Abstract. We continue the study of infinite geodesics in planar first-passage percolation, pioneered by Newman in the mid 1990s. Building on more recent work of Hoffman, and Damron and Hanson, we develop an ergodic theory for infinite geodesics via the study of what we shall call \textit{random coalescing geodesics}. Random coalescing geodesics have a range of nice asymptotic properties, such as asymptotic directions and linear Busemann functions. We show that random coalescing geodesics are (in some sense) dense in the space of geodesics. This allows us to extrapolate properties from random coalescing geodesics to obtain statements on all infinite geodesics. As an application of this theory we solve the ‘midpoint problem’ of Benjamini, Kalai and Schramm and address a question of Furstenberg on the existence of bigeodesics.

1. Introduction

In first-passage percolation the edges (or sites) of the \( \mathbb{Z}^2 \) nearest neighbor lattice are equipped with non-negative random weights, thus giving rise to a random metric space. Since the first works on spatial growth by Eden \cite{Ede61} and Hammersley and Welsh \cite{HW65}, first-passage percolation has attracted vast attention from mathematicians and physicists, see e.g. \cite{Kes86,KS91,ADH}, aiming to understand the large-scale behavior of distances, balls and geodesics in this random metric space. Despite its success in inspiring powerful theories, first-passage percolation has proven to be one of the more difficult statistical physics models to analyze, and many of the most important conjectures remain unsettled.

The study of first-passage percolation has led to the development of powerful mathematical tools, like a rigorous theory for subadditive ergodic processes by Kingman \cite{Kin68,Kin73}, as well as one of the most important avenues of study in mathematical physics, based on the predictions originating from the work of Kardar, Parisi and Zhang \cite{KPZ86}. The behavior of finite geodesics is an integral component in KPZ-theory. The theory predicts the existence of exponents \( \chi \) and \( \xi \), known as the \textit{fluctuation} and \textit{wandering} exponents, such that, with high probability, the distance between \((0,0)\) and \((n,0)\) deviates from its mean by \( n^{\chi+o(1)} \) and the vertical displacement of the geodesic between \((0,0)\) and \((n,0)\) scales as \( n^{\xi+o(1)} \). In two dimensions the two exponents \( \chi \) and \( \xi \) should equal \( 1/3 \) and \( 2/3 \) respectively, and thus be related through the equation \( \chi = 2\xi - 1 \). While these values are believed to differ in higher dimensions, the relation \( \chi = 2\xi - 1 \) is expected to prevail; see \cite{KS91}. Even the existence of these exponents remains a mystery in first-passage percolation. However, if they do exist then the scaling relation \( \chi = 2\xi - 1 \) should also hold; see \cite{Cha13,AD14}. There are closely related so-called ‘exactly solvable’ models for which such a behavior has been rigorously
established [BDJ99, Joh00]. The theme of the present paper is geodesics, and we aim to develop an ergodic theory for the study of infinite geodesics in first-passage percolation.

The study of infinite geodesics was pioneered in the mid 1990s by Newman and collaborators [New95, NP95, LN96]. The structure of infinite geodesics has been found to exhibit intriguing connections with other important probabilistic models. For instance, geodesics in first-passage percolation was closely linked to solutions of the Burgers equation in work of Bakhtin, Cator and Khanin [BCK14]. Inspired by the study of geodesics in a Euclidean version of first-passage percolation by Howard and Newman [HN01], Bakhtin, Cator and Khanin incorporated these ideas to study the space of solutions to the Burgers equation. They constructed space-time stationary solutions of the one dimensional Burgers equation with random forcing in the absence of periodicity or compactness assumptions and, showed that there is a unique global solution to the Burgers equation with any prescribed average velocity under a model where the forcing is given by a homogeneous Poissonian point field in space-time.

In Euclidean space the geodesic between two points is given by a line segment, and each line segment can be extended to a bi-infinite distance minimizing curve – the straight line. In first-passage percolation bi-infinite geodesics has been conjectured not to exist. Kesten [Kes86, p. 258] attributes the question of existence of bigeodesics in first-passage percolation to Hillel Furstenberg, and the question has since gained fame through its connection with the existence of non-trivial ground states of the two-dimensional Ising ferromagnet with random exchange constants; see [LN96, Weh97]. Newman has given a convincing heuristic argument, based in part on the scaling behavior predicted by KPZ-theory, ruling out the existence of bigeodesics. This argument has been reproduced in [ADH, Section 4.5]. This question is also related to hypersurfaces with minimal random weights [Kes87].

We shall in this paper continue the study of infinite geodesics in first-passage percolation initiated by Newman, and continued by Hoffman [Hof08] and Damron and Hanson [DH14]. Via the study of what we shall call random coalescing geodesics, we build an ergodic theory for the study of infinite geodesics, incorporating elements like coalescence, Busemann functions and subsequential limiting procedures present also in previous work. Random coalescing geodesics have a range of nice asymptotic properties. Although random coalescing geodesics do not account for all geodesics, we show that they are sufficiently dense in the space of infinite geodesics so that we may extrapolate certain properties of random coalescing geodesics to obtain global statements about geodesics. While the existence of exponents and bigeodesics remain open problems, we shall as a consequence of the theory we develop provide partial results in their direction.

We shall throughout the paper work under a stationary and ergodic assumption on the weight distribution, which enables us to obtain a very general theory. However, in this general setting different models of first-passage percolation are known to behave very differently. Consequently, our results will all be qualitative and not quantitative. That means that in order to obtain quantitative estimates in a specific setting, say for independent edge weights, one would have to incorporate the independence assumption in some fundamental way.

2. Statement of results

In this paper we consider a large number of models of first-passage percolation on \( \mathbb{Z}^2 \), including those with independent edge weights from a common continuous distribution with finite mean; see Section 3 below for a precise description. Let \( \mathcal{E}^2 \) denote the set of edges of
the $\mathbb{Z}^2$ nearest-neighbor lattice. For each $\omega \in \Omega := [0, \infty)^{\mathbb{Z}^2}$ we define a metric on $\mathbb{Z}^2$ via
\[
T(x, y) := \inf \left\{ \sum_{e \in \pi} \omega_e : \pi \text{ a path connecting } x \text{ and } y \right\}.
\]

A path attaining the infimum in (1) is referred to as a geodesic. In each of the first-passage models that we shall work with there will (i) exist a unique geodesic between any two points $x$ and $y$ – we shall denote this path by $\text{Geo}(x, y)$; and (ii) exist a compact and convex set $\text{Ball} \subseteq \mathbb{R}^2$ with non-empty interior such that $\frac{1}{t} \{ x : T(0, x) \leq t \}$ approaches $\text{Ball}$ as $t$ increases. Again, we refer to Section 3 below for a precise statement.

The focus of this paper lies on infinite geodesics. More precisely, we shall develop an ergodic theory around what we shall call random coalescing geodesics, which we define shortly. Our reasons for this are two-fold. First, we aim to describe the set of infinite geodesics originating at the origin, relating the number of geodesics and their directions to the asymptotic shape $\text{Ball}$. Second, we address questions related to the existence of bigeodesics and exponents. Random coalescing geodesics have, indeed, nice properties such as coalescence, asymptotic directions and asymptotically linear Busemann functions. By showing that these properties are in some sense ‘typical’ we obtain our results.

The first of our results relates to both exponents and bigeodesics. Based on the predictions of KPZ-theory it is widely believed that the probability that the geodesic between $(-n, 0)$ and $(n, 0)$ visits the origin should scale like $n^{-\xi + o(1)}$, where again $\xi = 2/3$. Our result takes a modest first step in this direction, and answers a longstanding open question of Benjamini, Kalai and Schramm [BKS03].

**Theorem 2.1.** For any sequence $(v_k)_{k \geq 1}$ in $\mathbb{Z}^2$ such that $|v_k| \to \infty$ we have
\[
P(0 \in \text{Geo}(-v_k, v_k)) \to 0.
\]

A semi-infinite path $(v_k)_{k \geq 1}$ will be referred to as an (infinite) geodesic if each finite segment is a geodesic, and a bi-infinite path $(v_k)_{k \in \mathbb{Z}}$ with the same property will be referred to as a bi-infinite geodesic, or a bigeodesic. It has been conjectured that there are almost surely no bigeodesics. We are able to show that in each fixed direction (except for an at most countable set determined by Ball) this is true. This is closely related to the fact that multiple geodesics do not occur in directions of differentiability.

**Theorem 2.2.** Let $\theta$ be a direction of differentiability of $\partial \text{Ball}$. Then,
\[
\begin{align*}
(a) \quad & P(\exists \text{ two geodesics in direction } \theta) = 0; \\
(b) \quad & P(\exists \text{ a bigeodesic in direction } \theta) = 0.
\end{align*}
\]

Let $\mathcal{G}_0 = \mathcal{G}_0(\omega)$ denote the set of infinite geodesics originating at the origin. That $\mathcal{G}_0$ is non-empty follows easily from compactness. Proving that
\[
P(\mathcal{G}_0 \text{ has size at least } 2) = 1
\]
is significantly more difficult, and was established through a series of papers [HP98, GM05, Hof05]. We shall interchangeably think of $\mathcal{G}_0$ as a set as well as the graph obtained as a union of its elements. Our standing assumption on unique passage times assures that this graph is a tree, almost surely.

Much of the early work on infinite geodesics by Newman and collaborators was carried out under additional assumptions on the asymptotic shape that remain unverified to this day. As will become clear throughout this paper, the lack of knowledge about Ball is the major
factor limiting our understanding of $\mathcal{T}_0$. Inspired by the work of Newman [New95], later work has aimed at obtaining rigorous results, without further assumptions on the limiting shape. Hoffman [Hof08] used Busemann functions to show that $\mathcal{T}_0$ contain at least one geodesic for each side of $\partial \text{Ball}$, almost surely, and thus showed that $\mathbb{P}(\mathcal{T}_0 \text{ has size at least } 4) = 1$.

Damron and Hanson [DH14] strengthened these results to show that these geodesics are coalescing and asymptotically directed in the intersection of $\partial \text{Ball}$ and one of its tangent lines. Recent work by Georgiou, Rassoul-Agha and Seppäläinen [GRASb, GRASa] parallels this development in the related setting of last-passage percolation. One of the main goals of this paper is to show that for many of the properties described in [DH14] not only some, but all geodesics have these properties. Among these properties we find asymptotic ‘generalized’ directions and linear Busemann function, and for other properties such as coalescence, we will show that the behavior described in these papers is in some sense typical.

The fundamental object that we study in this paper is a random coalescing geodesic.

**Definition 2.3.** Let $\mathcal{P}$ denote the set of infinite self-avoiding paths starting at the origin. We say that a measurable map $G : \Omega \to \mathcal{P}$ is a random coalescing geodesic if $G(\omega) \in \mathcal{T}_0$ and for every $v \in \mathbb{Z}^2$

$$G(\omega) \setminus \sigma_{-v}(G(\sigma_v \omega))$$

is finite, almost surely. We shall frequently write $G(v)$ for the map $\sigma_{-v} \circ G \circ \sigma_v$.

In this direction we have two major goals. The first is to classify all random coalescing geodesics. The second is to use this classification to make statements about all geodesics in $\mathcal{T}_0$. In order to accomplish the second goal it would be nice if every $g \in \mathcal{T}_0$ was the image of some random coalescing geodesic. However, widely believed conjectures (for example that there is a geodesic in every direction) imply that this is not the case. Instead we shall show that random coalescing geodesics are sufficiently dense in $\mathcal{T}_0$ that we can still use our knowledge of random coalescing geodesics to make statements about every geodesic in $\mathcal{T}_0$.

A first consequence of the ergodic theory that we develop is that the number of geodesic is almost surely constant.

**Theorem 2.4.** The cardinality of the set $\mathcal{T}_0$ is almost surely constant.

In order to state our remaining theorems precisely we introduce a few definitions. Given a geodesic $g = (v_0, v_1, \ldots)$ we define the direction $\text{Dir}(g)$ of $g$ as the set of limit points of the set $\{v_k: k \geq 1\}$. Hence, $\text{Dir}(g)$ is an arc of the unit circle $S^1$, and we shall identify this arc with a subset of $[0, 2\pi]$, when suitable.

Busemann functions were introduced to first-passage percolation in a couple of papers by Hoffman [Hof05, Hof08]. Given a geodesic $g \in \mathcal{T}_0$ we define the Busemann function $B_g : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ of $g = (v_0, v_1, v_2, \ldots)$ as the limit

$$B_g(x, y) := \lim_{k \to \infty} [T(x, v_k) - T(y, v_k)].$$

It is proved in [Hof05] that this limit exists for all $g \in \mathcal{T}_0$ and $x, y \in \mathbb{Z}^2$; see Lemma 4.1 below. We say that a Busemann function is asymptotically linear if there exists a linear functional $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\limsup_{|y| \to \infty} \frac{1}{|y|} |B_g(0, y) - \rho(y)| = 0.$$

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2The sides corresponds to tangent lines, of which there are $n$ if $\partial \text{Ball}$ is an $n$-gon and $\infty$ otherwise.

3The map $\sigma_v : \Omega \to \Omega$ denotes the usual shift along the vector $v \in \mathbb{Z}^2$. 
Theorem 2.6. There exists a closed set \( \mathcal{C} \subseteq \mathcal{I} \), containing all linear functionals tangent to Ball, such that \( B_g \) is asymptotically linear to \( \rho \) and \( \text{Dir}(G) \) is a subset of \( \{ x \in S^1 : \mu(x) = \rho(x) \} \).

The Busemann function of a geodesic \( g \) should be thought of as measuring the difference in distance to infinity along the geodesic \( g \). From the linearity of a Busemann function it is possible to obtain information on the direction of the geodesics used to define it, and hence to distinguish geodesics from one another. An exposition of this will be given in Section 6.

We shall call a linear functional \( \rho : \mathbb{R}^2 \to \mathbb{R} \) supporting to Ball if \( \{ x \in \mathbb{R}^2 : \rho(x) = 1 \} \) is a supporting line for \( \partial \text{Ball} \) at some point in \( \partial \text{Ball} \), and tangent to Ball if \( \{ x : \rho(x) = 1 \} \) is the unique supporting line – the tangent line – to \( \partial \text{Ball} \) at some point in \( \partial \text{Ball} \). It is well known that Ball can be expressed as \( \{ x \in \mathbb{R}^2 : \mu(x) \leq 1 \} \) for some norm \( \mu : \mathbb{R}^2 \to \mathbb{R} \). Consequently, the intersection of \( \partial \text{Ball} \) and a supporting line of the form \( \{ x \in \mathbb{R}^2 : \rho(x) = 1 \} \) can thus be represented by the arc \( \{ x \in S^1 : \mu(x) = \rho(x) \} \). The set of supporting functionals of Ball is naturally parametrized by the direction of its gradient, or its angle with the first-coordinate axis. The set of supporting functionals of Ball, which we denote henceforth by \( \mathcal{I} \), thus inherits the topology of \( S^1 \).

Theorem 2.5. With probability one there exists for each \( g \in \mathcal{I}_0 \) a linear functional \( \rho \in \mathcal{I} \) such that \( B_g \) is asymptotically linear to \( \rho \) and \( \text{Dir}(G) \) is a subset of \( \{ x \in S^1 : \mu(x) = \rho(x) \} \).

The above theorem describes the asymptotic properties of all infinite geodesics, but does not address existence and uniqueness of a given functional. Since for almost every realization all geodesics have linear Busemann functions we shall proceed and describe this set. Let

\[ \mathcal{C} := \{ \rho \in \mathcal{I} : \exists \text{ a geodesic in } \mathcal{I}_0 \text{ with Busemann function linear to } \rho \} . \]

The next result addresses the ergodic properties of \( \mathcal{I}_0 \) further by describing the topological properties of the random set \( \mathcal{C} \).

Theorem 2.6. There exists a closed set \( \mathcal{C}_\star \subseteq \mathcal{I} \), containing all linear functionals tangent to Ball, such that \( \mathcal{P}(\mathcal{C} = \mathcal{C}_\star) = 1 \). Moreover, for every functional \( \rho \in \mathcal{C}_\star \) we have

\[ \mathcal{P}(\exists \text{ two geodesics in } \mathcal{I}_0 \text{ with Busemann function linear to } \rho) = 0 . \]

The remainder of this paper is organized as follows. We first review the relevant background on first-passage percolation and describe in detail the class of models with which we shall work. In Section 4 we describe some fundamental properties of random coalescing geodesics, and at the same time illustrate the role of coalescence and Busemann functions. The existence of at least four random coalescing geodesics is derived in Section 5, based on previous work of Damron and Hanson [DH14]. We then aim, in Section 6, to introduce a shift invariant labeling of geodesics which is consistent with some natural ordering. This gives us a way to identify non-crossing families of geodesics by referring to their labels. In Section 7 we present a central geometric argument that will be crucial in order to develop our theory without further assumptions on the asymptotic shape. The set of labels obtained by the labeling procedure of Section 6 is in Section 8 found to be a deterministic closed set. This allows us to talk about a random non-crossing geodesic with a given label. These random non-crossing geodesics are in Section 9 proven to be coalescing, and we show that there are no random coalescing geodesics apart from these ones. This gives us a classification of all random coalescing geodesics. We end the paper by exploring some consequences of the theory we develop, and resolve in Section 11 the midpoint problem from [BKS03] and deduce our remaining results in Section 12.

Since comparisons between geodesics will recur throughout the paper, we end this section with a small glossary on infinite geodesics. We shall by \( \mathcal{I}_v = \mathcal{I}_v(\omega) \) denote the set of infinite geodesics originating from the vertex \( v \in \mathbb{Z}^2 \). Two geodesics \( g \in \mathcal{I}_v \) and \( g' \in \mathcal{I}_v \), starting from different vertices, will be said to intersect if \( g \) and \( g' \) both visit some vertex \( z \), and
to **coalesce** if the symmetric difference $g \Delta g'$ is finite. They are said to be **non-crossing** if they either coalesce or are disjoint, and are said to **cross** if they are not non-crossing.

### 3. Background on model and assumptions

We shall work under the standard assumptions on passage time distributions outlined by earlier work of Hoffman [Hof08] and Damron and Hanson [DH14]. As above, we will denote by $\Omega = [0, \infty)^{\mathbb{Z}^2}$ our state space, equipped with the product Borel sigma-algebra. $\mathbf{P}$ will throughout the paper be a shift invariant probability measure on $\Omega$ satisfying either of the following two sets of conditions:

**A1** $\mathbf{P}$ is a product measure whose common marginal distribution is continuous with

$$E\left[\min\{\omega_{e_1}, \omega_{e_2}, \omega_{e_3}, \omega_{e_4}\}\right] < \infty,$$

where $e_1, \ldots, e_4$ denote the four edges incident to the origin.

**A2** $\mathbf{P}$ is ergodic with respect to translations of $\mathbb{Z}^2$ and has the following properties:

(i) $\mathbf{P}$ has all the symmetries of $\mathbb{Z}^2$;

(ii) $E[\omega^2 + \varepsilon] < \infty$ for some $\varepsilon > 0$;

(iii) the asymptotic shape Ball is bounded;

(iv) for any two finite paths $\pi$ and $\pi'$ that differ for at least one edge we have

$$\mathbf{P}\left(\sum_{e \in \pi} \omega_e = \sum_{e' \in \pi'} \omega_{e'}\right) = 0;$$

(v) for any $e \in \mathcal{E}^2$ and $t > 0$ such that $\mathbf{P}(\omega_e > t) > 0$ we have

$$\mathbf{P}(\omega_e > t | \{\omega_f : f \in \mathcal{E}^2, f \neq e\}) > 0.$$

The conditions have been specified so to make sure the conditions of the shape theorem (see below) are satisfied and that for each pair of points $x$ and $y$ there is a unique geodesic. The assumption (iv) assures unique passage times, while (v), known as the **upward finite energy** condition, is assumed to allow for local modifications of an edge configuration; see e.g. [HJ06] for a further account on its relevance in the statistical mechanics literature.

**Remark 3.1.** While the total ergodicity condition (i) in **A2** makes many arguments easier it is not essential. With some extra effort it can be replaced by $\mathbf{P}$ is ergodic in any of our arguments or those in [DH14].

### 3.1. The shape theorem.

One of the most celebrated results in first-passage percolation is known as the shape theorem, and originates from Kingman’s [Kin68, Kin73] ergodic theory for subadditive processes. Under either of the conditions **A1** or **A2** Kingman’s theorem shows that for any $z \in \mathbb{Z}^2$ we have

$$\exists \mu(z) := \lim_{n \to \infty} \frac{T(0, nz)}{n} = \inf_{n \geq 1} \frac{E[T(0, nz)]}{n} \text{ almost surely and in } L^1.$$

Richardson [Ric73], and later Cox and Durrett [CD81] and Boivin [Boi90], extended the radial converge in (3) to obtain simultaneous convergence in all directions. Their results show that under either of the assumptions **A1** or **A2** we have

$$\limsup_{|z| \to \infty} \frac{1}{|z|} \left| T(0, z) - \mu(z) \right| = 0 \text{ almost surely.}$$

The function $\mu : \mathbb{Z}^2 \to \mathbb{R}$ extends to a function $\mu$ on $\mathbb{R}^2$ through homogeneity, and inherits the properties of a norm. The unit ball $\text{Ball} := \{x \in \mathbb{R}^2 : \mu(x) \leq 1\}$ in this norm is a good approximation of a rescaled version of a large ball $\{z \in \mathbb{Z}^2 : T(0, z) \leq t\}$ in the first-passage
metric. On this form the result in \[4\] is know as the shape theorem as takes the familiar form
\[
\mathbb{P}\left( (1 - \epsilon)\text{Ball} \subset \frac{1}{t}B(t) \subset (1 + \epsilon)\text{Ball} \text{ for all large } t \right) = 1.
\]
where \(\text{Ball}(t) := \{ z \in \mathbb{Z}^2 : T(0, z) \leq t \} + [-1/2, 1/2]^2 \). It is straightforward to show that the properties of a norm implies that Ball is compact, convex and has non-empty interior. The assumptions on \(\mathbb{P}\) further imposes that Ball necessarily has all the symmetries of \(\mathbb{Z}^2\).

3.2. Shapes and geodesics in ergodic first-passage percolation. The shape theorem gives a first-order approximation of large balls in the first-passage metric \(T\) with a compact and convex shape Ball, but does not provide further insight to the topological properties of that shape. The results of \[Hof08, DH14\] relate existence and properties of infinite geodesics to the number of sides of Ball, and conclude that there are at least four almost surely. It turns out that these results are sharp under the general ergodic assumption, but most likely not for independent models.

Häggström and Meester \[HM95\] have showed that for any compact and convex shape \(S \subset \mathbb{R}^2\) with the symmetries of \(\mathbb{Z}^2\) there is a model of ergodic first-passage percolation with \(\text{Ball} = S\). That is, the asymptotic shape can have as few sides as four in the case it equals either a square or a diamond.

Similarly, Brito and Hoffman \[BH\] have constructed a model of ergodic first-passage percolation which almost surely has exactly four coalescing geodesics. These geodesics have directions that span an angle of \(\pi/2\) each. This shows the results of \[Hof08, DH14\] are sharp.

Very little is known about the asymptotic shape for edge weights that are independent. In particular, it is unknown whether for some edge distribution \(\partial\text{Ball}\) may equal a circle or a square. Simulations indicate for exponential edge weights the shape is very close, but not equal to, a circle \[AD15\]. Getting better results about geodesics in independent first-passage percolation will require new techniques for the shape or geodesics which make use of independence in some fundamental way.

While our focus in this paper is strictly on first-passage percolation on the \(\mathbb{Z}^2\) nearest-neighbor lattice, we mention that it has been observed by Benjamini and Tessera \[BT\] that bigeodesics may exist on graph with different geometry.

3.3. An extended shape theorem. Sometimes we shall need to control the location of geodesics via the shape theorem. Occasionally the following ‘extended version’ of the shape theorem will be useful. Loosely speaking, it says that the shape theorem has ‘kicked in’ around a point at a time scale proportional to its distance from the origin, for all points sufficiently far from the origin.

**Proposition 3.2.** For every \(\epsilon > 0\) there exists an almost surely finite \(N \geq 1\) such that for all \(|z| \geq N\) we have

\[
|T(z, z + y) - \mu(y)| \leq \epsilon \max\{|z|, |y|\} \quad \text{for all } y \in \mathbb{Z}^2.
\]

**Proof.** Given \(\epsilon > 0\) and \(z \in \mathbb{Z}^2\), let \(C(\epsilon, z)\) denote the event

\[
C(\epsilon, z) := \{ |T(z, z + y) - \mu(y)| \leq \epsilon \max\{|z|, |y|\} \text{ for all } y \in \mathbb{Z}^2 \}.
\]

We first argue that \(\mathbb{P}(C(\epsilon, z)) \to 1\) as \(|z| \to \infty\). This is an immediate consequence of the shape theorem: For every \(\epsilon > 0\) we may find \(M \geq 1\) such that

\[
\mathbb{P}\left( |T(z, z + y) - \mu(y)| \leq \epsilon |y| \text{ for all } |y| \geq M \right) \geq 1 - \epsilon,
\]
and
\[ P\left( \max_{|y| \leq M} |T(z, z + y) - \mu(y)| \leq \varepsilon|z| \right) \geq 1 - \varepsilon \]
for $|z|$ large. Hence, for every $\varepsilon > 0$ we find $N$ so that $P(C(\varepsilon, z)) > 1 - 2\varepsilon$ when $|z| \geq N$.

Relying on the ergodic theorem we may for any $\delta > 0$ find almost surely finite $N \geq 1$ such that for every $n \geq N$ the density of $z$ within distance $n$ from the origin for which $C(\varepsilon/100, z)$ fails is at most $\delta$. Now, take $z$ with $|z| \geq N$. Either $C(\varepsilon/100, z)$ occurs, or we may find $x$ within distance $\sqrt{\delta}|z|$ of $z$ for which $C(\varepsilon/100, x)$ occurs. In the latter case we have
\[
|T(z, z + y) - \mu(y)| \leq |T(x, z + y) - \mu(z + y - x)| + T(x, z) + |\mu(z + y - x) - \mu(y)|
\leq \frac{\varepsilon}{100} \max\{|x|, |z + y - x|\} + \frac{\varepsilon}{100} \max\{|x|, |z - x|\} + 2\mu(z - x),
\]
where we in the second step have used the triangle inequality once and the fact that $C(\varepsilon/100, x)$ occurs twice. Using the fact that $\mu$ is comparable to Euclidean distance and that $|z - x| \leq \sqrt{\delta}|z|$, we obtain a constant $c$ and the further upper bound
\[
|T(z, z + y) - \mu(y)| \leq \frac{\varepsilon}{100}|y| + 2\frac{\varepsilon}{100}(1 + \sqrt{\delta})|z| + 2c\sqrt{\delta}|z|.
\]
In particular, if $\delta > 0$ was chosen small enough this is all bounded by $\varepsilon \max\{|z|, |y|\}$ and $C(\varepsilon, z)$ holds, for all $z$ at distance at least $M$ from the origin, as required.

\section{Properties of random coalescing geodesics}

In this section we collect some fundamental properties of random coalescing geodesics and simultaneously highlight the usefulness of Busemann functions and coalescence. First we give the statement of all the theorems and then we provide the proofs. Given a random coalescing geodesic $G$ we define the \textbf{Busemann function} $B_G : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ of $G = (v_0, v_1, v_2, \ldots)$ as the limit
\[
B_G(x, y) = \lim_{k \to \infty} \left[ T(x, v_k) - T(y, v_k) \right].
\]
Since $G$ is coalescing, the limit does not depend on the representative of $G$.

\begin{lemma}
Let $G$ be a random coalescing geodesic. The limit in (6) exists and satisfies
\begin{enumerate}[(a)]
    \item $B_G(x, z) = B_G(x, y) + B_G(y, z)$ for all $x, y, z \in \mathbb{Z}^2$;
    \item $|B_G(x, y)| \leq T(x, y)$ for all $x, y \in \mathbb{Z}^2$;
    \item $B_G(x, y) = T(x, y)$ for all $x, y \in \mathbb{Z}^2$ such that $y \in G(x)$.
\end{enumerate}
\end{lemma}

Although this definition is new, many of the arguments used to prove the theorems below have previously appeared in the literature. The above properties indeed hold for any geodesic in $\mathcal{Z}_0$. For a random coalescing geodesic $G$ the Busemann function has especially nice properties. Due to shift invariance and coalescence, it follows that
\[
B_G(x, y) \overset{d}{=} B_G(x + z, y + z) \quad \text{for all } x, y, z \in \mathbb{Z}^2.
\]
Translation invariance and additivity imply that $B_G$ grows like some linear functional $\rho_G$.

\begin{proposition}
Let $G$ be a random coalescing geodesic. There exists a linear functional $\rho_G : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying $\rho_G(z) = E[B_G(0, z)]$ for all $z \in \mathbb{Z}^2$, and
\[
P\left( \limsup_{|z| \to \infty} \frac{1}{|z|} |B_G(0, z) - \rho_G(z)| = 0 \right) = 1.
\]
\end{proposition}
Linearity of the Busemann function, together with the shape theorem, has the important consequence of providing a bound on the asymptotic direction of $G$: It is confined by the intersection of the line $\{\rho_G(x) = 1\}$ and the asymptotic shape $\partial \text{Ball} = \{\mu(x) = 1\}$. We define the set of limiting directions of $G$ as the set of directions $|x| = 1$ such that $v_n/|v_n| \to x$ for some subsequence $(v_n)_{n \geq 1}$ of $G$. We will identify this set with a subset of $[0, 2\pi]$ when appropriate.

**Proposition 4.3.** Let $G$ be a random coalescing geodesic. The line $\{x \in \mathbb{R} : \rho_G(x) = 1\}$ is a supporting line for Ball, and the set of asymptotic directions $\text{Dir}(G)$ is a deterministic subset of $\text{Arc}(G) := \{x \in S^1 : \mu(x) = \rho_G(x)\}$.

We say that a random coalescing geodesic $G$ **eventually moves into a half-plane** $H$ if for all parallel half-planes $H' \subseteq H$ we have

$$\mathbb{P}(\{|G(v) \cap H'| < \infty\} = 1).$$

Let $H_i$, for $i = 0, 1, 2, \ldots, 7$, denote the half-plane

$$H_i := \{x \in \mathbb{R}^2 : x \cdot (\cos(i\pi/4), \sin(i\pi/4)) \geq 0\}.$$ 

Due to convexity we have that $\text{Arc}(G)$ has width at most $\pi/2$. It may thus contain $i\pi/4$ for at most three values of $i$. It follows that any random coalescing geodesic $G$ eventually moves into one of the eight half-planes $H_i$.

Our next proposition says, in particular, that for any two distinct random coalescing geodesics $G$ and $G'$ we have that $\text{Arc}(G)$ and $\text{Arc}(G')$ share at most one point.

**Proposition 4.4.** Let $G$ and $G'$ be random coalescing geodesics with $\rho_G = \rho_{G'}$. Then, $G = G'$ almost surely.

Finally, we show that a given random coalescing geodesic cannot be part of a bigeodesic. We say that a random coalescing geodesic $G$ is **backwards finite** if for every $x \in \mathbb{Z}^2$ we have $x \in G(y)$ for at most finitely many $y \in \mathbb{Z}^2$.

**Proposition 4.5.** A random coalescing geodesic is almost surely backwards finite.

Lemma 4.1 has its origins in [10]. Results similar to Propositions 4.2, 4.3 and 4.5 has previously appeared in [11]. A result similar to Proposition 4.4 has previously been obtained in [12] and [13]. The proofs presented below differ from these in some details.

**4.1. Proof of Lemma 4.1**. For the existence of the limit in [14] we follow Hoffman [10]. Let $G = (v_0, v_1, v_2, \ldots)$ and note that

$$T(x, v_k) - T(y, v_k) = [T(x, v_k) - T(v_0, v_k)] - [T(y, v_k) - T(v_0, v_k)].$$

The two expressions on the right-hand side are decreasing in $k$ and bounded from below by $-T(x, v_0)$ and $-T(y, v_0)$. Hence, the limit as $k \to \infty$ exists almost surely and in $L^1$.

The remaining properties are easy consequences of the definition and subadditivity of $T$.

**4.2. Proof of Proposition 4.2**. Let $\rho_G : \mathbb{R}^2 \to \mathbb{R}$ be the linear functional defined as $\rho_G(x) = \langle E[B_G(0,e_1)], E[B_G(0,e_2)] \rangle \cdot x$, where $\cdot$ denotes inner product. Using translation invariance and additivity of $B_G$, we find that for $v = (v_1, v_2)$

$$E[B_G(0,v)] = E\left[\sum_{i=1}^{v_1} B_G((i-1)e_1, ie_1) + \sum_{j=1}^{v_2} B_G((j-1)e_2, je_2)\right] = \rho_G(v).$$
Using the ergodic theorem we have by translation invariance that, almost surely,
\[
\lim_{n \to \infty} \frac{1}{n} B_G(0, nv) = \rho_G(v) \quad \text{for all } v \in \mathbb{Z}^2.
\]
(7)

We wish to strengthen this radial convergence statement to simultaneous convergence in all directions. Given \( \varepsilon > 0 \), pick \( v_1, v_2, \ldots, v_m \) such that for every \( z \in \mathbb{Z}^2 \) we have \( |z - nv_k| < \varepsilon |z| \) for some \( n \) and \( k \); write \( v_2 \) for the point of the form \( nv_k \) that minimizes \( |z - nv_k| \). By (7) we may choose \( N < \infty \) large so that
\[
|B_G(0, v_2) - \rho_G(v_2)| \leq \varepsilon |v_2| \quad \text{for all } |z| \geq N.
\]

By additivity of \( B_G \) and the triangle inequality, we find that
\[
|B_G(0, z) - \rho_G(z)| \leq |B_G(0, v_2) - \rho_G(v_2)| + |B_G(v_2, z) - \rho_G(v_2 - z)|.
\]
Since by assumption \( |v_2 - z| < \varepsilon |z| \), and \( |B_G(v_2, z)| \leq T(v_2, z) \), we obtain for some constant \( C < \infty \)
\[
|B_G(0, z) - \rho_G(z)| \leq T(v_2, z) + C\varepsilon |z|.
\]
However, by Proposition 3.2 we find \( M < \infty \) such that for all \( |z| \geq M \)
\[
|T(z, z + y) - \mu(y)| \leq \varepsilon \max\{|z|, |y|\} \quad \text{for all } y \in \mathbb{Z}^2.
\]
Together with the above, we obtain for large \( |z| \) that
\[
|B_G(0, z) - \rho_G(z)| \leq (C + 2)\varepsilon |z|.
\]
Since \( \varepsilon > 0 \) was arbitrary, this completes the proof.

4.3. Proof of Proposition 4.3. By the properties of \( B_G \), we have for any sequence \((x_n)_{n \geq 1}\) such that \(|x_n| \to \infty \) and \( x_n/|x_n| \to x \), that
\[
\rho_G(x_n/|x_n|) = \frac{1}{|x_n|} \mathbb{E}[B_G(0, x_n)] \leq \frac{1}{|x_n|} \mathbb{E}[T(0, x_n)].
\]
Taking limits leaves \( \rho_G(x) \leq \mu(x) \).

If \( x \in \mathbb{R}^2, |x| = 1 \), is a limiting direction for \( G \), then there exists a subsequence \((x_n)_{n \geq 1}\) of points on \( G \) such that \( x_n/|x_n| \to x \), and
\[
\frac{1}{|x_n|} B_G(0, x_n) = \frac{1}{|x_n|} T(0, x_n).
\]
Taking limits leaves us with \( \rho_G(x) = \mu(x) \). It follows that \( \{\rho_G(x) = 1\} \) is a supporting line for \( \text{Ball} = \{\mu(x) = 1\} \), and that every limiting point of \( G \) is contained in \( \text{Arc}(G) \).

It remains to conclude that the set of limiting directions of \( G \) is almost surely constant. Since the set of limiting directions is a (closed) interval, it suffices to show that \( x \) is a limiting point with probability 0 or 1. Assume that \( x \) is a limiting point of \( G \) with positive probability. Then, by the ergodic theorem, \( x \) is a limiting point for \( G(y) \) for some \( y \in \mathbb{Z}^2 \) with probability one. But \( G(0) \) and \( G(y) \) coalesce, so \( x \) is a limiting point also for \( G(0) \).

4.4. Proof of Proposition 4.4. We will work with half-plane geodesics. Let \( G \) be a random coalescing geodesic and assume that \( G \) eventually moves into one of the eight half-planes \( H = H_i \). We will assume below that \( H \) is the right half-plane, remaining cases being similar. Let \( T_H \) denote the restriction of the first-passage metric to \( H \) (that is, set \( \omega_\varepsilon = \infty \) if \( \varepsilon \) has some endpoint outside \( H \)). For \( x \in H \) we define
\[
G_H(x) := \lim_{n \to \infty} G_H(x, v_n),
\]
where \( G = (v_0, v_1, v_2, \ldots) \) and \( G_H(x, y) \) denotes the geodesic between \( x \) and \( y \) with respect to \( T_H \). Finally, we let
\[
B^H_G(x, y) := \lim_{n \to \infty} |T_H(x, v_n) - T_H(y, v_n)|.
\]
Both these limits exist and satisfy the usual properties.

**Lemma 4.6.** Let \( G \) be a random coalescing geodesic. Then the family \( \{G_H(z) : z \in \mathbb{Z}^2\} \) is coalescing and \( \mathbb{E}[B^H_G(0, z)] = \rho_G(z) \).

**Proof.** First, we observe that as \( G \) eventually moves into \( H \) there will be a density of boundary points \( z \) of \( H \) for which \( G(z) \) is entirely contained in \( H \). For any \( x \) we may thus find \( y \) and \( z \) so that \( G_H(x) \) is sandwiched between \( G(y) \) and \( G(z) \). Since \( G \) is coalescing it follows that also \( G_H \) is coalescing.

The function \( B^H_G \) is translation invariant along the boundary of \( H \). So, using additivity and the ergodic theorem, we have that
\[
\exists \lim_{n \to \infty} \frac{1}{n} B^H_G(0, ne_2) \quad \text{almost surely,}
\]
and since \( B^H_G(me_2, ne_2) = B_G(me_2, ne_2) \) whenever \( G(me_2) = G_H(me_2) \) and \( G(ne_2) = G_H(ne_2) \), the limit equals \( \rho_G(e_2) \). In particular, \( \mathbb{E}[B^H_G(0, e_2)] = \rho_G(e_2) \).

Now, assume that \( \rho_G = \rho_G' \). By Proposition 4.3 the two geodesics are confined to the same sector of width at most \( \pi/2 \). There is thus a half-plane \( H = H_i \) for which both \( G \) and \( G' \) eventually moves into. We assume that \( H \) is the right half-plane; the remaining cases being similar. Define
\[
\Delta_H(x, y) := B^H_G(x, y) - B^H_G(x, y) \quad \text{for } x, y \in H.
\]
By assumption \( \mathbb{E}[\Delta_H(x, y)] = 0 \). We aim to show that \( \Delta_H(x, y) = 0 \).

Either \( G_H = G'_H \), or their intersection is finite. In either case, \( G_H \) will either lie asymptotically above \( G'_H \) in the sense that we may reach \( G_H \) from \( G'_H \) via a counterclockwise motion, or \( G'_H \) will lie asymptotically above \( G_H \). Since \( G_H \) and \( G'_H \) are coalescing, one of the two will hold with probability one. We assume that \( G_H \) almost surely lies asymptotically above \( G'_H \).

Given \( k < \ell \), define three points \( x, y, z \) as follows: Since \( G_H(ke_2) \) and \( G_H(\ell e_2) \), and \( G''_H(ke_2) \) and \( G''_H(\ell e_2) \), coalesce we have that \( G_H(ke_2) \) and \( G''_H(\ell e_2) \) must intersect at some point \( z \). Next, take \( x \) on \( G'_H(ke_2) \cap G''_H(\ell e_2) \) beyond \( z \), and \( y \) on \( G_H(ke_2) \cap G''_H(\ell e_2) \) beyond \( z \); see Figure 1.

![Figure 1](image_url)

**Figure 1.** The paths \( G_H \) and \( G'_H \) diverge before leaving the shaded regions.
By exploiting the intersection point $z$ we may construct two paths from $ke_2$ to $x$: One being the segment of $G'_H(ke_2)$ and one being the concatenation of the segments of $G_H(ke_2)$ from $ke_2$ to $z$ and of $G'_H(ke_2)$ from $z$ to $x$. Denote this latter path by $\pi_x$. Similarly, we can construct two paths from $\ell e_2$ to $y$: One being a segment of $G_H(\ell e_2)$ and one being the concatenation of paths to $z$, which we denote by $\pi_y$. We see that

$$\Delta_H(ke_2,\ell e_2) = [T_H(ke_2,y) - T_H(\ell e_2,y)] - [T_H(ke_2,x) - T_H(\ell e_2,x)]$$

$$= [T_H(\pi_y) - T_H(ke_2,y)] - [T_H(\pi_x) - T_H(\ell e_2,x)],$$

which is non-negative. Since $E[\Delta_H(ke_2,\ell e_2)] = 0$ we conclude that, almost surely,

(8) \[ \Delta_H(ke_2,\ell e_2) = 0 \text{ for all } k, \ell. \]

To complete the argument, we show that (8) implies that $G_H = G'_H$, which in turn implies that $G = G'$, almost surely. Assume, for a contradiction, that $G_H \neq G'_H$ with positive probability. Since $G_H$ and $G'_H$ are coalescing, they must then differ with probability one. Let $A_k$ denote the event that $|G_H(ke_2) \cap G'_H(ke_2)| \leq m$. For large $m$ the event $A_k$ has probability at least $2/3$ to occur. Using the ergodic theorem we may find $k$ and $\ell$, at distance greater than $2m$ apart, such that $A_k \cap A_\ell$ occurs. Now, let $x$ denote the first common point of $G'_H(ke_2)$ and $G_H(\ell e_2)$, and let $y$ be the first common point of $G_H(ke_2)$ and $G'_H(\ell e_2)$ (again, see Figure 1). Since $A_k \cap A_\ell$ occurs and $|k - \ell| > 2m$ we have that $x \neq y$. This gives us two paths between $ke_2$ and $\ell e_2$, one visiting $x$ the other visiting $y$, which by (8) have equal weight. This contradicts the assumption of unique passage times. Hence we must have $G_H = G'_H$, and consequently $G = G'$, with probability one.

4.5. Proof of Proposition 4.5. We follow the lines of Damron and Hanson [DH14]. Consider the subgraph of the $\mathbb{Z}^2$ lattice containing all edges crossed by $G(z)$ for some $z \in \mathbb{Z}^2$. The resulting graph is connected, due to the coalescence of $G$, and does not contain any cycles, due to the assumption of unique passage times.

Assume that $G(0)$ is backwards infinite with positive probability. In that case there will be a density of sites $z \in \mathbb{Z}^2$ for which $G(z)$ is backwards infinite. A density of points in the plane will therefore be so-called trifurcation points. However, the number of trifurcation points in a box of side-length $n$ cannot be larger than the number of points on the boundary, which is a contradiction.

5. Existence of random coalescing geodesics

This section will contain two parts; the first of which recap the main results of [DH14], based on which we in the second establish the existence of random coalescing geodesics.

5.1. Damron-Hanson geodesic measures. In order to obtain geodesics with certain properties, Damron and Hanson [DH14] worked on an enlarged probability space $\tilde{\Omega}$, in order to keep track of Busemann functions at the same time as a family of limiting geodesics were constructed. We will describe their procedure in some detail below.

Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be any linear functional tangent to the asymptotic shape $\text{Ball}$. Let $\ell_\alpha := \{ x \in \mathbb{R}^2 : \rho(x) \geq \alpha \}$ and consider the family $\mathcal{F}_\alpha = \{ \text{Geo}(z, \ell_\alpha) : z \in \mathbb{Z}^2 \}$ of finite geodesics from points in $\mathbb{Z}^2$ to the half-plane $\ell_\alpha$. Our goal will be to obtain a family of infinite geodesics by sending $\alpha$ to infinity. The family $\mathcal{F}_\alpha$ we may encode as an element $\eta_\alpha \in \{0,1\}^{\tilde{E}^2}$, where $\tilde{E}^2$ denotes the set of directed edges in $\mathbb{Z}^2$, as follows:

$$\eta_\alpha(\tilde{e}) := \begin{cases} 1 & \tilde{e} \in \text{Geo}(z, \ell_\alpha) \text{ for some } z \in \mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases}$$
In order to get their hands on the limiting object, Damron and Hanson encode alongside the finite geodesics their associated Busemann differences. Define for each \( z \in \mathbb{Z}^2 \) an element \( \theta_{\alpha}(z) \in \mathbb{R}^2 \) as follows:

\[
\theta_{\alpha}(z) := (T(z, \ell_{\alpha}) - T(z + e_1, \ell_{\alpha}), T(z, \ell_{\alpha}) - T(z + e_2, \ell_{\alpha})).
\]

Encoded in \( \theta_{\alpha} \) we find the difference in distance between any two points and the line \( \ell_{\alpha} \), and thus serves as a Busemann function for the finite geodesics in \( \mathcal{F}_{\alpha} \). Moreover, every site \( z \notin \ell_{\alpha} \) has out-degree one in the directed graph encoded by \( \eta_{\alpha} \), and seen as an undirected graph \( \eta_{\alpha} \) has no cycles.

Let \( \Omega_1 = \Omega = [0, \infty)^{\mathbb{Z}^2}, \Omega_2 = (\mathbb{R}^2)^{\mathbb{Z}^2} \) and \( \Omega_3 = \{0,1\}^{\mathbb{Z}^2} \). For each \( \alpha \geq 0 \) we obtain a measurable map \( \Psi_{\alpha} : \Omega_1 \to \Omega_1 \times \Omega_2 \times \Omega_3 \) via \( \omega \mapsto (\omega, \theta_{\alpha}, \eta_{\alpha}) \). Damron and Hanson use this map to push forward the measure \( \mathbb{P} \) to obtain a measure \( \nu_{\alpha} \) on \( \bar{\Omega} := \Omega_1 \times \Omega_2 \times \Omega_3 \). In order to obtain a measure which is increasingly invariant with respect to translations, Damron and Hanson considers the averages

\[
\nu^*_n(\cdot) := \frac{1}{n} \int_0^n \nu_\alpha(\cdot) \, d\alpha.
\]

From the observation that \( \theta_{\alpha}(z) \leq (\omega(z, z + e_1), \omega(z, z + e_2)) \) it follows that the sequence of measures \( (\nu^*_n)_{n \geq 1} \) is tight. Prokhorov’s theorem then implies that \( (\nu^*_n)_{n \geq 1} \) has a weakly convergent subsequence. Damron and Hanson move on to show that every subsequential limit \( \nu \) of the sequence \( (\nu^*_n)_{n \geq 1} \) is invariant with respect to translations and supported on families of geodesics with desirable properties. Damron and Hanson prove in [DH14], among other things, the following:

**Theorem 5.1.** Let \( \rho : \mathbb{R}^2 \to \mathbb{R} \) be a linear functional tangent to Ball. Every subsequential limit \( \nu \) is invariant with respect to translations and satisfies the following properties: For \( \nu \)-almost every \( (\omega, \theta, \eta) \in \bar{\Omega} \) and all \( y, z \in \mathbb{Z}^2 \) we have

(a) a unique forwards path \( \gamma_z \) which is a geodesic;
(b) \( \text{Dir}(\gamma_z) \subseteq \{ x \in S^1 : \mu(x) = \rho(x) \} \);
(c) \( \gamma_y \) and \( \gamma_z \) are coalescing.

We shall construct a set of four random coalescing geodesics based on Theorem 5.1

### 5.2. Existence of random coalescing geodesics

The existence of random coalescing geodesics is certainly hinted at from the geodesic measures considered by Damron and Hanson, but to construct them remains a non-trivial task even from their work. The goal of this section is to prove their existence.

**Theorem 5.2.** Let \( \rho : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear functional tangent to the asymptotic shape Ball. Then, there exists a random coalescing geodesic geodesic \( G \) such that

\[
\text{Dir}(G) \subseteq \{ x \in S^1 : \mu(x) = \rho(x) \}.
\]

Since the asymptotic shape has at least four sides, Theorem 5.2 proves the existence of at least four random coalescing geodesics.

Recall that \( \mathcal{F}_0 = \mathcal{F}_0(\omega) \) denotes the set of all one-sided geodesics starting at the origin. Given a linear functional \( \rho : \mathbb{R}^2 \to \mathbb{R} \) tangent to Ball, let \( \mathcal{F}_0^\rho = \mathcal{F}_0^\rho(\omega) \) denote the set of geodesics in \( \mathcal{F}_0 \) whose set of directions intersect the arc \( \{ x \in S^1 : \mu(x) = \rho(x) \} \).

**Lemma 5.3.** For any linear functional \( \rho \) tangent to Ball we have that \( \mathcal{F}_0^\rho \neq \emptyset \) and totally ordered almost surely.
Proof. The fact that $\mathcal{R}_0^p$ is non-empty follows from Theorem \ref{thm:existence}. By the same theorem we also have that there exists a geodesic in $\mathcal{R}_0 \setminus \mathcal{R}_0^p$. Due to the tree structure of $\mathcal{R}_0$ any two geodesics will share at most a finite number of edges, after which they diverge, never to intersect again. Hence, for any two geodesics $g$ and $g'$, one is attained via a counterclockwise motion from the other. We say $g \leq g'$ if we can move counterclockwise from $g$ to $g'$ staying in $\mathcal{R}_0^p$. It is easy to check that for any $g$ and $g'$ we have either $g \leq g'$ or $g' \leq g$. Finally, if both $g \leq g'$ and $g' \leq g$ then $g = g'$, so this is a total ordering.

In order to construct a random coalescing geodesic we will use the Damron-Hanson geodesic measures to put a measure on the totally ordered set $\mathcal{R}_0^p$.

Definition 5.4. Given a linear functional $\rho : \mathbb{R}^2 \to \mathbb{R}$ tangent to Ball and a Damron-Hanson geodesic measure $\nu$ we obtain, for almost every $\omega \in \Omega_1$, a probability measure $\hat{\nu} = \hat{\nu}(\omega)$ on $\Omega_3$ through conditional expectation.

It is important to note that $\hat{\nu}(\omega)$ is a function of the weight configuration $\omega$ and the linear functional $\rho$, but independent of everything else. Since $\hat{\nu}(\omega)$ is a probability measure on geodesic graphs in $\mathbb{Z}^2$, its marginal gives a measure on $\mathcal{R}_0^p$. We will interchangeably think of $\hat{\nu}$ as a measure on $\Omega_3$ and as a measure on $\mathcal{R}_0^p$.

We will next exhibit a function $\psi : \mathcal{R}_0^p \to [0, 1]$ such that Lesbesgue measure on $[0, 1]$, which we write as $\text{Leb}$, is the pushforward of $\hat{\nu}$ by $\psi$. Given a geodesic $g \in \mathcal{R}_0^p$ define $\mathcal{R}_0^{\leq g} \subseteq \mathcal{R}_0^p$ as the subtree consisting of all geodesics $g'$ in $\mathcal{R}_0^p$ such that $g' < g$. We similarly define $\mathcal{R}_0^{\leq g}$ with $<$ replaced by $\leq$. For every $\beta \in [0, 1]$ let

$$G_\beta := \sup \{ g \in \mathcal{R}_0^p : \hat{\nu}(\mathcal{R}_0^{\leq g}) < \beta \}. $$

Lemma 5.5. For $\mathbf{P}$-almost every $\omega \in \Omega_1$ we have for all $h \in \mathcal{R}_0^p$ that

\begin{equation}
\{ \beta \in [0, 1] : G_\beta \in \mathcal{R}_0^{\leq h} \} = [0, \hat{\nu}(\mathcal{R}_0^{\leq h})];
\end{equation}

and for any $v \in \mathbb{Z}^2$ and any coalescing pair of geodesics $h \in \mathcal{R}_0^p$ and $h' \in \mathcal{R}_0^p$ that

\begin{equation}
\{ \beta \in [0, 1] : G_\beta \in \mathcal{R}_0^{\leq h} \} = \{ \beta \in [0, 1] : G_\beta(v) \in \mathcal{R}_0^{\leq h'} \}.
\end{equation}

Proof. We first observe that $\{ \beta : G_\beta \in \mathcal{R}_0^{\leq h} \} = \{ \beta : G_\beta \leq h \}$ by definition of $\mathcal{R}_0^{\leq h}$. We first set out to prove \ref{lem:existence} and need for this to establish two claims.

We first claim that if $\beta > \hat{\nu}(\mathcal{R}_0^{\leq h})$, then $G_\beta > h$. To see this, consider the following two cases: Either there is a least element $h' \in \mathcal{R}_0^p$ strictly larger than $h$, or there is a decreasing sequence $(h_k)_{k \geq 1}$ approaching $h$. In the former case we have

$$\hat{\nu}(\mathcal{R}_0^{< h'}) = \hat{\nu}(\mathcal{R}_0^{< h}) < \beta,$$

which implies that $G_\beta \geq h' > h$. In the latter case, by continuity of measure, we have that

$$\hat{\nu}(\mathcal{R}_0^{< h_k}) \to \hat{\nu}(\mathcal{R}_0^{< h}) < \beta.$$

Hence, for some $h_k > h$ we have $\hat{\nu}(\mathcal{R}_0^{< h_k}) < \beta$, and thus that $G_\beta \geq h_k > h'$. This settles the first claim.

Second, we claim that if $\beta \leq \hat{\nu}(\mathcal{R}_0^{\leq h})$, then $G_\beta \leq h$. To see this, take $h' > h$. Then

$$\hat{\nu}(\mathcal{R}_0^{< h'}) \geq \hat{\nu}(\mathcal{R}_0^{< h}) \geq \beta.$$

That is, no $h' > h$ is contained in the set $\{ g \in \mathcal{R}_0^p : \hat{\nu}(\mathcal{R}_0^{\leq g}) < \beta \}$, and so $G_\beta \leq h$.

The two claims imply $\mathbf{P}$. Note that it holds for almost every $\omega \in \Omega_1$ because $\hat{\nu}$ is a conditional expectation and is only defined almost surely. To prove \ref{lem:coalescing} it will suffice to show that $\hat{\nu}(\mathcal{R}_0^{\leq h}) = \hat{\nu}(\mathcal{R}_0^{< h'})$ for almost every $\omega$. Assume, for a contradiction, that

\begin{equation}
\hat{\nu}(\mathcal{R}_0^{\leq h}) > \hat{\nu}(\mathcal{R}_0^{< h'})
\end{equation}

and let $\tilde{\nu}$ be the probability measure on $\Omega_3$ such that

\begin{equation}
\tilde{\nu}(\mathcal{R}_0^{\leq h}) = \hat{\nu}(\mathcal{R}_0^{\leq h}) < \hat{\nu}(\mathcal{R}_0^{< h'}) = \tilde{\nu}(\mathcal{R}_0^{< h'}). 
\end{equation}

Then

$$\tilde{\nu}(\mathcal{R}_0^{\leq h}) = \hat{\nu}(\mathcal{R}_0^{\leq h}) > \hat{\nu}(\mathcal{R}_0^{< h'}) = 1, $$

which is a contradiction. Hence, \ref{lem:coalescing} holds.
\[ \hat{\nu}(\mathcal{F}^{<h}_0) < \hat{\nu}(\mathcal{F}^{<h}_0') \] for some pair \( h, h' \) with positive probability. In this case, due to the total ordering, we must have

\[ \hat{\nu}(\mathcal{F}^{<h'}_v \cap \mathcal{F}^p \setminus \mathcal{F}^{<h}_0) > 0, \]

in which case \( \hat{\nu} \) puts positive mass on a non-coalescing family of geodesics. However, by Theorem 5.1 we know that this can only happen on a null set. \( \square \)

**Lemma 5.6.** For Lebesgue-almost every \( \beta \in [0, 1] \) we have that

\[ P(G_\beta \text{ is coalescing}) = 1. \]

**Proof.** We start by putting two measures on \( \mathcal{P} \) and showing that they are the same. The first measure is \( \nu \) projected onto \( \Omega_3 \), by which we obtain a measure on \( \mathcal{P} \). The second is the pushforward of \( P \times \text{Leb} \) via the map \( \psi : \Omega_1 \times [0, 1] \to \mathcal{P} \) given by \( (\omega, \beta) \mapsto G_\beta(\omega) \).

Observe first that by taking Lebesgue measure on both sides in (9) we obtain

\[ \nu(B) = \int_{\Omega_3} \hat{\nu}(B(\omega)) \, dP = \int_{\Omega_3} \text{Leb}(\{ \beta : G_\beta \in B(\omega) \}) \, dP = (P \times \text{Leb})(G_\beta \in B). \]

In particular, for any set \( B \) with full measure, \( P(G_\beta \in B) = 1 \) for Lebesgue almost every \( \beta \in [0, 1] \).

It remains to show that, for almost every \( \beta \), \( P(G_\beta(0) \setminus G_\beta(v) \text{ is finite}) = 1 \). Assume that \( G_\beta(0) \) and \( G_\beta(v) \) do not coalesce. As \( G_\beta \) is supported on coalescing families of geodesics (since \( \hat{\nu} \) is) there would then exist a pair of coalescing geodesics \( h \in \mathcal{F}^p_0 \) and \( h' \in \mathcal{F}^p_0 \) such that

\[ \{ \beta : G_\beta \in \mathcal{F}^{<h}_0 \} \neq \{ \beta : G_\beta(v) \in \mathcal{F}^{<h'}_v \}, \]

as the particular \( \beta \) is question would pertain to one of the sets but not the other. This would contradict \( \{ \beta \} \), and thus have probability zero to occur. \( \square \)

**Proof of Theorem 5.2.** It follows from Lemma 5.6 that there exists \( \beta \) for which \( G_\beta : \Omega_1 \to \mathcal{P} \) is a random coalescing geodesic. That it has the prescribed properties is a consequence of Theorem 5.1. \( \square \)

### 5.3. The tail of a random coalescing geodesic

We conclude this section with a result about random coalescing geodesics. It is similar in spirit to Proposition 4.10 in \[DH\].

**Proposition 5.7.** Let \( G \) be a random coalescing geodesic which eventually moves into some half-plane \( H \). Let \( \hat{\Sigma} \) be the \( \sigma \)-algebra that associates all configurations that agree on all edges with both endpoints in \( H \). Then the tail of \( G \) is almost surely measurable with respect to \( \hat{\Sigma} \).

**Proof.** We put a measure on geodesics restricted to the half-plane \( H \). For any \( \omega \) we can find \( G(\omega) = (0, v_1, v_2, \ldots) \). As argued in the proof of Proposition 4.4 the limit of Geo(0, \( v_k \)) restricted to \( H \) exists. Thus by taking conditional measure \( M_{G,H} \) with respect to \( \hat{\Sigma} \) we get a measure on half-plane geodesics. This measure is shift invariant in the natural sense.

We will argue as in Theorem 5.2. For almost every \( \alpha \) between 0 and 1 this will give us a random coalescing geodesic \( M_{G,H,\alpha} \) in the half-plane. All of these have a Busemann function which is asymptotically linear to \( \rho \). It is easy to extend them to a random coalescing geodesic in the whole plane that also have Busemann functions which are asymptotically linear to \( \rho \). By Proposition 4.4 these random coalescing geodesics are the same for all \( \alpha \). Thus the
measure $M_{G,H}$ is almost surely supported on one geodesic. $G$ differs from this geodesic in a finite number of edges almost surely. As $M_{G,H}$ is $\Sigma$ measurable, so is the tail of $G$. □

6. A shift invariant labeling of geodesics

In this section we define a flow on the tree $T_0 = T_0(\omega)$ of one-sided geodesics emanating from the origin, and use the resulting flow to label geodesics in a systematic way. For the labeling to be useful it will have to be consistent with some natural notion of order among geodesics. Loosely speaking, we will work with an ordering in which $g \leq g'$ if to reach $g'$ from some reference geodesic $g_\ast$, in a counterclockwise motion, we first cross $g$.

Since the asymptotic shape has at least four sides, Theorem 5.2 grants the existence of at least four random coalescing geodesics. However, we shall also note that from the existence of a single random coalescing geodesic $\Gamma_0$ we obtain an additional three distinct random coalescing geodesics via right angle rotation. This again gives a set of four random coalescing geodesics, whose asymptotic properties also them are related via right angle rotation. For the rest of this paper we will denote by $\Gamma_0$, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ the four random coalescing geodesics obtained by right angle rotation of a given random coalescing geodesic. We will further fix one of these four geodesics as our reference geodesic: call this geodesic $\Gamma_\ast$. Finally, we denote by $\Gamma_i(v)$ the translate of $\Gamma_i$ along the vector $v \in \mathbb{Z}^2$.

6.1. A total ordering of geodesics. Recall that $T_v$ denotes the tree of one-sided geodesics emanating from the vertex $v \in \mathbb{Z}^2$. There is a natural total ordering among geodesics in $T_v$, where $g \leq g'$ if we can reach $g'$ from $g$ in a counterclockwise motion without crossing $\Gamma_\ast(v)$. Since $\Gamma_\ast$ is coalescing, the total ordering on $T_\ast$ extends to a total ordering among all geodesics in $\{g \in T_v : v \in \mathbb{Z}^2\}$: Any two geodesics $g$ and $g'$ that intersect either coalesce or intersect in a finite connected set of edges. Given $g \in T_u$ and $g' \in T_v$, let $S \subset \mathbb{Z}^2$ be finite and connected with the property that $\Gamma_\ast(u)$ and $\Gamma_\ast(v)$ agree outside $S$ and $g$ and $g'$ either agree or are disjoint outside $S$. Indeed, if some set $S$ has this property, then every set $S$ containing $S$ has this property too. We say that $g \leq g'$ if we can reach $g'$ from $g$ in a counterclockwise motion along the boundary of $S$ without crossing $\Gamma_\ast$. This gives a well-defined total ordering with probability one. Note that if $g$ and $g'$ coalesce, then they are considered equal in the above ordering. With a slight abuse of notation we think of the ordering as cyclic, in which $\Gamma_\ast$ is not only as the minimal element of the ordering, but also as the maximal element.

Based on the above ordering, we shall say that a geodesic $g$ is a ccw limit if there exists a sequence of geodesics $g_1 < g_2 < g_3 \cdots$ such that $\lim_{k \to \infty} g_k = g$. Otherwise we say that $g$ is ccw isolated. The terms ccw limit and ccw isolated are defined analogously. We note that if a geodesic $g = (v_0, v_1, \ldots)$ is a ccw limit, then so is the geodesic $g' = (v_n, v_{n+1}, \ldots)$.

6.2. A single source flow. In a first step, we define, for every vertex $v \in \mathbb{Z}^2$, a flow from $v$ to $\infty$ along $T_v$. This flow has a source of magnitude one at $v$ and no sinks. We define the flow inductively starting at the root. Suppose we have defined the flow into a vertex $w$. The flow splits the mass flowing into $w$ equally among all edges in $T_v$ that emanate from $w$. We denote by $M_v(\Gamma_\ast, g)$ the mass that flows out along geodesics in $T_v$ that lie strictly above $\Gamma_\ast$ and below and including $g$, in the counterclockwise ordering. This assigned to each geodesic in $T_v$, a value between 0 and 1. In consistence with the convention of considering $\Gamma_\ast$ both minimal and maximal, we think of it as taking value both 0 and 1.

The cumulative flow $M_v(\Gamma_\ast, g)$ will not (necessarily) provide a labeling of geodesics consistent with the ordering. That is, even for $g \in T_u$ and $g' \in T_v$ that coalesce, and hence are equal as far as the ordering concerns, we may have $M_v(\Gamma_\ast, g) \neq M_v(\Gamma_\ast, g')$. To obtain a
labeling consistent with the ordering we will employ an averaging procedure over equivalence classes of \( \mathbb{Z}^2 \) and work with subsequential limits.

6.3. Averaging over equivalence classes. Let \( \{ \xi_z \}_{z \in \mathbb{Z}^2} \) be independent \([0, 1]\)-uniform random variables. For each \( i = 1, 2, \ldots \) let \( \{ V_i(z) : z \in \mathbb{Z}^2 \} \) be the partition of \( \mathbb{Z}^2 \) obtained as follows: Let \( S_i \) denote the set of all points in \( \mathbb{Z}^2 \) with \( \xi_z \leq 1/4^i \) and define \( f_i : \mathbb{Z}^2 \to S_i \) by mapping each point to the one in \( S_i \) at least \( \ell_1 \)-distance. (Choose the one with minimal \( \xi \)-value in case of a tie.) This induces an equivalence relation on \( \mathbb{Z}^2 \) in which two sites are equivalent in case they map to the same site in \( S_i \). Let \( \{ V_i(z) : z \in \mathbb{Z}^2 \} \) be the collection of equivalence classes of this equivalence relation.

**Lemma 6.1.** For every pair \( u, v \in \mathbb{Z}^2 \) we have

\[
P(V_i(u) = V_i(v) \text{ for all } i \text{ sufficiently large}) = 1.
\]

**Proof.** In case \( u \) and \( v \) belong to different equivalence classes, then there exists \( z \in S_i \setminus \{ f_i(u) \} \) such that \( \| z - u \| < \| f_i(u) - u \| + 2\| u - v \| \). To see this assume the contrary, in which case the triangle inequality, for any \( z \in S_i \setminus \{ f_i(u) \} \), gives that

\[
\| z - v \| \geq \| z - u \| - \| u - v \| > \| f_i(u) - u \| + \| u - v \| \geq \| f_i(u) - v \|,
\]

and hence \( V_i(u) = V_i(v) \).

For each \( i \geq 1 \) the expected distance from \( u \) to \( S_i \) is of order \( 2^i \), so with high probability we have \( \| f_i(u) - u \| < 3^i \). Since \( \| y - u \| \geq \| f_i(u) - u \| \) for all \( y \in S_i \), there are order \( 3^i \| u - v \| \) possible choices for the point \( z \). Since each has probability \( 1/4^i \) to belong to \( S_i \) we conclude that \( V_i(u) \neq V_i(v) \) with probability of order \( (3/4)^i \| u - v \| \). The result then follows from Borel-Cantelli.

We next use the equivalence classes above defined to obtain a labeling of geodesics which is consistent with the ordering among sites within the same equivalence class. Based on the total ordering we may define \( M_u(\Gamma, g) \) for any geodesic \( g \), not necessarily in \( \mathcal{T}_u \), as the mass (under the flow from \( u \)) along all geodesics between \( \Gamma \) and \( g \). That is, let

\[
M_u(\Gamma, g) := \sup \{ M_u(\Gamma, g') : g' \in \mathcal{T}_u \text{ and } g' \leq g \}.
\]

For each \( i = 1, 2, \ldots \) and \( v \in \mathbb{Z}^2 \) we define for \( g \in \mathcal{T}_v \)

\[
M'_i(v, \Gamma, g) := \frac{1}{|V_i(v)|} \sum_{u \in V_i(v)} M_u(\Gamma, g).
\]

It is straightforward to verify that the labeling generated by the averaged cumulative flow \( M'_i \) is consistent with the ordering of geodesics originating from the points in the same equivalence class; we save the details for the proof of Lemma 6.2 below.

6.4. Subsequential weak limits. The labels produced by \( M'_i : \mathcal{T}_v \to [0, 1] \) can be encoded as an element in \([0, 1]^{\mathcal{E}^2}\) in the following manner, where \( \mathcal{E}^2 \) denotes the set of (undirected) edges of the square lattice. For each \( v \in \mathbb{Z}^2 \) and \( e \in \mathcal{E}^2 \) define

\[
\varphi_i(v, e) := \sup \{ M'_{i}(\Gamma, g) : g \in \mathcal{T}_v \text{ and } e \in g \}.
\]

(Supremum of the empty set is interpreted as zero.) This defines, for each \( v \in \mathbb{Z}^2 \) and \( i \geq 1 \), an element \( \varphi_i(v, \cdot) \in [0, 1]^{\mathcal{E}^2} \). Note further that if \( g = (e_1, e_2, \ldots) \) is a geodesic in \( \mathcal{T}_v \), then we can recover the value of \( F'_i(g) \) from \( \varphi_i \) as the limit

\[
M'_i(\Gamma, g) = \lim_{n \to \infty} \varphi_i(v, e_n) = \inf_{e \in g} \varphi_i(v, e),
\]

as \( \varphi_i(v, e_n) \) is decreasing in \( n \).
Let $\Omega_1 = [0, \infty)^{\mathbb{Z}^2}$, $\Omega_2 = [0, 1]^{\mathbb{Z}^2}$ and $\Omega_3 = [0, 1]^{2\mathbb{Z} \times \mathbb{Z}^2}$. For each $i \geq 1$ we can exhibit a (measurable) map $\Psi_i : \Omega_1 \times \Omega_2 \to \Omega_1 \times \Omega_3$ as $(\omega, \xi) \mapsto (\omega, \varphi_i)$. The measure $\mathbf{P} \times \text{Leb}$ may be pushed forward through the mapping $\Psi_i$ to give a measure $\nu_i$ on $\Omega_1 \times \Omega_3$. Via a compactness argument and Prokhorov's theorem we conclude that $(\nu_i)_{i \geq 1}$ has a weakly converging subsequence.

The next lemma shows that the limit of the converging subsequence is well behaved, and consistent with the total ordering of geodesics. Given $v \in \mathbb{Z}^2$, let $\tilde{\sigma}_v$ denote the shift operator on $\Omega_1 \times \Omega_3$ for which

$$[\tilde{\sigma}_v(\omega, \varphi)](e, (z, f)) = (\omega_{e-v}, \varphi(z - v, f - v)).$$

**Lemma 6.2.** Every subsequential limit $\nu$ of $(\nu_i)_{i \geq 1}$ is invariant with respect to $\tilde{\sigma}_v$ and satisfies the following properties: For $\nu$-almost every $(\omega, \varphi) \in \Omega_1 \times \Omega_3$ and every $u, v \in \mathbb{Z}^2$ we have that

(a) $\varphi(u, e_n)$ is decreasing for every geodesic $g = (e_1, e_2, \ldots) \in \mathcal{T}_u$;

(b) for any $g = (e_1, e_2, \ldots) \in \mathcal{T}_u$ and $g' = (e'_1, e'_2, \ldots) \in \mathcal{T}_v$ for which $g \leq g'$ we have

$$\lim_{n \to \infty} \varphi(u, e_n) \leq \lim_{n \to \infty} \varphi(v, e'_n);$$

(c) for any decreasing sequence $(g_k)_{k \geq 1}$ in $\mathcal{T}_u$ the limit $\lim_{k \to \infty} g_k$ exists and

$$\lim_{k \to \infty} \lim_{n \to \infty} \varphi(u, e_k^n) = \lim_{n \to \infty} \varphi(u, e^n_k);$$

**Proof.** Let $\nu$ be a subsequential limit of $(\nu_i)_{i \geq 1}$. We first show that $\int f \, d\nu = \int f \, d\nu \circ \tilde{\sigma}_v$ for all bounded continuous functions $f : \Omega_1 \times \Omega_3 \to \mathbb{R}$, and thus that $\nu = \nu \circ \tilde{\sigma}_v$. Observe that the measure $\nu_i$ is invariant with respect to $\tilde{\sigma}_v$ due to the product structure of $\mathbf{P} \times \text{Leb}$. That is, if $\tilde{\sigma}_v$ denotes the operator on $\Omega_1 \times \Omega_2$ for which $[\tilde{\sigma}_v(\omega, \xi)](e, z) = (\omega_{e-v}, \xi_{z-v})$, then

$$\int f \, d\nu_i = \int f \circ \tilde{\sigma}_v \circ \Psi_i \, d(\mathbf{P} \times \text{Leb}) = \int f \circ \Psi_i \circ \tilde{\sigma}_v \, d(\mathbf{P} \times \text{Leb}) = \int f \, d\nu_i$$

for each bounded continuous function $f$, since $\tilde{\sigma}_v \circ \Psi_i = \Psi_i \circ \tilde{\sigma}_v$, and $\mathbf{P} \times \text{Leb}$ is invariant with respect to $\tilde{\sigma}_v$. Hence, $\nu_i = \nu_i \circ \tilde{\sigma}_v$ for every $i \geq 1$, and by continuity of $\tilde{\sigma}_v$ it follows that $\nu = \nu \circ \tilde{\sigma}_v$ by taking limits.

We proceed with the proof of the attributed properties. Since $\mathbb{Z}^2$ is countable it will suffice to prove each of the statements for a fixed pair of vertices $u, v \in \mathbb{Z}^2$. We start with part (a), and let $A_u$ denote the event

$$A_u = \{ (\omega, \varphi) : \varphi(u, e_n) \text{ is decreasing for every } g = (e_1, e_2, \ldots) \in \mathcal{T}_u \}. $$

By construction we have $\nu_i(A_u) = 1$ for every $i \geq 1$, and by the Portmanteau theorem $\nu(A_u) = \lim_{i \to \infty} \nu_i(A_u) = 1$, since $A_u$ is closed, proving part (a).

For part (b), let $A'_{u,v}$ denote the event that

$$\lim_{n \to \infty} \varphi(u, e_n) \leq \lim_{n \to \infty} \varphi(v, e'_n)$$

for every $g = (e_1, e_2, \ldots) \in \mathcal{T}_u$ and $g' = (e'_1, e'_2, \ldots) \in \mathcal{T}_v$ such that $g \leq g'$. According to the averaging procedure over equivalence classes we have for $u$ and $v$ for which $\Gamma(u) = \Gamma(v)$ and $V_i(u) = V_i(v)$ that

$$M^\nu_{\ast}(\Gamma, g) = \frac{1}{|V_i(u)|} \sum_{w \in V_i(u)} M(w, \Gamma(u), g) \leq \frac{1}{|V_i(u)|} \sum_{w \in V_i(u)} M_w(\Gamma(u), g') = M^\nu_{i}(\Gamma, g').$$

It follows that

$$\nu_i(A'_{u,v}) \geq \mathbf{P} \times \text{Leb}(\{(\omega, \xi) : V_i(u) = V_i(v)\}),$$
which by Lemma 6.1 tends to 1 as \( i \to \infty \). Consequently, \( \nu(A^i) = 1 \) and part (b) holds.

That the limit \( \lim_{k \to \infty} g_k \), in part (c), exists follows from the assumed monotonicity and the fact that \( g_k \geq \Gamma_x \) for all \( k \geq 1 \). The inequality \('\geq'\) in the double limit is thus a consequence of part (b) above, so it will suffice to show that the reversed inequality holds as well. Let \( A^i_n \) denote the event that for every sequence \( g_1 \geq g_2 \geq \ldots \) in \( \mathcal{T}_u \) we have
\[
\lim_{k \to \infty} \lim_{n \to \infty} \varphi(u, e^k_n) \leq \lim_{n \to \infty} \lim_{k \to \infty} \varphi(u, e^k_n).
\]
By definition of a limit it follows that if \( g' > g_\infty = \lim_{k \to \infty} g_k \) for some \( g' \in \mathcal{T}_w \), then also \( g' > g_k \) for all large \( k \). Consequently we have \( M_w(\Gamma_x, g_k) \to M_w(\Gamma_x, g_\infty) \), and hence \( \nu(A^i_n) = 1 \). So, also part (c) holds.

6.5. Global labeling of geodesics. Finally, given a subsequential limit \( \nu \) of \( (\nu_i)_{i \geq 1} \) we give each geodesic in the plane a label \( \alpha \in [0, 1] \) through the reconstructed cumulative flow obtained through \( \nu \). For each \( \omega \in \Omega_1 \) we obtain a probability measure \( \hat{\nu} = \hat{\nu}(\omega) \) on \( \Omega_3 \) through conditional expectation. For each \( \omega \in \Omega_1 \) and \( v \in \mathbb{Z}^2 \), define for each geodesic \( g = (e_1, e_2, \ldots) \) in \( \mathcal{T}_v(\omega) \), a label through averaging:
\[
F(g) := \lim_{n \to \infty} \int \varphi(v, e_n) \, d\hat{\nu}(\omega).
\]
Finally, we show that the labeling is well-defined and consistent with the ordering of geodesics.

**Proposition 6.3.** The limit in (11) exists and satisfies the following properties for \( \mathbb{P} \)-almost every \( \omega \in \Omega_1 \):

(a) For any two geodesics \( g \leq g' \) we have \( F(g) \leq F(g') \).

(b) If \( g \) and \( g' \) coalesce, then \( F(g) = F(g') \).

(c) For any \( v \) and any decreasing sequence \( (g_k)_{k \geq 1} \) in \( \mathcal{T}_v \) we have
\[
\lim_{k \to \infty} F(g_k) = F\left( \lim_{k \to \infty} g_k \right).
\]

**Proof.** That the limit in (11) exists for \( \mathbb{P} \)-almost every \( \omega \in \Omega_1 \) follows from part (a) of Lemma 6.2 and the (conditional) monotone convergence theorem. Subsequently, property (a) follows from part (b) of Lemma 6.2 Property (b) follows from (a) since if \( g \) and \( g' \) coalesce, then we have both \( g \leq g' \) and \( g' \leq g \). Finally, property (c) is the consequence of parts (b) and (c) of Lemma 6.2 and the monotone convergence theorem.

Note that for every random coalescing geodesic its labels are shift invariant and thus almost surely constant.

6.6. Multiplicity of labels. Finally we show that the labeling does a good job of distinguishing distinct geodesics.

**Lemma 6.4.** For any subsequential limit \( \nu \) and any \( \alpha \in [0, 1] \), each of the following events has probability zero to occur:

(a) there are at least three geodesics in \( \mathcal{T}_0 \) which have label \( \alpha \);

(b) there are two geodesics in \( \mathcal{T}_0 \) with label \( \alpha \) of which the cw-most is a cw-limit;

(c) there are two geodesics in \( \mathcal{T}_0 \) with label \( \alpha \) of which the ccw-most is a ccw-limit;

(d) there are two geodesics in \( \mathcal{T}_0 \) with label \( \alpha \) of which the ccw-most is ccw-isolated.

**Proof.** First we note that parts (b) and (c) follow from (a), since either of the two events implies the existence of at least three geodesics with label \( \alpha \). It will thus suffice to prove parts (a) and (d). During this proof, a set \( S_v \subseteq \mathcal{T}_v \) will be said to be ‘\( K \)-good’ if it contains at least three geodesics that diverge within \( K \geq 1 \) steps from \( v \).
It is a standard fact about Voronoi tessellations that there exists $c > 0$ such that for $i \geq 1$
\begin{equation}
P\left([-c^{2^i},c^{2^i}] \subseteq V_i(0) \subseteq [-c^{-1}2^i,c^{-1}2^i]\right) > 3/4.
\end{equation}
For $\alpha \in [0,1]$, $\varepsilon > 0$ and $i \geq 1$ let $A_i = A_i(\alpha, \varepsilon, K)$ denote the set of $v \in V_i(0)$ such that there exists a $K$-good set $S_v \subseteq \mathcal{T}_v$ for which
\[
M^{(1)}_v(\Gamma_*, g) \in (\alpha - \varepsilon, \alpha + \varepsilon) \text{ for all } g \in S_v.
\]

**Claim 6.5.** For every $\varepsilon > 0$, $K \geq 1$ and $i \geq 1$ we have
\[
P\left(\left|A_i(\alpha, \varepsilon, K)\right| < 2\varepsilon 4^{K+i+1} \text{ for all } \alpha \in [0,1]\right) > 3/4.
\]

**Proof of claim.** For every $v \in A_i$ let $a_v$ and $b_v$ denote the clockwise- and counterclockwise-most elements in $S_v$, and let $a_i$ and $b_i$ denote the least and largest elements among all geodesics in $\bigcup_{v \in A_i} S_v$ (in the total ordering). We then observe that for each $v$ mass of at least $4^{-K}$ escapes in between $a_v$ and $b_v$. Consequently, we obtain
\[
2\varepsilon > M^{(1)}_v(\Gamma_*, b_v) - M^{(1)}_v(\Gamma_*, a_v) > 4^{-K} \frac{|A_i|}{|V_i|},
\]
or that $|A_i| < 2\varepsilon 4^K |V_i|$. The claim then follows from (12). \hfill \square

We proceed via contradiction and assume that for some $\alpha \in [0,1]$ and $\delta > 0$ we have
\[
P\left(\exists \text{ at least three geodesics in } \mathcal{T}_0 \text{ which have label } \alpha \right) > \delta.
\]

Then we can choose $K$ large so that
\begin{equation}
P\left(\exists \text{ a } K\text{-good set } S_0 \subseteq \mathcal{T}_0 \text{ such that } F(g) = \alpha \text{ for all } g \in S_0 \right) > \delta.
\end{equation}

Let $(\nu_m)_{m \geq 1}$ be some subsequence converging weakly to $\nu$, and pick $\varepsilon > 0$ such that $\delta > 4\sqrt{2\varepsilon} 4^{K+i}$. By assumption we obtain that for almost every $\omega \in \Omega_i$
\begin{equation}
F(g) = \int_{\varepsilon \in \mathcal{Y}} \inf_{v \in E} \phi(v, e) \, d\hat{\nu}(\omega) = \lim_{m \to \infty} \int_{\varepsilon \in \mathcal{Y}} \inf_{v \in E} \phi(v, e) \, d\hat{\nu}_m(\omega) = \lim_{m \to \infty} E[M^{(1)}_v(\Gamma_*, g) | \omega].
\end{equation}

Let $C_i = C_i(\alpha, \varepsilon, c, K)$ denote the set of points $v \in [-c^{2^i},c^{2^i}]$ such that there exists a $K$-good set $S_v \subseteq \mathcal{T}_v$ for which $E[M^{(1)}_v(\Gamma_*, g) | \omega]$ is in $(\alpha - \varepsilon/2, \alpha + \varepsilon/2)$ for all $g \in S_v$. The ergodic theorem, from (13) and (14), implies that for some large $i$
\[
P\left(\left|C_i\right| > 2\delta c^{2^{4i}}\right) > 3/4.
\]

We now claim that then, for some large $i$, we have
\begin{equation}
P\left(\left|A_i(\beta, \sqrt{\varepsilon}, K)\right| > 2\delta c^{2^{4i}} \text{ for some } \beta \in [0,1]\right) > 1/4.
\end{equation}

To see this, note that the contrary would imply that either $|C_i| \leq 2\delta c^{2^{4i}}$ (which has probability at most $1/4$), $[-c^{2^i},c^{2^i}]$ is not contained in $V_i(0)$ (having probability at most $1/4$), or there are $u, v \in V_i(0)$ with $g \in \mathcal{T}_u$ and $g' \in \mathcal{T}_v$ for which $E[M^{(1)}_u(\Gamma_*, g) | \omega] - E[M^{(1)}_v(\Gamma_*, g) | \omega] < \varepsilon$ but $|M^{(1)}_u(\Gamma_*, g) - M^{(1)}_v(\Gamma_*, g')| > \sqrt{\varepsilon}$ (which occurs with probability at most $\sqrt{\varepsilon}$). Therefore (15) holds. However, (15) contradicts Claim 6.5.

The above proves (a). The proof of (d) is similar. \hfill \square
7. A central geometric argument

In this section we present a central geometric argument. This argument will effectively function as a 0-1 law, and will be used repeatedly for constructing geodesics that starts at some vertex \( x \) and have certain desired properties. Recall that a random coalescing geodesic has an almost surely constant label due to the coalescence property. We demonstrate the use of our geometric argument below and show that random non-crossing geodesics, defined next, have constant label.

**Definition 7.1.** A measurable map \( G : \Omega \rightarrow \mathcal{P} \) is a random non-crossing geodesic if \( G(\omega) \in \mathcal{T}_0 \) and for \( u, v \in \mathbb{Z}^2 \) the geodesics \( G(u) \) and \( G(v) \) are non-crossing almost surely.

Recall that we since Section 6 have fixed a set of four random coalescing geodesics \( \Gamma_i \), for \( i = 0, 1, 2, 3 \), obtained from one another through right angle rotation. The random non-crossing geodesics \( G \) that we shall encounter will almost surely be contained counterclockwise between \( \Gamma_i \) and \( \Gamma_{i+1} \) for some \( i \). By relabeling the geodesics \( \Gamma_i \) we may assume that \( G \) is between \( \Gamma_0 \) and \( \Gamma_1 \) a.s.

We first illustrate the idea of the geometric argument. We start with the cone determined by moving counterclockwise from \( \Gamma_0(0) \) to \( \Gamma_1(0) \). Then we find two geodesics \( g \in \mathcal{T}_u \) and \( g' \in \mathcal{T}_v \) such that (see Figure 2)

- \( u \in \Gamma_0 \) and \( g \) is in the cone counterclockwise between \( \Gamma_0 \) and \( \Gamma_1 \);
- \( v \in \Gamma_1 \) and \( g' \) is in the cone counterclockwise between \( \Gamma_0 \) and \( \Gamma_1 \);
- \( g' \) is counterclockwise of \( g \).

![Figure 2. Description of the central geometric argument.](image)

In this setup we are ensured that there exists a geodesic \( h \in \mathcal{T}_0 \) which is contained counterclockwise between \( g \) and \( g' \); let \( (u, u_1, u_2, \ldots) \) be an enumeration of the vertices in \( g \) and consider the limit of \( \text{Geo}(0, u_k) \) as \( k \) increases. This limit exists and is shielded off by \( g \) and \( g' \). The label of the geodesic is then contained between those of \( g \) and \( g' \), and if \( g \) and \( g' \) coalesce then so does \( h \).

We will next formulate a statement which will allow us to draw the above picture. We will apply this lemma a number of times in settings which differ somewhat one from another. In order to provide a result that encompasses these different settings we will describe the statement in terms of translation invariant subfamily \( \mathcal{G} \) of geodesics contained between \( \Gamma_0 \) and \( \Gamma_1 \). Below, we apply the lemma for \( \mathcal{G} \) being the image of a random non-crossing geodesic.
Proposition 7.2. Let \( \mathcal{G}(v) \) denote a translation invariant subfamily of geodesics in \( \mathcal{T}_v \) contained between \( \Gamma_0(v) \) and \( \Gamma_1(v) \). Let \( I \subseteq [0, 1] \) be an interval and let \( A(v) \) denote the event that \( \mathcal{G}(v) \) contains a geodesic with label in \( I \). If \( P(A(0)) > 0 \), then, almost surely,

- there exists \( u \in \Gamma_0 \) for which \( A(u) \) occurs, and
- there exists \( v \in \Gamma_1 \) for which \( A(v) \) occurs.

By applying the above proposition twice, we may obtain \( u \in \Gamma_0 \) and \( v \in \Gamma_1 \) and geodesics starting at \( u \) and \( v \) with different properties.

7.1. Labels of random non-crossing geodesics are constant. Before presenting a proof we give a typical application of Proposition 7.2 in Section 8 we shall see several more.

Corollary 7.3. Let \( G \) be a random non-crossing geodesic which with probability one is contained counterclockwise between \( \Gamma_0 \) and \( \Gamma_1 \). Then, \( F(G) \) is almost surely constant.

Proof. Suppose the lemma is not true. Then there are two disjoint intervals \([a, b]\) and \([c, d]\) such that \( b < c \) and the probability that \( F(G) \) is in either of those intervals is positive. We let \( A(v) \) be the event that \( G(v) \) has label in \([c, d]\) and \( B(v) \) be that \( G(v) \) has label in \([a, b]\).

Then by Proposition 7.2 we get a \( u \) on \( \Gamma_0 \) such that \( A(u) \) occurs. We also get a \( v \) on \( \Gamma_1 \) such that \( B(v) \) occurs. As the label of \( G(u) \) is greater than the label of \( G(v) \) we must have that \( G(v) < G(u) \). But this means that \( G(u) \) and \( G(v) \) must cross. This is a contradiction.

7.2. Proof of Proposition 7.2. We will divide the proof into two cases, depending on whether the set of limiting directions for the four geodesics \( \Gamma_i \) has width \( \pi/2 \) or not. The width cannot be larger than \( \pi/2 \), and if it indeed is as large as \( \pi/2 \) then the asymptotic shape is necessarily either a square or a diamond. The case when the width is strictly smaller than \( \pi/2 \) is easier as we in this case can find a half-plane \( H \) which both \( \Gamma_0 \) and \( \Gamma_1 \) eventually move into. In the remaining case we do not know that this is true, and we will require some additional arguments.

Case 1: Width less than \( \pi/2 \). Consider first the case that the width is strictly smaller than \( \pi/2 \). We may in this case find two half-planes \( H \) and \( H' \), both containing the origin as a boundary point, and such that \( \Gamma_0 \) and \( \Gamma_1 \) visits the complement of \( H \cap H' \) at most finitely many times almost surely. Fix \( m \geq 1 \) and let \( B_m(u) \) be the event that \( \Gamma_0(u) \) and \( \Gamma_1(u) \) visits at most \( m \) points in \( (H \cap H')^c + u \).\(^4\) We may make the probability of \( B_m(u) \) as close to 1 as we wish by increasing \( m \) if necessary. In particular, there exist \( \varepsilon > 0 \) and \( m \geq 1 \) such that

\[
P(A(u) \cap B_m(u)) > \varepsilon.
\]

According to the ergodic theorem we may find a density of sites in the symmetric difference \( H \Delta H' \) for which \( A(u) \cap B_m(u) \) occurs. Since \( \Gamma_0 \) and \( \Gamma_1 \) visits \( (H \cap H')^c \) at most finitely many times, almost surely, we may find \( u \) and \( v \) in \( H \Delta H' \), sufficiently far from the origin, such that (see Figure 3):

- \( \Gamma_0(u) \) and \( \Gamma_1(u) \) intersect \( \Gamma_0(0) \) and does not contain the origin ccw between them;
- \( \Gamma_0(v) \) and \( \Gamma_1(v) \) intersect \( \Gamma_1(0) \) and does not contain the origin ccw between them;
- there exists a geodesic \( g \in \mathcal{G}(u) \) ccw between \( \Gamma_0(u) \) and \( \Gamma_1(u) \) with label in \( I \);
- there exists a geodesic \( g' \in \mathcal{G}(v) \) ccw between \( \Gamma_0(v) \) and \( \Gamma_1(v) \) with label in \( I \).

\(^4\)Here \( S + u \) denotes the translate of the set \( S \) along the vector \( u \).
Then, there exists an almost surely finite width of the set of limit angles of $\Gamma$. The geodesics $g$ and $g'$ also intersect $\Gamma_0$ and $\Gamma_1$ respectively, and their subpaths, from these intersections and onwards, are contained between $\Gamma_0$ and $\Gamma_1$. We have thus found $u' \in \Gamma_0$ and $v' \in \Gamma_1$ and geodesics in $\mathcal{F}_u$ and $\mathcal{F}_v$ with the required properties.

Case 2: Width equal to $\pi/2$. We proceed with the proof in the second case, where the width of the set of limit angles of $\Gamma_0$ and $\Gamma_1$ is equal to $\pi/2$. In this case the asymptotic shape is necessarily either a square or a diamond. In either case the proof if the same, so we assume in the following that the asymptotic shape is a diamond, and hence strictly convex in the coordinate directions.

The difficulty that arises in the case the set of limiting angles has width $\pi/2$ is that we cannot guarantee that $\Gamma_0$ and $\Gamma_1$ are both contained in a half-plane. We may nevertheless pick two half-planes $H$ and $H'$ such that $\Gamma_0$ and $\Gamma_1$ visits the complement of $H \cup H'$ at most finitely many times with probability 1. If we choose $m \geq 1$ large and let $B_m(u)$ denote the event that $\Gamma_0(u)$ and $\Gamma_1(u)$ visits $(H \cup H')^c + u$ at most $m$ times, then we may again appeal to the ergodic theorem to obtain a density of points in $H \cap H'$ for which $A(u) \cap B_m(u)$ occurs. By the choice of the half-planes, if $A(u) \cap B_m(u)$ occurs and $u$ is at distance at least $m$ from the origin, then the origin is not contained in the region counterclockwise between $\Gamma(u)$ to $\Gamma_1(u)$; compare with Figure 3. In order to conclude that there exist $u$ and $v$ such that $\Gamma_0(u)$ and $\Gamma_1(u)$, and $\Gamma_0(v)$ and $\Gamma_1(v)$, intersect $\Gamma_0$ and $\Gamma_1$ respectively, we need to control the structure of $\Gamma_0$ and $\Gamma_1$ further.

Claim 7.4. Assume that the asymptotic shape is not flat in the first coordinate direction. Then, there exists an almost surely finite $N \geq 1$ such that if $\Gamma_i$, for some $i$, visits $[-\epsilon n, \epsilon n]^2 + n \epsilon e_1$ for some $n \geq N$, then it does not visit the intersection of $(H \cap H')^c + \epsilon n e_1$ and $[n/2, n/2]^2 \setminus [-10\epsilon n, 10\epsilon n]^2$.

Proof of claim. Let $C(\epsilon, z)$ be the event from the proof of Proposition 3.2. Given $\delta > 0$, let $N \geq 1$ be large so that both $|T'(0, z) - \mu(z)| \leq \delta |z|$ and $C(\delta, z)$ hold for $|z| \geq N$. If $\delta > 0$ is small enough, then for $n \geq N$ we have

$$T(z, 0) < T(z, [(H \cap H')^c + \epsilon n e_1] \setminus [-\epsilon n, \epsilon n]^2) \quad \text{for all } z \in [-\epsilon n, \epsilon n]^2 + n \epsilon e_1,$$

and $T(0, y) < T(z, y)$ for all $y$ in the intersection of $(H \cap H')^c + \epsilon n e_1$ and $[n/2, n/2]^2 \setminus [-10\epsilon n, 10\epsilon n]^2$. In particular, no geodesic in $\mathcal{F}_0$ that visits $[-\epsilon n, \epsilon n]^2 + n \epsilon e_1$ can also visit the intersection of $(H \cap H')^c + \epsilon n e_1$ and $[n/2, n/2]^2 \setminus [-10\epsilon n, 10\epsilon n]^2$, when $n \geq N$. \(\square\)
Theorem 8.1. The set $P$ will thus suffice to consider the case that a clockwise limit according to part (d) $\alpha g$ contained counterclockwise between the picture in Figure 2. If a geodesic $g$ a geodesic $g$ Proposition 7.2 to obtain $u$ may cross as that would produce a point with two geodesics labeled $\alpha \beta$ surely no $i$ coincides with the label of one of the $\Gamma$. Proof. First of all we recall that a random coalescing geodesic has constant label, so if for any $\Gamma$ the first two exemplifying further applications of Proposition 7.2. Label lemmata. 8.1. Unlike the sets $L$ and $L_*$, the sets $M$ and $M_*$ are generally not equal.

8.1. Label lemmata. In order to prove Theorem 8.1 we will require a range of lemmas, the first two exemplifying further applications of Proposition 7.2.

Lemma 8.2. For any $\alpha \in [0, 1]$ we have $P(\alpha \in L) \in \{0, 1\}$.

Proof. First of all we recall that a random coalescing geodesic has constant label, so if $\alpha$ coincides with the label of one of the $\Gamma$, geodesics, then there is nothing more to prove. It will thus suffice to consider the case that $P(\alpha \in L) > 0$ for some $\alpha \in (F(\Gamma_0), F(\Gamma_1))$.

We have two cases: either $\alpha \in M_*$ or not. In case $\alpha$ is not in $M_*$, then there are almost surely no $z \in \mathbb{Z}^d$ with two geodesics in $\mathcal{G}_z$ labeled $\alpha$. Consequently, no two geodesics labeled $\alpha$ may cross as that would produce a point with two geodesics labeled $\alpha$. We then apply Proposition 7.2 to obtain $u \in \Gamma_0$ and a geodesic $g \in \mathcal{G}_u$ with label $\alpha$, and $v \in \Gamma_1$ and a geodesic $g' \in \mathcal{G}_v$ with label $\alpha$. These two geodesics cannot cross and hence produce the picture in Figure 2. If $g = (u, u_1, u_2, \ldots)$, then the finite geodesics Geo$(0, u_k)$ are all contained counterclockwise between $g$ and $g'$. Sending $k$ to infinity we obtain a geodesic sandwiched between two geodesics with labels $\alpha$, and hence $P(\alpha \in L) = 1$.

Now suppose that $\alpha \in M_*$. The counterclockwise-most geodesic with label $\alpha$ has to be a clockwise limit according to part (d) of Proposition 6.4. By part (c) of Proposition 6.3
this sequence has labels converging to $\alpha$ from above. We conclude that for every $\varepsilon > 0$ we have $P(\mathcal{L} \cap (\alpha, \alpha + \varepsilon) \neq \emptyset) > 0$. We then appeal to Proposition 7.2 to obtain $u \in \Gamma_0$ and $g \in \mathcal{T}_u$ with label $\alpha$, and $v \in \Gamma_1$ and $g' \in \mathcal{T}_v$ with label in $(\alpha, \alpha + \varepsilon)$. The limit $h = \lim_{k \to \infty} \text{Geo}(0, u_k)$, where $g = (u, u_1, \ldots)$, exists and is sandwiched between $g$ and $g'$. It thus has label bounded between $\alpha$ and $\alpha + \varepsilon$. Since $\varepsilon > 0$ was arbitrary $h$ has to have label $\alpha$ almost surely. 

\textbf{Lemma 8.3.} Let $I \subseteq [0, 1]$ be an (open or closed) interval. Then

$$P(\mathcal{L} \cap I \neq \emptyset) \in \{0, 1\}.$$  

\textbf{Proof.} If $I$ contains $F(\Gamma_0), F(\Gamma_1), F(\Gamma_2)$ or $F(\Gamma_3)$, then there is nothing more to prove. We may therefore assume that $I$ is a subset of $(F(\Gamma_0), F(\Gamma_1))$. If for some $\alpha \in I$ we have

$$P(\alpha \in \mathcal{L}) > 0,$$

then by Lemma 8.2 we are done. If not we may split $I$ into two disjoint intervals $I_1$ and $I_2$ such that there is positive probability to find a geodesic with label in either of the two. We apply Proposition 7.2 to find a site $u \in \Gamma_0$ and a geodesic $g \in \mathcal{T}_u$ with label in $I_1$, and a site $v \in \Gamma_1$ and a geodesic $g' \in \mathcal{T}_v$ with label in $I_2$. We then have

$$F(\Gamma_0) < F(g) < F(g') < F(\Gamma_1).$$

Let $(u, u_1, u_2, \ldots)$ be an enumeration of the vertices in $g$. We may take the limit of the sequence of finite geodesics Geo$(0, u_k)$ to obtain a geodesic in $\mathcal{T}_0$ shielded of by $g$ and $g'$. This geodesic must therefore have a label in $[F(g), F(g')] \subseteq I$, as required. 

\textbf{Lemma 8.4.} Let $I_1 \supseteq I_2 \supseteq \ldots$ be some sequence of (open or closed) intervals nesting down to some value $\alpha \in [0, 1]$. If $P(\mathcal{L} \cap I_k \neq \emptyset) = 1$ for every $k \geq 1$, then $P(\alpha \in \mathcal{L}) = 1$.

\textbf{Proof.} If $P(\alpha \in \mathcal{L}) > 0$, then this is just a restatement of Lemma 8.2 Assume henceforth the contrary, that $P(\alpha \in \mathcal{L}) = 0$, in order to reach a contradiction. There are then two cases: Either $\mathcal{L}$ has positive probability to have nonempty intersection with $(\alpha - \varepsilon, \alpha)$ for all $\varepsilon > 0$; or, $\mathcal{L}$ has positive probability to have nonempty intersection with $(\alpha, \alpha + \varepsilon)$ for all $\varepsilon > 0$. By Lemma 8.3 positive probability implies probability 1.

In the case that $P(\mathcal{L} \cap (\alpha, \alpha + \varepsilon) \neq \emptyset) = 1$ for every $\varepsilon > 0$, we may find a decreasing sequence of geodesics $(g_k)_{k \geq 1}$ whose labels converge to $\alpha$. By the clockwise continuity of labels in part $(c)$ of Proposition 6.3, it follows that the geodesic $\lim_{k \to \infty} g_k$ has label $\alpha$.

In the remaining case we have $P(\mathcal{L} \cap (\alpha - \varepsilon, \alpha) \neq \emptyset) = 1$ for all $\varepsilon > 0$, and may now find an increasing sequence $(g_k)_{k \geq 1}$ with labels converging to $\alpha$. Let

$$g_{\alpha^-} := \sup\{g \in \mathcal{T}_0 : F(g) < \alpha\}.$$  

By assumption, $g_{\alpha^-} = \lim_{k \to \infty} g_k$, so $g_{\alpha^-}$ is almost surely a counterclockwise limit with label $F(g_{\alpha^-}) \geq \alpha$. In order to complete the proof it will suffice to show that $F(g_{\alpha^-}) = \alpha$ almost surely.

We first observe that $g_{\alpha^-}$ defines a random non-crossing geodesic. Suppose the contrary, that there are $u, v \in \mathbb{Z}^2$ such that the $g_{\alpha^-}$ geodesics originating from $u$ and $v$ cross at some point $z$. Both these geodesics have label at least $\alpha$, while as the counterclockwise most of the two geodesics that comes out of the point $z$ is a counterclockwise limit of geodesics with labels strictly less than $\alpha$. This is a contradiction to the monotonicity of the labels, so $g_{\alpha^-}$ is almost surely non-crossing.

Since $g_{\alpha^-}$ is a random non-crossing geodesic it follows from Lemma 7.3 that $F(g_{\alpha^-})$ is almost surely constant. (That $g_{\alpha^-}$ is contained between two consecutive $\Gamma$-geodesics
follows since neither of them may have label $\alpha$, by current assumptions.) Assume that $F(g_{\alpha-}) = \beta > \alpha$. In this case we must have a positive density of sites of $v$ for which there is a increasing sequence $(g_k)_{k \geq 1}$ in $\mathcal{T}$ with labels converging to $\alpha$, but

$$M_v(\Gamma_0, g_{\alpha-}) > \lim_{k \to \infty} M_v(\Gamma_0, g_k).$$

For this to happen, there has to exist an isolated geodesic $g' \in \mathcal{T}$, satisfying

$$g_{\alpha-} \geq g' \geq \lim_{k \to \infty} g_k.$$

This implies that there is positive probability to find two geodesics in $\mathcal{T}$ with label $\alpha$, namely $g'$ and $\lim_{k \to \infty} g_k$, of which the counterclockwise one is isolated. This contradicts Lemma 6.4, so we conclude that $F(g_{\alpha-}) = \alpha$. \hfill $\square$

**Lemma 8.5.** Let $I \subseteq [0,1]$ be an (open or closed) interval. If $P(\mathcal{L} \cap I \neq \emptyset) = 1$, then there exists $\alpha \in I$ such that $P(\alpha \in \mathcal{L}) = 1$.

**Proof.** Observe first that if $I$ is open, then we may pick a closed subinterval $I' \subseteq I$ for which $P(\mathcal{L} \cap I' \neq \emptyset) > 0$. According to Lemma 8.3, this probability is then 1, so it will suffice to prove the lemma for $I$ closed.

Now, assume that $I$ is closed and choose a decreasing sequence $(I_k)_{k \geq 1}$ of closed intervals such that $P(\mathcal{L} \cap I_k \neq \emptyset) = 1$ for all $k \geq 1$, by inductively breaking each $I_{k-1}$ in half. The intersection $\bigcap I_k$ is nonempty and contains a unique element $\alpha \in [0,1]$. By Lemma 8.4, we have $P(\alpha \in \mathcal{L}) = 1$. \hfill $\square$

### 8.2. Proof of Theorem 8.1

We first argue that $\mathcal{L}_*$ is a closed set. Let $\alpha \in [0,1]$ be a limiting point to $\mathcal{L}_*$. In this case there exists a (monotone) sequence $(\alpha_k)_{k \geq 1}$ of points in $\mathcal{L}_*$ such that $\alpha_k \to \alpha$. Then $P(\mathcal{L} \cap (\alpha - \varepsilon, \alpha + \varepsilon) \neq \emptyset) = 1$ for all $\varepsilon > 0$, and by Lemma 8.4, we conclude that also $\alpha$ is contained in $\mathcal{L}_*$.

Since $\mathcal{L}_*$ is closed, its complement $[0,1] \setminus \mathcal{L}_*$ is the union of at most countably many open intervals $I$. It follows by Lemma 8.3 that $P(\mathcal{L} \cap I \neq \emptyset) = 0$. Since there are at most countably many such sets, we conclude that $P(\mathcal{L} \subseteq \mathcal{L}_*) = 1$.

Finally, assume that $\mathcal{L}_* \setminus \mathcal{L} \neq \emptyset$. Then either this occurs for one of the countably many boundary point of $\mathcal{L}_*$, which has probability zero, or it occurs at an interior point $\alpha$. In the latter case we have a decreasing sequence of geodesics $(g_k)_{k \geq 1}$ with labels $F(g_k)$ converging to $\alpha$, but that $F(\lim_{k \to \infty} g_k) \neq \alpha$. This has probability zero to happen according to part (c) of Proposition 6.3. Hence, $P(\mathcal{L}_* \subseteq \mathcal{L}) = 1$, which concludes the proof of Theorem 8.1.

### 8.3. Continuity of Labels

The fact that $\mathcal{L}$ is almost surely closed suggests that the labeling is a continuous function.

**Corollary 8.6.** With probability one we have for any monotone sequence $(g_k)_{k \geq 1}$ in $\mathcal{T}$ that

$$\lim_{k \to \infty} F(g_k) = F\left(\lim_{k \to \infty} g_k\right).$$

**Proof.** For decreasing sequences this is the statement of Proposition 6.3 part (c), so it will suffice to consider increasing sequences. Let $(g_k)_{k \geq 1}$ be an increasing sequence in $\mathcal{T}$, and let $g_\infty$ denote the limiting geodesic $\lim_{k \to \infty} g_k$. In case $F(g_\infty) > \alpha := \lim_{k \to \infty} F(g_k)$ then one of the following has to occur: Either $\alpha$ is not in $\mathcal{L}$, or it is in $\mathcal{L}$ but $F(g_k)$ coincide for all large $k$. The former would imply that $\mathcal{L}$ is not closed and thus has probability zero. The latter implies that $\alpha$ is a boundary point of $\mathcal{L}$ and that there are at least three geodesics with label $\alpha$. Since there are at most countably many boundary points it follows
by Proposition 6.4 that there are almost surely no boundary points with multiplicity larger than two.

8.4. Labels identify random non-crossing geodesics. For $\alpha \in \mathcal{L}$, we define

$$G_{\alpha}^{ccw} := \sup \{ g \in T_0 : F(g) \leq \alpha \},$$

$$G_{\alpha}^{cw} := \inf \{ g \in T_0 : F(g) \geq \alpha \}.$$ 

For $\alpha \in \mathcal{L}$, there exist, almost surely, a one or two geodesics labeled $\alpha$, so both $G_{\alpha}^{ccw}$ and $G_{\alpha}^{cw}$ have label $\alpha$. If $P(\alpha \in \mathcal{M}) = 0$, then $G_{\alpha}^{ccw} = G_{\alpha}^{cw}$ almost surely.

**Lemma 8.7.** For every $\alpha \in \mathcal{L}$, we have that $G_{\alpha}^{ccw}$ is a random non-crossing geodesic.

**Proof.** We distinguish between two cases. We first assume that $\alpha$ is isolated from above, meaning that $(\alpha, \alpha + \varepsilon) \cap \mathcal{L} = \emptyset$ for all sufficiently small $\varepsilon > 0$. In this case $G_{\alpha}^{ccw}$ is almost surely cw-isolated, so $P(\alpha \in \mathcal{M}) = 0$ by Proposition 6.4. In case $G_{\alpha}^{ccw}(u)$ and $G_{\alpha}^{ccw}(v)$ cross with positive probability for some $u, v \in \mathbb{Z}^2$, then we may find $z \in G_{\alpha}^{ccw}(u) \cap G_{\alpha}^{ccw}(v)$ such that there are two geodesics with label $\alpha$ leaving $z$. This contradicts the fact that $P(\alpha \in \mathcal{M}) = 0$.

In the remaining case we may find a decreasing sequence $(\alpha_k)_{k \geq 1}$ in $\mathcal{L}$, converging to $\alpha$. In this case $G_{\alpha_k}^{ccw}$ is almost surely a cw-limit. If $G_{\alpha_k}^{ccw}(u)$ and $G_{\alpha_k}^{ccw}(v)$ cross with positive probability for some $u$ and $v$, then we may find $z$ at which there are two geodesics labeled $\alpha$ leaving $z$, and since both are cw-limits and can only cross once, there will have to be an infinite number of geodesics in between. This is again a contradiction to Proposition 6.4.

**Lemma 8.8.** If $P(\alpha \in \mathcal{M}) > 0$ and $G_{\alpha}^{ccw}$ is coalescing, then $P(\alpha \in \mathcal{M}) = 1$. If $P(\alpha \in \mathcal{M}) = 1$, then $G_{\alpha}^{cw}$ is a random non-crossing geodesic.

**Proof.** If $G_{\alpha}^{cw}$ is coalescing and $P(\alpha \in \mathcal{M}) > 0$, then we can use the geometric argument, Proposition 7.2 to obtain $u \in \Gamma_0(0)$ and $v \in \Gamma_1(0)$ at which there are multiple geodesics with label $\alpha$. Since the counterclockwise most of these geodesics must coalesce we can take two limits, one from the origin along sites of $G_{\alpha}^{cw}(v)$ and one from the origin along vertices of $G_{\alpha}^{cw}(u)$; see Figure 4: Since the counterclockwise-most coalesce, these limits have to be different, while both limits will have label $\alpha$. Hence, $P(\alpha \in \mathcal{M}) = 1$.

**Figure 4.** The ccw-most geodesics coalesce, the cw-most are non-crossing.

Finally, assume that there are $u$ and $v$ such that $G_{\alpha}^{cw}(u)$ and $G_{\alpha}^{cw}(v)$ cross with positive probability. Let $z$ be a point in the intersection. There are then two geodesics with label
Theorem 9.1. Every member of $G^c_{ccw}(u)$. This contradicts either of the facts that there are at most two geodesics with label $\alpha$ or that $G^c_{ccw}$ is coalescing, almost surely. \hfill $\square$

8.5. Multiplicity of labels revisited. We end this section with a description of the sets $\mathcal{M}$ and $\mathcal{M}^*$. We mention that we shall later prove that the set $\mathcal{M}^*$ is at most countable, but start here with a more modest statement.

Lemma 8.9. The set $\mathcal{M}$ is almost surely at most countable. The set $\mathcal{M}^*$ has Lebesgue measure zero. Either $\mathcal{L}^*$ is finite and $\mathcal{M}^*$ empty, or the set $\mathcal{L}^* \setminus \mathcal{M}^*$ is infinite.

Proof. For every $\alpha \in \mathcal{M}$ there exist $g, g' \in \mathcal{H}_0$ and a point $z$ such that $F(g) = F(g') = \alpha$ and $z$ is the last common point of $g$ and $g'$. For each $z \in \mathbb{Z}^2$ there can be at most four elements of $\mathcal{M}$ for which $z$ is the last common point of the clockwise most and counterclockwise most geodesic in $\mathcal{H}_0$ with that label. Since $\mathbb{Z}^2$ is countable, it follows that $\mathcal{M}$ is at most countable, almost surely.

By Fubini's theorem we obtain that

$$\int_0^1 P(\alpha \in \mathcal{M}) d\alpha = E \int_0^1 1\{\alpha \in \mathcal{M}\} d\alpha = 0.$$ 

Hence, the set $\{\alpha \in [0, 1] : P(\alpha \in \mathcal{M}) > \epsilon\}$ has Lebesgue measure zero. By writing $\mathcal{M}^*$ as a countable union of sets of this form we may conclude that $\mathcal{M}^*$ has Lebesgue measure zero.

Finally, note that if $\alpha \in \mathcal{M}^*$, then $\mathcal{L}^*$ is infinite as of Proposition 6.4. It will therefore suffice to consider the case when $\mathcal{L}^*$ is finite and $\text{Leb}(\mathcal{L}^*) = \text{Leb}(\mathcal{M}^*) = 0$. In this case the complement of $\mathcal{L}^*$ is the union of countably many open intervals. Given such an interval $I$ and $\beta \in I$ consider the supremum over points in $\mathcal{L}^*$ smaller than $\beta$. This points is a boundary point of $\mathcal{L}^*$ and cannot be part of $\mathcal{M}^*$ of of Proposition 6.4. Hence we find infinitely many point in $\mathcal{L}^*$ that are not in $\mathcal{M}^*$. \hfill $\square$

9. Coalescence

Let $\mathcal{G}$ denote the class of random geodesics of the form $G^c_{ccw}$ and $G^c_{cw}$ for $\alpha \in \mathcal{L}^*$. In the previous section we saw that many of these geodesics are non-crossing. In this section we prove that they are coalescing, and that there are no random coalescing geodesics outside of this class.

Theorem 9.1. Every member of $\mathcal{G}$ is a random coalescing geodesic, and for every random coalescing geodesic $G$ there exists $G' \in \mathcal{G}$ such that $G = G'$ almost surely. In particular,

$$\mathcal{M}^* = \{\alpha \in [0, 1] : P(\alpha \in \mathcal{M}) = 1\}.$$ 

Just as for coalescing geodesics, we say that a random non-crossing geodesic $G$ eventually moves into a half-plane $H$ if for every parallel half-plane $H' \subseteq H$ we have

$$P(|G(v) \cap H'| < \infty) = 1.$$ 

Moreover, a non-crossing geodesic $G$ defines an equivalence relation on $\mathbb{Z}^2$ by declaring $u \sim v$ if $G(u)$ and $G(v)$ coalesce. The equivalence classes of this relation will be referred to as the coalescing classes of $G$. If $G$ almost surely has a unique coalescing class then we say that $G$ is coalescing.

In order to prove Theorem 9.1 we mainly need to show that each member of $\mathcal{G}$ is coalescing. Ideally the outline of our proof would be as follows. We first show that every non-crossing geodesic eventually moves into a half-plane. We would also show that if $\omega'$ is an alteration of $\omega$ on finitely many edges then $G(v)(\omega) \Delta G(v)(\omega')$ contains finitely many edges for any
vertex \( v \). Then we let \( R \) be any shift invariant order on \( \mathbb{Z}^2 \). We show by the mass transport principle that for any coalescence class of \( G \) there is no least element according to the order \( R \). However, by the finite upwards energy property we show that if \( G \) is not coalescing then there exists a coalescence class with a least element. This is a contradiction and \( G \) is coalescing.

However, the above programme is problematic for the reason that there may be multiple geodesics with the same label with positive probability, which complicates the proof that each geodesic eventually moves into a half-plane. Again, this is not an issue in the case that the asymptotic shape is neither a square nor a diamond. In this case the shape has at least eight extreme points: two corresponding to angles in \( [0, \pi/2) \), and another six are produced by rotation of an angle \( \pi/2 \). For each consecutive pair of extreme points there is, by Theorem 5.2, a random coalescing geodesic directed in between. This gives a set of eight random coalescing geodesics, and each consecutive pair of geodesics are together directed in a sector of width at most \( \pi/2 \). Since this sector contains at most three angles of the form \( i\pi/4 \), we can produce a \( H_i \) half-plane with five of the remaining points. Since every other geodesic is contained in between two consecutive geodesics of the above form, each geodesic will eventually move into one of the eight half-planes \( H_i \).

In the general case, when we make no assumption on the limiting shape, we will need a bootstrapping argument to implement this plan, by first extending the set of random coalescing geodesics to eight (should eight exist). Hence, there will be two argument to obtain coalescence. We shall first prove that random non-crossing geodesics that eventually move into a half-plane are coalescing, and second, that every geodesic eventually moves into a half-plane.

9.1. **Half-plane geodesics are coalescing.** Recall that by \( H_i \), for \( i = 0, 1, \ldots, 7 \), we denote the eight half-planes with normal vector in direction \( i\pi/4 \).

**Proposition 9.2.** Every random non-crossing geodesic in \( \mathcal{G} \) that eventually moves into one of the eight half-planes \( H_i \), \( i = 0, 1, \ldots, 7 \), is a random coalescing geodesic.

Our proof of Proposition 9.2 will be adapted from an argument due to Licea and Newman [New95, LN96]; see also [DH14]. We will require a series of lemmas. The core of the proof is a 'local modification' argument, that shows that if \( G \) is not coalescing, then there exists a coalescing class with a least element according to some ordering of the elements in \( \mathbb{Z}^2 \). Our first lemma says that this cannot happen.

**Lemma 9.3.** Let \( G \in \mathcal{G} \) and let \( R \) be some shift invariant order on \( \mathbb{Z}^2 \). Every coalescence class for \( G \) has no least element for \( R \).

**Proof.** This lemma is an application of the mass transport principle. We define a function \( m(x, y) \) which is one if \( y \) is the least element in the coalescence class of \( x \), and otherwise zero. If \( y \) is the least element of its coalescence class then it gets infinite mass as all the points on \( G(y) \) have \( m(x, y) = 1 \). If the probability that \( y \) is the least element for some class is positive, then the expected amount of mass into \( y \) (\( \mathbb{E} \sum_x m(x, y) \)) is infinite while the expected amount of mass out of \( y \) (\( \mathbb{E} \sum_x m(y, x) \)) is at most one. This is a contradiction. Thus the probability that \( y \) is the least element of its coalescence class is zero, so by the union bound the probability that there exists a coalescence class which has a least element is zero. 

For the local modification argument we need a lemma that says that the modified picture has positive probability to occur. We say that an event \( A \) is *increasing* with respect to a
subset $E$ of the edge set if for each $\omega \in A$ and $\omega'$ satisfying $\omega'_e \geq \omega_e$ for $e \in E$ and $\omega'_e = \omega_e$ otherwise, then $\omega' \in A$.

**Lemma 9.4.** Let $E = E(\omega)$ be an almost surely finite random subset of the edges of $\mathbb{Z}^2$, and let $A$ be an increasing event with respect to $E$. If $\mathbb{P}(A) > 0$ and $\mathbb{P}(\omega_e > t) > 0$, then

$$\mathbb{P}\{A \cap \{\omega_e > t : e \in E\}\} > 0.$$  

**Proof.** Consider the following local modification coupling (see [HJ06, Couplings 2.4 and 12.3]) of two configurations $(\omega, \omega')$: Let $Q$ be some probability measure on finite subsets of the edges of $\mathbb{Z}^2$ that assigns a positive probability to each element. First, pick $F$ according to $Q$. Second, pick $\omega = \omega'$ on $F$ independently according to $\mathbb{P}$. Finally, conditioned on the outcome in the previous step, pick $\omega$ and $\omega'$ on $F$ independently according to the conditional law $\mathbb{P}(\cdot | \omega_F)$.

Denote the measure of $(\omega, \omega')$ by $P'$. Since $A$ occurs with positive probability and $Q$ has full support, we have that $P'(\omega \in A, E(\omega) = F) > 0$. By the upwards finite energy property we have

$$P'(\omega'_e > t : e \in E(\omega)) > 0.$$  

Since $A$ is $E$-increasing, it follows that also $P'(\omega' \in A \cap \{\omega'_e > t : e \in E(\omega)\}) > 0$. $\square$

Next we take a geodesic $g$ in a configuration $\omega$ and then increase that configuration in a finite region to get a new configuration $\omega'$. The tail of $g$ is a geodesic in $\omega'$. This next lemma shows that the label of $\omega$ under $g$ is the same as the label of the tail of $g$ under $\omega'$.

For any set of edges $S$ and configuration $\omega$ define $\tilde{\omega}$ to be the set of $\omega'$ such that $\omega \Delta \omega' \subseteq S$.

**Lemma 9.5.** Let $G \in \mathcal{G}$ and let $S$ be any finite set of edges. For almost all $\omega$ and almost all $\omega' \in \tilde{\omega}$ for all $v$ sufficiently far in the tail of $G(\omega)$ we have $G(v)(\omega) = G(v)(\omega')$.

The first step in the proof is to note that for almost every $\omega', \omega'' \in \tilde{\omega}$ we have that the tail of $G(0)(\omega'')$ is a geodesic in $\omega'$. Because of this we can write $F_{\omega'}(G(0)(\omega''))$ for the label (in $\omega'$) of the portion of that path which is a geodesic. Next we show that the label of $G$ does not change under a finite alteration. For this part of the argument we define $\text{Bad}(\alpha, \beta, \delta, S)$ to be the event that

- $\mathbb{P}_2 \times \mathbb{P}_2((\omega', \omega'') \in \tilde{\omega}^2 : F_{\omega'}(G(0)(\omega'')) \geq \beta) > \delta$; and
- $\mathbb{P}_2 \times \mathbb{P}_2((\omega', \omega'') \in \tilde{\omega}^2 : F_{\omega'}(G(0)(\omega'')) \leq \alpha) > \delta$.

If with positive probability the label of $G$ does change under a finite alteration then there exist $\alpha < \beta \in L^*$ and $\delta > 0$ such that

$$\mathbb{P}(\text{Bad}(\alpha, \beta, \delta, S)) > \delta.$$  

We will show that this is not possible.

If $L^* \cap [\alpha, \beta]$ is finite then we enumerate $L^* \cap [\alpha, \beta]$ by $\alpha_1 = \alpha < \alpha_2 < \cdots < \alpha_k = \beta$. For simplicity of notation we will only consider the case that the $\alpha_i$ are in $L^* \setminus \mathcal{M}$. This does not materially change the proof.[6] If $L^* \cap [\alpha, \beta]$ is infinite then we choose $\alpha = \alpha_1 < \alpha_2 < \cdots < \alpha_k = \beta$ in $L^* \setminus \mathcal{M} \cap [\alpha, \beta]$.

For the remainder of the proof of Lemma 9.5 we suppress the subscript and write $G_{\alpha_i}$ for $G_{\alpha_i}^{cw}$. Define $G_{\alpha_i}(v, n)$ to be the first $n$ steps of $G_{\alpha_i}(v)$. Define $\text{Prediction}(\alpha_i, v, n, N)$ to be the maximum likelihood estimate of $G_{\alpha_i}(v, n)$ given $\omega_{[v+1-N,N]^2}$. Define

$$\text{Accurate}(\alpha, \beta, v, n, N) := \{\text{Prediction}(\alpha_i, v, n, N) = G_{\alpha_i}(v, n) \text{ for all } i\}.$$  

[6] By assumption, only the endpoints $\alpha$ and $\beta$ may occur with multiplicity due to Lemma 6.4. If that happens we will below need to work with both $cw$ and $ccw$ geodesics labeled $\alpha$ and $\beta$, which gives an additional two geodesics. It is easy to adapt the following argument for the extra geodesics. We leave this to the reader.
Define $Q_{\omega,v}$ as the set of $\omega' \in \omega$ such that

- no edge of $S$ is in the cone ccw between $\Gamma_0(v)$ to $\Gamma_1(v)$ for $\omega'$;
- Accurate($\alpha, \beta, v, n, N)$ occurs for $\omega$; and
- Prediction($\alpha_i, v, n, N)$ are distinct for $i = 1, 2, \ldots, k$.

**Lemma 9.6.** Let $v \in \mathbb{Z}^2$ such that $(v + [-N, N]^2) \cap S = \emptyset$. Then there exist $\alpha', \alpha'', \beta''$, and $\beta'$ such that $\alpha \leq \alpha'' \leq \alpha' < \beta' \leq \beta'' \leq \beta$ and either

- $F_{\omega, \alpha''}(G(0)(\omega''))$ is less than or equal to $\alpha'$ for almost every $\omega', \omega'' \in Q_{\omega,v}$;
- $F_{\omega, \alpha''}(G(0)(\omega'')) \in (\alpha'', \beta'')$ for almost every $\omega', \omega'' \in Q_{\omega,v}$; or
- $F_{\omega, \alpha''}(G(0)(\omega''))$ is greater than or equal to $\beta''$ for almost every $\omega', \omega'' \in Q_{\omega,v}$.

**Proof.** By Proposition 5.7 for each $i = 0, 1$ we have that the $\Gamma_i(v)(\omega')$ and $\Gamma_i(v)(\omega'')$ coalesce. They start at the same vertex, eventually coalesce and do not go through $S$. Thus by unique geodesics $\Gamma(v)(\omega') = \Gamma(v)(\omega'')$. Then we have that the environments in the cone ccw between $\Gamma_0(v)$ to $\Gamma_1(v)$ for both $\omega'$ and $\omega''$ are the same. Thus by the uniqueness of geodesics we have that $\omega$ falls in is independent of the choice of $\omega$.

We first consider the case that $L_i \cap [\alpha, \beta]$ is finite. There are finitely many of these geodesics in the smaller cone between $G_{\alpha_i}$ and $G_{\alpha_i}$, and they are the $G_{\alpha_i}$. (Recall the previous footnote.) By the choice of $v$ sufficiently far from $S$, the local environments for $\omega'$ and $\omega''$ in $v + [-N, N]^2$ are the same. Thus Prediction($\alpha_i, v, n, N)$ agree for $\omega'$ and $\omega''$ for all $i$. As Accurate($\alpha, \beta, v, n, N)$ occurs for $\omega'$ and $\omega''$ and for all $i$, we have that for all $i$ the geodesics $G_{\alpha_i}(v)(\omega')$ and $G_{\alpha_i}(v)(\omega'')$ agree until they have diverged from the other $G_{\alpha_i}$.

Each pair of random non-crossing geodesics $G_{\alpha_i}$ and $G_{\alpha_i'}$ have zero probability of containing a geodesic between themselves by Theorem 8.1 and Lemma 6.4. As there are no intermediate geodesics we have that almost surely for $\omega', \omega'' \in Q_{\omega,v}$ and any $i$

$$G_{\alpha_i}(v)(\omega') = G_{\alpha_i}(v)(\omega'').$$

Now $G(0)(\omega')$ is comparable with $G_{\alpha_i}(v)(\omega'')$. Either $G(0)(\omega')$

- is counterclockwise of $G_{\alpha_i}(v)(\omega'')$;
- is clockwise of $G_{\alpha_i}(v)(\omega'')$;
- coalesces with one of the $G_{\alpha_i}(v)(\omega'')$; or
- is strictly between $G_{\alpha_i}(v)(\omega'')$ and $G_{\alpha_{i+1}}(v)(\omega'')$ and crosses $G_{\alpha_{i+1}}(v)(\omega'')$ but not $G_{\alpha_i}(v)(\omega'')$ for some $i$.

As the geodesics $G_{\alpha_i}(v)(\omega'')$ are independent of $\omega'' \in Q_{\omega,v}$. Which of the above categories it falls in is independent of the choice of $\omega''$.

Choose any $\alpha', \alpha'', \beta', \beta''$ such that $\alpha \leq \alpha' \leq \alpha' < \beta' \leq \beta'' \leq \beta$. For almost every $\omega''$ and all $i$ there no geodesic starting at $0$ strictly between $G_{\alpha_i}(0)(\omega'')$ and $G_{\alpha_{i+1}}(0)(\omega'')$. Thus if the fourth option holds by the the uniqueness of geodesics we have that $F_{\omega, \alpha''}(G(0)(\omega')) = \alpha_i$. Substituting $\omega'$ for $\omega''$ we get the desired results. We then check the other cases. If the third option holds then we also have that the label of $F_{\omega, \alpha''}(G(0)(\omega')) = \alpha_i$. If the first option holds then $F_{\omega, \alpha''}(G(0)(\omega')) \geq \beta$ while it the second option holds then $F_{\omega, \alpha''}(G(0)(\omega')) \leq \alpha$ and $F_{\omega, \alpha''}(G(0)(\omega')) \leq \alpha$. In each of these last three cases substituting $\omega'$ for $\omega''$ we get the desired results. This completes the proof in the case that $L_i \cap [\alpha, \beta]$ is finite.

Now we consider the case that $L_i \cap [\alpha, \beta]$ is infinite. For $i \geq 2$ let $g_i$ be the counterclockwise most geodesic through Prediction($\alpha_i, v, n, N$). As the passage times in the cone are the same these geodesics are independent of the choice of $\omega'$. Then the proof goes through as before if we take $\alpha' = \alpha_2$ and $\beta' = \alpha_4$ and $\omega'' = \alpha_1$ and $\beta'' = \alpha_5$. 

We next bound the probability of Bad($\alpha, \beta, \delta, S$).
Lemma 9.7. For all \( \alpha < \beta \) and \( \delta > 0 \)
\[
\mathbb{P}(\text{Bad}(\alpha, \beta, \delta, S)) < \delta.
\]

Proof. We have the measure \( \mathbb{P}_\omega^2 \) on \( \hat{\omega}^2 \) and the function \( R(\omega', \omega'') = F_{\omega'}(G(\omega'')(0)) \). By the previous lemma for any \( \nu \) this function \( R \) is constant on \((Q_{\omega,\nu})^2\). By the definition of \( \text{Bad}(\alpha, \beta, \delta, S) \), if it occurs for \( \omega \) then the function \( R \) differs from the value it takes on \((Q_{\omega,\nu})^2\) on a set of \( \mathbb{P}_\omega^2 \) measure at least \( \delta \). By properties of product measure
\[
\mathbb{P}_\omega^2((Q_{\omega,\nu})^2) > 1 - 2\mathbb{P}_\omega((Q_{\omega,\nu})^\cap)
\]
we have that if \( \text{Bad}(\alpha, \beta, \delta, S) \) occurs with probability \( \delta \), then for any \( \nu \)
\[
\mathbb{P}_\omega((Q_{\omega,\nu})^\cap) > \delta/2.
\]
Thus we only need to show that we can pick \( \nu \) such that for most \( \hat{\omega} \) we have that \( \mathbb{P}_\omega(Q_{\omega,\nu}) \) is close to 1. First pick \( n \) such that
\[
\mathbb{P}(\text{all } G_{\alpha_i}(v) \text{ diverge within } n \text{ steps of } v) \text{ is close to } 1.
\]
This will ensure that the third condition in the definition of \( Q_{\omega,\nu} \) is satisfied with high probability. By measurability considerations we can find an appropriate \( N \) to ensure that the second condition in the definition of \( Q_{\omega,\nu} \) is satisfied with high probability. By Theorem 5.2 we can find \( \nu \) such that the first condition in the definition of \( Q_{\omega,\nu} \) is satisfied with high probability and that \( (v + [N, N]^2) \cap S = \emptyset \).

Proof of Lemma 9.5. By Lemma 9.7 we get that
\[
\mathbb{P}(\text{Bad}(\alpha, \beta, \delta, S)) < \delta.
\]
Thus the label of \( G(v) \) is unchanged by a finite modification of \( \omega \) almost surely. To complete the proof we note that being a cw or ccw limit (or being cw or ccw isolated) do not change under a finite alteration. As there is almost surely a unique geodesic with label \( F(G) \) and is cw (or ccw) isolated we are done.

Proof of Proposition 9.2. Let \( G \) be an element in \( \mathcal{G} \) that eventually moves into some half-plane \( H_t \). For notational simplicity we shall assume that this half-plane is the right half-plane, and note that any other case is analogous. We will argue by contradiction and assume that \( G \) is not coalescing with positive probability. Given \( i, j \in \mathbb{Z} \), let \( A(i, j) \) denote the event that \( G(ie_2) \) and \( G(je_2) \) are entirely contained in the right half-plane and belong to different coalescing classes. Since \( G \) eventually moves into the right half-plane and is assumed to be non-coalescing (with positive probability), the event \( A(i, j) \) has positive probability to occur for some \( i < j \). An application of the ergodic theorem assures that \( A(i, i+k) \) will occur for some \( k \geq 1 \) and infinitely many \( i \in \mathbb{Z} \) with probability one. Consequently, there are infinitely many coalescing classes for \( G \) almost surely.

The remainder of the proof will be divided into two different cases, depending on whether the support of the edge weights is unbounded or not. We start with the significantly simpler case of unbounded support.

Case 1: Unbounded support. Given \( m \geq 1 \), let \( R_m \) denote the rectangle \( \{(i, j) \in \mathbb{Z}^2 : i \in \{0, -1\}, 0 \leq j \leq m\} \). We partition the set of edges with both their endpoints in \( R_m \) into interior edges, connecting \((-1, j) \) to \((0, j) \) for some \( 0 < j < m \), and boundary edges, being the remaining ones. For \( m \geq 1 \), \( t > 0 \) and \( 0 < j_1 < j_2 < j_3 < m \), let \( A = A(m, t, j_1, j_2, j_3) \) denote the event that (see Figure 5)
- \( G(j_1e_2) \), \( G(j_2e_2) \) and \( G(j_3e_2) \) belong to different coalescence classes;

\[
\text{Bad}(\alpha, \beta, \delta, S) \text{ occurs with positive probability. Given } \omega, v \
\]
\[
\text{if it occurs for } \omega \text{ then the function } R \text{ differs from the value it takes on } (Q_{\omega,\nu})^2 \text{ on a set of } \mathbb{P}_\omega^2 \text{ measure at least } \delta. \]

\[
\mathbb{P}_\omega^2((Q_{\omega,\nu})^2) > 1 - 2\mathbb{P}_\omega((Q_{\omega,\nu})^\cap) > \delta/2.
\]
• $G(j_1e_2)$, $G(j_2e_2)$ and $G(j_3e_2)$ are entirely contained in the right half-plane;
• every boundary edge in $R_m$ has weight at most $t$.

![Figure 5. The rectangle $R_m$ (shaded area) see three coalescing classes represented.](image)

It is clear that $A$ has positive probability for some choice of $m$, $j_1$, $j_2$ and $j_3$, and all $t$ large enough, since by making $m$ large we can make sure that three coalescing classes intersect $R_m$, and by making $m$ larger, some representative of each class will visit the vertical axis only inside $R_m$. Then choose $t$ large.

Let $A'$ denote the event that $A$ occurs and that all interior edges in $R_m$ have weight larger than $2mt$. It follows from Lemma 9.5 that the images of $G$ at the points $j_1e_2$, $j_2e_2$ and $j_3e_2$ do not change as we increase the interior edges. That is, $A$ is increasing with respect to the set of interior edges in $R_m$. By Lemma 9.4 it follows that also $P(A') > 0$, for some choice of parameters.

However, we note that on the event $A'$ there are three coalescence classes of $G$ represented in $R_m$, but that no vertex in the left half-plane belongs to the middle of the three. This is a contradiction to Lemma 9.3 which shows that our initial assumption, that $G$ has positive probability not to coalesce, must have been wrong.

**Case 2: Bounded support.** We proceed with the case that the edge weight distribution has bounded support, that is, that

\[ t_0 := \sup \{ t \geq 0 : P(\omega_e > t) > 0 \} \]

is finite. We will in this case need to replace the rectangle $R_m$ by a larger random selected region, delimited by two vertical lines and two finite geodesics. The finite geodesics will be segments of $G$, for two different starting points, and we need to know that we can choose the other endpoints not too far apart.

**Claim 9.8.** Assume that $G$ eventually moves into the right half-plane $H_0$. Then, for every $\varepsilon > 0$ and $x, y \in \mathbb{Z}^2$, there are infinitely many $n$ for which $G(x)$ and $G(y)$ intersect $H_0 + n\mathbf{e}_1$ within distance $\varepsilon n$ of each other, almost surely.

**Proof of claim.** It will suffice to show that the set of limiting points of $G$ is almost surely constant. Assume the contrary, that there is an angle $\theta \in [0, 2\pi]$ which is in the set of directions for $G$ with probability strictly between 0 and 1. Then the event $A_j$ that $G(j_je_2)$ is entirely contained in $H_0$ and has $\theta$ in its set of directions has positive probability. By the ergodic theorem we may find $j_1 < j_2 < j_3$ such that $A_{j_1} \cap A_{j_3}$ occurs, while $G(j_2e_2)$ does not have $\theta$ in its set of directions, while still is entirely contained in $H_0$. However, then $G(j_2e_2)$ must cross one of the others, which is a contradiction. \[\square\]
Fix $\varepsilon = (t_0 - \mathbb{E}[\omega_a])/6$. Given $m \geq 1$, $0 < j_1 < j_2 < j_3 < \varepsilon m/t_0$, and $-m < j_1' < j_2' < m$ with $j_3' - j_1' < \varepsilon m/t_0$, let $A = A(m, j_1, j_2, j_3, j_1', j_2')$ denote the event that (see Figure 6)

- $G(j_1, e_1)$, $G(j_2, e_2)$, and $G(j_3, e_2)$ belong to different coalescence classes;
- $G(j_1, e_2)$, $G(j_2, e_2)$, and $G(j_3, e_2)$ are entirely contained in the right half-plane;
- the top- respectively bottom-most intersections of $G(j_1, e_2)$ and $G(j_3, e_2)$ with the vertical line $\{(m, j) : j \in \mathbb{Z}\}$ occur at $(m, j_1')$ and $(m, j_3')$;
- $T((0, j_1), (m, j_1')) \leq (\mathbb{E}[\omega_a] + \varepsilon)\| (m, j_1' - j_1)\|_1$.

![Figure 6](image)

**Figure 6.** Three coalescing classes represented on the vertical axis, and touch the vertical line $\{(x, y) : y = m\}$ within distance $\varepsilon m/t$ from each other. The shaded region is the one to be resampled.

We argue that $A$ has positive probability to occur for some set of parameters. First, observe that by making $m$ large we can make sure to find three coalescing classes represented on the vertical axis within 0 and $\varepsilon m/t_0$, and that some representative in each class is entirely contained in the right half-plane. Second, using the shape theorem\(^6\) and Claim 9.8, we increase $m$ so that the fourth condition is met and the three representatives intersect $\{(m, j) : j \in \mathbb{Z}\}$ within distance $\varepsilon m/t_0$ of each other. We conclude that $A$ has positive probability to occur for some set of parameters.

Denote by $R_m$ the region confined by the two vertical segments connecting $(0, j_1)$ with $(0, j_3)$ and $(m, j_1')$ with $(m, j_3')$, and the two segments of $G(j_1, e_2)$ and $G(j_3, e_2)$ connecting $(0, j_1)$ and $(0, j_3)$ with $(m, j_1')$ and $(m, j_3')$. While the event $A$ defined above is not increasing with respect to the set of interior edges in $R_m$, we may easily modify $A$ slightly to become increasing. Let $A' = A'(m, j_1, j_3, j_1', j_2', j_3')$, where $j_1' < j_2' < j_3'$, be the event that (recall Figure 6)

- $G(j_1, e_2)$, $G(m e_1 + j_2' e_2)$, and $G(j_3, e_2)$ belong to different coalescence classes;
- $G(j_1, e_2)$ and $G(j_3, e_2)$ are entirely contained in the right half-plane, while $G(m e_1 + j_2' e_2)$ is entirely contained in the half-plane $\{(i, j) : i \geq m, j \in \mathbb{Z}\}$;
- the top- respectively bottom-most intersections of $G(j_1, e_2)$ and $G(j_3, e_2)$ with the vertical line $\{(m, j) : j \in \mathbb{Z}\}$ occur at $(m, j_1')$ and $(m, j_3')$;
- $T((0, j_1), (m, j_1')) \leq (\mathbb{E}[\omega_a] + \varepsilon)\| (m, j_1' - j_1)\|_1$.

It is clear that $A \subseteq A'$, so also $A'$ has positive probability to occur for some set of parameters. In addition, it follows from Lemma 9.5 that $A'$ is increasing with respect to the set of interior edges in $R_m$. Let $A''$ denote the event that $A'$ occurs and that the weight of each interior edge in $R_m$ has weight at least $t_0 - \varepsilon$. By Lemma 9.4, we conclude that $\mathbb{P}(A'') > 0$.

\(^6\)Note that the time constant satisfies $\mu(z) \leq \mathbb{E}[\omega_a]\|z\|_1$. 

On the event $A''$ there are three coalescing classes represented on the vertical segment $\{(m,j): |j| < m\}$, so our argument will be complete if we only can show that no vertex in the left half-plane is contained in the middle of the three, as that would be a contradiction to Lemma 4.10. To see this, note that on $A'$ we have that every path $\pi$ between a point $(0,j)$ to a point $(m,j')$, where $j_1 < j < j_3$ and $j'_1 < j' < j'_3$, that does not touch neither $G(j_1e_2)$ nor $G(j_3e_2)$, has to pick up at least $\|(m,j'-j)\|_1$ weights of size at least $t_0 - \epsilon$. Since $|j' - j| \geq |j'_1 - j_1| - \epsilon m/t_0$, this amounts to a total weight $T(\pi)$ satisfying

$$T(\pi) \geq (t_0 - \epsilon)\|(m,j'-j)\|_1 \geq (t_0 - 2\epsilon)\|(m,j'_1 - j_1)\|_1.$$ 

However, the path obtained by moving vertically from $(0,j)$ to $(0,j_1)$, then follow $G(j_1e_2)$ to $(m,j'_1)$, and finally move vertically to $(m,j')$, has weight at most

$$T((0,j_1),(m,j'_1)) + 2\epsilon m \leq (E[\omega] + 3\epsilon)\|(m,j'_1 - j_1)\|_1 = (t_0 - 3\epsilon)\|(m,j'_1 - j_1)\|_1,$$

as required. \hfill $\square$

9.2. Every geodesic eventually moves into a half-plane. We shall prove that every geodesic in the class $\mathcal{G}$ eventually moves into one of the eight half-planes $H_i$, $i = 0, 1, \ldots, 7$.

**Proposition 9.9.** For every $G \in \mathcal{G}$ there exists $i$ such that $G$ eventually moves into $H_i$.

As mentioned above, the argument is easy in the case that the limiting shape is neither a square nor a diamond. In the general case, where we make no assumption on the limiting shape, we will produce another four random coalescing geodesics to obtain a set of eight. Of course, this is only possible should the cardinality of $\mathcal{L}$ exceed four. More precisely, we prove that if there is a unique geodesic with label $\alpha$ almost surely, then this geodesic eventually moves into some half-plane almost surely. The bulk of the argument will be a subsequent limiting argument of Busemann measures.

**Lemma 9.10.** For every $\alpha \in \mathcal{L} \setminus \mathcal{M}$, there exists $i$ so that $G_{\alpha}^{ccw}$ eventually moves into $H_i$.

**Proof.** Let $\alpha \in \mathcal{L}$ but $\alpha \notin \mathcal{M}$ be fixed. Since we know each $\Gamma_i$ is contained in some $H_i$, there is no restriction to assume that $\alpha$ does not coincide with the label of one of the $\Gamma_i$’s, or that $F(\Gamma_0) < \alpha < F(\Gamma_1)$.

**Claim 9.11.** Let $x, y \in \mathbb{Z}^2$ and $G_{\alpha}^{cw}(y) = (y, y_1, \ldots)$. Then, almost surely,

$$\lim_{k \to \infty} \text{Geo}(x, y_k) = G_{\alpha}^{cw}(x).$$

**Proof of claim.** Given $x, y, z \in \mathbb{Z}^2$, we may due to coalescence of $\Gamma_0$ and $\Gamma_1$ find $u \in \Gamma_0(x) \cap \Gamma_0(y)$ and $v \in \Gamma_1(x) \cap \Gamma_1(y)$. Since $G_{\alpha}^{cw}$ is non-crossing and contained between $\Gamma_0$ and $\Gamma_1$ it follows that $G_{\alpha}^{cw}(y)$ and any subsequential limit of $(\text{Geo}(x, y_k))_{k \geq 1}$ is sandwiched between $G_{\alpha}^{cw}(u)$ and $G_{\alpha}^{cw}(v)$ (see Figure 7). Consequently, each subsequential limit has label $\alpha$. Since there is a unique geodesic with label $\alpha$, the limit $\lim_{k \to \infty} \text{Geo}(x, y_k)$ exists and equals $G_{\alpha}^{cw}(x)$, almost surely. \hfill $\square$

We will now move on to define a measure on Busemann functions associated to $G_{\alpha}^{cw}$. First, for $x, y, z \in \mathbb{Z}^2$ we let

$$B_{\alpha}^y(x, z) := \lim_{k \to \infty} \left[ T(x, y_k) - T(z, y_k) \right],$$

where $(y, y_1, \ldots)$ in an enumeration of the sites of $G_{\alpha}^{cw}(y)$. That the limit exists, almost surely and in $L^1$, follows from the argument used to prove Lemma 4.1. Moreover, $B_{\alpha}^y(x, z)$
is additive and hence completely determined by the configuration $\theta^\nu = (\theta^\nu(x))_{x \in \mathbb{Z}^2}$ defined as

$$\theta^\nu(x) = (B_\alpha^\nu(x, x + e_1), B_\alpha^\nu(x, x + e_2)).$$

Recall the sequence of equivalence relations on $\mathbb{Z}^2$ introduced in Section 6 based on $(V_i(x))_{x \in \mathbb{Z}^2}$, for $i = 1, 2, \ldots$. For each $i$ we define $\theta_i = (\theta_i(x))_{x \in \mathbb{Z}^2}$ by coordinate-wise averaging over elements in the same equivalence class. That is, let

$$\theta_i(x) := \frac{1}{|V_i(x)|} \sum_{y \in V_i(x)} \theta^\nu(y), \quad \text{for } x \in \mathbb{Z}^2. \quad (20)$$

Let $\Omega_1 = [0, \infty)^{\mathbb{Z}^2}$, $\Omega_2 = [0, 1]^{\mathbb{Z}^2}$ and $\Omega_4 = (\mathbb{R}^2)^{\mathbb{Z}^2}$. For each $i$ we may exhibit a measurable map $\Psi_i : \Omega_1 \times \Omega_2 \to \Omega_1 \times \Omega_4$ as $(\omega, \xi) \mapsto (\omega, \theta_i)$. The measure $P \times \text{Leb}$ may be pushed forward through the map $\Psi_i$ to give a measure $\nu_i$ on $\Omega_1 \times \Omega_4$. Because of tightness, the sequence $(\nu_i)_{i \geq 1}$ will have a subsequential limit, which we denote by $\nu$.

We next use $\nu$ to reconstruct a Busemann-like function $\hat{B}_\alpha : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ associated to $G_{ccw}^\alpha$. For each $\omega \in \Omega_1$ we obtain a probability measure $\hat{\nu} = \hat{\nu}(\omega)$ on $\Omega_4$ through conditional expectation. For $\theta \in \Omega_4$ we define for any neighboring pair of vertices $x$ and $y$

$$b(x, y) := \begin{cases} 
\theta_1(x) & \text{if } y = x + e_1, \\
\theta_2(x) & \text{if } y = x + e_2, \\
-\theta_1(y) & \text{if } y = x - e_1, \\
-\theta_2(y) & \text{if } y = x - e_2.
\end{cases}$$

Let $\{\pi(x, z) : x, z \in \mathbb{Z}^2\}$ be some predefined family of finite paths. For $x, z \in \mathbb{Z}^2$, let $x_0 = x, x_1, \ldots, x_m = z$ be the enumeration of the sited of $\pi(x, z)$, and define

$$\hat{B}_\alpha(x, z) := \sum_{i=0}^{m-1} \int b(x_i, x_{i+1}) \, d\hat{\nu}(\omega). \quad (21)$$

Claim 9.12. The function $\hat{B}_\alpha : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ has the following almost sure properties:

(a) $\hat{B}_\alpha(x, z) = \hat{B}_\alpha(x, y) + \hat{B}_\alpha(y, z)$ for all $x, y, z \in \mathbb{Z}^2$;

(b) $|\hat{B}_\alpha(x, z)| \leq T(x, z)$ for all $x, z \in \mathbb{Z}^2$;
(c) $\hat{B}_\alpha(x, z) = T(x, z)$ for all $x$ and $z \in G^\text{cw}_\alpha(x)$.

Proof of claim. We verify that the above properties are satisfied for the finite averages. First, the three properties are satisfied by $B^\text{cw}_\alpha(x, z)$ since it is additive, since $|T(x, y_k) - T(z, y_k)| \leq T(x, z)$, and since if $z \in G^\text{cw}_\alpha(x)$ then $\text{Geo}(z, y_k) \subseteq \text{Geo}(x, y_k)$ for all large $k$ by Claim 9.11. Second, for each choice of $x, y, z \in \mathbb{Z}^2$, properties (b) and (c) are clearly preserved also in the averaging in (20). For $i$ large enough, so that the paths $\pi(x, y), \pi(y, z)$ and $\pi(x, z)$ are contained in the same equivalence class induced by $(V_i(x))_{x \in \mathbb{Z}^2}$, also (a) is preserved. For fixed $x, y, z$ the three properties are given probability one by the limiting measure $\nu$. The averaging in (21) does not change that.

It is immediate from construction that the limiting measure $\nu$ is translation invariant, and hence that $\hat{B}_\alpha(x, z)$ equals $\hat{B}_\alpha(x + y, z + y)$ in distribution. Define $\hat{\rho}_\alpha : \mathbb{R}^2 \to \mathbb{R}$ as the linear functional given by

$$\hat{\rho}_\alpha(x) := \langle \mathbb{E}[\hat{B}_\alpha(0, e_1)], \mathbb{E}[\hat{B}_\alpha(0, e_2)] \rangle : x.$$  

By repeating the argument in the proof of Proposition 4.4, we obtain that

$$\limsup_{|z| \to \infty} \frac{1}{|z|} |\hat{B}_\alpha(0, z) - \hat{\rho}_\alpha(z)| = 0 \quad \text{almost surely.}$$  

Moreover, arguing as in the proof of Proposition 4.3, it follows that $\{x \in \mathbb{R}^2 : \hat{\rho}_\alpha(x) = 1\}$ is a supporting line for the asymptotic shape Ball, and that each limiting direction for $G^\text{cw}_\alpha$ is contained in the intersection of this supporting line and Ball. This set (possibly random) is thus contained in an arc of $S^1$ of width at most $\pi/2$. This arc may contain at most three angles of the form $\pm \pi/4$, and by choosing a consecutive set among the remaining five angles we obtain a half-plane of the form $H_i$. Hence, for some $i = 0, 1, \ldots, 7$ we have $G^\text{cw}_\alpha$ almost surely contained in $H_i$. 

We are now in position to prove Proposition 9.9.

Proof of Proposition 9.9. There are two cases. Either the cardinality of $\mathcal{Z}$ equals four, or it is at least eight. In the former case there can be no multiple labels by Lemma 6.4, so $\mathcal{G}$ consists of the four $\Gamma_i$ geodesics. Since each $\Gamma_i$ is already known to be contained in a half-plane there is in this case nothing more to prove. In the latter case we obtain from Lemma 8.9 that there are at least eight labels without multiplicity. By Lemma 9.10 the geodesics corresponding to these labels eventually move into some half-plane.

We conclude that there are either four geodesics almost surely, or at least eight geodesics eventually contained in some half-plane $H_i$. By Proposition 9.2 these geodesics are random coalescing geodesics. Combining Propositions 4.3 and 4.4 it follows that the set of limiting angles for each pair share at most one point. For each consecutive pair of geodesics (in the usual ordering of geodesics), their set of limiting directions are contained in some interval of length at most $\pi/2$, due to symmetry. As before, we find for each consecutive pair a half-plane of the form $H_i$ which they both eventually move into. Since each geodesic in $\mathcal{G}$ is contained in between two consecutive geodesics of the above form it follows that for each geodesic $G \in \mathcal{G}$ there exists a half-plane $H_i$ which it eventually moves into, almost surely.

$\square$
9.3. Proof of Theorem 9.1. We first argue that $G_{cw}^\alpha$ and $G_{ccw}^\alpha$ are random coalescing geodesics for each $\alpha \in \mathcal{L}$. By Lemma 8.7, we have that $G_{cw}^\alpha$ is a random non-crossing geodesic. By Proposition 9.9, there exists a half-plane $H_i$ such that $G_{cw}^\alpha$ eventually moves into $H_i$, so by Proposition 9.2 it follows that $G_{cw}^\alpha$ is coalescing. Since $G_{cw}^\alpha$ is coalescing we must then have $\mathbb{P}(\alpha \in \mathcal{M}) \in \{0, 1\}$. In case $\mathbb{P}(\alpha \in \mathcal{M}) = 1$ then $G_{cw}^\alpha$ is a random non-crossing geodesic. By Proposition 9.9 there exists a half-plane $H_i$ such that $G_{cw}^\alpha$ eventually moves into $H_i$, so by Proposition 9.2 it is coalescing. In conclusion, every member of $\mathcal{G}$ is a random coalescing geodesic.

We now take a random coalescing geodesic $G$. $F(G)$ is an almost sure constant; let’s call it $\alpha$. Suppose that $\mathbb{P}(G = G_{cw}^\alpha) > 0$. Since $G$ is coalescing we must then have $\mathbb{P}(G = G_{cw}^\alpha) = 1$. The only other possibility is that $\mathbb{P}(G = G_{cw}^\alpha) = 1$. In either case there exists $G'$ such that $G = G'$ almost surely.

9.4. Almost sure properties. The following is a simple consequence of our characterization of random coalescing geodesics in Theorem 9.1.

Proposition 9.13. For any two random coalescing geodesic $G$ and $G'$, the events \{ $G = G'$\}, \{ $G$ is a cw-limit\} and \{ $G$ is a ccw-limit\} are all 0-1 events.

Proof. Recall that every random coalescing geodesic is of the form $G_{cw}^\alpha$ or $G_{ccw}^\alpha$. That $G = G'$ or $G < G'$ occurs with probability zero or one is then a consequence of the consistency of the labeling with the ordering imposed on $\mathcal{T}_0$. We proceed with the remaining events.

If $\alpha \in \mathcal{L}$ is isolated from above, then $G_{cw}^\alpha$ is cw-isolated, while if $\alpha$ is not isolated from above then there is $(\alpha_k)_{k \geq 1}$ such that $\alpha_k \downarrow \alpha$ and $\alpha = \lim_{k \to \infty} F(G_{cw}^{\alpha_k}) = F(\lim_{k \to \infty} G_{cw}^{\alpha_k})$. Hence $\lim_{k \to \infty} G_{cw}^{\alpha_k} = G_{cw}^\alpha$, and $G_{cw}^\alpha$ is a cw-limit. That is, $\mathbb{P}(G_{cw}^\alpha$ is a cw-limit$) \in \{0, 1\}$.

Via a similar argument we conclude that $\mathbb{P}(G_{ccw}^\alpha$ is a ccw-limit$) \in \{0, 1\}$. So, if $G_{cw}^\alpha = G_{ccw}^\alpha$, then there is nothing more to prove. On the other hand, if $\mathbb{P}(\alpha \in \mathcal{M}) = 1$, then by Lemma 6.4 $G_{cw}^\alpha$ is a cw-limit and ccw-isolated, while $G_{cw}^\alpha$ is cw-isolated, which completes the proof.

10. Sparse non-crossing geodesics

In preparation for addressing the midpoint problem we shall in this section investigate properties of ‘sparse’ non-crossing geodesics, which generalizes the concept of random non-crossing geodesics. More precisely, a measurable map $G : \Omega \to \mathcal{H}$ is a sparse random non-crossing geodesic if, almost surely, either $G(\omega) \in \mathcal{T}_0$ or $G(\omega) = \{0\}$, and for every $u, v \in \mathbb{Z}^2$ either

$$|\sigma_u(G(\sigma_u\omega) \setminus \sigma_v(G(\sigma_v\omega)))| < \infty \quad \text{or} \quad \sigma_u(G(\sigma_u\omega) \cap \sigma_v(G(\sigma_v\omega))) = \emptyset,$$

where $\mathbb{P}(G(\omega) \in \mathcal{T}_0) > 0$. In particular, also a random non-crossing geodesic is a sparse random non-crossing geodesic. As before we shall write $G(v)$ as short for $\sigma_u \circ \sigma_v \circ G(v)$.

A sparse random non-crossing geodesic $G$ induces an equivalence relation on the set $V_G := \{v \in \mathbb{Z}^2 : G(v) \in \mathcal{T}_0\}$ by declaring $u \sim v$ if $G(u)$ and $G(v)$ coalesce. The equivalence classes of this relation will be referred to as the coalescing classes of $G$. If $G$ almost surely has a unique coalescing class then we say that $G$ is coalescing.

Theorem 10.1. A sparse random non-crossing geodesic has at most eight coalescing classes, almost surely. A sparse random non-crossing geodesic which is almost surely contained between $\Gamma_0$ and $\Gamma_1$ is coalescing.
Theorem 10.2. Let $\nu$ be a shift-invariant measure on $\Omega_1 \times \Omega_2$ that is supported on sparse non-crossing geodesics. Then $\nu$ is supported on finitely many coalescing classes a.s.

Proof. First we decompose $\nu$ into at most eight pieces $\nu_1, \ldots, \nu_k$ so that each piece is either equal to a $\Gamma_i$ a.s. or is between $\Gamma_i$ and $\Gamma_{i+1}$ a.s. Then as in Theorem 10.1 we have that for each $(\omega, \eta)$ the label of the geodesics is supported on a single value a.s. Thus we can project each $\nu_i$ onto a shift invariant measure $\pi(\nu_i, \omega)$ on $\mathbb{R}$. By ergodicity and the fact that $\pi(\nu_i, \omega)$ is shift-invariant we have that there exists a shift $\pi(\nu_i)$ such that $\pi(\nu_i) = \pi(\nu_i, \omega)$ for a.e. $\omega$. As in the proof of Theorem 10.1 we have that $\nu_i$ is a.s. supported on a subset of a random coalescing geodesic. Thus $\nu$ has finitely many coalescing classes a.s. \hfill $\Box$

11. The Midpoint Problem

In this section we apply the theory constructed around random coalescing geodesics to prove Theorem 2.3 which answers a question raised by the work of Benjamini, Kalai and Schramm [BKS03]. We begin by outlining our strategy in the case that $\nu_k = (k, 0)$. We argue by contradiction. If there does exist a $\delta > 0$ such that, uniformly in $k$,

$$\mathbb{P}\left((0, 0) \in \text{Geo}((-k, 0), (k, 0))\right) > \delta,$$

then we will construct a (sparse) random non-crossing geodesic which has infinitely many coalescing classes. This violates Theorem 10.1.

To construct a non-crossing geodesic we proceed as follows: For each $k$ we form a set

$$I_k = \{ i \in \mathbb{Z} : (0, i) \in \text{Geo}((-k, i), (k, i)) \}.$$
By the ergodic theorem and [22] this set $I_k$ has density at least $\delta$. Given $i, j \in I_k$ we then note, crucially, that if $\text{Geo}((-k, 1), (0, i))$ and $\text{Geo}((-k, j), (0, j))$ touch, then $\text{Geo}((0, i), (k, i))$ and $\text{Geo}((0, j), (k, j))$ are unlikely to touch too. If they did, then both geodesics would have to visit both $(0, i)$ and $(0, j)$ not to contradict unique passage times (see Figure 8). As this is unlikely to happen for $i$ and $j$ far apart we next use this observation to thin $I_k$ to get a subset $I_k^+ \subset I_k$ such that for all $i, j \in I_k^+$

$$\text{Geo}((0, i), (k, i)) \cap \text{Geo}((0, j), (k, j)) = \emptyset.$$ 

This gives us a family $\{\text{Geo}((0, i), (k, i)) : i \in I_k^+\}$ of finite non-crossing geodesics that do not coalesce. Importantly, because of the above observation, the geodesics in this family do not become ‘more coalescing’ as $k$ increases. That is, also $I_k^+$ will have density bounded away from zero.

![Figure 8. The intersection of two geodesics is a continuous path.]

We may now use the Damron-Hanson strategy to form (a sequence of) measures on finite geodesics. This sequence will have a sub-sequential limit. Since $I_k^+$ has positive density, each sub-sequential limit will be supported on (families of) non-crossing and non-coalescing geodesics. Then, we use this to construct a (sparse) random non-crossing geodesic which has infinitely many coalescing classes. This is a contradiction to Theorem 10.1.

The rest of this section will be dedicated to making the above outline rigorous. In the case when Ball is a polygon with a small number of sides (less than 16), then this will be mostly straightforward based on the above outline. In the case when Ball is a polygon with a small number of sides (less than 16), this will require a much more careful analysis. In either case, we will part from the following assumption, that there exist $\delta > 0$ and a sequence $(v_k)_{k \geq 1}$ in $\mathbb{Z}^2$ such that $|v_k| \to \infty$ and

$$P(0 \in \text{Geo}(-v_k, v_k)) > \delta. \tag{23}$$

By restricting to a further subsequence, we may assume that $v_k/|v_k| \to v$ for some $v \in S^1$.

11.1. The central argument. We start off by defining an event central for the construction of a family of finite geodesics. This event will involve a family of random coalescing geodesics to be used to control the direction of the finite geodesics. Given $x \in \mathbb{Z}^2$ and eight random coalescing geodesics $\{G^i : i = 1, 2, \ldots, 8\}$, define $\text{Good}_k(x) = \text{Good}_k(x, G^1, G^2, \ldots, G^8)$ to be the event that the following all occur (see Figure 9):

- $x \in \text{Geo}(x - v_k, x + v_k)$;
- $\text{Geo}(x, x + v_k)$ is counterclockwise between $G^1(x)$ and $G^2(x)$;
- $\text{Geo}(x - v_k, x)$ is counterclockwise between $G^3(x)$ and $G^4(x)$;
- $\text{Geo}(x - v_k, x)$ is counterclockwise between $G^5(x - v_k)$ and $G^6(x - v_k)$; and
- $\text{Geo}(x, x + v_k)$ is counterclockwise between $G^7(x + v_k)$ and $G^8(x + v_k)$.

Naturally, the $G^i$’s can only help to control $\text{Geo}(x - v_k, x + v_k)$ if they can be chosen suitably, while $\text{Good}_k(x)$ occurs with positive probability. Recall that $H_i$, $i = 0, 1, \ldots, 7$,

\footnote{Recall that the number of sides equals $n$ if Ball is an $n$-gon and $\infty$ otherwise.}
denote the eight half-planes in directions $(\pm 1,0)$, $(0,\pm 1)$ and $(\pm 1,\pm 1)$. About half of the effort in proving Theorem 2.1 will aim at establishing the following lemma.

**Lemma 11.1.** Assume that \( \lbrack 23 \rbrack \) holds. Then there exists \( \alpha > 0 \), a half-plane \( H_i \) and eight random coalescing geodesics \( G^1, G^2, \ldots, G^8 \) such that \( G^1, G^2 \) and \( G^5, G^6 \) eventually move into \( H_i, G^3, G^4 \) and \( G^7, G^8 \) eventually move into \( H_i^c \), and such that for all large \( k \) we have

\[
P(G_{\delta k}(0)) > \alpha.
\]

We now sketch a short proof of Lemma 11.1 under the additional assumption that the asymptotic shape has sufficiently many sides. Given \( v \) we can choose \( H_i \) such that the angle between \( v \) and \( \partial H_i \) is greater than \( \pi/4 \). If \( Ball \) has at least 24 sides then every cone of angle \( \pi/4 \) has at least three coalescing geodesic whose directions intersect the interior of the cone. Thus there is one geodesic whose direction is strictly contained in the cone. Thus we can find geodesics \( G^1 = G^5 \) and \( G^2 = G^6 \) which eventually move into \( H_i \) and \( v \) is strictly counterclockwise of \( \text{Dir}(G^1) \) and strictly clockwise of \( \text{Dir}(G^2) \). Also we can choose \( G^3 = G^7 \) and \( G^4 = G^8 \) which eventually move into \( H_i^C \) and \( -v \) is strictly counterclockwise of \( \text{Dir}(G^3) \) and strictly clockwise of \( \text{Dir}(G^4) \). Then for any choice of \( \alpha < \delta \) the conclusion of Lemma 11.1 follows from the directions of the \( G^i \) and that \( v_k/|v_k| \to v \).

The proof in the remaining case, when \( Ball \) has fewer than 24 sides, is considerably more involved. We postpone a full proof of Lemma 11.1 to Section 11.4 below. Now we proceed to show how to use Lemma 11.1 to construct a family of finite non-crossing and non-coalescing geodesics. For ease of notation we further assume that \( H_i = H_0 \), i.e. the right half-plane; the remaining cases are treated verbatim. This assumption, in particular, implies that the projection of \( v \) along the first coordinate axis is strictly positive.

First, let

\[
W^+_\ell(x) := \{ G^i(x) \cap (x + H_0) \subseteq x + [ -\ell/3, \ell/3 ] : i = 1, 2, 5, 6 \},
\]

\[
W^-\ell(x) := \{ G^i(x) \cap (x + H_0) \subseteq x + [ -\ell/3, \ell/3 ] : i = 3, 4, 7, 8 \},
\]

and \( W_\ell(x) := W^+_\ell(x) \cap W^-\ell(x) \). Next, we set

\[
Good_{k,\ell}(x) := Good_k(x) \cap W^+_\ell(x - v_k) \cap W^-\ell(x) \cap W^-\ell(x + v_k).
\]

Since the geodesics, by assumption, eventually move into the half-plane \( H_0 \) or its complement we can make the probability of both \( W^+_\ell(x) \) and \( W^-\ell(x) \) arbitrarily close to 1 by increasing \( \ell \). Hence, for some \( \ell \geq 1 \) and all sufficiently large \( k \) we have that

\[
P(Good_{k,\ell}(x) > \delta/4.
\]

**Lemma 11.2.** On the event that \( Good_{k,\ell}(i \mathbf{e}_2) \) and \( Good_{k,\ell}(j \mathbf{e}_2) \) occur, where \( |i - j| > \ell \), then at least one of the following occurs:

(a) \( \text{Geo}(i \mathbf{e}_2 - v_k, i \mathbf{e}_2) \cap \text{Geo}(j \mathbf{e}_2 - v_k, j \mathbf{e}_2) = \emptyset \);
It is immediate from the definition that for any \( x \)

Proof. First we note that the conditions of Good_k and \( W_k \) imply that \( i e_2 \notin Geo(j e_2 - v_k, j e_2 + v_k) \) and \( j e_2 \notin Geo(i e_2 - v_k, i e_2 + v_k) \). Then we note that for any two finite geodesics the set of points in their intersection is connected, since otherwise would contradict the assumption on unique passage times. Thus the intersection occurs on one side of the midpoint or the other, but not both; recall Figure 8. \( \square \)

Lemma 11.3. Suppose that Good_{k,\ell}(0), Good_{k,\ell}(i e_2) and Good_{k,\ell}(j e_2) occur for \( i, j - \ell > \ell \).

(a) If \( Geo(-v_k, 0) \cap Geo(i e_2 - v_k, i e_2) = \emptyset \), then \( Geo(-v_k, 0) \cap Geo(j e_2 - v_k, j e_2) = \emptyset \).

(b) If \( Geo(0, v_k) \cap Geo(i e_2, j e_2 + v_k) = \emptyset \), then \( Geo(0, v_k) \cap Geo(j e_2, j e_2 + v_k) = \emptyset \).

Proof. Our hypothesis imply that \( Geo(-v_k, 0) \) and \( Geo(j e_2 - v_k, j e_2) \) can only intersect if both intersect \( Geo(i e_2 - v_k, i e_2) \); see Figure 10. Hence, if \( Geo(-v_k, 0) \) does not intersect \( Geo(i e_2 - v_k, i e_2) \), then it cannot intersect \( Geo(j e_2 - v_k, j e_2) \) either. The other case is identical. \( \square \)

Figure 10. Layering of finite geodesics in \( I_{k,\ell} \).

Next define \( I_{k,\ell}(x) := \{ i \in \mathbb{Z} : Good_{k,\ell}(x + i e_2) \text{ occurs} \} \), and \( I_+^{k,\ell}(x) \) and \( I_-^{k,\ell}(x) \) as

\[
\{ j \in I_{k,\ell} : Geo(x + j e_2 - v_k, x + j e_2) \cap Geo(x + i e_2 - v_k, x + i e_2) = \emptyset \text{ for all } i \in I_{k,\ell}, i < j \}.
\]

\[
\{ j \in I_{k,\ell} : Geo(x + j e_2, x + j e_2 + v_k) \cap Geo(x + i e_2, x + i e_2 + v_k) = \emptyset \text{ for all } i \in I_{k,\ell}, i < j \}.
\]

It is immediate from the definition that for any \( x \in \mathbb{Z}^2 \) and \( i, j \in I^{\pm}_{k,\ell}(x) \) we have

\[
Geo(x + i e_2, x + i e_2 + v_k) \cap Geo(x + j e_2, x + j e_2 + v_k) = \emptyset.
\]

Hence, the set \( F_{k,\ell}(x) := \{ Geo(x + i e_2, x + i e_2 + v_k) : i \in I^{\pm}_{k,\ell}(x) \} \) defines a family of disjoint finite geodesics.

We shall require some knowledge about the density of geodesics in \( F_{k,\ell}(x) \). By Lemmas 11.2 and 11.3 it follows that \( I_{k,\ell}(x) = I_+^{k,\ell}(x) \cup I_-^{k,\ell}(x) \). Combined with (24) we conclude that for some \( \ell \) each large enough \( k \) we have that either \( P(0 \in I_+^{k,\ell}(x)) \) or \( P(0 \in I_-^{k,\ell}(x)) \) exceeds \( \delta/8 \). For one of the two this occurs for infinitely many \( k \). Since the remainder of the argument is identical in both cases, we proceed assuming, possibly after restricting to a subsequence, that there exists \( \ell \) such that for all large \( k \) we have

\[
(25) \quad P(0 \in I^{\pm}_{k,\ell}(x)) > \delta/8.
\]
Define, for \( m \geq 1 \) and \( y \in x + [0, v_k \cdot e_1] e_1 \),
\[
X_{k,\ell}^m(y) := \#\{ i \in I_{k,\ell}^+(x) : \text{Geo}(x + ie_2, x + ie_2 + v_k) \cap (x + (m\ell, m\ell)e_2) \neq \emptyset \}.
\]
Importantly, the following estimate holds uniformly in \( k \), and thus guarantees that the family \( \mathcal{F}_{k,\ell}(x) \) does not become sparser as \( k \) increases.

**Lemma 11.4.** Assume that \((25)\) holds. Then, there exists \( \ell \geq 1 \) so for all large \( k \) we have
\[
P(X_{k,\ell}^m(y) > \delta m/16) > \delta^2/1024\ell^2.
\]

**Proof.** By invariance with respect to shifts along the vector \( le_2 \) it follows from the ergodic theorem that, almost surely,
\[
\lim_{M \to \infty} \frac{1}{2M+1} \sum_{j=-M}^{M} X_{k,\ell}^m(y + j2m\ell e_2) = \mathbb{E}[X_{k,\ell}^m(y)].
\]
However, since every path from a site \( x + ie_2 \) to \( x + ie_2 + v_k \) has to cross the vertical line \( y + Ze_2 \) we also have, as \( M \to \infty \), that
\[
\frac{1}{2M+1} \sum_{j=-M}^{M} X_{k,\ell}^m(y + j2m\ell e_2) \geq \frac{1}{2M+1} X_{k,\ell}^{2M+1}(y) \to m \mathbb{P}(0 \in I_{k,\ell}^+(x)).
\]
Hence \( \mathbb{E}[X_{k,\ell}^m(y)] > \delta m/8 \). Since \( X_{k,\ell}^m(y) \) is bounded above by \( 2m\ell \), it follows from the Paley-Zygmund inequality that
\[
P(X_{k,\ell}^m(y) > \delta m/16) \geq \frac{1}{4} \frac{\mathbb{E}[X_{k,\ell}^m(y)]^2}{\mathbb{E}[X_{k,\ell}^m(y)^2]} > \frac{1}{4} \frac{(\delta m/8)^2}{(2m\ell)^2} = \frac{\delta^2}{1024\ell^2},
\]
as required. \( \square \)

### 11.2. Constructing a non-crossing geodesic
We will in this section take the set of finite and disjoint geodesics \( \mathcal{F}_{k,\ell}(x) := \{ \text{Geo}(x + ie_2, x + ie_2 + v_k) : i \in I_{k,\ell}^+(x) \} \) and construct a (sparse) random non-crossing geodesic with infinitely many coalescing classes.

Let \( J_{k,\ell} = \{-3r_k/4, \ldots, -r_k/4\} \times \{0, \ldots, \ell-1\} \), where \( r_k = v_k \cdot e_1 \). Recall that \( r_k \) is strictly positive by assumption. Denote by \( \mathbb{Z}^2 \) the set of oriented edges of the \( \mathbb{Z}^2 \) lattice. We encode the family of geodesics \( \mathcal{F}_{k,\ell}(x) \) as follows: Let \( \eta_{k,\ell}(x) = (\eta_{k,\ell}(x, e))_{e \in \mathbb{Z}^2} \) be defined as
\[
\eta_{k,\ell}(x, e) := \begin{cases} 1 & \text{if } e \text{ is crossed from left to right by some } g \in \mathcal{F}_{k,\ell}(x), \\ 0 & \text{otherwise.} \end{cases}
\]
We exhibit a measurable map \( \Psi_{k,\ell}(x) : \Omega_1 \to \Omega_1 \times \Omega_2 \) via \( \omega \mapsto (\omega, \eta_{k,\ell}(x)) \). We obtain a measure \( \nu_{k,\ell}(x) \) as the push-forward of \( \mathbb{P} \) through the mapping \( \Psi_{k,\ell}(x) \). Averaging over \( x \) in \( J_{k,\ell} \) we obtain
\[
\nu_{k,\ell}^* := \frac{1}{|J_{k,\ell}|} \sum_{x \in J_{k,\ell}} \nu_{k,\ell}(x).
\]
By compactness the sequence \( (\nu_{k,\ell}^*)_{k \geq 1} \) has a convergent subsequence. Let \( \nu \) be the limiting measure of some convergent subsequence. See Figure [11]

**Lemma 11.5.** Assume that \((25)\) holds. Then, every sub-sequential limit \( \nu \) is invariant with respect to translations, and for \( \nu \)-almost every \( (\omega, \eta) \in \Omega_1 \times \Omega_2 \) we have that
(a) every finite directed path in the graph encoded by \( \eta \) is a geodesic;
(b) every \( z \in \mathbb{Z}^2 \) has both in- and out-degree either 0 or 1 in \( \eta \);
(c) there are no cycles in \( \eta \).
Proof. We first aim to show that $\nu \circ \tilde{\sigma}_z = \nu$ for every $z \in \mathbb{Z}^2$. Hence, fix some sequence $(k_j)_{j \geq 1}$ such that $\nu_{k_j,\ell}^* \rightharpoonup \nu$ weakly. Then, by continuity of $\tilde{\sigma}_z$, also $\nu_{k_j,\ell}^* \circ \tilde{\sigma}_z \rightharpoonup \nu \circ \tilde{\sigma}_z$.

It follows from the periodicity of the set $I_{k,\ell}^+$ that $\nu_{k,\ell}(x) \circ \tilde{\sigma}_{e_2} = \nu_{k,\ell}(x + e_2) = \nu_{k,\ell}(x)$. Hence, for any bounded continuous function $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ we have that

$$\left| \nu_{k,\ell}^* \circ \tilde{\sigma}_z(f) - \nu_{k,\ell}^*(f) \right| = \frac{1}{|J_{k,\ell}|} \left| \sum_{x \in J_{k,\ell}} [\nu_{k,\ell}^*(x + z_1 e_1)](f) - [\nu_{k,\ell}(x)](f) \right| \leq \frac{2|z_1| \ell}{|J_{k,\ell}|} \max |f|,$$

which tends to zero as $k \to \infty$. It follows that $\nu \circ \tilde{\sigma}_z = \nu$.

We proceed with the proof of properties $(a)$-$(c)$. Let $\gamma$ be a finite directed path in $\mathbb{Z}^2$, and denote by $A_\gamma$, the event that $\gamma$ is a path in the (directed) graph encoded by $\eta$. $A_\gamma$ is defined in terms of the state of finitely many coordinates in $\eta$ and is thus both open and closed. So is its complement, and also the event $B_\gamma = \{ (\omega, \eta) : \gamma \text{ is a geodesic} \}$. Since each path in $F_{k,\ell}(x)$ is a geodesic, and each pair in $F_{k,\ell}(x)$ are non-crossing, it follows that $\nu_{k,\ell}^*(A^c_{\gamma} \cup (A_\gamma \cap B_\gamma)) = 1$ for each $k$. By Portmanteau’s lemma it follows that also $\nu(A^c_{\gamma} \cup (A_\gamma \cap B_\gamma)) = 1$. Since the number of finite directed paths is countable, this proves part $(a)$.

For part $(b)$, let $C_d(z)$ denote the event that $z$ has both in- and out-degree $d$ in the directed graph encoded by $\eta$. $C_d(z)$ are defined in terms of edges adjacent to $z$ and is thus closed. For fixed $z \in \mathbb{Z}^2$ and $k$ large enough we have $\nu_{k,\ell}^*(C_0(z) \cup C_1(z)) = 1$, since the contrary would imply that with positive probability $z$ would either be an endpoint to some path in $F_{k,\ell}(x)$ or a point of intersection for two paths in $\eta_{k,\ell}(x)$, for some large value of $|z|$. Hence, also $\nu(C_0(z) \cup C_1(z)) = 1$, and since $\mathbb{Z}^2$ is countable, part $(b)$ follows.

Finally, let $\gamma$ be a finite cycle and let $D_\gamma$ denote the event that $\gamma$ is a path in the graph encoded by $\eta$. Again, $D_\gamma$ is closed and so is its complement. Moreover, $\nu_{k,\ell}^*(D_\gamma) = 0$ since for each $k$ each path in $F_{k,\ell}(x)$ has almost surely no loops. Consequently, $\nu(D_\gamma) = 0$, and since the set of finite cycles in $\mathbb{Z}^2$ is countable, this proves part $(c)$. \[\square\]

It follows by Lemma 11.5 that each site in the graph encoded by $\eta$ has $\nu$-almost surely either out-degree 0 or a unique infinite forwards-path.\footnote{There would also have to be a unique infinite backwards-path, and hence a bigeodesic, but this observation will not be important in what follows.} In the case that $z$ has out-degree 1 in $\eta$, let $\gamma_\eta(\eta)$ denote the infinite forwards-path starting at $z$. In the case that $z$ has out-degree 0, then set $\gamma_\eta(\eta) = \{z\}$.
Lemma 11.6. Assume that (25) holds. For \( \nu \)-almost every \((\omega, \eta) \in \Omega_1 \times \Omega_2 \) we have that
(a) for every \( y, z \in \mathbb{Z}^2 \) either \( \gamma_y(\eta) \cup \gamma_z(\eta) \) is finite or \( \gamma_y(\eta) \cap \gamma_z(\eta) = \emptyset \);
(b) the set \( \{ z \in \mathbb{Z}^2 : \gamma_z(\eta) \neq \{ z \} \} \) has density at least \( \delta/8 \ell \);
(c) the set \( \{ z \in \mathbb{Z}^2 : \gamma_z(\eta) \neq \{ z \} \} \) contains infinitely many coalescing classes.

Proof. Assume that \( \gamma_y(\eta) \cap \gamma_z(\eta) \neq \emptyset \) for some \( y \) and \( z \). Let \( x \) be the first point in \( \gamma_y(\eta) \) that also lies on \( \gamma_z(\eta) \). Then \( x \) either equals \( y \) or \( z \), or has in-degree at least two. Part (a) thus follows from part (b) of Lemma 11.5.

Note that the measure \( \nu_{k,\ell}^* \) is invariant with respect to vertical shifts. Hence, for large enough \( k \) it follows that
\[
\nu_{k,\ell}^*(0 \text{ has out-degree at least } 1) \geq \frac{1}{\ell} \mathbb{P}(0 \in I^+_{k,\ell}(0)) > \frac{\delta}{8\ell}.
\]
Consequently, \( \nu(0 \text{ has out-degree at least } 1) > \delta/8\ell \). By the ergodic theorem it follows that the set \( \{ z \in \mathbb{Z}^2 : \gamma_z(\eta) \neq \{ z \} \} \) has density at least \( \delta/8\ell \), \( \nu \)-almost surely, and part (b) follows.

Given integers \( M > m \geq 1 \) and \( n \geq 1 \) let \( A_n^m,M \) denote the event that there are at least \( n \) points in \( V_m = \{ z \in \mathbb{Z}^2 : \gamma_z(\eta) \neq \{ z \} \} \cap [-m, m] \mathcal{E}_2 \) whose forward-paths remain pairwise disjoint inside \([ -M, M ]^2 \). By Lemma 11.4 we may for every \( n \geq 1 \) find an \( m \geq 1 \) such that for all \( M > m \) we have \( \nu_{k,\ell}(A_n^m,M) > \delta^2/1024\ell^2 \). Since \( A_n^m,M \) is defined in terms of finitely many edges it is closed, and hence also \( \nu(A_n^m,M) \geq \delta^2/1024\ell^2 \). By continuity, since \( A_n^m,M \) is decreasing in \( M \), we conclude that
\[
\nu(V_m \text{ contains at least } n \text{ coalescing classes}) \geq \lim_{M \to \infty} \nu(A_n^m,M) \geq \frac{\delta^2}{1024\ell^2}.
\]
Using the ergodic theorem we may conclude that the set \( \{ z \in \mathbb{Z}^2 : \gamma_z(\eta) \neq \{ z \} \} \) contains infinitely many coalescing classes \( \nu \)-almost surely.

We next anticipate the proof of Lemma 11.11 and show how to deduce Theorem 2.1.

Proof of Theorem 2.1 If Lemma 11.11 is not true then without loss of generality we can assume that (25) holds. By Lemma 11.6 there exists a measure \( \nu \) on non-crossing geodesics which has infinitely many coalescing classes. This is a contradiction to Theorem 10.2 and hence proves Theorem 2.1.

The remainder of this section is devoted to prove Lemma 11.11 and hence complete the proof of Theorem 2.1.

11.3. Interlude on neighboring geodesics. Two random coalescing geodesics \( G \) and \( G' \) satisfying \( G < G' \) are said to be **neighboring** if \( \mathbb{P}(\exists g \in \mathcal{G}_0 : G < g < G') = 0 \).

Lemma 11.7. Let \( G \) and \( G' \) be random coalescing geodesics such that \( G < G' \).

(a) If \( G \) and \( G' \) are not neighboring, then there exists a random coalescing geodesic \( G'' \) such that \( G < G'' < G' \).
(b) If \( G \) and \( G' \) are neighboring, then for every \( m \geq 1 \) we have
\[
\lim_{|z| \to \infty} \mathbb{P}(\text{Geo}(0, z) \text{ is ccw between } G \text{ and } G' \text{ and diverges within } m \text{ steps}) = 0.
\]

Proof. Assume that \( \mathbb{P}(\exists g \in \mathcal{G}_0 : G < g < G') > 0 \). Let \( \alpha = F(G) \) and observe that \( \alpha' = F(G') > \alpha \), since the contrary would give three geodesics labeled \( \alpha \) with positive probability – a contradiction to Lemma 6.4. There are three cases to consider: That with
positive probability either there exists a geodesic with label in \((\alpha, \alpha')\), with label \(\alpha\) or with label \(\alpha'\).

Assume first that there exists a geodesic with label in \((\alpha, \alpha')\) with positive probability. In that case, according to Lemma \[8.3\], there exists \(\beta \in (\alpha, \alpha')\) such that \(\beta \in \mathcal{Z}_z\). By Theorem \[9.1\] the geodesic \(G_{\beta \text{cw}}\) is a random coalescing geodesic and lies strictly between \(G\) and \(G'\).

Assume now that with positive probability there exists a geodesic \(g \in \mathcal{G}_0\), strictly larger than \(G\), which has label \(\alpha\). Then \(\alpha \in \mathcal{M}_z\). By Theorem \[9.1\] there are two geodesics with label \(\alpha\) with probability one, and \(G\) must coincide with the clockwise most geodesic with label \(\alpha\). Hence, \(G_{\alpha \text{cw}}\) lies strictly between \(G\) and \(G'\). The final case, when with positive probability there exists a geodesic with label \(\alpha'\), is similar. Hence, part \((a)\) has been proven.

In order to prove part \((b)\) we argue by contradiction. Suppose there are \(\varepsilon > 0\), \(m \geq 1\) and a sequence \((z_k)_{k \geq 1}\) such that \(|z_k| \to \infty\) for which

\[
\sup_{k \geq 1} P(\text{Geo}(0, z_k) \text{ ccw between } G \text{ and } G' \text{ and diverges in at most } m \text{ steps}) > \varepsilon.
\]

In that case there exists an infinite geodesic which coincides with either of \(G\) and \(G'\) for at most \(m\) steps with probability at least \(\varepsilon\). This contradicts the assumption that \(G\) and \(G'\) are neighboring. □

We counter the above lemma by showing that if \(\text{Geo}(-z, z)\) goes through the origin, then it cannot coincide with a given random coalescing geodesics for very long.

**Lemma 11.8.** Let \(\mathcal{F}\) be a finite family of random coalescing geodesics. For every \(\varepsilon > 0\) there exists \(m \geq 1\) such that for all large \(|z|\)

\[
P(0 \in \text{Geo}(-z, z) \text{ and Geo}(-z, z) \text{ coincides with some } G \in \mathcal{F} \text{ for at least } m \text{ steps}) < \varepsilon.
\]

**Proof.** Let \(G\) be a random coalescing geodesic and set

\[
V_G := \{ z \notin G(0) : \exists g \in \mathcal{G}_z \text{ such that } G(0) \subseteq g \},
\]

\[
U_G := \{ z \in G(0) : \exists y \notin V_G \text{ such that Geo}(0, z) \subseteq \text{Geo}(y, z) \}.
\]

In words, \(V_G\) is the set of vertices to which the geodesic \(G(0)\) can be extended backwards, and \(U_G\) is the set of vertices \(z\) on \(G(0)\) for which the finite segment \(\text{Geo}(0, z)\) can be extended backwards beyond \(V_G\).

By Proposition \[4.5\] \(V_G\) is almost surely finite. Consequently, the outer boundary of \(V_G\) is almost surely finite, and hence likewise \(U_G\). Next we note that if the event in the lemma occurs, then either \(-z\) or \(z\) is contained in \(V_G\) for some \(G \in \mathcal{F}\), or the first \(m\) points on either \(\text{Geo}(0, z)\) or \(\text{Geo}(-z, 0)\) belong to \(U_G\) for some \(G \in \mathcal{F}\). That this would happen is increasingly unlikely if both \(m\) and \(|z|\) are large. □

11.4. **Proof of Lemma** \[11.1\] In order to resolve the midpoint problem it only remains to prove Lemma \[11.1\]. We first recall that under the additional assumption that the asymptotic shape has at least 16 sides, then the proof is straightforward: From the work of Damron and Hanson \[DH14\] we obtain in Theorem \[5.2\] the existence of (at least) 16 random coalescing geodesics, each of which is directed in the region of intersection of the asymptotic shape and a tangent line. The direction \(v\) is contained in at most two of these regions. Let \(G^1\) and \(G^2\) denote the random coalescing geodesics associated with the counterclockwise- and clockwise-most regions not containing \(v\). The cone obtained via a counterclockwise movement from \(G^1\) to \(G^2\) contains \(v\) and at most four out of the mentioned regions. By symmetry these regions cannot span an angle larger than \(\pi/2\). Hence, there exists a half-plane \(H_v\) which \(G^1\)
and $G^2$ eventually moves into. Since neither $G^1$ or $G^2$ contains $v$ as a limiting direction it follows that
\[
\lim_{k \to \infty} P(\text{Geo}(0, v_k) \text{ is ccw between } G^1 \text{ and } G^2) = 1.
\]
By symmetry, we may choose $G^3$ and $G^4$ analogously, and finally let $G^5 = G^1$, $G^6 = G^2$, $G^7 = G^3$ and $G^8 = G^4$. This shows that under the assumption that (23) holds and Ball has at least 16 sides, then $P(\text{Good}_k(0)) > \delta/2$ for all large $k$.

We shall therefore, for the remainder of this section, assume that (23) holds and that the asymptotic shape has at most 16 sides, in which case Ball is a polygon. The argument will in this case be extensive and will have to accommodate a range of possibilities, such as if the geodesics we consider are clockwise isolated or clockwise limits.

Fix $\varepsilon \in (0, \delta/100)$. If $v$ is a direction of differentiability of Ball, then let $G^+ = G^-$ denote the (unique) random coalescing geodesic associated with the tangent line in direction $v$. If instead Ball has a corner at $v$, then let $G^-$ denote the random coalescing geodesic associated with the side clockwise of $v$ and define $G^+$ idem counterclockwise of $v$; see Figure 12. Next we choose an increasing sequence of geodesics $(G^-_n)_{n \geq 1}$ as follows: If $G^-$ is counterclockwise isolated, then let $G^-_n$ denote its clockwise neighbor. If $G^-$ is a counterclockwise limit, then pick $G^-_n < G^-$ so that
\[
P(G^-_n \text{ and } G^- \text{ diverge within } n \text{ steps}) < \varepsilon.
\]
By Proposition 4.4 there is a unique geodesic for each tangent line of Ball, so there is no restriction to assume that each geodesic in the sequence $(G^-_n)_{n \geq 1}$ has the same set of limiting directions; either they are all directed in the corner clockwise of the side associated to $G^-$, or they all coincide with the geodesic associated with the side clockwise of that corner. We choose similarly a decreasing sequence $(G^+_n)_{n \geq 1}$ counterclockwise of $G^+$.

The geodesics specified so far satisfy the following properties.

**Claim 11.9.** For every $n \geq 1$ we have
(a) $\lim_{k \to \infty} P(\text{Geo}(0, v_k) \text{ is ccw between } G^+_n \text{ and } G^-_n) = 0$;
(b) $\limsup_{|z| \to \infty} P(\text{Geo}(0, z) \text{ is ccw between } G^-_n \text{ and } G^- \text{ and diverges within } n \text{ steps}) < \varepsilon$;
(c) \( \limsup_{|z| \to \infty} P(\text{Geo}(0, z) \text{ is ccw between } G^+ \text{ and } G^+_n \text{ and diverges within } n \text{ steps}) < \varepsilon. \)

**Proof of claim.** Part (a) is an immediate consequence of \( v \) not being a limiting direction of neither \( G^-_n \) nor \( G^+_n \). For part (b), note that if \( G^- \) is a counterclockwise limit, then this follows from (26). If \( G^- \) is counterclockwise isolated, then it follows from part (b) of Lemma [11.7]. The proof of part (c) is identical. \( \Box \)

Combining Lemma [11.8] and Claim [11.9] we see that \( \text{Geo}(0, v_k) \) has nowhere to go but between \( G^- \) and \( G^+ \).

**Claim 11.10.** If (23) holds, then \( \text{Ball} \) has a corner in direction \( v \). Moreover, there exists \( m \geq 1 \) such that for all sufficiently large \( k \) we have

\[ P(0 \in \text{Geo}(-v_k, v_k) \text{ and } \text{Geo}(0, v_k) \text{ ccw between } G^-, G^+ \text{ and diverges in } m \text{ steps}) > \delta - 6\varepsilon. \]

**Proof of claim.** Let \( \mathcal{F} = \{G^-_1, G^-_n, G^+_1, G^+_n \} \). Recall Lemma [11.8] and pick \( m \) such that

\[ P(0 \in \text{Geo}(-v_k, v_k) \text{ and } \text{Geo}(0, v_k) \text{ coincides with some } G \in \mathcal{F} \text{ for at least } m \text{ steps}) < \varepsilon \]

for all large enough \( k \).

Now we observe that on the event that \( 0 \in \text{Geo}(-v_k, v_k) \) there are five possibilities for \( \text{Geo}(0, v_k) \): Either \( \text{Geo}(0, v_k) \) is ccw between \( G^+_m \) and \( G^-_m \), ccw between \( G^-_m \) and \( G^-_n \), and diverges within \( m \) steps, ccw between \( G^+_m \) and \( G^+_m \), and diverges within \( m \) steps, coincides with either of \( G^-_m, G^-_n, G^+_m, G^+_n \) for at least \( m \) steps, or \( \text{Geo}(0, v_k) \) lies ccw between \( G^- \) and \( G^+ \) and splits from both within \( m \) steps. The first three each have probability at most \( \varepsilon \) to occur by Claim [11.9] while the fourth has probability at most \( 3\varepsilon \) to occur by the choice of \( m \) and (26). The only other possibility is that \( \text{Geo}(0, v_k) \) lies counterclockwise between \( G^- \) and \( G^+ \) and splits from both within \( m \) steps. However, this can only happen if \( G^- \neq G^+ \), in which case Ball must have a corner in direction \( v \). \( \Box \)

Combining Lemma [11.7] and [11.10] we conclude that \( G^- \) and \( G^+ \) are not neighboring, and that there are random coalescing geodesics between \( G^- \) and \( G^+ \). We now choose a decreasing sequence \( (G''_n)_{n \geq 1} \) as follows: If \( G^- \) is clockwise isolated, then let \( G''_n \) denote its counterclockwise neighbor. If \( G^- \) is a clockwise limit, then pick \( G''_n \) strictly ccw between \( G^- \) and \( G^+ \) such that

\[ P(G^- \text{ and } G''_n \text{ diverge within } n \text{ steps}) < \varepsilon. \]

Choose an increasing sequence \( (G''''_n)_{n \geq 1} \) analogously, with the additional condition that \( G''''_1 \geq G''_1 \). All of these geodesics \( \{G''_n\} \) and \( \{G''''_n\} \) are directed in the direction of non-differentiability \( v \).

**Claim 11.11.** For every \( n \geq 1 \) we have

(a) \( \limsup_{|z| \to \infty} P(\text{Geo}(0, z) \text{ is ccw between } G^- \text{ and } G'_n \text{ and diverges within } n \text{ steps}) < \varepsilon; \)

(b) \( \limsup_{|z| \to \infty} P(\text{Geo}(0, z) \text{ is ccw between } G''''_n \text{ and } G^+ \text{ and diverges within } n \text{ steps}) < \varepsilon. \)

**Proof of claim.** Note that if \( G^- \) is a counterclockwise limit, then this follows from (27). If \( G^- \) is counterclockwise isolated, then it follows from part (b) of Lemma [11.7]. The remaining case is similar. \( \Box \)
Figure 13. The family $F_n$ of geodesics. The main gap is indicated. The remaining four gaps are minor.

Given a random coalescing geodesic $G$, let $\bar{G}$ denote the random coalescing geodesic obtained by rotation by $\pi$. We have now defined all geodesics with which we will work. Set

$$F_n := \{G_n^-, G_n^-, G'_n, G''_n, G_n^+, G_n^+\},$$

$$\bar{F}_n := \{\bar{G}_n^-, \bar{G}_n^-, \bar{G}'_n, \bar{G}''_n, \bar{G}_n^+, \bar{G}_n^+\}.$$  

We will refer to the counterclockwise gap between $G'_n(x)$ and $G''_n(x)$ as the main gap of $F_n$ at $x$, and say that $\text{Geo}(x, x + v_k)$ moves into the main gap of $F_n$ at $x$ if $\text{Geo}(x, x + v_k)$ is counterclockwise between $G'_n(x)$ and $G''_n(x)$. The gaps between $G^-_n(x)$ and $G^+_n(x)$, between $G''_n(x)$ and $G^+_n(x)$, and between $G^+_n(x)$ and $G^+_n(x)$ will be referred to as minor gaps. Define $\text{Mid}_{k,n}(0)$ to be the event that following three occur:

- $x \in \text{Geo}(x - v_k, x + v_k)$;
- Geo$(x, x + v_k)$ moves into the main gap of $F_n$ at $x$;
- Geo$(x - v_k, x)$ moves into the main gap of $\bar{F}_n$ at $x$.

Claim 11.12. Assume that (23) holds. Then, there exist $n_0 \geq 1$ such that for all $n \geq n_0$ and sufficiently large $k$ we have

$$P(\text{Mid}_{k,n}(0)) > \frac{\delta}{2}.$$

Proof of claim. Via Lemma 11.8 we know that for some $m \geq 1$ we have for all large $k$ that

$$P(0 \in \text{Geo}(-v_k, v_k) \text{ and Geo}(0, v_k) \text{ coincides with some } G \in F_1 \text{ for at least } m \text{ steps}) < \varepsilon.$$

As in the proof of Claim 11.10 but this time with seven possibilities for Geo$(0, v_k)$, we reach the conclusion that for all $n \geq m$ we have

$$\lim_{k \to \infty} P(0 \in \text{Geo}(-v_k, v_k) \text{ and Geo}(0, v_k) \text{ ccw between } G'_n \text{ and } G''_n) > \delta - 10\varepsilon.$$  

Due to symmetry we may rephrase the above conclusion as follows: For all $n \geq m$ we have

$$\lim_{k \to \infty} P(0 \in \text{Geo}(-v_k, v_k) \text{ and Geo}(0, -v_k) \text{ ccw between } \bar{G}'_n \text{ and } \bar{G}''_n) > \delta - 10\varepsilon.$$  

Iterating the argument then gives that

$$\lim_{k \to \infty} P(\text{Mid}_{k,n}(0)) > \delta - 20\varepsilon$$

for all $n \geq m$, as required. □
Define $OK_{k,n}(x)$ to be the event that
- $Mid_{k,n}(x)$ occurs;
- $Geo(x - \nu, x)$ moves into the main gap of $\mathcal{F}_n$ at $x - \nu$;
- $Geo(x, x + \nu)$ moves into the main gap of $\mathcal{F}_n$ at $x + \nu$.

Note that $OK_{k,n}(x)$ in nothing but a version of the event $Good_k(x)$ for a specific set of geodesics $G^i$ moving into direction either $v$ or $-v$.

We further define $OK^-_{k,n}(x)$ to be the event that
- $Mid_{k,n}(x)$ occurs;
- $Geo(x - \nu, x)$ moves into the main gap of $\mathcal{F}_n$ at $x - \nu$;
- $Geo(x, x + \nu)$ moves into a minor gap of $\mathcal{F}_n$ at $x + \nu$.

and define $OK^+_{k,n}(x)$ analogously, interchanging the location of ‘the main’ and ‘a minor’.

**Claim 11.13.** If \(33\) holds, then there exist \(\delta' > 0\) and \(n \geq 1\) such that for all large \(k\)

\[
P(OK_{k,n}(0) \cup OK^-_{k,n}(0) \cup OK^+_{k,n}(0)) > \delta'.
\]

**Proof of claim.** We may without restriction assume that \(v\) is strictly contained in \(H_0\), the right half-plane. Denote by \(A_{k,t,n}(x)\) the event that \(Mid_{k,n}(x)\) and the following occur:

- \((G^i_n(x) \cup G^m_n(x)) \cap (x + H_0) \subseteq x + [-\ell/3, \ell/3]^2\);  
- \((G^i_n(x) \cup G^m_n(x)) \cap (x + H_0) \subseteq x + [-\ell/3, \ell/3]^2\).

Since \(G^i_n\) and \(G^m_n\) have direction \(v\) they eventually move into \(H_0\). Hence, due to Claim 11.12, we can make the probability of \(A_{k,t,n}(x)\) as close to \(\delta/2\) as we like by increasing \(\ell\). Let \(Z_M = \{0, 1, \ldots, M - 1\}\) and \(S^M_{k,t,n} := \{i \in \ell Z_M : A_{k,t,n}(ie_i)\) occurs\}. We shall next fix a whole slew of parameters as follows:

(i) Fix \(\ell, n_0\) and \(k_0\) so that \(P(A_{k,t,n}(x)) > \delta/4\) for all \(n \geq n_0\) and \(k \geq k_0\).

(ii) Mimicking the argument of Lemma 11.4 we find \(M\) large so that \(P(\#S^M_{k,t,n} > 36) > 1/2\) for all \(n \geq n_0\) and \(k \geq k_0\).

(iii) Since the geodesics in \(\mathcal{F}_n\) are coalescing, we fix \(m\) and \(n_1 \geq n_0\) so that

\[P(G(ie_i)\text{ and } G(je_j)\text{ coalesce in } m \text{ steps } \forall i, j \in \ell Z_M, G \in \mathcal{F}_n, n \geq n_1)\]

is at least \(1 - \varepsilon\). Note that the bound is uniform in \(n \geq n_1\) since if \(G^<(x)\) and \(G^<(y)\) coalesce within \(m\) steps and \(G^<(x)\) and \(G^<(y)\) coincide with \(G^>(x)\) and \(G^>(y)\) for at least \(m\) steps, then \(G^<(x)\) and \(G^<(y)\) coalesce for all \(n \geq n_1\).

(iv) Pick \(n_2 \geq n_1\) and \(k_1 \geq k_0\) so that for all \(k \geq k_1\) the probability that for some \(i \in \ell Z_M\) Geo\((ie_i, ie_i + v_k)\) moves into a minor gap of \(\mathcal{F}_n\) at \(ie_i\) within \(m\) steps is at most \(\varepsilon\).

(v) Finally, select a \(k_2 \geq k_1\) large enough so that for all \(k \geq k_2\) the probability that there exists \(i \in \ell Z_M\) such that Geo\((ie_i - v_k, ie_i)\) is ccw between \(G^+_{n_2}(ie_i - v_k)\) and \(G^-_{n_2}(ie_i - v_k)\) is at most \(\varepsilon\).

We continue with an argument similar to that in Lemma 11.2. Consider the event that \#\(S^M_{k,t,n} \geq 37\) and that for all \(i, j \in S^M_{k,t,n}\) we have that

- Geo\((ie_i - v_k, ie_i)\) is not ccw between \(G^+_{n_2}(ie_i - v_k)\) and \(G^-_{n_2}(ie_i - v_k)\), nor is Geo\((ie_i, ie_i + v_k)\) ccw between \(G^+_{n_2}(ie_i + v_k)\) and \(G^-_{n_2}(ie_i + v_k)\);
- Geo\((ie_i - v_k, ie_i)\) does not move into a minor gap in \(\mathcal{F}_n\) at \(ie_i - v_k\), nor does Geo\((ie_i, ie_i + v_k)\) move into a minor gap of \(\mathcal{F}_n\) at \(ie_i + v_k\), in less than \(m\) steps;
- \(G(ie_i - v_k)\) and \(G(je_j - v_k)\) coalesce within \(m\) steps for all \(G \in \mathcal{F}_n\);
- \(G(ie_i + v_k)\) and \(G(je_j + v_k)\) coalesce within \(m\) steps for all \(G \in \mathcal{F}_n\);
For parameter $\ell$ and $M$ specified as above, and for all $n \geq n_2$ and $k \geq k_2$, the above event occurs with probability at least $1/2 - 6\varepsilon$. We further note that on the above event, then for no two $i \in S^M_{k,\ell,n}$ and no pair $G \in \mathcal{F}_n, G' \in \bar{\mathcal{F}}_n$ may $\text{Geo}(ie_2 - v_k, ie_2 + v_k)$ coincide with both $G(ie_2 - v_k)$ and $G'(ie_2 + v_k)$ for as long as $m$ steps. Indeed, the contrary would contradict the assumption of unique passage times; see Figure 14. Since there are 36 such combinations of elements from $\mathcal{F}_n$ and $\bar{\mathcal{F}}_n$, but $S^M_{k,\ell,n}$ contains at least 37 elements, then only possibility is that for some $i \in S^M_{k,\ell,n}$, one of the two ends of $\text{Geo}(ie_2 - v_k, ie_2 + v_k)$ moves into the main gap, while the other either moves into the main or one of the minor gaps. That is, for all $k \geq k_2$

$$
P(\exists i \in S^M_{k,\ell,n_2} : \text{OK}_{k,n_2}(ie_2) \cup \text{OK}^-_{k,n_2}(ie_2) \cup \text{OK}^+_{k,n_2}(ie_2)) > 1/4,$$

from which the conclusion of the claim follows. □

**Proof of Lemma 11.1.** By Lemma 11.13 possibly restriction to a subsequence, either of the three events $\text{OK}_{k,n}(0)$, $\text{OK}^-_{k,n}(0)$ or $\text{OK}^+_{k,n}(0)$ will occur with probability at least $\delta'/3$ for all large $k$. Recall that $\text{OK}_{k,n}(0)$ is a versions of the event $\text{Good}_k(0)$ for which the $G^i$’s move into direction $v$ and $-v$. So, if $P(\text{OK}_{k,n}(0)) > \delta'/3$ for all large $k$, then there is nothing more to prove.

In the remaining cases we need to verify that there exists a half-plane $H_i$ such that the random coalescing geodesics involved eventually move into either $H_i$ or $H_i^c$. The remaining two cases are symmetric to each other, so we only consider the case of $\text{OK}^-_{k,n}(0)$.

By restricting to a further subsequence we can specify which of the four minor gaps that the event $\text{OK}^-_{k,n}(0)$ occurs for. By Lemma 11.10 we have that Ball has a corner at $v$. By assumption $G^-$ and $G^+$ are directed in flat regions clockwise and counterclockwise of $v$. The geodesics $G'_n$ and $G''_n$ both move in direction $v$. Whether $G^-_n$ and $G^+_n$ are directed in corners or flat pieces may depend on the number of sides of the shape.

Consider first the case that Ball has four sides, i.e. that Ball is either a square or a diamond. By symmetry, there are then geodesics directed in all four corners, and both $G^-_n$ and $G^+_n$ are thus directed in (opposite) corners. Each of the minor gaps therefore spans an angle of directions at most $\pi/2$. For each minor gap we can then easily find a half-plane
Hi such that \( v \) is strictly contained in \( H_i \) and the geodesics constituting the minor gap eventually move into \( H_i \).

Consider next the remaining case, that Ball has at least eight sides. Each pair of consecutive sides may span an angle of at most \( \pi/2 \) due to symmetry. Hence, regardless of whether \( G_n^- \) and \( G_n^+ \) are directed in corners or not, each minor gap cannot span an angle of directions greater than \( \pi/2 \). So, also in this case we may find a half-plane \( H_i \) such that \( v \) is strictly contained in \( H_i \) and which the geodesics constituting the minor gap eventually move into.

This completes the proof of Lemma 11.1 and hence the proof of Theorem 2.1.

12. Random coalescing geodesics and the geodesic tree

In this section we investigate some of the consequences of the theory that we built in the previous sections. We saw in Section 9 that the set of random coalescing geodesics coincides with the class of random geodesics of the form \( G_{ccw}^\alpha \) and \( G_{cw}^\alpha \) for \( \alpha \in \mathcal{L}^\star \). Moreover, we saw in Section 8 that the set of labels we see almost surely coincides with \( \mathcal{L}^\star \), and in Section 6 that the probability of observing more than two geodesics with a given label is zero. Hence, if \( T_0 \) is at most countable, then there are almost surely no geodesics in \( T_0 \) that are not the image of a random coalescing geodesic.

However, the usual predictions (such as strict convexity of Ball) suggest that \( \mathcal{L}^\star \) is uncountable in the i.i.d. setting. This would imply that there may (and will) be exceptional labels attributed to more geodesics than expected. In this case the set of random coalescing geodesics will not account for all of \( T_0 \), but the set of random coalescing geodesics will be dense in the sense of Lemma 11.7. That is, for any two non-neighboring random coalescing geodesics \( G \) and \( G' \) there exists a random coalescing geodesics in between. In addition, the set of labels for which there are multiple geodesics is at most countable. This will allow us to extrapolate certain statements on random coalescing geodesics to all geodesics in \( T_0 \).

Recall that we by \( \mathcal{I} \) denote the set of linear functionals \( \rho : \mathbb{R}^2 \to \mathbb{R} \) supporting Ball. The following theorem is a precursor to many of the results we prove in this section.

**Theorem 12.1.** Let \( \mathcal{C}^\star \subseteq \mathcal{I} \) denote the set of linear functionals of the form \( \rho_G \) for some random coalescing geodesic \( G \).

(a) \( \mathcal{C}^\star \) contains every functional tangent to Ball.

(b) \( \mathcal{C}^\star \) is closed as a subset of \( S^1 \).

(c) For every \( \rho \in \mathcal{C}^\star \), there exists a unique random coalescing geodesic \( G \) with \( \text{Dir}(G) \) being an almost surely deterministic subset of \( \{ x \in S^1 : \mu(x) = \rho(x) \} \) and

\[
\limsup_{|y| \to \infty} \frac{1}{|y|} |B_G(0, y) - \rho(y)| = 0 \quad \text{almost surely.}
\]

**Proof.** That each linear functional tangent to Ball is present in \( \mathcal{C}^\star \) is a consequence of Theorem 5.2. By Proposition 4.4 there is at most one random coalescing geodesic for each element in \( \mathcal{C}^\star \), and by Propositions 4.2 and 4.3 its direction and Busemann function are described by this functional. It therefore remains to show that \( \mathcal{C}^\star \) is a closed set. (Recall that \( \mathcal{I} \) is parametrized by \( S^1 \), which induces a topology on \( \mathcal{I} \).)

Let \( \rho \) be a limiting element of \( \mathcal{C}^\star \). Then there exists a monotone sequence \( (\rho_k)_{k \geq 1} \) in \( \mathcal{C}^\star \) converging to \( \rho \). By repeating the argument in the proof of Proposition 4.4 we find that \( G < G' \) implies \( \rho_G < \rho_{G'} \), and hence that the parametrization of \( \mathcal{I} \) is consistent with the ordering of geodesics. That is, the sequence \( (G_k)_{k \geq 1} \) of random coalescing geodesics corresponding to the sequence \( (\rho_k)_{k \geq 1} \) is again monotone. Hence it has a limit \( G \) which is a
random coalescing geodesic; it is the (ccw- or cw-most) geodesic with label \( \lim_{k \to \infty} F(G_k) \).

To end the proof it will suffice to prove that \( \rho_G = \rho \).

However, by definition of \( \rho_G \) we have for every \( z \in \mathbb{Z}^2 \) that
\[
|\rho_G(z) - \rho_k(z)| = |E[B_G(0, z) - B_{G_k}(0, z)]|,
\]
which is arbitrarily small for \( k \) large. Hence \( \rho_G = \rho \). \( \square \)

### 12.1. Cardinality of the geodesic tree – Proof of Theorem 2.4

We first address the question of the number of topological ends of the geodesic tree.

**Proof of Theorem 2.4.** If the number of random coalescing geodesics is a finite value \( m \), then by Lemma [11.7] the cardinality of \( \mathcal{J}_0 \) equals \( m \) almost surely. If the number of random coalescing geodesics is countable then also by Lemma [11.7] the set \( \mathcal{J}_0 \) is countable. If the number of random coalescing geodesics is uncountable then also by the definition of random coalescing geodesics, or Theorem [8.1] the cardinality of \( \mathcal{J}_0 \) is uncountable. \( \square \)

### 12.2. Directions of differentiability – Proof of Theorem 2.2

Outside of corners of the asymptotic shape we rule out the existence of multiple geodesics and bigeodesics.

**Proof of Theorem 2.2.** Let \( v \in S^1 \) be a direction of differentiability. Then there is a unique linear functional \( \rho \) supporting \( \partial \text{Ball} \) in direction \( v \). This linear functional is tangent to \( \partial \text{Ball} \), so by Theorem [12.1] there is a unique random coalescing geodesic \( G \) for which \( \text{Dir}(G) \) is a subset of \( \{ x \in S^1 : \mu(x) = \rho(x) \} \). \( \text{Dir}(G) \) may or may not contain \( v \), but for every other random coalescing geodesic \( G' \) we have \( v \not\in \text{Dir}(G') \) almost surely.

Assume for a contradiction that there are two geodesics in direction \( v \) with positive probability. We may then find \( \delta > 0 \) and \( m \geq 1 \) such that
\begin{equation}
\mathbb{P}(\exists \text{ two geodesics in direction } v \text{ that diverge within } m \text{ steps}) > \delta.
\end{equation}

We define two random coalescing geodesics \( G^- \) and \( G^+ \) as follows: If \( G \) is ccw-isolated, then let \( G^- \) be its clockwise neighbor. If \( G \) is a ccw-limit, then pick \( G^- < G \) such that
\begin{equation}
\mathbb{P}(G \text{ and } G^- \text{ diverge within } m \text{ steps}) < \delta/4.
\end{equation}

Define \( G^+ > G \) similarly. Neither of the two geodesics may contain \( v \) in its set of directions.

The assumption in (28) implies that
\[ \mathbb{P}(\exists \text{ two geodesics ccw between } G^- \text{ and } G^+ \text{ that diverge within } m \text{ steps}) > \delta. \]

Then with probability \( \delta/2 \) there must be a geodesic counterclockwise between either \( G^- \) and \( G \) or \( G \) and \( G^+ \) that diverges within \( m \) steps. If \( G^- \) and \( G \) or \( G \) and \( G^+ \) are neighboring, then this is a contradiction to Lemma [11.7]. Otherwise it is a contradiction to (29).

This proves part (a). Part (b) follows from part (a) together with Proposition [4.5]. \( \square \)

### 12.3. Asymptotic directions and Busemann functions – Proof of Theorem 2.5

We shall prove the following result, which includes Theorem 2.5 and the first half of Theorem 2.6. Recall that \( \mathcal{C}_* \subseteq \mathcal{F} \) denotes the set of linear functionals of the form \( \rho_G \) for some random coalescing geodesic \( G \).

**Theorem 12.2.** With probability one we have that for every \( g \in \mathcal{J}_0 \) there exists \( \rho \in \mathcal{C}_* \) such that \( B_\rho \) is asymptotically linear to \( \rho \) and \( \text{Dir}(g) \) is a subset of \( \{ x \in S^1 : \rho(x) = \mu(x) \} \). Moreover, the set \( \mathcal{C}_* \) of functionals observed in a given realization equals \( \mathcal{C}_* \) almost surely.

Proving that every geodesic has an asymptotic direction is easily obtained by choosing a sufficiently dense set of geodesics and arguing as in the proof of Theorem 2.2. However, to get the full result we need a couple of lemmas.
Lemma 12.3. Suppose that $G^1 < G^2$ are random coalescing geodesics and that $y \in \mathbb{Z}^2$ satisfies

- $y$ is not in the cone ccw between $G^1(0)$ and $G^2(0)$;
- $0$ is not in the cone ccw between $G^1(y)$ and $G^2(y)$; and
- $G^2(y)$ intersects $G^1(0)$.

Then, for all $z$ which are in the intersection of the two cones we have

$$B_{G^2}(0, y) \leq T(0, z) - T(y, z) \leq B_{G^1}(0, y).$$

Similarly, if the first two conditions hold and $G^2(0)$ intersects $G^1(y)$, then

$$B_{G^1}(0, y) \leq T(0, z) - T(y, z) \leq B_{G^2}(0, y).$$

Proof. Let $z'$ be the first point of intersection of $\text{Geo}(0, z)$ and $\text{Geo}(y, z)$. Note that since $z$ is in the intersection of the two cones then $z'$ must be as well. Thus we can draw the picture in Figure 15. Note that some of the regions may be degenerate and some of the times may be zero.

Using the nomenclature of the figure we obtain the expression

$$T(0, z) - T(y, z) = T(0, z') - T(y, z') = t_2 + t_8 - t_5 - t_{11}.$$  

Moreover, we note that we have

$$t_1 \leq t_2 + t_7 \quad \text{and} \quad t_5 + t_{11} \leq t_4 + t_9 + t_8.$$  

Combining the above expression with the inequalities leaves us with

$$T(0, z) - T(y, z) \geq t_1 - t_7 + t_8 - t_5 - t_{11} \geq t_1 - t_7 - t_4 - t_9 = B_{G^2}(0, y).$$  

From the figure we also read out that

$$t_6 \leq t_5 + t_{12} \quad \text{and} \quad t_2 + t_8 \leq t_3 + t_{10} + t_{11}.$$  

These equalities together with the above expression for $T(0, z) - T(y, z)$ gives

$$T(0, z) - T(y, z) \leq t_3 + t_{10} - t_5 \leq t_3 + t_{10} + t_{12} - t_6 = B_{G^1}(0, y),$$  

Figure 15. Labeling of the path stubs.
as required.

We say that \( G \) has an \( \varepsilon \)-linear Busemann function if there exists a linear functional \( \rho : \mathbb{Z}^2 \to \mathbb{R} \) such that
\[
|B_G(0, y) - \rho(y)| < \varepsilon |y|
\]
for all but finitely many \( y \).

**Lemma 12.4.** There exists \( K \geq 1 \) such that for any \( \varepsilon > 0 \) and pair of random coalescing geometries \( G^1 \) and \( G^2 \) such that
- \( \text{Dir}(G^1) \cup \text{Dir}(G^2) \) is contained in an arc or length at most \( \varepsilon \); and
- \( \|\rho_{G^1} - \rho_{G^2}\| < \varepsilon \),
every \( g \in \mathcal{T}_0 \) with \( G^1 < g < G^2 \) has Busemann function \( K\varepsilon \)-linear to \( \rho_{G^1} \), almost surely.

**Proof.** We first choose an almost surely finite \( N_1 \geq 1 \) such that for all \( |y| \geq N_1 \) we have
\[
|T(y, y + z) - \mu(z)| < \varepsilon \max\{|y|, |z|\} \quad \text{for all} \quad z \in \mathbb{Z}^2.
\]
This can be done as of Proposition 3.2. Second, pick \( v_1, v_2, \ldots, v_m \) in \( \mathbb{Z}^2 \) so that for every \( y \in \mathbb{Z}^2 \) we have \(|y - n v_k| < 2\varepsilon |y|\) for some \( n \) and \( k \); write \( v_y \) for the point of the form \( n v_k \) minimizing \(|y - n v_k|\). By assumption of the lemma, we may further assume that neither of the \( v_k \) is directed in the arc obtained as the convex hull of \( \text{Dir}(G^1) \) and \( \text{Dir}(G^2) \), nor in its rotation by an angle \( \pi \). That is, for we may pick \( N_2 \geq N_1 \) such that neither \( v_y \) in contained in the cone ccw between \( G^1(0) \) and \( G^2(0) \), nor is \( 0 \) contained in the cone ccw between \( G^1(v_y) \) and \( G^2(v_y) \), for all \( |y| \geq N_2 \). Third, we pick \( N_3 \geq N_2 \) such that for \( i = 1, 2 \) we have
\[
|B_{G^i}(0, y) - \rho_{G^i}(y)| < \varepsilon |y| \quad \text{for all} \quad |y| \geq N_3.
\]

The above choices of \( v_1, v_2, \ldots, v_m \) and \( N_1 \leq N_2 \leq N_3 \) assures that for some \( K \geq 1 \) we have
\[
|B_y(0, y) - B_y(0, v_y)| \leq |B_y(y, v_y)| \leq T(y, v_y) \leq \varepsilon K |y|
\]
for all \( |y| \geq N_1 \). By Lemma 12.3 we conclude that for \( |y| \geq N_2 \) we have
\[
|B_y(0, v_y) - B_{G^1}(0, v_y)| \leq |B_{G^1}(0, v_y) - B_{G^2}(0, v_y)| \leq |\rho_{G^1}(v_y) - \rho_{G^2}(v_y)| + \varepsilon |v_y|,
\]
which by assumption is bounded above by \( \varepsilon 4 |y| \). Hence, combining (30) and (31) we conclude that for every \( g \in \mathcal{T}_0 \) such that \( G^1 < g < G^2 \) and every \( y \geq N_3 \) we have
\[
|B_y(0, y) - \rho_{G^1}(y)| < (K + 4) \varepsilon |y|,
\]
as required. \(\square\)

We now prove Theorem 12.2.

**Proof of Theorem 12.2.** We shall aim to find a nested sequence \( (\mathcal{F}_n)_{n \geq 1} \) of finite families of random coalescing geometries such that for each \( n \geq 1 \) and consecutive pair \( G^1 \) and \( G^2 \) in \( \mathcal{F}_n \) (in the counterclockwise ordering) we have
\[
P\left(\forall g \in \mathcal{T}_0 : G^1 < g < G^2 \text{ we have } |B_g(0, y) - \rho_{G^1}(y)| < |y|/n \text{ for large } |y|\right) = 1.
\]
We first see how the theorem follows from (32). For each \( n \) we let \( \rho_{g,n} \) denote the linear functional corresponding to largest element of \( \mathcal{F}_n \) smaller than \( g \). Since the families \( \mathcal{F}_n \) are nested the limit \( \rho_g = \lim_{n \to \infty} \rho_{g,n} \) exists for each \( g \in \mathcal{T}_0 \) and since \( C \) is closed it is contained in \( C \). By (32) it follows that
\[
P\left(\forall g \in \mathcal{T}_0 \text{ we have } |B_g(0, y) - \rho_g(y)| < |y|/n \text{ for all large } |y|\right) = 1.
\]
By the union bound it follows that almost surely every \( g \in \mathcal{T}_0 \) has a Busemann function linear to some element in \( C \). Moreover, for any \( g \in \mathcal{T}_0 \) and sequence \((v_k)_{k \geq 1}\) in \( g \) such that \( v_k / |v_k| \to x \) we have

\[
\rho_g(x) = \lim_{k \to \infty} \frac{B_g(0, v_k)}{|v_k|} = \lim_{k \to \infty} \frac{T(0, v_k)}{|v_k|} = \mu(x),
\]

almost surely, by \([33]\) and the shape theorem. If the families \( \mathcal{F}_n \) are chosen so that for every random coalescing geodesic \( G \) there exists an increasing sequence \((\rho_k)_{k \geq 1}\) in the union of all \( \mathcal{F}_n \) such that \( \rho_k \to \rho_G \), then the result follows.

We proceed with the proof of \([32]\). Let \( K \) be as in Lemma \([12.4]\). Recall that \( C \) is closed, so its complement is a countable union of open intervals \( I \). We define the finite set \( \mathcal{F}_n \) inductively as containing all members of \( \mathcal{F}_{n-1} \), all \( G \) for which the arc \( \{ x \in S^1 : \mu(x) = \rho(x) \} \) spans an angle at least \( 1/(6Kn) \), all \( G \) that are endpoints to some interval \( I \) of width larger than \( 1/(6Kn) \), and some additional finite number of random coalescing geodesics so that \( \mathcal{F}_n \) is \( 1/(6Kn) \)-dense in \( C \).

Assume the contrary, that \([32]\) fails for some pair \( G^1 \) and \( G^2 \). Then, we may find \( m \geq 1 \) and \( \delta > 0 \) such that with probability \( \delta \) there exists a geodesic \( g \in \mathcal{T}_0 \) with \( G^1 < g < G^2 \) that diverges from both \( G^1 \) and \( G^2 \) within \( m \) steps and such that \(|B_g(0, y) - \rho_G(0, y)| \geq |y|/n \) for some arbitrarily large \( y \).

The pair \( G^1 \) and \( G^2 \) are either neighboring, which would be an immediate contradiction, or they together span an angle at most \( 1/(6Kn) \), which would be a contradiction to Lemma \([12.4]\) or either of them (or both) span an angle at least as large as \( 1/(6Kn) \). In this final case we pick \( G' > G^1 \) as its counterclockwise neighbor or so that the two has probability at most \( \delta/4 \) to diverge within \( m \) steps. Pick \( G'' < G^2 \) similarly. In either case we conclude that with probability at least \( \delta/2 \) there exists a geodesic \( g \in \mathcal{T}_0 \) with \( G' < g < G'' \) such that \(|B_g(0, y) - \rho_G(0, y)| \geq |y|/n \), and hence \(|B_g(0, y) - \rho_G(0, y)| \geq |y|/(2n) \), for some arbitrarily large \( y \). However, the geodesics \( G' \) and \( G'' \) span together an angle no larger than \( 1/(2Kn) \), so this contradicts Lemma \([12.4]\). In conclusion, \([32]\) has to hold, as required.

Finally we notice that the \( \mathcal{F}_n \)'s are chosen so that for every random coalescing geodesic \( G \) there exists an increasing sequence \((\rho_k)_{k \geq 1}\) in the union of all \( \mathcal{F}_n \) such that \( \rho_k \to \rho_G \). \( \Box \)

### 12.4. Uniqueness – Proof of Theorem \([2.6]\)

Notice how the first half of Theorem \([2.6]\) is a consequence of Theorem \([12.2]\), so it will suffice to address the second half on uniqueness.

**Proof of Theorem \([2.6]\)** Any two random coalescing geodesics are associated to different linear functionals due to Proposition \([4.4]\). Either there are finitely many random coalescing geodesics, in which case every geodesic is a random coalescing geodesic, or there are infinitely many. In the former case we are done, so we assume the latter.

Fix a random coalescing geodesic \( G \) and our aim will be to show that almost surely no other geodesic has Busemann function linear to \( \rho_G \). That no other member of \( \mathcal{S} \) appears as the limit of a Busemann function was proved in Theorem \([12.2]\). Assume, for a contradiction, that

\[
P(\exists \text{two geodesics with Busemann function linear to } \rho_G) > \delta.
\]

In that case there exists \( m \geq 1 \) such that there is probability at least \( \delta \) to exist two geodesics that diverge within \( m \) steps that both have Busemann function linear to \( \rho_G \). We then choose \( G^- < G < G^+ \) as in the proof of Theorem \([2.2]\) either being neighboring of \( G \) or satisfying \([29]\). Together these choices and \([34]\) imply that there is probability at least \( \delta/2 \).
to exist a geodesic counterclockwise between $G^+$ and $G^-$ that has Busemann function linear to $\rho_G$. We will show that this cannot happen.

Select eight random coalescing geodesics, neither being $G$, such that any pair of consecutive geodesics (in the natural order) span a set of limiting directions of angle at most $\pi/2$. Hence, each consecutive pair will be eventually directed in some half-plane $H_i$, $i = 0, 1, \ldots, 7$. Let $G^1 < G^2$ be two random coalescing geodesics that eventually move into some $H_i$.

**Claim 12.5.** With probability one there exists a sequence $(y_k)_{k \geq 1}$ such that for every $k \geq 1$ and $g \in \mathcal{F}_h$ with $G^1 < g < G^2$ we have

$$B_{G^1}(0, y_k) \leq B_g(0, y_k) \leq B_{G^2}(0, y_k).$$

**Proof of claim.** We may assume that $H_i$ is the right half-plane, as the remaining cases are all similar. Let $(y_k)_{k \geq 1}$ be some sequence of vertices on the vertical axis such that $G^1(y_k)$ and $G^2(0)$ cross. Such a sequence exists almost surely. By Lemma 12.3, the conclusion follows. $\square$

Let $\mathcal{F}$ denote the set of these eight geodesics together with $G^-$ and $G^+$. For some consecutive pair $G^1$ and $G^2$ in $\mathcal{F}$ we must then have that with probability at least $\delta/20$ there exists a geodesic $g \in \mathcal{F}_h$ such that $G^1 < g < G^2$ and

$$\limsup_{|y| \to \infty} \frac{1}{|y|} \left| B_g(0, y) - \rho_G(y) \right| = 0. \tag{35}$$

This must occur for some pair apart from the pair $G^-$ and $G^+$.

Let $\varepsilon > 0$ be chosen so that $|\rho_G(y) - \rho(y)| > 2\varepsilon|y|$ for every $y \in \mathbb{Z}^2$ and every linear functional $\rho$ associated to some geodesic in $\mathcal{F}$. By linearity of $B_{G^1}$ and $B_{G^2}$ and Claim 12.5 there exists a sequence $(y_k)_{k \geq 1}$ such that for all large $k$ and $G^1 < g < G^2$ we have

$$\rho_{G^1}(y_k) - \varepsilon |y_k| < B_g(0, y_k) < \rho_{G^2}(y_k) + \varepsilon |y_k|.$$ 

However, this implies that either $B_g(0, y_k) < \rho_{G^1}(y_k) - \varepsilon |y_k|$ or $B_g(0, y_k) > \rho_{G^2}(y_k) + \varepsilon |y_k|$ for these values of $k$, which contradicts (35), and thus assumption (34). $\square$

## 13. Open Questions

We conclude with some open questions. As we mentioned in Section 3.2 the shape theorem and the results in [Hof08] and [DHL14] are the best possible results for ergodic first-passage percolation. Our first two questions ask whether our theorems is also sharp. These questions are closely related to the fact that we have very limited understanding for what goes on in corners of the asymptotic shape.

**Open Question 1.** Does there exist a model of FPP satisfying condition $A_1$ or $A_2$ and a label $\alpha$ such that there are two coalescing geodesics with label $\alpha$ almost surely?

**Open Question 2.** Does there exist a model of FPP satisfying condition $A_1$ or $A_2$ and a direction $v \in S^1$ such that there are two coalescing geodesics with direction $v$ almost surely?

Our theory of coalescing geodesics uses the fact that we are working with first-passage percolation on $\mathbb{Z}^2$ in a very strong way. We also use the uniqueness of geodesics, but in a much weaker way. The next two questions ask whether we can eliminate these conditions.

**Open Question 3.** Can we extend this theory to distributions with atoms?

**Open Question 4.** Can we develop a version of this theory for $d > 2$?

**Open Question 5.** Can we use this theory to rule out bi-infinite geodesics in a given direction $v \in S^1$, even if $\partial \text{Ball}$ has a sharp corner in the direction $v$?
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