Rare radiative exclusive $B$ decays in soft-collinear effective theory

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Abstract

We consider rare radiative $B$ decays such as $\bar{B} \to K^*\gamma$ or $\bar{B} \to \rho\gamma$ in soft-collinear effective theory, and show that the decay amplitudes are factorized to all orders in $\alpha_s$ and at leading order in $\Lambda_{\text{QCD}}/m_b$. By employing two-step matching, we classify the operators for radiative $B$ decays in powers of a small parameter $\lambda(\sim \sqrt{\Lambda_{\text{QCD}}/m_b})$ and obtain the relevant operators to order $\lambda$ in SCET$_I$. These operators are constructed with or without spectator quarks including the four-quark operators contributing to annihilation and $W$-exchange channels. And we employ SCET$_{II}$ where the small parameter becomes of order $\Lambda_{\text{QCD}}/m_b$, and evolve the operators in order to compute the decay amplitudes for rare radiative decays in soft-collinear effective theory. We show explicitly that the contributions from the annihilation channels and the $W$-exchange channels vanish at leading order in SCET. We present the factorized result for the decay amplitudes in rare radiative $B$ decays at leading order in SCET, and at next-to-leading order in $\alpha_s$.

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I. INTRODUCTION

The formalism of soft-collinear effective theory (SCET) [1–6] can be applied to high-energy processes in which massless quarks are emitted with large energy. In this case a light quark with large energy has three different scales $Q$, $Q\lambda$ and $Q\lambda^2$, where $Q$ is the large energy scale and $\lambda \sim p_\perp/Q$ is a small parameter. Here $p_\perp$ is a typical transverse momentum scale. SCET has been applied to many high-energy processes in which final-state particles include energetic light particles. For example, the hard contribution to the heavy-to-light form factor [1, 2, 5, 7, 8], the pion form factor and high-energy scattering [9], and radiative leptonic $B$ decays $B \to \gamma e\nu$ [10, 11] have been studied. Recently it has been applied to nonleptonic $B$ decays into two light mesons by the authors [12, 13] and nonleptonic decays are shown to factorize at leading order in SCET and to all orders in $\alpha_s$.

In this paper we apply SCET to rare radiative $B$ decays $B \to V\gamma$, where $V$ can be a $\rho$ meson or a $K^*$ meson. In this case, employing a two-step matching offers a convenient procedure to obtain the matrix elements of the relevant operators evaluated below the scale $\mu_0 \sim \sqrt{\Lambda_{QCD} m_b}$. First we consider an effective theory called SCET$_I$, in which the degrees of freedom with $p \sim m_b$ are integrated out and the effective theory describes physics below the scale $m_b$ and above $\mu_0$. In SCET$_I$, we can construct all the effective operators which are gauge invariant under collinear and ultrasoft (usoft) gauge transformations. But a typical scale in SCET$_I$ is of order $m_b \lambda \sim \sqrt{m_b \Lambda_{QCD}}$, which is still too big for the evaluation of the matrix elements of the effective operators. A next step is to go down to the second effective theory SCET$_{II}$, in which all the degrees of freedom with $p \sim \sqrt{m_b \Lambda_{QCD}}$ are integrated out and a typical scale in SCET$_{II}$ is of order $\Lambda_{QCD}$ and the small parameter becomes of order $\Lambda_{QCD}/m_b$. In SCET$_{II}$, we can evaluate the matrix elements of the operators between meson states. The two-step matching looks complicated, but the construction and the power counting of operators become manifest.

Radiative $B$ decays have been previously considered in different theoretical frameworks. Ali and Parkhomenko [14] calculated hard spectator contributions to radiative $B$ decays in the scheme of the large energy effective theory, and presented their results at next-to-leading order and to leading order in $\Lambda_{QCD}/m_b$. In this approach they calculated the decay amplitudes in the full theory, and expanded them in powers of $1/E$, where $E \sim m_b$. Radiative $B$ decays were also considered in the context of the heavy quark mass limit [15, 16],
in which the amplitudes in the full theory are expanded in powers of $1/m_b$ [17], employing the light-cone wave functions of the mesons convoluted with the hard scattering amplitudes. The results in Refs. [14–16] show that the radiative $B$ decay amplitudes are factorized at next-to-leading order in $\alpha_s$ and to leading order in $\Lambda_{\text{QCD}}/m_b$. Subleading effects in radiative $B$ decays such as the study of the topology of the weak annihilation and the $W$-exchanges channels [18], and the isospin breaking effects [19] were studied.

If we apply SCET to radiative $B$ decays, we can extend the proof of the factorization theorem to all orders in $\alpha_s$ at leading order in $\Lambda_{\text{QCD}}/m_b$. It is possible because we require that the effective operators in SCET$_{\text{II}}$ be gauge invariant under collinear and soft gauge transformations. The gauge invariance and the reparameterization invariance put a serious constraint on the possible forms of the effective operators in SCET [20, 21]. The gauge invariance is achieved by introducing the corresponding Wilson lines, and these Wilson lines include the effects of collinear and soft gluons to all orders in $\alpha_s$. The Wilson coefficients of the effective operators can be calculated to desired accuracy, say, next-to-leading order, or next-to-next-to-leading order, but the form of the effective operators remains the same.

Another advantage in applying SCET is that we can make the power counting explicit at each step of the effective theories SCET$_{\text{I}}$ and SCET$_{\text{II}}$. In SCET$_{\text{I}}$, we can obtain gauge invariant operators in powers of $\lambda$. When we integrate out an intermediate scale, we need not worry about operator mixing in different powers of $\lambda$ since the matching is perturbative. As we go down to SCET$_{\text{II}}$, the small parameter now becomes $\sim \Lambda_{\text{QCD}}/m_b$ and the power counting is explicit. To summarize, our procedure to employ SCET is first to match QCD onto SCET$_{\text{I}}$ at $\mu \sim m_b$, and factorize the usoft-collinear interactions with the field redefinitions. Finally we match SCET$_{\text{I}}$ onto SCET$_{\text{II}}$ at $\mu_0 \sim \sqrt{m_b\Lambda_{\text{QCD}}}$, and evaluate the matrix elements.

The organization of the paper is as follows: In Section II, we briefly review the basic ingredients of SCET in studying radiative $B$ decays. We explain which types of operators are needed in computing the decay amplitudes of radiative $B$ decays, and the outline of the procedure in using SCET is sketched. In Section III, all the relevant operators are derived in SCET$_{\text{I}}$. There are effective operators without spectator quarks, and if we include the effects of spectator quarks, new four-quark operators and four-quark operators with a photon are induced. These operators are constructed in a gauge invariant way. In Section IV, the operators obtained in SCET$_{\text{I}}$ are evolved down to SCET$_{\text{II}}$, and we evaluate all the matrix
elements of the operators at leading order in SCET. In Section V, all the contributions are combined to produce the decay amplitudes for the radiative $B$ decays. In the final section, we summarize all the effects of various operators and a conclusion is presented.

II. BASICS OF SCET FOR $B \rightarrow V \gamma$ DECAYS

In order to describe radiative $B$ decays in SCET, the first step is to construct effective operators in SCET$_I$ by integrating out the degrees of freedom of order $m_b$ from the full theory. The effective operators should be gauge invariant under collinear and usoft gauge transformations, and these operators can be systematically expanded in a power series of the small parameter $\lambda$, of order $\sqrt{\Lambda_{\text{QCD}}/m_b}$. After we construct all the operators contributing to radiative $B$ decays in SCET$_I$, we evolve these operators down to SCET$_{II}$. In obtaining SCET$_{II}$, we integrate all the off-shell modes of order $\sqrt{m_b \Lambda_{\text{QCD}}}$, and all the operators in SCET$_{II}$ can be expanded in a power series of the small parameter, which now becomes of order $\Lambda_{\text{QCD}}/m_b$. At each step of the effective theories, we match theories at the boundary requiring that matrix elements of a given operator be the same in both theories. And we go down to the scale of interest by using the renormalization group equation.

We begin with the effective Hamiltonian for $B$ decays in the full QCD, which is given by

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} V_{pd} V_{pb} \left( C_1 O_1^p + C_2 O_2^p + \sum_{i=3,\ldots,8} C_i O_i \right),$$

(1)

where

$$O_1^p = (\not{p}b_i)_{V-A}(\not{q}j \gamma_5)_{V-A}, \quad O_2^p = (\not{p}b_i)_{V-A}(\not{d}q_j)_{V-A},$$

$$O_3 = (\not{d}b_i)_{V-A} \sum_q (\not{q}j \gamma_5)_{V-A}, \quad O_4 = (\not{d}b_i)_{V-A} \sum_q (\not{q}j \gamma_5)_{V-A},$$

$$O_5 = (\not{d}b_i)_{V-A} \sum_q (\not{q}j \gamma_5)_{V+A}, \quad O_6 = (\not{d}b_i)_{V-A} \sum_q (\not{q}j \gamma_5)_{V+A},$$

$$O_7 = -\frac{em_b}{8\pi^2} \tilde{\sigma}_{\mu\nu} F_{\mu\nu} (1 + \gamma_5) b, \quad O_8 = -\frac{gm_b}{8\pi^2} d_i \sigma_{\mu\nu} G_{a\mu}^{\nu} (T_a)_{ij} (1 + \gamma_5) b_j.$$  

(2)

Here $p$ is an up-type quark $u$ or $c$ quark, and $d$ is a down-type quark $d$ (for $B \rightarrow \rho \gamma$) or $s$ (for $B \rightarrow K^* \gamma$) quark. The indices $i, j$ are color indices, $F_{\mu\nu}$ and $G_a^{\mu\nu}$ are the electromagnetic and the chromomagnetic field strength tensors respectively, and $V_{ij}$ are the Cabibbo-Kobayashi-Maskawa matrix elements. The sign convention for $O_{7,8}$ corresponds to negative $C_{7,8}$ and the covariant derivative is defined as $D^\mu = \partial^\mu - igT_a A_a^\mu - ieQ A^\mu$, where $A_a^\mu$ ($A^\mu$) is the gluon (photon) field.
The leading contribution to $B \to V\gamma$ comes from the operator $O_7$ in the full theory. In the effective theory, the leading-contribution comes from the leading-order operator derived from $O_7$. In the following computation, we will neglect the radiative corrections of the penguin operators since their coefficients are proportional to $\alpha_s C_i$ ($i = 3, \cdots, 6$), which are numerically small compared to $C_7$. But we include the loop correction from $O_1$ since $\alpha_s C_1$ is not negligible compared to $C_7$. When the penguin operators appear at leading order in $\alpha_s$, we include their effects.

Before we consider the effective operators for radiative $B$ decays in SCET$_I$, let us introduce the notations we use in this paper and discuss the power counting of operators. We choose the direction of the vector meson as $n^\mu$, and the direction of the photon as $\bar{\pi}^\mu$. And the vector mesons are regarded as massless at leading order in SCET. Therefore the $SU(3)$ breaking effects in $B \to \rho \gamma$ and $B \to K^* \gamma$ and the $SU(2)$ isospin breaking effects in $B \to \rho^0 \gamma$ and $B^- \to \rho^- \gamma$ arise at subleading order. The quark mass effect in SCET was discussed in Ref. [22], which can cause leading operators in SCET$_{II}$. However the matrix elements of these operators in radiative $B$ decays at leading order in SCET turn out to vanish. This is discussed in Appendix in detail.

We denote the collinear fields $\xi$ and $\chi$ as the collinear fields in the $n^\mu$ and the $\bar{\pi}^\mu$ directions respectively, satisfying

$$\not{n}\xi = 0, \quad \frac{\not{n}}{4}\xi = \xi, \quad \not{\bar{\pi}}\chi = 0, \quad \frac{\not{\bar{\pi}}}{4}\chi = \chi.$$  \hfill (3)

The collinear gluon field $A_n^\mu$ in the $n^\mu$ direction can be decomposed as

$$A_n^\mu = \frac{n^\mu}{2} \not{n} \cdot A_n + A_{n\perp}^\mu + \frac{\bar{\pi}^\mu}{2} n \cdot A_n = \mathcal{O}(\lambda^0) + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^2),$$  \hfill (4)

and the power counting of the operators involving collinear fields can be made explicit as shown in Ref. [2, 4]. On the other hand, the photon field $A^\mu$ cannot be decomposed as the collinear gluon to facilitate the power counting. The decomposition of the collinear gluon field in Eq. (4) is meaningful since quarks and gluons have small fluctuations due to strong interaction. If we consider an external photon, the photon field does not have such small fluctuations and the momentum of the photon is always of order $\lambda^0$, and only the transverse part contributes to decay amplitudes. Therefore we do not include the photon field in the power counting of operators. Though we assign any power of $\lambda$ to the photon field, it does not affect the relative power counting of various operators since all the operators for $B \to V\gamma$ will include a single photon.
There are two kinds of effective operators in SCET$_I$ from different channels, which can be classified into the operators without spectator quarks and those with spectator quarks. The leading operator for the first type of operators comes from $O_7$ by replacing the quarks in the full theory by those in the effective theory and making the resultant operator gauge invariant under collinear gauge transformations. We also have the contributions from the operators $O_1$ and $O_8$ by radiative corrections, whose matrix elements are proportional to the matrix element of $O_7$. The resultant operators are heavy-light quark bilinear operators with a photon field. These operators, after we make them gauge invariant by attaching Wilson lines, can produce operators with an additional external gluon. At order $g$, these operators can be obtained by attaching a collinear gluon to $O_7$, or a photon to $O_8$, or a gluon and a photon to $O_1$.

All these operators without spectator quarks can be obtained by attaching a photon or a gluon to fermion lines in the operators $O_7$, $O_1$ and $O_8$. When we attach a photon or a gluon to each operator, if any intermediate states have momenta of order $\lambda^0$, we integrate out these modes to obtain effective operators. If the intermediate states have momenta of order $\lambda m_b$, then these modes cannot be integrated out in SCET$_I$. Instead that channel should be considered in SCET$_{II}$. In order to make the operators gauge invariant under collinear gauge transformations, we also have to consider attaching collinear gluons and integrate out all the off-shell modes of order $\lambda^0$. We compute all the operators in SCET$_I$ to order $\lambda$ in order to obtain the leading-order result due to the possible enhancement in evolving the operators to SCET$_{II}$.

The second type of the operators is the operators with spectator quarks. These include the four-quark operators of the generic form $\chi \Gamma_1 h \cdot \bar{\xi} \Gamma_2 \xi$, and $\bar{\xi} \Gamma_1 h \cdot \chi \Gamma_2 \xi$. As mentioned in Ref. [13], we can use the Fierz transformation to make the operator $\bar{\xi} \Gamma_1 h \cdot \chi \Gamma_2 \xi$ of the form $\chi \Gamma_1 h \cdot \bar{\xi} \Gamma_2 \xi$, and we will consider the four-quark operators only in the form $\chi \Gamma_1 h \cdot \bar{\xi} \Gamma_2 \xi$ from now on. In these four-quark operators, the spectator quark is a collinear quark $\chi$. And we can make time-ordered products of these operators with the electromagnetic interaction with a collinear quark $\chi$ and an usoft quark. It is necessary to include the four-quark operators of the form $\chi \Gamma_1 h \cdot \bar{\xi} \Gamma_2 \xi$ in SCET$_I$ since the time-ordered products of the four-quark operators with the electromagnetic interaction contributes to radiative $B$ decays.

There are also four-quark operators with a photon field of the form $\bar{\eta}_{us} \Gamma_1 h \cdot \bar{\xi} \Gamma_2 \xi \cdot A^\mu$, where $\Gamma_i$ are Dirac matrices. These operators are obtained from the four-quark operators
in Eq. (1) by attaching a photon to all the quarks except the spectator quark of the $B$ meson. In this case, the intermediate state has momentum of $m_b$ and can be integrated out to produce effective operators. The effects of the operators with a photon attached to a spectator quark are already included in the four-quark operators mentioned above. In this case the intermediate state has the momentum of order $m_b \lambda$ and cannot be integrated out in SCET$_1$.

When we evaluate the matrix elements of all the operators in SCET$_1$, we consider the matrix elements of the operators without spectator quarks, the four-quark operators with a photon, and the time-ordered products of the four-quark operators with the electromagnetic interaction of an usoft quark and a collinear quark $\chi$. In addition, when we consider the hard scattering contribution to the form factor for $B \to V$, we also include the time-ordered products of the operators without spectator quarks with the subleading collinear effective Lagrangian. And these matrix elements are evolved down to SCET$_{II}$. By decoupling soft gluon interactions, we obtain gauge invariant sets of operators and time-ordered products under collinear and soft gauge transformations, and evaluate the matrix elements.

We can consider other forms of four-quark operators, say, $\bar{\xi} \Gamma_1 h \cdot \bar{\xi} \Gamma_2 \xi A^\mu$, in which all the light quarks are collinear fields in the $n^\mu$ direction. We can make time-ordered products of these operators with the interaction of collinear gluons with an usoft quark and a collinear quark $\xi$ [21, 23]. If there are no external gluons in the time-ordered products, all the collinear gluons should appear as internal gluons, and the final operators correspond to, at higher orders in $\alpha_s$, the four-quark operators with a photon of the form $\bar{q}_{us} \Gamma_1 h \cdot \bar{\xi} \Gamma_2 \xi A^\mu$ after we perform the time-ordered product with the interaction involving the usoft quark. If there are external gluons, these are collinear gluons in the $n^\mu$ direction. In this case, these gluons should belong to the final state of a vector meson. Therefore it involves higher Fock space for the vector meson state, which we will neglect since the contributions are subleading.
III. OPERATORS IN SCET$_1$

A. Operators without spectator quarks

In radiative $B$ decays, the leading operator is $O_7$ and the corresponding gauge invariant operators in SCET$_1$ to first order in $\lambda$ are given by

$$O_7 \rightarrow A_7^{(0)} O_7^{(0)} + A_7^{(1a)} O_7^{(1a)} + A_7^{(1b)} O_7^{(1b)},$$

(5)

where $A_7^{(0,1a,1b)}$ are the Wilson coefficients, which are operators in SCET. The operator $O_7^{(0)}$ is the leading operator, and $O_7^{(1a,1b)}$ are the subleading operators suppressed by $\lambda$ compared to $O_7^{(0)}$. These operators are obtained by attaching a collinear gluon $A_\mu$ to the heavy quark and integrate out off-shell modes, as shown in Fig. 1. The operators are given as

$$O_7^{(0)} = \frac{em_b^2}{8\pi^2} (\xi W)_{\bar{n}} \hat{A}(1 + \gamma_5)h = -\frac{em_b^2}{4\pi^2} (\xi W)_{\bar{n}} \hat{A}(1 - \gamma_5)h,$$

$$O_7^{(1a)} = \frac{em_b}{4\pi^2} (\xi W)_{\bar{n}} \hat{A}[W^\dagger i\bar{\psi}_{n\perp} W](1 + \gamma_5)h,$$

$$O_7^{(1b)} = \frac{em_b}{4\pi^2} (\xi W)_{\bar{n}} \hat{A}[W^\dagger i\bar{\psi}_{n\perp} W](1 + \gamma_5)h,$$

(6)

where $\hat{A}$ is the photon field, and $h$ is the heavy quark field in the heavy quark effective theory. The factor $W$ is the Wilson line defined as

$$W = \sum_{\text{perm}} \exp \left[ -g \frac{1}{P} \cdot \hat{P} \cdot A_\mu \right],$$

(7)

where $\hat{P} = \pi \cdot P n^\mu / 2 + \hat{P}_\perp$ is the label momentum operator for collinear fields in the $n^\mu$ direction.

FIG. 1: QCD diagram attaching a collinear gluon to the $b$ quark to make a gauge invariant operator. A wavy line represents a photon, and a curly line represents a gluon. The momentum of the photon (gluon) is outgoing (incoming).
When we actually compute the Feynman diagram in Fig. 1, we obtain $O_7^{(0)}$ and $O_7^{(1a)}$ at tree level. However, the operators $O_7^{(1a,1b)}$ are the most general operators at subleading order considering the Dirac structure and the collinear gauge invariance. When we match SCET to the full theory at tree level, the Wilson coefficient for $O_7^{(1b)}$ is zero, but in the matching at higher order in $\alpha_s$, a nonzero coefficient $A_7^{(1b)}$ can be developed. If we calculate decay amplitudes to next-to-leading order accuracy, we need to calculate $A_7^{(0)}$ to order $\alpha_s$ and $A_7^{(1a,1b)}$ to order $\alpha_s^0$ since the operators $O_7^{(1a,1b)}$ begin with order $g$ and these operators will be combined with the effective Lagrangian in the time-ordered products, which are of order $\alpha_s$. There are no other subleading operators by including the subleading correction to the collinear field $\xi$ in $O_7^{(0)}$, where the fermion field $\psi$ in the full theory changes to

$$\psi(x) \rightarrow \left(1 + \frac{1}{\pi \cdot (\mathcal{P} + g\mathcal{A}_n)}(\hat{\mathcal{P}} \perp + g\mathcal{A}_n \perp)\frac{\pi}{2}\right)\xi.$$ (8)

It is because the subleading correction to the collinear field $\xi$ is proportional to $\pi$ and this vanishes when contracted with $\pi$ in the operator $O_7^{(0)}$.

The Wilson coefficient $A_7^{(0)}$, to first order in $\alpha_s$, is given as

$$A_7^{(0)}(\mu) = 1 - \frac{\alpha_s C_F}{4\pi} \left[\frac{1}{2} \ln \frac{\mu^2}{m_b^2} + \frac{7}{2} \ln \frac{\mu^2}{m_b^2} - 2 \ln \frac{\mu^2}{m_b^2} \ln \pi \cdot \hat{\mathcal{P}}^\dagger \right. $$

$$+ \left. 2 \ln^2 \pi \cdot \hat{\mathcal{P}}^\dagger + 2 \text{Li}_2(1 - \pi \cdot \hat{\mathcal{P}}^\dagger) - 2 \ln \pi \cdot \hat{\mathcal{P}}^\dagger + \frac{\pi^2}{12} + 6\right],$$ (9)

where $\pi \cdot \hat{\mathcal{P}} = \pi \cdot \mathcal{P}/m_b$, and $\text{Li}_2(x)$ is the dilogarithmic function. The coefficient $A_7^{(0)}$ is the same as $C_9$ in Ref. [3]. The remaining Wilson coefficients to zeroth order in $\alpha_s$ are given by

$$A_7^{(1a)} = 1, \quad A_7^{(1b)} = 0.$$ (10)

Since the Wilson coefficients are operators, $A_7^{(0)}$ in Eq. (5) means that the Wilson coefficient is sandwiched by the operator, that is, it implies that

$$A_7^{(0)} O_7^{(0)} = \frac{em_b^2}{8\pi^2} \xi W \mathcal{P} A_7^{(0)}(\pi \cdot \mathcal{P}) A(1 + \gamma_5) h = A_7^{(0)}(\pi \cdot p, \pi \cdot q) \frac{em_b^2}{8\pi^2} \xi W \mathcal{P} A(1 + \gamma_5) h,$$ (11)

where $p$ ($q$) is the momentum of the quark $\xi$ (the gluon in $W$).

We can obtain operators from $O_1$ and $O_8$, proportional to $O_7$ by attaching an external photon and a gluon in a loop, whose matrix elements are proportional to the matrix element of $O_7$ through the radiative corrections. The leading-order contributions from the operators $O_1$ and $O_8$, which are proportional to $O_7^{(0)}$ will be included in calculating the decay amplitudes in Section IV.
Now let us consider the operators which consist of a heavy-to-light current with an external photon and an external gluon, in addition to $O_7^{(1a,1b)}$. From the operator $O_8$ in the full theory, we can construct such an operator. It is given by

$$O_8 \rightarrow Q_d A_{8}^{(1a)} O_7^{(1a)} + Q_d A_{8}^{(1b)} O_7^{(1b)},$$

(12)

where $Q_d$ is the electric charge of the down-type quark ($d$ or $s$). Note that there are only two independent operators $O_7^{(1a,1b)}$ at order $\lambda$ in SCET$_I$. Here the operator without a photon from $O_8$ is not included since it does not contribute to radiative $B$ decays, but we comment that the leading operator from $O_8$ without a photon field is suppressed by $\lambda^2$ compared to $O_7^{(0)}$. We can compute the Wilson coefficients $A_{8}^{(1a,1b)}$ to any desired accuracy, but at next-to-leading order, we need the tree-level Wilson coefficients since the operators begin with order $g$. Matching the operators with the full theory at tree level, as shown in Fig. 2, the Wilson coefficient $A_{8}^{(1b)}$ is zero, and $Q_d A_{8}^{(1a)} O_7^{(1a)}$ can be written as

$$Q_d A_{8}^{(1a)} O_7^{(1a)} = \frac{e Q_d m_b}{4\pi^2} \xi W A \frac{1}{\tau} \cdot \tau \cdot \mathcal{P}[W^\dagger i \not{D}_{a\perp} W] (1 + \gamma_5)h,$$

(13)

where $Q_d$ is the electric charge of the collinear quark $\xi$. The operator in Eq. (13) is obtained by attaching a photon to fermion lines in the operator $O_8$, as shown in Fig. 2. Note that the contribution of Fig. 2 (b) is suppressed by $\lambda$ compared to that of Fig. 2 (a), and it is neglected here.

We can also obtain an operator with a photon and a gluon from the operator $O_1$. We cannot make the operator from $O_2$ due to the color structure, and we neglect the contributions of penguin operators. The full QCD diagram is shown in Fig. 3 and the resultant

![FIG. 2: QCD diagrams attaching a photon to each fermion to the operator $O_8$ in order to generate the operator with an external photon and an external gluon. The momentum of the photon (gluon) is outgoing (incoming). Diagram (a) is of order $\lambda$ compared to $O_7^{(0)}$. Diagram (b) is suppressed by $\lambda^2$ and is neglected.](image-url)
operator can be written as [24, 25]

\[ \bar{d} A^\mu A^\nu I_{\mu\nu} b, \]

where

\[ I_{\mu\nu} = \frac{g e Q_u}{8 \pi^2} [i \epsilon_{\mu\nu\alpha\beta} (q^\alpha \Delta i_5 + q^\alpha \Delta i_6) + \frac{1}{q \cdot q_T} (i \epsilon_{\mu\nu\alpha\beta} q^\rho \Delta i_{23} + i \epsilon_{\rho\mu\alpha\beta} q^\rho q_T \Delta i_{26})] \gamma^\beta (1 - \gamma_5). \] (15)

Here the momentum of the gluon \( q \) is incoming to the loop and the momentum of the photon \( q_n \) is outgoing. The quantities \( \Delta i_n \equiv \Delta_i(z_0, z_1, z_2) \) are functions of the three variables \( z_0 \), \( z_1 \) and \( z_2 \), which are the invariant mass-squared of the fermion-antifermion pair in the loop, the gluon momentum squared, and the photon momentum squared, each divided by the internal fermion mass \( m_i^2 \), and they are given by

\[ z_0 = \frac{(q - q_T)^2}{m_i^2} = \frac{(p_1 - m_b \nu)^2}{m_i^2} = \frac{m_b^2}{m_i^2} (1 - u), \quad z_1 = \frac{q^2}{m_i^2}, \quad z_2 = \frac{q_T^2}{m_i^2} = 0, \] (16)

where \( u \) is the momentum fraction of the \( d \) quark in a vector meson. Replacing the quark fields with the fields in SCET, it turns out that \( I_{\mu\nu} \) starts from terms of order \( \lambda \) apart from \( \Delta i_n \)'s. Therefore we need to expand \( \Delta i_n \) to leading order in \( \lambda \), which are given by \( \Delta i_n(z_0, 0, 0) \). In this case, \( \Delta i_n \) have the relation

\[ \Delta i_{23}(z_0, 0, 0) = \Delta i_{26}(z_0, 0, 0) = \Delta i_6(z_0, 0, 0) = \Delta i_5(z_0, 0, 0), \] (17)

and \( \Delta i_5(z_0, 0, 0) \) is given by

\[ \Delta i_5(z_0, 0, 0) = -\frac{4}{z_0} \left[ \text{Li}_2 \left( \frac{2}{1 - \sqrt{1 - 4/z_0 + i\epsilon}} \right) + \text{Li}_2 \left( \frac{2}{1 + \sqrt{1 - 4/z_0 + i\epsilon}} \right) \right] + 2. \] (18)

![QCD diagrams](image)

**FIG. 3:** QCD diagrams generating the operator with an external photon and an external gluon from \( O_1 \), in which a photon and a gluon are attached to the internal fermion line. The gluon momentum \( q \) is incoming and the photon momentum \( q_T \) is outgoing. The same diagrams with the penguin operators are neglected.
Note that, in Ref. [25], the momenta of the gluon and the photon are both outgoing, while the momentum of the gluon is reversed in our case in order to keep a consistent convention that a collinear gluon is incoming. Because of this different choice of the directions of the momenta, $\Delta i_5$ and $\Delta i_23$ in Eqs. (17) and (18) have opposite signs compared to those in Ref. [25].

Replacing the fields in Eq. (14) by the fields in the effective theory, and making the resultant operator collinear gauge invariant, we obtain the effective operators at order $\lambda$, which are given by

$$O_1 \rightarrow Q_u A_1^{(1a)} O_7^{(1a)} + Q_u A_1^{(1b)} O_7^{(1b)},$$

where $Q_u$ is the electric charge of the up-type quark in the fermion loop. To next-to-leading order, the Wilson coefficients are given by

$$A_1^{(1a)} = -A_1^{(1b)} = \frac{1}{4} H(\pi \cdot P^\dagger, s_i),$$

where $H(\pi \cdot P^\dagger, s_i) = \Delta i_5 (z_0 \rightarrow (1 - \pi \cdot P^\dagger/m_b)/s_i, 0, 0)$ is now an operator with $s_i = m_i^2/m_b^2$.

Compared to $h(u, s)$ in Ref. [16], $H(u, s) = -uh(u, s)$. The operators in Eq. (19) can be written as

$$Q_u A_1^{(1a)} O_7^{(1a)} + Q_u A_1^{(1b)} O_7^{(1b)} = \frac{eQ_u m_b}{16\pi^2} [\zeta WH(\pi \cdot P^\dagger, s_i) \mathcal{A}[W^\dagger i\bar{\psi}_{n\perp} W](1 + \gamma_5)h - \bar{\zeta} WH(\pi \cdot P^\dagger, s_i)[W^\dagger i\bar{\psi}_{n\perp} W] \mathcal{A}(1 + \gamma_5)h].$$

So far, we have considered the effective operators from the operator $O_1$, in which a photon and a gluon are attached to the internal fermion loop. However, there are other possibilities to attach a photon and a gluon to fermions. For example, we can attach a gluon in the fermion loop and a photon to external fermions, as shown in Fig. 4. Without the photon,

![Diagram](image)

FIG. 4: QCD diagrams generating an operator with a gluon and a photon from $O_1$. The resultant operator is suppressed by $\lambda^2$ compared to $O_7^{(1a)}$, and is neglected.
FIG. 5: QCD diagrams in which a photon attached to the internal fermion line and a gluon is attached to one of the external fermions of the operator \( O_1 \). These diagrams vanish due to the electromagnetic gauge invariance.

The fermion loop with a gluon attached to it produces an operator [26]

\[
\frac{g}{16\pi^2} \left( \frac{4}{3} \ln \frac{m_i}{\mu} + \frac{2}{3} + 4B \right) \overline{\chi}(q^2\gamma_\mu - q_\mu\gamma_5) A^\mu(1 - \gamma_5) b, \tag{22}
\]

with

\[
B = \int_0^1 dz z(1 - z) \ln \left(1 - z(1 - z) \frac{q^2}{m_i^2} \right), \tag{23}
\]

where \( m_i \) is the fermion mass in the loop and \( q^\mu \) is the gluon momentum. For the on-shell collinear gluon \( q^2 = 0 \), when we replace the quark fields by the fields in the effective theory, the operator in Eq. (22) is proportional to \( \overline{\xi}q \cdot A_n q_\perp (1 - \gamma_5) h \), and it is suppressed by \( \lambda^3 \) compared to \( O_7^{(0)} \) since \( q \cdot A_n q_\perp \sim O(\lambda^3) \). When we attach a photon to external fermions to make the final effective operators, the power counting does not change, hence they are neglected.

The remaining possibility is to attach a photon to the fermion loop and a gluon is attached to external fermions of the operator \( O_1 \), as shown in Fig. 5. This operator is zero due to the electromagnetic gauge invariance. That is, the fermion loop in Fig. 5 is proportional to Eq. (22), replacing the gluon field by the photon field. Since the photon is on-shell, \( q^2 = 0 \) and \( q \cdot A = 0 \), the operator vanishes.

**B. Operators of the form \( (\overline{\chi}\Gamma_1 h)(\overline{\xi}\Gamma_2 \xi) \)**

The operators with spectator quarks involve four-quark operators and four-quark operators with a photon. The four-quark operators are generically of the form \( (\overline{\chi}\Gamma_1 h)(\overline{\xi}\Gamma_2 \xi) \). The inclusion of these operators in SCET\(_1\) is necessary because there is an electromagnetic interaction of a photon in the \( \overline{\xi}^\mu \) direction with an usoft quark \( q_{us} \) and a collinear quark \( \chi \).
The time-ordered products of the four-quark operators with the electromagnetic interaction correspond to the emission of an energetic photon from the spectator quark. At first sight, we can make an operator by attaching a photon to a spectator quark and integrate out the intermediate state to make an operator, but the intermediate state has the momentum of order $m_b \lambda$, which cannot be integrated out in SCET. Therefore we construct the four-quark operators of the form $(\bar{\psi} \Gamma_1 h)(\bar{\xi} \Gamma_2 \xi)$ in SCET and make the time-ordered products of these operators with the electromagnetic interaction. Note that, if a photon is attached to external fermions other than the spectator quark, the intermediate state has the momentum of order $m_b$, and it can be integrated out to produce four-quark operators with a photon. This will be considered in the next subsection.

The explicit form of the effective four-quark operators can be obtained from the four-quark operators for nonleptonic decays in Ref. [13] with a minor modification. We only have to change $n^\mu \leftrightarrow \overline{p}^\mu$ and $\chi \leftrightarrow \xi$ in the effective operators in Ref. [13]. These operators are given as

$$
O_{1R} = \left( (\bar{\psi}^{\dagger} W) h_\alpha \right)_{V-A} \left( (\bar{\xi}^d W) \beta (W^\dagger \xi^u) \beta \right)_{V-A}, \\
O_{2R} = \left( (\bar{\psi}^{\dagger} W) \beta h_\alpha \right)_{V-A} \left( (\bar{\xi}^d W) \alpha (W^\dagger \xi^u) \beta \right)_{V-A}, \\
O_{3R} = \left( (\bar{\psi}^{\dagger} W) h_\alpha \right)_{V-A} \sum_q \left( (\bar{\xi}^d W) \beta (W^\dagger \xi^q) \beta \right)_{V-A}, \\
O_{4R} = \left( (\bar{\psi}^{\dagger} W) \beta h_\alpha \right)_{V-A} \sum_q \left( (\bar{\xi}^d W) \alpha (W^\dagger \xi^q) \beta \right)_{V-A}, \\
O_{5R} = \left( (\bar{\psi}^{\dagger} W) h_\alpha \right)_{V-A} \sum_q \left( (\bar{\xi}^d W) \alpha (W^\dagger \xi^q) \beta \right)_{V-A}, \\
O_{6R} = \left( (\bar{\psi}^{\dagger} W) \beta h_\alpha \right)_{V-A} \sum_q \left( (\bar{\xi}^d W) \beta (W^\dagger \xi^q) \beta \right)_{V-A}, \\
O_{1C} = \left( (\bar{\psi}^{\dagger} W) \beta h_\alpha \right)_{V-A} \left( (\bar{\xi}^u W) \alpha (W^\dagger \xi^u) \beta \right)_{V-A}, \\
O_{2C} = \left( (\bar{\psi}^{\dagger} W) h_\alpha \right)_{V-A} \left( (\bar{\xi}^u W) \beta (W^\dagger \xi^u) \beta \right)_{V-A}, \\
O_{3C} = \sum_q \left( (\bar{\psi}^{\dagger} W) \beta h_\alpha \right)_{V-A} \left( (\bar{\xi}^d W) \alpha (W^\dagger \xi^q) \beta \right)_{V-A}, \\
O_{4C} = \sum_q \left( (\bar{\psi}^{\dagger} W) h_\alpha \right)_{V-A} \left( (\bar{\xi}^d W) \beta (W^\dagger \xi^q) \beta \right)_{V-A}.
$$

Here the summation over $q$ goes over to light massless quarks, say, $u$, $d$, and $s$ quarks, and $\overline{W}$ is the Wilson line which is given by

$$\overline{W} = \sum_{\text{perm}} \exp \left[ -g \frac{1}{n \cdot Q} n \cdot A \right], \\
(25)$$

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FIG. 6: QCD diagrams generating the four-quark operators from $O_1$ and $O_8$. But these operators are included in the four-quark operators in Eq. (24).

where $Q^\mu = n \cdot Q \tau^\mu / 2 + Q_\perp^\mu$ is the label momentum operator for the collinear fields in the $\tau^\mu$ direction. The operators in Eq. (24) with $\overline{W}$ and $W$ are collinear gauge invariant. The Wilson coefficients of the operators in Eq. (24) are the same as the corresponding operators in Ref. [13]. The time-ordered products of these operators with the electromagnetic interaction with an usoft quark and a collinear quark $\chi$ contributes to the radiative $B$ decays. These correspond to the contribution from the annihilation and the $W$-exchange channels.

The operators $O_1$ and $O_8$ can induce four-quark operators, as shown in Fig. 6. However, these operators are four-quark operators which can be expressed in terms of the four-quark operators in Eq. (24). We include these operators in the operators given by Eq. (24), and their effects appear in the effective Wilson coefficients of the operators.

C. Four-quark operators with a photon field

We can obtain four-quark operators with a photon field by attaching a photon to fermion lines in four-quark operators. First these operators can be obtained from the annihilation and the $W$-exchange channels from the four-quark operators in Eq. (1) by attaching a photon on each fermion except the spectator quark. The relevant Feynman diagrams are shown in

FIG. 7: The four-quark operator with a photon in SCET$_1$ arising from the four-quark operators, which correspond to annihilation channels or $W$-exchange channels.
FIG. 8: QCD diagrams to produce the four-quark operators with a photon arising from $O_1$ and $O_8$. These diagrams are subleading.

Fig. 7. The Feynman diagrams in Fig. 7 give the desired operators in SCET_I by integrating out the off-shell modes. But these are subleading since the effective operators involve an usoft quark. When we go down to SCET_{II}, there are no enhancements and the effect of these operators still remains subleading and we neglect these operators at leading order in SCET. In Fig. 7, the Feynman diagram in which a photon is attached to a spectator quark is missing, but it has been considered in the previous subsection in which we consider the four-quark operators of the form $(\bar{\chi_1} \Gamma_1 h)(\bar{\chi_2} \Gamma_2 \xi)$ and we calculate the time-ordered product of this operator with the electromagnetic interaction. This is because the intermediate state has momentum of order $m_b \lambda$, and it cannot be integrated out.

The effective four-quark operators with a photon can also arise from the operators $O_1$ and $O_8$, as shown in Fig. 8. However, the diagrams in Fig. 8 are subleading due to the interaction of a collinear gluon with an ultrasoft quark and a collinear quark. When we match this operator to SCET_{II}, there is no enhancement, hence they are still subleading and neglected here.

IV. OPERATORS IN SCET_{II} AND THEIR MATRIX ELEMENTS

A. Form factors and the leading contributions

We evaluate the matrix elements of all the relevant operators for radiative $B$ decays, which are obtained in SCET_I. And we decouple the usoft gluon interactions by redefining collinear fields and go down to SCET_{II}. There are various matrix elements we should consider. First we evaluate matrix elements of the operators without spectator quarks, namely
$O_7^{(0)}$ and the operators induced from $O_1$ and $O_8$, of which the matrix elements are proportional to the matrix element of $O_7^{(0)}$. Second, we consider the time-ordered products of the operators including external gluons without spectator quarks with the interaction of a collinear gluon with a collinear quark $\xi$ and a usoft quark. These matrix elements contribute to the form factor $B \to V$. As will be seen later, some of the matrix elements are factorized and they correspond to the contribution from the hard scattering amplitude, which is calculable in perturbation theory. The remaining nonfactorizable contributions involve endpoint singularities, but these are absorbed into the soft form factor. The time-ordered products from other operators, the time-ordered products of four-quark operators with the electromagnetic interaction, and the four-quark operators with a photon correspond to hard scattering amplitudes with the convolution of the light-cone wave functions of mesons. As discussed before, the contributions from the four-quark operators and the four-quark operators with a photon are subleading and they will not be included in the final result for the decay amplitudes at leading order in SCET.

In evaluating matrix elements of various contributions, the matrix elements of heavy-to-light currents are related to form factors. The form factors can be defined in SCET with a fewer number of independent form factors due to the symmetry of SCET. In general, the form factors for $B$ decays into vector mesons are defined as [27]

$$
\langle V(p, \xi^\nu) | \overline{T}_{\gamma^\mu} b | B(p_B) \rangle = \frac{2iV(q^2)}{m_B + m_V} \epsilon^{\mu\nu\alpha\beta} \eta^\alpha_p \eta^\beta_{p_B},
$$

$$
\langle V(p, \eta^\nu) | \overline{T}_{\gamma^\mu} \gamma_5 b | B(p_B) \rangle = 2m_V (q^2) \eta^\nu \frac{q^\mu}{q^2} + (m_B + m_V) A_1(q^2) \left[ \eta^\mu - \eta^\nu \frac{q^\mu}{q^2} \right],
$$

$$
\langle V(p, \eta^\nu) | \overline{T}_{\sigma^\mu\nu} q_\nu (1 + \gamma_5) b | B(p_B) \rangle = 2T_1(q^2) \epsilon^{\mu\rho\sigma\beta} \eta^\rho_{p_B} p_{\beta},
$$

$$
- i T_2(q^2) \left[ (m_B^2 - m_V^2) \eta^\mu - (\eta^\nu q^\mu) (p_B^\nu + p_B^\mu) \right],
$$

$$
- iT_3(q^2) \left[ q^\mu - \frac{q^2}{m_B^2 - m_V^2} (p_B^\nu + p_B^\mu) \right],
$$

where $q = p_B - p$, $m_V (\eta)$ is the mass (polarization vector) of the vector meson and we use the convention $\epsilon^{0123} = -1$. Note that the definition of the form factors are different from that defined by Ball and Braun [28], but if we multiply the right-hand sides of Eq. (26) by the factor $-i$, we obtain the definition in Ref. [28].
In SCET, we neglect the masses of the vector mesons, and the polarization vector is transverse in this limit. We choose the plane of the transverse polarization defined by $n \cdot \eta^* = \bar{n} \cdot \eta^* = 0$. In SCET, all the independent form factors defined in Eq. (26) are reduced to a few independent nonperturbative functions. Especially, at leading order in SCET, there is only one independent form factor $\zeta_\perp$, and the matrix elements in Eq. (26) reduce to [2, 7, 29]

$$\langle V(n, \eta^*)|\bar{\xi} W \gamma^\mu \gamma_5 S^\dagger h|B\rangle = iE\zeta_\perp(E)\epsilon^{\mu\nu\alpha\beta} \eta^*_\nu n_\alpha \bar{\eta}_\beta,$$

$$\langle V(n, \eta^*)|\bar{\xi} W \gamma^\mu \gamma_5 S^\dagger h|B\rangle = 2E\zeta_\perp(E)\eta^*_\mu,$$

$$\langle V(n, \eta^*)|\bar{\xi} W \sigma^{\mu\nu} q_\nu S^\dagger h|B\rangle = -Em_B\zeta_\perp(E)\epsilon^{\mu\nu\alpha\beta} \eta^*_\nu n_\alpha \bar{\eta}_\beta,$$

$$\langle V(n, \eta^*)|\bar{\xi} W \sigma^{\mu\nu} q_\nu \gamma_5 S^\dagger h|B\rangle = -2iEm_B\zeta_\perp(E)\eta^*_\mu,$$  \hspace{1cm} (27)

where the operators are replaced by the operators in SCET II in a gauge invariant form. Note that there is another nonperturbative function $\zeta_{\parallel}$, but since we choose $n \cdot \eta^* = \bar{n} \cdot \eta^* = 0$, it does not appear in Eq. (27) at leading order in SCET.

From now on, we express all the operators at next-to-leading order accuracy and replace the operators of the Wilson coefficients by the Wilson coefficients after we apply the operators of the Wilson coefficients inside the relevant operators. The leading-order contribution to the radiative $B$ decays comes from the operator $O_7^{(0)}$, and in SCET II, the gauge invariant form under the collinear and soft gauge transformations is given by

$$O_7^{(0)} = -\frac{em_B^2}{4\pi^2} \left(\bar{\xi} W\right) A(1 - \gamma_5) S^\dagger h.$$ \hspace{1.5cm} (28)

The matrix element of $O_7^{(0)}$ is given by

$$\langle V(n, \eta_\perp), \gamma(\bar{n}, \epsilon_\perp)|O_7^{(0)}|B\rangle = \frac{em_B^2}{8\pi^2} m_B \zeta_\perp(E) \left(2\epsilon^*_\perp \cdot \eta_\perp + i\epsilon^{\mu\nu\alpha\beta} \eta^*_\nu \epsilon^*_\alpha n_\alpha \bar{\eta}_\beta\right),$$  \hspace{1cm} (29)

where $|B\rangle$ is an appropriate $B$ meson state depending on the type of the vector meson $V$.

There are also radiative corrections at order $\alpha_s$ from the operators $O_1$ and $O_8$, in which there are an external photon and an internal gluon. The matrix elements of these operators are proportional to the matrix element of $O_7^{(0)}$. Denoting the operators induced from $O_1$ and $O_8$ in as $O_1^{(0)}$ and $O_8^{(0)}$, their matrix elements are proportional to the matrix element of $O_7^{(0)}$, and can be written as [16, 25]

$$\langle O_{1,8}^{(0)} \rangle = \langle O_7^{(0)} \rangle \frac{\alpha_s C_F}{4\pi} G_{1,8},$$ \hspace{1.5cm} (30)
where

\[ G_1(s_c) = \frac{104}{27} \ln \frac{m_b}{\mu} + g_1(s_c), \quad G_8 = -\frac{8}{3} \ln \frac{m_b}{\mu} + g_8, \]

\[ g_1(s) = -\frac{833}{162} - \frac{2}{27} i\pi + \frac{8\pi^2}{9} s^{3/2} \]
\[ + \frac{2}{9} [48 + 30i\pi - 5\pi^2 - 2i\pi^3 - 36\zeta(3) + (36 + 6i\pi - 9\pi^2) \ln s \]
\[ + (3 + 6i\pi) \ln^2 s + \ln^3 s] s \]
\[ + \frac{2}{9} [18 + 2\pi^2 - 2i\pi^3 + (12 - 6\pi^2) \ln s + 6i\pi \ln^2 s + \ln^3 s] s^2 \]
\[ + \frac{1}{27} [-9 + 112i\pi - 14\pi^2 + (182 - 48i\pi) \ln s - 126 \ln^2 s] s^3, \]
\[ g_8 = \frac{11}{3} - \frac{2\pi^2}{9} + \frac{2i\pi}{3}, \] (31)

and \( s_c = m_{\ell}^2/m_b^2 \). There can be subleading operators \( O_7^{(1a,1b)} \) from \( O_1 \) and \( O_8 \) in SCET, but they start with \( g^3 \).

**B. Contributions to the form factor**

We consider first the effects of the operators \( O_7^{(0,1a,1b)} \), which are obtained from \( O_7 \). We calculate the time-ordered products of these operators with the interaction of collinear quarks with an usoft quark and a collinear quark \( \xi \) in SCET\(_1\). The interaction of collinear and usoft gluons starts from order \( \lambda \), but since the propagator of the exchanged gluon is enhanced by \( 1/\lambda^2 \), we have to include the interaction of collinear and usoft gluons to order \( \lambda^2 \) to obtain the leading-order result. Among these contributions, the time-ordered products of the operator derived from \( O_7 \) contribute to the form factor, while those from the operators \( O_1 \) and \( O_8 \) contribute to the hard scattering amplitudes. Here we first consider the contributions to the form factor.

The Lagrangian for the interactions of collinear gluons with a collinear quark \( \xi \) and an usoft quark \( q_{us} \) is given by \([5, 8, 30]\)

\[ \mathcal{L}_{\xi q}^{(1)} = ig\bar{\xi} \frac{1}{im \cdot D_n} B_\perp W q_{us} + \text{h.c.}, \]
\[ \mathcal{L}_{\xi q}^{(2a)} = ig\bar{\xi} \frac{1}{im \cdot D_n} W q_{us} + \text{h.c.}, \]
\[ \mathcal{L}_{\xi q}^{(2b)} = ig\frac{\gamma_5}{2} i\not{\!D}_\perp \frac{1}{(im \cdot D_n)^2} B_\perp W q_{us} + \text{h.c.}, \] (32)

where

\[ ig\not{\!D}_\perp = [im \cdot D_n, i\not{\!D}_\perp], \quad ig\not{\!W} = [im \cdot D_n, i\not{\!D}_{us} + \frac{m}{2} gn \cdot A_n]. \] (33)
FIG. 9: Tree-level graphs in SCET\textsubscript{I} for the spectator contribution to the heavy-to-light form factor. The first diagram contributes to $T_{1,3}$, and the second diagram contributes to $T_{0,2,3,4,5}$.

At leading order in SCET\textsubscript{I}, the time-ordered products, which contribute to the form factor, are given as

\begin{align}
T^F_0 &= \int d^4x T[O^0_7(0)i\mathcal{L}^{(1)}_{\xi q}(x)], \quad T^F_1 = \int d^4x T[O^1_7(0)i\mathcal{L}^{(1)}_{\xi q}(x)], \\
T^F_2 &= \int d^4x T[O^0_7(0)i\mathcal{L}^{(2b)}_{\xi q}(x)], \quad T^F_{3} = \int d^4x T[O^0_7(0)i\mathcal{L}^{(2a)}_{\xi q}(x)], \\
T^{NF}_4 &= \int d^4xd^4y T[O^0_7(0)i\mathcal{L}^{(1)}_{\xi q}(x)i\mathcal{L}^{(1)}_{\xi q}(y)], \quad T^{NF}_5 = \int d^4xd^4y T[O^0_7(0)i\mathcal{L}^{(1)}_{c q}(x)i\mathcal{L}^{(1)}_{d q}(y)],
\end{align}

and these are shown in Fig. 9 schematically. After we evaluate the matrix elements of the time-ordered products in SCET\textsubscript{I}, we decouple the collinear-usoft interaction using the field redefinitions

\begin{align}
\xi^{(0)} &= Y^\dagger \xi, \quad A_n^{(0)} = Y^\dagger A_n Y, \quad Y(x) = \text{P exp}\left(ig\int_{-\infty}^x ds n \cdot A_{us}(ns)\right),
\end{align}

in order to go down to SCET\textsubscript{II}. Matching at $\mu_0 \sim \sqrt{m_b \Lambda}$, the usoft fields become soft ($Y \rightarrow S$) and the operators are matched onto the operators in SCET\textsubscript{II}. When we redefine the fields using the Wilson line $Y$, soft gluons decouple in $T^F_i$. It means that the contributions are factorizable to all orders in $\alpha_s$. On the other hand, soft gluons do not decouple in the time-ordered products $T^{NF}_i$. Furthermore these contributions have endpoint singularities [8]. They are absorbed in the soft form factor $\zeta_\perp$, as was done in Ref. [8].

Among the factorizable time-ordered products, $\langle T^F_0 \rangle = \langle T^F_2 \rangle = 0$. The matrix element of $T^F_1$ is given by

\begin{align}
\langle T^F_1 \rangle &= \langle V, \gamma | \int d^4x T[O^1_7(0)i\mathcal{L}^{(1)}_{\xi q}(x)] | B \rangle \\
&= \frac{\alpha_s C_F em_b}{\pi N} \int d\vec{m} \cdot x \int dn \cdot k e^{in-k\vec{\pi}x/2}.
\end{align}
\[
\times \langle V, \gamma | (\vec{\tau}_s S)(\vec{\tau} \cdot x) \frac{1}{n \cdot R} \slashed{\eta} (1 + \gamma_5) S^\dagger h \cdot \xi W \frac{1}{\Pi \cdot P} \slashed{\eta} (1 + \gamma_5) W \xi | B \rangle
\]
\[
= \frac{\alpha_s C_F}{8\pi} \int_B f_B m_B^2 \left( 2\eta_\perp \cdot \epsilon_\perp^* + i\epsilon_{\mu\alpha\beta} \eta_\perp^\mu \epsilon_\perp^\alpha \eta_\perp^\beta \right) \int dr_+ \frac{\phi_B^+(r_+)}{r_+} \int du \frac{\phi_\perp(u)}{u}, \quad (36)
\]
where \( u (\bar{u} \equiv 1 - u) \) is the momentum fraction of the quark (antiquark) inside the meson. \( R \) is the operator which extracts a soft momentum (of order \( \Lambda_{\text{QCD}} \)) from a soft particle. In evaluating Eq. (36), the matrix element involving the \( B \) meson can be calculated as
\[
\langle 0 | \bar{\tau}_s S(\vec{\tau} \cdot x) \slashed{\eta} (1 + \gamma_5) S^\dagger h | B \rangle
\]
\[
= \int dr_+ e^{-ir_+ \vec{\tau} \cdot x/2} \text{Tr} \left[ \Psi_B(r_+) \slashed{n} (1 + \gamma_5) \right]
\]
\[
= -\frac{i f_B m_B}{4} \int dr_+ e^{-ir_+ \vec{\tau} \cdot x/2} \text{Tr} \left[ \frac{1 + \gamma_5}{2} \slashed{n} \Psi_B(r_+) \right]
\]
\[
= if_B m_B \int dr_+ e^{-ir_+ \vec{\tau} \cdot x/2} \phi_B^+(r_+), \quad (37)
\]
where the leading-twist \( B \) meson light-cone wave function is defined through the projection of the \( B \) meson as [27, 31]
\[
\Psi_B(r_+) = -\frac{if_B m_B}{4} \left[ \frac{1 + \gamma_5}{2} \left( \bar{\eta}_\perp \phi_B^+(r_+) + \eta_\perp \phi_B^-(r_+) \right) \gamma_5 \right]. \quad (38)
\]
And the momentum-space representation of the vector meson light-cone projection at leading order is given by
\[
M_{\alpha\beta}^V = M_{\alpha\beta||}^V + M_{\alpha\beta\perp}^V, \quad (39)
\]
where, at leading twist and at leading order in SCET with \( n \cdot \eta_\perp = \vec{\eta} \cdot \eta_\perp = 0 \)
\[
M_{||}^V = 0, \quad M_{\perp}^V = -\frac{if_\perp}{4} E_{\eta_\perp}^\dagger \slashed{\eta} \phi_\perp(u), \quad (40)
\]
where \( \eta_\perp^\mu \) is the transverse polarization of the vector meson.

Among the nonfactorizable contributions, \( \langle T_5^{NF} \rangle = 0 \). In \( \langle T_3^{NF} \rangle \), there exist \( 1/u^2 \), and \( 1/r_+^2 \) singularities. However, since all endpoint singularities are regulated by \( \Lambda_{\text{QCD}} \) in the full QCD, if all the soft operators are included to cover the endpoint regions, the singularities will not arise. We absorb the nonfactorizable contributions to the form factor into the soft form factor, given by \( \zeta_\perp \).
C. Contributions to hard scattering amplitudes

There are other time-ordered products of the operators derived from $O_{1,8}$, which contribute to the nonfactorizable contributions to the radiative decays, which are defined as

$$U_1 = \int d^4 x T[O_{1}^{(1)}(0)i\mathcal{L}_{\xi q}^{(1)}(x)], \quad U_2 = \int d^4 x T[O_{8}^{(1)}(0)i\mathcal{L}_{\xi q}^{(1)}(x)],$$

where the operators $O_{1,8}^{(1)}$ are the effective operators derived from $O_{1,8}$ at order $\lambda$. These operators are given as

$$O_{1}^{(1)} = \frac{eQ_u m_b}{16\pi^2} \left( \xi \mathcal{W}_H(\pi \cdot \mathcal{P}, s_i) \mathcal{A}[W^\dagger i\mathcal{P}_n W] (1 + \gamma_5) \hbar \right)$$

$$O_{8}^{(1)} = \frac{eQ_d m_b}{4\pi^2} \xi \mathcal{W}_A \mathcal{A} \mathcal{P}[W^\dagger i\mathcal{P}_n W] (1 + \gamma_5) \hbar,$$

which can be obtained from Eqs. (21) and (13). Since the operators $O_{1,8}^{(1)}$ start with $\lambda$, we need time-ordered products only with $\mathcal{L}_{\xi q}^{(1)}$.

The matrix element of $U_1$ is written as

$$\langle U_1 \rangle = \langle V, \gamma | \int d^4 x T[O_{1}^{(1)}(0)i\mathcal{L}_{\xi q}^{(1)}(x)] | B \rangle$$

$$= \frac{\alpha_s C_F eQ_u}{4\pi N} \int d\vec{n} \cdot x \int dn \cdot k e^{i\vec{n} \cdot \vec{k} - x^2/2}$$

$$\times \langle V, \gamma | \left( \mathcal{S}(\vec{n}) S(\vec{n} \cdot x) \right) \mathcal{A} \mathcal{P} \mathcal{W}_H(\pi \cdot \mathcal{P}, s_i) \mathcal{A} \mathcal{P} (1 + \gamma_5) W^\dagger \xi | B \rangle$$

$$= \frac{\alpha_s C_F}{32\pi N} eQ_uf_B f_{1\perp} m_B^2 (2\eta_\perp^\star \epsilon_\perp^\star + ie_{\mu\nu\alpha\beta} \eta_\perp^\mu \epsilon_\perp^\nu n^\alpha \mathcal{P}^\beta)$$

$$\times \int dr_+ \frac{\phi_+^+(r_+)}{r_+} \int duH(m_b u, s_i) \phi_-(u) \frac{\phi_\perp^+(u)}{u}.$$  

(43)

If we use the leading-twist light-cone wave function $\phi_\perp(u) = 6u\pi$ for the vector meson, we can replace $\pi$ by $u$ in the last line of Eq. (43). But in general, if the valence quarks have different masses such that the wave function is not symmetric under $u \leftrightarrow \pi$, the present form should be kept.

Note that the matrix element of the time-ordered product with the second operator of $O_{1}^{(1)}$ in Eq. (42), or the operator $O_{7}^{(1b)}$ in Eq. (6), vanishes. To prove this, let us consider the matrix element of the operator

$$O_{7}^{(1b)} = \frac{eQ_u m_b}{4\pi^2} \xi \mathcal{W}[W^\dagger i\mathcal{P}_n W] \mathcal{A} (1 + \gamma_5) \hbar.$$  

(44)
The Wilson coefficients, when sandwiched between the heavy quark and the collinear quark fields, produce kinematic variables such as \( \mathbf{p} \cdot \mathbf{p}_1 \), where \( \mathbf{p}_1 \) is the momentum of the collinear quark \( \xi \). Then the matrix element of the operator \( O^{(1b)}_1 \) with the collinear Lagrangian \( \mathcal{L}^{(1)}_{\xi q} \) is written as

\[
\langle V, \gamma | \int d^4x T[O^{(1b)}_1(0)i\mathcal{L}^{(1)}_{\xi q}(x)]|B \rangle = \frac{\alpha_s C_F e m_b}{4N} \int d\mathbf{n} \cdot x \int d\mathbf{n} \cdot k e^{i n \cdot \mathbf{p} x/2} \times \langle \overline{\psi}_s S(\mathbf{p} \cdot x) \frac{1}{n \cdot R} \gamma \mathcal{A}(1 + \gamma_5) S^T h \cdot \mathbf{\xi} W \frac{1}{\mathbf{\pi} \cdot P} (1 - \gamma_5) W^T \mathbf{\xi} |B \rangle \quad (45)
\]

The matrix element in the last line in Eq. (45) is zero when we use the \( B \) mesons and the vector meson projections at leading order. Therefore the matrix element of the time-ordered products of \( O^{(1b)}_1 \) with \( \mathcal{L}^{(1)}_{\xi q} \) vanishes.

Similarly the matrix element of \( U_2 \) is written as

\[
\langle U_2 \rangle = \langle V, \gamma | \int d^4x T[O^{(1b)}_8(0)i\mathcal{L}^{(1)}_{\xi q}(x)]|B \rangle \quad (46)
\]

\[
= \frac{\alpha_s C_F e Q_{dmb}}{8\pi N} \int d\mathbf{n} \cdot x \int d\mathbf{n} \cdot k e^{i n \cdot \mathbf{p} x/2} \times \langle \overline{\psi}_s S(\mathbf{p} \cdot x) \frac{1}{n \cdot R} \gamma \mathcal{A}(1 + \gamma_5) S^T h \cdot \mathbf{\xi} W \frac{1}{\mathbf{\pi} \cdot P} (1 + \gamma_5) W^T \mathbf{\xi} |B \rangle
\]

D. Contributions of the operators \((\overline{\chi} \Gamma_1 \hbar)(\overline{\xi} \Gamma_2 \xi)\)

In SCET, the contributions of the operators \((\overline{\chi} \Gamma_1 \hbar)(\overline{\xi} \Gamma_2 \xi)\) can be calculated through the time-ordered product with the electromagnetic current interacting with an usoft quark and a collinear quark. The electromagnetic interaction with an usoft quark and a collinear quark is given by

\[
\mathcal{L}_{\chi q}^{em} = e Q_{sp} \overline{q}_{us} A W^T \chi,
\]

where \( e Q_{sp} \) is the electric charge of the spectator quark. To be complete, we need to include a nonlocal term and the electromagnetic interaction should read

\[
\overline{q}_A q \rightarrow \overline{q}_{us} A W^T \chi + T[\mathcal{T}_g, i\mathcal{L}^{(1)}_{\xi q}],
\]

where the operators in the nonlocal term are defined as

\[
\mathcal{T}_g = \overline{\chi} \left[ A \frac{1}{n \cdot (Q + g A_{\perp})} (\mathcal{Q}_{\perp} + g A_{\perp \perp}) + (\mathcal{Q}_{\perp} + g A_{\perp \perp}) \frac{1}{n \cdot (Q + g A_{\perp})} A_{\perp \perp} \right],
\]

\[
\mathcal{L}^{(1)}_{\chi q} = \overline{q}_{us} W^T (\mathcal{Q}_{\perp} + g A_{\perp \perp}) \chi + \text{h.c.}
\]

\[
(49)
\]
FIG. 10: Time-ordered products of the four-quark operators with the electromagnetic interaction of an usoft quark and a collinear quark $\chi$. The circle represents the four-quark operators, and the square represents the electromagnetic interaction.

The operators in Eq. (49) are the same as those defined in Ref. [10] except that $n^\mu$ and $\bar{m}^\mu$ are interchanged. The matching of the operators to the full theory, and the computation of the Wilson coefficients have been extensively discussed in Ref. [10]. But the main point here is that the electromagnetic interaction at leading order in SCET has the same operator structure as given in Eq. (47), but with a different Wilson coefficient including all the effects with a nonlocal term. We will show that the contribution of the time-ordered products of the four-quark operators with the electromagnetic interaction vanishes at leading order in SCET, and we only need the form of the operator in the argument.

All the contributions from the four-quark operators through the annihilation channels and the $W$-exchange channels vanish at leading order. The proof is the following. Neglecting the color indices, the four-quark operators can be generically written as

$$Q_i = (\chi \Gamma_1 h)(\bar{\chi} \Gamma_2 W^T \xi),$$

where $\Gamma_1$, $\Gamma_2$ are Dirac matrices. The time-ordered products with the electromagnetic current, which is schematically shown in Fig. 10, are given by

$$\langle V_i \rangle = \langle V, \gamma | \int d^4 x T [Q_i(0), i L_{em}^{\chi}(x)] | B \rangle$$

$$= -\frac{e Q_{sp} \gamma_5}{8 \pi} \int d n \cdot x \int \frac{d m \cdot k}{n \cdot k} e^{i n \cdot x/2} \langle V, \gamma | [\bar{\xi} \Gamma_2 \xi \cdot \bar{q}_s m_1 \Gamma_1 h] | B \rangle = 0. \quad (51)$$

The matrix element vanishes at leading order in SCET when we take the matrix element of the collinear sector $\bar{\xi} \Gamma_2 \xi$ for $\Gamma_2 = \bar{\pi}(1 \pm \gamma_5)$, or $1 \pm \gamma_5$ with the vector meson, they vanish using the leading projection for the vector meson. Therefore at leading order in SCET, the
annihilation channels and the $W$-exchange channels do not contribute to the radiative $B$ decays, though there may be subleading contributions. For example, the isospin breaking effects in $B \to V\gamma$ decays are explained by the effects of these operators.

V. DECAY AMPLITUDES FOR $B \to V\gamma$

Combining all the contributions considered in the previous section, we can write the decay amplitudes for radiative $B$ decays. In general, the decay amplitudes can be written as

$$A(B \to V\gamma) = \frac{G_F e m_b m_B^2}{\sqrt{2}} c_V \left( 2 \epsilon_\perp^* \cdot \eta_\perp^* + i e^{\mu\nu\alpha\beta} \eta_\perp^{*\mu} \epsilon_\perp^{*\nu} n_\alpha \pi_\beta \right) \sum_{p=u,c} V_{pd}^* V_{pb}$$

$$\times \left[ C_7(m_b) A_i^0(\mu_0) \frac{m_b}{m_B} \zeta_{\perp}(E, \mu) + \frac{\alpha_s C_F}{4\pi} C_i(m_b) G_i(m_b) A_i(\mu_0) \frac{m_b}{m_B} \zeta_{\perp}(E, \mu) + \frac{4}{m_b} \int d\mu_0 T_a(\mu_0) J_a(u, r_+, \mu_0, \mu) \phi_B^+(r_+, \mu) \phi_{\perp}(u, \mu) \right],$$

where $c_V = 1$ for $V = K^*$, $\rho$, and $c_V = 1/\sqrt{2}$ for $V = \rho^0$, and $i = 1, 8$. $C_i(m_b)$ are the Wilson coefficients in the full theory, evaluated at $\mu = m_b$, and $A_i(\mu_0)$ are the Wilson coefficients in SCET$_I$, evaluated at $\mu_0 = m_b \Lambda_{QCD}$. $G_i$ are the loop corrections, which are given in Eq. (31) at order $\alpha_s$. The functions $T_a(\mu_0)$ are the products of the Wilson coefficients in the full QCD at $\mu = m_b$, and the Wilson coefficients in SCET$_I$ at $\mu = \mu_0$. The functions $J_a$ are the jet functions which can be obtained in matching SCET$_I$ and SCET$_II$.

The decay amplitudes for radiative $B$ decays at leading order in SCET and at next-to-leading order in $\alpha_s$ are written as

$$A(B \to V\gamma) = \frac{G_F e m_b m_B^2}{\sqrt{2}} c_V \sum_{p=u,c} V_{pd}^* V_{pb} \left[ \left( C_7(m_b) A_i^0(\mu_0) + \frac{\alpha_s C_F}{4\pi} (C_1 G_1 + C_8 G_8) \right) \langle V, \gamma | O_i^0(0) | B \rangle + C_7(m_b) \langle T_i^F(0) | + C_1(m_b) \langle U_1 | + C_8(m_b) \langle U_2 | \right]$$

$$= \frac{G_F e m_b m_B^2}{\sqrt{2}} c_V \sum_{p=u,c} V_{pd}^* V_{pb} \left[ C_7(m_b) \left( A_i^0(\mu_0) \frac{m_b}{m_B} \zeta_{\perp}(E) + \pi \alpha_s \frac{C_F}{N} \frac{f_B}{m_b m_B} \int d\mu_0 \phi_B^+(r_+) \int du \frac{\phi_{\perp}(u)}{\pi} \right) + C_1(m_b) \left( \frac{\alpha_s C_F}{4\pi} \frac{m_b}{m_B} \zeta_{\perp}(E) G_1(s_p) \right) + \frac{\pi \alpha_s}{6} \frac{C_F}{N} \frac{f_B}{m_b m_B} \int d\mu_0 \frac{\phi_B^+(r_+)}{r_+} \int du H(m_b u, s_i) \frac{\phi_{\perp}(u)}{\pi} \right]$$

$$+ C_8(m_b) \left( \frac{\alpha_s C_F}{4\pi} \frac{m_b}{m_B} \zeta_{\perp}(E) G_8(s_p) \right),$$
The Wilson coefficient $A_7^{(0)}$ to next-to-leading order accuracy is given by Eq. (9), and all the other Wilson coefficients are 1 at this order because the Wilson coefficients appearing in the terms other than the first term in Eq. (53) is proportional to $\alpha_s$. In fact we have to use the evolution of the Wilson coefficient $A_7^{(0)}(\mu)$ from $m_b$ to $\mu_0$ using the renormalization group equation because there is a double logarithm. And the evolution of other Wilson coefficients should be performed, which is not available yet. However, since the evolution from $\mu = m_b$ to $\mu = \mu_0$ is small, the numerical difference would be small.

In the final expression of Eq. (53), the first term proportional to $C_7$ is the contribution to the form factor $B \to V$. The nonfactorizable part $\langle T_i^{\text{NF}} \rangle$ has been absorbed in the soft form factor $\zeta_\perp$, and the hard scattering contribution can be calculated. The form factor $F_V$ for $B \to V$ as

$$\langle V(n, \eta_\perp)\gamma(\Pi, \epsilon_\perp)|O_7|B \rangle = \frac{e m_b}{8\pi^2} m_B^2 C_V F_V$$

$$\times \left(2\epsilon_\perp \cdot \eta_\perp + i\epsilon^{\mu\alpha\beta} \eta_{\perp \mu} \epsilon_{\perp \nu} \eta_{\nu \perp} \right),$$

where the $B \to V$ form factor $F_V$ is evaluated at $q^2 = 0$, and in terms of the nonperturbative function $\zeta_\perp$, it is written, at leading order in SCET, as

$$F_V = A_7^{(0)}(\mu_0) \frac{m_b}{m_B} \zeta_\perp(E, \mu) + \pi\alpha_s(\mu) \frac{C_F}{N} m_b m_B \int dr_+ \frac{\phi_B^+(r_+/r_+)}{r_+} \int du \frac{\phi_\perp(u/\mu)}{u},$$

where nonfactorizable contributions to the form factor from $T_i^{\text{NF}}$, which include the singularities in $1/u^2$ and $1/r_+^2$ are absorbed to the soft form factor $\zeta_\perp$ [8]. This result is consistent with the result of Ref. [16], which was obtained at leading order in $\Lambda_{\text{QCD}}/m_b$ in the heavy quark limit. Therefore we conclude that the leading-order result in SCET corresponds to the leading-order result in the heavy quark mass limit, as it should be.

In Ref. [16], they consider the scale dependence of the matrix element of $O_7$ by including the running of the product of the $b$ quark mass and form factor as

$$(m_b F_V)[\mu] = (m_b F_V)[m_b] \left(1 + \frac{\alpha_s}{4\pi} 8C_F \ln \frac{m_b}{\mu} \right),$$

but in our formalism, we use the running $b$ quark mass at $\mu = m_b$ and the Wilson coefficient $A_7^{(0)}$ evaluated at $\mu_0$, and the scaling of $F_V$ is manifestly expressed in Eq. (55).
VI. CONCLUSION

We can apply SCET to the radiative $B$ decays such as $\bar{B} \to K^* \gamma$ or $\bar{B} \to \rho \gamma$, and the organization of the relevant operators can be systematically achieved using the power counting in SCET. Employing the two-step matching from SCET$_{I}$ to SCET$_{II}$, off-shell modes are integrated out successively and we can obtain gauge invariant operators under collinear and soft gauge transformations in powers of $\lambda$. The scaling of $\lambda$ changes from $\sqrt{\Lambda_{\text{QCD}}/m_b}$ in SCET$_{I}$ to $\Lambda_{\text{QCD}}/m_b$ in SCET$_{II}$, and the power counting of the operators can be achieved including the time-ordered products of the operators with the Lagrangian. The matrix elements of the operators can be evaluated in SCET$_{II}$ including the effects of the evolution of the operators. We considered four-quark operators through the annihilation channels and the $W$ exchange channels in SCET, and all these effects turn out to be at subleading order by our power counting method.

All the hard scattering amplitudes are shown to be factorized and the decay amplitudes can be written as the convolution of the hard scattering amplitudes with the light-cone wave functions of the mesons. For the contribution to the form factor, we can categorize the time-ordered products of $O_{\gamma}^{(0,1a)}$ with the interaction of a collinear gluon with a collinear quark $\xi$ and a soft quark into factorizable and nonfactorizable contributions. The factorizable contributions are calculable as a convolution of the hard scattering amplitude and light-cone wave functions of mesons. In the nonfactorizable contributions to the form factor, the effects of soft gluons are not decoupled and they are purely nonperturbative effects, and they have endpoint $1/u^2, 1/r_+^2$ singularities. These nonfactorizable contributions to the form factor are absorbed into the definition of the nonperturbative form factor $\zeta_\perp$.

We have explicitly shown that the decay amplitudes are factorized to all orders in $\alpha_s$ at leading order in SCET. This is an extension of the factorization theorem proved at next-to-leading order in the heavy quark limit [16]. It is also shown that the decay amplitudes at leading order in SCET coincide with the amplitudes obtained in the heavy quark limit at leading order in $\Lambda_{\text{QCD}}/m_b$. This should be true because the two limits at leading order are the same, and this correspondence was also shown in nonleptonic decays into two light mesons [12, 13]. We have calculated the decay amplitudes at next-to-leading order. But this does not mean that the factorization theorem is valid only to this order. The operators obtained in SCET are gauge invariant by attaching the Wilson lines, and these include
the effects of gluons to all orders. If we analyze radiative $B$ decays at next-to-next-to-leading order, the only changes occur in the Wilson coefficients and we have to include more time-ordered products, the form of the operators does not change. And the factorization properties remain intact. Therefore the factorization theorem works to all orders in $\alpha_s$ at leading order in SCET, and we have evaluated the Wilson coefficients at next-to-leading order in this paper.

The contributions of the annihilation channels and the $W$-exchange channels are subleading. This point was also pointed out in the heavy quark mass limit [16]. Therefore the isospin breaking effects due to the different contents of the spectator quarks in the $B$ meson appear only at subleading order in $\Lambda_{QCD}/m_b$ in the heavy quark limit [19]. The $SU(3)$ breaking effects which arise in the difference between the decay rates $\bar{B} \to K^{*}\gamma$ and $\bar{B} \to \rho\gamma$ also appear at subleading order. It would be challenging to consider radiative $B$ decays to order $\Lambda_{QCD}$ in SCET. In order to probe subleading effects such as the isospin breaking effects and the $SU(3)$ flavor breaking effects, we have to devise a systematic method to derive gauge-invariant subleading operators and the time-ordered products at a given order. Recent discussion [21, 22] on the field redefinitions to make the gauge invariance explicit to all orders, and the effects of the quark masses would help construct the operators.

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APPENDIX: QUARK MASS EFFECTS IN RADIATIVE $B$ DECAYS

In Ref. [22], quark masses are included in the SCET Lagrangian. If the strange quark mass scales as $\Lambda_{QCD}$, the strange quark mass terms are suppressed in SCET$_{I}$, but are leading order in SCET$_{II}$. The leading-order Lagrangian is given as

$$\mathcal{L}_0 = \bar{\xi} \left[ m \cdot D + \left( \not{p}_\perp + g_{\not{A}_n} \right) W \frac{1}{m \cdot \not{p}} W^\dagger \left( \not{p}_\perp + g_{\not{A}_n} \right) \right] \frac{m}{2} \bar{\xi}, \quad (A.1)$$
FIG. 11: Feynman diagram for the time-ordered product $T_m$ from $\mathcal{L}_m$. The square represents $O_7^{(0)}$ and the cross represents $\mathcal{L}_m$. The momentum $q_2$ of the gluon is incoming to $\mathcal{L}_m$.

and the mass terms are given by

$$\mathcal{L}_m = m\bar{\Sigma}_\mu \left[ (\slashed{p} + g\slashed{A}_{\perp}) W - \frac{1}{\pi} \frac{W^\dagger}{n^2} \right] \xi - m^2 \bar{\xi} W - \frac{1}{\pi} \frac{W^\dagger}{n^2} \xi.$$  \hspace{1cm} (A.2)

We will consider only the case of the strange quark since the up and the down quark masses are very small. As discussed in Ref. [22], if the strange quark mass scales as $\Lambda_{\text{QCD}}$, $\mathcal{L}_m$ is suppressed. However, when we go down to SCET$_{\pi}$, there is an enhancement and we have to consider the operators to order $\lambda$ in SCET$_{\text{I}}$ to obtain a leading-order result for radiative $B$ decays. Therefore we include the effect of the first term in $\mathcal{L}_m$.

The additional time-ordered product due to the mass terms at leading order in SCET$_{\text{I}}$ is given by

$$T_m = \int d^4x d^4y T\left[ O_7^{(0)}(0), i\mathcal{L}_m(x), i\mathcal{L}_{\xi q}^{(1)}(y) \right].$$  \hspace{1cm} (A.3)

The Feynman diagram for $T_m$ is given by Fig. 10. The time-ordered product in which there is no gluon from the vertex given by $\mathcal{L}_m$ becomes zero when we match the operator to SCET$_{\pi}$ because it is proportional to $p_{\perp}^4$. The time-ordered product is written as

$$T_m = \frac{em_B^2}{4\pi^2} g_m \frac{C_F}{N} \int \frac{d^4x d^4q_1 d^4y d^4q_2}{(2\pi)^4} \frac{q_2}{q_1} e^{-iq_1 \cdot x - iq_2 \cdot (y-x)} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \frac{A(1 - \gamma_5)h_j}{q_2^2} \frac{1}{q_2^2} \gamma_\mu \bar{\xi}_i \gamma_\nu \xi_j,$$  \hspace{1cm} (A.4)

where the momenta are specified in Fig. 11. Here $i, j$ are color indices, and we omit the Wilson lines, but after the calculation, we can always make the operator gauge-invariant.

In going down to SCET$_{\pi}$, after we decouple soft gluons as was done in Eq. (35), we can evaluate the matrix element of the resultant operator. However, the basic form of the
operator to be evaluated is given by
\[
\bar{q}_{i} \gamma_{\mu} (1 - \gamma_{5}) h_{j} \cdot \bar{q}_{s,j} \gamma_{\mu} (1 - \gamma_{5}) \xi_{i} = \frac{1}{2} \bar{q}_{i} \gamma_{\mu} (1 + \gamma_{5}) \bar{h}_{j} \cdot \bar{q}_{s,j} \gamma_{\mu} (1 - \gamma_{5}) \xi_{i} \\
+ \frac{1}{2} \bar{q}_{i} \gamma_{\mu} (1 + \gamma_{5}) \bar{h}_{j} \cdot \bar{q}_{s,j} \gamma_{\mu} (1 - \gamma_{5}) \xi_{i} \\
= - \bar{q}_{i} (1 - \gamma_{5}) \xi_{i} \cdot \bar{q}_{s,j} (1 + \gamma_{5}) \bar{h}_{j} \\
+ \frac{1}{2} \bar{q}_{i} \gamma_{\mu} (1 + \gamma_{5}) \xi_{i} \cdot \bar{q}_{s,j} \gamma_{\mu} (1 - \gamma_{5}) \bar{h}_{j} ,
\]
where the Wilson lines and the momentum factors are omitted for simplicity. When we take the matrix elements of the final operators in Eq. (A.5), they vanish at leading order in SCET. The quark mass effect may appear at higher orders in \( \alpha_s \) or at subleading order, but at the order we consider, there is no quark mass effect.

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