Abstract. After an introduction in which we review the fundamental difficulty in constructing lattice chiral gauge theories, we summarize the analytic and numerical evidence that abelian lattice chiral gauge theories can be nonperturbatively constructed through the gauge-fixing approach. In addition, we indicate how we believe that the method may be extended to nonabelian chiral gauge theories.

1. Introduction

Consider a collection of left-handed fermion fields transforming in a representation of some symmetry group. A gauge theory containing these fermions can be regulated by putting it on a (euclidean) space-time lattice. We may then ask about the anomaly structure of the theory by keeping the gauge fields external (and smooth), and it is clear that each fermion field will have to contribute its share to the expected chiral anomaly. This can happen in two ways: either the regulated theory is exactly invariant under the full symmetry group, and each fermion comes with its species doublers [1, 2], or the symmetry is somehow explicitly broken by the regulator (in this case the lattice), making it possible for the fermion field in each irre-
ducible representation to produce the correct contribution to the anomaly in the continuum limit (i.e., for smooth external gauge fields).

The Nielsen-Ninomiya theorem [2] tells us that fermion representations with doublers contain equally many left- and right-handed fermions transforming the same way under the symmetry group. This implies that if we make the gauge fields dynamical, a vector-like gauge theory will emerge. Hence, if we wish to construct a genuinely chiral theory on the lattice, we will have to resort to the second option: an explicit breaking, by the lattice regulator, of the symmetry group. The most well-known example of this is the formulation of lattice QCD with Wilson fermions [3]. In this method, a momentum-dependent mass term of the form

\[ -\frac{r}{2} \sum_{\mu} \left( \bar{\psi}_x \psi_{x+\mu} + \bar{\psi}_{x+\mu} \psi_x - 2 \bar{\psi}_x \psi_x \right) \]  

is added to the action, which removes the doublers by giving them a mass of the order of the cutoff $1/a$ (where $a$ is the lattice spacing, which we set equal to one in this talk).

For theories in which only vector-like symmetries are gauged, like QCD, this works fine. The Wilson mass term can be made gauge invariant (by inserting the SU(3)-color link variables on each hopping term). The global chiral symmetry is broken, but can be restored in the continuum limit by a subtraction of the quark mass.

However, the situation changes dramatically when we wish to gauge a chiral symmetry. We can still remove the doublers with a Wilson mass term as in Eq. (1), by introducing a right-handed “spectator” fermion for each left-handed fermion. (Other possibilities exist, but, because of the anomaly argument, the conclusions are similar in all cases [4].) Obviously, since we are now interested in gauging a chiral symmetry, the Wilson mass term does not respect gauge invariance. This means that, on the lattice, the longitudinal gauge field (which represents the gauge degrees of freedom) couples to the fermions. If we only have a Maxwell-like term ($\sim \text{tr} F_{\mu\nu}^2$) controlling the dynamics of the gauge field, the longitudinal modes are not suppressed at all, and they typically destroy the chiral nature of the fermion spectrum (see Refs. [5, 6] for reviews and references). Note that this phenomenon is nonperturbative in nature. The problem is invisible for smooth gauge fields, but the point is that longitudinal gauge fields do not have to be smooth, even for small gauge coupling, if all gauge fields on an orbit have equal weight in the partition function.

This is precisely where gauge fixing comes in. A renormalizable choice of gauge adds a term to the gauge-field action which controls the longitudinal part of the gauge field. In this talk, we will consider the Lorentz gauge, with gauge-fixing lagrangian $(1/2\xi)\text{tr} (\partial_{\mu} A_{\mu})^2$. The longitudinal part of the
gauge field \((\partial_\mu A_\mu)\) has now acquired the same “status” as the transverse part \((F_{\mu\nu})\).

Before we start the discussion of gauge fixing, it is instructive to see in more detail what goes wrong without it, using our example of a Wilson mass term. If we perform a gauge transformation on the left-handed fermion field, \(\psi_L \rightarrow \phi^\dagger \psi_L\), with \(\phi\) a group-valued scalar field, the Wilson mass term transforms into

\[
-\frac{r}{2} \sum_\mu \left( \psi_{Rx}^\dagger \phi_{x+\mu}^R \psi_{Lx+\mu} + \cdots \right).
\]

The parameter \(r\) is promoted to a Yukawa-like coupling, and the lattice regulator (which led to the introduction of the Wilson mass term in the first place) leads to couplings between the fermions and the longitudinal degrees of freedom through the scalar field \(\phi\). Note that the lattice theory is invariant under the symmetry \(\psi_{L,R} \rightarrow h_{L,R} \psi_{L,R}, \phi \rightarrow h_L \phi h_R^\dagger\) etc., with \(h_R\) global and \(h_L\) local \((h\)-symmetry). The \(h_L\)-symmetry is, however, not the same as that of the gauge theory we wish to construct, since \(\phi\) is supposed to decouple.

We can now explore the phase diagram \((i.e.\ all\ values\ of\ r)\) in order to see whether we might decouple these longitudinal modes, while retaining the fermion spectrum that we put in. (In a confining theory, of course the “chiral quarks” do not appear in the spectrum, but we can imagine first considering the theory with only the gauge degrees \(\phi\) dynamical, with external smooth transverse gauge fields. This is the so-called “reduced model” of Refs. \([7, 8, 9]\).)

It turns out that three things can happen (see Ref. \([6]\) and refs. therein). The \(h\)-symmetry can be spontaneously broken, and the doublers will be removed if \(\langle \phi \rangle \sim 1/a\). However, in that case also the gauge-field mass will be of order \(1/a\), which is not what we want. It follows that we would like the \(h\)-symmetry to be unbroken.

For small \(r\), we may read off the fermion spectrum by replacing \(\phi \rightarrow \langle \phi \rangle\). If \(\langle \phi \rangle = 0\), we find that the Wilson-Yukawa term does not lead to any fermion masses, and the doublers are degenerate with the massless physical fermion! (In the broken phase, this degeneracy is partially removed, but, as we already noticed, the doubler masses will be set by the scale of the gauge-field mass. An elegant way of doing this was reviewed in Ref. \([10]\).) There also exists a phase with unbroken \(h\)-symmetry at large \(r\), and it turns out that in that phase the only massless left-handed fermion is described by the composite field \(\phi^\dagger \psi_L\). This fermion, however, does not couple to the gauge field, since its gauge charge is “screened” by the longitudinal field \(\phi\) \([12, 13]\). Both this composite left-handed and the spectator right-handed fermion do not couple to the gauge field (in four dimensions; for two dimensions, see Ref. \([11]\)).
What we will show in the rest of this talk is that this conclusion, that there is no place in the phase diagram where a chiral gauge theory can be defined, changes completely when a gauge-fixing term is added, therewith enlarging the parameter space of the phase diagram. We mention here that another approach exists which aims to “tame the rough gauge fields” (interpolation, or two-cutoff approach), see Ref. [5] for a review and references.

Before we end this introduction, we would like to rephrase our conclusions thus far in a somewhat different way. Imagine that we have defined the fermionic partition function for an external lattice gauge field (not necessarily smooth!) in a certain attempt to construct a chiral gauge theory. This then yields an effective action $S_{\text{effective}}(A)$, where $A$ is the external gauge field, and we have, under a gauge transformation,

$$\delta S_{\text{effective}}(A) = \text{anomaly}(A) + \text{lattice artifacts}(A).$$

The anomaly part can be identified by choosing the external gauge field to be small and smooth. The lattice-artifact terms are generically not small. They cannot be, if the dynamics of the longitudinal modes is to change the theory into a vector-like theory (as described above), in which the gauge anomaly vanishes. This points to another way of avoiding the conclusions sketched above: by finding a fermion partition function for which the lattice-artifact term vanishes, if the fermion representation is anomaly-free in the continuum, for \textit{all} lattice gauge fields. A recent proposal along these lines, starting from a Dirac operator satisfying the Ginsparg–Wilson relation, is reviewed in Ref. [14].

2. Gauge fixing – the abelian case

The central idea of the gauge-fixing approach is to make gauge fixing part of the definition of the theory [15]. This contrasts with the case of lattice QCD, where, because of the compact nature of the lattice gauge fields, gauge fixing is not needed. The theory is defined by the action

$$S = S_{\text{gauge}} + S_{\text{fermion}} + S_{\text{g.f.}} + S_{\text{ghosts}} + S_{\text{c.t.}}.$$  \hspace{1cm} (4)

For $S_{\text{gauge}}$, we choose the standard plaquette term. For $S_{\text{fermion}}$ we use Wilson fermions, with only the left-handed fermions coupled to the gauge fields. We will choose the Wilson mass term as in Eq. (1), without any gauge fields in the hopping terms. Other choices are possible, but in a chiral gauge theory, all break the gauge symmetry. Our choice has the (technical) advantage of making the action invariant under shift symmetry, $\psi_R \rightarrow \psi_R + \epsilon_R$, with $\epsilon_R$ a constant, right-handed Grassmann spinor [16]. For $S_{\text{g.f.}}$, we will choose a lattice discretization of $\int d^4x (1/(2\xi))(\partial \mu A_\mu)^2$, to be discussed below. In
this section, we will be concerned only with abelian theories, and $S_{\text{ghosts}}$ can be omitted [17].

Since the lattice regulator breaks gauge invariance explicitly, counterterms are needed, and they are added through $S_{\text{c.t.}}$. These counterterms include one dimension 2 operator (the gauge-field mass counterterm), no dimension 3 operators (because of the shift symmetry), and a host of dimension-4 counterterms (see Refs. [15, 18] for a detailed discussion). Tuning these counterterms to the appropriate values (by requiring the Slavnov–Taylor identities of the continuum target theory to be satisfied) should then bring us to the critical point(s) in the phase diagram at which a chiral gauge theory can be defined. Because of the choice of a renormalizable gauge, it is clear that this can be done in perturbation theory (if the theory is anomaly free). The observation of Ref. [15] is that also nonperturbatively gauge-fixing will be needed in order to make the program described above work.

At the nonperturbative level, the following important questions arise [19]. First, what should we choose as the lattice discretization of $S_{\text{g.f.}}$? More precisely, given a certain choice, what does the phase diagram look like, and for which choices do we find a phase diagram with the desired critical behavior? Second, if we find that a suitable discretization exists, so that the fermion content is indeed chiral, how does this precisely happen? Note that, without gauge fixing, the action above is essentially just the Smit–Swift model [20], which, for the reasons summarized in the introduction, does not work. In this section, we will answer these two questions. We relegate the discussion of a third important question, namely the extension to nonabelian theories, to section 3.

Gauge fixing on the lattice

It was argued in Ref. [19] that the lattice gauge-fixing term

$$S_{\text{g.f.,naive}} = \frac{1}{2\xi g^2} \sum_x \left( \sum_{\mu} (\text{Im } U_{x,\mu} - \text{Im } U_{x-\mu,\mu}) \right)^2$$

is not the right choice, even though, expanding the link variables $U_{x,\mu} = \exp(igA_{x,\mu})$, it looks like the most straightforward discretization of the continuum form. This is because this choice admits an infinite set of lattice Gribov copies (which have no continuum counterpart) of the perturbative vacuum $U_{x,\mu} = 1$. This is dangerous, since our intuition that this approach should lead us to the lattice construction of chiral gauge theories is based on the fact that our regulator does work in perturbation theory. Therefore, we insist that lattice perturbation theory should be a reliable approximation of our lattice theory at weak coupling. In fact, we showed, through a combination of numerical and mean field techniques, that the naive choice
The gauge-fixing action of Eq. (5) does not lead to a phase diagram with the desired properties [18].

The vacuum degeneracy of $S_{g.f.,naive}$ can be lifted by adding irrelevant terms to it [19, 21], so that

$$S_{g.f.} = S_{g.f.,naive} + \tilde{r}S_{irrelevant},$$

where $\tilde{r}$ is a parameter very similar to the Wilson parameter $r$ multiplying the Wilson mass term. While we will not give any explicit form of $S_{irrelevant}$ here, it was shown [21] that $S_{irrelevant}$ can be chosen such that

$$S_{g.f.}(U) \geq 0 \quad \text{and} \quad S_{g.f.}(U) = 0 \iff U_{x,\mu} = 1.$$  

This means that $U_{x,\mu} = 1$ is the unique perturbative vacuum. Also, obviously, $S_{g.f.}(U) \to \int d^4x (1/2\xi)(\partial_\mu A_\mu)^2$ in the classical continuum limit. Our choice does not respect BRST symmetry, which will necessitate adjustment of the counterterms [17].

For small gauge coupling $g$, the classical potential should give us an idea of what the phase diagram looks like. Without fermions (which contribute to the gauge-field effective potential only at higher orders in lattice perturbation theory), including (only) a mass counterterm $-\kappa \sum_\mu (U_{x,\mu} + U_{x,\mu}^\dagger)$, and expanding $U_{x,\mu} = \exp(igA_{x,\mu})$, we have, for our choice of $S_{irrelevant}$,

$$V_{\text{classical}}(A) = \frac{\tilde{r}g^4}{4\xi} \left( \sum_\mu A^2_\mu \sum_\nu A^4_\nu + \ldots \right) + \kappa g^2 \left( \sum_\mu A^2_\mu + \ldots \right)$$

for a constant field. The dots indicate higher-order terms in $g^2$. While the precise form of the term proportional to $\tilde{r}$ is not important, it is clearly irrelevant (order $A^6$) and positive (i.e. it stabilizes the perturbative vacuum).

We can now distinguish two different phases, depending on the value of $\kappa$. For $\kappa > 0$, $A_\mu = 0$, and the gauge field has a positive mass $\sqrt{2\kappa g^2}$. For $\kappa < 0$, the gauge field acquires an expectation value $A_\mu = \pm \left( \frac{\kappa}{3\tilde{r}g^2} \right)^{1/4}$, for all $\mu$, and we encounter a new phase, in which the (hypercubic) rotational symmetry is spontaneously broken! These two phases are separated by a continuous phase transition (classically at $\kappa = \kappa_c = 0$), at which the gauge-field mass vanishes. In other words, we will take our continuum limit by tuning $\kappa \to \kappa_c$. (For a discussion including the other, dimension 4, counterterms, see Ref. [21].)

A detailed analysis of the phase diagram for the abelian theory without fermions was given in Ref. [18]. A complete description of the phase diagram in the four-parameter space spanned by the couplings $g$, $\xi$, $\tilde{r}$ and $\kappa$ can be found there, as well as a discussion of the other counterterms and a study...
of gauge-field propagators. In the region of interest (basically small $g$ and $\tilde{r} \approx 1$) good agreement was found between a high-statistics numerical study and lattice perturbation theory. In particular, the picture that emerges from the classical potential as described above was shown to be correct, as long as we choose $\tilde{r}$ away from zero, and $g^2$, $\xi g^2$ sufficiently small. As it should, the theory (without fermions) at the critical point describes free, relativistic photons.

**Fermions**

We now come to the behavior of the fermions in this gauge-fixed lattice theory. Employing a continuum-like notation for simplicity, our lattice lagrangian, including fermions, reads

\[ L = \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\xi} (\partial_\mu A_\mu)^2 + \tilde{r} L_{\text{irrelevant}}(A) \]

\[ + \bar{\psi} (D_\mu A_\mu) P_L + \tilde{\partial} P_R \psi - \frac{r}{2} \bar{\psi} \Box \psi \]

\[ + \kappa g^2 A_\mu^2 + \text{other counterterms} . \]  

In order to investigate the interaction between fermions and longitudinal modes, we can make the latter explicit by a gauge transformation

\[ A_\mu \rightarrow \phi_\mu A_\mu + \frac{i}{g} \phi_\mu \partial_\mu \phi \equiv -\frac{i}{g} \phi_\mu D_\mu \phi , \]

\[ \psi_L \rightarrow \phi L \psi_L . \]

This yields the lagrangian in the “Higgs” or “Stückelberg” picture,

\[ L = \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\xi g^2} \left( \partial_\mu (\phi^\dagger (-i\partial_\mu + gA_\mu) \phi) \right)^2 + \tilde{r} L_{\text{irrelevant}}(A, \phi) \]

\[ + \bar{\psi} (D_\mu A_\mu) P_L + \tilde{\partial} P_R \psi - \frac{r}{2} \left( \bar{\psi}_R \Box (\phi^\dagger \psi_L) + \bar{\psi}_L \phi \Box \psi_R \right) \]

\[ + \kappa (D_\mu (A) \phi)^\dagger (D_\mu (A) \phi) + \text{other counterterms} . \]

This action is invariant under the $h$-symmetry mentioned in the introduction.

In order to find out whether the longitudinal modes, which are represented by the field $\phi$ in the Higgs-picture lagrangian, change the fermion spectrum, we may now simplify the theory, by first considering the “reduced” model, in which we set $A_\mu = 0$ in Eq. (11). Expanding $\phi = \exp(i\sqrt{\xi} g \theta)$, which is appropriate for small gauge coupling because of the $1/g^2$ in front of the gauge-fixing term, gives the reduced-model lagrangian

\[ L_{\text{red}} = \frac{1}{2} (\Box \theta)^2 + \kappa \xi g^2 (\partial_\mu \theta)^2 + \theta \text{ self-interactions} \]
\[ + \overline{\psi} \partial \psi - \frac{r}{2} \overline{\psi} \Box \psi + i \sqrt{\xi g} (\overline{\psi}_L \theta \Box \psi_R - \overline{\psi}_R \Box (\theta \psi_L)) + O(g^2). \]

This lagrangian teaches us the following. First, \( \theta \) is a real scalar field with dimension 0, and inverse propagator \( p^2 (p^2 + 2 \kappa \xi g^2) \). Near the critical point (which is at \( \kappa = 0 \) to lowest order), this behaves like \( p^4 \). This actually implies \([7, 22]\) that

\[ \langle \phi \rangle \propto (\kappa - \kappa_c)^{\xi g^2/(32\pi^2)} \rightarrow 0 \]

for \( \kappa \rightarrow \kappa_c \). (This behavior is very similar to that of a normal scalar field in two dimensions in the massless limit.) This means that the \( h \)-symmetry is restored at the critical point.

Second, the fermion-scalar interactions in Eq. (12) are dimension 5, and therefore irrelevant. This (heuristically) implies that \( \theta \) decouples from the fermions near the critical point, which is saying that the longitudinal, or gauge degrees of freedom decouple. The doublers are removed by the Wilson mass term, which is present in Eq. (12). The conclusion is that a continuum limit exists (at the critical point of the reduced model) with free charged left-handed fermions (\textit{i.e.} fermions which couple to the transverse gauge field in the full theory) and free neutral right-handed fermions. In other words, the fermion spectrum is chiral. It is clear from the discussion here that gauge-fixing plays a crucial role: without it, the higher-derivative kinetic term for \( \theta \) would not be present. It is the infrared behavior of this scalar field that causes this novel type of critical behavior to occur. Note, finally, that the restoration of \( h \)-symmetry at the critical point and the decoupling of \( \theta \) and fermion fields together imply that the target gauge group is unbroken in the resulting continuum theory.

Of course, the description given here is quick and dirty. The unusual infrared properties of this theory were investigated perturbatively in much more detail in Ref. [22]. Fermion propagators were computed numerically in Ref. [8], and the agreement with perturbation theory was shown to be very good. (The numerical computations were done in the quenched approximation. However, the effects of quenching occur only at higher orders in perturbation theory, so the good agreement between numerical and perturbative results indicates that this is not a serious problem.) All these studies confirm the results in this talk, and we refer to them for more details. A somewhat more extensive, but still pedagogical account is given in Refs. [9, 23].

3. Nonabelian speculations

The fermionic results described in the previous section actually carry over to the nonabelian case. However, in the nonabelian case, we know that,
in perturbation theory already, the ghosts do not decouple. Omitting the
ghost determinant, or something equivalent, leads to the wrong Boltzmann
weight for the gauge fields, and the resulting theory will not be unitary.

However, nonperturbatively, ghosts cannot be added with impunity. While
they lead to the restoration of BRST symmetry in perturbation the-
ory, outside perturbation theory the existence of Gribov copies [24] most
likely will cause the theory to be ill-defined. In fact, a theorem was proven
some time ago [25], stating that, for a BRST-invariant lattice gauge theory,
the (unnormalized) expectation value of any BRST-invariant operator van-
ishes identically. The heuristic explanation is that Gribov copies contribute
to the partition function with opposite sign for the Fadeev–Popov deter-
minant, canceling their contributions in pairwise fashion. Even in a lattice
theory in which BRST symmetry is not exact on the lattice (as is the case
with our approach), one may still worry that a similar phenomenon would
occur in the continuum limit.

This particular problem would be solved if we would employ the absolu-
tate value of the Fadeev–Popov determinant. However, it is not presently
known whether in that case unitarity of the theory can be maintained at
the nonperturbative level.

Here, we would like to discuss a different, “ghost-free” approach to
gauge fixing, originally proposed in Refs. [26, 27]. First consider an ex-
actly gauge-invariant lattice gauge theory, with the expectation value of a
gauge-invariant operator schematically denoted by

\[ \int \mathcal{D}A \, O_{\text{inv}}(A) \exp(-S_{\text{inv}}(A)) . \]  \hspace{2cm} (14)

Insert unity into this expectation value in the form

\[ 1 = \frac{\int \mathcal{D}g \, \exp(-S_{\text{ni}}(A^g))}{\int \mathcal{D}h \, \exp(-S_{\text{ni}}(A^h))} \]  \hspace{2cm} (15)

where the integrals are over the (compact) gauge orbit of the field \( A \), and
\( A^g,h \) are gauge transformations of \( A \). \( S_{\text{ni}}(A) \) is a non-invariant functional
of the gauge field. Changing variables in the numerator, we find that the
expectation value in Eq. (14) is equal to

\[ \int \mathcal{D}A \, O_{\text{inv}}(A) \exp(-S_{\text{inv}}(A) - S_{\text{ni}}(A)) \frac{\int \mathcal{D}h \, \exp(-S_{\text{ni}}(A^h))}{\int \mathcal{D}h \, \exp(-S_{\text{ni}}(A^h))} , \]  \hspace{2cm} (16)

where we used \( \int \mathcal{D}g = 1 \). Note that in this construction

\[ \exp(-S_{\text{eff}}(A)) = \left[ \int \mathcal{D}h \, \exp(-S_{\text{ni}}(A^h)) \right]^{-1} \]  \hspace{2cm} (17)
replaces the Fadeev–Popov determinant.

This construction has two important properties. First, it is rigorously correct, in that it does not change the value of gauge-invariant quantities. Second, the gauge-field measure is positive definite, which is very important from a practical point of view.

Obviously, this gauge-fixing procedure does not apply directly to the case at hand, because the lattice action \( S(A, \psi) \) from which we start is not gauge invariant. In order to adapt this idea to our goal, we need to make the basic assumption that, when gauge invariance is not preserved by the regulator, this ghost-free approach can be made to work by adding counterterms, the form of which can be determined in perturbation theory. While this is an assumption, it is not unreasonable. First, this is exactly the way things work in the usual perturbative formulation with ghosts, if a non-BRST-invariant regulator is used. Second, and more important, the existence of continuum Gribov copies in the nonabelian case is a long-distance property of the theory, while the need to add counterterms in order to restore gauge invariance comes from the ultraviolet behavior of the regulator. Our assumption amounts to the expectation that the form of the counterterms is not affected by long-distance continuum Gribov copies.

However, this does not yet mean that we have solved the problem. Let us consider conventional perturbation theory, with ghosts. The action is, schematically,

\[
S = S_{\text{classical}}(A, \psi) + S_{\text{gaugefix}}(A) + \int d^4x \, \bar{\psi} \Omega(A) \psi + S_{\text{c.t.}}(A, \psi, c, \bar{c}),
\]  

(18)

where \( \Omega(A) \) is the Fadeev–Popov operator. A very important property of the action (18) is that it is local. The role of ghosts is crucial in this respect: they make it possible to write the Fadeev-Popov determinant as a local term in the action. The price we have to pay is that the counterterms can also depend on the ghost fields (and, in fact they do [28]).

In our ghost-free approach, the corresponding action is

\[
S = S_{\text{classical}}(A, \psi) + S_{\text{gaugefix}}(A) + S_{\text{eff}}(A) + S_{\text{c.t.}}(A, \psi),
\]

(19)

where \( S_{\text{eff}}(A) \) (cf. Eq. (17)) replaces the ghost action, as we saw above. However, \( S_{\text{eff}}(A) \) is nonlocal, and this will result in a nonlocal counterterm action! As soon as the theory becomes nonlocal, we “loose control,” and it becomes difficult, if not impossible, to even classify the counterterms. The problem is that \( S_{\text{eff}}(A) \), while formally the same as the (logarithm of the) ghost determinant, cannot be expressed in terms of a local action.

This, however, does not mean that perturbation theory cannot be formulated in a local way. We can make some progress by examining what
happens in more detail. For definiteness, choose a gauge-fixing lagrangian as in Ref. [26],
\[ \mathcal{L}_{\text{gaugefix}}(A) = M^2 \text{tr} \, A^2 , \]
with \( M \sim 1/a \), and \( \text{tr} \, T^a T^b = \frac{1}{2} \delta^{ab} \), with \( T^a \) the hermitian generators of the gauge group. Since \( M \) is not a physical mass, it is natural to choose it of order the cutoff. Expand \( h \) in Eq. (17) as
\[ h = \exp \left( i g \theta / M \right), \]
then
\[ \mathcal{L}_{\text{gaugefix}}(A_h) = \text{tr} \left[ M^2 A^2 - 2M \theta \partial_\mu A_\mu + (\partial \theta)^2 - ig A_\mu [\theta, \partial_\mu \theta] \right] + O \left( \frac{1}{M} \right). \]
(21)

Changing variables,
\[ A_\mu = A_\mu^T + \frac{1}{M} \partial_\mu \eta , \]
\[ \theta \to \theta - \eta , \]
with \( \partial_\mu A_\mu^T = 0 \), we have
\[ \mathcal{L} = \frac{1}{2} \text{tr} \left( F_{\mu\nu}^T \right)^2 + M^2 \text{tr} \, A^2 + \mathcal{L}_{\text{fermion}}(A^T, \psi) + \mathcal{L}_{\text{c.t.}}(A^T, \eta, \psi) + O \left( \frac{1}{M} \right) \]
(23)
in the numerator of Eq. (16), and
\[ \mathcal{L} = \text{tr} \left[ M^2 A^2 - (\partial_\mu \eta)^2 + (\partial_\mu \theta)^2 - ig A_\mu^T [\theta - \eta, \partial_\mu (\theta - \eta)] \right] + \mathcal{L}_{\text{c.t.}}(A^T, \eta, \theta) + O \left( \frac{1}{M} \right) \]
(24)
in the denominator. A few important facts: the \( M^2 A^2 \) term cancels between numerator and denominator, showing that this is not a (transverse) gauge-field mass term, the transverse gauge field is massless; using shift symmetry, one can show that there are no counterterms coupling fermions to the fields \( \theta \) and \( \eta \) (which both have the canonical dimension of a boson field, namely 1). Counterterms involving \( \theta \) are, by construction, part of the denominator; we do not have to worry about the detailed form of \( O(1/M) \) terms, since they are irrelevant \( (1/M \sim a) \).

We now make an important observation. In perturbation theory, the theory described by this (complicated) partition function, is equivalent to the simpler theory defined by the Feynman rules following from
\[ \mathcal{L}_{\text{pt}} = \frac{1}{2} \text{tr} \left( F_{\mu\nu}^T \right)^2 + \mathcal{L}_{\text{fermion}}(A^T, \psi) + \text{tr} \left[ (\partial_\mu \eta)^2 + (\partial_\mu \theta)^2 - ig A_\mu [\theta - \eta, \partial_\mu (\theta - \eta)] \right] + \mathcal{L}_{\text{c.t.}} + O \left( \frac{1}{M} \right) , \]
(25)
with the additional rule that each connected \( \theta \)-subdiagram gets an extra minus sign. This is actually very similar to what happens in the standard perturbative analysis with ghosts. The (technical) difference is that here we cannot obtain this extra minus sign from a change in statistics of the \( \theta \) field, since not all connected \( \theta \)-subdiagrams are single \( \theta \) loops. (\( S_{\text{eff}} \) is not the logarithm of a determinant.)

One more ingredient is needed. The lagrangians of Eqs. (23,24) are nonlocal because of the presence of \( A^T \), and the decomposition (22) is not straightforward on the lattice. We therefore now relax the constraint that \( A \) has to be transverse, and, instead, add a new term \((1/\xi)\text{tr} (\partial_\mu A_\mu)^2 \) to the lagrangian, accompanied by the rule that all correlation functions have to be calculated in the \( \xi \to 0 \) limit (which sets the longitudinal part of \( A \) equal to zero), before the continuum limit is taken. Our theory is now defined by the partition function (16) (with a functional integral over \( \eta \) added), with

\[
\mathcal{L}_{\text{num}} = \frac{1}{2} \text{tr} F_{\mu\nu}^2 + \text{tr} (\partial_\mu \eta)^2 + (1/\xi)\text{tr} (\partial_\mu A_\mu)^2 + \mathcal{L}_{\text{fermion}}(A, \psi) + \mathcal{L}_{\text{c.t.}}(A, \eta, \psi)
\]

in the numerator, and

\[
\mathcal{L}_{\text{den}} = \text{tr} \left[ (\partial_\mu \theta)^2 - igA_\mu [\theta - \eta, \partial_\mu (\theta - \eta)] \right] + \mathcal{L}_{\text{c.t.}}(A, \eta, \theta)
\]

in the denominator, with \( \mathcal{L}_{\text{den}} \) (including \( \mathcal{L}_{\text{c.t.}}(A, \eta, \theta) \)) containing only those counterterms which depend explicitly on \( \theta \), and with the limit \( \xi \to 0 \) implied. We note here that it can be proven that physical quantities do not depend on \( \xi \), and that therefore we may also define our theory keeping \( \xi \) finite. In addition, it is possible to show that, in perturbation theory, the partition function is equal to the one obtained in the standard Fadeev–Popov approach with gauge-fixing action \( \int d^4x (1/\xi)\text{tr}(\partial_\mu A_\mu)^2 \) [29].

In the abelian case the commutator term in \( \mathcal{L}_{\text{den}} \) vanishes, so that \( \theta \) decouples (just like abelian ghosts decouple in a linear gauge). The field \( \eta \) also decouples, and the theory simplifies to that considered in the previous section. (Obviously, if \( \theta \) and \( \eta \) decouple, no counterterms for these fields are required either.)

Now we can go back to the lattice, replacing \( A \to U, \theta \to h \), and using Eq. (6) in the lattice transcription of \((1/\xi)\text{tr} (\partial_\mu A_\mu)^2 \). The field \( \eta \) remains a Lie-algebra valued scalar field (it is important that Eq. (22) be a linear transformation).

Clearly, the above described procedure needs to be checked, at the very least to one loop in perturbation theory. In principle, by coupling the gauge field to a source, the Slavnov–Taylor identities of the target theory can be constructed, and they can be used to construct the counterterms. In order to carry this out efficiently, it would be nice to have a symmetry similar
in nature to BRST symmetry in the case with ghosts, in order to have an explicit handle on counterterms involving the field $\theta$.

4. Concluding remarks

In this talk, we gave an overview of recent progress with the gauge-fixing approach to the construction of lattice chiral gauge theories. The construction looks complicated, but may actually turn out to be (relatively) practical. There are quite a few counterterms [15, 18], but (in our present formulation) only one is of dimension 2, all others are of dimension 4. The dimension 2 counterterm (the gauge-field mass counterterm) will have to be tuned nonperturbatively, but this can be done. It is not unlikely that, to a given typical precision, all other counterterms can be calculated in perturbation theory to one loop (or even tree level, i.e. omitting them altogether!). Reference [18] contains more discussion of this point in the abelian case. In the nonabelian case, there is one more scalar field, $\eta$, but there are not many more counterterms involving this field, as a consequence of a shift symmetry $\eta \rightarrow \eta + \text{constant}$. Furthermore, apart from the chiral fermion determinant, the Boltzmann weight is positive.

An interesting feature is that we expect our approach to be independent of the choice of lattice fermions, and, in this sense, to be universal. This is because the original problem (lack of gauge invariance) is generic, as a consequence of the chiral anomaly. Gauge fixing plays the central role in overcoming this problem here, and not the choice of lattice fermions. For encouraging results with gauge-fixed domain-wall fermions, see Ref. [30].

While most of our previous work was limited to the case of abelian chiral gauge theories, we sketched an outline as to how the gauge-fixing approach may be extended to the nonabelian case as well, without ignoring Gribov copies.

Another relevant issue is that of a “spuriously” conserved fermion number in our approach, coming from the fact that our action is bilinear in $\psi$ and $\bar{\psi}$ [31]. The action (and also the fermion measure) are invariant under an exact global U(1) symmetry which, at first glance, seems to be in contradiction with fermion-number violation. However, Ref. [32] demonstrated, in a two-dimensional toy model, that fermion-number violation can actually occur. The central observation is that fermionic states are excitations relative to the vacuum. The global U(1) symmetry prohibits a given state to change fermion number, but nothing prevents the ground state to change when an external field is applied (see also Refs. [33, 22]). A similar phenomenon may explain how fermion-number violating processes take place in our four-dimensional dynamical theory.
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