Ward Identity Implies Recursion Relation at Tree and Loop Level

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Abstract

In this article, we use Ward identity to calculate tree and one loop level off shell amplitudes in pure Yang-Mills theory with a pair of external lines complexified. We explicitly prove Ward identity at tree and one loop level using Feynman rules, and then give recursion relations for the off shell amplitudes. We find that the cancellation details in the proof of Ward identity simplifies our derivation of the recursion relations. Then we calculate three and four point one loop off shell amplitudes as examples of our method.

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I. INTRODUCTION

At tree level, the amplitudes of pure Yang-Mills fields can be written as rational functions of external momenta and polarization vectors in spinor form \[2-7\]. Such rational functions can be analyzed in detail in algebra system. According to this, BCFW recursion relation was proposed and developed in \[8-10\], and then proved in \[11\] using the pole structure of the tree level on shell amplitudes. This has been an exiting progress on the amplitudes in pure Yang-Mills theory. For theory with massive fields \[12-16\], the amplitudes are also rational functions of external momenta and polarization vectors in spinor form.

At loop level, although the whole amplitudes are no longer rational functions in general, they can be decomposed into some basic scalar integrals with coefficients being rational functions of external spinors \[17, 18\]. The coefficient structures are studied in depth in \[19, 20, 22\]. On the other hand, the integrands of the amplitudes are rational functions of the external spinors and integral momenta. For the N=4 planar super Yang-Mills theory, \[23\] gives an explicit recursive formula for the all-loop integrand of scattering amplitudes.

The amplitudes in gauge theory are constrained by gauge symmetry. This leads to Ward identity which constrains the amplitudes at all loop level. Inspired by the BCFW momenta shift, we considered the Ward identity for tree level amplitudes with complexified momenta for a pair of external lines, and then obtained a recursion relation for the boundary terms using BCFW technique in our recent article \[1\]. However, in \[1\], we chose a particular momenta shift such that the external states of the complexified lines are independent of the complex parameter \(z\). Then a natural question is how to obtain a recursion relation for other possible momenta shifts. Furthermore, is it possible to obtain the full amplitudes from the Ward identity, and to extend the technique to one loop amplitudes? In this article, we will give positive answers to all these questions.

In section II, we first give the proof of Ward identity at tree level using Feynman rules directly, and then derive the recursion relation for off shell amplitudes, where the cancellation details in the proof of Ward identity helps to simplify the recursion relations. Section III is parallel to section II. We first extend the proof of Ward identity to one loop level and then derive the recursion relation for one loop off shell amplitudes. Our technique does not rely on the on-shell momenta shifts. Also, in our calculation using the recursion relation, four point vertexes are not used explicitly. We calculate three and four point one loop off shell amplitudes as examples in section III C.
II. WARD IDENTITY AND IMPLIED RECURSION RELATION AT TREE LEVEL

In [1], we directly proved complexified Ward identity for pure Yang-Mills fields at tree level, and then used it to deduce a recursion relation for the boundary terms of the complexified amplitudes. Here we generalize the method to deduce a recursion relation for tree level amplitudes with one external off shell line. This section will serve as a basis for our generalization to one loop level in the next section. We will call the external off shell line $L_{\text{off}}$ with momentum $k_{\text{off}}$, and the corresponding off shell amplitudes $A_{\mu}$.

A. Proof of Ward Identity at Tree Level

Although done in our previous paper [1], we briefly summarize some key points in the direct proof of tree level Ward identity, since these points are useful for deriving tree level recursion relation and also will be part of the proof at one loop level in the next section.

The amplitude is complexified by shifting the momenta of a pair of external lines. We choose $L_{\text{off}}$ and one on shell line $L_{s}$ with momentum $k_{s} = \lambda_{s} \bar{\lambda}_{s}$, and the shift is:

\[ k_{s} \rightarrow k_{s} - z \eta, \quad k_{\text{off}} \rightarrow k_{\text{off}} + z \eta, \quad (1) \]

where $z$ is the complexifying parameter and $\eta$ should satisfy $\eta^{2} = 0$ and $k_{s} \cdot \eta = 0$.

The color ordered Feynman rules of the gauge field are as in [6], with outgoing momenta. We also write the Feynman rules for ghost fields here in Figure 1, which will be used in the next section.

FIG. 1: Ghost field color ordered Feynman rules. Dashed line for ghost field and solid line for gauge field.

For a three point vertex with line 1, 2 and $L_{\text{off}}$ in anti-clockwise order, we write it in the following form:

\[ V_{\mu_{1} \mu_{2} \mu} \equiv S_{\mu_{1} \mu_{2} \mu} + R_{\mu_{1} \mu_{2} \mu} + M_{\mu_{1} \mu_{2} \mu}, \quad (2) \]
where
\[ S_{\mu_1\mu_2} = \frac{i}{\sqrt{2}} ( \eta_{\mu_1\mu_2} (k_1 - k_2) ) \]
\[ R_{\mu_1\mu_2} = \frac{i}{\sqrt{2}} (-2 \eta_{\mu_2\mu} (k_{\text{off}})_{\mu_1} + 2 \eta_{\mu_1\mu} (k_{\text{off}})_{\mu_2} ) \]
\[ M_{\mu_1\mu_2} = \frac{i}{\sqrt{2}} (-\eta_{\mu_2\mu} (k_1)_{\mu_1} + \eta_{\mu_1\mu} (k_2)_{\mu_2} ) \].

We will refer to these terms as S, R and M parts of the vertex. Contracting this vertex with \( k_{\text{off}} \), we get:
\[ k_{\text{off}}^\mu \cdot V_{\mu_1\mu_2} = \frac{i}{\sqrt{2}} \eta_{\mu_1\mu_2} k_2^2 - \frac{i}{\sqrt{2}} \eta_{\mu_1\mu_2} k_1^2 + \frac{i}{\sqrt{2}} k_2 \mu_2 k_3 \mu_1 - \frac{i}{\sqrt{2}} k_1 \mu_1 k_3 \mu_2 , \]
and we represent these terms by the symbols in Figure 2. These terms are frequently used throughout the paper, and we will call the terms in the first line of Figure 2 as solid triangle terms, and the second line terms as hollow triangle terms.

![Figure 2](image-url)  

FIG. 2: Notations for 4. We specialize \( L_{\text{off}} \) using photon line.

Then a proof of tree level Ward identity can be shown in several steps. Assume it holds for N-point and less than N-point amplitudes (for example, 3-point case can be immediately checked), we will show how it holds for (N+1) point amplitudes. We choose \( L_{\text{off}} \) as the (N+1)-th line. We can construct an (N+1)-point color ordered diagram from an N-point one by inserting \( L_{\text{off}} \) to an N-point diagram between Line 1 and Line N.

First, when \( L_{\text{off}} \) is inserted to a propagator or Line 1 or Line N, we denote the vertex as \( V_{\text{off}} \), and contract it with \( k_{\text{off}} \), the following two hollow triangle terms in Figure 3 vanish due to less-point Ward Identities or the on-shell conditions of Line 1 or N. The meaning of the symbols are in Figure 2.

Second, \( L_{\text{off}} \) is inserted to a three-point vertex in the N-point diagram. These terms and the remaining terms, ie. solid triangle terms, from the above case can be re-combined as in Figure 4 to cancel each other.

Figure 3 and Figure 4 constitute the proof of Ward identity at tree level.
FIG. 3: When $L_{\text{off}}$ is inserted to a propagator or Line 1 or Line N, these terms vanish due to the less point Ward identity or the on-shell conditions of Line 1 or N. $A_1$ and $A_2$ are sub amplitudes.

\[
\begin{align*}
A_1 &+ A_2 = 0 \\
A_1 &+ A_2 = 0
\end{align*}
\]

FIG. 4: A group of diagrams cancel. In (a) and (b), the cancellation is solely due to the vertex structures, not dependent on whether the legs are on shell or off shell. (c) and (d) are due to on shell conditions for Line 1 and Line N: $k_1^2 = 0$ and $k_N^2 = 0$.

\[
\begin{align*}
(a) &+ (b) = 0, \\
(c) &+ (d) = 0.
\end{align*}
\]

B. Recursion Relation for Tree Level Off Shell Amplitudes

As discussed in [1], from the complexified Ward identity $\hat{k}_{\text{off}}^{\mu} \cdot \hat{A}_\mu = 0$, by a derivative over $z$ we get:

\[
\hat{A}_\mu \eta^{\mu}|_{z \to 0} = -\frac{d \hat{A}_\mu \hat{k}_{\text{off}}^{\mu}|_{z \to 0}}{dz}.
\]

(5)

The symbol $\hat{\cdot}$ represents that the quantity is complexified, i.e. depends on the shift parameter $z$. Here $k_{\text{off}}$ is shifted as in $\hat{k}_{\text{off}} = k_{\text{off}} + z \eta$. Our destination is to calculate $A_\mu$,
and we will realize it by calculating the right hand side of 5.

We name the vertex which contains $L_{\text{off}}$ as $V_{\text{off}}$. At tree level, we have the following three cases:

1. the derivative acts on a propagator;

2. the derivative acts on a three point vertex which does not contain $L_{\text{off}}$;

3. the derivative acts on a three point vertex $V_{\text{off}}$.

In the first and second cases, when $V_{\text{off}}$ is a three point vertex, we write $k_{\text{off}}^\mu \cdot V_{\text{off}} \mu$ as in Figure 2 and take out the hollow triangle terms. These terms, together with the terms from the third case where the derivative acts on the M part of $V_{\text{off}} \mu$ as written in 3, add up to be 0 due to Ward identity for some sub amplitudes.

From above we know that in the first and second cases, we only need the solid triangle terms for $k_{\text{off}}^\mu \cdot V_{\text{off}} \mu$ as represented in Figure 2 when $V_{\text{off}}$ is a three point vertex; in the third case, $\frac{d}{dz}$ only need to act on the S and R part of $V_{\text{off}} \mu$ as written in 3. The first two cases can be further simplified. Due to (a) and (b) in Figure 4 the terms relevant for the first two cases are reduced to those with $k_{\text{off}}$ neighboring to the three point vertex or the propagator to be differentiated, as depicted in Figure 5.

Thus, for the first case, the diagrams are (a) and (b) in Figure 5. The contributions from (a) and (b) to $-\frac{dA_{\mu}}{dz}k_{\text{off}}|_{z \rightarrow 0}$ are:

for (a) $-\frac{i\sqrt{2}}{k_{A_1}^2 k_{A_2}^2} k_{A_1} \cdot \eta \ A_1 \cdot A_2$,

for (b) $\frac{i\sqrt{2}}{k_{A_1}^2 k_{A_2}^2} k_{A_2} \cdot \eta \ A_1 \cdot A_2$. (6)

As noted in Figure 4, $\{A_i\}$ are some less less point amplitudes. $k_{A_i}$ is the total momentum of the external legs contained in the sub amplitude $A_i$. If some $A_i$ just contains one external line $L_m$, we define this $A_i$ to be $i k_m^2 \epsilon_m$, and accompany it with a propagator $\frac{i}{k_{A_i}^2} = \frac{i}{k_m^2}$. Another point is that, although $k_s \cdot \eta = 0$, we keep it in the evaluations here and below as if it is not 0, as will be explained at the end of this subsection.

The second case corresponds to (c) (d) (e) and (f) in Figure 5 and the contributions to $-\frac{dA_{\mu}}{dz}k_{\text{off}}|_{z \rightarrow 0}$ are:

for (c) $\frac{1}{2k_{A_1}^2 k_{A_2}^2 k_{A_3}^2} (A_3 \cdot \eta \ A_1 \cdot A_2 + A_1 \cdot \eta \ A_2 \cdot A_3 - 2A_2 \cdot \eta \ A_1 \cdot A_3)$,
FIG. 5: Terms to be calculated for tree-level off-shell amplitudes. Here and following, the dark solid circle symbol • denotes where we act $\frac{d}{dz}$. We shift $L_{\text{off}}$ and some other line $L_s$. \{\hat{A}_i\} denote some sub-diagrams with less external states. \hat{A}_k includes $L_s$. In different diagrams, the same $A_k$ symbols do not mean the same sub amplitudes. They sum over all allowed sub amplitudes.

for (d) $\frac{1}{2k_{A_1}^2 k_{A_2}^2 k_{A_3}^2}(-A_3 \cdot \eta A_1 \cdot A_2 + 2A_1 \cdot \eta A_2 \cdot A_3 - A_2 \cdot \eta A_1 \cdot A_3)$,

for (e) $\frac{-1}{2k_{A_1}^2 k_{A_2}^2 k_{A_3}^2}(-2A_3 \cdot \eta A_1 \cdot A_2 + A_1 \cdot \eta A_2 \cdot A_3 + A_2 \cdot \eta A_1 \cdot A_3)$,

for (f) $\frac{-1}{2k_{A_1}^2 k_{A_2}^2 k_{A_3}^2}(-A_1 \cdot \eta A_2 \cdot A_3 + 2A_2 \cdot \eta A_1 \cdot A_3 - A_3 \cdot \eta A_1 \cdot A_2)$.

(7)

And the third case corresponds to (g) and (h) in Figure [5] whose contributions are:

for (g) $\frac{-i}{\sqrt{2}k_{A_1}^2 k_{A_2}^2}(k_{\text{off}} \cdot \eta A_1 \cdot A_2 + 2A_1 \cdot \eta k_{\text{off}} \cdot A_2 - 2A_2 \cdot \eta k_{\text{off}} \cdot A_1)$,
for \((h)\) \[ -\frac{i}{\sqrt{2k^2_{A_1}k^2_{A_2}}}(-k_{\text{off}} \cdot \eta A_1 \cdot A_2 + 2A_1 \cdot \eta k_{\text{off}} \cdot A_2 - 2A_2 \cdot \eta k_{\text{off}} \cdot A_1). \] (8)

As explained before, in this case \(\frac{d}{dz}\) only need to act on the S and R part of \(V_{\text{off}} \mu\) as written in 3.

It can be observed that, for \(L_s\) contained in \(A_1\) or \(A_2\) or \(A_3\), the expressions are the same. In the case when \(L_s\) is contained in \(A_2\) we should sum (d) and (e) in Figure 5 to see that the expression is the same as when \(L_s\) is contained in \(A_1\) or \(A_3\). The common expression is:

\[ \frac{1}{2k^2_{A_1}k^2_{A_2}k^2_{A_3}}(A_3 \cdot \eta A_1 \cdot A_2 + A_1 \cdot \eta A_2 \cdot A_3 - 2A_2 \cdot \eta A_1 \cdot A_3). \] (9)

\(6\) and \(7\) summed up also give a common expression, regardless of whether \(L_s\) is contained in \(A_1\) or \(A_2\):

\[ -\frac{i}{\sqrt{2k^2_{A_1}k^2_{A_2}}}((k_{A_1} - k_{A_2}) \cdot \eta A_1 \cdot A_2 + 2A_1 \cdot \eta k_{\text{off}} \cdot A_2 - 2A_2 \cdot \eta k_{\text{off}} \cdot A_1) \] (10)

The final tree level result for \(A_\mu \eta^\mu\) is the sum of 9 and 10, which can be written in the form of \(\tilde{A}_\mu \eta^\mu\). In the expressions we should sum over all allowed allocations of the on shell external legs into \(\{A_i\}\). It is easy to show that the off shell amplitude \(A_\mu = \tilde{A}_\mu\). In four dimensional spacetime, we only need to find 4 independent \(\eta_i\) such that \(A_\mu \eta^\mu = \tilde{A}_\mu \eta^\mu\). Since in the shift \(\eta^\mu\) is required to satisfy \(k_s \cdot \eta = 0\) and \(\eta^2 = 0\), the three choices of \(\eta_i\) as \(\epsilon^+_s\), \(\epsilon^-_s\) or \(k_s\) satisfy \(A_\mu \eta^\mu_i = \tilde{A}_\mu \eta^\mu_i\). The remaining choice of \(\eta_i\) can be chosen as \(k_{\text{off}}\).

This is not obvious to satisfy \(A_\mu \eta^\mu_i = \tilde{A}_\mu \eta^\mu_i\). However, in our calculations we have kept the terms \(k_s \cdot \eta\) as if it is not 0, and by this trick it comes out that \(\tilde{A}_\mu k_{\text{off}}^\mu = 0 = A_\mu k_{\text{off}}^\mu\).

In conclusion \(A_\mu = \tilde{A}_\mu\), where \(\tilde{A}_\mu\) is contained in the sum of 9 and 10 in form of \(\tilde{A}_\mu \eta^\mu\).

Compare to Berends-Giele recursion relation \([4]\), it is seen that 9 corresponds to \(k_{\text{off}}\) contained in a four point vertex, and 10 is equivalent to the contribution when \(k_{\text{off}}\) is contained in a three point vertex. This on one hand supports the correctness of our method, and on the other hand a little undermines the value of our method at tree level. There are also other recursion relations for off shell tree level amplitudes, eg. \([24]\). Yet we are going to extend our method to one loop level, where the situation is much more complicated and our method is new.

### III. WARD IDENTITY AND IMPLIED RECURSION RELATION AT 1-LOOP LEVEL

In this section we are going to extend our method to 1-loop level. We will show how complexified Ward identity holds at 1-loop level and then we deduce the corresponding
recursive calculation of 1-loop off shell amplitude. Using our method, we will calculate three and four point 1-loop off shell amplitudes as examples. In our calculation we use FDH scheme \[21\], in which only the loop momentum is continued to dimensionality different from 4.

We first explain some subtleties at loop level. First, after momentum shifting, some lines on the loop carry complex momenta. This brings ambiguities to the meaning of the loop integral and prevents us from translating the loop momentum \( l \to l + k \) or flip it \( l \to -l \). However, according to equation 5, what we need for our technique is the derivative of the integral at the value \( z \to 0 \). And it is easy to prove that:

\[
\int d^D l \frac{d}{dz} f(l^\mu, \hat{k}^\mu)|_{z \to 0} = \int d^D l \frac{d}{dz} f(-l^\mu, \hat{k}^\mu)|_{z \to 0}, \tag{11}
\]

\[
\int d^D l \frac{d}{dz} f(l^\mu, \hat{k}^\mu)|_{z \to 0} = \int d^D l \frac{d}{dz} f(l^\mu + \hat{k}^\mu, \hat{k}^\mu)|_{z \to 0}. \tag{12}
\]

Thus for our technique, we can translate or flip the loop momentum even when the integrand is complex.

Second, some attention should be paid to color orderings and symmetry factors. At tree level there is only one color ordering contributing to the the primitive part of the color ordered amplitudes. At one loop level, most diagrams also only have one color ordering. However, for gauge field loop diagrams, there are three kinds of diagrams having two color orderings. Those are diagrams with two vertexes on the loop: two three-point vertexes; two four-point vertexes; a three-point vertex and a four-point vertex. For the first and second cases, the contributions from the two color orderings are the same at integrand level. For the third case, the contributions from the two color orderings at integrand level differ by a translation and flip of the loop momentum, and due to 12 the two orderings contribute the same in our method after integration. In a word, these three kinds of diagrams have a factor of 2 from possible color orderings. At the same time, these three kinds of diagrams have symmetry factor \( \frac{1}{2} \), just canceling the doubling from color orderings. For ghost loop diagrams, those with two vertexes on the loop also have a doubling from two color orderings, while there is only either clockwise or anti clockwise ghost loop when there are only two vertexes on the ghost loop. We replace the doubling from color orderings by drawing both clockwise and anti-clockwise ghost loop diagrams, which are actually equal when there are only two vertexes on the ghost loop.

Finally, as our convention for the loop momentum for all our loop diagrams, we specify the loop momentum in the following way. For each external leg \( L_i \), when we want to make a path from it to the loop, there is one definite vertex \( V \) on the loop first encountered in the path, then we say the external leg \( L_i \) is associated with this loop vertex \( V \). We find
the vertex with which \( L_{\text{off}} \) is associated, call it \( V_0 \). Assume all the lines associated with \( V_0 \) in color ordering are \( L_j, L_{j+1}, \ldots, L_N, L_{\text{off}}, L_1, \ldots, L_i \), then we assign the momentum of the first loop propagator on the counter clockwise side of \( V_0 \) as \( \ell - k_1 - \cdots - k_i \), with the loop momentum flowing in counter clockwise direction. \( \ell \) is to be integrated. External leg momenta are outgoing.

### A. Proof of Ward Identity at 1-Loop Level

In this section we use \( A_l \) for 1-loop amplitudes, and \( A_t \) for tree level amplitudes.

Two point and three point 1-loop Ward identity is easy to verify directly. Similar to the proof at tree level, we use induction, assume Ward identity holds for \( N \) and less than \( N \) point one loop amplitudes, and construct an \( (N+1) \) point diagram from an \( N \) point one by inserting \( k_{\text{off}} \) in different places. We denote the vertex with \( k_{\text{off}} \) as \( V_{\text{off}} \) and when \( V_{\text{off}} \) is a three point vertex, we decompose \( k_{\text{off}} \cdot V_{\text{off}} \) as in Figure 2.

**Case 1.** When \( k_{\text{off}} \) is linked to a propagator (including gauge field loop propagator) or external line of the \( N \) point diagram, the solid triangle terms from \( k_{\text{off}} \cdot V_{\text{off}} \) mostly cancel the terms with \( k_{\text{off}} \) in a four point vertex, in the manner of Figure 4. Only the terms in Figure 6 remain.

![Figure 6](image.png)

**FIG. 6:** The remaining terms in the first case that does not cancel in the manner of Figure 4. The loop is ghost loop and has two directions.

**Case 2.** We need to consider the hollow triangle terms from \( k_{\text{off}} \cdot V_{\text{off}} \) remaining from the above case, and we divide them into two sub cases:

**Sub Case 1.** When \( V_{\text{off}} \) is not on the loop, these terms vanish due to Ward identity for less point amplitudes in the induction assumption, similar to the tree level counterpart Figure 8.

**Sub Case 2.** The remaining sub case is that \( V_{\text{off}} \) is on the gauge field loop. We analyze one of the hollow triangle terms in Figure 7. The Figure has considered all the
possible cases with the first right side vertex to be three or four point, and different types of second vertex relevant. When the first right side vertex is a three point vertex, acting on it with one of the factor in the hollow triangle term, we can again decompose it as in Figure 2 into solid and hollow triangle terms. (a) and (b) in Figure 7 are in fact the same diagrams as in Figure 4. (c) vanishes due to tree level Ward identity, and (d) is due to on shell condition for external legs besides $L_{\text{off}}$. Then the type of term in (e) of Figure 7 remains, which is a hollow triangle term staying on the loop, and it will act on the next vertex on the loop, repeating the same processes as in (a)-(d) of Figure 7 until it meets the final vertex on the loop. For this sub case, the remaining diagrams are in Figure 8.

\[
\begin{align*}
(a) \quad L_{\text{off}} + V_{\text{off}} A_{i_2} A_{i_1} A_{i_1} = 0 \\
(b) \quad L_{\text{off}} + V_{\text{off}} A_{i_3} A_{i_2} A_{i_2} = 0 \\
(c) \quad L_{\text{off}} = 0 \\
(d) \quad V_{\text{off}} A_{i_1} = 0, \text{ when } A_{i_1} \text{ is an (on shell) external leg.}
\end{align*}
\]

(e) \quad Repeat the processes in (a) (b) (c) and (d).

FIG. 7: Analysis of the action of the hollow triangle terms in **Sub Case 2.** The dashed line is not ghost field, but just part of the loop diagram not relevant.
FIG. 8: The terms from **Sub Case 2**. Except the hollow triangle terms at $V_{\text{off}}$, other hollow and solid triangle terms on the loop are induced from the hollow triangle term of the previous loop vertex, as described in the text of **Sub Case 2**. and Figure 7.

**Case 3.** The remaining case: $k_{\text{off}}$ is linked to a ghost propagator of the $N$ point diagram, as in Figure 9.

FIG. 9: Diagram for **Case 3**. The ghost loop can be in two directions.

By direct and simple calculations, the terms from Figure 6, Figure 8 and Figure 9, with same set of sub amplitudes $A_i$, add up to be 0. Combine **Case 1, 2, 3**, we have proven that Ward identity holds at $N + 1$ point one loop level. Thus by induction we have proven Ward identity holds at one loop level using Feynman rules in a direct way.
B. Recursion Relation for Loop Level Off Shell Amplitudes

Similar to the tree level off shell amplitudes calculation, we can use \( \hat{A}_\mu \eta^\mu |_{z \to 0} = -\frac{d}{dz} \hat{k}_\text{off} |_{z \to 0} \) to calculate one loop level off shell amplitudes. The experience at tree level, and the details of how Ward identity holds at one loop level discussed in the last subsection, help us to simplify our discussion and calculation of one loop level off shell amplitudes.

When the derivative acts on a gauge field propagator or a vertex which is not on the loop, we can use the expressions derived in section II B directly, ie. 9 and 10, for the contribution to \(-\frac{d}{dz} \hat{A}_\mu \eta^\mu |_{z \to 0}\):

\[
\frac{1}{2k_1^2 k_2^2 k_3^2} \left( A_1^{l/t} \cdot \eta \cdot A_1^{l/t} \cdot A_2^{l/t} \cdot A_3^{l/t} + A_1^{l/t} \cdot \eta \cdot A_2^{l/t} \cdot A_3^{l/t} - 2A_2^{l/t} \cdot \eta \cdot A_1^{l/t} \cdot A_3^{l/t} \right),
\]

\[
-\frac{i}{\sqrt{2k_1^2 k_2^2 k_2^2}} \left( (k_{A_1} - k_{A_2}) \cdot \eta \cdot A_1^{l/t} \cdot A_2^{l/t} + 2A_1^{l/t} \cdot \eta \cdot A_2^{l/t} - 2A_2^{l/t} \cdot \eta \cdot A_1^{l/t} \right).
\]

In 13, we allocate the on shell external legs into \( \{ A_i^{l/t} \} \) in color ordering, with one and only one \( A_i^{l/t} \) being one loop level. As in tree level, in each expression we should sum over all allowed allocations of the on shell external legs into \( \{ A_i^{l/t} \} \).

When the derivative acts on a gauge field loop propagator or a loop vertex, these are shown in Figure 10. For the same reasons as discussed in tree level recursion calculation, in (a) to (f), we only need to consider \( L_{\text{off}} \) next to the propagator or vertex differentiated and only need the solid triangle term. In (g), we only differentiate the S and R terms of the vertex. The M part of the vertex in (g) will be dealt with in the following. In Figure 10, we encounter tree level two line off shell amplitudes \( A_{\sigma \rho}^t \). This quantity can also be calculated recursively using our method, but in this paper we will not discuss it, and will use Feynman rules to calculate it in our example. Those \( A^t \) without sub indices are tree level one line off shell amplitudes, which can apply our method in the previous section. (a) is 0 due to our convention for the loop momentum, described in the paragraph before section III A.

Regardless of whether the other shifted line \( L_n \) is among \( \{ L_1, L_2, \cdots, L_j \} \) or among \( \{ L_{j+1}, \cdots, L_N \} \), the contributions to \(-\frac{d}{dz} \hat{k}_\text{off} |_{z \to 0}\) from Figure 10 are (we use \( K_{m,n} \) to represent for \( k_m + k_{m+1} + \cdots + k_n \)):

\[
(a) \quad : 0
\]

\[
(b) + (g) : \frac{-i}{\sqrt{2l^2(l + k_{\text{off}})^2}} \left( (2l + k_{\text{off}}) \cdot \eta \cdot A_{\sigma \rho}^t(1,2,\cdots,N) \ g^{\sigma \rho} + 2\eta^\sigma \ A_{\sigma \rho}^t(1,2,\cdots,N) \ k^\rho - 2\eta^\rho \ A_{\sigma \rho}^t(1,2,\cdots,N) \ k^{\sigma} \right)
\]
\[
(d) + (e) : \frac{1}{2l^2(l - K_{1,j})^2K_{j+1,N}^2} (A^t(j + 1, \cdots , N) - \eta^t A^t_{\sigma \rho}(1, 2, \cdots , j) g^{\sigma \rho} \\
- 2A^t(1, \cdots , j) \eta^t A^t_{\sigma \rho}(1, 2, \cdots , j)) \eta^t A^t_{\sigma \rho}(1, 2, \cdots , j) \\
(c) + (f) : \frac{1}{2(l + k_{\text{off}})^2(l - K_{1,j})^2K_{1,j}^2} (A^t(1, \cdots , j) - \eta^t A^t_{\sigma \rho}(j + 1, \cdots , N) g^{\sigma \rho} \\
- 2A^t(1, \cdots , j) \eta^t A^t_{\sigma \rho}(j + 1, \cdots , N) + A^t(1, \cdots , j) \eta^t A^t_{\sigma \rho}(j + 1, \cdots , N) )
\]

FIG. 10: Diagrams with derivative acting on the propagator or vertex on the loop, which cannot directly apply the tree level results.

The final contributions to \(- \frac{d}{dz} \hat{A}_\mu(z \to 0)|_{z=0}\) come from the derivatives in the diagrams of Figure 6, Figure 8 and Figure 9. Denoting the diagrams as \(D_i\), since \(\sum D_i \cdot k_{\text{off}} = 0\) from the last subsection, we have \(- \sum \frac{dD_i}{dz} \hat{k}_\mu|_{z=0} = \sum D_i \mu \eta^\mu\). This is like an opposite
operation compared to the method in the current paper to deal with the set of diagrams $D_i$, but it simplifies the local calculation, eg. the diagrams in Figure 6 turn out to be not contributing. We use $K_{A_{m,n}}$ to represent for $k_{A_m} + k_{A_{m+1}} + \cdots + k_{A_n}$, with $k_{A_i}$ the total momentum of the external legs contained in the sub amplitude $A_i$. The total momentum conservation is then $K_{A_{1,n}} + k_{\text{off}} = 0$. The contributions to $-\frac{dA_{\mu}}{dz}k_{\mu}|_{z \to 0}$ from derivatives in Figure 6 Figure 8 and Figure 9 are:

$$
(-i)^n (l - K_{A_{1,1}}) \cdot A_2 (l - K_{A_{1,2}}) \cdot A_3 \cdots (l - K_{A_{1,n-2}}) \cdot A_{n-1} \\
\frac{(\sqrt{2})^{n+1}k_{A_1}^2 \cdots k_{A_n}^2(l - K_{A_{1,1}})^2 \cdots (l - K_{A_{1,n-1}})^2}{2l \cdot A_1 (l - K_{A_{1,n-1}}) \cdot A_n (2l + k_{\text{off}}) \cdot \eta} \\
+ \frac{l \cdot A_1 A_n \cdot \eta}{l^2} + \frac{(l - K_{A_{1,n-1}}) \cdot A_n A_1 \cdot \eta}{l + k_{\text{off}}^2}
$$

(15)

This expression is well defined when $n \geq 2$. Especially when $n = 2$, one should multiply the pre-factor with each term in the bracket to see that it is well defined. When $n = 1$, the last two terms in the bracket vanish.

13, 14 and 15 constitute our expressions for recursively calculating one loop off shell amplitudes. In each expression, eg. in the above one 15 we should sum over all the allowed different allocations of the on shell external legs into $A_1, \cdots, A_n$, with $n = 1, 2, \cdots, N$. This summation is not written explicitly in the expressions. Similar to our statement in the tree level counterpart, summing over 13 14 and 15 we get a form $A_{\mu}^I \eta^\mu$, with the $A_{\mu}^I$ our wanted one loop off shell amplitude.

C. Examples of 1-loop Off Shell Amplitudes

As an application and verification of our method, we have computed three and four point one loop amplitudes with one off shell leg, ie. $A_{\mu}^I(k_1, k_2)$ and $A_{\mu}^I(k_1, k_2, k_3)$, by summing up the contributions from 13 14 and 15. We use the integral reduction method in [25] to reduce the integrals to scalar integrals. We use the following notations for the scalar integrals:

$$
B_0[1, 3] = \int \frac{d^Dl}{(2\pi)^D l^2(l - k_1 - k_2)^2}, \quad B_0[1, 4] = \int \frac{d^Dl}{(2\pi)^D l^2(l - k_1 - k_2 - k_3)^2}, \\
B_0[2, 4] = \int \frac{d^Dl}{(2\pi)^D l^2(l - k_2 - k_3)^2}, \quad C_0[1, 2, 3] = \int \frac{d^Dl}{(2\pi)^D l^2(l - k_1)^2(l - k_1 - k_2)^2}, \\
C_0[1, 2, 4] = \int \frac{d^Dl}{(2\pi)^D l^2(l - k_1)^2(l - k_1 - k_2 - k_3)^2},
$$
We start from the two point function:

\[ A^I_{\mu
u}(k) = \frac{2}{2(1-D)}(k^2 g_{\mu\nu} - k_\mu k_\nu) \int \frac{d^Dl}{(2\pi)^D} \frac{1}{l^2(l-k)^2}. \]  

Then we can calculate three point one loop off shell amplitude using our method:

\[
A^I_\mu(k_1, k_2) = \frac{1}{2\sqrt{2}} \left[ k_1 \cdot \epsilon_2 \epsilon_1 \mu - k_2 \cdot \epsilon_1 \epsilon_2 \mu - \frac{2D-5}{D-1} \epsilon_1 \cdot \epsilon_2 (k_1 - k_2)_\mu \\
+ \frac{D-4}{D-1} \frac{k_1 \cdot \epsilon_2 k_2 \cdot \epsilon_1}{k_1 \cdot k_2} (k_1 - k_2)_\mu \right] B0[1, 3] \\
+ \frac{1}{2\sqrt{2}} k_1 \cdot k_2 [-3 \epsilon_1 \cdot \epsilon_2 (k_1 - k_2)_\mu + 4k_1 \cdot \epsilon_2 \epsilon_1 \mu - 4k_2 \cdot \epsilon_1 \epsilon_2 \mu] C0[1, 2, 3].
\]  

At four point, the length of the expressions grow very quickly, and we will only give \( A^I_\mu(1^+, 2^+, 3^+). \) Instead of giving this expression directly, we will give \( A^I_\mu(1^+, 2^+, 3^+) \epsilon_1^\mu, \)
\( A^I_\mu(1^+, 2^+, 3^+) \epsilon_2^\mu \) and \( A^I_\mu(1^+, 2^+, 3^+) k_3^\mu \). Together with \( A^I_\mu(1^+, 2^+, 3^+) (k_1 + k_2 + k_3)_\mu = 0, \) the expressions are enough to determine all the 4 components of \( A^I_\mu(1^+, 2^+, 3^+). \) We choose the spinor representations for \( k_{1,2,3} \) and \( \epsilon_{1,2,3} \) to be:

\[ k_1 = \lambda_1 \bar{\lambda}_1, \ k_2 = \lambda_2 \bar{\lambda}_2, \ k_3 = \lambda_3 \bar{\lambda}_3, \ \epsilon_1 = \frac{\lambda_\nu \bar{\lambda}_1}{\langle \lambda_\nu \lambda_1 \rangle}, \ \epsilon_2 = \frac{\lambda_\nu \bar{\lambda}_2}{\langle \lambda_\nu \lambda_2 \rangle}, \ \epsilon_3 = \frac{\lambda_\nu \bar{\lambda}_3}{\langle \lambda_\nu \lambda_3 \rangle}. \]  

with \( \lambda_\nu \) an arbitrary reference spinor. We will use \( \langle \nu 1 \rangle \) to stand for \( \langle \lambda_\nu \lambda_1 \rangle \) and similarly others. We use \( (A^I_\mu(1^+, 2^+, 3^+) \epsilon_1^\mu)_{D0[1,2,3,4]} \) to denote for the coefficient of \( D0[1,2,3,4] \) in \( A^I_\mu(1^+, 2^+, 3^+) \epsilon_1^\mu, \) and similarly for others. We give the coefficients at \( D = 4. \) The off shell line makes the expressions much more complicated than that with all on shell lines. On one hand, when all lines are on shell, since the amplitudes are gauge invariant, we can choose some specific reference spinor, while in the off shell case we should keep the reference spinor \( \lambda_\nu \) arbitrary. On the other hand, there are many terms in the expressions below which is 0 when all lines are on shell. For example, the first coefficient below would be 0 due to \( (\langle 12 \rangle 12 + \langle 13 \rangle 13 + \langle 23 \rangle 23) = 0 \) when all lines were on shell.
Then for $A_1^i(1^+, 2^+, 3^+) e^{i1}$:

$\langle A_1^i(1^+, 2^+, 3^+) e^{i1} \rangle_{D_0[1, 2, 3, 4]}$

$= - \frac{(13)(2\nu - 2(12)(3\nu))[12((2\nu) + (3\nu))[13])(23)((12)[12] + (13)[13] + (23)[23])}{64(13)^2(1\nu)(2\nu)}$

$= - \frac{(2\nu)[12] + (3\nu)[13]}{64(13)(1\nu) + (1\nu)2(23)(3\nu)[12] + (1\nu)[13]} + \frac{(1\nu)^2(23)(12)(2\nu)[12]^2 + (2\nu)(12)3\nu[12]^2}{(1\nu)^2(23)(12)(2\nu)[12]^2 + (2\nu)(12)3\nu[12]^2}

$+ 4(12)(3\nu)[13][12] + (1\nu)^2(3\nu)[13]^2(23) - (1\nu)(12)[12] + (13)[13](23)(3\nu)[12]^2 + (1\nu)(3\nu)[12]((4\nu)[13] - (2\nu)(3\nu)[13]^2),

$- (2\nu)(12)[12]^2 - 2(1\nu)^2(2\nu)[12] + 3(3\nu)[13][12] - (1\nu)^3(3\nu)[13]^2),

$= - \frac{(2\nu)[12] + (3\nu)[13][13][23]}{64(13)(1\nu)[13] + (1\nu)^2(3\nu)[13]^2(23)}(23)(3\nu)[12]^2 + (1\nu)(3\nu)[13] + (2\nu)(3\nu)[23]),$

$+ 3(2\nu)(3\nu)[23][12] + (1\nu)^2(3\nu)[13][13] + (2\nu)(3\nu)[23][13] + 2(1\nu)^2(3\nu)[23]^2 - (3\nu)(3\nu)[23][13] + (1\nu)(1\nu)[13] + (2\nu)[23]),$

$= \frac{(2\nu)[12] + (3\nu)[13]}{64(13)(1\nu)[13] + (1\nu)^2(3\nu)[13]^2(23)}(2\nu)[12] + (1\nu)(3\nu)[13] + (2\nu)(3\nu)[23][13] + 2(1\nu)^2(3\nu)[23]^2 - (3\nu)(3\nu)[23][13] + (1\nu)(1\nu)[13] + (2\nu)[23]),$

$= \frac{(2\nu)[12] + (3\nu)[13]}{64(13)(1\nu)[13] + (1\nu)^2(3\nu)[13]^2(23)}(2\nu)[12] + (1\nu)(3\nu)[13] + (2\nu)(3\nu)[23][13] + 2(1\nu)^2(3\nu)[23]^2 - (3\nu)(3\nu)[23][13] + (1\nu)(1\nu)[13] + (2\nu)[23]),$

$= \frac{(2\nu)[12] + (3\nu)[13]}{64(13)(1\nu)[13] + (1\nu)^2(3\nu)[13]^2(23)}(2\nu)[12] + (1\nu)(3\nu)[13] + (2\nu)(3\nu)[23][13] + 2(1\nu)^2(3\nu)[23]^2 - (3\nu)(3\nu)[23][13] + (1\nu)(1\nu)[13] + (2\nu)[23]),$
\[+4(13)(23)^2(2\nu)[13][23][3] + 6(13)^2(23)(2\nu)[13]^2[23]^2 + (13)^3[13]^3((1\nu)[13] + 4(2\nu)[23]))],

\[
A_{\mu}^f(1^+, 2^+, 3^+)\epsilon_4^\mu \mid B\{1, 3\} \ni \frac{1}{64(12)(13)(2\nu)[12][23][3](1\nu)[13]((1\nu)[13] + 2(2\nu)[23])^3(13)}
\]

For \(A_{\mu}^f(1^+, 2^+, 3^+)\epsilon_4^\mu \mid C\{1, 2, 3, 4\} \ni \)

\[
A_{\mu}^f(1^+, 2^+, 3^+)\epsilon_4^\mu \mid C\{2, 3, 4\} \ni \frac{1}{64(12)(13)(2\nu)[12][23][3][13]((1\nu)[13] + 2(2\nu)[23])^3(13)}
\]

For \(A_{\mu}^f(1^+, 2^+, 3^+)\epsilon_4^\mu \mid C\{1, 3\} \ni \)

\[
A_{\mu}^f(1^+, 2^+, 3^+)\epsilon_4^\mu \mid D\{1, 2, 3, 4\} \ni \frac{1}{64(12)(13)(2\nu)[12][23][3][13]((1\nu)[13] + 2(2\nu)[23])^3(13)}
\]
We checked our four point amplitudes using the known simple results of $A^l(1^+, 2^+, 3^+, 4^+)\text{ and } A^l(1^+, 2^+, 3^+, 4^-)$ in [21, 26, 27, 29].

IV. CONCLUSION

We have discussed the Ward identity in detail for off shell amplitudes in pure Yang-Mills theory. We explicitly prove that the Ward identity with two complexified external lines holds at tree and one loop level using Feynman rules. Then we use the Ward identity to deduce recursion relations for off shell amplitudes at tree and one loop level. In this technique, three steps are important to simplify the calculation. First, according to the complexified Ward identity, we can convert the calculation of the amplitudes to the calculation of derivative of the amplitudes. Second, we decompose the three point vertex which contains the off shell line into three terms, which simplifies many steps in our calculation. Thirdly, according to the cancellation details in the proof of complexified Ward identity, we find most terms from different diagrams cancel with each other. The number of remaining effective terms or diagrams are reduced. It turns out that the recursion relation we derive at tree level is equivalent to Berends-Giele recursion relation [4]. However, our expressions at 1-loop level are new. And we present 1-loop off shell three and four point amplitudes as examples of applying our method at 1-loop level.

Comparing with our previous work [1], we find the technique in this article is more universal. Here we can obtain a recursion relation for the total amplitudes instead of just the boundary terms of the amplitudes, and we do not need to use BCFW recursion relation. Furthermore, for this technique, we do not need to avoid the unphysical poles from the polarization vectors of the shifted on shell leg which can also depend on $z$. Hence this technique works well for the amplitudes with any helicity structure and the momenta shifts are more general than the ones in [1]. In addition, this technique can be used for calculating one loop off shell amplitudes with any helicity structure.

In principle, it is possible to generalize our method to higher loop levels and to other theories such as QCD. The only obstruct is to classify all the cancellation details for the Ward identity with complexified external momenta. We leave this to future work. Another extension is to combine our technique with other methods, such as unitary cut, generalized unitary cut, BCFW, OPP [28] etc. to further simplify the calculation in pure Yang-Mills theory.

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