On the existence of extreme waves and
the Stokes conjecture with vorticity

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Abstract

This is a study of singular solutions of the problem of traveling gravity water waves on flows with vorticity. We show that, for a certain class of vorticity functions, a sequence of regular waves converges to an extreme wave with stagnation points at its crests. We also show that, for any vorticity function, the profile of an extreme wave must have either a corner of 120° or a horizontal tangent at any stagnation point about which it is supposed symmetric. Moreover, the profile necessarily has a corner of 120° if the vorticity is nonnegative near the free surface.

1 Introduction

This article addresses the classical hydrodynamical problem concerning traveling two-dimensional gravity water waves with vorticity. There has been considerable interest on this problem in recent years, starting with the systematic study of Constantin and Strauss [7].

When the water depth is finite, which is the setting of [7], the problem arises from the following physical situation. A wave of permanent form moves with constant speed on the surface of an incompressible, inviscid, heavy fluid, the bottom of the fluid domain being horizontal. With respect to a frame of reference moving with the speed of the wave, the flow is steady and occupies a fixed region \( \Omega \) in \((X, Y)\)-plane, which lies above a horizontal line \( B_F := \{(X, F) : X \in \mathbb{R}\} \), where \( F \) is a constant, and below some a priori unknown free surface \( S := \{(u(s), v(s)) : s \in \mathbb{R}\} \). Since the fluid is incompressible, the flow can be described by a stream function \( \psi \) which satisfies the following equations and
boundary conditions:

\[
\begin{align*}
\Delta \psi &= -\gamma(\psi) & \text{in } \Omega, \\
0 &\leq \psi \leq B & \text{in } \Omega, \\
\psi &= B & \text{on } \partial B_F, \\
\psi &= 0 & \text{on } \partial S, \\
|\nabla \psi|^2 + 2gY &= Q & \text{on } \partial S,
\end{align*}
\]

where \( Q \) is a constant, \( B, g \) are positive constants and \( \gamma : [0, B] \to \mathbb{R} \) is a function. The meaning of equation (1.1a) is that the vorticity of the flow \( \omega := -\Delta \psi \) and the stream function \( \psi \) are functionally dependent. It is customary \[7\] to assume that the constants \( g, B \) and the function \( \gamma \), called a vorticity function, are given. The problem consists in determining the curves \( S \) for which there exists a function \( \psi \) in \( \Omega \) satisfying (1.1) for some values of the parameters \( Q \) and \( F \). Any such solution quadruple \( (S, B_F, \psi, Q) \) of (1.1) gives rise to a traveling-wave solution of the two-dimensional Euler equations for a heavy fluid with a free surface, see \[7\] for details. In particular, the relative velocity of the fluid particles is given by \( (\psi_X, -\psi_Y) \). Among various types of waves, of main interest are the periodic waves, for which \( S \) is periodic in the horizontal direction, and the solitary waves, for which \( S \) is asymptotic to a horizontal line at infinity.

In the related problem of waves of infinite depth, one seeks a curve \( S \) such that in the domain \( \Omega \) below \( S \) there exists a function \( \psi \) which satisfies (1.1a), (1.1d), (1.1e) and

\[
\begin{align*}
\psi &\geq 0 & \text{in } \Omega, \\
\nabla \psi(X,Y) &\to (0, -C) & \text{as } Y \to -\infty, \text{ uniformly in } X,
\end{align*}
\]

where \( \gamma : [0, \infty) \to \mathbb{R} \) is a given function and \( C \) is a parameter. Of main interest are the periodic waves.

When \( \gamma \equiv 0 \), the corresponding flow is called irrotational. Nowadays the mathematical theory dealing with this situation contains a wealth of results, mostly obtained during the last three decades. The first existence result for waves of large amplitude was given by Krasovskii \[17\]. Then, global bifurcation theories for regular waves of various types were given by Keady and Norbury \[13\] and by Amick and Toland \[2, 3\]. Moreover, it was shown by Toland \[33\] and by McLeod \[22\] that in the closure of these continua of solutions there exist waves with stagnation points (i.e., points at which the relative fluid velocity is zero) at their crests. The existence of such waves, called extreme waves, was predicted by Stokes \[31\], who also conjectured that their profiles necessarily have corners with included angle of 120° at the crests. This conjecture was proved independently by Amick, Fraenkel, and Toland \[4\], and by Plotnikov \[26\]. In more recent developments, the method of \[4\] was simplified and generalized in \[38\], while Fraenkel \[14\] gave a direct proof of the existence of an extreme wave (of infinite depth), with corners of 120° at the crests, without relying on existence results for regular waves.
When $\gamma \neq 0$, the flow is called rotational or with vorticity, and advances in the mathematical theory have been made only in the last few years. The existence of global continua of solutions was proved by Constantin and Strauss [7] for the periodic finite depth problem, and by Hur [16] for the periodic infinite depth problem. The wave profiles in [7,16] have one crest and one trough per minimal period, are monotone between crests and troughs and symmetric with respect to vertical lines passing through any crest. The continuum of solutions in [7] contains waves for which the values of $\max_{0} \psi$ are arbitrarily close to 0 and, at least in certain situations [41], the values of $|\nabla \psi|$ at the crests are also arbitrarily close to 0. Thus it is natural to expect that, as in the irrotational case, waves with stagnation points at their crests, referred to as extreme waves, exist for many vorticity functions, and that they can be obtained as limits, in a suitable sense, of certain sequences of regular waves found in [7].

In the case of constant vorticity, numerical evidence [19, 29, 32, 35, 36, 37] strongly points to the existence of extreme waves for any negative vorticity and for small positive vorticity, and also indicates that, for large positive vorticity, continua of solutions bifurcating from a line of trivial solutions develop into overhanging profiles (a situation which is not possible in the irrotational case, see [40] for references) and do not approach extreme waves. The above mentioned numerical computations support the formal speculation in various places in the fluid mechanics literature [10], [23, §14.50] that extreme waves with vorticity must also have corners with angles of $120^\circ$ at the crests. We refer to this claim as the Stokes conjecture, although Stokes himself seems to have made it explicitly only for irrotational waves.

This article is, to the best of our knowledge, the first rigorous study of the existence of extreme waves with vorticity and their properties. Attention is restricted here to the case of periodic waves in water of finite depth, though it is clear that similar arguments can be used in related situations, such as solitary waves of finite depth or periodic waves of infinite depth.

A fundamental difficulty when trying to extend to the general case of waves with vorticity known results for irrotational waves is that new methods are needed. Indeed, the irrotational case is the only one in which conformal mappings can be used to equivalently reformulate the free-boundary problem as an integral equation [18, 3], originally due to Nekrasov [24], for a function which gives the angle between the tangent to the free boundary and the horizontal. The existence of large-amplitude regular waves, the existence of extreme waves and the Stokes conjecture are then proved by using hard analytic estimates for this integral equation [34]. For waves with vorticity, the existence of large-amplitude regular waves [7] is based on a study of another equivalent reformulation of the problem, originally due to Dubreil-Jacotin [11], as a quasilinear second order elliptic partial differential equation with nonlinear boundary conditions in a fixed domain. However, this reformulation of the problem does not seem suitable to describe extreme waves.

Our first task, pursued in Section 2, is thus to identify generalized formulations of problem (1.1), under minimal regularity assumptions, which are suitable for the description of extreme waves. We introduce two types of solutions, called
respectively **Hardy-space solutions** and **weak solutions**. An extensive theory of Hardy-space solutions has been given in the case of irrotational waves by Shar- gorodsky and Toland [28], and further developed in [38, 39, 40]. The notion of a weak solution of (1.1) is inspired by the article of Alt and Caffarelli [1], who considered a class of free boundary problems in bounded domains (in any number of dimensions) for harmonic functions satisfying simultaneously on a free boundary a Dirichlet boundary condition of type (1.1a) and a boundary condition of a more general type than (1.1b). Each of these solution types has certain advantages over the other, and the main result of Section 2 is that the two coincide. The material in this section pervades the rest of the article.

In Section 3 we prove, by means of the maximum principle, an a priori estimate concerning the pressure in the fluid. This result, which extends to the general case some very recent results in [41] for vorticity functions which do not change sign, plays a pivotal role in the investigation of the existence of extreme waves and the Stokes conjecture.

In Section 4 we study the existence of extreme waves. We consider a sequence of solutions \( \{(S_j, B_0, \psi^j, Q_j)\}_{j \geq 1} \) of (1.1), which have similar properties to the solutions in the continuum in [7]. In particular, for all \( j \geq 1 \), \( S_j = \{(X, \eta_j(X)) : X \in \mathbb{R}\} \), where

\[ \eta_j \in C^1(\mathbb{R}) \text{ is } 2L\text{-periodic, even and } \eta'_j < 0 \text{ on } (0,L). \]

In Theorem 4.1 we prove, under the assumption that

\[ \{Q_j\}_{j \geq 1} \text{ is bounded above,} \]

that a subsequence of \( \{(S_j, B_0, \psi^j, Q_j)\}_{j \geq 1} \) necessarily converges in a specified sense to a weak solution \( (\tilde{S}, B_0, \tilde{\psi}, \tilde{Q}) \) of (1.1). Moreover, the additional assumption that

\[ |\nabla \psi^j(0, \eta_j(0))| \to 0 \text{ as } j \to \infty, \]

ensures that \( (\tilde{S}, B_0, \tilde{\psi}, \tilde{Q}) \) is an extreme wave. This result is far from trivial. The most difficult steps in the proof are the definition of \( \tilde{S} \) as non-self-intersecting curve in the absence of any uniform bound on the slopes of \( \{S_j\}_{j \geq 1} \), and the recovery of the free-boundary condition (1.1e) in a weak sense along \( \tilde{S} \).

Combining Theorem 4.1 with existing results in the literature [7, 41] on the validity of (1.2) and (1.3) for a sequence in the continuum in [7], we obtain in Theorem 4.4 the existence of extreme waves arising as limits of regular waves in the case when \( \gamma(0) < 0, \gamma'(r) \leq 0 \text{ and } \gamma'(r) \geq 0 \text{ for all } r \in [0,B]. \) However, these assumptions on \( \gamma \) also ensure the existence of trivial extreme waves, for which \( S \) is a horizontal line consisting only of stagnation points and \( \psi \) is independent of the \( X \) variable. Unfortunately, it is not known at present whether the extreme waves we obtain as limits of regular waves are trivial or not.

Nevertheless, it is hoped that Theorem 4.4 may be useful in proofs of the existence of extreme waves in much more general situations than those in Theorem 4.4. A key open problem remains that of determining for what vorticity functions are (1.2) and (1.3) necessarily valid for a sequence of regular waves in
the continuum in [7]. Theorem 4.1 might also be useful in proving the existence of waves with stagnation points at the bottom or in the interior of the fluid domain, in situations when only (1.2), but not (1.3), holds for suitable sequences of regular waves.

In Section 5 we address the Stokes conjecture for extreme waves. We deal with symmetric wave profiles which are locally monotone on either side of a stagnation point (these assumptions were also required for the Stokes conjecture in the irrotational case). In Theorem 5.2 we show that at such a stagnation point the profile has either a corner of $120^\circ$ or a horizontal tangent. Moreover, we show that the profile necessarily has a corner of $120^\circ$ whenever the vorticity is nonnegative near the free surface.

The existence of trivial extreme waves shows that the possibility of a horizontal tangent cannot be ruled out in general. One should also point out that only smooth vorticity functions are considered here. For a specific unbounded vorticity function, there exists an explicit example, discovered by Gerstner in 1802, see [23, §14.40-14.41], of an extreme wave whose profile has cusps at the stagnation points. However, a study of waves with unbounded vorticity is beyond the scope of this article.

The proof given here of the Stokes conjecture for waves with vorticity is similar in spirit to that in [4] for the irrotational case, in that they are both based on a blow-up argument, which is a standard tool in the study of regularity of free boundaries [5]. But whilst in the irrotational case the blow-up is applied in Nekrasov’s integral equation to yield a new integral equation [4], here the blow-up is performed directly in the physical domain. More precisely, a blow-up sequence (i.e., a sequence of functions obtained from $\psi$ by rescaling) is shown in Theorem 5.5 to converge along a subsequence to the solution of a free-boundary problem for a harmonic function in an unbounded domain whose boundary is curve passing through, and globally monotone on either side of, the original stagnation point. Apart from a trivial solution where the free boundary is the real axis, this limiting problem has another explicit solution, for which the free boundary consists of two half-line with endpoints at the origin, enclosing an angle of $120^\circ$ which is symmetric with respect to the imaginary axis. It was the existence of this solution, nowadays called the Stokes corner flow [10], that led Stokes [31] to his conjecture. It is however the uniqueness, which is proved in Theorem 5.6 of this solution in the class of symmetric nontrivial solutions of the limiting problem, that leads to the proof of the conjecture. We show here that the limiting problem can be described by means of a nonlinear integral equation for a function $\theta^*$ which gives the angle between the tangent to the free boundary and the horizontal. This equation first arose in [4] as a blow-up limit of Nekrasov’s equation, but its connection to a free-boundary problem seems to have never been explicitly mentioned in the literature. The monotonicity of the free boundary means that $0 \leq \theta^* \leq \pi/2$ on $(0, \infty)$. In this generality, the uniqueness of the solution of this integral equation has been proven only very recently in [38]. Prior to that, a uniqueness result was known [4] only under the restriction that $0 \leq \theta^* \leq \pi/3$ on $(0, \infty)$. That result would not have been enough for a proof of the Stokes conjecture for waves with vorticity.
We also show, as a byproduct of our approach to the Stokes conjecture, that if a possibly nonsymmetric extreme wave with vorticity has lateral tangents at a stagnation point, then the tangents have to be symmetric with respect to the vertical line passing through that point and either enclose an angle of 120° or be horizontal. Moreover, the possibility of horizontal lateral tangents can be ruled out whenever the vorticity is nonnegative near the free surface.

Some problems left open by the present article are: the structure of the set of stagnation points for weak solutions of (1.1), the regularity of the wave profiles away from stagnation points, the extent of the validity of (1.2) and (1.3) for a sequence in the continuum in [7], the existence of nonsymmetric extreme waves and the Stokes conjecture in that case, the uniqueness of solutions of the limiting problem in the absence of symmetry, the existence of overhanging wave profiles.

2 Two generalized formulations of the problem

We consider throughout the rest of the article only the problem of periodic waves of finite depth. We now make precise the sense in which (1.1) is to hold.

It is required throughout that

\[ S \text{ is locally rectifiable}, \]
\[ \psi \in \text{Lip}(\Omega), \]
\[ S \text{ and } \psi \text{ are } 2L\text{-periodic in the horizontal direction}, \]

for some given \( L > 0 \). It is assumed that

\[ \gamma \in C^{1,\alpha}([0, B]) \text{ for some } \alpha \in (0, 1). \]

It is required that (1.1) be satisfied in the classical sense. The condition (1.1b) is to hold in the following sense:

\[ \int_{\Omega} \nabla \psi \nabla \zeta \, dL^2 = \int_{\Omega} \gamma(\psi) \zeta \, dL^2 \text{ for all } \zeta \in C^1_0(\Omega), \]

where \( L^2 \) denotes two-dimensional Lebesgue measure. Then, standard interior and boundary Hölder regularity estimates [15, Lemma 4.2 and Theorem 6.19] show that \( \psi \in C^{3,\alpha}_{\text{loc}}(\Omega \cup B_F) \), and that (1.1a) holds in the classical sense. In particular,

\[ \Delta \psi_x = -\gamma'(\psi) \psi_x \quad \text{in } \Omega, \]
\[ \Delta \psi_y = -\gamma'(\psi) \psi_y \quad \text{in } \Omega. \]

Several types of solutions of (1.1) are described below, depending on how (1.1c) is required to hold.

We say that \((S, B_F, \psi, Q)\) is a classical solution of (1.1) if \( S \) is a \( C^1 \) curve, \( \psi \in C^1(\Omega \cup S) \) and (1.1c) holds everywhere on \( S \).
We say that \((S, B_F, \psi, Q)\) is a weak solution of (1.1) if
\[
\int_{\Omega} \nabla \psi \nabla \zeta \, d\mathcal{L}^2 = \int_{\Omega} \gamma(\psi) \zeta \, d\mathcal{L}^2 - \int_{S} (Q - 2gY)^{1/2} \zeta \, d\mathcal{H}^1
\]
for all \(\zeta \in C^1_0(U_F)\),
where \(U_F := \{(X, Y) : X \in \mathbb{R}, Y > F\}\) and \(\mathcal{H}^1\) denotes one-dimensional Hausdorff measure.

We say that \((S, B_F, \psi, Q)\) is a Hardy-space solution of (1.1) if the partial derivatives of \(\psi\) have non-tangential limits \(\mathcal{H}^1\)-almost everywhere on \(S\) which satisfy (1.1e) \(\mathcal{H}^1\)-almost everywhere.

For the definition of a non-tangential limit and for a summary of notions and results concerning the classical Hardy spaces of harmonic functions, the reader is referred to the Appendix.

Obviously, any classical solution of (1.1) is both a Hardy-space solution and a weak solution. The main result of this section is that the Hardy-space solutions and the weak solutions of (1.1) coincide.

**Theorem 2.1.** Let \((S, B_F, \psi, Q)\) be such that (2.1)-(2.5) hold. Then \((S, B_F, \psi, Q)\) is a Hardy-space solution of (1.1) if and only if it is a weak solution.

The proof of Theorem 2.1 follows from a series of results concerning some properties of solutions \((S, B_F, \psi, Q)\) of (2.1)-(2.5).

In the irrotational case, the partial derivatives of \(\psi\) are harmonic functions, and their boundedness in \(\Omega\) ensures, by Fatou’s Theorem, that they have non-tangential limits \(\mathcal{H}^1\)-almost everywhere on \(S\). Here this result is extended to the general case of waves with vorticity.

**Proposition 2.2.** Let \((S, B_F, \psi, Q)\) be such that (2.1)-(2.5) hold. Then the partial derivatives of \(\psi\) have non-tangential limits \(\mathcal{H}^1\)-almost everywhere on \(S\).

The proof of Proposition 2.2 is based on the following simple observation, whose conclusion holds more generally.

**Lemma 2.3.** Let \(G \subset \mathbb{R}^2\) be a bounded open set whose boundary is a rectifiable Jordan curve \(J\). Let \(w \in C^{2,\alpha}_{\text{loc}}(G) \cap L^\infty(G)\) be such that
\[
\Delta w = q \quad \text{in} \ G,
\]
where \(q \in C^{0,\alpha}_{\text{loc}}(G) \cap L^\infty(G)\). Then \(w\) has non-tangential limits \(\mathcal{H}^1\)-almost everywhere on \(J\).

**Proof of Lemma 2.3.** Let us write \(w = u + v\), where \(u\) is the Newtonian potential of \(q\),
\[
u(x) = \frac{1}{2\pi} \int_{G} \log |x - y| q(y) \, d\mathcal{L}^2(y) \quad \text{for all} \ x \in \mathbb{R}^2.
\]
It is well known [15] Lemma 4.1 and Lemma 4.2] that \(u \in C^1(\mathbb{R}^2) \cap C^{2,\alpha}_{\text{loc}}(G)\) satisfies
\[
\Delta u = q \quad \text{in} \ G.
\]
Hence \( v \) is a bounded harmonic function in \( G \), and therefore has non-tangential limits \( \mathcal{H}^1 \)-almost everywhere on \( J \). Since \( u \) is continuous on \( \mathbb{R}^2 \), the required conclusion follows. \( \square \)

**Proof of Proposition 2.4.** It suffices to apply Lemma 2.3 with the partial derivatives of \( \psi \), which satisfy (2.6), in the role of \( w \) in an obvious domain \( G \). \( \square \)

Under the assumptions of Proposition 2.2 let, for \( \mathcal{H}^1 \)-almost every \( (X_0, Y_0) \in S \),
\[
\nabla \psi(X_0, Y_0) := \lim_{(X, Y) \to (X_0, Y_0)} \nabla \psi(X, Y),
\]
where the limit is taken non-tangentially within \( \Omega \). For \( \mathcal{H}^1 \)-almost every \( (X_0, Y_0) \in S \), let
\[
\frac{\partial \psi}{\partial n}(X_0, Y_0) := \nabla \psi(X_0, Y_0) \cdot n(X_0, Y_0),
\]
where \( \cdot \) denotes the standard inner product in \( \mathbb{R}^2 \) and \( n(X_0, Y_0) \) is the outward unit normal to \( \Omega \) at \( (X_0, Y_0) \).

**Proposition 2.4.** Let \((S, B_F, \psi, Q)\) be such that (2.1)–(2.5) hold, and suppose in addition that (1.14) is satisfied. Then, in the notation of (2.3) and (2.5), \( \psi \) satisfies (1.16) \( \mathcal{H}^1 \)-almost everywhere on \( S \) if and only if
\[
\frac{\partial \psi}{\partial n}(X, Y) = -(Q - 2gY)^{1/2} \quad \text{for} \quad \mathcal{H}^1 \text{-almost every } (X, Y) \in S.
\]

The proof of Proposition 2.4 depends on the following lemma.

**Lemma 2.5.** Let \( G \subset \mathbb{R}^2 \) be a bounded open set whose boundary is a rectifiable Jordan curve \( J \). Let \( w \in C^1(\overline{G}) \cap \operatorname{Lip}(\overline{G}) \) be such that the partial derivatives of \( w \) have non-tangential limits \( \mathcal{H}^1 \)-almost everywhere on \( J \). Suppose that \( w \) is a constant on a closed arc \( I \) of \( J \). Then
\[
\nabla w(X_0, Y_0) \cdot t(X_0, Y_0) = 0 \quad \text{for } \mathcal{H}^1 \text{-almost every } (X_0, Y_0) \in I,
\]
where \( \nabla w(X_0, Y_0) \) denotes the non-tangential limit within \( G \) of \( \nabla w \) at \( (X_0, Y_0) \) and \( t(X_0, Y_0) \) is a unit tangent to \( J \) at \( (X_0, Y_0) \).

**Proof of Lemma 2.5.** Let \( D \) be the unit disc in the plane, and let \( f : D \to \overline{G} \) be a conformal mapping from \( D \) onto \( G \). Since the boundary of \( G \) is a rectifiable Jordan curve, it is classical \( \square \) Theorem 3.11 and Theorem 3.12 \( f \) is a homeomorphism from the closure of \( D \) onto the closure of \( G \), \( f' \) belongs to the Hardy space \( H^2(\mathbb{D}) \), the mapping \( t \mapsto f(e^{it}) \) is locally absolutely continuous and
\[
\frac{d}{dt} f(e^{it}) = \lim_{r \to 1} ire^{it} f'(re^{it}) \quad \text{for almost every } t \in \mathbb{R},
\]
where \( ' \) denotes complex differentiation. Let \( a, b \in \mathbb{R} \) be such that \( t \mapsto f(e^{it}) \) is a bijection from \([a, b]\) onto \( I \). Then, for every \( t_1, t_2 \in [a, b] \) with \( t_1 \leq t_2 \) and for every \( r \in (0, 1) \),
\[
w(f(re^{it_2})) - w(f(re^{it_1})) = \int_{t_1}^{t_2} \nabla w(f(re^{it})) \cdot \frac{d}{dt} f(re^{it}) \, dt. \tag{2.10}
\]
We now pass to the limit as \( r \not\to 1 \) in (2.10) using the Dominated Convergence Theorem, with the integrands bounded in absolute value by the integrable function \(|\nabla w|_{L^\infty(G)} M_{\text{rad}}[f']\), where \( M_{\text{rad}}[f'] \) denotes the radial maximal function, see the Appendix, of the function \( f' \in H^1_{L^1}(D) \), to obtain (2.10) with \( r = 1 \). It is important in this argument that, for almost every \( t \in (a, b) \), \( f(re^{it}) \not\to f(e^{it}) \) non-tangentially within \( G \) as \( r \not\to 1 \), see [12] Section 3.5. Since \( \frac{d}{dt}f(e^{it}) \not\equiv 0 \) for almost every \( t \in (a, b) \), the required conclusion follows.

**Proof of Proposition 2.4.** The required result follows immediately by applying Lemma 2.5 to the function \( \psi \) in an obvious domain \( G \). Note also that, when it is assumed that \( \psi \) satisfies (1.16), the sign of the normal derivative of \( \psi \) can be determined from the fact that \( \psi = 0 \) on \( S \) and \( \psi \geq 0 \) in \( \Omega \). □

**Proposition 2.6.** Let \((S, B_F, \psi, Q)\) be such that (2.1)-(2.5) hold. Then, in the notation of (2.9),

\[
\int_{\Omega} \nabla \psi \nabla \zeta \, dL^2 = \int_{\Omega} \gamma(\psi) \zeta \, dL^2 + \int_{S} \frac{\partial \psi}{\partial n} \zeta \, dH^1 \tag{2.11}
\]

for all \( \zeta \in C_0^1(U_F) \).

**Proof of Proposition 2.6.** Fix \( \zeta \in C_0^1(U_F) \). Then one can find points \( Z_1, Z_2 \) on \( S \), \( W_1, W_2 \) on \( B_F \), and a bounded open set \( G \) contained in \( \Omega \), whose boundary is a rectifiable Jordan curve \( \mathcal{J} := \mathcal{I} \cup \mathcal{L}_2 \cup \mathcal{M} \cup \mathcal{L}_1 \), such that

\[
(\text{supp} \, \zeta) \cap \Omega \subset G, \tag{2.12}
\]

\[
\text{dist} (\text{supp} \, \zeta, \mathcal{J} \setminus \mathcal{I}) > 0, \tag{2.13}
\]

where \( \mathcal{I} \) is the arc of \( S \) joining \( Z_1 \) and \( Z_2 \), \( \mathcal{L}_2 \) is an arc contained in \( \Omega \) joining \( Z_2 \) and \( W_2 \), \( \mathcal{M} \) is the line segment joining \( W_2 \) and \( W_1 \), and \( \mathcal{L}_1 \) is an arc contained in \( \Omega \) joining \( W_1 \) and \( Z_1 \). To prove (2.11) is equivalent, by means of (2.12)-(2.13), to proving

\[
\int_{G} \nabla \psi \nabla \zeta \, dL^2 = \int_{G} \gamma(\psi) \zeta \, dL^2 + \int_{\mathcal{I}} \frac{\partial \psi}{\partial n} \zeta \, dH^1. \tag{2.14}
\]

Let \( D \) be the unit disc in the plane, and let \( f : D \to G \) be a conformal mapping from \( D \) onto \( G \) and a homeomorphism from the closure of \( D \) onto the closure of \( G \). Let \( a, b \in \mathbb{R} \) be such that \( t \mapsto f(e^{it}) \) is a bijection from \([a, b]\) onto \( \mathcal{I} \). For every \( r \in (0, 1) \), let \( D_r \) be the disc centred at 0 and of radius \( r \), and \( G_r := f(D_r) \). It follows from (2.13) and the standard Green’s Formula that, for all \( r \) sufficiently close to 1,

\[
\int_{G_r} \nabla \psi \nabla \zeta \, dL^2 = \int_{G_r} \gamma(\psi) \zeta \, dL^2 + \int_{a}^{b} \left[ \nabla \psi(f(re^{it})) \cdot \left( \frac{d}{dt}f(re^{it}) \right) \right] \zeta(f(re^{it})) \, dt \tag{2.15}
\]

Since \( f(re^{it}) \to f(e^{it}) \) non-tangentially within \( G \) as \( r \not\to 1 \), for almost every \( t \in (a, b) \), and since the integrands in the last term of (2.15) are bounded in absolute value by the integrable function \(|\nabla w|_{L^\infty(G)} ||\zeta||_{L^\infty(G)} M_{\text{rad}}[f']\), one can pass to the limit as \( r \not\to 1 \) in (2.15), using the Dominated Convergence Theorem, to get (2.11). This completes the proof of Proposition 2.6. □
Proof of Theorem 2.1. Suppose first that \((S, B_F, \psi, Q)\) is a Hardy-space solution of (1.1). It is immediate from Proposition 2.4 and Proposition 2.6 that \((S, B_F, \psi, Q)\) is a weak solution.

Suppose now that \((S, B_F, \psi, Q)\) is a weak solution of (1.1). By comparing (2.7) and (2.11), we deduce that, for all \(\zeta \in C_0^1(U_F)\),

\[
\int_S \left[ \frac{\partial \psi}{\partial n} + (Q - 2gY)^{1/2} \right] \zeta \, d\mathcal{H}^1 = 0. \tag{2.16}
\]

A simple approximation argument shows that (2.16) also holds for all \(\zeta \in C_0(U_F)\), from where it is immediate that

\[
\frac{\partial \psi}{\partial n}(X,Y) = -(Q - 2gY)^{1/2} \quad \text{for } \mathcal{H}^1\text{-almost every } (X,Y) \in S.
\]

It follows from Proposition 2.4 that \((S, B_F, \psi, Q)\) is a Hardy-space solution.

The proof of Theorem 2.1 is therefore completed. \(\square\)

We conclude this section with the following obvious observation.

**Proposition 2.7.** Let \((S, B_F, \psi, Q)\) be a classical/weak solution of (1.1). \(\Omega\) be the open set whose boundary consists of \(S\) and \(B_F\), and \(G \in \mathbb{R}\). Let \(\hat{\Gamma} : [0, B] \to \mathbb{R}\) be given by

\[
\hat{\Gamma}(r) = \int_0^r \gamma(t) \, dt \quad \text{for all } r \in [0, B]. \tag{3.1}
\]

The function \(R[\psi]\) given in \(\Omega\) by

\[
R[\psi] := \frac{1}{2}|\nabla \psi|^2 + gY - \frac{1}{2}Q + \hat{\Gamma}(\psi)
\]

is, up to a constant, the negative of the pressure in the fluid. Let \(T[\psi]\) be given in \(\Omega\) by

\[
T[\psi] := \frac{1}{2}|\nabla \psi|^2 + gy - \frac{1}{2}Q + \hat{\Gamma}(\psi) - \varpi \psi,
\]

3 An a priori estimate on the pressure in the fluid

In this section we use the maximum principle to derive an a priori estimate on the pressure in the fluid. Apart from being of interest in itself, this result plays an essential role in the investigation of the existence of extreme waves and the Stokes conjecture with vorticity.

Let \((S, B_F, \psi, Q)\) be a classical solution of (1.1). Let \(\hat{\Gamma} : [0, B] \to \mathbb{R}\) be given by

\[
\hat{\Gamma}(r) = \int_0^r \gamma(t) \, dt \quad \text{for all } r \in [0, B]. \tag{3.1}
\]

The function \(R[\psi]\) given in \(\Omega\) by

\[
R[\psi] := \frac{1}{2}|\nabla \psi|^2 + gY - \frac{1}{2}Q + \hat{\Gamma}(\psi)
\]

is, up to a constant, the negative of the pressure in the fluid. Let \(T[\psi]\) be given in \(\Omega\) by

\[
T[\psi] := \frac{1}{2}|\nabla \psi|^2 + gy - \frac{1}{2}Q + \hat{\Gamma}(\psi) - \varpi \psi,
\]
where 
\[ \varpi := \frac{1}{2} \max \left\{ 0, \max_{r \in [0,B]} \gamma(r) \right\}. \]

Obviously \( T[\psi] = R[\psi] \) whenever \( \gamma(r) \leq 0 \) for all \( r \in [0,B] \). Theorem 3.1 below is an extension of [41, Theorem 2.1 and Theorem 2.4], where the same result was proved under the assumption that \( \gamma : [0, B] \to \mathbb{R} \) does not change sign.

**Theorem 3.1.** Let \((S, B_F, \psi, Q)\) be a classical solution of (1.1) such that \( \psi_Y < 0 \) in \( \Omega \). Then \( T[\psi] \leq 0 \) in \( \Omega \).

**Proof of Theorem 3.1.** The proof is merely an application of a result in Sperb [30, Section 5.2]. I am grateful to John Toland for pointing out this reference to me.

The required result is obtained from a more general one. Let \( \lambda : [0, B] \to \mathbb{R} \) be a \( C^1 \) function, let \( \Lambda : [0, B] \to \mathbb{R} \) be given by \( \Lambda(r) = \int_0^r \lambda(t) \, dt \) for all \( r \in [0, B] \), and let \( S : \Omega \to \mathbb{R} \) be given by

\[ S := \frac{1}{2} |\nabla \psi|^2 + \hat{\Gamma}(\psi) + \Lambda(\psi). \tag{3.2} \]

Then \( S = 0 \) on \( S \). We seek conditions on \( \lambda \) which ensure that \( S \leq 0 \) in \( \Omega \).

Let \( W : \Omega \to \mathbb{R} \) be given by

\[ W = \frac{1}{2} |\nabla \psi|^2 + \hat{\Gamma}(\psi) + \Lambda(\psi). \tag{3.3} \]

It is easy to check, using the fact that (1.1a) holds, that \( W \) satisfies the following elliptic equation in \( \Omega \):

\[ \Delta W + \frac{L_1}{|\nabla \psi|^2} W_x + \frac{L_2}{|\nabla \psi|^2} W_y = \lambda'(\psi)|\nabla \psi|^2 + (2\lambda(\psi) + \gamma(\psi))\lambda(\psi), \tag{3.4} \]

where

\[ L_1 := -2[W_x - (2\lambda(\psi) + \gamma(\psi))\psi_x], \quad L_2 := -2[W_y - (2\lambda(\psi) + \gamma(\psi))\psi_y]. \tag{3.5} \]

Equation (3.3) is [30] equation (5.17), p. 69], which is correct, despite the fact that there is a misprint in [30] equation (5.16), p. 69]. Note also that

\[ W_Y = \lambda(\psi)\psi_Y \quad \text{on} \ B_F. \tag{3.6} \]

It is immediate from (3.4)-(3.6) that

\[ \Delta S + \frac{M_1}{|\nabla \psi|^2} S_x + \frac{M_2}{|\nabla \psi|^2} S_y = \lambda'(\psi)|\nabla \psi|^2 + (2\lambda(\psi) + \gamma(\psi))\lambda(\psi) + \frac{2g}{|\nabla \psi|^2} [g + (2\lambda(\psi) + \gamma(\psi))\psi_Y], \tag{3.7} \]

and

\[ S_Y = g + \lambda(\psi)\psi_Y \quad \text{on} \ B_F, \tag{3.8} \]
where

\[ M_1 := -2[S_X - (2\lambda(\psi) + \gamma(\psi))\psi_X], \]
\[ M_2 := -2[S_Y - 2g - (2\lambda(\psi) + \gamma(\psi))\psi_Y]. \]  

(3.9)

Since \( S = 0 \) on \( S \) and \( \psi_Y < 0 \) in \( \Omega \), the maximum principle shows that \( S \leq 0 \) in \( \Omega \) whenever

\[ \lambda(r) \leq 0, \quad 2\lambda(r) + \gamma(r) \leq 0, \quad \lambda'(r) \geq 0 \quad \text{for all } r \in [0, B]. \]  

(3.10)

In particular, \( T[\psi] \leq 0 \) in \( \Omega \). This completes the proof of Theorem 3.1.

Proposition 3.2. Let \((S, B_F, \psi, Q)\) be a classical solution of (1.1) such that \(|\nabla \psi| \neq 0\) in \( \Omega \cup B_F \). Then

\[ \min_S |\nabla \psi|^2 \leq |\nabla \psi(X, Y)|^2 + 2\tilde{\Gamma}(\psi(X, Y)) \leq \max_S |\nabla \psi|^2 \quad \text{for all } (X, Y) \in \overline{\Omega}. \]

Remark 3.3. The estimate in Proposition 3.2 holds with equalities for any solution of (1.1) for which \( S \) is a horizontal line and \( \psi \) does not depend on \( X \).

4 On the existence of extreme waves

Let \((S, B_F, \psi, Q)\) be a weak solution of (1.1). We say that a point \((X_0, Y_0)\) on \( S \) is a stagnation point if \( Q - 2gY_0 = 0 \). This would formally correspond to the fact that \( \nabla \psi(X_0, Y_0) = (0, 0) \). A weak solution of (1.1) with stagnation points on the free surface \( S \) is called an extreme wave.

In view of Proposition 2.7, there is no loss of generality in considering only solutions of (1.1) for which \( F = 0 \). In this section we are interested in solutions \((S, B_0, \psi, Q)\) of (1.1), for which

\[ \psi \text{ is even in the } X \text{ variable,} \quad \psi_Y < 0 \text{ in } \Omega. \]  

(4.1)

and, in some situations, also

\[ S = \{(X, \eta(X)) : X \in \mathbb{R}\}, \text{ with} \]
\[ \eta \in C^1(\mathbb{R}), \text{ 2L-periodic, even, and } \eta' < 0 \text{ on } (0, L). \]  

(4.2)

The following result gives general conditions under which a sequence of regular waves contains a subsequence converging in a certain sense to an extreme wave. Here and in what follows, for any (weak) solution \((S, B_0, \psi, Q)\) of (1.1), we extend \( \psi \) to \( \mathbb{R}_+^2 \), with the value 0 in \( \mathbb{R}_+^2 \backslash \overline{\Omega} \). The extension, denoted also by \( \psi \), is a Lipschitz function on \( \mathbb{R}_+^2 \).
**Theorem 4.1.** Let \( \{(S_j, B_0, \psi_j, Q_j)\}_{j \geq 1} \) be a sequence of classical solutions of (1.1) for which (4.1) and (4.2) hold. Suppose that the sequence \( \{Q_j\}_{j \geq 1} \) is bounded above. \( (4.3) \)

Then there exists a weak solution \((\tilde{S}, B_0, \tilde{\psi}, \tilde{Q})\) of (1.1) for which (4.1) holds, such that, along a subsequence (not relabeled),

\[
Q_j \rightarrow \tilde{Q}, \quad (4.4)
\]

\[
\psi_j \rightarrow \tilde{\psi}, \quad \text{uniformly on } \mathbb{R}^2_+ , \quad (4.5)
\]

\[
\nabla \psi_j \rightarrow \nabla \tilde{\psi}, \quad \text{weak* in } L^\infty(\mathbb{R}^2_+). \quad (4.6)
\]

If, in addition,

\[
|\nabla \psi_j(0, \eta_j(0))| \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad (4.7)
\]

then \((\tilde{S}, B_0, \tilde{\psi}, \tilde{Q})\) is an extreme wave.

**Remark 4.2.** The proof of Theorem 4.1 also provides a precise sense in which the sequence of curves \( \{S_j\}_{j \geq 1} \) converges along a subsequence to \( \tilde{S} \). For the sake of brevity, we have chosen not to include this in the statement of the theorem.

**Remark 4.3.** The proof of Theorem 4.1 would be simpler if it were assumed that the family \( \{\eta_j\}_{j \geq 1} \) is equi-Lipschitz on \( \mathbb{R} \).

Such an assumption would probably be difficult to verify in practice, so it is important that we do not need it in Theorem 4.1.

Constantin and Strauss [7, Theorem 1.1] proved that, if \( \gamma : [0, B] \rightarrow \mathbb{R} \) satisfies the condition

\[
\int_0^B \left[ \frac{\pi^2(B - r)^2}{L^2}(2\tilde{\Gamma}_{\max} - 2\tilde{\Gamma}(r))^{1/2} + (2\tilde{\Gamma}_{\max} - 2\tilde{\Gamma}(r))^{3/2} \right] \, dr < gB^2 , \quad (4.8)
\]

where \( \tilde{\Gamma} \) is given by (3.1) and \( \tilde{\Gamma}_{\max} := \max_{r \in [0, B]} \tilde{\Gamma}(r) \), then there exists a set \( \mathcal{C} \) (connected in a certain function space) of solutions of (1.1) of the form \((S, B_0, \psi, Q)\), satisfying (4.1) and (4.2), which contains a sequence \( \{(S_j, B_0, \psi_j, Q_j)\}_{j \geq 1} \) such that \( \max_{y \in \Omega} \psi_j^i \rightarrow 0 \) as \( j \rightarrow \infty \).

The following new result concerning the convergence of a sequence of regular waves in \( \mathcal{C} \) to an extreme wave is easily obtained by combining Theorem 4.1 with existing results in literature on the validity of (4.3) and (4.7) for a sequence in \( \mathcal{C} \).

**Theorem 4.4.** Let \( \gamma : [0, B] \rightarrow \mathbb{R} \) be such that (4.8) holds, and suppose in addition that \( \gamma(0) < 0, \gamma(r) \leq 0 \) and \( \gamma'(r) \geq 0 \) for all \( r \in [0, B] \).

Let \( \{(S_j, B_0, \psi_j, Q_j)\}_{j \geq 1} \) be a sequence in \( \mathcal{C} \) such that

\[
\max_{y \in \Omega} \psi_j^i \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (4.9)
\]
Then \( \{ (S_j, B_0, \psi_j, Q_j) \}_{j \geq 1} \) converges in the sense of Theorem 4.1 along a subsequence, to an extreme wave \((\tilde{S}, B_0, \tilde{\psi}, \tilde{Q})\).

**Remark 4.5.** Let \( \gamma : [0, B] \to \mathbb{R} \) be such that \( \gamma(0) < 0 \) and \( \gamma(r) \leq 0 \) for all \( r \in [0, B] \). Let \( \Upsilon : [0, B] \to \mathbb{R} \) be given by

\[
\Upsilon(r) = \int_0^r \frac{1}{(-2\hat{\Gamma}(t))^{1/2}} dt \quad \text{for all} \quad r \in [0, B],
\]

where the function \( \hat{\Gamma} \) is given by (3.1). Then \( \Upsilon \) is a bijection from \([0, B]\) onto \([0, \Upsilon(B)]\), with inverse \( \Upsilon^{-1} : [0, \Upsilon(B)] \to [0, B] \). Let \( S := \{(X, \Upsilon(B)) : X \in \mathbb{R}\} \) and \( \Omega \) be the strip whose boundary consists of \( S \) and \( B_0 \). Let \( Q := 2g \Upsilon(B) \) and \( \psi : \Omega \to \mathbb{R} \) be given by

\[
\psi(X, Y) := \Upsilon^{-1}(\Upsilon(B) - Y) \quad \text{for all} \quad (X, Y) \in \Omega.
\]

It is easy to check that \((S, B_0, \psi, Q)\) is a solution of (1.1) for which all the points of \( S \) are stagnation points. We call such a solution of (1.1) a *trivial extreme wave*.

**Remark 4.6.** It is not known whether the extreme wave obtained in Theorem 4.4 is trivial or not. It is natural to conjecture that the conclusion of Theorem 4.4 remains valid in the absence of the condition \( \gamma(0) < 0 \), but the present method of proof of Theorem 4.4 cannot handle this more general situation. The difficulty is to prove the validity of (4.3) for a suitable sequence in \( C \). This fact can be proved in the irrotational case, thus leading to the existence of a *nontrivial* extreme wave, but the only proof we know makes use of Nekrasov’s integral equation, and because this method cannot be used for rotational waves we refrain from giving any details here.

We now give the proof of Theorem 4.1, and then that of Theorem 4.4.

**Proof of Theorem 4.1.** Let \( \{ (S_j, B_0, \psi_j, Q_j) \}_{j \geq 1} \) be as in the statement of the theorem, with \( S_j = \{(X, \eta_j(X)) : X \in \mathbb{R}\} \) for all \( j \geq 1 \). The condition (4.3) means that

\[
\text{the sequence} \quad \{ \max_{S_j} |\nabla \psi_j| \} \quad \text{is uniformly bounded above}, \quad (4.10)
\]

\[
\text{the sequence} \quad \{ \eta_j \} \quad \text{is uniformly bounded above}. \quad (4.11)
\]

It follows from (4.10) and Proposition 3.2 that

\[
\text{the family} \quad \{ \psi_j \} \quad \text{is equi-Lipschitz on} \quad \mathbb{R}^2_+ \quad \text{.} \quad (4.12)
\]

We deduce from (4.12), using (4.2) and the relation

\[
-B = \psi^j(L, \eta_j(L)) - \psi^j(L, 0) = \int_0^{\eta_j(L)} \psi_y(L, V) dV,
\]

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that
\[ \{ \eta_j \}_{j \geq 1} \text{ is uniformly bounded away from 0.} \quad (4.13) \]

This implies that
\[ \{ Q_j \}_{j \geq 1} \text{ is bounded away from 0.} \quad (4.14) \]

Let \( \ell_j \) denote the length of the arc \( \{(X, \eta_j(X)) : X \in [0, L]\} \), for all \( j \geq 1 \).
It follows from (4.2) and (4.11) that
\[ \{ \ell_j \}_{j \geq 1} \text{ is bounded above and away from 0.} \quad (4.15) \]

For any \( j \geq 1 \), a parametrization of the curve \( S_j \) is given by \( S_j = \{(u_j(s), v_j(s)) : s \in \mathbb{R}\} \), where \( u_j, v_j : \mathbb{R} \to \mathbb{R} \) are \( C^1 \) functions, periodic of period 1, such that
\[ u_j(0) = 0, \quad v_j(0) = \eta_j(0), \quad u_j(1) = L, \quad v_j(1) = \eta_j(L), \quad (4.16) \]
\[ u_j \text{ is odd,} \quad v_j \text{ is even,} \quad (4.17) \]
\[ u_j'(s) \geq 0 \text{ for all } s \in \mathbb{R}, \quad v_j'(s) \leq 0 \text{ for all } s \in [0, 1], \quad (4.18) \]
\[ u_j'(s)^2 + v_j'(s)^2 = \ell_j^2 \text{ for all } s \in \mathbb{R}. \quad (4.19) \]

It follows from (4.3) and (4.11)-(4.19) that there exist constants \( \tilde{Q} > 0 \), \( \tilde{\ell} > 0 \), and functions \( \tilde{\psi} \in \text{Lip}(\mathbb{R}^2) \) and \( \tilde{u}, \tilde{v} \in \text{Lip}(\mathbb{R}) \), with \( \tilde{u}, \tilde{v} \) periodic of period 1, such that, along a subsequence (not relabeled), (4.4)-(4.6) hold and
\[ \ell_j \to \tilde{\ell}, \quad (4.20) \]
\[ u_j \to \tilde{u}, \quad v_j \to \tilde{v} \text{ uniformly on } \mathbb{R}, \quad (4.21) \]
\[ u_j' \to \tilde{u}', \quad v_j' \to \tilde{v}' \text{ weak* in } L^\infty(\mathbb{R}). \quad (4.22) \]

It is immediate from (4.16)-(4.19) that
\[ \tilde{u}(0) = 0, \quad \tilde{u}(1) = L, \quad (4.23) \]
\[ \tilde{u} \text{ is odd,} \quad \tilde{v} \text{ is even,} \quad (4.24) \]
\[ \tilde{u}'(s) \geq 0 \text{ for a.e. } s \in \mathbb{R}, \quad \tilde{v}'(s) \leq 0 \text{ for a.e. } s \in (0, 1), \quad (4.25) \]
\[ \tilde{u}'(s)^2 + \tilde{v}'(s)^2 \leq \tilde{\ell}^2 \text{ for almost every } s \in \mathbb{R}. \quad (4.26) \]

It is also a consequence of (4.18) and (4.19) that, for all \( j \geq 1 \) and for every \( a, b \in [0, 1] \) with \( a < b \),
\[ (b - a)\ell_j = \int_a^b (u_j'(s)^2 + v_j'(s)^2)^{1/2} \, ds \leq |u_j(b) - u_j(a)| + |v_j(b) - v_j(a)|. \]

This implies that
\[ (b - a)\tilde{\ell} \leq |\tilde{u}(b) - \tilde{u}(a)| + |\tilde{v}(b) - \tilde{v}(a)|. \quad (4.27) \]

Therefore,
\[ \tilde{\ell} \leq |\tilde{u}'(s)| + |\tilde{v}'(s)| \text{ for almost every } s \in (0, 1). \quad (4.29) \]
We would now like to prove that

the mapping \( s \mapsto (\hat{u}(s), \hat{v}(s)) \) is injective on \( \mathbb{R} \). \hspace{1cm} (4.30)

Let \( \sigma \in [0, 1) \) and \( \varsigma \in (0, 1] \), with \( \sigma < \varsigma \), be given by

\[
\sigma := \max\{s \in [0, 1] : \hat{u}(s) = 0\}, \hspace{1cm} (4.31a)
\]

\[
\varsigma := \min\{s \in [0, 1] : \hat{u}(s) = L\}. \hspace{1cm} (4.31b)
\]

To prove (4.30), it suffices, in view of (4.28) and (4.23)-(4.25), to show that \( \varsigma \) and \( B \), a weak solution of (1.1).

\[
\int_{0}^{L} \gamma(\varsigma) \zeta dL^2 = \int_{0}^{L} (Q - 2g \hat{v}(s))^{1/2} \zeta(\hat{u}(s), \hat{v}(s)) \ell_{ds}
\]

for all \( \zeta \in C_{0}^{1}(\mathbb{R}^2_{+}) \). \hspace{1cm} (4.35)

The validity of (4.31b) and (4.33) and (4.34) makes it possible to pass to the limit as \( j \to \infty \) in (4.35), to obtain

\[
\int_{\tilde{\Omega}} \nabla \hat{\psi} \nabla \zeta dL^2 = \int_{\tilde{\Omega}} (Q - 2g \hat{v}(s))^{1/2} \zeta(\hat{u}(s), \hat{v}(s)) \hat{\psi} \ell d\sigma
\]

for all \( \zeta \in C_{0}^{1}(\mathbb{R}^2_{+}) \). \hspace{1cm} (4.36)
With $\sigma$ defined in (4.31), we now claim that $\sigma = 0$. Suppose for a contradiction that this is not so. Let $D$ be the disc centred at $(0, \tilde{v}(0))$ and with the point $(0, \tilde{v}(\sigma))$ on its boundary. It follows from (4.36) and (4.32) that
\[
\int_{\tilde{v}(\sigma)}^{\tilde{v}(0)} (\bar{Q} - 2gY)^{1/2} \zeta(0, Y) \, dY = 0 \quad \text{for all } \zeta \in C^1_0(D).
\]
Since this is clearly not possible, it follows that $\sigma = 0$.

With $\varsigma$ defined in (4.31), we now claim that $\varsigma = 1$. Suppose for a contradiction that this is not so. Let $R := \{ (X,Y) \in \mathbb{R}^2 : -L < X < 3L, 0 < Y < \tilde{v}(\varsigma) \}$, and let $R_- := \{ (X,Y) \in R : X < L \}, \ R_+ := \{ (X,Y) \in R : X > L \}$ and $R_L := \{ (X,Y) \in R : X = L \}$. It follows from (4.36) and (4.32) that, for all $\zeta \in \text{Lip}(R)$,
\[
\int_R \nabla \tilde{v} \nabla \zeta \, d\mathcal{L}^2 = \int_{R_-} \gamma(\tilde{v}) \zeta \, d\mathcal{L}^2 - 2\int_{\tilde{v}(1)}^{\tilde{v}(\varsigma)} (\bar{Q} - 2gY)^{1/2} \zeta(L, Y) \, dY.
\] (4.38)
Let $M := \tilde{v}(1)$ and $N := \tilde{v}(\varsigma)$. Since $\tilde{\psi}$ is even with respect to the line $X = L$, it follows that
\[
\int_{R_-} \nabla \tilde{\psi} \nabla \zeta \, d\mathcal{L}^2 = \int_{R_-} \gamma(\tilde{\psi}) \zeta \, d\mathcal{L}^2 - \int_{M}^{N} (\bar{Q} - 2gY)^{1/2} \zeta(L, Y) \, dY,
\] (4.37)
for all $\zeta \in \text{Lip}(R_-)$ with $\zeta = 0$ on $(\partial R_- \setminus R_L$.

To show that this is not possible, we use a blow-up argument. Let $\{\varepsilon_k\}_{k \geq 1}$ be a sequence with $\varepsilon_k \searrow 0$ as $k \to \infty$. For any $k \geq 1$, let $\tilde{\psi}^k : \mathbb{R} \to \mathbb{R}$ be given by
\[
\tilde{\psi}^k(X,Y) := \frac{1}{\varepsilon_k} \tilde{\psi}(L + \varepsilon_k(X - L), M + \varepsilon_k(Y - M)) \quad \text{for all } (X,Y) \in R_-.
\]
Let $\zeta \in \text{Lip}(R_-)$ with $\zeta = 0$ on $(\partial R_- \setminus R_L$. We extend $\zeta$ to a Lipschitz function in $\{(X,Y) : X < L, Y \in \mathbb{R}\}$, with the value 0 outside of $R_-$. By applying (4.37) to the function $\zeta^k : \mathbb{R} \to \mathbb{R}$ given by
\[
\zeta^k(X,Y) := \zeta \left( L + \frac{1}{\varepsilon_k}(X - L), M + \frac{1}{\varepsilon_k}(Y - M) \right) \quad \text{for all } (X,Y) \in R_-,
\]
we deduce, after a change of variables in the integrals, that
\[
\int_{R_-} \nabla \tilde{\psi}^k \nabla \zeta \, d\mathcal{L}^2 = \int_{R_-} \varepsilon_k \gamma(\varepsilon_k \tilde{\psi}^k) \zeta \, d\mathcal{L}^2
\]
\[
- \int_{M}^{N} (\bar{Q} - 2gM - 2g\varepsilon_k(Y - M))^{1/2} \zeta(L, Y) \, dY.
\] (4.38)
Since the family \( \{ \hat{\psi}^k \}_{k \geq 1} \) is equi-Lipschitz on \( \mathcal{R}_- \), there exists a function \( \hat{\psi} \in \text{Lip}(\mathcal{R}_-) \) such that, along a subsequence (not relabeled),

\[
\hat{\psi}^k \rightarrow \hat{\psi} \quad \text{uniformly on } \mathcal{R}_-,
\]

\[
\nabla \hat{\psi}^k \rightarrow \nabla \hat{\psi} \quad \text{weak* in } L^\infty(\mathcal{R}_-).
\]

Since \( \tilde{\psi} = 0 \) on \( \mathcal{J} \), it follows that \( \hat{\psi} = 0 \) on \( \mathcal{R}_L \cap \mathcal{J} \). Also, by passing to the limit as \( k \to \infty \) in (4.38), we conclude that

\[
\int_{\mathcal{R}_-} \nabla \hat{\psi} \nabla \zeta \, d\mathcal{L}^2 = -\int_{\mathcal{M}} (\tilde{Q} - 2gM)^{1/2} \zeta(L, Y) \, dY
\]

(4.39)

for all \( \zeta \in \text{Lip}(\mathcal{R}_-) \) with \( \zeta = 0 \) on \( (\partial \mathcal{R}_-) \setminus \mathcal{R}_L \).

This shows in particular that \( \hat{\psi} \) is a harmonic function in \( \mathcal{R}_- \). Let \( \mathcal{J}_0 := \{ (\tilde{u}(s), \tilde{v}(s)) : s \in J \setminus \mathbb{Z} \} \). Since \( \hat{\psi} = 0 \) on \( \mathcal{R}_L \cap \mathcal{J}_0 \), the Reflection Principle shows that \( \hat{\psi} \) can be extended as a harmonic function, odd with respect to the line \( X = L \), in \( \mathcal{R}_- \cup (\mathcal{R}_L \cap \mathcal{J}_0) \cup \mathcal{R}_+ \). Let the extension be denoted also by \( \hat{\psi} \). Then the holomorphic function \( f := \hat{\psi}_x - i\hat{\psi}_y \) in \( \mathcal{R}_- \cup (\mathcal{R}_L \cap \mathcal{J}_0) \cup \mathcal{R}_+ \) satisfies

\[
f = -((\tilde{Q} - 2gM)^{1/2}) \quad \text{on } \mathcal{R}_L \cap \mathcal{J}_0.
\]

Since any holomorphic function on a connected domain is uniquely determined by its values on any set which has a limit point in that domain [27, Theorem 10.18], it follows that \( f(X + iY) = -((\tilde{Q} - 2gM)^{1/2}) \) for all \((X, Y) \in \mathcal{R}_- \cup (\mathcal{R}_L \cap \mathcal{J}_0) \cup \mathcal{R}_+ \). Hence necessarily

\[
\hat{\psi}(X, Y) = -((\tilde{Q} - 2gM)^{1/2})(X - L) \quad \text{for all } (X, Y) \in \mathcal{R}_-.
\]

But this contradicts (4.39), since \( \tilde{Q} - 2gM > 0 \) and \( 0 < M < N \). This shows that \( \zeta = 1 \).

Since \( \sigma = 0 \) and \( \zeta = 1 \), it has been therefore proved that (4.30) holds, \( \mathcal{J} \) and \( \mathcal{I} \) are empty, and that \( \tilde{S} = \{ (\hat{u}(s), \hat{v}(s)) : s \in \mathbb{R} \} \). Note now from (4.33) that, for any compact set \( \mathcal{K} \subset \mathbb{R}^2 \),

\[
\mathcal{K} \subset \tilde{\Omega} \cup \mathcal{B}_0 \text{ implies } \mathcal{K} \subset \tilde{\Omega}_j \cup \mathcal{B} \text{ for all } j \text{ sufficiently large.} \quad (4.40)
\]

It is a consequence of (4.30) that

\[
\int_{\tilde{\Omega}} \nabla \hat{\psi} \nabla \zeta \, d\mathcal{L}^2 = \int_{\tilde{\Omega}} \gamma(\hat{\psi}) \zeta \, d\mathcal{L}^2 \quad \text{for all } \zeta \in C_0^1(\tilde{\Omega}).
\]

It follows that \( \hat{\psi} \in C_0^{2,\alpha}(\tilde{\Omega} \cup \mathcal{B}) \) satisfies

\[
\Delta \hat{\psi} = -\gamma(\hat{\psi}) \quad \text{in } \tilde{\Omega}. \quad (4.41)
\]

Since \( \hat{\psi} \in \text{Lip}(\mathbb{R}^2_+) \), Proposition 2.2 ensures that the partial derivatives of \( \hat{\psi} \) have non-tangential limits \( \mathcal{H}^1 \)-almost everywhere on \( \tilde{S} \). Taking into account (4.29), we write (2.11) in the form

\[
\int_{\tilde{\Omega}} \nabla \hat{\psi} \nabla \zeta \, d\mathcal{L}^2 = \int_{\tilde{\Omega}} \gamma(\hat{\psi}) \zeta \, d\mathcal{L}^2
\]

\[
\quad + \int_{\mathbb{R}} \frac{\partial \hat{\psi}}{\partial n}(\hat{u}(s), \hat{v}(s))\zeta(\hat{u}(s), \hat{v}(s))(\hat{u}'(s)^2 + \hat{v}'(s)^2)^{1/2} \, ds
\]

for all \( \zeta \in C^1_0(\mathbb{R}^2_+) \). \quad (4.42)
By comparing (4.36) and (4.42), we deduce that
\[-\frac{\partial \tilde{\psi}}{\partial n}(\tilde{u}(s), \tilde{v}(s))(\tilde{u}'(s)^2 + \tilde{v}'(s)^2)^{1/2} = (\tilde{Q} - 2g\tilde{v}(s))^{1/2}\tilde{\ell} \text{ for a.e. } s \in \mathbb{R}. \tag{4.43}\]

Note now that, in view of (4.12), there is no loss of generality in assuming that
\[\psi^j \to \psi \text{ in } C^{0,\alpha}(\mathbb{R}^2). \tag{4.44}\]
This implies that
\[\gamma(\psi^j) \to \gamma(\psi) \text{ in } C^{0,\alpha}(\mathbb{R}^2). \tag{4.44}\]
Since (4.41), (4.40) and (4.44) hold, standard elliptic estimates [15, Theorem 4.6 and Theorem 4.11] show that
\[\psi^j \to \psi \text{ in } C^{2,\alpha}_{\text{loc}}(\tilde{\Omega} \cup \mathcal{B}). \tag{4.45}\]
Now, Theorem 3.1 shows that
\[T[\psi^j] \leq 0 \text{ in } \Omega_j \text{ for all } j \geq 1. \tag{4.46}\]
We deduce from (4.45) and (4.46) that
\[T[\psi] \leq 0 \text{ in } \tilde{\Omega}. \tag{4.47}\]
It follows from (4.47) that, in the notation of (2.28),
\[|\nabla \tilde{\psi}(X,Y)|^2 + 2gY - \tilde{Q} \leq 0 \text{ for } \mathcal{H}^1\text{-almost every } (X,Y) \in \tilde{S}. \tag{4.48}\]
Since \(\tilde{\psi} = 0\) on \(\tilde{S}\), it follows, by using (4.48) and Lemma 2.5 upon taking into account (1.29), that
\[0 \leq \frac{\partial \tilde{\psi}}{\partial n}(\tilde{u}(s), \tilde{v}(s)) \leq (\tilde{Q} - 2g\tilde{v}(s))^{1/2} \text{ for almost every } s \in \mathbb{R}. \tag{4.49}\]
It follows from (4.48), (4.26) and (4.49) that
\[\tilde{u}'(s)^2 + \tilde{v}'(s)^2 = \tilde{\ell}^2 \text{ for almost every } s \in \mathbb{R}, \]
\[\frac{\partial \tilde{\psi}}{\partial n}(X,Y) = -(\tilde{Q} - 2gY)^{1/2} \text{ for } \mathcal{H}^1\text{-almost every } (X,Y) \in \tilde{S}.\]
This completes the proof of the fact that \((\tilde{S}, \mathcal{B}_0, \tilde{\psi}, \tilde{Q})\) is a weak solution of (1.1).

We now recall for easy reference the following version of the maximum principle [13, Lemma 1, p. 519], in which we emphasize that there is no assumption on the sign of the coefficient \(c : \mathcal{G} \to \mathbb{R}\).

**Proposition 4.7.** Let \(\mathcal{G} \subset \mathbb{R}^n\), where \(n \geq 1\), be a connected open set. Let \(c \in L^\infty(\mathcal{G})\) and \(w \in C^2(\mathcal{G})\) with \(w \geq 0\) in \(\mathcal{G}\) be such that
\[\Delta w + cw \leq 0 \text{ in } \mathcal{G}.\]
Then either \(w \equiv 0\) in \(\mathcal{G}\), or \(w > 0\) in \(\mathcal{G}\).
Since $\psi_j < 0$ in $\Omega_j$ for all $j \geq 1$, it follows that $\tilde{\psi}_Y \leq 0$ everywhere in $\tilde{\Omega}$. Since

$$\Delta \tilde{\psi}_Y = -\gamma'(\tilde{\psi}) \tilde{\psi}_Y \quad \text{in } \tilde{\Omega},$$

(4.50)

Proposition 4.7 shows that $\tilde{\psi}_Y < 0$ in $\tilde{\Omega}$. As $\tilde{\psi}$ is clearly even in the $X$ variable, it follows that (4.1) holds.

If, in addition, (4.7) holds, then obviously $\tilde{Q} - 2g\tilde{v}(0) = 0$, so that $(\tilde{S}, B_0, \tilde{\psi}, \tilde{Q})$ is an extreme wave. This completes the proof of Theorem 4.1.

Proof of Theorem 4.4. Any solution $(S, B_0, \psi, Q)$ of (1.1) which belongs to $C$ has the properties (4.1) and (4.2). It has been proved in [41, Theorem 2.3], improving on an earlier result in [8], that, if $\gamma(r) \leq 0$ and $\gamma'(r) \geq 0$ for all $r \in [0, B]$, then any solution $(S, B_0, \psi, Q)$ of (1.1) with the properties (4.1) and (4.2) satisfies

$$\max_{\tilde{\Omega}} \psi_Y = \psi_Y(0, \eta(0)),$$

where $S = \{(X, \eta(X)) : X \in \mathbb{R}\}$ and $\Omega$ is the domain whose boundary consists of $S$ and $B_0$. Hence (4.7) follows from (4.9).

The fact that $Q$ is bounded above along $C$ whenever $\gamma(0) < 0$ and $\gamma(r) \leq 0$ for all $r \in [0, B]$ is an immediate consequence of an estimate in [9, Proof of Lemma 7.1]. I am grateful to Adrian Constantin for pointing this out to me.

Let $f : (2\hat{\Gamma}_{\max}, \infty) \to \mathbb{R}$ be given by

$$f(\lambda) = \lambda + 2g \int_0^B (\lambda - 2\hat{\Gamma}(r))^{-1/2} dr \quad \text{for all } \lambda \in (2\hat{\Gamma}_{\max}, \infty),$$

and let $(S, B_0, \psi, Q)$ be a solution of (1.1) which belongs to $C$. It is proven there that $\psi_Y^2(0, \eta(0)) < \lambda_0$, where $\lambda_0$ is the unique solution in $(2\hat{\Gamma}_{\max}, \infty)$ of the equation $f'(\lambda) = 0$. It is also proven there that, if $\psi_Y^2(0, \eta(0)) > 2\hat{\Gamma}_{\max}$, then

$$Q < f(\psi_Y^2(0, \eta(0))).$$

(4.51)

If $\gamma(r) \leq 0$ for all $r \in [0, B]$, then $\hat{\Gamma}_{\max} = 0$. Since the restriction of $f$ to the interval $(0, \lambda_0)$ is bounded above whenever $\gamma(0) < 0$ and $\gamma(r) \leq 0$ for all $r \in [0, B]$, it follows from (4.51) that $Q$ is bounded above along $C$ in that case. Therefore (4.3) holds.

The required conclusion follows from Theorem 4.1. This completes the proof of Theorem 4.4.

5 On the Stokes conjecture

In this section we study the shape of the profile of an extreme wave in a neighbourhood of a stagnation point. In view of Proposition 2.7, there is no loss of generality in considering only extreme waves for which $Q = 0$. With the origin a stagnation point, we are interested in the shape of $S$ close to the origin.
Let \((S, B_F, \psi, 0)\) be an extreme wave, where \(F < 0\), such that
\[
\psi_y < 0 \quad \text{in } \Omega,
\]
and \(S = \{(u(s), v(s)) : s \in \mathbb{R}\}\), where

- the mapping \(s \mapsto (u(s), v(s))\) is injective, \((5.2a)\)
- \(u(0) = v(0) = 0\), \((5.2b)\)
- \(s \mapsto u(s)\) is nondecreasing on \(\mathbb{R}\), \((5.2c)\)
- there exist \(d, e \in \mathbb{R}\) with \(d < 0 < e\) such that \(s \mapsto v(s)\) is nondecreasing on \([d, 0]\) and nonincreasing on \([0, e]\), \((5.2d)\)

We further assume that
\[
T[\psi] \leq 0 \text{ in } \Omega. \quad (5.3)
\]

**Remark 5.1.** Although Theorem 3.1 suggests that \((5.3)\) may be true for all weak solutions of \((1.1)\) for which \((5.1)\) holds, we have so far not been able to prove this. Note however that, as \((4.47)\) shows, \((5.3)\) holds for weak solutions of \((1.1)\) which arise as limits of sequences of classical solutions as in Theorem 4.1.

The main result of this section is a proof of the Stokes conjecture in the following form.

**Theorem 5.2.** Let \((S, B_F, \psi, 0)\) be an extreme wave which satisfies \((5.1)-(5.3)\). In addition, suppose that \(S\) and \(\psi\) are symmetric with respect to the vertical line \(X = 0\). Then

\[
\text{either } \lim_{s \to 0^\pm} \frac{v(s)}{u(s)} = \frac{1}{\sqrt{3}} \quad \text{or} \quad \lim_{s \to 0^\pm} \frac{v(s)}{u(s)} = 0.
\]

Moreover, if \(\gamma(r) \geq 0\) for all \(r \in [0, \delta]\), for some \(\delta \in (0, B]\), then

\[
\lim_{s \to 0^\pm} \frac{v(s)}{u(s)} = \frac{1}{\sqrt{3}}.
\]

The proof of Theorem 5.2 is obtained by combining Theorem 5.5, Theorem 5.6 and Proposition 5.8 below, and will be given after the proofs of those results.

**Remark 5.3.** We conjecture that the result of Theorem 5.2 continues to hold if the assumption of symmetry of \(S\) and \(\psi\) is dropped, but this is open even when \(\gamma \equiv 0\).

**Remark 5.4.** The existence of trivial extreme waves, noted in Remark 4.5, shows that the possibility that \(\lim_{s \to 0^\pm} \frac{u(s)}{v(s)} = 0\) in Theorem 5.2 cannot in general be ruled out under the assumptions there.

We study the asymptotics near the origin of extreme waves \((S, B_F, \psi, 0)\) satisfying \((5.1)-(5.3)\) by means of a blow-up argument fully described in the
The limiting problem obtained is the following: find a locally rectifiable curve \( \tilde{S} = \{(\tilde{u}(s), \tilde{v}(s)) : s \in \mathbb{R}\} \), where

\[
s \mapsto (\tilde{u}(s), \tilde{v}(s)) \text{ is injective on } \mathbb{R},
\]
\[
\tilde{u}(0) = 0, \quad \tilde{v}(0) = 0,
\]
\[
s \mapsto \tilde{u}(s) \text{ is nondecreasing on } \mathbb{R},
\]
\[
s \mapsto \tilde{v}(s) \text{ is nondecreasing on } (-\infty, 0] \text{ and nonincreasing on } [0, \infty),
\]
\[
\lim_{s \to \pm \infty} (|\tilde{u}(s)| + |\tilde{v}(s)|) = \infty.
\]

such that there exists a function \( \tilde{\psi} \) in the unbounded domain \( \tilde{\Omega} \) below \( \tilde{S} \), which satisfies

\[
\Delta \tilde{\psi} = 0 \quad \text{in } \tilde{\Omega},
\]
\[
\tilde{\psi} \in \text{Lip}_{\text{loc}}(\tilde{\Omega} \cup \tilde{S}),
\]
\[
\tilde{\psi} \geq 0 \quad \text{on } \tilde{\Omega} \quad \text{and} \quad \tilde{\psi}_r \leq 0 \quad \text{on } \tilde{\Omega},
\]
\[
\tilde{\psi} = 0 \quad \text{on } \tilde{S},
\]
\[
|\nabla \tilde{\psi}|^2 + 2gY = 0 \quad \mathcal{H}^1\text{-almost everywhere on } \tilde{S}.
\]

Note that, in view of (5.4f) and (5.4g), the partial derivatives of \( \tilde{\psi} \) have non-tangential limits \( \mathcal{H}^1\)-almost everywhere on \( \tilde{S} \). The requirement (5.4j) refers to these non-tangential boundary values.

**Theorem 5.5.** Let \((S, B_F, \psi, 0)\) be an extreme wave which satisfies (5.1)-(5.3).

Let

\[
Q := \{q \in [-\infty, 0] : \text{there exists a sequence } \{\varepsilon_j\}_{j \geq 1} \text{ with } \varepsilon_j \searrow 0 \text{ as } j \to \infty \text{ such that } \frac{u(\varepsilon_j)}{u(\varepsilon_j)} \to q \text{ as } j \to \infty \}.
\]

If \(q \in Q\) and \(q \neq -\infty\), then there exists a solution \((\tilde{S}, \tilde{\psi})\) of (5.4) with \(\tilde{v}(\hat{s}) = q\tilde{u}(\hat{s})\) for some \(\hat{s} \in (0, \infty)\).

If \(-\infty \in Q\), then there exists a solution \((\tilde{S}, \tilde{\psi})\) of (5.4) with \(\tilde{u}(\hat{s}) = 0\) for some \(\hat{s} \in (0, \infty)\).

Moreover, if \(S\) is symmetric with respect to the line \(X = 0\), then \(-\infty \notin Q\).

Note that problem (5.4) has a trivial solution \((\tilde{S}_0, \tilde{\psi}_0)\) where \(\tilde{S}_0 = \{(X, 0) : X \in \mathbb{R}\} \text{ and } \tilde{\psi}_0 = 0 \text{ in } \mathbb{R}^2_+\), the lower half-plane. Any other solution of (5.4) is called a nontrivial solution.

There also exists an explicit nontrivial solution of (5.4), known as the **Stokes corner flow**. Let \(\tilde{S}^* := \{(X, \eta^*(X)) : X \in \mathbb{R}\} \text{, where}

\[
\eta^*(X) := -\frac{1}{\sqrt{3}}|X| \quad \text{for all } X \in \mathbb{R}.
\]
Let $\tilde{\Omega}^*$ be the domain below $\tilde{S}^*$, and let the harmonic function $\tilde{\psi}^*$ in $\tilde{\Omega}^*$ be given, for all $(X,Y) \in \tilde{\Omega}^*$, by
\[ \tilde{\psi}^*(X,Y) := \frac{2}{3} g^{1/2} \text{Im} \left( i(iZ)^{3/2} \right) \text{ where } Z = X + iY. \quad (5.7) \]

Then $(\tilde{S}^*, \tilde{\psi}^*)$ is a nontrivial solution of (5.4).

**Theorem 5.6.** The only nontrivial solution $(\tilde{S}, \tilde{\psi})$ of (5.4) for which both $\tilde{S}$ and $\tilde{\psi}$ are symmetric with respect to the vertical line $X = 0$ is the Stokes corner flow $(\tilde{S}^* , \tilde{\psi}^*)$.

**Remark 5.7.** We conjecture that the result of Theorem 5.6 continues to hold if the assumption of symmetry of $\tilde{S}$ and $\tilde{\psi}$ is dropped. If this were the case, the validity of the conjecture in Remark 5.3 would immediately follow. It is conceivable that the moving-planes method could be used to prove the symmetry of all solutions of (5.4). This method has so far been successfully used to prove the symmetry of various types of hydrodynamic waves, see [6] for references. The main difficulty in the present situation is the lack of any estimates on the behavior of $(\tilde{S}^*, \tilde{\psi}^*)$ at infinity. If good enough estimates of this type were available, the desired result would follow, see [9] for a related situation and [38, Theorem 3.1] for how to deal with the presence of a stagnation point.

The following simple result, which will be used in the proofs of Theorem 5.2 and Theorem 5.9 below, is also of some interest in itself.

**Proposition 5.8.** Suppose that $\gamma(r) \geq 0$ for all $r \in [0, \delta]$, for some $\delta \in (0, B]$. Let $(S, B_F, \psi, 0)$ be an extreme wave, where (5.1), (5.2a)-(5.2c) hold, and $T[\psi] \leq 0$ in $\Omega$. Then $\Omega$ does not contain any truncated cone with vertex at the origin and opening angle greater that $120^\circ$.

The next result is new even for irrotational waves, in that the symmetry of $S$ and $\psi$ is not required. The drawback is that the existence of lateral tangents at the stagnation point is an assumption.

**Theorem 5.9.** Let $(S, B_F, \psi, 0)$ be an extreme wave which satisfies (5.1)- (5.3). Suppose that there exist $q_\pm \in [0, \infty]$ such that $\lim_{s \to \pm 0} \frac{|v(s)|}{|u(s)|} = q_\pm$. Then either $q_\pm = \frac{1}{\sqrt{3}}$ or $q_\pm = 0$. Moreover, if $\gamma(r) \geq 0$ for all $r \in [0, \delta]$, for some $\delta \in (0, B]$, then $q_\pm = \frac{1}{\sqrt{3}}$.

We now give the proofs of the results of this section.

**Proof of Theorem 5.3.** There is clearly no loss of generality in assuming that the properties (5.2) of are satisfied by a parametrization of $S$ by arclength, i.e., $S = \{(u(s), v(s)) : s \in \mathbb{R}\}$, where $u, v \in \text{Lip}(\mathbb{R})$ satisfy
\[ u'(s)^2 + v'(s)^2 = 1 \text{ for almost every } s \in \mathbb{R}. \]

We extend $\psi$ to $\mathbb{R}^2$ with the value 0 on the connected component of $\mathbb{R}^2 \setminus \Omega$ whose boundary is $S$, and with the value $B$ on the component whose boundary
is $B$. The extension, denoted also by $ψ$, is a Lipschitz function on $\mathbb{R}^2$. It is an immediate consequence of the assumption (5.3) that there exists a constant $K > 0$ such that

$$|\nabla ψ(X, Y)|^2 \leq K|Y| \quad \text{for } \mathcal{L}^2\text{-almost every } (X, Y) \in \mathbb{R}^2. \quad (5.8)$$

Let $q \in Q$ and let the sequence $\{ε_j\}_{j \geq 1}$ with $ε_j \downarrow 0$ as $j \to \infty$ be such that $v(ε_j)/u(ε_j) \to q$ as $j \to \infty$. Let us consider the following sequence of rescalings of the domain $Ω$ and the function $ψ$. For any $j \geq 1$, let

$$Ω_j := \frac{1}{ε_j}Ω, \quad (5.9)$$

and $ψ^j : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$ψ^j(X, Y) := \frac{1}{ε_j}ψ(ε_jX, ε_jY) \quad \text{for all } (X, Y) \in \mathbb{R}^2. \quad (5.10)$$

The boundary of the domain $Ω_j$ consists of the curve $S_j := ε_j^{-1}S$ and the horizontal line $B_{F/ε_j}$. The curve $S_j$ is $2Lε_j^{-1}$-periodic in the horizontal direction, and can be parametrized by arclength by means of the functions $u_j, v_j : \mathbb{R} \to \mathbb{R}$ given by

$$u_j(s) = \frac{1}{ε_j}u(ε_j s), \quad v_j(s) = \frac{1}{ε_j}v(ε_j s) \quad \text{for all } s \in \mathbb{R}. \quad (5.11)$$

The function $ψ^j$ is also $2Lε_j^{-1}$-periodic in the horizontal direction and is a weak solution of

$$\nabla ψ^j = -ε_j^{1/2}γ(ε_j^{3/2}ψ^j) \quad \text{in } Ω_j, \quad (5.11a)$$

$$ψ^j = Bε_j^{-3/2} \quad \text{on } B_{F/ε_j}, \quad (5.11b)$$

$$ψ^j = 0 \quad \text{on } S_j, \quad (5.11c)$$

$$|\nabla ψ^j|^2 + 2gY = 0 \quad \text{on } S_j. \quad (5.11d)$$

In particular, for any $ζ \in C^1_{0}(\mathbb{R}^2)$, the following holds for all $j$ sufficiently large:

$$\int_{Ω_j} \nabla ψ^j \nabla ζ \, d\mathcal{L}^2 = \int_{Ω_j} ε_j^{1/2}γ(ε_j^{3/2}ψ^j)ζ \, d\mathcal{L}^2$$

$$- \int_{\mathbb{R}} (-2gv_j(s))^{1/2}ζ(u_j(s), v_j(s)) \, ds. \quad (5.12)$$

It is immediate from (5.8) and (5.10) that the family $\{ψ^j\}_{j \geq 1}$ is equi-Lipschitz in any horizontal strip $G \subset \mathbb{R}^2. \quad (5.13)$

It follows that there exist functions $\tilde{ψ} \in \text{Lip}_{\text{loc}}(\mathbb{R}^2)$ and $\tilde{u}, \tilde{v} \in \text{Lip}(\mathbb{R})$ such that, along a subsequence (not relabeled),

$$ψ^j \to \tilde{ψ} \quad \text{uniformly on any compact set } K \subset \mathbb{R}^2, \quad (5.14)$$

$$\nabla ψ^j \to \nabla \tilde{ψ} \quad \text{weak* in } L^\infty(\mathcal{G}) \quad \text{for any horizontal strip } \mathcal{G} \subset \mathbb{R}^2, \quad (5.15)$$

$$u_j \to \tilde{u}, \quad v_j \to \tilde{v} \quad \text{uniformly on any compact subset of } \mathbb{R}, \quad (5.16)$$

$$u_j' \to \tilde{u}', \quad v_j' \to \tilde{v}' \quad \text{weak* in } L^\infty(\mathbb{R}). \quad (5.17)$$
It is immediate that
\[ \tilde{u}'(s)^2 + \tilde{v}'(s)^2 \leq 1 \text{ for almost every } s \in \mathbb{R}. \] (5.18)

By arguing as in the proof of (4.27), we deduce that, for every \( a, b \in \mathbb{R} \) having the same sign,
\[ |b - a| \leq |\tilde{u}(b) - \tilde{u}(a)| + |\tilde{v}(b) - \tilde{v}(a)|. \] (5.19)

Therefore,
the mapping \( s \mapsto (\tilde{u}(s), \tilde{v}(s)) \) is injective on \((−\infty, 0]\) and on \([0, \infty)\), (5.20)
\[ 1 \leq |\tilde{u}'(s)| + |\tilde{v}'(s)| \text{ for almost every } s \in \mathbb{R}. \] (5.21)

It is obvious that (5.4b)-(5.4e) hold.

We would now like to prove that (5.4a) holds. Let \( \sigma \in [0, \infty) \) be such that \( \sigma := \sup\{s \in [0, \infty) : \tilde{u}(\pm s) = 0\} \). (5.22)

To prove (5.4a) it suffices, in view of (5.4c), (5.4d) and (5.20), to show that \( \sigma = 0 \). Note from (5.4c), (5.4d), (5.18) and (5.21) that
\[ \tilde{u}'(s) = 0 \text{ for a.e. } s \in (−\sigma, \sigma) \] (5.23a)
\[ \tilde{v}'(s) = 1 \text{ for a.e. } s \in (−\sigma, 0), \quad \tilde{v}'(s) = −1 \text{ for a.e. } s \in (0, \sigma). \] (5.23b)

We now claim that \( \sigma \in [0, \infty) \). Suppose for a contradiction that \( \sigma = +\infty \). It is immediate from (5.16) and (5.23) that, for any compact set \( K \subset \mathbb{R}^2 \),
\[ \{0, Y : Y \leq 0\} \text{ implies } K \subset \tilde{\Omega} \text{ for all } j \text{ sufficiently large}, \] (5.24)
where, for any \( j \geq 1 \), \( \tilde{\Omega} \) is the component of \( \mathbb{R}^2 \setminus (\Omega_j \cup S_j) \) whose boundary is \( S_j \). It follows from (5.24) that \( \tilde{\psi} = 0 \) in \( \mathbb{R}^2 \setminus \{0, Y : Y \leq 0\} \) and hence, using the continuity of \( \tilde{\psi} \) in \( \mathbb{R}^2 \), that \( \tilde{\psi} = 0 \) in \( \mathbb{R}^2 \). Moreover, by passing to the limit as \( j \to \infty \) in (5.12), we obtain, taking also into account (5.23), that
\[ \int_{−\infty}^0 (−2gY)^{1/2} \zeta(0, Y) dY = 0 \text{ for all } \zeta \in C_0^1(\mathbb{R}^2). \]

Since this is clearly not possible, it follows that \( \sigma \in [0, \infty) \).

Let
\[ \mathcal{I} := \{(\tilde{u}(s), \tilde{v}(s)) : s \in (−\sigma, \sigma)\}, \quad \tilde{\Sigma} := \{(\tilde{u}(s), \tilde{v}(s)) : s \in \mathbb{R} \setminus (−\sigma, \sigma)\}. \]
Then \( \mathcal{I} \) is either empty or a half-open vertical segment, while \( \tilde{\Sigma} \) is a locally rectifiable curve. Let \( \tilde{\Omega} \) be the unbounded domain below \( \tilde{\Sigma} \). We first show that \( \mathcal{I} \) is empty, and then that \( (\tilde{\Sigma}, \tilde{\psi}) \) is a solution of (5.4).

It is immediate from (5.16) that, for any compact set \( K \subset \mathbb{R}^2 \),
\[ \mathcal{K} \subset \tilde{\Omega} \text{ implies } \mathcal{K} \subset \Omega_j \text{ for all } j \text{ sufficiently large}, \] (5.25)
\[ \mathcal{K} \subset \mathbb{R}^2 \setminus (\tilde{\Omega} \cup \tilde{\Sigma} \cup \mathcal{I}) \text{ implies } \mathcal{K} \subset \mathcal{V}_j \text{ for all } j \text{ sufficiently large}. \] (5.26)
It is obvious that \( \tilde{\psi} \geq 0 \) in \( \tilde{\Omega} \). Also, it follows from (5.26) that \( \tilde{\psi} = 0 \) in \( \mathbb{R}^2 \setminus (\tilde{\Omega} \cup \tilde{S} \cup \mathcal{I}) \), and hence, using the continuity of \( \tilde{\psi} \) in \( \mathbb{R}^2 \), that \( \tilde{\psi} = 0 \) on \( \mathbb{R}^2 \setminus \tilde{\Omega} \). The validity of (5.14)-(5.16) makes it possible to pass to the limit as \( j \to \infty \) in (5.12), to obtain

\[
\int_{\tilde{\Omega}} \nabla \tilde{\psi} \nabla \zeta \, d\mathcal{L}^2 = - \int_{\mathbb{R}} (-2g\tilde{v}(s))^{1/2} \zeta(\tilde{u}(s), \tilde{v}(s)) \, ds \quad \text{for all } \zeta \in C_0^1(\mathbb{R}^2). \tag{5.27}
\]

We now claim that \( \sigma = 0 \). Suppose for a contradiction that this is not so. It is a consequence of (5.23) that \( \tilde{v}(\sigma) = \tilde{v}(\sigma) \). Let \( D \) be the disc centred at \((0,0)\) and with the point \((0, \tilde{v}(\sigma))\) on its boundary. It follows from (5.27) and (5.23) that

\[
\int_{\tilde{v}(\sigma)}^0 (-2gY)^{1/2} \zeta(0, Y) \, dY = 0 \quad \text{for all } \zeta \in C_0^1(D).
\]

Since this is clearly not possible, it follows that \( \sigma = 0 \).

It has been therefore proved that \( \mathcal{I} \) is empty, \( \tilde{S} = \{ (\tilde{u}(s), \tilde{v}(s)) : s \in \mathbb{R} \} \), and that (5.34)-(5.46) hold. It is a consequence of (5.27) that

\[
\int_{\tilde{\Omega}} \nabla \tilde{\psi} \nabla \zeta \, d\mathcal{L}^2 = 0 \quad \text{for all } \zeta \in C_0^1(\tilde{\Omega}).
\]

It follows that \( \tilde{\psi} \in C^{\infty}(\tilde{\Omega}) \) satisfies

\[
\Delta \tilde{\psi} = 0 \quad \text{in } \tilde{\Omega}. \tag{5.28}
\]

The condition \( \tilde{\psi} \in \text{Lip}_{\text{loc}}(\mathbb{R}^2) \) ensures that the partial derivatives of \( \tilde{\psi} \) have non-tangential limits \( H^1 \)-almost everywhere on \( \tilde{S} \). It follows from (2.11), upon taking into account (5.21), that

\[
\int_{\tilde{\Omega}} \nabla \tilde{\psi} \nabla \zeta \, d\mathcal{L}^2 = \int_{\mathbb{R}} \frac{\partial \tilde{\psi}}{\partial n}(\tilde{u}(s), \tilde{v}(s))\zeta(\tilde{u}(s), \tilde{v}(s))(\tilde{u}'(s)^2 + \tilde{v}'(s)^2)^{1/2} \, ds
\]

for all \( \zeta \in C_0^1(\mathbb{R}^2) \). \tag{5.29}

By comparing (5.27) and (5.29), we deduce that

\[
- \frac{\partial \tilde{\psi}}{\partial n}(\tilde{u}(s), \tilde{v}(s))(\tilde{u}'(s)^2 + \tilde{v}'(s)^2)^{1/2} = (-2g\tilde{v}(s))^{1/2} \quad \text{for a.e. } s \in \mathbb{R}. \tag{5.30}
\]

It is a consequence of (5.13) that

\[
\varepsilon_j^{1/2} \gamma(\varepsilon_j^{3/2} \psi^j) \to 0 \quad \text{in } C^{0,\alpha}(K) \quad \text{for any compact set } K \subset \mathbb{R}^2. \tag{5.31}
\]

Since (5.11a), (5.28), (5.25) and (5.31) hold, a standard elliptic estimate [15, Theorem 4.6] shows that

\[
\psi^j \to \tilde{\psi} \quad \text{in } C^{2,\alpha}_{\text{loc}}(\tilde{\Omega}). \tag{5.32}
\]
Note now that (5.3) yields, for all \( j \geq 1, \)
\[
|\nabla \psi_j(X,Y)|^2 + 2gY + \frac{2}{\varepsilon_j} \tilde{\Gamma}(\varepsilon_j^{3/2} \psi_j(X,Y)) - 2\varepsilon_j^{1/2} \varpi \psi_j(X,Y) \leq 0
\]
for all \((X,Y) \in \Omega_j.\) (5.33)

We deduce from (5.32) and (5.33) that
\[
|\nabla \tilde{\psi}(X,Y)|^2 + 2gY \leq 0 \quad \text{for all } (X,Y) \in \tilde{\Omega}.
\] (5.34)

Since \( \tilde{\psi} = 0 \) on \( \tilde{S}, \) it follows, by using (5.34) and Proposition 2.5, upon taking into account (5.21), that
\[
0 \leq -\frac{\partial \tilde{\psi}}{\partial n}(\tilde{u}(s),\tilde{v}(s)) \leq (-2g\tilde{v}(s))^{1/2} \quad \text{for almost every } s \in \mathbb{R}.
\] (5.35)

It follows from (5.30), (5.34) and (5.35) that
\[
\tilde{u}'(s)^2 + \tilde{v}'(s)^2 = 1 \quad \text{for almost every } s \in \mathbb{R},
\]
\[
\frac{\partial \tilde{\psi}}{\partial n}(X,Y) = -(-2gY)^{1/2} \quad \text{for } \mathcal{H}^1\text{-almost every } (X,Y) \in \tilde{S}.
\] (5.37)

This completes the proof of the fact that \((\tilde{S},\tilde{\psi})\) is a solution of (5.4).

If \( q \neq -\infty, \) then obviously \( \tilde{v}(1) = q\tilde{u}(1), \) while if \( q = -\infty, \) then \( \tilde{u}(1) = 0. \)

If \( S \) is symmetric, the fact that \(-\infty \notin Q\) is an immediate consequence of the fact that \( \sigma = 0. \) This completes the proof of Theorem 5.5.

Proof of Theorem 5.6. We first show how solutions of (5.4) can be described by solutions of the nonlinear integral equation (5.52). The required result is then obtained by invoking a uniqueness result from [38] for the integral equation. In the process of deriving (5.52) we also give a theory of (not necessarily symmetric) solutions of (5.4), concerning the reduction of this free-boundary problem to a problem in a fixed domain and on the local regularity of solutions. Whilst problem (5.4) appears not to have been studied before, there are obvious similarities to problem (1.1) for irrotational waves of finite or infinite depth, treatments of Hardy-space solutions of which have been given in [28, 38, 39, 40]. To avoid inessential technicalities, proofs of results for (5.4) are sometimes not given in situations where they would be obtainable by routine modifications from proofs in [28, 38, 39, 40].

Let \((\tilde{S},\tilde{\psi})\) be any nontrivial solution of (5.4). It follows that (5.4h) holds in the form
\[
\tilde{\psi} > 0 \quad \text{on } \tilde{\Omega} \quad \text{and} \quad \tilde{\psi}_Y < 0 \quad \text{on } \tilde{\Omega}.
\] (5.38)

Since the non-tangential boundary values of any bounded holomorphic function in a bounded open set whose boundary is a rectifiable Jordan curve cannot vanish on a set of positive \( \mathcal{H}^1 \) measure unless the function is identically 0, it follows from (5.4d), (5.4j) and (5.38) that
\[
\tilde{v}(s) < 0 \quad \text{for all } s \in \mathbb{R} \setminus \{0\}.
\] (5.39)
Let us denote
\[ \tilde{S}_+ := \{(\tilde{u}(s), \tilde{v}(s)) : s \in (0, \infty)\} \quad \text{and} \quad \tilde{S}_- := \{(\tilde{u}(s), \tilde{v}(s)) : s \in (-\infty, 0)\}. \]

Let \( W_0 : \mathbb{C}_+ \to \tilde{\Omega} \) be a conformal mapping from the upper half-plane \( \mathbb{C}_+ \) onto \( \tilde{\Omega} \). By Carathéodory’s Theorem, \( W_0 \) has an extension as a homeomorphism between the closures in the extended complex plane of these domains. It is also classical that \( W_0 \) can be chosen such that it maps the origin onto itself, the positive real axis onto \( \tilde{S}_- \) and the negative real axis onto \( \tilde{S}_+ \). Then \( \tilde{\psi} \circ W_0 \) is a positive harmonic function in the upper half-plane and continuous on its closure, with \( \tilde{\psi} \circ W_0 = 0 \) on the real line. Hence there exists \( c > 0 \) such that
\[ (\tilde{\psi} \circ W_0)(z) = cy \quad \text{for all} \quad z = x + iy \in \mathbb{C}_+. \]

Let \( \tilde{\varphi} \) be a harmonic conjugate of \(-\tilde{\psi}\) in \( \tilde{\Omega} \), so that the function \( \tilde{\omega} := \tilde{\varphi} + i\tilde{\psi} \) is holomorphic in \( \tilde{\Omega} \) and satisfies
\[ (\tilde{\omega} \circ W_0)(z) = z \quad \text{for all} \quad z \in \mathbb{C}_+. \]

Let \( W : \mathbb{C}_+ \to \tilde{\Omega} \) be given by \( W(z) := W_0(c^{-1}z) \) for all \( z \in \mathbb{C}_+ \). Then \( W \) has the same conformal mapping properties as \( W_0 \), and \( \tilde{\omega} \) is the inverse conformal mapping of \( W \). Let us write, for all \( x + iy \in \mathbb{C}_+ \),
\[
\begin{align*}
W(x + iy) &= U(x, y) + iV(x, y), \\
W'(x + iy) &= -\exp(\tau(x, y) + i\theta(x, y)) \tag{5.40}
\end{align*}
\]
where \( \tau \) and \( \theta \) are harmonic functions on \( \mathbb{R}^2_+ \). It follows from (5.38) that
\[ -\frac{\pi}{2} < \theta(x, y) < \frac{\pi}{2} \quad \text{for all} \quad (x, y) \in \mathbb{R}^2_+. \tag{5.42} \]
The M. Riesz Theorem now implies that \( \tau \in h^p(\mathbb{R}^2_+) \) for all \( p \in (1, \infty) \). Therefore \( \tau \) and \( \theta \) have non-tangential boundary values almost everywhere on the real line, from which they can be recovered by Poisson Formula and which are related to one another by the Hilbert transform.

For any \( x_0 \in (0, \infty) \), let \( X_0 + iY_0 := W(x_0 + i0) \), so that \( Z_0 := X_0 + iY_0 \) is located on \( \tilde{S}_- \). Let \( Z_1 \) and \( Z_2 \) be located on \( \tilde{S}_- \) such that \( Z_0 \) is situated between \( Z_1 \) and \( Z_2 \) and that there exist non-tangential limits of \( \nabla \psi \) at \( Z_1 \) and \( Z_2 \). Let \( \mathcal{G} \) be a subdomain of \( \tilde{\Omega} \) such that the boundary of \( \mathcal{G} \) is a rectifiable Jordan curve \( J := \mathcal{I} \cup \mathcal{L} \), where \( \mathcal{I} \) is the arc of \( \tilde{\mathcal{S}} \) joining \( Z_1 \) and \( Z_2 \), and \( \mathcal{L} \) is an arc contained in \( \tilde{\mathcal{S}} \) joining \( Z_1 \) and \( Z_2 \) and which approaches \( \tilde{\mathcal{S}} \) non-tangentially at \( Z_1 \) and \( Z_2 \). By (5.4) and the construction of \( \mathcal{G} \), the non-tangential boundary values of the harmonic function \( \tau \circ W \) in \( h^p(\mathcal{G}) \) are essentially bounded, and therefore \( \tau \circ W \) is bounded in \( \mathcal{G} \). It follows that there exists a rectangle \( \Pi := (x_0 - \epsilon, x_0 + \epsilon) \times (0, \delta) \) in \( \mathbb{R}^2_+ \), where \( 0 < \epsilon < x_0 \) and \( \delta > 0 \), in which \( \tau \) is bounded. This shows that the partial derivatives of \( U, \ V \) in (5.40) are bounded in \( \Pi \), and therefore have non-tangential limits almost everywhere on \((x_0 - \epsilon, x_0 + \epsilon) \times \{0\} \). Since \( x_0 \in (0, \infty) \) was arbitrary, it follows that the partial derivatives of \( U, \ V \) have non-tangential
limits almost everywhere on the positive real axis. A similar statement can be made for the negative real axis.

By arguing as in [40, Lemma 4.2], we deduce that the mapping \( t \mapsto W(t+i0) \) is locally absolutely continuous on each of the intervals \((0, \infty)\) and \((-\infty, 0)\), and

\[
\frac{d}{dt}W(t+i0) = \lim_{(x,y) \to (t,0)} W'(x+iy) \quad \text{for almost every } t \in \mathbb{R},
\]

the above limit being taken non-tangentially within \( \mathbb{R}^2_+ \). But since the mappings \( t \mapsto U(t,0), t \mapsto V(t,0) \) are monotone on \([0, \infty)\) and on \((-\infty, 0)\), it follows that \( t \mapsto W(t+i0) \) is locally absolutely continuous on \( \mathbb{R} \).

For any harmonic function \( \xi \) in \( \mathbb{R}^2_+ \) which has non-tangential limits almost everywhere on the real axis, we use from now on the notation \( t \mapsto \xi(t) \) instead of either \( t \mapsto \xi(t,0) \) or \( t \mapsto \xi(t+i0) \) to denote the boundary values of \( \xi \).

We deduce from the free boundary condition (5.4j) that

\[
|W'(t)|^2(-2gV(t)) = 1 \quad \text{for almost every } t \in \mathbb{R},
\]

and therefore

\[
\tau(t) = -\log\{-2gV(t)^{1/2}\} \quad \text{for almost every } t \in \mathbb{R}.
\]

It is also obvious that, for almost every \( t \in \mathbb{R} \),

\[
-U'(t) = \frac{\cos \theta(t)}{(-2gV(t))^{1/2}}, \quad -V'(t) = \frac{\sin \theta(t)}{(-2gV(t))^{1/2}}.
\]

(Note that by (5.43) the notation \( U'(t), V'(t) \), for almost every \( t \in \mathbb{R} \), is unambiguous.) It follows that \( \theta(t) \) gives the angle between the tangent to the curve \( \tilde{S} \) at the point \((U(t),V(t))\) and the horizontal, for almost every \( t \in \mathbb{R} \). Note also that a consequence of the fact that \( \tau \in h^p_\Omega(\mathbb{R}^2_+) \) for all \( p \in (1, \infty) \) is that

\[
\int_\mathbb{R} |\tau(w)|^p \left( \frac{1}{1+w^2} \right) dw < +\infty \quad \text{for all } p \in (1, \infty).
\]

By a bootstrap argument as in [41, Theorem 3.5], see also [38, Theorem 2.3], we deduce that \( W, \tau, \theta \in C^\infty(\mathbb{R}^2_+ - \{(0,0)\}) \), which implies that \( \tilde{S}_+ \) and \( \tilde{S}_- \) are \( C^\infty \) curves and \( \tilde{\psi} \in C^\infty(\Omega \cup \tilde{S}_+ \cup \tilde{S}_-) \). A classical result of Lewy [21] then shows that \( \tilde{S}_+ \) and \( \tilde{S}_- \) are real-analytic curves, and \( \tilde{\psi} \) has a harmonic extension across \( \tilde{S}_+ \) and \( \tilde{S}_- \).

Integrating the second relation in (5.46) written in the form

\[
-V'(t)(-2gV(t))^{1/2} = \sin \theta(t) \quad \text{for almost every } t \in \mathbb{R},
\]

we obtain, since \( V(0) = 0 \), that

\[
(-2gV(y))^{1/2} = \frac{1}{(3g)^{1/3}} \left( \int_0^y \sin \theta(w) \, dw \right)^{1/3} \quad \text{for all } y \in \mathbb{R}.
\]
The geometric properties of $\tilde{S}$ expressed by (5.4c) and (5.4d) imply that

$$0 \leq \theta \leq \pi/2 \text{ on } (0, \infty) \quad \text{and} \quad -\pi/2 \leq \theta \leq 0 \text{ on } (-\infty, 0).$$

Moreover, note from (5.39) that $V(y) < 0$ for all $y \neq 0$. This means, in view of (5.48), that

$$\int_{y}^{\infty} \sin \theta(w) \, dw > 0 \quad \text{for all } y \neq 0.$$  

(5.50)

Suppose now that $\tilde{S}$ and $\tilde{\psi}$ are symmetric with respect to the line $X = 0$. It follows that $\tau$ is an even function and $\theta$ is an odd function on $\mathbb{R}$. The definition of a Hilbert transform then shows that

$$\theta(x) = \frac{1}{3\pi} \int_{0}^{\infty} \log \left| \frac{x+y}{x-y} \right| \{ -\tau'(y) \} \, dy \quad \text{for all } x \in (0, \infty).$$

(5.51)

Note from (5.45) and (5.48) that $\tau(y)/y \to 0$ as $y \to \infty$. Using this fact, (5.47) and the monotonicity of $\tau$ on $(0, \infty)$, an integration by parts (the validity of which can be justified as in [38, Proof of Proposition 4.3]) shows that

$$\theta(x) = \frac{1}{3\pi} \int_{0}^{\infty} \log \left| \frac{x+y}{x-y} \right| \frac{\sin \theta(y)}{\int_{y}^{\infty} \sin \theta(w) \, dw} \, dy \quad \text{for all } x \in (0, \infty).$$

(5.52)

The following result, which is [38, Theorem 4.5], is the key to the proof of Theorem 5.10.

**Theorem 5.10.** The only solution $\theta : (0, \infty) \to \mathbb{R}$ of (5.52) with $0 \leq \theta \leq \pi/2$ on $(0, \infty)$ and such that

$$0 < \inf_{x \in (0, \infty)} \theta(x)$$

is the function $\theta^* : (0, \infty) \to \mathbb{R}$ given by $\theta^*(x) = \pi/6$ for all $x \in (0, \infty)$.

The following new result shows that (5.53) is in fact not a restriction in Theorem 5.10.

**Proposition 5.11.** Let $\theta : (0, \infty) \to \mathbb{R}$ be any solution of (5.52) with $0 \leq \theta \leq \pi/2$ on $(0, \infty)$ and such that

$$\int_{0}^{y} \sin \theta(w) \, dw > 0 \quad \text{for all } y \in (0, \infty).$$

(5.54)

Then $\theta$ satisfies (5.53).
Proof of Proposition 5.11. It is obvious that
\[ \theta(x) \geq \frac{1}{3\pi} \int_{0}^{x} \log \left| \frac{x+y}{x-y} \right| \frac{1}{y} \sin(\theta(y)) \, dy \quad \text{for all } x \in (0, \infty). \]
Since for every \( x, y \in (0, \infty) \) with \( 0 < y < x \), the following inequality holds:
\[ \log \left| \frac{x+y}{x-y} \right| \geq \frac{2}{x}, \]
it follows that
\[ \theta(x) \geq \frac{2}{3\pi x} \int_{0}^{x} \sin(\theta(y)) \, dy \quad \text{for all } x \in (0, \infty). \]
From this it is immediate that
\[ \sin(\theta(y)) \geq \frac{4}{3\pi^2} \frac{1}{y} \int_{0}^{y} \sin(\theta(w)) \, dw \quad \text{for all } y \in (0, \infty). \]
We now deduce from (5.52) that
\[ \theta(x) \geq \frac{4}{9\pi} \int_{0}^{\infty} \log \left| \frac{x+y}{x-y} \right| \frac{1}{y} \, dy = \frac{2}{9\pi} \quad \text{for all } x \in (0, \infty), \]
which proves (5.53).

Since (5.50) and (5.49) hold, it follows from Theorem 5.10 and Proposition 5.11 that, for any symmetric nontrivial solution \((\tilde{S}, \tilde{\psi})\) of (5.4), the function \( \theta \) associated to it necessarily coincides with \( \theta^* \), the constant function \( \pi/6 \). It is then straightforward that \((\tilde{S}, \tilde{\psi})\) coincides with \((\tilde{S}^*, \tilde{\psi}^*)\) given by (5.6)-(5.7). This completes the proof of Theorem 5.6.

Proof of Proposition 5.8. We use the following particular case of a result of Oddson [25].

Proposition 5.12. Let \( r_0 > 0 \) and \( \mu > 1 \). Let
\[ \mathcal{G} := \{r e^{it} : 0 < r < r_0, |t| < \pi/(2\mu)\}. \]
Let \( w \in C^2(\mathcal{G}) \cap C(\overline{\mathcal{G}}) \) be a superharmonic function in \( \mathcal{G} \), such that \( w(0,0) = 0 \) and \( w > 0 \) in \( \mathcal{G} \setminus \{0,0\} \). Then there exists \( \kappa > 0 \) such that
\[ w(re^{it}) \geq \kappa r^{\mu} \cos \mu t \quad \text{in } \overline{\mathcal{G}}. \]
Suppose for a contradiction that \( \Omega \) contains such a truncated cone. Then there exist \( r_0 > 0 \) and \( \alpha_1, \alpha_2 \) with \( -\pi \leq \alpha_1 < \alpha_2 \leq 0 \) and \( \alpha_2 - \alpha_1 > 2\pi/3 \), such that \( \mathcal{G} \setminus \{(0,0)\} \subset \Omega_0 \), where \( \mathcal{G} := \{r e^{it} : 0 < r < r_0, \alpha_1 < t < \alpha_2\} \) and \( \Omega_0 := \{(X,Y) \in \Omega : 0 < \psi(X,Y) < \delta\} \). Since \( \psi \) is superharmonic in \( \mathcal{G} \),
\( \psi(0, 0) = 0 \) and \( \psi > 0 \) in \( \overline{G} \setminus \{(0, 0)\} \), Proposition 5.12 shows that there exists \( \kappa_0 > 0 \) such that

\[
\psi(0, Y) \geq \kappa_0 |Y|^\mu \quad \text{for all } Y \in (-r_0, 0),
\]

where \( \mu := \pi/(\alpha_2 - \alpha_1) \), so that \( \mu < 3/2 \). But this contradicts the estimate, see (5.8),

\[
|\nabla \psi(0, Y)|^2 \leq K|Y| \quad \text{for all } Y \text{ such that } (0, Y) \in \Omega,
\]

which is a consequence of the assumption \( T[\psi] \leq 0 \). This completes the proof of Proposition 5.8. \( \Box \)

**Proof of Theorem 5.9.** Let \( Q \) be given by (5.5). Obviously, \( Q \) is a closed subinterval of \([−\infty, 0]\). Since \( S \) and \( \psi \) are symmetric, it is immediate from Theorem 5.5 and Theorem 5.6 that \( Q \) is a subset of \([0, -1/\sqrt{3}]\). Hence either \( Q = \{0\} \) or \( Q = \{-1/\sqrt{3}\} \). When \( \gamma(r) \geq 0 \) for all \( r \in [0, \delta] \), the possibility that \( Q = \{0\} \) is ruled out by Proposition 5.8. This completes the proof of Theorem 5.2. \( \Box \)

**Proof of Theorem 5.9.** Suppose first that \( q_+ \neq \infty \) and \( q_- \neq \infty \). Let \((\tilde{S}, \tilde{\psi})\) be the solution of (5.4) whose existence is given by Theorem 5.5. Moreover, the proof of Theorem 5.5 shows that necessarily \( \tilde{S} = \{(X, \tilde{\psi}(X)) : X \in \mathbb{R}\} \), where

\[
\tilde{\psi}(X) := \begin{cases} 
- q_+ |X| & \text{for all } X \in [0, \infty), \\
- q_- |X| & \text{for all } X \in (-\infty, 0].
\end{cases}
\]

We now ask for what values of \( q_\pm \) there exist solutions \( \tilde{\psi} \) of (5.4) in the domain \( \tilde{\Omega} \) below the curve \( \tilde{S} \) described above. It is easy to see that, if \( \alpha_\pm := \arctan q_\pm \), then the only solutions of (5.4) are given, for all \((X, Y) \in \tilde{\Omega}\), by

\[
\tilde{\psi}(X, Y) := \beta \text{Im} \left[ i \left( ie^{i(\alpha_+ - \alpha_-)/2} Z \right)^{\pi/(\pi - (\alpha_+ + \alpha_-))} \right],
\]

where \( Z = X + iY \) and \( \beta \geq 0 \). It is straightforward to check that, apart from the cases when either \( q_\pm = 0 \) or \( q_\pm = 1/\sqrt{3} \), none of the above functions \( \tilde{\psi} \) satisfies (5.4). When \( q_\pm = 0 \), the only solution of (5.4) of the above type is \( \tilde{\psi}_0 := 0 \) in \( \tilde{\Omega} \). When \( q_\pm = 1/\sqrt{3} \), the only solution of (5.4) of the above type is the function \( \tilde{\psi}^* \) given by (5.4).

If \( q_+ \neq \infty \) and \( q_- = \infty \) then, for the solution \((\tilde{S}, \tilde{\psi})\) of (5.4) given by Theorem 5.5, \( \tilde{S} \) necessarily consists of the negative imaginary axis and the half-line \( \{(X, -q_+ X) : X \geq 0\} \). Arguing as before, a contradiction is reached. A similar argument shows that it is also not possible that \( q_+ = \infty \) and \( q_- \neq \infty \).

The possibility that \( q_\pm = \infty \) is ruled out by the argument used to show that \( \sigma = 0 \) in the proof of Theorem 5.6.

We conclude that necessarily either \( q_\pm = 1/\sqrt{3} \) or \( q_\pm = 0 \). When \( \gamma(r) \geq 0 \) for all \( r \in [0, \delta] \), the possibility that \( q_\pm = 0 \) is ruled out by Proposition 5.8. This completes the proof of Theorem 5.9. \( \Box \)
6 Appendix

We recall the definition of a non-tangential limit and some notions and results concerning the classical Hardy spaces of harmonic functions. More details can be found in [12, 20, 27]. In what follows, $\mathcal{D}$ denotes the unit disc in the plane and $\mathcal{D}_\pm := \mathcal{D} \cap \mathbb{R}_-^2$.

Let $\mathcal{G}$ be an open set in the plane. Let $(X_0, Y_0) \in \partial \mathcal{G}$ be such that there exist an open set $\mathcal{U}$ containing $(X_0, Y_0)$ and a homeomorphism $h : \mathcal{D} \to \mathcal{U}$ such that $h(\mathcal{D}_+ \cap \mathcal{D}) = \mathcal{G} \cap \mathcal{U}$, $h((-1, 1) \times \{0\}) = \partial \mathcal{G} \cap \mathcal{U}$ and the curve $\partial \mathcal{G} \cap \mathcal{U}$ has a tangent at $(X_0, Y_0)$. Let $n$ be the unit inner normal to $\mathcal{G}$ at $(X_0, Y_0)$. We say that a sequence $\{(X_n, Y_n)\}_{n \geq 1}$ of points in $\mathcal{G}$ tends to $(X_0, Y_0)$ non-tangentially if $(X_n, Y_n) \to (X_0, Y_0)$ as $n \to \infty$ and there exists $\kappa > 0$ such that

$$(X_n - X_0, Y_n - Y_0) \cdot n \geq \kappa [(X_n - X_0)^2 + (Y_n - Y_0)^2]^{1/2} \quad \text{for all } n \geq 1,$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^2$. Let $f : \mathcal{G} \to \mathbb{C}$ and $l \in \mathbb{C}$. We say that $f$ has non-tangential limit $l$ at $(X_0, Y_0)$ if $\lim_{n \to \infty} f(X_n, Y_n) = l$ for every sequence $\{(X_n, Y_n)\}_{n \geq 1}$ which tends to $(X_0, Y_0)$ non-tangentially.

For $p \in [1, \infty)$, the Hardy space $H^p_c(\mathcal{D})$ is usually defined as the class of harmonic functions $f : \mathcal{D} \to \mathbb{C}$ with the property that

$$\sup_{r \in (0, 1)} \int_{-\pi}^{\pi} |f(re^{it})|^p \, dt < +\infty. \quad (6.1)$$

The Hardy space $H^p_c(\mathcal{D})$ is the class of bounded harmonic functions in $\mathcal{D}$. For $p \in [1, \infty]$, the Hardy space $H^p_c(\mathcal{D})$ is the class of holomorphic functions in $H^p_c(\mathcal{D})$. Fatou’s Theorem states that any function in $H^p_c(\mathcal{D})$, $p \in [1, \infty]$, has non-tangential limits almost everywhere on the unit circle. The boundary values of any function in $H^p_c(\mathcal{D})$ cannot vanish on a set of positive measure unless the function is identically 0 in $\mathcal{D}$. The M. Riesz Theorem [12, Theorem 4.1] states that, if $u \in H^p_c(\mathcal{D})$ for some $p \in (1, \infty)$, and if $v$ is a harmonic function such that $u + iv$ is holomorphic, then $v \in H^p_c(\mathcal{D})$. For any function $f : \mathcal{D} \to \mathbb{C}$, the radial maximal function $M_{\text{rad}}[f]$ is defined [27, Definition 11.19] by

$$M_{\text{rad}}[f] := \sup_{r \in [0, 1)} |f(re^{it})| \quad \text{for all } t \in \mathbb{R}.$$

If $f \in H^p_c(\mathcal{D})$, where $p \in [1, \infty]$ then [27, Theorem 7.11] shows that $M_{\text{rad}}[f] \in L^p_{2\pi}$, the space of $2\pi$-periodic functions in $L^p_{\text{loc}}(\mathbb{R})$.

The definition of Hardy spaces in general domains [12, Ch. 10] is based on the fact that, for $p \in [1, \infty]$, a harmonic function $f$ belongs to $H^p_c(\mathcal{D})$ if and only if the subharmonic function $|f|^p$ has a harmonic majorant, i.e. there exists a positive harmonic function $w$ in $\mathcal{D}$ such that $|f|^p \leq w$ in $\mathcal{D}$. Let $\mathcal{G}$ be an open set. For $p \in [1, \infty]$, the space $H^p_c(\mathcal{G})$ is the class of harmonic functions $f : \mathcal{G} \to \mathbb{C}$ for which the subharmonic function $|f|^p$ has a harmonic majorant in $\mathcal{G}$. The Hardy space $h^p_c(\mathcal{G})$ is the class of bounded harmonic functions in $\mathcal{G}$. The spaces $H^p_c(\mathcal{G})$ consists of the holomorphic functions in $h^p_c(\mathcal{G})$, for $p \in [1, \infty]$. It is easy to check that the Hardy spaces are conformally invariant: if $\mathcal{G}_1$ and
\( \mathcal{G}_2 \) are open sets, and \( \sigma : \mathcal{G}_1 \to \mathcal{G}_2 \) is a conformal mapping, then \( f \in h^p_\mathcal{C}(\mathcal{G}_2) \) if and only if \( f \circ \sigma \in h^p_\mathcal{C}(\mathcal{G}_1) \), where \( p \in [1, \infty) \). For this reason, many properties of the Hardy spaces of the disc extend by conformal mapping to Hardy spaces of simply connected domains. If \( \mathcal{G} \) is a bounded open set whose boundary is a rectifiable Jordan curve, then any function in \( h^p_\mathcal{C}(\mathcal{G}) \), where \( 1 \leq p \leq \infty \), has non-tangential boundary values \( \mathcal{H}^1 \)-almost everywhere. A consequence of this is the existence of non-tangential boundary values \( \mathcal{H}^1 \)-almost everywhere for functions in \( h^p_\mathcal{C}(\mathcal{G}) \), \( 1 \leq p \leq \infty \), for any open set \( \mathcal{G} \) with the following property: for any \( (X_0, Y_0) \in \partial \mathcal{G} \) there exist an open set \( \mathcal{U} \) containing \( (X_0, Y_0) \) and a homeomorphism \( h : \mathcal{D} \to \mathcal{U} \) such that \( h(\mathcal{D}_+) = \mathcal{G} \cap \mathcal{U} \), \( h((-1, 1) \times \{0\}) = \partial \mathcal{G} \cap \mathcal{U} \) and the curve \( \partial \mathcal{G} \cap \mathcal{U} \) is rectifiable. If \( \mathcal{G} \) is a bounded open set whose boundary is a rectifiable Jordan curve, then the non-tangential boundary values of any function in \( \mathcal{H}^1_\mathcal{C}(\mathcal{G}) \) cannot vanish on a set of positive \( \mathcal{H}^1 \) measure unless the function is identically 0 in \( \mathcal{G} \).

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