The connection between the DRED and NSVZ Renormalisation Schemes

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We explore the relationship between the DRED and NSVZ schemes. Using certain exact results for the soft scalar mass $\beta$-function, we derive the transformation of $\alpha^{\text{NSVZ}}$ to $\alpha^{\text{DRED}}$ through terms of order $\alpha^4$. We thus incidentally determine $\beta_{\alpha}^{\text{DRED}}$ through four loops, and we compare our result to a previous Padé Approximant prediction.

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1 This paper is dedicated to the memory of Mark Samuel
In a recent series of papers we have explored the scheme–dependence associated with the renormalisation of the coupling constants and mass-parameters of a softly-broken supersymmetric theory. There are two schemes of particular interest, which we term the DRED scheme and the NSVZ scheme. The DRED scheme is defined by the procedure of minimal (or modified minimal) subtraction associated with regularisation by dimensional reduction; also known as DR (or DR). (The distinction between DR and DR is immaterial for our present purposes.) The NSVZ scheme is one such that the NSVZ formula [1] for the gauge \( \beta \)-function \( \beta_\alpha \) holds; we will define this scheme in more detail later. The NSVZ formula relates \( \beta_\alpha \) to the the anomalous dimension matrix \( \gamma \) of the chiral superfields as follows:

\[
\beta^\text{NSVZ}_\alpha = 2 \frac{\alpha^2}{16\pi^2} \left[ \frac{Q - 2r^{-1}\text{tr}[\gamma C(R)]}{1 - 2\alpha C(G)(16\pi^2)^{-1}} \right],
\]

where \( \alpha = g^2 \) and \( Q = T(R) - 3C(G) \). \( C(R) \) and \( C(G) \) are the quadratic matter and adjoint Casimirs respectively; \( T(R) = r^{-1}d_R\text{tr}[C(R)] \), where \( d_R \) is the dimension of the matter representation, and \( r \) is the number of generators of the gauge group.

The NSVZ scheme is related to the “holomorphic” scheme (wherein the one–loop \( \beta_\alpha \) is exact) by the transformation

\[
\frac{1}{\alpha^{\text{H}}} = \frac{1}{\alpha^{\text{NSVZ}}} + 2 \frac{C(G) \ln \alpha^{\text{NSVZ}}}{16\pi^2} - \frac{4}{16\pi^2}r^{-1}\text{tr}[ZC(R)],
\]

where \( \mu \frac{dZ}{d\mu} = \gamma \).

No relation of the form of Eq. (1) exists in DRED; on the other hand, DRED is a well-defined procedure for the calculation of radiative corrections. Thus if we wish to perform a calculation which involves

1. Running couplings and masses from \( 10^{16}\text{GeV} \) (say) to \( M_Z \) and then
2. Calculating radiative corrections to physical masses and processes,

we might well consider using NSVZ (or the holomorphic scheme) in the former procedure and DRED in the latter. In fact, the NSVZ scheme has been used for running the dimensionless couplings in Refs. [2], [3] (see also Ref. [4]). For this reason it is useful to know as precisely as possible the connection between the schemes.

In Refs. [5], [6] we constructed perturbatively a redefinition \( \alpha^{\text{NSVZ}} \to \alpha^{\text{DRED}} \) by comparing \( \beta_\alpha \) in the two schemes. The result was

\[
\alpha^{\text{DRED}} = \alpha^{\text{NSVZ}} + \sum_{L=1}^{\infty} \delta^{(L)}(\alpha^{\text{NSVZ}}, Y, Y^*),
\]
where $\delta^{(1)} = 0,$

$$(16\pi^2)^2 \delta^{(2)} = \alpha^2 \left[ r^{-1} \text{tr} [PC(R)] - \alpha QC(G) \right],$$

and

$$\delta^{(3)} = \rho_1 \Delta_1 + \rho_2 \Delta_2 + \rho_3 \Delta_3.$$  

Here

$$(16\pi^2)^3 \Delta_1 = \alpha^3 C(G) \left[ r^{-1} \text{tr} [PC(R)] - \alpha QC(G) \right]$$  

$$(16\pi^2)^3 \Delta_2 = r^{-1} \text{tr} \left[ \alpha^2 S_4 C(R) - 2\alpha^4 QC(R)^2 + 2\alpha^3 PC(R)^2 \right]$$  

$$(16\pi^2)^3 \Delta_3 = \alpha^2 r^{-1} \text{tr} [P^2 C(R)] - \alpha^4 Q^2 C(G),$$

and in Ref. [6] we showed that $\rho_2 = -\frac{4}{3}$ and $\rho_3 = \frac{1}{3}$. $P^i_j$ and $S^i_{4j}$ are defined as follows:

$$P^i_j = \frac{1}{2} Y^{ijkl} Y^j_{kl} - 2\alpha C(R)^i_j,$$

$$S^i_{4j} = Y^{imn} P^p_m Y^j_{pn},$$

where we have written the superpotential as

$$W(\Phi) = \frac{1}{6} Y^{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{2} \mu^{ij} \Phi_i \Phi_j.$$  

As usual we raise and lower indices by complex conjugation, e.g. $Y_{ijk} = (Y^{ijk})^*$. In principle the undetermined coefficient $\rho_1$ could be found by the same method as employed in Ref. [6] to find $\rho_2$ and $\rho_3$; that is, by calculating a relevant contribution to $\beta^{(4)}_{i\lambda DRED}$. This would be very tedious, however [7]. In this paper we show that our recent work on the soft supersymmetry-breaking $\beta$-functions leads to a determination of $\rho_1$ based on a remarkably simple three-loop calculation.

We take the soft breaking Lagrangian $L_{SB}$ as follows:

$$L_{SB}(\phi, \lambda) = \left[ \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} M \lambda^2 + \text{h.c.} \right] - (m^2)^i j \phi^i \phi^j.$$  

Here $M$ is the gaugino mass, and $\phi^i = \phi^*_i$. In Ref. [8] we showed that the soft scalar mass $\beta$-function is given by the following expression:

$$(\beta_{m^2})^i j = \left[ \Delta + \tilde{X}(\alpha, Y, Y^*, h, h^*, m, M) \frac{\partial}{\partial \alpha} \right] \gamma^i j.$$  

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1 In Ref. [7] a method based on Padé approximants was used to suggest that $\rho_1 \approx 4.9$
where
\[ \Delta = 2\mathcal{O} + 2MM^* \frac{\partial}{\partial \alpha} + \tilde{Y}_{lmn} \frac{\partial}{\partial Y_{lmn}} + \tilde{Y}^{lmn} \frac{\partial}{\partial Y^{lmn}} \],
(11)
\[ \mathcal{O} = \left( M\alpha \frac{\partial}{\partial \alpha} - h^{lmn} \frac{\partial}{\partial Y^{lmn}} \right) , \]
(12)
and
\[ \tilde{Y}^{ijk} = (m^2)^i Y^{ijk} + (m^2)^j Y^{ikl} + (m^2)^k Y^{ijl}. \]
(13)
The function \( \tilde{X} \) was introduced in Ref. [8]; it does not appear in a naive application of the spurion formalism [9], because (when using DRED) this fails to allow for the fact that the \( \epsilon \)-scalars associated with DRED acquire a mass through radiative corrections [10]. (For further discussion, see Refs. [11], [12].) Indeed, in DRED, \( \beta_{m^2} \) will actually depend on the \( \epsilon \)-scalar mass. It is, however, possible to define a scheme, DRED', related to DRED, such that \( \beta_{m^2} \) is independent of the \( \epsilon \)-scalar mass [13].

In Ref. [14] we claimed that Eq. (10) holds in both DRED' and the NSVZ scheme, the two schemes being related by the transformation Eq. (3) and an associated transformation on the gaugino mass \( M \), given by
\[ \alpha M = \alpha' M' \frac{\partial \alpha(\alpha', Y, Y^*)}{\partial \alpha'} - h^{ijk} \frac{\partial \alpha(\alpha', Y, Y^*)}{\partial Y^{ijk}}. \]
(14)
(These two transformations define precisely what we mean by the NSVZ scheme.) We also argued that in the NSVZ scheme we have simply
\[ \tilde{X}^{NSVZ} = -4 \frac{\alpha^2}{16\pi^2} \frac{S}{[1 - 2\alpha C(G)(16\pi^2)^{-1}]} \]
(15)
where
\[ S = r^{-1} \text{tr}[m^2 C(R)] - MM^* C(G), \]
(16)
whereas in the DRED' scheme, \( \tilde{X} \) is related to the \( \beta \)-function for the \( \epsilon \)-scalar mass, \( \tilde{m} \). Writing
\[ \beta_{\tilde{m}^2} = N_1 + N_2 \tilde{m}^2, \]
(17)
where \( N_1(\alpha, Y, Y^*, h, h^*, m, M) \) does not depend on \( \tilde{m} \), we have
\[ \tilde{X}^{DRED'} = - \sum_{L=1}^{\infty} \frac{\alpha}{L N_1^{(L)}} \]
(18)
where $N_1^{(L)}$ is the $L$-loop contribution to $N_1$. The distinction between DRED and DRED' has no influence on the calculation of $N_1$; the DRED → DRED' redefinition only changes $N_2$.

Clearly, given Eqs. (18) and (15), we can determine the relation between $\alpha^{\text{DRED}}$ and $\alpha^{\text{NSVZ}}$ by calculating $N_1$ if we know how $\tilde{X}$ transforms under a scheme redefinition. We showed in Ref. [14] that under a transformation $\alpha \to \alpha'$, with an associated transformation of $M$ given by Eq. (14), the transformed $\tilde{X}'$ is related to $\tilde{X}$ by

$$
\tilde{X}' = \tilde{X}' \frac{\partial \alpha}{\partial \alpha'} + 2M'M'^* \left[ \alpha'^2 \frac{\partial^2 \alpha}{\partial \alpha'^2} + 2\alpha' \frac{\partial \alpha}{\partial \alpha'} - 2 \frac{\alpha'^2}{\alpha} \left( \frac{\partial \alpha}{\partial \alpha'} \right)^2 \right] - \left[ 2M'\alpha' \frac{\partial^2 \alpha}{\partial Y_{ijk} \partial \alpha'} - \frac{\partial \alpha}{\partial Y_{ijk}} \frac{\partial \alpha}{\partial \alpha'} \right] - \tilde{Y}_{ijk} \frac{\partial \alpha}{\partial Y_{ijk}} + \text{c.c.} \right) - \tilde{Y}_{ijk} \frac{\partial \alpha}{\partial Y_{ijk}} + \text{c.c.} \right)
$$

and we also showed that Eq. (19) is consistent with Eq. (4). Moreover, we calculated the contributions of all tensor structures of the general form $\alpha m^2 Y^2 Y^\ast 2C(R)$ to $N_1$, which enabled us to test Eq. (19) against Eq. (4); this calculation did not involve tensor structures associated with $\rho_1$. Our confidence bolstered by this, we now proceed to determine $\rho_1$ by calculating the $N_1$ contributions from some tensor structures that are sensitive to $\rho_1$. This involves a three-loop calculation in the broken theory; far simpler than the four-loop calculation (albeit in the unbroken theory) required to determine $\rho_1$ from $\beta^{\text{DRED}}_\alpha$.

If we take the primed scheme in Eq. (19) to be NSVZ and the unprimed scheme to be DRED', then using Eqs. (3)-(5) we find that in DRED'
where

\[ W^j_i = \frac{1}{2} \left[ Y^{jkl} \tilde{Y}_{ikl} + \tilde{Y}^{jkl} Y_{ikl} \right] + h_{ipq} h^{jipq} - 8\alpha M M^* C(R)^j_i, \]  \hspace{1cm} (21)

and

\[ H^i_j = h^{ikl} Y_{jkl} + 4\alpha M C(R)^i_j. \]  \hspace{1cm} (22)

(The combination \( H^i_j \) was formerly known \cite{10} as \( X^i_j \).) We have not substituted for \( \rho_2 \) and \( \rho_3 \) in Eq. (20) above so that the contributions emanating from the first term on the RHS of Eq. (19) are easier to identify. As explained above, in Ref. \cite{14} we explicitly calculated the three-loop Feynman diagrams corresponding to contributions to \( \beta_{\tilde{m}^2} \) of the form \( \alpha m^2 Y^2 Y^* C(R) \). Here we report on analogous calculations involving the following tensor structures:

\[ T_1 = \alpha^2 h^* M Y C(R) C(G) + \text{c.c}, \]
\[ T_2 = \alpha^2 h h^* C(R) C(G), \]
\[ T_3 = \alpha^2 M M^* Y Y^* C(R) C(G). \]  \hspace{1cm} (23)

These all have coefficients that depend on \( \rho_1 \).

The Feynman graphs giving \( T_1 \)-type contributions to \( N_1 \) are shown in Fig. 1.

\[ T_1 = \alpha^2 h^* M Y C(R) C(G) + \text{c.c} \]

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The calculation is quite straightforward, especially when one realises that Fig. 1(a) does not contribute, because the two possible places for the \( M \)-insertion give opposite signs and cancel. Thus we are reduced to evaluating the simple pole in \( \epsilon = 4 - d \) from a single graph, and we obtain:

\[ 2 - \rho_1 = -\frac{10}{3} \]  \hspace{1cm} (24)
whence
\[ \rho_1 = \frac{16}{3}. \] (25)

As a check, we have also calculated the \( T_2 \) and \( T_3 \) contributions. Here we have more Feynman diagrams, including ones with vector fields; care must be taken with subtractions. In both cases we also obtain Eq. (25).

The determination of \( \rho_1 \) completes the NSVZ/DRED connection through terms of \( O(\alpha^4) \). In terms of physics at \( M_Z \) this facilitates a very accurate transformation between the two schemes in the MSSM. It also completes the determination of \( \beta_\alpha^{(4)\text{DRED}} \); see Eq. (3.18) of Ref. [6] (note that \( \rho_1 = 2\alpha_1 \)). For the special case of SQCD we have the following results:

\[ \beta_\alpha^{\text{DRED}} = 2\alpha \sum_{n=1}^{\infty} \beta_n \left( \frac{\alpha}{16\pi^2} \right)^n, \] (26)

where

\[ \begin{align*}
\beta_1 &= N_f - 3N_c, \quad \text{(27a)} \\
\beta_2 &= \left[ 4N_c - \frac{2}{N_c} \right] N_f - 6N_c^2, \quad \text{(27b)} \\
\beta_3 &= \left[ \frac{3}{N_c} - 4N_c \right] N_f^2 + \left[ 21N_c^2 - \frac{2}{N_c} - 9 \right] N_f - 21N_c^3, \quad \text{(27c)} \\
\beta_4 &= A + BN_f + CN_f^2 + DN_f^3. \quad \text{(27d)}
\end{align*} \]

Here \( N_c \) is the number of colours, and

\[ \begin{align*}
A &= -102N_c^4, \\
B &= 132N_c^3 - 66N_c - \frac{8}{N_c} - \frac{4}{N_c^2}, \\
C &= -[42 + 12\zeta(3)]N_c^2 + 44 + \frac{36\zeta(3) - 20}{3N_c^2}, \quad \text{(28)} \\
D &= -\frac{2}{3N_c}.
\end{align*} \]

In the case \( N_f = 0 \) it is interesting to compare the above DRED results with the exact NSVZ formula,

\[ \beta_\alpha^{\text{NSVZ}} = \frac{-6N_c\alpha^2}{16\pi^2 \left[ 1 - 2\alpha N_c(16\pi^2)^{-1} \right]}. \] (29)

In both cases the \( \beta \)-function coefficients have the same sign through four loops. In the NSVZ case, the series manifestly has a finite radius of convergence; it is not clear whether or not this is true in the DRED case.
Our result for $\rho_1$ represents what we at least regard as striking confirmation of the
Asymptotic Padé Approximant prediction $\rho_1/2 = 12/5$ or $5/2$. The error of $6.25 - 10\%$
is remarkably small and provides further evidence for the precocious convergence of APAPs
when applied to calculations of $\beta$–functions in non-abelian theories.

Finally we have confirmation of our exact result for $\tilde{\xi}^{\text{NSVZ}}$, Eq. (15). We will explore
the effect of this on the running analysis within the NSVZ scheme elsewhere.

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3 With hindsight (always reliable) we might have arrived at the correct result for $\rho_1$ from the
cue that $\rho_2$ and $\rho_3$ are both fractions with 3 in the denominator
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