A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$

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Abstract

We provide a new proof of the sharpness of the phase transition for nearest-neighbour Bernoulli percolation. More precisely, we show that

- for $p < p_c$, the probability that the origin is connected by an open path to distance $n$ decays exponentially fast in $n$.
- for $p > p_c$, the probability that the origin belongs to an infinite cluster satisfies the mean-field lower bound $\theta(p) \geq \frac{p - p_c}{p(1 - p_c)}$.

This note presents the argument of [DCT15], which is valid for long-range Bernoulli percolation (and for the Ising model) on arbitrary transitive graphs in the simpler framework of nearest-neighbour Bernoulli percolation on $\mathbb{Z}^d$.

1 Statement of the result

Notation. Fix an integer $d \geq 2$. We consider the $d$-dimensional hypercubic lattice $(\mathbb{Z}^d, E^d)$. Let $\Lambda_n = \{-n, \ldots, n\}^d$, and let $\partial \Lambda_n := \Lambda_n \setminus \Lambda_{n-1}$ be its vertex-boundary. Throughout this note, $S$ always stands for a finite set of vertices containing the origin. Given such a set, we denote its edge-boundary by $\Delta S$, defined by all the edges $\{x, y\}$ with $x \in S$ and $y \notin S$.

Consider the Bernoulli bond percolation measure $\mathbb{P}_p$ on $\{0,1\}^{E^d}$ for which each edge of $E^d$ is declared open with probability $p$ and closed otherwise, independently for different edges.

Two vertices $x$ and $y$ are connected in $S \subset V$ if there exists a path of vertices $(v_k)_{0 \leq k \leq K}$ in $S$ such that $v_0 = x$, $v_K = y$, and $\{v_k, v_{k+1}\}$ is open for every $0 \leq k < K$. We denote this event by $x \leftrightarrow_S y$. If $S = \mathbb{Z}^d$, we drop it from the notation. We set $0 \leftrightarrow \infty$ (resp. $0 \leftrightarrow \partial \Lambda_n$) if 0 is connected to infinity (resp. 0 is connected to a vertex in $\partial \Lambda_n$).
Phase transition. A new idea of this paper is to use a different definition of the critical parameter than the standard one. This new definition relies on the following quantity. For \( p \in [0,1] \) and \( 0 \in S \subset \mathbb{Z}^d \), define
\[
\varphi_p(S) := p \sum_{(x,y) \in \Delta S} \mathbb{P}_p[0 \leftrightarrow x]
\]
and introduce the following quantities:
\[
\tilde{p}_c := \sup \{ p \in [0,1] \text{ s.t. there exists a finite set } 0 \subset S \subset \mathbb{Z}^d \text{ with } \varphi_p(S) < 1 \},
\]
\[
p_c := \sup \{ p \text{ s.t. } \mathbb{P}_p[0 \leftrightarrow \infty] = 0 \}.
\]
We are now in a position to state our main result.

**Theorem 1.1.** For any \( d \geq 2 \), \( \tilde{p}_c = p_c \). Furthermore,
1. For \( p < p_c \), there exists \( c = c(p) > 0 \) such that for every \( n \geq 1 \),
\[
\mathbb{P}_p[0 \leftrightarrow \partial \Lambda_n] \leq e^{-cn}.
\]
2. For \( p > p_c \),
\[
\mathbb{P}_p[0 \leftrightarrow \infty] \geq \frac{p - p_c}{p(1 - p_c)}.
\]

**Remarks.**
1. We refer to [DCT15] for a detailed bibliography, and for a version of the proof valid in greater generality. The aim of this note is to provide a proof in the simplest framework.
2. Theorem 1.1 was proved by Aizenman and Barsky [AB87] in the more general framework of long-range percolation. In their proof, they consider an additional parameter \( h \) corresponding to an external field, and they derive the results from differential inequalities satisfied by the thermodynamical quantities of the model. A different proof, based on the geometric study of the pivotal edges, was obtained at the same time by Menshikov [Men86]. These two proofs are also presented in [Gri99].
3. In the definition of \( \tilde{p}_c \), the set of parameters \( p \) such that there exists a finite set \( 0 \subset S \subset \mathbb{Z}^d \) with \( \varphi_p(S) < 1 \) is an open subset of \([0,1]\). Thus, \( \tilde{p}_c \) do not belong to this set.
We obtain that the expected size of the cluster of the origin satisfies that for every $p > p_c$,
\[ \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p[0 \leftrightarrow x] \geq \sum_{n \geq 0} \varphi_p(\Lambda_n) = +\infty. \]

4. Since $\varphi_p(\{0\}) = 2dp$, we obtain $p_c \geq 1/2d$.

5. Item 2 provides a mean-field lower bound for the infinite cluster density.

6. Theorem 1.1 implies that $p_c \leq 1/2d$. Combined with Zhang's argument [Gri99, Lemma 11.12], this shows that $p_c = 1/2$.

2 Proof of the theorem

It is sufficient to show Items 1 and 2 with $p_c$ replaced by $\tilde{p}_c$ (since it immediately implies the equality $p_c = \tilde{p}_c$).

2.1 Proof of Item 1

The proof of Item 1 can be derived from the BK-inequality [vdBK85]. We present here an exploration argument, similar to the one in [Ham57], which does not rely on the BK-inequality. Let $p < \tilde{p}_c$. By definition, one can fix a finite set $S$ containing the origin, such that $\varphi_p(S) < 1$. Let $L > 0$ such that $S \subset \Lambda_{L-1}$.

Let $k \geq 1$ and assume that the event $0 \leftrightarrow \partial \Lambda_{kL}$ holds. Let
\[ \mathcal{C} = \{ z \in S : 0 \leftrightarrow z \}. \]

Since $S \cap \partial \Lambda_{kL} = \emptyset$, there exists an edge $\{x, y\} \in \Delta S$ such that the following events occur:
- $0$ is connected to $x$ in $S$,
- $\{x, y\}$ is open,
- $y$ is connected to $\partial \Lambda_{kL}$ in $\mathcal{C}^c$.

Using first the union bound, and then a decomposition with respect to possible values of $\mathcal{C}$, we find
\[
\mathbb{P}_p[0 \leftrightarrow \partial \Lambda_{kL}] \\
\leq \sum_{(x, y) \in \Delta S} \sum_{C \subset S} \mathbb{P}_p[\{0 \leftrightarrow S, \mathcal{C} = C\} \cap \{\{x, y\} \text{ is open}\} \cap \{y \leftrightarrow \partial \Lambda_{kL}\}] \\
= p \sum_{(x, y) \in \Delta S} \sum_{C \subset S} \mathbb{P}_p[0 \leftrightarrow S, \mathcal{C} = C] \mathbb{P}_p[y \leftrightarrow \partial \Lambda_{kL}].
\]
In the second line, we used the fact that the three events depend on different sets of edges and are therefore independent. Since \( y \in \Lambda_L \), one can bound \( P_p[0 \leftrightarrow \partial \Lambda_{kL}] \) by \( P_p[0 \leftrightarrow \partial \Lambda_{(k-1)L}] \) in the last expression. Hence, we find
\[
P_p[0 \leftrightarrow \partial \Lambda_{kL}] \leq \varphi_p(S) P_p[y \leftrightarrow \partial \Lambda_{(k-1)L}]
\]
which by induction gives
\[
P_p[0 \leftrightarrow \partial \Lambda_{kL}] \leq \varphi_p(S)^{k-1}.
\]
This proves the desired exponential decay.

2.2 Proof of Item 2

Let us start by the following lemma providing a differential inequality valid for every \( p \).

**Lemma 2.1.** Let \( p \in [0, 1] \) and \( n \geq 1 \),
\[
\frac{d}{dp} P_p[0 \leftrightarrow \partial \Lambda_n] \geq \frac{1}{p(1-p)} \cdot \inf_{S \subseteq \Lambda_n} \varphi_p(S) \cdot (1 - P_p[0 \leftrightarrow \partial \Lambda_n]). \tag{2.1}
\]

Let us first see how it implies Item 2 of Theorem 1.1. Integrating the differential inequality (2.1) between \( \tilde{p} \) and \( p > \tilde{p} \) implies that for every \( n \geq 1 \), \( P_p[0 \leftrightarrow \partial \Lambda_n] \geq \frac{p}{p(1-p)} \). By letting \( n \) tend to infinity, we obtain the desired lower bound on \( P_p[0 \leftrightarrow \infty] \).

**Proof of Lemma 2.1.** Recall that \( \{x,y\} \) is pivotal for the configuration \( \omega \) and the event \( \{0 \leftrightarrow \partial \Lambda_n\} \) if \( \omega(\{x,y\}) \notin \{0 \leftrightarrow \partial \Lambda_n\} \) and \( \omega(\{x,y\}) \in \{0 \leftrightarrow \partial \Lambda_n\} \). (The configuration \( \omega(\{x,y\}) \), resp. \( \omega(\{x,y\}) \), coincides with \( \omega \) except that the edge \( \{x,y\} \) is closed, resp. open.) By Russo’s formula (see [Gri99, Section 2.4]), we have
\[
\frac{d}{dp} P_p[0 \leftrightarrow \partial \Lambda_n] = \sum_{e \in \Lambda_n} P_p[e \text{ is pivotal}]
\]
\[
= \frac{1}{1-p} \sum_{e \in \Lambda_n} P_p[e \text{ is pivotal}, 0 \leftrightarrow \partial \Lambda_n].
\]

Define the following random subset of \( \Lambda_n \):
\[
\mathcal{S} := \{x \in \Lambda_n \text{ such that } x \leftrightarrow \partial \Lambda_n\}.
\]

The boundary of \( \mathcal{S} \) corresponds to the outmost blocking surface (which can be obtained by exploring from the outside the set of vertices connected to the boundary). When 0 is not connected to \( \partial \Lambda_n \), the set \( \mathcal{S} \) is always
a subset of $\Lambda_n$ containing the origin. By summing over the possible values for $\mathcal{S}$, we obtain

$$\frac{d}{dp} \mathbb{P}_p[0 \leftrightarrow \partial \Lambda_n] = \frac{1}{1 - p} \sum_{0 \in S} \sum_{e \in \Lambda_n} \mathbb{P}_p[e \text{ is pivotal}, \mathcal{S} = S]$$

Observe that on the event $\mathcal{S} = S$, the pivotal edges are the edges $\{x, y\} \in \Delta S$ such that $0$ is connected to $x$ in $S$. This implies that

$$\frac{d}{dp} \mathbb{P}_p[0 \leftrightarrow \partial \Lambda_n] = \frac{1}{1 - p} \sum_{0 \in S} \sum_{\{x, y\} \in \Delta S} \mathbb{P}_p[0 \xleftarrow{S} x, \mathcal{S} = S].$$

The event $\{\mathcal{S} = S\}$ is measurable with respect to the configuration outside $S$ and is therefore independent of $\{0 \xleftarrow{S} x\}$. We obtain

$$\frac{d}{dp} \mathbb{P}_p[0 \leftrightarrow \partial \Lambda_n] = \frac{1}{1 - p} \sum_{0 \in S} \sum_{\{x, y\} \in \Delta S} \mathbb{P}_p[0 \xleftarrow{S} x] \mathbb{P}_p[\mathcal{S} = S]$$

$$= \frac{1}{p(1 - p)} \sum_{S \subseteq \Lambda_n} \varphi_p(S) \mathbb{P}_p[\mathcal{S} = S]$$

$$\geq \frac{1}{p(1 - p)} \inf_{S \subseteq \Lambda_n} \varphi_p(S) \cdot \mathbb{P}_p[0 \leftrightarrow \partial \Lambda_n],$$

as desired. \(\square\)

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**References**

[AB87] Michael Aizenman and David J. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108(3):489–526, 1987.

[DCT15] Hugo Duminil-Copin and Vincent Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. preprint, 2015.

[Gri99] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.
[Ham57] J. M. Hammersley. Percolation processes: Lower bounds for the critical probability. *Ann. Math. Statist.*, 28:790–795, 1957.

[Men86] M. V. Menshikov. Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR*, 288(6):1308–1311, 1986.

[vdBK85] J. van den Berg and H. Kesten. Inequalities with applications to percolation and reliability. *J. Appl. Probab.*, 22(3):556–569, 1985.

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