ON HAMILTONIAN AND QUANTUM DYNAMICS OF MASSLESS PARTICLES.

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Abstract

A short review of special relativistic dynamics describing a particle acted upon by an arbitrary conservative external force is presented. If the mass of the particle is zero and the force is central then the equations of motion turn out to be completely integrable. A well-known result.

Hamiltonian flows on the twistor phase space $T$ are constructed which, for conservative forces and value of the helicity equal to zero, reproduce equations of motion of the classical massless particle. For helicities different from zero the same hamiltonian flows produce equations of motion showing a curious "Zitterbewegung" like behaviour.

A canonical Poincaré covariant quantization procedure on $T$ is suggested. One simple example describing a spinning and massless 3-D quantum mechanical harmonic oscillator is analysed in some detail.
INTRODUCTION AND NOTATION.

It is possible that elementary massive particles such as electron, proton, neutron etc. should be regarded as bound states of a finite number of massless and spinning interacting constituents.

In this article a mathematical formalism with its roots in the Twistor Theory of Penrose is investigated. The physical force in the model is external (and conservative). Therefore its future physical application (if any) aims only at a new phenomenological attempt to understand where the masses of elementary particles come from.

The author does not claim that the model (as it now stands) describes any known physical system. He wishes just to show that there are some concrete uninvestigated possibilities hidden in the mathematics of Twistors. The virtue of such models (when fully developed) is the simplicity and economy of thought they provide.

Earlier, exploring the idea of instantaneous relativistic action at a distance in the phase space of twistors we have shown how a free massive and spinning particle may be thought of as a relativistic rigid rotator (endowed with intrinsic spin) composed of two massless spinning parts. Instantaneous refers to the rest frame defined by the total time-like four-momentum of the rigid rotator the two massless and spinning particles happen to define.

By continuing these ideas and by taking a larger number of such massless constituents more complicated closed massive and spinning systems may be constructed.

However, with the increasing number of massless parts, calculations, although straightforward, become quite cumbersome.

This fact triggered the work presented in this paper. In order to get an idea of how a closed system, composed of a large finite number of massless spinning mutually interacting particles, might behave we investigate dynamics of just one massless spinning particle acted upon by an external conservative force. The latter may be thought of as an effective force coming from an inertial ”source” defined by the total freely moving composite system.

In other words we assume that there exists a special inertial frame in Minkowski space (the rest frame of the ”infinitely” heavy ”source” of the force) to which the massless particle is bounded.

One of the shortcomings of our approach is that there is nothing in the formalism which tells us how to choose the external force (or equivalently the corresponding hamiltonian) in order to describe a physical system. Future investigations will perhaps show how the external force should be chosen in concrete physical situations.

The work is organized as follows:

In the next section we review the relativistic Newton’s second law of dynamics with emphasis on the massless particle case. It is demonstrated again that when a conservative and central force acts on a massless particle then its relativistic equations of motion are completely integrable.

In the third section an Hamiltonian mechanics, reproducing equations of motion of a
the approach is generalized to be valid for non-vanishing values of the helicity.

Finally, in the fourth section a canonical quantization is performed. One relatively simple example, representing an analog of the 3-D harmonic oscillator, is studied in some detail.

Latin letters with lower case latin indices denote four-vectors and four-tensors. Latin letters with lower case greek indices within brackets denote three-vectors. In section 1, 2 the usual three-vector notation (with a line over a letter) will also be used. In section 2 and 3 a bar (not a line) over a letter or over an expression denotes complex conjugation. Lower case greek letters with upper case latin indices (either primed or unprimed) denote spinors. Upper case latin letters with lower case greek indices denote twistors. The physical units are so chosen that \( c = \hbar = 1 \). The signature of the metric \( g_{ij} \) in Minkowski space is taken to be \(+−−−\). The fully antisymmetric alternating four-tensor will be denoted by \( \eta_{ijkl} \). The fully antisymmetric alternating three-tensor will be denoted by \( \epsilon^{(\alpha)(\beta)(\gamma)} \). The usual summation convention over repeated indices will be assumed throughout.

1 A SHORT REVIEW OF RELATIVISTIC PARTICLE DYNAMICS.

In this paper we intend to achieve two goals. The first is to describe relativistic classical dynamics of a massless spinning particle acted upon by an external conservative force in terms of canonical flows on the twistor phase space. The second is to formulate the corresponding relativistic quantum dynamics and examine the case when the force is chosen to be of the 3-D harmonic oscillator type.

To define the context we first review the special relativistic version of Newton’s second law of dynamics in general and its massless limit in particular.

For a massive particle, Newton’s second law of dynamics may be written in the following Poincaré covariant form:

\[
\frac{dY^i}{d\tau} = \frac{P^i}{m},
\]

\[
\frac{dP_i}{d\tau} = F_i, \quad \text{where} \quad F^i P_i = 0.
\]

(1.1)

(1.2)

\( Y^i \) is the four-position of the particle in Minkowski space, \( P_i \) denotes its four-momentum, \( \tau \) its proper time, \( m \) its rest mass and \( F_i \) represents the so called four-force acting on the particle. Suppose that the world-line of an inertial frame is given by:

\[
X^i = X_0^i + (t - t_0)t^i,
\]

(1.3)

where \( X^i \) denotes its four-position, \( t^i \) its constant time-like four-velocity, \( t \) its proper time and where \( X_0^i \) represents a constant four-vector starting from some arbitrarily chosen origin in Minkowski space and ending at a point (an event) on the world-line of the inertial frame where we have put the value of its proper time to be equal to \( t_0 \).

Then, a space-like four-vector \( r^i \), which represents the particle’s instantaneous centre of energy with respect to the inertial frame defined by (1.3), is given by:
\[ r^i := R^i - (R^k t_k) t^i, \quad (1.4) \]

where

\[ R^i = Y^i - X^i. \quad (1.5) \]

Using (1.3) it yields:

\[ r^i = (Y^i - X^i_0) - [(Y^k - X^k_0) t_k] t^i. \quad (1.6) \]

Let us also introduce a space-like four-vector \( f_k \) which fulfils:

\[ f_k t^k = 0, \quad (1.7) \]

and represents the physical three-force, exerted on the particle under consideration, as "measured" by an observer following the world-line of the inertial frame, where

\[ - f_k \frac{P^n}{P_k t^k}, \quad (1.8) \]

represents the work done by this force. The four-force in (1.2) may now be split into two components. One along the four-velocity \( t^i \) and another projected onto the space-like three-plane orthogonal to \( t^i \):

\[ F_i = \frac{P_k t^k}{m} f_i - \frac{f_k P^k}{m} t_i. \quad (1.9) \]

The equations of motion in (1.1) and (1.2) may then be rewritten as follows:

\[ \dot{r}^i = \frac{P^i}{P^k t_k} - t^i, \quad (1.10) \]

\[ \dot{P}_i - (\dot{P}_k t^k) t_i = f_i \quad \text{and} \quad \dot{P}_i P^i = 0. \quad (1.11) \]

The dot over a letter denotes differentiation with respect to the proper time in the inertial frame defined by (1.3). If \( m \) differs from zero this is just a reformulation of the equations given in (1.1) and (1.2). However, in contrast to (1.1) and (1.2), the equations in (1.10) and (1.11) are also valid for \( m = 0 \). The physical three force represented by \( f_k \) may depend functionally on \( r^i \) i.e.:

\[ f_k = f_k (r^i). \quad (1.12) \]

The centre of energy space-like four-vector \( r^i \) depends on the location of the inertial frame and also on the location of the particle in Minkowski space. Therefore the assumption in (1.12) implies that the inertial frame, with the world-line given by (1.3), is not arbitrary but constitutes a source producing the force acting on the particle. In the inertial frame of the source we have:
\[ r^i = (0, \tau), \tag{1.14} \]
\[ f^i(r^k) = (0, \mathbf{\mathcal{F}}(\tau)), \tag{1.15} \]

where we have used the familiar three-vector notation. The equations of motion in (1.10) and (1.11) now read:
\[ \dot{r} = \frac{\mathbf{p}}{E}, \tag{1.16} \]
\[ \dot{\mathbf{p}} = \mathbf{\mathcal{F}}(\tau). \tag{1.17} \]

In addition we also have:
\[ m^2 := P_i P_i = E^2 - |p|^2 = E^2 - \mathbf{p}^2 = \text{constant}. \tag{1.18} \]

If the force \( \mathbf{\mathcal{F}}(\tau) \) is conservative then one has:
\[ \mathbf{\mathcal{F}} = -\nabla U(\tau), \tag{1.19} \]

where \( U(\tau) \) represents the potential energy of the particle. From (1.16)-(1.19) we obtain in the usual manner that:
\[ \dot{E} = -[\dot{\tau} \cdot \nabla U(\tau)], \tag{1.20} \]

which implies the energy conservation law:
\[ H := E + U(\tau) = \text{constant}. \tag{1.21} \]

The constant \( H \) represents the total energy of the particle. Differentiating (1.16), using (1.17), (1.20) and (1.21) yields:
\[ (H - U)\ddot{\tau} + \nabla U(\tau) - \dot{\tau} [\dot{\tau} \cdot \nabla U(\tau)] = 0. \tag{1.22} \]

Again, if \( m \) differs from zero and the non-relativistic condition is fulfilled i.e.:
\[ |\dot{\tau}| << 1, \tag{1.23} \]

then one has that:
\[ (H - U) \simeq m, \tag{1.24} \]

which implies that the equations of motion in (1.22) above acquire (as they of course should) the familiar Newtonian form:
\[ m\ddot{\tau} + \nabla U(\tau) \simeq 0. \tag{1.25} \]

Note that the relativistic non-linear equation in (1.22) is also valid for massless particles.
To proceed further we assume that the force is also central i.e. that:

\[ U(\mathbf{r}) = U(|\mathbf{r}|) = U(r). \]  

(1.26)

From (1.16), (1.17), (1.21) and (1.26) we then obtain (as in the non-relativistic case) that the orbital angular momentum does not change in time:

\[ \mathbf{\dot{L}} := \mathbf{\tau} \times \mathbf{p} = [H - U(r)][\mathbf{\tau} \times \dot{\mathbf{r}}] = \text{constant}, \]  

(1.27)

implying plane particle motion. \( \times \) denotes the usual vector product in the three dimensional space (Lorentz) "perpendicular" to the time-like direction given by \( t^i \).

Choosing the space origin at the location of the observer who follows the world-line of the inertial frame and the \( z \)-axis along the three-vector \( \mathbf{L} \) the equations of motion, in the familiar polar coordinates on the spatial plane perpendicular to the \( z \)-axis, read:

\[ \ddot{\rho} - \frac{U'}{(H - U)}(\dot{\rho}^2 - 1) - \frac{L^2}{(H - U)^2 \rho^3} = 0, \]  

(1.28)

\[ \dot{\phi} = \frac{L}{(H - U)\rho^2} \quad \dot{z} = z = 0, \]  

(1.29)

where

\[ L := |\mathbf{L}|, \]  

(1.30)

and where

\[ U' := \frac{dU}{d\rho}. \]  

(1.31)

In the massless case, which is our main concern here, we have that:

\[ |\mathbf{p}| = E, \]  

(1.32)

which, by the use of (1.16), implies:

\[ |\mathbf{\tau}| = 1. \]  

(1.33)

In the introduced polar coordinates this yields:

\[ \dot{\rho}^2 + \rho^2 \dot{\phi}^2 = 1, \]  

(1.34)

which inserted into the first part of (1.29) reduces the equations of motion to a simple expression:

\[ \dot{\rho}^2 = 1 - \frac{L^2}{(H - U)^2 \rho^2}. \]  

(1.35)

The equation above may be integrated giving \( t \) as a function of \( \rho \):
\[ t = \pm \int \frac{d\rho}{\sqrt{1 - \frac{L^2}{(H-U)^2}\rho^2}} \]  
(1.36)

and for the polar angle \( \phi \) as a function of \( \rho \) one obtains:

\[ \phi = \pm L \int \frac{d\rho}{(H-U)\rho^2 \sqrt{1 - \frac{L^2}{(H-U)^2}\rho^2}}. \]  
(1.37)

The equations of motion, describing massless particles acted upon by central conservative forces, are completely integrable.

The above results may be understood in terms of symplectic (phase space) geometry. The natural symplectic structure on the cotangent bundle \( T^* M \) (eight dimensions) of the Minkowski space \( M \) is given by:

\[ \Omega := dp_i \wedge dY^i. \]  
(1.38)

The Lorentz covariant and four-translation invariant coordinates of the momentum four-vector \( P_i \) and the Poincaré covariant coordinates of the position four-vector \( Y^i \) regarded as functions on \( T^* M \) are canonically conjugated to each other. The ten Poincaré covariant functions on \( T^* M \) given by:

\[ P_i \quad \text{and} \quad M^{ij} := 2Y^{[i} P^{j]} \]  
(1.39)

define the so called momentum mapping for the action of the Poincaré group on \( T^* M \). In other words the algebra of their Poisson brackets represents the commutation relations of the Poincaré algebra.

The particle’s instantaneous position relative to the inertial source, which is a uniformly moving point in \( M \), resides on space-like planes given by:

\[ Y^i t_i = 0. \]  
(1.40)

If \( M^{ij} \) is taken about this uniformly moving point in \( M \) then the position of the particle’s centre of energy is given by:

\[ r^i = \frac{M^{ij} t_j}{P^m t_m}. \]  
(1.41)

The equations of motion in (1.22) arise as a consequence of the canonical flow in \( T^* M \) (canonical with respect to the symplectic structure \( \Omega \) in (1.38)) generated by \( H \) in (1.21) treated as a real valued function on \( T^* M \).

Note that the flow generated by the mass squared function defined in (1.18) commutes with the flow generated by the function \( H \). \( m^2 \) is therefore a constant of the physical motion.

In the next section we are going to show that, for a massless particle, equations of motion in (1.22) may be regarded in a completely different way.
consequence of the canonical flow generated by \( H \) in (1.21) treated as a real valued function on the twistor phase space.

2 TWISTOR PHASE SPACE FORMULATION FOR \( m = 0 \).

The pair \( (X_0^{AA'}, t^{AA'}) \) (here we adopt Penrose’s abstract index notation\(^7\)) defining the straight world-line (of the inertial source producing the external force acting on the massless particle) in (1.3) may be represented by two fixed intersecting null-twistors \( V^\alpha, W^\alpha \) in \( T \) (and their twistor complex conjugates \( \bar{V}_\alpha, \bar{W}_\alpha \)) i.e. twistors which fulfil:

\[
V^\alpha \bar{V}_\alpha = W^\alpha \bar{W}_\alpha = V^\alpha \bar{W}_\alpha = 0,
\]

\[
I_{\alpha\beta} V^\alpha W^\beta = \frac{1}{\sqrt{2}},
\]

where \( I_{\alpha\beta} \) is the infinity twistor\(^8\). Using the spinor representation this yields:

\[
V^\alpha = (\sigma^A, \alpha_A), \quad \bar{V}_\alpha = (\bar{\alpha}_A, \bar{\sigma}^{A'}), \quad (2.3)
\]

\[
W^\alpha = (\varsigma^A, \beta_A), \quad \bar{W}_\alpha = (\bar{\beta}_A, \bar{\varsigma}^{A'}). \quad (2.4)
\]

Therefore the relation in (2.2) may be written as:

\[
I_{\alpha\beta} V^\alpha W^\beta = \alpha^{A'} \beta_A = \frac{1}{\sqrt{2}}, \quad (2.5)
\]

while \( X_0^{AA'} \) is explicitly given by (see Ref. 2):

\[
X_0^{AA'} := i \sqrt{2}(\sigma^A \beta^{A'} - \varsigma^A \alpha^{A'}) = -i \sqrt{2}(\bar{\sigma}^{A'} \bar{\beta}^A - \bar{\varsigma}^{A'} \bar{\alpha}^A), \quad (2.6)
\]

and \( t^{AA'} \) by:

\[
t^{AA'} := \alpha^{A'} \bar{\alpha}^A + \beta^{A'} \bar{\beta}^A. \quad (2.7)
\]

The two fixed twistors \( V \) and \( W \) also define three space-like directions rigidly attached to the inertial frame defined in (1.3):

\[
u^a_{(1)} \equiv u^a \equiv u^{AA'} := \alpha^{A'} \bar{\alpha}^A - \beta^{A'} \bar{\beta}^A, \quad (2.8)
\]

\[
u^a_{(2)} \equiv v^a \equiv v^{AA'} := \alpha^{A'} \bar{\beta}^A + \beta^{A'} \bar{\alpha}^A, \quad (2.9)
\]

\[
u^a_{(3)} \equiv w^a \equiv w^{AA'} := i(\alpha^{A'} \bar{\beta}^A - \beta^{A'} \bar{\alpha}^A). \quad (2.10)
\]

We may call the direction defined by \( u^a \) the \( z \)-axis direction, the direction defined by \( v^a \) the \( x \)-axis direction and the direction defined by \( w^a \) the \( y \)-axis direction.

In effect \( u^a_{(\alpha)} \) and \( t^a \) form a non-rotating fixed tetrad rigidly attached to the inertial source.
Coordinates of a variable point in \( T \) (i.e. a twistor) \( Z^\alpha \) and coordinates of its (twistor) complex conjugate point \( \bar{Z}_\alpha \) will be represented by two variable Weyl spinors and their conjugates:

\[
Z^\alpha = (\omega^A, \pi_A'), \quad \bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^A').
\] (2.11)

This spinor representation of a point in \( T \) is very convenient because it shows explicitly how the Poincaré group acts on \( T \). Coordinates of the two spinors represented by \( \pi_A' \) and \( \omega^A \) are covariant with respect to the (identity connected part of the) Lorentz group while four-translations \( T^a \) act only on the "\( \omega \)" spinor parts of the twistor \( Z \) and its (twistor) complex conjugate \( \bar{Z} \) according to the following simple rule:

\[
\bar{\omega}^A = \omega^A + iT^{AA'} \pi_A', \quad \bar{\omega}' = \bar{\omega}' = \bar{\omega}' = iT^{AA'} \bar{\pi}_A.
\] (2.12)

(Do not confuse \( (\alpha) \) indices which label the three mutually orthogonal physical space directions with \( \alpha \) indices which label twistors.)

The natural \( SU(2,2) \) invariant symplectic structure on \( T \) defines the following canonical Poisson bracket relations:

\[
\{Z^\alpha, \bar{Z}_\beta\} = i\delta^\alpha_\beta, \quad \{Z^\alpha, Z^\beta\} = \{\bar{Z}_\alpha, \bar{Z}_\beta\} = 0,
\] (2.13)

which in terms of spinor variables reads:

\[
\{\omega^A, \bar{\pi}_B\} = i\delta^A_B, \quad \{\pi_{B'}, \bar{\omega}'\} = i\delta^{A'}_{B'},
\] (2.14)

\[
\{\omega^A, \omega^B\} = \{\pi_A', \pi_{B'}\} = \{\pi_A', \bar{\pi}_B\} = \{\pi_{B'}, \bar{\pi}_A\} = 0,
\] (2.15)

\[
\{\bar{\omega}' , \bar{\omega}'\} = \{\bar{\omega}' , \bar{\pi}_A\} = \{\bar{\pi}_A, \bar{\pi}_B\} = \{\bar{\pi}_A, \bar{\omega}'\} = 0.
\] (2.16)

The linear four-momentum \( P_a \) and the angular four-momentum \( M_{ab} = -M_{ba} \) of a massless particle may be regarded as given by the following set of Poincaré covariant functions on \( T \):

\[
P_a := \pi_{A'} \bar{\pi}_A,
\] (2.17)

\[
M_{ab} := i\bar{\omega}_{(A'} \bar{\pi}_{B')} \epsilon_{AB} - i\omega_{(A} \bar{\pi}_{B)} \epsilon_{A'B'}.
\] (2.18)

The canonical Poincaré covariant Poisson brackets in (2.13)-(2.16) imply that \( P_a \) and \( M_{ab} \) in (2.17)-(2.18) fulfil the Poisson bracket relations of the Poincaré algebra:

\[
\{P_a, P_b\} = 0,
\] (2.19)

\[
\{M_{ab}, P_c\} = 2g_{[a}P_{b]} \epsilon_{c],
\] (2.20)

\[
\{M_{ab}, M_{cd}\} = 2(g_{[a}M_{b][d]} + g_{d[b}M_{a]c}).
\] (2.21)
The Poisson bracket relations in (2.19) - (2.21) define the momentum mapping for the action of the Poincaré group on \( T \).

The Pauli-Lubanski spin four-vector of a massless particle:

\[
S^a := \frac{1}{2} \eta^{abcd} P_b M_{cd} \tag{2.22}
\]

reduces itself to a simple expression (use spinor representation of \( \eta^{abcd} \) to prove it):

\[
S^a = s P^a, \tag{2.23}
\]

where

\[
s = \frac{1}{2} (Z^a \bar{Z}_a) = \frac{1}{2} (\omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'}). \tag{2.24}
\]

The kinetic energy of the massless particle, in the special inertial frame defined by the source, will be defined by the following function on \( T \).

\[
E := \pi_{C'} \bar{\pi}_C t^{CC'}, \tag{2.25}
\]

while the linear three-momentum will be given by:

\[
p_{(\alpha)} := -\pi_{C'} \bar{\pi}_C u_{(\alpha)}^{CC'}. \tag{2.26}
\]

If \( M_{ab} \) is taken about the inertial source producing the force (i.e. about a uniformly moving point in Minkowski space \( M \)) then the functions representing angular momentum of the massless particle about this source, are given by:

\[
J_{(\alpha)} := \frac{1}{2} \epsilon_{(\alpha)(\gamma)(\delta)} M_{ab} u_{(\gamma)}^a u_{(\delta)}^b, \tag{2.27}
\]

where

\[
\epsilon_{(\alpha)(\beta)(\gamma)} := \eta_{abcd} t^a u_{(\alpha)}^b u_{(\beta)}^c u_{(\gamma)}^d. \tag{2.28}
\]

while the three functions representing the position of the centre of energy relative to the inertial source are given by:

\[
y_{(\alpha)} := -\frac{M_{ab} t^b u_{(\alpha)}^a}{P_c t^c}. \tag{2.29}
\]

Explicitly for the three components of the total angular momentum it yields:

\[
J_z = J_{(1)} = -\left[ \frac{1}{\sqrt{2}} (\alpha_A \bar{\beta}^B + \bar{\beta}_A \alpha^B) \bar{\pi}_B \omega^A + \frac{1}{\sqrt{2}} (\alpha_{A'} \bar{\beta}^{B'} + \bar{\beta}_{A'} \alpha^{B'}) \pi_{B'} \bar{\omega}^{A'} \right], \tag{2.30}
\]
\[
J_y = J_3 = -\frac{\bar{\alpha}_A \alpha^B + \bar{\beta}_A \beta^B}{\sqrt{2}} \bar{\pi}_B \omega^A + \frac{\i}{\sqrt{2}} (\alpha_{A'} \alpha^{B'} + \beta_{A'} \beta^{B'}) \pi_{B'} \bar{\omega}^{A'},
\] (2.32)

and for the three components of the position vector of the centre of energy one obtains:

\[
z = y(1) = -\frac{\i}{\pi_{C'} \pi_{C'C'C}} \left[ \frac{1}{\sqrt{2}} (\bar{\alpha}_A \beta^B + \bar{\beta}_A \alpha^B) \bar{\pi}_B \omega^A - \frac{1}{\sqrt{2}} (\alpha_{A'} \beta^{B'} + \beta_{A'} \alpha^{B'}) \pi_{B'} \bar{\omega}^{A'} \right],
\] (2.33)

\[
x = y(2) = -\frac{\i}{\pi_{C'} \pi_{C'C'C}} \left[ \frac{1}{\sqrt{2}} (\bar{\beta}_A \beta^B - \bar{\alpha}_A \alpha^B) \bar{\pi}_B \omega^A - \frac{1}{\sqrt{2}} (\beta_{A'} \beta^{B'} - \alpha_{A'} \alpha^{B'}) \pi_{B'} \bar{\omega}^{A'} \right],
\] (2.34)

\[
y = y(3) = \frac{1}{\pi_{C'} \pi_{C'C'C}} \left[ \frac{1}{\sqrt{2}} (\bar{\alpha}_A \alpha^B + \bar{\beta}_A \beta^B) \bar{\pi}_B \omega^A + \frac{1}{\sqrt{2}} (\alpha_{A'} \alpha^{B'} + \beta_{A'} \beta^{B'}) \pi_{B'} \bar{\omega}^{A'} \right].
\] (2.35)

The helicity \(s\) in (2.24) is a conformal scalar and thereby also a Poincaré scalar function on \(T\). Therefore the function \(s\) Poisson commutes with all the functions introduced in (2.30)-(2.35).

Using (2.19)-(2.21) one obtains following Poisson bracket relations:

\[
\{ y(\alpha), y(\beta) \} = \frac{8 \epsilon(\alpha)(\beta)(\gamma) P(\gamma)}{E^3} ,
\] (2.36)

\[
\{ p(\beta), y(\alpha) \} = \delta(\alpha)(\beta) ,
\] (2.37)

\[
\{ E, y(\alpha) \} = \frac{p(\alpha)}{E} ,
\] (2.38)

\[
\{ E, p(\beta) \} = \{ E, J(\alpha) \} = 0 ,
\] (2.39)

\[
\{ J(\alpha), J(\beta) \} = \epsilon(\alpha)(\beta)(\gamma) J(\gamma) ,
\] (2.40)

\[
\{ J(\alpha), y(\beta) \} = \epsilon(\alpha)(\beta)(\gamma) y(\gamma) ,
\] (2.41)

\[
\{ J(\alpha), p(\beta) \} = \epsilon(\alpha)(\beta)(\gamma) p(\gamma) .
\] (2.42)

The above commutation relations are quite reasonable from the physical point of view. Apart from (2.36) they are what one should expect.

The energy \(H\) in (1.21) (for \(m = 0\)) may now be treated as a function on \(T\). The canonical flow which \(H\) generates on \(T\) is then explicitly given by:

\[
\dot{\omega}^A = \{ H, \omega^A \} = -i \frac{\partial H}{\partial \pi^A} .
\] (2.43)
\[ \hat{\pi}_{B'} = \{ H, \pi_{B'} \} = -i \frac{\partial H}{\partial \bar{\omega}_{B'}.} \]  

(2.44)

For functions describing physical variables in (2.27)-(2.35) it yields:

\[ \dot{y}_{(\alpha)} = \{ H, y_{(\alpha)} \} = \{ E + U, y_{(\alpha)} \} = \frac{p_{(\alpha)}}{E} - \frac{s\epsilon_{(\alpha)(\beta)(\gamma)}p_{(\gamma)}}{E^3} \frac{\partial U}{\partial y_{(\beta)}}, \]  

(2.45)

\[ \dot{p}_{(\alpha)} = \{ E + U, p_{(\alpha)} \} = \{ U, p_{(\alpha)} \} = \frac{\partial U}{\partial y_{(\beta)}} \{ y_{(\beta)}, p_{(\alpha)} \} = -\frac{\partial U}{\partial y_{(\alpha)}}, \]  

(2.46)

which for \( s = 0 \) implies the equations of motion in (1.22). For \( s \) not being equal to zero the above description is a generalization of the massless phase space dynamics. Due to (2.36) the velocity of the centre of energy and the velocity of the massless spinning particle cease to define the same physical quantity. This indicates the extended nature of the massless spinning particle. One possible interpretation of this Zitterbewegung-like behaviour is presented in Ref. 11. Another is that, when external forces are acting on a spinning massless particle, the direction of motion of the centre of its entire kinetic energy (including energy generated by the helicity) simply ceases to be parallel with the (null-) direction of motion of the centre of its translational kinetic energy. Relative to the rest frame of the source the velocity of the centre of entire kinetic energy of the interacting massless spinning particle exceeds the velocity of light. The velocity of the centre of its translational kinetic energy remains on the other hand always equal to the velocity of light. The two velocities coincide when the massless spinning particle is moving freely.

Using arguments having their origins in symplectic geometry we will now demonstrate that for central forces, irrespective whether the helicity is equal to zero or not, equations in (2.45)-(2.46) are completely integrable.

First we note that on the 8-D twistor phase space \( T \) the energy function \( H \) Poisson commutes with the helicity function \( s \). For a fixed value of \( s \) the energy function \( H \) may therefore be regarded as a function on the 6-D phase space obtained as a reduction of \( T \) by the function \( s \). In other words, points in the reduced 6-D phase space are represented by these curves on \( T \) of the hamiltonian flow generated by \( s \) which lie on the shell-surface given by a fixed value of the function \( s \).

In the same way components of the total angular momentum \( J_{(\alpha)} \) which Poisson commute with the helicity function \( s \) on \( T \) may be regarded as functions on the reduced 6-D phase space. For central forces they also commute with the energy function \( H \). So they also commute with \( H \) on the reduced 6-D phase space.

Any of the components of the total angular momentum \( J_{(\alpha)} \) say \( J_{(1)} \), the square of the total angular momentum \( J^2 := J_{(\alpha)}J_{(\alpha)} \) and \( H \) are mutually Poisson commuting functions on the 6-D reduced phase space

Reduction of the 6-D phase space by the two mutually Poisson commuting functions \( J_{(1)} \) and \( J^2 \) produces, for each fixed value of these functions, a 2-D phase space. The energy function \( H \) may now be regarded as a function on this 2-D phase space. But equations of motion on a 2-D phase space are always completely integrable.
The somewhat unexpected result reproduced in the introduction, proving the complete integrability of the equations of motion of a massless and spinless particle moving under the action of a conservative and central force, thus turns out to be a special case of the general feature as described above.

Consequently, for external central forces, there exists a general solution of the equations of motion in (2.45)-(2.46). For $s = 0$ this solution reduces itself to that presented in (1.36)-(1.37). To find the general solution by means of quadratures we proceed as follows. In the rest frame of the source we reintroduce the standard vector notation.

We recall that the translational kinetic energy of the massless spinning particle is denoted by:

$$ E = |\mathbf{p}|, \quad (2.47) $$

the distance from the source to the centre of the entire kinetic energy of the particle by:

$$ r = |\mathbf{r}| = \sqrt{\rho^2 + z^2} \quad (2.48) $$

($\rho$ and $z$ are plane polar coordinates of the corresponding position vector) and the total energy of the massless and spinning particle by:

$$ H = E + U(r). \quad (2.49) $$

For later convenience we introduce a function defined by:

$$ f(r) := -r \frac{dU}{dr}, \quad (2.50) $$

while we also note that in standard vector notation one has:

$$ \mathbf{J} = \mathbf{r} \times \mathbf{p} + \frac{s\mathbf{p}}{E} \quad (2.51) $$

where $\mathbf{J}$ is the total angular momentum vector of the massless spinning particle and where $\mathbf{p}$ is its linear momentum vector.

In three vector notation the equations of motion in (2.45)-(2.46) now read:

$$ \dot{\mathbf{p}} = \frac{f(r)}{r^2} \mathbf{r} \quad (2.52) $$

$$ \dot{\mathbf{r}} = \frac{\mathbf{p}}{E} - \frac{s}{E^3} (\mathbf{r} \times \mathbf{p}) \frac{f(r)}{r^2} \quad (2.53) $$

Standard calculations also give:

$$ (\mathbf{r} \cdot \mathbf{p})^2 = E^2 r^2 + s^2 - J^2. \quad (2.54) $$

Choosing the direction of the constant total angular momentum along the positive direction of the $z$ axis implies...
\( J = J \tau_z. \) \hfill (2.55)

(2.55), (2.51) and (2.54) then yield:

\[
z = \pm \frac{rs}{J} \sqrt{1 + \frac{s^2 - J^2}{E^2 r^2}}. \tag{2.56}
\]

From (2.52) follows that:

\[
\dot{E} = \frac{f(r)(\overline{r} \cdot \overline{p})}{r^2 E}, \tag{2.57}
\]

while from (2.49) follows that:

\[
\dot{E} = -\frac{dU}{dr} \dot{r}. \tag{2.58}
\]

Using (2.50), (2.54), (2.57), (2.58) we finally obtain:

\[
\dot{r} = \sqrt{1 + \frac{s^2 - J^2}{E^2 r^2}}. \tag{2.59}
\]

For \( s = 0 \) the above results imply that \( z = 0 \) and that (2.59) is equivalent to (1.36).

Using (2.51), (2.55) one obtains:

\[
p_\varphi = \frac{E^2 \rho J}{s^2 + r^2 E^2}, \tag{2.60}
\]

\[
p_z = \frac{E J (s^2 + z^2 E^2)}{s(s^2 + r^2 E^2)}, \tag{2.61}
\]

\[
p_\rho = \frac{E^3 J z \rho}{s(s^2 + r^2 E^2)}. \tag{2.62}
\]

The above results and the equation of motion in (2.53) yield:

\[
\dot{\varphi} = \frac{E J}{s^2 + r^2 E^2} (1 + \frac{s^2 f(r)}{r^2 E^2}) \tag{2.63}
\]

which for \( s = 0 \) imply (1.37).

For specific choices of the potential energy in (2.49) the above general equations may easily be integrated and plotted by the use of modern computer programs such as e.g. Maple V, Release 3. The concrete results of such calculations we hope to be able to present in a future publication.
3 MASSLESS RELATIVISTIC QUANTUM DYNAMICS.

In this section a quantization of the twistor phase space dynamics is suggested. The energy eigenvalue equation corresponding to the potential of the 3-D harmonic oscillator is studied in some detail.

A non-standard, as opposed to the standard procedure introduced by Penrose, canonical twistor quantization is obtained by means of a natural prescription à la Dirac given by:

\[ \hat{\omega}^A := -\frac{\partial}{\partial \bar{\pi}^A}, \quad \hat{\omega}^{A'} := \frac{\partial}{\partial \pi^{A'}}, \]
\[ \hat{\bar{\pi}}_A := \bar{\pi}_A, \quad \hat{\bar{\pi}}_{A'} := \pi_{A'}. \]

The Poisson brackets relations in (2.14) - (2.16) will hereby be replaced by the corresponding commutators turning the classical twistor phase space dynamics of a massless particle into its quantum mechanical analog.

So by the use of (3.1)-(3.2) the linear four-momentum functions in (2.17), the angular four-momentum functions in (2.18), the helicity function in (2.24) turn into the corresponding operators:

\[ \hat{P}_a := \bar{\pi}_A \pi^{A'}, \]
\[ \hat{M}^{ab} := i \bar{\pi}^{(A'} \frac{\partial}{\partial \pi^{B')} \epsilon^{AB} + i \pi^{(A} \frac{\partial}{\partial \bar{\pi}_{B)} \epsilon^{A'B'}}, \]
\[ \hat{s} := -\frac{1}{2} (\bar{\pi}_A \frac{\partial}{\partial \bar{\pi}_A} - \bar{\pi}_{A'} \frac{\partial}{\partial \pi_{A'}}). \]

The Poisson bracket relations in (2.19)-(2.21) ensure that operators in (3.3) and (3.4) obey commutation relations of the Poincaré algebra. In this sense the quantization suggested in (3.1)-(3.2) is Poincaré covariant. It is not conformally covariant, however. Besides, the helicity operator in (3.5) commutes with all the operators in (3.3) and (3.4).

Reassuming, the helicity function \( s \) in (2.24), the three functions representing components of the angular momentum in (2.30)-(2.32), the three functions representing components of the position vector of the centre of entire energy in (2.33)-(2.34) turn into linear differential operators acting on the infinitely dimensional space \( \Gamma \) of complex valued functions defined on the (two complex dimensional) configuration space \( \Pi \) spanned by Weyl spinors (multiplicative operators in (3.2) "acting" on \( \Gamma \) itself). It is obvious from (3.3) that the functions in (2.25) - (2.26) representing the energy and the linear three-momentum of the massless particle turn into multiplicative operators "acting" on \( \Gamma \). The Hamiltonian function \( H \) in (1.21) becomes a (possibly non-linear and non-local) differential operator acting on \( \Gamma \). (For free particles \( H \) is simply the kinetic energy in (2.25) i.e. a multiplicative operator).

A Poincaré invariant scalar product on the space of complex valued functions on \( \Pi \) will, tentatively, be defined as:
\begin{equation}
< g_1 | g_2 > := \int [g_1(\bar{\pi}_B, \pi_{B'}) g_2(\pi_{B'}, \bar{\pi}_B)] d\pi_{A'} \wedge d\pi_A \wedge d\bar{\pi}_A \wedge d\bar{\pi}_A. \tag{3.6}
\end{equation}

The subspace $\mathcal{K}$ of $\Gamma$ consisting of functions having finite norm (with respect to the scalar product in (3.6)) defines a Hilbert space of quantum states of a massless particle.

Note that, provided state functions vanish at infinity i.e. vanish for infinite values of the kinetic energy (see 3.9), the operators in (3.3) and (3.4) are hermitian with respect to the scalar product in (3.6). Our tentative choice of the scalar product is based on this fact.

The helicity operator in (3.5) may be regarded as "hermitian" with respect to the scalar product in (3.6) if the two functions $g_1$ and $g_2$ in (3.6) represent eigenstates corresponding to the same eigenvalue of the helicity operator in (3.5).

A point in the four-dimensional configuration space $\Pi$ has a relatively clear physical interpretation. Three of its coordinates combine to give components of the linear null four-momentum (the pole) while the fourth coordinate represents the phase (the flag) of the Weyl spinor.

This implies that instead of $\pi_{A'}$ and $\bar{\pi}_A$ we may choose more physical coordinates on $\Pi$ to label its points. These physical coordinates are $E = p$ - the kinetic energy of the massless quantum particle as "observed" in the inertial frame of the source moving along the world-line in (1.3), $\varphi$, $\theta$ - the angles denoting the direction of motion of the particle in this frame and $\psi$ - the angle representing the phase variable in this frame. The correspondence between the two types of coordinates is given by:

\begin{equation}
\bar{\alpha}^A \bar{\pi}_A = \pm \sqrt{E} e^{\frac{i\varphi}{2}} e^{-i\psi} \sin \frac{\theta}{2}, \quad \alpha^{B'} \pi_{B'} = \pm \sqrt{E} e^{-\frac{i\varphi}{2}} e^{i\psi} \sin \frac{\theta}{2}, \tag{3.7}
\end{equation}

\begin{equation}
\bar{\beta}^B \bar{\pi}_B = \pm \sqrt{E} e^{-\frac{i\varphi}{2}} e^{-i\psi} \cos \frac{\theta}{2}, \quad \beta^{A'} \pi_{A'} = \pm \sqrt{E} e^{\frac{i\varphi}{2}} e^{i\psi} \cos \frac{\theta}{2}, \tag{3.8}
\end{equation}

and inversely by:

\begin{equation}
E = (\alpha^{B'} \bar{\alpha}^B + \beta^{B'} \bar{\beta}^B) \pi_{B'} \bar{\pi}_B, \tag{3.9}
\end{equation}

\begin{equation}
e^{4i\psi} = \frac{\left(\alpha^{A'} \bar{\alpha}_{A'}\right) \left(\beta^{B'} \pi_{B'}\right)}{\left(\beta^{B'} \bar{\alpha}_{B}\right) \left(\alpha^{B'} \bar{\alpha}_{B}\right)}, \tag{3.10}
\end{equation}

\begin{equation}
e^{2i\varphi} = \frac{p_{(2)} + i p_{(3)}}{p_{(2)} - i p_{(3)}} = \frac{\left(\alpha^{A'} \bar{\alpha}_{A'}\right) \left(\beta^{B'} \pi_{B'}\right)}{\left(\beta^{B'} \bar{\alpha}_{B}\right) \left(\alpha^{B'} \bar{\alpha}_{B}\right)}, \tag{3.11}
\end{equation}

\begin{equation}
\cos \theta = \frac{p_{(1)}}{E} = \frac{\left(\beta^{A'} \bar{\beta}_{A} - \beta^{B'} \bar{\beta}_{B}\right) \pi_{A'} \bar{\pi}_A}{\left(\alpha^{B'} \bar{\alpha}^B + \beta^{B'} \bar{\beta}^B\right) \pi_{B'} \bar{\pi}_B}. \tag{3.12}
\end{equation}

From (3.4) and (2.27), (2.30)-(2.32) it follows that the hermitian differential operators corresponding to the three components of the total angular momentum $J_{(1)}$, $J_{(2)}$, $J_{(3)}$ are given by:

\begin{equation}
\hat{J}_{(1)} = \frac{1}{E} \frac{\left(\bar{\alpha}_{A} \beta^B + \bar{\beta}^B \beta_{A}\right) \pi_{B'}}{\bar{\alpha}_{B} \beta_{B'} + \bar{\beta}^B \beta_{B'}} \partial_{\bar{\alpha}_{A}} - \frac{1}{E} \frac{\left(\alpha^{A'} \beta^B + \alpha^{B'} \beta_{A}\right) \pi_{B'}}{\alpha^B \beta_{B'} + \alpha^{B'} \beta_{B'}} \partial_{\alpha^{A'}}, \tag{3.13}
\end{equation}
\[ \hat{J}_2 = \frac{1}{\sqrt{2}}(\beta_A\beta^B - \bar{\alpha}_A\bar{\alpha}^B)\pi_B \frac{\partial}{\partial \bar{\pi}_A} - \frac{1}{\sqrt{2}}(\beta_{A'}\beta^{B'} - \alpha_{A'}\alpha^{B'})\pi_{B'} \frac{\partial}{\partial \bar{\pi}_{A'}}. \quad (3.14) \]

\[ \hat{J}_3 = \frac{i}{\sqrt{2}}(\alpha_A\bar{\alpha}^B + \bar{\alpha}_A\beta^B)\pi_B \frac{\partial}{\partial \bar{\pi}_A} + \frac{i}{\sqrt{2}}(\alpha_{A'}\bar{\alpha}^{B'} + \bar{\alpha}_{A'}\beta^{B'})\pi_{B'} \frac{\partial}{\partial \bar{\pi}_{A'}.} \quad (3.15) \]

The hermitian differential operator representing the square of the absolute value of the total angular momentum:

\[ \hat{J}^2 := \hat{J}_1 \hat{J}_1 + \hat{J}_2 \hat{J}_2 + \hat{J}_3 \hat{J}_3 \quad (3.16) \]

may, as well-known, be written as:

\[ \hat{J}^2 = (\hat{J}_2 + i\hat{J}_3)(\hat{J}_2 - i\hat{J}_3) + \hat{J}_1 \hat{J}_1 - \hat{J}_1. \quad (3.17) \]

For the massless harmonic oscillator the classical (total energy) Hamilton function on \( T \) is given by:

\[ H = E + \frac{k}{2}r^2 \quad r^2 := y(\alpha)y(\alpha) \quad (3.18) \]

where \( y(1), y(2), y(3) \) are functions on \( T \) in accordance with (2.33)-(2.35). Using the quantization prescription in (3.1) and (3.2) (and the principle of normal ordering) these functions may easily be turned into differential operators acting on \( \Gamma \). However, from (2.54) we already know that on \( T \) one has:

\[ r^2 = y(\alpha)y(\alpha) = \frac{J^2 - s^2 + (y(\alpha)p(\alpha))^2}{E^2}. \quad (3.19) \]

Direct calculations also show that in terms of twistor variables we have:

\[ -i\kappa := y(\alpha)p(\alpha) = \frac{i}{2}(\bar{\pi}_A\omega^A - \pi_{A'}\bar{\omega}^{A'}). \quad (3.20) \]

Therefore normal ordering of terms yields:

\[ \hat{\kappa} := \frac{1}{2}(\bar{\pi}_A \frac{\partial}{\partial \pi_A} + \pi_{A'} \frac{\partial}{\partial \bar{\pi}_{A'}} + 4) \quad (3.21) \]

which implies that:

\[ [\hat{\kappa}, \pi_{A'}\bar{\pi}_A] = \pi_{A'}\bar{\pi}_A. \quad (3.22) \]

\[ [\hat{\kappa}, t^{A'A'}\pi_{A'}\bar{\pi}_A] = [\hat{\kappa}, E] = t^{A'A'}\pi_{A'}\bar{\pi}_A = E. \quad (3.23) \]

where by square brackets we denote the commutator.

(3.20)-(3.21), normal ordering of terms and quantization prescription in (3.1)-(3.2) imply that the hermitian differential operator corresponding to the function \( r^2 \) in (3.18)-(3.19) is given by:
\[ \hat{r}^2 := \frac{1}{E^2} (\hat{J}^2 - \hat{s}^2 - \hat{\kappa}^2). \]  

(3.24)

For the quantum mechanical harmonic oscillator the Hamilton (total energy) hermitian differential operator acting on \( \Gamma \) is represented by:

\[ \hat{H} := E + \frac{k}{2} \hat{r}^2. \]  

(3.25)

Common eigenfunctions and eigenvalues of the maximal set of mutually commuting hermitian differential operators \( \hat{s}, \hat{J}_1, \hat{J}_2, \hat{H} \) may now be determined, in the usual fashion, by the following set of differential equations:

\[ \hat{s} f(\pi_{A'}, \bar{\pi}_A) = s f(\pi_{A'}, \bar{\pi}_A), \]  

(3.26)

\[ \hat{J}_1 f(\pi_{A'}, \bar{\pi}_A) = m f(\pi_{A'}, \bar{\pi}_A), \]  

(3.27)

\[ \hat{J}_2 f(\pi_{A'}, \bar{\pi}_A) = J(J+1) f(\pi_{A'}, \bar{\pi}_A), \]  

(3.28)

\[ \hat{H} f(\pi_{A'}, \bar{\pi}_A) = \epsilon f(\pi_{A'}, \bar{\pi}_A). \]  

(3.29)

We will be looking for a set of solutions of the above equations in the form:

\[ f(\pi_{A'}, \bar{\pi}_A) = f_\sigma(\pi_{A'}, \bar{\pi}_A) Y_{\ln}(\theta, \varphi) f_\epsilon(E) \]  

(3.30)

where \( f_\sigma(\pi_{A'}, \bar{\pi}_A) \) is any of the following four holomorphic (either in \( \bar{\pi}_A \) or in \( \pi_{A'} \)) simple expressions:

\[ (\alpha^A \bar{\pi}_A)^{-\sigma} (\bar{\beta}^A \bar{\pi}_A)^{-\sigma} (\alpha^{A'} \pi_A)^{-\sigma} (\beta^{A'} \pi_A)^{-\sigma}. \]

The analysis in the sequel does not depend on which of these four options we assume so the second one is adopted for definiteness. We just note that complex conjugation “creates” a massless particle with reversed eigenvalue of the helicity operator while the two remaining options correspond to the different choices of the “direction” of the helicity relative to the “direction” of the orbital magnetic quantum number \( n \) (see 3.33).

\( \sigma \) is a positive integer (say) and \( Y_{\ln}(\theta, \varphi) \) represent the usual spherical harmonics where \( \varphi \) and \( \theta \) are functions of the spinor variables as defined in (3.11)-(3.12). Spherical harmonics are thus homogenous functions of degree 0 in spinor variables. Besides for any \( f_\epsilon(E) \) where \( E \) is given by (3.9) and \( \hat{s} \) by (3.5) one has:

\[ \hat{s} f_\epsilon(E) = 0. \]  

(3.31)

Therefore we obtain:

\[ \hat{s} f(\pi_{A'}, \bar{\pi}_A) = \hat{s}[(\beta^A \bar{\pi}_A)^\sigma Y_{\ln}(\theta, \varphi)f_\epsilon(E)] = Y_{\ln}(\theta, \varphi)f_\epsilon(E) \hat{s}(\beta^A \bar{\pi}_A)^\sigma = \]

\[ = \frac{\sigma}{\beta^A \bar{\pi}_A^\sigma} Y_{\ln}(\theta, \varphi)f_\epsilon(E) = s f(\pi_{A'}, \bar{\pi}_A) \]  

(3.32)
which shows that eigenvalues of the helicity operator are given by \( s = \frac{\sigma}{2} \).

Simple calculations also show that:

\[
\hat{J}_{(1)} f(\pi_A', \bar{\pi}_A) = Y_{l\ell}(\theta, \varphi) f(\sigma) \hat{J}_{(1)}(\bar{\beta}^A \bar{\pi}_A)^{-\sigma} + (\bar{\beta}^A \bar{\pi}_A)^{-\sigma} f(\sigma) \hat{J}_{(1)} Y_{l\ell}(\theta, \varphi)
\]

\[
= \left( \frac{\sigma}{2} + n \right)(\bar{\beta}^A \bar{\pi}_A)^{-\sigma} Y_{l\ell}(\theta, \varphi) f(\sigma) = m f(\pi_A', \bar{\pi}_A).
\]

(3.33)

The eigenvalues of \( \hat{J}_{(1)} \) are thus given by half-integer or integer numbers:

\[
m = \frac{\sigma}{2} + n.
\]

(3.34)

From standard considerations (see e.g. Ref. 14) it now follows that, for a given eigenvalue of the operator \( \hat{J}^2 \) the quantum numbers \( m \) are bounded by certain upper and lower limits. If we denote the upper limit by a positive number \( J \), then one obtains that the eigenvalue of \( \hat{J}^2 \) is \( J(J+1) \). Note that the positive number \( J = l + \frac{\sigma}{2} \), where \( l \) is the upper limit (for a given eigenvalues of \( \hat{J}^2 \) and of \( \hat{s} \) of \( n \) in (3.34), may assume half-integer or integer values.

Finally we use the eigenvalue equation in (3.29) which, given quantum numbers \( s, m \) and \( J \), determines the eigenfunctions \( f_{\epsilon} \) and the corresponding eigenvalues of the total energy operator \( \hat{H} \) in (3.25). When written out explicitly the equation in (3.29) reduces itself to an ordinary differential equation of second order:

\[
- \frac{d^2 f_{\epsilon}}{dE^2} - \frac{5}{E} \frac{df_{\epsilon}}{dE} + \left( \frac{a}{E^2} + bE \right) f_{\epsilon} = \lambda f_{\epsilon},
\]

(3.35)

where

\[
a = J(J+1) - \left( \frac{\sigma^2}{2} + 2\sigma + 12 \right), \quad b = \frac{2}{k}, \quad \lambda = \frac{2\epsilon}{k}.
\]

(3.36)

The energy levels \( \lambda \) and the corresponding eigenfunctions \( f_{\epsilon} \) may be calculated explicitly on a computer using standard methods e.g. the Ritz method\(^{15}\). The already mentioned computer program Maple V, Release 3 may be of great value here. The results of such explicit calculations we hope to be able to present at a later occasion.

4 CONCLUSIONS AND REMARKS.

If ”elementary” particles such as e.g. electron, proton, neutron etc. may be regarded as bound states of a finite number of massless spinning parts then twistor theory combined with the idea of relativistic action at a distance should provide a very powerful tool for construction of such models.

There arises a possibility to use a twistor phase space formulation. Such a reducible phase space may then be thought of as a direct product of a finite number of copies of an elementary twistor phase space \( T \). Dynamics will then be generated by an appropriately chosen Poincaré scalar hamiltonian function.


In this paper we therefore emphasized particle aspects of Penrose’s twistor formalism as opposed to the standard treatments where field aspects are at the front.

The suggested non-standard quantization in the previous section of this note corresponds to the real polarization of the twistor phase space as described by Woodhouse\textsuperscript{16}. Choosing this non-standard quantization we however loose some of the results of conventional twistor theory such as the twistor description of massless free fields in terms of holomorphic sheaf cohomology, the scalar product on such fields, geometrization of the concept of positive frequency of the field and the relationship between conformal curvature and the twistor ”position” (twistor variables) and ”momentum” (complex conjugates of the twistor variables) operators\textsuperscript{17}.

What we gain is that the real dimension of the relativistic configuration space of a massless spinning particle is one half of the real dimension of the configuration space obtained by means of the conventional holomorphic twistor quantization\textsuperscript{2}. Further, the configuration space obtained in our paper has a clear physical interpretation. Wave functions on such a configuration space define quantum states in (the ”square root” of) the linear momentum representation. However in our opinion the most important gain is the fact that using our formulation we are able to treat interacting massless spinning particles (not fields) forming a closed composite bound system.

Using ideas presented in this paper and in\textsuperscript{4} it would be interesting to investigate a fully relativistic closed system forming a massive and spinning particle composed of three or four directly interacting massless and spinning parts.

Similar in spirit, attempts to remodel the physics of elementary particles, have been made before by Hughston\textsuperscript{10}, Popovich\textsuperscript{18} and Perjê\textsuperscript{19}. These authors make use of conventional twistor quantization and as primary objects regard free fields which are then represented by elements of the holomorphic sheaf cohomology group of an appropriate twistor space.

Finally we note that commutation relation in (2.36) may have an experimental implication. The uncertainty principle following from (2.36) predicts that position of a massless and spinning particle can never be measured exactly. No sharp value of its position vector exists.

The speculations presented in this note are inconclusive with respect to their physical significance. Some calculations are in progress in order to find phenomenological support for the presented ideas.

Nevertheless, our attempts seem to comply with the Twistor Programme announced by Penrose\textsuperscript{20}.

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