Spectroscopy of weakly isolated horizon via adiabatic invariance

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Abstract

The spectroscopy of a weakly isolated horizon (WIH) has been studied via adiabatic invariance. We obtain an equally spaced entropy spectrum with its quantum to be equal to the one given by Bekenstein. We demonstrate that the quantization of entropy and area is a generic property of horizon, not only for a static, spherically symmetric black hole, and the result exists in a wide class of spacetime admitting weakly isolated horizons.

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I. INTRODUCTION

Since the first exact solution of Einstein equation was found out, studying black holes’ properties has become an important part of gravitational physics. Properties of black holes, for example, laws of black hole mechanics, Hawking radiation and black hole spectroscopy, arose deep, unsuspected connections among classical general relativity, quantum physics and statistical mechanics, which are greatly surprising us. However, the traditional definition of a black hole \[1\], is too global and idealized: it requires knowledge of the entire future of the space-time, this is often cumbersome to use for the requirements of practical research \[2\]. In recent years, a new, quasi-local framework was introduced by Ashtekar and his collaborators to analyze different facets of black holes in a unified way \[2–4\]. Compared with the event horizon, this framework doesn’t need the knowledge of overall space-time, and only involves quasi-local conditions, so it accords with the practical physical process. In this framework, black holes in equilibrium (no matter and energy flow across the horizon) are described by (weakly) isolated horizons (WIH).

In 1970s, Bekenstein proved that the quantum of the black hole horizon is given as \((\Delta A)_{\text{min}} = 8\pi l_p^2\) \[5\]. From then on, there has been much attention paid to the quantization of black hole entropy spectrum and area spectrum \[6–15\], and many methods are closely related with quasi-normal frequency which requires the knowledge of the global geometry of the space-time, not just the geometry near a horizon. Recently, Majhi and Vagenas \[16\] proposed a new approach to derive the entropy spectrum and the horizon area quantum utilizing solely the adiabaticity of black holes and the Bohr-Sommerfeld quantization rule, and there is no use at all of the quasinormal frequencies to obtain the result. Later on there were many works using their method to study the entropy spectrum of other kinds of black hole \[17\]. This method is closely related to Parikh and Wilczek’s tunneling method \[18–20\] and only concerns the physics around a horizon, so it is natural to guess that this method could be used to discuss the spectroscopy of weakly isolated horizons, which generalize Majhi and Vagenas’s results to a wide class of spacetime that admit WIHs.

This paper is organized as follows. In section 2, we briefly review the definition of weakly isolated horizon and the geometry near it in the Bondi-like coordinate system with Bondi-gauge. In section 3, we apply Majhi and Vagenas’s adiabatic invariance method \[16\] to quantize a weakly isolated horizon. Finally, some conclusion and discussion are given in
section 4.

II. GEOMETRY OF WEAKLY ISOLATED HORIZON

In this section we will briefly review some geometric properties of WIH \cite{2,4}. As in Refs. \cite{20,21}, it is very convenient to introduce the Bondi-like coordinates \((u,r,\theta,\varphi)\), which are well defined on the horizon, and choose a set of null tetrad, which satisfy Bondi gauge, to study the behavior in the neighborhood of WIH. The null tetrad can be expressed as

\[
l^a = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X \frac{\partial}{\partial \varsigma} + \bar{X} \frac{\partial}{\partial \bar{\varsigma}},
\]

\[
n^a = -\frac{\partial}{\partial r},
\]

\[
m^a = \omega \frac{\partial}{\partial r} + \xi_3 \frac{\partial}{\partial \varsigma} + \xi_4 \frac{\partial}{\partial \bar{\varsigma}},
\]

\[
m^a = \omega \frac{\partial}{\partial r} + \bar{\xi}_3 \frac{\partial}{\partial \varsigma} + \bar{\xi}_4 \frac{\partial}{\partial \bar{\varsigma}},
\]

(1)

where \(U \equiv X \equiv \omega \equiv 0\) on the horizon \(H\) (following the notation in Ref. \cite{2}, equalities restricted to \(H\) will be denoted by \(\equiv \)), and \(\varsigma = e^{i\phi \cot \frac{\theta}{2}}\). Note that \(n^a\) and \(l^a\) are future directed. We take the spacetime metric \(g_{ab}\) to have a signature \((-,,+,+),\) so the metric can be expressed as

\[
g_{ab} = m_a \bar{m}_b + \bar{m}_a m_b - n_a l_b - l_a n_b.
\]

(2)

The definition of WIH \cite{2,4} implies that there is a one form \(\omega_a\) on \(H\) which satisfy the following relationship, \(\mathcal{L}_l \omega^a \equiv 0\) and \(D_a b^b \equiv \omega_a b^b\), where \(D_a\) is the induced covariant derivative on \(H\). In terms of the Newman-Penrose formalism, \(\omega_a\) can be explicitly expressed as

\[
\omega_a = -(\varepsilon + \overline{\varepsilon}) n_a + (\alpha + \overline{\beta}) \overline{m}_a + (\overline{\alpha} + \beta) m_a = -(\varepsilon + \overline{\varepsilon}) n_a + \pi \overline{m}_a + \bar{\pi} m_a,
\]

(3)

which means \((\varepsilon + \overline{\varepsilon})\) is constant on \(H\) from \(\mathcal{L}_l \omega^a \equiv 0\). The commutators of the null tetrad \([l^a, n^a]\) and \([m^a, n^a]\) tell us that

\[
\frac{\partial U}{\partial r} = (\varepsilon + \overline{\varepsilon}) + \pi \overline{\omega} + \pi \omega, \quad \frac{\partial X}{\partial r} = \pi \overline{\xi}_4 + \pi \xi_3, \quad \frac{\partial \omega}{\partial r} = \pi + \bar{\pi} \bar{\omega} + \mu \omega,
\]

\[
\frac{\partial \xi_3}{\partial r} = \bar{\pi} \bar{\xi}_4 + \mu \xi_3, \quad \frac{\partial \xi_4}{\partial r} = \pi \xi_3 + \mu \xi_4,
\]

(4)

which means \(\frac{\partial U}{\partial r} \equiv (\varepsilon + \overline{\varepsilon})\). Then the behavior of functions \(U, X\) and \(\omega\) near \(H\) is

\[
U = (\varepsilon + \overline{\varepsilon}) r + O(r^2),
\]

\[
X = O(r), \omega = O(r).
\]

(5)
Using Eq. (1), Eqs. (2) and Eq. (5), the out-going null geodesic can be calculated as

\[0 = 2du^2\frac{\xi_3 X - \xi_4 X}{|\xi_4|^2 - |\xi_3|^2} - 2du^2 \left( U - \frac{\xi_4 \omega - \xi_3 \omega}{|\xi_4|^2 - |\xi_3|^2} X - \frac{\xi_3 \omega - \xi_4 \omega}{|\xi_4|^2 - |\xi_3|^2} X \right) + 2dudr,\]

which leads to

\[\frac{dr}{du} = U.\] (7)

Based on Ref. [2], not any choice of time direction can give a Hamiltonian evolution, and only some suitably chosen time direction can lead to a well-defined horizon mass. In Ref. [2], A. Ashtekar and B. Krishnan gave a canonical way to choose the time direction \( t^a \) for a WIH, and the restriction of \( t^a \) to \( H \) should be a linear combination of a null normal \( l^a \) and the axisymmetric vector \( \psi^a \),

\[ t^a \equiv B_l l^a - \Omega_t \psi^a, \] (8)

where \( B_l \) and \( \Omega_t \) are constant on the horizon. Compared with the Schwarzschild case, the parameter of \( t^a \) takes the place of the Killing time. Using Eq. (7) and Eq. (8), we get the time derivative of \( r \) along the outgoing geodesic [20],

\[ \dot{r} = \frac{du}{dt} \frac{dr}{du} = (B_t + O(r))U = B_t(\varepsilon + \bar{\varepsilon})r + O(r^2). \] (9)

With the canonical time direction \( t^a \), A. Ashtekar and B. Krishnan [2] established the zeroth and the first law of WIH. By definition, the surface gravity of \( H \) is \( \kappa_t := B_t l^a \omega_a = B_t(\varepsilon + \bar{\varepsilon}) \). Because \( B_t(\varepsilon + \bar{\varepsilon}) \) is constant on \( H \), the zeroth law of black hole mechanics is valid for WIH. The first law is expressed as

\[ \delta M_H^{(t)} = \frac{\kappa_t}{8\pi} \delta a_H + \Omega_t \delta J_H, \] (10)

where \( M_H^{(t)} \) is the horizon mass, \( a_H \) is the area of the cross section of WIH, \( \Omega_t \) is the angular velocity of the horizon and \( J_H = -\frac{1}{8\pi} \oint_S (\omega_a \psi^a) dS \) is the angular momentum. The first law of WIH is the generalization of the first law of stationary black holes. Because Ref. [20] studied the tunneling effect near WIH, so the laws of WIH mechanics are laws of WIH thermodynamics.
III. SPECTROSCOPY OF A WEAKLY ISOLATED HORIZON VIA ADIABATIC INVARIANCE

In this section, we consider the quantization of a weakly isolated horizon by applying Majhi and Vagenas’s adiabatic invariance method [16]. We shall use some geometric properties introduced in section 2. Let us consider an adiabatic invariant quantity of the form

\[ I = \int p_i dq_i = \int_0^{p_i} \int_0^H \frac{dH'}{q_i} dq_i, \]  

(11)

where \( p_i \) is the conjugate momentum of the coordinate \( q_i \) with \( i = 0, 1 \) for which \( q_0 = \tau \) and \( q_1 = r \). Please note that we use the Euclidean time \( q_0 = \tau \) and the Einstein summation convention. To get the last equation, we have used Hamilton’s equation \( \dot{q}_i = \frac{dH}{dp_i} \), where the Hamiltonian \( H \) is the total energy of the black hole. Write the Eq. (11) explicitly,

\[ I = \int p_i dq_i = \int_0^H dH' \dot{\tau} + \int_0^H \frac{dH'}{\dot{r}} dr. \]  

(12)

In order to calculate the adiabatic invariant quantity of a WIH, we shall obtain the quantity \( \dot{r} \) that appears in Eq. (12). As Ref. [16], let us consider the radial null paths. Our subsequent analysis will concentrate on the outgoing paths, since these are the ones related to the quantum mechanically nontrivial features [18]. Because \( \tau \) is the Euclidean time, Ref. [16] uses the transformation \( t \rightarrow -i\tau \) to Euclideanize the metric and gets the radial null paths. Unlike Ref. [16]’s method of Euclideanizing the metric, we substitute directly the transformation \( t \rightarrow -i\tau \) into the Eq. (9) to get the outgoing radial null path. This leads to

\[ \dot{r} \equiv \frac{dr}{d\tau} = -iB_t(\varepsilon + \bar{\varepsilon})r + O(r^2) = R_+(r). \]  

(13)

Now, using Eq. (13), we get that

\[ \int_0^H dH' \dot{\tau} = \int_0^H dH' \frac{dr}{R_+(r)} = \int_0^H dH' \frac{dr}{\dot{r}}. \]  

(14)

and the adiabatic invariant quantity (12) reads

\[ I = \int p_i dq_i = 2 \int_0^H dH' \dot{\tau} = 2 \int_0^r dH' \frac{dr}{\dot{r}}. \]  

(15)

In Ref. [16], authors perform the \( \tau \)-integration by considering the periodicity of imaginary time \( \tau \) for static black holes, but for weakly isolated horizons, we do not know if this is valid.
Fortunately, we can do the \( r \)-integration. Using the technology in Parikh and Wilczek’s tunneling method \([18,20]\), we can get

\[
\int \int H \frac{dH'}{r} = \int \int_{r_{\text{out}}}^{r_{\text{in}}} \frac{dr}{-iB_t(\varepsilon + \bar{\varepsilon})r + O(r^2)} = \pi \int \frac{dH'}{B_t(\varepsilon + \bar{\varepsilon})} = \pi \int_0^H \frac{dH'}{\kappa_t}, \tag{16}
\]

where \( \kappa_t \) is the surface gravity of WIH

\[
\kappa_t = B_t(\varepsilon + \bar{\varepsilon}). \tag{17}
\]

So the adiabatic invariant quantity is

\[
I = \int p_i dq_i = \frac{2\pi}{\kappa_t}, \tag{18}
\]

which is the same as static black holes \([16]\). As we all know that the temperature of a black hole is proportional to the surface gravity of the horizon

\[
T_{bh} = \frac{\hbar \kappa_t}{2\pi}, \tag{19}
\]

thus the adiabatic invariant quantity given in Eq. (18) becomes

\[
I = \int p_i dq_i = \hbar \int_0^H \frac{dH'}{T_{bh}} = \hbar S_{bh}, \tag{20}
\]

where we have used the first law of WIH (10) with \( \Omega_t = 0 \) in the last step

\[
dH = \delta M^{(t)}_H = \frac{\kappa_t}{8\pi} \delta a_H = T_{bh} dS_{bh}. \tag{21}
\]

At last, implementing the Bohr-Sommerfeld quantization rule

\[
\int p_i dq_i = n\hbar \tag{22}
\]

in Eq. (20), we derive the WIH entropy spectrum

\[
S_{bh} = 2\pi n, \tag{23}
\]

where \( n = 1, 2, 3, \ldots \), and it is straightforward to see that the spacing in the entropy is given by

\[
\Delta S_{bh} = S_{(n+1)bh} - S_{(n)bh} = 2\pi. \tag{24}
\]

Thus, the entropy spectrum is quantized and equidistant for a weakly isolated horizon.

Recalling that in the framework of Einstein’s theory of gravity, black hole entropy is proportional to the black hole horizon area \( \mathcal{A} \), \( S_{bh} = \frac{\mathcal{A}}{4G} \). It is evident that if we employ the
spacing of the entropy spectrum given in Eqs. (24), the quantum of the WIH area has the form

$$\Delta A = 8\pi l_p^2,$$

which is the same as the area quantum derived by Bekenstein [5].

For an axial symmetric horizon, using the method in Ref. [20, 22], the last equation of (15) can be modified as

$$I = 2\int \int \frac{dH}{\tilde{r}} \tilde{r} d\phi = 2\int \int \frac{dH - \dot{\phi} dp_\phi}{\tilde{r}} = 2\int \int \frac{dM - \Omega t dJ_H}{\tilde{r} B_t(\varepsilon + \tilde{\varepsilon})} = \hbar S_{bh},$$

where we have used the first law of WIH (10) and Eqs. (17, 19) in the last equality. The result is the same with that for the non-rotating WIH.

IV. SUMMARY AND CONCLUSION

In this paper, we have quantized the entropy and the horizon area of a weakly isolated horizon via adiabatic invariance and Bohr-Sommerfied quantization rule, and obtain the quantized entropy and area spectrum which are the same as Bekenstein’s original result. This result indicates that the quantization of entropy and area of the black hole horizon is a generic property of horizon, not only for stationary black hole, so we generalize the results to a more general class of spacetime which may conclude dynamical situations.

Acknowledgments

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