$p$-Laplace equations with singular weights

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Abstract

We study a class of $p$-Laplacian Dirichlet problems with weights that are possibly singular on the boundary of the domain, and obtain nontrivial solutions using Morse theory. In the absence of a direct sum decomposition, we use a cohomological local splitting to get an estimate of the critical groups.

\textsuperscript{*}MSC2010: Primary 35J75, Secondary 35J20, 35J25, 35J92

Key Words and Phrases: $p$-Laplacian, Dirichlet problem, singular weights, nontrivial solutions, Morse theory, critical groups, cohomological local splitting

\textsuperscript{†}This work was completed while the first-named author was visiting the Department of Mathematics at the University of Ulsan, and he is grateful for the kind hospitality of the department.

\textsuperscript{‡}This work was supported by the 2012-0117 Research Fund of the University of Ulsan.
1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial \Omega$. The purpose of this paper is to study the boundary value problem

$$
\begin{cases}
-\Delta_p u = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian of $u$, $p \in (1, \infty)$, and $f$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying a subcritical growth condition of the form

$$|f(x, t)| \leq \sum_{i=1}^{n} K_i(x) |t|^{q_i-1} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}$$

for some $q_i \in [1, p^*)$, $p^* = Np/(N-p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$, and measurable weights $K_i$ that are possibly singular on $\partial \Omega$.

Elliptic boundary value problems with singular weights have become an increasingly active area of research during recent years. Cuesta [3] studied the eigenvalue problem

$$
\begin{cases}
-\Delta_p u = \lambda K(x) |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where the weight $K \in L^s(\Omega)$ with $s > N/p$ if $p \leq N$ and $s = 1$ if $p > N$. In the ODE case $N = 1$, Kajikiya, Lee, and Sim [8] considered weights that are strongly singular on the boundary and may not be in $L^1$. Montenegro and Lorca [11] used the Hardy-Sobolev inequality to study the spectrum of (1.3) for a wide class of singular weights $A_p$ (see Definition 2.1). In the semilinear case $p = 2$, Kajikiya [7] showed that problem (1.1) with $f(x, u) = K(x) |u|^{q-1} u$, $q > 1$, and $K \in A_q$ has a positive solution and infinitely many solutions without positivity using variational methods. Kajikiya, Lee, and Sim [9] obtained positive and nodal solutions of (1.1) when $N = 1$ and $f$ is strongly singular on the boundary and $p$-superlinear at infinity using bifurcation arguments.

Here we study the critical groups of the variational functional associated with problem (1.1) and obtain nontrivial solutions using Morse theory. The admissible class of weights is defined in Section 2. After the preliminaries on a related eigenvalue problem in Section 3, we use a cohomological local splitting to get an estimate of the critical groups in the absence of a direct sum decomposition in Section 4. Our main existence result is for the $p$-superlinear case and is proved in Section 5 (see Theorem 5.6).

2 Variational setting

Let $|.|_p$ denote the norm in $L^p(\Omega)$, and let $\rho(x) = \inf_{y \in \partial \Omega} |x-y|$ be the distance from $x \in \Omega$ to $\partial \Omega$. We consider the following class of weights.

**Definition 2.1.** For $q \in [1, p^*)$, let $A_q$ denote the class of measurable functions $K$ such that $K \rho^a \in L^r(\Omega)$ for some $a \in [0, q-1]$ and $r \in (1, \infty)$ satisfying

$$
\frac{1}{r} + \frac{a}{p} + \frac{q-a}{p^*} < 1.
$$

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Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space, with the norm $\|u\| = |\nabla u|_p$, and let $C$ denote a generic positive constant.

**Lemma 2.2.** If $q \in [1, p^*)$ and $K \in A_q$, then there exists $b < p^*$ such that
\[
\int_{\Omega} |K(x)||u|^{q-1}|v| \, dx \leq C \|u\|^{q-1}|v|_b \quad \forall u, v \in W_0^{1,p}(\Omega).
\]

**Proof.** Let $a$ and $r$ be as in Definition 2.1. By the Hölder inequality,
\[
\int_{\Omega} |K(x)||u|^{q-1}|v| \, dx = \int_{\Omega} |K\rho^a| \left| \frac{|u|}{\rho} \right|^a |u|^{q-1-a}|v| \, dx \leq |K\rho^a| |u|^{a+b} |v|_b,
\]
where $1/r + a/p + (q-a)/b = 1$ and hence $b < p^*$ by (2.1). Since $|u/\rho|_p \leq C \|u\|$ by the Hardy inequality (see Nečas [12]) and $|u|_b \leq C \|u\|$ by the Sobolev imbedding, the conclusion follows. \qed

We assume that $K_i \in A_{q_i}$ for $i = 1, \ldots, n$. Recall that a weak solution of problem (1.1) is a function $u \in W_0^{1,p}(\Omega)$ satisfying
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f(x, u) v \, dx \quad \forall v \in W_0^{1,p}(\Omega).
\]
By (1.2) and Lemma 2.2, the integral on the right is well-defined. Weak solutions coincide with critical points of the $C^1$-functional
\[
\Phi(u) = \int_{\Omega} \left[ \frac{1}{p} |\nabla u|^p - F(x, u) \right] \, dx, \quad u \in W_0^{1,p}(\Omega), \tag{2.2}
\]
where $F(x, t) = \int_0^t f(x, \tau) \, d\tau$. Lemma 2.2 also gives the following compactness result.

**Lemma 2.3.** Every bounded sequence $(u_j) \subset W_0^{1,p}(\Omega)$ such that $\Phi'(u_j) \to 0$ has a convergent subsequence.

**Proof.** By Lemma 2.2,
\[
\left| \int_{\Omega} f(x, u_j) (u_j - u) \, dx \right| \leq \sum_{i=1}^n \int_{\Omega} K_i(x) |u_j|^{q_i-1} |u_j - u| \, dx \leq C \sum_{i=1}^n \|u_j\|^{q_i-1} |u_j - u|_b_i, \tag{2.3}
\]
where each $b_i < p^*$. Since $(u_j)$ is bounded in $W_0^{1,p}(\Omega)$, a renamed subsequence converges to some $u$ weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^{b_i}(\Omega)$ for $i = 1, \ldots, n$. Then
\[
\int_{\Omega} |\nabla u|^2 \nabla u \cdot \nabla (u_j - u) \, dx = \Phi'(u_j) (u_j - u) + \int_{\Omega} f(x, u_j) (u_j - u) \, dx \to 0
\]
by (2.3), and hence $u_j \to u$ in $W_0^{1,p}(\Omega)$ by the $(S)_+$ property of the $p$-Laplacian. \qed

## 3 An eigenvalue problem

In this section we consider the eigenvalue problem
\[
\begin{cases}
-\Delta_p u = \lambda h(x) |u|^{q-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \tag{3.1}
\]
where \( q \in [1,p^*) \) and \( h \in A_q \) is possibly sign-changing. We assume that \( h \) is positive on a set of positive measure and seek positive eigenvalues.

Let
\[
I(u) = \int_{\Omega} \frac{1}{p} |\nabla u|^p \, dx, \quad J(u) = \int_{\Omega} \frac{1}{q} h(x) |u|^q \, dx, \quad u \in W^{1,p}_0(\Omega),
\]
and set
\[
\Psi(u) = \frac{1}{J(u)}, \quad u \in \mathcal{M} = \{ u \in W^{1,p}_0(\Omega) : I(u) = 1 \text{ and } J(u) > 0 \}.
\]
Then \( \mathcal{M} \) is nonempty, and positive eigenvalues and associated eigenfunctions of problem \([3.1]\) on \( \mathcal{M} \) coincide with critical values and critical points of \( \Psi \), respectively. By Lemma \([2.2]\)
\[
0 < J(u) \leq C \|u\|^q \leq C \quad \forall u \in \mathcal{M}
\]
and hence \( \lambda_1 := \inf_{u \in \mathcal{M}} \Psi(u) > 0 \).

**Lemma 3.1.** For all \( c \in \mathbb{R} \), \( \Psi \) satisfies the (PS)_c condition, i.e., every sequence \( (u_j) \subset \mathcal{M} \) such that \( \Psi(u_j) \to c \) and \( \Psi'(u_j) \to 0 \) has a subsequence that converges to some \( u \in \mathcal{M} \).

**Proof.** We have \( c \geq \lambda_1 \), and there is a sequence \( (\mu_j) \subset \mathbb{R} \) such that
\[
\mu_j I'(u_j) - \frac{J'(u_j)}{J(u_j)^2} \to 0. \tag{3.2}
\]
Since \( I'(u_j) u_j = p I(u_j) = p \), \( J'(u_j) u_j = q J(u_j) \), and \( J(u_j) \to 1/c \), then \( \mu_j \to qc/p > 0 \). By Lemma \([2.2]\)
\[
|J'(u_j)(u_j - u)| \leq \int_{\Omega} |h(x)||u_j|^q |u_j - u| \, dx \leq C \|u_j\|^q |u_j - u|_b, \tag{3.3}
\]
where \( b < p^* \). Since \( (u_j) \) is bounded in \( W^{1,p}_0(\Omega) \), a renamed subsequence converges to some \( u \) weakly in \( W^{1,p}_0(\Omega) \) and strongly in \( L^b(\Omega) \). Then
\[
\int_{\Omega} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla (u_j - u) \, dx = I'(u_j)(u_j - u) \to 0
\]
by \([3.2]\) and \([3.3]\), and hence \( u_j \to u \) in \( W^{1,p}_0(\Omega) \) by the \((S)_+ \) property of the \( p \)-Laplacian. By continuity, \( J(u) = 1/c > 0 \) and hence \( u \in \mathcal{M} \). \( \square \)

We can now define an increasing and unbounded sequence of critical values of \( \Psi \) via a minimax scheme. Although the standard scheme is based on the Krasnosel’skii’s genus, here we use a cohomological index as in Perera \([14]\). This gives additional topological information about the associated critical points that is often useful in applications.

Let us recall the definition of the \( \mathbb{Z}_2 \)-cohomological index of Fadell and Rabinowitz \([5]\). Let \( W \) be a Banach space. For a symmetric subset \( M \) of \( W \setminus \{0\} \), let \( \overline{M} = M/\mathbb{Z}_2 \) be the quotient space of \( M \) with each \( u \) and \( -u \) identified, let \( f : \overline{M} \to \mathbb{R}P^{\infty} \) be the classifying map of \( \overline{M} \), and let \( f^* : H^*((\mathbb{R}P^{\infty}) \to H^*(\overline{M}) \) be the induced homomorphism of the Alexander-Spanier cohomology rings. Then the cohomological index of \( M \) is defined by
\[
i(M) = \begin{cases} \sup \{ m \geq 1 : f^*(\omega^{m-1}) \neq 0 \}, & M \neq \emptyset \\ 0, & M = \emptyset, \end{cases}
\]
where \( \omega \in H^1(\mathbb{R}P^\infty) \) is the generator of the polynomial ring \( H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega] \). For example, the classifying map of the unit sphere \( S^{m-1} \) in \( \mathbb{R}^m, m \geq 1 \), is the inclusion \( \mathbb{R}P^{m-1} \subset \mathbb{R}P^\infty \), which induces isomorphisms on \( H^q \) for \( q \leq m - 1 \), so \( i(S^{m-1}) = m \).

Let \( F \) denote the class of symmetric subsets of \( M \), and set

\[
\lambda_k := \inf_{M \in F} \sup_{i(M) \geq k} \Psi(u), \quad k \geq 1.
\]

Then \( (\lambda_k) \) is a sequence of positive eigenvalues of \( (3.1) \), \( \lambda_k \nearrow +\infty \), and

\[
i(\{ u \in M : \Psi(u) \leq \lambda_k \}) = i(\{ u \in M : \Psi(u) < \lambda_{k+1} \}) = k
\]

if \( \lambda_k < \lambda_{k+1} \) (see Perera, Agarwal, and O’Regan [15, Propositions 3.52 and 3.53]).

As an application consider the pure power problem

\[
\begin{cases}
- \Delta_p u = h(x) |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(3.5)

\[\text{Theorem 3.2. If } q \in [1, p^*), q \neq p, \text{ and } h \in A_{q, p} \text{ is positive on a set of positive measure, then problem (3.5) has a sequence of nontrivial weak solutions } (u_k) \text{ such that}
\]

\( \lambda_k \)

\[\begin{align*}
(1) & \text{ if } q < p, \text{ then } \|u_k\| \to 0; \\
(2) & \text{ if } q > p, \text{ then } \|u_k\| \to \infty.
\end{align*}
\]

Proof. Let \( v_k \) be a critical point of \( \Psi \) with \( \Psi(v_k) = \lambda_k \). Then \( u_k := \lambda_k^{1/(q-p)} v_k \) solves (3.5), and \( \|u_k\| = \lambda_k^{1/(q-p)} p^{1/p} \text{ since } I(v_k) = 1 \). \( \square \)

In the next section we will use the index information in (3.4) to compute certain critical groups when \( q = p \).

4 Critical groups

In this section we consider the problem

\[
\begin{cases}
- \Delta_p u = \lambda h(x) |u|^{p-2} u + g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(4.1)

where \( \lambda \geq 0 \) is a parameter, \( h \in A_p \) is positive on a set of positive measure, and \( g \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) satisfying the growth condition

\[
|g(x, t)| \leq \sum_{i=1}^n K_i(x) |t|^{q_i-1} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}
\]

(4.2)

for some \( q_i \in (p, p^*) \) and \( K_i \in A_{q_i} \). Problem (4.1) has the trivial solution \( u = 0 \), and we study the critical groups of the associated functional

\[
\Phi(u) = \int_{\Omega} \left[ \frac{1}{p} |\nabla u|^p - \frac{1}{p} \lambda h(x) |u|^p - G(x, u) \right] dx, \quad u \in W_0^{1,p}(\Omega),
\]

(3.4)
where \( G(x, t) = \int_0^t g(x, \tau) d\tau \), at 0.

Let us recall that the critical groups of \( \Phi \) at 0 are given by
\[
C^q(\Phi, 0) = H^q(\Phi^0 \cap U, \Phi^0 \cap U \setminus \{0\}), \quad q \geq 0,
\]
where \( \Phi^0 = \{ u \in W^{1,p}_0(\Omega) : \Phi(u) \leq 0 \} \), \( U \) is any neighborhood of 0, and \( H \) denotes Alexander-Spanier cohomology with \( \mathbb{Z}_2 \)-coefficients. They are independent of \( U \) by the excision property of the cohomology groups. They are also invariant under homotopies that preserve the isolatedness of the critical point by the following proposition (see Chang and Ghoussoub \cite{1} or Corvellec and Hantoute \cite{2}).

**Proposition 4.1.** Let \( \Phi_s, s \in [0,1] \) be a family of \( C^1 \)-functionals on a Banach space \( W \) such that 0 is a critical point of each \( \Phi_s \). If there is a closed neighborhood \( U \) of 0 such that

1. each \( \Phi_s \) satisfies the (PS) condition over \( U \),
2. \( U \) contains no other critical point of any \( \Phi_s \),
3. the map \([0,1] \to C^1(U, \mathbb{R}), s \mapsto \Phi_s\) is continuous,

then \( C_q(\Phi_0, 0) \approx C_q(\Phi_1, 0) \) for all \( q \).

In the absence of a direct sum decomposition, the main technical tool we use to get an estimate of the critical groups is the notion of a cohomological local splitting introduced in Perera, Agarwal, and O’Regan \cite{15}, which is a variant of the homological local linking of Perera \cite{13} (see also Li and Willem \cite{10}). The following slightly different form of this notion was given in Degiovanni, Lancelotti, and Perera \cite{4}.

**Definition 4.2.** We say that a \( C^1 \)-functional \( \Phi \) on a Banach space \( W \) has a cohomological local splitting near 0 in dimension \( k \geq 1 \) if there are symmetric cones \( W_{\pm} \subset W \) with \( W_+ \cap W_- = \{0\} \) and \( \rho > 0 \) such that
\[
i(W_+ \setminus \{0\}) = i(W \setminus W_+) = k
\]
and
\[
\Phi(u) \leq \Phi(0) \quad \forall u \in B_{\rho} \cap W_-, \quad \Phi(u) \geq \Phi(0) \quad \forall u \in B_{\rho} \cap W_+,
\]
where \( B_{\rho} = \{ u \in W : \|u\| \leq \rho \} \).

**Proposition 4.3** (Degiovanni, Lancelotti, and Perera \cite{4} Proposition 2.1). If \( \Phi \) has a cohomological local splitting near 0 in dimension \( k \), and 0 is an isolated critical point of \( \Phi \), then \( C^k(\Phi, 0) \neq 0 \).

Let \( \lambda_k \nearrow +\infty \) be the sequence of positive eigenvalues of the problem
\[
\begin{cases}
-\Delta_p u = \lambda h(x) |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]
that was constructed in the last section. The main result of this section is the following theorem.

**Theorem 4.4.** Assume that \( h \in \mathcal{A}_p \) is positive on a set of positive measure, \( g \) satisfies (4.2) for some \( q_i \in (p, p^*) \) and \( K_i \in \mathcal{A}_{q_i} \) for \( i = 1, \ldots, n \), and 0 is an isolated critical point of \( \Phi \).
(1) $C^0(\Phi, 0) \approx \mathbb{Z}_2$ and $C^q(\Phi, 0) = 0$ for $q \geq 1$ in the following cases:

(i) $0 \leq \lambda < \lambda_1$;
(ii) $\lambda = \lambda_1$ and $G(x, t) \leq 0$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$.

(2) $C^k(\Phi, 0) \neq 0$ in the following cases:

(i) $\lambda_k < \lambda < \lambda_{k+1}$;
(ii) $\lambda = \lambda_k < \lambda_{k+1}$ and $G(x, t) \geq 0$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;
(iii) $\lambda_k < \lambda_{k+1} = \lambda$ and $G(x, t) \leq 0$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$.

Proof. By (4.2) and Lemma 2.2

$$\left| \int_{\Omega} G(x, u) \, dx \right| \leq C \sum_{i=1}^{n} \|u\|^{q_i} = o(\|u\|^p) \quad \text{as} \quad \|u\| \to 0$$

since each $q_i > p$. So

$$\Phi(u) = I(u) - \lambda J(u) + o(\|u\|^p) \quad \text{as} \quad \|u\| \to 0. \quad (4.7)$$

(1) We show that 0 is a local minimizer of $\Phi$. Since $\Psi(u) \geq \lambda_1$ for all $u \in M$,

$$I(u) \geq \lambda_1 J(u) \quad \forall u \in W_0^{1,p}(\Omega). \quad (4.8)$$

(i) For sufficiently small $\rho > 0$,

$$\Phi(u) \geq \left( 1 - \frac{\lambda}{\lambda_1} + o(1) \right) \frac{\|u\|^p}{p} \geq 0 \quad \forall u \in B_\rho$$

by (4.7) and (4.8).

(ii) We have

$$\Phi(u) \geq -\int_{\Omega} G(x, u) \, dx \geq 0 \quad \forall u \in W_0^{1,p}(\Omega).$$

(2) We show that $\Phi$ has a cohomological local splitting near 0 in dimension $k$ and apply Proposition 4.3. Let

$$W_- = \{ u \in W_0^{1,p}(\Omega) : I(u) \leq \lambda_k J(u) \}, \quad W_+ = \{ u \in W_0^{1,p}(\Omega) : I(u) \geq \lambda_{k+1} J(u) \}.$$

Then $W_- \setminus \{0\}$ and $W \setminus W_+$ radially deformation retract to $\{ u \in M : \Psi(u) \leq \lambda_k \}$ and $\{ u \in M : \Psi(u) < \lambda_{k+1} \}$, respectively, so (4.4) holds by (3.4). It only remains to show that (4.5) holds for sufficiently small $\rho > 0$.

(i) For sufficiently small $\rho > 0$,

$$\Phi(u) \leq -\left( \frac{\lambda}{\lambda_k} - 1 + o(1) \right) \frac{\|u\|^p}{p} \leq 0 \quad \forall u \in B_\rho \cap W_-$$

and

$$\Phi(u) \geq \left( 1 - \frac{\lambda}{\lambda_{k+1}} + o(1) \right) \frac{\|u\|^p}{p} \geq 0 \quad \forall u \in B_\rho \cap W_+$$
by \((4.7)\).

(ii) We have

\[
\Phi(u) \leq -\int_{\Omega} G(x,u) \, dx \leq 0 \quad \forall u \in W_-, 
\]

and for sufficiently small \(\rho > 0\), \(\Phi(u) \geq 0\) for all \(u \in B_{\rho} \cap W_+\) as in (i).

(iii) For sufficiently small \(\rho > 0\), \(\Phi(u) \leq 0\) for all \(u \in B_{\rho} \cap W_-\) as in (i), and

\[
\Phi(u) \geq -\int_{\Omega} G(x,u) \, dx \geq 0 \quad \forall u \in W_+. \quad \Box
\]

When \(p > N\), it suffices to assume the sign conditions on \(G\) in Theorem 4.4 for small \(|t|\) by the imbedding \(W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)\), so we also have the following theorem.

**Theorem 4.5.** Assume that \(p > N\), \(h \in \mathcal{A}_p\) is positive on a set of positive measure, \(g\) satisfies (4.2) for some \(q_i \in (p, \infty)\) and \(K_i \in \mathcal{A}_{q_i}\) for \(i = 1, \ldots, n\), and 0 is an isolated critical point of \(\Phi\).

1. \(C^0(\Phi, 0) \approx \mathbb{Z}_2\) and \(C^q(\Phi, 0) = 0\) for \(q \geq 1\) if \(\lambda = \lambda_1\) and, for some \(\delta > 0\), \(G(x,t) \leq 0\) for a.a. \(x \in \Omega\) and \(|t| \leq \delta\).

2. \(C^k(\Phi, 0) \neq 0\) in the following cases:

   (i) \(\lambda = \lambda_k < \lambda_{k+1}\) and, for some \(\delta > 0\), \(G(x,t) \geq 0\) for a.a. \(x \in \Omega\) and \(|t| \leq \delta\);

   (ii) \(\lambda_k < \lambda_{k+1} = \lambda\) and, for some \(\delta > 0\), \(G(x,t) \leq 0\) for a.a. \(x \in \Omega\) and \(|t| \leq \delta\).

We close this section by showing that the conclusions of Theorem 4.5 also hold for \(p \leq N\) when the weights \(h\) and \(K_i\) belong to suitable subclasses of \(\mathcal{A}_p\) and \(\mathcal{A}_{q_i}\), respectively.

**Definition 4.6.** For \(p \leq N\) and \(q \in [1,p^*)\), let \(\tilde{\mathcal{A}}_q\) denote the class of measurable functions \(K\) such that \(K \rho^a \in L^r(\Omega)\) for some \(a \in [0, q-1]\) and \(r \in (1, \infty)\) satisfying

\[
\frac{1}{r} + \frac{a}{p} + \frac{q - 1 - a}{p^*} < \frac{p}{N}. \tag{4.9}
\]

Note that \(\tilde{\mathcal{A}}_q = \mathcal{A}_q\) when \(p = N\). When \(p < N\),

\[
\frac{1}{p^*} + \frac{p}{N} = 1 - \frac{(N-p)(p-1)}{Np} < 1
\]

and hence \(\tilde{\mathcal{A}}_q \subset \mathcal{A}_q\).

**Lemma 4.7.** If \(p \leq N\), \(q \in [1,p^*)\), and \(K \in \tilde{\mathcal{A}}_q\), then there exists \(s > N/p\) such that \(K(x) |u|^{q-1} \in L^s(\Omega)\) and

\[
|K(x) |u|^{q-1}|_s \leq C \|u\|^{q-1}
\]

for all \(u \in W_0^{1,p}(\Omega)\).

**Proof.** Let \(a\) and \(r\) be as in Definition 4.6 By (4.9), there exists \(b < p^*\) such that

\[
\frac{1}{r} + \frac{a}{p} + \frac{q - 1 - a}{b} < \frac{p}{N}. \tag{4.10}
\]
By the Hölder inequality,

\[ \int_\Omega |K(x)|^s |u|^{(q-1)s} \, dx = \int_\Omega |K \rho|^s \left| \frac{u}{\rho} \right|^{as} |u|^{(q-1-a)s} \, dx \leq |K \rho|^s \left| \frac{u}{\rho} \right|^{as} |u|^b_{b(q-1-a)s}, \]

where \( s/r + as/p + (q - 1 - a) s/b = 1 \) and hence \( s > N/p \) by (4.10). Since \( |u/\rho|_p \leq C \|u\| \) by the Hardy inequality (see Nečas [12]) and \( |u|^b \leq C \|u\| \) by the Sobolev imbedding, the conclusion follows.

Assume that \( p \leq N, \ h \in \tilde{A}_p, \) and \( K_i \in \tilde{A}_q \) for \( i = 1, \ldots, n. \) First we show that the critical groups of \( \Phi \) at \( 0 \) depend only on the values of \( g(x, t) \) for small \( |t|. \)

**Lemma 4.8.** Let \( \delta > 0 \) and let \( \vartheta : \mathbb{R} \to [-\delta, \delta] \) be a smooth nondecreasing function such that \( \vartheta(t) = -\delta \) for \( t \leq -\delta, \vartheta(t) = t \) for \( -\delta/2 \leq t \leq \delta/2, \) and \( \vartheta(t) = \delta \) for \( t \geq \delta. \) Set

\[ \Phi_1(u) = \int_\Omega \left[ \frac{1}{p} \| \nabla u \| - \frac{1}{p} \lambda h(x) \| u \|^p - G(x, (1-s)u + s \vartheta(u)) \right] \, dx, \quad u \in W^{1,p}_0(\Omega). \]

If \( 0 \) is an isolated critical point of \( \Phi \), then it is also an isolated critical point of \( \Phi_1 \) and \( C_q(\Phi, 0) \approx C_q(\Phi_1, 0) \) for all \( q. \)

**Proof.** We apply Proposition 1.1 to the family of functionals

\[ \Phi_s(u) = \int_\Omega \left[ \frac{1}{p} \| \nabla u \| - \frac{1}{p} \lambda h(x) \| u \|^p - G(x, (1-s)u + s \vartheta(u)) \right] \, dx, \quad u \in W^{1,p}_0(\Omega), \quad s \in [0, 1] \]

in a small ball \( B_\varepsilon(0) = \{ u \in W^{1,p}_0(\Omega) : \| u \| \leq \varepsilon \}, \) noting that \( \Phi_0 = \Phi. \) Lemma 2.3 implies that each \( \Phi_s \) satisfies the (PS) condition over \( B_\varepsilon(0), \) and it is easy to see that the map \( [0, 1] \to C^1(B_\varepsilon(0), \mathbb{R}), \) \( s \mapsto \Phi_s \) is continuous, so it only remains to show that for sufficiently small \( \varepsilon > 0, \) \( B_\varepsilon(0) \) contains no critical point of any \( \Phi_s \) other than \( 0. \)

Suppose \( u_j \to 0 \) in \( W^{1,p}_0(\Omega), \) \( \Phi'_{s_j}(u_j) = 0, \) \( s_j \in [0, 1], \) and \( u_j \neq 0. \) Then

\[ \begin{cases} -\Delta_p u_j = \lambda h(x) \| u_j \|^{p-2} u_j + g_j(x, u_j) & \text{in } \Omega \\ u_j = 0 & \text{on } \partial \Omega, \end{cases} \]

where

\[ g_j(x, t) = (1 - s_j + s_j \vartheta'(t)) g(x, (1-s_j) t + s_j \vartheta(t)). \]

Since \( (1-s_j) t + s_j \vartheta(t) = t \) for \( |t| \leq \delta/2 \) and \( |(1-s_j) t + s_j \vartheta(t)| \leq |t| + \delta < 3 |t| \) for \( |t| > \delta/2, \) (4.2) implies

\[ |g_j(x, t)| \leq C \sum_{i=1}^n K_i(x) |t|^{q_i-1} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}, \quad (4.11) \]

where \( C \) denotes a generic positive constant independent of \( j. \) By Lemma 4.7, there exists \( s > N/p \) such that \( h(x) |u_j|^{p-1} \in L^s(\Omega) \) and

\[ h(x) |u_j|^{p-1} \|_{s} \leq C \| u_j \|^{p-1}, \quad (4.12) \]

and there exists \( s_i > N/p \) such that \( K_i(x) |u_j|^{q_i-1} \in L^{s_i}(\Omega) \) and

\[ K_i(x) |u_j|^{q_i-1} \|_{s_i} \leq C \| u_j \|^{q_i-1} \quad (4.13) \]
for $i = 1, \ldots, n$. Let $s_0 = \min \{s, s_1, \ldots, s_n\} > N/p$. By \cite[4.13–4.14]{4}, $\lambda h(x) |u_j|^{p-2} u_j + g_j(x, u_j) \in L^{s_0}(\Omega)$ and

$$|\lambda h(x) |u_j|^{p-2} u_j + g_j(x, u_j)|_{s_0} \leq C \left( |h(x) |u_j|^{p-1}|_{s_0} + \sum_{i=1}^{n} |K_i(x) |u_j|^{q_i-1}|_{s_0} \right)$$

$$\leq C \left( |h(x) |u_j|^{p-1}|_{s} + \sum_{i=1}^{n} |K_i(x) |u_j|^{q_i-1}|_{s_i} \right) \leq C \left( \|u_j\|^{p-1} + \sum_{i=1}^{n} \|u_j\|^{q_i-1} \right) \to 0,$$

and hence $u_j \in L^\infty(\Omega)$ and $u_j \to 0$ in $L^\infty(\Omega)$ by Guedda and Véron \cite[Proposition 1.3]{6}. So for sufficiently large $j$, $|u_j| \leq \delta/2$ a.e. and hence $\Phi'(u_j) = \Phi'_{s_j}(u_j) = 0$, contradicting our assumption that 0 is an isolated critical point of $\Phi$.

The following theorem is now immediate from Lemma 4.8 and Theorem 4.4.

**Theorem 4.9.** Assume that $p \leq N$, $h \in \mathcal{A}_p$ is positive on a set of positive measure, $g$ satisfies (4.2) for some $q_i \in (p, p^*)$ and $K_i \in \mathcal{A}_{q_i}$ for $i = 1, \ldots, n$, and 0 is an isolated critical point of $\Phi$.

1. $C^0(\Phi, 0) \approx \mathbb{Z}_2$ and $C^q(\Phi, 0) = 0$ for $q \geq 1$ if $\lambda = \lambda_1$ and, for some $\delta > 0$, $G(x, t) \leq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$.

2. $C^k(\Phi, 0) \neq 0$ in the following cases:

   (i) $\lambda = \lambda_k < \lambda_{k+1}$ and, for some $\delta > 0$, $G(x, t) \geq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$;

   (ii) $\lambda_k < \lambda_{k+1} = \lambda$ and, for some $\delta > 0$, $G(x, t) \leq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$.

## 5 Nontrivial solutions

In this section we obtain a nontrivial solution of the problem

$$\begin{cases} -\Delta_p u = \lambda h(x) |u|^{p-2} u + K(x) |u|^{q-2} u + g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

(5.1)

where $q \in (p, p^*)$, $K \in \mathcal{A}_q$ satisfies

$$\text{ess inf}_{x \in \Omega} K(x) > 0,$$

(5.2)

and $g$ satisfies (4.2) with each $q_i \in (p, q)$. We will assume that the weights $h$ and $K_i$ belong to suitable subclasses of $\mathcal{A}_p$ and $\mathcal{A}_{q_i}$, respectively.

**Definition 5.1.** For $q \in (1, p^*)$ and $s \in [1, q)$, let $\mathcal{A}^q_s$ denote the class of measurable functions $K$ such that $K \in L^r(\Omega)$ for some $a \in [0, s - 1]$ and $r \in (1, \infty)$ satisfying

$$\frac{1}{r} + \frac{a}{p} + \frac{s - a}{q} \leq 1.$$

(5.3)

Clearly, $\mathcal{A}^q_s \subset \mathcal{A}_s$. 


Lemma 5.2. If \( q \in [1, p^*) \), \( s \in [1, q) \), and \( K \in \mathcal{A}_q^s \), then there exist \( t < p \) and, for every \( \varepsilon > 0 \), a constant \( C(\varepsilon) \) such that

\[
\int_{\Omega} |K(x)| |u|^q \, dx \leq C(\varepsilon) \|u\|^t + \varepsilon |u|^q \quad \forall u \in W_0^{1,p}(\Omega).
\]

Proof. Let \( a \) and \( r \) be as in Definition 5.1. By the Hölder inequality,

\[
\int_{\Omega} |K(x)| |u|^s \, dx = \int_{\Omega} |K \rho^a| \left| \frac{u}{\rho} \right|^a |u|^{s-a} \, dx \leq |K \rho^a|_r \left| \frac{u}{\rho} \right|^a |u|^{s-a},
\]

where \( 1/r + a/p + (s - a)/b = 1 \) and hence \( b \leq q \) by (5.3). Since \( |u/\rho|_p \leq C \|u\| \) by the Hardy inequality (see Nečas [12]) and \( |u|_b \leq C |u|_q \), the last expression is less than or equal to \( C \|u\|^a |u|^{s-a} \). By the Young inequality, the latter is less than or equal to \( C(\varepsilon) \|u\|^t + \varepsilon |u|^q \), where \( a/t + (s - a)/q = 1 \) and hence \( t < p \) by (5.3).

We assume that \( h \in \mathcal{A}_p^0 \) and \( K_i \in \mathcal{A}_q^1 \) for \( i = 1, \ldots, n \). First we verify that the associated functional

\[
\Phi(u) = \int_{\Omega} \left[ \frac{1}{p} |\nabla u|^p - \frac{1}{p} \lambda h(x) |u|^p - \frac{1}{q} K(x) |u|^q - G(x, u) \right] \, dx, \quad u \in W_0^{1,p}(\Omega),
\]

where \( G(x, t) = \int_0^t g(x, \tau) \, d\tau \), satisfies the (PS) condition.

Lemma 5.3. Every sequence \((u_j) \subset W_0^{1,p}(\Omega)\) such that \((\Phi(u_j))\) is bounded and \(\Phi'(u_j) \to 0\) has a convergent subsequence.

Proof. It suffices to show that \((u_j)\) is bounded by Lemma 2.3. We have

\[
\left( 1 - \frac{p}{q} \right) \int_{\Omega} K(x) |u_j|^q \, dx = p \Phi(u_j) - \Phi'(u_j) u_j + \int_{\Omega} [p G(x, u_j) - u_j g(x, u_j)] \, dx.
\] (5.4)

By (4.2) and Lemma 5.2, for every \( \varepsilon > 0 \), there exists \( C(\varepsilon) \) such that

\[
\left| \int_{\Omega} [p G(x, u_j) - u_j g(x, u_j)] \, dx \right| \leq \sum_{i=1}^n \left( 1 + \frac{p}{q_i} \right) \int_{\Omega} K_i(x) |u_j|^{q_i} \, dx \leq C(\varepsilon) \sum_{i=1}^n \|u_j\|^{t_i} + \varepsilon |u_j|_q^q,
\] (5.5)

where each \( t_i < p \). Combining (5.2), (5.4), and (5.5) gives

\[
|u_j|_q^q \leq C \left( \sum_{i=1}^n \|u_j\|^{t_i} + 1 \right) + o(\|u_j\|).
\] (5.6)

Now we use

\[
\left( \frac{q}{p} - 1 \right) \left( \|u_j\|^p - \lambda \int_{\Omega} h(x) |u_j|^p \, dx \right) = q \Phi(u_j) - \Phi'(u_j) u_j + \int_{\Omega} [q G(x, u_j) - u_j g(x, u_j)] \, dx.
\] (5.7)

By Lemma 5.2,

\[
\left| \int_{\Omega} h(x) |u_j|^p \, dx \right| \leq C \left( \|u_j\|^t + |u_j|_q^q \right),
\] (5.8)
where \( t < p \). As in (5.5),

\[
\left| \int_{\Omega} \left[ q G(x, u_j) - u_j g(x, u_j) \right] dx \right| \leq C \left( \sum_{i=1}^{n} \| u_j \|^{t_i} + |u_j|^q \right).
\]

(5.9)

Combining (5.6)–(5.9) gives

\[
\| u_j \|^p \leq C \left( \sum_{i=1}^{n} \| u_j \|^{t_i} + \| u_j \|^t + 1 \right) + o(\| u_j \|),
\]

which implies that \((u_j)\) is bounded since each \( t_i < p \) and \( t < p \). \( \square \)

Next we study the structure of the sublevel sets \( \Phi^\alpha = \{ u \in W^{1,p}_0(\Omega) : \Phi(u) \leq \alpha \} \) for \( \alpha < 0 \) with \( |\alpha| \) large.

**Lemma 5.4.** We have

1. \( \sup_{u \in W^{1,p}_0(\Omega)} \left( \phi'(u) u - \frac{p+q}{2} \phi(u) \right) < +\infty \);

2. \( \lim_{t \to +\infty} \phi(tu) = -\infty \quad \forall u \in W^{1,p}_0(\Omega) \setminus \{0\} \).

**Proof.** (1) We have

\[
\Phi'(u) u - \frac{p+q}{2} \Phi(u) = -\frac{q-p}{2} \int_{\Omega} \left[ \frac{1}{p} |\nabla u|^p - \frac{1}{p} \lambda \frac{1}{h(x)} |u|^p + \frac{1}{q} K(x) |u|^q \right] dx
\]

\[+ \int_{\Omega} \left[ \frac{p+q}{2} G(x, u) - u g(x, u) \right] dx. \quad (5.10)\]

By (4.2) and Lemma 5.2 for every \( \varepsilon > 0 \), there exists \( C(\varepsilon) \) such that

\[
\left| \int_{\Omega} \left[ \frac{p+q}{2} G(x, u) - u g(x, u) \right] dx \right| \leq C(\varepsilon) \sum_{i=1}^{n} \| u \|^{t_i} + \varepsilon |u|^q, \]

(5.11)

\[
\left| \int_{\Omega} h(x) |u|^p dx \right| \leq C(\varepsilon) \| u \|^t + \varepsilon |u|^q, \quad (5.12)
\]

where each \( t_i < p \) and \( t < p \). Combining (5.2) and (5.10)–(5.12) gives

\[
\Phi'(u) u - \frac{p+q}{2} \Phi(u) \leq -\frac{1}{2} \left( \frac{q}{p} - 1 \right) \| u \|^p + C \left( \sum_{i=1}^{n} \| u \|^{t_i} + \| u \|^t \right),
\]

from which the conclusion follows.

(2) This follows from (5.2) and (4.2) since \( p < q \) and each \( q_i < q \). \( \square \)

**Lemma 5.5.** There exists \( \alpha < 0 \) such that \( \Phi^\alpha \) is contractible in itself.
Proof. By Lemma 5.4 (1), there exists $\alpha < 0$ such that
\[ \Phi'(u) < 0 \quad \forall u \in \Phi^\alpha. \] (5.13)
For $u \in W^{1,p}_0(\Omega) \setminus \{0\}$, taking into account Lemma 5.4 (2), set
\[ t(u) = \min \{ t \geq 1 : \Phi(tu) \leq \alpha \}, \]
and note that the function $u \mapsto t(u)$ is continuous by (5.13). Then $u \mapsto t(u)$ is a retraction of $W^{1,p}_0(\Omega) \setminus \{0\}$ onto $\Phi^\alpha$, and the conclusion follows since $W^{1,p}_0(\Omega) \setminus \{0\}$ is contractible in itself.

We are now ready to prove our main existence result. Let $\lambda_k \nearrow +\infty$ be the sequence of positive eigenvalues of problem (4.6) considered in the last section.

**Theorem 5.6.** Assume that $\lambda \geq 0$, $q \in (p,p^*)$, $K \in A_q$ satisfies (5.2), $h \in A^q_0$ is positive on a set of positive measure, and $g$ satisfies (4.2) for some $q_i \in (p,q)$ and $K_i \in A^q_{q_i}$ for $i = 1, \ldots, n$. Then problem (5.1) has a nontrivial weak solution in each of the following cases:

1. $\lambda \notin \{ \lambda_k : k \geq 1 \}$;
2. $G(x,t) \geq 0$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;
3. $G(x,t) \leq 0$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;
4. $p > N$ and, for some $\delta > 0$, $G(x,t) \geq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$;
5. $p > N$ and, for some $\delta > 0$, $G(x,t) \leq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$;
6. $p \leq N$, $h \in \tilde{A}_p$, $K_i \in \tilde{A}_{q_i}$ for $i = 1, \ldots, n$, and, for some $\delta > 0$, $G(x,t) \geq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$;
7. $p \leq N$, $h \in \tilde{A}_p$, $K_i \in \tilde{A}_{q_i}$ for $i = 1, \ldots, n$, and, for some $\delta > 0$, $G(x,t) \leq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$.

Proof. Suppose that 0 is the only critical point of $\Phi$. Taking $U = W^{1,p}_0(\Omega)$ in (4.3), we have
\[ C^q(\Phi,0) = H^q(\Phi^0,\Phi^0 \setminus \{0\}). \]
Let $\alpha < 0$ be as in Lemma 5.5. Since $\Phi$ has no other critical points and satisfies the (PS) condition by Lemma 5.3, $\Phi^0$ is a deformation retract of $W^{1,p}_0(\Omega)$ and $\Phi^\alpha$ is a deformation retract of $\Phi^0 \setminus \{0\}$ by the second deformation lemma. So
\[ C^q(\Phi,0) \approx H^q(W^{1,p}_0(\Omega),\Phi^\alpha) = 0 \quad \forall q \]
since $\Phi^\alpha$ is contractible in itself, contradicting Theorem 4.4, Theorem 4.5 or Theorem 4.9.

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