INFINITE SETS OF $b$-ADDITIVE AND $b$-MULTIPLICATIVE RAMANUJAN-HARDY NUMBERS

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Abstract. Let $b$ be a numeration base. A $b$-additive Ramanujan-Hardy number $N$ is an integer for which there exists at least an integer $M$, called additive multiplier, such that the product of $M$ and the sum of base $b$ digits of $N$, added to the reversal of the product, gives $N$. We show that for any $b$ there exists an infinity of $b$-additive Ramanujan-Hardy numbers and an infinity of additive multipliers. A $b$-multiplicative Ramanujan-Hardy number $N$ is an integer for which there exists at least an integer $M$, called multiplicative multiplier, such that the product of $M$ and the sum of base $b$ digits of $N$, multiplied by the reversal of the product, gives $N$. We show that for an even $b$, $b \equiv 1 \pmod{3}$, and for $b = 2$, there exists an infinity of $b$-multiplicative Ramanujan-Hardy numbers and an infinity of multiplicative multipliers.

These results completely answer two questions and partially answer two other questions among those asked in V. Nițică, About some relatives of the taxicab number, arXiv:1805.10739v3.

1. Introduction

Let $b \geq 2$ be a numeration base. In Nițică [7], motivated by some properties of the taxicab number, 1729, we introduce the classes of $b$-additive Ramanujan-Hardy (or $b$-ARH) numbers and $b$-multiplicative Ramanujan-Hardy (or $b$-MRH) numbers. The first class consists of numbers for which there exists at least an integer $M$, called additive multiplier, such that the product of $M$ and the sum of base $b$ digits of $N$, added to the reversal of the product, gives $N$. The second class consists of numbers for which there exists at least an integer $M$, called multiplicative multiplier, such that the product of $M$ and the sum of base $b$ digits of $N$, multiplied by the reversal of the product, gives $N$.

It is asked [7, Question 6] if the set of $b$-ARH numbers is infinite and it is asked [7, Question 8] if the set of additive multipliers is infinite. It is shown [7, Theorems 12 and 15] that the answer is positive if $b$ is even. The case $b$ odd is left open. It is asked [7, Question 7] if the set of $b$-MRH numbers is infinite for all numeration bases and it is asked [7, Question 9] if the set of multiplicative multipliers is infinite. It is shown [7, Theorem 30] that the answer is positive if $b$ is odd. The case $b$ even is left open.

We recall that Niven (or Harshad) numbers are numbers divisible by the sum of their decimal digits. Niven numbers have been extensively studied.
See for instance Cai [1], Cooper and Kennedy [2], De Koninck and Doyon [4], and Grundman [5]. Of interest are also \( b \)-Niven numbers, which are numbers divisible by the sum of their base \( b \) digits. See, for example, Fredricksen, Ionascu, Luca, and Stanica [6]. A \( b \)-MRH-number is a \( b \)-Niven number. No degree \( b \)-Niven numbers are introduced in [8].

The goal of this paper is to show that, for any numeration base, there exist an infinity of \( b \)-ARH numbers and an infinity of distinct additive multipliers. We also show that for even \( b \), \( b \equiv 1 \pmod{3} \), and for \( b = 2 \), there exist an infinity of \( b \)-MRH numbers, and an infinity of distinct multiplicative multipliers. The results here overlap with some in [7], but with different sets of examples. They also completely answer the first two questions from [7] revisited above, and partially answer the other two. We observe that a trivial example of an infinity of \( b \)-MRH numbers is given by \( \{1(0)^\wedge k\}_b | k \in \mathbb{N} \} \). The examples we show here have at least two digits different from zero. Finding an infinity of \( b \)-MARH numbers with all digits different from zero remains an open question.

Our results about \( b \)-ARH numbers also give solutions to the dyophantine equation \( N \cdot M = \text{reversal}(N \cdot M) \). Motivated by this link, we show that the dyophantine equation has a solution for all fixed integers \( N \) not divisible by the base \( b \) and for any numeration base. Our final result shows that for any string of digits \( I \) there exists an infinity of \( b \)-Niven number that contains \( I \) in their base \( b \)-representation. We do not know how to prove a similar result for the classes of \( b \)-ARH and \( b \)-MRH numbers.

2. Statements of the main results

Let \( s_b(N) \) denote the sum of base \( b \) digits of integer \( N \). If \( x \) is a string of digits and let \( (x)^\wedge k \) denote the base 10 integer obtained by repeating \( x \) \( k \)-times. Let \( [x]_b \) denote the value of the string \( x \) in base \( b \). The reversal of an integer \( N \) is the number obtained from \( N \) writing its digits in reverse order. If \( N \) is an integer written in base \( b \), let \( N^R \) denote the reversal of \( N \). While the operations of addition and multiplication of integers are independent of the base, the operation of taking the reversal is not. In the definition of a \( b \)-ARH-number/b-MRH number \( N \) we take the reversal of the base \( b \) representation of \( s_b(N)M \).

**Theorem 1.** Let \( \alpha \geq 1 \) integer, \( b \geq \alpha + 1 \) integer, and \( k = (1 + \alpha)^\ell, \ell \geq 0 \). Assume \( b \equiv 2 + \alpha \pmod{2 + 2\alpha} \). Define

\[
N_k = [(1\alpha)^\wedge k]_b.
\]

Then there exists \( M \geq 0 \) integer such that

\[
s_b(N_k) \cdot M = (s_b(N_k) \cdot M)^R = \frac{N_k}{2}.
\]

In particular, the numbers \( N_k, k \geq 1 \), are \( b \)-ARH numbers and \( b \)-Niven numbers.

The proof of Theorem 1 is done in Section 3.
Remark. The particular case $b = 10, \alpha = 2$, of Theorem [1] which gives $N_k = (12)^{3^k}$, is covered by [7, Example 10]. Theorem [1] does not give any information if $b = 2$.

The following proposition gives positive answers to [7, Questions 5 and 6].

**Proposition 1.** For any numeration base, there exists an infinite set of $b$-ARH numbers and an infinite set of additive multipliers. The $b$-ARH numbers in the infinite set also are $b$-Niven numbers.

The proof of Proposition [1] is done in Section 4.

**Remark.** We observe that [7, Theorems 12 and 15] show, for all even bases, an infinity of $b$-ARH numbers that are not $b$-Niven numbers. The case of odd base is open. The question of finding an infinity of $b$-Niven numbers that are not $b$-ARH numbers is also open. It is shown in [7, Theorem 28] that for any base there exists an infinity of numbers that are not $b$-ARH numbers.

The result in Theorem [1] gives many base 10 solutions for the equation:

(1) \[ N \cdot M = (N \cdot M)^R. \]

One can try to solve, for any numeration base $b$, the equation:

(2) \[ N \cdot M = (N \cdot M)^R, \]

where $(N \cdot M)^R$ is the reversal of $N \cdot M$ written in base $b$.

Observe that if $N$ is divisible by $b$, then $(N \cdot M)^R$ has less digits then $N \cdot b \cdot M$, therefore $N$ is not a solution of (2). Note also that if $N = N^R$ and $N$ has $k$ digits then (2) always has an infinite set of solutions with

$M = [(1(0)^{\ell})^R]_b, \ell \geq k - 1, p \geq 0.$

Consequently, if $(N_0, M_0)$ is a solution of (2), then (2) also has infinite sets of solutions of types $(N_0, M)$ and $(N, M_0)$.

**Theorem 2.** Let $b \geq 2$ and $N \geq 1$ integer such that $b \not\mid N$. Then $N$ is a solution of (2).

The proof of Theorem [2] is done in Section 5. For base 10, a proof belonging to David Radcliffe can be found at [10]. We learned about the reference [10] from J. Shallit. We generalize the proof for an arbitrary numeration base. After this paper was written, we learned from J. Shallit [11] that he also has a proof of Theorem [2].

A $b$-numeric palindrome is a base $b$ integer $N$ such that $N = N^R$.

**Corollary 1.** The prime factors of $b$-numeric palindromes exhaust the set of prime numbers.

**Definition 1.** The multiplicity of a multiplier $M$ is the number of $N$ solutions of (2).
It was observed above that for any solution \((N, M)\) of (2), \(M\) has infinite multiplicity. The following theorem shows infinite sets of solutions of (2) that cannot be derived from the previous considerations.

**Theorem 3.** Let \(b \geq 2\) a numeration base. Then:
\[
[1(b - 1)]_b \cdot [(b - 1)^{\wedge k}]_b = [1(b - 2)(b - 1)^{\wedge k - 2}(b - 2)1]_b
\]
for all \(k \geq 0\).

The proof of Theorem 3 is done in Section 6.

In [9] we show more infinite sets of solutions of (2). We also show infinite sets of numbers that satisfy equation (2) up to a prescribed number of misplaced digits for which we know their position.

Our next results shows, for \(b\) even, more examples of infinite sets of \(b\)-ARH that are not \(b\)-Niven numbers.

**Theorem 4.** Let \(b \geq 2\) even. Let \(a \in \{1, 2, \ldots, b - 1\}\) and \(k \geq 0\) integer.

a) Let
\[
N_k = [[a(0)^{\wedge k}a]]_b.
\]

Then \(N_k\) is a \(b\)-ARH number, but not a \(b\)-Niven number.

b) Let
\[
N_k = [(1(0)^{\wedge k})^{\wedge b} 0 ((0)^{\wedge k}1)^{\wedge b}]_b.
\]

Then \(N_k\) is a \(b\)-ARH number, but not a \(b\)-Niven number

c) Let
\[
N_k = [[(0)^{\wedge k}1]^{\wedge b} 0 (1(0)^{\wedge k})^{\wedge b}]_b.
\]

Then \(N_k\) is a \(b\)-ARH number and a \(b\)-Niven number.

The proof of Theorem 4 is done in Section 7.

**Theorem 5.** a) Let \(b \geq 4\) even and \(b \equiv 1 \pmod{3}\). Let \(k \geq 1\) integer such that \(k \equiv 1 \pmod{3}\). Define

\[
\alpha_k = [1(0)^{\wedge k}(b - 2)]_b.
\]

Then \(N_k = \alpha_k \cdot (\alpha_k)^R\) is a \(b\)-MRH number.

b) Let \(b = 2\) and \(k \geq 1\) even integer. Define

\[
\alpha_k = [1(0)^{\wedge k}1]_2.
\]

Then \(N_k = \alpha_k \cdot (\alpha_k)^R\) is a \(b\)-MRH number.

The proof of Theorem 5 is done in Section 8.

The following proposition gives partial answers to [7, Questions 7 and 8].

**Proposition 2.** For any even numeration base \(b\), \(b \equiv 1 \pmod{3}\) and for \(b = 2\) there exist an infinite set of \(b\)-MRH numbers and an infinite set of multipliers.

The proof of Proposition 2 is done in Section 9.

Our next result lists several infinite sequences of 10-MRH-numbers.
**Proposition 3.** Assume $k \geq 1$ integer and define $N_k = \alpha_k \cdot (\alpha_k)^R$, where $\alpha_k$ is one of the following numbers:

- $[1(0)^{k}8]_{10}, k \equiv 1 \pmod{3}$,
- $[7(0)^{k}2]_{10}$,
- $[5(0)^{k}4]_{10}$,
- $[4(0)^{k}5]_{10}$

Then $N_k$ is an 10-MRH number.

The first item follows an a corollary of Theorem 5. The other items can be proved using the same approach as in the proof of Theorem 5.

**Theorem 6.** For any base $b$ and for any string of base $b$ digits $I$ there exists an infinity of $b$-Niven number that contains the string $I$ in their base $b$-representation.

The proof of Theorem 6 is done in section 10.

### 3. Proof of Theorem 1

**Proof.** The base $b$ representation for $N_k/2$ is $N_k/2 = \left(0_{\frac{b+\alpha}{2}}^\ell \right)_b$. One has that:

$$s_b(N_k) = k \cdot (1 + \alpha) = (1 + \alpha)^\ell + 1.$$

The value of $N_k/2$ in base 10 is obtained summing a geometric series.

$$
\frac{N_k}{2} = \frac{b + \alpha}{2} \cdot b^{2k-2} + \frac{b + \alpha}{2} \cdot b^{2k-4} + \ldots \\
= \frac{b + \alpha}{2} \cdot \frac{b^2 + b + \alpha}{2} = \frac{b + \alpha}{2} \cdot \frac{b^{2k} - 1}{b^2 - 1} \\
= \frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell} - 1}{b^2 - 1}.
$$

Note that $N_k/2 = (N_k/2)^R$. We finish the proof of the theorem if we show that:

$$
(1 + \alpha)^\ell + 1 = \frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell} - 1}{b^2 - 1}.
$$

We prove (5) by induction on $\ell$. For $\ell = 0$ equation (5) becomes

$$1 + \alpha \left| \frac{b + \alpha}{2} \right|,$$

which is true because $b \equiv 2 + \alpha \pmod{2 + 2\alpha}$.

Now we assume that (5) is true for $\ell$ and show that it is true for $\ell + 1$. 
\[
\frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)\ell + 1} - 1}{b^2 - 1} = \frac{b + \alpha}{2} \cdot \frac{\left(b^{2(1+\alpha)\ell}\right)^{1+\alpha} - 1}{b^2 - 1}
\]

\[
= \frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)\ell} - 1}{b^2 - 1} \left(B^\alpha + B^{\alpha-1} + \cdots + B^2 + B + 1\right),
\]

where

\[
B = b^{2(1+\alpha)\ell}.
\]

The congruence \(b \equiv 2 + \alpha \pmod{2 + 2\alpha}\) implies that

\[
b^2 \equiv (2 + \alpha)^2 \equiv \alpha^2 + 4\alpha + 4 \equiv \alpha^2 \equiv 1 \pmod{1 + \alpha},
\]

which implies that

\[
b^m \equiv 1 \pmod{1 + \alpha}, \ m \text{ even}.
\]

From (7) and (8) follows that \(B^p \equiv 1 \pmod{1 + \alpha}, 1 \leq p \leq \alpha\), so

\[
1 + \alpha | B^\alpha + B^{\alpha-1} + \cdots + B^2 + B + 1.
\]

Combining (4) (for \(\ell\)) and (9), and taking into account (6), it follows that (5) is true for \(\ell + 1\). \(\square\)

4. Proof of Proposition 1

Proof. The case \(b = 2\) is covered by [7, Theorem 12]. If \(b \geq 3\), choose \(\alpha = b - 2\) and apply Theorem 1. We show now that the multipliers appearing in the proof of Theorem 1 for a fixed base \(b\), are all distinct. It follows from (3) and (4) that the multiplier for \(N_k\) is given by:

\[
M = \frac{\frac{N_k}{2}}{s_b(N_k)} = \frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)\ell} - 1}{b^2 - 1} \left(1 + \alpha\right)^{\ell + 1}.
\]

Note that \(\alpha = b - 2\). After algebraic manipulations, equation (10) becomes

\[
M = \frac{b^{2(1+\alpha)\ell} - 1}{(b - 1)^\ell (b^2 - 1)}.
\]

In order to show that the multipliers are distinct it is enough to show that the sequence of multipliers for \(N_k\) is strictly increasing as a function of \(\ell\), that is, we need to show that:

\[
\frac{b^{2(1+\alpha)\ell} - 1}{(b - 1)^\ell (b^2 - 1)} < \frac{b^{2(1+\alpha)\ell + 1} - 1}{(b - 1)^{\ell + 1} (b^2 - 1)}.
\]

After algebraic manipulations (11) becomes

\[
(b - 1)(b^{2(1+\alpha)\ell} - 1) < b^{2(1+\alpha)\ell + 1} - 1.
\]

After denoting

\[
B = b^{2(1+\alpha)\ell} = b^{2(b-1)^\ell},
\]
right hand side of (12) factors as:

\[ b^{2(1+\alpha)^\ell+1} - 1 = (b^{2(1+\alpha)^\ell} - 1)(B^\alpha + B^{\alpha-1} + \cdots + B + 1). \]

Now (12) follows from (13) and the following inequality:

\[ b - 1 < b^{2(b-1)^\ell}, \ell \geq 0, \ell \geq 0, b \geq 3. \]

\[ \square \]

5. Proof of Theorem 2

**Proof.** Let \( b = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \alpha_i \geq 1, p_i \) prime, \( 1 \leq i \leq k \). We recall that a base \( b \) integer \( N \) is divisible by \( p_i^{\gamma} \) if the last \( \gamma \) digits of \( N \) form a base \( b \) integer divisible by \( p_i^{\gamma} \). Let \( N = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} w, \) where \( \gcd(w, b) = 1 \). Let \( m = \max(\beta_1, \beta_2, \cdots, \beta_k) \). Let \( L \) be the base \( b \) integer equal to \( p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \). As \( b \mid N \), the last digit of \( L \) is not 0. Let \( \ell \) be the length of \( L \). Consider the base \( b \) palindrome \( P = [L^R(0)^{m-\ell} L], \) where \( L^R \) is the reversal of base \( b \)-representation of \( L \). As \( P \) is divisible by \( p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}, \) this is the end of the proof if \( w = 1 \). As \( \gcd(w, b) = 1 \) Euler’s theorem implies that \( b^{\phi(w)} - 1 \) (mod \( w \)).

Let \( r \) be an integer divisible by \( \phi(w) \) and greater than \( l + m \), the length of \( P \). Let \( q \geq 1 \) a multiple of \( b^{\phi(w)} - 1 \). Consider the infinite family of integers:

\[
Q_{r,q} = [1((0)^r-1)^q]_b = 1 + b^r + b^{2r} + \cdots + b^{qr} = 1 + b^r + b^{2r} + \cdots + b^{qr} + q - q = (b^r - 1) + (b^{2r} - 1) + (b^{3r} - 1) + \cdots + (b^{qr} - 1) + q.
\]

All terms in the last part of (14) are divisible by \( b^{\phi(w)} - 1 \), so \( Q_{r,q} \) is divisible by \( b^{\phi(w)} - 1 \) and by \( w \). We finish the proof observing that \( P \cdot Q_{r,q} \) is a base \( b \) palindrome divisible by \( N \). \[ \square \]

6. Proof of Theorem 3

**Proof.** Observe that:

\[
(b - 1) \cdot (b - 1) = b(b - 2) + 1 = [(b - 2)1]_b
\]

\[
(b - 1)b^k + (b - 1)b^{k-1} = b^k + (b - 2)b^{k-1} = [1(b - 2)0^k]_b.
\]

Using (15) one has that
\[ [1(b - 1)]_b \cdot [(b - 1)^k]_b = (b + b - 1) \cdot \left( \sum_{i=0}^{k-1} (b - 1)b^i \right) \]
\[ = \sum_{i=0}^{k-1} \left( (b - 1)b^{i+1} + (b(b - 2) + 1)b^i \right) \]
\[ = \sum_{i=1}^{k} (b - 1)b^i + \sum_{i=0}^{k-1} (b(b - 2) + 1)b^i \]
\[ = (b - 1)b^k + \sum_{i=1}^{k-1} \left( (b - 1) + b(b - 2) + 1 \right)b^i + b(b - 2) + 1 \]
\[ = (b - 1)b^k + \sum_{i=1}^{k-2} (b - 1)b^{i+1} + b(b - 2) + 1 \]
\[ = b^k + (b - 2)b^{k-1} + \sum_{i=1}^{k-2} (b - 1)b^{i+1} + b(b - 2) + 1 \]
\[= 1(b - 2)(b - 1)^{k-2}(b - 2)1_b. \]

\[ \square \]

7. **Proof of Theorem 4**

*Proof.* a) Note that \( s_b(N_k) = 2a. \) As \( b \) is even, there exists an integer \( M \) such that:

\[ 2a \cdot M = [a(0)^{k+1}]_b. \]

The following computation shows that \( N_k \) is a \( b \)-ARH number:

\[ s_b(N_k) \cdot M + (s_b(N_k) \cdot M)^R \]
\[ = [a(0)^{k+1}]_b + [a]_b = [a(0)^kb]_b = N_k. \]

To show that \( N_k \) is not a \( b \)-Niven number observe that \( N_k/a = [1(0)^k]_b \) is not an odd number.

b) Note that \( s_b(N_k) = 2b. \) As \( b \) is even the multiplier \( M = [(1(0)^k)^b(0)^{k+b}]_b/2 \) is an integer.

The following computation shows that \( N_k \) is a \( b \)-ARH number:

\[ s_b(N_k) \cdot M + (s_b(N_k) \cdot M)^R \]
\[ = [(1(0)^k)^b(0)^{k+b+1}]_b + [(0)^{k+1}]_b = [(1(0)^k)^b0((0)^k1)^b]_b = N_k. \]

To show that \( N_k \) is not a \( b \)-Niven number observe that \( N_k \) is not divisible by \( b. \)
8. Proof of Theorem 5

Proof. a) Using that
\[(b - 2)^2 = b^2 - 4b + 4 = b(b - 4) + 4 = [(b - 4)4]_b,
\]
an equivalent base b representation for \(N_k\) is given by
\[N_k = [(b - 2)(0)^{k-1}(b - 4)5(0)^k(b - 2)]_b, \text{ if } b \neq 4,
\]
\[N_k = [2(0)^{k-1}11(0)^k2]_4, \text{ if } b = 4.
\]
If \(b \neq 4\) one has \(s_b(N_k) = 3(b - 1)\) and if \(b = 4\) one has \(s_4(N_k) = 6\). To finish the proof of case a) it is enough to show that \(\alpha_k\) is divisible by \(s_b(N_k)\).

If \(b \neq 4\) one has that:
\[\alpha_k = b^{k+1} + b - 2 = b^{k+1} - 1 + b - 1\]
\[= (b - 1)(b^k + b^{k-1} + \cdots + b^2 + b + 2)
\]
and
\[b^k + b^{k-1} + \cdots + b^2 + b + 2 \equiv k + 2 \pmod{3} \equiv 0 \pmod{3},
\]
where for the first congruence we use \(b \equiv 1 \pmod{3}\) and for the second congruence we use \(k \equiv 1 \pmod{3}\).

If \(b = 4\), then clearly \(\alpha_k\) is divisible by 2. Moreover
\[\alpha_k = 4^{k+1} + 2 = (3 + 1)^{k+1} + 2 \equiv 0 \pmod{3},
\]
which shows that \(\alpha_k\) is divisible by 6.

b) Assume now \(b = 2\). Then an equivalent base 2 representation for \(N_k\) is given by
\[N_k = [1(0)^{k-1}10(0)^k1]_2,
\]
so \(s_2(N_k) = 3\). To finish the proof we show that \(\alpha_k\) is divisible by 3:
\[\alpha_k = 2^{k+1} + 1 = (3 - 1)^{k+1} + 1 \equiv 0 \pmod{3}.
\]
The congruence follows because \(k\) is even.

9. Proof of Proposition 2

Proof. To prove the proposition we show that the multipliers from Theorem 5 corresponding to various values of \(k\) are distinct. This follows from the explicit formulas below, as all the sequences of multipliers are strictly increasing as functions of \(k\):

If \(b = 2\) the sequence of multipliers is given by \(M_k = \frac{2^{k+1}+1}{3}\).

If \(b = 4\) the sequence of multipliers is given by \(M_k = \frac{4^{k+1}+2}{6}\).

If \(b > 4\) the sequence of multipliers is given by \(M_k = \frac{b^{k+1}+b-2}{3(b-1)}\).
10. Proof of Theorem [6]

Proof. Let $I$ be a string of base $b$-digits. There exists an infinity of base $b$ strings $J$ such that $s_b([IJ]_b)$ is a power of $b$, say $b^k$, $k \geq 1$. Then the number $N_J = [IJ(0)^kb]_b$ is a $b$-Niven number.

□

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