The vortex dynamics in incompressible viscous turbulent flows

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Abstract

In this paper, we consider turbulence from a geometric perspective based on the vorticity equations for incompressible viscous fluid flows. We derive several quantitative statements about the statistics of turbulent flows. In particular we establish an entropy decay inequality which involves the turbulent kinetic energy and $L^1$-enstrophy, and we identify the small time scale of the vorticity broken down and the vorticity creation under the universality assumption of small time scales of turbulence flows.

key words: Navier-Stokes equation, rate-of-strain tensor, Reynolds number, turbulent flows, vorticity

MSC classifications: 76F05, 76F20

1 Introduction

The study of turbulence in fluids has been a prevalent topic as an unfulfilled scientific discipline, although, unlike other theoretical physics problems, the equations of motion even for turbulent flows have been known for over a century. The study of turbulence flows has been and will remain to be concentrated on methods of extracting both qualitative and quantitative results about turbulent flows from the fluid dynamic equations. The difficulty for attacking the so called turbulence problem lies in the non-linear and non-local characteristic of the motion equations. It has been known for quite long time that solutions of Navier-Stokes equations are unstable under even tiny perturbations of initial data, and turbulent flows must develop into chaotic movements if the Reynolds number becomes large. Statistical mechanics approaches were thus developed several decades ago, in which fluid dynamical variables were treated as random fields, see [6, 7, 14] for a comprehensive survey. In recent years, other ideas

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from field theories and statistical mechanics such as renormalization have been intro-
duced to address on the other hand possible coherent structures in chaotic turbulent
motions, while only limited quantitative results are achieved, see however [10, 11]
and the original literature therein for further information. In the last five decades, the
study of vorticity in fluids and vortex dynamics has become an active research area in
fluid mechanics and established as an important branch of fluid dynamics. The vortex
method has been applied to the study of turbulence in particular through numerical
simulations [12, 13, 9]. In this paper we aim to deduce a few quantitative statements
on the statistics of turbulent flows by applying the study of turbulent vortex dynamics.
We will consider incompressible viscous fluid flows in terms of the vorticity equations,
which are however equivalent to the Navier-Stokes equations, but have an appealing
feature that the vorticity actually dominates the turbulent motion of fluids, see [18] for
a comprehensive description on the relation between Navier-Stokes equations, vorti-
city and turbulence. The vorticity and the rate-of-strain tensor have been emphasized
in turbulence, and the production of the vorticity through the rate-of-strain field has
been considered as the cause of the energy cascade, see Davidson [5], Hiney [7] and
Monin and Yaglom [14] for its precise description.

The main results of this paper are presented in part 3 and 4, where we consider the
isotropic and non-isotropic flows respectively, and in part 2, we will introduce several
fluid dynamical variables and derive a few equations with respect to these variables,
which will be useful in the computation later. Finally, the Einstein summation con-
vention is used throughout this paper.

2 Fluid dynamical variables

We begin with a review of several fluid dynamical variables from a geometric per-
spective which may shed new light on turbulence in fluids. The motion of a turbulent
flow with viscosity $\nu > 0$ may be described by its velocity $u = (u^1, u^2, u^3)$, the fluid
density $\rho$ and the pressure $p$, through the Navier-Stokes equation. There are several
fluid dynamical variables which play important roles in understanding the underlying
flow motion. Among them, the vorticity $\omega = \nabla \times u$, whose components $\omega^i = \varepsilon_{kji}\frac{\partial u^j}{\partial x^k}$,
where $\varepsilon$ is the Levi-Civita symbol. The total derivative $\nabla u$ of $u$ is denoted by $A$. Its
components $A^i_j = \frac{\partial u^i}{\partial x^j}$, are the most important fluid dynamical variables in our study
below. $A$ is canonically decomposed into its symmetric part $S = (S^i_j)$ and its skew-
symmetric part whose components are $\frac{1}{2}\varepsilon_{kji}\omega^k$. In the turbulence literature, $|\omega|^2$ is
called the enstrophy, $S = (S^i_j)$ is known as the strain tensor or rate-of-strain, and
$2\nu|S|^2 = 2\nu S^i_j S^j_i$ is the energy dissipation. These dynamical variables have clear geo-
metric meanings. The vorticity $\omega$ is the exterior derivative of $u$, and the strain tensor
$S$ is the Ricci curvature in the sense of Bakry-Emery [1] associated with Taylor’s
diffusion of Brownian fluid particles. Therefore the vorticity equation is exactly the
Bochner identity in this context, and the helicity $u \times \omega$ is nothing but the Chern-Simon
invariant with respect to the material derivative $D/Dt$. Applications to isotropic tur-
bulence flows may be addressed therefore from this point of view.
Assume that the fluid is incompressible, hence $\nabla \cdot \mathbf{u} = 0$ which is equivalent to that $\text{tr} \mathbf{A} = 0$. The following vector identities

$\text{tr} \mathbf{A}^2 = \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$  \hspace{1cm} (1)

and

$\text{tr} \mathbf{A}^3 = \nabla \cdot \left[ \mathbf{u} \cdot \nabla (\mathbf{u} \cdot \nabla \mathbf{u}) - \frac{1}{2} (\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})) \mathbf{u} \right]$  \hspace{1cm} (2)

hold, which show that both $\text{tr} \mathbf{A}^2$ and $\text{tr} \mathbf{A}^3$ are exact, and $\text{tr} \mathbf{A}^2$, $\text{tr} \mathbf{A}^3$ are taken as two parameters, named by some authors as $Q$ and $R$, in the classification of turbulent flows, see [5] and [15] for example. Equation (1) is well known and (4) was discovered firstly, to the best knowledge of present authors, by Betchov [4]. On the other hand, a direct calculation shows that

$\text{tr} \mathbf{A}^2 = \text{tr} \mathbf{S}^2 - \frac{1}{2} |\mathbf{\omega}|^2$,  \hspace{1cm} (3)

$\text{tr} \mathbf{A}^3 = \text{tr} \mathbf{S}^3 + \frac{3}{4} \mathbf{\omega} \cdot \mathbf{S} \mathbf{\omega}$,  \hspace{1cm} (4)

and $\text{tr} \mathbf{S}^2$ is identified with its norm squared $|\mathbf{S}|^2$. The following is a remarkable relation about the strain tensor $\mathbf{S}$ and its vorticity $\mathbf{\omega}$, which appears new to the present authors:

$|\nabla \mathbf{S}|^2 - \frac{1}{2} |\nabla \mathbf{\omega}|^2 = \nabla \cdot [\text{tr} (\nabla ((\mathbf{u} \cdot \nabla) \nabla \mathbf{u})) - (\mathbf{u} \cdot \nabla) \Delta \mathbf{u}]$,  \hspace{1cm} (5)

where the trace on the right-hand side is taken over the two co-variant derivatives, that is

$\text{tr} (\nabla ((\mathbf{u} \cdot \nabla) \nabla \mathbf{u})) = \sum_k \frac{\partial}{\partial x^k} \left( (\mathbf{u} \cdot \nabla) \frac{\partial}{\partial x^k} \mathbf{u} \right)$.

Therefore not only $|\mathbf{S}|^2 - \frac{1}{2} |\mathbf{\omega}|^2$ is exact, but also is $|\nabla \mathbf{S}|^2 - \frac{1}{2} |\nabla \mathbf{\omega}|^2$.

After having discussed a few basic vector identities about the strain tensor and vorticity, we now want to provide a geometric interpretation of the strain tensor. Recall that Taylor’s diffusion for a viscous fluid flow with velocity $\mathbf{u}(x, t)$ is the process $X_t$ of Brownian fluid particles so that

$dX = \mathbf{u}(X, t) dt + \sqrt{2} v dB$,  \hspace{1cm} (6)

where $B$ is a 3D Brownian motion. The precise meaning of stochastic Itô’s equation (6) is not needed in the discussion below, but see Ikeda-Watanabe [8] for the theory of diffusion processes. What we need is the fact that the distribution of the Taylor diffusion $X$ is completely determined by the backward problem of the parabolic equation

$\left( \frac{\partial}{\partial t} - (v \Delta + \mathbf{u} \cdot \nabla) \right) f = 0$.

In this sense one says that the elliptic operator of second order $L = v \Delta + \mathbf{u} \cdot \nabla$ is the infinitesimal generator of Taylor’s diffusion. It is important to notice that for incompressible fluid the adjoint operator is given by $L^* = v \Delta - \mathbf{u} \cdot \nabla$ due to the divergence
free condition $\nabla \cdot \mathbf{u} = 0$. According to Bakry-Emery [1], the Ricci curvature of $L$ can be described by two step iteration: in the first iteration one recovers the metric via the equation

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf),$$

so that $\Gamma(f, g) = v \nabla f \cdot \nabla g$. The Ricci curvature is obtained by iterating the previous process to define

$$\Gamma_2(f, g) = \frac{1}{2}(L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)),$$

which yields that

$$\Gamma_2(f, f) = v^2 |\nabla^2 f|^2 - \nu \sum_i \partial_i \left( \frac{\partial f}{\partial x^i} \right)^2.$$

According to the Bochner’s equality [3], one may conclude that the Ricci curvature of $L$ is $-S = (-S_i^i)$, and the Ricci curvature of $L^*$ is the strain tensor $S = (S_i^j)$.

The Navier-Stokes equation may be written in terms of the adjoint operator

$$\left( \frac{\partial}{\partial t} - L^* \right) \mathbf{u} = -\nabla p, \quad \nabla \cdot \mathbf{u} = 0,$$

and the vorticity equation is given by

$$\left( \frac{\partial}{\partial t} - L^* \right) \boldsymbol{\omega} = S \boldsymbol{\omega},$$

where $S \omega^i = S_j^i \omega^j$ for $i = 1, 2, 3$, which is the Weitzenböck formula applying to $\boldsymbol{\omega}$. The energy balance equation is

$$\left( \frac{\partial}{\partial t} - L^* \right) |\mathbf{u}|^2 = -v |\boldsymbol{\omega}|^2 + v \nabla \cdot (\mathbf{u} \times \boldsymbol{\omega}) - 2 \nabla \cdot (p \mathbf{u}).$$

In the description of turbulence flows, the enstrophy $|\boldsymbol{\omega}|^2$ and the variation of the dissipation of energy $\text{tr} S^2 = S_i^j S_j^i$ are essential, and therefore one would like to understand their dynamics. The evolution of the enstrophy is well known and follows easily from the vorticity equation (9):

$$\left( \frac{\partial}{\partial t} - L^* \right) \frac{|\boldsymbol{\omega}|^2}{2} = \boldsymbol{\omega} \cdot S \boldsymbol{\omega} - v |\nabla \boldsymbol{\omega}|^2,$$

where the first term on the right-hand side is associated with the vortex stretching. A detailed analysis of the dissipation energy $|S|^2$ was initiated in the beautiful papers by Townsend [17] and Betchov [4], and there is an excellent account in the book Davidson [5] (pages 240 to 251). Perhaps a better way to understand the dissipation of turbulence flows is to start with the evolution equations for $S = (S_i^j)$ obtained by differentiating the Navier-Stokes equation (8), so that we obtain that for all $i, j$,

$$\left( \frac{\partial}{\partial t} - L^* \right) S_j^i = -\sum_k S_j^k S_k^j + \frac{1}{4} (\delta_{ij} |\boldsymbol{\omega}|^2 - \omega^j \omega^j) - \frac{\partial^2 p}{\partial x^i \partial x^j}.$$
and therefore
\[
\left( \frac{\partial}{\partial t} - L^* \right) \text{tr} S^2 = -2 \text{tr} S^3 - \frac{1}{2} \mathbf{\omega} \cdot \mathbf{S} \mathbf{\omega} - 2 \mathbf{\nabla} |\mathbf{S}|^2 - 2 \sum_{i,j} S^i_j \frac{\partial^2 p}{\partial x^i \partial x^j}. \tag{13}
\]

Now here is a non-trivial observation: the contraction between the strain tensor \( \mathbf{S} \) and the hessian of the pressure \( p \) is in fact exact. More precisely, we have
\[
\sum_{i,j} S^j_i \frac{\partial^2 p}{\partial x^i \partial x^j} = \mathbf{\nabla} \cdot (\mathbf{u} \cdot \mathbf{\nabla} (\mathbf{\nabla} p)) - \mathbf{\nabla} \cdot (\mathbf{u} \Delta p),
\]
and thus, by combining (2, 4, 5) together we obtain that
\[
\left( \frac{\partial}{\partial t} - L^* \right) \text{tr} S^2 = \mathbf{\omega} \cdot \mathbf{S} \mathbf{\omega} - 2 \mathbf{\nabla} |\mathbf{\nabla} \mathbf{\omega}|^2 - 2 \mathbf{\nabla} \cdot 2 \mathbf{\nabla} \cdot \left[ \mathbf{\nabla} (\mathbf{u} \cdot \mathbf{\nabla} \mathbf{\nabla} p) - (\Delta p) \mathbf{u} \right]
- \mathbf{\nabla} \cdot [2 \mathbf{u} \cdot \mathbf{\nabla} (\mathbf{u} \cdot \mathbf{\nabla} \mathbf{\nabla} p) - (\mathbf{\nabla} \cdot (\mathbf{u} \cdot \mathbf{\nabla}) \mathbf{u})] - 2 \mathbf{\nabla} \cdot \left[ \mathbf{\nabla} \cdot (\mathbf{u} \cdot \mathbf{\nabla} \mathbf{\nabla} p) - (\mathbf{u} \cdot \mathbf{\nabla}) \Delta \mathbf{u} \right]. \tag{14}
\]

Let us note that \( \text{tr} \mathbf{S} = \mathbf{\nabla} \cdot \mathbf{u} = 0 \). Since \( \mathbf{S} \) is a symmetric tensor, if we assume that its three real eigenvalues are denoted by \( a, b \) and \( c \), which are arranged so that \( a \geq b \geq c \), then \( a + b + c = 0 \), \( \text{tr} \mathbf{S}^2 \) equals \( a^2 + b^2 + c^2 \) and \( \text{tr} \mathbf{S}^3 \) coincides with \( a^3 + b^3 + c^3 = 3abc \). Note that the variance of three eigenvalues \( a, b \) and \( c \) is a dynamical quantity
\[
P = \frac{1}{3} ((a - b)^2 + (b - c)^2 + (c - a)^2)
\]
which measures the derivation from the mean, namely from zero. Then \( P = \frac{1}{3} \text{tr} \mathbf{S}^2 \) due to the fact that \( \text{tr} \mathbf{S} = 0 \), and therefore (14) is also the evolution equation for the variance of the eigenvalues of the strain tensor. This identity explains the stretching of strain tensor is proportional to that of its magnitude.

We also note that among \( a, b \) and \( c \), only two of them are free. Therefore in order to determine the evolution of these eigenvalues of the rate-of-strain, in addition to (14) one needs one more equation: naturally an evolution equation for \( \text{tr} \mathbf{S}^3 \), which is however a bit complicated and is given as the following:
\[
\left( \frac{\partial}{\partial t} - L^* \right) \text{tr} S^3 = -3 |\mathbf{S}|^4 + \frac{3}{4} (|\mathbf{S}|^2 |\mathbf{\omega}|^2 - |\mathbf{S} \mathbf{\omega}|^2)
- 6 \nu S^i_j \frac{\partial S^j_k}{\partial x^i} \frac{\partial S^k_i}{\partial x^j} - 3 S^i_k S^j_l \frac{\partial^2 p}{\partial x^i \partial x^j \partial x^l}.
\]

We will not explore this equation further in the present paper.

### 3 Isotropic turbulent flows

In this section we present several new results about homogeneous isotropic flows [2, 11, 14] in a developed turbulence region. Suppose \( \mathbf{u}(\mathbf{x}, t) \) is the random velocity field
of an isotropic turbulent flow, so that its mean velocity is constant (being zero without loss of generality).

The following convention is employed: if $Z$ be a dynamical variable of turbulent flow, then $\langle Z \rangle$ denotes the mean value of $Z$. Since the flow is isotropic, so that $\langle \nabla \cdot Z \rangle = 0$ as long as $Z$ is a tensor field depending only on the turbulent dynamical variables. Moreover, if $f$ is a dynamical scalar, then

$$L^* f = \nu \Delta f - \mathbf{u} \cdot \nabla f$$

$$= \nu \Delta f - \nabla \cdot [f \mathbf{u}]$$

so that $\langle L^* f \rangle = 0$.

**Theorem 1.** For an isotropic turbulent flow $\mathbf{u}(\mathbf{x}, t)$ we have

$$\langle |\mathbf{S}|^2 \rangle = \frac{1}{2} \langle |\mathbf{\omega}|^2 \rangle , \quad (15)$$

$$\langle \text{tr}(\mathbf{S}^3) \rangle = -\frac{3}{4} \langle \mathbf{\omega} \cdot \mathbf{S} \mathbf{\omega} \rangle \quad (16)$$

and

$$\langle |\nabla \mathbf{S}|^2 \rangle = \frac{1}{2} \langle |\nabla \mathbf{\omega}|^2 \rangle . \quad (17)$$

**Proof.** The first equality (15) follows from (3, 1), and similarly (16) is a consequence of (2, 4). From (5) one also deduces (17). \qed

For the dissipation rate of an isotropic turbulent flow, it follows immediately from (10) and (13) that

$$\frac{\partial}{\partial t} \langle |\mathbf{u}|^2 \rangle = -\nu \langle |\mathbf{\omega}|^2 \rangle ,$$

$$\frac{\partial}{\partial t} \langle |\mathbf{S}|^2 \rangle + \nu \langle |\nabla \mathbf{\omega}|^2 \rangle = \langle \mathbf{\omega} \cdot \mathbf{S} \mathbf{\omega} \rangle .$$

The following theorem provides a quantitative result about the energy dissipation in isotropic turbulent flows.

**Theorem 2.** For an isotropic turbulent flow then the following entropy functional

$$t \rightarrow \langle |\mathbf{\omega}(\cdot, t)|^2 \rangle + \frac{1}{\sqrt{2 \nu}} \langle |\mathbf{u}(\cdot, t)|^2 \rangle \quad (18)$$

is monotonically decreasing.

**Proof.** If $\Psi$ is differentiable on $(0, \infty)$, then

$$\left( \frac{\partial}{\partial t} - L^* \right) \Psi(|\mathbf{\omega}|^2) = 2\Psi' \mathbf{\omega} \cdot \mathbf{S} \mathbf{\omega} - 2\nu \Psi'' |\nabla \mathbf{\omega}|^2 - \nu \Psi''' \langle |\nabla \mathbf{\omega}|^2 \rangle . \quad (19)$$
Applying (19) to $\Psi_\delta(x) = (x + \delta)^{q/2}$, where $\delta > 0$ and $q \geq 1$ are two constants, and using

$$|\nabla \omega|^2 \geq \frac{|\nabla \omega|^2}{4|\omega|^2},$$

one obtains that

$$\left( \frac{\partial}{\partial t} - L^* \right) \Psi_\delta(|\omega|^2) \leq -2\nu \Psi_\delta' \frac{|\nabla \omega|^2}{4(|\omega|^2 + \delta) + 2\Psi_\delta' \omega \cdot S \omega.}$$

(20)

By letting $\delta \downarrow 0$ and taking mean value on both sides, one may deduce that

$$\frac{d}{dt} \langle |\omega|^q \rangle \leq -4 \left(1 - \frac{1}{q}\right) \nu \langle |\nabla \omega|^{q/2} \rangle + q \langle |\omega|^{q-2} \omega \cdot S \omega \rangle.$$  

(21)

Choosing $q = 1$ we thus obtain that

$$\frac{d}{dt} \langle |\omega| \rangle \leq -\frac{2}{\sqrt{2}} \nu \langle |\omega|^2 \rangle + \frac{1}{\sqrt{2}\nu} \frac{\partial}{\partial t} \langle |u|^2 \rangle,$$  

(22)

which yields that

$$\frac{d}{dt} \left[ \langle |\omega| \rangle + \frac{1}{\sqrt{2}\nu} \langle |u|^2 \rangle \right] \leq 0.$$  

4 Small scale of the vorticity under the similarity hypotheses

In this section we consider turbulent flows which are not necessary isotropic.

Consider a viscous turbulent flow (with viscosity $\nu > 0$) in a region with typical velocity $U$ being the maximum velocity, and $L$ the typical length so that the Reynolds number is $Re = UL/\nu$. As in the dimensionless analysis of fluid flows, let us set $\phi(x,t) = \frac{1}{U} u(Lx, \kappa t)$ and $\tilde{p}(x,t) = \frac{1}{U^2} \tilde{p}(Lx, \kappa t)$, where $\kappa = L/U$. Then the Navier-Stokes equation turns into

$$\left( \frac{\partial}{\partial t} + \phi \cdot \nabla - \frac{1}{Re} \Delta \right) \phi = -\nabla \tilde{p}, \quad \nabla \cdot \phi = 0.$$  

(23)

Let $\theta = \nabla \times \phi$ and $\Gamma_{ij} = \frac{1}{2} \left( \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i} \right)$, $1 \leq i, j \leq 3$, so that $\theta(x,t) = \frac{1}{U} \omega(Lx, \kappa t)$, $\Gamma(x,t) = \frac{1}{U^2} S(Lx, \kappa t)$. The dimensionless vorticity equation then is given by

$$\left( \frac{\partial}{\partial t} + \phi \cdot \nabla - \frac{1}{Re} \Delta \right) \theta = \Gamma \theta.$$  

(24)
The enstrophy equation is written as the following dimensionless form:

$$
\left( \frac{\partial}{\partial t} + \varphi \cdot \nabla - \frac{1}{\text{Re}} \Delta \right) \frac{\theta}{2} = \theta \cdot \Gamma - \frac{1}{\text{Re}} | \nabla \theta |^2.
$$

(25)

The similarity hypothesis claims that, in the developed turbulent region, the local small structures of turbulent flows with the same Reynolds number are the same statistically. A trivial solution to (23) with the maximum velocity is the the constant solution \( \varphi = (1, 1, 1) \), the fundamental solution associated with the corresponding elliptic operator \( \frac{1}{\text{Re}} \Delta \pm \nabla \) is given by

$$
\Gamma_{\pm}(\xi, t, x) = \sigma^3 \left( \frac{1}{2\pi t} \right)^{3/2} \exp \left[ -\frac{\left| \sigma (x - \xi) \mp \sigma t \right|^2}{2t} \right]
$$

for \( t > 0 \), where \( \sigma = \sqrt{\text{Re}/2} \) for simplicity. For a general \( \varphi \), there are explicit bounds for the fundamental solution \( \Gamma(s, \xi, t, x) \) associated with \( \frac{1}{\text{Re}} \Delta - \varphi \cdot \nabla \) obtained in [16]. For \( \beta \in \mathbb{R} \), let

$$
p^\beta(x, t, y) = \frac{1}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^\infty ze^{-(z-\beta \sqrt{t})^2/2}dz = e^{-\frac{\beta^2}{2}t+\beta|x-y|} \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2\beta^2}+\beta\Psi \left( \frac{|x-y|-\beta t}{\sqrt{t}} \right)},
$$

where \( \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-z^2/2}dz \) and \( \Psi(a) = 1 - \Phi(a) \). Then we have

$$
\sigma^3 \prod_{i=1}^3 p^\sigma(\sigma \xi_i, t - \tau, \sigma x_i) \leq \Gamma(\tau, \xi, t, x) \leq \sigma^3 \prod_{i=1}^3 p^{-\sigma}(\sigma \xi_i, t - \tau, \sigma x_i).
$$

Let \( \delta = t - \tau > 0 \) be the elapsing time. Then

$$
f_{\pm}(\sigma, \xi, \delta, x) = \sigma p^{\pm\sigma}(\sigma \xi, \delta, \sigma x)
$$

$$
= e^{-\frac{\sigma^2}{2} \delta \pm \sigma^2 |x-y|} \frac{1}{\sqrt{2\pi \delta \sigma^{-2}}} e^{-\frac{|x-y|^2}{2\delta \sigma^{-2}}} \pm \sigma \Psi \left( \frac{|x-y| \mp \delta}{\sqrt{\delta \sigma^{-2}}} \right).
$$

Now observe that, when \( \delta > 0 \) is small and \( \sigma > 0 \) is large, it follows that

$$
f_{\pm}(\sigma, \xi, \delta, x) \sim e^{-\frac{\sigma^2}{2} \delta \pm \sigma^2 |x-y|} \delta_\delta(d\xi) \pm \sigma \Psi \left( \frac{|x-y| \mp \delta}{\sqrt{\delta \sigma^{-2}}} \right)
$$

$$
\sim \begin{cases} 
  e^{-\frac{\sigma^2}{2} \delta} \delta_\delta(d\xi), & \text{if } |x-y| \mp \delta > 0; \\
  e^{-\frac{\sigma^2}{2} \delta} \delta_\delta(d\xi) \pm \sigma, & \text{if } |x-y| \mp \delta < 0; \\
  e^{-\frac{\sigma^2}{2} \delta} \delta_\delta(d\xi) \pm \frac{\sigma}{2}, & \text{if } |x-y| \mp \delta = 0.
\end{cases}
$$

In particular,

$$
f_{-}(\sigma, \xi, \delta, x) \sim e^{-\frac{\sigma^2}{2} \delta} \delta_\delta(d\xi),
$$

8
and
\[
f_{+}(\sigma, \xi, \delta, x) \sim \begin{cases} 
  e^{-\frac{\sigma^2}{2} \delta} \delta_{x}(d\xi), & \text{if } |x-y| > \delta; \\
  e^{-\frac{\sigma^2}{2} \delta} \delta_{x}(d\xi) + \sigma, & \text{if } |x-y| < \delta; \\
  e^{-\frac{\sigma^2}{2} \delta} \delta_{x}(d\xi) + \frac{\sigma}{2}, & \text{if } |x-y| = \delta.
\end{cases}
\]

Now we apply this to the vorticity equation (24). We are interested in the enstrophy transfer in a small scale. Therefore we assume that the tensor-of-strain \( \Gamma \) is a constant symmetric matrix with three eigenvalues \( a \geq b \geq c \) with \( a + b + c = 0 \). Then

\[
\theta(x, t) = e^{(t-\tau)\Gamma} \int \Gamma(\tau, \xi, t, x) \theta(\xi, \tau) d\xi
\]

for \( \tau < t \) with \( \delta = t - \tau > 0 \) being small.

Since \( e^{\delta \Gamma} \approx I + \delta \Gamma \), we obtain that

\[
\theta^i(x, t) \sim \int \Gamma(\tau, \xi, t, x) \theta^i(\xi, \tau) d\xi + \delta \int \Gamma(\tau, \xi, t, x) \Gamma^j \theta^j(\xi, \tau) d\xi.
\]

Let us replace \( \Gamma(\tau, \xi, t, x) \) by its bounds, and obtain

\[
\theta^i(x, t) \sim e^{-3\frac{\sigma^2}{2} \delta} \theta^i(x, \tau) + e^{-3\frac{\sigma^2}{2} \delta} \delta \Gamma^j \theta^j(x, \tau)
+ \sigma \int_{|\xi-x|<\delta} \theta^i(\xi, \tau) d\xi + \sigma \delta \int_{|\xi-x|<\delta} \Gamma^j \theta^j(\xi, \tau) d\xi.
\]

Therefore

\[
\theta^i(x, t) \sim e^{-3\frac{\sigma^2}{2} \delta} \theta^i(x, \tau) + e^{-\frac{\sigma^2}{2} \delta} \delta \Gamma^j \theta^j(x, \tau) + \sigma \int_{|\xi-x|<\delta} \theta^i(\xi, \tau) d\xi.
\]

This equation shows that the original vorticity in a turbulent flow will be quickly destroyed after \( k \delta \gg \frac{L^2}{\nu U^2} = \frac{2\nu}{U^2} \) and new vorticity created about the time \( k \delta \sim \frac{L^2}{\nu U^2} = \frac{2\nu}{U^2} \), which are independent of the size of the turbulent flows as one may expect.

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