulas for the recurrence coefficients:

\[ P_{n+1}(x) + (\rho_1 - A_n - C_n)P_n(x) + A_n - 1C_nP_{n-1}(x) = xP_n(x), \]

where

\[ A_n = \begin{cases} \frac{(n+1+2\rho_1-2r_1)(n+1+2\rho_1-2r_2)}{4(n+1-r_1-r_2+\rho_1+\rho_2)}, & n \text{ even} \\ \frac{(n+1-2r_1+2\rho_1+2\rho_2)}{4(n+1-r_1-r_2+\rho_1+\rho_2)}, & n \text{ odd} \end{cases} \]

(4.41)

and

\[ C_n = \begin{cases} \frac{-n(n-2r_1-2r_2)}{4(n-r_1-r_2+\rho_1+\rho_2)}, & n \text{ even} \\ \frac{(n-2r_2+2\rho_2)(n-2r_1+2\rho_2)}{4(n-r_1-r_2+\rho_1+\rho_2)}, & n \text{ odd} \end{cases} \]

(4.42)

Consider now the question of the positive definiteness of the polynomials \( P_n(x) \) obeying a recurrence relation of the form (4.31). Recall that polynomials \( P_n(x) \) are called positive definite if \( u_n > 0 \) for all \( n = 1, 2, \ldots \). This property is equivalent to the existence of a positive measure \( d\mu(x) \) on the real axis such that the orthogonality property reads

\[ \int_a^b P_n(x)P_m(x)d\mu(x) = h_n, \delta_{nm} \]

(4.43)

where \( h_n = u_1u_2\ldots u_n \), and where the integration limits \( a, b \) can be either finite or infinite.

In our case it is seen that \( u_n < 0 \) for sufficiently large \( n \). It is hence impossible to ensure positivity for all \( n = 1, 2, \ldots \). Nevertheless it is possible to obtain a finite set of positive definite orthogonal polynomials.

If \( u_i > 0 \) for \( i = 1, 2, \ldots, N \) and \( u_{N+1} = 0 \), it is well known that we have a finite system of orthogonal polynomials \( P_0(x), P_1(x), \ldots, P_N(x) \) satisfying the discrete orthogonality relation

\[ \sum_{s=0}^N w_s P_n(x_s)P_m(x_s) = h_n \delta_{nm}, \quad h_n = u_1u_2\ldots u_n, \]

(4.44)

where \( x_s, s = 0, 1, \ldots, N \) are the simple roots of the polynomial \( P_{N+1}(x) \). (The fact that the roots \( x_s \) are simple is guaranteed by the condition \( u_n > 0, \ n = 1, 2, \ldots, N \) [6].)

The discrete weights \( w_s \) are then given by the formula [6]

\[ w_s = \frac{P^{(1)}_N(x_s)}{P^{(1)}_{N+1}(x_s)} = \frac{h_N}{P_N(x_s)P_{N+1}(x_s)} > 0, \quad s = 0, 1, \ldots, N, \]

(4.45)
where \( P_n^{(1)}(x) \) are associative polynomials satisfying the recurrence relation

\[
P_{n+1}^{(1)}(x) + b_{n+1}P_n^{(1)}(x) + u_{n+1}P_{n-1}^{(1)}(x) = xP_n^{(1)}(x)
\]

with initial conditions \( P_0^{(1)} = 1, \ P_1^{(1)}(x) = x - b_1 \).

We thus have positive definite polynomials \( P_n(x) \) which are orthogonal on the finite set of distinct points \( x_0, x_1, \ldots x_N \).

Let us consider when such a situation takes place for our polynomials.

First, assume that \( N \) is even. We have then from (4.35) that the condition \( u_{N+1} = 0 \) is equivalent to one of 4 possible conditions

\[
2(r_i - \rho_k) = N + 1, \quad i, k = 1, 2.
\]

We restrict ourselves with the condition

\[
2(r_2 - \rho_2) = N + 1. \tag{4.46}
\]

Then it is sufficient to take

\[
r_2 = r_1 + e + \frac{N}{2}, \quad \rho_1 = r_1 + e + d + \frac{N - 1}{2}, \quad \rho_2 = r_1 - 1/2 + e, \tag{4.47}
\]

where \( r_1, e, d \) are arbitrary positive parameters. Assuming (4.47), we can present the recurrence coefficients \( u_n \) in the following form. For even \( n = 2, 4, \ldots, N \)

\[
u_n = \frac{n(n + 4r_1 - 2 + 4e + 2d + N)(-n + 4r_1 + 2d + N)(n - 2 + 2e + 2d)}{16(n - 1 + e + d)^2},
\]

for odd \( n = 1, 3, \ldots, N - 1 \)

\[
u_n = \frac{(n - 1 + 2e + 2d + N)(n - 1 + 2d)(n - 1 + 2e)(-n + 1 + N)}{16(n - 1 + e + d)^2}.
\]

From these expressions it is seen indeed that the positivity condition \( u_n > 0 \) is satisfied for \( n = 1, 2, \ldots, N \).

Consider now the case when \( N > 1 \) is odd. Then from (4.34), we see that the condition \( u_{N+1} = 0 \) is equivalent to one of three conditions:

(i) \( \rho_1 + \rho_2 = -(N + 1)/2 \),

(ii) \( r_1 + r_2 = (N + 1)/2 \),

(iii) \( \rho_1 + \rho_2 - r_1 - r_2 = -(N + 1)/2 \).

Condition (iii) leads to singular \( u_n \) for \( n = (N + 1)/2 \). Hence only conditions (i) and (ii) are admissible.

Consider condition (ii) for definiteness.

It is convenient to introduce the parametrization

\[
r_1 = (1 - \zeta - \eta)/2, \quad r_2 = (\eta + \zeta + N)/2, \quad \rho_1 = (\eta - \zeta)/2, \quad \rho_2 = (\zeta - \eta - 2\xi - N + 1)/2,
\]

where \( \xi, \eta, \zeta \) are positive parameters with the restriction \( \xi > \eta \). Condition (ii) holds automatically and the recurrence coefficients become

\[
u_n = \frac{n(-n + N - 1 + 2\xi)(-n + N + 1)(-n + 2N + 2\xi)}{16(-n + N + \xi)^2}
\]
for even $n$ and
\[ u_n = \frac{(n + 2\eta - 1)(-n + 2\zeta + N)(-n - 2\zeta + N + 2\xi)(-n + 2\eta + 2N - 1 + 2\xi)}{16(-n + N + \xi)^2} \]
for odd $n$. Again it is visible that $u_n > 0$ when $n = 1, 2, \ldots, N$ and $u_{N+1} = 0$.

5. Complementary Bannai-Ito polynomials and an alternative expression in terms of Racah polynomials

Let us start with the following simple lemma

**Lemma 1** Let $P_n(x)$ be monic orthogonal polynomials satisfying the recurrence relation
\[ P_{n+1}(x) + (\theta - A_n - C_n)P_n(x) + C_n A_{n-1}P_{n-1}(x) = xP_n(x), \] (5.1)
with the standard initial conditions
\[ P_0 = 1, \quad P_1(x) = x - \theta + A_0. \]

Assume that the real coefficients $A_n, C_n$ are such that $A_{n-1} > 0, C_n > 0, n = 1, 2 \ldots$ and that $C_0 = 0$. Take $\theta$ to be an arbitrary real parameter.

Define the new polynomials
\[ \tilde{P}_n(x) = \frac{P_{n+1}(x) - A_nP_n(x)}{x - \theta}. \] (5.2)

Then the monic polynomials $\tilde{P}_n(x)$ are orthogonal and satisfy the recurrence relation
\[ \tilde{P}_{n+1}(x) + (\theta - A_n - C_{n+1})\tilde{P}_n(x) + C_n A_n \tilde{P}_{n-1}(x) = x\tilde{P}_n(x). \] (5.3)

The inverse transformation from the polynomials $\tilde{P}_n(x)$ to the polynomials $P_n(x)$ is given by the formula
\[ P_n(x) = \tilde{P}_n(x) - C_n \tilde{P}_{n-1}(x). \] (5.4)

To the best of our knowledge this Lemma is due to Karlin and McGregor [16].

Another interpretation of this Lemma consists in the observation that formulas (5.2) and (5.4) are equivalent to the Christoffel and Geronimus transformations of orthogonal polynomials which in turn are special cases of the more general rational spectral (or Darboux) transformations [36], [4].

If the polynomials $P_n(x)$ are orthogonal with respect to a linear functional $\sigma$:
\[ \langle \sigma, P_n(x)P_m(x) \rangle = 0, \quad n \neq m, \] (5.5)
then the polynomials $\tilde{P}_n(x)$ are orthogonal with respect to the functional $(x - \theta)\sigma$, i.e.
\[ \langle \sigma, (x - \theta)\tilde{P}_n(x)\tilde{P}_m(x) \rangle = 0, \quad n \neq m. \] (5.6)
Note also that

$$A_n = \frac{P_{n+1}(\theta)}{P_n(\theta)}$$

which can easily be verified directly from (5.1).

The next Lemma will be useful in the identification of polynomial systems with known families of orthogonal polynomials.

**Lemma 2** Let $P_n(x)$ and $\tilde{P}_n(x)$ be two systems of orthogonal polynomials defined as in the previous Lemma. Assume moreover that $\theta = \chi^2 \geq 0$.

Define the following monic polynomials

$$S_{2n}(x) = P_n(x^2), \quad S_{2n+1}(x) = (x - \chi)\tilde{P}_n(x^2).$$

The polynomials $S_n(x)$ are orthogonal and satisfy the recurrence relation

$$S_{n+1} + (-1)^n\chi S_n(x) + v_n S_{n-1}(x) = x S_n(x),$$

where

$$v_{2n} = -C_n, \quad v_{2n+1} = -A_n.$$

In particular, when $\chi = 0$, the polynomials $S_n(x)$ become symmetric, i.e. $S_n(-x) = (-1)^n S_n(x)$.

Conversely, assume that polynomials $S_n(x)$ satisfy the recurrence relation (5.9) with some coefficients $v_n$ and a real constant $\chi$. Then the polynomials $P_n(x)$ and $\tilde{P}_n(x)$ defined by (5.8), are orthogonal and obey the recurrence relations (5.1) and (5.3) with $\theta = \chi^2$.

This Lemma is due to Chihara [5]. For further development and applications of this Lemma see, e.g. [23], [33].

We shall call the polynomials $\tilde{P}_n(x)$, the companion polynomials with respect to $P_n(x)$ (sometimes the polynomials $\tilde{P}_n(x)$ are referred to as the kernel polynomials [6]).

The importance of Lemma 1 lies in the fact that almost all known classical orthogonal polynomials in the Askey table [17] admit a representation of their recurrence relation in the form (5.1) with simple explicit coefficients $A_n, C_n$. The corresponding companion polynomials $\tilde{P}_n(x)$ satisfy similar recurrence relation (5.3) and belong to the same class of orthogonal polynomials (with shifted parameters). For the polynomials at the top of the Askey hierarchy - the Askey-Wilson (or $q$-Racah) polynomials, this was first observed by Chihara in [7]. Further development of this subject in terms of Darboux transformations can be found in [27].

We see that the Bannai-Ito polynomials admit as well this representation with the coefficients $A_n, C_n$ given by (4.41) and (4.42).

We can hence introduce their companion polynomials, which we denote by $W_n(x)$, as follows

$$W_n(x) = \frac{P_{n+1}(x) - A_n P_n(x)}{x - \rho_1}$$

(5.11)
with \( A_n \) given by (4.41). The polynomials \( W_n(x) \) satisfy the recurrence relation (5.3). Specifically, this recurrence relation reads

\[
W_{n+1}(x) + (-1)^n \rho_2 W_n(x) + v_n W_n(x) = x W_n(x),
\]

(5.12)

with

\[
v_{2n} = -\frac{n(n + \rho_1 - r_1 + 1/2)(n + \rho_1 - r_2 + 1/2)(n - r_1 - r_2)}{(2n + 1 + g)(2n + g)},
\]

\[
v_{2n+1} = -\frac{(n + g + 1)(n + \rho_1 + \rho_2 + 1)(n + \rho_2 - r_1 + 1/2)(n + \rho_2 - r_2 + 1/2)}{(2n + 1 + g)(2n + g + 2)},
\]

(5.13)

where we denote

\[ g = \rho_1 + \rho_2 - r_1 - r_2. \]

It is seen that the companion polynomials \( W_n(x) \) satisfy a recurrence relation of the type (5.9) with \( \chi = \rho_2 \). Hence a pair of polynomials \( U_n(x), V_n(x) \) can be constructed via

\[
U_n(x^2) = W_{2n}(x), \quad V_n(x^2) = \frac{W_{2n+1}(x)}{x - \rho_2}, \quad n = 0, 1, 2, \ldots
\]

(5.14)

(It is seen that \( W_{2n}(x) \) are polynomials depending only on \( x^2 \) and that the polynomials \( W_{2n+1}(x) \) all have in common the factor \( x - \rho_2 \)).

From the above Lemmas, it follows that the polynomials \( U_n(y), V_n(y) \) satisfy the following system of relations

\[
U_n(y) = V_n(y) + v_{2n} V_{n-1}(y), \quad (y - \rho_2^2) V_n(y) = U_{n+1}(y) + v_{2n+1} U_n(y)
\]

(5.15)

from which the three-term recurrence relations for the polynomials \( U_n(y) \) and \( V_n(y) \) can be derived:

\[
U_{n+1}(y) + (\rho_2^2 + v_{2n} + v_{2n+1}) U_n(y) + v_{2n+2} v_{2n-1} U_{n-1}(y) = y U_n(y)
\]

(5.16)

and

\[
V_{n+1}(y) + (\rho_2^2 + v_{2n+2} + v_{2n+1}) V_n(y) + v_{2n} v_{2n+1} V_{n-1}(y) = y V_n(y).
\]

(5.17)

From the explicit expressions (5.13) of the coefficients \( v_n \), it is possible to conclude that the polynomials \( U_n(x^2), V_n(x^2) \) are Wilson polynomials [17]. Omitting technical details we have

\[
U_n(x^2) = \kappa_n^{(1)} \mathbf{4F}_3 \left( \begin{array}{c}-n, n + g + 1, \rho_2 + x, \rho_2 - x \\ \rho_1 + \rho_2 + 1, \rho_2 - r_1 + 1/2, \rho_2 - r_2 + 1/2 \end{array} ; 1 \right)
\]

(5.18)

and

\[
V_n(x^2) = \kappa_n^{(2)} \mathbf{4F}_3 \left( \begin{array}{c}-n, n + g + 2, \rho_2 + 1 + x, \rho_2 + 1 - x \\ \rho_1 + \rho_2 + 2, \rho_2 - r_1 + 3/2, \rho_2 - r_2 + 3/2 \end{array} ; 1 \right),
\]

(5.19)

where the normalization coefficients (making the polynomials monic) are

\[
\kappa_n^{(1)} = \frac{(1 + \rho_1 + \rho_2)_n(\rho_2 - r_1 + 1/2)_n(\rho_2 - r_1 + 1/2)_n}{(n + g + 1)_n}, \quad \kappa_n^{(2)} = \frac{(2 + \rho_1 + \rho_2)_n(\rho_2 - r_1 + 3/2)_n(\rho_2 - r_1 + 3/2)_n}{(n + g + 2)_n}.
\]

(Note that the polynomials \( V_n(x^2) \) are obtained from the polynomials \( U_n(x^2) \) by simply shifting the parameter \( \rho_2 \) by 1: \( \rho_2 \rightarrow \rho_2 + 1 \).)
We thus have an explicit expression for the polynomials $W_n(x)$ (which are Christoffel transforms of the Bannai-Ito polynomials) in terms of the Wilson polynomials:

$$W_{2n}(x) = U_n(x^2), \quad W_{2n+1}(x) = (x - \rho_2)V_n(x^2),$$

(5.20)

where the polynomials $U_n(x^2), V_n(x^2)$ are given by (5.18), (5.19).

Hence, by (5.4), we get for the Bannai-Ito polynomials

$$P_n(x) = W_n(x) - C_nW_{n-1}(x),$$

(5.21)

where $C_n$ is given by (4.42). Formula (5.21) can be considered as an alternative explicit expression of the Bannai-Ito polynomials. Like in (3.15) and (3.16), we have a presentation of the Bannai-Ito polynomials $P_n(x)$ as a linear combination of two hypergeometric polynomials $4F_3(1)$. In (5.21) however, the hypergeometric functions are -1-balanced and are thus bona fide Wilson polynomials. Note that the representation (3.15), (3.16) is equivalent to the one obtained by Bannai and Ito for their polynomials by using a direct limit process from the q-Racah polynomials [3]. The representation (5.21) in terms of the Wilson polynomials seems to be new. When $N \to \infty$, the Bannai-Ito polynomials tend to the big-1 Jacobi polynomials [33]. Formula (5.21) then becomes the formula that gives the expression of the big-1 Jacobi polynomials in terms of the ordinary Jacobi polynomials [33].

We shall refer to the polynomials $W_n(x)$ as the complementary Bannai-Ito (CBI) polynomials. They are Christoffel transforms of the Bannai-Ito polynomials. They have a simpler structure (e.g. for every $n$ the polynomial $W_n(x)$ is expressed in terms of only one hypergeometric function $4F_3(1)$). However, in contrast to the Bannai-Ito polynomials, the CBI polynomials do not satisfy the Leonard duality property. In other words, they are not eigenpolynomials of a Dunkl shift operator of the form (2.4). Indeed, the CBI polynomials have recurrence coefficients that do not belong to the BI class, and hence they do not arise in the classification of all orthogonal polynomials with the Leonard duality property realized by Bannai and Ito [3].

6. Symmetry factor for the Bannai-Ito operator

Let us return to the Bannai-Ito operator (2.4), where $F(x)$ and $G(x)$ are given by (2.12). Consider the formal Lagrange adjoint $L^*$ of $L$:

$$L^* = F(x)I - R^*F(x) + (T^+R)^*G(x) - G(x)I.$$  

We will assume that $R^* = R$ and $T^{+*} = T^-$ (these are natural and conventional conditions for difference operators on the real line).

There are in addition, the obvious relations

$$T^-R = RT^+, \quad T^+Rf(x) = f(-x - 1)T^+R,$$
where \( f(x) \) is an arbitrary function of the real variable \( x \). From these relations it follows that the operator \( T + R \) is formally self-adjoint: \((T + R)^* = T + R\).

We then obtain
\[
L^* = F(x)I - F(-x)R + G(-x - 1)T^+R - G(x)I.
\]
It is visible that the Bannai-Ito operator is not symmetric, i.e. \( L^* \neq L \) for any choice of the parameters \( r_1, r_2, \rho_1, \rho_2 \).

Nevertheless, it is possible to find a symmetry factor \( \varphi(x) \) for the operator \( L \) such that
\[
(\varphi(x)L)^* = \varphi(x)L. \tag{6.1}
\]
An operator \( L \) is called symmetrizable if there exists a real-valued functions \( \varphi(x) \) ensuring that property (6.1) holds. The similar approach for differential operators and their polynomial solutions is familiar [22].

Let us show that the Bannai-Ito operator is symmetrizable for generic values of the parameters \( r_1, r_2, \rho_1, \rho_2 \).

We have
\[
\varphi(x)L = \varphi(x)F(x) - \varphi(x)F(x)R + \varphi(x)G(x)T^+R - \varphi(x)G(x) \tag{6.2}
\]
and
\[
(\varphi(x)L)^* = \varphi(x)F(x) - \varphi(-x)F(-x)R + \varphi(-x - 1)G(-x - 1)T^+R - \varphi(x)G(x). \tag{6.3}
\]
Comparing (6.3) and (6.2), we see that condition (6.1) is valid iff the following two conditions
\[
\varphi(-x)F(-x) = \varphi(x)F(x), \quad \varphi(-x - 1)G(-x - 1) = \varphi(x)G(x) \tag{6.4}
\]
are fulfilled.

Clearly, these conditions are equivalent to
\[
\varphi(x)F(x) = E_0(x), \quad \varphi(x)G(x) = E_1(x + 1/2), \tag{6.5}
\]
where \( E_0(x), E_1(x) \) are even functions, i.e. \( E_0(-x) = E_0(x), \ E_1(-x) = E_1(x) \). From (6.5) we have
\[
\frac{F(x)}{G(x)} = \frac{E_0(x)}{E_1(x + 1/2)} \tag{6.6}
\]
as well as
\[
\frac{F(-x)}{G(-x)} = \frac{E_0(x)}{E_1(x - 1/2)} \tag{6.7}
\]
and
\[
\frac{F(-x - 1)}{G(-x - 1)} = \frac{E_0(x + 1)}{E_1(x + 1/2)} \tag{6.8}
\]
From (6.6) and (6.7), we get an equation for the unknown function \( E_1(x) \)
\[
\frac{E_1(x + 1/2)}{E_1(x - 1/2)} = \frac{F(-x)G(x)}{F(x)G(-x)}. \tag{6.9}
\]
Similarly, from (6.6) and (6.8), we find an equation for the unknown function $E_0(x)$:

$$\frac{E_0(x+1)}{E_0(x)} = \frac{F(-x-1)G(x)}{F(x)G(-x-1)}.$$  \hspace{1cm} (6.10)

The general solution of equations (6.10) and (6.9) can easily be obtained from the expressions for the functions $F(x)$ and $G(x)$:

$$E_0(x) = \frac{\sigma_0(x)}{x} \frac{\Gamma(1/2 - r_1 + x)\Gamma(1/2 - r_1 - x)\Gamma(1 + \rho_1 + x)\Gamma(1 + \rho_1 - x)}{\Gamma(1/2 + r_2 + x)\Gamma(1/2 + r_2 - x)\Gamma(-\rho_2 + x)\Gamma(-\rho_2 - x)},$$  \hspace{1cm} (6.11)

$$E_1(x) = \frac{\sigma_1(x)}{x} \frac{\Gamma(1 - r_1 + x)\Gamma(1 - r_1 - x)\Gamma(1/2 + \rho_1 + x)\Gamma(1/2 + \rho_1 - x)}{\Gamma(r_2 + x)\Gamma(r_2 - x)\Gamma(1/2 - \rho_2 + x)\Gamma(1/2 - \rho_2 - x)},$$  \hspace{1cm} (6.12)

where $\sigma_0(x)$ and $\sigma_1(x)$ are arbitrary functions with period 1: $\sigma_{0,1}(x+1) = \sigma_{0,1}(x)$. The fact that both functions $E_0(x)$ and $E_1(x)$ must be even, requires that both $\sigma_0(x)$ and $\sigma_1(x)$ be odd:

$$\sigma_{0,1}(-x) = -\sigma_{0,1}(x).$$

From condition (6.6) it follows that $\sigma_1(x) = \sigma_0(x)$. Then, from (6.5), we find the function $\varphi(x)$

$$\varphi(x) = -2\sigma_0(x) \frac{\Gamma(r_1 - x)\Gamma(x - r_1 + 1/2)\Gamma(-x - r_1 + 1/2)\Gamma(x + 1 + \rho_1)}{\Gamma(x + 1 - \rho_2)\Gamma(x + r_2 + 1/2)\Gamma(r_2 + 1/2 - x)\Gamma(-x - \rho_2)}. $$  \hspace{1cm} (6.13)

Conversely, assume that the function $\varphi(x)$ is given by (6.13). It can then be easily verified that the function $\varphi(x)$ is a solution of conditions (6.4) iff the function $\sigma_0(x)$ satisfies the two conditions:

(i) be 1-periodic, i.e. $\sigma_0(x+1) = \sigma_0(x)$;

(ii) be odd, i.e. $\sigma_0(-x) = -\sigma_0(x)$.

Hence we succeeded in constructing the desired symmetry factor $\varphi(x)$.

It should be noted that the function $\varphi(x)$ can be presented in several equivalent forms if one exploits the classical formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

For example, one can present $\varphi(x)$ in the form

$$\varphi(x) = \sigma_2(x) \frac{\Gamma(\rho_1 - x)\Gamma(\rho_1 + 1 + x)\Gamma(\rho_2 - x)\Gamma(1 + \rho_2 + x)}{\Gamma(r_2 + 1/2 + x)\Gamma(r_2 + 1/2 - x)\Gamma(r_1 + 1/2 + x)\Gamma(r_1 + 1/2 - x)}.$$  \hspace{1cm} (6.14)

where the function $\sigma_2(x)$ has the same properties (i) and (ii) as $\sigma_0(x)$.

Let us introduce the operator $M = \varphi(x)L$, where $\varphi(x)$ is given by (6.13). This operator will thus be formally self-adjoint, or symmetric: $M^* = M$.

7. **Bannai-Ito grid and weight function for the Bannai-Ito polynomials**

One of the most important property of Dunkl shift operators like $L$ or $M$ is that there is a discrete set of points on the real line which is invariant under the action of these operators.
Indeed, let us introduce the Bannai-Ito (BI) grid of the first type \(x_s\) as a discrete set of real points defined by the two conditions

\[-x_{2s} = x_{2s-1}, \quad -1 - x_{2s} = x_{2s+1}, \quad s = 0, \pm 1, \pm 2, \ldots \]  

(7.1)

It is easy to verify that the general solution for the grid \(x_s\) satisfying conditions (7.1), is

\[x_s = \begin{cases} a + \frac{s}{2} & \text{if } s \text{ is even,} \\ -a - \frac{s+1}{2} & \text{if } s \text{ is odd,} \end{cases} \]  

(7.2)

where \(a\) is an arbitrary real parameter.

Equivalently,

\[x_s = -1/4 + (-1)^s(1/4 + a + s/2). \]  

(7.3)

Using properties (7.1), we can conclude that for any Dunkl shift operator

\[H = A(x)R + B(x)T^+ R + C(x) \]  

(7.4)

with arbitrary functions \(A(x), B(x), C(x)\), we have for any function \(f(x)\)

\[H f(x_s) = \xi_s f(x_{s+1}) + \eta_s f(x_s) + \zeta_s f(x_{s-1}), \]  

(7.5)

where

\[\xi_s = \begin{cases} B(x_s) & \text{if } s \text{ is even,} \\ A(x_s) & \text{if } s \text{ is odd,} \end{cases} \]  

(7.6)

\[\zeta_s = \begin{cases} A(x_s) & \text{if } s \text{ is even,} \\ B(x_s) & \text{if } s \text{ is odd,} \end{cases} \]  

(7.7)

\[\eta_s = C(x_s). \]  

(7.8)

In other words, the Dunkl shift operator becomes a 3-diagonal matrix (either finite or infinite) in the basis \(f(x_s)\).

Similarly, we can define the BI grid of the second type \(y_s\) by the two conditions

\[-y_{2s} = y_{2s+1}, \quad -1 - y_{2s} = y_{2s-1}, \quad s = 0, \pm 1, \pm 2, \ldots \]  

(7.9)

with the general solution

\[y_s = \begin{cases} b - \frac{s}{2} & \text{if } s \text{ is even,} \\ -b + \frac{s-1}{2} & \text{if } s \text{ is odd,} \end{cases} \]  

(7.10)

where \(b\) is an arbitrary real parameter.

Equivalently,

\[y_s = -1/4 + (-1)^s(1/4 + b - s/2). \]  

(7.11)

We then have that the operator \(H\) given by (7.4) is 3-diagonal as in (7.5) with

\[\xi_s = \begin{cases} A(y_s) & \text{if } s \text{ is even,} \\ B(y_s) & \text{if } s \text{ is odd,} \end{cases} \]  

(7.12)
In particular, the BI difference equation (2.11) for the BI polynomials can be presented in the form
\[ \xi_s P_n(x_{s+1}) + \eta_s P_n(x_s) + \zeta_s P_n(x_{s-1}) = \lambda_n P_n(x_s), \] (7.15)
where the coefficients \( \xi_s, \eta_s, \zeta_s \) are given by (7.6)-(7.8) (or by (7.12)-(7.14)) with
\[ A(x) = -F(x), \quad B(x) = G(x), \quad C(x) = F(x) - G(x). \]

In turn, property (7.15) means that the BI polynomials possess the Leonard duality property [3], [29]: they satisfy both a 3-term recurrence relation with respect to \( n \) and a 3-term difference equation with respect to the argument index \( s \). Bannai and Ito found all orthogonal polynomials satisfying the Leonard duality property [3]. The corresponding classification, referred to as the Bannai-Ito theorem, is a generalization of the Leonard theorem [21] that deals only with polynomials orthogonal on a finite set of points. We have thus derived the Leonard duality property of the BI polynomials starting from the Dunkl shift difference equation (2.11).

Note that the Bannai-Ito grid satisfies the linear equation
\[ x_{s+1} + x_{s-1} + 2x_s + 1 = 0 \] (7.16)
which is a special case of the generic equation
\[ x_{s+1} + x_{s-1} - (q + q^{-1})x_s = \text{const} \] (7.17)
characterizing the Askey-Wilson grids [14], [24]. As expected, the Bannai-Ito grid corresponds to the case \( q = -1 \) of the Askey-Wilson grid.

Consider now the special case when the operator \( H \) is symmetric \( H^* = H \), with real functions \( A(x), B(x), C(x) \). This is equivalent to the conditions
\[ A(-x) = A(x), \quad B(-x - 1) = B(x). \] (7.18)
It is easily verified that in this case the 3-diagonal matrix in (7.5) becomes symmetric as well, i.e. \( \xi_s = \zeta_{s+1} \), or in details:
\[ H f(x_s) = \zeta_{s+1} f(x_{s+1}) + \eta_s f(x_s) + \zeta_s f(x_{s-1}). \] (7.19)
This result does not depend on the type of the BI grid. We already know that the operator \( M = \varphi(x)L \)
is symmetric \( M^* = M \). Assume further that the function \( \varphi(x) \) is positive, i.e \( \varphi(x) > 0 \) for some interval \( x_0 < x < x_1 \). We can then introduce the operator
\[ \bar{L} = \varphi^{1/2}(x)L\varphi^{-1/2}(x). \] (7.20)
By condition (6.1), this operator will be symmetric \( \bar{L}^* = \bar{L} \) on the interval \([x_0, x_1]\).
Assume additionally that the corresponding 3-diagonal matrix is finite, say it has dimension \((N+1) \times (N+1)\).

This is equivalent to the conditions

\[ \zeta_0 = \zeta_{N+1} = 0 \]  
\[ (7.21) \]

and \(\zeta_s > 0, \ s = 1, 2, \ldots, N\). Let \(q_n(s)\) be a set of eigenvectors of the \(s\)-diagonal matrix \(\tilde{L}\):

\[ \zeta_{s+1} q_n(s+1) + \eta_s q_n(s) + \zeta_s q_n(s) = \lambda_n q_n(s), \quad s, n = 0, 1, 2, \ldots, N. \]  
\[ (7.22) \]

The operator \(\tilde{L}\) is a Hermitian matrix in the basis \(q_n(s)\). By elementary linear algebra, it is well known that the coordinates \(q_n(s)\) (under appropriate normalization) satisfy the orthonormality property

\[ \sum_{s=0}^{N} q_n(s) q_m(s) = \delta_{nm}. \]  
\[ (7.23) \]

From the definition (7.20) of the operator \(\tilde{L}\), it then follows that

\[ q_n(s) = \kappa_n^{-1} \varphi^{1/2}(x_s) P_n(x) \]  
\[ (7.24) \]

with some normalization factor \(\kappa_n\).

We thus arrive on the one hand, at the orthogonality relation for the BI polynomials:

\[ \sum_{s=0}^{N} w_s P_n(x_s) P_m(x_s) = \kappa_n^2 \delta_{nm}, \]  
\[ (7.25) \]

where the discrete weights are defined by the formula

\[ w_s = \varphi(x_s). \]  
\[ (7.26) \]

By construction, the weights are positive \(w_s > 0\) as should be for positive definite orthogonal polynomials \(P_n(x)\).

On the other hand, we have formula (4.44) giving the orthogonality relation of the BI polynomials in the finite case, where \(x_s\) are simple roots of the polynomial \(P_{N+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_N)\). This means that the roots \(x_s\) can be parametrized in terms of the BI grid.

Explicitly, the roots \(x_s\) of the polynomial \(P_{N+1}(x)\) can be found from formula (5.21) which expresses the BI polynomials in terms of the Wilson polynomials. Omitting technical details, we present the results.

When \(N\) is even and condition (4.46) is assumed then

\[ x_s = -1/4 + (-1)^s (\rho_2 + 1/4 + s/2), \quad s = 0, 1, 2, \ldots, N, \]  
\[ (7.27) \]

i.e. the roots for the case of even \(N\) belong to the BI grid of the first type (7.3) with \(a = \rho_2\).

When \(N\) is odd and condition \(r_1 + r_2 = (N + 1)/2\) is assumed then

\[ x_s = -1/4 + (-1)^s (-1/4 + r_1 - s/2), \quad s = 0, 1, 2, \ldots, N, \]  
\[ (7.28) \]

i.e. the roots for the case of odd \(N\) belong to the BI grid of the second type (7.11) with \(b = r_1 - 1/2\).
So for even \( N = 2, 4, 6, \ldots \) the orthogonality relation reads as in (7.25) with the spectral points \( x_s \) given by (7.27) and the discrete weights expressed as in (7.26), and where \( \varphi(x) \) can be taken in the form (6.13).

For odd \( N = 1, 3, 5, \ldots \) the spectral points \( x_s \) are given by (7.27) and the discrete weights expressed as in (7.26), where \( \varphi(x) \) can now be taken of the form (6.14) (this form is preferable for the case of odd \( N \) in order to avoid singularities when \( x = x_s \)).

It is easily verified that in all cases the function \( \sigma_0(x_s) \) and \( \sigma_2(x_s) \) are constant sign-alternating sequence, e.g.

\[
\sigma_0(x_s) = (-1)^s \text{const}, \quad s = 0, 1, 2, \ldots, N
\]

Note finally that the set of spectral points \( x_s \) is the union of two discrete subsets \( x_{2s} \) and \( x_{2s+1} \) corresponding to even and odd values of \( s \). These two subsets never overlap if the positivity conditions form the BI polynomials are fulfilled.

8. Bannai-Ito and complementary Bannai-Ito polynomials as limits

\( q \to -1 \) of the Askey-Wilson polynomials

The Bannai-Ito polynomials were introduced in [3] through a direct limit process from the \( q \)-Racah polynomials as \( q \to -1 \).

We show in this section how the Bannai-Ito polynomials can be obtained from the Askey-Wilson (AW) polynomials as \( q \to -1 \). We also indicate how the complementary Bannai-Ito polynomials can equally be derived from the AW-polynomials in the same limit.

Consider the Askey-Wilson polynomials \( R_n(x(z); a, b, c, d) \) [2], [17]

\[
R_n(z; a, b, c, d) = \kappa_n^{(1)} \Phi_3\left(q^{-n}, abcdq^{n-1}, az, az^{-1} \bigg| \frac{ab, ac, ad}{q} \right)
\]

(8.1)

where

\[
\kappa_n^{(1)} = a^{-n}(ab, ac, ad; q)_n.
\]

The polynomials \( R_n(x(z); a, b, c, d) \) depend on the argument \( x = (z + z^{-1})/2 \) and on 4 complex parameters \( a, b, c, d \). They satisfy the recurrence relation [17]

\[
A_n R_{n+1}(x) + (a + a^{-1} - A_n - C_n) R_n(x) + C_n R_{n-1}(x) = (z + z^{-1}) R_n(x) = 2x R_n(x)
\]

(8.2)

with coefficients

\[
A_n = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})},
\]

\[
C_n = \frac{a(1 - q^n)(1 - bq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-1})(1 - abcdq^{2n-2})}.
\]

(8.3)

In order to recover the BI-polynomials we choose the following parametrization

\[
q = -e^\varepsilon, \quad a = -ie^{\varepsilon_\alpha}, \quad b = -ie^{\varepsilon_\beta}, \quad c = ie^{\varepsilon_\gamma}, \quad d = ie^{\varepsilon_\delta},
\]

(8.4)
where \( \alpha, \beta, \gamma, \delta \) are real parameters. In what follows these parameters are more conveniently expressed in terms of the 4 parameters \( r_1, 2, \rho_1, 2 \):

\[
\alpha = 2\rho_1 + 1/2, \quad \beta = 2\rho_2 + 1/2, \quad \gamma = -2r_2 + 1/2, \quad \delta = -2r_1 + 1/2.
\] (8.5)

Moreover, we take

\[
z = ie^{-2\varepsilon y}
\] (8.6)

with \( y \) a real parameter.

It can be checked that the limit \( q \to -1 \) of the Askey-Wilson polynomials

\[
\lim_{q \to -1} R_n(x(z)) = R^{(-1)}(y)
\] (8.7)

does exist, and that \( R_n^{(-1)}(y) \) is a polynomial of degree \( n \) in the argument \( y \).

Dividing the recurrence relation (8.2) by \( 1 + q \) and taking afterwards the limit \( \varepsilon \to 0 \) (which is equivalent to the limit \( q \to -1 \)), we obtain a recurrence relation for the limiting polynomials \( R_n^{(-1)}(y) \):

\[
A^{(-1)}_n R^{(-1)}_{n+1}(y) + (1/4 + \rho_1 - A^{(-1)}_n - C^{(-1)}_n) R^{(-1)}_n(y) + C^{(-1)}_n R^{(-1)}_{n-1}(y) = y R^{(-1)}_n(y),
\] (8.8)

where the coefficients \( A^{(-1)}_n, C^{(-1)}_n \) have expressions coinciding with (4.41) and (4.42). The polynomials \( R_n^{(-1)}(y) \) are thus identified with the Bannai-Ito polynomials (up to a shift of the argument):

\[
\hat{R}_n^{(-1)}(y) = P_n(y - 1/4; r_1, r_2, \rho_1, \rho_2).
\] (8.9)

\( \hat{R}_n^{(-1)}(y) \) in (8.9) denotes the monic version of the polynomials \( R_n^{(-1)}(y) \), while \( P_n(y; r_1, r_2, \rho_1, \rho_2) \) stands for the monic Bannai-Ito polynomials defined in (4.40). The parameters \( r_1, r_2, \rho_1, \rho_2 \) for the Bannai-Ito polynomials stem under the limiting procedure (8.5), from the corresponding 4 parameters of the Askey-Wilson polynomials.

For the complementary Bannai-polynomials, we use the same procedure but with a slightly different parametrization:

\[
q = -e^{\varepsilon}, \quad a = ie^{\varepsilon(2\rho_1+3/2)}, \quad b = -ie^{\varepsilon(\rho_2+1/2)}, \quad c = ie^{\varepsilon(-2r_2+1/2)}, \quad d = ie^{\varepsilon(-2r_1+1/2)}, \quad z = ie^{-2\varepsilon y}.
\] (8.10)

The main difference between (8.4) and (8.10) is a change of sign in \( a \) and the shift \( \rho_1 \to \rho_1 + 1/2 \).

It is directly verified that in the limit \( \varepsilon \to 0 \), we get the recurrence relation (5.12) that defines the CBI polynomials and therefore that

\[
\lim_{\varepsilon \to 0} R_n(z; a, b, c, d) = W_n(y - 1/4; r_1, r_2, \rho_1, \rho_2).
\] (8.11)

Thus the BI and CBI polynomials are obtained through very similar \( q \to -1 \) limits of the Askey-Wilson polynomials.
9. The Bannai-Ito Dunkl shift operator as a limiting form of the
Askey-Wilson difference operator

In this section, we describe how the Bannai-Ito Dunkl shift operator appears in the limiting process \( q \to -1 \) from the difference Askey-Wilson operator.

The Askey-Wilson polynomials \( R_n(x(z)) \) satisfy the difference equation [17]

\[
\Omega(z)R_n(x(zq)) + \Omega(z^{-1})R_n(x(zq^{-1})) - (\Omega(z) + \Omega(z^{-1}))R_n(x(z)) = \Lambda_n R_n(x(z)), \quad n = 0, 1, 2, \ldots, \tag{9.1}
\]

where

\[
\Lambda_n = (q^{-n} - 1)(1 - abcdq^{n-1})
\]

is the eigenvalue and

\[
\Omega(z) = \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)}. \tag{9.2}
\]

The linear operator in the lhs of (9.1) (acting on the variable \( z \)) is the Askey-Wilson difference operator [2], [17].

Consider the limiting form of the difference equation (9.1) when \( q \to -1 \). As in the previous section, we choose the parametrization (8.5) and (8.6). We already showed that the Askey-Wilson polynomials \( R_n(x(z)) \) become the Bannai-Ito polynomials \( R_n^{(-)}(y) \). The operation \( R_n(z) \to R_n(zq) \) is reduced to the operation \( R_n^{(-)}(y) \to R_n^{(-)}(-y + 1/2) \) while the operation \( R_n(z) \to R_n(zq^{-1}) \) becomes the operation \( R_n^{(-)}(y) \to R_n^{(-)}(-y - 1/2) \).

It is then straightforward to verify that in the limit \( \varepsilon \to 0 \) we get the equation

\[
\Phi_1(y)R_n^{(-)}(-y + 1/2) + \Phi_2(y)R_n^{(-)}(-y - 1/2) - (\Phi_1(y) + \Phi_2(y))R_n^{(-)}(y) = \lambda_n R_n^{(-)}(y), \tag{9.3}
\]

where

\[
\Phi_1(y) = \lim_{\varepsilon \to 0} \frac{\Omega(z)}{4(1 + q)} = -\frac{2(y - \rho_1 - 1/4)(y - \rho_2 - 1/4)}{4y - 1},
\]

\[
\Phi_2(y) = \lim_{\varepsilon \to 0} \frac{\Omega(z^{-1})}{4(1 + q)} = \frac{2(y - r_1 + 1/4)(y - r_2 + 1/4)}{4y + 1} \tag{9.4}
\]

and that the eigenvalue

\[
\lambda_n = \lim_{\varepsilon \to 0} \frac{\Lambda_n}{4(1 + q)} \tag{9.5}
\]

has the expression (2.13).

This gives the Bannai-Ito polynomials \( R_n^{(-)}(y) \) as eigenfunctions of some Dunkl shift operator. In operator form this eigenvalue equation can be presented as follows:

\[
HR_n^{(-)}(y) = \lambda_n R_n^{(-)}(y). \tag{9.6}
\]

\( H \) in (9.6), acts on the space of functions \( f(y) \) of argument \( y \) according to

\[
H = \Phi_1(y)T^{-1/2}R + \Phi_2(y)T^{1/2}R - (\Phi_1(y) + \Phi_2(y)), \tag{9.7}
\]
and $T^h$ stands for the shift operator

$$T^h f(y) = f(y + h).$$

The Dunkl shift operator $H$ contains (apart from the identity operator) the operators $T^{1/2}, T^{-1/2}$ and $R$.

We can now reduce the operator (9.7) to the "canonical" form which only involves the operators $T^+$ and $R$. It goes as follows. The similarity transformation

$$\tilde{H} = T^h HT^{-h}$$

with an arbitrary real $h$ is a unitary transformation of the operator $H$ (recall that by $T^+$ and $T^-$ we mean the operators $T^{+1}$ and $T^{-1}$ respectively). Under this transformation $H$ goes into

$$\tilde{H} = \Phi_1(y + h)T^{2h-1/2}R + \Phi_2(y + h)T^{2h+1/2}R - \Phi(y + h) - \Phi_1(y + h).$$

(9.8)

For $h = 1/4$, we have

$$\tilde{H} = \Phi_1(y + 1/4)R + \Phi_2(y + 1/4)T^+R - \Phi_1(y + 1/4) - \Phi_2(y + 1/4),$$

(9.9)

where

$$\Phi_1(y + 1/4) = -\frac{(y - \rho_1)(y - \rho_2)}{2y} = -F(y), \quad \Phi_2(y + 1/4) = -\frac{(y - r_1 + 1/2)(y - r_2 + 1/2)}{2y + 1} = G(y).$$

(9.10)

The functions $G(x), F(x)$ in (9.10) are precisely those of (2.12). We have thus reduced the operator $H$ to our initial "canonical" Dunkl shift operator $L$ defined in (2.4).

Note that there is one more "canonical" form of the Dunkl shift operator where only $T^-$ and $R$ appear. This other form is obtained by choosing $h = -1/4$, so that

$$\tilde{H} = \Phi_1(y - 1/4)T^-R + \Phi_2(y - 1/4)R - \Phi_1(y - 1/4) - \Phi_2(y - 1/4).$$

(9.11)

We see that the Dunkl shift operator can be presented in 3 "canonical" form: the "+ form" given by (9.8), the "- form" given by (9.11) and the "symmetric" form given by (9.7). Clearly, all these forms are equivalent and correspond to shifts $y \to y + h$ in the argument of the polynomials.

As was already mentioned, the complementary BI polynomials do not satisfy a difference equation of Dunkl shift type. Indeed, under the choice (8.10), the corresponding Askey-Wilson difference operator does not survive in the limit $q \to -1$.

10. Symmetric BI polynomials and continuous dual Hahn polynomials

We already saw that for generic BI polynomials with real parameters $r_1, r_2, \rho_1, \rho_2$ it is impossible to obtain positive definite polynomials for all $n = 0, 1, 2, \ldots$.
There is however, a special case for which it is possible to construct such positive definite polynomials after the following change of argument: \( x \to ix \). This case arises for symmetric BI polynomials.

Recall that polynomials \( S_n(x) \) are said to be the symmetric if they satisfy the condition
\[
S_n(-x) = (-1)^n S_n(x). \tag{10.1}
\]
Orthogonal polynomials \( P_n(x) \) obeying a recurrence relation of the form
\[
P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x)
\]
are symmetric iff \( b_n = 0 \) [6] (i.e. iff the diagonal recurrence coefficient \( b_n \) vanishes for all \( n = 0, 1, 2, \ldots \)).

If \( b_n = b = \text{const}, n = 0, 1, 2, \ldots \) it is then the "shifted" polynomials \( \tilde{P}_n(x) = P_n(x+b) \) that are symmetric.

Let us determine when the (shifted) BI polynomials \( R_n^{(-1)}(x) \) (defined by (8.9)) are symmetric. This is equivalent to determining when the condition \( b_n = -1/4, n = 0, 1, 2, \ldots \) for the diagonal recurrence coefficient of the BI polynomials, is satisfied. It is easy to see that this condition holds iff

either

(i) \( \rho_1 = -r_1, \rho_2 = -r_2 \);

or

(ii) \( \rho_1 = -r_2, \rho_2 = -r_1 \).

In both cases we have
\[
b_n = -1/4, \quad n = 1, 2, 3, \ldots \tag{10.2}
\]
and
\[
u_n = \begin{cases} 
\frac{-n(n-4(r_1+r_2))}{16}, & \text{if } n \text{ even} \\
\frac{-(n-4r_1)(n-4r_2)}{16}, & \text{if } n \text{ odd} 
\end{cases} \tag{10.3}
\]
The corresponding symmetric polynomials \( S_n(x) = \tilde{R}_n^{(-1)}(x) = P_n(x - 1/4) \) are expressed in terms of dual Hahn polynomials. In order to obtain positive definite polynomials, we need to perform the change of variable \( x \to ix \). This leads to symmetric polynomials with positive coefficients \( u_n \). Indeed, for the new polynomials \( \tilde{S}_n(x) = i^{-n} S_n(ix) \), we have the 3-term recurrence relation
\[
\tilde{S}_{n+1}(x) + \tilde{u}_n \tilde{S}_n(x) = x \tilde{S}_n(x) \tag{10.4}
\]
with coefficients \( \tilde{u}_n = i^2 u_n = -u_n \) and thus
\[
\tilde{u}_n = \begin{cases} 
\frac{n(n-4(r_1+r_2))}{16}, & \text{if } n \text{ even} \\
\frac{(n-4r_1)(n-4r_2)}{16}, & \text{if } n \text{ odd} 
\end{cases} \tag{10.5}
\]
It is seen that under the restrictions
\[
r_1 < 1/4, \quad r_2 < 1/4,
\]
one has \( \tilde{u}_n > 0 \) for all \( n = 1, 2, 3, \ldots \). This means that the polynomials \( \tilde{S}_n(x) \) are orthogonal on the real axis with a positive measure.
Comparing the recurrence coefficients (10.5) with those for the continuous dual Hahn polynomials [17] and using Chihara’s standard procedure to relate symmetric and non-symmetric polynomials [6], we can derive an explicit expression for the polynomials $\tilde{S}_n(x)$:

$$
\tilde{S}_{2n}(x) = \kappa_n^{(0)} \binom{3}{2} F_2 \left( -n, a + 2ix, a - 2ix \mid a, a+b \right), \quad \tilde{S}_{2n+1}(x) = \kappa_n^{(1)} x \binom{3}{2} F_2 \left( -n, a + 2ix, a - 2ix \mid a+1, a+b \right), \quad n = 0, 1, 2, \ldots,
$$

(10.6)

where

$$
a = -2r_1 + 1/2, \quad b = -2r_2 + 1/2
$$

and the normalization coefficients are

$$
\kappa_n^{(0)} = (-1)^n 2^{-2n} (a)_n (a+b)_n, \quad \kappa_n^{(1)} = (-1)^n 2^{-2n} (a+1)_n (a+b)_n.
$$

Recall that the (unnormalized) continuous dual Hahn polynomials $C_n(x^2; a, b, c)$ depend on 3 parameters $a, b, c$ and are expressed as [17]

$$
C_n(x^2; a, b, c) = \binom{3}{2} F_2 \left( -n, a + ix, a - ix \mid a, a+c \right).
$$

(10.7)

We thus see that

$$
\tilde{S}_{2n}(x) = \kappa_n^{(0)} C_n(4x^2; a, b, 0), \quad \tilde{S}_{2n+1}(x) = \kappa_n^{(1)} x C_n(4x^2; a, b, 1).
$$

From the expression of the weight function of the continuous dual Hahn polynomials [17], we reconstruct the weight function of the symmetric BI polynomials

$$
w(x) = \left| \frac{\Gamma(a + 2ix)\Gamma(b + 2ix)}{\Gamma(1/2 + 2ix)} \right|^2,
$$

(10.8)

for which they are orthogonal on the whole real line

$$
\int_{-\infty}^{\infty} w(x) \tilde{S}_n(x) \tilde{S}_m(x) dx = 0, \quad n \neq m.
$$

The difference equation for these polynomials follows immediately from the corresponding difference equation (9.3) for the BI polynomials after replacing $y \rightarrow iy$:

$$
\Phi_1(y) \tilde{S}_n(y - i/2) + \Phi_2(y) \tilde{S}_n(y + i/2) - (\Phi_1(y) + \Phi_2(y)) \tilde{S}_n(y) = \lambda_n \tilde{S}_n(y),
$$

(10.9)

where

$$
\Phi_1(y) = -i \frac{(y + ia/2)(y + ib/2)}{2(y + i/4)}, \quad \Phi_2(y) = \Phi_1^*(y) = i \frac{(y - ia/2)(y - ib/2)}{2(y - i/4)}
$$

and the eigenvalue is

$$
\lambda_n = \left\{ \begin{array}{ll}
\frac{n}{2} & \text{if } n \text{ is even} \\
2(r_1 + r_2) - \frac{n+1}{2} & \text{if } n \text{ is odd}
\end{array} \right.
$$

(10.10)

Consider the special case when one of the parameters $r_1$ or $r_2$ vanishes, say $r_2 = 0$. The recurrence coefficients $\tilde{u}_n$ then become

$$
\tilde{u}_n = \frac{n(n-4r_1)}{16}, \quad n = 1, 2, 3, \ldots
$$

(10.11)
These correspond to the recurrence coefficients of the symmetric Meixner-Pollaczek polynomials [17]

\[ \tilde{S}_n(x) = i^{n-2}2^{-2n} (2a)_n \, {_2F_1} \left( \begin{array}{c} \frac{-n, a + 2ix}{2a} \\ \frac{2}{2} \end{array} ; 2 \right). \] (10.12)

The corresponding Dunkl shift eigenvalue equation is

\[-\frac{i}{4} (2y + ia) \tilde{S}_n(-y - i/2) + \frac{i}{4} (2y - ia) \tilde{S}_n(-y + i/2) + \frac{a}{2} \tilde{S}_n(y) = \lambda_n \tilde{S}_n(y). \] (10.13)

Taking into account the property \( \tilde{S}_n(-y) = (-1)^n S_n(y) \) of the symmetric polynomials, equation (10.13) can be cast in the form

\[-\frac{i}{4} (2y + ia) \tilde{S}_n(y + i/2) + \frac{i}{4} (2y - ia) \tilde{S}_n(y - i/2) = \frac{a + n}{2} \tilde{S}_n(y) \] (10.14)

which coincides with the ordinary difference equation of the Meixner-Pollaczek polynomials on a uniform imaginary grid [17].

The weight function in this case is obtained from the weight function (10.8) by putting \( b = 1/2 \):

\[ w(x) = |\Gamma(a + 2ix)|^2, \]

which coincides with the well known expression for the weight function of the symmetric Meixner-Pollaczek polynomials [17].

The symmetric Meixner-Pollaczek polynomials therefore belong, as a special case, to the class of BI polynomials. The eigenvalue equation for these polynomials can be presented either in the form (10.13) as an eigenvalue equation for a Dunkl shift difference operator of first order, or in the form (10.14) which is a standard Sturm-Liouville difference equation of the second order.

11. The big and little -1 Jacobi polynomials as limit cases

Changing the variable to \( x = y/h \) with \( h \) a real parameter, we can rewrite the BI operator in the form

\[ L f(y) = \frac{(y - \rho_1 h)(y - \rho_2 h)}{2hy} (f(y) - f(-y)) + \frac{(y - r_1 h + h/2)(y - r_2 h + h/2)}{h(2y + h)} (f(-y - h) - f(y)). \] (11.1)

Choose now the following parametrization

\[ r_1 = a_1/h, \ r_2 = a_2/h, \ \rho_1 = a_1/h + b_1, \ \rho_2 = a_2/h + b_2, \] (11.2)

where the real parameters \( a_i, b_i, i = 1, 2 \) do not depend on \( h \). Taking the limit \( h \to 0 \), we then arrive at the operator

\[ L_0 = \lim_{h \to 0} L = \frac{(y - a_1)(y - a_2)}{2y} \partial_y R - \frac{F(y)}{4y^2} (I - R), \] (11.3)

where

\[ F(y) = (2b_1 + 2b_2 + 1)y^2 - 2(a_1 b_2 + a_2 b_1)y - a_1 a_2. \]
The operator (11.3) is a first order differential operator of Dunkl type. It was shown in [34] that the operator $L_0$ as given in (11.3) with $a_1, a_2, b_1$ and $b_2$ real, represents the most general first order differential operator of Dunkl type that has orthogonal polynomials as eigenfunctions. In general the operator $L_0$ has 4 free parameters. However, if $a_2 \neq 0$, by scaling the argument $y \rightarrow \gamma y$ one can set $a_2 = 1$ and only 3 independent real parameters $a_1, b_1, b_2$ remain.

Using a slightly different parametrization, we thus have [33]

$$L_0 = g_0(y)(R - I) + g_1(y)\partial_y R,$$

where

$$g_0(y) = \frac{(\alpha + \beta + 1)y^2 + (c\alpha - \beta)y + c}{y^2}, \quad g_1(y) = \frac{2(y - 1)(y + c)}{y}.$$  (11.5)

The 3 independent real parameters of the operator $L_0$ are now $\alpha, \beta$ and $c$. The polynomial eigensolutions $P_n(y), n = 0, 1, 2, \ldots$ of $L_0$ are expressed in terms of the big -1 Jacobi polynomials (see [33] for details).

If $a_2 = 0$, the operator $L_0$ can be reduced to

$$L_0 = 2(1 - y)\partial_y R + (\alpha + \beta + 1 - \alpha y^{-1})(1 - R)$$  (11.6)

with only two independent real parameters $\alpha, \beta$. The polynomial eigensolutions $P_n(y), n = 0, 1, 2, \ldots$ of the operator (11.6) coincide with the little -1 Jacobi polynomials [32].

In [33] it was shown how the recurrence coefficients of the big -1 Jacobi polynomials could be recovered through a special large $N$ limit, from those of the Bannai-Ito polynomials. Clearly, the eigenvalue problem for the big -1 Jacobi polynomials can also be obtained from the corresponding eigenvalue problem for the BI polynomials. For $q \rightarrow 1$ this corresponds to the transition from the q-Racah to the big q-Jacobi polynomials [20].

**12. Conclusion**

The study of the polynomial eigenfunctions of first order Dunkl shift operator has thus led synthetically to a thorough characterization of the Bannai-Ito polynomials: weight function, structure and recurrence coefficients etc. It has brought to light a relation between certain Jordan algebras, their representations and orthogonal polynomials. It made natural the introduction of the complementary Bannai-Ito polynomials, their expression in terms of Wilson polynomials and their connection to the BI polynomials. It further allowed to see that the symmetric BI polynomials are related to the dual Hahn polynomials and to study finally, various limiting cases.

Note that polynomial eigenfunctions of certain linear operators with reflections were previously considered in the literature (see for instance [19]); these solutions however exhibit non-standard orthogonality properties. In contrast, the polynomial eigensolutions that have arisen in our treatment, are clearly orthogonal in the ordinary sense.
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