A robust one-step catalytic machine for high fidelity anti-cloning and W-state generation in a multi-qubit system

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We propose a physically realizable machine which can either generate multiparticle W-like states, or implement high fidelity $1 \rightarrow M$ (where $M = 1, 2, \cdots, \infty$) anti-cloning of an arbitrary qubit state, in a single step. Moreover this universal machine acts as a catalyst in that it is unchanged after either procedure, effectively resetting itself for its next operation. It also possesses an inherent immunity to decoherence. Most importantly in terms of practical multi-party quantum communication, the machine’s robustness in the presence of decoherence actually increases as the number of qubits $M$ increases.

The mathematical foundations of quantum mechanics yield two remarkable consequences in terms of what is possible in our Universe, and what isn’t. First, the linearity of quantum mechanics implies that it is impossible to make a perfect copy of an arbitrary quantum (qubit) state, no matter how ingenious the experimental scheme. Second, the unitarity of quantum mechanics implies that there is no quantum device, no matter how well-built, which can perfectly transform an arbitrary qubit state into its orthogonal complement. Despite these fundamental ‘laws of Nature’, we now know that cloning and the complementing of qubits can still be carried out with reasonably high fidelity. There is even the suggestion that these two processes might actually be closely related. In fact, recent experiments have demonstrated that optimal $1 \rightarrow 2$ cloning (i.e. partially copying a quantum state from one qubit onto two target qubits) and the universal-NOT operation on photon polarization states, can both be performed using the same unitary transformation. Indeed, it has been suggested that a combination of copying and complementing could lead to optimal entangling transformations. However, the connections between these two quantum processes are still not well understood either in theoretical or practical terms. In a seemingly unrelated development, researchers interested in building quantum information machines have begun to propose experimental schemes using ‘always-on’ Hamiltonian interaction terms, in order to avoid the need for switching on and off multiple quantum gates. However such schemes invariably assume that specific two-body, nearest-neighbor interactions can be engineered in some particular qubit geometry (e.g. chain) despite the fact that nanostructures, for example, may have long-range interactions due to residual electrostatic potentials.

In this paper, we bring together these two seemingly separate lines of research by proposing a multiqubit-cavity scheme in which the same unitary transformation can be used to produce multiqubit W-entangled states and high (in some cases optimal) fidelity $1 \rightarrow M$ anti-cloning, where $M$ is any arbitrary number of qubits.

As a result, our work provides a concrete connection between copying, complementing and entangling operations. From a practical point of view, the implementation of our scheme offers a number of outstanding advantages and features. First, the cavity acts as a catalyst in that its state is unchanged after either procedure – in short, our machine acts as its own reset button. Second, the machine has an inherent immunity to decoherence effects. In particular, our calculations show that entangling and anti-cloning operations become increasingly robust as the number of qubits increases, in contrast to typical quantum information schemes whose performance would deteriorate as the number of degrees-of-freedom increases. Third, our machine avoids the need for carefully engineered nearest-neighbor interactions, multiple cavities and/or gate operations. Moreover, our multiqubit-cavity machine could be built using current atom- or ion-cavity technology, or next-generation quantum-dot or SQUID-cavity technology.

The Hamiltonian for the $M$-qubit-plus-cavity system in the interaction picture and rotating-wave approximation ($\hbar = 1$) is

$$H_I = \sum_{j=1}^{M} \gamma_j \{ a^d_1 \sigma_j^- + \sigma_j^+ a \} ,$$

where $\sigma_j^+ = |1_j\rangle \langle 0_j|$, $\sigma_j^- = |0_j\rangle \langle 1_j|$ with $|1_j\rangle$ and $|0_j\rangle$ being the excited and ground states of the $j$’th qubit. Here $a^d$ and $a$ are cavity-photon creation and annihilation operators while $\gamma_j$ are the set of (in general unequal) qubit-cavity couplings. Since $[H_I,\mathcal{N}] = 0$ where $\mathcal{N} = a^d a + \sum_{j=1}^{M} \sigma_j^+ \sigma_j^-$ is the excitation number operator, the dynamics is separable into subspaces having a prescribed eigenvalue $N$ of $\mathcal{N}$. In particular, in the subspace with $N = 0$ there is only one state $|\phi_0\rangle = |0_1,0_2,0_3,\cdots,0_M;0\rangle$ while in the $N = 1$ subspace, the basis states are

$$|\phi_1\rangle = |1_1,0_2,0_3,\cdots,0_M;0\rangle = |Q_1\rangle \otimes |0\rangle$$

$$|\phi_2\rangle = |0_1,1_2,0_3,\cdots,0_M;0\rangle = |Q_2\rangle \otimes |0\rangle$$

$$\vdots = \vdots = \vdots$$
\[ |\phi_j\rangle = |0_1, 0_2 \cdots 1_j, \cdots 0_M; 0\rangle = |Q_j\rangle \otimes |0\rangle \] (2)
\[ |\phi_{M+1}\rangle = |0_1, 0_2 \cdots 0_M, 1\rangle \]

where the last label in each ket denotes the photon number in the cavity. In this \( N = 1 \) subspace, the system’s state at time \( t \) is \( |\Psi(t)\rangle = \hat{U}(t, 0)|\Psi(0)\rangle \) where \( \hat{U}(t, 0) \) in the basis of states \( \{|\phi_1\rangle \cdots |\phi_{M+1}\rangle\} \) becomes

\[
\hat{U}(t, 0) = \begin{pmatrix}
1 - 2\gamma_1^2\beta & -2\gamma_1\gamma_2^2\beta & \cdots & -2\gamma_1\gamma_M\beta & -i\gamma_1\sin(\omega t)/\omega \\
-2\gamma_1\gamma_2\beta & 1 - 2\gamma_2^2\beta & \cdots & -2\gamma_2\gamma_M\beta & -i\gamma_2\sin(\omega t)/\omega \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-2\gamma_M\gamma_1\beta & 2\gamma_M\gamma_2\beta & \cdots & 1 - 2\gamma_M^2\beta & -i\gamma_M\sin(\omega t)/\omega \\
-i\gamma_1\sin(\omega t)/\omega & -i\gamma_2\sin(\omega t)/\omega & \cdots & -i\gamma_M\sin(\omega t)/\omega & \cos(\omega t)
\end{pmatrix}.
\] (3)

The effective collective Rabi frequency of the \( M \) qubits is \( \omega^2 = \sum_{j=1}^{M} \gamma_j^2 \), and \( \beta = \sin^2(\omega t/2)/\omega^2 \). We now show how we build our machine using this temporal evolution, evaluated over a specially chosen time interval.

Consider an initial product state where one of the qubits (e.g. \( j = 1 \)) is in a coherent superposition \( |\phi_q(0)\rangle = \sin(\theta/2)|0_1\rangle + e^{i\alpha}\cos(\theta/2)|1_1\rangle \) with the others unexcited:

\[ |\Psi(0)\rangle = |\phi_q(0)\rangle \otimes |0_2, \cdots 0_M; 0\rangle = \sin(\theta/2)|\phi_0\rangle + e^{i\alpha}\cos(\theta/2)|\phi_1\rangle \] (4)

Using Eq. (4) yields

\[ |\Psi(t)\rangle = \sin(\theta/2)|\phi_0\rangle + e^{i\alpha}\cos(\theta/2)|\phi_1(t)\rangle \] (5)

with

\[ |\phi_1(t)\rangle = \sum_{j=1}^{M} U_j(t)|Q_j\rangle \otimes |0\rangle - i\gamma_1\sin(\omega t)/\omega |\phi_{M+1}\rangle \] (6)

When \( \omega t = m\pi \equiv \omega\tau^* \) \((m \) odd\) a vacuum trapping state condition is achieved: the cavity state is unchanged overall and becomes fully separable from the multiqubit sub-system. However, its catalytic action has induced entanglement into the initially unentangled multiqubit subsystem. Because of the cavity’s inertness at \( t = \tau^* \), we drop the cavity state notation from now on.

Consider the following two specific examples: (i) \( \theta = 0 \), which will yield one-step W-state generation; (ii) \( \theta = \pi/2 \), which will yield optimal quantum anti-cloning.

(i) Using \( \theta = 0 \) yields

\[ |\Psi(\tau^*)\rangle = (1 - 2\gamma_1^2/\omega^2)|Q_1\rangle - (2\gamma_1/\omega^2)\sum_{j=2}^{M} \gamma_j|Q_j\rangle \] (7)

In general, an \( M \)-qubit W-state cannot be generated using identical couplings \( \gamma_i \equiv \gamma \). However for non-identical couplings, the qubit-exchange symmetry is broken thereby allowing control over the degree of entanglement and the final state symmetry (see Ref. [12]).

(ii) Using \( \theta = \pi/2 \) enables us to anti-clone (i.e. copy the orthogonal complement) of the state of qubit-1 (i.e. the input qubit) to the target qubits (i.e. the \( M - 1 \) qubits initially in state |0\rangle). Since the initial state of qubit-1 is in the equatorial plane of the Bloch sphere (see Eq.(4)), we will call this process Phase-Covariant Anti-Cloning (PCAC) in analogy with phase-covariant cloning (PCC) [13]. An ideal anti-cloning process is defined as

\[ |q\rangle_{a}|0\rangle_b|D\rangle_{in} \rightarrow |q^+\rangle_a|q^-\rangle_b|\bar{D}\rangle_{out} \] (11)

for \( M = 2 \). Suppose \( \gamma_1 \neq \gamma_j = \gamma \) for all \( j > 1 \), and define \( r = \gamma_1/\gamma \). The collective qubit frequency is \( \omega = \gamma(r^2 + M - 1)/2 \) and

\[ |\Psi(\tau^*)\rangle = a_1(\tau^*)|Q_1\rangle + a(\tau^*)\sum_{j=2}^{M} |Q_j\rangle \] (8)

where

\[ a_1(\tau^*) = \frac{M - 1 - r^2}{M - 1 + r^2} \quad a(\tau^*) = \frac{-2r}{M - 1 + r^2} \] (9)

Two W states of \( M \) qubits can now be generated for \( a_1(\tau^*) = \pm a(\tau^*) \), yielding an optimal coupling ratio \( r_{W^*} = \sqrt{M \pm 1} \). Here \( r_{W^*} \) correspond to symmetric and antisymmetric states with respect to exchange of qubit-1 with any other. The corresponding state is

\[ |\Psi(\tau^*)\rangle = |W_{M}^\pm\rangle = \frac{e^{i\pi}}{\sqrt{M}} \left[ \pm|Q_1\rangle + \sum_{j=2}^{M} |Q_j\rangle \right] \] (10)

For \( M = 4 \), both \( r = 1 \) and \( r = 3 \) produce a fully symmetric W state. However in the many-qubit limit \( M \to \infty \), it is only for non-identical couplings \( (r_{W^*} \simeq \sqrt{M}) \) that we can generate a multi-qubit W entangled state. (N.B. A W-state is of interest for quantum information protocols since the excitation has identical probabilities of being found on any of the qubits). We note that for \( M \geq 3 \), a fully symmetric W state of \( M - 1 \) qubits can also be obtained when \( a_1(\tau^*) = 0 \), yielding \( r_{W'} = \sqrt{M - 1} \). The initial excitation gets transferred to, and shared among, the remaining \( M - 1 \) qubits.
where \(|q\rangle_a\) is the initial state of the input qubit, \(|0\rangle_a\) is the initial state of a target qubit, \(|q\rangle q^+\rangle = 0\), and \(|D\rangle_{in}\) and \(|D\rangle_{out}\) are the input and output states of the copying device. Ref.\([12]\) showed that the optimal fidelity for \(1 \rightarrow 2\) PCC is \(F^{opt} = \frac{1}{2}[1 + \frac{1}{\sqrt{M}}]\). Here we demonstrate that the fidelity of our \(1 \rightarrow 2\) anti-cloning equals this optimal value. We also show that there are two protocols to achieve this process, the main difference being the final state of the input qubit. For an arbitrary number of output qubits \(M\) with asymmetric couplings, we find that the fidelity of the anti-cloning operation is comparable to that obtained for a XY spin star network \([8]\) and reaches larger values than for the case of identical couplings. The state of the system is now \(|\Psi(t)\rangle = \frac{1}{\sqrt{2}}[|\phi_0\rangle + e^{i\alpha}|\phi_1(t)\rangle]\) and the reduced density matrix of the \(j^{th}\) qubit reads

\[
\rho_j(t) = \frac{1}{2} \left[ (2 - |U_j(t)|^2) |0\rangle\langle 0| + |U_j(t)|^2 |1\rangle\langle 1| \right] U_j(t) (e^{-i\alpha}|1\rangle\langle 1| + e^{i\alpha}|1\rangle\langle 0|) \]

(12)

The fidelity of copying \(|\tilde{q}\rangle\) is \(\mathcal{F}_j(t) = \langle \tilde{q}|\rho_j(t)|\tilde{q}\rangle = \frac{1}{2}(1 + U_j(t) \cos(\alpha - \mu))\). For a target qubit, at \(t = \tau^*\),

\[
\mathcal{F}_{j > 1}(\tau^*) = \frac{1}{2} \left[ 1 - 2\gamma_1 \gamma_j \cos(\alpha - \mu)/\omega^2 \right]
\]

(13)

hence the fidelity is greater than 1/2 when the state that has been copied corresponds to the orthogonal complement of the input state (anti-cloning), i.e. \(\alpha - \mu = \pi\). Figure 1 shows the fidelity of a target qubit as well as for the input qubit (inset) as a function of the number of qubits. For coupling ratio \(r_{W}^\ast\), the input qubit finishes entangled with the target qubits, i.e. \(|\Psi(\tau^*)\rangle = \frac{1}{\sqrt{2}}[|\phi_0\rangle + e^{i\alpha} |W_j\rangle]\) such that the fidelity of the input qubit (see inset) equals the fidelity of the target qubits. Hence, we obtain \(M\) outputs (including the input qubit) with fidelity \(\mathcal{F}^+ = \frac{1}{2}[1 + \frac{1}{\sqrt{M}}] = \mathcal{F}_{1\rightarrow 2}^+\). For \(M = 2\), we obtain \(\mathcal{F}_{1\rightarrow 2}^+ = \frac{1}{2}[1 + \frac{1}{\sqrt{2}}]\) which equals the optimal value for the \(1 \rightarrow 2\) PCC \([13]\). Interestingly, such optimal transformation combines two operations in one-step: complementing the original qubit’s state and copying. We also note that this optimal fidelity is achieved for the same conditions under which two-qubit maximally entangled states were found\([12]\), hence establishing a direct connection between optimal anti-cloning and maximal entanglement. For \(r_{W}^\ast\), the fidelity of the target qubits equals \(\mathcal{F}^+\) but the fidelity of the input qubit is always less than 1/2, i.e. \(\mathcal{F}^- = \frac{1}{2}[1 - \frac{1}{\sqrt{M}}]\) which is undesirable for a single qubit NOT operation. For \(r_{W}^\ast = \sqrt{M - 1}\), we obtain \(M - 1\) outputs with fidelity \(\mathcal{F}^{sep} = \frac{1}{2}[1 + \frac{1}{\sqrt{M}}]\) while the fidelity of the input qubit equals 1/2 irrespective of the number of qubits (see inset). This is because the input qubit ends in its ground state and separated from the rest, i.e. \(|\Psi(\tau^*)\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}[|\phi_0\rangle + e^{i\alpha} |W_{M - 1}\rangle]\) with \(|\phi_0\rangle = |0_2, 0_3, \ldots, 0_M\rangle\). For \(M = 3\) we obtain \(1 \rightarrow 2\) anti-cloning with optimal fidelity \(\mathcal{F}^{opt}\) for the target qubits. In general, \(\mathcal{F}^+(M) = \mathcal{F}^{sep}(M + 1)\) which means that there exist two protocols for obtaining \(M\) outputs with high fidelity: (i) \(M\) qubits and \(r = r_{W}^\ast\), and (ii) \(M + 1\) qubits and \(r = r_{W}^\ast\). The main difference between these two protocols is the time operation \(\tau^* = \pi/\omega\): it is shorter for the \(r_{W}^\ast\) case since \(\omega(r_{W}^\ast) > \omega(r_{W}^\ast)\). For a large number of outputs, this difference is negligible since \(r_{W} \sim r_{W}^\ast\). Interestingly, the operation time decreases with the number of anti-clones, implying that the protocols are robust in the presence of decoherence. We confirm this robustness in more detail below. In both cases \(r_{W}^\ast\) and \(r_{W}^\ast\), the fidelity of the one-step anti-cloning procedure is comparable with that reported for cloning operations using a XY spin network \([8]\) since it depends on the number of outputs \(M\) as \(1/\sqrt{M}\). In the case of identical couplings, the fidelity of the target qubits is \(\mathcal{F}^{iden} = \frac{1}{2}[1 + \frac{1}{\sqrt{M} \gamma_j}]\) which is always less than \(\mathcal{F}^{sep}\) as well as being less than \(\mathcal{F}^\pm\) for \(M > 4\). This behaviour is comparable with that of a Heisenberg spin network since it depends on the number of outputs \(M\) as \(1/M\) \([8]\).

Decoherence would take place through two main channels: qubit dipole decay at rate \(\Gamma\), and cavity decay with rate \(\kappa\). A single trajectory in the quantum jump model \([13]\) is well-suited to evaluate the effects on the fidelity at the trapping time. We suppose that the photon decays are continuously monitored, and that the single trajectory is specified by the evolution of the system condi-
For the situation in which $\Gamma = \kappa$, the fidelity $F_r$ equals unity for any value of $r$ and at any time. This is due to the fact that the non-Hermitian operator accounting for the dissipative interaction in $\tilde{H}$ is just the excitation number-operator (i.e. $-i\Gamma\hat{N}$) hence the conditional state becomes $|\Psi_{\text{cond}}(t)\rangle = e^{-i\Gamma t}|\Psi(t)\rangle$ and $P(0, t) = e^{-2\Gamma t}$. Therefore the decoherence sources can effectively be combined to produce a negligible net effect. This feature becomes more prominent as the number of qubits increases, as can be seen in Figure 2. The state fidelity $F_r$ with $\Gamma \neq \kappa$, is shown in the two cases in which it is possible to either generate symmetric $W$ entangled states or to obtain $M$ anti-clones with high fidelity: $r_W^+$ and $r_W^-$. In both situations the state fidelity moves closer to unity as the number of qubits increases, since the time interval required to achieve the desired state becomes shorter. It is also worth noting that higher values of fidelity are obtained for the symmetric case $r_W^-$ than in the $r_W^+$ case. This effect can be better appreciated for a small number of qubits. We note that the probability of not detecting a photon in $(0, \tau_c)$ does not fall below 0.97 for $M = 2$ and becomes even closer to unity for higher numbers of qubits (see Fig. 2 inset). This again shows how efficient our protocols for entangling/anti-cloning are, and concludes the justification of the claims in this paper.

In summary, we have shown how asymmetric cavity-qubit couplings can be exploited to perform very robust, high-fidelity entangling and anti-cloning operations, in a physically realizable multi-qubit system.

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