We study signaling in Bayesian ad auctions, in which bidders’ valuations depend on a random, unknown state of nature. The auction mechanism has complete knowledge of the actual state of nature, and it can send signals to bidders so as to disclose information about the state and increase revenue. For instance, a state may collectively encode some features of the user that are known to the mechanism only, since the latter has access to data sources unaccessible to the bidders. We study the problem of computing how the mechanism should send signals to bidders in order to maximize revenue. While this problem has already been addressed in the easier setting of second-price auctions, to the best of our knowledge, our work is the first to explore ad auctions with more than one slot. In this paper, we focus on public signaling and VCG mechanisms, under which bidders truthfully report their valuations. We start with a negative result, showing that, in general, the problem does not admit a PTAS unless $P = NP$, even when bidders’ valuations are known to the mechanism. The rest of the paper is devoted to settings in which such negative result can be circumvented. First, we prove that, with known valuations, the problem can indeed be solved in polynomial time when either the number of states $d$ or the number of slots $m$ is fixed. Moreover, in the same setting, we provide an FPTAS for the case in which bidders are single minded, but $d$ and $m$ can be arbitrary. Then, we switch to the random valuations setting, in which these are randomly drawn according to some probability distribution. In this case, we show that the problem admits an FPTAS, a PTAS, and a QPTAS, when, respectively, $d$ is fixed, $m$ is fixed, and bidders’ valuations are bounded away from zero.

1 Introduction

Nowadays, worldwide spending in digital advertising is skyrocketing, and this growth is primarily driven by ad auctions. These account for almost all market share, since they are at the core of popular advertising platforms, such as, e.g., those by Google, Amazon, and Facebook. According to a recent report by [eMarketer](https://www.emarketer.com) [2021], digital ad spending will reach over $490 billion in 2021 and zoom past half a trillion in 2022.

We study signaling in ad auction settings by means of the Bayesian persuasion framework [Kamenica and Gentzkow](https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2147320) [2011]. Over the last years, this framework has received considerable attention from the computer science community, due to its applicability to many real-world scenarios, such as, e.g., online advertising [Bro Miltersen and Sheffet](https://link.springer.com/article/10.1007/s00474-012-1048-2) [2012]; [Emek et al.](https://arxiv.org/abs/1807.00046) [2014]; [Badanidiyuru et al.](https://arxiv.org/abs/1802.07180) [2018], voting [Alonso and Câmara](https://link.springer.com/article/10.1007/s00474-019-1063-9) [2016]; [Cheng et al.](https://dl.acm.org/doi/10.1145/2725451.2754532) [2015]; [Castiglioni et al.](https://arxiv.org/abs/2001.02183) [2020a]; [Castiglioni and Gatti](https://arxiv.org/abs/2108.02886) [2021], traffic routing [Vasserman et al.](https://dl.acm.org/doi/10.1145/2568897.2568959) [2013]; [Bhaskar et al.](https://dl.acm.org/doi/10.1145/3158141.3158144) [2016]; [Castiglioni et al.](https://arxiv.org/abs/2112.10628) [2021], recommendation systems [Mansour et al.](https://dl.acm.org/doi/10.1145/2759654.2759708) [2016], security [Rabinovich et al.](https://dl.acm.org/doi/10.1145/3087841.3088014) [2015]; [Xu et al.](https://dl.acm.org/doi/10.1145/2806848.2806944) [2016], and product marketing [Babichenko and Barman](https://link.springer.com/article/10.1007/s00474-017-1802-2) [2017]; [Candogan](https://dl.acm.org/doi/10.1145/3188745.3188748) [2019].

In a standard ad auction, the advertisers (also called bidders) compete for displaying their ads on a limited number of slots, and each bidder has their own private valuation representing how much they value a click on their ad. In this work, we study Bayesian ad auctions, which are characterized by the fact that bidders’ valuations depend on a random, unknown state of nature. The auction mechanism has complete knowledge of the actual state of nature, and it can send signals to bidders so as to disclose information about the state and increase revenue. In particular, the auction mechanism commits to a signaling scheme, which is defined as a randomized mapping from states of nature to signals being sent to the bidders. Our model fits many real-world applications that are not captured by classical ad auctions.
For instance, a state of nature may collectively encode some features of the user visualizing the ads—such as, e.g., age, gender, or geographical region—that are known to the mechanism only, since the latter has access to data sources unaccessible to the bidders.

We study the problem of computing a revenue-maximizing signaling scheme for the mechanism. In particular, in this paper we focus on public signaling, in which the mechanism can only send a single signal that is observed by all the bidders. Moreover, we restrict our attention to VCG mechanisms, which are widely used in practice and have the appealing property of inducing bidders to truthfully report their valuations. While the signaling problem studied in this paper has already been addressed in the easier setting of second-price auctions Badanidiyuru et al. (2018), to the best of our knowledge, our work is the first to explore algorithmic signaling in general ad auctions with more than one slot.

1.1 Original Contributions

We start our analysis with a negative result, showing that, in general, the revenue-maximizing problem with public signaling does not admit a PTAS unless \( P = NP \), even when bidders’ valuations are known to the mechanism. Thus, in the rest of the paper, we address settings in which we can prove that such a negative result can be circumvented.

First, we show that, in the known valuations setting, the problem admits a polynomial-time algorithm when either the number of slots \( m \) or the number of states \( d \) is fixed. The proposed algorithms work by solving suitably-defined linear programs (LPs) of polynomial size, thanks to the crucial property that, when either \( m \) or \( d \) is fixed, there always exists an optimal signaling scheme using a polynomial number of different signals. Moreover, we also study special instances in which the bidders are single minded, but \( m \) and \( d \) can be arbitrary. In this case, each bidder positively values a click on their ad only when the actual state of nature is a specific (single) state, and all the bidders interested in the same state value a click on their ad for the same amount. By exploiting a particular combinatorial structure of the set of bidders’ posterior distributions induced by signaling schemes, we are able to provide an FPTAS in such setting. The algorithm works by applying the ellipsoid method in a non-trivial way, with only access to an approximate polynomial-time separation oracle. The latter is implemented by a rather involved dynamic programming algorithm, which works thanks to the particular structure of the set of bidders’ posteriors.

Then, we switch the attention to the random valuations setting, where bidders’ valuations are unknown to the mechanism, but randomly drawn according to some probability distribution. In this case, we first provide some preliminary results that establish useful connections between the optimal value of the revenue-maximizing problem and that of optimal signaling schemes restricted to suitably-defined finite sets of posterior distributions. These sets are defined so that the expected revenue of the mechanism is “stable”, meaning that it does not decrease too much when restricting signaling schemes to use posteriors in such sets. In particular, for our results we use sets of \( q \)-uniform posteriors, for suitable values of \( q \). As a preliminary step, we also show that it is possible to compute an approximately-optimal signaling scheme having only access to a finite number of samples from the distribution of bidders’ valuations. In conclusion, all the preliminary results described so far allow us to prove that, in the random valuations setting, the problem admits an FPTAS, a PTAS, and a QPTAS, when, respectively, \( d \) is fixed, \( m \) is fixed, and bidders’ valuations are bounded away from zero\(^1\).

1.2 Related Works

To the best of our knowledge, the algorithmic study of signaling in auctions is limited to the second-price auction, which can be seen as a special ad auction with a single slot.

Emek et al. [2014] study second-price auctions in the known valuations setting. They provide an LP to compute an optimal public signaling scheme. Moreover, they show that it is NP-hard to compute an optimal signaling scheme in the random valuations setting. In our work, we generalize their positive result, in order to provide our polynomial-time algorithm working when the number of slots \( m \) is fixed.

Cheng et al. [2015] complement the hardness result of Emek et al. [2014] by providing a PTAS for the random valuations setting. This result cannot be extended to ad auctions, as we show in our first negative result. However, we provide two generalizations of the result by Cheng et al. [2015]: we provide a PTAS for the random valuations setting with a fixed number of slots \( m \), and a QPTAS when the bidder’s valuations are bounded away from zero.

Finally, Badanidiyuru et al. [2018] study algorithms whose running time does not depend on the number of states of nature. Moreover, they initiate the study of private signaling schemes, showing that, in second-price auctions, private signaling introduces non-trivial equilibrium selection problems.

\(^1\) All the proofs are in the Supplementary Material.
2 Preliminaries

In a standard ad auction (see also the book by Nisan and Ronen [2001] for more details), there is a set \( \mathcal{N} := \{1, \ldots, n\} \) of advertisers (or bidders) who compete for displaying their ads on a set \( \mathcal{M} := \{1, \ldots, m\} \) of slots, with \( m \leq n \). Each bidder \( i \in \mathcal{N} \) is characterized by a private valuation \( v_i \in [0, 1] \), which represents how much they value a click on their ad. Moreover, each slot \( j \in \mathcal{M} \) is associated with a click through rate parameter \( \lambda_j \in [0, 1] \), which is the probability with which the slot is clicked by a user. W.l.o.g., we assume that the slots are ordered so that \( \lambda_1 \geq \ldots \geq \lambda_m \). The auction goes on as follows: first, each bidder \( i \in \mathcal{N} \) separately reports a bid \( b_i \in [0, 1] \) to the auction mechanism; then, based on the bids, the latter allocates an ad to each slot and defines how much each bidder has to pay the mechanism for a click on their ad. We focus on truthful mechanisms, and the VCG mechanism in particular (see the book by Mas-Colell et al. [1995] for a complete description of the mechanism). In truthful mechanisms, allocation and payments are defined so that it is a dominant strategy for each bidder to report their true valuation to the mechanism, namely \( b_i = v_i \) for every \( i \in \mathcal{N} \). In particular, the allocation implemented by the VCG mechanism orders the first \( m \) bidders in decreasing value of \( b_i \) to the first \( m \) slots (those with the highest click through rates). At the same time, assuming w.l.o.g. that bidder \( i \) is assigned to slot \( i \), the mechanism defines an expected payment \( p_i := \sum_{j=i+1}^{m+1} b_j (\lambda_{j-1} - \lambda_j) \) for each bidder \( i \in \{1, \ldots, m\} \), where, for the ease of notation, we let \( \lambda_{m+1} = 0 \). The payment is zero for all the other bidders. In practice, each bidder \( i \in \{1, \ldots, m\} \) has to pay \( \frac{p_i}{\lambda_i} \) whenever a user clicks on their ad, so that their utility is \( \lambda_i v_i - p_i \) in expectation over the clicks. The expected utility of all the other bidders is zero.

![Figure 1: Time-line of a Bayesian ad auction.](image)

We study Bayesian ad auctions, which are characterized by a set \( \Theta := \{\theta_1, \ldots, \theta_d\} \) of \( d \) states of nature. Each bidder \( i \in \mathcal{N} \) has a valuation vector \( v_i \in [0, 1]^d \), with \( v_i(\theta) \) being bidder \( i \)'s valuation in state \( \theta \in \Theta \), and all such vectors are arranged in a matrix of bidders' valuations \( V \in [0, 1]^{n \times d} \), whose entries are defined as \( V(i, \theta) := v_i(\theta) \) for all \( i \in \mathcal{N} \) and \( \theta \in \Theta \). We model signaling by means of the Bayesian persuasion framework [Kamenica and Gentzkow 2011]. We consider the case in which the auction mechanism knows the state of nature and acts as a sender by issuing signals to the bidders (the receivers), so as to partially disclose information about the state and increase revenue. As customary in the literature, we assume that the state is drawn from a common prior distribution \( \mu \in \Delta_\Theta \), with \( \mu_\theta \) denoting the probability of state \( \theta \in \Theta \). The mechanism publicly commits to a signaling scheme \( \phi \), which is a randomized mapping from states of nature to signals for the bidders. We focus on the case of public signaling in which all the bidders receive the same signal from the auction mechanism. Formally, a signaling scheme is a function \( \phi : \Theta \to \Delta_S \), where \( S \) is a set of available signals. For the ease of notation, we let \( \phi_\theta(s) \) be the probability of sending signal \( s \in S \) when the state is \( \theta \in \Theta \).

A Bayesian ad auction goes on as follows (see Figure 1 for a picture): (i) the auction mechanism commits to a signaling scheme \( \phi \), and the bidders observe it; (ii) the mechanism gets to know the state of nature \( \theta \sim \mu \) and draws signal \( s \sim \phi(\theta) \); and (iv) the bidders observe the signal \( s \) and rationally update their prior belief over states according to Bayes rule. After observing signal \( s \in S \), all the bidders infer a posterior distribution \( \xi_s \in \Delta_\Theta \) over states (also called posterior for short) such that the posterior probability of state \( \theta \in \Theta \) is

\[
\xi_s(\theta) := \frac{\mu_\theta \phi_\theta(s)}{\sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{\theta'}(s)}.
\]

Finally, each bidder \( i \in \mathcal{N} \) truthfully reports to the mechanism their expected valuation given the posterior \( \xi_s \), namely \( \xi_s^i v_i = \sum_{\theta \in \Theta} v_i(\theta) \xi_s(\theta) \), and the mechanism allocates slots and defines payments as in a standard ad auction.

Representing Signaling Schemes. It is oftentimes useful to represent signaling schemes as convex combinations of the posteriori they can induce [Dughmi 2014; Cheng et al. 2015]. Formally, a signaling scheme \( \phi : \Theta \to \Delta_S \) induces

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2 In this work, for the ease of presentation, we assume that the click through rate only depends on the slot and not on the ad being displayed. In general, each slot may have its own prominence value—the probability with which a user observes it—and each bidder may have their own ad quality—the probability with which their ad is clicked once observed—, so that the click through rate is defined as the product of these two quantities. All the results in this paper can be easily extended to such general model.

3 Given a finite set \( X \), we denote with \( \Delta_X \) the \((|X| - 1)\)-dimensional simplex defined over the elements of \( X \).
a probability distribution $\gamma$ over posteriors in $\Delta_\Theta$, with $\gamma(\xi)$ denoting the probability of posterior $\xi \in \Delta_\Theta$, defined as

$$\gamma(\xi) := \sum_{s \in s_i: \xi = \xi} \sum_{\theta \in \Theta} \mu_\theta \phi_\theta(s).$$

Indeed, we can directly reason about distributions $\gamma$ over $\Delta_\Theta$ rather than about signaling schemes, provided that they are consistent with the prior. By letting $\text{supp}(\gamma) := \{\xi \in \Delta_\Theta | \gamma(\xi) > 0\}$ be the support of $\gamma$, this requires that

$$\sum_{\xi \in \text{supp}(\gamma)} \gamma(\xi) \xi(\theta) = \mu_\theta \quad \forall \theta \in \Theta. \tag{2}$$

In the rest of the paper, we will use the term signaling scheme to refer to a consistent distribution $\gamma$ over $\Delta_\Theta$.

Computational Problems. We focus on the problem of computing an optimal signaling scheme, i.e., one maximizing the revenue of the mechanism. We study two settings:

- the known valuations (KV) setting in which the matrix of bidders’ valuations $V$ is known to the mechanism; and
- the random valuations (RV) setting in which the matrix of bidders’ valuations $V$ is unknown, but randomly drawn according to a probability distribution $\mathcal{V}$.

As it is customary in the literature (see, e.g., [Badanidiyuru et al. (2013)] in the RV setting we assume that algorithms have access to a black-box oracle returning i.i.d. samples drawn from $\mathcal{V}$ (rather than actually knowing such distribution). We denote by $\text{REV}(V, \xi)$ the expected revenue of the mechanism when the bidders’ valuations are given by $V$ and the posterior induced by the mechanism is $\xi \in \Delta_\Theta$. Formally, given that bidders truthfully report their expected valuations and assuming w.l.o.g. that bidder $i$ is assigned by the mechanism to slot $i$, we can write $\text{REV}(V, \xi) := \sum_{j = 1}^{m} j \xi^\top v_{j+1}(\lambda_j - \lambda_{j+1})$. Then, given a signaling scheme $\gamma$, the expected revenue of the mechanism is $\sum_{\xi \in \text{supp}(\gamma)} \gamma(\xi) \text{REV}(V, \xi)$. When the valuations are unknown, we let $\text{REV}(V, \xi) := \mathbb{E}_{V \sim \mathcal{V}}\text{REV}(V, \xi)$ and define the expected revenue analogously. Notice that, given a distribution of valuations $\mathcal{V}$ (or, in the KV setting, a matrix of bidders’ valuations $V$) and a finite set $\Xi \subseteq \Delta_\Theta$ of posteriors, it is possible to formulate the problem of computing an optimal signaling scheme as an LP, as follows:

$$\max_{\gamma \in \Delta_\Theta} \sum_{\xi \in \Xi} \gamma(\xi) \text{REV}(V, \xi) \quad \text{s.t.}$$

$$\sum_{\xi \in \Xi} \gamma(\xi) \xi(\theta) = \mu_\theta \quad \forall \theta \in \Theta. \tag{3a}$$

In the following, we let $\text{OPT}_\Xi$ be the optimal value of LP $\text{OPT}_\Xi$ while we denote with $\text{OPT}$ the optimal expected revenue of the mechanism over all the possible signaling schemes $\gamma$.

3 A General Inapproximability Result

We start our analysis with the following negative result:

**Theorem 1.** The problem of computing an optimal signaling scheme does not admit a PTAS unless $P = \text{NP}$, even when it is restricted to the KV setting.

Theorem 1 is proved by a reduction from the VERTEX COVER problem in cubic graphs [Alimonti and Kann (2000)].

In the rest of this work, we study several settings in which the negative result in Theorem 1 can be circumvented, by either fixing some parameters of the problem (see Sections 4 and 6.1) or considering instances with a specific structure (see Sections 5 and 6.2).

4 KV Setting: Parametrized Complexity

In this section, we study the parametrized complexity of the problem of computing an optimal signaling scheme, showing that it admits a polynomial-time algorithm when either the number of slots $m$ or the number of states of nature $d$ is fixed.

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4LP $3$ is written for the RV setting, its analogous for the KV setting can be obtained by substituting $\text{REV}(V, \xi)$ with $\text{REV}(V, \xi)$. The dependence of $\text{OPT}_\Xi$ and $\text{OPT}$ from either $V$ or $\mathcal{V}$ is omitted, as it will be clear from context.
In the following, we let $\Pi_l \subseteq 2^N$ be the set of all the $l$-possible permutations of $l \leq n$ bidders taken from $N$, with $\pi = (i_1, ..., i_l) \in \Pi_l$ denoting a tuple made by bidders $i_1, ..., i_l \in N$, in that order. We also let $\Xi_\pi \subseteq \Delta_\Theta$ be the (possibly empty) polytope of posteriors in which the expected valuations of bidders in $\pi \in \Pi_l$ are ordered (from the highest to the lowest) according to $\pi$; formally, it holds $\Xi_\pi := \{ \xi \in \Delta_\Theta \mid \xi^\top v_{i_1} \geq \xi^\top v_{i_2} \geq ... \geq \xi^\top v_{i_l} \}$. Notice that, given a permutation $\pi \in \Pi_l$ of $l \geq m + 1$ bidders, the expected revenue of the mechanism in any posterior $\xi \in \Xi_\pi$ is $\text{REV}(V, \xi) = \xi^\top \sum_{j=1}^{m} j v_{i_{j+1}} (\lambda_j - \lambda_{j+1})$, since the bidders truthfully report their expected valuations to the mechanism, and, thus, the latter allocates slots to bidders in $\pi$ according to their order in the permutation. Thus, for any fixed $\pi \in \Pi_l$ with $l \geq m + 1$, the term $\text{REV}(V, \xi)$ is linear in $\xi$ over $\Xi_\pi$.

### 4.1 Fixing the Number of Slots $m$

In this case, the problem can be solved in polynomial time by formulating it as an LP, thanks to the following lemma:

**Lemma 1.** There always exists an optimal signaling scheme $\gamma$ such that $|\Xi_\pi \cap \text{supp}(\gamma)| \leq 1$ for every $\pi \in \Pi_{m+1}$.

Intuitively, the lemma follows from the fact that, given any signaling scheme $\gamma$ and two posteriors $\xi, \xi' \in \text{supp}(\gamma)$ such that $\xi, \xi' \in \Xi_\pi$ for some $\pi \in \Pi_{m+1}$, it is always possible to define a new signaling scheme that replaces $\xi$ and $\xi'$ with a suitably-defined convex combination of them, without decreasing the expected revenue (since it is linear over $\Xi_\pi$).

By Lemma 1, we can re-write the revenue maximization problem as $\max \sum_{\pi \in \Pi_{m+1}} \gamma(\xi) \text{REV}(V, \xi)$ subject to constraints ensuring that each $\xi \in \Xi_\pi$ belongs to $\Xi_\pi$ (for $\pi \in \Pi_{m+1}$) and that $\gamma$ is a consistent probability distribution over such posteriors (see Equation 2). This problem can be formulated as an LP by introducing a variable for each $\pi \in \Pi_{m+1}$ and a vector $\theta \in \Theta$, encoding the products $\gamma(\xi)\xi(\theta)$ that define the expected revenue. Overall, the resulting LP (see LP8 in the Supplementary Material) has a number of variables and constraints that is $O(n^m)$, which, after fixing $m$, is polynomial in the size of the input. Thus, we conclude that:

**Theorem 2.** In the KV setting, if the number of slots $m$ is fixed, then an optimal signaling scheme can be computed in polynomial time.

### 4.2 Fixing the Number of States $d$

Our polynomial-time algorithm exploits the fact that an optimal signaling scheme can be computed by restricting the attention to distributions supported on a finite set of posteriors whose cardinality is polynomial in all the parameters, except from $d$. In particular, it is sufficient to focus on the set $\Xi^* := \bigcup_{\pi \in \Pi_n} V(\Xi_\pi)$, where $V(\cdot)$ denotes the set of vertices of the polytope given as input. Formally:

**Lemma 2.** It holds that $\text{OPT}_{\Xi^*} = \text{OPT}$.

The lemma follows from the fact that, given any signaling scheme $\gamma$ and posterior $\xi \in \text{supp}(\gamma)$ such that $\xi \in \Xi_\pi$ for some $\pi \in \Pi_n$, by Carathéodory’s theorem it is always possible (since $\Xi_\pi$ is a polytope) to decompose $\xi$ into a convex combination of the vertices of $\Xi_\pi$, obtaining a new signaling scheme that provides the mechanism with an expected revenue at least as large as that of $\gamma$ (since $\text{REV}(V, \xi)$ is linear over $\Xi_\pi$). By observing that $|\Xi^*| = O((n^2 + d)^{d-1})$, it is easy to show that an optimal signaling scheme can be computed by means of LP8 instantiated for the set $\Xi^*$, which has a number of variables and constraints that is polynomial once $d$ is fixed. This proves the following:

**Theorem 3.** In the KV setting, if the number of states $d$ is fixed, then an optimal signaling scheme can be computed in polynomial time.

### 5 KV Setting: Single-Minded Bidders

In this section, we focus on particular Bayesian ad auctions where the bidders are single minded. Intuitively, in our setting, by single mindedness we mean that each bidder is interested in displaying their ad only when the realized state of nature is a specific (single) state, and that all the bidders interested in the same state value a click on their ad for the same amount. We introduce the following formal definition:

**Definition 1** (Single-minded bidders). In a Bayesian ad auction, we say that bidders are single minded if there exist $N_0 \subseteq N$ and $\theta_0 \in \{0, 1\}$ for all $\theta \in \Theta$ such that:

1. $N = \bigcup_{\theta \in \Theta} N_0$ and $N_0 \cap N_{0'} = \emptyset$ for all $\theta \neq \theta' \in \Theta$;

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6 We remark that LP8 in the Supplementary Material is a generalization of the LP presented by Emek et al. [2014] for the easier case of second price auctions.
(ii) for every $\theta \in \Theta$ and $i \in N_0$, it holds $v_i(\theta) = \delta_0$ and $v_i(\theta') = 0$ for all $\theta' \in \Theta : \theta' \neq \theta$.

Notice that, given a posterior $\xi \in \Delta_\Theta$, induced by the mechanism, all the bidders $i$ belonging to the same set $N_0$ have the same expected valuation, namely $\xi v_i = \delta_0 \xi(\theta)$ for all $\theta \in \Theta$ and $i \in N_0$. As a result, given that bidders truthfully report their expected valuations, the mechanism will always receive at most $d$ different bids, one per set $N_0$.

The last observation implies that, given $\xi \in \Delta_\Theta$, in order to unequivocally define an allocation of bidders to slots (and, thus, also define the expected payments) it is sufficient to know the relative ordering of the (at most) $d$ different expected valuations associated to sets $N_0$. This allows us to tackle the problem with an approach analogous to the one of Section\textsuperscript{4} with the only difference that, in this case, we will reason about permutations of the groups of bidders $N_0$, rather than about permutations of all the individual bidders.

In the following, we let $\Pi \subseteq 2^{\Theta}$ be the set of all the permutations of the states of nature $\Theta = \{\theta_1, \ldots, \theta_d\}$, while we let $\pi = (\theta_{k_1}, \ldots, \theta_{k_d}) \in \Pi$ be an ordered tuple made by states $\theta_{k_1}, \ldots, \theta_{k_d} \in \Theta$, where $k_1, \ldots, k_d \in \{1, \ldots, d\}$. Moreover, $\Xi_\pi := \{\xi \in \Delta_\Theta \mid \delta_{\theta_{k_1}} \xi(\theta_{k_1}) \geq \cdots \geq \delta_{\theta_{k_d}} \xi(\theta_{k_d})\}$ is the polytope of posteriors in which the expected valuations associated to sets $N_0$ are ordered according to $\pi$.

The first preliminary result that we need in order to derive our approximation algorithm is a characterization of the vertices of the sets $\Xi_\pi$ for $\pi \in \Pi$, as follows.

**Lemma 3.** Given $\pi \in \Pi$ and $\xi \in \Xi_\pi$, it holds that $\xi \in V(\Xi_\pi)$ if and only if there exists $\ell \in \{1, \ldots, d\}$ such that:

(i) $\delta_{\theta_{k_1}} \xi(\theta_{k_1}) = \cdots = \delta_{\theta_{k_\ell}} \xi(\theta_{k_\ell}) > 0$; and

(ii) $\delta_{\theta_{k_{\ell+1}}} \xi(\theta_{k_{\ell+1}}) = \cdots = \delta_{\theta_{k_d}} \xi(\theta_{k_d}) = 0$.

Intuitively, Lemma\textsuperscript{3} states that the vertices of a set $\Xi_\pi$ are all the posteriors $\xi \in \Delta_\Theta$ such that, for some $\ell \in \{1, \ldots, d\}$, only the first $\ell$ states according to the ordering defined by $\pi$ are assigned a positive probability, while all the remaining states have zero probability. Moreover, the positive probabilities of the posterior $\xi$ are defined so that all the bidders belonging to the first $\ell$ sets $N_0$, according to the ordering defined by $\pi$, are the same. Notice that, in the special case in which all the values $\delta_\theta$ are equal to one, the vertices of all the sets $\Xi_\pi$ are all the uniform probability distributions over subsets of $\ell$ states of nature, for any $\ell \in \{1, \ldots, d\}$.

By letting $\Xi^* = \bigcup_{\pi \in \Pi} V(\Xi_\pi)$, since the term $\text{REV}(V, \xi)$ is linear in $\xi$ over $\Xi_\pi$ for every permutation $\pi \in \Pi$, we can conclude that $OPT_{\Xi^*} = OPT$ (the proof is analogous to that of Lemma\textsuperscript{2}). Thus, Lemma\textsuperscript{3} allows us to find an optimal signaling scheme by solving LP\textsuperscript{4} for the set $\Xi^*$ and the matrix of bidders’ valuations $V$. However, notice that, since the size of $\Xi^*$ is exponential in $d$, the resulting LP has exponentially-many variables. Nevertheless, since the LP has polynomially-many constraints, we can still solve it in polynomial time by applying the ellipsoid algorithm to its dual, provided that a polynomial-time separation oracle is available.

In order to design a polynomial-time separation oracle, we apply the procedure described above to a relaxed version of LP\textsuperscript{4} whose optimal value is sufficiently “close” to that of the original LP. Given $\beta \in \mathbb{R}_+$, the relaxed LP reads as follows:

\begin{equation}
\max_{\gamma \in \Delta_{\Xi^*} : z \leq 0} \sum_{\xi \in \Xi} \gamma(\xi) \text{REV}(V, \xi) + \beta z \tag{4a}
\end{equation}
\begin{equation}
\sum_{\xi \in \Xi^*} \gamma(\xi) \xi(\theta) - z \geq \mu_\theta \quad \forall \theta \in \Theta. \tag{4b}
\end{equation}

The dual problem of LP\textsuperscript{4} reads as follows:

\begin{equation}
\min_{y \leq 0, t} \sum_{\theta \in \Theta} y_\theta \mu_\theta + t \quad \text{s.t.} \tag{5a}
\end{equation}
\begin{equation}
\sum_{\theta \in \Theta} y_\theta \xi(\theta) + t \geq \text{REV}(V, \xi) \quad \forall \xi \in \Xi^*. \tag{5b}
\end{equation}
\begin{equation}
\sum_{\theta \in \Theta} y_\theta \geq -\beta, \tag{5c}
\end{equation}

where $y_\theta$ for $\theta \in \Theta$ are dual variables associated to Constraints\textsuperscript{3b}, while $t$ is a dual variable for $\sum_{\xi \in \Xi^*} \gamma(\xi) = 1$. Notice that, by relaxing the LP, in the dual LP\textsuperscript{5} we get the additional Constraint\textsuperscript{5c} and that $y_\theta \leq 0$ for all $\theta \in \Theta$. This is crucial to design a polynomial-time separation oracle.

The separation problem associated to Problem\textsuperscript{5} reads as:
Definition 2 (Separation problem). Given values for the dual variables \( y_\theta \in [-\beta, 0] \) for all \( \theta \in \Theta \), compute:

\[
\max_{\xi \in \Xi} \text{REV}(V, \xi) - \sum_{\theta \in \Theta} y_\theta \xi(\theta).
\] (6)

The following Lemma 4 shows that Problem 6 can be solved optimally up to any given additive loss \( \lambda > 0 \), by means of a dynamic programming algorithm that runs in time polynomial in the size of the input, in \( \frac{1}{\lambda} \), and in \( \beta \). Formally:

**Lemma 4.** Given \( \lambda > 0 \), there exists an algorithm that finds an additive \( \lambda \)-approximation to Problem 6 in time polynomial in the size of the input, in \( \frac{1}{\lambda} \), and in \( \beta \).

The crucial observation that allows us to solve Problem 6 by means of dynamic programming is that, in any posterior \( \xi \in \Xi \), bidders’ expected valuations are either a positive, bidder-independent value or zero (see Lemma 3). This allows us to build a discretized range of possible bidders’ valuation values, so that, for each discretized value, we can compute an optimal posterior \( \xi \in \Xi \) inducing that value by adding states of nature incrementally in a dynamic programming fashion.

Since the algorithm in Lemma 4 only returns an approximate solution to Problem 6, we need to carefully apply the ellipsoid algorithm to solve LP 3, so that it correctly works even with an approximated oracle. Some non-trivial duality arguments allow us to prove that, indeed, this can be achieved by only incurring in a small additive loss on the quality of the returned solution, and without degrading the running time of the algorithm. Overall, this allows us to conclude that:

**Theorem 4.** In the KV setting, if the bidders are single minded, then the problem of computing an optimal signaling scheme admits an (additive) FPTAS.

6 RV Setting

In this setting, as stated in Section 2, we assume that the auction mechanism has access to the distribution of bidders’ valuations \( \mathcal{V} \) only through a black-box sampling oracle. In the following, given \( s \in \mathbb{N}_{>0} \) i.i.d. samples of matrices of bidders’ valuations, namely \( V_1, \ldots, V_s \in [0, 1]^{n \times d} \), we let \( \mathcal{V}^s \) be their empirical distribution, which is such that \( \mathbb{P}_{\mathcal{V} \sim \mathcal{V}^s} \left\{ V = \tilde{V} \right\} = \frac{1}{s} \sum_{i=1}^s 1_{(V_i = \tilde{V})} \) for all \( \tilde{V} \in [0, 1]^{n \times d} \).

In this section, we first study the parametrized complexity of the problem of computing an optimal signaling scheme in general auctions (Section 6.1), and, then, we address special auction settings in which the bidders’ valuations are bounded away from zero, namely \( v_i(\theta) > \delta \) for all \( i \in N \) and \( \theta \in \Theta \), for some threshold \( \delta > 0 \). In the latter case, we show that the problem admits a QPTAS and the result is tight (Section 6.2).

Before stating our main results (Theorems 5, 6, 7, and 8), we introduce some preliminary useful lemmas. The first one (Lemma 5) works under the true distribution of bidders’ valuations \( \mathcal{V} \), and it establishes a connection between the optimal expected revenue \( (OPT) \) and the optimal value of LP 5 for suitably-defined finite sets \( \Xi \subseteq \Delta_\Theta \) of posteriors \( (OPT_\Xi) \). In particular, we look at sets \( \Xi \subseteq \Delta_\Theta \) for which the function \( \text{REV}(\mathcal{V}, \cdot) \) is “stable” according to the following definition:

**Definition 3 ((\( \alpha, \varepsilon \))-stability).** Given \( \alpha, \varepsilon \geq 0 \) and a finite set \( \Xi \subseteq \Delta_\Theta \), we say that \( \text{REV}(\mathcal{V}, \cdot) \) is \( (\alpha, \varepsilon) \)-stable for \( \Xi \) if, for every \( \xi \in \Delta_\Theta \), there exists a distribution \( \gamma_\xi \in \Delta_\Xi \) such that:

\[
\sum_{\xi' \in \Xi} \gamma_\xi(\xi') \text{REV}(\mathcal{V}, \xi') \geq (1 - \alpha)\text{REV}(\mathcal{V}, \xi) - \varepsilon.
\] (7)

For any finite set \( \Xi \subseteq \Delta_\Theta \) such that \( \text{REV}(\mathcal{V}, \cdot) \) is \( (\alpha, \varepsilon) \)-stable for \( \Xi \), starting from an optimal signaling scheme \( \gamma \) one can recover an optimal solution to LP 5 only incurring in “small” multiplicative and additive losses in the expected revenue, respectively of \( 1 - \alpha \) and \( \varepsilon \). This can be accomplished by decomposing each posterior \( \xi \in \text{supp}(\gamma) \) into \( \gamma_\xi \in \Delta_\Xi \) and, then, putting such distributions together. These observations allow us to prove the following lemma:

**Lemma 5.** Given \( \alpha, \varepsilon \geq 0 \) and \( \Xi \subseteq \Delta_\Theta \) such that \( \text{REV}(\mathcal{V}, \cdot) \) is \( (\alpha, \varepsilon) \)-stable for \( \Xi \), it holds \( OPT_\Xi \geq (1 - \alpha)OPT - \varepsilon \).

The second lemma (Lemma 6) deals with the approximation error introduced by using an empirical distribution of bidders’ valuations \( \mathcal{V}^s \), rather than the actual distribution \( \mathcal{V} \). Given a finite set \( \Xi \subseteq \Delta_\Theta \) of posteriors, let \( \gamma_{\mathcal{V}^s} \in \Delta_\Xi \) be an optimal solution to LP 5 for distribution \( \mathcal{V}^s \) and set \( \Xi \). Moreover, let \( OPT_\Xi,s := \mathbb{E} \left[ \sum_{\xi \in \Xi} \gamma_{\mathcal{V}^s}(\xi) \text{REV}(\mathcal{V}, \xi) \right] \) be the average expected revenue of signaling schemes \( \gamma_{\mathcal{V}^s} \) under the true distribution of valuations \( \mathcal{V} \), where the expectation is with respect to the sampling procedure that determines \( \mathcal{V}^s \). Then, a concentration argument proves the following:

\[\text{Notions of stability analogous to that in Definition 3 have already been used in the literature; see, e.g., Cheng et al., 2015.}\]
Lemma 6. Given \( \rho, \tau > 0 \), let \( \Xi \subseteq \Delta_\Theta \) be finite and \( s := \left\lfloor \frac{2(\lambda \eta m)\log \frac{2}{\tau}}{\tau} \right\rfloor \). \( OPT_{\Xi, s} \geq (1 - \rho|\Xi|) OPT_{\Xi} - \tau \).

Finally, the last lemma (Lemma 7) exploits Lemma 5 to provide two useful bounds on the value of \( OPT_{\Xi, q} \), where \( \Xi_q \subseteq \Delta_\Theta \) (for a given \( q \in \mathbb{N}_{>0} \)) is the finite set of all the \( q \)-uniform posteriors, according to the following definition:

Definition 4 \((q\text{-uniform posterior})\). Given \( q \in \mathbb{N}_{>0} \), a posterior \( \xi \in \Delta_\Theta \) is \( q \)-uniform if each \( \xi(\theta) \) is a multiple of \( \frac{1}{q} \).

Notice that the set \( \Xi_q \) has size \( |\Xi_q| \leq \min\{d^q, q^d\} \). The two points in the following lemma are readily proved by applying Lemma 5 after noticing that the sets \( \Xi_q \) in the statement are such that the function \( REV(\mathcal{V}, \cdot) \) is \((\alpha, \varepsilon)\)-stable for them, with suitable values of \( \alpha \geq 0 \) and \( \varepsilon \geq 0 \). Formally:

Lemma 7. Given \( \eta > 0 \) and \( q := \left\lfloor \frac{1}{2\eta^2} \log \frac{m+1}{\eta} \right\rfloor \), it holds:

(i) \( OPT_{\Xi_q} \geq OPT - 2\eta m \);

(ii) if, for some \( \delta > 0 \), it is the case that \( v_i(\theta) > \delta \) for all \( i \in \mathcal{N} \) and \( \theta \in \Theta \), then \( OPT_{\Xi_q} \geq (1 - \frac{q}{\eta})^2 OPT \).

6.1 Parametrized Complexity

First, we study the computational complexity of the problem of computing an optimal signaling scheme when the number of states \( d \) is fixed. We provide an (additive) FPTAS that works by performing the following two steps: (i) it collects a suitable number \( s \in \mathbb{N}_{>0} \) of matrices of bidders’ valuations, by invoking the sampling oracle; and (ii) it solves \( \text{LP}_s \) for the resulting empirical distribution \( \mathcal{V}^s \) and a suitably-defined set of \( q \)-uniform posteriors. In particular, given a desired (additive) error \( \lambda > 0 \), the algorithm works on the set \( \Xi_q \) for \( q = \left\lfloor \frac{md}{\lambda} \right\rfloor \) and its approximation guarantees rely on the following Lemma 8 proved again by means of Lemma 5.

Lemma 8. For \( \lambda > 0 \) and \( q = \left\lfloor \frac{md}{\lambda} \right\rfloor \), \( OPT_{\Xi_q} \geq OPT - \lambda \).

Thanks to Lemmas 6 and 8 (the former applied for suitable values \( \rho, \tau > 0 \)), we can prove that the procedure described in steps (i) and (ii) above gives a signaling scheme achieving an expected revenue at most a function of \( \lambda \) lower than \( OPT \), provided that the number of samples \( s \) is defined as in Lemma 8. Moreover, let us notice that, since \( |\Xi_q| = O(q^d) = O((\frac{1}{\lambda} md)^d) \), if \( d \) is fixed, then the overall procedure runs in time polynomial in the input size and in \( \frac{1}{\lambda} \). Thus, we can conclude that:

Theorem 5. In the RV setting, if the number of states \( d \) is fixed, then the problem of computing an optimal signaling scheme admits and (additive) FPTAS.

Next, we switch the attention to the case in which the number of slots \( m \) is fixed. We provide an (additive) PTAS that works as the FPTAS in Theorem 5 but whose approximation guarantees follow from Lemma 5 and point (i) in Lemma 7 (rather than Lemma 6). Thus, the only difference with respect to the previous case is that the algorithm works on the set \( \Xi_q \) of \( q \)-uniform posteriors for \( q \) defined as in Lemma 7. As a result, \( |\Xi_q| = O(d^q) \) and \( q \) depends on a parameter \( \eta > 0 \) that is related to the quality of the obtained approximation, the algorithm is only a PTAS rather than an FPTAS. Formally, we can prove the following:

Theorem 6. In the RV setting, if the number of slots \( m \) is fixed, then the problem of computing an optimal signaling scheme admits and (additive) PTAS.

6.2 Valuations Bounded Away From Zero

We conclude the section by studying the case in which the bidders’ valuations are bounded away from zero. This case is dealt with an algorithm identical to the one in Theorem 6 but carrying on the approximation analysis by using Lemma 6 and point (ii) in Lemma 7 (rather than point (i)). Thus, since the value of \( q \) in Lemma 7 is related to the quality of the approximation through a parameter \( \eta > 0 \) and also depends logarithmically on the number of slots \( m \), we obtain:

Theorem 7. In the RV setting, if \( v_i(\theta) \geq \delta \) for all \( i \in \mathcal{N} \) and \( \theta \in \Theta \) for some \( \delta > 0 \), then the problem of computing an optimal signaling scheme admits a (multiplicative) QPTAS.

The following theorem shows that the result is tight.

Theorem 8. Assuming the ETH, there exists a constant \( \omega > 0 \) such that finding a signaling scheme that provides an expected revenue at least of \((1 - \omega)OPT\) requires \( f^I(\log I) \) time, where \( I \) is the size of the problem instance. This holds even when \( v_i(\theta) > \frac{1}{2} \) for all \( i \in \mathcal{N} \) and \( \theta \in \Theta \)^8.

^8The \( \Omega \) notation hides poly-logarithmic factors.
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Supplementary Material

**Theorem 1.** The problem of computing an optimal signaling scheme does not admit a PTAS unless P = NP, even when it is restricted to the KV setting.

**Proof.** We reduce from vertex cover in cubic graphs. Formally, it is NP-Hard to approximate the minimum size vertex cover in cubic graph with an approximation $(1 + \epsilon)$, for a given constant $\epsilon > 0$ [Alimonti and Kann 2000]. Let $\eta = \epsilon/7$ and $\delta = \eta/4$. We show that for $\delta$, an $1 - \delta$ approximation to the signaling problem can be used to provide a $(1 + \epsilon)$ approximation to vertex cover in polynomial time.

Given an instance of vertex cover $(L, E)$ with nodes $\rho = |L|$ and edges $E$. For each $z \in \{1, \ldots, \rho\}$, we build an instance as follows. There are $m_z = z + \rho |E| - 1$ slots and $\lambda_j = 1$ for each $j \in \{1, \ldots, m\}$. The set of states is $\Theta = \{\theta_i\}_{i \in L}$ and the set of receivers is $N = \{r_{e,i}\}_{e \in E, i \in \{1, \ldots, \rho\}} \cup \{r_l\}_{l \in L} \cup \{r_i\}_{i \in \{1, \ldots, m + 1\}}$. The valuation of a receiver $r_{e,i}, e \in E$ and $i \in \{1, \ldots, \rho\}$, is $v_{r_{e,i}}(\theta_v) = 1$ if $v \in e$, i.e., $e$ is an edge that includes $v$, and 0 otherwise. The valuation of a receiver $r_l, l \in L$, is $v_{r_l}(\theta_v) = 1$ and $v_{r_l}(\theta_v) = 0$ for each $l' \neq l$. Moreover the valuation of a receiver $\{r_i\}, i \in \{1, \ldots, m + 1\}$ is $v_{r_i}(\theta_v) = (1 - \eta)/z$ for each state $\theta_v \in \Theta$. Finally, the prior is uniform over all the states.

Let $L^*$ be the minimum vertex cover and $z^*$ be its size. We show how to build a vertex cover of size at most $z^*(1 + \epsilon)$ from the solutions to the signaling problems instantiated with $z \in \{1, \ldots, \rho\}$. For all $z \in \{1, \ldots, \rho\}$, given a signaling scheme we recover a vertex cover $L(z)$ as follows. Take the posterior with larger sender’s utility and add to the vertex cover $L(z)$ all the vertices $l \in L$ such that the receiver $r_l$ has valuation at least $1/2(1 - \frac{3}{4}\eta)$. Then, for each edge that is not covered, we add to $L(z)$ one arbitrary adjacent vertex. It is easy to see that the resulting solution $L(z)$ is a vertex cover. Finally, the algorithm returns the smallest among the vertex covers $L(z), z \in \{1, \ldots, \rho\}$.

We show that the vertex cover $L(z^*)$ has size at most $z^*(1 + \epsilon)$, concluding the proof. First, we show that the optimal solution of the signaling problem is at least $\frac{m_z}{z^*}(1 - \frac{1}{2}\eta)$. Consider the signaling scheme with two signals $s_1$ and $s_2$ with $\phi_0(s_1) = 1$ for each $l \in L^*$ and $\phi_0(s_2) = 1$ for each $l \notin L^*$. In the posterior induced by $s_1$, the revenue is at least $\frac{m_z}{z^*}$ since all the receivers $r_{e,i}$, have expected at least $1/z^*$, while all the receivers $\{r_l\}_{l \in L^*}$ have utility at least $1/z^*$. Hence, there are at least $z^* + \rho |E| = m_z + 1$ agents with valuation at least $1/z^*$. Moreover, in the posterior induced by $s_2$ the revenue is at least $(1 - \eta)\frac{m_z}{z^*}$ since all the receivers $\{r_i\}_{i \in \{1, \ldots, m + 1\}}$ have expected valuation $(1 - \eta)/z^*$. Since $z^* \geq |E|/3$ and $\rho = \frac{|E|}{2}$, the signal $s_1$ is sent with probability at least $z^*/\rho \geq \frac{1}{2}$ and the solution has value at least $\frac{1}{2}\frac{m_z}{z^*} + \frac{1}{2}(1 - \eta)\frac{m_z}{z^*} = \frac{m_z}{z^*}(1 - \frac{1}{2}\eta)$. Hence, a $1 - \delta$ approximation algorithm for the signaling problem must return a signaling scheme with value at least $\frac{m_z}{z^*}(1 - \frac{1}{2}\eta)(1 - \delta) \geq \frac{m_z}{z^*}(1 - \frac{3}{4}\eta)$. Since the expected revenue is of the signaling scheme is at least $\frac{m_z}{z^*}(1 - \frac{3}{4}\eta)$, this signaling scheme sends a signal that induces a posterior $\xi \in \Delta_\Theta$ with revenue at least $\frac{m_z}{z^*}(1 - \frac{3}{4}\eta)$. This implies that there are at least $z^*$ receivers $r_l$ with utility greater or equal to $\frac{1}{2}(1 - \frac{3}{4}\eta)$ and that the utility of all the receivers $r_{e,i}$ is at least $\frac{1}{2}(1 - \frac{3}{4}\eta)$. We show that our algorithm recovers a vertex cover with size at most $z^*(1 + 7\eta) = z^*(1 + \epsilon)$ from this posterior. Consider the set of vertexes $L^1$ with utility at least $\frac{1}{2}(1 - \frac{3}{4}\eta)$. This set has size at most $z^*/(1 - \frac{3}{4}\eta)$ since in each state only one receiver $r_l$ has valuation 1 and all the other receivers $r_{l'}, l' \neq l$, have valuation 0. Consider the set $L^2 = L \setminus L^1$ of vertexes not in this set. We have that $\sum_{l \in L^2} \xi(\theta_l) \leq \frac{1}{4}\eta$ since $\sum_{l \in L^1} \xi(\theta_l) = \sum_{l \in L} \xi_l v_{r_l} \geq (1 - \frac{3}{4}\eta)$. Let $\bar{E}$ be the set of edges not covered by $L^1$. Since each vertex has three edges, we have that $\sum_{e \in \bar{E}} \xi_{r_{e,i}} \leq 3\frac{1}{4}\eta$ for each $i \in \{1, \ldots, m + 1\}$.

Moreover, since for each edge in $\bar{E}$, $\xi_{r_{e,i}} \geq (1 - \frac{3}{4}\eta)/z^*$, we have that $|\bar{E}| \leq \frac{3\frac{1}{4}\eta}{(1 - \frac{3}{4}\eta)/z^*}$. Then, the vertex cover built by the algorithm for $z = z^*$ includes at most $z^*(1 + \frac{1}{1 - \frac{3}{4}\eta} + \frac{3\frac{1}{4}\eta}{1 - \frac{3}{4}\eta}) \leq z^*(1 + \frac{9}{4}\eta)(1 + \frac{3}{4}\eta) \leq z^*(1 + 7\eta) = z^*(1 + \epsilon)$.

\[\square\]

**Lemma 1.** There always exists an optimal signaling scheme $\gamma$ such that $|\Xi_\pi \cap \text{supp}(\gamma)| \leq 1$ for every $\pi \in \Pi_{m+1}$.

**Proof.** To prove the result we show that, given a signaling scheme $\gamma$ that induces two posteriors $\xi, \xi' \in \Xi_\pi$, for a $\pi \in \Pi_{m+1}$, we can recover a signaling scheme $\gamma^*$ with at least the same revenue that replaces the two posteriors $\xi$ and $\xi'$ with a convex combination of them. Let $\xi, \xi' \in \Xi_\pi$ be two elements belonging to the support of an optimal signaling scheme $\gamma$. In order to show the result we introduce a posterior probability $\xi^*$ as follows: $\xi^* = z\xi_1 + (1 - z)\xi_2$ with $z = \gamma(\xi_1)/(\gamma(\xi_1) + \gamma(\xi_2))$. Since $\Xi_\pi$ is a convex polytope each convex combination of a subset of its elements belongs to it. Hence, $\xi^* \in \Xi_\pi$. Moreover, we will define a new signaling scheme $\gamma^*$ as follows: $\gamma^*(\xi^*) = \gamma(\xi_1) + \gamma(\xi_2)$ and $\gamma^*(\xi_1) = \gamma^*(\xi_2) = 0$ while $\gamma^*(\xi) = \gamma(\xi) \forall i \neq 1, 2$. To conclude the proof, we observe that the two signaling schemes
gain the same revenue. Indeed, by linearity we have: \( \gamma(\xi^*) \text{REV}(V,\xi^*) = (\gamma(\xi_1) + \gamma(\xi_2)) \text{REV}(V, z\xi_1 + (1 - z)\xi_2) = \gamma(\xi_1) \text{REV}(V,\xi_1) + \gamma(\xi_2) \text{REV}(V,\xi_2) \).

**Theorem 2.** In the KV setting, if the number of slots \( m \) is fixed, then an optimal signaling scheme can be computed in polynomial time.

**Proof.** Given a tuple \( \pi \in \Pi_{m+1} \) and \( \xi \in \Xi_\pi \), we define \( x_\pi(\theta) := \gamma(\xi) \xi(\theta) \) for each \( \theta \in \Theta \) as the posterior probability multiplied by the probability of state \( \theta \) in \( \xi \). Notice that by Lemma 1 there is at most one \( \xi \in \Xi_\pi \) belonging to the support of an optimal \( \gamma \) for each possible tuple \( \pi \in \Pi \). Thus, we can represent our optimization problem with the following LP.

\[
\begin{align*}
\max_{x \in [0,1]^{\Pi_{m+1}^{(\pi)}}} & \sum_{\pi = (i_1, \ldots, i_{m+1}) \in \Pi_{m+1}} \sum_{\theta \in \Theta} x_\pi(\theta) \sum_{j=1}^m j v_{i_{j+1}}(\theta) (\lambda_j - \lambda_{j+1}) \\
\text{s.t.} & \sum_{\pi \in \Pi_{m+1}} x_\pi(\theta) = \mu_\theta \quad \forall \theta \in \Theta \\
& \sum_{\theta \in \Theta} x_\pi(\theta) [v_{i_{j+1}}(\theta) - v_{i_j}(\theta)] \geq 0 \quad \forall \pi = (i_1, \ldots, i_{m+1}) \in \Pi_{m+1}, \ j \in \{1, \ldots, m\}. 
\end{align*}
\]

(8a)\hspace{1cm} (8b)\hspace{1cm} (8c)

Note that LP [8] is solvable in polynomial time as long as \( m \) is fixed. To conclude the proof, we show that from a solution of LP [8] we can always recover a signaling scheme setting \( \gamma(\xi_\pi) = \sum_{\theta \in \Theta} x_\pi(\theta) \) for each \( \pi \in \Pi_{m+1} \) and \( \xi_\pi(\theta) = x_\pi(\theta)/\gamma(\xi_\pi) \) for each \( \pi \in \Pi_{m+1} \) and \( \theta \in \Theta \) if \( \gamma(\xi_\pi) \neq 0 \). Moreover, given a signaling scheme \( \gamma \) we can compute a solution to LP [8] using the same relation. Finally, we show that an optimal solution to LP [8] provides the same value of the relative distribution \( \gamma \). In particular, we have

\[
\sum_{\pi \in \Pi_{m+1}} \gamma(\xi_\pi) \sum_{\theta \in \Theta} \sum_{j=1}^m j \xi_\pi(\theta) v_{i_{j+1}}(\theta)(\lambda_j - \lambda_{j+1}) = \sum_{\pi = (i_1, \ldots, i_{m+1}) \in \Pi_{m+1}} \sum_{\theta \in \Theta} \gamma(\xi_\pi) \xi_\pi(\theta) \sum_{j=1}^m j v_{i_{j+1}}(\theta)(\lambda_j - \lambda_{j+1}) = \sum_{\pi = (i_1, \ldots, i_{m+1}) \in \Pi_{m+1}} x_\pi(\theta) \sum_{j=1}^m j v_{i_{j+1}}(\theta)(\lambda_j - \lambda_{j+1})
\]

(8d)

\[\square\]

**Lemma 2.** It holds that \( OPT_{\Xi^-} = OPT \).

**Proof.** First, we observe that for each \( \xi \in \Delta_\theta \) there exists a tuple \( \pi \in \Pi_\pi \) such that \( \xi \in \Xi_\pi \), this easily follow from the fact that \( \bigcup_{\pi \in \Pi_\pi} \Xi_\pi = \Delta_\theta \). As observed before in such regions the revenue is a linear function. Thus, it is possible to decompose each posterior \( \xi \in \Xi_\pi \) by Caratheodory’s theorem as a convex combination of the vertexes of \( \Xi_\pi \) without decreasing the revenue. Formally, for each \( \pi \in \Pi_\pi \) and each posterior \( \xi \in \Xi_\pi \), there exists a distribution \( \gamma_\xi \in \Delta V(\Xi_\pi) \) such that

\[\xi(\theta) = \sum_{\xi' \in \Xi_\pi} \gamma_\xi(\xi') \xi(\theta) \quad \forall \theta \in \Theta.\]

We show that such a decomposition does not affect the final revenue. Indeed, by linearity we get the following:

\[
\sum_{\xi \in \Xi_\pi} \gamma_\xi(\xi) \text{REV}(V,\xi) = \text{REV} \left( V, \sum_{\xi \in \Xi_\pi} \gamma(\xi) \xi \right) = \text{REV}(V,\xi).
\]

To conclude the proof, we show that given the optimal distribution \( \gamma \), we can recover a distribution \( \gamma^* \in \Delta_\Xi^- \) with the same revenue. In particular, \( \gamma^* \in \Delta_\Xi^- \) is such that:

\[\gamma^*(\xi) = \sum_{\xi' \in \text{supp}(\gamma)} \gamma(\xi) \gamma_\xi(\xi') \quad \forall \xi \in \Xi_.\]

Since \( \gamma \) satisfies the consistency constraints, it easy to see that also \( \gamma^* \in \Delta_\Xi^- \) satisfies the consistency constraints. Moreover, the two distribution provide the same revenue. Indeed, we have

\[
\sum_{\xi \in \Xi_\pi} \gamma^*(\xi) \text{REV}(V,\xi) = \sum_{\xi' \in \text{supp}(\gamma)} \gamma(\xi') \sum_{\xi \in \Xi_\pi} \gamma_\xi(\xi') \text{REV}(V,\xi) = \sum_{\xi \in \text{supp}(\gamma)} \gamma(\xi) \text{REV}(V,\xi),
\]
and $OPT = OPT_Ξ$. □

**Theorem 3.** In the KV setting, if the number of states $d$ is fixed, then an optimal signaling scheme can be computed in polynomial time.

**Proof.** We first observe that the vertexes of each region $Ξ_π$ are identify by the intersection of $d - 1$ linear independent hyperplanes for each $π ∈ Π_n$. Moreover, we note that each of these vertexes is identified by a subset of the $O(n^2)$ constraints $ξ^T v_i ≥ ξ^T v_j$ for each $i ≠ j ∈ N$ and the $d$ constraint $ξ(θ) ≥ 0$ for each $θ ∈ Θ$. Hence, the total number of vertexes defining the previous discussed regions will be equal to $|Ξ'| = O((n^2 + d)^{d-1})$. Finally, we notice that, as long as $d$ is a fixed parameter, it is possible to find an optimal signaling scheme in polynomial time solving LP with set of posteriors $Ξ^*$.

□

**Lemma 3.** Given $π ∈ Π$ and $ξ ∈ Ξ_π$, it holds that $ξ ∈ V(Ξ_π)$ if and only if there exists $ℓ ∈ \{1, \ldots, d\}$ such that:

(i) $δ_{θ_{k,j}} ξ(θ_{k,j}) = \ldots = δ_{θ_{k,ℓ}} ξ(θ_{k,ℓ}) > 0$; and

(ii) $δ_{θ_{k,ℓ+1}} ξ(θ_{k,ℓ+1}) = \ldots = δ_{θ_{k,n}} ξ(θ_{k,n}) = 0$.

**Proof.** First, we show that if a posterior $ξ ∈ Ξ_π$ satisfies (i) and (ii), then $ξ ∈ V(Ξ_π)$, i.e., it is a vertex of $Ξ_π$. In particular, $ξ$ satisfies the linear independent equality $δ_{θ_{k,j}} ξ(θ_{k,j}) = δ_{θ_{k,j+1}} ξ(θ_{k,j+1})$ for each $j ∈ [ℓ - 1]$. Moreover, it satisfies $δ_{θ_{k,j}} ξ(θ_{k,j}) = 0$ for each $j ∈ \{ ℓ, \ldots, d\}$ and the simplex equality $θ_{k,j} ξ(θ_{k,j}) = 1$. Hence, $ξ ∈ Ξ_π$ is at the intersection of $d$ linear independent hyperplanes defining $Ξ_π$ and it is a vertex of $Ξ_π$. To conclude the proof, we show that each vertex $ξ ∈ Ξ_π$ satisfies (i) and (ii). In particular, we show that given a posterior $ξ ∈ Ξ_π$ such that $δ_{θ_{k,j}} ξ(θ_{k,j}) > δ_{θ_{k,j+1}} ξ(θ_{k,j+1}) > 0$ for a $j^∗ ∈ [d]$, i.e., it does not satisfies (i) and (ii), the posterior is at the the intersection of at most $d - 1$ linear independent hyperplanes. Consider the hyperplanes $δ_{θ_{k,j}} ξ(θ_{k,j}) = δ_{θ_{k,j+1}} ξ(θ_{k,j+1})$ for $j ≤ j^* - 1$. Notice that $ξ$ satisfies at most all the $j^* - 1$ inequalities of this kind. Moreover, consider all the $j > j^*$ with $δ_{θ_{k,j}} ξ(θ_{k,j}) > 0$. Let $j^**$ be the largest $j$ that satisfies this condition. By a similar argument as above, we can show that there are at most $j^** - j^* - 1$ linear inequalities $δ_{θ_{k,j}} ξ(θ_{k,j}) = δ_{θ_{k,j+1}} ξ(θ_{k,j+1})$ with $j^* + 1 ≤ j ≤ j^** - 1$. Finally, for all the $j > j^**$, the equality $δ_{θ_{k,j}} ξ(θ_{k,j}) = 0$ is satisfied. Hence, including the simplex constraint there are at most $j^* - 1 + j^** - j^* - 1 + d - j^** + 1 = d - 1$ linear independent equalities, concluding the proof.

□

**Lemma 4.** Given $λ > 0$, there exists an algorithm that finds an additive $λ$-approximation to Problem 5 in time polynomial in the size of the input, in $\frac{1}{λ}$, and in $β$.

**Proof.** Let $f(v, j)$ be the revenue when $j ∈ [n]$ bidders have expected valuation $v ∈ [0, 1]$ and all the other bidders have expected valuation 0. Moreover, given a set $E ⊆ R$ and an $x ∈ R$, let $|x|_E$ be equal to the largest element $e ∈ E$ such that $e ≤ x$. Similarly, we define $|x|_E$ be equal to the smallest element $e ∈ E$ such that $e ≥ x$.

We show that Algorithm 6 provides the desired guarantees. It is easy to see that the algorithm runs in polynomial time. Let $ξ^*$ be the optimal solution to $\max_{ξ ∈ Ξ} \rev(V, ξ) - \sum_{θ ∈ Θ} y_θ ξ(θ)$. Notice that by the definition of $Ξ$, there exists a subset of states $Θ^* ⊆ Θ$ and a value $v^*$ such that $ξ^*(θ)δ_θ = v^*$ for each $θ ∈ Θ^*$ and $ξ^*(θ) = 0$ for each $θ ∈ Θ^*$. Our first step is to show that this solution $ξ^*$ corresponds to a feasible solution to the algorithm. Let $v = [v^*]_G$ and $w = \sum_{θ ∈ Θ} [v^*]_θ |N_θ|$. Formally, we show that $Θ^*$ is a feasible subset of states for $Θ(v, d, w, \sum_{θ ∈ Θ} |N_θ|)$. In particular, it is sufficient to prove that $w = \sum_{θ ∈ Θ} [v^*]_θ |N_θ| ≥ \sum_{θ ∈ Θ} v^*δ_θ ≥ \sum_{θ ∈ Θ} v^*/δ_θ = 1$.

Since this solution is feasible, it provides a lower bound on the value $\max_{v ∈ G, j ∈ [n], w ∈ E} f(v, j) + M(v, d, w, j)$. Let $θ^∗ ∈ Θ^*$ be the state in $Θ^*$ with smallest $δ_θ$. First, we provide a bound on $v^*$. It holds $v^* ≤ δ_θ^∗$, otherwise $ξ^*(θ) = v^*/δ_θ > 1$. Moreover, $v^* ≥ δ_θ^*/d$, otherwise $\sum_{θ ∈ Θ} v^*/δ_θ < 1$. This implies that for each $θ ∈ Θ^*$, $v^*/δ_θ ≥ [v^*]_G/δ_θ ≥ (v^* - dδ_θ)/δ_θ ≥ v^*/δ_θ - d$ and $v^* ≥ v^* - ed$. Now, we can provide our lower bound

\[ \text{Given } n ≥ 0 \text{ we denote with } [n] = \{1, \ldots, n\} \]

\[ \text{Notice that the equality } δ_{θ_{k,j}} ξ(θ_{k,j}) = δ_{θ_{k,j}} ξ(θ_{k,j}) \text{ with } j > j' \text{ and } |j - j'| ≥ 2 \text{ is linear dependent from the equalities } δ_{θ_{k,j}} ξ(θ_{k,j}) = δ_{θ_{k,j+1}} ξ(θ_{k,j+1}) \text{ for each } j ∈ \{j, \ldots, j - 1\}. \]
Algorithm 1 Dynamic programming algorithm in the proof of Lemma 4

Require: $\epsilon > 0$
1: $c \leftarrow \lceil 1/\epsilon \rceil$
2: $E \leftarrow \{i/c \mid i \leq 0\}$
3: $G \leftarrow \cup_{\theta \in \Theta} \{\delta \theta i/c \mid i \leq 0\}$
4: initialize empty matrices $M$ and $\Theta$ with dimension $cd \times d \times c \times n$
5: for $v \in E$ do
6:     for $w \in E, w \geq [v/\delta \theta], E$ do
7:         $M(v, 1, w, [N \theta]) \leftarrow -y_\theta v/\delta \theta$
8:         $\Theta(v, 1, w, [N \theta]) \leftarrow \{0\}$
9:     end for
10: end for
11: for $i \in [d], w \in E, j \in [n]$ do
12:     if $M(v, i - 1, w, j) \geq M(v, \theta_i - 1, w - [v/\delta \theta], E, j - [N \theta]) - y_\theta v/\delta \theta$ then
13:         $M(v, i, w, j) \leftarrow M(v, i - 1, w, j)$
14:     else
15:         $M(v, i, w, j) \leftarrow M(v, i - 1, w - [v/\delta \theta], E, j) - y_\theta v/\delta \theta$
16:     end if
17: end for
18: end for
19: $(\hat{v}, j, \hat{w}) \leftarrow \arg \max_{v \in E, j \in [n], w \in E} f(v, j) + M(v, d, w, j)$
20: $\hat{\Theta} \leftarrow \Theta(\hat{v}, d, \hat{w}, j)$
21: $w_{\text{real}} = \sum_{\theta \in \hat{\Theta}} \hat{v}/\delta \theta$
22: for $\theta \in \hat{\Theta}$ do
23:     $\hat{\xi}(\theta) = \frac{\hat{v}}{\delta \theta w_{\text{real}}}$
24: end for
25: return $\hat{\xi}$

on $\max_{v \in G, j \in [n], w \in E} f(v, j) + M(v, d, w, j)$. In particular, the solution that takes states $\Theta^*$, $v = [v^*]_G$, and $w = \sum_{\theta \in \Theta^*} [v/\delta \theta]_E$ has value at least and has value at least
\[
\frac{f(v, \sum_{\theta \in \Theta^*} |N \theta|) - \sum_{\theta \in \Theta^*} y_\theta v}{\delta \theta} \geq f(v^*, \sum_{\theta \in \Theta^*} |N \theta|) - \epsilon dm - \sum_{\theta \in \Theta^*} y_\theta v
\]
\[
\geq f(v^*, \sum_{\theta \in \Theta^*} |N \theta|) - \sum_{\theta \in \Theta^*} y_\theta \xi(\theta) - \epsilon dm - d^2 \beta \epsilon
\]
\[
= \max_{v \in E} \text{REV}(V, \xi) - \sum_{\theta \in \Theta} y_\theta \xi(\theta) - \epsilon dm - d^2 \beta \epsilon,
\]
where the first inequality comes from Lipschitz continuity of $f(\cdot, x)$, i.e., $f(v^*, x) - f(v, x) \leq \epsilon |v^* - v|$ for each $x \in \mathbb{N}$ and $v = [v^*]_G \geq v^* - \epsilon$. The second inequality comes from $-\frac{y_\theta v}{\delta \theta} \geq -y_\theta (v^* + \epsilon - d) \geq -y_\theta v^* + \delta \theta - d \epsilon$ for each $\theta \in \Theta^*$.

To conclude the proof, we show that from a solution $(\hat{v}, j, \hat{w})$ and $\hat{\Theta}$ the algorithm find a posterior $\hat{\xi}$ with value at least $f(\hat{v}, j) + M(\hat{v}, d, \hat{w}, j) - \epsilon dm - 2 \beta de$. First, we bound the value of $w_{\text{real}}$. In particular, $w_{\text{real}} \leq 1 + \epsilon d$ since for each state $\theta$, $\hat{v}/\delta \theta - [\hat{v}/\delta \theta]_E \leq \epsilon$ and $\hat{w} \leq 1$. Hence, in the posterior $\hat{\Theta}$ all the bidders have valuation at least $\hat{v}/(1 + \epsilon d) \geq \hat{v} - \epsilon d$ and $\text{REV}(V, \xi) = f(\hat{v} - \epsilon d, \sum_{\theta \in \hat{\Theta}} |\hat{N}(\theta)|) \geq f(\hat{v}, \sum_{\theta \in \hat{\Theta}} |\hat{N}(\theta)|) - \epsilon dm$ by the Lipschitz continuity of $f(\cdot, x)$. Now, we consider the component $M(\hat{v}, d, \hat{w}, j)$. In particular, we show that $\sum_{\theta \in \Theta} -\xi(\theta)y_\theta \geq -2 \beta \epsilon$. Since $y_\theta \leq 0$ for each $\theta$, it holds
\[
\sum_{\theta \in \Theta} -\xi(\theta)y_\theta = \sum_{\theta \in \Theta} -\frac{\hat{v}}{\delta \theta w_{\text{real}}} y_\theta = M(\hat{v}, d, \hat{w}, j)/w_{\text{real}} \geq M(\hat{v}, d, \hat{w}, j)/(1 + \epsilon d) \geq M(\hat{v}, d, \hat{w}, j) + 2 \beta \epsilon,
\]
where the last inequality comes from $M(\hat{v}, d, \hat{w}, j) \leq \beta w_{\text{real}} \leq \beta (1 + \epsilon d)$ and $1/(1 + \epsilon d) \geq 1 - \epsilon d$.

To conclude, the value of the solution $\hat{\xi}$ is an additive $(\epsilon dm + d^2 \beta \epsilon + \epsilon dm + 2 \beta \epsilon)$-approximation. For $\epsilon$ small enough, we obtain the desired approximation.
Theorem 4. In the KV setting, if the bidders are single minded, then the problem of computing an optimal signaling scheme admits an (additive) FPTAS.

Proof. Our FPTAS is described in Algorithm 2.

Algorithm 2 FPTAS in the proof of Theorem 4

\textbf{Input}: parameter of the relaxed LP $\beta$, approximation factor of the approximation oracle $\lambda$, error $\eta$.

1: \textbf{Initialization}: $\rho_1 \leftarrow 0$, $\rho_2 \leftarrow 1$, $H \leftarrow \emptyset$, $H^* \leftarrow \emptyset$.
2: \textbf{while} $\rho_2 - \rho_1 > \eta$ \textbf{do}
3: \hspace{0.5cm} $\rho_3 \leftarrow (\rho_1 + \rho_2)/2$
4: \hspace{0.5cm} $H \leftarrow \{\text{posteriors relative to the violated constraints returned by the ellipsoid method on } \mathbb{F}\}$
5: \hspace{0.5cm} with objective $\rho_3$ and approximation error $\delta$
6: \hspace{0.5cm} \textbf{if} unfeasible \textbf{then}
7: \hspace{1cm} $\rho_1 \leftarrow \rho_3$
8: \hspace{1cm} $H^* \leftarrow H$
9: \hspace{0.5cm} \textbf{else}
10: \hspace{1cm} $\rho_2 \leftarrow \rho_3$
11: \hspace{0.5cm} \textbf{end if}
12: \hspace{0.5cm} $(\gamma, z) \leftarrow \text{solution to LP 18 with only posteriors in } H^*$
13: \hspace{0.5cm} \textbf{return} the solution $\tilde{\gamma}$ corresponding to the solution of the relaxed problem $(\gamma, z)$
14: \textbf{end while}

We start providing the following relaxation to LP 3 for a value $\beta \in \mathbb{R}_+$ defined in the following.

\begin{equation}
\max_{\gamma \in \Delta_{\Xi^*}, z \leq 0} \sum_{\xi \in \Xi} \gamma(\xi) \text{REV}(V, \xi) + \beta z \quad \text{s.t.}
\end{equation}

\begin{equation}
\sum_{\xi \in \Xi} \gamma(\xi)(1 - dm/\beta) + \sum_{\xi \in \Xi} [\mu_\theta - \sum_{\xi \in \Xi} \gamma(\xi)(1 - dm/\beta)]
\end{equation}

\begin{equation}
\forall \theta \in \Theta.
\end{equation}

Given a solution to $\gamma, z$ to LP 11 we can find an approximate solution to LP 3 as follows. First, notice that by the optimality of $\gamma, z$, we have $\sum_{\xi \in \Xi} \gamma(\xi) \text{REV}(V, \xi) + \beta z \geq 0$ and $z \geq -m/\beta$. Let $\mu = \sum_{\xi \in \Xi} \gamma(\xi)$ be the mean of $\gamma$. We have that $|\mu_\theta - \mu_\theta| \leq dm/\beta$ for each $\theta \in \Theta$. Consider a distribution $\tilde{\gamma}$ such that

\begin{equation}
\tilde{\gamma}(\xi) = \gamma(\xi) (1 - dm/\beta) + \sum_{\theta} [\mu_\theta - \sum_{\xi} \gamma(\xi)(1 - dm/\beta)],
\end{equation}

where $\mathbb{I}_{\xi(\theta) = 1} = 1$ iff $\xi(\theta) = 1$ and 0 otherwise. $\tilde{\gamma}$ is a feasible solution to LP 3 since $\sum_{\xi \in \Xi} \tilde{\gamma}(\xi) \mu(\xi) = \sum_{\xi \in \Xi} \gamma(\xi) \mu(\xi)(1 - dm/\beta) + \mu_\theta - \sum_{\xi} \gamma(\xi)(1 - dm/\beta)] = \mu_\theta$ and $\tilde{\gamma}(\xi) \geq 0$ for each $\xi \in \Xi^*$. Moreover, it has value at least $\text{OPT}_{\Xi^*} - dm^2/\beta$ since the distribution $\gamma$ is scaled by a factor $(1 - dm/\beta)$ and $\text{OPT}_{\Xi^*} \leq m$.

Hence, to provide an approximation to LP 3 it is sufficient to provide an approximation to LP 14 for a sufficiently large $\beta$. Since LP 11 has an exponential number of variables, the algorithm works by applying the ellipsoid method to the following dual problem.

\begin{equation}
\min_{y \leq 0, t} \sum_{\theta} y_\theta \mu_\theta + t \quad \text{s.t.}
\end{equation}

\begin{equation}
\sum_{\theta} y_\theta \xi(\theta) + t \geq \text{REV}(V, \xi) \quad \forall \xi \in \Xi^*
\end{equation}

\begin{equation}
\sum_{\theta} y_\theta \geq -\beta,
\end{equation}

where the dual variables are $y \in \mathbb{R}_d$ and $t \in \mathbb{R}$. Instead of an exact separation oracle, we use an approximate separation oracle that employs Algorithm 1 with a suitably-defined approximation $\lambda > 0$. We use a binary search scheme to find
a value $\rho^* \in [0, 1]$ such that the dual problem with objective $\rho^*$ is unfeasible, while the dual with objective $\rho^* + \eta$ is approximately feasible, for some $\eta \geq 0$ defined in the following. The algorithm requires $\log(\eta)$ steps and, at each step, it works by determining, for a given value $\rho_3$, whether there exists a feasible solution for the following feasibility problem that we call $\mathcal{P}$:

$$\sum_{\theta \in \Theta} y_{\theta} \mu_{\theta} + t \leq \rho_3$$  \hspace{1cm} (14a)$$

$$\sum_{\theta \in \Theta} y_{\theta} \xi(\theta) + t \geq \text{REV}(V, \xi) \quad \forall \xi \in \Xi^*$$  \hspace{1cm} (14b)$$

$$\sum_{\theta} y_{\theta} \geq -\beta$$  \hspace{1cm} (14c)$$

$$y_{\theta} \leq 0 \quad \forall \theta \in \Theta.$$  \hspace{1cm} (14d)$$

At each iteration of the bisection algorithm, the feasibility problem $\mathcal{P}$ is solved via the ellipsoid method. To do so, we need a separation oracle. We focus on an approximate separation oracle that returns a violated constraint. The max solution

$$\sum \rho \leq \rho^*$$

Notice that any solution to LP 3 is also a feasible solution to the previous modified problem. Since in any feasible solution $\sum_{\xi \in \Xi^*} \gamma(\xi) = 1$ and LP 17 has value at most $\rho^* + \eta$, then $OPT \leq \rho^* + \eta + \lambda$. 

Notice, that we guarantee that Algorithm 1 is called with $y_{\theta} \geq -\beta$ for each $\theta \in \Theta$. Let $\xi$ be the returned posterior. If there returned posterior has value at least $t$, the separation oracle returns the constraint relative to posterior $\xi$. Otherwise, it returns feasible. The bisection procedure terminates when it determines a value $\rho^*$ such that on $\mathcal{P}$ the ellipsoid method returns unfeasible for $\rho^*$, while returning feasible for $\rho^* + \eta$. Then, the algorithm solves a modified primal LP with only the subset of posteriors in $H^*$, where $H^*$ is the set of posteriors relative to the violated constraints returned by the ellipsoid method applied on the unfeasible problem with objective $\rho^*$. Finally, it computes a solution $\gamma$ from the solution $\gamma$ of LP 18 using 12.

Now, we prove the approximation guarantees of the algorithm. The algorithm finds a $\rho^*$ such that the problem is unfeasible, i.e., the value of $\rho_1$ when the algorithm terminates, and a value smaller than or equal to $\rho^* + \eta$ such that the ellipsoid method returns feasible, i.e., the value of $\rho_2$ when the algorithm terminates. In particular, we show that $OPT \leq \rho^* + \beta + \delta$, where $OPT$ is the value of LP 11. Since, the bisection algorithm returns that $\mathcal{P}$ is feasible with objective $\rho^* + \eta$, it finds a solution $(y, t)$ such that the approximate separation oracle did not find a violated constraint. We show that $(y, t)$ is a solution to the following LP.

$$\sum_{\theta \in \Theta} y_{\theta} \mu_{\theta} + t \leq \rho^* + \eta$$  \hspace{1cm} (16a)$$

$$\sum_{\theta \in \Theta} y_{\theta} \xi(\theta) + t \geq \text{REV}(V, \xi) - \lambda \quad \forall \xi \in \Xi^*$$  \hspace{1cm} (16b)$$

$$\sum_{\theta} y_{\theta} \geq -\beta$$  \hspace{1cm} (16c)$$

$$y_{\theta} \leq 0 \quad \forall \theta \in \Theta.$$  \hspace{1cm} (16d)$$

This holds because we have shown that, when the separation oracle returns feasible, it holds $\max_{\xi \in \Xi^*} [\text{REV}(V, \xi) - \sum_{\theta \in \Theta} y_{\theta} \xi(\theta)] \leq t + \lambda$ by the approximation guarantees of Algorithm 1, implying that all the Constraints 16b are satisfied. Moreover, when the separation oracle returns feasible all the other constraints are satisfied. Then, by strong duality the value of the following LP is at most $\rho^* + \eta$.

$$\max_{\gamma \in \Delta_{\Xi^*}, z \leq \rho_3} \sum_{\xi \in \Xi^*} \gamma(\xi) \left(\text{REV}(V, \xi) - \lambda\right) + \beta z \quad \text{s.t.}$$  \hspace{1cm} (17a)$$

$$\sum_{\xi \in \Xi^*} \gamma(\xi) \xi(\theta) - z \geq \mu_{\theta} \quad \forall \theta \in \Theta.$$  \hspace{1cm} (17b)$$

Notice that any solution to LP 3 is also a feasible solution to the previous modified problem. Since in any feasible solution $\sum_{\xi \in \Xi^*} \gamma(\xi) = 1$ and LP 17 has value at most $\rho^* + \eta$, then $OPT \leq \rho^* + \eta + \lambda$. 


Let $H^*$ be the set of posteriors relative to the constraints returned by the ellipsoid method run with objective $\rho^*$. Since the ellipsoid method with the approximate separation oracle returns unfeasible, by strong duality LP\textsuperscript{[11]} with only the variables $\gamma(\xi)$ relative to constraints in $H^*$ has value at least $\rho^*$. Moreover, since the ellipsoid method guarantees that $H^*$ has polynomial size, the LP can be solved in polynomial time. Hence, solving the following LP, i.e., the primal LP\textsuperscript{[13]} with only the variables $\gamma(\xi)$ in $H^*$, we can find a solution with value at least $\rho^*$.

\[
\max \sum_{\xi \in H^*} \gamma(\xi) \ REV(V, \xi) + \beta z \quad \text{s.t.} \quad \sum_{\xi \in H^*} \gamma(\xi) \xi(\theta) - z \geq \mu_\theta \quad \forall \theta \in \Theta. \tag{18b}
\]

To conclude the proof, notice that the algorithm provides an $dn^2/\beta + \eta + \lambda$, where the term $dn^2/\beta$ is due to the relaxation of the primal and $\eta + \lambda$ to the use of an approximate separation oracle. Since the algorithm runs in time polynomial in $\beta$, $1/\eta$ and $1/\lambda$, we can provide an arbitrary good approximations choosing sufficiently small values of $\eta$ and $\lambda$, and a sufficiently large value for $\beta$.

\[\square\]

**Lemma 5.** Given $\alpha, \varepsilon \geq 0$ and $\Xi \subseteq \Delta_\Theta$ such that $\REV(V, \cdot)$ is $(\alpha, \varepsilon)$-stable for $\Xi$, it holds $OPT_\Xi \geq (1 - \alpha)OPT - \varepsilon$.

**Proof.** Let $\gamma$ be an optimal distribution over $\Delta_\Theta$ satisfying the consistency constraints. We define $\gamma^* \in \Delta_\Xi$ as follow:

\[\gamma^*(\xi) = \sum_{\xi \in \supp(\gamma)} \gamma(\xi) \gamma_\xi(\xi) \quad \forall \xi \in \Xi,
\]

where $\gamma_\xi$ is the distribution that satisfies Definition\textsuperscript{[3]} It easy to see that $\gamma^* \in \Delta_\Xi$ satisfies the consistency constraints. Moreover, since $\REV(V, \hat{\xi})$ is $(\alpha, \varepsilon)$-stable for $\Xi$, we get:

\[
\sum_{\xi \in \Xi} \gamma^*(\xi)\REV(V, \xi) = \sum_{\xi \in \supp(\gamma)} \gamma(\xi) \sum_{\xi \in \Xi} \gamma_\xi(\xi)\REV(V, \hat{\xi}) \geq \sum_{\xi \in \supp(\gamma)} \gamma(\xi)((1 - \alpha)\REV(V, \xi) - \varepsilon) = (1 - \alpha) \sum_{\xi \in \supp(\gamma)} \gamma(\xi)\REV(V, \xi) - \varepsilon = (1 - \alpha)OPT - \varepsilon
\]

\[\square\]

**Lemma 6.** Given $\rho, \tau > 0$, let $\Xi \subseteq \Delta_\Theta$ be finite and $s := \left\lceil \frac{2(\lambda_1 m)^2}{\tau^2} \log \left( \frac{2}{\rho} \right) \right\rceil$, $OPT_\Xi, s \geq (1 - \rho|\Xi|)OPT_\Xi - \tau$.

**Proof.** We first observe that for each $\xi \in \Delta_\Theta$ we have:

\[
\REV(V, \xi) \leq \sum_{j=1}^m j(\lambda_j - \lambda_{j+1}) \leq m \sum_{j=1}^m (\lambda_j - \lambda_{j+1}) = \lambda_1 m
\]

So that by Hoeffding bound we have:

\[Pr\left( |\REV(V, \xi) - \REV(V, \xi)| \leq \tau/2 \right) \geq 1 - 2e^{\frac{-\tau^2}{2(\lambda_1 m)^2}} = 1 - \rho
\]

where the inequality is attained considering a number of samples $s = \frac{2(\lambda_1 m)^2}{\tau^2} \log \left( \frac{2}{\rho} \right)$. Moreover, by union bound and De Morgan’s laws, we get:

\[Pr\left( \bigcap_{\xi \in \Xi} \left\{ |\REV(V, \xi) - \REV(V, \xi)| \leq \tau/2 \right\} \right) \geq 1 - \rho|\Xi|
\]

Let $\gamma^* \in \Delta_\Xi$ be an optimal solution of the problem when the auctioneer can observe the actual receivers’ valuation distribution while let $\gamma_{V_s} \in \Delta_\Xi$ an optimal solution of the problem when the distribution is the empirical one. We
observe that the expected revenue provided by \( \gamma_{V} \in \Delta_{\Xi} \) will be greater or equal to the one obtained with \( \gamma^{*} \in \Delta_{\Xi} \) minus a fixed parameter with a probability of at least \( 1 - \rho|\Xi| \). Formally it holds:

\[
\sum_{\xi \in \Xi} \gamma_{V}(\xi)\text{REV}(V, \xi) \geq \sum_{\xi \in \Xi} \gamma_{V}(\xi)\text{REV}(V, \xi) - \tau \geq \sum_{\xi \in \Xi} \gamma^{*}(\xi)\text{REV}(V, \xi) - \tau / 2 \geq \sum_{\xi \in \Xi} \gamma^{*}(\xi)\text{REV}(V, \xi) - \tau
\]

Finally, we indicate with \( E[\sum_{\xi \in \Xi} \gamma_{V}(\xi)\text{REV}(V, \xi)] \) the expectation over the sampling procedure of the expected revenue archived by a solution of the LP considering the empirical receivers distribution. Finally, we have that:

\[
E[\sum_{\xi \in \Xi} \gamma_{V}(\xi)\text{REV}(V, \xi)] \geq \sum_{\xi \in \Xi} \gamma^{*}(\xi)\text{REV}(V, \xi) - \tau
\]

**Lemma 7.** Given \( \eta > 0 \) and \( q := \left[ \frac{1}{2\eta^2} \log \frac{m+1}{\eta} \right] \), it holds:

(i) \( \text{OPT}_{\Xi_{q}} \geq \text{OPT} - 2\eta m \);

(ii) if, for some \( \delta > 0 \), it is the case that \( v_{i}(\theta) > \delta \) for all \( i \in N \) and \( \theta \in \Theta \), then \( \text{OPT}_{\Xi_{q}} \geq (1 - \frac{\eta}{\delta})^{2} \text{OPT} \).

**Proof.** We show that there exists a distribution \( \gamma \in \Delta_{\Xi_{q}} \) over \( q \)-uniform posterior that provides an expected revenue that satisfies the conditions in the statement. For a \( \xi \in \Delta_{\Theta} \), let \( \xi^{\dagger} \in \Xi^{q} \) be the empirical mean of \( q \) vectors built form \( q \) i.i.d. samples drawn from the given posterior \( \xi \). In particular, each sample is obtained by randomly drawing a state of nature, with each state \( \theta \in \Theta \) having probability \( \xi(\theta) \) of being selected, and, then, a \( d \)-dimensional vector is built by letting all its components equal to \( 0 \), except for that one corresponding to \( \theta \), which is set to \( 1 \). Notice that \( \xi^{\dagger} \) is a random vector supported on \( q \)-uniform posteriors, whose expected value is posterior \( \xi \). Then, we let \( \xi_{q} \in \Delta_{\Xi_{q}} \) be such that, for every \( \xi \in \Xi_{q} \), it holds \( \xi_{q}(\xi) = \text{Pr} \{ \xi^{\dagger} = \xi \} \). In the following, given a \( \xi \in \Xi_{q} \) and a \( j \in [m + 1] \) we write \( \xi^{\dagger}v_{ij} \) without specifying that \( \pi = (i_{1}, ..., i_{m+1}) \in \Pi_{m+1} \) is the vector such that \( \xi_{q} \in \Xi_{\pi} \). Then, by Hoeffding bound we have that:

\[
Pr\left( \bigcap_{j=1}^{m+1} \{ \xi^{\dagger}v_{ij} \geq \xi^{\dagger}v_{ij} - \eta \} \right) \geq 1 - e^{-2\eta^2} = 1 - \frac{\eta}{m + 1} \forall j \in \{1, ..., m + 1\}
\]

where the equality follows from the definition of \( q \). Thanks to union bound and De Morgan’s laws we get:

\[
Pr\left( \bigcap_{j=1}^{m+1} \{ \xi^{\dagger}v_{ij} \geq \xi^{\dagger}v_{ij} - \eta \} \right) \geq 1 - \eta,
\]

Now, we prove that the revenue is \((0, 2\eta m)\)-stable over \( \Xi_{q} \).

\[
E_{\xi \sim \gamma_{q}}\left[ \text{REV}(V, \xi) \right] \geq E_{\xi \sim \gamma_{q}}\left[ \text{REV}(V, \xi) \bigcap \{ \xi^{\dagger}v_{ij} \geq \xi^{\dagger}v_{ij} - \eta \} \right] Pr\left( \bigcap_{j=1}^{m+1} \{ \xi^{\dagger}v_{ij} \geq \xi^{\dagger}v_{ij} - \eta \} \right)
\]

\[
\geq (1 - \eta)E_{\xi \sim \gamma_{q}}\left[ \text{REV}(V, \xi) \bigcap \{ \xi^{\dagger}v_{ij} \geq \xi^{\dagger}v_{ij} - \eta \} \right]
\]

\[
\geq (1 - \eta)\left( \sum_{j=1}^{m} \{ \lambda_{j} - \lambda_{j+1} \} \xi^{\dagger}v_{ij+1} - \eta \right)
\]

\[
\geq (1 - \eta)\left( \text{REV}(V, \xi) - \eta m \right)
\]

Proving that the revenue is \((0, 2\eta m)\)-stable over \( \Xi_{q} \) with \( q \geq \frac{1}{2\eta^2} \log \left( \frac{m+1}{\eta} \right) \). Hence, by Lemma 5, \( \text{OPT}_{\Xi_{q}} \geq \text{OPT} - 2\eta m \) proving the first point of the lemma.
Now, we prove that the revenue is $((1 - \frac{q}{3})^2, 0)$-stable over $\Xi_q$. In particular, it holds:

$$
\mathbb{E}_{\xi \sim \gamma_{\xi}} \left[ \text{REV}(V, \hat{\xi}) \right] \geq \mathbb{E}_{\xi \sim \gamma_{\xi}} \left[ \text{REV}(V, \hat{\xi}) \right] \bigg| \sum_{j=1}^{m+1} \{ \xi^T v_i \geq \xi^T v_i - \eta \} \right] \mathbb{P} \left\{ \sum_{j=1}^{m+1} \{ \xi^T v_i \geq \xi^T v_i - \eta \} \right\} \geq (1 - \eta)\mathbb{E}_{\xi \sim \gamma_{\xi}} \left[ \text{REV}(V, \hat{\xi}) \right] \bigg| \sum_{j=1}^{m+1} \{ \xi^T v_i \geq \xi^T v_i - \eta \} \right] \\
\geq (1 - \eta) \left( \sum_{j=1}^{m} j (\alpha_j - \alpha_{j+1}) (\xi^T v_{i,j} - \eta) \right) \\
\geq (1 - \eta) \left( (1 - \frac{q}{3}) \sum_{j=1}^{m} j (\alpha_j - \alpha_{j+1}) \xi^T v_{i,j} \right) \\
\geq \left( 1 - \frac{q}{3} \right)^2 \text{REV}(V, \xi) \, .
$$

Proving that the revenue is $((1 - \frac{q}{3})^2, 0)$-stable over $\Xi_q$ with $q \geq \frac{1}{\lambda^2} \log \left( \frac{m+1}{\eta} \right)$. Thus, by Lemma 5, $OPT_{\Xi_q} \geq (1 - \frac{q}{3})^2 OPT$ proving the second point of the lemma. \qed

**Lemma 8.** For $\lambda > 0$ and $q = \left\lceil \frac{md}{\lambda} \right\rceil$, $OPT_{\Xi_q} \geq OPT - \lambda$.

**Proof.** First, we show that the revenue is a Lipschitz continuous function in the posterior probability $\xi \in \Delta_\Theta$ with respect to the infinity norm. In particular, it holds:

$$
|\text{REV}(V, \xi) - \text{REV}(V, \xi')| \leq md \|\xi - \xi'||_{\infty} \quad \forall \xi, \xi' \in \Delta_\Theta
$$

This follows from the fact that $\text{REV}(V, \xi)$ is a piecewise linear, continuous function and the partial derivative of $\text{REV}(V, \xi)$ with respect each component of $\xi$ is bounded almost everywhere by $m$. Then, we show that each posterior $\xi$ can be decomposed in a probability distribution $\gamma_{\xi} \in \Delta_{\Xi_q}$ with a small loss of revenue. We define $\text{I}_\lambda(\xi) = \{ \xi' \in \Delta_\Theta \mid \|\xi - \xi'||_{\infty} \leq \lambda / md \}$ as the neighbourhood of a given posterior $\xi \in \Delta_\Theta$ and $\Xi(\xi) = \text{I}_\lambda(\xi) \cap \Xi_q$ its intersection with the set $\Xi_q$. It is easy to see that $\xi \in \text{co}(\Xi(\xi))$. Hence, by Caratheodory’s theorem we can decompose each $\xi$ as follow:

$$
\sum_{\xi \in \Xi(\xi)} \gamma_{\xi}(\xi')(\theta) = \xi(\theta) \quad \forall \theta \in \Theta
$$

with $\gamma_{\xi} \in \Delta_{\Xi(\xi)}$. We show now that such a decomposition will decrease the expected revenue under $\xi \in \Delta_\Theta$ of at most a fixed parameter. Formally, we have that:

$$
\mathbb{E}_{\xi \sim \gamma_{\xi}} \left[ \text{REV}(V, \hat{\xi}) \right] = \sum_{\xi \in \Xi(\xi)} \gamma_{\xi}(\xi') \text{REV}(V, \hat{\xi}) \\
\geq \sum_{\xi \in \Xi(\xi)} \gamma_{\xi}(\xi')(\text{REV}(V, \xi) - \lambda) \\
= \text{REV}(V, \xi) - \lambda,
$$

where the inequality comes from the Lipschitz continuity of $\text{REV}(V, \xi)$ and $\|\xi - \xi'||_{\infty} \leq \lambda / md$ for all $\xi \in \Xi(\xi)$. This proves that the revenue is $(0, \lambda)$-stable over $\Xi_q$. By lemma 5 we have that $OPT_{\Xi_q} \geq OPT - \lambda$ \qed

**Theorem 5.** In the RV setting, if the number of states $d$ is fixed, then the problem of computing an optimal signaling scheme admits and (additive) FPTAS.

**Proof.** Let $\eta$ be the desired approximation and let $\tau, \alpha,$ and $q$ be three suitable values defined in the following. Applying Lemma 6 for $\Xi = \Xi_q$, $\rho = \alpha / m$, $r = 2m^2 \log(2m)$, we get:

$$
\mathbb{E} \left[ \sum_{\xi \in \Xi_q} \gamma_{\xi}(\xi') \text{REV}(V, \xi) \right] \geq \left( 1 - \frac{\alpha|\Xi_q|}{m} \right) \text{OPT}_{\Xi_q} - \tau \\
\geq \text{OPT}_{\Xi_q} - \tau - \alpha|\Xi_q|, \quad \text{OPT}_{\Xi_q} \geq (1 - \frac{q}{3})^2 OPT - \lambda.
$$

18
By Lemma 8 for a value \( \lambda \) defined in the following and \( q = \left\lceil \frac{m \cdot d}{\lambda} \right\rceil \) we have that:

\[
\mathbb{E} \left[ \sum_{\xi \in \Xi_q} \gamma_{\nu}(\xi) \text{REV}(V, \xi) \right] \geq OPT_{\Xi_q} - \tau - \alpha|\Xi_q| \\
\geq OPT - \lambda - \tau - \alpha|\Xi_q| \\
= OPT - \eta, 
\]

where the last equality holds taking \( \lambda = \eta/3, \varepsilon = \eta/6, \alpha = \eta/(3|\Xi_q|) \)

\textbf{Theorem 6.} In the RV setting, if the number of slots \( m \) is fixed, then the problem of computing an optimal signaling scheme admits an (additive) PTAS.

\textit{Proof.} Let \( \nu \) be the desired approximation and let \( \tau, \alpha, \) and \( q \) be three suitable values defined in the following. Applying Lemma 6 for \( \Xi = \Xi_q, \rho = \alpha/m, \tau \) and \( s = \frac{2m^2}{\nu^2} \log \frac{2m}{\alpha} \), we get:

\[
\mathbb{E} \left[ \sum_{\xi \in \Xi_q} \gamma_{\nu}(\xi) \text{REV}(V, \xi) \right] \geq \left(1 - \frac{\alpha|\Xi_q|}{m}\right)OPT_{\Xi_q} - \tau \\
\geq OPT_{\Xi_q} - \tau - \alpha|\Xi_q|,
\]

By Lemma 7 we will have that for a value \( \eta \) defined in the following and \( q = \frac{1}{2\nu^2} \log \frac{m+1}{\eta} \) it holds:

\[
\mathbb{E} \left[ \sum_{\xi \in \Xi_q} \gamma_{\nu}(\xi) \text{REV}(V, \xi) \right] \geq OPT_{\Xi_q} - \tau - \alpha|\Xi_q| \\
\geq OPT - 2\eta m - \tau - \alpha|\Xi_q| \\
= OPT - \nu
\]

Where the last equality holds for \( \eta = \nu/6m, \tau = \nu/3, \) and \( \alpha = \nu/3|\Xi_q| \).

\textbf{Theorem 7.} In the RV setting, if \( v_i(\theta) \geq \delta \) for all \( i \in \mathcal{N} \) and \( \theta \in \Theta \) for some \( \delta > 0 \), then the problem of computing an optimal signaling scheme admits a (multiplicative) QPTAS.

\textit{Proof.} Let \( \beta \) be the desired approximation. Moreover, let \( \alpha \) and \( \eta \) be values defined in the following, \( \tau = \nu \delta \lambda_1, q = \frac{1}{2\nu^2} \log \left( \frac{m+1}{\eta} \right), \) and \( s = \frac{2m^2}{(\nu^2)^2} \log \left( \frac{2}{\alpha} \right) \). By Lemma 8 it holds

\[
\mathbb{E} \left[ \sum_{\xi \in \Xi_q} \gamma_{\nu}(\xi) \text{REV}(V, \xi) \right] \geq \left(1 - \alpha|\Xi_q|\right)OPT_{\Xi_q} - \nu \delta \lambda_1 \\
\geq \left(1 - \nu - \alpha|\Xi_q|\right)OPT_{\Xi_q},
\]

By Lemma 7 we have:

\[
\mathbb{E} \left[ \sum_{\xi \in \Xi_q} \gamma_{\nu}(\xi) \text{REV}(V, \xi) \right] \geq \left(1 - \nu - \alpha|\Xi_q|\right)OPT_{\Xi_q} \\
\geq (1 - \nu - \alpha|\Xi_q|) \left(1 - \frac{\eta}{\delta}\right)^2 OPT \\
= (1 - \beta)OPT,
\]

where the last inequality holds for \( \nu = \beta/4, \alpha = \frac{\beta}{4|\Xi_q|}, \) and \( \eta = \delta \beta/2 \).

\textbf{Theorem 8.} Assuming the ETH, there exists a constant \( \omega > 0 \) such that finding a signaling scheme that provides an expected revenue at least of \( (1 - \omega)OPT \) requires \( t^{\Omega(\log I)} \) time, where \( I \) is the size of the problem instance. This holds even when \( v_i(\theta) > \frac{1}{2} \) for all \( i \in \mathcal{N} \) and \( \theta \in \Theta \).

\textsuperscript{11}The \( \Omega \) notation hides poly-logarithmic factors.
Proof. We reduce from public signaling in elections with a k-voting rule. In particular, each receiver $i \in N$ has an utility $u_i^\theta \in [-1, 1]$ in a state $\theta$, where $u_i^\theta$ represent the difference between the utility of receiver $i$ in voting $c_0$ with respect to $c_1$. The sender’s utility is 1 if the at least $k$-voters vote for $c_0$, i.e., the induced posterior $\xi \in \Delta_\Theta$ is such that $\sum_\theta \xi(\theta)u_i^\theta \geq 0$. Otherwise, the sender’s utility is 0. See Castiglioni et al. [2020b] for a more detailed description of the problem. Castiglioni et al. [2020b] show that assuming the ETH, there exists a constant $\epsilon > 0$ such that distinguish between this two cases requires $n^{O(\log(n))}$ time:

- there exists a signaling scheme such that in any induced posterior $\xi$ at least $k$ voters have $\sum_\theta \xi(\theta)u_i^\theta \geq 0$;
- in all the posteriors $\xi$ there are strictly less than $k$ receiver with $\sum_\theta \xi(\theta)u_i^\theta \geq -\epsilon$.

To prove the theorem, we show how to reduce the $k$-voting problem to our revenue maximization problem. In particular, given an instance of $k$-voting, we build an instance of signaling with the same number of receivers. The valuation of receiver $i$ in a state $\theta$ is $v_i(\theta) = \frac{u_i^\theta}{3} + \frac{2}{3}$. Moreover, there are $m = k - 1$ slots with $\lambda_j = 1$ for each $j \in [m]$. Finally, we set the required approximation $\omega = \epsilon/2$.

We show that when the first case holds, the revenue is at least $(k - 1)\frac{2}{3}$, while in the second case it is strictly less than $(k - 1)\frac{2}{3} - \epsilon$. Hence, a $\frac{2}{3} - \epsilon/2 = 1 - \omega$ approximation to the signaling problem can be used to provide an $\epsilon$ approximation to $k$-voting. Since we provide a polynomial time reduction from $k$-voting to the revenue maximization problem, this is sufficient to prove the theorem.

soundness. Suppose that there exists a signaling scheme such that in any induced posterior $\xi$ at least $k$ voters have $\sum_\theta \xi(\theta)u_i^\theta \geq 0$. Consider the same signaling scheme in the revenue maximization problem. Then, in all the induced posteriors $\xi$ there are at least $k$ receivers with expected valuation at least $\frac{2}{3}$ and the total revenue is at least $(k - 1)\frac{2}{3}$.

completeness. Suppose that in all the posteriors $\xi$ there are strictly less than $k$ receiver with $\sum_\theta \xi(\theta)u_i^\theta \geq -\epsilon$. Notice that the revenue of a posterior is given by $(k - 1)x$, where $x$ is the k-th largest expected valuation. Hence, the maximum revenue is strictly less than $(k - 1)\frac{2}{3} - \epsilon$.\qed