Total intrinsic spin
for plane gravity waves

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March 28, 1997

Abstract

A quantity which measures total intrinsic spin along the z axis is constructed for planar gravity (fields dependent on z and t only), in both the Ashtekar complex connection formalism and in geometrodynamics. The total spin is conserved but (surprisingly) is not a surface term. This constant of the motion coincides with one of four observables previously discovered by Husain and Smolin. Two more of those observables can be interpreted physically as raising and lowering operators for total spin.

PACS categories: 04.60, 04.30
I Introduction

This paper derives a constant of the motion which gives the total spin angular momentum along the z axis, for planar symmetric gravitational waves moving in the directions ±z. This constant of the motion was derived as part of an ongoing investigation [1, 2, 3] of the properties of the complex connection formalism proposed by Ashtekar [4], and the derivation is carried out within that formalism; however, the final result will also be stated in terms of the geometrodynamical variables ($g_{ij}$, $\pi^{ij}$).

In four spacetime dimensions, when the manifold is $\mathbb{R} \times \Sigma$ and the spatial slice $\Sigma$ is asymptotically flat at infinity, the expression for total angular momentum is well-known; it is a two-dimensional integral over the surface at spatial infinity [5]. In contrast, conserved quantities in special relativity are typically three-dimensional integrals over the volume of $\Sigma$. The intuitive reason why conserved quantities in general relativity are associated with surfaces, rather than volumes, is that conserved quantities are associated with coordinate transformations, via Noether’s Theorem; and in general relativity coordinate transformations within the volume have little physical meaning because of the diffeomorphism invariance of the theory. The transformations at the surface must preserve the asymptotic flatness of the theory, and are reminiscent of transformations in special relativity; consequently the associated conserved quantities depend only on degrees of freedom at the surface. In the planar case (effectively one-dimensional because there is no x,y dependence) a “volume” integral is a one-dimensional integral along z, from a left boundary $z_l$ to a right boundary $z_r$. The “surface” is the two points $z_l$ and $z_r$. Surprisingly, the total spin momentum in the planar case has the form of a volume integral.

In certain respects, however, the planar case resembles special relativity more than general relativity. In practice one fixes the variables x,y (also the variables X,Y in the local Lorentz frame) so that the symmetry in the xy plane is manifest; consequently tensors which have only x, y, X, and Y indices ("transverse" tensors) are left invariant by the surviving diffeomorphisms and the local Lorentz transformations which have not been fixed. The transverse tensors therefore resemble scalar fields in special relativity, rather
than tensors in general relativity. It turns out that the expression for spin angular momentum contains only transverse tensors.

The remainder of this section introduces the Ashtekar notation and writes out the expression for spin angular momentum in both Ashtekar and geometrodynamical language. Section II derives this expression; section III examines the connection between spin angular momentum and four conserved quantities previously derived by Husain and Smolin [6].

The basic variables of the Ashtekar approach are an inverse densitized triad $\tilde{E}^a_A$ and a complex $\text{SU}(2)$ connection $A^A_a$.

$$\tilde{E}^a_A = e e^a_A;$$ (1)

$$[\tilde{E}^a_A, A^B_B] = \hbar \delta(x - x') \delta^B_A \delta^a_b.$$(2)

All quantities are three-dimensional unless explicitly indicated otherwise. Upper case indices $A, B, ...$, $I, J, K, ...$ denote local Lorentz indices ("internal" $\text{SU}(2)$ indices) ranging over $X, Y, Z$ only. Lower case indices $a, b, ..., i, j, ...$ are also three-dimensional and denote global coordinates on the three-manifold. The quantity $e$ is the determinant of the 3x3 spatial subblock of the tetrad matrix $e^A_a$; similarly $e^a_A$ is an inverse tetrad. I use Levi-Civita symbols of various dimensions: $\epsilon_{TXYZ} = \epsilon_{XYZ} = \epsilon_{XY} = +1$.

The planar symmetry (two spacelike commuting Killing vectors, $\partial_x$ and $\partial_y$ in appropriate coordinates) allows Husain and Smolin [6] to solve and eliminate four constraints (the $x$ and $y$ vector constraint and the $X$ and $Y$ Gauss constraint) and correspondingly eliminate four pairs of $(\tilde{E}^a_A, A^A_a)$ components. The 3x3 $\tilde{E}^a_A$ matrix then assumes a block diagonal form, with one 1x1 subblock occupied by $\tilde{E}^Z_Z$ plus one 2x2 subblock which contains all the “transverse” $\tilde{E}^a_A$, those with $a = x, y$ and $A = X, Y$. The 3x3 matrix of connections $A^A_a$ assumes a similar block diagonal form. None of the surviving fields depends on $x$ or $y$.

In the Ashtekar notation the total spin angular momentum around the $z$ axis is given by the quantity

$$L_z = i \int dz [\tilde{E}^y_T (A^1_x - ReA^1_x) - (x \leftrightarrow y)]$$ (3)

The integral is over the entire wave packet, that is from $z_l$ to $z_r$. I have chosen the notation $L_z$ rather than $L_z$ deliberately, since $L_z$
suggests a covariant vector, whereas eq. (3) defines an invariant. The corresponding integral in geometrodynamical notation is

\[ L_Z = 2 \int dz [g_{xj} \pi^{yj} - g_{yj} \pi^{xj}], \]  

(4)

where \( g_{ij} \) is the three-metric and \( \pi^{ij} \) its conjugate momentum.

The integrals in eqs. (3) and (4) range over \( z_l \leq z \leq z_r \), where the left and right boundary points \( z_l \) and \( z_r \) are a finite distance from the origin. In three spatial dimensions it is usual to place the boundary surface at spatial infinity. Bringing the surface at infinity in to finite points is a major change, because at infinity the metric goes over to flat space, and flat space is a considerable simplification. In the present case (effectively one dimensional because of the planar symmetry) the space does not become flat at \( z \) goes to infinity, and nothing is lost by considering an arbitrary location for the boundary surface. The result that the space does not become flat as \( z \) goes to infinity was established in paper II. Note that this result agrees with one’s intuition from Newtonian gravity, where the potential in one spatial dimension due to a bounded source does not fall off, but grows as \( z \) at large \( z \).

The planar metrics considered here and in previous work \([1, 2, 3]\) admit two null vectors \( k \) and \( l \) which have the right hypersurface orthogonality properties to be the propagation vectors for right-moving \( (k) \) and left-moving \( (l) \) gravitational waves along the \( z \) axis. It is possible to choose \( z \) and \( t \) coordinates so that the equations defining these hypersurfaces become especially simple: \( u = (ct - z)/\sqrt{2} = \text{constant} \) and \( v = (ct + z)/\sqrt{2} = \text{constant} \) \([7]\). I have not used these special coordinates in the proof, and one might ask whether the angular momentum exists for more general metrics which are planar symmetric but do not admit hypersurface orthogonal null congruences \([8]\). I believe the answer is yes, but have not checked whether the gauge fixing employed by Husain and Smolin \([6]\) continues to go through for the more general cases.

I have also used the assumption that the gravitational wave is confined to a wave packet which lies entirely inside the boundary and has not reached the boundary points \( z_l, z_r \). This assumption is used in a subtle way. The derivation in section \([4]\) uses the Noether procedure, which in turn assumes that a Lagrangian formulation
exists. When the space is non-compact, as it is here (the z axis is the entire real line, not a circle), then the Lagrangian must contain surface terms, or the Euler-Lagrange procedure is not well-defined \[2, 5\]. In reference \[2\] I constructed the appropriate surface term, assuming that certain degrees of freedom are absent at the boundaries $z_l, z_r$. These degrees of freedom produce the transverse displacements of test particles characteristic of gravitational radiation. In the language of reference \[2\], the fields B and W were assumed to vanish at the boundaries. (The assumption that B and W vanish at the boundaries may be restated in a more covariant language: the Weyl tensor has components which produce transverse deviations of geodesics, and these components vanish at boundaries.) I was unable to construct a suitable surface term, therefore was unable to define a Lagrangian, for B or W non-zero at the boundaries.

II Derivation

To prove eq. (3), I use the Noether procedure, with some modifications \[5\]. In the usual, three-dimensional case, one exploits the invariance of the action under general coordinate transformations $x^\mu \rightarrow x^\mu + \xi^\mu$, where $\xi^\mu$ at infinity must reduce to a rotation around $z$:

$$\xi^b = \epsilon_{bac}\delta \phi^c x^c;$$

$$b, c = x, y;$$

$$\delta \phi^z \rightarrow \text{constant.} \quad (5)$$

In the present, one-dimensional case, $\delta \phi^z$ must be a rigid rotation, that is, a constant for all $z$, or the transformation generated by $\xi$ will destroy the gauge conditions on the fields $A^A$. (The matrix $A^A_a$ is block diagonal, with off-diagonal elements $A^Z_x = A^Z_y = A^X_z = A^Y_z = 0$.) The change in $A^I_i$ is given by (minus one times) the Lie derivative.

$$\delta A^I_i = -\xi^b \partial_b A^I_i - (\partial_i \xi^b) A^I_b. \quad (6)$$

If $\xi$ depends on $z$, say, then the second term on the right will allow $\delta A^I_z$ to be non-zero for $I = X, Y$, which violates the gauge conditions. The first term on the right cannot cancel the second, because the
first term vanishes: \( b = x \) or \( y \), and the fields do not depend on \( x \) or \( y \). Thus \( \xi \) must be a constant for all \( z \). This result is consistent with the observation made in section \([\text{I}]\) the \( x,y \) sector of the theory resembles special relativity rather than general relativity.

Now continue with the usual Noether procedure. The variation in the action is (suppressing the obvious indices for simplicity)

\[
\delta S = \int \! dtdz \left[ i \tilde{E} \delta \dot{A} + i \delta \tilde{E} \dot{A} - (\delta H / \delta \tilde{E}) \delta \tilde{E} - (\delta H / \delta A) \delta A \right]. \tag{7}
\]

Because of the classical equations of motion, the second term in the square brackets cancels the third, while the fourth term provides an \( i \dot{\tilde{E}} \delta A \) term which combines with the first term. Also, because the action is invariant under the transformation generated by eq. (5), \( \delta S = 0 \). Eq. (7) becomes

\[
0 = \int \! dtdz \partial_t \left[ i \tilde{E} \delta A \right]
= \int \! dtdz \partial_t \left[ i \tilde{E}_t^a (-A_t^b \partial_a \xi^b) \right]
= \int \! dz [i \tilde{E}_t^a (-A_t^b \epsilon_{bza})]_{t_1}^{t_2} \delta \phi^z
=: [L_Z(t_2) - L_Z(t_1)] \delta \phi^a. \tag{8}
\]

On the second and third lines I have used eqs. (5) and (6), with \( i = x,y,z \) replaced by \( a = x,y \) only.

The quantity \( L_Z \) on the final line is not quite the desired constant of the motion, eq. (3): since the original action \( S \) was complex, the \( L_Z \) in eq. (8) is also complex. I wish to argue that I can and should drop the imaginary part of \( L_Z \). Normally the complex action is assumed to depend on both the connection \( A \) and the densitized triad \( \tilde{E} \), which are varied as independent fields. Before the 3+1 reduction, the complex action depends on the four-dimensional spin connection \( \omega^I_1 \) and the tetrad \( e^I_i \), also varied as independent fields. So long as \( \omega \) and \( e \) are treated as independent fields, the imaginary part of \( S \) is non-trivial. If one invokes the classical equations of motion, however, these give the usual relation \( \omega = \omega(e) \) between tetrad and connection. Once this relation is inserted back into the action, \( \text{Im} \ S \) vanishes trivially because of the Bianchi identity. In the Noether procedure I assume the classical equations of motion.
Therefore Im S vanishes, and I should discard the imaginary part of the constant of motion in eq. (3). This leaves me with the constant of the motion given at eq. (3).

I should perhaps point out that in the usual, three-dimensional derivations of conserved quantities, the $\partial_i$ and $\partial_b$ in eq. (6) are manipulated until the integrand in eq. (8) becomes a total derivative $d^3x \partial_i [\cdots]$. Integration by parts with respect to $dx_i$ then leads to a constant of the motion in the form of a surface term. This procedure is not possible here because $i = x, y$ only, and there are no $dx$ or $dy$ integrations.

I have tacitly dropped an “orbital” contribution to $L^Z$, so that this quantity is pure spin angular momentum. At eq. (4) I dropped the $\xi^b \partial_b$ term because $b = x, y$ only. From eq. (6) $\xi^b \partial_b$ is essentially $(r \times \nabla)_z$, the orbital angular momentum.

The expression eq. (3) can be quantized readily. In references [2, 3] I constructed solutions \( \psi \) which were annihilated by all the constraints and which depended only on the transverse fields $\hat{E}_a^A$ and the connection $A^Z_z$. In order to represent the commutation relations eq. (2) correctly, one can make the remaining fields into functional derivatives in the standard manner,

\[
\begin{align*}
\hat{E}_Z^a &\to -\hbar \delta / \delta A_z^Z; \\
A_a^A &\to +\hbar \delta / \delta \hat{E}_a^A,
\end{align*}
\]

(for $a = x, y$ and $A = X, Y$),

then order functional derivatives to the right in eq. (3).

From the discussion so far, it is not obvious that $L^Z$ is the intrinsic spin operator for a helicity two field. The pattern of $x$ and $y$ indices in eq. (3) looks like a spin one cross product. To clarify the helicity content, it is helpful to express $L^Z$ in terms of fields which are eigenstates of rotations around the $Z$ and $z$ axes. In three dimensions, the local Lorentz symmetry is $O(3)$ (rather than $SU(2)$, because there are only vector, not spinor indices). After the $X$ and $Y$ internal Gauss constraints are fixed, $O(3)$ reduces to $O(2)$, the group of rotations about $Z$. It is convenient to shift to transverse fields which are eigenstates under these rotations:

\[
\hat{E}_Z^a = [\hat{E}_X^a \pm i \hat{E}_Y^a] / \sqrt{2},
\]
where \( a = x, y \); and similarly for \( A_a^\pm \). Expand not only the local indices \( X, Y \), but also the global indices \( x, y \), because after gauge fixing the latter indices possess a residual \( O(2) \) symmetry. This symmetry, a rigid rotation around the \( z \) axis which makes no distinction between contravariant and covariant indices \( x \) and \( y \), is just the symmetry which was used originally to construct \( L_Z \) via the Noether procedure. When the fields in eq. (3) are expanded in \( O(2) \) eigenstates, one gets (after some manipulations; see the next paragraph)

\[
L_Z = 2 \int dz [\tilde{E}^+_x A^-_y - \tilde{E}^-_x A^+_y]
\]
\[
= 2\hbar \int dz [\tilde{E}^+_x \delta / \delta \tilde{E}^-_x - \tilde{E}^-_x \delta / \delta \tilde{E}^+_x].
\]  

(11)

This expression counts the number of \( \tilde{E}^\pm_x \) fields in the wavefunctional \( \psi \), assigning each field a value \( \pm 2\hbar \). This looks very much helicity two, and implies that the helicity content of \( \psi \) is determined by the number of transverse \( \tilde{E}^\pm_x \) fields that it contains, but not by its \( \tilde{E}^{\pm}_x \) fields. Similarly, if one uses a connection representation rather than a momentum representation, the helicity content is determined by the number of transverse \( A^\pm_x \) fields.

Proof of eq. (11): I wish to manipulate the \( \text{Re} A \) term in eq. (3) extensively, while leaving alone the first two terms. Accordingly I introduce the following notation which allows me to abbreviate the first two terms.

\[
G^a_b = \int \tilde{E}^a_x A^I_b.
\]

(12)

Then

\[
L_Z = -i\epsilon_{ab} G^a_b + i \int dz \epsilon_{ab} \tilde{E}^a_x \text{Re} A^I_b.
\]

(13)

Before this expression can be quantized, the \( \text{Re} A \) term must be written out in terms of \( \tilde{E} \) fields. The following formulas are useful.

\[
2^{(4)} A^{MN}_b = \omega^{MN}_b + i\epsilon^{MN}_{PQ} \omega^{PQ}_b / 2;
\]

\[
A^I_b = (4) A^MN_b \epsilon_{MN};
\]

\[
\text{Re} A^I_b = \epsilon_{IJ} \omega^{ZJ}_b;
\]

\[
\omega^{IJ}_b = [\Omega_{ij[b]} + \Omega_{j[bi]} - \Omega_{b[ij]}] e^I e^J;
\]

\[
\Omega_{ij[b]} = \epsilon_{IM} [\partial_j e^M_b - \partial_b e^M_j] / 2.
\]  

(14)
The first two lines relate the four dimensional complex connection to the corresponding three dimensional connection, and to the four dimensional Lorentz connection $\omega^M_N$. I use the same sign conventions as in reference [2]. The third line is the reality condition for the transverse connections $A^I_b$. $e^iM$ is the tetrad, and $e^M_i$ is its inverse.

The standard [9] Lorentz gauge fixing condition, $e^iZ = 0$, may be used to simplify the sums over $i$ in the definition of $\omega^I_J$, when $I = Z$; one gets

$$\omega^Z_J = -\partial_z g_{bj} e^Ze^jJ/2;$$
$$\bar{E}_a^I \text{Re} A^I_b = e e^a_b e^iJ \omega^Z_J = -\partial_z g_{bj} e^{aj}/2,$$

which turns out to be a total derivative. Now convert back to the basic Ashtekar fields, the densitized triads. A useful identity is $g_{bj} = \epsilon_{bm} \epsilon_{jn} \delta^{m(2)}$, where all metric components and determinants are in the 2x2 transverse sector. One needs also the definition of the densitized triads, eq. (2). The Re A term in eq. (13) becomes

$$\bar{E}_a^I \text{Re} A^I_b = -\partial_z [\bar{E}_m^m \bar{E}_M^M \bar{E}_Z^Z/(2\bar{E})] \epsilon_{bm}/2,$$

(16)

$\bar{E}$ is the determinant of the matrix $\bar{E}_a^I$ in the 2x2 transverse sector. When this expression is inserted into eq. (13), the Re A term becomes

$$-i \int dz \epsilon_{MN} \bar{E}_m^m \bar{E}_M^M / (2\bar{E}) = -i \int \partial_z \bar{E}_Z^Z dz = -i \int \epsilon_{MN} \bar{E}_M^c A^N_c,$$

(19)
where I have used the surviving Gauss constraint, \( \partial_z \tilde{E}^z - \epsilon_{MN} \tilde{E}^a_M A^N_a = 0 \), to replace the \( \tilde{E}^z \) by transverse fields. Now insert this back into \( L_Z \), eq. (13). At this point the expression no longer involves \( \text{Re} A \).

Expand in terms of \( O(2) \) eigenstates, remembering to contract every + index with a - index, as \( \epsilon_{MN} \tilde{E}^a_M A^N_a = \epsilon_+ \tilde{E}^a A^+_a + \cdots \), with

\[
\epsilon_{\pm} = \mp i.
\]

The result is eq. (11) \( \square \).

In eq. (3) or eq. (13), the term containing \( \text{Re} A \) is separately a constant of the motion. Proof: the term is obviously a scalar under internal Gauss rotations about \( Z \). When the term is commuted with the remaining, scalar and \( z \) diffeomorphism constraints, those constraints must be smeared with a functions which represent small changes in the lapse and shift. The smearing functions must vanish at boundary points \([2, 3]\), since lapse and shift are required to reduce to fixed constants there, and cannot change at boundaries. Hence the scalar and diffeomorphism constraints commute with any expression which depends only on fields evaluated at boundary points. But the integrand of the \( \text{Re} A \) term is a total derivative, eq. (17), therefore the term depends only on fields evaluated at the boundaries, \( z = z_l \) and \( z_r \). In fact any surface term which is Gauss invariant is automatically a constant of the motion. \( \square \)

If the \( \text{Re} A \) term in eq. (13) is separately a constant of the motion, then the quantity \( \epsilon_{ab} G^a_b \) in that equation is also separately a constant of the motion, where \( G^a_b \) is the integral defined at eq. (12). Husain and Smolin have shown that all four \( G^a_b \) integrals are constants of the motion \([3]\). These authors apparently did not use Noether’s theorem, so were unaware of the connection with total spin. It is possible to recover two more of the Husain-Smolin conserved quantities by using a Noether procedure, since the Lagrangian happens to be form-invariant under the larger group \( SL(2, R) \). (The group acts on the \((x, y)\) indices, like the \( O(2) \) group used in the derivation of \( L_Z \). Covariant indices are transformed by an \( SL(2, R) \) matrix, while contravariant indices are transformed by another matrix which is the transposed inverse of the matrix for the covariant indices.)

I have tried to derive the fourth Husain-Smolin conserved quantity by enlarging \( SL(2, R) \) to \( GL(2, R) \), i.e. by adding dilatations.
However, there is a technical difficulty. Since the Lagrangian is a density rather than a scalar, it is not form-invariant under dilatations. Normally the Lagrangian is multiplied by a dxdy integration which is also non-invariant, so that the action as a whole is invariant. In the present case, however, there is no dxdy integration, and it is not clear how to apply the Noether procedure. The Husain-Smolin quantities are discussed further in the next section.

III Discussion

Husain and Smolin have shown that all four of the integrals $G^a \cdot b$, eq. (12), are conserved [6]. The present work gives a physical interpretation to one linear combination of these integrals, $\epsilon_{ab} G^a \cdot b = G^x_y - G^y_x$ (more precisely, to the imaginary part of this linear combination). From eq. (13), $L_Z$ equals

$$L_Z = -i[G^x_y - G^y_x - \Re(G^x_y - G^y_x)].$$

(21)

It is possible to find physical interpretations for two other linear combinations of the $G^a \cdot b$. From section II, after quantization eq. (21) can be rewritten as at eq. (11):

$$L_Z = 2\hbar \int dz [\tilde{E}^+ \delta / \delta \tilde{E}^- - \tilde{E}^- \delta / \delta \tilde{E}^+].$$

(22)

The $\pm$ indices refer to the O(2) eigenstates defined at eq. (14). As remarked in section II, this expression implies that the spin content of any wavefunctional $\psi$ is determined by the number of transverse $\tilde{E}^\pm$ fields that $\psi$ contains; $\tilde{E}^\mp$ fields do not contribute to the total spin. Now express two other linear combinations of $G^a \cdot b$ components in terms of O(2) eigenstates:

$$G^x_x - G^y_y \pm i(G^x_y + G^y_x) = 2 \int dz \tilde{E}^\pm A^\pm$$

$$= 2\hbar \int dz \tilde{E}^\pm \delta / \delta \tilde{E}^\mp.$$

(23)

These operators are raising and lowering operators for intrinsic spin. The upper sign (for example) replaces

$$\tilde{E}^+_+ \rightarrow 2\hbar \tilde{E}^+_+;$$

$$\tilde{E}^- - \rightarrow 2\hbar \tilde{E}^- -.$$

(24)
when acting on a solution $\psi$, hence raises the $L_z$ eigenvalue by $2\hbar$ units.

The remaining linear combination, $G_x^x + G_y^y$, is a two dimensional version of a three-dimensional operator which plays a key role in Thiemann’s regularization scheme [10]. It is also a number operator for the number of transverse $\tilde{E}$ fields in $\psi$: if $\psi$ is a string of $n$ $\tilde{E}$ fields, then the eigenvalue of $G_x^x + G_y^y$ is $n\hbar$. Presumably $n$ characterizes the background geometry. To clarify this, one can write out the Ashtekar connections in $G_x^x + G_y^y$ in terms of the four-dimensional Lorentz connection $\omega_{\alpha}^{IJ}$; then use the classical equations of motion to express $\omega$ in terms of the tetrads. Use a conformally flat gauge to shorten the (lengthy but straightforward) algebra. The necessary formulas relating $A_a^A$ to $\omega$, and $\omega$ to the tetrads, are given at eq. (14). The final result is

$$G_x^x + G_y^y = i \int dz \frac{\partial\partial (e^{(2)})}{\partial t}. \tag{25}$$

In words, the average rate of change of the area operator is a constant of the motion.

As already stressed in section [1], if one uses a connection rather than a momentum representation, then the transverse $A_a^I$ fields play the role formerly played by the transverse $E_a^a$. In particular, the spin is determined by the $A_{\pm}^a$ fields; and $G_x^x + G_y^y$ counts the number of transverse $A$ fields.

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