Dimensional crossover in non-relativistic effective field theory

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Received 18 October 2018, revised 26 November 2018
Accepted for publication 4 December 2018
Published 4 January 2019

Abstract
Isotropic scattering in various spatial dimensions is considered for arbitrary finite-range potentials using non-relativistic effective field theory. With periodic boundary conditions, compactifications from a box to a plane and to a wire, and from a plane to a wire, are considered by matching S-matrix elements. The problem is greatly simplified by regulating the ultraviolet divergences using dimensional regularization with minimal subtraction. General relations among (all) effective-range parameters in the various dimensions are derived, and the dependence of bound states on changing dimensionality are considered. Generally, it is found that compactification binds the two-body system, even if the uncompacted system is unbound. For instance, compactification from a box to a plane gives rise to a bound state with binding momentum given by $\ln \left( \frac{1}{\sqrt{2}} \right)$ in units of the inverse compactification length. This binding momentum is universal in the sense that it does not depend on the two-body interaction in the box. When the two-body system in the box is at unitarity, the S-matrices of the compactified two-body system on the plane and on the wire are given exactly as universal functions of the compactification length.

Keywords: few body systems, dimensional crossover, compactification, atomic systems, Bose–Einstein condensation

(Some figures may appear in colour only in the online journal)

1. Introduction

Recent experimental advances have brought remarkable control to bear on atomic systems [1]. Both the strength of the inter-atomic interaction and the dimensionality of space can be altered in ways that require an understanding of non-relativistic quantum mechanics as the interaction potential and the dimensionality of space are varied. Recent investigations of the phase diagram of Bose gases as the number of spatial dimensions is continuously varied have made use of relations among the scattering lengths in various dimensions [2–6]. In addition, there has been interest in investigating the three-body system and the Efimov effect as spatial dimensions are varied [7, 8]. In cold-atom experiments, confinement of a dimension is typically achieved using trapping potentials. However, here toroidal confinement will be considered in a general setup, where the fundamental assumption made is that the relevant potentials are of finite range. This allows a formulation of the problem in terms of an effective non-relativistic action which is an expansion in local operators. A primary difficulty in dealing with local operators is the renormalization which is necessary to deal with the highly singular nature of the delta-function interactions. While the observable physics which results from an investigation of these interactions is, of course, not dependent on the manner in which the theory is regulated, it is highly beneficial to choose the regularization and renormalization scheme wisely, particularly if one is interested in a general analysis which holds for any finite range potential. The technology of effective field theory (EFT) is well known to be suited to the task1.

A useful way of expressing the S-matrix for two-body scattering is via the effective range expansion, which is valid at momenta small compared to the inverse of the range of the

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1 For reviews, see, [9, 10].
interaction. Effective range theory is therefore a natural means of expressing observables calculated from EFT. It is a straightforward exercise to obtain the effective range expansions to all orders in various spatial dimensions. The question of interest here is what occurs in the presence of boundaries which continuously interpolate between dimensions. An elegant way of doing this is by imposing a boundary with periodic boundary conditions and then shrinking the boundary. In this manner one compactifies a three-dimensional box to a two-dimensional plane or a one-dimensional wire and expresses the one- and two-dimensional effective range parameters in terms of the three-dimensional parameters. In similar fashion, the one-dimensional effective range parameters can be expressed in terms of the two-dimensional parameters. Of course, all of these relations are dependent on the initial geometry. The main mathematical characteristic of the toroidal compactifications is the presence of the Riemann zeta function and related functions and several interesting approximate relations among Riemann zeta functions of odd integer and half-integer argument are found to emerge naturally from the compactification scheme. An interesting theoretical scenario which is straightforward to investigate in the general formulation occurs when the initial system in three spatial dimensions is at unitarity. In this case, due to the absence of a scale in the initial configuration, the presence of a boundary gives rise to S-matrices that are universal in the sense that they are exactly calculable in terms of the confinement length.

This paper is organized as follows. Section 2 reviews the EFT relevant to the interactions of non-relativistic identical bosons in \( d \) spacetime dimensions. The general form of the two-body isotropic scattering amplitude is constructed. This analysis is greatly simplified by regulating the theory using dimensional regularization (DR) with minimal subtraction (MS). In section 3, the special cases with \( d = 2, \ d = 4, \) and \( d = 3 \) are reviewed and the effective range expansions are defined. Special attention is given to the case \( d = 3 \) as only this case experiences non-trivial renormalization. Section 4 contains the main results of the paper. Starting from a \( d = 4 \) box with periodic boundary conditions, compactification to \( d = 3 \) and to \( d = 2 \) is considered. All effective range parameters in \( d = 3 \) and \( d = 2 \) are expressed in terms of the \( d = 4 \) effective range parameters. The special case of compactification when the \( d = 4 \) theory possesses Schrödinger symmetry is considered. Then, starting from a \( d = 3 \) square with periodic boundary conditions, compactification to \( d = 2 \) is considered. All effective range parameters in \( d = 2 \) are expressed in terms of the \( d = 3 \) effective range parameters. With the various results in hand, a comparison is performed of the one-step versus two-step compactification from \( d = 4 \) to \( d = 2 \). Finally, section 5 summarizes the main points of the paper.

## 2. Effective field theory

This section reviews EFT technology which is helpful in deriving a general expression for the isotropic scattering phase shift in any number of spatial dimensions [11]. If one is interested in non-relativistic scattering at low energies, an arbitrary interaction potential that is of finite range may be replaced by an tower of contact operators\(^2\), whose coefficients are determined either by matching to some known underlying theory, or by fitting to experimental data [12–14]. The crucial point is that at low energies only a few of the contact operators will be important. The EFT of bosons, interacting isotropically, has the following Lagrangian:

\[
\mathcal{L} = \psi^\dagger \left( \frac{i}{\hbar} \partial_t + \frac{\hbar^2}{2M} \nabla^2 \right) \psi - \frac{C_0}{4} (\psi^\dagger \psi)^2 - \frac{C_2}{8} (\nabla \psi^\dagger \nabla (\psi^\dagger \psi)) - \frac{D_0}{36} (\psi^\dagger \psi)^3 + \ldots, \tag{2.1}
\]

where the field operator \( \psi \) destroys a boson. The operators in this Lagrangian are constrained by Galilean invariance, parity and time-reversal invariance, and describe bosons which interact at low-energies via an arbitrary potential of finite range. The \( C_{2n} \) are coefficients of two-body operators and \( D_0 \) is the coefficient of a three-body operator. This Lagrangian is valid in any number of spacetime dimensions, \( d \). The mass dimensions of the boson field and of the operator coefficients depend on \( d \) as follows: \( [\psi] = (d - 1)/2, \ [C_{2n}] = 2 - d - 2n \) and \( [D_0] = 3 - 2d \). In this paper bosons living in \( d = 4, \ 3 \) and \( 2 \) spacetime dimensions will be considered. While the coefficients of the operators are \( d \)-dependent there is no need to label them as they are not observable quantities and therefore no ambiguity will be encountered. By contrast, as will be seen, the S-matrix takes a distinct form for each spacetime dimension. Units with \( \hbar = 1 \) are used throughout and the boson mass, \( M \), is kept explicit.

Consider \( 2 \rightarrow 2 \) scattering, with incoming momenta labeled \( \mathbf{p}_1, \ \mathbf{p}_2 \) and outgoing momenta labeled \( \mathbf{p}'_1, \ \mathbf{p}'_2 \). In the center-of-mass frame, \( \mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2 \), and the sum of Feynman diagrams –illustrated in figure 1–computed in the EFT gives the two-body scattering amplitude

\[
A_2(p) = \frac{-\sum C_{2n} p'^{2n}}{1 - I_0(p) \sum C_{2n} p^{2n}}, \tag{2.2}
\]

\(^2\) In coordinate space this corresponds to a sequence of delta-functions and their derivatives.
where
\[
I_0(p) = \frac{M}{2} \left( \frac{\mu}{2} \right) \Gamma \left( \frac{D+1}{2} \right) \frac{1}{p^2 - q^2 + i\delta}. 
\] (2.3)

It is understood that the ultraviolet divergences in the EFT are regulated using DR\(^3\). In equation (2.3), \(\mu\) and \(D\) are the DR scale and dimensionality, respectively, and \(\epsilon \equiv d - D\). A useful integral is:
\[
I_\ell(p) = \frac{M}{2} \left( \frac{\mu}{2} \right) \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{1}{p^2 - q^2 + i\delta};
\]
\[
= -\frac{M}{2} \beta_0^2 (-p^2 - i\delta)^{(D-3)/2} \Gamma \left( \frac{3 - D}{2} \right) \left( \frac{\mu/2\gamma}{(4\pi)^{(D-3)/2}} \right). 
\] (2.4)

In this paper, the EFT coefficients will be defined in DR with \(\overline{MS}\). This choice is valid if the renormalized EFT coefficients are of natural size with respect to the distance scale \(\ell\), which characterizes the range of the interaction. In systems with a scattering length in three spatial dimensions which is large compared to \(\ell\), it is convenient to use DR with the PDS scheme \([13]\), which keeps the renormalized coefficients of natural size in the presence of a large scattering length. However, it is important to emphasize that there is no barrier to working in \(\overline{MS}\) for a scattering length of any size as physics is independent of the regularization and renormalization scheme.

The scattering amplitude can be parametrized via \([11]\)
\[
A_2(p) = \frac{-1}{\text{Im}(I_0(p))} \frac{1}{\cot \delta(p) - i},
\] (2.5)

with
\[
\cot \delta(p) = \frac{1}{\text{Im}(I_0(p))} \left[ \frac{1}{\sum C_{2n} \beta_0^{2n} - \text{Re}(I_0(p))} \right]. 
\] (2.6)

Bound states are present if there are poles of the scattering amplitude on the positive, imaginary, momentum axis. That is, if \(\cot \delta(p) = i\) with binding momentum satisfying \(\gamma > 0\). Evaluating \(I_0(p)\) in DR, it is convenient to consider even- and odd-spacetime dimensions separately. For \(d\) even the Gamma function has no poles and one finds the finite result
\[
I_0(p) = -\frac{M}{2(4\pi)^{(d-1)/2}} \frac{\pi i}{\Gamma \left( \frac{d-1}{2} \right)} p^{d-3}. 
\] (2.7)

Hence the \(\overline{MS}\) EFT coefficients do not run with \(\mu\) in even spacetime dimensions and the bare parameters are

\(^3\) In particular, the separation of the potential from the loop integral in equation (2.2) relies on special properties of DR, and would not, for instance, hold generally using cutoff regularization.

\(\overline{MS}\) parameters. For \(d\) odd, one finds
\[
I_0(p) = \frac{M}{2(4\pi)^{(d-1)/2}} \Gamma \left( \frac{d-1}{2} \right) \frac{1}{\Gamma \left( \frac{d-1}{2} \right)} \left[ \ln \left( \frac{p^2}{\mu^2} \right) - \psi_0 \left( \frac{d-1}{2} \right) - \ln \pi - \frac{2}{d} \right], 
\] (2.8)

where \(\psi_0(n)\) is the digamma function. Here there is a single logarithmic divergence which is hidden in the \(1/\epsilon\) pole. Therefore in this scheme, at least one EFT coefficient will depend on the scale \(\mu\). The general expression for the isotropic phase shift in \(d\) spacetime dimensions is:
\[
p^{d-3} \cot \delta(p) = -\frac{(4\pi)^{(d-1)/2}}{\pi M} \frac{\Gamma \left( \frac{d-1}{2} \right)}{2} \sum_{n=2} \frac{2}{C_{2n} \beta_0^{2n}} 
\]
\[
+ (1 - (-1)^d) \frac{p^{d-3}}{2\pi} \ln \left( \frac{p^2}{\mu^2} \right), 
\] (2.9)

where \(\overline{M}\) is defined by equating the logarithm in equation (2.9) with the content of the square brackets in equation (2.8). This is, of course, an unrenormalized equation as the \(C_{2n}\) coefficients are bare parameters and there is a logarithmic divergence for odd spacetime dimensions.

3. Isotropic scattering in the continuum

3.1. \(d = 2\): one spatial dimension

In one spatial dimension, equation (2.9) gives
\[
p^{-1} \cot \delta(p) = -a_1 + \gamma_1 p^2 + \sum_{n=2} \infty u_{(n)} p^{2n}; 
\] (3.1)

with scattering length and volume, respectively,
\[
a_1 = \frac{4}{MC_0}; \quad \gamma_1 = \frac{4C_2}{MC_0}. 
\] (3.2)

The \(u_{(n)}\) are shape parameters which are easily matched to the \(C_{2n}\) coefficients. Neglecting the scattering volume, for \(a_1 < 0\) there is a bound state with binding momentum \(\gamma_1 = -1/a_1\).

3.2. \(d = 4\): three spatial dimensions

For three spatial dimensions, equation (2.9), yields the familiar effective range expansion,
\[
p \cot \delta(p) = -\frac{1}{a_3} + r_3 p^2 + \sum_{n=2} \infty v_{(n)} p^{2n}; 
\] (3.3)

with scattering length\(^4\) and effective range, respectively,

\(^4\) Note that in the PDS scheme \([13]\), the relationship between the scattering length and the renormalized coefficient is modified to
\[
\frac{8\pi}{MC_0(\mu)} = \frac{1}{a} - \mu, 
\] (3.4)

where \(\mu\) is the PDS renormalization scale. The \(\overline{MS}\) scheme is recovered as \(\mu \to 0\).
The $v_{(n)}$ are shape parameters. Neglecting the effective range, for $a_3 > 0$ there is a bound state with binding momentum $\gamma_3 = 1/a_3$.

### 3.3. $d = 3$: two spatial dimensions

In this section, the case $d = 3$ will be considered in some detail. From the general formula, equation (2.9), one finds

$$\cot \delta(p) = \frac{1}{\pi} \ln \left( \frac{p^2}{\mu^2} \right) - \frac{1}{\alpha_2(\mu)} + \sigma_2 p^2 + \sum_{n=2}^{\infty} v_{(n)} p^{2n}$$

with coupling constant and effective area, respectively,

$$\alpha_2(\mu) = \frac{MC_0(\mu)}{8}; \quad \sigma_2 = \frac{8C_2(\mu)}{MC_0^2(\mu)}$$

Note that $\sqrt{\sigma_2}$ is the effective range. The $v_{(n)}$ are shape parameters. Neglecting all higher-order range corrections, for $\alpha_2(\mu)$ of either sign, there is a bound state with binding momentum $\gamma_2 = \mu \exp(\pi/2\alpha_2(\mu))$. This occurs because quantum mechanical effects generate an attractive logarithmic contribution which always dominates at long distances. However, in the repulsive case this bound state is not physical.

The scale dependence of the leading EFT coefficient is determined by the condition that the scattering amplitude be independent of the scale $\mu$:

$$\frac{d}{d \mu} C_0(\mu) = \frac{M}{4\pi} C_0^2(\mu)$$

This equation is readily integrated to give the renormalization group evolution equation

$$\alpha_2(\mu) = \frac{\alpha_2(\nu)}{1 - \frac{2}{\pi} \alpha_2(\nu) \ln \left( \frac{\mu}{\nu} \right)}.$$

It is clear from equation (3.9) that the attractive case, $\alpha_2(\mu) = -|\alpha_2(\mu)|$, corresponds to an asymptotically free coupling, while the repulsive case, $\alpha_2(\mu) = +|\alpha_2(\mu)|$, has a Landau pole and the coupling grows weaker in the infrared.

The position of the bound state in the repulsive case is the position of the Landau pole, which sets the cutoff scale of the EFT; this is the scale at which new ultraviolet physics should make its appearance and effectively remove the singularity. Therefore, the bound state in the repulsive case is unphysical.

A more common parametrization of the phase shift is given by

$$\cot \delta(p) = \frac{1}{\pi} \ln(p^2a_2^2) + \sigma_2 p^2 + \sum_{n=2}^{\infty} w_{(n)} p^{2n},$$

where $a_2$ is the scattering length in two spatial dimensions. By matching with equation (3.6), one finds $a_2^{-1} = \mu \exp(\pi/2\alpha_2(\mu))$, which in the repulsive case is the position of the Landau pole. Hence, in the repulsive case, $a_2^{-1}$ is the momentum cutoff scale. This suggests that $a_2$ is not an optical parameter for describing low-energy physics since it is unnatural with respect to the characteristic interaction length scale, $\ell$. By contrast, one expects that the dimensionless parameter $\alpha_2(\mu)$ will take a natural size when $\mu \sim \ell^{-1}$.

### 4. Compactification

In this section, toroidal compactifications of space, which interpolate between the various cases at fixed spatial dimension outlined above, will be considered. The procedure is simple and intuitive. In the two-body scattering problem, all the effects that arise from placing an infrared boundary on a dimension, appear through the loop integral, $I_{(d)}$. If the characteristic range of the interaction is taken to be $\ell$, then placing the two-body system in a box of sides $[L_x, L_y, L_z]$ with periodic boundary conditions quantizes the momenta that can run around the loop such that

$$q_i = \frac{2\pi n_i}{L_i},$$

where $n_i \in \mathbb{Z}^{d-1} = (n_x, n_y, ..., n_{d-1})$. In figure 2 the various patterns of compactification considered here are illustrated\(^5\) in

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\(^5\) An alternative means of moving continuously among various dimensions is to constrain two-particle scattering to a cylinder of radius $R$ and to consider the limit of a plane $(R \to \infty)$ and of a wire $(R \to 0)$ as is relevant in the case of a carbon nanotube [15].
order of consideration. The distinct regimes depend on the ratio of the size of the ‘compactified’ dimension and the physical scales of the problem. If $L_x, L_y, L_z \gg \ell$ then the continuum results are recovered.

4.1. $d = 4$ to $d = 3$

Consider the case of three spatial dimensions with one dimension compactified on a circle. One has a box with sides $[L_x, L_y, L_z]$ and $L_x, L_y \gg \ell$ where again $\ell$ represents the characteristic range of the interaction. Near the continuum limit in the x- and y-directions, and assuming that $L_z$ is finite, the topology of space is $\mathbb{R}^2 \times S^1$. Hence, one can write

$$l_n^L(p) = \frac{M}{2} L_z \sum_{q_n} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q_n^2 - q^2 + i\delta},$$

(4.2)

where here $q^2 \equiv q_x^2 + q_y^2$. Noting that putting the infrared boundary in the $z$-direction has not altered the ultraviolet behavior, evaluating this expression in $\overline{MS}$ yields

$$l_n^L(p) = -\frac{iM}{8\pi} 2^{\gamma_E} \sin \left( \frac{L_z}{2} \sqrt{p^2 + i\delta} \right).$$

(4.3)

In the continuum limit, $L_z \to \infty$, and tracking the parameter $\delta$, one finds

$$l_n^L(p) = -\frac{iM}{8\pi} 2^{\gamma_E} \ln(L_z p) - \frac{1}{n} \left[ \frac{L_z}{2\pi} \right]^{2\gamma_E} \zeta(2n),$$

(4.4)

as expected from equation (2.7). When $L_z p \ll 1$, separating off the non-analytic piece yields

$$l_n^L(p) = \frac{M}{4\pi L_z} \left( -\frac{i\pi}{2} + \ln(L_z p) - \sum_{n=1}^{\infty} \frac{1}{n} \left[ \frac{L_z}{2\pi} \right]^{2\gamma_E} \zeta(2n) \right),$$

(4.5)

where $\zeta(s)$ is the Riemann zeta function. In this limit of scattering in three spatial dimensions equation (2.6) gives

$$\cot \delta(p) = -8L_z M \left[ \frac{1}{C_0} - \frac{C_2}{C_0^2} p^2 \right]$$

$$- \frac{M}{4\pi L_z} \left( \ln(L_z p) - \frac{L_z^2 p^2}{24} \right) + O(p^4),$$

(4.6)

where the $C_{2n}$ are bare $d = 4$ coefficients. Using equation (3.5) to express the EFT coefficients in terms of the effective range parameters in three spatial dimensions and matching to the general form for the phase shift in two spatial dimensions, equation (3.10), yields

$$a_2 = L_z \exp \left( -\frac{L_z}{2\alpha_3} \right), \quad \sigma_2 = \frac{L_z}{2\pi} (r_3 - \frac{L_z}{6}).$$

(4.7)

The expression matching the scattering lengths is in agreement with previous results found in [5, 6]. The matching of higher-order effective range parameters gives (for $n \geq 2$)

$$w(n) = \frac{L_z}{\pi} v(n) - \frac{2}{\pi n} \left( \frac{L_z}{2\pi} \right)^{2\gamma_E} \zeta(2n).$$

(4.8)

The condition for a bound state in the presence of the boundary is

$$1 = 2 \sin \left( \frac{L_z \gamma_2}{2} \right) \exp \left( -\frac{L_z}{2} \left( \frac{1}{\alpha_3} + \ldots \right) \right),$$

(4.9)

where $\gamma_2$ is the binding momentum and the dots signify higher order terms in the effective range expansion. Hence, in the limit of two spatial dimensions where $L_z \to 0$ the binding momentum tends to the universal value

$$\gamma_2 = \ln \left( \frac{1}{2} (3 + \sqrt{5}) \right) L_z^{-1} = (0.962 423 650 \ldots) L_z^{-1}.$$  

(4.10)

This value is independent of details of the finite-range potential in three spatial dimensions. For instance, even if the two-body attraction in the three-dimensional theory is not enough for binding, the presence of the boundary and the compactification to two dimensions binds the system. This realizes in practice for this particular geometry the observation made in section 3 that there is always a bound state in two dimensions due to strong infrared effects.

If there is a bound state in the three-dimensional theory with binding momentum $\gamma_3$, then in the presence of the boundary the binding momentum is given by the approximate formula

$$\gamma_2 = \frac{2}{L_z} \ln \left( \frac{1}{2} \exp \left( \frac{L_z \gamma_3}{2} \right) + \sqrt{1 + \frac{1}{4} \exp(L_z \gamma_3)} \right),$$

(4.11)

which smoothly interpolates between the universal value of equation (4.10) in the limit of two spatial dimensions and the binding momentum in the box as the boundary is removed. If there is no bound state in the three-dimensional theory, then the appearance of the boundary signals the presence of a bound state at threshold which again tends toward the universal value of equation (4.10) in the limit of two spatial dimensions.

Now consider the special case where the original theory in three spatial dimensions is at or near unitarity; this corresponds to $\alpha_3 \to \infty$, while $r_3, v(n) \to 0$. In this case, the original (two-body) theory is at a non-trivial fixed point of the renormalization group and therefore has a non-relativistic conformal invariance, i.e. Schrödinger symmetry. One consequence of this symmetry is that the bound state in the system has zero energy (as there is no scale). The exact phase

6 Note that in the PDS scheme, equation (4.3) has an additional piece, $-M/8\pi$, which then gives the correct relationship between the scattering length and the renormalized coefficient in equation (4.6) so that all relations involving the effective range parameters are independent of the renormalization scheme.

7 This point has been made previously in [7].

8 See, for instance, [16].
shift in the presence of the boundary is then given by [7]

$$\cot \delta(p) = \frac{2}{\pi} \ln \left( 2 \sin \left( \frac{L_p}{2} \right) \right),$$  \hspace{1cm} (4.12)

for \(L_p \sim 1\), with the restriction \(0 < \frac{L_p}{2} < \pi\). This phase shift is plotted in figure 3 and compared with effective range theory.

As the boundary is brought in from infinity where there is a bound state at threshold, one expects that the binding momentum should scale as \(1/L_c\), and indeed one finds from equation (4.12) (and equation (4.9)) that the binding momentum is given by equation (4.10). In the limit of two spatial dimensions where \(L_z \to 0\), all effective range parameters, and therefore the EFT, is entirely fixed by the one parameter, \(L_c\), which breaks the scale invariance. Evidently this two-dimensional theory is repulsive and indeed in this limit the binding energy rises to its largest possible value which is of course the position of the Landau pole. Hence this EFT in two spatial dimensions with repulsive interactions has as its ultraviolet completion (for \(p \gg L_z^{-1}\)) a conformal field theory in three spatial dimensions.

4.2. \(d = 4\) to \(d = 2\)

Now consider the case of three spatial dimensions with two dimensions compactified on a sphere. Take a box with sides \(L_x, L_y, L_z\) and choose \(L_x \gg L\). Near the continuum limit in the \(x\)-direction, and assuming that \(L_x = L_y \equiv L\) is finite, the topology of space is \(\mathbb{R}^1 \times S^2\). Hence, the loop integral becomes

$$I_0^2(p) = \frac{M}{2L^2} \sum_{q, q_c} \int_{-\infty}^{\infty} \frac{dq}{(2\pi)^3} \frac{1}{p^2 - q^2 - q_c^2 - q_c^2 + i\epsilon^2}. $$ \hspace{1cm} (4.13)

Evaluating the integral in \(\overline{MS}\) yields

$$I_0^2(p) = -\frac{M}{4L^2} \left[ \frac{i}{p} + \frac{L}{2\pi} \sum_{n, n_c = 0}^{\infty} \frac{1}{n^2 + n_c^2 - \rho^2} - 2\pi \Lambda_n \right],$$ \hspace{1cm} (4.14)

where \(\rho \equiv pL/2\pi\), and \(\Lambda_n \to \infty\) in an integer cutoff. This two-dimensional sum is tractable in the sense that it can be expressed as a one-dimensional sum over special functions. It is straightforward to find \([11, 17]\)

$$\sum_{n, n_c = 0}^{\infty} \frac{1}{n^2 + n_c^2 - \rho^2} = -2\pi \Lambda_n$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k^2 + k} 4\zeta \left( \frac{1}{2} + k \right) \beta \left( \frac{1}{2} + k \right),$$ \hspace{1cm} (4.15)

where \(\beta(s)\) is the Dirichlet beta function. The phase shift is now given by

$$p^{-1} \cot \delta(p) = -\frac{4L^2}{M} \left[ \frac{1}{C_0} - \frac{C_2}{C_0} \rho^2 + \frac{M}{8\pi L} \left( 4\zeta \left( \frac{1}{2} \right) \beta \left( \frac{1}{2} \right) \right. \right.
\left. + \frac{L^2}{2\pi^2} \zeta \left( \frac{3}{2} \right) \beta \left( \frac{3}{2} \right) \rho^2 \right] + O(p^4),$$ \hspace{1cm} (4.17)

where again the \(C_{2n}\) are bare \(d = 4\) coefficients. Proceeding as before one finds

$$a_1 = \frac{L^2}{2\pi^2} \left( \frac{1}{a_3} \right) \left( 1 - \frac{1}{4\zeta \left( \frac{1}{2} \right) \beta \left( \frac{1}{2} \right)} \right)$$
\hspace{1cm} and
$$\tau_1 = \frac{L^2}{4\pi} \left( \frac{1}{\tau_3} \right) \left( 1 - \frac{1}{\pi^2} \zeta \left( \frac{3}{2} \right) \beta \left( \frac{3}{2} \right) \right).$$ \hspace{1cm} (4.18)

where

$$4\zeta \left( \frac{1}{2} \right) \beta \left( \frac{1}{2} \right) = -3.900 264 920 001 955 882 845 475 336$$
\times 604 973 219 209 047 865 477 5;
$$\zeta \left( \frac{3}{2} \right) \beta \left( \frac{3}{2} \right) = 2.258 405 420 775 237 576 432 628$$
\times 819 829 264 634 875 791 346 143 6.

(4.19)

The reason for quoting so many digits will be made clear below. The expression matching the scattering lengths is in agreement with the result found in \([6]\). It is straightforward to match higher-order effective range parameters giving (for \(n \geq 2\))

9 Equivalently, one can express the two-dimensional sum as a one-dimensional sum over a single special function

$$\sum_{n, n_c = 0}^{\infty} \frac{1}{n^2 + n_c^2 - \rho^2} = 2\pi \Lambda_n = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}} \left( \frac{1}{2} - \frac{\rho^2}{(2k+1)} \right)$$ \hspace{1cm} (4.16)

where \(\zeta(s, a)\) is the Hurwitz zeta function.

10 Note that \([6]\) uses the opposite sign convention for the one-dimensional scattering length.
\[ u(\alpha) = \frac{L^2}{2\pi} \psi(\alpha) - (-1)^{\nu} \left( -\frac{1}{n^2} \right) \left( \frac{L}{2\pi} \right)^{2n+1} \times 4\zeta \left( \frac{1}{2} + n \right) \beta \left( \frac{1}{2} + n \right). \] (4.20)

The one-dimensional limit, \( L \to 0 \), supports a bound state with binding momentum determined by the roots of

\[ -1 = \sum_{k=0}^{\infty} \left( \frac{1}{2} \right) \left( \frac{L \gamma}{2\pi} \right)^{2k+1} 4\zeta \left( \frac{1}{2} + k \right) \beta \left( \frac{1}{2} + k \right). \] (4.21)

One finds a bound state with binding momentum

\[ \gamma = (1.511 955 584...) L^{-1}. \] (4.22)

As in the previous case, this result is universal in the sense that it is independent of whether the initial system is bound.

One can again consider the special case where the original theory in three spatial dimensions is at unitarity. Here the exact phase shift in the one-dimensional theory with compactified dimensions is

\[ p^{-1} \cot \delta(p) = -\left( \frac{L}{2\pi} \right) \sum_{k=0}^{\infty} (-1)^{k} \left( \frac{1}{2} \right) \left( \frac{L \gamma}{2\pi} \right)^{2k} \times 4\zeta \left( \frac{1}{2} + k \right) \beta \left( \frac{1}{2} + k \right). \] (4.23)

which supports a bound state with binding momentum given by equation (4.22). This phase shift is plotted in figure 4 and compared with effective range theory.

4.3. \( d = 3 \) to \( d = 2 \)

Lastly consider the case of two spatial dimensions with one dimension compactified on a circle. Take a square with sides \( (L_x, L_y) \) and choose \( L_x \gg \ell \). Near the continuum limit in the \( x \)-direction, and assuming that \( L_y \) is finite, the topology of space is \( \mathbb{R}^1 \times S^1 \). Hence, the loop integral becomes

\[ L^L_0(p) = \frac{M}{2} \frac{1}{L_y} \sum_{q_y} \int_{-\infty}^{\infty} dq_x \left( \frac{2\pi}{p} \right)^2 \left( q_x^2 - q^2 + i\delta \right). \] (4.24)

This case is somewhat more involved than the other cases as there is a logarithmic divergence which is not affected by the infrared boundary. As the integral is linearly convergent, this may be evaluated first, giving

\[ L^L_0(p) = -\frac{M}{4L_y} \left( \frac{i}{p} + \sum_{q_y \neq 0} \frac{1}{q_y^2 - p^2} \right). \] (4.25)

where the logarithmically-divergent sum must be evaluated in

\[ Figure 4. \text{Exact S-matrix in the presence of a boundary when two-body system is at unitarity in } d = 4. \text{ Units are chosen with } L = 1. \text{ The red curve is the low-momentum expansion up to effective area corrections, and the blue curve includes eight orders in the effective range expansion.} \]
Note that in the one-dimensional limit, there is again always a bound state approaching threshold. However, the bound state that is always present in the initial two-dimensional theory can be lost in the presence of the boundary only to reappear again when a bound state appears at threshold in the approach to the one-dimensional limit when $L_y \sim 4\pi a_z e^{-\gamma}$. 

4.4. $d = 4$ to $d = 2$ in two steps

Combining the results of sections 4.1 and 4.3, the compactification from $d = 4$ to $d = 2$ can be achieved in two steps. Setting $L_x = L_y = L$, one obtains

$$a_i = \frac{L^2}{2\pi} \left( \frac{1}{d_3} + \frac{1}{L} (\gamma_E - \ln 4\pi) \right),$$

$$\tau_i = \frac{L^2}{4\pi} \left( \frac{1}{d_3} - \frac{L}{\pi^2} \left( \zeta(3) + \frac{\pi^2}{3} \right) \right).$$

(4.31)

where

$$2(\gamma_E - \ln 4\pi) = -3.907 617 164 135 515 864 742 759 008 374 018 833 512 271 491 391 8;$$

$$\frac{1}{2} \left( \zeta(3) + \frac{\pi^2}{3} \right) = 2.245 962 518 428 023 579 172 284 247 401 750 184 601 443 047 377 0. \quad (4.32)$$

Note that these expressions differ at the one-part-per-mil level from the expressions obtained in the one-step compactification, equation (4.19). It is straightforward to match higher-order effective range parameters giving (for $n \geq 2$)

$$u_{(n)} = \frac{L^2}{2\pi} \varrho_{(n)} - \frac{2}{L} \left( \frac{L}{2\pi} \right)^{2n+1} \left( \frac{1}{n} \zeta(2n) + (-1)^n \left( \frac{1}{n} \right)^2 \zeta(2n+1) \right).$$

(4.33)

This difference with the one-step compactification equation (4.20) increases with $n$. The discrepancy between the one- and two-step compactifications to the wire is not surprising as the initial geometries differ; in the two-step case, the initial compactification to the plane assumed that the $\hat{y}$-direction was infinite, whereas in the one-step case both the $\hat{y}$- and $\hat{x}$-directions were finite in extent and equal. The near equality, particularly between the expressions equations (4.19) and (4.32), is intriguing given that there are no known expressions of the Riemann and Dirichlet beta functions of half integer argument in terms of fundamental constants.

5. Conclusion

Interesting quantum mechanical phenomena, like resonance effects that occur in few-body systems and phase transitions which occur in many-body systems, depend critically on the strength and form of the quantum mechanical potential and on environmental constraints like temperature and spatial dimensionality. Given recent experimental progress in controlling spatial dimensionality, it is of interest to consider properties of general quantum mechanical systems as the spatial dimensionality is varying. This paper has considered a very general type of non-relativistic quantum mechanical system of bosons that interact entirely via finite-range interactions. Starting from a world with three spatial dimensions, it is straightforward to perform toroidal compactifications to worlds with two and one spatial dimensions. It is somewhat counter intuitive that in some sense the most difficult aspect of this general problem is properly accounting for the renormalization of ultraviolet divergences which are unaffected by the infrared boundaries that enter through the compactification procedure. The use of DR with minimal subtraction greatly simplifies the computations, primarily because power-law divergences do not appear in this scheme. General relations among effective range parameters were obtained between various dimensions. These relations may be useful in computing non-universal corrections to Bose gas thermodynamic variables in various dimensional crossover schemes. For instance, the effective range (area) corrections to the weakly interacting Bose gas in two spatial dimension were recently computed in [18]. That result, together with the expression for the effective range given in equation (4.7), yields the leading non-universal correction due to the effective range to the energy of the quasi-two dimensional Bose gas given in [6].

An interesting consequence of compactification found in this paper is that even if the initial two-body system is not bound in three dimensions, as the boundary is removed and the system is compactified to a plane or a wire, the resulting two-body system always ends up bound. This paper also considered the theoretical scenario where scattering in three spatial dimensions is at unitarity, and this conformal system is compactified to a plane or to a wire. The resulting S-matrices in the reduced dimensionality are then known exactly, and are, of course, universal functions of the compactified length scale, since the underlying theory has no scale. This provides an interesting example of an EFT in two-spatial dimensions with repulsive interactions where the presence of the Landau pole is traced to the underlying theory which is given by a three dimensional system at unitarity.

Finally, it should be mentioned that the EFT methods used here to obtain the relations among all effective range parameters for the case of toroidal compactification can also be fruitfully applied to the case of compactification achieved via the presence of atomic traps, which effectively confine the particles using harmonic potentials [8]. In addition, the
consideration of three- and four-body systems as dimensionality is altered [7, 8] is also a straightforward extension of EFT methods, although the analysis is significantly more involved than in the two-body sector.

Acknowledgments

We would like to thank Dmitry Petrov, Varese Timóteo and Nikolaj Zinner for valuable comments on the manuscript. This work was supported in part by the US Department of Energy grant DE-SC001347.

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