Correction to the Probability Distribution of Induced Gluon

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Abstract. We present a novel technique for the calculation of the probability for emission of gluon radiated from a high-$p_T$ quark through a medium (QGP). Our work is an extension of the maximal helicity violating (MHV) method to compute the non-abelian correction for 2 gluon emissions.

1. Introduction

It is believed that shortly after the big bang the universe was filled by mixture of quarks and gluons which are the fundamental building blocks of matters. The mixture is known as the quark gluon plasma (QGP) where quarks and gluons are no longer confined. And as the universe expanded the temperature decreases and the interaction increases, then quarks and gluons start to combine into baryons and mesons which will become atoms then molecules and eventually stars and galaxies will emerge.

In heavy ion collision, at RHIC and LHC, experiments have been conducted in order to reproduce the condition at early universe where QGP can be probed \[1\]. In these experiments heavy ions like Au or Pb are accelerated and collide them at very high speed, so every single proton and neutron will melt into free quark and gluon. One of the challenge in the experimental side is to detect any formation of QGP from the collision, since QGP has a very short lifetime and cannot be directly observed. This is one of the reasons why we are interested on “jets”, since they carry information on any medium produced in heavy ion collisions.

Jets are the manifestation of high-$p_T$ partons produced from nucleus-nucleus collision. These partons experience multiple interaction inside the collision region prior to fragmentation and hadronisation, especially in presence of medium, jet looses energy through scattering and mainly gluon radiation \[2, 3, 4\]

\[
\text{Jet}(E) \rightarrow \text{Jet}(E - \Delta E) + \text{Radiation}(\Delta E).
\]

Practically, we cannot detect those radiations since they are either emitted at low energy compare to the resolution of the detectors (soft), or they are emitted at a very small angle and they cannot be resolved from the parent jet (collinear). However in the theoretical side, the soft-collinear behaviour of these radiations provide the necessary tools to compute the probability...
distribution for such emission. But due to the complexity of QCD (quantum chromodynamic) the probability of gluon radiation was approximated to be poisson, in another word the gluon interferences was neglected and the individual emission are independent from each other. Then the main goal of this work is to perform a full QCD calculation and find the non-abelian correction that comes from gluon interferences to the poissonian approximation,

\[ dP_{\text{improved}} = dP_{\text{poisson}} + dP_{\text{correction}}. \]  

In this proceeding, we consider a light quark passing through a medium tickled by a gluon and compute the probability distribution two gluon emission. In the next section, we are going to give a short introduction to the maximal helicity violating (MHV) techniques that we are going to use. Then in section 2 we are going to compute the two gluon emission in the poisson approximation, and last in section 4 we present the non-abelian correction for the two gluon emissions.

2. Maximal Helicity Violating

This section is short introduction for the maximal helicity violating techniques in order to compute scattering amplitudes [7,8,9]. As physicist, it is very natural for us to use Feynman diagrams techniques to compute amplitudes, but we know that the computations get complicated as the number of particle increases. For example in the case of photon to quark anti-quark production, we can see in the table below how the number of diagram increases exponentially with the number of gluon corrections.

| # Gluon | 1 | 2 | 3 | \cdots | n |
|---------|---|---|---|-------|---|
| # Diagram | 2 | 8 | 48 | \cdots | 2^n n! |

But it turns out, the analytical expression of the amplitudes are extremely simple using the spinor helicity formalism and with a simple change of variables especially when the conservation of helicity is maximally violated, i.e. \( \Delta h = |h_i - h_f| \) is maximal. The spinor helicity formalism is one of the main ingredient of the MHV techniques, and in that formalism instead of the four vector representation the momenta will be represented with a 2 by 2 matrices given by \( P^{\mu \hat{a}} = (\sigma_\mu)^{\hat{a} \hat{\alpha}} p^\hat{\alpha} \). And the invariant mass of the individual particle will be the determinant of the respective momenta in that representation. For massless particles, since the invariant mass is zero then the momenta can be parametrized as follow

\[ P^{\mu \hat{a}} = (\sigma_\mu)^{\hat{a} \hat{\alpha}} p^\hat{\alpha} = \lambda^{\hat{\alpha}} \tilde{\lambda}^\hat{\alpha}, \]  

where \( \lambda \) and \( \tilde{\lambda} \) are called the spinor helicity variables and transform respectively as left and right handed spinors. Out of those new variables, we can define two different Lorentz invariant
products which are both antisymmetric
\[
\langle 12 \rangle = \epsilon_{ab} \lambda^a_1 \lambda^b_2 \quad \text{and} \quad [12] = \epsilon_{ab} \tilde{\lambda}^a_1 \tilde{\lambda}^b_2,
\]  
(4)
those products, angle and square brackets, can be related to the four vector scalar product by \(2p_1.p_2 = \langle 12 \rangle [12]\). The MHV amplitude corresponding to a tickled quark that radiate \(n\) gluon is given by the partial amplitude below,

\[
p^{n+1} = g^{n+1} \frac{\langle pq \rangle^3 \langle qp' \rangle}{\langle pp' \rangle \langle p'q \rangle \langle q1 \rangle \langle 12 \rangle \cdots \langle np \rangle}
\]  
(5)

The full amplitudes can be computed by summing the permutation of gluon indices weighted by the color string basis \((T_{a_1} T_{a_2} \cdots T_{a_n})\).

3. Two gluon emissions in the poisson approximation
In the poisson approximation it is assumed that gluon emissions are independent then the distribution of multiple gluon emission is given by the product of individual emissions. Technically, independent emission is satisfied when the emission angles of the gluon emitted are strongly ordered also known by angular ordering, \(\theta_1 \gg \theta_2 \gg \cdots \gg \theta_n\) where \(\theta_i\) is the emission angle of the \(i\)-th gluon. To begin let us recall the soft-collinear factorization where the information on the momentum distribution is factorized from the born amplitude as below,

\[
\mathcal{M}_{n\text{-gluon}} = \mathcal{M}_{\text{born}} \times J_n(\{k_i\}),
\]  
(6)
where \(\mathcal{M}_{\text{born}}\) is the born amplitude and \(J_n\) is the eikonal factor. Now using this soft-collinear factorization using the MHV techniques, we can compute the single gluon distribution \(dP^{(1)}\) to be the square of the eikonal factor \(J_1\) and we obtain

\[
\frac{dP^{(1)}}{d\Phi_1} = \frac{C_A}{2} \frac{p.q}{2p.k_1 q.k_1},
\]  
(7)
where \(d\Phi_1\) is the differential phase space of single emission of momentum \(k_1\), and \(C_A\) is the color factor and equal to 3 for \(SU(3)\). And if have independent emission the 2 gluon distribution is given by the following expression,

\[
\frac{dP^{(2)}}{d\Phi_{1,2}} = \frac{1}{2} \left( \frac{C_A}{2} \right)^2 \prod_{i=1,2} \frac{p.q}{2p.k_1 q.k_1},
\]  
(8)
\(d\Phi_{1,2}\) is the differential phase space of two gluons of momenta \(k_1\) and \(k_2\). In the next section we will use MHV to fully compute the two gluon emission distribution and show that the poisson approximation is only valid in some region of the phase space.

4. Non-abelian correction for the two gluon emissions
To compute the full amplitude in the MHV, we need perform a sum over the permutation of gluon indices weighted by the color string bases in which we apply the factorization. After
computation, the eikonal factor $J_2$ can be written in two terms which are respectively symmetric and anisymmetric under the permutation of 1 and 2,

$$J(1, 2) = C^{a}_{12} J^a(1, 2) + C^{a}_{12} J^a(1, 2),$$  (9)

where the symmetric part contains the independent emission as in the poisson and the antisymmetric contains information on the gluon interference that break the poisson approximation,

$$|J_2(1, 2)|_{\text{new}}^2 = \frac{1}{4}[3 + F(1, 2)]|J_2(1, 2)|_{\text{poisson}}^2.$$  (10)

The function $F(1, 2)$ is what we call non-abelian correction or gluon interference, this correction depends only on the cross ratio of the four momenta $p$, $q$, $k_1$, $k_2$ respectively the in coming momentum, the tickling momentum, the two radiated momenta. A cross ratio is scale invariant quantity such as

$$\frac{p.k_1 q.k_2}{k_1.k_2 p.q},$$  (11)

which means $F(1, 2)$ depends only on the various angle between them, and since the angle between $p$ and $q$ are fixed then it turns out it depends only on three angles

$$F(\theta_1, \theta_2, \phi) = \frac{1 - \cos \theta_1 \cos \theta_2}{1 - \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \phi},$$  (12)

$\theta_1$ and $\theta_2$ are the emission angles of the emitted gluons while $\phi$ is the angle between them emission plane, see image below

![Figure 2: Two gluon emitted in two different plane](image)

To see the effect of the non-abelian correction, let us consider the ratio between the *new* result and the *old* result where it is assumed to be poisson. Then plot them for a fixed $\theta_2 = 0.01$ and $\phi$, at first we consider that the gluon are emitted in the same plane $\phi = 0$, then second for different value of $\phi$:

- In fig[3a]: We can see two different regions, the part in right where $\theta_1 \gg \theta_2$ we have a poissonian behaviour where there is no interference between the two emission. However when $\theta_1$ approaches the value of $\theta_2$, in the left part, the ratio grows exponentially and diverges at $\theta_1 = \theta_2$.
- In fig[3b]: Here we can see that we recover the poisson distribution for $\theta_1 \gg \theta_2$. For $\phi = 0$, in *blue*, is the same as in fig[3a] where $F(1, 2)$ is singular at $\theta_1 = \theta_2$. For $\phi = \frac{\pi}{4}$, in *orange*, the $F(1, 2)$ still grows as $\theta_1$ approaches $\theta_2$ up to some finite value. For $\phi = \frac{\pi}{2}$, in *green*, $F(1, 2)$ becomes constant and equal to 1, i.e. the interference between the two gluons vanishes and the distribution becomes poisson. And last for $\phi = \pi$, in *red*, $F(1, 2)$ decreases as $\theta_1$ approaches the value of $\theta_2$ with a minimal value 0.5 at $\theta_1 = \theta_2$. 
5. Conclusion
To conclude let us remind ourselves that we want to improve the momentum distribution for emitting soft-collinear radiation for a quark passing through a QGP. For that we performed a full QCD calculation using the maximal helicity violating techniques in order to avoid diagrammatic complications. And for the two gluon emissions we computed the non-abelian piece $F(1, 2)$ which is a correction to the poissonian distribution and it depends only on three variables: the two emission angles and the angle between the emission planes. It has been shown that we can recover the poissonian distribution behaviour at some region of the phase space where the angles are strongly ordered or at $\phi = \frac{\pi}{2}$. However outside of these region the interference between gluon become more important and the non-abelian piece (correction) cannot be ignored. Now in order to resum all emission we still need to compute the result for the tree gluons emissions and find more patterns to exponentiate the non-abelian correction.

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