Robust unidirectional transport in a one-dimensional metacrystal with long-range hopping

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Abstract – In two- and three-dimensional structures, topologically protected chiral edge modes offer a powerful mean to realize robust light transport. However, little attention has been paid so far to robust one-way transport in one-dimensional systems. Here it is shown that unidirectional transport, which is immune to disorder and backscattering, can occur in certain one-dimensional metacrystals with long-range hopping without resorting to topological protection. Such metacrystals are described by an effective Hermitian Hamiltonian with broken time-reversal symmetry, and transport does not require adiabatic (Thouless) pumping. A simple implementation in optics of such one-dimensional metacrystals, based on transverse light dynamics in a self-imaging optical cavity with phase gratings, is suggested.

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Introduction. – Topological photonic structures, a new class of optical systems inspired by quantum Hall effect and topological insulators, have attracted a huge attention in recent years owing to their rather unique property of permitting robust transport via topologically protected chiral edge modes [1]. Such two-dimensional (2D) or three-dimensional (3D) optical structures are usually realized by breaking time-reversal symmetry, e.g., using magneto-optic media [2–7], or by the introduction of synthetic gauge fields [8–17]. Other examples of chiral edge transport in 2D or 3D optical media include photonic Floquet topological insulators in helical waveguide lattices [18,19], gyroid photonic crystals [20], bianisotropic metamaterials [21,22], chiral hyperbolic metamaterials [23], and optomechanical lattices [24]. In one-dimensional (1D) systems, the possibility of realizing robust one-way transport has received less attention so far. Proposals of 1D transport include “Thouless pumping” in quasicrystals [25,26], Landau-Zener transport in binary lattices [27], the use of “synthetic” dimensions in addition to the physical spatial dimension [28–30], and non-Hermitian transport [31]. Topologically protected transport found in 2D systems using synthetic gauge fields cannot be trivially applied to 1D lattices with short-range hopping because the action of a synthetic gauge field is generally trivial in 1D. For example, application of a uniform magnetic field to a 1D tight-binding lattice just corresponds to a shift in momentum space of the band dispersion curve. For such a reason, topological protection in 1D requires adiabatic change of some parameter (Thouless pumping) or the addition of a “synthetic” dimension [28–30]. However, 2D lattices can be mapped into 1D chains with long-range hoppings [32], so that loops with nonvanishing magnetic fluxes and quantum Hall physics become possible even in 1D systems without additional synthetic dimensions [33]. An implementation of 1D lattices with long-range hopping and synthetic gauge fields, based on periodically driven spin chains with special driving protocols, has been recently suggested in ref. [33]. However, its practical realization remains challenging.

In this letter it is shown that in a wide class of 1D metacrystals, i.e., synthetic crystals described by an effective Hermitian Hamiltonian with long-range hopping and broken time reversal symmetry, one can realize unidirectional and robust transport which is not assisted by topological protection. A simple physical implementation of such metacrystals in optics, based on transverse light dynamics in a self-imaging optical resonator with phase gratings, is suggested.

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Robust unidirectional transport in a one-dimensional metacrystal. – Let us consider the motion of a quantum particle on a 1D lattice, subjected to an external potential $U(x)$ which varies slowly over the lattice period $a$. In our analysis, the potential $U(x)$ accounts for lattice defects or disorder of site energies. In the single-band approximation and after expanding the wave function $\psi(x,t)$ of the particle on the basis of displaced Wannier functions $W(x-na)$, i.e., after setting $\psi(x,t) = \sum_n f(n,t)W(x-na)$, it is well known that the envelope function $f(n,t)$ is obtained as $f(n,t) = \phi(x=na,t)$, where $\phi(x,t)$ satisfies the Schrödinger equation (taking $\hbar = 1$)

$$i\partial \phi = \hat{H}_{eff}\phi$$  \hspace{1cm} (1)

with an effective Hamiltonian $\hat{H}_{eff} = \hat{H}_0 + U(x)$ [34], where

$$\hat{H}_0 = E(-i\partial_x), \hspace{1cm} (2)$$

$E(k) = E(k+2\pi/a)$ is the energy dispersion curve of the lattice band, and $-\pi/a \leq k < \pi/a$ is the Bloch wave number (quasi-momentum). After introduction of the Fourier coefficients $J_n$ of $E(k)$, $E(k) = \sum_n J_n \exp(inak)$, the Schrödinger equation (1) reads explicitly

$$i\partial \phi(x,t)/\partial t = \sum_n J_n \phi(x+na,t) + U(x)\phi(x,t). \hspace{1cm} (3)$$

Note that $J_n$ corresponds to the hopping amplitude between two sites in the lattice spaced by $n$. For a Hermitian lattice with time-reversal symmetry, the energy $E(k)$ is real and with the even symmetry $E(-k) = E(k)$, which implies $J_n$ real and $J_{-n} = J_n$. For example, in the nearest-neighbor tight-binding approximation (short-range hopping), $E(k) = -J\cos(ka)$, where $J/2 = -J_1$ is the hopping amplitude between adjacent lattice sites. The even symmetry of the dispersion curve $E(k)$ is responsible for backscattering of a particle wave packet, propagating along the lattice, in the presence of defects or disorder. In fact, for a forward-propagating wave packet with carrier quasi-momentum $k_0$, moving with a group velocity $v_g = (dE/dk)_k = 0$, the scattering potential can excite the excite the energy-degenerate state with quasi-momentum $-k_0$, corresponding to a backward propagating wave packet $\partial E/dk < 0$; see figs. 1(a) and (b). By breaking time-reversal symmetry, one can in principle synthesize a lattice band with a dispersion curve $E(k)$ which is an increasing (or decreasing) function of $k$ over the entire Brillouin zone $-\pi/a < k < \pi/a$, with a rapid (abrupt) change at the Brillouin zone edges $k = \pm \pi/a$; see fig. 1(c). In this way, back reflections are forbidden, since at any quasi-momentum $k_0$ the group velocity has the same sign (apart from the Brillouin zone edges of negligible measure). In such a metacrystal, long-range hopping is necessary, together with a proper engineering of the phases of hopping amplitudes to break time-reversal symmetry. For example, in a metacrystal with a sawtooth-shaped

$$E(k) = J\sin(kl)/\pi, \quad -\pi/a < k < \pi/a, \hspace{1cm} (4)$$

the hopping amplitudes $J_n$, as obtained from the Fourier series expansion $E(k) = 2J\sum_{n=1}^{\infty} [(-1)^{n+1}/n\pi] \sin(nka)$, are given by

$$J_n = \begin{cases} 0, & n = 0, \\ (-1)^n J/(\pi n), & n \neq 0. \end{cases} \hspace{1cm} (5)$$

Note that while $J_{-n} = J_n^*$ (Hermitian lattice), $J_n$ is imaginary, indicating that time-reversal symmetry is broken. An interesting property of the sawtooth metacrystal is that, besides of ensuring one-way propagative states, the group velocity $v_g = Ja/\pi$ is uniform, corresponding to vanishing of group velocity dispersion and distortionless wave packet propagation. To highlight the robust propagation properties of the sawtooth metacrystal as compared to an ordinary tight-binding crystal with short-range hopping and time-reversal symmetry, described by a sinusoidal band $E(k) = -J\cos(ka)$, we numerically computed the evolution of a Gaussian wave packet in two lattices in the presence of either a potential site defect (fig. 2(a)) and site-energy disorder (fig. 2(b)), assuming the same bandwidth $2J$ and lattice period $a$. The figure clearly shows that while back reflections and deceleration of motion is observed in the sinusoidal lattice band, wave packet propagation turns out to be robust in the sawtooth metacrystal. It is interesting to comment on the limiting case of the sawtooth band corresponding to a
The initial condition is $f(n, 0) \propto \exp\left[-(n + 20)^2/16 + i\pi n/2\right]$, corresponding to a group velocity $v_g = J_a/\pi$ in the sinusoidal crystal, and $v_g = J_a/\pi$ in the sawtooth metacrystal. In (a) a potential defect at site $n = 0$ is introduced, namely $U(x = na) = U_0\delta_{n,0}$ with $U_0 = 2J$. In (b) on-site potential energy disorder is introduced, corresponding to $U(x = na)$ random variable uniformly distributed in the range $(-J/2, J/2)$.

Resonator optics realization of a metacrystal. – A main challenging is the physical implementation of a 1D metacrystal, which requires long-range hopping and breaking of time-reversal symmetry. A possible platform is provided, at least in principle, by spin chains and trapped ions with synthetic gauge fields [33]. However, the precise tailoring of hopping rates in amplitude and phase remains a rather challenging task. Here we suggest a rather simple optical implementation of a 1D metacrystal, which is based on transverse beam dynamics in a self-imaging passive optical resonator with a phase grating. In a few recent works, it has been suggested that light waves propagating back and forth in an optical resonator can emulate synthetic magnetism [16,17] and can realize diffraction management [35,36], i.e., the optical analogue of kinetic-energy operator management. Here we show that a self-imaging ring resonator with an intracavity phase grating can emulate for light waves the effective Schrödinger equation (3) of a metacrystal. A schematic of the passive optical resonator is shown in fig. 3. It consists of a four-lens ring cavity of total length $L = 8f$ in the so-called 4-$f$ self-imaging configuration [36]. Planes $\gamma$ and $\gamma_F$ at the longitudinal coordinates $z = 0$ and $z = 2f$, shown in fig. 3, are Fourier conjugate planes. A thin phase grating with spatial period $A$ and transmission function $t_1(x) = \exp[-i\varphi_1(x)]$, with $\varphi_1(x + A) = \varphi_1(x)$, is placed at the Fourier plane $\gamma_F$, whereas a second phase mask with transmission function $t_2(x) = \exp[-i\varphi_2(x)]$ is placed at the plane $\gamma$. Light propagation inside the optical ring can be readily obtained by application of the generalized Huygens-Fresnel integral. Assuming one transverse spatial dimension $x$ and following a rather standard procedure [37,38], the spatial-temporal dynamics of the intracavity optical field $\psi(x, z, t)$, which depends on space $(x, z)$ and time $t$, can be reduced to a discrete time system for the field $\psi_m(x) \equiv \psi(x, z = 0, t = mT_R)$ at the $\gamma$ transverse plane and at discretized times $t = mT_R$, where $T_R$ is the round-trip time of the cavity and $m = 0, 1, 2, ...$ is the round-trip number. In addition, for a high-finesse passive resonator probed by an injected pulsed beam with carrier frequency tuned in resonance with one longitudinal mode of the cavity and with pulse duration $\tau$ much longer than the cavity round trip time $T_R$, a single longitudinal mode
of the cavity is excited and the envelope $\psi(x, z, t)$ varies slowly in time over the round-trip time $T_R$. To understand how the transverse beam dynamics in the resonator can emulate the Schrödinger equation (1) in a metacrystal, let us disregard at this stage of the analysis cavity losses and external beam injection. The evolution of the field envelope $\psi_m(x)$ at plane $\gamma$ in the cavity and at the $m$-th round trip is governed by the following map:

$$\psi_{m+1}(x) = t_2(t_1 \left( \frac{\lambda f}{2\pi} \frac{\partial}{\partial x} \right) \psi_m(x) \equiv \hat{L} \psi_m(x), \quad (6)$$

where $\lambda$ is the wavelength of the circulating optical field and $\hat{L}$ is the round-trip cavity propagator. The two terms $t_1$ and $t_2$ that define the round-trip cavity propagator basically account for the phase shifts introduced by the phase grating and phase mask in the Fourier ($\gamma_F$) and direct ($\gamma$) planes inside the cavity, respectively. Note that diffraction in the cavity is suppressed because it is operated in a self-imaging condition [36]. The recurrence relation (6) can be viewed as a stroboscopic map of the solution of a Schrödinger-like wave equation, with effective Hamiltonian $\hat{H}_{\text{eff}}$, at discretized times $m T_R$. In fact, let us write the cavity field propagator $\hat{L}$ in the exponential form $L = \exp(-i \hat{H}_{\text{eff}})$. In this way it readily follows that $\psi_m(x)$ is given by $\psi_m(x) = \psi(x, t = m)$, where $\psi(x, t)$ satisfies the Schrödinger equation $i \partial_t \psi = \hat{H}_{\text{eff}} \psi$ and time $t$ is measured in units of the round-trip time. In the single-longitudinal mode approximation, the condition $|\varphi_{1,2}(x)| \ll 1$ should be satisfied, so as the optical field varies slowly over each round trip in the resonator. In this limit $t_1, t_2(x)$ can be expanded up to first order as $t_{1,2}(x) \simeq 1 - i \varphi_{1,2}(x)$ and the operator $\hat{L}$ can be approximated as $\hat{L} \approx 1 - i \hat{H}_{\text{eff}}$. Hence $1 - i \hat{H}_{\text{eff}} \simeq 1 - \varphi_1(x) - i \varphi_2(i \lambda f / 2\pi \partial_x)$, which yields

$$\hat{H}_{\text{eff}} = E(-i \partial_x) \psi + U(x) \psi \quad (7)$$

where we have set

$$U(x) = \varphi_2(x), \quad E(k) = \varphi_1 \left( -\frac{\lambda f k}{2\pi} \right). \quad (8)$$

Note that eq. (7) is precisely the Hamiltonian of a metacrystal with band dispersion curve $E(k)$ and external potential $U(x)$, defined by eq. (8). Hence, the transverse beam motion along the spatial coordinate $x$ at the resonator plane $\gamma$ emulates the motion of a quantum particle in an arbitrary 1D crystal under an external potential. The profile of the phase grating in the Fourier plane $\gamma_F$ defines the dispersion curve $E(k)$ of the lattice band, and can be thus tailored to realize a metacrystal, i.e., a crystal with long-range hopping and broken time-reversal symmetry. For instance, a sawtooth band dispersion curve can be simply implemented by using a surface relief grating with a sawtooth shape manufactured in fused silica. In this way, long-range hopping defined by eq. (5) with time-reversal symmetry breaking can be emulated without resorting to synthetic gauge fields nor special modulation of parameters [33], making the method rather simple.

The profile of the phase mask at the plane $\gamma$ determines the external potential $U(x)$, and can be designed to emulate lattice defects or disorder. Note that the equivalent spatial period $a$ of the metacrystal in real space $x$ is given by $a = \lambda f / A$. Since the cavity operates in a self-imaging condition and the phase mask and grating act on the $x$ spatial coordinate solely, in the orthogonal $y$ transverse coordinate the beam profile is not affected by propagation inside the resonator.

In an optical experiment, wave packet dynamics and robustness against back reflections can be observed by considering the freely decaying beam dynamics in the passive resonator initially loaded with a pulsed Gaussian beam $E(x, t) = F(t) G(x)$, which is injected through one of the four cavity mirrors. Assuming that the carrier frequency of the injected beam is in resonance with one of the cavity axial modes, the map (6) is replaced by the following one:

$$\psi_{m+1}(x) = t_2(t_1 \left( \frac{\lambda f}{2\pi} \frac{\partial}{\partial x} \right) \psi_m(x) + \sqrt{T} E_m(x) - \frac{T}{2} \psi_m(x), \quad (9)$$

where $T \ll 1$ is the transmittance of the coupling mirror, and $E_m(x) = F(m) G(x)$ is the spatial profile of the injected beam at plane $\gamma$ and at the $m$-th round trip. The free decay of light in the cavity, following the pulse excitation with the external beam, basically emulates wave packet evolution in a metacrystal. In an experiment, transverse light evolution at successive transits in the cavity can be detected by time-resolved beam profile measurements using a gated camera, as demonstrated, e.g., in refs. [39,40]. As an example, fig. 4 shows the beam evolution of the intracavity field at plane $\gamma$, as obtained by numerical integration of the map (9), assuming either a siniusoidal grating profile $\varphi_1(x) = -J \cos(2\pi x / A)$, or a sawtooth grating profile with the same period $A = 30 \mu m$ and amplitude $J = 0.15$. Parameter values used in the simulations are $\lambda = 633 \mu m$, $f = 2 \mu m$, and $T = 2 \%$, corresponding to a spatial period $a = \lambda f / A = 422 \mu m$ of the metacrystal. The injected field is a pulsed and tilted Gaussian beam with transverse profile (at plane $\gamma$) $G(x) = \exp[-x^2/w^2 + i k_{0x} x]$ ($w = 800 \mu m$, transverse momentum $k_{0x} = \pi / (2a)$) and pulse envelope $F(t) = \exp[-(t - t_0)^2/r^2]$, $t_0 = 20$, $\tau = 10$ in units of the round trip time $T_R = 8 f / c \approx 0.53$ ns. The external potential $U(x)$ is assumed to be either a localized defect of the lattice ($U(x) = -U_0 \exp[-(x - d)^2/s^2]$), $U_0 = 0.2$, $d = 1600 \mu m$, $s = 600 \mu m$), fig. 4(b); or a random potential ($U(x)$ random variable with uniform distribution in the range $(-0.1, 0.1)$), fig. 4(c). The freely evolving optical beam, in the absence of lattice defects and disorder, is shown for comparison in fig. 4(a). The behavior of the intracavity power before and after injection with the external beam is also shown in fig. 4(d). The decay of the optical power in the cavity after initial pulse excitation is
due to cavity losses at the output coupler. The numerical results clearly indicate that, after excitation of the passive cavity with the tilted external pulsed beam, transverse beam propagation is robust in the case of the sawtooth phase grating, while back reflections are well visible in the case of the sinusoidal phase grating according to the scenario of fig. 2. It is worth mentioning that scattering of the propagating wave packet with the disordered or defective phase mask in the near-field plane $\gamma$ creates momentum components $k_i$ which differ from the initial transverse momentum $k_{0x}$ by integer multiplies than $2\pi/a$, i.e., $k_i = k_{0x} + 2\pi l/a$ ($l = \pm 1, \pm 2, \ldots$); see the schematic picture shown in fig. 1(c). Backscattering is not observed because all scattered wave number components $k_i$ have the same group velocity. However, in the near-field plane the superposition of the various plane-wave components $\sim \exp(\imath k_{lx} x)$ yields a characteristic interference pattern, the shortest spatial period of the pattern being ultimately limited by aperture effects in the far-field plane $\gamma_F$ or by diffraction effects arising from slight deviations of the cavity from exact self-imaging configuration. Such an interference pattern is clearly observed as rapid oscillations of the field intensity distribution in the pseudo color maps of fig. 4, right panels. Note that, if the phase grating at plane $\gamma_F$ were replaced by a prism, i.e., in the limit $A \to \infty$ and $a \to 0$, a linear dispersion curve $E(k) \propto k$, rather than a periodic sawtooth-shaped curve, would be implemented. In this case the above-mentioned interference pattern will not be observed since, as $a \to 0$, scattering does not occur at all, requiring an infinite spectral momentum component in the Fourier spectrum of the scattering potential.

As a final comment, it should be noticed that the proposed optical set-up to emulate robust unidirectional transport in a metacrystal only involves refractive optical elements that do not break time reversal of the optical structure. Nevertheless, the transverse beam dynamics in the resonator can emulate an effective Hamiltonian in which time reversal is broken, yielding unidirectional transport along the transverse spatial coordinate $x$. How can we reconcile these two seemingly contradictory facts? It is clear from the optical implementation of the Schrödinger equation that breaking of the time-reversal symmetry $E(-k) \neq E(k)$ of the band dispersion curve is obtained in the far-field plane $\gamma_F$ by suitably tailoring the phase profile $\varphi(x)$ of the grating, which defines the kinetic-energy operator in the Schrödinger equation according to eq. (8). Yet, in the optical resonator with only reciprocal elements (lenses and phase masks) it is true that time reversal is not broken. Time reversal of the beam dynamics in the resonator corresponds to reversing the circulation direction of light in the ring (either clockwise or counterclockwise). Correspondingly, the direction of the transverse motion of the beam in the $x$ plane would be reversed. Therefore, the optical structure remains reciprocal and if we reverse the circulation direction of light in the ring (time reversal) the unidirectional transverse motion of the beam can be reversed. It is just because we restrict our analysis to unidirectional circulation in the ring that the dynamics seems apparently to break time reversal. Unidirectional circulation in the ring is forced by external beam injection, that determines the circulation direction of light.

Conclusions. — Robust unidirectional transport can occur in a wide class of 1D Hermitian metacrystals with engineered lattice band. However, long-range hopping and broken time-reversal symmetry are needed to implement such metacrystals. While 1D matter-wave systems, such as spin chains or trapped ions, could be a potential platform to implement long-range hopping and synthetic gauge fields [33], their experimental realization remains challenging. Here we have shown that transverse beam dynamics in a self-imaging optical resonator with a phase grating provides a rather simple and experimentally accessible system in optics to implement a metacrystal, in which time-reversal symmetry breaking and long-range hopping are readily realized without the need for synthetic gauge fields nor special modulation of parameters. The present results disclose an important strategy to realize robust transport in 1D lattices, without resorting to adiabatic (Thouless) pumping [24,25] or non-Hermitian transport [31], and suggest resonator optics as a suitable platform to implement a metacrystal. The underlying analysis also suggests that robust unidirectional transport is expected to arise in a rather general class of synthetic quantum systems, i.e., not necessarily
crystals, with engineered kinetic energy $E = E(p)$ such that $E(p)$ is made of intervals of monotonously increasing (or decreasing) functions of $p$, connected each other by rapid jumps.

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