On the vertex degrees of the skeleton of the matching polytope of a graph

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Abstract

The convex hull of the set of the incidence vectors of the matchings of a graph $G$ is the matching polytope of the graph, $\mathcal{M}(G)$. The graph whose vertices and edges are the vertices and edges of $\mathcal{M}(G)$ is the skeleton of the matching polytope of $G$, denoted $G(\mathcal{M}(G))$. Since the number of vertices of $G(\mathcal{M}(G))$ is huge, the structural properties of these graphs have been studied in particular classes. In this paper, for an arbitrary graph $G$, we obtain formulae to compute the degree of a vertex of $G(\mathcal{M}(G))$ and prove that the minimum degree of $G(\mathcal{M}(G))$ is equal to the number of edges of $G$. Also, we identify the vertices of the skeleton with the minimum degree and characterize regular skeletons of the matching polytopes.

Keywords: graph, matching polytope, degree of matching.

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1 Introduction

Let $G := G(V(G), E(G))$ be a simple and connected graph with vertex set $V := V(G) = \{v_1, v_2, \ldots, v_n\}$ and set of edges $E := E(G) = \{e_1, e_2, \ldots, e_m\}$. For each $k$, $1 \leq k \leq m$, $e_k = v_ivi_j$ is an edge incident to the adjacent vertices $v_i$ and $v_j$, $1 \leq i < j \leq n$. The set of adjacent vertices of $v_i$ is $N_G(v_i)$, called the neighborhood of $v_i$, whose cardinality $d(v_i)$ is the degree of $v_i$.

If two edges have a common vertex they are said to be adjacent edges. For a given edge $e_k$, $I(e_k)$ denotes the set of adjacent edges of $e_k$. Two non adjacent edges are disjoint and a set of pairwise disjoint edges $M$ is a matching of $G$. An unitary edge set is an one-edge matching and the empty set is the empty matching, $\emptyset$. A vertex $v \in V(G)$ is said to be $M$-saturated if there is an edge of $M$ incident to $v$. Otherwise, $v$ is said an $M$-unsaturated vertex. A perfect matching $M$ is one for which every vertex of $G$ is an $M$-saturated.

For a natural number $k$, a path with length $k$, $P_{k+1}$ (or simply $P$), is a sequence of distinct vertices $v_1v_2\ldots v_kv_{k+1}$ such that, for $1 \leq i \leq k$, $e_i = v_iv_{i+1}$ is an edge of $G$. A cycle $C = C_k$, with length $k$, is obtained by path $P_k$ adding the edge $v_kv_k$. If $k$ is odd, $C_k$ is said to be an odd cycle. Otherwise, $C_k$ is an even cycle. When it is clear, we denote a path and a cycle by a sequence of their respective edges $e_1e_2\ldots e_k$ instead of their respective sequences of vertices. Given a matching $M$ in $G$, a path $P$ (or, cycle $C$) is $M$-alternating path (or, $M$-alternating cycle) in $G$ if given two adjacent edges of $P$ (or, $C$), one belongs to $M$ and one belongs to $E \setminus M$. Naturally, the set of edges of $P \setminus M$ (and $C \setminus M$) is also a matching in $G$. For more basic definitions and notations of graphs, see [2, 5] and, of matchings, see [9].

A polytope of $\mathbb{R}^n$ is the convex hull $P = \text{conv}\{x_1, \ldots, x_r\}$ of a finite set of vectors $x_1, \ldots, x_r \in \mathbb{R}^n$. Given a polytope $P$, the skeleton of $P$ is a graph $G(P)$ whose vertices and edges are, respectively, vertices and edges of $P$.

Ordering the set $E$ of $m$ edges of $G$, we denote by $\mathbb{R}^E$ the vectorial space of real-valued functions in $E$ whose $\text{dim}(\mathbb{R}^E) = m$. For $F \subset E$, the incidence vector of $F$ is defined as follows:

$$
\chi_F(e) = \begin{cases} 
1, & \text{if } e \in F; \\
0, & \text{otherwise.}
\end{cases}
$$

In general, we identify each subset of edges with its respective incidence vector. The matching polytope of $G$, $\mathcal{M}(G)$, is the convex hull of the incidence vectors of the matchings in $G$. 

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In this paper, we are interested in studying the skeleton of a polytope obtained from matchings of a given graph. For more definitions and notations of polytopes, see [8].

The graph \( G \equiv K_3 \) has the following matchings: \( \emptyset, \{e_1\}, \{e_2\} \) and \( \{e_3\} \), where \( e_1, e_2 \) and \( e_3 \) are edges of \( G \). Figure 1 displays the matching polytope of \( K_3, \mathcal{M}(K_3) = \text{conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\} \) which corresponds to a tetrahedron in \( \mathbb{R}^3 \). Its skeleton is \( \mathcal{G}(\mathcal{M}(K_3)) \equiv K_4 \).

Two matchings \( M \) and \( N \) are said adjacent, \( M \sim N \), if and only if their correspondent vertices \( \chi_M \equiv M \) and \( \chi_N \equiv N \) are adjacent in the skeleton of the matching polytope. The degree of a matching \( M \), denoted \( d(M) \), is the degree of the correspondent vertex in \( \mathcal{G}(\mathcal{M}(G)) \). Next theorems characterize the adjacency of two matchings, \( M \) and \( N \), by their symmetric difference \( M \Delta N = (M \setminus N) \cap (N \setminus M) \).

**Theorem 1.1.** (3) Let \( G \) be a graph. Two distinct matchings \( M \) and \( N \) of \( G \) are adjacent in the matching polytope \( \mathcal{M}(G) \) if and only if \( M \Delta N \) is a connected subgraph of \( G \).

**Theorem 1.2.** (10) Let \( G \) be a graph. Two distinct matchings \( M \) and \( N \) in \( G \) are adjacent in the matching polytope \( \mathcal{M}(G) \) if and only if \( M \Delta N \) is a path or a cycle in \( G \).

Note that the symmetric difference \( M \Delta N \) of two adjacent vertices of \( \mathcal{M}(G) \) is an \( M \) (and \( N \))-alternating path (or, cycle). And if it is a cycle, it is an even cycle.

Figure 2 shows the skeleton of the matching polytope of \( C_4 \). Since \( M_1 \Delta M_2 \) is a path and \( M_5 \Delta M_6 \) is a cycle, \( M_1 \sim M_2 \) and \( M_5 \sim M_6 \). However, once \( M_1 \Delta M_3 \) is a disconnected subgraph of \( G \), \( M_1 \not\sim M_3 \).

Let \( T \) be a tree with \( n \) vertices. The acyclic Birkhoff polytope, \( \Omega_n(T) \), is the set of \( n \times n \) doubly stochastic matrices \( A = [a_{ij}] \) such that the diagonal

![Figure 1: K_3 and G(M(K_3)) ≡ K_4](image-url)
Figure 2: $C_4$ and $\mathcal{G}(\mathcal{M}(C_4))$

entries of $A$ correspond to the vertices of $T$ and each positive entry of $A$ is either on the diagonal or on a correspondent position of each edge of $T$. The matching polytope $\mathcal{M}(T)$ and the acyclic Birkhoff polytope $\Omega_n(T)$ are affinely isomorphic, see [4]. The skeleton of $\Omega_n(T)$ was studied in [1, 6] where some results about the structure of such graph were presented. In the sequence, we highlight the following contributions given in those papers.

**Theorem 1.3.** ([1]) If $T$ is a tree with $n$ vertices, the minimum degree of $\mathcal{G}(\mathcal{M}(T))$ is $n - 1$.

**Theorem 1.4.** ([1]) Let $T$ be a tree with $n$ vertices and $M$ be a matching of $T$. The degree of $M$ in $\mathcal{G}(\mathcal{M}(T))$ is $d(M) = n - 1$ if and only if $M$ is either the empty matching or every edge of $M$ is a pendant edge of $T$.

In this paper, we generalize the results above for an arbitrary graph $G$. In next section, we obtain a formulae to compute the degree of a vertex of $\mathcal{G}(\mathcal{M}(G))$. In Section 3, we prove that the degree of a matching $M$ is non decreasing function of the cardinality of $M$. As a consequence, we conclude that the minimum degree of $\mathcal{G}(\mathcal{M}(G))$ is equal to the number of edges of $G$. Section 4 ends the paper with two theorems of characterization: the first gives a necessary and sufficient condition under $G$ in order to have $\mathcal{G}(\mathcal{M}(G))$ as a regular graph and, the second identifies those matchings of $G$ for which their correspondent vertices of the skeleton have the minimum degree.

## 2 The degree of a matching in a graph.

As we have pointed in the previous section, the first advances on the computation of degree of a matching of tree were due to Abreu et al. [1] and
In the present section we generalize those results by presenting formulae for the computation of the degree of a vertex of the skeleton of the matching polytope of an arbitrary graph. First, we enunciate two simple results that are immediate consequences of Theorem 1.2. In these, we denote the cardinality of a set $M$ by $|M|$. 

**Proposition 2.1.** Let $M$ and $N$ be matchings of a graph $G$. If $M$ is adjacent to $N$ in $G(M(G))$ then $|M| - |N| \in \{0, 1\}$.

**Proof.** Let $M$ and $N$ be matchings of $G$ such that $M \sim N$ in $G(M(G))$. From Theorem 1.2, $M \Delta N$ is a $M$-alternating path (or, cycle) in $G$. If the length of the path (cycle) is even, then $|M| = |N|$. Otherwise, $|M| - |N| = 1$. □

**Proposition 2.2.** Let $G$ be a graph with $m$ edges. The degree of the empty matching is equal to the number of edges of $G$, that is, $d(\emptyset) = m$.

**Proof.** Let $G$ be a graph with $m$ edges and suppose $M \neq \emptyset$ be a vertex of $G(M(G))$ such that $M \sim \emptyset$. We know that $M \Delta \emptyset = M$ and, so, from Proposition 2.1, $|M| = 1$. For some $e \in E(G)$, $M = \{e\}$ is an one-edge matching and, then, $d(\emptyset) \leq m$. The reciprocal, $m \leq d(\emptyset)$, comes from Theorem 1.2 once the one-matching edge is a path. □

Let $G$ be a graph with a matching $M$ and $P$ be an $M$-alternating path with at least two vertices. We say that:

(i) $P$ is an oo-$M$-path if its pendent edges belong to $M$;
(ii) $P$ is a cc-$M$-path if its pendent vertices are both $M$-unsaturated;
(iii) $P$ is an oc-$M$-path if one of its pendent edges belongs to $M$ and one of its pendent vertex is $M$-unsaturated.

An $M$-alternating path $P$ is called an $M$-good path if and only if $P$ is one of those paths defined above.

The concept of $M$-good path, introduced by [1], has a important role to determining the degree of a matching of a tree. Since it is our goal to deduce a formulae for the degree of a matching $M$ of any graph, we must consider $M$-alternating cycles. In order to facilitate the writing of the proofs that follow, we will call such cycles by $M$-good cycles. Finally, note that an $M$-good cycle $C$ has a perfect matching, given by $M \cap E(C)$.

In Figure 3, $M = \{e_1, e_3, e_6\}$ is a perfect matching of the graph. Hence, there are neither cc-$M$-paths nor oc-$M$-paths in $G$. However, $e_1e_2e_3$ is an oo-$M$-path of $G$ but $e_1e_2$ is not an $M$-good path. Moreover, $e_1e_2e_3e_4$ is an $M$-good cycle.

Denote $\nu_{oo}(M)$, $\nu_{cc}(M)$ and $\nu_{oc}(M)$ the numbers of oo-$M$-paths, cc-$M$-paths and oc-$M$-paths, respectively. Denote $\nu_P(M)$ the number of $M$-good
paths of $G$. So, $\nu_P(M) = \nu_{oo}(M) + \nu_{cc}(M) + \nu_{oc}(M)$. Similarly, the number of $M$-good cycles of $G$ is denoted by $\nu_C(M)$.

**Theorem 2.3.** Let $M$ be a matching of a graph $G$. The degree of $M$ in the skeleton $G(M(G))$ is given by

$$d(M) = \nu_P(M) + \nu_C(M).$$

(1)

Proof. Let $M$ and $N$ be adjacent vertices in $G(M(G))$. From Theorem 1.2, $M \Delta N$ is an $M$-alternating path or a cycle in $G$. If $M \Delta N$ is a path $P$, then it is an $M$-good path. In fact, supposing that $P = e_1e_2 \ldots e_k$ is not an $M$-good path. Then, two cases can happen. In the first one, $e_1 \notin M$ and $e_k \notin M$. Besides, one of these edges has an $M$-saturated terminal vertex $u$. So, there is an edge $f \in M \setminus P$ such that $f$ is incident to $u$. Once $P = M \Delta N$ and $f \notin M \Delta N$, then $f \in N$. Consequently, $N$ is not a matching of $G$. In the second case, suppose $e_1 \in M$ and $e_k \notin M$. Similarly, we get the same contradiction. So, $P$ is an $M$-good path.

Now, assume $P$ as an $M$-good path (or cycle) in $G$ and, do $N = M \Delta P$. If $P$ is a cycle, it is clear that $N$ is also a matching of $G$. If $P$ is a path, it is also occurs. Indeed, in this case, if $N$ is not a matching of $G$, there is two adjacent edges $f, g \in N$ such that $f \in M \setminus P$ and $g \in P \setminus M$. Since $P$ is an $M$-alternating path, $g$ is pendent edge of $P$ and its pendent vertex is $M$-saturated, which leads us to a contradiction. So, $N$ is a matching of $G$. As $M \Delta N = P$ is a path (or cycle) of $G$, from Theorem 1.2 $M \sim N$.

Finally, for $N$ and $N'$ matchings of $G$, we have $M \Delta N = M \Delta N'$ if and only if $N = N'$. Hence, $d(M) = \nu_{oo}(M) + \nu_{cc}(M) + \nu_{oc}(M) + \nu_C(M)$.

Proposition 2.2 is also a consequence of Theorem 2.3. Assume $M = \emptyset$ and $N \sim M$. Then, $M \Delta N$ is one-edge matching and, so, it is a $cc$-$M$-path of $G$.

A matching $M = \{e_1, \ldots, e_k\}$ is said to be a matching without common neighbors of $G$ if and only if

$$\forall i, j \in \{1, \ldots, k\}, \ e_i, e_j \in M, \ e_i \neq e_j \Rightarrow I(e_i) \cap I(e_j) = \emptyset.$$
**Lemma 2.4.** Let $M$ be a matching without common neighbors in a graph $G$. If $C$ is a cycle of $G$ then $C$ is not an $M$-good cycle.

Proof. Let $M$ be a matching of $G$ without common neighbors. Suppose there is an $M$-good cycle $C$ of $G$ given by sequence $e_1f_1e_2f_2\ldots e_tf_t$, where $f_j \notin M$ and $e_j \in M$, $1 \leq j \leq t$. Since $E(C) \cap M$ is a perfect matching in $C$, there is $i, 1 \leq i \leq t - 1$, such that $e_i f_i e_{i+1}$ is a path. Hence, $\{f_i\} \subseteq I(e_i) \cap I(e_{i+1})$. Once $M$ is a matching without common neighbors, this is an absurd. \hfill \Box

**Lemma 2.5.** Let $G$ be a graph and $M$ be a matching without common neighbors. If $P$ is an $M$-good path then $P$ is an $M$-alternating path with length is at most 3 in which at most one edge belongs to $M$.

Proof. The proof follows straightforward from $M$ to be a matching without common neighbors and $P$ be an $M$-alternating path of $G$. \hfill \Box

**Theorem 2.6.** Let $M = \{e_i = u_iv_i | u_i, v_i \in V, 1 \leq i \leq s\}$ be a matching without common neighbors of a graph $G$. The degree of $M$ is

$$d(M) = k + \sum_{i=1}^{s} (d(u_i)d(v_i) - |N(u_i) \cap N(v_i)|),$$

(2)

where $k \geq 0$ is the number of edges of $G$ which neither incides to $u_i$ nor to $v_j$, for all $1 \leq i < j \leq s$.

Proof. Let $M = \{e_i = u_iv_i | u_i, v_i \in V, 1 \leq i \leq s\}$ be a matching without common neighbors of $G$. From Lemma 2.4 $G$ does not have $M$-good cycles and, from Lemma 2.5 if $P$ is an $M$-good path, $P$ has at most length 3 with at most one edge of $M$. Hence, if $P$ is an $M$-good path of $G$, $P$ has to take one of the cases bellow.

(1) $P$ is an oo-M-path. Then, $P = e$ such that $e \in M$. There are $s$ of these paths in $G$;

(2) $P$ is an oc-M-path. So, $P = ef$ such that $e \in M$ and $f \notin M$. Of course, for some $i, 1 \leq i \leq s$ we have $e = u_iv_i$ and $f$ is incident to $u_i$ or to $v_i$. There are $(d(u_i) - 1) + (d(v_i) - 1)$ of these paths;

(3) If $P$ is a cc-M-path, we have to consider two possibilities for $P$. Firstly, $P = feg$ with $e \in M$ and $f, g \notin M$. So, for some $i, 1 \leq i \leq s$, $e = u_iv_i$ such that $f$ is incident to $u_i$ and $g$ is incident to $v_i$. In this case, there are $(d(u_i) - 1) \cdot (d(v_i) - 1) - |N(u_i) \cap N(v_i)|$ of such paths. The second possibility is $P = f$ with $f \notin M$. Since $P$ is an oc-M-path, for each $e \in E(G)$ that is incident to $f$, $e \notin M$. We can admit that there are $k$ of these edges in the graph that satisfy this last case.
From the items (1), (2) and (3) and, by applying Theorem 2.3, we obtain
\[
d(M) = s + k + \sum_{i=1}^{s} ((d(u_i)−1) + (d(v_i)−1) + (d(u_i)−1)(d(v_i)−1) - |N(u_i) \cap N(v_i)|) = \\
= k + \sum_{i=1}^{s} (d(u_i)d(v_i) - |N(u_i) \cap N(v_i)|).
\]

□

3 Vertices with minimum degree in the skeleton.

We begin this section proving that, if a matching within the other, the degree of the first is at most equal to the degree of the last. Based on this, we prove that the minimum degree of \(G(M(G))\) is equal to the number of edges of \(G\). The section follows by determining the degree of a matching of a graph whose components are stars or triangles.

**Theorem 3.1.** Let \(M\) and \(N\) be matchings of a graph \(G\). If \(M \subset N\), then \(d(M) \leq d(N)\).

Proof. Let \(M\) and \(N\) be matchings of a graph \(G\) such that \(N = M \cup \{e\}\), where \(e \in E(G)\). Consider \(\mathbb{B}_M\) the sets of \(M\)-good paths and \(M\)-good cycles of \(G\). Similarly, define \(\mathbb{B}_N\). From Theorem 2.3, \(d(M) = |\mathbb{B}_M|\) and \(d(N) = |\mathbb{B}_M|\). Build the function \(\varphi_e : \mathbb{B}_M \to \mathbb{B}_N\) such that, for every cycle \(C \in \mathbb{B}_M\), \(\varphi_e(C) = C\) and, for every path \(P \in \mathbb{B}_M\),

\[
\varphi_e(P) = \begin{cases} 
  P, & \text{if } e \in P; \\
  P, & \text{if } e \notin P \text{ and if } f \in P \text{ then not } f \sim e; \\
  P \cup \{e\}, & \text{otherwise.}
\end{cases}
\]

By construction, for distinct paths or cycles belonging to \(\mathbb{B}_M\), we have distinct images in \(\mathbb{B}_N\). Then, \(\varphi_e\) is a injective function and so, \(d(M) \leq d(N)\).

In the general case, do \(N \setminus M = \{e_1, e_2, \ldots, e_k\}\), \(N_1 = M \cup \{e_1\}\), \(N_2 = M \cup \{e_1, e_2\}, \ldots, \) and \(N_k = N\). By the same argument used before, we get \(d(M) \leq d(N_1), d(N_1) \leq d(N_2), \ldots,\) and \(d(N_{k-1}) \leq d(N)\). Consequently, \(d(M) \leq d(N)\). □

**Theorem 3.2.** Let \(G\) be a graph with \(m\) edges. The minimum degree of \(G(M(G))\) is equal to \(m\).
Proof. Let $M$ be a matching of a graph $G$. Since $\emptyset \subseteq M$, from Theorem 3.1, $d(\emptyset) \leq d(M)$. Besides, by Proposition 2.2, $d(\emptyset) = m$ and, the result follows. □

An edge $e = uv$ of a graph $G$ is called a bond if $d(u) = d(v) = 2$ and $|N(u) \cap N(v)| = 1$. Note that if $e$ is a bond of $G$, $e$ is an edge of a triangle of graph. However, the reciprocal is not necessarily true.

The following lemma allows us to obtain, in the next section, a characterization of graphs (connected or not) for which the respective skeletons are regular. See that the skeleton of a polytope is always a connected graph even if the original graph is disconnected, [7].

**Lemma 3.3.** Let $G$ be a graph with $m$ edges such that $G$ is a disjoint union of stars and triangles. For every $M$, a matching of $G$, $d(M) = m$.

Proof. Let $G$ be a graph with $m$ edges. Suppose $M$ a matching of $G$. If $M = \emptyset$, the result follows straightforward from Proposition 2.2.

From hypotheses, $G$ is union of $r$ triangles $K_3$ and $p$ stars $S_{1,t_j}$, $1 \leq j \leq p$. So, it can be written as

$$G = rK_3 \bigcup_{j=1}^{p} S_{1,t_j},$$

for some non negative integers $r$, $p$ and $t_j$, $1 \leq j \leq p$ such that

$$m = 3r + \sum_{j=1}^{p} t_j.$$  \(\text{(4)}\)

Since (3) holds, each two distinct edges of $M$ belong to distinct components of $G$. Also, $M$ is a matching without common neighbors of $G$. Hence, the expression (3) can be rewritten as

$$G = r_1K_3 \bigcup_{j=1}^{p_1} S_{1,t_j} \cup r_2K_3 \bigcup_{z=1}^{p_2} S_{1,t_z},$$

where $r_1 + p_1$ is the cardinality of $M$ and $r_2 + p_2$ is the number of components without any edges of $M$. So, the number of edges that not are incident to any edge of $M$ is

$$k = 3r_2 + \sum_{z=1}^{p_2} t_z.$$  \(\text{(6)}\)
For each $i, 1 \leq i \leq r_1 + p_1$, let $e_i = u_iv_i \in M$, denote $s_i = d(u_i)d(v_i) - |N(u_i) \cap N(v_i)|$. If $e_i \in E(K_3)$ then $s_i = 3$. Otherwise, $e_i \in E(S_1,t_j)$ is a pendent edge of $G$ and $s_i = t_j$.

As $M$ is a matching without common neighbors in $G$, from (6) and applying Theorem 2.6, we get

$$d(M) = (3r_2 + \sum_{z=1}^{p_2} t_z) + \sum_{i=1}^{r_1+p_1} s_i = 3(r_1 + r_2) + \sum_{j=1}^{p_1} t_j + \sum_{z=1}^{p_2} t_z. \quad (7)$$

Once $r_1 + r_2 = r$ and $p_1 + p_2 = p$, from (4), we get $d(M) = m$. \hfill \Box

Figure 4 displays the graph $G = K_3 \cup S_{1,1}$ and its skeleton, $G(M(K_3 \cup S_{1,1}))$. For $M = \{e_1, e_2\}$, we have $r = r_1 = 1$ and $r_2 = 0$, $p = p_1 = 1$ and $p_2 = 0$. Besides, $k = 0$, $s_1 = 1$ and $s_2 = 3$. So, $d(M) = k + s_1 + s_2 = 0 + 1 + 3 = 4$.

![Figure 4: $G = K_3 \cup S_{1,1}$ and $G(M(G))$](image)

**Proposition 3.4.** Let $G$ be a graph with $m$ edges and $M = \{e\}$ be an one-edge matching of $G$. The degree of $M$ is $d(M) = m$ if and only if $e$ is either a bond or a pendent edge of $G$.

**Proof.** Let $G$ be a graph with $m$ edges and $e = uv$ an edge of $G$. We know that $m - d(u) - d(v) + 1$ is the number of edges that neither is incident to $u$ nor to $v$. From Theorem 2.6, $d(\{e\}) = m - d(u) - d(v) + 1 + d(u)d(v) - t$, where $t = |N(u) \cap N(v)|$. But, $d(\{e\}) = m$ if and only if $d(u)d(v) - t = d(u) + d(v) - 1$, i.e., $(d(u) - 1)(d(v) - 1) = t$. Since $d(u) > t$ and $d(v) > t,$
\[(d(u) - 1)(d(v) - 1) \geq t^2.\] Moreover, \((d(u) - 1)(d(v) - 1) = t\) if and only if \(t = 0\) or \(t = 1\). In the first case, \(e\) is a pendent edge and, otherwise, \(d(u) = d(v) = 2\) and so, \(e\) is a bond.

Note that, if \(G\) is a graph without bonds and pendent edges, \(G(M(G))\) has only one vertex \(M = \emptyset\) with the minimum degree. It is not difficult to see that all 2-connected graphs satisfy this property.

4 Characterizing regular skeletons and matchings with minimum degree

We close this paper with two characterizations. The first gives a necessary and sufficient condition under a graph in order to have the skeleton of its matching polytope as a regular graph. The second identifies all matchings of a graph which correspondent vertices in the skeleton have the minimum degree.

**Proposition 4.1.** A graph \(G\) with \(m\) edges is a disjoint union of stars and triangles if and only if \(G(M(G))\) is an \(m\)-regular graph.

Proof. From Lemma 3.3, if \(G\) is a disjoint union of stars or triangles, the skeleton of its matching polytope is a regular graph. Suppose \(G(M(G))\) an \(m\)-regular graph. Then, for every \(e \in E(G)\), we have \(d(\{e\}) = d(\emptyset)\) and, from Proposition 2.2, \(d(\{e\}) = m\). Besides, by Proposition 3.4, this occurs only if \(e\) is a bond or a pendent edge of \(G\). 

**Theorem 4.2.** Let \(M \neq \emptyset\) is a matching of a graph \(G\) with \(m\) edges. The degree of \(M\) is \(m\), that is, \(d(M) = m\) if and only if every edge of \(M\) is a bond or a pendent edge of \(G\).

Proof. Let \(G\) be a graph with a matching \(M\). Suppose there is \(e \in M\) such that \(e\) is neither a bond nor a pendent edge of \(G\). So, from Proposition 3.4 and Theorem 3.1, \(m < d(\{e\}) \leq d(M)\). Consequently, \(m \neq d(M)\). By the contrapositive, if \(d(M) = m\), every edge of \(M\) is a pendent edge or a bond of the graph.

Suppose now that \(M\) is a matching of \(G\) with \(s\) edges such that if \(e \in M\), \(e\) is a bond or \(e\) is a pendent edge of \(G\). Let \(N\) be a matching such that \(N \sim M\) in \(G(M(G))\). From here and by Theorem 1.2, \(M \Delta N\) is an \(M\)-good path \(P\) or an \(M\)-good cycle \(C\) of \(G\). Since \(C\) is an even cycle, \(C \neq K_3\). So, \(C\) does not have bonds. Consequently, \(M \Delta N\) is a path.

Concerning the path \(P\), only their pendent edges can belong to \(M\). Moreover, once \(P\) is an alternated path, it has length at most length 3. Hence, there are only the following possibilities to \(P\):
1. If $P$ is an $oo$-$M$-path, then $P = e$, where $e \in M$, or $P = e_1f e_2$, where $f \notin M$ and $I(f) \cap M = \{e_1, e_2\}$. In the first case, there are $s$ possibilities to $P$ and, in the second, there are $t_1$ possibilities to $P$, where $t_1$ is the number of edges of $E(G) \setminus M$ such that the both terminal vertices are incident to an edge of $M$;

2. If $P$ is a $cc$-$M$-path, $P = f$, where $f \notin M$ and $I(f) \cap M = \emptyset$. Here, there are $t_2$ possibilities to $P$, where $t_2$ is the number of edges of $E(G) \setminus M$ for which every edge of $M$ does not incident to the any end vertices of those edges;

3. Finally, if $P$ is an $oc$-$M$-path, then $P = ef$, where $f \notin M$ and $I(f) \cap M = \{e\}$. In this last case, there are $t_3$ possibilities to $P$, where $t_3$ is the number of edges of $E(G) \setminus M$ which only one end vertex of edge is incident to some edge of $M$.

From (1), (2) and (3) possibilities above, $s + \sum_{i=1}^{3} t_i = s + |E(G) \setminus M| = m$ is the number of the possibilities to have $P$ as an $M$-good path of $G$. From Theorem 2.3 it follows that $d(M) = m$. □

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