ASYMPTOTIC EXPANSION OF THE WAVELET TRANSFORM WITH ERROR TERM

R S Pathak and Ashish Pathak

Abstract

Using Wong’s technique asymptotic expansion for the wavelet transform is derived and thereby asymptotic expansions for Morlet wavelet transform, Mexican Hat wavelet transform and Haar wavelet transform are obtained.

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1 Introduction

The wavelet transform of \( f \) with respect to the wavelet \( \psi \) is defined by

\[
(W_\psi f)(b,a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt, \quad b \in \mathbb{R}, a > 0,
\]

provided the integral exists \[1\]. Using Fourier transform it can also be expressed as

\[
(W_\psi f)(b,a) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{ib\omega} \overline{\hat{\psi}(a\omega)} d\omega,
\]

where

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t) dt.
\]

Asymptotic expansion with explicit error term for the general integral

\[
I(x) = \int_{0}^{\infty} f(t) h(xt) dx,
\]

where \( h(t) \) is an oscillatory function, was obtained by Wong \[3\], \[4\] under different conditions on \( g \) and \( h \). Then the asymptotic expansion for \[2\] can be obtained by setting \( g(t) = e^{ibt} \hat{f}(t) \) for fixed \( b \in \mathbb{R} \). Let us recall basic results from \[4\] which will be used in the present investigation. Here we assume that \( g(t) \) has an expansion of the form

\[
g(t) \sim \sum_{s=0}^{\infty} c_s t^{s+\lambda-1} \quad \text{as} \quad t \to 0,
\]

\[
= \sum_{s=0}^{n-1} c_s t^{s+\lambda-1} + g_n(t) \quad (4)
\]

\[1\] This work is contained in the research monograph "The Wavelet Transform" by Prof. R S Pathak and edited by Prof. C. K. Chui (Stanford University, U.S.A.) and published by Atlantis Press/World Scientific (2009), ISBN: 978-90-78677-26-0, pp:154-164
where $0 < \lambda \leq 1$. Regarding the function $h$, we assume that as $t \to 0^+$,

$$h(t) = O(t^\rho), \quad \rho + \lambda > 0,$$

and that as $t \to +\infty$,

$$h(t) \sim \exp(i\tau t^p) \sum_{s=0}^{\infty} b_s t^{-s-\beta},$$

where $\tau \neq 0$ is real, $p \geq 1$ and $0 < \beta \leq 1$. Let $M[h; z]$ denote the generalized Mellin transform of $h$ defined by

$$M[h; z] = \lim_{\varepsilon \to 0^+} \int_0^\infty t^{z-1} h(t) \exp(-\varepsilon t^p) dt.$$  

This, together with (48) and [4, p.216], gives

$$I(x) = \sum_{s=0}^{n-1} c_s M[h; s + \lambda] x^{-s-\lambda} + \delta_n(x),$$

where

$$\delta_n(x) = \lim_{\varepsilon \to 0^+} \int_0^\infty g_n(t) h(xt) \exp(-\varepsilon t^p) dt.$$  

If we now define recursively $h^0(t) = h(t)$ and

$$h^{(-j)}(t) = - \int_t^\infty h^{(-j+1)}(u) du, \quad j = 1, 2, \ldots,$$

Repeated integration by part, we have

$$h^{(-j)}(t) \sim \exp(i\tau t^p) \sum_{s=0}^{\infty} b_s^{(j)} t^{-s-\beta}, \quad as \to \infty,$$

where $b_s^{(j)}$ are some constants and for each $j$, and $\mu_{s,j}$ is a monotonically increasing sequence of positive numbers depending on $p$ and $\beta$.

Then conditions of validity of aforesaid results are given by the following [4, Theorem 6, p.217]:

**Theorem 1.** Assume that (i) $g^{(m)}(t)$ is continuous on $(0, \infty)$, where $m$ is a non-negative integer; (ii) $g(t)$ has an expansion of the form (4), and the expansion is $m$ times differentiable; (iii) $h(t)$ satisfies (2) and (3) and (iv) and as $t \to \infty$, $t^{-\beta} g^{(j)}(t) = O(t^{1-\varepsilon})$ for $j = 0, 1, \ldots, m$ and for some $\varepsilon > 0$. Under these conditions, the result (8) holds with

$$\delta_n(x) = \frac{(-1)^m}{x^m} \int_0^\infty g_n^{(m)}(t) h^{(-m)}(xt) dt,$$

where $n$ is the smallest positive integer such that $\lambda + n > m$. 

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Proof. Integrating by part (9) we get
\[
\int_{0}^{\infty} f_n(t)h(xt)e^{-\epsilon t^p} dt = -\frac{1}{x} \int_{0}^{\infty} f_n(t)e^{-\epsilon t} h^{(-1)}(xt) dt + \frac{\epsilon p}{x} \int_{0}^{\infty} f_n(t)h^{(-1)}(xt)t^{p-1} e^{-\epsilon t} dt
\]
(12)
the integrated term vanishing due to \(\rho + \lambda > 0\) and condition (iv) and the asymptotic behaviour in (10). The same reasoning, together with Lemma1 and Lemma2, ensures that the second term on the right-hand side of (12) tends to zero as \(\epsilon \to 0^+\).

Thus
\[
\delta_n(x) = \left( -\frac{1}{x} \right) \lim_{\epsilon \to 0^+} \int_{0}^{\infty} f'_n(t)h^{(-1)}(xt)e^{-\epsilon t^p} dt.
\]
(13)
Repeated application of this technique shows that
\[
\delta_n(x) = \left( -\frac{1}{x} \right) \lim_{\epsilon \to 0^+} \int_{0}^{\infty} f^{(m)}_n(t)h^{(-m)}(xt)e^{-\epsilon t^p} dt
\]
(14)
The last equality again follows from Lemma1.

The aim of the present paper is to derive asymptotic expansion of the wavelet transform given by (2) for large values of \(a\), using formula (8). We also obtain asymptotic expansions for the special transforms corresponding to Morlet wavelet, Mexican hat wavelet and Haar wavelet.

2 Asymptotic expansion for large \(a\)

In this section using aforesaid technique, we obtain asymptotic expansion of \((W_\psi f)(b,a)\) for large values of \(a\), keeping \(b\) fixed. We have
\[
(W_\psi f)(b,a) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} e^{ib\omega} \overline{\tilde{\psi}(a\omega)} \tilde{f}(\omega) d\omega
\]
\[
= \frac{\sqrt{a}}{2\pi} \left\{ \int_{0}^{\infty} e^{ib\omega} \overline{\tilde{\psi}(a\omega)} \tilde{f}(\omega) d\omega + \int_{0}^{\infty} e^{-ib\omega} \overline{\tilde{\psi}(-a\omega)} \tilde{f}(-\omega) d\omega \right\}
\]
\[
= \frac{\sqrt{a}}{2\pi} (I_1 + I_2), \text{ say.}
\]
(15)
Let us set
\[
h(\omega) = \overline{\tilde{\psi}(\omega)}.
\]
(16)
Assume that
\[ \overline{\psi}(\omega) \sim \exp(i\tau \omega^p) \sum_{r=0}^{\infty} b_r \omega^{-r-\beta}, \quad \beta > 0, \ \omega \to +\infty, \ \tau \neq 0, \ p \geq 1, \] (17)
and
\[ \hat{\psi}(\omega) \sim \sum_{s=0}^{\infty} c_s \omega^{s+\lambda-1} \quad \text{as} \ \omega \to 0. \] (18)
where \( 0 < \lambda \leq 1. \) Also assume that as \( \omega \to 0, \)
\[ h(\omega) = \overline{\psi}(\omega) = O(\omega^\rho), \ \rho + \lambda > 0. \] (19)
Then, as \( \omega \to 0, \)
\[ g(\omega) := e^{ib\omega} \hat{\psi}(\omega) \]
\[ \sim \sum_{s=0}^{\infty} c_s \omega^{s+\lambda-1} \sum_{r=0}^{\infty} \frac{(ib\omega)^r}{r!} \]
\[ = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} c_s \frac{(ib)^r}{r!} \omega^{s+\lambda-1+r} \]
\[ = \sum_{s=0}^{\infty} \left\{ \sum_{r=0}^{s} \frac{(ib)^r}{r!} c_{s-r} \right\} \omega^{s+\lambda-1} \]
\[ = \sum_{s=0}^{\infty} d_s \omega^{s+\lambda-1}, \] (20)
where
\[ d_s = \sum_{r=0}^{s} \frac{(ib)^r}{r!} c_{s-r}. \] (21)
For each \( n \geq 1, \) we write
\[ g(\omega) = \sum_{s=0}^{n-1} d_s \omega^{s+\lambda-1} + g_n(\omega). \] (22)
The generalized Mellin transform of \( h \) is defined by
\[ M[h; z_1] = \lim_{\varepsilon \to 0^+} \int_{0}^{\infty} \omega^{z_1-1} h(\omega) e^{-\varepsilon \omega} d\omega \] (23)
Then by (8),
\[ I_1(a) = \sum_{s=0}^{n-1} d_s M[h; s + \lambda] a^{-s-\lambda} + \delta_n^1(a), \] (24)
where
\[
\delta_1^n(a) = \lim_{\varepsilon \to 0^+} \int_0^\infty g_n(\omega)h(a\omega)e^{-\varepsilon\omega}d\omega.
\]  
(25)

Also, from (19) we have
\[
h(-\omega) = O(\omega^\rho), \quad \omega \to 0, \quad \rho + \lambda > 0
\]  
(26)

and
\[
M[h(-\omega); z_1] = \lim_{\varepsilon \to 0^+} \int_0^\infty \omega^{z_1 - 1}h(-\omega)e^{-\varepsilon\omega}d\omega.
\]  
(27)

Hence
\[
I_2(a) = \sum_{s=0}^{n-1} d_s(-1)^{s+\lambda+1}M[h(-\omega); s + \lambda]a^{-s-\lambda} + \delta_2^n(a),
\]  
(28)

where
\[
\delta_2^n(a) = \lim_{\varepsilon \to 0^+} \int_0^\infty g_n(-\omega)h(-a\omega)e^{-\varepsilon\omega}d\omega.
\]  
(29)

Finally, from (15), (24) and (28) we get the asymptotic expansion:
\[
(W_\psi f)(b,a) = \sqrt{a} \cdot \frac{\pi}{2} \int_{-\infty}^{\infty} \left\{ \sum_{s=0}^{n-1} d_s(-1)^{s+\lambda+1}M[\hat{\psi}(\omega); s + \lambda] + (-1)^{s+\lambda+1}
\right.
\]  
\[
\times M[\hat{\psi}(-\omega)s + \lambda]a^{-s-\lambda} + \delta_n(a) \left. \right\}
\]  
(30)

where
\[
\delta_n(a) = \lim_{\varepsilon \to 0^+} \left( \int_0^\infty g_n(\omega)h(a\omega)e^{-\varepsilon\omega}d\omega 
\right.
\]  
\[
+ \int_0^\infty g_n(-\omega)h(-a\omega)e^{-\varepsilon\omega}d\omega \right) 
\]  
(31)

Since \(g(\omega) = e^{ib\omega}\hat{f}(\omega)\), the continuity of \(\hat{f}(m)(\omega)\) implies continuity of \(g^{(m)}(\omega)\). Using Theorem 1 we get the following existence theorem for formula (31).

**Theorem 2.** Assume that (i) \(\hat{f}(m)(\omega)\) is continuous on \((-\infty, \infty)\), where \(m\) is a nonnegative integer; (ii) \(\hat{f}(\omega)\) has asymptotic expansion of the form (18) and the expansion is \(m\) times differentiable, (iii) \(\hat{\psi}(\omega)\) satisfies (10) and (17) and (iv) as \(\omega \to \infty, \omega^{-\beta}\hat{f}(j)(\omega) = O(\omega^{-1-\varepsilon})\) for \(j = 0, 1, 2, ..., m\) and for some \(\varepsilon > 0\). Under these conditions, the result (30) holds with
\[
\delta_n(a) = \frac{(-1)^m}{a^m} \int_{-\infty}^{\infty} g_n^{(m)}(\omega)(\hat{\psi}(a\omega))(-m)d\omega,
\]  
(32)

where \(n\) is the smallest positive integer such that \(\lambda + n > m\).

In the following sections we shall obtain asymptotic expansions for certain special cases of the general wavelet transform.
3 MORLET WAVELET TRANSFORM

In this section we choose

$$\psi(t) = e^{i\omega_0 t - t^2/2}. $$

Then from [1] p. 373 we have

$$\hat{\psi}(\omega) = \sqrt{2\pi} e^{-\frac{(\omega - \omega_0)^2}{2}},$$

which is exponentially decreasing. Therefore, Theorem 1 is not directly applicable, but a slight modification of the technique works well. Assume that $\hat{f}$ has an asymptotic expansion of the form (18). In this case we have

$$h(\omega) = \hat{\psi}(\omega) = \sqrt{2\pi} e^{-\frac{(\omega - \omega_0)^2}{2}}, \quad (33)$$

and

$$h(\omega) = O(1) \quad as \quad \omega \to 0. \quad (34)$$

Then from (24) and (33), we get

$$I_1(a) = \sum_{s=0}^{n-1} d_s M\left[\sqrt{2\pi} e^{-\frac{(\omega - \omega_0)^2}{2}}; s + \lambda\right] a^{-s-\lambda}$$

$$+ \lim_{\epsilon \to 0+} \int_0^\infty g_n(\omega) \sqrt{2\pi} e^{-\frac{(\omega - \omega_0)^2}{2}} e^{-\epsilon\omega} d\omega, \quad (35)$$

where

$$M\left[\sqrt{2\pi} e^{-\frac{(\omega - \omega_0)^2}{2}}; s + \lambda\right] = \sqrt{2\pi} \int_0^\infty \omega^{s+\lambda-1} e^{-\frac{(\omega - \omega_0)^2}{2}} d\omega$$

$$= \sqrt{2\pi} e^{-\frac{\omega_0^2}{2}} \int_0^\infty \omega^{s+\lambda-1} e^{-\frac{\omega^2}{2} + \omega\omega_0} d\omega.$$ 

Evaluating the last integral by means of formula [2] (31), p.320;]

$$\int_0^\infty x^{s-1} e^{-\frac{x^2}{2} - \beta x} dx = e^{\beta^2/4} \Gamma(s) D_{-s}(\beta), \quad Re(s) > 0,$$

where $D_{-\nu}(x)$ denotes parabolic cylinder function, we get

$$M\left[\sqrt{2\pi} e^{-\frac{(\omega - \omega_0)^2}{2}}; s + \lambda\right] = \sqrt{2\pi} e^{-\frac{\omega_0^2}{2}} \Gamma(s + \lambda) D_{-(s+\lambda)}(-\omega_0), \quad s + \lambda > 0. \quad (36)$$

From (35) and (36), we get

$$I_1(a) = \sqrt{2\pi} e^{-\frac{\omega_0^2}{2}} \sum_{s=0}^{n-1} d_s \Gamma(s + \lambda) D_{-(s+\lambda)}(-\omega_0) a^{-s-\lambda}$$

$$+ \int_0^\infty g_n(\omega) \sqrt{2\pi} e^{-\frac{(\omega - \omega_0)^2}{2}} d\omega. \quad (37)$$
Similarly, we get
\[
I_2(a) = \sqrt{2\pi} e^{-\frac{\omega_0^2}{2}} \sum_{s=0}^{n-1} d_s \Gamma(s + \lambda)(-1)^{s+\lambda-1} D_{-(s+\lambda)}(\omega_0) \ a^{s-\lambda} + \int_0^\infty g_n(-\omega) \sqrt{2\pi e^{-\frac{(\omega+\omega_0)^2}{2}}} \ d\omega.
\]
(38)

Finally, using (15), (37) and (38) we get
\[
(W_\psi f)(b, a) = e^{-\frac{\omega_0^2}{2}} \sum_{s=0}^{n-1} d_s \Gamma(s + \lambda) \ [D_{-(s+\lambda)}(-\omega_0)] a^{s-\lambda+\frac{1}{2}} + \delta_n(a),
\]
(39)

where
\[
\delta_n(a) = \sqrt{a} \int_0^\infty g_n(\omega) e^{-(a\omega-\omega_0)^2} \ d\omega + \sqrt{a} \int_0^\infty g_n(-\omega) e^{-(a\omega+\omega_0)^2} \ d\omega.
\]

Using Theorem 2 we get the following existence theorem for formula (39).

**Theorem 3.** Assume that \( \hat{f}(\omega) \) satisfies conditions of Theorem 2. Then the result (39) holds with
\[
\delta_n(a) = (-1)^m a^{-m+1/2} \int_{-\infty}^{\infty} g_n^{(m)}(\omega) \left( e^{-\frac{(a\omega-\omega_0)^2}{4}} \right)^{(-m)} \ d\omega,
\]
where \( n \) is the smallest positive integer such that \( \lambda + n > m \).

### 4 MEXICAN HAT WAVELET TRANSFORM

In this section we choose
\[
\psi(t) = (1 - t^2)e^{-t^2/2}.
\]

Then from [11, p.372]
\[
h(\omega) := \hat{\psi}(\omega) = \sqrt{2\pi} \omega^2 e^{-\omega^2/2};
\]
(40)

so that
\[
h(\omega) = O(\omega^2), \quad \omega \to 0.
\]
(41)

Assume that \( \hat{f} \) has an asymptotic expansion of the form (18), and satisfies
\[
\hat{f}(\omega) = O(e^{a\omega^2}), \quad \omega \to +\infty;
\]
(42)
for some $\sigma > 0$. Therefore,

$$g(\omega) := e^{ib\omega} \hat{f}(\omega) = O(e^{\sigma \omega^2}), \quad \omega \to +\infty. \quad (43)$$

Then by (23) and (40), we get

$$I_1(a) = \sum_{s=0}^{n-1} d_s M[\sqrt{2\pi} \omega^2 e^{-\omega^2/2}; s + \lambda] a^{s-\lambda} + \delta^1_n(a), \quad (44)$$

where

$$M[\sqrt{2\pi} \omega^2 e^{-\omega^2/2}; s + \lambda] = \sqrt{2\pi} \int_0^{\infty} \omega^{s+\lambda+1} e^{-\omega^2/2} d\omega$$

$$= \sqrt{\pi} 2^{(s+\lambda+1)/2} \Gamma \left( \frac{s + \lambda + 2}{2} \right),$$

and

$$\delta^1_n = \int_0^{\infty} g_n(\omega) \sqrt{2\pi}(a\omega)^2 e^{-(a\omega)^2/2} d\omega. \quad (45)$$

Similarly, we get

$$I_2(a) = \sqrt{\pi} 2^{(\lambda+1)/2} \sum_{s=0}^{n-1} d_s (-1)^{s+\lambda-1} 2^{s/2} \Gamma \left( \frac{s + \lambda + 2}{2} \right) a^{s-\lambda}$$

$$+ \delta^2_n(a), \quad (46)$$

where

$$\delta^2_n(a) = \int_0^{\infty} g_n(-\omega) \sqrt{2\pi}(a\omega)^2 e^{-(a\omega)^2/2} d\omega. \quad (47)$$

Finally, using (15), (44) and (46), we have

$$(W_{\psi f})(b, a) = \frac{2(\lambda+1)/2}{\sqrt{\pi}} \sum_{s=0}^{n-1} d_s 2^{s/2} \Gamma \left( \frac{s + \lambda + 2}{2} \right) \left\{ (1 + (-1)^{s+\lambda-1} \right\}$$

$$\times a^{s-\lambda+1/2} + \delta_n(a), \quad (48)$$

where

$$\delta_n(a) = 2^{3/2} \sqrt{\pi} \int_0^{\infty} g_n(\omega)(a\omega)^2 e^{-(a\omega)^2/2} d\omega.$$ 

Existence theorem for formula (48) is as follows:

**Theorem 4.** Assume that $\hat{f}(\omega)$ satisfies conditions of Theorem 2. Then the result (48) holds with

$$\delta_n(a) = \left( \frac{-1}{a} \right)^m 2^{3/2} \sqrt{\pi} \int_{-\infty}^{\infty} g_n^{(m)}(\omega)((a\omega)^2 e^{-(a\omega)^2/2})^{-m} d\omega,$$

where $n$ is the smallest positive integer such that $\lambda + n > m.$
5 Haar wavelet transform

In this section we choose

\[ \psi(t) = \begin{cases} 
1, & 0 \leq t < 1/2 \\
-1, & 1/2 \leq t < 1 \\
0, & \text{otherwise.} 
\end{cases} \]

Then from [1, p.368],

\[ \widehat{\psi}(\omega) = \frac{4i e^{-i\omega/2} \sin^2 \omega/4}{\omega} = \frac{i}{\omega} \left( 1 - 2 e^{i\omega/2} + e^{i\omega} \right). \] (49)

Although the condition \( \beta > 0 \) of (16) is not satisfied in this case but the result (6)- (7) remains valid, cf. [3, p.753].

Clearly,

\[ h(\omega) = O(\omega), \quad \text{as} \quad \omega \to 0. \] (50)

Assume that \( \hat{f}(\omega) \) has an asymptotic expansion of the form (18). Using (15) and (49) we get

\[ I_1(a) = \int_0^\infty e^{ib\hat{f}(\omega)} \frac{1}{a\omega} (1 - 2 e^{ia\omega/2} + e^{ia\omega}) d\omega \]

\[ = \frac{i}{a} F(b) - 2i \int_0^\infty e^{ib\hat{f}(\omega)} \frac{e^{ia\omega/2}}{a\omega} d\omega \]

\[ + i \int_0^\infty e^{ib\hat{f}(\omega)} \frac{e^{ia\omega}}{a\omega} d\omega, \]

where

\[ F(b) = \int_0^\infty e^{ib\hat{f}(\omega)} \frac{d\omega}{\omega}. \]

Then, (20) and the generalized Mellin transform formula [4, Lemma 2, p.198]:

\[ M[e^{it}; z] = e^{i\pi z/2} \Gamma(z) \]

we get

\[ I_1(a) = \frac{i}{a} F(b) - \frac{2i}{a} \int_0^\infty \left[ \sum_{s=1}^\infty d_s \omega^{s+\lambda-2} + g_n(\omega) \right] e^{ia\omega/2} d\omega \]

\[ + \frac{i}{a} \int_0^\infty \left[ \sum_{s=1}^\infty d_s \omega^{s+\lambda-2} + g_n(\omega) \right] e^{ia\omega} d\omega \]

\[ = \frac{i}{a} F(b) + \frac{2i}{a} \left\{ \sum_{s=1}^{n-1} d_s \Gamma(s + \lambda - 1)(a/2)^{-s-\lambda+1} e^{i\pi(s+\lambda)/2} \right. \]

\[ - (2i/a)^n \int_0^\infty g_n^{(n)}(\omega) e^{ia\omega/2} d\omega \]
\[
+ \frac{i}{a} \left\{ - \sum_{s=1}^{n-1} d_s \Gamma(s + \lambda - 1)(a)^{-s-\lambda+1} e^{i\pi(s+\lambda)/2} \right.
\]
\[
+ (i/a)^n \int_0^\infty g^{(n)}_n(\omega) e^{ia\omega} d\omega \right\}
\]
\[
= \frac{i}{a} F(b) + i \sum_{s=1}^{n-1} d_s \Gamma(s + \lambda - 1)a^{-s-\lambda}(2^{s+\lambda} - 1)e^{i\pi(s+\lambda)/2}
\]
\[
+ (i/a)^{n+1} \int_0^\infty g^{(n)}_n(\omega) (e^{ia\omega} - 2^{n+1} e^{ia\omega/2}) d\omega. \tag{51}
\]

Notice that for existence of the Mellin transform in the above case we have to assume that \(d_0 = 0\). Similarly,

\[
I_2(a) = \frac{i}{a} \int_{-\infty}^0 e^{ib\omega} \frac{\hat{f}(\omega)}{\omega} d\omega + i \sum_{s=0}^{n-1} d_s \Gamma(s + \lambda - 1)a^{-s-\lambda}(-1)^{s+\lambda-1}
\]
\[
\times (2^{s+\lambda} - 1)e^{i\pi(s+\lambda)/2} + (i/a)^{n+1} \int_0^\infty g^{(n)}_n(-\omega)
\]
\[
\times (e^{-ia\omega} - 2^{n+1} e^{-ia\omega/2}) d\omega. \tag{52}
\]

Finally, using formula \([2, (15), p.152]\), from (15), (51) and (52) we get

\[
(W_{\psi} f)(b, a) = \frac{i}{\sqrt{a}} f^{(-1)}(b) + \frac{i}{\pi} \sum_{s=0}^{n-1} d_s \Gamma(s + \lambda - 1)a^{-s-\lambda+1/2}
\]
\[
\times \{1 + (-1)^{s+\lambda-1}(2^{s+\lambda} - 1)e^{i\pi(s+\lambda)/2} + \delta_n(a), \tag{53}
\]

where

\[
f^{(-1)}(b) = (D^{-1} f)(b)
\]

and

\[
\delta_n(a) = (i/a)^{n+1} \sqrt{a} \int_0^\infty g^{(n)}_n(\omega) (e^{ia\omega} - 2^{n+1} e^{ia\omega/2}) d\omega. \tag{54}
\]

Existence theorem for (53) is as follows:

**Theorem 5.** Assume that \(\hat{f}(\omega)\) satisfies conditions of Theorem\([2]\). Then the result (53) holds with

\[
\delta_n(a) = (i/a)^{m+1} \sqrt{a} \int_{-\infty}^\infty g^{(m)}_n(\omega) (e^{ia\omega} - 2^{m+1} e^{ia\omega/2}) d\omega,
\]

where \(n\) is the smallest positive integer such that \(\lambda + n > m.\)
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R S Pathak
DST Center for Interdisciplinary Mathematical Sciences
Banaras Hindu University
Varanasi-221005, India
e-mail: ranshankarpathak@yahoo.com

Ashish Pathak
Department of Mathematics and Statistics
Dr. Harisingh Gour Central University
Sagar-470003, India.
e-mail: ashishpathak@dhsgsu.ac.in