Gauge from holography

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Abstract

In a spacetime divided into two regions $U_1$ and $U_2$ by a hypersurface $\Sigma$, a perturbation of the field in $U_1$ is coupled to perturbations in $U_2$ by means of the first-order holographic imprint that it leaves on $\Sigma$. The linearized glueing field equation constrains perturbations on the two sides of a dividing hypersurface. This linear operator may have a nontrivial null space; a non-trivial perturbation of the field leaving a holographic imprint on a dividing hypersurface which does not affect perturbations on the other side should be considered physically irrelevant.

This condition, together with a locality requirement, leads to the notion of gauge equivalence in first-order Lagrangian field theory.
Motivated by the development of blackhole thermodynamics \[3, 2\], more than two decades ago pioneers of modern physics put forward a holographic principle that sparked immense interest in the community \[8, 11\]. More recently, it was discovered that the behavior of gauge theories confined to bounded domains is holographic in a precise sense: that the presence of boundaries affects the notion of gauge leading to “would be gauge degrees of freedom” living at the boundary \[6\]. Entanglement entropy, gauge and locality turned out to be interestingly intertwined \[5, 7\]. Certainly these results provide important insights into the nature of spacetime as the carrier of the fields that exhibit such behavior. It is expected that a deeper understanding of quantum field theory, and probably quantum spacetime, will follow from further developments in this direction. The aim of this essay is to review the basic principles of classical field theory providing a different point of view of the relation between field theories with gauge freedom and holography.

Einstein’s theory of general relativity sets spacetime geometry as a dynamical field interacting with matter fields. Since causal structure follows from spacetime geometry, in the context of our argument there is no prefixed causal structure. Physical properties of the field are assumed to be spacetime localized, but measuring devices live in spacetime as well. Measurement will be understood as the interaction between the system of interest (a certain field) and the measuring device (a field or a detector modelled otherwise). Think for example of a beam interacting with a screen for a short period of time; a spacetime description of the situation takes place in a bounded portion of spacetime where part of its boundary is the world history of the screen. Measurement did not take place inside a spacetime region where the field lives but at a boundary where it interacts with another system. A different type of measurement is the one taking place in a bubble chamber where the field of interest interacts with a probe field within the spacetime domain of interest; we will ignore this type of measurement at first and come back to it at the end of the essay.

Our argument is in the context of first-order formulations of Lagrangian field theory (following the notation of \[10\]). The variation of the action \(dS_U[\delta \phi^a] = \int_U E_a \delta \phi^a + \int_{\partial U} \theta^a \delta \phi^a n_a\) together with Hamilton’s principle give us field equations and a boundary term leading to a presymplectic structure whose current is \(\omega^a_{ab} = (d \theta)^a_{ab}\).

Jordan’s curve theorem states that a closed curve in the plane splits it into two disconnected components enclosing some points inside a bounded region that cannot be reached from far away without crossing the curve \[9\]. Consider a solution of the field equations \(\phi\) and a hypersurface \(\Sigma\) separating spacetime into two connected components \(U_1\) and \(U_2\) that intersect along \(\Sigma\). Clearly, if we restrict the solution to one of the components we get a solution in a restricted domain \(\phi_i = \phi|_{U_i}\). Consider a different solution \(\phi_1'\) over subdomain \(U_1\) such that its restriction to \(\Sigma\) coincides with \(\phi_1\) up to its first-order partial derivatives. It is fact that we could cut and paste to replace the portion of solution over \(U_1\) with \(\phi_1'\) and generate a different solution \(\phi' = \phi_1' \#_\Sigma \phi_2\). As far as the field over \(U_2\) is concerned, the cut and paste operation
is not physically relevant.

Let us justify the cut and paste operation just described. Consider two domains intersecting at a hypersurface $\Sigma$ in such a way that they form a composite domain $U = U_1 \cup U_2$. If the field is considered as differentiable at each subdomain but only continuous over $\Sigma$, the variation of the action for the composite domain $dS_U = dS_{U_1} + dS_{U_2}$ includes a term $\int_{\Sigma} \theta^a_\mu (\delta \phi_1 - \delta \phi_2) n_\mu$. A physical field must satisfy the field equations over each subdomain $U_i$ and also must be an extremum of the given integral when the value of the field at $\partial \Sigma$ is kept fixed. The resulting glueing field equation (a part from an integration by parts which would not be present if $\Sigma$ has no boundary) is a momentum matching condition at $\Sigma$. The momentum crossing $\Sigma$ coming from $U_1$ is calculated from $\theta^a_\mu|_\Sigma$, the boundary term of $dS_{U_1}$, and since $\phi_1$ and $\phi'_1$ coincide up to first-order at $\Sigma$, the glueing field equation sees no difference between those two fields.

Because of our interest in the type of measurements described above, we are particularly interested in the propagation of variations of the field through hypersurfaces. Consider a solution of the field over $U = U_1 \cup U_2$ that is perturbed on the side of $U_1$. The perturbation, which may be encoded in a vector field in the space of first-order data $\delta \phi^a_1$, modifies the momentum at $\Sigma$ on the $U_1$ side. Since a change in the momentum is recorded by its differential, the change of momentum turns out to be $\omega^a_{ab} \delta \phi^a_1|_\Sigma$. Thus, the linearized glueing field equation for the perturbation $\delta \phi^a = \delta \phi^a_1 + \#_\Sigma \delta \phi^a_2$ reveals itself as $\omega^a_{ab} \delta \phi^a_1|_\Sigma = -\omega^a_{ab} \delta \phi^a_2|_\Sigma$ (a part from integration by parts). The linearized glueing field equation is a linear operator which may have a nontrivial null space; if this is the case, a part from the required continuity at the hypersurface, all the information regarding the partial derivatives of the perturbation in the direction transversal to the hypersurface is lost and is not transmitted to $U_2$. A perturbation $\delta \phi^a$ such that

$$\omega^a_{ab} \delta \phi^a$$

is a pure divergence hits any hypersurface $\Sigma$ leaving a holographic imprint that has no impact on the other side. Perturbations of this class should be regarded as physically equivalent to the zero perturbation. Elevating this to a definition is described in the title of this essay with the phrase gauge from holography.

Locality is the only other ingredient needed in the definition of gauge equivalence. In the first-order Lagrangian formalism, there are three forms $F$ leading to observables through integration $f_{\Sigma}[\phi] = \int_{\Sigma} F(\phi^a, \phi'^a_\mu)$, also the presymplectic current acting on (gauge invariant) variations $\omega^a_{ab} \delta \phi^a_1$ leading to $\omega_{\Sigma}[\delta \phi^a, \delta \phi'^b]$ is of interest. The locality condition amounts to requiring that for any hypersurface with $\partial \Sigma \subset \partial U$ integrals of the type $f_{\Sigma}[\phi]$ and $\omega_{\Sigma}[\delta \phi^a, \delta \phi'^b]$ can be evaluated and are gauge invariant.

**Definition** (Gauge vector fields). A solution of the linearized field equation $\delta \phi^a$ is declared to be a gauge vector field if and only if
(i) $\delta \phi^a$ is in the null space of the linearized glueing field equation, and
(ii) $\delta \phi^a |_{\partial U} = 0$.

This is the definition of gauge vector fields in first-order Lagrangian field theory. Condition (i) is derived following a different route in [12], and the origin of condition (ii), relevant in the presence of boundaries, is a contribution of [6]. It can be verified that this definition of gauge vector fields leads to a Lie subalgebra of vector fields and to a notion of gauge equivalence among the fields; see [4] for a proof.

In domains of the type $U = \Sigma \times [0, 1]$ which are endowed with a foliation $\Sigma_t$, condition (ii) could be replaced by $\delta \phi^a |_{\partial \Sigma \times [0, 1]} = 0$. In this way all leaves $\Sigma_t$, including those corresponding to initial and final conditions, would be analogous. This way of working introduces an asymmetry regarding glueing domains in the “time” direction and in the other direction of the product. Alternatively, we use condition (ii) while restricting to leaves in the foliation with $t \in (1, 0)$. In a spacelike region $R_t = U \cap \Sigma_t$ the amount information stored in the first-order field restricted to $R_t$ is not necessarily bounded by the area of a surface $S_t$ enclosing $R_t$. However, information is filtered out by (the world history of) surface $S_t$ and the only information relevant for physics outside $R_t$ turns out to be bounded by the area of $S_t$ [?]. Moreover, everywhere in spacetime gauge invariant information is declared to be that information that is not filtered out by dividing hypersurfaces.

Isolated gravitational systems may be modelled over a spacetime domain of the type $U = \Sigma \times [0, 1]$ with the boundary $\partial \Sigma \times [0, 1]$ representing a world tube at spatial infinity (and possibly an inner boundary modelling a horizon). Our framework applies to Palatini’s formulation of General Relativity, where it is known that all generators of diffeomorphisms induce perturbations such that $\omega^{ab}_{\alpha} \delta \phi^a$ is a pure divergence [1], which implies that $\delta \phi^a$ satisfies condition (i) of the definition of gauge vector fields. However, regarding variations that do not vanish at infinity as gauge is inappropriate since they modify the reference frame needed to define energy, momentum and angular momentum. Thus, preserving a reference frame at the boundary that may be used as a reference for measurements is another motivation for condition (ii) of the definition of gauge.

Finally, we comment on the type of measurements of the bubble chamber. If the field of interest $\phi$ couples with the measuring field $\psi$ in such a way that gauge equivalent fields $\phi_A \sim \phi_B$ are resolved, it should be claimed that the measuring field significantly disturbs the system, and the actual field being measured is the one corresponding to the composite system.

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