Remarks on Quantum Aspects of 3D-Gravity in the First-Order Formalism

L. M. de Moraes*, J. A. Helayël-Neto†, V. J. Vásquez Otoya ‡
Centro Brasileiro de Pesquisas Físicas (CBPF), R. Dr Xavier Sigaud, 150 Urca, Rio de Janeiro, RJ, Brazil - 22290-180

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Abstract

In this paper, we reassess the issue of working out the propagators and identifying the spectrum of excitations associated to the vielbein and spin connection of (1+2)-D gravity in the presence of torsion by adopting the first-order formulation. A number of peculiarities is pointed out whenever the Chern-Simons term is taken into account along with the possible bilinear terms in the torsion tensor. We present a procedure to derive the full set of propagators, based on a set of spin-type operators, and we discuss under which conditions the pole of these tree-level 2-point functions correspond to physical excitations.

1 Introduction

The need to better understand gauge fields has lead to an widespread use of local transformations due the natural manner gauge fields appear in it. In the attempt to write (1+2)-D gravity as a gauge theory, the formulation requires some specific technicalities, by virtue of the possibility of including the so-called (topological) Chern-Simons term. Adopting the Poincaré group as the local gauge group, one naturally obtains the curvature and torsion tensor by means of the Cartan’s structure equations. The translational part of the Poincaré group is represented by the vielbein gauge fields, $e_\alpha^a$, which are also diffeomorphic invariant under general coordinate transformations, and the Lorentzian part — realizing the equivalence principle — given by the spin connection gauge fields, $\omega_\alpha^{ab}$. The vielbein fields associate to each point a locally flat coordinate system and the spin connection relates any two local Lorentz coordinate systems at the given point.

* moraes@cbpf.br
† helayel@cbpf.br
‡ vjose@cbpf.br
This formalism is believed to be completely equivalent to the formalism that employs affine connections and define curvature and torsion by means of it. There is a great deal of results that confirm this, mainly at the level of expressions to the curvature and torsion. At the classical level, this equivalence is indeed true. However, where we go over to the quantum field-theoretic version, there appear remarkable differences and we must indeed adopt the vielbein and spin connection as the independent fundamental degrees of freedom [1].

In order to investigate this further, we begin with the analysis of a traditional (1+2)-D gravity model previously done by two of us [2], where we studied the inclusion of torsion in three-dimensional Einstein-Chern-Simons gravity and added up higher-derivative terms, all in the affine connection formalism. In this work, due to invertibility problems that appear in the theory using the local formalism, we are forced to change to a simpler Lagrangian, where we consider the Einstein-Chern-Simons Lagrangian with torsion algebraic terms only.

Due to the importance of the torsion terms, it is worthwhile to remember that torsion was introduced by E. Cartan in 1922, as the antisymmetric part of the affine connection and was recognized by him as a geometric object related to an intrinsic angular moment of matter. After the introduction of the spin concept, it was suggested that torsion should mediate a contact interaction between spinning particles without propagation in matter-free space [3],[4]. Later, due the fact that at the microscopic level particles are classified by their mass and spin according to the Poincaré group, gauge theories of General Relativity were developed that brings in it dynamic torsion [5],[6]. These theories are motivated by the requirement that the Dirac equation in a gravitational field preserves local invariance under Lorentz transformations which yields, across the minimal coupling, a direct interaction between torsion and fermions. Observational constraints for a propagating torsion and its matter interactions are discussed in [7],[8],[9],[10].

Our work is organised according to the following outline: in Section 2, we present a quick review of the Einstein-Cartan formalism, with the purpose of fixing notation and setting our conventions. Next, the general model and the decomposition of the action in terms of spin operators is the subject of Section 3, where we point out a serious problem related to a spin-2 excitation. This motivates us to introduce and to analyse a number of torsion terms in the action, which is done in Section 4. In Section 5, we come to the task of computing the propagators and we analyse thereby their poles with the corresponding residues, in order to locate the physically relevant regions in the parameter space. Finally, in Section 6, we present our Concluding Comments, with a critical discussion on our main results and possible issues for future investigation.
2 Well-known Results on the Einstein-Cartan Approach

A Riemann-Cartan space-time \([3],[10]\) is defined as a manifold where the covariant derivative of the metric field exists and is given by:

\[
\nabla_\gamma g_{\alpha\beta}(x) = 0, \tag{1}
\]

where this equation defines the so called metric-compatible affine connection, \(\Gamma_{\alpha\beta}^\gamma\); it allows the presence of torsion, given by the antisymmetric part of the affine connection,

\[
T_{\alpha\beta}^\gamma = 2\Gamma_{[\alpha\beta]}^\gamma. \tag{2}
\]

We then have:

\[
\Gamma_{\alpha\beta}^\gamma = \left\{ \gamma_{\alpha\beta} \right\} + K_{\alpha\beta}^\gamma, \tag{3}
\]

where \(\left\{ \gamma_{\alpha\beta} \right\}\) is the Christoffel symbol, which is completely determined by the metric,

\[
\left\{ \gamma_{\alpha\beta} \right\} = \frac{1}{2} g^{\gamma\lambda}(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) \tag{4}
\]

and

\[
K_{\alpha\beta}^\gamma = \frac{1}{2}(T_{\alpha\beta}^\gamma + T_{\gamma\alpha\beta}^\gamma - T_{\beta}^\gamma_{\alpha\lambda}). \tag{5}
\]

is the contortion tensor, antisymmetric in the last two indices.

In order to study local properties one introduces (in our specific (1+2)-D case) the dreinbein vector fields, \(e_\alpha^a(x)\), that spans at any given point the local Minkowski space-time, which in this work has metric: \(\eta_{ab} = diag(1,-1,-1)\).

The introduction of the tangent Minkowski space-time allows local Lorentz transformations on geometrical objects (with Latin index). In order to render these objects invariant under local Lorentz rotations, one introduces the spin connection \(\omega^\gamma_{\alpha\beta}\). The covariant derivative of the dreinbein then reads:

\[
\nabla_\gamma e_\alpha^a = D_\gamma e_\alpha^a - \Gamma_{\gamma\alpha}^\lambda e_\lambda^a = 0, \tag{6}
\]

where \(D_\gamma e_\alpha^a = \partial_\gamma e_\alpha^a + \omega_\gamma^\alpha e_\alpha^i\) is the Lorentz covariant derivative.

One finds from eq. (6) that the affine connection can then be written as:

\[
\Gamma_{\alpha\beta}^\gamma = e_j^\gamma D_\alpha e_\beta^j, \tag{7}
\]

and the torsion tensor, eq. (2), reads

\[
T_{\alpha\beta}^\gamma = 2\Gamma_{[\alpha\beta]}^\gamma = e_j^\gamma(\partial_\alpha e_\beta^j - \partial_\beta e_\alpha^j + \omega_\alpha^i e_\beta^j - \omega_\beta^i e_\alpha^j). \tag{8}
\]
As known, the curvature tensors and scalar are given in terms of the affine connection by the expressions:

\[ R_{\mu \alpha \beta \nu} = \partial_\mu \Gamma_{\alpha \beta \nu} - \partial_\alpha \Gamma_{\mu \beta \nu} + \Gamma_{\mu \rho}^{\nu} \Gamma_{\alpha \beta \rho} - \Gamma_{\alpha \rho}^{\nu} \Gamma_{\mu \beta \rho}, \quad (9) \]

\[ R_{\alpha \beta} = R_{\mu \alpha \beta \mu} = \partial_\mu \Gamma_{\alpha \beta \mu} - \partial_\alpha \Gamma_{\mu \beta \mu} + \Gamma_{\mu \rho \mu} \Gamma_{\alpha \beta \rho} - \Gamma_{\alpha \rho \mu} \Gamma_{\mu \beta \rho}, \quad (10) \]

and

\[ R = g^{\alpha \beta} R_{\alpha \beta}. \quad (11) \]

In terms of the spin connection,

\[ R_{\mu \alpha \beta \nu} = e^\beta_i e^j_\nu (\partial_\mu \omega_{\alpha \iota}^{\iota} - \partial_\alpha \omega_{\mu \iota}^{\iota} + \omega_{\mu k}^{\iota} \omega_{\alpha \iota}^{k} - \omega_{\alpha k}^{\iota} \omega_{\mu \iota}^{k}), \quad (12) \]

\[ R_{\alpha \beta} = e^\beta_i e^j_\mu (\partial_\mu \omega_{\alpha \iota}^{\iota} - \partial_\alpha \omega_{\mu \iota}^{\iota} + \omega_{\mu k}^{\iota} \omega_{\alpha \iota}^{k} - \omega_{\alpha k}^{\iota} \omega_{\mu \iota}^{k}) \quad (13) \]

and

\[ R = \eta^i a^\alpha_i e^j_\alpha (\partial_\mu \omega_{\alpha \iota}^{\iota} - \partial_\alpha \omega_{\mu \iota}^{\iota} + \omega_{\mu k}^{\iota} \omega_{\alpha \iota}^{k} - \omega_{\alpha k}^{\iota} \omega_{\mu \iota}^{k}). \quad (14) \]

### 3 A Problem Related to a Spin-2 Excitation

We start off with the three-dimensional action for topologically massive gravity:

\[ S = \int d^3x \, e (a_1 R + a_2 R^2 + a_3 R_{\alpha \beta} R^{\alpha \beta} + a_4 L_{CS}), \quad (15) \]

where

\[ L_{CS} = \varepsilon^{\alpha \beta \gamma} \Gamma^{\lambda \delta}_\gamma \left( \partial_\alpha \Gamma_{\beta \delta}^{\iota} + \frac{2}{3} \Gamma^{\iota \delta}_\alpha \Gamma_{\beta \lambda}^{\iota} \right), \quad (16) \]

is the topological Chern-Simons term and

\[ \varepsilon^{\alpha \beta \gamma} = \frac{\epsilon^{\alpha \beta \gamma}}{e} \quad (17) \]

is the completely antisymmetric tensor in (1+2)-D, with \( \epsilon^{\alpha \beta \gamma} \) the Levi-Civita tensor density in the flat space and \( e = \sqrt{g} \) where \( g = \det(g_{\alpha \beta}) = \eta \varepsilon^2 \). \( a_1 \), \( a_2 \) and \( a_3 \) are free coefficients, whereas \( a_4 \) is the Chern-Simons parameter.

For a beauty discussion of theories with Chern-Simons term see [11].

As the Riemann tensor, \( R_{\mu \alpha \beta \nu} \), has the same number of independent components as the Ricci tensor, \( R_{\alpha \beta} \), in three dimensions, a term squared in \( R_{\mu \alpha \beta \nu} \) is not necessary in the action.

In [2], we wrote the affine connection as in eq. (3), further decomposing the torsion in its \( SO(1,2) \) irreducible components: a scalar from the totally antisymmetric part, a three-vector from the trace and a symmetric traceless rank-2
tensor. With this procedure, we have obtained a particle spectrum where only massive excitations of spin-2 associated with the linearized gravitational field $h^{\alpha\beta}$ and with the symmetric part of the torsion field had dynamics that preserved the unitarity of the theory for some values of the action parameters.

In this section, we reconsider the action (15) but, contrary to what we have done in [2], we propose to adopt in the first-order formulation, dropping the torsion as our fundamental excitation and electing the dreinbein and the spin connection as the fundamental quantum fields.

Now, making use of equations (7),(13) and (14), along with the weak field approximation to the gravitational field,

$$e^a_\alpha = \delta^a_\alpha + \frac{k}{2} h^a_\alpha \Rightarrow g^a_{\alpha\beta} = \eta^a_{\alpha\beta} + k h^a_{\alpha\beta}, \quad (18)$$

and the spin connection decomposition,

$$\omega_a^{bc} = \epsilon^{bcd} Y_{ad}, \quad (19)$$

which can be further split in,

$$Y_{ab} = y_{ab} + \mathcal{Y}_{ab} : y_{ab} = Y_{(ab)} \quad \mathcal{Y}_{ab} = Y_{[ab]} \quad (20)$$

and

$$\mathcal{Y}_{ab} = \epsilon_{abc} y^c \Rightarrow y_a = \frac{1}{2} \epsilon_{abc} \mathcal{Y}^{bc}, \quad (21)$$

we can write the action (15), to which the gauge-fixing terms have been added,

$$\mathcal{L}_{GF-diff} = \lambda F_a F^a, \quad F_a = \partial_b \left( h^b_a - \frac{1}{2} \delta^b_a h^c \right), \quad (22)$$

and

$$\mathcal{L}_{GF-LL} = \xi \left( \partial^\mu \omega_a^{\mu b} \partial^\nu \omega_{\nu ab} \right), \quad (23)$$

in the convenient linearized form

$$S = \int d^3 x \frac{1}{2} \Phi^T M \Phi, \quad \Phi = \begin{pmatrix} y^c_d \\ y^c \\ h^{cd} \end{pmatrix}. \quad (24)$$

The wave operator $M$, being expressed in an extension of the spin-projection operator formalism introduced in [12],[13],[2]. Five additional operators coming from the $y^a$ and Chern-Simons terms are needed. The six operator for a rank-2 symmetric tensor in 3D are given by:

$$P^{(2)}_{ab,cd} = \frac{1}{2} (\theta_{ac} \theta_{bd} + \theta_{ad} \theta_{bc}) - \frac{1}{2} \theta_{ab} \theta_{cd}, \quad (25)$$

$$P^{(1-m)}_{ab,cd} = \frac{1}{2} (\theta_{ac} \omega_{bd} + \theta_{ad} \omega_{bc} + \theta_{bc} \omega_{ad} + \theta_{bc} \omega_{ad}), \quad (26)$$

$$P^{(1+m)}_{ab,cd} = \frac{1}{2} (\theta_{ac} \omega_{bd} - \theta_{ad} \omega_{bc} + \theta_{bc} \omega_{ad} - \theta_{bc} \omega_{ad}), \quad (27)$$

$$P^{(2)}_{ab,cd} = \frac{1}{2} (\theta_{ac} \omega_{bd} + \theta_{ad} \omega_{bc} - \theta_{bc} \omega_{ad} - \theta_{bc} \omega_{ad}), \quad (28)$$

$$P^{(3)}_{ab,cd} = \frac{1}{2} (\theta_{ac} \omega_{bd} - \theta_{ad} \omega_{bc} + \theta_{bc} \omega_{ad} - \theta_{bc} \omega_{ad}), \quad (29)$$

$$P^{(4)}_{ab,cd} = \frac{1}{2} (\theta_{ac} \omega_{bd} + \theta_{ad} \omega_{bc} - \theta_{bc} \omega_{ad} - \theta_{bc} \omega_{ad}), \quad (30)$$

$$P^{(5)}_{ab,cd} = \frac{1}{2} (\theta_{ac} \omega_{bd} - \theta_{ad} \omega_{bc} + \theta_{bc} \omega_{ad} - \theta_{bc} \omega_{ad}), \quad (31)$$

$$P^{(6)}_{ab,cd} = \frac{1}{2} (\theta_{ac} \omega_{bd} + \theta_{ad} \omega_{bc} - \theta_{bc} \omega_{ad} - \theta_{bc} \omega_{ad}), \quad (32)$$

The wave operator $M$, being expressed in an extension of the spin-projection operator formalism introduced in [12],[13],[2]. Five additional operators coming from the $y^a$ and Chern-Simons terms are needed. The six operator for a rank-2 symmetric tensor in 3D are given by:
\begin{align}
\mathcal{P}^{(0-s)}_{ab,cd} &= \frac{1}{2} \theta_{ab} \theta_{cd}, \\
\mathcal{P}^{(0-w)}_{ab,cd} &= \omega_{ab} \omega_{cd}, \\
\mathcal{P}^{(0-sw)}_{ab,cd} &= \frac{1}{\sqrt{2}} \theta_{ab} \omega_{cd},
\end{align}

and

\begin{align}
\mathcal{P}^{(0-ws)}_{ab,cd} &= \frac{1}{\sqrt{2}} \omega_{ab} \theta_{cd},
\end{align}

where $\theta_{ab}$ is the transverse and $\omega_{ab}$ is the longitudinal projector operators for vectors. The others five operators are:

\begin{align}
S_{ab,cd}^{(2a)} &= \left( \epsilon_{ace} \theta_{bd} + \epsilon_{ade} \theta_{bc} + \epsilon_{bce} \theta_{ad} + \epsilon_{bce} \theta_{ad} \right) \partial^e, \\
R_{ab,cd}^{(1a)} &= \left( \epsilon_{ace} \omega_{bd} + \epsilon_{ade} \omega_{bc} + \epsilon_{bce} \omega_{ad} + \epsilon_{bce} \omega_{ad} \right) \partial^e, \\
A_{ab} &= \epsilon_{abc} \partial^c,
\end{align}

\begin{align}
B_{a, bc} &= \eta_{ab} \partial_c + \eta_{ac} \partial_b
\end{align}

and

\begin{align}
D_{a, bc} &= A_{ab} \partial_c + A_{ac} \partial_b.
\end{align}

We recall that the usual Barnes-Rivers operators obey the algebra:

\begin{align}
\mathcal{P}_{ab,kl}^{(i-a)} \mathcal{P}_{ab,cd}^{(j-b)} &= \delta^{ij} \delta^{ab} \mathcal{P}_{ab,cd}^{(j-b)}, \\
\mathcal{P}_{ab,kl}^{(i-ab)} \mathcal{P}_{ab,cd}^{(j-cd)} &= \delta^{ij} \delta^{bc} \mathcal{P}_{ab,cd}^{(j-a)}, \\
\mathcal{P}_{ab,kl}^{(i-a)} \mathcal{P}_{ab,cd}^{(j-ac)} &= \delta^{ij} \delta^{bc} \mathcal{P}_{ab,cd}^{(j-ac)}, \\
\mathcal{P}_{ab,kl}^{(i-ab)} \mathcal{P}_{ab,cd}^{(j-c)} &= \delta^{ij} \delta^{bc} \mathcal{P}_{ab,cd}^{(j-ac)},
\end{align}

and satisfy the tensor identity,

\begin{align}
\mathcal{P}_{ab,cd}^{(2)} + \mathcal{P}_{ab,cd}^{(1m)} + \mathcal{P}_{ab,cd}^{(0a)} + \mathcal{P}_{ab,cd}^{(0w)} + \mathcal{P}_{ab,cd}^{(0w)} &= \frac{1}{2} \left( \eta_{bc} \eta_{bd} + \eta_{ad} \eta_{bc} \right).
\end{align}

The new set of spin operators that comes about displays, besides the operators $S_{ab,cd}^{(2a)}$, $R_{ab,cd}^{(1a)}$, $A_{ab}$, and $B_{a, bc}$ (already known from [2]), one new operator,
\(D_{a,bc}\), given in \((20)\). These five operators have their own multiplicative table; we quote below only some of the relevant products amongst them:

\[
S_{ab,ef}^{(2a)} S_{cd,ef}^{(2a)} = -16 \Box P_{ab,cd}^{(2)}
\]
\[
R_{ab,ef}^{(1a)} R_{cd,ef}^{(1a)} = -4 \Box P_{ab,cd}^{(1m)}
\]
\[
P_{ab,ef}^{(2)} S_{cd,ef}^{(2a)} = S_{ab,ef}^{(2a)} P_{cd,ef}^{(2)} = S_{ab,cd}^{(2a)}
\]
\[
P_{ab,ef}^{(1m)} R_{cd,ef}^{(1a)} = R_{ab,ef}^{(1a)} P_{cd,ef}^{(1m)} = R_{ab,cd}^{(1m)}
\]
\[
A_{ae} A_{b}^{+} = -\Box \theta_{ab},
\]
\[
B_{a,ef} B_{c}^{+} = 2 \Box (\theta_{ac} + 2 \omega_{ac}),
\]
\[
B_{e,ab} B_{e}^{+} = 2 \Box (P_{ab,cd}^{(1m)} + 2 P_{ab,cd}^{(0w)}),
\]
\[
D_{a,ef} D_{c}^{+} = 2 \Box 2 \theta_{ac}
\]

and

\[
D_{e,ab} D_{e}^{+} = 2 \Box 2 P_{ab,cd}^{(1m)}
\]

Thus, the wave operator acquires the form:

\[
M = \begin{pmatrix}
    yy_{ab,cd} & yy_{ab,c} & yh_{ab,cd} \\
    yy_{cd,ab} & yy_{a,c} & yh_{a,cd} \\
    h_{yab,cd} & h_{yab,c} & hh_{ab,cd}
\end{pmatrix},
\]

(30)

where

\[
yy_{ab,cd} = (2a_{1} - 2a_{0} \Box) P_{ab,cd}^{(2)} + (2a_{1} - a_{0} \Box - 2 \xi \Box) P_{ab,cd}^{(1m)} - (2a_{1} + 2a_{0} \Box) P_{ab,cd}^{(0s)}
\]
\[-4 \xi P_{ab,cd}^{(0w)} - 2 \sqrt{2} a_{1} (P_{ab,cd}^{(0ws)} + P_{ab,cd}^{(0ws)}) + \frac{3}{2} (S_{ab,cd}^{(2a)} + R_{ab,cd}^{(1a)}),
\]
\[
yy_{ab,c} = a_{4} B_{c,ab} + (2 \xi - a_{3}) D_{c,ab},
\]
\[
yh_{ab,cd} = \frac{k \Box}{2} a_{4} (P_{ab,cd}^{(2)} - P_{ab,cd}^{(0s)}) + \frac{k}{4} a_{1} (S_{ab,cd}^{(2a)} + R_{ab,cd}^{(1a)}),
\]
\[
yy_{a,cd} = -a_{4} B_{a,bc} + (2 \xi - a_{3}) D_{a,bc},
\]


\[ g_{a,c} = -(4a_1 + 2a_3 \Box + 4\xi \Box)\theta_{a,c} - (4a_1 + 32a_2 \Box + 12a_3 \Box)\omega_{a,c} + 2a_4 A_{a,c}, \tag{31} \]

\[ yh_{a,c} = -\frac{k}{2} a_1 B_{a,bc} + ka_4 (\theta_{bc} + \omega_{bc}) \partial_a, \]

\[ hy_{ab,cd} = \frac{k \Box}{2} a_4 (P_{ab,cd}^{(2)} - P_{ab,cd}^{(0)}) + k a_1 (S_{cd,ab}^{(2a)} + R_{ab,cd}^{(1a)}), \]

\[ hy_{ab,c} = \frac{k}{2} a_1 B_{a,bc} - ka_1 (\theta_{bc} + \omega_{bc}) \partial_a \]

and

\[ hh_{ab,cd} = -\lambda \Box \left( P_{ab,cd}^{(1m)} + P_{ab,cd}^{(0s)} + \frac{1}{2} P_{ab,cd}^{(0w)} - \sqrt{2}/2 \left( P_{ab,cd}^{(0sw)} + P_{ab,cd}^{(0sw)} \right) \right). \]

In order to calculate the propagators of the theory,

\[ \langle 0 | T[F(x) F(y)] | 0 \rangle = i M^{-1} \delta^{(3)}(x - y), \tag{32} \]

we need to calculate the inverse matrix, \( M^{-1} \), of the wave operator, but here we find a problem: the matrix element \( hh_{ab,cd} \) has not a term in \( P_{ab,cd}^{(2)} \), and we cannot find the inverse element of this fundamental term (to compute the inverse we need to close the relation given in eq. (28), that does not occur).

We can see in this manner that a completely invertible theory, when decomposed in terms of one gauge field and its torsion tensor components, loses this property when we focus in the version where we do not adopt the torsion as the fundamental field, but rather work with the gauge field associated to Lorentz local transformation that incorporates the torsion information (in an Einstein-Cartan theory \( \omega_{abc} = \gamma_{abc} - K_{abc} \), where \( \gamma_{abc} \) is the "pure Riemannian", without torsion, part and \( K_{abc} \) is the contortion term). The missing spin-2 term of the gravitational gauge field is incorporated into the "Riemannian part" of the spin connection gauge field.

**4 Introducing the Torsion Terms**

In order to try to obtain a pure gauge theory of planar gravitation, and yet understand the role of torsion in it, we change our study to the following action:

\[ S = \int d^3 x \left( a_1 R + a_2 T_{\alpha \beta \gamma} T^{\alpha \beta \gamma} + a_3 T_{\alpha \beta \gamma} T^{\beta \gamma \alpha} + a_4 T_{\alpha \beta} T^{\alpha \beta} + a_5 L_{CS} \right), \tag{33} \]

where we explicitly introduce torsion terms in the action, with \( L_{CS} \) being the usual Chern-Simons term given in eq. (16). \( a_1, a_2, a_3 \) and \( a_4 \) are free coefficients, whereas \( a_5 \) is the Chern-Simons parameter. See reference [14] for these
specific torsion terms. From now on, all our calculations and results refer to the action (33). In our final section, we shall make a comment on the possibility of introducing a term linear in the torsion \( [11] \).

By means of equations (14), (8) and (7), but the decompositions (19), (20) and (21) with the following weak expansion:

\[
e^{a}_\alpha = \delta^{a}_\alpha + \frac{k}{2} H^{a}_\alpha \quad (\Rightarrow g^{a}_{\alpha\beta} = \eta^{a}_{\alpha\beta} + kh^{a}_{\alpha\beta}, \quad h^{a}_{\alpha\beta} = \frac{1}{2}(H^{a}_{\alpha\beta} + H^{a}_{\beta\alpha})).
\]

(34)

With the new decomposition,

\[
H_{ab} = h_{ab} + \mathcal{H}_{ab}, \quad h_{ab} = H_{(ab)} \quad \epsilon \quad \mathcal{H}_{ab} = H_{[ab]}
\]

(35)

and

\[
\mathcal{H}_{ab} = \epsilon^{abc} h^{c} \Rightarrow h_{a} = \frac{1}{2} \epsilon^{abc} \mathcal{H}^{bc}.
\]

(36)

We can rewrite the action (33), introducing the gauge-fixing terms

\[
\mathcal{L}_{GF-diff} = \lambda F_a F^a, \quad F_a = k \partial^b \left( H_{ba} - \frac{1}{2} \eta_{ba} H^c_{ac} \right),
\]

(37)

in the linearized form:

\[
S = \int d^3x \Phi^T M \Phi, \quad \Phi = \begin{pmatrix} h^{cd} \\ h^{c} \\ y^{cd} \\ y^{c} \end{pmatrix}.
\]

(38)

As before, we express the wave operator, \( M \), in terms of the extended spin-projection operator formalism. In addition to the operators listed above, there appear two new operators:

\[
\theta_{ab} \partial_{c} \quad \text{and} \quad \omega_{ab} \partial_{c},
\]

(39)

which, together with the old ones, completely close the algebra.

This yields the form below for the wave operator:

\[
M = \begin{pmatrix} hh_{ab,cd} & hh_{ab,c} & hy_{ab,cd} & hy_{ab,c} \\ hh_{a,cd} & hh_{a,c} & hy_{a,cd} & hy_{a,c} \\ yh_{ab,cd} & yh_{ab,c} & yy_{ab,cd} & yy_{ab,c} \\ yh_{a,cd} & yh_{a,c} & yy_{a,cd} & yy_{a,c} \end{pmatrix},
\]

(40)

where
\[
h_{ab,cd} = \frac{k^2}{2} \Box (a_3 - 2a_2)p^{(2)}_{ab,cd} + \frac{k^2}{4} \Box (a_3 - 2a_2 - a_4 - 4\lambda)p^{(1m)}_{ab,cd}
\]
\[
+ \frac{k^2}{2} \Box (a_3 - 2a_2 - 2a_4 - 2\lambda)p^{(0x)}_{ab,cd} - \left( \frac{k^2}{2} \Box \lambda \right)p^{(0w)}_{ab,cd}
\]
\[
- \left( \frac{\sqrt{2}}{2}k^2 \Box \lambda \right)(p^{(0sw)}_{ab,cd} + p^{(0sw)}_{ab,cd}) - \frac{k^2}{2}a_5(S^{(2a)}_{ab,cd} + R^{(1\alpha)}_{ab,cd}),
\]
\[
h_{ab,c} = -\left( \frac{k^2}{2}a_5 \right)B_{c,ab} + \frac{k^2}{4}(a_3 - 2a_2 - a_4 + 4\lambda)D_{c,ab},
\]
\[
y_{ab,cd} = \frac{k}{2}(\Box a_6 - 2a_5)p^{(2)}_{ab,cd} - (ka_5)p^{(1m)}_{ab,cd} - \frac{k}{2}(\Box a_6 + 2a_5)p^{(0s)}_{ab,cd}
\]
\[
- (ka_5)p^{(0w)}_{ab,cd} + \frac{k}{4}(a_1 + 2a_2 - 2a_3)(S^{(2a)}_{ab,cd} + R^{(1\alpha)}_{ab,cd})
\]
\[
h_{a,cd} = \left( \frac{k^2}{2}a_5 \right)B_{a,cd} + \frac{k^2}{4}(a_3 - 2a_2 - a_4 + 4\lambda)D_{a,cd},
\]
\[
h_{a,c} = \frac{k^2}{2} \Box (a_3 - 2a_2 - a_4 - 4\lambda)\theta_{a,c} - (k^2 \Box)(2a_2 + a_3)\omega_{a,c} - (k^2 a_5)A_{a,c},
\]
\[
y_{a,cd} = -\frac{k}{2}(a_1 + 2a_2)B_{a,cd} + k(a_1 - 2a_2 - 2a_3)(\theta_{bc} + \omega_{bc})\theta_a,
\]
\[
y_{a,c} = (2ka_5)\theta_{a,c} + k(2a_5 - \Box a_6)\omega_{a,c} + k(a_1 - 2a_2 - 2a_4)A_{a,c},
\]
\[
y_{ab,cd} = \frac{k}{2}(\Box a_6 - 2a_5)p^{(2)}_{ab,cd} - (ka_5)p^{(1m)}_{ab,cd} - \frac{k}{2}(\Box a_6 + 2a_5)p^{(0s)}_{ab,cd}
\]
\[
- (ka_5)p^{(0w)}_{ab,cd} + \frac{k}{4}(a_1 + 2a_2 - 2a_3)(S^{(2a)}_{ab,cd} + R^{(1\alpha)}_{ab,cd}),
\]
\[
y_{ab,c} = \frac{k}{2}(a_1 + 2a_2)B_{c,ab} - k(a_1 - 2a_2 - 2a_3)(\theta_{ab} + \omega_{ab})\theta_c,
\]
\[
y_{ab,cd} = 2(a_1 + 2a_2 - a_3)p^{(2)}_{ab,cd} + 2(a_1 + 2a_2 - a_3)p^{(1m)}_{ab,cd}
\]
\[
+ 2(6a_2 + 5a_3 - a_1)p^{(0s)}_{ab,cd} + 4(2a_2 + a_3)p^{(0w)}_{ab,cd}
\]
\[
+ 2\sqrt{2}(2a_2 + 3a_3 - a_1)(p^{(0sw)}_{ab,cd} + p^{(0sw)}_{ab,cd}) + \left( \frac{a_6}{2} \right)(S^{(2a)}_{ab,cd} + R^{(1\alpha)}_{ab,cd}),
\]
\[10\]
\[ yy_{ab,c} = a_6 B_{c,ab}, \]
\[ y h_{a,cd} = -\frac{k}{2} (a_1 - 2a_2 - 2a_4) B_{a,bc} + k(a_1 - 2a_2 - 2a_4)(\theta_{bc} + \omega_{bc}) \partial_s, \]
\[ y h_{a,c} = (2k a_5) \theta_{a,c} + k(2a_5 - a_6) \omega_{a,c} + k(a_1 - 2a_2 - 2a_4) A_{a,c}, \]
\[ yy_{a,cd} = -a_6 B_{a,cd} \]
and
\[ yy_{a,c} = 4(2a_1 + 2a_4 - a_1 - a_3) \theta_{a,c} + 4(2a_2 + 2a_4 - a_1 - a_3) \omega_{a,c} + (2a_6) A_{a,c}. \]

Once all operators have been identified and worked out, we finally come to the task of computing the inverses. This is what we shall do next.

5 Propagators and Excitation Modes

In order to calculate the propagators, eq. (32), we use a straightforward, but lengthy, procedure in terms of which we decompose the matrix \( M \) into four sectors, namely:

\[ M = \begin{pmatrix} hh & hy \\ yh & yy \end{pmatrix}. \]  

(42)

Thus the inverse matrix \( M^{-1} \) can be written as:

\[ M^{-1} = \begin{pmatrix} HH & HY \\ YH & YY \end{pmatrix}. \]  

(43)

where its blocks are given by:

\[ HH = [hh - hy(yy)^{-1}yh]^{-1}. \]
\[ HY = -(hh)^{-1}hyYY. \]  
\[ YH = -(yy)^{-1}yhHH. \]
\[ YY = [yy - yh(hh)^{-1}hy]^{-1}. \]  

(44)

Once whit the propagators, we must check the tree-level unitarity of the theory. To this, we have to analyse the residues of the current-current transition amplitude in momentum space, given by the saturated propagator after
where sources, and vectors that satisfy the conditions:

\[ \text{Sources}_{\mu \nu} = c_1 p_\mu p_\nu + c_2 p_\mu q_\nu + c_3 p_\mu \varepsilon_\nu + c_4 q_\mu p_\nu + c_5 q_\mu q_\nu + c_6 q_\mu \varepsilon_\nu + c_7 \varepsilon_\mu p_\nu + c_8 \varepsilon_\mu q_\nu + c_9 \varepsilon_\mu \varepsilon_\nu, \]  

(45)

where \( p_\mu = (p_0, -\vec{p}) \), \( q_\mu = (p_0, \vec{p}) \) and \( \varepsilon_\mu = (0, -\vec{\varepsilon}) \) are linearly independent vectors that satisfy the conditions:

\[ p_\mu p^\mu = q_\mu q^\mu = m^2. \]
\[ p_\mu q^\mu = p_0^2 + \vec{p}^2 \neq 0. \]
\[ p_\mu \varepsilon^\mu = q_\mu \varepsilon^\mu = 0. \]
\[ \varepsilon_\mu \varepsilon^\mu = -1. \]

These conditions and the symmetry requirements of the theory split the sources, \( S_{\mu \nu} \), in a symmetric and an antisymmetric part:

\[ S_{\mu \nu} = S_{(\mu \nu)} = c_1 p_\mu p_\nu + c_2 (p_\mu q_\nu + q_\mu p_\nu) + c_3 (p_\mu \varepsilon_\nu + \varepsilon_\mu p_\nu) \]
\[ + c_4 q_\mu q_\nu + c_5 (q_\mu \varepsilon_\nu + \varepsilon_\mu q_\nu) + c_6 \varepsilon_\mu \varepsilon_\nu \]  

(47)

and

\[ A_{\mu \nu} = S_{[\mu \nu]} = d_1 (p_\mu q_\nu - q_\mu p_\nu) + d_2 (p_\mu \varepsilon_\nu - \varepsilon_\mu p_\nu) \]
\[ + d_3 (q_\mu \varepsilon_\nu - \varepsilon_\mu q_\nu), \]

where \( c_1 = c_1 \), \( c_2 = \frac{c_3 + c_4}{2} \), \( c_3 = \frac{c_5 + c_6}{2} \), \( c_4 = c_4 \), \( c_5 = \frac{c_7 + c_8}{2} \), \( c_6 = c_6 \), \( d_1 = \frac{c_9 - c_4}{2} \), \( d_2 = \frac{c_9 - c_6}{2} \), and \( d_3 = \frac{c_9 - c_5}{2} \).

The currente-current transition amplitude is written as:

\[ \mathcal{A} = (\tau^+ \rho^+ \begin{pmatrix} HH & HY \\ YH & YY \end{pmatrix}) (\begin{pmatrix} \tau^- \\ \rho^- \end{pmatrix}) \Rightarrow \]
\[ \mathcal{A} = \tau^+ HH \tau + \tau^+ HY \rho + \rho^+ Y H \tau + \rho^+ Y Y \rho, \]

(49)

where \( \tau \) is the source to the \( h \) fields and \( \rho \) the source to the \( y \) fields.

\( \mathcal{A} \) can then be cast into the form below:

\[ \mathcal{A} = t^{ab}_{cd} HH_{ab,cd} \tau^+ + t^{ab}_{cd} HH_{ab,ce} \tau^+ + t^{ab}_{cd} HH_{ab,cd} \rho^+ + t^{ab}_{cd} HH_{ab,cd} \rho^+ \]
\[ + t^{ab}_{cd} HY_{ab,cd} \tau^+ + t^{ab}_{cd} HY_{ab,cd} \rho^+ + t^{ab}_{cd} HY_{ab,cd} \rho^+ + t^{ab}_{cd} HY_{ab,cd} \rho^+ \]
\[ + Y^{ab}_{cd} HY_{ab,cd} \tau^+ + Y^{ab}_{cd} Y H_{ab,cd} \tau^+ + Y^{ab}_{cd} Y H_{ab,cd} \rho^+ + Y^{ab}_{cd} Y H_{ab,cd} \rho^+ \]
\[ + Y^{ab}_{cd} Y Y_{ab,cd} \tau^+ + Y^{ab}_{cd} Y Y_{ab,cd} \rho^+ + Y^{ab}_{cd} Y Y_{ab,cd} \rho^+ + Y^{ab}_{cd} Y Y_{ab,cd} \rho^+, \]

(50)
where $\tau^{cd} = \tau^{(cd)}$, $c^e = \frac{1}{2} c^{de} T_{de}$ with $T_{de} = \tau^{[de]}$ and $r^{cd} = \rho^{(cd)}$, $r^e = \frac{1}{2} c^{de} R_{de}$ with $R_{de} = \rho_{[de]}$.

Due to the source constraints, $p_\mu t^{cd} = 0$, $p_\mu T^{cd} = 0$, $p_\mu r^{cd} = 0$ and $p_\mu R^{cd} = 0$, only the projectors $P^{(2)}_{ab,cd}$, $P^{(0s)}_{ab,cd}$, $S^{(2a)}_{ab,cd}$, $\theta_{ab}\partial\tau$ and $\omega_{a,b,c,d}$ give non-vanishing contributions to the amplitude.

For a massless pole, or for a massive pole in the rest frame (where $p_\mu = (m, 0, q_\mu = (m, 0)$ and $\varepsilon_\mu = (0, -\varepsilon)$), only the projectors $P^{(2)}_{ab,cd}$ and $P^{(0s)}_{cd}$ survive and contribute.

With the restrictions above, the amplitude reads:

$$
A = < H^2 H^2_{(2)} > t^{ab} P^{(2)}_{ab,cd} t^{cd} + < H^2 H^2_{(0s)} > t^{ab} P^{(0s)}_{ab,cd} t^{cd} \\
+ < H^2 Y^2_{(2)} > r^{ab} P^{(2)}_{ab,cd} r^{cd} + < H^2 Y^2_{(0s)} > r^{ab} P^{(0s)}_{ab,cd} r^{cd} \\
+ < Y^2 H^2_{(2)} > r^{ab} P^{(2)}_{ab,cd} r^{cd} + < Y^2 H^2_{(0s)} > r^{ab} P^{(0s)}_{ab,cd} r^{cd},
$$

(51)

where $< H^2 H^2_{(2)} >$ is the symmetric rank-2 ($H^2$ in $H^2 H^2_{(2)}$) gravitational field propagator associated to the operator $P^{(2)}_{ab,cd}$ ($2$ in $H^2 H^2_{(2)}$). The other coefficients have analogous meaning. Explicitly writing the sources, we get:

$$
A = \frac{1}{2} ( < H^2 H^2_{(2)} > + < H^2 H^2_{(0s)} > ) |c_{6r}|^2 \\
+ \frac{1}{2} ( < H^2 Y^2_{(2)} > + < H^2 Y^2_{(0s)} > ) c^*_6 c_{6r} \\
+ \frac{1}{2} ( < Y^2 H^2_{(2)} > + < Y^2 H^2_{(0s)} > ) c^*_6 c_{6t} \\
+ \frac{1}{2} ( < Y^2 Y^2_{(2)} > + < Y^2 Y^2_{(0s)} > ) |c_{6r}|^2
$$

(52)

where $t$ and $r$ in the $c$ mean the source associated to the particular term.

We must now replace the results obtained by the procedure described in [14] into (52). Before, explicitly we put our results, the following comments should be done:

1. With the whole set of action parameters, $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$ plus $\lambda$, different from zero, our computational algebraic facilities failed in attaining a result due the extension of the resulting expressions.

2. Considering the Chern-Simons term, $a_5$, we obtained the following behaviour in the denominator of the propagator:

- With $a_1 = 0$, we have terms proportional to $p^{22}$.
- The lowest power, $p^6$, occurs with $a_1 = a_2 = a_4 = 0$, only $a_3$ and $a_5$ are considered.
• With \( a_3 = 0 \), we do not have an invertible case.

3. Without the Chern-Simons term, \( a_5 = 0 \), we obtain, in all invertible cases, a power \( p^2 \). This is not a straightforward result; we may justify it by pointing out that Chern-Simons contributes a term quadratic in the spin connection with a space-time derivative, whereas the scalar curvature contributes a term that mixes \( H \) with \( \omega \). Setting \( a_5 \) to zero, we suppress \( \omega - \omega \) terms with a derivative, and so we unavoidably reduces the powers of the momentum.

We then consider in (52) only the cases with \( a_5 = 0 \).

The least invertible case occurs by considering only \( a_3 \) different from zero in the action. In this case, the relevant propagators read:

\[
H2H2_{(2)} = \frac{2}{3k^2p^2a_3}i. \\
H2H2_{(0s)} = -\frac{2}{k^2p^2a_3}i. \\
H2Y2_{(2)} = H2Y2_{(0s)} = Y2H2_{(2)} = Y2H2_{(0s)} = 0. \\
Y2Y2_{(2)} = \frac{1}{6a_3}i. \\
Y2Y2_{(0s)} = 0
\]  

(53)

and the saturated amplitude is as given below,

\[
A = \left( \frac{2}{3k^2p^2a_3} |c_6|_{tt}^2 + \frac{1}{12a_3} |c_6|_{rr}^2 \right) i. 
\]  

(54)

We notice in this expression that the massless pole comes from the \( h \)-block and has contributions from the spin-0 and the spin-2 sectors.

Then, by calculating the imaginary part of the residue of the amplitude at the massless pole, we get:

\[
\text{Im}(\text{res}A) = \text{Im} \left( \lim_{p^2 \to 0} [p^2 A] \right) = -\frac{2|c_6|_{tt}^2}{3k^2a_3}. 
\]  

(55)

From the requirement of having positive-definite residue at the pole, we must have \( a_3 < 0 \).

Considering now the addition of the scalar of curvature term \( a_1 \), we get:
\[ H_2H_2^{(2)} = \frac{2(a_3 - a_1)}{k^2 p^2 (3a_3^2 + a_1^2 - 3a_3 a_1)} i \]
\[ H_2H_2^{(0s)} = \frac{2(a_3 + a_1)}{k^2 p^2 (a_3^2 - a_1^2 + 3a_3 a_1)} i \]
\[ H_2Y_2^{(2)} = H_2Y_2^{(0s)} = Y_2H_2^{(2)} = Y_2H_2^{(0s)} = 0 \quad (56) \]
\[ Y_2Y_2^{(2)} = Y_2Y_2^{(0s)} = Y_2H_2^{(2)} = Y_2H_2^{(0s)} = 0 \]

and the amplitude becomes:

\[ A = \left( -\frac{2}{k^2 p^2} \times \frac{a_3^3}{3a_3^4 - 5a_3^2 a_1^2 + 4a_3 a_1^3 - a_1^4} \right|_{c_6}^2 + \frac{a_3}{2(3a_3^2 + a_1^2 - 3a_3 a_1)} \right|_{c_6}^2 \right) i. \]

We can see that the structure of the amplitude is not changed, with the pole having contributions from the same spin sectors. The parameters relations now reads:

\[ \text{Im}(resA) = \text{Im} \left( \lim_{p^2 \to 0} [p^2 A] \right) = -\frac{2}{k^2} \frac{a_3^3}{3a_3^4 - 5a_3^2 a_1^2 + 4a_3 a_1^3 - a_1^4} \right|_{c_6}^2. \quad (57) \]

The denominator in (57) can be written as:

\[ (a_3^2 + a_3 a_1 - a_1^2)(3a_3^2 - 3a_3 a_1 + a_1^2). \quad (58) \]

The binomial \( 3a_3^2 - 3a_3 a_1 + a_1^2 \) has complex roots and is greater than zero.

The requirement of having positive-definite residue at the pole implies (with \( a_3 < 0 \)) \( a_3^2 - a_3 a_1 - a_1^2 < 0 \). And the scalar term must obey \( \frac{1 + \sqrt{5}}{2} a_3 \approx 1.618 a_3 < a_1 < \frac{1 - \sqrt{5}}{2} a_3 \approx -0.618 a_3 \).

The case where all parameters (with exception to \( a_5 \)) are different from zero brings only new algebraic corrections to the amplitude, without changing its structure. The relations among the parameters become very cumbersome, due to the considerable number of parameters involved, so that many hypotheses must be done.

6 Concluding Comments

In the course of the calculations we report on here, if we complete the action [34] by adjoining the term \( a_6 \epsilon^\mu \nu \lambda \mathcal{T}_{\mu \nu} a_7 \epsilon_\lambda \eta_{ab} = a_6 \epsilon^\mu \nu \lambda \mathcal{T}_{\mu \nu} a_7 \epsilon_\lambda \eta_{ab} = a_6 \epsilon^\mu \nu \lambda \mathcal{T}_{\mu \nu} a_7 \epsilon_\lambda \eta_{ab} = a_6 \epsilon^\mu \nu \lambda \mathcal{T}_{\mu \nu} a_7 \epsilon_\lambda \eta_{ab} \) [11], a problem shows up: though our procedure of introducing the spin operators works, the propagators could not be found in their generality (with all...
the six coefficients $a_i$) even with the help of algebraic computation techniques. However, we found out that, once any of the $a_i$ are set to zero, we succeed in reading off the propagators, even if they display higher powers in the momentum. It is worthwhile to mention here that this linear term in the torsion combines with the Chern-Simons action to give a rich structure of poles in the propagators. We do not report these results here because this investigation is the matter of a forthcoming publication [15]. The situation gets better when we discovered that, ruling out the Chern-Simons term, we get only simple poles in the terms that contribute to the amplitude. Very surprising was the discovery of the very different role the torsion terms ($a_2$ and $a_3$) play, being $a_3$ fundamental to compute the inverse matrix, which is not the case for $a_2$.

We see that the physical poles are all massless. It is worthy to note that, in [2], we get only physical mass poles. The unitarity condition for the physical poles demand that $a_3 < 0$ and this implies in that the parameter that governs the scalar curvature must obey the condition $\frac{1+\sqrt{5}}{2}a_3 < a_1 < \frac{1-\sqrt{5}}{2}a_3$.

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