The symmetric periodic orbits for the two-electron atom

Jaume Llibre\textsuperscript{2} · Durval José Tonon\textsuperscript{1}

Abstract We analyse the existence of symmetric periodic orbits of the two-electron atom. The results obtained show that there exist six families of periodic orbits that can be prolonged from a continuum of periodic symmetric orbits. The main technique applied in this study is the continuation method of Poincaré.

Keywords Symmetric periodic orbits · Helium atom · Poincaré continuation method

Mathematics Subject Classification Primary 34C14 · 34C20 · 34C25

1 Introduction

The estimate of the ground state of the energy of the helium atom was a cornerstone in the evolution of quantum mechanics. In 1980s and 1990s (see [5,6]), it was presented some shortcomings of the old quantum theory, and one of the most relevant subject was: to understand the behaviour of the periodic trajectories when the classical dynamics is non-integrable or even chaotic.

In this context, we study in this paper the existence of periodic trajectories of the system that governs the dynamics of the collinear two-electron atom and
investigate when this periodic orbits can be prolonged, depending on a small parameter.

Many distinct techniques can be applied for studying the existence of periodic orbits, for example, numerical analysis, averaging theory, Melnikov functions, normal forms, variational methods, among others. However, one of the first analytical studies of the existence of periodic orbits was done by Poincaré in [9].

The continuation method of Poincaré, originally presented in [9], consists in providing conditions for extending a given periodic solution with a parameter equal to zero to a solution with small but positive values of the parameter. For more details about this method, see [3], for example.

In [4], [12] and [11] was presented the model of collinear helium atom, that consists of two electrons of mass $m_e$ and charge $-e$ moving on a straight line with respect to a fixed positively charged nucleus of charge $+2e$. The Hamiltonian modelling the collinear helium system presented in [4], [12] and [11] is given by

$$H_1 = \frac{1}{2} \left( \dot{r}_1^2 + \dot{r}_2^2 \right) - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_1 + r_2},$$

here $\dot{r}_1 = p_1, \dot{r}_2 = p_2$ and $Z$ represents the charge of the nucleus of the helium atom.

Considering the change of coordinates given by $\tilde{r}_1 = 2/Z r_1, \tilde{r}_2 = 2/Z r_2$ and the rescaling of time $\tilde{t} = (Z/2)^2 t$, we obtain that the Hamiltonian $H_1$ becomes the Hamiltonian

$$H = \frac{1}{2} \left( \dot{\tilde{r}}_1^2 + \dot{\tilde{r}}_2^2 \right) - \frac{2}{\tilde{r}_1} - \frac{2}{\tilde{r}_2} + \frac{\mu}{\tilde{r}_1 + \tilde{r}_2},$$

where $\mu = 2/Z$, the derivative is computed considering the new time $\tilde{t}$ and we omit the tilde over the variables.

In what follows we study the Hamiltonian system defined by the Hamiltonian $H$, assuming that $\mu = 0$ that represents the critical case when the charge of the nucleus of the helium tends to infinity, and we obtain results about the existence of families of periodic orbits for small values of $\mu$ (that corresponds to large values of charge $Z$).

In the previous paper [7], we applied the same approach to study the symmetric periodic orbits for the charged collinear 3-body problem, which consists of three different masses each one with different charge, and we do not have any of the three masses fixed, as the collinear helium system has. This is a strong difference between these two systems. This difference originates completely different Hamiltonian models. In fact, the Hamiltonian that models the collinear charged 3-body problem is of the form

$$H_2 = \frac{1}{2} \left( \mu_1 \dot{r}_1^2 + \mu_2 \dot{r}_2^2 + \mu_3 \dot{r}_1 \dot{r}_2 \right) - \frac{e_{12}}{r_1} - \frac{e_{23}}{r_2} - \frac{e_{13}}{r_1 + r_2},$$

where the $\mu_i$’s are related to the masses of the three bodies and the $e_{ij}$’s with their masses and charges. Moreover, we note that the terms in the Hamiltonian $H_2$ different from the term $\frac{1}{2} \mu_3 \dot{r}_1 \dot{r}_2$ cannot be non-dimensionalized as in the Hamiltonian $H$. 
because the masses $m_1$ and $m_3$, as their charges are not equal as in $H$. In this way, the results about the existence and continuation of periodic orbits obtained in the paper [7] cannot be applied to the collinear two-electron atom.

We organize this paper as follows: In Sect. 2 we provide the equations that model the dynamics of the two-electron atom; in Sect. 3 we study the symmetries of the periodic solutions of this system; and in Sect. 4 we present the main results. Considering the parameter $\mu = 0$, the symmetric periodic orbits are studied in Sect. 5, and in Sect. 6 we apply the continuation method of Poincaré to extend the periodic solutions obtained in the previous section for $\mu = 0$ to small positive values of the parameter $\mu$. A brief conclusion and some comments comparing the periodic orbits of the uncharged, charged symmetric three-body problem with the ones of the two-electron atom are presented in Sect. 7.

## 2 Differential equations that govern the classical motion of the two-electron atom

In the literature, there are few rigorous results about the general three-body Coulomb problem and the main reason for this is that the equation that governs the motion are multidimensional, non-integrable and singular. In order to obtain a model of the equations that govern the motion, an essential ingredient for the classical analysis of the three-body Coulomb problem is the regularization of its equations of motion, see [12] and [11].

We assume that the particles are in position $x_1, x_2, x_3 \in \mathbb{R}$, respectively, such that $0 < x_1 < x_2 < x_3$, and we consider the change of coordinates given by $z_1 = x_2 - x_1$ and $z_2 = x_3 - x_2$, that denotes the distance between $x_2$ and $x_1$ and $x_3$ and $x_2$. Let $L = T - U$ be the Lagrangian associated with this system, where $T, U$ denotes the kinetic and potential energy, respectively. We introduce the variables

$$p_1 = \frac{\partial L}{\partial \dot{z}_1} \quad \text{and} \quad p_2 = \frac{\partial L}{\partial \dot{z}_2}.$$  

Then the Hamiltonian that governs the two-electron atom is given by

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \frac{2}{z_1} - \frac{2}{z_2} + \frac{\mu}{z_1 + z_2}. \quad (1)$$

Associated with Hamiltonian (1), we have the system

$$\begin{align*}
\frac{dz_1}{dt} &= p_1, \\
\frac{dz_2}{dt} &= p_2, \\
\frac{dp_1}{dt} &= \frac{\mu}{(z_1 + z_2)^2} - \frac{2}{z_1^2}, \\
\frac{dp_2}{dt} &= \frac{\mu}{(z_1 + z_2)^2} - \frac{2}{z_2^2}.
\end{align*} \quad (2)$$

Springer
Applying the Levi-Civita transformation, see [8], given by

$$z_1 = \xi_1^2, \quad z_2 = \xi_2^2, \quad p_1 = \frac{\eta_1}{2\xi_1}, \quad p_2 = \frac{\eta_2}{\xi_2} \quad \text{and} \quad dt = 4\xi_1^2\xi_2^2 ds,$$

the trajectories of the two-electron atom (2) in the new coordinates are the solution of the system

\begin{align*}
\frac{d\xi_1}{ds} &= \eta_1 \xi_2^2, \\
\frac{d\xi_2}{ds} &= \eta_2 \xi_1^2, \\
\frac{d\eta_1}{ds} &= \frac{\xi_2}{\xi_1} \left( \eta_1^2 + \frac{8\mu \xi_1^4}{(\xi_1^2 + \xi_2^2)^2} - 16 \right), \\
\frac{d\eta_2}{ds} &= \frac{\xi_1}{\xi_2} \left( \eta_2^2 + \frac{8\mu \xi_2^4}{(\xi_1^2 + \xi_2^2)^2} - 16 \right),
\end{align*}

(3)

on the energy level \( H = h \) for some constant \( h \). System (3) is a Hamiltonian system with a Hamiltonian \( G \) given by

$$G = 4\xi_1^2\xi_2^2 \left( \frac{\eta_1^2 - 16}{8\xi_1^2} + \frac{\eta_2^2}{8\xi_2^2} - h + \frac{\mu}{\xi_1^2 + \xi_2^2} - \frac{2}{\xi_2^2} \right),$$

with \( G = 0 \) if and only if \( H = h \). System (3) is analytic except when \( \xi_1^2 + \xi_2^2 = 0 \), that corresponds to the triple collision.

We want to study the periodic orbits of the two-electron atom with binary collisions that correspond to the parameter \( \mu = 0 \). Considering these periodic solutions, our objective is to study the periodic solutions of system (3) for \( \mu > 0 \) sufficiently small, satisfying the energy relation \( G = 0 \). In this way in the following section we explore some symmetries involving this system.

### 3 Symmetries of the two-electron atom system

The results obtained in this section are similar to the ones given in [2, 7] for the collinear uncharged and charged three-body systems, respectively. Consider the involutions

\begin{align*}
Id : (\xi_1, \xi_2, \eta_1, \eta_2, s) &\mapsto (\xi_1, \xi_2, \eta_1, \eta_2, s), \\
S_1 : (\xi_1, \xi_2, \eta_1, \eta_2, s) &\mapsto (-\xi_1, \xi_2, \eta_1, -\eta_2, -s), \\
S_2 : (\xi_1, \xi_2, \eta_1, \eta_2, s) &\mapsto (\xi_1, -\xi_2, -\eta_1, \eta_2, -s), \\
S_3 : (\xi_1, \xi_2, \eta_1, \eta_2, s) &\mapsto (\xi_1, \xi_2, -\eta_1, -\eta_2, s), \\
S_4 : (\xi_1, \xi_2, \eta_1, \eta_2, s) &\mapsto (-\xi_1, -\xi_2, -\eta_1, -\eta_2, s), \\
S_5 : (\xi_1, \xi_2, \eta_1, \eta_2, s) &\mapsto (-\xi_1, \xi_2, -\eta_1, \eta_2, s), \\
S_6 : (\xi_1, \xi_2, \eta_1, \eta_2, s) &\mapsto (\xi_1, -\xi_2, \eta_1, -\eta_2, s), \\
S_7 : (\xi_1, \xi_2, \eta_1, \eta_2, s) &\mapsto (-\xi_1, -\xi_2, \eta_1, \eta_2, -s).
\end{align*}
We say that the solution \( \varphi(s) = (\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s)) \) of system (3) is invariant under the symmetry \( S_i \) if \( S_i(\varphi(s)) \) is also a solution of system (3) with \( i \in \{1, \ldots, 7\} \).
We say that \( \varphi(s) \) is \( S_i \)-symmetric if \( S_i(\varphi(s)) = \varphi(s) \).
Observe that the set of involutions \( \{Id, S_1, \ldots, S_7\} \) with the usual composition forms an abelian group isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \). These symmetries usually appear in Hamiltonian systems, see, for example, [1], [10] and [7]. Note that the symmetries \( S_1, S_2 \) and \( S_3 \) generate the other ones. So we only consider the symmetric periodic orbits with respect to the symmetries \( S_1, S_2 \) and \( S_3 \). Using similar arguments to the ones presented in [1] and [7], the following proposition holds.

**Proposition 1** Let be \( \varphi(s) \) a solution of system (3). The following statements hold.

(a) A solution \( \varphi(s) \) is a \( S_{12} \)-symmetric periodic solution of period \( S \) if and only if either \( \xi_1(s_0) = \eta_2(s_0) = 0 \) and \( \xi_2(s_0 + S/4) = \eta_1(s_0 + S/4) = 0 \) and there is no \( s \in (s_0, s_0 + S/4) \) such that \( \xi_2(s) = \eta_1(s) = 0 \), or \( \xi_2(s_0) = \eta_1(s_0) = 0 \) and \( \xi_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0 \) and there is no \( s \in (s_0, s_0 + S/4) \) such that \( \xi_1(s) = \eta_2(s) = 0 \).

(b) A solution \( \varphi(s) \) is a \( S_{13} \)-symmetric periodic solution of period \( S \) if and only if either \( \xi_1(s_0) = \eta_2(s_0) = 0 \) and \( \eta_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0 \) and there is no \( s \in (s_0, s_0 + S/4) \) such that \( \eta_1(s) = \eta_2(s) = 0 \), or \( \eta_1(s_0) = \eta_2(s_0) = 0 \) and \( \xi_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0 \) and there is no \( s \in (s_0, s_0 + S/4) \) such that \( \xi_1(s) = \eta_2(s) = 0 \).

(c) A solution \( \varphi(s) \) is a \( S_{23} \)-symmetric periodic solution of period \( S \) if and only if either \( \xi_2(s_0) = \eta_1(s_0) = 0 \) and \( \eta_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0 \) and there is no \( s \in (s_0, s_0 + S/4) \) such that \( \eta_1(s) = \eta_2(s) = 0 \) or \( \eta_1(s_0) = \eta_2(s_0) = 0 \) and \( \xi_2(s_0 + S/4) = \eta_1(s_0 + S/4) = 0 \) and there is no \( s \in (s_0, s_0 + S/4) \) such that \( \xi_2(s) = \eta_1(s) = 0 \).

Moreover the following result, proved in [2], shows that there are no symmetric periodic solutions having more than two symmetries.

**Proposition 2** There are no periodic solutions of system (3) which are simultaneously \( S_i \)-symmetric for \( i = 1, 2, 3 \).

### 4 Main results about the symmetric periodic orbits

As explored in [7], the periodic solutions of the differential equation (3) of the two-electron atom, which are simultaneously \( S_1 \) and \( S_2 \) symmetric, are denoted by \( S_{12} \)-symmetric periodic solutions. Similarly, we have \( S_{13} \)- and \( S_{23} \)-periodic solutions.

The results on the \( S_{12} \)-symmetric periodic solutions for small and positive values of \( \mu \) are given in the next theorem.

**Theorem 1** Consider \( h = h_1 + h_2 < 0 \), \( p \) and \( q \) odd positive integers. Then the \( S_{12} \)-symmetric periodic solutions of the two-electron atom (3) for \( \mu = 0 \) and with the initial conditions

\[
\begin{align*}
(a) & \quad \xi_1(0) = 0, \quad \xi_2(0) = \sqrt{-2/h_2}, \quad \eta_1(0) = 4, \quad \eta_2(0) = 0, \\
(b) & \quad \text{or } \xi_1(0) = \sqrt{-2/h_1}, \quad \xi_2(0) = 0, \quad \eta_1(0) = 0, \quad \eta_2(0) = 4,
\end{align*}
\]
where \( h_1 = \left( \frac{p}{q} \right)^{\frac{2}{3}} h_2 \), can be continued to a \( \mu \)-parameter family of \( S_{12} \)-symmetric periodic orbits of the two-electron atom (3) for \( \mu > 0 \) small.

The \( S_{13} \)-symmetry periodic solutions are given by the next result.

**Theorem 2** Consider \( h = h_1 + h_2 < 0 \), \( p \) odd and \( q \) even positive integers. Then the \( S_{13} \)-symmetric periodic solutions of the two-electron atom (3) for \( \mu = 0 \) with initial conditions

(a) \( \xi_1(0) = 0, \xi_2(0) = \sqrt{-2/h_2}, \eta_1(0) = 4, \eta_2(0) = 0 \),
(b) or \( \xi_1(0) = \sqrt{-2/h_1}, \xi_2(0) = \sqrt{-2/h_2}, \eta_1(0) = 0, \eta_2(0) = 0 \),

where \( h_1 = \left( \frac{p}{q} \right)^{\frac{2}{3}} h_2 \), can be continued to a \( \mu \)-parameter family of \( S_{13} \)-symmetric periodic orbits of the two-electron atom (3) for \( \mu > 0 \) small.

Finally for the \( S_{23} \)-symmetric periodic solutions, we obtain:

**Theorem 3** Consider \( h = h_1 + h_2 < 0 \), \( p \) even and \( q \) odd positive integers. Then the \( S_{23} \)-symmetric periodic solutions of the two-electron atom (3) for \( \mu = 0 \) with initial conditions

(a) \( \xi_1(0) = \sqrt{-2/h_1}, \xi_2(0) = 0, \eta_1(0) = 0, \eta_2(0) = 4 \),
(b) or \( \xi_1(0) = \sqrt{-2/h_1}, \xi_2(0) = \sqrt{-2/h_2}, \eta_1(0) = 0, \eta_2(0) = 0 \),

where \( h_1 = \left( \frac{p}{q} \right)^{\frac{2}{3}} h_2 \) can be continued to a \( \mu \)-parameter family of \( S_{23} \)-symmetric periodic orbits of the two-electron atom (3) for \( \mu > 0 \) small.

5 Symmetric periodic orbits for \( \mu = 0 \)

In the Levi-Civita coordinates \( (\xi_1, \xi_2, \eta_1, \eta_2) \), system (3) with \( \mu = 0 \) is given by

\[
\begin{align*}
\frac{d\xi_1}{ds} &= \eta_1 \xi_2^2, \\
\frac{d\xi_2}{ds} &= \eta_2 \xi_1^2, \\
\frac{d\eta_1}{ds} &= \xi_1 \left( 16 - \eta_2^2 + 8h\xi_2^2 \right), \\
\frac{d\eta_2}{ds} &= \xi_2 \left( -\eta_1^2 + 8h\xi_1^2 + 16 \right),
\end{align*}
\]

(4)

with the Hamiltonian

\[
H = \frac{1}{2} \left( \eta_1^2 - 16 \right) \xi_2^2 + 4\xi_1^2 \xi_2^2 \left( \frac{\eta_2^2}{8\xi_2^2} - h - \frac{2}{\xi_2^2} \right),
\]
that can be rewritten as $H = H_1(\xi_1, \eta_1) + H_2(\xi_2, \eta_2)$ where

$$H_1(\xi_1, \eta_1) = \frac{\eta_1^2}{8\xi_1^2} - \frac{2}{\xi_1^2},$$

$$H_2(\xi_2, \eta_2) = \frac{\eta_2^2}{8\xi_2^2} - \frac{2}{\xi_2^2},$$

if we rescale the time by $4\xi_1^2\xi_2^2$. Note that the level of energy $H = h$ satisfies that $h = h_1 + h_2$ where $H_1 = h_1$ and $H_2 = h_2$. Consider $\varphi(s) = (\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s))$ the solution of system (4) satisfying the energy condition $H = h$, we define the new times $\sigma$ and $\tau$:

$$\frac{d\sigma}{ds} = \xi_2^2, \quad \text{or equivalently} \quad \frac{dt}{d\sigma} = 4\xi_1^2,$$

$$\frac{d\tau}{ds} = \xi_1^2, \quad \text{or equivalently} \quad \frac{dt}{d\tau} = 4\xi_2^2. \quad (5)$$

Considering these new times, we obtain the system of differential equations in coordinates $(\xi_1, \eta_1)$

$$\frac{d\xi_1}{d\sigma} = \eta_1,$$

$$\frac{d\eta_1}{d\sigma} = 8h_1\xi_1, \quad (6)$$

and $(\xi_2, \eta_2)$ satisfies the system of differential equations

$$\frac{d\xi_2}{d\tau} = \eta_2,$$

$$\frac{d\eta_2}{d\tau} = 8h_2\xi_2. \quad (7)$$

Consider the functions $G_1 = G/\xi_2^2$ and $G_2 = G/\xi_1^2$, i. e.

$$G_1 = \frac{1}{2} \left( \eta_1^2 - 8\left(h_1\xi_1^2 + 2\right) \right),$$

$$G_2 = \frac{1}{2} \left( \eta_2^2 - 8\left(h_2\xi_2^2 + 2\right) \right).$$

These two functions are the Hamiltonians of system (6) and (7), respectively. Now we will study the periodic solutions of systems (6) and (7). Fixing $h_1 < 0$ we can integrate system (6) directly with the initial conditions $\xi_1(0) = \xi_{10}$ and $\eta_1(0) = \eta_{10}$, obtaining the solution $(\xi_1(\sigma), \eta_1(\sigma))$ given by

$$\xi_1(\sigma) = \xi_{10} \cos(\omega_1\sigma) + \eta_{10} \frac{\omega_1}{\omega_1^2} \sin(\omega_1\sigma),$$

$\omega_1$ Springer
Table 1  Period of periodic solution \( \varphi(s) \) of system (4)

| Time | \( \sigma^* \) | \( \tau^* \) | \( t \) | \( s^* \) |
|------|----------------|----------------|---------|---------|
| Period | \( \sigma^* = p \sigma \) | \( \tau^* = q \tau \) | \( T = pT_1(h_1, \alpha, v) = qT_2(h_2, \gamma, v) \) | \( S^* = s(T) \) |
| Period/4 | \( \sigma^*/4 \) | \( \tau^*/4 \) | \( T/4 \) | \( S^*/4 \) |

\[
\eta_1(\sigma) = \eta_{10} \cos(\omega_1 \sigma) - \xi_{10} \omega_1 \sin(\omega_1 \sigma), \tag{8}
\]

where \( \omega_1 = \sqrt{-8h_1} \). Note that solution (8) is a periodic solution of system (6) with period \( \sigma = 2\pi/\omega_1 \). Assuming that solution (8) satisfies the energy relation \( G_1 = 0 \), we obtain the following relationship between the initial conditions \( \xi_{10} \) and \( \eta_{10} \):

\[
\frac{\eta_{10}^2}{2} - 4h_1\xi_{10}^2 - 8 = 0.
\]

Moreover using (5), we obtain the period of this solution in terms of the time \( t \), i.e.

\[
T_1(h_1, \alpha, v) = \int_0^{\sigma} 4\xi_1^2(\sigma)d\sigma = \frac{\sqrt{8\pi}}{h_1\sqrt{-h_1}}.
\]

In the same way, fixing \( h_2 < 0 \) we can integrate system (7) with the initial conditions \( \xi_2(0) = \xi_{20} \) and \( \eta_2(0) = \eta_{20} \), getting the solution \((\xi_2(\tau), \eta_2(\tau))\) given by

\[
\xi_2(\tau) = \xi_{20} \cos(\omega_2 \tau) + \frac{\eta_{20}}{\omega_2} \sin(\omega_2 \tau), \tag{9}
\]

\[
\eta_2(\tau) = \eta_{20} \cos(\omega_2 \tau) - \xi_{20} \omega_2 \sin(\omega_2 \tau),
\]

where \( \omega_2 = \sqrt{-8h_2} \). Solution (9) is periodic with period \( \tau = 2\pi/\omega_2 \). Assuming that solution (9) satisfies the energy relation \( G_2 = 0 \), the initial conditions \( \xi_{20} \) and \( \eta_{20} \) satisfy the equation

\[
\frac{\eta_{20}^2}{2} - 4h_2\xi_{20}^2 - 8 = 0,
\]

and by (5) the period of solution (9) in time \( t \) is given by

\[
T_2(h_2, \gamma, v) = \int_0^{\tau} 4\xi_2^2(\tau)d\tau = \frac{\sqrt{8\pi}}{h_2\sqrt{-h_2}}.
\]

In Table 1, we summarize the relationships between the times \( \sigma, \tau, t \) and \( s \):

The next result provides some relationship between the energy levels \( h_1 \) and \( h_2 \), the times \( \sigma, \tau, t \) and \( s \) and the initial conditions of the periodic solutions of systems (6) and (7). The proof is similar to the proof of Proposition 3 given in [7].
Proposition 3  Consider the periodic solutions \((\xi_1(\sigma), \eta_1(\sigma))\) and \((\xi_2(\tau), \eta_2(\tau))\) of systems (6) and (7) with periods \(\sigma\) and \(\tau\), satisfying the energy conditions \(G_1 = 0\) and \(G_2 = 0\), respectively. Assume that the functions \(\sigma(s)\) and \(\tau(s)\) given in (5) satisfy \(\sigma(0) = \tau(0) = 0\) and there is no \(s \in \mathbb{R}\) such that \(\xi_1(\sigma(s)) = \xi_2(\tau(s)) = 0\). Then the following statements hold.

(a) The solution \(\varphi(s) = (\xi_1(\sigma(s)), \xi_2(\tau(s)), \eta_1(\sigma(s)), \eta_2(\tau(s)))\) of system (4) with initial condition \(\xi_1(0) = \xi_{10}, \xi_2(0) = \xi_{20}, \eta_1(0) = \eta_{10}, \eta_2(0) = \eta_{20}\) satisfies the energy relation \(G(\xi_{10}, \xi_{20}, \eta_{10}, \eta_{20}) = 0\).

(b) If \(h_1 = \left(\frac{p}{q}\right)^\frac{2}{3} h_2\) for some \(p, q \in \mathbb{N}\) coprime, then \(\varphi(s)\) is a periodic solution of system (4).

(c) Let \(s(t)\) be the inverse function of

\[
t(s) = \int_0^s 4\xi_1^2(\theta)\xi_2^2(\theta) d\theta.
\]

For the \(h_1\) given in (b) the period and the quarter of period in times \(\sigma, \tau, t\) and \(s\) are given in Table 1.

Observe that by statement (c) of Proposition 3 we have that \(dt/ds > 0\) when there are no collisions, and zero in the binary collisions. Therefore the inverse function \(s = s(t)\) exists always that the system has no triple collision, and it is differentiable if there is no binary collisions. The number \(p\) in Proposition 3 represents the number of binary collisions between \(m_1\) and \(m_2\), and \(q\) is the number of binary collisions between the particles \(m_2\) and \(m_3\).

We emphasize that our interest in this paper is to study the symmetric periodic orbits of system (3) satisfying the energy relation \(G = 0\). So in the following we exhibit conditions under the initial points, given in Proposition 3, to get symmetric periodic orbits.

Proposition 4  The following statements hold.

(a) If \(p\) and \(q\) are odd, then the solution \(\varphi(s)\) given in Proposition 3 with initial conditions

\[
either \quad \xi_{10} = 0, \xi^*_{20} = \sqrt{\frac{2}{-h_2}}, \eta^*_{10} = 4, \eta_{20} = 0,
\]

or \(\xi^*_{10} = \sqrt{\frac{2}{-h_1}}, \xi_{20} = 0, \eta_{10} = 0, \eta^*_{20} = 4\),

is a \(S_{12}\)-symmetric periodic solution.

(b) If \(p\) is odd and \(q\) is even, then the solution \(\varphi(s)\) given in Proposition 3 with initial conditions

\[
either \quad \xi^*_{10} = 0, \xi^*_{20} = \sqrt{\frac{2}{-h_2}}, \eta^*_{10} = 4, \eta^*_{20} = 0,
\]
or \( \xi_{10}^* = \sqrt{\frac{2}{-h_1}}, \xi_{20}^* = \sqrt{\frac{2}{-h_2}}, \eta_{10}^* = 0, \eta_{20}^* = 0, \)

is a \( S_{13} \)-symmetric periodic solution.

(c) If \( p \) is even and \( q \) is odd, then the solution \( \varphi(s) \) given in Proposition 3 with initial conditions

\[
\begin{align*}
\xi_{10}^* &= \sqrt{\frac{2}{-h_1}}, \xi_{20}^* = 0, \eta_{10}^* = 0, \eta_{20}^* = 4, \\
\text{or} \quad \xi_{10}^* &= \sqrt{\frac{2}{-h_1}}, \xi_{20}^* = \sqrt{\frac{2}{-h_2}}, \eta_{10}^* = 0, \eta_{20}^* = 0,
\end{align*}
\]

is a \( S_{23} \)-symmetric periodic solution.

Proof Let \( \varphi(s) = (\xi_1(\sigma(s)), \xi_2(\tau(s)), \eta_1(\sigma(s)), \eta_2(\tau(s))) \) be the solution of system (3). The proof follows evaluating \( \varphi(s) \) at times \( s = 0 \) and \( s = S^*/4 \), where \( S^* \) is given in Table 1.

\( \square \)

Remark 1 Note that the Levi-Civita transformation duplicates the number of orbits. Hence it is sufficient to consider the positive square roots of the initial conditions given in Proposition 4.

6 Applying the continuation method of Poincaré to obtain symmetric periodic solutions for \( \mu > 0 \) small

In Proposition 4, we have the values for the initial conditions to get \( S_{12}, S_{13}, S_{23} \)-symmetric periodic orbits for system (3) when \( \mu = 0 \). We study the symmetric periodic orbits for small positive values of \( \mu \) using the continuation method of Poincaré.

6.1 The \( S_{12} \)-symmetric periodic solutions

Fixing values of \( h = h_1 + h_2 \), by Proposition 4 the solution \( \varphi(s; \xi_{10}, \xi_{20}, \eta_{10}, \eta_{20}, \mu) \) of system (3) for \( \mu = 0 \) is a \( S_{12} \)-symmetric if the initial conditions satisfies

(a) Either \( \xi_1(0) = 0, \xi_2(0) = \xi_{20}^*, \eta_1(0) = \eta_{10}^*, \eta_2(0) = 0; \)

(b) Or \( \xi_1(0) = \xi_{10}^*, \xi_2(0) = 0, \eta_1(0) = 0, \eta_2(0) = \eta_{20}^*. \)

6.1.1 Case (a)

By statement (a) of Proposition 1 we have that the solution \( \varphi \) is a \( S_{12} \)-symmetric periodic solution of the two-electron atom with period \( S \) satisfying the energy condition \( G = 0 \) if and only if

\[
\xi_2(S/4; \xi_{20}, \eta_{10}, (\alpha, \beta, \gamma, \mu)) = 0,
\]
The solution of the last equation in terms of $\eta$ is given by $\eta_{10} = 4$. Thus the solution $\varphi(s; 0, \xi_{20}, \eta_{10}, 0, \mu)$ is a $S_{12}$-symmetric periodic solution of system (3) satisfying $G = 0$ if and only if

$$\begin{align*}
\xi_2(S/4; \xi_{20}, \mu) &= 0, \\
\eta_1(S/4; \xi_{20}, \mu) &= 0.
\end{align*}$$

Statement (a) of Proposition 4 provides additional information on the initial conditions and on the numbers $p$ and $q$, see Table 1, for these $S_{12}$-symmetric periodic solutions. Thus if $p = 2m + 1$, $q = 2k + 1$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$, $\xi_{20} = \sqrt{\frac{-2}{h_2}}$ and $\eta_{10} = 4$, with $m$ and $k$ positive integers, then $\varphi(s; 0, \xi_{20}, \eta_{10}, 0, 0)$ is a $S_{12}$-symmetric solution of system (3) with $\mu = 0$ and the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\frac{p}{q}\right)^2 \frac{2}{3} h_2$ given in statement (b) of Proposition 3. Now we extend this solution to $\mu > 0$ and small. Applying the Implicit Function Theorem to system (10) in a neighbourhood of a known solution we have that if

$$\begin{vmatrix}
\frac{\partial \xi_2}{\partial s} & \frac{\partial \xi_2}{\partial \eta_1} \\
\frac{\partial \xi_20}{\partial \eta_1} & \frac{\partial \xi_20}{\partial \xi_20}
\end{vmatrix}
\neq 0,$$

then there exist unique analytic functions $\xi_{20} = \xi_{20}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

1. $\xi_{20}(0) = \xi_{20}^*$ and $S(0) = S^*$,
2. and $\varphi(s; 0, \xi_{20}^*, \eta_{10}, 0, \mu)$ is a $S_{12}$-symmetric periodic solution of (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

Note that the derivatives $\frac{\partial \xi_2}{\partial s}$ and $\frac{\partial \eta_1}{\partial s}$ are obtained evaluating the right hand of system (3) for $\mu = 0$, $s = S^*/4$ and with the initial conditions $(0, \xi_{20}^*, \eta_{10}, 0)$. So

$$\left.\frac{\partial \xi_2}{\partial s}\right|_{\xi_{20} = \xi_{20}^*} = \eta_{20}^2 \xi_1^2 \left.\frac{\partial \xi_2}{\partial s}\right|_{\xi_{20} = \xi_{20}^*} = (-1)^{k+1} \frac{8}{h_1} \neq 0,$$

and

$$\left.\frac{\partial \eta_1}{\partial s}\right|_{\xi_{20} = \xi_{20}^*} = 8h_1 \xi_1 \xi_2^2 \left.\frac{\partial \eta_1}{\partial s}\right|_{\xi_{20} = \xi_{20}^*} = 0.$$

Therefore it remains to calculate $\frac{\partial \eta_1}{\partial \xi_{20}}$ evaluated at $s = S^*/4$, $\xi_{20} = \xi_{20}^*$ and $\mu = 0$. We obtain this value derivating the solution $\eta_1(\sigma(s); 0, \xi_{20}, \eta_{10}, 0)$ of system

\[ \odot \text{ Springer} \]
with respect to the variable \( \xi_{20} \), where the initial conditions \( \xi_1(0) = 0, \xi_2(0) = \xi_{20}, \eta_1(0) = \eta_{10}, \eta_2(0) = 0 \) satisfy the energy relation \( G = 0 \). So

\[
\left. \frac{\partial \eta_1(s)}{\partial \xi_{20}} \right|_{s = \frac{s^*}{4}, \xi_{20} = \xi_{20}^*, \mu = 0} = \left. \left( \frac{\partial \eta_1}{\partial \sigma} \frac{\partial \sigma}{\partial \xi_{20}} + \frac{\partial \eta_1}{\partial \xi_{20}} \right) \right|_{s = \frac{s^*}{4}, \xi_{20} = \xi_{20}^*, \mu = 0}.
\]

By (5) we get the relationship between \( \xi_{20}, \tau \) and \( \sigma \)

\[
\xi_1^2(\sigma)d\sigma = \xi_2^2(\tau)d\tau.
\]

Integrating this equation and assuming that \( \sigma(0) = 0 \) and \( \tau(0) = 0 \), we obtain the relation between \( \sigma(s) \) and \( \tau(s) \), i.e.

\[
\frac{2\sqrt{2} \sin (2\omega_1 \sigma(s))}{(-h_1)^{3/2}} + \frac{16\sigma(s)}{h_1} + \frac{\sqrt{2}\xi_{20}^2 \sin (2\omega_2 \tau(s))}{\sqrt{-h_2}} + 8\xi_{20}^2 \tau(s) = 0, \tag{12}
\]

where \( \omega_1 = \sqrt{-8h_1}, \omega_2 = \sqrt{-8h_2} \) and \( h = h_1 + h_2 \). Derivating implicitly equation (12) with respect to the variable \( \xi_{20} \), we obtain

\[
\left. \left( \frac{\partial \sigma(s)}{\partial \xi_{20}} \right) \right|_{s = \frac{s^*}{4}, \xi_{20} = \xi_{20}^*, \mu = 0} = (-1)^{m+1} 8\sqrt{-2h_1}.
\]

From (6) and (8), we get

\[
\left. \left( \frac{\partial \eta_1}{\partial \sigma} \right) \right|_{s = \frac{s^*}{4}, \xi_{20} = \xi_{20}^*, \mu = 0} = (-1)^m 16\sqrt{2\pi q}\sqrt{-h_1}.
\]

and

\[
\left. \left( \frac{\partial \eta_1}{\partial \xi_{20}} \right) \right|_{s = \frac{s^*}{4}, \xi_{20} = \xi_{20}^*, \mu = 0} = (-1)^m \frac{\sqrt{2\pi(-h_1)^{3/2}}}{h_2}
\]

So

\[
\left. \left( \frac{\partial \eta_1(s)}{\partial \xi_{20}} \right) \right|_{s = \frac{s^*}{4}, \xi_{20} = \xi_{20}^*, \mu = 0} = (-1)^m \frac{2\sqrt{2\pi(-h_1)^{3/2}}}{h_2}
\]

Finally the determinant (11) is given by

\[
(-1)^{k+m} \frac{16\sqrt{2\pi q}\sqrt{-h_1}}{h_2} \neq 0.
\]
6.1.2 Case (b)

Similarly to the previous case the solution $\varphi$ is a $S_{12}$-symmetric periodic solution with period $S$ satisfying the energy condition $G = 0$ if and only if

$$
\xi_1(S/4; \xi_{10}, \eta_{20}, \mu) = 0,
\eta_2(S/4; \xi_{10}, \eta_{20}, \mu) = 0,
G(\xi_{10}, \eta_{20}, \mu) = 0.
$$

The solution of $G = 0$ in terms of $\eta_{20}$ is given by $\eta_{20} = 4$. So the solution $\varphi(s; \xi_{10}, 0, 0, \eta_{20}, \mu)$ is a $S_{12}$-symmetric periodic solution of system (3) satisfying $G = 0$ if and only if

$$
\xi_1(S/4; \xi_{10}, \mu) = 0,
\eta_2(S/4; \xi_{10}, \mu) = 0.
$$

Again by statement (a) of Proposition 4 we obtain the initial conditions and the numbers $p$ and $q$ to have $S_{12}$-symmetric periodic solutions (see Table 1), i.e. $p = 2m+1$, $q = 2k+1$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$, $\xi_{10} = \sqrt{-2/h_1}$, $\eta_{20} = 4$, for $\mu = 0$ and the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\frac{p}{q}\right)^{\frac{3}{2}} h_2$ given in statement (b) of Proposition 3. As in the previous case applying the Implicit Function Theorem to system (13) in a neighbourhood of a known solution, we have that

$$
\left| \begin{array}{cc}
\frac{\partial \xi_1}{\partial s} & \frac{\partial \xi_1}{\partial \eta_2} \\
\frac{\partial \eta_2}{\partial s} & \frac{\partial \eta_2}{\partial \xi_{10}}
\end{array} \right| \neq 0,
$$

then there exist unique analytic functions $\xi_{10} = \xi_{10}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\xi_{10}(0) = \xi_{10}^*$ and $S(0) = S^*$,
(ii) and $\varphi(s; \xi_{10}, 0, 0, \eta_{20}, \mu)$ is a $S_{12}$-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

In the same way as we work in the previous case, we obtain

$$
\left. \frac{\partial \xi_1}{\partial s} \right|_{s = s^*/4, \xi_{10} = \xi_{10}^*, \mu = 0} = (-1)^{m+1} \frac{8}{h_2} \neq 0,
$$

and

$$
\left. \frac{\partial \eta_2}{\partial s} \right|_{s = s^*/4, \xi_{10} = \xi_{10}^*, \mu = 0} = 0.
$$
So it remains to calculate $\partial \eta_2 / \partial \xi_{10}$ evaluated at $s = S^*/4$, $\xi_{10} = \xi_{10}^*$ and $\mu = 0$. As in the previous case, this derivative is obtained derivating the solution $\eta_2(\tau(s); \xi_{10}, 0, 0, \eta_{20}, 0)$ of system (4) evaluated in $\xi_{10} = \xi_{10}^*$, $\xi_2(0) = 0$, $\eta_1(0) = 0$, $\eta_2(0) = \eta_{20}$, $s = S^*/4$ and satisfying the energy relation $G = 0$. Then

$$\left. \frac{\partial \eta_2(\tau(s); \xi_{10}, 0, 0, \eta_{20}, 0)}{\partial \xi_{10}} \right|_{\xi_{10} = \xi_{10}^*} = \left( \frac{\partial \eta_2}{\partial \tau} \frac{\partial \tau}{\partial \xi_{10}} + \frac{\partial \eta_2}{\partial \xi_{10}} \right) \bigg|_{\xi_{10} = \xi_{10}^*} = (-1)^{k+12} \frac{p \sqrt{2h_2}}{h_1}$$

and determinant (14) is given by

$$(-1)^{k+m+1} \frac{16p \sqrt{-2h_2}}{h_1} \neq 0.$$

### 6.2 The $S_{13}$-symmetric periodic orbits

By Proposition 4 the solution $\varphi(s; \xi_{10}, \xi_{20}, \eta_{10}, \eta_{20}, \mu)$ of system (3) for $\mu = 0$ and fixed values of $h = h_1 + h_2$ is a $S_{13}$-symmetric if the initial conditions satisfies

(a) either $\xi_1(0) = 0$, $\xi_2(0) = \xi_{20}^*$, $\eta_1(0) = \eta_{10}^*$, $\eta_2(0) = 0$,
(b) or $\xi_1(0) = \xi_{10}^*$, $\xi_2(0) = \xi_{20}^*$, $\eta_1(0) = 0$, $\eta_2(0) = 0$.

#### 6.2.1 Case (a)

By statement (b) of Proposition 1 we have that the solution $\varphi$ is a $S_{13}$-symmetric periodic solution with period $S$ satisfying the energy condition $G = 0$ if and only if

$$\eta_1(S/4; \xi_{20}, \eta_{10}, \mu) = 0,$$
$$\eta_2(S/4; \xi_{20}, \eta_{10}, \mu) = 0,$$
$$G(\xi_{20}, \eta_{10}, \mu) = 0.$$ 

Solving the last equation in terms of $\eta_{10}$ we obtain $\eta_{10} = 4$. Then the solution is $S_{13}$-symmetric periodic satisfying $G = 0$ if and only if

$$\eta_1(S/4; \xi_{20}, \mu) = 0,$$
$$\eta_2(S/4; \xi_{20}, \mu) = 0.$$ 

Statement (b) of Proposition 4 provides the initial conditions and numbers $p$ and $q$, see Table 1 in order that these periodic orbits are $S_{13}$-symmetric, i.e. $p = 2m + 1$, $q = 2k$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$, $\xi_{20} = \sqrt{-2h_2}$ and $\eta_{10} = 4$, with $m$ and $k$ positive integers, then $\varphi(s; 0, \xi_{20}, \eta_{10}, 0, 0)$ is a $S_{13}$-symmetric solution for system (3) with $\mu = 0$. Moreover, the energy level $H = h = h_1 + h_2$ satisfies the condition

$$h_1 = \left( \frac{p}{q} \right)^2 h_2$$

given in statement (b) of Proposition 3. Now we extend this solution
for $\mu > 0$ and small. Applying the Implicit Function Theorem in system (15) in a neighbourhood of a known solution we have that if

$$\begin{vmatrix} \frac{\partial \eta_1}{\partial s} & \frac{\partial \eta_1}{\partial \xi_{20}} \\ \frac{\partial \eta_2}{\partial s} & \frac{\partial \eta_2}{\partial \xi_{20}} \end{vmatrix} = 0,$$

then there exist unique analytic functions $\eta_{20} = \eta_{20}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\eta_{20}(0) = 0$ and $S(0) = S^*$,

(ii) and $\varphi(s; 0, \xi_{20}, 0, 0, \mu)$ is a $S_{13}$-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

As in the previous case we get that

$$\left. \frac{\partial \eta_1}{\partial s} \right|_{s = S^*/4, \xi_{20} = \xi_{20}^*, \mu = 0} = (-1)^m \frac{16\sqrt{-2h_1}}{h_2} \neq 0,$$

and

$$\left. \frac{\partial \eta_2}{\partial s} \right|_{s = S^*/4, \xi_{20} = \xi_{20}^*, \mu = 0} = (-1)^{k+1} \frac{16\sqrt{-2h_2}}{h_1} \neq 0.$$ 

Now we have to calculate the other two derivatives $\partial \eta_1/\partial \xi_{20}$ and $\partial \eta_2/\partial \xi_{20}$ evaluated at $s = S^*/4, \xi_{20} = \xi_{20}^*$ and $\mu = 0$. We obtain these values derivating the solutions $\eta_1(\sigma(s); 0, \xi_{20}, 0, 0)$ and $\eta_2(\tau(s); 0, \xi_{20}, 0, 0)$ of system (4) with respect to the variable $\xi_{20}$, where the initial conditions $\xi_1(0) = 0, \xi_2(0) = \xi_{20}, \eta_1(0) = \eta_{10}, \eta_2(0) = 0$ satisfy the energy relation $G = 0$. So

$$\left. \frac{\partial \eta_1(\sigma(s); 0, \xi_{20}, 0, 0)}{\partial \xi_{20}} \right|_{s = S^*/4, \xi_{20} = \xi_{20}^*, \mu = 0} = \left( \frac{\partial \eta_1}{\partial \xi_{20}} + \frac{\partial \eta_1}{\partial \xi_{20}} \right)_{s = S^*/4, \xi_{20} = \xi_{20}^*, \mu = 0} = (-1)^m \sqrt{8(-h_1)^{3/2}} \left( 8\frac{\partial \sigma}{\partial \xi_{20}} \sqrt{\frac{2}{-h_2}} + \pi q \right),$$

and

$$\left. \frac{\partial \eta_2(\tau(s); 0, \xi_{20}, 0, 0)}{\partial \xi_{20}} \right|_{s = S^*/4, \xi_{20} = \xi_{20}^*, \mu = 0} = \left( \frac{\partial \eta_2}{\partial \xi_{20}} + \frac{\partial \eta_2}{\partial \xi_{20}} \right)_{s = S^*/4, \xi_{20} = \xi_{20}^*, \mu = 0} = 0.$$ 

Therefore, determinant (16) is

$$(-1)^{k+m+1} 64\pi q \sqrt{h_1/h_2} \neq 0.$$
6.2.2 Case (b)

From the second part of statement (b) of Proposition 1 the solution $\varphi$ is a $S_{13}$-symmetric periodic solution with period $S$ satisfying the energy condition $G = 0$ if and only if

$$\xi_1(S/4; \xi_{10}, \xi_{20}, \mu) = 0, $$
$$\eta_2(S/4; \xi_{10}, \xi_{20}, \mu) = 0,$$
$$G(\xi_{10}, \xi_{20}, \mu) = 0.$$

The solution of the equation $G = 0$ in terms of $\xi_{20}$ is given by $\xi_{20} = -\sqrt{2/\eta_2}$. So the solution satisfies $G = 0$ if and only if

$$\xi_1(S/4; \xi_{10}, \mu) = 0, $$
$$\eta_2(S/4; \xi_{10}, \mu) = 0. $$

(17)

Again by statement (b) of Proposition 4 the solutions are $S_{13}$-symmetric taking $p = 2m + 1$, $q = 2k$, $S = S^* = s(p T_1(h_1)) = s(q T_2(h_2))$, $\xi_{10}^* = -2/\eta_1$, $\eta_{20}^* = 4$, for $\mu = 0$. Furthermore, the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\frac{p}{q}\right)^{1/2} h_2$ given in statement (b) of Proposition 3. As in the previous case applying the Implicit Function Theorem to system (17) in a neighbourhood of a known solution we have that if

$$\left|\begin{array}{cc}
\frac{\partial \xi_1}{\partial s} & \frac{\partial \xi_1}{\partial \xi_{10}} \\
\frac{\partial \eta_2}{\partial s} & \frac{\partial \eta_2}{\partial \xi_{10}}
\end{array}\right|_{s = S^*/4, \xi_{10} = \xi_{10}^*, \mu = 0} \neq 0, $$

then there exist unique analytic functions $\xi_{10} = \xi_{10}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\xi_{10}(0) = \xi_{10}^*$ and $S(0) = S^*$,

(ii) and $\varphi(s; \xi_{10}, 0, \eta_{20}, \mu)$ is a $S_{13}$-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

Analogously to the previous case we obtain

$$\left.\frac{\partial \xi_1}{\partial s}\right|_{s = S^*/4, \xi_{10} = \xi_{10}^*, \mu = 0} = (-1)^m \frac{8}{h_2} \neq 0,$$

and

$$\left.\frac{\partial \eta_2}{\partial s}\right|_{s = S^*/4, \xi_{10} = \xi_{10}^*, \mu = 0} = 0.$$

So it remains to calculate $\partial \eta_2/\partial \xi_{10}$ evaluated at $s = S^*/4$, $\xi_{10} = \xi_{10}^*$ and $\mu = 0$. This derivative is obtained by derivating the solution $\eta_2(\tau(s); \xi_{10}, \xi_{20}, 0, 0, 0)$ of system
The symmetric periodic orbits for the two-electron atom evaluated in $\xi_{10} = \xi_{10}^*$ and $s = S^*/4$ with initial conditions $\xi_1(0) = \xi_{10}, \xi_2(0) = \xi_{20}, \eta_1(0) = 0$ and $\eta_2(0) = 0$, satisfying the energy relation $G = 0$. Then

$$\frac{\partial \eta_2(s; \xi_{10}, \xi_{20}, 0, 0, 0)}{\partial \xi_{10}} \bigg|_{\xi_{10} = \xi_{10}^*} = \left( \frac{\partial \eta_2(s) \partial \tau(s)}{\partial \tau \partial \xi_{10}} + \frac{\partial \eta_2}{\partial \xi_{10}} \right) \bigg|_{\xi_{10} = \xi_{10}^*} = (-1)^m \frac{\pi p}{h_1},$$

and determinant (18) is $8\pi p \neq 0$.

### 6.3 The $S_{23}$-symmetric periodic solutions

By Proposition 4 the solution $\varphi(s; \xi_{10}, \xi_{20}, \eta_{10}, \eta_{20}, \mu)$ of system (3) for $\mu = 0$ is a $S_{23}$-symmetric if the initial conditions satisfy

1. either $\xi_1(0) = \xi_{10}^*, \xi_2(0) = 0, \eta_1(0) = 0, \eta_2(0) = \eta_{20}^*$
2. or $\xi_1(0) = \xi_{10}^*, \xi_2(0) = \xi_{20}^*, \eta_1(0) = 0, \eta_2(0) = 0$

for fixed values of $h = h_1 + h_2$.

#### 6.3.1 Case (a)

By statement (c) of Proposition 1 the solution $\varphi$ is a $S_{23}$-symmetric periodic solution with period $S$ satisfying the energy condition $G = 0$ if and only if

$$\xi_1(S/4; \xi_{10}, \eta_{20}, \mu) = 0,$$
$$\eta_2(S/4; \xi_{10}, \eta_{20}, \mu) = 0,$$
$$G(\xi_{20}, \eta_{10}, \mu) = 0.$$

Solving the last equation in terms of $\eta_{10}$ we obtain $\eta_{20} = 4$. So the solution $\varphi(s; \xi_{10}, 0, 0, \eta_{20}, \mu)$ is a $S_{23}$-symmetric periodic solution of system (3) satisfying $G = 0$ if and only if

$$\xi_1(S/4; \xi_{10}, \mu) = 0,$$
$$\eta_2(S/4; \xi_{10}, \mu) = 0.$$

(19)

Statement (c) of Proposition 4 provides the initial conditions and the numbers $p$ and $q$, see Table 1 in order that the periodic orbits be $S_{23}$-symmetric. More precisely, if $p = 2m, q = 2k + 1, S = S^* = s(pT_1(h_1)) = s(qT_2(h_2)), \xi_{10}^* = \sqrt{-2/h_1}$ and $\eta_{20}^* = 4$, with $m$ and $k$ positive integers, then $\varphi(s; \xi_{10}^*, 0, 0, \eta_{20}^*, 0)$ is a $S_{23}$-symmetric solution for system (3) with $\mu = 0$. Moreover the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left( \frac{p}{q} \right)^{\frac{3}{2}} h_2$ given in statement (c) of Proposition 3. We
extend this solution to $\mu > 0$ and small. Applying the Implicit Function Theorem in system (19) in a neighbourhood of a known solution we have that if

$$\begin{vmatrix} \frac{\partial \xi_1}{\partial s} & \frac{\partial \xi_1}{\partial \eta_2} \\ \frac{\partial \xi_1}{\partial \eta_2} & \frac{\partial \xi_1}{\partial \xi_1} \end{vmatrix}_{s = S^* / 4, \xi_10 = \xi^*_1, \mu = 0} \neq 0,$$

then there exist unique analytic functions $\eta_1 = \eta_1(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\eta_1(0) = 0$ and $S(0) = S^*$,

(ii) and $\varphi(s; \xi_10, 0, 0, \eta_20, \mu)$ is a $S_{23}$-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

We have

$$\frac{\partial \xi_1}{\partial s}igg|_{s = S^* / 4, \xi_10 = \xi^*_1, \mu = 0} = 0,$$

and

$$\frac{\partial \eta_2}{\partial s}igg|_{s = S^* / 4, \xi_10 = \xi^*_1, \mu = 0} = (-1)^k \frac{16\sqrt{-2h_2}}{h_1} \neq 0.$$

So we have to calculate the derivative $\partial \xi_1 / \partial \xi_1$ evaluated at $s = S^* / 4, \xi_10 = \xi^*_1$ and $\mu = 0$. As in the previous cases we obtain that

$$\frac{\partial \xi_1(\sigma(s); \xi_10, 0, 0, \eta_20)}{\partial \xi_10} \bigg|_{s = S^* / 4, \xi_10 = \xi^*_1, \mu = 0} = \left( \frac{\partial \xi_1}{\partial \sigma} \frac{\partial \sigma(s)}{\partial \xi_10} + \frac{\partial \xi_1}{\partial \xi_10} \right) \bigg|_{s = S^* / 4, \xi_10 = \xi^*_1, \mu = 0} = (-1)^m.$$

Therefore, determinant (18) is $(-1)^{k+m} \frac{16\sqrt{-2h_2}}{h_1} \neq 0$.

6.3.2 Case (b)

From the second part of statement (c) of Proposition 1 the solution $\varphi$ is a $S_{23}$-symmetric periodic solution with period $S$ satisfying the energy condition $G = 0$ if and only if

$$\xi_1(S/4; \xi_10, \xi_20, \mu) = 0,$$

$$\xi_2(S/4; \xi_10, \xi_20, \mu) = 0,$$

$$G(\xi_10, \xi_20, \mu) = 0.$$
Solving $G = 0$ in terms of $\xi_{20}$ we obtain $\xi_{20} = \sqrt{-2/h_2}$. So the solution $\varphi(s; \xi_{10}, \xi_{20}, 0, 0, \mu)$ is a $S_{23}$-symmetric periodic solution of system (3) that satisfies $G = 0$ if and only if

$$
\begin{align*}
\xi_1(S/4; \xi_{10}, \mu) &= 0, \\
\xi_2(S/4; \xi_{10}, \mu) &= 0.
\end{align*}
$$

(20)

Again by statement (c) of Proposition 4 we have that if $p = 2m, q = 2k + 1$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$ and $\xi_{10}^* = \sqrt{-2/h_1}, \xi_{20}^* = \sqrt{-2/h_2}$ then the solution is $S_{23}$-symmetric periodic, for $\mu = 0$. Furthermore the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left( \frac{p}{q} \right) \frac{2}{3} h_2$ given in statement (c) of Proposition 3. As in the previous case applying the Implicit Function Theorem to system (20) in a neighbourhood of a known solution we have that if

$$
\begin{bmatrix}
\frac{\partial \xi_1}{\partial s} & \frac{\partial \xi_1}{\partial \xi_{10}} \\
\frac{\partial \xi_2}{\partial s} & \frac{\partial \xi_2}{\partial \xi_{10}}
\end{bmatrix}_{\xi_{10} = \xi_{10}^*, \mu = 0} \neq 0,
$$

(21)

then there exist unique analytic functions $\xi_{10} = \xi_{10}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\xi_{10}(0) = \xi_{10}^*$ and $S(0) = S^*$,

(ii) and $\varphi(s; \xi_{10}, \xi_{20}, 0, 0, \mu)$ is a $S_{23}$-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

Working as in the previous cases we get

$$
\frac{\partial \xi_1}{\partial s} \bigg|_{\xi_{10} = \xi_{10}^*, \mu = 0} = 0,
$$

and

$$
\frac{\partial \xi_2}{\partial s} \bigg|_{\xi_{10} = \xi_{10}^*, \mu = 0} = (-1)^k \frac{8}{h_1}.
$$

So it remains to calculate $\partial \xi_1/\partial \xi_{10}$ evaluated at $s = S^*/4$. $\xi_{10} = \xi_{10}^*$ and $\mu = 0$. This derivative is obtained deriving the solution $\xi_1(\sigma(s); \xi_{10}, \xi_{20}, 0, 0, 0)$ of system (4) evaluated at $\xi_{10} = \xi_{10}^*$ and $s = S^*/4$ with initial conditions $\xi_1(0) = \xi_{10}, \xi_2(0) = \xi_{20}, \eta_1(0) = 0, \eta_2(0) = 0$ satisfying the energy relation $G = 0$. Then

$$
\frac{\partial \xi_1(\sigma(s); \xi_{10}, \xi_{20}, 0, 0, 0)}{\partial \xi_{10}} \bigg|_{\xi_{10} = \xi_{10}^*, \xi_{10} = \xi_{10}^*} = \left( \frac{\partial \xi_1}{\partial \sigma} \frac{\partial \sigma}{\partial \xi_{10}} + \frac{\partial \sigma_1}{\partial \xi_{10}} \right) \bigg|_{\xi_{10} = \xi_{10}^*}.
$$
As the previous case, we obtain

$$\left. \frac{\partial \xi_1(\sigma(s); \xi_{10}, \xi_{20}, 0, 0)}{\partial \xi_{10}} \right|_{s = S^*/4} = \frac{(-1)^m}{\xi_{10}}$$

and determinant (21) is \((-1)^{k+m} \frac{8}{h_1}\).

7 Concluding remarks

In this paper, we have studied the periodic solutions of the two-electron atom which are $S_{12-}$, $S_{13-}$ and $S_{23-}$ symmetric. Applying the continuation method of Poincaré, we obtain that six families of symmetric periodic orbits can be extend from $\mu = 0$ to small positive values of $\mu$. In [2] and [7], it was studied similar problems, but in that papers the authors studied the collinear 3-body problem in the cases where the bodies are uncharged and charged, respectively. In both cases, it was applied the continuation method of Poincaré. In the first one only three families of periodic orbits can be extended from $\mu = 0$ to $\mu$ small and positive and in the case where the bodies are charged, similar as we obtain in this paper, six families of symmetric periodic orbits can be extended.

Acknowledgements  The first author is partially supported by the MINECO-FEDER Grant MTM2016-77278-P, a MINECO Grant MTM2013-40998-P and an AGAUR Grant Number 2014SGR-568. The second author is supported by PROCAD/CAPES Grant 88881.0 68462/2014-01 and by CNPq-Brazil.

References

1. Casasayas, J., Llibre, J.: Qualitative analysis of the anisotropic Kepler problem. Mem. Am. Math. Soc 52(312) (1984). https://doi.org/10.1090/memo/0312
2. Corbera, M., Llibre, J.: Developments in Mathematical and Experimental Physics: Volume C: Hydrodynamics and Dynamical Systems, Chapter Symmetric Periodic Orbits for the Collinear 3-body Problem Via the Continuation Method, pp. 117–141. Springer, Berlin (2003)
3. Corbera, M., Llibre, J.: Periodic orbits of a collinear restricted three-body problem. Celest. Mech. Dyn. Astron. 86, 163–183 (2003)
4. Ezrat, G.S., Richter, K., Tanner, G., Wintgen, D.: Semiclassical cycle expansion for the helium atom. J. Phys. B 24, 413–420 (1991)
5. Gutzwiller, M.C.: Chaos in Classical and Quantum Mechanics, 1 of Interdisciplinary Applied Mathematics. Springer, New York (1990)
6. Leopold, J.G., Richards, D.: The effect of a resonant electric field on a one-dimensional classical hydrogen atom. J. Phys. B 18, 3369–3394 (1985)
7. Llibre, J., Tonon, D.J.: Symmetric periodic orbits for the collinear charged 3-body problem. J. Math. Phys. 58, 022702 (2017)
8. McGehee, R.: Triple collision in the collinear three-body problem. Invent. Math. 27, 191–227 (1974)
9. Poincaré, H.: Les méthodes nouvelles de la mécanique céleste, Tome III. In: Invariants intégraux. Solutions périodiques du deuxième genre. Solutions doublement asymptotiques. Dover Publications, Inc., New York(1957)
10. Santoprete, M.: Symmetric periodic solutions of the anisotropic Manev problem. J. Math. Phys. 43, 3207–3219 (2002)
11. Tanner, G., Ezra, G.S., Wintgen, D., Richter, K.: Semiclassical cycle expansion for the helium atom. J. Phys. B 24, 413–420 (1991)
12. Tanner, G., Wintgen, D., Richter, K.: The semitwo-electron atom, Chaos: an interdisciplinary. J. Nonlinear Sci. 19, 19–33 (1992)