Simultaneous Diagonalization and SVD of Commuting Matrices

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Abstract. We present a matrix version of a known method of constructing common eigenvectors of two diagonalizable commuting matrices, thus enabling their simultaneous diagonalization. The matrices may have simple eigenvalues of multiplicity greater than one. The singular value decomposition (SVD) of a class of commuting matrices also is treated. The effect of row/column permutation is examined. Examples are given.

1 Introduction

It is well known that if two diagonalizable matrices have the same eigenvectors, then they commute. The converse also is true and a construction for the common eigenvectors (enabling simultaneous diagonalization) is known. If one of the matrices has distinct eigenvalues (multiplicity one), it is easy to show that its eigenvectors pertain to both commuting matrices. The case of matrices with simple eigenvalues of multiplicity greater than one requires a more complicated construction of their common eigenvectors. Here, we present a matrix version of a construction procedure given by Horn and Johnson [1, Theorem 1.3.12] and in a video by Sadun [4]. The eigenvector construction procedure also is applied to the singular value decomposition of a class of commuting matrices that includes the case where at least one of the matrices is real and symmetric. In addition, we consider row/column permutation of the commuting matrices. Three examples illustrate the eigenvector construction procedure.

2 Eigenvector Construction

Let $A$ and $B$ be diagonalizable square matrices that commute, i.e.

$$AB = BA$$

and their Jordan canonical forms read

$$A = S_A D_A S_A^{-1}, \quad B = S_B D_B S_B^{-1}.$$  \hspace{1cm} (2)

Here, the columns of $S_A$ are the eigenvectors $s_i$ of $A$ corresponding to the eigenvalues $\lambda_i$ in the diagonal matrix $D_A$, i.e.

$$A s_i = \lambda_i s_i.$$  \hspace{1cm} (3)

If $A$ is normal ($AA^* = A^* A$), then $S_A$ can be made unitary ($S_A^{-1} = S_A^*$). From (3) and (1) we have

$$B A s_i = A (B s_i) = \lambda_i (B s_i),$$

whence $B s_i$ also is an eigenvector of $A$ for the eigenvalue $\lambda_i$.

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When \( \lambda_i \) are distinct, it follows from (8) and (9) that each eigenvector \( \mathbf{B} \mathbf{s}_i \) must be a scalar multiple of \( \mathbf{s}_i \) and, noting (2), we have

\[
\mathbf{B} \mathbf{A} = \mathbf{S}_A \mathbf{D}_B \quad \text{or} \quad \mathbf{D}_B = \mathbf{S}_A^{-1} \mathbf{B} \mathbf{A}, \quad \mathbf{B} = \mathbf{S}_A \mathbf{D}_B \mathbf{S}_A^{-1}
\]

which shows that \( \mathbf{S}_A \) is an eigenvector matrix for \( \mathbf{B} \) as well as \( \mathbf{A} \) and thus \( \mathbf{S}_A \) diagonalizes both of them.

In the case that \( \mathbf{A} \) has \( k \) simple eigenvalues of multiplicity greater than or equal to one, \( \mathbf{D}_A \) can be written in block matrix form as

\[
\mathbf{D}_A = \text{diag} \left[ \mathbf{D}_{A1}, \mathbf{D}_{A2}, \ldots, \mathbf{D}_{Ak} \right], \quad \mathbf{D}_{Ai} = \text{diag} [\lambda_i, \lambda_i, \ldots, \lambda_i] = \lambda_i \mathbf{I}_i,
\]

where \( \mathbf{I}_i \) is the identity matrix of order equal to the number of \( \lambda_i \)'s. Then \( \mathbf{S}_A \) has the block form

\[
\mathbf{S}_A = \left[ \mathbf{S}_{A1}, \mathbf{S}_{A2}, \ldots, \mathbf{S}_{Ak} \right],
\]

where the columns of \( \mathbf{S}_{Ai} \) are the eigenvectors corresponding to \( \lambda_i \). In view of (4), the columns of \( \mathbf{B} \mathbf{S}_{Ai} \) also are eigenvectors corresponding to \( \lambda_i \) and therefore they must be linear combinations of the eigenvectors in \( \mathbf{S}_{Ai} \), i.e.

\[
\mathbf{B} \mathbf{S}_{Ai} = \mathbf{S}_A \mathbf{T}_i,
\]

where \( \mathbf{T}_i \) is a square matrix of the same order as \( \mathbf{I}_i \). Thus, we may write \( \mathbf{B} \mathbf{S}_A \) as

\[
\mathbf{B} \mathbf{S}_A = \left[ \mathbf{B} \mathbf{S}_{A1}, \mathbf{B} \mathbf{S}_{A2}, \ldots, \mathbf{B} \mathbf{S}_{Ak} \right] = \left[ \mathbf{S}_{A1} \mathbf{T}_1, \mathbf{S}_{A2} \mathbf{T}_2, \ldots, \mathbf{S}_{Ak} \mathbf{T}_k \right]
\]

\[
= \mathbf{S}_A \mathbf{T}, \quad \mathbf{T} = \text{diag} [\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_k] = \mathbf{S}_A^{-1} \mathbf{B} \mathbf{S}_A.
\]

(9)

Since \( \mathbf{B} \) is diagonalizable, each \( \mathbf{T}_i \) is diagonalizable and its Jordan form reads

\[
\mathbf{T}_i = \mathbf{S}_{Ti} \mathbf{D}_{Ti} \mathbf{S}_{Ti}^{-1},
\]

where we take \( \mathbf{S}_{Ti} = \mathbf{I}_i \) if \( \lambda_i = 0 \) in order to make \( \mathbf{S}_{Ti}^{-1} = \mathbf{I}_i \). Then \( \mathbf{T} \) has the Jordan form

\[
\mathbf{T} = \mathbf{S}_T \mathbf{D}_T \mathbf{S}_T^{-1},
\]

(11)

where

\[
\mathbf{S}_T = \text{diag} [\mathbf{S}_{T1}, \mathbf{S}_{T2}, \ldots, \mathbf{S}_{Tk}], \quad \mathbf{D}_T = \text{diag} [\mathbf{D}_{T1}, \mathbf{D}_{T2}, \ldots, \mathbf{D}_{Tk}].
\]

(12)

From (9) and (11) we have

\[
\mathbf{B} = \mathbf{S}_A \mathbf{T} \mathbf{S}_A^{-1} = (\mathbf{S}_A \mathbf{S}_T) \mathbf{D}_T (\mathbf{S}_A \mathbf{S}_T)^{-1}
\]

(13)

which is a Jordan form for \( \mathbf{B} \). It follows that \( \mathbf{S}_A \mathbf{S}_T \) is an eigenvector matrix for \( \mathbf{B} \), and \( \mathbf{D}_T \) must contain the same eigenvalues as \( \mathbf{D}_B \), but they may be in a different order as shown by the examples below.

From (2) we form

\[
\mathbf{A} = (\mathbf{S}_A \mathbf{S}_T) (\mathbf{S}_T^{-1} \mathbf{D}_A \mathbf{S}_T) (\mathbf{S}_A \mathbf{S}_T)^{-1}.
\]

(14)

By (12) we find that

\[
\mathbf{S}_T^{-1} \mathbf{D}_A \mathbf{S}_T = \text{diag} [\mathbf{S}_{T1}^{-1}, \mathbf{S}_{T2}^{-1}, \ldots, \mathbf{S}_{Tk}^{-1}] \text{diag} [\mathbf{D}_{A1}, \mathbf{D}_{A2}, \ldots, \mathbf{D}_{Ak}]
\]

\[
\times \text{diag} [\mathbf{S}_{T1}, \mathbf{S}_{T2}, \ldots, \mathbf{S}_{Tk}]
\]

\[
= \text{diag} [\mathbf{S}_{T1}^{-1} \lambda_1 \mathbf{I}_1 \mathbf{S}_{T1}, \mathbf{S}_{T2}^{-1} \lambda_2 \mathbf{I}_2 \mathbf{S}_{T2}, \ldots, \mathbf{S}_{Tk}^{-1} \lambda_k \mathbf{I}_k \mathbf{S}_{Tk}]
\]

\[
= \text{diag} [\lambda_1 \mathbf{I}_1, \lambda_2 \mathbf{I}_2, \ldots, \lambda_k \mathbf{I}_k]
\]

\[
= \text{diag} [\mathbf{D}_{A1}, \mathbf{D}_{A2}, \ldots, \mathbf{D}_{Ak}]
\]

(15)

where

\[
\mathbf{D}_A = \mathbf{S}_A \mathbf{D}_B \mathbf{S}_A^{-1}
\]
and (14) becomes
\[ A = (S_A S_T) D_A (S_A S_T)^{-1} \]  
which shows that \( S_A S_T \) is an eigenvector matrix for \( A \) as well as \( B \). Thus, \( A \) and \( B \) can be simultaneously diagonalized by \( S_A S_T \), i.e.
\[ \text{D}_A = (S_A S_T)^{-1} A (S_A S_T), \quad \text{D}_B = (S_A S_T)^{-1} B (S_A S_T). \]  

We note that the role of \( A \) and \( B \) can be interchanged in the above construction process. However, this results in essentially the same common eigenvalue matrix as \( S_A S_T \). To see this, we rewrite (17) and (13) as
\[ A = (S_A S_{TA}) D_A (S_A S_{TA})^{-1}, \quad B = (S_A S_T) D_T (S_A S_T)^{-1}, \]  
where, as noted above, \( D_T \) contains the same eigenvalues as the original \( D_B \) but in a different position on the diagonal. Thus
\[ D_T = P_B D_B P_B^T, \quad B = (S_A S_{TA} P_B) D_B (S_A S_{TA} P_B)^{-1}, \]  
where \( P_B \) is a permutation matrix. Similarly the construction starting with \( B \) results in
\[ D_A = P_A D_A P_A^T, \quad A = (S_B S_{TB} P_A) D_A (S_B S_{TB} P_A)^{-1}, \quad B = (S_B S_{TB}) D_B (S_B S_{TB})^{-1}. \]

On comparing these results, we see that
\[ S_B S_{TB} = S_A S_{TA} P_B, \quad S_A S_{TA} = S_B S_{TB} P_A, \quad P_A = P_B^T, \]  
i.e. \( S_B S_{TB} \) is a reordering of the eigenvectors (columns) of \( S_A S_{TA} \) according to the reordering of the eigenvalues in \( D_T \) via \( P_B \) as an example below illustrates.

Furthermore, if \( A \) has distinct eigenvalues, by (9) and (5) (11) we have
\[ T = S_A^{-1} B S_A = S_A^{-1} S_A D_B S_A^{-1} S_A = \text{ID}_T I = S_T D_T S_T^{-1}, \]  
\[ \therefore S_T = I, \quad S_A S_T = S_A, \]  
so nothing is gained from the construction of \( S_A S_T \) in this case.

3 Singular Value Decomposition

The construction of matrices in the singular value decompositions (SVD) of two commuting matrices also is of interest. The SVD of \( A \) and \( B \) read
\[ A = U_A \Sigma_A V_A^*, \quad B = U_B \Sigma_B V_B^*, \]  
where \( U_A, V_A, U_B, \) and \( V_B \) are unitary matrices, whereas \( \Sigma_A \) and \( \Sigma_B \) are diagonal matrices with non-negative real numbers (called singular values) on the diagonal. It follows from (23) that
\[ AA^* = U_A \Sigma_A^2 U_A^*, \quad BB^* = U_B \Sigma_B^2 U_B^* \]  

3
which are Jordan forms of $AA^*$ and $BB^*$. These Jordan forms enable the determination of $U_A$, $U_B$, $\Sigma_A$, and $\Sigma_B$ after which $V_A$ and $V_B$ can be determined from (23). If $\Sigma_A$ and $\Sigma_B$ are nonsingular, then (23) leads to

$$V_A = A^* U_A \Sigma_A^{-1}, \quad V_B = B^* U_B \Sigma_B^{-1}. \quad (25)$$

and if they are singular, an alternate approach is given by Meyer [2].

If $A$ and $B$ commute and if $A^*B = BA^*$, then (26)

$$AA^*BB^* = ABA^*B^* = BAB^*A^* = BB^*AA^*, \quad (27)$$

i.e. $AA^*$ and $BB^*$ commute. Thus, they have a common left-singular vector matrix $U_A = U_B$ which can be found by the foregoing eigenvector construction procedure. If $A$ and $B$ also are normal with common unitary matrix $S$, then $U_A = U_B = S$. Note that (27) is satisfied if $A$ is real and symmetric. The construction of SVD matrices is illustrated in the examples below.

4 Permutation

We can form two new matrices by row/column permutations of the restricted form

$$\hat{A} = PAP^T, \quad \hat{B} = PBP^T, \quad (28)$$

where $P$ is a permutation matrix. Since $P$ is orthogonal \(PP^T = I\), it follows that $\hat{A}$ and $\hat{B}$ commute when $A$ and $B$ commute as seen from

$$\hat{A}\hat{B} = PAP^TPBP^T = PABP^T = PBP^TPAP^T = \hat{B}\hat{A}. \quad (29)$$

Furthermore, the Jordan forms of $\hat{A}$ and $\hat{B}$, with (2), read

$$\hat{A} = \hat{S}_A D_A \hat{S}_A^{-1}, \quad \hat{B} = \hat{S}_B D_B \hat{S}_B^{-1},$$

$$\hat{S}_A = PS_A, \quad \hat{S}_B = PS_B \quad (30)$$

which show that the eigenvalues are unchanged by the permutation (28) and the eigenvectors are permuted. Similar formulas apply to the SVD’s of $\hat{A}$ and $\hat{B}$ and their singular values also are unchanged by the permutation (28). Indeed, it is known that the singular values of any matrix $A$ are invariant under row/column permutations of the more general form

$$\hat{A} = PAQ, \quad (31)$$

where $Q$ is a second permutation matrix. Next, we present three examples to illustrate our theoretical results.

5 Examples

Example 1. We start with the normal matrix $A$ and the magic square matrix $B$ given by

$$A = \begin{bmatrix} 1 + i & 1 & 1 \\ 1 & 1 + i & 1 \\ 1 & 1 & 1 + i \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 0 & 5 \\ 2 & 4 & 6 \\ 3 & 8 & 1 \end{bmatrix} \quad (32)$$
which commute and whose Jordan-form matrices are

\[ S_A = \frac{1}{6} \begin{bmatrix} 2\sqrt{3} & 3\sqrt{3} & \sqrt{6} \\ 2\sqrt{3} & 0 & -2\sqrt{6} \\ 2\sqrt{3} & -3\sqrt{2} & \sqrt{6} \end{bmatrix}, \quad S_B = \begin{bmatrix} 1 & 5 & 1 \\ 1 & 2 \left(1 + \sqrt{6}\right) & 2 \left(1 - \sqrt{6}\right) \\ 1 & -\left(7 + 2\sqrt{6}\right) & -\left(7 - 2\sqrt{6}\right) \end{bmatrix}, \]

\[ D_A = \text{diag} \left[ 3 + i, i, i \right], \quad D_B = \text{diag} \left[ 12, -2\sqrt{6}, 2\sqrt{6} \right], \]

where \( S_A \) is orthogonal. Since \( B \) has distinct eigenvalues, \( S_B \) is a (nonorthogonal) eigenvector matrix for \( A \) as well as \( B \). However, \( S_A \) is not an eigenvector matrix for \( B \) since \( A \) has multiple eigenvalues.

In the SVD matrices for \( A \) and \( B \), noting that (26) is satisfied, we find that \( U_A = U_B = S_A \) and

\[ \Sigma_A = \text{diag} \left[ \sqrt{10}, 1, 1 \right], \quad V_A = \frac{1}{30} \begin{bmatrix} \sqrt{30} \left(3 - i\right) & -15i\sqrt{2} & -5i\sqrt{6} \\ \sqrt{30} \left(3 - i\right) & 0 & 10i\sqrt{6} \\ \sqrt{30} \left(3 - i\right) & 15i\sqrt{2} & -5i\sqrt{6} \end{bmatrix}, \]

\[ \Sigma_B = \text{diag} \left[ 12, 4\sqrt{3}, 2\sqrt{3} \right], \quad V_B = \frac{1}{6} \begin{bmatrix} 2\sqrt{3} & \sqrt{6} & 3\sqrt{2} \\ 2\sqrt{3} & -2\sqrt{6} & 0 \\ 2\sqrt{3} & \sqrt{6} & -3\sqrt{2} \end{bmatrix}. \]

Example 2. The example given in the video by Sadun [3] has the commuting matrices

\[ A = \begin{bmatrix} 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \]

whose Jordan-form matrices are

\[ S_A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad S_B = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \]

\[ D_A = \text{diag} \left[ 2, 2, -2, -2 \right], \quad D_B = \text{diag} \left[ 1, 1, -1, -1 \right], \]

where \( S_B \) is orthogonal.

On following the matrix construction procedure for the common eigenvector matrix from \( A \), we find that

\[ T_A = S_A^{-1} B S_A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S_{TA} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \]

\[ S_A S_{TA} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & -1 & -1 \\ -2 & 2 & -2 & 2 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \quad D_A = \text{diag} \left[ 2, 2, -2, -2 \right], \quad D_B = \text{diag} \left[ 1, 1, -1, -1 \right]. \]

It can be verified that \( S_A S_{TA} \) is an eigenvalue matrix for both \( A \) and \( B \). Note that the eigenvalues in \( D_B \) are in a different order than those in \( D_B \) and they are related
by (19) with $D_T \equiv \hat{D}_B$ and

$$P_B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (38)$$

On following the matrix construction procedure for the common eigenvector matrix from $B$ (instead of $A$) we find that

$$T_B = S_B^{-1}AS_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \quad S_{TB} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix},$$

$$S_B S_{TB} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ -1 & 1 & -1 & 1 \\ 2 & 2 & -2 & -2 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \quad \tilde{D}_A = \text{diag } [-2, 2, -2, 2], \quad D_B = \text{diag } [1, 1, -1, -1], \quad (39)$$

and (21) can be verified.

Noting that $AA^*$ and $BB^*$ commute since $B$ is real and symmetric, the SVD matrices for $A$ and $B$ are

$$U_A = U_B = I, \quad V_A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Sigma_A = \text{diag } [4, 1, 4, 1], \quad \Sigma_B = I, \quad V_B = B. \quad (40)$$

An alternate set of SVD matrices for $A$ and $B$ is

$$U_A = U_B = S_B, \quad \Sigma_A = \text{diag } [1, 4, 1, 4], \quad \Sigma_B = I, \quad V_A = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad V_B = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad (41)$$

where $V_A$ is determined from (25) and $V_B$ is formed by changing the sign of the two eigenvectors (last two columns) in $S_B$ associated with its negative eigenvalues ($-1, -1$) in (36).

Example 3. Following a method given by Nordgren [3], we construct the symmetric commuting matrices

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 3 & 0 & 0 & 7 & 0 \\ 2 & 0 & 1 & 4 & 0 & 3 \\ 3 & 0 & 4 & 1 & 0 & 2 \\ 0 & 7 & 0 & 0 & 3 & 0 \\ 4 & 0 & 3 & 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} i & i & i & 1 & 1 & 1 \\ i & i & i & 1 & 1 & 1 \\ i & i & i & 1 & 1 & 1 \\ 1 & 1 & 1 & i & i \end{bmatrix} \quad \begin{bmatrix} i & i & i & 1 & 1 & 1 \\ 1 & 1 & 1 & i \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & i \end{bmatrix}, \quad (42)$$
with orthogonal Jordan-form matrices

\[
S_A = \frac{1}{4} \begin{bmatrix}
-2 & \sqrt{2} & 0 & 2 \\
0 & 2 & 0 & -2 \\
0 & -2 & 0 & 2 \\
-2 & -\sqrt{2} & 0 & 2 \\
\end{bmatrix},
\]

\[
D_A = \text{diag}[0, -4, -4, -2, 10],
\]

\[
S_B = \frac{\sqrt{6}}{12} \begin{bmatrix}
2 & 2 & 2 & 2 \\
-\sqrt{3} - 1 & \sqrt{3} - 1 & 2 & -\sqrt{3} - 1 \\
2 & \sqrt{3} - 1 & -\sqrt{3} - 1 & -\sqrt{3} - 1 \\
-2 & -2 & 2 & 2 \\
-2 & \sqrt{3} + 1 & -\sqrt{3} + 1 & 2 \\
-2 & -\sqrt{3} + 1 & \sqrt{3} + 1 & 2 \\
\end{bmatrix},
\]

\[
D_B = \text{diag}[-3 + 3i, 0, 0, 3 + 3i, 0, 0].
\]

The matrix construction procedure leads to the following common eigenvector matrix and corresponding eigenvalues:

\[
S_AS_{TA} = \frac{\sqrt{3}}{6} \begin{bmatrix}
-\sqrt{3} & \sqrt{2} & 1 & -\sqrt{3} & \sqrt{2} & 1 \\
0 & \sqrt{2} & -2 & 0 & \sqrt{2} & -2 \\
\sqrt{3} & \sqrt{2} & 1 & \sqrt{3} & \sqrt{2} & 1 \\
\sqrt{3} & -\sqrt{2} & -1 & -\sqrt{3} & \sqrt{2} & 1 \\
0 & -\sqrt{2} & 2 & 0 & \sqrt{2} & -2 \\
-\sqrt{3} & -\sqrt{2} & -1 & \sqrt{3} & \sqrt{2} & 1 \\
\end{bmatrix},
\]

\[
D_A = \text{diag}[-2, -2, -2, 0, 8, 8], \quad D_B = \text{diag}[0, -3 + 3i, 0, 0, 3 + 3i, 0].
\]

Since \( A \) and \( B \) are symmetric and \( S_AS_{TA} \) is orthogonal, suitable SVD matrices are

\[
U_A = U_B = S_AS_{TA},
\]

\[
V_A = \frac{\sqrt{3}}{6} \begin{bmatrix}
\sqrt{3} & -\sqrt{2} & -1 & \sqrt{3} & \sqrt{2} & 1 \\
0 & -\sqrt{2} & 2 & 0 & \sqrt{2} & -2 \\
-\sqrt{3} & -\sqrt{2} & -1 & -\sqrt{3} & \sqrt{2} & 1 \\
-\sqrt{3} & \sqrt{2} & 1 & \sqrt{3} & \sqrt{2} & 1 \\
0 & \sqrt{2} & -2 & 0 & \sqrt{2} & -2 \\
\sqrt{3} & \sqrt{2} & 1 & -\sqrt{3} & \sqrt{2} & 1 \\
\end{bmatrix},
\]

\[
V_B = \frac{\sqrt{3}}{6} \begin{bmatrix}
-\sqrt{3} & -1 - i & 1 & -\sqrt{3} & 1 - i & 1 \\
0 & -1 - i & 2 & 0 & 1 - i & -2 \\
\sqrt{3} & -1 - i & 1 & \sqrt{3} & 1 - i & 1 \\
\sqrt{3} & 1 + i & -1 & -\sqrt{3} & 1 - i & 1 \\
0 & 1 + i & 2 & 0 & 1 - i & -2 \\
-\sqrt{3} & 1 + i & -1 & \sqrt{3} & 1 - i & 1 \\
\end{bmatrix},
\]

\[
\Sigma_A = \text{diag}[0, 4, 4, 2, 10], \quad \Sigma_B = \text{diag}[0, 3\sqrt{2}, 0, 0, 3\sqrt{2}, 0],
\]

where \( V_A \) is formed by changing the sign of the first three columns of \( S_AS_{TA} \) associated with the negative eigenvalues in \( D_A \).
To illustrate row/column permutation, let

\[ P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \] (48)

By (28), we have

\[ \hat{A} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 3 & 4 & 2 & 0 \\ 0 & 3 & 1 & 2 & 4 & 0 \\ 0 & 4 & 2 & 1 & 3 & 0 \\ 0 & 2 & 4 & 3 & 1 & 0 \\ 7 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} i & i & i & i & 1 & 1 \\ 1 & i & 1 & i & i & i \\ 1 & i & i & i & 1 & 1 \\ 1 & i & i & i & 1 & 1 \\ i & i & 1 & 1 & i & i \\ 1 & i & 1 & i & i & i \end{bmatrix}. \] (49)

By (28), we have

It can be verified that \( \hat{A} \) and \( \hat{B} \) commute and that their eigenvalues and singular values are the same as those of \( A \) and \( B \).

6 Conclusion

The three examples illustrate the efficacy of the presented matrix construction procedure for the common eigenvectors of two commuting matrices, thereby enabling their simultaneous diagonalization. The construction also is useful in finding the SVD of matrices when at least one of them is real and symmetric. A restricted row/column permutation of two commuting matrices produces two computing matrices with unchanged eigenvalues and singular values.

References

[1] R. A. Horn and C. R. Johnson, Matrix Analysis, 2nd Edition, Cambridge University Press, Cambridge (2013).
[2] C. D. Meyer, Matrix Analysis and Applied Linear Algebra. Society for Industrial and Applied Mathematics (2000).
[3] R. P. Nordgren, Compounding Commuting Matrices, Journal of Advances in Mathematics and Computer Science, 30 (2019), 1-8.
[4] L. Sadun, Simultaneous diagonalization II: The general case, https://www.youtube.com/watch?v=4qtV566HE.