UNIQUENESS OF PERCOLATION ON PRODUCTS WITH $\mathbb{Z}$

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ABSTRACT. We show that there exists a connected graph $G$ with subexponential volume growth such that critical percolation on $G \times \mathbb{Z}$ has infinitely many infinite clusters. We also give some conditions under which this cannot occur.

This paper begins with the observation that if $G$ is any connected graph and $p$ is any number in $[0, 1]$, then the number of infinite clusters in $p$-percolation on $G \times \mathbb{Z}$ is deterministic, and is either 0, 1 or $\infty$. The proof is an easy consequence of the fact that one can take any finite set of vertices and translate it along the $\mathbb{Z}$ axis and get a set of variables disjoint from the one you started with.

In view of this, Sznitman asked whether the Burton-Keane argument [BK89] applies. Namely, assume $G$ is amenable, does it follow that $G \times \mathbb{Z}$ has only finitely many infinite clusters? The definition of amenability used here is that the Cheeger constant is 0, namely, for every $\epsilon > 0$ there is some finite set of vertices $A$ such that $|\partial A| \leq \epsilon |A|$ where $\partial A$ is the edge boundary of $A$.

As stated the answer is no. A binary tree with a infinite path added at the root serves as a counterexample. We suggest a slight modification.

Say that $G$ is strongly amenable if $G$ contains no nonamenable subgraph.

Assume $G$ is strongly amenable, can one find an interval $[p_1, p_2]$ such that percolation on $G \times \mathbb{Z}$ has infinitely-many infinite clusters for every $p$ in this interval? What if we further assume that $G$ has polynomial volume growth?

Our main result is to construct an example of a strongly amenable graph of the form $G \times \mathbb{Z}$ with non uniqueness at $p_c$. We do not see yet any example of such a graph in which no percolation occurs at $p_c$, but non uniqueness occurs for some $p > p_c$.

It is tempting to reformulate this question as $p_c = p_u$ (where $p_u$ is the threshold for uniqueness, see [LP] for the precise definition) but there is no monotonicity of uniqueness for graphs of the type $G \times \mathbb{Z}$. Indeed, connect the root of a $\mathbb{Z}^{99}$ lattice to the root of a 10 regular tree $T$, denote this graph by $G$. The parameters were chosen so as to satisfy

$$p_c(\mathbb{Z}^{100}) < p_c(T \times \mathbb{Z}) < p_u(T \times \mathbb{Z})$$

(\text{the first inequality follows from the fact that } p_c(\mathbb{Z}^d) \leq C/d, \text{ see [K90] or [ABS04, §4]; and from the bound } p_c(T \times \mathbb{Z}) \geq \frac{1}{10} \text{ which holds for any graph with degree 12. The second inequality follows from [S01]). It is not hard to see that on } G \times \mathbb{Z} \text{ for small } p \text{ no percolation occurs. Then between } p_c(\mathbb{Z}^{100}) \text{ and } p_c(T \times \mathbb{Z}) \text{ there is a unique infinite cluster. Between } p_c(T \times \mathbb{Z}) \text{ and } p_u(T \times \mathbb{Z}) \text{ there are infinitely many infinite clusters. Finally, above } p_u(T \times \mathbb{Z}) \text{ again one has a unique infinite

\footnote{For background on percolation see [G99] or [LP].}

\footnote{Personal communication.}
cluster. This example can be generalized to an arbitrary, even infinite number of transitions.

Regarding strong amenability we ask the following. There is no known example of an exponentially growing Cayley graph which is strongly amenable and the existence of such is still open, (see \[CT08\] for recent related work and a review of what is known for groups). A graph \(G\) has uniform growth if all balls with the same radius have the same size up to a fixed multiplicative constant. Is there a graph with a uniform exponential growth which is strongly amenable?

This note has two results on this problem. The first is a counterexample:

**Theorem 1.** There exists a connected graph \(G\) with subexponential volume growth such that critical percolation on \(G \times \mathbb{Z}\) has infinitely many infinite clusters.

The second is a positive result, a family of graphs \(G\) for which we can prove that \(G \times \mathbb{Z}\) does not have infinitely-many infinite clusters at any \(p\). The result is not very satisfying, and calls for strengthening.

**Theorem 2.** Let \(G\) be a connected graph such that each finite set can be disconnected from infinity by removing a bounded number of edges. Then \(G \times \mathbb{Z}\) does not have infinitely-many infinite components.

In other words, we require from \(G\) that there exists some constant \(K\) such that for every finite set of vertices \(A\) one can find \(K\) edges \(e_1(A), \ldots, e_K(A)\) such that removing these edges will make all the components of all \(v \in A\) finite.

**Proof of Theorem 1**

Let \(d\) be some sufficiently large number to be fixed later. The graph is constructed as follows. Take a tree of degree \(4d\). Let \(l_1 = 1\) and \(l_{n+1} = l_n + \lceil d^2 \log(n + 1) \rceil\). Now, for each \(n \geq n_0\) (\(n_0\) to be fixed later too, depending on \(d\)) and for each edge \((x, y)\) where \(x\) is in level \(l_n - 1\) and \(y\) is in level \(l_n\), disconnect \((x, y)\) and instead take a copy of \(\mathbb{Z}^d\) (considered as a graph with the usual structure) and connect \(x\) with the vertex \((0, \ldots, 0)\) and \(y\) with the vertex \((n, \ldots, n)\). All copies of \(\mathbb{Z}^d\) (for all such \((x, y)\)) are disjoint. This terminates the definition of the graph \(G\).

We will show that at \(p = p_c(\mathbb{Z}^{d+1})\) the graph \(G \times \mathbb{Z}\) has infinitely many infinite clusters. One can rather easily convince oneself that in fact below \(p\) our graph \(G \times \mathbb{Z}\) has no infinite clusters, so \(p = p_c(G \times \mathbb{Z})\), but we will not do it here. Note that \(p = (1 + o(1))/2d\) where \(o(1)\) is as \(d \to \infty\) \([K90]\).

**Subexponential growth.** Examine the ball \(B\) of radius \(r\) around the root of the tree we started with. Now, for any \(x\) in level \(l_n\) of the tree, \(d(x, 0) \approx n^2\) because the shortest path wastes \(k\) steps between levels \(l_k - 1\) and \(l_k\) for each \(k < n\). Therefore \(B\) contains tree elements up to level \(l_h\) for \(h \approx \sqrt{r}\). Since \(l_h \approx \sqrt{r} \log r\) we get that \(B\) contains \(\leq \exp(C \sqrt{r} \log r)\) tree vertices. The non-tree vertices of \(B\) are contained in \(\leq \exp(C \sqrt{r} \log r)\) copies of a \(d\)-dimensional ball of radius \(r\), so all in all we get

\[ |B| \leq Cr^d e^{C\sqrt{r} \log r} \leq Ce^{C\sqrt{r} \log r} \]

which is subexponential, as needed.
Existence. We now turn to show that there are infinite clusters. Let $\gamma$ be some path in $G \times Z$. We say that $\gamma$ is “between levels $l_{n-1}$ and $l_n$” if for each vertex $(v,n)$ of $\gamma$, either $v$ is in the tree, and its level is between $l_{n-1}$ and $l_n$, or $v$ belongs to one of the copies of $Z^d$ that were connected between levels $l_n - 1$ and $l_n$. Further we require that only the first and last vertices of $\gamma$ may have their $v$ in levels $l_{n-1}$ and $l_n$. The interior vertices need to be in levels strictly between, or in the copies of $Z^d$. With this definition we have

Lemma 1. Let $d$ be sufficiently large, $n > n_0$ and let $x$ be a tree element in level $l_{n-1}$. Let $Z$ be the set of vertices $z$ in level $l_n$ such that $(z,0)$ is connected to $(x,0)$ by an open path between $l_{n-1}$ and $l_n$. Then $|Z|$ stochastically dominates a variable $U$, independent of $n$, with $\mathbb{E}U > 1$.

Proof. Examine the set $Y$ of vertices $y$ of $G$ in level $l_n - 1$ such that $(x,0) \in G \times Z$ is connected to $(y,0)$ inside the “slice” $G \times \{0\}$. This is just a problem on supercritical branching processes (for $d$ sufficiently large $p_c$ of the tree, which is $1/(4d - 1)$, is smaller than $p_c(Z^{d+1}) = (1 + o(1))/2d$ and a standard second moment argument gives that

$$\mathbb{P}\left(|Y| > \frac{1}{2}(4d - 1)p)^{d^2\log n} > c\right)$$

where the term $d^2\log n$ is simply $l_n - 1 - l_{n-1}$, the height of the tree we are examining. Here and below $c$ denotes positive constants which are allowed to depend only on $d$. For $d$ sufficiently large we may replace the term $(4d - 1)p$ with $3/2$ and drop the the $1/2$ before it. We get

$$\mathbb{P}(|Y| > nd^{d^2}) > c.$$ 

Examine next the set $Z(y)$ of vertices $z$ in level $l_n$ such that $(y,0)$ is connected to $(z,0)$ by an open path that starts by moving from $(y,0)$ into an element $(0,\ldots,0)$ in one of the copies of $Z^d \times Z$ “below” it, then winds around in that copy and finally takes the last step from $(n,\ldots,n,0)$ to $(z,0)$ (this time we allow the path to use the extra dimension, i.e. it is not restricted to the slice $G \times \{0\}$). By [H08], the probability that $(0,\ldots,0) \leftrightarrow (n,\ldots,n,0)$ in $Z^d \times Z = Z^{d+1}$ is $\geq cn^{2-d}$, recall that we are examining $p_c(Z^{d+1})$. Further, all these events (for different $y$) are independent, because they examine disjoint copies of $Z^{d+1}$. With the argument of the previous paragraph we see that $(x,0)$ has probability $> c$ to have $n^{d^2}$ “children” in level $l_n - 1$ and each one has probability $> cn^{2-d}$ to have a child in level $l_n$, independently. In other words, $Z = \bigcup_{y \in Y} Z(y)$ dominates a random variable which is with probability $1 - c$ empty, and with probability $c$ a sum of $n^{d^2}$ independent Bernoulli variables with probability $cn^{2-d}$. Hence, for $n$ sufficiently large, $Z$ dominates a variable $U$ which is empty with probability $1 - c/2$ and with probability $c/2$ is $4/c$. The lemma is thus proved. 

The existence of an infinite cluster at $p$ now follows. Examine a vertex $(x,0)$ with $x$ in level $n$ for $n > n_0$. Define inductively sets of vertices $X_i$ with $X_0 = \{(x,0)\}$ and $X_i$ being all vertices $(y,0)$ with $y$ in level $l_{n+i}$ which are connected to some vertex in $X_{i-1}$ by an open path between levels $l_{n+i-1}$ and $l_{n+i}$. By lemma 1, the number of elements in $X_i$ which are connected to a given element in $X_{i-1}$, stochastically dominates the variable $U$. Further, all these connection events are independent i.e. if $x$ and $x'$ are different elements in $\bigcup X_i$, then the set of their
decendants are independent events, because the connections use different edges. Hence the process $X_t$ dominates an independent branching process with offspring distribution $U$. By lemma 1, $\mathbb{E} U > 1$ so by standard results, a branching process with distribution $U$ survives with positive probability. Hence the process $X_t$ also survives with positive probability. But if $X_t$ survives to infinity then the cluster of $(x,0)$ is infinite. So the probability that an infinite cluster exists is positive. As remarked above, this is a 0-1 event, so in fact the probability is 1.

Below we will also need that the probability has a uniform lower bound, so let us note it now: there exists some constant $c > 0$ such that

$$\mathbb{P}((x,k) \text{ is in an infinite cluster}) > c \quad \forall x \in I_n \forall n > n_0 \forall k.$$  

(1)

**Non-uniqueness.** To see that there are infinitely many clusters we apply the approach of Benjamini and Schramm [BS96, Theorem 4] by comparing to a branching random walk. We will not use usual branching random walk but a slightly different process. Let us describe it.

1. If we have a particle in some vertex $(x,k)$ for $x$ in the tree, it sends one particle to each neighbour of $(x,k)$ in $G \times \mathbb{Z}$ with probability $p$. In particular, if $x$ is in level $l_n-1$ then a particle is sent to each copy of $\mathbb{Z}^{d+1}$ “below” it, and if it is in level $l_n$ then one particle is sent to the copy of $\mathbb{Z}^{d+1}$ “above” it.

2. Now assume we have a particle in $(0,\ldots,0,k)$ in some copy of $\mathbb{Z}^{d+1}$ in the $n^{th}$ level. It sends two kinds of particles. First, one particle with probability $p$ to its tree neighbour $(x,k)$ (which is “above” it in level $l_n-1$). Second, it sends particles to all vertices $(y,1)$ with $y$ being in the same copy of $\mathbb{Z}^d$ and equal to either $(0,\ldots,0)$ or $(n,\ldots,n)$, with the distribution of descendants identical to that of vertices connected to $(0,\ldots,0,k)$ by independent percolation at $p$ in $\mathbb{Z}^{d+1}$.

3. A particle in $(n,\ldots,n,k)$ does the same, sending one particle with probability $p$ to its tree neighbour $(x,k)$ “below” it, and extra particles to $(0,\ldots,0,1)$ and $(n,\ldots,n,1)$ with the percolation distribution.

This ends the description of the process. Denote the step of particles at time $t$ by $X_t$. The proof of non-uniqueness now follows from the following two claims

**Lemma 2.** For any $x \in G \times \mathbb{Z}$, the set of vertices visited by $X_t$ at some $t$ stochastically dominates $\mathcal{C}(x) \setminus Z$ where $Z$ is the union of all copies of $\mathbb{Z}^{d+1}$.

(here and below $\mathcal{C}(x)$ denotes the cluster of $x$).

**Lemma 3.** For $d$ and $n_0$ sufficiently large ($n_0$ depending on $d$), $X_t$ is transient i.e. the expected number of returns to the starting point is finite.

**Proof of non-uniqueness given lemmas 2 and 3.** Assume by contradiction that there is only one infinite cluster. That would imply, for any $x$ and $y$ in $G$.

$$\mathbb{P}((x,0) \leftrightarrow (y,0)) \geq \mathbb{P}\left(\{|\mathcal{C}((x,0))| = \infty\} \cap \{|\mathcal{C}((y,0))| = \infty\}\right) \geq \mathbb{P}(|\mathcal{C}((x,0))| = \infty) \mathbb{P}(|\mathcal{C}((y,0))| = \infty)$$

where the second inequality follows from FKG (see [G99, §2.2]). Assuming $x$ and $y$ belong to level $l_n$ (not necessarily the same $n$ for $x$ and $y$), (1) would give

$$\mathbb{P}((x,0) \leftrightarrow (y,0)) \geq c.$$
On the other hand, our process $X$ is transient (lemma 3) and symmetric, i.e. the probability to reach $x$ from $y$ is the same as reaching $y$ from $x$. Any such process must satisfy that, when we fix the starting point of $X$,\
\[
\lim_{y \to \infty} \mathbb{P} \left( y \in \bigcup X_t \right) = 0
\] (2)\
since otherwise you will have a sequence $y_n \to \infty$ such that you can return to your the starting point with probability $> c$ after visiting $y_n$. This clearly contradicts transience.

Now apply the domination result. If we also assume $y$ is not in the copies of $\mathbb{Z}^{d+1}$ then we get\
\[
\lim_{y \to \infty} \mathbb{P}((x,0) \leftrightarrow (y,0)) = 0. (3)
\]
We have reached a contradiction, demonstrating that one cannot have a unique infinite cluster, and thus proving the theorem.  

$\square$

Proof of lemma 2. This is completely standard: one simply explores the cluster using breadth-first search and note that the “past” of the algorithm only blocks you from exploring some vertices, while the branching process has no such restriction. One has to adapt the breadth-first search to our branching process i.e. when it enters a copy of $\mathbb{Z}^{d+1}$, search all neighbours in the two lines which connect outside in one step, but other than that there is no change necessary in the standard proof (see e.g. [BS96]). $\square$

Proof of lemma 3. We will show that even the projection of $X_t$ on $G$ is transient. Since $X_t$ avoids the copies of $\mathbb{Z}^d$ (except for the points directly connected to the tree), let us consider the graph $H$ which is the tree of degree $4d$, with every edge between level $l_n - 1$ and level $l_n$ “stretched” i.e. replaced by a line with three edges and two vertices. The projection of $X_t$ to $G$ is equivalent to a process on $H$ that, from every particle, sends particles to all neighbours with probability $p$, and sometimes sends additional particles to itself and to one of its neighbours (above or below, depending on whether you are in one of the stretched levels, and where exactly you are in them). By lemma 4 below, the expected number of the additional particles that remain in place is $< \frac{1}{2}$, if only $d$ is chosen sufficiently large. As for the additional particles sent to the neighbouring vertices, these can be bounded directly from the two-point function [HHS03] i.e. from $\mathbb{P}(0 \leftrightarrow x) \leq C(d)|x|^{2d-d}$, and if only $n_0$ is sufficiently large (as a function of $d$), the expectation of these can be bounded by $1/4d$.

As a final simplification, embed $H$ in a $4d$-regular tree by “filling” the two sparse rows between $l_n - 1$ and $l_n$. Namely, level $l_n - 1$ of $H$ goes to a subset of level $l_n - 1 + 2(n - n_0)$ of the tree, the vertices on the stretched edges go to levels $l_n + 2(n - n_0)$ and $l_n + 1 + 2(n - n_0)$ of the tree, level $l_n$ of $H$ goes to level $l_n + 2 + 2(n - n_0)$ of the tree etc. The process on $H$ is now stochastically dominated by a process on the tree, which sends from each vertex of the tree to each of its neighbours a particle with probability $p$ (like the “usual” vertices of $H$), and also sends to itself and to all of its neighbors additional particles (like the vertices of $H$ on the stretched edges). In short, each vertex sends both the particles it would have sent if it were a vertex of the tree and the particles it would have sent if it were a vertex of the stretched edges.
We have reached now a very well-understood process: a branching random walk on a $4d$-regular tree, where each particle sends to each of its neighbours an expected $p + 1/4d < 1/d$ offspring, and an expected $< 1/4$ offspring remain in place. Showing that this process is transient can be done with a straightforward calculation. Fix some $t$ and examine the number of particles still at the origin at time $t$. Any such particles must have done $s$ steps on the tree (for some $s \leq t$) and stayed in place $t - s$ steps. For a fixed $t$ and $s$ the expected number of offspring is less than or equal to

$$\# \{\text{paths in the tree of length } s \text{ returning to } x\} \cdot \frac{1}{d^s} \cdot \frac{1}{2^{t-s}} = (4d)^{s/2} \cdot \frac{1}{d^s} \cdot \frac{1}{2^{t-s}}$$

and summing over all $t$ and $s$ shows that the process is transient, if only $d$ is sufficiently large (this last step requires $d > 4$, but we also rely on $p_c(Z^d)$ being sufficiently close to $1/2$ and on lemma 4, both which requires larger $d$).

**Lemma 4.** Let $d$ be sufficiently large. Then critical percolation on $\mathbb{Z}^d$ satisfies

$$\sum_{n \neq 0} \mathbb{P}(\vec{0} \leftrightarrow (0, \ldots, 0, n)) < \frac{C}{\sqrt{d}}$$

(here $\vec{0} = (0, \ldots, 0) \in \mathbb{Z}^d$ and $C$ is a constant independent of the dimension)

We assume that the correct asymptotic behaviour is $C/d$, but we do not need it in this paper. It is well known that in $d = 2$ this sum is $\infty$, for example, it follows from the estimate $\mathbb{P}(0 \leftrightarrow \partial B(n)) \geq cn^{-1/3}$, see [K87, equation (5.1)]. We will not give more details on this fact, as it will take us too far off course.

**Proof.** We follow Heydenreich, van der Hofstad and Sakai [HHS08]. Let us recall formula (1.41) ibid., in their notation:

$$\hat{G}_{z_c}(k) = \frac{1 + O(\beta)}{1 - D(k)}.$$ (4)

Let us explain the notation. The $z_c$ is $2d \cdot p_c$ [HHS08, §1.2.3] and $G$ is the connection probability

$$G_z(x) = \mathbb{P}(0 \leftrightarrow x)$$

where the probability is with respect to percolation at $z/2d$ [HHS08, §1.2.3 and equation (1.19)] so $G_{z_c}$ is the critical connection probability. $\beta = K/d$ where $K$ is some absolute constant [HHS08, first line of §1.3] and the constant implicit in the $O(\cdot)$ is also dimension independent. $D(x) = \frac{1}{d} \mathbf{1}_{\{|x| = 1\}}$ [HHS08, (1.1)]. Finally $\hat{\cdot}$ is the usual Fourier transform and $k \in [-\pi, \pi)^d$. In particular

$$\hat{D}(k) = \frac{1}{d} \sum_{l=1}^{d} \cos(k_l).$$
With (4) explained, let us calculate first the sum including the term \( P(\vec{0} \leftrightarrow \vec{0}) \),

\[
\sum_n P(\vec{0} \leftrightarrow (0, \ldots, 0, n)) = \frac{1}{(2\pi)^{d-1}} \int \hat{G}_c^z(k_1, \ldots, k_{d-1}, 0) \, dk_1 \cdots dk_{d-1}
\]

\[
\leq \frac{1}{(2\pi)^{d-1}} \int \frac{1 + C/d}{d-1 - \frac{1}{d} \sum_{i=1}^{d-1} \cos(k_i)} \, dk_1 \cdots dk_{d-1}
\]

\[
= \frac{d + C}{d-1} \cdot \frac{1}{(2\pi)^{d-1}} \int \frac{dk}{1 - \hat{D}_{d-1}(k)}.
\]

Removing the term \( P(\vec{0} \leftrightarrow \vec{0}) = 1 = \frac{1}{(2\pi)^{d-1}} \int 1 \) gives

\[
\sum_{n \neq 0} \leq d + C \cdot \frac{1}{d-1} \cdot \frac{1}{(2\pi)^{d-1}} \int \frac{\hat{D}_{d-1}(k)}{1 - \hat{D}_{d-1}(k)} \, dk + \frac{C + 1}{d-1}.
\]

To estimate the integral, apply Cauchy-Schwarz and get

\[
\int \frac{\hat{D}}{1 - D} \leq \left( \int \hat{D}^2 \right)^{1/2} \left( \int \frac{1}{(1 - D)^2} \right)^{1/2}.
\]

The first integral (with the \((2\pi)^{1-d}\) which we have omitted from the formula above) is simply the probability that simple random walk returns to zero after two steps, and hence it is simply \(1/2(d-1)\). The second integral is shown in \([HHS08, (3.4)-(3.6)]\) to be bounded independently of the dimension (\([HHS08]\) have an additional parameter in the calculation, \(s\), which in our case is 1). All in all we get

\[
\int \frac{\hat{D}_{d-1}(k)}{1 - \hat{D}_{d-1}(k)} \, dk \leq \left( \frac{1}{2(d-1)} \right)^{1/2} C^{1/2}
\]

which we plug into (5) and get

\[
\sum_{n \neq 0} P(\vec{0} \leftrightarrow (0 \ldots, 0, n)) \leq d + C \cdot \frac{C}{d-1} \leq \sqrt{\frac{d}{2d-1}} \leq \frac{C}{\sqrt{d}}
\]

as required. \( \Box \)

**Proof of Theorem 2**

Fix some vertex \( v \) and examine the number \( N \) of infinite components that intersect \( \{v\} \times \mathbb{Z} \). \( N \) is invariant to the translations of \( \mathbb{Z} \) so it is constant almost surely. Further, the standard modification argument \([BK89]\) shows that \( N \) cannot take any finite value \( > 1 \). The main step is to preclude the possibility that \( N = \infty \). An infinite cluster which intersects \( \{v\} \times \mathbb{Z} \) could intersect it either at finitely many vertices or in infinitely many vertices. We start with the first case.

**Lemma 5.** For every vertex \( v \) of \( G \), the probability that there exists an infinite cluster intersecting \( \{v\} \times \mathbb{Z} \) at finitely many vertices only is 0.

**Proof.** The idea is simple: if such clusters exist then they have some positive density. However, as you trace the cluster from \( v \) further and further in \( G \), its boundary
must increase, eventually increasing beyond \(K/(\text{the density})\), leading to a contradiction to the disconnection property of \(G\).

Let us make this more formal. Using the disconnection property of \(G\) repeatedly, one may find a sequence of \(v \in Q_1 \subset Q_2 \subset \cdots\) with \(|\partial Q_i| \leq K|\) and such that every vertex \(w\) which belong to some edge in \(\partial Q_i\) belongs to \(Q_{i+1}\). Denote by \(\partial_v\) the internal vertex boundary i.e. \(\partial_v, X\) is the set of all vertices in \(X\) with neighbours outside \(X\). Then the events

\[
E_i := \{(v, 0) \leftrightarrow \partial_v Q_i \times \mathbb{Z}\} \setminus \{(v, 0) \leftrightarrow \partial_v Q_{i+1} \times \mathbb{Z}\}
\]

are disjoint, and hence \(\sum \mathbb{P}(E_i) \leq 1\) and in particular \(\mathbb{P}(E_i) \to 0\). Fix now some \(L\) and let \(F_i\) be the event that

\[
0 < |\{n \in \mathbb{Z} : \exists x \in \partial_v Q_i \text{ s.t. } (v, 0) \leftrightarrow (x, n) \text{ in } Q_i \times \mathbb{Z}\}| \leq L
\]

(as usual, “\(a \leftrightarrow b\) in \(X\)” means that there exists an open path from \(a\) to \(b\) using only vertices in \(X\)). Then if \(F_i\) happened then there are at most \(KL\) edges through which the cluster may continue, and if they are all closed this would imply \(E_i\). Hence \(\mathbb{P}(E_i) \geq (1 - p)^{KL}\mathbb{P}(F_i)\). Hence \(\mathbb{P}(F_i) \to 0\).

Assume now that the probability that the cluster of \((v, 0)\) is infinite, but its intersection with \(\{v\} \times \mathbb{Z}\) is finite, is positive. Let \(r\) be some number such that

\[
q := \mathbb{P}\left(\|\mathcal{C}(v, 0)\| = \infty, (\mathcal{C}(v, 0) \cap \{v\} \times \mathbb{Z}\) \subset \{v\} \times [-r, r]\right) > 0.
\]

Fix \(L = 8Kr/q\) (recall that \(K\) is the constant in the disconnection property of \(G\)) and with this \(L\) define the event \(F_i\) above. Since \(\mathbb{P}(F_i) \to 0\), let \(i\) be sufficiently large such that \(\mathbb{P}(F_i) < \frac{1}{4q}\). Subtracting we get

\[
\mathbb{P}\left(\|\mathcal{C}(v, 0)\| = \infty, (\mathcal{C}(v, 0) \cap \{v\} \times \mathbb{Z}\) \subset \{v\} \times [-r, r], |\mathcal{C}(v, 0) \cap \partial_v Q_i \times \mathbb{Z}| > L\right) > \frac{3}{4q}
\]

(we used here that if \(\mathcal{C}(0)\) is infinite then it cannot be contained in \(Q_i \times \mathbb{Z}\), except with probability 0, because \(p_c\) (finite graph \(\times \mathbb{Z}\) = 1). Finally, strengthen the last requirement to \(|\mathcal{C}(0) \cap \partial_v Q_i \times [-N, N]| > L\) for some \(N\) so large so that it only decreases the probability by \(\frac{1}{4q}\). We get

\[
\mathbb{P}\left(\|\mathcal{C}(v, 0)\| = \infty, (\mathcal{C}(v, 0) \cap \{v\} \times \mathbb{Z}\) \subset \{v\} \times [-r, r], |\mathcal{C}(v, 0) \cap \partial_v Q_i \times [-N, N]| > L\right) > \frac{1}{2q}.
\]

Denote this event by \(B\) and its translation by \(n\) by \(B_n\). We finish by ergodicity of the translations by \(2r\mathbb{Z}\). Indeed, we know that

\[
|\{n \in 1, \ldots, a : B_{2rn} \text{ occurred}\}| > \frac{1}{2}aq
\]

for \(a\) sufficiently large (random). But this means that in \(\partial_v Q_i \times [2r - N, 2ar + N]\) there are \(> \frac{1}{2}aq \cdot L = 4Kar\) distinct points, since each cluster has \(> L\) points, and the clusters are disjoint since their intersections with \(\{v\} \times \mathbb{Z}\) belong to disjoint intervals. But there is no room for \(4Kar\) points, only to \(K(2ar - 2r + 1 + 2N) = 2Kar + o(a)\), for \(a\) sufficiently large, leading to a contradiction and establishing the lemma.

\[
\square
\]

**Lemma 6.** It is not possible for infinitely-many clusters to intersect \(\{v\} \times \mathbb{Z}\) at infinitely many vertices each.
Proof. The idea is as follows: we construct a 3-regular tree of trifurcation points in \( \{v\} \times \mathbb{Z} \) (as was done in [BLPS99]), and show that this contradicts the amenability of \( \mathbb{Z} \). Let us give the details.

We first show that there are trifurcation points. Following [BK89] we note that if there are infinitely-many clusters that intersect \( \{v\} \times \mathbb{Z} \), then there are trifurcation points. Indeed, for some \( r \geq 1 \) there is positive probability that three different infinite components intersect \( \{v\} \times [-r,r] \). Let \( y_1, y_2, y_3 \) be three different points in \( \{v\} \times [-r,r] \) which are connected to infinity by three simple open paths which do not intersect \( \{v\} \times [-r,r] \) again and are contained in three different components. Now modify the environment as follows: open all edges of \( \{v\} \times [-r,r] \) and close all edges of \( \{v\} \times [-r,r] \) for all edges \( e \ni v \) except the three edges which connect each \( y_i \) to the next vertex in \( \gamma_i \). It is clear that after this modification one of the \( y_i \) (the middle one) is a trifurcation point. We showed that there is positive probability that there is a trifurcation point in \( \{v\} \times [-r,r] \) and hence by translation invariance the probability that \((v,n)\) is a trifurcation point is positive for any \( n \).

We remark that, from lemma 5 we can deduce that for any trifurcation point, each of the clusters one would get by removing the trifurcation point must intersect \( \{v\} \times \mathbb{Z} \) at infinitely-many vertices.

We now follow [BLPS99, §4], which shows that under these conditions one may find a 3-regular tree of trifurcation points. Since [BLPS99] explains the argument clearly, here we will brief. The first step is

**Claim.** If \((v,n)\) is a trifurcation point then each infinite cluster left after removal of \((v,n)\) has at least one other trifurcation point \((v,m)\).

**Proof.** Define a mass transport function \( M(n,m) \) for any \( m, n \in \mathbb{Z} \) as follows: \( M(n,m) = 1 \) if \((v,n) \leftrightarrow (v,m)\) and \((v,n)\) is the unique closest trifurcation point to \((v,m)\) (the distance is the graph distance on the cluster). Let \( M = 0 \) otherwise. Then each \( m \) sends at most 1 unit of mass, and by the mass transport principle the expected amount of mass received by \( n \) should also be no more than 1 (we are using the mass transport principle on \( \mathbb{Z} \) here). But the negation of the statement of the claim means that \( n \) receives an infinite amount of mass (here is where we use the remark above that each of the three clusters remaining after removal of \((v,n)\) is not just infinite, but intersects \( \mathbb{Z} \) at infinitely-many vertices), so this must happen with probability 0. \( \square \)

As promised, we now construct an auxiliary graph \( T \) over the trifurcation points as follows: \( m \) will be connected to \( n \) (denoted by \( m \sim n \)) if both are trifurcation points and if \((v,m)\) is the closest trifurcation point to \((v,n)\) in one of the clusters remaining after removal of \((v,n)\), or vice versa. Again “closest” means in the graph distance, and we break ties by adding i.i.d. variables \( X_n \) uniform in \([0,1]\) and choosing the one with the larger \( X \). As in [BLPS99], \( T \) is a 3-regular tree.

Finally we derive a contradiction to the amenability of \( \mathbb{Z} \). Denote \( q = \mathbb{P}(0 \text{ is a trifurcation}) \). Let \( N \) be some parameter and define the event \( E(N) \) that 0 is a trifurcation point and in addition the three \( v \sim 0 \) satisfy \( v \in [-N,N] \). Taking \( N \) to be sufficiently large one may assume that \( \mathbb{P}(E(N)) > \frac{1}{2}q \). We fix \( N \) at this value (and remove it from the notation \( E(N) \), so that we just denote it by \( E \) from now). Denote \( E_n \) the translation of \( E \) by \( n \) i.e. the event that \( n \) is a trifurcation point with the condition on the neighbours as above.
Using ergodicity we know that as $r \to \infty$, the number of trifurcation points in $[-r, r)$ is $2r(q + o(1))$, while the number of $E_n$ for $n \in [-r, r)$ is $> 2r \cdot \frac{2}{4}$. Denote by $A$ the set of these trifurcation points. Since $T$ is a 3-regular tree we know that

$$\left| \{v \in T \setminus A \text{ such that } \exists w \in A, w \sim v \} \right| \geq |A| + 2.$$  

But this is clearly impossible, since any such $v$ must be in $[-r - N, r + N]$, and there are $< 2N + 2rq\left(\frac{1}{4} + o(1)\right)$ trifurcation points in this interval which are not in $A$. This is a contradiction and the lemma is proved. With lemma 5, and since $v$ was arbitrary, the theorem has been demonstrated.

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