LEVEL CURVES OF MINIMAL GRAPHS

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Abstract. We consider minimal graphs \( u = u(x, y) > 0 \) over domains \( D \subset \mathbb{R}^2 \) bounded by an unbounded Jordan arc \( \gamma \) on which \( u = 0 \). We prove an inequality on the curvature of the level curves of \( u \), and prove that if \( D \) is concave, then the sets \( u(x, y) > C \) \((C > 0)\) are all concave. A consequence of this is that solutions, in the case where \( D \) is concave, are also superharmonic.

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1. INTRODUCTION

Let \( D \) be a plane domain bounded by an unbounded Jordan arc \( \gamma \). In this paper we consider the boundary value problem for the minimal surface equation

\[
\begin{align*}
\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} &= 0 \quad \text{and} \quad u > 0 \quad \text{in} \quad D \\
u &= 0 \quad \text{on} \quad \gamma
\end{align*}
\]

(1.1)

We shall study the curvature \( \kappa = \pm |d\varphi/ds| \) for level curves \( u = C \) \((C > 0)\) where \( \varphi \) is the angle of the tangent vector to the curve, and the sign will be taken to be + when the curve bends away from the set where \( u > C \).

Theorem 1. There exists a constant \( K \) depending on \( u \) such that, if \( u \) as in (1.1) and \( C > 0 \), the curvature \( \kappa = \kappa(C) \) of the level curve \( u = C \) satisfies the inequality

\[
|\kappa| \leq \frac{K}{C}.
\]

Further comments regarding the constant \( K \) are given in \S 6.

Our next result concerns solutions whose domains are concave. There is a literature (see [3] and references cited there) regarding the propagation of convexity for level curves of solutions to partial differential equations over convex domains.

However, regarding the possible geometry of \( D \) in (1.1), it follows from a theorem of Nitsche [6, p.256] that \( D \) cannot be convex unless \( D \) is a halfplane since (1.1) cannot have nontrivial solutions over domains contained in a sector of opening less than \( \pi \). On the other hand, amongst the examples given in [3], there is a continuum of graphs...
which do have concave domains; specifically those given parametrically in the right half plane $H$ by

$$(1.3) \quad z(\zeta) = (\zeta + 1)^{\gamma} - \frac{1}{\gamma(2 - \gamma)}(\zeta + 1)^{2 - \gamma} \quad (\zeta \in H, \ 1 < \gamma < 2)$$

together with the height function $2 \text{Re}\zeta$. A concave domain $D$ is taken to be one whose complement is an unbounded convex domain. The boundary of $D$ is then a curve which bends away from the domain.

In §6 we will verify that the domains for the graphs of (1.3) are concave. In this note we shall prove the following

**Theorem 2.** If $u$ is a solution to (1.1) with $D$ concave and bounded by a $C^2$ curve $\gamma$, then the sets where $u > C$ are concave for each $C > 0$.

This has the curious consequence

**Corollary.** If $u$ is as in Theorem 2 above, then $u$ is also superharmonic in $D$.

### 2. PRELIMINARIES

For a solution $u$ to the minimal surface equation over a simply connected domain $D$ we shall slightly abuse notation by using $u$ to also denote the solution to (1.1) when given in parametric form. We shall make use of the parametrization of the surface given by $u$ in isothermal coordinates using Weierstrass functions $(x(\zeta), y(\zeta), u(\zeta))$ with $\zeta$ in the right half plane $H$. Our notation will then be given by

$$(2.1) \quad f(\zeta) = x(\zeta) + iy(\zeta) \quad \zeta = \sigma + i\tau \in H.$$ 

Then $f(\zeta)$ is univalent and harmonic, and since $D$ is simply connected it can be written in the form

$$(2.2) \quad f(\zeta) = h(\zeta) + \overline{g(\zeta)} \quad \zeta = \sigma + i\tau \in H$$

where $h(\zeta)$ and $g(\zeta)$ are analytic in $H$,

$$(2.3) \quad |h'(\zeta)| > |g'(\zeta)|,$$

and

$$(2.4) \quad u(\zeta) = 2\text{Re} i \int \sqrt{h'(\zeta)g'(\zeta)} \, d\zeta.$$ 

(cf. [2, §10.2]).

Now, $u(\zeta)$ is harmonic and positive in $H$ and vanishes on $\partial H$. Thus, (cf. [7, p. 151]),

$$(2.5) \quad u(\zeta) = k_0 \text{Re} \zeta,$$

where $k_0$ is a positive constant. This with (2.4) gives
(2.6) \[ g'(\zeta) = -\frac{k}{h'(\zeta)} \quad (k = k_0^2/4). \]

Then from (2.3) we have, in particular, that

(2.7) \[ |h'(\zeta)| \geq \sqrt{k}. \]

It follows from (2.5) that the level curves of \( u \) can be parametrized by \( f(\sigma_0 + i\tau) \) for \(-\infty < \tau < \infty\) and fixed values \( \sigma_0 \). Then the curvature \( \kappa \) corresponding to height \( \sigma_0 \) with the sign convention given at the begining for

\[ \varphi = \arctan(y_x/x_x) \]

is given by

(2.8) \[ \kappa = \kappa(\sigma_0, \tau) = \frac{d\varphi}{ds} = \frac{1}{(x_x^2 + y_x^2)^{3/2}}(x_x y_{x\tau} - y_x x_{x\tau}). \]

To compute (2.8) we use (2.1) and (2.6) to write

(2.9) \[ x_x = \frac{\partial}{\partial \tau} \Re(h + \bar{g}) = \Re i(h' - k/h') = -\Im(m(h' - k/h')) = -(|h'|^2 + k)\Im\frac{1}{h} \]

(2.10) \[ x_{x\tau} = -\frac{\partial}{\partial \tau} \Im(m(h' - k/h')) = -\Re(h'' + kh''/h'^2) \]

(2.11) \[ y_x = \frac{\partial}{\partial \tau} \Im(m(h + \bar{g})) = \Im m(ih' + k/h') = \Re(h' + k/h') = (|h'|^2 + k)\Re\frac{1}{h} \]

(2.12) \[ y_{x\tau} = \frac{\partial}{\partial \tau} \Re(h' + k/h') = -\Im(h'' - kh''/h'^2) \]

Substituting (2.9)-(2.12) into (2.8) we get

\[ \kappa = \frac{|h'|^3}{4(|h'|^2 + k)^2} \left( -\left(\frac{1}{h} - \frac{1}{h'}\right)(h'' - k \frac{h''}{h'^2} - \frac{h'''}{h'^2}) + \left(\frac{1}{h} + \frac{1}{h'}\right)(h'' + k \frac{h''}{h'^2} + \frac{h'''}{h'^2}) \right) \]

which simplifies down to

(2.13) \[ \kappa = \frac{|h'|}{|h'|^2 + k} \Re \frac{h''}{h'}. \]

Summarizing this, we have

**Lemma 1.** With \( u \) as in (1.1) and \( k_0 \) as in (2.5), then the locus of \( u = C \) is the set \( \zeta = \sigma_0 + i\tau \), where \( \sigma_0 = C/k_0 \) and \(-\infty < \tau < \infty\). The curvature \( \kappa \) at each point of this level set satisfies (2.13).
The proof of Theorem 2 uses the comparison of \( \kappa \) in (2.13) with the corresponding curvature \( \kappa_1 \) of the image of the line \( \sigma_0 + i\tau \) \((-\infty < \tau < \infty)\) under \( h \). Since \( \arg h' = \Im \log h' \), the formula (2.8) gives

\[
(2.14) \quad \kappa_1 = \frac{1}{|h'|} \Re \frac{h''}{h'}.
\]

3. PROOF OF THEOREM 1

Since \( f \) in (2.2) is a univalent harmonic mapping, we may convert the estimate from [1, Lemma 1] (cf. also [2, p. 153]) for a univalent harmonic mapping \( F = H + \overline{G} \) in the unit disk \( U \) to a mapping of the half plane \( H \).

**Lemma 2.** Let \( u \) be as in (1.1) and \( f = h + \overline{g} \) as in (2.2). Then

\[
\left| \frac{h''(\zeta)}{h'(\zeta)} \right| \leq A/\sigma
\]

for some absolute constant \( A \).

**Proof of Lemma 2.** For the univalent harmonic mapping \( F = H + \overline{G} \) of \( U \), the estimate of [1] is

\[
\left| \frac{H''(w)}{H'(w)} \right| \leq \frac{A_1}{1 - |w|}, \quad w \in U
\]

for some absolute constant \( A_1 \). Now, for \( f(\zeta) = h(\zeta) + \overline{g(\zeta)} \), let

\[
F(w) = f \left( \frac{1 + w}{1 - w} \right), \quad w \in U.
\]

Then,

\[
h(\zeta) = H \left( \frac{\zeta - 1}{\zeta + 1} \right),
\]

\[
h'(\zeta) = H' \left( \frac{\zeta - 1}{\zeta + 1} \right) \frac{2}{(\zeta + 1)^2},
\]

and

\[
h''(\zeta) = H'' \left( \frac{\zeta - 1}{\zeta + 1} \right) \frac{4}{(\zeta + 1)^4} - H' \left( \frac{\zeta - 1}{\zeta + 1} \right) \frac{4}{(\zeta + 1)^3}.
\]

Thus,

\[
\left| \frac{h''(\zeta)}{h'(\zeta)} \right| \leq \frac{2}{|\zeta + 1|} \left( \frac{1}{|\zeta + 1| - \frac{\zeta - 1}{\zeta + 1}} + 1 \right) \leq \frac{2}{|\zeta + 1|} \left( \frac{A_1}{|\zeta + 1| - |\zeta - 1|} + 1 \right) \leq \frac{2}{|\zeta + 1|} \left( \frac{A_2(|\zeta + 1| + |\zeta - 1|)}{4\sigma} + 1 \right)
\]

where \( A_2 \) is an absolute constant.
\[ \leq \frac{A}{\sigma} \]

for some absolute constant \( A \). \( \square \)

**Proof of Theorem 1.** From Lemma 1, Lemma 2, and (2.7) it follows that, on the level set \( u = C \),
\[
|\kappa| \leq \frac{A}{\sqrt{kC}}.
\]
\( \square \)

4. PROOF OF THEOREM 2

For convenience, we dismiss the trivial case where \( u \) is planar, and hence we may assume that \( h' \) is nonconstant.

From the given hypothesis, it follows that \( \gamma \) must have asymptotic angles in both directions as \( z \to \infty \). By a rotation we may assume that the asymptotic tangent vectors have directions \( \pm \alpha \) for some \( 0 \leq \alpha \leq \pi/2 \).

From the concavity of \( D \) and the assumption that the asymptotic tangents to \( \gamma \) have angles \( \pm \alpha \), it follows that \( y_{\tau} \geq 0 \) for \( \sigma = 0 \). Thus, from (2.11) it follows that for \( \sigma = 0 \), \( \Re 1/h' \geq 0 \), and hence \( \Re 1/h' \geq 0 \). Since, by (2.7) \( 1/h' \) is bounded in \( \mathbb{H} \), this means that \( \Re 1/h' > 0 \) throughout \( \mathbb{H} \). This in turn gives
\[
\Re h'(\zeta) > 0 \quad \zeta \in \mathbb{H}.
\]

Let \( \psi(\tau) = \arg h'(i\tau) \). It follows from (2.13) and (2.14) that \( 0 \leq \kappa_{1} \neq 0 \) on \( \partial\mathbb{H} \) so that
\[
\frac{d\psi}{d\tau} = \frac{\partial}{\partial \tau} \Im m(\log h') = \Re \frac{h''}{h'} \geq 0 \quad \text{when} \ \tau = 0.
\]

By (4.1)
\[
(4.3) \quad -\pi/2 \leq \psi(\tau) \leq \pi/2.
\]

Now, \(-\pi/2 < \Im m(h') < \pi/2 \) in \( \mathbb{H} \), and in particular is a bounded harmonic function in \( \mathbb{H} \). So for \( \zeta = \sigma + i\tau \in \mathbb{H} \),
\[
\Im m \log h'(\zeta) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\psi(t)dt}{\sigma^{2} + (t-\tau)^{2}}.
\]

Then
\[
\Re \frac{h''(\zeta)}{h'(\zeta)} = \frac{\partial}{\partial \tau} \Im m \log h'(\zeta) = \frac{\partial}{\partial \tau} \left( \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\psi(t)dt}{\sigma^{2} + (t-\tau)^{2}} \right) = \frac{2\sigma}{\pi} \int_{-\infty}^{\infty} \frac{(t-\tau)\psi(t)dt}{(\sigma^{2} + (t-\tau)^{2})^{2}}.
\]
An integration by parts yields

\[
\Re \frac{h''}{h'} = \frac{\sigma}{\pi} \left( \frac{-\psi(t)}{\sigma^2 + (t - \tau)^2} \right)_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\psi'(t)dt}{\sigma^2 + (t - \tau)^2}.
\]

By (4.3) it follows that the first term on the right vanishes, and by (4.2) the second term is positive. Thus \( \kappa_1 \) in (2.14) and hence \( \kappa \) in (2.13) are positive in \( H \).

5. PROOF OF THE COROLLARY

We may write the minimal surface equation for \( u \) as

\[
\Delta u + \frac{F}{|\nabla u|^3} = 0
\]

where \( F = F(u, x, y) = u_y^2 u_{xx} + u_x^2 u_{yy} - 2u_x u_y u_{xy} \).

Now, for a given function \( v(x, y) > 0 \) the curvature of the level set \( v(x, y) = 0 \) is given by \( F(v, x, y)/|\nabla v|^3 \) [1, p. 72] which is positive when the curve bends away from the interior of the domain. Since Theorem 2 shows that the level sets \( u = c \) which bound the sets \( u > c \) each have positive curvature, then applying this to \( F(u - c, x, y) \) we find that \( \Delta u < 0 \) and hence \( u \) is superharmonic in \( D \).

6. CONCLUDING REMARKS.

For the examples (1.3) of §1,

\[
\Re \frac{h''}{h'} = \Re \frac{\gamma - 1}{\zeta + 1} > 0.
\]

for \( 1 < \gamma < 2 \) so that by (2.13) these have concave domains.

Furthermore, using (2.13), this shows that Theorem 1 is sharp. Regarding the constant \( K \) in Theorem 1, the scaling factor \( k \) in (3.1) is consistent with the fact that \( \kappa \) would be rescaled by replacing \( u(x, y) \) by \( cu(x/c, y/c) \) for \( 0 < c < \infty \).

REFERENCES

1. Y. Abu-Muhanna and A. Lyzzaik, _The boundary behaviour of harmonic univalent maps_, Pacific Jour. Math. 141 (1990) 1-20.
2. P. Duren, _Harmonic mappings in the plane_, Cambridge Tracts in Mathematics, 2004.
3. A.-K. Gallagher, J. Lebl, K. Ramachandran, Convexity of level lines of Martin functions and applications, Analysis and Mathematical Physics, 2019, Volume 9, Issue 1, 443-452.
4. A. Gray, _Modern differential geometry of curves and surfaces_, Studies in Advanced Mathematics, 1993.
5. E. Lundberg, A. Weitsman, _On the growth of solutions to the minimal surface equation over domains containing a half plane_, Calc Var. Partial Differential Equations 54 (2015) 3385-3395.
6. J.C.C. Nitsche, _On new results in the theory of minimal surfaces_, Bull. Amer. Mat. Soc. 71 (1965), 195-270.
7. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co., Ltd., Tokyo (1959).

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