Ricci curvature of Cayley graphs for dihedral groups, generalized quaternion groups, and cyclic groups

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Abstract

Lin, Lu, and Yau formulated the Ricci curvature of edges in simple undirected graphs[3]. Using their formulation, we calculate the Ricci curvatures of Cayley graphs for the dihedral groups, the general quaternion groups, and the cyclic groups with some generating sets chosen so that their cardinal numbers are less than or equal to four. For the dihedral group and the general quaternion group, we obtained the Ricci curvatures of all edges of the Cayley graph with generator sets consisting of the four elements that are the two generators defining each group and their inverses elements. For the cyclic group \((\mathbb{Z}/n\mathbb{Z}, +)\), we have the Ricci curvatures of edges of the Cayley graph generating by \(S_{1,k} = \{+1, -1, +k, -k\}\).

1 Introduction

Yann Ollivier defined a new way to obtain the Ricci curvature on the metric space in his paper[6]. The Ollivier Ricci curvature method was also applied to graph theory by Lin, Lu, and Yau[3]. Lin, Lu, and Yau convert the Ollivier Ricci curvature into the Ricci curvature. In the papers[3][9], they provided the Ricci curvature of graphs such as a path, a cycle, and a complete graph.

In this paper, we calculate the Ricci curvatures of Cayley graphs for the dihedral groups, the general quaternion groups, and the cyclic groups with some generating sets chosen so that their cardinal numbers are less than or equal to four. The results are shown in the following table 1 - 10.

The dihedral groups are defined by \(D_n =< \sigma, \tau | \sigma^n = \tau^2 = e, \tau \sigma = \sigma^{n-1} \tau >\), where \(n \geq 3\). We investigate the Ricci curvatures of Cayley graphs for the dihedral group of the minimum generator set \(S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}\). \(S\) makes two types of edges in the Cayley graphs. One type of the edge set is \(A = \{ (g, g\sigma) | g \in D_n \}\), and the other is the edge set \(B = \{ (g, g\tau) | g \in D_n \}\). The results of the Ricci curvatures are given in Table 1.
Table 1: The Ricci curvature of the Cayley graphs of the dihedral group generated set $S$.

| Cayley graph | $\Gamma(D_3, S)$ | $\Gamma(D_4, S)$ | $\Gamma(D_5, S)$ | $\Gamma(D_n, S)(n \geq 6)$ |
|--------------|------------------|------------------|------------------|------------------|
| Ricci curvature (type A) | 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | 0 |
| Ricci curvature (type B) | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |

The generalized quaternion groups $Q_{4m}$ are defined by $Q_{4m} = \langle \sigma, \tau \mid \sigma^{2m} = e, \tau^2 = \sigma^m, \tau^{-1}\sigma \tau = \sigma^{-1} \rangle$, where $m \geq 2$. We consider the Cayley graphs for the generalized quaternion groups with the generating set $S = \{\sigma, \tau, \sigma^{-1}, \tau^{-1}\}$. We distinguish between the two sets of edges. One is the edge set $A = \{(g, g\sigma) \mid g \in Q_{4m}\}$, and the other is the edge set $B = \{(g, g\tau) \mid g \in Q_{4m}\}$. The results are given in Table 2.

Table 2: The Ricci curvature of the Cayley graphs of $Q_{4m}$ generated set $S$.

| Cayley graph | $\Gamma(Q_8, S)$ | $\Gamma(Q_{12}, S)$ | $\Gamma(Q_{4m}, S)(m \geq 4)$ |
|--------------|------------------|------------------|------------------|
| Ricci curvature (type A) | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| Ricci curvature (type B) | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |

We consider the Cayley graph for the cyclic group $(\mathbb{Z}/n\mathbb{Z}, +)$ with a generating set $S_{1,k} = \{+1, +k, -1, -k\}$, where $k$ is a positive integer not equal to 1. We distinguish between the two sets of edges. One is the edges $A = \{(g, g+1) \mid g \in \mathbb{Z}/n\mathbb{Z}\}$, the other is edges $B = \{(g, g+k) \mid g \in \mathbb{Z}/n\mathbb{Z}\}$. The Ricci curvatures of the Cayley graphs for the cyclic group $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,2})$ ($6 \leq n \leq 10$) with the generating set $S_{1,2} = \{+1, +2, -1, -2\}$ are given in Table 3.

Table 3: The Ricci curvature of the Cayley graphs of $\mathbb{Z}/n\mathbb{Z}$ with $S_{1,2}$.

| $n$ | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|----|
| Ricci curvature (type A) | 1 | 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| Ricci curvature (type B) | 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |

The Ricci curvatures of the Cayley graphs $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,2})$ for $n \geq 11$ are given in Table 4.

Table 4: The Ricci curvature of the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ ($n \geq 11$) with $S_{1,2}$.

| Cayley graph | $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,2})(n \geq 11)$ |
|--------------|------------------|
| Ricci curvature (type A) | $\frac{2}{3}$ |
| Ricci curvature (type B) | 0 |

The Ricci curvatures of the Cayley graphs $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,3})$ with the generating set $S_{1,3} = \{+1, +3, -1, -3\}$ ($6 \leq n \leq 15$) are given in Table 5.

Table 5: The Ricci curvature of the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ ($6 \leq n \leq 15$) generated by $S_{1,3}$.

| $n$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|----|----|----|----|----|----|
| Ricci curvature (type A) | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| Ricci curvature (type B) | $\frac{2}{3}$ | 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 0 | $\frac{1}{2}$ |
The Ricci curvatures of the Cayley graphs $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,3})$ with the generating set $S_{1,3} = \{+1, +3, -1, -3\}$ ($n \geq 16$) are given in Table 6.

Table 6: The Ricci curvature of the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ ($n \geq 16$) generated set $S_{1,3}$.

| Cayley graph $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,3})$ ($n \geq 16$) | Ricci curvature (type A) | Ricci curvature (type B) |
|---|---|---|
| | | |

The Ricci curvatures of the Cayley graphs $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,4})$ of generating set $S_{1,4} = \{+1, +4, -1, -4\}$ ($6 \leq n \leq 22$) are given in Table 7.

Table 7: The Ricci curvature of the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ generated by $S_{1,4}$.

| $n$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ricci curvature (type A) | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| Ricci curvature (type B) | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ |

The Ricci curvatures of the Cayley graphs $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,4})$ of generating set $S_{1,4} = \{+1, +4, -1, -4\}$ ($n \geq 23$) are given in Table 8.

Table 8: The Ricci curvature of the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ ($n \geq 23$) generated set $S_{1,4}$.

| Cayley graph $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,4})$ ($n \geq 23$) | Ricci curvature (type A) | Ricci curvature (type B) |
|---|---|---|
| | | |

The Ricci curvature of the edges of the Cayley graph $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,5})$ with generating set $S_{1,5} = \{+1, +5, -1, -5\}$ for $7 \leq n \leq 25$ are given in Table 9.

Table 9: The Ricci curvature of the Cayley graph that can be generated by $S_{1,5}$.

| $n$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Type A | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Type B | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The Ricci curvatures of the Cayley graphs $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,k})$ with conditions written in Theorem 5 are zero in Table 10. The conditions for type A are $k \geq 5$, $n \neq 3k - 2$, and $n \geq 2k + 4$. The Ricci curvatures of type B edges vanish in the following 4 cases: 1. $k \geq 5$ and $3k + 3 \leq n \leq 4k - 2$, 2. $k \geq 3$ and $4k + 2 \leq n \leq 5k - 1$, 3. $k \geq 3$ and $n \geq 5k + 1$, 4. $k \geq 6$ and $2k + 4 \leq n \leq 3k - 3$.

Table 10: The Ricci curvature of the Cayley graph of the cyclic group generated set $S_{1,k}$.

| Cayley graph $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,k})$ with the above conditions | Ricci curvature (type A) | Ricci curvature (type B) |
|---|---|---|
| | | |

3
There are edges that do not satisfy the above any conditions 1-4 in Theorem 5. Such edges have non-zero Ricci curvature in general and their values depend on relations between $n$ and $k$. Therefore, determining the curvature of these edges does not seem to be an easy task, as it requires an in-depth discussion of the detailed structure of the Cayley graph, which is determined by $n$ and $k$. It remains a future work.

This paper is organized as follows. In Section 2 we prepare some definitions and some theorems used in this paper. The Ricci curvatures of Cayley graphs are calculated for dihedral groups in Section 3 for generalized quaternion groups in Section 4 and for cyclic groups in Section 5.

2 Preparations

We review the definitions of the Ricci curvature and the Cayley graphs and related theorems to prepare for the calculations in the following sections.

2.1 Ricci curvature of graph

Let $G$ be a simple undirected graph with vertex set $V$ and edge set $E$, $N_G(x)$ be the set of neighbors of $x$, $\text{deg}_G(x) = |N_G(x)|$ be degree of vertex $x$, and $d(x, y)$ be the distance between $x$ and $y \in V$.

**Definition 1** \cite{6} \cite{3} \cite{7} Let $G$ be a connected graph. A probability measure $\mu$ on $G$ is defined as a function $\mu : V \rightarrow [0, 1]$ with

$$\sum_{x \in V} \mu(x) = 1.$$  
(1)

The set of all probability measure on $G$ is denoted by $P(V)$.

**Definition 2** \cite{6} \cite{3} \cite{7} Let $G = (V, E)$ be a connected graph and $\mu, \nu \in P(V)$ be two probability measures. The transport plan from $\mu$ to $\nu$ is a map $\pi : V \times V \rightarrow [0, 1]$ such that

$$\mu(x) = \sum_{y \in V} \pi(x, y) \quad \forall x \in V,$$  
(2)

and

$$\nu(y) = \sum_{x \in V} \pi(x, y) \quad \forall y \in V.$$  
(3)

The value $\pi(x, y)$ is the mass transported from $x$ to $y$ along a geodesic (at cost $d(x, y) \pi(x, y)$), and total cost of the transport plan $\pi$ is defined as

$$\text{cost}(\pi) := \sum_{x, y \in V} d(x, y) \pi(x, y).$$  
(4)
We call \( d(x, y)\pi(x, y) \) “transport cost from \( x \) to \( y \)”, and we describe it as \( x \to y : d(x, y)\pi(x, y) \) for \( x, y \) in \( V \). We denote the set of all transport plans from \( \mu \) to \( \nu \) by \( \Pi(\mu, \nu) \). We define 1-Wasserstein distance between \( \mu \) to \( \nu \) as
\[
W_1(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \text{cost}(\pi). 
\]
(5)

A transport plan \( \pi \in \Pi \) is called optimal if we have
\[
W_1(\mu, \nu) = \text{cost}(\pi). 
\]
(6)

**Definition 3** [6][3][7] Let \( G = (V, E) \) be a connected graph. A function \( f \) over the vertex set \( V \) of \( G \) is said to 1-Lipschitz if
\[
|f(u) - f(v)| \leq d(u, v) \quad \forall u, v \in V. 
\]
(7)

The following theorem plays an essential role in this study.

**Theorem 1** [6][3] Let \( G = (V, E) \) be a connected graph. For \( \mu, \nu \in P(V) \),
\[
W_1(\mu, \nu) = \sup_{f \in 1-Lip(G)} \sum_{x \in V} f(x)(\mu(x) - \nu(x)). 
\]
(8)

**Definition 4** [6][3][7] A probability measure is a function \( \mu : V \to [0, 1] \) with \( \sum_{x \in V} \mu(x) = 1 \).

To define Ricci curvature for each edge of a graph in this paper, we only consider probability measures \( \mu^\alpha \) in the following form,
\[
\mu^\alpha_x(v) = \begin{cases} 
\alpha & (v = x), \\
\frac{1-\alpha}{\deg_G(x)} & (v \in N_G(x)), \\
0 & (\text{otherwise})..
\end{cases}
\]
(9)

Here, \( x, v \in V \) and \( \alpha \in [0, 1] \).

**Definition 5** [6] Let \( G = (V, E) \) be a connected graph. For any value \( \alpha \in [0, 1] \) and \( x, y \in V \), we define the Ollivier Ricci curvature \( \kappa^\alpha(x, y) \) as
\[
\kappa^\alpha(x, y) := 1 - \frac{W_1(\mu^\alpha_x, \mu^\alpha_y)}{d(x, y)}. 
\]
(10)

The Ricci curvature is definable on any undirected pair of vertices \( x \) and \( y \).

**Definition 6** [3] The Ricci curvature is defined from the Ollivier Ricci curvature;
\[
\kappa(x, y) = \lim_{\alpha \to 1} \frac{\kappa^\alpha(x, y)}{1 - \alpha}. 
\]
(11)
2.2 Cayley graph

This paper deals with the Cayley graph as an undirected graph.

Definition 7 Let $\mathcal{G}$ be a group. Let $S \subset \mathcal{G}$ be a generating set of $\mathcal{G}$ such that $e \notin S$ and $S = S^{-1}$. Here $S = S^{-1}$ means that $s^{-1} \in S$ if $s \in S$, and we call this $S$ symmetric. If a graph satisfies the following conditions, then the graph is called a Cayley graph, and the graph is written by $\Gamma(\mathcal{G}, S)$:

1. There exist a bijection between the group $\mathcal{G}$ and the set of vertices of graph $\Gamma(\mathcal{G}, S)$. The vertex corresponding to $g$ is labeled as $g$.
2. The set of edges of graph $\Gamma(\mathcal{G}, S)$ is defined as $\{(g, gs) \mid g \in \mathcal{G}, s \in S\}$.

For example, refer to Magnus’ book, which provides a detailed introduction to the properties of Cayley graphs [4].

In this paper, we calculate the Ricci curvature of Cayley graphs defined by dihedral groups, general quaternion groups, and cyclic groups.

3 The Ricci curvature of the Cayley graph of dihedral groups

The purpose of this section is to calculate the Ricci curvatures for edges in Cayley graphs for dihedral groups.

A dihedral group is related to the congruent transformation of regular polygons. The dihedral group is generated by two generators $\sigma, \tau$. $\sigma$ represents the action of rotation of the angle $\frac{\pi}{n}$ around the center of gravity, and $\tau$ represents the transformation of reflection around a certain axis.

Definition 8 The dihedral group is defined by $D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = e, \tau\sigma = \sigma^{n-1}\tau \rangle$, where $n \geq 3$.

We investigate Ricci curvatures of Cayley graphs for the dihedral group of the generating set $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$. Here, the generating set $S$ makes two types of edges in Cayley graphs. One type of the edge set is $A = \{(g, g\sigma) \mid g \in D_n\}$, and the other is the edge set $B = \{(g, g\tau) \mid g \in D_n\}$. Now, we call the edge in set $A$ and $B$ type $A$ and type $B$, respectively.

At first, let us consider Ricci curvatures of the Cayley graph for the dihedral group $D_3$ with the generating set $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$.

Proposition 1 Let $\Gamma(D_3, S)$ be the Cayley graph of the dihedral group $D_3$ with $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$. The Ricci curvature of any type $A$ edge in $\Gamma(D_3, S)$ is $\kappa = 1$. The Ricci curvature of any type $B$ edge in $\Gamma(D_3, S)$ is $\kappa = \frac{2}{3}$. 
Proof. First, we prove that the Ricci curvature of the edge $xy$ in Figure 1 as a type A edge is $\kappa = 1$.

We consider the transport cost from $\mu_x^\alpha$ to $\mu_y^\alpha$. By Definition 4, the vertex $x$ has probability $\alpha$ for $\mu_x^\alpha$. Each probability of vertex $y$, $a$ and $b$ is $\frac{1-\alpha}{3}$. On the other hand, vertex $y$ has probability $\alpha$ for $\mu_y^\alpha$. Each probability of vertex $x$, $b$ and $c$ is $\frac{1-\alpha}{3}$. Satisfying Definition 2, we can provide the following transport cost: $x \rightarrow y : 1 \times \pi(x,y) = \alpha - \frac{1-\alpha}{3}$, $a \rightarrow c : 1 \times \pi(a,c) = \frac{1-\alpha}{3}$, and the amount of transport between the other vertices is zero.

From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha. \quad (12)$$

From Definition 5 we have the following result about the Ollivier Ricci curvature.

$$\kappa_\alpha(x,y) \geq (1 - \alpha). \quad (13)$$

By Definition 6 we have the following result of the lower bound of the Ricci curvature.

$$\kappa(x,y) \geq 1. \quad (14)$$

Next, using a 1-Lipschtiz function, we estimate the upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschtiz function as Figure 1. The number in each box beside each vertex is the value of the 1-Lipschiz function. We have the following result from Theorem 1 and this 1- Lipschiz function.

$$W_1 \geq \alpha. \quad (15)$$

From Definition 5 we have the following the Ollivier Ricci curvature.

$$\kappa_\alpha(x,y) \leq (1 - \alpha). \quad (16)$$

Therefore, we have the upper bound of the Ricci curvature by Definition 6.

$$\kappa(x,y) \leq 1. \quad (17)$$
By (14) and (17),

$$\kappa(x, y) = 1.$$  \hspace{1cm} (18)

Thus, the curvature of all type A edges is $$\kappa = 1$$.

Next, we prove that the Ricci curvature of the edge $$xy$$ in Figure 2 as a type B edge is $$\kappa = \frac{2}{3}$$. We consider the transport cost from $$\mu_x^\alpha$$ to $$\mu_y^\alpha$$. By Definition 4, vertex $$x$$ has probability $$\alpha$$ for $$\mu_x^\alpha$$. Each probability of vertex $$y$$, $$a$$ and $$b$$ is $$\frac{1-\alpha}{3}$$. On the other hand, vertex $$y$$ has probability $$\alpha$$ for $$\mu_y^\alpha$$. Each probability of vertex $$x$$, $$c$$ and $$d$$ is $$\frac{1-\alpha}{3}$$. Satisfying Definition 2, we can provide the following transport cost: $$x \to y : 1 \times \pi(x, y) = \alpha - \frac{1-\alpha}{3}, a \to c : 1 \times \pi(a, c) = \frac{1-\alpha}{3}, b \to d : 1 \times \pi(b, d) = \frac{1-\alpha}{3}$$, and the amount of transport between the other vertices is zero. From (19), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + \frac{1}{3}(1 - \alpha).$$  \hspace{1cm} (19)

From Definition 5 we have the following result about the Ollivier Ricci curvature.

$$\kappa_\alpha(x, y) \geq \frac{2}{3}(1 - \alpha).$$  \hspace{1cm} (20)

By Definition 6 we obtain a lower bound of the Ricci curvature.

$$\kappa(x, y) \geq \frac{2}{3}.$$  \hspace{1cm} (21)

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 2. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

$$W_1 \geq \alpha + \frac{1}{3}(1 - \alpha).$$  \hspace{1cm} (22)

From Definition 5 we have an upper bound for the Ollivier Ricci curvature

$$\kappa_\alpha(x, y) \leq \frac{2}{3}(1 - \alpha).$$  \hspace{1cm} (23)

Therefore, we have an upper bound of the Ricci curvature by Definition 6

$$\kappa(x, y) \leq \frac{2}{3}.$$  \hspace{1cm} (24)

By (21) and (24),

$$\kappa(x, y) = \frac{2}{3}.$$  \hspace{1cm} (25)
Thus, the curvature of all type B edges is $\kappa = \frac{2}{3}$.

Similarly, let us consider the Ricci curvature of the Cayley graph for dihedral group $D_4$, $D_5$, $D_6$ with a minimum generating set $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$.

**Proposition 2** Let $\Gamma(D_4, S)$ be a Cayley graph of dihedral group $D_4$ with $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$. The Ricci curvature $\kappa$ of any edge in $\Gamma(D_4, S)$ is $\frac{2}{3}$. The proof of this proposition is given in Appendix A.1.

**Proposition 3** Let $\Gamma(D_5, S)$ be a Cayley graph of dihedral group $D_5$ with $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$. The Ricci curvature of any type A edge in $\Gamma(D_5, S)$ is $\kappa = \frac{1}{3}$. The Ricci curvature of any type B edge in $\Gamma(D_5, S)$ is $\kappa = \frac{2}{3}$. The proof of this proposition is given in Appendix A.2.

**Proposition 4** Let $\Gamma(D_6, S)$ be a Cayley graph of dihedral group $D_6$ with $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$. The Ricci curvature of the edges in Cayley graph $\Gamma(D_6, S)$.

Ricci curvature of type A is $\kappa = 0$.

Ricci curvature of type B is $\kappa = \frac{2}{3}$.

The proof of this proposition is given in Appendix A.3.

**Theorem 2** Let $\Gamma(D_n, S)$ be a Cayley graph of dihedral group $D_n$ with $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$ for $n \geq 6$. The Ricci curvature of any type A edge in $\Gamma(D_n, S)$ is $\kappa = 0$. The Ricci curvature of any type B edge in $\Gamma(D_n, S)$ is $\kappa = \frac{2}{3}$.

**Proof.** First, we prove that the Ricci curvature of the edge $bc$ in Figure 3 as a type A edge in $\Gamma(D_n, S)$ is $\kappa = 0$.

Figure 3: Type A in the Cayley graph of $D_n$  Figure 4: Type B in the Cayley graph of $D_n$

Let us consider transport costs. By Definition 3, vertex $b$ has probability $\alpha$, and each vertex $a, c$ and $g$ has probability $\frac{1-\alpha}{3}$ for $\mu_b^\alpha$. $\mu_c^\alpha$ is determined by the similar way. Satisfying
Definition 2, we can provide the following transport costs: $a \rightarrow b : \frac{1-\alpha}{3}$, $b \rightarrow c : \alpha$, $c \rightarrow d : \frac{1-\alpha}{3}$, $g \rightarrow h : \frac{1-\alpha}{3}$, and the amount of transport between the other vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + (1 - \alpha).$$  \hspace{1cm} (26)

From Definition 5, $\kappa_\alpha \geq 0$ about the Ollivier Ricci curvature. By Definition 6, we have the following result of a lower bound of the Ricci curvature.

$$\kappa \geq 0.$$  \hspace{1cm} (27)

Next, using a 1-Lipschtiz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschtiz function as Figure 3. The number in each box beside each vertex is the value of the 1-Lipschiz function. The condition $n \geq 6$ is used to define this 1-Lipschiz function. We have the following result from Theorem 1 and this 1-Lipschiz function.

$$W_1 \geq \alpha + (1 - \alpha).$$  \hspace{1cm} (28)

From Definition 5, $\kappa_\alpha \leq 0$ about the Ollivier Ricci curvature. Therefore, we have the following upper bound of the Ricci curvature.

$$\kappa \leq 0.$$  \hspace{1cm} (29)

By (27) and (29), the curvature of all type A edges is $\kappa = 0$.

Next, we prove that the Ricci curvature of type B is $\kappa = \frac{2}{3}$. Now, we calculate the Ricci curvature of the edge $bg$, which is the type B, in Figure 4. We consider transport costs. By Definition 4, vertex $b$ has probability $\alpha$, and each vertex $a, c$ and $g$ has probability $\frac{1-\alpha}{3}$ for $\mu_\alpha$. $\mu_\alpha^g$ is determined similarly. Satisfying Definition 2, we can provide the following transport costs: $b \rightarrow g : \alpha - \frac{1-\alpha}{3}$, $a \rightarrow f : \frac{1-a}{3}$, $c \rightarrow h : \frac{1-a}{3}$, and the amount of transport between the other vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W \leq \alpha + \frac{1-\alpha}{3}.$$  \hspace{1cm} (30)

From Definition 5, $\kappa_\alpha \geq 2 \times \frac{1-\alpha}{3}$ about the Ollivier Ricci curvature. By Definition 6, we have the following result of the lower bound of the Ricci curvature for type B.

$$\kappa \geq \frac{2}{3}.$$  \hspace{1cm} (31)

Next, using a 1-Lipschtiz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschtiz function as Figure 4. The number in each box beside each vertex is the value of the 1-Lipschiz function. We have the following result from Theorem 1 and this 1-Lipschiz function.
\[ W_1 \geq \alpha + \frac{1}{3} (1 - \alpha). \]  
(32)

From Definition 5 \( \kappa_\alpha \leq \frac{2}{3} (1 - \alpha) \) about the Ollivier Ricci curvature. Therefore, we have the following upper bound of the Ricci curvature by Definition 6,

\[ \kappa \leq \frac{2}{3}. \]  
(33)

By (31) and (33), the Ricci curvature of all type B edges is \( \kappa = \frac{2}{3} \).

\[ \square \]

4 The Ricci curvature of the Cayley graph for generalized quaternion groups

4.1 generalized quaternion group

\( Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle \) is known as the quaternion group. Its generalization is as follows.

Definition 9 [4] The generalized quaternion group \( Q_{4m} \) is defined by

\[ Q_{4m} = \langle \sigma, \tau \mid \sigma^{2m} = e, \tau^2 = \sigma^m, \tau^{-1} \sigma \tau = \sigma^{-1} \rangle, \text{where } m > 1. \]

We consider the Cayley graph for the generalized quaternion group with the generating set \( S = \{ \sigma, \tau, \sigma^{-1}, \tau^{-1} \} \).

We distinguish between the two sets of edges. One is the edge set \( A = \{ (g, g\sigma) \mid g \in Q_{4m} \} \), and the other is edge set \( B = \{ (g, g\tau) \mid g \in Q_{4m} \} \). We call the edge in sets \( A \) and \( B \) type \( A \) and type \( B \), respectively.

4.2 Ricci curvature of Cayley graph for generalized quaternion group \( Q_8 \) with the generating set \( S = \{ \sigma, \tau, \sigma^{-1}, \tau^{-1} \} \)

Proposition 5 For the Cayley graph \( \Gamma(Q_8, S) \), The Ricci curvature \( \kappa \) of any edge in \( \Gamma(Q_8, S) \) is \( \frac{1}{2} \).

The proof of this proposition is given in Appendix A.4.

Proposition 6 For the Cayley graph \( \Gamma(Q_{12}, S) \), The Ricci curvature of any type A edge in \( \Gamma(Q_{12}, S) \) is \( \kappa = \frac{1}{4} \), and the Ricci curvature of any type B edge in \( \Gamma(Q_{12}, S) \) is \( \kappa = \frac{1}{2} \).

The proof of this proposition is given in Appendix A.5.

The Ricci curvature of Cayley graph for generalized quaternion group \( Q_{4m}(m \geq 4) \) with the generating set \( S = \{ \sigma, \tau, \sigma^{-1}, \tau^{-1} \} \) is given as follows:
Theorem 3 For the Cayley graph $\Gamma(Q_{4m}, S)$ with $m \geq 4$, the Ricci curvature of any type A edge in $\Gamma(Q_{4m}, S)$ is $\kappa = 0$, and the Ricci curvature of any type B edge in $\Gamma(Q_{4m}, S)$ is $\kappa = \frac{1}{2}$.

Proof.

First, we prove that the Ricci curvature of type A is $\kappa = 0$.

Now, we calculate the Ricci curvature of the edge $xy$, which is the type A, i.e. $y = x\sigma$ in Figure 5. Satisfying Definition 2, we can provide the following transport cost between $\mu_x$ and $\mu_y$:

- $a \rightarrow x$: $\frac{1-\alpha}{4}$, $x \rightarrow y$: $\alpha$, $y \rightarrow e$: $\frac{1-\alpha}{4}$, $c \rightarrow f$: $\frac{1-\alpha}{4}$, $b \rightarrow d$: $\frac{1-\alpha}{4}$, and the amount of transport between the other vertices is zero.

From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result:

$$W_1 \leq \alpha + (1 - \alpha).$$

(34)

From Definition 5, $\kappa \geq 0$ about the Ollivier Ricci curvature. By Definition 6, we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq 0.$$ 

(35)

Next, using a 1-Lipschtiz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 5. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

$$W_1 \geq \alpha + (1 - \alpha).$$

(36)

From Definition 5, $\kappa \leq 0$ about the Ollivier Ricci curvature. Therefore, we have the following upper bound of the Ricci curvature.

$$\kappa \leq 0.$$ 

(37)

By (35) and (37), the curvature of all type A edges is $\kappa = 0$. 

Figure 5: Type A in the Cayley graph of $Q_{4m}$  Figure 6: Type B in the Cayley graph of $Q_{4m}$
Next, we prove that the Ricci curvature of type B is \( \kappa = 0 \). Now, we calculate the Ricci curvature of the edge \( xy \), which is the type B as in Figure 6. For \( \mu_x^0 \) and \( \mu_y^0 \), satisfying Definition 2, we can provide the following transport cost:

- \( x \rightarrow y : \alpha - \frac{1 - \alpha}{4} \),
- \( c \rightarrow d : \frac{1 - \alpha}{4} \),
- \( a \rightarrow f : \frac{1 - \alpha}{4} \),
- \( b \rightarrow e : \frac{1 - \alpha}{4} \),

and the amount of transport between the other vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

\[
W_1 \leq \alpha + \frac{1 - \alpha}{2}.
\] (38)

From Definition 5 \( \kappa_\alpha \geq \frac{1 - \alpha}{2} \) about the Ollivier Ricci curvature. By Definition 6 we have the following result of a lower bound of the Ricci curvature.

\[
\kappa \geq \frac{1}{2}.
\] (39)

Next, using a 1-Lipschtiz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschtiz function as Figure 6. The number in each box beside each vertex is the value of the 1-Lipschiz function. We have the following result from Theorem 1 and this 1- Lipschiz function.

\[
W_1 \geq \alpha + \frac{1}{2} (1 - \alpha).
\] (40)

From Definition 5 \( \kappa_\alpha \leq \frac{1}{2} (1 - \alpha) \) about the Ollivier Ricci curvature. Therefore, we have the following upper bound of the Ricci curvature.

\[
\kappa \leq \frac{1}{2}.
\] (41)

By (39) and (41), the curvature of all type B edges is \( \kappa = \frac{1}{2} \). This is the same result for all other type B edges. \( \square \)

5 The Ricci curvature of the Cayley graph for cyclic groups

5.1 Cyclic group and Cayley graph

Every infinite cyclic group is isomorphic to the additive group of \( \mathbb{Z} \), the integers. Every finite cyclic group of order \( n \) is isomorphic to the additive group of \( \mathbb{Z}/n\mathbb{Z} \), the integers modulo \( n \). Every cyclic group is an abelian group meaning that its group operation is commutative, and every finitely generated abelian group is a direct product of cyclic groups.

We consider the Cayley graph for the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) with a generating set \( S_{1,k} = \{+1,+k,-1,-k\} \), where \( k \) is a positive integer not equal to 1. We distinguish between the two sets of edges. One is the edges \( A = \{(g,g+1) \mid g \in \mathbb{Z}/n\mathbb{Z}\} \), the other is edges \( B = \)
\[(g, g + k) \mid g \in \mathbb{Z}/n\mathbb{Z}\]. We call the edge in sets \(A\) and \(B\) type \(A\) and type \(B\), respectively.

In the following, Ricci curvatures of the type \(A\) and type \(B\) edges for these Cayley graphs are discussed. It shall be noted that results for the Ricci curvature of certain Cayley graphs related to \(\mathbb{Z}_n\) have been obtained in [1]. However, the Cayley graphs discussed therein are different from those in this paper.

5.2 The Ricci curvature of the Cayley graph for the cyclic group \(\Gamma(\mathbb{Z}/n\mathbb{Z}, S_1)\) with the generating set \(S_1 = \{+1, -1\}\)

Before considering the cases for \(S_1, k\), we make comments about the Ricci curvatures of Cayley graphs for \(\Gamma(\mathbb{Z}/n\mathbb{Z})\) with \(S_1 = \{+1, -1\}\). For the case of \(\Gamma(\mathbb{Z}/n\mathbb{Z})\) with \(S_1\), if \(n\) is equal to 1, then the Cayley graph consists of an isolated vertex. At this time, there are no edges. For \(n = 2, 3\) cases, we know a useful result.

Theorem 4 [9] The complete graph \(K_n\) has constant the Ricci curvature \(\kappa = \frac{n}{n-1}\).

By Theorem [4] we can calculate the following cases. If \(n = 2\), the Cayley graph is a complete graph \(K_2\) with 2 vertices having the Ricci curvature \(\kappa = 2\). If \(n = 3\), the Cayley graph is a complete graph \(K_3\) with 3 vertices having the Ricci curvature \(\kappa\) is equal to \(\frac{3}{2}\). Also, We know the following results in [3][7]. If \(n = 4\), the Cayley graph is a cycle graph \(C_4\) with 4 vertices, then the Ricci curvature \(\kappa = 1\). If \(n = 5\), the Cayley graph is a cycle graph \(C_5\) with 5 vertices having the Ricci curvature \(\kappa = \frac{1}{2}\). If \(n \geq 6\), the Cayley graph is a cycle \(C_n\) with length \(n\) having the Ricci curvature \(\kappa = 0\).

5.3 The Ricci curvature of the Cayley graph for the cyclic group \(\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,2})\) with the generating set \(S_{1,2} = \{+1, +2 - 1, -2\}\)

For \(n = 3\), \(S_{1,2} = S_1\), and then the Cayley graph is given by \(K_3\) discussed in section 5.2. Let us consider \(\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,2})\) for the cases \(n = 4, 5\), at first. For these cases, Theorem [4] gives the Ricci curvatures. If \(n = 4\), the Cayley graph is a complete graph \(K_4\) with 4 vertices having the Ricci curvature \(\kappa = \frac{1}{3}\). If \(n = 5\), the Cayley graph is a complete graph \(K_5\) with 5 vertices having the Ricci curvature \(\kappa = \frac{1}{4}\). For \(n \geq 6\), we have to distinguish between type \(A\) and type \(B\) edges when we calculate the Ricci curvature of their graphs.

Proposition 7 The Ricci curvatures of the Cayley graphs for the cyclic group \(\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,2})\) for \(6 \leq n \leq 10\) with the generating set \(S_{1,2} = \{+1, +2, -1, -2\}\) are given in Table 11.

![Table 11: The Ricci curvature of the Cayley graph generated by \(S_{1,2}\).](image)

The proof is omitted because it is obtained easily in similar ways as them in the previous sections.
Proposition 8 Let $\Gamma(Z/nZ, S_{1,2})$ be a Cayley graph of $Z/nZ$ for $n \geq 11$ with the generating set $S_{1,2}$. The Ricci curvature of any type A edge in $\Gamma(Z/nZ, S_{1,2})$ is $\kappa = \frac{1}{2}$. The Ricci curvature of any type B edge in $\Gamma(Z/nZ, S_{1,2})$ is $\kappa = 0$.

The proof of this proposition is given in Appendix A.6.

5.4 The Ricci curvature of the Cayley graph for the cyclic group $\Gamma(Z/nZ, S_{1,3})$ with the generating set $S_{1,3} = \{+1, +3, -1, -3\}$

By Theorem 4, if we can calculate the following case $n = 5$, the Cayley graph is a complete graph $K_5$ with 5 vertices having the Ricci curvature $\kappa = \frac{5}{4}$.

Proposition 9 The Ricci curvatures of the Cayley graphs for the cyclic group $\Gamma(Z/nZ, S_{1,3})$ with the generating set $S_{1,3} = \{+1, +3, -1, -3\}$ for $6 \leq n \leq 15$ are given in Table 12.

| n  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|----|----|----|----|----|----|----|----|----|----|----|
| Type A | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ |
| Type B | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

The proof is obtained from direct calculations.

Proposition 10 Let $\Gamma(Z/nZ, S_{1,3})(n \geq 16)$ be a Cayley graph with the generating set $S_{1,3}$. The Ricci curvature of is any type A edge in $\Gamma(Z/nZ, S_{1,3})$ is $\kappa = \frac{1}{2}$. The Ricci curvature of any type B edge in $\Gamma(Z/nZ, S_{1,3})$ is $\kappa = 0$.

The proof of this proposition is given in Appendix A.7.

5.5 The Ricci curvature of the Cayley graph for the cyclic group $\Gamma(Z/nZ, S_{1,4})$ with the generating set $S_{1,4} = \{+1, +4, -1, -4\}$

For the case $n = 5$ and $S_{1,4}$, the Cayley graph is a cycle $C_5$ with 5 vertices having $\kappa = \frac{1}{2}$ as mentioned in Subsection 5.2.

Proposition 11 The Ricci curvatures of the Cayley graph for the cyclic group $\Gamma(Z/nZ, S_{1,4})$ of generating set $S_{1,4} = \{+1, +4, -1, -4\}$ with $6 \leq n \leq 22$ are given in Table 13.

| n  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Type A | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ |
| Type B | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

The proof is obtained from direct calculations.
Proposition 12 Let $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,4}) (n \geq 23)$ be a Cayley graph with the generating set $S_{1,4}$. The Ricci curvature of any type A edge in $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,4})$ is $\kappa = \frac{1}{4}$. The Ricci curvature of any type B edge in $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,4})$ is $\kappa = 0$.

The proof of this proposition is given in Appendix A.8.

5.6 The Ricci curvature of the Cayley graph for the cyclic group $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,5})$ with the generating set $S_{1,5} = \{+1, +5, -1, -5\}$

Proposition 13 The Ricci curvature of the edges of the Cayley graph for the cyclic group $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,5}) (7 \leq n \leq 25)$ with the generating set $S_{1,5} = \{+1, +5, -1, -5\}$ are given in Table 14.

Table 14: The Ricci curvature of the Cayley graph that can be generated by $S_{1,5}$

| $n$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|-----|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Type A | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| Type B | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

The proof is obtained from direct calculations. For $n \geq 26$, Theorem 5 gives the value of Ricci curvature in the following subsection.

5.7 The Ricci curvature of the Cayley graph for the cyclic group $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,k})$ with the generating set $S_{1,k} = \{+1, +k, -1, -k\}$

Theorem 5 Let $n$ and $k$ be positive integers satisfying $n > k$. Consider the Cayley graph for the cyclic group $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,k})$.

If $k \geq 5$, $n \neq 3k-2$, and $n \geq 2k+4$, then the Ricci curvature of any type A edge in $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,k})$ is $\kappa = 0$. The Ricci curvature of any type B edge in $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,k})$ is $\kappa = 0$ if $n$ and $k$ satisfy any one of the following conditions:

1. $k \geq 5$ and $3k + 3 \leq n \leq 4k - 2$, (42)
2. $k \geq 3$ and $4k + 2 \leq n \leq 5k - 1$, (43)
3. $k \geq 3$ and $n \geq 5k + 1$, (44)
4. $k \geq 6$ and $2k + 4 \leq n \leq 3k - 3$. (45)
Proof.
First, we calculate the curvature of edge $V_i V_{i+1}$ as type A in Figure 7. For $\mu_{V_i}$ and $\mu_{V_{i+1}}$, we calculate cost ($\pi$) under the following transport plan:

- $V_i \rightarrow V_i - k$: $\frac{1}{1 - \alpha}$, $V_i - k \rightarrow V_i + k - 2$: $\frac{1}{1 - \alpha}$, and the amount of transport between the other vertices is zero in Figure 7. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + (1 - \alpha).$$

From Definition 5, $\kappa_{\alpha} \geq 0$. By Definition 6, we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq 0.$$ (47)

Now, we find sufficient conditions under which the 1-Lipschitz function in Figure 7 can be defined. Let $P_{V_{i+k}, V_{i+k+2}}$ be a path that consists of type A edges and some vertices:

$$P_{V_{i+k}, V_{i+k+2}} := V_{i+k} \cdots V_{i+k-1} V_{i+k+2}.$$ Then,

$$|P_{V_{i+k}, V_{i+k+2}}| = (i + k) - (i + 2) \geq 3 \iff k \geq 5.$$ (48)

Also, from Figure 7,

$$d(V_{i-1}, V_{i-k+2}) \geq 2 \iff (i - 1) - (i - k + 2) \geq 2 \iff k \geq 5.$$ (49)

Next, consider the distance between $V_{i-k}, V_{i+k+2}$ in Figure 7. Let $P_{V_{i-k}, V_{i+k+2}}$ be a path that consists of type A edges and some vertices:

$$P_{V_{i-k}, V_{i+k+2}} := V_{i-k} V_{i-k-1} \cdots V_{i+k+3} V_{i+k+2}.$$ Then,

$$|P_{V_{i-k}, V_{i+k+2}}| = (i - k) - (i + k + 2 - n) \geq 2 \iff n \geq 2k + 4.$$ (50)

By $d(V_{i-k}, V_{i+k+2}) \geq 2$, they do not connect by $\pm k$ (by a type B edge), so

$$d(V_{i-k}, V_{i+k+2}) = (i - k) - (i + k + 2 - n) \neq k \iff n \neq 3k - 2.$$ (51)
From the above, we have found all the sufficient conditions (48), (50) and (51) for the 1-Lipschitz function defined in Figure 7. In short, the sufficient conditions are

\[ k \geq 5, \quad n \neq 3k - 2, \quad \text{and} \quad n \geq 2k + 4. \]  

These sufficient conditions (52) are conditions for type A in Theorem 5. In other words, when the conditions of Theorem 5 for type A are satisfied, the upper bound of the Ricci curvature can be calculated using the Lipschitz function in Figure 7. So, using the 1-Lipschitz function in Figure 7 we estimate an upper bound of Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 7. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

\[ W_1 \geq \alpha + (1 - \alpha). \]  

From Definition 5 we have the following the Ollivier Ricci curvature.

\[ \kappa_\alpha \leq 0. \]  

Therefore, we have the following upper bound of the Ricci curvature by Definition 6.

\[ \kappa \leq 0. \]

By (47) and (55), the curvature of all type A edges is zero.

Next, we investigate the curvature of edge \( V_iV_{i+k} \) as type B. At first, we find sufficient conditions under which the 1-Lipschitz function in Figure 8 can be defined.

(i) Let \( P_{V_{i+1},V_{i+k-1}} \) be a path that consists of type A edges and some vertices:

\[ P_{V_{i+1},V_{i+k-1}} := V_{i+1}V_{i+2} \cdots V_{i+k-2}V_{i+k-1}. \]

Then,

\[ |P_{V_{i+1},V_{i+k-1}}| = (i + k - 1) - (i + 1) \geq 1 \iff k \geq 3. \]  

(ii) Let \( P_{V_{i+k+1},V_{i+2k}} \) be a path that consists of type A edges and some vertices:

\[ P_{V_{i+k+1},V_{i+2k}} := V_{i+k+1}V_{i+k+2} \cdots V_{i+2k-1}V_{i+2k}. \]

Then,

\[ |P_{V_{i+k+1},V_{i+2k}}| = (i + 2k) - (i + k + 1) \geq 2 \iff k \geq 3. \]  

(iii) Let \( P_{V_{i-1},V_{i-k}} \) be a path that consists of type A edges and some vertices:

\[ P_{V_{i-1},V_{i-k}} := V_{i-1}V_{i-2} \cdots V_{i-k+1}V_{i-k}. \]

\[ |P_{V_{i-1},V_{i-k}}| \text{ denotes the length of the path. Then,} \]

\[ |P_{V_{i-1},V_{i-k}}| = (i - k) - (i + k + 1) \geq 0 \iff k \geq 1. \]  

(iv) Here, we show that \( d(V_{i-k},V_{i+2k}) = 3 \) in Figure 8 when any of the conditions (42), (43), or (44) in Theorem 5 is satisfied. In other words, we prove that any of (42), (43), or (44) is sufficient condition for \( d(V_{i-k},V_{i+2k}) = 3 \), in the following.
The condition $3k + 3 \leq n \leq 4k - 2$ of 1 in Theorem 5 is equivalent to

$$(i - k) - (i + 2k - n) \geq 3, \quad (i + 2k - n) - (i - 2k) \geq 2.$$  \hfill (59)

Consider the following paths:

$$P_1 := V_i - k V_i - k + 1 \cdots V_i - 2k + 1 V_{i+2k}, \quad (60)$$

$$P_2 := V_i + 2k V_i + 2k - 1 \cdots V_i - 2k + 1 V_i - 2k V_{i-k}, \quad (61)$$

(see Figure 9). From (59), we find that

$$|P_1| \geq 3, \quad \text{and} \quad |P_2| \geq 3.$$  \hfill (62)

Figure 9: condition 1

From (62)

$$d(V_{i-1}, V_{i+2k}) = 3.$$  \hfill (63)

The condition $4k + 2 \leq n \leq 5k - 2$ of 2 in Theorem 5 is equivalent to

$$(i - 2k) - (i + 2k - n) \geq 2, \quad (i + 2k - n) - (i - 3k) \geq 1.$$  \hfill (64)

Consider the following paths:

$$P_3 := V_i - k V_i - 2k V_i - 2k + 1 \cdots V_{i+2k}, \quad (65)$$

$$P_4 := V_i - k V_i - 2k V_i - 3k V_i - 3k + 1 \cdots V_{i+2k}, \quad (66)$$

(see Figure 10). From (64), we find that

$$|P_3| \geq 3, \quad |P_4| \geq 3.$$  \hfill (67)
From (67),

\[ d(V_{i-1}, V_{i+2k}) = 3. \]  \hspace{1cm} (68)

The condition \( n \geq 5k + 1 \) of 3 in Theorem 5 is equivalent to

\[ (i - 3k) - (i + 2k - n) \geq 1. \]  \hspace{1cm} (69)

Let us consider the path:

\[ P_5 := V_{i-k}V_{i-2k}V_{i-3k}V_{i-3k-1} \cdots V_{i+2k}, \]  \hspace{1cm} (70)

(see Figure 11). From (69), we find that

\[ |P_5| \geq 3. \]  \hspace{1cm} (71)

From (71),

\[ d(V_{i-1}, V_{i+2k}) = 3. \]  \hspace{1cm} (72)

Thus it was shown that any one of (42), (43), or (44) in Theorem 5 derives \( d(V_{i-k}, V_{i+2k}) = 3. \)
From the above, we have found all the sufficient conditions for the 1-Lipschitz function defined in Figure 8 are (56), (57), (58) and any of (42), (43), or (44):

\[ k \geq 3 \quad \text{and any of (42), (43), or (44)}. \] (73)

In other words, when one of the conditions 1 through 3 of Theorem 5 is satisfied, the upper bound of Ricci curvature can be calculated using the 1-Lipschitz function in Figure 8.

![Figure 12: Condition 4 in type B](image)

Also, we find sufficient conditions under which the 1-Lipschitz function in Figure 12 can be defined.

(v) Let \( P_{\nu_i+k-1, \nu_{i+1}} \) be a path that consists of type A edges and some vertices:

\[ P_{\nu_i+k-1, \nu_{i+1}} := \nu_{i+k-1} \nu_{i+k-2} \cdots \nu_{i+2} \nu_{i+1}. \]

\[ |P_{\nu_i+k-1, \nu_{i+1}}| = (i + k - 1) - (i + 1) \geq 1 \iff k \geq 3. \] (74)

This condition implies \( d(\nu_{i+1}, \nu_{i+k-1}) \geq 1. \)

(vi) Let \( P_{\nu_{i-k}, \nu_{i+k+1}} \) be a path that consists of type A edges and some vertices:

\[ P_{\nu_{i-k}, \nu_{i+k+1}} := \nu_{i-k} \nu_{i-k-1} \cdots \nu_{i+k+2} \nu_{i+k+1}. \]

Then,

\[ |P_{\nu_{i-k}, \nu_{i+k+1}}| = (i - k) - (i + k + 1 - n) \geq 1 \iff n \geq 2k + 2. \] (75)

Here we use \( \nu_{i-k+1} = \nu_{i+k-1}. \) This condition implies \( d(\nu_{i+k+1}, \nu_{i-k}) \geq 1. \)

(vii) Let \( P_{\nu_{i-1}, \nu_{i+2k}} \) be a path that consists of type A edges and some vertices:

\[ P_{\nu_{i-1}, \nu_{i+2k}} := \nu_{i-1} \nu_{i-2} \cdots \nu_{i+2k+1} \nu_{i+2k}. \]

Then,

\[ |P_{\nu_{i-1}, \nu_{i+2k}}| = (i - 1) - (i + 2k - n) \geq 3 \iff n \geq 2k + 4. \] (76)

Here we use \( \nu_{i+2k} = \nu_{i+2k-n}. \) This condition implies \( d(\nu_{i+2k}, \nu_{i-1}) \geq 3. \)

(viii) Let \( P_{\nu_{i+2k}, \nu_{i-k}} \) be a path that consists of type A edges and some vertices:

\[ P_{\nu_{i+2k}, \nu_{i-k}} := \nu_{i+2k} \nu_{i+2k-1} \cdots \nu_{i-k+1} \nu_{i-k}. \]

Then,

\[ |P_{\nu_{i+2k}, \nu_{i-k}}| = (i + 2k - n) - (i - k) \geq 3 \iff n \leq 3k - 3. \] (77)
Here we use $V_{i+2k} = V_{i+2k-n}$. This is one of the sufficient conditions.

From the above, we have found all the sufficient conditions (74), (75), (76) and (77) for the 1-Lipschitz function defined in Figure 12. In short, the sufficient conditions are

$$k \geq 3 \quad \text{and} \quad 2k + 4 \leq n \leq 3k - 3. \quad (78)$$

However for $k = 3, 4, 5$ there is no $n$ that satisfies (78). Thus, we found that the sufficient condition to define the 1-Lipschitz function in Figure 12:

$$k \geq 6 \quad \text{and} \quad 2k + 4 \leq n \leq 3k - 3. \quad (79)$$

In other words, when condition 4 of Theorem 5 is satisfied, the upper bound of Ricci curvature can be calculated using the 1-Lipschitz function in Figure 12.

We want to adopt the following transportation cost between $\mu_{V_i}^\alpha$ and $\mu_{V_{i+k}}^\alpha$ for any case 1-4:

$$V_i \rightarrow V_{i+k} : \alpha - \frac{1}{4}, V_{i+1} \rightarrow V_{i+k+1} : \frac{1}{4}, V_{i+1} \rightarrow V_{i+k+1} : \frac{1}{4}, \quad (80)$$

$$V_{i-k} \rightarrow V_{i+2k} : \frac{1}{4} \times 3. \quad (81)$$

All other transport masses are zero.

In (80) above, since

$$d(V_i, V_{i+k}) = d(V_{i+1}, V_{i+k+1}) = d(V_{i-1}, V_{i+k-1}) = 1, \quad (83)$$

transport mass in (80) is proper. On the other hand, it is unclear if transport (81) is proper. Since the path $V_{i-k}V_iV_{i+k}V_{i+2k}$ exists in Figure 8 and Figure 12, $d(V_{i-k}, V_{i+2k}) \leq 3$ holds.

For cases 1, 2, or 3, as we already saw in (iv) $d(V_{i-k}, V_{i+2k}) = 3$. (81) is correct. For the case 4, we have to check $d(V_{i-k}, V_{i+2k}) = 3$ to use (81).

Let us consider $d(V_{i-k}, V_{i+2k})$ in Figure 12. The condition $2k + 4 \leq n \leq 3k - 3$ of 4 in Theorem 5 is equivalent to

$$i - (i + 2k - n) + 1 \geq 5, \quad (i + 2k - n) - (i - k) \geq 3. \quad (84)$$

Let us consider the following paths:

$$P_6 := V_{i-k}V_{i-k+1}V_{i-k+2} \cdots V_{i+2k-1}V_{i+2k}, \quad |P_6| \geq 3. \quad (85)$$

$$P_7 := V_{i+2k}V_{i+2k+1} \cdots V_{i-1}V_{i-k}. \quad (86)$$
(See Figure 13). Note that $P_7$ consists not only of type A edges but also type B edge $V_iV_{i-k}$.

From (84) we find that

$$|P_6| \geq 3,$$  \hspace{1cm} and \hspace{1cm} $$|P_7| \geq 5.$$  \hspace{1cm} (87)

From (87), we can conclude

$$d(V_{i-k}, V_{i+2k}) = 3.$$  \hspace{1cm} (88)

Thus it was shown that (15) in Theorem 5 derives $d(V_{i-k}, V_{i+2k}) = 3$ in Figure 12.

Therefore, for any one of the conditions 1, 2, 3, or 4 in Theorem 5 the transportation plan (80), (81) and (82) is consistent.

By Definition 2 we can estimate Wasserstein distance for the transport between probability measures.

The transport cost is given by (80), (81), and (82). From (5), we can estimate an upper bound of the Wasserstein distance.

$$W_1 \leq \alpha + (1 - \alpha).$$  \hspace{1cm} (89)

From Definition 5 we have $\kappa_\alpha \geq 0$. By Definition 6 we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq 0.$$  \hspace{1cm} (90)

We introduced a 1-Lipschitz function as Figure 8 or 12. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

$$W_1 \geq \alpha + (1 - \alpha).$$  \hspace{1cm} (91)
From Definition 5 we have $\kappa_\alpha \leq 0$. Therefore, we have the following upper bound of the Ricci curvature by Definition 6.

$$\kappa \leq 0.$$  

(92)

By (90) and (92), the curvature of all type B edges is $\kappa = 0$.  \qed

A proof of propositions

A.1 Proof of Proposition 2

Proof. Cayley graph of $D_4$ with $S$ is a regular tetrahedral Figure 14. In the current case, there is no distinction between type A and type B edges. First, we calculate the Ricci curvature of the edge $ae$ as the type A or B.

![Figure 14: The Cayley graph of $D_4$](image)

We consider the transport cost from $\mu_\alpha^a$ to $\mu_\alpha^e$. By Definition 4, vertex $a$ has probability $\alpha$ for $\mu_\alpha^a$. Each probability of vertex $b, d$ and $e$ is $\frac{1-\alpha}{3}$. On the other hand, vertex $e$ has probability $\alpha$, and $a, f$ and $h$ have $\frac{1-\alpha}{3}$ for $\mu_\alpha^e$. Satisfying Definition 2, we can provide the following transport cost: $a \to e$: $\pi(a, e) = \alpha - \frac{1-\alpha}{3}$, $b \to f$: $\pi(b, f) = \frac{1-\alpha}{3}$, $d \to h$: $\pi(d, h) = \frac{1-\alpha}{3}$, and the amount of transport between the other vertices is zero. From (19), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures:

$$W_1 \leq \alpha + \frac{1-\alpha}{3}.$$  

(93)

From Definition 5 we have the following result about the Ollivier Ricci curvature.

$$\kappa_\alpha(a, e) \geq \frac{2}{3}(1 - \alpha).$$  

(94)

By Definition 6 we have the following result of the lower bound of the Ricci curvature.

$$\kappa(a, e) \geq \frac{2}{3}.$$  

(95)
Next, using a 1-Lipschtiz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschtiz function as Figure 14. The number in each box beside each vertex is the value of the 1-Lipschiz function. We have the following result from Theorem 1 and this 1-Lipschiz function.

\[ W_1 \geq \alpha + \frac{1 - \alpha}{3}. \]  

(96)

From Definition 5, the Ollivier Ricci curvature value is bounded as

\[ \kappa(\alpha, e) \leq \frac{2}{3}(1 - \alpha). \]  

(97)

Therefore, we have the following upper bound of the Ricci curvature by Definition 6

\[ \kappa(a, e) \leq \frac{2}{3}. \]  

(98)

By (95) and (98),

\[ \kappa(a, e) = \frac{2}{3}. \]  

(99)

Thus, the curvature of all edges is \( \kappa = \frac{2}{3} \).

\[ \Box \]

A.2 Proof of Proposition 3

Proof.

Figure 15: Type A in the Cayley graph of \( D_5 \)  
Figure 16: Type B in the Cayley graphs of \( D_5 \)

At first, we calculate the Ricci curvature of the edge \( ab \) as type A in Figure 15. By Definition 4, vertex \( a \) has probability \( \alpha \) for \( \mu_a^a \), on the other hand, vertex \( b \), \( f \) and \( e \) each has probability \( \frac{1}{3} \). \( \mu_b^b \) is defined similarly. Satisfying Definition 2, we provide the following transport cost:

\[ a \rightarrow b : \alpha - \frac{\alpha}{1} \quad f \rightarrow g : \frac{1}{3} \quad e \rightarrow c : \frac{1}{3} \times 2 \]

and the amount of transport between the other

25
vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

\[ W_1 \leq \alpha + \frac{2}{3}(1 - \alpha) \]

From Definition 5, we have \( \kappa_\alpha \geq \frac{1 - \alpha}{3} \). By Definition 6, we have the following result of the upper bound of the Ricci curvature.

\[ \kappa \geq \frac{1}{3}. \] (100)

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 15. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

\[ W_1 \geq \alpha + \frac{2}{3}(1 - \alpha). \] (101)

From Definition 5, we have \( \kappa_\alpha \leq \frac{1}{3}(1 - \alpha) \) for the Ollivier Ricci curvature value. Therefore, we have the following upper bound of the Ricci curvature by Definition 6.

\[ \kappa \leq \frac{1}{3}. \] (102)

By (100) and (102), the curvature of all type A edges is \( \kappa = \frac{1}{3} \).

Next, we calculate the Ricci curvature of the edge \( af \) as type B in Figure 16. We consider the transport cost. By Definition 4, vertex \( a \) has probability \( \alpha \), and vertex \( b, f \) and \( e \) each has probability \( \frac{1 - \alpha}{3} \) in \( \mu_a \). \( \mu_f \) is defined similarly. Satisfying Definition 2, we can provide the following transport cost: \( a \to f : \alpha - \frac{1 - \alpha}{3}, \ b \to g : \frac{1 - \alpha}{3}, \ e \to j : \frac{1 - \alpha}{3} \), and the amount of transport between the other vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures.

\[ W_1 \leq \alpha + \frac{1 - \alpha}{3}. \] (103)

From Definition 5, we have \( \kappa_\alpha \geq 2 \times \frac{1 - \alpha}{3} \). By Definition 6, we have the following result of the lower bound of the Ricci curvature.

\[ \kappa \geq \frac{2}{3}. \] (104)

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 16. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.
\[ W_1 \geq \alpha + \frac{(1 - \alpha)}{3}. \]  

(105)

From Definition 5, we have \( \kappa_\alpha \leq 2 \times \frac{1 - \alpha}{3} \). Therefore, we have the following upper bound of the Ricci curvature by Definition 6,

\[ \kappa \leq \frac{2}{3}. \]  

(106)

By (104) and (106), the curvature of all type B edges is \( \kappa = \frac{2}{3} \).

A.3 Proof of Proposition 4

\textit{Proof.} First, we prove that the Ricci curvature of type A is \( \kappa = 0 \).

![Figure 17: Cayley graph of $D_6$ with $S$](image)

Now, we calculate. Ricci curvature of the edge $ab$ in Figure 5.

We are considering a transport plan. By considering 4, vertex $a$ has probability $\alpha$. on the other hand, vertex $b$, $f$ and $g$ each has probability $\frac{1 - \alpha}{3}$. Satisfying Definition 2, we can provide the following transport plan:

- $f \rightarrow a; \frac{1 - \alpha}{3}$,
- $a \rightarrow b; \alpha$,
- $b \rightarrow c; \frac{1 - \alpha}{3}$,
- $g \rightarrow k; \frac{1 - \alpha}{3}$, but the amount of transport between the other vertices is zero. The following results can be calculated from these transport plans.

\[ W_1 \leq \alpha + (1 - \alpha). \]  

(107)

Next, by Definition 5, we get the following result of the Ollivier Ricci curvature value.

\[ \kappa_\alpha \geq 0. \]  

(108)
By Definition 7, we get the following lower bound of the Ricci curvature.

\[ \kappa \geq 0. \]  \hspace{1cm} (109)

We define a 1–Lipschtiz function as figure 5. The function in each beside each vertex is the value of the 1-Lipschiz function. We have the following result from Theorem 1 and this 1–Lipschiz function.

\[ W_1 \geq \alpha + (1 - \alpha). \]  \hspace{1cm} (110)

We use Definition 5 and 7, and we get the following the Ollivier Ricci curvature value.

\[ \kappa_a \leq 0. \]  \hspace{1cm} (111)

Therefore, we get the following upper bound of the Ricci curvature.

\[ \kappa \leq 0. \]  \hspace{1cm} (112)

By (109) and (112), the curvature of all edges is as follows.

\[ \kappa = 0. \]  \hspace{1cm} (113)

Next, we prove that the Ricci curvature of type B is \( \kappa = \frac{2}{3}. \)

Now, we calculate. Ricci curvature of the edge \( ab \) in Figure 6. We are considering a transport plan. By Definition 4, vertex \( a \) has probability \( \alpha \). on the other hand, vertex \( b, f \) and \( g \) each has probability \( \frac{1 - \alpha}{3} \). Satisfying Definition 2, we can provide the following transport plan:

- \( a \rightarrow f; \alpha - \frac{1 - \alpha}{3} \)
- \( b \rightarrow h; \frac{1 - \alpha}{3} \)
- \( f \rightarrow l; \frac{1 - \alpha}{3} \)

but the amount of transport between the other vertices is zero. The following results can be calculated from these optimal transport plans.

\[ W_1 \leq \alpha + \frac{1 - \alpha}{3}. \]  \hspace{1cm} (114)

Figure 18: Cayley graphs of the dihedral group of regular Icosahedrons
Next, by Definition 5, we get the following result of the Ollivier Ricci curvature value.

$$\kappa_{\alpha} \geq 2 \times \frac{1 - \alpha}{3}. \quad (115)$$

By Definition 7, we get the following lower bound of the Ricci curvature.

$$\kappa \geq \frac{2}{3}. \quad (116)$$

We define a 1-Lipschitz function as figure 6. The number in each beside each vertex is the value of the 1-Lipschiz function. We have the following result from Theorem 1 and this 1–Lipschiz function.

$$W_1 \geq \alpha + \frac{1}{3}(1 - \alpha). \quad (117)$$

We use Definition 5 and 7, and we get the following the Ollivier Ricci curvature value.

$$\kappa_{\alpha} \leq \frac{2}{3}(1 - \alpha). \quad (118)$$

Therefore, we get the following upper bound of the Ricci curvature.

$$\kappa \leq \frac{2}{3}. \quad (119)$$

By (49) and (52), the curvature of all edges is as follows:

$$\kappa = \frac{2}{3}. \quad (120)$$

### A.4 Proof of Proposition 5

**Proof.**

Figure 19: Type A in the Cayley graph of $Q_8$  
Figure 20: Type B in the Cayley graphs of $Q_8$
First, we prove that the Ricci curvature of type A is $\kappa = \frac{1}{2}$. We calculate the Ricci curvature of the edge $V_6V_7$ in Figure 19 as a type A. By Definition 4, vertex $V_6$ has probability $\alpha$ for $\mu_{V_6}^\alpha$, on the other hand, vertex $V_1, V_5, V_7$ and $V_3$ each has probability $\frac{1-\alpha}{4}$. $\mu_{V_7}^\alpha$ is determined similarly. Satisfying Definition 2, we can provide the following transport cost: $V_6 \to V_7 : \alpha - \frac{1-\alpha}{4}$, $V_1 \to V_4 : \frac{1-\alpha}{4}$, $V_5 \to V_8 : \frac{1-\alpha}{4}$, $V_3 \to V_2 : \frac{1-\alpha}{4}$, and the amount of transport between the other vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + \frac{1}{2}(1-\alpha). \tag{121}$$

From Definition 5, we have $\kappa \alpha \geq \frac{1-\alpha}{2}$. By Definition 6 we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq \frac{1}{2}. \tag{122}$$

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 19. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

$$W_1 \geq \alpha + \frac{1-\alpha}{2}. \tag{123}$$

From Definition 5, we have $\kappa \alpha \leq \frac{1-\alpha}{2}$. Therefore, we have the following upper bound of the Ricci curvature by Definition 6.

$$\kappa \leq \frac{1}{2}. \tag{124}$$

By (122) and (124), the curvature of all type A edges is $\kappa = \frac{1}{2}$.

We prove that the Ricci curvature of type B is $\kappa = \frac{1}{2}$. We calculate the Ricci curvature of the edge $V_4V_7$ in Figure 20 as a type B edge. By Definition 4, vertex $V_7$ has probability $\alpha$, and vertex $V_2, V_4, V_6$ and $V_8$ each has probability $\frac{1-\alpha}{4}$. $\mu_{V_4}^\alpha$ is defined similarly. Satisfying Definition 2, we can provide the following transport cost: $V_7 \to V_4 : \alpha - \frac{1-\alpha}{4}$, $V_2 \to V_1 : \frac{1-\alpha}{4}$, $V_6 \to V_5 : \frac{1-\alpha}{4}$, $V_8 \to V_7 : \frac{1-\alpha}{4}$, and the amount of transport between the other vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + \frac{1-\alpha}{2}. \tag{125}$$

From Definition 5, we have $\kappa \alpha \geq 2 \times \frac{1-\alpha}{2}$. By Definition 6, we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq \frac{1}{2}. \tag{126}$$
Next, using a 1-Lipschtiz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschtiz function as Figure 20. The number in each box beside each vertex is the value of the 1-Lipschiz function. We have the following result from Theorem 1 and this 1- Lipschiz function.

\[ W_1 \geq \alpha + \frac{1}{2}(1 - \alpha). \] (127)

From Definition 5, we have \( \kappa_\alpha \leq \frac{1}{2}(1 - \alpha) \). Therefore, we have the following upper bound of the Ricci curvature by Definition 6.

\[ \kappa \leq \frac{1}{2}. \] (128)

By (126) and (128), the curvature of all type B edges is \( \kappa = \frac{1}{2} \).

\[ \Box \]

A.5 Proof of Proposition 6

Proof.

![Figure 21: Type A in the Cayley graph of \( Q_{12} \)](image1)

![Figure 22: Type B in the Cayley graphs of \( Q_{12} \)](image2)

Figure 21: Type A in the Cayley graph of \( Q_{12} \) Figure 22: Type B in the Cayley graphs of \( Q_{12} \)

First, we prove that the Ricci curvature of type A is \( \kappa = \frac{1}{2} \). We calculate the Ricci curvature of type A as \( V_9V_{10} \) in Figure 21. Satisfying Definition 2, we can provide the following transport cost from \( \mu_9 \) to \( \mu_{10} \): \( V_9 \rightarrow V_{10} : \alpha - \frac{1}{4} \alpha \), \( V_6 \rightarrow V_1 : \frac{1}{2} \alpha \), \( V_3 \rightarrow V_4 : \frac{1}{4} \alpha \), \( V_8 \rightarrow V_{11} : \frac{1}{4} \alpha \times 2 \), and the amount of transport between the other vertices is zero. From (13), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

\[ W_1 \leq \alpha + \frac{3}{4}(1 - \alpha). \] (129)
From Definition 5, we have \( \kappa \alpha \geq 1 - \frac{\alpha}{4} \). By Definition 6, we have the following result of the lower bound of the Ricci curvature.

\[
\kappa \geq \frac{1}{4}.
\] (130)

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 21. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

\[
W_1 \geq \alpha + \frac{3}{4}(1 - \alpha).
\] (131)

From Definition 5, we have \( \kappa \alpha \leq \frac{1}{4}(1 - \alpha) \). Therefore, we have the following upper bound of the Ricci curvature by Definition 6.

\[
\kappa \leq \frac{1}{4}.
\] (132)

By (130) and (132), the curvature of all type A edges is \( \kappa = \frac{1}{4} \). This is the same result for all other type A edges.

Next, we prove that the Ricci curvature of type B is \( \kappa = \frac{1}{2} \). Now, we calculate the Ricci curvature of \( V_1V_7 \) in Figure 22 as a type B edge. Satisfying Definition 2, we can provide the following transport cost from \( \mu_{V_1}^{\alpha} \) to \( \mu_{V_7}^{\alpha} \):

\[
\begin{align*}
V_1 \to V_7 : & \quad \alpha - \frac{1 - \alpha}{4}, \\
V_2 \to V_12 : & \quad \frac{1 - \alpha}{4}, \\
V_6 \to V_8 : & \quad \frac{1 - \alpha}{4}, \\
V_{10} \to V_4 : & \quad \frac{1 - \alpha}{4}, \\
\end{align*}
\]

and the amount of transport between the other vertices is zero. From 5, we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

\[
W_1 \leq \alpha + \frac{1 - \alpha}{2}.
\] (133)

From Definition 5, we have \( \kappa \alpha \geq \frac{1 - \alpha}{2} \). By Definition 6, we have the following result of the lower bound of the Ricci curvature.

\[
\kappa \geq \frac{1}{2}.
\] (134)

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 22. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

\[
W_1 \geq \alpha + \frac{1}{2}(1 - \alpha).
\] (135)
From Definition 5 we have $\kappa_\alpha \leq \frac{1}{2}(1 - \alpha)$. Therefore, we have the following upper bound of the Ricci curvature by Definition 6,

$$\kappa \leq \frac{1}{2}. \quad (136)$$

By (134) and (136), the curvature of all type B edges is $\kappa = \frac{1}{2}$. This is the same result for all other type B edges. \(\square\)

A.6 Proof of Proposition 8

Proof.

We prove that the curvature of the edge type A as $V_i V_{i+1}$ in Figure 23 is $\frac{1}{2}$. Satisfying Definition 2, we can provide the following transport cost: $V_{i-2} \to V_i : \frac{1-\alpha}{4}$, $V_i \to V_{i+1} : \alpha$, $V_{i+1} \to V_{i+3} : \frac{1-\alpha}{4}$, and the amount of transport between the other vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + \frac{1}{2}(1 - \alpha). \quad (137)$$

From Definition 3 we have $\kappa_\alpha \geq \frac{1}{2}(1 - \alpha)$. By Definition 6 we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq \frac{1}{2}. \quad (138)$$

We define a 1-Lipschitz function as Figure 23. The number in each box beside each vertex is the value of the 1-Lipschitz function. To define this 1-Lipschitz function, we use the condition $n \geq 11$. We have the following result from Theorem 1 and this 1- Lipschitz function.

$$W_1 \geq \alpha + \frac{1}{2}(1 - \alpha). \quad (139)$$
From Definition 5, we have $\kappa \leq \frac{1}{2}(1 - \alpha)$. Therefore, we have the following upper bound of the Ricci curvature by Definition 6:

$$\kappa \leq \frac{1}{2}.$$  \hfill (140)

By (138) and (140), the curvature of all type A edges is $\kappa = \frac{1}{2}$.

Next, we prove that the Ricci curvature of the edge $V_{i-1}V_{i+1}$ as a type B is $\kappa = 0$. Satisfying Definition 2, we can provide the following transport cost:

$V_{i-1} \rightarrow V_{i+1} : \alpha - \frac{1}{4} \alpha, V_{i-2} \rightarrow V_{i+2} : \frac{1-\alpha}{4} \times 2, V_{i-3} \rightarrow V_{i+3} : \frac{1-\alpha}{4} \times 3$, and the amount of transport between the other vertices is zero. Here, we use the fact that $d(V_{i-3}, V_{i+3}) = 3$. This fact is derived from $n \geq 11$ as follows. Let us define the path in Figure 23 $P_{12b} := V_{i+3}V_{i+4} \cdots V_{i-4}V_{i-3}$. Here, $|P_{12b}| = n - (i - 3) - (i + 3) = n - 6$. If $n \geq 11$, $|P_{12b}| \geq 2 + 2 + 1$, this means that there is no shortcut through the vertices in this path. Then we obtain $d(V_{i-3}, V_{i+3}) = 3$.

From (13), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result:

$$W_1 \leq \alpha + (1 - \alpha).$$  \hfill (141)

From Definition 5, we have $\kappa \alpha \geq 0$. By Definition 6, we have the following result of a lower bound of the Ricci curvature:

$$\kappa \geq 0.$$  \hfill (142)

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 23. Here, we use $d(V_{i-3}, V_{i+3}) = 3$, again. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function:

$$W_1 \geq \alpha + (1 - \alpha).$$  \hfill (143)

From Definition 5, we have $\kappa \alpha \leq 0$. Therefore, we have the following upper bound of the Ricci curvature by Definition 6:

$$\kappa \leq 0.$$  \hfill (144)

By (142) and (144), the curvature of all type B edges is $\kappa = 0$. \hfill $\square$

This is the same result for all other edges.

A.7 Proof of Proposition 10
Proof.

Figure 24: Type A in Cayley graphs of $\mathbb{Z}/n\mathbb{Z}$. Figure 25: Type B in Cayley graphs of $\mathbb{Z}/n\mathbb{Z}$.

We prove that the Ricci curvature of the edge $V_iV_{i+1}$ as a type A edge in Figure 24. We calculate the transport cost defined by $V_i \to V_{i+1}: \alpha - \frac{1 - \alpha}{4}$, $V_{i-1} \to V_{i+2} : \frac{1 - \alpha}{4}$, $V_{i-3} \to V_{i-2} : \frac{1 - \alpha}{4}$, $V_{i+3} \to V_{i+4} : \frac{1 - \alpha}{4}$, and the amount of transport between the other vertices is zero. From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + \frac{1}{2}(1 - \alpha). \quad (145)$$

From Definition 5 we have $\kappa_\alpha \geq \frac{1}{2}(1 - \alpha)$. By Definition 6 we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq \frac{1}{2}. \quad (146)$$

We define a 1-Lipschitz function as figure 21. Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 25. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

$$W_1 \geq \alpha + \frac{1}{2}(1 - \alpha). \quad (147)$$

From Definition 5 we have $\kappa_\alpha \leq \frac{1}{2}$. Therefore, we have the following upper bound of the curvature by Definition 6.

$$\kappa \leq \frac{1}{2}. \quad (148)$$

By (146) and (148), the curvature of all type A edges is $\kappa = \frac{1}{2}$.

Next, we investigate the curvature of the edge $V_{i-1}V_{i+2}$ as type B in Figure 25. We calculate the transport cost by $V_{i-1} \to V_{i+2} : \alpha - \frac{1 - \alpha}{4}$, $V_{i-2} \to V_{i+1} : \frac{1 - \alpha}{4}$, $V_{i} \to V_{i+3} : \frac{1 - \alpha}{4}$, $V_{i-4} \to V_{i+5} : \frac{1 - \alpha}{4}$. 

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$1 - \alpha \times 3$, and the amount of transport between the other vertices is zero. Here, we use the fact that $d(V_{i-4}, V_{i+5}) = 3$. This fact is derived from $n \geq 16$ as follows. Let us define the path in Figure 25 $P_{a13b} := V_{i+6}V_{i+7} \cdots V_{i-5}V_{i-4}$. From $n \geq 16$, $|P_{a13b}| = n - (i + 5) - (i - 4) \geq 3 + 3 + 1$. This means that there is no shortcut through the vertices in this path, and we obtain $d(V_{i-4}, V_{i+5}) = 3$.

From (13), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + (1 - \alpha). \quad (149)$$

From Definition 5, we have $\kappa_\alpha \geq 0$. By Definition 6, we have the following result of a lower bound of the Ricci curvature.

$$\kappa \geq 0. \quad (150)$$

We define a 1-Lipschitz function as figure 25. The number in each box beside each vertex is the value of the 1-Lipschitz function. From Theorem 1 and this 1-Lipschitz function, we have the following result.

$$W_1 \geq 0. \quad (151)$$

From Definition 5, we have $\kappa_\alpha \leq 0$. Therefore, we have the following upper bound of the Ricci curvature by Definition 6.

$$\kappa \leq 0. \quad (152)$$

By (150) and (152), Thus, the curvature of all type B edges is $\kappa = 0$. □

A.8 Proof of Proposition 12

Proof.

We investigate the curvature of $V_iV_{i+1}$ as a type A edge. We calculate the transport cost by $V_i \rightarrow V_{i+1} : \alpha - \frac{1 - \alpha}{4}$, $V_{i-4} \rightarrow V_{i-3} : \frac{1 - \alpha}{4}$, $V_{i+4} \rightarrow V_{i+5} : \frac{1 - \alpha}{4}$, $V_{i-1} \rightarrow V_{i+2} : \frac{1 - \alpha}{4} \times 2$, and the amount of transport between the other vertices is zero.

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From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + \frac{3}{4}(1 - \alpha). \quad (153)$$

From Definition 5 we have \(\kappa_\alpha \geq \frac{1}{4}(1 - \alpha)\). By Definition 6 we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq \frac{1}{4}. \quad (154)$$

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 26. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.

$$W_1 \geq \alpha + \frac{3}{4}(1 - \alpha). \quad (155)$$

From Definition 5 we have \(\kappa_\alpha \leq \frac{1}{4}\). Therefore, we have the following upper bound of the Ricci curvature by Definition 6.

$$\kappa \leq \frac{1}{4}. \quad (156)$$

By (154) and (156), the curvature of all type A edges is \(\kappa = \frac{1}{4}\).

Next, we investigate the curvature of \(V_iV_{i+4}\) as a type B edge. We calculate the transport cost by \(V_i \rightarrow V_{i+4} : \alpha - \frac{1-\alpha}{4}, V_{i-1} \rightarrow V_{i+3} : \frac{1-\alpha}{4}, V_{i+1} \rightarrow V_{i+5} : \frac{1-\alpha}{4}, V_{i-4} \rightarrow V_{i+8} : \frac{1-\alpha}{4} \times 3\), and the amount of transport between the other vertices is zero. Here, we use the fact that \(d(V_{i-4}, V_{i+8}) = 3\). This fact is derived from \(n \geq 23\) as follows. Let us define the path in Figure 27 \(P_{12b} := V_{i-4}V_{i-5} \cdots V_{i+7}V_{i+8}\). Then, \(|P_{12b}| = n - (i + 8) - (i - 5) \geq 4 + 4 + 3\), and we get \(d(V_{i-4}, V_{i+5}) = 3\).

From (5), we can estimate an upper bound of the Wasserstein distance for the above transport between probability measures. We have the following result.

$$W_1 \leq \alpha + (1 - \alpha). \quad (157)$$

From Definition 5 we have \(\kappa_\alpha \geq 0\). By Definition 6 we have the following result of the lower bound of the Ricci curvature.

$$\kappa \geq 0. \quad (158)$$

Next, using a 1-Lipschitz function, we estimate an upper bound of the Ricci curvature by Theorem 1. We define a 1-Lipschitz function as Figure 27. The number in each box beside each vertex is the value of the 1-Lipschitz function. We have the following result from Theorem 1 and this 1-Lipschitz function.
\[ W_1 \geq \alpha + (1 - \alpha). \]  
(159)

From Definition 5, we have \( \kappa_\alpha \leq 0 \). Therefore, we have the following upper bound of the Ricci curvature by Definition 6,
\[ \kappa \leq 0. \]  
(160)

By (156) and (160), the curvature of all type B edges is \( \kappa = 0 \).

\[ \square \]

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