Tensor gauge fields in arbitrary representations of $GL(D, \mathbb{R})$: duality & Poincaré lemma

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Abstract

Using a mathematical framework which provides a generalization of the de Rham complex (well-designed for $p$-form gauge fields), we have studied the gauge structure and duality properties of theories for free gauge fields transforming in arbitrary irreducible representations of $GL(D, \mathbb{R})$. We have proven a generalization of the Poincaré lemma which enables us to solve the above-mentioned problems in a systematic and unified way.

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1 Introduction

The surge of interest in string field theories has refocused attention on the old problem of formulating covariant field theories of particles carrying arbitrary representation of the Lorentz group. These fields appear as massive excitations of string (for spin $S > 2$). It is believed that in a particular phase of M-theory, all such excitations become massless. The covariant formulation of massless gauge fields in arbitrary representations of the Lorentz group has been completed for $D = 4$ [1]. However, the generalization of this formulation to arbitrary values of $D$ is a difficult problem since the case $D = 4$ is a very special one, as all the irreps of the little group $SO(2)$ are totally symmetric. The covariant formulation for totally antisymmetric representations in arbitrary spacetime dimension has been easily obtained using differential forms. For mixed symmetry type gauge fields, the problem was partially solved in the late eighties [2] (for recent works, see for instance [3]). A recent approach [4] has shed new light on higher-spin gauge fields, showing how it is possible to formulate the free equations while foregoing the trace conditions of the Fang-Fronsdal formalism. In this formulation, the higher-spin gauge parameters are then not constrained to be irreducible under $SO(D - 1, 1)$, it is sufficient for them to be irreducible under $GL(D, \mathbb{R})$.

Dualities are crucial in order to scrutinize non-perturbative aspects of gauge field and string theories, it is therefore of relevance to investigate the duality properties of arbitrary tensor gauge fields. It is well known that the gravity field equations in four-dimensional spacetime are formally invariant under a duality rotation (for recent papers, see for example [5, 6]). As usual the Bianchi identities get exchanged with field equations but, as for Yang-Mills theories, this duality rotation does not appear to be a true symmetry of gravity: the covariant derivative involves the gauge field which is not inert under the duality transformation. A deep analogy with the self-dual D3-brane that originates from the compactified M5-brane is expected to occur when the six-dimensional (4,0) superconformal gravity theory is compactified over a 2-torus [7]. Thus, a $SL(2, \mathbb{Z})$-duality group for $D = 4$ Einstein gravity would be geometrically realized as the modular group of the torus. In any case, linearized gravity does not present the problem mentioned previously and duality is thus a true symmetry of this theory. Dualizing a free symmetric gauge field in $D > 4$ generates new irreps of $GL(D, \mathbb{R})$.

This paper provides a systematic treatment of the gauge structure and duality properties of tensor gauge fields in arbitrary representations of $GL(D, \mathbb{R})$. Review and reformulation of known results ([6, 8] and the references therein) are given in a systematic unified mathematical framework and are presented together with new results and their proofs.

Section 2 is a review of massless spin-two gauge field theory and its dualisation. The obtained free dual gauge fields are in representations of $GL(D, \mathbb{R})$ corresponding to Young diagrams with one row of two columns and all the other rows of length one. Section 3 “N-complexes” gathers together the mathematical background needed for the following sections. Based on the works [9, 10, 11, 12], it includes definitions and propositions together with a review of linearized gravity gauge structure in the language of $N$-complexes. Section 4 discusses linearized gravity field equations and their duality properties in the introduced mathematical framework. The section 5 presents our theorem, which general-
izes the standard Poincaré lemma. This theorem is then used in section 6 to elucidate the
gauge structure and duality properties of tensor gauge fields in arbitrary representations
of \( GL(D, \mathbb{R}) \). The proof of the theorem, contained in the appendix, is iterative and simply
proceeds by successive applications of the standard Poincaré lemma.

2 Linearized gravity

2.1 Pauli-Fierz action

A free symmetric tensor gauge field \( h_{\mu \nu} \) in \( D \) dimensions has the gauge symmetry
\[
\delta h_{\mu \nu} = 2 \partial_{[\mu} \xi_{\nu]) .
\]  
(2.1)
The linearized Riemann tensor for this field is
\[
R_{\mu \nu \sigma \tau} \equiv \frac{1}{2} (\partial_\mu \partial_\sigma h_{\nu \tau} + \ldots) = -2 \partial_{[\mu} h_{\nu][\sigma, \tau]} .
\]  
(2.2)
It satisfies the property
\[
R_{\mu \nu \sigma \tau} = R_{\sigma \tau \mu \nu}
\]  
(2.3)
and the second Bianchi identity
\[
\partial_{[\rho} R_{\mu \nu \sigma]} = 0
\]  
(2.4)
and the second Bianchi identity
\[
\partial_{[\rho} R_{\mu \nu \sigma]} = 0 .
\]  
(2.5)

It has been shown by Pauli and Fierz \[13\] that there is a unique, consistent action
that describes a pure massless spin-two field. This action is the Einstein action linearized
around a Minkowski background\[^2\]
\[
S_{EH}[g_{\mu \nu}] = \frac{2}{\kappa^2} \int d^D x \sqrt{-g} R_{\text{full}} , \quad g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu} ,
\]  
(2.6)
where \( R_{\text{full}} \) is the scalar curvature for the metric \( g_{\mu \nu} \). The constant \( \kappa \) has mechanical
dimensions \( L^{D/2-1} \). The term of order \( 1/\kappa^2 \) in the expansion of \( S_{EH} \) vanishes since the
background is flat. The term of order \( 1/\kappa \) is equal to zero because it is proportional to the
(sourceless) Einstein equations evaluated at the Minkowski metric. The next order term
in the expansion in \( \kappa \) is the action for a massless spin-2 field in \( D \)-dimensional spacetime
\[
S_{PF}[h_{\mu \nu}] = \int d^D x \left[ -\frac{1}{2} (\partial_\mu h_{\nu \rho}) (\partial^\rho h^\nu_\nu) + (\partial_\mu h^\nu_\nu) (\partial_\rho h^\rho_\mu)
\right.
\[
- (\partial_\mu h^\rho_\mu) (\partial_\rho h^\nu_\nu) + \frac{1}{2} (\partial_\mu h^\rho_\nu) (\partial^\nu h^\rho_\mu) \right] .
\]  
(2.7)
\[^2\]Notice that the way back to full gravity is quite constrained. It has been shown that there is no
local consistent coupling, with two or less derivatives of the fields, that can mix various gravitons \[14\]. In
other words, there are no Yang-Mills-like spin-2 theories. The only possible deformations are given by a
sum of individual Einstein-Hilbert actions. Therefore, in the case of one graviton, \[14\] provides a strong
proof of the uniqueness of Einstein’s theory.
Since we linearize around a flat background, spacetime indices are raised and lowered with the flat Minkowskian metric $\eta_{\mu\nu}$. For $D = 2$ the Lagrangian is a total derivative so we will assume $D \geq 3$. The (vacuum) equations of motion are the natural free field equations

$$R^\sigma_{\mu\sigma\nu} = 0$$

(2.8)

which are equivalent to the linearized Einstein equations. Together with (2.2) the previous equation implies that

$$\partial^\mu R_{\mu\nu\sigma\tau} = 0.$$  

(2.9)

### 2.2 Minimal coupling

The Euler-Lagrange variation of the Pauli-Fierz action is

$$\frac{\delta S_{PF}}{\delta h_{\mu\nu}} = R^\sigma_{\mu\sigma\nu} - \frac{1}{2} \eta_{\mu\nu} R^{\sigma\tau}_{\sigma\tau}.$$  

(2.10)

It can be shown that the second Bianchi identity (2.5) implies on-shell

$$\partial^\mu R_{\mu\nu\rho} + \partial_\rho R_{\nu\mu}^\mu - \partial_\nu R_{\rho\mu}^\mu = 0,$$

(2.11)

and taking the trace again this leads to

$$\partial^\mu R^\sigma_{\mu\sigma\nu} - \frac{1}{2} \partial_\nu R^{\sigma\tau}_{\sigma\tau} = 0.$$  

(2.12)

From another perspective, the equations (2.12) can be regarded as the Noether identities corresponding to the gauge transformations (2.1).

Let us introduce a source $T_{\mu\nu}$ which couples minimally to $h_{\mu\nu}$ through the term

$$S_{\text{minimal}} = -\kappa \int d^Dx h_{\mu\nu} T_{\mu\nu}.$$  

(2.13)

We add this term to the Pauli-Fierz action (2.7), together with a kinetic term $S_K$ for the sources, to obtain the action

$$S = S_{PF} + S_K + S_{\text{minimal}}.$$  

(2.14)

The field equations for the symmetric gauge field $h_{\mu\nu}$ are the linearized Einstein equations

$$R^\sigma_{\mu\sigma\nu} - \frac{1}{2} \eta_{\mu\nu} R^{\sigma\tau}_{\sigma\tau} = \kappa T_{\mu\nu}.$$  

(2.15)

Consistency with (2.12) implies that the linearized energy-momentum tensor is conserved

$$\partial^\mu T_{\mu\nu} = 0.$$  

The simplest example of a source is that of a free particle of mass $m$ following a worldline $x^\mu(s)$ with $s$ the proper time along the worldline. The Polyakov action for the massive particle reads

$$S_{\text{Polyakov}}[x^\mu(s)] = -m \int ds g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}.$$  

(2.16)
It results as the sum of the two actions

\[ S_K = -m \int ds \eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}, \quad (2.17) \]

\[ S_{\text{minimal}} = -m\kappa \int ds h_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}, \quad (2.18) \]

from which it can be inferred that the (matter) source \( T_{\mu\nu} \) for a massive particle is equal to

\[ T^{\mu\nu}(x) = m \int ds \delta^{D}(x - x(s)) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}. \quad (2.19) \]

This relationship is conserved if and only if \( \frac{d^2 x^\mu}{ds^2} = 0 \), which means that the test particle follows a straight worldline. In general, when considering a free massless spin-two theory coupled with matter, the latter has to be constrained in order to be consistent with the conservation of the linearized energy-momentum tensor\(^3\). At first sight, it is however inconsistent with the natural expectation that matter reacts to the gravitational field. Anyway, the constraint \( \frac{d^2 x^\mu}{ds^2} = 0 \) is mathematically inconsistent with the e.o.m. obtained from varying (2.17) and (2.18) with respect to the worldline \( x^\mu(s) \) which constrains the massive particle to follow a geodesic for \( g_{\mu\nu} \) (and not a straight line). In fact, for matter to respond to the gravitational field, it is necessary to add a source \( \kappa T_{\mu\nu}^{\text{self}} \) for the gravitational field itself, in such a way that the sum \( T_{\mu\nu} + T_{\mu\nu}^{\text{self}} \) is conserved if the matter obeys its own equation to first order in \( \kappa \) and if the gravitational field obeys (2.15). This gravitational self-energy must come from a first order (in \( \kappa \)) deformation of the Pauli-Fierz action. This modification was the starting point of Feynman\(^4\) and others in their derivation of the Einstein-Hilbert action by consistent deformation of the Pauli-Fierz action with back reaction \([16]\). At the end of the perturbative procedure, the result obtained is that the free-falling particle must follow a geodesic for consistency with the (full) Einstein equations.

### 2.3 Duality in linearized gravity

Let us mention for further purpose that the equation (2.9) can be directly deduced from the equations (2.4)-(2.5)-(2.8) for the linearized Riemann tensor without using its explicit expression (2.2). To simplify the proof and initiate a discussion about duality properties, let us introduce the tensor

\[ (\ast R)_{\mu_1...\mu_{D-2}\mid\rho\sigma} = \frac{1}{2} \varepsilon_{\mu_1...\mu_D} R^{\mu_{D-1}\mu_D \mid \rho\sigma}. \quad (2.20) \]

The linearized second Bianchi identity and the Einstein equations can be written in terms of this new tensor respectively as

\[ \partial^\mu \left[ (\ast R)_{\mu...\nu\mid\rho\sigma} \right] = 0 \quad (2.21) \]

---

\(^3\)This should not be too surprising since it is well known that the Einstein equations simultaneously determine the gravity field and the motion of matter.

\(^4\)In 1962, Feynman presented this derivation in his sixth Caltech lecture on gravitation \([15]\). One of the intriguing features of this viewpoint is that the initial flat background is no longer observable in the full theory. In the same vein, the fact that the self-interacting theory has a geometric interpretation is “not readily explainable - it is just marvelous”, as Feynman expressed.
\[(\ast R)_{[\mu \ldots \nu | \rho] \sigma} = 0 \, . \quad (2.22)\]

Taking the divergence of \((2.22)\) with respect to the first index \(\mu\), and applying \((2.21)\), we obtain

\[\partial^{\mu} [(\ast R)_{\rho \ldots \nu | \mu} \sigma] = 0 \quad (2.23)\]

which is equivalent to

\[\partial^{\mu} R_{\alpha \beta \mu \sigma} = 0 \quad (2.24)\]

as follows from the definition \((2.20)\). Using the symmetry property \((2.3)\) of the Riemann tensor we recover \((2.9)\).

In Corollary \((1)\) we will prove that the equations

\[R_{\sigma \mu \sigma \nu} = 0, \quad \partial^{\mu} R_{\mu \nu \sigma \tau} = \partial^{\sigma} R_{\mu \nu \sigma \tau} = 0 \quad (2.25)\]

are (locally) equivalent to the following equation \([6]\)

\[(\ast R)_{[\mu_{1} \ldots \mu_{D-2} | \rho \sigma] = \partial_{[\mu_{1}} h_{\mu_{2} \ldots \mu_{D-2}]} [\rho, \sigma] \, , \quad (2.26)\]

which defines the tensor field \(\tilde{h}_{[\mu_{1} \ldots \mu_{D-3} | \rho} \) called the dual gauge field of \(h_{\mu \nu}\) and which is said to have mixed symmetry because it is neither (completely) antisymmetric nor symmetric. In fact, it obeys the identity

\[\tilde{h}_{[\mu_{1} \ldots \mu_{D-3} | \rho} \equiv 0 \, . \quad (2.27)\]

However, for \(D = 4\) the dual gauge field is a symmetric tensor \(\tilde{h}_{\mu \nu}\), which signals a possible duality symmetry. The curvature dual \((2.26)\) remains unchanged by the transformations

\[\delta h_{[\mu_{1} \ldots \mu_{D-3} | \rho} = \partial_{[\mu_{1}} S_{\mu_{2} \ldots \mu_{D-3}]} [\rho + \partial_{\rho} A_{\mu_{2} \ldots \mu_{D-3} \mu_{1}} + A_{\rho [\mu_{2} \ldots \mu_{D-3}, \mu_{1}]} \quad (2.28)\]

where complete antisymmetrization of the gauge parameter \(S_{[\mu_{1} \ldots \mu_{D-4}]} \mu_{D-3}\) vanishes and the other gauge parameter \(A_{\mu_{1} \ldots \mu_{D-3}}\) is completely antisymmetric.

### 2.4 Mixed symmetry type gauge fields

Let us consider the general case of massless gauge fields \(M_{[\mu_{1} \mu_{2} \ldots \mu_{n} | \mu_{n+1}}\) having the same symmetries as the above-mentioned dual gauge field \(\tilde{h}_{[\mu_{1} \ldots \mu_{D-3} | \rho} \). These can be represented by the Young diagram

\[
\begin{array}{c}
1 \\
2 \\
3 \\
\vdots \\
n+1
\end{array}
\]

which implies that the field obeys the identity

\[M_{[\mu_{1} \mu_{2} \ldots \mu_{n} | \mu_{n+1}} \equiv 0 \, . \quad (2.30)\]

Such tensor gauge fields were studied two decades ago by the authors of \([8, 17, 18]\) and appear in the bosonic sector of some odd-dimensional CS supergravities \([19]\). Here, \(n\) is
used to denote the number of antisymmetric indices carried by the field $M_{\mu_1\mu_2...\mu_n | \mu_{n+1}}$, which is also the number of boxes in the first column of the corresponding Young array. The tensors $M_{\mu_1\mu_2...\mu_n | \mu_{n+1}}$ have $\frac{n(D+1)!}{(n+1)!(D-n)!}$ components in $D$ dimensions.

The field equations derived from (2.32) are equivalent to

$$\eta^{\mu_1\nu_1} K_{\mu_1\mu_2...\mu_{n+1} | \nu_1\nu_2} = 0$$

(2.33)

where

$$K_{\mu_1\mu_2...\mu_{n+1} | \nu_1\nu_2} = \partial_{[\mu_1} M_{\mu_2...\mu_{n+1} | \nu_1\nu_2]}$$

(2.34)

is the curvature and obeys the algebraic identity

$$K_{[\mu_1...\mu_{n+1} | \nu_1]} = 0.$$ 

(2.35)

The action (2.31) and the curvature (2.34) are invariant under the following gauge transformations

$$\delta S A M_{\mu_1...\mu_{n+1}} = \partial_{[\mu_1} S_{\mu_2...\mu_n]} | \mu_{n+1} + \partial_{[\mu_1} A_{\mu_2...\mu_n]} | \mu_{n+1} + \partial_{\mu_{n+1}} A_{\mu_2...\mu_n,\mu_1}$$

(2.36)

where the gauge parameters $S_{\mu_2...\mu_n,\mu_1}$ and $A_{\mu_2...\mu_n,\mu_1}$ have the symmetries

$$\begin{pmatrix}
\frac{2}{n+1} \\
\frac{3}{n+1} \\
\vdots \\
\frac{n+1}{n+1}
\end{pmatrix}$$

and

$$\begin{pmatrix}
\frac{2}{n+1} \\
\frac{3}{n+1} \\
\vdots \\
\frac{n+1}{n+1}
\end{pmatrix},$$

respectively.

These gauge transformations are accompanied by a chain of $n-1$ reducibilities on the gauge parameters. These reducibilities read, with $1 \leq i \leq n$

$$S_{\mu_1...\mu_{n-1-i} | \mu_{n-i+1}} = \partial_{[\mu_1} (i-1) S_{\mu_2...\mu_{n-i-1} | \mu_{n-i+1}} +$$

$$\frac{(n+1)}{(n-i+1)} \left[ \partial_{[\mu_1} (i-1) A_{\mu_2...\mu_{n-i} | \mu_{n-i+1}} + \partial_{\mu_{n-i+1}} (i-1) A_{\mu_2...\mu_{n-i-1} | \mu_1} \right],$$

(2.37)

$$A_{\mu_1...\mu_{n-i+1} | \mu_{n-i}} = \partial_{[\mu_1} (i-1) A_{\mu_2...\mu_{n-i+1} | \mu_{n-i}}$$

(2.38)

Notice that, for $n = 1$, the Lagrangian reproduces (2.7).
with the conventions that
\[(1)\quad S_{\mu_1...\mu_{n-1}\mid\mu_n} = S_{\mu_1...\mu_{n-1}\mid\mu_n}, \quad (n)\quad S_\mu = 0, \quad (1)\quad A_{\mu_1...\mu_n} = A_{\mu_1...\mu_n}.
\]
The reducibility parameters at reducibility level \(i\) have the symmetry
\[(i+1)\quad S_{\mu_1...\mu_{n-i-1}\mid\mu_{n-i}} \simeq \begin{array}{c} 1 \\ 2 \\ \vdots \\ n-i \\ \vdots \\ \end{array}
\quad \text{and} \quad (i+1)\quad A_{\mu_1...\mu_{n-i}} \simeq \begin{array}{c} 1 \\ 2 \\ \vdots \\ n-i \\ \vdots \\ \end{array}.
\]
Note that \( S_{\mu\nu} \simeq [\mu\nu] \). These gauge transformations and reducibilities have already been introduced and discussed in references [17, 18, 8]. The problem of investigating all the possible consistent couplings among the fields \( M_{\mu_1\mu_2\mid\mu_3} \) will be treated in [20]. Our theorem will provide a systematic tool for the investigation of mixed symmetry type gauge field theories.

The number of physical degrees of freedom for the theory \((2.31), (2.36)\), is equal to
\[
\frac{(D - 2)! D (D - n - 2) n}{(D - n - 1)! (n + 1)!}.
\]
This number is manifestly invariant under the exchange \( n \leftrightarrow D - n - 2 \) which corresponds to a Hodge duality transformation. This confirms that the dimension for which the theory is dual to a symmetric tensor is equal to \( D = n + 3 \), which is also the critical dimension for the theory to have local physical degrees of freedom. The theory \((2.31), (2.36)\) is then dual to Pauli-Fierz’s action \((2.7)\) for \( D = n + 3 \).

## 3 N-complexes

The objective of the works presented in [9, 11, 12] was to construct complexes for irreducible tensor fields of mixed Young symmetry type, thereby generalizing to some extent the calculus of differential forms. This tool provides an elegant formulation of symmetric tensor gauge fields and their Hodge duals (such as differential form notation within electrodynamics).

### 3.1 Young diagrams

A **Young diagram** \( Y \) is a diagram which consists of a finite number \( S > 0 \) of columns of identical squares (referred to as the **cells**) of finite non-increasing lengths \( l_1 \geq l_2 \geq \ldots \geq l_S \geq 0 \). For instance,
\[
Y \equiv \begin{array}{c} \square \\ \square \\ \square \\ \end{array}
\]
The total number of cells of the Young diagram $Y$ is denoted by

$$|Y| = \sum_{i=1}^{S} l_i.$$  

(3.1)

Order relations

For future reference, the subset $\mathcal{Y}(S)$ of $\mathbb{N}^S$ is defined by

$$\mathcal{Y}(S) \equiv \{(n_1, \ldots, n_S) \in \mathbb{N}^S | n_1 \geq n_2 \geq \ldots \geq n_S \geq 0\}.$$  

(3.2)

For two columns, the set $\mathcal{Y}(2)$ can be depicted as the following set of points in the plane $\mathbb{R}^2$:

\[
\begin{array}{c}
(3,3) \quad \ldots \\
(2,2) \quad (3,2) \quad \ldots \\
(1,1) \quad (2,1) \quad (3,1) \\
(0,0) \quad (1,0) \quad (2,0) \quad (3,0) \quad \ldots \\
\end{array}
\]

(3.3)

Let $Y$ be a diagram with $S$ columns of respective lengths $l_1, l_2, \ldots, l_S$. If $Y_p$ is a well-defined Young diagram, then $(l_1, l_2, \ldots, l_S) \in \mathcal{Y}(S)$. Conversely, a Young diagram $Y$ with $S$ columns is uniquely determined by the gift of an element of $\mathcal{Y}(S)$, and can therefore be labeled unambiguously as $Y_{(l_1, l_2, \ldots, l_S)}$ $(S \neq 0)$. We denote\(^6\) by $Y^{(S)}$ the set of all Young diagrams $Y_{(l_1, l_2, \ldots, l_S)}$ with at most $S$ columns of respective length $0 \leq l_S \leq \ldots \leq l_1 \leq D - 1$. This identification between $\mathcal{Y}(S)$ and $Y^{(S)}$ suggests obvious definitions of sums and differences of Young diagrams.

There is a natural definition of inclusion of Young diagrams

$$Y_{(m_1, \ldots, m_S)}^{(S)} \subseteq Y_{(n_1, \ldots, n_S)}^{(S)} \iff m_1 \leq n_1, m_2 \leq n_2, \ldots, m_S \leq n_S.$$  

(3.4)

We can develop a stronger definition of inclusion. Let $Y_{(m_1, \ldots, m_S)}^{(S)}$ and $Y_{(n_1, \ldots, n_S)}^{(S)}$ be two Young diagrams of $Y^{(S)}$. We say that $Y_{(m_1, \ldots, m_S)}^{(S)}$ is well-included into $Y_{(n_1, \ldots, n_S)}^{(S)}$ if $Y_{(m_1, \ldots, m_S)}^{(S)} \subseteq Y_{(n_1, \ldots, n_S)}^{(S)}$ and $n_i - m_i \leq 1$ for all $i \in \{1, \ldots, S\}$. In other words, if no column of the Young diagrams differs by more than a single box. We denote this particular inclusion by $\subseteq$, i.e.

$$Y_{(m_1, \ldots, m_S)}^{(S)} \subseteq Y_{(n_1, \ldots, n_S)}^{(S)} \iff m_i \leq n_i \leq m_i + 1 \ \forall i \in \{1, \ldots, S\}.$$  

(3.5)

\(^6\)We will sometimes use the symbol $\mathcal{Y}(S)$ instead of $Y^{(S)}$. 

This new inclusion suggests the following pictorial representation of $\mathcal{Y}(S)$:

where all the arrows represent maps $\in$. This diagram is completely commutative.

The previous inclusions $\subset$ and $\in$ provide partial order relations for $\mathcal{Y}(S)$. The order is only partial because all Young tableaux are not comparable.

We now introduce a total order relation $\ll$ for $\mathcal{Y}(S)$. If $(m_1, \ldots, m_S)$ and $(n_1, \ldots, n_S)$ belong to $\mathcal{Y}(S)$, then

$$Y_{(m_1, \ldots, m_S)} \ll Y_{(n_1, \ldots, n_S)} \Leftrightarrow \exists K \in \{1, \ldots, S\}: \begin{cases} m_i = n_i, & \forall i \in \{1, \ldots, K\}, \\ m_{K+1} \leq n_{K+1}. \end{cases}$$

(3.7)

This ordering simply provides the lexicographic ordering for $\mathcal{Y}(S)$.

**Maximal diagrams**

A sequence of $\mathcal{Y}(S)$ which is of physical interest is the **maximal sequence** denoted by $Y^S \equiv (Y^S_p)_{p \in \mathbb{N}}$. This is defined as the naturally ordered sequence of maximal diagrams (the ordering is induced by the inclusion of Young tableaux). **Maximal diagrams** are diagrams with maximally filled rows, that is to say, Young diagrams $Y^S_p$ with $p$ cells defined in the following manner: we add cells to a row until it contains $S$ cells and then we proceed in the same way with the row below, and continue until all $p$ cells have been used. Consequently all rows except the last one are of length $S$ and, if $r_p$ is the remainder of the division of $p$ by $S$ ($r_p \equiv p \mod S$) then the last row of the Young diagram $Y^S_p$ will contain $r_p \leq S$ cells (if $r_p \neq 0$). For two columns ($S = 2$) the maximal sequence is

The subsequent notations for maximal sequences are different from the ones of [11, 12]. We have shifted the upper index by one unit.
represented as the following path in the plane depicting $Y^{(2)}$:

![Diagram](image)

Diagrams for which all rows have exactly $S$ cells are called **rectangular diagrams**. These are those represented by the leftmost diagonal of the diagram $Y^{(S)}$.

**Duality**

Let $Y^{(S)}_{(l_1,\ldots,l_S)}$ be a Young diagram in $Y^{(S)}$ and $I$ a non-empty subset of $\{1,\ldots,S\}$.

The diagram $D_{(l_1,\ldots,l_S)}^{(S)}$ with $S$ columns of respective lengths

$$\ell_i \equiv \begin{cases} l_i & \text{if } i \notin I, \\ D - l_i & \text{if } i \in I, \end{cases}$$

is, in general, *not* a Young diagram. We define the **dual Young diagram** $\tilde{Y}^I_{(\lambda_1,\ldots,\lambda_S)} \subset Y^{(S)}$ associated to the set $I$ as the Young diagram obtained by reordering the columns of $D_{(l_1,\ldots,l_S)}^{(S)}$. In other words, its $i$-th column has length

$$\lambda_i = \ell_{\pi(i)}, \quad \lambda_j \leq \lambda_i \text{ for } i \leq j,$$

where $\pi$ is a permutation of the elements of $\{1,\ldots,S\}$.

**Schur module**

Multilinear applications with a definite symmetry are associated with a definite Young diagram \(^8\). Let $V$ be a finite-dimensional vector space of dimension $D$ and $V^*$ its dual. The dual of the $n$-th tensor power $V^n$ of $V$ is canonically identified with the space of multilinear forms: $(V^n)^* \cong (V^*)^n$. Let $Y$ be a Young diagram and let us consider that each of the $|Y|$ copies of $V^*$ in the tensor product $(V^*)^{|Y|}$ is labeled by one cell of $Y$. The **Schur module** $V^Y$ is defined as the vector space of all multilinear forms $T$ in $(V^*)^{|Y|}$ such that:

(i) $T$ is completely antisymmetric in the entries of each column of $Y$,

(ii) complete antisymmetrization of $T$ in the entries of a column of $Y$ and another entry of $Y$ which is on the right-hand side of the column vanishes.

\(^8\)This set of definitions essentially comes from [12].
$V^Y$ is an irreducible subspace invariant for the action of $GL(D, \mathbb{R})$ on $V^{[Y]}$.

Let $Y$ be a Young diagram and $T$ an arbitrary multilinear form in $(V^*)^{[Y]}$, one defines the multilinear form $\mathcal{Y}(T) \in (V^*)^{[Y]}$ by

$$\mathcal{Y}(T) = T \circ \mathcal{A} \circ \mathcal{S}$$

with

$$\mathcal{A} = \sum_{c \in C} (-)^{\varepsilon(c)} c, \quad \mathcal{S} = \sum_{r \in R} r$$

where $C$ is the group of permutations which permute the entries of each column, $\varepsilon(c)$ is the parity of the permutation $c$, and $R$ is the group of permutations which permute the entries of each row of $Y$. Any $\mathcal{Y}(T) \in V^Y$ and the application $\mathcal{Y}$ of $V^{[Y]}$ satisfies the condition $\mathcal{Y}^2 = \lambda \mathcal{Y}$ for some number $\lambda \neq 0$. Thus $\mathcal{Y} = \lambda^{-1} \mathcal{Y}$ is a projection of $V^{[Y]}$ onto itself, i.e. $\mathcal{Y}^2 = \mathcal{Y}$, with image $\text{Im}(\mathcal{Y}) = V^Y$. The projection $\mathcal{Y}$ will be referred to as the Young symmetrizer of the Young diagram $Y$.

### 3.2 Differential $N$-complex

Let $(Y) = (Y_p)_{p \in \mathbb{N}}$ be a given sequence of Young diagrams such that the number of cells of $Y_p$ is $p$, $\forall p \in \mathbb{N}$. For each $p$, we assume that there is a single shape $Y_p$ and $Y_p \subset Y_q$ for $p < q$. We define $\Omega^p_{(Y)}(M)$ as the vector space of smooth covariant tensor fields of rank $p$ on the pseudo-Riemannian manifold $M$ which have the Young symmetry type $Y_p$ (i.e. their components $T(x)$ belong to the Schur module $V^{Y_p}$ associated to $Y_p$). More precisely they obey the identity $Y_p T(x) = T(x), \forall x \in M$, with $Y_p$ the Young symmetrizer on tensor of rank $p$ associated to the Young symmetry $Y_p$. Let $\Omega_{(Y)}(M)$ be the graded vector space $\oplus_p \Omega^p_{(Y)}(M)$ of irreducible tensor fields on $M$.

The exterior differential can then be generalized by setting \[ d \equiv Y_{p+1} \circ \partial : \Omega^p_{(Y)}(M) \rightarrow \Omega^{p+1}_{(Y)}(M), \tag{3.10} \]

that is to say, first taking the partial derivative of the tensor $T \in \Omega^p_{(Y)}(M)$ and applying the Young symmetrizer $Y_{p+1}$ to obtain a tensor in $\Omega^{p+1}_{(Y)}(M)$. Examples are provided in subsection 3.3.

Let us briefly mention that there are no $dx^\mu$ in this definition of the operator $d$. The operator $d$ is not nilpotent in general, therefore $d$ does not always endow $\Omega_{(Y)}(M)$ with the structure of a standard differential complex.

If we want to generalize the calculus of differential forms, we have to use the extension of the structure of differential complex with higher order of nilpotency. An $N$-complex is defined as a graded space $V = \oplus_i V_i$ equipped with an endomorphism $d$ of degree 1 that is nilpotent of order $N \in \mathbb{N} - \{0, 1\}$: $d^N = 0$. It is important to stress that the operator $d$ is not necessarily a differential because, in general, $d$ is neither nilpotent nor a derivative (for instance, even if one defines a product in $\Omega_{(Y)}(M)$, the non-trivial projections affect the Leibnitz rule).

A sufficient condition for $d$ to endow $\Omega_{(Y)}(M)$ with the structure of an $N$-complex is that the number of columns of any Young diagram be strictly smaller than $N$. \[ d \equiv Y_{p+1} \circ \partial : \Omega^p_{(Y)}(M) \rightarrow \Omega^{p+1}_{(Y)}(M), \tag{3.10} \]
Let $S$ be a non-vanishing integer and assume that the sequence $(Y)$ is such that the number of columns of the Young diagram $Y_p$ is strictly smaller than $S + 1$ (i.e. $\leq S$) for any $p \in \mathbb{N}$. Then the space $\Omega_{(Y)}(\mathcal{M})$, endowed with the operator $d$, is a $(S + 1)$-complex.

Indeed, the condition $d^{S+1}\omega = 0$ for all $\omega \in \Omega_{(Y)}(\mathcal{M})$ is fulfilled since the indices in one column are antisymmetrized and $d^{S+1}\omega$ necessarily involves at least two partial derivatives $\partial$ in one of the columns (there are $S + 1$ partial derivatives involved and a maximum of $S$ columns).

**Notation:** The space $\Omega_{(Y(S))}(\mathcal{M})$ is a $(S + 1)$-complex that we denote $\Omega_{(S)}(\mathcal{M})$. The subcomplex $\Omega_{(Y(S))}^{(t_1,t_2,...,t_S)}(\mathcal{M})$ is denoted by $\Omega_{(S)}^{(t_1,t_2,...,t_S)}(\mathcal{M})$.

This complex $\Omega_{(S)}(\mathcal{M})$ is the generalization of the differential form complex $\Omega(\mathcal{M}) = \Omega(\mathcal{M})$ we are seeking for because each proper space is invariant under the action of $GL(D, \mathbb{R})$. For example, the previously mentioned mixed symmetry type gauge field $M$ belongs to $\Omega_{(2)}(\mathcal{M})$.

### 3.3 Symmetric gauge tensors and maximal sequences

A Young diagram sequence of interest in theories of spin $S \geq 1$ is the maximal sequence $Y^S = (Y_p^S)_{p \in \mathbb{N}} \mathbb{Z}_{\geq 1}[\mathbb{Z}_{\leq 12}]$. This sequence is defined as the sequence of diagrams with maximally filled rows naturally ordered by the number of boxes.

**Notation:** In order to simplify the notation, we shall denote $\Omega_{(Y^S)}(\mathcal{M})$ by $\Omega_S(\mathcal{M})$ and $\Omega_{(Y^S)}^{(t_1,t_2,...,t_S)}(\mathcal{M})$ by $\Omega_{(S)}^{(t_1,t_2,...,t_S)}(\mathcal{M})$.

If $D$ is the dimension of the manifold $\mathcal{M}$ then the subcomplexes $\Omega_S(\mathcal{M})$ with $p > SD$ are trivial since, for these values of $p$, the Young diagrams $Y_p^S$ have at least one column containing more than $D$ cells.

### Massless spin-one gauge field

It is clear that $\Omega_1(\mathcal{M})$ with the differential $d$ is the usual complex $\Omega(\mathcal{M})$ of differential forms on $\mathcal{M}$. The connection between the complex of differential forms on $\mathcal{M}$ and the theory of classical $q$-form gauge fields is well known. Indeed the subcomplex

$$
\Omega^0(\mathcal{M}) \xrightarrow{d} \Omega^1(\mathcal{M}) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^q(\mathcal{M}) \xrightarrow{d} \Omega^{q+1}(\mathcal{M}) \xrightarrow{d} \Omega^{q+2}(\mathcal{M})
$$

with $d_p \equiv d : \Omega^p \rightarrow \Omega^{p+1}$, has the following interpretation in terms of $q$-form gauge field $A_{[q]}$ theory. The space $\Omega^q(\mathcal{M})$ is the space which the field strength $F_{[q]}$ lives in. The space $\Omega^{q+1}(\mathcal{M})$ is the space of Hodge duals to magnetic sources $* J_m$ (at least if we extend the space of “smooth” $(q-2)$-forms to de Rham currents) since $dF_{[q+1]} = ([* J_m]_{q+2]}$. If there is no magnetic source, the field strength belongs to the kernel of $d_{q+1}$. The Abelian gauge field $A_{[q]}$ belongs to $\Omega^q(\mathcal{M})$. The subspace $\text{Ker } d_q$ of $\Omega^q(\mathcal{M})$ is the space of pure gauge configurations (which are physically irrelevant). The space $\Omega^{q-1}(\mathcal{M})$ is the space of infinitesimal gauge parameters $A_{[q-1]}$ and $\Omega^{q-2}(\mathcal{M})$ is the space of first reducibility...
parameters $\Lambda_{[q-2]}$, etc. If the manifold $\mathcal{M}$ has the topology of $\mathbb{R}^D$ then (3.11) is an exact sequence.

**Massless spin-two gauge field**

As another example, we demonstrate the correspondence between some Young diagrams in the maximal sequence with at most two columns and their corresponding spaces in the differential 3-complex $\Omega_2(\mathcal{M})$

| Young tableau | Vector space | Example | Components |
|---------------|--------------|---------|------------|
| $\begin{array}{c}1\end{array}$ | $\Omega^1_2(\mathcal{M})$ | lin. diffeomorphism parameter | $\xi_\mu$ |
| $\begin{array}{c}1\end{array}$ | $\Omega^2_2(\mathcal{M})$ | graviton | $h_{\mu\nu}$ |
| $\begin{array}{c}2\end{array}$ | $\Omega^3_2(\mathcal{M})$ | mixed symmetry type field | $M_{\mu\nu\rho}$ |
| $\begin{array}{c}2\end{array}$ | $\Omega^4_2(\mathcal{M})$ | Riemann tensor | $R_{\mu\nu\rho\sigma}$ |
| $\begin{array}{c}3\end{array}$ | $\Omega^5_2(\mathcal{M})$ | Bianchi identity | $\partial_{[\lambda}R_{\mu\nu]\rho\sigma}$ |

Table 1: Two-column maximal sequence and its physical relevance.

The interest of $\Omega_2(\mathcal{M})$ is its direct applicability in free spin-two gauge theory. Indeed, in this case the analog of the sequence (3.11) is

$$
\Omega^1_2(\mathcal{M}) \xrightarrow{d} \Omega^2_2(\mathcal{M}) \xrightarrow{d^2} \Omega^4_2(\mathcal{M}) \xrightarrow{d} \Omega^5_2(\mathcal{M})
$$

(3.12)

where $\Omega^1_2(\mathcal{M})$ is the space of covariant vector fields $\xi_\mu$ on $\mathcal{M}$, $\Omega^2_2(\mathcal{M})$ is the space of covariant rank 2 symmetric tensor fields $h_{\mu\nu}$ on $\mathcal{M}$, $\Omega^4_2(\mathcal{M})$ the space of covariant tensor fields $R_{\mu\nu\rho\sigma}$ of rank 4 having the symmetries of the Riemann curvature tensor, and $\Omega^5_2(\mathcal{M})$ is the space of covariant tensor fields of degree 5 having the symmetries of the left-hand side of the Bianchi II identity. The action of the operator $d$, whose order of nilpotency is equal to 3, is written explicitly in terms of components:

$$
(d\xi)_{\mu\nu} = \frac{1}{2}(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)
$$

(3.13)

$$
(d^2h)_{\lambda\mu\nu\rho} = \frac{1}{4}(\partial_\lambda \partial_\rho h_{\mu\nu} + \partial_\mu \partial_\nu h_{\lambda\rho} - \partial_\mu \partial_\rho h_{\lambda\nu} - \partial_\lambda \partial_\nu h_{\mu\rho})
$$

(3.14)

$$
(dR)_{\lambda\mu\nu\alpha\beta} = \frac{1}{3}(\partial_\lambda R_{\mu\nu\alpha\beta} + \partial_\mu R_{\nu\lambda\alpha\beta} + \partial_\nu R_{\lambda\mu\alpha\beta}).
$$

(3.15)
The generalized 3-complex \( \Omega_2(M) \) can be pictured as the commutative diagram

\[
\begin{array}{c}
\cdots \\
\Omega^6_2(M) \\
\downarrow \\
\cdots \\
\Omega^5_2(M) \\
\downarrow \\
\cdots \\
\Omega^4_2(M) \\
\downarrow \\
\cdots \\
\Omega^3_2(M) \\
\downarrow \\
\cdots \\
\Omega^2_2(M) \\
\downarrow \\
\cdots \\
\Omega^1_2(M) \\
\downarrow \\
\cdots \\
\Omega^0_2(M)
\end{array}
\]

In terms of this diagram, the higher order nilpotency \( d^3 = 0 \) translates into the fact that (i) if one takes a vertical arrow followed by a diagonal arrow, or (ii) if a diagonal arrow is followed by a horizontal arrow, it always maps to zero.

### 3.4 Rectangular diagrams

The generalized cohomology \([21]\) of the \(N\)-complex \( \Omega_{N-1}(M) \) is the family of graded vector spaces \( H_{(k)}(d) \) with \( 1 \leq k \leq N-1 \) defined by \( H_{(k)}(d) = \ker(d^k)/\im(d^{N-k}) \). In general the cohomology groups \( H^{(k)}(d) \) are not empty, even when \( M \) has a trivial topology. Nevertheless there exists a generalization of the Poincaré lemma for \( N \)-complexes of interest in gauge theories.

Let \( Y^S \) be a maximal sequence of Young diagrams. The (generalized) Poincaré lemma states that for \( M \) with the topology of \( \mathbb{R}^D \) the generalized cohomology of \( d \) on tensors represented by rectangular diagrams is empty in the space of maximal tensors \([9, 11, 12]\).

**Proposition 1.** (Generalized Poincaré lemma for rectangular diagrams)

- \( H^{0}_{(k)}(\Omega^S(\mathbb{R}^D)) \) is the space of real polynomial functions on \( \mathbb{R}^D \) of degree strictly less than \( k \) (\( 1 \leq k \leq N-1 \)) and
- \( H^{nS}_{(k)}(\Omega^S(\mathbb{R}^D)) = 0 \) \( \forall n \) such that \( 1 \leq n \leq D-1 \).

This is the first theorem of \([12]\), the proof of which is given therein. This theorem strengthens the analogy between the two complexes \((3.11)\) and \((3.12)\) since it implies that \((3.12)\) is also an exact sequence when \( M \) has a trivial topology.

---

\( ^9 \)Strictly speaking, the generalized Poincaré lemma for rectangular diagrams was proved in \([11, 12]\) with an other choice of convention where one first antisymmetrizes the columns. This other choice is more convenient to prove the theorem in \([12]\) but is inappropriate for considering Hodge dualization properties. This explains our choice of convention; still, as we will show later, the generalized Poincaré lemma for rectangular diagrams remains true with the definition \((3.10)\).
Exactness at $\Omega^2_2(\mathcal{M})$ means $H^2_{(2)}(\Omega_2(\mathbb{R}^D)) = 0$ and exactness at $\Omega^4_2(\mathcal{M})$ means $H^4_{(1)}(\Omega_2(\mathbb{R}^D)) = 0$. These properties have a physical interpretation in terms of the linearized Bianchi identity II and gauge transformations. Let $R_{\mu\nu\rho\sigma}$ be a tensor that is antisymmetric in its two pairs of indices $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$, namely it has the symmetry of the Young diagram $\square \otimes \square$. This latter decomposes according to

$$
\square \otimes \square = \square \oplus \square \oplus \square.
$$

If we impose the condition that $R$ obeys the first Bianchi identity (2.15), we eliminate the last two terms in its decomposition (3.17) hence the tensor $R$ has the symmetries of the Riemann tensor and belongs to $\Omega^4_2(\mathcal{M})$. Furthermore, from (3.15) it is apparent that the linearized second Bianchi identity (2.5) for $R$ reads $dR = 0$. As the Riemann tensor has the symmetries of a rectangular diagram, we obtain $R = d^2h$ with $h \in \Omega^2_2(\mathcal{M})$ from the exactness of the sequence (3.12). This means that $R$ is effectively the linearized Riemann tensor associated to the spin-two field $h$, as can be directly seen from (3.14). However, the definition of the metric fluctuation $h$ is not unique: the gauge field $h + \delta h$ is physically equivalent to $h$ if it does not affect the physical linearized Riemann tensor, i.e. $d^2(\delta h) = 0$. Since the sequence (3.11) is exact we find: $\delta h = d\xi$ with $\xi \in \Omega^1_2(\mathcal{M})$. As a result we recover the standard gauge transformations (2.1).

### 3.5 Multiforms, Hodge duality and trace operators

A good mathematical understanding of the gauge structure of free symmetric tensor gauge field theories is provided by the maximal sequence and the vanishing of the rectangular diagrams cohomology. However, several new mathematical ingredients are needed as well as an extension of Proposition 1 to capture their dynamics. A useful new ingredient is the obvious generalization of Hodge’s duality for $\Omega_S(\mathbb{R}^D)$, which is obtained by contracting the columns with the epsilon tensor $\varepsilon^{\mu_1 \cdots \mu_D}$ of $\mathcal{M}$ and lowering the indices with the Minkowskian metric. For rank $S$ symmetric tensor gauge theories there are $S$ different Hodge operations since the corresponding diagrams may contain up to $S$ columns. A simple but important point to note is the following: generically the Hodge duality is not an internal operation in the space $\Omega_S^1(\mathcal{M})$. For this reason, we define a new space of tensors in the next subsection.

**Multiforms**

A key ingredient is the graded tensor product of $C^\infty(\mathcal{M})$ with $S$ copies of the exterior algebra $\Lambda \mathbb{R}^D$ where $\mathbb{R}^D$ is the dual space of basis $dx^\mu$ ($1 \leq i \leq S$, thus there are $S$ times $D$ of them). Elements of this space will be referred to as **multiforms** (12). They are sums of products of the generators $dx^\mu$ with smooth functions on $\mathcal{M}$. The components of a multiform define a tensor with the symmetry properties of the product of $S$ columns.
**Notation:** We shall denote this multigraded space \((\otimes^S \Lambda(\mathbb{R}^{D^s}) \otimes C^\infty(M))\) by \(\Omega_{[S]}(M)\). The subspace \(\Omega^{l_1, l_2, \ldots, l_S}_{[S]}(M)\) is defined as the space of multiforms whose components have the symmetry properties of the diagram \(D_{l_1, l_2, \ldots, l_S} := \bigotimes_{i=1}^{S} Y^{(1)}_{(l_i)}\) which represents the above product of \(S\) columns with respective lengths \(l_1, l_2, \ldots, l_S\).

The tensor field \(\alpha[\mu_1 \ldots \mu_{l_1} \ldots \mu_{l_S} \ldots \mu_{l_S}](x)\) defines a multiform \(\alpha \in \Omega^{l_1, \ldots, l_S}_{[S]}(M)\) which explicitly reads

\[
\alpha = \alpha[\mu_1 \ldots \mu_{l_1} \ldots \mu_{l_S} \ldots \mu_{l_S}](x) \; d_1 x^{\mu_1} \wedge \ldots \wedge d_1 x^{\mu_{l_1}} \ldots d_S x^{\mu_{l_S}} \wedge \ldots \wedge d_S x^{\mu_{l_S}} .
\] (3.18)

In the sequel, when we refer to the multiform \(\alpha\) we will speak either of (3.18) or of its components. More accurately, we will identify \(\Omega_{[S]}(M)\) with the space of the smooth tensor field components.

We endow \(\Omega_{[S]}(M)\) with the structure of a (multi)complex by defining \(S\) anticommuting differentials

\[
d_i : \Omega^{l_1, \ldots, l_i, \ldots, l_S}_{[S]}(M) \to \Omega^{l_1, \ldots, l_{i+1}, \ldots, l_S}_{[S]}(M) , \quad 1 \leq i \leq S ,
\] (3.19)

defined by adding a box containing the partial derivative in the \(i\)-th column. For instance, \(d_2\) acting on the previous diagrammatic example is

**Summary of notations:** The multicomplex \(\Omega_{[S]}(M)\) is the subspace of \(S\)-uple multiforms. It is the direct sum of subcomplexes \(\Omega^{l_1, l_2, \ldots, l_S}_{[S]}(M)\). The space \(\Omega(S)(M)\) is the \((S+1)\)-complex of tensors represented by Young tableaux with at most \(S\) columns. It is the direct sum of subcomplexes \(\Omega^{l_1, l_2, \ldots, l_S}_{(S)}(M)\). The space \(\Omega_S(M) = \oplus_p \Omega^p_S(M)\) is the space of maximal tensors. Thus we have the chain of inclusions \(\Omega_S(M) \subset \Omega(S)(M) \subset \Omega_{[S]}(M)\).

**Hodge and trace operators**

We introduce the following notation for the \(S\) possible Hodge dual definitions

\[
*_{i} : \Omega^{l_1, \ldots, l_i, \ldots, l_S}_{[S]}(M) \to \Omega^{l_1, \ldots, D-l_i, \ldots, l_S}_{[S]}(M) , \quad 1 \leq i \leq S .
\] (3.20)

The operator \(*_{i}\) is defined as the action of the Hodge operator on the indices of the \(i\)-th column. To remain in the space of covariant tensors requires the use of the flat metric to lower down indices.
Using the metric, another simple operation that can be defined is the trace. The convention is that we always take the trace over indices in two different columns, say the \(i\)-th and \(j\)-th. We denote this operation by
\[
\text{Tr}_{ij} : \Omega_{[S]}^{l_1, \ldots, l_i, \ldots, l_S}(\mathcal{M}) \to \Omega_{[S]}^{l_1, \ldots, l_i - 1, \ldots, l_j - 1, \ldots, l_S}(\mathcal{M}), \quad i \neq j.
\] (3.21)

The Schur module definition (see subsection 3.1) gives the necessary and sufficient set of conditions for a (covariant) tensor \(T_{\mu_1 \mu_2 \ldots \mu_p}(x)\) of rank \(p\) to be in the irreducible representation of \(GL(D, \mathbb{R})\) associated with the Young diagram \(Y\) (with \(|Y| = p\)). Each index of \(T_{\mu_1 \mu_2 \ldots \mu_p}(x)\) is placed in one box of \(Y\). The set of conditions is the following:

(i) \(T_{\mu_1 \mu_2 \ldots \mu_p}(x)\) is completely antisymmetric in the entries of each column of \(Y\),

(ii) complete antisymmetrization of \(T_{\mu_1 \mu_2 \ldots \mu_p}(x)\) in the entries of a column of \(Y\) and another entry of \(Y\) which is on the right of the column, vanishes.

Using the previous definitions of multiforms, Hodge dual and trace operators, this set of conditions gives

**Proposition 2. (Schur module)**

Let \(\alpha\) be a multiform in \(\Omega_{[S]}^{l_1, \ldots, l_i, \ldots, l_S}(\mathcal{M})\). If
\[
l_j \leq l_i < D, \quad \forall i, j \in \{1, \ldots, S\} : i \leq j,
\]
then one obtains the equivalence
\[
\text{Tr}_{ij}\{\ast_i \alpha\} = 0 \quad \forall i, j : 1 \leq i < j \leq S \quad \iff \quad \alpha \in \Omega_{(S)}^{l_1, \ldots, l_S}(\mathcal{M}).
\]

Indeed, condition (i) is satisfied since \(\alpha\) is a multiform. Condition (ii) is simply rewritten in terms of tracelessness conditions.

Another useful property, which generalizes the derivation followed in the chain of equations (2.20)-(2.24), is for any \(i, j \in \{1, \ldots, S\}\)

\[
\bullet \quad \begin{cases} 
\text{Tr}_{ij} \alpha = 0 \\
 d_i \alpha = 0
\end{cases} \quad \implies \quad d_j (\ast_j \alpha) = 0.
\] (3.22)

The following property on powers of the trace operator will also be useful later on. We state it as

**Proposition 3.** Let \(\alpha \in \Omega_{[S]}^{l_1, \ldots, l_S}(\mathcal{M})\) be a multiform. For any \(m \in \mathbb{N}\) such that \(0 \leq m \leq \min(D - l_i, D - l_j)\), one has the equivalence
\[
(T_{\text{Tr}ij})^m\{\ast_i \ast_j \alpha\} = 0 \quad \iff \quad (T_{\text{Tr}ij})^{m + l_i + l_j - D}\{\alpha\} = 0.
\]

**Proof:** The proof of the proposition is inductive, the induction parameter being the number of traces, and is mainly based on the rule for contractions of epsilon tensors.
We start the proof of the necessity by a preliminary lemma:

For any given integer $p \in \mathbb{N}$,

\[
\begin{align*}
&(\text{Tr}_{ij})^{D-l_i-p}\{\ast_i \ast_j \alpha\} = 0 \\
&(\text{Tr}_{ij})^{l_j-n+1}\{\alpha\} = 0, \ \forall n \geq p
\end{align*}
\]

\Rightarrow \ (\text{Tr}_{ij})^{l_j-p}\{\alpha\} = 0. \quad (3.23)

This is true because it can be checked explicitly that $(\text{Tr}_{ij})^{D-l_i-p}\{\ast_i \ast_j \alpha\}$ is equal to a sum of terms proportional to $(\text{Tr}_{ij})^{l_j-k}\{\alpha\}$ for all $k \geq p$. The second hypothesis says that these last terms vanish for $k \geq p+1$. As a result, $(\text{Tr}_{ij})^{D-l_i-p}\{\ast_i \ast_j \alpha\} \propto (\text{Tr}_{ij})^{l_j-p}\{\alpha\}$. Therefore the vanishing of $(\text{Tr}_{ij})^{D-l_i-p}\{\ast_i \ast_j \alpha\}$ implies the vanishing of $(\text{Tr}_{ij})^{l_j-p}\{\alpha\}$.

Now that this preliminary lemma is given, we can turn back to our inductive proof.

The induction hypothesis $I_m$ is the following :

\[(\text{Tr}_{ij})^m\{\ast_i \ast_j \alpha\} = 0 \ \Rightarrow \ (\text{Tr}_{ij})^{m+l_i+l_j-D}\{\alpha\} = 0. \quad (3.24)\]

The starting point of the induction is $I_{D-l_i+1}$ [considering without loss of generality that $D-l_i = \min(D-l_i, D-l_j)$] which is obviously true since in this case where $m = D-l_i+1$, both traces in (3.24) vanish. What we have to show now is that, if $I_n$ is true $\forall n \geq m$, then $I_{m-1}$ is also true.

It is obvious that $(\text{Tr}_{ij})^{m-1}\{\ast_i \ast_j \alpha\} = 0$ implies that $(\text{Tr}_{ij})^n\{\ast_i \ast_j \alpha\} = 0$ for all $n \geq m-1$. The induction hypothesis thus implies that $(\text{Tr}_{ij})^{n+l_i+l_j-D}\{\alpha\} = 0$ for all $n \geq m$. Together with $(\text{Tr}_{ij})^{m-1}\{\ast_i \ast_j \alpha\} = 0$ and the help of the lemma (3.23), we eventually obtain

\[(\text{Tr}_{ij})^{m-1+l_i+l_j-D}\{\alpha\} = 0, \text{ which ends the proof of the induction hypothesis.}\]

In this case, the sufficiency is a consequence of the necessity. In other words, since we proved that the implication $I_m$ is valid from the left to the right in (3.24) we will show that then, it is also valid from the right to the left.

Indeed, the relation $\ast_i \ast_j (\ast_i \ast_j \alpha) = \pm \alpha$ allows to write

\[(\text{Tr}_{ij})^{m+l_i+l_j-D}\{\alpha\} = \pm (\text{Tr}_{ij})^{m+(l_i-D)+(l_j-D)+D}\{\ast_i \ast_j (\ast_i \ast_j \alpha)\}. \quad (3.25)\]

The (proven) implication $I_m$ of Proposition 3 applied to the multiform $\ast_i \ast_j \alpha \in \Omega^I_{(S)}$ is

\[(\text{Tr}_{ij})^{m+(l_i-D)+(l_j-D)+D}\{\ast_i \ast_j (\ast_i \ast_j \alpha)\} = 0 \ \Rightarrow \ (\text{Tr}_{ij})^m\{\alpha\} = 0. \quad (3.26)\]

Combined with the relation (3.25), the previous implication is precisely the (reversed) implication in Proposition 3.

\[\Box\]
3.6 Generalized nilpotency

Let $Y_p^p$ be well-included into $Y_p^{p+q}$, that is $Y_p^p \subseteq Y_{p+q}^p$. Let $I$ be the subset of $\{1, 2, \ldots, S\}$ containing the $q$ elements ($\# I = q$) corresponding to the difference between $Y_{p+q}^p$ and $Y_p^p$. We “generalize” the definition (3.10) by introducing the differential operators $d^I$ as follows (see also [9, 10])

$$d^I \equiv c^p_I Y_{p+q}^p \circ \left( \prod_{i \in I} \partial_i \right) : \Omega^p_{(Y)}(\mathcal{M}) \to \Omega^{p+q}_{(Y)}(\mathcal{M}) \quad (3.27)$$

where $\partial_i$ indicates that the index corresponding to this partial derivative is placed at the bottom of the $i$-th column and $c^p_I$ are normalization factors so that we have strict equalities in the next Proposition 4.11. When $I$ contains only one element ($q = 1$) we recover the definition $d$. The tensor $d^I Y_p^p$ will be represented by the Young diagram $Y_{p+q}^p$ where we place a partial derivative symbol $\partial$ in the $q$ boxes which do not belong to the subdiagram $Y_p^p \subseteq Y_{p+q}^p$.

The product of operators $d^I$ is commutative: $d^I \circ d^J = d^J \circ d^I$ for all $I, J \subset \{1, \ldots, S\}$ ($\# I = q, \# J = r$) such that the product maps to a well-defined Young diagram $Y_{p+q+r}^p$. The following proposition gathers all these properties

**Proposition 4.** Let $I$ and $J$ be two subsets of $\{1, \ldots, S\}$. Let $\alpha$ be an irreducible tensor belonging to $\Omega_{(2)}(\mathcal{M})$. The following properties are satisfied

- If $I \cap J = \phi$, then $(d^I \circ d^J) \alpha = d^{I \cup J} \alpha$. Therefore, $d^I \alpha = d^{\# I} \alpha$
- If $I \cap J \neq \phi$, then $(d^I \circ d^J) \alpha = 0$.

Proposition 4 is proved in [9]. The last property states that the product of $d^I$ and $d^J$ identically vanishes if it is represented by a diagram $Y_{p+q+r}^p$ with at least one column containing two partial derivatives. Proposition 4 proves that the operator $d^I$ provides the most general non-trivial way of applying partial derivatives in $\Omega_{(S)}(\mathcal{M})$.

Proposition 4 is also helpful because it makes contact with the definition (3.10) in that the operator $d^I$ can be identified, up to a constant factor, with the (non-trivial) $\# I$-th power of the operator $d$. Despite this identification, we frequently use the notation $d^I$ because it contains more information than the notation $d^{\# I}$.

The space $\Omega_{(2)}(\mathcal{M})$ can be pictured analogously to the representation (3.6) of the set

---

10 See subsection 3.1
11 The precise expression for the constants $c^p_I$ was obtained in [9].
12 According to the terminology of [9], these properties mean that the set $Y^{(S)}$ is endowed with the structure of hypercomplex by means of the maps $d^I$. 

of Young diagrams $Y^{(2)}$

\[
\begin{array}{cccccc}
\Omega^{(0,0)}(\mathcal{M}) & \Omega^{(1,0)}(\mathcal{M}) & \Omega^{(2,0)}(\mathcal{M}) & \Omega^{(3,0)}(\mathcal{M}) & \cdots \\
\Omega^{(1,1)}(\mathcal{M}) & \Omega^{(2,1)}(\mathcal{M}) & \Omega^{(3,1)}(\mathcal{M}) & \cdots \\
\Omega^{(2,2)}(\mathcal{M}) & \Omega^{(3,2)}(\mathcal{M}) & \cdots \\
\Omega^{(3,3)}(\mathcal{M}) & \cdots \\
\end{array}
\]

From the previous discussions, the definitions of the arrows should be clear:

\(\rightarrow\) : Horizontal arrows are maps $d = d^{(1)}$.

\(\uparrow\) : Vertical arrows are maps $d = d^{(2)}$.

\(\nearrow\) : Diagonal arrows are maps $d^2 = d^{(1,2)}$.

Proposition 4 translates in terms of this diagram into the fact that

- this diagram is completely commutative, and
- the composition of any two arrows with at least one common direction maps to zero identically.

Of course, these diagrammatic properties hold for arbitrary $S$ (the corresponding picture would be simply a higher-dimensional generalization since $Y(S) \subset \mathbb{R}^S$).

4 Linearized gravity field equations

From now on, we will restrict ourselves to the case of linearized gravity, i.e. rank-2 symmetric gauge fields. There are two possible Hodge operations, denoted by $\ast$, acting on the first column if it is written on the left, and on the second column if it is written on the right. Since we are no longer restricted to maximal Young diagrams the notation $d$ is ambiguous (we do not know a priori on which Young symmetry type we should project in the definition (3.10)). Instead we use the above mentioned differentials $d_i$ of multiform theory. There are only two of these in the case of linearized gravity: $d_1$ called the (left) differential, denoted by $d_L$, and $d_2$, the (right) differential, denoted by $d_R$. With these differentials it is possible to rewrite (2.25) in the compact form $d_L \ast R = 0 = d_R R \ast$. The second Bianchi identity reads $d_L R = 0 = d_R R$. 
The convention that we use is to take the trace over indices in the first row, using the flat background metric \( \eta_{\mu \nu} \). We denote this operation by \( \text{Tr} \) (which is \( \text{Tr}_{12} \) according to the definition given in the previous section). In this notation the Einstein equation (2.8) takes the form \( \text{Tr} R = 0 \), while the first Bianchi identity (2.4) reads \( \text{Tr} \ast R = 0 \). From Proposition 2 it is clear that the following property holds: let \( B \) be a “biform” in \( \Omega^{p,q} (\mathcal{M}) \) which means \( B \) is a tensor with symmetry

\[
\begin{array}{c}
\begin{array}{c}
\mu \otimes \nu \\
\vdots \\
\sigma
\end{array}
\end{array},
\]

(4.1)

then, \( B \) obeys the (first) “Bianchi identity”

\[
\text{Tr} (\ast B) = 0
\]

(4.2)

if and only if \( B \in \Omega^{p,q} (\mathcal{M}) \). This is pictorially described by the diagram

\[
\begin{array}{c}
\begin{array}{c}
\mu \otimes \nu \\
\vdots \\
\sigma
\end{array}
\end{array}
\]

(4.3)

that is, the two columns of the product are attached together.

With all the new artillery introduced in the previous section, it becomes easier to extend the concept of electric-magnetic duality for linearized gravity. First of all we emphasize the analogy between the Bianchi identities and the field equations by rewriting them respectively as

\[
\begin{align*}
\begin{aligned}
\text{Tr} & \ast R = 0 \\
d_L R & = 0 = d_R R
\end{aligned}
\end{align*}
\]

(4.4)

and

\[
\begin{align*}
\begin{aligned}
\text{Tr} & R = 0 \\
d_L (\ast R) & = 0 = d_R (R \ast)
\end{aligned}
\end{align*}
\]

(4.5)

where \( R_{\mu \nu \rho \sigma} \equiv \frac{\mu}{\nu} \otimes \frac{\rho}{\sigma} \). We recall that \( d_L (\ast R) = 0 = d_R (R \ast) \) was obtained in section 2 by using the second Bianchi identity.

As discussed in subsection 3.5 the first Bianchi identity implies that \( R \) effectively has the symmetry properties of the Riemann tensor, i.e. \( R_{\mu \nu \rho \sigma} \equiv \frac{\mu}{\nu} \otimes \frac{\rho}{\sigma} \). Using this symmetry property the two equations \( d_L R = 0 = d_R R \) can now be rewritten as the single equation \( dR = 0 \). Therefore, if the manifold \( \mathcal{M} \) is of trivial topology then, for a given multiform \( R \in \Omega^{2,2} (\mathcal{M}) \), one obtains the equivalence

\[
\begin{align*}
\begin{aligned}
\text{Tr} & \ast R = 0 \\
d_L R & = 0 = d_R R
\end{aligned} \iff \begin{aligned}
R & = d^2 h \\
h & \in \Omega^2 (\mathcal{M})
\end{aligned}
\end{align*}
\]

(4.6)

due to Proposition 1 and Proposition 2.
4.1 Dual linearized Riemann tensor

By Proposition 2, the (vacuum) Einstein equation $\text{Tr}R = 0$ can then be translated into the assertion that the dual of the Riemann tensor has (on-shell) the symmetries of a diagram $(D - 2, 2)$, in other words $*R \in \Omega^{(D-2,2)}_2(M)$. As explained in subsection 2.3, the second Bianchi identity $d_R R = 0$, together with the linearized Einstein equations, implies the equation $d_L * R = 0$. Furthermore the second Bianchi identity $d_R R = 0$ is equivalent to $d_R * R = 0$, therefore we have the equivalence

\[
\begin{cases}
\text{Tr}R = 0 \\
\text{d}_R R = 0
\end{cases} \iff \begin{cases}
* R \in \Omega^{(D-2,2)}_2(M) \\
\text{d}_L * R = 0 = \text{d}_R * R
\end{cases}.
\] (4.7)

In addition $d_L * R = 0 = d_R * R$ now implies $* R = \text{d}_2^2 h$ (where we denote the non-ambiguous product $d_L d_R$ by $\text{d}^2$). The tensor field $\tilde{h} \in \Omega^{(D-3,1)}_2(M)$ is the dual gauge field of $h$ obtained in (2.26). This property (2.26), which holds for manifolds $M$ with the topology of $\mathbb{R}^D$, is a direct application of Corollary 1 of the generalized Poincaré lemma given in the following section; we anticipate this result here in order to motivate the theorem by using a specific example. We have an equivalence analogous to (4.6),

\[
\begin{cases}
\text{Tr}R = 0 \\
\text{d}_L * R = 0 = \text{d}_R * R
\end{cases} \iff \begin{cases}
* R = \text{d}_2^2 \tilde{h} \\
\tilde{h} \in \Omega^{(D-3,1)}_2(M)
\end{cases}.
\] (4.8)

Therefore linearized gravity exhibits a duality symmetry similar to the electric-magnetic duality of electrodynamics, which interchanges Bianchi identities and field equations [6]. Tensor gauge fields in $\Omega^{(D-3,1)}_2(M)$ have mixed symmetry and were discussed above in section 2.4. The right-hand-side of (2.26) is represented by

\[
\begin{array}{c}
\partial \partial \\
\partial & \\
\end{array} \quad \approx \quad \begin{array}{c}
\partial \\
\partial \\
\partial \\
\end{array}.
\]

The appropriate symmetries are automatically implemented by the antisymmetrizations in (2.26) since the dual gauge field $\tilde{h}$ already has the appropriate symmetry

\[
\begin{array}{c}
\partial \\
\partial \\
\end{array} \quad \approx \quad \begin{array}{c}
\partial \\
\partial \\
\end{array}.
\]

In other words, the two explicit antisymmetrizations in (2.26) are sufficient to ensure that the dual tensor $* R$ possesses the symmetries associated with $\gamma^{(2)}_{(D-2,2)}$. A general explanation of this fact will be given at the end of the next section.
The dual linearized Riemann tensor is invariant under the transformation
\[ \delta \tilde{h} = d(S + A) \quad \text{with} \quad S \in \Omega^{(D-4,1)}_{(2)}(\mathcal{M}), \quad A \in \Omega^{(D-3,0)}_{(2)}(\mathcal{M}) \cong \Omega^{D-3}(\mathcal{M}). \] (4.9)

The right-hand side of this gauge transformation, explicitly written in (2.28), is represented by

\[
\begin{array}{ccc}
\bigoplus & \bigoplus^2 & \\
\cdot & \cdot & \cdot \\
\bigotimes & \\
\end{array}
\]

In this formalism, the reducibilities (2.37) and (2.38) respectively read (up to coefficient redefinitions)

\[
(i-1) S = d_L (i) S + d (i) A, \quad (i-1) A = -d_L (i) A, \quad (i = 2, \ldots, D - 2),
\]
\[ S \in \Omega^{(D-3-i,1)}_{(2)}(\mathcal{M}), \quad A \in \Omega^{D-2-i}(\mathcal{M}). \] (4.10)

These reducibilities are a direct consequence of Corollary 2.

4.2 Comparison with electrodynamics

Compared to electromagnetism, linearized gravity presents several new features. First, there are now two kinds of Bianchi identities, some of which are algebraic relations (Bianchi I) while the others are differential equations (Bianchi II). In electromagnetism, only the latter are present. Second (and perhaps more importantly), the equation of motion of linearized gravity theory is an algebraic equation for the curvature (more precisely, \(\text{Tr}R = 0\)). This is natural since the curvature tensor already contains two derivatives of the gauge field. Moreover, for higher spin gauge fields \(h \in \Omega^{(1,\ldots,1)}_{(1)}(\mathcal{M}) (S \geq 3)\) the natural gauge invariant curvature \(d^S h \in \Omega^{(2,\ldots,2)}_{(S,\ldots,2)}(\mathcal{M})\) contains \(S\) derivatives of the completely symmetric gauge field, hence local second order equations of motion cannot contain this curvature. Third, the current conservation in electromagnetism is a direct consequence of the field equation while for linearized gravity the Bianchi identities play a crucial role.

In relation to the first remark, the introduction of sources for linearized gravity seems rather cumbersome to deal with. A natural proposal is to replace the Bianchi I identities by equations

\[
\text{Tr} \ast R = \hat{T}, \quad \hat{T} \in \Omega^{D-3,1}_{(2)}(\mathcal{M}). \] (4.11)

If one uses the terminology of electrodynamics it is natural to call \(\hat{T}\) a “magnetic” source. If such a dual source is effectively present, i.e. \(\hat{T} \neq 0\), the tensor \(R\) is no longer irreducible under \(GL(D, \mathbb{R})\), that is to say \(R\) becomes a sum of tensors of different symmetry types and only one of them has the Riemann tensor symmetries. This seems a difficult starting point. The linearized Einstein equations read

\[
\text{Tr}R = \overline{T}, \quad \overline{T} \in \Omega^{1,1}_{(2)}(\mathcal{M}). \quad \text{(4.12)}
\]
The sources $T$ and $\hat{T}$ respectively couple to the gauge fields $h$ and $\tilde{h}$. The “electric” source $T$ is a symmetric tensor (related to the energy-momentum tensor) if the dual source $\hat{T}$ vanishes, since $R \in \Omega_2^2(M)$ in that case. An other intriguing feature is that a violation of Bianchi II identities implies a non-conservation of the linearized energy-momentum tensor because

$$\partial^\mu T_{\mu\nu} = \frac{3}{2} \partial_{[\mu} R_{\nu\rho]}^{\mu\rho},$$

according to the linearized Einstein equations (2.15).

Let us now stress some peculiar features of $D = 4$ dimensional spacetime. From our previous experience with electromagnetism and our definition of Hodge duality, we naturally expect this dimension to be privileged. In fact, the analogy between linearized gravity and electromagnetism is closer in four dimensions because less independent equations are involved: $\ast R$ has the same symmetries as the Riemann tensor, thus the dual gauge field $\tilde{h}$ is a symmetric tensor in $\Omega_2^2(M)$. So the Hodge duality is a symmetry of the theory only in four-dimensional spacetime. The dual tensor $\ast R$ is represented by a Young diagram of rectangular shape and Proposition 1 can be used to derive the existence of the dual potential as a consequence of the field equation $d \ast R = 0$.

5 Generalized Poincaré lemma

Even if we restrict our attention to completely symmetric tensor gauge field theories, the Hodge duality operation enforced the use of the space $\Omega_{(S)}(M)$ of tensors with at most $S$ columns in the previous section. This unavoidable fact requires an extension of the Proposition 1 to general irreducible tensors in $\Omega_{(S)}(M)$.

5.1 Generalized cohomology

The generalized cohomology\(^{13}\) of the generalized complex $\Omega_{(S)}(M)$ is defined to be the family of graded vector spaces $H_{(m)}(d) = \bigoplus_{Y(S)} H_{(m)}^{(l_1, \ldots, l_S)}(d)$ with $1 \leq m \leq S$ where $H_{(m)}^{(l_1, \ldots, l_S)}(d)$ is the set of $\alpha \in \Omega_{(S)}^{(l_1, \ldots, l_S)}(M)$ such that

$$d^I \alpha = 0 \quad \forall I \subset \{1, 2, \ldots, S\} \mid \#I = m, \quad d^I \alpha \in \Omega_{(S)}(M)$$

with the equivalence relation

$$\alpha \sim \alpha + \sum_{J \subset \{1, 2, \ldots, S\}} d^J \beta_J, \quad \beta_J \in \Omega_{(S)}(M).$$

Let us stress that each $\beta_J$ is a tensor in an irreducible representation of $GL(D, \mathbb{R})$ such that $d^J \beta_J \in \Omega_{(S)}^{(l_1, \ldots, l_S)}(M)$. In other words, each irreducible tensor $d^J \beta_J$ is represented by a specific diagram $Y_{(l_1, \ldots, l_S)}^{(J)}$ constructed in the following way:

\(^{13}\)This definition of generalized cohomology extends the definition of “hypercohomology” introduced in [2].
1st. Start from the Young diagram $Y^{(s)}_{(l_1, \ldots, l_S)}$ of the irreducible tensor field $\alpha$.

2nd. Remove the lowest cell in $S - m + 1$ columns of the diagram, making sure that the reminder is still a Young diagram.

3rd. Replace all the removed cells with cells containing a partial derivative.

The irreducible tensors $\beta_J$ are represented by a diagram obtained at the second step.

A less explicit definition of the generalized cohomology is by the following quotient

$$H_{(m)}(d) = \frac{\bigcap \text{Ker } d^m_{m}}{\sum \text{Im } d^{s-m+1}}.$$  (5.3)

We can now state a generalized version of the Poincaré lemma, the proof of which will be postponed to the next subsection because it is rather lengthy and technical.

**Theorem (Generalized Poincaré lemma)**

Let $Y^{(l_1, \ldots, l_S)}_{(S)}$ be a Young diagram with $l_S \neq 0$ and columns of lengths strictly smaller than $D : l_i < D, \forall i \in \{1, 2, \ldots, S\}$. For all $m \in \mathbb{N}$ such that $1 \leq m \leq S$ one has that

$$H^{(l_1, \ldots, l_S)}_{(m)}(\Omega_{(S)}(\mathbb{R}^D)) \simeq 0.$$  

The theorem extends Proposition 1; the latter can be recovered retrospectively by the fact that, for rectangular tensors, there exists only one $d'\alpha$ and one $\beta_J$.

### 5.2 Applications to gauge theories

In linearized gravity, one considers the action of nilpotent operators $d_i$ on the tensors instead of the distinct operators $d^{(i)}$. However, it is possible to show the useful

**Proposition 5.** Let $\alpha$ be an irreducible tensor of $\Omega_{(S)}(\mathcal{M})$. We have the implication

$$\left( \prod_{i \in I} d_i \right) \alpha = 0, \quad \forall I \subset \{1, 2, \ldots, S\} \mid \# I = m$$

$$\implies \quad d'\alpha = 0, \quad \forall I \subset \{1, 2, \ldots, S\} \mid \# I = m.$$  

Therefore, the conditions appearing in symmetric tensor gauge theories are stronger than the cocycle condition of $H_{(m)}(d)$ and the coboundary property also applies.

Now we present the following corollary which is a specific application of the theorem together with the Proposition 5. Its interest resides in its applicability in linearized gravity field equations (we anticipated the use of this corollary in the previous subsection).
Corollary 1. Let \( \kappa \in \Omega_{(S)}(\mathcal{M}) \) be an irreducible tensor field represented by a Young diagram with at least one row of \( S \) cells and without any column of length \( \geq D - 1 \). If the tensor \( \kappa \) obeys

\[
d_i \kappa = 0 \quad \forall i \in \{1, \ldots, S\},
\]

then

\[
\kappa = \left( \prod_{i=1}^{S} d_i \right) \lambda
\]

where \( \lambda \) belongs to \( \Omega_{(S)}(\mathcal{M}) \) and the tensor \( \kappa \) is represented by a Young diagram where all the cells of the first row are filled by partial derivatives.

Proof: The essence of the proof is that the two tensors with diagrams

\[
\begin{array}{c}
\partial & \partial & \ldots & \partial \\
\vdots & \vdots & \ddots & \vdots \\
\partial & \vdots & \vdots & \partial
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\quad & \vdots & \ddots & \vdots \\
\partial & \vdots & \vdots & \partial
\end{array}
\]

are proportional since the initial symmetrization of the partial derivatives will be cancelled out by the antisymmetrization in the columns that immediately follows for two attached columns of different length (if they have the same length the partial derivatives are in the same row and the symmetrization is automatic). By induction, starting from the left, one proves that this is true for an arbitrary number of columns. This argument remains true if we add smaller columns on the right of the Young diagram.

The last subtlety in the corollary is that antisymmetrization in each column \( (\kappa = \left( \prod d_i \right) \lambda) \) automatically provides the appropriate Young symmetrization since \( \lambda \) has the appropriate symmetry. This can be easily checked by performing a complete antisymmetrization of the tensor \( \kappa \) in the entries of a column and another entry which is on its right. The result automatically vanishes because the index in the column at the right is either

- attached to a partial derivative, in which case the antisymmetrization contains two partial derivatives, or
- attached to the tensor \( \lambda \). In this case, antisymmetrization over the indices of the column except the one in the first row (corresponding to a partial derivative) causes an antisymmetrization of the tensor \( \lambda \) in the entries of a column and another entry which is on the right-hand side. The result vanishes since \( \lambda \) has the symmetry properties corresponding to the diagram obtained after eliminating the first row of \( \kappa \).

\[\square\]
This last discussion can be summarized by the operator formula
\[
\prod_{i=1}^{S} d_i \circ Y^{(S)}_{(l_1, \ldots, l_S)} \propto Y^{(S)}_{(l_1+1, \ldots, l_S+1)} \circ \partial^{S} \circ Y^{(S)}_{(l_1, \ldots, l_S)},
\]
(5.4)
where \(\partial^{S}\) are \(S\) partial derivatives with indices corresponding to the first row of a given Young diagram.

We now present another corollary, which determines the reducibility identities for the mixed symmetry type gauge field.

**Corollary 2.** Let \(\lambda \in \Omega^{(2)}(\mathcal{M})\) be a sum of two irreducible tensors \(\lambda_1 \in \Omega^{(l_1-1,l_2)}(\mathcal{M})\) and \(\lambda_2 \in \Omega^{(l_1,l_2-1)}(\mathcal{M})\) with \(l_1 \geq l_2\) (\(\lambda_1 = 0\) if \(l_1 = l_2\)). Then,
\[
\sum_{i=1}^{2} d^{(i)} \lambda_i = 0 \Rightarrow \lambda_i = \sum_{j=1}^{2} d^{(j)} \mu_{ij} \quad (i = 1, 2),
\]
where
- \(\mu_{11} \in \Omega^{(l_1-2,l_2)}(\mathcal{M})\) (which vanishes if \(l_1 \leq l_2 + 1\)),
- \(\mu_{12}, \mu_{21} \in \Omega^{(l_1-1,l_2-1)}(\mathcal{M})\) (\(\mu_{1,2} = 0\) if \(l_1 = l_2\)), and
- \(\mu_{22} \in \Omega^{(l_1,l_2-2)}(\mathcal{M})\).

Furthermore, if \(l_1 > l_2\) we can assume, without loss of generality, that
\[
\mu_{21} = -\mu_{12}.
\]

**Proof:** We apply \(d^{(1)}\) and \(d^{(2)}\) to the equation \(\sum_{i=1}^{2} d^{(i)} \lambda_i = 0\) and obtain \(0 = d^{1,2} \lambda_i \propto d_1 d_2 \lambda_i\) in view of the remarks following Corollary 1. From the theorem, we deduce that \(\lambda_i = \sum_{j=1}^{2} d^{(j)} \mu_{ij}\) with tensors \(\mu_{ij}\) in the appropriate spaces given in Corollary 2. The fact they vanish agrees with the rule given above.

To finish the proof we should consider the case \(l_1 > l_2\). Assembling the results together, \(\sum_{i=1}^{2} d^{(i)} \lambda_i = d^{(1,2)} (\mu_{12} + \mu_{21}) = 0\) due to Proposition 1. Thus, \(d_1 d_2 (\mu_{12} + \mu_{21}) = 0\). Using Corollary 2 again, one obtains \(\mu_{12} + \mu_{21} = \sum_{k=1}^{2} d^{(k)} \nu_k\) with \(\nu_1 \in \Omega^{(l_1-2,l_2-1)}(\mathcal{M})\) and \(\nu_2 \in \Omega^{(l_1-1,l_2-2)}(\mathcal{M})\) (\(\nu_1 = 0\) if \(l_1 = l_2\)). Hence we can make the redefinitions \(\mu_{12} \rightarrow \mu'_{12} = \mu_{12} - d^{(2)} \nu_2\) and \(\mu_{21} \rightarrow \mu'_{21} = \mu_{21} - d^{(1)} \nu_1\) which do not affect \(\lambda\), in such a way that we have \(\mu'_{21} = -\mu'_{12}\). □

This proposition can be generalized to give a full proof of the gauge reducibility rules given in 2] and will be reviewed in subsection 6.2.
6 Arbitrary Young symmetry type gauge field theories

We now generalize the results of section 4 to arbitrary irreducible tensor representations of $GL(D, \mathbb{R})$. The discussion presented below fits into the approach followed by [18] for two columns ($S = 2$) and by [8, 6] for an arbitrary number of columns. The interest of this section lies in the translation of these old results in the present mathematical language and in the use of the generalized Poincaré lemma for a more systematic mathematical foundation.

6.1 Bianchi identities

Firstly, we generalize our previous discussion on linearized gravity by introducing a tensor $K$, which is the future curvature. A priori, $K$ is a multiform of $\Omega_{[s]}^{l_1, \ldots, l_S} (\mathcal{M})$ ($l_s \neq 0$) with $1 \leq l_j \leq l_i < D$ for $i \leq j$. Secondly, we suppose the (algebraic) Bianchi I relations to be

$$\text{Tr}_{ij} \{ \ast_i K \} = 0, \quad \forall i, j : 1 \leq i < j \leq S,$$

(6.1)

in order to obtain, from Proposition 2, that $K$ is an irreducible tensor under $GL(D, \mathbb{R})$ belonging to $\Omega_{(S)}^{(l_1, \ldots, l_S)} (\mathcal{M})$. Thirdly, we define the (differential) Bianchi II relations as

$$d_i K = 0, \quad \forall i : 1 \leq i \leq S,$$

(6.2)

in such a way that, from Corollary 1, one obtains

$$K = d_1 d_2 \ldots d_S \kappa.$$

(6.3)

In this case, the curvature is indeed a natural object for describing a theory with gauge fields $\kappa \in \Omega_{(l_1-1, \ldots, l_{i-1})}^{(l_1, \ldots, l_S-1)} (\mathcal{M})$. The gauge invariances are then

$$\kappa \rightarrow \kappa + d^{(i)} \beta_i,$$

(6.4)

where the gauge parameters $\beta_i$ are irreducible tensors $\beta_i$ in $\Omega_{(S)}^{(l_1-1, \ldots, l_{i-2}, \ldots, l_S-1)} (\mathcal{M})$ for any $i$ such that $l_i \geq 2$ (and $l_i > l_{i-1}$), as follows from our theorem and Proposition 5.

6.2 Reducibilities

The gauge transformations (6.4) are generally reducible, i.e. $d^{(j)} \beta_j \equiv 0$ for non-vanishing irreducible tensors $\beta_j \neq 0$. The procedure followed in the proof of Corollary 2 can be applied to the general case. This generates the inductive rules of 8 to form the $(i+1)$-th generation reducibility parameters $\beta_{j_1 j_2 \ldots j_{i+1}}$ from the $i$-th generation parameters $\beta_{j_1 j_2 \ldots j_i}$. 
• $i = 1$

(A) Start with the Young diagram $Y^{(S)}_{(l_1-1, \ldots, l_S-1)}$ corresponding to the tensor gauge field $\kappa$.

(B) Remove a box from a row such that the result is a standard Young diagram. In other words, the gauge parameters are taken to be the first reducibility parameters: $\beta_j^{(1)} = \beta_j$.

• $i \to i+1$

(C) Remove a box from a row which has not previously had a box removed (in forming the lower generations of reducibility parameters) such that the result is a standard Young diagram.

(D) There is one and only one reducibility parameter for each Young diagram.

The Labastida-Morris rules (A)-(D) provide the complete BRST spectrum with the full tower of ghosts of ghosts. More explicitly, the chain of reducibilities is

$$\beta^{(i)}_{j_1j_2 \ldots j_i} = d^{(j_{i+1})} \beta^{(i+1)}_{j_1j_2 \ldots j_{i+1}} = 0, \quad (i = 1, 2, \ldots, r) \quad (6.5)$$

where $r = l_1 - 1$ is the number of rows of $\kappa$. The chain is of length $r$ because at each step one removes a box from a row which has not been chosen before. We can see that the order of reducibility of the gauge transformations (6.4) is equal to $l_1 - 2$. For $S = 1$, we recover the fact that a $p$-form gauge field theory $(l_1 = p + 1)$ has its order of reducibility equal to $p - 1$.

The subscripts of the $i$-th reducibility parameter $\beta^{(i)}_{j_1j_2 \ldots j_i}$ belong to the set $\{1, \ldots, S\}$. These determine the Young diagram corresponding to the irreducible tensor $\beta^{(i)}_{j_1j_2 \ldots j_i}$: reading from the left to the right, the subscripts give the successive columns from which to remove the bottom box following the rules (A)-(C). A reducibility parameter $\beta^{(i)}_{j_1j_2 \ldots j_i}$ vanishes if these rules are not fulfilled. Furthermore, they are antisymmetric for any pair of different indices

$$\beta^{(i)}_{j_1 \ldots j_k \ldots j_i} = - \beta^{(i)}_{j_1 \ldots j_i \ldots j_k}, \quad \forall j_i \neq j_k. \quad (6.6)$$

This property ensures the rule (D) and provides the correct signs to fulfill the reducibilities. Indeed,

$$d^{(j_{i+1})} \beta^{(i)}_{j_1j_2 \ldots j_i} = d^{(j_{i+1})} d^{(j_i)} \beta^{(i+1)}_{j_1j_2 \ldots j_{i+1}} = 0, \quad (6.7)$$

due to Proposition 4 and equation (6.6).
6.3 Field equations and dualisation properties

We make the important following assumption concerning the positive integers \( l_i \) \( (i = 1, \ldots, S) \) associated to \( K \in \Omega^{l_1, \ldots, l_S}_{[S]} \):

\[
l_i + l_j \leq D, \quad \forall i, j
\]

and take the field equations to be in that case

\[
\text{Tr}_{ij}\{K\} = 0, \quad \forall i, j.
\]

Indeed, if \( l_1 + l_2 > D \) and if the equation (6.9) holds, then the curvature \( K \) identically vanishes, as is well known when studying irreps of \( O(D-1, 1) \). This property is a particular instance of Proposition 3 (for \( m = 0 \)).

The field equations (6.9) combined with the Bianchi I identities (6.1) state that the curvature \( K \) is a tensor irreducible under \( O(D-1, 1) \).

To any non-empty subset \( I \subset \{1, 2, \ldots, S\} \) \( (#I = m) \), we associate a Hodge duality operator

\[
*_{I} \equiv \prod_{k \in I} *_{k}.
\]

The dual \( *_{I}K \) of the curvature is a multiform in \( \Omega^{l_1, \ldots, l_S}_{[S]}(\mathcal{M}) \), where the lengths \( \ell_i \) are defined in equation (3.8).

The Bianchi I identities (6.1) together with the field equations (6.9) imply the relations

\[
\text{Tr}_{ij}\{*_{i}(*_{J}K)\} = 0, \quad \forall i, j : \ell_j \leq \ell_i
\]

where \( \ell_i \) is the length (3.8) of the i-th column of the dual tensor \( *_{I}K \). Indeed, let be \( i \) and \( j \) such that \( \ell_j \leq \ell_i \). There are essentially four possibilities:

- \( i \notin I \) and
  - \( j \notin I \): Then \( l_j \leq l_i \) and the Bianchi I identities (6.1) are equivalent to (6.11) since \( \text{Tr}_{ij} \) and \( *_{k} \) commute if \( i \neq k \) and \( j \neq k \).
  - \( j \in I \): Then one should have \( D - l_j \leq l_i \) which means that \( D \leq l_i + l_j \), in contradiction with the hypothesis (6.8) except for the case where \( l_i + l_j = D \). From Proposition 3 we deduce that in such a case the field equations (6.9) are equivalent to (6.11).

- \( i \in I \) and
  - \( j \notin I \): We have \( l_j \leq D - l_i \) which is equivalent to \( l_i + l_j \leq D \). The field equations (6.9) are of course equivalent to (6.11) since \( *_{i}K = \pm K \).
  - \( j \in I \): We have \( D - l_j \leq D - l_i \) which is equivalent to \( l_i \leq l_j \). The Bianchi I identities \( \text{Tr}_{j}i\{*_{J}K\} = 0 \) are therefore satisfied and equivalent to (6.11) because \( \text{Tr}_{ij} = \text{Tr}_{ji} \).
Let $\tilde{Y}^{(S)}_{(\lambda_1, \ldots, \lambda_S)}$ be the Young diagram dual to $Y^{(S)}_{(1, \ldots, S)}$. We define $\tilde{K}_I$ to be the multiform in $\Omega^{(\lambda_1, \ldots, \lambda_S)}(\mathcal{M})$ obtained after reordering the columns of $*_I K$. The identity (6.11) can then be formulated as

$$\text{Tr}_{ij} \{ *_I \tilde{K}_I \} = 0, \forall \, i, j : 1 \leq i < j \leq S. \quad (6.12)$$

Due to Proposition 2, it follows from (6.12) that the tensor $\tilde{K}_I$ is irreducible under $GL(D, \mathbb{R})$:

$$\tilde{K}_I \in \Omega^{(\lambda_1, \ldots, \lambda_S)}(\mathcal{M}). \quad (6.13)$$

Now we use the property (3.22) to deduce from the Bianchi II identities (6.2) and the field equations (6.9) that $d_i (*_I K) = 0$ for any $i$. Therefore

$$d_i (*_I K) = 0, \forall \, i \in \{1, \ldots, S\}, \quad (6.14)$$

because $d_i$ and $*_j$ commute if $i \neq j$, and either

- $i \notin I$ so (6.14) follows from $d_i K = 0$, or
- $i \in I$ and then (6.14) is a consequence of $d_i *_i K = 0$.

In other words, any dual tensor $\tilde{K}_I$ satisfies (on-shell) its own Bianchi II identity (6.14) which, together with (6.13), implies the (local) existence of a dual gauge field $\tilde{\kappa}_I$ such that the Hodge dual of the curvature is itself a curvature

$$\tilde{K}_I = d_1 d_2 \ldots d_S \tilde{\kappa}_I \quad (6.15)$$

for some gauge field

$$\tilde{\kappa}_I \in \Omega^{(\lambda_1 - 1, \ldots, \lambda_S - 1)}(\mathcal{M}). \quad (6.16)$$

The Hodge operators $*_I$ therefore relate different free field theories of arbitrary tensor gauge fields, extending the electric-magnetic duality property of electrodynamics.

In the same way, we obtain the field equations of the dual theory

$$\text{Tr}_{ij} \{ *_I K \} = 0, \forall \, i, j : i < j \quad (6.17)$$

where

$$m_{ij} \equiv \begin{cases} 1 + D - l_i - l_j & \text{if } i \text{ and } j \in I, \\ 1 & \text{if } i \text{ or } j \notin I. \end{cases} \quad (6.18)$$

Indeed, since the trace is symmetric in $i$ and $j$ we must consider only three distinct cases:

- $i \notin I$ and $j \notin I$: The starting field equation (6.9) is naturally equivalent to the dual field equation (6.17).
- $i \in I$ and
  - $j \notin I$: If $i < j$ the Bianchi I relation (6.1) is satisfied and it implies (6.17).
  - $j \in I$: A direct use of Proposition 3 leads from the field equation (6.9) to (6.17).
We can summarize the algebraic part of the previous discussions in terms of a remark on tensorial irreps of $O(D-1,1)$.

**Remark:** Let $I \subset \{1, \ldots, S\}$ be a non-empty subset. Let $\tilde{Y}^I_{(\lambda_1, \ldots, \lambda_S)}$ be the Young diagram dual to $Y^I_{(l_1, \ldots, l_S)}$. If $\alpha \in \Omega^{(l_1, \ldots, l_S)}(\mathcal{M})$ is a tensor in the irreducible representation of $O(D-1,1)$ associated to the Young diagram $Y^I_{(l_1, \ldots, l_S)}$, then the dual tensor $\tilde{\alpha}^I \in \Omega^{(\lambda_1, \ldots, \lambda_S)}(\mathcal{M})$ is in the irreducible representation of $O(D-1,1)$ associated to the Young diagram $\tilde{Y}^I_{(\lambda_1, \ldots, \lambda_S)}$.

As one can see, a seemingly odd feature of some dual field theories is that their field equations (6.17) are not of the same type as (6.9). In fact, the dual field equations are of the type (6.9) for all $I$ only in the exceptional case where $D$ is even and $l_i = l_j = D/2$. Note that this condition is satisfied for free gauge theories of completely symmetric tensors in $D = 4$ flat space. The point is that $\ell_i + \ell_j = 2D - l_i - l_j \geq D$ for $i, j \in I$, therefore the property (6.8) is generally not satisfied by the dual tensor $\tilde{K}^I$.

To end up, we generalize the field equation (6.9) to the case where the hypothesis (6.8) is not satisfied. A natural idea is that when $l_i + l_j > D$ for a curvature tensor $K \in \Omega^{(l_1, \ldots, l_S)}(\mathcal{M})$ ($l_S \neq 0$), the corresponding fields equations are

$$\text{Tr}_{ij}^{1+l_i+l_j-D} \{ K \} = 0.$$  (6.19)

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**A Inductive proof of the generalized Poincaré lemma**

The proof of the generalized Poincaré lemma that we give hereafter is inductive in several directions. The first induction parameter is the number $S$ of columns; in section A.1 we start from the standard Poincaré lemma, i.e. $S = 1$, and compute the generalized cohomologies when a cell is added in a new, second column, i.e. $S = 2$. The second induction parameter is the number $\ell$ of cells in the new (second) column. Thus section A.1 which gives a (pictorial) proof that the cohomological groups $\Omega^{(s,1)}(\mathbb{R}^D)$ are trivial, also provides the starting point for the induction on $\ell$, keeping $S = 2$ fixed.

The inductive proof of the vanishing of $H^{(s,s)}_{(\ell)}(\Omega(\mathbb{R}^D))$ is then given in sections A.2. This proof of $H^{(s,s)}_{(\ell)}(\Omega(\mathbb{R}^D)) \cong 0$ is purely algebraic and does not contain any
pictorial description, this time. However, for a better understanding of the algebraic demonstration, a pictorial translation of most of the results obtained in section A.2 is furnished in section A.3.

The inductive progression we have sketched above is the one used to obtain the proof of the vanishing \( H^{(\ast,\ldots,\ast)}(\Omega_{(s)}(\mathbb{R}^D)) \cong 0 \), for diagrams obeying the assumptions of the Theorem (section 5). This time, instead of progressing from \( S = 1 \) to \( S = 2 \) and then from a length-(\( \ell - 1 \)) to a length-\( \ell \) second column, we go from \( S \) to \( S + 1 \) columns and subsequently, keeping the number of columns fixed to \( S + 1 \), we increase the length of the last \( (S + 1) \)-th column.

Since this progression, exposed in detail in sections A.1, A.2 and A.3 provides the proof of our generalized Poincaré lemma, we only summarize those results in section A.4 and cast our Theorem in precise mathematical terms.

**A.1 Generalized cohomology in \( \Omega_{(2)}^{(\ast,1)}(\mathbb{R}^D) \)**

Using the standard Poincaré lemma \( (S = 1) \), we begin by providing a pictorial proof that the two cohomologies \( H^{(n,1)}_{(1)}(\Omega_{(2)}(\mathbb{R}^D)) \) and \( H^{(n,1)}_{(2)}(\Omega_{(2)}(\mathbb{R}^D)) \) are trivial for \( 0 < n < D \), i.e. that

- (1)
  \[
  d^{(i)} \begin{array}{c}
  \frac{1}{2} \\
  \vdots \\
  \frac{n}{2} \\
  \end{array} = 0 , \quad i = 1, 2 \quad (A.1)
  \]
  implies
  \[
  \begin{array}{c}
  \frac{1}{2} \\
  \vdots \\
  \frac{n}{2} \\
  \end{array} = \begin{array}{c}
  1 \\
  \vdots \\
  0 \\
  \end{array} \quad (A.2)
  \]
  and
  - (2)
  \[
  d^{(1,2)} \begin{array}{c}
  \frac{1}{2} \\
  \vdots \\
  \frac{n}{2} \\
  \end{array} = 0 \quad (A.3)
  \]
  implies
  \[
  \begin{array}{c}
  \frac{1}{2} \\
  \vdots \\
  \frac{n}{2} \\
  \end{array} = \begin{array}{c}
  1 \\
  \vdots \\
  0 \\
  \end{array} + \begin{array}{c}
  1 \\
  \vdots \\
  0 \\
  \end{array} \quad (A.4)
  \]
The numbers in the cells are irrelevant, they simply signal the length of the columns. For clarity we recall the following convention that, whenever a Young tableau $Y$ appears with certain boxes filled in with partial derivatives $\partial$, one takes a field with the representation of the Young tableau obtained by removing all the $\partial$-boxes from $Y$. One differentiates this new field as many times as there are derivatives in $Y$ and then project the result on the Young symmetry of $Y$.

**First cohomology group**

For the two different possible values of $i$ in (A.1) we have the following two conditions on the field $(n, 1)$:

- $\begin{bmatrix} 1 & n+1 \\ 2 \\ : \\ n \\ \partial \end{bmatrix} = 0$ for $i = 1$

and

- $\begin{bmatrix} 1 & n+1 \\ : \\ \partial \\ n \end{bmatrix} = 0$ for $i = 2$.

The first condition is treated now: one considers the index of the second column as a spectator, which yields

$$\begin{bmatrix} 1 & n+1 \\ 2 \\ : \\ n \\ \partial \end{bmatrix} \simeq \begin{bmatrix} 1 \\ 2 \\ : \\ n \\ \partial \end{bmatrix} = 0$$

where the symbol $\simeq$ means that there is an implicit projection using $Y$ on the right-hand side in order to agree with the left-hand side (in other words the symbol $\simeq$ replaces the expression $= Y$). The Poincaré Lemma is used for the first column to write, using branching rules for $GL(D, \mathbb{R})$:

$$\begin{bmatrix} 1 & n+1 \\ 1 \\ 2 \\ : \\ n \\ \partial \end{bmatrix} \simeq \begin{bmatrix} 1 \\ 2 \\ : \\ n \\ \partial \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \\ : \\ n \\ \partial \end{bmatrix} = 0$$

In the last step, we have undone the manifest antisymmetrization with the index carrying the partial derivative; we are more interested in the symmetries of the tensor under the derivative.
We first perform the product in the brackets to obtain a sum of different types of irreducible tensors. Then, we perform the product with the partial derivative to get

\[
\begin{array}{cccc}
1 & n+1 \\
2 & \\
\vdots & \\
n & \\
\end{array} \quad \sim \quad \begin{array}{cccc}
1 & \partial \\
2 & \\
\vdots & \\
n & \partial \\
\end{array} \quad \oplus \quad \begin{array}{cccc}
1 & n \\
2 & \\
\vdots & \\
n & \partial \\
\end{array} \quad \oplus \quad \begin{array}{cccc}
1 & \\
2 & \\
\vdots & \\
n & \\
\end{array}.
\]

(A.5)

The last term in the above equation (A.5) does not match the symmetry of the left-hand-side, so it must vanish. Using the Poincaré lemma, which is applicable since one is not in top form degree: \( n < D \), one obtains

\[
\begin{array}{cccc}
1 & \\
2 & \\
\vdots & \\
n & \\
\end{array} = \begin{array}{cccc}
1 & \\
2 & \\
\vdots & \\
n & \\
\end{array}.
\]

(A.6)

Substituting this result in the decomposition (A.5) yields

\[
\begin{array}{cccc}
1 & n+1 \\
2 & \\
\vdots & \\
n & \\
\end{array} \quad \sim \quad \begin{array}{cccc}
1 & \partial \\
2 & \\
\vdots & \\
n & \partial \\
\end{array} \quad \oplus \quad \begin{array}{cccc}
1 & n \\
2 & \\
\vdots & \\
n & \partial \\
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
1 & n \\
2 & \\
\vdots & \\
n & \\
\end{array}
\]

(A.7)

where the arrow indicates that we performed a field redefinition. Thus, without loss of generality, the right-hand-side can be assumed to contain a partial derivative in the first column. With this preliminary result, the second condition expressed in (A.1),

\[
d^{(2)} \begin{array}{cccc}
1 & n+1 \\
2 & \\
\vdots & \\
n & \\
\end{array} \equiv \begin{array}{cccc}
1 & n+1 \\
2 & \partial \\
\vdots & \\
n & \\
\end{array} = 0,
\]

(A.8)

gives

\[
\begin{array}{cccc}
1 & n \\
2 & \partial \\
\vdots & \\
n-1 & \\
\end{array} = 0.
\]

(A.9)

The Poincaré lemma on the second column leads to

\[
\begin{array}{cccc}
1 & n \\
2 & \\
\vdots & \\
n-1 & \\
\end{array} \quad \sim \quad \partial \otimes \begin{array}{cccc}
1 & \\
2 & \\
\vdots & \\
n & \partial \\
\end{array} \quad \sim \quad \begin{array}{cccc}
1 & \\
2 & \\
\vdots & \\
n & \partial \\
\end{array} \quad \oplus \quad \begin{array}{cccc}
1 & \partial \\
2 & \\
\vdots & \\
n & \\
\end{array}
\]

(A.10)
The first totally antisymmetric component vanishes since there is no component with the same symmetry on the left-hand-side, implying that

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
n-1 \\
n \\
\partial
\end{array} = \begin{array}{c}
1 \\
2 \\
\vdots \\
n-1 \\
n \\
\partial
\end{array} = 0
\] (A.11)

which in turn, substituted into (A.10), gives

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
n-1 \\
n \\
\partial
\end{array} = \begin{array}{c}
1 \\
2 \\
\vdots \\
n-1 \\
n \\
\partial
\end{array} = 0
\] (A.12)

Substituting this result in (A.7) proves (A.2).

**Second cohomology group**

We now turn to the proof that (A.3) implies (A.4). The condition (A.3) reads

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n+1 \\
\partial
\end{array} = 0
\] (A.13)

whose type was already encountered in (A.9) above. We use our previous result (A.12) and write

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n+1 \\
\partial
\end{array} = \begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n \\
\partial
\end{array} = 0
\] (A.14)

or

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n+1 \\
\partial
\end{array} - \begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n \\
\partial
\end{array} = 0
\] (A.15)

This kind of equation was also found before, in (A.1), when i=1. Then we are able to write

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n+1 \\
\partial
\end{array} - \begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n \\
\partial
\end{array} = \begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n \\
\partial
\end{array} = 0
\] (A.16)
which is the analogue of (A.7). Equivalently,

\[
\begin{array}{c}
1 \quad n_1 \\
2 \quad n_2 \\
\vdots \quad \vdots \\
n \quad n_n \\
\end{array}
\begin{array}{c}
1 \quad \partial \\
2 \quad \partial \\
\vdots \quad \vdots \\
\partial \quad \partial \\
\end{array}
= \begin{array}{c}
1 \quad n_1 \\
2 \quad n_2 \\
\vdots \quad \vdots \\
n \quad n_n \\
\end{array}
\begin{array}{c}
1 \quad \partial \\
2 \quad \partial \\
\vdots \quad \vdots \\
\partial \quad \partial \\
\end{array}
\begin{array}{c}
1 \quad n_1 \\
2 \quad n_2 \\
\vdots \quad \vdots \\
n \quad n_n \\
\end{array}
\begin{array}{c}
1 \quad \partial \\
2 \quad \partial \\
\vdots \quad \vdots \\
\partial \quad \partial \\
\end{array}
\]

(A.17)

which is the desired result.

### A.2 Generalized cohomology in \( \Omega^{(*,*)}_{(2)}(\mathbb{R}^D) \)

Here we proceed by induction on the number of boxes in the last (second) column. We will temporarily leave the diagrammatic exposition. For an easier understanding of the following propositions, we sketch a pictorial translation of the proof that \( H_{(2)}^{(n,l)}(\Omega_{(2)}(\mathbb{R}^D)) \cong 0 \), \( 0 < l < n < D \) in subsection A.3.

**Induction hypothesis \( S_\ell \) :** Suppose that the three following statements hold :

- \( d^{(1)} \mu(l_1, l_2) = 0 \Rightarrow \mu(l_1, l_2) = d^{(1)} \nu(l_1 - 1, l_2) \), \( (A.18) \)
- \( H^{(l_1, l_2)}_{(1)}(\Omega_{(2)}(\mathbb{R}^D)) \cong 0 \), \( (A.19) \)
- \( H^{(l_1, l_2)}_{(2)}(\Omega_{(2)}(\mathbb{R}^D)) \cong 0 \), \( (A.20) \)

where \( 0 < l_1 < D \), \( 0 < l_2 < \ell \leq l_1 \) and where the notation \( \mu(l_1, l_2) \) indicates that \( \mu \in \Omega^{(l_1, l_2)}_{(2)}(\mathbb{R}^D) \), similarly \( \nu \in \Omega^{(l_1 - 1, l_2)}_{(2)}(\mathbb{R}^D) \). The integer \( \ell \) is fixed and is our induction parameter.

The induction hypothesis \( S_\ell \) is that one knows the cohomology of \( d^{(1)} \) and the generalized cohomology for all tensors whose second column has length strictly smaller than \( \ell \). What we showed in the above section A.1 constitutes the “initial conditions \( S_2 \)” of our induction proof. The Poincaré lemma actually constitutes \( S_1 \).

To prove \( S_\ell \Rightarrow S_{\ell + 1} \) amounts to show that we have the three assertions \( (A.18), (A.19) \) and \( (A.20) \) with the new conditions \( 0 < l_1 < D \), \( 0 < l_2 \leq \ell \leq l_1 \), i.e. where the second column is now allowed to have length \( l_2 = \ell \). 14

These three assertions (with the new conditions on the lengths of the columns) are proved in the following and lead to the result that \( H^{(l_1, l_2)}_{(s)}(\Omega_{(2)}(\mathbb{R}^D)) \cong 0 \) for any \( (l_1, l_2) \in \mathcal{Y}(2) \).

Before starting these three proofs and for later purposes, we introduce a total order relation in the space \( \Omega_{(2)}(\mathbb{R}^D) \), naturally induced by the total order relation (3.7) for \( \mathcal{Y}(S) \).

If \( \alpha(l_1, \ldots, l_S) \) and \( \beta(l'_1, \ldots, l'_S) \) belong to \( \Omega^{(l_1, \ldots, l_S)}_{(S)}(\mathbb{R}^D) \) and \( \Omega^{(l'_1, \ldots, l'_S)}_{(S)}(\mathbb{R}^D) \), respectively, then

\[
\alpha(l_1, \ldots, l_S) \ll \beta(l'_1, \ldots, l'_S)
\]

(A.21)

14 The case where the fixed induction parameter satisfies \( \ell = l_1 \) is a little bit particular, so will have to be treated separately.
if and only if

\[ l_k = l'_k, \ 1 \leq k \leq K, \ \text{and} \]
\[ l_{K+1} \leq l'_{K+1}, \]

where \( K \) is an integer satisfying \( 1 \leq K \leq S \). This lexicographic ordering induces a grading in \( \Omega(S)(\mathbb{R}^D) \), that we call “L-grading”. This L-grading is a generalization to \( \Omega(S)(\mathbb{R}^D) \) of the form-grading in \( \Omega(1)(\mathbb{R}^D) \). In the sequel we will use hatted symbols to denote multiforms of \( \Omega(S)(\mathbb{R}^D) \), while the unhatted tensors belong to \( \Omega(S)(\mathbb{R}^D) \).

Turning back to our inductive proof, we begin with the

**Lemma 2.** If \( S_l \) is satisfied, then

\[ d^{(1)} \mu(l_1, \ell) = 0, \ 0 < l_1 < D \implies \mu(l_1, \ell) = d^{(1)} \nu(l_1 - 1, \ell). \]  

**Proof:**

(1) \( \ell < l_1 \).

Applying the Poincaré lemma to the cocycle condition in Eqn [A.23], viewing the second column as a spectator, yields \( \mu(l_1, \ell) \simeq d_1 \nu(l_1 - 1, \ell) \), where \( \nu(l_1 - 1, \ell) \in \Omega^{l_1-1,\ell} \). Decomposing the right-hand-side (expressed in terms of multiforms) into irrep. of \( GL(D, \mathbb{R}) \) gives

\[ \mu(l_1, \ell) \simeq d^{(1)} \nu(l_1 - 1, \ell) + d^{(2)} \nu(l_1, \ell - 1) + d^{(1)} \nu(l_1, \ell - 1) + d^{(2)} \nu(l_1 + 1, \ell - 2) + (\ldots), \]

where \((\ldots)\) denotes tensors of higher L-grading. Because the third and fourth terms do not belong to \( \Omega^{l_1,\ell} \), they must cancel:

\[ d^{(1)} \nu(l_1, \ell - 1) + d^{(2)} \nu(l_1 + 1, \ell - 2) = 0. \]  

Applying the operator \( d^{(2)} \) to [A.25] gives \( d^{(1,2)} \nu(l_1, \ell - 1) = 0 \). The induction hypothesis \( S_l \) allows us to write \( \nu(l_1, \ell - 1) = d^{(1)} \nu(l_1 - 1, \ell - 1) + d^{(2)} \nu(l_1, \ell - 2) \). Substituting back in the decomposition of \( \mu(l_1, \ell) \), we find, after the redefinition \( \nu'(l_1 - 1, \ell) = \nu(l_1 - 1, \ell) + d^{(2)} \nu(l_1 - 1, \ell - 1) \), the result we were looking for:

\[ \mu(l_1, \ell) = d^{(1)} \nu'(l_1 - 1, \ell). \]  

(2) The case \( \ell = l_1 \) can be analyzed in the same way.

The equation

\[ d^{(1)} \mu(l_1, l_1) = 0, \ 0 < l_1 < D \]

gives, after applying the standard Poincaré lemma, that \( \mu(l_1, l_1) \simeq d_1 \nu(l_1 - 1, l_1) \), \( \nu(l_1 - 1, l_1) \in \Omega^{l_1-1,l_1} \simeq \Omega^{l_1-1,l_1-1} \). In terms of irrep. of \( GL(D, \mathbb{R}) \), we get \( \mu(l_1, l_1) \simeq d^{(2)} \nu(l_1, l_1 - 1) + d^{(1)} \nu(l_1, l_1 - 1) + d^{(2)} \nu(l_1 + 1, l_1 - 2) + (\ldots) \) where \((\ldots)\) denote terms of higher L-grading. The sum of the second and third terms must vanish, and applying \( d^{(2)} \) gives \( d^{(1,2)} \nu(l_1, l_1 - 1) = 0 \). By virtue of our hypothesis of induction
Starting with $S_\ell$, we obtain $\nu(l_1, l_1 - 1) = d^{(1)}\nu(l_1 - 1, l_1 - 1) + d^{(2)}\nu(l_1, l_1 - 2)$. Here, the result which emerges after substituting the above equation in the decomposition of $\mu(l_1, l_1)$ and preforming a field redefinition, is

$$\mu(l_1, l_1) = d^{(1, 2)}\nu(l_1 - 1, l_1 - 1). \quad (A.28)$$

Note that the case $\ell = l_1$ gave us for free

**Proposition 6.** If $S_\ell$ is satisfied, then $H^{(\ell, \ell)}(\Omega_2(\mathbb{R}^D)) \cong 0$.

Having the Lemma 2 at our disposal, we now proceed to prove

**Proposition 7.** If $S_\ell$ is satisfied, then $H^{(l_1, \ell)}(\Omega_2(\mathbb{R}^D)) \cong 0$.

It states that the cocycle conditions

$$d^{(i)}\mu(l_1, \ell) = 0, \quad i \in \{1, 2\}, \quad 0 < l_1 < D \quad (A.29)$$

imply that

$$\mu(l_1, \ell) = d^{(1, 2)}\nu(l_1 - 1, \ell - 1). \quad (A.30)$$

**Proof:**

1. $\ell < l_1$.

   In the case $i = 1$, the conditions $(A.29)$ give, using Lemma 2, that
   $$\mu(l_1, \ell) = d^{(1)}\nu(l_1 - 1, \ell). \quad (A.31)$$

   Substituting this into the condition $(A.29)$ for $i = 2$ yields
   $$d^{(1, 2)}\nu(l_1 - 1, \ell) = 0. \quad (A.32)$$

   Using the Poincaré lemma on the second column, we have
   $$d^{(1)}\nu(l_1 - 1, \ell) \cong d_1\nu(l_1 - 1, \ell) \cong d^{(2)}\nu(l_1, \ell - 1) + d^{(1)}\nu(l_1, \ell - 1) + d^{(2)}\nu(l_1 + 1, \ell - 2) + (\ldots), \quad (A.33)$$

   where, as before, we used the branching rules for $GL(D, \mathbb{R})$ and $(\ldots)$ corresponds to terms of higher L-grading. The sum of the second and third terms of the right-hand side must vanish, as does the action of $d^{(2)}$ on it. As a consequence, $\nu(l_1, \ell - 1) = d^{(1)}\nu(l_1 - 1, \ell - 1) + d^{(2)}\nu(l_1, \ell - 2)$, hence $d^{(1)}\nu(l_1 - 1, \ell) = d^{(1, 2)}\nu(l_1 - 1, \ell - 1)$. Substituting this into $(A.31)$ we finally have
   $$\mu(l_1, \ell) = d^{(1, 2)}\nu(l_1 - 1, \ell - 1). \quad (A.34)$$

   which proves the proposition.

2. $\ell = l_1$.

   This case was already obtained in Proposition 6. \[\square\]
The following vanishing of cohomology still remains to be shown:

**Proposition 8.** If $S_\ell$ is satisfied, then $H_{(2)}^{(l_1,\ell)}(\Omega_2(\mathbb{R}^D)) \cong 0$.

**Proof:** The cocycle condition with $\ell < l_1$ has in fact already been encountered in (A.32). We can then use the results already obtained in the proof of the Proposition 7 to write that the cocycle condition

$$d^{(1,2)}\mu(l_1, \ell) = 0, \quad 0 < l_1 < D, \ \ell \neq l_1$$

leads to $d^{(1)}\mu(l_1, \ell) = d^{(1,2)}\mu(l_1, \ell - 1)$. Rewriting this equation as $d^{(1)}\mu(l_1, \ell) - d^{(2)}\mu(l_1, \ell - 1) = d^{(1)}\mu(l_1 - 1, \ell)$, i.e.

$$\mu(l_1, \ell) = d^{(2)}\mu(l_1, \ell - 1) + d^{(1)}\mu(l_1 - 1, \ell).$$

Had we started with the cocycle condition $d^{(1,2)}\mu(l_1, \ell = l_1) = 0, \ 0 < l_1 < D$, we would have found $d^{(1)}\mu(l_1, l_1) = d^{(1,2)}\mu(l_1, l_1 - 1)$, then $\mu(l_1, l_1) - d^{(2)}\mu(l_1, l_1 - 1) = d^{(1,2)}\mu(l_1 - 1, l_1 - 1)$, and after a field redefinition, the result

$$\mu(l_1, l_1) = d^{(2)}\mu(l_1, l_1 - 1)$$

which is the coboundary condition analogous to (A.36) in the case of maximally filled tensors in $\Omega_2(\mathbb{R}^D)$.

**Conclusions**

Our inductive proof provided us with the following results about the generalized cohomologies of $d^{(i)}$, $i \in \{1, 2\}$, in the space $\Omega_2$:

$$H_{(s)}^{(l_1,l_2)}(\Omega_2(\mathbb{R}^D)) \cong 0, \ \forall (l_1, l_2) \in \mathbb{Y}(2), l_2 \neq 0.$$  \hfill (A.38)

**A.3 Diagrammatical presentation**

The pictorial translation of Lemma 2, which proved to be crucial in proving $H_{(1)}^{(n,l)}(\Omega_2(\mathbb{R}^D)) \cong 0$, reads

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ 1 \\
\vdots \\
1 \\
\vdots \\
0
\end{array}
\end{array}
\end{array}
\end{array}
\quad = 0 \quad \Rightarrow \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ 1 \\
\vdots \\
1
\end{array}
\end{array}
\end{array}
\end{array}
\quad =
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ 1 \\
\vdots \\
1 \\
\vdots \\
0
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}.$$  \hfill (A.39)
For practical purposes and for simplicity, we fix \( l = 2 \). Using the standard Poincaré lemma, one obtains

\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n \\
\partial \\
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n \\
\partial \\
\end{array}
\end{array}
\simeq \partial \otimes \left( \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array} \right)
\]
\tag{A.40}
\]

i.e.

\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array}
\simeq \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array} + \partial \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array} + \partial \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array}
\]
\tag{A.41}
\]

The condition that the sum of the third and fourth terms of the right-hand side vanishes, implies, due to the induction hypothesis, that the tensor in the second term can be written as

\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array}
\simeq \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array} + \partial \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array}
\]
\tag{A.42}
\]

which, substituted into (A.41), gives

\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array}
\simeq \begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n-1 \\
\partial \\
\end{array}
\end{array}
\]
\tag{A.43}
\]

Once Lemma 2 is obtained, we turn to the pictorial description of the proof for \( H^{(n,l)}(\Omega^{(2)}(D)) \cong 0 \). The first cocycle condition

\[
d^{(1)}\alpha(n,l) = 0
\]
\tag{A.44}
\]

is represented by

\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n \\
\partial \\
\end{array}
\end{array}
\]
\tag{A.45}
\]

Using Lemma 2, its solution is (as we showed pictorially in the case where \( l = 2 \)),

\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{2} \\
\vdots \\
n \\
\partial \\
\end{array}
\end{array}
\]
\tag{A.46}
\]
Substituting in the second cocycle condition

\[ d^{(2)} \alpha(n, l) = 0 \quad (A.47) \]
yields

\[ \begin{array}{c c c c c c c c c c c}
1 & 1 & \\
\vdots & \vdots & \\
1 & l & \\
\vdots & \vdots & \\
n-1 & \partial & \\
\end{array} = 0 . \quad (A.48) \]

Applying the Poincaré lemma on the second column, viewing the first one as a spectator, gives

\[ \begin{array}{c c c c c c c c c c}
1 & 1 & \\
\vdots & \vdots & \\
1 & l & \\
\vdots & \vdots & \\
n-1 & \partial & \\
\end{array} \cong \begin{array}{c c c c c c c c c c}
\partial \otimes \left( \begin{array}{c c c c c c c c c c}
1 & 1 & \\
\vdots & \vdots & \\
1 & n & \\
\vdots & \vdots & \\
1 & l-1 & \\
\end{array} \right) \\
\end{array} \cong \begin{array}{c c c c c c c c c c}
1 & 1 & \\
\vdots & \vdots & \\
1 & l-1 & \\
\vdots & \vdots & \\
n-1 & \partial & \\
\end{array} \oplus \begin{array}{c c c c c c c c c c}
1 & 1 & \\
\vdots & \vdots & \\
1 & l-2 & \\
\vdots & \vdots & \\
n-1 & \partial & \\
\end{array} \oplus \cdots \quad (A.49) \]

where the dots in the above equation correspond to tensors of higher order \( L \)-grading (whose first column has a length greater or equal to \( n + 2 \)). The second and third terms must cancel because they do not have the symmetry of the left-hand side. Applying \( d^{(2)} \) on the sum of the second and third term and using our hypothesis of induction, we obtain

\[ \begin{array}{c c c c c c c c c c}
1 & 1 & \\
\vdots & \vdots & \\
1 & l & \\
\vdots & \vdots & \\
n-1 & \partial & \\
\end{array} \cong \begin{array}{c c c c c c c c c c}
1 & 1 & \\
\vdots & \vdots & \\
1 & l-1 & \\
\vdots & \vdots & \\
n-1 & \partial & \\
\end{array} \quad (A.50) \]

This, substituted back into (A.46), gives us the vanishing of \( H_{(1)}^{(n, l)}(\Omega_{(2)}(\mathbb{R}^D)) \) for \( n \neq D \), \( l \neq D \) and \( l \neq 0 \).

### A.4 Generalized Poincaré lemma in \( \Omega_{(\ast)}^{(\ast, \ldots, \ast)}(\mathbb{R}^D) \)

Here we present the final result concerning our generalized Poincaré lemma:

\[ H_{(\ast)}^{(\ast, \ldots, \ast)}(\Omega_{(\ast)}(\mathbb{R}^D)) \cong 0 , \quad (A.51) \]
for diagrams obeying the assumption of the Theorem (cfr. section 5).
Eqn (A.51) is really proved if one has the following inductive progression:

Under the assumption that
\[
H(l_1, \ldots, l_{S-1}, l_S)(\Omega_{(S)}(\mathbb{R}^D)) \simeq 0
\]  
(A.52)

\[
\forall (l_1, \ldots, l_{S-1}) \in \mathcal{Y}^{(S-1)} \ 	ext{and} \ l_S \ \text{fixed such that} \ 0 < l_S < l_{S-1},
\]

the following holds:
\[
H(l_1, \ldots, l_{S-1}, l_S+1)(\Omega_{(S)}(\mathbb{R}^D)) \simeq 0,
\]  
(A.53)

and
\[
H(l_1, \ldots, l_{S-1}, l_S+1)(\Omega_{(S+1)}(\mathbb{R}^D)) \simeq 0,
\]  
(A.54)

\[0 < l < S + 2.\]

More explicitly, if the following statements are satisfied:
\[
d^I \mu(l_1, \ldots, l_{S-1}, l_S) = 0 \ \forall I \subset \{1, 2, \ldots, S\} \ | \ # I = m
\]

\[
\Rightarrow \mu(l_1, \ldots, l_{S-1}, l_S) = \sum_J d^J \nu_J
\]

\[
\forall J \subset \{1, 2, \ldots, S\} \ | \ # J = S+1-m \ \text{and} \ d^J \nu_J \in \Omega_{(S)}(\mathbb{R}^D),
\]

it can be showed that these statements are also true:

- \[d^I \mu(l_1, \ldots, l_{S-1}, l_S + 1) = 0 \ \forall I \subset \{1, 2, \ldots, S\} \ | \ # I = m\]

\[
\Rightarrow \mu(l_1, \ldots, l_{S-1}, l_S + 1) = \sum_J d^J \nu_J
\]

\[
\forall J \subset \{1, 2, \ldots, S\} \ | \ # J = S+1-m \ \text{and} \ d^J \nu_J \in \Omega_{(S)}(\mathbb{R}^D)
\]

and

- \[d^I \mu(l_1, \ldots, l_{S-1}, l_S, 1) = 0 \ \forall I \subset \{1, 2, \ldots, S, S+1\} \ | \ # I = m\]

\[
\Rightarrow \mu(l_1, \ldots, l_{S-1}, l_S, 1) = \sum_J d^J \nu_J
\]

\[
\forall J \subset \{1, \ldots, S, S+1\} \ | \ # J = S+2-m \ \text{and} \ d^J \nu_J \in \Omega_{(S)}(\mathbb{R}^D).
\]

We just have to follow the same lines as in sections A.1, A.2 and A.3.

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