Abstract—The problem of finding a unique low dimensional decomposition of a given matrix has been a fundamental and recurrent problem in many areas. In this paper, we study the problem of seeking a unique decomposition of a low rank matrix $Y \in \mathbb{R}^{p \times n}$ that admits a sparse representation. Specifically, we consider $Y = AX$ where the matrix $A \in \mathbb{R}^{p \times r}$ has full column rank, with $r < \min\{n, p\}$, and the matrix $X \in \mathbb{R}^{r \times n}$ is element-wise sparse. We prove that this low rank, sparse decomposition of $Y$ can be uniquely identified, up to some intrinsic signed permutation. Our approach relies on solving a nonconvex optimization problem constrained over the unit sphere. Our geometric analysis for its nonconvex optimization landscape shows that any strict local solution is close to the ground truth, and can be recovered by a simple data-driven initialization followed with any second order descent algorithm. Our theoretical findings are corroborated by numerical experiments.

Index Terms—Matrix factorization, low-rank decomposition, nonconvex optimization, second-order geometry, sparse representation, unsupervised learning.

I. INTRODUCTION

THE problem of matrix decomposition has been a popular and fundamental topic under extensive investigations across several disciplines, including signal processing, machine learning, natural language processing [7], [9], [10], [31], [32], [47], just to name a few. From the decomposition, one can construct useful representation of the original data matrix. However, for any matrix $Y \in \mathbb{R}^{p \times n}$ that can be factorized as a product of two matrices $A \in \mathbb{R}^{p \times r}$ and $X \in \mathbb{R}^{r \times n}$, there exist infinitely many decompositions, simply because one can use any $r \times r$ invertible matrix $Q$ to construct $A' = AQ$ and $X' = Q^{-1}X$ such that $Y = AX = A'X'$, while $A' \neq A$ and $X' \neq X$. Thus, in various applications, additional structures and priors are imposed for a unique representation [14], [21]. For example, principal component analysis (PCA) aims to find orthogonal representations which retain as much variations in $Y$ as possible [16], [22], whereas independent component analysis (ICA) targets the representations of statistically independent non-Gaussian signals [25].

In this paper, we are interested in finding a unique, sparse, low-dimensional representation of $Y$. To this end, we study the decomposition of a low rank matrix $Y \in \mathbb{R}^{p \times n}$ that satisfies

$$Y = AX,$$  

where $A \in \mathbb{R}^{p \times r}$ is an unknown deterministic matrix, with $r < \min\{n, p\}$, and $X \in \mathbb{R}^{r \times n}$ is an unknown sparse matrix.

Finding a unique sparse decomposition of (I.1) turns out to be important in many applications. For example, if the matrix $X$ is viewed as a low-dimensional sparse representation of $Y$, finding such representation is ubiquitous in signal recovery, image processing and compressed sensing, just to name a few. If columns of $Y$ are viewed as linear combinations of columns of $A$ with $X$ being the sparse coefficient, then (I.1) can be used to form overlapping clusters of the $n$ columns of $Y$ via the support of $X$ with columns of $A$ being viewed as $r$ cluster centers [6], [11]. Furthermore, we could form a $p \times r$ low-dimensional representation of $Y$ via linear combinations with sparse coefficients. Such sparse coefficients greatly enhance the interpretability of the resulting representations [3], [20], [27], in the same spirit as the sparse PCA, but generalizes to the factorization of non-orthogonal matrices.

To motivate our approach, consider the simple case that $A$ has orthonormal columns, namely, $A^TA = I_r$. Then it is easy to see that the sparse coefficient matrix $X$ is recovered by multiplying $Y$ on the left by $A^T$,

$$A^TY = A^TAX = X.$$  

(I.2)

The problem of finding such orthonormal matrix $A$ boils down to successively finding a unit-norm direction $q$ that renders $q^TY$ as sparse as possible [34], [35], [39],

$$\min_q \|q^TY\|_{\text{sparsity}} \quad \text{s. t.} \quad \|q\|_2 = 1.\tag{I.3}$$

However, the natural choice of the sparsity penalty, such as the $\ell_p$ norm for $p \in [0, 1]$, leads to trivial and meaningless solutions, as there always exists $q$ in the null space of $A^T$ such that $q^TY = 0$.

To avoid the null space of $A^T$, we instead choose to find the unit direction $q$ that maximizes the $\ell_4$ norm of $q^TY$ as

$$\max_q \|q^TY\|_4 \quad \text{s. t.} \quad \|q\|_2 = 1.\tag{I.4}$$
The above formulation is based on the key observation that the objective value is maximized when \( q \) coincides with one column of \( A \) (see, Section II, for details) while the objective value is zero when \( q \) lies in the null space of \( A^T \). The \( \ell_1 \) norm objective function and its variants have been adopted as a sparsity regularizer in a line of recent works [30], [35], [43], [44], [45], arguably because solving (I.4) requires a milder restriction on sparsity level of \( X \) to recover \( A \) comparing to solving (I.3). However, all previous works study the setting where \( A \) has full row rank and, to the best of our knowledge, the setting where \( A \) has full column rank has not been studied elsewhere. As we will elaborate below, when \( A \) has full column rank, analysis of the optimization landscape of (I.4) becomes more difficult since the null space of \( A^T \) persists as a challenge for solving the optimization problem: they form a flat region of local optimal solutions.

This paper characterizes the nonconvex optimization landscape of (I.4) and proposes a guaranteed procedure that avoids the flat null region and provably recovers the global solution to (I.4), which corresponds to one column of \( A \). More specifically, we demonstrate that, despite the nonconvexity, (I.4) still possesses benign geometric property in the sense that any strict local solution with large objective value is globally optimal and recovers one column of \( A \) up to its sign. See, Theorem 3.1 in Section III-A for the population level result and Theorem 3.4 for the finite sample result.

We further extend these results to the general case when \( A \) only has full column rank in Theorem 3.6 of Section III-B. To recover a general \( A \) with full column rank, our procedure first resorts to a preconditioning procedure of \( Y \) proposed in Section II-C and then solves an optimization problem similar to (I.4). From our analysis of the optimization landscape, the intriguing problem boils down to developing algorithms to recover the nontrivial local solutions by avoiding regions with small objective values. We thus propose a simple initialization scheme in Section IV-A and prove in Theorem 4.3 that such initialization, proceeded with any second order descent algorithm [19], [26], suffices to find the global solution, up to some statistical error. Our theoretical analysis provides explicit rates of convergence of the statistical error and characterizes the dependence on various dimensions, such as \( p, r \) and \( n \), as well as the sparsity of \( X \).

Numerical simulation results are provided in Section V. Finally in Section VI, we conclude our results and discuss several future directions of our work. All the proofs and supplementary simulations are deferred to the Appendix.

Notations: Throughout this paper, we use bold lowercase letters, like \( a \), to represent vectors and bold uppercase letters, like \( A \), to represent matrices. For matrix \( X \), \( X_{ij} \) denotes the entry at the \( i \)-th row and \( j \)-th column of \( X \), with \( X_i \) and \( X_j \) denoting the \( i \)-th row and \( j \)-th column of \( X \), respectively. Oftentimes, we write \( X_{ij} = X_{ji} \) for simplicity. We use \( grad \) and \( Hess \) to represent the Riemannian gradient and Hessian. For any vector \( v \in \mathbb{R}^d \), we use \( \|v\|_q \) to denote its \( \ell_q \) norm, for \( 1 \leq q \leq \infty \). The notation \( v^{eq} \) stands for \( \{v_{i,j}^q\} \). For matrices, we use \( \|\cdot\|_F \) and \( \|\cdot\|_{op} \) to denote the Frobenius norm and the operator norm, respectively. For any positive integer \( d \), we write \( \{d\} = \{1, 2, \ldots, d\} \). The unit sphere in \( d \)-dimensional real space \( \mathbb{R}^d \) is written as \( S^{d-1} \). For two sequences \( a_n \) and \( b_n \), we write \( a_n \lesssim b_n \) if there exists some constant \( C > 0 \) such that \( a_n \leq C b_n \) for all \( n \). Both uppercase \( C \) and lowercase \( c \) are reserved to represent numerical constants, whose values may vary line by line.

A. Related Work

Finding the unique factorization of a matrix is an ill-posed problem in general due to infinitely many solutions.

There exist several strands of studies from different contexts on finding the unique decomposition of \( Y \) by imposing additional structures on \( A \) and \( X \). We start by reviewing the literature which targets the sparse decomposition of \( Y \).

1) Dictionary Learning: The problems of dictionary learning (DL) [1], [18], [38], [39] and sparse blind deconvolution or convolutional dictionary learning [12], [29] study the unique decomposition of \( Y = AX \) where \( X \) is sparse and \( A \) has full row rank. In this case, the row space of \( Y \) lies in the row space of \( X \), suggesting to recover the sparse rows of \( X \) via solving the following problem,

\[
\min_{q} \|q^T Y\|_1 \quad \text{s.t.} \quad q \neq 0. \quad (I.5)
\]

Under certain scaling and incoherence conditions on \( A \), the objective achieves the minimum value when \( q \) is equal to one column of \( A \), at the same time \( q^T Y \) recovers one sparse row of \( X \). This idea has been considered and generalized in a strand of papers when \( A \) has full row rank [30], [35], [37], [38], [39], [43], [45], [46], [48]. In our context, the major difference rises in the matrix \( A \), which has full column rank rather than row rank, therefore minimizing \( \|q^T Y\|_1 \) as before only leads to some vector in the null space of \( A^T \), yielding the trivial zero objective value.

On the other hand, [35], [44] consider the same objective function in (I.4) to study the problem of overcomplete/complete dictionary learning (where \( A \) has full row rank). However the optimization landscape when \( A \) has full column rank is significantly different from that in the (over)complete setting. The more complicated optimization landscape in our setting brings additional difficulty of the analysis and requires a proper initialization in our proposed algorithm. We refer to Appendix A for detailed technical comparison with [35] and [44].

2) Sparse PCA: Sparse principal component analysis (SPCA) is a popular method that recovers a unique decomposition of a low-rank matrix \( Y \) by utilizing the sparsity of its singular vectors. However, as being said, under \( Y = AX \), SPCA is only applicable when \( X \) coincides with the right singular vectors of \( Y \). Indeed, one formulation of SPCA is to solve

\[
\max_{U \in \mathbb{R}^{n \times r}} \text{tr} \left( U^T Y^T Y U \right) - \lambda \| U \|_1, \quad (I.6)
\]

s.t. \( U^T U = I_r \),

which is promising only if \( X \) corresponds to the right singular vectors of \( Y \). It is worth mentioning that among the various
approaches of SPCA, the following one might be used to recover one sparse row of $X$,
\[
\min_{u,v} \| Y - uv^T \|^2 + \lambda \|v\|_1 \quad \text{s.t.} \quad \|u\|_2 = 1. \quad (I.7)
\]
This procedure was originally proposed by [50] and [36] together with an efficient algorithm by alternating the minimization between $u$ and $v$. However, there is no guarantee that the resulting solution recovers the ground truth.

3) Factor Analysis: Factor analysis is a popular statistical tool for constructing low-rank representations of $Y$ by postulating
\[
Y = AX + E \quad (I.8)
\]
where $A \in \mathbb{R}^{p \times r}$ is the so-called loading matrix with $r = \text{rank}(A) < \min\{n, p\}$, $X \in \mathbb{R}^{r \times n}$ contains $n$ realizations of a $r$-dimensional factor and $E$ is some additive noise. Factor analysis is mainly used to recover the low-dimension column space of $A$ or the row space of $X$, rather than to identify and recover the unique decomposition. Recently, [6] studied the unique decomposition of $Y$ when the columns of $X$ are i.i.d. realizations of a $r$-dimensional latent random factor. The unique decomposition is further used for (overlapping) clustering the rows of $Y$ via the assignment matrix $A$.

To uniquely identify $A$, [6] assumes that $A$ contains at least one $r \times r$ identity matrix, coupled with other scaling conditions on $A$ (we refer to [6] for detailed discussions of other existing conditions in the literature of factor models that ensure the unique decomposition of $Y$ but require strong prior information on either $A$ or $X$). By contrast, we rely on the sparsity of $X$, which is more general than requiring the existence of a $r \times r$ identity matrix in $A$.

4) NMF and Topic Models: Such existence condition of identity matrix in either $A$ or $X$ has a variant in non-negative matrix factorization (NMF) [13] and topic models [2], [7], [8], also see the references therein, where $Y$, $A$ and $X$ have non-negative entries. Since all $Y$, $A$ and $X$ from model (I.1) are allowed to have arbitrary signs in our context, the approaches designed for NMF and topic models are inapplicable.

II. FORMULATION AND ASSUMPTIONS

The decomposition of $Y = AX$ is not unique without further assumptions. To ensure the uniqueness of such decomposition, we rely on two assumptions on the matrices $A$ and $X$, stated in Section II-A.

Our goal is to uniquely recover $A$ from $Y$, up to some signed permutation. More precisely, we aim to recover columns of $AP$ for some signed permutation matrix $P \in \mathbb{R}^{r \times r}$. To facilitate understanding, in Section II-B we first state our procedure for uniquely recovering $A$ when $A$ has orthonormal columns. Its theoretical analysis is presented in Section III. Later in Section II-C, we discuss how to extend our results to the case where $A$ is a general full column rank matrix under Assumption 2.2.

For now, we only focus on the recovery of one column of $A$ as the remaining columns can be recovered via the same procedure after projecting $Y$ onto the complement space spanned by the recovered columns of $A$ (see Section IV-B for detailed discussion).

A. Assumptions

We first resort to the matrix $X \in \mathbb{R}^{r \times n}$ being element-wise sparse. The sparsity of $X$ is modeled via the Bernoulli-Gaussian distribution, stated in the following assumption.

Assumption 2.1: Assume $X_{ij} = B_{ij}Z_{ij}$ for $i \in [r]$ and $j \in [n]$, where
\[
B_{ij} \overset{i.i.d.}{\sim} \text{Ber}(\theta), \quad (II.1)
\]
\[
Z_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2). \quad (II.2)
\]
The Bernoulli-Gaussian distribution is popular for modeling sparse random matrices [1], [38], [39]. The overall sparsity level of $X$ is controlled by $\theta$, the parameter of the Bernoulli distribution. We remark that the Gaussianity is assumed only to simplify the proof and to obtain more transparent deviation inequalities between quantities related with $X$ and their population counterparts. Both our approach and analysis can be generalized to cases where $Z_{ij}$ are centered i.i.d. sub-Gaussian random variables.

We also need another condition on the matrix $A$. To see this, note that even when $A$ were known, recovering $X$ from $Y = AX$ requires $A$ to have full column rank. We state this in the following assumption.

Assumption 2.2: Assume the matrix $A \in \mathbb{R}^{r \times r}$ has rank$(A) = r$ with $\|A\|_\text{op} = 1$.

The unit operator norm of $A$ is assumed without loss of generality as one can always re-scale $\sigma^2$, the variance of $X$, by $\|A\|_\text{op}$.

B. Recovery of the Orthonormal Columns of $A$

In this section, we consider the recovery of one column of $A$ when $A$ is a semi-orthogonal matrix satisfying the following assumption.

Assumption 2.3: Assume $A^T A = I_r$.

Our approach recovers columns of $A$ one at a time by adopting the $\ell_4$ maximization in (II.4) to utilize the sparsity of $X$ and orthogonality of $A$. Its rationale is based on the following lemma.

Lemma 2.4: Under Assumption 2.3, solving the following problem
\[
\max_q \left\| A^T q \right\|_4^4 \quad \text{s.t.} \quad \|q\|_2 = 1 \quad (II.3)
\]
recovers one column of $A$, up to its sign.

Intuitively, under Assumption 2.3, we have $\|A^T q\|_2 \leq 1$ for any unit vector $q$. Therefore, criterion (II.3) seeks a vector $A^T q$ within the unit ball to maximize its $\ell_4$ norm. When $q$ corresponds to one column of $A$, that is, $q = a_i$, for any $i \in [r]$, we have the largest objective $\|A^T a_i\|_4^4 = 1$. This $\ell_4$ norm maximization approach has been used in several related literature, for instance, sparse blind deconvolution [30], [45], complete and over-complete dictionary learning [35], [43], [44], independent component analysis [23], [24] and tensor decomposition [17].

The appealing property of maximizing the $\ell_4$ norm is its benign geometry landscape under the unit sphere constraint. Indeed, despite of the non-convexity of (II.3), our result in Theorem 3.1 implies that any strict location solution to (II.3)
is globally optimal. This enables us to use any second order gradient accent method to solve (II.3).

Motivated by Lemma 2.4, since we only have access to \( Y \in \mathbb{R}^{p \times n} \), we propose to solve the following problem to recover one column of \( A \),

\[
\min_q F(q) = -\frac{1}{12\theta \sigma^n} \| Y^T q \|_4^4 \quad \text{s.t.} \quad \| q \|_2 = 1. \tag{II.4}
\]

The scalar \((12\theta \sigma^n)\) is a normalization constant. The following lemma justifies the usage of (II.4) and also highlights the role of the sparsity of \( X \).

**Lemma 2.5:** Under model (I.1) and Assumption 2.1, we have

\[
\mathbb{E}[F(q)] = -\frac{1}{4} \left[ (1 - \theta) \| A^T q \|_4^4 + \theta \| A^T q \|_2^2 \right]
\]

where the expectation is taken over the randomness of \( X \).

**Remark 2.6 (Role of the Sparsity Parameter \( \theta \)):**

Lemma 2.5 implies that, for large \( n \), solving (II.4) approximately finds the solution to

\[
\min_q f(q) = -\frac{1}{4} \left[ (1 - \theta) \| A^T q \|_4^4 + \theta \| A^T q \|_2^2 \right]
\]

\[\text{s.t.} \quad \| q \|_2 = 1\]  \tag{II.5}

The objective function is a convex combination of \( \| A^T q \|_4^4 \) and \( \| A^T q \|_2^2 \) with coefficients depending on the magnitude of \( \theta \). In view of Lemma 2.4, it is easy to see that solving (II.5) recovers one column of \( A \), up to the sign, as long as \( \theta < 1 \). However, the magnitude of \( \theta \) controls the benignity of the geometry landscape of (II.5). When \( \theta \) is small, equivalently, \( X \) is sufficiently sparse, we essentially solve (II.3) which has the most benign landscape. On the other hand, when \( \theta \rightarrow 1 \), the landscape of (II.5) is mostly determined by the eigenvalue problem\(^3\) which maximizes \( \| A^T q \|_2^2 \) subject to \( \| q \|_2 = 1 \). We will demonstrate that when \( X \) is sufficiently sparse, second order descent algorithm with a simple initialization finds the globally optimal solution to (II.4) in Section III.

### C. Recovery of the Non-Orthogonal Columns of \( A \)

In this section, we discuss how to extend our procedure to recover \( A \) from \( Y = AX \) when \( A \) is a general full column rank matrix satisfying Assumption 2.2. The main idea is to first resort to a preconditioning procedure of \( Y \) such that the preconditioned \( Y \) has the decomposition \( AX \), up to some small perturbation, where \( A \) satisfies Assumption 2.3 and \( X \) satisfies Assumption 2.1 with \( \sigma^2 = 1 \). Then we apply our procedure in Section II-B to recover \( A \). The recovered \( A \) is further used to recover the original \( A \).

To precondition \( Y \), we propose to left multiply \( Y \) by the following matrix

\[
D = \left[ \left( YY^T \right)^+ \right]^{1/2} \in \mathbb{R}^{p \times p} \tag{II.6}
\]

where \( M^+ \) denotes the Moore-Penrose inverse of any matrix \( M \). The resulting preconditioned \( Y \) satisfies

\[
\tilde{Y} = DY = AX + E
\]

with \( A \) satisfying Assumption 2.3, \( X = X/\sqrt{\theta \sigma^2} \) and \( E \) being a perturbation matrix with small entries. We refer to Proposition 3.5 below for its precise statement.

Analogous to (II.4), we propose to recover one column of \( \tilde{A} \) by solving the following problem

\[
\min_q F_q(q) = -\frac{\theta n}{12} \| Y^T q \|_4^4 \quad \text{s.t.} \quad \| q \|_2 = 1. \tag{II.7}
\]

Theoretical guarantees of this procedure are provided in Section III-B. After recovering one column of \( \tilde{A} \), the remaining columns of \( A \) can be successively recovered via the procedure in Section IV-B. In the end, \( A \) can be recovered by first inverting the preconditioning matrix \( D \) as \( D^{-1} \tilde{A} \) and then re-scaling its largest singular value to 1.

### III. Theoretical Guarantees

We provide theoretical guarantees for our procedure (II.4) in Section III-A when \( A \) has orthonormal columns. The theoretical guarantees of (II.7) for recovering a general full column rank \( A \) are stated in Section III-B.

#### A. Theoretical Guarantees for Semi-Orthonormal \( A \)

In this section, we provide guarantees for our procedure by characterizing the solution to (II.4) when \( A \) satisfies Assumption 2.3.

As the objective function \( F(q) \) in (II.4) concentrates around \( f(q) \) in (II.5), it is informative to first analyze the solution to (II.5). Although (II.5) is a nonconvex problem and has multiple local solutions, Theorem 3.1 below guarantees that any strict local solution to (II.5) is globally optimal, in the sense that, it recovers one column of \( A \), up to its sign. We introduce the null region \( R_0 \) of our objective in (II.5),

\[
R_0 = \left\{ q \in \mathbb{S}^{p-1} : \| A^T q \|_2 = 0 \right\}. \tag{III.1}
\]

**Theorem 3.1 (Population Case):** Under Assumption 2.3, assume \( \theta \leq 1/6 \). Any local solution \( \bar{q} \) to (II.5), that is not in \( R_0 \), satisfies

\[
\bar{q} = AP_{\bar{1}} \tag{III.2}
\]

for some signed permutation matrix \( P \in \mathbb{R}^{r \times r} \).

The detailed proof of Theorem 3.1 is deferred to Appendix C-C. We only offer an outline of our analysis below.

The proof of Theorem 3.1 relies on analyzing the optimization landscape of (II.5) on disjoint partitions of \( \mathbb{S}^{p-1} = \{ q \in \mathbb{R}^p : \| q \|_2 = 1 \} \), defined as

\[
R_1 = R_1(C_\ast) = \left\{ q \in \mathbb{S}^{p-1} : \| A^T q \|_2^2 \geq C_\ast \right\},
\]

\[
R_2 = \mathbb{S}^{p-1} \setminus (R_0 \cup R_1). \tag{III.3}
\]

Here \( C_\ast \) is any fixed constant between 0 and 1. The region \( R_0 \) can be easily avoided by choosing an initialization such that the objective function \( f(q) \) is not equal to zero. For
Condition (III.5) puts restrictions on the upper bound of $\theta$. It is easy to see that (III.5) holds for any $\theta \leq 1/6$. As discussed in Remark 2.6, a smaller $\theta$ leads to a more benign optimization landscape. Figure 1 illustrates our results by depicting the landscape of (II.5) in a 3-dimensional case. As shown there, the region $R_1$ contains saddle points as well as all local solutions which recover the ground truth up to the sign.

In light of Theorem 3.1, we now provide guarantees for the solution to the finite sample problem (II.4) in the following theorem. Define the sample analogue of the null region $R_0$ in (III.1) as

$$R'_0(c_\ast) = \{ q \in S^{p-1} : \| A^T q \|_\infty \leq c_\ast \}$$

(III.6)

for any given value $c_\ast \in [0, 1)$.

In our analysis, we are mainly interested in the order of rates of convergence of our estimator and do not pursue derivation of explicit expression of the involved constants.

**Theorem 3.4 (Finite Sample Case):** Under Assumptions 2.1 and 2.3, assume $\theta \in (0, 1/9]$ and

$$n \geq C \max \left\{ \frac{r^2}{c_\ast^2 \log^2 n}, \frac{\log n}{\theta c_\ast} \right\} \left( \frac{r \log n}{n} \right)$$

(III.7)

for some sufficiently large constant $C > 0$ and any $c_\ast \in (0, 1/4]$. Then with probability at least $1 - cn^{-c'}$, any local solution $q$ to (II.4) that is not in $R'_0(c_\ast)$ satisfies

$$\| q - AP_1 \|_2 \lesssim \sqrt{\frac{r^2 \log n}{\theta n}} + \left( \frac{\theta r^2 + \log^2 n}{\theta} \right) \frac{r \log n}{n}$$

(III.8)

for some signed permutation matrix $P$. Here $c$ and $c'$ are some absolute positive numeric constants.

The proof of Theorem 3.4 can be found in Appendix C-D. The geometric analysis of the landscape of the optimization problem (II.4) is in spirit similar to that of Theorem 3.1, but has additional technical difficulty of taking into account the deviations between the finite sample objective $F(q)$ in (II.4) and the population objective $f(q)$ in (II.5), as well as the deviations of both their gradients and hessian matrices. Such deviations also affect both the size of $R'_0(c_\ast)$ in (III.6), an enlarged region of $R_0$ in (III.1), via condition (III.7), and the estimation error of the local solution $\tilde{q}$.

In Lemmas 3.14, 3.15 and 3.16 of Appendix C-I, we provide finite sample deviation inequalities of various quantities between $F(q)$ and $f(q)$. Our analysis characterizes the explicit dependency on dimensions $n$, $p$ and $r$, as well as on the sparsity parameter $\theta$. In particular, our analysis is valid for fixed $p$, $r$ and $\theta$, as well as growing $p = p(n)$, $r = r(n)$ and $\theta = \theta(n)$.

The estimation error of our estimator in (III.8) depends on both the rank $r$ and the sparsity parameter $\theta$, but is independent of the higher dimension $p$. The smaller $\theta$ is, the larger estimation error (or the stronger requirement on the sample size $n$) we have. This is as expected since one needs to observe enough information to accurately estimate the population-level objective in (II.5) by using (II.4). On the other hand, recalling from Remark 2.6 that a larger $\theta$ could lead to worse geometry landscape. Therefore, we observe an interesting trade-off of the magnitude of $\theta$ between the optimization landscape and the statistical error.

**B. Theoretical Guarantees for General Full Column Rank $A$**

In this section, we provide theoretical guarantees for our procedure of recovering a general full column rank matrix $A$ under Assumption 2.2.
Recall from Section II-C that our approach first precondition the matrix $Y$ by using $D$ from (II.6). The following proposition provides guarantees for the preconditioned $Y$, denoted as $\tilde{Y} = DY$. The proof is deferred to Appendix C-E. Write the SVD of $A = U_A D_A V_A^T$ with $U_A \in \mathbb{R}^{p \times r}$ and $V_A \in \mathbb{R}^{r \times r}$ being, respectively, the left and right singular vectors.

**Proposition 3.5:** Under Assumptions 2.1 and 2.2, assume $n \geq C r / \theta^2$ for some sufficiently large constant $C > 0$. With probability greater than $1 - 2e^{-c' r}$, one has

$$\tilde{Y} = \tilde{A} \tilde{X} + E$$

where $\tilde{A} = U_A V_A^T$, $\tilde{X} = X / \sqrt{\theta} n \sigma^2$ and $E = \tilde{A} \Delta \tilde{X}$ with

$$\|\Delta\|_\infty \leq c' \frac{1}{\theta} \sqrt{\frac{r}{n}}.$$  \hspace{1cm} (III.9)

Here $c'$ and $c''$ are some absolute positive constants.

Proposition 3.5 implies that, when $n \geq C r / \theta^2$, the preconditioned $Y$ satisfies

$$\tilde{Y} = \tilde{A}(I_r + \Delta) \tilde{X} \approx \tilde{A} \tilde{X}$$

with $\tilde{A}^T \tilde{A} = I_r$. This naturally leads us to apply our procedure in Section II-B to recover columns of $\tilde{A}$ via (II.7). We formally show in Theorem 3.6 below that any local solution to (II.7) approximates recover one column of $\tilde{A}$ up to a signed permutation matrix. Similar to (III.6), define

$$R_0''(c_*) = \left\{ q \in \mathbb{S}^{p-1} : \|A^T q\|_\infty \leq c_* \right\}$$  \hspace{1cm} (III.10)

for some given value $c_* \in [0, 1)$.

**Theorem 3.6:** Under Assumption 2.1 and 2.2, assume $\theta \in (0, 1/9]$ and

$$n \geq C \frac{r}{c_* \theta} \max \left\{ \log^2 n, \frac{\log n}{c_* \theta}, \frac{\log^2 n}{c_* \theta}, \frac{r \log n}{c_* \theta} \right\}.$$  \hspace{1cm} (III.11)

Then with probability at least $1 - cn^{-c'} - 4e^{-c'' r}$, any solution $q$ to (II.7) that is not in Region $R_0''(c_*)$ satisfies

$$\|q - \bar{A} P_1\|_2 \leq \sqrt{\frac{r \log n}{\theta^2 n}} + \sqrt{\frac{\log^2 n}{\theta n}} + \left( \frac{\theta r^2}{\theta^2 n} + \frac{\log^2 n}{\theta n} \right) \frac{r \log n}{n}$$

for some signed permutation matrix $P$. Here $c, c', c''$ are some absolute numeric constants.

The proof of Theorem 3.6 can be found in Appendix C-F. Due to the preconditioning step, the requirement of the sample size in (III.11) is slightly stronger than (III.7), whereas the estimation error of $q$ only has an additional $\sqrt{r \log n / (\theta^2 n)}$ term comparing to (III.8).

Theorem 3.6 requires to avoid the null region $R_0''(c_*)$ in (III.10). We provide a simple initialization in the next section that provably avoids $R_0''$. Furthermore, every iterate of any descent algorithm based on such initialization is provably not in $R_0''$ either.

### IV. Complete Algorithm and Provable Recovery

In this section, we present a complete pipeline for recovering $A$ from $Y$. So far we have established that every local solution to (II.7), that is not in $R_0''(c_*)$, temporarily recovers one column of $\bar{A}$; To our end, we will discuss: (1) a data-driven initialization in Section IV-A which, together with Theorem 3.6, provably recovers one column of $A$; (2) a deflation procedure in Section IV-B that sequentially recovers all remaining columns of $A$.

#### A. Initialization

Our goal is to provide a simple initialization such that solving (II.7) via any second order descent algorithm provably recovers one column of $A$. According to Theorem 3.6, such an initialization needs to guarantee the following conditions.

- **Condition I:** The initial point $q^{(0)}$ does not fall into region $R_0''(c_*)$ for some $c_*$ satisfying (III.11) in Theorem 3.6.

- **Condition II:** The updated iterates $q^{(k)}$, for all $k \geq 1$, stay away from $R_0''(c_*)$ as well.

We propose the following initialization

$$q^{(0)} = \frac{Y 1_n}{\|Y 1_n\|_2} \in \mathbb{S}^{p-1}.$$  \hspace{1cm} (IV.1)

The following two lemmas guarantee that both **Condition I** and **Condition II** are met for this choice. Their proofs can be found in Appendices C-G and C-H.

**Lemma 4.1:** Under Assumption 2.1 and 2.2, assume $\theta \in (0, 1/9]$ and

$$n \geq C \left( \log^3 n, \frac{r \log n}{\theta \sqrt{n}}, \frac{r^2 \log n}{\theta}, \frac{r^2}{\theta n}, \frac{r^3 \log n}{\theta} \right)$$

holds, then, with probability at least $1 - 2e^{-c'' r}$, the initialization $q^{(0)}$ in (IV.1) is not in region $R_0''(c_*)$ with $c_* = 1/(2r)$.

**Lemma 4.2:** Let $q^{(k)}$, for $k \geq 1$, be any updated iterate from solving (II.4) by using any descent algorithm with the initial point $q^{(0)}$ chosen as (IV.1). If

$$n \geq C \left( \log^3 n, \frac{r \log n}{\theta \sqrt{n}}, \frac{r^2}{\theta n}, \frac{t_2^2 \log n}{\theta} \right)$$

holds, then, with probability at least $1 - cn^{-c'} - 4e^{-c'' r}$ for some absolute numeric constants $c, c', c'' > 0$, one has

$$q^{(k)} \notin R_0'' \left( \frac{1}{2r} \right), \quad \forall k \geq 1.$$  \hspace{1cm} (IV.3)

Combining Lemmas 4.1 and 4.2 together with Theorem 3.6 readily yields the following theorem.

**Theorem 4.3:** Under Assumptions 2.1 and 2.2, assume $\theta \in (0, 1/9]$ and (IV.2) holds. Let $\bar{q}$ be any local solution to (II.7) from any second order descent algorithm with the initial point chosen as (IV.1). With probability at least $1 - cn^{-c'} - 4e^{-c'' r}$ for some absolute numeric constants $c, c', c'' > 0$, one has

$$\|q - \bar{A} P_1\|_2 \leq \sqrt{\frac{r \log n}{\theta^2 n}} + \sqrt{\frac{\log^2 n}{\theta n}} + \frac{r \log^3 n}{\theta n}$$

for some signed permutation matrix $P$. 

Algorithm 1 Sparse Low Rank Decomposition

Data: a matrix $Y \in \mathbb{R}^{p \times n}$ with rank $r \leq \min\{n, p\}$
Result: matrix $A \in \mathbb{R}^{p \times r}$

Compute $D$ from (II.6) and obtain $Y = DY$;
Set $A_j = \emptyset$ and initialize $q^{(0)}$ as (IV.1);
for $j = \{1, 2, \ldots, r\}$ do
  Solve $a_j^{(0)}$ from (IV.4) by using $q^{(0)}$ and any second order descent algorithm;
  Update $\tilde{A}_j = a_j^{(0)}$;
  Set $A_j = \text{span}(\tilde{A}_1, \ldots, \tilde{A}_j)$;
end
Compute $A = D^{-1} \tilde{A} / \| D^{-1} \tilde{A} \|_{\text{op}}$.

Theorem 4.3 provides the guarantees for using any second order descent algorithms [4], [33] to solve (II.7) with the initialization chosen in (IV.1).

B. Recovering the Full Matrix $A$

Theorem 4.3 provides the guarantees for recovering one column of $A$. In this section, we discuss how to recover the remaining columns of $A$ by using the deflation method [35], [38], [39].

Suppose that solving (II.7) recovers $\bar{a}_1$, the first column of $\bar{A}$. For any $k \in \{2, \ldots, r\}$, write $A_k = \text{span}(\bar{a}_1, \ldots, \bar{a}_{k-1})$, the space spanned by all previously recovered columns of $A$ at step $k$. Further define $P_{A_k}$ as the projection matrix onto $A_k$ and write $P_{A_k}^\perp = I_p - P_{A_k}$. We propose to solve the following problem to recover a new column of $A$,

$$\min_q \frac{-\theta n}{12} \| q^TP_{A_k}^\perp Y \|_4^4, \quad \text{s.t. } \| q \|_2 = 1. \quad \text{(IV.4)}$$

To facilitate the understanding, consider $k = 2$ and $P_{A_k}^\perp = P_{A_1}^\perp$. Then (IV.4) becomes

$$\min_q \frac{-\theta n}{12} \| q^TP_{A_1}^\perp Y \|_4^4, \quad \text{s.t. } \| q \|_2 = 1. \quad \text{(IV.5)}$$

From Proposition 3.5, we observe that

$$P_{A_1}^\perp Y \approx P_{A_1}^\perp \bar{A} \bar{X} = \bar{A}_{(-1)} \bar{X}_{(-1)}, \quad \text{(IV.6)}$$

where we write $\bar{A}_{(-1)} \in \mathbb{R}^{p \times (r-1)}$ and $\bar{X}_{(-1)} \in \mathbb{R}^{(r-1) \times n}$ for $\bar{A}$ and $\bar{X}$ with the 1th column and the 1 row removed, respectively. Then it is easy to see that recovering one column of $A_{(-1)}$ from $P_{A_1}^\perp Y$ is the same problem as recovering one column of $A$ from $\bar{Y}$ with $r$ replaced by $r - 1$, hence can be done via solving (IV.5). Similar reasoning holds for any $2 \leq k \leq r$.

As soon as we recover $\bar{A}$, the original $A$ is recovered by $D^{-1} \tilde{A} / \| D^{-1} \tilde{A} \|_{\text{op}}$ (under Assumption 2.2) with $D$ defined in (II.6). For the reader’s convenience, we summarize our whole procedure of recovering $A$ in Algorithm 1.

V. EXPERIMENTS

In this section we verify the empirical performance of our proposed algorithm for recovering $A$ under model (I.1) in different scenarios.

A. Experiment Setup

We start by describing our data generating mechanism of $Y = AX$. The general full rank $A$ is generated as $A_{ij} \sim \mathcal{N}(0, 1)$ with its operator norm scaled to 1. On the other hand, entries of the sparse coefficient matrix $X \in \mathbb{R}^{r \times n}$ are generated i.i.d. from Bernoulli-Gaussian with parameter $\theta$ and $\sigma^2 = 1$.

Recall that $A = U_A V_A^T$ with $U_A$ and $V_A$ consisting of the left and right singular vectors of $A$, respectively. We first evaluate the performance of our procedure (II.7) in terms of the probability of successfully recovering one column of $\bar{A}$. Specifically, let $\bar{q} \in \mathbb{S}^{p-1}$ be our estimate from (II.7), we compute

$$\text{Err}(\bar{q}) = \min_{1 \leq i \leq r} (1 - | \langle \bar{q}, \tilde{a}_i \rangle |) \quad \text{(V.1)}$$

with $\tilde{A} = (\bar{a}_1, \ldots, \bar{a}_r)$. If $\text{Err}(\bar{q}) \leq \rho_c$ for some small value $\rho_c > 0$, we say the vector $\bar{q}$ successfully recovers one column of $\bar{A}$. We consider two scenarios in Section V-B to evaluate the probability of recovering one column of $A$. In the first case, we vary simultaneously $\theta$ and $r$ while in the second case we change $n$ and $r$.

In Section V-C we also examine the performance of our proposed Algorithm 1 for recovering the whole matrix $A$ by using the following normalized Frobenius norm between any estimate $A_{\text{est}}$ and the true $A$:

$$\min_P \frac{1}{\sqrt{p}} \| A_{\text{est}} - AP \|_F \quad \text{s.t. } P \text{ is a signed permutation matrix.} \quad \text{(V.2)}$$

In Section V-D, we further compare the performance of our proposed method with two algorithms that are popular in solving the sparse principal component analysis (SPCA) problem.

In the aforementioned settings, we choose the projected Riemannian gradient descent (for the one-column recovery result in Figure 2) and the general power method [28] (for the full matrix recovery result in Figures 3, 4a and 4b) in Algorithm 1. Detailed specifications of these two descent methods as well as the comparison between them are stated in Appendix B.

B. Successful Rate of One Column Recovery

In this part we examine the probability of successfully recovering one column in $A$. Here we use (V.1) for computing recovering error and $\rho_c$ is set to 0.01 in our simulation.

1) Varying $\theta$ and $r$: We fix $p = 100$ and $n = 5 \times 10^3$ while vary $\theta \in \{0.01, 0.04, \ldots, 0.58\}$ and $r \in \{10, 30, \ldots, 70\}$. For each pair of $(\theta, r)$, we repeatedly generate 200 data sets and apply our procedure in (II.7). The averaged recovery probability of our procedure over the 200 replicates is shown in Figure 2. The recovery probability gets larger as $r$ decreases, in line with Theorem 3.6. We also note that the recovery increases for smaller $\theta$. This is because smaller $\theta$ renders a more benign geometric landscape of the proposed non-convex problem, as detailed in Remark 2.6. On the other hand, the recovery probability decreases when $\theta$ is approaching to 0.
As suggested by Theorem 3.4, the statistical error of estimating \( A \) gets inflated as \( \theta \) gets too small.

2) Varying \( n \) and \( r \): Here we fix \( p = 100 \) and the sparsity parameter \( \theta = 0.1 \). We vary \( r \in \{10, 30, \ldots, 70\} \) and \( n \in \{2000, 3000, \ldots, 12000\} \).

Figure 2 shows the averaged recovery probability of our procedure over 200 replicates in each setting. Our procedure performs increasingly better as \( n \) increases, as expected from Theorem 3.4.

C. Estimation Error of a General Full Column Rank \( A \)

To generate a full column rank matrix \( A \), we use \( A_{ij} \sim \mathcal{N}(0, 1) \). We follow the same procedure in Section V for generating \( X \). We choose \( n = 1.2 \times 10^4 \) and \( p = 100 \), and vary \( \theta \in \{0.01, 0.04, \ldots, 0.58\} \) and \( r \in \{10, 20, \ldots, 50\} \).

Figure 3 depicts the performance of our method under different choices of \( \theta \) and \( r \). The error of estimating \( A \) gets smaller when either \( \theta \) or \( r \) decreases for \( \theta \geq 0.1 \). Also for relatively small \( \theta \) \((\theta < 0.1)\) we find that the error increases when \( \theta \) gets smaller.

These findings are in line with our result in Theorem 4.3. The tradeoff of \( \theta \) that we observe here agrees with our discussion in Remark 2.6.

D. Comparison With Algorithms Used for SPCA

In this section we compare the performance of our method with two SPCA procedures. Among the various algorithms for solving the SPCA problem, we compare with the LARS algorithm [49], [50] and the alternating direction method (ADM) [5], [40]. The LARS solves

\[
\min_{v_i} \|Z_i - Y v_i\|_2 + \lambda \|v_i\|_1 + \lambda' \|v_i\|_2,
\]

for each \( 1 \leq i \leq r \), to recover rows of \( X \). Here, \( Z_i \) is the \( i \)th principle component of \( Y \). Denote by \( \hat{X} \) the
Our setting implies that the setting in [35], the matrix which brings fundamental differences in both the rationale of setting of over-complete dictionary learning, that is, \( A \) has full rank with \( p \leq r \). Although our criterion in (II.4) is similar to that used in [35], the low-rank structure of \( Y \) in our setting implies that \( A \) has full column rank with \( p > r \), which brings fundamental differences in both the rationale of using (II.4) and its subsequent analysis. To be specific, under the setting in [35], the matrix \( A = (a_1, \ldots, a_r) \in \mathbb{R}^{p \times r} \), with \( r \geq p \), is assumed to be unit norm tight frame (UNTF), in the sense that

\[
AA^T = \frac{r}{p} I_p, \quad \|a_i\|_2 = 1, \quad \mu = \max_{i \neq j} |\langle a_i, a_j \rangle| \ll 1.
\]

Under this condition and Assumption 2.1, the objective \( F(q) \) in (II.4) satisfies (see, display (2.4) in [35])

\[
\mathbb{E}[F(q)] = -\frac{1}{4}(1 - \theta)\|A^T q\|_4^2 - C \tag{A.1}
\]

where \( C \) is some numerical value that does not depend on \( q \). Therefore, solving (II.4), for large \( n \), approximately maximizes \( \|A^T q\|_4^2 \) over the unit sphere in the context of [35].

There are at least three major differences to be noted. First, as \( A \) is UNTF in the setting of [35], columns of \( A \) are not orthonormal, or equivalently, \( \mu > 0 \). As a result, Lemma 2.4 does not hold for their setting. In another word, even one can directly solve (II.3), the solution does not exactly recover one column of \( A \). Indeed, Proposition B.1 in [35] shows that the difference between the solution to (II.3) and one column of \( A \) is small when \( \mu \ll 1 \) but is not exactly equal zero unless \( \mu = 0 \). By contrast, when \( A \) satisfies \( A^T A = I_r \) in our setting, the exact recovery of columns of \( A \) is achievable via solving (II.3) as shown in Lemma 2.4.

Second, due to \( \text{rank}(A) = r \), solving (II.4) in our setting approximately maximizes (II.5), the objective of which is a convex combination of \( \|A^T q\|_2^4 \) and \( \|A^T q\|_2^4 \) with coefficients depending on the sparsity parameter \( \theta \). Thus, the expected objective in our setting no longer coincides with that in [35] and in fact is more complicated due to the extra term \( \|A^T q\|_2^4 \). This additional term brings more complications in our analysis of the geometry landscape of (II.4) and requires more delicate arguments. Indeed, in view of (II.4), although it is clear that columns of \( A \) are still the global maximizers regardless of the presence of \( \theta \|q^T A\|_2^4 \), it is non-trivial to establish how \( \theta \|q^T A\|_2^4 \) affects the geometric landscape, such as properties of all stationary points. Our population-level results fully characterize the benign regime at the presence of \( \theta \|q^T A\|_2^4 \) (see Lemmas 3.1, 3.3 and 3.4). To account the existence of \( \theta \|q^T A\|_2^4 \), we have to base on not only a different partition of the unit sphere but also a different metric, the sup-norm, of the partition, from the analysis used in [35]. As we move to the sample-level analysis, extra cares need to be taken, see, for instance, Lemmas 3.6, 3.9 and 3.10.

Third, the low-rank structure of \( Y \) leads to a null region of solving (II.4), that is, the region of \( q \) such that \( q^T Y = q^T A X = 0 \) (see, (III.1) for the population-level analysis and (III.6) – (III.10) for the finite sample results). This null region does not appear when the matrix \( A \) is UNTF. In the presence of such a region, to provide guarantees for any second-order gradient-based algorithm, we need to provide a proper initialization outside of this region and to further carefully prove that every iterate does not fall into the null region. Such analysis brings more technical challenges to our analysis than [35] (see, for instance Lemmas 4.1 and 4.2).

Appendix A

Technical Comparison With [35] and [44]

[35] studies the unique recovery of \( Y = AX \) under the setting of over-complete dictionary learning, that is, \( A \) has full row rank with \( p \leq r \). Although our criterion in (II.4) is similar to that used in [35], the low-rank structure of \( Y \) in our setting implies that \( A \) has full column rank with \( p > r \), which brings fundamental differences in both the rationale of using (II.4) and its subsequent analysis. To be specific, under the setting in [35], the matrix \( A = (a_1, \ldots, a_r) \in \mathbb{R}^{p \times r} \), with \( r \geq p \), is assumed to be unit norm tight frame (UNTF), in the sense that

\[
AA^T = \frac{r}{p} I_p, \quad \|a_i\|_2 = 1, \quad \mu = \max_{i \neq j} |\langle a_i, a_j \rangle| \ll 1.
\]

[44] studies the problem of recovering a complete, orthonormal, matrix \( A \in \mathbb{O}_p \) via

\[
\max_{A \in \mathbb{O}_p} \|A^T Y\|_4^4.
\]

[44, Theorem 1] shows that the maximizer of the above problem is close to the desired target. However, due to the intrinsic difficulty of analyzing the stiefel manifold, there is
no global convergence analysis and it is still an open problem that whether local solutions or saddle points exist. By contrast, based on our landscape analysis of stationary points, we are able to provide global guarantees for any second-order descent algorithm.

APPENDIX B
EMPIRICAL STUDIES ON COMPARISON OF THE GENERAL POWER METHOD AND THE PROJECTED RIEMANNIAN GRADIENT DESCENT

In this section we compare the performance of Algorithm 1 by using the general power method (PM) [28], [35] and the projected Riemannian gradient descent (PRGD) [35] in terms of the one-column recovery error and algorithmic convergence rates.

For simplicity, we consider the matrix $A$ with orthonormal columns. Specifically, the ground-truth matrix $A$ is set as the left singular vectors of random matrix $Z \in \mathbb{R}^{r \times r}$ with $Z_{ij} \sim \mathcal{N}(0, 1)$ and $X \in \mathbb{R}^{r \times n}$ is i.i.d Bernoulli-Gaussian with varying parameter $\theta$ and $\sigma^2 = 1$.

We compare the performance of using projected gradient descent and power method for solving problem II.4, as detailed below.

- **Projected Riemannian Gradient Descent (PRGD):** We base on the update

$$q^{(k+1)} = P_{S^{p-1}} \left( q^{(k)} - \gamma^{(k)} \nabla F(q^{(k)}) \right) \tag{B.1}$$

for $k \in \{1, 2, \ldots\}$, where $\gamma^{(k)} := \gamma^{(0)} k^{-0.9}$ is the step size of the $k$-th iteration with $\gamma^{(0)} = 0.01$ being the initial step size and $\nabla F(q^{(k)})$ being the Riemannian gradient of $F(q^{(k)})$. Here $P_{S^{p-1}}$ represents the projection matrix onto the unit sphere.

- **Power Method (PM):** We base on the update

$$q^{(k+1)} = P_{S^{p-1}} \left( \nabla F(q^{(k)}) \right), \tag{B.2}$$

for all $k \in \{1, 2, \ldots\}$, with $\nabla F(q^{(k)})$ being the gradient of $F(q^{(k)})$ at each step $k$.

**A. Comparison on One Column Recovery Errors**

In this section we compare the one-column recovery error, defined in (IV.1), of solving (II.4) by using (B.1) and (B.2). We consider $r \in \{10, 20, \ldots, 80\}$ and $\theta = 0.1$ in Figure 5a while $r = 10$, $\theta \in \{0.05, 0.1, 0.15, \ldots, 0.6\}$ in Figure 6a when fix $n = 5 \times 10^3$ for both cases. For comparison, both (B.1) and (B.2) use the same initialization $q^{(0)}$ in (IV.1) as well as the same stopping criterion, $k \leq 5000$. Figure 5a and 5b depict the one-column recovery errors of both methods, averaged across 50 repetitions, for each choice of $r$ and $\theta$ respectively. These two methods have nearly the same recovery errors for all $r$ and $\theta$, implying that they both reach the same global minimum. This observation is in line with Theorem 4.3.

**B. Comparison on Algorithmic Convergence Rates**

In this part we check the time of reaching convergence for PRGD and PM under the same data generating setup as described in the previous section. Here both methods base on the same initialization and use $\text{Err}(q^{(k)}) \leq 0.01$ as the stopping criteria. Figure 6a and 6b show the averaged running time before convergence for both methods across 50 repetitions. Clearly, PM has a faster convergence rate comparing to PRGD, especially for large $r$. This empirical efficiency of using PM has also been observed by [35], [42], and [44]. However, theoretical justifications for under-complete model on this aspect are still an open problem, deserving future investigation.

APPENDIX C
MAIN PROOFS

**A. Proof of Lemma 2.4**

**Proof:** First, note that, for any $q \in S^{p-1}$,

$$\|A^T q\|_4^4 = \sum_{j=1}^{r} (a_j^T q)^4$$

Fig. 5. Comparison of recovery errors for PRGD and PM: Above two panels compare the recovery errors where we varies $r$ in figure 5a and $\theta$ in figure 5b. We set $p = 100$ and $n = 5 \times 10^3$ for all figures.

(b) One column recovery errors of using PRGD and PM when varying $\theta$
This choice of $q$ leads to $\|A^T q\|_2^4 = 1/s$ which is equal to one if and only if $s = 1$.

B. Proof of Lemma 2.5

Proof: Pick any $q$ and $C$. One has
\[
\mathbb{E} [F(q)] = -\frac{1}{12 \theta \sigma^4} \mathbb{E} [\|q^T A X\|_2^4]
= -\frac{1}{12 \theta \sigma^4} \mathbb{E} [\|q^T A X|_i^4]
\] (C.2)
by the i.i.d. assumption of columns of $X$. Write $\zeta = A^T q$ and use Assumption 2.1 to obtain
\[
\mathbb{E} [F(q)] = -\frac{1}{12 \theta \sigma^4} \mathbb{E} \left[ \left( \sum_{j=1}^r \zeta_j B_{ji} Z_{ji} \right)^4 \right].
\] (C.3)
Since $B_{ji}$ is independent of $Z_{ji}$ and
\[
\sum_{j=1}^r \zeta_j B_{ji} Z_{ji} |_{B_i} \sim N \left( 0, \sigma^2 \sum_{j=1}^r \zeta_j^2 B_{ji}^2 \right)
\] (C.4)
from Assumption 2.1, we obtain
\[
\mathbb{E} \left[ \left( \sum_{j=1}^r \zeta_j B_{ji} Z_{ji} \right)^4 \right]
= 3 \sigma^4 \mathbb{E} \left[ \left( \sum_{j=1}^r \zeta_j^2 B_{ji}^2 \right)^2 \right]
= 3 \sigma^4 \mathbb{E} \left[ \sum_{j=1}^r \zeta_j^4 B_{ji}^2 \right] + 3 \sigma^4 \mathbb{E} \left[ \sum_{j \neq \ell} \zeta_j^2 \zeta_\ell^2 B_{ji}^2 B_{\ell i} \right]
= 3 \sigma^4 \theta \sum_{j=1}^r \zeta_j^4 + 3 \sigma^4 \theta^2 \sum_{j \neq \ell} \zeta_j^2 \zeta_\ell^2
= 3 \sigma^4 \theta \left[ \left( 1 - \theta \right) \sum_{j=1}^r \zeta_j^4 + \theta \left( \sum_{j=1}^r \zeta_j^2 \right)^2 \right]
= 3 \sigma^4 \theta \left[ \left( 1 - \theta \right) \|\zeta\|_4^4 + \theta \|\zeta\|_2^4 \right].
\] (C.5)
The result then follows.

C. Proof of Theorem 3.1

We prove Theorem 3.1 by proving Lemmas 3.2 and 3.3 in Sections C-C.2 and C-C.1, respectively.

To analyze the solution to (II.5), we need the following Riemannian gradient and Hessian matrix of $f(q)$ constrained on the sphere $\|q\|_2 = 1$
\[
\text{grad } f(q)
= - P_q \left[ (1 - \theta) \sum_{j=1}^r a_j (q^T a_j)^3 + \theta \|q^T A\|_2^2 A^T q \right]
\] (C.6)
Hess $f(q)$
\[
= \text{Hess}_{a_i} f(q) + \text{Hess}_{a_2} f(q)
\] (C.7)
Recall that, for any $A$, we partition $\mathbb{S}^{p-1}$ into

$$R_1(A) = \left\{ q \in \mathbb{S}^{p-1} : \| A^T q \|_\infty \geq C_* \right\},$$

$$R_2(A) = \mathbb{S}^{p-1} \setminus \left( R_1(A) \cup R_0 \right).$$

1) Geometric Analysis for $q \in R_2$: We prove the following lemma which shows the existence of negative curvature for any $q \in R_2$.

**Lemma 3.1:** Assume $\theta < 1/3$. For any point $q \in R_2(C_*)$ with

$$C_* \leq \frac{1 - 3\theta}{2},$$

there exists $v$ such that

$$v^T \text{Hess}_q f(q) v < 0. \quad \text{(C.10)}$$

In particular, if $\theta \leq 1/6$, for any point $q \in R_2(C_*)$ with

$$C_* \leq \frac{1}{3\sqrt{2}},$$

there exists $v$ such that

$$v^T \text{Hess}_q f(q) v < -\frac{11 - 5\sqrt{2}}{9} \| \zeta \|_\infty^2. \quad \text{(C.11)}$$

**Proof:** Fix $C_*$. Pick any $q \in R_2(C_*)$ and write $\zeta = A^T q$ for simplicity. Assume $\| \zeta \|_\infty = \| \zeta \|_\infty^i$ for some $i \in [r]$.

Recall that $D\zeta^2 = \text{diag} (\zeta^{2i})$ with $\zeta^{2i} = (\zeta^i_j)_{j \in [r]}$. From (C.8), we have

$$a^T_i \text{Hess}_q f(q) a_i = (1 - \theta) \left[ -3a^T_i A D\zeta^2 A^T a_i + 6\zeta_i^4 \zeta^T D\zeta^2 A^T a_i \right] - 3\zeta_i^4 \| \zeta \|_4 - \| \zeta \|_2^2 (\zeta_i^2 - \| a_i \|_2^2) \right]$$

$$= (1 - \theta) \left[ -3\zeta_i^2 + 6\zeta_i^4 + 3\zeta_i^4 - \| \zeta \|_2^2 (\zeta_i^2 - 1) \right]$$

$$= (1 - \theta) \left[ -2\zeta_i^2 + 6\| \zeta \|_\infty^2 \zeta_i^2 \zeta_i^4 - 4\| \zeta \|_\infty^2 \| \zeta \|_2^4 \right]$$

where in the last line we used $\| \zeta \|_4 \leq \| \zeta \|_2 \| \zeta \|_\infty \leq \| \zeta \|_\infty^2$ and $\| \zeta \|_4 \geq \| \zeta \|_\infty^2$. On the other hand, we obtain

$$a^T_i \text{Hess}_q f(q) a_i = \theta \left[ -2\zeta_i^2 + 6\| \zeta \|_\infty^2 \zeta_i^2 \zeta_i^4 - 4\| \zeta \|_\infty^2 \| \zeta \|_2^4 \right]$$

- $\| \zeta \|_2^4 \| a_i^T A \|_2^2 + \| \zeta \|_2^4 \right]$$

$$\leq \theta \left[ -2\| \zeta \|_\infty^2 + 6\| \zeta \|_2^4 - 4\| \zeta \|_\infty^2 \right]$$

$$+ \| \zeta \|_2^4 \left( \| \zeta \|_2^4 - 1 \right)$$

$$\leq \theta \left[ 4\| \zeta \|_2^4 - 4\| \zeta \|_\infty^2 \right], \quad \text{(C.13)}$$

where in the second and third lines we used $\| \zeta \|_2^4 \leq 1$. Combine (C.12) and (C.13) to obtain

$$a^T_i \text{Hess}_q f(q) a_i \leq -4\| \zeta \|_\infty^2 - 6(1 - \theta)\| \zeta \|_\infty^4 - 2(1 - 3\theta)\| \zeta \|_\infty^2.$$ (C.14)

Define

$$g(x) = x^2 - \phi x + \omega,$$

with $\phi = \frac{3(1 - \theta)}{2}$, $\omega = \frac{1 - 3\theta}{2}. \quad \text{(C.15)}$$

It remains to prove $a^T_i \text{Hess}_q f(q) a_i \leq -4\| \zeta \|_\infty^2 g \left( \| \zeta \|_\infty^2 \right) < 0$. To this end, note that $\omega > 0$ under $\theta < 1/3$. Since

$$\phi^2 - 4\omega = \frac{9(1 - \theta)^2}{4} - 2 + 6\theta = \left( \frac{3\theta + 1}{2} \right)^2 > 0, \quad \text{(C.16)}$$

we know that, for all

$$\| \zeta \|_\infty^2 \leq \phi - \sqrt{\phi^2 - 4\omega} = \frac{1 - 3\theta}{2},$$

$$g(\| \zeta \|_\infty^2) \geq 0 \text{ and } g(\| \zeta \|_\infty^2) \text{ increases as } \| \zeta \|_\infty^2 \text{ gets smaller.}$$

Recall that $q \in R_2(C_*)$ implies $\| \zeta \|_\infty^2 < C_*$. Thus, as long as

$$C_* \leq \frac{1 - 3\theta}{2},$$

we conclude $g(\| \zeta \|_\infty^2) > g(C_*) \geq 0$ hence

$$a^T_i \text{Hess}_q f(q) a_i \leq -4\| \zeta \|_\infty^2 g \left( \| \zeta \|_\infty^2 \right) < 0. \quad \text{(C.18)}$$

This completes the proof of the first statement. The second one follows by taking $C_* \leq 1/(3\sqrt{2}). \quad \square$

2) Geometric Analysis for $q \in R_1$: In this section we prove that any local solution to (II.5) in $R_1$ recovers one column of $A$, as stated in the following lemma.

**Lemma 3.2:** Assume $\theta < 1$. Any local solution $\bar{q} \in R_1(C_*)$ to (II.5) with

$$C_* > \frac{1}{2} \sqrt{\frac{\theta}{1 - \theta}},$$

recovers one column of $A$, that is,

$$\bar{q} = \pm A e_i$$

for some standard basis vector $e_i$.

**Proof:** We prove the result by showing that any critical point of (II.5) in $R_1(C_*)$ is either a saddle point, or one of the ground-truth that satisfies the second order optimality condition and is equal to one column of $A$. 

\[ \text{where} \]

\[ \text{Hess}_q f(q) = -(1 - \theta) P_{q^+} \left[ 3 \sum_{j=1}^r a_j a_j^T (q^T a_j)^2 - \| q^T A \|_4^2 I \right] P_{q^+}, \quad \text{(C.8)} \]

\[ \text{Hess}_q f(q) = -\theta P_{q^+} \left[ \| q^T A \|_2^2 A A^T + 2 A A^T q q^T A A^T - \| q^T A \|_4^2 I \right] P_{q^+}. \quad \text{(C.9)} \]
Our proof starts by characterizing all critical points of (II.5). For any critical point \( \zeta \) of (II.5), by writing \( \zeta = A^T q \), letting the gradient (C.6) equal zero gives
\[
(1 - \theta) A \zeta_4^{\circ 3} - (1 - \theta) q \| \zeta_4 \|_2^4 + \theta \| \zeta_4 \|_2^2 A \zeta - \theta q \| \zeta_4 \|_2^4 = 0.
\]
(C.19)

Pick any \( 1 \leq i \leq r \). Multiply both sides by \( a_i^T \) to obtain
\[
(1 - \theta) a_i^T A \zeta_4^{\circ 3} - (1 - \theta) a_i \| \zeta_4 \|_2^4 + \theta \| \zeta_4 \|_2^2 a_i^T A \zeta - \theta a_i \| \zeta_4 \|_2^4 = 0
\]
(C.20)

with \( \zeta_4^{\circ 3} \) means \( \{ \zeta_i^4 \}_{j \in \{r\}} \). By using
\[
\begin{align*}
& a_i^T A \zeta_4^{\circ 3} = \| a_i \|_2^2 \zeta_i^3 + \sum_{j \neq i} (a_i, a_j) \zeta_j^3 = \zeta_i^3 \\
& a_i^T A \zeta = \| a_i \|_2^2 \zeta_i + \sum_{j \neq i} (a_i, a_j) \zeta_j = \zeta_i
\end{align*}
\]
(C.21)
under Assumption 2.3, after a bit algebra and rearrangement, we obtain
\[
\zeta_i^3 - \alpha \zeta_i = 0
\]
where
\[
\alpha = \| \zeta \|_4^4 + \frac{\theta}{1 - \theta} \left( \| \zeta \|_2^2 - 1 \right) \| \zeta \|_2^2.
\]
(C.23)

We then have that, for any critical point \( q \in R_1 \), \( \zeta = A^T q \) satisfies (C.22) for all \( 1 \leq i \leq r \). Furthermore, since Lemma 3.5, stated and proved in Section C-C.3, shows that \( \alpha > 0 \), we conclude that \( \zeta \) belongs to one of the following three cases:

1) Case 1: \( \| \zeta \|_\infty = 0 \);
2) Case 2: There exists \( i \in [r] \) such that
\[
|\zeta_i| = \sqrt{\alpha}, \quad |\zeta_j| = 0, \quad \forall j \in [r] \setminus \{i\};
\]
3) Case 3: There exists at least \( i, j \in [r] \) with \( i \neq j \) such that
\[
|\zeta_i| = |\zeta_j| = \sqrt{\alpha}.
\]

Note that the definition of \( R_1 \) excludes \( R_0 \) defined in (III.3), hence rules out Case 1. We then provide analysis for the other two cases separately. Specifically, for any \( \zeta \) belonging to Case 2, Lemma 3.3 below proves that \( \zeta \) satisfies the second order optimality condition, hence is a local solution. Furthermore, \( \zeta \) is equal to one column of \( A \) up to the sign.

**Lemma 3.3:** Let \( q \) be any critical point in \( R_1(C_\ast) \) and let \( \zeta = A^T q \). If there exists \( i \in [r] \) such that
\[
|\zeta_i| = \sqrt{\alpha}, \quad |\zeta_j| = 0, \quad \forall j \in [r] \setminus \{i\},
\]
with \( \alpha \) defined in (C.23), then there exists some signed permutation \( P \) such that
\[
q = AP_i^{-1}
\]
(C.24)
Furthermore,
\[
v^T \text{Hess } f(q)v \geq (1 - \theta) \| P_i^\perp v \|_2^2,
\]
\( \forall v \) such that \( P_i^\perp v \neq 0 \).
(C.25)

**Proof:** Lemma 3.3 is proved in Section C-C.4.

Finally, we show in Lemma 3.4 below that any \( \zeta \) belonging to Case 3 is a saddle point, hence is not a local solution.

**Lemma 3.4:** For any critical point \( q \in R_1(C_\ast) \) with \( \zeta = A^T q \) and \( \alpha \) as defined in (C.23), if there exists \( k (k \geq 2) \) non-zero elements such that
\[
|\zeta_{(1)}| = |\zeta_{(2)}| = \cdots = |\zeta_{(k)}| = \sqrt{\alpha}.
\]

For some permutation \( \pi : [r] \rightarrow [r] \), then there exists \( v \) with \( P_i^\perp v \neq 0 \) such that
\[
v^T \text{Hess } f(q)v \leq \frac{(-1 - \theta)}{k} \| P_i^\perp v \|_2^2 < 0.
\]
(C.26)

**Proof:** Lemma 3.4 is proved in Section C-C.5.

Summarizing the above two lemmas concludes that all local solutions in \( R_1 \) lie in Case 2, hence completes the proof of Lemma 3.2.

3) **Additional Lemmas Used in Section C-C.2 and C-D:**

The following lemma gives the upper and lower bounds for \( \alpha \) defined in equation (C.23).

**Lemma 3.5:** For any \( q \in R_1(C_\ast) \), let \( \zeta = A^T q \) and \( \alpha \) be defined in (C.23). We have
\[
\| \zeta \|_4^4 \left[ 1 - \frac{\theta}{4(1 - \theta) C_\ast^2} \right] \leq \alpha \leq \| \zeta \|_4^4.
\]
(C.27)

As a result, when
\[
C_\ast^2 > \frac{\theta}{4(1 - \theta)},
\]
we have \( \alpha > 0 \).

**Proof:** The upper bound of \( \alpha \) follows from
\[
\alpha = \| \zeta \|_4^4 \left[ 1 + \frac{\theta}{(1 - \theta) \| \zeta \|_4^4 \left( \| \zeta \|_2^4 - \| \zeta \|_2^2 \right) } \right] \leq \| \zeta \|_4^4.
\]
by using \( \| \zeta \|_2^2 \leq 1 \) and \( \| \zeta \|_2^2 \leq \| \zeta \|_2^2 \). To prove the lower bound, we have
\[
\alpha = \| \zeta \|_4^4 \left[ 1 + \frac{\theta}{(1 - \theta) \| \zeta \|_4^4 \left( \| \zeta \|_2^4 - \| \zeta \|_2^2 \right) } \right] \geq \| \zeta \|_4^4 \left[ 1 - \frac{\theta}{(1 - \theta) \| \zeta \|_\infty^4 \left( 1 - \| \zeta \|_2^2 \right) } \right] \geq \| \zeta \|_4^4 \left[ 1 - \frac{\theta}{(1 - \theta) \| \zeta \|_\infty^4 \left( 1 - \| \zeta \|_2^2 \right) } \right] \geq \| \zeta \|_4^4 \left[ 1 - \frac{\theta}{4(1 - \theta) C_\ast^2} \right].
\]
(C.28)

Here we use \( \| \zeta \|_2^2 \leq 1 \) in first inequality and \( \| \zeta \|_\infty^2 (1 - \| \zeta \|_2^2) \leq 1/4 \) in second inequality.

4) **Proof of Lemma 3.3:**

**Proof:** Let \( q \) be any critical point in \( R_1(C_\ast) \) with \( C_\ast > 0 \).
Write \( \zeta = A^T q \) and suppose
\[
|\zeta_i| = \sqrt{\alpha}, \quad |\zeta_j| = 0, \quad \forall j \in [r] \setminus \{i\},
\]
with \( \alpha \) defined in (C.23). Our proof contains two parts. We first show that \( q = a_i \) (we assume \( P \) is identity for simplicity) and then show that \( q \) satisfies the second order optimality condition.
a) Recovery of $a_{\ell}$: First notice that

$$
\zeta_{\ell}^2 = \alpha = \frac{1}{\|a\|_4^4} \left[ 1 + \frac{\theta}{\|a\|_4^4 (1 - \theta)} \left( \frac{\|a\|_2^2}{\|a\|_4^2} - \|a\|_2^2 \right) \right].
$$

(C.29)

Since $\|a\|_2^2 = \zeta_{\ell}^2$ and $\|a\|_4^4 = \zeta_{\ell}^4$, we immediately have

$$
\alpha = \alpha^2 \left[ 1 + \frac{\theta}{\alpha^2 (1 - \theta)} (\alpha^2 - \alpha) \right].
$$

(C.30)

Solving it gives $\alpha = 1$, which implies $\zeta_{\ell}^2 = |\langle a_{\ell}, q \rangle|^2 = 1$, as desired.

b) Second order optimality: We prove

$$
v^T \text{Hess} f(q)v = v^T [\text{Hess}_{\theta} f(q) + \text{Hess}_{\theta} f(q)] v > 0
$$

for all $v$ such that $P_q v \neq 0$.

Recall from (C.8) that

$$
\text{Hess}_{\theta} f(q) = -(1 - \theta) P_q = \begin{bmatrix}
3 \sum_{j=1}^r a_j^T (q^T a_j)^2 & -a^T A a \\
-a^T A & I
\end{bmatrix} P_q.
$$

(C.31)

Without loss of generality, let $v \in \mathbb{S}^{p-1}$ be any vector such that $v \perp q$. Recall that $\zeta = A^T q$. Then

$$
v^T \text{Hess}_{\theta} f(q)v = (1 - \theta) \left[ -3 \sum_{j=1}^r (a_j^T v)^2 \zeta_j^2 + \|\zeta\|_2^4 \right]
$$

(C.32)

where we used $\zeta_j^2 = 1$ and $\zeta_j = 0$ for all $j \neq \ell$ together with $\|\zeta\|_2^4 = 1$ in the second line. In addition, we find

$$
(a^T v)^2 = |\langle a, v \rangle|^2 = \|q, v\|^2 = 0.
$$

(C.33)

so that

$$
v^T \text{Hess}_{\theta} f(q)v = 1 - \theta.
$$

(C.34)

On the other hand,

$$
v^T \text{Hess}_{\theta} f(q)v = \theta \left[ -2 (v^T A \zeta)^2 - \|A v\|_2^2 + \|\zeta\|_2^2 \right]
$$

$$
= \theta \left[ -2 (a^T v \zeta) \right] - \|A v\|_2^2 + 1
$$

$$
\geq 0,
$$

(C.35)

where we used $\lambda_1(A A^T) \leq 1$ in the last line. Combine equation (C.34) and inequality (C.106) to obtain

$$
v^T \text{Hess} f(q)v \geq 1 - \theta > 0,
$$

(C.36)

completing the proof.

5) Proof of Lemma 3.4:

Proof: Let $q$ be any critical point $q \in R_{\theta}(C_{\ast})$ with $C_{\ast} > 0$ and $\zeta = A^T q$ having at least $k$ non-zero entries for $2 \leq k \leq r$. Without loss of generality, we assume

$$
\|\zeta\|_2 = \sqrt{\alpha} \quad \forall j \leq k, \quad \zeta_j = 0 \quad \forall j > k.
$$

(C.37)

We show there exists $v$ such that

$$
v^T \text{Hess} f(q)v = v^T (\text{Hess}_{\theta} f(q) + \text{Hess}_{\theta} f(q)) v = -2 (1 - \theta) \|P_q v\|^2_2 < 0.
$$

Without loss of generality, pick any vector $v \in \mathbb{S}^{p-1}$ satisfying $v \perp q$ and $v$ lies in the span of $\{a_1, a_2, \ldots, a_k\}$. Write $v = \sum_{j=1}^k c_j a_j$. From (C.7), we have

$$
v^T \text{Hess}_{\theta} f(q)v = (1 - \theta) \left[ -3 v^T A D_c^2 A v + \|\zeta\|_4^4 \right]
$$

(C.38)

Recall from the definition of $\alpha$ in (C.23) that

$$
\alpha = \|\zeta\|_4^4 \left[ 1 + \frac{\theta}{\|\zeta\|_4^4 (1 - \theta)} (\|\zeta\|_2^2 - \|\zeta\|_2^2) \right],
$$

(C.39)

using $\|\zeta\|_4^4 = \sum_{j=1}^r \zeta_j^4 = \sum_{j=1}^r \zeta_j^4 = k \alpha^2$ and $\|\zeta\|_2^2 = \sum_{j=1}^r \zeta_j^2 = k \alpha$ yields

$$
\alpha = k \alpha^2 \left[ 1 + \frac{\theta}{k \alpha^2 (1 - \theta)} (k \alpha^2 - k \alpha) \right].
$$

(C.40)

Solve the equation above to obtain $\alpha = 1/k$, hence

$$
\|\zeta\|_4^4 = \frac{1}{k}, \quad \|\zeta\|_2^4 = \frac{1}{k}, \quad \forall j \leq k.
$$

Plugging this into (C.38) gives

$$
v^T \text{Hess}_{\theta} f(q)v = (1 - \theta) \left[ -3 \sum_{j=1}^k (a_j^T v)^2 + \frac{1}{k^2} \right]
$$

$$
= (1 - \theta) \left[ -3 \sum_{j=1}^k c_j^2 + \frac{1}{k} \right]
$$

$$
= -2 (1 - \theta),
$$

(C.41)

where the second equality used $\sum_{j=1}^k (a_j^T v)^2 = \sum_{j=1}^k c_j^2 = 1$. On the other hand, we have

$$
v^T \text{Hess}_{\theta} f(q)v = \theta \left[ -2 (v^T A \zeta)^2 - \|A v\|_2^2 + \|\zeta\|_2^2 \right]
$$

$$
\leq \theta \left[ -\|\zeta\|_2^2 \|A^T v\|_2^2 + \|\zeta\|_2^4 \right]
$$

$$
= \theta \left[ -\|\zeta\|_2^2 \sum_{j=1}^k c_j^2 + \|\zeta\|_2^4 \right]
$$

$$
= \theta \left[ -\|\zeta\|_2^2 \sum_{j=1}^k c_j^2 + \|\zeta\|_2^4 \right]
$$
In particular, if the objective \( q \) critical point in region \( \mathbb{S}^p \) and (II.4), respectively, at any point \( \zeta \),

\[
\sup_{q \in \mathbb{S}^p} \| A^T q \|_2 \leq c_* ,
\]

and \( \sup_{q \in \mathbb{S}^p} \| \text{Hess}(q) \|_{op} \leq \delta \),

**(D. Proof of Theorem 3.4)**

To prove Theorem 3.4, analogous to (III.3), we give a new partition of \( \mathbb{S}^p \) as

\[
R'_0 = R'_0(c_*) = \left\{ q \in \mathbb{S}^p : \| A^T q \|_2^2 \leq c_* \right\} ,
\]

\[
R'_1 = R_1(C_*) = \left\{ q \in \mathbb{S}^p : \| A^T q \|_2^2 \geq C_* \right\} ,
\]

\[
R'_2 = \mathbb{S}^{p-1} \setminus \left( R'_0 \cup R'_1 \right) .
\]

Here \( c_* \) and \( C_* \) are positive constants satisfying \( 0 \leq c_* \leq C_* < 1 \).

Let \( \delta_1 \) and \( \delta_2 \) be some positive sequences to be determined later. Define the random event

\[
E = \left\{ \sup_{q \in \mathbb{S}^p} \| \text{grad}(q) - \text{grad}(F(q)) \|_2 \leq \delta_1 , \sup_{q \in \mathbb{S}^p} \| \text{Hess}(q) - \text{Hess}(F(q)) \|_{op} \leq \delta_2 \right\} .
\]

Here \( \text{grad}(q) \) and \( \text{grad}(F(q)) \) are the gradients of (II.5) and (II.4), respectively, at any point \( q \in \mathbb{S}^p \). Similarly, \( \text{Hess}(q) \) and \( \text{Hess}(F(q)) \) are the corresponding Hessian matrices.

On the event \( E \), the results of Theorem 3.4 immediately follow from the two lemmas below. Lemma 3.6 shows that the objective \( F(q) \) in (II.4) exhibits negative curvature at any point \( q \in R'_2 \). Meanwhile, Lemma 3.7 proves that any critical point in region \( R'_1 \) is either a solution that is close to the ground-truth, or a saddle point with negative curvature that is easy to escape by any second-order descent algorithm.

**Lemma 3.6 (Optimization Landscape for \( R'_2 \)):** Assume \( \theta \leq 6/25 \). For any point \( q \in R'_2(C_*) \) with

\[
C_* \leq \frac{12}{25} - 2\theta ,
\]

if \( \delta_2 \leq c_* / 25 \), for some constant \( c_* \in (0, C_*) \), then there exists \( v \) such that

\[
v^T \text{Hess}(F(q)) v < 0 .
\]

In particular, if \( \theta \leq 1/9 \), for any point \( q \in R'_2(C_*) \) with \( C_* \leq 1/4 \), there exists \( v \) such that

\[
v^T \text{Hess}(F(q)) v < -\frac{21}{100} \| \zeta \|_2^2 < -\frac{1}{5} c_* .
\]

**Lemma 3.7 (Optimization Landscape for \( R'_1 \)):** Assume

\[
\theta \leq 1/9 , \quad \delta_1 \leq 5 \times 10^{-5} , \quad \delta_2 \leq 10^{-3} .
\]

Any local solution \( \bar{q} \in R'_1(C_*) \) with \( C_* \geq 1/5 \) satisfies

\[
\| \bar{q} - AP_1 \|_2^2 \leq C\delta_1 ,
\]

for some signed permutation matrix \( P \) and some constant \( C > 0 \).

Finally, the proof of Theorem 3.4 is completed by invoking Lemmas 3.15 and 3.16 and using condition (III.7) to establish that \( \delta_2 \leq \varepsilon_* / 25, \delta_2 \leq 10^{-3} \) and \( \delta_1 \leq 5 \times 10^{-5} \). Indeed, we have

\[
\delta_1 = \sqrt{\frac{r^2 \log(M_n)}{\theta n}} + \frac{M_n r \log(M_n)}{n} ,
\]

\[
\delta_2 = \sqrt{\frac{r^3 \log(M_n)}{\theta n}} + \frac{M_n r \log(M_n)}{n} ,
\]

where \( M_n = C(n + r) (\theta r^2 + \log^2 n / \theta) \). Under (III.7), we have \( \log(M_n) \leq \log n \) whence \( \delta_2 \leq \min \{ \varepsilon_* / 25, 10^{-3} \} \),

\[
\delta_2 \leq C \max \left\{ \frac{r^3 \log n}{\theta^2 C_*}, \left( \frac{\theta r^2 + \log^2 n}{\theta} \right) \frac{\log n}{C_*} \right\} ,
\]

for sufficiently large \( C > 0 \), which holds under (III.7).

**1) Proof of Lemma 3.6**

**Proof:** Fix \( C_* \). Pick any \( q \in R'_2(C_*) \) and write \( \zeta = A^T q \) for simplicity. Assume \( |g_i| = \| \zeta \|_\infty \) for some \( i \in [r] \). Note that, on the event \( E \),

\[
\alpha_i^T \text{Hess}(F(q)) a_i \leq \alpha_i^T \text{Hess}(F(q)) a_i + \alpha_i^T [\text{Hess}(F(q)) - \text{Hess}(f(q))] a_i \leq \alpha_i^T \text{Hess}(f(q)) a_i + \frac{1}{29} \| \zeta \|_\infty^2 ,
\]

where in the last inequality we used \( \delta_2 \leq \varepsilon_* / 25 \leq \| \zeta \|_\infty^2 / 25 \) as \( q \in R'_2(C_*) \). By inequality (C.14), we obtain

\[
\alpha_i^T \text{Hess}(f(q)) a_i \leq -4 \| \zeta \|_\infty^2 \left\{ \| \zeta \|_\infty^4 - \frac{3 (1 - \theta)}{2} \| \zeta \|_\infty^2 + \frac{1 - 3\theta}{2} \right\} ,
\]

hence

\[
\alpha_i^T \text{Hess}(F(q)) a_i \leq -4 \| \zeta \|_\infty^2 \left\{ \| \zeta \|_\infty^4 - \frac{3 (1 - \theta)}{2} \| \zeta \|_\infty^2 + \frac{1 - 3\theta}{2} \right\} .
\]

Define

\[
g(x) = x^2 - \phi x + \omega ,
\]

with \( \phi = \frac{3 (1 - \theta)}{2} , \omega = \frac{1 - 3\theta}{2} - \frac{1}{100} \)

It remains to prove \( \alpha_i^T \text{Hess}(F(q)) a_i \leq -4 \| \zeta \|_\infty^2 g(\| \zeta \|_\infty^2) < 0 \). To this end, note that \( \omega > 0 \) under \( \theta < 1/4 \). Since

\[
\phi^2 - 4\omega = \frac{9 (1 - \theta)^2}{4} - 2 + 6\theta + \frac{1}{25} = \left( \frac{3\theta + 1}{2} \right)^2 + \frac{1}{25} > 0 ,
\]

(56)
we know that, for all
\[
\frac{\|\zeta\|_\infty^2}{2} \geq \frac{\phi - \sqrt{\phi^2 - 4\omega}}{2} \geq \frac{3 - \theta}{2} - \frac{(3\theta + 1)}{2} 1 + \frac{1}{25} \frac{\phi^2}{2} \geq r_-	ag{C.58}
\]
g(\|\zeta\|_\infty^2) \geq 0 and g(\|\zeta\|_\infty^2) increases as \|\zeta\|_\infty^2 gets smaller. Recall that \( q \in R_2(C_*) \) implies \( \|\zeta\|_\infty^2 < C_* \). We then have
\[
g(\|\zeta\|_\infty^2) > g(C_*)
\]
We proceed to show \( C_* \leq r_- \) by noticing that
\[
r_- \geq \frac{3 - \theta}{2} - \frac{(3\theta + 1)}{2} 1 + \frac{1}{25} \phi^2 \geq 1 - 3\theta - \frac{12}{25} \theta - \frac{1}{25} \geq \frac{12}{25} - 2\theta.	ag{C.59}
\]
Thus, provided that
\[
C_* \leq \frac{12}{25} - 2\theta,
\]
we conclude \( g(\|\zeta\|_\infty^2) > g(C_*) \geq 0 \) hence
\[
a_i^T \text{Hess} \, F(q) a_i \leq -4 \|\zeta\|_\infty^2 g(\|\zeta\|_\infty^2) < 0. \tag{C.60}
\]
In particular, taking \( \theta \leq 1/9 \) and \( C_* \leq 1/4 \) yields
\[
a_i^T \text{Hess} \, F(q) a_i \leq -\frac{21}{100} \|\zeta\|_\infty^2.	ag{C.61}
\]
This completes the proof.

2) Proof of Lemma 3.7:

Proof: The proof of this lemma is similar in spirit to that of Lemma C-C.2. Follows the notations there, any critical point \( q \in \tilde{R}_1^*(C_*) \) satisfies
\[
\text{grad} \, f(q) + \text{grad} \, F(q) = \text{grad} \, f(q) = 0. \tag{C.62}
\]
Following the same procedure of proving Lemma C-C.2, analogous to (C.22), we obtain
\[
\zeta_i^3 - \alpha \zeta_i + \beta = 0 \tag{C.63}
\]
for any \( i \in [r] \), where \( \zeta = \alpha \zeta_i \). Note that \( \alpha > 0 \) from Lemma 3.5 and we also prove in Lemma 3.11, stated and proved in Section C-D.3, that \( 4\beta < \alpha^{3/2} \). In conjunction with Lemma 3.12 in Section C-D.3, we conclude that \( \zeta \) belongs to one of the following three cases:

- **Case 1:**
  \[
  |\zeta_i| \leq \frac{2|\beta|}{\alpha}, \quad \forall 1 \leq i \leq r;
  \]
- **Case 2:** There exists \( i \in [r] \) such that
  \[
  |\zeta_i| \geq \sqrt{\alpha - \frac{2|\beta|}{\alpha}}, \quad |\zeta_j| < \sqrt{\alpha - \frac{2|\beta|}{\alpha}}, \quad \forall j \in [r] \setminus \{i\}; \tag{C.65}
  \]
- **Case 3:** There exists at least \( i, j \in [r] \) with \( i \neq j \) such that
  \[
  |\zeta_i| \geq \sqrt{\alpha - \frac{2|\beta|}{\alpha}}, \quad |\zeta_j| \geq \sqrt{\alpha - \frac{2|\beta|}{\alpha}}.
  \]
We provide analysis case by case. **Case 1** is ruled out by Lemma 3.8 below. For any \( \zeta \) belonging to **Case 2**, Lemma 3.9 below proves that \( \zeta \) satisfies the second order optimality condition, hence is a local solution. Furthermore, \( q \) is close to one column of \( A \). Finally, Lemma 3.10 shows that any \( \zeta \) belonging to **Case 3** is a saddle point, hence is not a local solution. Summarizing the Lemmas 3.8 – 3.10 concludes that all local solutions in \( R_1 \) lie in **Case 2**, hence concludes the proof of lemma 3.7. Lemmas 3.8 – 3.10 are proved in Sections C-D.4, C-D.6 and C-D.5, respectively.

**Lemma 3.8:** Assume \( \theta \leq 1/9, \quad \delta_1 \leq 10^{-4}. \)

For any critical point \( q \in \tilde{R}_1^*(C_*) \) with \( C_* \geq 1/5 \), there exists at least one \( i \in [r] \) such that
\[
\|\zeta_i\| > \frac{2|\beta|}{\alpha}
\]
where \( \zeta = \alpha^T q \) and \( \alpha \) and \( \beta \) are defined in (C.64).

**Lemma 3.9:** Assume
\[
\theta \leq 1/9, \quad \delta_1 \leq 5 \times 10^{-5}, \quad \delta_2 \leq 10^{-3}. \tag{C.66}
\]
Let \( q \) be any critical point in \( \tilde{R}_1^*(C_*) \) with \( C_* \geq 1/5 \). If there exists \( i \in [r] \) such that
\[
|\zeta_i| \geq \sqrt{\alpha - \frac{2|\beta|}{\alpha}}, \quad |\zeta_j| \leq \frac{2|\beta|}{\alpha}, \quad \forall j \in [r] \setminus \{i\},
\]
with \( \zeta = \alpha^T q \) and \( \alpha \) and \( \beta \) defined in (C.64), then
\[
\|q - AP_1^2\|_2^2 \leq C\delta_1. \tag{C.67}
\]
for some signed permutation matrix \( P \) and some constant \( C > 0 \). Furthermore,
\[
v^T \text{Hess} \, f(q) v > 0, \quad \forall v \text{ such that } P_q^2 v \neq 0. \tag{C.68}
\]

**Lemma 3.10:** Assume
\[
\theta \leq 1/9, \quad \delta_1 \leq 10^{-4}, \quad \delta_2 \leq 10^{-3}. \tag{C.69}
\]
For any critical point \( q \in \tilde{R}_1^*(C_*) \) with \( C_* \geq 1/5 \), if there exists \( i, j \in [r] \) with \( i \neq j \) such that
\[
|\zeta_i| \geq \sqrt{\alpha - \frac{2|\beta|}{\alpha}}, \quad |\zeta_j| \geq \sqrt{\alpha - \frac{2|\beta|}{\alpha}},
\]
where \( \zeta = \alpha^T q \) and \( \alpha \) and \( \beta \) are defined in (C.64), then there exists \( v \) with \( P_q^2 v \neq 0 \) such that
\[
v^T \text{Hess} \, f(q) v \leq -0.00315 \|P_q^2 v\|_2^2 < 0. \tag{C.70}
\]
3) Lemmas Used in Section C-D.2:

Lemma 3.11: Assume $\theta \leq 1/9$. For any critical point $q \in R'_1(C_\ast)$ with $C_\ast \geq 1/5$, on the event $\mathcal{E}$ in (C.45), we have

$$4|\beta| < \alpha^{3/2}$$

where $\beta$ and $\alpha$ are defined in (C.64).

Proof: By definition and the event $\mathcal{E}$,

$$|\beta| = |\langle \text{grad} \ f(q) - \text{grad} \ F(q), a_i \rangle| \leq \delta_1||a_i||_2 = \delta_1.$$ (C.71)

Then by Lemma 3.5,

$$\frac{|\beta|}{\alpha^{3/2}} \leq \frac{\delta_1}{\alpha^{3/2}} \leq \frac{\delta_1}{\|\zeta\|^2 \left[1 - \frac{\theta}{4(1 - \theta) C_\ast^2}\right]^{1/2}}.$$ (C.72)

Since $q \in R'_1(C_\ast)$ implies $\|\zeta\|_\infty \geq C_\ast$, using $\|\zeta\|^6_\infty \geq \|\zeta\|^6_\infty \geq C^2_\ast$ together with $C_\ast \geq 1/5$ and $\theta < 1/9$ gives

$$\|\zeta\|^6_\infty \left[1 - \frac{\theta}{4(1 - \theta) C_\ast^2}\right]^{1/2} \geq \left[ C_\ast^2 - \frac{\theta}{4(1 - \theta)} \right]^{1/2} \geq \left( \frac{7}{800} \right)^{1/2}. $$ (C.72)

The result follows from $\delta_1 < 2 \times 10^{-4}$. □

Lemma 3.12 (Lemma B.3, [35]): Considering the cubic function

$$f(x) = x^3 - \alpha x + \beta$$

When $\alpha \geq 0$ and $4|\beta| \leq \alpha^{3/2}$, the roots of the function $f(\cdot)$ are contained in the following union of the intervals.

$$\left\{ |x| \leq \frac{2|\beta|}{\alpha} \right\} \bigcup \left\{ |x - \sqrt{\alpha}| \leq \frac{2|\beta|}{\alpha} \right\}$$

$$\bigcup \left\{ |x + \sqrt{\alpha}| \leq \frac{2|\beta|}{\alpha} \right\}. $$ (C.75)

4) Proof of Lemma 3.8:

Proof: We prove that for any critical point $q \in R'_1$, there exists at least one $i \in [r]$ such that $|\zeta_i| > 2|\beta|/\alpha$ with $\alpha$ and $\beta$ being defined in (C.64). Suppose

$$|\zeta_i| \leq \frac{2|\beta|}{\alpha}, \quad \forall i \in [r].$$

Assume $|\zeta_k| = \|\zeta\|_\infty$ for some $k \in [r]$. We obtain

$$\|\zeta\|_\infty \leq \frac{2|\beta|}{\alpha},$$

hence, by also using $\|\zeta\|_2 \leq 1$,

$$\|\zeta\|^2_2 \leq \|\zeta\|^2_\infty \|\zeta\|_2^2 \leq \frac{4|\beta|^2}{\alpha^2} \leq \frac{4\delta^2}{\|\zeta\|^2 \left[1 - \frac{\theta}{4(1 - \theta) C_\ast^2}\right]^{1/2}} \|\zeta\|^4_4.$$ (C.76)

This is a contradiction whenever

$$\delta_1 \leq \frac{1}{2} \left( \frac{7}{800} \right)^{-3/2},$$

which is the case if $\delta_1 \leq 10^{-4}$. □

5) Proof of Lemma 3.10:

Proof: Let $q$ be any critical point $q \in R'_1(C_\ast)$ with $C_\ast \geq 1/5$ and write $\zeta = A^T q$. Suppose there exists $l, m \in [r]$ with $l \neq m$ such that

$$|\zeta_l| > \sqrt{\alpha - \frac{2|\beta|}{\alpha}}, \quad |\zeta_m| > \sqrt{\alpha - \frac{2|\beta|}{\alpha}}.$$ (C.77)

We prove there exist $v$ such that.

$$v^T \text{Hess} F(q) v \leq \langle v^T \text{Hess} f(q) v + v^T \text{Hess}_1 f(q) v \rangle < 0.$$ (C.78)

Pick any vector $v \in S^{p-1}$ such that $v \perp q$ and $v$ lies in the span of $\{a_l, a_m\}$, that is, $v = c_l a_l + c_m a_m$ for some $c_l^2 + c_m^2 = 1$. Recall from (C.7) that

$$v^T \text{Hess} f(q) v = v^T \text{Hess}_1 f(q) v + v^T \text{Hess}_2 f(q) v.$$ (C.79)

By (C.8), we first have

$$\langle v^T \text{Hess}_1 f(q) v \rangle = (1 - \theta) \left[ -3 \left( a_l^T v \right)^2 \zeta_l^2 - 3 \left( a_m^T v \right)^2 \zeta_m^2 \right. \right.$$ \left. \right.$$ - \sum_{k \neq l, k \neq m} \left[ A^T v \right]^2 \zeta_k^2 + \|\zeta\|^4_4 \right.$$

$$\leq (1 - \theta) \left[ -3 \left( a_l^T v \right)^2 + \left( a_m^T v \right)^2 \right] \min \{\zeta_l^2, \zeta_m^2 \} + \|\zeta\|^4_4 \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \right.$$ \r...
\[
\begin{align*}
\leq 4\delta_1 \left( \frac{800}{\theta} \right)^{1/2} \\
\leq \frac{4\delta_1}{C_*^4} \left( \frac{800}{\theta} \right)^{1/2} \|\zeta\|_4^4,
\end{align*}
\] (C.80)

where the second inequality is due to (C.72) and the last one uses \(\|\zeta\|_4^4 \geq \|\zeta\|_\infty^4 \geq C_*^4\). By writing
\[
\eta = \frac{4\delta_1}{C_*^4} \left( \frac{800}{\theta} \right)^{1/2},
\]
(C.81)

it follows that
\[
\zeta_l^2 \geq \|\zeta\|_4^4 (1 - \eta) - \frac{\theta}{(1 - \theta)} \left[ \|\zeta\|_2^2 - \|\zeta\|_4^4 \right].
\] (C.82)

This lower bound also holds for \(\min\{\zeta_l^2, \zeta_m^2\}\). Plugging it in (C.78) yields
\[
\begin{align*}
v^T \text{Hess}_{\zeta_l} f(q) v \\
\leq (1 - \theta) \left\{ -3 \left( \|\zeta\|_4^4 (1 - \eta) - \frac{\theta}{(1 - \theta)} \left[ \|\zeta\|_2^2 - \|\zeta\|_4^4 \right] \right) \\
+ \|\zeta\|_4^4 \right\} \\
= (1 - \theta) \left\{ (-2 + 3\eta) \|\zeta\|_4^4 + \frac{3\theta}{1 - \theta} \left[ \|\zeta\|_2^2 - \|\zeta\|_4^4 \right] \right\}.
\end{align*}
\] (C.83)

On the other hand, from (C.8), we have
\[
\begin{align*}
v^T \text{Hess}_{\zeta_l} f(q) v \\
= \theta \left[ -2 (v^T A\zeta)^2 - \|\zeta\|_2^2 \right] A^T v\|_2^2 + \|\zeta\|_2^2 \\
\leq \theta \left[ - \|\zeta\|_2^2 \right] A^T v\|_2^2 + \|\zeta\|_2^2 \\
= \theta \left[ - \|\zeta\|_2^2 + \|\zeta\|_2^2 \right] \\
= \|v\|_2^2 \\
\end{align*}
\] (C.84)

Third inequality uses \(\|A^T v\|_2^2 = \sum_{j=1}^r (v^T a_j)^2 = \zeta_1^2 + \zeta_m^2 = \|v\|_2^4 = 1\). Combine (C.84) and (C.83) to obtain
\[
\begin{align*}
v^T \text{Hess}_{\zeta_l} f(q) v \\
\leq (1 - \theta) \left\{ (-2 + 3\eta) \|\zeta\|_4^4 + \frac{3\theta}{1 - \theta} \left[ \|\zeta\|_2^2 - \|\zeta\|_4^4 \right] \right\} \\
+ \theta \left[ - \|\zeta\|_2^2 + \|\zeta\|_2^2 \right] \\
\leq (1 - \theta) \left\{ (-2 + 3\eta) \|\zeta\|_4^4 + \frac{3\theta}{1 - \theta} \left[ \|\zeta\|_2^2 - \|\zeta\|_4^4 \right] \right\} \\
\leq (1 - \theta) \left\{ (2 - 3\eta) \right\} \|\zeta\|_4^4 + \frac{\theta}{2} \\
\leq \Delta_l \delta_1,
\] (C.85)

Here \(\|\zeta\|_2^2 - \|\zeta\|_4^4 \leq 1/4\) is used in last step. We thus conclude that
\[
\begin{align*}
v^T \text{Hess} F(q) v \\
\leq v^T \text{Hess}_{\zeta_l} f(q) v + \|\text{Hess} F(q) - \text{Hess} f(q)\|_{op} \\
\leq (1 - \theta) \left\{ (2 - 3\eta) \right\} \|\zeta\|_4^4 + \frac{\theta}{2} + \delta_2,
\] (C.86)

on the event \(\mathcal{E}\). Note that \(-2 + 3\eta \leq 0\) from \(C_* \geq 1/5\) and \(\delta_1 < 10^{-4}\). By using \(\|\zeta\|_4^4 \geq \|\zeta\|_\infty^4 \geq C_*^4\), we obtain
\[
v^T \text{Hess} F(q) v \leq (1 - \theta) (2 - 3\eta) C_*^4 + \frac{\theta}{2} + \delta_2 \\
= (1 - \theta) \left( -2C_*^2 + 12\delta_1 \left( \frac{800}{\theta} \right)^{1/2} \right) + \frac{\theta}{2} + \delta_2. \] (C.87)

Here we use (C.81) in last step.

Recalling that
\[
\begin{align*}
\theta \leq 1/9, \quad C_* \geq 1/5, \quad \delta_1 \leq 10^{-4}, \quad \delta_2 \leq 10^{-3},
\end{align*}
\]
we further have
\[
v^T \text{Hess} F(q) v \leq -0.00315 \|P^4 v\|_2^2 = -0.00315 < 0.
\] (C.88)

This completes the proof. \(\square\)

6) Proof of Lemma 3.9:

Proof: This proof contains two parts: the first part shows that any critical \(q \in R_1\) is close to the ground truth vector \(a_l\) for some \(l \in [m]\), and the second part proves the second order optimality for this \(q\).

a) Closeness to the target ground-truth vector: Pick any critical point \(q \in R_1\) and suppose that, for some \(l \in [m]\),
\[
Q \geq \sqrt{\alpha - \frac{2\beta}{\alpha}}, \quad \zeta_l \geq \frac{2\beta}{\alpha}, \quad \forall j \neq l.
\]

On the one hand, we bound \(\zeta_l^2\) from below as
\[
\begin{align*}
\zeta_l^2 \\
= \|\zeta\|_4^4 - \sum_{k \neq l} \zeta_k^2 \\
\leq \|\zeta\|_4^4 - \sum_{k \neq l} \zeta_k^2 \max \zeta_k^2 \\
\geq \|\zeta\|_4^4 - \|\zeta\|_2^2 \frac{4\beta^2}{\alpha^2} \\
\geq \|\zeta\|_4^4 - 4\delta_1^2 \frac{800}{\theta} \|\zeta\|_4^4
\end{align*}
\] (C.89)

The last inequality also uses \(\sum_{k \neq l} \zeta_k^2 \leq \|\zeta\|_2^2 \leq 1\). On the other hand, since Lemma 3.13, stated and proved below, ensures that
\[
|\beta_l| \leq \|a_l - q\|_2 \delta_1 := \Delta_l \delta_1,
\]
the upper bound of \(\zeta_l^2\) follows from
\[
\zeta_l^2 \\
\leq \left( \sqrt{\alpha + \frac{2\beta_l}{\alpha}} \right)^2 \\
= \alpha + \frac{4\beta^2}{\alpha^2} + 4\zeta_l^2 \\
\leq \|\zeta\|_4^4 + 4\delta_1^2 \frac{800}{\theta} \|\zeta\|_4^4
\] (C.90)

where we also use Lemma 3.5, (C.76) and (C.80) and recall from (C.81) that
\[
\eta = \frac{4\delta_1}{C_*^4} \left( \frac{800}{\theta} \right)^{1/2}.
\]

Define
\[
\xi = \frac{4\delta_1^2}{C_*^4} \left( \frac{800}{\theta} \right)^{3/2}.
\] (C.91)
Combine (C.89) and (C.90) to obtain
\[ \zeta_i^2 = \frac{\zeta_i^4}{\zeta_i^2} \geq \frac{1 - \xi}{1 + \xi + \Delta \eta} = 1 - \frac{2\xi + \Delta \eta}{1 + \xi + \Delta \eta} \] (C.92)
which implies
\[ 1 - |\zeta_i| \leq 2\xi + \Delta \eta. \]
Consequently, assuming \( \zeta_i = a_i^T q > 0 \) without loss of generality, we have
\[
\begin{align*}
|a_i - q|_2^2 &\geq \frac{|a_i|^2}{\zeta_i} + \frac{|q|^2}{\zeta_i} - 2\zeta_i \\
&= 2(1 - |\zeta_i|) \\
&\leq 4\zeta_i + 2|a_i - q|^2\eta_i,
\end{align*}
\] (C.93)
implying the desired result.

b) Second order optimality: We show that
\[ v^T \text{Hess}_F(q) v \geq 0. \] (C.94)
for any \( v \in S^{p-1} \) that \( P_q v \neq 0 \). Pick any \( v \in S^{p-1} \) such that \( v \perp q \). We have
\[
v^T \text{Hess}_F(q) v = v^T \text{Hess}_f(q) v + v^T [\text{Hess}_F(q) - \text{Hess}_f(q)] v \geq v^T \text{Hess}_f(q) v - \|\text{Hess}_F(q) - \text{Hess}_f(q)\|_\text{op} v
\] = \( v^T [\text{Hess}_f(q) + \text{Hess}_{\xi_i} f(q)] v - \|\text{Hess}_F(q) - \text{Hess}_f(q)\|_\text{op}. \) (C.95)
We bound from below \( v^T \text{Hess}_{\xi_i} f(q) v \) and \( v^T \text{Hess}_{\xi_i} f(q) v \) respectively.
Recall from (C.8) that
\[
\text{Hess}_{\xi_i} f(q) = -(1 - \theta)P_q v_i \left[ 3 \sum_{j=1}^r a_j^T (q^T a_j)^2 - \|q^T A\|_4 I \right] P_q v + \]
\[
\begin{align*}
&\leq 2 \left( v^T A \zeta_i - \sum_{k \neq l} a_k^T v_{\zeta_k} \right)^2 \\
&\leq 2 \left( v^T A \zeta_i^2 + \sum_{k \neq l} a_k^T v_{\zeta_k} \right)^2 \\
&\leq 2 \left( 2(1 - \zeta_i^2) \zeta_i^2 + \sum_{k \neq l} a_k^T v_{\zeta_k}^2 \right) \\
&\leq 2 \left( 2(1 - \zeta_i^2) \zeta_i^2 + \|\zeta_i\|_2^2 - \zeta_i^2 \right),
\end{align*}
\] (C.103)
where we used (C.98) and Cauchy-Schwarz inequality in the second line, and the fact that \( \sum_{k \neq l} (A^T v)_k^2 \leq \|A^T v\|_2^2 \leq 1 \) in the third line. By observing that
\[
\|\zeta_i\|_2^2 = \frac{\|\zeta_i\|_4^2}{\zeta_i^2} \|\zeta_i\|_4^2 \leq \frac{\|\zeta_i\|_4^2}{\zeta_i^2} \|\zeta_i\|_4^2 \leq \frac{\|\zeta_i\|_4^2}{\zeta_i^2},
\] (C.104)
and
\[ \zeta_i^2 \leq \frac{\|\zeta_i\|_4^2}{\zeta_i^2}, \]
we have
\[
\|\zeta_i\|_2^2 - \zeta_i^2 \leq \frac{\|\zeta_i\|_4^2}{\zeta_i^2} (1 - \zeta_i^2),
\] which further yields
\[
(v^T A \zeta_i)^2 \leq 2 \left( v^T A \zeta_i^2 + \sum_{k \neq l} a_k^T v_{\zeta_k}^2 \right).
\] (C.102)
\begin{align*}
\leq & \frac{2(2\xi + \eta)}{1 + \xi + \eta} \left[ 2(1 + \xi + \eta) + \left( 1 - \frac{2\xi + \eta}{1 + \xi + \eta} \right)^2 \right] \| \xi \|_F^2 \\
& \leq 2(2\xi + \eta) \left[ 2 + \frac{1}{1 - 3\xi - \eta} \right] \| \xi \|_F^2.
\end{align*}
(C.105)

Here we use lemma (C.92) in second inequality and (C.92), (C.100) in third inequality. On the other hand, we have
\[ \| \xi \|_F^2 \| Av \|_2^2 \leq \| \xi \|_F^2 \leq \frac{1}{(1 - 2\xi + \eta)} \| \xi \|_F^2, \]
by (C.104) and (C.92), and \( \| \xi \|_F^2 \geq \| \xi \|_2^2 \). It then follows that
\begin{align*}
\nu^T \text{Hess}_{\xi}(q) \nu \\
\geq & \theta \| \xi \|_2^2 \left[ 4(2\xi + \eta) \left[ 2 + \frac{1}{1 - 3\xi - \eta} \right] \\
& - \frac{1}{(1 - 2\xi + \eta)} + 1 \right] \\
\geq & -16.5\theta \| \xi \|_2^2, 
\end{align*}
whence, on the event \( \mathcal{E} \),
\begin{align*}
\nu^T \text{Hess} F(q) \nu \\
\geq & \| \xi \|_2^2 \left[ (1 - \theta) (1 - 15\xi - 6\eta) - 16.5\theta (2\xi + \eta) \right], 
\end{align*}
whence, on the event \( \mathcal{E} \),
\begin{align*}
\nu^T \text{Hess} F(q) \nu \\
\geq & \| \xi \|_2^2 \left[ (1 - \theta) (1 - 15\xi - 6\eta) - 16.5\theta (2\xi + \eta) \right] \delta_2 \\
\geq & C^2 \left[ (1 - \theta) (1 - 15\xi - 6\eta) - 16.5\theta (2\xi + \eta) \right] \delta_2 \\
& > 0
\end{align*}
(C.108)
by using \( \delta_1 \leq 5 \times 10^{-5} \) and \( \delta_2 < 10^{-3} \).

The proof is complete.

Lemma 3.13: Let \( \beta_i \) and \( \delta_1 \) be defined in (C.64) and (C.45), respectively. Then,
\[ |\beta_i| \leq \|a_i - q\|_2 \delta_1. \]

Proof: By the gradients in (C.144) and (C.145), both \( \text{grad} \, F(q) \) and \( \text{grad} \, f(q) \) lie in the space of \( P_q \). We immediately have
\begin{align*}
|\beta_i| & = \| \langle \text{grad} \, f(q) - \text{grad} \, F(q), a_i \rangle \| \\
& = \| \langle \text{grad} \, f(q) - \text{grad} \, F(q), a_i - q \rangle \| \\
& \leq \| a_i - q \|_2 \delta_1,
\end{align*}
(C.109)
as desired.

E. Proof of Proposition 3.5

Proof: Write the eigenvalue decomposition of \( YY^T = U\Lambda U^T \) with \( U = [u_1, \ldots, u_r] \) and \( \Lambda \) contains the first \( r \) eigenvalues (in non-increasing order). By the definition of the Moore-Penrose inverse, we have
\[ D = U\Lambda^{-1/2}U^T \]
such that
\[ \tilde{Y} = DY = U\Lambda^{-1/2}U^T AX. \]

Here \( \Lambda^{-1/2} \) is the diagonal matrix with diagonal elements equal to the reciprocals of the square root of those of \( \Lambda \). Further write the SVD of \( A \) as \( A = U_A D_A V_A^T \) with \( U_A^T U_A = I_r \) and \( D_A \) being diagonal and containing non-increasing singular values. Since \( U_A = UQ \) for some orthogonal matrix \( Q \in \mathbb{R}^{r \times r} \), we obtain
\begin{align*}
D & = U_A Q^T \Lambda^{-1/2} Q U_A \\
& = U_A D_A^{-1} U_A^T + U_A \left( Q^T \Lambda^{-1/2} Q - D_A^{-1} \right) U_A^T A.
\end{align*}
(C.110)

It then follows that
\[ \tilde{Y} \]
\[ = U_A \frac{D_A^{-1}}{\sqrt{n\sigma^2\theta}} U_A^T A X \\
+ U_A \left( \frac{Q^T \Lambda^{-1/2} Q - D_A^{-1}}{\sqrt{n\sigma^2\theta}} \right) U_A^T A X \\
= U_A V_A^T A X \\
+ U_A \left( \frac{Q^T \Lambda^{-1/2} Q - D_A^{-1}}{\sqrt{n\sigma^2\theta}} \right) D_A V_A^T A X \\
= U_A V_A^T A X \\
+ U_A V_A^T V_A \left( \frac{\sqrt{n\sigma^2\theta} Q^T \Lambda^{-1/2} Q D_A - I_r}{\sqrt{n\sigma^2\theta}} \right) V_A^T A X \\
= A X + A V_A \left( \frac{\sqrt{n\sigma^2\theta} Q^T \Lambda^{-1/2} Q D_A - I_r}{\sqrt{n\sigma^2\theta}} \right) V_A^T A X \\
= A X + A \Delta X
\]
(C.111)
where we used \( \Delta = X / \sqrt{n\sigma^2\theta} \) and \( A = U_A V_A^T \) and write
\[ \Delta = V_A \left( \sqrt{n\sigma^2\theta} Q^T \Lambda^{-1/2} Q D_A - I_r \right) V_A^T \]
and it remains to bound from above \( \| \Delta \|_{op} \). Note that
\[ U_A U^T = YY^T \\
= A \left( \sigma^2 \theta I_r + XX^T - \sigma^2 \theta I_r \right) A^T \\
= U_A D_A V_A^T \left( \sigma^2 \theta I_r + XX^T - \sigma^2 \theta I_r \right) V_A D_A U_A^T.
\]
(C.112)

It then follows by using \( U_A = UQ \) that
\[ Q^T A - \sigma^2 \theta D_A^2 = D_A V_A^T \left( XX^T - \sigma^2 \theta I_r \right) V_A D_A, \]
hence
\[
\frac{1}{\theta n \sigma^2} D_A^{-1} Q^T \Lambda Q D_A^{-1} - I_r = V_A^T \left( \frac{1}{\theta n \sigma^2} X X^T - I_r \right) V_A.
\]
(C.113)

Let \( \lambda_k \) denote the largest \( k \)th eigenvalue of the left hand side of the above equation, for \( 1 \leq k \leq r \). Then Weyl’s inequality guarantees
\[
\max_k |\lambda_k - 1| \leq \left\| \frac{1}{\theta n \sigma^2} X X^T - I_r \right\|_{op}.
\]
Clearly,
\[
\|\Delta\|_{op} = \left\| \sqrt{n} \sigma^2 Q^T \Lambda^{-1/2} Q D_A - I_r \right\|_{op} = \max_k \left| \frac{1}{\sqrt{\lambda_k}} - 1 \right| = \max_k \frac{|1 - \lambda_k|}{\sqrt{\lambda_k} (1 + \sqrt{\lambda_k})} \leq \max_k \frac{|1 - \lambda_k|}{\sqrt{\lambda_k}}.
\]
(C.114)

It remains to bound from above the operator norm of \( (\theta n \sigma^2)^{-1} X X^T - I_r \). It is easy to see that
\[
E \left[ \frac{1}{\theta n \sigma^2} X X^T \right] = I_r.
\]
Since \( X_n \) for \( 1 \leq i \leq r \) and \( 1 \leq t \leq n \) are i.i.d. sub-Gaussian random variables with sub-Gaussian constant no greater than 1, classical deviation inequality of the operator norm of the sample covariance matrices for i.i.d. sub-Gaussian entries [41, Remark 5.40] gives
\[
\left\| \frac{1}{n \sigma^2} X X^T - \theta I_r \right\|_{op} \leq c \left( \frac{\sqrt{r}}{n} + \frac{r}{n} \right) \tag{C.115}
\]
with probability \( 1 - 2e^{-c' r} \) for some constants \( c, c' > 0 \). Using
\[
\frac{1}{\theta} \sqrt{\frac{r}{n}} \leq c''
\]
for some small constant \( c'' > 0 \) concludes
\[
\left\| \sqrt{n} \sigma^2 Q^T \Lambda^{-1/2} Q D_A - I_r \right\|_{op} \leq c'' \frac{1}{\theta} \sqrt{\frac{r}{n}} \tag{C.116}
\]
with probability \( 1 - 2e^{-c' r} \). This completes the proof. \( \square \)

F. Proof of Theorem 3.6

In this section we provide the proof of Theorem 3.6. Our proof is similar to Section C-D. Recall that \( \tilde{A} = U_A V_A^T \). We define a new partition of \( \mathbb{S}^{p-1} \) as
\[
R_0'' = R_0''(c_*) = \left\{ q \in \mathbb{S}^{p-1} : \right\| \tilde{A}^T q \right\|_{\infty} \leq c_*, \right\}
\]
\[
R_1'' = R_1''(C_*) = \left\{ q \in \mathbb{S}^{p-1} : \right\| \tilde{A}^T q \right\|_{\infty} \geq C_* \right\},
\]
\[
R_2'' = \mathbb{S}^{p-1} \setminus (R_0'' \cup R_1''). \tag{C.117}
\]

Here \( c_* \) and \( C_* \) are positive constants satisfying \( 0 \leq c_* \leq C_* < 1 \). Further define
\[
\bar{f}_g(q) := E \left[ \frac{1}{12 \theta n \sigma^2} \left\| q^T \tilde{A} X \right\|_{1}^4 \right]
\]
\[
= -\frac{1}{4} \left( (1 - \theta) \left\| A^T q \right\|_{4}^4 + \theta \left\| \tilde{A}^T q \right\|_{2}^4 \right). \tag{C.118}
\]

The equality uses Lemma 2.5. Let \( \delta_1 \) and \( \delta_2 \) be some positive sequences and define the random event \( \mathcal{E} \)
\[
\mathcal{E} = \left\{ \sup_{q \in \mathbb{S}^{p-1}} \left\| \text{grad} \bar{f}_g(q) - \text{grad} F_g(q) \right\|_2 \leq \delta_1, \sup_{q \in \mathbb{S}^{p-1}} \left\| \text{Hess} \bar{f}_g(q) - \text{Hess} F_g(q) \right\|_{op} \leq \delta_2 \right\}. \tag{C.119}
\]

Here \( \text{grad} \bar{f}_g(q) \) and \( \text{grad} F_g(q) \) are the gradients of (C.118) and (II.7), respectively, at any point \( q \in \mathbb{S}^{p-1} \). Similarly, \( \text{Hess} \bar{f}_g(q) \) and \( \text{Hess} F_g(q) \) are the corresponding Hessian matrices.

We observe that Lemmas 3.6 and 3.7 continue to hold by replacing \( F(q), f(q) \) and \( A \) by \( F_g(q), \bar{f}_g(q) \) and \( \tilde{A} \), respectively, and by using \( R_1' \) and \( R_2' \) in lieu of \( R_1' \) and \( R_2' \). The proof is then completed by verifying that \( \delta_2 \leq c_* / 25 \), \( \delta_2 \leq 10^{-3} \) and \( \delta_1 \leq 5 \times 10^{-3} \). These are guaranteed by invoking Lemmas 3.19 and 3.20 and using condition (III.11).

G. Proof of Lemma 4.1

Proof: It suffices to prove
\[
\left\| \tilde{A}^T q^{(0)} \right\|_{\infty} \geq c_*
\]
for some \( c_* \) such that (III.11) holds. To this end, we work on the event where Proposition 3.5 holds such that
\[
\tilde{Y} = \tilde{A} (I_r + \Delta) \tilde{X}.
\]
It then follows that
\[
\left\| \tilde{A}^T q^{(0)} \right\|_{\infty} \geq \frac{1}{r} \left\| \tilde{A}^T q^{(0)} \right\|_2 = \frac{1}{r} \left\| \tilde{A}^T \tilde{Y} 1_n \right\|_2^2 = \frac{1}{r} \left\| \left( I_r + \Delta \right) \tilde{X} 1_n \right\|_2^2 \geq \frac{1}{r} \left\| \tilde{A} (I_r + \Delta) \tilde{X} 1_n \right\|_2^2 \tag{C.120}
\]
by using \( \tilde{A}^T \tilde{A} = I_r \), provided that \( \left\| \tilde{Y} 1_n \right\|_2 \neq 0 \) which holds only one a set with zero measure. The proof is completed by invoking condition (IV.2) to ensure (III.11) holds for \( c_* = 1/(2r) \). \( \square \)
H. Proof of Lemma 4.2

Proof: Recall that $F_g(q)$ and $\tilde{f}_g(q)$ are defined in (II.7) and (C.118), respectively. We work on the event

$$E_g := \left\{ \sup_{q \in S^{p-1}} |F_g(q) - \tilde{f}_g(q)| \lesssim \delta_n \right\},$$  \hspace{1cm} (C.121)

with

$$\delta_n = \left( \sqrt{r \theta} + \sqrt{\log n} \right) \sqrt{\frac{r}{\theta^2 \sqrt{n}}} + \left( \theta r^2 + \frac{\log^2 n}{\theta} \right) \frac{r \log n}{n}.$$  \hspace{1cm} (C.122)

According to Lemma 3.17, $E_g$ holds with probability at least $1 - cn^{-c} - 2e^{-c' r}$. We aim to prove

$$\left\| \tilde{A}^T \theta(q^{(k)}) \right\|_\infty^2 \geq c_r = \frac{1}{2r}, \quad \forall k \geq 1.$$  \hspace{1cm} (C.123)

Pick any $k \geq 1$. On the event $E_g$, we have

$$F_g(q^{(k)}) \geq \tilde{f}_g(q^{(k)}) - \delta_n.$$  \hspace{1cm} (C.124)

For any $q \in S^{p-1}$, we write $\tilde{\zeta} = \tilde{A}^T \theta$. Since

$$\tilde{f}_g(q) = -\frac{1}{4} \left[ (1 - \theta) \left| \tilde{\zeta} \right|_4^4 + \theta \left| \tilde{\zeta} \right|_2^4 \right] \geq -\frac{1}{4} \left[ (1 - \theta) \left| \tilde{\zeta} \right|_\infty^2 + \theta r \left| \tilde{\zeta} \right|_\infty^2 \right] = -\frac{1}{4} (1 - \theta + \theta r) \left| \tilde{\zeta} \right|_\infty^2,$$  \hspace{1cm} (C.125)

where we used $\left| \tilde{\zeta} \right|_4^2 \leq \left| \tilde{\zeta} \right|_\infty^2 \left| \tilde{\zeta} \right|_2^2 \leq \left| \tilde{\zeta} \right|_\infty^2$, $\left| \tilde{\zeta} \right|_2^2 \leq r \left| \tilde{\zeta} \right|_\infty^2$ and $\left| \tilde{\zeta} \right|_\infty^2 \leq 1$. It then follows that

$$F_g(q^{(k)}) \geq -\frac{1}{4} (1 - \theta + \theta r) \left\| \tilde{A}^T \theta(q^{(k)}) \right\|_\infty^2 - \delta_n.$$  \hspace{1cm} (C.126)

on the event $E_g$. On the other hand, any gradient descent algorithm ensures

$$F_g(q^{(k)}) \leq F_g(q^{(0)}).$$

We thus have

$$-\frac{1}{4} (1 - \theta + \theta r) \left\| \tilde{A}^T \theta(q^{(k)}) \right\|_\infty^2 \geq -F_g(q^{(0)}) - \delta_n \geq -\tilde{f}_g(q^{(0)}) - 2\delta_n$$  \hspace{1cm} (C.127)

by using $E_g$ again in the last inequality. To bound from below $-\tilde{f}_g(q^{(0)})$, recalling the definition of $\tilde{f}_g$ from (C.118), we have

$$-\tilde{f}_g(q^{(0)}) = \frac{1}{4} \left[ (1 - \theta) \left\| \tilde{A}^T \theta(q^{(0)}) \right\|_4^4 + \theta \left\| \tilde{A}^T \theta(q^{(0)}) \right\|_2^4 \right]$$  \hspace{1cm} (C.128)

Since, on the event where Proposition 3.5 holds such that $Y = \tilde{A}(I_r + \Delta)X$,

$$\left\| \tilde{A}^T \theta(q^{(0)}) \right\|_4^4 = \left\| \tilde{A}^T \frac{Y_1}{\|Y1\|_2} \right\|_4^4 = \left\| (I_r + \Delta)X \frac{1}{\|Y1\|_2} \right\|_4^4 \left\| (I_r + \Delta)X1 \right\|_2^4$$  \hspace{1cm} (C.129)

and, similarly,

$$\left\| \tilde{A}^T \theta(q^{(0)}) \right\|_2^4 = 1,$$  \hspace{1cm} (C.130)

we conclude

$$-\tilde{f}_g(q^{(0)}) \geq \frac{1}{4} \left[ (1 - \theta) \left\| \tilde{A}^T \theta(q^{(0)}) \right\|_4^4 + \theta \left\| \tilde{A}^T \theta(q^{(0)}) \right\|_2^4 \right] \geq \frac{1}{4} \left[ (1 - \theta + \theta r) \right.$$  \hspace{1cm} (C.131)

with probability $1 - 2e^{-c r}$. Here we used the basic inequality $\|v\|_2^4 \leq r \|v\|_2^4$ for any $v \in \mathbb{R}^r$. With the same probability, we further have

$$\left\| \tilde{A}^T \theta(q^{(k)}) \right\|_\infty^2 \geq \frac{1}{r} - \frac{8\delta_n}{1 - \theta + \theta r} \geq \frac{1}{2r},$$  \hspace{1cm} (C.132)

provided that

$$\frac{8\delta_n}{1 - \theta + \theta r} \leq \frac{1}{2r}.$$  \hspace{1cm} (C.133)

This is guaranteed by condition (IV.3). The proof is complete. \(\Box\)

I. Concentration Inequalities When $A$ Is Semi-Orthonormal

In this section, we provide deviation inequalities for different quantities between the population-level problem $f(q)$ in (II.5) and its sample counterpart $F(q)$ in (II.4), including the objective function, the Riemannian gradient and the Riemannian Hessian matrix. Our analysis adapts some technical results in [45] and [35] to our setting.

1) Deviation Inequalities of the Objective Value: Recall that

$$F(q) = -\frac{1}{128 \sigma^2 n} \left\| q^TAX \right\|_4^4,$$  \hspace{1cm} (C.134)

$$f(q) = -\frac{1}{4} \left[ (1 - \theta) \left\| A^Tq \right\|_4^4 + \theta \left\| A^Tq \right\|_2^4 \right] = \frac{1}{4} \left[ (1 - \theta + \theta r) \left\| A^Tq \right\|_4^4 \right.$$  \hspace{1cm} (C.135)

Define

$$M_n = C(n + r) \left( \theta^2 + \frac{\log^2 n}{\theta} \right)$$  \hspace{1cm} (C.136)

for some constant $C > 0$.

Lemma 3.14: Under Assumptions 2.1 and 2.3, with probability greater than $1 - cn^{-c'}$ for some constants $c, c' > 0$, one has

$$\sup_{q \in S^{p-1}} |F(q) - f(q)| \lesssim \sqrt{\frac{r \log(M_n)}{\theta n}} + \frac{M_n r \log(M_n)}{n}.$$  \hspace{1cm} (C.137)

Proof: Pick any $q \in S^{p-1}$. Note that the result holds trivially if $A^Tq = 0$. For $A^Tq \neq 0$, we define

$$\tilde{q} = \frac{A^Tq}{\|A^Tq\|_2}, \quad \text{with} \quad \tilde{q} \in S^{r-1}.$$  \hspace{1cm} (C.138)

Note that

$$\frac{F(q)}{\|A^Tq\|_2^2} = \frac{1}{n} \sum_{k=1}^{n} F_q(x_k),$$  \hspace{1cm} (C.139)
with \( F_q(x_k) = - \frac{1}{12\theta \sigma^4} (q^T x_k)^4 \). \hfill (C.132)

The proof of Lemma 2.5 shows that

\[
\mathbb{E}[F_q(x_k)] = \frac{f(q)}{\|A^T q\|_2^4} = -\frac{1}{4} \left[ (1 - \theta) \|q\|_4^4 + \theta \|q\|_2^4 \right] \leq g(q), \quad \forall 1 \leq k \leq n.
\] \hfill (C.133)

We thus aim to invoke Lemma 4.6 with \( n_1 = r, d_1 = 1, n = r \) and \( p = n \) to bound from above

\[
\sup_{q \in \mathbb{S}^{p-1}} \left| \frac{1}{n} \sum_{k=1}^{n} F_q(x_k) - g(q) \right|.
\]

Consequently, the result follows by noting that

\[
\sup_{q \in \mathbb{S}^{p-1}} \left| F(q) - f(q) \right| = \sup_{q \notin \mathbb{S}^{p-1} \setminus R_n} \left| \frac{1}{n} \sum_{k=1}^{n} F_q(x_k) - g(q) \right|
\]

and using \( \|A^T q\|_2 \leq 1 \) which holds uniformly over \( q \in \mathbb{S}^{p-1} \).

Since the entries of \( x_k \) are i.i.d. Bernoulli-Gaussian random variables with parameter \( (\theta, \sigma^2) \), each \( x_{ki} \), for \( 1 \leq i \leq r \), is sub-Gaussian with the sub-Gaussian parameter equal to \( \sigma^2 \).

It thus suffices to verify Conditions 1 – 2 in Lemma 4.6. For simplicity, we write \( x = x_k \).

**Verification of Condition 1:** Since \( \mathbb{E}[F_q(x)] = g(q) \), we observe

\[
\mathbb{E}[F_q(x)] = \frac{1}{4} \left[ (1 - \theta) \|q\|_4^4 + \theta \|q\|_2^4 \right] \leq \frac{1}{4} \|q\|_2^4 = \frac{1}{4}
\] \hfill (C.135)

where we used \( \|q\|_4^4 \leq \|q\|_2^4 \). Thus \( B_f = 1/4 \). For any \( q_1 \neq q_2 \in \mathbb{S}^{p-1} \), we have

\[
\mathbb{E} \left[ F_{q_1}(x) - F_{q_2}(x) \right] \leq \frac{1}{4} \left[ (1 - \theta) \|q_1\|_4^4 - \|q_2\|_4^4 + \theta \|q_1\|_2^4 - \|q_2\|_2^4 \right] \leq \frac{1}{4} \left( \|q_1\|_2^2 + \|q_2\|_2^2 \right) 
\]

This gives \( L_f = 1 \).

**Verification of Condition 2:** We define

\[
x = \bar{x} + \bar{x}
\]

as (D.28) with \( B = 2\sigma \sqrt{\log(nr)} \). For the similar fashion, we define \( x_k = \bar{x}_k + \bar{x}_k \) for \( 1 \leq k \leq n \). We verify Condition 2 on the event

\[
\mathcal{E}' := \left\{ \|x_k\|_2 \leq \sigma(\sqrt{\theta} + \sqrt{\log n}) \right\}.
\] \hfill (C.137)

Lemma 4.3 ensures that \( \mathbb{P}(\mathcal{E}') \geq 1 - 2n^{-c} \) for some \( c > 0 \). Note that on the event \( \mathcal{E}' \),

\[
\|\bar{x}\|_2 \leq \sigma(\sqrt{\theta} + \sqrt{\log n}).
\] \hfill (C.138)

Pick any \( q \in \mathbb{S}^{r-1} \), we have

\[
\|F_q(\bar{x})\|_2 = \frac{1}{12\theta \sigma^4} \left( q^T \bar{x} \right)^4 \leq \frac{\|q\|_2^2 \|\bar{x}\|_2^4}{12\theta \sigma^4} \leq C \left( \frac{\theta r^2 + \frac{\log^2 n}{\theta}} \right)
\] \hfill (C.139)

for some constant \( C > 0 \). Thus,

\[
R_1 = C \left( \frac{\theta r^2 + \frac{\log^2 n}{\theta}} \right).
\] \hfill (C.140)

On the other hand, we have

\[
\sup_{q \in \mathbb{S}^{p-1}} \mathbb{E} \left[ \|F_q(\bar{x})\|_2^2 \right] \leq \sup_{q \in \mathbb{S}^{p-1}} \mathbb{E} \left[ \|F_q(x)\|_2^2 \right] \leq c\theta^{-1} = R_2
\] \hfill (C.141)

for some constant \( c > 0 \). Here Lemma 4.4 is used in the last inequality.

On the other hand, pick \( q_1 \neq q_2 \in \mathbb{S}^{r-1} \). We obtain

\[
\|F_{q_1}(\bar{x}) - F_{q_2}(\bar{x})\|_2 = \frac{1}{12\theta \sigma^4} \left\| (q_1^T \bar{x})^4 - (q_2^T \bar{x})^4 \right\|_2
\]

\[
= \frac{1}{12\theta \sigma^4} \left\| (q_1^T \bar{x}) - (q_2^T \bar{x}) \right\| \cdot \left\| (q_1^T \bar{x}) + (q_2^T \bar{x}) \right\| \cdot \left\| (q_1^T \bar{x})^2 + (q_2^T \bar{x})^2 \right\|
\]

\[
\leq 4 \frac{1}{12\theta \sigma^4} \left\| \bar{x} \right\|_2 \|q_1 - q_2\|_2.
\] \hfill (C.142)

Combine with (C.138) to conclude

\[
\|F_{q_1}(\bar{x}) - F_{q_2}(\bar{x})\|_2 \leq R_1 \|q_1 - q_2\|_2,
\] \hfill (C.143)

hence \( L_f = R_1 \).

Finally, invoke Lemma 4.6 with \( M = cR_1 n = M_n \) to obtain the desired result and complete the proof. \( \square \)

2) **Deviation Inequalities of the Riemannian Gradient:** In this part, we derive the deviation inequalities between the Riemannian gradient of \( F(q) \) and that of function \( f(q) \). From (C.6), note that, for any \( q \in \mathbb{S}^{p-1} \),

\[
\nabla F(q) = \frac{1}{\|q\|_2} \sum_{k=1}^{n} (q^T A x_k)^3 A x_k,
\] \hfill (C.144)

\[\quad \nabla f(q) = \frac{1}{\|q\|_2} f(q)\]
Recall that

\[ F(q) = \left(1 - \theta \right) \sum_{j=1}^{r} a_j (q^T a_j)^3 + \theta \| q^T A \|_2^2 A A^T q \].

(C.145)

Direct calculation shows that

\[ \mathbb{E} [\text{grad} F(q)] = \text{grad} f(q). \]  
(C.146)

The following lemma provides deviation inequalities between \( F(q) \) and \( f(q) \) by invoking Lemma 4.6, stated in Appendix D. Recall that \( M_n \) is defined in (C.130).

**Lemma 3.15:** Under Assumptions 2.1 and 2.3, with probability greater than \( 1 - cn^{-c'} \) for some constants \( c, c' > 0 \), one has

\[
\sup_{q \in S^{r-1}} \| \text{grad} F(q) - \text{grad} f(q) \|_2 
\leq \sqrt{\frac{r^2 \log(M_n)}{\theta n}} + M_n r \log(M_n) \frac{1}{n}.
\]  
(C.147)

**Proof:** Pick any \( q \in S^{r-1} \). As the result trivially holds for \( A^T q = 0 \), we only focus on when \( A^T q \neq 0 \). Define

\[ \bar{q} = \frac{A^T q}{\| A^T q \|_2}, \quad \text{with} \quad \bar{q} \in S^{r-1}. \]

By writing \( \zeta = A^T q \), observe that

\[
\| \text{grad} F(q) - \text{grad} f(q) \|_2 
= \left\| P_{\mathbb{R}^+} \left[ \frac{1}{3\theta^4n} \sum_{k=1}^{n} (q^T A x_k)^3 A x_k - \left(1 - \theta \right) A (\zeta)^3 + \theta \| \zeta \|_2^2 A \zeta \right] \right\|_2 
= \left\| \frac{1}{3\theta^4n} \sum_{k=1}^{n} (q^T A x_k)^3 A x_k - \left(1 - \theta \right) A (\zeta)^3 + \theta \| \zeta \|_2^2 A \zeta \right\|_2 
\leq \left\| A^T q \right\|_2 \left\| \frac{1}{3\theta^4n} \sum_{k=1}^{n} (q^T A x_k)^3 x_k - \left(1 - \theta \right) (\bar{q})^3 + \theta \| \bar{q} \|_2^2 \bar{q} \right\|_2 
\leq \left\| A^T q \right\|_2 \left\| \frac{1}{3\theta^4n} \sum_{k=1}^{n} (q^T x_k)^3 x_k - \left(1 - \theta \right) (\bar{q})^3 + \theta \| \bar{q} \|_2^2 \bar{q} \right\|_2 \],

where we have used \( \| A \|_{op} \leq 1 \) and \( \| A^T q \|_2^3 \leq 1 \) in the last two steps. Define

\[ F_q(x) := \frac{1}{3\theta^4} \left( q^T x \right)^3 x. \]  
(C.149)

It is easy to verify that

\[ \mathbb{E} \left[ F_q(x) \right] = \left(1 - \theta \right) (\bar{q})^3 + \theta \| \bar{q} \|_2^3 \bar{q} \]  
(C.150)

We thus aim to invoke Lemma 4.6 with \( n_1 = r, d_1 = r, n = r \) and \( p = n \), to bound from above

\[
\sup_{\bar{q} \in S^{r-1}} \left\| \frac{1}{n} \sum_{k=1}^{n} F_q(x_k) - g(\bar{q}) \right\|_2.
\]

Recall that \( x_{ij} \) is sub-Gaussian with parameter \( \sigma^2 \), for \( 1 \leq i \leq n, 1 \leq j \leq r \).

**Verification of Condition 1:** By \( \| \bar{q} \|_2 = 1 \), notice that

\[
\| g(\bar{q}) \|_2
\leq (1 - \theta) \| (\bar{q})^3 - (\bar{q})^3 \|_2 + \theta \| \bar{q} - \bar{q} \|_2
\leq 3(1 - \theta) \| \bar{q} - \bar{q} \|_2 + \theta \| \bar{q} \|_2
\leq 1.
\]  
(C.151)

Further note that, for any \( \bar{q}_1, \bar{q}_2 \in S^{r-1} \),

\[
\| g(\bar{q}_1) - g(\bar{q}_2) \|_2
\leq (1 - \theta) \left\| (\bar{q}_1)^3 - (\bar{q}_2)^3 \right\|_2 + \theta \| \bar{q}_1 - \bar{q}_2 \|_2
\leq 3(1 - \theta) \| \bar{q}_1 - \bar{q}_2 \|_2 + \theta \| \bar{q}_1 - \bar{q}_2 \|_2
\leq 3 \| \bar{q}_1 - \bar{q}_2 \|_2.
\]  
(C.152)

Here \( \| (\bar{q}_1)^3 - (\bar{q}_2)^3 \|_2 \leq 3 \| \bar{q}_1 - \bar{q}_2 \|_2 \) is used in the second step. As a result, \( B_1 = 1 \) and \( L_f = 6 \).

**Verification of Condition 2:** We still work on the event \( \mathcal{E}' \) in (C.137) such that \( (C.138) \) holds for all \( 1 \leq k \leq n \). In this case,

\[
\| F_q(x_i) \|_2
\leq \left\| \frac{1}{3\theta^3} \left( q^T x_i \right)^3 x_i \right\|_2
\leq \left\| x_i \right\|_2^3 \left( \frac{1}{3\theta^3} \right)
\leq C \left( \theta r^2 + \frac{\log^2 n}{\theta} \right).
\]  
(C.153)

Hence

\[ R_1 = C \left( \theta r^2 + \frac{\log^2 n}{\theta} \right). \]
(C.154)

Also from Lemma 4.8 with some straightforward modifications, we know

\[
\sup_{\bar{q} \in S^{r-1}} \mathbb{E} \left[ \| F_q(x_i) \|_2^2 \right] \leq \sup_{\bar{q} \in S^{r-1}} \mathbb{E} \left[ \| F_q(x_i) \|_2 \right] \leq c \theta^{-1} r,
\]  
(C.155)

for some constant \( c > 0 \). We thus have \( R_2 = c \theta^{-1} r \).

To calculate \( L_f \), we have

\[
\| F_{\bar{q}_1}(x_i) - F_{\bar{q}_2}(x_i) \|_2
\leq \frac{1}{3\theta^4} \left\| (\bar{q}_1^T x_i)^3 x_i - (\bar{q}_2^T x_i)^3 x_i \right\|_2
\leq \frac{1}{3\theta^4} \left\| (\bar{q}_1)^3 - (\bar{q}_2)^3 \right\|_2 \| x_i \|_2^4
\]
Here \( \|\tilde{q}_1^3 - \tilde{q}_2^3 \|_2 \leq 3 \|\tilde{q}_1 - \tilde{q}_2 \|_2 \) is used in the last step. We thus conclude
\[
\|F_{q_1}(\tilde{x}) - F_{q_2}(\tilde{x})\|_2 \leq R_1 \|q_1 - q_2\|_2
\]
(C.157)
hence \( \bar{L}_f = R_1 \).

Finally invoke Lemma 4.6 with \( M = C'(n + r)R_1 \) to complete the proof. \( \square \)

3) Deviation Inequalities of the Riemannian Hessian: In this part we will show that the Hessian of \( F(q) \) concentrates around that of \( f(q) \). Notice that, for any \( q \in \mathbb{S}^{p-1} \) with \( \zeta = A^T q \),

\[
\text{Hess } F(q) = \sum_{k=1}^{n} P_{q^\perp} \left[ 3 \left( \zeta^T x_k \right)^2 A x_k (A x_k)^T - \left( \zeta^T x_k \right)^4 I_p \right] P_{q^\perp},
\]
 \[
\text{Hess } f(q) = \sum_{k=1}^{n} P_{q^\perp} \left[ 3 \left( \zeta^T x_k \right)^2 A x_k (A x_k)^T - \left( \zeta^T x_k \right)^4 I_p \right] P_{q^\perp} + \theta P_{q^\perp} \left[ \left\| \zeta \right\|_2^2 A A^T + 2A \zeta \zeta^T - \|\zeta\|_2^4 I_p \right] P_{q^\perp} \right].
\]
(C.158)

Straightforward calculation shows that
\[
\mathbb{E}[\text{Hess } F(q)] = \text{Hess } f(q).
\]

The following lemma provides the deviation inequalities between \( \text{Hess } F(q) \) and \( \text{Hess } f(q) \) via an application of Lemma 4.5, stated in Appendix D. Recall that \( M_n \) is defined in (1.30).

Lemma 3.16: Under Assumptions 2.1 and 2.3, with probability greater than \( 1 - cn^{-c'} \) for some constants \( c, c' > 0 \), one has
\[
\sup_{q \in \mathbb{S}^{p-1}} \left\| \text{Hess } F(q) - \text{Hess } f(q) \right\|_2 \leq \sqrt{\frac{r^3 \log(M_n)}{\theta n} + \frac{M_n r \log(M_n)}{n}}.
\]
(C.160)

Proof: Pick any \( q \in \mathbb{S}^{p-1} \) and consider \( A^T q \neq 0 \). Recall that
\[
\tilde{q} = \frac{A^T q}{\left\| A^T q \right\|_2}, \quad \text{with } \tilde{q} \in \mathbb{S}^{r-1}.
\]
Observe that
\[
\left\| \text{Hess } F(q) - \text{Hess } f(q) \right\|_2 = \left\| \frac{1}{3 \theta \sigma^2} \sum_{k=1}^{n} P_{q^\perp} \left[ 3 \left( \zeta^T x_k \right)^2 A x_k (A x_k)^T - \left( \zeta^T x_k \right)^4 I_p \right] P_{q^\perp} - (1 - \theta) P_{q^\perp} \left[ 3A \text{diag}(\zeta^2) A^T - \left\| \zeta \right\|_2^4 I_p \right] P_{q^\perp} \right. \\
\left. - \theta P_{q^\perp} \left[ \left\| \zeta \right\|_2^2 A A^T + 2A \zeta \zeta^T - \left\| \zeta \right\|_2^4 I_p \right] P_{q^\perp} \right)
\]
(C.162)
Notice that the second term has been studied in Appendix C-I.1. It suffices to invoke Lemma 4.5 with $n_1 = d_1 = d_2 = r$, $n_2 = 1$ and $p = n$ to bound from above

$$\sup_{\bar{q} \in \mathbb{S}^{r-1}} \left\| \frac{1}{n} \sum_{k=1}^{n} F_{L^2}^{-1}(x_k) - g_{L^2}(\bar{q}) \right\|_{\text{op}} \leq \frac{1}{3\theta^3n} \sum_{k=1}^{n} E_{i}^{-1}(x_k) - g_{L^2}(\bar{q}).$$  \hspace{1cm} (C.166)

We note that $x_k$ is sub-Gaussian for all $1 \leq k \leq n$ and $1 \leq i \leq r$. W.L.O.G., we assume $\sigma^2 = 1$.

Verification of Condition 1: By $\|\bar{q}\|_2 = 1$, notice that

$$\left\| E \left[ F_{L^2}^{-1}(x) \right] \right\|_{\text{op}} = \left\| 3(1 - \theta) \text{diag}(\bar{q}^2^2) + \theta \left( \|\bar{q}\|_2^2 I_r + 2\bar{q}q^T \right) \right\|_{\text{op}} \leq 3(1 - \theta) \|\bar{q}\|_2^2 + 3\theta \leq 3.$$

For any $\tilde{q}_1, \tilde{q}_2 \in \mathbb{S}^{-1}$,

$$\left\| E \left[ F_{L^2}^{-1}(x) \right] - E \left[ F_{L^2}^{-1}(x) \right] \right\|_{\text{op}} \leq 3 \left( 1 - \theta \right) \text{diag}(\tilde{q}_1^2^2) - \text{diag}(\tilde{q}_2^2^2) + 2\theta \bar{q}_1 \tilde{q}_1^T \|\tilde{q}_1\|_2^2 I_r - \theta \|\bar{q}_2\|_2^2 I_r \leq 6 \left( 1 - \theta \right) \|\bar{q}_2 - \bar{q}_1\|_{\infty} + 4\theta \|\bar{q}_2 - \bar{q}_1\|_{\text{op}} + 2\theta \|\bar{q}_2 - \bar{q}_1\|_{\text{op}} \leq 6 \|\bar{q}_2 - \bar{q}_1\|_{\text{op}}.$$

We thus have $L_f = 6$ and $B_f = 3$.

Verification of Condition 2: We again work on the event $\mathcal{E}'$ in (C.173) such that, for each $i \in [n]$,

$$\left\| F_{L^2}^{-1}(x_i) \right\|_{\text{op}} \leq \frac{1}{\theta^3} \left( \left( \frac{q^T x_i}{x_i} \right)^2 \right)^{\frac{1}{2}} \leq \theta^{-1} \|\bar{q}\|_2^2 \|x_i\|^4 \leq C \left( \theta r^2 \frac{\log n}{\theta} \right)^\frac{1}{2}.$$

Lemma 4.9 in Appendix D with some straightforward modifications ensures

$$\sup_{\bar{q} \in \mathbb{S}^{r-1}} \left\| E \left[ F_{L^2}^{-1}(x_i) \right] \right\|_{\text{op}} \leq \sup_{\bar{q} \in \mathbb{S}^{r-1}} \left\| E \left[ F_{L^2}^{-1}(x_i) \right] \right\|_{\text{op}} \leq \frac{1}{3\theta^3n} \sum_{k=1}^{n} E_{i}^{-1}(x_k) - g_{L^2}(\bar{q}) \leq c\theta^{-1}r^2.$$  \hspace{1cm} (C.170)

for some constant $c > 0$. Therefore, we have

$$R_1 = C \left( \theta r^2 + \frac{\log n}{\theta} \right),$$

$$R_2 = c\theta^{-1}r^2.$$  \hspace{1cm} (C.171)

On the other hand, for any $\tilde{q}_1, \tilde{q}_2 \in \mathbb{S}^{r-1}$,

$$\left\| F_{L^2}^{-1}(x_i) - F_{L^2}^{-1}(x_i) \right\|_{\text{op}} \leq \frac{1}{\theta^3} \left( \left( \frac{q^T x_i}{x_i} \right)^2 \right)^{\frac{1}{2}} \|x_i\|_{\text{op}} \leq \frac{2}{\theta} \|x_i\|_{\text{op}} \|\bar{q}_1 - \bar{q}_2\| \leq 2R_1 \|\bar{q}_1 - \bar{q}_2\|_2 \leq 2R_1 \|\bar{q}_1 - \bar{q}_2\|_2.$$

on the event $\mathcal{E}'$, which implies $L_f = 2R_1$.

Finally invoke Lemma 4.5 with $M = C' R_1(n + r)$ to conclude the proof.

### J. Concentration Inequalities When $A$ Is Full Column Rank

In this section we provide deviation bounds for the objective values, Riemannian gradients and Hessian matrices between $F_g(q)$ and $f_g(q)$ defined as

$$F_g(q) := -\frac{\theta n}{12} \left\| Y^T \right\|_4^4,$$

$$f_g(q) := -\frac{1}{4} \left( 1 - \theta \right) \left\| A^T \right\|_4^4$$

where

$$Y = \left( \left( Y^T \right)^2 \right)^{\frac{1}{2}} Y = D Y,$$

$$A = \left( \left( A^T \right)^2 \right)^{\frac{1}{2}} A = U_A V_A^T.$$

1) Deviation Inequalities Between the Function Values:

Recall that $M_n$ is defined in (C.130).

Lemma 3.17: Under Assumptions 2.1 and 2.2, assume

$$n \geq C \max \left\{ \frac{r \log n}{\theta}, \theta r^3 \log n, \frac{r^2}{\theta \sqrt{\theta}}, \frac{r \log n}{\theta \sqrt{\theta}} \right\}$$

for some constant $C > 0$. With probability greater than $1 - cn^{-c} - 2e^{-cn}$,

$$\sup_{q \in \mathbb{S}^{r-1}} \left| F_g(q) - f_g(q) \right| \leq \left( \sqrt{r \theta} + \sqrt{\log n} \right) \left\| A \right\|_2 \frac{r}{\theta \sqrt{\theta}} + \left( \theta r^2 + \frac{\log n}{\theta} \right) \frac{r \log n}{\theta}.$$

Proof: First we introduce

$$f_g(q) := -\frac{1}{12\theta n^2} \left\| q^T A X \right\|_4^4.$$

The proof of Lemma 2.5 yields

$$\mathbb{E} [f_g(q)] = \bar{f}_g(q).$$

$$f_g(q) := -\frac{1}{12\theta n^2} \left\| q^T A X \right\|_4^4.$$
Triangle inequality gives
\[
\sup_{q \in \mathbb{S}^{p-1}} |F_g(q) - \tilde{f}_g(q)| \leq \sup_{q \in \mathbb{S}^{p-1}} |F_g(q) - f_g(q)| + \sup_{q \in \mathbb{S}^{p-1}} |f_g(q) - \tilde{f}_g(q)|.
\] (C.180)

a) Controlling \( \Gamma_1 \): Define
\[
v_0 := \sqrt{n} \sigma^2 Y^T q \quad \text{and} \quad v_1 := (AX)^T q
\] (C.181)
We have
\[
\Gamma_1 = \frac{1}{12 \sigma^2 \theta n} \sup_{q \in \mathbb{S}^{p-1}} \left| \theta^2 n^2 \sigma^4 \left\| q^T \bar{Y} \right\|_4^4 - \left\| q^T \bar{A}X \right\|_4^4 \right|
\]
\[
= \sup_{q \in \mathbb{S}^{p-1}} \frac{1}{12 \sigma^2 \theta n} \left| \left( \left\| v_0 \right\|_4 - \left\| v_1 \right\|_4 \right) \left( \left\| v_0 \right\|_4 + \left\| v_1 \right\|_4 \right) \right|
\]
\[
\times \left( \left\| v_0 \right\|_4^2 + \left\| v_1 \right\|_4^2 \right)
\]
\[
\lesssim \sup_{q \in \mathbb{S}^{p-1}} \frac{1}{\sigma^4 \theta n} \left( \left\| v_0 - v_1 \right\|_4 \left( \left\| v_0 \right\|_4^2 + \left\| v_1 \right\|_4^2 \right) \right). \quad (C.182)
\]
Invoking Lemma 3.18 gives
\[
\mathbb{P} \left\{ \Gamma_1 \lesssim \left( \sqrt{r \theta} + \sqrt{\log n} \right) \frac{1}{\theta \sqrt{n \theta n}} \right\}
\]
\[
\geq 1 - 2e^{-cr} - c'n^{-c'r}.
\] (C.183)

b) Controlling \( \Gamma_2 \): Notice that that \( \bar{A}^T \bar{A} = I_r \). We can thus apply Lemma 3.14 by replacing \( F(q) \) and \( f(q) \) with \( f_g(q) \) and \( f_g(q) \), respectively, to obtain
\[
\mathbb{P} \left\{ \Gamma_2 \lesssim \sqrt{\log(M_n)} + M_n r \log(M_n) \right\} \geq 1 - cn^{-c'}. \quad (C.186)
\]
Combining the bounds of \( \Gamma_1 \) and \( \Gamma_2 \) and using (C.176) to simplify the expressions complete the proof. \( \Box \)

Recall that, for any \( q \in \mathbb{S}^{p-1} \),
\[
v_0 := \sqrt{n} \sigma^2 Y^T q \quad \text{and} \quad v_1 := (AX)^T q
\]
with \( \bar{A} = U \bar{A}V_A^T \) and \( \bar{Y} = DY \).

Lemma 3.18: Assume \( n \geq C \theta^2 / c' \) for some constant \( C > 0 \). With probability \( 1 - 2e^{-cr} - 2n^{-c'} \) for some constant \( c, c' > 0 \), one has
\[
\sup_{q \in \mathbb{S}^{p-1}} \left\| v_0 - v_1 \right\|_\infty \lesssim \sigma \left( \sqrt{r \theta} + \sqrt{\log n} \right) \frac{1}{\theta \sqrt{n \theta n}}.
\] (C.184)

Furthermore, if additionally (C.176) holds, then with probability \( 1 - 2e^{-cr} - c'n^{-c'r} \),
\[
\sup_{q \in \mathbb{S}^{p-1}} \left\| v_1 \right\|_4 \lesssim (\theta n \sigma^4)^{1/4},
\]
\[
\sup_{q \in \mathbb{S}^{p-1}} \left\| v_0 \right\|_4 \lesssim (\theta n \sigma^4)^{1/4}.
\] (C.185)

Proof: We work on the event \( \mathcal{E}' \), defined in (C.137), intersecting with
\[
\mathcal{E}'' := \left\{ \left\| \sqrt{\theta n \sigma^2} DA - \bar{A} \right\|_\text{op} \lesssim \frac{1}{\theta \sqrt{n \theta n}} \right\},
\] (C.186)
which, according to Lemmas 4.3 and 4.1, holds with probability \( 1 - 2e^{-cr} - 2n^{-c'} \). Recall \( Y' = DY' = DAX \).
By definition,
\[
\left\| v_0 - v_1 \right\|_\infty = \max_{t \in [n]} \left\| q^T (A - \sqrt{n} \sigma^2 DA) Y_t \right\|_\infty \leq \max_{t \in [n]} \left\| x_t \right\|_2 \|q\|_2 \left\| \sqrt{n} \sigma^2 DA - \bar{A} \right\|_\text{op}
\]
\[
\lesssim \sigma \left( \sqrt{r \theta} + \sqrt{n \theta n} \right) \frac{1}{\theta \sqrt{n \theta n}} \quad \text{(by \( \mathcal{E}' \cap \mathcal{E}'' \)).}
\] (C.187)
To bound from above \( \|v_1\|_4 \), by recalling \( F(q) \) and \( f(q) \) from (C.129) with \( A \) in lieu of \( A \), we observe that
\[
\left\| v_1 \right\|_4 \leq \left\| q^T \bar{A}X \right\|_4 \leq 12 \theta n \sigma^4 \left| f(q) \right|
\]
\[
\leq 12 \theta n \sigma^4 \left( |F(q) - f(q)| + |f(q)| \right).
\] (C.188)
By Lemma 3.14 and \( |f(q)| \leq 1 \) from its proof, we obtain
\[
\sup_{q \in \mathbb{S}^{p-1}} \left\| v_1 \right\|_4 \leq 12 \theta n \sigma^4 \left( 1 + \sqrt{r \log(M_n)} + M_n r \log(M_n) \right) \frac{1}{\theta \sqrt{n \theta n}} \quad \text{(C.189)}
\]
with probability at least \( 1 - (nr)^{-2} - cM_n^{-c'} \) for some constants \( c, c' > 0 \). Here \( M_n \) is defined in (C.130). The result then follows by invoking condition (C.176) and noting that \( \log M_n \lesssim \log n \).
Finally, since
\[
\left\| v_0 \right\|_4 \leq \left\| v_1 \right\|_4 + \left\| v_0 - v_1 \right\|_4 \leq \left\| v_1 \right\|_4 + n^{1/4} \left\| v_0 - v_1 \right\|_\infty,
\] the last result follows by combining the previous two results. \( \Box \)

2) Deviation Inequalities Between the Riemannian Gradients: In this section, we derive the deviation inequalities between the Riemannian gradient of \( F_g(q) \) and that of \( \tilde{f}_g(q) \).
Note that, for any \( q \in \mathbb{S}^{p-1} \),
\[
\text{grad} F_g(q) \quad \text{and} \quad \text{grad} \|q\|_2^2 = F_g(q)
\]
\[
= - \frac{\theta n}{3} \sum_{k=1}^n (q^T \bar{Y} \bar{Y}_k)^3 \bar{Y}_k,
\] (C.190)
\[
\text{grad} \tilde{f}_g(q) \quad \text{and} \quad \text{grad} \|q\|_2^2 = \tilde{f}_g(q)
\]
\[
= - \sum_{j=1}^r A_j \left( q^T \bar{A}_j \right)^3 + \theta \|q^T \bar{A}\|^2 \bar{A} \bar{A}^T q).
\] (C.191)
Here $\mathbf{Y}$ and $\mathbf{A}$ are defined in (C.175).

**Lemma 3.19:** Under Assumptions 2.1 and 2.2, assume
\[
n \geq C \max \left\{ \frac{r \log^3 n}{\theta}, \frac{\theta r^3 \log n}{\theta \sqrt{r}}, \frac{r^2 \log n}{\theta \sqrt{r}} \right\}
\]  
for some constant $C > 0$. With probability greater than $1 - cn^{-c' - 2e^{-c''r}}$,
\[
\sup_{q \in S^{r-1}} \| \nabla f_G(q) - \nabla \bar{f}_g(q) \|_2 \lesssim \sqrt{\frac{r \log n}{\theta^2}} + \frac{r^2 \log n}{\theta} + \left( \frac{\theta r^2 + \log^2 n}{\theta} \right) \frac{r \log n}{n}
\]  

Observe that $\theta_n \sigma_0$ with probability at least $1 - cn^{-c'}$, where $M_n$ is defined in (C.130). We used condition (C.192) to simplify the expressions in the second step and $\| A^T q \|^3 \lesssim 1$ in the second step. Invoke $E''$ in (C.186) to conclude
\[
\Gamma_{11} \lesssim \frac{1}{\theta} \sqrt{\frac{r}{n} (1 + \Gamma_{12})},
\]  
with probability at least $1 - cn^{-c' - 2e^{-c''r}}$.

To control $\Gamma_{12}$, we have
\[
\Gamma_{12} = \sup_{q \in S^{r-1}} \frac{1}{3 \theta \sigma^4 n} \left\| \sum_{k=1}^{n} x_k v_{0k}^3 - v_{1k}^3 \right\|_2
\]  
\[
\lesssim \frac{1}{\theta n \sigma^2} \left\| X \right\|_{op} \sup_{q \in S^{r-1}} \left( \left\| v_1^2 \circ (v_0 - v_1) \right\|_2 + \left\| v_0^2 \circ (v_0 - v_1) \right\|_2 \right)
\]  
\[
\lesssim \sup_{q \in S^{r-1}} \frac{1}{\theta n \sigma^2} \left\| \sum_{k=1}^{n} x_k (v_{0k} - v_{1k}) v_{1k}^2 \right\|_2
\]  
\[
+ \left\| \sum_{k=1}^{n} x_k v_{0k} (v_{0k} - v_{1k}) (v_{0k} + v_{1k}) \right\|_2
\]  
\[
\lesssim \frac{1}{\theta n \sigma^2} \left\| X \right\|_{op} \sup_{q \in S^{r-1}} \left( \left\| v_1^2 \circ (v_0 - v_1) \right\|_2 + \left\| v_0^2 \circ (v_0 - v_1) \right\|_2 \right)
\]  
\[
\lesssim \frac{1}{\theta n \sigma^2} \left\| X \right\|_{op} \sup_{q \in S^{r-1}} \left( \left\| v_1^2 \circ (v_0 - v_1) \right\|_2 + \left\| v_0^2 \circ (v_0 - v_1) \right\|_2 \right)
\]
\[
\leq \frac{1}{\theta n\sigma^2} \|X\|_{\text{op}} \sup_{q \in S^{p-1}} \left( \|v_0\|_4^2 + \|v_1\|_4^2 \right) \|v_0 - v_1\|_{\infty} \tag{C.199}
\]

where in the penultimate step we used
\[
\|v^2 \circ v'\|_2^2 = \sum_i v_i^4(v'_i)^2 \leq \|v'\|_{\infty} \|v\|_4^2.
\]

Invoking Lemma 4.2 and Lemma 3.18 yields
\[
\Gamma_{12} \lesssim \left( \sqrt{r\theta} + \sqrt{\log n} \right) \frac{1}{\theta} \sqrt{\frac{r}{n}} \tag{C.200}
\]

with probability at least \(1 - cn^{-c'} - 2e^{-c''r}\).

**b) Controlling \(\Gamma_2\):** Since \(\bar{A}^T\bar{A} = I_c\) and direct calculation gives
\[
E[\text{grad} f_g(q)] = \text{grad} \bar{f}_g(q). \tag{C.201}
\]

Applying lemma 3.15 with \(F(q)\) and \(f(q)\) replaced by \(f_g(q)\) and \(\bar{f}_g(q)\), respectively, gives

\[
\mathbb{P}\left\{ \Gamma_2 \lesssim \sqrt{\frac{r^2 \log(M_n)}{\theta n}} + \frac{M_n r \log(M_n)}{n} \right\} \geq 1 - cn^{-c'}. \tag{C.202}
\]

Finally collecting (C.198), (C.200) and (C.202) and using (C.192) to simplify the expression finish the proof. \(\square\)

3) Deviation Inequalities of the Riemannian Hessian: In this part we will show that the Hessian of \(f_g(q)\) concentrates around that of \(f_g(q)\). Notice that, for any \(q \in S^{-1}\),
\[
\text{Hess} f_g(q) = -\frac{\theta n}{3} \sum_{k=1}^n P_{q_k} \left(3(q^T \bar{Y} k)^2 \bar{Y} k (\bar{Y} k)^T - (q^T \bar{Y} k)^4 I_p \right) P_{q_{\perp}},
\]

\[
\text{Hess} \bar{f}_g(q) = -\left(1 - \theta P_{q_{\perp}} \right) \left[3\bar{A} \text{diag}((\bar{A}^T q)^{\circ 2}) \bar{A}^T \right. \\
- \left. \|A^T q\|_4^4 I_p \right] P_{q_{\perp}}, \tag{C.203}
\]

Here \(\bar{Y}\) and \(\bar{A}\) are defined in (C.175).

**Lemma 3.20:** Under Assumptions 2.1 and 2.2, assume
\[
n \geq C \max \left\{ \frac{r \log n}{\theta}, \frac{r \log n}{\theta^2 \sqrt{\theta}}, \frac{r^2 \log n}{\theta \sqrt{\theta}}, \frac{r^3 \log n}{\theta} \right\} \tag{C.205}
\]

for some constant \(C > 0\). With probability greater than \(1 - cn^{-c'} - 4e^{-c''r}\),
\[
\sup_{q \in S^{p-1}} \|\text{Hess} f_g(q) - \text{Hess} \bar{f}_g(q)\|_{\text{op}} \lesssim \left( \sqrt{r \sqrt{\theta} + \log n} \right) \sqrt{\frac{r}{\theta^2 n}} \tag{C.206}
\]

\[
+ \left( \frac{r \log n}{\theta} + \left( r \log n \right)^2 \right) \frac{r \log n}{n}.
\]

**Proof:** Recall \(f_g(q)\) from (C.178). Notice that
\[
\sup_{q \in S^{p-1}} \|\text{Hess} f_g(q) - \text{Hess} \bar{f}_g(q)\|_{\text{op}} \leq \sup_{q \in S^{p-1}} \|\text{Hess} f_g(q) - \text{Hess} f_g(q)\|_{\text{op}} + \sup_{q \in S^{p-1}} \|\text{Hess} f_g(q) - \text{Hess} \bar{f}_g(q)\|_{\text{op}}. \tag{C.207}
\]

Straightforward calculation gives
\[
\text{Hess} f_g(q) = -\frac{1}{3\theta^2 n} \sum_{k=1}^n P_{q_{\perp}} \left[3(q^T \bar{A}^T X k)^2 \bar{A}^T X k (\bar{A}^T X k)^T \\
- (q^T \bar{A}^T X k)^4 I_p \right] P_{q_{\perp}}. \tag{C.208}
\]

It remains to bound from above \(\Gamma_1\) and \(\Gamma_2\) respectively.

**a) Controlling \(\Gamma_1\):** Using the definition of \(v_0\) and \(v_1\) in (C.181), we have:
\[
\Gamma_1 \leq \sup_{q \in S^{p-1}} \|\text{Hess} f_g(q) - \text{Hess} f_g(q)\|_{\text{op}} \leq \frac{1}{3\theta^2 n} \sum_{q \in S^{p-1}} \left(\theta^2 n^2 \sigma^4 \left[3\bar{Y} \text{diag}((\bar{Y}^T q)^{\circ 2}) \bar{Y}^T \\
- \|\bar{Y}^T q\|_4^4 I_p \right] \right) \tag{C.209}
\]

Here
\[
\beta_1 := \frac{1}{\theta^2 n} \sup_{q \in S^{p-1}} \|\theta n \sigma^2 \bar{Y} \text{diag}(v_0^{\circ 2}) \bar{Y}^T \\
- \bar{A} \text{diag}(v_1^{\circ 2}) (\bar{A}^T X)^T \|_{\text{op}}, \tag{C.210}
\]

**b) Upper bound for \(\beta_2\):** By adding and subtracting terms, we have
\[
\beta_2 := \frac{1}{\theta^2 n} \left(\sup_{q \in S^{p-1}} \|v_0\|_4^4 - \|v_1\|_4^4 \right) \tag{C.211}
\]

where
\[
\beta_{11} = \frac{1}{\theta^2 n} \sup_{q \in S^{p-1}} \left(\sqrt{\theta n} \sigma^2 \bar{Y} - \bar{A} X \right) \text{diag}(v_1^{\circ 2}) X^T \|_{\text{op}}, \tag{C.212}
\]

\[
\beta_{12} = \frac{1}{\theta^2 n} \sup_{q \in S^{p-1}} \sqrt{\theta n} \sigma^4 \bar{Y} \text{diag}(v_1^{\circ 2}) \times (\sqrt{\theta n} \bar{Y} - \bar{A} X)^T \|_{\text{op}}.
\]
For $\beta_{11}$, by recalling that (C.175), we have

$$
\beta_{11} = \frac{1}{\theta \sigma^4 n} \sup_{q \in \mathcal{S}^{p-1}} \left\| \sqrt{\theta n \sigma^4} \bar{Y} \text{diag}(v_1^2 - v_0^2) \right\|_{\infty}.
$$

Recalling (C.162) and (C.163), we have

$$
\beta_{11} \leq \frac{1}{\theta \sigma^4 n} \sup_{q \in \mathcal{S}^{p-1}} \left\| (\sqrt{\theta n \sigma^4} \tilde{D} A - \bar{A}) X \text{diag}(v_1^2) \right\|_{\infty}
$$

$$
\beta_{11} \leq \frac{1}{\theta \sigma^4 n} \sup_{q \in \mathcal{S}^{p-1}} \left\| \frac{1}{n} \sum_{t=1}^{n} v_t^2 x_t x_t^T \right\|_{\infty} \sup_{q \in \mathcal{S}^{p-1}} \left\| \sqrt{\theta n \sigma^4} \tilde{D} A - \bar{A} \right\|.
$$

(C.213)

Since on the event $\mathcal{E'}$ in (C.137),

$$
\|v_1\|_{\infty} = \max_{t \in [n]} \|q^T \tilde{A} x_t\|_{\infty} \lesssim \sigma(\sqrt{r \theta} + \sqrt{\log n}),
$$

and

$$
\|v_0\|_{\infty} \leq \|v_1\|_{\infty} + \|v_0 - v_1\|_{\infty},
$$

we have

$$
\beta_{11} \lesssim (r \theta + \log n) \frac{1}{\theta} \sqrt{\frac{r}{n}} = \sqrt{\frac{r \theta n}{n} + \frac{\log n}{\theta^2 n}}.
$$

(C.220)

With probability at least $1 - 4e^{-c' r} - c'n^{-c'}$.

c) Upper bound for $\beta_2$: Notice that

$$
\beta_2 = \frac{1}{4} \sup_{q \in \mathcal{S}^{p-1}} |F_g(q) - f_g(q)|.
$$

(C.221)

Display (C.183) yields

$$
P \left\{ \beta_2 \lesssim \left( \sqrt{r \theta} + \sqrt{\log n} \right) \frac{1}{\theta} \sqrt{\frac{r}{n}} \right\} \geq 1 - 2e^{-c' r} - c'n^{-c'}.
$$

(C.222)

d) Controlling $\Gamma_2$: Notice that $\tilde{A}^T \bar{A} = I_r$ and simple calculation gives

$$
E [\text{Hess} f_g(q)] = \text{Hess} \tilde{f}_g(q).
$$

(C.223)

Apply Lemma 3.16 with $F(q)$ and $f(q)$ replaced by $f_g(q)$ and $\tilde{f}_g(q)$, respectively, to obtain

$$
P \left\{ \Gamma_2 \lesssim \sqrt{\frac{r \log(M_n)}{\theta n} + \frac{M_n}{n} \frac{r \log(M_n)}{n}} \right\} \geq 1 - cn^{-c'}.
$$

(C.224)

Finally, collecting (C.215), (C.218), (C.220), (C.222) and (C.224) and using condition (C.205) to simplify expressions complete the proof.

APPENDIX D

AUXILIARY LEMMAS

Recall that the SVD of $A$ is $U_A D_A V_A^T$ and $D$ is defined in (II.6).

Lemma 4.1: Under Assumptions 2.1 and 2.2, assume $n \geq C r / \theta^2$ for some constant $C > 0$. With probability at least $1 - 2e^{-c'}$ for some constant $c > 0$, we have:

$$
\left\| \sqrt{\theta n \sigma^2} D A - U_A V_A^T \right\|_{\infty} \lesssim \frac{1}{\theta} \sqrt{\frac{r}{n}}.
$$

(D.1)

Proof: From (C.110), we have

$$
\sqrt{\theta n \sigma^2} \bar{D} = U_A D_A^{-1} U_A^T
$$

$$
+ U_A \left( \sqrt{\theta n \sigma^2} Q^T A^{-1/2} Q - D_A^{-1} \right) U_A^T.
$$

(D.2)

where $U$ contains the left $r$ singular vectors of $Y$ and $U_A = U Q$. It then follows

$$
\left\| \sqrt{\theta n \sigma^2} D A - U_A V_A^T \right\|_{\infty}
$$

$$
= \left\| U_A \left( \sqrt{\theta n \sigma^2} Q^T A^{-1/2} Q D_A - I_r \right) V_A^T \right\|_{\infty}
$$
\begin{align}
\mathbb{E} \left[ \| F_q(x) \|_2^2 \right] & \leq C \theta^{-1} \tag{D.8}
\end{align}
for some constant \( C > 0 \).

Proof: Note that \( x = b \odot g \) where \( b \overset{i.i.d.}{\sim} \text{Ber}(\theta) \) and \( g \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \). We assume \( \sigma = 1 \) for simplicity. Define \( \mathcal{I} \) as the nonzero support of \( x \) such that we could write \( x = \mathcal{P}_\mathcal{I}(g) \).

We have
\begin{align}
\mathbb{E} \left[ \| F_q(x) \|_2^2 \right] &= \left( \frac{1}{12\theta} \right)^2 \mathbb{E} \left[ (q^T x)^8 \right] = \left( \frac{1}{12\theta} \right)^2 \mathbb{E} \left[ (\mathcal{P}_\mathcal{I}(q), g)^8 \right]. \tag{D.9}
\end{align}

Since
\begin{align}
\mathbb{E} \left[ (\mathcal{P}_\mathcal{I}(q), g)^8 \right] &= (7!!) \mathbb{E} \left[ \| \mathcal{P}_\mathcal{I}(q) \|_2^8 \right], \tag{D.10}
\end{align}
we further obtain
\begin{align}
\mathbb{E} \left[ \| \mathcal{P}_\mathcal{I}(q) \|_2^8 \right] &= \sum_{k_1, k_2, k_3, k_4} q_{k_1}^2 \mathbbm{1}_{k_1 \in \mathcal{I}} q_{k_2}^2 \mathbbm{1}_{k_2 \in \mathcal{I}} q_{k_3}^2 \mathbbm{1}_{k_3 \in \mathcal{I}} q_{k_4}^2 \mathbbm{1}_{k_4 \in \mathcal{I}}. \tag{D.11}
\end{align}

We consider four scenarios.

- Only one index among \( k_1, k_2, k_3, k_4 \) in \( \mathcal{I} \). In this case we have
  \begin{align}
  \mathbb{E} \left[ \| \mathcal{P}_\mathcal{I}(q) \|_2^8 \right] &= \theta \sum_{k_1} q_{k_1}^8 \leq \theta \| q \|_2^8 \tag{D.12}
  \end{align}

- Two index among \( k_1, k_2, k_3, k_4 \) in \( \mathcal{I} \). In this case we have
  \begin{align}
  \mathbb{E} \left[ \| \mathcal{P}_\mathcal{I}(q) \|_2^8 \right] &= \theta^2 \sum_{k_1, k_2} [q_{k_1}^2 q_{k_2}^6 + q_{k_1}^6 q_{k_2}^4 + q_{k_1}^4 q_{k_2}^2] \leq 3\theta^2 \| q \|_2^8 \tag{D.13}
  \end{align}

- Three index among \( k_1, k_2, k_3, k_4 \) in \( \mathcal{I} \). In this case we have
  \begin{align}
  \mathbb{E} \left[ \| \mathcal{P}_\mathcal{I}(q) \|_2^8 \right] &= \theta^3 \sum_{k_1, k_2, k_3} [q_{k_1}^2 q_{k_2}^2 q_{k_3}^4] \leq \theta^3 \| q \|_2^8 \tag{D.14}
  \end{align}

- All four index among \( k_1, k_2, k_3, k_4 \) in \( \mathcal{I} \). In this case we have
  \begin{align}
  \mathbb{E} \left[ \| \mathcal{P}_\mathcal{I}(q) \|_2^8 \right] &= \theta^4 \sum_{k_1, k_2, k_3, k_4} [q_{k_1}^2 q_{k_2}^2 q_{k_3}^2 q_{k_4}] \leq \theta^4 \| q \|_2^8 \tag{D.15}
  \end{align}

Use \( \| q \|_2 = 1 \) and collect the above four results to obtain
\begin{align}
\mathbb{E} \left[ \| \mathcal{P}_\mathcal{I}(q) \|_2^8 \right] &= \theta + 3\theta^2 + 3\theta^3 + \theta^4 \leq c_1 \theta \tag{D.16}
\end{align}

Here \( c_1 > 6 \). Plugging back into (D.10) yields
\begin{align}
\| f_q(x) \|_2^2 \leq (7!!) \frac{\theta}{144\theta^2} \leq C \theta^{-1} \tag{D.17}
\end{align}
and finishes our proof. \( \square \)

The following results provide deviation inequalities of the average of i.i.d. functionals of a sub-Gaussian random vector / matrix. They are proved in [45] and we offer a modified version here.

Lemma 4.5 (Theorem F.1, [35]): Let \( Z_1, Z_2, \ldots, Z_p \) be i.i.d realizations of a random matrix \( Z \in \mathbb{R}^{n_1 \times n_2} \) satisfying \( \mathbb{E} [Z] = 0 \),
\begin{align}
\mathbb{P} (|Z_{ij}| > t) \leq 2 \exp \left( -\frac{t^2}{2\sigma^2} \right), \quad \forall 1 \leq i \leq n_1, 1 \leq j \leq n_2. \tag{D.18}
\end{align}

For any fixed \( q \in \mathbb{S}^{n_1-1} \), define a function \( f_q : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{d_1 \times d_2} \) such that the following conditions hold.

- **Condition 1:** There exists some positive numbers \( B_f \) and \( L_f \) such that
  \begin{align}
  \| f_q(Z) \|_\text{op} \leq B_f, \quad \| f_q(Z) - \mathbb{E} [f_{q_2}(Z)] \|_\text{op} \leq L_f \| q_1 - q_2 \|_2, \quad \forall q_1, q_2 \in \mathbb{S}^{n_1-1}. \tag{D.19}
  \end{align}
Then with probability greater than 

\[ \Pr \left( \sum_{i=1}^{p} f_q(Z_i) - E[f_q(Z)] \right) \leq B \]

where 

\[ B = 2\sigma \sqrt{\log(pMn^2)} \]

there exists some positive quantities 

\[ R_1, R_2, L_f \]

such that

\[ \left\| f_q(Z) \right\|_{op} \leq R_1, \]

\[ \max \left\{ \left\| \mathbb{E} \left[ f_q(Z) \left( f_q(Z) \right)^T \right] \right\|_{op}, \right. \]

\[ \left. \left\| \mathbb{E} \left[ (f_q(Z))^T f_q(\tilde{Z}) \right] \right\|_{op} \right\} \leq R_2, \]

\[ \left\| f_q(Z) - f_q(\tilde{Z}) \right\|_{op} \leq L_f \left\| q_1 - q_2 \right\|_2, \]

\[ \forall q_1, q_2 \in S^{n-1}. \]

Lemma 4.9 (Corollary F.7, [35]): Suppose \( x \in \mathbb{R}^r \) contains i.i.d Bernoulli-Gaussian random variables with parameter \( (\theta, \sigma^2) = 1 \). For any \( q \in S^{n-1} \), let

\[ f_q(x) = \frac{1}{\theta} (q^T x)^2 x x^T. \]

We have

\[ \sup_{q \in S^{n-1}} \mathbb{E} \left[ \left\| f_q(x) \right\|_2 \right] \leq C \theta^{-1} r, \]

for some constant \( C > 0 \).

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