THE KRULL–GABRIEL DIMENSION OF DISCRETE DERIVED CATEGORIES

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Abstract. We compute the Krull–Gabriel dimension of the category of perfect complexes for finite dimensional algebras which are derived discrete.

INTRODUCTION

Let $k$ be an algebraically closed field and $\Lambda$ a finite dimensional $k$-algebra. We denote by $\text{mod} \Lambda$ the category of finitely presented $\Lambda$-modules and by $\text{proj} \Lambda$ the full subcategory of finitely generated projective $\Lambda$-modules.

The Krull–Gabriel dimension of the representation theory of $\Lambda$ is an invariant first studied by Geigle [11]. For this invariant one considers the abelian category $\mathcal{C} = \text{Ab}(\text{mod} \Lambda)$ of finitely presented functors $\text{mod} \Lambda \to \text{Ab}$ into the category of abelian groups. The Krull–Gabriel dimension $\text{KGdim} \mathcal{C}$ of $\mathcal{C}$ is by definition the smallest integer $n$ such that $\mathcal{C}$ admits a filtration by Serre subcategories

$$0 = C_{-1} \subseteq C_0 \subseteq \ldots \subseteq C_n = C,$$

where $C_i/C_{i-1}$ is the full subcategory of all objects of finite length in $C/C_{i-1}$.

We have $\text{KGdim} \mathcal{C} = 0$ if and only if $\Lambda$ is of finite representation type by a classical result of Auslander [1], and $\text{KGdim} \mathcal{C} \neq 1$ by a result of Herzog [14] and Krause [16]. In his thesis [11], Geigle proved that $\text{KGdim} \mathcal{C} = 2$, when $\Lambda$ is tame hereditary.

In this work we investigate the category of perfect complexes which is by definition the bounded derived category $\mathcal{D}^b(\text{proj} \Lambda)$. We compute the Krull–Gabriel dimension $\text{KGdim} \mathcal{C}$ of $\mathcal{C}$ is by definition the smallest integer $n$ such that $\mathcal{C}$ admits a filtration by Serre subcategories

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The rest of this note is devoted to proving this theorem. For an elementary description of the Krull–Gabriel dimension, see Proposition 2.2.

Main Theorem. Let $\Lambda$ be a finite dimensional $k$-algebra.

1. If $\Lambda$ is derived discrete and piecewise hereditary, then

$$\text{KGdim} \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda)) = 0.$$ 

2. If $\Lambda$ is derived discrete and not piecewise hereditary, then

$$\text{KGdim} \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda)) = \begin{cases} 1 & \text{if } \text{gl. dim} \Lambda = \infty, \\ 2 & \text{if } \text{gl. dim} \Lambda < \infty. \end{cases}$$

3. If $\Lambda$ is not derived discrete, then

$$\text{KGdim} \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda)) \geq 2.$$ 

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Conventions. By \( \mathbb{Z} \), \( \mathbb{N} \), and \( \mathbb{N}_+ \), we denote the sets of integers, nonnegative integers, and positive integers, respectively. For \( i, j \in \mathbb{Z} \), set
\[
[i, j] := \{ l \in \mathbb{Z} \mid i \leq l \leq j \}.
\]
Furthermore, \([i, \infty) := \{ l \in \mathbb{Z} \mid i \leq l \} \) and \((\neg \infty, j] := \{ l \in \mathbb{Z} \mid l \leq j \} \).

1. Derived discrete algebras

Let \( \Lambda \) be a finite dimensional \( k \)-algebra. The algebra \( \Lambda \) is called derived discrete if for each sequence \( (h_n)_{n \in \mathbb{Z}} \) of nonnegative integers there are only finitely many isomorphism classes of indecomposable objects \( X \) in \( \mathcal{D}^b(\text{proj} \, \Lambda) \) such that \( \dim_h H^n(X) = h_n \) for each \( n \in \mathbb{Z} \). Note that Vossieck’s original definition [19] uses the category \( \mathcal{D}^b(\text{mod} \, \Lambda) \), but he has shown that both versions are equivalent. The one we use is more adequate in our setup.

In [19], it is shown that an algebra \( \Lambda \) is derived discrete if and only if either \( \Lambda \) is piecewise hereditary of Dynkin type or \( \Lambda \) is a one-cycle gentle algebra not satisfying the clock condition. Recall from [13] that \( \Lambda \) is piecewise hereditary if it is derived equivalent to a finite dimensional hereditary algebra. The class of one-cycle gentle algebras not satisfying the clock condition has been further studied in [4]. There it is shown that if \( \Lambda \) is a derived discrete algebra and not piecewise hereditary of Dynkin type, then \( \Lambda \) is derived equivalent to an algebra of the form \( \Lambda(r, n, m) \), for some triple \((r, n, m) \in \Omega \). Here, \( \Omega \) denotes the set of all triples \((r, n, m) \) of nonnegative integers such that \( 1 \leq r \leq n \), and \( \Lambda(r, n, m) \) is the path algebra of the quiver

\[
\begin{array}{ccccccc}
\bullet & \overset{\alpha_m}{\rightarrow} & \overset{\alpha_{m+1}}{\rightarrow} & \cdots & \overset{\alpha_{n-1}}{\rightarrow} & \overset{\alpha_0}{\leftarrow} & \overset{\alpha_n}{\downarrow} \\
& & & & & & \overset{\alpha_{n-r}}{\leftarrow} \cdots \overset{\alpha_{n-2}}{\leftarrow} \overset{\alpha_{n-1}}{\leftarrow} \overset{\alpha_{n-r+1}}{\rightarrow} \\
\end{array}
\]

bound by the relations
\[
\alpha_{n-r+1}\alpha_{n-r}, \ldots, \alpha_{n-2}\alpha_{n-1}, \alpha_0\alpha_{n-1}.
\]

Prototypical examples to have in mind are the algebra \( \Lambda(1, 1, 0) \) which equals the algebra \( k[e] \) of dual numbers \( (e^2 = 0) \), and its Auslander algebra \( \Lambda(1, 2, 0) \). Note that \( \text{gl} \, \dim \Lambda(1, 1, 0) = \infty \) while \( \text{gl} \, \dim \Lambda(1, 2, 0) = 2 \).

2. Krull–Gabriel dimension

Let \( C \) be an abelian category. A full subcategory \( C' \subseteq C \) is called a Serre subcategory if it is closed under subobjects, quotients and extensions. If \( C' \subseteq C \) is a Serre subcategory, then one defines the quotient category \( C/C' \) as follows. The objects of \( C/C' \) coincide with the objects of \( C \), and if \( X \) and \( Y \) are objects of \( C \), then
\[
\text{Hom}_{C/C'}(X, Y) := \varinjlim \text{Hom}_C(X', Y/Y'),
\]
where \( X' \) and \( Y' \) run through all subobjects of \( X \) and \( Y \), respectively, such that \( X/X' \) and \( Y/Y' \) belong to \( C' \).

Following Gabriel [10, IV.1] and Geigle [11, §2], the Krull–Gabriel dimension \( \text{KDim} \, C \) of \( C \) is defined as follows. Let \( C_{-1} := 0 \). and for each \( n \in \mathbb{N} \) denote by \( C_n \) the full subcategory of all objects \( X \) in \( C \) which are of finite length, when viewed as objects of \( C/C_{n-1} \). Then \( \text{KDim} \, C \) equals the smallest \( n \) such that \( C_n = C \) (and \( \infty \) when such \( n \) does not exist).
Let \( \mathcal{T} \) be a triangulated category. Following Freyd [9, §3] and Verdier [18, II.3], we consider the abelianisation \( \text{Ab}(\mathcal{T}) \) of \( \mathcal{T} \) which is the abelian category of finitely presented functors \( F : \mathcal{T} \to \text{Ab} \) into the category \( \text{Ab} \) of abelian groups. Recall that a functor \( F : \mathcal{T} \to \text{Ab} \) is finitely presented (finitely generated, respectively) if there exists an exact sequence of the form \( H_Y \to H_X \to F \to 0 \) (\( H_X \to F \to 0 \), respectively). Here, for an object \( X \) in \( \mathcal{T} \), we denote by \( H_X \) the representable functor \( \text{Hom}(X, -) : \mathcal{T} \to \text{Ab} \). Similarly, if \( f : X \to Y \) is a morphism in \( \mathcal{T} \), then we denote by \( H_f \) the induced morphism \( H_Y \to H_X \). The cohomological functor \( \iota : \mathcal{T} \to \text{Ab}(\mathcal{T}) \) sending \( X \in \mathcal{T} \) to \( H_X \) is universal in the following sense. If \( \varphi : \mathcal{T} \to \mathcal{A} \) is a contravariant cohomological functor, then there exists a unique exact functor \( \varphi^\prime : \text{Ab}(\mathcal{T}) \to \mathcal{A} \), such that \( \varphi = \varphi^\prime \circ \iota \).

Now let \( \Lambda \) be any ring. We wish to compute the Krull–Gabriel dimension of \( \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda)) \) and begin with an elementary observation. To this end fix a modular lattice \( L \). Denote by \( L' \) the quotient which is obtained by collapsing all finite length intervals in \( L \). Set \( L_{-1} = L \) and \( L_n = (L_{n-1})' \) for \( n \in \mathbb{N} \). The dimension of \( L \) is the smallest \( n \) such that \( L = 0 \).

**Lemma 2.1** ([16, Lemma 1.1]). Let \( C \) be an abelian category and \( X \) an object. For the lattice \( L_C(X) \) of subobjects we have \( L_C(X)_n \cong L_{C/C_n}(X) \) for all \( n \in \mathbb{N} \). □

This lemma suggests an alternative description of the Krull–Gabriel dimension which avoids the formation of quotient categories.

**Proposition 2.2.** Let \( \Lambda \) be any ring.

1. The finitely generated subfunctors of the forgetful functor \( \text{mod} \Lambda \to \text{Ab} \) form a modular lattice and its dimension equals \( \text{KGdim} \text{Ab}(\text{mod} \Lambda) \).

2. The finitely generated subfunctors of \( H^0 : \mathcal{D}^b(\text{proj} \Lambda) \to \text{Ab} \) form a modular lattice and its dimension equals \( \text{KGdim} \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda)) \).

**Proof.** (1) The lattice of finitely generated subfunctors of \( F = \text{Hom}_\Lambda(\Lambda, -) \) equals the lattice of subobjects of \( F \) in \( \text{Ab}(\text{mod} \Lambda) \). Given a Serre subcategory \( \mathcal{C} \subseteq \text{Ab}(\text{mod} \Lambda) \), we have \( F \in \mathcal{C} \) if and only if \( \mathcal{C} = \text{Ab}(\text{mod} \Lambda) \). Now apply Lemma 2.1.

(2) The lattice of finitely generated subfunctors of \( H^0 \) equals the lattice of subobjects of \( H_\Lambda \) in \( \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda)) \), where \( \Lambda \) is viewed as a complex concentrated in degree zero. Now apply Lemma 2.1, keeping in mind that \( \Lambda \) generates \( \mathcal{D}^b(\text{proj} \Lambda) \) as a triangulated category. □

From now on suppose that \( \Lambda \) is a finite dimensional \( k \)-algebra and set \( \mathcal{C} := \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda)) \). For the description of \( \mathcal{C}_0 \) one uses the well-known fact that the simple objects in \( \mathcal{C} \) correspond to the Auslander–Reiten triangles in \( \mathcal{D}^b(\text{proj} \Lambda) \). Namely, if \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\eta} \Sigma X \) is an Auslander–Reiten triangle in \( \mathcal{D}^b(\text{proj} \Lambda) \), then \( H^0_Z/\text{Im}H_f \) is a simple object in \( \mathcal{C} \), and every simple object in \( \mathcal{C} \) is of this form. This follows directly from the definition of an Auslander–Reiten triangle. Consequently, if \( F \in \mathcal{C} \), then \( F \in \mathcal{C}_0 \) if and only if

\[
\sum_{X \in \text{ind} \mathcal{D}^b(\text{proj} \Lambda)} \dim_k F(X) < \infty,
\]

where \( \text{ind} \mathcal{D}^b(\text{proj} \Lambda) \) denotes a fixed set of representatives of the indecomposable objects in \( \mathcal{D}^b(\text{proj} \Lambda) \); see also [1, §2] for the above description of simple and finite length objects. This condition immediately implies the first part of the Main Theorem.

**Proposition 2.3.** Let \( \Lambda \) be a derived discrete algebra which is piecewise hereditary. Then

\[
\text{KGdim} \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda)) = 0.
\]
Proof. If \( \Lambda \) is a derived discrete algebra, which is piecewise hereditary, then \( \Lambda \) is piecewise hereditary of Dynkin type. The well-known description of \( \mathcal{D}^b(\text{proj } \Lambda) \) in this case (see for example [13]), immediately implies that
\[
\sum_{X \in \text{ind } \mathcal{D}^b(\text{proj } \Lambda)} \dim_k H_M(X) < \infty
\]
for each complex \( M \) in \( \mathcal{D}^b(\text{proj } \Lambda) \). Consequently,
\[
\sum_{X \in \text{ind } \mathcal{D}^b(\text{proj } \Lambda)} \dim_k F(X) < \infty
\]
for each \( F \in \text{Ab}(\mathcal{D}^b(\text{proj } \Lambda)) \), hence the claim follows. \( \square \)

3. Krull–Gabriel dimension and generic objects

Let \( \Lambda \) be a finite dimensional \( k \)-algebra. Generic modules were introduced by Crawley-Boevey in order to describe the representation type of an algebra [7, 8]. Following [12], an indecomposable object \( X \) of the unbounded derived category \( \mathcal{D}(\text{Mod } \Lambda) \) of all \( \Lambda \)-modules is called \textit{generic}, if \( H^i(X) \) is a finite length \( \text{End}(X) \)-module, for each \( i \in \mathbb{Z} \), but \( X \) is not in \( \mathcal{D}^b(\text{mod } \Lambda) \). Derived discrete algebras can be characterised in terms of generic complexes. This follows from work of Bautista [2] and we recall the following result.

Proposition 3.1 ([2, Theorem 1.1]). Let \( \Lambda \) be a finite dimensional \( k \)-algebra which is not derived discrete. Then there exists a generic object \( X \) in \( \mathcal{D}(\text{Mod } \Lambda) \) such that the division ring \( \text{End}(X)/\text{rad } \text{End}(X) \) contains an element which is transcendental over \( k \). \( \square \)

Note that the description of the endomorphism ring in [2, Theorem 1.1] is a consequence of the proof which uses [8, Theorem 9.5].

Next we combine Bautista’s result with an argument due to Herzog [14]. To be precise, Herzog proves a result about the abelianisation \( \text{Ab}(\text{mod } \Lambda) \), but the same argument works for \( \text{Ab}(\mathcal{D}^b(\text{proj } \Lambda)) \) and yields the following.

Proposition 3.2 ([14, Theorem 3.6]). Let \( \Lambda \) be a finite dimensional \( k \)-algebra. If there exists a generic object \( X \) in \( \mathcal{D}(\text{Mod } \Lambda) \) such that \( \text{End}(X)/\text{rad } \text{End}(X) \) contains an element which is transcendental over \( k \), then
\[
\text{KGdim } \text{Ab}(\mathcal{D}^b(\text{proj } \Lambda)) \geq 2.
\]
(\( \square \)

Our claim about the Krull–Gabriel dimension of an algebra which is not derived discrete is an immediate consequence.

Corollary 3.3. Let \( \Lambda \) be a finite dimensional \( k \)-algebra which is not derived discrete. Then
\[
\text{KGdim } \text{Ab}(\mathcal{D}^b(\text{proj } \Lambda)) \geq 2.
\]
(\( \square \)

4. The category of perfect complexes

Throughout this section we fix \( (r, n, m) \in \Omega \) such that \( r < n \) and we put \( \Lambda := \Lambda(r, n, m) \). Note that the condition \( r < n \) is equivalent to \( \text{gl.dim } \Lambda < \infty \). In this section we follow [3] and describe a quiver \( \Gamma \) together with a set \( R \) of relations such that the category \( \mathcal{D}^b(\text{proj } \Lambda) \) is equivalent to the path category \( k\Gamma \) modulo the given relations (see for example [17, §2.1] for the definition of \( k\Gamma \)). We refer to [5] for a detailed study of morphisms in \( \mathcal{D}^b(\text{proj } \Lambda) \); the diagrams in there might help to understand our calculations.
For $i \in [0, r - 1]$ we set

$$I_i := \mathbb{Z}^2,$$

$$I'_i := \{(a, b) \in \mathbb{Z}^2 \mid a \leq b + \delta_{i,0} \cdot m\},$$

$$I''_i := \{(a, b) \in \mathbb{Z}^2 \mid a + \delta_{i,0} \cdot n \leq b\},$$

where $\delta_{x,y}$ is the Kronecker delta. The set of vertices of $\Gamma$ is

$$\Gamma_0 := \{X^{(i)}_v \mid i \in [0, r - 1], v \in I'_i\} \cup \{Y^{(i)}_v \mid i \in [0, r - 1], v \in I''_i\} \cup \{Z^{(i)}_v \mid i \in [0, r - 1], v \in I_i\}.$$  

Now we describe the arrows in $\Gamma$ and associate to each arrow a degree. There are three cases.

1. Fix $i \in [0, r - 1]$ and $v := (a, b) \in I'_i$. We put

$$\mathcal{I}^{(i)}_v := [a, b + \delta_{i,0} \cdot m] \times [b, \infty),$$

$$\mathcal{X}^{(i)}_v := [a, b + \delta_{i,0} \cdot m] \times \mathbb{Z},$$

$$\mathcal{X}''^{(i)}_v := (-\infty, a + \delta_{i,r-1} \cdot m] \times [a, b + \delta_{i,0} \cdot m].$$

For $u \in \mathcal{I}^{(i)}_v$, $u \neq v$, there is an arrow $f^{(i)}_{v,u} : X^{(i)}_v \to X^{(i)}_u$ of degree 0. Next, for $u \in X^{(i)}_v$ there is an arrow $g^{(i)}_{v,u} : X^{(i)}_v \to Z^{(i)}_u$ of degree 1. Finally, for $u \in X''^{(i)}_v$ there is an arrow $e^{(i)}_{v,u} : X^{(i)}_v \to X^{(i+1)}_u$ of degree 2, where we always change the upper index modulo $r$.

2. Fix $i \in [0, r - 1]$ and $v := (a, b) \in I''_i$. We put

$$\mathcal{I}''^{(i)}_v := [a, b - \delta_{i,0} \cdot n] \times [b, \infty),$$

$$\mathcal{Y}^{(i)}_v := \mathbb{Z} \times [a, b - \delta_{i,0} \cdot n],$$

$$\mathcal{Y}''^{(i)}_v := (-\infty, a - \delta_{i,r-1} \cdot n] \times [a, b - \delta_{i,0} \cdot n].$$

For $u \in \mathcal{I}''^{(i)}_v$, $u \neq v$, there is an arrow $f^{(i)}_{v,u} : Y^{(i)}_v \to Y^{(i)}_u$ of degree 0. Next, for $u \in Y^{(i)}_v$ there is an arrow $g^{(i)}_{v,u} : Y^{(i)}_v \to Z^{(i)}_u$ of degree 1. Finally, for $u \in Y''^{(i)}_v$ there is an arrow $e^{(i)}_{v,u} : Y^{(i)}_v \to Y^{(i+1)}_u$ of degree 2.

3. Fix $i \in [0, r - 1]$ and $v := (a, b) \in I_i$. We put

$$\mathcal{I}^{(i)}_v := [a, \infty) \times [b, \infty),$$

$$\mathcal{Z}^{(i)}_v := (-\infty, a + \delta_{i,r-1} \cdot m] \times [a, \infty),$$

$$\mathcal{Z}''^{(i)}_v := (-\infty, b - \delta_{i,r-1} \cdot n] \times [b, \infty),$$

$$\mathcal{Z}''^{(i)}_v := (-\infty, a + \delta_{i,r-1} \cdot m] \times [a, \infty),$$

$$\mathcal{Z}^{(i)}_v := (-\infty, b - \delta_{i,r-1} \cdot n] \times [b, \infty).$$

For $u \in \mathcal{I}^{(i)}_v$, $u \neq v$, there is an arrow $f^{(i)}_{v,u} : Z^{(i)}_v \to Z^{(i)}_u$ of degree 0. Next, for $u \in Z^{(i)}_v$ there is an arrow $h^{(i)}_{v,u} : Z^{(i)}_v \to X^{(i+1)}_u$ of degree 1. Similarly, for $u \in Z''^{(i)}_v$ there is an arrow $h^{(i)}_{v,u} : Z^{(i)}_v \to Y^{(i+1)}_u$ of degree 1. Finally, for $u \in Z''^{(i)}_v$ there is an arrow $e^{(i)}_{v,u} : Z^{(i)}_v \to Z^{(i+1)}_u$ of degree 2.

Now we describe the set $\mathcal{R}$ of relations. Let $f : X \to Y$ and $g : Y \to Z$ be arrows of degree $p$ and $q$, respectively. If there is an arrow $h : X \to Z$ of degree $p + q$, then we have the relation $gf = h$, otherwise we have the relation $gf = 0$.

We summarise our construction.

**Proposition 4.1.** There exists a $k$-linear equivalence $k\Gamma/\mathcal{R} \sim \mathcal{D}^b(\text{proj } \Lambda)$.

**Proof.** The Auslander–Reiten quiver of $\mathcal{D}^b(\text{proj } \Lambda)$ has been described in [4]. It consists of $2r$ components of type $\mathbb{Z}A_{\infty}$ (they correspond to $X$- and $Y$-vertices of $\Gamma$) and $r$ components of type $\mathbb{Z}A_{\infty}^c$ (they correspond to $Z$-vertices of $\Gamma$). Moreover,
under the action of the shift $\Sigma$ the components fall into three orbits, consisting of $r$ components each. The objects lying on the border of $\mathbb{Z}A_\infty$ components have been also identified. Using string combinatorics [6, 15] it is straightforward to verify the description of $H_X$ for $X$ lying on the border of a component of type $\mathbb{Z}A_\infty$.

By induction, using Auslander–Reiten triangles, the description of $H_X$ follows for each $X$ in the components of type $\mathbb{Z}A_\infty$. Then we verify this description for two (cleverly) chosen neighbouring objects in a component of type $\mathbb{Z}A_\infty$ and proceed again by induction (using Auslander–Reiten triangles) to finish the proof. □

For future use we introduce the following notation. For $i \in [0, r - 1]$ set

$$f^{(i)}_{v\cdot u} := \text{id}_{X^{(i)}_v}, \quad f^{(i)}_{v\cdot0} := \text{id}_{v^{(i)}_v}, \quad f^{(i)}_{0\cdot v} := \text{id}_{Z^{(i)}_v}, \quad v \in I_i.$$ 

Also, we let $f^{(i)}_{v\cdot u}$ denote the zero morphism $X^{(i)}_v \to 0$, if $v \in I_i'$, and $u \notin I_i'$. The same convention applies to $f^{(i)}_{v\cdot u}$, $h^{(i)}_{v\cdot u}$, and $h^{(i)}_{v\cdot u}$.

5. The Krull–Gabriel dimension for algebras of finite global dimension

Let $(r, n, m) \in \Omega$ be such that $r < n$. We consider $\mathcal{C} := \text{Ab}(\mathcal{D}^b(\text{proj} \Lambda))$ for $\Lambda := \Lambda(r, n, m)$. Our aim is to prove that $C_1 \neq \mathcal{C}$, but $C_2 = \mathcal{C}$. In order to show the latter claim, it is enough to prove that $H_U \in C_2$ for each indecomposable $U \in \mathcal{D}^b(\text{proj} \Lambda)$. In fact, we will prove that $H_U$ is either zero or simple in $\mathcal{C}/C_1$.

Note that in order to prove that $H_U$ (more generally, $H_U/\text{Im} H_g$, where $g : U \to M$ is a morphism) is either zero or simple in $\mathcal{C}/C_1$ for some $n \in \mathbb{N}$, it is enough to prove that, for every non-zero map $f : U \to V$ with $V$ indecomposable, either $\text{Im} H_f \mid_{\mathcal{C}/C_{n-1}} \cong 0$ or $\text{Im} H_f/\text{Im} H_g \mid_{\mathcal{C}/C_{n-1}} \cong H_U/\text{Im} H_g$, where the subscript means that the equalities hold in $\mathcal{C}/C_{n-1}$.

We begin with the description of the simple objects in $\mathcal{C}$.

Lemma 5.1. The simple object in $\mathcal{C}$ are

1. $A^{(i)}_v := H_{\mathcal{X}^{(i)}_v}/\text{Im} H_{f_{v\cdot v + \{0, 1\}}^{(i)}, f_{v\cdot v + \{0, 1\}}^{(i)}}, i \in [0, r - 1], v \in I_i'$

2. $A^{(i)}_{v\cdot0} := H_{\mathcal{Y}^{(i)}_v}/\text{Im} H_{f_{v\cdot v + \{0, 1\}}^{(i)}, f_{v\cdot v + \{0, 1\}}^{(i)}}, i \in [0, r - 1], v \in I_i'$

3. $A^{(i)}_0 := H_{\mathcal{Z}^{(i)}_v}/\text{Im} H_{f_{v\cdot v + \{0, 1\}}^{(i)}, f_{v\cdot v + \{0, 1\}}^{(i)}}, i \in [0, r - 1], v \in I_i.$

Proof. This follows from the well-known description of the Auslander–Reiten triangles in $\mathcal{D}^b(\text{proj} \Lambda)$; see Section 2. □

Now we move to the category $\mathcal{C}/C_0$. We first describe the simple objects.

Lemma 5.2. The objects

1. $B^{(i)}_{a, b, b'} := H_{\mathcal{X}^{(i)}_{(a, b)}}, f_{(a, b), (a+1, b)}^{(i)}, g_{(a, b), (a, b')}^{(i)}}, i \in [0, r - 1], (a, b) \in I_i', b' \in \mathbb{Z},$

2. $B^{(i)}_{a, b, b'} := H_{\mathcal{Y}^{(i)}_{(a, b)}}, f_{(a, b), (a+1, b)}^{(i)}, g_{(a, b), (a, b')}^{(i)}}, i \in [0, r - 1], (a, b) \in I_i', b' \in \mathbb{Z},$

3. $C^{(i)}_{a, b, a'} := H_{\mathcal{Z}^{(i)}_{(a, b)}}, f_{(a, b), (a+1, b)}^{(i)}, g_{(a, b), (a, 0)}^{(i)}}, i \in [0, r - 1], (a, b) \in I_i, b' \in (\pm \infty, a + \delta, a + \delta, a), a' \in (\pm \infty, b - \delta, b - \delta),$
Remark 5.3. One may easily show that every simple object in $C/\mathcal{C}_0$ is (up to isomorphism) of the above form. It is also not difficult to describe the isomorphism classes of the above objects.

Proof. We only prove the first claim; the remaining ones are proved similarly. Let $i \in [0, r - 1]$, $(a, b) \in I'_i$ and $b' \in \mathbb{Z}$. It is clear that $B^{(i)}_{a, b, b'} \neq 0$. This follows, since for each $n \in \mathbb{N}$ we have in $C$ a short exact sequence

$$0 \to B^{(i)}_{a, b + n + 1, b'} \to B^{(i)}_{a, b + n, b'} \to A^{(i)}_{a, b + n} \to 0.$$ 

For a non-zero morphism $f : X^{(i)}_{(a, b)} \to V$ with $V$ indecomposable we put

$$B'_f := \text{Im} H_f / (\text{Im} H_f \cap \text{Im} H_{f'_{(a, b), (a + 1, b')} g^{(i)}_{(a, b), (a, b')} v}) .$$

We have to show that either $B'_f = 0$ or $B'_f = B^{(i)}_{a, b, b'}$ for every $f$ as above. We may assume that we are in one of the following cases:

(1) $V = X^{(i)}_{(a, b)}$ and $f = f'_{(a, b), (c, d)}$ for some $(c, d) \in I^{(i)}_{(a, b)}$,

(2) $V = Z^{(i)}_{(a, b)}$ and $f = g_{(a, b), (c, d)}$ for some $(c, d) \in X^{(i)}_{(a, b)}$,

(3) $V = X^{(i+1)}_{(c, d)}$ and $f = f'_{(c, d), (a, b)}$ for some $(c, d) \in A^{(i)}_{(a, b)}$.

Case (1). If $c > a$, then $f$ factors through $f'_{(a, b), (a + 1, b)}$, hence $B'_f$ is the zero subobject of $B^{(i)}_{a, b, b'}$. If $c = a$, then we prove by induction on $d$ that $B'_f \cong B^{(i)}_{a, b, b'}$. Indeed, if $d = b$, then the claim is obvious. If $d > b$, then we have a short exact sequence

$$0 \to B'_f \to B'_{f'_{(a, b), (a, d - 1)}} \to A^{(i)}_{(a, d - 1)} \to 0,$$

hence $B'_f \cong B'_{f'_{(a, b), (a, d - 1)}}$ by Lemma 5.1. Moreover, $B'_{f'_{(a, b), (a, d - 1)}} \cong B^{(i)}_{a, b, b'}$ by induction.

Case (2). We prove that $B'_f = 0$ in this case. Again, we may assume that $c = a$. If $d \geq b'$, then $f$ factors through $g_{(a, b), (a, b')}$, hence $B'_f$ is again zero. Finally, if $d < b'$, then we have a short exact sequence

$$0 \to B'_f \to B'_{g_{(a, b), (c, d + 1)}} \to B'_{f} \to A_{(a, d)} \to 0,$$

hence the claim follows by an obvious induction.

Case (3). In this case $f$ factors through $(f'_{(a, b), (a + 1, b)}, g^{(i)}_{(a, b), (a, b')})^T$, hence $B'_f$ is zero.

Now we show that some representable functors have finite length in $C/\mathcal{C}_0$.

Lemma 5.4. Let $i \in [0, r - 1]$.

(1) If $v \in I'_i$, then $H_{X^{(i)}_{v}} \in \mathcal{C}_1$.

(2) If $v \in I''_i$, then $H_{X^{(i)}_v} \in \mathcal{C}_1$.

Proof. Again, we only prove the first claim. Let $v = (a, b)$. The claim is shown by induction on $b - a$. If $b - a = \delta_{0,0} \cdot m$, then we have a short exact sequence

$$0 \to C^{(i)}_{a, b, a + \delta_{0, r-1}, m+1} \to H_{X_v^{(i)}} \to B^{(i)}_{a, b, 0} \to 0,$$

hence the claim follows from Lemma 5.2. If $b - a > \delta_{0, 0} \cdot m$, then we have exact sequences

$$H_{X_{(a, b+1, b)}^{(i)}} \to H_{X_v^{(i)}} \to H_{X_{v'}^{(i)}} / \text{Im H}_{f'_{(a, b), (a + 1, b)}} \to 0.$$
and
\[ 0 \to C^{(i)}_{a,0,a+\delta_{r-1},m+1} \to H_{X^{(i)}_r} / \text{Im } H_{f^{(i)}_{(a,b),(a+n,b)}} \to B^{(i)}_{(a,b),0} \to 0, \]
and the claim follows by induction and Lemma 5.2.

Next we show that the remaining representable functors corresponding to the indecomposable objects in $\mathcal{D}^b(\text{proj } \Lambda)$ are not of finite length in $\mathcal{C}/\mathcal{C}_0$.

**Lemma 5.5.** If $i \in [0, r - 1]$ and $v \in I_i$, then $H_{Z^{(i)}_v} \not\in \mathcal{C}_1$.

**Proof.** Let $v = (a, b)$. For each $n \in \mathbb{N}$ we have the following exact sequence
\[ 0 \to \text{Im } H_{f^{(i)}_{(a,b),(a+n,b)}} \to \text{Im } H_{f^{(i)}_{(a,b),(a+n,b)}} \to C^{(i)}_{a+n,b,a+\delta_{r-1},m+1} \to 0, \]
which implies the claim (note that $\text{Im } H_{f^{(i)}_{(a,b),(a,b)}} = H_{Z^{(i)}_v}$).

Finally, we show that $\mathcal{C}_2 = \mathcal{C}$. For this we only need to prove the following.

**Lemma 5.6.** If $i \in [0, r - 1]$ and $v \in I_i$, then $H_{Z^{(i)}_v} \in \mathcal{C}_2$.

**Proof.** Let $v = (a, b)$. We know from Lemma 5.5 that $H_{Z^{(i)}_v}$ is a non-zero object in $\mathcal{C}_1$. In order to prove it is simple we fix a non-zero morphism $f : Z^{(i)}_v \to V$. We may assume that we are in one of the following cases:

1. $V = Z_{(c,d)}^{(i)}$ and $f = f_{(a,b),(c,d)}^{(i)}$ for some $(c, d) \in T^{(i)}_{(a,b)}$,
2. $V = X_{(c,d)}^{(i+1)}$ and $f = h_{(a,b),(c,d)}^{(i)}$ for some $(c, d) \in Z^{(i)}_{(a,b)}$,
3. $V = Y_{(c,d)}^{(i+1)}$ and $f = h_{(a,b),(c,d)}^{(i)}$ for some $(c, d) \in Z^{(i)}_{(a,b)}$,
4. $V = Z_{(c,d)}^{(i+1)}$ and $f = e_{(a,b),(c,d)}^{(i)}$ for some $(c, d) \in Z^{(i)}_{(a,b)}$.

Case (1). We prove by induction on $c + d$ that $\text{Im } H_{f_{C/C_1}} = H_{Z^{(i)}_v}$. If $c + d = a + b$ (i.e., $c = a$ and $d = b$), the claim is obvious. Assume $c > a$. Then we have an exact sequence
\[ 0 \to \text{Im } H_{f_{(a,b),(c-1,d)}} \to C^{(i)}_{c-1,d,a+\delta_{r-1},m+1} \to 0, \]
hence $\text{Im } H_{f_{C/C_1}^{(i)}_{(a,b),(c-1,d)}}$ by Lemma 5.2. Moreover, we have by induction
\[ \text{Im } H_{f^{(i)}_{(a,b),(c-1,d)}}_{C/C_1} = H_{Z^{(i)}_v}. \]
We proceed similarly if $d > b$.

Case (2). We have an epimorphism $H_{X_{(c,d)}^{(i+1)}} \to \text{Im } H_{f_{C/C_1}}$, hence $\text{Im } H_{f_{C/C_1}} = 0$ by Lemma 5.4.

Case (3). Analogous to Case (2).

Case (4). Again, we have an epimorphism $H_{X_{(c,d)}^{(i+1)}} \to \text{Im } H_{f_{C/C_1}}$, hence we get $\text{Im } H_{f_{C/C_1}} = 0$ (in fact one may even prove that $\text{Im } H_{f_{C/C_1}} \in \mathcal{C}_0$ in this case).

6. The algebras of infinite global dimension

Throughout this section fix $n \in \mathbb{N}_+$ and $m \in \mathbb{N}$. We put $\Lambda := \Lambda(n, n, m)$. The aim of this section is to prove that $\text{Kdim } \mathcal{C} = 1$, where $\mathcal{C} := \text{Ab} (\mathcal{D}^b (\text{proj } \Lambda))$. The basic idea is to extract the $X$-part of the arguments in the finite global dimension case. We explain this in more detail.

First we describe the category $\mathcal{D}^b (\text{proj } \Lambda)$. Let $\Gamma$ be the quiver with the vertices $X^{(i)}_v$ for $i \in [0, n - 1]$ and $v \in I_i$, where
\[ I_i := \{(a, b) \in \mathbb{Z}^2 \mid a \leq b + \delta_{i,0} \cdot m\}. \]
For \( i \in [0, n-1] \) and \( v = (a, b) \in I_i \) we define
\[
\mathcal{I}_v^{(i)} := [a, b + \delta_{0,0} \cdot m] \times [b, \infty)
\]
\[
\mathcal{X}_v^{(i)} := (-\infty, a + \delta_{i,n-1} \cdot m] \times [a, b + \delta_{i,0} \cdot m].
\]

Then for each \( i \in [0, n-1] \), \( v \in I_i^{(i)} \), and \( u \in I_i^{(i)}, u \neq v \), we have an arrow \( f_{v,u}^{(i)} : X_v^{(i)} \to X_u^{(i)} \) of degree 0, and for each \( i \in [0, n-1] \), \( v \in I_i^{(i)} \), and \( u \in \mathcal{X}_v^{(i)} \), we have an arrow \( e_{v,u}^{(i)} : X_v^{(i)} \to X_u^{(i+1)} \) of degree 1. Finally, by \( \mathcal{R} \) we denote the set of the following relations. Let \( f : X \to X' \) and \( g : X' \to X'' \) be arrows of degree \( p \) and \( q \), respectively. If there is an arrow \( h : X \to X'' \) of degree \( p + q \), then we have the relation \( gf = h \), otherwise we have the relation \( gf = 0 \) (an explicit list of relations can be found in [3, \S 5]).

**Proposition 6.1.** There exists a \( k \)-linear equivalence \( k\Gamma/\mathcal{R} \sim \mathcal{D}^b(\text{proj } \Lambda) \).

**Proof.** Analogous to the proof of Proposition 4.1. \( \Box \)

It is obvious that \( \mathcal{C}_0 \neq \mathcal{C} \). In order to prove \( \mathcal{C}_1 = \mathcal{C} \), it suffices to show that \( H_U \in C_1 \) for each indecomposable \( U \in \mathcal{D}^b(\text{proj } \Lambda) \). The arguments are similar to those used in Section 5 and we state the analogues of Lemmas 5.1, 5.2 and 5.4 without proofs. Again, we use the convention that \( f_{v,u}^{(i)} (e_{v,u}^{(i)}) \) denotes the zero morphism \( X_v^{(i)} \to 0 \) if \( i \in [0, n-1] \), \( v \in I_i \), and \( u \notin I_i \) (or \( I_{i+1} \), respectively).

**Lemma 6.2.** The simple objects in \( \mathcal{C} \) are
\[
A_v^{(i)} := H_{X_v^{(i)}} / \text{Im } H_{(f_{v,v+1}^{(i)}, f_{v,v+(0,1)}) v}
\]
for \( i \in [0, n-1] \) and \( v \in I_i \).

**Lemma 6.3.** The objects
\[
B_{a,b,a'}^{(i)} := H_{X_{a,b}^{(i)}} / \text{Im } H_{(f_{a,b}, (a+1,b), v_{a,b},(a',a)) v}
\]
for \( i \in [0, n-1] \), \( (a, b) \in I_i \), and \( (a', b) \in (-\infty, a + \delta_{i,n-1} \cdot m] \), are simple in \( \mathcal{C}/\mathcal{C}_0 \).

**Lemma 6.4.** If \( i \in [0, n-1] \) and \( v \in I_i \), then \( H_{X_v^{(i)}} \in \mathcal{C}_1 \).

7. **Concluding remarks**

There are two other important triangulated categories, which one often studies for a finite dimensional algebra \( \Lambda \): the bounded derived category \( \mathcal{D}^b(\text{mod } \Lambda) \) and the stable category \( \text{mod } \hat{\Lambda} \) of the repetitive algebra \( \hat{\Lambda} \). Thus one may also ask about the Krull–Gabriel dimensions of the abelianisations of these two categories. We have the following result.

**Theorem 7.1.** Let \( \Lambda \) be a finite dimensional \( k \)-algebra and denote by \( \mathcal{C} \) either \( \text{Ab}(\mathcal{D}^b(\text{mod } \Lambda)) \) or \( \text{Ab}(\text{mod } \hat{\Lambda}) \). Then \( \text{KGdim } \mathcal{C} \neq 1 \), and \( \text{KGdim } \mathcal{C} = 0 \) if and only if \( \Lambda \) is piecewise hereditary of Dynkin type.

**Proof.** We apply the Main Theorem and use the chain of fully faithful exact functors
\[
\mathcal{D}^b(\text{proj } \Lambda) \to \mathcal{D}^b(\text{mod } \Lambda) \to \text{mod } \hat{\Lambda}.
\]

If \( \Lambda \) is piecewise hereditary of Dynkin type, then \( \Lambda \) is derived discrete and \( \mathcal{D}^b(\text{proj } \Lambda) = \mathcal{D}^b(\text{mod } \Lambda) = \text{mod } \hat{\Lambda} \). Thus \( \text{KGdim } \mathcal{C} = 0 \).

If \( \Lambda \) is not derived discrete, then Lemma 7.2 below yields
\[
2 \leq \text{KGdim } \text{Ab}(\mathcal{D}^b(\text{proj } \Lambda)) \leq \text{KGdim } \text{Ab}(\mathcal{D}^b(\text{mod } \Lambda)) \leq \text{KGdim } \text{Ab}(\text{mod } \hat{\Lambda}).
\]

Finally, assume \( \Lambda \) is derived discrete, but not piecewise hereditary. If \( \text{gl. dim } \Lambda < \infty \), then again \( \mathcal{D}^b(\text{proj } \Lambda) = \mathcal{D}^b(\text{mod } \Lambda) = \text{mod } \hat{\Lambda} \). Thus assume \( \text{gl. dim } \Lambda = \infty \).
In this case the description of $\text{mod} \Lambda$ is the same as the description of $D^b(\text{proj} \Lambda)$ given in Section 4, hence the arguments from Section 5 apply. On the other hand, $D^b(\text{mod} \Lambda)$ lies strictly between $D^b(\text{proj} \Lambda)$ and $\text{mod} \Lambda$, but some of the $\mathbb{Z}$-modules from Section 4 survive (see [4] for details) and the corresponding representable functors are not of finite length in $\mathcal{C}/\mathcal{C}_0$ when $\mathcal{C} = \text{Ab}(D^b(\text{mod} \Lambda))$.

In the above proof the following lemma is used.

**Lemma 7.2.** Let $S$ be a thick subcategory of a triangulated category $\mathcal{T}$. Then $\text{KGdim} \text{Ab}(S) \leq \text{KGdim} \text{Ab}(\mathcal{T})$.

**Proof.** The universal property of the abelianisation yields an exact embedding $\text{Ab}(S) \to \text{Ab}(\mathcal{T})$. An easy induction shows that $\text{Ab}(\mathcal{T})_n \cap \text{Ab}(S) \subseteq \text{Ab}(S)_n$ for each $n \in \mathbb{N}$.

We observe that

$$\text{KGdim} \text{Ab}(D^b(\text{mod} \Lambda)) = \text{KGdim} \text{Ab}(\text{mod} \Lambda)$$

if $\Lambda$ is derived discrete. We do not know whether this equality holds for arbitrary finite dimensional $k$-algebra $\Lambda$.

**References**

[1] M. Auslander, *Representation theory of Artin algebras. II*, Comm. Algebra 1 (1974), 269–310.
[2] R. Bautista, *Derived endo-discrete Artin algebras*, Colloq. Math. 105 (2006), 297–310.
[3] G. Bobiński, *The graded centers of derived discrete algebras*, J. Algebra 333 (2011), 55–66.
[4] G. Bobiński, Ch. Geiß, and A. Skowroński, *Classification of discrete derived categories*, Cent. Eur. J. Math. 2 (2004), 19–49.
[5] N. Broomhead, D. Pauksztello, and D. Ploog, *Discrete derived categories I: homomorphisms, autoequivalences and t-structures*, available at arXiv:1312.5203.
[6] W. W. Crawley-Boevey, *Maps between representations of zero-relation algebras*, J. Algebra 126 (1989), 259–263.
[7] , *Tame algebras and generic modules*, Proc. London Math. Soc. (3) 63 (1991), 241–265.
[8] , *Modules of finite length over their endomorphism rings*, Representations of algebras and related topics (Kyoto, 1990), London Math. Soc. Lecture Note Ser., vol. 168, Cambridge Univ. Press, Cambridge, 1992, pp. 127–184.
[9] P. Freyd, *Stable homotopy*, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 121–172.
[10] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962), 323–448 (French).
[11] W. Geigle, *The Krull-Gabriel dimension of the representation theory of a tame hereditary Artin algebra and applications to the structure of exact sequences*, Manuscripta Math. 54 (1985), 83–106.
[12] Ch. Geiss and H. Krause, *On the notion of derived tameness*, J. Algebra Appl. 1 (2002), 133–157.
[13] D. Happel, *Triangulated Categories in the Representation Theory of Finite-dimensional Algebras*, London Math. Soc. Lecture Note Ser., vol. 119, Cambridge Univ. Press, Cambridge, 1988.
[14] I. Herzog, *The endomorphism ring of a localized coherent functor*, J. Algebra 191 (1997), 416–426.
[15] H. Krause, *Maps between tree and band modules*, J. Algebra 137 (1991), 186–194.
[16] , *Generic modules over Artin algebras*, Proc. London Math. Soc. (3) 76 (1998), 276–306.
[17] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math., vol. 1099, Springer, Berlin, 1984.
[18] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque 239 (1996), xii+253 pp.
[19] D. Vossieck, *The algebras with discrete derived category*, J. Algebra 243 (2001), 168–176.
