A general framework to study the extremal phase transition of black holes

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Abstract

We investigate the universality of some features for the extremal phase transition of black holes and unify all the approaches which have been applied in different spacetimes. Unlike the other existing approaches where the information of the spacetime and its dimension is directly used to get various results, we provide a general formulation in which those results are obtained for any arbitrary black hole spacetime having an extremal limit. Calculating the second order moments of fluctuations of some thermodynamic quantities we show that, the phase transition occurs only in the microcanonical ensemble. Without considering any specific black hole we calculate the values of critical exponents for this type of phase transition. These are shown to be in agreement with the values obtained earlier for metric specified cases. Finally we extend our analysis to the geometrothermodynamics (henceforth GTD) formulation. We show that for any black hole, if there is an extremal point, the Ricci scalar for the Ruppeiner metric must diverge at that point.

1 Introduction

The remarkable discovery of Bekenstein \cite{1} and Hawking \cite{2} in the seventies laid the foundation of black hole thermodynamics, which has been the subject of ardent research in the following decades till date. Identifying the thermodynamic parameters (such as entropy, temperature, energy \textit{etc.}) from the geometrical quantities of the black hole spacetime (such as the area of the horizon, surface gravity of the black hole horizon \textit{etc.}), four laws of black hole mechanics were formulated in 1973 \cite{3}. These works clearly imply the existence of thermodynamic structure of the black hole horizon. Since then, many thermodynamic phenomena have been observed in black hole spacetime. The study of phase transition, which is an important phenomenon in ordinary thermodynamics, has also been explored in black hole mechanics since seventies. It was introduced by Davies \cite{4} and subsequently followed by many other researchers \cite{5,8}. Davies endorsed that a black hole goes through a second order phase
transition when it passes through a point (Davies’ point) where the heat capacity becomes infinitely discontinuous. However, later Kaburaki et al. [9–12] claimed that Davies’ point is not a critical point. Instead, it is merely a turning point, where stability changes.

Although, Davies’ claim was later falsified, other groups argued that a second order phase transition indeed takes place when a non-extremal black hole transforms to an extremal one and the extremal limit was identified as a critical point. It was first concluded by Curir in [13,14]. Later Pavón and Rubí [15,16] calculated second order moments of fluctuation of mass, angular momentum etc. using Landau-Lifshitz hydrodynamic fluctuation theory (see chapter 17 of [17]) and have shown that those second order moments diverge in the extremal limit of Kerr and Reissner-Nordström (RN) black holes but those moments are finite in the non-extremal limit and for the Schwarzschild black hole. Also, those second order moments remain finite at the Davies’ point. Both the analysis are in agreement with each other and suggest that the extremal limit of the black hole is a critical point, and the divergence of second order moments of fluctuation should signal a second order phase transition of the black holes which are changing from its non-extremal phase to the extremal phase. Later, this phase transition in the extremal limit has been rigorously studied for different (Kerr-Newman [12], BTZ [18–20] etc.) black holes and critical exponents were obtained. These exponents satisfy the well known scaling laws [21,22] of thermodynamics.

The works, which are mentioned above, are performed in different spacetimes to come to the same central conclusion that the extremal limit is a critical point and the transformation from a non-extremal to an extremal black hole is a second order phase transition. Moreover, in those cases, the information of the spacetime has directly been used to obtain the results. One question naturally appears: is it really necessary to start with a particular spacetime to reach this conclusion? The results present in different papers suggest us to believe that probably the conclusion is true irrespective of spacetime metric and its dimension. But till now there has not been any such proof. Moreover, there are few major questions which has not been addressed properly. Some of these are: Are the critical exponents universal? Is the effective spatial dimension one in every extremal black holes etc. In this paper we address all these issues systematically.

Our analysis is valid for all the black holes which are extremal at certain limit. Without introducing any particular spacetime we show that the transformation of black hole from non-extremal to extremal is a second order phase transition with the extremal limit being the critical point. To prove that, we calculate the second order moments of fluctuation modes of some thermodynamic quantities using equilibrium fluctuation theory of statistical mechanics [11,12,23] and show that those moments diverge in the microcanonical ensemble. Thereby we show that the phase transition is well described only by the micro-canonical ensemble instead of the canonical or the grand canonical ensembles. Later, we proceed our analysis to obtain the values of critical exponents in a general way. These exponents match with the results, obtained earlier by considering the explicit form of the spacetime. Also these have been shown to satisfy the scaling laws. We emphasize that in our whole analysis the only underlying information one requires is: one should consider the particular class of black hole spacetimes which exhibit such non-extremal to extremal transition at certain limit and additionally, the thermodynamics of those black holes are governed by the usual first law of black hole mechanics at the non-extremal limit.

We also analyze another interesting aspect in our paper. It is known for a long time that classical thermodynamics can also be studied by geometric method. This is the GTD formulation. In Weinhold’s approach the metric is defined as the Hessian of the internal energy and in the Ruppeiner’s approach the metric is defined as the Hessian of the entropy. It has been shown that Ruppeiner
curvature scalar diverges at the extremal limit of the BTZ black hole [19,20]. In the present paper we have proved this result for any arbitrary black hole which has an extremal point.

Very recently it has been claimed that neither the Weinhold nor Ruppeiner formulation is Legendre invariant and, hence, they are inappropriate to analyze the thermodynamics. So, we proceed one step further to find the thermodynamic behaviour at the extremal point using Legendre-invariant metric. We do this for two Quevedo GTD metrics and find that the Ricci scalar for both of those metrics are finite at the extremal point. Thus, our work connects all the previous diverse conclusions about extremal phase transitions, all of which are black hole specific. In this sense, our work is unique and fills an important gap in the literature.

Before we proceed further, let us mention the organization of our paper. In the second section we discuss the black hole thermodynamics at the extremal point without using any particular form of spacetime. Second order moments of fluctuation are calculated for microcanonical, canonical and grand canonical ensembles in three subsections. It is observed that the phase transition is compatible with the first ensemble. Next section is dedicated to calculate the values of different critical exponents. Then in section 3 thermogeometric analysis has been performed separately for Weinhold, Ruppeiner and two Legendre invariant metrics. It is shown that the curvature scalar diverges only for the Ruppeiner metric. Finally, in the last section, we draw conclusions of our work.

2 Thermodynamic analysis of extremal point in different ensembles

We have already mentioned that, the extremal phase transition is regarded as a second order phase transition. This was first claimed by Curir [13, 14]. According to Pavón and Rubí [15, 16], the divergence of the second order moments of fluctuations of thermodynamic quantities is a signature of this phase transition. Following this argument, here we calculate these second order moments in different ensembles. We show that, only in microcanonical ensemble extremal limit of black hole (if it exists) is a second order phase transition.

Here, we calculate the second order moments using the well-defined equilibrium fluctuation theory of statistical mechanics. In that case, the required thermodynamical quantities are obtained from Massieu function, which are the Legendre transformations of the entropy. In that formalism, the state of a given environment is completely characterized by the Massieu function $\Phi$, whose variation is given by

$$d\Phi = \lambda_i dY^i.$$  \hspace{1cm} (1)

Here, the summation convention has been adopted. In the above relation, Massieu function is a function of the intrinsic variables $Y^i$. $\lambda_i$, which is the conjugate variables of $Y^i$, is defined as $\lambda_i = (\partial \Phi / \partial Y^i)_{\bar{Y}^i}$. In our notation, $\bar{Y}^i$ is the set of all intrinsic variables excluding $Y^i$. Throughout our analysis, a bar overhead will imply similar thing. Now for a given environment, the spontaneous fluctuation from the equilibrium occurs only in the conjugate variables $\lambda_i$. This is because, the reservoirs are considered to be large compared to the system and as a result, the intrinsic variables are fixed. Then the probability of the deviation from the equilibrium is proportional to $\exp[-\Sigma \lambda_i (\delta \lambda^i)^2 / (2k_B)]$ [12], where $k_B$ is the Boltzmann constant. The eigenvalues of the fluctuation modes are defined as

$$\lambda_i = \left. \frac{\partial Y_i}{\partial \lambda^i} \right|_{\bar{Y}^i} = \left( \frac{\partial^2 \Phi}{\partial Y^i} \right)^{-1}.$$  \hspace{1cm} (2)
Here it should be mentioned that the probability is accurate only up to the second order. The averages of modes of fluctuations always vanish \(^{23}\) and the second order moments are given by

\[
M_{ij} = \langle \delta X_i \delta X_j \rangle = k_B \left( \frac{\partial^2 \Phi}{\partial Y^2} \right)_{Y_i} \delta_{ij} = \frac{k_B}{\lambda_i} \delta_{ij} .
\]

(3)

In the following analysis, we investigate the behaviour of these quantities in each ensemble. Since the extremal limit is not a turning point \(^{12}\), the divergence of the second order moments will imply the presence of second order phase transition.

2.1 Microcanonical ensemble

Let us consider an isolated black hole by definition which exchanges nothing with the environment. In this case, the proper Massieu function \(\Phi_1\) is the entropy \(S\). Its change is given by the first law of black hole mechanics

\[
dS = \beta dM - \tilde{X}^i dY_i ,
\]

(4)

where \(\beta = 1/T\) and \(\tilde{X}^i = \beta X^i\). According to our notations \(X^i\) are potential, angular velocity etc., whereas \(Y_i\) are charge, angular momentum etc. Then the eigenvalues of the fluctuations are given by

\[
\lambda_M^{(1)} = \left( \frac{\partial^2 S}{\partial M^2} \right)_{Y_i}^{-1} = \left( \frac{\partial M}{\partial \beta} \right)_{Y_i} = - T^2 C_Y
\]

(5)

and

\[
\lambda_{Y_i}^{(1)} = \left( \frac{\partial^2 S}{\partial Y^2_i} \right)_{M, \tilde{Y}_i}^{-1} = \left( \frac{\partial Y_i}{\partial \tilde{X}_i} \right)_{M, \tilde{Y}_i} = - T I_M^{(i)} .
\]

(6)

Here we used the following definitions: \(C_Y = (\partial M/\partial T)_{Y_i} = - \beta^2 (\partial M/\partial \beta)_{Y_i}\) and \(I_M^{(i)} = (\partial Y_i/\partial X^i)_{M, \tilde{Y}_i} = \beta (\partial Y_i/\partial \tilde{X}_i)_{M, \tilde{Y}_i}\). Therefore the second order moments are given by

\[
\langle \delta \beta \delta \beta \rangle = k_B \left( \frac{\partial^2 S}{\partial M^2} \right)_{Y_i} = - k_B \frac{\beta^2}{C_Y}
\]

(7)

and

\[
\langle \delta \tilde{X}^i \delta \tilde{X}^i \rangle = k_B \left( \frac{\partial^2 S}{\partial Y^2_i} \right)_{M, \tilde{Y}_i} = - k_B \frac{\beta}{I_M^{(i)}} .
\]

(8)

In the following section, where we obtain the critical exponents in a general way, we show that both \(\langle \partial^2 S/\partial M^2 \rangle_{Y_i}\) and \(\langle \partial^2 S/\partial Y^2_i \rangle_{M, \tilde{Y}_i}\) diverge at the extremal limit (see \((30)\) and \((35)\)). Therefore, we can conclude from \((5)\) and \((6)\) that all the eigenvalues \(\lambda_M^{(1)}\) and \(\lambda_{Y_i}^{(1)}\) vanish. As a result, from \((7)\) and \((8)\) we see that all the second order moments diverge, which is the signature of phase transition. Thus, in the microcanonical ensemble, an extremal phase transition is a second order phase transition with the extremal limit being the critical point.

\(^{1}\)This is one of the inputs of our present discussion; whereas the other input is the existence of extremal limit in the black hole thermodynamics.
2.2 Canonical ensemble

In canonical ensemble, black hole can exchange only energy with the environment. The proper Massieu function ($\Phi_2$) in this ensemble is $\Phi_2 = S - \beta M = -\beta F$, where $F = M - TS$ is the Helmholtz free energy. Note that $dF = -SdT + X^i dY_i$ and $d\Phi_2 = -M d\beta - \tilde{X}^i dY_i$. Therefore, in this case, the intrinsic variables are $\beta$ and $Y_i$ whereas the conjugate quantities are $(-M)$ and $(-\tilde{X}^i)$. The eigenvalues are given by

$$\lambda^{(2)}_{\beta} = \left( \frac{\partial^2 \Phi_2}{\partial \beta^2} \right)^{-1} Y_i = -\left( \frac{\partial \beta}{\partial M} \right) Y_i = \frac{\beta^2}{C_Y} .$$

and

$$\lambda^{(2)}_{Y_i} = \left( \frac{\partial^2 \Phi_2}{\partial Y_i^2} \right)^{-1} \beta, \tilde{X}_i Y_i = -\left( \frac{\partial Y_i}{\partial \tilde{X}_i} \right) \beta, \tilde{Y}_i = -T I^{(i)}_{\beta} .$$

In the above, we have used $I^{(i)}_{\beta} = (\partial Y_i / \partial X^i)_{\beta, \tilde{Y}_i} = \beta (\partial Y_i / \partial \tilde{X}_i)_{\beta, \tilde{Y}_i}$. The second order moments, in this case, are found to be

$$\langle \delta M \delta M \rangle = k_B \left( \frac{\partial^2 \Phi_2}{\partial \beta^2} \right) Y_i = k_B T^2 C_Y$$

and

$$\langle \delta \tilde{X}_i \delta \tilde{X}_i \rangle = k_B \left( \frac{\partial^2 \Phi_2}{\partial Y_i^2} \right) \beta, \tilde{X}_i Y_i = -k_B \frac{\beta}{I^{(i)}_{\beta}} .$$

In the appendix A, we show that $\left( \frac{\partial^2 \Phi_2}{\partial \beta^2} \right) Y_i$ vanishes and $\left( \frac{\partial^2 \Phi_2}{\partial Y_i^2} \right) \beta, \tilde{X}_i Y_i$ diverges. As a result $\lambda^{(2)}_{\beta}$ in (9) diverges and $\lambda^{(2)}_{Y_i}$ in (10) vanishes. Also, the nature of the second order moments are evident: $\langle \delta M \delta M \rangle$ of (11) vanishes and $\langle \delta \tilde{X}_i \delta \tilde{X}_i \rangle$ of (12) diverges. Therefore the extremal limit is not a critical point in the canonical ensemble.

2.3 Grand canonical ensemble

Finally we consider the black hole in grand canonical ensemble. It means, the black hole not only exchanges energy with the environment but also performs work on the surroundings. The proper Massieu function in this case is $\Phi_3 = \Phi_2 + \tilde{X}^i Y_i = S - \beta M + \tilde{X}^i Y_i = -\beta G$. Wher, $G = M - TS - X^i Y_i$ is Gibbs free energy. The variation of $G$ is $dG = -SdT - Y_idX^i$ and the variation of Massieu function $\Phi_3$ is $d\Phi_3 = -Md\beta + Y_id\tilde{X}_i$ . Therefore in this ensemble, the intrinsic variables are $\beta$ and $\tilde{X}_i$ whereas the conjugate variables are $(-M)$ and $Y_i$. The eigenvalues of the fluctuation modes are

$$\lambda^{(3)}_{\beta} = \left( \frac{\partial^2 \Phi_3}{\partial \beta^2} \right)^{-1} \tilde{X}_i = -\left( \frac{\partial \beta}{\partial M} \right) \tilde{X}_i = \frac{\beta^2}{C_{\tilde{X}}} ,$$

and

$$\lambda^{(3)}_{\tilde{X}_i} = \left( \frac{\partial^2 \Phi_3}{\partial \tilde{X}_i^2} \right)^{-1} \beta, \tilde{X}_i = \left( \frac{\partial \tilde{X}_i}{\partial Y_i} \right) \beta, \tilde{X}_i = \frac{\beta}{I^{(i)}_{\beta}} .$$
In the above, we have used $C \tilde{X} = (\partial M / \partial T)_{\tilde{X}} = -\beta^2 (\partial M / \partial \beta)_{\tilde{X}}$. The second order moments in grand canonical ensemble are

$$\langle \delta M \delta M \rangle = k_B \left( \frac{\partial^2 \Phi_3}{\partial \beta^2} \right)_{\beta, \tilde{X}} = k_B T^2 C \tilde{X} \quad (15)$$

and

$$\langle \delta Y_i \delta Y_i \rangle = k_B \left( \frac{\partial^2 \Phi_3}{\partial \tilde{X}_i^2} \right)_{\beta, \tilde{X}} = k_B T \tilde{f}_i^{(i)} \quad (16)$$

In the appendix B, we show that both $(\partial^2 \Phi_3 / \partial \beta^2)_{\tilde{X}}$ and $(\partial^2 \Phi_3 / \partial \tilde{X}_i^2)_{\beta, \tilde{X}}$ vanish. As a result, we conclude that both the eigenvalues of the fluctuation modes $\lambda^{(3)}_{\beta}$ and $\lambda^{(3)}_{\tilde{X}_i}$ diverge. Naturally both the second order moments $\langle \delta M \delta M \rangle$ and $\langle \delta Y_i \delta Y_i \rangle$ vanish. As a result, the extremal limit is not a second order phase transition in the grand canonical ensemble.

3 Obtaining the critical exponents in a general way

In the earlier section, we have generally shown that the extremal phase transition is indeed a second order thermodynamic phase transition in the microcanonical ensemble. In this section we obtain the values of the critical exponents in a general manner. There are several works which studied extremal criticality and obtained the critical exponents case by case. For example, in [12] the extremal phase transition of Kerr-Newman black hole was studied and critical exponents were obtained. Similar studies were done for BTZ black hole in [18–20]. In our general framework, we obtain the values of critical exponents in a metric independent way.

The critical exponents are defined for the response coefficients and for the order parameters to show how those quantities diverge near the critical point [24]. The response coefficients are defined as the inverse of the eigen values $\lambda_i$’s [11]. For the extremal phase transition and in the microcanonical ensemble, the response coefficients are defined as:

$$\zeta_Y = \left( \frac{\partial^2 S}{\partial M^2} \right)_{Y_i} \quad (17)$$

$$\zeta_M^{i} = \left( \frac{\partial^2 S}{\partial Y_i^2} \right)_{M, \tilde{Y}_i} \quad (18)$$

In the first definition, $Y_i$ includes all the charges present in the theory, whereas, in the second definition, $\tilde{Y}_i$ includes all the charges except $Y_i$. In classical thermodynamics, the order parameters are the difference of some extensive quantities of the two different phases. For black hole, the order parameters are defined as the difference of the conjugate quantities on the inner and the outer horizon [18,24–26]. For the presence of multiple charge and angular momentum, we define the order parameters in a general manner,

$$\eta_{Y_i} = \tilde{X}_+^i - \tilde{X}_-^i \quad (19)$$

where, $\tilde{X}_i = (X_i / T) = -(\partial S / \partial Y_i)_{M, \tilde{Y}_i}$ as we have defined earlier. The subscripts “+” and “−” stands for the outer horizon ($r_+$) and inner horizon ($r_−$) respectively. Now, the critical exponents are defined
\[ \zeta_Y \sim m^{-\alpha} \quad \text{(for} \quad Y_i = Y_{ic} \text{)} \tag{20} \]
\[ \zeta_Y \sim y_i^{-\phi_i} \quad \text{(for} \quad M = M_c \text{ and} \quad \bar{Y}_i = \bar{Y}_{ic} \text{)} \tag{21} \]
\[ \zeta_i^M \sim m^{-\gamma_i} \quad \text{(for} \quad Y_i = Y_{ic} \text{)} \tag{22} \]
\[ \zeta_i^M \sim y_i^{-\sigma_i} \quad \text{(for} \quad M = M_c \text{ and} \quad \bar{Y}_i = \bar{Y}_{ic} \text{)} \tag{23} \]
\[ \eta_i \sim m^{\beta_i} \quad \text{(for} \quad Y_i = Y_{ic} \text{)} \tag{24} \]
\[ \eta_i \sim y_i^{\delta_i-1} \quad \text{(for} \quad M = M_c \text{ and} \quad \bar{Y}_i = \bar{Y}_{ic} \text{)} \tag{25} \]

Here we use the notation \( m = 1 - M/M_c \) and \( y_i = 1 - Y_i/Y_{ic} \), whereas \( c \), in the subscript, signifies the corresponding values at the critical point. Remember that the critical point, in our present discussion, is the extremal point where temperature \( T \) vanishes.

Now we expand the mass as a function of entropy \( S \) and charge \( Y_i \) near the critical point. Then
\[
M = a_{00} + a_{20}s^2 + a_{30}s^3 + a_{40}s^4 + ... \\
+ a_{01}y_1 + a_{02}y_1^2 + a_{03}y_1^3 + a_{04}y_1^4 + ... \\
+ a_{02}y_2 + a_{03}y_2^2 + a_{04}y_2^3 + a_{04}y_2^4 + ... \\
+ ... + a_{11}sy_1 + a_{12}sy_2 + ... + a_{(k)ij}y_i^jy_j^k 	ag{26}
\]

Note that here \( a_{10} \sim (\partial M/\partial S)_c = T_c = 0 \). Therefore, it has not appeared in the expansion of the mass. Now the contribution up to first order is
\[
\left( \frac{\partial M}{\partial s} \right)_{Y_i} \sim A_{10}s + A_{01}^{(k)}y_k. \tag{27}
\]

Here we have rescaled the coefficients as \( A_{ij}^{(k)} = (i + 1)a_{i+1,j}^{(k)} \). One can keep higher order terms in the above equation without any change of conclusion. Thus first order contribution serves our purpose.

Now, we calculate \( (\partial^2 S/\partial M^2)_{Y_i} \) in the following way.
\[
\left( \frac{\partial^2 S}{\partial M^2} \right)_{Y_i} \sim \left( \frac{\partial}{\partial M} \left( \frac{\partial M}{\partial S} \right)^{-1}_{Y_i} \right)_{Y_i} \sim \left( \frac{\partial}{\partial M} \left[ \frac{1}{A_{10}s + A_{01}^{(k)}y_k} \right] \right)_{Y_i}. \tag{28}
\]

Therefore using (27) we finally obtain,
\[
\left( \frac{\partial^2 S}{\partial M^2} \right)_{Y_i} \sim \frac{1}{(A_{10}s + A_{01}^{(k)}y_k)^2} \frac{\partial s}{\partial M} \sim \frac{1}{(A_{10}s + A_{01}^{(k)}y_k)^3}. \tag{29}
\]

When \( Y_i = Y_{ic} \) we find \( s \sim m^{1/2} \) (from (26)). Thus from (29), taking the leading order contribution we get,
\[
\left( \frac{\partial^2 S}{\partial M^2} \right)_{Y_i} \sim m^{-\frac{3}{2}} \quad \text{(for} \quad Y_i = Y_{ic} \text{)} \tag{30}
\]

Therefore from the definition of the critical exponent \( \alpha \) (see (20)), we find \( \alpha = 3/2 \).
Again when \( M = M_c \) and \( \tilde{Y}_i = \tilde{Y}_{ic} \), we obtain \( s \sim y_i^{1/2} \) (from \[26\]). Thus, from \[29\] we get
\[
(\partial^2 S / \partial M^2)_{Y_i} \sim (A_{10} y_i^{1/2} + A_{01}^{(i)} y_i)^{-3}.
\]
This implies that the quantity diverges as
\[
\left( \frac{\partial^2 S}{\partial M^2} \right)_{Y_i} \sim y_i^{-\frac{3}{2}} \quad \text{for } M = M_c \text{ and } \tilde{Y}_i = \tilde{Y}_{ic}.
\]

Therefore from the definition \[21\], we get \( \phi_i = 3/2 \).

Next we expand \( Y_i \) as a function of \( S, M \) and other charge \( \tilde{Y}_i \):
\[
Y_i = a_{000} + a_{200} s^2 + a_{300} s^3 + a_{400} s^4 + \ldots
\]  
\[+ a_{010} m + a_{020} m^2 + a_{030} m^3 + \ldots
\]  
\[+ \ldots + a_{jkl}^{(p)} s^j m^k y_p + \ldots
\]
\[ (32) \]

Similar to the earlier case, here \( a_{100} \sim T_c = 0 \). Note that \( Y_p \) includes all the charges except \( Y_i \).

Therefore, from \[32\] we obtain up to the first order
\[
\frac{\partial Y_i}{\partial s} \bigg|_{M, \tilde{Y}_i} \sim A_{100} s + A_{010} m + A_{001}^{(p)} y_p.
\]
\[ (33) \]

Again, we have rescaled the coefficients as \( A_{jkl}^{(p)} = (j + 1) a_{j+1, k+l}^{(p)} \). It should be mentioned that first order contribution is enough to serve our purpose. Now, following the similar approach as was done earlier, we obtain
\[
\frac{\partial^2 S}{\partial Y_i^2} \bigg|_{M, \tilde{Y}_i} \sim \frac{1}{(\partial Y_i / \partial s)^3} \bigg|_{M, \tilde{Y}_i} \sim \frac{1}{(A_{100} s + A_{010} m + A_{001}^{(p)} y_p)^3}.
\]
\[ (34) \]

Now, for all \( Y_i = Y_{ic} \), we obtain from \[32\] \( s \sim m^{1/2} \). This when substituted in \[34\] gives \( (\partial^2 S / \partial Y_i^2)_{M, \tilde{Y}_i} \sim (A_{100} m^{1/2} + A_{010} m)^{-3} \). Therefore, the leading order contribution gives
\[
\frac{\partial^2 S}{\partial Y_i^2} \bigg|_{M, \tilde{Y}_i} \sim m^{-\frac{3}{2}} \quad \text{for } Y_i = Y_{ic}.
\]
\[ (35) \]

Therefore from the definition of \( \gamma_i \) (see \[22\]), we find \( \gamma_i = 3/2 \).

Again when \( M = M_c \) and \( \tilde{Y}_i = \tilde{Y}_{ic} \), we obtain from \[32\] \( s \sim y_i^{1/2} \). Therefore from \[34\] we get the result
\[
\frac{\partial^2 S}{\partial Y_i^2} \bigg|_{M, \tilde{Y}_i} \sim y_i^{-\frac{3}{2}} \quad \text{for } M = M_c \text{ and } \tilde{Y}_i = \tilde{Y}_{ic}.
\]
\[ (36) \]

Therefore, from the definition of the critical exponent \( \sigma_i \) (in eq. \[23\]) we obtain \( \sigma_i = 3/2 \).

Again from \[33\], the leading order contribution provides
\[
X_i \sim \frac{\partial Y_i}{\partial S} \bigg|_{M, \tilde{Y}_i} \sim \frac{1}{A_{100}} m^{-\frac{1}{2}} \quad \text{for } Y_i = Y_{ic}.
\]
\[ (37) \]

The above equation implies
\[
\eta_{Y_i} = X_i - X_i^* \sim \left( \frac{1}{A_{100}} + \frac{1}{A_{100}} \right) m^{-\frac{1}{2}} \quad \text{for } Y = Y_c.
\]
\[ (38) \]
Thus, from the definition of $\beta_i$ (see (24)), we get the value $\beta_i = -1/2$.

Furthermore, when $M = M_c$ and $\bar{Y}_i = \bar{Y}_{ic}$, we obtain

$$\tilde{X}^i \sim \frac{1}{A_{001}^{(i)}} y_i^{-\frac{1}{2}} \quad \text{(for } M = M_c \text{ and } \bar{Y}_i = \bar{Y}_{ic}).$$

(39)

In that case,

$$\eta Y_i \sim y_i^{-\frac{1}{2}} \quad \text{(for } M = M_c \text{ and } \bar{Y}_i = \bar{Y}_{ic}).$$

(40)

Therefore from the definition of $\delta_i$ in (25), we get $\delta_i = -2$.

The numerical values of critical exponents obtained so far is given in the following table

| Table 1: Values of first set of critical exponents |
|---------------------------------------------|
| $\alpha$ | $\phi_i$ | $\gamma_i$ | $\sigma_i$ | $\beta_i$ | $\delta_i$ |
| $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $-\frac{1}{2}$ | $-2$ |

One can easily check above exponents satisfy the following scaling laws of “first kind”.

$$\alpha + 2\beta + \gamma = 2,$$

(41)

$$\beta(\delta - 1) = \gamma,$$

(42)

$$\phi(\beta + \gamma) = \alpha.$$

(43)

The same values of the critical exponents were obtained earlier in [12,18] considering specific form of metrics. On the contrary, here we obtained those without the explicit information of the black hole spacetime by taking into account two inputs: (a) the black holes we considered here belong to the class, which exhibit extremal phase transition and (b) those black holes satisfy the first law of black hole mechanics. This shows the universality of this type of critical phenomenon.

Apart from these critical exponents which were obtained above, there are a few others which are studied in the context of the extremal criticality. In the following, we shall discuss those critical exponents and shall obtain their values in a general manner. Near the critical point, the asymptotic form of the two point correlation function for large $r$ is defined by [22],

$$G(r) \sim \frac{e(-r/\xi)}{r^{d-2-\eta}}.$$

(44)

Here, $\eta$ is called as the Fisher’s exponent, $d$ is the effective spatial dimension and $\xi$ is called the correlation length. Near the critical point, the behaviour of $\xi$ is given as

$$\xi \sim m^{-\nu} \quad \text{(for all } Y_i = Y_{ic});$$

(45)

$$\xi \sim y_i^{-\mu_i} \quad \text{(for } M = M_c \text{ and } \bar{Y}_i = \bar{Y}_{ic}).$$

(46)

In the theory of quantum gravity, we do not have much knowledge about the two point correlation function defined in [14]. However, for extremal Reissner-Nordstrom black hole, the inverse of the surface gravity is argued to play the role of the correlation length [27]. This result also holds for BTZ
black hole \cite{18,28,29} and black p-branes \cite{26,30}. If we assume, this to be true in the presence of multiple charges in arbitrary dimensions, we get \( \xi \sim 1/\kappa \sim 1/T \). Using (27), we can further conclude \( \xi \sim (\partial M/\partial s)_{Y_i}^{-1} \). Therefore, from (26), the leading order contribution gives

\[ \xi \sim m^{-\frac{1}{2}} \quad (\text{for all } Y_i = Y_{ic}) \quad (47) \]

From the definition of \( \nu \) in (45), we get the value \( \nu = 1/2 \). Now, when \( M \) and all \( Y \) are at their critical values except the \( i \)-th charge \( Y_i \), we obtain from (26)

\[ \xi \sim y_i^{-\frac{1}{2}} \quad (\text{for } M = M_c \text{ and } \bar{Y}_i = \bar{Y}_{ic}) \quad (48) \]

Therefore, from (46) we see that all \( \mu_i \)'s are the same and \( \mu_i = \mu = 1/2 \).

Now, these critical exponents are supposed to satisfy the scaling laws of “second kind”, which are given by \cite{21,22},

\[ \nu(2 - \eta) = \gamma \quad (49) \]
\[ \nu d = 2 - \alpha \quad (50) \]
\[ \mu(\beta + \gamma) = \nu \quad (51) \]

Using the obtained value of \( \alpha, \beta, \gamma, \mu \) and \( \nu \) in the scaling law of the second kind, we get the value of remaining critical exponent \( \eta \) and effective spacetime dimension \( d \). These are \( \eta = -1 \) and \( d = 1 \).

Following table shows these values of exponents.

| \( \nu \) | \( \mu_i \) | \( \eta \) |
|---------|---------|---------|
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | \(-1\) |

Remember, in the above analysis we have assumed that the correlation length is given by the inverse of the surface gravity. This has been checked and accepted for several instances \cite{18,26,30}. However, we are not sure if this is true in general. Therefore, it would be interesting if the same conclusion can be drawn from a general argument. For the time being, we leave that analysis for future.

4 GTD in extremal phase transition

The concepts of differential geometry is used in thermodynamics for a long time. The underlying motivation to pursue in this direction is to study various thermodynamic phenomena in terms of the geometric properties of the phase space of the system. For non-extremal black holes, there are two major approaches of studying the phase transition of black hole— one approach deals with the divergence of heat capacity and inverse of isothermal compressibility \cite{31,38}. The other approach \cite{39,42} is for the black holes in the AdS background, in which the cosmological constant is treated as the thermodynamic pressure. The latter approach exactly resembles the phase transition of the van der Waals fluid system. It must be mentioned that both these phase transitions have been studied
extensively under the light of the GTD \[43–45\]. Here people have formulated thermogeometrical metrics in the thermodynamic phase space of black hole and have shown that the corresponding Ricci-scalar diverges at the phase transition point.

In this section, we incorporate those ideas to study the extremal phase transition. Here, we comment that there are several ways to formulate the thermogeometrical metric. First Weinhold \[46\] introduced a metric, the components of which are given by the Hessian of the internal thermodynamic energy. Later, Ruppeiner \[47, 48\] introduced another metric, which is defined as the negative of the Hessian of the entropy, and is conformal to the Weinhold metric with the conformal factor being the inverse temperature. Later, Quevedo \[49–56\] came up with the idea of defining the thermogeometrical metric in a Legendre-invariant way.

In our general procedure of analyzing the extremal phase transition, we study the behaviour of Ricci-scalar near the critical point for all these metrics.

### 4.1 The Weinhold metric

To write the Weinhold metric, one has to write mass (which plays the role of internal energy) as the function of entropy and the charges i.e.,\[ M \equiv M(S, Y_i) \]. Now for the sake of simplicity we consider the dependence of mass on a particular charge \( Y \) and keep all other charges fixed. Therefore the first law of thermodynamics is written as,

\[
dM = TdS + XdY .
\]

Here \( T = (\partial M/\partial S)_Y \) and \( X = (\partial M/\partial Y)_S \).

Now the Weinhold metric is given by,

\[
ds^2_W = \frac{\partial^2 M}{\partial x_i \partial x_j} dx_i dx_j \quad \{x_1 = S, \; x_2 = Y\} .
\]

The expanded form of the Weinhold metric is

\[
ds^2_W = -f(S, Y)dS^2 + g(S, Y)dY^2 + 2h(S, Y)dSdY ,
\]

where \( f(S, Y) = -M_{SS}, \; g(S, Y) = M_{YY} \) and \( h(S, Y) = M_{SY} = M_{YS} \). The Ricci scalar corresponding to the Weinhold metric \[54\] is given by,

\[
R(W) = \frac{1}{2(fg + h^2)^2} \left[ f(fgY - g^2 + 2gyhS) + g\left( f_Y^2 + fs(2hY - gs) - 2f(fY + hSY - gSS) \right) \right. \\
+ h\left\{ -gYfS + fY(2hY + gs) + 4hYhS - 2gS + 2h(fY + 2hSY - gSS) \right\} ,
\]

where \( f, g, h \) and so on. Now, from the expansion of \( M \) (given in \[26\]) we can conclude that \( f, g, h \) and their derivatives are finite. Therefore, the Ricci scalar of the Weinhold metric is a finite quantity near the critical point.

### 4.2 The Ruppeiner metric

We first write the first law of thermodynamics\[52\] as \( dS = \beta dM - \bar{X}dY \). In this form, the conjugate quantities are taken as \( \beta = (\partial S/\partial M)_Y \) and \( \bar{X} = -(\partial S/\partial Y)_M \). Now, the Ruppeiner metric is defined

\[
\frac{1}{2(gf + h^2)^2} \left[ f(fgY - g^2 + 2gyhS) + g\left( f_Y^2 + fs(2hY - gs) - 2f(fY + hSY - gSS) \right) \right. \\
+ h\left\{ -gYfS + fY(2hY + gs) + 4hYhS - 2gS + 2h(fY + 2hSY - gSS) \right\} .
\]
as
\[ ds^2_R = -\frac{\partial^2 S}{\partial x'_i \partial x'_j} dx'_i dx'_j \quad \{x'_1 = M, \ x'_2 = Y\}. \] (56)

Here, \( g_{11} = -S_{MM}, \ g_{22} = -S_{YY} \) and \( g_{12} = g_{21} = -S_{MY} \). It implies that the expansion of the Ruppeiner metric is
\[ ds^2_R = -f'(M,Y)dM^2 + g'(M,Y)dY^2 + 2h'(M,Y)dMdY, \] (57)
where \( f' = S_{MM}, \ g' = -S_{YY} \) and \( h' = -S_{MY} \). The Ricci scalar of the metric \( (57) \) is found to be,
\[
R_{(R)} = \frac{1}{2(f'g' + h'^2)^2} \left[ f'(f'_Y g'_Y - g'_M^2 + 2g'_Y h'_M) \\
+ g' \left\{ f''_M + f'_M(2h'_Y - g'_M) - 2f'_Y(g'_Y + h'_M - g'_MM) \right\} \\
+ h' \left\{ -g'_Y f'_M + f'_M(2h'_Y + g'_M) + 4h'_Y h'_M - 2g'_M h'_M - 2h'(f'_Y + 2h'_M - g'_MM) \right\} \right] \] (58)

Now, we have to calculate each term of the Ricci scalar of \( (58) \) to see its dependence on \( s \). To do that, we find out the leading order contribution of \( f', \ g' \) and their derivatives. From \( (29) \) we see that \( f' = -(\partial^2 S/\partial M^2) \sim 1/s^3 \). Therefore, \( f'_M = (\partial f'/\partial M)_Y \sim (1/s^4)(\partial s/\partial M)_Y \). Using \( (27) \), one obtains \( f'_M \sim s^{-5} \). In a similar way, \( f''_{MM} \sim s^{-7} \). Now, \( f''_Y = (\partial f'/\partial Y)_M \sim (1/s^4)(\partial s/\partial Y)_M \). Again, using \( (33) \) one gets \( f'_Y \sim s^{-5} \). The same arguments yield \( f''_{YY} \sim s^{-7} \) and \( f''_{MY} = f''_{YM} \sim s^{-7} \). Following the same procedure, one similarly obtains \( g' \sim s^{-3}, \ g'_M \sim s^{-5} \) and \( g'_M g'_Y \sim s^{-7} \). Also, \( h' \sim s^{-3}, \ h'_M \sim s^{-5} \) and \( h'_M h'_Y \sim s^{-7} \). As a result, we see that the denominator goes as \( s^{-12} \) and each term in the numerator goes as \( s^{-13} \). Therefore, the Ricci scalar diverges as
\[ R_{(R)} \sim s^{-1}. \] (59)

The property of the Ruppeiner metric has also been studied in a different way \[19, 20\] while studying the extremal phase transition of BTZ black holes. It has been there argued that the Ruppeiner metric should diverge as \( R_{(R)} \sim \xi^d \). Since, in our case \( \xi \sim s^{-1} \) near the critical point, we obtain \( R_{(R)} \sim \xi^4 \).

Therefore, we can again conclude that the effective spatial dimension \( d = 1 \) for any extremal black hole, which is in agreement with the claim of the recent papers \[57, 58\]. Thus, from the thermogeometric approach, we can again generally prove that the effective spatial dimension of an extremal black hole is one.

### 4.3 Legendre invariant metric

Above two thermogeometrical metrics, namely the Wienhold and the Ruppeiner metric are not Legendre-invariant. Moreover in some cases, conclusions derived from the Weinhold metric and the Ruppeiner metric are not consistent with each other. Later Quevedo et al. claimed that those inconsistencies appear because these metrics are not Legendre-invariant and hence they came up with Legendre invariant metric formalism \[49, 56\]. In the following, we discuss two types of Legendre invariant thermogeometrical metric. One of them (Quevedo metric: 1) is mostly used as a Legendre-invariant metric. Here, we see that the Ricci scalar of the first type of the Legendre-invariant metric is a finite quantity at the critical point. So we discuss another type of Legendre-invariant metric (Quevedo metric: 2). The second metric is not that familiar but we see that the Ricci scalar corresponding to this metric vanishes. The formalism which we adopt here was originally developed by Hermann \[59\] and Mrugala \[60, 61\], which was later followed extensively by Quevedo.
4.3.1 Quevedo metric: 1

We define a thermodynamic phase space $\mathcal{T}$ with coordinates $Z^A = \{S, q^a, p^a\}$ where $q^a = \{M, Y\}$ are the variables and $p^a = \{S_M = \beta, S_Y = -\bar{X} = -\beta \bar{X}\}$ are the conjugate variables. Therefore, in the entropy representation, the fundamental one form in $\mathcal{T}^*$ (where, $\mathcal{T}^*$ is the cotangent space of $\mathcal{T}$) is given by,

$$\Theta_S = dS - \beta dM + \bar{X} dY,$$

which is invariant under the Legendre transformation

$$M(q) = \tilde{M}(\tilde{q}) - \delta_{ab} \tilde{q}^a \tilde{p}^b$$

with $q^a = -\tilde{p}^a$ and $p^a = \tilde{q}^a$.

Now, following Quevedo’s formalism, one possible form of the Legendre invariant thermogeometrical metric (on $\mathcal{T}$) is (Eq. (39) of [49])

$$G_1 = \Theta_S^2 + (\beta M + \bar{X} Y)(d\beta dM + dY d\bar{X}).$$

Expanding the conjugate quantities ($\beta$ and $\bar{X}$) as a function of the variables ($M$ and $Y$), one finds the the expression of $G_1$ in the space of equilibrium ($\Theta_S = 0$) as

$$G_1 = -f_1(M,Y)dM^2 + g_1(M,Y)dY^2,$$

where $f_1(M,Y) = -(\beta M + \bar{X} Y)S_{MM}$ and $g_1(M,Y) = -(\beta M + \bar{X} Y)S_{YY}$. The Ricci scalar of the metric [63] is given by,

$$R_1 = \frac{1}{2(f_1 g_1)^2} \left[ f_1(f_{1Y} g_{1Y} - g_{1M}) + g_1 \left( f_{1Y}^2 - f_{1M} g_{1M} - 2f_1(f_{1YY} - g_{1MM}) \right) \right].$$

Again, we check the order of each term in the Ricci scalar. $f_1 \sim \beta S_{MM} \sim (\partial S/\partial M)_Y (\partial^2 S/\partial M^2)_Y$. This implies $f_1 \sim s^{-4}$. Similarly $g_1 \sim s^{-4}$. Following the same procedure as was done in the Ruppeiner case, we obtain $f_{1x} \sim s^{-6}$, $g_{1x} \sim s^{-6}$, $f_{1xx} \sim s^{-8}$ and $g_{1xx} \sim s^{-8}$. Therefore, we see that the denominator goes as $\sim s^{-16}$ and the numerator also goes as $\sim s^{-16}$. Therefore, the Ricci scalar is finite in this case.

4.3.2 Quevedo metric: 2

As the choice of Legendre invariant metric is not unique, we can formulate other Legendre invariant metric. Following Quevedo’s formalism (Eq. (37) of [49]) we see

$$G_2 = \Theta_S^2 + c_1 \beta M d\beta dM + c_2 \bar{X} Y d\bar{X} dY + d\beta^2 + dM^2 + d\bar{X}^2 + dY^2$$

is Legendre invariant for any value of the real constants $c_1$ and $c_2$. For the simplicity of calculation, we take $c_1 = c_2 = 1$. Now using $d\beta = S_{MM} dM + S_{MY} dY$ and $d\bar{X} = -S_{YM} dM - S_{YY} dY$ in (65) we get, in equilibrium space

$$G_2 = -f_2(M,Y)dM^2 + g_2(M,Y)dY^2 + 2h_2(M,Y)dMdY,$$
where \( f_2 = -[1 + \beta MS_{MM} + S^2_{MM} + S^2_{MY}], \) \( g_2 = 1 - \bar{X}Y S_{YY} + S^2_{YY} + S^2_{MY} \) and \( h_2 = \frac{1}{2}(\beta M - \bar{X}Y)S_{MY} + S_{MM}S_{MY} + S_{YM}S_{YY} \). Thus the Ricci scalar is given by,

\[
R_2 = \frac{1}{2(f_2g_2 + h_2^2)} \left[ f_2(f_2Yg_2Y - g_2^2M + 2g_2Yh_2M) + g_2 \left\{ f_2^2Y + f_2M(2h_2Y - g_2M) - 2f_2(f_2YY + h_2MY - g_2MM) \right\} + h_2 \left\{ -g_2Yf_2M + f_2Y(2h_2Y + g_2M) + 4h_2Yh_2M - 2g_2Mh_2M - 2h_2(f_2YY + 2h_2MY - g_2MM) \right\} \right]
\]

Now, \( f_2 = O(s^0) + O(s^{-4}) + O(s^{-6}) \). The leading order contribution near the critical point will be \( f_2 \sim s^{-6} \). As a result, \( f_{2x_i} \sim s^{-8} \) and \( f_{2x_i x_j} \sim s^{-10} \). Leading order contributions of \( g_2 \) and \( h_2 \) are same as \( f_2 \). Therefore, the denominator goes as \( \sim s^{-24} \) and the numerator goes as \( \sim s^{-22} \). As a result,

\[
R_2 \sim s^2.
\]

Consequently, we see that the Ricci-scalar vanishes near the critical point.

In this section, we have studied the behaviour of the Ricci-scalar for different thermogeometrical metrics and have shown that the Ricci-scalar of the Ruppeiner metric diverges at the extremal limit. On the contrary, the Ricci-scalar of other thermogeometrical metrics remains finite (or vanishes) at that point. Therefore, we conclude that the extremal phase transition shows the behaviour of the second order phase transition not only in the specific ensemble of thermodynamics (i.e. the micro-canonical ensemble), but also for a specific thermogeometric manifold as well (the Ruppeiner one). Note that the Legendre-invariant thermogeometrical metrics, which are mostly used nowadays, cannot confirm the second order phase transition in the present case. A plausible explanation to that might be as follows. Remember that the Legendre-invariant metrics are constructed on the line of arguments that a proper thermogeometrical metric should be Legendre invariant as the thermodynamic features are invariant in all ensembles. Since one thermodynamic potential, by which an ensemble is characterized, is connected to the same in the other ensemble by the Legendre transformation, the entire thermodynamic description is invariant due to the Legendre transformation, which should reflect on the thermogeometrical metric. However, as we have noticed in the present case, the identification of the non-extremal to extremal transformation with the second order phase transition is valid only in the microcanonical ensemble. As a result, the present thermodynamic description is not invariant across all ensembles. Therefore, the use of a Legendre-invariant metric might not be suitable in this case. Nonetheless, we have checked the behaviour of the Ricci-scalar of all the thermogeometrical metrics which are popular in GTD and from that analysis we found that the Ruppeiner metric is the ideal one for the thermogeometric description of the extremal phase transition. Interestingly, here entropy \( S \) plays the central role both in microcanonical ensemble (“S is chosen as the Massieu function) and in Ruppeiner geometrical description (the metric is constructed by considering \( S \) as the thermodynamic potential).

5 Conclusions

In this work, we have studied the extremal phase transition of black hole in a general framework. There are several works [12,16,18,20,62,63] to show that the extremal phase transition is a second
order phase transition. These earlier works were done case by case for a particular spacetime and dimension. The obtained results in different spacetimes (such as the critical exponents, scaling laws etc.) are in accordance with each other and strongly suggest that there must be a metric independent way to establish those earlier results. This has been the major motivation for this work.

We have proved that the transformation of the black hole from a non-extremal to an extremal one is a second order phase transition. For that, we have calculated the second order moments of fluctuations in different ensembles and have shown that those moments diverge for a black hole in microcanonical ensemble, which is a sign of a second order phase transition as per the prescription of Pavón and Rubí [15,16]. Afterwards, we have generally obtained the critical exponents for this phase transition and have shown that the critical exponents satisfy the scaling laws. While proving those results, we have not accounted any particular spacetime, which implies our results are valid for all the black hole spacetimes which become extremal at certain limit. Thus, the universality of results, which were predicted by earlier works, is proved by our analysis and hence from now on one need not check the critical behaviour case by case.

Finally, we have extended our analysis to GTD, which is a recent formalism to describe the phase transition geometrically. We have shown that the extremal critical point of black holes can be identified as a particular point where the Ricci scalar corresponding to the Ruppeiner metric diverges. In addition, we have also shown that the Ricci scalar of the Weinhold metric and of one type of Legendre-invariant metric (Quevedo metric: 1) is a finite quantity and does not show any special behaviour. In another Legendre invariant metric (Quevedo metric: 2), the Ricci-scalar vanishes on the critical point. In this analysis we observed that extremal phase transition is properly explained in microcanonical ensemble and by Ruppeiner geometry. Note that in both the descriptions, entropy plays the central role: $S$ acts as Massieu function in microcanonical ensemble and thermodynamical potential in GTD. At this moment, the actual reason for this is not known to us; hope we shall be able to find the precise reason in future.

Thus our paper covers different thermodynamics aspects of extremal black hole. Other previous works in this field confined their analysis to specific cases and hence can not explain questions regarding universality. The novelty of our work is, it is very general and does not require any specific metric. In this sense our paper unifies all other work on extremal phase transition in an elegant manner.

Appendix

A Obtaining the values of $\left( \partial^2 \Phi_2 / \partial \beta^2 \right)_Y$, and $\left( \partial^2 \Phi_2 / \partial Y_i^2 \right)_{\beta, Y_i}$

We take the canonical ensemble in which Helmholtz function is $F \equiv F(T, Y_i)$ . Equivalently one can write $T \equiv T(F, Y_i)$ . As we have done earlier, we expand $T$ around the critical point $T_c = 0$ which yields,

$$T = b_{10} f + b_{20} f^2 + b_{30} f^3 + b_{40} f^4 + ...$$

$$+ b_{01}^{(1)} y_1 + b_{02}^{(1)} y_1^2 + b_{03}^{(1)} y_1^3 + b_{04}^{(1)} y_1^4 + ...$$

$$+ b_{01}^{(2)} y_2 + b_{02}^{(2)} y_2^2 + b_{03}^{(2)} y_2^3 + b_{04}^{(2)} y_2^4 + ...$$

$$... + b_{ij}^{(k)} f^i y_k^j ,$$

(69)
where, \( f = F - F_c \) and so on. In the above expansion, we have used \( T_c = 0 \). Now keeping terms up to first order we get,

\[
\left. \frac{\partial F}{\partial T} \right|_{Y_i} = \left. \frac{\partial T}{\partial F} \right|_{Y_i}^{-1} \sim \frac{1}{B_{00} + B_{10}f + B_{11}^{(i)} y_i}. \tag{70}
\]

and

\[
\left. \frac{\partial^2 F}{\partial T^2} \right|_{Y_i} \sim \frac{\partial}{\partial T} \left( \frac{1}{B_{00} + B_{10}f} \right) \left. \right|_{Y_i} \sim \frac{1}{(B_{00} + B_{10}f)^2 + B_{11}^{(i)} y_i} \left. \frac{\partial F}{\partial T} \right|_{Y_i} \sim \frac{1}{(B_{00} + B_{10}f + B_{11}^{(i)} y_i)^3}. \tag{71}
\]

It implies that \( (\partial^2 F / \partial T^2)_{Y_i} \) is a non-zero finite quantity at the critical point and near that point, it goes as \( (\partial^2 F / \partial T^2)_{Y_i} \sim B_{00}^{-3} \).

Now to obtain \( (\partial^2 F / \partial Y_i^2)_{T,Y_i} \), we expand \( Y_i \) near the critical point as a function of \( T, F \) and \( \bar{Y}_i \).

This is

\[
Y_i = Y_{ic} + b_{100}f + b_{200}f^2 + b_{300}f^3 + b_{400}f^4 + \ldots \nonumber
\]

\[
b_{010}T + b_{020}T^2 + b_{030}T^3 + b_{040}T^4 + \ldots \nonumber
\]

\[
\ldots + b_{jkl}f^j T^k \bar{y}_i. \tag{72}
\]

In the above equation, we have used \( T_c = 0 \). Again, adopting the similar method as earlier, it can be shown straightforwardly that \( (\partial^2 F / \partial Y_i^2)_{T,Y_i} \) is also a non-zero finite quantity at the critical point.

As \( \Phi_2 = -\beta F \), one can straightforwardly obtain \( (\partial^2 \Phi_2 / \partial \beta^2)_{Y_i} = -T^3(\partial^2 F / \partial T^2)_{Y_i} \). Therefore at the critical point, \( (\partial^2 \Phi_2 / \partial \beta^2)_{Y_i} \) vanishes as

\[
(\partial^2 \Phi_2 / \partial \beta^2)_{Y_i} \sim T^3. \tag{73}
\]

Again, \( (\partial^2 \Phi_2 / \partial Y_i^2)_{\beta,\bar{Y}_i} = \beta (\partial^2 F / \partial Y_i^2)_{T,Y_i} \). Therefore, at the critical point, \( (\partial^2 \Phi_2 / \partial Y_i^2)_{\beta,\bar{Y}_i} \) diverges as

\[
(\partial^2 \Phi_2 / \partial Y_i^2)_{\beta,\bar{Y}_i} \sim T^{-1}. \tag{74}
\]

B Obtaining the values of \( (\partial^2 \Phi_3 / \partial \beta^2)_{\bar{X}} \) and \( (\partial^2 \Phi_3 / \partial \bar{X}_i^2)_{\beta,\bar{X}_i} \)

Let us take the Gibbs free energy \( G \equiv G(T, X^i) \). Alternatively temperature is written as \( T \equiv T(G, X^i) \). Now expanding \( T \) near the critical point, as we have done earlier, it can be shown that \( (\partial^2 G / \partial T^2)_{X_i} \) is a non-zero finite quantity. Similarly, expanding \( X^i \) in terms of \( T, G \) and \( \bar{X}_i \), one finds that \( (\partial^2 G / \partial X_i^2)_{T,\bar{X}_i} \) is also a non-zero finite quantity. Now, as \( \Phi_3 = -\beta G \), we obtain \( (\partial^2 \Phi_3 / \partial \beta^2)_{\bar{X}_i} = -T^3(\partial^2 G / \partial T^2)_{X_i} \). Therefore, we conclude that near the critical point \( (\partial^2 \Phi_3 / \partial \beta^2)_{\bar{X}_i} \) vanishes as

\[
(\partial^2 \Phi_3 / \partial \beta^2)_{\bar{X}_i} \sim T^3. \tag{75}
\]
Now using $\tilde{X}^i = \beta X^i$, one can show \((\partial^2 \Phi_3 / \partial \tilde{X}^i \tilde{X}^i)_{\beta, \tilde{X}^i} = T (\partial^2 G / \partial X^i \tilde{X}^i)_{T, \tilde{X}^i} \). Hence, near the critical point, \((\partial^2 \Phi_3 / \partial \tilde{X}^i \tilde{X}^i)_{\beta, \tilde{X}^i} \) vanishes as

\[
\left( \frac{\partial^2 \Phi_3}{\partial \tilde{X}^2} \right)_{\beta, \tilde{X}^i} \sim T. \tag{76}
\]

References

[1] J. D. Bekenstein, “Black holes and entropy,” Phys. Rev. D 7, 2333 (1973).

[2] S. W. Hawking, “Particle Creation by Black Holes,” Commun. Math. Phys. 43, 199 (1975) Erratum: [Commun. Math. Phys. 46, 206 (1976)].

[3] J. M. Bardeen, B. Carter and S. W. Hawking, “The Four laws of black hole mechanics,” Commun. Math. Phys. 31, 161 (1973).

[4] P. C. W. Davies, “Thermodynamics of Black Holes,” Proc. Roy. Soc. Lond. A 353, 499 (1977).

[5] C. O. Lousto, “The Fourth law of black hole thermodynamics,” Nucl. Phys. B 410, 155 (1993) Erratum: [Nucl. Phys. B 449, 433 (1995)] [gr-qc/9306014].

[6] C. O. Lousto, “Effective two-dimensional description from critical phenomena in black holes,” Gen. Rel. Grav. 27, 121 (1995).

[7] C. O. Lousto, “The Emergence of an effective two-dimensional quantum description from the study of critical phenomena in black holes,” Phys. Rev. D 51, 1733 (1995) [gr-qc/9405048].

[8] J. P. Munian and D. D. Piriz, “Critical behavior of dimensionally continued black holes,” Phys. Rev. D 53, 816 (1996) [gr-qc/9502029].

[9] O. Kaburaki, I. Okamoto and J. Katz, “Thermodynamic Stability of Kerr Black holes,” Phys. Rev. D 47, 2234 (1993).

[10] J. Katz, I. Okamoto and O. Kaburaki, “Thermodynamic stability of pure black holes,” Class. Quant. Grav. 10, 1323 (1993).

[11] O. Kaburaki, “Should entropy be concave?,” Phys. Lett. A 185, 21 (1994).

[12] O. Kaburaki, “Critical behavior of extremal Kerr-Newman black holes,” Gen. Rel. Grav. 28, 843 (1996).

[13] A. Curir, “Rotating black holes as dissipative spin-thermodynamical systems,” Gen. Rel. Grav. 13, 417 (1981).

[14] A. Curir, “Black hole emissions and phase transitions,” Gen. Rel. Grav. 13, 1177 (1981).

[15] D. Pavon and J. M. Rubi, “Nonequilibrium Thermodynamic Fluctuations of Black Holes,” Phys. Rev. D 37, 2052 (1988).

[16] D. Pavon, “Phase transition in Reissner-Nordstrom black holes,” Phys. Rev. D 43, 2495 (1991).
[17] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, New York, 1966).

[18] R. G. Cai, Z. J. Lu and Y. Z. Zhang, “Critical behavior in (2+1)-dimensional black holes,” Phys. Rev. D 55, 853 (1997) [gr-qc/9702032].

[19] R. G. Cai and J. H. Cho, “Thermodynamic curvature of the BTZ black hole,” Phys. Rev. D 60, 067502 (1999) [hep-th/9803261].

[20] Y. H. Wei, “Thermodynamic critical and geometrical properties of charged BTZ black hole,” Phys. Rev. D 80, 024029 (2009).

[21] H. E. Stanley, “Introduction to Phase Transition and Critical Phenomena,” Oxford University Press, New York (1987).

[22] N. Goldenfeld, “Lectures on Phase Transitions and the Renormalization Group,” (Westview Press, New York, 1992).

[23] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, New York, 1977).

[24] O. Kaburaki, “Scaling laws at the critical point black hole equilibrium series,” Phys. Lett. A 217, 315 (1996).

[25] R. K. Su, R. G. Cai and P. K. N. Yu, “Nonequilibrium thermodynamic fluctuations and phase transition in black holes,” Phys. Rev. D 50, 2932 (1994).

[26] R. G. Cai and Y. S. Myung, “Critical behavior for the dilaton black holes,” Nucl. Phys. B 495, 339 (1997) [hep-th/9702159].

[27] J. H. Traschen, “Discrete selfsimilarity and critical point behavior in fluctuations about extremal black holes,” Phys. Rev. D 50, 7144 (1994) [gr-qc/9403016].

[28] G. Lifschytz and M. Ortiz, “Scalar field quantization on the (2+1)-dimensional black hole background,” Phys. Rev. D 49, 1929 (1994) [gr-qc/9310008].

[29] I. Ichinose and Y. Satoh, “Entropies of scalar fields on three-dimensional black holes,” Nucl. Phys. B 447, 340 (1995) [hep-th/9412144].

[30] R. G. Cai, “Effective spatial dimension of extremal non-dilatonic black p-branes and the description of entropy on the world volume,” Phys. Rev. Lett. 78, 2531 (1997) [hep-th/9702142].

[31] R. Banerjee and D. Roychowdhury, “Critical phenomena in Born-Infeld AdS black holes,” Phys. Rev. D 85, 044040 (2012) [arXiv:1111.0147 [gr-qc]].

[32] R. Banerjee and D. Roychowdhury, “Critical behavior of Born Infeld AdS black holes in higher dimensions,” Phys. Rev. D 85, 104043 (2012) [arXiv:1203.0118 [gr-qc]].

[33] B. R. Majhi and D. Roychowdhury, “Phase transition and scaling behavior of topological charged black holes in Horava-Lifshitz gravity,” Class. Quant. Grav. 29, 245012 (2012) [arXiv:1205.0146 [gr-qc]].

[34] A. Lala, “Critical phenomena in higher curvature charged AdS black holes,” Adv. High Energy Phys. 2013, 918490 (2013) [arXiv:1205.6121 [gr-qc]].
[35] M. S. Ma, F. Liu and R. Zhao, “Continuous phase transition and critical behaviors of 3D black hole with torsion,” Class. Quant. Grav. 31, 095001 (2014) [arXiv:1403.0449 [gr-qc]].

[36] M. Azreg-Anou, G. T. Marques and M. E. Rodrigues, “Phantom black holes and critical phenomena,” JCAP 1407, 036 (2014) [arXiv:1405.5745 [gr-qc]].

[37] J. X. Mo and W. B. Liu, “Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly,” Adv. High Energy Phys. 2014, 739454 (2014) [arXiv:1312.0679 [hep-th]].

[38] A. Mandal, S. Samanta and B. R. Majhi, “Phase transition and critical phenomena of black holes: A general approach,” Phys. Rev. D 94, no. 6, 064069 (2016) [arXiv:1608.04176 [gr-qc]].

[39] D. Kubiznak and R. B. Mann, “P-V criticality of charged AdS black holes,” JHEP 1207, 033 (2012) [arXiv:1205.0559 [hep-th]].

[40] D. Kubiznak, R. B. Mann and M. Teo, “Black hole chemistry: thermodynamics with Lambda,” arXiv:1608.06147 [hep-th].

[41] B. R. Majhi and S. Samanta, “P-V criticality of AdS black holes in a general framework,” arXiv:1609.06224 [gr-qc].

[42] K. Bhattacharya, B. R. Majhi and S. Samanta, “Van der Waals criticality in AdS black holes: a phenomenological study,” Phys. Rev. D 96, no. 8, 084037 (2017) [arXiv:1709.02650 [gr-qc]].

[43] R. Banerjee, B. R. Majhi and S. Samanta, “Thermogeometric phase transition in a unified framework,” Phys. Lett. B 767, 25 (2017) [arXiv:1611.06701 [gr-qc]].

[44] K. Bhattacharya and B. R. Majhi, “Thermogeometric description of the van der Waals like phase transition in AdS black holes,” Phys. Rev. D 95, no. 10, 104024 (2017) [arXiv:1702.07174 [gr-qc]].

[45] A. Dehyadegari, B. R. Majhi, A. Sheykhi and A. Montakhab, “Universality class of alternative phase space and Van der Waals criticality,” Phys. Lett. B 791, 30 (2019) [arXiv:1811.12308 [hep-th]].

[46] F. Weinhold, J. Chem. Phys. 63, 2479 (1975); 63, 2484 (1975); 63, 2488 (1975); 63, 2496 (1975); 65, 559 (1976).

[47] G. Ruppeiner, Phys. Rev. A 20, 1608 (1979).

[48] G. Ruppeiner, “Riemannian geometry in thermodynamic fluctuation theory,” Rev. Mod. Phys. 67, 605 (1995) Erratum: [Rev. Mod. Phys. 68, 313 (1996)].

[49] H. Quevedo, “Geometrothermodynamics,” J. Math. Phys. 48, 013506 (2007) [physics/0604164].

[50] H. Quevedo, “Geometrothermodynamics of black holes,” Gen. Rel. Grav. 40, 971 (2008) [arXiv:0704.3102 [gr-qc]].

[51] H. Quevedo and A. Sanchez, “Geometrothermodynamics of asymptotically de Sitter black holes,” JHEP 0809, 034 (2008) [arXiv:0805.3003 [hep-th]].
[52] H. Quevedo and A. Sanchez, “Geometric description of BTZ black holes thermodynamics,” Phys. Rev. D 79, 024012 (2009) [arXiv:0811.2524 [gr-qc]].

[53] J. L. Alvarez, H. Quevedo and A. Sanchez, “Unified geometric description of black hole thermodynamics,” Phys. Rev. D 77, 084004 (2008) [arXiv:0801.2279 [gr-qc]].

[54] H. Quevedo and M. N. Quevedo, “Fundamentals of Geometrothermodynamics,” [arXiv:1111.5056 [math-ph]].

[55] H. Quevedo, M. N. Quevedo and A. Sanchez, “Homogeneity and thermodynamic identities in geometrothermodynamics,” [arXiv:1701.06702 [gr-qc]].

[56] H. Quevedo, M. N. Quevedo and A. Sanchez, “Geometrothermodynamics of phantom AdS black holes,” Eur. Phys. J. C 76, no. 3, 110 (2016) [arXiv:1601.07120 [gr-qc]].

[57] G. T. Horowitz, D. A. Lowe and J. M. Maldacena, “Statistical entropy of nonextremal four-dimensional black holes and U duality,” Phys. Rev. Lett. 77, 430 (1996) [hep-th/9603195].

[58] A. Ghosh, “Effectiveness of one-dimensional gas models for black holes,” Phys. Lett. B 425, 269 (1998) [hep-th/9801064].

[59] R. Hermann, “Geometry, physics and systems,” Marcel Dekker, New York, 1973.

[60] R. Mrugala, “Geometrical formulation of equilibrium phenomenological thermodynamics,” Rep. Math. Phys. 14, 419 (1978).

[61] R. Mrugala, “Submanifolds in the thermodynamic phase space,” Rep. Math. Phys. 21, 197 (1985).

[62] M. S. Ma and R. Zhao, “Phase transition and entropy spectrum of the BTZ black hole with torsion,” Phys. Rev. D 89, no. 4, 044005 (2014) [arXiv:1310.1491 [gr-qc]].

[63] R. G. Cai, “Critical behavior in black hole thermodynamics,” J. Korean Phys. Soc. 33, 5477 (1998) [gr-qc/9901020].