Between Arrow and Gibbard-Satterthwaite: A representation theoretic approach

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Abstract

A central theme in social choice theory is that of impossibility theorems, such as Arrow’s theorem [Arr63] and the Gibbard-Satterthwaite theorem [Gib73, Sat75], which state that under certain natural constraints, social choice mechanisms are impossible to construct. In recent years, beginning in Kalai [Kal01], much work has been done in finding robust versions of these theorems, showing “approximate” impossibility remains even when most, but not all, of the constraints are satisfied. We study a spectrum of settings between the case where society chooses a single outcome (à-la-Gibbard-Satterthwaite) and the choice of a complete order (as in Arrow’s theorem). We use algebraic techniques, specifically representation theory of the symmetric group, and also prove robust versions of the theorems that we state. Our relaxations of the constraints involve relaxing of a version of “independence of irrelevant alternatives”, rather than relaxing the demand of a transitive outcome, as is done in most other robustness results.

1 Introduction

Social choice deals with the aggregation of opinions of individuals in a society into a single opinion. There are several important impossibility theorems in the field, stating that aggregation mechanisms satisfying some natural conditions, are dictatorial (dependent on the opinion of a single voter).

The first of these theorems is Arrow’s theorem. Let there be a set of \( n \) individuals, who wish to decide on a ranking of \( m \) alternatives. Each individual has their own full ranking of the alternatives. Let \( L_m \) be the set of full transitive linear order on \([m]\) and \( O_m \) be the set of all anti symmetric relations on \([m]\). A social welfare function (SWF) is a function \( f : L_m^n \rightarrow O_m \), that maps the individual rankings of the \( n \) voters into an aggregated relation.

**Definition 1.1:** A SWF \( f \) is called

- *Independent of Irrelevant Alternatives (IIA)*, if for every 2 alternatives \( a, b \), the aggregated preference between \( a \) and \( b \) depends only on the individual preferences between \( a \) and \( b \).
• Consistent, if it always returns a transitive order (is into $L_m$).

**Theorem 1.2**: (Arrow) For $m \geq 3$, every function that is consistent and IIA, and agrees with unanimous votes, is dictatorial.

Another theorem of similar flavor is Gibbard-Satterthwaite’s theorem (GS), known to be strongly connected to Arrow’s theorem. It deals with a setting in which the voters only wish to choose one of the $m$ alternatives. A social choice function (SCF) is a function $f : L^n_m \rightarrow [m]$, that maps the individual rankings of $n$ voters into an aggregated choice. GS deals with the game-theoretic notion of strategy proofness, where no voter has an incentive to misreport their true opinion and obtain a better result from her perspective.

For the formal definition of strategy-proofness, we introduce some notations. For a profile $x \in L^n_m$, $x = (x_1, ..., x_n)$ and a voter $i \in [n]$, we will denote $x = (x^{-i}, x_i)$, where $x^{-i}$ indicates the votes of all voters except the $i$'th. For $y \in L_m$, we will use $<y$ to indicate the corresponding order. We similarly define $>_y, \geq_y, \leq_y$.

**Definition 1.3**: A SCF $f$ is called strategy-proof, iff

$$\forall i \in [n], x^{-i} \in L^{n-1}_m, x_i, y \in L_m, f(x^{-i}, x_i) \geq x_i, f(x^{-i}, y)$$

i.e. no voter, under any circumstance, has an incentive to misreport their true preference.

**Theorem 1.4**: (Gibbard-Satterthwaite) for $m \geq 3$, a social aggregator $f : L^n_m \rightarrow [m]$ that is onto and strategy-proof is dictatorial.

The connection between the notions of strategy-proofness and IIA was demonstrated in [NP07, DL07], and the connection between the proofs of these theorems was demonstrated in, e.g., [Ren01]. However we are not aware of previous work which presents a single scheme that unites the different settings (SWF vs. SCF) and the different constraints (IIA vs. strategy-proofness).

In the past decade there has been a flurry of work done in providing analytical proofs of these theorems, and finding robust versions of them - i.e. showing that aggregators that almost satisfy the constraints (consistency, IIA, strategy-proofness) are close to fitting the classification (dictatorial). The relaxation of consistency in Arrow’s theorem was initiated in [Kal01] and culminated in [Mos11], which finally provided a robust version of the unmodified Arrow’s theorem. See also [Mos09, Kel10b, Kel10a]. The same was done for several examples in the judgment aggregation setting in [Neh10]. See also [Xia08].

Relaxing the strategy-proofness constraint in GS has some important computational implications. In [FKKN11] such a result was achieved for functions with $m = 3$, and in [IKM10] for neutral functions with $m > 3$. Recently [MR11] provided the final word on this theme, proving a robust version of the unmodified Gibbard-Satterthwaite theorem.

Our work continues this line of research. A word or two on the novelty of our approach. The basic and beautiful idea in Kalai’s paper [Kal01], which was followed in most of the subsequent
work, was using the IIA condition to translate a SWF to a set of Boolean functions, thus enabling the application of Fourier analysis on $\{0,1\}^n$, a technique that is very useful for proving robustness results. The robustness referred to the distance between a given SWF to a function which is consistent, i.e. where the output is always transitive. In the current paper we chose to insist on the output being of the same form as the input (e.g. being a complete order in the SWF setting), and measure robustness with respect to the number of violations of a special set of constraints.

To this end, and in order to allow a spectrum of results between SWF (Arrow) and SCF (Gibbard-Satterthwaite), we use a notion similar to IIA, called Independence of Rankings (IR) and show a robust impossibility theorem for it. IR first appeared in [DH10b], where they characterize impossibility domains for non-binary judgment aggregation and IR is their prime example.

We complete the analogue to Gibbard-Satterthwaite’s theorem by showing that the impossibility theorem for the IR definition implies an impossibility theorem for some proper definition of strategy-proofness. This definition is an adaptation of a strategy-proofness definition introduced in [DL07] for binary judgment aggregation.

Our approach leads naturally to representation theory of the symmetric group, which replaces the Abelian Fourier analysis that arises in the previous analytical works cited above. As in most applications of spectral techniques to combinatorial problems, this approach includes two components: The encoding of a combinatorial quantity as a quadratic form, and the extraction of combinatorial information from the spectral analysis of that form. In this work the algebraic encoding entailed the use of block matrices. The usage of block matrices encompasses substantial expressive power, as it enables the encoding of every Constraint Satisfaction Problem for constraints that relate to pairs of input points, on any size of alphabet. The canonical uses of the spectral method, utilizing standard 0–1 matrices, are usually limited to a certain type of constraints on alphabet of size 2, which mainly enables the treatment of notions related to expansion in graphs. In this paper the spectral analysis involves tensor algebra, allowing us to take advantage of the block structure of the matrices. This enables the extraction of the combinatorial information encoded inside the block structure.

2 Structure of the Paper

The paper is organized as follows:

- In section 3 we define the constraints we are using and state the robust impossibility theorem.
- In section 4 we provide a bird’s-eye view of the proof.
- In section 5 we recall some essentials of representation theory.
- In section 6 we present the formal structure of the proof, divided into short lemmas, which are proved in the subsequent section. The section is divided into two subsections, subsection 6.1 which deals with the case of a single voter, and subsection 6.2 for an arbitrary number of voters.
• Section 7 provides the proofs of the lemmas from section 6.

• The final robustness result is derived from the conclusions made in section 6 combined with an extension of a result by Friedgut, Kalai and Naor ([FKN02]). Section 8 provides the proof of the extension of [FKN02] with the necessary adaptations to our setting. The main analytic tool this proof uses is a hypercontractive inequality of Beckner and of Bonami, which is also adapted to our setting in subsection 8.1.

• In section 9 we introduce a strategy-proofness definition and show its connection to IR and an appropriate impossibility theorem.

3 Results

In this paper we present a robust impossibility theorem in the flavor of Arrow’s and Gibbard-Satterthwaite’s theorems. The constraint we will use is a variant of IIA. We present a single proof dealing with functions in a spectrum of ranges, from functions returning a full ranking (SWFs) to functions returning one alternative (SCFs), including a plethora of ranges in between. The result can be interpreted as a 2-query dictatorship test with full completeness.

In our setting, we deal with aggregation of rankings of \( m \) alternatives. A ranking is a permutation \( x \in S_m \). We will use the convention \( x(rank) = name \).

For presentation sake, we shall begin with the definition of the constraint when used for functions returning a full ranking (SWFs) and state the corresponding impossibility theorem without robustness. The more complicated definitions and theorems will follow.

3.1 Main Theorem

Definition 3.1: A social aggregator \( f : S^n_m \rightarrow S_m \) satisfies Independence of Rankings (IR) iff the aggregated ranking of the \( j \)’th alternative is dependent only on the individual rankings of the \( j \)’th alternative
\[
\forall x, y \in S^n_m, j \in [m], (\forall i \in [n], x_i^{-1}(j) = y_i^{-1}(j)) \Rightarrow f(x)^{-1}(j) = f(y)^{-1}(j)
\]

This constraint requires independence of rankings instead of independence of pairwise preferences required in IIA. This constraint was discussed in [DH10b], in the context of non-binary judgment aggregation.

As in IIA, this definition compares voting profiles which may differ in any number of votes. Throughout this paper, we shall use an alternative, equivalent definition, that compares inputs that differ in a single vote (as is the case in the definition of strategy-proofness):

Definition 3.2: A social aggregator \( f : S^n_m \rightarrow S_m \) satisfies Independence of Rankings (IR) iff
\[
\forall i \in [n], j \in [m], x^{-i} \in S^i_m, x_i, y_i \in S_m, x_i^{-1}(j) = y_i^{-1}(j) \Rightarrow f(x^{-i}, x_i)^{-1}(j) = f(x^{-i}, y_i)^{-1}(j)
\]
It is easy to show that these two definitions are equivalent, via a hybrid argument. The corresponding impossibility theorem is

**Theorem 3.3:** For \( m \geq 3 \), a social aggregator \( f : S^n_m \rightarrow S_m \) that is IR is either a constant function or dictatorial of the following form: there exists a voter \( i \) and a constant permutation \( y \) of the rankings such that \( f(x) = y \circ x_i \).

In [DH10b], a similar impossibility theorem was shown, using purely combinatorial arguments. Their result deals with a larger range of possible formats of input and output, yet demands further constraints on the function in question. We do not see a way to extend their techniques to achieve robustness.

### 3.2 Robust Impossibility Theorem

A robust impossibility theorem means that when the constraint is *almost* satisfied, then the function is *almost* dictatorial.

To the best of our knowledge, most previous robustness results regarding SWF’s focused on relaxation of the rationality (the transitivity of the outcome), and measure the distance to a function which is rational. We, instead, demand rationality, and relax (our variant) of IIA. There is something satisfying about relaxing this specific constraint, rather than others, as it seems to be slightly less natural than rationality and unanimity.

**Definition 3.4:** A social aggregator \( f : S^n_m \rightarrow S_m \) is called \( \epsilon - IR \) if the rate of constraints that are not satisfied is smaller than \( \epsilon \), i.e.

\[
\sum_{i \in [n], j \in [m]} Pr_{x^{-i} \in S^n_m, x_i, y_i \in S_m} \left[ (x_i^{-1}(j) = y_i^{-1}(j)) \land (f(x^{-i}, x_i)^{-1}(j) \neq f(x^{-i}, y_i)^{-1}(j)) \right] \leq \epsilon
\]

**Theorem 3.5:** For \( m \geq 3 \), a social aggregator \( f : S^n_m \rightarrow S_m \) that is \( \epsilon - IR \) is \( O(m^8\epsilon) \) close to a function that is either a constant function or dictatorial of the following form: there exists a voter \( i \) and a constant permutation \( y \) of the rankings such that \( f(x) = y \circ x_i \).

### 3.3 A Spectrum of Ranges

As stated earlier, we will also deal with a setting where the aggregated opinion is not a full ranking, but a partial ranking. Let \( H \subseteq S_m \) be a subgroup of \( S_m \). We call it a *fixing* subgroup if it consists of all permutations respecting a given partition of the \( m \) rankings into 2 or more parts. An \( H \)-social aggregator is a function \( f : S^n_m \rightarrow S_m/H \), where \( S_m/H \) refers to right cosets of \( H \) (We use this notation even though \( H \) is not a normal subgroup). Many types of functions fall under this scheme. Examples are:

- For \( H \) as the trivial group, \( H \)-social aggregators are SWFs.
- For \( H \) as the group of permutations fixing the element 1, \( H \)-social aggregators are SCFs.
• For \( H \) as the group of permutations fixing the set \( \{1, 2, 3\} \), \( H \)-social aggregators are functions returning triumvirates.

• For \( H \) as the group of permutations fixing the sets \( \{1\} \) and \( \{2, 3\} \), \( H \)-social aggregators are functions returning a president and two vice-presidents.

The definition of IR can be extended to \( H \)-social aggregators in the following manner:

**Definition 3.6:** Let \( H \subseteq S_m \) be a fixing subgroup of \( S_m \). For \( H_1 \) a right coset of \( H \) in \( S_m \), and \( j \in [m] \) define the \( j \)-profile of \( H_1 \) as the multiset \( H_1^{-1}(j) = \{ y^{-1}(j) | y \in H_1 \} \).

**Definition 3.7:** Let \( H \subseteq S_m \) be a fixing subgroup of \( S_m \). An \( H \)-social aggregator \( f \) satisfies Independence of Rankings (IR) iff the aggregated \( j \)-profile is dependent only on the individual rankings of the \( j \)'th alternative.

\[
\forall i \in [n], j \in [m], x^{-i} \in S_m^{-1}, x_i, y_i \in S_m, x_i^{-1}(j) = y_i^{-1}(j) \Rightarrow f(x^{-i}, x_i)^{-1}(j) = f(x^{-i}, y_i)^{-1}(j)
\]

**Example 3.8:** Let \( H = S_{m,1} \), the group of permutations that fix the element 1. In that case, an \( H \) social aggregator \( f \) is a function that returns a single winner in an election. When the winner of the election is \( k \), then the function returns the coset \( H^k \), which is the coset that includes all permutations that assign \( k \) to 1. The \( k \)-profile of \( H^k \) is \( |H| \) copies of 1. The \( j \)-profile of \( H^k \) for every \( j \neq k \) has \( \frac{|H|}{m-1} \) copies of each number between 2 and \( m \).

\( f \) is IR in that case if, for an alternative \( j \), when given the individual rankings of \( j \) by all voters in a voting profile \( x \), we are able to determine whether \( j \) is the winner of the election (i.e., the \( j \)-profile of \( f(x) \) is \( |H| \) copies of 1) or not (i.e. the \( j \)-profile of \( f(x) \) has \( \frac{|H|}{m-1} \) copies of each number between 2 and \( m \)).

We shall leave the exact definition of an \( \epsilon \)-IR \( H \)-social aggregator to a later part of the paper, see definition [6.4]. The impossibility theorems also extend to \( H \)-social aggregators.

**Theorem 3.9:** Let \( H \subseteq S_m \) be a fixing subgroup of \( S_m \). For \( m \geq 3 \), an \( H \)-social aggregator \( f \) that is IR is either a constant function or dictatorial of the following form: there exists a voter \( i \) and a constant permutation \( y \) of the rankings such that \( f(x) = Hy \circ x_i \).

**Theorem 3.10:** Let \( H \subseteq S_m \) be a fixing subgroup of \( S_m \). For \( m \geq 3 \), an \( H \)-social aggregator \( f \) that is \( \epsilon \)-IR is \( O_H(\text{poly}(m)\epsilon) \) close to a function that is either a constant function or dictatorial of the following form: there exists a voter \( i \) and a constant permutation \( y \) of the rankings such that \( f(x) = y \circ x_i \).
4 Structure of the proof

We give here a short exposition of the proof. For simplicity, we shall refer here to the basic form of the Main theorem (Theorem 3.3), where the function is a SWF. To simplify the notation, we shall also use in this section definition 3.1 for IR, instead of 3.2, which is the definition we shall use in the rest of the paper.

We shall treat this problem as a constraint satisfaction problem (CSP). We shall use the definition

\[
\text{Find all functions } f : S^n_m \to S_m \text{ s.t.}
\]

\[
\text{IR: } j \in [m], x, y \in S^n_m, \quad x^{-1}(j) = y^{-1}(j) \Rightarrow (f(x))^{-1}(j) = (f(y))^{-1}(j)
\]

A CSP has a generic algebraic encoding. The function \(f\) can be encoded as a function returning a vector in \(\mathbb{R}^{S^n_m}\), which is the characteristic vector of the singleton \(\{f(x)\}\). This encoding can be interpreted as a tensor \(F \in \mathbb{R}^{S^n_m \times S_m}\), with 2 indices \(x, v \in S_m\)

\[
F_{x,v} = 1_{v = f(x)}.
\]

The constraints can be algebraically encoded using a matrix that represents their truth table, or, in our case, since we want to count the number of violated constraints, the truth table of their negation. We use a matrix of matrices. For every two inputs \(x, y \in S^n_m\), the \((x, y)\)'th entry of the matrix will be a matrix in \(\mathbb{R}^{S_m \times S_m}\). This matrix will be the truth table of the negation of the constraints concerning \(x, y\) and \(j\). This encoding can also be interpreted as a tensor:

\[
\left( (L_j)_{xy} \right)_{v_x v_y} = 1_{[x^{-1}(j) = y^{-1}(j) \wedge [v_x^{-1}(j) \neq v_y^{-1}(j)]}
\]

where \(j \in [m], x, y \in S^n_m\) and \(v_x, v_y \in S_m\).

We can use these tensors in a quadratic form to count the number if unsatisfied constraints. Since \(L\) is the truth table of the negation of the constraints, the quadratic form

\[
\sum_j F L_j F^t
\]

counts the number of violated constraints.

The CSP under this encoding takes the form

\[
\text{Find all } F \in \mathbb{R}^{S^n_m \times S_m} \text{ s.t.}
\]

\[
\text{Consistency: } \forall x \in S^n_m, F_{fx} \text{ is a characteristic vector of a singleton}
\]

\[
\text{IR: } \sum_j F L_j F = 0
\]

The proof unfolds as follows:

- We show that \(L \succeq 0\) (PSD), i.e. \(\sum_j F L_j F^t \geq 0\) for every \(F\). Therefore, the functions that satisfy IR are precisely the kernel of \(L\).
• Explicitly find the kernel of $L$, using diagonalization.

• Show that all consistent functions in the kernel of $L$ are dictatorships.

As for the robustness, we will show that functions that are $\varepsilon$-IR are $L_2$ close to the kernel of $L$. We shall generalize the result of [FKN02] to prove that such functions, that are also consistent, are $L_2$ close to dictatorships.

For an $H$ social aggregator, we shall encode $f$ to return characteristic vectors of cosets of $H$, normalized so that their $L_1$ norm equals 1. We shall call such vectors $H$ coset vectors. As a tensor $F$, this encoding takes the form:

$$F_{x,v} = \frac{1}{|H|} 1_{v \in f(x)}$$

A very convenient feature of our definitions and approach is that the introduction of $H$ social aggregators does not insert any new elements to the proof. Essentially, the same Laplacian $L$ encodes the notions of IR and $\varepsilon$-IR for $H$ social aggregators, and only the consistency constraint changes. The algebraic CSP for $H$ social aggregators is:

Find all $F \in \mathbb{R}^{S_m \times S_m}$ s.t.

Consistency: $\forall x \in S_m, F_{x,*}$ is an $H$-coset vector

IR: $\sum_j F L^j F^t = 0$

5 Representation Theory

In this section we recall some basic notions of representation theory that are necessary for our proof.

A representation is a Homomorphism $\rho$ from a group $G$ to $GL_d(\mathbb{C})$, the group of complex $d$-dimensional square matrices. $d$ is called the dimension of the representation $d(\rho)$. A representation is called irreducible if it is not similar to a direct sum of 2 representations.

For a finite group, there is a one-to-one correspondence between the conjugacy classes of the group and irreducible representations (up to similarity). We shall denote the number of conjugacy classes of $G$ as $[G]$. For a conjugacy class $k \in [[G]]$, its corresponding irreducible representation will be denoted as $\rho^k$. We will sometimes consider $\rho^k$ as a function, and sometimes treat it as a vector with $|G|$ entries all of which are $d(\rho^k) \times d(\rho^k)$ matrices.

In this paper we will deal exclusively with the symmetric group or direct products of the symmetric group. It is well known that in the symmetric group one can choose a basis for which all irreducible representations (also known as irreps) have real, unitary matrices as values. Henceforth we will assume we are dealing with such a basis.

The defining representation of the symmetric group $S_m$ is the permutation representation $P$ of dimension $m$.

$$P(x)_{ij} = 1_{x(i)=j}$$
It is well known that $P = \rho^0 \oplus \rho^1$, i.e. it is the direct sum of two irreducible representations: the trivial one, which we denote by $\rho^0$, and the $(n-1)$-dimensional $\rho^1$. Specifically, if one chooses a basis for which $\rho^0, \rho^1$ are real and unitary then there exists an orthonormal $m \times m$ matrix $U$ such that

$$P(x) = U \left( \rho^0(x) \oplus \rho^1(x) \right) U^t$$

where here the $\oplus$ refers to a matrix composed of blocks.

The all ones vector spans the one dimensional eigenspace of $P$ corresponding to the trivial representation component of $P$, hence $U$ can be written in the following form (where $C$ is a $m \times (m-1)$ matrix, and the $C_i$’s are its rows)

$$U = \begin{pmatrix} \frac{1}{\sqrt{m}} & \cdots & C_1 \\ \vdots \\ \frac{1}{\sqrt{m}} & \cdots & C_m \end{pmatrix}$$

The character of a representation $\rho$, denoted by $\chi_\rho$, is the trace of the representation: $\chi_\rho(x) = \text{tr}(\rho(x))$. It is easy to see the the character of similar representations $\rho$ and $U \rho U^{-1}$ are the same.

A final tool we wish to recall is Schur’s orthogonality, which states that the vectors of the form $\rho^k_{ij}$ are orthogonal, (but not necessarily orthonormal)

$$\sum_x \rho^k_{i_1 j_1}(x) \rho^{k_2}(x)_{i_2 j_2} = \delta_{k_1 k_2} \delta_{i_1 i_2} \delta_{j_1 j_2} \frac{m!}{d(\rho^k)}$$

For a finite group $G$, these vectors form a complete orthogonal basis for the set of functions from $G$ to $\mathbb{C}$.

Schur’s orthogonality implies that the characters of irreducible representations are orthonormal: $< \chi_\rho^k, \chi_\rho^l > = \delta_{k,l}$. This implies that for a representation $\tau$ and an irreducible representation $\rho$, $< \chi_\tau, \chi_\rho >$ is the multiplicity of $\rho$ in the decomposition of $\tau$ to irreducible representations.

A complete set of irreducible representations for $G^n$ is the set of tensors of the irreps of $G$:

$$\{ \rho^\sigma = \bigotimes_{i=1}^n \rho^{\sigma_i} \}_{\sigma \in [G]^n}$$

### 5.1 Fourier transform and diagonalization

For a finite group $G$, given a function $f : G \to \mathbb{C}$, its Fourier transform at a representation $\rho$ is

$$\widehat{f}(\rho) = \mathbb{E}_{x \in G} f(x) \rho(x).$$

We shall sometimes use the abbreviated notation $\widehat{f}(k) = \widehat{f}(\rho^k)$
For such a function $f$, define a matrix $M \in \mathbb{C}^{G \times G}$ whose values are $M_{x,y} = f(x^{-1}y)$, then $M$ can be partially diagonalized (decomposed into eigenspaces) using representation theory. In a partial diagonalization, an eigenspace does not have a corresponding eigenvalue, but rather a corresponding eigenblock. Given a chosen set of basis vectors $\{v_i\}$, for an eigenspace of a matrix $A$, the $(i,j)$th entry of the corresponding eigenblock is $v_i Av_j^*$. 

In the case of $M$, each irrep $\rho^k$ defines an eigenspace spanned by the aforementioned orthogonal vectors $(\rho_{i,j}(x))_{x \in G}$. The eigenblock corresponding to $\rho^k$ is $I_d(\rho^k) \otimes \widehat{f}(k)$.

Another formulation of this is: $M$ is a $|G| \times |G|$ matrix, whose entries are given by 

$$M_{x,y} = \sum_{k \in [\|G\|]} d(\rho^k) tr \left( \left( \rho^k \widehat{f}(k) \bar{\rho}^k \right)_{x,y} \right)$$

Where $\bar{\rho}^k$ is a column vector of size $|G|$ whose $x$'th entry is the matrix $\rho^k(x)$. The multiplication $\widehat{f}(k) \bar{\rho}^k$ means multiplying the entries of $\bar{\rho}^k$ by $\widehat{f}(k)$, hence $\rho^k \widehat{f}(k) \bar{\rho}^k$ is a $|G|$ dimensional matrix whose entries are $d(\rho^k)$ dimensional matrices.

For a function $g$, $gM$ is known as the convolution of $g$ and $f$ and is denoted as $g \ast f$. The partial diagonalization discussed above shows that:

$$\widehat{g} \ast \widehat{f}(k) = \widehat{g}(k) \widehat{f}(k)$$

$$(gMg^t)_{x,y} = \sum_{k \in [\|G\|]} d(\rho^k) tr \left( \left( \widehat{g}(k) \widehat{f}(k) \bar{g}^t(k) \right)_{x,y} \right)$$

If $f$ is a characteristic function of a set $T$ of generators of $G$, then $M$ is the adjacency matrix of the Cayley graph $\Gamma(G,T)$. For convenience, we shall call such a graph a Cayley graph even when $f$ is a characteristic function of any subset $T$ of $G$, as the property of $T$ generating $G$ is irrelevant for our uses.

6 The Proof

In this section we present a more detailed version of the proof, divided into lemmas. The actual proofs of the lemmas will appear in section 7.

6.1 One Voter Functions

We begin our analysis by treating the case of a single voter, since this contains the analytical seed from which the multi-voter case grows. The combinatorial problem for a single voter is not very difficult, although if one is interested in a robustness theorem it seems that straightforward elementary techniques are insufficient.

In order to treat a social welfare functions on one voter $f : S_m \rightarrow S_m$, we construct in this section a quadratic form encoding the constraints, and, as mentioned in section 4, we will diagonalize it.
For $f : S_m \rightarrow S_m$, denote by $IR(f)$ the rate of unsatisfied constraints

$$IR(f) = \sum_{j \in [m]} Pr_{x,y \in S_m}[(x^{-1}(j) = y^{-1}(j)) \land (f(x)^{-1}(j) \neq f(y)^{-1}(j))].$$

Let $X^j$ be the matrix $X \in \mathbb{R}^{S_m \times S_m}$,

$$X^j_{xy} = 1_{x^{-1}(j) = y^{-1}(j)} = 1_{x^{-1}y(j) = j}.$$  

Let $\tilde{X}^j$ be its complement $\tilde{X}^j_{xy} = 1 - X^j_{xy}$. We will use the vector encoding described in \cite{4} $F_{x,v} = 1_{v = f(x)}$ (or the corresponding definition for $H$-social aggregators).

The following lemma describes a quadratic form in the values of $f$ that equals $IR(f)$:

**Lemma 6.1:** Let $f$ be a social aggregator and $F$ its encoding as described above. Let $L^j = X^j \otimes \tilde{X}^j$, and $L' = \sum_j L^j$, then

$$IR(f) = \frac{1}{|S_m|^2} tr(FL'F^t).$$

$X^j$ is the adjacency matrix of the Cayley graph $\Gamma(S_m, S_m,j)$, where $S_m,j$ is the subgroup of $S_m$ of permutations fixing $j$. A quadratic form based on the Laplacian of that same graph is more suitable for our purposes, because it is PSD. The Laplacian of that graph, $Y^j$, is given by

$$Y^j = (m-1)!I - X^j$$

The corresponding quadratic form is given in the following lemma. The quadratic forms given in lemmas 6.1 and 6.2 are equivalent when $F$ represents a consistent function.

**Lemma 6.2:** Let $f$ be a social aggregator and $F$ its encoding as described above. Let $L''^j = Y^j \otimes X^j$, and $L'' = \sum_j L''^j$, then

$$IR(f) = \frac{1}{|S_m|^2} tr(FL''F^t).$$

Since $X$ the adjacency matrix of a Cayley graph, it can be partially diagonalized via the representations of the symmetric group, as explained in subsection 5.1. As will be shown in the proof of the following lemma, all the information relevant to the computation of $IR(f)$ lies in $X$’s $\rho^1$ component. This leads to a simplified quadratic form, used with a different encoding for $f$. Let $g : S_m \rightarrow \mathbb{R}^{(m-1) \times (m-1)}$ be a an encoding of $f$ such that $g(x) = \rho^1(f(x))$. A vector form of $g$ is a vector $G$ whose each entry is a $m-1 \times m-1$ matrix $G_x = g(x)$. (For $H$ social aggregators, $g(x) = E_{y \in f(x)}(g(y)).$

The corresponding quadratic form is as follows.

**Lemma 6.3:** Let $f$ be a social aggregator and $G$ its encoding as described above. Let $L^j = Y^j \otimes D^j$, where $D^j = C_j^tC_j$ (See 2), and $L = \sum_j L^j$, then

$$IR(f) = \frac{1}{|S_m|^2} tr(GLG^t) \quad (3)$$
For an $H$ social aggregator, $\text{IR}(f)$ is defined as such: it averages, for pairs of inputs that agree on the ranking of $j$, the square of the $\ell_2$ distance of the characteristic vector of the $j$-profile of the outputs.

**Definition 6.4:** For an $H$-social aggregator $f : S_m^n \rightarrow S_m / H$, define $\text{IR}(f)$ to be

$$\text{IR}(f) = \sum_{i,j} \mathbb{E}_{x^{-1},x_i,y_i} \left( 1_{x^{-1}(j) = y_i^{-1}(j)} \right) \left\| \frac{n_f(x^{-1},x_i)^{-1}(j) - n_f(x^{-1},y_i)^{-1}(j)}{|H|} \right\|^2$$

Where for a multiset $S$, $n_S$ is its characteristic vector, i.e., for an element $x$, $n_S$ at the index $x$ equals the number of occurrences of $x$ in $S$.

In the following we claim that the quadratic form we have defined in 6.3 indeed equals the value $\text{IR}(f)$, as defined in definition 6.4. As usual, we only address the 1 voter case in this subsection:

**Claim 6.5:** Let $H \subseteq S_m$ be a fixing subgroup of $S_m$. Let $f$ be an $H$-social aggregator over 1 voter, and $G$ be its encoding as defined above, then

$$\text{IR}(f) = \frac{1}{|S_m|^2} \text{tr}(GLG^t)$$

We partially diagonalize $L$ in the following lemma, decomposing it to eigenspaces. For the purposes of this lemma, we shall define the operator $\tilde{\text{tr}}$ that operates on block matrices. The operator returns a matrix whose $(x, y)$'th entry is the trace of $(x, y)$'th block in the original matrix: $\tilde{\text{tr}}(M)_{xy} = \text{tr}(M_{xy})$.

**Lemma 6.6:**

$$L = \sum_{r \in [[S_m]]} d(\rho^r) \tilde{\text{tr}} \left( (\rho^r \otimes I) \tilde{L}(r) (\rho^r \otimes I)^t \right)$$

where

$$\tilde{L}(0) = I \cdot 0 , \quad \tilde{L}(r > 1) = I \otimes I \cdot \frac{1}{m}$$

$$\tilde{L}(1) = \frac{1}{m-1} \left( \frac{m-1}{m} I \otimes I - \sum_j D^j \otimes D^j \right)$$

The diagonalization of $\tilde{L}(1)$ is given by:

**Lemma 6.7:** $\tilde{L}(1)$ has 3 orthogonal eigenspaces whose dimensions are 1, $m-1$, $(m-1)^2 - m$. Denote their corresponding basis matrices as $U_0, U_1, U_2$. The corresponding eigenvalues are $0, \frac{1}{m(m-1)}, \frac{1}{m}$. The eigenvectors are vectors in $\mathbb{R}^{(m-1)(m-1)}$. When read as a $(m-1) \times (m-1)$ matrix, $U_0$ is the identity matrix.
This diagonalization proves that $L$ is PSD and hence, in light of Lemma 6.3, determines that its kernel is the space of IR functions. This is summarized in this corollary:

**Corollary 6.8:** Let $f : S_m \rightarrow S_m/H$ be an IR function, and let $g : S_m \rightarrow \mathbb{R}^{m-1 \times m-1}$ be defined by $g(x) = \rho^1(f(x))$ then there exist vectors $a$ and $b$ in $\mathbb{R}^{(m-1) \times (m-1)}$, that can be read as $(m-1) \times (m-1)$ matrices $A$ and $B$) such that

$$g_x = b \rho^0(x) + \text{tr}((aU_0^t)(I \otimes \rho^1(x))) = B + A \rho^1(x)$$

To complete the characterization of 1-voter IR functions, we present the following claim and lemma. We shall not prove the lemma as it is a consequence of the $n$ voter case.

**Claim 6.9:** Let $H$ be a subgroup of $S_m$, and let $f$ be an $H$-social aggregator and $g$ be $g(x) = \rho^1(f(x))$, then

$$\forall x, g(x)g^t(x) = M$$

When $H$ is a fixing subgroup, then $M \neq 0$.

**Lemma 6.10:** Let $g : S_m \rightarrow \mathbb{R}^{m-1 \times m-1}$ be a function of the form $g(x) = B + A \rho^1(x)$ that satisfies the constraint $\forall x, g(x)g^t(x) = M$ for some constant matrix $M \neq 0$. Then either $B = 0$ or $A = 0$.

### 6.2 Many Voter Functions

The quadratic form capturing the notion of $IR(f)$ for functions on $n$ voters is constructed using the quadratic form for 1 voter, in the following lemma.

**Lemma 6.11:** Let $H$ be a fixing subgroup of $S_m$. For a function $f : S_m \rightarrow S_m/H$, let $G$ be as before, the encoding of $f$ in terms of its $\rho^1$ component:

$$G(x) = \rho^1(f(x)) = \mathbb{E}_{y \in f(x)} \rho^1(y),$$

where the expectation with respect to $y$ refers to the case of $H$-social-aggregators. Let

$$L^{n,i,j} = I^{\otimes i-1} \otimes Y^j \otimes I^{\otimes n-i} \otimes D_i^j \quad L^{n,i} = \sum_j L^{n,i,j} \quad L^n = \sum_i L^{n,i}$$

Then the number of unsatisfied constraints is

$$IR(f) = \frac{1}{|S_m|^{n+1}} \text{tr}(GL^nG^t)$$

We can diagonalize $L^n$ based on our diagonalization of $L$.
Corollary 6.12: The diagonalization of $L^n$ is given by:

$$L^{n,i} = \frac{1}{|S_m|^n} \sum_{\bar{r} \in [S_m]^n} d(\rho^\bar{r}) \text{Tr} \left( (\rho^\bar{r} \otimes I) \hat{L}^{n,i}(\bar{r})(\rho^\bar{r} \otimes I)^\dagger \right)$$

The $\hat{L}^{n,i}$’s are derived from the 1 voter $\hat{L}$’s, as follows.

$$\left( \hat{L}^{n,i}(\rho^\bar{r}) \right)_{k_1 \ldots k_n l_1 \ldots l_n} = \prod_{t \in [n], t \neq i} \left( I_{d(\rho^r_t)} \right)_{k_t l_t} \cdot \left( \hat{L}(\rho^{r_t}) \right)_{k_t l_t}$$

The $\hat{L}^n$ coefficients are matrices which are not necessarily diagonal. The following lemma partly characterizes their diagonalization, in a manner that suffices for our needs.

Lemma 6.13:

1. If there exists any coordinate $i \in [n]$ for which $r_i > 1$ then $\hat{L}^n(\bar{r}) \succeq \frac{1}{m} I$ ($A \succeq B$ means that $A - B$ is PSD).

2. Otherwise, if there exist at least 2 coordinates $i \in [n]$ for which $r_i = 1$ then $\hat{L}^n(\bar{r}) \succeq O\left( \frac{1}{m^2} \right) I$

3. Otherwise, if there exists exactly 1 coordinate $i \in [n]$ for which $r_i = 1$ then $\hat{L}^n(\bar{r}) = \hat{L}(1)$.

   As shown in lemma 6.7, it has a 0 eigenvalue corresponding to the eigenvector $U_0$ and a smallest nonzero eigenvalue of $O\left( \frac{1}{m^2} \right)$.

4. Otherwise, $\bar{r}$ is all zeros, and $\hat{L}^n(\bar{r}) = 0$.

For a PSD matrix, its spectral gap is its smallest non-zero eigenvalue. We conclude the following from the diagonalization of $L^n$:

Corollary 6.14:

- The kernel of $L^n$, which is the set of all IR functions, consists exclusively of functions of the form $g(x_1, \ldots, x_n) = B + \sum_{i=1}^n A^i \cdot \rho^1(x_i)$

- The spectral gap of $\frac{1}{|S_m|} L^n$ is $\frac{1}{O(m^2)}$.

To finish the proof of theorem 3.3, we need to show that the intersection of the kernel of $L^n$ with the consistency constraint, includes only dictatorships. We don’t need to use the consistency constraint to its full capacity. All we need to use is the quadratic constraint that $\forall x, g(x)g^f(x) = M$, where $M$ is some constant matrix. In the proofs section, we shall show that this constraint is valid for any $H$. For instance, if $f$ is a SWF ($H$ is the trivial group), then since $\rho^1$ is unitary, $\forall x, g(x)g^f(x) = I$.

Corollary 6.15: IR functions which are consistent are dictatorships.
6.3 Robustness

By using the quadratic form, we are able to connect the combinatorial notion of $IR(f)$ to the analytical notion of the distance between $f$ and the kernel of $L^n$. This connection depends on the spectral gap of $L$:

**Corollary 6.16:** If $IR(f) \leq \epsilon$, and $g$ is the encoding of $f$ as above, then there exists a function $h$ in the kernel of $L^n$ such that $\|h - g\|_2^2 \equiv \mathbb{E}_{x \in \mathbb{G}_m} \|h(x) - g(x)\|_2^2 \leq O(m^2)\epsilon$

In corollary 6.15 we characterized $IR$ functions, using the fact that the function satisfies two constraints:

- Being IR, which a linear constraint, because it is equivalent to being in the kernel of $L$.
- Being consistent, which a quadratic constraint.

It is clear that the intersection of a linear and a quadratic constraint can contain only a few points. Indeed, we showed that consistent functions which satisfy the linear constraint are dictatorial.

For $\epsilon$-IR functions, corollary 6.16 shows that the first (linear) constraint is relaxed to being $L_2$ close to the linear constraint. We wish to show that consistent functions that are $L_2$ close to the linear constraint are $L_2$ close to being dictatorial.

In [FKNO2] a similar result was shown for Boolean functions on Boolean variables. It was shown that Boolean functions that are linear are dictatorial, and that Boolean functions that are $L_2$ close to being linear are $L_2$ close to a dictatorial function. Being Boolean is, naturally, a quadratic constraint. We adapt this theorem to our setting. From it we deduce our main theorem.

7 Proofs for section 6

7.1 Proofs of the Lemmas for 1 voter functions

Proof of lemma 6.1: This is the straightforward definition of the anti-constraints.

$$\frac{1}{|S_m|^2} \sum_{jxvy} F_{vx} L'_{xvy} F_{vy} = \frac{1}{|S_m|^2} \sum_{jxvy} F_{vx} X^j y X^j v y F_{vy} =$$

$$\frac{1}{|S_m|^2} \sum_{jxvy} 1_{v = f(x)} 1_{x^{-1}(j) = y^{-1}(j)} 1_{v^{-1}(j) \neq v_y^{-1}(j)} 1_{v = f(x)} =$$

$$\sum_j \mathbb{E}_{xy} 1_{x^{-1}(j) = y^{-1}(j)} 1_{f(x)^{-1}(j) \neq f(y)^{-1}(j)}$$

$\square$
Proof of Lemma 6.2

Recall
\[ L'' = \sum_j X_j \otimes (J - X_j) = \sum_j X_j \otimes J - X_j \otimes X_j \]
\[ L' = \sum_j ((m-1)!I - X_j) \otimes X_j = \sum_j (m-1)!I \otimes X_j - X_j \otimes X_j \]

Therefore, we need to show that
\[ F \left( \sum_j X_j \cdot J \right) F^t = F \left( \sum_j (m-1)!I \otimes X_j \right) F^t \quad (4) \]

when \( F \) is consistent.

Since \( F \) is consistent, \( \forall x, \sum_v F_{xv} = 1 \), so the left hand side of (4) is
\[ \sum_{jxyvy} F_{xvy} (X_{jxy}^j \cdot 1) F_{yvy} = \sum_{jxy} X_{xy}^j = m \cdot m! \cdot (m-1)! = m!^2 \]

Since the diagonal of \( X_j \), for every \( j \), is all ones, the right hand side of (4) is
\[ \sum_{jx} \left( (m-1)!X_{f(x)f(x)}^j \right) = \sum_{jx} ((m-1)!1) = m \cdot m! \cdot (m-1)! = m!^2 \]

\[ \square \]

Before we carry on, this is good place to recall \( U \) and \( C \) from equation 2. The orthonormality of \( U \) implies the following:

Claim 7.1:
\[ CC^t = I - \frac{J}{m} \]
\[ C^tC = I \]
\[ 1C = 0 \]

Proof of Lemma 6.3 We will need two simple claims and their corollary.

Claim:
\[ X_{xy}^j = (U \left( \rho^0(xy^{-1}) \oplus \rho^1(xy^{-1}) \right) U^t)_{jj} \]

Proof: Recall \( P \), the defining representation of \( S_m \). A permutation \( x \) has a fixed point \( j \) iff \( (P_x)_{jj} \) is 1. Therefore,
\[ X_{xy}^j = (P_{x^{-1}y})_{jj} \]

Recall that \( P_x = U(\rho^0 \oplus \rho^1)U^t \), and that \( U \) is orthonormal. Therefore
\[ X_{xy}^j = (P_{xy^{-1}})_{jj} = (U \left( \rho^0(xy^{-1}) \oplus \rho^1(xy^{-1}) \right) U^t)_{jj} \]
Claim: \[ X_{xy}^j = \left( \frac{1}{m} \rho^0(xy^{-1}) \mathbf{1}_1^t + C \rho^1(xy^{-1}) C^t \right)_{jj} \] (5)

Proof: Follows from the expansion of \( U \).

Next denote \( X_j = X_j^0 + X_j^1 \), where \( X_j^0 = \left( \frac{1}{m} \rho^0(xy^{-1}) \right)_{jj} \) and \( X_j^1 = \left( C \rho^1(xy^{-1}) C^t \right)_{jj} \).

Likewise, denote \( L_j = L_j^0 + L_j^1 \) where \( L_j^0 = Y_j \otimes X_j^0 \) and \( L_j^1 = Y_j \otimes x_j^1 \).

Corollary: \[ IR(f) = FL'F^t = \sum_j (FL_j^0F^t + FL_j^1F^t) = \sum_j FL_j^1F^t \]

Proof: The first equalities follow from the expansion of \( IR(f) \) and \( L' \). We now show that \( FL_j^0F^t = 0 \).

\[
FL_j^0F^t = \sum_{xyv_x y} \frac{1}{|S_m|^2} F_{xv_x y} \left( Y_{xy}^j \left( \frac{1}{m} \rho^0(v_x) \rho^0(v_y^{-1}) \mathbf{1}_1^t \right)_{jj} \right) F_{yv_y}
\]

Since \( F \) is consistent, and \( \rho^0_2 = 1 \), we have \( \sum_{v_x} F_{xv_x y} \rho^0(v_x) = 1 \). Therefore

\[
FL_j^0F^t = \sum_{xy} \frac{1}{|S_m|^2} 1_y \left( Y_{xy}^j \frac{1}{m} \mathbf{1}_1^t \right)_{yy} = 0
\]

It follows that

\[
IR(f) = \sum_j FL_j^1F^t = \sum_{jxyv_x y} \frac{1}{|S_m|^2} F_{xv_x y} \left( Y_{xy}^j \left( C \rho^1(v_x) \rho^1(v_y^{-1}) C^t \right)_{jj} \right) F_{yv_y}
\]

In light of the above we can define \( G \), a vector whose entries are \((m-1)\) dimensional matrices, as \( G_x = \sum_v F_{xv} \rho^1(v) \), and the quadratic form becomes

\[
IR(f) = \sum_{jxyv_x y} \frac{1}{|S_m|^2} (G_x)_{klx} (Y_{xy}^j C_{klx} C_{ylx}) (G_y)_{klx} = \sum_j \frac{1}{|S_m|^2} \text{tr} \left( G \left( Y_j \otimes D_j \right) G^t \right)
\]

This completes the Proof of Lemma 6.3.
Proof of claim 6.5: Recall definition 6.4

$$IR(f) = \sum_j \mathbb{E}_{x,y} \left( 1_{x^{-1}(j) = y^{-1}(j)} \left\| \frac{n_f(x)^{-1}(j) - n_f(y)^{-1}(j)}{|H|} \right\|_2^2 \right)$$

We will show the equivalency via two simple claims:

Claim a:

$$\frac{1}{|S_m|} tr(GL^j G^t) = \sum_j \mathbb{E}_{x,y} \left( 1_{x^{-1}(j) = y^{-1}(j)} \left\| C^j g^t(x) - C^j g^t(y) \right\|_2^2 \right)$$

Claim b:

$$\left\| \frac{n_f(x)^{-1}(j) - n_f(y)^{-1}(j)}{|H|} \right\|_2^2 = \left\| C^j g^t(x) - C^j g^t(y) \right\|_2^2$$

Proof of Claim a: Quadratic forms based on matrices which are Laplacians of graphs, such as $Y^j$, naturally differentiate values across edges of the graph. This can be seen via the following simple decomposition of $Y^j$. Define the matrix $Z^{j,x,y}$ to be the Laplacian of the graph whose vertices are the elements of $S_m$ and is either the empty graph or contains a single edge connecting $x$ with $y$ in case $x^{-1}(j) = y^{-1}(j)$.

Clearly, $Y^j = \sum_{(x,y) \in (S_m/2)} Z^{j,x,y}$. It is also easy to see that the diagonalization of the $Z$’s is given by $Z^{j,x,y} = 1_{xy^{-1}(j) = j} d^{x,y} \cdot d^{x,y}$ where $d^{x,y}$ is a vector that has 1 in $x$, -1 in $y$ and 0 otherwise.

Using this, we get

$$tr \left( GL^j G^t \right) = \sum_{(x,y) \in (S_m/2)} tr \left( G \left( Z^{j,x,y} \otimes D^j \right) G^t \right) =$$

$$\sum_{(x,y) \in (S_m/2)} 1_{xy^{-1}(j) = j} tr \left( G \left( d^{x,y} \otimes C^j \right) \cdot \left( d^{x,y} \otimes C^j \right) G^t \right) =$$

$$\sum_{(x,y) \in (S_m/2)} 1_{xy^{-1}(j) = j} \left( C^j (g(x) - g(y)) \cdot C^j (g(x) - g(y))^t \right) =$$

$$\sum_{(x,y) \in (S_m/2)} 1_{xy^{-1}(j) = j} \left( C^j (g(x) - g(y))^t, C^j (g(x) - g(y))^t \right)$$

\[ \square \]

Proof of claim b: Clearly, the normalized characteristic vector of the $j$-profile of $g(x)$ is $\frac{1}{|H|} n_f(x)^{-1}(j) = e_j (\mathbb{E}_{y \in g(x)} P_y^t)$, where $e_j$ is the $j$’th unit vector. When transforming this vector using the orthonormal matrix $U$, we get:

$$\frac{1}{|H|} n_f(x)^{-1}(j) U = \left( e_j \mathbb{E}_{y \in g(x)} P_y^t \right) U = \left( e_j U \left( 1 \oplus g^t(x) \right) U^t \right) U = e_j U \left( 1 \oplus g^t(x) \right) =$$
\[
\left( \frac{1}{\sqrt{m}} \ C_j \right) (1 \oplus g^i(x)) = \left( \frac{1}{\sqrt{m}} \ C^i g^j(x) \right)
\]

Therefore, since \( U \) is orthonormal,
\[
\left\| \frac{n_f(x)^{-1}(j) - n_f(y)^{-1}(j)}{|H|} \right\|_2^2 = \left\| \frac{n_f(x)^{-1}(j) - n_f(y)^{-1}(j)}{|H|} U \right\|_2^2 = \left\| \frac{1}{\sqrt{m}} \ C^i g^j(x) - \frac{1}{\sqrt{m}} \ C^i g^j(y) \right\|_2^2
\]

\[\square\]

**Proof of lemma 6.6**: Recall that
\[
L = \sum_j ((m - 1)! I - X^j) \otimes D^j
\]

The identity matrix can be decomposed using the irreps of \( S_m \) and Selur orthogonality.
\[
(m - 1)! I_{xy} = \frac{(m - 1)!}{m!} \left( \sum_{r \in \mathbb{S}_m} d(r^r)(x) \rho^r(y^{-1}) \right)
\]

Recall the decomposition of \( X^j \) from (5):
\[
X^j_{xy} = \frac{\rho^0(x)\rho^0(y^{-1})}{m} + C^i_j \rho^1(x)\rho^1(y^{-1})C^i_j
\]

Summing these decompositions, and using the fact that \( \sum_j D^j = \sum_j C^i_j C^i_j = C^i C = I \) yields that for a fixed pair \( x, y \), one has
\[
L_{xy} = \rho^0(x) \otimes I \cdot 0 + \sum_{k>1} d^k \tilde{t} \cdot 0
\]

as required.
Proof of lemma 6.7: Denote by $E$, a $m \times (m-1)^2$ matrix whose $j$’s row is $C_j \otimes C_j$. We need to diagonalize

$$
\sum_j D^j \otimes D^j = \sum_j (C^t_j C_j) \otimes (C^t_j C_j) = \sum_j (C^t_j \otimes C^t_j) (C_j \otimes C_j) = E^t E
$$

The nonzero eigenvalues of $E^t E$ are the nonzero eigenvalues of $E E^t$ (this can be deduced from the SVD decomposition of $E$). Recall $CC^t = I - \frac{J}{m}$. Therefore,

$$
(EE^t)_{ij} = (C_i \otimes C_i) (C^t_j \otimes C^t_j) = (C_i C^t_j)^2 = (CC^t)^2_{ij} = 
\left( \delta_{ij} - \frac{1}{m} \right)^2 = \left( 1 - \frac{2}{m} \right) \delta_{ij} + \left( \frac{1}{m} \right)^2
$$

Therefore, $EE^t = \frac{m-2}{m} I + \frac{J}{m^2}$, and its eigenvalues are $\frac{m-1}{m}$ with multiplicity 1 and $\frac{m-2}{m}$ with multiplicity $m - 1$.

We can verify that the eigenvector of $E^t E$ corresponding to the $\frac{m-1}{m}$ eigenvalue is $U_0$, which is the identity matrix parsed as a vector. We need to use two simple facts:

- For 3 matrices $A, B$ and $C$, the term $A \cdot B \cdot C^t$ when parsed as a vector, is equal to the term $(A \otimes C) \cdot B$, when $B$ is parsed as a vector.

- For a matrix $C$ whose rows are $\{C_j\}_j$, the term $\sum_j C_j \otimes C_j$ equals to the term $C^t C$ parsed as a row vector. This is because $C^t C = \sum_j C^t_j C_j$.

Therefore, we get that

$$
(E \cdot U_0)_{ij} = (C_j \otimes C_j) U_0 = C_j IC^t_j = C_j C^t_j = (CC^t)_{jj} = \left( I - \frac{J}{m} \right)_{jj} = \frac{m-1}{m}
$$

Which means that $EU_0 = 1 \frac{m-1}{m}$ and

$$
(E^t E) U_0 = E^t \cdot \frac{m-1}{m} = \frac{m-1}{m} \left( \sum_j C_j \otimes C_j \right)^t
$$

Since $\sum_j C_j \otimes C_j$ is $C^t C$ parsed as a vector and $C^t C = I$ and $U_0$ is $I$ parsed as a vector, we get that $(E^t E) U_0 = \frac{m-1}{m} U_0$.

\[ \square \]

Proof of corollary 6.8: We have shown that eigenspaces of $L$ with eigenvalue 0 are $\rho^0$ and $\rho^1 U^0$. All the other eigenvalues are positive, so $L$ is PSD, and IR functions are in the kernel of $L$.

Therefore, if $G$ is an IR function:
1. Its $\rho^0$ Fourier coefficient can be anything.

2. Its $\rho^1$ Fourier coefficient must be of the form $A(U^0)^t$ for some vector $A$. Recall that this Fourier coefficient has 4 indices, and that $U^0$ is the identity matrix, parsed as a vector, and also $A$ is some matrix parsed as a vector.

3. All of its other Fourier coefficients must be 0

Explicitly,

$$g(x)_{kl} = B_{kl} \rho^0(x) + \sum_{ts} \rho_{ts}^1(x)\delta_{st}A_{kt} = B_{kl} + \sum_t A_{kt}\rho_{tt}^1(x)$$

This means that the function $g$ is $g(x) = B + A \cdot \rho^1(x)$ (where $A$ and $B$ are parsed as $(m-1) \times (m-1)$ matrices), so $g$ is a linear function in $\rho^1(x)$, and, according to lemma 6.10 since $g$ is consistent, either $A$ or $B$ are 0.

Proof of claim 6.9: Let $M_H$ be $M_H = \mathbb{E}_{x \in H} \rho^1(x)$. Since $g$ is consistent, it must be of the form $\forall x, \exists y, g(x) = M_H \rho^1(y)$. Therefore,

$$g(x)g(x)^\dagger = M_H \rho^1(y)\rho^1(y)^\dagger M_H^\dagger = \mathbb{E}_{x \in H} \rho^1(x)\mathbb{E}_{y \in H} \rho^1(y^{-1}) = \mathbb{E}_{x,y \in H} \rho^1(xy^{-1}) = \mathbb{E}_{z \in H} \rho^1(z) = M_H$$

We now turn to show that $M_H \neq 0$ when $H$ is fixing. Denote $P_H = \mathbb{E}_{x \in H} P(x)$. Clearly $P_H = U(1 \oplus M_H)U^t$. Therefore, if $M_H = 0$, then $P_H \simeq J$. However, let $i$ and $j$ be 2 indices from 2 different parts of the partition that $H$ fixes, then clearly $(P_H)_{ii} \neq (P_H)_{ij}$, because for every $x \in H$, $P_{ij}(x) = 0$, and for some $x \in H$, $P_{ii}(x) = 1$.

Remark 7.2: Notice that when $H$ is not a fixing subgroup, it might be the case that $M_H = 0$, and then $g \equiv 0$, and every consistent $H$ social aggregator satisfies IR. This is the case, for instance, when $H$ is the group of even permutations.

7.2 Proofs of Lemmas for many voter functions

Proof of lemma 6.11: We show that applying the $n$ voter quadratic form is similar to applying the 1 voter quadratic form but over pairs of inputs that differ in only one voter:

$$GL^{n-i}G^t = \sum_{x,y} G(x) \left( I^{\otimes i-1} \otimes Y^j \otimes I^{\otimes n-i} \otimes D^j \right) G^t(y) =$$
\[ \sum_{x,y} G(x) \left( \prod_{t=1}^{i-1} I_{x_t,y_t} \otimes Y^j \prod_{t=i+1}^{n} I_{x_t,y_t} \otimes D^j \right) G(y^{-1}) = \]
\[ \sum_{x^{-i},x_i,y_i} G(x^{-i}, x_i, y_i) \left( Y^j \otimes D^j \right) G(x^{-i}, y_i) \]

From here, the proof follows exactly in the path of the proof of claim 6.5.

\[\square\]

**Proof of lemma 6.13:**

1. Let \( i \) be such that \( r_i > 1 \), then, by corollary 6.12

\[ \hat{L}^{n,j} (\bar{r}) = \frac{1}{m} \bigotimes_{k=1}^{n} I_{d(r_k)} \otimes I_{m-1} \]

Since \( \hat{L}^n(\bar{r}) = \sum_j \hat{L}^{n,j} (\bar{r}) \) and for every \( j, \bar{r}, \hat{L}^{n,j} (\bar{r}) \succeq 0 \), we have

\[ \hat{L}^n(\bar{r}) = \hat{L}^{n,i} (\bar{r}) + \sum_{j \neq i} \hat{L}^{n,j} (\bar{r}) \succeq \hat{L}^{n,i} (\bar{r}) \]

2. We shall focus only on the case where there are exactly 2 distinct \( i \) and \( j \) such that \( r_i = r_j = 1 \), as this case produces the minimal eigenvalue. Indeed, assume that there are \( k > 2 \) such indices \( i_1, i_2, ... i_k \) for which \( r_{i_j} = 1 \), and denote \( r' \) to be \( 1 \) in \( i_1 \) and \( i_2 \) and \( 0 \) otherwise, then \( \hat{L}^{n,i_1} (r) + \hat{L}^{n,i_2} (r) = \hat{L}^n (r') \otimes I_{m-1} \). Since the rest of the terms \( \hat{L}^{n,i_k} (r) \) are PSD, The minimal eigenvalue of \( \hat{L}^n \) is at least as large as the minimal eigenvalue of \( \hat{L}^n (r') \).

Recall the diagonalization of \( \hat{L}(1) \) (from lemma 6.7)

\[ \hat{L}(1) \succeq \frac{1}{m(m-1)} \left( I \otimes I - \frac{1}{m-1} U^0 U^{0t} \right) \]

From this, it is easy to deduce the following:

\[ \hat{L}^n (\bar{r}) = \hat{L}^{n,i} (\bar{r}) + \hat{L}^{n,j} (\bar{r}) \succeq \frac{1}{m(m-1)} \left( 2I \otimes I - \frac{1}{m-1} \left( A^i A^{i^t} + A^j A^{j^t} \right) \right) \] (6)

Where \( A^i \) is a \((m-1)^3 \times (m-1)^3 \) matrix of the form \( A_{(pqst)} = \delta_{pq}\delta_{st} \) \( p,q,s \) and \( t \) are indices going from 1 to \( m-1 \). \((pqst)\) forms the row index of \( A^i \) and \( t \) is its column index. Likewise, \( A^j \) is a \((m-1)^3 \times (m-1)^3 \) matrix of the form \( A_{(pqst)} = \delta_{ps}\delta_{qt} \). Denote the right hand side of (6) as \( Q \).

Denote \( B = A^i + A^j \) and \( C = A^i - A^j \). It is easy to see that \( A^i A^{i^t} + A^j A^{j^t} = \frac{1}{2} \left( BB^t + CC^t \right) \). The following are easy to verify:

- \( B^t C = C^t B = 0 \)
• $BB^t = 2m \cdot I$
• $CC^t = (2m - 4) \cdot I$

Therefore, we may deduce that the columns of $B$ and the columns of $C$ are orthogonal eigenvectors of $A^tA^t + A^tA^j$ with eigenvalues $m$ and $m - 2$, respectively. Plugging this into the expression for $Q$, we get that the minimal eigenvalue of $Q$, corresponding to the columns of $B$, is $\frac{1}{m(m-1)}(2 - \frac{m}{m-1}) = \frac{m^2 - 2}{m(m-1)^2}$.

**Important Note:** Notice that if $m = 2$, $Q$ has eigenvalues equal to 0, and therefore we cannot deduce that the function is a dictatorship for $m = 2$.

Items 3 and 4 are trivial.

**Proof of corollary 6.15:** We need to show that only one of $B, A_1, ..., A_n$ is not zero. We begin by showing that w.l.o.g., we may assume that $E(xg(x) = 0$ and therefore $B = 0$. Indeed, we introduce a dummy variable $y \in S_m$ and define

$$g'(x, y) = g(xy^{-1})\rho^1(y)$$

Where for $x \in S_m^n$ we denote $xy = (x_1y_1, x_2y_2, ..., x_ny_n)$. Note that $E_xg'(x, y) = 0$ because

$$E_xg'(x, y) = E_{x,y}g'(x, y) = E_xg(x)E_y\rho^1(y) = 0$$

and that $IR(g') = IR(g)$. Assume the claim is true for $g$ such that $Eg = 0$, apply it to $g'$ to get that either $g'(x, y) = A_i\rho^1(x_i)$ for some $i$, or $g'(x, y) = B\rho^1(y)$. In the first case, it follows that $g(x) = A_i\rho^1(x_i)$, and in the second case, it follows that $g(x) = B$.

We now assume that $B = 0$. Recall that by claim 6.9 we have $g(x)g(x)^t = M$ for some $M \neq 0$. On the other hand,

$$g(x)g(x)^t = \sum_{ij} A_i^t \rho^1(x_i)\rho^1(x_j)A^t$$

The summand for $i, j$, translates to

$$(A_i^t \rho^1(x_i)\rho^1(x_j)A^t)_{tu} = \sum_{pq,w,s} p_{pq}^1(x_i)p_{ws}^1(x_j)(A_{tp}^iA_{wu}^j)\delta_{qs}$$

From this expansion we may deduce the Fourier coefficient of $gg^t$ at $\vec{r}$ where $\vec{r}$ is 1 at $i$ and $j$ and 0 otherwise:

$$\hat{gg}^t_{tpwuqs}(\vec{r}) = (m - 1)^2 \left( \left( A_{tp}^iA_{wu}^i\delta_{qs} \right) + \left( A_{tp}^jA_{wu}^j \delta_{qs} \right) \right)$$

Equation (7) implies that $A_{tp}^iA_{wu}^i = -A_{tp}^jA_{wu}^j \neq 0$. Therefore, $A_{tp}^j \neq 0$ and $\hat{gg}^t_{tpwpq}(\vec{r}) \sim A_{tp}^iA_{tp}^j \neq 0$
Proof of corollary 6.16:

The result follows from the following general statement regarding quadratic forms.

Claim: Let $K$ be a finite set. Equip the linear space $\mathbb{R}^K$ with the $L_2$ metric $d(u,v) = \mathbb{E}_{x \in K} (v_x - u_x)^2$. Let $M \in \mathbb{R}^{K \times K}$ be a PSD matrix with spectral gap $\lambda$, then for any vector $u$,

$$\frac{1}{|K|} uM u^t \leq \lambda d(u, \ker(M))$$

where $\ker(M)$ is the kernel of $M$ and $d(u, \ker(M))$ is the minimal distance of $u$ from an element of $\ker(M)$ according to $d$.

Proof: Assume that $M$ has eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{|K|}$, with corresponding eigenvectors $v_1, \ldots, v_{|K|}$, and that $s$ is the first index for which $\lambda_s > 0$. Denote by $\hat{u}_i$ the projection of $u$ on $v_i$. Clearly, the distance of $u$ from the kernel of $M$ is $\frac{1}{|K|} \sum_{i=1}^{|K|} \hat{u}_i^2$, and the value of the quadratic form on $u$ is

$$u^t M u = \sum_{i=1}^{|K|} \hat{u}_i^2 \lambda_i = \sum_{s} \hat{u}_s^2 \lambda_s \leq \lambda_s \left( \sum_{s} \hat{u}_s^2 \right)$$

In our case, we have shown in lemma 6.11 that $IR(f) = \frac{1}{|S_m|^m} tr(GLG^t)$. When the elements of $G$ are parsed as vectors instead of $(m-1) \times (m-1)$ matrices, this translates to

$$IR(f) = \frac{1}{|S_m|^m} G \cdot (I \otimes L) G^t$$

Since the spectral gap of $\frac{1}{|S_m|^m} L$ is $O\left(\frac{1}{m^2}\right)$, so is the spectral gap of $\frac{1}{|S_m|^m} I \otimes L$.

Therefore, if $IR(f) \leq \epsilon$, then there exists $h$ in the kernel of $I \otimes L$ such that $\|G-h\|_2 \leq O(m^2)\epsilon$ (when the elements of $G$ and $h$ are parsed as vectors).

8 Adapted version of [FKN02]

For the sake of self-containedness, we present here the proof from FKN with minor modifications, needed for the application of the theorem to our setting.

The adapted theorem goes as follows

**Theorem 8.1:** Let $\text{Lin}(S^n_m)$ be the space of functions of the form $\sum_i A^i p^1(x_i)$. Let $g : S^n_m \rightarrow \mathbb{R}^{m-1 \times m-1}$ be a function such that
• $\mathbb{E}g = 0$

• There exists a matrix $M$ such that $\text{tr}(M) = 1$ and $\forall x \in S^n_m, g(x)g^t(x) = M$.

• $\|g - \text{Lin}(S^n_m)\|_2^2 \leq \tau$.

then there exists $i$ such that $\mathbb{E}\|g - A^i\rho^1(x_i)\|_2^2 \leq O(m^5\tau)$.

Before we carry on with the proof of theorem 8.1 we shall show how to apply it to get the proof of the main theorem:

**Proof of theorem 3.10**: Let $f$ be an $e$-IR H social aggregator and let $g$ be $g(x) = \rho^1(f(x))$, as before. We assume, w.l.o.g., that $\mathbb{E}g = 0$ (see the proof of 6.15). Since $g$ is consistent, by claim 6.9 it holds that there exists some matrix $M$ such that for every $x$, $g(x)g^t(x) = M$. Let $\text{K} = \text{tr}(M)$. By corollary 6.14 we have $\mathbb{E}\|g - \text{Lin}(S^n_m)\|_2^2 \leq O(m^2\epsilon)$. Therefore, we may apply theorem 8.1 on $\frac{1}{\text{K}}g$ and get that there exists $i$ such that $\mathbb{E}\|g(x) - A^i\rho^1(x_i)\|_2^2 \leq O(\text{K}m^7\epsilon)$. This is almost what we need, except that we have no guarantee that $A^i\rho^1(x_i)$ is consistent. However, we show in lemma 8.2 that we can round it to a consistent function without losing more than a multiplicative constant factor in the distance.

The value of $\text{K}$ ranges between 1 (for SCF’s) and $m$ (for SWF’s).

Next we show, as promised, that we can ”round off” our approximation to an approximation which is consistent without losing too much.

**Lemma 8.2**: Let $g : S^n_m \rightarrow \mathbb{R}^{(m-1) \times (m-1)}$ be a function such that there exists a matrix $M$ such that $\text{range}(g) \subseteq M \cdot \text{range}(\rho^1)$. Assume that there exists a function $h : S^n_m \rightarrow \mathbb{R}^{(m-1) \times (m-1)}$, where $h(x) = A\rho^1(x_i)$ for some $i$, and $\|g - h\|_2 \leq \delta$, then there exists a function $h' = A'\rho^1(x_i)$ such that $\text{range}(h') \subseteq M \cdot \text{range}(\rho^1)$, and $\|g - h'\|_2 \leq 2\delta$

**Proof**: Assume that there does not exist $A' \in M \cdot \text{range}(\rho^1)$ such that $\|A - A'\|_2 \leq \delta$, then

$$|g - h|_2 = |g\rho^1(x_i^{-1}) - h\rho^1(x_i^{-1})|_2 = |g\rho^1(x_i^{-1}) - A|_2 \geq d(M \cdot \text{range}(\rho^1), A) \geq \delta$$

which is a contradiction. Therefore, there exists such $A'$. Define $h'$ as above $h'(x) = A'\rho^1(x_i)$, then

$$|g - h'|_2 \leq |g - h|_2 + |h - h'|_2 \leq 2\delta$$

**Proof of theorem 8.1**

Let $h$ be the projection of $g$ on $\text{Lin}(S^n_m)$ and let

$$q = g - h$$

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and
\[ r = f \cdot f^t - M \]
Since \( E_g = 0 \), there exist \( A^i \)'s such that \( h = \sum_i A^i \rho_1(x_i) \). We will show that since \( h \cdot h^t \) is close to \( M \), \( r \) is typically close to 0, and we will deduce some information on the \( A^i \)'s.

In the proof of corollary 6.13 we used the fact that when \( \epsilon = 0 \), \( r \equiv 0 \). For the case when \( \epsilon \) is positive, we will try and show that \( r \) is close to the 0 function.

Remark 8.3: Note that the entries of \( r \) are of the form
\[
r_{k,l}(x) = \left( C + \sum_{i,j} A_i \rho_1(x_i) \rho_1^t(x_j) A_j^t \right)_{k,l}
\]
Which makes functions of degree 2. For a formal definition of the degree of a function, see definition 8.11 to come in subsection 8.1

Remark 8.4: In the following lemma, we use a constant \( C = \frac{1}{m} \) that is defined in corollary 8.13. We shall call it \( C \) from now on so that it will be easier to trace its origin.

Lemma 8.5:
\[ E(r^2) \leq K \epsilon = 108(m - 1)^4 C^8 \epsilon. \]

Corollary 8.6: There exists \( i \) such that \( \|A^i\|^2 \geq 1 - \left( 1 + \frac{(m-1)K}{1-\epsilon} \right) \epsilon = 1 - O(m^3) \epsilon. \)

Clearly, this corollary implies theorem 8.1.

Proof of corollary:
Because of the orthogonality of \( h \) and \( q \), and because \( E\|g\|^2 = 1, E\|h\|^2 = 1 - \epsilon. \)

Also, because of the orthogonality of the \( \rho_1(x_i) \)'s,
\[
E\|h\|^2 = \sum_i E\|A^i \rho_1(x_i)\|^2 = \sum_i \|A^i\|^2
\]
\[
E\|r\|^2 \geq \sum_{i \neq j} E\|A^i \rho_1(x_i)(A^j \rho_1(x_j))^t\|^2
\]
Expanding this expression we get:
\[
E\|A^i \rho_1(x_i)(A^j \rho_1(x_j))^t\|^2 =
\]
\[
\mathbb{E}_{x_1 x_2} \sum_{\alpha \beta \gamma \eta \beta \gamma \eta_2} A_{i \beta}^j \rho_{\beta \gamma}^1 (x_i) \rho_{\eta \gamma}^1 (x_j) A_{i \mu}^j A_{i \beta \eta} \rho_{\beta \gamma}^1 (x_i) \rho_{\eta \gamma}^1 (x_j) A_{i \mu}^j = \\
\frac{1}{(m - 1)^2} \sum_{\alpha \mu \beta \gamma \eta \beta \gamma \eta_2} A_{i \beta}^i A_{j \mu}^j A_{i \alpha} \rho_{\beta \gamma}^1 (x_i) \rho_{\eta \gamma}^1 (x_j) A_{i \mu}^j = \\
\frac{1}{m - 1} \| A_i^j \|_2^2 \| A_j^i \|_2^2
\]

Therefore, if \( \max \| A_i^j \|_2^2 = t \)

\[
(\mathbb{E} \| h \|_2^2)^2 = (1 - \epsilon)^2 = \sum_i \| A_i^j \|_2^2 \| A_j^i \|_2^2 + \sum_{i,j, i \neq j} \| A_i^j \|_2^2 \| A_j^i \|_2^2 \leq \\
(1 - \epsilon) t + (m - 1) \mathbb{E} \| r \|_2^2 \leq (1 - \epsilon) t + (m - 1) K \epsilon
\]

which gives the desired bound on \( t \).

\[\square\]

**Proof of Lemma 8.5**: The proof consists of two parts: first we will show that typically \( r \) obtains values close to 0. Then we will use a hypercontractive estimate due to Beckner and Bonami to bound higher moments of \( r \) in terms of its second moment showing that its tail decays fast enough.

**Lemma 8.7**: Let \( 0 < \alpha < 1/4 \) be a constant to be chosen later. Let

\[
p = \text{Prob}(\| r \|_2^2 > \alpha^2).
\]

Then

\[
p \leq \frac{16 (m - 1)^2 \epsilon}{\alpha^2}.
\]

We defer the proof of this lemma for the moment.

**Lemma 8.8**: 

\[
E(\| r \|_2^2) \leq \frac{(m - 1)^2 \alpha^2}{1 - 4(m - 1) C^4 \sqrt{\epsilon/\alpha}}.
\]

Choosing the optimal value of \( \alpha \) (which is \( 6(m - 1) C^4 \sqrt{\epsilon} \)) immediately proves Lemma 8.5.

So, to finish the proof we now present the proofs of Lemmas 8.7 and 8.8.

**Proof of Lemma 8.7**: Recall that \( q = g - h \) and that \( \mathbb{E} \| q \|_2^2 = \epsilon \). Using the fact that \( gg^t = M \) yields

\[
r = hh^t - M = qq^t - gg^t - qq^t
\]
By the triangle inequality,

\[ \|r(x)\|_2 \leq \|r(x)\|_1 \leq \|q(x)q^T(x)\|_1 + 2\|g(x)q^T(x)\|_1 \]

We shall prove in claim 8.9 that, for \(d \times d\) matrices \(A\) and \(B\), \(\|AB\|_1 \leq d\|A\|_2\|B\|_2\). This implies

\[ \|r(x)\|_2 \leq (m-1)(\|q(x)\|_2^2 + 2\|g(x)\|_2\|q(x)\|_2) \]

denote \(t = \|q(x)\|_2\). We use that fact that \(\|g(x)\|_2^2 = \text{tr}(g(x)g^T(x)) = \text{tr}(M) = 1\)

\[ \|r(x)\|_2 \leq (m-1)(2t + t^2) \]

A simple analysis shows that if \(t < \alpha/(4(m-1))\), then \(\|r(x)\|_2 < \alpha\). Hence by Markov’s inequality

\[ \Pr[\|r(x)\|_2 > \alpha] \leq \Pr[\|q(x)\|_2^2 > (\alpha/(4(m-1)))^2] \leq \frac{(4(m-1))^2 \epsilon}{\alpha^2} \]

\[ \square \]

**Claim 8.9:** The following is the version of Cauchy-Schwartz that we used in the previous proof.
Let \(A\) and \(B\) be two \(d\) dimensional real matrices, then

\[ \|AB\|_1 \leq d\|A\|_2\|B\|_2 \]

**Proof:** Let \(J\) be the all ones matrix. It is easy to see that for a matrix \(A\), \(\|A\|_1 =\langle A, J \rangle\). \(J\) can be decomposed to a sum of \(d\) permutation matrices \(P^1, ... P^d\). Then,

\[ \|AB\|_1 = \langle AB, J \rangle = \sum_{i=1}^{d} \langle AB, P^i \rangle = \sum_{i=1}^{d} \langle A, B^i P^i \rangle \leq \sum_{i=1}^{d} \|A\|_2 \|B^i P^i\|_2 = d\|A\|_2\|B\|_2 \]

When \(A = B = J\), this inequality is tight.

\[ \square \]

**Proof of Lemma 8.8:** For convenience of notation let \(X = E\|r\|_2^2\), \(X_{ij} = E(r_{ij}^2)\) and \(Y_{ij} = E(r_{ij}^4)\). Also denote \(p_{ij} = P(|r_{ij}| \leq \alpha)\). In corollary 8.13 we show a Beckner type inequality of the form:

\[ Y_{ij} \leq C^8 X_{ij}^2 \]

(Notice that, as mentioned in remark 8.3, the entries of \(r\) are real functions of degree 2 with Fourier coefficients only in \(\rho^0\) and \(\rho^1\), and therefore conform to the requirements of corollary 8.13) Using this we obtain

\[ X = \sum_{ij} X_{ij} = \sum_{ij} E(r_{ij}^2) = \sum_{ij} ((1-p_{ij})E(r_{ij}^2|r_{ij}^2 \leq \alpha^2) + p_{ij}E(r_{ij}^2|r_{ij}^2 > \alpha^2)) \leq \]

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\[
\sum_{ij} \left( (1 - p_{ij}) \alpha^2 + p_{ij} \sqrt{E(r_{ij}^4 | r_{ij}^2 > \alpha^2)} \right) \leq \\
\sum_{ij} \left( \alpha^2 + p_{ij} \sqrt{\frac{Y_{ij}}{p_{ij}}} \right) \leq \\
\sum_{ij} (\alpha^2 + \sqrt{p_{ij} C^4 X_{ij}}) \leq \\
(m - 1)^2 \alpha^2 + \sqrt{p C^4 X} \leq \\
(m - 1)^2 \alpha^2 + 4(m - 1) \frac{\sqrt{\epsilon}}{\alpha} C^4 X
\]

This yields
\[
X \leq \frac{(m - 1)^2 \alpha^2}{1 - 4(m - 1) C^4 \sqrt{\epsilon/\alpha}}.
\]

\[\square\]

8.1 Beckner’s inequality

This subsection is an adaptation of results from \cite{Wol07} for our case.

8.1.1 Introduction to hypercontractivity

Let \( \Omega \) be an arbitrary finite set endowed with the measure \( \mu \). Assume that \( \mu \) has at least 2 atoms with non-zero measure. Let \( \mathcal{S} \) be a linear subspace of the space of all functions from \( \Omega \) to \( \mathbb{R} \), that includes the constant functions.

Define the operator \( L \) to be the orthogonal projection to functions with zero mean \( L = \text{Id} - E_{\mu} \). Define the semigroup \( T_t = e^{-tL} \quad (t \geq 0) \). We shall use the explicit formula
\[
T_t = E_{\mu} + e^{-t}L = (1 - e^{-t})E_{\mu} + e^{-t}\text{Id}
\]

**Definition 8.10:** The semigroup \( (T_t)_{t \geq 0} \) is \((p, q)\)-hypercontractive (for \( 1 < q < p < \infty \)) over \( \mathcal{S} \) if there exists \( t_0 \) such that for all \( t \geq t_0 \) and \( f \in L_q(\mu) \cap \mathcal{S} \)
\[
\|T_t f\|_{L_p(\mu)} \leq \|f\|_{L_q(\mu)}
\]

If such a \( t_0 \) exists and is the least possible, then \( \sigma_{p, q}(\mu, \mathcal{S}) = e^{-t_0} \) is called the \((p, q)\)-hypercontractive constant for the measure \( \mu \) over \( \mathcal{S} \).
Multivariate functions are functions from $\Omega^n$ to $\mathbb{R}$. The operators $L$ and $T_t$ have multivariate parallels. Define

$$L^i = Id - E_{x_i} \mu, \quad T_t^i = e^{-tL^i}, \quad T_t = \prod_{i=1}^n T_t^i$$

From supermultiplicativity of norms, it is easy to observe that for a linear space of functions $\mathcal{S}$ as above, $\sigma_{p,q}(\mu, \mathcal{S}) \leq \sigma_{p,q}(\mu, \mathcal{S}^n)$. However, when $q < p$ one can apply Minkowski’s inequality to get the inequality in the opposite direction, and deduce $\sigma_{p,q}(\mu, \mathcal{S}) = \sigma_{p,q}(\mu, \mathcal{S}^n)$.

We now wish to consider the degree of multivariate functions. A monomial of degree $d$ is a function $f$ such that there exists a set of indices $S \subseteq [n]$ of size $d$ such that for $i \in S$, $L^i f = f$ and for $i \notin S$, $L^i f = 0$.

**Definition 8.11:** A function of degree $d$ is a function in the linear span of all monomials of degree $\leq d$ and not in the span of monomials of degree $\leq d - 1$.

We will use the following corollary of the hypercontractivity for functions of degree $d$:

**Claim 8.12:** If $\sigma_{p,2}(\mu, (S)) = e^{-t_0}$, then for $f \in \mathcal{S}$ of degree $d$,

$$\|f\|_{L_p(\mu)} \leq e^{d \cdot t_0} \|f\|_{L_2(\mu)}$$

**Proof:** Write $f = T_t T_t^{-1} f$. Therefore,

$$\|f\|_{L_p(\mu)} \leq \|T_t^{-1} f\|_{L_2(\mu)}$$

Now write $f = f_0 + f_1 + ... + f_d$, where $f_i$ is the projection of $f$ onto the monomials of degree $d$. Then $T_t^{-1} f = \sum_{i=1}^d e^{t_0 \cdot i} f_i$. Because the monomials are orthogonal to each other, we have

$$\|T_t^{-1} f\|_2^2 = \sum_{i=1}^d e^{t_0 \cdot i} \|f_i\|_2^2 = d \sum_{i=1}^d \|e^{t_0 \cdot i} f_i\|_2^2 \leq e^{2 \cdot t_0 \cdot d} \sum_{i=1}^d \|f_i\|_2^2 = e^{2 \cdot t_0 \cdot d} \|f\|_2^2$$

$\square$

In the next subsubsection, we will calculate $\sigma$ for our needs and arrive at the following lemma:

**Corollary 8.13:** Let $m \geq e^4$. Let $r$ be a function $r : \mathbb{S}^n_m \rightarrow \mathbb{R}$, of degree 2, all of whose Fourier coefficients are supported on tensors of $\rho^0$ and $\rho^1$. Then we have

$$\|r\|_4^4 \leq C^8 \|r\|_2^4$$

for $C = \frac{1}{m^{\frac{3}{2}}}$. 

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8.1.2 Calculating \( \sigma \)

In this section, we shall calculate \( \sigma = \sigma_{p,q}(\mu, \mathcal{S}_m) \), where \( \mathcal{S}_m \) is the space of functions from \( \mathbb{S}_m \) to \( \mathbb{R} \), whose Fourier transform is supported on \( \rho^0 \) and \( \rho^1 \) (and \( \mu \) is the uniform measure). Our main result is the following.

**Theorem 8.14:** There exists some \( m_0 \) such that for all \( m > m_0 \),

\[
\sigma_{4,2}(\mu, \mathcal{S}_m) \geq m^{-\frac{1}{2}}
\]

In the following claim, we present the functions in \( \mathcal{S}_m \) in a form that will help us in the analysis:

**Claim 8.15:** For \( f \in \mathcal{S}_m \), \( f(x) = \text{tr}(A \cdot P(x)) \), where \( A \) is the matrix \( U \left( \hat{f}(0) \oplus (m - 1) \hat{f}^t(1) \right) U^t \) (See the definition of \( U \) in (1)).

**Proof:** Since \( f \in \mathcal{S} \), we have \( f(x) = \hat{f}(0) \rho^0(x) + (m - 1) \text{tr}(\hat{f}^t(1) \rho^1(x)) \). This equals to

\[
f(x) = \hat{f}(0) \rho^0(x) + (m - 1) \text{tr}(\hat{f}^t(1) \rho^1(x)) = \text{tr}(U \left( \hat{f}(0) \oplus (m - 1) \hat{f}^t(1) \right) U^{-1} \cdot U \left( \rho^0(x) \oplus \rho^1(x) \right) U^{-1}) = \text{tr}(A \cdot P(x))
\]

\( \Box \)

Note that by the definition of \( U \), we have \( 1A = (A1^t)^t = 1\hat{f}(0) \).

We define the following moment-like operators on \( A \):

\[
M_1(A) = \sum_{i,j} A_{ij} \quad M_2(A) = \sum_{i,j} A_{ij}^2 \quad M_3(A) = \sum_{i,j} A_{ij}^3 \quad M_4(A) = \sum_{i,j} A_{ij}^4
\]

\[
M_r(A) = \sum_i \left( \sum_j A_{ij}^2 \right)^r \quad M_c(A) = \sum_j \left( \sum_i A_{ij}^2 \right)^2
\]

\[
M_q(A) = \text{tr}(AA^tAA^t) = \sum_{i,j} \left( \sum_k A_{ik}A_{kj} \right)^2
\]

We can express the 2’nd and 4’th norms of functions in \( \mathcal{S}_m \) using these operators, as shown in the following lemmas:

**Lemma 8.16:** Let \( f \) be a function in \( \mathcal{S}_m \) and let \( f(x) = \text{tr}(A P(x)) \) as in claim 8.15, then

\[
|f|^2 = \mathbb{E}_x f(x)^2 = \frac{1}{m - 1} \left( (m - 2)M_2(A) + \frac{M_2^2(A)}{m^2} \right)
\]

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**Lemma 8.17:** Let \( f \) be a function in \( S_m \) and let \( f(x) = \text{tr}(AP(x)) \) as in claim 8.15. For some \( t > 0 \), denote \( \sigma = e^{-t} \). Then

\[
|T_t f|_4 = E_x f(x)^4 = \frac{1}{(m-1)(m-2)(m-3)} \left( \left( \frac{m-12}{m^2} + O \left( \frac{\sigma^4}{m^3} \right) \right) M_1(A) \right.

- O(\sigma^4) M_1(A) M_3(A)

- O \left( \frac{\sigma^4}{m} \right) M_2(A) M_2(A)

+ O \left( \sigma^4 m \right) M_2(A)

+ \sigma^4 (m^2 + m) M_4(A)

- O \left( \sigma^4 m \right) M_r(A)

- O \left( \sigma^4 m \right) M_c(A)

+ O \left( \frac{\sigma^4}{m} \right) M_q(A) \left.
\right)

The proofs of these lemmas will be given later.

Next, we shall assume that \( f \) is normalized so that \( |f|^2 = 1 \) and bound the terms appearing in the expression for \( |T f|_4 \) under this constraint:

**Claim 8.18:** Let \( f \) be a function in \( S_m \) such that \( \|f\|^2 = 1 \), then:

1. \( |M_1(A)| \leq O(m) \)
2. \( M_2(A) \leq m - 1 \)
3. \( |M_3 A| \leq O \left( \frac{m^2}{2} \right) \)
4. \( M_4(A) \leq O(m^2) \)
5. \( -M_r(A) \leq 0, -M_c(A) \leq 0 \)
6. \( M_q(A) \leq M_2^2(A) \leq O(m^2) \)

Also, the leading coefficient in all of these bounds is 1.

Combining all of this information yields Theorem 8.14.

**Proof:** Assigning the bounds from claim 8.18 in the expression from lemma 8.17, we see that all the term except for three are \( o(\frac{1}{m}) \). Addressing the remaining terms we have

\[
\|T_t f\|^4_4 = \frac{m-12}{m^2} M_1^4(A) + O \left( \sigma^4 m \right) M_2^2(A) + \sigma^4 m^2 M_4(A) + o \left( \frac{1}{m} \right) =

\frac{m^2(m-12) + O \left( \sigma^4 m^3 \right) + \sigma^4 m^4}{(m-1)(m-2)(m-3)} + o \left( \frac{1}{m} \right)
\]

When choosing \( \sigma = m^{-\frac{1}{2}} \) we get

\[
\|T_t f\|^4_4 = \frac{m^2(m-12) + O \left( m \right) + m^2}{(m-1)(m-2)(m-3)} + o \left( \frac{1}{m} \right)
\]

This expression is asymptotically smaller than 1.
We shall now present the proofs of the lemmas. We begin with a complicated proof of the following easy lemma.

**Lemma 8.19:** Let $f$ be a function in $\mathcal{S}_m$ and let $f(x) = \text{tr}(AP(x))$ as in claim 8.15, then

$$E_x f(x) = \frac{M_1(A)}{m}$$

**Proof:**

$$E_x f(x) = E_x \text{tr}(AP(x)) = \text{tr}(AE_x P(x))$$

Remember that $P$ is a representation and is reducible to a copy of $\rho^0$ and $\rho^1$, so

$$E_x P(x) = U (E_x \rho^0(x) \oplus E_x \rho^1(x)) U^t$$

Because of Schur’s orthonormality, we have

$$E_x \rho^0(x) = <\rho^0, 1> = 1$$

$$E_x \rho^1(x) = <\rho^1, 1> = 0$$

Therefore, $E_x P(x) = U (1 \oplus 0) U^t = \frac{J}{m}$. We can now conclude that

$$E_x f(x) = \text{tr} \left( A \frac{J}{m} \right) = \frac{M_1(A)}{m}$$

We shall use a similar technique to prove lemma 8.16 regarding the 2nd norm of $f$.

**Proof of lemma 8.16:** We have

$$f^2(x) = \text{tr}(AP(x)) \cdot \text{tr}(AP(x)) = \text{tr}((A \otimes A)(P(x) \otimes P(x)))$$

and therefore,

$$E_x f^2(x) = \text{tr}((A \otimes A)E_x(P(x) \otimes P(x)))$$

Denote $Q = E_x[P(x) \otimes P(x)]$. We need a closed formula for $Q$, which we shall obtain by diagonalizing it. Notice that $P \otimes P$ is a reducible representation. When taking the expectation $E_x[P(x) \otimes P(x)]$, all the copies of $\rho^0$ will become 1, and the copies of other representations will zero out. The rank of $Q$ is the multiplicity of $\rho^0$ in $P \otimes P$, which is:

$$<\chi_{\rho^0}, \chi_{P \otimes P} > = E_x \chi_{P \otimes P} = E_x \left( \sum_i \sum_j P_{ii}(x)P_{jj}(x) \right) = \sum_i E_x P_{ii}^2(x) + \sum_{i,j,i \neq j} E_x P_{ii}(x)P_{jj}(x) = m \cdot \frac{1}{m} + m(m-1) \cdot \frac{1}{m(m-1)} = 2$$

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So $Q$’s only nonzero eigenvalue is 1 with multiplicity 2. Before computing the eigenvectors of $Q$, notice that $Q$ is symmetric, because $Q^t = E_x P(x) \otimes P^t(x) = E_x P(x^{-1}) \otimes P(x^{-1}) = Q$. The eigenvectors of $Q$ are of size $m^2$. We can guess two eigenvectors of $Q$. We denote them by $u$ and $v$, and list them by their entries, indexed by $i, j \in [m]$ (Recall that $1$ is the all ones vector, and $1_i$ is its $i$th entry, which equals 1):

$$u_{ij} = 1_i 1_j$$

$$v_{ij} = \delta_{ij}$$

It is easy to see that for every $x$, $u$ and $v$ are eigenvectors of $P(x) \otimes P(x)$. Therefore, these vectors are also eigenvectors of $Q$. However, these vectors are not orthogonal. We need to find linear combinations of them which are orthonormal. In other words, let $E$ be the $2 \times (m^2)$ matrix whose rows are $u$ and $v$. We wish to find a $2 \times 2$ matrix $O$ of coefficients such that $OE \cdot (OE)^t = OEE^tO^t = I$. In that case, we will have $Q = (OE)^t OE = E^t O^t O E$. Denote $C = EE^t$, then,

$$OCO^t = I \Rightarrow O^{-1} = CO^t \Rightarrow CO^t O = I \Rightarrow O^t O = C^{-1}$$

Since $C = EE^t$ is symmetric and PSD, we may choose $O$ to be $O = C^{-\frac{1}{2}}$ and then $O$ is also symmetric. It is easy to verify that

$$C = EE^t = \begin{pmatrix} uu^t & vv^t \\ vu^t & vv^t \end{pmatrix} = \begin{pmatrix} m^2 & m \\ m & m \end{pmatrix}$$

and then

$$O^t O = OO = C^{-1} = \frac{1}{(m-1)m} \begin{pmatrix} 1 & -1 \\ -1 & m \end{pmatrix}$$

Now

$$tr( (A \otimes A) Q ) = tr( (A \otimes A) E^t O^t O E ) = tr( E (A \otimes A) E^t OO ) = tr( E (A \otimes A) E^t C^{-1} )$$

One can verify that

$$E (A \otimes A) E^t = \begin{pmatrix} u (A \otimes A) u^t & u (A \otimes A) v^t \\ v (A \otimes A) u^t & v (A \otimes A) v^t \end{pmatrix} = \begin{pmatrix} M_1^2 (A) & \frac{1}{m} M_1^2 (A) \\ \frac{1}{m} M_2^2 (A) & M_2 (A) \end{pmatrix}$$

Finally,

$$\mathbb{E}_x f^2 (x) = \frac{1}{(m-1)m} \tr \left( \begin{pmatrix} M_1^2 (A) & \frac{1}{m} M_1^2 (A) \\ \frac{1}{m} M_2^2 (A) & M_2 (A) \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -1 & m \end{pmatrix} \right) = \frac{1}{m-1} \left( \frac{m-2}{m^2} M_1^2 (A) + M_2 (A) \right)$$

□

We shall now use the same technique to compute $\mathbb{E} f^4 (x)$:
Lemma 8.20: Let $f$ be a function in $\mathfrak{S}_m$ and let $f(x) = \text{tr}(AP(x))$ as in claim 8.15, then
\[
\|f\|^4_4 = \mathbb{E}_x f^4(x) = \frac{1}{(m-1)(m-2)(m-3)} \left( \frac{m^4 - 12}{m^2} + o \left( \frac{1}{m} \right) \right) M_1^4(A)
\]
\[
\begin{align*}
&- O(1)M_1(A)M_3(A) \\
&- O \left( \frac{1}{m} \right) M_2^2(A) \\
&+ O \left( \frac{1}{m} \right) M_3(A) \\
&+ O \left( \frac{1}{m} \right) M_4(A) \\
&- O(m) M_6(A) \\
&- O(m) M_8(A) \\
&+ O \left( \frac{1}{m} \right) M_9(A)
\end{align*}
\]

Proof: We have
\[
f^4(x) = \text{tr}(AP(x))^4 = \text{tr} \left( (A^\otimes 4) \left( P^\otimes 4(x) \right) \right)
\]
and therefore,
\[
\mathbb{E}_x f^4(x) = \text{tr} \left( (A^\otimes 4) \mathbb{E} \left( P^\otimes 4(x) \right) \right)
\]

Denote $Q = \mathbb{E}_x P^\otimes 4(x)$. We need a closed formula for $Q$, which we shall obtain by diagonalizing it. Notice that $P^\otimes 4$ is a reducible representation. When taking the expectation $\mathbb{E}_x P^\otimes 4(x)$, all the copies of $\rho^0$ will become 1, and the copies of other representations will zero out. The rank of $Q$ is the multiplicity of $\rho^0$ in $P^\otimes 4$, which is:
\[
\langle \chi_{\rho^0}, \chi_{P^\otimes 4} \rangle = \mathbb{E}_x \chi_{P^\otimes 4}(x) = \mathbb{E}_x \left( \sum_{i,j,k,l} P_{ii}(x)P_{jj}(x)P_{kk}(x)P_{ll}(x) \right) = 
\]
\[
\sum_i \mathbb{E}_x P_{ii}^4(x) + 4 \cdot \sum_{i,j,i \neq j} \mathbb{E}_x P_{ii}^2(x)P_{jj}(x) + 6 \cdot \sum_{i,j,k, \text{ distinct}} \mathbb{E}_x P_{ii}^2(x)P_{jj}(x)P_{kk}(X) +
\]
\[
3 \cdot \sum_{i,j,i \neq j} \mathbb{E}_x P_{ii}^2(x)P_{jj}^2(x) + \sum_{i,j,k,l, \text{ distinct}} \mathbb{E}_x P_{ii}(x)P_{jj}(x)P_{kk}(x)P_{ll}(x) =
\]
\[
\frac{m}{m} + 4 \cdot \frac{m(m-1)}{m(m-1)} + 6 \cdot \frac{m(m-1)(m-2)}{m(m-1)(m-2)} +
\]
\[
3 \cdot \frac{m(m-1)}{m(m-1)} + \frac{m(m-1)(m-2)(m-3)}{m(m-1)(m-2)(m-3)} = 15
\]

So $Q$’s only nonzero eigenvalue is 1 with multiplicity 15. As before, $Q$ is symmetric. Eigenvectors of $Q$ are of size $m^4$. We now show 15 eigenvectors of $Q$, divided into 5 groups. We list them by their entries, indexed by $i, j, k, l \in [m]$:
\[
E_1 = \{1,1,1,1\}
\]
\[
E_2 = \{\delta_{i,j,1,1}, \delta_{i,j,1,1}, \delta_{i,1,1,1}, \delta_{j,1,1,1}, \delta_{j,1,1,1}, \delta_{j,1,1,1}, \delta_{j,1,1,1}, \delta_{j,1,1,1}\}
\]
\[
E_3 = \{\delta_{i,j,k,l}, \delta_{i,k,j,l}, \delta_{i,l,j,k}\}
\]
\[
E_4 = \{\delta_{i,j,1,1}, \delta_{i,j,1,1}, \delta_{i,1,1,1}, \delta_{j,1,1,1}, \delta_{j,1,1,1}\}
\]
\[
E_5 = \{\delta_{i,j,1,1}\}
\]

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It is easy to see that for every $x$, These vectors are eigenvectors of $\mathbb{P}^{\otimes 4}(x)$. Therefore, these vectors are also eigenvectors of $Q$. However, these vectors are not orthogonal. As before, let $E$ be the $15 \times (m^2)$ matrix whose rows are these vectors. Let $C = EE^t$. Then there exists a symmetric $15 \times 15$ matrix $O$ such that $OE$ is orthonormal, and $OO = C^{-1}$. In appendix A, we give the expression for $C$ and $E$ ($A \otimes A$ or $A^t$). We calculated $C^{-1}$ using a mathematical software called Sage. The expression is too large to show it here. Using this result, we computed $\mathbb{E}_x f^4(x) = tr \left( E \left(A \otimes A \right) E^t C^{-1} \right)$ and got the result.

\[ \square \]

We now wish to apply the result of lemma 8.20 on $T_t f$ to obtain lemma 8.17. We shall use the following two claims:

**Claim 8.21:** Let $f$ be a function in $\mathbb{S}_m$ and let $f(x) = tr(\mathbb{A}P(x))$ as in claim 8.15 then $T_t f(x) = tr(A^t P(x))$, where $A' = \sigma A + (1 - \sigma) \frac{M_1(A)}{m^2} J$, and $\sigma = e^{-t}$.

**Proof:**

\[
tr(A^t P(x)) = tr \left( \left( \sigma A + (1 - \sigma) \frac{M_1(A)}{m^2} J \right) P(x) \right) = 
\]

\[
tr(\sigma A P(x)) + tr \left( (1 - \sigma) \frac{M_1(A)}{m^2} JP(x) \right) = \sigma f(x) + (1 - \sigma) \mathbb{E} f = T_t f(x)
\]

where we have used the fact that $tr(JP(x)) = m$ and $M_1(A) = m \cdot \mathbb{E} f$.

\[ \square \]

**Claim 8.22:** For $A$ as in claim 8.15 and $A' = \sigma A + (1 - \sigma) \frac{M_1(A)}{m^2} J$, denote by $\tau = \frac{(1-\sigma)}{m^2}$. We have:

1. $M_1(A') = M_1(A)$
2. $M_2(A') = \sigma^2 M_2(A) + 2\sigma \tau M_1^2(A) + q^2 M_1^2(A)$
3. $M_3(A') = \tau^3 M_3(A) + 3 \tau^2 M_1 M_2(A) + 3 \tau^2 M_1^2 + \tau^3 M_1^2(A)m^2$
4. $M_4(A') = \sigma^4 M_4(A) + 4\sigma^3 \tau M_1(A) M_3(A) + 6\sigma^2 \tau^2 M_1^2(A) M_2(A) + 4\sigma \tau^3 M_1^4(A) + \tau^4 M_1^4(A)m^2$
5. $M_r(A') = \sigma^4 M_r(A) + 4\sigma^3 \tau M_1^2(A) M_2(A) \frac{1}{m} + 2\sigma^2 \tau^2 M_1^2(A) M_2(A) m + 4\sigma^2 \tau^2 M_1^4(A) \frac{1}{m} + 4\sigma \tau^3 M_1^4(A) m + \tau^4 M_1^4(A)m^3$
6. $M_c(A') = \sigma^4 M_c(A) + 4\sigma^3 \tau M_1^2(A) M_2(A) \frac{1}{m} + 2\sigma^2 \tau^2 M_1^2(A) M_2(A) m + 4\sigma^2 \tau^2 M_1^4(A) \frac{1}{m} + 4\sigma \tau^3 M_1^4(A) m + \tau^4 M_1^4(A)m^3$
7. $M_q(A') = \sigma^4 M_q(A) + 4\sigma^3 \tau M_1^2(A) \frac{1}{m^2} + 6\sigma^2 \tau^2 M_1^4(A) + 4\sigma \tau^3 M_1^4(A) m^2 + \tau^4 M_1^4(A)m^4$
Proof: Since all the cases in this lemma are straightforward, we make do with demonstrating the proof of item number 4, and do not include the tedious, but simple calculations of the other cases. For item 4 we have:

\[ M_4(A') = \sum_{i_{11},i_{22}} \delta_{i_{11}k_{11}} \left( A' \right)^{\otimes 4}_{(i_{11}k_{11})(i_{22}k_{22})} \delta_{i_{22}k_{22}} = \]

\[ \sum_{i_{11},i_{22}} \delta_{i_{11}k_{11}} \left( \sigma A + \tau M_1(A)J \right)^{\otimes 4}_{(i_{11}k_{11})(i_{22}k_{22})} \delta_{i_{22}k_{22}} \]

The above summation can be expanded into six types of summands:

- 1 summand of the form
  \[ \sigma^4 \sum_{i_{11},i_{22}} \delta_{i_{11}k_{11}} \left( A \right)^{\otimes 4}_{(i_{11}k_{11})(i_{22}k_{22})} \delta_{i_{22}k_{22}} = \sigma^4 M_4(A) \]

- 4 summands of the form
  \[ \sigma^3 \tau M_1(A) \sum_{i_{11},i_{22}} \delta_{i_{11}k_{11}} \left( A^{\otimes 3} \otimes J \right)_{(i_{11}k_{11})(i_{22}k_{22})} \delta_{i_{22}k_{22}} = \sigma^3 \tau M_1(A) M_3(A) \]

- 4 summands of the form
  \[ \sigma^2 \tau^2 M_2(A) \sum_{i_{11},i_{22}} \delta_{i_{11}k_{11}} \left( A^{\otimes 2} \otimes J^{\otimes 2} \right)_{(i_{11}k_{11})(i_{22}k_{22})} \delta_{i_{22}k_{22}} = \]
  \[ = \sigma^2 \tau^2 M_2(A) M_2(A) \]

- 2 summands of the form
  \[ \sigma^2 \tau^2 M_1^2(A) \sum_{i_{11},i_{22}} \delta_{i_{11}k_{11}} \left( A \otimes J \otimes A \otimes J \right)_{(i_{11}k_{11})(i_{22}k_{22})} \delta_{i_{22}k_{22}} = \]
  \[ = \sigma^2 \tau^2 M_1^2(A) M_2(A) \]

- 4 summands of the form
  \[ \sigma \tau^3 M_1^3(A) \sum_{i_{11},i_{22}} \delta_{i_{11}k_{11}} \left( A \otimes J^{\otimes 3} \right)_{(i_{11}k_{11})(i_{22}k_{22})} \delta_{i_{22}k_{22}} = \]
  \[ = \sigma \tau^3 M_1^3(A) M_1(A) = \sigma \tau^3 M_1^4(A) \]

- 1 summand of the form
  \[ \tau^4 M_1^4(A) \sum_{i_{11},i_{22}} \delta_{i_{11}k_{11}} \left( J^{\otimes 4} \right)_{(i_{11}k_{11})(i_{22}k_{22})} \delta_{i_{22}k_{22}} = \tau^4 M_1^4(A) m^2 \]

\[ \square \]
Proof of lemma 8.17: This lemma is simply an application of lemma 8.20 to $T_1 f$ and an assignment of the values of the moments of $A'$ given in claim 8.22. Since there are many terms involved in the assignment, we used Sage to get to the actual expression here as well.

\[ \square \]

Proof of claim 8.18:

1. $|M_1(A)| \leq O(m)$:

   \[ \mathbb{E} f^2(x) = 1 = \frac{1}{m - 1} \left( M_2(A) + (m - 1) \frac{M_1^2(A)}{m^2} \right) \geq \frac{1}{m - 1} \left( (m - 1) \frac{M_2^2(A)}{m^2} \right) \Rightarrow \]
   \[ |M_1(A)| \leq O(m) \]

2. $M_2(A) \leq m - 1$:

   \[ \mathbb{E} f^2(x) = 1 = \frac{1}{m - 1} \left( M_2(A) + (m - 1) \frac{M_1^2(A)}{m^2} \right) \geq \frac{1}{m - 1} (M_2(A)) \Rightarrow \]
   \[ M_2(A) \leq m - 1 \]

3. $|M_3A| \leq O \left( \frac{m^3}{2} \right)$: Let $a = \mathbb{E}_{i,j} A_{i,j}$, $B = A - aJ$, $b^2 = \mathbb{E} B^2_{i,j}$. We take $b$ to be positive.

   The items above imply that $|a| \leq O \left( \frac{1}{m} \right)$ and that $b \leq \sqrt{\frac{1}{m}}$.

   \[ \sum_{i,j} A_{i,j}^3 = \sum_{i,j} (a + B_{i,j})^3 = m^2 a^3 + 3aM_2(B) + 3a^2 M_1(B) + M_3(B) = \]
   \[ m^2 a^3 + 3am^2 b^2 + M_3(B) \]

   Given that the 2nd moment of $B$ is $m^2 b^2$, Jensen’s inequality implies that $|M_3(B)| \leq m^3 b^3$. Applying the bounds on $a$ and $b$ yields the result.

4. $M_4(A) \leq O(m^2)$: Let $a, B$ and $b$ be as in the previous item.

   \[ \sum_{i,j} A_{i,j}^4 = \sum_{i,j} (a + B_{i,j})^4 = m^2 a^4 + 4aM_3(B) + 6a^2 M_2(B) + 4a^3 M_1(B) + M_4(B) = \]
   \[ m^2 a^4 + 4aM_3(B) + 6a^2 m^2 b^2 + M_4(B) \]

   Again, given that the 3rd moment of $B$ is $m^2 b^2$, Jensen’s inequality implies that $|M_3(B)| \leq m^3 b^3$ and $|M_4(B)| \leq m^4 b^4$. Applying the bounds on $a$ and $b$ yields the result.

5. $-M_e(A) \leq 0, -M_v(A) \leq 0$: By definition, these operators are nonnegative.

6. $M_q(A) \leq M_2^2(A) \leq O(m^2)$: Notice that $AA^t$ is a symmetric PSD matrix. Let $\lambda_i$ be the vector of eigenvalues of $AA^t$ (which are all nonnegative). We have $M_2(A) = \text{tr}(AA^t) = \sum_i \lambda_i$ and $M_q(A) = \text{tr} \left( AA^t AA^t \right) = \sum_i \lambda_i^2$. Therefore, $M_q = \sum_i \lambda_i^2 \leq (\sum_i \lambda_i)^2 = M_2^2$.

\[ \square \]
9 Strategy Proofness

Before discussing further work we wish to briefly discuss the notion of IR, that arises naturally when considering the spectrum of outcomes between SCF’s and SWF’s. The connection between IR and IIA is easy to understand, despite the fact that neither of these constraints implies the other. In this section we wish to touch upon the other end of the spectrum and discuss the connection between IR and strategy proofness.

We present a definition of strategy proofness that is closely related to IR. We will show how our robust impossibility theorem for IR implies a robust impossibility theorem for this definition of strategy proofness. In this sense, our technique provides a single proof for the analogues of both Arrow’s theorem (SWF’s with independence) and Gibbard-Satterthwaite’s theorem (SCF with strategy proofness).

The definition we give here is based on the definition of Dietrich and List [DL07] for strategy proofness on judgment aggregation, which is a general framework that includes the setting of Arrow’s theorem. In this framework, there is a permissible opinion space \( X \subseteq \{k\}^m \) and we are interested in social aggregators of the form \( f : X^n \rightarrow X \). A widely studied topic was the characterization of spaces \( X \) for which independence implies dictatorship, as in Arrow’s theorem and our setting. This characterization can be found in [NP10] and [DH10a] for \( k = 2 \) and in [DH10b] for general \( k \). In [DL07], a general definition of strategy proofness was given for this framework was for \( k = 2 \), and was connected to independence. The definition we present here follows from an extension of the definition in [DL07] for any \( k \), which also deals with the case where the output space of the function is different than the input space.

Let \( H \) be a fixing subgroup of \( S_m \). Let \( f \) be \( f : S_m^n / H \rightarrow S_m/H \). We assume that for any alternative \( j \in [m] \), given the ranking of \( j \) by a certain voter, we are able to rank the voter’s preference over all possible \( j \)-profiles of the outcome.

Formally, define the set \( Q_j \) to be the set of all possible \( j \)-profiles of a coset of \( H \), \( Q_j = \{K^{-1}(j)\}_{K \in S_m/H} \). For every alternative \( j \in [m] \) and ranking \( r \in [m] \), we assume that there exists some full transitive order relation on \( Q_j \). We denote it by \( <_{r,j} \). It is natural to assume that for a specific ranking of the alternatives \( x \), the top of the order \( <_{x^{-1}(j),j} \) will be the \( j \)-profile of the coset that includes \( x \). However, we do not demand that.

A manipulation is a situation where for any alternative \( j \), a voter \( i \in [n] \) can report a false opinion and get better \( j \)-profile of the outcome, according to his preference order \( <_{x_i^{-1}(j),j} \). Formally, a manipulation exists when the following holds:

\[
\exists j, i, x^{-i}, x_i, y_i f^{-1}(x^{-i}, x_i)(j) <_{x_i^{-1},j} f^{-1}(x^{-i}, y_i)(j)
\]

The manipulation power of a voter \( i \) is the rate of manipulations he can make:

\[
M_i(f) = \sum_j \mathbb{E}_{x^{-i}, x_i, y_i} \left( f^{-1}(x^{-i}, x_i)(j) <_{x_i^{-1},j} f^{-1}(x^{-i}, y_i)(j) \right)
\]

The total manipulation power of \( f \) is \( M(f) = \sum_i M_i(f) \). \( f \) will be called strategy-proof if \( M(f) = 0 \). The rationale behind this definition is that as the designers of a social aggregation
mechanism, we do not wish to specify up front for which alternatives \( j \) there might be a voter \( i \) who wishes to manipulate, and therefore we wish to be immune against all such manipulations.

We shall now connect \( IR(f) \) to \( M(f) \). Denote \( c = \max_{K_1, K_2 \in \mathbb{S}_m / H} \left\| \frac{n_{K_1} - n_{K_2}}{|H|} \right\|_2^2 \) (Recall \( n_K \) is the characteristic vector of a multiset \( K \)). For example, when \( H \) is the group of permutations fixing \( \{1\} \), as in SCFs, we have \( c = \left\| (1, 0, \ldots, 0) - (0, \frac{1}{m-1}, \ldots, \frac{1}{m-1}) \right\|_2^2 = \frac{m}{m-1} \).

Claim 9.1: \( c M(f) \geq IR(f) \)

Proof: Examine all pairs of profiles \((x^{-i}, x_i)\) and \((x^{-i}, y_i)\), such that \( x_i^{-1}(j) = y_i^{-1}(j) \) and \( f(x^{-i}, x_i)^{-1}(j) \neq f(x^{-i}, y_i)^{-1}(j) \). By definition, those pairs of inputs contribute at most \( c \) to \( IR(f) \).

Denote \( r = x_i^{-1}(j) = y_i^{-1}(j) \), \( K_x = f(x^{-i}, x_i)^{-1}(j) \), \( K_y = f(x^{-i}, y_i)^{-1}(j) \). Since \( K_x \neq K_y \), we must have either \( K_x <_{r,j} K_y \) or \( K_y <_{r,j} K_x \). Therefore, this pair of inputs contribute 1 to \( M(f) \).

\[ \square \]

As a corollary of this claim and theorem 3.10, we have

Corollary 9.2: Let \( H \subseteq \mathbb{S}_m \) be a fixing subgroup of \( \mathbb{S}_m \). For \( m \geq 3 \), an \( H \)-social aggregator \( f \) for which \( M(f) \leq \epsilon \) is \( O_H(\text{poly}(m)\epsilon) \) close to a function that is either a constant function or dictatorial of the following form: there exists a voter \( i \) and a constant permutation \( y \) of the rankings such that \( f(x) = y \circ x_i \).

10 Further Work

We have began to apply the techniques of this paper to Arrow’s theorem (with relaxed independence constraint) and to Gibbard-Satterthwaite’s theorem, with partial success. GS seems to offer many more new challenges for this scheme, as the Laplacian is not PSD and not Symmetric.

It is also interesting to see how this technique applies to the generalized problem of judgment aggregation (see its definition in section 9). There are known combinatorial characterization results of functions satisfying independence in this setting, depending on the opinion space \( X \), and our technique may be useful in finding robust versions of these theorems, wherever possible.

Another direction is to study how our proof could be modified for groups other than \( \mathbb{S}_m \).

As mentioned earlier, our result can be interpreted as a 2 query dictatorship tester, for functions \( \mathbb{S}_m \rightarrow \mathbb{S}_m / H \). It is interesting to see whether this has any computational implications.

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### A Appendix: matrices for the proof of lemma 8.20

\[ C = EE^t = \begin{pmatrix} m^4 & m^3 & m^3 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^4 & m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\ m^3 & m^2 & m^2 & m^2 & m^2 & m^2 & m^2 & m^1 \\
\end{pmatrix} \]

We compute \( C^{-1} \) using an algebraic software called *Sage*. The expressions are too large to write here. Since all the entries of \( C \) are monomials in \( m \), the determinant of \( C \) is a polynomial of a bounded degree in \( m \), and therefore \( C \) is regular for all but a finite number of values of \( m \). Actually, using *Sage*, we find the determinant of \( C \) to be \( m^{15}(m-1)^{14}(m-2)^7(m-3) \), so \( C \) is singular only for \( m = 0, 1, 2, 3 \).

For the expression for \( E(\mathcal{A}^{\otimes 4})E \) we abuse the notation and use the sets \( E_i \) as matrices whose rows are the elements of \( E_i \). We have:

\[
E(\mathcal{A}^{\otimes 4}) = \begin{pmatrix} E_1(\mathcal{A}^{\otimes 4})E_1^t & E_1(\mathcal{A}^{\otimes 4})E_2^t & \cdots & E_1(\mathcal{A}^{\otimes 4})E_5^t \\ \vdots & \vdots & \ddots & \vdots \\ E_5(\mathcal{A}^{\otimes 4})E_1^t & E_5(\mathcal{A}^{\otimes 4})E_2^t & \cdots & E_5(\mathcal{A}^{\otimes 4})E_5^t \end{pmatrix}
\]
Where:

\[ E_1 (A^{\otimes 4}) E_1 = \left( M_1^4 \right) \]

\[ E_1 (A^{\otimes 4}) E_2 = \frac{1}{m^2} \left( \begin{array}{cccccc} M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \end{array} \right) \]

\[ E_1 (A^{\otimes 4}) E_3 = m \left( \begin{array}{cccc} M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 \end{array} \right) \]

\[ E_1 (A^{\otimes 4}) E_4 = \left( M_1^4 \right) \]

\[ E_1 (A^{\otimes 4}) E_5 = \frac{1}{m^2} \left( M_1^4 \right) \]

\[ E_2 (A^{\otimes 4}) E_2 = \frac{1}{m^2} \left( \begin{array}{cccccc} M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 & M_1^4 \end{array} \right) \]

\[ E_2 (A^{\otimes 4}) E_3 = \frac{1}{m} \left( \begin{array}{cccc} M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 \end{array} \right) \]

\[ E_2 (A^{\otimes 4}) E_4 = \frac{1}{m} \left( \begin{array}{cccc} M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 \end{array} \right) \]

\[ E_2 (A^{\otimes 4}) E_5 = \frac{1}{m^2} \left( \begin{array}{c} M_1^4 \end{array} \right) \]

\[ E_3 (A^{\otimes 4}) E_3 = \left( \begin{array}{ccc} M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 \end{array} \right) \]

\[ E_3 (A^{\otimes 4}) E_4 = \frac{1}{m^2} \left( \begin{array}{cccc} M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 \\ M_1^4 & M_1^4 & M_1^4 & M_1^4 \end{array} \right) \]

\[ E_3 (A^{\otimes 4}) E_5 = \left( \begin{array}{c} M_1^4 \end{array} \right) \]
\[ E_4(A^\otimes 4) E_4^t = \frac{1}{m^2} \begin{pmatrix} M_1 M_3 \frac{1}{m} & M_1^2 M_2 & M_1^2 M_2 & M_1^2 M_2 \\ \frac{1}{m} M_1 M_3 & M_2^2 M_2 & M_2^2 M_2 & M_2^2 M_2 \\ \frac{1}{m} M_2^2 M_2 & \frac{1}{m} M_1 M_3 & M_2^2 M_2 & M_2^2 M_2 \\ \frac{1}{m} M_2^2 M_2 & \frac{1}{m} M_2^2 M_2 & M_2^2 M_2 & M_2^2 M_2 \end{pmatrix} \]

\[ E_4(A^\otimes 4) E_5^t = \frac{1}{m} \begin{pmatrix} M_1 M_3 \\ M_1 M_3 \\ M_1 M_3 \\ M_1 M_3 \end{pmatrix} \]

\[ E_5(A^\otimes 4) E_3^t = (M_c \quad M_c \quad M_c) \]

\[ E_5(A^\otimes 4) E_5^t = (M_4) \]

For the blocks that weren’t explicitly mentioned, \( E_i(A^\otimes 4) E_j^t = \left(E_i(A^\otimes 4) E_j^t\right)^t \)

We add here the protocol of the Sage code we used, for completeness:

```python
R.<m,a1,a13,a112,a22,a4,r22,c22,q,sig>=QQ[]
C=matrix(R,15,15,
[ m^4, m^3,m^3,m^3,m^3,m^3,m^3,m^3,m^3,m^2,m^2,m^2,m^2,m^2,m^2,m^1,
  m^3,m^3,m^2,m^2,m^2,m^2,m^2,m^2,m^2,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^3,m^2,m^3,m^2,m^2,m^2,m^2,m^1,m^1,m^1,m^2,m^2,m^1,m^1,m^1,m^1,
  m^2,m^2,m^2,m^3,m^2,m^2,m^2,m^1,m^1,m^1,m^2,m^1,m^2,m^1,m^1,m^1,
  m^2,m^2,m^2,m^2,m^3,m^2,m^2,m^2,m^1,m^1,m^2,m^1,m^1,m^2,m^1,m^1,
  m^2,m^2,m^2,m^2,m^2,m^3,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^2,m^2,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^2,m^1,m^1,m^1,m^1,m^2,m^2,m^2,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,
  m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,m^1,]
]

\( t = (1-sig)/m^2 \)
# the variables $a_1, a_{13}, a_{112}, a_2, a_4, r_2, c_2, q$ represent, respectively
\[ M_1^1(A), M_1(A)M_3(A), M_2^2(A)M_2(A), M_4(A), M_c(A), M_r(A), M_q(A) \]

# the variables $a, b, c, d, e, f, g, h$ represent, respectively
\[ M_1^1(A'), M_1(A')M_3(A'), M_2^2(A)M_2(A'), M_4(A'), M_c(A'), M_r(A'), M_q(A') \]

\[
\begin{align*}
\text{a} &= a_1 \\
\text{b} &= \text{sig}^3 a_{13} + 3 \text{sig}^2 (1-\text{sig}) a_{112} - 2 a_1 m^2 + t^3 a_1 m^2 \\
\text{c} &= \text{sig}^2 a_{112} + (2 \text{sig}^3 - \text{t}^2) a_1 \\
\text{d} &= \text{sig}^4 a_2 + (2 \text{sig}^3 - \text{t}^2) a_1 + 6 \text{sig}^2 t - 2 a_1 + 4 \text{sig}^3 t a_{13} + t^4 a_1 m^2 \\
\text{e} &= \text{sig}^4 a_4 + 2 \text{sig}^3 t^3 a_1 + 8 \text{sig}^2 t^2 a_1 + 4 \text{sig}^3 t a_{13} + t^4 a_1 m^2 \\
\text{f} &= \text{sig}^4 r_2 + 4 \text{sig}^3 t a_{112} + (2 a_{112} m + 4 a_1 m) + 4 \text{sig}^3 t a_{13} + t^4 a_1 m^2 \\
\text{g} &= \text{sig}^4 c_2 + 4 \text{sig}^3 t a_{112} + (2 a_{112} m + 4 a_1 m) + 4 \text{sig}^3 t a_{13} + t^4 a_1 m^2 \\
\text{h} &= \text{sig}^4 q + 4 \text{sig}^3 t a_1 m^2 + 6 \text{sig}^2 t^2 a_1 + 4 \text{sig}^3 t a_{13} m^2 + t^4 a_1 m^4
\end{align*}
\]

Sage can not work with matrices with rational entries. Therefore, we multiply all the variables by $m^{-10}$ and the matrix $E$ by $m^{-3}$.

\[
\begin{align*}
\text{a} &= a_m^{10}; \text{b} = b_m^{10}; \text{c} = c_m^{10}; \text{d} = d_m^{10}; \text{e} = e_m^{10}; \text{f} = f_m^{10}; \text{g} = g_m^{10}; \text{h} = h_m^{10}; \\
E &= \text{matrix(R,15,15,} \\
& \begin{bmatrix}
a_m^{10}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{1}, a_m^{2}, c, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, a_m^{2}, \end{bmatrix}
\end{align*}
\]

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\begin{verbatim}
a*m^1, a*c^m^2, c^m^2, a, a, c^m^2, c^m, c^m, c^m, b, c^m, b*m^2, a*m^1, a, a, a, c^m^2, c^m^2, c^m, c^m, c^m, c^m, b, c^m, b^m^2, a, c^m, c^m, c^m, c^m, c^m, g*m^3, g*m^3, g*m^3, b*m^2, b*m^2, b*m^2, b*m^2, e*m^3 ]
\)
\)
l=(C.inverse()*E).trace()/m^13
l(m,1,0,0,0,0,0,0,sig)
l(m,0,1,0,0,0,0,0,sig)
l(m,0,0,1,0,0,0,0,sig)
l(m,0,0,0,1,0,0,0,sig)
l(m,0,0,0,0,1,0,0,sig)
l(m,0,0,0,0,0,1,0,sig)
l(m,0,0,0,0,0,0,1,sig)
\end{verbatim}