Generalized Nonlinear Equation and Solutions for Fluid Contour Deformations

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Abstract

We generalize the nonlinear one-dimensional equation for a fluid layer surface to any geometry and we introduce a new infinite order differential equation for its traveling solitary waves solutions. This equation can be written as a finite-difference expression, with a general solution that is a power series expansion with coefficients satisfying a nonlinear recursion relation. In the limit of long and shallow water, we recover the Korteweg-de Vries equation together with its single-soliton solution.

I. INTRODUCTION

Shape deformations are important for an understanding of diverse many-body systems like the dynamics of suspended liquid droplets [1], long-lying excitations of atomic nuclei or "rotation-vibration" excitations of deformed nuclei and their fission or cluster emission modes [2] hydrodynamics of vortex patches [3], evolution of atmospheric plasma clouds [4], formation of patterns in magnetic fluids and superconductors [5], electronic droplets and quantum Hall effect [6], as well as resonant formation of symmetric vortex waves [7], etc.

The theoretical description of such systems is often realized in terms of collective modes, such as large amplitude collective oscillations in nuclei, sound waves in solids, collective excitations in BEC, plasmons in charged systems, or surface moves in tubifex worms populations. Collective modes are especially important when their energies are lower than competing single-particle degrees of freedom. Sometimes, however, single-particle or collective modes in the bulk of a system show particularly dense or sparse spectra. Systems of latter type are often referred as incompressible [1,5-7]. The incompressibility can be hard or relatively soft and still serves as a convenient limit for describing large differences in relevant length or time scales, as for the macroscopic motion of a liquid drop. Under these conditions, one can usually focus attention on the motion of the boundaries of the system, which usually has softer modes with frequencies that are lower than those of the bulk (e.g. Rayleigh waves
for a solid or surface waves for a liquid drop travel at speeds considerable slower than the bulk sound waves).

Concentrating on the motion of the boundary of the system has considerable advantage: a reduction in the dimensionality of the problem that leads to relatively simple analytical treatments or a tremendous reduction in the effort needed to numerically solve or simulate the problem. Furthermore, associated with the incompressibility one usually finds microscopic conservation laws that expand into global constrains on the whole systems, even when the microscopic dynamics is completely local, as for example when the volume of a liquid drop is conserved.

These shape deformations and their dynamics have played an important role in gaining a better understanding of numerous problems in diverse fields of physics. The incompressibility is normally reflected in the existence of a field that is piecewise constant, so that there is a sharp boundary between two or more distinct regions of the space with different physical properties. This field can be of classical origin like the density of a liquid or the charge density of a plasma, or it can originate in the quantum-mechanical properties of the system, like the magnetization of a type-II superconductor. Among the various examples where these questions are relevant, the waves on the surface of a liquid represent a unique opportunity to study the dynamics of shape deformations in a clean and controlled environment. We showed recently [1] that the surface dynamics of an incompressible liquid drop can be modeled by the Korteweg-de Vries (KdV) or the modified KdV (mKdV) equations for the circular shape [8,9]. The KdV equation was proposed a century ago, [9], to explain the dynamics of water waves observed in a channel by Scott Russel in 1834.

The main property of this equation is the near-equal importance of nonlinear and dispersion effects. Some of its most important solutions, like solitary waves, are single, localized, and non-dispersive structures that have a localized finite energy density. Among these solutions, the solitons are the solitary waves with special added requirements concerning their behavior at infinity \((x \to \infty, t \to \infty)\) and having special properties associated with scattering with other such single-solitary wave solutions. Despite this nonlinearity, the KdV
equation is an infinite dimensional Hamiltonian system [10] and, remarkably, when the KdV solution evolves in time, the eigenvalues of the associated Sturm-Liouville operator $\partial^2 + \eta$ (where $\eta$ is the solution of the KdV equation) remain constant. Several extended and complex methods have been developed to study and to solve the KdV equation and other nonlinear wave equations like the nonlinear Schrödinger equation (NLS), the sine-Gordon equation (SG), etc. Among these there is the inverse scattering theory (IST), theoretical group methods, numerical approaches. For a review of some of these techniques one can look into the book by Witham [11] or the paper by Scott et al [12].

Due to its properties the KdV equation, and its quadratic extension mKdV, were the source of many applications and results in a large area of non-linear physics (for a recent review see [13] and the references herein). However, in all these applications, the KdV equation and its various generalizations (mKdV, KdV hierarchy, KP equation, supersymmetric generalizations, etc.) appear as the consequence of certain simplifications of the physical systems, especially due to some perturbation techniques involved.

In the present paper we use only one of the two well-known necessary conditions for obtaining the KdV equation from a one-dimensional shallow water channel, i.e. only the smallness of the amplitude of the soliton $\eta_0$ with respect to the depth of the channel, $h$. This is the single condition which we use, with $h$ taken to be an arbitrary parameter. In order to obtain the KdV equation from shallow water channel models, a second condition is imposed, i.e. the depth $h$ of the channel should be smaller than the half-width $\lambda$ of the solitary wave. In the following, in our generalized approach, we do not impose this second condition nor the assumption of an infinite length channel; the channel length $L$, is taken to be an arbitrary parameter. By doing this we are actually investigating exact solutions for nonlinear modes in finite domains. What we actually do is replace the fixed-length boundary condition by a periodic requirement and in this way model general surface dynamics equations for liquid drops or liquid drop-like systems. This generalizations is important in connection with atomic and nuclear physics applications where the height of the perturbation may be comparable to the atomic or nuclear radius (big clusters formation...
on the surface of symmetric molecules or nuclear molecules) so the dynamics of a shallow fluid layer becomes inappropriate. Therefore, we study, as a first step, the non-linear dynamics of a fluid of arbitrary depth in a bounded domain. This different starting point leads us to a new type of equation which generalizes in some sense the KdV equation (higher order in the derivatives and higher order nonlinearity) and also reduces to it the shallow liquid case.

As we shall show, this generalized result (infinite order differential equation and higher order nonlinearity) can also be written in terms of a finite-difference equation. In this context we introduce here another possible physical interpretation for the translation operator in fluid dynamics, by relating its associate parameter to the depth of the fluid layer. In the present model we take into account not only the existence of a uniform force field (like an electric field or the gravitational field) but also the influence of the surface pressure acting on the free surface of the fluid. This implies that the KdV-like structure for the dynamically equation arises in first order in the smallness parameter and one does not have to rely on a second order effect to see its consequences on the dynamics. We also note that, despite the general tendency followed by papers concerning applications of differential-finite-difference equations in physics of first introducing the equation and then searching possible applications, we obtained an infinite order partial differential equation and its connections to a finite-difference equation, in a natural way, starting from a traditionally physical one-dimensional hydrodynamic model.

II. THE GENERALIZED KDV EQUATION

Consider a one-dimensional ideal incompressible fluid layer with depth \( h \) and constant density \( \rho \), in an uniform force field. We suppose irrotational motion and consequently the velocity field is obtained from a potential function \( \Phi(x, y, t) \), that is \( \vec{V}(x, y, t) = \nabla \Phi(x, y, t) \) where we denote the two components of the velocity field \( \vec{V} = (u, v) \). The continuity equation for the fluid results in the Laplace equation for \( \Phi \), \( \Delta \Phi(x, y, t) = 0 \). This Laplace equation should be solved with appropriate boundary conditions for the physical problem of interest.
We take a 2-dimensional domain: $x \in [x_0 - L, x_0 + L]$ (as the "horizontal" coordinate) and $y \in [0, \xi(x,t)]$ (as the "vertical" coordinate), where $x_0$ is an arbitrary parameter, $L$ is an arbitrary length, and $\xi(x,t)$ is the shape of the free surface of the fluid. The boundary conditions on the lateral walls $x = x_0 \pm L$ and on the bottom $y = 0$ consist of the vanishing of the normal component of the velocity. The free surface fulfills the kinematic condition

$$v|_\Sigma = (\xi_t + \xi_x u)|_\Sigma,$$

(1)

where we denote by $\Sigma$ the free surface of equation $y = \xi(x,t)$ and the subscript indicates the derivative. Eq. (1) expresses the fact that the fluid particles which belong to the surface remain on the surface during time evolution. By taking into account the above boundary conditions on the lateral walls and on the bottom, we can write the potential of the velocities in the form:

$$\Phi(x, y, t) = \sum_{k=0}^{\infty} \cosh\left(\frac{k\pi y}{L}\right) \left(\alpha_k(t) \cos\left(\frac{k\pi x}{L}\right) + \beta_k(t) \sin\left(\frac{k\pi x}{L}\right)\right),$$

(2)

where $\alpha_k$ and $\beta_k$ are time dependent coefficients fulfilling the condition

$$\alpha_k \sin \frac{k\pi(x_0 \pm L)}{L} = \beta_k \cos \frac{k\pi(x_0 \pm L)}{L},$$

for any positive integer $k$. This restriction introduces a special time dependence of $\alpha_k$ and $\beta_k$, i.e. $\frac{\alpha_k}{\beta_k} = \gamma_k = \text{constant}$ for any $k$, $\beta \neq 0$. If $\beta_k = 0$ we simply equate with 0 the inverse of the above fraction. The $\alpha_k(t)$ and $\beta_k(t)$ functions also depend on $x_0$. This special time dependence doesn’t affect the general nature of the potential $\Phi$, but does affect the balance between the two terms in the RHS of eq. (2). If we set all $\beta_k = 0$, we can take arbitrary values for $x_0$ and general form for $\alpha_k$. In the infinite channel limit, $L \to \infty$, there are no more restrictions concerning $\alpha_k$ and $\beta_k$ functions, and we can choose $x_0 = 0$ without loss of generality. We introduce the function:

$$f(x, t) = \sum_{k=0}^{\infty} \left(-\alpha_k(t) \sin\left(\frac{k\pi x}{L}\right) + \beta_k(t) \cos\left(\frac{k\pi x}{L}\right)\right) \frac{k\pi}{L},$$

(3)

so the velocity field can be written like:
\[ u = \Phi_x = \cos(y\partial)f(x,t) \]
\[ v = \Phi_y = -\sin(y\partial)f(x,t). \]

(4)

where, for simplicity, the operator \( \partial \) represents the partial derivative with respect to the \( x \) coordinate. Equations (4) do not depend on \( L \) and therefore any approach toward the long channel limit must include the \( L \to \infty \) (unbounded) limit. In the following we describe excitations of small height compared to the depth, and not necessarily large widths. Also, in the boundary condition eq.(1) we use velocities evaluated at \( y = \xi(x,t) = h + \eta(x,t) \) to the first order in \( \eta \). Eqs.(4) reads:

\[ u(x,\xi(x,t),t) = \left[ \cos(h\partial) - \eta(x,t)\partial \sin(h\partial) \right] f(x,t) \]
\[ v(x,\xi(x,t),t) = -\left[ \sin(h\partial) + \eta(x,t)\partial \cos(h\partial) \right] f(x,t). \]

(5)

The dynamics of the fluid is described by the Euler equation at the free surface. The equation that results is written on the surface \( \Sigma \) in terms of the potential and differentiated with respect to \( x \). By imposing the condition \( y = \xi(x,t) \), and by using a constant force field we obtain the form

\[ u_t + uu_x + vv_x + g\eta_x + \frac{1}{\rho}P_x = 0, \]

(6)

where \( g \) represents the force field constant and \( P \) is the surface pressure. Following the same approach as used in the calculation of surface capillary waves [17], we have for our one-dimensional case

\[ P|_\Sigma = \frac{\sigma}{\mathcal{R}} = \frac{\sigma\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \simeq -\sigma\eta_{xx}, \quad \text{for small} \ \eta, \]

(7)

where \( \mathcal{R} \) is the local radius of curvature of the surface (in this case, the curvature radius of the curve \( y = \xi(x,t) \)) and \( \sigma \) is the surface pressure coefficient. Inside the fluid the pressure is given by the Euler equation. The nonlinearities appear in the dynamics through the nonlinear terms in eqs. (1), (5), (6) and (7). Consequently, we have a system of two differential equations (1 and 6) for the two unknown functions: \( f(x,t) \) and \( \eta(x,t) \), with \( u \)
and \( v \) depending on \( \eta \) and \( f \) from eqs.(5). With \( f \) and \( \eta \) determined and introduced in the expression for \( u \) and \( v \), we can finally find the coefficients \( \alpha_k \) and \( \beta_k \). In the following we treat these equations in the approximation of small perturbations of the surface \( \Sigma \), with respect to the depth, \( a = \max |\eta^{(k)}(x,t)| << h \), where \( k = 0, ..., 3 \) are the orders of differentiation.

In the linear approximation eqs. (1) and (6) become, respectively

\[
- \sin(h\partial)f = \eta_t \\
\cos(h\partial)f_t = -g\eta_x + \frac{\sigma}{\rho} \eta_{xxx},
\]

and we obtain, by eliminating \( \xi \) from eqs.(8)

\[
\cos(h\partial)\eta_{tt} = \sin(h\partial) \left( \frac{c_0^2}{h^2} \eta_x - \frac{\sigma}{\rho} \eta_{xxx} \right),
\]

which, in the lowest order of approximation in \( (h\partial) \) for the sine and cosine functions, and in the absence of the surface pressure, gives us the familiar wave equation \( \eta_{tt} = c_0^2 \eta_{xx} \), where \( c_0 = \sqrt{gh} \) is the sound wave velocity. By introducing the solution \( \eta = e^{i(kx-\omega t)} \) in the linearized eq.(9) we obtain a nonlinear dispersion relation

\[
\omega^2 = c_0^2k^2 \left( 1 + \frac{\sigma}{\rho g} \right) \frac{\tanh(kh)}{kh}.
\]

In the limit of shallow waters we find for the dispersion relation, in the \( \sigma \neq 0 \) case of an acoustical wave as well as the \( \sigma >> \rho g \) surface capillary wave limit, \( \omega^2 = \frac{h\sigma}{\rho} k^4 \). In the absence of the surface pressure (\( \sigma = 0 \)) the function \( f \) is given in this linear approximation, at least formally, by

\[
f_{\text{lin}} = \frac{c_0}{h} \left( \frac{\sin(2h\partial)}{2h\partial} \right)^{-\frac{1}{2}} \eta,
\]

which in the limit of a shallow fluid has a particular solution of the form

\[
f^0(x,t) = \frac{c_0}{h} \eta.
\]

For the time derivative of \( f \) we have, from the second equation of eq.(8)

\[
f_{\text{lin}}^t(x,t) = \left( \cos(h\partial) \right)^{-1} \left( \frac{c_0^2}{h} \eta_x + \frac{\sigma}{\rho} \eta_{xxx} \right),
\]
which in the limit of a shallow fluid reduces to

\[ f^0_t(x, t) = -\frac{c_0^2}{h} \eta_x + \frac{\sigma}{\rho} \eta_{xxx}. \]

Following [10] we look for the solution of eqs.(1 and 6) in the form

\[ f = \frac{a}{h} c_0 \tilde{\eta} + \left( \frac{a}{h} \right)^2 f_2, \]

\[ f_t = -c_0^2 \frac{a}{h} (\cos (h \partial))^{-1} \tilde{\eta}_x + \frac{a \sigma}{\rho} (\cos (h \partial))^{-1} \tilde{\eta}_{xxx} + \left( \frac{a}{h} \right)^2 g_2, \]

which represents a sort of perturbation technique in \( \frac{a}{h} \), where \( \eta = a \tilde{\eta} \). Of course a functional connection exists between the perturbation \( f_2(x, t) \) and \( g_2(x, t) \). Eq.(1) yields in the lowest order in \( \frac{a}{h} \)

\[- c_0 \sin (h \partial) \tilde{\eta} = h \tilde{\eta}_t + ac_0 (\tilde{\eta}_x \cos (h \partial) \tilde{\eta} + \tilde{\eta} \cos (h \partial) \tilde{\eta}_x). \]  

\[ (14) \]

If we approximate \( \sin(h \partial) \approx h \partial - \frac{1}{6} (h \partial)^3, \cos(h \partial) \approx 1 - \frac{1}{2} (h \partial)^2 \), we obtain from eq.(14) the polynomial differential equation

\[ a \tilde{\eta}_t + 2c_0 \epsilon^2 h \tilde{\eta} \tilde{\eta}_x + c_0 \epsilon h \tilde{\eta}_x - c_0 \epsilon^3 h^3 \tilde{\eta}_{xxx} = 0, \]  

\[ (15) \]

where \( \epsilon = \frac{a}{h} \). The first four terms in eq.(15) correspond to the zeroth approximation for terms in \( f \), obtained from boundary conditions at the free surface, in eq.(6.1.15a) from Lamb’s book [10], i.e., the traditional way of obtaining the KdV equation in shallow channels. In this case all the terms are first and second order in \( \epsilon \). If we apply the second restriction with respect to the solution, that is the half-width is much larger than \( h \), we can neglect the last parenthesis in eq.(15) and we obtain exactly the KdV equation for the free surface boundary conditions. In other words we understand the condition \( h \partial \) is ”small” like \( (h \partial) f (x, t) \ll 1 \) over the entire domain of definition of \( f \). This means that the spatial extension of the perturbation \( f(x, t) \) is large compared to \( h \), which is exactly the case in which the KdV equation arises from the shallow water model (see Chapter 6 in [3], \( h \partial f(x, t) \) of order \( \approx \frac{h}{\delta} = \delta \ll 1 \).

By using again the approximations given by eq.(5) we can write the Euler equation eq.(6) in the form

\[ 9 \]
\[
\partial_t \Omega f + \Omega (\partial \Omega f) + \Omega f (\partial_t \Omega f) + \omega f (\partial_t \Omega f) = -g \eta_x + \frac{\sigma}{\rho} \eta_{xxx}, \tag{16}
\]

where we use the notation

\[
\Omega = \cos(h\partial) - \eta \partial \sin(h\partial), \tag{17}
\]

\[
\omega = \sin(h\partial) + \eta \partial \cos(h\partial).
\]

Note that the operators given in eq. (17) satisfy the following interesting relation

\[
\Omega + i\omega = e^{i\partial |\Sigma} + \mathcal{O}_2(h\partial) + \mathcal{O}_2(h\eta). \tag{18}
\]

Following the same procedure as for the free surface boundary condition eq.(3) namely using an approximation for small \( \eta \) we obtain from eq.(16)

\[
\cos(h\partial) f_t = -g \eta_x + \frac{\sigma}{\rho} \eta_{xxx}, \tag{19}
\]

which, in the lowest order in \( \frac{\sigma}{h} \) and by using eq.(12 and 13) reduces to an identity.

Before further analysis we would like to note that in the shallow water case, following the same notation as in Chapter 6 of Lamb [10], that is \( \epsilon = \frac{\sigma}{h} \), \( \delta = \frac{l}{h} \), where \( l \) gives the order of magnitude of the half-width of the perturbation \( \eta \), and introducing a new parameter \( \alpha = -\frac{\sigma}{g l^2 \rho} \), we obtain for the Euler equation the form

\[
\tilde{\eta}_{tt} + \tilde{\eta}_{xx} + \frac{3}{2} \tilde{\eta} \tilde{\eta}_{xx} + \alpha \frac{\epsilon}{2} \tilde{\eta}_{x'x'x'} = 0, \tag{20}
\]

which is again the KdV equation. The primes attached to the subscripts denote dimensionless units [10]. The difference between eq.(20) and the corresponding eq.(1.15.b) from Lamb’s book is given by the inclusion of the surface pressure effects. If the coefficient \( \alpha \) is large enough one can ignore second order terms, like for example \( \delta^2 \), to obtain the KdV equation. Of course, the above approach introduces changes in the differential equations which involve higher order perturbations like \( f^{(1)} \) and \( f^{(2)} \) from [10] or \( f_2 \), \( g_2 \) in our case. The reduction of eq.(1) to KdV equation occurs if in eq.(14) we limit the expression to terms that are at most third order in \( h\partial \)
\[ \eta_t + c_0 \eta_x - c_0 \frac{h^2}{6} \eta_{xxx} + \frac{2c_0}{h} \eta \eta_x = 0. \] (21)

In the following we shall investigate this generalized KdV equation (gKdV) obtained from eq.(14) by keeping all terms in sin and cos, namely
\[ \eta_t + \frac{c_0}{h} \sin(h \partial) \eta + \frac{c_0}{h} (\eta_x \cos(h \partial) \eta + \eta \cos(h \partial) \eta_x) = 0. \] (22)

Eqs.(1) and (16) yield, to higher orders in \( \frac{\alpha}{h} \) and in \( h \partial \), corresponding differential equations for the functions \( f_2 \) and \( g_2 \), but here we shall study only eq.(21).

III. THE FINITE-DIFFERENCE FORM OF THE GKD V EQUATION

In this section we limit ourselves to the steady-state translational waves and consider only solutions of the form \( \eta(x, t) = \eta(x + Ac_0 t) = \eta(X) \) where \( A \in \mathbb{R} \) and \( X = x + Ac_0 t \). Eq.(21) can be written in the form
\[ Ah\eta_X(X) + \frac{\eta(X + ih) - \eta(X - ih)}{2i} + \frac{\eta(X) \eta(X + ih) + \eta(X - ih)}{2} + \frac{\eta(X) \eta(X + ih) + \eta(X - ih)}{2} = 0, \] (23)
if we suppose that \( \eta \) is an analytic function in the domain \( \text{Re}(z) \in (-\infty, \infty), \text{Im}(z) \in (-h, h) \), \( z = x \pm ih \) of the complex plane. We study rapidly decreasing solutions at infinity and we make the substitution \( v = e^{Bx} \) for \( x \in (-\infty, 0) \) and \( v = e^{-Bx} \) for \( x \in (0, \infty) \), where \( B \) is a positive constant. By introducing \( \eta(X) = -hA + f(v) \) we obtain a differential-finite-difference equation (DFDE) for the function \( f(v) \)
\[ f(v) \frac{\delta f^2(v)}{\delta f(v)} + f(v) \frac{\delta f^2(v)}{\delta f(v)} + 2 \sin(Bh) \frac{f(v)}{B} \delta f(v) = 0, \] (24)
where we define the finite-difference operator as
\[ \delta f(v) = \frac{f(e^{iBh}v) - f(e^{-iBh}v)}{e^{iBh}v - e^{-iBh}v}. \] (25)
We can write the solution of eq.(23) (or eq.(24)) as a power series in \( v \).
\[ f(v) = \sum_{n=0}^{\infty} a_n v^n, \]

and we choose \( a_0 = hA \) in order to have \( \lim_{x \to \pm \infty} \eta(x) = 0 \). Equation (25) results in a non-linear recursion relation for the coefficients \( a_n \), that is

\[ \left[ Ahk + \frac{1}{B} \sin (Bhk) \right] a_k = - \sum_{n=1}^{k-1} n \left( \cos (Bh (k - n)) + \cos (Bh (k - 1)) \right) a_{n} a_{k-n}. \]  

(27)

By taking \( k = 1 \) in the above relation, we obtain \( a_1 \left[ Ah + \frac{1}{B} \sin (Bh) \right] = 0 \). Without loss of generality

and because of the arbitrarily of \( B \) we can write

\[ A = -\frac{\sin (Bh)}{Bh}. \]  

(28)

This relation fixes the velocity of the envelope of the perturbation if its asymptotic behavior is fulfilled. In order to have \( A \neq 0 \), we need \( Bh \neq k\pi \) for \( k \) integer. In this condition \( a_1 \) is still arbitrary and by writing \( a_k = \alpha_k a_1^k \) we have \( \alpha_1 = 1 \) and the recursion relation

\[ \alpha_k = \frac{2B \cos \frac{Bh(k-1)}{2}}{k \sin (Bh) - \sin (kBh)} \sum_{n=1}^{k-1} n \cos \frac{Bh (2k - n - 1)}{2} \alpha_n \alpha_{k-n}, \]  

(29)

for \( k \geq 2 \). This recursion relation gives the coefficient for \( k \) in terms of those for \( k-1 \) and lesser values. For a smooth behavior of the solution \( \eta(X) \) at \( X = 0 \), i.e. continuity of its derivative, we must introduce the condition

\[ f_v(1) = \sum_{n=1}^{\infty} n \alpha_n a_1^{n-1} = 0, \]  

(30)

or require that the derivative of the power series \( f(v) \) with coefficients given in eq.(28) to be zero in \( z \in \mathbb{R}, z = a_1 \). This set the value for \( a_1 \).

In the following we study a limiting case of the relation eq.(28), by replacing the \( \sin \) and \( \cos \) expressions with their lowest nonvanishing terms in their power expansions

\[ \alpha_k = \frac{6}{B^2 h^3 k (k^2 - 1)} \sum_{n=1}^{k-1} n \alpha_n \alpha_{k-n}. \]  

(31)
It is straightforward exercise to prove that

$$\alpha_k = \left( \frac{1}{2B^2h^3} \right)^{k-1} k, \quad (32)$$

is a solution of the recursion equation. This can be done using mathematical induction and by taking into account the relations

$$\sum_{n=1}^{k-1} n^2 = \frac{k(k-1)(2k-1)}{6},$$
$$\sum_{n=1}^{k-1} n^3 = \left( \frac{k(k-1)}{2} \right)^2. \quad (33)$$

We can write the power expansion

$$g(z) = \sum_{k=1}^{\infty} k \left( \frac{1}{2B^2h^3} \right)^{k-1} z^k, \quad (33)$$

which has the radius of convergence $R = 2B^2h^3$ (due to the Cauchy-Hadamard criteria). The function $g(z)$ can be written in the form

$$g(z) = z \left( \frac{1}{1 - \frac{z}{2B^2h^3}} \right) 2B^2h^3 = -\frac{z}{1 - \frac{z}{2B^2h^3}}^2. \quad (34)$$

Condition eq.(31) results in $a_1 = -2B^2h^3$ and

$$\alpha_k = k \left( \frac{1}{2B^2h^3} \right)^{k-1} (-2B^2h^3)^k = 2B^2h^3 (-1)^k, \quad (35)$$

which provides

$$\eta(x) = 2B^2h^3 \sum_{k=1}^{\infty} k \left( -e^{-B|x|} \right)^k$$

$$2B^2h^3 \frac{e^{-B|x|}}{(1 + e^{-B|x|})^2} = \frac{B^2h^3}{2} \frac{1}{\left( \cosh \left( \frac{Bx}{2} \right) \right)^2}. \quad (36)$$

As expected, this solution is exactly the single-soliton solution of the KdV equation and it was indeed obtained by assuming $h$ small in the recursion relation eq.(28). So, the gKdV equation has two general features: in the reduction given by eq.(16) it yields the KdV
equation, and for eq.(230, namely in the limit $h \partial$ ”small” the differential equation and one of its solutions, $\eta(X)$ go into the KdV equation and its single-soliton solution.

In general we do not obtain a simple solution like the relation (33) but we have all the necessary information from eq.(28). It seems that the power series $g(z)$ with the coefficients given by the recursion relation (28) has a nonvanishing radius of convergence. The problem of the existence of a real point $z_o$ in the disk of convergence or on its edge ($g'(z_o) = 0$) needs further study. For the KdV equation, this point is on the edge of the disk of convergence of the power series with coefficients given by the recursion relation (28). We mention that $g'(z_o) = 0$ means that the function is not univalent (it is not injective in the neighborhood of this point).

An interesting observation concerning a conjecture formulated by Bieberbach in 1916 seems to be worth noting: if $f(z) = z + a_2 z^2 + \ldots$ is analytic and univalent in the unit disk, then $|a_n| \leq n$ for all $n$, with equality occurring only for rotations of the Koebe function $k(z) = \frac{z}{(1-z)^2}$ [14]. The form of our solution for the KdV equation, eq.(34), is exactly the Koebe function up to a scaling of the variable $z$. This conjecture was proved in 1986 by de Branges [15]. This fact suggests that one could obtain information about the possible zeros of the $g'(z)$ if it is possible to obtain estimates for the radius of convergence of the power series with coefficients given by eq.(30).

The change of the variable that we used to obtain the nonlinear DFDE eq.(25) can be used in the time-dependent equation obtained from eq.(23). If $\eta(x, t) = f(v, t)$ where $v = e^{Bx}$ or $v = e^{-Bx}$, then we obtain

$$2\frac{h}{c_0} \frac{1}{v} f_t^\mp(v, t) \pm B f_v^\mp(v) \frac{\delta f^{\mp 2}(v, t)}{\delta f^\mp(v, t)} \pm 2 \sin (B h) \delta f^\mp(v, t) = 0,$$

(37)

where $\pm$ refers to the positive or negative real semiaxis. If we try a solution of the form $f^\pm(v, t) = \sum_{k=0}^{\infty} a_k^\pm v^k$, we obtain the following infinite system of ODE:

$$f \frac{d}{d t} a_k^\mp(t) = \pm \sin (k B h) a_k^\mp(t)$$
\[ 2B \cos \frac{Bh(n-1)}{2} \sum_{n=0}^{k} na_n(t)a_{k-n}(t) \cos \left( \frac{(2k-n-1) Bh}{2} \right). \] (38)

We note that if \( a_0(t_0) = 0 \) this relation is preserved in time. The above system can be resolved step-by-step when the equation for \( a_n \) involves only the coefficients \( a_k \) with \( k \leq n \). However, the smooth behavior of the solution in \( x = 0 \) at any moment of time is a nontrivial problem.

IV. CONCLUSIONS

We have shown that the KdV equation used to describe fluids in shallow channels can be generalize to liquids flowing at any depth or length. As a consequence the role of the depth parameter in various applications that use such a model can be explored, like for example the dynamics of cluster formation and fission. The present paper announces two key results: a generalization of the KdV equation starting from a physical model; and an embedding of this nonlinear equation into a DFDE.

By using a nonlinear hydrodynamic approach for a fluid layer of arbitrary depth and length, we obtained a differential equation of infinite order which is the generalization of the KdV equation for the corresponding shallow water pattern. We succeeded in rewriting this equation in the form of an finite-difference multiscale formalism, and consequently we have obtained a nonlinear recursion relation for the coefficients of its general traveling solution. We stress the importance of the introduction of the surface pressure term, eqs.(8,9) which yields the dispersion term, that is the term proportional to \( n_{xxx} \) in the equation. This term is essential in two respects: first it introduces the dispersion in a lower order of magnitude than in the traditional case [1], and second, it is responsible for dispersion in the case of the cylindrical geometry [10]. As it should be, the generalized KdV equation and its formal general traveling solution approach the KdV ones, in the shallow layer limit of the model.

Based on these results we conjectured that there may exists a deep connection between NPDE, infinite order linear ODE and finite-difference equation, together with their symme-
tries. In short, we have shown that by starting from a one-dimensional model for an ideal incompressible irrotational fluid layer, a more general differential structure than the KdV equation is obtained and it has interesting properties, not all explored, that reduce as they approach the KdV equation in the shallow layer limit.

We also stress that the result of section 3, namely the realization of the gKdV equation in terms of a DFDE could be the starting point for searching for more interesting symmetries. Finite-difference equations are related to self-similarity and wavelet bases, too. The $B$ parameter in our model is similar with the scale parameter in multiresolution analysis, and we can obtain in this way a connection with this field.

We anticipate many possible applications of such a formalism, especially in the field of clusters, fusion, fission, drops, bubbles and shells. If the gKdV equation could be shown to arise from a Lagrangean, or Hamiltonian formalism, one can apply this result in physical models which involve nonlinear shapes or contours.

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