Exact solution of dispersion equation corresponding to ellipsoidal statistical equation from Stokes’ second problem

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Abstract

In the present work zero of dispersive function from Stokes’ second problem are investigated. Stokes’ second problem is a problem about behaviour of the rarefied gas filling half-space. A plane, limiting half-space, makes harmonious oscillations in the plane. The linearization kinetic ellipsoidal the statistical equation with parametre is used. The factorization formula of dispersion function is proved. By means of the factorization formula zero of dispersion function in

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an explicit form and their research is carried out. Dependence on dimensionless quantity collision frequency of a plane limiting gas and on parameter equation are investigated.

Keywords: Stokes’ second problem, kinetic ellipsoidal statistical equation, separation of variables, zero of dispersion function, eigen solutions, continuous and discrete spectrum, factorization of dispersion function.

Introduction

Problem of generation of shift waves by fluctuating plate or Stokes’ second problem for the continuous matter has been formulated in the middle of XIX century [1]. Then, after introduction by Maxwell and Boltzmann the kinetic equations, Stokes’ second problem began to be studied for the rarefied gas.

The detailed history of this problem is stated in our works [2]–[4]. In these works the Stokes’ second problem was solved analytically. The known kinetic BGK–equation (Bhatnagar, Gross, Krook) thus was used.

In work [5] zero of the dispersion function corresponding to BGK–equation have been investigated. The range of values of frequencies of oscillations of the plane is found out, in which to within one percent exact expression of zero of dispersion function can replace it by asymptotic approximation taken by means of expansion in asymptotic series of dispersion function in neighbourhood of infinitely remote point.

Necessity of this replacement of exact expression of zero of dispersion function by it asymptotic representation speaks that fact, that calculation of macrocharacteristics of this problem demands variety calculation of values of composite functions in zero of dispersion function.

Let’s underline, that this problem draws to itself wide attention of many authors (see, for example, [6]–[11]). It has been solved by the numerical and (or) by approximate methods. In works [7, 8] Stokes’ second problem has been successfully applied in nanotechnology.

In work [12] Stokes’ second problem has been solved analytically with help of ellipsoidal statistical equation.
Methods of calculation of zero of dispersion functions for the transport
equations of neutrons have been put in pawn in work [13]. Then these
methods were applied and developed for various problems in works [14–
[20].

In the present work in explicit form eigen solutions of ellipsoidal
statistical equation are presented. These solutions correspond to discrete
spectrum. For this purpose in the explicit form zero of the dispersion
equation are found. For finding of zero it is used factorization of dispersion
function. For this proved boundary value Riemann problem from the
theory of functions of complex variable. Coefficient of Riemann problem
is the relation of boundary values of dispersion function from above and
from below on the real semiaxis.

At small values of frequency of oscillations of a plane limiting rarefied
gas the simple asymptotic formula for calculation of zero of dispersion
function is found. Graphic research of modules of zero of dispersion
function is carried out, and also the real and imaginary parts calculated
by exact and asymptotic formulas. The function of errors representing
a relative deviation of the module of asymptotic representations of zero
from the module of its exact representation is entered.

The interval of values of frequency of oscillations of a plane, in which
the value of function of errors does not exceed one percent is found out.

In item 2 of the present work properties of dispersion function are
studied in complex plane. In item 3 exact formulas for calculation of zero
of dispersion function are deduced. Properties of these zero as functions
of dimensionless frequency of oscillations of the plane limiting rarefied
gas also are investigated.

In work [12] Stokes’ second problem has been analytically solved with
using ellipsoidal statistical equation, which after linearization and of
some simplifications it is reduced to the equation

\[
\mu \frac{\partial h}{\partial x_1} + z_0 h(x_1, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2)(1 - a\mu\mu')h(x_1, \mu')d\mu', \quad (1.1)
\]
where

$$z_0 = 1 - i\Omega,$$

$x_1$ is the dimensionless coordinate, $x_1 = x/l$, $l$ is the mean free path of gaseous molecules, $\Omega = \omega \tau = \frac{\omega}{\nu}$, $\tau = 1/\nu$, $\nu$ is the frequency of collision of gaseous molecules, $\omega$ is the oscillation frequency of plates, limiting half-space filling of rarefied gas, $a$ is the number parameter of problem, $0 \leq a \leq 1$.

## 1 Statement problem

Let rarefied one-atomic gas fills half-space $x > 0$ over plane solid surface, laying in the plane $x = 0$. Surface $(y, z)$ makes harmonical oscillations lengthwise an axes $y$ under the law $u_s(t) = u_0 e^{-i\omega t}$.

We will be linearize the distribution function of gaseous molecules believing

$$f(x, t, v) = f_0(v)(1 + \varphi(x, t, v)).$$

Here $f_0(v) = n(\beta/\pi)^{3/2} \exp(-\beta v^2)$ is the absolute Maxwellian, $\beta = m/(2kT)$, $k$ is the Boltzmann constant, $T$ is the temperature of gas, $n$ is the concentration (number density) of gas, $m$ is the mass of molecule of gas.

Let further $\nu = 1/\tau$ is the collision frequency of gaseous molecules, $\tau$ is the time between two consecutive collisions of molecules, $u_y(x, t)$ is the mass velocity of gas, $\sigma_{xy}(x, t)$ is the component of viscous stress tensor,

$$u_y(x, t) = \frac{1}{n} \int v_y f(x, t, v) d^3v,$$

$$\sigma_{xy}(x, t) = m \int v_x v_y f(x, t, v) d^3v.$$

Concentration of gas and its temperature are considered as constants in linearization statement of problem.

We introduce dimensionless velocities and parameters: dimensionless velocity of molecules $C = \sqrt{\beta} v$ ($\beta = m/(2kT)$), dimensionless velocity
of gas $U_y(x, t) = \sqrt{\beta} u_y(x, t)$, dimensionless time $t_1 = \nu t$ and dimensionless velocity of surface $U_s(t) = U_0 e^{-i\omega t}$, dimensionless component of viscous stress tensor

$$P_{xy}(x, t) = \frac{\beta}{p} \sigma_{xy}(x, t),$$

where $U_0 = \sqrt{\beta} u_0$ is the dimensionless amplitude of oscillation velocity of half-space border. Then the linearization ellipsoidal statistical kinetic equation (see, for example, [21]) (short: the ES–equation) can be written down in the form

$$\frac{\partial \varphi}{\partial t_1} + C_x \frac{\partial \varphi}{\partial x_1} + \varphi(x_1, t_1, C) = 2C_y U_y(x_1, t_1) - 2a C_x C_y P_{xy}(x_1, t_1). \quad (1.1)$$

Here $a$ is the parametre of equation, and at $a = 1$ Prandtl number is true ($\text{Pr} = 2/3$). Let’s notice, that for dimensionless time $U_s(t_1) = U_0 e^{-i\Omega t_1}$.

Let’s underline, that the problem about gas fluctuations resolves in to linearization statement. Linearization of problems it is spent on dimensionless mass speed $U_y(x_1, t_1)$ provided that $|U_y(x, t_1)| \ll 1$. This inequality is equivalent to the inequality

$$|u_y(x_1, t_1)| \ll v_T,$$

where $v_T = 1/\sqrt{\beta}$ is the heat velocity of molecules, having an order of velocity of a sound.

These quantities of dimensionless mass velocity and components of the viscous stress tensor through function $\varphi$ are expressed as follows

$$U_y(x_1, t_1) = \frac{1}{\pi^{3/2}} \int \exp(-C^2) C_y \varphi(x_1, t_1, C) d^3 C, \quad (1.2)$$

and

$$P_{xy}(x_1, t_1) = \frac{1}{\pi^{3/2}} \int \exp(-C^2) C_x C_y \varphi(x_1, t_1, C) d^3 C. \quad (1.3)$$

Considering, that plate oscillations are considered along an axis $y$, we will search function $\varphi$ in the form

$$\varphi(x_1, t_1, C) = C_y e^{-i\Omega t_1} \tilde{h}(x_1, C_x). \quad (1.4)$$
By means of (1.4) it is received the following boundary problem

$$
\mu \frac{\partial h}{\partial x_1} + z_0 h(x_1, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2)(1 - a\mu'h)(x_1, \mu')d\mu',
$$

(1.5)

$$
h(0, \mu) = 2U_0, \quad \mu > 0, \quad z_0 = 1 - i\Omega,
$$

(1.6)

$$
h(+\infty, \mu) = 0.
$$

(1.7)

It is easy to show, that equation parametre $a$ and Prandtl number are connected by equality

$$
Pr = \frac{2}{2 + a}, \quad \text{from which} \quad a = \frac{2(1 - Pr)}{Pr}.
$$

To the correct (true) Prandtl number $Pr = 2/3$ is answered value of parametre $a = 1$. At $a = 0$ the ellipsoidal statistical equation passes in to BGK–equation with Prandtl number $Pr = 1$, i.e. at $Pr = 1$ $a = 0$.

**2 Eigen solutions of continuous spectrum**

Separation of variables in the equation (1.5) is carried out to the following substitution

$$
h_\eta(x_1, \mu) = \exp\left(-\frac{x_1z_0}{\eta}\right)\Phi(\eta, \mu),
$$

(2.1)

where $\eta$ is the parametre of separation, or spectral parametre, general speaking, it is complex one.

Substituting (2.1) in the equation (1.5) it is received the characteristic equation

$$
z_0(\eta - \mu)\Phi(\eta, \mu) = \frac{1}{\sqrt{\pi}}\eta m_0(\eta) - \frac{1}{\sqrt{\pi}}a\mu\eta m_1(\eta),
$$

(2.2)

where

$$
n_k(\eta) = \int_{-\infty}^{\infty} \exp(-\mu'^2)\mu'^k\Phi(\eta, \mu')d\mu', \quad k = 0, 1.
$$
From the equation (2.2) we find, that it is possible to present it in the form

\[(\eta - \mu)\Phi(\eta, \mu) = \frac{1}{\sqrt{\pi z_0}} \eta n_0(\eta)(1 - b\mu \eta), \quad (2.3)\]

where

\[b = -\frac{i\Omega a}{z_0}.\]

Further we will accept the following normalizing condition

\[n_0(\eta) \equiv \int_{-\infty}^{\infty} \exp(-\mu'^2)\Phi(\eta, \mu')d\mu' \equiv z_0.\]

Then the equation (2.3) has at \(\eta, \mu \in (-\infty, +\infty)\) the following solution

\[\Phi(\eta, \mu) = \frac{1}{\sqrt{\pi}} P \eta(1 - b\mu \eta) + e^{\eta^2} \lambda(\eta)\delta(\eta - \mu), \quad (2.4)\]

where \(\delta(x)\) is the Dirac delta function, the symbol \(Px^{-1}\) means an integral principal value at integration \(x^{-1}\), \(\lambda(z)\) is the dispersion function, introducing by equality

\[\lambda(z) = 1 - i\Omega + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2}(1 - b\tau z)d\tau}{\tau - z}.\]

This function can be transformed to the form

\[\lambda(z) = -i\Omega + (1 - bz^2)\lambda_0(z),\]

where \(\lambda_0(z)\) is the known function from theory of plasma,

\[\lambda_0(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau - z}.\]

Eigen functions (2.4) are called as eigen functions of continuous spectrum for the spectral parametre \(\eta\) fills continuously all real axis.

Thus, eigen solution of the equation (1.5) look like (2.1), in which function \(\Phi(\eta, \mu)\) is defined by equality (2.4).
On the condition of our problem we search the solution which is not increasing far from the wall. In this connection the spectrum of the boundary problem we will name positive real half-axes of parameter \( \eta \).

Let's result Sokhotsky formulas from above and from below on the real axis for dispersion function

\[
\lambda^\pm(\mu) = \pm i \sqrt{\pi} \mu e^{-\mu^2} (1 - b\mu^2) - i\Omega + \frac{1 - b\mu^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2} \tau d\tau / (\tau - \mu).
\]

\textbf{Fig. 1.} Real part of boundary values of dispersion function \( \lambda^+(x) \), frequency is equal to \( \Omega = 0.637 \); curve 1 corresponds to Prandtl number \( \text{Pr} = 2/3 \); curve 2 corresponds to BGK–equation, \( \text{Pr} = 1 \).
Fig. 2. Real part of boundary values of dispersion function $\lambda^{-}(x)$, frequency is equal to $\Omega = 0.637$; curve 1 corresponds to Prandtl number $Pr = 2/3$; curve 2 corresponds to BGK–equation, $Pr = 1$.

Fig. 3. Imaginare part of boundary values of dispersion function $\lambda^{+}(x)$; curve 1 corresponds to Prandtl number $Pr = 2/3$, $\Omega = 0.1$; curve 2 corresponds to $Pr = 2/3$, $\Omega = 1$; curve 3 corresponds to BGK–equation, $Pr = 1$, $\Omega = 0.637$. 
Fig. 4. Imaginare part of boundary values of dispersion function $\lambda^-(x)$, curve 1 corresponds to Prandtl number $\text{Pr} = 2/3, \Omega = 0.1$; curve 2 corresponds to $\text{Pr} = 2/3, \Omega = 1$; curve 3 corresponds to BGK-equation, $\text{Pr} = 1, \Omega = 0.637$.

Difference of boundary values from above and from below on the real axis to dispersion function from here it is equal

$$\lambda^+(\mu) - \lambda^-(\mu) = 2\sqrt{\pi}\mu e^{-\mu^2}(1 - b\mu^2)i,$$

the half-sum of boundary values is equal

$$\frac{\lambda^+(\mu) + \lambda^-(\mu)}{2} = -i\Omega + \frac{1 - b\mu^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2}}{\tau - \mu} d\tau.$$

The singular integral in these equalities is understood in sense of the principal value.

3 Structure of discrete spectrum

Let’s show, that the discrete spectrum consisting of zero of the dispersion
equations $\lambda(z) = 0$, contains two zero $-\eta_0$ and $\eta_0$, from which are designated through $\eta_0$ that zero, at which $\text{Re } \eta_0 > 0$.

At first we will consider the case of small values $\Omega$. Let’s expand dispersion function in asymptotic series on negative degrees variable $z$ at the vicinity of infinitely remote point

$$\lambda(z) = -i\Omega + \frac{b}{2} - \frac{1}{2z^2} + \frac{3b}{4z^4} + \ldots, \quad z \to \infty. \quad (3.1)$$

From expansion (3.1) it is visible, that at small values $\Omega$ dispersion function has two complex zero differing only signs. We will replace a number (3.1) its partial sum

$$\lambda^{as}(z) = -i\Omega + \frac{b}{2} - \frac{1}{2z^2}. $$

Then from the equation $\lambda^{as}(z) = 0$ we will find asymptotic representation of zero of the dispersion equation

$$\pm \eta_0^{as}(\Omega) = \sqrt{i \frac{1 + 3i\Omega a/2z_0}{\Omega(2 + a/z_0)}} = \sqrt{i \frac{1 - i\Omega + 3i\Omega a/2}{2\Omega(1 - i\Omega + a/2)}}, \quad 0 \leq a \leq 1.$$

From here it is visible, that at $\Omega \to 0$ both zero of dispersion function have as limit one infinitely remote point $\eta_i = \infty$ of two order.

Now we investigate the case of any values $\Omega$. Further is required to us the function

$$G(\tau) = \frac{\lambda^+(\tau)}{\lambda^-(\tau)} = \frac{-i\Omega + (1 - b\tau^2)\lambda^+_0(\tau)}{-i\Omega + (1 - b\tau^2)\lambda^-_0(\tau)}. \quad (3.2)$$

Let’s separate at function $G(\tau)$ the real and imaginary parts. Let’s notice, that

$$b = b_1 + ib_2, \quad b_1 = \frac{a\Omega^2}{1 + \Omega^2}, \quad b_2 = -\frac{a\Omega}{1 + \Omega^2},$$

$$\lambda^\pm_0(\tau) = l(\tau) \pm is(\tau), \quad s(\tau) = \sqrt{\pi \tau} e^{-\tau^2},$$

$$l(\tau) = 1 - 2\tau^2 \int_0^1 e^{-\tau^2(1-x^2)} dx.$$
Now equality (3.2) can be written follows
\[
G(\tau) = \frac{-i\Omega + [(1 - b_1\tau^2) - ib_2\tau^2](l(\tau) + is(\tau))}{-i\Omega + [(1 - b_1\tau^2) - ib_2\tau^2](l(\tau) - is(\tau))},
\]
or
\[
G(\tau) = \frac{p + q - i(\Omega - p_1 + q_1)}{p - q - i(\Omega + p_1 + q_1)},
\]
where
\[
p(\tau) = (1 - b_1\tau^2)l(\tau), \quad q(\tau) = b_2\tau^2s(\tau),
\]
\[
p_1(\tau) = (1 - b_1\tau^2)s(\tau), \quad q_1(\tau) = b_2\tau^2l(\tau).
\]
Now the function \( G(\tau) \) it is possible to present in the form
\[
G(\tau) = G_1(\tau) + iG_2(\tau),
\]
where
\[
G_1(\tau) = \frac{g_1(\tau)}{g_0(\tau)}, \quad G_2(\tau) = \frac{g_2(\tau)}{g_0(\tau)},
\]
\[
g_1(\tau) = p^2 - q^2 + \Omega^2 - p_1^2 + q_1^2,
\]
\[
g_2(\tau) = 2[pp_1 + q(\Omega + q_1)],
\]
\[
g_0(\tau) = (p - q)^2 + (\Omega + p_1 + q_1)^2.
\]
These functions \( g_j(\tau)(j = 0, 1, 2) \) will be necessary in the explicit form
\[
g_1(\tau) = \Omega^2 - [s^2(\tau) - l^2(\tau)][(1 - b_1\tau^2)^2 + b_2^2\tau^4],
\]
\[
g_2(\tau) = 2s(\tau)\{\Omega b_2\tau^2 + l(\tau)[(1 - b_1\tau^2)^2 + b_2^2\tau^4]\},
\]
\[
g_0(\tau) = \Omega^2 + 2\Omega[(1-b_1\tau^2)s(\tau)+b_2\tau^2l(\tau)]+[l^2(\tau)+s^2(\tau)][(1-b_1\tau^2)^2+b_2^2\tau^4].
\]
In these equalities
\[
(1 - b_1\tau^2)^2 + b_2^2\tau^4 = \frac{1 + \Omega^2(1 - a\tau^2)^2}{1 + \Omega^2}.
\]
Thus, it is definitively received
\[
g_1(\tau) = \frac{\Omega^4 - \Omega^2s_1(\tau) - s_0(\tau)}{1 + \Omega^2},
\]
where
\[
s_0(\tau) = s^2(\tau) - l^2(\tau), \quad s_1(\tau) = s_0(\tau)(1 - a\tau^2)^2 - 1,
and

\[
g_2(\tau) = \frac{2s(\tau)}{1 + \Omega^2} \left\{ -a\Omega^2\tau^2 + l(\tau)[1 + \Omega^2(1 - a\tau^2)^2] \right\},
\]

\[
g_0(\tau) = \Omega^2 + \left[l^2(\tau) + s^2(\tau)\right] \frac{1 + \Omega^2(1 - a\tau^2)^2}{1 + \Omega^2} + \\
+ 2\Omega \left[ \left(1 - \frac{a\Omega^2\tau^2}{1 + \Omega^2}\right)s(\tau) - \frac{a\Omega^2\tau^2}{1 + \Omega^2}l(\tau) \right].
\]

It is possible to show by means of principle of argument similarly that, as it is made in [2], that number of zero of dispersion function it is equal

\[
N = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\ln G(\tau) = \frac{1}{\pi i} \int_{0}^{\infty} d\ln G(\tau) =
\]

\[
= \frac{1}{\pi} \left[ \arg G(\tau) \right]_{0}^{+\infty} = \frac{1}{\pi} \arg G(+\infty) = 2\kappa(G),
\]

i.e. to the doubled index of function \( G(\tau) \).

Let’s enter the angle \( \theta(\tau) = \arg G(\tau) \), which is the principal value of argument, fixed in zero by condition \( \theta(0) = 0 \),

\[
\theta(\tau) = \arctg \frac{\Re G(\tau)}{\Im G(\tau)} = \arctg \frac{g_1(\tau)}{g_2(\tau)}.
\]  \hspace{1cm} (3.3)

From equation \( g_1(\tau) = 0 \) we find its positive root

\[
\Omega(a) = \sqrt{\frac{s_1(\tau)}{2} + \sqrt{\left(\frac{s_1(\tau)}{2}\right)^2 + s_0(\tau)}} \equiv \Omega(\tau, a).
\]

Let’s enter the allocated frequency of oscillations of the plate limiting gas

\[
\underbrace{\Omega^*(a) = \max_{0<\tau<\infty} \Omega(\tau, a)}_{(3.4)}.
\]

This frequency of oscillations we will name \textit{critical}.

Similarly [2] it is possible to show, that in the case, when frequency plate oscillations less than critical, i.e. at \( 0 \leq \omega < \Omega^*(a) \), the index of function \( G(t) \) is equal to unit. It means, that number of complex zero of dispersion function in a plane with cut along real axis, equally to two.
In the case when frequency of oscillations of the plate exceeds the critical \( (\omega > \Omega^*(a)) \) the index of function \( G(t) \) is equal to zero: \( \kappa(G) = 0 \). It means, that dispersion function has no zero in top and bottom of half-planes. In this case discrete (partial) solutions the initial equation (1.9) has no.

Thus, the discrete spectrum of the characteristic equation, consisting of zero of dispersive function, in the case \( 0 \leq \Omega < \Omega^*(a) \) there is a set from two points \( \eta_0 \) and \( -\eta_0 \). At \( \Omega > \Omega^*(a) \) the discrete spectrum is the empty set. At \( 0 \leq \Omega < \Omega^*(a) \) decreasing eigen solution of the equation (1.9) looks like \( h_{\eta_0}(x_1, \mu) = e^{-x_1z_0/\eta_0}\Phi(\eta_0, \mu) \), where

\[
\Phi(\eta_0, \mu) = \frac{1}{\sqrt{\pi}} \frac{\eta_0(1 - b\mu\eta_0)}{\eta_0 - \mu}
\]

is the eigen solution of characteristic equation.

It means, that the discrete spectrum of the considered boundary problem consists of one point \( \eta_0 \) in the case \( 0 < \Omega < \Omega^*(a) \).

At \( \Omega \to 0 \) both zero \( \pm\eta_0 \) as already it was specified above, move to one and same infinitely remote point. It means, that in the case \( \Omega = 0 \)
the discrete spectrum of this problem consists of one infinitely remote point of frequency rate two also is attached to the continuous spectrum. In this case discrete (partial) solutions exactly two:

\[ h_1(x_1, \mu) = 1, \quad h_2(x_1, \mu) = x_1 - \frac{2}{2 + a} \mu. \]

Let’s result the table of critical frequencies depending on values of Prandtl number and equation parametre \( a \) according to (3.4).

**Table** of critical frequency values .

| Prandtl number Pr | Parametre \( a \) | Critical frequency \( \Omega^* \) |
|-------------------|-------------------|-------------------------------|
| 1                 | 0                 | 0.733                         |
| 0.952             | 0.1               | 0.717                         |
| 0.909             | 0.2               | 0.717                         |
| 0.870             | 0.3               | 0.691                         |
| 0.833             | 0.4               | 0.681                         |
| 0.800             | 0.5               | 0.672                         |
| 0.769             | 0.6               | 0.662                         |
| 0.741             | 0.7               | 0.654                         |
| 0.714             | 0.8               | 0.648                         |
| 0.690             | 0.9               | 0.642                         |
| 2/3               | 1                 | 0.637                         |

4  **Factorization of dispersion function**

Here we deduce the formula representing factorization of dispersion function in the top and bottom half-planes, and also the formula for factorization boundary values of dispersion function from above and from below on the real axis is deduced. Such factorization it is given in terms of function \( X(z) \).

At the heart of the analytical solution of boundary problems the kinetic theory lays the solution of the homogeneous boundary value
Riemann problem (see [23]) with coefficient $G(\mu) = \lambda^+(\mu)/\lambda^-(\mu)$

$$\frac{X^+(\mu)}{X^-(\mu)} = G(\mu), \quad \mu > 0.$$  

Homogeneous boundary value Riemann problem is called also (see [23]) the factorization problem of coefficient $G(\mu)$.

The Riemann problem mean that relationship $\lambda^+(\mu)/\lambda^-(\mu)$ can replace by relation $X^+(\mu)/X^-(\mu)$. Here $X^\pm(\mu)$ are boundary value of function $X(z)$, analytical in complex plane $\mathbb{C}$ and having as jumps line the real positive half-axes. Dispersion function have as line of jumps all real axis.

We consider corresponding homogeneous boundary value Riemann problem

$$X^+(\mu) = G(\mu)X^-(\mu), \quad \mu > 0, \quad (4.1)$$

where coefficient of problem $G(\tau)$ is defined by equality (3.2).

The solution of Riemann problem Римана (4.1) is carried out similarly [3] and given by integral of Cauchy type

$$X(z) = \frac{1}{z^\kappa} \exp V(z), \quad (4.2)$$

where $\kappa = \kappa(G)$ is the index of coefficient of problem, entered in item 3, and $V(z)$ is understood as integral Cauchy type

$$V(z) = \frac{1}{2\pi i} \int_0^\infty \frac{\ln G(\tau) - 2\pi i\kappa}{\tau - z} d\tau. \quad (4.3)$$

Here $\ln G(\tau) = \ln |G(\tau)| + i\theta(\tau)$ is the principal branch of logarithm, fixed at zero by condition $\ln G(0) = 0$, angle $\theta(\tau) = \arg G(\tau)$ is the principal value of argument, entered by equality (3.3). The integral (4.3) is more convenient to consider in the form

$$V(z) = \frac{1}{\pi} \int_0^\infty \frac{q(\tau) - \pi\kappa}{\tau - z},$$

where

$$q(\tau) = \frac{\theta(\tau)}{2} - i \frac{1}{2} \ln |G(\tau)|,$$
or in the form

\[ V(z) = \frac{1}{\pi} \int_0^\infty \frac{\zeta(\tau)d\tau}{\tau - z}, \quad \kappa = 0, 1, \]

where

\[ \zeta(\tau) = q(\tau) - \pi \kappa. \]

Let first \( \kappa(G) = 1 \), i.e. \( \Omega \in [0, \Omega^*(a)) \). We show that for dispersion function \( \lambda(z) \) everywhere in the complex plane \( \mathbb{C} \), excepting the real axis \( \mathbb{R} \), is true the formula

\[ \lambda(z) = -\lambda(\infty)(z^2 - \eta_0^2)X(z)X(-z). \]  

Here

\[
\lambda(\infty) = -i\Omega + \frac{b}{2} = -i\Omega \left[ 1 + \frac{a}{2(1 - i\Omega)} \right] = \\
= -\frac{i\Omega}{\Pr} \frac{1 - i\Omega}{1 - i\Omega}, \quad \frac{2}{3} \leq \Pr \leq 1.
\]

From this formula follows that for its boundary values on \( \mathbb{R} \) are carried out the following relations

\[ \lambda^{\pm}(\mu) = -\lambda(\infty)(\mu^2 - \eta_0^2)X^{\pm}(\mu)X(-\mu), \quad \mu \geq 0, \]  

(4.5)

\[ \lambda^{\mp}(\mu) = -\lambda(\infty)(\mu^2 - \eta_0^2)X(\mu)X^{\mp}(-\mu), \quad \mu \leq 0. \]  

(4.6)

For the proof of the formula (4.4) we will enter auxiliary function

\[ R(z) = \frac{\lambda(z)}{-\lambda(\infty)(z^2 - \eta_0^2)X(z)X(-z)}. \]  

(4.7)

This function analytic everywhere in the complex plane, except points of cuts \( \mathbb{R}_+ \) and \( \mathbb{R}_- \). These points \( z = \pm \eta_0 \) are removable, because at these points \( \lambda(\pm \eta_0) = 0 \).

Each point of cuts \( \mathbb{R}_+ \) and \( \mathbb{R}_- \) is removable. Really, if \( \mu > 0 \), then on the basis of equality (4.4) and (4.7) it is had

\[ \frac{\lambda^+(\mu)}{\lambda(\infty)(\mu^2 - \eta_0^2)X^+(\mu)X(-\mu)} = \frac{\lambda^-(\mu)}{\lambda(\infty)(\mu^2 - \eta_0^2)X^-(\mu)X(-\mu)}, \]

where

\[ \lambda(\infty) = -i\Omega + \frac{b}{2} = -i\Omega \left[ 1 + \frac{a}{2(1 - i\Omega)} \right] = \\
= -\frac{i\Omega}{\Pr} \frac{1 - i\Omega}{1 - i\Omega}, \quad \frac{2}{3} \leq \Pr \leq 1. \]
from which $R^+(\mu) = R^-(\mu)$, $\mu > 0$. If $\mu < 0$, then on the basis of equality (4.4), in which $\mu$ is replaced on $-\mu$, we have

$$\frac{X^+(-\mu)}{X^-(\mu)} = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \quad \mu < 0.$$ 

It easy to see that $\lambda^+(\mu) = \lambda^-(\mu)$, $\lambda^-(\mu) = \lambda^+(\mu)$. Hence

$$\frac{X^+(-\mu)}{X^-(\mu)} = \frac{\lambda^-(\mu)}{\lambda^+(\mu)}, \quad \mu < 0,$$

from which

$$\frac{\lambda^+(\mu)}{\lambda(\infty)(\mu^2 - \eta_0^2)X(\mu)X^-(\mu)} = \frac{\lambda^-(\mu)}{\lambda(\infty)(\mu^2 - \eta_0^2)X(\mu)X^+(-\mu)}, \quad \mu < 0,$$

or

$$R^+(\mu) = R^-(\mu), \quad \mu < 0.$$

To prove equalities

$$R^+(\mu) = \frac{\lambda^+(\mu)}{-\lambda(\infty)(\mu^2 - \eta_0^2)X(\mu)X^-(\mu)}$$

and

$$R^-(\mu) = \frac{\lambda^-(\mu)}{-\lambda(\infty)(\mu^2 - \eta_0^2)X(\mu)X^+(-\mu)},$$

we notice that if point $z$ tends to point $\mu$ ($\mu < 0$) from upper or below half-plane, then functions $R^+(\mu)$ or $R^-(\mu)$ are calculated according to the previous equalities.

Hence, it is possible to consider this function as analytical function everywhere in $\mathbb{C}$, and in cut points, having predetermined it on the cut by continuity. It is necessary to notice, that function $R(z)$ is analytical everywhere in $\overline{\mathbb{C}}$ and $R(\infty) = 1$. Under Liouville theorem this function is identically constant: $R(z) \equiv 1$, whence the formula (4.4) is proved.

Formulas (4.5) and (4.6) obviously follow from the formula (4.4).

From the formula (4.6) we will find in the explicit form the formula for calculation of zero of dispersion function

$$\eta_0(\Omega) = \sqrt{z^2 + \frac{\lambda(z)}{\lambda(\infty)X(z)X(-z)}}.$$
In this formula as a point \( z \) it is convenient to take the point on the imaginary axis: \( z = Ni, N = 1, 2, \ldots \). Then we will receive the formula

\[
\eta_0(\Omega) = \sqrt{-N^2 + \frac{\lambda(Ni)}{\lambda(\infty)X(Ni)X(-Ni)}}.
\]

We will calculate both parts of equality (4.4) at the point \( z = i \). As a result for zero of dispersion function it is received the following formula

\[
\eta_0(\Omega) = \sqrt{-1 + \frac{\lambda(i)}{\lambda(\infty)} \exp \left[ -V(i) - V(-i) \right]} . \quad (4.8)
\]

Let’s consider the case of zero index: \( \kappa(G) = 0 \), i.e. \( \Omega \in (\Omega^*, +\infty) \). Similarly previous are proved formulas

\[
\lambda(z) = -\lambda(\infty)X(z)X(-z), \quad \text{Im} \ z \neq 0 .
\]

\[
\lambda^\pm(\mu) = -\lambda(\infty)X^\pm(\mu)X(-\mu), \quad \mu \leq 0 .
\]

\[
\lambda^\pm(\mu) = -\lambda(\infty)X(\mu)X^\mp(-\mu), \quad \mu < 0 .
\]

Research of properties of zero of dispersion function is carry out on to the formula (4.8). On fig. 5 it is spent comparison of modules of exact value of zero \( \eta_0(\Omega) \) (curve 1) and asymptotic representations zero \( \eta_0^{as} \) (curve 2).

Let’s enter function of errors, which is the function of relative deviation asymptotic representations of the module of zero from the module of its exact representation

\[
O(\Omega) = \frac{|\eta_0(\Omega)| - |\eta_0^{as}(\Omega)|}{|\eta_0(\Omega)|} \cdot 100%.
\]

On fig. 7 the behaviour of function of errors is presented as function of dimensionless frequencies of oscillations of the plane limiting rarefied gas in Stokes’ second problem. From fig. 7 it is visible, that in an interval \( 0 \leq \Omega \leq 0.2 \) quantity of function of errors not exceeds one percent. This fact allows in applied questions to use asymptotic representation of zero of dispersion function.

5. Conclusions
In the present work zero of dispersion function from the Stokes’ second problem are investigated. Stokes’ second problem is the problem about behaviour of the rarefied gas filling half-space. A plane, limiting the half-space, makes harmonious oscillations in own plane. It is used the linearization kinetic ellipsoidal statistical equation. By means of the solution of boundary value Riemann problem the formula of factorization of dispersion function is proved. By means of the factorization formula in explicit form there are zero of dispersion function and their research of dependence on quantity of dimensionless frequency of the plane limiting gas is carried out. The interval of values of frequency of oscillations of the plane $0 \leq \Omega \leq 0.2$ in which the quantity of function of errors does not exceed one percent is found out.

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Fig. 7. Relative deviations of the module of zero $|\eta_0(\Omega)|$ from $|\eta_0(\Omega)|$, $Pr = 2/3$.

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