BIRATIONAL EQUIVALENCES OF VORTEX MODULI *

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Abstract. We construct a finite dimensional Kähler manifold with a holomorphic, symplectic circle action whose symplectic reduced spaces may be identified with the $\tau$-vortex moduli spaces (or $\tau$-stable pairs). The Morse theory of the circle action induces natural birational maps between the reduced spaces for different values of $\tau$ which in the case of rank two bundles can be canonically resolved in a sequence of blow-ups and blow-downs.

* Revised – April 6, 1993.
1 Supported in part by NSF grant DMS-9103950 and an NSF-NATO Postdoctoral Fellowship.
2 Supported in part by NSF Mathematics Postdoctoral Fellowship DMS-9007255.
1. Introduction

For holomorphic bundles over a Riemann surface there is essentially one notion of stability, and hence a single moduli space for bundles of fixed rank and degree. This rigidity can disappear when one considers moduli of bundles over higher dimensional varieties or when one considers bundles with additional structure, such as parabolic bundles. The concept of stability can then depend on parameters, and one can get families of moduli. In this paper we explore this phenomenon in the case of holomorphic bundles with prescribed global sections — the so-called holomorphic pairs. The point of view we take is inspired by Morse Theory and symplectic geometry.

In [B1] and [B-D1] we introduced a notion of stability for a pair \((E, \phi)\) consisting of a holomorphic bundle together with a holomorphic section. The definition involves a real valued parameter and can be stated as follows:

**Definition 1.1.** Let \(E \to \Sigma\) be a holomorphic vector bundle over a compact Riemann surface \(\Sigma\). Let \(\phi \in H^0(\Sigma, E)\) be a holomorphic section, and let \(\tau\) be a real number. We say that the pair \((E, \phi)\) is \(\tau\)-stable (resp. \(\tau\)-semistable) if the following two conditions hold:

(i) \(\frac{\text{degree}(F)}{\text{rank}(F)} < \tau\) (resp. \(\leq \tau\)), for every holomorphic subbundle \(F \subset E\);

(ii) \(\frac{\text{degree}(E/F)}{\text{rank}(E/F)} > \tau\) (resp. \(\geq \tau\)), for every proper holomorphic subbundle \(F \subset E\) such that \(\phi\) is a section of \(F\).

Throughout the paper, we shall denote the rank of \(E\) by \(R\), the degree of \(E\) by \(d\), and the genus of \(\Sigma\) by \(g\). We shall also assume that \(g \geq 2\), \(R \geq 2\), and that \(d > R(2g - 2)\) (cf. Assumption 2 of [B-D1]). These assumptions may be relaxed, giving rise to interesting special cases; for the presentation of the general theory, however, it is convenient to make them.

Definition 1.1, and specifically the origin of the parameter \(\tau\), is motivated by a correspondence between stability criteria and the existence of special bundle metrics. In the case of pure holomorphic bundles, this is the Hitchin-Kobayashi correspondence between stability and the Hermitian-Einstein condition [Ko]. The Hermitian-Einstein condition is expressed in the form of a set of partial differential equations called the Hermitian-Einstein equations. These amount to a pointwise constraint on the curvature of a unitary connection. But curvature forms are always globally constrained, via Chern-Weil formulae, by the topology of the bundle. For bundles over closed Riemann surfaces, this removes any ambiguity in the Hermitian-Einstein condition and hence in the definition of stability. Now the Hermitian-Einstein equations admit a natural modification which is appropriate when
a global section is prescribed [B1]. The new equations, called the vortex equations, are obtained by adding extra terms which involve only the global section. These terms are not subject to any topological constraint, and thus unlike the Hermitian-Einstein equations, the vortex equations involve a true parameter. Tracing back the Hitchin-Kobayashi correspondence from the equations to a constraint on the holomorphic structure, one is led from the vortex equations to the above notion of stability. Since the equations have a parameter, so does the notion of stability.

The parameter \( \tau \) can be explained in another way, which depends on a correspondence between holomorphic pairs on a compact Riemann surface \( \Sigma \) and a certain holomorphic extension on \( \Sigma \times \mathbb{P}^1 \). This correspondence was observed by Garcia-Prada, who used it to relate the stability of a pair to the stability of the corresponding extension [G-P]. But on \( \Sigma \times \mathbb{P}^1 \), the definition of stability depends on a choice of polarization. This introduces a single parameter which is essentially the relative weights of the polarizations on \( \Sigma \) and \( \mathbb{P}^1 \). When transferred back to the pair on \( \Sigma \), the parameter is no longer in the polarization, but in the stability criterion itself.

The impact of the parameter \( \tau \) is shaped primarily by two things. Firstly, for purely numerical reasons, at almost all values of \( \tau \), the strict inequalities in Definition 1.1 are equivalent to weak inequalities. At only a discrete set of values (specifically, rational numbers whose denominator is strictly between 0 and \( \text{Rank}(E) \)) is equality possible. Let us call these values the critical values of \( \tau \). Secondly, for values of \( \tau \) between any two successive critical values, the definitions of stability are entirely equivalent. Furthermore, for these “generic” values of \( \tau \) we get good moduli spaces of \( \tau \)-stable pairs. Specifically, we have

**Theorem 1.2.** (cf. [B-D1], [B-D2], [Be], [G-P], [Th]) Let \( \mathcal{B}_\tau \) denote the set of isomorphism classes of \( \tau \)-stable pairs on \( E \). If \( \tau \) is not a positive rational with denominator less than \( R \), then \( \mathcal{B}_\tau \) naturally has the structure of a compact Kähler manifold of dimension \( d + (R^2 - R)(g - 1) \). Indeed, \( \mathcal{B}_\tau \) is an algebraic variety (see Theorem 6.3), and the same result holds for \( \mathcal{B}_\tau(L) \), where \( L \to \Sigma \) is a degree \( d \) line bundle and \( \mathcal{B}_\tau(L) \) denotes the space of \( \tau \)-stable pairs with fixed determinant \( L \).

If we think of the parameter as a “height function” and the moduli spaces as “level sets”, then these features strongly suggest a Morse Theory interpretation. After all, the range of the height function is partitioned into intervals between critical values, and the levels sets at levels strictly between successive critical heights are all equivalent.
In this paper we will show that this is more than simply an analogy. We will give a precise way of realizing just such a picture. In fact the parameter $\tau$ can be realized as a height with respect to a Morse function of a very special kind, namely one arising from a symplectic moment map. This is not too surprising since it is well known that the equations corresponding to stability criteria (i.e. the Hermitian-Einstein and Vortex equations) have moment map interpretations. By using this aspect of the problem, we relate the present situation to a phenomenon studied in symplectic geometry, i.e. the variation of symplectically reduced level sets of moment maps [G-S].

In essence what we do is to construct a single large “master space” (the terminology is due to Bertram) which contains the stable and semistable pairs for all values of the parameter $\tau$. The space has a symplectic structure and a symplectic circle action. We detect $\tau$ as the value of the moment map for this circle action, and we recover the moduli spaces of $\tau$-stable pairs as the Marsden-Weinstein reductions for different values of this moment map. Stated more precisely, we prove the following

**Theorem 1.3.** Consider the holomorphic pairs on $E$. There is a compact topological space $\hat{B}$ whose points correspond to holomorphic pairs which are $\tau$-semistable for at least one value of $\tau$. Furthermore, there is an open set $\hat{B}_0 \subset \hat{B}$ which has a natural Kähler manifold structure. The space $\hat{B}$ and $\hat{B}_0$ have the following properties:

(i) There is a quasi-free $U(1)$-action on $\hat{B}$, i.e. an action for which the isotropy subgroup is either trivial or the whole $U(1)$. On $\hat{B}_0$ the action is holomorphic and symplectic.

(ii) There is a moment map $f : \hat{B}_0 \to \mathbb{R}$ for this $U(1)$-action which extends continuously to $\hat{B}$. The critical values for $f$ are precisely the critical values of the parameter $\tau$.

(iii) The level sets $f^{-1}(\tau)$ are $U(1)$-invariant. At regular values, the orbit spaces $f^{-1}(\tau)/U(1)$ inherit a Kähler structure and can be identified with the moduli spaces $B_\tau$. At critical values the orbit spaces correspond to the spaces of isomorphism classes of semistable pairs.

In the case of rank two bundles of odd degree, $\hat{B}$ itself has the structure of a Kähler V-manifold with at most $\mathbb{Z}_2$ singularities along the minimum value of $f$. Furthermore, $f$ is a perfect Morse function in the sense of Bott.

An important feature of our picture is that the master space $\hat{B}$ is compact, and thus the critical values of the $U(1)$ moment map include an absolute maximum and an absolute
minimum. The level sets at these extremal values correspond to moduli spaces of semistable bundles. If the degree $d$ of the underlying bundle $E$ is coprime to both $R$ and $(R - 1)$, then at one extreme we obtain precisely the moduli space of rank $R$ degree $d$ stable bundles, while at the other extreme we obtain precisely the moduli space of rank $(R - 1)$ degree $d$ stable bundles.

In the case of rank two bundles we can use our construction to recover some of the beautiful results of Thaddeus [T] (see also Theorem 6.4). Using techniques from Geometric Invariant Theory, Thaddeus constructed and analyzed the moduli spaces of $\tau$-stable pairs with fixed determinant and rank two. He showed that for values of $\tau$ separated by a critical value, the moduli spaces are related by flip in the sense of Mori theory. That is, the spaces are birationally equivalent projective varieties, and yield the same space if each is blown up along the locus where it differs from the other. Thus the one is transformed into the other by first blowing up, and then blowing down the exceptional divisor “along a different direction”. In our master space construction this phenomenon has an explanation both from the symplectic point of view as well as in terms of the Morse theory. From the symplectic point of view it corresponds exactly to the relationship between reduced level sets of moment maps as described by Guillemin and Sternberg in [G-S]. In terms of the Morse theory, the birationality of the level sets comes from a map induced by flows along the gradient lines of the Morse function. The centers of the blow-up in the two spaces are given by the stable and unstable manifolds in the sense of Morse theory (i.e. the points on flow lines which terminate at critical points). Furthermore, by again using the gradient flow, the exceptional divisors of both blow-ups can be identified as the projectivized normal bundle of the critical submanifold in the critical level set.

Acknowledgement. The authors are pleased to acknowledge the warm hospitality of the Mathematics Institute at the University of Warwick where part of this work was completed.
2. Moment maps and master spaces

\subsection{Outline of the construction}

In this section we carry out the construction of the space $\hat{B}$ described in Theorem 1.3. The construction is formally very similar to that given in [B-D1] for the moduli spaces of $\tau$-stable pairs. We begin with a brief overview of both the construction of the $B_\tau$ and the modifications required for the new space $\hat{B}$.

As in the Introduction, let $E \rightarrow \Sigma$ be a fixed complex vector bundle of rank $R$ and degree $d$. Also (cf. [B-D1] for more details) let $C$ denote the space of $\overline{\partial}$-operators on $E$ (or equivalently, the space of holomorphic structures on $E$), and let $\Omega^0(E)$ denote the space of smooth sections of $E$. The space of holomorphic pairs is then given by

\begin{equation}
\mathcal{H} = \{ (\bar{\partial}E, \phi) \in C \times \Omega^0(E) : \bar{\partial}E \phi = 0 \}
\end{equation}

(cf. [B-D1], Definition 1.1, and note that here we allow the case $\phi \equiv 0$). The complex gauge group $G^C$, i.e. the group of bundle automorphisms, acts on $\mathcal{H}$ by

\begin{equation}
g(\bar{\partial}E, \phi) = (g \circ \bar{\partial}E \circ g^{-1}, g\phi).
\end{equation}

The $G^C$-orbits correspond to isomorphism classes of holomorphic pairs. For the construction of the moduli spaces $B_\tau$, we need to identify the orbits corresponding to the $\tau$-stable pairs. We use

\textbf{Theorem 2.1.} (see [B1]) Let $E \rightarrow \Sigma$ be a fixed complex vector bundle over a closed Riemann surface, and let $(\bar{\partial}E, \phi)$ be a holomorphic pair as in Definition 1.1. Suppose that $(\bar{\partial}E, \phi)$ is $\tau$-stable for a given value of the parameter $\tau$. Then the $\tau$-Vortex equation

\begin{equation}
\sqrt{-1} \Lambda F_{\bar{\partial}E,H} + \frac{1}{2} \phi \otimes \phi^* = \frac{\tau}{2} I
\end{equation}

considered as an equation for the metric $H$, has a unique smooth solution. Here $F_{\bar{\partial}E,H}$ is the curvature of a metric connection, $\Lambda F_{\bar{\partial}E,H}$ is a section in $\Omega^0(\text{End} E)$ and is obtained by a contraction of $F_{\bar{\partial}E,H}$ against the Kähler form on $\Sigma$, $\phi \otimes \phi^*$ is a section of $\Omega^0(E \otimes E^*) \simeq \Omega^0(\text{End} E)$, and $I$ is the identity section in $\Omega^0(\text{End} E)$. Conversely, suppose that for a given value of $\tau$ there is a Hermitian metric $H$ on $E$ such that the $\tau$-vortex equation is satisfied by $(\bar{\partial}E, \phi, H)$. Then $E$ splits holomorphically as $E = E_\phi \oplus E_s$, where

(i) $E_s$, if not empty, is a direct sum of stable bundles, each of slope $\tau \cdot \frac{\text{Vol}(\Sigma)}{4\pi}$;
(ii) $E_\phi$ contains the section $\phi$ and $(E_\phi, \phi)$ is $\tau$-stable, where $E_\phi$ has the holomorphic structure induced from $\bar{\partial}_E$.

Notice that the split case $E = E_\phi \oplus E_s$ cannot occur unless $\tau \cdot \frac{\text{Vol}(\Sigma)}{4\pi}$ corresponds to the slope of a subbundle, i.e. unless $\tau \cdot \frac{\text{Vol}(\Sigma)}{4\pi}$ is a rational number with denominator less than the rank of $E$. Hence, for generic values of $\tau$ the summand $E_s$ is empty, and the $\tau$-stable pairs comprise the set

$$\mathcal{V}_\tau = \left\{ (\bar{\partial}_E, \phi) \in \mathcal{H} : \Lambda F_{\bar{\partial}_E, H} - \frac{\sqrt{-1}}{2} \phi \otimes \phi^* = -\sqrt{-1} \tau \mathbb{I} \right\}.$$  

An important feature of the vortex equation is its interpretation as a symplectic moment map condition. This comes about as follows. If a Hermitian bundle metric, $H$ say, is fixed on $E$, then $\mathcal{H}$ acquires a natural symplectic structure, i.e. the one coming from the usual symplectic structures on $\mathcal{C}$ and $\Omega^0(E)$ (cf. [B-D1]). Moreover, the unitary gauge group $\mathfrak{g}$ acts symplectically and the moment map for this action is exactly the left hand side of the vortex equation, viz.

$$\Psi(\bar{\partial}_E, \phi) = \Lambda F_{\bar{\partial}_E, H} - \frac{\sqrt{-1}}{2} \phi \otimes \phi^*.$$  

It follows that for generic $\tau$, the space $\mathcal{V}_\tau$ is the saturation of $\Psi^{-1}(\frac{\sqrt{-1} \tau}{2} \mathbb{I})$, i.e. consists of all the $\mathfrak{g}^\mathbb{C}$-orbits through the holomorphic pairs in $\Psi^{-1}(\frac{\sqrt{-1} \tau}{2} \mathbb{I})$. We thus get two descriptions of the moduli spaces

$$\mathcal{B}_\tau = \mathcal{V}_\tau / \mathfrak{g}^\mathbb{C} = \Psi^{-1} \left(-\frac{\sqrt{-1} \tau}{2} \mathbb{I}\right) / \mathfrak{g}$$

of $\tau$-stable pairs. The first description gives the complex structure and the second gives the symplectic structure.

Our new space $\tilde{\mathcal{B}}$ will similarly have two descriptions; one as a complex orbit space, and one as a symplectic reduction from a moment map. To see what the moment map should be we start from the fact that the space is designed to contain all holomorphic pairs that are $\tau$-stable for some value of $\tau$. Hence, using the vortex equation characterization of $\tau$-stability, it is evident that the pairs in $\tilde{\mathcal{B}}$ are characterized by the condition that for some metric $H$,

$$\Lambda F_{\bar{\partial}_E, H} - \frac{\sqrt{-1}}{2} \phi \otimes \phi^* = \text{const.} \mathbb{I}.$$
If we let $\Omega^0(\text{End } E)_0$ denote the $L^2$-orthogonal complement of the constant multiples of the identity in $\Omega^0(\text{End } E)$, and let $\pi^\perp : \Omega^0(\text{End } E) \to \Omega^0(\text{End } E)_0$ denote the orthogonal projection, then the defining condition for $\hat{B}$ becomes

$$\pi^\perp(\Lambda F_{\bar{\partial}E,H} - \frac{\sqrt{-1}}{2} \phi \otimes \phi^*) = \pi^\perp \Psi(\bar{\partial}E, \phi) = 0.$$  

This can be realized as a moment map condition if we replace the full unitary gauge group $\mathfrak{g}$ by a subgroup $\mathfrak{g}_0 \subset \mathfrak{g}$ which has a $U(1)$ quotient and whose Lie algebra is the ortho-complement (with respect to the $L^2$-metric) of the constant multiples of the identity. Denoting the new moment map by $\Psi_0$, the “master space” $\hat{B}$ will then correspond to the reduced zero set $\Psi_0^{-1}(0)/\mathfrak{g}_0$. Finally, to obtain the complex description of the space we must now consider the saturation of $\Psi_0^{-1}(0)$, not with respect to orbits of the full complex gauge group, but with respect to orbits of the subgroup corresponding to the complexification of $\mathfrak{g}_0$.

§2.2 The subgroups of $\mathfrak{g}$ and $\mathfrak{g}^C$

A key element in the construction of $\hat{B}$ is thus the choice of an appropriate subgroup of $\mathfrak{g}$ (or $\mathfrak{g}^C$). At least in the connected component of the identity, the discussion above makes it clear that the subgroup we want is the unique connected subgroup corresponding to the Lie subalgebra $\Omega^0(\text{End } E)_0$. In the image of the exponential map, this subgroup can also be described as the kernel of the homomorphism

$$\chi(\exp(u)) = \exp \left( \int_X \text{Tr}(u) \right).$$

In order to define the full subgroup we need to extend this homomorphism to all of $\mathfrak{g}$. The technical details are as follows, beginning with

**Lemma 2.2.** Let $f : \Sigma \to \mathbb{C}^*$ be a smooth map in the connected component of the identity in $\text{Map}(\Sigma, \mathbb{C}^*)$. Then there exists a unique $\chi_1(f) \in \mathbb{C}^*$ and $u : \Sigma \to \mathbb{C}$ satisfying $f = \chi_1(f) \exp u$ and $\int_\Sigma u = 0$. Moreover, if $f'$ is another such map, $\chi_1(ff') = \chi_1(f)\chi_1(f')$.

**Proof.** Let us first prove uniqueness. If

$$f = \chi_1(f) \exp u_1 = \chi_2(f) \exp u_2,$$

then $u_1 - u_2$ must be constant. But then $\int_\Sigma u_1 - u_2 = 0$ implies the constant is zero, and so $u_1 = u_2$ and $\chi_1(f) = \chi_2(f)$. Similarly, $\chi_1(ff') = \chi_1(f)\chi_1(f')$. To prove existence, let $\Sigma = \mathbb{H}/\Gamma$ where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$ and $\Gamma \subset \text{PSL}(2, \mathbb{R})$ is the uniformizing...
group. Then $f$ lifts to $\tilde{f} : \mathbb{H} \to \mathbb{C}^*$ satisfying $\tilde{f}(\gamma z) = \tilde{f}(z)$ for all $\gamma \in \Gamma$. Choose a point $z_0 \in \mathbb{H}$, and let $u_0 : \mathbb{H} \to \mathbb{C}$ be defined by

$$u_0(z) = \int_{z_0}^{z} \frac{d\tilde{f}}{\tilde{f}}. \tag{2.8}$$

Since $d\tilde{f}/\tilde{f}$ is closed, this is independent of the path from $z_0$ to $z$. Moreover, since $\Gamma \subset \text{PSL}(2, \mathbb{R})$ we have

$$\int_{\gamma z_0}^{\gamma z} \frac{d\tilde{f}}{\tilde{f}} = \int_{z_0}^{z} \frac{d\tilde{f}}{\tilde{f}},$$

from which it follows that

$$u_0(\gamma z) = u_0(z) + \int_{z_0}^{\gamma z_0} \frac{d\tilde{f}}{\tilde{f}},$$

for all $z \in \mathbb{H}$, $\gamma \in \Gamma$. The second term is just the winding number of $f$ about the cycle defined by $\gamma$, and this vanishes since $f$ is assumed to be connected to the identity. Thus $u_0$ descends to a map $u_0 : \Sigma \to \mathbb{C}$, and clearly $f = \text{const. \exp} u_0$. Normalizing $u = u_0 - \frac{1}{\text{Vol}(\Sigma)} \int_{\Sigma} u_0$, we obtain $\chi_1(f)$. It is now easily verified that both $u$ and $\chi_1(f)$ are independent of the choice of point $z_0$.

Let $\mathfrak{g}^C_1 \subset \mathfrak{g}^C$ denote the connected component of the identity, and let $\Upsilon$ denote the quotient group of components. Then $\Upsilon$ is a free abelian group on $2g$ generators corresponding to $H_1(\Sigma, \mathbb{Z})$ (see [A-B], p. 542). We can find a splitting of the exact sequence

$$1 \to \mathfrak{g}^C_1 \to \mathfrak{g}^C \to \Upsilon \to 1 \tag{2.9}$$

which realizes $\mathfrak{g}^C$ as a direct product

$$\mathfrak{g}^C \simeq \mathfrak{g}^C_1 \times \Upsilon, \tag{2.10}$$

where the isomorphism is given by $(g, h) \mapsto gh$. Using Lemma 2.2 and the isomorphism (2.10), we can now define a character on $\mathfrak{g}^C$ as follows: First, for $g \in \mathfrak{g}^C_1$, $\det g : \Sigma \to \mathbb{C}^*$ is in the connected component of the identity, and so we may set $\chi(g) = \chi_1(\det g)$. Then we extend $\chi$ to $\mathfrak{g}^C_1 \times \Upsilon$ by $\chi(g, h) = \chi(g)$. This defines a homomorphism $\mathfrak{g}^C \to \mathbb{C}^*$.

**Definition 2.3.** Let $\mathfrak{g}^C_0$ be the kernel of the character $\chi : \mathfrak{g}^C \to \mathbb{C}^*$ defined as above. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be defined by $\mathfrak{g}_0 = \mathfrak{g}^C_0 \cap \mathfrak{g}$.

Note that a different choice of splitting, or isomorphism (2.10), will give rise to an isomorphic group $\tilde{\mathfrak{g}}^C_0$ with the same connected component of the identity as $\mathfrak{g}^C_0$. The following is immediate from the definition:
Proposition 2.4. The groups $G_0$ and $G^C_0$ have the structure of Fréchet Lie groups with Lie algebras

\begin{equation}
\text{Lie } G^C_0 = \Omega^0(\text{End } E)_0 = \{ u \in \Omega^0(\text{End } E) : \int_{\Sigma} Tr u = 0 \} ,
\end{equation}

and

\begin{equation}
\text{Lie } G_0 = \Omega^0(ad E)_0 = \{ u \in \Omega^0(ad E) : \int_{\Sigma} Tr u = 0 \} .
\end{equation}

§2.3 Local complex structure

We now proceed with the construction of the master space and begin with the construction of $\hat{B}$ as a complex manifold. As a complex manifold, $\hat{B}$ is essentially the orbit space for the $G^C_0$ action on $H$. The construction is thus a relatively small modification of the procedures used for the orbit space $H/G^C$. In that case, the obstructions to having a smooth manifold structure, as well as the description of the tangent spaces, come from the cohomology of the deformation complex $C_{\phi}^\bar{\partial} E$:

\begin{equation}
0 \longrightarrow \Omega^0(\text{End } E) \xrightarrow{d_1} \Omega^{0,1}(\text{End } E) \oplus \Omega^0(E) \xrightarrow{d_2} \Omega^{0,1}(E) \longrightarrow 0 ,
\end{equation}

where the maps $d_1$, $d_2$ are given by

\begin{equation}
\begin{aligned}
d_1(u) &= (-\bar{\partial}_E u, u\phi) \\
d_2(\alpha, \eta) &= \bar{\partial}_E \eta + \alpha \phi .
\end{aligned}
\end{equation}

The salient features of this complex are contained in the following

Proposition 2.5. Let $(\bar{\partial}_E, \phi)$ be an element of $H$. Then

(i) $C_{\phi}^\bar{\partial} E$ is an elliptic complex;

(ii) If $d > R(2g-2)$, then $H^2(C_{\phi}^\bar{\partial} E) = 0$ whenever $\phi \neq 0$ or $E^{\bar{\partial} E}$ is semistable;

(iii) The Euler characteristic of the complex $C_{\phi}^\bar{\partial} E$ is given by

$$
\chi(C_{\phi}^\bar{\partial} E) = \chi(\text{End } E) - \chi(E) .
$$

Proof. This is proven in [B-D1], except there the case $\phi = 0$ is excluded and an extra assumption is made which ensures that $E^{\bar{\partial} E}$ is always semistable. It is clear that, under the assumption that $d > R(2g-2)$, the case $\phi = 0$ can be included without any modification
to the proof. Furthermore, as shown in [Th], when $\phi \neq 0$, the vanishing of $H^2(C_{\phi}^{\tilde{E}}) = 0$ follows in general since the map

$$H^0(K \otimes E^*) \xrightarrow{\otimes \phi} H^0(K \otimes \text{End } E)$$

is injective. By Serre duality this is equivalent to the surjectivity of the map $H^1(\text{End } E) \rightarrow H^1(E)$ in the long exact sequence

\begin{equation}
0 \rightarrow H^0 \rightarrow H^0(\text{End } E) \rightarrow H^0(E) \rightarrow H^1 \rightarrow H^1(\text{End } E) \rightarrow H^2 \rightarrow 0,
\end{equation}

where $H^i = H^i(C_{\phi}^{\tilde{E}})$. The proofs of (i) and (iii) are as in [B-D1].

For the construction of $\hat{B}$, we need to restrict to the following subcomplex of $C_{\phi}^{\tilde{E}}$:

\begin{equation}
0 \rightarrow \Omega^0(\text{End } E)_0 \xrightarrow{d_1} \Omega^{0,1}(\text{End } E) \oplus \Omega^0(E) \xrightarrow{d_2} \Omega^{0,1}(E) \rightarrow 0,
\end{equation}

which we will refer to as $C_{\phi,0}^{\tilde{E}}$. It is worth pointing out that the adjoint $d_1^{*0}$ in $C_{\phi,0}^{\tilde{E}}$ is related to the $d_1^*$ in $C_{\phi}^{\tilde{E}}$ by

\begin{equation}
d_1^{*0} = \pi^* d_1,
\end{equation}

where $\pi^*$ gives the orthogonal projection onto $\Omega^0(\text{End } E)_0$. This affects the determination of the harmonic 1-cocycles in the two complexes. In fact, we have

**Proposition 2.6.** Let $(\bar{\partial}_E, \phi)$ be an element of $\mathcal{H}$. Then:

(i) $C_{\phi,0}^{\tilde{E}}$ is a Fredholm complex;

(ii) $H^2(C_{\phi,0}^{\tilde{E}}) = H^2(C_{\phi}^{\tilde{E}})$;

(iii) Either

\begin{align*}
H^1(C_{\phi,0}^{\tilde{E}}) &\cong H^1(C_{\phi}^{\tilde{E}}) \oplus \mathbb{C} \\
H^0(C_{\phi,0}^{\tilde{E}}) &\cong H^0(C_{\phi}^{\tilde{E}})
\end{align*}

or,

\begin{align*}
H^1(C_{\phi,0}^{\tilde{E}}) &\cong H^1(C_{\phi}^{\tilde{E}}) \\
H^0(C_{\phi,0}^{\tilde{E}}) \oplus \mathbb{C} &\cong H^0(C_{\phi}^{\tilde{E}})
\end{align*}

(iv) $\chi(C_{\phi,0}^{\tilde{E}}) = \chi(\text{End } E) - \chi(E) - 1$.

**Proof.** (i), (ii) follow immediately from the definition of $C_{\phi,0}^{\tilde{E}}$.

(iii) $H^1(C_{\phi}^{\tilde{E}})$ and $H^1(C_{\phi,0}^{\tilde{E}})$ are related by the short exact sequence

$$0 \rightarrow H^1(C_{\phi}^{\tilde{E}}) \rightarrow H^1(C_{\phi,0}^{\tilde{E}}) \rightarrow \mathbb{C} \rightarrow 0.$$
Similarly, the zero-th cohomology groups are related by

\[ 0 \to H^0(C_{\phi,0}) \to H^0(C_{\phi}) \to \mathbb{C} \to 0, \]

where the map is orthogonal projection in \( \Omega^0(\text{End} \ E) \). Here by \( \mathbb{C} \) we mean the constant multiples of the identity in \( \Omega^0(\text{End} \ E) \). The desired conclusion now follows from the fact that the map \( d_1 : H^1(C_{\phi,0}) \to \mathbb{C} \) is surjective if and only if \( \pi : H^0(C_{\phi}) \to \mathbb{C} \) is zero. (iv) follows from (ii) and (iii).

**Corollary 2.7.** Let \( H^* \subset H \) denote the subspace of all \((\bar{\partial}E, \phi) \in H\) such that \( E_{\bar{\partial}E} \) is semistable if \( \phi = 0 \). Then \( H^* \) is a smooth submanifold of \( \mathcal{C} \times \Omega^0(E) \).

**Proof.** Consider the map \( F : \mathcal{C} \times \Omega^0(E) \longrightarrow \Omega^1(E) \) defined by \( F(\bar{\partial}E, \phi) = \bar{\partial}E(\phi) \). The derivative of \( F \) at \((\bar{\partial}E, \phi)\) is given by

\[ (2.18) \quad \delta F_{\bar{\partial}E, \phi}(\alpha, \eta) = \bar{\partial}E \eta + \alpha \phi = d_2(\alpha, \eta). \]

Let \( C_{ss} \) denote the semistable holomorphic structures on \( E \). Then provided that \((\bar{\partial}E, \phi)\) does not belong to the closed subspace \((\mathcal{C} - C_{ss}) \times \{0\} \subset C \times \Omega^0(E)\), it follows from Proposition 2.5 (ii) that \( \delta F_{\bar{\partial}E, \phi} \) is onto. Hence by the Inverse Function Theorem

\[ (2.19) \quad H^* = F^{-1}(0) \cap \{\mathcal{C} \times \Omega^0(E) - (\mathcal{C} - C_{ss}) \times \{0\}\} \]

is a smooth submanifold of \( \mathcal{C} \times \Omega^0(E) \).

**Definition 2.8.** A pair \((\bar{\partial}E, \phi) \in H^*\) is called simple if \( H^0(C_{\phi,0}) = 0 \). Let \( H_\sigma \) denote the subspace of simple pairs in \( H^* \).

Clearly, \( H_\sigma \) is an open subset in \( H^* \) and is therefore a submanifold. Now by identifying \( H^1(C_{\phi,0}) \) with the tangent space to the space of orbits of \( \mathcal{S}^C_0 \), we have the following theorem whose proof is in all essentials the same as the proof of the analogous result in Section 3 of [B-D1]:

**Theorem 2.9.** \( H_\sigma / \mathcal{S}^C_0 \) is a complex \( V \)-manifold (possibly non-Hausdorff) of complex dimension \( d + 1 + (R^2 - R)(g - 1) \). Moreover, we have the identification

\[ (2.20) \quad T_{[\bar{\partial}E, \phi]}(H_\sigma / \mathcal{S}^C_0) \simeq H^1(C_{\phi,0}). \]
The space $\mathcal{H}/\mathfrak{G}_0^C$ is almost, but not quite, the master space $\hat{\mathcal{B}}$. The problem is the possibly non-Hausdorff nature of the space. This can be traced back (as in all such moduli space problems) to the fact that the definition of a “simple” pair is too weak; it needs to be replaced by a concept of “stability”, i.e. we need to restrict from $\mathcal{H}$ to some suitable analogue of the $\mathcal{V}$ used in the construction of $\mathcal{B}_\tau$. We recall briefly the procedure for the $\mathcal{B}_\tau$. For generic values of $\tau$, i.e. $\tau$ not equal to a rational number with denominator less than $R$, the set of $\tau$-stable pairs in $\mathcal{H}$ is given precisely by the set $\mathcal{V}_\tau$ defined in (2.4). Furthermore for such $\tau$, the set $\mathcal{V}_\tau$ is open subset in, and therefore a submanifold of, $\mathcal{H}^\ast$. The quotient $\mathcal{V}_\tau/\mathfrak{G}^C$ is homeomorphic to $\Psi^{-1}(-\sqrt{-1}/2\tau \cdot \mathbf{I})/\mathfrak{G}$, which is a Hausdorff, smooth, symplectic manifold. In [B-D1] these last assertions are proven in the case where the degree of $E$ is large and $\tau$ is small (Assumptions (1) and (2)). However, in view of Corollary 2.7 it is clear that the conclusion holds more generally. That is,

**Proposition 2.10.** For any degree and any generic value of $\tau$, the space $\mathcal{V}_\tau/\mathfrak{G}^C \simeq \Psi^{-1}(-\sqrt{-1}/2\tau \cdot \mathbf{I})/\mathfrak{G}$, if nonempty, is a smooth compact Kähler manifold. It is $\mathcal{B}_\tau$, the moduli space of $\tau$-stable pairs.

**Proof.** The purpose of the two assumptions in [B-D1] was primarily to ensure that the subset of $\mathcal{H}$ used in the construction of the moduli spaces was a submanifold of $C \times \Omega^0(E)$. However, by Corollary 2.7 we see that this can be achieved without any restriction on $\tau$ by using the space $\mathcal{H}^\ast$. The constraint on the degree of the bundle can also be relaxed; for the construction of the moduli spaces of $\tau$-stable pairs all that is required is the vanishing of $H^2(C_{\phi}^E)$ when $\phi \neq 0$, or equivalently the surjectivity of the map

$$\delta F_{\bar{\partial}E,\phi} : \Omega^{0,1}(\text{End } E) \to \Omega^1(E)$$

when $\phi \neq 0$.

It will also be important to know what the allowed range for $\tau$ is. By taking the trace of the vortex equation and integrating over the base manifold $\Sigma$, one sees that there can be no solutions unless

$$\frac{\tau \text{Vol}(\Sigma)}{4\pi} \geq \frac{d}{R}.$$

Furthermore, an upper bound on $\tau$ can be obtained by looking at the quotient $E/[_\phi]$, where $[_\phi]$ is the line subbundle generated by the section $\phi$. If the $\tau$-vortex equation is satisfied, then (cf. [B1])

$$\frac{\tau \text{Vol}(\Sigma)}{4\pi} \leq \frac{\deg (E/[_\phi])}{\text{rank}(E/[_\phi])},$$
and hence, since \( \deg([\phi]) \geq 0 \),

\[
(2.23) \quad \frac{\tau \text{Vol}(\Sigma)}{4\pi} \leq \frac{d}{R - 1}.
\]

In fact, solutions corresponding to all intermediary values of \( \tau \) between these extreme points can be constructed (see Propositions 3.2 and 2.18), and we thus have

**Proposition 2.11.** There is a solution to the \( \tau \)-Vortex equation if and only if \( \frac{\tau \text{Vol}(\Sigma)}{4\pi} \) is in the closed interval \([d/R, d/R - 1]\).

§2.4 Symplectic structure and global complex structure

We now consider the symplectic (Kähler) structure on \( H^*/G_0^C \). In particular we need the \( G_0 \)-moment map, the symplectic reduction of its zero set, and the \( G_0^C \)-saturation of this zero set.

**Proposition 2.12.** A moment map for the action of \( G_0 \) on \( H^* \) is given by

\[
(2.24) \quad \Psi_0(\partial E, \phi) = \pi^*\Psi(\partial E, \phi) = \Psi(\partial E, \phi) - \frac{1}{R \cdot \text{Vol}(\Sigma)} \int_{\Sigma} \text{Tr} \Psi(\partial E, \phi) \cdot I.
\]

**Proof.** Let \( j : G_0 \to G \) denote the inclusion. Then a moment map for \( G_0 \) is given by \( j^*\Psi \), where \( j^* : (\text{Lie } G)^* \to (\text{Lie } G_0)^* \). Using the \( L^2 \)-inner product on \( \Omega^0(\text{End } E) \) to identify the Lie algebras with their duals, we obtain \( \Psi_0 \).

**Definition 2.13.** Let

\[
(2.25) \quad \hat{B} = \Psi_0^{-1}(0)/G_0
\]

denote the Marsden-Weinstein reduction of \( H^* \) by the symplectic action of \( G_0 \). In addition, let \( H_0 \subset H^* \) denote the subspace of \( H^* \) where \( G_0 \) acts with at most finite stabilizer. We then define

\[
(2.26) \quad \hat{B}_0 = \Psi_0^{-1}(0) \cap H_0/G_0.
\]

**Proposition 2.13.** The quotient \( \hat{B} \) is a compact, Hausdorff topological space. The quotient \( \hat{B}_0 \) is a Hausdorff symplectic V-manifold.

**Proof.** To prove the compactness of \( \hat{B} \), let \( (D_i, \phi_i) \) be a sequence in \( \Psi_0^{-1}(0) \). We will have to find \( g_i \in G_0 \) such that \( (g_i(D_i), g_i\phi_i) \to (D, \phi) \in \Psi_0^{-1}(0) \). Let us write

\[
\Lambda F_{D_i} + \frac{\sqrt{-1}}{2} \phi_i \otimes \phi_i^* = -\frac{\sqrt{-1}}{2} \tau_i
\]
for real numbers $\tau_i \in [d/R, d/R - 1]$. By passing to a subsequence we may assume that $\tau_i \to \tau$. The rest of the compactness argument follows as in [B-D1], Proposition 5.1, and the observation that $g_i$ can be chosen to be in $\mathfrak{G}_0$, since the constant central elements of the gauge group act trivially on $\mathcal{C}$. The Hausdorff property follows exactly the same way as in [B-D1], Proposition 5.4. Finally, for the symplectic structure we use the symplectic reduction theorem for the group $\mathfrak{G}_0$ (cf. [B-D1], Theorem 4.5). The only difference is that one must replace the complex $\mathcal{M}_{\hat{}\phi^e}$ of [B-D1] by the subcomplex

$$0 \to \Omega^0(\text{ad} E)_0 \xrightarrow{D_1} \Omega^{0,1}(\text{End} E) \oplus \Omega^0(E) \to \Omega^0(\text{ad} E) \oplus \Omega^{0,1}(E) \to 0.$$  

This completes the proof.

To complete the construction of the master space as a complex manifold we make the following

**Definition 2.15.** Let $\mathcal{V}_0 \subset \mathcal{H}$ denote the subset of $\mathfrak{G}_0^\mathbb{C}$-orbits through points in $\Psi_0^{-1}(0)$, i.e.

$$\mathcal{V}_0 = \{ (\bar{\partial}_E, \phi) \mid \Psi_0(g(\bar{\partial}_E, \phi)) = 0 \text{ for some } g \in \mathfrak{G}_0^\mathbb{C} \}.$$  

It is easily seen that $\mathcal{V}_0 \cap \mathcal{H}_\sigma$ is an open subset of $\mathcal{H}_\sigma$, and thus is a submanifold. Also, using the same techniques as those applied to $\mathcal{V}_\tau$ in [B-D2] it can be shown that $\mathcal{V}_0$ and $\mathcal{V}_0 \cap \mathcal{H}_\sigma$ are connected. Indeed, the only essential difference in the argument is that now the projection to the set of holomorphic structures includes unstable structures. Since these add sets of small codimension they do not affect the connectedness. Finally, there is clearly a bijective correspondence between $\hat{\mathcal{B}}_0$ and $\mathcal{V}_0 \cap \mathcal{H}_\sigma / \mathfrak{G}_0^\mathbb{C}$. Combining Theorems 2.9 and 2.13 we thus obtain

**Theorem 2.16.** $\hat{\mathcal{B}}_0 = \mathcal{V}_0 \cap \mathcal{H}_\sigma / \mathfrak{G}_0^\mathbb{C}$ is a smooth, Hausdorff, Kähler manifold of dimension $d + 1 + (R^2 - R)(g - 1)$.

§2.5 $S^1$-action, Morse function, and reduced level sets

The most important feature of the master space $\hat{\mathcal{B}}$ is the fact that it carries an $S^1$ action. This comes from the quotient $U(1) \simeq \mathfrak{G}/\mathfrak{G}_0$, and the action on $\hat{\mathcal{B}}$ is given by

$$(2.27) \quad e^{i\theta} \cdot [\bar{\partial}_E, \phi] = [\bar{\partial}_E, g_\theta \phi].$$  

Here $g_\theta$ denotes the gauge transformation $\text{diag}(e^{i\theta/R}, \ldots, e^{i\theta/R})$. Notice that $g_\theta$ itself depends on the choice of an $R$-th root of unity but that the action is well-defined and independent of this choice, since if $h = \text{diag}(e^{2\pi i/R}, \ldots, e^{2\pi i/R})$, then $h \in \mathfrak{G}_0$ and

$$[\bar{\partial}_E, h_\phi] = [h^{-1}(\bar{\partial}_E), \phi] = [\bar{\partial}_E, \phi].$$  


The following summarizes the basic properties of this action:

**Proposition 2.17.** (i) The action of $U(1)$ on $\hat{\mathfrak{B}}_0$ is holomorphic and symplectic; the moment map for the action is given by

\[
\hat{f}[\bar{\partial}_E, \phi] = -2\pi i \left( \frac{\|\phi\|^2}{4\pi R} + \mu(E) \right).
\]

(ii) The action extends continuously to $\hat{\mathfrak{B}}$ as does the moment map $\hat{f}$.

*Proof.* The only statement that needs to be verified is the computation of the moment map. First, observe that there is no natural splitting of the exact sequence of groups

\[
1 \longrightarrow \mathfrak{G}_0 \longrightarrow \mathfrak{G} \xrightarrow{\chi} U(1) \longrightarrow 1.
\]

However, on the level of Lie algebras there is a splitting $\lambda : \text{Lie} U(1) \rightarrow \text{Lie} \mathfrak{G}$ given by $\lambda(\xi) = \xi/R$, where $\xi \in \text{Lie} U(1)$ is identified with the constant infinitesimal gauge transformations. Under the identification with dual Lie algebras given by the $L^2$ metric,

\[
\lambda^* : (\text{Lie} \mathfrak{G})^* \longrightarrow (\text{Lie} U(1))^*
\]

is given by

\[
\lambda^*(g) = \int_{\Sigma} \text{Tr} \, g.
\]

Now given $[\bar{\partial}_E, \phi] \in \hat{\mathfrak{B}}_0$, choose a representative $(\bar{\partial}_E, \phi)$ satisfying $\Psi_0(\bar{\partial}_E, \phi) = 0$. It is straightforward to compute that

\[
\hat{f}[\bar{\partial}_E, \phi] = \frac{1}{R} \lambda^* \circ \Psi_0(\bar{\partial}_E, \phi) = -2\pi i \left( \frac{\|\phi\|^2}{4\pi R} + \mu(E) \right).
\]

Notice that if we represent a point $[\bar{\partial}_E, \phi]$ in $\hat{\mathfrak{B}}_0$ by $(\bar{\partial}_E, \phi) \in \Psi_0^{-1}(0)$, then

\[
\frac{-\hat{f}[\bar{\partial}_E, \phi]}{2\pi i} = \frac{\tau \text{Vol}(\Sigma)}{4\pi}
\]

if and only if

\[
\Psi(\bar{\partial}_E, \phi) = -\frac{\sqrt{-1}}{2} \tau I.
\]

For convenience we let $f : \hat{\mathfrak{B}} \rightarrow \mathbb{R}$ denote the function

\[
f = \frac{1}{2\pi i} \hat{f}.
\]

We now have
Proposition 2.18. (i) The image of \( f \) is the interval \([d/R, d/R - 1]\).
(ii) The critical points of \( f \) on \( \hat{B}_0 \) are precisely the fixed points of the \( U(1) \) action. The critical values of \( f \) coincide with the image under \( f \) of the fixed point set of the \( U(1) \) action on \( \hat{B} \).
(iii) Let \( \hat{\tau} = \tau \cdot \frac{\text{Vol}(\Sigma)}{4\pi} \) be a regular value of \( f \). Then the reduced space \( f^{-1}(\hat{\tau})/U(1) \) is a Kähler manifold which can be identified with the moduli space of \( \tau \)-vortices in degree \( d \) and rank \( R \), i.e.
\[
f^{-1}(\hat{\tau})/U(1) = B_\tau.
\]

Proof. (i) By the comment after Proposition 2.17, \( \tau \) is in the image of \( f \) if and only if the equation \( \Psi(\bar{\partial}_E, \phi) = -\sqrt{-1} \tau I \) has a solution. Hence, by Theorem 2.1 the range for \( \tau \) is in \([d/R, d/R - 1]\). Now the endpoints of this interval are included in the image, since explicit elements of the pre-image can be constructed (see Proposition 3.2). The result then follows from the connectedness of \( V_0 \cap H_\sigma \) (see the remark following Definition 2.15).
(ii) This follows from the fact that \( \hat{f} \) is a moment map for the \( U(1) \) action. (iii) Given \( [\bar{\partial}_E, \phi] \in f^{-1}(\hat{\tau})/U(1) \), choose a representative \( (\bar{\partial}_E, \phi) \in \Psi^{-1}_0(0) \). Then by definition of the moment maps we obtain the pair of equations
\[
\begin{align*}
\Lambda F_{\bar{\partial}_E,H} - \frac{\sqrt{-1}}{2} \phi \otimes \phi^* &= \frac{1}{R} \int_\Sigma \text{Tr} \left( \Lambda F_{\bar{\partial}_E,H} - \frac{\sqrt{-1}}{2} \phi \otimes \phi^* \right) \cdot I \\
\frac{\|\phi\|^2}{4\pi R} + \mu(E) &= \hat{\tau},
\end{align*}
\]
which are clearly equivalent to the \( \tau \)-vortex equation (2.3). Thus, taking the class of \( (\bar{\partial}_E, \phi) \) in \( B_\tau \) defines a map
\[
f^{-1}(\hat{\tau})/U(1) \longrightarrow B_\tau.
\]
The inverse map, i.e. mapping the class of \( (\bar{\partial}_E, \phi) \) in \( B_\tau = \Psi^{-1}(\frac{-\sqrt{-1} \tau}{2} I)/\mathfrak{G} \) to the class of \( (\bar{\partial}_E, \phi) \) in \( f^{-1}(\hat{\tau})/U(1) \), is well defined, and hence the map is a bijection. It is easily checked that for non-critical values of \( \tau \) this map is indeed an isomorphism of Kähler manifolds.

Let \( T \) denote the set of critical values of \( f : \hat{B}_0 \rightarrow \mathbb{R} \). We shall see in the next section that \( T \) consists of the rational numbers in \([d/R, d/R - 1]\) which appear as slopes of subbundles of \( E \). Implicit in Proposition 2.18 is the statement that \( \hat{B} \setminus f^{-1}(T) = \hat{B}_0 \setminus f^{-1}(T) \) is smooth. This is clear, since a point \( (\bar{\partial}_E, \phi) \in \Psi^{-1}_0(0) \) can have non-trivial isotropy in \( \mathfrak{G}_0 \) only if the bundle \( E \) splits holomorphically or if \( \phi \equiv 0 \). In rank two we can say more:
Proposition 2.19. Suppose $R = 2$. Then

\begin{equation}
\hat{\mathcal{B}} \setminus f^{-1}(d/2) = \hat{\mathcal{B}}_0 \setminus f^{-1}(d/2),
\end{equation}

and this space is smooth. If the degree $d$ is odd, then $\hat{\mathcal{B}} = \hat{\mathcal{B}}_0$, and this space has at most $\mathbb{Z}_2$ quotient singularities along $f^{-1}(d/2)$.

Proof. Suppose $(\bar{\partial}_E, \phi) \in \Psi_0^{-1}(0)$. Then $(\bar{\partial}_E, \phi) \in \Psi^{-1}(-\sqrt{-1} \tau^2 I)$ for some $\tau$. By Theorem 2.1, $\phi \equiv 0$ if and only if $\hat{\tau} = d/2$. Now suppose that $g \in \mathfrak{g}_0$, $g \neq I$, and $g \cdot (\bar{\partial}_E, \phi) = (\bar{\partial}_E, \phi)$. If $\phi \neq 0$, then $E$ must split holomorphically as $E = E_\phi \oplus E_s$ with $\phi \in H^0(E_\phi)$ and $g = (1, \tilde{g})$. But rank$(E_s) = 1$ implies $\tilde{g}$ is constant, and since $g \in \mathfrak{g}_0$ we must have $\det(g) = \tilde{g} = 1$. This proves that the stabilizer for points away from $\Psi_{d/2}^{-1}(0)$ is trivial, and therefore

\begin{equation}
\hat{\mathcal{B}} \setminus f^{-1}(d/2) = \hat{\mathcal{B}}_0 \setminus f^{-1}(d/2)
\end{equation}

is smooth. If in addition we assume that $d$ is odd, then $(\bar{\partial}_E, \phi) \in \Phi_{d/2}^{-1}(0)$ implies that $\phi \equiv 0$ and $E$ is stable; hence the stabilizer consists of $\pm I$. This completes the proof.

3. Critical sets

To further our understanding of the master space $\hat{\mathcal{B}}$ we now describe the level sets $f^{-1}(\hat{\tau})$ for the critical values $\hat{\tau}$ in the interval $[d/R, d/R - 1]$.

Definition 3.1. Let $\text{Fix}(\hat{\mathcal{B}})$ denote the $U(1)$ fixed point set in $\hat{\mathcal{B}}$. For a critical value $\hat{\tau} = \tau \cdot \frac{\text{Vol}(\Sigma)}{4\pi}$, let

\begin{equation}
Z_\tau = f^{-1}(\hat{\tau}) \cap \text{Fix}(\hat{\mathcal{B}}).
\end{equation}

Proposition 3.2. The level set corresponding to the minimum is precisely the moduli space of semistable bundles of degree $d$ and rank $R$, i.e.

\begin{equation}
f^{-1}(d/R) = \mathcal{M}(R, d).
\end{equation}

The level set corresponding to the maximum is precisely the moduli space of semi-stable bundles of degree $d$ and rank $R - 1$, i.e.

\begin{equation}
f^{-1}(d/R - 1) = \mathcal{M}(R - 1, d).
\end{equation}

Proof. By Proposition 2.18, $f^{-1}(d/R)$ consists precisely of $\tau$-semistable pairs where $\phi \equiv 0$. But then $\tau$-semistable is equivalent to semistable. For $f^{-1}(d/R - 1)$, suppose $(\bar{\partial}_E, \phi)$
supports a solution to the \( \tau \)-vortex equation with \( \tau = d/R - 1 \). By Theorem 2.1, \( (\bar{\partial}_E, \phi) \) is either a stable pair, or splits holomorphically. The former is not possible, as can be seen by applying the second condition of Definition 1.1 to \([\phi]\), the line subbundle generated by \( \phi \). This yields \( \deg([\phi]) < 0 \). In fact the only possibility is that \( \deg([\phi]) = 0 \) and thus \( \mu(E/[\phi]) = \tau \). It follows (cf. Theorem 2.1 and Section 2.4 of [B1]) that \( \phi \) is a constant section of a trivial line subbundle and \( E = \mathcal{O} \oplus E_s \), where \( E_s \) is a semistable bundle of degree \( d \), rank \( R - 1 \). In fact, the summand \( E_s \) is a direct sum of stable bundles all of the same slope, i.e. \( E_s \) is equal to the graded bundle in its S-equivalence in \( \mathcal{M}(R-1, d) \). As above, one can check that the map \( \mathcal{O} \oplus E_s \mapsto E_s \) does indeed give a bijective correspondence between \( f^{-1}(d/(R-1)) \) and \( \mathcal{M}(R-1, d) \).

Next, we describe the level sets for the intermediate values of \( \hat{\tau} \). Let \( \hat{\tau} = p/q \in (d/R, d/(R-1)) \) be a critical value. Then any pair \( (\bar{\partial}_E, \phi) \in \mathcal{Z}_\tau \) has \( \mu_M = p/q = \mu_n(\phi) \), and the bundle splits holomorphically as \( E_{\bar{\partial}_E} = E_\phi \oplus E_{ss} \), where

1. \( \phi \in H^0(E_\phi) \),
2. \( (E_\phi, \phi) \) is a \( \tau \)-stable pair,
3. \( E_{ss} \) is a direct sum \( \bigoplus_i E_i \) of stable bundles, all of slope \( \hat{\tau} \).

**Lemma 3.3.** Fix a critical value \( p/q \in (d/R, d/(R-1)) \). Suppose that \( E_{\bar{\partial}_E} = E_\phi \oplus E_{ss} \) is part of a non-\( \tau \)-stable pair in \( f^{-1}(p/q) \) as above. Let the degree and rank of \( E_\phi \) be \( (R_\phi, d_\phi) \) and those of \( E_i \) be \( (R_i, d_i) \). Then we have the following constraints:

1. \( d_i/R_i = (d - d_\phi)/(R - R_\phi) = p/q \);
2. \( \sum_i R_i = R - R_\phi \);
3. \( d_\phi/R_\phi < p/q < d_\phi/(R_\phi - 1) \).

Conversely, given any stable pair \( (E_\phi, \phi) \) and set of stable bundles \( E_i \) such that the conditions above are satisfied, we obtain a representative for a fixed point \( (E_\phi \oplus \bigoplus_i E_i, \phi) \) in the critical level set corresponding to \( p/q \).

We see that to a fixed point \( (\bar{\partial}_E, \phi) \in \mathcal{Z}_\tau \) we can assign an \( (n+2) \)-tuple of integers \( (d_\phi, R_\phi, R_1, \ldots, R_n) \). The degrees \( d_i \) are then determined by condition (i) of Lemma 3.3. For \( \hat{\tau} = p/q \in (d/R, d/(R-1)) \), let

\[
I_\tau = \left\{ (d_\phi, R_\phi, R_1, \ldots, R_n) \in \mathbb{Z}^{n+2} : \text{ with } \frac{p}{q} = \hat{\tau} = \frac{d - d_\phi}{R - R_\phi} \right\}
\] (3.4)

with conditions (ii) and (iii) of Lemma 3.3 are satisfied.
and set
\begin{equation}
Z(d_\phi, R_\phi, R_1, \ldots, R_n) = \left\{ (\partial E, \phi) \in Z_\tau : E^{\partial E} = E_\phi \oplus \bigoplus_{i=1}^n E_i \right\}
\end{equation}

where degree($E_\phi$) = $d_\phi$, rank($E_\phi$) = $R_\phi$, and rank($E_i$) = $R_i$ for $i = 1, \ldots, n$. We can extend this notation to include the extreme values $\hat{\tau} = d/R$ and $\hat{\tau} = d/(R - 1)$. This requires the convention that

(a) if $\hat{\tau} = d/R$, then $d_\phi = R_\phi = 0$ and $\sum_i R_i = R$,
(b) if $\hat{\tau} = d/(R - 1)$, then $d_\phi = 0$, $R_\phi = 1$ and conditions (ii) and (iii) of Lemma 3.3
are satisfied with $p/q = \hat{\tau} = d/(R - 1)$.

Then Lemma 3.3 can be rephrased as

**Proposition 3.4.** Let $\hat{\tau}$ be as in Lemma 3.3. Then
\begin{equation}
Z_\tau = \bigcup_{(d_\phi, R_\phi, R_1, \ldots, R_n) \in I_\tau} Z(d_\phi, R_\phi, R_1, \ldots, R_n).
\end{equation}

**Example 3.5.** In the case of rank two the only possibility for split bundles is $E_\phi \oplus E_s$, where $E_s$ is a line bundle of degree $\hat{\tau}$ and $\phi \in H^0(E_\phi)$, where $E_\phi$ is a line bundle of degree $d - \hat{\tau}$. The pair $(E_\phi, \phi)$ is determined up to equivalence by the divisor class of $\phi$ (see [B2]); hence, the space of equivalence classes of pairs $(E_\phi, \phi)$ is simply the $d - \hat{\tau}$ symmetric product $\text{Sym}^{d - \hat{\tau}}\Sigma$. Since $E_s$ is arbitrary, we have
\begin{equation}
Z_\tau \simeq \text{Sym}^{d - \hat{\tau}}\Sigma \times J_{\hat{\tau}},
\end{equation}

where $J_{\hat{\tau}}$ denotes the component of the Jacobian variety of $\Sigma$ corresponding to line bundles of degree $\hat{\tau}$. 
4. The algebraic stratification of $\hat{B}$

In this and the next section we examine two stratifications of the master space $\hat{B}$. From one point of view these are a consequence of the fact that the holomorphic pairs in $\hat{B}$ can be characterized by a stability property, and as such admits two natural filtrations. The filtrations are the analogs of the Seshadri filtration for semistable bundles, and just as in that case, can be used to associate “gradings” to the stable pairs. These lead to stratifications according to grading type. From a different perspective, the gradings and filtrations can be understood in terms of the Morse theory of the moment map $f$ on $\hat{B}$. In that context the gradings correspond to the critical points at infinity on flow lines either up or down the gradient of the moment map. The stratifications thus correspond to the stratifications given by the stable (or unstable) manifolds in the sense of Morse Theory. This will be explained in the next section. We first give a purely algebraic description. Let $(E, \phi)$ be a holomorphic pair, where $E$ is here understood as a holomorphic bundle, i.e. the underlying smooth bundle with a $\bar{\partial}_E$-operator. The two filtrations are characterized by the parameters $\mu_+(E)$ and $\mu_-(E)$, where

$$
\mu_+(E) = \max \{ \mu(E') : E' \subset E \text{ is a holomorphic subbundle} \}
$$

$$
\mu_-(E, \phi) = \min \{ \mu(E/E'') : E'' \subset E \text{ is a holomorphic subbundle and } \phi \in H^0(E'') \}.
$$

Generalizing Definition 1.1, we call the pair $(E, \phi)$ stable if

$$
\mu_+(E) < \mu_-(E, \phi).
$$

This is clearly equivalent to the pair being $\tau$-stable for all $\mu_+(E) < \tau < \mu_-(E, \phi)$. Thus the set of isomorphism classes of stable pairs is the union over all $\tau$ of the $B_\tau$, or equivalently, the union of the reduced level sets $f^{-1}(\tau)/U(1)$ (cf. Proposition 2.18). Conversely, all points in $\hat{B}$ are represented by stable pairs. We will refer to the two filtrations associated to a stable pair as the $\mu_-$-filtration and the $\mu_+$-filtration.

**Proposition 4.1.** (The $\mu_-$-filtration) Let $(E, \phi)$ be a stable holomorphic pair. There is a filtration of $E$ by subbundles

$$
0 \subset E_\phi = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E
$$

such that the following properties hold:

(i) $\phi \in H^0(E_\phi)$, the pair $(E_\phi, \phi)$ is a stable pair, and $\mu_+(E_\phi) < \mu_-(E, \phi) < \mu_-(E_\phi, \phi)$,
(ii) for $i = 1, \ldots, n$ the quotients $F_i/F_{i-1}$ are stable bundles each of slope $\mu(F_i/F_{i-1}) = \mu_-(E, \phi)$,

(iii) $E_{\phi}$ has minimal rank among filtrations satisfying (i) and (ii).

The subbundle $E_{\phi}$ is uniquely determined, and the graded object

$$gr^-(E, \phi) = E_{\phi} \oplus F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus E/F_{n-1}$$

is unique up to isomorphism of $F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus E/F_{n-1}$.

Using this result, we define the $\mu_-$-grading for the pair $(E, \phi)$ by

**Definition 4.2.** The $\mu_-$-grading for a stable pair $(E, \phi)$ is given by

$$gr^-(E, \phi) = (gr^-(E), \phi)$$

where $gr^-(E)$ is as above.

We obtain a convenient way to interpret this grading if we adopt the convention that the slope of a stable pair is $\mu(E, \phi) = \tau$ where $\tau$ is any number such that $\mu_+(E) < \tau < \mu_-(E, \phi)$. Then the pair $(E_{\phi}, \phi)$ has slope $\mu_-(E, \phi)$. By rewriting

$$gr^-(E, \phi) = (E_{\phi}, \phi) \oplus F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus E/F_{n-1},$$

this then becomes a direct sum of stable objects all of the same slope.

In a similar way, the second filtration will be characterized by the fact that all quotients are stable objects of slope $\mu_+(E)$. Such a filtration is well known in the case when $\mu_+(E) = \mu(E)$, i.e. when $E$ is a semistable bundle. Indeed the usual Seshadri filtration has this property. If the bundle is not semistable, and thus $\mu_+(E) > \mu(E)$, we obtain the filtration a follows:

**Proposition 4.3.** (The $\mu_+$-filtration) Let $(E, \phi)$ be a stable holomorphic pair. There is a filtration of $E$ by subbundles

$$(4.2) \quad 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset F_{n+1} = E$$

such that the following properties hold: If $E$ is semistable then this is a Seshadri filtration and the quotients $F_i/F_{i-1}$ are all stable bundles of slope $\mu_+(E) = \mu(E)$. Otherwise,

(i) for $i = 1, \ldots, n$ the quotients $F_i/F_{i-1}$ are stable bundles each of slope $\mu(F_i/F_{i-1}) = \mu_+(E)$,
(ii) \( \phi \) has a non-zero projection, \( \varphi \), into \( H^0(E/F_n) \), and the pair \( (E/F_n, \varphi) \) is a stable pair with \( \mu_+(E/F_n) < \mu_+(E) < \mu_-(E/F_n) \),

(iii) \( E/F_n \) has minimal rank among filtrations satisfying (i) and (ii).

In the case where \( \mu_+(E) > \mu(E) \), the quotient \( Q = E/F_n \) is uniquely determined, and the graded object

\[
gr^+(E) = F_1/F_0 \oplus F_2/F_1 \oplus \ldots F_n/F_{n-1} \oplus Q
\]

is unique up to isomorphism of \( F_1/F_0 \oplus F_2/F_1 \oplus \ldots F_n/F_{n-1} \).

**Definition 4.4.** For a stable pair \( (E, \phi) \) for which \( \mu_+(E) > \mu(E) \), the \( \mu_+(E) \)-grading is defined to be

\[
\text{gr}^+(E, \phi) = (\text{gr}^+(E), \varphi).
\]

For \( \mu_+(E) = \mu(E) \), we set

\[
\text{gr}^+(E, \phi) = (\text{Gr}(E), 0),
\]

where \( \text{Gr}(E) \) is the grading for \( E \) coming from the Seshadri filtration.

Notice that the filtration for the semistable subbundle \( F_n \) is precisely the Seshadri filtration. The case \( \mu_+(E) = \mu(E) \) thus corresponds to the case where \( Q = 0 \). We now prove Propositions 4.1 and 4.3. We begin with the \( \mu_+(E) \)-filtration.

**Proposition 4.5.** Given a stable pair \( (E, \phi) \) there is a unique quotient \( Q \) of \( E \) arising from an exact sequence

\[
0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0,
\]

with the properties:

(i) \( F \) is a semi-stable bundle,

(ii) \( \mu(F) = \mu_+(E) \),

(iii) if \( Q \neq 0 \), then under projection of \( E \) onto \( Q \) the section \( \phi \) has a nontrivial image, \( \varphi \), and the holomorphic pair \( (Q, \varphi) \) is stable.

(iv) If \( Q \neq 0 \), then \( \mu_+(Q) < \mu_+(E) < \mu_-(Q) \),

(v) \( Q \) has minimal rank among quotients satisfying (i) - (iv).

**Proof.** If \( E \) is a semistable bundle then \( \mu_+(E) = \mu(E) \), and we take \( F = E, Q = 0 \). Otherwise, for \( F \) we take the unique maximal semistable subbundle of \( E \). Properties (i) , (ii) and (v) follow immediately from this choice of \( F \). Properties (iii) and (iv) are consequences of the following
Lemma 4.6. Let \((E, \phi)\) be a stable pair with \(\mu_+(E) > \mu(E)\). Let \(F\) be the unique maximal semistable subbundle of \(E\), and let \(Q\) be the quotient \(E/F\). Let \(\varphi \in H^0(Q)\) be the image of \(\phi\) under the projection of \(E\) onto \(Q\). Then

(i) \(\varphi \neq 0\),
(ii) \(\mu_+(Q) < \mu_+(E)\),
(iii) \(\mu_-(Q, \phi) \geq \mu_-(E, \phi)\).

Proof. (i) If \(\varphi = 0\) then \(\phi \in H^0(F)\). But then, as \((E, \phi)\) is stable, \(\mu(E/F) \geq \mu_-(E, \phi) > \mu_+(E) = \mu(F)\), i.e. \(\mu(Q) > \mu(F)\). This is incompatible with \(\mu(F) = \mu_+(E) > \mu(E)\). Thus \(\varphi \neq 0\).

(ii) Let \(Q' \subset Q\) be any holomorphic subbundle. Lift \(Q'\) to a subbundle \(E' \subset E\). This gives a short exact sequence

\[ 0 \to F \to E' \to Q' \to 0. \]

By definition, \(\mu(E') \leq \mu_+(E) = \mu(F)\), but in fact the inequality must be strict since \(\text{rank}(E') > \text{rank}(F)\). It follows from this and the above short exact sequence that \(\mu(Q') < \mu_+(E)\). Thus \(\mu_+(Q) < \mu_+(E)\).

(iii) Suppose in addition that \(\varphi \in Q'\). Then \(\phi \in E'\), and thus \(\mu(E/E') \geq \mu_-(E, \phi)\). But \(\mu(E/E') = \mu(Q/Q')\), and thus it follows that \(\mu_-(Q, \phi) \geq \mu_-(E, \phi)\).

Proof of Proposition 4.3. The Proposition follows immediately from Proposition 4.5, plus the usual Seshadri filtration for the subbundle \(F\).

We now turn to the proof of Proposition 4.1. The key result here is

Lemma 4.7. Let \((E, \phi)\) be a stable pair. Let \(E_\phi \subset E\) be a holomorphic subbundle such that \(\phi \in H^0(E_\phi)\) and \(\mu(E/E_\phi) = \mu_-(E, \phi)\). Then

(i) \(\mu_+(E_\phi) \leq \mu_+(E)\),
(ii) \(\mu_-(E_\phi, \phi) \geq \mu_-(E, \phi)\), and the inequality is strict if \(E_\phi\) has minimal rank among all subbundles satisfying the hypotheses of the Lemma,
(iii) \((E_\phi, \phi)\) is a stable pair,
(iv) \(E/E_\phi\) is a semi-stable bundle,
(v) \(\mu(E_\phi) < \mu_-(E, \phi)\)
(vi) Suppose that \(E_\phi\) has minimal rank among all subbundles satisfying the hypotheses of the Lemma, and that \(E'_\phi\) is any other subbundle such that \(\phi \in H^0(E'_\phi)\) and \(\mu(E/E'_\phi) = \mu_-.\) Then \(E_\phi \subseteq E'_\phi\).

Proof. (i) The first inequality is clear, since \(E_\phi\) is a subbundle of \(E\).
(ii) Let $E''$ be such that $E'' \subset E_\phi \subset E$, and $\phi \in E''$. Use the following notation:

\[
\begin{array}{cccc}
E'' & E_\phi & E \\
\text{degree} & d'' & d_\phi & d \\
\text{rank} & R'' & R_\phi & R
\end{array}
\]

Then

\[
\mu(E\phi/E'') - \mu(E/E'') = \frac{R(d_\phi - d) + R''(d - d_\phi) + R_\phi(d'' - d)}{(R - R'')(R_\phi - R'')},
\]

and

\[
\mu(E/E'') - \mu(E/E_\phi) = \frac{R(d_\phi - d) + R''(d - d_\phi) + R_\phi(d'' - d)}{(R - R'')(R - R_\phi)}.
\]

Hence,

\[
\mu(E\phi/E'') - \mu(E/E'') = \left(\frac{R - R_\phi}{R_\phi - R''}\right) (\mu(E/E'') - \mu(E/E_\phi)).
\]

The right hand side of this equality is non-negative by the definition of $\mu_-(E, \phi)$, and it is strictly positive if $E_\phi$ has minimal rank among all subbundles satisfying the hypotheses of the Lemma. This is because $\mu(E/E'') = \mu(E/E_\phi)$ would imply that $E''$ is a subbundle satisfying the hypotheses but with rank less than that of $E_\phi$. The result now follows from the fact that $\mu(E/E_\phi) = \mu_-(E, \phi)$.

(iii) This follows immediately from (i) and (ii), and the fact that $(E, \phi)$ is stable.

(iv) Suppose that $E/E_\phi$ is not semistable. Pick a subbundle $F \subset E/E_\phi$ such that $\mu(F) = \mu_+(E/E_\phi)$. Let $E' \subset E$ be the lift of $F$ to $E$, i.e. such that

\[
0 \longrightarrow E_\phi \longrightarrow E' \longrightarrow F \longrightarrow 0.
\]

Now $\mu(E/E') = \mu((E/E_\phi)/F)$, and if $\mu(F) > \mu(E/E_\phi)$ then $\mu((E/E_\phi)/F) < \mu(E/E_\phi)$. Hence

\[
\mu(E/E') < \mu(E/E_\phi).
\]

However, since $\phi \in H^0(E')$, we have

\[
\mu(E/E') \geq \mu_-(E, \phi) = \mu(E/E_\phi).
\]

Thus $E/E_\phi$ must be semistable.

(v) Since $(E, \phi)$ is stable, we have $\mu(E_\phi) \leq \mu_+(E) < \mu_-(E, \phi)$. 
(vi) Let $E_\phi$ and $E'_\phi$ satisfy the hypotheses, and suppose that $E_\phi$ is of minimal rank. Now consider the map $E_\phi \to E/E'_\phi$, and let $K$ and $L$ be its kernel and image respectively. We thus have

\begin{equation}
0 \to K \to E_\phi \to L \to 0
\end{equation}

Suppose that $K \neq E_\phi$. Since $\phi$ is a section of $K$, we have

$$\mu(E_\phi/K) \geq \mu_-(E_\phi, \phi).$$

Also, since by (iv) $E/E'_\phi$ is semistable, we have

$$\mu(L) \leq \mu(E/E'_\phi) = \mu_-(E, \phi).$$

By (ii) we have $\mu_-(E, \phi) < \mu_-(E_\phi, \phi)$, and from (4.3) we have $\mu(E_\phi/K) = \mu(L)$. We thus get

$$\mu(L) \leq \mu_-(E, \phi) < \mu_-(E_\phi, \phi) \leq \mu(E_\phi/K) = \mu(L),$$

which is impossible. We conclude that $K = E_\phi$, i.e. $E_\phi \subset E'_\phi$.

**Proposition 4.8.** Given a stable pair $(E, \phi)$ there is a unique subbundle $E_\phi \subset E$ such that

(i) $\phi \in H^0(E_\phi)$ and $(E_\phi, \phi)$ is a stable pair,

(ii) $E/E_\phi$ is a semi-stable bundle,

(iii) $\mu(E/E_\phi) = \mu_-(E, \phi)$,

(iv) $\mu_+(E_\phi) < \mu_-(E, \phi) < \mu_-(E_\phi, \phi)$

(v) $E_\phi$ has minimal rank among all subbundles satisfying (i)-(iv)

**Proof.** By parts (i)-(v) of Lemma 4.7, any subbundle $E_\phi \subset E$ such that $\phi \in H^0(E_\phi)$ and $\mu(E/E_\phi) = \mu_-$ will satisfy (i)-(iv). By part (vi) of Lemma 4.7, there is a unique such $E_\phi$ of minimal rank.

**Proof of Proposition 4.1.** The required filtration is constructed as follows. Let

$$0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_n = E/E_\phi$$

be the Seshadri filtration for $E/E_\phi$. Set $F_i = \pi^{-1}(Q_i)$, where $\pi : E \to E/E_\phi$ is the projection map.

**Remark.** An important feature of the two gradings, $gr^- (E, \phi)$ and $gr^- (E, \phi)$ is that, as pairs, they are both semistable and thus correspond to points in the masterspace $\hat{B}$. Indeed
for \( gr^{-}(E, \phi) \) we have \( \mu_{-}(gr^{-}(E)) = \mu_{+}(gr^{-}(E)) = \mu_{-}(E) \), while for \( gr^{+}(E, \phi) \) we have \( \mu_{-}(gr^{+}(E)) = \mu_{+}(gr^{+}(E)) = \mu_{+}(E) \).

It is worth pointing out that there are other filtrations associated to a stable pair, but none of the resulting gradings are semistable as pairs and thus are not represented in \( \mathcal{B} \). These other gradings arise by successive applications of Propositions 4.5 and 4.8. For example, one can successively apply Proposition 4.8 to \( E, \phi \) in the extension

\[
0 \to E_{\phi} \to E \to Q \to 0 ,
\]

until the subbundle containing \( \phi \) is of rank one. This yields a grading of the form \( ([\phi] \oplus Gr(E/[\phi]), \phi) \), where \([\phi]\) is the line subbundle generated by \( \phi \), and \( Gr(E/[\phi]) \) is the Seshadri grading for \( E/[\phi] \). Similarly, successive application of Proposition 4.5 to the quotient pair \((Q, \varphi)\) leads in all cases (i.e. not just when \( E \) is semistable) to the grading \((Gr(E), 0)\). Such gradings and their relation to \( gr^{\pm}(E, \phi) \) will be discussed in a future publication.

We now use the gradings \( gr^{\pm}(E, \phi) \) to define a stratification of \( \mathcal{B} \). Notice firstly that for any pair \((\partial E, \phi)\), the gradings \( gr^{\pm}(E, \phi) \) are each characterised by \((n+2)\)-tuples of integers \((d_{\phi}, R_{\phi}, R_{1}, \ldots, R_{n})\) which satisfy the constraints in Lemma 3.3. The notation is such that the pair \((d_{\phi}, R_{\phi})\) refers to the degree and rank of the summand in \( gr^{\pm}(E, \phi) \) which contains the section. In the case of \( gr^{-}(E, \phi) \) this is \( E_{\phi} = F_{0} \) in the notation of (4.1), while for \( gr^{+}(E, \phi) \) this is the quotient \( Q \) of Proposition 4.5. In both cases the \( R_{i} \) give the ranks of the quotients \( F_{i}/F_{i-1} \).

**Definition 4.9.** Given a stable pair \((E, \phi)\) and an \((n+2)\)-tuple \((d_{\phi}, R_{\phi}, R_{1}, \ldots, R_{n}) \in I_{\tau}\), let

\[
W^{\pm}(d_{\phi}, R_{\phi}, R_{1}, \ldots, R_{n}) = \left\{ (E, \phi) \in \mathcal{B} : (E, \phi) \text{ is a stable pair, and} \right. \\
\left. gr^{\pm}(E, \phi) \in Z(d_{\phi}, R_{\phi}, R_{1}, \ldots, R_{n}) \right\} \cup Z(d_{\phi}, R_{\phi}, R_{1}, \ldots, R_{n}) .
\]

Set

\[
W^{\pm}_{\tau} = \bigcup_{(d_{\phi}, R_{\phi}, R_{1}, \ldots, R_{n}) \in I_{\tau}} W^{\pm}(d_{\phi}, R_{\phi}, R_{1}, \ldots, R_{n}) .
\]

The next proposition justifies our definition of \( W^{\pm}(d_{\phi}, R_{\phi}, R_{1}, \ldots, R_{n}) \). Its proof is straightforward, and since we shall not have need of the statement in the sequel we omit the details.
Proposition 4.10. With respect to the obvious ordering of critical values of \( \hat{\tau} \) in 
\([d/R, d/(R-1)]\), the subspaces \( \mathcal{W}_\tau^\pm \) form a piecification of \( \hat{\mathcal{B}} \) in the sense of Goresky-MacPherson (see [G-M]). A similar result holds for \( \mathcal{W}_\tau^- \). Moreover, the subspaces

\[
\mathcal{W}^\pm(d_\phi, R_\phi, R_1, \ldots, R_n) \cap \hat{\mathcal{B}}_0
\]

are V-manifolds for all \((d_\phi, R_\phi, R_1, \ldots, R_n) \in I_\tau\).

We end this section with the following proposition which will be used in §6:

Proposition 4.11. If \( R > 2 \), then for critical values \( \hat{\tau} \in (d/R, d/(R-1)) \) the complex codimension of \( \mathcal{W}_\tau^\pm \) in \( \hat{\mathcal{B}} \) is \( \geq 2 \).

For the proof we shall need the following simple

Lemma 4.12. Suppose \((E_\phi, \phi)\) is a \( \tau \)-stable pair and \( E_s \) is a semistable bundle with slope \( \tau \). Then \( H^0(E_\phi \otimes E_s^*) = \{0\} \).

Proof. Suppose \( \alpha \in H^0(E_\phi \otimes E_s^*) \) and \( \alpha \neq 0 \). Then \( \alpha \) defines a map of sheaves \( E_s \to E_\phi \), and since \( \alpha \neq 0 \), \( \text{rank}(\ker \alpha) < \text{rank}(E_s) \). But then the semistability of \( E_s \) and the \( \tau \)-stability of \( E_\phi \) imply

\[
\tau = \mu(E_s) \leq \mu(E_s/\ker \alpha) = \mu(\text{image} \alpha) < \tau,
\]

which is a contradiction. This proves the Lemma.

Proof of Proposition 4.11. It suffices to compute the codimension of the largest stratum \( \mathcal{W}^\pm(d_\phi, R_\phi, R_s) \), where \((d_\phi, R_\phi, R_s) \in I_\tau\). We first consider \( \mathcal{W}^- \). Let \((E, \phi)\) be a stable pair such that \( gr^+(E, \phi) \in Z(d_\phi, R_\phi, R_s) \). Then we have an exact sequence

\[
0 \to E_\phi \to E \to E_s \to 0.
\]

The tangent space to \( \mathcal{W}^- (d_\phi, R_\phi, R_s) \) at \((E, \phi)\) naturally splits

\[
T_{(E, \phi)} \mathcal{W}^- (d_\phi, R_\phi, R_s) \cong T_{(E_\phi, \phi; E_s)} Z(d_\phi, R_\phi, R_s) \oplus \text{Ext}^1(E_s, E_\phi).
\]

The dimension of \( Z(d_\phi, R_\phi, R_s) \) is computed as in Section 3 of [B-D1]:

\[
\dim Z(d_\phi, R_\phi, R_s) = d - \hat{\tau} R_s + 1 + (R - R_s - 1)(R - R_s)(g - 1) + R_s^2(g - 1),
\]

and \( \text{Ext}^1(E_s, E_\phi) \cong H^1(E_\phi \otimes E_s^*) \). By Lemma 4.12 and Riemann-Roch we have that
\begin{align*}
\dim T_{(E, \phi)} \mathcal{W}^-(d_\phi, R_\phi, R_s) &= \dim \mathcal{Z}(d_\phi, R_\phi, R_s) + (\hat{\tau} R - d + (R - R_s)(g - 1)) R_s \\
&= d + 1 + (R^2 - R)(g - 1) + R_s(R_s - R + 1)(g - 1) \\
&\quad + (\hat{\tau}(R - 1) - d) R_s.
\end{align*}

By Theorem 2.16,

\[ p_-(R_s, \hat{\tau}) \equiv \text{codim } \mathcal{W}^-(d_\phi, R_\phi, R_s) = (d - \hat{\tau}(R - 1)) R_s + R_s(R - R_s - 1)(g - 1). \]

For \( 1 < R_s < R - 1 \) the last term in the expression above is \( \geq 2 \) (we assume \( g > 1 \)), and since \( \hat{\tau} < d/(R - 1) \) the first term is positive. Now we check the case where \( R_s = 1 \). Then

\[ p_-(1, \hat{\tau}) = d - \hat{\tau}(R - 1) + (R - 2)(g - 1). \]

Since we assume \( R > 2 \) and \( g > 1 \) the last term is \( \geq 1 \). Also, \( d - \hat{\tau}(R - 1) \) is a positive integer and so must also be \( \geq 1 \). Thus, \( p_-(1, \hat{\tau}) \geq 2 \). For \( R_s = R - 1 \),

\[(4.4) \quad p_-(R - 1, \hat{\tau}) = (d - \hat{\tau}(R - 1))(R - 1). \]

Again, \( d - \hat{\tau}(R - 1) \) is a positive integer and \( R - 1 \geq 2 \). Therefore, in all cases we have \( \text{codim } \mathcal{W}^-(d_\phi, R_\phi, R_s) \geq 2 \). Now consider \( \mathcal{W}^+(d_\phi, R_\phi, R_s) \). Let \( (E, \phi) \) be a stable pair such that \( \text{gr}^+(E, \phi) \in \mathcal{Z}(d_\phi, R_\phi, R_s) \). Then \( E \) may be written (see Proposition 4.3):

\[ 0 \rightarrow F \rightarrow E \xrightarrow{\pi} Q \rightarrow 0. \]

As in the case of \( \mathcal{W}^+ \) the tangent space

\[ T_{(E, \phi)} \mathcal{W}^+(d_\phi, R_\phi, R_s) \supset T_{(Q, \pi(\phi); F)} \mathcal{Z}(d_\phi, R_\phi, R_s) \]

as a summand. The complement is naturally isomorphic to the space of extensions of \( Q \) by \( F \) direct sum with the equivalence classes of liftings of \( \pi(\phi) \). The liftings are parameterized by \( H^0(F) \), and two liftings are equivalent if and only if they differ by an element of \( H^0(Q^* \otimes F) \). Therefore,

\begin{align*}
\dim T_{(E, \phi)} \mathcal{W}^+(d_\phi, R_\phi, R_s) &= \dim T_{(Q, \pi(\phi); F)} \mathcal{Z}(d_\phi, R_\phi, R_s) + \dim H^1(Q^* \otimes F) \\
&\quad + \dim H^0(F) - \dim H^0(Q^* \otimes F).
\end{align*}
Since $F$ is stable with slope $\hat{\tau} > d/R > 2g - 2$, $H^1(F) = 0$. Therefore, by Riemann-Roch

$$\dim T_{(E, \phi)} W^+(d_\phi, R_\phi, R_s) = d - \hat{\tau}R_s + 1 + (R - R_s - 1)(R - R_s)(g - 1) + R_s^2(g - 1)$$

$$+ (d - \hat{\tau}(R - 1) + (R - R_s - 1)(g - 1)) R_s$$

$$= d + 1 + (R^2 - R)(g - 1) + (d - R\hat{\tau}) R_s$$

$$+ R_s(R_s - R)(g - 1).$$

By Theorem 2.16, this implies

$$p_+(R_s, \hat{\tau}) \equiv \text{codim } W^+(d_\phi, R_\phi, R_s) = (R\hat{\tau} - d) R_s + R_s(R - R_s)(g - 1).$$

Since $\hat{\tau} > d/R$,

$$p_+(R_s, \hat{\tau}) > R_s(R - R_s)(g - 1),$$

and it is easily checked that the latter expression is always $\geq 2$ for $R > 2$, $g > 1$, and $1 \leq R_s \leq R - 1$. This completes the proof of Proposition 4.11.

5. The Morse theory of $f$

We now turn to the description of the Morse theory of the function $f$. We shall write down solutions to the gradient flow of $f$ and describe the stable and unstable manifold stratifications of $\hat{B}$. Furthermore, we show that the Morse theoretical stratification of $\hat{B}$ coincides with the algebraic stratification of the previous section. The results of this section are similar in spirit to the results of [D]. However, the situation here is technically simpler because we are dealing with a finite dimensional problem and an abelian group action (see also [K1]).

**Proposition 5.1.** Let $\Phi : \hat{B} \times [0, \infty) \to \hat{B}$ be the flow

$$\Phi_t[\bar{\partial}_E, \phi] = [\bar{\partial}_E, e^{-t/2\pi R}\phi].$$

Then $\Phi$ is continuous. Moreover, $\Phi$ preserves $\hat{B}_0$ and coincides with the gradient flow of $f$ on $\hat{B}_0$.

**Proof.** We must verify that

$$\frac{d\Phi_t}{dt} = -\nabla_{\Phi_t} f.$$
First recall that after identifying $T_{[\partial_E,\phi]}\hat{B}$ with $H^1(C_{\phi,0})$, the infinitesimal vector field of the $U(1)$ action on $\hat{B}$ is given by

$$\xi^#([\partial_E,\phi]) = \frac{i}{R}(0,\phi).$$  

Indeed,

$$\xi^#([\partial_E,\phi]) = \left. \frac{d}{d\theta} \right|_{\theta=0} [\partial_E, e^{i\theta/R}\phi] = \frac{i}{R}(0,\phi).$$

Moreover,

$$\nabla_{\Phi_t[\partial_E,\phi]} f = \frac{-1}{2\pi i} \nabla_{\Phi_t[\partial_E,\phi]} \Psi = \frac{1}{2\pi i} \xi^#(\Phi_t[\partial_E,\phi])$$

$$= \frac{1}{2\pi i} \left. \frac{d}{dt} \right|_{t=0} (0,e^{-t/2\pi R}\phi) = -\frac{d}{dt} \left(0,e^{-t/2\pi R}\phi\right) = -\frac{d\Phi_t[\partial_E,\phi]}{dt},$$

which is what was to be shown.

**Definition 5.2.** Given a critical $\hat{\tau}$ and

$$(d\phi, R\phi, R_1, \ldots, R_n) \in I_{\hat{\tau}}$$

as in §3, let

$$W^s(d\phi, R\phi, R_1, \ldots, R_n) = \left\{ [\partial_E,\phi] \in \hat{B} : \lim_{t \to \infty} \Phi_t[\partial_E,\phi] \in \mathcal{Z}(d\phi, R\phi, R_1, \ldots, R_n) \right\},$$

and let $W^u(d\phi, R\phi, R_1, \ldots, R_n)$ be defined similarly as $t \to -\infty$. Also, for a critical value of $\hat{\tau}$, we set

$$W^s_{\hat{\tau}} = \bigcup_{(d\phi, R\phi, R_1, \ldots, R_n) \in I_{\hat{\tau}}} W^s(d\phi, R\phi, R_1, \ldots, R_n)$$

$$W^u_{\hat{\tau}} = \bigcup_{(d\phi, R\phi, R_1, \ldots, R_n) \in I_{\hat{\tau}}} W^u(d\phi, R\phi, R_1, \ldots, R_n).$$

We call $\{W^s_{\hat{\tau}}\}$ and $\{W^u_{\hat{\tau}}\}$ the stable and unstable Morse stratifications of $\hat{B}$, respectively.

**Theorem 5.3.** For each critical value $\tau$, $W^s_{\tau} = W^+_{\tau}$, and $W^u_{\tau} = W^-_{\tau}$. Consequently, the Morse stratification of $\hat{B}$ coincides with the algebraic stratification of §3.

**Proof.** We shall show that $W^u_{\tau} = W^-_{\tau}$. Indeed, since both $\{W^u\}$ and $\{W^-\}$ are stratifications of $\hat{B}$, it suffices to prove the inclusion $W^-_{\tau} \subset W^u_{\tau}$ for all $\tau$. In fact, we are going to show that

$$W^-(d\phi, R\phi, R_1, \ldots, R_n) \subset W^u(d\phi, R\phi, R_1, \ldots, R_n)$$
for all \((d, R, R_1, \ldots, R_n) \in I_\tau\). Fix \([\partial E, \phi] \in \mathcal{W}^- (d, R, R_1, \ldots, R_n)\). Let
\[
0 = E_\phi = F_0 \subset F_1 \subset \cdots \subset F_n = E
\]
denote the \(\mu_\cdot\) filtration of the pair \((E, \phi)\). Fix real numbers
\[
0 < \mu_1 < \mu_2 < \cdots < \mu_n, \quad \sum_{i=1}^n R_i \mu_i = R_\phi,
\]
and consider the following 1-parameter subgroup of gauge transformations in \(\mathfrak{G}_0^C\),
\[
(5.3) \quad g_t = \begin{pmatrix}
e^{t/2\pi R} & 0 & \cdots & 0 \\
0 & e^{-t \mu_1/2\pi R} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{-t \mu_n/2\pi R}
\end{pmatrix}
\]
written diagonally with respect to the filtration above. Then
\[
\lim_{t \to -\infty} \Phi_t[\partial E, \phi] = \lim_{t \to -\infty} [\partial E, e^{-t/2\pi R} \phi] = \lim_{t \to -\infty} [\partial E, g_t^{-1} \phi] = \lim_{t \to -\infty} [g_t(\partial E), \phi] = [gr^-(E, \phi), \phi].
\]
The last equality follows the same way as in [D], p. 716. Hence,
\[
[\partial E, \phi] \in \mathcal{W}^u (d, R, R_1, \ldots, R_n).
\]
The case of the stable manifolds is similar. To prove that \(\mathcal{W}^+(d, R, R_1, \ldots, R_n) \subset \mathcal{W}^s (d, R, R_1, \ldots, R_n)\) for all \((d, R, R_1, \ldots, R_n) \in I_\tau\) one must show that for all \([\partial E, \phi] \in \mathcal{W}^+(d, R, R_1, \ldots, R_n),\)
\[
\lim_{t \to \infty} \Phi_t[\partial E, \phi] = [gr^+(E, \phi), \phi],
\]
where \([gr^+(E, \phi), \phi]\) is the \(\mu_+\)-grading as defined in §4. The above method can be used, but now the complex gauge transformations \(g_t \in \mathfrak{G}_0^C\) must be defined as follows. Let
\[
0 = F_0 \subset F_1 \subset \cdots \subset F_{n+1} = E
\]
denote the $\mu_+$-filtration of the pair $(E, \phi)$. If $E$ is a semistable bundle, fix real numbers
\[ 1 > \mu_1 > \mu_2 > \cdots > \mu_{n+1} , \quad \sum_{i=1}^{n} R_i \mu_i = 0 , \]
and let $g_t$ be
\[(5.3)\]
\[ g_t = \begin{pmatrix}
  e^{t \mu_1 / 2\pi R} & 0 & \cdots & 0 \\
  0 & e^{t \mu_2 / 2\pi R} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & e^{t \mu_{n+1} / 2\pi R}
\end{pmatrix} \]
written diagonally with respect to the filtration above. If $E$ is not semistable, then one must take $\mu_{n+1} = 1$ and impose the constraint $R_{n+1} + \sum_{i=1}^{n} R_i \mu_i = 0$. The rest of the argument proceeds as before.

6. Birational equivalence of stable pairs

In this section we describe how the moduli of vortices $\mathcal{B}_\tau$ change with respect to $\tau$. The analogous situation has been studied in the symplectic category by Guillemin and Sternberg [G-S] and in the algebraic category by Goresky and MacPherson [G-M]. However, since $\mathcal{B}$ has singularities and no obvious embedding in projective space compatible with the $U(1)$ action, the results of [G-S] and [G-M] are not directly applicable to the case at hand. We thus prove the following directly:

**Theorem 6.1.** (i) Suppose the interval $[\hat{\tau}, \hat{\tau} + \varepsilon]$ contains no critical value of the function $f$. Then the Morse flow induces a biholomorphism between $\mathcal{B}_{\tau+\varepsilon}$ and $\mathcal{B}_\tau$. (ii) Suppose that $\hat{\tau}$ is the only critical value of $f$ in the interval $[\hat{\tau}, \hat{\tau} + \varepsilon]$. Then the Morse flow defines a continuous map from $\mathcal{B}_{\tau+\varepsilon}$ onto $\mathcal{B}_\tau$ which restricts to a biholomorphism between $\mathcal{B}_{\tau+\varepsilon} \setminus \mathbb{P}(W^+)_{\tau}$ and $\mathcal{B}_\tau \setminus Z_\tau$, where
\[ \mathbb{P}_\varepsilon(W^+_{\tau}) = W^+_{\tau} \cap f^{-1}(\tau + \varepsilon)/U(1) . \]
(iii) In the case $R = 2$, the restriction of the Morse flow to $\mathbb{P}_\varepsilon(W^+_{\tau})$ induces a map
\[ \mathbb{P}_\varepsilon(W^+_{\tau}) \rightarrow Z_\tau \]
which is a holomorphic projective bundle (unless $d$ is even and $\hat{\tau} = d/2$). In particular, in rank two $\mathbb{P}_\varepsilon(W^+_{\tau})$ is a smooth subvariety of $\mathcal{B}_{\tau+\varepsilon}$.

**Proof.** (i) For the sake of notational simplicity we denote the equivalence class of the pair $[\partial_E, \phi] \in \mathcal{B}$ by $x$. Let
\[(6.1)\]
\[ F : f^{-1}(\hat{\tau} + \varepsilon) \times [0, \infty) \rightarrow \mathbb{R} \]
denote the map \( F(x, t) = f(\Phi_t(x)) \). By our assumption on \([\hat{\tau}, \hat{\tau} + \varepsilon]\), \( F \) is smooth. Moreover, given \((x, t) \in F^{-1}(\hat{\tau}) \) we have
\[
\frac{\partial F}{\partial t} \bigg|_{(x,t)} = df_{\Phi_t(x)} \left( \frac{\partial \Phi_t}{\partial x} \bigg|_x \right) = \| \nabla_{\Phi_t(x)} f \|^2 \neq 0 ,
\]
since \( \hat{\tau} = f(\Phi_t(x)) \) is not a critical value of \( f \). Then by the implicit function theorem we can solve \( F(x, t) = \hat{\tau} \) as \( t = t(x) \), where \( t \) is a smooth function of \( x \). We define
\[
(6.2) \quad \hat{\sigma}_+ : f^{-1}(\hat{\tau} + \varepsilon) \rightarrow f^{-1}(\hat{\tau})
\]
by \( \hat{\sigma}_+(x) = f(x, t(x)) \). It follows that \( \hat{\sigma}_+ \) is a diffeomorphism between \( f^{-1}(\hat{\tau} + \varepsilon) \) and \( f^{-1}(\hat{\tau}) \).

Next we show that \( \hat{\sigma}_+ \) is a CR-map with respect to the induced CR-structure on the level sets \( f^{-1}(\hat{\tau} + \varepsilon) \) and \( f^{-1}(\hat{\tau}) \). Indeed, let
\[
X \in T^{1,0} \hat{B} \cap T f^{-1}(\hat{\tau} + \varepsilon) \otimes \mathbb{C} ,
\]
and let \( \overline{X} \) denote the complex conjugate. Then
\[
d\hat{\sigma}_+(\overline{X}) = \frac{\partial \Phi_t}{\partial t} \left( \frac{\partial t}{\partial x}(X) \right) + \frac{\partial \Phi_t}{\partial x}(X) = \nabla f \left( \frac{\partial t}{\partial x}(X) \right) + \frac{\partial \Phi_t}{\partial x}(X) .
\]
Since \( \Phi_t \) is holomorphic in \( x \), \( \partial \Phi_t / \partial x(x) = 0 \). Hence,
\[
d\hat{\sigma}_+(\overline{X}) = \nabla f \left( \frac{\partial t}{\partial x}(X) \right)
\]
is both tangential and normal to \( f^{-1}(\hat{\tau}) \), and therefore \( d\hat{\sigma}_+(\overline{X}) = 0 \). Thus, \( \hat{\sigma}_+ \) is a CR-map. Since \( \hat{\sigma}_+ \) and the CR-structures on \( f^{-1}(\hat{\tau} + \varepsilon) \) and \( f^{-1}(\hat{\tau}) \) are \( U(1) \)-invariant, \( \hat{\sigma}_+ \) induces a biholomorphism
\[
(6.3) \quad \sigma_+ : B_{\tau+\varepsilon} = f^{-1}(\hat{\tau} + \varepsilon)/U(1) \rightarrow f^{-1}(\hat{\tau})/U(1) = B_\tau .
\]

(ii) The same argument as in (i) gives a smooth map
\[
(6.4) \quad \hat{\sigma}_+ : f^{-1}(\hat{\tau} + \varepsilon) \setminus \mathcal{W}_\tau^+ \rightarrow f^{-1}(\hat{\tau}) \setminus \mathcal{Z}_\tau .
\]
We extend \( \hat{\sigma}_+ \) across \( \mathcal{W}_\tau^+ \) by setting \( \hat{\sigma}_+(x) = \lim_{t \rightarrow \infty} \Phi_t(x) \) for \( x \in \mathcal{W}_\tau^+ \). We are going to show that \( \hat{\sigma}_+ \) is continuous. It is easily seen (e.g. from the uniqueness of the filtration) that
the restrictions of $\hat{\sigma}_+ \to f^{-1}(\hat{\tau}_\varepsilon)\backslash W^+_\tau$ and $W^+_\tau$ are continuous. Therefore, it suffices to prove that if $\{x_l\}$ is a sequence in $f^{-1}(\hat{\tau} + \varepsilon)$, $x \in W^+_\tau$, and $x_l \to x$, then $\hat{\sigma}_+(x_l) \to \hat{\sigma}_+(x)$. Let $\rho$ be a metric compatible with the topology of $\hat{\mathcal{B}}$. Set $t_l = t(x_l)$, where $\hat{\sigma}_+(x_l) = f(\Phi_{t_l}(x_l))$. Then

$$
\rho(\hat{\sigma}_+(x_l), \hat{\sigma}_+(x)) \leq \rho(\Phi_{t_l}(x_l), \Phi_{t_l}(x)) + \rho(\Phi_{t_l}(x), \hat{\sigma}_+(x)).
$$

Since clearly $t_l \to \infty$ and $\Phi_t$ is uniformly continuous, both terms on the right hand side of the above inequality go to zero, and this proves the continuity of $\hat{\sigma}_+$. Since $\hat{\sigma}_+$ is also $U(1)$-invariant, it induces a continuous map

$$(6.5) \quad \sigma_+: \mathcal{B}_{\tau + \varepsilon} = f^{-1}(\hat{\tau} + \varepsilon)/U(1) \longrightarrow f^{-1}(\hat{\tau})/U(1) = \mathcal{B}_\tau.$$

On the other hand, by the same argument as in (i), $\sigma_+$ defines a biholomorphism onto its image away from

$$
P_{\varepsilon}(W^+) = W^+ \cap f^{-1}(\hat{\tau} + \varepsilon)/U(1).$$

(iii) First, we suppose that $\hat{\tau} > d/2$ since otherwise $P_{\varepsilon}(W^+) = \mathcal{B}_{\tau + \varepsilon}$. Then the fixed point sets $Z_\tau$ in $\hat{\mathcal{B}}$ are smooth, and hence $W^+ \cap f^{-1}(\hat{\tau} + \varepsilon)$ is a smooth submanifold of $\hat{\mathcal{B}} \backslash f^{-1}(d/2)$. The Morse flow clearly induces a continuous map

$$(6.6) \quad W^+ \cap f^{-1}(\hat{\tau} + \varepsilon) / U(1) \longrightarrow Z_\tau,$$

which is an odd dimensional sphere bundle (say with fiber $S^{2n+1}$) over $Z_\tau$. Since $W^+$ is an analytic subvariety, the CR-structure on $f^{-1}(\hat{\tau} + \varepsilon)$ induces a CR-structure on the intersection with $W^+$, and as in the proof of part (ii) above, the map $\pi$ is a CR-map. Since $\pi$ is also $U(1)$-invariant and the $U(1)$ action is CR, $\pi$ descends to a holomorphic map

$$(6.7) \quad W^+ \cap f^{-1}(\hat{\tau} + \varepsilon)/U(1) \longrightarrow Z_\tau,$$

with fiber $S^{2n+1}/U(1) \simeq \mathbb{P}^n$. This completes the proof of Theorem 6.1.

By reversing the orientation of the flow lines, we obtain a similar result relating $\mathcal{B}_\tau$ and $\mathcal{B}_{\tau - \varepsilon}$. Combining the two results immediately proves the

**Corollary 6.2.** If $\hat{\tau}$ is the only critical value of $\Psi$ is $[\hat{\tau} - \varepsilon, \hat{\tau} + \varepsilon]$, then $\mathcal{B}_{\tau - \varepsilon}$, $\mathcal{B}_{\tau + \varepsilon}$ are related by the diagram

$$
\begin{array}{ccc}
\mathcal{B}_{\tau - \varepsilon} & \sigma_- \searrow & \mathcal{B}_{\tau + \varepsilon} \\
& \mathcal{B}_\tau & \\
\end{array}
$$

with $\sigma_-$ and $\sigma_+$ as indicated.
where $\sigma_\pm$ are continuous maps. Moreover,

$$\sigma_\pm : \mathcal{B}_{\tau \pm \varepsilon} \setminus \sigma_\pm^{-1}(Z_\tau) \longrightarrow \mathcal{B}_\tau \setminus Z_\tau$$

are biholomorphisms.

Before continuing, we digress to prove that the $\mathcal{B}_\tau$ are in fact projective varieties.

**Theorem 6.3.** (see also [B-D2]) For all non-critical values of $\hat{\tau}$, $\mathcal{B}_\tau$ is a non-singular projective variety.

**Proof.** According to [M], since $\mathcal{B}_\tau$ is a Kähler manifold we need only prove that $\mathcal{B}_\tau$ is Moishezn. This is equivalent, by a theorem of Siu (see [S], Theorem 1), to proving that $\mathcal{B}_\tau$ admits a hermitian, holomorphic line bundle which is semipositive and positive at at least one point. From the codimension estimate given in (4.4) and (4.5) it suffices to prove this for $\hat{\tau}$ close to $d/R$ (for rank two and $d - 1 < \hat{\tau} < d$ we use a separate, easier argument). In this case, according to [B-D1], Theorem 6.4, there is a holomorphic map

$$\mathcal{B}_\tau \longrightarrow \mathcal{M}(d, R),$$

where $\mathcal{M}(d, R)$ denotes the Seshadri compactification of stable bundles. Moreover, the restriction of $\pi$ to the open set $\mathcal{M}^s(d, R)$ consisting of stable bundles is a fibration with fiber $\mathbb{P}^N$:

$$\mathcal{B}_\tau^s \longrightarrow \mathcal{M}^s(d, R),$$

From [B-D2] there is an hermitian, holomorphic line bundle $\gamma$ on $\mathcal{B}_\tau^s$ whose restriction to the fiber is $\mathcal{O}_{\mathbb{P}^N}(R)$ (an $\mathcal{O}(1)$ is not always possible, due to the Brauer obstruction on $\mathcal{M}(d, R)$). By pulling back a sufficiently high power $k$ of an ample bundle $H \rightarrow \mathcal{M}^s(d, R)$ we can arrange $L = \gamma \otimes \pi^*H^k$ to be positive at a point. It is easily seen that $L$ extends to a semipositive line bundle on $\mathcal{B}_\tau$ which is positive at a point of $\mathcal{B}_\tau^s$. Siu’s theorem then completes the proof.

We then have the following

**Corollary 6.4.** For all noncritical values of $\hat{\tau}$ in $(d/R, d/(R - 1))$, the spaces $\mathcal{B}_\tau$ are all birational.

**Proof.** According to Corollary 6.2, the complex manifolds $\mathcal{B}_{\tau \pm \varepsilon} \setminus \sigma_\pm^{-1}(Z_\tau)$ are biholomorphic. Thus their fields of meromorphic functions $\mathcal{M}(\mathcal{B}_{\tau \pm \varepsilon} \setminus \sigma_\pm^{-1}(Z_\tau))$ are isomorphic. On
the other hand, since for $R > 2$, $\sigma_{\pm}^{-1}(Z_\tau)$ has codimension at least 2 in $B_{\tau \pm \epsilon}$ (see Proposition 4.11) and $B_{\tau \pm \epsilon}$ are smooth, it follows from the Kontinuitätssatz for meromorphic functions (cf. [K-K], 53A.9) that

$$\mathcal{M}(B_{\tau \pm \epsilon \setminus \sigma_{\pm}^{-1}(Z_\tau)}) \simeq \mathcal{M}(B_{\tau \pm \epsilon}).$$

By Theorem 6.3, $B_{\tau \pm \epsilon}$ are projective varieties, hence by GAGA (cf. [G-H], p. 171) $\mathcal{M}(B_{\tau \pm \epsilon}) \simeq K(B_{\tau \pm \epsilon})$, where $K(B_{\tau \pm \epsilon})$ denotes the field of rational functions. It follows that $K(B_{\tau - \epsilon}) \simeq K(B_{\tau + \epsilon})$, and hence $B_{\tau - \epsilon}$ is birational to $B_{\tau + \epsilon}$.

It is interesting to apply Corollary 6.4 to the case of $B_{d + \epsilon}$ and $B_{d - \epsilon}$. Assume that $d$ is coprime to both $R$ and $R - 1$ and $d > R(2g - 2)$. Let $U(d, R) \rightarrow \Sigma \times \mathcal{M}(d, R)$ denote the universal bundle over the moduli space of vector bundles of rank $R$ and degree $d$, and let $\pi : \Sigma \times \mathcal{M}(d, R) \rightarrow \mathcal{M}(d, R)$ denote the projection onto the second factor. It follows from §3 that $B_{d + \epsilon}$ is biholomorphic to the projectivization of the vector bundle $\pi_*U(d, R)$. On the other hand, let $\text{Ext}^1(U(d, R - 1), \mathcal{O})$ denote the bundle over $\Sigma \times \mathcal{M}(d, R - 1)$ whose fiber over a point $(p, E^\partial_\epsilon)$ consists of the extensions of $U(d, R - 1)|_{\Sigma \times \{E^\partial_\epsilon\}}$ by $\mathcal{O}$. It follows again from §3 that $B_{d - \epsilon}$ is biholomorphic to the projectivization of the restriction of $\text{Ext}^1(U(d, R - 1), \mathcal{O})$ to $\{\text{point}\} \times \mathcal{M}(d, R - 1)$. By combining with Corollary 6.4, we obtain

**Corollary 6.5.** Assume that $d > R(2g - 2)$ is coprime to both $R$ and $R - 1$. Then $\mathbb{P}(\pi_*U(d, R))$ over $\mathcal{M}(d, R)$ is birational to $\mathbb{P}(\text{Ext}^1(U(d, R - 1), \mathcal{O}))$ over $\mathcal{M}(d, R - 1)$.

Presumably, Corollary 6.5 may also be obtained by carrying out a GIT construction of these spaces as in [Be] and [Th].

In the case of rank two our theorem combined with the result of [G-S] implies the following theorem of Thaddeus [Th]:

**Theorem 6.6.** Let $R = 2$ and $d > 4(g - 1)$. Suppose that $\hat{\tau} \in (d/2, d)$ is the only critical value of $f$ in the interval $[\hat{\tau} - \epsilon, \hat{\tau} + \epsilon]$. Then there is a projective variety $\tilde{B}_\tau$ and holomorphic maps

$$\begin{array}{ccc}
\tilde{B}_\tau & \xrightarrow{\rho_-} & B_{\tau - \epsilon} \\
\downarrow & & \downarrow \\
B_{\tau + \epsilon} & \xleftarrow{\rho_+} & B_{\tau + \epsilon}
\end{array}$$

Moreover, for $\hat{\tau} < d - 1$, $\rho_\pm$ are blow-down maps onto the smooth subvarieties $\mathbb{P}_\epsilon(W^\pm_\tau)$. For $\hat{\tau} = d - 1$, $\rho_+$ is the blow-down map onto $\mathbb{P}_\epsilon(W^+_{\hat{\tau}})$ and $\rho_-$ is the identity.
Proof. By Theorem 6.1 (iii), $\mathbb{P}_\varepsilon(\mathcal{W}_\tau^\pm)$ are smooth subvarieties of $\mathcal{B}_\tau^\pm$, respectively. Their codimensions are given by

$$p_+(\hat{\tau}) = 2\hat{\tau} - d + g - 1$$
$$p_-(\hat{\tau}) = d - \hat{\tau}$$

(see (4.4) and (4.5)). For $\hat{\tau} \in (d/2, d-1)$, $p_\pm(\hat{\tau}) \geq 2$ and so we may let $\tilde{\mathcal{B}}_\tau^\pm$ denote the blow-ups of $\mathcal{B}_\tau^\pm$ along $\mathbb{P}_\varepsilon(\mathcal{W}_\tau^\pm)$ and $\rho_\pm$ denote the corresponding blow-down maps. If $\hat{\tau} = d - 1$, $p_-(d-1) = 1$, and $\mathbb{P}_\varepsilon(\mathcal{W}_\tau^-)$ is already a divisor. In this case we let $\tilde{\mathcal{B}}_{\tau^-}$ and $\tilde{\mathcal{B}}_{\tau^+}$ the blow-up of $\mathcal{B}_{\tau^-}$ and $\mathcal{B}_{\tau^+}$ along $\mathbb{P}_\varepsilon(\mathcal{W}_\tau^\pm)$, which may be identified with $\Sigma \times J_{d-1}$ by Example 3.5. In any case, the biholomorphism

$$\sigma_+^{-1} \circ \sigma_- : \mathcal{B}_{\tau^-} \setminus \sigma_-^{-1}(Z_{\tau}) \longrightarrow \mathcal{B}_{\tau^+} \setminus \sigma_+^{-1}(Z_{\tau})$$

from Corollary 6.2 clearly lifts to a bimeromorphic map $\tilde{\sigma} : \tilde{\mathcal{B}}_{\tau^-} \rightarrow \tilde{\mathcal{B}}_{\tau^+}$. On the other hand, the result of [G-S] proves that $\tilde{\sigma}$ extends to a continuous bijection on all of $\tilde{\mathcal{B}}_{\tau^-}$. Now the Riemann extension theorem implies that $\tilde{\mathcal{B}}_{\tau^-}$ is biholomorphic to $\tilde{\mathcal{B}}_{\tau^+}$. We therefore set $\tilde{\mathcal{B}}_{\tau} = \tilde{\mathcal{B}}_{\tau^-} \simeq \tilde{\mathcal{B}}_{\tau^+}$.

7. Concluding remarks

In [B-D2] we introduced the moduli space of stable pairs of fixed determinant (see also [Th]). These are defined as follows: Let $J_d$ denote component of the Jacobian of $\Sigma$ corresponding to degree $d$ line bundles, and let

$$(7.1) \quad \det : \mathcal{B}_\tau \longrightarrow J_d$$

denote the map $\det(E^{\hat{\phi}}, \phi) = \Lambda^R E^{\hat{\phi}}$. It was shown in [B-D2] that $\det$ is a holomorphic map of maximal rank and thus a fibration. Let $\mathcal{B}_\tau(L)$ denote the fiber of $\det$ over $L \in J_d$. More generally, let

$$(7.2) \quad \det : \hat{\mathcal{B}} \longrightarrow J_d$$

denote the map $\det(E^{\hat{\phi}}, \phi) = \Lambda^R E^{\hat{\phi}}$. For $L \in J_d$ let $\hat{\mathcal{B}}(L) = \det^{-1}(L)$ and $\hat{\mathcal{B}}_0(L) = \hat{\mathcal{B}}(L) \cap \hat{\mathcal{B}}_0$. Clearly, $\hat{\mathcal{B}}(L)$ and $\hat{\mathcal{B}}_0(L)$ are preserved by the $U(1)$ action, and for any non-critical value of $\hat{\tau}$, $f^{-1}(\hat{\tau}) \cap \hat{\mathcal{B}}(L)/U(1)$ is biholomorphic to $\mathcal{B}_\tau(L)$.

It is easily seen that all the constructions performed in the previous sections commute with the map $\det$ and thus one has the analogous theorems for $\mathcal{B}_\tau(L)$. In particular, Theorem 6.1 and and Corollaries 6.2, 6.4 and 6.5 remain valid by replacing $\mathcal{B}_\tau$ by $\mathcal{B}_\tau(L)$.
Perhaps the most important question is how to resolve the birational maps of Corollary 6.4. The problem is that the master space $\hat{B}$ is singular along some of the critical sets. This means that $\mathbb{P}_\epsilon(W_\pm^\tau)$, the centers along which we wish to blow-up, are singular in general. One way to proceed might be to desingularize the master space $\hat{B}$ as in Kirwan [K2] and extend the circle action. However, one would still have to deal with finite quotient singularities. Such a description is desirable because by Corollary 6.5 one would then have a relationship between the moduli spaces of rank $R$ bundles which is inductive on the rank. This could be used to compute, for example, Verlinde dimensions as in [Th] or perhaps even the cohomology ring structure of these spaces in a manner similar to [Be-D-W].

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