Positivity of Kac polynomials and DT-invariants for quivers

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Abstract

We give a cohomological interpretation of both the Kac polynomial and the refined Donaldson-Thomas-invariants of quivers. This interpretation yields a proof of a conjecture of Kac from 1982 and gives a new perspective on recent work of Kontsevich–Soibelman. This is achieved by computing, via an arithmetic Fourier transform, the dimensions of the isotypical components of the cohomology of associated Nakajima quiver varieties under the action of a Weyl group. The generating function of the corresponding Poincaré polynomials is an extension of Hua’s formula for Kac polynomials of quivers involving Hall-Littlewood symmetric functions. The resulting formulae contain a wide range of information on the geometry of the quiver varieties.

1 The main results

Let $\Gamma = (I, \Omega)$ be a quiver: that is, an oriented graph on a finite set $I = \{1, \ldots, r\}$ with $\Omega$ a finite multiset of oriented edges. In his study of the representation theory of quivers, Kac [17] introduced $A_v(q)$, the number of isomorphism classes of absolutely indecomposable representations of $\Gamma$ over the finite field $\mathbb{F}_q$ of dimension $v = (v_1, \ldots, v_r)$ and showed they are polynomials in $q$. We call $A_v(q)$ the Kac polynomial for $\Gamma$ and $v$. Following ideas of Kac [17], Hua [16] proved the following generating function identity:

$$
\sum_{v \in \mathbb{Z}^r_+\setminus\{0\}} A_v(q) T^v = (q - 1) \log \left( \sum_{v = (v_1, \ldots, v_r) \in \mathcal{P}} \prod_{i < j \in \Omega} q^{(v_i, v_j)} \prod_{i \in I} q^{(v_i, v_i)} \prod_k \prod_{j=1}^m (1 - q^{-j}) T^{[v]} \right),
$$

(1.1)

where $\mathcal{P}$ denotes the set of partitions of all positive integers, Log is the plethystic logarithm (see [14, §2.3.3]), $\langle , \rangle$ is the pairing on partitions defined by

$$
\langle \lambda, \mu \rangle = \sum_{i,j} \min(i,j) m_i(\lambda)m_j(\mu)
$$
with $m_j(\lambda)$ the multiplicity of the part $j$ in the partition $\lambda$, $T^\gamma := T_1^{r_1} \cdots T_r^{r_r}$ for some variables $T_i$ and finally $|\pi| := (|\pi_1| \cdots |\pi_s|)$

Using such generating functions Kac \cite{17} proved that in fact $A_\epsilon(q)$ has integer coefficients and formulated two main conjectures. First, he conjectured that for quivers with no loops the constant term $A_\epsilon(0)$ equals the multiplicity of the root $v$ in the corresponding Kac-Moody algebra. The proof of this conjecture was completed in \cite{13}. We will give a general proof of Kac’s second conjecture here:

**Conjecture 1.1** ([\cite{17} Conjecture 2]). The Kac polynomial $A_\epsilon(q)$ has non-negative coefficients.

This conjecture was settled for indivisible dimension vectors and any quiver by Crawley-Boevey and Van den Bergh \cite{7} in 2004; they gave a cohomological interpretation of the Kac polynomial for indivisible dimension vectors in terms of the cohomology of an associated Nakajima quiver variety. More precisely \cite{7} (1.1), they showed that for $v$ indivisible

$$A_\epsilon(q) = \sum \dim \left( H^2_{\text{c}}(Q_{\epsilon}; \mathbb{C}) \right) q^{-d_v}, \quad (1.2)$$

where $Q_{\epsilon}$ is a certain smooth generic complex quiver variety of dimension $2d_v$. Similarly, our proof of the general case will follow by interpreting the coefficients of $A_\epsilon(q)$ as the dimensions of the sign isotypical component of cohomology groups of a smooth generic quiver variety $Q_{\epsilon}$ attached to an extended quiver (see (1.9)).

Recently, Mozgovoy \cite{26} proved Conjecture 1.1 for any dimension vector for quivers with at least one loop at each vertex. His approach uses Efimov’s proof \cite{10} of a conjecture of Kontsevich–Soibelman \cite{20} which implies positivity for certain refined DT-invariants of symmetric quivers with no potential.

The goal of Kontsevich–Soibelman’s theory is to attach refined (or motivic, or quantum) Donaldson–Thomas invariants (or DT-invariants for short) to Calabi–Yau 3-folds $X$. The invariants should only depend on the derived category of coherent sheaves on $X$ and some extra data; this raises the possibility of defining DT-invariants for certain Calabi-Yau 3-categories which share the formal properties of the geometric situation, but are algebraically easier to study. The simplest of such examples are the Calabi-Yau 3-categories attached to quivers (symmetric or not) with no potential (c.f. \cite{12}).

Denote by $\overline{\Gamma} = (I, \overline{\Omega})$ the double quiver, that is $\overline{\Omega} = \Omega \coprod \Omega^{\text{opp}}$, where $\Omega^{\text{opp}}$ is obtained by reversing all edges in $\Omega$. The refined DT-invariants of $\overline{\Gamma}$ (a slight renormalization of those introduced by Kontsevich and Soibelman \cite{19}) are defined by the following combinatorial construction. For $v = (v_1, \ldots, v_r) \in \mathbb{Z}_{\geq 0}^r$ let

$$\delta(v) := \sum_{i=1}^r v_i, \quad \gamma(v) := \sum_{i=1}^r v_i^2 - \sum_{i \neq j} v_i v_j.$$ 

Then

$$\sum_{v \in \mathbb{Z}_{\geq 0}^r \backslash \{0\}} \text{DT}_v(q) (-1)^{\delta(v)} T^v := (q - 1) \log \sum_{v \in \mathbb{Z}_{\geq 0}^r} (-1)^{\delta(v)} q^{-\frac{1}{2}(\gamma(v) + \delta(v))} \prod_{i=1}^r \left( 1 - q^{-v_i} \right) T^v \quad (1.3)$$

It was proved in \cite{20} that $\text{DT}_v(q) \in \mathbb{Z}[q, q^{-1}]$. In fact, as a consequence of Efimov’s proof \cite{10} of \cite{20} Conjecture 1, $\text{DT}_v(q)$ actually has non-negative coefficients. We will give an alternative proof of this in (1.10) by interpreting its coefficients as dimensions of cohomology groups of an associated quiver variety.

**Remark 1.2.** We should stress that we have restricted to double quivers for the benefit of exposition; our results extend easily to any symmetric quiver. We outline how to treat the general case in \S 3.2.
The technical starting point in this paper is a common generalization of (1.3) and Hua’s formula (1.1). Namely we consider

$$H(x_1, \ldots, x_r; q) := (q-1) Log \left( \sum_{\pi=(\pi', \ldots, \pi')} \frac{\prod_{i=j \in \Omega} q^{\langle \pi, \pi' \rangle}}{\prod_{i=j \in I} q^{\langle \pi'', \pi' \rangle} \prod_{k} \prod_{j=1}^{s} (1 - q^{-j})} \prod_{i=1}^{r} \tilde{H}_{\pi'}(x_i; q) \right),$$

(1.4)

where $x_i = (x_{i,1}, x_{i,2}, \ldots)$ is a set of infinitely many independent variables, and $\tilde{H}_{\pi'}(x_i; q)$ denote the (transformed) Hall-Littlewood polynomial (see [14, §2.3.2]), which is a symmetric function in $x_i$ and polynomial in $q$. From (1.4) we can extract many rational functions in $q$ just by pairing against other symmetric functions. For a multi-partition $\mu = (\mu^1, \ldots, \mu^r) \in \mathcal{P}^r$ we let $s_\mu := s_{\mu^1}(x_1) \cdots s_{\mu^r}(x_r)$, where $s_{\mu^i}(x_i)$ is the Schur symmetric function attached to the partition $\mu^i$. Define

$$H^r_\mu(q) := \left( H(x_1, \ldots, x_r; q), s_\mu \right),$$

(1.5)

where $\langle , \rangle$ is the natural extension to $r$ variables of the Hall pairing on symmetric functions, defined by declaring the basis $s_\mu$ orthonormal (see [14, (2.3.1)]). Note that a priori the $H^r_\mu$ are rational functions in $q$; we will prove that they are actually polynomials with non-negative integer coefficients. More precisely, we will show (Theorem 1.4(i)) that the coefficients of $H^r_\mu(q)$ are the dimensions of certain cohomology groups.

Our first formal observation is the following

**Proposition 1.3.** For any $v = (v_1, \ldots, v_r) \in \mathbb{Z}_{\geq 0}$ we have

(i)

$$A_v(q) = H^r_{v^1}(q),$$

(1.6)

where $v^1 := ((v_1), \ldots, (v_r)) \in \mathcal{P}^r$ and

(ii)

$$DT_v(q) = H^r_{1^r}(q),$$

(1.7)

where $1^r := ((1^{v_1}), \ldots, (1^{v_r})) \in \mathcal{P}^r$.

We will prove Part (i) as a consequence of Theorem 2.12 since $h_{v^1} = s_{v^1}$ and part (ii) is a special case of Proposition 3.5.

Fix a non-zero multi-partition $\mu \in \mathcal{P}^r$ and let $v = |\mu| := (|\mu^1|, \ldots, |\mu^r|)$. Associated to the pair $(\Gamma, v)$ we construct a new quiver by attaching a leg of length $v_i - 1$ at the vertex $i$ of $\Gamma$ where $v_i := |\mu^i|$. We denote it by $\tilde{\Gamma}_v = (\tilde{\Gamma}_v, \tilde{\Omega}_v)$. We extend the dimension vector $v : I \to \mathbb{Z}_{\geq 0}$ to $\tilde{v} : \tilde{\Gamma}_v \to \mathbb{Z}_{\geq 0}$ by placing decreasing dimensions $v_i - 1, v_i - 2, \ldots, 1$ at the extra leg starting with $v_i$ at the original vertex $i$. We also consider the subgroup $W_v < W$ of the Weyl group of the quiver generated by the reflections at the extra vertices $\tilde{I} \setminus I$. We may identify $W_v$ with $S_{v_1} \times \cdots \times S_{v_r}$, the Weyl group of the group $GL_v := GL_{v_1} \times \cdots \times GL_{v_r}$.

Because by construction $\tilde{v}$ is indiscernible we can define the corresponding smooth generic complex quiver variety $Q_v$. Note that $\tilde{v}$ is left invariant by $W_v$ and thus $W_v$ acts on $H^r_{\tilde{\Gamma}_v}(Q_v; \mathbb{C})$ by work of Nakajima [28, 29], Lusztig [23], Maffei [25] and Crawley-Boevey–Holland [5]. We denote by $\chi^\mu = \chi^{\mu^1} \cdots \chi^{\mu^r} : W_v \to \mathbb{C}^\times$ the product of the irreducible characters $\chi^{\mu^i}$ of the symmetric groups $S_{v_i}$ in the notation of [24, §1.7]. In particular, $\chi^{(v_i)}$ is the trivial character and $\chi^{(1^{v_i})}$ is the sign character $\epsilon_i : S_{v_i} \to \{ \pm 1 \} < \mathbb{C}^\times$. If $\mu' := ((\mu^1)', \ldots, (\mu^r)')$, where $(\mu^i)'$ is the dual partition of $\mu^i$, then $\chi^{\mu'} = \epsilon \chi^\mu$ with $\epsilon := \epsilon_1 \cdots \epsilon_r$ the sign character of $W_v$. 
We may decompose the representation of \( W_v \) on \( H^i_c(Q_v; \mathbb{C}) \) into its isotypical components

\[
H^i_c(Q_v; \mathbb{C}) \cong \bigoplus_{\mu \in \mathcal{P}_v} H^i_c(Q_v; \mathbb{C})_{\lambda^\mu},
\]

where \( \mathcal{P}_v \) denotes the set of multi-partitions \( \mu = (\mu^1, \ldots, \mu^r) \) of size \( v = (v_1, \ldots, v_r) \).

More generally, for a multi-partition \( \mu = (\mu^1, \ldots, \mu^r) \in \mathcal{P}_v \), with \( \mu^i = (\mu^i_1, \mu^i_2, \ldots, \mu^i_{l_i}) \) and \( l_i \) the length of \( \mu^i \), denote by \( \Gamma_{\mu} \) the quiver obtained from \((\Gamma, \mu)\) by adding at each vertex \( i \in I \) a leg with \( l_i - 1 \) edges. We denote by \( v_\mu \) the dimension vector of \( \Gamma_{\mu} \) with coordinates \( v_i, v_i - \mu^i_1, v_i - \mu^i_1 - \mu^i_2, \ldots, \mu^i_{l_i} \) at the \( i \)-th leg. Define

\[
d_\mu := 1 - \frac{1}{2} v_\mu C_\mu v_\mu
\]

with \( C_\mu \) the Cartan matrix of \( \Gamma_{\mu} \). Notice that if \( \mu = (1^r) \) then \( \tilde{\Gamma}_v = \Gamma_1 \), \( v_\mu = \tilde{v} \); we will write \( d_v \) instead of \( d_{1^r} \). The quiver variety \( Q_v \) is non-empty if and only if \( \tilde{v} \) is a root of \( \Gamma_1 \) in which case it has dimension \( 2d_v \) \([3]\) Theorem 1.2).

Our main geometric result is the following

**Theorem 1.4.** We have

(i) \[
\mathbb{H}_{\mu}^i(q) = \sum_i \dim \left( H^2_{\mu}(Q_v; \mathbb{C})_{\lambda^\mu} \right) q^{i-d_\mu}.
\]

(ii) \( \mathbb{H}_{\mu}^i(q) \) is non-zero if and only if \( v_\mu \) is a root of \( \Gamma_{\mu} \), in which case it is a monic polynomial of degree \( d_\mu \). Moreover, \( \mathbb{H}_{\mu}^i(q) = 1 \) if and only if \( v_\mu \) is a real root.

**Remark 1.5.** 1. Another way to phrase Theorem 1.4(i) is as follows. Let \( V_{v,i} \) be the representation \( H^1_c(Q_v; \mathbb{C}) \) of \( W_v \) tensored with the sign representation and consider the graded representation \( V := \bigoplus_{v,i} V_{v,i} \). Then the Frobenius-Hilbert series of \( V \) is \( \mathbb{H} \). In other words,

\[
\sum_{v,i} \text{ch}(V_{v,i}) q^{-d_v} = \mathbb{H}(x_1, \ldots, x_r; q),
\]

where \( \text{ch} \) is the Frobenius characteristic map (naturally extended to the product of symmetric groups \( W_v \)).

2. Theorem 1.4(ii) provides a criterion (in terms of root systems) for the appearance of an irreducible character of \( W_v \) in \( \bigoplus_i V_{v,i} \).

In combination with Proposition 1.3 Theorem 1.4 implies the following.

**Corollary 1.6.** We have

(i) \[
A_v(q) = \sum_i \dim \left( H^2(\mu^\mu_1(Q_v; \mathbb{C})_{\lambda^\mu} \right) q^{i-d_\mu},
\]

and

\[
\text{DT}_v(q) = \sum_i \dim \left( H^2(Q_v; \mathbb{C})_{\lambda^\mu} \right) q^{i-d_\mu},
\]

(ii) In particular, \( A_v(q) \) and \( \text{DT}_v(q) \) have non-negative integer coefficients.

(iii) Conjecture 1.1 holds for any quiver and dimension vector \( v \).

(iv) The polynomial \( \text{DT}_v(q) \) is non-zero if and only if \( \tilde{v} \) is a root of \( \tilde{\Gamma}_v \), in which case it is monic of degree \( d_\mu \). Moreover, \( \text{DT}_v(q) = 1 \) if and only if \( \tilde{v} \) is a real root.
Remark 1.7. 1. In the case when the dimension vector $v$ is indivisible, we could have proved Theorem 1.4 (i) using the ideas of [22] based on the theory of perverse sheaves. For more details see Remark 2.13.

2. As mentioned above, the fact that $\text{DT}_v(q)$ has non-negative coefficients also follows from [20, Conjecture 1] proved by Efimov [10].

3. Mozgovoy in [27] shows that $A_v(q)$ can also be interpreted as the refined DT-invariant of an associated quiver $\tilde{\Gamma}$ with a non-trivial potential. It raises the question of what is the meaning of our more general polynomials $H^\mu_s(q)$ in refined DT-theory.

4. For a star-shaped quiver the series (1.4) is shown in [14] to be the pure part (the specialization $(q, t) \mapsto (0, q)$) of a series in two variables $q, t$. We expect that there is an analogous $t$-deformation of (1.4) for any quiver. Under this deformation the Hall-Littlewood polynomials would be replaced by the Macdonald polynomials; the challenge is to $t$-deform the rest of the terms. Conjecturally, the geometrical meaning of such a $(q, t)$ formula should involve the mixed Hodge polynomials of multiplicative quiver varieties of Crawley-Boevey and Shaw [6]. We also expect that the natural Weyl group action on the cohomology of such multiplicative quiver varieties extends, at least in some special cases, to representations of rational Cherednik algebras.

5. Finally, we mention that in some cases such $(q, t)$-formulas were found recently by Chuang-Diaconescu-Pan [2] to conjecturally describe refined BPS invariants and, in particular, refined DT-invariants of local curve Calabi-Yau 3-fold geometries.

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2 Proof of Theorem 1.4

2.1 The quiver varieties $Q_v$

Let $\Gamma, I, \Omega$ be as in the introduction. Let $\mathbb{K}$ be any algebraically closed field. For a dimension vector $v = (v_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$ put

$$\mathbb{K}^v := \bigoplus_{i \in I} \mathbb{K}^{v_i}, \quad \text{GL}_v := \prod_{i \in I} \text{GL}_{v_i}(\mathbb{K}), \quad \mathfrak{gl}_v := \bigoplus_{i \in I} \mathfrak{gl}_{v_i}(\mathbb{K}).$$

By a graded subspace of $V \subseteq \mathbb{K}^v$ we will mean a subspace of the form

$$V = \bigoplus_{i \in I} V_i, \quad V_i \subseteq \mathbb{K}^{v_i}.$$

The group $\text{GL}_v$ acts on $\mathfrak{gl}_v$ by conjugation. For an element $X = (X_i)_{i \in I} \in \mathfrak{gl}_v$ we put $\text{Tr}(X) := \sum_{i \in I} \text{Tr}(X_i)$. We denote by $T_v$ the maximal torus of $\text{GL}_v$ whose elements are of the form $(g_i)_{i \in I}$ with $g_i$ a diagonal matrix for each $i \in I$. The Weyl group $W_v := N_{\text{GL}_v}(T_v)/T_v$ of $\text{GL}_v$ with respect to $T_v$ is isomorphic to $\prod_{i \in I} S_{v_i}$ where
\( S_v \) denotes the symmetric group in \( v \) letters. Recall that a semisimple element \( X \in \mathfrak{gl}_v \) is \emph{regular} if \( C_{GL_v}(X) \) is a maximal torus of \( GL_v \), i.e., the eigenvalues of the coordinates of \( X \) are all with multiplicity 1.

We say that an adjoint orbit \( O \) of \( \mathfrak{gl}_v \) is \emph{generic} if \( \text{Tr}(X) = 0 \) and if for any graded subspace \( V \subseteq \mathbb{K}^v \) stable by some \( X \in \mathcal{O} \) such that

\[
\text{Tr}(X|_V) = 0
\]

then either \( V = 0 \) or \( V = \mathbb{K}^v \). We fix such a generic regular semisimple adjoint orbit \( O \subset \mathfrak{gl}_v \) (we can prove as in [14] §2.2] that such a choice is always possible).

Let \( \Gamma \) be the \emph{double quiver} of \( \Gamma \); namely, \( \Gamma \) has the same vertices as \( \Gamma \) but for each arrow \( \gamma \in \Omega \) going from \( i \) to \( j \) we add a new arrow \( \gamma^* \) going from \( j \) to \( i \). We denote by \( \Omega^* = \{ \gamma, \gamma^* \mid \gamma \in \Omega \} \) the set of arrows of \( \Gamma \).

Consider the space

\[
\text{Rep}_\mathbb{K}(\Gamma, \mathbf{v}) := \bigoplus_{i \rightarrow j \in \Omega} \text{Mat}_{v_i, v_j}(\mathbb{K})
\]

of representations of \( \Gamma \) with dimension \( \mathbf{v} \). Recall that \( GL_v \) acts on \( \text{Rep}_\mathbb{K}(\Gamma, \mathbf{v}) \) as

\[
(g \cdot \varphi)_{i \rightarrow j} = g_j \varphi_{i \rightarrow j} g_i^{-1}
\]

for any \( g = (g_i)_{i \in I} \in GL_v, \varphi = (\varphi_{\gamma})_{\gamma \in \Omega} \in \text{Rep}_\mathbb{K}(\Gamma, \mathbf{v}) \) and any arrow \( i \rightarrow j \in \Omega \).

Let \( \tilde{I}_v \) on vertex set \( \tilde{I}_v \) be the quiver obtained from \( (\Gamma, \varphi) \) by adding at each vertex \( i \in I \) a leg of length \( v_i - 1 \) with the edges all oriented toward the vertex \( i \). Define \( \tilde{v} \in \mathbb{Z}_{\geq 0}^{|I|} \) as the dimension vector with coordinate \( v_i \) at \( i \in I \subset \tilde{I}_v \) and with coordinates \( (v_i - 1, v_i - 2, \ldots, 1) \) on the leg attached to the vertex \( i \in I \).

Let \( Q_v \) be the quiver variety over \( \mathbb{K} \) attached to the quiver \( \tilde{\Gamma} \) and parameter set defined from the eigenvalues of \( O \) (see [14] and the reference therein).

Concretely, define the moment map

\[
\mu_v : \text{Rep}_\mathbb{K}(\tilde{\Gamma}, \mathbf{v}) \rightarrow \mathfrak{gl}_v^0 \tag{2.1}
\]

\[
(x_{\gamma})_{\gamma \in \Omega^*} \mapsto \sum_{\gamma \in \Omega} [x_\gamma, x_{\gamma^*}], \tag{2.2}
\]

where

\[
\mathfrak{gl}_v^0 := \{ X \in \mathfrak{gl}_v \mid \text{Tr}(X) = 0 \}.
\]

Then \( Q_v \) is the affine GIT quotient

\[
\mu_v^{-1}(O)//GL_v := \text{Spec} \left( \mathbb{K}[\mu_v^{-1}(O)]^{GL_v} \right). \tag{2.3}
\]

Note that the one-dimensional torus \( \mathbb{G}_m \) embeds naturally in \( GL_v \) as \( t \mapsto (t \cdot I_v)_{i \in I} \) where \( I_v \) is the identity matrix of \( GL_v \). The action of \( GL_v \) on \( \mu_v^{-1}(O) \) factorizes through an action of \( G_v := GL_v/\mathbb{G}_m \).

We have the following theorem whose proof is similar to that of [14] Theorem 2.2.4].

**Theorem 2.1.** (i) The variety \( Q_v \) is non-singular and the quotient map \( \mu_v^{-1}(O) \rightarrow Q_v \) is a principal \( G_v \)-bundle in the \( \text{étale topology} \).

(ii) The odd degree cohomology of \( Q_v \) vanishes.
2.2 Weyl group action

2.2.1 Weyl group action

Let $K$ denote an arbitrary algebraically closed field as before. Let $\ell$ be a prime different from the characteristic char $K$ of $K$, and for an algebraic variety over $K$ denote by $H^i_c(X; \overline{Q}_\ell)$ the compactly supported $\ell$-adic cohomology.

Denote by $t^\text{gen}_v$ the generic regular semisimple elements of the Lie algebra $t_v$ of $T_v$. For $\sigma \in t^\text{gen}_v$ define

$$M_\sigma := \{ (\varphi, X, gT_v) \in \text{Rep}_{K}(\Gamma, v) \times gl_v \times (GL_v/T_v) \mid g^{-1}Xg = \sigma, \mu_v(\varphi) = X \} \sslash \text{GL}_v$$

where $\text{GL}_v$ acts by

$$g \cdot (\varphi, X, hT_v) = (g \cdot \varphi, gXg^{-1}, ghT_v).$$

The following lemma is immediate.

Lemma 2.2. The projection $(\varphi, X, gT_v) \to \varphi$ induces an isomorphism from $M_\sigma$ onto the quiver variety associated to the adjoint orbit of $\sigma$ as in (2.3).

For $w \in W_v$, denote by $w : M_\sigma \to M_{\sigma w^{-1}}$ the isomorphism $(\varphi, X, gT_v) \mapsto (\varphi, X, gw^{-1}T_v)$. The first aim of this section is to prove the following theorem.

Theorem 2.3. There exists a prime $p_0$ such that if char $K \geq p_0$ or if $K = \mathbb{C}$, then for any $\sigma, \tau \in t^\text{gen}_v$ there exists a canonical isomorphism $i_{\sigma, \tau} : H^i_c(M_\sigma; \overline{Q}_\ell) \to H^i_c(M_\tau; \overline{Q}_\ell)$ such that the following diagram commutes

$$
\begin{array}{ccc}
H^i_c(M_\sigma; \overline{Q}_\ell) & \to & H^i_c(M_{\sigma w^{-1}}; \overline{Q}_\ell) \\
i_{\sigma, w} \downarrow & & \downarrow \text{id}_{H^i_c(M_{\sigma w^{-1}}; \overline{Q}_\ell)} \\
H^i_c(M_{\sigma w^{-1}}; \overline{Q}_\ell) & \to & H^i_c(M_{\sigma w^{-1} \tau w^{-1}}; \overline{Q}_\ell)
\end{array}
$$

Moreover for all $\sigma, \tau, \xi \in t^\text{gen}_v$ we have $i_{\sigma, \tau} \circ i_{\xi, \sigma} = i_{\xi, \tau}$.

Before writing the proof let us explain the rough strategy. Put

$$M := \{ (\varphi, X, gT_v, \sigma) \in \text{Rep}_{K}(\Gamma, v) \times gl_v \times (GL_v/T_v) \times t^\text{gen}_v \mid g^{-1}Xg = \sigma, \mu_v(\varphi) = X \} \sslash \text{GL}_v$$

where $\text{GL}_v$ acts on the first three coordinates as before and trivially on the last one. Denote by $\pi : M \to t^\text{gen}_v$ the projection to the last coordinate. Note that the stalk at $\sigma$ of the sheaf $R^i\pi_!\overline{Q}_\ell$ is $H^i_c(\pi^{-1}(\sigma); \overline{Q}_\ell) = H^i_c(M_\sigma; \overline{Q}_\ell)$. Since $\pi$ commutes with Weyl group actions, to prove Theorem 2.3 we need to prove that the sheaf $R^i\pi_!\overline{Q}_\ell$ is constant. Unfortunately we do not know any algebraic proof of this last statement so we do not have a proof which works independently from char $K$. We follow the same strategy as in [7, Proof of Proposition 2.3.1] proving first the statement with $K = \mathbb{C}$ using the hyperkähler structure on quiver varieties and then proving the positive characteristic case by reducing modulo $p$.

We prefer to work with étale $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaves instead of $\ell$-adic sheaves. We will show that the sheaves $R^i\pi_!\mathbb{Z}/\ell^n\mathbb{Z}$ are constant for all $n \geq 1$ assuming that char $K$ is either sufficiently large or equal to 0. This will prove Theorem 2.3 for the étale cohomology $H^i_c(M_\sigma; \mathbb{Z}/\ell^n\mathbb{Z})$ with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$. We then pass to the direct limit to get the statement for $\ell$-adic cohomology

$$H^i_c(M_\sigma; \overline{Q}_\ell) = \left( \lim_{\longrightarrow} H^i_c(M_\sigma; \mathbb{Z}/\ell^n\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \overline{Q}_\ell.$$
Proof of Theorem\textsuperscript{2.3} (i) Assume that $\mathbb{K} = \mathbb{C}$. From the equivalence of categories between constructible étale sheaves on a complex variety and constructible sheaves on the underlying topological space (see \textsuperscript{25}, §6 for more details) we are reduced to prove that if $\mathbb{Z}/\ell^n\mathbb{Z}$ is the constant sheaf (for the analytic topology) on the complex variety $\mathcal{M}$, then the analytic sheaf $R^i\pi_!\mathbb{Z}/\ell^n\mathbb{Z}$ is constant on $t^\text{gen}_v$. But this is exactly what is proved in \textsuperscript{[25], Lemma 48} where the hyperkähler structure on quiver varieties is used as an essential ingredient.

We note that in \textsuperscript{[25]} it is assumed that the quiver does not have loops. The proof of \textsuperscript{[25], Lemma 48} however remains valid in the general case. The only non-trivial ingredient is the proof of the surjectivity \textsuperscript{[25], CON3 §5} of the hyperkähler moment map over the regular locus in the case for quivers with possible loops. More recent result \textsuperscript{[4], Theorem 2} implies that as long as the dimension vector is a root the complex moment map is surjective for any quiver. By hyperkähler rotation we get that the corresponding hyperkähler moment map is also surjective.

(ii) The $\mathbb{C}$-schemes $\mathcal{M}, t^\text{gen}_v$ and the projection $\pi : \mathcal{M} \to t^\text{gen}_v$ to the last coordinate can actually be defined over $\mathbb{Z}$ (see \textsuperscript{[7], Appendix B}). We will denote these by $\mathcal{M}/\mathbb{Z}, t^\text{gen}_v/\mathbb{Z}, \pi/\mathbb{Z} : \mathcal{M}/\mathbb{Z} \to t^\text{gen}_v/\mathbb{Z}$ and denote by $\mathcal{F} = \mathcal{F}_\pi$ the sheaf $R^i\pi_!\mathbb{Z}/\ell^n\mathbb{Z}$. Recall that if $f : X \to \text{Spec}\mathbb{Z}$ denotes the structure map of a $\mathbb{Z}$-scheme, then by Deligne \textsuperscript{[8], Theorem 1.9}, for any constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf $\mathcal{E}$ on $X$, there exists an open dense subset $U$ of $\text{Spec}\mathbb{Z}$ such that for any base change $S \to U \subset \text{Spec}\mathbb{Z}$ we have $(f_*\mathcal{E})_S \simeq (f_\sharp)_*(\mathcal{E}_S)$. Denote by $t/\mathbb{Z}$ the structure map of $t^\text{gen}_v/\mathbb{Z}$. We are thus reduced to prove that the canonical map

$$\eta : (t/\mathbb{Z})^*(t/\mathbb{Z}), \mathcal{F} \to \mathcal{F}$$

given by adjointness is an isomorphism over an open subset $U$ of $\text{Spec}\mathbb{Z}$. Indeed, this will prove that for any prime $p$ such that $p\mathbb{Z} \subset U$, the map $\eta/\mathbb{Z}_p$ obtained from $\eta$ by base change is an isomorphism which is equivalent to say that the sheaf $\mathcal{F}_\eta \simeq R^i(\pi/\eta)_!\mathbb{Z}/\ell^n\mathbb{Z}$ is constant. By (i) we know that the sheaf $\mathcal{F}_\pi$ is constant, i.e., the isomorphism $\eta_C : (t/\mathbb{C})^*(t/\mathbb{C}), \mathcal{F}_\pi \to \mathcal{F}_C$ of étale sheaves obtained from $\eta$ by base change is an isomorphism. Hence if $\mathcal{K}$ and $C$ denote respectively the kernel and co-kernel of $\eta$, then $\mathcal{K}_C = 0$ and $C_C = 0$. Since by Deligne \textsuperscript{[8], Theorem 1.9} the sheaf $(t/\mathbb{Z})^*(t/\mathbb{Z}), \mathcal{F}$ is constructible over an open subset $V$ of $\text{Spec}\mathbb{Z}$, the sheaves $\mathcal{K}$ and $C$ are also constructible over $V$ and so the support of $\mathcal{K}_V$ and $C_V$ are constructible sets. Since $\mathcal{K}_C = 0$ and $C_C = 0$, the supports do not contain the generic point and so there exists an open subset $U$ of $V$ such that $\mathcal{K}_U = C_U = 0$. \hfill $\square$

Assume that char $\mathbb{K}$ is as in Theorem\textsuperscript{2.3}. For $w \in W_v$ and $\tau \in t^\text{gen}_v$ define

$$\rho^j(w) : H^j_c(\mathcal{M}_\tau; \overline{\mathbb{Q}}_l) \to H^j_c(\mathcal{M}_\tau; \overline{\mathbb{Q}}_l)$$

as the composition $i_{w\tau w^{-1}, \tau} \circ (w^{-1})^*$. The following proposition is a straightforward consequence of Theorem\textsuperscript{2.3}

\begin{proposition}
The map

$$\rho^j = \rho^j_{\mathbb{K}} : W_v \to \text{GL}
\left(H^j_c(\mathcal{M}_\tau; \overline{\mathbb{Q}}_l)\right)
\quad w \mapsto \rho^j(w)$$

is a representation of $W_v$ which, does not depend on the choice of $\tau \in t^\text{gen}_v$.
\end{proposition}

\begin{theorem}
Assume that char $\mathbb{K} \gg 0$. Then the $\overline{\mathbb{Q}}_l$-representations $\rho^j_{\mathbb{K}}$ and $\rho^j_{\mathbb{C}}$ are isomorphic.
\end{theorem}

\begin{proof}
Let $U$ be the open subset of $\text{Spec}\mathbb{Z}$ as in the proof of Theorem\textsuperscript{2.3} over which $\eta$ is an isomorphism and $(t/\mathbb{Z}), \mathcal{F}$ is constructible. Namely, if we denote by $\mathcal{E}$ the restriction of $(t/\mathbb{Z}), \mathcal{F}$ to $U$, then $\mathcal{E}$ is constructible and $\mathcal{F}_U$ is canonically isomorphic to $(t/\mathbb{Z})^*\mathcal{E}$. Since $\mathcal{E}$ is constructible, it is locally constant over an open
subset of $U$ and so we may assume (taking an open subset of $U$ if necessary) that there is an étale covering $s : U' \to U$ over $U$ with $U'$ irreducible such that the sheaf $s^*\mathcal{E}$ is constant. Now let $p$ be a prime such that $p\mathbb{Z} \in U$ and assume that $\text{char} \mathbb{K} = p$. Consider the geometric points $\overline{x}_0 : \text{Spec} \mathbb{C} \to U$ and $\overline{x}_p : \text{Spec} \mathbb{K} \to U$ induced by the maps $\mathbb{Z} \subset \mathbb{C}$ and $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{K}$, and let $\overline{x}_0$ and $\overline{x}_p$ be geometric points of $U'$ lying over $x_0$ and $x_p$ respectively. Since the sheaf $s^*\mathcal{E}$ is constant, we have a canonical isomorphism $(s^*\mathcal{E})\overline{x}_0 \simeq (s^*\mathcal{E})\overline{x}_p$ and so we get an isomorphism $h : \mathcal{E}_{x_0} \simeq \mathcal{E}_{x_p}$. Since any two geometric points $\overline{x}_0 : \text{Spec} \mathbb{C} \to \mathbb{C}^\text{gen}/\mathbb{C} \to \mathbb{C}^\text{gen}/\mathbb{Z}$ and $\overline{x}_p : \text{Spec} \mathbb{K} \to \mathbb{K}^\text{gen}/\mathbb{C} \to \mathbb{K}^\text{gen}/\mathbb{Z}$ lie over $x_0$ and $x_p$ respectively we have $\mathcal{F}_{x_0} \simeq \mathcal{E}_{x_0} \simeq \mathcal{E}_{x_p} \simeq \mathcal{F}_{x_p}$. Identifying $\tau_0$ and $\tau_p$ with their respective images in $\mathbb{C}^\text{gen}/\mathbb{C}$ and $\mathbb{K}^\text{gen}/\mathbb{C}$, we deduce an isomorphism

$$\sigma_{\tau_0,\tau_p} : H^i_c(M_{x_0} ; \overline{\mathbb{Q}}_\ell) = \mathcal{F}_{x_0} \simeq \mathcal{F}_{x_p} = H^i_c(M_{x_p} ; \overline{\mathbb{Q}}_\ell).$$

If $\tau'_0$ is another geometric point $\text{Spec} \mathbb{C} \to \mathbb{C}^\text{gen}/\mathbb{C} \to \mathbb{C}^\text{gen}/\mathbb{Z}$, then the isomorphism $i_{\tau_0,\tau'_0} : \mathcal{F}_{x_0} \simeq \mathcal{F}_{x_0}$ of Theorem 2.3 is the composition of the canonical isomorphisms $\mathcal{F}_{x_0} \simeq \mathcal{E}_{x_0} \simeq \mathcal{F}_{x'_0}$, and same thing with geometric points over $\text{Spec} \mathbb{K}$. The following commutative diagram summarizes what we just said

$$
\begin{array}{ccc}
H^i_c(M_{x_0} ; \overline{\mathbb{Q}}_\ell) & \xrightarrow{\sigma_{\tau_0,\tau_p}} & H^i_c(M_{x_p} ; \overline{\mathbb{Q}}_\ell) \\
\downarrow{i_{\tau_0,\tau'_0}} & & \downarrow{i_{\tau_p,\tau'_p}} \\
H^i_c(M_{x'_0} ; \overline{\mathbb{Q}}_\ell) & \xrightarrow{\sigma_{\tau'_0,\tau'_p}} & H^i_c(M_{x'_p} ; \overline{\mathbb{Q}}_\ell) \\
\end{array}
$$

where the maps are all isomorphisms. Now the fact that $\sigma_{\tau_0,\tau_p}$ commutes also with the isomorphisms

$$w_C : H^i_c(M_{x_0} ; \overline{\mathbb{Q}}_\ell) \to H^i_c(M_{x^{-1}x_0} ; \overline{\mathbb{Q}}_\ell)$$

and

$$w_K : H^i_c(M_{x_0} ; \overline{\mathbb{Q}}_\ell) \to H^i_c(M_{x^{-1}x_0} ; \overline{\mathbb{Q}}_\ell)$$

follows from the fact that the action of $W_v$ is defined over $\mathbb{Z}$. This proves that the $\overline{\mathbb{Q}}_\ell$-representations $\rho^i_C$ and $\rho^i_K$ are isomorphic. \qed

Note that the representation $\rho^i_C$ of $W_v$ on $H^i_c(M_{\ell} ; \overline{\mathbb{Q}}_\ell)$ is defined so that it agrees via the comparison theorem with the action $\rho^i$ of $W_v$ on the compactly supported cohomology $H^i_c(M_{\ell} ; \mathbb{C})$ as defined from [25, Lemma 48].

2.2.2 Introducing Frobenius

Here $\mathbb{K}$ is an algebraic closure of a finite field $\mathbb{F}_q$. We use the same letter $F$ to denote the Frobenius endomorphism on $\text{Rep}_\mathbb{K}(\Gamma, \mathfrak{v})$ and $\mathfrak{gl}_v$ that raises entries of matrices to their $q$-th power. Let us first recall how we define a map from the set of $F$-stable regular semisimple orbits of $\mathfrak{gl}_v$ onto the set of conjugacy classes of the Weyl group $W_v$ of $\text{GL}_v$.

Take any regular semisimple $X \in \mathfrak{gl}_v(\mathbb{F}_q) = (\mathfrak{gl}_v)^F$. By definition the centralizer $C_{\text{GL}_v}(X)$ is an $F$-stable maximal torus $T_X$ of $\text{GL}_v$. Since maximal tori are all $\text{GL}_v$-conjugate to each other, we may write $T_X = gT_vg^{-1}$ for some $g \in \text{GL}_v$. Applying $F$ to the identity and using the fact that both $T_v$ and $T_X$ are $F$-stable we find that $g^{-1}F(g) \in N_{\text{GL}_v}(T_v)$. Taking the image of $g^{-1}F(g)$ in $W_v$ gives a well-defined map from the set of
regular semisimple elements of $\mathfrak{gl}_v(\mathbb{F}_q)$ to $W_v$ and so a map from $F$-stable regular semisimple orbits of $\mathfrak{gl}_v$ to conjugacy classes of $W_v$ (as $F$-stable orbits always contain a representative in $\mathfrak{gl}_v(\mathbb{F}_q)$).

Given $w \in W_v$ choose a generic regular semisimple adjoint orbit $O$ of $\mathfrak{gl}_v(\mathbb{F}_q)$ mapping to $w$ and denote by $Q_w^v$ the corresponding quiver variety associated to $O$ defined by (2.3). If $w = 1$ then we will denote these simply by $Q_w$ instead of $Q_w^v$; note that in this case the orbit $O$ has all its eigenvalues in $\mathbb{F}_q$. Since $O$ is $F$-stable the quiver variety $Q_w^v$ inherits an action of the Frobenius endomorphism which we also denote by $F$.

By Lemma 2.2 and Proposition 2.4 we have a well-defined representation $\rho^\ell$ of $W_v$ in $H^i_c(Q_w, \overline{Q}_\ell)$ assuming that the characteristic of $\mathbb{K}$ is large enough.

The aim of this section is to prove the following theorem.

**Theorem 2.6.** Assume that $\text{char } \mathbb{K} \gg 0$. We have

$$
\#Q_w^v(\mathbb{F}_q) = \sum_i \text{Tr}
\left(\rho^{2i}(w), H^c_c(Q_w; \overline{Q}_\ell)\right)q^i.
$$

We keep the notation introduced in §2.2.1.

The Frobenius morphism $(\varphi, X, gT_v) \mapsto (F(\varphi), F(X), F(g)T_v)$ defines a bijective morphism $F : \mathcal{M}_\sigma \to \mathcal{M}_{F(\sigma)}$. Since the map $\pi : M \to \mathfrak{t}_v^{\text{gen}}$ commutes with $F$, the following diagram commutes

$$
\begin{array}{ccc}
H^c_c(\mathcal{M}_\sigma; \overline{Q}_\ell) & \xrightarrow{F} & H^c_c(\mathcal{M}_{F(\sigma)}; \overline{Q}_\ell) \\
\downarrow{i_{\sigma,\tau}} & & \downarrow{i_{F(\sigma), F(\tau)}} \\
H^c_c(\mathcal{M}_\tau; \overline{Q}_\ell) & \xrightarrow{F} & H^c_c(\mathcal{M}_{F(\sigma)}; \overline{Q}_\ell)
\end{array}
$$

for all $\sigma, \tau \in \mathfrak{t}_v^{\text{gen}}$, where $i_{\sigma, \tau}$ is as in Theorem 2.3.

For $w \in W_v$ consider the $w$-twisted Frobenius endomorphism

$$
wF : \begin{array}{ccc}
\mathfrak{gl}_v & \to & \mathfrak{gl}_v \\
X & \mapsto & wF(X)\hat{w}^{-1}
\end{array}
$$

where $\hat{w}$ is a representative of $w$ in $N_{\text{GL}_v}(T_v)$.

Let $\sigma \in (\mathfrak{t}_v^{\text{gen}})^{wF}$. Since $wF(\sigma) = \sigma$ we get a Frobenius endomorphism

$$
wF : \begin{array}{ccc}
\mathcal{M}_\sigma & \to & \mathcal{M}_\sigma \\
(\varphi, X, gT_v) & \mapsto & (F(\varphi), F(X), F(g)\hat{w}^{-1}T_v).
\end{array}
$$

Let $\tau \in (\mathfrak{t}_v^{\text{gen}})^F$. By Theorem 2.3 the following diagram commutes

$$
\begin{array}{ccc}
H^c_c(\mathcal{M}_\sigma; \overline{Q}_\ell) & \xrightarrow{\rho^\ell(w)} & H^c_c(\mathcal{M}_\tau; \overline{Q}_\ell) \\
\downarrow{i_{\sigma,\tau}} & & \downarrow{i_{F(\sigma), F(\tau)}} \\
H^c_c(\mathcal{M}_\tau; \overline{Q}_\ell) & \xrightarrow{w^*} & H^c_c(\mathcal{M}_{F(\sigma)}; \overline{Q}_\ell) \\
\downarrow{F_{\sigma,\tau}} & & \downarrow{F_{F(\sigma), F(\tau)}} \\
H^c_c(\mathcal{M}_{F(\sigma)}; \overline{Q}_\ell) & \xrightarrow{F^{-1}} & H^c_c(\mathcal{M}_{F(\sigma)}; \overline{Q}_\ell).
\end{array}
$$

Note that the arrow labelled by $w^*$ is well-defined as $F(\sigma) = \hat{w}^{-1}\sigma\hat{w}$.

Applying the Grothendieck trace formula to $wF : \mathcal{M}_\tau \to \mathcal{M}_\sigma$ we find that

$$
\#\mathcal{M}_\sigma(\mathbb{K})^{wF} = \sum_i (-1)^i \text{Tr}
\left((wF)^*H^c_c(\mathcal{M}_\tau; \overline{Q}_\ell)\right) = \sum_i (-1)^i \text{Tr}
\left(F^* \circ \rho^\ell(w), H^c_c(\mathcal{M}_\tau; \overline{Q}_\ell)\right) = \sum_i \text{Tr}
\left(F^* \circ \rho^\ell(w), H^c_c(\mathcal{M}_\tau; \overline{Q}_\ell)\right)
$$

as desired.
Theorem 2.7. The automorphism $F^*$ on $H^2(M_v; \overline{Q}_q)$ has a unique eigenvalue $q^i$.

It is not difficult to verify that the two automorphisms $\rho^i(w)$ and $F^*$ commute for all $i$ and $w \in W_v$. We have

$$\text{Tr} \left( F^* \circ \rho^i(w), H^2(M_v; \overline{Q}_q) \right) = \text{Tr} \left( \rho^i(w), H^2(M_v; \overline{Q}_q) \right) q^i,$$

from which we deduce that

$$\#M_v(\mathbb{K})^{wF} = \sum_i \text{Tr} \left( \rho^i(w), H^2(M_v; \overline{Q}_q) \right) q^i.$$  \hspace{1cm} (2.5)

Hence Theorem 2.6 follows from (2.5) and the following lemma.

Lemma 2.8. Let $w \in W_v$ and let $\sigma$ be a representative of the orbit $O^w$ in $V = (\mathbb{K})^{wF}$, then

$$\#M_v(\mathbb{K})^{wF} = \#Q_v^w(\mathbb{K})^{F}.$$  \hspace{1cm}

Proof. It follows from the fact that the isomorphism $M_v \rightarrow Q^w_v$, $(\varphi, X, gT_v) \mapsto \varphi$ of Lemma 2.2 commutes with $wF$ and $F$.  \hfill \Box

2.3 Counting points of $Q_v^w$ over finite fields

In this section we will evaluate $\#Q_v^w(\mathbb{F}_q)$. As in §1 we label the vertices of $\Gamma$ by $1, \ldots, r$ and we denote by $P_v$ the set of all multi-partitions $(\mu^1, \ldots, \mu^r)$ of size $(v_1, \ldots, v_r)$. The conjugacy class of an element $w = (w_1, \ldots, w_r) \in W_v$ determines a multi-partition $\lambda = (\lambda^1, \ldots, \lambda^r) \in P_v$, where $\lambda^i$ is the cycle type of $w_i \in S_v$. We will call $\lambda$ the cycle type of $w$.

Let

$$p_{\lambda} := p_{\lambda^1}(x_1) \cdots p_{\lambda^r}(x_r),$$

where for a partition $\lambda$, $p_{\lambda}(x_i)$ is the corresponding power symmetric function in the variables of $x_i = \{x_{i,1}, x_{i,2}, \ldots\}$ (see [24] Chapter I, §2). Recall that $\varepsilon$ denotes the sign character of $W_v$. Denote by $\check{C}_v$ the Cartan matrix of the quiver $\check{\Gamma}_v$, then

$$d_{\check{\epsilon}} := 1 - \frac{1}{2} \text{dim} Q_\check{\epsilon} \text{ if } Q_\check{\epsilon} \text{ is non-empty.}$$

The aim of this section is to prove the following theorem.

Theorem 2.9. Let $w \in W_v$ have cycle type $\lambda \in P_v$. Then

$$\#Q_v^w(\mathbb{F}_q) = q^{d_{\check{\epsilon}}} \varepsilon(w) \langle \mathcal{H}(x_1, \ldots, x_r; q), p_{\lambda^1} \rangle.$$  \hspace{1cm} (2.6)
Fix a non-trivial additive character \( \Psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times \) and for \( X, Y \in \mathfrak{gl}_n \) put \( \langle X, Y \rangle := \text{Tr}(XY) \). Denote by \( C(\mathfrak{gl}_n) \) the \( \mathbb{C} \)-vector space of all functions \( \mathfrak{gl}_n^F \rightarrow \mathbb{C} \) and define the Fourier transform \( \mathcal{F} : C(\mathfrak{gl}_n) \rightarrow C(\mathfrak{gl}_n) \) by
\[
\mathcal{F}(f)(X) := \sum_{Y \in \mathfrak{gl}_n^F} \Psi(\langle X, Y \rangle) f(Y),
\]
with \( f \in C(\mathfrak{gl}_n) \) and \( X \in \mathfrak{gl}_n^F \). Basic properties of \( \mathcal{F} \) can be found for instance in [21]. For an \( F \)-stable adjoint orbit \( O \) of \( \mathfrak{gl}_n \) we denote by \( 1_O \in C(\mathfrak{gl}_n) \) the characteristic function of the adjoint orbit \( O^F \) of \( \mathfrak{gl}_n \), i.e., \( 1_O(X) = 1 \) if \( X \in O^F \) and \( 1_O(X) = 0 \) if \( X \notin O^F \).

**Proposition 2.10.** For any \( F \)-stable generic adjoint orbit \( O \) of \( \mathfrak{gl}_n^0 \) we have
\[
\#(\mu^{-1}(O)/\text{GL}_n^\mathbb{C})^F = \frac{q - 1}{|\text{GL}_n^\mathbb{C}|} \#\mu^{-1}(O)^F = \frac{(q - 1)|\text{Rep}_{\mathbb{C}}(\Gamma, \mathfrak{v})|}{|\text{GL}_n^F| \cdot |\mathfrak{gl}_n^F|} \sum_{X \in \mathfrak{gl}_n^F} \# \left\{ \varphi \in \text{Rep}_{\mathbb{C}}(\Gamma, \mathfrak{v}) \mid [X, \varphi] = 0 \right\} \mathcal{F}(1_O)(X)
\]
where \( [X, \varphi] = 0 \) means that for each arrow \( \gamma = i \rightarrow j \) in \( \Omega \) we have \( X_i \varphi = \varphi X_i \).

**Proof.** The first equality comes from the fact that \( G_v = \text{GL}_n^\mathbb{C}/\mathbb{G}_m \) is connected and acts freely on \( \mu^{-1}(O) \) (see Theorem 2.1(i)). For the second write
\[
\#\mu^{-1}(O)^F = \sum_{z \in O^F} \#\mu^{-1}(z)^F
\]
and use [13] Proposition 2.

In order to compute the right hand side of Formula (2.7) we need to introduce some notation. Denote by \( \mathcal{O} \) the set of all \( F \)-orbits of \( \mathbb{K} \). The adjoint orbits of \( \mathfrak{gl}_n^F \) are parametrized by the maps \( h : \mathcal{O} \rightarrow \mathcal{P} \) such that
\[
\sum_{\gamma \in \mathcal{O}} |\gamma| \cdot |h(\gamma)| = n.
\]
Denote by \( 0 \in \mathcal{P} \) the unique partition of 0. The type of an adjoint orbit \( O \) of \( \mathfrak{gl}_n^F \) corresponding to \( h : \mathcal{O} \rightarrow \mathcal{P} \) is defined as the map \( \omega_O \) that assigns to a positive integer \( d \) and a non-zero partition \( \lambda \) the number of Frobenius orbits \( \gamma \in \mathcal{O} \) of degree \( d \) such that \( h(\gamma) = \lambda \).

It is sometimes also convenient (see [14]) to write a type as follows. Choose a total ordering \( \preceq \) on partitions which we extend to a total ordering on the set \( \mathbb{Z}_{\geq 0} \times (\mathcal{P} \setminus \{0\}) \) as \( (d, \lambda) \preceq (d', \lambda') \) if \( d \geq d' \) and \( \lambda \geq \lambda' \). Then we may write the type \( \omega_O \) as a the strictly decreasing sequence \( (d_1, \lambda_1^n)(d_2, \lambda_2^n) \cdots (d_s, \lambda_s^n) \) with \( n_i = \omega_O(d_i, \lambda_i) \). The set of all non-increasing sequences \( (d_1, \lambda_1)(d_2, \lambda_2) \cdots (d_s, \lambda_s) \) of size \( n \) (i.e., \( \sum_{i=1}^s d_i |\lambda_i| = n \)) denoted by \( \mathbb{T}_n \) parametrizes the types of the adjoint obits of \( \mathfrak{gl}_n^F \).

It is easy to extend this to adjoint orbits of \( \mathfrak{gl}_n^F \). They are parametrized by the set of all maps \( h = (h_1, \ldots, h_r) : \mathcal{O} \rightarrow \mathcal{P}^r \) such that for each \( i = 1, \ldots, r \), we have
\[
\sum_{\gamma \in \mathcal{O}} |\gamma| \cdot |h_i(\gamma)| = v_i.
\]
A type of an adjoint orbit \( O \) of \( \mathfrak{gl}_n^F \) corresponding to \( h : \mathcal{O} \rightarrow \mathcal{P}^r \) is now a map \( \omega_O \) that assigns to a positive integer \( d \) and a non-zero multi-partition \( \lambda = (\lambda_1^{d_1}, \ldots, \lambda_r^{d_r}) \) the number of Frobenius orbits \( \gamma \in \mathcal{O} \) of degree \( d \) such that \( h(\gamma) = \lambda \).
As above, after choosing a total ordering on the set $\mathbb{Z}_{>0} \times (\mathcal{P}^r \setminus \{0\})$ we may write $\omega_\mathcal{O}$ as a (strictly) decreasing sequence $(d_1, \lambda_1)^{\nu_1} (d_2, \lambda_2)^{\nu_2} \cdots (d_s, \lambda_s)^{\nu_s}$ with $n_i = \omega_\mathcal{O}(d_i, \lambda_i)$. We denote by $\mathbb{T}_\mathcal{V}$ the set of all non-increasing sequences $(d_1, \lambda_1) \cdots (d_s, \lambda_s)$ of size $\mathbf{v}$ so that $\mathbb{T}_\mathcal{V}$ parametrizes the types of the adjoint orbits of $\mathfrak{gl}_\mathcal{V}^F$. We may also write a type $\omega \in \mathbb{T}_\mathcal{V}$ as $(\omega_1, \ldots, \omega_r)$, where $\omega_i = (d_i, \lambda_i^1)(d_i, \lambda_i^2) \cdots$, with $\lambda_i^j$ the $i$-th coordinate of $\lambda_j$, is a type in $\mathbb{T}_\mathcal{V}$.

Given any family $\{A_\mu(x_1, \ldots, x_r ; q)\}_{\mu \in \mathcal{P}^r}$ of functions separately symmetric in each set $x_1, \ldots, x_k$ of infinitely many variables with $A_0 = 1$, we extend its definition to types $\omega = (d_1, \lambda_1) \cdots (d_s, \lambda_s) \in \mathbb{T}_\mathcal{V}$ as

$$A_\omega(x_1, \ldots, x_r ; q) := \prod_{i=1}^s A_{d_i}(x_i^{d_1}, \ldots, x_i^{d_s}; q^{\mathbf{v}}),$$

where $x_i^{d}$ stands for all the variables $x_1, x_2, \ldots \in x$ replaced by $x_1^{d_1}, x_2^{d_2}, \ldots$.

For $\pi = (\pi_1, \ldots, \pi^r) \in \mathcal{P}^r$ put

$$\mathcal{A}_\pi(q) = \prod_{i=1}^{\infty} q^{(\pi_i, \pi^r)}, \quad \mathcal{Z}_\pi(q) := \prod_{i=1}^{\infty} q^{(\pi_i, \pi^r)} \prod_{i=1}^{\infty} \prod_{j=1}^{n_i} (1 - q^{-j}) \quad \mathcal{H}_\pi(q) := \frac{\mathcal{A}_\pi(q)}{\mathcal{Z}_\pi(q)},$$

where we use the same notation as in §A Then by [16, Theorem 3.4], for any element $X$ in an adjoint orbit of $\mathfrak{gl}_\mathcal{V}^F$ of type $\omega \in \mathbb{T}_\mathcal{V}$ we have

$$\frac{1}{|\text{GL}_{\mathcal{V}}^F|} \sum_{\mathbf{v} \in \mathcal{V}} \# \left\{ \varphi \in \text{Rep}_{\mathcal{V}}(\Gamma, \mathbf{v}) \mid [X, \varphi] = 0 \right\} = \mathcal{A}_\omega(q), \quad |C_{\text{GL}_{\mathcal{V}}^F}(X)| = \mathcal{Z}_\omega(q).$$

Hence

$$\frac{1}{|\text{GL}_{\mathcal{V}}^F|} \sum_{\mathbf{v} \in \mathcal{V}} \# \left\{ \varphi \in \text{Rep}_{\mathcal{V}}(\Gamma, \mathbf{v}) \mid [X, \varphi] = 0 \right\} \mathcal{F}(1_\mathcal{O})(X) = \sum_{\omega \in \mathbb{T}_\mathcal{V}} \mathcal{H}_\omega(q) \sum_{\mathcal{O}'} \mathcal{F}(1_\mathcal{O})(\mathcal{O}'),$$

where the last sum is over the adjoint orbits $\mathcal{O}'$ of $\mathfrak{gl}_\mathcal{V}^F$ of type $\omega$ and $\mathcal{F}(1_\mathcal{O})(\mathcal{O}')$ denotes the common value $\mathcal{F}(1_\mathcal{O})(X)$ for $X \in \mathcal{O}'$

For a type $\omega = (d_1, \lambda_1) \cdots (d_s, \lambda_s) \in \mathbb{T}_\mathcal{V}$ put

$$C_\omega := \begin{cases} \mu(d), (-1)^{r-1} \frac{(r-1)!}{\prod d_i m_{d_i}(\omega)} & \text{if } d_1 = d_2 = \cdots = d_s = d \\ 0 & \text{otherwise.} \end{cases}$$

where $m_{d_1}(\omega)$ denotes the multiplicity of the pair $(d_1, \lambda)$ in $\omega$ and where $\mu$ denotes the ordinary Möbius function.

Recall that we defined a map (see the beginning of §2.2) from regular semisimple adjoint orbits of $\mathfrak{gl}_\mathcal{V}(\mathbb{F}_q)$ to $W_\mathcal{V}$.

**Proposition 2.11.** Let $w = (w_1, \ldots, w_r) \in W_\mathcal{V}$ have cycle type $\lambda = (\lambda^1, \ldots, \lambda^r) \in \mathcal{P}_\mathcal{V}$. Let $O$ be a generic regular semisimple adjoint orbit of $\mathfrak{gl}_\mathcal{V}(\mathbb{F}_q)$ mapping to $w$ and $\omega = (\omega_1, \ldots, \omega_r)$ is any type in $\mathbb{T}_\mathcal{V}$. Then

$$\sum_{\mathcal{O}'} \mathcal{F}(1_\mathcal{O})(\mathcal{O}') = e(w) q^{1+\delta r/2} C_\omega \prod_{i=1}^{r} \langle \tilde{H}_{\omega_i}(x_i ; q), p_{\lambda_i}(x_i) \rangle,$$

where the sum is over the adjoint orbits $\mathcal{O}'$ of $\mathfrak{gl}_\mathcal{V}^F$ of type $\omega$, $\mathcal{F}(1_\mathcal{O})(\mathcal{O}')$ denotes the common value $\mathcal{F}(1_\mathcal{O})(X)$ for $X \in \mathcal{O}'$ and $\delta_\mathcal{V} = \dim \text{GL}_{\mathcal{V}} - \dim \mathcal{T}_\mathcal{V} = \sum_i v_i^2 - \sum_i v_i$. 
Proof. For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $d \in \mathbb{Z}_{>0}$, define $d \cdot \lambda := (d\lambda_1, \ldots, d\lambda_m)$, and for a type $\tau = (d_1, \tau^1) \cdots (d_s, \tau^s) \in \mathbb{T}_n$ put $[\tau] := \cup_i d_i \cdot \tau^i$, a partition of $n$. We will say that two types $\nu = (d_1, \nu^1) \cdots (d_s, \nu^s)$ and $\omega = (e_1, \omega^1) \cdots (e_l, \omega^l)$ are compatible, which we denote $\nu \sim \omega$, if $s = l$ and for each $i = 1, \ldots, s$ we have $d_i = e_i$ and $|\nu^i| = |\omega^i|$. For two partitions of same size $\lambda, \mu$ let $Q^\omega_{\mu}(q)$ be the Green polynomial as defined for instance in [24, Chap. III §7]. For two compatible types $\nu$ and $\omega$ we put $Q^\omega_{\nu}(q) := \prod_i Q^\omega_{\nu^i}(q^{d_i})$ and $Q^\omega_{\nu}(q) := 0$ otherwise. Let $z_\lambda$ be the order of the centralizer of an element of cycle type $\lambda$ in $S_{\lambda^1}$. For a type $\nu = (d_1, \nu_1), \ldots, (d_s, \nu_s)$ set $z_\nu = \prod_i z_{\nu^i}$.

Notice that for $\lambda$ a partition $(d_1, d_2, \ldots, d_s) \in \mathbb{T}_n$ we have $p_\lambda(x) = s_\tau(x)$ where $\tau \in \mathbb{T}_n$ is the type $(d_1, 1)(d_2, 1) \cdots (d_s, 1)$. Hence by [14, Lemma 2.3.5] for any $\omega \in \mathbb{T}_n$ and any partition $\lambda$ of $n$.

$$\langle \hat{H}_\omega(x; q), p_\lambda(x) \rangle = z_\lambda \sum_{\nu \in \mathbb{T}_n \mid |\nu| = |\lambda|} \frac{Q^\omega_{\nu}(q)}{z_\nu}. $$

We are therefore reduced to prove that

$$\sum_{\mathcal{O}'} \mathcal{F}(1_\mathcal{O})(\mathcal{O}') = \epsilon(w) q^{1+\Delta/2} C^{z_\lambda}_{\nu} \prod_{i=1}^r z_{\nu^i} \sum_{\nu \in \mathbb{T}_n \mid |\nu| = |\lambda|} \frac{Q^\omega_{\nu}(q)}{z_\nu}. \quad (2.9)$$

The proof of this formula is similar to that of [14, Theorem 4.3.1(2)] although the context is different and will require some new calculations. Embed $GL_N$ in $GL_N$ with $N = \sum_{i=1}^r \nu_i$. Write $\omega = (d_1, \mu_1) \cdots (d_s, \mu_s)$ and define $\omega = (d_1, \cup \mu_1) \cdots (d_s, \cup \mu_s) \in \mathbb{T}_N$ where for a multi-partition $\mu = (\mu^1, \ldots, \mu^r) \in \mathcal{P}^\omega$, we put $\cup \mu = \cup \mu^i$. If $\mathcal{O}'$ is an $F$-stable adjoint orbit of $gl_N$ of type $\omega$, then the unique $GL_N$-adjoint orbit of $gl_N$ which contains $\mathcal{O}'$ is of type $\omega$. Consider a representative of an adjoint orbit of $gl_N' \subset gl_N$ of type $\omega$ with Jordan form $\sigma + u$ where $\sigma$ is semisimple and $u$ is nilpotent. Put $L := C_{GL_N}(\sigma)$ and, denote by $l$ the Lie algebra of $L$ and by $z_l$ the center of $l$. Note that $l$ is not contained in $gl_N$ unless each for each $i$ the multi-partition $\mu_i$ has a unique non-zero coordinate in which case $L = M := C_{GL_N}(\sigma)$. However we always have $z_l \subset gl_N$. Put $(z_l)_{\text{reg}} := \{ y \in z_l \mid C_{GL_N}(y) = L \}$.

The map that sends $\omega \in (z_l)_{\text{reg}}$ to the $GL_N$-orbits of $\sigma + u$ surjects onto the set of adjoint orbits $\mathcal{O}'$ of $gl_N'$ of type $\omega$. The fibers of this map can be identified with $\{ g \in GL_N' \mid gLg^{-1} = L, gC_Ng^{-1} = C_u \} = M/L$, where $C_u$ is the $M$-orbit of $u$. Hence we may in turn identify these fibers with

$$W(\omega) := \prod_{i,d \lambda} (\mathbb{Z}/d\mathbb{Z})^{m_{d,\lambda}(\omega)} \times S_{m_{d,\lambda}(\omega)}. $$

It follows that

$$\sum_{\mathcal{O}'} \mathcal{F}(1_\mathcal{O})(\mathcal{O}') = \frac{1}{|W(\omega)|} \sum_{\omega \in (z_l)_{\text{reg}}} \mathcal{F}(1_\mathcal{O})(\omega + u) $$

$$= \frac{1}{|W(\omega)|} \sum_{\omega \in (z_l)_{\text{reg}}} \prod_{i=1}^r \mathcal{F}_{\omega}(1_\mathcal{O}_i)(z_i + u_i), $$

where $\mathcal{F}_{\omega}$ denotes the Fourier transform on $gl_N'$, $O_i$ is the $i$-th coordinate of $O$ (a $GL_N'$-orbit of $gl_N'$) and $z_i, u_i$ are the $i$-th coordinates of $z, u$ respectively.

It is known [21, Theorem 7.3.3][14, Formulas (2.5.4), (2.5.5)] that

$$\mathcal{F}_{\omega}(1_\mathcal{O}_i)(z_i + u_i) = \epsilon(\omega_i) q^{(\lambda_i - \nu_i)} |M_i^{F_{\ell}}|^{-1} \sum_{\omega \in GL_{\nu_i}(\mathbb{Z}/\mathcal{O}_i) \cdot h \in H^1_{\nu_i}(\mathcal{O}_i) \cdot u_i + 1} Q_{hT_i, h^{-1}}(u_i + 1) \Psi \left( \left( X_i, h^{-1}z_i \right) \right).$$
where $X_i$ is a fixed element in $O_{\nu}^F$, $T_{\nu}$ is the unique $F$-stable maximal torus of $GL_{\nu}$ whose Lie algebra $t_{\nu}$ contains $X_i$, $M_i = C_{GL_{\nu}}(z_i)$ and where $Q_{hT_{\nu}h^{-1}}^M$ is the Green function defined by Deligne and Lusztig [9] (the values of these functions are products of usual Green polynomials).

It follows that

$$\sum_{O} F(1_O)(O') = \frac{1}{|W(\omega)|} \varepsilon(w) q^{\frac{1}{2}(\sum_i v_i^2 - \sum_i v_i)} \sum_{h = (h_1, \ldots, h_r)} \Phi_h(u) \sum_{z \in (\mathbb{Z})_{\text{reg}}} r \Psi \left( \langle X, h_i^{-1} z h_i \rangle \right)$$

where $h = (h_1, \ldots, h_r)$ runs over the set

$$\{ h \in GL_{\nu}^F \mid hT_{\nu}h^{-1} \subset M \} = \{ h \in GL_{\nu}^F \mid z \in hT_{\nu}h^{-1} \}$$

with $T_A := \prod_{i=1}^r T_{\nu_i}$, $X := (X_i)_{i=1,\ldots,r} \in t_A \cap O$ and where to simplify we set

$$\Phi_h(u) := \prod_{i=1}^r |M_i^F|^{-1} Q_{hT_{\nu}h^{-1}}^M(u_i + 1).$$

To finish the proof it suffices to check the following two formulas

$$\sum_{z \in (\mathbb{Z})_{\text{reg}}} \Psi \left( \langle X, h^{-1} z h \rangle \right) = \begin{cases} (-1)^{s-1} \mu(d) q(s - 1)! & \text{if } d_i = d \text{ for all } i = 1, \ldots, s \\ 0 & \text{otherwise}, \end{cases}$$

$$\sum_{h} \Phi_h(u) = \prod_{i=1}^r \sum_{\{v \in \mathbb{Z} \mid |v| = \nu_i \}} \frac{Q_{\nu_i}(q)}{z_v}. \quad \text{(15)}$$

where recall that $\omega = (d_1, \mu_1), (d_2, \mu_2), \ldots$.

The proof of the second formula is contained in the proof of [14, Theorem 4.3.1(2)]. For the first formula, by [22, Proposition 6.8.3], it is enough to prove that the linear character $\Theta : t^F_\lambda \to \mathbb{C}^\times$, $z \mapsto \Psi(\langle X, z \rangle)$ is a generic character, i.e., the restriction of $\Theta$ to $z^F_{\text{gl}_N}$ is trivial and for any $F$-stable Levi subgroup $L$ (of some parabolic subgroup) of $GL_N$ which contains $T_A$ the restriction of $\Theta$ to $z^F_\lambda \subset t^F_\lambda$ is non-trivial unless $L = GL_N$.

But $\Theta$ is generic because the adjoint orbit $O$ is generic. Indeed, since $\text{Tr}(O) = 0$ we have $\Theta|_{\text{reg}} = 1$. Now assume that $L \supset T_A$ satisfies $\Theta|_{L^F} = 1$. There is a decomposition $\mathbb{K}^N = W_1 \oplus W_2 \oplus \cdots \oplus W_s$, with $W_j \not= 0$, such that $t$ is $GL_N$-conjugate to $\bigoplus_j \text{gl}(W_j)$. An element $z \in z_\lambda$ is of the form $(\xi_1, \ldots, \xi_s)$ where $\xi_1, \ldots, \xi_s \in \mathbb{K}$ and where $\iota_j$ denotes the identity endomorphism of $W_j$. Denote by $X^j$ the $\text{gl}(W_j)$ coordinate of $X$. Since $\Theta|_{L^F} = 1$, we must have $\langle X, z \rangle = \sum_{j=1}^s \xi_j \text{Tr}(X^j) = 0$ for all $z = (\xi_1, \ldots, \xi_s)$ and so $\text{Tr}(X^j) = 0$ for all $j = 1, \ldots, s$. Now $\text{gl}_\nu$ and $l$ are two Levi sub-algebras of $\text{gl}_N$ that contains $t_A$, hence $\text{gl}_\nu \cap l \simeq \bigoplus_{i,j} \text{gl}(U_{i,j})$ where $W_j = \bigoplus U_{i,j}$. For each $j = 1, \ldots, s$, the space $\bigoplus_{i,j} U_{i,j}$ is also a graded subspace of $\mathbb{K}^N = \mathbb{K}^\times$ on which $X$ acts by $X^j$ and so by the genericity assumption we must have $W_j = \mathbb{K}^N$ i.e. $L = GL_N$.

\[\square\]

**Proof of Theorem 2.9** By definition (1.8)

$$d_\nu = \sum_{j \in \Omega} v_j \lambda_j - \sum_i v_i^2 + \delta_\nu + 2$$

$$= \dim \text{Rep}_{\mathbb{K}}(\Gamma, \nu) - \dim \text{gl}_\nu + 1 + \delta_\nu/2.$$
We prove a general fact about how to extract Kac polynomials of the quivers $\Gamma_\mu$ from the generating function $H(x_1, \ldots, x_r; q)$ defined in [14]. The statement and proof are along the same lines as [15, Theorem 3.2.7]. For any multi-partition $\mu \in P^r$ denote by $A_\mu(q)$ the Kac polynomial associated with $(\Gamma_\mu, v_\mu)$ where $v_\mu$ is defined as in [14] with $v = |\mu|$. For a partition $\lambda$, denote by $h_\lambda$ the complete symmetric function as in [24].

**Theorem 2.12.** For any $\mu \in P_\mu$, we have

$$\left\langle \tilde{H}(x_1, \ldots, x_r; q), h_\mu \right\rangle = A_\mu(q).$$

**Proof.** Denote by $\Omega_\mu$ the arrows of $\Gamma_\mu$. The starting idea is that for any indecomposable representation $\varphi$ of $(\Gamma_\mu, v_\mu)$, the coordinate $\varphi_{i \rightarrow j}$ of $\varphi$ at any arrow $i \rightarrow j \in \Omega_\mu \backslash \Omega$ (i.e., $i \rightarrow j$ is an arrow on one of the added leg) must be injective (see [15, Lemma 3.2.1]). Denote thus by $\text{Rep}_k(\Gamma_\mu, v_\mu)^*$ the space of all representations of $\Gamma_\mu$ of dimension $v_\mu$ whose coordinates at the legs are all injective. For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ of $n$, denote by $\mathcal{F}_\mu$ the set of partial flag of $k$-vector spaces

$$\{0\} \subset E^{r-1} \subset \cdots \subset E^1 \subset E^0 = k^n,$$

with $\dim E^i - \dim E^{i+1} = \mu_{i+1}$. Then we have a natural bijection from the isomorphism classes of $\text{Rep}_k(\Gamma_\mu, v_\mu)^*$ onto the orbit space

$$\delta_\mu := \left( \text{Rep}_k(\Gamma, \mu) \times \prod_{i=1}^r \mathcal{F}_\mu \right) / \text{GL}_v.$$ 

Put $G_\mu(q) := \#\delta_\mu(F_q)$. Then as in [15, Theorem 3.2.3] we prove that

$$\log \left( \sum_{\mu \in P^r} G_\mu(q)m_\mu \right) = \sum_{\mu \in P^r \cup \{0\}} A_\mu(q)m_\mu,$$

where $m_\mu$ denotes the monomial symmetric functions as in [24]. Recall that the basis $\{m_\mu\}_\mu$ is dual to $\{h_\mu\}_\mu$ with respect to the Hall pairing. The analogue of Proposition [15, Proposition 3.2.5] reads

$$\sum_\mu G_\mu(q) = \prod_{d=1}^\infty \Omega(x^d_1, \ldots, x^d_r, q^d)\phi_d(q),$$

(2.10)
where \( \Omega(x_1, \ldots, x_r; q) \) is the series inside the brackets of Formula (1.4), i.e.
\[
\Omega(x_1, \ldots, x_r; q) = \text{Exp}((q - 1)^{-1} H(x_1, \ldots, x_r; q))
\]
and where \( \phi_d(q) \) is the number of Frobenius orbits of \( \overline{\mathbb{F}_q} \setminus \{0\} \) of size \( d \). Taking \( \text{Log} \) on both side of Formula (2.10) and using the properties of \( \text{Log} \) [15, Lemma 2.1.2] gives
\[
\sum_{\mu} A_{\mu}(q)m_{\mu} = \mathbb{H}(x_1, \ldots, x_r; q),
\]
hence the result. \( \Box \)

2.5 Proof of Theorem 1.4

Proof of Theorem 1.4. We start by proving (i). Let us denote here by \( Q_v/\mathbb{C} \) and \( Q_v/\overline{\mathbb{F}_q} \) the associated quiver varieties over the indicated field. Assume also that the characteristic of \( \mathbb{F}_q \) is large enough so that the results of §2.2 apply. Combining Theorem 2.5 and Theorem 2.9 we find that
\[
\langle \mathbb{H}(x_1, \ldots, x_r; q), p_A \rangle = \epsilon(w) \sum_i \text{Tr} \left( \rho^{2i}(w), H^{2i}(Q_v/\mathbb{C}_v) \right) q^{i-dv}.
\]
(2.11)

We deduce from Theorem 2.5 and the comment below the proof of Theorem 2.5 that
\[
\langle \mathbb{H}(x_1, \ldots, x_r; t), p_A \rangle = \epsilon(w) \sum_i \text{Tr} \left( \rho^{2i}(w), H^{2i}(Q_v/\mathbb{C}_v) \right) t^{i-dv}
\]
(2.12)
is an indentity in \( \mathbb{Q}[t] \).

The Schur functions \( s_\mu \), with \( \mu \in P_v \), decompose into power symmetric functions as [24, Chapter I, Proof of (7.6)]
\[
s_\mu = \sum_{A \in P_v} \chi^\mu_A p_A
\]
where \( \chi^\mu_A = \chi^\mu(w) \) is the value of the irreducible character \( \chi^\mu \) of \( W_v \) at an element \( w \in W_v \) of cycle type \( A \). Hence
\[
\langle \mathbb{H}(x_1, \ldots, x_r; t), s_\mu \rangle = q^{-dv} \sum_i \langle \chi^\mu \otimes \epsilon, \rho^{2i} \rangle_{W_v} t^i
= t^{-dv} \sum_i \langle \chi^\mu, \rho^{2i} \rangle_{W_v} t^i,
\]
and Theorem 1.4(i) follows. Note that \( \mathbb{H}^\mu_\mu(t) \) is a polynomial since \( H^{i}(Q_v; \mathbb{C}) = 0 \) unless \( d_v \leq i \leq 2d_v \) as the variety \( Q_v \) is affine.

We now proceed with the proof of (ii).

Consider the partial ordering \( \preceq \) on partitions defined as \( \lambda \preceq \mu \) if for all \( i \) we have \( \sum_i \lambda_i \leq \sum_i \mu_i \). Extend this ordering on multi-partitions by declaring that \( \alpha \preceq \beta \) if and only if for all \( i \), we have \( \alpha^i \preceq \beta^i \). A simple calculation shows that if \( \alpha \preceq \beta \) and \( \alpha \neq \beta \) for any two multi-partitions in \( P^r \), then \( d_\beta < d_\alpha \).

Using the relations between Schur functions and complete symmetric functions [24, page 101] together with Theorem 2.12 we find that
\[
A_\lambda(q) = \sum_{\mu \subseteq \lambda} K^{e}_{\mu \lambda} A_\mu(q), \quad \mathbb{H}^\mu_\mu(q) = \sum_{\lambda \subseteq \mu} K^{s}_{\mu \lambda} A_\lambda(q),
\]
(2.13)
where \( K = (K_{i\mu}) \) is the matrix of Kostka numbers, \( K' \) and \( K^* \) are respectively the transpose and the transpose inverse of \( K \). Recall [17, §1.15] that, if non-zero, \( A_\mu(q) \) is monic of degree \( d_\mu \) and \( A_\mu(q) \) is non-zero if and only if \( v_\mu \) is a root of \( \Gamma_\mu \), with \( A_\mu(q) = 1 \) if and only if \( v_\mu \) is real [17, §1.10]. Since \( K^*_{\mu\nu} = 1 \) and since the polynomials \( A_\mu(q) \), with \( \lambda \geq \mu, \lambda \neq \mu \), are of degree strictly less than \( d_\mu \), we deduce from the second equality (2.13) that if \( v_\mu \) is a root then \( H^i_{\mu}(q) \) is monic of degree \( d_\mu \). Conversely if \( H^i_{\mu}(q) \) is non-zero then by the first formula (2.13) the polynomial \( A_\mu(q) \) must be non-zero (i.e. \( v_\mu \) is a root) as the Kostka numbers are non-negative, \( K_{\mu\nu} = 1 \) and \( H^i_{\mu}(q) \) has non-negative coefficients. This completes the proof. \( \square \)

Remark 2.13. 1. When the dimension vector \( v \) is indivisible we can prove Theorem 1.4 using the ideas in [22] based on the theory of perverse sheaves (although the context in [22] is different). More precisely when \( v \) is indivisible and \( \mu = (\mu^1, \ldots, \mu^r) \in \mathcal{P}_v \) there exists a generic adjoint orbit \( O \) of \( \mathfrak{gl}_v(\mathbb{C}) \) (see §2.1) such that each component \( O_i \) of \( O \) in \( \mathfrak{gl}_v \) is of the form \( \zeta_i \cdot I_{v_i} + N_i \) where \( \zeta_i \in \mathbb{C} \) and \( N_i \) is nilpotent with Jordan form given by the dual partition \( (\mu^i)' \) of \( \mu^i \). Consider the associated singular complex quiver variety \( Q_{v_\mu} := \mu^{-1}(O)/\text{GL}_v \) (where \( \mu_v \) is the moment map defined in §2.1). Then following the strategy in [22, Corollary 7.3.5] we can prove that
\[
H^i_{\mu}(q) = \sum \dim \left( IH^2_c(Q_{v_\mu}; \mathbb{C}) \right) q^{-d_\mu}
\]
is up to a power of \( q \) the Poincaré polynomial of \( Q_{v_\mu} \) for the compactly supported intersection cohomology.

2. While in this paper we construct the action of \( W_v \) on \( H^i_c(Q_v; \mathbb{C}) \) using the hyperkähler structure on quiver varieties, in the case where \( v \) is indivisible it is possible to give an alternative construction of this Weyl group action as in [22] based on the theory of perverse sheaves and then give an alternative proof of Theorem 1.4.

3 DT-invariants for symmetric quivers

3.1 Preliminaries

Denote by \( \Lambda \) the ring of symmetric functions in the variables \( x = \{x_1, x_2, \ldots \} \) with coefficients in \( \mathbb{Q}(q) \) and \( \Lambda_n \) those functions in \( \Lambda \) homogeneous of degree \( n \). We define the \( u \)-specialization of symmetric functions as the ring homomorphism \( \Lambda \to \mathbb{Q}[u] \) that on power sums behaves as follows
\[
p_v(x) \mapsto 1 - u^v.
\]
In plethystic notation this is denoted by \( f \mapsto f[1 - u] \).

Note that for any \( f \in \Lambda_n \) the \( u \)-specialization \( f[1 - u] \) is a polynomial in \( u \) of degree at most \( n \). We will need to consider the effect of taking top degree coefficients in \( u \) after \( u \)-specialization. Define the top degree of \( f \in \Lambda_n \) as
\[
[f] := u^n f \left[1 - u^{-1}\right]_{u=0}.
\]

It is a crucial fact for what follows that \( u \)-specialization and taking its top degree coefficient commute with the Log map. More precisely we have the following.

Proposition 3.1. Let \( \Omega(x; T) = \sum_{n \geq 0} A_n(x) T^n \in \Lambda[[T]] \) be a power series with \( A_n(x) \in \Lambda_n \) and let \( V_n(x) \in \Lambda \) be defined by
\[
\sum_{n \geq 1} V_n(x) T^n := \text{Log} \, \Omega(x; T).
\]
Then we have
(i) \[ \sum_{n \geq 1} V_n[1 - u] T^n = \log \sum_{n \geq 0} A_n[1 - u] T^n. \]

and

(ii) \[ \sum_{n \geq 1} [V_n] T^n = \log \sum_{n \geq 0} [A_n] T^n. \]

**Proof.** Define \( U_n(x) \in \Lambda \) by
\[
\sum_{n \geq 1} U_n(x) \frac{T^n}{n} := \log \Omega(x; T).
\]
The relation between \( U_n \) and \( V_n \) is
\[
V_n(x) := \frac{1}{n} \sum_{d|n} \mu(d) U_{n/d}(x^d),
\]
where \( \mu \) is the ordinary Möbius function.

Since the \( u \)-specialization is a ring homomorphism we have
\[
\sum_{n \geq 1} U_n[1 - u] \frac{T^n}{n} = \log \sum_{n \geq 0} A_n[1 - u] T^n.
\]

It is clear that \( U_n(x) \) is homogeneous of degree \( n \) and hence we may write it as \( \sum_{|\lambda|=n} \mu_A \lambda P_\lambda(x) \) for some coefficients \( \mu_A \). Therefore,
\[
V_n(x) = \frac{1}{n} \sum_{d|n} \mu(d) \sum_{|\lambda|=n/d} \mu_A \lambda p_\lambda(x^d).
\]

Note that \( p_\lambda(x^d) = p_{d \lambda}(x) \); so applying the \( u \)-specialization to both sides we get
\[
V_n[1 - u] = \frac{1}{n} \sum_{d|n} \mu(d) \sum_{|\lambda|=n/d} \mu_A \lambda p_{d \lambda}[1 - u].
\]

Similarly, since \( p_{d \lambda}[1 - u] = 1 - u^{d \lambda} = p_r[1 - u^d] \), the inner sum on the right hand side equals \( U_{n/d}[1 - u] \) evaluated at \( u^d \), proving (i).

To prove (ii) replace in (3.2) \( u \) by \( u^{-1} \), \( T \) by \( uT \) and set \( u = 0 \) to obtain
\[
\sum_{n \geq 1} [U_n] \frac{T^n}{n} = \log \sum_{n \geq 0} [A_n] T^n.
\]

Now the claim follows from (3.1). \( \square \)

**Proposition 3.2.** For any partition \( \lambda \in \mathcal{P} \) we have

(i) \[ \tilde{H}_\lambda(q)[1 - u] = (u)_l \]

where \( l := l(\lambda) \) is the length of \( \lambda \) and \( (u)_l := \prod_{i=1}^l (1 - q^{-1} u). \)

(ii) \( [\tilde{H}_\lambda] \) is zero unless \( \lambda = (1^n) \) when it equals \( (-1)^n q^n(\mathcal{S}) \).

**Proof.** The specialization (i) follows from the corresponding result for Macdonald polynomials, see [11, Corollary 2.1]. The second claim is an immediate consequence of (i). \( \square \)

We will need one last fact.
Lemma 3.3. For the Schur function $s_\lambda$ we have that $s_\lambda[1 - u]$ is zero unless $\lambda = (r, 1^{n-r})$ with $1 \leq r \leq n$ is a hook, in which case it equals $(-u)^{n-r}(1 - u)$. In particular, for $f \in \Lambda_n$ we have

$$[f] = (-1)^n \langle f, s_{(1^n)} \rangle.$$  

(3.3)

Proof. The $u$-specialization of the Schur functions is given in [11, (2.15)]. The identity (3.3) follows immediately.  

\[\square\]

3.2 DT-invariants

In this section we prove a somewhat more general case of Proposition 1.3 (ii). We work with a symmetric quiver (a quiver with as many arrows going from the vertex $i$ to $j$ as arrows going from $j$ to $i$) instead of the double of a quiver (see Remark 1.2). The only difference is that the double of a quiver has an even number of loops at every vertex whereas a symmetric quiver may not. We deal with this by attaching an arbitrary number of legs to each vertex instead of just one. In general, the parity of the number of legs required at a vertex $i$ is the opposite of that of the number of loops at $i$.

Concretely, attach $k_i \geq 1$ infinite legs to each vertex $i \in I$ of $\Gamma$. The orientation of the arrows ultimately does not matter but say all the arrows on the new legs point towards the vertex. Consider the following generalization of (1.4)

$$\mathcal{H}(x; q) := (q - 1) \log \left( \sum_{x \in \mathcal{P}} \mathcal{H}_\pi(q) \tilde{H}_\pi(x; q) \right),$$  

(3.4)

where to simplify we let

$$\tilde{H}_\pi(x; q) := \prod_{i=1}^{r} \prod_{j=1}^{k_i} H_\pi(x^{i,j}; q)$$

and $x^{i,j} = (x_{1}^{i,j}, x_{2}^{i,j}, \ldots)$ for $i = 1, \ldots, r$ and $j = 1, \ldots, k_i$ are independent sets of infinitely many variables.

Given a multi-partition $\mu = (\mu^{i,j})$ where $i = 1, \ldots, r$ and $j = 1, \ldots, k_i$ define

$$s_\mu(x) := \prod_{i=1}^{r} \prod_{j=1}^{k_i} s_{\mu_{i,j}}(x^{i,j})$$

and

$$\mathcal{H}_{\mu}(q) := \langle \mathcal{H}(x; q), s_\mu(x) \rangle.$$  

(3.5)

Note that $\mathcal{H}_{\mu}(q)$ is zero unless $|\mu^{i,1}| = |\mu^{i,2}| = \cdots = |\mu^{i,k_i}|$ for each $i = 1, \cdots, r$.

For $v \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}$, denote by $1^v$ the multi-partition $(\mu^{i,j})$ where for every $j = 1, \ldots, k_i$ either $\mu^{i,j} = (1^v)$ if $v_i > 0$ or $\mu^{i,j} = 0$ otherwise.

Proposition 3.4. We have

$$(q - 1) \log \left( \sum_{v \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}} \frac{q^{-\gamma(v) + \delta(v)} - 1}{(q - 1)^{\gamma(v)}} \right) = \sum_{v \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}} \mathcal{H}_{\mu}(q)(-1)^{\delta(v)} T^v,$$  

(3.6)

where

$$\gamma(v) := \sum_{i=1}^{r} (2 - k_i)v_i^2 - 2 \sum_{i=1}^{r} v_i v_j, \quad \delta(v) := \sum_{i=1}^{r} k_i v_i,$$

and $(q)_v := (q)_{v_1} \cdots (q)_{v_r}$ with $(q)_v := (1 - q) \cdots (1 - q^v)$.
Proof. By (3.5) we have
\[ \mathbb{H}(x; q) = \sum_{\mu} \mathbb{H}_{\mu}(q)s_{\mu}(x). \]

Apply Proposition 3.1 (ii) to all the variables \( x^{i,j} \) in (3.4) to get
\[ \sum_{\mu} \mathbb{H}_{\mu}(q)[s_{\mu}(x)T^{i|\mu]} = (q - 1) \log \left( \sum_{\sigma \in \psi} \mathcal{H}_{\sigma}(q)[\bar{H}_{\sigma}(x; q)]T^{i|\sigma]} \right), \]
where \( \mu \) runs through the non-zero multi-partitions \( (\mu^{i,j}) \) with \( |\mu^{i,j}| = v_i \) for some \( v_i \in \mathbb{Z}_{\geq 0} \) independent of \( j \), \( T^{i|\mu} := \prod_i T_i^{v_i} \) and \( T^{i|\sigma]} = \prod_i T_i^{v_i} \). A calculation using Proposition 3.2 (ii) shows that the right hand side equals
\[ (q - 1) \log \left( \sum_{v \in \mathbb{Z}_{\geq 0}} q^{\frac{1}{2}(\gamma(v) + \delta(v))} (q^{-1})_v (-1)^{\delta(v)}T^v \right) \]
Finally, (3.3) shows that the left hand side equals
\[ \sum_{v \in \mathbb{Z}_{\geq 0}} \mathbb{H}_{1,v}(q)(-1)^{\delta(v)}T^v \]
and our claim is proved. \( \square \)

Now let \( \Gamma' = (I, \Omega') \) be any symmetric quiver with \( r \) vertices. The Donaldson–Thomas invariants for a symmetric quiver \( \Gamma' \), as defined by Kontsevich and Soibelman, are given as follows in an equivalent formulation.

Let \( c_{v,k} \) be the coefficients in the generating function identity
\[ \log \sum_v \left( -q^{1/2} \right)^{\gamma'(v)} \left( q \right)_v T^v = (1 - q)^{-1} \sum_v \sum_k (-1)^k c_{v,k} q^{k/2} T^v, \] (3.7)
where
\[ \gamma'(v) := \sum_{i=1}^r v_i^2 - \sum_{i,j \in \Omega'} v_i v_j. \]

Put \( \Omega_v(q) := \sum_k c_{v,k} q^{k/2} \); it is a Laurent polynomial in \( q^{1/2} \) (See [10] p.15,)

Efimov [10] Thm. 4.1] proves that if \( c_{v,k} \) is non-zero then \( k \equiv \gamma'(v) \mod 2 \). Since \( \Gamma' \) is symmetric we also have \( \gamma'(v) \equiv \delta'(v) \mod 2 \), where \( \delta' \) is a fixed linear form
\[ \delta'(v) := \sum_{i=1}^r k'_i v_i, \quad k'_i \in \mathbb{Z}_{\geq 0}, \quad k'_i \equiv a'_{i,i} - 1 \mod 2 \]
with \( a'_{i,i} \) the number loops at the vertex \( i \) of \( \Gamma' \). Hence we may write (3.7) as
\[ (1 - q) \log \sum_v q^{\frac{1}{2} \gamma'(v)} \left( q \right)_v (-1)^{\delta'(v)} T^v = \sum_v \Omega_v(q) (-1)^{\delta'(v)} T^v. \] (3.8)

Changing \( q \mapsto q^{-1} \) and then \( T_i \mapsto q^{-k'_i/2} T_i \), we find that
\[ (q - 1) \log \sum_v q^{\frac{1}{2} \gamma'(v) + \delta'(v)} \left( q^{-1} \right)_v (-1)^{\delta'(v)} T^v = \sum_v q^{1-k'_i} \Omega_v(q^{-1}) (-1)^{\delta'(v)} T^v \] (3.9)
We extend the definition of $\text{DT}_v$ given in [1, 3] to $\Gamma'$ by setting

$$(q - 1) \log \sum_v q^{-\frac{1}{2}(\gamma'(v) + \delta'(v))} (-1)^{\delta'(v)} T^v = \sum_{v \in \mathbb{Z}^r_\geq 0} \text{DT}_v(q) (-1)^{\delta'(v)} T^v. \quad (3.10)$$

Up to powers of $q$ the definition of $\text{DT}_v(q)$ is independent of the choice of linear form $\delta'$ and we do not include it in the notation. Note that then

$$\text{DT}_v(q) = q^{1 - \frac{1}{2}\delta'(v)} \Omega_v(q^{-1}).$$

We would like to match (3.10) with (3.6) by making appropriate choices for $\Gamma$ and $k_i$. Denote by $a_{i,j}$ (resp. $a'_{i,j}$) the number of arrows of $\Gamma$ (resp. $\Gamma'$) going from $i$ to $j$. To match $\gamma$ with $\gamma'$ requires that

$$a'_{i,j} + a_{j,i} = 2(a_{i,j} + a_{j,i}), \quad i \neq j, \quad k_i - 2 + 2a_{i,i} = -1 + a'_{i,i}. \quad (3.11)$$

This we can always do (typically in more than one way) because $\Gamma'$ is symmetric. We have then

**Proposition 3.5.** With the above notation let $\Gamma$ be a quiver and $k_i$ be integers satisfying (3.11). Then

$$\text{DT}_v(q) = q^{\delta'(v) - \delta'(v)_{\Gamma'}} \mathbb{X}_1(q) \quad (3.12)$$

for all $v \in \mathbb{Z}^r_\geq 0 \setminus \{0\}$. A special case of Proposition 3.5 is when $\Gamma' = \overline{\Gamma}$ for some quiver $\Gamma$. In this case we may take $k_i = k'_i = 1$ for all $i$ and (3.12) is claim (ii) of Proposition 1.3.

### 4 Examples

**Example 1.** Let $\Gamma' = S_m$ be the quiver with one node and $m$ loops. We can take $\Gamma$ to be the quiver with one node and no arrows and take $k = k' = m + 1$. Then $\gamma'(n) = \gamma(n) = (1 - m)n^2$, $\delta(n) = \delta'(n) = (m + 1)n$ and hence

$$\sum_{n \geq 1} \text{DT}_n(q) (-1)^{(m-1)n} T^n = (q - 1) \log \sum_{n \geq 0} \frac{q^{(m-1)(\cdot) - n}}{(q-1)_n} (-1)^{(m-1)n} T^n.$$ 

Up to a power of $q$ the invariants $\text{DT}_n(q)$ are those considered by Reineke [30]. Here is a list of the first few values. For $m = 0, 1$ we have $\text{DT}_1(q) = 1$ and $\text{DT}_n(q) = 0$ for all $n > 1$.

For $m = 2$,

| $n$ | $\text{DT}_n(q)$ |
|-----|------------------|
| 1   | $1$              |
| 2   | $1$              |
| 3   | $q$              |
| 4   | $q^2 + q$        |
| 5   | $q^4 + q^3 + q^2 + q$ |
| 6   | $q^6 + q^5 + 2q^4 + q^3 + 3q^2 + 2q + q$ |

For $m = 3$,
n  |  \(DT_n(q)\)  
---|---
1  | 1  
2  | 1 \((q)\)  
3  | 1 \((q)^3\)  
4  | 1 \((q)^4\)  
5  | 1 \((q)^5\)  

\[DT_n(q) = \frac{1}{n!} \sum_{\lambda} (-1)^\lambda T_\lambda(q)\]

**Example 2.** Let \(\Gamma\) be the \(A_2\) quiver, i.e., two nodes connected by an arrow, and let \(k_1 = k_2 = 1\). Then \(\Gamma'\) is its double; the quiver with two nodes and an arrow between them in each direction. In this case, \(\tilde{\Gamma}\) is actually a finite quiver (of type \(A\)). It follows that \(DT_n(q)\) is zero unless \(v\) is a root of \(A_2\); i.e. \(v = (0,1), (1,0), \) or \((1,1)\) and \(DT_v(q) = 1\) (see Corollary 1.6).

Using that \((q) = (\frac{1}{q-1})\), we can write (3.6) as

\[DT_n(q) = (-1)^n q^{\binom{n}{2} + n(q^{-1})_n}\]  

which can be proved directly in an elementary way using the \(q\)-binomial theorem (see [18 Prop. 1]). This identity is related to the quantum pentagonal identity.

This case is the only case, other than the trivial quiver with one vertex and no arrows, where \(\tilde{\Gamma}\) is a finite quiver, or, equivalently, where the right hand side of (3.6) has only finitely many terms.

**Example 3.** Let \(\Gamma\) be the quiver with \(r\) nodes and \((m - 1) \min(i,j)\) arrows pointing from vertex \(i\) to \(j\) and let \(k_i = 2\) for all \(i\). Then up to a power of \(q\) the \(DT_v(q)\) invariants equal the truncated form \(A_\lambda(q)\) of the Kac polynomial for the \(S_m\) quiver considered in [31]. In particular, this shows that these truncations \(A_\lambda(q)\) are indeed, as conjectured, polynomials in \(q\) and have non-negative integer coefficients.

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