SUFFICIENCY CLASS FOR GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS

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Abstract. Let $\Omega$ be an open domain of class $C^2$ contained in $\mathbb{R}^3$, let $L^2(\Omega)^3$ be the Hilbert space of square integrable functions on $\Omega$ and let $H[\Omega] := H$ be the completion of the set, $\{ u \in (C_0^\infty(\Omega))^3 \mid \nabla \cdot u = 0 \}$, with respect to the inner product of $L^2(\Omega)^3$. A well-known unsolved problem is the construction of a sufficient class of functions in $H$ which will allow global, in time, strong solutions to the three-dimensional Navier-Stokes equations. These equations describe the time evolution of the fluid velocity and pressure of an incompressible viscous homogeneous Newtonian fluid in terms of a given initial velocity and given external body forces. In this paper, we use the analytic nature of the Stokes semigroup to construct an equivalent norm for $H$, which provides strong bounds on the nonlinear term. This allows us to prove that, under appropriate conditions, there exists a number $u_+$, depending only on the domain, the viscosity, the body forces and the eigenvalues of the Stokes operator, such that, for all functions in a dense set $D$ contained in the closed ball $B(\Omega) =: B$ of radius $\frac{1}{2}u_+$ in $H$, the Navier-Stokes equations have unique, strong, solutions in $C^1((0, \infty), H)$.

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Introduction

Let \( \Omega \) be an open domain of class \( \mathbb{C}^k \) contained in \( \mathbb{R}^n \), \( n \geq 2 \), let \((L^2(\Omega))^n\) be the Hilbert space of square integrable functions on \( \Omega \) with values in \( \mathbb{C}^n \), let \( D[\Omega] \) be \( \{ u \in (C_0^\infty[\Omega])^n \mid \nabla \cdot u = 0 \} \), let \( \mathbb{H} \) be the completion of \( D[\Omega] \) with respect to the inner product of \((L^2(\Omega))^n\), and let \( \mathbb{V}[\Omega] \) be the completion of \( D[\Omega] \) with respect to the inner product of \( \mathbb{H}^1[\Omega] \), the functions in \( \mathbb{H} \) with weak derivatives in \((L^2(\Omega))^n\).

The global in time classical Navier-Stokes initial-value problem (for \( \Omega \subset \mathbb{R}^n \), and all \( T > 0 \)) is to find functions \( u : [0, T] \times \Omega \rightarrow \mathbb{R}^n \), and \( p : [0, T] \times \Omega \rightarrow \mathbb{R} \), such that

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= f(t) \text{ in } (0, T) \times \Omega, \\
\nabla \cdot u &= 0 \text{ in } (0, T) \times \Omega, \\
u(t, x) &= 0 \text{ on } (0, T) \times \partial \Omega, \\
u(0, x) &= u_0(x) \text{ in } \Omega.
\end{align*}
\]

(1)

The equations describe the time evolution of the fluid velocity \( u(x, t) \) and the pressure \( p \) of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient \( \nu \) in terms of a given initial velocity \( u_0(x) \) and given external body forces \( f(x, t) \).

The existence of global weak solutions of (1) was proved by Leray [Le] in 1934, for \( \Omega = \mathbb{R}^3 \) and later, in 1951, Hopf [Ho] solved the problem for a bounded open domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), with homogeneous Dirichlet conditions on smooth boundaries, \( \partial \Omega \). These results were subsequently extended to include functions \( f(x, t) \in L^2([0, T); \mathbb{V}[\Omega]^{-1}] \), where \( \mathbb{V}[\Omega]^{-1} = \mathbb{V}[\Omega]^* \) is the dual of \( \mathbb{V}[\Omega] \) (see [Li, T1, vW]). In 1962, Kato and Fujita [KF] proved the existence of strong,
global in time, smooth three-dimensional solutions, provided that the body forces are small (in an appropriate sense) and the initial data is small in the Sobolev space $H^{1/2}[\Omega]$ (see also [CH] and Temam [T1] pages 205-208). (As noted by Temam [T1, see page 344-345], the importance of their work is that it points out the dependence of the existence of global solutions on the size of the initial data, the body forces and possibly the spectral properties of these quantities.) In another (related) direction, Raugel and Sell showed that one can get stronger results for thin 3D domains (see [RS]). In this case, they show that the Navier-Stokes equations have strong solutions and that the long-time dynamics has a global attractor.

For $n = 3$, let $P$ be the (Leray) orthogonal projection of $(L^2[\Omega])^3$ onto $H$ and define the Stokes operator by: $Au := -P\Delta u$, for $u \in D(A) \subset H^2[\Omega] \cap H^1_0[\Omega]$, the domain of $A$. The purpose of this paper is to prove that there exists a number $u_+$, depending only on $A$, $f$, $\nu$ and $\Omega$, such that, for all functions in $\mathcal{D} = D(A) \cap \mathbb{B}$, where $D(A)$ is the domain of $A$ and $\mathbb{B}$ is the closed ball of radius $\frac{1}{2}u_+$, in $H$, the Navier-Stokes equations have unique, strong, solutions in $u \in L^\infty_{\text{loc}}([0, \infty); \mathbb{V}(\Omega)) \cap C^1([0, \infty); \mathbb{H})$.

**Preliminaries**

Applying the Leray projection to equation (1), with $B(u, u) = P(u \cdot \nabla)u$, we can recast equation (1) in the standard form:

$$\partial_t u = -\nu Au - B(u, u) + Pf(t) \text{ in } (0, T) \times \Omega,$$

$$u(t, x) = 0 \text{ on } (0, T) \times \partial \Omega, \quad u(0, x) = u_0(x) \text{ in } \Omega,$$

(2)
where we have used the fact that the orthogonal complement of $H$ relative to $(L^2[\Omega])^3$ is \( \{ v : v = \nabla q, \ q \in (H^1[\Omega])^3 \} \) to eliminate the pressure term (see Galdi [GA] or [SY, T1, T2]).

**Definition 1.** We say that the operator $J(\cdot, t)$ is (for each $t$)

1. 0-Dissipative if $\langle J(u, t), u \rangle_H \leq 0$.
2. Dissipative if $\langle J(u, t) - J(v, t), u - v \rangle_H \leq 0$.
3. Strongly dissipative if there exists an $\alpha > 0$ such that

$$\langle J(u, t) - J(v, t), u - v \rangle_H \leq -\alpha \| u - v \|_H^2.$$

Note that, if $J(\cdot, t)$ is a linear operator, definitions (1) and (2) coincide. Theorem 2 below is essentially due to Browder [B], see Zeidler [Z, Corollary 32.27, page 868 and Corollary 32.35 page 887, in Vol. IIB], while Theorem 3 is from Miyadera [M, p. 185, Theorem 6.20], and is a modification of the Crandall-Liggett Theorem [CL] (see the appendix to the first section of [CL]).

**Theorem 2.** Let $B$ be a closed, bounded, convex subset of $H$. If $J(\cdot, t) : B[\Omega] \rightarrow H$ is strongly dissipative for each fixed $t \geq 0$, then for each $b \in B$, there is a $u \in B$ with $J(u, t) = b$ (i.e., the range, $\text{Ran}[J(\cdot, t)] \supset B$).

**Theorem 3.** Let $\{ A(t), t \in I = [0, \infty) \}$ be a family of operators defined on $H$ with domains $D(A(t)) = D$, independent of $t$. (We assume that the closure of $D = D(A) \cap B$, in the $H$ norm equals $B$):

1. The operator $A(t)$ is the generator of a contraction semigroup for each $t \in I$.
2. The function $A(t)u$ is continuous in both variables on $I \times D$. 
Then, for every \( u_0 \in D \), the problem
\[
\partial_t u(t, x) = A(t) u(t, x), \quad u(0, x) = u_0(x),
\]
has a unique solution \( u(t, x) \in C^1(I; \mathbb{H}) \).

**Stokes Equation.** The difficulty in proving the existence of global-in-time strong solutions for equation (2) can be directly linked to the problem of getting good estimates for the nonlinear term \( B(u, u) \). For example, the following theorem is one of the major estimates used to study this equation (see equation 61.22 on page 366, in Sell and You [SY] and Constantin and Foias [CF]). (We assume that \( u, v \in D(A) \).)

**Theorem 4.** Let \( \Omega \) be a bounded open set of class \( C^k \) in \( \mathbb{R}^3 \). Let \( \alpha_i, 1 \leq i \leq 3 \), satisfy
\[
0 \leq \alpha_1 \leq k, \quad 0 \leq \alpha_2 \leq k - 1, \quad 0 \leq \alpha_3 \leq k, \quad \text{with} \quad \alpha_1 + \alpha_2 + \alpha_3 \geq 3/2 \quad \text{and}
\]
\[
(\alpha_1, \alpha_2, \alpha_3) \notin \{(3/2, 0, 0), (0, 3/2, 0), (0, 0, 3/2)\}.
\]

Then there is a positive constant \( c = c(\alpha_i, \Omega) \) such that

\[
|\langle B(u, v), w \rangle_{\mathbb{H}}| \leq c \left\| A^{\alpha_1/2} u \right\|_{\mathbb{H}} \left\| A^{(1+\alpha_2)/2} v \right\|_{\mathbb{H}} \left\| A^{\alpha_3/2} w \right\|_{\mathbb{H}}.
\]

We plan to show that, by renorming \( \mathbb{H} \), we can prove a very strong inequality for equation (3). First we need to investigate the Stokes equation.

If we drop the nonlinear term, we get the well-known Stoke’s equation (\( Pf(t) = 0 \)):

\[
\partial_t u = -\nu A u \quad \text{in} \quad (0, T) \times \Omega,
\]

\[
u u(t, x) = 0 \quad \text{on} \quad (0, T) \times \partial \Omega,
\]

\[
u u(0, x) = u_0(x) \quad \text{in} \quad \Omega.
\]
A proof of the next theorem holds and may be found in Sell and You [SY] (page 114):

**Theorem 5.** Let $\Omega$ be a open bounded domain of class $C^2$ in $\mathbb{R}^3$, and let $A$ be the Stokes operator on $\Omega$. Then the following holds:

1. The operator $A$ is a positive selfadjoint generator of a contraction semigroup $S(t)$.
2. The inverse of $A$, $A^{-1}$, is a compact linear operator from $H$ onto $D(A)$.
3. The operator $A$ is sectorial and there exist eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_n \to \infty$, as $n \to \infty$ and, $\|S(t)\| \leq e^{-\lambda_1 t}$.
4. The eigenfunctions, $\{e_1, e_2, \cdots\}$, of $A$ form an orthonormal basis for $H$.

**Equivalent Norms.**

**Example 6.** In order to see how we can use the analytic properties of a semigroup to bound the generator, let $\mathcal{H} = L^2[\mathbb{R}^3, d\mu]$, where $d\mu = (2\pi)^{-3/2}e^{-\frac{1}{2}|x|^2}d\mathbf{x}$, and consider the problem:

$$\frac{\partial}{\partial t}u(t, \mathbf{x}) = \Delta u(t, \mathbf{x}) - \mathbf{x} \cdot \nabla u(t, \mathbf{x}), \quad u(0, \mathbf{x}) = u_0(\mathbf{x}).$$

This is the well-known Ornstein-Uhlenbeck equation with solution $(T(t)u_0)(\mathbf{x}) = u(t, \mathbf{x})$, where:

$$(T(t)u_0)(\mathbf{x}) = \frac{1}{\sqrt{[2\pi(1-e^{-t})]^3}} \int_{\mathbb{R}^3} \exp \left\{ -\frac{(e^{-t/2}\mathbf{x} - \mathbf{y})^2}{2(1-e^{-t})} \right\} u_0(\mathbf{y})d\mathbf{y}.$$
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analytic semigroup, \( \|S(t)u_0\|_2 \leq \|u_0\|_2 \). It follows that there is a positive constant \( M \) such that for any fixed \( t \)

\[ \|S(t)u_0\|_2 \leq \|u_0\|_2 \leq M \|S(t)u_0\|_2. \]

Thus, if we set \( \|w\|_{2,\gamma} = \|S(\gamma)w\|_2 \leq \|w\|_2 \) for any \( \gamma \in (0,1) \), we obtain an equivalent norm on \( \mathcal{H} \). Moreover, since \( S(t) \) is analytic, if \( w \) is in the domain of \( D^2 \), there is a constant \( c > 0 \) such that

\[ \|D^2w\|_{2,\gamma} = e^{\omega\gamma} \|D^2T(\gamma)w\|_2 \leq \|w\|_2 \leq \frac{Mc}{\gamma} \|w\|_{2,\gamma}. \]

We thus conclude that this equivalent norm on \( \mathcal{H} \) makes \( D^2 \) bounded (on its domain) without using the graph norm. Note that \( \Delta = D^2 + x \cdot \nabla \). From here, it is easy to see that the equivalent norm also makes \( \Delta \) and \( \nabla \) bounded in the above sense. In this latter case, we obtain the reverse of the Poincaré inequality.

In our case, we let \( T(t) = \exp\{-tA\} \) be the analytic semigroup generated by the Stokes operator \( A \), with \( \|T(t)u\|_{\mathcal{H}} \leq e^{-\omega t} \|u\|_{\mathcal{H}} \). Let \( S(t) = e^{\omega T(t)} \) and choose \( M \) as in our example, so that \( \|u\|_{\mathcal{H},1} = \|S(r)u\|_{\mathcal{H}} \) is an equivalent norm, where \( r \) is to be determined. Since \( A \) is analytic, there is a constant \( c_2 \) such that, for \( u \in D(A^2) \),

\[ \|A^2u\|_{\mathcal{H},1} = e^{\omega r} \|A^2T(r)u\|_{\mathcal{H}} \leq \frac{Mc}{(r^2)} \|u\|_{\mathcal{H}} \leq \frac{Mc}{(r^2)} \|u\|_{\mathcal{H},1}. \]

Since the norms are equivalent, we also have (for \( u \in D(A^2) \)):

\[ \|A^2u\|_{\mathcal{H}} \leq M \|A^2u\|_{\mathcal{H},1} = e^{\omega r} \|A^2T(r)u\|_{\mathcal{H}} \leq \frac{Mc}{(r^2)} \|u\|_{\mathcal{H}}. \]

From Theorem 4, we have the following result:

**Theorem 7.** Let \( u \in D(A) \), set \( S = S(r) \) and renorm \( \mathcal{H} \) so that \( \|u\|_{\mathcal{H},1} = \|Su\|_{\mathcal{H}} \). Then, with \( k = 2 \):
(1) If we let $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = 1/2$, there are positive constants $c = c(\alpha_i, \Omega)$ and $c_1$ such that

\[ |\langle A^{-1}B(u,v), w\rangle_{H,1}| \leq \frac{M^3 cc_1}{p^{1/4}} \|u\|_{H,1} \|w\|_{H,1} \|v\|_{H,1}. \]

(2) If we let $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = 1/2$, there are positive constants $c = c(\alpha_i, \Omega) c_1$ and $c_2$ such that

\[ |\langle B(u,v), w\rangle_{H,1}| \leq \frac{M^4 cc_1 c_2}{p^{5/4}} \|u\|_{H,1} \|v\|_{H,1} \|w\|_{H,1}. \]

Proof. To prove (4), first note that $A$ and $S$ commute on $D(A)$ so, if we set $Sw = w_1$, we have:

\[ b(A^{-1}u, v, w)_{H,1} = \langle A^{-1}SB(u,v), Sw\rangle_{H} = b(u, v, SA^{-1}w_1)_{H}. \]

Using the selfadjoint property of $A$, and integration by parts, we have

\[ b(u, v, SA^{-1}w_1)_{H} = -b(u, SA^{-1}w_1, v). \]

It now follows from Theorem 4 that:

\[ |\langle A^{-1}B(u,v), w\rangle_{H,1}| \leq c \|A^{\alpha_1/2}u\|_{H} \|SA^{(1+\alpha_2)/2}A^{-1}w_1\|_{H} \|A^{\alpha_3/2}v\|_{H}. \]

If we set $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = 1/2$ we have

\[ |\langle A^{-1}B(u,v), w\rangle_{H,1}| \leq c \|u\|_{H} \|AA^{-1}w_1\|_{H} \|A^{1/4}v\|_{H} \]

\[ \leq \frac{M cc_1}{p^{1/4}} \|u\|_{H} \|w_1\|_{H} \|v\|_{H} \]

\[ \leq \frac{M^3 cc_1}{p^{1/4}} \|u\|_{H,1} \|w\|_{H,1} \|v\|_{H,1}. \]

To prove (5), as before, set $Sw = w_1$, to obtain:

\[ b(u, v, w)_{H,1} = \langle SB(u,v), Sw\rangle_{H} = b(u, v, Sw_1)_{H}. \]
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As before, using the selfadjoint property of $A$, and integration by parts, we have

$$b(u, v, Sw_1)_H = -b(u, Sw_1, v)_H.$$  

It follows that:

$$|\langle B(u, v), w \rangle_{H,1}| \leq c \left\| A^{\alpha_1/2} u \right\|_H \left\| SA^{(1+\alpha_2)/2} w_1 \right\|_H \left\| A^{\alpha_3/2} v \right\|_H.$$  

(6)

Setting $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = 1/2$ we have:

$$\left\| \langle B(u, v), w \rangle_{H,1} \right\| \leq c \left\| A^{1/4} v \right\|_H \left\| Aw_1 \right\|_H \left\| u \right\|_H$$

$$\leq \frac{Mcc_1c_2}{r^{5/4}} \left\| u \right\|_H \left\| w \right\|_H \left\| v \right\|_H$$

$$\leq \frac{M^4cc_1c_2}{r^{5/4}} \left\| u \right\|_{H,1} \left\| w \right\|_{H,1} \left\| v \right\|_{H,1}.$$  

□

Corollary 8.

$$\max\{\left\| B(u, v) \right\|_{H,1} , \left\| B(v, u) \right\|_{H,1} \} \leq \frac{M^4cc_1c_2}{r^{5/4}} \left\| u \right\|_{H,1} \left\| v \right\|_{H,1}.$$  

M-Dissipative Conditions

In the remainder of the paper, we assume that $f(t) \in L^\infty([0, \infty); \mathbb{H}(\Omega))$ and is Hölder continuous in $t$, with $\|f(t) - f(\tau)\|_{H,1} \leq d|t - \tau|^\theta$, $d > 0$, $0 < \theta < 1$. We can now rewrite equation (2) in the form:

$$\partial_t u = \nu AJ(u, t) \text{ in } (0, T) \times \Omega,$$

(7)

$$J(u, t) = -u - \nu^{-1}A^{-1}B(u, u) + \nu^{-1}A^{-1}Pf(t).$$
We begin with a study of the operator $J(\cdot, t)$, for fixed $t$, and seek conditions depending on $A$, $\nu$, $\Omega$ and $f(t)$ which guarantee that $J(\cdot, t)$ is m-dissipative for each $t$. Clearly $J(\cdot, t) : D(A) \rightarrow D(A)$ and, since $\nu A = \nu P[-\Delta]$ is a closed positive (m-accretive) operator, so that $-A$ generates a linear contraction semigroup, we expect that $\nu A J(\cdot, t)$ will be m-dissipative for each $t$.

**Theorem 9.** For $t \in I = [0, \infty)$ and, for each fixed $u \in D(A)$, $J(u, t)$ is Hölder continuous, with $\|J(u, t) - J(u, \tau)\|_{H_{1,1}} \leq d'|t - \tau|^{\theta}$, where $d' = d\nu^{-1}(\lambda_1)^{-1}$, $d$ is the Hölder constant for the function $f(t)$ and $\lambda_1$ is the first eigenvalue of $A$.

**Proof.** For fixed $u \in D(A)$,

$$\|J(u, t) - J(u, \tau)\|_{H_{1,1}} = \nu^{-1}\|A^{-1}S[f(t) - f(\tau)]\|_{H_{1,1}} \leq d\nu^{-1}(\lambda_1)^{-1}|t - \tau|^{\theta}.$$ 

We have used the fact that every function $h(t) \in H(\Omega)$ has an expansion in terms of the eigenfunctions of $A$, so that $A^{-1} h(t) = \sum_{k=1}^{\infty} \lambda_k^{-1} h_k(t) e_k(x)$ and, from here, it is easy to see that $\|A^{-1} h(t)\|_{H_{1,1}} \leq \lambda_1^{-1} \| h(t)\|_{H_{1,1}}$. \square

**Main Results**

**Theorem 10.** Let $f = \sup_{t \in \mathbb{R}^+} \|Pf(t)\|_{H_{1,1}} < \infty$, then there exists a positive constant $u_+$, depending only on $f$, $A$, $\nu$ and $\Omega$, such that for all $u$, with $\|u\|_{H_{1,1}} \leq \frac{1}{2} u_+$, $J(\cdot, t)$ is strongly dissipative.

**Proof.** The proof of our first assertion has two parts. First, we require that the nonlinear operator $J(\cdot, t)$ be 0-dissipative, which gives us an upper bound $u_+$, in
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terms of the norm (i.e., $\|u\|_{H^1} \leq u_+$). We then use this part to show that $J(\cdot, t)$
is strongly dissipative on the closed ball, $\mathcal{B} = \{ u \in D(A) : \|u\|_{H^1} \leq \frac{1}{2} u_+ \}$.

Part 1) From equation (7), we get that

$$
\langle J(u, t), u \rangle_{H^1} = - \langle u, u \rangle_{H^1} + \nu^{-1} \langle -A^{-1} B(u, u) + A^{-1} P f(t), u \rangle_{H^1}
$$

$$
= - \|u\|_{H^1}^2 - \nu^{-1} \langle A^{-1} B(u, u), u \rangle_{H^1} + \nu^{-1} \langle A^{-1} P f(t), u \rangle_{H^1}
$$

$$
= - \|u\|_{H^1}^2 + \nu^{-1} \langle B(u, A^{-1} u_1), u \rangle_{H^1} + \nu^{-1} \langle A^{-1} P f(t), u \rangle_{H^1}
$$

It follows that

$$
\langle J(u, t), u \rangle_{H^1} \leq - \|u\|_{H^1}^2 + \nu^{-1} \|B(u, A^{-1} u_1)\|_{H^1} + \nu^{-1} f \|A^{-1} u\|_{H^1}
$$

$$
\leq - \|u\|_{H^1}^2 + \nu^{-1} \frac{M^3 cc_1}{r^{1/4}} \|u\|_{H^1}^3 + (\nu \lambda_1)^{-1} f \|u\|_{H^1}.
$$

In the last line, we used our estimate from Theorem 7. Thus, $J(\cdot, t)$ will be 0-
dissipative if

$$
- \|u\|_{H^1}^2 + \nu^{-1} \frac{M^3 cc_1}{r^{1/4}} \|u\|_{H^1}^3 + (\nu \lambda_1)^{-1} f \|u\|_{H^1} \leq 0,
$$

so that

$$
\|u\|_{H^1} \left[ \nu^{-1} \frac{M^3 cc_1}{r^{1/4}} \|u\|_{H^1}^2 - \|u\|_{H^1} + (\nu \lambda_1)^{-1} f \right] \leq 0.
$$

Since $\|u\|_{H^1} > 0$, we have that $J(\cdot, t)$ is 0-dissipative if ($\delta = \frac{\nu r^{1/4}}{M^3 cc_1}$)

$$(\delta)^{-1} \|u\|_{H^1}^2 - \|u\|_{H^1} + (\nu \lambda_1)^{-1} f \leq 0.$$

Solving, we get that

$$
u_{\pm} = \frac{1}{2} \delta \left( 1 \pm \sqrt{1 - 4f/(\delta \nu \lambda_1)} \right) = \frac{1}{2} \delta \left( 1 \pm \sqrt{1 - \gamma} \right),
$$

where $\gamma = \frac{4M^3 cc_1}{\nu^2 \lambda_1 r^{1/4}}$. Since we want real distinct solutions, we must require that

$$
\gamma < 1 \Rightarrow \nu > \left[ \frac{4M^3 cc_1}{r^{1/4} \lambda_1} \right]^{1/2}.
$$
It follows that, if $Pf \neq 0$, then $u_- < u_+$, and our requirement that $J$ is 0-dissipative implies that, since our solution factors as $(\|u\|_{H,1} - u_+) (\|u\|_{H,1} - u_-) \leq 0$, we must have that:

$$\|u\|_{H,1} - u_+ \leq 0, \quad \|u\|_{H,1} - u_- \geq 0.$$ 

This means that whenever $u_- \leq \|u\|_{H,1} \leq u_+$, $\langle J(u, t), u \rangle_{H,1} \leq 0$. (It is clear that when $Pf(t) = 0, u_- = 0$, and $u_+ = \delta$.)

Part 2): Now, for any $u, v \in D$ with $u - v \in D$ and $\max(\|u\|_{H,1}, \|v\|_{H,1}) \leq (1/2)u_+$, we have that

$$\langle J(u, t) - J(v, t), u - v \rangle_{H,1} = -\|(u - v)\|_{H,1}^2$$

$$- \nu^{-1} \langle A^{-1}[B(u, u - v) + B(u - v, v)], (u - v) \rangle_{H,1}$$

$$\leq -\|u - v\|_{H,1}^2 + \nu^{-1}[1/(\nu^{1/4})]M^3cc_1 \|u - v\|_{H,1}^2 \left(\|u\|_{H,1} + \|v\|_{H,1}\right)$$

$$\leq -\|u - v\|_{H,1}^2 + \nu^{-1}[1/(\nu^{1/4})]M^3cc_1 \|u - v\|_{H,1}^2 \cdot u_+$$

$$= -\|u - v\|_{H,1}^2 + \nu^{-1}[1/(\nu^{1/4})]M^3cc_1 \|u - v\|_{H,1}^2 \left[\frac{1}{2} \delta \left(1 + \sqrt{1 - \gamma}\right)\right]$$

$$= -\frac{1}{2} \|u - v\|_{H,1}^2 \left\{1 - \sqrt{1 - \gamma}\right\}$$

$$= -\alpha \|u - v\|_{H,1}^2, \quad \alpha = \frac{1}{2} \left\{1 - \sqrt{1 - \gamma}\right\}.$$ 

\[\Box\]

**Theorem 11.** The operator $A(t) = \nu AJ(\cdot, t)$ is closed, strongly dissipative and jointly continuous in $u$ and $t$. Furthermore, for each $t \in \mathbb{R}^+$ and $\omega > 0$, $\text{Ran}[I - \omega A(t)] \supset B$, so that $A(t)$ is m-dissipative on $\mathbb{D}$.

**Proof.** Since $J(\cdot, t)$ is strongly dissipative on $B[\Omega]$, it follows from Theorem 2 that $\text{Ran}[J(\cdot, t)] \supset B$. 

To show that $A(t) = \nu A J(\cdot, t)$ is strongly dissipative, for $u, v \in \mathbb{B}$, we have

$$\langle A(t)u - A(t)v, (u - v) \rangle_{\mathcal{H}, 1} = -\nu \left\| A^{1/2}(u - v) \right\|^2_{\mathcal{H}, 1} - \langle [B(u, u - v) + B(u - v, v)], (u - v) \rangle_{\mathcal{H}, 1}$$

Now, from equation (5) and Corollary 8,

$$\left| \langle B(u, u - v) + B(u - v, v), (u - v) \rangle_{\mathcal{H}, 1} \right| \leq \left\| M^4 c_1 c_2 \right\| \left\| u - v \right\|^2_{\mathcal{H}, 1} \left\{ \left\| u \right\|_{\mathcal{H}, 1} + \left\| v \right\|_{\mathcal{H}, 1} \right\}.$$

We now have that

$$\langle A(t)u - A(t)v, u - v \rangle_{\mathcal{H}, 1} \leq -\nu \left\| A^{1/2}(u - v) \right\|^2_{\mathcal{H}, 1} + \left\| M^4 c_1 c_2 \right\| \left\| u - v \right\|^2_{\mathcal{H}, 1} \left\{ \left\| u \right\|_{\mathcal{H}, 1} + \left\| v \right\|_{\mathcal{H}, 1} \right\}$$

$$\leq -\nu \lambda_1 \left\| u - v \right\|^2_{\mathcal{H}, 1} + \left\| M^4 c_1 c_2 \right\| \left\| u - v \right\|^2_{\mathcal{H}, 1} \left\{ \left\| u \right\|_{\mathcal{H}, 1} + \left\| v \right\|_{\mathcal{H}, 1} \right\}$$

$$\leq -\nu \lambda_1 \left\| u - v \right\|^2_{\mathcal{H}, 1} + \left\| M^4 c_1 c_2 \right\| \left\| u - v \right\|^2_{\mathcal{H}, 1} \left\{ -\nu \lambda_1 + \frac{1}{2} c_2 \left[ 1 + \sqrt{1 - \gamma} \right] \right\}.$$

Thus, if we set $r = \hat{r} = c_2/\lambda_1$, we can set $a = \nu \lambda_1 \left[ 1 - \sqrt{1 - \gamma} \right]$, so that

$$\langle A(t)u - A(t)v, u - v \rangle_{\mathcal{H}, 1} \leq -a \left\| (u - v) \right\|^2_{\mathcal{H}, 1}.$$

It follows that $A(t)$ is strongly dissipative. Since $-A$ is m-dissipative, for $\omega > 0$,

$\text{Ran}(I + \omega A) = \mathbb{H}$. As $J$ is strongly dissipative, with $\text{Ran}[J] \supset \mathbb{B}$, and $J(\cdot, t) : \mathbb{D} \rightarrow \mathbb{D}$, $A(t)$ is maximal dissipative, and hence closed, so that $\text{Ran}[I - \omega A(t)] \supset \mathbb{B}[\Omega]$. It follows that $A(t)$ is m-dissipative on $\mathbb{B}$ for each $t \in \mathbb{R}^+$ (since $\mathbb{H}$ is a Hilbert space). To see that $A(t)u$ is continuous in both variables, let $u_n, u \in \mathbb{B}$,
\[ \| (u_n - u) \|_{\mathcal{H}} \rightarrow 0 , \text{ with } t_n, t \in I \text{ and } t_n \rightarrow t. \text{ Then (see Corollary 8)} \]

\[ \| A(t_n)u_n - A(t)u \|_{\mathcal{H},1} \leq \| A(t_n)u - A(t)u \|_{\mathcal{H},1} + \| A(t_n)u_n - A(t_n)u \|_{\mathcal{H},1} \]

\[ = \| [Pf(t_n) - Pf(t)] \|_{\mathcal{H},1} + \| \nu (u_n - u) + B(u_n, u_n - u) + B(u, u_n - u) \|_{\mathcal{H},1} \]

\[ \leq d |t_n - t|^\theta + \nu \| A(u_n - u) \|_{\mathcal{H},1} + \| B(u_n, u_n - u) + B(u, u_n - u) \|_{\mathcal{H},1} \]

\[ \leq d |t_n - t|^\theta + \nu \frac{M c_3}{F} \| (u_n - u) \|_{\mathcal{H},1} + \nu \frac{M^4 c_1 c_2}{F^5/4} \| (u_n - u) \|_{\mathcal{H},1} + \| (u_n - u) \|_{\mathcal{H},1} \]

\[ \leq d |t_n - t|^\theta + \nu \frac{M c_3}{F} \| (u_n - u) \|_{\mathcal{H},1} + \nu \frac{M^4 c_1 c_2}{F^5/4} \| (u_n - u) \|_{\mathcal{H},1} + 2 \| (u_n - u) \|_{\mathcal{H},1} + u. \]

It follows that \( A(t)u \) is continuous in both variables. \( \square \)

When \( f = 0 \), \( B \) is a ball about \( 0 \). Thus, we can equip \( B \) with the closure of \( D \) in the \( \mathcal{H} \) norm. In this case, it follows that \( B \) is a closed, bounded, convex set, so that:

**Theorem 12.** For each \( T \in \mathbb{R}^+ \), \( t \in (0, T) \) and \( u_0 \in D \subset B \), the global in time Navier-Stokes initial-value problem in \( \Omega \subset \mathbb{R}^3 \):

\[ \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0 \text{ in } (0, T) \times \Omega, \]

\[ \nabla \cdot u = 0 \text{ in } (0, T) \times \Omega, \]

\( (8) \)

\[ u(t, x) = 0 \text{ on } (0, T) \times \partial \Omega, \]

\[ u(0, x) = u_0(x) \text{ in } \Omega. \]

has a unique strong solution \( u(t, x) \), which is in \( L^2_{loc}([0, \infty); \mathcal{H}) \) and in \( L^\infty_{loc}([0, \infty); \mathcal{V}] \cap C^1([0, \infty); \mathcal{H}]. \)

**Proof.** Theorem 3 allows us to conclude that when \( u_0 \in D \), the initial value problem is solved and the solution \( u(t, x) \) is in \( C^1([0, \infty); D] \). Since \( D \subset \mathcal{H}^2 \), it follows that
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$u(t, x)$ is also in $V$, for each $t > 0$. It is now clear that for any $T > 0$,

$$
\int_0^T \|u(t, x)\|_{H^1}^2 dt < \infty, \quad \text{and} \quad \sup_{0 < t < T} \|u(t, x)\|_V^2 < \infty.
$$

This gives our conclusion. □

When $f \neq 0$, $u \neq 0$. Let $k = \{u : \|u\|_{H^1} \leq u\}$ and set $B_\pm = B \cap k^c$, where $k^c$ is the complement of $k$. Thus, we can take the closure of $B_\pm \cap D$ in the $H$ norm and use the largest closed convex set containing the initial data, inside this set.

DISCUSSION

One interesting aspect of Theorem 12 is that it is impossible to restrict ourselves to a ball if the body forces are nonzero (e.g., the initial fluid velocity can never be zero). This result is expected on physical grounds.

It is known that if $u_0 \in V$, and $f(t)$ is $L^\infty((0, \infty); H)$ then there is a time $T > 0$ such that a weak solution with this data is uniquely determined on any subinterval of $[0, T)$ (see Sell and You page 396, [SY]). Thus, we also have that:

**Corollary 13.** For each $t \in \mathbb{R}^+$ and $u_0 \in \mathbb{D}$ the Navier-Stokes initial-value problem in $\Omega \subset \mathbb{R}^3$:

$$\begin{align*}
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= f(t) \text{ in } (0, T) \times \Omega, \\
\nabla \cdot u &= 0 \text{ in } (0, T) \times \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \text{ on } (0, T) \times \partial \Omega, \\
u(t, x) &= \mathbf{0} \text{ on } (0, T) \times \partial \Omega, \\
\mathbf{u}(0, x) &= u_0(x) \text{ in } \Omega.
\end{align*}$$

(9)

has a unique weak solution $\mathbf{u}(t, x)$, which is in $L^2_{\text{loc}}([0, \infty); \mathbb{H})$ and in $L^\infty_{\text{loc}}([0, \infty); V) \cap C^1([0, \infty); \mathbb{H})$. 

Since we require that our initial data be in $H^2$, the conditions for the Leray-Hopf weak solutions are not satisfied. However, it was an open question as to whether these solutions developed singularities, even if $u_0 \in C_0^\infty$ (see Giga [G], and references therein). The above Corollary shows that it suffices that $u_0(x) \in H^2$ to insure that the solutions develop no singularities.

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