Horizon thermodynamics in $f(R, R^\mu_\nu R^\nu_\mu)$ Theory

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We investigate whether the new horizons first law still holds in $f(R, R^\mu_\nu R^\nu_\mu)$ theory. We obtain the formula via the new horizons first law to compute the entropy of a static spherically symmetric black hole in $f(R, R^\mu_\nu R^\nu_\mu)$ theory. Using the degenerate Legendre transformation, we also obtain the formula to compute the energy of black hole. For application, we consider the quadratic-curvature gravity and firstly calculate the entropy and the energy for a static spherically symmetric black hole, especially for a Schwarzschild-(A)ds black hole.

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I. INTRODUCTION

Black hole, predicted in general relativity, is an object of long-standing interest to physicists. Bekenstein firstly proposed that black holes actually possess entropy [1]. In sharp contrast to standard thermodynamic notions where entropy is supposed to be a function of volume, he suggested that the entropy of black hole is proportional to the horizon area. Since then a variety of different theoretical methods have been used to calculate the Bekenstein-Hawking entropy, such as starting from quantum fields near the horizon [2], from quantum field theory in a fixed background [3], from entanglement entropy [4], from string theory [5–10], from loop quantum gravity [11, 12], from Noether charge [13, 14], from induced gravity [15], from causal set theory [16], from symmetry near horizon [17, 18], from inherently global characteristics of a black hole spacetime [19], etc. All these methods are complicated. Finding a simpler way to calculate the entropy of a black hole is an important task.

Energy is another important issue besides entropy in black hole physics, especially in higher order gravitational theories, the energy of black hole is a controversial issue. Several efforts to find a satisfactory answer to this issue have been carried out [20–24]. It was shown that the entropy and energy of black hole can be simultaneously obtained in Einstein’s gravity via the horizon first law [25]. Recently a new horizon first law, in which both the entropy and the free energy are derived concepts, was suggested in Einstein’s gravity and Lovelock’s gravity, the standard horizon first law can be recovered by a Legendre projection [26]. In [27], it was found that the new horizon first law still work in $f(R)$ theories: it can give not only the energy but also the entropy of black holes, which reproduce the known results in literatures. Here we will consider the new horizon first law and the entropy and the energy issues in $f(R, R^\mu_\nu R^\nu_\mu)$ gravity.

The paper is organised as follows. In section II, we briefly review the new horizon first law. In section III, we discuss the entropy and the energy of black holes in $f(R, R^\mu_\nu R^\nu_\mu)$ Theory. In section IV, applications are considered. Finally, We will briefly summarize our results in section V.

II. THE NEW HORIZON FIRST LAW

According the suggestion proposed in [28], that the source of thermodynamic system is also that of gravity, the radial component of the stress-energy tensor can act as the thermodynamic pressure, $P = T^r_r|_{r_+}$, then at the horizon of Schwarzschild black hole the radial Einstein equation can be written as

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2},$$

(1)
which can be rewritten as a horizon first law after a imaginary displacement of the horizon, $\delta E = T\delta S - P\delta V$, with $E$ the quasilocal energy and $S$ the horizon entropy of the black hole \[25\]. Since the temperature $T$ in the equation \[11\] is identified from thermal quantum field theory, independent of any gravitational field equations \[26\], while the pressure $P$ in \[11\], according to the conjecture proposed in \[28\], is identified as the radial component of the matter stress-energy, so it is reasonable to assume that the radial field equation of a gravitational theory under consideration takes the form \[26\]

$$P = D(r_+) + C(r_+)T,$$

(2)

where $C$ and $D$ are analytic functions of the radius of black hole, $r_+$, in general they depend on the gravitational theory under considering. Varying the equation \[11\] and multiplying the geometric volume $V(r_+)$, it is straightforward to have a new horizon first law \[26\]

$$\delta G = -S\delta T + V\delta P,$$

(3)

with the Gibbs free energy as

$$G = \int_{r_{+}}^{r_{+}} V(r)D'(r)dr + T\int_{r_{+}}^{r_{+}} V(r)C'(r)dr = PV - ST - \int_{r_{+}}^{r_{+}} V'(r)D(r)dr,$$

(4)

and the entropy as \[26\]

$$S = \int_{r_{+}}^{r_{+}} V'(r)C(r)dr.$$

(5)

Under the degenerate Legendre transformation $E = G + TS - PV$, yields the energy as \[27\]

$$E = -\int_{r_{+}}^{r_{+}} V'(r)D(r)dr.$$

(6)

This procedure was firstly discussed in Einstein gravity and Lovelock gravity which only give rise to second-order field equation \[26\]. It was generalized to $f(R)$ gravity with a static spherically symmetric black hole \[27\] or with a general spherically symmetric black hole \[29\] and was also applied to $D$-dimensional $f(R)$ Theory \[30\].

III. THE ENTROPY AND THE ENERGY OF BLACK HOLES IN $f(R, R^{\mu\nu}R_{\mu\nu})$ THEORY

As shown in Section II, the new horizon first law works well in Einstein’s theory and Lovelock gravity \[26\] and $f(R)$ theory \[27\] \[29\] \[30\]. Whether it still works in other gravitational theories, such as $f(R, R^{\mu\nu}R_{\mu\nu})$ theory? We consider this question in this Section. In four-dimensional spacetime, the general action of $f(R, R^{\mu\nu}R_{\mu\nu})$ theory with source is given by

$$I = \int d^4x\sqrt{-g}\left[\frac{f(R, R^{\mu\nu}R_{\mu\nu})}{16\pi} + L_m\right],$$

(7)

where $L_m$ is the matter Lagrangian and $f(R, R^{\mu\nu}R_{\mu\nu})$ is a general function of the Ricci scalar $R$ and the square of the Ricci tensor $R_{\mu\nu}$. We take the units $G = c =\hbar = 1$. Varying the action (7) with respect to metric $g_{\mu\nu}$, yields the gravitational field equations as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\left[\frac{T_{\mu\nu}}{f_R} + \frac{1}{8\pi}\Omega_{\mu\nu}\right],$$

(8)

where $X = R^{\mu\nu}R_{\mu\nu}$, $f_R = \frac{\partial f}{\partial R}$ and $f_X = \frac{\partial f}{\partial X}$. $G_{\mu\nu}$ is the Einstein tensor. $T_{\mu\nu} = \frac{2\delta I}{\delta g_{\mu\nu}}$ is the energy-momentum tensor of matter. $\Omega_{\mu\nu}$ is the tress-energy tensor of the effective curvature fluid, it is given by

$$\Omega_{\mu\nu} = \frac{1}{f_R}\left[\frac{1}{2}g_{\mu\nu}(f - f_R) + \nabla_{\mu}\nabla_{\nu}f_R - g_{\mu\nu}\Box f_R - 2f_X R^a_\mu R_{a\nu} - \Box (f_X R_{\mu\nu}) - g_{\mu\nu}\nabla_\alpha\nabla_\beta(f_X R^{\alpha\beta}) + 2\nabla_\alpha\nabla_{(\mu}(R^a_\nu)f_X\right).$$

(9)
where $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$. Thanking of the following two derivational relations
\[
\nabla_\alpha \nabla_\beta (fx R^{\alpha\beta}) = R^{\alpha\beta} \nabla_\alpha \nabla_\beta fx + (\nabla_\beta R)(\nabla_\alpha fx) + \frac{1}{2}fx \Box R, \tag{10}
\]
and
\[
\nabla_\alpha \nabla_\mu (fx R^\mu_\nu) + \nabla_\alpha \nabla_\nu (fx R^\nu_\mu) = R^\nu_\mu \nabla_\alpha \nabla_\mu fx + R^\mu_\nu \nabla_\alpha \nabla_\nu fx + \frac{1}{2}(\nabla_\mu fx)(\nabla_\nu R) + \frac{1}{2}(\nabla_\nu fx)(\nabla_\mu R)
+ (\nabla_\alpha fx)(\nabla_\mu R^\mu_\nu) + (\nabla_\alpha fx)(\nabla_\nu R^\nu_\mu) + fx \nabla_\mu \nabla_\nu R + 2fx R_{\alpha\mu\lambda}R^{\alpha\lambda}
+ \frac{1}{2}fx R_{\lambda\nu} R^\lambda_\nu, \tag{11}
\]
inserting them into the equation (9), the tress-energy tensor of the effective curvature fluid $\Omega_{\mu\nu}$ is simplified as
\[
\Omega_{\mu\nu} = \frac{1}{fR} \left[ \frac{1}{2}g_{\mu\nu}(f - Rf_R) + \nabla_\mu \nabla_\nu f_R - g_{\mu\nu} \Box f_R
- fx \Box R_{\mu\nu} - R_{\mu\nu} \Box fx - g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta fx - \frac{1}{2}fx g_{\mu\nu} \Box R
+ R^\mu_\nu \nabla_\alpha \nabla_\mu fx + R^\mu_\nu \nabla_\alpha \nabla_\nu fx - fx \nabla_\mu \nabla_\nu R - 2fx R_{\alpha\mu\lambda} R^{\alpha\lambda} \right]. \tag{12}
\]
For a static spherically symmetric black hole whose geometry is given by
\[
ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \tag{13}
\]
where the event horizon is located at $r = r_+$ the largest positive root of $B(r_+) = 0$ with $B'(r_+) \neq 0$, the (1) components of the Einstein tensor is
\[
G^1_1 = \frac{1}{r^2} (-1 + rB' + B), \tag{14}
\]
with the primes denoting the derivative with respect to $r$. At the horizon, since $B(r_+) = 0$, it reduces to
\[
G^1_1 = \frac{1}{r^2} (-1 + rB'). \tag{15}
\]
While the radial components of the tress-energy tensor of the effective curvature fluid $\Omega_{\mu\nu}$ at the horizon takes the following form
\[
\Omega^1_1 = \frac{1}{fR} \left[ \frac{1}{2}(f - Rf_R) - \frac{1}{2}B'f_R' - B'(fx R^1_1)' - \frac{2fx B'R^1_1}{r_+} + \frac{2fx B'R^1_1}{r_+} \right]. \tag{16}
\]
Substituting the equations (15) and (16) into the equation (8), and thinking of $P = T^r_{r\mid r_+}$, we derive
\[
8\pi p = Rf_R - 2f(R, R^{\alpha\mu}R_{\mu\nu}) + 3\Box f_R + 4fx \Box R + R\Box fx - 2fx R^{\alpha\lambda} R_{\alpha\lambda} - 2R^{\alpha\mu} \nabla_\alpha \nabla_\mu fx. \tag{17}
\]
Thinking of Hawking temperature $T = \frac{B'(r_+)}{4\pi}$, the equation (17) can be rewritten as
\[
P = T \left[ \frac{f_R}{2r_+} + \frac{1}{4}f_R' + \frac{1}{2}(R^1_1 f_X)' + \frac{fx R^1_1}{r_+} \right]
- \frac{1}{8\pi} \left[ \frac{f_R}{r_+^2} + \frac{1}{2}(f - Rf_R) + \frac{2fx B'R^3_1}{r_+} - 2fx (R^1_1)^2 \right]. \tag{18}
\]
Comparing this equation with the equation (2), we have
\[
D(r_+) = -\frac{1}{8\pi} \left[ \frac{f_R}{r_+^2} + \frac{1}{2}(f - Rf_R) + \frac{2fx B'R^3_1}{r_+} - 2fx (R^1_1)^2 \right], \tag{19}
\]
and
\[
C(r_+) = \left[ \frac{f_R}{2r_+} + \frac{1}{4}f_R' + \frac{1}{2}(R^1_1 f_X)' + \frac{fx R^1_1}{r_+} \right]. \tag{20}
\]
Considering the geometric volume of black hole $V(r_+) = \frac{4\pi r_+^3}{3}$, and substitute equations (20) into equation (15), we obtain the entropy as follows

$$S = \int_0^{r_+} V'(r)C(r)dr = \int_0^{r_+} 4\pi r^2 \left[ \frac{f_R}{2r} + \frac{1}{4} f'_R + \frac{1}{2} R_1^1 f_X' + \frac{f_X R_1^1}{r} \right] dr = \frac{A}{4}(f_R + 2f_X R_1^1) \quad (21)$$

where $A$ is the area of black hole. Inserting the geometric volume $V$ and the equation (19) into the equation (16), and then we have the energy as

$$E = - \int_0^{r_+} V'(r)D(r)dr = \frac{1}{2} \int_0^{r_+} 2r \left[ \frac{f_R}{r^2} + \frac{1}{2}(f - Rf_R) + \frac{2f_X B'R_3^3}{r} - 2f_X (R_1^1)^2 \right] dr. \quad (22)$$

When $f_X = 0$, the equations (21) and (22) return to the results obtained in $f(R)$ theory (27). Using these two equations, we can calculate the entropy and the energy of black hole in $f(R, R^\mu\nu R_{\mu\nu})$ theory.

**IV. APPLICATION: QUADRATIC-CURVATURE GRAVITY**

For application, we consider a simple but important example: the most general quadratic-curvature gravity theory with a cosmological constant in four dimensions, its Lagrangian density is given by an arbitrary combination of scalar curvature-squared and Ricci-squared terms, namely

$$f(R, R^\mu\nu R_{\mu\nu}) = R + \alpha R^\mu\nu R_{\mu\nu} + \lambda R^2 - 2\Lambda, \quad (23)$$

where $\alpha$ and $\lambda$ are constants, $\Lambda$ is the cosmological constant. For this theory, we have $f_R = 1 + 2\lambda R$ and $f_X = \alpha$. After some calculations, we find from (18) that in spacetime with metric (13)

$$P = T \left[ \frac{1}{2r_+} + \frac{\lambda R}{r_+} + \frac{1}{2} \lambda R' + \frac{1}{2} \alpha (R_1^1)' + \frac{\alpha R_1^1}{r_+} \right]$$

$$- \frac{1}{8\pi} \left[ \frac{1}{r_+^2} + \frac{2 \lambda R}{r_+^2} + \frac{1}{2} \alpha R^\mu R_{\mu\nu} - \frac{1}{2} \lambda R^2 + \frac{2\alpha B'R_3^3}{r_+} - 2\alpha R_1^1 - \Lambda \right], \quad (24)$$

From this equation or from the equations (19) and (20), it’s not hard to see that

$$D(r_+) = \frac{1}{8\pi} \left[ \frac{1}{r_+^2} + \frac{2 \lambda R}{r_+^2} + \frac{1}{2} \alpha R^\mu R_{\mu\nu} - \frac{1}{2} \lambda R^2 + \frac{2\alpha B'R_3^3}{r_+} - 2\alpha R_1^1 - \Lambda \right], \quad (25)$$

and

$$C(r_+) = \left[ \frac{1}{2r_+} + \frac{\lambda R}{r_+} + \frac{1}{2} \lambda R' + \frac{1}{2} \alpha (R_1^1)' + \frac{\alpha R_1^1}{r_+} \right]. \quad (26)$$

Inserting the equation (25) into the equation (21), yields the entropy as

$$S = \int 4\pi r^2 \left[ \frac{1}{2r_+} + \frac{\lambda R}{r_+} + \frac{1}{2} \lambda R' + \frac{1}{2} \alpha (R_1^1)' + \frac{\alpha R_1^1}{r_+} \right] dr_+ = \pi r_+^2 (1 + 2\lambda R + 2\alpha R_1^1). \quad (27)$$

Substituting the equation (25) into the equation (22), the energy of the black hole is given by

$$E = \frac{1}{2} \int r^2 \left[ \frac{1}{r_+^2} + \frac{2 \lambda R}{r_+^2} + \frac{1}{2} \alpha R^\mu R_{\mu\nu} - \frac{1}{2} \lambda R^2 + \frac{2\alpha B'R_3^3}{r_+} - 2\alpha R_1^1 - \Lambda \right] dr_+ \quad (28)$$

$$= - \frac{1}{4} \int \left[ \left( 6\alpha R_1^1 + 2R_1^1 \alpha B'' + 2\alpha R_2^2 + 2\lambda R^2 \right) r_+^2 + 4R_1^1 \alpha r_+ B' - 2(2\alpha R_2^2 + 2\lambda R + 1) \right] dr_+.$$

In the case of a Schwarzschild-(A)dS black hole, for example, whose metric is given by $B(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}$, the entropy (27) and the energy (28) respectively returns to

$$S = \pi r_+^2 (1 + 2\lambda R + 2\alpha R_1^1) = \pi r_+^2 [1 + 2(\alpha + 4\lambda)\Lambda], \quad (29)$$

and

$$E = [1 + 2(\alpha + 4\lambda)\Lambda]M, \quad (30)$$

where we have used $B(r_+)=0$. These results are consistent with those obtained in [31, 32] by using other methods. When $\alpha = 0$, we obtain the results in $R + \lambda R^2 - 2\Lambda$. While for $\alpha = 0$ and $\lambda = 0$, we get the results in Einstein’s gravity with cosmological constant.
V. CONCLUSION

We have discussed whether the new horizons first law is still valid in $f(R,R^{\mu\nu}R_{\mu\nu})$ theory. We have assumed a general form for the horizon equation of state, $P = P(V,T)$, and have obtained the formula via the new horizons first law to compute the entropy of a static spherically symmetric black hole in $f(R,R^{\mu\nu}R_{\mu\nu})$ theory. Using the degenerate Legendre transformation, $E = G + TS - PV$, we also have obtained the formula to compute the energy of black hole. For application, we have considered quadratic-curvature gravity and have presented the entropy and the energy for a static spherically symmetric black hole, especially for a Schwarzschild-(A)ds black hole where the results consistent with those obtained in literature. Whether the proposed method here can be applied to other higher order gravity is still an interesting topic for future research.

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