Rickart gamma rings

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Abstract. We introduce the concepts of Rickart gamma ring as a generalized of a Rickart ring and Baer gamma ring. Also, For an $\Gamma$—ring $R$, we show that: (1) $R$ is a right Rickart if and only if it is right p.p.. (2) $R$ is a Baer, quasi-Baer if and only if $R$ is Rickart, p.q. Baer (respectively) and $\mathcal{R} = \{etR | e^2 = e \in R\}$, under inclusion, is complete lattice. (3) $R$ is prime if and only if it is quasi-(or right p.q.) Baer and semicentral reduced. (4) If $R$ is Rickart then: (a) $R$ is both right and left nonsingular. (b) $R$ has no nonzero central nilpotent elements. (c) The image isomorphic of $R$ is Rickart. (d) If $R$ is reduced then the idempotent element which is generated the right annihilator of any element in $R$ is unique. (5) The direct product of $\Gamma$—rings is Rickart, quasi-(right p.q.) Baer iff $R_i$ is Rickart quasi-(right p.q.) Baer for all $i \in I$, respectively. (6) The corner and the center of Quasi- Baer, p.q. Baer and Rickart are Quasi-Baer and Baer, p.q. and Rickart and, by conditions, Rickart and Rickart, respectively. (7) We give some conditions to show that (a) If $R$ is Rickart, Quasi-Baer then Baer and p.q.- Baer, Rickart, respectively. (b) If $R$ is Rickart, then the following statements of $R$ are equivalent: reduced, abelian, idempotents of $R$ commute, the set of idempotents is closed under multiplication, $R$ is commutative at 0, $RFI(x) = LF1(x)$ for every $x \in R$ and $r_p(x) = l_p(x)$ for all $x \in R$.

Key words: gamma rings; Baer and Rickart gamma rings; Quasi and p.q. Baer gamma rings; Annihilators; Idempotents.

0. Introduction and Preliminaries

In this paper all gamma rings with identity. N. Nobusawa [9] was the first who introduced the concept of $\Gamma$-rings. Later the concept of Nobusawa’s $\Gamma$-rings was generalized and given a another definition of a $\Gamma$-ring by Barnes [14]. Many Mathematicians studied $\Gamma$-rings and obtained some fruitful results which are considered a generalization of many classical ring theories. In (1955), Kaplansky postulated a Baer ring $R$ as: for every non empty subset of a ring $R$, the right annihilator is generated by an idempotent element [7]. Clark, in (1967), presented a quasi-Baer ring as the right annihilator of every two sided ideal of a ring $R$ is generated by an idempotent element in $R$ [15]. In (1960) S. Maeda a ring $R$ is said to be right (left) Rickart if the right (left) annihilator of any single element in $R$ is generated by an...
idempotent element [12]. A. Hattori in (1960) [5] introduced p.p. (Rickart) ring as every principle ideal is projective which equivalent to the Rickart ring.

G.F. Birkenmeier, J. Y. Kim, J.K. Park, in (2001) generalized a quasi-Baer ring to be a p.q.-Baer ring R [6]. A ring R is said to be p.q.-Baer if an idempotent element generated the right annihilator of every principle right ideal of a ring R as an R-module. A. C.Paul and MD Sabur [1], in (2012) presented a Baer gamma ring and some characterizations of this gamma ring was obtained. Recently, there are studies that have dealt with these types of rings and their relations. The aim of this research is to raise the concept of Rickart from the category of rings to the category of gamma rings and present it as a generalization to both concept of Rickart ring and Baer gamma ring. As well as presenting some of the concepts close to this concept such as Quasi and p.q. Baer gamma rings and studying the relationship between them. In addition, a number of important issues have been submitted.

Let R and Γ be two abelian groups. If (1) there is a mapping \( \cdot : R \times \Gamma \times R \rightarrow R \), \((x, y, \gamma) \mapsto xy\gamma \) such that \( (i)(x + y)yz = xy + yz \cdot (ii)x(x + \beta)y = xy + x\beta y \cdot (iii)x(y + \beta) = xy + x\beta y \cdot (iv)(x\gamma)y\beta z = xy(y\beta z) \) for all \( x, y, z \in R \) and \( \gamma, \beta \in \Gamma \). (2) there is a mapping \( \cdot : R \times \Gamma \times \Gamma \rightarrow \Gamma \), \((y, \alpha, \beta) \mapsto y\alpha \beta \) such that \( (i)(x\gamma)y\beta z = x(y\gamma)y\beta z \cdot (ii)xy\gamma = 0 \) implies \( \gamma = 0 \) for all \( x, y, z \in R \) and \( \gamma, \beta \in \Gamma \). Then R is called \( \Gamma - \text{ring} \). This definition was given by N. Nobusawa [9], while satisfied (1) suffices to be \( \Gamma - \text{ring} \), according to Barnes [14]. The \( \Gamma - \text{ring} \) in this paper is according to Barnes unless it is defined differently. An \( \Gamma - \text{ring} \) is said to be with identity, if there exist elements, say 1 in R and \( \gamma_0 \in \Gamma \), such that, \( 1\gamma_0 r = r\gamma_0 1 = r \), for all \( r \in R \) [14]. Recall that, A gamma ring R is called Baer if the right annihilator in R of any nonempty subset of R is generated by an idempotent of R [1]. An \( \Gamma - \text{ring} \) R is called regular if for each \( a \in R \) there exist \( \gamma \in \Gamma \) such that \( a\gamma a = a \) [11]. Let R be an \( \Gamma - \text{ring} \) R, \( a \in R \), then \( r_{\gamma}(a) = \{ r \in R : r\gamma_{r} = 0 \} \) is called the right annihilator of an element \( a \) in R. The left annihilator is defined similarly. An element \( e \) in \( \Gamma - \text{ring} \) R is called idempotent if \( e\gamma e = e \) for some \( \gamma \in \Gamma \) [4].

1. Basic structure of Rickart Gamma Rings

Definition 1.1. A gamma ring \( R \) is said to be right (left) Rickart if there is an idempotent generates the right (left) annihilator of any element. A gamma ring is Rickart if it is right and left Rickart.

Examples 1.2.

1- Every Baer gamma ring is Rickart.

2- Every gamma ring is Rickart if it is regular.

Proof: Assume that \( \Gamma - \text{ring} \) R is a regular and \( a \in R \), then there exist \( \gamma \in \Gamma \) such that \( a\gamma a = a \) for some \( \gamma \in \Gamma \), take \( e = a \Rightarrow e\gamma e = e, r_{\gamma}(a) = r_{\gamma}(e) = (1 - e)\Gamma R \). Thus \( R \) is Rickart.

3- Every semisimple gamma ring is Rickart.

4- If an \( \Gamma - \text{ring} \) R is right Rickart, then any right ideal \( I \in R \) is also right Rickart.

Proof: Let \( a \in I \). As R is right Rickart, then \( r_{\gamma}(a) = e\Gamma R \) for some idempotent \( e\gamma_0 e = e \in R \). Claim that \( r_{\gamma}(a) = I \cap r_{\gamma}(a) = e\Gamma I \). Let \( y \in I \). Then \( a\gamma y = 0 \). Hence \( y \in r_{\gamma}(a) \). Now let \( x \in r_{\gamma}(a) \), then \( x \in r_{\gamma}(a) \) and so \( x = e\gamma_0 x \in e\Gamma I \). Hence \( I \) is right Rickart.

5- A sub \( \Gamma - \text{ring} \) of Rickart gamma ring \( R \) which is containing all idempotent elements in \( R \) is a Rickart.

Proof: Let \( B \) be a sub \( \Gamma - \text{ring} \) of a Rickart gamma ring \( R \) which containing all idempotents in \( R \). Let \( x \in B \). Then \( r_{\gamma}(x) = e\Gamma R \) for some idempotent idempotent \( e \in R \). But \( e \in B \). Then \( r_{\gamma}(x) = e\Gamma R \cap B = e\Gamma B \). Hence \( B \) is Rickart.

Theorem 1.3. Let \( R \) be an \( \Gamma - \text{ring} \) (as the sense of nobasawa). Then the following are equivalent:

1- \( R \) is a regular.

2- \( R \) is a right Rickart such that \( l_{\gamma}(r_{\gamma}(R\Gamma x)) = R\Gamma x \) for all \( x \) in \( R \).

3- \( R \) is a left Rickart such that \( r_{\gamma}(l_{\gamma}(x\Gamma R)) = x\Gamma R \) for all \( x \) in \( R \).
Proof: (1) ⇒ (2): By regular and from examples (1.2.(2)) we have \( R \) is Rickart and \( R \subseteq R^\prime \). From Rickart \( r_\ell(x) = eR \) and hence \( R^\prime x \subseteq l_\ell(eR) = l_\ell(r_\ell(x)) \subseteq l_\ell(r_\ell(R^\prime x)). \) Thus \( l_\ell(r_\ell(R^\prime x)) = R^\prime x. \) (2) ⇒ (1): Let \( x \in R, \) if \( r_\ell(x) = 0 \) ⇒ \( r_\ell(R^\prime x) = 0 \) ⇒ \( l_\ell(r_\ell(R^\prime x)) = l_\ell(r_\ell(0)) = R \Rightarrow R^\prime x = R, \) then \( r_\ell x = 1 \) (for some \( r \in R, y \in \Gamma \)) ⇒ \( xy_0 r yx = xy_0 1 = x \) for some \( y_\ell \in \Gamma \) hence \( R \) is regular. If \( r_\ell(x) \neq 0 \neq r_\ell(y) = eR \Rightarrow x \in eR(e) = R^\prime (1 - e) = R^\prime x \subseteq l_\ell(eR) \subseteq l_\ell(r_\ell(R^\prime x)) = R^\prime x, \) hence \( R^\prime x = R^\prime (1 - e) \Rightarrow r_\ell x = 1 - e \) (for some \( r \in R, y \in \Gamma \)) ⇒ \( xy_0 r yx + xy e = xy_0 1 = x \) for some \( y_0 \in \Gamma \) ⇒ \( xy_0 r yx = x, \) hence \( R \) is regular. The proof is complete since 2 and 3 are symmetric.

Proposition 1.4. An \( \Gamma^\prime - \text{ring} \) \( R \) is right Rickart if and only if \( R \) is right Rickart \( \Gamma - \text{ring}.\)

Proof: Suppose that \( R \) is a right Rickart ring and let \( a \in R. \) Then \( r_\ell(a) = eR \) with \( e^2 = e \in R, \) this implies \( r_\ell(a) = eR \) with \( e e e = e \in R, \) thus \( R \) is Rickart \( R - \text{ring}. \) Conversely, suppose that \( R \) is a right Rickart \( \Gamma - \text{ring} \) and take \( a \in R. \) So \( r_\ell(a) = eR \) with \( e e e = e \in R \) for some \( f \in R, \) this implies \( r_\ell(a) = eR \) with \( e f e = eF \in R, \) thus \( R \) is Rickart.

It can be seen easily that the category of Rickart gamma rings is generalization of the categories of Rickart rings and Baer gamma rings.

Recall that An \( R_f - \text{module} \) \( M \) is called free if it has basis [2]. An \( R_f - \text{module} \) \( M \) is called \( R_f - \text{projective} \) if there is an \( R_f - \text{module} \) \( \mathbb{N} \) such that \( M \oplus \mathbb{N} \) is \( R_f - \text{free} \) [8].

Definition 1.5. A gamma ring \( \Gamma \) is called a right (left) principal projective (simply, right (left) p.p.) if each right (left) principal ideal of \( R \) is projective. A gamma ring \( \Gamma \) is called a p.p. gamma ring if it is right and left.

Proposition 1.6. Let \( R \) be an \( \Gamma - \text{ring}. \) Then \( R \) is a right Rickart if and only if it is right p.p..

Proof: Suppose that \( R \) is a right Rickart gamma ring and let \( a \in R. \) Then \( r_\ell(a) = eR \) with \( e e = e \in R \) for some \( \gamma \in \Gamma. \) Thus, \( (1 - e) \Gamma R \cong R/\tau_\ell(a) \cong a R \) as right \( R_\ell - \text{modules}, \) so \( a R \ell R_\ell \) is projective. Conversely, suppose that \( R \) is right p.p.. Take \( x \in R. \) Then the homomorphism \( \theta : R_\ell \rightarrow x R \ell R_\ell, \) defined by \( \theta(r) = x \beta r \) for some \( \beta \in \Gamma, \) splits because \( x R \ell R_\ell \) is projective. So, \( Ker(\theta) = r_\ell(x) = f R \ell, \) \( f \ell f = f \in R, \) for some \( f \in \Gamma. \) Thus, \( R \) is right Rickart.

Recall that, For a gamma ring \( \Gamma, \) right ideal \( I \) is essential in \( \Gamma \) if, for any a nonzero right ideal \( J \) in \( \Gamma, \) \( I \cap J \neq 0. \) Similarly for essential left ideals. A gamma ring \( \Gamma \) is said to be right nonsingular if there exists right ideal \( J \neq 0 \) such that \( r_\ell(x) \cap J = 0 \) whenever \( 0 \neq x \in \Gamma (\text{in other words, if for all } x \in \Gamma, \) \( r_\ell(x) \) is an essential right ideal is in \( \Gamma \) implies \( x = 0). \) Similarly for left nonsingularity [11].

Proposition 1.7. Every \( \Gamma - \text{ring} \) \( R \) is both right and left nonsingular if it is Rickart.

Proof: Let \( 0 \neq x \in \Gamma. \) Then \( r_\ell(x) = eR \) with \( e e = e \in R \) for some \( \gamma \in \Gamma, \) \( e \neq 1, \) and \( J = (1 - e) \Gamma R \) is a nonzero right ideal with \( r_\ell(x) \cap J = 0. \)

Proposition 1.8. For a Rickart \( \Gamma - \text{ring} \) \( R, \) the corner \( eR \ell R_\ell \) is Rickart where \( e R \ell = e \).

Proof: Let \( x \in eR \ell R_\ell, \) then \( r_\ell(x) = f R \ell \) with \( f R_\ell f = f \in R \) for some \( \gamma \in \Gamma. \) Claim \( r_\ell eR \ell R_\ell e(x) = r_\ell(x) \cap eR \ell R_\ell e = e_0 R_\ell y_0 \ell e \ell R_\ell e, \) clearly \( e_0 R_\ell y_0 \ell e \ell R_\ell e \subseteq eR \ell R_\ell e, \) since \( x \in eR \ell R_\ell e \) and \( e e = e \) for all \( \gamma \in \Gamma, \) then \( e y e = x \) and hence \( x \ell (e_0 y_0 e R_\ell e) = 0 \) which is given \( e_0 R_\ell y_0 R_\ell R_\ell e \subseteq r_\ell(x) \) because \( e_0 R_\ell y_0 R_\ell R_\ell e \subseteq r_\ell(x) \cap eR \ell R_\ell e. \) Let \( a \in r_\ell(x) \cap eR \ell R_\ell e = a R_\ell R_\ell e \) and \( a = e_0 y_0 e_0 e_0 \) since \( e e = e \) for all \( \gamma \in \Gamma, \) then \( a = e_0 y_0 R_\ell R_\ell e = e_0 R_\ell y_0 R_\ell e = e_0 R_\ell y_0 R_\ell y_0 e_0 \ell e \subseteq e_0 R_\ell R_\ell e \) and hence \( r_\ell(x) \cap eR \ell R_\ell e \subseteq e_0 R_\ell R_\ell R_\ell e \) which is given the claim, now we need
only prove that \( ey_0f y_0e \) is idempotent in \( e\Gamma R e \), to show this we note that
\( x\Gamma (ey_0f - f y_0e) = 0 \), then
\( ey_0f - f y_0e \in R(x) \Leftrightarrow ey_0f - f y_0e = f y_0(e y_0f - f y_0e) \Rightarrow ey_0f = f y_0ey_0f \),
thus
\((ey_0f y_0e)^2 = (ey_0f y_0e)(ey_0f y_0e) = ey_0f y_0ey_0f y_0e = ey_0ey_0f y_0e \in e\Gamma R e \).
Recall that, the center of \( \Gamma - \text{ring } R \) (\( \text{Cen}(R) \)) is the set of all elements \( x \in R \) such that \( a xy = xy a \) for every \( a \in R \) [3]. A central element in \( \text{Cen}(R) \) is an idempotent contained in \( \text{Cen}(R) \).

**Definition 1.9.** An idempotent element \( e \) in \( \Gamma - \text{ring } R \) is called a left (resp. right) semicentral if \( xye = e xy e \) (resp. \( eyx = ey x e \)) for all \( x \in R \) and \( y, \hat{y} \in \Gamma \). The set of all a left (resp. right) semicentral elements is denoted by \( S_l(R) \) (resp. \( S_r(R) \)).

**Remark 1.10.**
1. For the set of all central idempotent elements \( B(R) \), \( S_l(R) \cap S_r(R) \subseteq B(R) \). The covers \( (B(R) \subseteq S_l(R) \cap S_r(R)) \) satisfied if \( e^{\Gamma} = e \) for all idempotent \( e \in R \).
2. If \( e = e ye f = e y e \delta \in S_l(R) \) (or \( S_r(R) \)) for some \( \gamma, \delta \in \Gamma \), then \( eyf \in S_l(R) \) (or \( S_r(R) \)). and \( e\Gamma R \cap f \Gamma R = eyf \Gamma R (\Gamma \Gamma e \cap \Gamma f = R \Gamma eyf) \).

**Proof:** Easy.

**Proposition 1.11.** Let \( e = ey_0e \) idempotent in \( \Gamma - \text{ring } R \) and \( 1\Gamma x = x\Gamma 1 = x \) for all \( x \in R \). Then the following are equivalent.
(a) \( e\Gamma R \) is an ideal in \( R \).
(b) \( (1 - e)\Gamma R e = 0 \).
(c) \( e \in S_l(R) \).
(d) \( \Gamma (1 - e) \) is an ideal in \( R \).
(e) \( 1 - e \in S_r(R) \).

**Proof:** (a) implies (b): let \( x \in e\Gamma R e \Rightarrow x = x y_0e \in R\Gamma e \), then \( e\Gamma R e \subseteq R\Gamma e \). Let \( x \in R\Gamma e \Rightarrow x = x y_0e = x y_0e y_0e \in e\Gamma R e \), since \( x y_0e \in e\Gamma R \) by ideal, then \( R\Gamma e \subseteq e\Gamma R e \), hence \( e\Gamma R \Gamma e = R\Gamma e \Rightarrow R\Gamma e - e\Gamma R \Gamma e = 0 \Rightarrow (1 - e)\Gamma R \Gamma e = 0 \). (b) implies (c): for any \( x \in R \), then \( (1 - e)xye = 0 \Rightarrow x ye = x ye x e \) for all \( y \in \Gamma \), therefore \( e \in S_l(R) \). Similarly, (c) implies (d), (d) implies (e) and (e) implies (a).

**Proposition 1.12.** For the Rickart \( \Gamma - \text{ring } R \), the Center of \( \Gamma - \text{ring } R \) is a Rickart if \( 1\Gamma x = x\Gamma 1 = x \) for all \( x \in R \).

**Proof:** Put \( C = \text{Cen}(R) \). Say \( a \in C \). Then \( r_{\Gamma}(a) = e\Gamma R \) is an ideal, hence \( e \in S_l(R) \) by Proposition (1.11). Say \( z \in R \). Then \( r_{\Gamma}(eyz - zye) = f \Gamma R \) with \( f\Gamma f = f \) is idempotent in \( R \) and for any \( y \in \Gamma \). Since \( (eyz - zye) \rho a = 0 \) for any \( y \in \Gamma \), \( a \in f\Gamma R \) and thus \( a = f \delta a \). Hence \( (1 - f )\delta a = a\delta (1 - f ) = 0 \), so \( 1 - f \in r_{\Gamma}(a) = e\Gamma R \), \( (1 - f ) = e\beta (1 - f ) \) .Therefore, \( (1 - e)\beta (1 - f ) = 0 \). Because \( e \in S_l(R) \), \( yz = eyz = eyz \) for all \( y \in \Gamma \) and we have that \( (eyz - zye) \rho (1 - f ) = (eyz - eyz) \rho (1 - f ) = (eyz)\gamma (1 - e) \rho (1 - f ) = 0 \). Therefore, we see that \( 1 - f \in r_{\Gamma}(eyz - zye) = f \Gamma R \), thus \( 1 - f = 0 \), and so \( f = 1 \). Hence, \( r_{\Gamma}(eyz - zye) = f \Gamma R = R \), therefore \( eyz - zye = 0 \) and \( e \in C \). From \( r_{\Gamma}(a) = e\Gamma R \), \( r_{\Gamma}(a) = e\Gamma C \) with \( e \in C \). Consequently, \( C \) is a Rickart ring.

**Proposition 1.13.** For the \( \Gamma - \text{rings } R_i \) \( i \in I \), the direct product \( R = \Pi_{i \in I} R_i \) is Rickart iff \( R_i \) is Rickart for all \( i \in I \).

**Proof:** Let \( a = (a_i) \in R = \Pi_{i \in I} R_i \). If each \( R_i \) is Rickart ring, then \( r_{\Gamma}(a_i) = e_i\Gamma R_i \) for some \( e_i \rho e_i = e_i \in R_i \) and \( \gamma_i \in \Gamma \). Let \( b = (b_j) \in r_{\Gamma}(a) \). Hence \( e\rho b = 0 \) if and only if \( a_i \rho b_j = 0 \) for all \( i,j \in I \) if and only if \( b_j \in r_{\Gamma}(a_i) \). This implies \( b_j = e_i \rho b_j \) and \( a_i \Gamma e_i = 0 \). Now, if we Put \( e = (e_i) \) and \( y = (\gamma_i) \) then \( eyb = (e_i)(\gamma_i b_j) = (e_i)\gamma b_j = (b_j) = b \). It is easy to see that \( a\Gamma e = 0 \) since \( (a_i)\Gamma (e_i) = 0 \) if and only if \( a_i \Gamma e_i = 0 \) for all \( i \in I \). Hence \( r_{\Gamma}(a) = e\Gamma R \) and so \( R = \Pi_{i \in I} R_i \) is Rickart. For the direction opposite, suppose that \( R = \Pi_{i \in I} R_i \) is Rickart gamma ring and \( a_i \in R_i \). Then for any
$a \in R$ is $a_i$ in ith component and 1 otherwise. Hence $r_R(a) = e_i \Gamma R$ for some idempotent $e \in R$. Then $e = (e_i)$ is also $e_i$ in ith component and 1 otherwise. So $r_R(a_i) = e_i \Gamma R_i$ for some idempotent $e_i \in R_i$. Therefore $R_i$ is Rickart.

**Proposition 1.14.** For the $\Gamma - isomorphic$, an image of Rickart $\Gamma - ring$ is Rickart.

**Proof:** Take $h: G \rightarrow H$ be an isomorphism of $\Gamma - rings$ and $G$ is a Rickart. Let $x \in H$. Due to the epimorphism, $x = h(a)$ for some $a \in G$. Therefore $r_G(a) = e_i \Gamma G$ for some idempotent $e \in G$. It is claimed that $r_H(a) = e_i \Gamma H$, where $\hat{e} = h(e)$. Firstly, it is shown that $h(r_G(a)) \subseteq r_H(a)$. For that take $z = h(n) \in h(r_G(a))$, where $a \Gamma n = 0$ and $n = e\gamma y_0 n$. Now, $x y z = h(a) y h(n) = h(a \gamma y_0 n) = h(0) = 0$. Hence $z \in r_H(x)$. But $(r_G(a)) = h(e \Gamma G) = h(e) \Gamma h(G) = h(e) \Gamma H = e \Gamma H$. Hence $e \Gamma H \subseteq r_H(x)$. Then $e \Gamma H = h(r_G(a)) \subseteq r_H(h(a))$. Now, let $n \in r_H(x)$. So $0 = x \Gamma n = h(a) \Gamma h(t) = h(a \Gamma t)$ for some $t \in G$ such that $n = h(t)$. Since $h$ is monomorphism and $h(a \Gamma t) = 0$, then $a \Gamma t = 0$. Hence $t \in r_G(a)$. So $t = e\gamma y_0 t$ and this yields $n = h(t) = h(e \gamma y_0 t) = h(e)\gamma y_0 h(t) \in e \Gamma H$. Therefore $r_H(x) = e \Gamma H$ for some idempotent $\hat{e} y_0 \gamma \hat{e} = \hat{e} \in H$.

Theorem 1.15. let $R = \{e \Gamma R \mid e^2 = e \in R\}$ where $R$ is a $a \Gamma - ring$. Then $R$ is a Baer if and only if $R$ is Rickart and $\mathcal{R}$, under inclusion, is complete lattice.

**Proof:** Suppose that $R$ is a Baer. Hence $R$ is a Rickart. Let $\{e_i \Gamma R \mid i \in A\}$ is a subset of $\mathcal{R}$. Since $R$ is Baer, $r_R(\sum e_i \Gamma R) = h \Gamma R$ where $h$ is an idempotent in $R$. Note that $e_i \Gamma R \subseteq h \Gamma R$ for each $i$. Next, let $g^2 = g \in R$ such that $e_i \Gamma g \subseteq g \Gamma R$ for all $i$. Then $\sum e_i \Gamma R \subseteq h \Gamma R$, so $h \Gamma R = r_R(\sum e_i \Gamma R) \subseteq g \Gamma R$. Hence $h \Gamma R = \sup \{e_i \Gamma R \mid i \in A\}$. Since $\mathcal{R}$, under inclusion, is a partially ordered set, then $\mathcal{R}$ is a complete lattice under inclusion. Conversely, let $\mathcal{F} \neq \emptyset \subset \sum x_i \Gamma R \subseteq A$. As $R$ is Rickart, there is $e_i \Gamma R \subseteq e_i \Gamma R$ for all $i$. Hence $e \Gamma R \subseteq \sum e_i \Gamma R$. Next, let $X \subseteq \sum e_i \Gamma R$. Then $r_R(X) = \cap \{r_R(x_i) \mid i \in A\}$. As $\mathcal{R}$ is complete, then there is $e^2 = e \in R$ with $e \Gamma R = \inf \{e_i \Gamma R \mid i \in A\}$. It is shown that $e \Gamma R = \cap \{e_i \Gamma R \mid i \in A\}$. As $e \Gamma R \subseteq e_i \Gamma R$ for all $i$, $e \Gamma R \subseteq \cap \{e_i \Gamma R \mid i \in A\}$. Next, let $x \in \cap \{e_i \Gamma R \mid i \in A\}$. Then $l_R(x) = R^H X$ for some $g^2 = g$ since $R$ is Rickart. Therefore, $r_R(l_R(x)) = (1 - h) \Gamma R \in \mathcal{R}$. As $x \in e_i \Gamma R$, $r_R(l_R(x)) \subseteq r_R(l_R(e_i \Gamma R)) = e_i \Gamma R$. So $r_R(l_R(x)) \subseteq e_i \Gamma R$ for each $i \in A$. As $r_R(l_R(x)) \subseteq e_i \Gamma R$, we get $r_R(l_R(x)) \subseteq e_i \Gamma R$. Therefore $x \in r_R(l_R(x)) \subseteq e_i \Gamma R$, and hence $\cap \{e_i \Gamma R \mid i \in A\} \subseteq e_i \Gamma R$. Accordingly, $e \Gamma R \cap \{e_i \Gamma R \mid i \in A\} = \cap \{e_i \Gamma R \mid i \in A\}$. Thus $R$ is a Baer gamma ring.

**Lemma 1.16.** Let $R$ be a right Rickart $\Gamma - ring$ with $1e = e \Gamma 1 = e$ for all idempotent $e \in R$. Then there exists a nonzero idempotent in any nonzero left annihilator.

**Proof:** Let a left annihilator $L \neq \emptyset$, write $L = l_R(A)$ with $\emptyset \neq A \subseteq R$. Take $0 \neq a \in l_R(A)$. Then $r_R(l_R(A)) \subseteq r_R(a) = e \Gamma R$ for some $e^2 = e \in R$. Hence, we have that $R^H (1 - e) = l_R(e \Gamma R) = l_R(r_R(a)) \subseteq l_R(r_R(l_R(A))) = l_R(A)$. Let $e = 1$, then $r_R(a) = e \Gamma R = R$, so $a = 0$, a contravalence. Thus, $L$ has a nonzero idempotent $1 - e$.

**Definition 1.17.** Idempotents $e$ and $f$ are called orthogonal idempotents if $e \gamma f = f \gamma e = 0$ for all $\gamma \in \Gamma$.

**Definition 1.18.** A gamma ring $R$ is called orthogonal finite if there are no infinite sets of orthogonal idempotents in $R$.

**Theorem 1.19.** If a Rickart $\Gamma - ring$ is orthogonally finite with $e \Gamma e = e$ and $1e = e \Gamma 1 = e$ for all idempotent $e \in R$, then it is Baer.

**Proof:** Assume that the conditions exist of $R$. As $R$ is orthogonally finite and from lemma (2.16.), we can choose an idempotent $e \in l_R(e)$ with $l_R(e)$ minimal in $\{l_R(h) \mid h^2 = h \in L\}$. We claim that $l_R(e) \cap L = 0$. Assume on the contrary that $l_R(e) \cap L \neq 0$. As $l_R(e) \cap L$ is a left annihilator, $l_R(e) \cap L$ contains an idempotent $f \neq 0$ from the above lemma. Let $b = e + f - e \gamma f$ for some $\gamma \in \Gamma$. Then $b \in L$ and $b^2 = b$ since $f \gamma e = 0$. Note that $b \Gamma R = e \Gamma R + f \Gamma R$, so $e \Gamma R \subseteq b \Gamma R$ and hence $l_R(b) \subseteq$
l_R(e). But, f \Gamma e = 0 and f \gamma b = f \neq 0. Thus l_R(b) \subset l_R(e) and b \in L, a contradiction to the choice of e. So l_R(e) \cap L = 0. If x \in L, then (x - x\gamma e)y \gamma e = 0 for all \gamma \in \Gamma and x - x\gamma e \in L. Hence x - x\gamma e \in l_R(e) \cap L = 0, so x = x\gamma e \in R\gamma e. Thus L = R\gamma e. So R is Baer.

2. Rickart gamma rings with Reduced gamma rings

Recall that, an element x in \Gamma - ring R is said to be nilpotent if for some \gamma \in \Gamma, there exist integer n = n(\gamma) such that x^n = 0 where x^n = (x\gamma)^{n-1} = (x\gamma x \ldots x\gamma x)x [3].

**Proposition 2.1.** Any right Rickart \Gamma - ring R has no nonzero central nilpotent elements.

**Proof:** Let R be a right Rickart \Gamma - ring and assume that there exists 0 \neq a \in Z(R) such that (ay)^{n-1}a = 0 and (ay)^{n-2}a \neq 0 for some positive integer n. Then since R is a right Rickart gamma ring, r_R((ay)^{n-1}a) = e\Gamma R for some idempotent e in R. But a \in r_R((ay)^{n-2}a), (ay)^{n-2}a \neq 0 and so neither e = 0 nor e = 1. eya = a = aye because a \in Z(R) and a \in r_R((ay)^{n-2}a). hence we obtain 0 = (ay)^{n-2}a ye = ey(ay)^{n-2}a = (eya)(ay)^{n-3}a = a y(ay)^{n-2}a = (ay)^{n-2}a, a contradiction.

The following definition is the image of the reduced rings in the category of gamma rings.

**Definition 2.2.** A \Gamma - ring R which has no nonzero nilpotent elements is called reduced. That implies there is no element x in R such that x^n = (x\gamma)^{n-1} = 0 for some n \in Z and \gamma \in \Gamma. Recall that, A \Gamma - ring R which every idempotent in it is central is called abelian [1], A \Gamma - ring which ayb = bya for all a, b \in R and \gamma \in \Gamma is called commutative [10].

**Proposition 2.3.** Let S be a subset of a Rickart \Gamma - ring R and \hat{S} = \{x \in R : x\gamma s = x\gamma x \forall s \in S and \gamma \in \Gamma \}, which is a sub \Gamma - ring of R. For \in \hat{S}, write r_R(x) = e\Gamma R, e idempotent. Then for all s \in S and \gamma, \rho \in \Gamma,

(i) spe = eyspe , and
(ii) (1 - e)ys = (1 - e)ys\rho(1 - e).

**Proof:** (i) For all s \in S one has x\gamma(spe) = s\gamma(xpe) = s\gamma 0 = 0 , thus spe \in \{x\}^\gamma = e\Gamma R, whence e\gamma(spe) = spe .

(ii ) For all s \in S, by (i) we have (1 - e)ys\rho(1 - e) = 1ys\rho1 - ey\gamma s1 - spe + eyspe = (1 - e)ys.

**Definition 2.4.** A gamma ring R is called commutative at 0 if ayb = 0 then bya = 0 for all a, b \in R and \gamma \in \Gamma.

**Corollary 2.5.** Any commutative gamma ring is commutative at 0.

**Proposition 2.6.** For any \Gamma - ring R, then

(1) If R is reduced then it is abelian. The convers is true if R right Rickart.

(2) If R is abelian then idempotents of R are commute. The convers is true if e\Gamma e = e for all idempotent e \in R.

(3) If the idempotents of R are commute then multiplying any two idempotent elements is an idempotent. The convers is true if e\Gamma e = e and e\Gamma 1 = 1\Gamma e = e for all idempotent e \in R.

(4) If R is reduced then R is commutative at 0. The convers is true when R is right Rickart and e\Gamma e = e for all idempotent e \in R.

**Proof:** (1) let an idempotent e = e\delta e \in R, (e\gamma(a - a\delta e))^2 = e\gamma(a - a\delta e)\delta e\gamma(a - a\delta e) = e\gamma\delta e\gamma a - e\gamma\delta e\gamma a = e\gamma\delta e\gamma a - e\gamma\delta e\gamma a = 0, by reduced, e\gamma(a - a\delta e) = 0 \Rightarrow e\gamma a = e\gamma a, also similarly we get a\gamma e = e\gamma a, then e\gamma a = a\gamma e for all \gamma \in \Gamma. For convers, let a \in R such that a^2 = 0 this implies a \in r_R(a) = e\Gamma R where e = e\delta e \in R, then a = e\delta a = a\delta e = 0 by using an abelian, hence R is reduced.
(2) The first is trivial. For convers, let \( a \in R \) and an idempotents \( e = e\delta e \) for all \( \delta \in \Gamma \), if \( e\delta a = 0 \), then \( e + a\delta e \) is idempotent, since \( e\delta(a - e\delta a) = 0 \) hence \( e + (a - e\delta a)e\delta e \) is idempotent. By commutation from (c) and multiplying by \( e \) we have \( e + e\delta a e - e\delta a e = e = e + a\delta e - e\delta a e \) therefore \( a\delta e = e\delta a e \), similarly we get \( e\delta a = e\delta a e \) and thus \( e\delta a = a\delta e \). (3) Let an idempotents \( e = e\delta e, f\beta f = f \in R \), \( (e\beta f)^2 = e\beta f\delta e\beta f = f\beta e\delta e\beta f = f\beta e\beta f = e\beta f \). For convers, let an idempotents \( e = e\delta e, f\beta f = f \in R \), by closed of idempotents we get \( (e\beta f)\delta(1 - e) \) is idempotent and since \( ((e\beta f)\delta(1 - e))^2 = 0 \) (same the start of proof) then \( (e\beta f)\delta(1 - e) = 0 \) \( e\beta f = e\delta f \). Similarly we have \( f\beta e = e\beta f \delta \) and hence \( e\beta f = f\beta e \) for all \( \beta \in \Gamma \). (4) Let \( ayb = 0 \) for any \( a, b \in R \) and \( y \in \Gamma \), since \( (by)^2 = (by)y(by) = by(by)ya = 0 \), then by reduced we have \( by = 0 \). For convers, let \( a \in R \), such that \( a^2 = 0 \) this implies \( a \in r_R(a) = e\Gamma R, e = e\delta e \in R \) for all \( e \in \Gamma \), then \( a = e\delta a \) and \( (1 - e)\beta a = 0 \) where \( 1\beta a = a\beta 1 = a, i.e. a\beta(1 - e) = 0 \) and \( a = 0 \) since \( a\beta e = 0 \), hence \( R \) is reduced.

**Proposition 2.7.** For a right Rickart \( \Gamma - ring \) \( R \), \( r_R(x) = l_R(x) \) for all \( x \in R \) if \( R \) is abelian. The convers is true if \( x\Gamma 1 = 1\Gamma x = x \ \forall x \in R \).

**Proof:** Take \( x \in R \), then by Rickart \( r_R(x) = e\Gamma R = R\delta e = l_R(x) \) from abelian. For convers, let \( e \) is idempotent in \( R \) such that \( e\gamma e = e \Rightarrow r_R(e) = l_R(e) = e\Gamma(e - 1) = (e - 1)e\Gamma R \). If \( x \in R \), then for all \( y \in \Gamma \), \( (1 - e)\gamma x = [(1 - e)\gamma]y(1 - e) \) \( = xy(1 - e) \) and \( xy(1 - e) = (1 - e)\gamma (1 - e) \), so \( (1 - e)\gamma x = x\gamma (1 - e) \) \( \Rightarrow e\gamma x = x\gamma e \), thus \( R \) is abelian.

**Corollary 2.8.** All commutative Rickart gamma rings with \( e\Gamma e = e \) for all idempotent \( e \in R \) are reduced.

**Proof:** By Corollary (2.5) and Proposition (2.6) which includes \( (e) \) implies \( (a) \).

**Definition 2.9.** For an element \( a \in \Gamma - ring \) \( R \), the right focal idempotent is any idempotent \( e \) provided that the right ideal generated by it, is right annihilator of \( a \). \( RFI(a) \) is denoted for the set of all right focal idempotent of an \( a \) in \( \Gamma - ring \) \( R \). The definition of the left focal idempotent of \( a \) is obtained in a similar way.

**Remark 2.10.** For any gamma ring \( R \), \( RFI(a) = \emptyset \ \forall a \in R \) if and only if \( R \) is right Rickart.

The next proposition given the relation between left zero divisors and right focal idempotents.

**Proposition 2.11.** For a non-zero element \( a \) of a right Rickart \( \Gamma - ring \) \( R \), the following are equivalent:

(i) \( a \) is a left zero divisor,

(ii) \( 0 \notin RFI(a) \),

(iii) \( RFI(a) \) has a non-zero element.

**Proof:** (i) \( \Rightarrow \) (ii) Assume that, inversely, \( 0 \in RFI(a) \); then \( 0\gamma y = y \) when-ever \( a\gamma y = 0 \) for all \( y, \gamma \in \Gamma \). Since \( 0\gamma y = y \) iff \( y = 0 \); then a is a non left zero divisor. (ii) \( \Rightarrow \) (iii) By the above remark, the proof is trivial. (iii) \( \Rightarrow \) (i) Evidently \( a\gamma a' = 0 \); it follows that if \( a' \neq 0 \), then a is a left divisor of 0.

Note that, in a right Rickart gamma ring \( R \), \( RFI(a) = \{0\} \) for ever a \( \neq 0 \) if and only if \( R \) has a no proper left zero divisor.

**Proposition 2.12.** Let \( R \) be a reduced \( \Gamma - ring \). Then \( RFI(a) = LFI(a) \) For all \( a \in R \). Consequently, in reduced gamma ring the condition of Rickart is left-right symmetric.

**Proof:** Suppose that \( R \) is a reduced \( \Gamma - ring \), and \( e \in RFI(a) \) for some element \( a \) and idempotent \( e \), this implies \( a\gamma y = 0 \) iff \( e\gamma e y = y \) for every \( y \in R \). As \( R \) is commutative at 0 and \( e \) is central (from proposition 2.6.), then \( y\gamma a = 0 \) iff \( y\gamma e = y \). So, when \( e \in RFI(a) \) then also \( e \in LFI(a) \). The proof of the opposite direction is similar.
From the previous propositions, the next proposition can be obtained.

**Proposition 2.13.** Let \( R \) be a Rickart \( \Gamma \)-ring. If \( e \Gamma e = e \) and \( x \Gamma 1 = 1 \Gamma x = x \) for all idempotent \( e \) and element \( x \) in \( R \), then the following are equivalent:

1. \( R \) is reduced.
2. \( R \) is abelian.
3. Idempotents of \( R \) commute.
4. Multiplying any two idempotent elements is an idempotent.
5. \( R \) is commutative at 0.
6. \( RFI(x) = LF1(x) \) for every \( x \in R \)
7. \( r_R(x) = l_R(x) \) for all \( x \in R \).

**Proposition 2.14.** For a reduced Rickart \( \Gamma \)-ring \( R \). The idempotent \( e \) which is generated the right annihilator of any element in \( R \) is unique.

**Proof:** Take any \( a \in R \), and assume that \( e \) and \( f \) are idempotents in \( R \) such that \( e \gamma_0 y = y \) iff \( ay_0 = 0 \) iff \( f \gamma_0 y = y \) for all \( y \in R, \gamma \in \Gamma \). Then \( e \gamma_0 f = f \) and \( f \gamma_0 e = e \) from the choice of \( e \) and \( f \). So \( e = f \) from commutativity in \( E \).

**Lemma 2.15.** Let \( R \) be a right Rickart \( \Gamma \)-ring. Then \( R \) is reduced if and only if \( e \gamma a = 0 \) for every \( a \in R, \gamma \in \Gamma \) and any \( e \in RFI(a) \).

**Proof:** Suppose that \( R \) be a right Rickart \( \Gamma \)-ring, \( a \in R \) and \( r_R(a) = e \Gamma R \) (i.e. \( e \in RFI(a) \)) such that \( a^2 = 0 \Rightarrow a = e \gamma a = 0 \), by condition. Conversely, let \( a \in R, e \in RFI(a) \) this implies \( r_R(a) = e \Gamma R \Rightarrow 0 = a \gamma e = 0 \), since \( R \) reduced (i.e. abelian), then \( e \gamma a = 0 \).

**Definition 2.16.** A ring \( R \) satisfies the insertion of -factors property (shortly, IFP) if \( ayb = 0 \) implies that \( aRy=b=0 \) for all \( a, b \in R \) and \( y \in \Gamma \).

**Proposition 2.17.** All Rickart gamma ring is satisfied IFP if any generated idempotent of the right(left) annihilator of any element is semicentral.

**Proof:** Let \( a, b \in R \) such that \( ayb = 0 \), hence, \( b \in r_R(a) = e \Gamma R \) with \( e \gamma_0 e = e \) for some \( \gamma_0 \in \Gamma \Rightarrow b = e \gamma_0 b \), then for any \( \gamma \in Gamma \), \( aRy = aRy \), \( aRy = aRy \) for all \( e \) is semicentral \( rye = eRy \).

**Remark 2.18.** For a \( \Gamma \)-ring \( R \) satisfied IFP, \( r_R(a) = r_R(aR) = r_R(RGammaR) \) for all \( a \in R \).

**Proof:** Let \( a \in r_R(a) \) \( \Rightarrow ayx = 0 \Rightarrow (by IFP)ax \gamma x = 0 \Rightarrow x \in r_R(aR) \), hence \( r_R(a) \subseteq (a \Gamma R) \). Let \( x \in r_R(aR) \) \( \Rightarrow ax = a \gamma 1 \Rightarrow 0 \) (where \( a \gamma 1 = a \)), then \( x \in r_R(a) \) and \( (a \Gamma R) \subseteq r_R(a) \) therefore \( r_R(a) = r_R(aR) \). Similarly we get to \( r_R(a) = r_R(RGammaR) \).

**Definition 2.19.** A \( \Gamma \)-ring \( R \) in which \( r_R(a) = \{ x \in R | a \Gamma x = 0 \} \) is an ideal in \( R \) for all \( a \in R \), is called semicommutative.

**Proposition 2.20.** A Rickart \( \Gamma \)-ring \( R \) is semicommutative iff it is satisfied IFP.

**Proof:** Suppose that \( R \) satisfied IFP. For any \( a \in R, r_R(a) = e \Gamma R \) where \( e \beta e = e \in R \) for some \( \beta \in \Gamma \), let \( x \in r_R(a) \) and \( b \in R \), then \( ax = 0 \) for all \( \gamma \in \Gamma \), clearly \( ayb = 0 \) for all \( y \in \Gamma \) this implies \( x \gamma b \in r_R(a) \), also by IFP we get to \( ayb = 0 \) then \( byx \in r_R(a) \) hence \( r_R(a) \) is ideal, thus \( R \) semicommutative. Conversely, assume that \( R \) is semicommutative and \( ayb = 0 \) for all \( a, b \in R \) and \( y \in \Gamma \), this implies \( b \in r_R(a) \) and then \( ryb \in r_R(a) \) by use semicommutative, therefore \( aRy = 0 \) for all \( y \in \Gamma \), thus \( R \) satisfied IFP.
Lemma 2.21. Any Rickart gamma ring is a semicommutative if it is reduced.

Proof: Suppose that $R$ is reduced $\Gamma -ring$. For any $a \in R$, $r_R(a) = e\Gamma R$ where $e\beta e = e \in R$ for some $\beta \in \Gamma$, let $x \in r_R(a)$ and $b \in R$, then for all $\gamma, \delta \in \Gamma$, $\alpha \gamma b\delta x = \alpha \gamma b\delta e\beta x = \alpha \gamma b\delta e\beta e \in R$ since $x = e\beta x$ and by equivalent in Proposition (2.6), this implies $b\delta x \in r_R(a)$ and similarly we have $x\beta b \in r_R(a)$ for any $\delta \in \Gamma$, therefore for any $a \in R$, $r_R(a)$ is ideal in $R$, hence $R$ is semicommutative. The converse is given in next proposition.

Colloray 2.22. Any reduced Rickart gamma ring satisfied IFP.

Proof: Clearly from Proposition (2.20) and Lemma (2.21).

Proposition 2.23. A semicommutative (or satisfied IFP) $\Gamma -ring$ is reduced if $e\Gamma e = e$ and $a\Gamma 1 = 1\Gamma a = a$ for all idempotent $e$ and element $a$ in $R$.

Proof: By the condition we have $e(\gamma \Gamma e) = 0$ for all idempotent $e \in R$ and $\gamma \in \Gamma$, this implies $(1 - e) \in r_R(e)$, since $R$ is semicommutative, then $\alpha(1 - e) \in r_R(e)$ for all $a \in R$, therefore $e\delta \gamma (1 - e) = 0$ for all $\delta \in \Gamma$, hence $e\delta a = e\delta \gamma e$, similarly we have $a\delta e = e\delta \gamma e$ and then $e\delta a = a\delta e$ which implies the set of idempotents in central and by equivalent in Proposition (2.6.) $R$ is reduced. The proof when $R$ satisfied IFP is clearly by use Proposition (2.20).

Recall that An element $a$ in a $\Gamma - ring R$ is called right nonzero divisor if $bya = 0$ implies $b = 0$ and for any $\gamma \in \Gamma$. The definition of left nonzero divisor is obtained in a similar way; an element is zero divisor if it is right or left zero divisor [10].

Proposition 2.24. Any Rickart $\Gamma -ring R$ have the idempotents 0 and 1 only, then it has no divisors of 0 (hence is a Baer).

Proof: Let $x \in R, x \neq 0$. Write $r_R(x) = e\Gamma R, e \text{idempotent.}$ By hypothesis, $e = 0 \ or \ 1 \ ;$ since $x \neq 0$, necessarily $e = 0 \ , \text{thus} \ r_R(x) = \{0\}.$

Lemma 2.25. Let $R$ be an abelian right Rickart $\Gamma -ring$ such that $e\Gamma e = e \ and \ r\Gamma 1 = 1\Gamma r = r$ for all idempotent $e$ and element $r$ in $R$. Then for every element $c$ of $R$ there exist an idempotent $e$ and a left nonzero divisor $a$ of $R$ such that $c = a\gamma e$ for all $\gamma \in \Gamma$.

Proof: Let $c \in R$, then by Rickart, $c = \gamma \Gamma R$ for some idempotent $\gamma$ in $R$. Put $e = 1 - \gamma \ and \ a = \delta + c$, if $a\delta e = 0$, then $e\gamma a\delta e = e\gamma \delta e + e\gamma c\delta = \gamma e = 0 \ and \ e\delta a\gamma = e\gamma \delta e + e\gamma c\delta = \gamma e = 0 \ \Rightarrow \ d = 0 \ , \text{thus} \ a \ \text{is left non zero divisor and also} \ a\gamma e = (\delta + c)\gamma e = c.$

Lemma 2.26. Let $R$ be an abelian right Rickart $\Gamma -ring$ such that $e\Gamma e = e \ and \ r\Gamma 1 = 1\Gamma r = r$ for all idempotent $e$ and element $r$ in $R$. Then every left or right nonzero divisor of $R$ is a nonzero divisor of $R$.

Proof: Take $b$ be a left non zero divisor of $R$ and assume that $c\gamma b = 0, c \in R$. From Lemma (2.25), $c = a\gamma e$ for a left non zero divisor $a$ of $R$ and an idempotent $e$ in $R$. Then it is $a\gamma e\gamma b = 0 \Rightarrow e\gamma b = 0 \Rightarrow bye = 0 \Rightarrow e = 0 \Rightarrow c = 0$. Therefore $b$ is a (right) non zero divisor of $R$. Next, take $b$ be $a$ right non zero divisor of $R$. Also, by Lemma (2.25), $b = a\gamma e$ for a left non zero divisor $a$ of $R$ and an idempotent $e$ in $R$. So $e\gamma b = e\gamma a\gamma e = a\gamma e = a\gamma e = b$, so $(1 - e)\gamma b = 0 \Rightarrow e = 1 \Rightarrow b = a$. Therefore $b$ is a (right) non zero divisor of $R$.

Proposition 2.27. Any right Rickart $\Gamma -ring R$, which is $1\Gamma f = f\Gamma 1 = f$ for all idempotent $f$ in it, is reduced if and only if no right zero divisor in it can be written as a sum $a + e$ with $e$ in RFI$(a)$.

Proof: Suppose that $R$ is reduced and let $a \in R$ such that $b(\gamma + e) = 0$ for some $b \in R$, then $0 = by(\gamma + e)\gamma e = bye$ and also $0 = by(\gamma + e)\gamma e = bya$ where $\gamma = 1 - e \ , \Rightarrow a\gamma b = 0 \Rightarrow e\gamma b = 0$, therefore the condition hold. For convers, Let $a\gamma = a\gamma a = 0$ for all $\gamma \in \Gamma$ and $e \in RFI(a)$, then $r_R(a) = e\Gamma R, a\gamma (\gamma + e) = 0$, hence $a = 0$ by the condition, thus $R$ is reduced.
3. Rickart gamma rings with Quasi (p.q.)-Baer gamma rings

**Definition 3.1** A gamma ring $R$ is called Quasi-Baer gamma ring if the right annihilator in $R$ of any ideal of $R$ is generated by an idempotent of $R$.

**Proposition 3.2.** For a $\Gamma - ring$ $R$ is Quasi Baer if and only if the left annihilator in $R$ of any ideal of $R$ is generated by an idempotent of $R$.

**Proof:** The proof is conventional.

The condition of Quasi Baer (from proposition (3.2.)), is right-left symmetric.

Recall that, An ideal $I$ of a $\Gamma - ring$ $R$ is said to be prime if for any two ideals $A$ and $B$ of $R$, $A\cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$ [14]. Also, a gamma ring $R$ is said to be prime if the zero ideal is prime [13].

In the following definition we will introduce the concept reduced semicentral in the gamma rings, which was previously defined in the rings.

**Definition 3.3.** A left semicentral reduced element $e$ of a $\Gamma -ring$ $R$ is a nonzero idempotent provided that $S_1(e^{\Gamma}R\Gamma e) = \{0, e\}$. A right semicentral reduced element is defined similarly. A $\Gamma -ring$ $R$ is said to be left (right) semicentral reduced if $R$ is left (right) semicentral reduced.

**Proposition 3.4.** Let $R$ be a $\Gamma -ring$. Then $R$ is prime if and only if it is quasi-Baer and semicentral reduced.

**Proof:** Suppose that $R$ is a prime. Let $e \in S_1(R)$. Then $(1 - e)\Gamma R\Gamma e = 0$ by Proposition (1.11.). Thus $e = 0$ or $e = 1$, and then $R$ is semicentral reduced. Let $I$ ideal in $R$. When $I = 0$, we have $r_R(I) = R$. If $I \neq 0$, then $r_R(I) = 0$ as $R$ is prime. Therefore $R$ is quasi-Baer. The opposite direction, assume that $R$ is quasi-Baer and semicentral reduced. Let $I$ and $J$ ideal in $R$ with $IJ = 0$. So $f \subseteq r_R(I) = fR$ for some idempotent $f \in R$. As $f\Gamma R$ is ideal in $R$, $f \in S_1(R)$ from Proposition (1.11.). Thus $f = 0$ or $f = 1$. If $f = 0$, then $I = 0$. If $f = 1$, then $I = 0$. So $R$ is prime.

**Proposition 3.5.** For the $\Gamma - rings$ $R_i \ i \in I$, the direct product $R = \Pi_{i \in I} R_i$ is quasi-Baer iff $R_i$ is quasi-Baer for all $i \in I$.

**Proof:** The proof is effortless. (Similar to proof of prop. 1.13).

For the Rickart, in Proposition (1.8), we needed the requirement $e\Gamma e = e$ while here , in the Quasi, it is achieved unconditionally.

**Theorem 3.6.** For any idempotent $e$ in a quasi-Baer $\Gamma -ring$ $R$, $e\Gamma R\Gamma e$ is a quasi-Baer.

**Proof:** Let $R$ be a quasi-Baer ring and $I$ ideal in $e\Gamma R\Gamma e$ where $e\Gamma e = e \in R$. Then $l_R(R\Gamma I) = R\Gamma f$ for some idempotent $f \in R$. We first show that $l_{e\Gamma R\Gamma e}(I) = e\Gamma l_R(R\Gamma I)e$. Let $x \in l_{e\Gamma R\Gamma e}(I)$. Then $x\Gamma R\Gamma I \subseteq x\Gamma e\Gamma R\Gamma \Gamma e\Gamma I \subseteq x\Gamma I = 0$. Thus, $x \in l_R(R\Gamma I)$ and hence $x = e\Gamma y\Gamma e \in e\Gamma l_R(R\Gamma I)e$. Next, if $y \in e\Gamma l_R(R\Gamma I)e$, then $y = e\Gamma y_1\Gamma e\Gamma y_2\Gamma e$ with $z \in l_R(R\Gamma I)$ and so $y\Gamma I = e\Gamma y_1\Gamma e\Gamma y_2\Gamma R\Gamma I = 0$. Hence, $y \in l_{e\Gamma R\Gamma e}(I)$. This shows that $l_{e\Gamma R\Gamma e}(I) = e\Gamma l_R(R\Gamma I)e = e\Gamma R\Gamma f\Gamma e$. Because $R\Gamma f$ is an idempotent in $R$, $f \in S_1(R)$ by Proposition (1.11.), hence $e\Gamma f\Gamma e$ is idempotent in $R$. Let $g = e\Gamma f\Gamma e$. Then $e\Gamma R\Gamma g = e\Gamma R\Gamma e\Gamma e\Gamma f\Gamma e \subseteq e\Gamma R\Gamma f\Gamma e$. Further, for $t \in e\Gamma R\Gamma f\Gamma e$, $t = e\Gamma y_1\Gamma y_2\Gamma f\gamma_3\Gamma e$ for some $r \in R$ and $r_1, r_2, r_3 \in e\Gamma f$. Since $b \in S_1(R)$, then $t = e\Gamma y_1\Gamma y_2\Gamma f\gamma_3\Gamma e\Gamma e \subseteq e\Gamma R\Gamma f\Gamma e\Gamma f\Gamma e$. Therefore, $e\Gamma R\Gamma f\Gamma e \subseteq e\Gamma R\Gamma e\Gamma g$. Thus $l_{e\Gamma R\Gamma e}(I) = e\Gamma R\Gamma f\Gamma e = e\Gamma R\Gamma e\Gamma g$ and hence, $e\Gamma R\Gamma e$ is quasi-Baer.

**Theorem 3.7.** For any quasi-Baer gamma ring, the center is a Baer.
Proof: Suppose that a $\Gamma$–ring $R$ is a quasi-Baer. Let $C = \text{Cen}(R)$. Take $\emptyset \neq Y \subseteq C$. So $r_R(Y) = r_R(Y \cap R) = e \Gamma R$ with $e \in S_R(R)$. We observe that $r_R(Y) = L_p(Y) = L_p(R \Gamma \gamma) = R \Gamma f$ for some $f \in S_R(R)$. Because $e \Gamma R = R \Gamma f$, $e = f \in S(R) \cap S_R(R) \subseteq B(R)$ by Remark (1.10). Therefore $e \in C$, so $r_C(Y) = r_R(Y \cap C) = e \Gamma R \cap C = e \Gamma C$. Hence $\text{Cen}(R)$ is Baer.

Corollary 3.8. A quasi-Baer $\Gamma$–ring $R$ is Rickart if it satisfies the IFP.

Proof: Let $a \in R$, since $R$ has IFP then by Remmark (2.18). $r_R(a) = r_R(R \Gamma a \Gamma R) = e \Gamma R$ for some idempotent $e \in R$.

Definition 3.9. A gamma ring $R$ is called right principally quasi-Baer (simply, right p.q.-Baer) if there is an idempotent generates the right annihilator of any principal ideal as a right ideal. A left principally quasi-Baer (simply, left p.q.-Baer) ring is defined in a similar way. Gamma rings which are right and left principally quasi-Baer are called principally quasi-Baer (simply, p.q.-Baer).

Proposition 3.10. A Rickart $\Gamma$–ring $R$ is right (left) p.q–Baer if any generated idempotent of the right(left) annihilator of any element is semicentral.

Proof: Let $R$ be Rickart. Take $a \Gamma R$ is a principle right ideal in $R$ for $a \in R$. Then, $r_R(a) = e \Gamma R$ for some idempotent $e \in R$. So, by Proposition (2.17.) and Remmark (2.18.), we have $r_R(a) = r_R(a \Gamma R) = e \Gamma R$. Thus $R$ is right p.q–Baer.

Proposition 3.11. Let $R$ be a $\Gamma$–ring. Then $R$ is prime if and only if it is right p.q.-Baer ring and semicentral reduced.

Proof: The proof is similar to that of Proposition (3.4.).

Proposition 3.12. Let $R$ be a $\Gamma$–ring. Then $R$ is right p.q.-Baer if and only if for each finitely generated ideal $I$ of $R$, there exists idempotent $e \in R$ such that $r_R(I) = e \Gamma R$.

Proof: Assume that $R$ is a right p.q.-Baer $\Gamma$–ring and let $I = \bigcap_{i=1}^{n} R \Gamma a_i \Gamma R$. Then $r_R(I) = \bigcap_{i=1}^{n} r_R(R \Gamma a_i \Gamma R) = \bigcap_{i=1}^{n} e_i \Gamma R$, where each $r_R(R \Gamma a_i \Gamma R) = e_i \Gamma R$ and $e_i$ is idempotent in $R$ for all $i$. So each $e_i \in S(R)$. From remark (1.10). $r_R(I) = \bigcap_{i=1}^{n} e_i \Gamma R = e \Gamma R$ for some $e \in S(R)$. The converse is obvious.

Proposition 3.13. For the $\Gamma$–rings $R_i$ $i \in I$, the direct product $R = \prod_{i \in I} R_i$ is right p.q.-Baer iff $R_i$ is right p.q.-Baer for all $i \in I$.

Proof: The proof is explicit. (Similar to proof of prop. 1.13.)

Theorem 3.14. Let $\mathcal{R} = \{e \Gamma R \mid e \in S(R)\}$ where $R$ be a $\Gamma$–ring. Then $R$ is a quasi-Baer if and only if $R$ is a p.q.-Baer and $\mathcal{R}$, under inclusion, is complete lattice.

Proof: Suppose that $R$ is a quasi-Baer ring. Hence $R$ is a p.q.-Baer ring. Let $\{e_i \Gamma R \mid i \in \Lambda\}$ is a subset of $\mathcal{R}$. Since $\sum e_i \Gamma R$ is an ideal and $R$ is quasi-Baer, $r_R(\bigcap \{e_i \Gamma R\}) = f \Gamma R$ where $f \in S(R)$. It can be easily checked that $f \Gamma R = \sup \{e_i \Gamma R \mid i \in \Lambda\}$. Since $\mathcal{R}$, under inclusion, is partially ordered set, then $\mathcal{R}$ is a complete lattice under inclusion. Conversely, let $I$ ideal $R$ and take $I = \sum_{i \in A} R \Gamma x_i \Gamma R$, $x_i \in R$. As $R$ is p.q.-Baer, then there is $e_i \in S(R)$ for each $i$ with $r_R(I) = \bigcap_{i \in A} r_R(R \Gamma x_i \Gamma R) = \bigcap_{i \in A} e_i \Gamma R$. By completeness, there exists $e \in S(R)$ such that $e \Gamma R = \inf \{e_i \Gamma R \mid i \in A\}$. Then, $e \Gamma R \subseteq \bigcap_{i \in A} e_i \Gamma R$. Next, let $y \in \bigcap_{i \in A} e_i \Gamma R$. Thus $r_R(R \Gamma y) = R \Gamma h$ for some $h \in S(R)$ since $R$ is p.q.-Baer. Because $y \in e_i \Gamma R$ and $e_i \Gamma R$ is ideal in $R$, $R \Gamma y \subseteq e_i \Gamma R$, then we have that $(1 - h) \Gamma R = r_R(\bigcap \{r_R(R \Gamma y)\}) \subseteq r_R(\bigcap \{e_i \Gamma R\}) = e_i \Gamma R$ for each $i$. From Proposition (1.11), $(1 - h) \Gamma R \subseteq e_i \Gamma R$. Therefore, $(1 - h) \Gamma R \subseteq e_i \Gamma R$, then $y \in R \Gamma y \subseteq r_R(\bigcap \{r_R(R \Gamma y)\}) = (1 - h) \Gamma R \subseteq e \Gamma R$. Hence, $\bigcap_{i \in A} e_i \Gamma R \subseteq e \Gamma R$. Therefore, $e \Gamma R = \bigcap_{i \in A} e_i \Gamma R$, and so $r_R(I) = e \Gamma R$. Hence, $R$ is quasi-Baer.

Theorem 3.15. If a $\Gamma$–ring $R$ is right p.q.-Baer, then:
(i) $e\Gamma R\Gamma e$ is a right p.q.-Baer for each idempotent $e\Gamma e = e \in R$.

(ii) The center $\text{Cen}(R)$ is a Rickart.

Proof: (i) Suppose that $R$ be a right p.q.-Baer. Let $x \in e\Gamma R\Gamma e$. Then $\rho_{e\Gamma R\Gamma e}(e) = e\Gamma R\Gamma e((e\Gamma y_{1}x_{2}y_{2})\Gamma(R)) = \rho_{e\Gamma R\Gamma e}(e)\Gamma(R)$ for all $y_{1}, y_{2} \in \Gamma$. As $R$ is right p.q.-Baer, $\rho_{e\Gamma R\Gamma e}(e) = f\Gamma R$ with $f \in S_{1}(R)$. So $\rho_{e\Gamma R\Gamma e}(e) = f\Gamma R\Gamma e$. Because $f \in S_{1}(R)$, $e\Gamma f\Gamma e$ is an idempotent. Also, $f\Gamma R\Gamma e = (e\Gamma f\Gamma e)\Gamma(R)\Gamma e$, thus $\rho_{e\Gamma R\Gamma e}(e) = (e\Gamma f\Gamma e)\Gamma(R)\Gamma e$. Therefore, $e\Gamma R\Gamma e$ is right p.q.-Baer. (ii) Suppose that $R$ is a right p.q.-Baer and Put $C = \text{Cen}(R)$. Take $a \in C$. Then $l_{R}(a) = r_{R}(\Gamma a) = r_{E}(a\Gamma R) = e\Gamma R$ with $e \in S_{1}(R)$ from Proposition (2.10). Note that $l_{R}(R\Gamma a) = l_{R}(l_{R}(R\Gamma a)) = l_{R}(r_{R}(e\Gamma R))$. Say $r_{R}(e\Gamma R) = f\Gamma R$ with $f \in S_{1}(R)$. Therefore $1 - f \in S_{1}(R)$ by Proposition (2.10). Hence, we have that $e\Gamma R = l_{R}(R\Gamma a) = l_{R}(r_{R}(e\Gamma R)) = l_{R}(f\Gamma R) = R\Gamma(1 - f)$. So $= 1 - f$. Because $a \in S_{1}(R)$ and $1 - f \in S_{2}(R)$, $e = 1 - f \in B(R)$ by (Remark 2.9.). Thus $r_{C}(a) = r_{C}(a) \cap \in C = e\Gamma R \cap C = e\Gamma C$. So $\text{Cen}(R)$ is Rickart.

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