Semi-direct product of groups, filter banks and sampling

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Abstract

An abstract sampling theory associated to a unitary representation of a countable discrete non abelian group \( G \), which is a semi-direct product of groups, on a separable Hilbert space is studied. A suitable expression of the data samples and the use of a filter bank formalism allows to fix the mathematical problem to be solved: the search of appropriate dual frames for \( \ell^2(G) \). An example involving crystallographic groups illustrates the obtained results by using average or pointwise samples.

Keywords: Semi-direct product of groups; unitary representation of a group; LCA groups; dual frames; sampling expansions.
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1 Statement of the problem

In this paper an abstract sampling theory associated to non abelian groups is derived for the specific case of a unitary representation of a semi-direct product of groups on a separable Hilbert space. Semi-direct product of groups provide important examples of non abelian groups such as dihedral groups, infinite dihedral group, euclidean motion groups or crystallographic groups. Concretely, let \((n,h) \mapsto U(n,h)\) be a unitary representation on a separable Hilbert space \(\mathcal{H}\) of a semi-direct product \(G = N \rtimes_\phi H\), where \(N\) is a countable discrete LCA (locally compact abelian) group, \(H\) is a finite group, and \(\phi\) denotes the action of the group
H on the group N (see Section 2 infra for the details); for a fixed a ∈ H we consider the U-invariant subspace in H

\[ \mathcal{A}_a = \left\{ \sum_{(n,h) \in G} \alpha(n,h) U(n,h)a : \{\alpha(n,h)\}_{(n,h) \in G} \in \ell^2(G) \right\}, \]

where we assume that \{U(n,h)a\} is a Riesz sequence for H, i.e., a Riesz basis for \mathcal{A}_a (see Ref. 2 for a necessary and sufficient condition). Given K elements \(b_k\) in H, which do not belong necessarily to \(\mathcal{A}_a\), the main goal in this paper is the stable recovery of any \(x \in \mathcal{A}\) from the given data (generalized samples)

\[ \mathcal{L}_k x(n) := \langle x, U(n,1_H)b_k \rangle_H, \quad n \in \mathbb{N} \text{ and } k = 1, 2, \ldots, K, \]

where \(1_H\) denotes the identity element in H. These samples are nothing but a generalization of average sampling in shift-invariant subspaces of \(L^2(\mathbb{R}^d)\); see, among others, Refs. 11 6 8 9 13 14 15 16. The case where G is a discrete LCA group and the samples are taken at a uniform lattice of \(G\) has been solved in Ref. 11; this work relies on the use of the Fourier analysis in the LCA group G. In the case involved here a Fourier analysis is not available and, consequently, we need to overcome this drawback.

Having in mind the filter bank formalism in discrete LCA groups (see, for instance, Refs. 3 5 10), the given data \(\{\mathcal{L}_k x(n)\}_{n \in \mathbb{N}; k=1,2,\ldots,K}\) can be expressed as the output of a suitable K-channel analysis filter bank corresponding to the input \(\alpha = \{\alpha(n,h)\}_{(n,h) \in G}\) in \(\ell^2(G)\). As a consequence, the problem consists of finding a synthesis part of the former filter bank allowing perfect reconstruction; besides only Fourier analysis on the LCA group \(N\) is needed. Then, roughly speaking, substituting the output of the synthesis part in \(x = \sum_{(n,h) \in G} \alpha(n,h) U(n,h)a\) we will obtain the corresponding sampling formula in \(\mathcal{A}_a\).

This said, as it could be expected the problem can be mathematically formulated as the search of dual frames for \(\ell^2(G)\) having the form

\[ \{T_n h_k\}_{n \in \mathbb{N}; k=1,2,\ldots,K} \text{ and } \{T_n g_k\}_{n \in \mathbb{N}; k=1,2,\ldots,K}. \]

Here \(h_k, g_k \in \ell^2(G), T_n h_k(m,h) = h_k(m-n,h)\) and \(T_n g_k(m,h) = g_k(m-n,h), (m,h) \in G,\) where \(n \in \mathbb{N}\) and \(k = 1, 2, \ldots, K.\) Besides, for any \(x \in \mathcal{A}_a\) we have the expression for its samples

\[ \mathcal{L}_k x(n) = \langle \alpha, T_n h_k \rangle_{\ell^2(G)}, \quad n \in \mathbb{N} \text{ and } k = 1, 2, \ldots, K. \]

Needless to say that frame theory plays a central role in what follows; the necessary background on Riesz bases or frame theory in a separable Hilbert space can be found, for instance, in Ref. 4. Finally, sampling formulas in \(\mathcal{A}_a\) having the form

\[ x = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n,1_H)c_k \quad \text{in } H, \]

for some \(c_k \in \mathcal{A}_a, k = 1, 2, \ldots, K,\) will come out by using, for \(g \in \ell^2(G)\) and \(n \in \mathbb{N},\) the shifting property \(\mathcal{T}_{U,a}(T_n g) = U(n,1_H) (\mathcal{T}_{U,a} g)\) that satisfies the natural isomorphism \(\mathcal{T}_{U,a} : \ell^2(G) \to \mathcal{A}_a\) which maps the usual orthonormal basis \(\{\delta_{(n,h)}\}_{(n,h) \in G}\) for \(\ell^2(G)\) onto the Riesz basis \(\{U(n,h)a\}_{(n,h) \in G}\) for \(\mathcal{A}_a\). All these steps will be carried out throughout the
remaining sections. For the sake of completeness, Section 2 includes some basic preliminaries on semi-direct product of groups and Fourier analysis on LCA groups. The paper ends with an illustrative example involving the quasi regular representation of a crystallographic group on $L^2(\mathbb{R}^d)$; sampling formulas involving average or pointwise samples are obtained for the corresponding $U$-invariant subspaces in $L^2(\mathbb{R}^d)$.

2 Some mathematical preliminaries

In this section we introduce the basic tools in semi-direct product of groups and in harmonic analysis in a discrete LCA group that they will be used in the sequel.

2.1 Preliminaries on semi-direct product of groups

Given groups $(N, \cdot)$ and $(H, \cdot)$, and a homomorphism $\phi : H \to Aut(N)$ their semi-direct product $G := N \rtimes_\phi H$ is defined as follows: The underlying set of $G$ is the set of pairs $(n, h)$ with $n \in N$ and $h \in H$, along with the multiplication rule

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 \phi_{h_1}(n_2), h_1h_2), \quad (n_1, h_1), (n_2, h_2) \in G,$$

where we denote $\phi(h) := \phi_h$; usually the homomorphism $\phi$ is referred as the action of the group $H$ on the group $N$. Thus we obtain a new group with identity element $(1_N, 1_H)$, and inverse $(n, h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1})$.

Besides, we have the isomorphisms $N \cong N \times \{1_H\}$ and $H \cong \{1_N\} \times H$. Unless $\phi_h$ equals the identity for all $h \in H$, the group $G = N \rtimes_\phi H$ is not abelian, even for abelian $N$ and $H$ groups. In case $N$ is an abelian group, it is a normal subgroup in $G$. Next we list some examples of semi-direct product of groups:

1. The dihedral group $D_{2N}$ is the group of symmetries of a regular $N$-sided polygon; it is the semi-direct product $D_{2N} = \mathbb{Z}_N \rtimes_\phi \mathbb{Z}_2$ where $\phi_0 \equiv Id_{\mathbb{Z}_N}$ and $\phi_1(\bar{n}) = -\bar{n}$ for each $\bar{n} \in \mathbb{Z}_N$. The infinite dihedral group $D_{\infty}$ defined as $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ for the similar homomorphism $\phi$ is the group of isometries of $\mathbb{Z}$.

2. The Euclidean motion group $E(d)$ is the semi-direct product $\mathbb{R}^d \rtimes_\phi O(d)$, where $O(d)$ is the orthogonal group of order $d$ and $\phi(A)(x) = Ax$ for $A \in O(d)$ and $x \in \mathbb{R}^d$. It contains as a subgroup any crystallographic group $M\mathbb{Z}^d \rtimes_\phi \Gamma$, where $M\mathbb{Z}^d$ denotes a full rank lattice of $\mathbb{R}^d$ and $\Gamma$ is any finite subgroup of $O(d)$ such that $\phi_\gamma(M\mathbb{Z}^d) = M\mathbb{Z}^d$ for each $\gamma \in \Gamma$.

3. The orthogonal group $O(d)$ of all orthogonal real $d \times d$ matrices is isomorphic to the semi-direct product $SO(d) \rtimes_\phi C_2$, where $SO(d)$ consists of all orthogonal matrices with determinant 1 and $C_2 = \{I, R\}$ a cyclic group of order 2; $\phi$ is the homomorphism given by $\phi_1(A) = A$ and $\phi_R(A) = RAR^{-1}$ for $A \in SO(d)$.

Suppose that $N$ is an LCA group with Haar measure $\mu_N$ and $H$ is a locally compact group with Haar measure $\mu_H$. Then, the semi-direct product $G = N \rtimes_\phi H$ endowed with the product topology becomes also a topological group. For the left Haar measure on $G$ see Ref. 2.
2.2 Some preliminaries on harmonic analysis on discrete LCA groups

The results about harmonic analysis on locally compact abelian (LCA) groups are borrowed from Ref. [7]. Notice that, in particular, a countable discrete abelian group is a second countable Hausdorff LCA group.

For a countable discrete group \((N, \cdot)\), non necessarily abelian, the convolution of \(x, y : N \to \mathbb{C}\) is formally defined as \((x * y)(m) := \sum_{n \in N} x(n)y(n^{-1}m), \ m \in N\). If in addition the group is abelian, therefore denoted by \((N, +)\), the convolution reads as

\[
(x * y)(m) := \sum_{n \in N} x(n)y(m - n), \ m \in N.
\]

Let \(\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}\) be the unidimensional torus. We said that \(\xi : N \mapsto \mathbb{T}\) is a character of \(N\) if \(\xi(n + m) = \xi(n)\xi(m)\) for all \(n, m \in N\). We denote \(\xi(n) = \langle n, \xi \rangle\). Defining \((\xi + \gamma)(n) = \xi(n)\gamma(n)\), the set of characters \(\hat{N}\) with the operation \(+\) is a group, called the dual group of \(N\); since \(N\) is discrete \(\hat{N}\) is compact [7, Prop. 4.4]. For \(x \in \ell^1(N)\) we define its Fourier transform as

\[
X(\xi) = \hat{x}(\xi) := \sum_{n \in N} x(n)\langle n, \xi \rangle = \sum_{n \in N} x(n)(-n, \xi), \ \xi \in \hat{N}.
\]

It is known [7, Theorem 4.5] that \(\hat{\mathbb{T}} \cong \mathbb{T}\), with \(\langle n, z \rangle = z^n\), and \(\hat{\mathbb{Z}}_s \cong \mathbb{Z}_s := \mathbb{Z}/s\mathbb{Z}\), with \(\langle n, m \rangle = W^{nm}\), where \(W = e^{2\pi i/s}\).

There exists a unique measure, the Haar measure \(\mu\) on \(\hat{N}\) satisfying \(\mu(\xi + E) = \mu(E)\), for every Borel set \(E \subset \hat{N}\) [7, Section 2.2], and \(\mu(\hat{N}) = 1\). We denote \(\int_{\hat{N}} X(\xi)d\xi = \int_{\hat{N}} X(\xi)d\mu(\xi)\). If \(N = \mathbb{Z}\),

\[
\int_{\hat{N}} X(\xi)d\xi = \int_{\mathbb{T}} X(z)dz = \frac{1}{2\pi} \int_0^{2\pi} X(e^{iw})dw,
\]

and if \(N = \mathbb{Z}_s\),

\[
\int_{\hat{N}} X(\xi)d\xi = \int_{\mathbb{Z}_s} X(n)dn = \frac{1}{s} \sum_{n \in \mathbb{Z}_s} X(n).
\]

If \(N_1, N_2, \ldots N_d\) are abelian discrete groups then the dual group of the product group is \((N_1 \times N_2 \times \ldots \times N_d)^\wedge \cong \hat{N}_1 \times \hat{N}_2 \times \ldots \times \hat{N}_d\) (see [7, Prop. 4.6]) with

\[
\langle (n_1, n_2, \ldots, n_d), (\xi_1, \xi_2, \ldots, \xi_d) \rangle = \langle n_1, \xi_1 \rangle \langle n_2, \xi_2 \rangle \cdots \langle n_d, \xi_d \rangle.
\]

The Fourier transform on \(\ell^1(N) \cap \ell^2(N)\) is an isometry on a dense subspace of \(L^2(\hat{N})\); Plancherel theorem extends it in a unique manner to a unitary operator of \(\ell^2(N)\) onto \(L^2(\hat{N})\) [7, p. 99]. The following lemma, giving a relationship between Fourier transform and convolution, will be used later:

**Lemma 1.** Assume that \(a, b \in \ell^2(N)\) and \(\hat{a}(\xi)\hat{b}(\xi) \in L^2(\hat{N})\). Then the convolution \(a * b\) belongs to \(\ell^2(N)\) and \(a * b(\xi) = \hat{a}(\xi)\hat{b}(\xi), \ \text{a.e.} \ \xi \in \hat{N}\).

**Proof.** By using Plancherel theorem [7, Theorem 4.25] we obtain

\[
(a * b)(n) = \sum_{m \in N} a(m)b(n - m) = \langle \hat{a}(\cdot - n), \hat{b}(\cdot - n) \rangle_{L^2(N)} = \langle \hat{a}(\cdot - n), \hat{b}(\cdot - n) \rangle_{L^2(\hat{N})} = \int_{\hat{N}} \overline{\hat{a}(\xi)}\hat{b}(\xi)\langle -n, \xi \rangle d\xi = \int_{\hat{N}} \overline{\hat{a}(\xi)}\hat{b}(\xi)\langle -n, \xi \rangle d\xi.
\]
Since \( \{ (-n, \xi) \}_{n \in \mathbb{N}} \) is an orthonormal basis for \( L^2(\hat{\mathbb{N}}) \) \[7\] Theorems 4.26 and 4.31] (we are assuming that \( \mu_{\mathbb{N}}(\hat{\mathbb{N}}) = 1 \)) we finally obtain
\[
\hat{a}(\xi) \hat{b}(\xi) = \sum_{n \in \mathbb{N}} (a \ast b)(n)(-n, \xi) = a \ast b(\xi), \quad \text{a.e. } \xi \in \hat{\mathbb{N}}.
\]

3 Filter banks formalism on semi-direct product of groups

In what follows we will assume that \( G = N \rtimes_{\phi} H \) where \( (N, +) \) is a countable discrete abelian group and \( (H, \cdot) \) is a finite group. Having in mind the operational calculus \( (n, h) \cdot (m, l) = (n + \phi_h(m), hl), \quad (n, h)^{-1} = (\phi_{h^{-1}}(-n), h^{-1}) \) and \( (n, h)^{-1} \cdot (m, l) = (\phi_{h^{-1}}(m - n), h^{-1}l) \), the convolution \( \alpha \ast h \) of \( \alpha, h \in \ell^2(G) \) can be expressed as

\[
(\alpha \ast h)(m, l) = \sum_{(n,h) \in G} \alpha(n,h) h[(n, h)^{-1} \cdot (m, l)]
= \sum_{(n,h) \in G} \alpha(n,h) h(\phi_{h^{-1}}(m - n), h^{-1}), \quad (m, l) \in G.
\]

For a function \( \alpha : G \to \mathbb{C} \), its \( H \)-decimation \( \downarrow_H \alpha : N \to \mathbb{C} \) is defined as \( (\downarrow_H \alpha)(n) := \alpha(n, 1_H) \) for any \( n \in N \). Thus we have
\[
\downarrow_H (\alpha \ast h)(m) = (\alpha \ast h)(m, 1_H) = \sum_{(n,h) \in G} \alpha(n,h) h(\phi_{h^{-1}}(m - n), h^{-1})
= \sum_{(n,h) \in G} \alpha(n,h) h[(n - m, h)^{-1}], \quad m \in N.
\]

Defining the polyphase components of \( \alpha \) and \( h \) as \( \alpha_h(n) := \alpha(n, h) \) and \( h_h(n) := h[(-n, h)^{-1}] \) respectively, we write
\[
\downarrow_H (\alpha \ast h)(m) = \sum_{h \in H} \sum_{n \in N} \alpha_h(n) h_h(m - n) = \sum_{h \in H} (\alpha_h \ast h_h)(m), \quad m \in N.
\]

For a function \( c : N \to \mathbb{C} \), its \( H \)-expander \( \uparrow_H c : G \to \mathbb{C} \) is defined as
\[
(\uparrow_H c)(n, h) = \begin{cases} 
  c(n) & \text{if } h = 1_H \\
  0 & \text{if } h \neq 1_H.
\end{cases}
\]

In case \( \uparrow_H c \) and \( g \) belong to \( \ell^2(G) \) we have
\[
(\uparrow_H c \ast g)(m, l) = \sum_{(n,h) \in G} (\uparrow_H c)(n, h) g[(n, h)^{-1} \cdot (m, l)]
= \sum_{(n,h) \in G} (\uparrow_H c)(n, h) g(\phi_{h^{-1}}(m - n), h^{-1}l)
= \sum_{n \in N} c(n) g(m - n, l) = (c \ast \mathcal{N} g)(m), \quad m \in N, l \in H,
\]

\[5\]
\[ \alpha(m, l) \quad \begin{array}{c} \downarrow H \quad c_1(m) \quad \uparrow H \quad g_1 \\ \downarrow H \quad c_2(m) \quad \uparrow H \quad g_2 \\ \vdots \\ \downarrow H \quad c_K(m) \quad \uparrow H \quad g_K \end{array} \quad \beta(m, l) \]

Figure 1: The K-channel filter bank scheme

where \( g_l(n) := g(n, l) \) is the polyphase component of \( g \).

From now on we will refer to a \( K \)-channel filter bank with analysis filters \( h_k \) and synthesis filters \( g_k \), \( k = 1, 2, \ldots, K \) as the one given by (see Fig. 1)

\[ c_k := \downarrow H (\alpha * h_k) , \quad k = 1, 2, \ldots, K , \quad \text{and} \quad \beta = \sum_{k=1}^{K} (\uparrow H c_k) * g_k , \tag{3} \]

where \( \alpha \) and \( \beta \) denote, respectively, the input and the output of the filter bank. In polyphase notation,

\[ c_k(m) = \sum_{h \in H} (\alpha_h * N h_k,h)(m) , \quad m \in N , \quad k = 1, 2, \ldots, K , \]

\[ \beta_l(m) = \sum_{k=1}^{K} (c_k * N g_l,k)(m) , \quad m \in N , \quad l \in H , \tag{4} \]

where \( \alpha_h(n) := \alpha(n, h), \beta_l(n) := \beta(n, l), h_{k,h}(n) := h_k[(-n, h)^{-1}] \) and \( g_{l,k}(n) := g_k(n, l) \) are the polyphase components of \( \alpha, \beta, h_k \) and \( g_k \), \( k = 1, 2, \ldots, K \), respectively. We also assume that \( h_k, g_k \in \ell^2(G) \) with \( \hat{h}_{k,h}, \hat{g}_{k,l} \in L^\infty(\hat{N}) \) for \( k = 1, 2, \ldots, K \) and \( h \in H \); from Lemma the filter bank (3) is well defined in \( \ell^2(G) \).

The above \( K \)-channel filter bank (3) is said to be a perfect reconstruction filter bank if and only if it satisfies \( \alpha = \sum_{k=1}^{K} (\uparrow H c_k) * g_k \) for each \( \alpha \in \ell^2(G) \), or equivalently, \( \alpha_h = \sum_{k=1}^{K} (c_k * N g_{h,k}) \) for each \( h \in H \).

Since \( N \) is an LCA group where a Fourier transform is available, the polyphase expression (4) of the filter bank (3) allows us to carry out its polyphase analysis.

### 3.1 Polyphase analysis: Perfect reconstruction condition

For notational ease, we denote \( L := |H| \), the order of the group \( H \), and its elements as \( H = \{h_1, h_2, \ldots, h_L \} \). Having in mind Lemma the \( N \)-Fourier transform in \( c_k(m) = \sum_{h \in H} (\alpha_h * N h_k,h)(m) \) gives \( \hat{c}_k(\gamma) = \sum_{h \in H} \hat{h}_{k,h}(\gamma) \hat{\alpha}_h(\gamma) \) a.e. \( \gamma \in \hat{N} \) for each \( k = 1, 2, \ldots, K \). In matrix notation,

\[ C(\gamma) = H(\gamma) A(\gamma) \quad \text{a.e.} \quad \gamma \in \hat{N} , \]
where \( \mathbf{C}(\gamma) = (\hat{c}_1(\gamma), \hat{c}_2(\gamma), \ldots, \hat{c}_K(\gamma))^\top \), \( \mathbf{A}(\gamma) = (\hat{\alpha}_{h_1}(\gamma), \hat{\alpha}_{h_2}(\gamma), \ldots, \hat{\alpha}_{h_L}(\gamma))^\top \), and \( \mathbf{H}(\gamma) \) is the \( K \times L \) matrix

\[
\mathbf{H}(\gamma) = \begin{pmatrix}
\hat{h}_{1,h_1}(\gamma) & \hat{h}_{1,h_2}(\gamma) & \cdots & \hat{h}_{1,h_L}(\gamma) \\
\hat{h}_{2,h_1}(\gamma) & \hat{h}_{2,h_2}(\gamma) & \cdots & \hat{h}_{2,h_L}(\gamma) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{h}_{K,h_1}(\gamma) & \hat{h}_{K,h_2}(\gamma) & \cdots & \hat{h}_{K,h_L}(\gamma)
\end{pmatrix}, \tag{5}
\]

where \( \hat{h}_{k,h_i} \in L^2(\hat{N}) \) is the Fourier transform of \( h_{k,h_i}(n) := h_k((-n, h_i)^{-1}) \in \ell^2(N) \).

The same procedure for \( \beta_l(m) = \sum_{k=1}^K (c_k * N \mathbf{g}_{l,k})(m) \) gives \( \hat{\beta}_l(\gamma) \) is the Fourier transform of \( \hat{\beta}_l(\gamma) = \sum_{k=1}^K \hat{c}_{l,k}(\gamma) \hat{c}_k(\gamma) \) a.e. \( \gamma \in \hat{N} \). In matrix notation,

\[
\mathbf{B}(\gamma) = \mathbf{G}(\gamma) \mathbf{C}(\gamma) \quad \text{a.e.} \quad \gamma \in \hat{N},
\]

where \( \mathbf{B}(\gamma) = (\hat{\beta}_{h_1}(\gamma), \hat{\beta}_{h_2}(\gamma), \ldots, \hat{\beta}_{h_L}(\gamma))^\top \), \( \mathbf{C}(\gamma) = (\hat{c}_1(\gamma), \hat{c}_2(\gamma), \ldots, \hat{c}_K(\gamma))^\top \) and \( \mathbf{G}(\gamma) \) is the \( L \times K \) matrix

\[
\mathbf{G}(\gamma) = \begin{pmatrix}
\hat{g}_{h_1,1}(\gamma) & \hat{g}_{h_1,2}(\gamma) & \cdots & \hat{g}_{h_1,K}(\gamma) \\
\hat{g}_{h_2,1}(\gamma) & \hat{g}_{h_2,2}(\gamma) & \cdots & \hat{g}_{h_2,K}(\gamma) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{g}_{h_L,1}(\gamma) & \hat{g}_{h_L,2}(\gamma) & \cdots & \hat{g}_{h_L,K}(\gamma)
\end{pmatrix}, \tag{6}
\]

where \( \hat{g}_{h_i,k} \in L^2(\hat{N}) \) is the Fourier transform of \( g_{h_i,k}(m) := g_k(n, h_i) \in \ell^2(N) \).

Thus, in terms of the polyphase matrices \( \mathbf{G}(\gamma) \) and \( \mathbf{H}(\gamma) \) the filter bank can be expressed as

\[
\mathbf{B}(\gamma) = \mathbf{G}(\gamma) \mathbf{H}(\gamma) \mathbf{A}(\gamma) \quad \text{a.e.} \quad \gamma \in \hat{N}. \tag{7}
\]

As a consequence of (7) we have:

**Theorem 2.** The \( K \)-channel filter bank given in (3), where \( h_k, g_k \) belong to \( \ell^2(G) \) and \( \hat{h}_{k,h_i}, \hat{g}_{h_i,k} \) belong to \( L^\infty(\hat{N}) \) for \( k = 1, 2, \ldots, K \) and \( i = 1, 2, \ldots, L \), satisfies the perfect reconstruction property if and only if \( \mathbf{G}(\gamma) \mathbf{H}(\gamma) = \mathbf{I}_L \) a.e. \( \gamma \in \hat{N} \), where \( \mathbf{I}_L \) denotes the identity matrix of order \( L \).

**Proof.** First of all, note that the mapping \( \mathbf{\alpha} \in \ell^2(G) \mapsto \mathbf{A} \in L_L^2(\hat{N}) \) is a unitary operator. Indeed, for each \( \mathbf{\alpha}, \mathbf{\beta} \in \ell^2(G) \) we have the isometry property

\[
\langle \mathbf{\alpha}, \mathbf{\beta} \rangle_{\ell^2(G)} = \sum_{(m,h) \in G} \alpha(m,h) \overline{\beta(m,h)} = \sum_{h \in H} \langle \hat{\mathbf{\alpha}}_h, \hat{\mathbf{\beta}}_h \rangle_{\ell^2(N)} = \sum_{h \in H} \langle \mathbf{\alpha}_h, \mathbf{\beta}_h \rangle_{L^2(\hat{N})} = \langle \mathbf{A}, \mathbf{B} \rangle_{L_L^2(\hat{N})}.
\]

It is also surjective since the \( N \)-Fourier transform is a surjective isometry between \( \ell^2(N) \) and \( L^2(\hat{N}) \). Having in mind this property, Eq. (7) tell us that the filter bank satisfies the perfect reconstruction property if and only if \( \mathbf{G}(\gamma) \mathbf{H}(\gamma) = \mathbf{I}_L \) a.e. \( \gamma \in \hat{N} \).

Notice that, in the perfect reconstruction setting, the number of channels \( K \) must be necessarily bigger or equal to the order \( L \) of the group \( H \), i.e., \( K \geq L \).
4 Frame analysis

For \( m \in \mathbb{N} \) the translation operator \( T_m : \ell^2(G) \to \ell^2(G) \) is defined as

\[
T_m \alpha(n, h) := \alpha((m, 1_H)^{-1} \cdot (n, h)) = \alpha(n - m, h), \quad (n, h) \in G.
\]

The involution operator \( \alpha \in \ell^2(G) \mapsto \bar{\alpha} \in \ell^2(G) \) is defined as \( \bar{\alpha}(n, h) := \overline{\alpha((n, h)^{-1} \cdot (n, h))} \). As expected, the classical relationship between convolution and translation operators holds. Thus, for the \( K \)-channel filter bank \( \mathcal{F}_G \) we have (see (2)):

\[
c_k(m) = \downarrow_H (\alpha \ast h_k)(m) = \langle \alpha, T_m \tilde{h}_k \rangle_{\ell^2(G)}, \quad m \in \mathbb{N}, \quad k = 1, 2, \ldots, K.
\]

Besides,

\[
(\uparrow_H c_k \ast g_k)(m, h) = \sum_{n \in \mathbb{N}} c_k(n) g_k(m - n, h) = \sum_{n \in \mathbb{N}} \langle \alpha, T_n \tilde{h}_k \rangle_{\ell^2(G)} T_n g_k(m, h).
\]

In the perfect reconstruction setting, for any \( \alpha \in \ell^2(G) \) we have

\[
\alpha = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} \langle \alpha, T_n \tilde{h}_k \rangle_{\ell^2(G)} T_n g_k \quad \text{in} \quad \ell^2(G).
\]

Given \( K \) sequences \( f_k \in \ell^2(G), \quad k = 1, 2, \ldots, K \), our main tasks now are: (i) to characterize the sequence \( \{T_n f_k\}_{n \in \mathbb{N} ; k = 1, 2, \ldots, K} \) as a frame for \( \ell^2(G) \), and (ii) to find its dual frames having the form \( \{T_n g_k\}_{n \in \mathbb{N} ; k = 1, 2, \ldots, K} \).

To the first end we consider a \( K \)-channel analysis filter bank with analysis filters \( h_k := \tilde{f}_k, \quad k = 1, 2, \ldots, K \); let \( H(\gamma) \) be its associated \( K \times L \) polyphase matrix \( \mathcal{F}_H \). First, we check that \( \mathcal{F}_H \) is:

\[
H(\gamma) = \left( \begin{array}{c}
\tilde{f}_{k,i} \left( \gamma \right) \\
\vdots \\
\tilde{f}_{k,L} \left( \gamma \right)
\end{array} \right)_{k=1,2,\ldots,K}, \quad i=1,2,\ldots,L.
\]

Indeed, for \( k = 1, 2, \ldots, K \) and \( i = 1, 2, \ldots, L \) we have

\[
\tilde{h}_{k,i}(\gamma) = \sum_{n \in \mathbb{N}} h_{k,i}(n) (-n, \gamma) = \sum_{n \in \mathbb{N}} h_k((-n, h_i)^{-1}) (-n, \gamma) = \sum_{n \in \mathbb{N}} \tilde{f}_k((-n, h_i)^{-1}) (-n, \gamma)
\]

\[
= \sum_{n \in \mathbb{N}} \tilde{f}_k(-n, h_i) (-n, \gamma) = \sum_{n \in \mathbb{N}} f_k(n, h_i) (-n, \gamma) = \tilde{f}_{k,i} \left( \gamma \right), \quad \gamma \in \hat{\mathbb{N}}.
\]

Next, we consider its associated constants

\[
A_H := \text{ess inf}_{\gamma \in \hat{\mathbb{N}}} \lambda_{\min} \left[ H^{	ext{T}}(\gamma) H(\gamma) \right] \quad \text{and} \quad B_H := \text{ess sup}_{\gamma \in \hat{\mathbb{N}}} \lambda_{\max} \left[ H^{	ext{T}}(\gamma) H(\gamma) \right].
\]

**Teorema 3.** For \( f_k \in \ell^2(G), \quad k = 1, 2, \ldots, K \), consider the associated matrix \( H(\gamma) \) given in (10). Then,

1. The sequence \( \{T_n f_k\}_{n \in \mathbb{N}; k = 1, 2, \ldots, K} \) is a Bessel sequence for \( \ell^2(G) \) if and only if \( B_H < \infty \).
2. The sequence \( \{T_n f_k\}_{n \in \mathbb{N}; k=1,\ldots,K} \) is a frame for \( \ell^2(G) \) if and only if the inequalities \( 0 < A_H \leq B_H < \infty \) hold.

**Proof.** Using Plancherel theorem [7] Theorem 4.25, for each \( \alpha \in \ell^2(G) \) we get

\[
\langle \alpha, T_n f_k \rangle_{\ell^2(G)} = \sum_{h \in H} \langle \alpha_h, f_{k,h}(-n) \rangle_{\Omega} = \sum_{h \in H} \int_{\tilde{N}} \hat{\alpha}_h(\gamma) \hat{f}_{k,h}(\gamma)(-n,\gamma) d\gamma
\]

\[
= \int_{\tilde{N}} \sum_{h \in H} \hat{\alpha}_h(\gamma) \hat{f}_{k,h}(\gamma)(-n,\gamma) d\gamma = \int_{\tilde{N}} H_k(\gamma) A(\gamma)(-n,\gamma) d\gamma ,
\]

where \( A(\gamma) = (\hat{\alpha}_{h_1}(\gamma), \hat{\alpha}_{h_2}(\gamma), \ldots, \hat{\alpha}_{h_L}(\gamma))^T \) and \( H_k(\gamma) \) denotes the \( k \)-th row of \( H(\gamma) \).

Since \( \{(-n,\gamma)\}_{n \in \mathbb{N}} \) is an orthonormal basis for \( L^2(\tilde{N}) \), in case that \( H(\gamma) A(\gamma) \in L^2_K(\tilde{N}) \) we have

\[
\sum_{k=1}^{K} \sum_{n \in \mathbb{N}} |\langle \alpha, T_n f_k \rangle|^2 = \sum_{k=1}^{K} \int_{\tilde{N}} \|H_k(\gamma) A(\gamma)\|^2 d\gamma = \int_{\tilde{N}} \|H(\gamma) A(\gamma)\|^2 d\gamma.
\]

If \( B_H < \infty \), having in mind that \( \|\alpha\|^2_{\ell^2(G)} = \|A\|^2_{L^2(\tilde{N})} = \int_{\tilde{N}} \|A(\gamma)\|^2 d\gamma \), the above equality and the Rayleigh-Ritz theorem [12] Theorem 4.2.2] prove that \( \{T_n f_k\}_{n \in \mathbb{N}; k=1,\ldots,K} \) is a Bessel sequence for \( \ell^2(G) \) with Bessel bound less or equal than \( B_H \).

On the other hand, if \( K < B_H \) then there exists a set \( \Omega \subset \tilde{N} \) having null measure such that \( \lambda_{\max}[H^*(\gamma) H(\gamma)] > K \) for \( \gamma \in \Omega \). Consider \( \alpha \) such that its associated \( A(\gamma) \) is 0 if \( \gamma \notin \Omega \), and \( A(\gamma) \) is a unitary eigenvector corresponding to the largest eigenvalue of \( H^*(\gamma) H(\gamma) \) if \( \gamma \in \Omega \). Thus we have that

\[
\sum_{k=1}^{K} \sum_{n \in \mathbb{N}} |\langle \alpha, T_n f_k \rangle|^2 = \int_{\tilde{N}} \|H(\gamma) A(\gamma)\|^2 d\gamma > K \int_{\tilde{N}} \|A(\gamma)\|^2 d\gamma = K \|\alpha\|^2_{\ell^2(G)}
\]

As a consequence, if \( B_H = \infty \) the sequence is not Bessel, and if \( B_H < \infty \) the optimal bound is precisely \( B_H \).

Similarly, by using inequality \( \|H(\gamma) A(\gamma)\|^2 \geq \lambda_{\min}[H^*(\gamma) H(\gamma)] \|A(\gamma)\|^2 \), and that equality holds whenever \( A(\gamma) \) is a unitary eigenvector corresponding to the smallest eigenvalue of \( H^*(\gamma) H(\gamma) \) one proves the other inequality in part 2. \( \square \)

**Corollary 4.** The sequence \( \{T_n f_k\}_{n \in \mathbb{N}; k=1,\ldots,K} \) is a Bessel sequence for \( \ell^2(G) \) if and only if for each \( k = 1,2,\ldots,K \) and \( i = 1,2,\ldots,L \) the function \( \hat{f}_{k,h_i} \) belongs to \( L^\infty(\tilde{N}) \).

**Proof.** It is a direct consequence of the equivalence between the spectral and Frobenius norms for matrices [12]. \( \square \)

To the second end, a \( K \)-channel filter bank formalism allows, in a similar manner, to obtain properties in \( \ell^2(G) \) of the sequences \( \{T_n f_k\}_{n \in \mathbb{N}; k=1,\ldots,K} \) and \( \{T_n g_k\}_{n \in \mathbb{N}; k=1,\ldots,K} \). In case they are Bessel sequences for \( \ell^2(G) \), the idea is to consider a \( K \)-channel filter bank [3] where the analysis filters are \( h_k := \hat{f}_k \) and the synthesis filters are \( \hat{g}_k, k = 1,2,\ldots,K \). As a consequence, the corresponding polyphase matrices \( H(\gamma) \) and \( G(\gamma) \), given in [5] and [6] are,

\[
H(\gamma) = \left( \hat{f}_{k,h_i}(\gamma) \right)_{k=1,\ldots,K \atop i=1,\ldots,L} \quad \text{and} \quad G(\gamma) = \left( \hat{g}_{h_i,k}(\gamma) \right)_{i=1,\ldots,L \atop k=1,\ldots,K}, \quad \gamma \in \tilde{N}.
\]
Theorem 5. Let \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) and \( \{ T_n g_k \}_{n \in N; k=1,2,...K} \) be two Bessel sequences for \( \ell^2(G) \), and \( H(\gamma) \) and \( G(\gamma) \) their associated matrices \( (\Pi) \). Under the above circumstances we have:

(a) The sequences \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) and \( \{ T_n g_k \}_{n \in N; k=1,2,...K} \) are dual frames for \( \ell^2(G) \) if and only if condition \( G(\gamma)H(\gamma) = I_L \) a.e. \( \gamma \in \hat{N} \) holds.

(b) The sequences \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) and \( \{ T_n g_k \}_{n \in N; k=1,2,...K} \) are biorthogonal sequences in \( \ell^2(G) \) if and only if condition \( H(\gamma)G(\gamma) = I_K \) a.e. \( \gamma \in \hat{N} \) holds.

(c) The sequences \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) and \( \{ T_n g_k \}_{n \in N; k=1,2,...K} \) are dual Riesz bases for \( \ell^2(G) \) if and only if \( K = L \) and \( G(\gamma) = H(\gamma)^{-1} \) a.e. \( \gamma \in \hat{N} \).

(d) The sequence \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) is an A-tight frame for \( \ell^2(G) \) if and only if condition \( H^*(\gamma)H(\gamma) = AI_L \) a.e. \( \gamma \in \hat{N} \) holds.

(e) The sequence \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) is an orthonormal basis for \( \ell^2(G) \) if and only if \( K = L \) and \( H^*(\gamma) = H(\gamma)^{-1} \) a.e. \( \gamma \in \hat{N} \).

Proof. Having in mind \( (\Pi) \) and Corollary \( (\Pi) \) part (a) is nothing but Theorem \( (\Pi) \).

The output of the analysis filter bank \( (\Pi) \) corresponding to the input \( g_{k'} \) is a \( K \)-vector which \( k' \)-entry is

\[
c_{k,k'}(m) = \bar{H}(g_{k'} \ast_N h_k)(m) = \langle g_{k'}, T_m \tilde{h}_k \rangle_{\ell^2(G)} = \langle g_{k'}, T_m f_k \rangle_{\ell^2(G)},
\]

and whose \( N \)-Fourier transform is \( C_{k'}(\gamma) = H(\gamma) G_{k'}(\gamma) \) a.e. \( \gamma \in \hat{N} \), where \( G_{k'} \) is the \( k' \)-column of the matrix \( G(\gamma) \). Note that \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) and \( \{ T_n g_k \}_{n \in N; k=1,2,...K} \) are biorthogonal if and only if \( \langle g_{k'}, T_m f_k \rangle_{\ell^2(G)} = \delta(k-k')\delta(m) \). Therefore, the sequences \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) and \( \{ T_n g_k \}_{n \in N; k=1,2,...K} \) are biorthogonal if and only if \( H(\gamma)G(\gamma) = I_K \). Thus, we have proved (b).

Having in mind \( (\Pi) \) Theorem 7.1.1, from (a) and (b) we obtain (c).

We can read the frame operator corresponding to the sequence \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \), i.e.,

\[
S(\alpha) = \sum_{k=1}^{K} \sum_{n \in N} \langle \alpha, T_n f_k \rangle_{\ell^2(G)} T_n f_k, \quad \alpha \in \ell^2(G),
\]

as the output of the filter bank \( (\Pi) \), whenever \( h_k = \tilde{f}_k \) and \( g_k = f_k \), for the input \( \alpha \). For this filter bank, the \((k, h_l)\)-entry of the analysis polyphase matrix \( H(\gamma) \) is \( f_{k,l}(\gamma) \) and the \((h_l,k)\)-entry of the synthesis polyphase matrix \( G(\gamma) \) is \( \hat{f}_{k,l}(\gamma) \); in other words, \( G(\gamma) = H^*(\gamma) \).

Hence, the sequence \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) is an A-tight frame for \( \ell^2(G) \), i.e.,

\[
\alpha = \frac{1}{A} \sum_{k=1}^{K} \sum_{n \in N} \langle \alpha, T_n f_k \rangle_{\ell^2(G)} T_n f_k, \quad \alpha \in \ell^2(G),
\]

if and only if \( H^*(\gamma)H(\gamma) = AI_L \) for all \( \gamma \in \hat{N} \). Thus, we have proved (d).

Finally, from (c) and (d) the sequence \( \{ T_n f_k \}_{n \in N; k=1,2,...K} \) is an orthonormal system if and only if \( H^*(\gamma) = H(\gamma)^{-1} \) a.e. \( \gamma \in \hat{N} \).
5 Getting on with sampling

Suppose that \( \{U(n, h)\}_{(n,h)\in G} \) is a unitary representation of the group \( G = N \rtimes \Phi \) on a separable Hilbert space \( \mathcal{H} \), and assume that for a fixed \( a \in \mathcal{H} \) the sequence \( \{U(n, h)a\}_{(n,h)\in G} \) is a Riesz sequence for \( \mathcal{H} \) (see Ref. [2, Theorem A]). Thus, we consider the \( U \)-invariant subspace in \( \mathcal{H} \)

\[
\mathcal{A}_a = \left\{ \sum_{(n,h)\in G} \alpha(n,h)U(n,h)a : \{\alpha(n,h)\}_{(n,h)\in G} \in \ell^2(G) \right\}.
\]

For \( K \) fixed elements \( b_k \in \mathcal{H} \), \( k = 1,2,\ldots,K \), non necessarily in \( \mathcal{A} \), we consider for each \( x \in \mathcal{A} \) its generalized samples defined as

\[
\mathcal{L}_k x(m) := \langle x, U(m,1_H)b_k \rangle_{\mathcal{H}}, \quad m \in N \text{ and } k = 1,2,\ldots,K.
\]

The problem is the stable recovery of any \( x \in \mathcal{A} \) from the data \( \{\mathcal{L}_k x(m)\}_{m\in N; k=1,2,\ldots,K} \).

In what follows, we propose a solution involving a perfect reconstruction \( K \)-channel filter bank. First, we express the samples in a more suitable manner. Namely, for each \( x = \sum_{(n,h)\in G} \alpha(n,h)U(n,h)a \) in \( \mathcal{A}_a \) we have

\[
\mathcal{L}_k x(m) = \sum_{(n,h)\in G} \alpha(n,h)\langle U(n,h)a,U(m,1_H)b_k \rangle = \sum_{(n,h)\in G} \alpha(n,h)\langle a,U[(n,h)^{-1} \cdot (m,1_H)]b_k \rangle = \downarrow_H (\alpha * h_k)(m), \quad m \in N,
\]

where \( \alpha = \{\alpha(n,h)\}_{(n,h)\in G} \in \ell^2(G) \), and \( h_k(n,h) := \langle a,U(n,h)b_k \rangle_{\mathcal{H}} \) also belongs to \( \ell^2(G) \) for each \( k = 1,2,\ldots,K \).

Suppose also that there exists a perfect reconstruction \( K \)-channel filter-bank with analysis filters the above \( h_k \) and synthesis filters \( g_k \), \( k = 1,2,\ldots,K \), such that the sequences \( \{T_n h_k\}_{n\in N; k=1,2,\ldots,K} \) and \( \{T_n g_k\}_{n\in N; k=1,2,\ldots,K} \) are Bessel sequences for \( \ell^2(G) \). Having in mind (9), for each \( \alpha = \{\alpha(n,h)\}_{(n,h)\in G} \in \ell^2(G) \) we have

\[
\alpha = \sum_{k=1}^K \sum_{n\in N} \downarrow_H (\alpha * h_k)(n) T_n g_k = \sum_{k=1}^K \sum_{n\in N} \mathcal{L}_k x(n) T_n g_k \quad \text{in } \ell^2(G).
\]

In order to derive a sampling formula in \( \mathcal{A}_a \), we consider the natural isomorphism \( \mathcal{T}_{U,a} : \ell^2(G) \rightarrow \mathcal{A}_a \) which maps the usual orthonormal basis \( \{\delta_{(n,h)}\}_{(n,h)\in G} \) for \( \ell^2(G) \) onto the Riesz basis \( \{U(n,h)a\}_{(n,h)\in G} \) for \( \mathcal{A}_a \), i.e.,

\[
\mathcal{T}_{U,a} : \delta_{(n,h)} \longrightarrow U(n,h)a \quad \text{for each } (n,h) \in G.
\]

This isomorphism \( \mathcal{T}_{U,a} \) possesses the following shifting property:

**Lemma 6.** For each \( m \in N \), consider the translation operator \( T_m \) operator defined in (\ref{eq:translation}). For each \( m \in N \), the following shifting property holds

\[
\mathcal{T}_{U,a}(T_m f) = U(m,1_H)(\mathcal{T}_{U,a} f), \quad f \in \ell^2(G).
\]
Proof. For each \( \delta_{(n,h)} \) it is easy to check that \( T_m \delta_{(n,h)} = \delta_{(m+n,h)} \). Hence,
\[
T_{U,a} (T_m \delta_{(n,h)}) = U(m+n,h) a = U(m,1_H) U(n,h) a = U(m,1_H) (T_{U,a} \delta_{(n,h)}) .
\]
A continuity argument proves the result for all \( f \) in \( \ell^2(G) \).

Now for each \( x = T_{U,a} \alpha \in \mathcal{A}_a \), applying the isomorphism \( T_{U,a} \) and the shifting property \( (14) \) in \( (13) \), we get for each \( x \in \mathcal{A}_a \) the expansion
\[
x = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) T_{U,a} (T_n g_k) = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n,1_H) T_{U,a} g_k
\]
\[
= \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n,1_H) c_{k,g} \quad \text{in } \mathcal{H} ,
\]
where \( c_{k,g} = T_{U,a} g_k; k = 1, 2, \ldots, K \). In fact, the following sampling theorem in the subspace \( \mathcal{A}_a \) holds:

**Theorem 7.** For \( k \) fixed \( b_k \in \mathcal{H} \), let \( \mathcal{L}_k : N \to \mathbb{C} \) be its associated \( U \)-system defined in \( (12) \) with corresponding \( h_k \in \ell^2(G) \), \( k = 1, 2, \ldots, K \). Assume that its polyphase matrix \( H(\gamma) \) given in \( (5) \) has all its entries in \( L^\infty(\tilde{\mathbb{N}}) \). The following statements are equivalent:

1. The constant \( A_H = \inf_{\gamma \in \tilde{\mathbb{N}}} \lambda_{\min} [H^*(\gamma) H(\gamma)] > 0 \).
2. There exist \( g_k \) in \( \ell^2(G) \), \( k = 1, 2, \ldots, K \), such that the associated polyphase matrix \( G(\gamma) \) given in \( (6) \) has all its entries in \( L^\infty(\tilde{\mathbb{N}}) \), and it satisfies \( G(\gamma) H(\gamma) = I_L \) a.e. \( \gamma \in \tilde{\mathbb{N}} \).
3. There exist \( K \) elements \( c_k \in \mathcal{A}_a \) such that the sequence \( \{ U(n,1_H) c_k \}_{n \in \mathbb{N}} \) is a frame for \( \mathcal{A}_a \) and for each \( x \in \mathcal{A}_a \) the sampling formula
\[
x = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n,1_H) c_k \quad \text{in } \mathcal{H} \]
holds.
4. There exists a frame \( \{ C_{k,n} \}_{n \in \mathbb{N}; k=1,2,\ldots,K} \) for \( \mathcal{A}_a \) such that for each \( x \in \mathcal{A}_a \) the expansion
\[
x = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) C_{k,n} \quad \text{in } \mathcal{H} \]
holds.

**Proof.** (1) implies (2). The \( L \times K \) Moore-Penrose pseudo-inverse \( H^\dagger(\gamma) \) of \( H(\gamma) \) is given by \( H^\dagger(\gamma) = [H^*(\gamma) H(\gamma)]^{-1} H^*(\gamma) \). Its entries are essentially bounded in \( \tilde{\mathbb{N}} \) since the entries of \( H(\gamma) \) belong to \( L^\infty(\tilde{\mathbb{N}}) \) and \( \det^{-1} [H^*(\gamma) H(\gamma)] \) is essentially bounded \( \tilde{\mathbb{N}} \) since \( 0 < A_H \). Besides, \( H^\dagger(\gamma) H(\gamma) = I_L \) a.e. \( \gamma \in \tilde{\mathbb{N}} \). The inverse \( N \)-Fourier transform in \( L^2(\tilde{\mathbb{N}}) \) of the \( k \)-th column of \( H^\dagger(\gamma) \) gives \( g_k \), \( k = 1, 2, \ldots, K \).
(2) implies (3). According to Theorems 3 and 5 the sequences \( \{T_n\hat{h}_k\}_{n \in \mathbb{N}, k = 1,2,\ldots,K} \) and \( \{T_n\tilde{g}_k\}_{n \in \mathbb{N}, k = 1,2,\ldots,K} \) form a pair of dual frames for \( \ell^2(G) \). We deduce the expansion sampling as by \( \lfloor 15 \rfloor \). Besides, the sequence \( \{U(n,1_H)c_k\}_{n \in \mathbb{N}, k = 1,2,\ldots,K} \) is a frame for \( \mathcal{A}_a \).

Obviously, (3) implies (4). Finally, (4) implies (1). Applying \( \mathcal{T}_U^{-1} \) we get that the sequences \( \{T_n\hat{h}_k\}_{n \in \mathbb{N}, k = 1,2,\ldots,K} \) and \( \{\mathcal{T}_U^{-1}(C_k,n)\}_{n \in \mathbb{N}, k = 1,2,\ldots,K} \) form a pair of dual frames for \( \ell^2(G) \); in particular, by using Theorem 5 we obtain that \( 0 < A_H \).

All the possible solutions of \( \mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L \) a.e. \( \gamma \in \hat{N} \) with entries in \( L^\infty(\hat{N}) \) are given in terms of the Moore-Penrose pseudo inverse by the \( L \times K \) matrices \( \mathbf{G}(\gamma) := \mathbf{H}^\dagger(\gamma) + \mathbf{U}(\gamma)[\mathbf{I}_K - \mathbf{H}(\gamma)\mathbf{H}^\dagger(\gamma)] \), where \( \mathbf{U}(\gamma) \) denotes any \( L \times K \) matrix with entries in \( L^\infty(\hat{N}) \).

Notice that \( K \geq L \) where \( L \) is the order of the group \( H \). In case \( K = L \), we obtain:

**Corollary 8.** In the case \( K = L \), assume that its polyphase matrix \( \mathbf{H}(\gamma) \) given in \( \lfloor 5 \rfloor \) has all entries in \( L^\infty(\hat{N}) \). The following statements are equivalent:

1. The constant \( A_H = \text{ess inf}_{\gamma \in \hat{N}} \lambda_{\min}(\mathbf{H}^\dagger(\gamma)\mathbf{H}(\gamma)) > 0 \).

2. There exist \( L \) unique elements \( c_k, k = 1,2,\ldots,L \), in \( \mathcal{A}_a \) such that the associated sequence \( \{U(n,1_H)c_k\}_{n \in \mathbb{N}, k = 1,2,\ldots,L} \) is a Riesz basis for \( \mathcal{A}_a \) and the sampling formula

\[
x = \sum_{k=1}^{L} \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n,1_H)c_k \quad \text{in} \ \mathcal{H}
\]

holds for each \( x \in \mathcal{A}_a \).

Moreover, the interpolation property \( \mathcal{L}_k c_k(n) = \delta_{k,k'} \delta_{n,0_N} \), where \( n \in \mathbb{N} \) and \( k,k' = 1,2,\ldots,L \), holds.

**Proof.** In this case, the square matrix \( \mathbf{H}(\gamma) \) is invertible and the result comes out from Theorem 5. From the uniqueness of the coefficients in a Riesz basis we get the interpolation property. \( \Box \)

Denote \( H = \{h_1,h_2,\ldots,h_L\} \); for a fixed \( b \in \mathcal{H} \), we consider the samples

\[
\mathcal{L}_k x(m) := \langle x, U(m,h_k)b \rangle, \quad m \in \mathbb{N} \text{ and } k = 1,2,\ldots,L,
\]

of any \( x \in \mathcal{A}_a \). Since \( U(m,h_k)b = U(m,1_H)U(0_N,h_k)b = U(m,1_H)b_k \), where \( b_k := U(0_N,h_k)b \), \( k = 1,2,\ldots,L \), we are in a particular case of \( \lfloor 12 \rfloor \) with \( K = L \).

### 5.1 An example involving crystallographic groups

The Euclidean motion group \( E(d) \) is the semi-direct product \( \mathbb{R}^d \rtimes_{\phi} O(d) \) corresponding to the homomorphism \( \phi : O(d) \to \text{Aut}(\mathbb{R}^d) \) given by \( \phi_A(x) = Ax \), where \( A \in O(d) \) and \( x \in \mathbb{R}^d \).

The composition law on \( E(d) = \mathbb{R}^d \rtimes_{\phi} O(d) \) reads \( (x,A) \cdot (x',A') = (x + A x', AA') \).

Let \( M \) be a non-singular \( d \times d \) matrix and \( \Gamma \) a finite subgroup of \( O(d) \) of order \( L \) such that \( A(MZ^d) = MZ^d \) for each \( A \in \Gamma \). We consider the crystallographic group \( \mathcal{C}_{M,\Gamma} := MZ^d \rtimes_{\phi} \Gamma \) and its quasi regular representation (see Ref. \( \lfloor 24 \rfloor \)) on \( L^2(\mathbb{R}^d) \)

\[
U(n,A)f(t) = f[A^\top(t-n)], \quad n \in MZ^d, A \in \Gamma \text{ and } f \in L^2(\mathbb{R}^d).
\]
For a fixed \( \varphi \in L^2(\mathbb{R}^d) \) such that the sequence \( \{U(n, A)\varphi\}_{(n, A) \in \mathcal{C}_{M, \Gamma}} \) is a Riesz sequence for \( L^2(\mathbb{R}^d) \) we consider the \( U \)-invariant subspace in \( L^2(\mathbb{R}^d) \)

\[
\mathcal{A}_\varphi = \left\{ \sum_{(n, A) \in \mathcal{C}_{M, \Gamma}} \alpha(n, A) \varphi[A^T(t-n)] : \{\alpha(n, A)\} \in \ell^2(\mathcal{C}_{M, \Gamma}) \right\} = \left\{ \sum_{(n, A) \in \mathcal{C}_{M, \Gamma}} \alpha(n, A) \varphi(At-n) : \{\alpha(n, A)\} \in \ell^2(\mathcal{C}_{M, \Gamma}) \right\}.
\]

Choosing \( K \) functions \( b_k \in L^2(\mathbb{R}^d), \ k = 1, 2, \ldots, K \), we consider the average samples of \( f \in \mathcal{A}_\varphi \)

\[
L_k f(n) = \langle f, U(n, I)b_k \rangle = \langle f, b_k(\cdot-n) \rangle, \quad n \in MZ^d.
\]

Under the hypotheses in Theorem \[ \] there exist \( K \geq L \) sampling functions \( \psi_k \in \mathcal{A}_\varphi \) for \( k = 1, 2, \ldots, K \), such that the sequence \( \{\psi_k(\cdot-n)\}_{n \in MZ^d; k=1,2,\ldots,K} \) is a frame for \( \mathcal{A}_\varphi \), and the sampling expansion

\[
f(t) = \sum_{k=1}^{K} \sum_{n \in MZ^d} \langle f, b_k(\cdot-n) \rangle_{L^2(\mathbb{R}^d)} \psi_k(t-n) \quad \text{in} \quad L^2(\mathbb{R}^d)
\]

holds.

If the generator \( \varphi \in C(\mathbb{R}^d) \) and the function \( t \mapsto \sum_n |\varphi(t-n)|^2 \) is bounded on \( \mathbb{R}^d \), a standard argument shows that \( \mathcal{A}_\varphi \) is a reproducing kernel Hilbert space (RKHS) of continuous functions in \( L^2(\mathbb{R}^d) \). As a consequence, convergence in \( L^2(\mathbb{R}^d) \)-norm implies pointwise convergence which is absolute and uniform on \( \mathbb{R}^d \).

Notice that the infinite dihedral group \( D_\infty = \mathbb{Z} \rtimes_\phi \mathbb{Z}_2 \) is a particular crystallographic group with lattice \( \mathbb{Z} \) and \( \Gamma = \mathbb{Z}_2 \). Its quasi regular representation on \( L^2(\mathbb{R}) \) reads

\[
U(n, 0)f(t) = f(t-n) \quad \text{and} \quad U(n, 1)f(t) = f(-t+n), \quad n \in \mathbb{Z} \text{ and } f \in L^2(\mathbb{R}).
\]

So we could obtain sampling formulas as \[ \] for \( K \geq 2 \) average functions \( b_k \).

5.2 The case of pointwise samples

Let \( \{U(n, h)\}_{(n, h) \in G} \) be a unitary representation of the group \( G = N \rtimes_\phi H \) on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^d) \). If the generator \( \varphi \in L^2(\mathbb{R}^d) \) satisfies that, for each \( (n, h) \in G \), the function \( U(n, h)\varphi \) is continuous on \( \mathbb{R}^d \), and the condition

\[
\sup_{t \in \mathbb{R}^d} \sum_{(n, h) \in G} |U(n, h)\varphi|(t)^2 < \infty,
\]

then the subspace \( \mathcal{A}_\varphi \) is a RKHS of continuous functions in \( L^2(\mathbb{R}^d) \); proceeding as in \[ \] one can prove that the above conditions are also necessary.

For \( K \) fixed points \( t_k \in \mathbb{R}^d, \ k = 1, 2, \ldots, K \), we consider for each \( f \in \mathcal{A}_\varphi \) the new samples given by

\[
L_k f(n) := [U(-n, 1_H)f](t_k), \quad n \in N \text{ and } k = 1, 2, \ldots, K.
\]
For each \( f = \sum_{(m,h) \in G} \alpha(m,h) U(m,h) \varphi \) in \( A_\varphi \) and \( k = 1, 2, \ldots, K \) we have

\[
\mathcal{L}_k f(n) = \left[ \sum_{(m,h) \in G} \alpha(m,h) U([-n,1_H] \cdot (m,h)) \varphi \right](t_k)
\]

\[
= \sum_{(m,h) \in G} \alpha(m,h) \left[ U(m - n,h) \varphi \right](t_k) = \langle \alpha, T_n h_k \rangle_{\ell^2(G)}, \quad n \in N,
\]

where \( \alpha = \{\alpha(m,h)\}_{(m,h) \in G} \) and \( h_k(m,h) := \left[ U(m,h) \varphi \right](t_k), \quad (m,h) \in G \). Notice that \( h_k \) belongs to \( \ell^2(G) \), \( k = 1, 2, \ldots, K \). As a consequence, under the hypotheses in Theorem 7 (on these new \( h_k \in \ell^2(G) \), \( k = 1, 2, \ldots, K \)) a sampling formula as (16) holds for the data sequence \( \{\mathcal{L}_k f(n)\}_{n \in N; k=1,2,\ldots,K} \) defined in (18).

In the particular case of the quasi regular representation of a crystallographic group \( C_{M,\Gamma} = M\mathbb{Z}^d \rtimes_\varphi \Gamma \), for each \( f \in A_\varphi \) the samples (18) read

\[
\mathcal{L}_k f(n) = [U(-n,1)f](t_k) = f(t_k + n), \quad n \in M\mathbb{Z}^d \text{ and } k = 1, 2, \ldots, K.
\]

Thus (under hypotheses in Theorem 7), there exist \( K \) functions \( \psi_k \in A_\varphi, \quad k = 1, 2, \ldots, K \), such that for each \( f \in A_\varphi \) the sampling formula

\[
f(t) = \sum_{k=1}^{K} \sum_{n \in M\mathbb{Z}^d} f(t_k + n) \psi_k(t - n), \quad t \in \mathbb{R}^d
\]

holds. The convergence of the series in the \( L^2(\mathbb{R}^d) \)-norm sense implies pointwise convergence which is absolute and uniform on \( \mathbb{R}^d \).

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References

[1] A. Aldroubi, Q. Sun and W. S. Tang. Convolution, average sampling, and a Calderon 
resolution of the identity for shift-invariant spaces. J. Fourier Anal. Appl., 11(2):215–
244, 2005.

[2] D. Barbieri, E. Hernández, J. Parcet. Riesz and frame systems generated by unitary 
actions of discrete groups. Appl. Comput. Harmon. Anal., 39(3): 369–399, 2015.

[3] H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger. Frame-theoretic analysis of oversam-
pled filter banks. IEEE Trans. Signal Process., 46(12):3256–3268, 1998.

[4] O. Christensen. An Introduction to Frames and Riesz Bases. Second Edition. Birkhäuser, 
Boston, 2016.

[5] Z. Cvetković and M. Vetterli. Oversampled filter banks. IEEE Trans. Signal Process., 
46:1245–1255, 1998.

[6] H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-
Bouzo. Generalized sampling: from shift-invariant to \( U \)-invariant spaces. Anal. Appl., 
13(3):303–329, 2015.
[7] G. B. Folland. *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.

[8] A. G. García and G. Pérez-Villalón. Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces. *Appl. Comput. Harmon. Anal.*, 20(3):422–433, 2006.

[9] A. G. García and G. Pérez-Villalón. Multivariate generalized sampling in shift-invariant spaces and its approximation properties. *J. Math. Anal. Appl.*, 355:397–413, 2009.

[10] A. G. García, M. A. Hernández-Medina y G. Pérez-Villalón. Filter Banks on Discrete Abelian Groups. *Internat. J. Wavelets Multiresolut. Inf. Process.*, doi.org/10.1142/s0219691318500297, 2018.

[11] A. G. García, M. A. Hernández-Medina and G. Pérez-Villalón. Sampling in unitary invariant subspaces associated to LCA groups. *Results Math.*, 72:1725–1745, 2017.

[12] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1999.

[13] S. Kang and K. H. Kwon. Generalized average sampling in shift-invariant spaces. *J. Math. Anal. Appl.*, 377:70–78, 2011.

[14] T. Michaeli, V. Pohl and Y. C. Eldar. $U$-invariant sampling: extrapolation and causal interpolation from generalized samples. *IEEE Trans. Signal Process.*, 59(5):2085–2100, 2011.

[15] V. Pohl and H. Boche. $U$-invariant sampling and reconstruction in atomic spaces with multiple generators. *IEEE Trans. Signal Process.*, 60(7):3506–3519, 2012.

[16] W. Sun and X. Zhou. Average sampling in shift-invariant subspaces with symmetric averaging functions. *J. Math. Anal. Appl.*, 287:279–295, 2003.

[17] X. Zhou and W. Sun. On the sampling theorem for wavelet subspaces. *J. Fourier Anal. Appl.*, 5(4):347–354, 1999.