New Non-Separable Diagonal Cosmologies.

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Abstract

We find all the perfect fluid $G_2$ diagonal cosmologies with the property that the quotient of the norms of the two orthogonal Killing vectors is constant along each fluid world-line. We find four different families depending each one on two or three arbitrary parameters which satisfy that the metric coefficients are not separable functions. Some physical properties of these solutions including energy conditions, kinematical quantities, Petrov type, the existence and nature of the singularities and whether they contain Friedman-Robertson-Walker cosmologies as particular cases are also included.

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1 Introduction

The study of exact solutions of Einstein’s field equations for a perfect-fluid energy-momentum tensor is a difficult task even when we assume the existence of a two-dimensional abelian isometry group acting on the manifold. Indeed, for these spacetimes the Einstein equations constitute a system of highly non-linear coupled partial differential equations with two independent variables which cannot be solved at present in the general case without making further hypotheses.

As we are interested in studying models which could represent inhomogeneous cosmologies we will assume that the isometry group two-dimensional orbits are spacelike everywhere (the so-called $G_2$ on $S_2$ metrics) and we will also make the usual hypothesis that the fluid velocity vector of the perfect fluid is irrotational. Under these hypotheses, the models can be classified [1] into four classes depending on whether the orbits of the group are orthogonally transitive (which means that the set of two-planes orthogonal at each point to the group orbits are themselves surface forming) and on the existence of integrable Killing fields. The simplest (and in fact the most studied up to now) of these classes is the so-called diagonal cosmologies in which the group orbits are orthogonally transitive and both the Killing vectors are integrable (which in turn implies that they can be chosen mutually orthogonal everywhere) so that it can be seen that there exist coordinates $\{t, x, y, z\}$ in which the metric take the Einstein-Rosen’s form

$$ds^2 = F(x, t) \left( -dt^2 + dx^2 \right) + G(x, t) \frac{dy^2}{P(x, t)} + G(x, t) \frac{P(x, t)}{G(x, t)} dz^2.$$  (1)

The first of the known perfect-fluid solutions under these hypotheses not satisfying an equation of state for a stiff fluid was found by Wainwright & Goode [2] some time ago (1980). This family depends on three arbitrary parameters and the perfect fluid satisfies a linear equation of state with the pressure proportional to the density. It has the particularity that the metric coefficient $P$ depends only on $t$, or equivalently the three-spaces orthogonal to the fluid velocity are conformally flat. This solution has also the property that all the metric functions are separable in the coordinates $t$ and $x$, that is to say, they are products of one function depending only on $t$ and another function depending only on $x$. New metrics were later found in [3], [4] and [5], all of them being also separable metrics with explicit assumptions on the form of the functions involved. Finally, the case of separable diagonal cosmologies was extensively treated in a remarkable paper [6], where the general equations for separable cosmologies in co-moving coordinates (not including the family due to Wainwright & Goode nor the solutions satisfying the equation of state for a stiff fluid) were written and studied. Explicit families were also found showing that all kinds of singular behaviours were possible even when the energy conditions were satisfied everywhere (model with a big-bang-like singularity and/or big crunch singularity, metrics with timelike singularities, models with pressure and density regular everywhere but with singularities in the Weyl
tensor or complete models with no singularities at all). An extensive discussion of the relation between the existence of non-singular solutions satisfying energy and causality conditions and the powerful singularity theorems [7] was published in [8]. More recently, all the models with pressure equal to density and assuming separation of variables for diagonal cosmologies have been published in [9].

Thus, we see that the seek of exact solutions for diagonal cosmologies has been performed mainly assuming the Ansatz of separation of variables in co-moving coordinates. However, another approach has also been used in order to find new diagonal cosmologies. The use of the generalised Kerr-Schild transformation using Friedman-Robertson-Walker cosmologies as original base metric [10] has proven to be fruitful to find new diagonal solutions which are non-separable in the metric coefficients (some of these metrics have in fact only one Killing vector field). All of them, however, are of Petrov type D everywhere.

Before stating clearly the assumptions we make in this paper, let us note that any Friedman-Robertson-Walker cosmology (which has a six dimensional group of isometries acting on three-dimensional spacelike hypersurfaces) possesses also a two-dimensional isometry subgroup acting orthogonally-transitively on spacelike two-surfaces and with both Killing vectors integrable, so that the metric can be written in Einstein-Rosen’s diagonal form. In fact, these metrics can be explicitly written as (see for instance [11])

\[ ds^2 = R^2(\tau) \left( -d\tau^2 + d\rho^2 + \Sigma^2(\rho, k)d\phi^2 + \Sigma'^2(\rho, k)dz^2 \right) , \]

where \( R \) is an arbitrary function usually called scale factor, the function \( \Sigma(\rho, k) \) is given by

\[ \Sigma(\rho, k) \equiv \begin{cases} 
\sin \rho & \text{if } k = 1 \\
\rho & \text{if } k = 0 \\
\sinh \rho & \text{if } k = -1 
\end{cases} \]

and the prime denotes derivative with respect to \( \rho \). Thus, for the Friedman-Robertson-Walker cosmologies the function \( P \) in (1) is given by

\[ P = \frac{\Sigma(\rho, k)}{\Sigma'(\rho, k)} , \]

which has the particularity that it does not depend on the coordinate \( \tau \). Thus, the diagonal cosmologies such that the metric coefficient \( P \) in (1) does not depend on \( t \) obviously contain the Friedman-Robertson-Walker as a particular case and therefore it is worth considering such metrics without assuming other hypotheses on the form of the metric potentials. In fact, this is the aim of this paper where we find out all the diagonal \( G_2 \) cosmologies with the property that the function \( P \) depends only on the spacelike coordinate. We find five families of exact solutions, each one depending on
arbitrary parameters. However, one of these families is in fact separable so that we do not study it here. All the other solutions are of Petrov type I and, consequently, they are not included in the non-separable Kerr-Schild families in [10] so that they are, to our knowledge, new. For each family there exist ranges of the parameters such that the solutions satisfy energy conditions and also three of the four families contain Friedman-Robertson-Walker models for particular values of the arbitrary constants. One of the families represents a stiff fluid and the solution can be seen to be generated from a vacuum solution using the procedure due to Wainwright, Ince & Marshman [12].

The behaviour of the solutions is analysed and the existence of singularities and their nature is also studied. One of the families is complete and singularity-free, while other two have a big-bang-like singularity while they are regular in the future and in the spacelike directions.

The plan of the paper is as follows. In section 2 we state the assumptions of this paper and give a coordinate independent characterisation, we also write the Einstein field equations and find the general solution of these equations. In section 3, we write down explicitly all the found solutions and analyse some of their physical properties, such as the energy density and pressure of the solution, whether they satisfy energy conditions, the Petrov type, the existence and nature of the singularities and the kinematical quantities of the fluid velocity vector.

2 Field equations and their solutions.

As stated in the introduction we will seek exact perfect-fluid solutions of Einstein’s field equations with an abelian two-dimensional group of isometries acting orthogonally transitively on spacelike two-surfaces and such that both Killing vectors are integrable. Thus, there exist coordinates \{t, x, y, z\} in which the metric takes the Einstein-Rosen’s form

\[
ds^2 = F(x,t) \left( -\frac{dt^2}{M(t)} + \frac{dx^2}{N(x)} \right) + G(x,t)P(x,t)dy^2 + \frac{G(x,t)}{P(x,t)}dz^2. \tag{2}
\]

where the two Killing vector fields are given by

\[
\xi = \frac{\partial}{\partial y}, \quad \eta = \frac{\partial}{\partial z}.
\]

Note that we could have set \(M = 1\) and \(N = 1\) by means of a coordinate change of the type

\[
t' = t'(t), \quad x' = x'(x),
\]

but in some cases is preferable to maintain these two functions in order to integrate some of the field equations and, in fact, this will be the case in our situation.
Our objective in this paper is to find out the co-moving perfect-fluid diagonal cosmologies with the property that the function $P$ of the metric does not depend on the variable $t$, that is to say, we will assume $P(x)$. Rewriting this in a coordinate-independent way, we will find the perfect-fluid diagonal cosmologies which satisfy

$$\vec{u} \left( \vec{\xi}_1 \cdot \vec{\xi}_1 \right) = 0,$$  \hspace{1cm} (3)

where $\vec{u}$ is the fluid velocity vector of the perfect fluid and $\vec{\xi}_1$ and $\vec{\xi}_2$ are two arbitrary mutually orthogonal Killing vector fields. In order to see the equivalence of the two assumptions, we first note that in a perfect-fluid solution with a two-dimensional group of isometries acting orthogonally transitively on spacelike surfaces, the fluid velocity vector is necessarily hypersurface orthogonal and, in consequence, there exist coordinates \{\(t, x, y, z\)\} in which the metric takes the form (2) with the fluid velocity vector given by

$$\vec{u} = \sqrt{\frac{M}{F}} \frac{\partial}{\partial t},$$

(co-moving coordinates). Except in the case that the function $P$ in (2) is a constant (so that the metric possesses a three-dimensional group of isometries and is out of the interest of this paper), it is trivial to see that the only two mutually orthogonal Killing vector fields are given by

$$\vec{\xi}_1 = a \frac{\partial}{\partial y}, \quad \vec{\xi}_2 = b \frac{\partial}{\partial z},$$

where $a$ and $b$ are arbitrary non-vanishing constants. The condition (3) now reads

$$\frac{a^2}{b^2} \sqrt{\frac{M}{F}} \frac{\partial}{\partial t} \left( P^2 \right) = 0 \iff \partial_t (P) = 0,$$

which is exactly the original assumption of $P$ depending only on $x$.

In order to write down the Einstein tensor (and all the other tensor components in this paper) we will use the orthonormal tetrad given by

$$\theta^0 = \sqrt{\frac{F}{M}} dt, \quad \theta^1 = \sqrt{\frac{F}{N}} dx, \quad \theta^2 = \sqrt{G} P dx, \quad \theta^3 = \sqrt{\frac{G}{P}} dy,$$ \hspace{1cm} (4)

so that the Einstein field equations for a co-moving perfect fluid energy-momentum tensor read (in units $c = 8\pi G = 1$)

$$S_{00} = \rho, \quad S_{11} = S_{22} = S_{33} = p, \quad S_{\alpha\beta} = 0 \quad (\alpha \neq \beta),$$

where $S_{\alpha\beta}$ (\(\alpha = 0, 1, 2, 3\)) stands fo the Einstein tensor and $\rho$ and $p$ are the energy density and the pressure of the fluid respectively. Obviously, there are combinations of
these equations involving the Einstein tensor alone and the calculation of the density
and pressure can be performed once these equations are solved. In our case, the three
equations which are non-identically satisfied are

\[ S_{01} = 0, \quad S_{11} - S_{22} = 0, \quad S_{22} - S_{33} = 0, \]

and they read explicitly

\[ S_{01} = \frac{\sqrt{NM}}{2F} \left( \frac{F_x G_{,x}}{G} + \frac{F_{,t} G_{,t}}{F} \right) \]

\[ S_{22} - S_{33} = \frac{N}{F} \left( \frac{-G_{,x} P'}{P} - \frac{1}{2} \frac{P'}{P} N - \frac{P''}{P} + \frac{P'^2}{P^2} \right) = 0, \]

\[ S_{11} - \frac{1}{2} (S_{22} + S_{33}) = \frac{1}{2F} \left( \frac{F_{,tt} M - \frac{F_{,t}^2}{F^2} M + \frac{F_x G_{,t}}{F} M + \frac{1}{2} \frac{F_{,t}}{F} \dot{M} - \frac{F_{xx}}{F} N + \frac{F_{,x}^2}{F^2} N + \frac{F_x G_{,x}}{G} N - \frac{1}{2} \frac{F_x}{G} N^2 - \frac{P''}{P^2} \right) = 0, \]

where the comma means partial derivative with respect to the variable indicated, the
prime stands for ordinary derivative with respect to \( x \) and the dot indicates ordinary
derivative with respect to \( t \).

In order to simplify this system of differential equations, it is convenient to define a
new function \( K(x) \) through

\[ \frac{P'}{P} \equiv \frac{K}{\sqrt{N}} \]

We can assume \( K \) non-identically vanishing because we are not considering solutions
with \( P \) constant. Thus, equation (8) can be integrated to give

\[ \frac{G_{,x}}{G} K + K' = 0 \iff G(x, t) = \frac{L(t)}{K(x)}, \]

where \( L(t) \) is an arbitrary non-vanishing function depending only on the variable \( t \).
Substituting this expression for \( G \) into the equation (5) we get

\[ \frac{F_{,t} K'}{F K} - \frac{F_x \dot{L}}{F L} - \frac{K' \dot{L}}{K L} = 0. \]

In order to handle this equation some different cases have to be considered. First, let us
assume \( \dot{L} = 0 \) so that we have

\[ \frac{F_{,t} K'}{F K} = 0. \]
As \( F_t = 0 \) would imply that no function in the metric (2) would depend on \( t \) and consequently \( \partial_t \) would be a Killing vector, it follows necessarily that \( K \) is a constant, but in this case the calculation of the density gives

\[
\rho = -\frac{K^2}{4F} < 0
\]

and this solution is by no means physically reasonable. Thus, we must restrict our study to the case \( \dot{L} \neq 0 \). If in addition we had \( K' = 0 \) equation (9) would imply that the function \( F \) would not depend on \( x \) and then it can be easily seen that the metric (2) possesses a third Killing vector field given by

\[
\vec{\xi}_3 = 2\sqrt{N} \frac{\partial}{\partial x} + K \left(z \frac{\partial}{\partial z} - y \frac{\partial}{\partial y}\right)
\]

so that we can discard this case. Thus we can assume \( K' \neq 0 \) and \( \dot{L} \neq 0 \) and use the freedom that we still have in the coordinates \( t \) and \( x \) to fix these two functions locally to

\[
K = K_0 e^x, \quad L = K_0 e^t
\]

where we have introduced a constant \( K_0 \neq 0 \) because nothing prevents the function \( K \) to be negative. With this choice, equation (9) takes the simple form

\[
F_t - F_x - F = 0,
\]

which can be trivially integrated to give

\[
F = e^t H(t + x), \quad \quad (10)
\]

where \( H \) is an arbitrary non-vanishing function depending only on the variable \( u \equiv t + x \). Let us define a new function \( Z(u) \) by

\[
\frac{1}{H} \frac{dH}{du} \equiv Z \quad (11)
\]

so that (11) takes the form

\[
\left(\frac{dZ}{du} + Z\right)(M - N) + \frac{Z}{2} \left(M - N'\right) + \frac{N'}{2} - K^2 = 0, \quad (12)
\]

which is a linear first order ordinary differential equation for the function \( Z \) with the particularity (which in fact makes it difficult to study) that the coefficients do not depend on \( u \) but on other coordinates \( t \) and \( x \). In order to handle this equation, we can consider \( Z \) as a function of the two variables \( t \) and \( x \) and impose the known condition that this
function depends only on the variable $t + x$. Thus, the equivalent system of partial differential equations for the function $Z(t, x)$ is

$$\left[ \frac{1}{2} (Z, t + Z, x) + Z \right] (M - N) + \frac{Z}{2} \left( \dot{M} - N' \right) + \frac{N'}{2} - K^2 = 0,$$

$$Z_t - Z_{xx} = 0.$$  \hspace{1cm} (13)

In order to solve this system, let us first note that if $M - N$ vanished, we would necessarily have both $M$ and $N$ constant and then the first equation in (13) would be clearly incompatible. Thus, we can define a new function $Z_0(t, x)$ by the relation

$$Z \equiv Z_0 e^{-(t+x)}$$

chosen in such a way that the system (13) takes the simpler form

$$\partial_t Z_0 + \partial_x Z_0 = e^{t+x} \left( 2K^2 - N' \right),$$

$$\partial_t Z_0 - \partial_x Z_0 = Z_0 \frac{\dot{M} + N'}{M - N}.$$ \hspace{1cm} (15)

We can evaluate the integrability condition $\partial_t \partial_x Z_0 - \partial_x \partial_t Z_0 = 0$ which gives

$$\left( 4K^2 - N'' \right) (M - N) + \left( 2K^2 - N' \right) \left( \dot{M} + N' \right) +$$

$$+ Z_0 e^{-(t+x)} \left( N'' + \ddot{M} + \frac{N'^2 - \dot{M}^2}{M - N} \right) = 0,$$  \hspace{1cm} (16)

This equation suggests two possibilities, either with a vanishing or a nonvanishing coefficient of $Z_0$. In the first case, the following two equations must be satisfied

$$\left( 4K^2 - N'' \right) (M - N) + \left( 2K^2 - N' \right) \left( \dot{M} + N' \right) = 0,$$

$$(M - N) \left( N'' + \ddot{M} \right) + N'^2 - \dot{M}^2 = 0.$$ \hspace{1cm} (17)

In order to find the general solution of this system of two ordinary differential equations that mixes functions of $x$ and functions of $t$, we take the second derivative with respect to $x$ and $t$ of the second equation to get

$$\ddot{M} N''' - \ddot{M} N' = 0,$$

from what it follows that either $M$ or $N$ are constants or they satisfy

$$N''' = bN', \hspace{1cm} \ddot{M} = b\dddot{M},$$

where $b$ is an arbitrary constant. Using this information, it is indeed very easy to find the general solution of (17). There exist two different possibilities given by
Solution I \[ \begin{align*}
N &= N_0 + R_0 e^{2x} \\
M &= N_0 + S_0 e^{-2t}
\end{align*} \]

Solution II \[ \begin{align*}
N &= N_0 + K^2_0 e^{2x} \\
M &= N_0 + S_0 e^{2t}
\end{align*} \]

where \( N_0, R_0 \) and \( S_0 \) are arbitrary constants. With these explicit forms for the functions \( M \) and \( N \), the partial differential equations (13) can be integrated to give \( Z_0 \) and therefore we can immediately calculate \( Z \) from its definition (14) and \( H \) from (11). The results are given by the following expressions (note that in the Solution I we have to distinguish between three subcases in order to perform explicitly the integral for \( H \)):

- Solution I with \( S_0 = 0 \).
  \[
  Z = \frac{R_0 - K^2_0 - ce^{-2(t+x)}}{2R_0} \implies H = H_0 \exp \left( \frac{R_0 - K^2_0}{2R_0} (t + x) + \frac{c}{4R_0} e^{-2(t+x)} \right).
  \]

- Solution I with \( R_0 = 0 \).
  \[
  Z = \frac{K^2_0 e^{2(t+x)} + c}{2S_0} \implies H = H_0 \exp \left( \frac{K^2_0}{4S_0} e^{2(t+x)} + \frac{c}{2S_0} (t + x) \right).
  \]

- Solution I with \( R_0 \neq 0 \) and \( S_0 \neq 0 \).
  \[
  Z = \frac{(K^2_0 - R_0) e^{2(t+x)} + \frac{S_0}{R_0} (R_0 - K^2_0 - 4bR_0)}{2S_0 - 2R_0 e^{2(t+x)}} \implies \\
  H = H_0 e^{\left( \frac{1}{2} - \frac{K^2_0 - 2b}{R_0} \right) (t+x)} \left| S_0 - R_0 e^{2(t+x)} \right|^b.
  \]

- Solution II.
  \[
  Z = -2ce^{-2(t+x)} \implies H = H_0 \exp \left( ce^{-2(t+x)} \right).
  \]

where the arbitrary constants \( b, c \) and \( H_0 \) come from the integration of the differential equations (14) and (11).

Thus, we have already found the general solution of the system (13) in the case that the integrability condition is identically satisfied. We can now move to the second case when we have

\[
N'' + \ddot{M} + \frac{N'^2 - \dot{M}^2}{M - N} \neq 0.
\]
Now, the integrability condition (16) gives explicitly $Z_0$ in terms of $M$ and $N$ as

$$Z_0 = e^{t+x} (M - N) \frac{(N'' - 4K^2) (M - N) + (N' - 2K^2) (\dot{M} + N')}{(M - N) (N'' + \ddot{M}) + N'^2 - \ddot{M}^2},$$

(19)

and substituting this expression in each of the equations in (15) we get two differential relations involving the functions $M$, $N$ and their derivatives. The second of these equations reads explicitly

$$\left(\dddot{M} \dddot{M} - \ddot{M}^2\right) A + \left(\dddot{M}M - \ddot{M} \dddot{M}\right) A' + \left(\ddot{M}M - \ddot{M}^2\right) A'' + \ddot{M} \left(AN' - A'N\right) + \ddot{M} \left(A'N' - A''N\right) + \ddot{M} \left(A'N'' - AN''\right) + 4MK^2 \left(N''' - 2N''\right) + B = 0,$$

(20)

where we have set

$$A \equiv N' - 2K^2, \quad B \equiv 2K^2 \left(N'''N' - 2NN'' - N''^2 + 2N'N'' + 4NN'' - 4N'^2\right).$$

The first equation in (15) gives an even more complicated relation between $M$ and $N$ but we will not need its explicit expression here.

The study of the differential equation (20) is much more difficult than in the previous case and in fact some different subcases must be considered again. We will not make here the whole detailed study of these possibilities, and we will only indicate how to handle them in order to write down explicitly the solutions that arise and which also satisfy the first equation in (15). This is due to the fact that most of the calculations have been performed with the aid of computer algebra so that they are impossible to reproduce here. To begin with, we must note that if $A$ vanishes identically, we have from (19) that $Z_0 = 0$ and therefore we also have $Z = 0$ and $F = F_0 e^t$, where $F_0$ is an arbitrary positive constant. It is not difficult to see that in this case, the metric (4) possesses more than two Killing vector fields. Thus, we can assume that $A$ is non-vanishing and divide equation (20) by $A$ and take the derivative with respect to the variable $x$ so that the term in $\left(\dddot{M}M - \ddot{M}^2\right)$ disappears. Consequently, the equation

$$\left(\dddot{M} \dddot{M} - \ddot{M}^2\right) \left(\frac{A'}{A}\right)' + \left(\dddot{M}M - \ddot{M}^2\right) \left(\frac{A''}{A}\right)' + \ddot{M} \left(\frac{A'N' - A''N}{A}\right)' + \ddot{M} \left(\frac{A'N'' - AN''}{A}\right)' + M \left(\frac{4K^2 (N''' - 2N'')}{A}\right)' + \left(\frac{B}{A}\right)' = 0,$$

(21)

must be satisfied. From this equation we learn that either

$$\left(\frac{A'}{A}\right)' = 0 \iff A' = bA, \quad b = \text{ const}$$
or we can divide the equation (21) by \((A'/A)'\) and perform the derivative with respect to \(x\) so that the quadratic term \((\ddot{M}M - \dot{M}M)\) also disappears. It is now obvious that continuing with this procedure we finally get a linear third order differential equation for \(M\)

\[
\alpha_1 \ddot{M} + \alpha_2 \dot{M} + \alpha_3 M + \alpha_4 M + \alpha_5 = 0,
\]

where \(\alpha_i\), \((i=1, \ldots, 5)\) are some complicated expressions depending only on \(x\) which contain some derivatives of quotients of the functions of \(x\) appearing in the original equation (20). Except in the particular case that all the coefficients \(\alpha_i\) in (22) are identically vanishing it follows that \(M\) must satisfy a relation of the type

\[
\ddot{M} = a_1 \dddot{M} + a_2 \ddot{M} + a_3 M + a_4,
\]

where \(a_1, a_2, a_3\) and \(a_4\) are now constants. This equation can be trivially solved for \(M\) although the explicit solution takes different forms depending on the multiplicity and the real or complex character of the roots of the polynomial

\[
X^4 - a_1 X^3 - a_2 X^2 - a_3 X - a_4 = 0.
\]

Using these explicit forms for the function \(M\) some long but straightforward calculations allow us to restrict the values of \(a_1, a_2, a_3\) and \(a_4\) which make the system (15) compatible and then find the solution for \(N\). Considering also the particular cases that we have been letting aside in the way to the equation (22) and which are not contained in (23) we finally get the following set of solutions which constitute the general solution of the system (15) satisfying the condition (18)

**Solution III**

\[
\begin{cases}
N = N_0 \\
M = N_0 + S_0 e^{-2t} + S_1 e^{-6t}
\end{cases}
\]

**Solution IV**

\[
\begin{cases}
N = N_0 + R_0 e^{-6x} \\
M = N_0 + S_0 e^{-2t}
\end{cases}
\]

**Solution V**

\[
\begin{cases}
N = N_0 + e^{2x} \left( R_0 e^{cx} - \frac{4K^2}{c} - K_0^2 \right) \\
M = N_0 + S_1 e^{-2t}
\end{cases}
\]

where the constants \(N_0, R_0, S_0\) and \(S_1\) are arbitrary and the remaining constants \(S_0\) and \(c\) are restricted to be non-vanishing.

Once the explicit forms for the metric functions \(M\) and \(N\) are known, we can substitute in (19) and back into (14) to find the expressions for the functions \(Z(t + x)\). Performing the integration of (11) we finally find the function \(H(t + x)\). The results are trivially found to be
Solution III \[ Z = \frac{K_0^2}{2S_0}e^{2(t+x)} \implies H = H_0 \exp\left(\frac{K_0^2}{4S_0}e^{2(t+x)}\right). \]

Solution IV \[ Z = \frac{3S_0 + K_0^2e^{2(t+x)}}{2S_0} \implies H = H_0e^{3/2(t+x)} \exp\left(\frac{K_0^2}{4S_0}e^{2(t+x)}\right). \]

Solution V \[ Z = \frac{c+2}{c+4} \implies H = H_0e^{c/2(t+x)}. \]

From the expression for \( H \) in the solution V we note that all the metric coefficients \( F \), \( G \) and \( P \) are separable functions of the coordinates \( t \) and \( x \) and therefore this solution is contained in the general case of diagonal cosmologies separable in co-moving coordinates which was exhaustively studied by Ruiz & Senovilla [6]. Thus we will not study this solution here any further and we concentrate in the four families I, II, III and IV.

3 Properties of the Solutions.

In this section we will list all the solutions that we have found and give some of their physical properties such as the energy density and pressure of the perfect fluid, the kinematical quantities of the fluid velocity vector and the Petrov type. We will also discuss whether the solutions contain Friedman-Robertson-Walker models as particular subcases, if the energy conditions are satisfied for some ranges of the values of the parameters and the existence of singularities and their nature.

For each of the four families of the solutions found in the previous section (Solutions I, II, III and IV) we have already written the functions \( M(t) \), \( N(x) \) and \( H(t+x) \), while the metric coefficient \( G(x,t) \) is common for all the cases and is given by
\[ G = e^{t-x}. \]

In order to get all the metric components we still have to determine \( F(x,t) \) and \( P(x) \). The first of these functions is trivially obtained from \( H(x+t) \) using (111) and \( P(x) \) is found by solving the equation (8) which reduces to a quadrature.

For the general metric (2) with \( P \) depending only on \( x \) the non-vanishing components (in the tetrad (4)) of the kinematical quantities for the fluid velocity are given by the following expressions

- **Expansion** \[ \theta = \sqrt{\frac{M}{F}} \left( \frac{F_t}{F} + \frac{G_t}{G} \right) \]

- **Shear tensor** \[ \sigma_{11} = \frac{1}{3} \sqrt{\frac{M}{F}} \left( \frac{F_t}{F} - \frac{G_t}{G} \right) \quad \sigma_{22} = \sigma_{33} = -\frac{1}{2} \sigma_{11} \]

- **Acceleration** \[ a = a_x \, dx = \frac{1}{2} \frac{F_x}{F} \, dx \]
while the vorticity obviously vanishes. The explicit form for the function $G$ and the fact that $F$ depends on the variable $t + x$ except for a global factor $e^t$ imply the relation

$$
\frac{F_x}{F} = \frac{F_t}{F} - \frac{G_t}{G}.
$$

In consequence, it is enough to list the acceleration and the expansion of the fluid for each solution in order to know all the kinematical quantities because the shear tensor components can be evaluated from

$$
\sigma_{22} = \sigma_{33} = -\frac{1}{2} \sigma_{11} = -\frac{1}{3} \sqrt{M} a_x.
$$

(24)

Before listing the possible non-separable families of solutions under the assumptions of this paper, let us note that some of the constants in $M$, $N$ and $H$ above can be reabsorbed by means of a linear coordinate change of the type

$$
x = aX + x_0,
$$

$$
t = aT + t_0.
$$

In the list that follows, this freedom has always been used to absorb all the spurious constants so that the parameters appearing in the solution are relevant (if occasionally some irrelevant constants still remain, we explicitly state this fact as well as the reasons why this is so).

**Solution I**

As indicated in the previous section we have to distinguish between three subcases depending on whether some of the constants $R_0$ or $S_0$ vanish or not.

**$R_0 = 0$**

The complete set of metric potentials for this solution is given by

$$
N = 1, \quad M = 1 + \epsilon e^{-2at}, \quad F = \exp \left( at + \beta a (t + x) + \epsilon e^{2a(t+x)} \right),
$$

$$
G = e^{a(t-x)}, \quad P = \exp (2ce^{ax}),
$$

where $\epsilon$ is a sign. In this family, however, not all the constants $a$, $\beta$ and $c$ are true parameters identifying a particular solution. In fact, a change of the type (25) allows us to fix $a$ to any desired value whenever $\beta$ is non-vanishing, while $a$ is a true parameter when $\beta = 0$. However, we prefer to maintain the constants as they stand in order to study the cases $\beta = 0$ and $\beta \neq 0$ together.
The energy density and pressure are given by
\[ p = \rho = \frac{\epsilon a^2 (2\beta + 3)}{4 \exp (3at + \beta a (t + x) + \epsilon c^2 e^{2a(t+x)})} \]
so that the perfect fluid satisfies a stiff equation of state. The energy (and pressure) is non-negative everywhere if and only if \( \beta \) and \( \epsilon \) satisfy
\[ \epsilon (2\beta + 3) \geq 0. \]

The kinematical quantities of the fluid velocity vector are
\[ \theta = \frac{a \sqrt{1 + \epsilon e^{2ax}} (2e^2 e^{2a(t+x)} + \beta + 3)}{2 \exp \left( \frac{a}{2} t + \frac{\beta}{2} a (t + x) + \epsilon c^2 e^{2a(t+x)} \right)}, \quad a = \frac{a}{2} \left( 2e^2 e^{2a(t+x)} + \beta \right) dx \]

The Petrov type is I except for the following particular cases
- when \( c = 0 \) and \( \beta \neq 0 \) the solution is type D everywhere.
- when \( c = 0, \beta = 0 \) the solution degenerates to conformally flat. Thus, the metric represents a Friedman-Robertson-Walker cosmology with density and pressure (we choose \( \epsilon = +1 \) so that the energy is positive) given by
\[ p = \rho = \frac{3}{4} a^2 e^{-3at}. \]

\( S_0 = 0 \)

The list of metric coefficients is
\[ N = 1 + \epsilon e^{2ax}, \quad M = 1, \quad F = e^{at} \exp \left( \frac{a}{2} \left( 1 - \epsilon \beta^2 \right) (t + x) + \epsilon c^{-1} e^{2a(t+x)} \right), \quad G = e^{a(t-x)}, \quad P = \begin{cases} \left( e^{ax} + \sqrt{1 + e^{2ax}} \right)^\beta & \text{if } \epsilon = +1 \\ \exp \left( \beta \arcsin (e^{ax}) \right) & \text{if } \epsilon = -1 \end{cases} \]

where \( \epsilon \) is a sign and the same considerations made in the previous case regarding the parameters \( a, \beta \) and \( c \) hold; when \( 3 - \epsilon \beta^2 = 0 \) the constants \( a \) and \( c \) identify uniquely a particular solution in the family while when \( 3 - \epsilon \beta^2 \neq 0 \), the constant \( a \) can be fixed to any desired value by means of the coordinate change \( (24) \).

The energy density and pressure read
\[
p = \rho = \frac{\epsilon a^2 c}{\exp\left(3at + \frac{a}{2} (1 - \epsilon \beta^2) (t + x) + ce^{-2a(t+x)}\right)}
\]

so that again the perfect fluid obeys a stiff equation of state. The density (and consequently the pressure) is non-negative in the whole spacetime whenever we have

\[
\epsilon c \geq 0.
\]

The kinematical quantities of the fluid velocity vector take the form

\[
\theta = \frac{a \left(7 - \epsilon \beta^2 - 4e^{-2a(t+x)}\right)}{4 \exp\left(\frac{1}{2}at + \frac{a}{4} (1 - \epsilon \beta^2) (t + x) + \frac{c}{2}e^{-2a(t+x)}\right)}, \quad a = \frac{a}{4} \left(1 - \epsilon \beta^2 - 4ce^{-2a(t+x)}\right) \, dx,
\]

and the Petrov type is I everywhere except for the following particular cases:

- either when \(\beta = 0\) or when \(1 - \epsilon \beta^2 = 0\) and \(c \neq 0\) or when \(9 - \epsilon \beta^2 = 0\) and \(c = 0\) the metric is Petrov type D everywhere.

- when \(1 - \epsilon \beta^2 = 0\) and \(c = 0\) the solution is conformally flat. In this case the density vanishes and the solution is in fact the Minkowski spacetime.

\(S_0 \neq 0\) and \(R_0 \neq 0\)

The metric coefficients for this solution can be written in the following form

\[
N = \sigma + \epsilon_1 e^{2ax}, \quad M = \sigma + \epsilon_2 e^{-2at}, \quad F = e^{at+a\left(\frac{1}{2}-2b-\epsilon_1 \frac{c^2}{2}\right)}(t+x) \left| e^{2a(t+x)} - \epsilon_1 \epsilon_2 \right|^b,
\]

\[
G = e^{a(t-x)}, \quad P = \left\{ \begin{array}{l}
\left(e^{ax} + \sqrt{\sigma + e^{2ax}}\right)^c \\
\exp\left(c \arcsin\left(e^{ax}\right)\right)
\end{array} \right\}
\]

where both \(\epsilon_1\) and \(\epsilon_2\) are signs and \(\sigma\) can take the values +1, -1 or 0 (obviously when any of the two signs \(\epsilon_1\) or \(\epsilon_2\) are -1 then we necessarily have \(\sigma = +1\)). The density and pressure are

\[
p = \rho = \frac{a^2 \epsilon_2 (4 - 4b - \epsilon_1 c^2)}{4 \left| e^{2a(t+x)} - \epsilon_1 \epsilon_2 \right|^b \exp\left(\frac{1}{2}at + \frac{1}{2}ax - a \left(2b + \epsilon_1 \frac{c^2}{2}\right) (t + x)\right)},
\]

which satisfy the energy conditions whenever the constants are restricted to

\[
\epsilon_2 \left(4 - 4b - \epsilon_1 c^2\right) \geq 0.
\]

The kinematical quantities read

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\[ \theta = \frac{a \sqrt{a + \epsilon_2 e^{-2at}} \left( e^{2a(t+x)} \left[ 7 - \epsilon_1 c^2 \right] + \epsilon_2 c^2 - 7 \epsilon_2 \epsilon_1 + 4b \epsilon_1 \epsilon_2 \right)}{4 \exp \left( \frac{3at + \frac{1}{4}ax}{2} - a \left( b + \epsilon_1 \frac{c^2}{4} \right) (t + x) \right) \left| e^{2a(t+x)} - \epsilon_1 \epsilon_2 \right|^2 \left( e^{2a(t+x)} - \epsilon_1 \epsilon_2 \right)}, \]

\[ a = \frac{a \left( e^{2a(t+x)} \left[ 1 - \epsilon_1 c^2 \right] + \epsilon_2 c^2 - \epsilon_1 \epsilon_2 + 4b \epsilon_1 \epsilon_2 \right)}{4 \left( e^{2a(t+x)} - \epsilon_1 \epsilon_2 \right)} \, dx \]

and the Petrov type is again I except for the following particular values of the constants

- when \( c = 0 \) the solution degenerates to type D everywhere.
- when \( \epsilon_1 c^2 = 1 \) and \( b = 0 \) the solution is conformally flat and, therefore, it represents a Friedman-Robertson-Walker cosmology with density (and pressure) given by (choosing \( \epsilon_2 = +1 \) so that the energy conditions are fulfilled)

\[ p = \rho = a^2 e^{-3at}. \]

Before finishing with the study of this Solution I let us indicate that all these \( p = \rho \) solutions can be seen to be generated from vacuum solutions using the procedure due to Wainwright, Ince & Marshman \[12\]. However, as far as we know, they were previously unknown.

**Solution II**

The metric coefficients are

\[ M = \alpha + \epsilon e^{2at}, \quad N = \alpha + e^{2ax}, \quad F = \exp \left( at + ce^{-2a(t+x)} \right), \]

\[ G = e^{a(t-x)}, \quad P = e^{ax} + \sqrt{e^{2ax} + \alpha}, \]

where \( \epsilon \) is a sign and the coordinate change \[25\] still allows us to fix the constant \( \alpha \) equal to \( \pm 1 \) (depending on its sign) when \( \alpha \neq 0 \) or set \( a \) equal to any desired value when \( \alpha = 0 \). We allow this spurious constant in the family in order to consider the two subcases together.

The expressions for the energy density and pressure read

\[ \rho = \frac{a^2 \left( 3\epsilon e^{4at+2ax} - 4\epsilon ce^{2at} + 4ce^{2ax} \right)}{4 \exp \left( 3at + 2ax + ce^{-2a(t+x)} \right)}, \quad p = \rho - \frac{2a^2 \epsilon e^{at}}{\exp \left( ce^{-2a(t+x)} \right)} \]
so that the condition for positive energy density is, obviously
\[ 3\epsilon e^{4at + 2ax} - 4\epsilon ce^{2at} + 4ce^{2ax} \geq 0. \] (25)

In order to determine the ranges for the parameters \( \alpha \) and \( c \) which make this condition possible everywhere we must distinguish between two cases depending on the sign of \( \alpha \). If \( \alpha < 0 \) it follows from the positivity of \( M \) that \( \epsilon = 1 \) and then the allowed ranges for the coordinates \( t \) and \( x \) are (we are choosing the constant \( a \) positive without loss of generality because a global change of sign in the coordinates \( t \) and \( x \) also changes the sign of \( a \))
\[ t > \frac{1}{2a} \log \left(-\alpha\right), \quad x > \frac{1}{2a} \log \left(-\alpha\right). \]

The condition (25) is fulfilled everywhere in this region whenever we have
\[ \epsilon = 1, \quad c \geq -\frac{3\alpha^2}{4}, \quad \alpha < 0. \]

However, from the expression for the pressure it follows that there always exist regions of the spacetime which do not fulfil the condition \( \rho + p \geq 0 \) so that we will no longer consider this subcase as physically admissible.

When \( \alpha \geq 0 \), the coordinate \( x \) can take arbitrarily large negative values and the dominant term in the relation (25) when \( x \to -\infty \) implies \( \epsilon c \leq 0 \). If \( \epsilon = +1 \) the coordinate \( t \) can also take arbitrarily large negative values and then (25) in the limit \( t \to -\infty \) clearly shows that the only possibility is \( c = 0 \). On the other hand, if \( \epsilon = -1 \), the expression for \( M \) implies that \( \alpha \) must be strictly positive so that we can use the freedom mentioned above to fix \( \alpha = 1 \) and then the ranges for the coordinates \( t \) and \( x \) are given by
\[ t < 0, \quad -\infty < x < +\infty. \]

It is not difficult to see that (25) is fulfilled in this region whenever \( c \) is restricted to
\[ \epsilon = -1 \quad c \geq \frac{3}{4}. \]

The expression for the pressure shows that it is always bigger than the density and therefore positive everywhere. Thus, the only energy condition which is not fulfilled is \( p \leq \rho \).

It can be easily seen that when we approach \( t = 0 \) along any timelike curve, the proper time remains finite. Consequently, either the spacetime is singular at \( t = 0 \) or it
is extendible and our coordinate system only covers a portion of the whole inextendible spacetime. An analysis of the curvature invariants near \( t = 0 \) shows that all of them remain finite so that this solution is likely to be extendible. The same consideration holds when \( x \to +\infty \) where the proper length remains finite and all the curvature invariants have also a finite limit there. In order to show that this metric is in fact extendible, let us perform the change of variables

\[
\begin{align*}
  t &= -\frac{1}{a} \log \cosh (aT) \quad \text{and} \quad x = -\frac{1}{a} \log \sinh (ar) \\
\end{align*}
\]

where the range of variation of the new coordinates is given by

\[
T < 0, \quad r > 0.
\]

The metric in this new coordinates take the following simple form

\[
\begin{align*}
  ds^2 &= \frac{1}{\cosh^2(2aT)} \left[ e^{c \cosh^2(2aT) \sinh^2(2ar)} \left( -dT^2 + dr^2 \right) + \cosh^2(ar) \, dz^2 + \frac{\sinh^2(ar)}{a^2} \, d\phi^2 \right],
\end{align*}
\]

where we have redefined \( a \to 2a \) and we have renamed \( z \to \phi \) and \( y \to z \) because this metric is cylindrically symmetric with a regular axis of symmetry at \( r = 0 \) where the so-called elementary flatness condition is satisfied. Thus, we need not extend the coordinate \( r \) as it represents a radial coordinate and the metric is already complete in the spacelike coordinates. Regarding the coordinate \( T \), nothing prevents us to extend its variation range to the whole real line so that we have in fact extended the original spacetime which is included as the \( T < 0 \) region. This solution is inextendible and the density and pressure in the new coordinates read

\[
\begin{align*}
  \rho &= \frac{a^2 \left( 4c \cosh^4(2aT) + 4c \cosh^2(2aT) \sinh^2(2ar) - 3 \right)}{\cosh(2aT) \exp \left( c \cosh^2(2aT) \sinh^2(2ar) \right)}, \\
  p &= \rho + \frac{8a^2}{\cosh(2aT) \exp \left( c \cosh^2(2aT) \sinh^2(2ar) \right)}.
\end{align*}
\]

which are now obviously non-negative (provided \( c \geq 3/4 \) as stated above).

The kinematical quantities of this solution are

\[
\begin{align*}
  \theta &= \frac{a \sinh(2aT) \left( 2c \cosh^2(2aT) \sinh^2(2ar) - 3 \right)}{\sqrt{\cosh(2aT) \exp \left( \frac{2}{5} \cosh^2(2aT) \sinh^2(2ar) \right)}}, \\
  a &= ac \cosh^2(2aT) \cosh(4ar) \, dr.
\end{align*}
\]

This solution is Petrov type I everywhere (provided \( c \neq 0 \)) except on the axis of symmetry \( r = 0 \) where it degenerates to type D as requires the axial symmetry.
From the expressions above we see that both the density and pressure are regular everywhere and also the kinematical quantities are non-singular in the whole spacetime. In fact, it is easy to see that all the Riemann invariants of this solution are regular everywhere so that this metric contains no singularity at all. Thus we have found a cosmological solution which satisfies the Strong Energy Condition everywhere and which contains no singularity at all. The first solution satisfying energy conditions and being complete and regular everywhere was found by Senovilla [4], this solution was generalized later to a rather large family in [6] where diagonal separable metrics in co-moving coordinates were studied. Some recent work has tried to isolate that family as being in some sense unique [13]. However, very recently the author has found a solution which is also complete and non-singular (obviously satisfying energy conditions) and such that the line-element is non-diagonal and with separable metric functions [14]. The family that we present in this paper is the first known non-singular model with non-separable metric coefficients.

Finally, let us note that in the particular case $c = 0$ (which does not satisfy the energy conditions) all the kinematical quantities of this solution except the expansion vanish identically and the perfect fluid satisfies a linear equation of state $p = \frac{-5}{3}\rho$. Thus, this family contains a Friedman-Roberson-Walker model (with a physically unreasonably equation of state) when the constant $c$ vanishes.

**Solution III**

The metric functions for this solution read

$$M = 1 + \epsilon e^{-2at} + \beta e^{-6at}, \quad N = 1, \quad F = \exp \left( at + \epsilon c^2 e^{2a(t+x)} \right),$$

$$G = e^{a(t-x)}, \quad P = \exp \left( 2ce^{ax} \right),$$

with energy density and pressure given by

$$\rho = \frac{a^2 \left( 3\epsilon e^{4at} + 4\epsilon c^2 \beta e^{2a(t+x)} + 3\beta \right)}{4 \exp \left( 7at + \epsilon c^2 e^{2a(t+x)} \right)}, \quad p = \rho + \frac{2a^2 \beta}{\exp \left( 7at + \epsilon c^2 e^{2a(t+x)} \right)}.$$

From the form of the function $M$ it follows that the coordinate $t$ can take arbitrarily large values so that the expression for the density clearly imposes $\epsilon = 1$ in order to fulfil the energy conditions there. Similarly, the coordinate $x$ can take arbitrarily large values and, given that the dominant term in the density when $x \to +\infty$ is $\beta c^2 e^{2a(t+x)}$, we have to impose $\beta \geq 0$ in order to satisfy the non-negativity of the energy everywhere. Summing up, we restrict the constants of the solution to satisfy

$$\epsilon = +1, \quad \beta \geq 0$$
so that the density is positive everywhere. Unfortunately, the pressure is always bigger than the density except for $\beta = 0$ (this particular case is exactly the solution $\epsilon = +1$, $\beta = 0$ in the subcase $R_0 = 0$ of the Solution I, so that we will not consider it here any longer).

The ranges of variation for the coordinates $t$ and $x$ are

$$-\infty < t < +\infty, \quad -\infty < x < \infty.$$  

The density and pressure diverge when $t$ tends to $-\infty$ and it is very easy to show that this singularity is at finite proper time in the past. On the other hand, the density and pressure tend to zero when the time coordinate tends to $+\infty$ and it can be trivially seen that this happens at an infinite proper time in the future so that this solution presents a big-bang-like singularity in the past and it is complete in the future. It is also easy to prove that this solution contains no other singularities and that it is complete in the coordinate $x$.

The kinematical quantities of the fluid velocity vector read

$$\theta = \frac{a}{4} \sqrt{1 + e^{-2at} + \beta e^{-6at}} \frac{4e^2 e^{2a(t+x)} + 6}{\exp \left( \frac{1}{2}at + \frac{a^2}{4}e^{2a(t+x)} \right)}, \quad a = ac^2 e^{2a(t+x)}dx.$$  

It follows that the expansion is everywhere positive and it diverges when we approach the big-bang-like singularity while it tends to zero when $t$ tends to $+\infty$. It is interesting to note that the acceleration (and therefore also the shear tensor) and the Weyl tensor tend to zero near the big-bang and the density and pressure satisfy a linear equation of state in the limit near the big bang. In fact, the asymptotic behaviour of the density and pressure when $t \to -\infty$ is given by

$$\rho \to \frac{3}{4} a^2 \beta e^{-7at}, \quad p \to \frac{11}{4} a^2 \beta e^{-7at}$$

so that near the big bang they satisfy approximately $p = \frac{11}{4} \rho$. Consequently, the metric is nearly homogeneous and isotropic (a nearly Friedman-Robertson-Walker geometry) in the vicinities of the big-bang and this geometry inhomogenizes and anisotropizes as the time goes on. In fact, the curvature singularity at $t = -\infty$ is Ricci dominated in the sense that

$$\lim_{t \to -\infty} \frac{C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}}{R_{\alpha\beta} R^{\alpha\beta}} = 0,$$

where obviously $C_{\alpha\beta\gamma\delta}$ stands for the Weyl tensor and $R_{\alpha\beta}$ is the Ricci tensor. Furthermore, in the particular case $\beta = 0$ (which implies the equation of state $p = \rho$) the singularity at $t = -\infty$ is an isotropic singularity in the sense that the metric is conformally related to a metric which is regular at the $t = -\infty$ hypersurface (see [15] and
for a precise definition). The Weyl tensor \( C^\alpha_{\beta\gamma\delta} \) in the natural coordinate cobasis \( \{ t, x, y, z \} \) is regular but non-vanishing at \( t = -\infty \) so that the so called Weyl tensor hypothesis is not satisfied. This hypothesis (see Penrose [17]) assures that the appropriate thermodynamic boundary condition for the Universe is that the Weyl tensor should vanish at any initial singularity. It has been conjectured and partially proved by Tod (see [18], [19]) that a perfect fluid model satisfying a barotropic equation of state with an isotropic singularity and satisfying the Weyl tensor hypothesis is necessarily an exact Friedman-Robertson-Walker cosmology everywhere. Thus, the particular solution \( \beta = 0 \) here presented supports this conjecture due to Tod.

Regarding the Petrov type of this solution, it can be easily seen that it is Petrov type I everywhere, except when \( c \) vanishes. In this particular case, the acceleration and the shear tensor also vanish and the density and pressure depend only on \( t \) so that they obey an equation of state. Thus, when \( c = 0 \), the solution is an exact Friedman-Roberson-Walker geometry.

**Solution IV**

The metric coefficients of this solution are given by

\[
M = 1 + \epsilon e^{-2at}, \quad N = 1 - e^{-6ax}, \quad F = \exp \left( at + \frac{3}{2}a (t + x) + \epsilon c^2 e^{2a(t+x)} \right),
\]

\[
G = e^{a(t-x)}, \quad P = \exp \left( \int \frac{2ace^{ax} dx}{\sqrt{1 - e^{-6ax}}} \right),
\]

where \( \epsilon \) is a sign and the expressions for the energy density and pressure are

\[
\rho = \frac{a^2 (2\epsilon c^2 e^{4at+2ax} + 9e^{2at} + 3\epsilon c^2 e^{6ax})}{2 \exp \left( \frac{9}{2}at + \frac{15}{2}ax + \epsilon c^2 e^{2a(t+x)} \right)}, \quad p = \rho - \frac{4a^2}{\exp \left( \frac{9}{2}at + \frac{15}{2}ax + \epsilon c^2 e^{2a(t+x)} \right)}.
\]

The range of variation of the coordinate \( x \) is

\[ x > 0 \]

so that the positivity condition for the density when \( x \to +\infty \) implies \( \epsilon = +1 \). With this value for \( \epsilon \) it follows that the \( \rho \) is positive everywhere and the pressure is also positive everywhere and satisfies

\[ p \leq \rho. \]

Thus, this solution satisfies both the Strong and Dominant energy conditions everywhere.
Given that $\epsilon = +1$ the range of variation for the coordinate $t$ is the whole real line. When this coordinate tends to $-\infty$, both the density and pressure diverge as

$$p, \rho \to \frac{3}{2}a^2 e^{-\frac{3}{2}a(3t+x)}.$$ 

This singularity at $t \to -\infty$ is at a finite proper time in the past and therefore this solution presents a big-bang-like singularity. On the other hand, when $t$ approaches $+\infty$, the density and pressure tend to zero and the proper time also diverges so that this solution contains no singularity in the future. Regarding the coordinate $x$, the proper length diverges when this coordinate tends to $+\infty$ so that the metric is complete in that direction. However, the proper length of the apparent singularity at $x = 0$ is finite so that either the solution is truly singular there or we must extend it beyond this value by means of an appropriate coordinate change. In order to perform this extension we define new coordinates $X$ and $T$ as

$$t = \frac{1}{a} \log [\sinh(aT)],$$

$$x = \frac{1}{3a} \log [\cosh(3aX)],$$

in which the metric takes the form

$$ds^2 = \sinh(aT) \left[ \sinh^{\frac{2}{3}}(aT) \cosh^{\frac{1}{3}}(3aX) e^{\frac{2}{3} \sinh^2(aT) \cosh^{\frac{2}{3}}(3aX)} \left(-dT^2 + dX^2\right) + \frac{e^{\frac{2}{3}ac} \int \cosh^{\frac{1}{3}}(3aX) dX}{\cosh^{\frac{1}{3}}(3aX)} dy^2 + \frac{e^{-\frac{2}{3}ac} \int \cosh^{\frac{1}{3}}(3aX) dX}{\cosh^{\frac{1}{3}}(3aX)} dz^2 \right].$$

This metric is regular at $X = 0$ (which corresponds to $x = 0$) and the range of variation of the new coordinate $X$ can be extended to the whole real line. An analysis of this metric shows that the only singularity of this extended manifold is the big-bang singularity (which is now situated at $T = 0$) while the solution is complete in the future and in the spacelike coordinates. The density and pressure in the new coordinates read

$$\rho = \frac{a^2 \left(2c^2 \sinh^4(aT) \cosh^{\frac{2}{3}}(3aX) + 9 \sinh^2(aT) + 3 \cosh^2(3aX)\right)}{2 \sinh^{\frac{2}{3}}(aT) \cosh^{\frac{2}{3}}(3aX) e^{\frac{2}{3} \sinh^2(aT) \cosh^{\frac{2}{3}}(3aX)}},$$

$$p = \rho - \frac{4a^2}{\sinh^{\frac{2}{3}}(aT) \cosh^{\frac{2}{3}}(3aX) e^{\frac{2}{3} \sinh^2(aT) \cosh^{\frac{2}{3}}(3aX)}},$$

which are obviously positive everywhere and satisfying all the energy conditions at any point of the extended spacetime. Finally, the kinematical quantities of the fluid velocity vector are given by
\[
\theta = \frac{a \cosh(aT) \left(9 + 4c^2 \sinh^2(aT) \cosh^2(3aX)\right)}{4 \sinh^2(aT) \cosh^2(3aX) e^{c^2 \sinh^2(aT) \cosh^2(3aX)}}.
\]

\[
a = \frac{a \sinh(3aX)}{4 \cosh(3aX)} \left(3 + 4c^2 \sinh^2(aT) \cosh^2(3aX)\right) dX.
\]

This solution can be seen of Petrov type I independently of the values of the arbitrary parameters of the solution and, consequently, this family does not contain a Friedman-Robertson-Walker model as a particular case.

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