Discrete-time Calogero–Moser system and Lagrangian 1-form structure

Sikarin Yoo-Kong, Sarah Lobb and Frank Nijhoff

School of Mathematics, Department of Applied Mathematics, University of Leeds, LS2 9JT, UK

E-mail: syookong@gmail.com, sarahlobb@gmail.com and nijhoff@maths.leeds.ac.uk

Received 4 February 2011, in final form 5 July 2011
Published 18 August 2011
Online at stacks.iop.org/JPhysA/44/365203

Abstract
We study the Lagrange formalism of the (rational) Calogero–Moser (CM) system, both in discrete time and continuous time, as a first example of a Lagrangian 1-form structure in the sense of the recent paper (Lobb and Nijhoff 2009 J. Phys. A: Math. Theor. 42 454013). The discrete-time model of the CM system was established some time ago arising as a pole reduction of a semi-discrete version of the Kadomtsev–Petviashvili (KP) equation, and was shown to lead to an exactly integrable correspondence (multivalued map). In this paper, we present the full KP solution based on the commutativity of the discrete-time flows in the two discrete KP variables. The compatibility of the corresponding Lax matrices is shown to lead directly to the relevant closure relation on the level of the Lagrangians. Performing successive continuum limits on both the level of the KP equation and the level of the CM system, we establish the proper Lagrangian 1-form structure for the continuum case of the CM model. We use the example of the three-particle case to elucidate the implementation of the novel least-action principle, which was presented in Lobb and Nijhoff (2009), for the simpler case of Lagrangian 1-forms.

PACS numbers: 45.10.Db, 45.20.Jj

1. Introduction

The Calogero–Moser (CM) model [1, 2] is an integrable one-dimensional many-particle system with long-range interactions, originally given as a continuous system with pairwise inverse square potential, but which has also been generalized to the trigonometric [3] and the elliptic case [4], cf also [6], and later to a relativistic model (the Ruijsenaars–Schneider model [7, 8]). The model has been extensively studied in both the classical and quantum cases [9] and is integrable on both levels. From a physical perspective, the emphasis is on the dynamics of real particles subject to pairwise repulsive potentials. However, in recent years the study of CM models has attained a wider mathematical significance, from the seminal paper [10] considering the symplectic geometry associated with the model to more recently its role in the representation theory of Lie and quantum algebras [11]. In this context, the investigation of
CM systems with attractive potential, allowing particles to collide, has led to interesting new perspectives from the point of view of algebraic geometry, cf e.g. [15].

An integrable discrete-time version of the (rational) CM model was presented in [17], where it was obtained from a semi-discrete Kadomtsev–Petviashvili (KP)-type equation with two discrete and one continuous independent variable. The construction, which we summarize in section 2, follows closely Krichever’s pole reduction of the continuous KP equation leading to a connection with the CM system [13]. In the discrete case, this leads to a rather complicated system of ordinary difference equations (OΔEs) which constitutes an integrable Lagrangian correspondence, i.e. a multivalued symplectic map [16]. The construction provides a Lax pair, whilst the classical $R$ matrix is essentially the same as for the continuous case (as given by [14]), from which the Liouville integrability (in the sense of [16]) follows. The structure of the Lax matrices was used to obtain the solution of the initial-value problem (IVP) in an adaptation of the standard way, cf [9, 10]. It was shown that a ‘naive’ continuum limit of the discrete equations reduces them to the equations of motion of the continuous CM system in the case of an attractive, rather than repulsive, potential. In spite of the fact that this makes the discrete-time CM model perhaps less relevant from a physical perspective, it is certainly of interest from the mathematical point of view in that it provides us with an insight into the basic structures underlying discrete integrability. Thus, in the same vein, we will use the model here to better understand the novel insights from our recent paper [18] of the Lagrangian multi-form structure underlying systems which are integrable in the sense of multidimensional consistency.

Our focus in this paper on Lagrangian structures has many motivations, most notably the possibility of quantizing the discrete-time model via a path integral formalism. Furthermore, whilst many integrable discrete systems admit a Lagrangian description, in the discrete-time case the Hamiltonian no longer provides such a natural framework as it does in the continuous-time case. Recently, two of the authors observed a new fundamental property of Lagrangians for integrable (in the sense of multidimensionally consistent) systems which reflects the multidimensional consistency of the lattice systems [18]. This is given by a closure relation which holds for Lagrangians when the system is embedded in a higher dimensional lattice (meaning that the same equations hold in a multitude of sublattices of the multidimensional lattice together with their corresponding Lagrangians) and leads to the interpretation of the Lagrangians as closed forms on the extended lattice. This observation, based on the elaboration of various explicit examples, prompted the proposal of a new variational principle for integrable systems which involves explicitly the geometries in the space of independent variables. In [18], we focused on establishing and interpreting the closure relation of Lagrangian 2-forms for the cases of integrable discrete equations with two independent variables, as well as of their continuous analogues, for lattice equations in the Adler-Bobenko-Suris (ABS) list [19]. Subsequently, the closure property was also established for Lagrangians describing multicomponent lattice systems in the so-called lattice Gel’fand–Dikii hierarchy [20] and for Lagrangian 3-forms in the case of the discrete bilinear KP equation in [21], whilst in [24] it was shown that all equations in the ABS list possessed this property (cf also [27] for a ‘universal’ Lagrange multi-form description for quadrilateral affine linear equations and their continuous counterparts).

An important case which has not yet been covered is that of Lagrangian 1-forms, which applies to the case of integrable ordinary differential and difference equations. Natural candidates for exhibiting such a structure are integrable finite-dimensional many-body systems, such as the CM systems. Thus, in this paper we establish the first example of what we call a Lagrangian 1-form structure in the context of the discrete-time rational CM system. This paper is very much an explorative study of what the features of such a structure are, so we
will not start with giving axiomatic definitions but we will use the example at hand to guide us towards an understanding of this novel concept. Nevertheless, in section 3, we present basic ideas and explain what we mean by the notion of a Lagrangian 1-form structure. For this sake, we have chosen the rational CM system as a model to illustrate the conceptual features because, both on the discrete and continuous levels, we have for this model an exact solution at our disposal which guarantees the validity of all claims. Starting with the discrete-time case, we will establish the Lagrangian structure also for the continuous case by continuum limits, rather than using the (known) Hamiltonians for the higher CM flows. This is necessary because, as it turns out, the proper Lagrangians cannot be obtained by performing Legendre transformations separately on each of the Hamiltonians. This leads, beyond the second-order flow, to Lagrangians containing rather complicated algebraic expressions of the higher time derivatives. Instead, we obtain a hierarchy of mixed polynomial Lagrangians in terms of these higher order derivatives from the discrete-time case by systematic expansions (indicating en passant that Legendre transformations should probably not be implemented order by order in the flows but rather altogether in one stroke). In all these considerations, the link with the underlying KP system is instrumental in deciding how to perform the higher order continuum limits by systematic expansions and in terms of the proper parameters.

The organization of this paper is as follows. In section 2, we review briefly the derivation from [17] of the discrete-time CM model on the basis of the pole reduction of the semi-discrete KP equation. Whereas in [17] we concentrated on the single-time flow, here we consider the full system of commuting discrete-time flows which are actually needed to construct the corresponding solution of the KP equation. Thus, our presentation here amounts to a prolongation of the solution on the full set of KP independent discrete variables. In section 3, we establish the closure relation of the discrete-time Lagrangian 1-form of the discrete-time CM model through the compatibility of the time-part Lax matrices, which are of Cauchy type. In section 4, we investigate the semi-continuum limit, or skew limit, of the semi-discrete KP equation. This limit leads to an equation which we refer to as the semi-continuous KP equation (defined in terms of two continuous and one discrete independent variable), from which we obtain by pole expansion the corresponding differential-difference system which we coin the semi-continuous CM system. In section 5, the latter, which acts as a generating system for the CM hierarchy, is a Lagrangian system in its own right obeying a closure relation of differential-difference type with the original discrete-time Lagrangian of the CM system. In section 6, the full continuum limit is performed, and we thus recover the usual fully continuous CM hierarchy in mixed form, together with continuous Lagrangians exhibiting a closed 1-form structure in section 7. We concentrate in section 8 on the three-particle case, governed by the CM flow and its first higher order counterpart, in order to implement the geometric variational principle which was formulated in [18]. Thus, we show how the principle leads in this case to the derivation of both the closure relation and the pertinent Euler–Lagrange (EL) equations. We conclude the paper with some discussion of open problems and possible extensions.

2. Pole reduction of the semi-discrete KP equation

In this section, we review the connection between the semi-discrete KP equation, following on from [12, 22], and the discrete-time CM model as in [17, 23]. This gives us an occasion to introduce appropriate notation which we will use throughout the paper. Furthermore, it helps

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1 Some results demonstrating that the considerations extend to the hyperbolic/trigonometric and elliptic case as well are given in appendix B. Admittedly, there exist other integrable systems of ODEs for which similar structures are expected to hold but their study will be a matter for future considerations.
us to identify the commuting flows needed to establish the corresponding nontrivial solution of the semi-discrete KP equation.

The semi-discrete KP equation that was used in [17] is

$$\partial_\xi (\tilde{u} - \tilde{u}) = (p - q + \tilde{u} - \tilde{u})(u + \tilde{u} - \tilde{u} - \tilde{u}),$$  

(2.1)

where $p$ and $q$ are two lattice parameters, $u$ is the classical field and $\tilde{u}$ represents the discrete shift of $u$ corresponding to a transition in the ‘time’ direction, while $\tilde{u}$ represents the discrete shift of $u$ corresponding to a translation in the ‘spatial’ direction. There are two discrete variables $n$ and $m$ to which the lattice parameters $p$, $q$ and shifts $\tilde{u}, \tilde{u}$ correspond, respectively, and one continuous variable $\xi$.

We observe that equation (2.1) is a consequence of the compatibility condition of a scalar Lax pair which reads

$$\tilde{\phi} = \phi_\xi + (p + u - \tilde{u})\phi,$$

(2.2a)

$$\hat{\phi} = \phi_\xi + (q + u - \tilde{u})\phi.$$

(2.2b)

Taking $u$ to have the form

$$u = \sum_{i=1}^{N} \frac{1}{\xi - x_i(n, m)},$$

(2.3)

we find that the corresponding solution of the linear Lax equations is

$$\phi = \left(1 - \frac{1}{k} \sum_{l=1}^{N} \frac{b_l(n, m)}{\xi - x_l(n, m)}\right) (p + k)^n (q + k)^m e^{k\xi},$$

(2.4)

where $k$ is a new spectral parameter and the $b_i$ are yet to be determined. Inserting expressions (2.3) and (2.4) for $u$ and $\phi$ into the first Lax equation (2.2a), we obtain the relations

$$(p + k)b_i = k + \left(\sum_{l=1}^{N} \frac{1}{x_i - x_l} - \sum_{l=1}^{N} \frac{1}{x_l - x_i}\right) b_i - \sum_{j=1}^{N} \frac{b_j}{x_i - x_j},$$

(2.5)

$$(p + k)\tilde{b}_i = k - \sum_{j=1}^{N} \frac{b_j}{\tilde{x}_i - x_j}, \quad i = 1, 2, ..., N.$$  

(2.6)

Introducing the vectors $b = (b_1, b_2, ..., b_N)^T, e = (1, 1, ..., 1)^T$ and the Lax matrices

$$L = \sum_{i,j=1}^{N} \frac{E_{ii}}{x_i - x_j} - \sum_{i,j=1}^{N} \frac{E_{ii} + E_{ij}}{x_i - x_j},$$

(2.7)

$$M = -\sum_{i,j=1}^{N} \frac{E_{ij}}{x_i - x_j},$$

(2.8)

where $E_{ij}$ are the generators of $GL_N$, i.e. matrices with entries $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, we can now rewrite equations (2.5) and (2.6) as

$$(p + k)b = ke + Lb,$$  

(2.9a)

$$(p + k)\tilde{b} = ke + Mb.$$  

(2.9b)
which forms an \( N \times N \) matricial Lax pair. The compatibility of equations (2.9a) and (2.9b) gives us the relation
\[
(\tilde{L}M - ML)b + k(\tilde{L} - M)e = 0
\]
and since this must hold for arbitrary value of \( k \), we deduce the discrete-time equations of motion of an \( N \)-particle system:
\[
\sum_{j=1}^{N} \left( \frac{1}{x_i - \tilde{x}_j} + \frac{1}{x_i - x_j} \right) - 2 \sum_{j=1}^{N} \frac{1}{x_i - x_j} = 0, \quad \text{where} \quad i = 1, \ldots, N, \tag{2.11}
\]
in which the under-tilde \( \tilde{x}_j \) is shorthand for a shift in the variable \( n \) over one unit in the negative direction, i.e. \( \tilde{x}_j(n, m) = x_i(n - 1, m) \). Furthermore, if (2.11) holds, both the following matrix relations are satisfied:
\[
\tilde{L}M = ML \quad \text{and} \quad (\tilde{L} - M)e = 0, \tag{2.12}
\]
and consequently the inhomogeneous Lax system (2.9) leads to an isospectral problem in terms of the Lax matrices \( L \) and \( M \). As was shown in [17], equation (2.11) is an integrable symplectic correspondence, in the sense of Liouville [16], whose exact solutions \( x_i(n, m) \), which denote the position of the particles \( x_i \) at the \( n \)th time step (with fixed \( m \)), are obtained by a linearization procedure implementing the Lax pair (for details, see appendix A). The resulting solution of the IVP, imposing the initial values \( x_i(0, m) \) and \( x_i(1, m) = \tilde{x}_i(0, m) \), can be obtained by solving the secular problem given by the characteristic equation of the matrix
\[
Y(n, m) = \Lambda_L^{-p}Y(0, m)\Lambda_L^p - n\Lambda_L^{-1}, \tag{2.13}
\]
subject to the constraint on the initial value matrix
\[
[Y(0, m), \Lambda_L] = I + \text{rank } 1,
\]
in which \( I \) denotes the \( N \times N \) unit matrix). The eigenvalues of \( Y(n, m) \) given by (2.13) are the particle positions \( x_i(n, m) \), and they are determined up to permutations of the particles, which accounts for the multivaluedness of the system of \( O(\Delta X) \). Here \( \Lambda_L \) denotes the diagonal matrix of eigenvalues of the Lax matrix \( L \) and the matrix \( Y(n, m) \) is related to the diagonal matrix \( X(n, m) = \sum_{i=1}^{N} x_i(n, m)E_{ii} \) of the particle positions by a similarity transformation. It follows from (2.12) that the matrix \( \Lambda_L \) is invariant under time shifts, i.e. \( \tilde{\Lambda}_L = \Lambda_L \).

We note that (2.13) resolves only the \( n \)-dependence of the solution of the KP equation, and to provide the full solution we need also to consider the dependence on \( m \). Obviously, the latter should come from the consideration of (2.2b), which can be treated in much the same way, and leads to the following relations:
\[
(q + k)b = ke + Kb, \tag{2.14a}
\]
\[
(q + k)\tilde{b} = ke + Nb, \tag{2.14b}
\]
where \( K \) and \( N \) take exactly the same form as \( L \) and \( M \), with just the ‘\(\tilde{\ }\)’ shift replaced by the ‘\(\sim\)’ shift. The compatibility condition leads to
\[
\tilde{K}N = NK \quad \text{and} \quad (\tilde{K} - N)e = 0, \tag{2.15}
\]
and the equations of the motion can be directly obtained from (2.11) by replacing the \(\sim\) shift by the \(\sim\) shift, 

\[
\sum_{j=1}^{N} \left( \frac{1}{x_i - \hat{x}_j} + \frac{1}{x_i - \tilde{x}_j} \right) - 2 \sum_{j=1}^{N} \frac{1}{x_i - x_j} = 0, \quad \text{where} \quad i = 1, \ldots, N, \tag{2.16}
\]

in which the under-hat \(\hat{x}\) is shorthand for a shift in the variable \(m\) over one unit in the negative direction, i.e. \(\hat{x}(n, m) = x(n, m - 1)\). The exact solution of the system of ODEs (2.16) can be obtained in the same way as (2.13), namely by considering

\[
Y(n, m) = \tilde{A}_K^m Y(n, 0) \tilde{A}_K^m - m \tilde{A}_K^{-1}, \tag{2.17}
\]

subject to the constraint on the initial value matrix

\[
[Y(n, 0), \tilde{A}_K] = I + \text{rank 1},
\]

where the matrix \(\tilde{A}_K\) is invariant under time shift, so that \(\hat{\tilde{A}}_K = \tilde{A}_K\).

The \(n\) - and \(m\) -parts of the solutions of course can only be combined if the corresponding flows are compatible. In particular, this requires that the Lax matrices \(L\) and \(K\) commute, which in turn suggests that these matrices can be simultaneously diagonalized. Furthermore, it requires the compatibility of the flows in the two lattice directions, which leads to the relations

\[
(p - q)b = (L - K)b, \tag{2.18}
\]

\[
(p - q)ke = (\tilde{M} - \tilde{N})ke + (\tilde{M} - \tilde{N})b. \tag{2.19}
\]

Equation (2.18) leads to the relation

\[
P - q = \sum_{l=1}^{N} \left( \frac{1}{x_i - \hat{x}_l} - \frac{1}{x_i - \tilde{x}_l} \right), \quad i = 1, \ldots, N, \tag{2.20}
\]

whereas (2.19) yields subsequently

\[
\sum_{i,j=1}^{N} \left[ p - q + \sum_{l} \left( \frac{1}{x_i - \hat{x}_l} - \frac{1}{x_i - \tilde{x}_l} \right) \right] \left( \frac{k}{N} \tilde{E}_{ij} - \frac{E_{ij}}{\hat{x}_i - \tilde{x}_j} b \right) = 0,
\]

which yields in addition

\[
P - q = \sum_{j=1}^{N} \left( \frac{1}{\hat{x}_i - \hat{x}_j} - \frac{1}{\tilde{x}_i - \tilde{x}_j} \right), \quad i = 1, \ldots, N. \tag{2.21}
\]

Furthermore, the relation

\[
\sum_{l=1}^{N} \left( \frac{1}{x_i - \hat{x}_l} - \frac{1}{\hat{x}_i - \hat{x}_l} \right) = \sum_{l=1}^{N} \left( \frac{1}{x_i - \tilde{x}_l} - \frac{1}{\tilde{x}_i - \tilde{x}_l} \right), \quad i, j = 1, \ldots, N, \tag{2.22}
\]

which is a consequence of (2.20) and (2.21), guarantees that the zero-curvature condition \(\tilde{M}N = \tilde{N}M\) holds, which in turn implies that \((\tilde{M} - \tilde{N})e = (p - q)e\).

Equations (2.20) and (2.21), which we will refer to as the constraint equations, guarantee that the discrete flows in the variables \(n\) and \(m\) commute, and hence that the corresponding linear equations can be simultaneously solved. In fact, the downward shift of (2.21) yields

\[
P - q = \sum_{l=1}^{N} \left( \frac{1}{x_i - \hat{x}_l} - \frac{1}{x_i - \tilde{x}_l} \right), \quad i = 1, \ldots, N, \tag{2.23}
\]
which implies
\[ \sum_{j=1}^{N} \left( \frac{1}{x_i - \tilde{x}_j} + \frac{1}{x_i - \tilde{x}_j} \right) = \sum_{j=1}^{N} \left( \frac{1}{x_i - \tilde{x}_j} + \frac{1}{x_i - \tilde{x}_j} \right), \]  
(2.24)
which is also a consequence of (2.11) and (2.16), and which expresses this compatibility with the sets of ODEs. It is, thus, the consistency of equations (2.13) and (2.17), which leads us to the complete solution of the semi-discrete KP equation. In particular, from equation (2.20), it follows that the diagonal parts of the matrices \( L \) and \( K \) differ by \((p - q)I\), and hence we can identify
\[ A_L = pI + \Lambda, \quad A_K = qI + \Lambda, \]  
(2.25)
with the common diagonal matrix \( \Lambda \). Using this identification, we can now combine the solution given by (2.13) for the CM system of ODEs in the \( n \)-direction and the solution (2.17) in the \( m \)-direction into a simultaneous solution which satisfies also the commutativity constraints (2.20) and (2.21), leading to the following statement.

**Proposition.** The eigenvalues \( x_1(n, m), \ldots, x_N(n, m) \) of the \( N \times N \) matrix
\[ Y(n, m) = (pI + \Lambda)^{-n}(qI + \Lambda)^{-m}Y(0, 0)(pI + \Lambda)^n(qI + \Lambda)^m - n(pI + \Lambda)^{-1} \]
\( - m(qI + \Lambda)^{-1} \)  
(2.26a)
in which the initial value matrix \( Y(0, 0) \) is subject to the condition
\[ [Y(0, 0), \, \Lambda] = I + \text{rank} \, 1 \]  
(2.26b)

obey both the discrete-time CM systems given by equations (2.11) and (2.16) and the systems of constraint equations given by (2.20) and (2.21).

The proof is presented in appendix A. In order to make a connection with an IVP, we mention that the initial value matrix \( Y(0, 0) \) can be obtained from the diagonal matrix of initial values \( X(0, 0) \) by a similarity transformation with a matrix \( U(0, 0) \) which is an invertible matrix diagonalizing the initial Lax matrices \( L(0, 0) \) and \( K(0, 0) \). We note that the secular problem can, hence, be reformulated as one for the following matrix:
\[ \mathcal{X}(n, m) = X(0, 0) - nL^{-1}(0, 0) - mK^{-1}(0, 0), \]  
(2.27)
and hence the solution is provided by the roots of the characteristic equation:
\[ p_{\mathcal{X}}(x) = \det(xI - \mathcal{X}(n, m)) = \prod_{i=1}^{N} (x - x_i(n, m)). \]  
(2.28)

The solution for \( x_i(n, m) \) obtained in this way contains \( 2N \) free parameters, namely the \( N \) matrix invariants \( \text{tr}(Y') = \text{tr}(\mathcal{X}') \) as well as the \( N \) components of the diagonal matrix \( \Lambda \). This corresponds to the number of initial data associated with the IVP. In fact, in order to pose the IVP for the multidimensional system of ODEs in each of the discrete-time variables, it suffices to pose initial conditions for each component \( x_i \) on two initial points in one single time direction. The system of constraints then allows one to find the initial data points in the other time directions, illustrated in figures 1(a) and (b): \( \{x_i(0, 0), x_i(1, 0)\} \) and \( \{x_i(0, 0), x_i(1, 0)\} \). Suppose we impose the initial condition \( (\mathbf{y}, x) \) as in figure 1(c). We use the constraint equation (2.23) to find the point \( \mathbf{x} \) and then we use the equations of motion (2.11) to find the point \( \mathbf{y} \). At this stage, we see that the point \( \mathbf{y} \) can be solved by using either the constraint (2.20) or the equations of motion (2.16). The consistency of these relations is a consequence of the compatibility of the Lax system, comprising the Lax equations in both discrete-time
variables, which in turn results in the equations of motion in each time variable as well as the constraints. The verification that the Lax system is compatible with the exact solution of the Proposition follows from the proof given in appendix A.

Returning now to the solution of the semi-discrete KP equation (2.1), inserting the roots obtained from (2.28) into expression (2.3), we obtain the required pole solutions. Furthermore, for the solution of the corresponding eigenfunction \( \phi(n, m) \) as given in (2.4) for the eigenvalue problem, the vector \( b(n, m) \) must be determined. This is obtained, for arbitrary value of the spectral parameter \( k \), from (2.9a), or equivalently (2.14a), and is given by

\[
\begin{align*}
\mathbf{b} = (p + k - L)^{-1} \mathbf{ke}, \quad \text{or equivalently} \quad \mathbf{b} = (q + k - K)^{-1} \mathbf{ke}.
\end{align*}
\]

Plugging all the results into (2.4), we obtain the explicit form of the eigenfunction \( \phi(n, m) \) solving equations (2.2a) and (2.2b). Furthermore, eliminating the function \( x \) from the latter, we obtain the following nonlinear differential-difference equation for \( \phi \) itself:

\[
\phi(\tilde{\phi} \tilde{\phi}_{\ell} - \tilde{\phi} \tilde{\phi}_{\ell}) = (\tilde{\phi} \phi - \phi \tilde{\phi}) (\tilde{\phi} - \tilde{\phi}) ,
\]

whose pole solutions are given by the construction outlined above.

### 3. The Lagrangian 1-form and its closure relation

In this section, in order to derive the Lagrangian 1-form structure, we first establish a direct connection between the Lagrangian of the discrete-time CM model, as was presented in [17], and the temporal part of the Lax representation. We will show that this connection, through the compatibility of the matrices \( M \) and \( N \) of the previous section, will immediately lead to a closure relation for the corresponding Lagrangians.

Let \( \mathbf{x} = (x_1, x_2, \ldots, x_N) \). Equation (2.11) can be computed from the variation of a discrete action given in [17] as

\[
S_{(n)}[\mathbf{x}(n, m)] = \sum_{n} \mathcal{L}_{(n)}(\mathbf{x}, \mathbf{\tilde{x}}) = \sum_{n} \left( - \sum_{i,j=1}^{N} \log |x_i - \tilde{x}_j| + \sum_{i,j=1}^{N} \log |x_i - x_j| \right) ,
\]

where \( \mathcal{L}_{(n)}(\mathbf{x}, \mathbf{\tilde{x}}) \) is the Lagrangian corresponding to the ‘~’ direction, and the sum over \( n \) represents the sum over all discrete-time ‘~’ iterates. Similarly, the action corresponding to the ‘~’ direction takes the form

\[
S_{(m)}[\mathbf{x}(n, m)] = \sum_{m} \mathcal{L}_{(m)}(\mathbf{x}, \mathbf{\tilde{x}}) = \sum_{m} \left( - \sum_{i,j=1}^{N} \log |x_i - \tilde{x}_j| + \sum_{i,j=1}^{N} \log |x_i - x_j| \right) ,
\]

where \( \mathcal{L}_{(m)}(\mathbf{x}, \mathbf{\tilde{x}}) \) is the Lagrangian corresponding to the ‘~’ direction, and the sum over \( m \) represents the sum over all discrete-time ‘~’ iterates.
The matrices $M$ and $N$ are Cauchy-type matrices (using the terminology of e.g. [30]), and so the exact forms of the determinant can be written as

$$\det(M) = \frac{\prod_{i<j} (x_i - x_j)(\tilde{x}_i - \tilde{x}_j)}{\prod_{i,j} (x_i - \tilde{x}_j)},$$

(3.3)

$$\det(N) = \frac{\prod_{i<j} (x_i - x_j)(\tilde{x}_i - x_j)}{\prod_{i,j} (x_i - \tilde{x}_j)},$$

(3.4)

and we also have that

$$\log |\det(M)| = \sum_{i,j=1}^{N} \log |x_i - x_j| + \log |\tilde{x}_i - \tilde{x}_j| - \sum_{i,j=1}^{N} \log |x_i - \tilde{x}_j|,$$

(3.5a)

$$\log |\det(N)| = \sum_{i,j=1}^{N} \log |x_i - x_j| + \log |\tilde{x}_i - \tilde{x}_j| - \sum_{i,j=1}^{N} \log |x_i - \tilde{x}_j|.$$

(3.5b)

Remarkably, action (3.1) can be obtained by considering the infinite chain product of the matrix $M$ in the following way:

$$S_{(n)} = \log |\det \left( \prod_{n=1}^{\infty} M(n) \right)|$$

$$= \sum_{n} \left( \sum_{i,j=1}^{N} \log |x_i - x_j| + \log |\tilde{x}_i - \tilde{x}_j| - \sum_{i,j=1}^{N} \log |x_i - \tilde{x}_j| \right)$$

$$= \sum_{n} \left( - \sum_{i,j=1}^{N} \log |x_i - \tilde{x}_j| + \sum_{i,j=1}^{N} \log |x_i - x_j| \right) = \sum_{n} \mathcal{L}_{(n)}(x, \tilde{x}).$$

(3.6)

The same procedure can be performed on the matrix $N$ in order to obtain action (3.2):

$$S_{(m)} = \log |\det \left( \prod_{m=1}^{\infty} N(m) \right)|$$

$$= \sum_{m} \left( \sum_{i,j=1}^{N} \log |x_i - x_j| + \log |\tilde{x}_i - \tilde{x}_j| - \sum_{i,j=1}^{N} \log |x_i - \tilde{x}_j| \right)$$

$$= \sum_{m} \left( - \sum_{i,j=1}^{N} \log |x_i - \tilde{x}_j| + \sum_{i,j=1}^{N} \log |x_i - x_j| \right) = \sum_{m} \mathcal{L}_{(m)}(x, \tilde{x}).$$

(3.7)

We note that the connection between the time-part Lax matrix and the Lagrangian also exists in the cases of the elliptic and trigonometric discrete-time CM systems (see appendix B).

We now consider the compatibility relation $\hat{M}N = \hat{N}M$ which we recall is satisfied if both (2.20) and (2.21) hold, and we rewrite this as

$$\log |\det(\hat{M})| + \log |\det(\hat{N})| - \log |\det(\hat{N})| - \log |\det(\hat{M})| = 0,$$

(3.8)

or

$$\mathcal{L}_{(n)}(x, \tilde{x}) - \mathcal{L}_{(n)}(x, \tilde{x}) - \mathcal{L}_{(m)}(\hat{x}, \tilde{x}) + \mathcal{L}_{(m)}(\hat{x}, \tilde{x}) = 0,$$

(3.9)
where we define \( \Xi = \sum_{i=1}^{N} x_i \) to be the centre of mass and

\[
\mathcal{L}_{(i)}(x, \vec{x}) = \log | \det(M)| + p(\Xi - \vec{x}), \tag{3.10a}
\]

\[
\mathcal{L}_{(j)}(x, \vec{x}) = \log | \det(N)| + q(\Xi - \vec{x}). \tag{3.10b}
\]

The last terms in (3.10a) and (3.10b) are the total derivative terms involving the centre of mass motion which can be separated from the relative motion. The extra equation comes directly from (3.9),

\[
\Xi + \tilde{\Xi} = \Xi + \tilde{\Xi}, \tag{3.11}
\]

which holds on the solutions. The additional terms in (3.10a) and (3.10b), containing the differences of the centre of mass \( \Xi \), are needed in order to account for the constraint equations (2.20) and (2.23) as they arrive from the EL equations on discrete curves, which is a connected collection of line segments (i.e. elementary links on the lattice) with or without end points (i.e. closed or non-closed), involving corners (vertices connecting line segments with different directions). These extra terms, however, do not affect the EL equation associated with the straight sections of the discrete curve. We will demonstrate this by means of two explicit examples of discrete curves indicated in figures 3 and 4.

The discrete Lagrangian 1-form. We will now describe what we mean by a Lagrangian 1-form structure in general terms. This is based on the interpretation of the closure relation, like equation (3.9), as the closedness of a difference 1-form, which suggests that the Lagrangians, embedded in a space of multi-time variables, could (and should) be interpreted as components of a difference (i.e. a discrete differential) form. To make this more precise, let us introduce the following notation: let \( e_i \) represent the unit vector in the lattice direction labelled by \( i \) and let any position in the lattice be identified by the vector \( n \), so that an elementary shift in the lattice can be created by the operation \( n \mapsto n + e_i \). Since the Lagrangian depends on \( x \) and its elementary shift in one discrete direction, it can be associated with an oriented vector \( e_i \) on a curve \( \Gamma_i(n) = (n, n + e_i) \), and we can treat these Lagrangians as defining a discrete 1-form \( \mathcal{L}_i(n) \),

\[
\mathcal{L}_i(n) = \mathcal{L}_i(x(n), x(n + e_i)). \tag{3.12}
\]

Proposition. As functions on the two-dimensional lattice, the discrete-time Lagrangians given in equation (3.10) satisfy the following relation:

\[
\mathcal{L}_i(x(n + e_j), x(n + e_i + e_j)) - \mathcal{L}_i(x(n), x(n + e_i)) - \mathcal{L}_j(x(n + e_i), x(n + e_j + e_i)) + \mathcal{L}_j(x(n), x(n + e_j)) = 0. \tag{3.13}
\]

Equation (3.9) represents the closure relation of the Lagrangian 1-form. The point is that the closure relation (3.9) is derived from the Lax equation (the compatibility of the matrices \( M \) and \( N \)), together with the property of determinants of Cauchy matrices. In contrast, in those cases of Lagrangian 2-forms and 3-forms in [18, 20, 21], the equation of the motion must be invoked in order to show that the closure relation holds. Also in those cases, the EL equation stemming from the variational principle was not exactly the equation of motion needed to verify the closure relation; rather, it was a discrete derivative, or a sum of copies, of the original equation of motion.
Choosing a discrete curve $\Gamma$ consisting of connected elements $\Gamma_i$, we can define an action on the curve by summing up the contributions $L_i$ from each of the oriented links $\Gamma_i$ in the curve to obtain

$$S(x(n); \Gamma) = \sum_{i \in \Gamma} L_i(x(n), x(n + e_i)). \quad (3.14)$$

The closure relation (3.9) is actually equivalent to the invariance of the action under local deformations of the curve. To see this, suppose we have an action $S$ evaluated on a curve $\Gamma$, and we deform this (keeping end points fixed) to obtain a curve $\Gamma'$ on which an action $S'$ is evaluated, such as in figure 2.

Then $S'$ is related to $S$ as follows:

$$S' = S - L_i(x(n + e_i), x(n + e_i + e_j)) + L_i(x(n), x(n + e_i)) + L_j(x(n + e_i), x(n + e_j + e_i)) - L_j(x(n), x(n + e_j)). \quad (3.15)$$

Equation (3.15) shows that the independence of the action under such a deformation is locally equivalent to the closure relation. The invariance of the action under the local deformation is a crucial aspect of the underlying variational principle.

Next, it is interesting to investigate further how to derive the discrete EL equation from the variational principle. For general discrete curves, it is cumbersome to implement the variational principle because of the notation it would require. We will, however, demonstrate how the principle works for a few simple cases: (a) the curve shown in figure 3; (b) the curve shown in figure 4.

(a) The curve shown in figure 3. We now introduce a new variable $N = n + m$ (which will play an important role in the next section) together with the change of notation

$$x(n, m) \rightarrow x(N, m), \quad \tilde{x} := x(N + 1, m) \quad \text{and} \quad \tilde{x} := x(N, m + 1),$$

and so we work with the curve given in figure 3(b). The action evaluated on this curve can be written in the form

$$S[x; \Gamma'] = \sum_{m = m_0}^{m_1 - 1} -L_{N_i}(x(N_0, m), x(N_0 - 1, m))$$

$$+ \sum_{m = m_0}^{m_1 - 1} L_{m} (x(N_0 - 1, m), x(N_0, m + 1)), \quad (3.16)$$
where

$$L_{(m)}(x, y) = \sum_{i,j=1 \atop i \neq j}^{N} \left( \frac{1}{2} \log |y_i - y_j| + \frac{1}{2} \log |x_i - x_j| \right) - \sum_{i,j=1}^{N} \log |y_i - x_j| + p \sum_{i=1}^{N} (y_i - x_i),$$  \hspace{1cm} (3.17)

$$L_{(n)}(x, y) = \sum_{i,j=1 \atop i \neq j}^{N} \left( \frac{1}{2} \log |x_i - x_j| + \frac{1}{2} \log |y_i - y_j| \right) - \sum_{i,j=1}^{N} \log |x_i - y_j| + q \sum_{i=1}^{N} (x_i - y_i).$$  \hspace{1cm} (3.18)
The minus sign in (3.16) indicates the reverse direction of the Lagrangian $\mathcal{L}_{(N)}$ along the horizontal links. Performing the variation $x \mapsto x + \delta x$, we have

$$
\delta S = 0 = \sum_{m=m_0}^{m_1-1} \left( - \frac{\partial \mathcal{L}_{(N)}(x(N_0, m), x(N_0 - 1, m))}{\partial x(N_0, m)} \delta x(N_0, m) - \frac{\partial \mathcal{L}_{(N)}(x(N_0, m), x(N_0 - 1, m))}{\partial x(N_0 - 1, m)} \delta x(N_0 - 1, m) \right) 
+ \sum_{m=m_0}^{m_1-1} \left( \frac{\partial \mathcal{L}_{(m)}(x(N_0 - 1, m), x(N_0, m + 1))}{\partial x(N_0, m + 1)} \delta x(N_0, m + 1) 
+ \frac{\partial \mathcal{L}_{(m)}(x(N_0 - 1, m), x(N_0, m + 1))}{\partial x(N_0 - 1, m)} \delta x(N_0 - 1, m) \right).
$$

(3.19)

We now obtain the EL equations

$$
- \frac{\partial \mathcal{L}_{(N)}(x(N_0, m), x(N_0 - 1, m))}{\partial x(N_0, m)} + \frac{\partial \mathcal{L}_{(m)}(x(N_0 - 1, m), x(N_0, m))}{\partial x(N_0, m)} = 0,
$$

(3.20a)

$$
- \frac{\partial \mathcal{L}_{(N)}(x(N_0, m), x(N_0 - 1, m))}{\partial x(N_0 - 1, m)} + \frac{\partial \mathcal{L}_{(m)}(x(N_0 - 1, m), x(N_0, m + 1))}{\partial x(N_0 - 1, m)} = 0,
$$

(3.20b)

which produce

$$
p - q = \sum_{j=1}^{N} \left( \frac{1}{x_i(N_0, m) - x_j(N_0 + 1, m - 1)} - \frac{1}{x_i(N_0, m) - x_j(N_0 - 1, m)} \right),
$$

(3.21a)

$$
p - q = \sum_{j=1}^{N} \left( \frac{1}{x_i(N_0, m) - x_j(N_0 + 1, m)} - \frac{1}{x_i(N_0, m) - x_j(N_0 - 1, m + 1)} \right),
$$

(3.21b)

which are equivalent to (2.20) and (2.21), respectively.

(b) The curve shown in figure 4. Introducing the variable $N' = n - m$, the corresponding curve is given in figure 4(b). The action evaluated on the curve $\Gamma'$ reads

$$
S[x; \Gamma'] = \sum_{m=m_0}^{m_1-1} \mathcal{L}_{(N)}(x(N_0, m), x(N_0 + 1, m)) 
+ \sum_{m=m_0}^{m_1-1} \mathcal{L}_{(m)}(x(N_0 + 1, m), x(N_0, m + 1)),
$$

(3.22)

where

$$
\mathcal{L}_{(N)}(x, y) = \sum_{i,j=1}^{N} \left( \frac{1}{2} \log |y_i - y_j| + \frac{1}{2} \log |x_i - x_j| \right) - \sum_{i,j=1}^{N} \log |x_i - y_j| 
+ p \sum_{i=1}^{N} (x_i - y_i),
$$

(3.23)
\[ \mathcal{L}_{(m)}(x, y) = \sum_{i,j=1}^{N} \left( \frac{1}{2} \log |x_i - x_j| + \frac{1}{2} \log |y_i - y_j| \right) \]

\[- \sum_{i,j=1}^{N} \log |x_i - x_j| + q \sum_{i=1}^{N} (x_i - y_i). \quad (3.24)\]

Performing the variation \( x \mapsto x + \delta x \), we have

\[ \delta S = 0 = \sum_{m=m_0}^{m=1} \left( \frac{\partial \mathcal{L}_{(N)}(x(N_0, m), x(N_0' + 1, m))}{\partial x(N_0, m)} \delta x(N_0, m) \right. \]

\[ + \frac{\partial \mathcal{L}_{(N)}(x(N_0, m), x(N_0' + 1, m))}{\partial x(N_0' + 1, m)} \delta x(N_0' + 1, m) \]

\[ + \sum_{m=m_0}^{m=1} \left( \frac{\partial \mathcal{L}_{(m)}(x(N_0' + 1, m), x(N_0', m + 1))}{\partial x(N_0', m + 1)} \delta x(N_0', m + 1) \right. \]

\[ + \frac{\partial \mathcal{L}_{(m)}(x(N_0' + 1, m), x(N_0', m + 1))}{\partial x(N_0', m)} \delta x(N_0', m) \right). \quad (3.25)\]

We now obtain the EL equations

\[ \frac{\partial \mathcal{L}_{(N)}(x(N_0, m), x(N_0' + 1, m))}{\partial x(N_0, m)} + \frac{\partial \mathcal{L}_{(m)}(x(N_0' + 1, m - 1), x(N_0', m))}{\partial x(N_0, m)} = 0, \quad (3.26a)\]

\[ \frac{\partial \mathcal{L}_{(N)}(x(N_0, m), x(N_0' + 1, m))}{\partial x(N_0' + 1, m)} + \frac{\partial \mathcal{L}_{(m)}(x(N_0' + 1, m), x(N_0', m + 1))}{\partial x(N_0', m + 1)} = 0, \quad (3.26b)\]

which produce

\[ q - p = \sum_{j=1}^{N} \frac{2}{x_j(N_0, m) - x_j(N_0' + 1, m)} - \sum_{j=1}^{N} \frac{1}{x_j(N_0, m) - x_j(N_0' + 1, m)} \]

\[ + \frac{1}{x_j(N_0', m) - x_j(N_0' + 1, m + 1)} \right). \quad (3.27a)\]

\[ p - q = \sum_{j=1}^{N} \frac{2}{x_j(N_0' + 1, m) - x_j(N_0 + 1, m)} - \sum_{j=1}^{N} \frac{1}{x_j(N_0' + 1, m) - x_j(N_0, m)} \]

\[ + \frac{1}{x_j(N_0 + 1, m) - x_j(N_0', m + 1)} \right). \quad (3.27b)\]

Using the equations of motion (2.11) and (2.16), in terms of new variables \((N_0', m)\), (3.27) can be rewritten in the form

\[ q - p = \sum_{j=1}^{N} \left( \frac{1}{x_j(N_0', m) - x_j(N_0' - 1, m + 1)} - \frac{1}{x_j(N_0', m) - x_j(N_0' + 1, m)} \right), \quad (3.28a)\]

\[ p - q = \sum_{j=1}^{N} \left( \frac{1}{x_j(N_0' + 1, m) - x_j(N_0' - 2, m)} - \frac{1}{x_j(N_0' + 1, m) - x_j(N_0' + 1, m + 1)} \right), \quad (3.28b)\]

which are again equivalent to (2.20) and (2.21), respectively.
The above two examples of discrete curves given in figures 3 and 4 demonstrate that the variational principle on those curves implies not only the equations of motion (2.11) and (2.16), but also the constraint relations (2.20) and (2.21). In what follows, we consider these equations holding \textit{locally} on the curves and in a way that cannot be detached from the choice of curve. Furthermore, the example of figure 3 is particularly useful, because this curve, and the corresponding change of variables from \((n, m)\) to \((N, m)\), is the one we need to perform the semi-continuum limit in the next section, which will form the starting point for the transition to the fully continuous case of the CM model. In contrast, the case of the curve in figure 4, described in terms of the new variables \((N', m)\), is not suitable for performing a continuum limit, as can be seen from the corresponding plane-wave factors of the KP solution (2.4). (In fact, in that case the change of variables does not produce forms of the plane-wave factors which are close to the identity, and hence does not lead to a natural limit.)

It is not \textit{a priori} clear whether the variational procedures, namely the one of varying the curve of the independent variables (from which we obtain the closure relation) and the one of varying the dependent variables (from which we obtain EL equations), commute, taking into account that the closure relation only holds on solutions of the equations of motion. This is an important question regarding the general consistency of the variational scheme, which as a mathematical question we defer to a future study. However, what we will show on the basis of the example of the CM system is that all the resulting relations (closure, EL equations and additional constraints) hold true for this particular case, and can be verified by explicit computations.

4. The semi-continuum limit: skew limit

In this section, we study a continuum analogue of a previous construction in section 2 by considering a particular semi-continuous limit. Since the semi-discrete KP equation (2.1) contains two discrete variables \(n\) and \(m\), we could perform a continuum limit on one of these variables separately while leaving the other discrete variable intact, and thus obtain a semi-continuous equation with one remaining discrete and two continuous independent variables. Alternatively, we can first perform a change of independent variables on the lattice as we have done in the previous section in connection with the curve of figure 4, and subsequently perform a limit on one of the new variables. The advantage of the latter approach over the former is that it often leads in a more direct way to a hierarchy of higher order flows. Adopting the latter approach in this section, we use a new discrete variable \(N := n + m\) and perform the transformation on the dependent variables by setting \(u(n, m) \mapsto u(N, m) =: \tilde{u}\), which leads to the following expressions for the shifted variables:

\[
\begin{align*}
    u &= u(n + 1, m) \quad \mapsto u(N + 1, m) =: \tilde{u}, \\
    \tilde{u} &= u(n, m + 1) \quad \mapsto u(N + 1, m + 1) =: \tilde{\tilde{u}}, \\
    \tilde{\tilde{u}} &= u(n + 1, m + 1) \quad \mapsto u(N + 2, m + 1) =: \tilde{\tilde{\tilde{u}}}. 
\end{align*}
\]

Rearranging the discrete exponential factor in (2.4), we have

\[
\phi(n, m) = \left(1 - \frac{1}{k} \sum_{i=1}^{N} \frac{b_i(n, m)}{x_i(n, m)}\right) (p + k)^m \left(\frac{q + k}{p + k}\right)^m e^{\xi},
\]

\[
= \left(1 - \frac{1}{k} \sum_{i=1}^{N} \frac{b_i(n, m)}{x_i(n, m)}\right) (p + k)^m \left(1 - \frac{p - q}{p + k}\right)^m e^{\xi}. \quad (4.1)
\]
We perform the limit $n \to -\infty, m \to \infty, \epsilon \to 0$ while keeping $N$ fixed and setting $\epsilon \equiv p - q$, such that $\epsilon m = \tau$ remains finite. Focusing on the penultimate factor in (4.1), we have that

$$\lim_{m \to \infty} \left(1 - \frac{\epsilon}{p + k}\right)^m = \lim_{m \to \infty} \left(1 - \frac{\tau}{m(p + k)}\right)^m = e^{-\frac{\tau}{2p}},$$

so that the solution to the Lax equations (2.2a) and (2.2b) takes the form

$$\phi(N, \tau) = \left(1 - \frac{1}{k} \sum_{i=1}^{N} \frac{b_i(N, \tau)}{\tau - x_i(N, \tau)}\right) (p + k)^N e^{k\tau} e^{-\frac{\tau}{2p}}.$$

We would now like to see the effect of this limit on the semi-discrete KP equation. We rewrite equation (2.1) as

$$\partial_t \ln(p - q + \hat{u} - \hat{u}) = \hat{u} - \hat{u} + \hat{u}.$$ (4.4)

Taking the limit, we have

$$\partial_t \ln(\epsilon + \hat{u} - \hat{u}) = u - \hat{u} - \hat{u} + \hat{u}.$$ (4.5)

Setting $\tau = t_0 + \epsilon$, where $\epsilon_0$ is a background constant of the continuous variable, and applying the Taylor expansion, we obtain

$$u(\tau + \epsilon) = u(\tau) + \epsilon \frac{\partial}{\partial \tau} u(\tau) + \frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial \tau^2} u(\tau) + \cdots$$

where $\hat{u} = \frac{\partial}{\partial \tau} u$. The leading order $O(\delta^0)$ gives

$$\partial_t \ln(1 + \hat{u}) = y + \hat{u} - 2u.$$ (4.7)

We call equation (4.7), which contains two continuous and one discrete variable, the semi-continuous KP equation to distinguish it from the semi-discrete KP equation which contains one continuous and two discrete variables).

**Remark.** The connection to the Toda system.

Introducing the variable $1 + \hat{u} =: \exp(v)$, we can write equation (4.7) in the form

$$\nu_{\tau \epsilon} = 2 \exp(v) - \exp(y) - \exp(\hat{y}).$$ (4.8)

Introducing next another variable $y$ through the relation $v = \tilde{y} - y$, we obtain

$$\tilde{y}_{\tau \epsilon} - y_{\tau \epsilon} = (\exp(\tilde{y} - y) - \exp(\tilde{y} - \hat{y})) + (\exp(y - \hat{y}) - \exp(\tilde{y} - y)),$$ (4.9)

from which we can identify

$$y_{\tau \epsilon} = \exp(y - \hat{y}) - \exp(\tilde{y} - y).$$ (4.10)

This is the well-known two-dimensional Toda lattice, cf [26]. A Lagrangian exists for this system and is given by

$$L_{2D-Toda} = \frac{1}{2} y_{\tau \epsilon} y_{\tau \epsilon} + \tilde{y}^2 y.$$ (4.11)

To follow the reductions of the Lagrangian structure for the CM system, it would be desirable to have a Lagrangian for the semi-discrete KP equation (2.1) and for the semi-continuous KP (4.7) equation, but these are strangely elusive. Obviously, the fully continuous KP equation does possess a Lagrangian structure which is easy to establish by inspection, and furthermore the 2D Toda lattice (4.10) reduces to the KP equation through some specific continuum limits, see e.g. [22]. Thus, it is conceivable that CM Lagrangians could be established through the ‘Toda route’, but we will not pursue this in this paper. We would also like to mention Krichever’s
elliptic analogue of the one-dimensional Toda lattice [28], which remarkably contains a term corresponding to the elliptic discrete-time CM given in [23]. We will give some results on the Lagrangian 1-form structure of the discrete-time elliptic CM model in appendix B, but our main concern in this paper is to study the Lagrangian 1-form structure in the simplest possible case, for the sake of transparency.

The skew limit on the Lax equations. To obtain the Lax representation for the semi-continuous KP equation (4.7), we perform a similar limit on the Lax equations (2.2a), (2.2b) by making the transformation \( \phi(n, m, \xi) \mapsto \phi(N, m, \xi) \) with similar replacements as above for its lattice shifts, i.e.

\[
\phi(n + 1, m, \xi) \mapsto \phi(N + 1, m, \xi) =: \tilde{\phi}, \quad \phi(n, m + 1, \xi) \mapsto \phi(N + 1, m + 1, \xi) =: \tilde{\phi}.
\]

Thus, we obtain

\[
\dot{\phi} = \phi_t + (p + u - \tilde{u})\phi, \quad \tilde{\phi} = \phi_t + (q + u - \tilde{u})\phi.
\]

Using a Taylor expansion as in the previous case, we can write (4.13) as

\[
\dot{\phi} + \varepsilon \dot{\phi} + \cdots = \phi_t + (p + \varepsilon + u - \tilde{u} - \varepsilon \tilde{u} + \cdots)\phi,
\]

where \( \dot{\phi} = \frac{\partial \phi}{\partial t} \). Then to leading order \( O(\varepsilon) \), we have

\[
\dot{\phi} = -(1 + \tilde{u})\phi.
\]

Equation (4.15) is a mixed equation, with one discrete and one continuous variable. It is easy to check that equation (4.7) arises as the compatibility condition of the Lax pair consisting of (4.15) and (4.12).

Inserting the special form of the solution, equation (4.3), into equation (4.12), we recover the set of equations (2.5) and (2.6), obviously with the replacements of \( x_i(n, m) \) by \( x_i(N, m) \) and with \( \tilde{x}_i = x_i(N + 1, m) \). For the sake of self-containedness, we write the Lax corresponding matrices (which are essentially the same as before) in the new notation as follows:

\[
L = \sum_{i,j=1}^{N} \frac{E_{ij}}{x_i - \tilde{x}_j} = \sum_{i,j=1}^{N} \frac{E_{ij} + E_{ji}}{x_i - x_j},
\]

\[
M = -\sum_{i,j=1}^{N} \frac{E_{ij}}{\tilde{x}_i - x_j}.
\]

The compatibility between the matrices \( L \) and \( M \) produces the equations of motion

\[
\sum_{j=1}^{N} \left(\frac{1}{x_i - \tilde{x}_j} + \frac{1}{x_i - \tilde{x}_j}\right) - 2 \sum_{j=1}^{N} \frac{1}{x_i - x_j} = 0, \quad \text{where} \quad i = 1, \ldots, N,
\]

implying as before the system of equations (2.16), but now viewed as part of a semi-continuous CM system. This set of equations will be complemented by equations involving derivatives with respect to the variable \( \tau \), which are obtained by inserting solution (4.3) into equation (4.15) and obtaining the complementary equations for \( b_i \). Equating to zero the resulting coefficients of \( (\xi - x_i)^{-1}, (\xi - x_i)^{-2} \) and \( (\xi - \tilde{x}_i)^{-1} \), we obtain

\[
(p + k)b_i = b_i + \sum_{j=1}^{N} \frac{\dot{x}_i \tilde{b}_j}{(x_i - \tilde{x}_j)^2},
\]

(4.19)
\[ (p + k) b_i = k - \sum_{j=1}^{N} \frac{b_j}{x_i - \bar{x}_j}, \quad (4.20) \]

\[ -1 = \sum_{j=1}^{N} \frac{\dot{x}_j}{(x_j - \bar{x}_j)^2}. \quad (4.21) \]

Introducing the matrix
\[ A = \sum_{i,j=1}^{N} \frac{\dot{x}_i}{(x_i - \bar{x}_j)^2} E_{ij}, \quad (4.22) \]

and the matrix as given above, we can rewrite equations (4.19) and (4.20) as
\[ (p + k) \dot{b} = b + A \dot{b}, \quad (4.23) \]
\[ (p + k) \ddot{b} = k \dot{e} + Mb. \quad (4.24) \]

Note that (4.23) and (4.24) can be directly obtained by taking the skew limit on (2.14b) in the orders $O(\delta^0)$ and $O(\delta)$, respectively. The compatibility condition between (4.23) and (4.27a) leads to
\[ (p + k)(\ddot{A} - \dot{M} - M A M^{-1}) b + k(1 + M A M^{-1}) \dot{e} = 0, \]

which, using (4.22) and (4.17), leads to
\[ \sum_{i,j=1}^{N} \left[ -\frac{E_{ij}}{\dot{x}_i - \bar{x}_j} \left( \sum_{l=1}^{N} \frac{\dot{x}_l}{(\bar{x}_l - x_l)^2} - \sum_{l=1}^{N} \frac{\dot{x}_l}{(x_l - \bar{x}_l)^2} \right) M^{-1} ((p + k) b - k \dot{e}) \right] \frac{k}{N} \left( 1 + \sum_{l=1}^{N} \frac{\dot{x}_l}{(x_l - \bar{x}_l)^2} \right) E_{ij} \dot{e} = 0. \]

This, using (4.21), implies that we also have
\[ 1 + \sum_{j=1}^{N} \frac{\dot{x}_j}{(x_j - \bar{x}_j)^2} = 0, \quad (4.25) \]

and thus, combining these two relations, we obtain
\[ 0 = \sum_{j=1}^{N} \left( \frac{\dot{x}_j}{(x_j - \bar{x}_j)^2} - \frac{\ddot{x}_j}{(x_j - \bar{x}_j)^2} \right). \quad (4.26) \]

We observe that if we use (4.25), we recover (4.21) from (4.26). Furthermore, we can show that equation (4.26) is a consequence of the skew limit of equation (2.22) of order $O(\delta)$. On the level of the matrix relations, this implies
\[ (1 + \ddot{A} - M) \dot{e} = 0, \quad (4.27a) \]
\[ M A M^{-1} - \ddot{A} + \dot{M} = 0. \quad (4.27b) \]

**Remark.** We observe that (4.21) and (4.25) can be obtained by taking the skew limit on (2.20) and (2.21) in the order $O(\delta)$, respectively.
The skew limit on the solution. We conclude this section by performing the skew limit on the full solution (2.17). Let us observe that due to the fact that the matrices $L$ and $K$ are related linearly through constraint (2.20) (this relation amounts to a shift over $(p - q)$ times the unit matrix), we can identify the diagonal matrices of eigenvalues as follows:

$$A_L = p + \Lambda \quad \text{and} \quad A_K = q + \Lambda,$$

where $\Lambda$ is the matrix which is independent of both $p$ and $q$ (i.e. independent of the direction of the lattice). Then the full solution (2.17) can be expressed in the form

$$Y(n, m) = (p + \Lambda)^{-n}(q + \Lambda)^{-m}Y(0, 0)(q + \Lambda)^m(p + \Lambda)^n - n(p + \Lambda)^{-1} - m(q + \Lambda)^{-1}.$$  \hspace{1cm} (4.29)

Using the definitions of the variables $N = n + m$ and $p - q = \epsilon$, we obtain

$$Y(N, m) = (p + \Lambda)^{-N}(1 - \epsilon(p + \Lambda)^{-1})^{-m}Y(0, 0)(1 - \epsilon(p + \Lambda)^{-1})^{-m}(p + \Lambda)^N - N(p + \Lambda)^{-1} - m((p + \Lambda)^{-1} - (q + \Lambda)^{-1}).$$ \hspace{1cm} (4.30)

Taking the limit, we have

$$\lim_{\substack{N \to \infty \\ \epsilon \to 0}} Y(N, m) = Y(N, \tau) = (p + \Lambda)^{-N}e^{\tau(p + \Lambda)^{-1}}Y(0, 0)e^{-\tau(p + \Lambda)^{-1}}(p + \Lambda)^N - N(p + \Lambda)^{-1} - \tau(p + \Lambda)^{-2}.$$ \hspace{1cm} (4.31)

This equation represents the full solution after taking the skew limit. The positions of the particles $x_i(N, m)$ can be determined by computing the eigenvalues of (4.31).

5. Semi-continuous Lagrangian and closure relation

Having obtained, in the previous section, the skew limit of the Lax equation and the corresponding differential-difference system comprising equations (4.21) and (4.25), together with equation (4.18), we now proceed to present the corresponding Lagrange form. First, we observe that equation (4.18) can once again be obtained by implementing the usual variational principle on the action $S_{(N)}$ given by

$$S_{(N)}[\mathbf{x}(N, \tau)] = \sum_{N} \mathcal{L}_{N} = \sum_{N} \left( -\sum_{i,j=1}^{N} \log |x_i - \bar{x}_j| + \sum_{i \neq j}^{N} \left( \frac{1}{2} \log |x_i - x_j| + \frac{1}{2} \log |\bar{x}_i - \bar{x}_j| \right) \right) + p \sum_{i=1}^{N} (x_i - \bar{x}_i),$$ \hspace{1cm} (5.1)

where now the Lagrangian $\mathcal{L}_{(N)}$ involves the variables $\bar{x}_i$ shifted in the discrete variable $N$ instead of the original variable $n$, and the corresponding discrete EL equation reads

$$\frac{\partial \mathcal{L}_{(N)}}{\partial x_i} + \frac{\partial \mathcal{L}_{(N)}}{\partial \bar{x}_i} = 0,$$ \hspace{1cm} (5.2)

yielding (4.18).

Next we observe that the combined differential-difference equation (4.26) can be obtained from the two-dimensional action

$$S_{(\tau)}[\mathbf{x}(N, \tau)] = \sum_{N} \int d\tau \left( \sum_{i,j=1}^{N} \frac{\dot{x}_i - \dot{\bar{x}}_j}{x_i - \bar{x}_j} \right),$$ \hspace{1cm} (5.3)
yielding (4.26) from the two-dimensional EL equations

\[
\frac{\partial \mathcal{L}}{\partial x_i} + \frac{\partial \tilde{\mathcal{L}}}{\partial \xi_j} - \frac{\partial}{\partial \tau} \left( \frac{\partial \mathcal{L}}{\partial x_i} \right) = 0. \tag{5.4}
\]

However, we will argue that this is not the correct point of view to take in the context of the CM systems, because they consist after all of ODEs rather than PDEs or differential-difference equations. In fact, this action, containing a summation over the variable \( n \) and an integration over the variable \( \tau \) can never be obtained as a continuum limit of the one-dimensional discrete actions presented in section 3, which contain one single summation.

The skew limit on the action. We note that the actions in (5.1) and (5.3) are obtained directly from the corresponding equations of motion (4.18), (4.21) and (4.25), respectively. An interesting question is whether we can obtain the mixed discrete and continuous 1-form structure directly by taking the skew limit on the fully discrete 1-form action (3.14). This will be rather tricky because it will sensitively depend on the parametrization of the curve. Furthermore, in order to perform a systematic expansion on the action functional, one would need to employ not only a Taylor expansion on the relevant lattice shifts, but at the same time also the opposite of the Taylor expansion on the sums. For the latter, we can employ the following ‘anti-Taylor’ expansion of the sum of a discrete function \( f(n) = F(\tau) \), where \( \tau = \tau_0 + n \varepsilon \) with \( \tau_0 \) fixed: the formula

\[
\sum_{n=n_0}^{n_2-1} f(n) = \frac{1}{\varepsilon} \int_{\tau_1}^{\tau_2} F(\tau) - \frac{1}{2} (F(\tau_2) - F(\tau_1)) + \frac{\varepsilon}{12} \left( (F'(\tau_2) - F'(\tau_1)) \right) + \cdots,
\]

which guarantees that the analogue of the fundamental theorem of integration holds, namely

\[
\sum_{n=n_0}^{n_1-1} (f(n + 1) - f(n)) = f(n_2) - f(n_1) + F(\tau_2) - F(\tau_1).
\]

We now would like to pursue this task by considering a simple discrete curve \( \Gamma \) given in figure 3(a). The action corresponding to the curve has been given by (3.14). Performing the skew limit, the curve has been transformed to figure 3(b) with the new variables \((N, m)\). Using the anti-Taylor expansion, action (3.16) becomes

\[
S[\mathbf{x}(N, m); \Gamma'] = \frac{1}{\varepsilon} \int_{\tau_1}^{\tau_2} \mathcal{L}_\text{(m)}(\mathbf{x}(N_0 - 1, \tau), \mathbf{x}(N_0, \tau - \varepsilon))
\]

\[
- \frac{1}{2} \mathcal{L}_\text{(m)}(\mathbf{x}(N_0 - 1, \tau_2), \mathbf{x}(N_0, \tau_2 - \varepsilon))
\]

\[
- \mathcal{L}_\text{(m)}(\mathbf{x}(N_0 - 1, \tau_1), \mathbf{x}(N_0, \tau_1 - \varepsilon)) + \cdots
\]

\[
- \frac{1}{\varepsilon} \int_{\tau_1}^{\tau_2 - \varepsilon} d\tau \mathcal{L}_\text{(N)}(\mathbf{x}(N_0, \tau), \mathbf{x}(N_0 - 1, \tau))
\]

\[
+ \frac{1}{2} \mathcal{L}_\text{(N)}(\mathbf{x}(N_0, \tau_2 - \varepsilon), \mathbf{x}(N_0 - 1, \tau_2))
\]

\[
- \mathcal{L}_\text{(N)}(\mathbf{x}(N_0, \tau_1 - \varepsilon), \mathbf{x}(N_0 - 1, \tau_1)) + \cdots. \tag{5.5}
\]

We also use the fact that the Lagrangian is an antisymmetric function under interchange of the arguments

\[
\mathcal{L}(\mathbf{x}(N_0, \tau), \mathbf{x}(N_0 + 1, \tau)) = -\mathcal{L}(\mathbf{x}(N_0 + 1, \tau), \mathbf{x}(N_0, \tau)),
\]
together with expansion with respect to $\varepsilon$; then the action becomes

$$S[x(N, \tau); \Gamma] = \int_{\tau_1}^{\tau_2} d\tau \mathcal{L}(\tau) \left( x(N_0 - 1, \tau), \frac{\partial x(N_0, \tau)}{\partial \tau} \right), \quad (5.6)$$

where $\mathcal{L}(\tau)$ is given in (5.8).

By using a specific choice of curve given in figure 3, we can perform the skew limit on the discrete action leading to what is called the ‘semi-continuous’ action in (5.6).

**The skew limit on the discrete-time closure relation.** We would like to consider the closure relation of the semi-continuous Lagrangians. Taking, as we did in the equations of motion, the skew limit on the discrete-time closure relation (3.9), implementing a change of variables on the lattice first, we obtain

$$\tilde{\mathcal{L}}(N) - \mathcal{L}(N) = \tilde{\mathcal{L}}(m) - \mathcal{L}(m) \Rightarrow$$

$$\left( \tilde{\mathcal{L}}(N) + \varepsilon \tilde{\mathcal{L}}(N) + \cdots \right) - \mathcal{L}(N) = \left( \tilde{\mathcal{L}}(N) - \varepsilon \tilde{\mathcal{L}}(\tau) + \cdots \right) - \left( \mathcal{L}(N) - \varepsilon \mathcal{L}(\tau) \right).$$

Equating the coefficients of $\varepsilon$ gives

$$\frac{\partial \mathcal{L}(N)}{\partial \tau} = \mathcal{L}(\tau) - \tilde{\mathcal{L}}(\tau), \quad (5.7)$$

where the Lagrangian $\mathcal{L}(\tau)$ is obtained by performing the skew limit on the Lagrangian $\mathcal{L}(m)$ in equation (3.10b) to give

$$\mathcal{L}(\tau) = \text{Tr}(M^{-1}\tilde{A}) - \frac{\partial \mathcal{L}(N)}{\partial \tau} = \sum_{i \neq j} \tilde{x}_j \left( \tilde{x}_i - \tilde{x}_j \right) + \sum_{i,j=1}^{N} \tilde{x}_j - \tilde{x}_i - \tilde{\mathcal{L}} - p \tilde{\mathcal{L}}, \quad (5.8)$$

which produces the semi-continuous equations of motion. The derivation of (5.8) is given in appendix C.

Furthermore, the variational derivatives with respect to $\tilde{x}(\tau)$ and $x(\tau)$ yield

$$\frac{\partial \mathcal{L}(\tau)}{\partial \tilde{x}_i} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}(\tau)}{\partial x_i} = 0, \quad (5.9)$$

which yield the equations

$$\frac{\partial \mathcal{L}(\tau)}{\partial \tilde{x}_i} = \sum_{j=1}^{N} \frac{\tilde{x}_j}{(\tilde{x}_j - \tilde{x}_i)^2} + 1 = 0, \quad (5.10)$$

$$\frac{\partial \mathcal{L}(\tau)}{\partial x_i} = \sum_{j=1}^{N} \frac{\dot{x}_j}{(\dot{x}_j - x_i)^2} + 1 = 0, \quad (5.11)$$

which are identical to equations (4.21) and (4.25).

Relation (5.7) represents a closure relation between the semi-continuous Lagrangian $\mathcal{L}(\tau)$ and the semi-discrete Lagrangian $\mathcal{L}(N)$. This points again to a Lagrangian 1-form structure but now of a mixed, part discrete and part continuous, type where the total action is a combination of a discrete part of the action $S(N)$ and the continuous part $S(\tau)$, defined on a semi-discrete curve in the space of the discrete independent variable $N$ and the continuous variable $\tau$. This is most easily visualized in the case where we have a piecewise linear curve as in figure 5, where in the horizontal direction we have stepwise jumps over units, while vertically the curve is continuous. The closure relation guarantees, nevertheless, that any such semi-discrete curve may be ‘locally’ deformed without changing the action functional consisting of sums of the form (5.1) over the horizontal iterates and of the form (5.6) in the vertical segments.
Variational principle for a semi-continuous 1-form structure. In (5.7), we obtain the semi-continuous closure relation. It is interesting to see how the variational principle works in this case. We will conclude this section with a scheme for deriving the closure relation. Now, suppose we have a curve $\Gamma$ living on the semi-continuous $(N - \tau)$-configuration shown in figure 6. The action associated with the curve $\Gamma$ can be written in the form

$$
S(x(N, \tau); \Gamma) = \int_{\tau}^{\tau+\delta \tau} L(\tau) (x(\tau, N), x(\tau + \delta \tau, N)) d\tau \\
+ L(N)(x(\tau, N), x(\tau + \delta \tau, N + \Delta N)),
$$

(5.12)

where the first term is the Lagrangian living on the vertical (continuous) line and the second term corresponds to the Lagrangian living on the horizontal (discrete) line of the curve $\Gamma$.

We now consider the action along the dashed curve

$$
S'(x(N, \tau); \Gamma') = L(N)(x(\tau, N), x(\tau, N + \Delta N)) \\
+ \int_{\tau}^{\tau+\delta \tau} L(\tau) (x(\tau, N + \Delta N), x(\tau + \delta \tau, N + \Delta N)) d\tau,
$$

(5.13)

where the first term is the Lagrangian living on the horizontal (discrete) line and the second term corresponds to the Lagrangian living on the vertical (continuous) line of the dashed curve.
We now impose the condition that \( \delta S = S' - S = 0 \), yielding
\[
\delta S = \mathcal{L}_{(N)}(x(\tau, N), x(\tau, N + \Delta N)) - \mathcal{L}_{(N)}(x(\tau + \delta \tau, N), x(\tau + \delta \tau, N + \Delta N))
\]
\[+ \int_t^{\tau + \delta \tau} \left( \mathcal{L}_{(t)}(x(\tau, N + \Delta N), x(\tau + \delta \tau, N + \Delta N)) - \mathcal{L}_{(t)}(x(\tau, N), x(\tau + \delta \tau, N)) \right) \, dt.\]

Using the Taylor expansion with respect to the variable \( \tau \), we have
\[
\delta S = \delta \tau \left[ \mathcal{L}_{(t)}(x(\tau, N + \Delta N), x(\tau + \delta \tau, N + \Delta N)) - \mathcal{L}_{(t)}(x(\tau, N), x(\tau + \delta \tau, N)) \right]
\]
\[= \frac{\partial}{\partial \tau} \mathcal{L}_{(N)}(x(\tau, N), x(\tau, N + \Delta N)).\]  

The term inside the bracket is required to be zero according to the condition \( \delta S = 0 \), yielding the semi-continuous closure relation.

6. The full continuum limit

In the previous section, we took the continuum limit on the discrete variable \( m \), leading to a system of differential-difference equations. The full continuum limit, performed on the remaining discrete variable \( N \), will lead to a coupled system of poles in the first instance, from which a hierarchy of ODEs can be retrieved, which is the CM hierarchy. How to perform this limit is inspired by the structure of the solutions of (4.3). Performing the following computation,
\[
(p + k)^N e^{\frac{tk}{p}} = p^N \exp \left\{ k \xi + N \ln \left( 1 + \frac{k}{p} \right) \frac{\tau}{1 + \frac{k}{p}} \right\},
\]
\[
= p^N \exp \left\{ k \xi + N \left( \frac{k}{p} - \frac{k^2}{2p^2} + \frac{k^3}{3p^3} + \cdots \right) - \frac{\tau}{p} \left( 1 - \frac{k}{p} + \frac{k^2}{p^2} - \cdots \right) \right\},
\]
we can identify the full continuum variables as the coefficients of the various powers of \( k \), namely
\[
t_1 = \xi + \frac{\tau}{p^2} + \eta, \quad t_2 = -\frac{\tau}{p^2} - \frac{\eta}{2p^2}, \quad t_3 = \frac{\tau}{p^4} + \frac{\eta}{3p^4}, \ldots,
\]
where \( \eta = \frac{N}{p} \). This reduces the KP eigenfunction given in (2.4) to the following form:
\[
\phi = \left( 1 - \frac{1}{k} \sum_{i=1}^{N} \frac{b_i}{t_1 - X_i} \right) p^N e^{-\frac{tk}{p}} e^{\xi_0 + k^2 t_2 + k^3 t_3 + \cdots}, \quad \text{where} \quad \xi_0 = X_i - \frac{\tau}{p^2} - \eta.
\]

The corresponding reduction to CM removes the centre of mass, which can be identified with the \( t_1 \) flow, from the system of equations. Since \( X_i = X_i(t_1, t_2, t_3, \ldots) \) and we take the assumption that \( \frac{\partial X_i}{\partial t_1} = 0 \), we find that
\[
\dot{x}_i = \frac{\partial x_i}{\partial \tau} = -\frac{1}{p^2} - \frac{\partial X_i}{\partial t_1} \frac{\partial t_1}{\partial \tau} + \frac{\partial X_i}{\partial t_2} \frac{\partial t_2}{\partial \tau} + \frac{\partial X_i}{\partial t_3} \frac{\partial t_3}{\partial \tau} + \cdots
\]
\[= -\frac{1}{p^2} - \frac{1}{p^3} \frac{\partial X_i}{\partial t_2} + \frac{1}{p^4} \frac{\partial^2 X_i}{\partial t_3^2} + \cdots.\]  

We also find that
\[
\ddot{x}_i = x_i - \frac{1}{p} - \frac{1}{2p^2} \frac{\partial X_i}{\partial t_2} + \frac{1}{3p^3} \frac{\partial X_i}{\partial t_3} + \frac{1}{8p^4} \frac{\partial^2 X_i}{\partial t_4^2} - \frac{1}{6p^5} \frac{\partial^2 X_i}{\partial t_2^2 \partial t_3} + \frac{1}{18p^6} \frac{\partial^3 X_i}{\partial t_2^3} + \cdots.\]
The full limit on the solution. Next, we would like to perform the full limit on the exact solution. We now rewrite (4.31) in the form
\[
\mathcal{Y}(N, \tau) = \left(1 + \frac{\Lambda}{p}\right)^{-N} e^{\frac{i}{p}(1 + \frac{\tau}{2})^{-1}} \mathcal{Y}(0, 0) e^{-\frac{i}{p}(1 + \frac{\tau}{2})^{-1}} \left(1 + \frac{\Lambda}{p}\right)^{N} - \frac{N}{p} \left(1 + \frac{\Lambda}{p}\right)^{-1} - \frac{\tau}{p} \left(1 + \frac{\Lambda}{p}\right)^{-2} - \frac{N}{p} \left(1 + \frac{\Lambda}{p}\right)^{-1} - \frac{\tau}{p^2} \left(1 + \frac{\Lambda}{p}\right)^{-2}.
\]
Using the definitions of \( t_1, t_2 \) and \( t_3 \) in (6.2), we now have the solution in the form
\[
\mathcal{Y}(t_1, t_2, t_3, \ldots) = e^{-\Lambda(t_1 - \xi) + \Lambda(t_2 - \xi) + \cdots} \mathcal{Y}(0, 0, \ldots) e^{\Lambda(t_1 - \xi) + \Lambda(t_2 - \xi) + \cdots} - (t_1 - \xi) - 2\Lambda t_2 - 3\Lambda^2 t_3 + \cdots, \tag{6.8}
\]
which is a function of time variables \( t_1, t_2, t_3, \ldots \). The positions of the particles \( X_i(t_1, t_2, t_3, \ldots) \) can be computed by looking for the eigenvalues of (6.8). Since the term \( t_1 - \xi \) represents the centre of mass flow, the explicit expression of the solution for the CM can be obtained from the secular problem for the matrix
\[
X(0, 0) - 2L(0, 0)t_2 - 3L^2(0, 0)t_3, \tag{6.9}
\]
where \( X(0, 0) = U^{-1}(0, 0)Y(0, 0)U(0, 0) \) and \( L(0, 0) = U^{-1}(0, 0)A U(0, 0) \). We note that (6.9) is identical to what was obtained in [9] up to the time variable \( t_2 \) (involving a scaling of the variable). Solution (6.8) involves \( N \)-time flows for the CM system. The next solutions in the hierarchy can be generated by pushing further on with the expansion.

The full limit on the equations of motion. We now would like to see what would result from taking the limit on the equations of motion. We know that the combination of equations (4.21) and (4.25) gives us the equations of motion in the skew limit case. We look at the full limit for each of these.

Using equations (6.4)–(6.6), we find that equation (4.21) gives
\[
\mathcal{O} \left( \frac{1}{p^2} \right) : \frac{1}{4} \left( \frac{\partial X_i}{\partial t_2} \right)^2 + \frac{1}{3} \frac{\partial X_i}{\partial t_3} - \sum_{j=1 \neq i}^{N} \frac{1}{(X_i - X_j)^2} = 0, \tag{6.10a}
\]
\[
\mathcal{O} \left( \frac{1}{p^3} \right) : \frac{1}{4} \left( \frac{\partial^2 X_i}{\partial t_2^2} \right)^2 + \frac{1}{3} \frac{\partial X_i}{\partial t_2} \frac{\partial X_j}{\partial t_3} - \frac{1}{4} \left( \frac{\partial X_i}{\partial t_2} \right)^3 - \frac{1}{2} \frac{\partial X_i}{\partial t_3} - \sum_{j=1 \neq i}^{N} \left( \frac{\partial X_i}{(X_i - X_j)^2} - \frac{1}{2} (X_i - X_j)^3 \right) = 0, \tag{6.10b}
\]
\[
\mathcal{O} \left( \frac{1}{p^4} \right) : \frac{1}{3} \left( \frac{\partial X_i}{\partial t_3} \right)^3 - \frac{1}{3} \frac{\partial^2 X_i}{\partial t_2^2} \frac{\partial X_i}{\partial t_3} + \frac{1}{4} \left( \frac{\partial X_i}{\partial t_2} \right)^2 \frac{\partial X_i}{\partial t_3} - \frac{1}{8} \frac{\partial X_i}{\partial t_2} \frac{\partial^2 X_i}{\partial t_2^2} + \frac{1}{2} \left( \frac{\partial X_i}{\partial t_2} \right)^4 + \frac{3}{4} \frac{\partial X_i}{\partial t_2} \frac{\partial X_i}{\partial t_3} - \sum_{j=1 \neq i}^{N} \left( \frac{3}{(X_i - X_j)^2} - \frac{1}{4} \frac{\partial X_i}{(X_i - X_j)^2} - \frac{2}{(X_i - X_j)^3} \right) = 0, \tag{6.10c}
\]
and (4.25) yields
\[ \mathcal{O} \left( \frac{1}{p^2} \right) : \frac{1}{4} \left( \frac{\partial X_i}{\partial t_2} \right)^2 + \frac{1}{3} \frac{\partial X_i}{\partial t_3} - \sum_{j=1}^{N} \frac{1}{(X_i - X_j)^2} = 0, \]  
(6.11a)

\[ \mathcal{O} \left( \frac{1}{p^3} \right) : -\frac{1}{4} \frac{\partial^2 X_i}{\partial t_2^2} - \frac{2}{3} \frac{\partial X_i}{\partial t_2} \frac{\partial X_i}{\partial t_3} - \frac{1}{4} \left( \frac{\partial X_i}{\partial t_2} \right)^3 - \frac{1}{2} \frac{\partial X_i}{\partial t_3} - \sum_{j=1}^{N} \frac{\partial X_i}{(X_i - X_j)^2} + \frac{2}{(X_i - X_j)^3} = 0, \]  
(6.11b)

\[ \mathcal{O} \left( \frac{1}{p^4} \right) : \frac{1}{3} \left( \frac{\partial X_i}{\partial t_3} \right)^3 + \frac{3}{4} \frac{\partial X_i}{\partial t_2} \frac{\partial X_i}{\partial t_3} + \frac{3}{8} \left( \frac{\partial X_i}{\partial t_2} \right)^2 \frac{\partial X_i}{\partial t_3} + \frac{1}{8} \frac{\partial X_i}{\partial t_3} \frac{\partial^2 X_i}{\partial t_2 \partial t_3} + 2 \frac{\partial X_i}{\partial t_3} + \frac{\partial X_i}{(X_i - X_j)^2} + \frac{\partial X_i}{(X_i - X_j)^3} = 0. \]  
(6.11c)

The difference between (6.10) and (6.11) gives the following.

\[ \frac{\partial^2 X_i}{\partial t_2^2} = -\sum_{j=1}^{N} \frac{8}{(X_i - X_j)^3}, \]  
(6.12)

which is the equation of motion for the continuous CM system [17].

\[ \frac{2}{3} \frac{\partial^2 X_i}{\partial t_2 \partial t_3} + \frac{1}{4} \frac{\partial X_i}{\partial t_2} \frac{\partial^3 X_i}{\partial t_2 \partial t_3} = \sum_{j=1}^{N} \frac{2 \frac{\partial X_i}{\partial t_2} + 4 \frac{\partial X_i}{\partial t_3}}{(X_i - X_j)^3}. \]  
(6.13)

We now use equation (6.12) to simplify equation (6.13), and we obtain
\[ \frac{\partial^2 X_i}{\partial t_2 \partial t_3} = 6 \sum_{j=1}^{N} \frac{\frac{\partial X_i}{\partial t_2} + \frac{\partial X_i}{\partial t_3}}{(X_i - X_j)^3}. \]  
(6.14)

This equation represents the next equation of motion beyond the usual continuous CM in the hierarchy. We will stop at this equation, but we can actually obtain the higher terms of the equation in which the variable \( t_4 \) and higher order time flows must be taken into account.

Note finally that summing (6.10) and (6.11) gives the following equations:

\[ \mathcal{O} \left( \frac{1}{p^2} \right) : \frac{1}{4} \left( \frac{\partial X_i}{\partial t_2} \right)^2 + \frac{1}{3} \frac{\partial X_i}{\partial t_3} - \sum_{j=1}^{N} \frac{1}{(X_i - X_j)^2} = 0, \]  
(6.15a)

\[ \mathcal{O} \left( \frac{1}{p^3} \right) : -\frac{2}{3} \frac{\partial X_i}{\partial t_2} \frac{\partial^2 X_i}{\partial t_3} - \frac{1}{4} \left( \frac{\partial X_i}{\partial t_2} \right)^3 - \frac{1}{2} \frac{\partial X_i}{\partial t_3} - \sum_{j=1}^{N} \frac{\partial X_i}{(X_i - X_j)^2} = 0, \]  
(6.15b)
superintegrability of the CM system, cf [5], and their validity can be checked on solutions.

The occurrence of these constraints seems to be related to the superintegrability of the CM system, cf [5], and their validity can be checked on solutions.

7. The continuous Lagrangian 1-form and its closure relation

In the previous section, we obtained directly the equations of motion of the continuous CM from the full limit. In this section, we obtain the continuous Lagrangians from the full limit as well. We will restrict ourselves to the case of three particles involving only the time flows \( t_2 \) and \( t_3 \), to establish the simplest example of a Lagrangian 1-form structure for the CM system. Both the Lagrangians \( \mathcal{L}(\tau) \) and \( \mathcal{L}(\eta) \) produce the hierarchy of the continuous CM Lagrangians corresponding to the equations of motion in the previous section. Rather than performing the full limit on the individual Lagrangians, we prefer to take the limit on the action itself.

The full limit on the action. In section 5, we performed the skew limit on the discrete action. In the present section, we continue to perform the full continuum limit on the semi-continuous action. We now take the action to be of the form

\[
S[\mathbf{x}(N, \tau); \Gamma] = \int_{t_1}^{t_2} dt \mathcal{L}(\tau) (\mathbf{x}(N, \tau), \dot{\mathbf{x}}(N, \tau)) + \sum_{N} \mathcal{L}(\eta) (\mathbf{x}(N, \tau), \mathbf{x}(N + 1, \tau)),
\]

(7.1)

where the first term belongs to the vertical part and the second term belongs to the horizontal part of the curve given in figure 5.

Using an anti-Taylor expansion, the action now becomes

\[
S[\mathbf{x}(N, \tau); \Gamma] = \int_{t_1}^{t_2} dt \mathcal{L}(\tau) (\mathbf{x}(N, \tau), \dot{\mathbf{x}}(N, \tau)) + p \int_{t_1}^{t_2} d\eta \mathcal{L}(\eta) (\mathbf{x}(N, \tau), \mathbf{x}(N + 1, \tau)),
\]

(7.2)

where we do not need to take into account the boundary terms coming from the expansion, because they are constant at the end points and do not contribute to the variational process.

We now perform a change of variables \( (\tau, \eta) \mapsto (t_2, t_3) \) by using (6.2)

\[
d\tau = 2p^3 dt_2 + 3p^4 dt_3,
\]

(7.3a)

\[
d\eta = -6p dt_2 - 6p^2 dt_3,
\]

(7.3b)

and also expand the Lagrangians with respect to the variable \( p \). We obtain

\[
S[\mathbf{X}(t_2, t_3); \Gamma] = \int_{t_2(1)}^{t_2(2)} dt_2 \mathcal{L}(t_3) \left( \mathbf{X}(t_2, t_3), \frac{\partial \mathbf{X}(t_2, t_3)}{\partial t_2} \right)
\]

\[
+ \int_{t_1(1)}^{t_1(2)} dt_3 \mathcal{L}(t_3) \left( \mathbf{X}(t_2, t_3), -\frac{\partial \mathbf{X}(t_2, t_3)}{\partial t_2}, -\frac{\partial \mathbf{X}(t_2, t_3)}{\partial t_3} \right),
\]

(7.4)

where \( \mathcal{L}(t_2) \) and \( \mathcal{L}(t_1) \) are given by

\[
\mathcal{L}(t_2) = \mathcal{L}_{CM} = \sum_{i=1}^{N} \frac{1}{2} \left( \frac{\partial X_i}{\partial t_2} \right)^2 + \sum_{i \neq j}^{N} \frac{2}{(X_i - X_j)^2},
\]

(7.5)
of which the EL equation
\[ \frac{\partial L_{(t_2)}}{\partial x_i} - \frac{\partial}{\partial t_2} \left( \frac{\partial L_{(t_2)}}{\partial \left( \frac{\partial x_i}{\partial t_2} \right)} \right) = 0 \] (7.6)
gives exactly equation (6.12) and
\[ L_{(t_1)} = \sum_{i=1}^{N} \left( \frac{\partial x_i}{\partial t_2} \frac{\partial x_i}{\partial t_3} + \frac{1}{4} \left( \frac{\partial x_i}{\partial t_2} \right)^3 \right) - \sum_{i\neq j}^{N} \frac{3}{2} \frac{\partial x_i}{\partial t_2} (X_i - X_j)^2. \] (7.7)
We see that the Lagrangian \( L \) of which the EL equation
\[ \frac{\partial L_{(t_1)}}{\partial t_2} - \frac{\partial}{\partial t_3} \left( \frac{\partial L_{(t_1)}}{\partial \left( \frac{\partial x_i}{\partial t_2} \right)} \right) = 0. \] (7.8)
Furthermore, we find that
\[ \frac{\partial L_{(t_1)}}{\partial \left( \frac{\partial x_i}{\partial t_2} \right)} = \frac{\partial x_i}{\partial t_3} + \frac{3}{4} \left( \frac{\partial x_i}{\partial t_2} \right)^2 - \sum_{i\neq j}^{N} \frac{3}{2} (X_i - X_j)^2 = 0, \] (7.9)
which is identical to equation (6.15a).

Here we obtained the hierarchy of Lagrangians for the CM model through the full continuum limit. Obviously, higher Lagrangians in the family can be generated by pushing further on with the expansion.

The full limit on the closure relation. Using the results regarding the limits of the Lagrangians \( L_{(r)} \) and \( L_{(N)} \) in the previous section, and also the relations of equation (6.2), we can write
\[ \frac{\partial L_{(N)}}{\partial \tau} \rightarrow - \frac{1}{p^3} \frac{\partial L_{(t_2,t_3,...)}}{\partial t_2} + \frac{1}{p^3} \frac{\partial L_{(t_2,t_3,...)}}{\partial t_3}. \] (7.10)
Substituting equation (7.10) into the closure relation (5.7) together with the full limit of \( L_{(r)} - L_{(1)} \), we find that the leading term is of order \( O(1/p^3) \), and is
\[ \frac{\partial L_{(t_1)}}{\partial t_2} \rightarrow \frac{\partial L_{(t_1)}}{\partial t_3}. \] (7.11)
This equation represents the closure relation for the continuous case of the Lagrangian 1-form.

We are now taking into account not only the usual CM Lagrangian, but also a higher order Lagrangian; these have up until now not been considered. The verification of the closure relation (7.11) is presented in appendix D.

8. The variational principle for Lagrangian 1-forms

In the previous section, we obtained the Lagrangian hierarchy of the CM system along with the corresponding EL equations. In this section, we are interested in investigating the variational principle for continuous Lagrangian 1-forms in its own right. For simplicity, we limit ourselves to the first two time flows in the CM hierarchy, \( t_2 \) and \( t_3 \), corresponding to a three-particle system, in order to exhibit the simplest example. Let us consider a (piecewise) smooth curve \( \Gamma : (t_2(s), t_3(s)) \) as in figure 7(a), parametrized by a variable \( s \) taking values in some interval \([s_0, s_1]\), i.e. the time variables \( t_2 \) and \( t_3 \) are functions of this new variable \( s \), \( t_i = t_i(s) \) for \( s_0 \leq s \leq s_1 \). Thus, on the parametrized curve the Lagrangian components of the 1-form can be written as
\[ L_{(t_2)} = L_{(t_2)}(x(t_2, t_3), x(t_2, t_3), x(t_2, t_3)), \] (8.1a)
Extremizing the action, we have

\[ S = \int_0^1 (L(t_s) dt_s + L(t_0) dt_3) \]

where

\[ x_c(t_2, t_3) = \frac{\partial x}{\partial t_2} \quad \text{and} \quad x_0(t_2, t_3) = \frac{\partial x}{\partial t_3}. \]

The action evaluated on the curve \( \Gamma \) can be defined as

\[ S(x(t_2, t_3); \Gamma) = \int_0^1 (L(t_s) dt_s + L(t_0) dt_3), \]

\[ = \int_{t_0}^{t_1} \left( L(t_2(s), t_3(s)) \frac{dq_2}{ds} + L(t_0(t_2), t_3(t_3)) \frac{dq_3}{ds} \right) ds. \]  

(8.2)

We now perform variation on the curve of the time variables \( t_2 \) and \( t_3 \) in the following way. Suppose we have an action \( S \) defined on a curve \( \Gamma \), and the curve is deformed: \( t_2 \mapsto t_2 + \delta t_2 \) and \( t_3 \mapsto t_3 + \delta t_3 \) in figure 7(b), keeping the end points fixed, so we have the conditions

\[ \delta t_2(s_0) = \delta t_2(s_1) = 0, \quad \delta t_3(s_0) = \delta t_3(s_1) = 0. \]  

(8.3)

The action \( S' \) is defined as

\[ S' \equiv S(x(t_2 + \delta t_2, t_3 + \delta t_3); \Gamma'). \]  

(8.4)

Extremizing the action, we have \( \delta S = 0 \), where \( \delta S \) is defined by

\[ \delta S = S(x(t_2 + \delta t_2, t_3 + \delta t_3); \Gamma') - S(x(t_2, t_3); \Gamma). \]  

(8.5)

Using the Taylor expansion, we obtain

\[ \delta S = \int_{t_0}^{t_1} ds \left[ \frac{\partial L(t_2)}{\partial t_2} \frac{\partial S}{\partial t_2} + \frac{\partial L(t_3)}{\partial t_3} \frac{\partial S}{\partial t_3} \right] \frac{dq_2}{ds} + L(t_0) \frac{dq_3}{ds} \]

\[ + \int_{t_0}^{t_1} ds \left[ \frac{\partial L(t_2)}{\partial t_2} \frac{\partial S}{\partial t_3} \right] \frac{dq_2}{ds} + L(t_0) \frac{dq_3}{ds} \]

\[ + \int_{t_0}^{t_1} ds \left[ \frac{\partial L(t_3)}{\partial t_3} \frac{\partial S}{\partial t_3} \right] \frac{dq_2}{ds} - L(t_0) \frac{dq_3}{ds} \]

\[ + \int_{t_0}^{t_1} ds \left[ \frac{\partial L(t_2)}{\partial t_2} \frac{\partial S}{\partial t_3} \right] \frac{dq_2}{ds} - L(t_0) \frac{dq_3}{ds} \]

\[ + \frac{\partial L(t_3)}{\partial t_3} \frac{\partial S}{\partial t_3} \frac{dq_2}{ds} - L(t_0) \frac{dq_3}{ds}. \]  

(8.6)
Using the fact that the end points are fixed, the first line of equation (8.6) is zero. We can also use
\[
\frac{d\mathcal{L}_{(t_2)}}{ds} = \frac{\partial \mathcal{L}_{(t_2)}}{\partial t_2} \frac{dr_2}{ds} + \frac{\partial \mathcal{L}_{(t_2)}}{\partial t_3} \frac{dr_3}{ds},
\]
\[
\frac{d\mathcal{L}_{(t_3)}}{ds} = \frac{\partial \mathcal{L}_{(t_3)}}{\partial t_2} \frac{dr_2}{ds} + \frac{\partial \mathcal{L}_{(t_3)}}{\partial t_3} \frac{dr_3}{ds},
\]
to obtain
\[
\delta \mathcal{S} = \int_{s_0}^{s_1} ds \left[ \delta t_2 \left( \frac{\partial \mathcal{L}_{(t_2)}}{\partial t_3} - \frac{\partial \mathcal{L}_{(t_3)}}{\partial t_2} \right) \frac{dr_2}{ds} + \delta t_3 \left( \frac{\partial \mathcal{L}_{(t_2)}}{\partial t_2} - \frac{\partial \mathcal{L}_{(t_3)}}{\partial t_3} \right) \frac{dr_3}{ds} \right].
\]
Both terms in the integrand vanish if the closure relation
\[
\frac{\partial \mathcal{L}_{(t_1)}}{\partial t_2} = \frac{\partial \mathcal{L}_{(t_1)}}{\partial t_3}
\]
is satisfied, leading to \(\delta \mathcal{S} = 0\). The closure relation (8.9) guarantees that any such continuous curve may be locally deformed without changing the value of the action functional corresponding to the time variables \(t_2\) and \(t_3\).

Next, we perform a variation on the field variables \(x\), where we now fix the curve \(\Gamma\) on the \(t_2-t_3\) plane and examine the corresponding curve \(\mathcal{E}_\Gamma\) of the variables \(x\) living in the \(x-t_2-t_3\) space as in figure 8(a).

The action can be expressed in the form
\[
\mathcal{S}(x, y_2, y_3; \mathcal{E}_\Gamma) = \int_{\Gamma} \left( \mathcal{L}_{(t_1)}(x(t_2, t_3), x_1(t_2, t_3), x_0(t_2, t_3)) \, dt_1 \right.
\]
\[
+ \mathcal{L}_{(t_2)}(x(t_2, t_3), x_0(t_2, t_3), x_1(t_2, t_3)) \, dt_2)
\]
\[
= \int_{s_0}^{s_1} ds \left( \mathcal{L}_{(t_1)}(x(t_2(s), t_3(s)), x_1(t_2(s), t_3(s)), x_0(t_2(s), t_3(s))) \frac{dr_2}{ds}
\]
\[
+ \mathcal{L}_{(t_2)}(x(t_2(s), t_3(s)), x_0(t_2(s), t_3(s)), x_1(t_2(s), t_3(s)) \frac{dr_3}{ds} \right).\]
\]

We now consider the variations
\[
\frac{d}{ds}(\delta x) = \delta x_i \frac{dr_i}{ds} + \delta x_j \frac{dr_j}{ds},
\]

\(8.11a\)
\[ \frac{\delta y}{\delta x} = \frac{d r_2}{d s} - \frac{d r_3}{d s}. \]  

(8.11b)

where \( \delta y \) denotes the variation of the derivative of \( x \) in a direction perpendicular to the curve at each point. From (8.11a), we have

\[ \frac{1}{2} \left( \frac{d}{d s} (\delta x) + \delta y \right) = \delta x_2 \frac{d r_2}{d s}, \]

(8.12a)

and, thus, performing the variation on the action, we have

\[ \delta S = \int_{t_1}^{t_2} ds \left[ \delta x \frac{\partial L_{(t_2)}}{\partial x} \frac{d r_2}{d s} + \delta x_2 \frac{\partial L_{(t_2)}}{\partial x_2} \right] \frac{d r_2}{d s} \]

\[ + \left[ \delta x \frac{\partial L_{(t_2)}}{\partial x} + \delta x_2 \frac{\partial L_{(t_2)}}{\partial x_2} + \delta x_3 \frac{\partial L_{(t_2)}}{\partial x_3} \right] \frac{d r_2}{d s}. \]

(8.13)

Using (8.12), (8.13) becomes

\[ \delta S = \int_{t_1}^{t_2} ds \left[ \delta x \frac{\partial L_{(t_2)}}{\partial x} \frac{d r_2}{d s} + \delta x_2 \frac{\partial L_{(t_2)}}{\partial x_2} \right] \frac{d r_2}{d s} \]

\[ + \frac{1}{2} \frac{d}{d s} (\delta x) \left[ \frac{\partial L_{(t_2)}}{\partial x} + \frac{\partial L_{(t_2)}}{\partial x_2} + \frac{\partial L_{(t_2)}}{\partial x_3} + \frac{\partial L_{(t_2)}}{\partial x_3} \right] \frac{d r_2}{d s} \]

\[ + \frac{1}{2} \delta y \left[ \frac{\partial L_{(t_2)}}{\partial x} - \frac{\partial L_{(t_2)}}{\partial x_2} - \frac{\partial L_{(t_2)}}{\partial x_3} + \frac{\partial L_{(t_2)}}{\partial x_3} \right] \frac{d r_2}{d s}. \]

(8.14)

Integrating by parts the second line in (8.14) while fixing the variations at the end points of the curve, i.e. setting

\[ \delta x(t_2(s_0)) = \delta x(t_3(s_1)) = 0, \quad \delta x(t_2(s_1)) = \delta x(t_3(s_1)) = 0, \]

we obtain

\[ \delta S = \int_{t_1}^{t_2} ds \left[ \delta x \left[ \frac{\partial L_{(t_2)}}{\partial x} \frac{d r_2}{d s} + \frac{\partial L_{(t_2)}}{\partial x_2} \frac{d r_2}{d s} \right] \right. \]

\[ - \frac{1}{2} \frac{d}{d s} (\delta x) \left[ \frac{\partial L_{(t_2)}}{\partial x} + \frac{\partial L_{(t_2)}}{\partial x_2} + \frac{\partial L_{(t_2)}}{\partial x_3} + \frac{\partial L_{(t_2)}}{\partial x_3} \right] \frac{d r_2}{d s} \]

\[ + \frac{1}{2} \delta y \left[ \frac{\partial L_{(t_2)}}{\partial x} - \frac{\partial L_{(t_2)}}{\partial x_2} - \frac{\partial L_{(t_2)}}{\partial x_3} + \frac{\partial L_{(t_2)}}{\partial x_3} \right] \frac{d r_2}{d s}. \]

(8.15)

The condition \( \delta S = 0 \) is satisfied once

\[ \frac{\partial L_{(t_2)}}{\partial x} \frac{d r_2}{d s} + \frac{\partial L_{(t_2)}}{\partial x_2} \frac{d r_2}{d s} = 0, \]

(8.16)

with

\[ \frac{\partial L_{(t_2)}}{\partial x_2} \left( \frac{d r_2}{d s} \right)^2 + \left( \frac{\partial L_{(t_2)}}{\partial x_2} - \frac{\partial L_{(t_2)}}{\partial x_2} \right) \frac{d r_2}{d s} \frac{d r_3}{d s} - \frac{\partial L_{(t_2)}}{\partial x_3} \left( \frac{d r_2}{d s} \right)^2 = 0. \]

(8.17)

We consider equation (8.16) to be a generalized version of the EL equations on the parametrized curve, whilst (8.17) represents constraints arising from the variational principle describing the dependence of the system on directional derivatives which are not tangential to the given curve. In (8.16), it is obviously assumed that neither \( \frac{d r_2}{d s} \) nor \( \frac{d r_3}{d s} \) vanishes. If they do, then it is understood that the constraint (8.17) should be used in (8.16) to avoid zero divisors. This set
of equations, together with the closure relation (8.9), constitutes the generalized EL system of equations associated with the 1-form structure.

To summarize, performing the variation on the curve $\Gamma_1$ with respect to the time variables on the $t_2-t_3$ plane, the computation leads to the closure relation (8.9). By fixing the curve $\Gamma_1$ and varying the curve $E_{\Gamma_1}$, the computation leads to the EL equations corresponding to the Lagrangian 1-form. However, the closure relation allows the deformation of the curve $\Gamma_1$ to a simpler curve, as in figure 9. If we now are on the curve $\Gamma_2$ where the time variable $t_2$ is a constant leading to $\frac{dt_2}{ds} = 0$, we obtain

$$\frac{\partial L_{(t_3)}}{\partial x} = \frac{\partial}{\partial t_3} \left( \frac{\partial L_{(t_3)}}{\partial x_{t_3}} \right) = 0,$$

(8.18a)

$$\frac{\partial L_{(t_2)}}{\partial x_{t_2}} = 0.$$

(8.18b)

If we are on the curve $\Gamma_1$, where the time variable is a constant leading to $\frac{dt_3}{ds} = 0$, we obtain exactly the same equation (8.18) with interchange of indices $2 \leftrightarrow 3$. This is entirely consistent with the results obtained in the previous section.

9. Concluding remarks

In this paper, we present what we believe to be the treatment of the Lagrangian structure for the Calogero–Moser (CM) system which best captures the essential integrability characteristics. It is well known that the continuous CM system is Liouville integrable [2] and also superintegrable [5] and this remains the case on the discrete-time level [25, 17]. However, we focused on another aspect of the integrability, namely multidimensional consistency, meaning in this context of finite-dimensional systems the existence of commuting flows. We believe that in this paper, we have given convincing arguments for the assertion that the proper Lagrangian structure should be the one in terms of Lagrangian 1-forms. Intriguingly, this is most manifest on the discrete-time level where it follows directly from the relevant Lax equations. In the continuous case, the relevant Lagrangians (apart from the obvious Lagrangian for the second-order CM flow) are more difficult to establish, but they follow through systematic continuum limits performed on the discrete Lagrangian. However, the latter is quite a subtle computation, and it is here that the connection between the CM system and the KP system (the former arising from the pole reduction of the latter) is essential. The construction starts from the
semi-discrete KP equation, and it is the dependence of this equation on (lattice) parameters which guides the choice of continuum limits. We believe that the CM system forms a first important example for the study of the new variational principle which applies to the case of Lagrangian 1-forms, cf [18, 20, 21, 27], as preparation for similar structures in the case of higher Lagrangian multiforms.

We conclude with a few observations.

• Since the construction is based on the connection with a semi-discrete KP equation, it would have been more satisfactory if a Lagrangian for that equation were at our disposal. However, at this stage, such a Lagrangian seems to be elusive, even though it exists for the bilinear discrete KP case [21].

• Another issue is the connection with the Hamiltonian hierarchy associated with the CM system. Usually, the connection between the Hamiltonian and the Lagrangian is given through the Legendre transformation, which for the second flow in the CM hierarchy is the standard one. However, when we move to the higher Hamiltonians in the hierarchy, these are no longer of Newtonian type, and hence lead to complicated expressions when implementing the Legendre transformation. What the treatment in section 7 reveals, however, is that the higher flows fit naturally in the 1-form structure through mixed, but polynomial, Lagrangians in terms of the higher time derivative.

• The natural framework for quantization of the discrete-time CM model seems to be that of the path integral, which can be interpreted as a quantum version of the least-action principle. One may conjecture that the quantization scheme exploiting the new Lagrange structure could lead to a rigorous approach to constructing the relevant path integral.

Our motivation in this paper was to present the most simple example of a Lagrangian 1-form structure, hence our restriction to the rational case. However, most of our results can be readily expanded to the trigonometric and hyperbolic and also elliptic cases. The formulae for those cases are presented in appendix B. A natural question is how our results extend to the relativistic case, i.e. the Ruijsenaars–Schneider model [29]. We intend to address that problem in a future paper.

Acknowledgments

SYK was supported by the Royal Thai Government and King Mongkut’s University of Technology Thonburi. SBL was supported by the UK Engineering and Physical Sciences Research Council (EPSRC).

Appendix A. The construction of the exact solution

In this appendix, we review the construction of the exact solution (2.13), which follows a procedure similar to the continuous case, cf e.g. [9]. The basic relations following from the Lax pair, i.e. (2.12) together with definitions (2.7) and (2.8), lead to

\[(\tilde{L} - M) E = 0, \quad (A.1a)\]
\[E (L - M) = 0, \quad (A.1b)\]
\[\tilde{X} M - MX = -E, \quad (A.1c)\]
\[XL - LX = I - E, \quad (A.1d)\]
where \( X = X(n, m) = \sum_{i=1}^{N} x_i(n, m) E_i \) is the diagonal matrix of particle positions. We concentrate first on the part of the Lax pairs associated with the dynamics in terms of the variable \( n \). From the Lax equation (2.12), we may write

\[
    M = \overline{U} U^{-1} \quad \text{and} \quad L = U \Lambda_L U^{-1},
\]

where \( U = U(n, m) \) is the matrix used to diagonalize \( L \). Then (A.1) becomes

\[
    \overline{U}^{-1} E = \Lambda_L^{-1} U^{-1} E, \quad (A.3a)
\]
\[
    E \overline{U} = E U \Lambda_L, \quad (A.3b)
\]
\[
    \widetilde{Y} - Y = -\overline{U}^{-1} E U, \quad (A.3c)
\]
\[
    Y \Lambda_L - \Lambda_L Y = U^{-1} (I - E) U, \quad (A.3d)
\]

where \( Y = U^{-1} X U \), which leads to the following expression:

\[
    Y \Lambda_L - \Lambda_L \widetilde{Y} = I. \quad (A.4)
\]

Noting the invariance of \( \Lambda_L \) under the discrete-time shift, we can easily solve (A.4), and its general solution is

\[
    Y(n, m) = \Lambda_L^{-n} Y(0, m) \Lambda_L^n - n \Lambda_L^{-1}, \quad (A.5)
\]

with \( Y(0, m) \) determined from the initial data \( X(0, m) \). A similar analysis can be applied to create the solution associated with the ‘\( \sim \)’ shift.

Conversely, we can start from a given \( N \times N \) diagonal matrix \( A \) with distinct entries, and an initial value matrix \( Y(0, 0) \) subject to the condition that

\[
    [Y(0, 0) , \ A] = I + \text{rank} 1, \quad (A.6)
\]

where \([ , \] denotes the matrix commutator bracket. Let \( U^{-1}(0, 0) \) be the matrix that diagonalizes \( Y(0, 0) \), i.e. such that

\[
    Y(0, 0) = U^{-1}(0, 0) X(0, 0) U(0, 0), \quad X(0, 0) = \text{diag}(x_1(0, 0), \ldots, x_N(0, 0)). \quad (A.7)
\]

If the eigenvalues of \( Y(0, 0) \) are distinct (which we can take as an assumption on the initial condition), then \( U^{-1}(0, 0) \) is determined up to multiplication from the right by a diagonal matrix times a permutation matrix of the columns. (Fix an ordering of the eigenvalues \( x_1(0, 0), \ldots, x_N(0, 0) \) unique only up to multiplication by a diagonal matrix from the right.) We can fix \( U^{-1}(0, 0) \) up to an overall multiplicative factor by demanding that

\[
    [Y(0, 0) , \ A] = I - U^{-1}(0, 0) E U(0, 0). \quad (A.8)
\]

Next we consider the matrix function given by

\[
    \widetilde{Y}(n, m) = \Lambda_L^{-n} \Lambda_K^{-m} Y(0, 0) \Lambda_L^n \Lambda_K^m - n \Lambda_L^{-1} - m \Lambda_K^{-1}, \quad (A.9)
\]

with \( \Lambda_L = pl + A \) and \( \Lambda_K = ql + A \). Let \( U^{-1}(n, m) \) be the matrix diagonalizing \( Y(n, m) \); by an appropriate choice of an overall factor (as a function of \( n \) and \( m \)), this matrix can be fixed such that it obeys

\[
    U^{-1}(n, m) E = \Lambda_L^{-n} \Lambda_K^{-m} U^{-1}(0, 0) E, \quad E U(n, m) = E U(0, 0) \Lambda_L^n \Lambda_K^m. \quad (A.10)
\]

and

\[
    [Y(n, m) , \ A] = I - U^{-1}(n, m) E U(n, m). \quad (A.11)
\]

From expression (A.9), we can now derive the relations

\[
    \Lambda_L \widetilde{Y} - Y \Lambda_L = -I, \quad (A.12a)
\]
\[
    \Lambda_K \widetilde{Y} - Y \Lambda_K = -I. \quad (A.12b)
\]
with the usual notation for the shifts in \( n \) and \( m \) over one unit. Together with relation (A.11), this subsequently yields
\[
\tilde{Y} - Y = -\tilde{U}^{-1} EU, \quad \hat{Y} - Y = -\hat{U}^{-1} EU.
\] (A.13)

Reversing these relations by rewriting them in terms of \( X(n, m) = U(n, m)Y(n, m)U^{-1}(n, m) \) and now defining the Lax matrices by
\[
L := U \Lambda U^{-1}, \quad K := U \Lambda^{-1},
\] (A.14)
together with
\[
M := \tilde{U}^{-1}, \quad N := \hat{U}^{-1},
\] (A.15)
we recover the relations
\[
[X, L] = [X, K] = I - E, \quad \tilde{X}M - MX = \tilde{X}N - NX = -E,
\] (A.16)
which determine the matrices \( M \) and \( N \) as functions of the \( x_i(n, m) \) as well as the off-diagonal parts of the matrices \( L \) and \( K \). From the definitions of \( L \) and \( K \), we have that
\[
L - K = (p - q)I.
\]
Furthermore, from (A.12) we obtain
\[
\tilde{L}\tilde{X}M - MXL = -M, \quad \hat{K}\hat{X}N - NXK = -N,
\] (A.17)
which, when combined with the latter relations of (A.16), yield
\[
(\tilde{L}M - ML)(X - E) = 0, \quad (\hat{K}N - NK)(X - E) = 0.
\] (A.18)
On the other hand, using the first of relations (A.16), we also obtain
\[
\tilde{X}(\tilde{L}M - ML) = E(L - M) \quad \text{and} \quad \hat{X}(\hat{K}N - NK) = E(K - N),
\] (A.19)
as well as
\[
(\tilde{L}M - ML)X = (\tilde{L} - M)E \quad \text{and} \quad (\hat{K}N - NK)X = (\hat{K} - N)E.
\] (A.20)
From relations (A.18)–(A.20), it follows that the Lax equations hold and their form is determined up to the diagonal part of the matrices \( L \) and \( K \).

Appendix B. The elliptic and trigonometric discrete-time CM

In this appendix, we show that there is a connection between the time-part Lax matrix and the Lagrangian for the trigonometric/hyperbolic and elliptic cases of the discrete-time CM system, similar to that which we have established for the rational case.

B.1. The elliptic case

The Lax matrices in this case read [23]
\[
L = \sum_{i=1}^N \left( \sum_{j=1}^N \xi(x_i - \tilde{x}_j) - \sum_{j \neq i}^N \xi(x_i - x_j) \right) E_{ii} - \sum_{i \neq j}^N \Phi_x(x_i - x_j)E_{ij}, \quad \text{(B.1)}
\]
\[
M = \sum_{i,j=1}^N \Phi_x(\tilde{x}_i - x_j)E_{ij}, \quad \text{(B.2)}
\]
where $\zeta(x)$ is the Weierstrass zeta function and $\Phi_\kappa(x) = \frac{\sigma(x+\kappa)}{\sigma(x)\sigma(\kappa)}\sigma(x)$, where $\sigma(x)$ is the Weierstrass sigma function. The Lax equation $\tilde{L}M = ML$ produces the equation of motion

$$\sum_{j=1}^{N} \left( \zeta(x_i - \tilde{x}_j) + \zeta(x_i - \xi_j) \right) - 2 \sum_{j \neq i} \zeta(x_i - x_j) = 0, \quad (B.3)$$

and the corresponding Lagrangian is

$$\mathcal{L} = -\sum_{i,j=1}^{N} \log |\sigma(x_i - \tilde{x}_j)| + \sum_{i \neq j}^{N} \log |\sigma(x_i - x_j)|. \quad (B.4)$$

The determinant of the matrix $M$ is

$$\det |M| = \Phi_\kappa(\Sigma)\sigma(\kappa) \prod_{i<j} \sigma(x_i - x_j) \prod_{i} \sigma(\tilde{x}_i - x_j), \quad (B.5)$$

where $\Sigma = \sum_{i=1}^{N} (x_i - \tilde{x}_i)$, and then we can also write

$$\log |\det (M)| = \log |\Phi_\kappa(\Sigma)\sigma(\kappa)| + \sum_{i<j}^{N} \log |\sigma(x_i - x_j)| + \log |\sigma(\tilde{x}_i - \tilde{x}_j)|$$

$$- \sum_{i}^{N} \log |\sigma(\tilde{x}_i - x_j)|. \quad (B.6)$$

The action of the system can be expressed by considering the chain product of the matrix $M$,

$$S = \log \left| \prod_{n=+\infty} \det \left( \sum_{i}^{N} \log |\Phi_\kappa(\Sigma)\sigma(\kappa)| + \sum_{i \neq j}^{N} \log |\sigma(x_i - x_j)| - \sum_{i}^{N} \log |\sigma(\tilde{x}_i - x_j)| \right) \right|.$$ 

$$= \sum_{n}^{N} \left( \log |\Phi_\kappa(\Sigma)\sigma(\kappa)| + \sum_{i \neq j}^{N} \log |\sigma(x_i - x_j)| - \sum_{i}^{N} \log |\sigma(\tilde{x}_i - x_j)| \right). \quad (B.7)$$

The last term in equation (B.7) is related to the motion of the centre of mass of the system and it can be separated from the relative motion.

### B.2. The trigonometric case

The Lax matrices in this case read [23]

$$L = \sum_{j=1}^{N} \left( \sum_{i=1}^{N} \coth(q_i - \tilde{x}_j) - \sum_{j \neq i}^{N} \coth(x_i - x_j) \right) E_{ii} - \sum_{i \neq j}^{N} \frac{\sinh(x_i - x_j + \kappa)E_{ij}}{\sinh(x_i - x_j)\sinh(\kappa)}, \quad (B.8)$$

$$M = \sum_{i,j=1}^{N} \frac{\sinh(\tilde{x}_i - x_j + \kappa)E_{ij}}{\sinh(\tilde{x}_i - x_j)\sinh(\kappa)}. \quad (B.9)$$

The Lax equation $\tilde{L}M = ML$ produces the equation of motion

$$\sum_{j=1}^{N} \left( \coth(x_i - \tilde{x}_j) + \coth(x_i - \xi_j) \right) - 2 \sum_{j \neq i}^{N} \coth(x_i - x_j) = 0, \quad (B.10)$$
and the corresponding Lagrangian is
\[ \mathcal{L} = - \sum_{i,j=1}^{N} \log |\sinh(x_i - \tilde{x}_j)| + \sum_{i \neq j}^{N} \log |\sinh(x_i - x_j)|. \] (B.11)

The determinant of the matrix \( M \) is
\[ \det |M| = \frac{\sinh(\Sigma + \kappa) \prod_{i<j} \sinh(x_i - x_j) \sinh(\tilde{x}_i - \tilde{x}_j)}{\prod_{i,j} \sinh(x_i - x_j)}. \] (B.12)

and then we can also write
\[ \log |\det (M^\epsilon)| = \log \left| \sinh(\Sigma + \kappa) \prod_{i<j} \sinh(x_i - x_j) \sinh(\tilde{x}_i - \tilde{x}_j) \right| \]
\[ - \sum_{i \neq j}^{N} \log |\sinh(\tilde{x}_i - x_j)|. \] (B.13)

The action of the system can be expressed by considering the chain product of the matrix \( M \),
\[ S = \log \left| \det \left( \prod_{n=\infty}^{\infty} M(n) \right) \right|, \]
\[ = \sum_{n} \left( \log \left| \frac{\sinh(\Sigma + \kappa)}{\sinh(\kappa)} \right| + \sum_{i \neq j}^{N} \log |\sinh(x_i - x_j)| - \sum_{i,j}^{N} \log |\sigma(\tilde{x}_i - x_j)| \right), \]
\[ = \sum_{n} \left( \mathcal{L} + \frac{\sinh(\Sigma + \kappa)}{\sinh(\kappa)} \right). \] (B.14)

Again the last term in equation (B.14) is related to the centre of mass motion of the system, and it is not going to affect the equations of the relative motion.

Appendix C. The derivation of the Lagrangian \( \mathcal{L}(\tau) \)

In this appendix, we show the derivation of the Lagrangian \( \mathcal{L}(\tau) \) (5.8) from the skew limit of the Lagrangian \( \mathcal{L}(\tau) \) (3.10b). We recall the Lagrangian \( \mathcal{L}(\tau) \) again here
\[ \mathcal{L}(\tau) = \log |\det(N)| + q(\Xi - \hat{\Xi}). \] (C.1)

Performing the skew limit, we obtain
\[ \mathcal{L}(\tau) \Rightarrow \log |\det(M - \epsilon \tilde{A})| + (p - \epsilon)(\Xi - \hat{\Xi} - \epsilon \hat{\Xi}) \]
\[ \Rightarrow \log |\det(M)| + \log |\det(1 - \epsilon M^{-1} \hat{A})| + \cdots + p(\Xi - \hat{\Xi}) - \epsilon(\Xi - \hat{\Xi}) - p\epsilon \hat{\Xi} \]
\[ \Rightarrow \mathcal{L}(\tau) + \log |1 - \epsilon \text{Tr}(M^{-1} \hat{A})| + \cdots | - \epsilon(\Xi - \hat{\Xi}) - p\epsilon \hat{\Xi} \]
\[ \Rightarrow \mathcal{L}(\tau) + \epsilon \mathcal{L}(\tau) + \cdots, \] (C.2)

where
\[ \mathcal{L}(\tau) = -\text{Tr}(M^{-1} \tilde{A}) - \Xi + \hat{\Xi} - p\hat{\Xi}. \] (C.3)

The inverse of the matrix \( M \) is given by
\[ M^{-1} = -\sum_{i,j=1}^{N} \frac{\tilde{\Psi}(x_i)\Psi(x_j)}{x_i - x_j} E_{ij}, \] (C.4)
where
\[ \tilde{\Psi}(x_i) = \prod_{l=1}^{N}(x_i - \tilde{x}_l), \quad (C.5a) \]
and
\[ \Psi(\tilde{x}_j) = \prod_{l=1}^{N}(\tilde{x}_j - x_l). \quad (C.5b) \]

We now can show that
\[ \text{Tr}(\mathbf{M}^{-1}\tilde{\mathbf{A}}) = \sum_{i,j=1}^{N} \frac{\tilde{\Psi}(x_i)\psi(\tilde{x}_j)\dot{x}_j}{(\tilde{x}_j - x_i)^3}. \quad (C.6) \]

We now consider the identity
\[ \prod_{l=1}^{N} \frac{\xi - \tilde{x}_l}{\xi - x_l} = 1 + \sum_{l=1}^{N} \frac{\tilde{\Psi}(x_l)}{\xi - x_l}, \quad (C.7) \]
which we differentiate with respect to \( \xi \) at a given value \( \xi = \tilde{x}_j \), yielding
\[ -\frac{1}{\Psi(\tilde{x}_j)} = \sum_{l=1}^{N} \frac{\tilde{\Psi}(x_l)}{(\tilde{x}_j - x_l)^2}. \quad (C.8) \]

We now differentiate (C.8) with respect to \( \tilde{x}_j \), yielding
\[ \frac{\partial}{\partial \tilde{x}_j} \frac{1}{\Psi(\tilde{x}_j)} = \sum_{l=1}^{N} \frac{\tilde{\Psi}(x_l)}{(\tilde{x}_j - x_l)^3}. \quad (C.9) \]

Using (C.9), we can rewrite (C.6) in the form
\[ \text{Tr}(\mathbf{M}^{-1}\tilde{\mathbf{A}}) = \sum_{j=1}^{N} \frac{\tilde{x}_j\psi(\tilde{x}_j)}{\tilde{x}_j - x_i} \frac{\partial}{\partial \tilde{x}_j} \frac{1}{\Psi(\tilde{x}_j)}, \]
\[ = -\sum_{j=1}^{N} \frac{\tilde{x}_j}{\tilde{x}_j - x_i} \ln |\Psi(\tilde{x}_j)|; \quad (C.10) \]
then the Lagrangian \( \mathcal{L}(\tau) \) can be expressed in the form
\[ \mathcal{L}(\tau) = \sum_{i=1}^{N} \frac{\tilde{x}_j}{x_i - \tilde{x}_j} + \sum_{i,j=1}^{N} \frac{\tilde{x}_j}{x_i - x_j} = \mathcal{L} + \tilde{\mathcal{L}}. \quad (C.11) \]

Appendix D. The direct proof of the closure relation (7.11)

In this section, we show that the closure relation (7.11) holds on the equations of motion. Here we recall the equations of motion for the time variables \( t_2 \) and \( t_3 \),
\[ \frac{\partial^2 X_i}{\partial t_2^2} = -\sum_{j=1}^{N} \frac{8}{(X_i - X_j)^3}, \quad (D.1) \]
\[ \frac{\partial^2 X_i}{\partial t_2 \partial t_3} = 6 \sum_{j=1}^{N} \frac{\partial X_i}{\partial t_2} \frac{\partial X_j}{\partial t_3} \frac{1}{(X_i - X_j)^3}. \quad (D.2) \]
The Lagrangians corresponding to (D.1) and (D.2) take the form

\[
\mathcal{L}_{(t_2)} = \sum_{i=1}^{N} \left( \frac{\partial X_i}{\partial t_2} \right)^2 + \sum_{i \neq j}^{N} \frac{2}{(X_i - X_j)^2},
\]

(D.3)

\[
\mathcal{L}_{(t_3)} = \sum_{i=1}^{N} \left( \frac{\partial X_i}{\partial t_3} + \frac{1}{4} \left( \frac{\partial X_i}{\partial t_2} \right)^2 \right) - \sum_{i \neq j}^{N} \frac{\partial X_i}{\partial t_2} + 2 \frac{\partial X_j}{\partial t_2},
\]

(D.4)

respectively.

We find that

\[
\frac{\partial \mathcal{L}_{(t_2)}}{\partial t_3} = \sum_{i \neq j}^{N} 6 \frac{(\partial X_i)}{(X_i - X_j)^2} - 8 \frac{\partial X_i}{(X_i - X_j)^3},
\]

(D.5)

and

\[
\frac{\partial \mathcal{L}_{(t_3)}}{\partial t_2} = \sum_{i \neq j}^{N} 6 \frac{(\partial X_i)}{(X_i - X_j)^2} - 8 \frac{\partial X_i}{(X_i - X_j)^3} - 3 \sum_{i \neq j}^{N} \frac{\partial X_i}{(X_i - X_j)^2}.
\]

(D.6)

We see that the first terms of equations (D.5) and (D.6) are identical. The remaining work is to show that the last term in equation (D.6) is zero. Using equation (D.1), we may rewrite the last term in equation (D.6) with the help of the identities

\[
\frac{1}{8} \sum_{i \neq j}^{N} \frac{\partial^2 X_i}{(X_i - X_j)^2} = \sum_{i \neq j}^{N} \sum_{k=1}^{N} \frac{1}{(X_i - X_j)^2(X_i - X_k)^2} = \sum_{i \neq j}^{N} \frac{1}{(X_i - X_j)^2} + \sum_{i \neq j \neq k}^{N} \frac{1}{(X_i - X_j)^2(X_i - X_k)^2}.
\]

(D.7)

We see that the first term is an antisymmetric function and hence vanishes, while the second term

\[
\sum_{i \neq j}^{N} \frac{1}{(X_i - X_j)^2(X_i - X_k)^2} = \sum_{i \neq j \neq k}^{N} \left( \frac{1}{X_i - X_j} - \frac{1}{X_i - X_k} \right)^2 \frac{1}{(X_j - X_k)^2} + \frac{1}{(X_i - X_j)^2(X_i - X_k)^2}.
\]

\[
= \sum_{i \neq j \neq k}^{N} \left( \frac{1}{(X_i - X_j)^2(X_i - X_k)^2} - \frac{2}{(X_i - X_j)(X_i - X_k)(X_j - X_k)^2} \right).
\]

(D.8)

The first and third terms are the antisymmetric functions, hence vanish leaving the middle term which is actually the opposite of the term on the left-hand side, i.e.

\[
\sum_{i \neq j \neq k}^{N} \frac{1}{(X_i - X_j)^2(X_i - X_k)^2} = - \sum_{i \neq j \neq k}^{N} \frac{1}{(X_k - X_j)^2(X_k - X_i)^2} = 0.
\]

(D.9)

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