Wormholes and solitonic shells in five-dimensional DGP theory

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We build five-dimensional spherically symmetric wormholes within the DGP theory. We calculate the energy localized on the shell, and we find that the wormholes could be supported by matter not violating the energy conditions. We also show that solitonic shells characterized by zero pressure and zero energy can exist; thereafter we make some observations regarding their dynamic on the phase plane. In addition, we concentrate on the mechanical stability of wormholes under radial perturbation preserving the original spherical symmetry. In order to do that, we consider linearized perturbations around static solutions. We obtain that for certain values of the mass $\mu$ and crossover scale $r_c$, stable wormholes exist with very small values of squared speed sound. Unlike the case of Einstein’s gravity, this type of wormholes fulfills the energy conditions. Finally, we show that the gravitational field associated with these wormhole configurations is attractive for $\mu > 0$.

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I. INTRODUCTION

Traversable Lorentzian wormholes [1, 2] are topologically non trivial solutions of the equations of gravity which would imply a connection between two regions of the same universe, or of two universes, by a traversable throat. In the case that such geometries actually exist they could show some interesting peculiarities as, for example, the possibility of using them for time travel [3, 4]. A basic difficulty with wormholes is that the flare-out condition [5] to be satisfied at the throat requires the presence of matter which violates the energy conditions (“exotic matter”) [1, 2, 5, 6]. It was recently shown [7], however, that the amount of exotic matter necessary for supporting a wormhole geometry can be made infinitesimally small. Thus, in subsequent works special attention has been devoted to quantifying the amount of exotic matter [8, 9], and this measure of the exoticity has been pointed to as an indicator of the physical viability of a traversable wormhole [10].

A central aspect of any solution of the equations of gravitation is its mechanical stability. The stability of wormholes has been thoroughly studied for the case of small perturbations preserving the original symmetry of the configurations. In particular, Poisson and Visser [11] developed a straightforward approach for analyzing this aspect for thin-shell wormholes; that is, those which are mathematically constructed by cutting and pasting two manifolds to obtain a new manifold [12, 13]. In these wormholes the associated supporting matter is located on a shell placed at the joining surface; so the theoretical tools for treating them is the Darmois-Israel formalism, which leads to the Lanczos equations [14, 15]. The solution of the Lanczos equations gives the dynamical evolution of the wormhole once an equation of state for the matter on the shell is provided. Such a procedure has been subsequently followed to study the stability of more general spherically symmetric configurations (see, for example, Refs. [16–25]). Moreover, the junction conditions were also used to construct plane symmetric thin-shell wormholes with cosmological constant [26, 27].

Wormholes in theories beyond Einstein framework have gained a lot of interest in the last years because they seem to possess some curious properties regarding the kind of matter that could support them. A few examples of these alternatives theories are the Einstein-Gauss-Bonnet picture [28–31], scalar-tensor theories [32–35], $F(R)$ theory, or massive gravity [36–40]. In particular, for the Einstein-Gauss-Bonnet theory, it was shown that static thin-shell wormholes could be supported by ordinary matter respecting the energy conditions [28]. Moreover, $C^2$-type wormholes with the latter property can also exist once the nonlinear Gauss-Bonnet term is included in the field equations [29, 30]. Of course, this feature is not only exclusive of the Gauss-Bonnet paradigms; the Brans-Dicke gravity is another set up where the thin-shell wormholes fulfill weak and null energy conditions [32].

In addition, a new type of gravitational model was widely studied in the context of cosmology as well as particle physics, the so-called Dvali, Gabadadze, and Porrati (DGP) theory. It predicts deviations from the standard 4D gravity over large distances. The transition between four- and higher-dimensional gravitational potentials in the DGP model arises because of the presence of both the brane and the bulk Einstein-Hilbert (E-H) terms in the action [41, 42]. Cosmological considerations of the DGP model were first discussed in [43, 44], where it was shown that in a Minkowski bulk spacetime we can obtain self-accelerating solutions. In the original DGP model it is known that 4D general relativity (GR) is not recovered at the linearized level. However, some authors have shown that at short distances we can recover the 4D general relativity in a spherically symmetric configuration...
It is worth mentioning that an interesting feature of the original DGP model is the existence of ghostlike excitations \[46, 49\]. Further, the viability of the self-accelerating cosmological solution in the DGP gravity was carefully studied in \[54\]. For a comprehensive review of the existence of 4D ghosts on the self-accelerating branch of solutions in DGP models, see \[51, 52\].

A common feature among alternative theories is that the junction conditions for the thin-shell wormholes are modified considerably, adding new types of geometrical objects besides the usual extrinsic curvature. The contributions from the curvature tensors, theoretically, seem to allow the existence of wormholes supported by ordinary matter. For all these reasons, we consider that the construction of wormholes within DGP gravity deserves to be examined in detail to conclude whether they could fulfill or not the energy conditions.

Another consequence of the nonlinearity introduced by the DGP theory is related to the way in which the stability analysis is carried out for the dynamic case; that is, in this context it is not completely clear how to obtain the stability zones.

The aim of the present paper is twofold. On the one hand, we explore the existence of five-dimensional wormholes within the DGP gravity theory. Our research is focused on configurations supported by nonexotic matter which satisfies the energy conditions. Then, we show the existence of solitonic vacuum shells and make some comments about their dynamic. On the other hand, our goal is to perform a study of the linear stability of wormholes preserving the original symmetry. We only examine configurations supported by ordinary matter. Moreover, we shall show that there exist stable wormholes with squared speed sound within the range \(0 \leq c_s^2 \leq 0.1\) indicating that the matter located at the throat of the wormhole could be nonrelativistic.

II. FIVE-DIMENSIONAL BULK SOLUTION

We start from the action for the DGP theory in five-dimensional manifold \(\mathcal{M}_5\) with four-dimensional boundary \(\partial \mathcal{M}_5 = \Sigma\) (cf. \[48, 49, 51, 54\]),

\[
S = 2M_5^3 \int_{\mathcal{M}_5} d^5x \sqrt{-g}R(g_{\mu\nu}) + 2M_4^2 \int_{\Sigma} d^4x \sqrt{-\gamma}\mathcal{R}(\gamma_{ab}) + \int_{\Sigma} d^4x \sqrt{-\gamma}\left(-4M_5^3\mathcal{K}(\gamma_{ab}) + \mathcal{L}_m\right),
\]

where \(g_{\mu\nu}\) is the five-dimensional metric, \(\gamma_{ab}\) is the four-dimensional induced metric on the boundary \(\Sigma\), and \(\mathcal{K}\) is the trace of extrinsic curvature. The extra term in the boundary introduces a mass scale \(m_c = 2M_5^3/M_4^2 = r_c^{-1}\); that is, the model has one adjustable parameter, namely, \(m_c\) which determines a scale that separates two different regimes of the theory. For distances much smaller than \(m_c^{-1}\) one would expect the solutions to be well approximated by general relativity and the modifications to appear at larger distances. This is indeed the case for distributions of matter and radiation which are homogeneous and isotropic at scales \(\gtrsim r_c\). Typically, \(m_c \sim 10.42\text{GeV}\), so it sets the distance/time scale \(r_c = m_c^{-1}\) at which the Newtonian potential significantly deviates from the conventional one (cf. \[53, 54\]).

It is a well-known fact that the DGP scheme is a five-dimensional model where gravity propagates throughout an infinite bulk, and matter fields in \(\mathcal{L}_m\) are confined to a 4-dimensional boundary. The action for gravity at lowest order in the derivate expansion is a bulk Einstein-Hilbert term and a boundary one, generically with two different Planck masses \(M_5, M_4\), plus a suitable Gibbons-Hawking term. In the bulk the DGP equations are the Einstein ones in vacuum: \(G_{\mu\nu}^{(5)} = 0\). Then in this case, Birkhoff’s theorem forces the bulk metric to be static, and of the Schwarzschild form:

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega_4^2, \quad f(r) = 1 - \frac{\mu}{r^2} \tag{1}
\]

where the parameter \(\mu\) is related to the five-dimensional Arnowitt-Deser-Misner (ADM) mass, \(M_{\text{ADM}} = 3\pi^2\mu M_5^3\). The above spacetime has only one horizon placed at \(r_+ = \sqrt{\mu}\) with \(\mu > 0\). Besides, when \(\mu < 0\) the manifold only presents a naked singularity at the origin \(r = 0\). Now, we are going to make some important remarks about this solution: (i) notice that the \(r\) does not measure the 4D distance from the origin because it has a spatial component in the extra dimension labeled as \(y\). Therefore, one can write the 5D distance as \(r^2 := r_{3D}^2 + y^2\); (ii) second, at large enough distance the five-dimensional Schwarzschild solution naturally appears in DGP gravity with correct boundary conditions. However, the behavior of the bulk solution and the corresponding black holes localized on a brane within DGP gravity are quite different in some regimes \[67\]. Further, regarding the four-dimensional point of view (localized on a DGP brane) in \[57\] it was found that the theory admits a 4D anti-de Sitter (AdS-)Schwarzschild solution: there the four-dimensional cosmological constant is given by \(\Lambda_{4D} = -3m_c^2 < 0\). In that work, the authors also showed that in the limit \(r_{4D} >> m_c^{-1}\) the 5D Schwarzschild solution of radius \(r_0\) emerges, whereas for \(r_{4D} << m_c^{-1}\) the metric can be accommodated in 4D Schwarzschild geometry. Interestingly enough, there is an interpolating solution between these two regimes (regular branch) together with a second solution that becomes 5D de Sitter-Schwarzschild at a large distance (accelerated branch)(cf. \[57\]).
III. WORMHOLES IN DGP THEORY

A. Thin-shell construction

Employing the metric Eqs. (1-2) we build a spherically thin-shell wormhole in DGP theory. We take two copies of the spacetime and remove from each manifold the five-dimensional regions described by

\[ \mathcal{M}_\pm = \{ x/r_\pm \leq a, a > r_h \}. \]  

(3)

The resulting manifolds have boundaries given by the timelike hypersurfaces

\[ \Sigma_\pm = \{ x/r_\pm = a, a > r_h \}. \]  

(4)

Then we identify these two timelike hypersurfaces to obtain a geodesically complete new manifold \( \mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^- \). We take values of \( a \) large enough to avoid the presence of singularities and horizons in the case that the geometry \( \Sigma \) has any of them. The manifold \( \mathcal{M} \) represents a wormhole with a throat placed at the surface \( r = a \), where the matter supporting the configuration is located. This manifold is constituted by two regions which are asymptotically flat (see Fig. 1). The wormholes throat time for studying the dynamics evolution of the wormholes, and then in general we have that the boundary hypersurface reads

\[ \Sigma : \mathcal{H}(r, \tau) = r - a(\tau) = 0. \]  

(5)

It is important to remark that the geometry remains static outside the throat, regardless of the fact that the radius \( a(\tau) \) can vary with time, so no gravitational waves are present. This is naturally guaranteed because the Birkhoff theorem holds for the original manifold.

Our starting point is to list the main geometric objects which shall appear in the junction condition associated with the field equation for \( \Sigma \). The extrinsic curvature, namely, \( K_{ab} \), associated with the two sides of the shell is defined as follows:

\[ K_{ab} = -n^i_k \left( \frac{\partial^2 X^a}{\partial \xi^i \partial \xi^b} + \Gamma^k_{i\nu} \frac{\partial X^i}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \right)_{r=a}, \]  

(6)

where \( n^\pm_k \) are the units normal \((n_\pm n^\pm = 1)\) to the surface \( \Sigma \) in \( \mathcal{M} \):

\[ n^\pm_k = \pm \left| g^{\mu\nu} \frac{\partial \mathcal{H}}{\partial X^\nu} \frac{\partial \mathcal{H}}{\partial X^\mu} \right| \frac{\partial \mathcal{H}}{\partial X^k}. \]  

(7)

The field equations projected on the shell \( \Sigma \) are the generalized junction (or Darmois–Israel) conditions \cite{48, 49, 51, 52}

\[ r_c \left( R_{ab} - \frac{1}{2} \gamma_{ab} R \right) - 2 \left( \langle K_{ab} - K_{\gamma ab} \rangle \right) = \frac{\mathcal{S}_{ab}}{8M_5^2}, \]  

(8)

where the bracket \( \langle, \rangle \) stands for the jump of a given quantity across the hypersurface \( \Sigma \) and \( \gamma_{ab} \) is the induced metric on \( \Sigma \). Notice that the first term in (8) is not enclosed with the brackets because this contribution comes from the four-dimensional E-H term in the DGP action (11) which already lives in the boundary so it does not need to be projected on \( \Sigma \). By taking the limit \( r_c \rightarrow 0 \) we recover the standard Darmois-Israel junction condition found in \cite{14}.

Now, let us calculate some quantities that we shall need later. The mixed components of the four-dimensional Einstein tensor are given by

\[ G^0_0 = -3 \left( \alpha^2 + \frac{1}{a^2} \right), \]  

(9)

\[ G^i_j = - \left( \frac{1}{a^2} + \frac{a^2}{a^2} + 2 \frac{a}{a} \right) \delta^i_j, \]  

(10)

where the dot means derivate with respect to the proper time on \( \Sigma \). The extrinsic curvature components read

\[ \langle K^0_0 \rangle = \frac{2 \dot{a} + f'(a)}{\sqrt{f(a) + \dot{a}^2}} \]  

(11)

\[ \langle K^i_j \rangle = \frac{2}{a} \sqrt{f(a) + \dot{a}^2} \delta^i_j, \]  

(12)
where the prime indicates the derivatives with respect to \( a \). The most general form of the stress-energy tensor on shell compatible with the symmetries is

\[
S^a_b = \text{diag} \left( -\sigma, p \delta^a_j \right)
\]

where \( \sigma \) is the energy density and \( p \) is the pressure. Replacing Eqs. (10-13) into the DGP junction condition (8) we obtain that the energy density and the pressure can be recast as

\[
\frac{\sigma}{8M_5^3} = 3r_c \left( \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} \right) - \frac{12}{a} \sqrt{f(a)} + \dot{a}^2, \tag{14}
\]

\[
\frac{p}{8M_5^3} = -r_c \left( \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} + \frac{2\dot{a}}{a} \right) + \frac{8}{a} \sqrt{f(a)} + \dot{a}^2, \tag{15}
\]

\[
+ 2 \frac{2\dot{a} + f'}{\sqrt{f(a)} + \dot{a}^2}. \tag{16}
\]

where the DGP contributions are encoded in the \( r_c \) factor of the above equations. If we take \( r_c \to 0 \) in both Eqs. (14) and (15) we recover the expression for the energy density \( \sigma \) and the pressure \( p \) found in [28], ignoring the Gauss-Bonnet contribution.

In order to carry on our comment that we still have the usual energy conservation, \( \nabla_a S^{ab} = 0 \) by virtue of \( \nabla_a (K_{ab} - \gamma_{ab} K) = 0 \), coming from the momentum constraint implicit in the five-dimensional Einstein equations. Further it is easy to see from \( \sigma \) and \( p \) that the energy conservation equation is fulfilled:

\[
\frac{d(\dot{a}^2\sigma)}{d\tau} + p \frac{da^3}{d\tau} = 0, \tag{17}
\]

The first term in Eq. (17) represents the internal energy change of the shell and the second the work by internal forces of the shell. The dynamical evolution of the wormhole throat is governed by the generalized Lanczos equations and to close the system we must supply an equation of state \( p = p(\sigma) \) that relates \( p \) and \( \sigma \). Notice that the reason why one obtains exact conservation, i.e., no energy flow to the bulk, is that the normal-tangential components of the stress tensor in the bulk are the same on both sides of the junction hypersurface.

\section{IV. \ MATTER SUPPORTING THE WORMHOLES}

Recently, classical solutions within the DGP model were found when the stress-energy tensor on the brane satisfies the dominant energy condition, yet the brane has negative energy from the bulk point of view (see [48, 49]). Within this frame, the study of superluminal propagation indicates that superluminosity occurs whenever the stress tensor on the shell is a pure cosmological constant, irrespective of the value of the shell density (cf. [48, 49]). All these elements are good reasons to consider a careful discussion about the nature of matter supporting wormholes in the DGP model. Moreover, motivated by the results within Einstein-Gauss-Bonnet gravity (i.e. with \( R^2 \)-like terms) in [32], here we evaluate the amount of exotic matter and the energy conditions, following the approach presented above where the four-dimensional EH term generalizes the standard junction, adding a few geometrical terms, which indeed represents the Einstein tensor projected on the shell. Consequently, coming the DGP contribution from the curvature tensor, the next approach is clearly the most suitable to give a precise meaning to the characterization of matter supporting the wormhole.

The \textit{weak} energy condition (WEC) states that for any timelike vector \( U^\xi \) it must be \( T_{\xi\eta} U^\xi U^\eta \geq 0 \); the WEC also implies, by continuity, the \textit{null} energy condition (NEC), which means that for any null vector \( k^\xi \) it must be \( T_{\xi\eta} k^\xi k^\eta \geq 0 \). In an orthonormal basis the WEC reads \( \rho \geq 0 \), \( \rho + p_t \geq 0 \) \( \forall \ l \) while the NEC takes the form \( \rho + p_t \geq 0 \) \( \forall \ l \). Besides, the \textit{strong} energy condition (SEC) states that \( \rho + p_t \geq 0 \) \( \forall \ l \), and \( \rho + 3p_t \geq 0 \) \( \forall \ l \).

In the case of thin-shell wormholes the radial pressure \( p_r \) is zero, within Einstein gravity, and the surface energy density must fulfill \( \sigma < 0 \), so that both energy conditions would be violated. The sign of \( \sigma + p_t \) where \( p_t \) is the transverse pressure is not fixed, but it depends on the values of the parameters of the system. In what follows we restrict to static configurations. The surface energy density \( \sigma_0 \) and the transverse pressure \( p_0 \) for a static configuration \((a = a_0, \dot{a} = 0, \ddot{a} = 0) \) are given by

\[
\frac{\sigma_0}{8M_5^3} = \frac{3r_c}{a_0^2} - \frac{12}{a_0} \sqrt{f(a_0)}, \tag{18}
\]

\[
\frac{p_0}{8M_5^3} = -\frac{r_c}{a_0^2} + \frac{8}{a_0} \sqrt{f(a_0)} + 2 \frac{f'(a_0)}{\sqrt{f(a_0)}}. \tag{19}
\]

Now the sign of the surface energy density as well as the pressure is, in principle, not fixed. The most usual choice for quantifying the amount of exotic matter in a Lorentzian wormhole is the integral [9]:

\[
\Omega = \int (\rho + p_r) \sqrt{-g_5} d^4 x. \tag{20}
\]

We can introduce a new radial coordinate \( R = \pm (r - a_0) \) with \( \pm \) corresponding to each side of the shell. Then, because in our construction the energy density is located on the surface, we can also write \( \rho = \delta(R) \sigma_0 \), and because the shell does not exert radial pressure the amount of exotic matter reads

\[
\Omega = \int \int \int \int \delta(R) \sigma_0 \sqrt{-g_5} dR d\xi d\theta d\phi = 2\pi^2 a_0^3 \sigma_0. \tag{21}
\]

Replacing the explicit form of \( \sigma_0 \) and \( g_5 \), we obtain the exotic matter amount as a function of the parameters that characterize the configurations:

\[
\Omega = 16M_5^3 \pi^2 \left( 3r_c a_0 - 12a_0^2 \sqrt{f(a_0)} \right). \tag{22}
\]
where $f$ is given by the bulk solution. For $r_c \to 0$ we obtain the exotic amount for Schwarzschild geometries as if it were calculated with the standard junction conditions. Far away from the general relativity limit we now find that there exist positive contributions to $\sigma_0$; these come from the different signs in the expression (22) for the surface energy density, because it is proportional to $\sigma_0$. We stress that this would not be possible if the standard Darmois-Israel formalism was applied, treating the DGP contribution as an effective energy-momentum tensor, because this leads to $\sigma_0 \propto -\sqrt{f(a_0)/a_0}$. Now, once the explicit form of the function $f(a_0)$ is introduced in Eq. (22), we focus on what are the conditions that lead to wormholes with $\sigma_0 > 0$ or $\Omega > 0$. Then, it can be proved that wormholes with a non-negative surface density located at the shell are allowable when the following inequalities are simultaneously satisfied:

\[
\frac{r_c}{a_0} - \frac{4}{a_0} \left( \frac{1 - \mu}{a_0^2} \right)^{1/2} > 0,
\]

(23)

\[
a_0^2 - \mu > 0,
\]

(24)

so it is always possible to choose $a_0$ such that the existence of thin-shell wormholes is compatible with positive surface energy density (see Fig. 2); more precisely its radius must belong to the interval given as

\[
\sqrt{\mu} < a_0 \leq \left( \mu + \frac{r_c^2}{16} \right)^{1/2}
\]

(25)

Notice that the $r_c$ term is essential to have positive energy density; as one would expect, in the limit $r_c \to 0$, this possibility completely vanishes. Despite that in most of our discussions we are going to keep $r_c$ and $\mu$ as free parameters; we wish to discuss some order of magnitude for $r_c^2/\mu$, so that from Eq. (25) one can immediately see whether there is a fine-tuning or not. Using constraints from type 1A supernovae [58], it turns out to be that the best fit for the crossover scale is $r_c = (1.21 \pm 0.09)H_0^{-1}$, where $H_0$ is today’s Hubble scale. Taking $H_0 = 70$km$^{-1}$Mpc$^{-1}$, it implies that $r_c \sim 5$Gpc; just to have an idea of this magnitude it is useful to remember that the distance to largest structures in the distribution galaxies is 100Mpc while the distance to the edge of the visible universe is 14Gpc.

On the other hand, four-dimensional Planck mass is given by $M_4 = 1.22 \times 10^{19}$GeV so using the definition of $r_c = M_4^2/2M_5^2$ we obtain that the five-dimensional Planck mass (scale) is $M_5 \sim 45$MeV. This value for the fundamental scale was also found in [59]; there are other interesting works that show more constraints on what is the value that it should take the scale $M_5$ (see [60], [61]).

In addition, a bound for the scale $M_5$ can be also obtained by using the Solar System itself [62] or outer planets of the Solar System [63]. Coming back to our main analyses, we are in a position to estimate in broad terms the ratio $r_c^2/\mu$. Taking $M_{ADHM} \sim M_{Sun}$ and $r_c \sim 5$Gpc implies that $r_c^2/\mu \sim O(10^{24})$ or $r_c^2 > > \mu$; consequently it indicates that there is no fine-tuning at all. Using the latter results we get an estimation of $\mu \sim 4 \times 10^{59}$GeV$^{-2}$, where in order to obtain the $M_{ADHM}$ we have assumed that at enough large distance the metric should correspond to asymptotically flat spacetime. For a full treatment about the role of the boundary conditions for obtaining a numerically black hole solution within the Randall-Sundrum infinite braneworld see [64].

Besides, from Eq. (10) and Eq. (11) we have that the sum of the pressure and energy density takes the form

\[
\sigma_0 + p_0 = 8M_5^3 \left( \frac{2r_c}{a_0^2} + \frac{2a_0 f'(a_0) - 4f(a_0)}{a_0 \sqrt{f(a_0)}} \right)
\]

(26)

because the first term in (26) is positive the sign of $\sigma_0 + p_0$ depends on the second term, implying that the sum is positive for $\sqrt{\mu} < a_0 \leq \sqrt{2\mu}$. Therefore, the remarkable result is that we have a region with $\sigma_0 \geq 0$ and besides $\sigma_0 + p_0 \geq 0$, so the WEC and the NEC are satisfied (see Figs. 3 and 4). Additionally, it is easy to corroborate that $\sigma_0 + 3p_0 = 12 \times 8M_5^3/(a_0 \sqrt{f(a_0)})$, then SEC holds in the interval $a_0 \in (\sqrt{\mu}, \sqrt{2\mu})$ (see Figs. 3 and 4). Thus, by treating the DGP contribution as a geometric object, the generalized junction conditions [8] provide a clear meaning to the matter in the shell leading to a central finding that in the DGP gravity the violation of the energy conditions could be avoided and wormholes could be supported by ordinary matter.

However, note that one could choose another route because Eq. (8) can be formally recast as follows:

\[
-16M_5^3 \left( \kappa_{ab} - \kappa \gamma_{ab} \right) = S_{ab}^{\text{eff}},
\]

(27)

\[
S_{ab} - 8M_5^3 r_c \left( R_{ab} - \frac{1}{2} \gamma_{ab} R \right) = S_{ab}^{\text{eff}}
\]

(28)
is possible to have dynamical solitonic wormholes/shells characterized by a zero pressure ($p = 0$) and zero energy density ($\sigma = 0$). Unlike the standard Darmois-Israel junction condition, nontrivial solutions may be possible even when $S^0_0 = 0$. That is, the extrinsic curvature can be discontinuous across the throat with no matter on the shell to serve a source, turning the discontinuity into a self-supported gravitational system. Of course, these configurations are impossible in the Einstein gravity but not in the Einstein-Gauss-Bonnet gravity (cf. [31]).

For $\dot{a} \neq 0$ Eq. (17) shows that if $\sigma = 0$ then $p = 0$; so we are going to work with the most useful expression which in this case is given by $\sigma$. Following the procedure mentioned in [51] we shall plot trajectories in the phase space spanned by $(\dot{a}, a)$. Because the energy constraint ($\sigma = 0$) is invariant under the symmetry $\dot{a} \leftrightarrow -\dot{a}$ we can work on a two-dimensional plane which is defined as a non-compact domain, namely, $\mathcal{B} = (0, +\infty) \times (r_+, +\infty)$. The curves which represent the dynamics of solitonic wormholes are obtained by imposing the following conditions:

$$r_c (\dot{a}^2 + 1) - 4 \left( a^2 - \mu + a^2 \dot{a}^2 \right)^{\frac{1}{2}} = 0, \quad (29)$$
$$a^2 - \mu > 0 \quad (30)$$

such that the first inequality guarantees zero energy, whereas the second one ensures that the wormhole radius is larger than the event horizon. According to Fig. 5, the phase diagrams show that for small or large $\mu$ and with a DGP scale covering the interval $[0.5, 50]$, the shell velocity is a monotone increasing function for $\dot{a} \in (0, +\infty)$ (or decreasing one when $\dot{a} \in (-\infty, 0]$). Notice that the same conclusion is obtained when the parameter $r_c$ takes larger values. In order to see if these types of shells speed up or decelerate we use the zero pressure condition to get a functional relation $\dot{a} = \mathcal{N} [a, \dot{a}]$ which determines the sign of $\dot{a}$:

$$\mathcal{N} = \frac{-8af(a) - 8a\dot{a}^2 - 2a^2 f'(a) + r_c (1 + \dot{a}^2) \sqrt{f(a) + \dot{a}^2}}{2a (2a - r_c \sqrt{f(a) + \dot{a}^2})} \quad (31)$$

For all $\mu$ and $r_c$ considered in this section we obtain that the kinematic of the shell has four possible types of dynamical evolution. More precisely, the solitonic solution could suffer an accelerated ($\dot{a} > 0$) or decelerated ($\dot{a} < 0$) expansion ($\dot{a} > 0$) as well as an accelerated or decelerated contraction ($\dot{a} < 0$) regimes.

Unlike the Einstein-Gauss-Bonnet case studied in [31] it turns out that the existence of solitonic shells in DGP gravity does not require the presence of a cosmological constant term in the bulk spacetime.

In the next section we are going to study the stability of five-dimensional wormholes against homogenous perturbations preserving the original symmetry.
VI. THE STABILITY ANALYSIS

In general to obtain the dynamic picture of the wormholes within the DGP gravity is a very complicated task. As can be seen from Eqs. [11, 65] the nonlinear character of these expressions makes the idea of obtaining exact solutions very hard to implement. However, we can follow another route and study the stability of static solutions by linearizing the field equation. A physically interesting wormhole geometry should last enough so that its traversability makes sense. Thus the stability of a given wormhole configuration becomes a central aspect of its study. Here we shall analyze the stability under small perturbations preserving the spherical symmetry of the configuration; for this we shall proceed as [11, 65]. As we said, the dynamical evolution is determined by Eqs. [11] and [65], or by any of them and Eq. [67], and to complete the system we must add an equation of state that relates \( p \) with \( \sigma \).

Our first move to address the stability issue is to recast Eq. [11] in such a way that it allows us to get \( \dot{a} = F(a, \sigma(a)) \). Then, by squaring appropriately the energy density, we obtain a quadratic polynomial in the variable \( X = a^2(1 + \dot{a}^2) \) as it reads

\[ \nu_c X^2 - 2X \left( r_c \bar{\sigma} + 8 \right) + \bar{\sigma}^2 - 16G(a) = 0 \] (32)

with \( \bar{\sigma} = \sigma/3 \), and \( G(a) = -\mu/a^4 \). From the master equation (32) we get a single dynamical equation which completely determines the motion of the wormhole throat after the energy density is selected:

\[ \dot{a}^2 = -V(a) \] (33)

\[ V(a) = 1 - \frac{a^2}{r_c} \left[ \bar{\sigma} + \frac{4}{r_c} \left( 2 + \epsilon Y(a) \right) \right] \] (34)

\[ Y(a) = \left( 4 + r_c^2 G + r_c \bar{\sigma} \right)^{\frac{4}{3}} \] (35)

where \( \epsilon \) denotes either +1 or -1. For \( \mu = 0 \), Eq. (33) is similar to the Friedmann one found by Maeda et al in the context of brane world cosmology with induced gravity [66]. In order to keep the potential defined on a real domain the following reality condition must hold:

\[ a^4 \left( 4 + r_c \bar{\sigma} \right) - r_c^2 \mu \geq 0 \] (36)

Now, making a Taylor expansion to second order of the potential \( V \) around the static solution yields

\[ V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2} V''(a_0)(a - a_0)^2 + O[(a - a_0)^3] \]

From Eq. (34) we get that the first derivative of \( V \) is

\[ V' = -\frac{2a}{r_c} \left[ \bar{\sigma} + \frac{4}{r_c} \left( 2 + \epsilon Y(a) \right) \right] - \frac{a^2}{r_c} \left[ \bar{\sigma}' + \frac{4}{r_c} \epsilon Y'(a) \right] \]

whereas the second derivate is given by

\[ V'' = -\frac{2}{r_c} \left( \bar{\sigma} + \frac{4}{r_c} \left( 2 + \epsilon Y(a) \right) \right) - \frac{4a}{r_c} \left( \bar{\sigma}' + \frac{4}{r_c} \epsilon Y'(a) \right) \]

\[ - \frac{2a^2}{r_c} \left( \bar{\sigma}'' + \frac{4}{r_c} \epsilon Y''(a) \right) \]

Now, it is useful to rewrite the energy conservation as

\[ a \bar{\sigma}'(a) = -3(\bar{p} + \bar{\sigma}) \]

where \( \bar{p} = p/3 \). Using the latter identity we can obtain the second derivative of the energy density

\[ \bar{\sigma}''(a) = \frac{3}{a^2}(4 + 3\eta)(\bar{\sigma} + \bar{p}) \] (37)

where the parameter \( \eta \) is defined by the relation

\[ \eta(\sigma) = \frac{\partial p}{\partial \sigma} \] (38)

which for ordinary matter would represent the squared speed sound: \( \nu_s^2 = \eta \). Here, however, we simply consider \( \eta \) as a parameter entering the equations of state. Besides, we shall be interested in the stability of wormholes supported by ordinary matter like those found in the last section. To study the stability of the static solutions under perturbations preserving the spherical symmetry we linearize the equation of state around the static solution as follows

\[ p - p_0 = \eta(\sigma - \sigma_0) \] (39)
where the surface energy density $\sigma_0$ and the transverse pressure $p_0$ for a static configuration ($a = a_0$, $\dot{a} = 0$, and $\ddot{a} = 0$) are given by

\[
\sigma_0 = \frac{3r_c}{a_0^2} - \frac{12}{a_0} \sqrt{f(a_0)},
\]

\[
p_0 = -\frac{r_c}{a_0^2} + \frac{2}{a_0} \left( \frac{4f(a_0) + a_0 f'(a_0)}{\sqrt{f(a_0)}} \right)
\]

When evaluating the potential at the static solution $a = a_0$ it is easy to see that $V(a_0) = V'(a_0) = 0$, so the potential is

\[
V(a) = \frac{1}{2} V''(a_0) (a - a_0)^2 + \mathcal{O}((a - a_0)^3)
\]

where the second derivatives has three parts as it can be seen below

\[
V''(a_0) = V_{II}''(a_0) + V_{III}''(a_0) + V_{IIII}''(a_0),
\]

with

\[
V_{II}'' = -\frac{2}{r_c} \left( \bar{a}_0 + \frac{4}{r_c} (2 + \epsilon Y(a_0)) \right),
\]

\[
V_{III}'' = -\frac{4a_0}{r_c} \left( -\frac{3}{a_0} (\bar{\sigma}_0 + \bar{p}_0) + \frac{4\epsilon}{r_c} Y'(a_0) \right),
\]

\[
V_{IIII}'' = -\frac{a_0^2}{r_c} \left( \frac{3}{a_0^2} (4 + 3\eta_0)(\bar{\sigma}_0 + \bar{p}_0) + \frac{4\epsilon}{r_c} Y''(a_0) \right).
\]

Before proceeding with the stability issue we must verify that the following constraints hold at the same time:

I. $\sigma_0 > 0$, 
II. $\sigma_0 + p_0 > 0$, 
III. $a^4 \left( 4 + r_c \frac{\sigma_0}{3} \right) - r_c^2 \mu \geq 0$.

These conditions must be supplemented with IV: $a_0 > \sqrt{7}$ to ensure the existence of the wormholes manifold. Notice that I and II refer to the energy conditions; that is, we are only interested in wormholes supported by ordinary matter, while inequality III is completely necessary to keep the potential defined on a real domain. From Fig. 6 we obtain that the space spanned by the above restrictions is quite representative, in the sense that it covers a vast region. So the wormholes are physically relevant and do not correspond to a finely tuned value of mass $\mu$ or crossoverscale $r_c$. Of course, there are others types of solutions but they are not as interesting as these.

To sum up, wormholes are stable if and only if $V''(a_0) > 0$, while for $V''(a_0) < 0$ perturbations can grow, at least until the nonlinear regime is reached.

From now on, without loss of generality, we consider the stability zones with $\mu = 1$. Let us investigate how the regions of stability change with the parameters $r_c$ and $\epsilon$. In the case of $\epsilon = -1$ and $r_c = 10$ we get that there are stable wormholes with $\eta_0 \geq 2$ which would correspond to superluminal sound velocity in the wormhole throat (see Fig. 7). For $\epsilon = \pm 1$ and $r_c = 100$ wormholes are stable only if $\eta_0 < 0$. However, for $\epsilon = +1$ and $r_c = 10$ the model exhibits stable wormholes with $\eta_0 \geq 0$ only for small radii close to $a_0 \gtrsim 1$. So, the theory admits classical stable wormholes with $0 \leq \eta_0 \leq 0.1$, indicating that $\eta_0$ could represent the speed of sound of nonrelativistic matter (see Fig. 7). Interestingly, we get that stable wormholes are achieved with values of the $r_c$ parameter far away from the general relativity limit ($r_c \to 0$).

Another novel characteristic introduced by the induced gravity theory called DGP is that these wormholes not only seem to be stable for many different choices of $\mu$, $r_c$, and $\epsilon$, but also they exhibit energy conditions (see Fig. 8), being these conditions completely independent of $\epsilon$. Notice that wormholes fulfilling energy conditions are not possible within Einstein’s gravity [2]. Besides, our results about stable wormholes are in agreement with the stable solutions found in [49] for the full DGP theory which supports superluminal excitations.

As a final comment, let us consider the attractive or repulsive character of the wormhole geometry. The wormholes studied could be either attractive or repulsive. To characterize this aspect of the configurations we analyze the force on a test particle at rest in the geometry described above. For this, we evaluate the radial acceleration given by

\[
A^r = -\Gamma_{tt}^r \left( \frac{dt}{d\tau} \right)^2
\]

The sign of the acceleration of a particle initially at rest is then given by minus the sign of the component $\Gamma_{tt}$ of the connection, which for the metric considered is equal to $f'f/2$. Thus we have an attractive gravitational field for $f' > 0$ and a repulsive field for $f' < 0$ (of course we consider only the possibility $f > 0$). In our case, we find that $f'(r) = 2\mu/r^3$ so the gravitational field turns out to be attractive, indicating that the wormholes are always attractive as long as $\mu > 0$. 

\[
FIG. 6: We show zones, in the plane $r_c - a_0$, where the constrains I-IV are satified for $\mu = 1, 100$.
\]
The generalization of Einstein gravity in the way proposed by Dvali, Gabadadze, and Porrati introduces a new parameter, which allows for more freedom in the framework of determining the most viable wormhole configurations. If wormholes could actually exist, one would be interested in those which require as little amount of exotic matter as possible. Of course, the case could be that a given change of the theory leads to a worse situation, i.e., that configurations require more matter violating the energy conditions as the departure from the standard theory becomes relevant. However, for suitable wormhole radius, this seems not to be the case with DGP gravity. Here we have examined the “exotic” matter content of thin-shell wormholes using the generalized junction condition, and we have found that for large values of the DGP parameter, corresponding to a situation far away from the general relativity limit, the amount of exotic matter is reduced in relation to the standard case because it can be positive definite. Moreover, the remarkable result is that we have a region with $\sigma_0 \geq 0$ and besides $\sigma_0 + p_0 \geq 0$, so the WEC and the NEC are satisfied. Further the SEC condition holds also. Thus if the requirement of exotic matter is considered as the hardest objection against wormholes, our results suggest that in a physical scenario with small crossover scale $[r_c \sim O(1)]$ or far away from the general relativity limit where the DGP becomes dominant $[r_c \gtrsim 10^5]$ these types of wormholes could be possible. In addition, we showed the existence of gravitational solitonic wormholes/shell characterized by $\sigma = p = 0$ within the DGP model. Unlike the case of Einstein-Gauss-Bonnet theory we found that the existence of solitonic shells in DGP gravity does not require the presence of a cosmological constant term in the bulk solution.

Besides, we also focused on the mechanical stability of wormhole configurations under radial perturbations preserving the spherical symmetry. We found stable wormholes supported by ordinary matter, that is, configurations which verify the energy conditions for many different choices of the parameter space spanned by $\mu$, $r_c$, and $\epsilon$. Further, the stability analysis shows that in a scenario with a crossover scale $r_c \sim O(10)$ (far away from the general relativity limit) stable wormholes with very small squared speed sound $\epsilon_s^2 \in [0,0.1]$ are obtained. Finally, studying the radial acceleration experimented by a test particle on the gravitational field we obtained that the latter one turns out to be attractive, indicating that the wormholes are always attractive as long as $\mu > 0$.

**VII. SUMMARY**

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static sources of mass $M$ and radius $r_0$ are taken into account, such that $r_M < r_0 \ll r_c$ ($r_M = 2MG$ is the Schwarzschild radius), a new scale, a combination of $r_c$ and $r_g$, emerges (the so-called Vainshtein scale): $r_\ast = (r^2 r_M)^{1/3}$. Below this scale the predictions of the theory are in good agreement with the GR results and above it they deviate considerably (cf. [55], [56]).