An Infinite Dimensional Model for
A Many Server Priority Queue

Neal Master, Zhengyuan Zhou, and Nicholas Bambos
Department of Electrical Engineering, Stanford University
Stanford, CA 94305
{nmaster, zyzhou, bambos}@stanford.edu

Abstract—We consider a Markovian many server queueing
system in which customers are preemptively scheduled according
to exogenously assigned priority levels. The priority levels are
randomly assigned from a continuous probability measure rather
than a discrete one and hence, the queue is modeled by an
infinite dimensional stochastic process. We analyze the equilib-
rium behavior of the system and provide several results. We
derive the Radon-Nikodym derivative (with respect to Lebesgue
measure) of the measure that describes the average distribution
of customer priority levels in the system; we provide a formula
for the expected waiting time of a customer as a function of his
priority level; and we provide a formula for the expected
sojourn time of a customer. Consequently, standard Markov
chain techiques that apply to finitely many priority levels, e.g.
[8], do not apply. Our recent previous work [9] also considered
a continuous distribution of priority levels but only for the
single server case. The current paper generalizes these results
to many servers and is hence distinct from both [9] and
[10].

Because of the complexity that arises due to having a
continuum of priority levels, we opt to simplify other aspects
of the model. We note that all customers in our model
experience the same service rate regardless of their priority
level. This differs from other priority queueing models, e.g.
[11], and restricts our attention to models in which priority
levels only impact scheduling and not service rate. We also
focus on preemptive scheduling as in [12] rather than non-
preemptive scheduling as in [13]. By focusing on preemptive
scheduling we know that the customer who is being serviced
is always the customer with the highest priority. Both of these
assumptions (uniform service rate and preemptive scheduling)
were also exploited in [9] and [10].

We note that the use of infinite dimensional stochastic
processes is itself not novel to queueing. Measure-valued
processes have been used to study the earliest-deadline-first
discipline [14] and the processor-sharing discipline [15], as
well as many server [16] and infinite server models [17]. In
these contexts, the state of the system varies continuously
as the dynamic properties of the jobs change. In our model,
the priority levels are static and so the state only changes
at arrival and departure events. Consequently, our model is
substantially more tractable. Indeed, while these other works
focus on diffusion approximations, we will only present exact
results.

With this motivation and background in mind, the remainder
of the paper is organized as follows. In Section II we fully
describe our model and discuss different choices for the state.
In Section III we analyze the steady state behavior of the
system. We compute the measure that tells us the average
distribution of customer priority levels in the system. We
derive formulae for the expected sojourn and waiting times
of a customer as a function of his priority level. We note how
these results generalize our previous work [9]. In Section IV
we provide some simulation results that verify our analytical
results. In Section V we discuss potential future work and we
conclude in Section VI.

I. INTRODUCTION

Priority queueing models arise in several applications. In
packet switched communication networks, priority levels are
used to deliver differentiated levels of quality of service, e.g.
[1], [2]. In emergency medicine, priority queueing models
are also used to study triage policies, e.g. [3]. Priority queueing
models are also used in financial engineering to model order
books in which limit orders are given priority for matching
with other orders based on their price and time of arrival
[4]. Because priority queueing models are useful in so many
domains, several priority queueing models exist; see [5] for a
standard reference on stochastic priority queueing.

In this paper, we formulate and analyze an $M/M/c$ priority
queueing model in which priority levels are drawn from a
continuum. Unlike previous models that allow for finitely
many priority levels, e.g. [6], [7], our model requires an infinite
dimensional state process. Consequently, standard Markov
chain techiques that apply to finitely many priority levels, e.g.
[8], do not apply. Our recent previous work [9] also considered
a continuous distribution of priority levels but only for the
single server case. The current paper generalizes the results
in our previous work [9] by extending the results to a many
server queue.

The idea of using a continuous distribution for randomly
assigning priority levels was also recently proposed as a
scheduling mechanism for the $M/G/1$ queue [10]. Although
the preemptive priority scheduling mechanism is the same
for both our work and the work in [10], a major difference
is that our work (and our previous work [9]) provides a
characterization of the distribution of customer priority levels
in the system in equilibrium. In contrast, [10] focuses more
on the effect of the randomized scheduling on the overall
population. Another major difference is that we do not assume
that the system is stable. Our current work considers a system
with many servers and is hence distinct from both [9] and
[10].
II. Model Formulation

In this section we formally describe our model and explain our modeling assumptions. We highlight the fact that certain seemingly limiting assumptions are actually without loss of generality. We present three state representations and explain their equivalence. This model is very similar to the model from our previous work \cite{19}; the key difference is that here we allow for more than one server.

We consider an infinite buffer queue with $c$ servers. Customers arrive according to a Poisson process with rate $\alpha > 0$. Customers have independent and identically distributed (IID) service times that are exponentially distributed. Since time can be scaled arbitrarily, we assume that the service times have unit mean. Therefore, the load is $\alpha/c$. Customers are also assigned IID priority levels that are uniformly distributed on the unit interval. The priority levels are independent of all other random quantities in the model. Customers are scheduled preemptively according to their priorities. When there are at most $c$ customers in the system, each customer is assigned to a server; when there are more than $c$ customers in the system, the $c$ with the highest priority levels are assigned to the $c$ servers while the rest wait. When a new customer arrives and no servers are available, he may immediately preempt the lowest priority customer who is in service. The preempted customer waits in the buffer. In summary, we have an $M/M/c$ queue (not necessarily stable) in which customers are preemptively scheduled according to exogenously assigned IID $U([0,1])$ priority levels.

Note that customers are scheduled based on their relative order rather than their absolute value and consequently, the fact that the priority levels are drawn from $U([0,1])$ (as opposed to some other distribution) is actually without loss of generality. Because the scheduling decisions only depend on the relative order of the priority levels, the dynamics would be unchanged if the priorities were transformed by any order-preserving map. In particular, suppose we want to consider priority levels that are drawn from some other distribution with cumulative distribution function (CDF) $F(\cdot)$. Consider two distinct customers $i$ and $j$ with priority levels $p_i$ and $p_j$ drawn from $U([0,1])$. Consider the transformed priority levels $\tilde{p}_i = F^{-1}(p_i)$ and $\tilde{p}_j = F^{-1}(p_j)$ where $F^{-1}(\cdot)$ is the quantile function associated with $F(\cdot)$:

$$F^{-1}(p) = \inf \{ x \in \mathbb{R} : p \leq F(x) \} \quad (1)$$

If $p_i > p_j$ then $\tilde{p}_i \geq \tilde{p}_j$ and we also have that $\tilde{p}_i$ and $\tilde{p}_j$ are distributed according to the CDF $F(\cdot)$ \cite{18} Theorem 2.1. So if $F(\cdot)$ is strictly increasing then using $\tilde{p}_i$ and $\tilde{p}_j$ yields the same scheduling dynamics as using $p_i$ and $p_j$. If $F(\cdot)$ is not strictly increasing, then with non-zero probability we could have $\tilde{p}_i = \tilde{p}_j$. However, in this situation customers $i$ and $j$ are indistinguishable and these ties can be broken in an arbitrary fashion, e.g. randomly. Consequently, our model encompasses arbitrary distributions of priority levels. For simplicity, we will focus having priority levels drawn from $U([0,1])$.

We also note that because of the memorylessness property of the exponential distribution, if a customer is preempted then his residual service time is still exponentially distributed with unit mean. As a result, any choice for the state does not need to include the residual service time of each customer in the system, merely the priority level of each customer. Since the priority levels are drawn from a continuum, almost surely no two customers will have the same priority. Consequently, the state needs to encode the unique priority level of each customer in the system. We find it convenient to encode this list of priority levels as a point measure on $[0,1]$. Let $\mathcal{B}([0,1])$ be the $\sigma$-algebra of Borel sets on $[0,1]$. Given $B \in \mathcal{B}([0,1])$ let $x_t(B)$ be the number of customers in the system at time $t$ with priority levels contained in $B$. To write this symbolically, let $\delta_z$ denote a Dirac measure at $z \in [0,1]$. If there are $N$ customers in the system at time $t$ and their priority levels are $\{p_1, \ldots, p_N\} \subset [0,1]$, then $x_t$ can be written as a sum of Dirac measures:

$$x_t = \sum_{i=1}^{N} \delta_{p_i} \quad (2)$$

Now consider the (non-normalized) CDF or the complement CDF:

$$X_t(p) = x_t([0,p]), \quad \bar{X}_t(p) = x_t((p,1]) \quad (3)$$

These two function-valued stochastic processes are actually equivalent to the measure-valued process defined above. The equivalence follows from the fact that $\{[0,p] : p \in [0,1]\}$ and $\{(p,1] : p \in [0,1]\}$ each form $\pi$-systems that generate $\mathcal{B}([0,1])$. We know that $x_t(\cdot)$ is finite because it is a counting measure. Hence, an elementary application the $\pi$-$\lambda$ Theorem shows that $\{X_t(p) : p \in [0,1]\}$ and $\{\bar{X}_t(p) : p \in [0,1]\}$ each uniquely define $x_t(\cdot)$. The definitions of $\pi$-systems and $\lambda$-systems along with the method of uniquely extending a measure from a $\pi$-system to a $\sigma$-algebra are standard; see \cite{19} Chapter 3 for details.

III. Some Theoretical Results

We now analyze the equilibrium behavior of the system. First we characterize the steady state distribution of $\bar{X}_t(p)$ for each $p \in [0,1]$. We provide a corollary that partially characterizes the steady state distribution of $x_t(\cdot)$. We then provide formulae for the expected sojourn time and the expected waiting time of a customer as functions of its priority level. Each of these results generalizes our previous results regarding the single server case \cite{19}. As in the previous section, we rely on standard results regarding the extension of measures from $\pi$-systems to $\sigma$-algebras \cite{19} Chapter 3.

**Theorem 1.** Fix any $p \in [0,1]$. $X_t(p)$ converges weakly to a random variable $\bar{X}(p)$. If $(1-p)\alpha < c$, then $\bar{X}(p)$ has the following probability mass function (PMF) on the non-negative integers:

$$\mathbb{P}(\bar{X}(p) = k) = \begin{cases} \prod_{i=0}^{c-1} \left( \frac{(1-(p)\alpha)^i}{\alpha^i} + \frac{(1-(p)\alpha)^c}{\alpha^c} \right)^{-1} & , k = 0 \\ \prod_{k=1}^{c} \left( \frac{(1-(p)\alpha)^k}{\alpha^k} \right) & , 1 \leq k \leq c \\ 0 & , k > c \end{cases} \quad (4)$$
If \((1 - p)\alpha \geq c\), then \(\bar{X}(p) = \infty\) almost surely.

**Proof.** As in [9], the key is to notice that because of the preemptive scheduling, the customers with priority levels in \([p, 1]\) are not affected in any way by customers with priority levels in \([0, p]\). Moreover, because the priority levels are independent of the inter-arrival times, the customers with priority levels in \([p, 1]\) arrive according to a Poisson process with rate \((1 - p)\alpha\). As a result, \(\bar{X}_t(p)\) is stochastically equivalent to the population in an \(M/M/c\) queue with unit service rate and arrival rate \((1 - p)\alpha\). As a result, \(\bar{X}_t(p)\) converges weakly to a random variable with the given PMF [20, Chapter 3]. Because there is no upper bound on \(\alpha\), it is possible that \((1 - p)\alpha \geq c\). In this case, the equivalent \(M/M/c\) queue is not stable and hence \(\bar{X}_t(p)\) diverges to infinity. \(\square\)

**Remark 1.** When \(c = 1\) and \((1 - p)\alpha < c\) we have that
\[
P(\bar{X}(p) = k) = (1 - (1 - p)\alpha)((1 - p)\alpha)^k
\]
for all non-negative integers \(k\). In other words, \(\bar{X}(p)\) is a geometric random variable on the non-negative integers with mean \((1 - p)/\alpha\). Hence, this result generalizes our previous work [9, Theorem 1].

**Definition 1.** For convenience, we define
\[
P_0(p) = \mathbb{P}(\bar{X}(p) = 0)
\]
when \((1 - p)\alpha < c\). We also define
\[
p_0(p) = -\frac{d}{dp} P_0(p)
\]
\[
= -P_0(p)^2 \sum_{i=1}^{c-1} \frac{(1 - p)^{i-1} \alpha^i}{i!} + \frac{(1 - p)\alpha^c}{c!(1 - (1 - p)\alpha/c)}
\]
\[
+ \frac{(1 - p)\alpha^{c+1}}{c! c(1 - (1 - p)\alpha/c)^2}
\]

**Corollary 1.** Fix \(B \in \mathcal{B}([0, 1])\). Then \(x_t(B)\) converges weakly to a random variable \(x(B)\) with mean
\[
\mu(B) = \mathbb{E}[x(B)] = \int_B m(p) dp
\]
where
\[
m(p) = \alpha + \frac{\alpha^{c+1}}{c \times c!} \left[ \frac{(c + 1)(1 - p)^c P_0(p) + (1 - p)^{c+1} P_0(p)}{(1 - (1 - p)\alpha/c)^2} \right]
\]
\[
+ \frac{2(1 - p)^{c+1} P_0(p)/(\alpha/c)}{(1 - (1 - p)\alpha/c)^3}
\]
when \((1 - p)\alpha < c\) and \(m(p) = \infty\) otherwise.

**Proof.** We can use the PMF from the previous theorem to show that
\[
\mathbb{E}[\bar{X}(p)] = (1 - p)\alpha + \frac{\alpha^{c+1}(1 - p)^{c+1} P_0(p)}{c \times c! (1 - (1 - p)\alpha/c)^2}.
\]
This is the average number of customers in an \(M/M/c\) queue with arrival rate \((1 - p)\alpha\) and unit service rate. Therefore, if \(B = [a, b]\) for some \(0 \leq a < b \leq 1\), then \(x_t(B) = \bar{X}_t(a) - \bar{X}_t(b)\). Since \(x_t([a, b])\) converges weakly to \(x(a) - x(b)\), performing the integration gives us the same result subtracting \(\mathbb{E}[\bar{X}(b)]\) from \(\mathbb{E}[\bar{X}(a)]\). Indeed, note that for \(p\) such that \((1 - p)\alpha < c\),
\[
m(p) = -\frac{d}{dp} \mathbb{E}[\bar{X}(p)].
\]
Now note that intervals of this form are a \(\pi\)-system that generates \(\mathcal{B}([0, 1])\). Consequently, if \(\alpha < c\) then \(\mu([0, 1]) < \infty\) and so this defines a unique measure on \(\mathcal{B}([0, 1])\). On the other hand, if \(\alpha \geq c\), we can still extend the measure from the \(\pi\)-system to \(\mathcal{B}([0, 1])\), but uniqueness is no longer guaranteed. However, we can apply the same reasoning as above to define a unique measure on \(\mathcal{B}([1-c/\alpha, 1])\) where \(\mu(\cdot)\) is finite. The fact that \(\mu(B) = \infty\) for any \(B\) such that \(B \cap [0, 1-c/\alpha]\) has non-zero Lebesgue measure follows from the instability argument in the previous theorem. Hence, regardless of the value of \(\alpha\), we can conclude that the expression for the mean equilibrium behavior of \(x_t(B)\) holds for any \(B \in \mathcal{B}([0, 1])\). \(\square\)

**Remark 2.** When \(c = 1\) and \((1 - p)\alpha < c\) we have that
\[
m(p) = \frac{\alpha}{(1 - (1 - p)\alpha)^2}
\]
so the corollary generalizes the results in our previous work [9].

Because service can be preempted and hence customers can enter service multiple times, we formally define the terms “sojourn time” and “waiting time”. In particular, we note that the amount of time a customer spends in service before being preempted is considered waiting. We used the same definitions in our prior work [9].

**Definition 2.** A customer’s sojourn time is the amount of time from when the customer arrives to when it departs after completing service.

**Definition 3.** A customer’s waiting time is the amount of time from when the customer arrives to the beginning of the last time the customer enters service.

**Theorem 2.** Fix any \(p \in [0, 1]\) and let \(s(p)\) be the expected sojourn time for a customer with priority \(p\) in steady state. Then if \((1 - p)\alpha < c\) then
\[
s(p) = \frac{1}{\alpha} m(p)
\]
and if \((1 - p)\alpha \geq c\) then \(s(p) = \infty\).

**Proof.** The case for which \(s(p) = \infty\) follows trivially from the instability argument in Theorem 1.\]
The corollary gives us a formula for $E \alpha$ have infinite sojourn times. This seems a bit paradoxical: time while only customers with priority levels equal to zero

Remark 5. Customers in $[0, 1]$ arrive at a rate $(1 - p)\alpha$ so we have that

$$E[X(p)] = (1 - p)\alpha S(p) = \alpha \int_p^1 s(q) dq. \quad (15)$$

The corollary gives us a formula for $E[X(p)]$:

$$\int_p^1 s(q) dq = \frac{1}{\alpha} E[X(p)] = \frac{1}{\alpha} \int_p^1 m(q) dq \quad (16)$$

Since this holds for any $p$, we have that $s(p) = m(p)/\alpha$. □

**Corollary 2.** Fix any $p \in [0, 1]$ and let $w(p)$ be the expected waiting time for a customer with priority $p$ in steady state to receive service. If $(1 - p)\alpha < c$ then

$$w(p) = s(p) - 1 = \frac{1}{\alpha} m(p) - 1 \quad (17)$$

and if $(1 - p)\alpha \geq c$ then $w(p) = \infty$.

Proof. The sojourn time is the sum of the waiting time and the service time. Since we have a unit service rate, we merely subtract 1 from $s(p)$ to get $w(p)$. □

**Remark 3.** We note that the relationships between $m(\cdot)$, $s(\cdot)$, and $w(\cdot)$ are the same as they were in the single server case [9]. Consequently, the previous theorem and corollary generalize the results from our previous work.

**Remark 4.** If $\alpha \geq c$ then $m(\cdot)$ (and hence both $s(\cdot)$ and $w(\cdot)$) exhibit a bifurcation, i.e. a qualitative change in behavior, at

$$p^* = 1 - \frac{c}{\alpha}. \quad (18)$$

It is intuitive that when the system is overloaded, lower priority customers will be ignored so that higher priority customers can be served. The quantity $p^*$ makes this intuition precise: when the system is overloaded, customers with priority levels in $[0, p^*]$ will have infinite expected waiting times while customers in $(p^*, 1]$ will have finite expected waiting times.

**Remark 5.** The aforementioned bifurcation makes the case of $\alpha = c$ particularly interesting. We know that when $\alpha = c$ the $M/M/c$ is unstable. However, in this case $p^* = 0$ so all customers with priority levels in $(0, 1]$ have a finite sojourn time while only customers with priority levels equal to zero have infinite sojourn times. This seems a bit paradoxical: the queue is unstable but almost every customer has a finite sojourn time. This counterintuitive result arises because $\alpha = c$ is the critical point between a stable $M/M/c$ queue and an unstable $M/M/c$ queue.

**Remark 6.** The previous remarks highlight the fact that this infinite dimensional priority scheduling scheme can be used to “partially stabilize” an unstable single class queueing system in the following sense. If we have a single class $M/M/c$ system with $\alpha \geq c$ that is scheduled in either a last-come-first-serve (LCFS) or first-come-first-serve (FCFS) manner, then we know that the overall population of the queue will be unstable and we cannot provide any guarantee of reasonable service to any of the customers. If we instead randomly assign priority levels to arriving customers and schedule preemptively according to these priority levels, then we can guarantee that $c/\alpha$ of the customers can expect to have finite waiting times. Moreover, upon arrival we can say with certainty exactly which customers will have this guarantee.

**IV. Simulation Verification**

In this section, we report the results of two discrete event simulations of the system: one with $\alpha < c$ and one with $\alpha \geq c$. In both cases, we use the simulated data to estimate $m(\cdot)$, $s(\cdot)$, and $w(\cdot)$. In general, we see that the estimates match our theoretical results, thus supporting our analysis.

**A. Estimation Methods**

For each of the functions that we estimate, we first get local estimates and then linearly interpolate to estimate the entire function. The details for each function are outlined below and are the same as in our previous work [9]. For all functions, we assume a discretization of $0 < \delta < 1$ with an integer $N_\delta = \delta^{-1}$.

We compute our estimate of $m(\cdot)$, which we denote $\hat{m}(\cdot)$, as follows:

1. Because “Poisson Arrivals See Time Averages” [22], we record $x_i(\cdot)$ as observed immediately before each arrival.
2. For $p_i \in \{\delta/2 + i\delta\}_{i=0}^{N_\delta - 1}$, we average the number of customers with priority levels in the half-open interval $[p_i - \delta/2, p_i + \delta/2)$ across our observations. We scale this average by $N_\delta$ to get $\hat{m}(p_i)$.
3. We linearly interpolate $\{\hat{m}(p_i)\}_{i=0}^{N_\delta - 1}$ to get $\hat{m}(\cdot)$.

We compute our estimate of $s(\cdot)$, which we denote $\hat{s}(\cdot)$, in a similar fashion:

1. We record the arrival time, the departure time, and the priority level of each customer. If a customer does not depart in the time horizon, then his departure time is infinite.
2. For $p_i \in \{\delta/2 + i\delta\}_{i=0}^{N_\delta - 1}$, we average the sojourn times for customers with priority levels in the half-open interval $[p_i - \delta/2, p_i + \delta/2)$ across our observations. We scale this average by $N_\delta$ to get $\hat{s}(p_i)$.
3. We linearly interpolate $\{\hat{s}(p_i)\}_{i=0}^{N_\delta - 1}$ to get $\hat{s}(\cdot)$.

We compute our estimate of $w(\cdot)$, which we denote $\hat{w}(\cdot)$, in a similar fashion:

1. We record the arrival time, the last time that the customer enters service before departing, and the priority level of each customer. If the customer never departs then the departure time is infinite.
2. For $p_i \in \{\delta/2 + i\delta\}_{i=0}^{N_\delta - 1}$, we average the waiting times for customers with priority levels in the half-open interval $[p_i - \delta/2, p_i + \delta/2)$ across our observations. We scale this average by $N_\delta$ to get $\hat{w}(p_i)$.
3. We linearly interpolate $\{\hat{w}(p_i)\}_{i=0}^{N_\delta - 1}$ to get $\hat{w}(\cdot)$.
B. Estimation Results

We use $\delta = 0.05$ and a time horizon of $T = 2 \times 10^3$. We fix $c = 2$ servers and consider two values of $\alpha$. When $\alpha = 1.5$ we have a stable system and when $\alpha = 5.0$ we have an unstable system.

First we consider the stable case in which $m(\cdot)$, $s(\cdot)$, and $w(\cdot)$ are finite. The results are plotted in Fig. 1. Though a bit “noisy”, the estimates generally agree with our theoretical analysis. Moreover, we see that the estimates have roughly the same shape and merely differ by constant factors. This confirms our previous analysis regarding the mean equilibrium behavior of $x_t(\cdot)$, the expected sojourn time, and the expected waiting time.

Now consider the unstable case for which $m(\cdot)$, $s(\cdot)$, and $w(\cdot)$ are finite only for $p \in (p^*, 1] = (0.6, 1]$. As a result, we do not plot the functions for $p < p^*$. Because of the vertical asymptote at $p^*$, we use a log-scale for the vertical axis. The results are plotted in Fig. 2. In Fig. 2a, we see that $\hat{m}(\cdot)$ and $m(\cdot)$ seem to agree on $(p^*, 1]$. Fig. 2a also depicts the bifurcation at $p^*$. We see that $\hat{m}(p^* - \delta/2)$ is roughly 10 times the value of $\hat{m}(p^* + \delta/2)$. This reflects the fact that $\hat{m}(p)$ will diverge to infinity as $T \uparrow \infty$ for $p < p^*$. We see similar results regarding $\hat{s}(\cdot)$ in Fig. 2b. For $p \in (p^*, 1]$, $\hat{s}(p)$ and $s(p)$ agree. Note that for $p < p^*$, neither $\hat{s}(p)$ nor $s(p)$ appear on the plot because both quantities are infinite. Hence, we see that $\hat{s}(\cdot)$ and $s(\cdot)$ agree for all $p \in [0, 1]$. We see the same results for $\hat{w}(\cdot)$: the estimate agrees with the analytic result where both are finite and also where both are infinite.

V. Future Work

Our work points to several potential directions of future work. One is to derive more results about the current model. For example, it would be interesting to know more about the higher order statistics of $x_t(\cdot)$. It would also be interesting to extend this model to a network setting. With a single queue, $x_t(\cdot)$ is a point measure on $[0, 1]$ but for a system with $n$ queues we would need to have $x_t(\cdot)$ be a point measure on $[0, 1]^n$. It seems reasonable to expect that the steady state distribution would have a product-form as in Jackson’s Theorem [23], but the details of the analysis are not immediately clear. In particular, although the arriving priority levels are IID $U([0, 1])$ we need to know how customers’ priority levels are correlated after they depart.

As noted in our previous work [9], it may also be interesting to consider a heavy traffic analysis. Priority queues are an example of a system that exhibits “state-space collapse” in heavy traffic [24]. In brief, we would see that upon appropriate rescaling, the diffusion limit associated with $X_t(p)$ for $p < p^*$ would be zero. However, it may be possible to consider a diffusion limit for which $p^* \downarrow 0$ so that the diffusion limit does not collapse to zero. This idea is not yet well developed but since our analysis applies to overloaded queues, it may fruitful to consider.

Fig. 1. Estimates of $m(\cdot)$, $s(\cdot)$, and $w(\cdot)$ based on the data generated by simulating the system with $c = 2$ and $\alpha = 1.5$. For these values of $c$ and $\alpha$, the queue is stable and so we use a linear scale for both axes.
VI. Conclusions

We have presented an infinite dimensional model for a many server priority queue in which customers are scheduled preemptively according to priority levels that are drawn from a continuous probability distribution. Our steady state analysis characterizes the first-order statistics of the measure-valued process that describes the priority levels of the customers in the queue. We have used derived formulae for the expected sojourn and waiting times of a function of customer priority level. These results generalize our previous work [9] and contribute to a broader literature on preemptive scheduling with random priorities [10]. Discrete event simulations verify our analytical results and we have discussed some areas of future work.

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