The sub-fractional CEV model

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Abstract

The sub-fractional Brownian motion (sfBm) could be considered as the intermediate step between
the standard Brownian motion (Bm) and the fractional Brownian motion (fBm). By the way, sub-
fractional diffusion is a candidate to describe stochastic processes with long-range dependence and
non-stationarity in their increments. In this note, we use sfBm for financial modeling. In particular,
we extend the results provided by Araneda [Axel A. Araneda. The fractional and mixed-fractional
CEV model. Journal of Computational and Applied Mathematics, 363:106–123, 2020] arriving at the
option pricing under the sub-fractional CEV model.

Keywords: sub-fractional diffusion, CEV model, option pricing, sub-fractional Fokker-Planck.

1 Introduction

The sub-fractional Brownian motion, in short sfBm, is a stochastic process which emerges from the occu-
pation time fluctuations of branching particle systems [2]. It owns the main properties of the fractional
Brownian motion (fBm) as long-range dependence, self-similarity and H\ölder paths (see [2] [3] for details
and properties of sfBm). However, a key difference among them is the sfBm has non-stationary incre-
ments. Besides, the sfBm has more weakly correlated increments and their covariance decays at a higher
rate, in comparison to the fBm.

Then, diffusion processes under sfBm could be considered to model some financial time-series which
exhibits long-range dependency and non-stationarity increments [4, 5]. Some attempts has been ad-
dressed in the literature extending the Black-Scholes (B-S) model under sfBm [6–10]. However, in this
communication, we consider a sub-fractional extension for Constant Elasticity of Variance (CEV) model,
which is capable to address some shortcomings of the B-S approach as the leverage effect and the implied
volatility skew [11].

Following the procedure given by Araneda [1] to fractional case, and the Itô calculus for sfBm [12],
we derive the sub-fractional Fokker-Planck equation and the transition probability density function for
the sub-fractional CEV (sfCEV) is obtained, leading to the price formula for an European Call option in
terms of the non-central chi-squared distribution and the M-Whittaker function.

2 The model

We consider that the asset price $S$ is ruled by following stochastic differential equation:

$$dS = rSdt + \sigma S^{\frac{\alpha}{2}} dB^H_t$$

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where \( r \) is the constant-risk-free rate of interest, \( \sigma \) a positive constant and \( \alpha \in [0, 2] \) the constant elasticity parameter. \( B_H^t \) is a sub-fractional Brownian motion with Hurst exponent \( H > 1/2 \). At the limit case \( H \to 1/2 \), the Eq. (1) becomes the standard CEV model proposed by Cox [13, 14]. If \( \alpha \to 2 \), the Eq. (1) goes to the sub-fractional Black-Scholes model addressed by Tudor [7].

Defining the new variable \( x = S^{2-\alpha} \), and using sub-fractional Itô rules [7, 12]:

\[
\begin{align*}
\frac{dx}{(2-\alpha)\left[rx + Ht^{2H-1}(2-2^{2H-1})(1-\alpha)\sigma^2\right]} + (2-\alpha)\sigma\sqrt{x}dB_H^t
\end{align*}
\]

3 Fokker-Planck equation and option pricing

3.1 Derivation of the Fokker-Planck equation for sub-fractional diffusions

We will follow and extend the procedure given in [15] for diffusion processes under fractional Brownian motion, but this time applied to sub-fractional case.

First, we start with the following SDE:

\[
\begin{align*}
dy = f(y, t) dt + g(y, t) dB_H^t
\end{align*}
\]

being \( dB_H^t \) a sub-fractional Brownian motion.

Let \( h = h(x) \) an scalar function. Using the Itô formula for sBm [12], we have:

\[
\begin{align*}
dh = \left[ f \frac{\partial h}{\partial x} + Ht^{2H-1}(2-2^{2H-1})g^2 \frac{\partial^2 h}{\partial x^2} \right] dt + g \frac{\partial h}{\partial x} dB_H^t \end{align*}
\]

Then, taking expectations over (4):

\[
\begin{align*}
E\left(\frac{dh}{dt}\right) = E\left(f \frac{\partial h}{\partial x}\right) + E\left[Ht^{2H-1}(2-2^{2H-1})g^2 \frac{\partial^2 h}{\partial x^2}\right]
\end{align*}
\]

Later, by the definition of expectations:

\[
\begin{align*}
E[h(x)] = \int h(x) P(x, t) dx
\end{align*}
\]

where \( P \) is the transition probability density function at time \( t \); the relations (5) and (6) yields to:

\[
\begin{align*}
\int_{-\infty}^{\infty} h \frac{\partial P}{\partial t} dx = \int_{-\infty}^{\infty} \left[ f \frac{\partial h}{\partial x} + Ht^{2H-1}(2-2^{2H-1})g^2 \frac{\partial^2 h}{\partial x^2}\right] Pdx
\end{align*}
\]

After that, using the following results:

\[
\begin{align*}
\int_{-\infty}^{\infty} f \frac{\partial h}{\partial x} Pdx = -\int_{-\infty}^{\infty} h \frac{\partial (fP)}{\partial x} dx \quad \int_{-\infty}^{\infty} g^2 \frac{\partial^2 h}{\partial x^2} Pdx = \int_{-\infty}^{\infty} h \frac{\partial (g^2P)}{\partial x} dx
\end{align*}
\]

the Eq. (7) goes to:

\[
\begin{align*}
\int_{-\infty}^{\infty} h \left[ \frac{\partial P}{\partial t} + \frac{\partial (fP)}{\partial x} - Ht^{2H-1}(2-2^{2H-1}) \frac{\partial (g^2P)}{\partial x^2}\right] dx = 0
\end{align*}
\]
Finally, the Fokker-Planck equation related to the process \([3]\), emerges from \([8]\):

\[
\frac{\partial P}{\partial t} = Ht^{2H-1} (2 - 2^{2H-1}) \frac{\partial}{\partial x^2} \Bigg[ (g^2 P) - \frac{\partial (f P)}{\partial x} \Bigg] \tag{9}
\]

### 3.2 Transition probability density function for the sub-fractional CEV model and the European Call price

We comeback to the process defined in the Eq. \([2]\). The evolution from \(x(t = 0) = x_0\) to \(x(t = T) = x_T\) is given by the transition probability density function \(P = P(x_T, t|x_0, 0)\), which obeys the the related sub-fractional Fokker-Planck equation. Since \(f = (2 - \alpha) \left[r x + Ht^{2H-1} (2 - 2^{2H-1}) (1 - \alpha) \sigma^2 \right]\) and \(g = (2 - \alpha) \sigma \sqrt{x}\), and replacing in \([9]\), the Fokker-Planck equation related to the process \([2]\) is given by:

\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial x^2} \Bigg[ Ht^{2H-1} (2 - 2^{2H-1}) (2 - \alpha)^2 \sigma^2 x P \Bigg] - \frac{\partial}{\partial x} \Bigg\{ [(2 - \alpha) r x + Ht^{2H-1} (2 - 2^{2H-1}) (2 - \alpha) (1 - \alpha) \sigma^2] P \Bigg\} \tag{10}
\]

Later, defining the rescaled time \(\tau = -(2 - \alpha) rt\), the relation \([10]\) becomes:

\[
\frac{\partial P}{\partial \tau} = \frac{\partial^2}{\partial x^2} [a(\tau) x P] + \frac{\partial}{\partial x} [(x + b(\tau)) P]
\]

with,

\[
a(\tau) = -\frac{\sigma^2}{r} (2 - \alpha) (1 - 2^{2H-2}) 2H \left( 1 - \frac{\tau}{b} \right)^{2H-1} \\
b(\tau) = -\frac{\sigma^2}{r} (1 - \alpha) (1 - 2^{2H-2}) 2H \left( 1 - \frac{\tau}{b} \right)^{2H-1}
\]

Since the ratio \(a(\tau)/b(\tau)\) is time-independent (constant), the PDE \([10]\) could be solved using the Feller’s lemma with time varying coefficients \([11]\, [16]\). Thus:

\[
P (x, \tau | x_0, 0) = \frac{1}{\phi(\tau)} \left( \frac{x e^{-\tau}}{x_0} \right)^{\frac{b-a}{2a}} \exp \left[ -\frac{(x + x_0 e^{-\tau})}{\phi(\tau)} \right] I_{1-b/a} \left[ \frac{2}{\phi(\tau)} \sqrt{e^{-\tau} x_0} \right]
\]

where \(I_\nu\) is the modified Bessel function of first kind of order \(\nu\), and \(\phi\) is defined by:

\[
\phi(\tau) = -\frac{\sigma^2}{r} (2 - \alpha) \int_0^\tau 2H (1 - 2^{2H-2}) \left( 1 - \frac{s}{b} \right)^{2H-1} e^{-s} ds \\
= -\frac{\sigma^2}{r} (2 - \alpha) \int_0^\tau 2H (1 - 2^{2H-2}) \left( 2H + 1 + e^{-\frac{1}{2} \tau} (-\tau)^{-H} M_{H,H+1/2} (-\tau) \right)
\]

being \(M_{k,v} (t)\) the M-Whittaker function.

Later, in terms of the original variables \((S, t)\), the transition probability density related to the process \([1]\) is written as:

\[
P (S_T, T | S_0, 0) = (2 - \alpha) k_s^{\frac{1-\alpha}{\sigma^2}} \left( y_s w_s^{1-2\alpha} \right)^{\frac{1-\alpha}{\sigma^2}} e^{-y-w} I_{1/(2-\alpha)} (2 \sqrt{y_s w_s}) \tag{11}
\]
with:

\[ k_s = \left[ \varphi ( - (2 - \alpha) rT) \right]^{-1}, \quad \text{(12)} \]
\[ y_s = k_s S_0^{2-\alpha} e^{r(2-\alpha)T}, \quad \text{(13)} \]
\[ w_s = k_s S_T^{2-\alpha}, \quad \text{(14)} \]

Using the same arguments supplied in [1], and defining \( z_s(t) = k_s(t)E^{2-\alpha} \), the European Call price at the inception time is given by:

\[
C_H (S_0, 0) = e^{-rT} \int_{-\infty}^{\infty} \max \{ S_T - E, 0 \} P(S_T, T|S_0, 0) \, dS_T
\]

\[
= S_0 Q \left( 2z_s; 2 + \frac{2}{2-\alpha}, 2y_s \right) - E e^{-rT} \left[ 1 - Q \left( 2y_s; \frac{2}{2-\alpha}, 2z_s \right) \right]
\]

being \( Q(\cdot) \) the non-central chi-squared complementary distribution function.

The Fig. 1 shows the price of an at-the-money European Call option under the sub-fractional CEV model (blue), in function of the elasticity parameter, for short (3 months, 1a) and long (2 years, 1b) maturities, with \( H = \{0.5, 0.7, 0.9\} \). Besides, the price under the fractional CEV is also drawn (red) by way of comparison. In all the cases, the sub-fractional pricing is lower than the fractional one for a fix \( H > 0.5 \). When the Hurst exponent is equal to 0.5, both models fit with the standard CEV. For \( \alpha = 2 \), the sub-fractional (fractional) CEV converges to the sub-fractional (fractional) Black-Scholes.

4 Summary

In this note, we consider the Constant Elasticity of Variance model under sub-fractional diffusion. This approach allow to us the model of assets where their prices presents features as long-range dependency, non-stationarity increments leverage effect and in the case of their options, the volatility skew pattern. Using sub-fractional Itô rules, the transition probability density function is obtained solving the corresponding sub-fractional Fokker-Planck equation in terms of the M-Whittaker function. Then, the price of a call option is obtained.

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Figure 1: Price of a European Call option under both fractional and sub-fractional CEV model as a function of the elasticity (\( \alpha \)) and the Hurst exponent (\( H \)). We fix \( \sigma = 30\% \), \( S_0 = E = 100 \) and \( r = 5\% \).
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