Influence of fractional calculus model parameters on nonlinear forced vibrations of suspension bridges

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Abstract. Nonlinear forced vibrations of suspension bridges, when the frequency of an external force is approaching one of the natural frequencies of the suspension system, which, in its turn, undergoes the conditions of the one-to-one internal resonance, are investigated. The generalized method of multiple time scales is used as the method of solution. The damping features are described by the fractional derivative, which is interpreted as the fractional power of the differentiation operator. An approximate analytical solution has been obtained by the method of successive approximations.

1. Introduction
The experimental data obtained in [1] during ambient vibrations studies of the Golden Gate Bridge show that different vibrational modes possess different amplitude damping factors, and the order of smallness of these coefficients tells about low damping capacity of suspension combined systems, resulting in prolonged energy transfer from one partial subsystem to another.

Free damped vibrations of suspension bridges in the cases of the one-to-one and two-to-one internal resonance have been investigated in [2] when damping features of the system are prescribed by the first derivative of the displacement with respect to time. It has been shown that for both types of the internal resonance the damping coefficients do not depend on the natural frequency of vibrations. However, this result is in conflict with the experimental data. To lead the theoretical investigations in line with the experiment, fractional derivatives were introduced in [3] for describing the processes of internal friction proceeding in suspension combined systems at free vibrations. The suspension bridge model put forward allows one to obtain the damping coefficient dependent on the natural frequency of vibrations. This model has been generalized further in [4] by using two different fractional parameters for analyzing vertical and torsional modes.

In the present paper, the model analyzed in [4] is used for investigating nonlinear forced vibrations of suspension bridges, when the frequency of an external force is approaching one of the natural frequencies of the suspension system, which undergoes the conditions of the one-to-one internal resonance.

2. Problem Formulation
The suspension bridge scheme presents a bisymmetrical thin-walled stiffening girder, which is connected with two suspended cables by virtue of vertical suspensions. The cables are thrown over the pylons and are tensioned by anchor mechanisms. The suspensions are considered as inextensible and uniformly distributed along the stiffening girder. The cables are parabolic, and the contour of the girder’s
cross-section is undeformable. It is assumed that the girder’s contour translates as a rigid body vertically (in the $y$-axis direction) on the value of $\eta(z,t)$ and rotates with respect to the girder’s axis (the $z$-axis) through the angle of $\varphi(z,t)$. The origin of the frame of references is in the center of gravity of the cross-section.

It is known for suspension bridges [2,5] that some natural modes belonging to different types of vibrations could be coupled with each other, i.e., the excitation of one natural mode gives rise to another one. Two modes interact more often than not, although the possibility for interaction of a greater number of modes is not ruled out.

If only two modes predominate in the vibrational process, namely: the vertical $n$-th mode with linear natural frequency $\omega_n$ and the torsional $m$-th mode with the natural frequency $\Omega_m$, then the functions $\eta(z,t)$ and $\varphi(z,t)$ could be approximately defined as

$$\eta(z,t) = \nu_n(z)x_{1n}(t), \quad \varphi(z,t) = \Theta_m(z)x_{2m}(t), \quad (1)$$

where $x_{1n}$ and $x_{2m}$ are the generalized displacements, and $\nu_n(z)$ and $\Theta_m(z)$ are the natural modes of the two interacting modes of vibrations.

The resolving set of equations describing forced vibrations is written in a dimensionless form as

$$\ddot{x}_{1n} + \omega^2_0 x_{1n} + \beta D^\gamma_0 x_{1n} + a^m_{12} x_{1n}^2 + a^m_{22} x_{2m}^2 + \left(b^m_{11} x_{1n}^2 + b^m_{22} x_{2m}^2\right)x_{1n} = \hat{F}\cos(\alpha_1 t), \quad (2)$$

$$\ddot{x}_{2n} + \Omega_0 x_{2n} + \beta D^\gamma_0 x_{2n} + a^m_{12} x_{1n}x_{2m}^2 + \left(c^m_{11} x_{1n}^2 + c^m_{22} x_{2m}^2\right)x_{2m} = 0,$$

where the coefficients $a_{ij}$, $b_{ij}$ and $c_{ij}$ ($i,j=1,2$) are defined in [5] (subsequently the indices $n$ and $m$ will be omitted for ease of presentation), $\hat{F}=\text{const}$ is the amplitude of the external harmonic force, the terms $\beta D^\gamma_0 x_i$ and $\beta D^\gamma_0 x_i$ describe inelastic reaction of the system, $\beta$ is the viscosity coefficient, $D^\gamma x (\gamma=\gamma_1 \text{ or } \gamma_2)$ is the fractional derivative which is interpreted as the fractional power of the differentiation operator [3,4] with the fractional parameter $\gamma$ (order of the fractional derivative).

$$D^\gamma x = \left(\frac{d}{dt}\right)^\gamma \quad (0 < \gamma \leq 1). \quad (3)$$

Let us consider the case of the one-to-one internal resonance, as well as suppose that the frequency of the external force is close to the natural frequencies of the interacting modes, i.e.,

$$\omega_0 = \Omega_0 + \varepsilon^2 \sigma, \quad \omega_1 = \omega_0 - \varepsilon^2 \sigma_1, \quad (4)$$

where $\varepsilon$ is a small parameter that is of the same order of magnitudes as the amplitudes of vibrations, and $\sigma$ and $\sigma_1$ are detuning parameters.

3. Method for determining the governing equations

An approximate solution of (2) for small but finite amplitudes weakly varying with time can be represented by an expansion in terms of different time scales in the form

$$x_1(t) = e\psi_1(T_0, T_2, \ldots) + \varepsilon^2 x_{12}(T_0, T_2, \ldots) + \varepsilon^3 x_{13}(T_0, T_2, \ldots) + \ldots \quad (5)$$

$$x_2(t) = e\psi_2(T_0, T_2, \ldots) + \varepsilon^2 x_{22}(T_0, T_2, \ldots) + \varepsilon^3 x_{23}(T_0, T_2, \ldots) + \ldots$$

Here, $T_0 = t$ is a fast scale characterizing motions with the natural frequencies $\omega_0$ and $\Omega_0$, and $T_2 = \varepsilon^2 t$ is a slow scale, characterizing the modulations of the amplitudes and phases. Moreover, we let $\beta = \varepsilon^2 \mu$, and $\hat{F} = \varepsilon^3 F$, where $\mu = \text{const}$ and $F = \text{const}.$

Considering that $\frac{d}{dt} = D_0 + \varepsilon^2 D_2 + \ldots$, $\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon^2 D_0 D_2 + \ldots$, where $D_0 = \partial / \partial t$, and substituting (4) into (2), after equating the coefficients at like powers of $\varepsilon$ to zero, one obtains

- To order $\varepsilon$:
  $$D_0^2 x_{11} + \omega_0^2 x_{11} = 0, \quad D_0^2 x_{21} + \Omega_0^2 x_{21} = 0, \quad (6)$$
To order $\varepsilon^2$:
\[
D_0^2x_{12} + \omega_0^2x_{12} = -a_{11}x_{11}^2 - a_{22}x_{22}^2, \quad D_0^2x_{22} = -a_{11}x_{11}x_{12},
\]
(7)

To order $\varepsilon^3$:
\[
\begin{align*}
D_0^2x_{13} + \omega_0^2x_{13} &= -2D_0D_2x_{11} - 2a_{11}x_{11}x_{12} - 2a_{22}x_{22}x_{22} - b_{11}x_{11}^3 - b_{22}x_{22}x_{11} \\
&\quad - \mu D_0^2x_{13} + F \cos(\omega_0 T), \\
D_0^2x_{23} + \Omega_0^2x_{23} &= -2D_0D_2x_{21} - a_{12}(x_{11}x_{22} + x_{12}x_{21}) - c_{22}x_{22}^3 - c_{11}x_{11}x_{21} - \mu D_0^2x_{23},
\end{align*}
\]
(8)

During the deduction of (6)-(8) it was assumed, following [3], that
\[
(\frac{d}{dt})^2 = + \Omega = \Omega,
\]
(9)

where $D_0^2$ is obtained from (3) by replacing $t$ with $0$. The solution of equations (6) will be sought in the form
\[
x_{12} = A_1(T_2) \exp(i\omega_0 T_0) + A_2 \exp(-i\omega_0 T_0), \quad x_{21} = A_2(T_2) \exp(i\Omega_0 T_0) + A_1 \exp(-i\Omega_0 T_0),
\]
(10)

where $A_1$ and $A_2$ are unknown complex functions, and $\tilde{A}_1$ and $\tilde{A}_2$ are the complex conjugates of $A_1$ and $A_2$, respectively. Substituting (10) in (7) yields
\[
\begin{align*}
x_{12} &= \frac{1}{3\omega_0} a_{11}A_1 \exp(2i\omega_0 T_0) + \frac{a_{12}A_2^* \exp(2i\Omega_0 T_0)}{4\Omega_0^2 - \omega_0^2} - (a_{11}\tilde{A}_1 + a_{22}A_2 \tilde{A}_2) \omega_0^2 + cc, \\
x_{22} &= \frac{a_{12}}{\omega_0(2\Omega_0 + \omega_0)} A_1 \tilde{A}_2 \exp[iT_0(\Omega_0 - \omega_0)] - \frac{a_{12}}{\omega_0(2\Omega_0 - \omega_0)} A_2 \tilde{A}_1 \exp[iT_0(\Omega_0 - \omega_0)] + cc,
\end{align*}
\]
(11)

where $cc$ is complex conjugate part to the preceding terms.

Substituting (11) in equations (8) with due account for the conditions (4) and $D_0^2 \exp(i\omega_0 T) = a^* \exp(i\omega_0 T)$, and eliminating the terms that produce secular terms, we obtain the solvability conditions
\[
\begin{align*}
-iD_2A_1 - \frac{1}{2} \mu \omega_0^{-1}(i\omega_0)^n A_1 - \lambda_1 A_1^2 \tilde{A}_1 - \lambda_2 A_2 A_2 \tilde{A}_2 + \frac{1}{4} \Gamma_1 A_1 A_1^2 \exp(-2i\sigma T_2) \\
+ \frac{1}{4} F \exp(-i\sigma T_1) \omega_0^{-1} &= 0, \\
-iD_2A_2 - \frac{1}{2} \mu \Omega_0^{-1}(i\Omega_0)^n A_2 - \lambda_1 A_1 A_1^2 \tilde{A}_1 - \lambda_2 A_2^2 A_2 \tilde{A}_2 + \frac{1}{4} \Gamma_2 A_1 A_1^2 \exp(2i\sigma T_2) &= 0,
\end{align*}
\]
(12)

where coefficients $\lambda_i$ and $\Gamma_j$ ($i = 1, 2, 3, 4; j = 1, 2$) are presented in [3, 5]. Substitution of functions $A_1$ and $A_2$ for $A_1 = A \exp(-i\sigma T_2)$ and $A_2 = A \exp[i(\sigma - \sigma_t) T_2]$, respectively, in equations (12) leads to eliminating $\exp(\pm i\sigma T_2)$ and $\exp(\pm i\sigma T_2)$.

Representing the functions $A_1$ and $A_2$ in (12) in their polar forms, i.e., $A_1 = a_1 \exp(i\phi_1)$ and $A_2 = a_2 \exp(i\phi_2)$, as a result we obtain the modulation equations
\[
\begin{align*}
\dot{a}_1 + \frac{1}{2} \mu \omega_0^{-1} \sin\left(\frac{1}{2} \pi \gamma_1\right)a_1 - \frac{1}{4} \Gamma_1 a_1 a_1^2 \sin \delta + \frac{1}{4} F \omega_0^{-1} \sin \phi_1 &= 0, \\
\dot{a}_2 + \frac{1}{2} \mu \Omega_0^{-1} \sin\left(\frac{1}{2} \pi \gamma_2\right)a_2 - \frac{1}{4} \Gamma_2 a_1 a_2^2 \sin \delta &= 0, \\
\dot{\phi}_1 - \frac{1}{2} \mu \omega_0^{-1} \cos\left(\frac{1}{2} \pi \gamma_1\right) - \sigma_1 - \lambda_1 a_1^2 - \lambda_2 a_2^2 + \frac{1}{4} \Gamma_1 a_1^2 \cos \delta + \frac{1}{4} F \omega_0^{-1} a_1^2 \cos \phi_1 &= 0, \\
\dot{\phi}_2 - \frac{1}{2} \mu \Omega_0^{-1} \cos\left(\frac{1}{2} \pi \gamma_2\right) - (\sigma_1 - \sigma_2) - \lambda_1 a_1^2 - \lambda_2 a_2^2 + \frac{1}{4} \Gamma_2 a_1^2 \cos \delta &= 0,
\end{align*}
\]
(13)
where a dot denotes differentiation with respect to $T_2$, and $\delta = 2(\varphi_2 - \varphi_1)$ is the phase difference.

The solution of the set of equations (13), when $F = 0$, has been considered for $\gamma_1 \neq \gamma_2$ and $\gamma_1 = \gamma_2$ in [4] and [3], respectively. For the case $\gamma_1 = \gamma_2$, the analytical solution has been obtained in [6].

An approximate solution of the set of nonlinear equations (13) will be found in the next section by the method of successive approximations.

4. Method of solution
As the initial approximation, let us consider the solution of the homogeneous part of the system of equations (13):

\begin{align}
\dot{a}_1 + \frac{1}{2} \mu \omega_0^{\gamma_1} \sin \left( \frac{1}{2} \pi \gamma_1 \right) a_1 &= 0, \\
\dot{a}_2 + \frac{1}{2} \mu \Omega_0^{\gamma_2} \sin \left( \frac{1}{2} \pi \gamma_2 \right) a_2 &= 0, \\
\dot{\varphi}_1 - \frac{1}{2} \mu \omega_0^{\gamma_1} \cos \left( \frac{1}{2} \pi \gamma_1 \right) - \sigma_1 &= 0, \\
\dot{\varphi}_2 - \frac{1}{2} \mu \Omega_0^{\gamma_2} \cos \left( \frac{1}{2} \pi \gamma_2 \right) - (\sigma_1 - \sigma) &= 0,
\end{align}

which has the form

\begin{align}
a_1 &= a_{10} e^{-S_{1T}}, & a_2 &= a_{20} e^{-S_{2T}}, & \varphi_1 &= S_{1T} + \varphi_{10}, & \varphi_2 &= S_{2T} + \varphi_{20},
\end{align}

where $a_{10}$ and $\varphi_{10}$ ($i=1,2$) are, respectively, the initial values of amplitudes and phases to be found from the initial conditions, $\delta_0 = 2(\varphi_{20} - \varphi_{10})$ is the initial phase difference, and

\begin{align}
S_1 &= \frac{1}{2} \mu \omega_0^{\gamma_1} \sin \left( \frac{1}{2} \pi \gamma_1 \right), & S_2 &= \frac{1}{2} \mu \Omega_0^{\gamma_2} \sin \left( \frac{1}{2} \pi \gamma_2 \right), \\
S_3 &= \frac{1}{2} \mu \omega_0^{\gamma_1} \cos \left( \frac{1}{2} \pi \gamma_1 \right) + \sigma_1, & S_4 &= \frac{1}{2} \mu \Omega_0^{\gamma_2} \cos \left( \frac{1}{2} \pi \gamma_2 \right) + (\sigma_1 - \sigma).
\end{align}

Now substituting (15) in equations (13) yields

\begin{align}
\dot{a}_1 + S_1 a_1 &= \frac{1}{4} \Gamma_1 a_{10} e^{-(S_{1T}+S_{2T})} a_2^2 \sin(S_{1T} + \delta_0) - \frac{1}{4} F \omega_1^{\gamma_1} \sin(S_{1T} + \varphi_{10}), \\
\dot{a}_2 + S_2 a_2 &= -\frac{1}{4} \Gamma_2 a_{20} e^{-(S_{1T}+S_{2T})} a_2 \sin(S_{2T} + \delta_0), \\
\dot{\varphi}_1 - S_3 &= \lambda_1 a_{10}^2 e^{-2S_{1T}} + \lambda_2 a_{20}^2 e^{-2S_{2T}} - \frac{1}{4} \Gamma_1 a_{10}^2 e^{-2S_{1T}} \cos(S_{1T} + \delta_0) - \frac{1}{4} F \omega_1^{\gamma_1} a_{10}^2 e^{S_{1T}} \cos(S_{1T} + \varphi_{10}), \\
\dot{\varphi}_2 - S_4 &= \lambda_1 a_{20}^2 e^{-2S_{2T}} + \lambda_2 a_{20}^2 e^{-2S_{2T}} - \frac{1}{4} \Gamma_2 a_{20}^2 e^{-2S_{2T}} \cos(S_{2T} + \delta_0),
\end{align}

where $\Sigma = 2(S_{1T} - S_{2T})$.

To solve the first two equations in (17), we will use the method of variation of arbitrary functions, and assume the proposed solution in the form

\begin{align}
a_1(T_2) &= C_1(T_2) e^{-S_{1T}}, & a_2(T_2) &= C_2(T_2) e^{-S_{2T}},
\end{align}

where $C_1(T_2)$ and $C_2(T_2)$ are arbitrary functions to be found.

Substituting the proposed solution (18) in equations (17) yields
\[ \dot{C}_1(T_z) = \frac{1}{4} \Gamma_1 a_{10}^2 a_{20}^2 e^{-2S_sT_z} \sin(\Sigma T_z + \delta_0) - \frac{1}{4} \Gamma_1 a_{10}^2 e^{S_sT_z} \sin(S_1T_z + \varphi_{10}), \]  
\[ \dot{C}_2(T_z) = -\frac{1}{4} \Gamma_2 a_{10}^2 a_{20}^2 e^{-2S_sT_z} \sin(\Sigma T_z + \delta_0). \]  

Integrating equations (19), we have

\[ C_1(T_z) = -\frac{1}{4} \Gamma_1 a_{10}^2 a_{20}^2 \frac{2S_s \sin(\Sigma T_z + \delta_0) + \Sigma \cos(\Sigma T_z + \delta_0)}{4S_s^2 + \Sigma^2} e^{-2S_sT_z} \]
\[ - \frac{F S_s \sin(S_1T_z + \varphi_{10}) - S_s \cos(S_1T_z + \varphi_{10})}{S_1^2 + S_s^2} e^{S_sT_z} + C_{10}, \]  
\[ C_2(T_z) = \frac{1}{4} \Gamma_2 a_{10}^2 a_{20}^2 \frac{2S_s \sin(\Sigma T_z + \delta_0) + \Sigma \cos(\Sigma T_z + \delta_0)}{4S_s^2 + \Sigma^2} e^{-2S_sT_z} + C_{20}, \]  

where \( C_{10} \) and \( C_{20} \) are constants of integration.

Considering relationships (20), the amplitude functions take the form

\[ a_1 = a_{10} e^{-S_sT_z} - \frac{1}{4} \Gamma_1 a_{10}^2 a_{20}^2 \frac{2S_s \sin(\Sigma T_z + \delta_0) + \Sigma \cos(\Sigma T_z + \delta_0)}{4S_s^2 + \Sigma^2} e^{-2S_sT_z}, \]
\[ - \frac{F S_s \sin(S_1T_z + \varphi_{10}) - S_s \cos(S_1T_z + \varphi_{10})}{S_1^2 + S_s^2} e^{S_sT_z} + C_{10} e^{-S_sT_z}, \]  
\[ a_2 = a_{20} e^{-S_sT_z} + \frac{1}{4} \Gamma_2 a_{10}^2 a_{20}^2 \frac{2S_s \sin(\Sigma T_z + \delta_0) + \Sigma \cos(\Sigma T_z + \delta_0)}{4S_s^2 + \Sigma^2} e^{-2S_sT_z} + C_{20} e^{-S_sT_z}. \]  

Integrating the third and fourth equations in (17), we obtain the \( T_z \)-functions of the phases of vibration

\[ \varphi_1 = S_1T_z + \varphi_{10} - \frac{\lambda_1 a_{10}^2}{2S_s} e^{-2S_sT_z} \]
\[ + \frac{1}{4} \Gamma_1 a_{10}^2 a_{20}^2 \frac{2S_s \cos(\Sigma T_z + \delta_0) + \Sigma \sin(\Sigma T_z + \delta_0)}{4S_s^2 + \Sigma^2} e^{-2S_sT_z}, \]
\[ - \frac{1}{4} \Gamma_1 a_{10}^2 \frac{S_s \cos(S_1T_z + \varphi_{10}) + S_s \sin(S_1T_z + \varphi_{10})}{S_1^2 + S_s^2} e^{S_sT_z} + C_{30}, \]  
\[ \varphi_2 = S_1T_z + \varphi_{20} - \frac{\lambda_2 a_{20}^2}{2S_s} e^{-2S_sT_z} \]
\[ + \frac{1}{4} \Gamma_2 a_{10}^2 a_{20}^2 \frac{2S_s \cos(\Sigma T_z + \delta_0) + \Sigma \sin(\Sigma T_z + \delta_0)}{4S_s^2 + \Sigma^2} e^{-2S_sT_z} + C_{40}, \]  

where \( C_{30} \) and \( C_{40} \) are constants of integration to be determined from the initial conditions.

Since the general solution of the system under consideration is the sum of the particular solution of the inhomogeneous set of equations and the general solution of the corresponding homogeneous system, then the arbitrary constants could be chosen in such a way that the initial conditions of all successive approximations would be zero. Thus, for the first approximation the constants to be found take the form
Substitution of the found constants of integration (23) in relationships (21) and (22) results in the approximate analytical solution of the formulated problem.

5. Conclusion
Nonlinear forced vibrations of suspension bridges, when the frequency of an external force is approaching one of the natural frequencies of the suspension system, which, in its turn, undergoes the conditions of the one-to-one internal resonance, are investigated.

The approximate analytical solution has been obtained for the case $\gamma_1 \neq \gamma_2$ by the method of successive approximations.

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