PARTIAL CONVEXITY TO THE HEAT EQUATION

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Abstract. In this paper, we study the partial convexity of smooth solutions to the heat equation on a compact or complete non-compact Riemannian manifold $M$ or on Kaehler-Ricci flow. We show that under a natural assumption, a new partial convexity property for smooth solutions to the heat equation is preserved.

1. Introduction

It is always interesting for people to find invariant sets for evolution equations. Invariant sets of geometric evolution equations can be defined by related monotonicity quantities. In the recent study of Ricci flow, G. Perelman [17] finds new monotonicity formulae. In this paper, we study invariant sets of the heat flow by the maximum principle trick. More precisely, we are concerned with partial convexity of solutions to the heat equation (associated to Ricci flow or Ricci-Kahler flow) by using the maximum principle method. Consider a smooth solution to the heat equation

\[ u_t = \Delta u, \quad \text{on } M \times [0, T) \]  

on the compact or complete Riemannian manifold $(M^n, g)$ of dimension $n$. Here $\Delta$ is the Laplacian operator of the metric $g$ (with the sign such that $\Delta u = u''$ on the real line $\mathbb{R}$). By definition, for some positive integer $1 \leq k \leq n$, we say that $u$ is partially $k$-convex on $M \times [0, T)$ if we have that the positivity of the function

\[ \sigma_j(u) > 0, \quad j = 1, \ldots, k, \]

where $\sigma_j(u)$ is the $j$-th elementary symmetric polynomial of the hessian matrix $D^2 u$ on $(M, g)$, of the solution $u$ is preserved along the heat equation [1]. We also study the corresponding problem for heat equation associated to the Kaehler-Ricci flow.

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Let’s just mention some earlier related research. In the previous study, R.Hamilton [8] has extended the famous Li-Yau [12] gradient estimate of the heat equation and has obtained the matrix Harnack inequality for the heat equation on Riemannian manifolds. Cao and Ni [4] have studied the matrix Harnack inequality for the heat equation along the Kahler-Ricci flow. Some years ago, Brascamp and E.Lieb etc [2] have studied the log convexity of the solution to the heat equation on convex domains, and soon after, S.T.Yau has found the continuity argument for convexity of solution (see the appendix in [20], also [10], [22] for this point and we shall use this kind of argument). L.Caffarelli and A.Friedman [3], P.L.Lions [14], etc, have studied the convexity of more general elliptic and parabolic equations. We refer to the works of Korevaar [10] and Kawohl [11] for more results and references.

To state our results, we make some conventions. Let $A = (u_{ij})$ be the Hessian matrix of the smooth function $u$. Here, we have denoted by $\nabla_i f = f_i$, in local coordinates $(x^i)$, the $i$-th covariant derivative of a smooth function $f$ on $M$, and $f_{ij}$ the corresponding second covariant derivatives.

Take a fixed point $x \in M$ and normal coordinates $(x^i)$ at $x \in M$. We now consider some elementary algebra on the tangent space $T_x M$. Note that at this point $x$ we have $(g_{ij}) = I$, which is the identity. For $k = 1, 2, ..., n$, let $\sigma_k(A)$ be the $k$-th elementary symmetric polynomial of $A$. For example,

$$\sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j$$

for the diagonal matrix $A = (\lambda_1 \oplus ... \oplus \lambda_n)$. The $k$-th Newton transformation associated to $A$ is

$$T_k(A) = \sigma_k(A) I - \sigma_{k-1}(A) A + ... + (-1)^k A^k.$$

In particular, we have

$$T_1(A) = \sigma_1(A) I - A.$$

Note that $\sigma_1(A) = \Delta u$.

Let $A(s)$ be a smooth one-parameter family of symmetric matrices on $T_x M$. Then we have

$$\frac{d}{ds} \sigma_k(A(s)) = \text{trace}(T_{k-1}(A(s)) \circ \frac{d}{ds} A(s)).$$
Hence, we have
\[
\frac{d}{ds}\sigma_2(A(s)) = \text{trace}(T_1(A(s)) \circ \frac{d}{ds}A(s))
\]
\[
= \sigma_1(A)\frac{d}{ds}\sigma_1(A) - \text{trace}(A(s) \circ \frac{d}{ds}A(s))
\]

One may see [18] for more relations. Sometimes, we also denote by \(\sigma_k(u) = \sigma_k(A)\). Clearly, one can define similar concepts on Kaehler manifolds.

For the heat equation associated to the Kaelher-Ricci flow, we have the following result

**Theorem 1.** Let \((M, g(t))\) be a compact or complete non-compact Kaehler manifold of dimension \(n\), where \((g(t)), 0 \leq t < T,\) is a Kaehler-Ricci flow with bounded curvature. Assume that the holomorphic bi-sectional curvature of each \(g(t)\) is non-negative such that
\[
-u_{\beta\bar{\alpha}}R_{\alpha\beta\gamma\delta}u_{\gamma\delta} + u_{\beta\alpha}R_{\alpha\delta}u_{\gamma\delta} \geq 0,
\]
for any hermitian matrix \((u_{\gamma\delta})\). Let \(\Delta_{g(t)}\) be the Laplacian of \(g(t)\), \(0 \leq t < T\). Let \(u\) be a smooth solution to
\[
u_t = \Delta_{g(t)}u
\]
on \(M^n \times [0, T)\) with nice decay at infinity when \(M\) noncompact. Let \(A = (u_{\alpha\beta})\) be the hessian of \(u\) with \(T_\alpha\) and \(\sigma_\alpha, \alpha = 1, 2,\) defined as above. Then we have
\[
\sigma_1(A) \geq (>)0, \text{ for } t > 0
\]
provided \(\sigma_1(A) \geq (>)0\) at \(t = 0\). Furthermore, we have partial 2-convexity of the solution \(u\); that is, the positivity of the function \(\sigma_2(u)\) is also preserved provided it is positive at \(t = 0\).

By definition, a smooth function \(f \in C^2(M)\) has a nice decay at infinity if \(|f|(x) + |\nabla f|(x) + |D^2 f(x)| \to 0\) as the distance \(d(x, 0) \to \infty\) for some fixed point \(o \in M\). We recall here that \(g(t)\) is a Kaehler-Ricci flow on the manifold \(M\) if in local complex coordinates \((z^\alpha)\), we have
\[
\partial_t g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}, \text{ on } M \times (0, T).
\]

We remark that in the proof of Theorem 1 in section 2, we only use condition 2 for any hermitian symmetric matrix \((u_{ij})\) in place of the non-negative holomorphic bi-sectional curvature. In particular, the assumption 5 is automatically true on standard \(n\) dimensional complex projective space \(CP^n\). We believe that one may apply Theorem 1 to the study of Kaehler-Ricci flow.
To explain the idea in the proof of Theorem 1, let’s assume that we are studying periodic or nice decay solutions to the heat equation on $\mathbb{R}^n$. In this case, we have

$$A_t = \Delta A.$$ 

Then we have

(3) $$\sigma_1(A)_t = \Delta \sigma_1(A).$$ 

Hence, assuming $\sigma_1(A) > 0$ at $t = 0$, by the standard maximum principle, the positivity $\sigma_1(u)$ is preserved along the heat equation. Here we have assumed that $u(., t)$ has a nice decay at infinity for each $t \in [0, T)$. In fact, this has been proved in [5]. We may make similar computation at least formally for higher order symmetric elementary functions. Since,

$$\partial_t \sigma_k(A) = \text{trace}(T_{k-1}(A) \circ \partial_t A),$$

see (2.3) in [7], and

$$\Delta \sigma_k(A) = \text{trace}(\nabla_i T_{k-1}(A) \circ \nabla_i A) + \text{trace}(T_{k-1}(A) \circ \Delta A),$$

we have

$$(\partial_t - \Delta) \sigma_k(A) = -\text{trace}(\nabla_i T_{k-1}(A) \circ \nabla_i A) + \text{trace}(T_{k-1}(A) \circ (\partial_t - \Delta) A)$$

$$= -\text{trace}(\nabla_i T_{k-1}(A) \circ \nabla_i A).$$

Then by the maximum principle, we have the following

**Proposition 2.** Let $u$ be a smooth solution to (1) on $\mathbb{R}^n \times [0, T)$ with nice decay at infinity. Let $A = (u_{ij})$ be the hessian of $u$ with $T_k$ and $\sigma_k$ defined above. Assume that

(4) $$\text{trace}(\nabla_i T_{k-1}(A) \circ \nabla_i A) \leq 0, \quad \text{on } M \times [0, T).$$

Then we have

$$\sigma_k(A) \geq (>0), \quad \text{for } t > 0$$

provided $\sigma_k(A) \geq (>0)$ at $t = 0$.

We remark that the assumption (4) may be difficult to verify in applications. However, we can find some partial convexity condition for the heat equation. In fact, we have

$$(\partial_t - \Delta) \sigma_2(A) = -\text{trace}(\nabla_i T_1(A) \circ \nabla_i A)$$

$$= -\text{trace}(\nabla_i \sigma_1(A) \nabla_i A) + \text{trace}(\nabla_i A \circ \nabla_i A)$$

$$= -|\nabla \sigma_1(A)|^2 + \text{trace}(\nabla_i A \circ \nabla_i A)$$

$$\geq -|\nabla \sigma_1(A)|^2.$$ 

By (5), we have

$$(\partial_t - \Delta) \sigma_1(A)^2 / 2 = -|\nabla \sigma_1(A)|^2.$$
Set \( F = \sigma_1(A)^2/2 \) and \( H = \sigma_2(A)/F \). Then, using the fact that \( \sigma_1^2 \geq 2\sigma_2 \) and the formula

\[
L\left(\frac{u}{v}\right) = \frac{Lu}{v} - \frac{uLv}{v^2} - \frac{2u}{v^3} |\nabla v|^2 + \frac{2}{v^2}(\nabla u, \nabla v)
\]

where \( L = \partial_t - \Delta \), we obtain that

\[
(\partial_t - \Delta)H \geq 2 < \nabla H, \nabla F > /F,
\]

and we get by the maximum principle that

\[
\sigma_2(A)/F \geq (>0), \quad \text{for } t > 0
\]

provided \( \sigma_2(A)/F \geq (>0) \) at \( t = 0 \). Hence, assuming \( \sigma_1(A) > 0 \), the positivity property of \( \sigma_2(A) \) is preserved along the heat equation. Hence, we have

**Proposition 3.** Let \( u \) be a smooth solution to (7) on \( \mathbb{R}^n \times [0, T) \) with nice decay at infinity. Let \( A = (u_{ij}) \) be the hessian of \( u \) with \( T_k \) and \( \sigma_k \) defined above. Assume that \( \sigma(A) > 0 \) and \( \sigma_1(A) > 0 \) at \( t = 0 \). Then we have

\[
\sigma_2(A) \geq (>0), \quad \text{for } t > 0.
\]

For the heat equation in the Riemannian case when \( g \) being a fixed metric, we have the following result whose assumption is similar to Corollary 4.4 in \[8\].

**Theorem 4.** Let \( (M, g) \) be a compact or complete noncompact Riemannian manifold of dimension \( n \) with non-negative sectional curvature and parallel Ricci curvature tensor. Assume further that

\[
-2u_{ij}R_{ikjl}u_{kl} + 2u_{ij}R_{ji}u_{il} \geq 0, \quad \text{on } M,
\]

for any symmetric matrix \( (u_{ij}) \). Let \( \Delta \) be the Laplacian of \( g \). Let \( u \) be a smooth solution to

\[
u_t = \Delta u
\]

on \( M^n \times [0, T) \) with nice decay at infinity when \( M \) is complete and noncompact. Let \( A = (u_{ij}) \) be the hessian of \( u \) with \( T_k \) and \( \sigma_k \) defined above. Then we have

\[
\sigma_1(A) \geq (>0), \quad \text{for } t > 0
\]

provided \( \sigma_1(A) \geq (>0) \) at \( t = 0 \). Furthermore, we have partial 2-convexity of the solution \( u \); that is, the positivity of the function

\[
\sigma_2(u)
\]

is preserved provided it is positive at \( t = 0 \).
We remark that in the proof of Theorem 4, we only use condition 5 for any symmetric matrix \((u_{ij})\), not the non-negative section curvature. In particular, the assumption \((5)\) is automatically true on standard \(n\)-sphere \(S^n\).

2. PROOF OF THEOREM 1

Let \((M^m, g(t))\) be a compact Kaehler manifold, where \(g(t)\) is a Kaehler-Ricci flow in the sense that
\[
\partial_t g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}.
\]
Let \(\Delta = \Delta_{g(t)}\) be the Laplacian of the metric \(g(t)\). Assume that \(u \in C^2(M \times [0, T])\) satisfies the heat equation
\[
(\partial_t - \Delta)u = 0, \text{ on } M_T
\]
where \(M_T = M \times [0, T]\). Doing the computation as in Lemma 2.1 in \((16)\), we have
\[
(\partial_t - \Delta)u_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}\gamma\delta}u_{\gamma\delta} - \frac{1}{2}(R_{\alpha\bar{s}\gamma\delta}u_{\gamma\delta} + u_{\alpha\bar{s}}R_{\bar{s}\beta})
\]
Let \(A = (u_{\alpha\bar{\beta}})\). Then we have
\[
(\partial_t - \Delta)\text{trace}(A) = 0,
\]
which implies that \(\sigma_1(0) > 0(\geq 0)\) is preserved on the heat equation, and then
\[
\text{trace}(A \circ (\partial_t - \Delta)A) = u_{\beta\bar{\alpha}}R_{\alpha\bar{\beta}\gamma\delta}u_{\gamma\delta} - u_{\beta\bar{\alpha}}R_{\alpha\bar{s}}u_{s\beta}.
\]
Hence, we have
\[
(\partial_t - \Delta)\sigma_2(A)
= -\text{trace}(\nabla_a T_1(A) \circ \nabla_{\bar{a}}A) + \text{trace}(T_1(A) \circ (\partial_t - \Delta)A)
= -|\nabla \sigma_1(A)|^2 + \text{trace}(\nabla_a (A) \circ \nabla_{\bar{a}}A) + \sigma_1(A)\text{trace}((\partial_t - \Delta)A)
- \text{trace}(A \circ (\partial_t - \Delta)A)
= -|\nabla \sigma_1(A)|^2 + \text{trace}(\nabla_a (A) \circ \nabla_{\bar{a}}A) - u_{\beta\bar{\alpha}}R_{\alpha\bar{\beta}\gamma\delta}u_{\gamma\delta} + u_{\beta\bar{\alpha}}R_{\alpha\bar{s}}u_{s\beta}.
\]
Note that by our assumption \((2)\),
\[
-u_{\beta\bar{\alpha}}R_{\alpha\bar{\beta}\gamma\delta}u_{\gamma\delta} + u_{\beta\bar{\alpha}}R_{\alpha\bar{s}}u_{s\beta} \geq 0.
\]
Then, using the same trick as in what we did in Proposition 3, we get the partial convexity for \(\sigma_2\) by the maximum principle.
3. Proof of Theorem 4

Recall the second contracted Bianchi identity that
\[ \nabla_k R_{ikl} + \nabla_j R_{il} - \nabla_l R_{ij} = 0. \]

Using the Ricci formula and the identity above, we can compute that
\[ \nabla_i \nabla_j (\Delta u) = \Delta \nabla_i \nabla_j u + 2 R_{kijl} \nabla_k \nabla_l u - R_{il} \nabla_j \nabla_l u - R_{jl} \nabla_i \nabla_l u \]
\[ - (\nabla_j R_{gl} + \nabla_j R_{il} - \nabla_l R_{ij}) \nabla_l u. \]

In fact, we have
\[ \nabla_i \nabla_j (\Delta u) = \nabla_i (u_{jkk} - R_{jli} u_l) \]
\[ = u_{jkk} - \nabla_i R_{jli} u_l - R_{jli} u_i \]
\[ + u_{jkk} - R_{ikjm} u_{km} - R_{im} u_{mj} - \nabla_i R_{jli} u_l - R_{jli} u_i \]
\[ = \nabla_k (u_{jik} - R_{ikjm} u_{km}) - R_{ikjm} u_{km} - R_{im} u_{mj} - \nabla_i R_{jli} u_l - R_{jli} u_i \]
\[ = u_{iikk} - \nabla_k R_{ikjm} u_{km} - R_{ikjm} u_{mk} \]
\[ - R_{ikjm} u_{km} - R_{im} u_{mj} - \nabla_i R_{jli} u_l - R_{jli} u_i. \]

So, we get (6) by using the second Bianchi identity above.

By our assumption that \( \nabla_j R_{il} = 0 \), we have
\[ \nabla_i \nabla_j (\Delta u) = \Delta \nabla_i \nabla_j u + 2 R_{kijl} \nabla_k \nabla_l u - R_{il} \nabla_j \nabla_l u - R_{jl} \nabla_i \nabla_l u. \]

Hence we have
\[ \partial_t u_{ij} = \Delta u_{ij} + 2 R_{kijl} u_{kl} - R_{il} u_{jl} - R_{jl} u_{il}. \]

Set \( A = (u_{ij}) \). Again we have that
\[ \text{tr}((\partial_t - \Delta) A) = 0, \]
which also gives us that \( \sigma_1(u) > 0 \) is preserved on the heat equation. We now have that
\[ (\partial_t - \Delta) \sigma_2(A) \]
\[ = -\text{tr}((\nabla_i T_1(A) \circ \nabla_i A) + \text{tr}(T_1(A) \circ (\partial_t - \Delta) A) \]
\[ = -|\nabla \sigma_1(A)|^2 + \text{tr}(\nabla_i A \circ \nabla_i A) + \sigma_1(A) \text{tr}((\partial_t - \Delta) A) \]
\[ - \text{tr}(A \circ (\partial_t - \Delta) A) \]
\[ = -|\nabla \sigma_1(A)|^2 + \text{tr}(\nabla_i A \circ \nabla_i A) - 2 u_{ij} R_{ikjl} u_{kl} + 2 u_{ij} R_{jl} u_{il}. \]
Note that by our assumption (5), we always have that
\[-2u_{ij}R_{ikjl}u_{kl} + 2u_{ij}R_{jl}u_{il} \geq 0.\]

Hence, we have
\[(\partial_t - \Delta)\sigma_2(A) \geq -|\nabla \sigma_1(A)|^2.\]

Then, using again the same trick as in what we did in Proposition 3, we get the desired partial convexity by the maximum principle.

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