ON THE INVERSE POLETSKY INEQUALITY IN METRIC SPACES AND PRIME ENDS

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(Received March 6, 2022; revised July 7, 2022; accepted July 17, 2022)

Abstract. We study mappings defined in the domain of a metric space that distort the modulus of families of paths by the type of the inverse Poletsky inequality. It is proved that such mappings have a continuous extension to the boundary of the domain in terms of prime ends. Under some additional conditions, the families of such mappings are equicontinuous in the closure of the domain with respect to the space of prime ends.

1. Introduction

The paper is devoted to the study of mappings in metric spaces, in particular, their extension to the boundary of the domain in terms of prime ends (see, for example, [1], [2], [12], [16], [17], [35], [10] and [28]). In [24], we investigated the problem of the continuous extension of mappings satisfying Poletsky inequality in metric spaces. The key point of the above article is the connection between the cuts of the domain and the families of paths joining the concentric spheres that is possible due to [2, Theorem 10.8]. This article deals with mappings whose inverse satisfy modulus inequalities. Note that the corresponding case of homeomorphisms was partially considered in [24]. However, we will consider here a more general case when the mapping, generally speaking, does not have the inverse, but only satisfies a modulus estimate of a certain form (which coincides with Poletsky inequality for inverse mappings, if they exist). For this purpose, consider the following definition.

Everywhere further \((X, d, \mu)\) and \((X', d', \mu')\) are metric spaces with metrics \(d\) and \(d'\) and locally finite Borel measures \(\mu\) and \(\mu'\), correspondingly.
We will assume that \( \mu \) is a Borel measure such that \( 0 < \mu(B) < \infty \) for all balls \( B \) in \( X \). Let \( y_0 \in X', \ 0 < r_1 < r_2 < \infty \) and

\[
(1.1) \quad A(y_0, r_1, r_2) = \left\{ y \in X' : r_1 < d'(y, y_0) < r_2 \right\},
B(y_0, r) = \left\{ y \in X' : d'(y, y_0) < r \right\}, \quad S(y_0, r) = \left\{ y \in X' : d'(y, y_0) = r \right\}.
\]

Given sets \( E, F \subset X' \) and a domain \( D \subset X' \) denote by \( \Gamma(E, F, D) \) the family of all paths \( \gamma : [a, b] \to X' \) such that \( \gamma(a) \in E, \gamma(b) \in F \) and \( \gamma(t) \in D \) for \( t \in [a, b] \). Given a domain \( D \subset X \), a mapping \( f : D \to X' \) is an arbitrary continuous transformation \( x \mapsto f(x) \). Let \( f : D \to X' \), let \( y_0 \in f(D) \) and let \( 0 < r_1 < r_2 < d_0 = \sup_{y \in f(D)} d'(y, y_0) \). Now, we denote by \( \Gamma_f(y_0, r_1, r_2) \) the family of all paths \( \gamma \in D \) such that \( f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2)) \).

Given a continuous path \( \gamma : [a, b] \to X \) in \((X, d)\), we define its length as the supremum of the sums

\[
\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1}))
\]

over all partitions \( a = t_0 \leq t_1 \leq \cdots \leq t_k = b \) of the segment \([a, b] \). A path \( \gamma \) is called \textit{rectifiable} if its length is finite. Given a family of paths \( \Gamma \) in \( X \), a Borel function \( \rho : X \to [0, \infty] \) is called \textit{admissible} for \( \Gamma \), abbr. \( \rho \in \text{adm} \Gamma \), if

\[
(1.2) \quad \int_{\gamma} \rho \, ds \geq 1
\]

for all (locally rectifiable) \( \gamma \in \Gamma \). Given \( p \geq 1 \), the \( p \)-modulus of \( \Gamma \) is a number

\[
(1.3) \quad M_p(\Gamma) = \inf_{\rho \in \text{adm} \Gamma} \int_{X} \rho^p \, d\mu(x).
\]

Should \( \text{adm} \Gamma \) be empty, we set \( M_p(\Gamma) = \infty \). A family of paths \( \Gamma_1 \) in \( X \) is said to be \textit{minorized} by a family of paths \( \Gamma_2 \) in \( X \), abbr. \( \Gamma_1 > \Gamma_2 \), if, for every path \( \gamma_1 \in \Gamma_1 \), there is a path \( \gamma_2 \in \Gamma_1 \) such that \( \gamma_2 \) is a restriction of \( \gamma_1 \). In this case,

\[
(1.4) \quad \Gamma_1 > \Gamma_2 \implies M_p(\Gamma_1) \leq M_p(\Gamma_2)
\]

(see [11, Theorem 1]). Let \( Q : X' \to [0, \infty] \) be a measurable function such that \( Q(y) \equiv 0 \) for \( y \in X' \setminus f(D) \). Assume that \( D \) and \( X' \) have finite Hausdorff dimensions \( \alpha \) and \( \alpha' \geq 1 \), respectively. We will say that \( f \) satisfies the \textit{inverse Poletsky inequality} at a point \( y_0 \in f(D) \), if the relation

\[
(1.5) \quad M_\alpha(\Gamma_f(y_0, r_1, r_2)) \leq \int_{\Gamma_f(y_0, r_1, r_2)} Q(y) \cdot \eta^\alpha \left( d'(y, y_0) \right) \, d\mu'(y)
\]

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holds for any Lebesgue measurable function \( \eta: (r_1, r_2) \to [0, \infty] \) such that

\[
(1.6) \quad \int_{r_1}^{r_2} \eta(r) \, dr \geq 1.
\]

Note that the inequalities (1.5) are well known in the theory of quasiregular mappings and hold for \( Q = N(f, D) \cdot K \), where \( N(f, D) \) is the maximal multiplicity of \( f \) in \( D \), and \( K \geq 1 \) is some constant that may be calculated in the following way: \( K = \text{ess sup} K_O(x, f), \) \( K_O(x, f) = \| f'(x) \|^n / |J(x, f)| \) for \( J(x, f) \neq 0; K_O(x, f) = 1 \) for \( f'(x) = 0 \), and \( K_O(x, f) = \infty \) for \( f'(x) \neq 0 \), where \( J(x, f) = 0 \) (see, e.g., [19, Theorem 3.2] or [22, Theorem 6.7.II]).

Let \( X \) and \( X' \) be metric spaces. A mapping \( f: X \to X' \) is discrete if \( f^{-1}(y) \) is discrete for all \( y \in X' \) and \( f \) is open if \( f \) maps open sets onto open sets. A mapping \( f: G \to X' \) is closed in \( G \subset X \) if \( f(A) \) is closed in \( f(G) \) whenever \( A \) closed in \( G \). From now on we assume that the space \( X' \) is complete and supports \( \alpha'-\text{Poincaré} \) inequality, and that the measure \( \mu' \) is doubling (see [2]). In this case, a space \( X' \) is locally connected (see [2, Section 2]), and proper (see [4, Proposition 3.1]).

The definition and construction of prime ends used in this paper corresponds to the publications [2], cf. [24]. Suppose \( \Omega \) is a bounded open connected set in \( X' \). We call a bounded connected set \( E \subset \Omega \) an acceptable set if \( \overline{E} \cap \partial \Omega \neq \emptyset \). By discussion in [2], we know that boundless and connectedness of an acceptable set \( E \) implies that \( \overline{E} \) is compact and connected. Furthermore, \( E \) is infinite, as otherwise we would have \( \overline{E} = E \subset \Omega \). Therefore, \( \overline{E} \) is a continuum. Recall that a continuum is a connected compact set containing at least two points. In what follows, given \( A, B \subset X' \),

\[
\text{dist}(A, B) = \inf_{x \in A, y \in B} d'(x, y).
\]

We call a sequence \( \{E_k\}_{k=1}^\infty \) of acceptable sets a chain if it satisfies the following conditions:

1. \( E_{k+1} \subset E_k \) for all \( k = 1, 2, \ldots, \)
2. \( \text{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) > 0 \) for all \( k = 1, 2, \ldots, \)
3. The impression \( \bigcap_{k=1}^\infty \overline{E_k} \subset \partial \Omega \).

We say that a chain \( \{E_k\}_{k=1}^\infty \) divides the chain \( \{F_k\}_{k=1}^\infty \) if, for each \( k \in \mathbb{N} \) there exists \( l_k \) such that \( E_{l_k} \subset F_k \). Two chains are equivalent if they divide each other. A collection of all mutually equivalent chains is called an end and denoted by \( [E_k] \), where \( \{E_k\}_{k=1}^\infty \) is any of the chains in the equivalence class. The impression of \( [E_k] \), denoted by \( I[E_k] \), is defined as the impression of any representative chain. The collection of all ends is called the end boundary and is denoted by \( \partial_E \Omega \). We say that an end \( [E_k] \) is a prime end if it is not divisible by any other end. The collection of all prime ends is called the prime end boundary and is denoted by \( E_\Omega \).
Remark 1.1. By [2, Remark 4.5] we may consider that the sets $E_k$ are open. By the assumption $X'$ is complete and supports a $\alpha'$-Poincaré inequality, and that the measure $\mu'$ is doubling (see [2]). In this case, a space $X'$ is quasiconvex (see [8, Theorem 17.1]) and, consequently, $X'$ is locally path connected. By Mazurkiewicz–Moore–Menger theorem, $X'$ is locally arcwise connected, see [18, Theorem 1, Ch. 6, § 50, item II]. Since $E_k$ are domains, they are path connected for any $k,l \in \mathbb{N}$ (see [18, Theorem 2.I.50, Ch.6]).

In what follows, we set $\overline{\Omega}_P := \Omega \cup E_\Omega$. We say that $\Omega$ is finitely connected at a point $x_0 \in \partial \Omega$ if for any $r > 0$ there is an open set $G$ (open in $X'$) such that $x_0 \in G \subset B(x_0, r)$ and $G \cap \Omega$ has only finitely many components. If $\Omega$ is finitely connected at every boundary point, then it is called finitely connected at the boundary. The following results have been proved in [2].

Proposition 1.2 (see [2, Theorem 10.8]). Assume that $\Omega$ is finitely connected at the boundary. Then all prime ends have singleton impressions, and every $x \in \partial \Omega$ is the impression of a prime end and is accessible.

We say that a sequence of points $\{x_n\}_{n=1}^\infty$ in $\Omega$ converges to the end $[E_k]$, and write $x_n \to [E_k]$ as $n \to \infty$, if for any $k \in \mathbb{N}$ there exists $n_k$ such that $x_n \in E_k$ whenever $n \geq n_k$. The following most important statement is true.

Proposition 1.3 (see [2, Corollary 10.9]). Assume that $\Omega$ is finitely connected at the boundary. Then the prime end closure $\overline{\Omega}_P$ is metrizable with some metric $m_P: \overline{\Omega}_P \times \overline{\Omega}_P \to \mathbb{R}$ such that the topology on $\overline{\Omega}_P$ given by this metric is equivalent to the topology given by the sequential convergence discussed above.

Let us give the following definition (see [20, Section 13.3], cf. [36, Definition 17.5(4)] and [21, Definition 2.8]). Let $(X,d,\mu)$ be a metric space with finite Hausdorff dimension $\alpha \geq 1$. Following [23, Section 9], we say that a boundary of $D$ is weakly flat at a point $x_0 \in \overline{D}$ if, for any number $P > 0$ and any neighborhood $U$ of the point $x_0$, there is a neighborhood $V \subset U$ such that $M_\alpha(\Gamma(E,F,D)) \geq P$ for all continua $E$ and $F$ in $D$ intersecting $\partial U$ and $\partial V$. We say that the boundary $\partial D$ is weakly flat if the corresponding property holds at every point of the boundary. More generally, we will say that $D$ is weakly flat (as a metric space) if it is so at each of its point $x_0 \in D$. The above can be fully carried over to the space $X$ or $X'$, depending on the context.

One of the main statements of the article is the following theorem.

Theorem 1.4. Let $D$ and $D'$ be domains with finite Hausdorff dimensions $\alpha$ and $\alpha' \geq 2$ in spaces $(X,d,\mu)$ and $(X',d',\mu')$, respectively. Assume that $X$ is locally connected, $\overline{D}$ is compact, $X'$ is complete and supports $\alpha'$-Poincaré inequality, and that the measures $\mu$ and $\mu'$ are doubling. Let $D' \subset X'$ be a bounded domain which is finitely connected at the boundary, and
Let \( Q : X' \to (0, \infty) \) be integrable function in \( D' \), \( Q(y) \equiv 0 \) for \( y \in X' \setminus D' \). Suppose that \( f : D \to D' \), \( D' = f(D) \), is an open discrete and closed mapping satisfying the relation (1.5) at any point \( y_0 \in \partial D' \), moreover, suppose that \( D \) has a weakly flat boundary. Then \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D}'_p \) such that \( \overline{f}(\partial D) = \overline{D}'_p \).

A homeomorphism \( f : D' \to \mathbb{R}^n \), \( D' \subset X' \), is called a quasiconformal mapping if there is \( K \geq 1 \) such that

\[
(1.7) \quad M_n(f(\Gamma)) \leq K \cdot M_{\alpha'}(\Gamma)
\]

holds for any family of paths \( \Gamma \) in \( D' \), where \( \alpha' \) is a Hausdorff dimension of \( D' \). The most commonly discussed version of inequality (1.7) is when \( n = \alpha' \). Some authors, like Caraman and Cristea are also discussing classes of mappings with different powers, see e.g. [5], [6,7,9]. We say that the boundary of the domain \( D_0 \) in \( \mathbb{R}^n \) is locally quasiconformal, if each point \( x_0 \in \partial D_0 \) has a neighborhood \( U \) in \( \mathbb{R}^n \), which can be mapped by a quasiconformal mapping \( \varphi \) onto the unit ball \( \mathbb{B}^n \subset \mathbb{R}^n \) so that \( \varphi(\partial D_0 \cap U) \) is the intersection of \( \mathbb{B}^n \) with the coordinate hyperplane. We say that a bounded domain \( D' \) in \( X' \) is regular, if \( D' \) can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in \( \mathbb{R}^n \). Note that, this definition is slightly different from that given in [15].

Now let us talk on the equicontinuity of families of mappings in the closure of a domain. Given \( \delta > 0, M > 0 \) domains \( D \subset X, D' \subset X' \), and a continuum \( A \subset D' \) denote by \( \mathcal{G}_{\delta,A,M}(D,D') \) a family of all open discrete and closed mappings \( f \) of \( D \) onto \( D' \) such that the condition (1.5) holds for some \( Q = Q_f \) for any \( y_0 \in D' \) and such that \( d(f^{-1}(A), \partial D) \geq \delta \) and \( \|Q_f\|_{L^1(D')} \leq M \). The statements like the ones below have been established in various situations in [33, Theorem 1.2] and [25, Theorem 2]. To such a great degree of generality, this statement is proved for the first time.

**Theorem 1.5.** Let \( D \) and \( D' \) be domains with finite Hausdorff dimensions \( \alpha \) and \( \alpha' \geq 2 \) in spaces \( (X,d,\mu) \) and \( (X',d',\mu') \), respectively. Assume that \( X \) is locally connected, \( \overline{D} \) and \( \overline{D}' \) are compact sets, \( D \) has a weakly flat boundary, \( D \) is weakly flat as a metric space, \( X' \) is complete and supports \( \alpha' \)-Poincaré inequality, and that the measures \( \mu \) and \( \mu' \) are doubling. Let \( D' \subset X' \) be a regular domain which is finitely connected at the boundary. Then any \( f \in \mathcal{G}_{\delta,A,M}(D,D') \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D}'_p \), wherein \( \overline{f}(\partial D) = \overline{D}'_p \) and, in addition, the family \( \mathcal{G}_{\delta,A,M}(\overline{D},\overline{D}') \) of all extended mappings \( \overline{f} : \overline{D} \to \overline{D}'_p \) is equicontinuous in \( \overline{D} \).

Let us briefly describe the further structure of the article. Section 1 is Introduction. In Section 2, we formulate and prove the main assertion related to the continuous extension of mappings of the form (1.5) to the boundary of the domain in terms of prime ends. In particular, we present
the proof of Theorem 1.4. Section 3 is auxiliary to the subsequent sections. In particular, the statements contained in it refer to families of paths of a special form and estimates of their moduli, which are used below to prove the equicontinuity of mappings inside a domain and its closure. Section 4 is devoted to the equicontinuity of families of mappings satisfying estimates of the form (1.5) at inner points of the given domain. The exposition of this section is very short and is limited only to the formulation of the corresponding result. Section 5 is devoted to the proof of Theorem 1.5. Here we establish the equicontinuity of families of mappings in the closure of a domain, when the mappings satisfy condition (1.5), in addition, the pre-image of some fixed continuum under them is distant from the boundary of the domain on a fixed distance. Section 6 is devoted to the proof of the so-called continuum lemma. Namely, given a sequence of mappings satisfying (1.5) and some sequence of continua diameter of which is lower bounded and its image is a fixed continuum, we prove that these continua are located at a positive distance from the boundary of the domain. This lemma is important for the last Section 7. The final Section 7 is devoted to the equicontinuity of families of mappings with one normalization condition. To a considerable extent, this section builds on Sections 5 and 6. Also here are some examples of mappings and their families that satisfy the main statements of the manuscript.

2. Continuous extension of mappings to the boundary

Before proving the main result, we give some more definitions, and also prove one important statement.

Given a domain $D \subset X$, the cluster set $C(f, b)$ of $f: D \to Y$ at $b \in \partial D$ is the set of all points $z \in Y$ for which there exists a sequence $\{b_k\}_{k=1}^{\infty}$ in $D$ such that $b_k \to b$ and $f(b_k) \to z$ as $k \to \infty$. For a non-empty set $E \subset \partial D$ let $C(f, E) = \bigcup C(f, b)$, where $b$ ranges over set $E$.

Let $D \subset X$, $f: D \to X'$ be a discrete open mapping, $\beta: [a, b) \to X'$ be a path, and $x \in f^{-1}(\beta(a))$. A path $\alpha: [a, c) \to D$ is called a maximal $f$-lifting of $\beta$ starting at $x$, if

1. $\alpha(a) = x$;
2. $f \circ \alpha = \beta|_{[a, c)}$;
3. for $c < c' \leq b$, there is no path $\alpha': [a, c') \to D$ such that $\alpha = \alpha'|_{[a, c)}$ and $f \circ \alpha' = \beta|_{[a, c')}$.  

If $X$ and $X'$ are locally compact, $X$ is locally connected, and $f: D \to X'$ is discrete and open, then there is a maximal $f$-lifting of $\beta$ starting at $x$, see [29, Lemma 2.1]. We also prove an even more general statement (for the space $\mathbb{R}^n$ see, for example, [37, Theorem 3.7]).
Lemma 2.1. Let $X$ and $X'$ be metric spaces, let $X$ be locally connected, let $X'$ be locally compact, let $D$ be a domain in $X$, and let $f: D \to X'$ be a discrete open and closed mapping of $D$ onto $D' \subset X'$. Assume that $\overline{D}$ is compact. If $\beta: [a,b] \to X'$ is a path such that $x \in f^{-1}(\beta(a))$, then there is a whole $f$-lifting of $\beta$ starting at $x$; in other words, there is a path $\alpha: [a,b] \to X$ such that $f(\alpha(t)) = \beta(t)$ for any $t \in [a,b)$. Moreover, if $\beta(t)$ has a limit $\lim_{t \to b-0} \beta(t) := B_0 \in D'$, then $\alpha$ has a continuous extension to $b$ and $f(\alpha(b)) = B_0$.

Proof. Since $f|_D \to X'$ is a mapping of the locally compact space $D$ to $X'$, in addition, $X$ is locally connected and $X'$ is locally compact, the existence of a maximal $f$-lifting $\alpha: [a,b) \to X$ follows from Lemma [29, Lemma 2.1]. Let us prove that this lifting is whole, for which we use the general scheme from [24, Proof of Lemma 2.1] (cf. [32, Lemma 1]). Suppose the opposite, namely that $c \neq b$. Note that $\alpha$ can not tend to the boundary of $D$ as $t \to c - 0$, since $C(f, \partial D) \subset \partial D'$ by Proposition 2.1 in [24]. Then $C(\alpha, c) \subset D$.

Consider

$$G = \left\{ x \in X : x = \lim_{k \to \infty} \alpha(t_k), \quad t_k \in [a,c), \quad \lim_{k \to \infty} t_k = c. \right\}$$

Letting to subsequences, we may restrict us by monotone sequences $t_k$. For $x \in G$, by continuity of $f$, $f(\alpha(t_k)) \to f(x)$ as $k \to \infty$, where $t_k \in [a,c)$, $t_k \to c$ as $k \to \infty$. However, $f(\alpha(t_k)) = \beta(t_k) \to \beta(c)$ as $k \to \infty$. Thus, $f$ is a constant on $G$. As usual, given a path $\alpha: [a,b] \to X$ we define

$$|\alpha| = \left\{ x \in X : \exists t \in [a,b] : \alpha(t) = x \right\}$$

a locus of $\alpha$. Observe that $\overline{|\alpha|}$ is a compact set, because $\overline{|\alpha|}$ is a closed subset of the compact space $\overline{D}$ (see [18, Theorem 2.II.4, § 41]). Now, by Cantor condition on the compact $\overline{\alpha}$, by monotonicity of $\alpha([t_k,c))$,

$$G = \bigcap_{k=1}^{\infty} \overline{\alpha([t_k,c))} \neq \emptyset,$$

see [18, 1.II.4, § 41]. Now, by [18, Theorem 5.II.5, § 47], $G$ is connected. By discreteness of $f$, $G$ is a single-point set, and $\alpha: [a,c) \to D$ extends to a closed path $\alpha: [a,c] \to D$ and $f(\alpha(c)) = \beta(c)$. Then, by [29, Lemma 2.1] there is a new $f$-lifting $\alpha' : [c,c') \to D$ of $\beta$ starting at $\alpha(c)$. Uniting liftings $\alpha$ and $\alpha'$ we obtain a new $f$-lifting $\alpha: [a,c') \to D$ starting at $x$ that contradicts the maximality of $\alpha$. The contradiction obtained above proves that $c = b$.

Let there exists $\lim_{t \to b-0} \beta(t) := B_0 \in D'$. Arguing similarly to what was proved above, we obtain that the set $G$ consists of one point $p_0 \in D$ and,
therefore, the path $\alpha$ extends to the closed path $\alpha: [a, b] \to D$. The equality $f(\alpha(b)) = B_0$ follows from the continuity of the mapping $f$. \hfill \Box

PROOF OF THEOREM 1.4. I. Put $x_0 \in \partial D$. It is necessary to show the possibility of continuous extension of the mapping $f$ to $x_0$. Assume that the conclusion about the continuous extension of the mapping $f$ to the point $x_0$ is not correct. Then any prime end $P_0 \in E_{D'}$ is not a limit of $f$ at $x_0$. It follows that, there is some sequence $x_k \in D$, $k = 1, 2, \ldots$, $x_k \to x_0$ as $k \to \infty$, and a number $\varepsilon_0 > 0$ such that $m_P(f(x_k), P_0) \geq \varepsilon_0$ for any $k \in \mathbb{N}$, where $m_P$ is defined in Proposition 1.3. By [2, Theorem 10.10], $(\overline{D'}, m_P)$ is a compact metric space. Thus, we may assume that $f(x_k)$ converges to some $P_1 \neq P_0$, $P_1 \in \overline{D'}$ as $k \to \infty$. Since $f$ has no a limit at $x_0$ by the assumption, there is some sequence $y_k$, $y_k \to x_0$ as $k \to \infty$, such that $m_P(f(y_k), P_1) \geq \varepsilon_1$ for any $k \in \mathbb{N}$ and some $\varepsilon_1 > 0$. Since the space $(\overline{D'}, m_P)$ is compact, we may assume that $f(y_k) \to P_2$ as $k \to \infty$, $P_1 \neq P_2$, $P_2 \in \overline{D'}$. Since $f$ is closed, $f$ is boundary preserving, as well (see e.g. [24, Proposition 2.1]. Thus, $P_1, P_2 \in E_{D'}$.

II. Let $P_1 = [E_k], k = 1, 2, \ldots$, and $P_2 = [G_l], l = 1, 2, \ldots$ By Remark 1.1 we may consider that $E_k$ and $G_l$ are open and path connected for any $k, l \in \mathbb{N}$. Let us show that there exists $k_0 \in \mathbb{N}$ such that

\begin{equation}
E_k \cap G_k = \emptyset \quad \text{for all } k \geq k_0.
\end{equation}

Suppose the contrary, i.e., suppose that for every $l = 1, 2, \ldots$ there exists $k_l \in \mathbb{N}$, $l = 1, 2, \ldots$, such that $x_{k_l} \in E_{k_l} \cap G_{k_l}$. Now $x_{k_l} \to P_1$ and $x_{k_l} \to P_2$, $l \to \infty$. Let $m_P$ be the metric on $\overline{D'}$ defined in Proposition 1.3. By triangle inequality,

\begin{equation}
m_P(P_1, P_2) \leq m_P(P_1, x_{k_l}) + m_P(x_{k_l}, P_2) \to 0, \quad l \to \infty.
\end{equation}

It follows that $P_1 = P_2$, which contradicts the choice of $P_1$ and $P_2$. Thus, (2.1) holds, as required.

III. Denote $y_0 := I([E_k])$ (see Proposition 1.2). Arguing as in the proof of [24, Lemma 2.1], we may show that for every $r > 0$ there exists $N \in \mathbb{N}$ such that

\begin{equation}
E_k \subset B(y_0, r) \cap D' \quad \text{for all } k \geq N.
\end{equation}

Since $D'$ is connected and $E_{k_0 + 1} \neq D'$, we obtain that $\partial E_{k_0 + 1} \cap D' \neq \emptyset$ (see [18, Ch. 5, § 46, item I]). Set

\begin{equation}
r_0 := d'(y_0, \partial E_{k_0 + 1} \cap D').
\end{equation}

Due to the condition 2 on the page 263 we may assume that $r_0 > 0$. By (2.2), there is $m_0 \in \mathbb{N}$, $m_0 > k_0 + 1$, such that

\begin{equation}
E_k \subset B(y_0, r_0/2) \cap D' \quad \text{for all } k \geq m_0.
\end{equation}

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IV. Set $D_0 := E_{m_0 + 1}$, $D_* := G_{m_0 + 1}$. Let us to show that

$$(2.5) \quad \Gamma(D_0, D_*, D') > \Gamma(S(y_0, r_0/2), S(y_0, r_0), A(y_0, r_0/2, r_0)),$$

where $A(y_0, r_1, r_2)$ is defined in (1.1). Assume that $\gamma \in \Gamma(D_0, D_*, D')$, $\gamma: [0, 1] \to D'$. By (2.1), $|\gamma| \cap E_{k_0 + 1} \neq \emptyset \neq |\gamma| \cap (D' \setminus E_{k_0 + 1})$. Thus,

$$(2.6) \quad |\gamma| \cap \partial E_{k_0 + 1} \neq \emptyset$$

(see [18, Theorem 1, § 46, item I]). Moreover, observe that

$$(2.7) \quad \gamma(1) \notin \partial E_{k_0 + 1}.$$ 

Suppose the contrary, i.e. that $\gamma(1) \in \partial E_{k_0 + 1}$. By definition of a prime end, $\partial E_{k_0 + 1} \cap D' \subset E^{k_0}$. Since dist $(D' \cap \partial E_{k_0 + 1}, D' \cap \partial E_k) > 0$ for all $k = 1, 2, \ldots$, we obtain that $\partial E_{k_0 + 1} \cap D' \subset E_{k_0}$. Now, we have that $\gamma(1) \in E_{k_0}$ and, simultaneously, $\gamma(1) \in G_{m_0 + 1} \subset G_{k_0}$. The last relations contradict with (2.1). Thus, (2.7) holds, as required.

By (2.4), we obtain that $|\gamma| \cap B(y_0, r_0/2) \neq \emptyset$. We prove that

$$|\gamma| \cap (D' \setminus B(y_0, r_0/2)) \neq \emptyset.$$ 

In fact, if it is not true, then $\gamma(t) \in B(y_0, r_0/2)$ for every $t \in [0, 1]$. However, by (2.6) we obtain that $(\partial E_{k_0 + 1} \cap D') \cap B(y_0, r_0/2) \neq \emptyset$, that contradicts to the definition of $r_0$. Thus, $|\gamma| \cap (D' \setminus B(y_0, r_0/2)) \neq \emptyset$, as required. Now, by [18, Theorem 1, § 46, item I], there exists $t_1 \in (0, 1]$ with $\gamma(t_1) \in S(y_0, r_0/2)$. We may consider that $t_1 = \max\{t \in [0, 1] : \gamma(t) \in S(y_0, r_0/2)\}$. We prove that $t_1 \neq 1$. Suppose the contrary, i.e., suppose that $t_1 = 1$. Now, we obtain that $\gamma(t) \in B(y_0, r_0/2)$ for every $t \in (0, 1)$. On the other hand, by (2.6) and (2.7), we obtain that $\partial E_{k_0 + 1} \cap B(y_0, r_0/2) \neq \emptyset$, which contradicts to the definition of $r_0$. Thus, $t_1 \neq 1$, as required. Set $\gamma_1 := \gamma|_{[t_1, 1]}$.

By definition, $|\gamma_1| \cap B(y_0, r_0) \neq \emptyset$. We prove that

$$|\gamma_1| \cap (D' \setminus B(y_0, r_0)) \neq \emptyset.$$ 

In fact, assume the contrary, i.e., assume that $\gamma_1(t) \in B(y_0, r_0)$ for every $t \in [t_1, 1]$. Since $\gamma(t) \in B(y_0, r_0/2)$ for $t < t_1$, by (2.6) we obtain that $|\gamma_1| \cap \partial E_{k_0 + 1} \neq \emptyset$. Consequently, $B(y_0, r_0) \cap (\partial E_{k_0 + 1} \cap D') \neq \emptyset$, that contradicts to the definition of $r_0$. Thus, $|\gamma_1| \cap (D' \setminus B(y_0, r_0)) \neq \emptyset$, as required. Now, by [18, Theorem 1, § 46, item I], there exists $t_2 \in (t_1, 1]$ with $\gamma(t_2) \in S(y_0, r)$. We may consider that $t_2 = \min\{t \in [t_1, 1] : \gamma(t) \in S(y_0, r_0)\}$. We put $\gamma_2 := \gamma|_{[t_1, t_2]}$. Observe that $\gamma > \gamma_2$ and

$$\gamma_2 \in \Gamma(S(y_0, r_0/2), S(y_0, r_0), A(y_0, r_0/2, r_0)).$$

Thus, (2.5) has been proved.
\[ M_\alpha(\Gamma(|\alpha_k|, |\beta_k|, D)) > P \quad \text{for all } k \geq k_P. \]

VI. We show that the condition (2.8) contradicts (1.5). Indeed, since \( f(\Gamma(|\alpha_k|, |\beta_k|, D)) \subset \Gamma(D_0, D_*, D') \), by (2.5) we obtain that

\[ \Gamma(|\alpha_k|, |\beta_k|, D) > \Gamma_f(y_0, r_0/2, r_0). \]
From the last relation, by minorization principle of the modulus (see, e.g., [11, Theorem 1(c)])

\[ M_\alpha(\Gamma(|\alpha_k|, |\beta_k|, D)) \leq M_\alpha(\Gamma_f(y_0, r_0/2, r_0)). \]

Set

\[ \eta(t) = \begin{cases} \frac{2}{r_0}, & t \in [r_0/2, r_0], \\ 0, & t \not\in [r_0/2, r_0]. \end{cases} \]

Observe that \( \eta \) satisfies the relation (1.6) for \( r_1 := r_0/2 \) and \( r_2 := r_0 \). Now, by (1.5) and (2.9) we obtain that

\[ M_\alpha(\Gamma(|\alpha_k|, |\beta_k|, D)) \leq \frac{2}{r_0^{\alpha'}} \int_{D'} Q(y) d\mu'(y) := c < \infty \quad \text{for all} \quad k \in \mathbb{N}, \]

because \( Q \in L^1(D') \). The relation (2.10) contradicts the condition (2.8). The contradiction obtained above refutes the assumption that there is no limit of the mapping \( f \) at the point \( x_0 \).

It remains to show that \( \overline{f(D)} = \overline{D}'P \). Obviously \( f(D) \subset \overline{f(D)} \). Let us to show that \( D'_P \subset \overline{f(D)} \). Indeed, let \( y_0 \in D'_P \). Then either \( y_0 \in D' \), or \( y_0 \in E_{D'} \). If \( y_0 \in D' \), then \( y_0 = f(x_0) \) and \( y_0 \in f(D) \), because \( f \) maps \( D \) onto \( D' \). Finally, let \( y_0 \in E_{D'} \). Then there is a sequence \( y_k \in D' \) such that \( m_P(y_k, y_0) \to 0 \) as \( k \to \infty \), \( y_k = f(x_k) \) and \( x_k \in D \), where \( m_P \) is a metric defined in Proposition 1.3. Since \( D \) is a compactum, we may assume that \( x_k \to x_0 \), where \( x_0 \in D \). Note that \( x_0 \in \partial D \), because \( f \) is open. Thus \( f(x_0) = y_0 \in \overline{f(\partial D)} \subset \overline{f(D)} \). Theorem is completely proved. \( \square \)

**3. Auxiliary lemmas**

Our immediate goal is the equicontinuity (global behavior) of mappings in the closure of a domain in a metric space. We note that, for mappings with a direct Poletsky inequality, similar results were established in [24, Section 5]. Our goal is to investigate similar mappings with the inverse Poletsky inequality. For this purpose, we use some approach used under consideration of similar problems in the Euclidean space, see [15]. The following almost obvious statement is true, see e.g. [30, Proposition 1].

**Proposition 3.1.** Let \( n \geq 2 \) and let \( D \) be a domain in \( \mathbb{R}^n \) that is locally connected on its boundary. Then every two pairs of points \( a \in D, b \in \overline{D} \) and \( c \in D, d \in \overline{D} \) can be joined by non-intersecting paths \( \gamma_1: [0, 1] \to \overline{D} \) and \( \gamma_2: [0, 1] \to \overline{D} \) so that \( \gamma_i(t) \in D \) for all \( t \in (0, 1) \) and all \( i = 1, 2 \), while \( \gamma_1(0) = a, \gamma_1(1) = b, \gamma_2(0) = c \) and \( \gamma_2(1) = d \).

The proof of the following statement completely repeats the proof of [36, Theorem 17.10], and therefore is omitted.
**Proposition 3.2.** Let $D \subset \mathbb{R}^n$ be a domain with a locally quasiconformal boundary. Then the boundary of this domain is weakly flat. Moreover, in the definition of a locally quasiconformal boundary, we may take an arbitrarily small neighborhood $U$, wherein $\varphi(x_0) = 0$.

**Lemma 3.3.** Let $D'$ be a bounded regular domain in $X$ with a finite Hausdorff dimension $\alpha'$ which is finitely connected at the boundary and, besides that, $\overline{D'}$ is compact. Assume that $X'$ is complete and supports $\alpha'$-Poincaré inequality, and that the measure $\mu'$ is doubling. Let $h$ be a quasiconformal mapping of $D'$ onto $D_0 \subset \mathbb{R}^n$, where $D_0$ is a domain with a locally quasiconformal boundary. If $P \in E_{D'}$, then $h(P) \in E_{D_0}$.

**Proof.** Let $D = \{E_k\}_{k=1}^\infty$, where $E_k$, $k = 1, 2, \ldots$, is a corresponding chain of acceptable sets, so that $E_k$ is connected set such that $E_k \subset D'$, $\overline{E_k} \cap \partial D' \neq \emptyset$, in addition,

1. $E_{k+1} \subset E_k$ for all $k = 1, 2, \ldots$,
2. $\text{dist}(D' \cap \partial E_{k+1}, D' \cap \partial E_k) > 0$ for all $k = 1, 2, \ldots$,
3. $I(P) := \bigcap_{k=1}^\infty \overline{E_k} \subset \partial D'$.

Let us show, first of all, that the above homeomorphism $h$ preserves the above mentioned properties 1–3. Since $h$ is a homeomorphism, $h(E_k)$ is connected for any $k = 1, 2, \ldots$ (see [18, Ch. 5, § 46, item I, Theorem 3]), $h(E_k) \neq D_0$ and $h(E_k) \cap \partial D_0 \neq \emptyset$ (see [20, Proposition 13.5]). The relations $h(E_{k+1}) \subset h(E_k)$, $k = 1, 2, \ldots$, are obvious. Let us to prove that

$$\text{(3.1)} \quad \text{dist}(D_0 \cap \partial h(E_{k+1}), D_0 \cap \partial h(E_k)) > 0 \quad \text{for any } k = 1, 2, \ldots. $$

Suppose the opposite, namely, let

$$\text{(3.2)} \quad \text{dist}(D_0 \cap \partial h(E_k), D_0 \cap \partial h(E_{k+1})) = 0. $$

Set

$$\rho(x) = \begin{cases} (\text{dist}(D' \cap \partial E_k, D' \cap \partial E_{k+1}))^{-1}, & x \in D', \\ 0, & \text{otherwise}. \end{cases}$$

Let $\Gamma = \Gamma(D' \cap \partial E_k, D' \cap \partial E_{k+1}, D')$. Obviously that, $\rho \in \text{adm}\Gamma$ and, by definition of modulus in (1.3),

$$\text{(3.3)} \quad M_{\alpha'}(\Gamma) \leq \frac{1}{(\text{dist}(D' \cap \partial E_k, D' \cap \partial E_{k+1}))^{\alpha'}} \cdot \mu(D') < \infty. $$

Here we took into account that $\overline{D'}$ is compact and $\mu(B) > 0$ for any ball $B \subset X$ so that $\mu(D') < \infty$. On the other hand, since $h$ is a homeomorphism we obtain that

$$\text{(3.4)} \quad M_n(\Gamma(D_0 \cap \partial h(E_k), D_0 \cap \partial h(E_{k+1}), D_0)) = M_n(h(\Gamma)). $$
It follows from (3.2) that there are $x_m \in D_0 \cap \partial h(E_k)$ and $y_m \in D_0 \cap \partial h(E_{k+1})$ such that $|x_m - y_m| \to 0$ as $m \to \infty$. Since $D_0$ is bounded, we may consider that $x_m \to x_0 \in D_0 \cap \partial h(E_k)$ and $y_m \to x_0 \in D_0$ as $m \to \infty$. Now, by the triangle inequality, $d(x_0, y_0) \leq d(x_0, x_m) + d(x_m, y_m) + d(y_m, y_0) \to 0$ as $m \to \infty$. Thus, $x_0 = y_0 \in D_0 \cap \partial h(E_k) \cap D_0 \cap \partial h(E_{k+1})$. By Proposition 3.2, $D_0$ has a weakly flat boundary. By [36, Theorem 17.10], the modulus of families of paths joining the sets with a common point in a domain with a weakly flat boundary equals to infinity. Thus, by (3.4)

\[ M_n(h(\Gamma)) = \infty. \]

The relation (3.5) contradicts with (3.3), because, by the quasiconformality of $h$, we obtain that

\[ M_n(h(\Gamma)) \leq K \cdot M_\alpha(\Gamma) < \infty. \]

Thus, (3.1) holds, as required.

It remains to show the validity of property 3 on a page 272 for the mapped family of acceptable sets $h(E_k)$, $k = 1, 2, \ldots$. Let $y \in \bigcap_{k=1}^{\infty} \overline{h(E_k)}$, then $y \in \overline{h(E_k)}$ for any $k \in \mathbb{N}$. Now, $y = \lim_{l \to \infty} y_{l,k}^k$, where $y_{l,k}^k = h(x_{l,k}^k)$, and $x_{l,k}^k \in E_k$, $l = 1, 2, \ldots$. Therefore, for any $k \in \mathbb{N}$ there is a number $l_k$, $k = 1, 2, \ldots$, such that $|h(x_{l_k,k}^k) - y| < 1/2^k$. Since $\overline{E_1}$ is compact (see [2]), we may consider that $x_{l_k,k}^k$ converge to some $x_0 \in \overline{E_1}$ as $k \to \infty$. Thus, $y \in \partial D_0$ (see [20, Proposition 13.5]) and, consequently, $\bigcap_{k=1}^{\infty} \overline{h(E_k)} \subset \partial D_0$. Thus, the chain of cuts $h(E_k)$, $k = 1, 2 \ldots$, defines some end $h(P)$.

It remains to show that the end $h(P)$ is prime. Let us show that the set $I(h(P))$ consists of exactly one point. To do this, we show that a mapping $h$ is the so-called ring $Q$-mapping with $Q := K$ (see, e.g., [24]). Let $0 < r_1 < r_2 < \infty$ and let $\eta: (r_1, r_2) \to [0, \infty]$ be a Lebesgue measurable function such that $\int_{r_1}^{r_2} \eta(r) \, dr \geq 1$, let $x_0 \in \partial D'$, $S_i = S(x_0, r_i)$ and let $A = A(x_0, r_1, r_2) = \{x \in X': r_1 < d(x, x_0) < r_2\}$. Put $\Gamma = \Gamma(S_1, S_2, A \cap D')$. Set

\[ \rho(x) = \begin{cases} \eta(d'(x, x_0)), & x \in A \cap D', \\ 0, & \text{otherwise}. \end{cases} \]

Let $\gamma$ be a locally rectifiable path in $\Gamma$. Then by [20, Proposition 13.4]

\[ \int_{\gamma} \rho(x) \, |dx| \geq \int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \]

So, $\rho \in \text{adm} \, \Gamma$ and, consequently, by definition of $h$ in (1.7) and by definition of the modulus of families of paths in (1.3)

\[ M_n(h(\Gamma)) \leq K \cdot M_{\alpha'}(\Gamma) \leq \int_{A} K \eta^{\alpha'}(d'(x, x_0)) \, d\mu'(x). \]

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The latter means that $h$ is a ring $K$-mapping at $x_0$, as required. Since $D_0$ has a quasiconformal boundary, $\partial D_0$ is weakly flat by Proposition 3.2 and, therefore, is strongly accessible (see e.g. [20, Proposition 13.6]). Besides that, $\overline{D_0}$ is a compactum by the assumption of the lemma. Then, by [24, Theorem 1.1] $h$ has a continuous extension $h : \overline{D}_p \to \overline{D}_0$.

Let now

$$y \in I(h(P)) := \bigcap_{k=1}^{\infty} h(E_k),$$

then by proving above $y = \lim_{k \to \infty} h(x_k^k)$, where $x_k^k$ is some sequence converging to $P$ as $k \to \infty$. Since $h$ has a continuous extension $h : \overline{D}_p \to \overline{D}_0$, $y$ is a one-point set, as required. Now, $h(P)$ is a prime end (see e.g. [2, Proposition 7.1]). □

Let $D'$ be a domain in a locally connected space $X$ and let $a \in D'$. Then we may define a sequence $V_k, k = 1, 2, \ldots$, of neighborhoods of a point $a$ such that $V_{k+1} \subset V_k$, $\text{dist}(\partial V_{k+1}, \partial V_k) > 0$ and $\bigcap_{k=1}^{\infty} V_k = a$. Two arbitrary such sequences $\{V_k\}_{k=1}^{\infty}$ and $\{U_k\}_{k=1}^{\infty}$ will be considered equivalent. In what follows, by a “prime end” corresponding to the point $a$, we mean the equivalence class of the sequences of “admissible sets” $V_k, k = 1, 2, \ldots$ indicated above.

Let us establish the following statement (see, for example, [15, Lemma 2.1]).

**Lemma 3.4.** Let $D'$ be a regular domain in $X'$ with a finite Hausdorff dimension $\alpha'$ which is finitely connected at the boundary. Assume that $X'$ is complete and supports $\alpha'$-Poincaré inequality, and that the measure $\mu'$ is doubling. Let $x_m \to P_1, y_m \to P_2$ as $m \to \infty$, $P_1, P_2 \in \overline{D}_p$, $P_1 \neq P_2$. Suppose that $d_m, g_m, m = 1, 2, \ldots$, are sequences of acceptable sets corresponding to $P_1$ and $P_2$, $d_1 \cap g_1 = \emptyset$, and $x_0, y_0 \in D' \setminus (d_1 \cup g_1)$. Then there are arbitrarily large $k_0 \in \mathbb{N}$, $M_0 = M_0(k_0) \in \mathbb{N}$ and $0 < t_1 = t_1(k_0), t_2 = t_2(k_0) < 1$ for which the following condition is fulfilled: for each $m \geq M_0$ there are non-intersecting paths $\gamma_{1,m} : [0, 1] \to D'$ and $\gamma_{2,m} : [0, 1] \to D'$ such that

$$\gamma_{1,m}(t) = \begin{cases} \tilde{\alpha}(t), & t \in [0, t_1], \\ \tilde{\alpha}_m(t), & t \in [t_1, 1] \end{cases}, \quad \gamma_{2,m}(t) = \begin{cases} \tilde{\beta}(t), & t \in [0, t_2], \\ \tilde{\beta}_m(t), & t \in [t_2, 1] \end{cases}$$

wherein

1. $\gamma_{1,m}(0) = x_0, \gamma_{1,m}(1) = x_m, \gamma_{2,m}(0) = y_0$ and $\gamma_{2,m}(1) = y_m$;
2. $|\gamma_{1,m}| \cap \overline{g_{k_0}} = \emptyset = |\gamma_{2,m}| \cap \overline{d_{k_0}}$;
3. $\tilde{\alpha}_m(t) \in d_{k_0+1}$ for $t \in [t_1, 1]$ and $\tilde{\beta}_m(t) \in g_{k_0+1}$ for $t \in [t_2, 1]$ (see Figure 2).

**Proof.** Let $P_1, P_2 \in E'_p$. By Remark 1.1 we may consider that the sets $d_k$ and $g_k, k = 1, 2, \ldots$, are open and path connected. If $P_1 \in D'$ or $P_2 \in D'$, then the sets $d_k$ and $g_k, k = 1, 2, \ldots$, are well defined, in addition, we may consider that $d_k$ and $g_k$ are open and path connected, as well.
We use a slightly modified scheme for the proof of [15, Lemma 2.1]. Since, by condition, \( D' \) is a regular domain, it can be mapped onto some domain with a locally quasiconformal boundary by (some) quasiconformal mapping \( h: D' \rightarrow D_0 \). Note that the domain \( D_0 \) is locally connected on its boundary, which follows directly from the definition of local quasiconformality. By Lemma 3.3 \( h(P_1) \) and \( h(P_2) \) are prime ends in \( E_{D_0} \). Observe that \( I(P_1) \) and \( I(P_2) \) are different points \( a \) and \( b \) in \( \partial D_0 \) whenever \( P_1 \neq P_2 \) (see e.g. [24, Lemma 4.1]). If \( P_1 \) or \( P_2 \) are inner points of \( D' \), then \( h(P_1) \) (or \( h(P_2) \)) are inner points of \( D_0 \), which we denote by \( a \) or \( b \), respectively.

Since by assumption \( x_0, y_0 \in D' \setminus (d_1 \cup g_1) \), then, in particular, \( P_1 \neq x_0 \neq P_2 \), \( P_1 \neq y_0 \neq P_2 \). This implies that \( a, b, h(x_0) \) and \( h(y_0) \) are four different points in \( \overline{D_0} \), at least two of which are inner points of \( D_0 \) (see Figure 3). By Proposition 3.1, one can join pairs of points \( a, h(x_0) \) and \( b, h(y_0) \) by disjoint paths.
$\alpha: [0, 1] \to \overline{D_0}$ and $\beta: [0, 1] \to \overline{D_0}$ so that $|\alpha| \cap |\beta| = \emptyset$, $\alpha(t), \beta(t) \in D_0$ for all $t \in (0, 1)$, $\alpha(0) = h(x_0), \alpha(1) = a$, $\beta(0) = h(y_0)$ and $\beta(1) = b$. Since $\mathbb{R}^n$ is a normal topological space, the loci $|\alpha|$ and $|\beta|$ have non-intersecting open neighborhoods $U, V$ such that

$$|\alpha| \subset U, \quad |\beta| \subset V.$$  

(3.7)

Here two cases are possible: either $h(P_1)$ is a prime end in $E_{D_0}$, or a point in $D_0$. Let $h(P_1)$ be a prime end in $E_{D_0}$. Since $I(h(P_1)) = a$, then there is a number $k_1 \in \mathbb{N}$ such that $\overline{h(d_k)} \subset U$ for $k \geq k_1$. The same is true if $h(P_1)$ is a point in $D_0$. In any of these two cases, $\overline{h(d_k)} \subset U$ for $k \geq k_1$. Similarly, there is a number $k_2 \in \mathbb{N}$ such that $\overline{h(g_k)} \subset V$ for all $k \geq k_2$. Then for $k_0 := \max\{k_1, k_2\}$ we obtain that

$$\overline{h(d_k)} \subset U, \quad \overline{h(g_k)} \subset V, \quad U \cap V = \emptyset, \quad k \geq k_0.$$  

(3.8)

Since a sequence $x_m$ converges to $P_1$ as $m \to \infty$, then $h(x_m)$ converges to $a$. Therefore, there is a number $m_1 \in \mathbb{N}$ such that $h(x_m) \in h(d_{k_0+1})$ for $m \geq m_1$. Similarly, since $y_m$ converges to $P_2$ as $m \to \infty$, then $h(y_m)$ converges to $b$. Therefore, there is a number $m_2 \in \mathbb{N}$ such that $h(y_m) \in h(g_{k_0+1})$ for $m \geq m_2$. Put $M_0 := \max\{m_1, m_2\}$. Show that

$$|\alpha| \cap h(d_{k_0+1}) \neq \emptyset, \quad |\beta| \cap h(g_{k_0+1}) \neq \emptyset.$$  

(3.9)

It suffices to establish the first of these relations, since the second relation can be proved similarly. If $a = h(P_1)$ is an inner point of $D_0$, then this inclusion is obvious. Now suppose that $h(P_1) \in E_{D_0}$. Since $D_0$ has a locally quasiconformal boundary, there is a sequence of spheres $S(0, 1/2^k)$, $k = 0, 1, 2, \ldots$, a decreasing sequence of neighborhoods $U_k$ of the point $a$ and some quasiconformal mapping $\varphi: U_0 \to \mathbb{B}^n$, for which

$$\varphi(U_k) = B(0, 1/2^k), \quad \varphi(\partial U_k \cap D_0) = S(0, 1/2^k) \cap \mathbb{B}^n_+,$$

where $\mathbb{B}_+^n = \{x = (x_1, \ldots, x_n): |x| < 1, x_n > 0\}$ (see the arguments given in the proof of [21, Lemma 3.5]). Note that $U_k \cap D_0$ is a domain, since $U_k \cap D_0 = \varphi^{-1}(B_+(0, 1/2^k))$, $B_+(0, 1/2^k) = \{x = (x_1, \ldots, x_n): |x| < 1/2^k, x_n > 0\}$, and $\varphi$ is a homeomorphism.

Observe that the sequence of domains $U_k \cap D_0$ corresponds to some prime end, the impression of which is the point $a$. Moreover, since $D_0$ is a domain with a locally quasiconformal boundary, $D_0$ is locally connected on the boundary and, therefore, the prime end $h(P_1)$ with an impression $a$ is unique (see [2, Corollary 10.14]). Therefore any domain $h(d_m)$ contains all domains $U_k \cap D_0$, except for a finite number, and vice versa. In particular, there is $s_0 \in \mathbb{N}$ such that $U_k \cap D_0 \subset h(d_{k_0+1})$ for all $k \geq s_0$. Since
a ∈ |α|, there is \( t_1 \in (0, 1) \) such that \( p := \alpha(t_1) \in U_{s_0} \cap D_0 \). But then also \( p \in h(d_{k_0+1}) \), since \( U_{s_0} \cap D_0 \subset h(d_{k_0+1}) \). The first relation in (3.9) is proved. As we said above, the second relation may be proved in exactly the same way.

So, let \( p := \alpha(t_1) \in |\alpha| \cap h(d_{k_0+1}) \). Fix \( m \geq M_0 \) and join the point \( p \) with the point \( h(x_m) \) using the path \( \alpha_m: [t_1, 1] \to h(d_{k_0+1}) \) so that \( \alpha_m(t_1) = p \), \( \alpha_m(1) = h(x_m) \), what is possible because \( h(d_{k_0+1}) \) is a domain. Set

\[
(3.10) \quad \gamma^*_{1,m}(t) = \begin{cases} \alpha(t), & t \in [0, t_1], \\ \alpha_m(t), & t \in [t_1, 1]. \end{cases}
\]

Note that the path \( \gamma^*_{1,m} \) completely lies in \( U \). Reasoning similarly, we have the point \( t_2 \in (0, 1) \) and the point \( q := \beta(t_2) \in |\beta| \cap h(g_{k_0+1}) \). Fix \( m \geq M_0 \) and join the point \( q \) with the point \( h(y_m) \) using the path \( \beta_m: [t_2, 1] \to h(g_{k_0+1}) \) so that \( \beta_m(t_2) = q \), \( \beta_m(1) = h(y_m) \), that is possible, because \( h(g_{k_0+1}) \) is a domain. Set

\[
(3.11) \quad \gamma^*_{2,m}(t) = \begin{cases} \beta(t), & t \in [0, t_2], \\ \beta_m(t), & t \in [t_2, 1]. \end{cases}
\]

Note that the path \( \gamma^*_{2,m} \) completely lies in \( V \). Set

\[
(3.12) \quad \gamma_{1,m} := h^{-1}(\gamma^*_{1,m}), \quad \gamma_{2,m} := h^{-1}(\gamma^*_{2,m}).
\]

Note that the paths \( \gamma_{1,m} \) and \( \gamma_{2,m} \) satisfy all the conditions of Lemma 3.4 for \( m \geq M_0 \). In fact, by definition, these paths join the points \( x_m, x_0 \) and \( y_m, y_0 \), respectively. The paths \( \gamma_{1,m} \) and \( \gamma_{2,m} \) do not intersect, since their images under the mapping \( h \) belong to non-intersecting neighborhoods \( U \) and \( V \), respectively. Note also that \( |\gamma_{1,m}| \cap g_{k_0} = \emptyset \) for \( m \geq M_0 \). Indeed, if \( x \in |\gamma_{1,m}| \cap g_{k_0} \), then either \( x \in |\gamma_{1,m}| \cap g_{k_0} \) or \( x \in |\gamma_{1,m}| \cap \partial g_{k_0} \). In the first case, \( h(x) \in |\gamma^*_{1,m}| \cap h(g_{k_0}) \subset U \cap h(g_{k_0}) \), which is impossible due to the relation (3.8). In the second case, if \( x \in |\gamma_{1,m}| \cap \partial g_{k_0} \), then there is a sequence \( z_m \in g_{k_0} \) such that \( z_m \to x \) as \( m \to \infty \). Now \( h(z_m) \to h(x) \) as \( m \to \infty \) and, therefore, \( h(x) \in \overline{h(g_{k_0})} \). At the same time, \( h(x) \in U \), and this is impossible by virtue of relation (3.8). Thus, the relation \( |\gamma_{1,m}| \cap g_{k_0} = \emptyset \) for \( m \geq M_0 \) is established. Similarly, \( |\gamma_{2,m}| \cap g_{k_0} = \emptyset \) for \( m \geq M_0 \). Finally, defining paths \( \tilde{\alpha}, \tilde{\alpha}_m, \tilde{\beta} \) and \( \tilde{\beta}_m \) by means of relations \( \tilde{\alpha} = h^{-1}(\alpha) \), \( \tilde{\alpha}_m = h^{-1}(\alpha_m) \), \( \tilde{\beta} = h^{-1}(\beta) \) and \( \tilde{\beta}_m = h^{-1}(\beta_m) \), we see that these paths correspond to the construction of \( \gamma_{1,m} \) and \( \gamma_{2,m} \) and also satisfy conditions 3) from the formulation of the lemma. Lemma 3.4 is proved. □

Consider the family of paths joining \( |\gamma_{1,m}| \) and \( |\gamma_{2,m}| \) in \( D' \) from the previous lemma. The following statement contains the upper estimate of
the modulus of the mapped family of paths under the mapping \( f \) with the inequality (1.5).

**Lemma 3.5.** Let \( D \) and \( D' \) be domains with finite Hausdorff dimensions \( \alpha \) and \( \alpha' \geq 2 \) in spaces \((X,d,\mu)\) and \((X',d',\mu')\), respectively. Assume that \( X \) is locally connected, \( \overline{D} \) is compact, \( X' \) is complete and supports \( \alpha' \)-Poincaré inequality, and that the measures \( \mu \) and \( \mu' \) are doubling. Let \( D' \subset X' \) be a regular domain which is finitely connected at the boundary, and let \( Q \colon X' \to (0,\infty) \) be integrable function in \( D' \), \( Q(y) \equiv 0 \) for \( y \in X' \setminus D' \). Suppose that \( f \colon D \to D' \), \( D' = f(D) \), is an open discrete and closed mapping satisfying the relation (1.5) at any point \( y_0 \in \overline{D'} \). Then, under conditions and notation of Lemma 3.4, we may choose a number \( k_0 \in \mathbb{N} \) for which there is \( 0 < N = N(k_0, \|Q\|_1, D') < \infty \), independent on \( m \) and \( f \), such that

\[
M_\alpha(\Gamma_m) \leq N, \quad m \geq M_0 = M_0(k_0),
\]

where \( \Gamma_m \) is a family of paths \( \gamma \colon [0,1] \to D \) such that

\[
f(\gamma) \in \Gamma(\|\gamma_1,m\|, \|\gamma_2,m\|, D').
\]

**Proof.** Denote \( y_0 := I(P_1) \) (see [2, Theorem 10.8]). Arguing as in the proof of [24, Lemma 2.1], we may show that, for every \( r > 0 \) there exists an \( N \in \mathbb{N} \) such that

\[
d_k \subset B(y_0,r) \cap D' \quad \text{for all } k \geq N. \tag{3.13}
\]

Since \( D' \) is connected and \( d_1 \neq D' \), we obtain that \( \partial d_1 \cap D' \neq \emptyset \) (see [18, Ch. 5, § 46, item I]). Set

\[
r_0 := \min\{d'(y_0, \partial d_1 \cap D'), d'(y_0, |\tilde{\beta}|)\}. \tag{3.14}
\]

Due to the condition 2 on the page 263 we may consider that \( r_0 > 0 \). By (3.13), there exists \( k_1 \in \mathbb{N} \) such that

\[
d_k \subset B(y_0,r_0/2) \cap D' \quad \text{for all } k \geq k_1. \tag{3.15}
\]

Set \( D_0 := d_{k_1+1} \), \( D_* := g_{k_1+1} \). Arguing similarly to the proof of relation (2.5), we may prove that

\[
\Gamma(D_0, D_*, D') > \Gamma(S(y_0, r_0/2), S(y_0, r_0), A(y_0, r_0/2, r_0)), \tag{3.16}
\]

where \( A(y_0, r_1, r_2) = \{ y \in X' \colon r_0/2 < d'(y, y_0) < r_0 \} \). Let \( k_0 \) be an arbitrary number for which the statement of Lemma 3.4 holds. Without loss of generality, we may assume that \( k_0 \geq k_1 + 1 \). By definition of \( \gamma_1,m \) and of the family \( \Gamma_m \) we may write that

\[
\Gamma_m = \Gamma_m^1 \cup \Gamma_m^2, \tag{3.17}
\]
where $\Gamma^1_m$ is a family of paths $\gamma \in \Gamma_m$ such that $f(\gamma) \in \Gamma(|\bar{\alpha}|, |\gamma_2, m|, D')$ and $\Gamma^2_m$ is a family of paths $\gamma \in \Gamma_m$ such that $f(\gamma) \in \Gamma(|\bar{\alpha}_m|, |\gamma_2, m|, D')$.

Taking into account the notation of Lemma 3.4, we put

$$
\varepsilon_0 := \min\{\text{dist}(|\bar{\alpha}|, f_{k_0}), \text{dist}(|\bar{\alpha}|, |\bar{\beta}|)\} > 0.
$$

Let us consider the covering $\bigcup_{x \in |\alpha|} B(x, \varepsilon_0/4)$ of $|\bar{\alpha}|$. Since $|\bar{\alpha}|$ is a compactum in $D'$, there are numbers $i_1, \ldots, i_{N_0}$ such that $|\bar{\alpha}| \subset \bigcup_{i=1}^{N_0} B(z_i, \varepsilon_0/4)$, where $z_i \in |\alpha|$ for $1 \leq i \leq N_0$. By [18, Theorem 1.1.5.46]

$$
(3.18) \quad \Gamma(|\bar{\alpha}|, |\gamma_2, m|, D') > \bigcup_{i=1}^{N_0} \Gamma(S(z_i, \varepsilon_0/4), S(z_i, \varepsilon_0/2), A(z_i, \varepsilon_0/4, \varepsilon_0/2)).
$$

Fix $\gamma \in \Gamma^1_m$, $\gamma : [0, 1] \to D$, $f(\gamma(0)) \in |\bar{\alpha}|$, $f(\gamma(1)) \in |\gamma_2, m|$. It follows from (3.18) that $f(\gamma)$ has a subpath $f(\gamma)_1 := f(\gamma)|_{[p_1, p_2]}$ in

$$
\Gamma(S(z_i, \varepsilon_0/4), S(z_i, \varepsilon_0/2), A(z_i, \varepsilon_0/4, \varepsilon_0/2))
$$

for some $1 \leq i \leq N_0$. Then $\gamma|_{[p_1, p_2]}$ is a subpath of $\gamma$ in $\Gamma_f(z_i, \varepsilon_0/4, \varepsilon_0/2)$. Thus

$$
(3.19) \quad \Gamma^1_m > \bigcup_{i=1}^{N_0} \Gamma_f(z_i, \varepsilon_0/4, \varepsilon_0/2).
$$

Put

$$
\eta(t) = \begin{cases} 
4/\varepsilon_0, & t \in [\varepsilon_0/4, \varepsilon_0/2], \\
0, & t \not\in [\varepsilon_0/4, \varepsilon_0/2]. 
\end{cases}
$$

Observe that the function $\eta$ satisfies the relation (1.6). Then, by definition of $f$ in (1.5), by the relation (3.19) and due to the subadditivity of the modulus of families of paths (see [36, Theorem 6.2]), we obtain that

$$
(3.20) \quad M_\alpha(\Gamma^1_m) \leq \sum_{i=1}^{N_0} M_\alpha(\Gamma_f(z_i, \varepsilon_0/4, \varepsilon_0/2)) \leq \sum_{i=1}^{N_0} \frac{N_0 4\alpha'}{\varepsilon_0'}, \quad m \geq M_0,
$$

where $\|Q\|_1 = \int_{D'} Q(y) d\mu'(y)$.

Let $\gamma \in \Gamma^2_m$. Then $\tilde{\gamma} \in f(\gamma) \in \Gamma(|\bar{\alpha}_m|, |\gamma_2, m|, D')$, $\tilde{\gamma} : [0, 1] \to D'$, $\tilde{\gamma}(0) \in |\bar{\alpha}_m|$ and $\tilde{\gamma}(1) \in |\gamma_2, m|$. Due to the definition of $r_0$ in (3.14), and since $\bar{\alpha}_m(t) \in d_{k_0+1}$ for $t \in [t_1, 1]$ and $\bar{\beta}_m(t) \in g_{k_0+1}$ for $t \in [t_2, 1]$, by (3.16) we obtain that $\Gamma^2_m > \Gamma_f(y_0, r_0/2, r_0)$. Arguing similarly as above, we put

$$
\eta(t) = \begin{cases} 
2/r_0, & t \in [r_0/2, r_0], \\
0, & t \not\in [r_0/2, r_0]. 
\end{cases}
$$
Now, by the last relation we obtain that

(3.21) \[ M_\alpha(\Gamma_m^2) \leq \frac{2\|Q\|_1}{r_0^{\alpha'}}, \quad m \geq M_0. \]

Thus, by (3.17), (3.20) and (3.21), due to the subadditivity of the modulus of families of paths (see [11, Theorem 1]), we obtain that

\[ M_\alpha(\Gamma_m) \leq \left( \frac{N_0 4^{\alpha'}}{\varepsilon_0^{\alpha'}} + \frac{2}{r_0^{\alpha'}} \right)\|Q\|_1, \quad m \geq M_0. \]

The right part of the last relation does not depend on \( m \), so we may put

\[ N := \left( \frac{N_0 4^{\alpha'}}{\varepsilon_0^{\alpha'}} + \frac{2}{r_0^{\alpha'}} \right)\|Q\|_1. \]

Lemma 3.5 is proved. \( \square \)

**Lemma 3.6.** Let \( D' \) be a regular domain in \( X' \), which is finitely connected on the boundary. Then any two pairs of (different) points \( a \in D', b \in \overline{D'} \) and \( c \in D', d \in \overline{D'} \) may be joined by disjoint paths \( \alpha: [0, 1] \to \overline{D'} \) and \( \beta: [0, 1] \to \overline{D'} \) such that \( \alpha(t), \beta(t) \in D' \) for any \( t \in [0, 1] \).

**Proof.** By the regularity of \( D' \), there is a quasiconformal mapping \( \varphi \) of \( D' \) onto some domain \( D_0 \subset \mathbb{R}^n \) with a locally quasiconformal boundary. Fix \( a \in D', b \in \overline{D'} \) and \( c \in D', d \in \overline{D'} \). Since \( D' \) is finitely connected on the boundary, by [2, Lemma 10.6] there are \( P_1, P_2 \in E_{D'} \) such that \( I(P_1) = b \) and \( I(P_2) = d \). Arguing as in the proof of Lemma 3.4, we may prove that \( \varphi \) satisfies the estimate similar to (3.6), in addition, we observe that \( \varphi \) has a continuous extension \( \varphi: \overline{D'} \to D_0 \). Moreover, by [25, Theorem 4.1] \( \varphi^{-1} \) has a continuous extension \( \varphi^{-1}: \overline{D}_0 \to \overline{D'} \). Let \( \tilde{a} = \varphi(a), \tilde{b} = \varphi(P_1), \tilde{c} = \varphi(c) \) and \( \tilde{d} = \varphi(P_2) \). By Proposition 3.1 the points \( \tilde{a} \in D_0, \tilde{b} \in \overline{D}_0 \) and \( \tilde{c} \in D_0, \tilde{d} \in \overline{D}_0 \) may be joined by disjoint paths \( C_1: [0, 1] \to \overline{D}_0 \) and \( C_2: [0, 1] \to \overline{D}_0 \) such that \( C_i(t) \in D_0 \) for any \( t \in [0, 1] \) and any \( i = 1, 2 \). Let \( \tilde{C}_i := \varphi^{-1}(C_i|_{[0,1)}) \). Since \( \varphi^{-1}: \overline{D}_0 \to \overline{D'} \) is continuous, \( \tilde{C}_1(t) \to P_1 \) and \( \tilde{C}_2(t) \to P_2 \) as \( t \to 1 - 0 \). Since \( I(P_1) = b \) and \( I(P_2) = d \), \( \tilde{C}_1(t) \to b \) and \( \tilde{C}_2(t) \to d \) as \( t \to 1 - 0 \). Putting

\[ \begin{align*}
\alpha(t) &= \begin{cases} 
\varphi^{-1}(C_1(t)), & t \in [0, 1), \\
\lim_{t \to 1-0} \varphi^{-1}(C_1(t)), & t = 1,
\end{cases} \\
\beta(t) &= \begin{cases} 
\varphi^{-1}(C_2(t)), & t \in [0, 1), \\
\lim_{t \to 1-0} \varphi^{-1}(C_2(t)), & t = 1,
\end{cases}
\end{align*} \]

we obtain the desired paths \( \alpha \) and \( \beta \). \( \square \)
4. On equicontinuity of families at inner points

The presentation of this section is conceptually close to [34]. The only significant difference is the presence of weak sphericalization in the space under consideration, which is present in [34], but is not necessary when presenting the material in this article. The proof of the result below is very similar to Theorem 1.1 in [34], so we omit it in the text.

We say that a space \((X, d, \mu)\) is upper \(\alpha\)-regular at a point \(x_0 \in X\) if there is a constant \(C > 0\) such that

\[
\mu(B(x_0, r)) \leq Cr^\alpha
\]

for the balls \(B(x_0, r)\) centered at \(x \in X\) with all radii \(r < r_0\) for some \(r_0 > 0\). We will also say that a space \((X, d, \mu)\) is upper \(\alpha\)-regular if the above condition holds at every point \(x_0 \in X\). It follows from [2, formulae (2.1) and (2.2)] that doubling measure spaces are locally \(\alpha\)-Ahlfors regular, moreover, arguing similarly to [14, Section 8.7, p. 61] we may show that the number \(\alpha\) equals to Hausdorff dimension of \(X\).

Given \(M > 0\) and domains \(D \subset X, D' \subset X'\), denote by \(\mathcal{S}_M(D, D')\) a family of all open discrete and closed mappings \(f\) of \(D\) onto \(D'\) such that the condition (1.5) holds for any \(y_0 \in D'\) for some \(Q = Q_f\) and \(\|Q_f\|_{L^1(D')} \leq M\).

The following result holds.

**Theorem 4.1.** Let \(D\) and \(D'\) be domains with finite Hausdorff dimensions \(\alpha\) and \(\alpha' \geq 2\) in spaces \((X, d, \mu)\) and \((X', d', \mu')\), respectively. Assume that \(X\) is locally connected, \(X'\) is complete and supports \(\alpha'\)-Poincaré inequality, and that the measures \(\mu\) and \(\mu'\) are doubling. Suppose that \(\overline{D}\) and \(\overline{D}'\) are compact sets, and \(D\) is weakly flat as a metric space. Let \(D' \subset X'\) be a regular domain which is finitely connected at the boundary. Then the family \(\mathcal{S}_M(D, D')\) is equicontinuous in \(D\).

5. Proof of Theorem 1.5

The possibility of continuous extension of the mapping \(f \in \mathcal{S}_M(D, D')\) to \(\partial D\) follows by Theorem 1.4. Since \(\mathcal{S}_{\delta, A, M}(D, D') \subset \mathcal{S}_M(D, D')\), the equicontinuity of \(\mathcal{S}_{\delta, A, M}(D, D')\) at inner points of \(D\) follows by Theorem 4.1.

Let us to show the equicontinuity of \(\mathcal{S}_{\delta, A, M}(\overline{D}, \overline{D}')\) on \(\partial D\). Assume the contrary.

Now, there is a point \(z_0 \in \partial D\), a number \(\varepsilon_0 > 0\), a sequence \(z_m \in \overline{D}\) and a mapping \(\overline{f}_m \in \mathcal{S}_{\delta, A, M}(\overline{D}, \overline{D}')\) such that \(z_m \to z_0\) as \(m \to \infty\) and

\[
m_P(\overline{f}_m(z_m), \overline{f}_m(z_0)) \geq \varepsilon_0, \quad m = 1, 2, \ldots ,
\]

where \(m_P\) is some of possible metrics in \(\overline{D}'_P\) defined in Proposition 1.3. Since \(f_m = \overline{f}_m|_D\) has a continuous extension to \(\overline{D}_P\), we may assume that
$z_m \in D$ and, in addition, there is one more sequence $z'_m \in D$, $z'_m \to z_0$ as $m \to \infty$ such that $m_P(f_m(z'_m), \overline{f}_m(z_0)) \to 0$ as $m \to \infty$. In this case, it follows from (5.1) that

$$m_P(f_m(z_m), f_m(z'_m)) \geq \varepsilon_0/2, \quad m \geq m_0.$$  

By [2, Theorem 10.10], $(D_P, m_P)$ is a compact metric space. Thus, we may assume that $f_m(z_m)$ and $f_m(z'_m)$ converge to some $P_1, P_2 \in D_P$, $P_1 \neq P_2$, as $m \to \infty$. Let $d_m$ and $g_m$ be sequences of decreasing domains corresponding to prime ends $P_1$ and $P_2$, respectively. Put $x_0, y_0 \in A$ such that $x_0 \neq y_0$ and $x_0 \neq P_1 \neq y_0$, where the continuum $A \subset D'$ is taken from the conditions of Theorem 1.5. Without loss of generality, we may assume that $d_1 \cap g_1 = \emptyset$ and $x_0, y_0 \not\in d_1 \cup g_1$.

By Lemmas 3.4 and 3.5, we may find disjoint paths $\gamma_{1,m} : [0,1] \to D'$ and $\gamma_{2,m} : [0,1] \to D'$ and a number $N > 0$ such that $\gamma_{1,m}(0) = x_0$, $\gamma_{1,m}(1) = f_m(z_m)$, $\gamma_{2,m}(0) = y_0$, $\gamma_{2,m}(0) = f_m(z'_m)$, wherein

$$(5.2) \quad M_\alpha(\Gamma_m) \leq N, \quad m \geq M_0,$$

where $\Gamma_m$ consists of those and only those paths $\gamma$ of $D$ for which $f_m(\gamma) \in \Gamma([\gamma_{1,m}], [\gamma_{2,m}], D')$ (see Figure 4).

On the other hand, let $\gamma_{1,m}^*$ and $\gamma_{2,m}^*$ be the total $f_m$-lifting of the paths $\gamma_{1,m}$ and $\gamma_{2,m}$ starting at the points $z_m$ and $z'_m$, respectively (such liftings exist by Lemma 3.6). Now, $\gamma_{1,m}(1) \in f_m^{-1}(A)$ and $\gamma_{2,m}(1) \in f_m^{-1}(A)$. Since by the condition $d(f_m^{-1}(A), \partial D) > \delta > 0, m = 1, 2, \ldots$, we obtain that

$$(5.3) \quad d(\gamma_{1,m}^*) \geq d(z_m, \gamma_{1,m}^*(1)) \geq (1/2) \cdot d(f_m^{-1}(A), \partial D) > \delta/2,$$

$$(5.3) \quad d(\gamma_{2,m}^*) \geq d(z'_m, \gamma_{2,m}^*(1)) \geq (1/2) \cdot d(f_m^{-1}(A), \partial D) > \delta/2$$
for sufficiently large $m \in \mathbb{N}$. Choose the ball

$$U := B(z_0, r_0) = \{ z \in X : d(z, z_0) < r_0 \},$$

where $r_0 > 0$ and $r_0 < \delta/4$. Observe that

$$\left| \gamma^*_{1,m} \right| \cap U \neq \emptyset \neq \left| \gamma^*_{1,m} \right| \cap (D \setminus U)$$

for sufficiently large $m \in \mathbb{N}$, because $d(f_m(\left| \gamma^*_{1,m} \right|)) \geq \delta/2$ and $z_m \in \left| \gamma^*_{1,m} \right|$, $z_m \to z_0$ as $m \to \infty$. Arguing similarly, we may conclude that $\left| \gamma^*_{2,m} \right| \cap U \neq \emptyset \neq \left| \gamma^*_{2,m} \right| \cap (D \setminus U)$. Since $\left| \gamma^*_{1,m} \right|$ and $\left| \gamma^*_{2,m} \right|$ are continua, by [18, Theorem 1.I.5.46]

$$\left(5.4\right) \quad \left| \gamma^*_{1,m} \right| \cap \partial U \neq \emptyset, \quad \left| \gamma^*_{2,m} \right| \cap \partial U \neq \emptyset.$$

Put $P := N > 0$, where $N$ is a number from the relation (5.2). Since $D$ has a weakly flat boundary, we may find a neighborhood $V \subset U$ of $z_0$ such that

$$\left(5.5\right) \quad M_{\alpha}(\Gamma(E, F, D)) > N$$

for any continua $E, F \subset D$ with $E \cap \partial U \neq \emptyset \neq E \cap \partial V$ and $F \cap \partial U \neq \emptyset \neq F \cap \partial V$. Observe that

$$\left(5.6\right) \quad \left| \gamma^*_{1,m} \right| \cap \partial V \neq \emptyset, \quad \left| \gamma^*_{2,m} \right| \cap \partial V \neq \emptyset.$$

for sufficiently large $m \in \mathbb{N}$. Indeed, $z_m \in \left| \gamma^*_{1,m} \right|$ and $z'_m \in \left| \gamma^*_{2,m} \right|$, where $z_m, z'_m \to z_0 \in V$ as $m \to \infty$. Thus, $\left| \gamma^*_{1,m} \right| \cap V \neq \emptyset \neq \left| \gamma^*_{2,m} \right| \cap V$ for sufficiently large $m \in \mathbb{N}$. Besides that, $d(V) \leq d(U) = 2r_0 < \delta/2$ and $\left| \gamma^*_{1,m} \right| \cap (D \setminus V) \neq \emptyset$ because $d(\left| \gamma^*_{1,m} \right|) > \delta/2$ by (5.3). Then $\left| \gamma^*_{1,m} \right| \cap \partial V \neq \emptyset$ (see [18, Theorem 1.I.5.46]). Similarly, $d(V) \leq d(U) = 2r_0 < \delta/2$. Now, since by (5.3) $d(\left| \gamma^*_{2,m} \right|) > \delta/2$, we obtain that $\left| \gamma^*_{2,m} \right| \cap (D \setminus V) \neq \emptyset$. By [18, Theorem 1.I.5.46], $\left| \gamma^*_{1,m} \right| \cap \partial V \neq \emptyset$. Thus, (5.6) is proved. By (5.4), (5.5) and (5.6), we obtain that

$$\left(5.7\right) \quad M_{\alpha}(\Gamma(\left| \gamma^*_{1,m} \right|, \left| \gamma^*_{2,m} \right|, D)) > N.$$

The inequality (5.7) contradicts with (5.2), since $\Gamma(\left| \gamma^*_{1,m} \right|, \left| \gamma^*_{2,m} \right|, D) \subset \Gamma_m$ and consequently

$$M_{\alpha}(\Gamma(\left| \gamma^*_{1,m} \right|, \left| \gamma^*_{2,m} \right|, D)) \leq M_{\alpha}(\Gamma_m) \leq N, \quad m \geq M_0.$$

The obtained contradiction indicates the incorrectness of the assumption in (5.1). Theorem 1.5 is proved. □
6. Lemma on the continuum

One of the versions of the following statement is established in [30, item v, Lemma 2] for homeomorphisms and “good” boundaries, see also [31, Lemma 4.1]. Let us also point out the case relating to mappings with branching and good boundaries, see [33, Lemma 6.1], as well as the case of bad boundaries and homeomorphisms, see [15, Lemma 2.13]. The statement below seems to refer to the most general situation when a mapping acts between domains of metric spaces. To the indicated degree of generality, this statement is proved for the first time.

**Lemma 6.1.** Let $D$ and $D'$ be domains with finite Hausdorff dimensions $\alpha$ and $\alpha' \geq 2$ in spaces $(X, d, \mu)$ and $(X', d', \mu')$, respectively. Assume that $X$ is locally connected, $\overline{D}$ is compact, $X'$ is complete and supports $\alpha'$-Poincaré inequality, and that the measures $\mu$ and $\mu'$ are doubling. Assume that $D$ has a weakly flat boundary, none of the components of which degenerates into a point, $\overline{D}$ is a compact set and $D'$ be a bounded regular domain in $X'$, which is finitely connected on the boundary. Let $A$ be a non-degenerate continuum in $D'$ and $\delta > 0$. Assume that $f_m$ is a sequence of open discrete and closed mappings of $D$ onto $D'$ satisfying the following condition: for any $m = 1, 2, \ldots$ there is a continuum $A_m \subset D$, $m = 1, 2, \ldots$, such that $f_m(A_m) = A$ and $d(A_m) \geq \delta > 0$. Let $\partial D \neq \emptyset$. If there is $0 < M_1 < \infty$ such that $f_m$ satisfies (1.5) at any $y_0 \in D'$ and $m = 1, 2, \ldots$ with some $Q = Q_m(y)$ for which $\|Q_m\|_{L^1(D')} \leq M_1$, then there exists $\delta_1 > 0$ such that

$$d(A_m, \partial D) > \delta_1 > 0 \text{ for all } m \in \mathbb{N}.$$ 

**Proof.** Since $\partial D \neq \emptyset$, $d(A_m, \partial D)$ is well-defined. Let us prove this statement by contradiction. Suppose that the conclusion of the lemma is not true. Then for each $k \in \mathbb{N}$ there is a number $m = m_k$ such that $d(A_{m_k}, \partial D) < 1/k$. We may assume that the sequence $m_k$ is increasing by $k$. Since $A_{m_k}$ is compact, there are $x_k \in A_{m_k}$ and $y_k \in \partial D$ such that $d(A_{m_k}, \partial D) = d(x_k, y_k) < 1/k$ (see Figure 5). Since by the assumption $\overline{D}$ is a compact set, $\partial D$ is a compact set, as well. Then may consider that $y_k \to y_0 \in \partial D$ as $k \to \infty$. Now we also have that $x_k \to y_0 \in \partial D$ as $k \to \infty$. Let $K_0$ be a component of $\partial D$ containing $y_0$. Obviously, $K_0$ is a continuum in $X$. Since $\partial D$ is weakly flat, by Theorem 1.4 $f_{m_k}$ has a continuous extension $\overline{f}_{m_k} : \overline{D} \to \overline{D'}$. It means that, for any $k = 1, 2, \ldots$, any $\varepsilon > 0$ and any $x_0 \in \overline{D}$ there is $\delta^* = \delta_k^*(\varepsilon, x_0) > 0$ such that

$$m_P(f_{m_k}(x), \overline{f}_{m_k}(x_0)) < \varepsilon$$

for any $x \in D$ such that $d(x, x_0) < \delta^*$. Let, as usual, $I(\overline{f}_{m_k}(x_0))$ denotes the impression of the prime end $\overline{f}_{m_k}(x_0)$ and let $\overline{f}_{m_k}(x_0) = [E_{kn}]$, $n = 1, 2, \ldots$,
where $E_{k_n}$ is a sequence of acceptable sets for the prime end $\overline{f_{m_k}(x)}$. Using (6.1) and the relation

$$E_{k_n} \subset D' \cap B(I(\overline{f_{m_k}(x)}), r), \quad r > 0, \ n = n(r, k) \in \mathbb{N},$$

see [24, relation (2.2), proof of Lemma 2.1], we conclude that

$$d'(f_{m_k}(x), I(\overline{f_{m_k}(x)})) \to 0$$
as $x \to x_0$. Now we conclude that there is (possibly, some another) $\delta = \delta_k(\varepsilon, x_0) > 0$ such that

$$d'(f_{m_k}(x), I(\overline{f_{m_k}(x)})) < \varepsilon$$

for any $x \in D$ such that $d(x, x_0) < \delta$. It follows from (6.2) that any mapping $f_{m_k}$ is also continuous in $\overline{D}$ as a mapping from $\overline{D}$ in $\overline{D'}$. Moreover, the mapping $\overline{f_{m_k}}$ is uniformly continuous in $\overline{D}$ for any fixed $k$, because $\overline{f_{m_k}}$ is continuous on the compact set $\overline{D}$. Now, for any $\varepsilon > 0$ there is $\delta_k = \delta_k(\varepsilon) < 1/k$ such that

$$d'(f_{m_k}(x), I(\overline{f_{m_k}(x)})) < \varepsilon$$

for all $x \in D$ and $x_0 \in \overline{D}$ such that $d(x, x_0) < \delta_k < 1/k$. Let $\varepsilon > 0$ be some number such that

$$\varepsilon < (1/2) \cdot d'(\partial D', A),$$

where $A$ is a continuum from the conditions of the lemma and $g: D_0 \to D'$ is a quasiconformal mapping of $D_0$ onto $D'$, while $D'$ is a domain with a
Let $\Gamma$ be a quasiconformal boundary corresponding to the definition of the metric $\rho$. Given $k \in \mathbb{N}$, we set

$$B_k := \bigcup_{x_0 \in K_0} B(x_0, \delta_k), \quad k \in \mathbb{N}.$$  

Since $B_k$ is a neighborhood of a continuum $K_0$, by [30, Lemma 2(iii)] there is a neighborhood $U_k$ of $K_0$ such that $U_k \subset B_k$ and $U_k \cap D$ is connected. We may consider that $U_k$ is open, so that $U_k \cap D$ is linearly path connected (see [20, Proposition 13.1]). Let $d(K_0) = m_0$. Then we may find $z_0, w_0 \in K_0$ such that $d(K_0) = d(z_0, w_0) = m_0$. Thus, there are sequences $\overline{y_k} \in U_k \cap D$, $z_k \in U_k \cap D$ and $w_k \in U_k \cap D$ such that $z_k \to z_0$, $\overline{y_k} \to y_0$ and $w_k \to w_0$ as $k \to \infty$. We may consider that

$$d(z_k, w_k) > m_0/2 \quad \text{for all } k \in \mathbb{N}.  \tag{6.5}$$

Since the set $U_k \cap D$ is linearly path connected, we may joint the points $z_k$, $\overline{y_k}$ and $w_k$ using some path $\gamma_k$ in $U_k \cap D$. As usually, we denote by $|\gamma_k|$ the locus of the path $\gamma_k$ in $D$. Then $f_{m_k}(|\gamma_k|)$ is a compact set in $D'$. If $x \in |\gamma_k|$, then we may find $x_0 \in K_0$ such that $x \in B(x_0, \delta_k)$. Fix $\omega \in A \subset D'$. Since $x \in |\gamma_k|$ and, in addition, $x$ is an inner point of $D$, we may use the notation $f_{m_k}(x)$ instead $\overline{f_{m_k}(x)}$. By (6.3), (6.4) and by the triangle inequality, we obtain that

$$d'(f_{m_k}(x), \omega) \geq d'(\omega, I(\overline{f_{m_k}(x_0)})) - d'(I(\overline{f_{m_k}(x_0)}), f_{m_k}(x))$$  

$$\geq (1/2) \cdot d'(\partial D', A) > \varepsilon$$

for $k \in \mathbb{N}$. Passing to $\inf$ in (6.6) over all $x \in |\gamma_k|$ and $\omega \in A$, we obtain that

$$d'(f_{m_k}(|\gamma_k|), A) > \varepsilon, \quad k = 1, 2, \ldots.  \tag{6.7}$$

We cover the set $A$ with balls $B(x, \varepsilon/4)$, $x \in A$. Since $A$ is compact, we may assume that $A \subset \bigcup_{i=1}^{M_0} B(x_i, \varepsilon/4)$, $x_i \in A$, $i = 1, 2, \ldots, M_0$, $1 \leq M_0 < \infty$. By definition, $M_0$ depends only on $A$, in particular, $M_0$ does not depend on $k$. Put

$$\Gamma_k := \Gamma(A_{m_k}, |\gamma_k|, D).$$

Let $\Gamma_{ki} := \Gamma_{f_{m_k}}(x_i, \varepsilon/4, \varepsilon/2)$, in other words, $\Gamma_{ki}$ consists of all paths $\gamma: [0, 1] \to D$ such that $f_{m_k}(\gamma(0)) \in S(x_i, \varepsilon/4)$, $f_{m_k}(\gamma(1)) \in S(x_i, \varepsilon/2)$, $\gamma(t) \in A(x_i, \varepsilon/4, \varepsilon/2)$ for $0 < t < 1$. Using [18, Ch. 5, § 46, item 1], we may prove that

$$\Gamma_k > \bigcup_{i=1}^{M_0} \Gamma_{ki}.  \tag{6.8}$$

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Set
\[ \eta(t) = \begin{cases} 
4/\varepsilon, & t \in [\varepsilon/4, \varepsilon/2], \\
0, & t \not\in [\varepsilon/4, \varepsilon/2].
\end{cases} \]

Observe that \( \eta \) satisfies the relation (1.6) for \( r_1 = \varepsilon/4 \) and \( r_2 = \varepsilon/2 \). Since \( f_{m_k} \) satisfies the relation (1.5), we obtain that
\begin{equation}
(6.9) \quad M_\alpha(\Gamma_{f_{m_k}}(x_i, \varepsilon/4, \varepsilon/2)) \leq (4/\varepsilon)^{\alpha'} \cdot \|Q\|_1 < M_0 < \infty.
\end{equation}

By (6.8) and (6.9) and due to the subadditivity of the modulus of families of paths, we obtain that
\begin{equation}
(6.10) \quad M_\alpha(\Gamma_k) \leq \frac{4^{\alpha'} M_0}{\varepsilon^{\alpha'}} \int_{D'} Q(y) d\mu'(y) \leq M_1 \cdot M_0 < \infty.
\end{equation}

Arguing similarly to the proof of relations (5.3) and using the condition (6.5), we obtain that \( M_\alpha(\Gamma_k) \to \infty \) as \( k \to \infty \), which contradicts with (6.10). The resulting contradiction proves the lemma. \( \square \)

7. Equicontinuity of families of mappings fixing a point

Given domains \( D \subset X, D' \subset X' \), points \( a \in D, b \in D' \) and a number \( M_0 > 0 \) denote by \( \mathcal{G}_{a,b,M_0}(D, D') \) the family of open discrete and closed mappings \( f \) of \( D \) onto \( D' \) satisfying the relation (1.5) for some \( Q = Q_f \), \( \|Q\|_{L^1(D')} \leq M_0 \) for any \( y_0 \in f(D) \), such that \( f(a) = b \). The following statement was proved in [33, Theorem 7.1] in the case of a fixed function \( Q \) and for the Euclidean space.

**Theorem 7.1.** Let \( D \) and \( D' \) be domains with finite Hausdorff dimensions \( \alpha \) and \( \alpha' \geq 2 \) in spaces \( (X, d, \mu) \) and \( (X', d', \mu') \), respectively. Assume that \( X \) is locally connected, \( \overline{D} \) is compact, \( X' \) is complete and supports \( \alpha' \)-Poincaré inequality, and that the measures \( \mu \) and \( \mu' \) are doubling. Assume that \( D \) has a weakly flat boundary, none of the components of which degenerates into a point and \( D' \) be a regular domain in \( X' \), which is finitely connected on the boundary. Then any \( f \in \mathcal{G}_{a,b,M_0}(D, D') \) has a continuous extension \( \overline{f}: \overline{D} \to \overline{D'}_P \), while \( \overline{f}(\overline{D}) = \overline{D'}_P \) and, in addition, the family \( \mathcal{G}_{a,b,M_0}(\overline{D}, \overline{D'}) \) of all extended mappings \( \overline{f}: \overline{D} \to \overline{D'}_P \) is equicontinuous in \( \overline{D} \).

**Proof.** The possibility of continuous extension of \( f \in \mathcal{G}_{a,b,M_0}(D, D') \) to a continuous mapping \( \overline{f}: \overline{D} \to \overline{D'}_P \) is a statement of Theorem 1.4, as well as the equality \( \overline{f}(\overline{D}) = \overline{D'}_P \). The equicontinuity of \( \mathcal{G}_{a,b,M_0}(D, D') \) at inner points of \( D \) is a result of Theorem 4.1.

It remains to establish the equicontinuity of the family of extended mappings \( \overline{f}: \overline{D} \to \overline{D'}_P \) at the boundary points of \( D \).
We prove this statement from the opposite. Assume that the family \( \mathcal{S}_{a,b,M_0}(\overline{D}, \overline{D}') \) is not equicontinuous at some point \( x_0 \in \partial D \). Then there are points \( x_m \in D \) and mappings \( f_m \in \mathcal{S}_{a,b,M_0}(\overline{D}, \overline{D}) \), \( m = 1, 2, \ldots \), such that \( x_m \to x_0 \) as \( m \to \infty \), moreover,

\[
(7.1) \quad m_P(f_m(x_m), f_m(x_0)) \geq \varepsilon_0, \quad m = 1, 2, \ldots
\]

for some \( \varepsilon_0 > 0 \), where \( m_P \) is a metric in \( \overline{D}_P \) defined in Proposition 1.3.

We choose in an arbitrary way a point \( y_0 \in D', y_0 \neq b \), and join it with the point \( b \) by some path in \( D' \), which we denote by \( \alpha \). Let \( A := |\alpha| \) and let \( A_m \) be a total \( f_m \)-lifting of \( \alpha \) starting at \( a \) (it exists by Lemma 2.1). Observe that \( d(A_m, \partial D) > 0 \) due to the closeness of \( f_m \). Now, the following two cases are possible: either \( d(A_m) \to 0 \) as \( m \to \infty \), or \( d(A_m) \geq \delta_0 > 0 \) as \( k \to \infty \) for some increasing sequence of numbers \( m_k \) and some \( \delta_0 > 0 \).

In the first of these cases, obviously, \( d(A_m, \partial D) \geq \delta > 0 \) for some \( \delta > 0 \). Then, by Theorem 1.5, the family \( \{f_m\}_{m=1}^{\infty} \) is equicontinuous at the point \( x_0 \), however, this contradicts the condition (7.1).

In the second case, if \( d(A_{m_k}) \geq \delta_0 > 0 \) for sufficiently large \( k \), we also have that \( d(A_{m_k}, \partial D) \geq \delta_1 > 0 \) for some \( \delta_1 > 0 \) by Lemma 6.1. Again, by Theorem 1.5, the family \( \{f_{m_k}\}_{k=1}^{\infty} \) is equicontinuous at the point \( x_0 \), and this contradicts the condition (7.1).

Thus, in both of the two possible cases, we came to the contradiction (7.1), and thus the family \( \mathcal{S}_{a,b,M_0}(D, D') \) is equicontinuous in \( \overline{D} \). Theorem 7.1 is proved. \( \square \)

8. Examples

To construct examples of mappings that satisfy the conditions of the main results of the article, we will use the constructions related to publications [25], [26] and [34].

Example 8.1. Let \( D \) be the half-disk

\[
D := \{ z \in \mathbb{C} : z = x + iy, \ |z| < 1, \ x > 0 \}
\]

in \( \mathbb{C} \). Put \( f_1(z) = z^2 \). In this case, \( f_1(D) \) is the punctured disk \( D' := \mathbb{D} \setminus I \), \( I := \{ z \in \mathbb{C} : z = x + iy, \ y = 0, \ x \in [0, 1) \} \), see Figure 6.

Let \( X = X' = \mathbb{C} \), let \( \mu \) and \( \mu' \) be Lebesgue measures in \( \mathbb{C} \), and let \( d(x, y) = d'(x, y) = |x - y| \). Let us verify that for the domains \( D \) and \( D' \), the metric space \( X = X' = \mathbb{C} \), containing them, and also for the mapping \( f_1 \), all conditions of Theorem 1.4 are satisfied. Obviously, \( \mathbb{C} \) is locally connected and complete, and \( \overline{D} \) is compact. Observe that \( \mathbb{C} \) satisfies 1-Poincaré inequality (see [13, Theorem 10.5]) and, consequently, \( n \)-Poincaré inequality. Obviously, the Lebesgue measure is doubling, in addition, \( D' \) is finitely connected on the boundary. Finally, the relation (1.5) holds with \( Q \equiv 1 \) because
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Figure 6: To Example 8.1

f_1 is 1-quasiconformal mapping (see [19, Theorem 3.2]). Thus, all conditions of Theorem 1.4 are satisfied, so that the mapping f_1 can be extended to a continuous mapping \( f_1 : \overline{D} \to \overline{D}' \).

In each subsequent example below, we retain the notation of the previous examples, in particular, we retain the notation for the domains D and D'.

**Example 8.2.** By slightly modifying Example 8.1, we may construct a similar mapping in the unit disk. For this purpose, we will map the unit disk to the unit semidisk, guided by the sequential application of the mappings

- \( f_2(z) = -\frac{iz-1}{z+1} \) (\( f_2 : \mathbb{D} \to \mathbb{H}^+ \), \( \mathbb{H}^+ := \{ z = x + iy \in \mathbb{C}, x > 0 \} \)),
- \( f_3(z) = \sqrt{z} = \sqrt{re^{i\varphi}/2} \), \( z = re^{i\varphi} \) (\( f_3 : \mathbb{H}^+ \to D_1 \), \( D_1 = \{ z \in \mathbb{C} : z = re^{i\varphi}, 0 < \varphi < \pi/2 \} \)) and \( f_4(z) = \frac{z+1}{z+1} \) (\( f_4 : D_1 \to D \)), we obtain a conformal mapping \( f_5 := f_4 \circ f_3 \circ f_2 \) of the unit disk onto a domain D. Now, the mapping \( F_1 := f_1 \circ f_5 \) maps \( \mathbb{D} \) onto \( D' \) conformally, besides that, \( F_1 \) satisfies all the conditions of Theorem 1.4.

**Example 8.3.** It is fairly easy to provide a similar example of a mapping with branching. For this purpose, we additionally set \( f_6(z) = z^2 \), \( z \in \mathbb{D} \). Put \( F_2 := F_1 \circ f_6 \). Now, \( F_2 \) is open discrete and closed mapping of the unit disk onto a domain \( D' \). Let \( \Gamma \) be a family of paths in \( \mathbb{D} \) and let \( M(\Gamma) \) be the Euclidean modulus of family of paths \( \Gamma \) in \( \mathbb{C} \). Now, by Theorem [19, Theorem 3.2] we obtain that \( M(\Gamma) \leq 2 \cdot M(f_6(\Gamma)) = 2 \cdot M(F_2(\Gamma)) \) because \( F_2 \) is a quasiregular mapping. Thus, the relation (1.5) holds for \( F_2 \) with \( Q \equiv 2 \). In addition, by Theorem 1.4 the mapping \( F_2 \) can be extended to a continuous mapping \( \overline{F_2} : \overline{\mathbb{D}} \to \overline{D}' \).

**Example 8.4.** Let us indicate a similar example of a mapping with unbounded characteristic. For this purpose, fix a number \( p \geq 1 \) satisfying the condition \( 2/p < 1 \). Put \( m \in \mathbb{N} \), \( \alpha \in (0, 2/p) \). We define the sequence of map-
pings $g_m$ of $B(0, 2)$ onto the unit disk $\mathbb{D}$ as follows:

$$
g_m(z) := \begin{cases} 
\frac{z}{|z|(|z| - 1)^{1/\alpha}}, & 1 + \frac{1}{m^\alpha} \leq |z| < 2, \\
\frac{1}{1+(1/m)^{1/\alpha}} \cdot z, & 0 < |z| < 1 + \frac{1}{m^\alpha}.
\end{cases}
$$

Note that $g_m$ satisfies the condition (1.5) for $Q = \frac{1+|z|^\alpha}{\alpha|z|^\alpha}$ at any $z_0 \in \mathbb{D}$, moreover, $Q \in L^p(\mathbb{D})$ (see, for example, the reasonings obtained under the consideration of [20, Proposition 6.3]). Putting $f_8(z) = 2z$, we observe that the mappings

$$
h_m := g_m \circ f_8 \circ F_2
$$

transform $\mathbb{D}$ onto $D'$, in addition, $h_m$ satisfies (1.5) for $Q = 2\frac{1+|z|^\alpha}{\alpha|z|^\alpha}$. By Theorem 1.4 the mapping $h_m$ can be extended to a continuous mapping $h_m : \overline{\mathbb{D}} \rightarrow \overline{D'}_P$ for any $m = 1, 2, \ldots$.

Note that, for the constructed sequence of mappings $h_m$, $m = 1, 2, \ldots$, all conditions and the conclusion of Theorem 1.5 are also satisfied. For this purpose, first of all, we note that there are infinitely many continua $A$ satisfying condition $d(h_m^{-1}(A), \partial \mathbb{D}) \geq \delta$, $m = 1, 2, \ldots$, with some $\delta > 0$, since all the mappings $g_m$ are a fixed mapping by $m$ for $m > 2^{1/\alpha}$, and equals to $\frac{z}{|z|(|z| - 1)^{1/\alpha}}$ for $3/2 < |z| < 2$, besides that, the mapping $f_8 \circ F_2$ does not depend on $m$. Observe that $\mathbb{D}$ is a weakly flat at inner and boundary points (see [36, Theorem 10.12, Theorem 17.10], cf. [31, Lemma 2.2]). Observe that $D'$ is regular by Riemann’s mapping theorem. Thus, all the conditions of Theorem 1.5 are fulfilled, so that the family of mappings $h_m : \overline{\mathbb{D}} \rightarrow \overline{D'}_P$ is equicontinuous in $\overline{\mathbb{D}}$.

Let $h_m^* = g_m \circ f_8 \circ F_1$. It can be shown that the sequence of inverse mappings $h_m^{-1}$ have continuous extension to $\overline{D'}_P$, but is not an equicontinuous family in $\overline{D'}_P$ (for this purpose it is necessary to construct a prime end with an impression at the point 0, and then apply similar arguments using in the consideration of [31, Example 1]).

EXAMPLE 8.5. Finally, consider a similar example of mappings in a metric space. For this purpose, we essentially use our construction from [27], cf. [34]. Regarding this example, we will need some definitions related to the factor space, discontinuously acting groups of mappings, and the normal neighborhood of a point. For these definitions, we also refer the reader to [27].

Let $G$ and $G^*$ be groups of Möbius transformations of the unit disk $\mathbb{D}$ onto itself, acting discontinuously and not having fixed points in $\mathbb{D}$. Suppose also that $\mathbb{D}/G$ and $\mathbb{D}/G^*$ are complete 2-Ahlfors regular spaces with 2-Poincaré inequality. Let $\pi : \mathbb{D} \rightarrow \mathbb{D}/G$ and $\pi_* : \mathbb{D} \rightarrow \mathbb{D}/G^*$ be the natural projections of $\mathbb{D}$ onto $\mathbb{D}/G$ and $\mathbb{D}/G^*$, respectively, and let $p_0 \in \mathbb{D}/G$ and
$p_0^* \in \mathbb{D}/G^*$ be such that $\pi(0) = p_0$ and $\pi_*(0) = p_0^*$. In what follows, $h$ denotes the hyperbolic metric in $\mathbb{D}$, $\tilde{h}$ and $\tilde{h}_*$ denote the metric in $\mathbb{D}/G$ and $\mathbb{D}/G^*$, respectively, $dv$ denotes the element of hyperbolic area in $\mathbb{D}$, and $d\tilde{v}$ and $d\tilde{v}_*$ denote the elements of the area in $\mathbb{D}/G$ and $\mathbb{D}/G^*$ (see [27]). Namely,

\begin{align}
(8.2) \quad & dv(x) = \frac{4dm(x)}{(1 - |x|^2)^4}, \\
(8.3) \quad & h(x, y) = \log \frac{1 + t}{1 - t}, \quad t = \frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}}, \\
(8.4) \quad & \tilde{h}(p_1, p_2) := \inf_{g_1, g_2 \in G} h(g_1(z_1), g_2(z_2)), \\
(8.5) \quad & \tilde{v}(A) := v(P \cap \pi^{-1}(A)),
\end{align}

where $P = \{x \in \mathbb{D} : h(x, x_0) < h(x, T(x_0))$ for all $T \in G \setminus \{I\}\}$. Similarly we may define $\tilde{h}_*$ and $\tilde{v}_*$ in $\mathbb{D}/G^*$. Observe that the length element $ds_h$ in the metric space $(\mathbb{D}, h)$ equals to $\frac{2|dz|}{1 - |z|^2}$.

Let $r_1 > 0$ be the radius of a disk $\tilde{B}(p_0, r_1) = \{p \in \mathbb{D}/G : \tilde{h}(p, p_0) < r_1\}$ centered at a point $p_0$, lying in some normal neighborhood $U$ of $p_0$ with its closure, and let $R_1 > 0$ be the radius of a disk

$$\tilde{B}(p_0^*, R_1) = \{p_* \in \mathbb{D}/G^* : \tilde{h}_*(p_*, p_0^*) < R_1\}$$

centered at a point $p_0^*$, entirely (with its closure) lying in some normal neighborhood $U_*$ of $p_0^*$.

Then by definition of the natural projection $\pi$, as well as the definition of the hyperbolic metric $h$ and the metrics $\tilde{h}$ in (8.4) we have

$$\pi(B(0, r_0)) = \tilde{B}(p_0, r_1) \quad \text{and} \quad \pi_*(B(0, R_0)) = \tilde{B}(p_0, R_1),$$

where $r_0 := (e^{r_1} - 1)/(e^{r_1} + 1)$ and $R_0 := (e^{R_1} - 1)/(e^{R_1} + 1)$. Let $V$ be a neighborhood of the origin containing $B(0, r_0)$ such that $\pi$ maps $V$ onto $U$ homeomorphically, and let $V_*$ be a neighborhood of the origin containing $B(0, R_0)$ such that $\pi_*$ maps $V_*$ onto $U_*$ homeomorphically.

Put $f_9(z) = R_0z$ and $f_{10}(z) = \frac{z}{r_0}$. In this case, the family of mappings

$$\tilde{H}_m(z) = (f_9 \circ h_m \circ f_{10})(z)$$

map the ball $B(0, r_0)$ onto $B(0, R_0) \setminus J_0$,

\begin{equation}
(8.6) \quad J_0 := \{z \in \mathbb{C} : z = x + iy, x \in [0, R_0), y = 0\}.
\end{equation}
Recall that by (8.1)

\[(8.7)\quad \tilde{H}_m = f_9 \circ g_m \circ f_8 \circ F_1 \circ f_6 \circ f_{10},\]

where
\[f_6(z) = z^2, \quad f_8(z) = 2z, \quad f_9(z) = R_0 z, \quad f_{10}(z) = \frac{z}{r_0},\]

\[g_m(z) := \begin{cases} \frac{z}{|z| - 1}^{1/\alpha}, & 1 + 1/m^\alpha \leq |z| < 2, \\ \frac{1/(1/m)\alpha}{1+(1/m)^\alpha} \cdot z, & 0 < |z| < 1 + 1/m^\alpha. \end{cases}\]

and \(F_1(z)\) is some conformal mapping. Put
\[H_m = \pi_* \circ \tilde{H}_m \circ (\pi_V)^{-1}.\]

Observe that the \(H_m\) maps \(\pi(B(0, r_0)) \subset \mathbb{D}/G\) onto \(\pi^*(B(0, R_0) \setminus J_0) \subset \mathbb{D}/G^*\), where \(J_0\) is defined in (8.6). Arguing similarly to [34, Example 4.4], we may show that the mappings \(H_m, m = 1, 2 \ldots\) satisfy the condition (1.5) with some general integrable function \(Q_1 = Q_1(p_*)\).

Let us check that all conditions of Theorems 1.4 and 1.5 are satisfied for the family of mappings \(H_m, m = 1, 2, \ldots\). Observe that \(\mathbb{D}/G\) is locally connected (see e.g. [27, Proposition 1.1], cf. [3, Proposition 3.14]). By the construction, \(H_m\) act between domains \(B(p_0, r_1)\) and \(\tilde{B}(p_0^*, R_1) \setminus \pi_*(J_0)\) with compact closures, in addition, all \(H_m\) are discrete, open and closed. By the assumption, the spaces \(\mathbb{D}/G\) and \(\mathbb{D}/G^*\) are complete 2-Ahlfors regular spaces with 2-Poincaré inequality. In particular, the measures \(\mu := \tilde{\nu}\) and \(\mu' := \tilde{\nu}_*\) are doubling. Obviously, \(\tilde{B}(p_0^*, R_1) \setminus \pi_*(J_0)\) is bounded and, in addition, finitely connected on the boundary because \(B(0, R_0) \setminus J_0\) is that. All of the conditions of Theorem 1.4 are fulfilled, thus, any \(H_m, m = 1, 2, \ldots\), has a continuous extension \(\overline{H_m}: \overline{B(p_0, r_1)} \to \overline{\tilde{B}(p_0^*, R_1)} \setminus \pi_*(J_0)_P\) such that
\[\overline{H_m}(B(p_0, r_1)) = \overline{B(p_0^*, R_1)} \setminus \pi_*(J_0)_P.\]

Note that for the constructed sequence of mappings \(H_m, m = 1, 2 \ldots\), satisfies and the conclusion of Theorem 1.5, as well. Indeed, observe that there are infinitely many continua \(A\) satisfying condition \(d(H^{-1}_m(A), \partial \mathbb{D}) \geq \delta, \ m = 1, 2, \ldots\), with some \(\delta > 0\), since all the mappings \(g_m\) in (8.7) are a fixed mapping by \(m\) for \(m > 2^{1/\alpha}\) and \(3/2 < |z| < 2\), and the remaining mappings that make up \(H_m\), are fixed and do not depend on the index \(m = 1, 2, \ldots\). Observe that \(\tilde{B}(p_0, r_1)\) is a weakly flat at inner and boundary points, because \(\tilde{B}(p_0, r_1)\) is hyperbolic isometric to the disk \(B(0, r_0)\) and, in addition, \(B(0, r_0)\) is weakly flat with the respect to hyperbolic metric and hyperbolic measure because the hyperbolic and Euclidean metrics are equivalent on.

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compact sets in \( \mathbb{D} \), and \( B(0, r_0) \) is weakly flat with the respect to Euclidean metric and Lebesgue measure by \([36, \text{Theorem 10.12, Theorem 17.10}]\), cf. \([31, \text{Lemma 2.2}]\). Observe that \( \overline{B}(p_0^*, R_1) \setminus \pi_*(J_0) \) is regular by Riemann’s mapping theorem. Thus, all the conditions of Theorem 1.5 are fulfilled, so that the family of mappings \( \overline{H}_m: \overline{B}(p_0, r_1) \rightarrow \overline{B}(p_0^*, R_1) \setminus \pi_*(J_0)_P \) is equicontinuous in \( \overline{B}(p_0, r_1) \).

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