Learning scattering amplitudes by heart

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Abstract

The canonical forms associated to scattering amplitudes of planar Feynman diagrams are interpreted in terms of masses of projectives, defined as the modulus of their central charges, in the hearts of certain \( t \)-structures of derived categories of quiver representations and, equivalently, in terms of cluster tilting objects of the corresponding cluster categories.

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1 Introduction

The amplituhedron program of $\mathcal{N} = 4$ supersymmetric Yang–Mills theories [1] culminating in the ABHY construction [2] has provided renewed impetus to the study of computation of scattering amplitudes in quantum field theories, even without supersymmetry, using geometric ideas. While homological methods for the evaluation of Feynman diagrams have been pursued for a long time [3], the ABHY program associates the geometry of Grassmannians to the amplitudes [4]. Relations of amplitudes to a variety of mathematical notions and structures have been unearthed [5–7]. For scalar field theories the amplitudes are expressed in terms of Lorentz-invariant Mandelstam variables. The space of Mandelstam variables is known as the kinematic space. The combinatorial structure of the amplitudes associated with the planar Feynman diagrams of the cubic scalar field theory is captured by writing those as differential forms associated to a polytope in the kinematic space, called the associahedron, and their integrals. The form is named canonical form.

In an attempt to categorify the ideas we extract the canonical form from cluster tilting objects in the cluster category of a quiver of type $A$ and, equivalently, from torsion pairs for the category of quiver representations and their intermediate $t$-structures. Such a connection may be anticipated from the existing literature, but here we highlight the categorical framework available in the representation-theoretic literature, which we expect to prove instructive in generalizing the computation of scattering amplitudes in general. We restrict our attention to the planar tree level Feynman diagrams in a cubic scalar field theory. These diagrams can be thought of as rooted binary trees. The Feynman diagrams are dual to triangulated polygons in the sense that they are obtained by drawing lines intersecting the edges of a triangulated polygon as indicated in Fig. 1.

![Binary tree, Feynman diagram and triangulation of polygon](image)

Figure 1: Binary tree, Feynman diagram and triangulation of polygon

The various triangulations of a plane polygon, and hence the Feynman diagrams, are in one-to-one correspondence with the vertices of an associahedron, which also appears in the representation theory of quivers [8] in the form of an exchange graph [9–11]. The canonical form is identified as the volume of the polytope dual to the associahedron [2,5]. The vertices of the associahedron are also associated to the set of cluster variables of cluster algebras [12], two adjacent vertices being related by a mutation [13]. In a categorical framework cluster algebras are associated to cluster categories which can be realized as triangulated orbit categories of the bounded derived categories of quiver representations. We show that the canonical form is directly obtained from the central charges of the projective indecomposable objects of hearts of intermediate $t$-structures. In other words, we obtain the red arrows in Fig. 2.

Let us describe the proposed interpretation at the outset. We shall then work out two examples, which are easily generalized. In a perturbative treatment of the scattering of $N$ particles in quantum field theory, the conservation of total momenta carried by the
particles as well as the nature of interactions are encoded in Feynman diagrams. The contribution to the scattering amplitude corresponding to the diagrams are expressed in terms of Lorentz-invariants formed out of the momenta of the particles. Consideration of the diagrams beyond tree level, which involve integration over momenta, will be postponed to future work. A set of such invariants, called planar variables, are denoted $X_{ij}$, with $i$ and $j$ running over the labels of $N$ particles. The indices are defined modulo $N$. This, in addition to the symmetry of the planar variables under the exchange of the indices, gives the set of planar variables a periodic structure. A certain combination of the planar variables, namely, the discrete Laplacian operating on $X_{ij}$, relates to the Mandelstam variables. This combination may be presented as a diamond-like relation with arrows indicating the sign of terms in the discrete Laplacian. Considered seriatim for all the particles, the diamond can be knit into a mesh. If the mesh is continued ad infinitum it yields the Auslander–Reiten (AR) quiver of the bounded derived category $\mathcal{D}_Q = \mathcal{D}^b(\text{rep } Q)$ of finite-dimensional representations of a quiver $Q$ of type $A$ [14,15], the planar variables corresponding one-to-one with indecomposable objects and the arrows in the mesh taken to represent the irreducible morphisms in $\mathcal{D}_Q$. The periodicity of the planar variables further restricts the structure by dictating an identification of objects in the AR quiver. We observe that this identification happens to correspond precisely to the passage from the derived category $\mathcal{D}_Q$ to the cluster category $\mathcal{C}_Q$, which is the triangulated category of orbits under a certain autoequivalence [13].

Planar tree level Feynman diagrams of $N$ particles are dual to triangulations of an $N$-gon, which in turn can be related to the cluster algebra or the cluster category of the $A_{N-3}$ quiver [16,17], the combinatorial structure of the set of all triangulations being described by mutations in the cluster algebra or cluster category. In the categorical setting, these combinatorics can be expressed via a range of different structures. In this article we focus on two such structures: intermediate $t$-structures for the derived category $\mathcal{D}_Q$ and cluster
tilting objects for the cluster category $C_Q$. Our goal is to use the correspondences

Feynman diagrams $\longleftrightarrow$ intermediate $t$-structures in $D_Q \longleftrightarrow$ cluster tilting objects in $C_Q$

to obtain the explicit expression of the canonical form.

The similarity between the mesh and the AR quiver is formal so far. While the objects in the cluster category are orbits of complexes of quiver representations in the derived category, the planar variables are real numbers. In order to relate these we remark that the central charge of objects are complex numbers associated to equivalence classes of the objects in the Grothendieck group of the derived category. These are related through the mesh relations, which coincide with the mesh relations among the planar variables $X_{ij}$. By identifying the modulus of the central charge of an object in the AR quiver of the cluster category with the corresponding planar variable $X_{ij}$ we re-derive, not surprisingly, the relations among the latter from the mesh relations of the derived category. The modulus of central charge is called the mass [18]. In here it relates to the squared invariant mass of a collection of particles. In the categorical parlance $X_{ij}$ are interpreted as the masses of indecomposable objects in the derived category. Their relations as derived from momentum conservations descend from the central charges on the stability lines. Each cluster tilting object in the cluster category corresponds to a Feynman diagram. The contribution of each Feynman diagram to the amplitude is thus given by a term in the canonical form. Each term is a logarithmic $(N - 3)$-form written in terms of the planar variables. We show that from the point of view of derived categories each such term is given by the masses of projective objects of the hearts of intermediate $t$-structures.

Categorical formulation facilitates organizing computations. The present formulation, while not as geometric as the associahedron or the Grassmannians, has the virtue of being algebraic and seems to be of use in developing computer algorithms for the evaluation of the tree diagrams. We now exemplify the program chalked out above for two examples, $N = 5$ and $N = 6$, the latter being completely generic. In the next section we review the derivation of the mesh relations among the planar variables from momentum conservation. The case of $N = 5$ particles is worked out explicitly. In section 3 we recall the notions of derived categories, intermediate $t$-structures and cluster categories and central charges, exemplified for the case of $A_2$ quiver, corresponding to the case of $N = 5$ particles. The canonical form for this case is then obtained for this case. In section 4 we present the case of $N = 6$ particle scattering corresponding to the $A_3$ quiver. We obtain the mesh relations, the cluster category and retrieve the canonical form, before concluding in section 5. The two examples are described in the following two sections. In each case we first identify the cluster category from the mesh relations of the planar variables. We then obtain the cluster tilting objects and identify their direct summands, which may be viewed equivalently as projective objects of hearts of intermediate $t$-structures. Their masses are then shown to correspond to terms of the canonical form.

2 Kinematics

Let us recall the definition of kinematic variables for the scattering of a system of $N$ scalar particles [2]. Their momenta are vectors in $R^{1,3}$, denoted $p_i$, for $i = 1, 2, \ldots, N$, satisfying
the conservation equation
\[ \sum_{i=1}^{N} p_i = 0. \]
(1)
This is solved by writing the momenta in terms of another set of \( N \) four-vectors \( x \) as
\[ p_i = x_{i+1} - x_i, \]
(2)
where from now on we define the indices modulo \( N \), in particular, \( x_{N+1} = x_1 \). Mandelstam variables are quadratic invariants for a pair of particles
\[ s_{ij} = (p_i + p_j)^2 \]
(3)
where the norm of a four-vector \( p = (p_0, p_1, p_2, p_3) \) is defined as \( p^2 = -p_0^2 + p_1^2 + p_2^2 + p_3^2 \). Entities defined similarly with the \( x \)'s as
\[ X_{ij} = (x_i - x_j)^2 \]
(4)
are called planar variables. These are symmetric with respect to exchange of indices by definition, \( X_{ji} = X_{ij} \). Using (2) and the periodicity of the indices the planar variables are related to the Mandelstam variables as
\[ s_{ij} = p_i^2 + p_j^2 + X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1}. \]
(5)
If we now assume that the particles are massless, that is, the momentum vectors are null, \( p_i^2 = 0 \) for each \( i \), then
\[ s_{ii} = 2p_i^2 = 0, \quad X_{i,i+1} = p_i^2 = 0 \]
(6)
and the relation (5) becomes
\[ s_{ij} = X_{i,j+1} + X_{i+1,j} - X_{ij} - X_{i+1,j+1}. \]
(7)
The right hand side is the negative discrete Laplacian operating on the planar variables. In particular, we have
\[ X_{N,N+1} = X_{N,1} = X_{1,N} = 0, \quad X_{2,N+1} = X_{2,1} = X_{1,2} = 0. \]
(8)
Equation (7) can be pictorially presented as
\[ s_{ij} = X_{ij} = X_{i+1,j+1} \]
(9)
For any given value of \( N \) this unit can be used to weave a mesh [14], which, upon using the periodicity of the indices, the symmetry of \( X \)'s and equation (8), gives rise to the cluster category of the \( A_{N-3} \) quiver. In the next two sections we work out the examples of \( N = 5 \) and \( N = 6 \) and obtain the canonical forms from the central charges.
2.1 Example of $N = 5$

In this section we recall various notions pertaining to the categorical description of the scattering amplitude. In order to be explicit we shall often use the example of an $A_2$ quiver arising in the case of scattering of five particles. The mesh diagram knit from (9) is

![Mesh Diagram](image)

Using the periodicity of the indices modulo 5 and the symmetry $X_{ji} = X_{ij}$, for example $X_{59} = X_{54} = X_{45}$, the ones in red blocks are null by (6) and (8). The canonical form associated to the $N = 5$ amplitude is [2]

$$
\Omega_5 = d \log X_{14} \wedge d \log X_{13} - d \log X_{35} \wedge d \log X_{13} + d \log X_{35} \wedge d \log X_{25} - d \log X_{24} \wedge d \log X_{25} + d \log X_{24} \wedge d \log X_{14}. \quad (11)
$$

The scattering amplitude is obtained from the canonical form using relations among the planar variables [2]. These relations are derived from the mesh relations in the derived category. Let us indicate the combinatorial scheme for fixing the relative signs of the terms. The set of planar variables appearing in each term of the canonical form are identified first and one term is fixed, say, the first one in (11) with a positive sign. We then replace one of the planar variables with a new one. The new term is given a negative sign, as in the second term in (11), where $X_{14}$ of the first term is replaced with $X_{35}$. Repeating this we obtain the full canonical form.

The main observation described in this article is that after dispensing with the red blocks along with the arrows to and from the corresponding nodes we are left with the Auslander–Reiten (AR) quiver of the bounded derived category of the finite-dimensional representations of the $A_2$ quiver. Moreover, the green block in (10) is, after identifying the objects as indicated, precisely the cluster category of $A_2$, and the terms of the canonical form are in one-to-one correspondence with the cluster tilting objects and, on the level of the bounded derived category, with the intermediate $t$-structures. The next three subsections digress to recall various notions pertaining to the categorification of the canonical form. The discussions are brief and often through examples. We refer to [13, 26, 27] for further elaborations.
3 Categorical picture

A quiver is a directed graph, that is, a collection of vertices and arrows amongst them. A representation of a quiver is given by associating a finite-dimensional vector space to the vertices and linear maps between these vector spaces to the arrows. An \( A_n \) quiver, is a quiver whose underlying undirected graph is a Dynkin diagram of type \( A_n \). In this article we work with the \( A_n \) quiver with linear orientation:

\[
1 \rightarrow 2 \rightarrow \cdots \rightarrow n,
\]

the natural numbers labelling the vertices. Such representations form an Abelian category and every representation is isomorphic to a direct sum of indecomposable representations. For a quiver of type ADE, Gabriel’s Theorem states that there are only finitely many indecomposable representations, which for the linearly oriented \( A_n \) quiver are of the simple form:

\[
0 \rightarrow \cdots \rightarrow 0 \rightarrow \mathbb{C} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbb{C} \rightarrow 0 \rightarrow \cdots \rightarrow 0
\]

where the vector space at each node is taken to be either the 1-dimensional vector space of complex numbers \( \mathbb{C} \), or the trivial vector space 0 and the linear maps between copies of \( \mathbb{C} \) are the identity maps.

For example, the \( A_2 \) quiver is \( 1 \rightarrow 2 \). The Abelian category of finite-dimensional representations \( \text{rep} A_2 \) contains three indecomposable representations

\[
\mathbb{C} \rightarrow 0, \quad 0 \rightarrow \mathbb{C} \quad \text{and} \quad \mathbb{C} \xrightarrow{\text{id}} \mathbb{C}.
\]

These representations are also denoted \( 1 \), \( 2 \) and \( \frac{1}{2} \), respectively. Here a single number \( i \) denotes the simple representation with a one-dimensional vector space at the vertex labelled \( i \) in the quiver and \( \frac{i}{j} \) denotes an extension of \( i \) by \( j \). These three representations fit into a short exact sequence

\[
0 \rightarrow 2 \rightarrow \frac{1}{2} \rightarrow 1 \rightarrow 0
\]

in \( \text{rep} A_2 \).

A derived category is obtained from an Abelian category by promoting short exact sequences to triangles. Elements of the derived category are \( \mathbb{Z} \)-graded cochain complexes of elements of the Abelian category, where two complexes are considered isomorphic in the derived category if there is a chain of morphisms of complexes between the two inducing an isomorphism in cohomology. The bounded derived category is formed by considering only complexes which are zero in all but finitely many degrees. The derived category is equipped with a shift functor \([1]\) which, when applied to a complex, shifts all the elements of the complex down by one degree. If \( C^n \) denotes the degree \( i \) component of a complex \( C^\bullet \), then the degree \( i \)th chain in the shifted complex is \( C^\bullet[n]^i = C^{i+n} \), where \([n]\) is the \( n \)-fold composition of \([1]\).

Given an element \( M \) in the Abelian category, one can view it as a complex with \( M \) placed in degree 0 and \( M[n] \) denotes its shift by \( n \), e.g.

\[
\begin{array}{ccc}
-2 & -1 & 0 & 1 \\
M & \leftrightarrow (\cdots \rightarrow 0 \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots) \\
M[1] & \leftrightarrow (\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow 0 \rightarrow \cdots).
\end{array}
\]

6
Given a short exact sequence $0 \rightarrow A \overset{f}{\rightarrow} B \overset{g}{\rightarrow} C \rightarrow 0$ in an Abelian category, the two complexes

$$
\begin{array}{cccccc}
-2 & -1 & 0 & 1 \\
\cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\end{array}
$$

(16)

and

$$
\begin{array}{cccccc}
\cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\end{array}
$$

are isomorphic in the derived category, as the maps in the short exact sequence give a map of complexes which induces an isomorphism in cohomology. Hence, there is a map $C \rightarrow A[1]$ in the derived category, given by

$$
\begin{array}{cccccc}
\cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\end{array}
$$

(17)

where the lower complex is $A[1]$. This is expressed by drawing a triangle

$$
\begin{array}{c}
B \\
\downarrow \\
A \\
\downarrow \\
C \\
\end{array}
$$

(18)

and such triangles equip the derived category with its triangulated structure. Another way to encode the information of the triangle is writing the complex

$$
\cdots \rightarrow A \rightarrow B \rightarrow C \rightarrow A[1] \rightarrow B[1] \rightarrow \cdots
$$

(19)

in the derived category corresponding to the short exact sequence in the Abelian category. The complex continues on both sides. In this fashion the Abelian category $\text{rep} A_n$ gives rise to the bounded derived category, denoted $\mathcal{D}_{A_n} = \text{D}^b(\text{rep} A_n)$.

Since the Abelian category $\text{rep} A_n$ is hereditary (i.e. $\text{Ext}_{\text{rep} A_n}^k$ vanishes for $k \geq 2$) and only has only finitely many indecomposable objects, any object in the bounded derived category can be decomposed into complexes of the form $M[n]$ for some indecomposable representation $M$ of $\text{rep} A_n$ and some integer $n$. All the essential information of the derived category $\mathcal{D}_{A_n}$ is then captured diagrammatically by its AR quiver, which encodes the morphisms and extensions between the indecomposable objects of the derived category. For example, starting from (14) we have, by (19), the long sequence in $\mathcal{D}_{A_2}$, which is depicted as the AR quiver

$$
\begin{array}{cccccc}
\frac{1}{2} & \cdots & \tau & \cdots & 2[1] & \cdots & \tau & \cdots & 1[1] \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2 & \cdots & \tau & \cdots & 1 & \cdots & \tau & \cdots & \frac{1}{2}[1] & \cdots & \tau & \cdots & 2[2] \\
\end{array}
$$

(20)

The diagram continues $ad\ infinitum$ on both sides and the dashed arrows $\tau$ are the AR translations, which encode the triangles of $\mathcal{D}_{A_2}$. This diagram can be identified with the middle portion, between the omitted red blocks, of the mesh (10), where the nodes in the AR quiver correspond to the planar variables $X_{ij}$ in (10).
3.1 Intermediate \( t \)-structures

Given the derived category \( D = \mathbb{D}^b(A) \) of an Abelian category \( A \), one can recover \( A \) as the complexes concentrated in degree 0. This process of obtaining an Abelian category from a derived category is generalized and axiomatized in the notion of a \( t \)-structure which provide a way of vivisecting the derived category, called truncations, realizing the Abelian categories as the portion common to the truncated parts, called hearts. The original Abelian category is usually referred to as the standard heart and the corresponding \( t \)-structure as the standard \( t \)-structure.

A \( t \)-structure on \( D \) is a pair \((D^{\leq 0}, D^{\geq 0})\) of strictly full subcategories, satisfying

1. \( D^{\leq 0} \subset D^{\leq 1} \) and \( D^{\geq 1} \subset D^{\geq 0} \)
2. \( \text{Hom}_D(D^{\leq 0}, D^{\geq 1}) = 0 \)
3. for each object \( M \) of the triangulated category \( D \), there exists a distinguished triangle

\[
\begin{array}{ccc}
M_{\geq 1} & \longrightarrow & M \\
\downarrow & & \downarrow \\
M & \leftarrow & M_{\leq 0}
\end{array}
\]

(21)

where we set \( D^{\geq m} = D^{\geq 0}[-m] \) and \( D^{\leq n} = D^{\leq n}[-n] \) for any integers \( m \) and \( n \). Further, a \( t \)-structure is called bounded if each \( M \) in \( D \) is contained in \( D^{\geq m} \cap D^{\leq n} \) for some integers \( m \) and \( n \). All \( t \)-structures in here are bounded.

When \( D^{\leq 0} \) is the full subcategory of complexes whose cohomology vanishes in positive degrees, and \( D^{\geq 0} \) the subcategory of complexes whose cohomology vanishes in negative degrees, then \((D^{\leq 0}, D^{\geq 0})\) is called the standard \( t \)-structure and its heart \( D^{\geq 0} \cap D^{\leq 0} \) is the standard heart and recovers the Abelian category \( A \).

So far, our discussion of \( t \)-structures have been rather general. We now briefly recall the notion of intermediate \( t \)-structures. A \( t \)-structure on \( D \) is called intermediate with respect to the standard \( t \)-structure \((D^{\leq 0}, D^{\geq 0})\) if \( D^{\leq 0}[1] \subset D^{\geq 0} \subset D^{\leq 0} \). These intermediate \( t \)-structures yield the terms of the canonical form and can be obtained by tilting with respect to torsion pairs, which are pairs of subcategories for the initial Abelian category \( A \) [9]. In order to be concrete we shall return to the specific examples of the \( A_2 \) and \( A_3 \) quivers.

Let \( D_{A_2} = \mathbb{D}^b(\text{rep} A_2) \) and let \((D_{A_2}^{\leq 0}, D_{A_2}^{\geq 0})\) denote the standard \( t \)-structure on \( D_{A_2} \) with heart \( D_{A_2}^{\leq 0} \cap D_{A_2}^{\geq 0} \simeq \text{rep} A_2 \). A torsion pair \((T, \mathcal{F})\) for \text{rep} \( A_2 \) may be obtained by choosing \( T \) to be the Abelian subcategory generated by any collection of indecomposable representations which is closed under extensions and quotients, and then letting \( \mathcal{F} = T^\perp = \{ Y \in \text{rep} A_2 | \text{Hom}(X, Y) = 0, \forall X \in T \} \) be its right orthogonal complement. The Abelian category \text{rep} \( A_2 \) has five torsion pairs which are presented in Fig. 3 and the corresponding intermediate \( t \)-structures of \( D_{A_2} \) in Fig. 5 where we omitted the arrows (cf. (22) and (23)). The black dots correspond to the indecomposable objects of \text{rep} \( A_2 \). In each diagram of Fig. 5 the blue part corresponds to \( D_{A_2}^{\leq 0} \) and the red part to \( D_{A_2}^{\geq 1} \) of the \( t \)-structure. The vertices in the shaded part correspond to the indecomposable objects in
the heart of the \( t \)-structure, which is given by \( D_{A_2}^{\leq 0} \cap D_{A_2}^{\geq 0} \). Diagrammatically, the heart is obtained as the intersection of the blue part and the shift (given by glide reflection to the right) of the red part.

Given a \( t \)-structure \( (D^{\leq 0}, D^{\geq 0}) \) on a triangulated category \( D \) with heart \( \mathcal{H} = D^{\leq 0} \cap D^{\geq 0} \), an object \( P \in D \) is called a projective of the heart if for all \( M \in \mathcal{H} \) and all \( k \neq 0 \) one has \( \text{Hom}_D(P, M[k]) = 0 \). For type \( A \) quivers the dimension of the Hom space between indecomposable objects can be read off the AR quiver \([8, \S 3.1.4]\) as \( \text{Hom}(P, M) \neq 0 \) precisely when \( M \) lies in the maximal slanted (possibly degenerate) rectangle \( \mathcal{R}(P) \) whose left-most point is \( P \) as illustrated for an \( A_3 \) quiver in Fig. 4. A bounded \( t \)-structure is completely described by its heart, which, in turn, is labelled by the projective objects of the heart.

![Diagram](image)

**Figure 3:** The 5 torsion pairs \( (\mathcal{T}, \mathcal{F}) \) of \( \text{rep} \ A_2 \) with torsion class \( \mathcal{T} \) (blue) and torsion-free class \( \mathcal{F} \) (red)

**Figure 4:** Three maximal slanted rectangles in the AR quiver of \( D_{A_3} \) indicating the nonzero Hom spaces from the objects marked by filled vertices to the objects in the rectangles

### 3.2 Cluster category

As mentioned earlier, the mesh (10) obtained from the planar variables upon effecting the various identifications coincides with the cluster category of type \( A \) quivers. We now recall the notion of the cluster category for a quiver of type \( A_n \) which is an orbit category of the derived category \( D_{A_n} \), obtained by quotienting with an automorphism of the derived category. The example pertinent for us in the present article is the automorphism \( F = \tau^{-1} \circ [1] : D_{A_2} \to D_{A_2} \), called the cluster automorphism. Quotienting the derived category by \( F \) yields a cluster category. We shall discuss the notion with the example of \( D_{A_2} \).

Including the AR translation \( \tau \) indicating the existence of an extension of 1 by \( \tau(1) = 2 \) furnishes the following diagrammatic picture of \( \text{rep} \ A_2 \)

![Diagram](image)

\[(22)\]
Repeating this unit on both left and right to include the shift functors (given diagrammatically by a glide reflection) yields the AR quiver (20) corresponding to the bounded derived category \( D_{A_2} = D^b(\text{rep} \ A_2) \) whose objects are bounded complexes of representations in \( \text{rep} \ A_2 \). Identifying the AR translation \( \tau \) with the shift functor \([1]\) in \( D_{A_2} \) one obtains the cluster category \( \mathcal{C}_{A_2} \) [17, 19]. It is a triangulated category [13] depicted as

![Diagram](image)

so that in the cluster category \( \mathcal{C}_{A_2} \), 2 is isomorphic to \( F(2) = 1[1] \) and \( \frac{1}{2} \) is isomorphic to \( F(\frac{1}{2}) = 2[2] \). Here the nodes are labelled by indecomposable representations in \( \text{rep} \ A_2 \) and their shifts as in (15). In (23) we have also indicated the AR label\(^1\) above and the corresponding planar variable below each node. Objects in the cluster category satisfy mesh relations compatible with its triangulated structure. The mesh relations written in terms of the AR labels are [20, 21]

\[
\tau(p,i) = (p, i - 1) \rightarrow (p, i) \rightarrow (p + 1, i).
\]  

(24)

While the objects in the cluster category are orbits of complexes of quiver representations under the cluster automorphism, we just need to work with the indecomposable objects, which we continue to denote simply by the indecomposable representations and their shifts.

The cluster category captures the combinatorics of the intermediate \( t \)-structures or torsion pairs in terms of cluster tilting objects, which in the cluster category of a type \( A \) quiver are precisely given by direct sums of projectives of the intermediate \( t \)-structures. Each term in the canonical form can thus be obtained from the cluster tilting objects, the indecomposable objects in the cluster category corresponding in some way to the planar variables.

We have remarked that the mesh (10) possesses the same structure of vertices and arrows as the cluster category of \( A_2 \). However, the planar variables are real numbers, satisfying the mesh relations obtained from (9), while the vertices in the cluster category are modules satisfying the mesh relations (24). In order to relate them we interpret the planar variables as the modulus of central charges of these objects. The planar variables then satisfy the 0-deformed mesh relations [15]. The central charge is defined as a map from the Grothendieck group of the derived category to the complex plane, \( Z : K_0(D_{A_2}) \rightarrow \mathbb{C} \). We label the central charge by the objects as well as the AR labels, for example, \( Z_{\frac{1}{2}} = Z_{(0,2)} \) etc., and use the notations interchangeably. The mesh relations then give rise to relations among central charges as

\[
Z_{(p,i)} = Z_{(p,i-1)} + Z_{(p+1,i)}.
\]  

(25)

\(^1(\rho, i)\) referring to the \( i \)th vertex in the \( \rho \)th copy of \( Q \) in the translation quiver of \( Q \)
For the present instance, in $D_{A_2}$, these are, with middle node $0,2$, $(1,1), (1,2), (2,1), (2,2), \quad (26)$

$$
egin{align*}
Z_{\frac{1}{2}} &= Z_2 + Z_1, \\
Z_1 &= Z_{\frac{1}{2}} + Z_{2[1]}, \\
Z_{2[1]} &= Z_1 + Z_{\frac{1}{2}[1]}, \\
Z_{\frac{1}{2}[1]} &= Z_{2[1]} + Z_{1[1]}, \\
Z_{1[1]} &= Z_{\frac{1}{2}[1]} + Z_{2[2]},
\end{align*}
$$

respectively. We have five relations among seven central charges. Thus two of them are “independent” and the rest can be expressed as their linear combinations. The choice of the two independent ones correspond to a choice of intermediate $t$-structure, whose heart has in the case of $A_2$ two indecomposable projective objects. Once the independent ones are chosen, the assignment of central charges is made according to

$$
Z_{A[1]} = -Z_A \quad (27)
$$

for any object $A$.

The reason for defining the charge on the Grothendieck group of the derived category is that the Grothendieck group of the cluster category with its standard triangulated structure is too small, as was brought to our attention by Yann Palu. This may be remedied by working with so-called extriangulated structures on the cluster category [22]. For the present article we shall freely use the two equivalent perspectives of projectives hearts of intermediate $t$-structures on the one hand and direct summands of cluster tilting objects on the other hand, and content ourselves with defining the charge only from the projectives of hearts.

### 3.3 Canonical form from hearts

We have introduced the notions relevant for the categorification of the canonical form. We now present the connection between its terms and the projective objects of the hearts of intermediate $t$-structures restricted to the cluster category $C_{A_n}$.

The standard heart in $D_{A_2}$ is given by the Abelian category $\operatorname{rep} A_2$ (22) and the indecomposable projective objects of the heart are the representations 2 and $\frac{1}{2}$ corresponding to the planar variables $X_{13}$ and $X_{14}$, respectively, as indicated in (23). The corresponding contribution to the canonical form (11) is $d \log X_{14} \wedge d \log X_{13}$.

For the intermediate $t$-structures of $D_{A_2}$ the projective objects of their hearts are marked in Fig. 5 as the circled vertices. From the point of view of cluster categories, the direct sums of the images of the projectives of the hearts in the cluster category are precisely the cluster tilting objects of the cluster category $C_{A_2}$. The corresponding planar variables contribute to the canonical form as follows.

- For the standard $t$-structure corresponding to the torsion pair $(\mathcal{T}, \mathcal{F}) = (\operatorname{rep} A_2, 0)$ (the first diagram in Fig. 3), the heart is the standard heart $\operatorname{rep} A_2 = \langle 2, \frac{1}{2}, 1 \rangle$ with (indecomposable) projectives 2 and $\frac{1}{2}$ so that $2 \oplus \frac{1}{2}$ is a cluster tilting object of $C_{A_2}$, where we have written also 2 and $\frac{1}{2}$ for their images in the cluster category.
Figure 5: The 5 intermediate \( t \)-structures of \( \mathcal{D}_{A_2} \) obtained by tilting with respect to a torsion pair from the standard heart (filled vertices), their hearts (shaded), projective objects (circled vertices) and their contribution to the canonical form.

Their central charges are chosen to be the independent ones. All the others can be expressed in terms of them using (26). We have, for example,

\[
Z_1 = Z_{\frac{1}{2}} - Z_2.
\]

(28)

Using (27) all the equations (26) coincide with this, leaving only two independent central charges. Writing \( Z_A = X_A e^{i \phi} \), for objects and choosing \( X \)'s according to (26), we can write this as a relation among the \( X \)'s,

\[
X_{24} = X_{14} - X_{13},
\]

(29)

where we have ignored the common phases of the central charges, tantamount to choosing a line of stability. All the other \( X \)'s can be similarly expressed in terms of \( X_{13} \) and \( X_{14} \). Contribution to the canonical form is \( d \log X_{24} \wedge d \log X_{13} \).

- The heart of the \( t \)-structure obtained by tilting with respect to the torsion pair \( (\mathcal{T}, \mathcal{F}) \) with \( \mathcal{T} = \langle \frac{1}{2}, 1 \rangle \) and \( \mathcal{F} = \langle 2 \rangle \) is given by \( \langle \frac{1}{2}, 1, 2[1] \rangle \) whose projectives are 1 and \( \frac{1}{2} \) so that \( 1 \oplus \frac{1}{2} \) is the corresponding cluster tilting object. (Note that here 2 is no longer a projective of the heart of the corresponding intermediate \( t \)-structure, since \( 2[1] \) is in the heart and \( \text{Hom}_{\mathcal{D}_{A_2}}(2, 2[1][1]) = \text{Hom}(2, 2) \cong \mathbb{C} \neq 0 \).) In this case the independent charges are \( Z_1 \) and \( Z_{\frac{1}{2}} \). Correspondingly, the independent \( X \)'s are \( X_{24} \) and \( X_{14} \), the rest being expressed in terms of these. Contribution to the canonical form is \( d \log X_{24} \wedge d \log X_{14} \).

- The contribution to the canonical form is determined completely analogously for the remaining three hearts, whose cluster tilting objects are \( 1 \oplus 2[1], 2[1] \oplus \frac{1}{2}[1] \) and \( 2 \oplus \frac{1}{2}[1] \). Matching their direct summands with the planar variables, their contributions to the canonical form are given in Fig. 5, where we have omitted \( \wedge \) and written \( [ij] \) for \( d \log X_{ij} \). Note that the heart of the \( t \)-structure of the last diagram in Fig. 5 is not equivalent to \( \text{rep} \ A_2 \) and that one of the two indecomposable projectives of the heart does not lie inside the heart. Indeed, this kind of phenomenon was part of the motivation for developing \( \tau \)-tilting theory [23,24] which generalizes the notion of tilting modules in a way that is compatible with mutations.

Collecting all these contributions we recover the canonical form (11). The signs of the terms are fixed, up to an overall factor, by demanding invariance under simultaneous
scaling of the planar variables \(2\). As mentioned in the introduction, each term in the canonical form, corresponding to a topologically different Feynman diagram, corresponds to an intermediate \(t\)-structure, thereby justifying the use of the latter in the scheme.

## 4 Example of \(N = 6\)

We proceed similarly for the case of \(N = 6\) particles. We identify the mesh diagram of the momenta with a portion of the AR quiver of \(\mathcal{D}_{A_3}\) corresponding to the cluster category \(\mathcal{C}_{A_3}\) which can be obtained by identifying the indecomposable objects \(3 \simeq F(3) = 2[1]\), 
\[
\begin{align*}
\frac{2}{3} \simeq F\left(\frac{2}{3}\right) &= \frac{1}{2}[1] \quad \text{and} \quad \frac{1}{3} \simeq F\left(\frac{1}{3}\right) &= 3[2] \quad \text{in the green block}
\end{align*}
\]

where the shaded part is the heart of the standard \(t\)-structure of \(\mathcal{D}_{A_3}\). There are now 14 torsion pairs, giving 14 intermediate \(t\)-structures whose hearts have three projective objects each, which in turn correspond to the 14 cluster tilting objects. The canonical form has 14 terms

\[
\Omega_{(6)} = -13[14][15] + 15[25][24] - 15[14][24] + 24[25][26] + 26[25][35] - 26[36][35] + 26[36][46] - 15[25][35] + 13[14][46] - 13[15][35] + 46[14][24] - 13[36][46] + 24[26][46] - 35[36][13]
\]

where we denoted \([ij] = d\log X_{ij}\). The hearts correspond to the fourteen vertices of an associahedron \([10, 11]\).

### 4.1 Cluster category, hearts and canonical form

The torsion pairs for rep \(A_3\) and their corresponding \(t\)-structures for \(\mathcal{D}_{A_3}\) are illustrated in Fig. 6. (In each diagram the torsion class is the Abelian category generated by the black vertices in the blue part and the torsion-free class the one generated by the black vertices in the red part.) Let us remark that the torsion pairs of rep \(A_3\) can be “pasted” from the torsion pairs of rep \(A_2\) using the following rule. First, a choice of a torsion pair on the three lower left vertices and a torsion pair on the three lower right vertices has to be made such that the overlap agrees. Since both \(\mathcal{T}\) and \(\mathcal{F}\) are closed under extensions and there can be no nonzero morphisms from objects in \(\mathcal{T}\) to objects in \(\mathcal{F}\). This determines in most cases uniquely whether the top vertex belongs to \(\mathcal{T}\), to \(\mathcal{F}\), or to neither. When there is a choice, both choices (\textit{i.e.} including the top vertex in \(\mathcal{T}\) or in \(\mathcal{F}\) define torsion
Figure 6: The 14 intermediate t-structures of $\mathcal{D}_{A_3}$ obtained by tilting from the standard heart (filled vertices), their hearts (shaded), projective objects (circled vertices) and their contribution to the canonical form.

pairs. In this way one can obtain torsion pairs for $\text{rep } A_{n+1}$ recursively from “pasting” torsion pairs of $\text{rep } A_n$ and $\text{rep } A_2$ (or equivalently from pasting $n$ torsion pairs of $\text{rep } A_2$).

There are nine mesh relations ensuing from (24) among the twelve objects of the cluster category, namely,

$$
\begin{align*}
3 & \rightarrow \frac{2}{3} \rightarrow 2, \\
\frac{1}{3}[1] & \rightarrow \frac{1}{2}[1] \rightarrow 3[2], \\
\frac{1}{2} & \rightarrow \frac{1}{2} \rightarrow 3[1], \\
2 & \rightarrow \frac{1}{2} \rightarrow 3[1], \\
\frac{1}{3} & \rightarrow \frac{1}{3}[1] \rightarrow 2[1], \\
\frac{2}{3} & \rightarrow \frac{1}{3} \oplus 2 \rightarrow \frac{1}{2}, \\
\frac{1}{2} & \rightarrow 3[1] \oplus 1 \rightarrow \frac{2}{3}[1], \\
\frac{2}{3}[1] & \rightarrow 2[1] \oplus \frac{1}{3}[1] \rightarrow \frac{1}{2}[1].
\end{align*}
$$

Accordingly, the twelve central charges are related by nine equations similar to (26). The general rule of assignment of charges are

$$
\begin{align*}
Z_{A[1]} &= -Z_A, \\
Z_{A \oplus B} &= Z_A + Z_B,
\end{align*}
$$

leaving three of them independent. As before, choosing the independent ones as the central charges of the projectives of the hearts from Fig. 6 their masses furnish the terms of the canonical form (31). The signs are again fixed by the scheme described in section 2.1, which guarantees its invariance under the scaling of the planar variables.
5 Conclusion

In this note we have considered the canonical form appearing in the computation of planar tree level Feynman diagrams of a cubic scalar field theory. The canonical form is a means to encode the contribution of these Feynman diagrams to the scattering amplitude. Its relation to various mathematical structures have been studied earlier. In here we interpret the terms of the canonical form as arising from the cluster tilting objects of the cluster categories of quivers of type $A$ which correspond to projectives of hearts of intermediate $t$-structures. This approach is categorical and makes no allusion to associahedrons and triangulations of polygons, although there is a precise general relation, the details of which we presented in two generic examples. As mentioned in the introduction, each $N$-particle planar diagram for any $N$ in a cubic theory corresponds to a triangulation of an $N$-gon by non-intersecting diagonals. The category of diagonals of an $N$-gon is identified with the cluster category of an $A_{N-3}$ quiver $[17, 28]$. The associahedron is then obtained as the exchange graph of the hearts of the intermediate $t$-structures of the cluster category $[10, 11]$. The objects in the cluster category, which are representations of the $A_{N-3}$ quiver, are then mapped to real numbers using the central charge, as described in the text. This furnishes the rationale of identifying the planar variables $X_{ij}$ arising in the kinematics of scattering with the mass of the central charges via mesh relations. The terms in the canonical form is then shown to be expressed in terms of the projectives of the hearts of intermediate $t$-structures, or equivalently of the direct summands of cluster tilting objects in the cluster category. The present treatment also generalizes to quadratic and higher order scalar field theories $[25]$ in terms of higher cluster categories.

Each Feynman diagram contributes a term in the canonical form or scattering amplitude. As demonstrated here, each of these corresponds to an intermediate $t$-structure and hence to a specific stability regime. We have illustrated this in two examples. The first one is for $N = 5$ particles, which is simpler, if somewhat restricted. The second example of $N = 6$ particles corresponding to the cluster category of $A_3$ quivers is generic. These considerations can be generalized to arbitrary number of particles. The categorification is expected to help the organization of the canonical form, especially in their evaluation using computer programs.

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References

[1] N. Arkani-Hamed and J. Trnka, “The Amplituhedron”, JHEP 10 (2014) 030.

[2] N. Arkani-Hamed, Y. Bai, S. He, G. Yan, “Scattering Forms and the Positive Geometry of Kinematics, Color and the Worldsheet”, JHEP 05 (2018) 096.
[3] R. C. Hwa, V. L. Teplitz, “Homology and Feynman Integrals”, W. A. Benjamin, Inc., New York, 1966.

[4] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov, J. Trnka, “Scattering Amplitudes and the Positive Grassmannian”, arXiv:1212.5605 [hep-th] (2012)

[5] J. Drummond, J. Foster, Ö. Gürdoğan, C. Kalousios, “Tropical Grassmannians, Cluster Algebras and Scattering Amplitudes”, JHEP 04 (2020) 146.

[6] S. Mizera, “Aspects of Scattering Amplitudes and Moduli Space Localization”, arXiv:1906.02099 [hep-th] (2019)

[7] P. Banerjee, A. Laddha, P. Raman, “Stokes Polytopes: The Positive Geometry for $\phi^4$ Interactions”, JHEP 08 (2019) 067.

[8] R. Schiffler, “Quiver Representations”, Canadian Mathematical Society, 2010.

[9] T. Brüstle, D. Yang, “Ordered Exchange Graphs”, Advances in Representation Theory of Algebras, 135–193, European Mathematical Society, 2013.

[10] A. King and Y. Qiu, “Exchange graphs and Ext quivers”, Adv. Math. 285 (2015) 1106.

[11] Y. Qiu, “Ext-Quivers of Hearts of A-Type and the Orientations of Associahedrons”, J. Algebra 393 (2013) 60.

[12] D. Chicherin, J. M. Henn, “Cluster Algebras for Feynman Integrals”, Phys. Rev. Lett. 126 (2021) 091603.

[13] B. Keller, “Cluster Algebras, Quiver Representations and Triangulated Categories”, Triangulated Categories, 76–160, Cambridge University Press, 2010.

[14] N. Arkani-Hamed, S. He, G. Salvatori, H. Thomas, “Causal Diamonds, Cluster Polytopes and Scattering Amplitudes”, arXiv:1912.12948 [hep-th] (2019)

[15] V. Bazier-Matte, G. Douville, K. Mousavand, H. Thomas, E. Yıldırım, “ABHY Associahedra and Newton Polytopes of $F$-Polynomials for Finite Type Cluster Algebras”, arXiv:1808.09986 [math.RT] (2018)

[16] S. Fomin, A. Zelevinsky, “Y-Systems and Generalized Associahedra”, Ann. Math. 158 (2003) 977.

[17] P. Caldero, F. Chapoton, R. Schiffler, “Quivers with Relations Arising From Clusters (A_n Case)” Trans. AMS 358 (2006) 1347.

[18] T. Bridgeland, “Stability Conditions on Triangulated Categories”, Ann. Math. 166 (2007) 317.

[19] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten, G. Todorov, “Tilting Theory and Cluster Combinatorics”, Adv. Math. 204 (2006) 572.
[20] S. Gratz, K. Peter, L. Hugentobler, “Presentation of the Derived Category of a Dynkin Quiver and the Cluster Category”, unpublished notes, 2009.

[21] T. Seynnaeve, “The Derived Category of a Dynkin Quiver”, unpublished notes.

[22] A. Padrol, Y. Palu, V. Pilaud, P.-G. Plamondon, “Associahedra for Finite Type Cluster Algebras and Minimal Relations Between $g$-Vectors”, arXiv:1906.06861 [math.RT] (2019)

[23] T. Adachi, O. Iyama, I. Reiten, “$\tau$-Tilting Theory”, Compos. Math. 150 (2014) 415.

[24] H. Treffinger, “$\tau$-Tilting Theory — An Introduction”, lectures notes for the LMS Algebra Autumn School 2020.

[25] S. Barmeier, P. Oak, A. Pal, K. Ray, H. Treffinger, in preparation.

[26] S. Mukhopadhyay, K. Ray, “Branes in hearts with perverse sheaves”, Indian J. Phys. A 80 (2006) 1109–1122.

[27] R. P. Thomas, “Derived categories for the working mathematician”, arXiv:math/0001045 [math.AG] (2000)

[28] K. Baur, “Cluster categories, $m$-cluster categories and diagonals in polygons”, arXiv:0912.3131 [math.RT] (2009)