I show that the expectation value of the composite field $T\bar{T}$, built from the components of the energy-momentum tensor, is expressed exactly through the expectation value of the energy-momentum tensor itself. The relation is derived in two-dimensional quantum field theory under broad assumptions, and does not require integrability.
1 Introduction

Determination of one-point expectation values of local fields is an important problem of quantum field theory (see e.g. [1] [2]). The expectation values \( \langle O_i \rangle \) control linear reaction of the system to external forces which couple to the fields \( O_i(z) \). Also, in view of the operator-product expansions (OPE)

\[
O_i(z) O_j(z') = \sum_k C_{ij}^k (z - z') O_k(z')
\]

the two-point correlation functions \( \langle O_i(z) O_j(z') \rangle \) (and, by repeated application of (1), the multipoint correlation functions) are expressed through the OPE structure functions \( C_{ij}^k (z - z') \) and the one-point expectation values \( \langle O_k \rangle \). But while the structure functions (which describe local dynamics of the field theory) usually admit perturbative expansions [3], the one-point expectation values (incorporating information about the vacuum state of the theory) are typically nonperturbative quantities [1] [2] [3], and no general approach to their systematic evaluation is known 1.

In recent years, some progress was made in determination of the one-point expectation values in 2D integrable quantum field theories [5] [6] [7] [8] [9] [10]. In particular, in Ref. [7] exact expectation values of the lowest nontrivial descendant fields were obtained in the cases of the sine-Gordon model and the \( \Phi(1,3) \)-perturbed minimal CFT. Simplest of these descendants is the composite field \( T \bar{T} \) built from the chiral components \( T, \bar{T} \) of the energy-momentum tensor \( T_{\mu\nu} \). Remarkably, the result of [7] shows that in these cases the expectation value of \( T \bar{T} \) relates to the expectation value of the trace component \( \Theta = \frac{\pi}{2} T_{\mu\mu} \) as follows

\[
\langle T \bar{T} \rangle = - \langle \Theta \rangle^2.
\]

Subsequently, expectation values of the lowest descendant fields were obtained in few other integrable models, including the Bullough-Dodd model and the \( \Phi(1,2) \)-perturbations of the minimal CFT [10], and again in all these cases the expectation values of \( T \bar{T} \) and \( \Theta \) turned out to fulfill the relation (2).

In this note I will show that the relation (2) (and indeed somewhat more general relation (3) below) holds in 2D quantum field theory under rather broad assumptions; in particular, the theory is not required to be integrable.

1 In two dimensions, rather accurate numerical estimates can be obtained in many cases through some version of the Truncated Conformal Space Approach, see Ref. [4]
In the Refs [7] and [10] the field theories were assumed to live on an infinite Euclidean plane. One can consider instead a field theory on an infinite cylinder, with one of the Euclidean axis compactified on a circle (this of course is the Matsubara representation of the field theory at finite temperature). I will show that in this case the relation (2) generalizes as follows,

\[ \langle T \bar{T} \rangle = \langle T \rangle \langle \bar{T} \rangle - \langle \Theta \rangle \langle \bar{\Theta} \rangle. \]  

(3)

When the circumference of the cylinder goes to infinity (equivalently, the temperature goes to zero) the global rotational symmetry is restored, making the expectation values of the chiral components \( T \) and \( \bar{T} \) vanish - in this limit (3) reduces to (2).

I will also argue that the relation (3) remains valid if the vacuum expectation values \( \langle \ldots \rangle = \langle 0 | \ldots | 0 \rangle \) there are replaced by more general diagonal matrix elements \( \langle n | \ldots | n \rangle \), where \( | n \rangle \) is any non-degenerate eigenstate of the energy and momentum operators (in the case of cylinder, to make this statement precise one has to take the Hamiltonian picture in which the coordinate along the cylinder is taken as the Euclidean time).

I present the arguments in sections 2 through 4 below. In section 5 I give another form of the relation (3), which can be useful in analysis of critical singularities in 2D statistical systems, and in other applications.

Throughout this paper I consider quantum field theory in flat 2D space, and my discussion is in terms of the Euclidean version of the theory. The points \( z \) of the 2D space can be labeled by the Cartesian coordinates \( (x, y) \), but I will usually use complex coordinates \( z = (z, \bar{z}) = (x + iy, x - iy) \).

I assume usual normalization of the energy-momentum tensor \( T_{\mu\nu} \): for instance, in the picture where \( y \) is taken as the Euclidean time, \( -T_{yy} \) coincides with the energy density. The chiral components \( T, \bar{T}, \Theta \) are normalized according to the CFT convention [11], namely \( T = -(2\pi) T_{zz}, \bar{T} = -(2\pi) T_{\bar{z}z}, \Theta = (2\pi) T_{z\bar{z}}. \)

2 Assumptions and sketch of the argument

In this section I list basic assumptions about the field theory and display main idea of my arguments. More subtle points, including precise definition of the field \( TT \), are discussed in the next two sections. Some of the assumptions concern with the local dynamics of the field theory, others are about the global settings (the geometry of the space). I will stress the distinction by giving them additional labels (L) or (G).
My basic assumptions are as follows:

1 (L). Local translational and rotational symmetry. This implies existence of local field $T_{\mu\nu}$ (the energy-momentum tensor) which is symmetric, $T^{\mu\nu}(z) = T^{\nu\mu}(z)$, and satisfies the continuity equation $\partial_{\mu}T^{\mu\nu}(z) = 0$. In terms of the conventional chiral components $T = -2\pi T_{zz}$, $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$ and $\Theta = 2\pi T_{z\bar{z}} = 2\pi T_{\bar{z}z}$ the continuity equation is written as

\begin{equation}
\partial_{z}T(z) = \partial_{\bar{z}}\Theta(z),
\end{equation}

\begin{equation}
\partial_{\bar{z}}\bar{T}(z) = \partial_{\bar{z}}\Theta(z).
\end{equation}

This assumption is already taken into account in writing the OPE, where the structure functions $C_{ij}^k$ are assumed to depend on the separations $z - z'$ only.

2 (G). Global translational symmetry. I assume that for any local field $O_i(z)$ the expectation value $\langle O_i(z) \rangle$ is a constant independent of $z$. It follows from (1) that the two-point correlation functions depend only on the separations,

\begin{equation}
\langle O_i(z)O_j(z') \rangle = G_{ij}(z - z').
\end{equation}

3 (G). Infinite separations. I assume that at least one direction (i.e. Euclidean vector $e = (e, \bar{e})$) exists, such that for any $O_i$ and $O_j$

\begin{equation}
\lim_{t \to \infty} \langle O_i(z + e t)O_j(z') \rangle = \langle O_i \rangle \langle O_j \rangle.
\end{equation}

The “global” assumptions 2 and 3 imply that the underlying geometry of 2D space is either an infinite plane, or an infinitely long cylinder.

4. (L) CFT limit at short distances. I will assume that the short-distance behavior of the field theory is governed by a conformal field theory, and that certain no-resonance condition is satisfied. I will detail the content of this assumption in section 4 below. Here I just mention that this assumption is needed in order to make definition of the composite field $TT$ essentially unambiguous.\(^2\)

The main idea of my arguments stems from simple identity involving two-point correlation functions of the energy-momentum tensor, consequence of

\(^2\)There is intrinsic ambiguity in adding certain total derivatives, which does not affect the expectation value $\langle TT \rangle$. I discuss it in section 4.
the assumptions 1-3 alone. Consider the following combination of two-point correlation functions

\[ C = \langle T(z)\bar{T}(z') \rangle - \langle \Theta(z)\Theta(z') \rangle. \] (8)

Take \( \partial_z \) derivative of (8) and transform it as follows. In the first term in (8), use the equation (4) to replace the derivative \( \partial_z \bar{T}(z) \) by \( \partial_z \Theta(z) \), and then apply the Eq.(6) to move the derivative to the second entry \( \bar{T}(z') \). When the derivative \( \partial_z \Theta(z) \) in the second term is also moved to \( \Theta(z') \), one finds

\[ \langle \partial_z \bar{T}(z) \bar{T}(z') - \partial_z \Theta(z) \Theta(z') \rangle = \langle -\Theta(z) \partial_{z'} \bar{T}(z') + \Theta(z) \partial_{z'} \Theta(z') \rangle = 0, \] (9)

where the equation (5) was used in the last step. By similar transformations one shows that the \( \partial_z \) derivative of (8) also vanishes, and hence the combination (8) is a constant, independent of the coordinates.

Note that in this derivation only the first two assumptions 1 and 2 are used. Adding the assumption 3 allows one to relate this constant to the one-point expectation values of the fields involved. Taking the limit (7) of the right-hand side of Eq.(8) one finds

\[ C = \langle T \rangle \langle \bar{T} \rangle - \langle \Theta \rangle \langle \Theta \rangle. \] (10)

On the other hand, some meditation about the equation (8) makes it plausible that the constant \( C \) also coincides with the expectation value of appropriately defined composite operator \( T\bar{T} \). Indeed, one expects that the composite field \( T\bar{T} \) can be obtained in some way from the product \( T(z)\bar{T}(z') \) by bringing the points \( z \) and \( z' \) together. The main obstacle is in the presence of singular terms in the operator product expansion of \( T(z)\bar{T}(z') \), which make straightforward limit impossible. As we will see in the next section, the second term in the combination \( T(z)\bar{T}(z') - \Theta(z)\Theta(z') \) exactly subtracts these singular terms, so that the limit \( z \to z' \) in (8) can be taken, leading to (3).

3 Operator product expansions

It is not difficult to repeat manipulations of the previous section, this time working not with the two-point functions (8) but with the combination of the operator products \( T(z)\bar{T}(z') - \Theta(z)\Theta(z') \) itself. Using only (4) and (5), one finds

\[ \partial_z (T(z)\bar{T}(z') - \Theta(z)\Theta(z')) = \]

\[ (\partial_z + \partial_{z'})\Theta(z)\bar{T}(z') - (\partial_z + \partial_{z'})\Theta(z)\Theta(z'), \] (11)
and
\[ \partial_z (T(z) \bar{T}(z') - \Theta(z) \Theta(z')) = \] \[ (\partial_{\bar{z}} + \partial_z) T(z) \bar{T}(z') - (\partial_{\bar{z}} + \partial_z) T(z) \Theta(z'). \] (12)

The meaning of these equations becomes clearer after inserting the operator product expansions
\[ \Theta(z) \bar{T}(z') = \sum_i B_i (z - z') O_i(z'), \] (13)
\[ T(z) \Theta(z') = \sum_i A_i (z - z') O_i(z'), \] (14)

and
\[ T(z) \bar{T}(z') = \sum_i D_i (z - z') O_i(z'), \] (15)
\[ \Theta(z) \Theta(z') = \sum_i C_i (z - z') O_i(z'), \] (16)

where the sums involve complete set of local fields \( \{ O_i \} \). The equations (11), (12) then read
\[ \sum_i \partial_{\bar{z}} F_i (z - z') O_i(z') = \] \[ \sum_i \left( B_i (z - z') \partial_{z'} O_i(z') - C_i (z - z') \partial_{\bar{z}} O_i(z') \right), \] (17)
\[ \sum_i \partial_z F_i (z - z') O_i(z') = \] \[ \sum_i \left( D_i (z - z') \partial_{z'} O_i(z') - A_i (z - z') \partial_{\bar{z}} O_i(z') \right), \] (18)

where
\[ F_i (z - z') = D_i (z - z') - C_i (z - z'). \] (19)

Note that the right-hand sides of the Eq's (17), (18) involve only coordinate derivatives of local fields. It follows that any operator \( O_i \) appearing in the expansion
\[ T(z) \bar{T}(z') - \Theta(z) \Theta(z') = \sum_i F_i (z - z') O_i(z'), \] (20)
unless itself is a coordinate derivative of another local operator, comes with a constant (i.e., coordinate-independent) coefficient $F_i$. In other words, the operator product expansion (20) can be written as

$$T(z)T(z') - \Theta(z)\Theta(z') = O_{TT}(z') + \text{derivative terms},$$

where $O_{TT}(z)$ is some local operator. At this point it is possible to define the composite field $\bar{T}T$ through the Eq. (21):

$$\bar{T}T(z) := O_{TT}(z).$$

then the desired relation follows immediately. Note that although in this way one defines $\bar{T}T$ only modulo derivative terms, in view of the assumption 2 those terms bring no contribution to the left-hand side of (3). However, this definition may look a bit too formal to bring much insight into the meaning of (3). To understand the nature of the limit $z \to z'$ in Eq. (21), and thus to make contact with more constructive definition of the composite field $TT$, one needs to know more about short-distance behavior of the field theory. In the the present discussion, this information is furnished through the assumption 4 (see section 2). Let me now describe its content and implications.

4 Dimensional analysis

As was mentioned in section 2, I assume that the short-distance limit of the field theory is controlled by certain conformal field theory, which I will refer to simply as the CFT. More precisely, I will assume that the field theory at hand is the CFT perturbed by its relevant operators. To avoid unnecessarily complex expressions, let me first assume that the perturbation is by a single operator $\Phi_\Delta$ of the dimensions $(\Delta, \bar{\Delta})$ with $\Delta < 1$; then the theory is described by the action

$$\mathcal{A} = \mathcal{A}_{CFT} + \mu \int \Phi_\Delta(z) d^2z,$$  

where $\mu$ is a coupling constant which has the dimension $[\text{length}]^{2\Delta - 2}$. This formulation of the theory makes it possible to carry out dimensional analysis of the structure functions in (15).

Let $\{O_i\}$ be complete set of local fields of the CFT, including primary fields as well as their descendents, and let $(\Delta_i, \bar{\Delta}_i)$ be the left and right scale dimensions of the fields $O_i$. This set includes the field $TT$ (of the
dimensions \((2, 2)\), which in CFT is just the descendant \(T\overline{T} = L_{-2}L_{-2}\) of the identity operator. Equivalently, this field can be defined as \(T\overline{T}(z') = \lim_{z \to z'} T(z)\overline{T}(z')\), where the limit is straightforward since in CFT the above operator product has no singularity at \(z = z'\).

As was explained in Ref. [12], the fields \(O_i\) of the perturbed theory are in one-to-one correspondence with the fields of the CFT (hence I use the same notations). The field \(O_i\) has the spin \(s_i = \Delta_i - \overline{\Delta}_i\) and the mass dimension \(d_i = \Delta_i + \overline{\Delta}_i\), and \(O_i\) coincides with the corresponding CFT field in the limit \(\mu \to 0\). Unless certain resonance conditions are met (see below), these properties characterize the field \(O_i\) uniquely [12] (see also [7]). One says that the field \(O_i\) has \(n\)-th order resonance with the field \(O_j\) if these fields have the same spins, \(s_j = s_i\), and their dimensions satisfy the equation \(d_i = d_j + 2n(1 - \Delta)\) (the resonance condition) with some positive integer \(n\). When this resonance condition is fulfilled the above characterization of the field \(O_i\) allows for the ambiguity \(O_i \to O_i + \text{Const} \mu^n O_j\).

The field \(T\overline{T}\) always has intrinsic ambiguity of the form \(T\overline{T} \to T\overline{T} + \text{Const} \partial_z \overline{\partial}_{\overline{z}} \Theta\), where \(\Theta\) is the trace component of the energy-momentum tensor of the perturbed theory. Since in \(23\) \(\Theta = (1 - \Delta) \pi \mu \Phi_{\Delta}\), the ambiguity is due to the first-order resonance of \(T\overline{T}\) with the derivative \(\partial_z \overline{\partial}_{\overline{z}} \Phi_{\Delta}\). However, this ambiguity has no effect on the expectation value of \(T\overline{T}\). For the present analysis the danger is in possible resonances with non-derivative fields. Since at this time I do not know how to handle the resonance cases, I accept the following no-resonance assumption:

4’. Dimensions \(\Delta_i\) of the fields \(O_i\) of the CFT satisfy the condition

\[
\Delta_i - 2 + n(1 - \Delta) \neq 0 \quad \text{for} \quad n = 1, 2, 3, \ldots, \tag{24}
\]

with the only exception of \(\Delta_i = \Delta + 1\) (which is the dimension of \(\partial_z \overline{\partial}_{\overline{z}} \Phi_{\Delta}\)).

According to [3], the OPE structure functions in \(1\) admit power-series expansions in \(\mu\), with the coefficients computable (in principle) through the conformal perturbation theory. Thus, the structure functions \(D_i(z - z')\) in \(1\) can be written as

\[
D_i(z - z') = \sum_{n=0}^{\infty} (z - z')^{\Delta_i - 2 + n(1 - \Delta)} (\bar{z} - \bar{z}')^{\overline{\Delta}_i - 2 + n(1 - \Delta)} D_i^{(n)} \mu^n. \tag{25}
\]

The zero-order coefficients \(D_i^{(0)}\) are taken from the unperturbed CFT, hence \(D_i^{(0)} = 0\) unless \(O_i\) is the field \(T\overline{T}\) or one of its derivatives, and \(D_{T\overline{T}}^{(0)} = 1\). Then it follows from \(24\) that the only terms in the expansions \(25\) which
carry vanishing powers of both $z - z'$ and $\bar{z} - \bar{z}'$ are the zero-order term of $D_{T\bar{T}}$, and the first-order term associated with $\mathcal{O}_i = \partial_x \partial_x \Phi_\Delta$.

Similar expansion can be written down for the structure functions $C_i(z - z')$ in the OPE (16),

$$C_i(z - z') = \sum_{n=2}^{\infty} (z - z')^{\Delta_i - 2 + n(1 - \Delta)} (\bar{z} - \bar{z}')^{\bar{\Delta}_i - 2 + n(1 - \bar{\Delta})} C_i^{(n)} \mu^n. \quad (26)$$

Note that the sum here starts from $n = 2$, consequence of the fact that $\Theta \sim \mu \Phi_\Delta$. In this case the no-resonance condition [24] implies that there are no terms with vanishing powers of both $z - z'$ and $\bar{z} - \bar{z}'$ at all.

Consider now the differences $F_i(z - z') = D_i(z - z') - C_i(z - z')$. It follows from [17], [18] that, unless $\mathcal{O}_i$ is a derivative of another local field, all terms with nonzero powers of $z - z'$ or $\bar{z} - \bar{z}'$ must cancel out in this difference \(^3\). Therefore

$$F_i(z - z') = 0 \quad \text{unless} \quad \mathcal{O}_i = T\bar{T} \quad \text{or} \quad \mathcal{O}_i = \text{derivative}, \quad (27)$$

and

$$F_{T\bar{T}}(z - z') = 1. \quad (28)$$

One concludes that the definition of $T\bar{T}$ through the conformal perturbation theory agrees with the formal definition \([22]\).

It is not difficult to generalize this analysis to the case when the CFT is perturbed by a mixture $\sum a \mu a \int \Phi_{\Delta a}(z) d^2 z$ of relevant operators $\Phi_{\Delta a}$. The dimensional analysis can be carried out in a similar straightforward way provided the no-resonance condition is modified as follows:

4”. Dimensions $\Delta_i$ of the fields $\mathcal{O}_i$ of the CFT satisfy the conditions

$$\Delta_i - 2 + \sum a n_a (1 - \Delta_a) \neq 0 \quad (29)$$

for any non-negative integers $n_a$ such that $\sum a n_a > 0$, with the only exceptions of $\Delta_i = \Delta_a + 1$.

\(^3\)This implies for instance $D_{T\bar{T}}^{(1)} = 0$, the statement which is easily verified in conformal perturbation theory.
5 Further remarks

Let the 2D space be a cylinder, with one of the Cartesian coordinates compactified on a circle of circumference $R$, $(x, y) \sim (x + R, y)$, and let $\mathbb{H}$ and $\mathbb{P}$ be the Hamiltonian and the momentum operators in the picture where the coordinate $y$ along the cylinder is taken as the Euclidean time. The arguments of the previous sections validate the relation (3) with $\langle \ldots \rangle$ standing for the matrix element $\langle 0 \mid \ldots \mid 0 \rangle$, where $\mid 0 \rangle$ is the ground state of the Hamiltonian $\mathbb{H}$ (and I assume that $\langle 0 \mid 0 \rangle = 1$). But it is not difficult to show that the same relation (3) remains valid if the vacuum expectation values there are replaced by generic diagonal matrix elements $\langle n \mid \ldots \mid n \rangle$, where $\mid n \rangle$ is an arbitrary non-degenerate eigenstate of the energy and momentum operators,

$$\mathbb{H} \mid n \rangle = E_n \mid n \rangle, \quad \mathbb{P} \mid n \rangle = P_n \mid n \rangle,$$

and again the normalization $\langle n \mid n \rangle = 1$ is assumed. Indeed, of the assumptions listed in section 2, the local ones (1 and 4) are independent on the choice of matrix element, while the assumption 2 (global translational invariance) certainly remains valid when any diagonal matrix element between energy-momentum eigenstates is taken. Hence one can repeat the calculation at the end of section 2 (which uses only the assumptions 1 and 2) and show that again the combination

$$C(n) = \langle n \mid T(z)\bar{T}(z') \mid n \rangle - \langle n \mid \Theta(z)\Theta(z') \mid n \rangle$$

is a constant, independent of the points $z$ and $z'$. In general, the asymptotic factorization no longer holds, since the two-point function $\langle n \mid \mathcal{O}_i(x, y)\mathcal{O}_j(x', y') \mid n \rangle$ can pick up contributions from the intermediate states $\mid n' \rangle$ with $E_{n'} < E_n$ which give rise to terms growing exponentially with $|y - y'|$. However, one can write down the spectral decompositions of the two-point functions in the right-hand side of (31), i.e.

$$\langle n \mid T(z)\bar{T}(z') \mid n \rangle = \sum_{n'} \langle n \mid T(z) \mid n' \rangle \langle n' \mid \bar{T}(z') \mid n \rangle \times$$

$$e^{(E_n - E_{n'})|y - y'| + i(P_n - P_{n'})(x - x')}$$

and similar decomposition of $\langle n \mid \Theta(z)\Theta(z') \mid n \rangle$, where $(x, y)$ and $(x', y')$ are Cartesian coordinates of the points $z$ and $z'$, respectively. Clearly, for the combination (31) to be independent of the coordinates, all terms in these
decompositions with \( n' \neq n \) must cancel out between the two correlators in the right-hand side of (31). If \( |n\rangle \) is non-degenerate, it follows that

\[
\mathcal{C}(n) = \langle n | T(z) | n \rangle \langle n | \bar{T}(z') | n \rangle - \langle n | \Theta(z) | n \rangle \langle n | \Theta(z') | n \rangle ,
\]

and by taking the limit \( z \to z' \) one arrives at the desired relation

\[
\langle n | T \bar{T} | n \rangle = \langle n | T | n \rangle \langle n | \bar{T} | n \rangle - \langle n | \Theta | n \rangle \langle n | \Theta | n \rangle .
\]

It is useful to rewrite the relation (34) in somewhat different form. In terms of Cartesian components of the energy-momentum tensor \( T_{\mu\nu} \) the Eq. (34) reads

\[
(34) = -\pi^2 \left( \langle n | T_{yy} | n \rangle \langle n | T_{xx} | n \rangle - \langle n | T_{xy} | n \rangle \langle n | T_{xy} | n \rangle \right),
\]

On the other hand, by the very meaning of the energy-momentum tensor we have

\[
\langle n | T_{yy} | n \rangle = -\frac{1}{R} E_n(R), \quad \langle n | T_{xx} | n \rangle = -\frac{d}{dR} E_n(R),
\]

and

\[
\langle n | T_{xy} | n \rangle = \frac{i}{R} P_n(R).
\]

where I have indicated explicitly the \( R \)-dependence of the energy-momentum eigenvalues (of course, the \( R \)-dependence of \( P_n \) is fixed by the momentum quantization condition: \( P_n(R) = 2\pi p_n/R \), where \( p_n \) are \( R \)-independent integers). Thus the expectation value (34) can be expressed in terms of the eigenvalues \( E_n(R), P_n(R) \) as follows

\[
\langle n | T \bar{T} | n \rangle = -\frac{\pi^2}{R} \left( E_n(R) \frac{d}{dR} E_n(R) + \frac{1}{R} P_n^2(R) \right).
\]

Suppose the field theory (23) is massive, with \( M_0 \) being the mass of its lightest particle. Then for \( R >> M_0^{-1} \) the ground-state energy \( E_0(R) \) approaches its asymptotic linear form with exponential accuracy, i.e.

\[
E_0(R) = F_0 R + O(e^{-M_0 R}),
\]

\[\text{The factor } \pi^2 \text{ here is due to the factor } 2\pi \text{ in the definition of the chiral components } T \text{ and } \bar{T}, \text{ see sect.1}\]
where $F_0$ is the vacuum energy density in infinite space. In the same limit, the first excited state $|1\rangle$ corresponds to the one-particle state with zero momentum, hence

$$E_1(R) = F_0 R + M_0 + O(e^{-M_0 R}).$$  \hspace{1cm} (40)

Then it follows from (34) that (up to terms $\sim e^{-M_0 R}$)

$$\frac{1}{\pi^2} \langle 0 | T \bar{T} | 0 \rangle = -F_0^2, \quad \frac{1}{\pi^2} \langle 1 | T \bar{T} | 1 \rangle = -F_0^2 - \frac{1}{R} F_0 M_0. \hspace{1cm} (41)$$

These expressions can be useful in analysis of subleading singularities in statistical systems near criticality, in the situations where the irrelevant operator $T \bar{T}$ plays significant role. This is the case, for instance, for the Ising phase transition near the Ising tri-critical point, because the RG flow from the tricritical fixed point (the $c = 7/10$ minimal CFT) down to the Ising fixed point (the $c = 1/2$ minimal CFT) arrives at the latter along direction which contains the field $T \bar{T}$ as its most significant (i.e. least irrelevant) component \cite{13}. Another example is the Ising field theory with pure imaginary magnetic field, taken near the Yang-Lee singularity. In such cases the relations (41) lead to predictions about amplitudes of subleading singular terms in the expansions of the free energy and correlation length near the critical point. I intend to discuss these applications elsewhere.

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