Generalized Quantum Turing Machine and its Application to the SAT Chaos Algorithm

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April 1, 2022

Abstract
Ohya and Volovich have proposed a new quantum computation model with chaotic amplification to solve the SAT problem, which went beyond usual quantum algorithm. In this paper, we generalize quantum Turing machine, and we show in this general quantum Turing machine (GQTM) that we can treat the Ohya-Volovich (OV) SAT algorithm.

1 Introduction

The problem whether NP-complete problems can be P problem has been considered as one of the most important problems in theory of computational complexity. Various studies have been done for many years [1]. Ohya and Volovich [2, 3] proposed a new quantum algorithm with chaotic amplification process to solve the SAT problem, which went beyond usual quantum algorithm. This quantum chaos algorithm enabled to solve the SAT problem in a polynomial time [2, 3, 4].

In this paper we generalize quantum Turing machine so that it enables to describe non-unitary evolution of states. This study is based on mathematical studies of quantum communication channels [5, 6]. It is discussed in this generalized quantum Turing machine (GQTM) that we can treat the OV SAT algorithm.

In Section 2, we generalize QTM by rewriting usual QTM in terms of channel transformation so that it contains both dissipative and unitary dynamics. In Section 3, the SAT problem is reviewed and fundamental quantum unitary gates are presented. In Section 4, based on the papers [4, 7], we concretely construct the fundamental gates needed for computation of the SAT problem. In Section 5, we rewrite the total process including a measurement process and amplifier process with chaotic dynamics by GQTM.
Generalized Quantum Turing Machine

Classical Turing machine (TM or CTM) $M_{cl}$ is defined by a triplet $(Q, \Sigma, \delta)$, where $\Sigma$ is a finite alphabets with an identified blank symbol $\#$, $Q$ is a finite set of states (with an initial state $q_0$ and a set of final states $q_f$) and $\delta : Q \times \Sigma \rightarrow Q \times \Sigma \times \{-1, 0, 1\}$ is a transition function. Note that $\{-1, 0, 1\}$ indicates moving direction of the tape head of TM. The deterministic TM has a deterministic transition function $\delta : Q \times \Sigma \rightarrow 2^{Q \times \Sigma \times \{-1, 0, 1\}}$, that is, $\delta$ is a non-branching map, in other words, the range of $\delta$ for each $(q, a) \in Q \times \Sigma$ is unique. A TM $M$ is called non-deterministic if it is not deterministic.

Quantum Turing machine (QTM) was introduced by Deutsch [8] and has been extensively studied by Bernstein and Vazirani [9]. In this section, we introduce a generalized quantum Turing machine (GQTM), which contains QTM as a special case.

The Hilbert space $H$ of QTM consists from complex functions defined on the space of classical configurations.

**Definition 1** Usual Quantum Turing machine $M_q$ is defined by a quadruplet $M_q = (Q, \Sigma, H, U)$, where $H$ is a Hilbert space described below in (2.1) and $U$ is a unitary operator on the space $H$ of the special form described below in (2.2).

Let $C = Q \times \Sigma \times \mathbb{Z}$ be the set of all classical configurations of the Turing machine $M_{cl}$, where $\mathbb{Z}$ is the set of all integers. It is a countable set and one has

$$H = \left\{ \phi : C \rightarrow \mathbb{C}; \sum_{C \in C} |\phi(C)|^2 < \infty \right\}. \quad (2.1)$$

Since the configuration $C \in C$ can be written as $C = (q, A, i)$ one can say that the set of functions $\{|q, A, i>\}$ is a basis in the Hilbert space $H$. Here $q \in Q$, $i \in \mathbb{Z}$ and $A$ is a function $A : \mathbb{Z} \rightarrow \mathbb{C}$. We will call this basis the computational basis.

By using the computational basis we now state the conditions to the unitary operator $U$. We denote the set $\Gamma \equiv \{-1, 0, 1\}$. One requires that there is a function $\delta : Q \times \Sigma \times Q \times \Sigma \times \Gamma \rightarrow \mathbb{C}$ which takes values in the field of computable numbers $\mathbb{C}$ and such that the following relation is satisfied:

$$U |q, A, i\rangle = \sum_{p,b,\sigma} \delta(q, A(i), p, b, \sigma) |p, A_b^i, i + \sigma\rangle. \quad (2.2)$$

Here the sum runs over the states $p \in Q$, the symbols $b \in \Sigma$ and the elements $\sigma \in \Gamma$. Actually this is a finite sum. The function $A_b^i : \mathbb{Z} \rightarrow \mathbb{C}$ is defined as

$$A_b^i(j) = \begin{cases} b & \text{if } j = i, \\ A(j) & \text{if } j \neq i. \end{cases}$$

The restriction to the computable number field $\mathbb{C}$ instead of all the complex number $\mathbb{C}$ is required since otherwise we can not construct or design a quantum Turing Machine.
Note that if, for some integer \( t \in \mathbb{N} \equiv \{1, 2, \ldots \} \), the quantum state \( U^t |q_0, A, 0\rangle \) is a final quantum state, i.e. \( \|E_Q(q_{FR})U^s |q_0, A, 0\rangle\| = 1 \) and for any \( s < t, s \in \mathbb{N} \) one has \( \|E_Q(q_{FR})U^s |q_0, A, 0\rangle\| = 0 \), then one says that the quantum Turing machine halts with running time \( t \) on input \( A \).

Now we define the generalized quantum Turing machine (GQTM) by using of a channel \( \Lambda \) (see below) instead of a unitary operator \( U \).

**Definition 2** Generalized Quantum Turing machine \( M_{gq} \) (GQTM) is defined by a quadruplet \( M_{gq} = (Q, \Sigma, \mathcal{H}, \Lambda) \), where \( Q \) and \( \Sigma \) are two alphabets, \( \mathcal{H} \) is a Hilbert space and \( \Lambda \) is a channel on the space of states on \( \mathcal{H} \) of the special form described below.

Let us explain GQTM in more detailed. GQTM \( M_{gq} \) is defined by quadruplet \( (Q, \Sigma, \mathcal{H}, \Lambda) \), where \( Q \) is a processor configuration, \( \Sigma \) is a set of alphabet including a blank symbol and \( \Lambda \) is a quantum transition function sending a quantum state to a quantum state. \( Q \) and \( \Sigma \) are represented by a density operator on Hilbert space \( \mathcal{H}_Q \) and \( \mathcal{H}_\Sigma \), which are spanned by canonical basis \( \{|q\rangle : q \in Q\} \) and \( \{|a\rangle : a \in \Sigma\} \), respectively. A tape configuration \( A \) is a sequence of elements of \( \Sigma \) represented by a density operator on Hilbert space \( \mathcal{H}_\Sigma \) spanned by a canonical basis \( \{|A\rangle : A \in \Sigma^*\} \), where \( \Sigma^* \) is the set of sequences of alphabets in \( \Sigma \). A position of tape head is represented by a density operator on Hilbert space \( \mathcal{H}_Z \) spanned by a canonical basis \( \{|i\rangle : i \in \mathbb{Z}\} \). Then a configuration \( \rho \) of GQTM \( M_{gq} \) is described by a density operator in \( \mathcal{H} = \mathcal{H}_Q \otimes \mathcal{H}_\Sigma \otimes \mathcal{H}_Z \). Let \( \mathcal{S}(\mathcal{H}) \) be the set of all density operators in Hilbert space \( \mathcal{H} \). A quantum transition function \( \Lambda \) is given by a completely positive (CP) channel

\[
\Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}).
\]

For instance, given a configuration \( \rho = \sum_k \lambda_k |\psi_k\rangle \langle \psi_k| \), where \( \sum \lambda_k = 1, \lambda_k \geq 0 \) and \( \psi_k = |q_k \rangle \otimes |A_k \rangle \otimes |i_k \rangle \) \( (q_k \in Q, A_k \in \Sigma^*, i_k \in \mathbb{Z}) \) is a vector in a basis of \( \mathcal{H} \). This configuration changes to a new configuration \( \rho' \) by one step transition as \( \rho' = \Lambda(\rho) = \sum_k \mu_k |\psi_k\rangle \langle \psi_k| \) with \( \sum \mu_k = 1, \mu_k \geq 0 \).

One requirement on GQTM \( M_{gq} = (Q, \Sigma, \mathcal{H}, \Lambda) \) is the correspondence with QTM. If the channel \( \Lambda \) in GQTM will be a unitary operator \( U \) then GQTM \( M_{gq} = (Q, \Sigma, \mathcal{H}, \Lambda = U \cdot U^*) \) reduces to QTM \( M_q = (Q, \Sigma, \mathcal{H}, U) \).

Several studies have been done on QTM whose transition function is represented by unitary operator, in which various theorems and computational classes in QTM were discussed in [9, 10].

Let us explain how to construct a QTM. Let \( \delta \) be a function

\[
\delta : Q \times \Sigma \times Q \times \Sigma \times \{-1, 0, 1\} \rightarrow \mathbb{C}.
\]

For any \( q \in Q, a \in \Sigma \), it holds

\[
\sum_{p \in Q, b \in \Sigma, d \in \{-1, 0, 1\}} |\delta(q, a, p, b, d)|^2 = 1.
\]

For any \( q \in Q, a \in \Sigma, q' (\neq q) \in Q, a' (\neq a) \in \Sigma \), it holds
Given QTM $M_q$ and its configuration $\rho = |\varphi\rangle \langle \varphi|$ with $|\varphi\rangle = |q, A, i\rangle$, after one step, this configuration is changed by a transition function $\delta$ as

$$\Lambda_\delta(|q, A, i\rangle \langle i, A, q|) = \sum_{p,b,\sigma,p',b',\sigma'} \delta(q, A(i), p, b, \sigma) \delta^*(q, A(i), p', b', \sigma')$$

$$|p, A^b_i, i + \sigma\rangle \langle i + \sigma, A'^b_i, p'|$$

**Remark 3** For any $q, p \in Q, a, b \in \Sigma, d \in \{-1, 0, 1\}$, let $\delta(q, a, p, b, d) = \{0, 1\}$, then QTM is a reversal TM.

A transition of GQTM is regarded as a transition of amplitude of each configuration vector. We categorize GQTMs by a property of CP channel $\Lambda$ as below.

**Definition 4** A GQTM $M_{gq}$ is called unitary QTM (UQTM, i.e., usual QTM), if all of quantum transition function $\Lambda$ in $M$ are unitary CP channel.

For all configuration $\rho = \sum_n \lambda_n \rho_n$ ($\sum_n \lambda_n = 1, \lambda_n \geq 0$), a GQTM $M_{gq}$ is called LQTM $M_{lq}$ if $\Lambda$ is affine ; $\Lambda(\sum_n \lambda_n \rho_n) = \sum_n \lambda_n \Lambda(\rho_n)$. Since a measurement defined by $\Lambda_M \rho = \sum_k P_k \rho P_k$ with a PVM $\{P_k\}$ on $\mathcal{H}$ is a linear CP channel, LQTM may include a measurement process.

For a more general channel the state change is expressed as

$$\Lambda(|q, A(i), i\rangle \langle q, A(i), i|) = \sum_{p,b,\sigma,p',b',\sigma'} \delta(q, A(i), p, b, \sigma) \delta^*(q, A(i), p', b', \sigma')$$

$$|p, A^b_i, i + \sigma\rangle \langle i + \sigma, A'^b_i, p'|$$

with some function $\delta(q, A(i), p, b, \sigma, p', b', \sigma')$ such that the RHS of this relation is a state.

Thus we define two more classes of GQTM for non-unitary CP channels.

**Definition 5** A GQTM $M_{gq}$ is called a linear QTM(LQTM) if its quantum transition function $\Lambda$ is a linear quantum channel.

Unitary operator is linear, hence UQTM is a sub-class of LQTM. moreover, classical TM is a special class of LQTM.

**Definition 6** A GQTM $M_{gq}$ is called non-linear QTM (NLQTM) if its quantum transition function $\Lambda$ contains non-linear CP channel.

A chaos amplifier used in [2, 3] is a non-linear CP channel, the details of this channel and its application to the SAT problem will be discussed in the sequel.
2.1 Computational class for GQTM

Let us state some language classes which classical Turing machine recognizes.

Definition 7 The class of languages is in \( P \) if its language is recognized by a deterministic Turing machine in polynomial time of input size.

Definition 8 The class of languages is in \( NP \) if there is a deterministic Turing machine, called the verifier, which recognize languages with some informations in polynomial time of input size. Besides, if a language \( L_1 \in NP \) and \( L_1 \) reduces to \( L_2 \in NP \) in polynomial time, a language \( L_1 \) is \( NP \)-complete.

Definition 9 If languages are accepted by non-deterministic Turing machine in polynomial time of input size with a certain probability, this class of languages are called the class of bounded probability polynomial time (BPP).

A \( NP \)-complete language is the most difficult one in \( NP \). If there is a polynomial time algorithm to solve it in the above sense, it implies \( P=NP \). The existence of such a algorithm is demonstrated in [2, 3] in an extended quantum domain, as is reviewed in the next section. In this paper we will show that this OV algorithm can be written by GQTM in the sequel section.

Given a GQTM \( M_{gq} = (Q, \Sigma, \delta) \) and an input configuration \( \rho_0 = |v_{in}\rangle \langle v_{in}| \), \( (|v_{in}\rangle = |q_0\rangle \otimes |T\rangle \otimes |0\rangle) \), a computation process is described as the following product of several different types of channels

\[ \Lambda_1 \circ \cdots \circ \Lambda_t (\rho_0) = \rho_f \equiv |v_f\rangle \langle v_f| \]

where \( \Lambda_1, \cdots, \Lambda_t \) are CP channels. Applying the CP channels to an initial state, we obtain a final state \( \rho_f \) and we measure this state by a projection (or PVM)

\[ P_f = |q_f\rangle \langle q_f| \otimes I_\Sigma \otimes I_Z, \]

where \( I_\Sigma, I_Z \) are identity operators on \( H_\Sigma, H_Z \), respectively. Let \( p \geq 0 \) be a halting probability such that

\[ tr_{H_\Sigma \otimes H_Z} (P_f \rho_f) = p |q_f\rangle \langle q_f|. \]

Then, we define the acceptance (rejection) of GQTM and some classes of languages.

Definition 10 Given GQTM \( M_{gq} \) and a language \( L \), if there exists \( N \) steps when we obtain the configuration of acceptance (or rejection) by the probability \( p \), we say that the GQTM \( M_{gq} \) accepts (or rejects) \( L \) by the probability \( p \), and its computational complexity is \( t \).

Definition 11 A language \( L \) is bounded quantum probability polynomial time GQTM (BGQPP) if there is a polynomial time GQTM \( M_{gq} \) which accepts \( L \) with probability \( p \geq \frac{1}{2} \).
Similarly, we can define the class of languages BUQPP (= BQPP), BLQPP, BNLQPP (= BGQPP) corresponding to UQTM, LQTM and NLQTM, respectively.

In Section 2, it is pointed out that LQTM includes classical TM, which it may imply: BPP ⊆ BLQPPL ⊆ BNLQPP ⊆ BGQPP. Moreover, if NLQTM accepts the SAT OV algorithm in polynomial time with probability $p \geq \frac{1}{2}$, then we may have the inclusion

$$NP \subseteq BGQPP$$

We will discuss this inclusion in Sec. 4 by constructing GQTM which accepts the SAT OV algorithm.

3 SAT Problem

Let $X \equiv \{x_1, \ldots, x_n\}, n \in \mathbb{N}$ be a set. $x_k$ and its negation $\overline{x}_k (k = 1, \ldots, n)$ are called literals. Let $\overline{X} \equiv \{\overline{x}_1, \ldots, \overline{x}_n\}$ be a set, then the set of all literals is denoted by $X' \equiv X \cup \overline{X} = \{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$. The set of all subsets of $X'$ is denoted by $\mathcal{F}(X')$ and an element $C \in \mathcal{F}(X')$ is called a clause. We take a truth assignment to all variables $x_k$. If we can assign the truth value to at least one element of $C$, then $C$ is called satisfiable. When $C$ is satisfiable, the truth value $t(C)$ of $C$ is regarded as true, otherwise, that of $C$ is false. Take the truth values as "true $\leftrightarrow 1$, false $\leftrightarrow 0$". Then $C$ is satisfiable iff $t(C) = 1$.

Let $L = \{0, 1\}$ be a Boolean lattice with usual join $\lor$ and meet $\land$, and $t(x)$ be the truth value of a literal $x$ in $X$. Then the truth value of a clause $C$ is written as $t(C) \equiv \bigvee_{x \in C} t(x)$.

Moreover the set $\mathcal{C}$ of all clauses $C_j (j = 1, 2, \ldots, m)$ is called satisfiable iff the meet of all truth values of $C_j$ is 1; $t(\mathcal{C}) \equiv \bigwedge_{j=1}^{m} t(C_j) = 1$. Thus the SAT problem is written as follows:

**Definition 12** SAT Problem: Given a Boolean set $X \equiv \{x_1, \ldots, x_n\}$ and a set $\mathcal{C} = \{C_1, \ldots, C_m\}$ of clauses, determine whether $\mathcal{C}$ is satisfiable or not.

That is, this problem is to ask whether there exists a truth assignment to make $\mathcal{C}$ satisfiable. It is known in usual algorithm that it is polynomial time to check the satisfiability only when a specific truth assignment is given, but we cannot determine the satisfiability in polynomial time when an assignment is not specified.

In [4] we discussed the quantum algorithm of the SAT problem, which was rewritten in [2] with showing that the OM SAT-algorithm is combinatorial. In [2, 3] it is shown that the chaotic quantum algorithm can solve the SAT problem in polynomial time.

Ohya and Masuda pointed out [4] that the SAT problem, hence all other NP problems, can be solved in polynomial time by quantum computer if the superposition of two orthogonal vectors $|0\rangle$ and $|1\rangle$ is physically detected. However this detection is considered not to be possible in the present technology.
The problem to be overcome is how to distinguish the pure vector $|0\rangle$ from the superposed one $\alpha |0\rangle + \beta |1\rangle$, obtained by the OM SAT-quantum algorithm, if $\beta$ is not zero but very small. If such a distinction is possible, then we can solve the NPC problem in the polynomial time. In [2, 3] it is shown that it can be possible by combining nonlinear chaos amplifier with the quantum algorithm, which implies the existence of a mathematical algorithm solving NP=P. The algorithm of Ohya and Volovich is not known to be in the framework of quantum Turing algorithm or not. This aspect is studied in this paper.

### 3.1 Quantum computation

In this subsection, we review fundamentals of quantum computation (see, for instance, [11]). Let $\mathbb{C}$ be the set of all complex numbers, and $|0\rangle$ and $|1\rangle$ be the two unit vectors $\left| \begin{array}{c} 1 \\ 0 \end{array} \right\rangle$ and $\left| \begin{array}{c} 0 \\ 1 \end{array} \right\rangle$, respectively. Then, for any two complex numbers $\alpha$ and $\beta$ satisfying $|\alpha|^2 + |\beta|^2 = 1$, $\alpha |0\rangle + \beta |1\rangle$ is called a qubit. For any positive integer $N$, let $\mathcal{H}$ be the tensor product Hilbert space defined as $(\mathbb{C}^2)^\otimes N$ and let $\{ |e_i\rangle : 0 \leq i \leq 2^{N-1} \}$ be the basis whose elements are denoted as

- $|e_0\rangle = |0\rangle \otimes |0\rangle \cdots \otimes |0\rangle \equiv |0,0,\ldots,0\rangle$,
- $|e_1\rangle = |1\rangle \otimes |0\rangle \cdots \otimes |0\rangle \equiv |1,0,\ldots,0\rangle$,
- $|e_2\rangle = |0\rangle \otimes |1\rangle \cdots \otimes |0\rangle \equiv |0,1,\ldots,0\rangle$, 
- $\vdots$
- $|e_{2^{N-1}}\rangle = |1\rangle \otimes |1\rangle \cdots \otimes |1\rangle \equiv |1,1,\ldots,1\rangle$.

For any two qubits $|x\rangle$ and $|y\rangle$, $|x,y\rangle$ and $|x^N\rangle$ is defined as $|x\rangle \otimes |y\rangle$ and $|x\rangle \otimes \cdots \otimes |x\rangle$, respectively.

The usual (unitary) quantum computation can be formulated mathematically as the multiplication by unitary operators. Let $U_{NOT}$, $U_{CN}$ and $U_{CCN}$ be the three unitary operators defined as

- $U_{NOT} \equiv |1\rangle \langle 0| + |0\rangle \langle 1|$,
- $U_{CN} \equiv |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes U_{NOT}$,
- $U_{CCN} \equiv |0\rangle \langle 0| \otimes I \otimes I + |1\rangle \langle 1| \otimes |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \langle 1| \otimes U_{NOT}$.

$U_{NOT}$, $U_{CN}$ and $U_{CCN}$ represent the NOT-gate, the Controlled-NOT gate and the Controlled-Controlled-NOT gate, respectively. Moreover, Hadamard transformation $H$ is defined as the transformation on $\mathbb{C}^2$ such as

$$H |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \quad H |1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle).$$

These four operators $U_{NOT}$, $U_{CN}$, $U_{CCN}$ and $H$ are called the elementary gates here. For any $k \in \mathbb{N}$, $U_{H}^{(N)}(k)$ denotes the $k$-tuple Hadamard transformation on $(\mathbb{C}^2)^\otimes N$ defined as
\[ U_{H}^{(N)}(k) |0^{N}\rangle = \frac{1}{2^{k/2}} (|0\rangle + |1\rangle)^{\otimes k} |0^{N-k}\rangle = \frac{1}{2^{k/2}} \sum_{i=0}^{2^{k-1}} |\epsilon_{i}\rangle \otimes |0^{N-k}\rangle. \]

The above unitary operators can be extended to the unitary operators on \((\mathbb{C}^{2})^{\otimes N}\):

\[
U_{NOT}^{(N)}(u) \equiv I^{\otimes u-1} \otimes (|0\rangle \langle 1| + |1\rangle \langle 0|) I^{\otimes N-u-1} \\
U_{CN}^{(N)}(u, v) \equiv I^{\otimes u-1} \otimes |0\rangle \otimes I^{\otimes N-u-1} + I^{\otimes u-1} \otimes |1\rangle \langle 1| \\
\otimes I^{\otimes v-u-1} \otimes U_{NOT} \otimes I^{\otimes N-v-1} \\
U_{CCN}^{(N)}(u, v, w) = I^{\otimes u-1} \otimes |0\rangle \otimes I^{\otimes N-u-1} + I^{\otimes u-1} \otimes |1\rangle \langle 1| \\
\otimes I^{\otimes v-u-1} \otimes |0\rangle \otimes I^{\otimes N-v-1} \\
+ I^{\otimes u-1} \otimes |1\rangle \langle 1| \otimes I^{\otimes v-u-1} \otimes |1\rangle \langle 1| \\
I^{\otimes w-t-1} \otimes U_{NOT} \otimes I^{\otimes N-w-1},
\]

where \(u, v\) and \(w\) be a positive integers satisfying \(1 \leq u < v < w \leq N\).

Furthermore we have the following three unitary operators \(U_{AND}, U_{OR}\) and \(U_{COPY}\), called the logical gates; (see [7])

\[
U_{AND} \equiv \sum_{\epsilon_{1}, \epsilon_{2} \in \{0, 1\}} \{|\epsilon_{1}, \epsilon_{2}, \epsilon_{1} \wedge \epsilon_{2}\} \langle \epsilon_{1}, \epsilon_{2}, 0| + |\epsilon_{1}, \epsilon_{2}, 1 - \epsilon_{1} \wedge \epsilon_{2}\} \langle \epsilon_{1}, \epsilon_{2}, 1|angle
\]

\[
= |0, 0, 0\rangle \langle 0, 0, 0| + |0, 0, 1\rangle \langle 0, 0, 1| + |1, 0, 0\rangle \langle 1, 0, 0| + |1, 0, 1\rangle \langle 1, 0, 1| \\
+ |0, 1, 0\rangle \langle 0, 1, 0| + |0, 1, 1\rangle \langle 0, 1, 1| + |1, 1, 0\rangle \langle 1, 1, 0| + |1, 1, 1\rangle \langle 1, 1, 1|. 
\]

\[
U_{OR} \equiv \sum_{\epsilon_{1}, \epsilon_{2} \in \{0, 1\}} \{|\epsilon_{1}, \epsilon_{2}, \epsilon_{1} \vee \epsilon_{2}\} \langle \epsilon_{1}, \epsilon_{2}, 0| + |\epsilon_{1}, \epsilon_{2}, 1 - \epsilon_{1} \vee \epsilon_{2}\} \langle \epsilon_{1}, \epsilon_{2}, 1|angle
\]

\[
= |0, 0, 0\rangle \langle 0, 0, 0| + |0, 0, 1\rangle \langle 0, 0, 1| + |1, 0, 0\rangle \langle 1, 0, 0| + |1, 0, 1\rangle \langle 1, 0, 1| \\
+ |0, 1, 0\rangle \langle 0, 1, 0| + |0, 1, 1\rangle \langle 0, 1, 1| + |1, 1, 0\rangle \langle 1, 1, 0| + |1, 1, 1\rangle \langle 1, 1, 1|. 
\]

\[
U_{COPY} \equiv \sum_{\epsilon_{1} \in \{0, 1\}} \{|\epsilon_{1}, \epsilon_{1}\} \langle \epsilon_{1}, 0| + |\epsilon_{1}, 1 - \epsilon_{1}\} \langle \epsilon_{1}, 1|\}
\]

\[
= |0, 0\rangle \langle 0, 0| + |0, 1\rangle \langle 0, 1| + |1, 0\rangle \langle 1, 0| + |1, 1\rangle \langle 1, 1|. 
\]

We call \(U_{AND}, U_{OR}\) and \(U_{COPY}\), AND gate, OR gate and COPY gate, respectively, whose extensions on \((\mathbb{C}^{2})^{\otimes N}\) are denoted by \(U_{AND}^{(N)}, U_{OR}^{(N)}\) and \(U_{COPY}^{(N)}\), which are expressed as
 propose. Let $U$ where $\mu$ unitary quantum algorithm. The detail of this section is given in the papers [4, 3, 7, 9]. In this section, we explain the algorithm of the SAT problem which has been introduced by Ohya-Masuda [4] and developed by Accardi-Sabbadin [7]. The operators can be written, in terms of elementary gates, as

$$U_{\text{AND}}^{(N)}(u, v, w) = \sum_{\varepsilon_1, \varepsilon_2 \in \{0, 1\}} I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2\rangle \langle \varepsilon_2|$$

$$I^{\otimes w-v-u-1} \otimes |\varepsilon_1 \land \varepsilon_2\rangle \langle 0| I^{\otimes N-w-v-u} +$$

$$I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes$$

$$|\varepsilon_2\rangle \langle \varepsilon_2| I^{\otimes w-v-u-1} \otimes |1 - \varepsilon_1 \land \varepsilon_2\rangle \langle 1| I^{\otimes N-w-v-u}.$$  

$$U_{\text{OR}}^{(N)}(u, v, w) \equiv \sum_{\varepsilon_1, \varepsilon_2 \in \{0, 1\}} I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2\rangle \langle \varepsilon_2|$$

$$I^{\otimes w-v-u-1} \otimes |\varepsilon_1 \lor \varepsilon_2\rangle \langle 0| I^{\otimes N-w-v-u} +$$

$$I^{\otimes u-1} \otimes |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} \otimes |\varepsilon_2\rangle \langle \varepsilon_2|$$

$$I^{\otimes w-v-u-1} \otimes |1 - \varepsilon_1 \lor \varepsilon_2\rangle \langle 1| I^{\otimes N-w-v-u}.$$  

$$U_{\text{COPY}}^{(N)}(u, v, w) \equiv \sum_{\varepsilon_1 \in \{0, 1\}} I^{\otimes u-1} |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} |\varepsilon_1\rangle \langle 0| I^{\otimes N-v-u}$$

$$+ I^{\otimes u-1} |\varepsilon_1\rangle \langle \varepsilon_1| I^{\otimes v-u-1} |1 - \varepsilon_1\rangle \langle 1| I^{\otimes N-v-u}.$$  

where $u, v$ and $w$ are positive integers satisfying $1 \leq u < v < w \leq N$. These operators can be written, in terms of elementary gates, as

$$U_{\text{OR}}^{(N)}(u, v, w) = U_{\text{AND}}^{(N)}(u, w) \cdot U_{\text{CN}}^{(N)}(v, w) \cdot U_{\text{CCN}}^{(N)}(u, v, w),$$

$$U_{\text{AND}}^{(N)}(u, v, w) = U_{\text{CCN}}^{(N)}(u, v, w),$$

$$U_{\text{COPY}}^{(N)}(u, v, w) = U_{\text{CN}}^{(N)}(u, v).$$

## 4 SAT Algorithm

In this section, we explain the algorithm of the SAT problem which has been introduced by Ohya-Masuda [4] and developed by Accardi-Sabbadin [7]. The computation of the truth value can be done by a combination of the unitary operators on a Hilbert space $\mathcal{H}$, so that the computation is described by the unitary quantum algorithm. The detail of this section is given in the papers [4, 3, 7, 9], so we will discuss just the essence of the OM algorithm. Throughout this section, let $n$ be the total number of Boolean variables used in the SAT problem. Let $\mathcal{C}$ be a set of clauses whose cardinality is equal to $m$. Let $\mathcal{H} = (C^2)^{\otimes n+\mu+1}$ be a Hilbert space and $|v_0\rangle$ be the initial state $|v_0\rangle = |0^n, 0^\mu, 0\rangle$, where $\mu$ is the number of dust qubits which is determined by the following proposition. Let $U_{\mathcal{C}}^{(n)}$ be a unitary operator for the computation of the SAT:

$$U_{\mathcal{C}}^{(n)} |v_0\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |e_i, x^\mu, t_e, (\mathcal{C})\rangle \equiv |v_f\rangle.$$
where \( x^\mu \) denotes the \( \mu \) strings in the dust bits and \( t_{e_i}(C) \) is the truth value of \( C \) with \( e_i \). In [4,7], \( U^{(n)}_C \) was constructed.

Let \( \{ s_k; k=1,\ldots, m \} \) be the sequence defined as

\[
\begin{align*}
    s_1 &= n + 1, \\
    s_2 &= s_1 + \text{card}(C_1) + \delta_{1,\text{card}(C_1)} - 1, \\
    s_i &= s_{i-1} + \text{card}(C_{i-1}) + \delta_{1,\text{card}(C_{i-1})}, \quad 3 \leq i \leq m,
\end{align*}
\]

where \( \text{card}(C_i) \) means the cardinality of a clause \( C_i \). And let define \( s_f \) as

\[
    s_f = s_m - 1 + \text{card}(C_m) + \delta_{1,\text{card}(C_m)}.
\]

Note that the number \( m \) of the clause is at most \( 2n \). Then we have the following proposition and theorem [7].

**Proposition 13** For \( m \geq 2 \), the total number of dust qubits \( \mu \) is

\[
    \mu = s_f - 1 - n = \sum_{k=1}^{m} \text{card}(C_k) + \delta_{1,\text{card}(C_k)} - 2.
\]

Determining \( \mu \) and the work spaces for computing \( t(C_k) \), we can construct \( U^{(n)}_C \) concretely. We use the following unitary gates for this concrete expression:

\[
    U^{(x)}_{AND}(k) = \begin{cases} 
        U^{(x)}_{AND}(s_k + 1, s_{k+2} - 2, s_{k+2} - 1), & 1 \leq k \leq m - 2 \\
        U^{(x)}_{AND}(s_m - 1, s_f - 1, s_f), & k = m - 1 
    \end{cases},
\]

\[
    U^{(x)}_{OR}(k) = \tilde{U}^{(x)}_{OR}(l_4, s_k - \text{card}(C_k) - 1, s_k - \text{card}(C_k) - 2) \cdots \tilde{U}^{(x)}_{OR}(l_3, s_k, s_k + 1) \tilde{U}^{(x)}_{OR}(l_1, l_2, s_k),
\]

\[
    \tilde{U}^{(x)}_{OR}(u, v, w) = \begin{cases} 
        U^{(x)}_{OR}(u, v, w), & x_u \in C_k \\
        U^{(x)}_{NOT}(u) \cdot U^{(x)}_{OR}(u, v, w) \cdot U^{(x)}_{NOT}(u), & \bar{x}_u \in C_k
    \end{cases},
\]

where \( l_1, l_2, l_3, l_4 \) are positive integers such that \( x_z \in C_k \) or \( \bar{x}_z \in C_k \), \( (z = l_1, \ldots, l_4) \).

**Theorem 14** The unitary operator \( U^{(n)}_C \), is represented as

\[
    U^{(n)}_C = U^{(n+\mu+1)}_{AND}(m - 1) \cdot U^{(n+\mu+1)}_{AND}(m - 2) \cdots U^{(n+\mu+1)}_{AND}(1) \\
    \cdot U^{(n+\mu+1)}_{OR}(m) \cdot U^{(n+\mu+1)}_{OR}(m - 1) \cdots U^{(n+\mu+1)}_{OR}(1) \cdot U^{(n+\mu+1)}_{H}(n).
\]

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4.1 The resulting state in the SAT algorithm

Applying the above unitary operator to the initial state, we obtain the final state $\rho$. The result of the computation is registered as $|t(C)\rangle$ in the last section of the final vector, which will be taken out by a projection $P_{n+\mu,1} \equiv I^{\otimes n+\mu} \otimes |1\rangle \langle 1|$ onto the subspace of $H$ spanned by the vectors $|\varepsilon^n, \varepsilon^{\mu}, 1\rangle$.

The following theorem is easily seen.

**Theorem 15** $C$ is SAT if and only if

$$P_{n+\mu,1}U_C^{(n)}|v_0\rangle \neq 0$$

According to the standard theory of quantum measurement, after a measurement of the event $P_{n+\mu,1}$, the state $\rho = |v_f><v_f|$ becomes

$$\rho \rightarrow \frac{P_{n+\mu,1}\rho P_{n+\mu,1}}{Tr\rho P_{n+\mu,1}} =: \rho'$$

Thus the solvability of the SAT problem is reduced to check that $\rho' \neq 0$. The difficulty is that the probability

$$Tr\rho P_{n+\mu,1} = \|P_{n+\mu,1}|v_f\rangle \|^2 = \frac{|T(C_0)|}{2^n}$$

is very small in some cases, where $|T(C_0)|$ is the cardinality of the set $T(C_0)$, of all the truth functions $t$ such that $t(C_0) = 1$.

We put $q \equiv \sqrt{\frac{r}{2^n}}$ with $r \equiv |T(C_0)|$. Then if $r$ is suitably large to detect it, then the SAT problem is solved in polynomial time. However, for small $r$, the probability is very small so that we in fact do not get an information about the existence of the solution of the equation $t(C_0) = 1$, hence in such a case we need further deliberation.

Let go back to the SAT algorithm. After the quantum computation, the quantum computer will be in the state

$$|v_f\rangle = \sqrt{1-q^2} |\varphi_0\rangle \otimes |0\rangle + q |\varphi_1\rangle \otimes |1\rangle$$

where $|\varphi_1\rangle$ and $|\varphi_0\rangle$ are normalized $n (=n + \mu)$ qubit states and $q = \sqrt{r/2^n}$. Effectively our problem is reduced to the following 1 qubit problem: The above state $|v_f\rangle$ is reduced to the state

$$|\psi\rangle = \sqrt{1-q^2} |0\rangle + q |1\rangle,$$

and we want to distinguish between the cases $q = 0$ and $q > 0$(small positive number). Let us denote the correspondence from $\rho_0 \equiv |v_0\rangle \langle v_0|$ with $\rho$ by a channel $\Lambda_I; \rho = \Lambda_I\rho_0$.

It is argued in [13] that quantum computer can speed up NP problems quadratically but not exponentially. The no-go theorem states that if the inner product of two quantum states is close to 1, then the probability that a measurement distinguishes which one of the two is exponentially small. And one
may claim that amplification of this distinguishability is not possible in usual quantum algorithm. At this point we emphasized that we do not propose to make a measurement which will be overwhelmingly likely to fail. What we did is a proposal to use the output $|\psi\rangle$ of the quantum computer as an input for another device which uses chaotic dynamics. The amplification would be not possible if we use the standard model of quantum computations with a unitary evolution. However the idea of the paper [2, 3] is different. In [2, 3] it is proposed to combine quantum computer with a chaotic dynamics amplifier. Such a quantum chaos computer is a new model of computations and we demonstrate that the amplification is possible in the polynomial time.

One could object that we do not suggest a practical realization of the new model of computations. But at the moment nobody knows of how to make a practically useful implementation of the standard model of quantum computing ever. It seems to us that the quantum chaos computer considered in [3] deserves an investigation and has a potential to be realizable.

4.2 Chaotic dynamics

Various aspects of classical and quantum chaos have been the subject of numerous studies ([5, 11] and ref’s therein). Here we will briefly review how chaos can play a constructive role in computation (see [2, 3] for the details).

Chaotic behavior in a classical system usually is considered as an exponential sensitivity to initial conditions. It is this sensitivity we would like to use to distinguish between the cases $q = 0$ and $q > 0$ discussed in the previous subsection.

Consider the so called logistic map which is given by the equation

$$x_{n+1} = ax_n(1 - x_n) \equiv g(x), \quad x_n \in [0, 1].$$

The properties of the map depend on the parameter $a$. If we take, for example, $a = 3.71$, then the Lyapunov exponent is positive, the trajectory is very sensitive to the initial value and one has the chaotic behavior [3]. It is important to notice that if the initial value $x_0 = 0$, then $x_n = 0$ for all $n$.

The state $|\psi\rangle$ of the previous subsection is transformed into the density matrix of the form

$$\overline{\rho} = q^2 P_1 + (1 - q^2) P_0$$

where $P_1$ and $P_0$ are projectors to the state vectors $|1\rangle$ and $|0\rangle$. One has to notice that $P_1$ and $P_0$ generate an Abelian algebra which can be considered as a classical system. The density matrix $\rho$ above is interpreted as the initial data, and we apply the channel $\Lambda \equiv \Lambda_{CA}$ due to the logistic map as

$$\Lambda_{CA}(\overline{\rho}) = \frac{(I + g(\overline{\rho}) \sigma_3)}{2},$$

where $I$ is the identity matrix and $\sigma_3$ is the z-component of Pauli matrices.

$$\overline{\rho}_k = \Lambda_{CA}^k(\overline{\rho})$$
To find a proper value \( k \) we finally measure the value of \( \sigma_3 \) in the state \( \rho_k \) such that

\[
M_k \equiv \text{tr} \rho_k \sigma_3.
\]

We obtain \[3\]

**Theorem 16**

\[
\rho_k = \left(I + g^k(q^2)\sigma_3\right)^2, \quad M_k = g^k(q^2).
\]

Thus the question is whether we can find such a \( k \) in polynomial steps of \( n \) satisfying the inequality

\[
M_k \geq \frac{1}{2}
\]

for very small but non-zero \( q \). Here we have to remark that if one has \( q = 0 \) then \( \rho = P_0 \) and we obtain \( M_k = 0 \) for all \( k \). If \( q \neq 0 \), the chaotic dynamics leads to the amplification of the small magnitude \( q \) in such a way that it can be detected. The transition from \( \rho \) to \( \rho_k \) is nonlinear and can be considered as a classical evolution because our algebra generated by \( P_0 \) and \( P_1 \) is abelian. The amplification can be done within at most \( 2n \) steps due to the following propositions. Since \( g^k(q^2) \) is \( x^k \) of the logistic map

\[
x_{k+1} = g(x_k)
\]

with \( x_0 = q^2 \), we use the notation \( x_k \) in the logistic map for simplicity.

**Theorem 17** For the logistic map \( x_{n+1} = ax_n (1 - x_n) \) with \( a \in [0, 4] \) and \( x_0 \in [0, 1] \), let \( x_0 = \frac{1}{2} \) and a set \( J \) be \( \{0, 1, 2, \ldots, n, \ldots, 2n\} \). If \( a \) is 3.71, then there exists an integer \( k \) in \( J \) satisfying \( x_k > \frac{1}{2} \).

**Theorem 18** Let \( a \) and \( n \) be the same in above theorem. If there exists \( k \) in \( J \) such that \( x_k > \frac{1}{2} \), then \( k > \frac{n-1}{\log_2 3.71 - 1} \).

**Corollary 19** If \( x_0 = \frac{r}{2^r} \) with \( r = |T(C)| \) and there exists \( k \) in \( J \) such that \( x_k > \frac{1}{2} \), then there exists \( k \) satisfying the following inequality if \( C \) is SAT.

\[
\left\lfloor \frac{n - 1 - \log_2 r}{\log_2 3.71 - 1} \right\rfloor \leq k \leq \left\lfloor \frac{5}{4} (n - 1) \right\rfloor.
\]

From these theorems, for all \( k \), it holds

\[
M_k \begin{cases} 
= 0 & \text{iff } C \text{ is not SAT} \\
> 0 & \text{iff } C \text{ is SAT}
\end{cases}
\]

## 5 SAT algorithm in GQTM

In this section, we construct a GQTM for the OV SAT algorithm. The GQTM with the chaos amplifier belongs to NLQTM because the chaos amplifier is represented by non-linear CP channel. The OV algorithm runs from an initial state \( \rho_0 \equiv |v_0\rangle \langle v_0| \) to \( \rho_k \) through \( \rho \equiv |v_f\rangle \langle v_f| \). The computation from \( \rho_0 \equiv |v_0\rangle \langle v_0| \) to \( \rho \equiv |v_f\rangle \langle v_f| \) is due to unitary channel \( \Lambda_C \equiv U_C \cdot U_C \), and that from \( \rho \equiv |v_f\rangle \langle v_f| \) to \( \rho_f \) is due to a non-unitary channel \( \Lambda_C^\dagger \circ \Lambda_I \), so that all
computation can be done by $\Lambda_{CA}^i \circ \Lambda_I \circ \Lambda_C$, which is a completely positive, so the whole computation process is deterministic. It is a multi-track (actually 4 tracks) GQTM that represents this whole computation process.

A multi-track GQTM has some workspaces for calculation, whose tracks are independent each other. This independence means that the TM can operate only one track at one step and all tracks do not affect each other. Let us explain our computation by a multi-track GQTM. The first track stores the input data and the second track stores the value of literals. The third track is used for the computation of $t(C_i), (i = 1, \ldots, m)$ described by unitary operators. The fourth track is used for the computation of $t(C)$ denoting the result. The work of GQTM is represented by the following 8 steps:

- **Step 1**: Store the counter $c = 0$ in Track 1. Calculate $\left\lfloor \frac{n}{2} (n - 1) \right\rfloor + 1$, we take this value as the maximum value of the counter. Then, store it in Track 4.
- **Step 2**: Calculate $c + 1$ and store it in Track 4.
- **Step 3**: Apply the Hadamard transform to Track 2.
- **Step 4**: Calculate $t(C_1), \ldots, t(C_m)$ and store them in Track 3.
- **Step 5**: Calculate $t(C)$ by using the value of the third track, and store $t(C)$ in Track 4.
- **Step 6**: Empty the first, second and third Tracks.
- **Step 7**: Apply the chaos amplifier to the result state obtained up to the step 6.
- **Step 8**: If $c = \left\lfloor \frac{n}{2} (n - 1) \right\rfloor + 1$ or GQTM is in the final state, GQTM halts. If GQTM is not in the final state, GQTM runs the step 2 to the step 8 again.

Let us explain the above steps for unitary computation (OM algorithm; i.e., up to the steps 6 above) by an example. Let the number of literals be $n$ and that of clauses be $m$. Then the language is represented by the following strings

$$0^n X \prod_{i=1}^{m} C_S G(C_i) C_E,$$

where

$$G(C_i) = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n Y_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n}$$

$$\varepsilon_k = \begin{cases} 0 & k \notin I_i \\ 1 & k \in I_i \end{cases}$$

$$\overline{\varepsilon_k} = \begin{cases} 0 & k \notin \overline{I_i} \\ 1 & k \in \overline{I_i} \end{cases}$$
and $X, C_S, Y, C_E$ are used as particular symbols of clauses. For example, given $X = \{1, 2, 3\}, C = \{C_1, C_2, C_3\}, C_1 = (\{1, 2\}, \{3\}), C_2 = (\{3\}, \{2\}), C_3 = (\{1\}, \{2, 3\}),$ the input tape will be 

$000XC_S110Y001C_EC_S001Y010C_EC_S100Y011C_E$

First, our GQTM applies DFT to a part of literals on the track 2. The transition function for DFT is written by the following table. Put the vector in $\mathcal{H}_Q$ by $q$ instead of $|q\rangle$ and denote the direction moving the tape head by $R$ for the right and $L$ to the left (Note that $O$ is the starting position).

| $#$ | 0          | 1          | $X$          |
|-----|------------|------------|--------------|
| $q_0$ | $q_a, 0, R$ | $q_a, 1, R$ | $q_b, X, L$ |
| $q_f$ | $\frac{1}{\sqrt{2}}q_b, 0, L + \frac{1}{\sqrt{2}}q_b, 1, L$ | $\frac{1}{\sqrt{2}}q_b, 0, L - \frac{1}{\sqrt{2}}q_b, 1, L$ | $q_a, 0, R$ |

The tape head moves to the right until it reads a symbol $C_S$. When the tape head reads $C_S$, GQTM increases a program counter by one, while moves to the right until it reads 1. Then GQTM stops increasing the counter and the tape head moves to the top of the tape. According to the program counter, the tape head moves to the right as reducing the counter by one. When the counter becomes zero, GQTM reads the data and calculates OR with the data in the track 2, then GQTM writes the result in the track 3. GQTM goes back to the top of the track 1 and repeats the above processes until it reads $Y$.

When GQTM reads $Y$, it calculates OR with the negation and repeats the processes as above. When it reads $C_E$, it writes down $f_{C_k}$ in the track 3 and clean the workspace for the next calculation. Then GQTM reads the blank symbol $#$, and it begins to calculate AND. The calculation of AND is done on the track 4. GQTM calculates them as moving to the left because the position of the tape head is at the end of the track 3 when the OR calculation is finished. Then the result of the calculation is showed on the top of the track 4.

The transition function of OR calculation is described, similar as classical TM, by the following three tables:
| State  | Transition | Action  |
|--------|------------|---------|
| $q_0$  | $q_a, 0, R$ | $C_S$   |
| $q_a$  | $q_a, 0, R$ | $q_a, 1, R$ |
| $q_b$  | $q_{b, 0}, 0, R$ | $q_{c, 1}, 0, L$ |
| $q_{b, 1}$ | $q_{b, 0}, 0, R$ | $q_{c, 1}, 0, L$ |
| $q_{b, k}$ | $q_{b, k+1}, 0, R$ | $q_{c, k+1}, 0, L$ |
| $q_{b, n}$ | $q_{b, n}, 0, R$ | $q_{c, 0}, 0, L$ |
| $q_{b, n+1}$ | $q_{b, n}, 0, R$ | $q_{c, n}, 0, L$ |
| $q_{c, 1}$ | $q_{c, 0}, 0, L$ | $q_{c, 1}, 0, L$ |
| $q_{c, n}$ | $q_{c, n}, 0, L$ | $q_{c, n}, X, L$ |
| $q_{d, 1}$ | $q_{d, 0}, 0, R$ | $q_{d, 1}, 1, R$ |
| $q_{d, n}$ | $q_{d, n-1}, 0, R$ | $q_{d, n-1}, 0, R$ |
| $q_{d, n+1}$ | $q_{d, n}, 0, R$ | $q_{d, n}, 0, L$ |
| $q_{e}$ | $q_{e}, 0, R$ | $q_{e}, 1, R$ |
| $q_{g}$ | $q_{g}, 0, R$ | $q_{g}, 1, R$ |
| $q_{g, k}$ | $q_{g, k+1}, 0, R$ | $q_{g, k+1}, 0, L$ |
| $q_{g, n}$ | $q_{g, n}, 0, R$ | $q_{g, n}, 0, L$ |
| $q_{h, 1}$ | $q_{h, 0}, 0, L$ | $q_{h, 1}, 0, L$ |
| $q_{h, n}$ | $q_{h, 0}, 0, L$ | $q_{h, n}, 0, L$ |
| $q_{i, 1}$ | $q_{i, 0}, 0, R$ | $q_{i, 1}, 0, R$ |
| $q_{i, n}$ | $q_{i, n-1}, 0, R$ | $q_{i, n-1}, 0, R$ |
| $q_j$  | $q_j, 0, L$ | $q_j, 0, L$ |
| $q_{j, 0}$ | $q_{j, 0}, 0, R$ | $q_{j, 0}, 0, L$ |

...
The transition function of AND calculation is described by the following table:

|   | 0   | 1   | #   |
|---|-----|-----|-----|
| q_{t3,0} | q_{t3,0,0}, R | q_{t3,1,1}, R | q_{a,0}, N |
| q_{t3,1} | q_{t3,1,0}, R | q_{t3,1,1}, R | q_{a,1}, N |
| q_{t3,a} | q_{t4,0,0}, N | q_{t4,1,1}, N | q_{a,0}, N |
| q_{t3,b} | q_{t3,b,1}, L | q_{t3,b,1}, L | q_{3,3,c, #}, R |
| q_{t3,c} | q_{t3,c, #}, R | q_{a,0}, N |

Let \( q_6 \) be the processor state of GQTM after the step 6 and \( T_i, i = 1, \ldots, 4 \) be the strings of the \( i \)-th track. Then the OM algorithm showed that the computation of the SAT problem of the example given above gives us the resulting state \( \rho_6 \) expressed as

\[
\rho_6 = \frac{g^2}{2} |q_6\rangle \langle q_6| \otimes |T_1, T_2, T_3, T_4 (1)\rangle \langle T_1, T_2, T_3, T_4 (1)| \otimes |O\rangle \langle O|
+ \frac{1 - g^2}{2} |q_6\rangle \langle q_6| \otimes |T_1, T_2, T_3, T_4 (0)\rangle \langle T_1, T_2, T_3, T_4 (0)| \otimes |O\rangle \langle O|
\]

where \( T_4 (1) \) (resp. \( T_4 (0) \)) indicates that the value in the track 4 is 1 (resp. 0).

Next step, as the three tracks (1,2,3) can be empty, we can apply the chaos amplifier to the above \( \rho_6 \) in the following manner:

The transition function of the step 7 denoted by the chaos amplifier is formally written as

\[
\Lambda_{CA}^k (\rho_6) = g^k (q^2) |q_7\rangle \langle q_7| \otimes |T_4 (1)\rangle \langle T_4 (1)| \otimes |O\rangle \langle O|
+ (1 - g^k (q^2)) |q_7\rangle \langle q_7| \otimes |T_4 (0)\rangle \langle T_4 (0)| \otimes |O\rangle \langle O|
\]

where \( g \) is the logistic map explained in Section 4.2. According to GQTM halts in at most \( \left[ \frac{n}{4} (n - 1) \right] \) steps with the probability \( p \geq \frac{1}{2} \), by which we can claim that \( C \) is SAT.

### 5.1 Computational complexity of the SAT algorithm

We define the computational complexity of the OV SAT algorithm as the product of \( T_Q \left( U_{C}^{(n)} \right) \) and \( T_{CA} (n) \), where \( T_Q \left( U_{C}^{(n)} \right) \) is the complexity of unitary computation and \( T_{CA} (n) \) is that of chaos amplification.

The following theorem is essentially discussed in [12 3 4].
Theorem 20 For a set of clauses $C$ and $n$ Boolean variables, the computational complexity of the OV SAT algorithm including the chaos amplifier, denoted by $T(C, n)$, is obtained as follows.

$$T_{GQT,M}(C, n) = T_Q \left( U_C^{(n)} \right) T_{CA}(n) = O(\text{poly}(n)),$$

where $\text{poly}(n)$ denotes a polynomial of $n$.

The computational complexity of quantum computer is determined by the total number of logical quantum gates. This inequality implies that the computational complexity of SAT algorithm is bounded by $O(n)$ for the size of input $n$ while a classical algorithm is bounded by $O(2^n)$.

5.2 Acknowledgment

One (MO) of the authors thanks IIAS and SCAT for finatial supports.

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