On integrating the left-flat vacuum Einstein equations

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Abstract
Considering the spin-coefficient version of the left-flat vacuum Einstein equations, all but one of the fifty equations can be explicitly integrated via the introduction of five spin-weight $s = -2$ complex potentials. The final equation is a nonlinear wave equation for the last of the potentials. Solutions to this equation determine solutions for the entire system. Solutions for several special cases are obtained.

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1. Introduction

1.1. Background

The self-dual/left-flat (or anti-self-dual/right-flat) Ricci-flat equations and their spaces with associated metrics has been a subject of considerable interest since the late 1970s. The very large literature has been associated with a variety of linked interests.

They play a fundamental role in the properties of real asymptotically flat space-times via solutions to the ‘good-cut’ equation [1–7], they define Penrose’s asymptotic twistor space [8–10], they (the solutions) are intimately related to (in fact are) Penrose’s nonlinear graviton fields, the equations themselves are studied as a beautiful example of a nonlinear integrable system [11–13] and finally the Euclidean versions, the so-called gravitational instantons [14, 15] (the general relativistic generalization of the Yang–Mills instantons) have played a role in attempts at understanding quantum gravity.

From the beginning there have been a variety of attempts at integrating the field equations and even now—40 years after their popularity began—only a relatively small number of solutions appear to be known. The first solutions—a very limited number—were obtained via the ‘good-cut’ equation. The best know is the Sparling–Tod metric [6] Eguchi–Hansen [7] metric. Many others were discovered by imposing special conditions as for example algebraic specialness [16] or (Killing) symmetries [17]. In addition there is a large literature on self-dual gravitational instantons [14].

There have been several attacks on the problem of finding (or even of studying) the general solution of the self-dual Einstein equations on general complex four dimensional
manifolds—either globally or locally [3, 4, 20, 21]. None were totally successful. It is the purpose of this note to return to this issue.

Although all H-spaces (i.e., spaces arising from solutions of the good-cut equation) are self-dual spaces, it is not yet clear whether all self-dual spaces are H-spaces. Nevertheless we will not make a distinction here between the two.

1.2. Modus operandi

We begin, in section 2, with the fully general Einstein equations in the spin-coefficient formalism written as complex equations on a thickened region of $\mathbb{C}^4$ in a neighborhood of the real ‘slice’ $\mathbb{R}^4$. In the standard ‘real’ version of the formalism, the basic field variables, i.e., the spin-coefficients, the metric variables and Weyl tensor components, are, in general, complex. In the field equations themselves, their complex conjugates appear symmetrically so that the equations themselves, as a complete set are real. In our present (SD) version however, the previous ‘complex conjugates’ are now completely freed up from their conjugate counterparts and are independent variables. These field equations are then reduced by setting the anti-self-dual part of the Weyl tensor to zero, thereby imposing the self-dual condition on the equations.

A coordinate system is introduced, in section 2, by choosing a world-line, $L$, in the thickened region of $\mathbb{C}^4$; with the (complex) $u$, being the ‘complex time’ along the world-line. The null cones with apex on the line are labeled by $u$. The complex sphere of null directions at the apex of each cone (and their associated null geodesics) are labeled by the complex stereographic coordinates $(\zeta, \tilde{\zeta})$ while the affine parameter along the null geodesics is given by $r$. Note that both $u$ and $r$, though complex, take values close to the real while the complex $\tilde{\zeta}$ takes values close to the complex conjugate $\zeta$.

From these imposed conditions we easily show, section 2, that the complex divergence, $\rho$, of the null geodesics with apex on $L$ is given by $\rho = -r^{-1}$ and the left shear by $\sigma = 0$.

At this point the entire set of field equations, 58, are divided into three sets: [1] the 29 radial equations which involve the $r$ derivative, [2] (7) of them involving the angular derivatives, i.e., $(\zeta, \tilde{\zeta})$ and [3] those, 22, that contain the time-derivative, $u$.

1.3. Results

What is a bit surprising is that every one of the $r$-derivative equations can be exactly integrated in terms of four spin-weight (spin-wt) $(s = -2)$, potentials, $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$, where each can be expressed as $r$-derivatives of the next one, so that the only independent one is $\gamma_4$.

The angular equations are then used to establish relationships between some of the free ‘constants’ of the $r$-integrations.

Finally when all these results are inserted into the 22 time-derivative equations we discover that they are all closely related and can be reduced to a single complex spin-wt $(s = -2)$, nonlinear wave equation for the last potential $\gamma_4$. This last equation carries all the evolution information for the entire set of equations.

Though we do not know of a way to give general solutions to this equation, it is quite easy to produce many special solutions.

2. The field equations

2.1. Complexification of the spin-coefficient equations

For the ease and clarity of the process of the complexification of the spin-coefficient equations we should first have in mind the full set of real spin-coefficient equations. However to avoid
cluttering the manuscript with too many equations upfront, we relegate the full set to an appendix and just explain the complexification process here.

The $\lambda^a_i \equiv (l^a, n^i, m^a, \bar{m}^i)$ are the components of a null tetrad, the $\Psi'$'s and $\bar{\Psi}'$'s are the components of the Weyl tensor, $\lambda^a_i \nabla_a = (D, \Delta, \delta, \bar{\delta})$ are the directional derivatives. All the other variables are the spin-coefficients. The Einstein equations are already built into the system by virtue of the Ricci tensor having been set to zero.

In the full set of the real spin-coefficient equations some of the equations are real while others are complex. In the real setting, the statement of a complex equation obviously implies the statement of the complex conjugate equation. This avoids the need to write both the complex equation and its complex conjugate.

For the complexification of the spin-coefficient equations we leave the original 'real' equations unchanged (though the variables become complex) but now the original complex equations with their complex conjugates become independent equations. The four coordinates are independent complex variables and the functions are now all locally holomorphic. There is no operation of complex conjugation. This has the formal effect of greatly increasing the number of equations we need to deal with.

As an example, consider the first three equations of the appendix, namely:

\[
\begin{align*}
\Delta l^a - Dn^a &= (\gamma + \bar{\gamma})l^a + (\epsilon + \bar{\epsilon})n^a - (\tau + \bar{\tau})\bar{m}^a - (\bar{\tau} + \tau)m^a, \\
\bar{\delta} l^a - Dm^a &= (\alpha + \bar{\beta} - \pi)l^a + \kappa n^a - \sigma \bar{m}^a - (\bar{\sigma} + \epsilon - \epsilon)m^a, \\
\tilde{\delta} l^a - D\bar{m}^a &= (\alpha + \bar{\beta} - \pi)l^a + \bar{\kappa} n^a - \bar{\sigma} m^a - (\rho + \bar{\epsilon} - \epsilon)m^a.
\end{align*}
\]

The first one, which was real, remains unchanged while the second and third are complex conjugates in the real case become independent equations after the complexification. We avoid the suggestion of complex conjugation by the notational change: an overbar becomes a tilde, e.g., $\bar{m}^a \mapsto \tilde{m}^a$, $\bar{\alpha} \mapsto \tilde{\alpha}$ or $\bar{\zeta} \mapsto \tilde{\zeta}$.

These complexified spin-coefficient equations become the starting point for our study of the self-dual Einstein equations. We avoid writing them out in detail since they can be easily inferred from the appendix. In the next section the self-dual condition, with coordinate and tetrad conditions, are imposed that greatly simplify the equations.

3. Left-flat spin-coefficient version of Einstein equations

As just remarked, several different conditions that are now imposed on these equations to restrict them to left-flat equations and in addition to greatly simplify them.

The first—and most basic—is to impose the left-flat (or self-dual) condition on the equations. This is accomplished by simply taking

\[
\begin{align*}
\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4 &= 0,
\end{align*}
\]

which is equivalent to imposing on the Weyl tensor $C_{abcd}$ that

\[
C_{abcd}e^{def} = iC_{ab}^{ef}.
\]

All the barred variables (e.g., $\sigma, \bar{\lambda}, \ldots$) are independent and freed from their dual counterparts. This is denoted by replacing the bars by tildes, (e.g., $\bar{\sigma}, \tilde{\lambda}, \ldots$).

The other conditions (both coordinate and tetrad conditions) are imposed on the tetrad vectors, $\lambda^a_i$. We choose a world-line, $l$, in the complex thickened $R^4$ and a one parameter family of (complex) light-cones with apex on $l$ labeled by the complex coordinate $\lambda^0 = u$. The null generators of the cones (null geodesics) are labeled by the points of the thickened sphere with complex stereographic coordinates $x^A = (\zeta, \bar{\zeta})$, with $\zeta \approx \bar{\zeta}$. The affine parameter of the null
geodesics is the complex radial coordinate \( x^1 = r \). These conditions allow us [18] to write the directional derivatives (or tetrad) as

\[
D = l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial r}
\]

\[
\nabla = n^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X^A \frac{\partial}{\partial x^A}, \quad (x^2, x^3) = (\zeta, \tilde{\zeta}),
\]

\[
\delta = m^a \frac{\partial}{\partial x^a} = \omega \frac{\partial}{\partial r} + \xi^A \frac{\partial}{\partial x^A},
\]

\[
\tilde{\delta} = \tilde{m}^a \frac{\partial}{\partial x^a} = \tilde{\omega} \frac{\partial}{\partial r} + \tilde{\xi}^A \frac{\partial}{\partial x^A},
\]

and associated metric, \( g^{ab} = l^a n^b + n^a \tilde{m}^b - \tilde{m}^a m^b \),

\[
\begin{bmatrix}
0, & 1, & 0 \\
1, & g^{11}, & g^{1A} \\
0, & g^{1A}, & g^{AB}
\end{bmatrix}
\]

with

\[
g^{22} = 2(U - \omega \tilde{\omega}), \quad g^{2A} = X^A - (\omega \xi^A + \omega \tilde{\xi}^A), \quad g^{AB} = - (\xi^A \tilde{\xi}^B + \tilde{\xi}^A \xi^B).
\]

By appropriate tetrad transformations [18] (parallel propagation of \( n^a, m^a, \tilde{m}^a \) and appropriate scaling of \( l^a \)) we can put the following conditions on the spin-coefficients:

\[
\kappa = \pi = \epsilon = \tilde{\kappa} = \tilde{\pi} = \tilde{\epsilon} = \rho - \tilde{\rho} = 0.
\]

We also have, but do not use immediately,

\[
\tau = \tilde{\alpha} + \beta, \quad \tilde{\tau} = \alpha + \tilde{\beta}.
\]

These conditions are then inserted into our Einstein field equations to obtain our final set of field equations. The equations are displayed in groups: first, \( 17 \) of them, that contain the \( D \) derivatives, then, \( 7 \), containing only the angular derivatives, \( (\delta, \tilde{\delta}) \) and finally, \( 20 \), containing the time-derivative, \( \Delta \).

\[
D\rho = \rho^2 + \sigma \tilde{\sigma}
\]

\[
D\sigma = 2 \rho \sigma
\]

\[
D\tilde{\sigma} = 2 \rho \tilde{\sigma} + \tilde{\Psi}_0
\]

\[
D\tau = \tau \rho + \tilde{\tau} \sigma
\]

\[
D\tilde{\tau} = \tau \tilde{\rho} + \tau \tilde{\sigma} + \tilde{\Psi}_1
\]

\[
D\alpha = \rho \alpha + \beta \tilde{\sigma}
\]

\[
D\tilde{\alpha} = \rho \tilde{\alpha} + \tilde{\pi} \sigma
\]

\[
D\beta = \alpha \sigma + \rho \beta
\]

\[
D\tilde{\beta} = \tilde{\alpha} \sigma + \tilde{\rho} \beta + \tilde{\Psi}_1
\]

\[
D\gamma = \tau \alpha + \tilde{\tau} \beta
\]

\[
D\tilde{\gamma} = \tau \tilde{\alpha} + \tau \tilde{\beta} + \tilde{\Psi}_2
\]
\[ D\lambda = \rho\lambda + \delta\mu \]  
(23)

\[ D\tilde{\lambda} = \rho\tilde{\lambda} + \sigma\tilde{\mu} \]  
(24)

\[ D\mu = \rho\mu + \sigma\lambda \]  
(25)

\[ D\tilde{\mu} = \rho\tilde{\mu} + \tilde{\sigma}\lambda + \tilde{\Psi}_2 \]  
(26)

\[ D\nu = \tau\mu + \tau\lambda \]  
(27)

\[ D\tilde{\nu} = \tau\tilde{\mu} + \tilde{\tau}\lambda + \tilde{\Psi}_3 \]  
(28)

\[ DU = \tau\omega + \tilde{\tau}\omega - (\gamma + \tilde{\gamma}) \]  
(29)

\[ DX^A = \tau\tilde{\xi}^A + \tilde{\tau}\xi^A, \]  
(30)

\[ D\omega = \sigma\tilde{\omega} + \rho\omega - \tau, \]  
(31)

\[ D\tilde{\omega} = \tilde{\sigma}\omega + \rho\tilde{\omega} - \tilde{\tau}, \]  
(32)

\[ D\xi^A = \sigma\tilde{\xi}^A + \rho\xi^A, \]  
(33)

\[ D\tilde{\xi}^A = \tilde{\sigma}\xi^A + \rho\tilde{\xi}^A, \]  
(34)

\[ \delta\tilde{\Psi}_0 - D\tilde{\Psi}_1 = 4\tilde{\alpha}\tilde{\Psi}_0 - 4\rho\tilde{\Psi}_1, \]  
(35a)

\[ \delta\tilde{\Psi}_1 - D\tilde{\Psi}_2 = \tilde{\lambda}\tilde{\Psi}_0 + 2\tilde{\alpha}\tilde{\Psi}_1 - 3\rho\tilde{\Psi}_2, \]  
(35b)

\[ \delta\tilde{\Psi}_2 - D\tilde{\Psi}_3 = 2\tilde{\lambda}\tilde{\Psi}_1 - 2\rho\tilde{\Psi}_3, \]  
(35c)

\[ \delta\tilde{\Psi}_3 - D\tilde{\Psi}_4 = 3\tilde{\lambda}\tilde{\Psi}_2 - 2\tilde{\alpha}\tilde{\Psi}_3 - \rho\tilde{\Psi}_4, \]  
(35d)

\[ \delta\rho - \delta\sigma = \rho(\tilde{\alpha} + \tilde{\beta}) - \sigma(3\tilde{\alpha} - \tilde{\beta}) \]  
(36)

\[ \tilde{\delta}\rho - \tilde{\delta}\tilde{\sigma} = \rho(\alpha + \tilde{\beta}) - \tilde{\sigma}(3\tilde{\alpha} - \beta) - \tilde{\Psi}_1 \]  
(37)

\[ \delta\alpha - \tilde{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\tilde{\alpha} + \beta\tilde{\beta} - 2\alpha\beta \]  
(38)

\[ \tilde{\delta}\alpha - \tilde{\delta}\tilde{\beta} = \tilde{\mu}\rho - \tilde{\lambda}\tilde{\sigma} + \alpha\tilde{\alpha} + \beta\tilde{\beta} - 2\tilde{\alpha}\tilde{\beta} - \tilde{\Psi}_2 \]  
(39)

\[ \delta\lambda - \tilde{\delta}\mu = \mu(\alpha + \tilde{\beta}) + \lambda(\tilde{\alpha} - 3\beta) \]  
(40)

\[ \tilde{\delta}\lambda - \tilde{\delta}\tilde{\mu} = \tilde{\mu}(\tilde{\alpha} + \tilde{\beta}) + \tilde{\lambda}(\alpha - 3\tilde{\beta}) - \tilde{\Psi}_3 \]  
(41)

\[ \tilde{\delta}\omega - \delta\tilde{\omega} = (\tilde{\mu} - \mu) - ((\tilde{\alpha} - \beta)\tilde{\omega} + (\alpha - \tilde{\beta})\omega) \]  
(42)

\[ \tilde{\delta}\xi^A - \delta\tilde{\xi}^A = -(\tilde{\alpha} - \beta)\xi^A + (\alpha - \tilde{\beta})\tilde{\xi}^A \]  
(43)

\[ \delta U - \Delta\omega = -\nu + \tilde{\lambda}\tilde{\omega} + (\mu - \gamma + \tilde{\gamma})\omega, \]  
(44)

\[ \tilde{\delta}U - \tilde{\Delta}\tilde{\omega} = -\tilde{\nu} + \lambda\omega + (\tilde{\mu} - \tilde{\gamma} + \gamma)\tilde{\omega}, \]  
(45)

\[ \delta X^A - \Delta\xi^A = \tilde{\lambda}\tilde{\xi}^A + (\mu - \gamma + \tilde{\gamma})\xi^A \]  
(46)

\[ \tilde{\delta}X^A - \tilde{\Delta}\tilde{\xi}^A = \lambda\tilde{\xi}^A + (\tilde{\mu} - \tilde{\gamma} + \gamma)\tilde{\xi}^A \]  
(47)
\[
\Delta \lambda - \delta v = -(\mu + \tilde{\mu})\lambda - (3\gamma - \tilde{\gamma}) + 2\alpha \lambda \\
\Delta \tilde{\lambda} - \delta \tilde{v} = -(\mu + \tilde{\mu})\tilde{\lambda} - (3\tilde{\gamma} - \gamma) + 2\tilde{\alpha} \lambda - \tilde{\Psi}_4 \\
\delta v - \Delta \mu = \mu^2 + \lambda \lambda + \mu(\gamma + \tilde{\gamma}) - 2\beta v \\
\delta \tilde{v} - \Delta \tilde{\mu} = \tilde{\mu}^2 + \lambda \lambda + \tilde{\mu}(\gamma + \tilde{\gamma}) - 2\tilde{\beta} \tilde{v} \\
\delta \gamma - \Delta \beta = \mu \tau - \sigma \nu - \beta(\gamma - \tilde{\gamma} - \mu) + \alpha \lambda \\
\delta \tilde{\gamma} - \Delta \tilde{\beta} = \tilde{\mu} \tau - \sigma \tilde{\nu} - \tilde{\beta}(\tilde{\gamma} - \gamma - \tilde{\mu}) + \tilde{\alpha} \lambda \\
\delta \tau - \Delta \sigma = (\mu \sigma + \rho \tilde{\lambda}) + \tau (\tau + \beta - \tilde{\alpha} - \sigma (3\gamma - \tilde{\gamma}) \\
\delta \tilde{\tau} - \Delta \tilde{\sigma} = (\tilde{\mu} \tilde{\sigma} + \rho \lambda) + \tilde{\tau} (\tilde{\tau} + \tilde{\beta} - \alpha - \tilde{\sigma} (3\tilde{\gamma} - \gamma) \\
\Delta \rho - \tilde{\sigma} \tau = -(\rho \tilde{\mu} + \sigma \lambda) + \tau (\tilde{\beta} - \alpha - \tilde{\tau}) + (\gamma + \tilde{\gamma}) \rho \\
\Delta \rho - \delta \tilde{\tau} = -(\rho \mu + \tilde{\sigma} \lambda) + \tilde{\tau} (\beta - \tilde{\alpha} - \tau) + (\gamma + \tilde{\gamma}) \rho - \tilde{\Psi}_2 \\
\Delta \alpha - \delta \gamma = \nu \rho - \lambda (\tau + \beta) + \alpha (\tilde{\gamma} - \tilde{\mu}) + \gamma (\tilde{\beta} - \tilde{\tau}) \\
\Delta \tilde{\alpha} - \delta \tilde{\gamma} = \nu \rho - \tilde{\lambda} (\tilde{\tau} + \tilde{\beta}) + \tilde{\alpha} (\gamma - \mu) + \tilde{\gamma} (\beta - \tau) - \tilde{\Psi}_3 \\
\Delta \tilde{\Psi}_0 - \delta \tilde{\Psi}_1 = (4\tilde{\gamma} - \tilde{\mu}) \tilde{\Psi}_0 - 2(2\tilde{\tau} + \tilde{\beta}) \tilde{\Psi}_1 + 3\tilde{\sigma} \tilde{\Psi}_2, \\
\Delta \tilde{\Psi}_1 - \delta \tilde{\Psi}_2 = \nu \tilde{\Psi}_0 + 2(\tilde{\gamma} - \tilde{\mu}) \tilde{\Psi}_1 - 3\tilde{\tau} \tilde{\Psi}_2 + 2\tilde{\sigma} \tilde{\Psi}_3, \\
\Delta \tilde{\Psi}_2 - \delta \tilde{\Psi}_3 = 2\nu \tilde{\Psi}_1 - 3\tilde{\mu} \tilde{\Psi}_2 + 2(\tilde{\beta} - \tilde{\tau}) \tilde{\Psi}_3 + \tilde{\sigma} \tilde{\Psi}_4, \\
\Delta \tilde{\Psi}_3 - \delta \tilde{\Psi}_4 = 3\nu \tilde{\Psi}_2 - 2(\tilde{\gamma} + 2\tilde{\mu}) \tilde{\Psi}_3 + (4\tilde{\beta} - \tilde{\tau}) \tilde{\Psi}_4.
\]

In the following section we integrate explicitly all the \(D\) equations. The solution to each equation will have an associated ‘constant’ (actually a function independent of \(r\)) of integration. These ‘constants’ are determined by either the angular equations or by the fact of all the null surfaces being light-cones with their origin on a world-line. The evolution equations (i.e., those with \(\Delta\)) turn out—initially—to be extremely complicated to deal with. Nevertheless—and quite surprising—in the end they all are equivalent to one single nonlinear wave equation.

4. Integration

4.1. Radial integration

4.1.1. Preliminary integration. The first thing to note is that equations (12) and (13) can be integrated immediately, (with the coordinate origin for \(r\) taken at the apex of the light-cone) as
\[
\rho = -\frac{1}{r}, \\
\sigma = 0.
\]

The later follows from the fact that \(\sigma\) vanishes at a light-cone apex.
The field equations are greatly simplified by these results. The spin-coefficient equations (15), (18), (19), (24), (25) decouple from the remainder and using the results from flat space light-cones with origins on a world-line [1] we have

\[ \tau = \tilde{\alpha} + \beta = 0, \]  
\[ \tilde{\alpha} = r^{-1} \tilde{\alpha}^0 = -\frac{1}{2} r^{-1} \partial P, \quad P = 1 + \zeta \tilde{\zeta}, \]  
\[ \beta = -\tilde{\alpha}, \]  
\[ \tilde{\lambda} = 0, \]  
\[ \mu = -r^{-1}. \]  

The form of the function \( P \), (67), which determines the 2-sphere metric at the cones apex via equation (9), is set as a coordinate condition.

In a similar fashion, with \( \tau = 0 \), the metric equations (31), (33) integrate to

\[ \xi^A = r^{-1}(\xi^0, \xi^\tilde{0}) = -r^{-1}(P, 0), \]  
\[ \omega = 0. \]  

4.1.2. The radial Bianchi identities and Weyl tensor. Turning now to our first non-trivial integrations, using the preceding results, the four radial Bianchi identities, (35a)–(35d) become

\[ \partial_r \tilde{\Psi}_1 = -4r^{-1} \tilde{\Psi}_1 - r^{-1} \partial \tilde{\Psi}_0, \]  
\[ \partial_r \tilde{\Psi}_2 = -3r^{-1} \tilde{\Psi}_2 - r^{-1} \partial \tilde{\Psi}_1, \]  
\[ \partial_r \tilde{\Psi}_3 = -2r^{-1} \tilde{\Psi}_3 - r^{-1} \partial \tilde{\Psi}_2, \]  
\[ \partial_r \tilde{\Psi}_4 = -r^{-1} \tilde{\Psi}_4 - r^{-1} \partial \tilde{\Psi}_3, \]

where equations (4) and (7), along with \( \partial \eta_s = P^{1-s} \eta_s(P \eta_s) \), were used.

Writing these equations, with \( n = 1, 2, 3, 4 \), succinctly as

\[ \partial_r \tilde{\Psi}_{5-n} = -nr^{-1} \tilde{\Psi}_{5-n} - r^{-1} \partial \tilde{\Psi}_{4-n}, \]

the solutions are given by

\[ \tilde{\Psi}_{5-n} = r^{-n} \tilde{\Psi}_{5-n}^{(0)} - r^{-n} \partial \int r^{n-1} \tilde{\Psi}_{4-n} \, dr, \]

the integral being indefinite, with \( \tilde{\Psi}_{5-n}^{(0)} \) the ‘constants of integration’.

Our integration process continues with the introduction of the set of five, spin-wt \( s = -2 \), potentials, \( \tilde{\gamma}_{5-n}(r) \), \( n = 1, 2, 3, 4, 5 \) by

\[ \tilde{\Psi}_{5-n} = (1)^{n-1} r^{-n+2} \tilde{\gamma}^{(5-n)} \partial_r \tilde{\gamma}_{5-n} \]

and inserting them into the integral of equation (75).

Integration leads to

\[ \tilde{\Psi}_{5-n} = r^{-n} \tilde{\Psi}_{5-n}^{(0)} - (1)^n r^{-n} \tilde{\gamma}^{(5-n)} \tilde{\gamma}_{5-n}(r), \]

where, from regularity at \( r = 0 \), we have

\[ \tilde{\Psi}_{5-n}^{(0)} = 0. \]
Remark 1. We emphasize that there are two different types of situations. In one case the potentials, $\tilde{\Upsilon}$, vanish sufficiently fast at $r = 0$ so that the origin is a regular point of the manifold. In the other case there are intrinsic singularities hidden in the $\tilde{\Upsilon}(r)$ at $r = 0$. See the following section for examples.

Written explicitly, equation (77) are

$$\tilde{\psi}_0 = r^{-5} \tilde{\Upsilon}_0,$$  \hspace{1cm} (79)

$$\tilde{\psi}_1 = -r^{-4} \partial_r \tilde{\Upsilon}_1,$$  \hspace{1cm} (80)

$$\tilde{\psi}_2 = r^{-3} \partial_r^2 \tilde{\Upsilon}_2,$$  \hspace{1cm} (81)

$$\tilde{\psi}_3 = -r^{-2} \partial_r^3 \tilde{\Upsilon}_3,$$  \hspace{1cm} (82)

$$\tilde{\psi}_4 = r^{-1} \partial_r^4 \tilde{\Upsilon}_4.$$  \hspace{1cm} (83)

By equating equations (76) and (77), we find the ‘ladder’ relationship between the different potentials,

$$\tilde{\Upsilon}_5 = r^2 \partial_r \tilde{\Upsilon}_6,$$  \hspace{1cm} (84)

or

$$\tilde{\Upsilon}_0 = r^3 \partial_r \tilde{\Upsilon}_1,$$  \hspace{1cm} (85)

$$\tilde{\Upsilon}_1 = r^3 \partial_r \tilde{\Upsilon}_2,$$  \hspace{1cm} (86)

$$\tilde{\Upsilon}_2 = r^3 \partial_r \tilde{\Upsilon}_3,$$  \hspace{1cm} (87)

$$\tilde{\Upsilon}_3 = r^3 \partial_r \tilde{\Upsilon}_4.$$  \hspace{1cm} (88)

Knowledge of $\tilde{\Upsilon}_4$ (which we can consider as free initial data) determines the others or equivalently, from $\tilde{\Upsilon}_0$ with the four constants of integration, from going up the ladder to $\tilde{\Upsilon}_4$, determines the others.

In the case of the regularity of the Weyl components at $r = 0$, we must impose conditions on the $\tilde{\Upsilon}_n$ so that near $r = 0$

$$\tilde{\Upsilon}_n = O(r^{5-n}).$$  \hspace{1cm} (89)

This, with suitable differentiability, follows from $\tilde{\Upsilon}_4 = A r + O(r^2)$.

For conventional asymptotic flatness (peeling) we require that, as $r \to \infty$, all the potentials tend to a constant, $\tilde{\Upsilon}_n = c_n + O(r^{-1})$.

From equation (84) we have the asymptotic behavior of the potentials:

$$\tilde{\Upsilon}_0 = r^3 \partial_r \tilde{\Upsilon}_1 = c_0 + 0(r^{-1}),$$  \hspace{1cm} (90)

$$\tilde{\Upsilon}_1 = r^3 \partial_r \tilde{\Upsilon}_2 = c_1 - c_0 r^{-1} + 0(r^{-2}),$$  \hspace{1cm} (91)

$$\tilde{\Upsilon}_2 = r^3 \partial_r \tilde{\Upsilon}_3 = c_2 - c_1 r^{-1} + \frac{1}{2} c_0 r^{-2} + 0(r^{-3}),$$  \hspace{1cm} (92)

$$\tilde{\Upsilon}_3 = r^3 \partial_r \tilde{\Upsilon}_4 = c_3 - c_2 r^{-1} + \frac{1}{2} c_1 r^{-2} - \frac{1}{6} c_0 r^{-3} + 0(r^{-4}),$$  \hspace{1cm} (93)

$$\tilde{\Upsilon}_4 = c_4 - c_3 r^{-1} + \frac{1}{2} c_2 r^{-2} - \frac{1}{6} c_1 r^{-3} + \frac{1}{24} c_0 r^{-4} + 0(r^{-5}).$$  \hspace{1cm} (94)
4.1.3. Remaining radial equations. The remaining spin-coefficient and metric equations can be integrated and expressed in terms of the potentials. We first investigate equation (14) using $\Psi_0 = r^{-3}\tilde{Y}_0$, i.e.,
\[ D\tilde{\sigma} = -2r^{-1}\tilde{\sigma} + r^{-3}\tilde{Y}_0. \] (95)
Its solution is given by
\[ \tilde{\sigma} = \tilde{\sigma}^0 r^{-2} + r^{-2} \int r^{-3}\tilde{Y}_0 \, dr. \] (96)
with (again) an indefinite integral and $\tilde{\sigma}^0$ being the constant of integration.

The integral term can be greatly simplified via repeated use of equation (84). From
\[ \int r^{-3}\tilde{Y}_0 \, dr = \int r^{-1}\tilde{\alpha}_1 \, dr = \int [\partial_i (r^{-1}\tilde{Y}_1) + r^{-2}\tilde{Y}_1] \, dr \]
\[ = r^{-1}\tilde{Y}_1 + \int r^{-2}\tilde{Y}_1 \, dr = r^{-1}\tilde{Y}_1 + \int \tilde{\alpha}_2 \, dr \]
\[ = r^{-1}\tilde{Y}_1 + \tilde{Y}_2 \] (97)
we have
\[ \tilde{\sigma} = \tilde{\sigma}^0 r^{-2} + r^{-3}\tilde{Y}_1 + r^{-2}\tilde{Y}_2. \] (98)

From the vanishing of the $\tilde{Y}$ at $r = 0$ and the regularity of the light-cones at the origin, we take $\tilde{\sigma}^0 = 0$, so that
\[ \tilde{\sigma} = r^{-3}\tilde{Y}_1 + r^{-2}\tilde{Y}_2. \] (99)

Using the same methods, i.e., the formal radial integration of the equations followed by the repeated use of the ladder relations, equation (84), all the remaining radial equations can be integrated and expressed in terms of the $\tilde{Y}_n$. Many of the associated 'constants of integration' are determined by the behavior near $r = 0$, several are determined by use of the angular equation. (It is likely that even those determined from the angular equations could have been determined from their $r = 0$ behavior.)

The following is the full set of solutions to all the spin-coefficient and metric radial equations.

Spin-coefficients:
\[ \sigma = \tau = \tilde{\alpha} + \beta = 0, \quad \tilde{\tau} = \alpha + \tilde{\beta}, \] (100)
\[ \rho = \tilde{\rho} = -r^{-1}, \] (101)
\[ \tilde{\alpha} = r^{-2}\tilde{Y}_2 + r^{-3}\tilde{Y}_1, \] (102)
\[ \tilde{\beta} = -r^{-2} \partial_i \tilde{Y}_2 - r^{-1} \partial_i \tilde{Y}_3, \] (103)
\[ \tilde{\sigma} = r^{-3} \tilde{\sigma}^0 = -\frac{1}{2} r^{-1} \partial_i P, \] (104)
\[ \beta = r^{-1} \beta^0 = -\tilde{\alpha}, \] (105)
\[ \alpha = -r^{-1} \left( \frac{1}{2} \partial_i P - \partial_i (P \tilde{Y}_3) \right) + r^{-2} \frac{3}{2} \partial_i P \tilde{Y}_2, \] (106)
\[ \tilde{\beta} = r^{-1} \left[ \frac{2}{3} \partial_i P - \partial_i (P \tilde{Y}_3) - \partial_i (P \tilde{Y}_3) \right] - r^{-2} \left( \partial_i \tilde{Y}_2 + \frac{3}{2} \partial_i P \tilde{Y}_2 \right) \] (107)
\[ \tilde{\gamma} = -\partial_i P \partial_i \tilde{Y}_4 - r^{-1} \frac{3}{2} \partial_i P \partial_i \tilde{Y}_3 \] (108)
\[ \gamma = \tilde{\gamma} = \partial_i P \partial_i \tilde{Y}_4 + r^{-1} \left( \partial_i \tilde{Y}_3 + \frac{3}{2} \partial_i P \partial_i \tilde{Y}_3 \right) \] (109)
\( \tilde{\lambda} = 0 \) \quad (110) \\
\( \mu = -r^{-1} \) \quad (111) \\
\( \lambda = -2r^{-1}\tilde{\gamma}_3 - r^{-2}\tilde{\gamma}_2 \) \quad (112) \\
\( \tilde{\mu} = -r^{-1} + r^{-1}\tilde{\theta}^2\tilde{\gamma}_3 \) \quad (113) \\
\( \tilde{\nu} = -\tilde{\theta}^3\tilde{\gamma}_4 \) \quad (114) \\
\( \nu = 2\tilde{\theta}^3\tilde{\gamma}_4 + r^{-1}\tilde{\theta}\tilde{\gamma}_3 \) \quad (115) \\

Metric Variables:

\( \xi^A = (\xi^\zeta, \xi^\tilde{\zeta}) = r^{-1}\xi^0 = -r^{-1}(P, 0), \quad P = 1 + \xi \tilde{\zeta} \) \quad (116) \\
\( \tilde{\xi}^A = (\tilde{\xi}^\zeta, \tilde{\xi}^\tilde{\zeta}) = -r^{-1}(0, P) - (r^{-1}\tilde{\gamma}_3 + r^{-2}\tilde{\gamma}_2) \) \quad (117) \\
\( \omega = r^{-1}\omega^0 = 0 \) \quad (118) \\
\( \tilde{\omega} = -r^{-1}\omega^0 + \tilde{\theta}\tilde{\gamma}_3 = \tilde{\theta}\tilde{\gamma}_3 \) \quad (119) \\
\( X^A = (X^\zeta, X^\tilde{\zeta}) = (P, 0)[r^{-1}\tilde{\theta}\tilde{\gamma}_3 + 2\tilde{\theta}\tilde{\gamma}_4] \) \quad (120) \\
\( U = -1 - r\tilde{\theta}^2\tilde{\gamma}_4 \). \quad (121) \\

From these results we have the derivative operators which are needed in the angular and evolution equations, namely

\( D = \partial_r, \) \quad (122) \\
\( \nabla = \partial_r - (1 + r\tilde{\theta}^2\tilde{\gamma}_4)\partial_{\tilde{\zeta}} + (r^{-1}P\tilde{\theta}\tilde{\gamma}_3 + 2P\tilde{\theta}\tilde{\gamma}_4)\partial_{\zeta}, \) \quad (123) \\
\( \delta = -r^{-1}P\partial_{\zeta}, \) \quad (124) \\
\( \tilde{\delta} = \tilde{\theta}\tilde{\gamma}_3\partial_r - r^{-1}P\tilde{\theta}\partial_{\tilde{\zeta}} - P(r^{-2}\tilde{\gamma}_2 + 2r^{-1}\tilde{\gamma}_3)\partial_{\zeta}. \) \quad (125) \\

### 4.2. Angular equations

Several of the seven angular derivative spin-coefficient equations turn out to be identities when the results of the previous section are used. Some determine the ‘constants of integration’. Several examples, (already used in the previous subsection) are:

\( \delta X^A - \Delta \xi^A = \xi^A(\mu + \tilde{\gamma} - \gamma) + \tilde{\lambda} \tilde{\xi}^A \implies \gamma^0 = 0, \) \quad (126) \\
\( \delta \tilde{\xi}^A - \tilde{\lambda} \xi^A = (\tilde{\beta} - \alpha)\xi^A + (\tilde{\alpha} - \beta)\tilde{\xi}^A \implies \tilde{\alpha}^0 = -\frac{1}{2}\partial_r P, \) \quad (127) \\
\( \delta \tilde{\omega} - \tilde{\delta} \omega = (\tilde{\beta} - \alpha)\omega + (\tilde{\alpha} - \beta)\tilde{\omega} + (\mu - \tilde{\mu}) \implies \text{Identity,} \) \quad (128) \\
\( \delta U - \Delta \omega = (\mu + \tilde{\gamma} - \gamma)\omega + \tilde{\lambda}\tilde{\omega} - \tilde{\nu} \implies \tilde{\nu}^0 = 0. \) \quad (129)
4.3. Evolution equations

Analyzing the twenty evolution equations for the five potentials, equations (44)–(63), initially presented a very difficult challenge. The process of inserting all the previous results, i.e., equations (79)–(83) and equations (100)–(121), into the evolution equations was a daunting task. Analyzing any one could take several days. The algebra was long and complicated with the easy production of many errors. The saving aspect of it was the fact that the equations were interrelated, in the sense that final forms of many of the equations could be compared or reduced to others, thereby giving a method of checking consistency and thus finding the errors.

Towards the end of the analysis all the evolution equations depended on just the four evolutionary Bianchi identities, equations (44)–(47). They however, using equations (85)–(88), are further reduced to a single equation for the last of the potentials, i.e., \( \dot{\Upsilon}_4 \).

Taking the four evolutionary Bianchi identities, equations (60)–(63), substituting the results of the radial integrations—with lengthy calculations using equations (85)–(88),—we obtain the following results:

\[
\begin{align*}
\dot{\Upsilon}_0' - 2\dot{\Upsilon}_0 \partial^2 \dot{\Upsilon}_1 - 20\dot{\Upsilon}_3 \partial^2 \dot{\Upsilon}_1 - 3\dot{\Upsilon}_2 \partial^2 \dot{\Upsilon}_2 + 20\dot{\Upsilon}_4 \partial^2 \dot{\Upsilon}_0 + \partial^2 \dot{\Upsilon}_4 \dot{\Upsilon}_0 - \frac{\partial}{\partial r} \dot{\Upsilon}_0 \\
- r^{-1}(3\dot{\Upsilon}_0 \partial^2 \dot{\Upsilon}_0 - 2\dot{\Upsilon}_3 \partial^2 \dot{\Upsilon}_0 - 4\dot{\Upsilon}_0 + \partial^2 \dot{\Upsilon}_1 - 60\dot{\Upsilon}_2 \partial^2 \dot{\Upsilon}_1 + 30\partial^2 \dot{\Upsilon}_2 \dot{\Upsilon}_1) \\
- \partial^2 \dot{\Upsilon}_4 \partial \dot{\Upsilon}_0 = 0, \\
(130)
\end{align*}
\]

\[
\begin{align*}
- \partial \dot{\Upsilon}_1 + 20\dot{\Upsilon}_3 \partial^2 \dot{\Upsilon}_3 - 20\dot{\Upsilon}_4 \partial^2 \dot{\Upsilon}_3 + 2\dot{\Upsilon}_3 \partial^2 \dot{\Upsilon}_2 + 2\dot{\Upsilon}_4 \partial^2 \dot{\Upsilon}_2 \\
+ r^{-1}(-2\dot{\Upsilon}_1 - 30\dot{\Upsilon}_3 \partial^2 \dot{\Upsilon}_3 + 2\dot{\Upsilon}_3 \partial^2 \dot{\Upsilon}_3 - 20\dot{\Upsilon}_2 \partial^2 \dot{\Upsilon}_3 + \partial^2 \dot{\Upsilon}_0 \partial \dot{\Upsilon}_0 \\
+ \partial^3 \dot{\Upsilon}_4 \dot{\Upsilon}_0): + r^{-2} \partial \dot{\Upsilon}_0 = 0, \\
(131)
\end{align*}
\]

\[
\begin{align*}
- \partial^3 \dot{\Upsilon}_2 - 2\partial^3 \dot{\Upsilon}_3 - 2\partial^3 \dot{\Upsilon}_4 - 2\partial^3 \dot{\Upsilon}_4 \partial \dot{\Upsilon}_0 + 2\partial^3 \dot{\Upsilon}_4 \partial \dot{\Upsilon}_0 \\
+ r^{-1}(2\partial^3 \dot{\Upsilon}_2 \partial \dot{\Upsilon}_0 + 3\partial^3 \dot{\Upsilon}_2 \partial \dot{\Upsilon}_0 - \partial \dot{\Upsilon}_0 \partial \dot{\Upsilon}_1 - 20\partial^3 \dot{\Upsilon}_4 \partial \dot{\Upsilon}_1 \\
- \partial^2 \dot{\Upsilon}_2 \partial^2 \dot{\Upsilon}_1): - r^{-2} \partial^2 \dot{\Upsilon}_1 = 0, \\
(132)
\end{align*}
\]

\[
\begin{align*}
- \partial^3 \dot{\Upsilon}_3 + 2\partial^3 \dot{\Upsilon}_4 - 2\partial^3 \dot{\Upsilon}_4 \partial \dot{\Upsilon}_0 - 4\partial^2 \dot{\Upsilon}_4 \partial^2 \dot{\Upsilon}_3 + 4\partial^2 \dot{\Upsilon}_4 \partial^2 \dot{\Upsilon}_3 + 2\partial^2 \dot{\Upsilon}_4 \dot{\Upsilon}_3 \\
: + r^{-1}(2\partial^3 \dot{\Upsilon}_3 - 60\partial^2 \dot{\Upsilon}_3 \partial^2 \dot{\Upsilon}_3 + \partial^3 \dot{\Upsilon}_2 \partial^2 \dot{\Upsilon}_4 + 3\partial^3 \dot{\Upsilon}_2 \partial^2 \dot{\Upsilon}_4 + 3\partial \dot{\Upsilon}_2 \partial^2 \dot{\Upsilon}_4 \\
+ \dot{\Upsilon}_2 \partial^2 \dot{\Upsilon}_4 - 20\partial^2 \dot{\Upsilon}_4 \partial^2 \dot{\Upsilon}_3): + r^{-2} \partial^2 \dot{\Upsilon}_2 = 0. \\
(133)
\end{align*}
\]

These equations have the following relationship to each other: the application of the operator \( \partial^2 \partial_r \) to the lower of each of the three consecutive pairs, (using equations (85)–(88)), leads to the equality with the operator \( \partial \) applied to the upper member of the pair, e.g., \( r^2 \partial_r \) applied to equation (133) equals \( \partial \) applied to equation (132).

Furthermore these four evolutionary Bianchi identities are closely related to all the other evolution equations by similar identities. For example, from equation (45), i.e.,

\[
\tilde{\delta} U - \Delta \tilde{\omega} = (\tilde{\mu} + \gamma - \tilde{\nu}) \tilde{\omega} - \nu,
(134)
\]

we have

\[
\begin{align*}
- \partial \dot{\Upsilon}_1 + 2\partial \dot{\Upsilon}_4 + 2\dot{\Upsilon}_4 \partial \dot{\Upsilon}_0 - 2\partial \dot{\Upsilon}_4 \partial \dot{\Upsilon}_3 \\
: + r^{-1}(2\partial \dot{\Upsilon}_1 - 2\partial \dot{\Upsilon}_3 \partial \dot{\Upsilon}_3 + \partial \dot{\Upsilon}_4 \partial \dot{\Upsilon}_4 + \partial \dot{\Upsilon}_2 \partial \dot{\Upsilon}_4) + r^{-2} \partial \dot{\Upsilon}_2 = 0. \\
(135)
\end{align*}
\]

The application of \( \partial \) to (135) yields equation (51),

\[
\tilde{\delta} \tilde{\nu} - \Delta \tilde{\mu} = \tilde{\mu}^2 + \tilde{\mu}(\gamma + \tilde{\tau}) - 2\tilde{\nu} \tilde{\beta},
(136)
\]
or
\[
-\tilde{\partial}^2 \tilde{Y}_1' + \tilde{\partial}^2 \tilde{\partial} \tilde{Y}_1 + 2 \tilde{Y}_3 \tilde{\partial}^2 \tilde{Y}_4 + 2 \tilde{\partial}^2 \tilde{Y}_4 - 20 \tilde{Y}_4 \tilde{\partial}^2 \tilde{Y}_3 - 20 \tilde{\partial}^2 \tilde{Y}_4 \tilde{\partial}^2 \tilde{Y}_1 + 20 \tilde{\partial}^2 \tilde{Y}_3 \tilde{\partial}^2 \tilde{Y}_4
+ r^{-1}(20 \tilde{Y}_3 \tilde{\partial}^2 \tilde{Y}_3 - 20 \tilde{\partial}^2 \tilde{Y}_3 \tilde{\partial}^2 \tilde{Y}_1 + 20 \tilde{\partial}^2 \tilde{Y}_3 \tilde{\partial}^2 \tilde{Y}_4 + \tilde{Y}_2 \tilde{\partial}^2 \tilde{Y}_4)
+ \tilde{\partial}^2 \tilde{\partial} \tilde{\partial}^2 \tilde{Y}_4) = -r^{-2} \tilde{\partial}^2 \tilde{Y}_2. \tag{137}
\]

Finally, in turn, the application of \(\tilde{\partial}\) to (137) yields the Bianchi identity (133). The commutator on a spin-wt-\(s\) function \(W_s\), i.e.,
\[
\tilde{\partial} \tilde{\partial} W_s - \tilde{\partial} \tilde{\partial} W_s = -2s W_s \tag{138}
\]
has been used several times.

Returning to the evolutionary Bianchi identities—leaving the first one, (130), unchanged—we can see that the last three are simplified by performing the angular integrations. They can each be rewritten as
\[
\begin{align*}
\tilde{\partial} B_1 &= 0, \quad \tag{139} \\
\tilde{\partial}^2 B_2 &= 0, \quad \tag{140} \\
\tilde{\partial}^3 B_3 &= 0, \quad \tag{141}
\end{align*}
\]
with
\[
\begin{align*}
B_1 &= -\tilde{Y}_1' + \tilde{\partial} \tilde{\partial} \tilde{Y}_1 - 2 \tilde{Y}_2 - 20 \tilde{Y}_4 \tilde{\partial} \tilde{Y}_1 + 2 \tilde{Y}_3 \tilde{\partial}^2 \tilde{Y}_2 - 20 \tilde{\partial}^2 \tilde{Y}_3 \tilde{\partial} \tilde{Y}_2 + 20 \tilde{\partial}^2 \tilde{Y}_3 \tilde{\partial} \tilde{Y}_2
+ r^{-1}(\tilde{\partial}^2 \tilde{Y}_2 - 20 \tilde{Y}_4 \tilde{\partial} \tilde{Y}_2 + 2 \tilde{Y}_3 \tilde{\partial}^2 \tilde{Y}_1 - 20 \tilde{\partial}^2 \tilde{Y}_3 \tilde{\partial} \tilde{Y}_1 + \tilde{Y}_2 \tilde{\partial}^2 \tilde{Y}_4)
+ r^{-2} \tilde{Y}_0, \tag{142}
\end{align*}
\]
\[
\begin{align*}
B_2 &= -\tilde{\partial} \tilde{\partial} \tilde{Y}_4 - 2 \tilde{Y}_3 + \tilde{\partial} \tilde{\partial} \tilde{Y}_4 - 20 \tilde{Y}_4 \tilde{\partial} \tilde{Y}_3 - \tilde{\partial} \tilde{\partial} \tilde{Y}_3 + 2 \tilde{\partial} \tilde{\partial} \tilde{Y}_3
+ r^{-1}(\tilde{\partial} \tilde{\partial} \tilde{Y}_4 - 20 \tilde{Y}_4 \tilde{\partial} \tilde{Y}_3 + \tilde{\partial} \tilde{\partial} \tilde{Y}_3) + r^{-2} \tilde{Y}_1, \tag{143}
\end{align*}
\]
\[
\begin{align*}
B_3 &= -\tilde{\partial} \tilde{\partial} \tilde{Y}_4 + 2 \tilde{\partial} \tilde{\partial} \tilde{Y}_4 - 20 \tilde{\partial} \tilde{\partial} \tilde{Y}_3 + r^{-1}(2 \tilde{\partial} \tilde{\partial} \tilde{Y}_4 - \tilde{\partial} \tilde{\partial} \tilde{Y}_3 + \tilde{\partial} \tilde{\partial} \tilde{Y}_4) + r^{-2} \tilde{Y}_2, \tag{144}
\end{align*}
\]
so that they immediately integrate to
\[
\begin{align*}
B_1 &= K_1, \quad \tag{145} \\
B_2 &= K_2, \quad \tag{146} \\
B_3 &= K_3, \quad \tag{147}
\end{align*}
\]
with \((K_1, K_2, K_3)\), the kernels of the operators \((\tilde{\partial}, \tilde{\partial}^2, \tilde{\partial}^3)\), i.e., \((\tilde{\partial} K_1, \tilde{\partial}^2 K_2, \tilde{\partial}^3 K_3) = 0\).

The kernels, using equation (84), are given by
\[
\begin{align*}
K_3 &= A(\tilde{\zeta}) + B(\tilde{\zeta}) P + C(\tilde{\zeta}) P^2 + r^{-1}(D(\tilde{\zeta}) + E(\tilde{\zeta}) P + r^{-2} F(\tilde{\zeta})), \tag{148} \\
K_2 &= -D(\tilde{\zeta}) - E(\tilde{\zeta}) P - 2 r^{-1} F(\tilde{\zeta}), \tag{149} \\
K_1 &= 2 F(\tilde{\zeta}), \tag{150}
\end{align*}
\]
with six arbitrary functions of \((\tilde{\zeta})\). All have angular (or wire) singularities. Since we wish to restrict ourselves to regular solutions, the \(K_i\) are set to zero here.

Our final evolution equations for the potentials then are \(B_3 = B_2 = B_1 = B_0 = 0\) or
\[
\begin{align*}
\tilde{Y}_0' - \tilde{\partial} \tilde{\partial} \tilde{Y}_1 - 2 \tilde{Y}_3 \tilde{\partial} \tilde{Y}_1 + 20 \tilde{Y}_3 \tilde{\partial} \tilde{Y}_1 - 3 \tilde{Y}_2 \tilde{\partial} \tilde{Y}_2 + 20 \tilde{\partial} \tilde{Y}_4 \tilde{Y}_0 + \tilde{\partial} \tilde{\partial} \tilde{Y}_0
- r^{-1}(3 \tilde{Y}_0 \tilde{\partial} \tilde{Y}_2 - 20 \tilde{Y}_3 \tilde{\partial} \tilde{Y}_0 - 4 \tilde{Y}_0 + \tilde{Y}_2 \tilde{\partial} \tilde{Y}_1 - 60 \tilde{\partial} \tilde{Y}_2 \tilde{Y}_1 + 3 \tilde{\partial} \tilde{Y}_2 \tilde{Y}_1)
: - \frac{\partial}{\partial r} \tilde{Y}_0 - r \tilde{\partial} \tilde{Y}_4 \frac{\partial}{\partial r} \tilde{Y}_0 &= 0. \tag{151}
\end{align*}
\]
even any one of the equations (151), (152), (153)) to Plebanski’s second Heavenly equation,
Assuming that the potential 4.4. The metric
constructed via
one could use either for integration.
We thus conjecture that such a relationship does exist.
Note that the left side of equation (155) is the wave operator in null coordinates on a
spin-wt, s = −2, function.
Since, with the ladder relations, equation (155) is equivalent to the previous four equations,
one could use either for integration.
Remark 2. A referee raised the question of the possible relationship of our equation (155), (or
even any one of the equations (151), (152), (153)) to Plebanski’s second Heavenly equation,
\[ u_{xz} + u_{tc} = u_s^2 + u_t u_3. \] (156)
To explore this issue in detail is a relatively long and complicated process. It would involve
rather lengthy coordinate transformations between the Plebanski coordinates (t, x, y, z) and our
(a, r, ζ, ξ). It is not clear that the gain is worth that effort. However we do remark that
since our equations are all of spin-wt, s = −2, while the Plebanski equation is spin-wt, s = 0,
we should apply the \( \partial^a \) operator to our equation to make the convergence. When this is done,
without carefully studying the details, one can see that there is a very strong resemblance
between our linear terms and the Plebanski linear terms (both sets of linear terms can be
transformed into the wave operator). We thus conjecture that such a relationship does exist.

4.4. The metric
Assuming that the potential \( \tilde{\gamma}_4 \) is known then the entire metric, from equation (9), can be
constructed via
\[ g^{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & g^{11} & g^{1A} \\ 0 & g^{A1} & g^{AB} \end{pmatrix} \] (157)
\[ g^{11} = 2U \]
\[ g^{1A} = X^A - \tilde{\omega}^A \]
\[ g^{AB} = -(\xi^A \xi^B + \xi^B \xi^A). \]
with
\[ \xi^A = (\xi^a, \xi^\bar{a}) = r^{-1} \xi^{\alpha A} = -r^{-1}(P, 0), \quad P = 1 + \xi^\bar{a} \] (158)
A small set of solutions can be found by making the starting ansatz

\[ \hat{\xi}^A = (\hat{\xi}^1, \hat{\xi}^2) = -r^{-1}(0, P) - (P, 0)(2r^{-1}\hat{\gamma}_3 + r^{-2}\hat{\gamma}_2) \]  
(159)

\[ \omega = 0 \]  
(160)

\[ \tilde{\omega} = \partial \hat{\gamma}_3 \]  
(161)

\[ X^A = (P, 0)[r^{-1}\partial \hat{\gamma}_3 + 2\partial \hat{\gamma}_4] \]  
(162)

\[ U = -1 - r\partial^2 \hat{\gamma}_4 \]  
(163)

\[ \hat{\gamma}_3 = r^2\partial_r \hat{\gamma}_4, \quad \hat{\gamma}_2 = r^2\partial_r \hat{\gamma}_3. \]  
(164)

5. Examples

A small set of solutions can be found by making the starting ansatz

\[ \hat{\gamma}_4 = c_4 - c_3r^{-1} + \frac{1}{2}c_2r^{-2} - \frac{1}{6}c_1r^{-3} + \frac{1}{24}c_0r^{-4}. \]  
(165)

This leads immediately to

\[ \hat{\gamma}_0 = c_0, \]  
\[ \hat{\gamma}_1 = c_1 - c_0r^{-1}, \]  
(166)

\[ \hat{\gamma}_2 = c_2 - c_1r^{-1} + \frac{1}{2}c_0r^{-2}, \]  
\[ \hat{\gamma}_3 = c_3 - c_2r^{-1} + \frac{1}{2}c_1r^{-2} - \frac{1}{12}c_0r^{-3}. \]

the \( c \)'s being independent of \( r \). Inserting \( \hat{\gamma}_4 \) into (154), and equating powers of \( r^{-1} \), yields the evolution equations for the individual \( c \)'s:

\[
\begin{align*}
    c_0' - \tilde{\omega}c_1 - 2c_3\partial^2c_1 + 2\partial c_3\partial c_1 - 3c_2\partial^2c_2 + 2\partial c_2\partial c_2 + c_0\partial^2c_4 &= 0, \\
    -c_1' + \tilde{\omega}c_2 - 2c_2 - 2\partial c_4\partial c_1 + 2c_3\partial^2c_2 - 2\partial c_3\partial c_2 &= 0, \\
    -c_2' + \tilde{\omega}c_3 - 2c_3 + c_2\partial^2c_4 - 2\partial c_4\partial c_2 - \partial c_3\partial c_3 &= 0, \\
    -c_3' + \tilde{\omega}c_4 + 2c_3\partial^2c_4 - 2\partial c_4\partial c_3 &= 0.
\end{align*}
\]  
(167)

We obtain a system easily integrated if we make the further ansatz,

\[ c_4 = 0. \]  
(168)

The (167) reduce to

\[
\begin{align*}
    c_0' &= \tilde{\omega}c_1 + 2c_3\partial^2c_1 - 2\partial c_3\partial c_1 + 3c_2\partial^2c_2, \\
    c_1' &= \tilde{\omega}c_2 - 2c_2 + 2c_3\partial^2c_2 - 2\partial c_3\partial c_2 + 2c_2\partial^2c_3, \\
    c_2' &= \tilde{\omega}c_3 - 2c_3 - \partial c_3\partial c_3 + 2c_3\partial^2c_3, \\
    c_3' &= 0
\end{align*}
\]  
(169)

leading to

\[
\begin{align*}
    c_1 &= c_1^0(\zeta, \bar{\zeta}), \\
    c_2 &= c_2^0(\zeta, \bar{\zeta}) + u(\tilde{\omega}c_1^0 - 2c_2^0 - \partial c_3^0 + 2c_2^0\partial^2c_3^0), \\
    c_1 &= c_1^0(\zeta, \bar{\zeta}) + \int u(\tilde{\omega}c_2 - 2c_2 + 2c_3\partial^2c_2 - 2\partial c_3\partial c_2 + 2c_3\partial^2c_3^0), \\
    c_0 &= c_0^0(\zeta, \bar{\zeta}) + \int u(\tilde{\omega}c_1 + 2c_3\partial^2c_1 - 2\partial c_3\partial c_1 + 3c_2\partial^2c_2).
\end{align*}
\]  
(170)
These solutions have in general cubic \( u \) dependence. The special case of \( c_3 = c_2 = c_1 = 0 \), leaves \( c_0 = c_0^0(\xi, \tilde{\xi}) \), a time independent solution

\[
\tilde{\Upsilon}_0 = c_0^0, \quad \tilde{\Upsilon}_1 = -c_0^0 r^{-1}, \quad \tilde{\Upsilon}_2 = +\frac{1}{2} c_0^0 r^{-2}, \quad \tilde{\Upsilon}_3 = -\frac{1}{6} c_0^0 r^{-3}.
\]

(171)

Another class of solutions of (167) starts with the ansatz, \( c_0 = c_1 = c_2 = c_3 = 0 \). The last of (167), becomes

\[
\tilde{\delta}\phi c_4 = \tilde{\delta}(P^i \delta_k (P^{-2} c_k)) = 0.
\]

(172)

The solutions, all type \( N \), given by

\[
c_4 = P^2 G(\tilde{\zeta}) + P^2 \int P^{-4} \delta_k F(\zeta) \, d\zeta,
\]

(173)

are all singular, and have, unfortunately, wire singularities.

6. Discussion

In the interests of honesty, we describe the thought perturbations that led to the present investigation. Several months ago we were working on aspects of the complexified full Einstein equations—in particular—on asymptotic shear-free null geodesic congruences, and simply noticed that we could, with great ease, integrate the radial Bianchi identities via the introduction of the five potentials (described in the text) with their ladder relations. Though we initially had no interest in investigating the self-dual Einstein metrics, the apparent simplicity, coming from the use of the potentials, charmed us into going for the ‘entire thing’. Briefly it even seemed possible that we might go the entire way and be able to construct explicit vacuum self-dual metrics directly from the initial data. It turned out that this hope was very much dashed: first of all by the complexity of the remaining equations—they involved long and hard calculations—and more important, we ended with the unpleasant nonlinear wave equation not easily solvable.

So in the end the question remains: though we have found some pretty results, what did we really accomplish. These results certainly clarify the structure of the self-dual equations—but do the results have applications or are they of interest in other investigations? They appear to potentially have use for our own investigations—but we are not certain.

There are several further issues to be mentioned.

The first is a mild mystery. The data needed for solutions to our nonlinear wave equation are two complex functions, one the value of \( \tilde{\Upsilon}_4 \) on an initial null cone \( \omega_0 \), i.e., an arbitrary function of \( (r, \xi, \tilde{\xi}) \) and the second, \( c_4 \), (the asymptotic value of \( \tilde{\Upsilon}_4 \)), a news-like function of \( (u, \xi, \tilde{\xi}) \) that drives the evolution. On the other hand, from the original formulation of the self-dual Einstein metrics, via the so-called ‘good-cut’ equation, one needs only one complex function as the needed data, namely the asymptotic Bondi shear \( \sigma^0(u_B, \xi, \tilde{\xi}) \), with \( u_B \) the Bondi time. It is probable that by the proper counting of the ranges of the arguments it can be shown that the two sets contain the same amount of data.

A second is that there should be a geometric meaning to the potentials, \( \tilde{\Upsilon}_n \). We have not attempted to investigate that.

For whatever it might be worth, and it is suggestive that it is worth something, we point out that studies [18] of the self-dual Yang–Mills equations in the spin-coefficient form show that, for any gauge group, the entire set of Yang–Mills equations can be encapsulated into a single nonlinear wave equation very similar to our nonlinear equation, (155). The basic variable, \( \tilde{\xi}_2 \), is one of three matrix valued potentials, \( \tilde{\xi}_n \), all connected to each other by a ladder relationship analogous to the one for GR.
The Yang–Mills wave equation [19] is
\[ \partial_t \mathcal{G}_2 - \partial^2 \mathcal{G}_2 - 2r^{-1} \partial_r \mathcal{G}_2 - r^{-2} \partial \mathcal{G}_2 = [\partial \mathcal{G}_2, \partial \mathcal{G}_2], \] (174)
with ladder relations among the potentials
\[ \mathcal{G}_1 = r^2 \partial_t \mathcal{G}_2, \] (175)
\[ \mathcal{G}_0 = r^2 \partial_r \mathcal{G}_1 = r^2 \partial_r (r^2 \partial_t \mathcal{G}_2). \] (176)

The tetrad components of the self-dual Yang–Mills field are given by
\[ \tilde{\chi}_0 = r^{-3} \mathcal{G}_0, \] (177)
\[ \tilde{\chi}_1 = r^{-2} \partial \mathcal{G}_1, \] (178)
\[ \tilde{\chi}_2 = r^{-1} \partial^2 \mathcal{G}_2, \] (179)
in an almost perfect analogy with the self-dual gravitational case.

**Appendix**

**A.1. Spin-coefficient version of Einstein equations**

The familiar spin-coefficient version of the Einstein equations [18]—with their ‘conjugates’ explicitly inserted and written out in detail—are:

the metric equations:
\[
\begin{align*}
\Delta \lambda &= - (\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) \lambda + (3\alpha + \bar{\beta} + \pi - \tau) v - \Psi_4 \\
\Delta \bar{\lambda} &= - (\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) \bar{\lambda} + (3\alpha + \bar{\beta} + \pi - \tau) \bar{v} - \bar{\Psi}_4 \\
\delta \rho &= \delta \sigma = \rho (\alpha + \bar{\beta}) - \sigma (3\alpha - \bar{\beta}) + (\rho - \bar{\rho}) \tau + (\mu + \bar{\mu}) \kappa - \Psi_4 \\
\delta \bar{\rho} &= \delta \bar{\sigma} = \bar{\rho} (\alpha + \beta) - \bar{\sigma} (3\alpha - \beta) + (\rho - \bar{\rho}) \bar{\tau} + (\mu + \bar{\mu}) \bar{\kappa} - \bar{\Psi}_4 \\
\delta \alpha &= \delta \bar{\beta} = \mu \rho - \lambda \alpha + \alpha \bar{\alpha} + \beta \bar{\beta} - 2\alpha \beta + \gamma (\rho - \bar{\rho}) + \epsilon (\mu - \bar{\mu}) - \Psi_2 \\
\delta \bar{\alpha} &= \delta \bar{\beta} = \bar{\mu} \bar{\rho} - \bar{\lambda} \alpha + \alpha \bar{\alpha} + \beta \bar{\beta} - 2\alpha \beta - \bar{\gamma} (\rho - \bar{\rho}) - \bar{\tau} (\mu + \bar{\mu}) - \bar{\Psi}_2 \\
\delta \lambda &= \delta \bar{\mu} = (\rho - \bar{\rho}) \nu + (\mu - \bar{\mu}) \pi + \mu (\alpha + \bar{\beta}) + \lambda (\bar{\alpha} - 3\beta) - \Psi_3 \\
\delta \bar{\lambda} &= \delta \bar{\mu} = - (\rho - \bar{\rho}) \bar{\nu} - (\mu + \bar{\mu}) \bar{\pi} + \bar{\mu} (\bar{\alpha} + \beta) + \bar{\lambda} (\alpha - 3\bar{\beta}) - \bar{\Psi}_3 \\
\delta \nu &= \Delta \mu = \mu^2 + \lambda \bar{\lambda} + \mu (\gamma + \bar{\gamma}) - \nu \pi + \pi (\tau - 3 \beta - \bar{\alpha}) \\
\delta \bar{\nu} &= \Delta \bar{\mu} = \bar{\mu}^2 + \lambda \bar{\lambda} + \bar{\pi} (\gamma + \bar{\gamma}) - \bar{\nu} \pi + \bar{\pi} (\bar{\tau} - 3 \bar{\beta} - \alpha) \\
\delta \gamma &= \Delta \beta = \gamma (\tau - \bar{\beta} - \bar{\alpha}) + \pi \tau - \sigma \nu - \bar{\sigma} \bar{\nu} - \beta (\bar{\gamma} - \gamma - \mu) + \alpha \bar{\lambda} \\
\delta \bar{\gamma} &= \Delta \bar{\beta} = \bar{\gamma} (\tau - \alpha - \beta) + \bar{\pi} \bar{\tau} - \bar{\sigma} \bar{\nu} - \bar{\tau} \nu - \bar{\beta} (\bar{\gamma} - \gamma - \mu) + \bar{\alpha} \lambda \\
\delta \tau &= \Delta \sigma = \mu \sigma + \rho \bar{\lambda} + \tau (\alpha + \beta - \bar{\alpha}) - \sigma (3 \gamma - \bar{\gamma}) - \kappa \bar{v} \\
\delta \bar{\tau} &= \Delta \bar{\sigma} = \bar{\mu} \bar{\sigma} + \bar{\rho} \lambda + \tau (\bar{\alpha} + \beta - \alpha) - \bar{\sigma} (3 \bar{\gamma} - \gamma) - \kappa \bar{v}
\end{align*}
\] (A.1)

The spin-coefficient equations:
\[ \delta \varphi = \Delta \varphi = \mu \sigma + \rho \lambda + \tau (\tau + \beta - \alpha) - \sigma (3\varphi - \gamma) - \kappa \nu \]
\[ \Delta \rho - \delta \tau = -\rho \mu - \sigma \lambda + \tau (\beta - \alpha - \tau) + (\gamma + \varphi) \rho + \kappa \nu - \Psi_2 \]
\[ \Delta \sigma - \delta \tau = -\sigma \mu - \tau (\sigma + \tau - \alpha) + (\gamma + \varphi) \sigma + \kappa \nu - \Psi_2 \]
\[ \Delta \alpha - \delta \gamma = \nu (\rho + \epsilon) - \lambda (\tau + \beta) + \alpha (\gamma - \tau) + (\gamma + \beta) \tau - \Psi_3 \]
\[ \Delta \delta - \delta \gamma = \nu (\rho + \epsilon) - \lambda (\tau + \beta) + \alpha (\gamma - \tau) + (\gamma + \beta) \tau - \Psi_3 \]
\[ \Delta \rho - \delta \kappa = \rho^2 + \sigma \varphi + (e + \tau) \rho - \kappa \tau - \kappa (3\alpha + \beta - \pi) \]
\[ \Delta \varphi - \delta \tau = \rho^2 + \sigma \varphi + (e + \tau) \rho - \kappa \tau - \kappa (3\alpha + \beta - \pi) \]
\[ \Delta \tau - \delta \kappa = (\tau + \pi) \rho + (\tau + \pi) \sigma + (e + \tau) \tau - (3\gamma + \varphi) \kappa + \Psi_1 \]
\[ \Delta \tau - \delta \varphi = (\tau + \pi) \varphi + (\tau + \pi) \sigma - (e + \tau) \tau - (3\gamma + \varphi) \kappa + \Psi_1 \]
\[ D\alpha - \delta \epsilon = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \pi \]
\[ \Delta \sigma - \delta \epsilon = (\tau + \epsilon) - 2\epsilon \alpha + \beta \sigma - \beta \epsilon - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \pi \]
\[ \Delta \beta - \delta \epsilon = (\alpha + \pi) \sigma + (\rho + \varphi) (\beta - \mu - \gamma) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ \Delta \tau - \delta \beta = (\tau + \pi) \varphi + (\tau + \pi) \sigma - (e + \tau) \tau - (3\gamma + \varphi) \kappa + \Psi_1 \]
\[ \Delta \delta - \delta \epsilon = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \pi \]
\[ \Delta \tau - \delta \epsilon = (\tau + \epsilon) - 2\epsilon \alpha + \beta \sigma - \beta \epsilon - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \pi \]
\[ \Delta \beta - \delta \epsilon = (\alpha + \pi) \sigma + (\rho + \varphi) (\beta - \mu - \gamma) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ \Delta \tau - \delta \beta = (\tau + \pi) \varphi + (\tau + \pi) \sigma - (e + \tau) \tau - (3\gamma + \varphi) \kappa + \Psi_1 \]
\[ D\lambda - \delta \pi = \rho \lambda + \sigma \mu + \pi^2 + (\alpha - \beta) \pi - \kappa (3\epsilon - \tau) \lambda \]
\[ D\varphi - \delta \pi = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\mu - \delta \pi = \sigma \mu + \sigma \lambda + \pi \pi - (e + \tau) \mu - \pi (\beta - \beta) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\varphi - \delta \pi = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\mu - \delta \pi = \sigma \mu + \sigma \lambda + \pi \pi - (e + \tau) \mu - \pi (\beta - \beta) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\varphi - \delta \pi = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\mu - \delta \pi = \sigma \mu + \sigma \lambda + \pi \pi - (e + \tau) \mu - \pi (\beta - \beta) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\varphi - \delta \pi = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\mu - \delta \pi = \sigma \mu + \sigma \lambda + \pi \pi - (e + \tau) \mu - \pi (\beta - \beta) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\varphi - \delta \pi = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\mu - \delta \pi = \sigma \mu + \sigma \lambda + \pi \pi - (e + \tau) \mu - \pi (\beta - \beta) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\varphi - \delta \pi = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\mu - \delta \pi = \sigma \mu + \sigma \lambda + \pi \pi - (e + \tau) \mu - \pi (\beta - \beta) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]
\[ D\varphi - \delta \pi = (\rho + \epsilon) (\alpha + \beta) (\rho + \epsilon) - \kappa \lambda - \kappa \gamma + (\epsilon + \rho) \varphi + \Psi_1 \]

and finally the Bianchi identities:

\[ \delta \Psi_0 - D\Psi_1 = (4\alpha + \pi) \Psi_0 - 2(2\rho + \epsilon) \Psi_1 + 3\kappa \Psi_2 \]
\[ \delta \Psi_1 - D\Psi_2 = \lambda \Psi_0 + 2(\alpha + \pi) \Psi_1 - 3\rho \Psi_2 + 2\kappa \Psi_3 \]
\[ \delta \Psi_2 - D\Psi_3 = 2\lambda \Psi_1 - 3\tau \Psi_2 + 2(\rho + \epsilon) \Psi_3 + \kappa \Psi_4 \]
\[ \delta \Psi_3 - D\Psi_4 = 3\lambda \Psi_2 - 2(\alpha + \pi) \Psi_3 + (4\epsilon - \rho) \Psi_4 \]

\[ \delta \Psi_0 - D\Psi_1 = (4\alpha + \pi) \Psi_0 - 2(2\rho + \epsilon) \Psi_1 + 3\kappa \Psi_2 \]
\[ \delta \Psi_1 - D\Psi_2 = \lambda \Psi_0 + 2(\alpha + \pi) \Psi_1 - 3\rho \Psi_2 + 2\kappa \Psi_3 \]
\[ \delta \Psi_2 - D\Psi_3 = 2\lambda \Psi_1 - 3\tau \Psi_2 + 2(\rho + \epsilon) \Psi_3 + \kappa \Psi_4 \]
\[ \delta \Psi_3 - D\Psi_4 = 3\lambda \Psi_2 - 2(\alpha + \pi) \Psi_3 + (4\epsilon - \rho) \Psi_4 \]

\[ \Delta \Psi_0 - \delta \Psi_1 = (4\gamma - \mu) \Psi_0 - 2(2\tau + \beta) \Psi_1 + 3\sigma \Psi_2 \]
\[ \Delta \Psi_1 - \delta \Psi_2 = \nu \Psi_0 + 2(\gamma - \mu) \Psi_1 - 3\tau \Psi_2 + 2\pi \Psi_3 \]
\[ \Delta \Psi_2 - \delta \Psi_3 = 2\nu \Psi_1 - 3\mu \Psi_2 + 2(\beta - \tau) \Psi_3 + \sigma \Psi_4 \]
\[ \Delta \Psi_3 - \delta \Psi_4 = 3\nu \Psi_2 - 2(\gamma + 2\mu) \Psi_3 + (4\beta - \tau) \Psi_4 \]
\[ \Delta \Psi_0 - \delta \Psi_1 = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma \Psi_2, \]
\[ \Delta \Psi_1 - \delta \Psi_2 = \tau \Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau \Psi_2 + 2\tau \Psi_3, \]
\[ \Delta \Psi_2 - \delta \Psi_3 = 2\tau \Psi_1 - 3\tau \Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma \Psi_4, \]
\[ \Delta \Psi_3 - \delta \Psi_4 = 3\tau \Psi_2 - 2(\gamma + 2\tau)\Psi_3 + (4\beta - \tau)\Psi_4. \] (A.14)

The \( \lambda^a_i \equiv (l^a, n^a, m^a, \overline{m}^a) \) are the components of a null tetrad, the \( \Psi \)'s and \( \overline{\Psi} \)'s are the components of the Weyl tensor, \( \lambda^a_i \nabla_a = (D, \Delta, \delta, \overline{\delta}) \) are the directional derivatives. All the other variables are the spin-coefficients. The Einstein equations are already built into the system by virtue of the Ricci tensor having been set to zero.

References

[1] Adamo T M, Kozameh C and Newman E T 2012 Null geodesic congruences, asymptotically-flat spacetimes and their physical interpretation Living Rev. Rel. 15 1
[2] Newman E T 1975 H-space Gen. Rel. Grav. 7 107–11
[3] Hansen R O, Newman E T, Penrose R and Tod K P 1978 The metric and curvature properties of H-space Proc. R. Soc. A 363 445–68
[4] Ludvigsen M, Newman E T and Tod K P 1981 Asymptotically flat H spaces J. Math. Phys. 22 4
[5] Ko M, Ludvigsen M, Newman E T and Tod K P 1961 The theory of H-space Phys. Rep. 71 51–139
[6] Sparling G A J and Tod K P 1981 An example of an H-space J. Math. Phys. 22 331–3326
[7] Eguchi T and Hanson A J 1979 Self-dual solutions to euclidean gravity Ann. Phys. 120 82–105
[8] Penrose R 1976 Nonlinear gravitons and curved twistor theory Gen. Relativ. Gravit. 7 31–52
[9] Tod K P 1980 Curved twistor spaces and H-space Surv. High Energy Phys. 1 299–312
[10] Ita E E 2011 Nonlinear gravitons from the initial value constraints of GR in Ashtekar variables J. Phys.: Conf. Ser. 314 012115
[11] Mason L J 1990 H-space, a universal integrable system? Twistor Newsletter 30 14–17
[12] Dunajski M, Mason L J and Woodhouse N M J 1998 From 2D integrable systems to self-dual gravity (arXiv:solv-int/9809006v1)
[13] Nutku Y, Sheftel M B, Kalayci J and Yazıcı D 2008 Self-dual gravity is completely integrable J. Phys. A: Math. Theor. 41 395206
[14] Wikipedia contributors Gravitational instanton Wikipedia http://en.wikipedia.org/w/index.php?title=Gravitational_instanton&oldid=516316194
[15] Gibbons G W and Hawking S W 1978 Gravitational multi-instantons Phys. Lett. B 78 430–2
[16] Fette C W, Janis A I and Newman E T 1976 Algebraically special H-spaces J. Math. Phys. 17 660
[17] Tod K P 1997 The Toda Field Equation and Special Metrics (Geometry and Physics) ed J E Andersen, J Dupont, H Pedersen and A Swann (New York: Dekker)
[18] Newman E T and Penrose R 2009 Spin-coefficient formalism Scholarpedia 4 7445
[19] Newman E T 1978 Source-free Yang–Mills theories Phys. Rev. D 18 8
[20] Tafel J 2007 Reductions of self-dual Einstein equations J. Phys.: Conf. Ser. 12th Conf. on Recent Developments in Gravity (NEB XII); 68 012037
[21] Plebanski J 1975 Some solutions of complex Einstein equations J. Math. Phys. 16 2395