LOCAL TIME IN PARISIAN WALKWAYS

JIRÔ AKAHORI

Abstract. In the present paper, Itô formula and Tanaka formula for a special kind of symmetric random walk in the complex plane are studied. The random walk is called Parisian walk, and its local time is defined to be the number of exit from some regions.

1. Parisian Walks

Let $\tau_1, ..., \tau_n, ...$ be an i.i.d. sequence with $P(\tau = 1) = P(\tau = \zeta) = P(\tau = \zeta^2) = 1/3$, where we denote $\zeta = (-1 + \sqrt{3})/2$. The filtration generated by $\{\tau\}$ will be denoted by $F \equiv \{F_t\}$.

Definition 1. An $F$-adapted complex valued process $\{Z_t\}$ is called Parisian (walk) if (i) it is a martingale starting from a point in $\mathbb{Z}[\zeta] \equiv \{a + b\zeta : a, b \in \mathbb{Z}\}$, and (ii) $Z_{t+1} - Z_t \in \{1, \zeta, \zeta^2\}$ for all $t$.

Thus, a Parisian walk is a random walk on $\mathbb{Z}[\zeta]$. Note that there are a lot of Parisian walks as functions of $\{\tau\}$, but the law is unique up to the initial point.

We classify the points in $\mathbb{Z}[\zeta]$ as

$P = \{0, 1, 1 + \zeta\}$,

$A_1 = \{k_1 + k_2\zeta^2 | 1 < k_2 + 1 < k_1\}$,

$A_2 = \{-k_1\zeta^2 - k_2 | 0 \leq k_2 < k_1\}$,

$A_3 = \{-k_1 - k_2\zeta | 0 < k_2 < k_1\}$,

$A_4 = \{k_1\zeta + k_2\zeta^2 | 0 \leq k_2 < k_1\}$,

$A_5 = \{-k_1\zeta^2 - k_2 | 1 < k_2 + 1 < k_1\}$,

$A_6 = \{k_1 + k_2\zeta | 0 < k_2 < k_1\}$,

Figure 1. Parisian walkways $\mathbb{Z}[\zeta]$. 
Further we define closure of $A_1$, $A_3$ and $A_5$, respectively, as

\[
\bar{A}_1 := A_1 \cup B_1 \cup B_5 \cup \{1\},
\bar{A}_3 := A_3 \cup B_3 \cup B_4 \cup \{0\},
\bar{A}_5 := A_5 \cup B_2 \cup B_6 \cup \{1 + \zeta\}.
\]

In this paper, local time of a Parisian walk is defined to be the number of exit from $\bar{A}_1$, $\bar{A}_3$ and $\bar{A}_5$. More precisely, we set

\[
\ell_j^t := \# \{ u \in \mathbb{Z} \geq 0, u < t \mid Z_u \in \bar{A}_j, Z_{u+1} \notin \bar{A}_j \},
\]

for $j = 1, 3, 5$ and local time $\{L_t\}$ is defined by $L_t = \ell_1^t + \ell_3^t + \ell_5^t$.

In addition, an $\mathbb{F}$-martingale $\{X_t\}$ with $X_0 \in \mathbb{Z}$ and

\[
X_t - X_{t-1} = \begin{cases} 
1 & \text{with probability } 1/3 \\
0 & \text{with probability } 1/3 \\
-1 & \text{with probability } 1/3
\end{cases}
\]

will be called simple (walk). Of course, the law of simple walks is unique up to the initial points.

In this paper, we present the following

**Theorem 1.** For a Parisian walk $Z$, $\|Z\| - L_t$ is a simple walk. Here, $\| \cdot \|$ is the graph distance from $P$ in $\mathbb{Z}[\zeta]$, namely it is the length of the shortest path from $P$.

**Remark.** Our results heavily rely on the arguments found in Fujita’s paper [F] on 1-dimensional random walks.

2. An Itô formula for Parisian walks

We begin with the following lemma.

**Lemma 2.** Let $Z$ be a Parisian walk. Then the two dimensional process $(Z, \bar{Z})$ enjoys martingale representation property; every complex valued $\mathbb{F}$-martingale is represented as a stochastic integral with respect to $(Z, \bar{Z})$.

**Proof.** Denote $\Delta Z_t := Z_t - Z_{t-1}$ for $t \in \mathbb{Z}_{>0}$. Fix $t$ and set

\[
\Delta Z_S := \prod_{s_i = \zeta} \Delta Z_{i} \prod_{s_i = \zeta^2} \Delta Z_{i}
\]

for $S = (s_1, ..., s_t) \in \{1, \zeta, \zeta^2\}^t$. Then we have $\mathbb{E}[\Delta Z_S \overline{\Delta Z_{S'}}] = 1$ if $S = S'$ and $= 0$ otherwise because of the martingale property and of the fact that $(\Delta Z_S)^2 = \overline{\Delta Z_S}$. Therefore $\{\Delta Z_S \mid S \in \{1, \zeta, \zeta^2\}^t\}$ forms an orthonormal basis (ONB) of $L^2(\mathcal{F}_t)$ since $\|\{1, \zeta, \zeta^2\}^t\| = \dim L^2(\mathcal{F}_t) = 3'$. 


For an adapted \{X_t\}, expanding \(X_t - X_{t-1}\) with respect to this ONB and denoting \(E[(X_t - X_{t-1})\Delta Z_S]\) = \(x_S\), we have

(1) \(X_t - X_{t-1}\)

\[
\sum_{s_i = \zeta} x_S \Delta Z_S + \sum_{s_i = \zeta^2} x_S \Delta Z_S + \sum_{s_i = 1} x_S \Delta Z_S = \left(\sum_{s_i = \zeta} x_S \Delta Z_{(s_1,\ldots,s_{t-1})}\right) \Delta Z_t + \sum_{s_i = 1} x_S \Delta Z_{(s_1,\ldots,s_{t-1})}.
\]

By summing up the above equation, we obtain the Doob decomposition of \(X\), and this completes the proof. □

**Remark.** The above lemma can be easily extended to general unit root cases. The point here is that Parisian walk is the right discrete analogue of planer Brownian motion.

Let \(\{Z_t\}\) be a Parisian walk, and let \(f\) be a complex valued function on \(\mathbb{Z}[\zeta]\). Then we have the following formula, which would correspond to an Itô’s formula in \(F\).

**Proposition 3.** For \(t = 0, 1, 2, \ldots\), we have

\[
f(Z_{t+1}) - f(Z_t) = \frac{1}{3}(Z_{t+1} - Z_t)[f(Z_t + 1) + \zeta f(Z_t + \zeta) + \zeta^2 f(Z_t + \zeta^2)]
\]

(2)

\[
+ \frac{1}{3}(\Delta Z_{t+1} - \Delta Z_t)[f(Z_t + 1) + \zeta f(Z_t + \zeta) + \zeta^2 f(Z_t + \zeta^2)]
\]

\[
+ \frac{1}{3}[f(Z_t + 1) + f(Z_t + \zeta) + f(Z_t + \zeta^2) - 3f(Z_t)].
\]

**Proof.** As in the expression (1),

\[
f(Z_{t+1}) - f(Z_t) = \alpha \Delta Z_{t+1} + \beta \Delta \tilde{Z}_{t+1} + \gamma
\]

for some \(F\)-measurable \(\alpha, \beta\) and \(\gamma\). On the set of \(\Delta Z_{t+1} = 1\), \(\Delta Z_{t+1} = \zeta\), and \(\Delta Z_{t+1} = \zeta^2\) respectively, we have

\[
f(Z_t + 1) - f(Z_t) = \alpha + \beta + \gamma,
\]

(3)

\[
f(Z_t + \zeta) - f(Z_t) = \alpha \zeta + \beta \zeta^2 + \gamma,
\]

and \(f(Z_t + \zeta^2) - f(Z_t) = \alpha \zeta^2 + \beta \zeta^2 + \gamma\).

Solving (3) in terms of \((\alpha, \beta, \gamma)\), we obtain (2). □

3. A Tanaka formula for Parisian walks

For a Parisian walk \(Z\), set

\[
\varphi_t := 1_{\{Z_t \in A_1\}} + \zeta 1_{\{Z_t \in \bar{A}_1\}} + \zeta^2 1_{\{Z_t \in \bar{A}_2\}},
\]

\[
\psi_t := 1_{\{Z_t \in A_2\}} + \zeta 1_{\{Z_t \in A_3\}} + \zeta^2 1_{\{Z_t \in \bar{A}_2\}}.
\]

Then, we have the following
**Proposition 4** (A Tanaka formula for Parisian walks). For \( t = 0, 1, 2, \ldots \), we have

\[
(4) \quad \|Z_{t+1}\| - \|Z_t\| = \frac{2}{3} \text{Re} \left((1 - \zeta^2)(\varphi_t \Delta Z_t + \psi_t \Delta \bar{Z}_t)\right) + L_{t+1} - L_t.
\]

**Proof.** Applying Itô formula (2) to \( f(z) = \|z\| \), we have

\[
\|Z_{t+1}\| - \|Z_t\| = \frac{1}{3}(\|Z_t + 1\| + \zeta^2 \|Z_t + \zeta\| + \zeta \|Z_t + \zeta^2\|)\Delta Z_t
\]

\[+ \frac{1}{3}(\|Z_t + 1\| + \zeta \|Z_t + \zeta\| + \zeta^2 \|Z_t + \zeta^2\|)\Delta \bar{Z}_t
\]

\[+ \frac{1}{3}(\|Z_t + 1\| + \|Z_t + \zeta\| + \|Z_t + \zeta^2\| - 3\|Z_t\|).
\]

Set

\[
g_1(z) = \|z + 1\| + \zeta^2 \|z + \zeta\| + \zeta \|z + \zeta^2\|
\]

\[
g_2(z) = \|z + 1\| + \zeta \|z + \zeta\| + \zeta^2 \|z + \zeta^2\|
\]

\[
g_3(z) = \|z + 1\| + \|z + \zeta\| + \|z + \zeta^2\| - 3\|z\|.
\]

Then, we have the following tables.

**Table 1.**

|   | \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) | \(A_5\) | \(A_6\) | \(B_1\) | \(B_2\) | \(B_3\) | \(B_4\) | \(B_5\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \(g_1\) | 1 - \zeta^2 | \zeta - \zeta^2 | \zeta - 1 | \zeta^2 - \zeta | 1 - \zeta | 1 | \zeta^2 | \zeta | \zeta - 2 | -2\zeta | -2\zeta^2 |
| \(g_2\) | 1 - \zeta | \zeta^2 - \zeta | \zeta - 1 | \zeta^2 - \zeta | 1 - \zeta^2 | 1 | \zeta | \zeta | \zeta - 2 | -2\zeta | -2\zeta^2 |
| \(g_3\) | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 2.**

|   | 0 | 1 | 1 + \zeta |
|---|---|---|---|
| \(g_1\) | -1 | -\zeta^2 | -\zeta |
| \(g_2\) | -1 | -\zeta | -\zeta^2 |
| \(g_3\) | 2 | 2 | 2 |

Using these tables, we obtain, for example,

\[
\|Z_{t+1}\| - \|Z_t\| = \frac{1}{3}(\Delta Z_t + \Delta \bar{Z}_t + 1) = \frac{1}{3}\left[(1 - \zeta^2)\Delta Z_t + (1 - \zeta)\Delta \bar{Z}_t\right] + L_{t+1} - L_t
\]

on the set \( \{Z_t \in B_1\} \), and similar expressions on \( B_2, ..., B_6 \) and \( \{0\}, \{1\}, \{1 + \zeta\} \) lead to the formula (4).

Now Theorem 1 follows easily from this version of Tanaka formula, by observing that \( \sum(\varphi \Delta Z + \psi \Delta \bar{Z}) \) is a Parisian walk, and real part of \( 2(1 - \zeta^2)Z'/3 \) is a simple walk whenever \( Z' \) is Parisian.

**Remark.** As we have seen in the proof of Lemma 2, \( Z \) and \( \bar{Z} \) are mutually orthogonal martingale so that, after taking appropriate scaling limit, \( Z \)
converges to a so-called planer Brownian motion. Then, we can prove the standard Itô’s formula from our Itô’s formula (2), noting that (2) can be rewritten as follows if \( f \) is real valued.

\[
\Delta f = \text{Re}(Df)\text{Re}(\Delta Z) + \text{Im}(Df)\text{Im}(\Delta Z) + Lf,
\]

where we have set \( Lf = \sum_{j=0,1,2} (f(z + \zeta^j) - f(z))/3 \) so that \( L \) corresponds to the Discrete Laplacian operator in the context of random walks on graphs, and \( Df = \sum_{j=0,1,2} \zeta^j f(z + \zeta^j)/3 \), which corresponds to a differential operator on \( \mathbb{Z}[\zeta] \).

Moreover, a continuous time version of our theorem is given in the following way.

Let \((B^1, B^2)\) be a 2-dimensional Brownian motion. Then there exists an increasing process \( \{L_t\} \) which increases only on (continuous time limit of) the lines \( B^1_t, \ldots, B_6 \) such that

\[
\max(|B^2_t|, |B^1_t - B^2_t/\sqrt{3}|, |B^1_t + B^2_t/\sqrt{3}|) - L_t \quad \text{is a standard Brownian motion.}
\]

This fact is verified by using the standard Tanaka formula again and again.

**Remark.** We can extend the discrete Itô formula to more general cases from a perspective of Fourier expansion. See [A].

**References**

[A] Akahori, J, “Discrete Ito Formulas and Their Applications to Stochastic Numerics”, RIMS kokyuroku 1462 202-210 (2006). arXiv math.PR/0603341

[F] Fujita, T. “A random walk analogue of Lévy’s theorem.” preprint, 2003.