Stronger Schrödinger-like uncertainty relations

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In a recent work [Phys. Rev. Lett. 113, 260401 (2014)], L. Maccone and A. K. Pati presented two stronger uncertainty relations and an amended Heisenberg-Robertson uncertainty relation for incompatible observables. In this work we derive a pair of Schrödinger-like uncertainty relations for the product and sum of two variances. We also obtain a uncertainty relation for three observables and investigate its property for spin-1 particle state, which indicates that the new uncertainty relation may provide a stronger lower bound than the trivial extension of Schrödinger uncertainty relation.

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I. INTRODUCTION

One of the distinct features of quantum mechanics is quantum uncertainty. The initial spirit of quantum uncertainty was postulated by Heisenberg [1]. The Heisenberg’s uncertainty relation was mathematically derived by Kennard [2] and Weyl [3]. Quantum uncertainty includes uncertainty principle and uncertainty relation. The former places a restriction upon the degree to which one can constrain the likelihoods of future measurements made on a quantum system [4]. The latter refers to the repulsive nature of incompatible observables which induces a spread in the measurement outcomes, and does not refer to the disturbance induced by the measurement or to joint measurements [5].

In recent years, a growing number of theoretical [6–10] and experimental [11–17] studies have claimed to violate Heisenberg-type error-disturbance relation. Nevertheless, these claims have been criticized strongly [4, 18–21].

The best known modern formulation of uncertainty relation, the Heisenberg-Robertson uncertainty relation [22], bounds the product of the variances through the expectation value of the commutator

$$\Delta A \Delta B \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle |^2 ,$$

for any observables A, B, and any state |ψ⟩, where the variances of an observable O in state |ψ⟩ is defined as

$$\Delta O^2 = \langle \psi | O^2 | \psi \rangle - \langle \psi | O | \psi \rangle^2$$

and the commutator is defined as [A, B] = AB - BA. A stronger extension of the uncertainty relation [1] was made by Schrödinger [23], which is generally formulated as

$$\Delta A^2 \Delta B^2 \geq \left| \frac{1}{2} \langle \psi | [A, B] | \psi \rangle \right|^2 + \left| \frac{1}{2} \langle A, B \rangle - \langle A \rangle \langle B \rangle \right|^2 .$$

where the anticommutator is defined as {A, B} = AB + BA, and ⟨O⟩ is equal to ⟨ψ|O|ψ⟩ for any operator O.

The above two uncertainty relations want to quantitatively express the impossibility of jointly sharp preparation of incompatible observables. However, in practice, they cannot achieve this and not capture the notion of incompatible observables [24]. Recently, Maccone and Pati derived two stronger uncertainty relations based on the sum of variances $\Delta A^2 + \Delta B^2$ [24], which to a large extent can avoid the triviality problem, i.e. to be null on both sides of the inequality, and provide more stringent bounds for observables being incompatible on the quantum state. The first one reads

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle |^2 + |\langle \psi | A \pm i B | \psi^\perp \rangle |^2 ,$$

which is valid for arbitrary states |ψ^\perp⟩ orthogonal to the state of the system |ψ⟩, where the sign should be chosen so that ±i⟨[A, B]| (a real quantity) is positive. The second uncertainty relation is

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2} |\langle \psi | A + B | \psi \rangle |^2 .$$

Here $|\psi_{A+B}⟩$ is a state orthogonal to $|ψ⟩$. Maccone and Pati also derived an amended Heisenberg-Robertson uncertainty relation

$$\Delta A \Delta B \geq \frac{\pm i \langle [A, B] \rangle}{1 - \frac{1}{2} |\langle \psi | A \pm i B | \psi^\perp \rangle |^2} ,$$

which is stronger than the Heisenberg-Robertson uncertainty relation.

In this work we derive an improved Schrödinger-like uncertainty relation and an even stronger uncertainty relation than Maccone and Pati’s relations for the sum of variances. Using the same procedure, we also obtain an uncertainty relation for three observables and discuss briefly the implication of this new uncertainty relation in spin-1 particle system.
II. AMENDED SCHRÖDINGER UNCERTAINTY RELATION

The amended Schrödinger uncertainty relation reads
\[
\Delta A^2 \Delta B^2 \geq \frac{\left[ \frac{1}{2} \langle \{ A, B \} \rangle^2 + \frac{1}{2} \langle \{ A, B \} \rangle - \langle A \rangle \langle B \rangle \right]^2}{(1 - \frac{1}{2} \langle \psi \| \Delta_A \| \psi \rangle - e^{i \alpha} \frac{B}{\Delta_B} \langle \psi \| \psi^\perp \rangle^2)^2}, \tag{6}
\]
which is valid for arbitrary states \( |\psi^\perp \rangle \) orthogonal to the state of the system \( |\psi \rangle \) and stronger than the Schrödinger uncertainty relation \( (2) \). Here \( \alpha \) is a real constant, if \( \langle \{ A, B \} \rangle - 2\langle A \rangle \langle B \rangle > 0 \), then \( \alpha = \arctan \left( \frac{-i\langle [A, B] \rangle}{\langle [A, B] \rangle - 2\langle A \rangle \langle B \rangle} \right) \); if \( \langle \{ A, B \} \rangle - 2\langle A \rangle \langle B \rangle < 0 \), then \( \alpha = \pi + \arctan \left( \frac{-i\langle [A, B] \rangle}{\langle [A, B] \rangle - 2\langle A \rangle \langle B \rangle} \right) \); and while \( \langle \{ A, B \} \rangle - 2\langle A \rangle \langle B \rangle = 0 \), the relation \( (10) \) will reduce to \( (3) \).

**Proof:** To prove uncertainty relation \( (6) \), we start by introducing a general inequality
\[
||m\tilde{A}|\psi\rangle - ne^{i\tau}\tilde{B}|\psi\rangle + k(|\psi\rangle - |\tilde{\psi}\rangle)|^2 \geq 0, \quad (7)
\]
with \( \tilde{A} = A - \langle \psi|A|\psi\rangle, \tilde{B} = B - \langle \psi|B|\psi\rangle, \) and \( m, n, k \) and \( \tau \) being arbitrary real numbers. By expanding the square modulus, we have
\[
m^2\Delta A^2 + n^2\Delta B^2 \geq -\lambda k^2 - mn\beta k + mn\pi. \quad (8)
\]
Here, \( \Delta A^2 \) and \( \Delta B^2 \) are the variances of \( A \) and \( B \) calculated on \( |\psi\rangle \), respectively. \( \lambda = \frac{1}{2}(1 - \text{Re}[\langle \psi|\psi\rangle]), \beta = \frac{1}{2}\text{Re}[\langle \psi|(-A/n + e^{-i\tau}B/m)|\psi\rangle], \) and \( \pi = 2\text{Re}[e^{i\tau}\langle \psi|AB|\psi\rangle]. \) We choose the value of \( k \) that maximizes the right-hand-side of \( (8) \), namely \( k = -mn\beta/2\lambda \), and then get
\[
m^2\Delta A^2 + n^2\Delta B^2 \geq (mn\beta)^2/4\lambda + mn\pi. \quad (9)
\]
We can further choose \( m = \Delta B \) and \( n = \Delta A \) in above inequality, it then becomes
\[
\Delta A\Delta B \geq \frac{\Delta A\Delta B\{\text{Re}[\langle \psi| - A \Delta A + e^{-i\tau}B \Delta B|\tilde{\psi}\rangle]\}^2}{4(1 - \text{Re}[\langle \psi|\psi\rangle])} + \text{Re}[e^{i\tau}\langle \psi|\tilde{A}\tilde{B}|\psi\rangle]. \tag{10}
\]
Suppose \( |\tilde{\psi}\rangle = \cos\theta|\psi\rangle + e^{i\phi}\sin\theta|\psi^\perp\rangle \), where \( |\psi^\perp\rangle \) is orthogonal to \( |\psi\rangle \), by taking the limit \( \theta \to 0 \), the state \( |\tilde{\psi}\rangle \) reduces to \( |\psi\rangle \) and then the above inequality can be reexpressed as
\[
\Delta A\Delta B \geq \frac{1}{2}\Delta A\Delta B\{\text{Re}[e^{i\theta}|\psi\rangle - A \Delta A + e^{-i\tau}B \Delta B|\psi^\perp\rangle]\}^2 + \text{Re}[e^{i\tau}\langle \psi|\tilde{A}\tilde{B}|\psi\rangle]. \tag{11}
\]
There exists \( \tau = -\alpha \), so that \( e^{i\tau}\langle \psi|\tilde{A}\tilde{B}|\psi\rangle \) is real and can be written as \( \langle \tilde{A}\tilde{B} \rangle \), then the second term of \( (11) \) becomes \( \text{Re}[e^{i\theta}|\psi\rangle - A \Delta A + e^{-i\tau}B \Delta B|\psi^\perp\rangle]\}^2 \). Choosing proper phase \( \phi \) which makes this term in the square brackets to be real, so that this term can be expressed as \( |\langle \psi|\tilde{A}\tilde{B}|\psi^\perp\rangle|^2 \). Hence, the inequality \( (11) \) turns out to be
\[
\Delta A\Delta B \geq \frac{\langle \tilde{A}\tilde{B} \rangle}{1 - \frac{1}{2}\langle \psi|\Delta A + e^{-i\tau}B \Delta B|\psi^\perp\rangle} \tag{12}
\]
Of the quantity \( \langle \tilde{A}\tilde{B} \rangle \), it is easy to see that
\[
\langle \tilde{A}\tilde{B} \rangle = \frac{1}{2}\langle \{ A, B \} \rangle + \frac{1}{2}\langle \{ A, B \} \rangle - \langle A \rangle \langle B \rangle \tag{13}
\]
Therefore, the inequality \( (12) \) can be represented as
\[
\Delta A\Delta B \geq \frac{\langle \{ A, B \} \rangle + \frac{1}{2}\langle \{ A, B \} \rangle - \langle A \rangle \langle B \rangle}{1 - \frac{1}{2}\langle \psi|\Delta A + e^{-i\tau}B \Delta B|\psi^\perp\rangle} \tag{14}
\]
Hence, we get the improved Schrödinger-like uncertainty relation \( (10) \).

Given \( m = 1 \) and \( n = 1 \) in \( (17) \), using the same procedure described above, we can obtain a even stronger version of the uncertainty relation than \( (3) \), i.e.
\[
\Delta A^2 + \Delta B^2 \geq |\langle \{ A, B \} \rangle + \langle \{ A, B \} \rangle - 2\langle A \rangle \langle B \rangle | + |\langle A | - e^{-i\tau}B \rangle |^2 \tag{15}
\]
which is similar to Eq. (4) of Ref. \( (25) \) and Eq. (55) of Ref. \( (26) \). In case of \( \langle \{ A, B \} \rangle - 2\langle A \rangle \langle B \rangle = 0 \), the above inequality \( (15) \) then reduces to inequality \( (3) \). Removing the last term in \( (15) \), we find the inequality
\[
\Delta A^2 + \Delta B^2 \geq |\langle \{ A, B \} \rangle + \langle \{ A, B \} \rangle - 2\langle A \rangle \langle B \rangle | \tag{16}
\]
which is implied by the Schrödinger uncertainty relation \( (2) \).

III. UNCERTAINTY RELATION FOR THREE OBSERVABLES

A. New uncertainty relation

One may generalize the Schrödinger uncertainty relation \( (2) \) to three observables trivially, that is
\[
\Delta A^2 + \Delta B^2 + \Delta C^2 \geq \frac{1}{2} \left[ |\langle A, B \rangle + \langle A, B \rangle - 2\langle A \rangle \langle B \rangle | + |\langle B, C \rangle + \langle B, C \rangle - 2\langle B \rangle \langle C \rangle | + |\langle C, A \rangle + \langle C, A \rangle - 2\langle C \rangle \langle A \rangle | \right], \tag{17}
\]
which is simply the sum of the inequality \( (16) \). However, this inequality meets the triviality problem, as the lower bound can be null. Instead, we will prove that
following more stringent inequality exists:
\[
\Delta A^2 + \Delta B^2 + \Delta C^2 \geq \frac{1}{3} \langle \psi_{ABC}^\dagger | A + B + C | \psi \rangle^2
\]
\[
+ i \frac{\sqrt{3}}{3} \langle [A, B, C] \rangle + \frac{2}{3} \langle \psi | A + e^{+i\frac{\pi}{4}} B + e^{+i\frac{\pi}{4}} C | \psi^\perp \rangle^2 ,
\]
which is valid for arbitrary states $| \psi^\perp \rangle$ orthogonal to the state of the system $| \psi \rangle$, where
$| \psi_{ABC}^\perp \rangle \propto (A + B + C - (A + B + C)) | \psi \rangle,
\langle [A, B, C] \rangle \equiv \langle [A, B] \rangle + \langle [B, C] \rangle + \langle [C, A] \rangle$ and
the sign should be chosen so that $\pm i \frac{\sqrt{3}}{3} \langle [A, B, C] \rangle$ (a real quantity) is positive.

\textbf{Proof:} To prove this uncertainty relation, we start by a general inequality
\[
\| A | \psi \rangle + e^{\pm i \frac{\pi}{4}} B | \psi \rangle + e^{\pm i \frac{\pi}{4}} C | \psi \rangle + k(| \psi \rangle - | \psi^\perp \rangle) \|^2 \geq 0 ,
\]
where $\bar{A} = A - \langle \psi | A | \psi \rangle$, $\bar{B} = B - \langle \psi | B | \psi \rangle$, $\bar{C} = C - \langle \psi | C | \psi \rangle$, and $k$ is a real number. By expanding the square modulus, we have
\[
\Delta A^2 + \Delta B^2 + \Delta C^2 \geq - \lambda k^2 + \beta k - \pi ,
\]
where we define
\[
\beta = 2 \text{Re} (\langle \psi | A | \psi \rangle + e^{\pm i \frac{\pi}{4}} \langle \psi | B | \psi \rangle + e^{\pm i \frac{\pi}{4}} \langle \psi | C | \psi \rangle)
\]
\[
= 2 \text{Re} (\langle \psi | \bar{A} + e^{\pm i \frac{\pi}{4}} \bar{B} + e^{\pm i \frac{\pi}{4}} \bar{C} | \psi \rangle) ,
\]
\[
\lambda = 2(1 - \text{Re} \langle \psi | \psi \rangle) ,
\]
and
\[
\pi = 2 \text{Re} (e^{\pm i \frac{\pi}{4}} \langle AB \rangle + e^{\pm i \frac{\pi}{4}} \langle AC \rangle + e^{\pm i \frac{\pi}{4}} \langle BC \rangle)
\]
\[
= - \frac{1}{2} \langle [A, B, C] \rangle \pm i \frac{\sqrt{3}}{2} \langle [A, B, C] \rangle
\]
\[
+ \langle A | B \rangle + \langle A | C \rangle + \langle B | C \rangle ,
\]
with $\langle [A, B, C] \rangle \equiv \langle [A, B] \rangle + \langle [A, C] \rangle + \langle [B, C] \rangle$. Calculating $\Delta (A + B + C)^2$, we find
\[
\Delta (A + B + C)^2
\]
\[
= \Delta A^2 + \Delta B^2 + \Delta C^2 + 2 \langle [A, B, C] \rangle - 2 \langle (A \langle B \rangle + \langle A | C \rangle + \langle B | C \rangle \rangle .
\]
\[
(24)
\]
Using the above equality, Eq. (24) can be rewritten as
\[
\pi = - \frac{1}{2} \Delta (A + B + C)^2 - (\Delta A^2 + \Delta B^2 + \Delta C^2)
\]
\[
\pm i \frac{\sqrt{3}}{2} \langle [A, B, C] \rangle .
\]
Assuming $| \psi \rangle = \cos \theta | \psi \rangle + e^{i\phi} \sin \theta | \psi^\perp \rangle$ and using the same techniques employed to derive (15), we obtain
\[
\Delta A^2 + \Delta B^2 + \Delta C^2 \geq \frac{1}{3} \Delta (A + B + C)^2
\]
\[
+ i \frac{\sqrt{3}}{3} \langle [A, B, C] \rangle + \frac{2}{3} \langle \psi | A + e^{\pm i \frac{\pi}{4}} B + e^{\pm i \frac{\pi}{4}} C | \psi^\perp \rangle^2 ,
\]
which is equivalent to relation (18) since $\Delta (A + B + C)^2 = \langle \psi_{ABC}^\dagger | A + B + C | \psi \rangle^2$.

Recently, the variance-based uncertainty equalities were introduced by Yao et al. [26] for all pairs of incompatible observables $A$ and $B$. Similarly, the uncertainty equality for three observables can be written as follows
\[
\Delta A^2 + \Delta B^2 + \Delta C^2 \geq \frac{1}{3} \langle \psi_{ABC}^\dagger | A + B + C | \psi \rangle^2
\]
\[
+ i \frac{\sqrt{3}}{3} \langle [A, B, C] \rangle + \frac{2}{3} \sum_{n=1}^{d-1} \langle (A + e^{\pm i \frac{\pi}{4}} B + e^{\pm i \frac{\pi}{4}} C | \psi^\perp \rangle^2 ,
\]
(27)
where $\{ | \psi \rangle, | \psi^\perp \rangle \}_{n=1}^{d-1}$ form an orthonormal and complete basis in $d$-dimensional Hilbert space. If we retain only one term associated with $| \psi^\perp \rangle \in \{ | \psi^\perp \rangle \}_{n=1}^{d-1}$ in the summation and discard the others, the uncertainty equality (27) reduces to the uncertainty inequality (18).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure1.png}
\caption{(Color online) Uncertainty relation for three observables. The figures illustrate how the different uncertainty relations \[17\] and \[18\] restrict the possible values of the sum of variances in different values of $\phi$ (\(\phi = 0\) in (a) and \(\phi = \pi/4\) in (b)). The upper red curve is $\Delta J^2 + \Delta J^2 + \Delta J^2$, the blue points are the calculation of the bound \[18\] for 15 randomly chosen states $| \psi^\perp \rangle$ for each of the 200 values of the phase $\theta$ depicted, dash-dotted green curve is the bound of the trivially generalized Schrödinger uncertainty relation \[17\] for three observables.}
\end{figure}

\section{B. Application to spin-1 particle state}

As an illustration of the new uncertainty relation (18), we consider a simple case of spin-1 particle state. Let
A = J_x, B = J_y and C = J_z to be the three components of angular momentum. We may choose

\[ |\psi\rangle = \sin \theta \cos \phi |1\rangle + \sin \theta \sin \phi |0\rangle + \cos \theta |-1\rangle, \tag{28} \]

and

\[ |\psi^\perp\rangle = (\cos \theta \cos \phi \cos \beta e^{i\gamma} - \sin \phi \sin \beta)|1\rangle + (\cos \theta \sin \phi \cos \beta e^{i\gamma} + \cos \phi \sin \beta)|0\rangle - (\sin \theta \cos \beta e^{i\gamma})|\pm 1\rangle, \tag{29} \]

with |±1⟩ and |0⟩ eigenstates of Jz corresponding to the eigenvalues ±1 and 0. Then there will be infinite number of orthogonal states of system depending on the values of β and γ.

For such a choice of |ψ⟩, |ψ^\perp⟩, A, B and C, we compare numerically the new uncertainty [15] with the trivially generalized Schrödinger uncertainty relation [17] for three observables in Fig. 4.

For \( \phi = 0 \), [17] can be represented as

\[ \frac{1}{2} (3 - \cos (4\theta)) \geq |\cos (2\theta)| \tag{30} \]

Removing the last term in [18], it then writes

\[ \frac{1}{2} (3 - \cos (4\theta)) \geq \frac{1}{6} \left( 2\sqrt{3} |\cos (2\theta)| - \cos (4\theta) + 3 \right), \tag{31} \]

because state |ψ^\perp⟩ in [18] is arbitrary state orthogonal to the state of the system |ψ⟩. The blue points in Fig. 4 illustrate the bound of [18] for 15 randomly taking state |ψ^\perp⟩ for each of the 200 values of the phase \( \theta \) depicted. We find the uncertainty [18] is non-trivial for all \( \theta \) and stronger than the trivially generalized Schrödinger uncertainty relation [17].

For \( \phi = \pi/4 \), while \( \theta \in (0, 0.3067) \cup (0.6991, \pi) \), the uncertainty relation [18] is also stronger than the generalized Schrödinger uncertainty relation [17]. And further more, if one chooses properly the |ψ^+⟩, relation [18] is always stronger than [17] for any values of \( \theta \).

IV. CONCLUSIONS

In this work, we derived two new Schrödinger-like uncertainty relations for product and sum of variances of a pair of incompatible observables, which are in general stronger than previous ones. Using the same procedure, we also derived an uncertainty relation for three observables and discussed its implication for spin-1 particle system, which states that the new uncertainty relation is stronger than the trivially generalized Schrödinger uncertainty relation.

Note added: As this work was finished, there appears a comment [22] mentioning that Eq. (3) of Ref. [24] may still experience triviality problem in special case while the state |ψ^±⟩ = \frac{A − \langle A |\psi⟩}{\Delta A} or |ψ^±⟩ = \frac{B − \langle B |\psi⟩}{\Delta B}, which means the uncertainty relations in this work also have such drawback. These uncertainty relations are state dependent, but some uncertainty relations [27, 28] are quantum state independent and hence immune from the triviality problem.

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