We study three different experiments that involve dry friction and periodic driving, and which employ both single- and many-particle systems. These experimental set-ups, besides providing a playground for investigation of frictional effects, are relevant in broad areas of science and engineering. Across all these experiments, we monitor the dynamics of objects placed on a substrate that is being moved in a horizontal manner. The driving couples to the degrees of freedom of the substrate and this coupling in turn influences the motion of the objects. Our experimental findings suggest emergence of stationary-states with non-trivial features. We invoke a minimalistic phenomenological model to explain our experimental findings. Within our model, we treat the injection of energy into the system to be dependent on its dynamical state, whereby energy injection is allowed only when the system is in its suitable-friction state. Our phenomenological model is built on the fact that such a state-dependent driving results in a force that repeatedly toggles the frictional states in time and serves to explain our experimental findings.

1. Introduction

A common parlour trick to demonstrate inertia is to place a coin on a postcard and then quickly pull the card. If the pull is quick enough, the coin remains almost in the same place while the card gets pulled out.
While the trick and its many variations (e.g. pulling a book out of a stack of books, or the game Jenga) are themselves very instructive, they nevertheless leave a few pertinent questions related to the stationary-state dynamics unanswered. The coin is attached to the card by forces of friction. Common wisdom tells us that if the acceleration of the card is larger than a critical value, the coin will begin to slip with respect to the card. However, as soon as the coin starts to slip, a crack opens up on the interface [1,2], and this results in weakening of the frictional coupling between the card and the coin. Since there is dissipation, the coin loses momentum and eventually heals the crack, and the frictional coupling regains its strength. The whole process of weakening and strengthening of the frictional coupling undergoes repeated cycles in time. In such a scenario, does the coin eventually move with uniform velocity or does it accelerate? In this paper, we will show that the qualitative aspects of this particularly simple experiment and some variants of it can be explained in terms of a phenomenological model in which the rate of energy injection into a system depends on its dynamical state.

(a) Degrees of freedom and internal states of a system

In a more general setting, the aforementioned scenario would be an example of an interacting system evolving in the presence of drive and dissipation. In our studied systems, input of energy through the external drive at the level of the substrate (e.g. the card) affects the evolution of the frictional states of the objects (e.g. the coin) placed on the substrate and in time may cause random transitions between the different frictional states. These transitions in turn would act back on the time evolution of the objects, resulting in an intricate (and intriguing, in many-body complex systems) interplay between the dynamics of the objects and that of the frictional states [3,4]. In the context of the coin and the card, the qualitatively different frictional states are that of slipping (low-friction state) and sticking (high-friction state). If the coin is replaced by a sphere, then the frictional states may be identified with rolling (low-friction state) and sliding (high-friction state) motion.

It is evidently of interest to ask: what is the long-time behaviour of the aforementioned class of systems? Does the system achieve a stationary state with a time-independent distribution of the degrees of freedom of the objects? We may in general anticipate that a balance between drive and dissipation does in fact lead the system to a stationary state. The presence of an external drive precludes the possibility for the stationary state to be an equilibrium one. Consequently, any stationary state the system relaxes to over a long time would be a generic non-equilibrium stationary state (NESS) [5]. In this work, we highlight the subtleties that come into play in the dynamics, by presenting three experiments: (i) a single coin on a platform oscillating along one axis, and (ii) a single sphere and (iii) a collection of spheres placed on an orbiting platform [6]. The first experiment is one dimensional in nature and from its detailed study, we establish that at a phenomenological level, the coupling between the plate and the coin can be modelled by a viscous force-like velocity-dependent term. In the second experiment, the centre of mass of the sphere covers a two-dimensional space and it does so under the influence of two orthogonal forces. From this experiment, we establish that the stochasticity in the trajectory of the sphere comes from the randomness in the amplitudes of the two orthogonal forces. In the third experiment, we introduce interactions between the particles and establish that the resulting velocity distribution attains a unique form with the tail, which is exponential in nature, gaining weight with increasing density of spheres. This observation is surprising, since collision-induced cooling is at the heart of many observable NESSs in granular systems.

Following the experimental results, we present a minimalistic one-particle theoretical model that captures the essence of the experimental findings and provides us with a framework to understand such scenarios. Within our model, we treat the injection of energy into the system to be dependent on its dynamical state, whereby energy injection is allowed only when the system is in a suitable-friction state. The idea of state-dependent driving is rooted in various biological systems, where the functionality of a state is maintained by an organism constantly regulating the activity of its various internal processes. In the context of active systems, this would translate
to the idea of prescribing a local density-dependent mobility to the particles. It is known that such quorum-sensing interactions can give rise to motility-induced phase transitions [7]. Another example is that of enzymes locally enhancing diffusion by self-regulating the phoretic [8] and hydrodynamic forces generated by themselves. This can lead to functional behaviour such as antichemotaxis [9].

(b) Dry friction models and their limitations

The experiments and numerics presented in this paper are considered for frictional systems. At a first glance, the experimental scenarios considered here might appear commonplace and well studied. It is thus important that we discuss the conventional method of treating apparently similar problems using the well-established framework of friction and point out the differences between the ones that are exhaustively discussed in the literature and the ones studied here.

(i) The problem with dry friction

A solid that is frictionally coupled to a substrate is put to motion when the force $F$ applied to the body exceeds $\mu|N|$, where $N$ is the normal force exerted by the solid on the substrate and $\mu$ is the coefficient of friction. When the relative velocity between the substrate and the body is zero, then $\mu = \mu_s$, the coefficient of static friction, and $|F_F| \leq \mu_s N$, where $F_F$ is the force of friction. However, when the object is in motion, then $\mu = \mu_k$, the coefficient of kinetic friction, and $F_F = -\mu_k N$. Here, $\mu_s$ is a positive real number that characterizes the transition from a stationary to a moving phase and $\mu_k$ is also a positive real number that characterizes the friction forces when the object is slipping with respect to the substrate. The above description is the Coulomb description of dry friction. For other phenomenological models of dry friction, see [10]. In general, the coefficients of friction are defined for a pair of materials and $\mu_s \geq \mu_k$. Thus, $F_F$ forms an admissible set $\mathbb{C}$, i.e. $\mathbb{C} = \{F_F| - \mu_s N \leq F_F \leq \mu_s N\}$. Unlike viscous friction, where $F_F \to 0$ as $v \to 0$, there exists a velocity gap $\Delta v$ in dry friction, i.e. the velocity becomes zero if the magnitude of the applied force $F$ becomes smaller than a threshold value [11]. Dry friction breaks the time-reversible symmetry associated with the dynamics of bodies. This loss of symmetry is sometimes reflected in the nonlinear (often chaotic) dynamics exhibited by these systems [12,13]. The presence of an admissible set of friction values and the velocity gap makes the problem of dry friction not amenable to an analysis within the framework of deterministic dynamics.

(ii) Modelling friction as a mechanical instability

Friction is considered to be a nonlinear phenomenon [14] and models like the Prandtl–Tomlinson associate this nonlinearity with mechanical instabilities (for an historical account, see [15,16]). Within the framework of these models, the overall response of the mechanical system is understood in terms of the ratio of two potentials: one associated with the substrate–body interaction, and the other associated with elastic coupling between the body and an external anchor point [17,18]. This anchor point can either remain stationary or have a well-defined trajectory. If the ratio of these two potentials is less than one, the overall motion of the body is smooth. On the other hand, when this ratio is large, the sliding motion becomes jagged (slip–stick motion) and the system continuously exhibits a series of elastic instability transitions [18, ch. 10]. Thus, Prandtl–Tomlinson is essentially a two-body model, which in itself or whose close variants like the ones that incorporate stress-augmented thermal activation process [19] has been successfully used to understand a wide range of frictional problems encountered in experimental studies using a variety of friction force microscopes (FFM) [15,16,20]. The dynamics of the model that comprise a single particle being dragged with velocity $v$ on a one-dimensional substrate considered to be a sinusoidal potential are described by the equation of motion

$$m\ddot{v} = -\gamma v + F(x, t) + \eta(t),$$

(1.1)
where \( \eta(t) \) represents Gaussian white noise satisfying \( \langle \eta(t) \rangle = 0; \langle \eta(t)\eta(t') \rangle = 2\gamma k_B T \delta(t-t') \) and \( T \) is the temperature. In the above equation, \( \gamma \) is the damping coefficient, while the force \( F(x, t) = -\partial_x U(x, t) \) is obtained from the potential \( U(x, t) = U_0 \cos(2\pi x/a) + (K/2)(x - vt)^2 \). Here, \( U_0 \) and \( a \) are, respectively, the amplitude and the periodicity of the sinusoidal potential, while \( K \) represents the strength of the harmonic coupling between the particle and the substrate. Friction associated with extended objects is treated within the framework of models like Frenkel–Kontorova (FK) [21]. Here, the elastic energy associated with the coupling between the body and the anchor point is replaced by the energy associated with the deformation of the body itself. As a result, the response of soft objects in the presence of frictional force is very different from that of the more rigid ones—a phenomenon that is known as the Aubry transition [22]. Rigid objects tend to move smoothly, while soft objects exhibit slip–stick motion.

Almost all experimental realizations in friction are classified into two types: one where the force is directly applied to the object, e.g. an atomic force microscope tip being dragged over a substrate, or one where the body is held with a spring to a point in space and the substrate moves below it [23–26]. In either case, the two competing energy scales can be easily identified and hence these situations are amenable to detailed analysis. However, there may be situations, e.g. a coin on a horizontally vibrating plate [27], where identifying two separate energies, one of which is due to the application of a force on the system of interest and the other is due to its motion against a frictionally coupled substrate, may not be possible. This happens when the processes by which energy is injected and dissipated are not separated in space and time. This is precisely the dynamical scenario that we address in this work.

(iii) Stochastic aspects of friction

Though the force of friction is an easily measured quantity, its value is highly sensitive to experimental conditions. Often, measured friction forces vary from one experiment to another and also they are known to strongly age with time [28,29]. For dry friction, the local structure of the interlocking interfaces plays a key role. Since the area of this interlocking region is small, each patch of the interface corresponds to different values of friction. This lack of self-averaging necessitates the use of stochastic dynamics to study the problem. In doing so, one may invoke stochasticity in the configuration space through the modelling of making and breaking of contacts [30,31]. Alternatively, one may invoke stochasticity in the energy space, whereby energy injection and dissipation are modelled as stochastic processes. In this paper, in contrast to earlier work, we follow the second approach and propose a model in which the energy transfer between a moving substrate and a body is treated as a stochastic variable whose value depends on the ‘state’ of the system. Here, we use the term ‘state’ in the same sense as described in the leading paragraph of §1a. One may recall that phenomenological models of friction, like the ‘rate-state model’ [32–35], also invoke the concept of states in a different sense, without however explicitly identifying the states. However, many recent works have attempted to do so by treating friction as a localized electro-plastic response of the contact zone [36].

2. Results

(a) A coin on an oscillating plate

A coin is placed on a plate that is being vibrated in the \( x \)-direction (figure 1a). The motion of both the coin and the plate is measured simultaneously. The plate moves sinusoidally, \( x_p(t) = x_{p0} \cos(2\pi ft) \), where \( x_{p0} \) is the position of a point on the plate, \( x_{p0} \) is the amplitude of vibration and \( f \) is the driving frequency.

(i) Experimental results

For small values of \( f (\leq f_c = 4.4 \text{ Hz}) \), the coin remains stuck to the plate, while for larger values of \( f \), it begins to slip with respect to the plate. Typical time traces of the velocity of the centre of mass of
Figure 1. (a) First experimental set-up: a coin on a plate that is being vibrated sinusoidally in time. (b) Second experimental set-up: Stainless steel balls of diameter 2a = 800 µm confined in a cylindrical container of diameter D = 150 mm and height 1 mm, with a glass plate used as a lid for the container to enforce a two-dimensional geometry for the problem. The container is placed on a square platform and the entire assembly along with a camera is made to perform an orientation-preserving horizontal circular motion. The net motion of the assembly may be imagined as a combination of two circular rigid-body motions of the platform: (i) a rotation about shaft 1 with angular frequency ω, and (ii) an opposing rotation with angular frequency −ω about shaft 2 that passes through the centre of the platform. (c) Typical temporal variation of the dichotomous noise ξ(t) used in the phenomenological dynamics (2.2) and (2.4) to model both the experiments. (Online version in colour.)

the coin and a point fixed on the plate for f = 6 Hz are shown in figure 2a. In the phase in which the coin slips with respect to the plate, its centre of mass drifts slowly in time. However, for small time intervals, its response x_c(t), namely, the location of its centre of mass, is approximately sinusoidal and has the same frequency of variation as that of the plate. The probability distribution of |v|, the magnitude of the velocity of the centre of mass of the coin, is plotted in figure 2c; the distribution is obtained by sampling the trajectories in the long-time limit over a time window. The variation of the time-averaged amplitude of displacement of the coin and its phase difference φ with respect to the plate in the long-time limit are plotted in figure 2, panels (d) and (e), respectively, as a function of the driving frequency.

Our main experimental observations may be summarized as follows:

(i) At long times, the velocity as well as the displacement of the coin are approximately sinusoidal in time (figure 2a).
(ii) The velocity distribution of the coin has a tail that is exponential (figure 2c).
(iii) The time-averaged amplitude of the displacement of the coin x_c0 decreases with the drive frequency (figure 2d).
(iv) The phase difference φ saturates to a large value with increases in the drive frequency (figure 2e). This is in contrast to what is seen in the case of a damped, driven harmonic oscillator with a real damping constant, which in turn precludes use of a noisy harmonic oscillator to model our experimental scenario. The case of a complex damping constant is discussed in the electronic supplementary material, SI.

(ii) Discussion

The most appropriate way to describe the system would be to use the Coulomb friction model [18]. If the driving frequency is such that the amplitude of the sinusoidal forcing exerted by the plate is larger than the maximum static friction force, i.e. |m(2πf)^2x_p0| > μ_smg, with μ_s being...
Figure 2. A coin on an oscillating plate: The trace of the velocity of the coin (black) and a point fixed on the plate (red) at driving frequency $f = 6 \text{ Hz}$ are both shown in panel (a). Panel (b) shows results from numerical simulation of the phenomenological model (equation 2.2). The probability distribution of the magnitude of the velocity of the coin in the laboratory frame is plotted in panel (c) at the representative values of $f$. The filled symbols correspond to values obtained from the experiment, while the solid lines correspond to values obtained from the model. The time-averaged amplitude of the displacement of the coin ($x_{c0}$) and the phase difference $\phi$ of the coin with respect to the drive are shown in panels (d) and (e), respectively, as a function of the driving frequency $f$. The red symbols correspond to values obtained from the experiment, while the grey symbols correspond to values obtained from the model. The dotted lines in (d) and (e) separate the stuck phase ($f \leq f_c$) from the slipping phase ($f > f_c$). Here, $f_c = 4.4 \text{ Hz}$ is the critical frequency below which the coin is completely stuck to the plate. For the simulation, for $f > f_c$ we took $\frac{\gamma \tau_p/m_p}{\beta^2} \sim 12$, $\beta^{-1} \sim (f - f_c)$ and $\gamma$ was taken to vary inversely with $f - f_c$ (see inset of (d) and electronic supplementary material, SI, for additional information on the parameters). (Online version in colour.)

The coefficient of static friction between the coin and the plate and $m$ is the mass of the coin, the coin will undergo both stick and slip motion periodically. In the stuck state, i.e. when $|m(2\pi f)^2x_{p0}\cos(2\pi ft)| < \mu_smg$, the trajectory of the coin will follow the sinusoidal nature of the trajectory of the plate. However, as soon as the driving force exceeds $\mu_km_g$, the coin will start to slip with respect to the plate and it will then move under a constant acceleration due to the force of kinetic friction $\mu_km_g$. Here, $\mu_k$ is the coefficient of kinetic friction between the coin and the plate. Hence, the velocity of the coin will be partly sinusoidal and partly linear in a single period. This model gives rise to a deterministic velocity profile. This is in stark contrast to the noisy sinusoidal nature of the velocity observed in the experiments. In addition, the velocity distribution, the amplitude and the phase response of the coin as obtained from the Coulomb friction model of friction are very different from what is observed in the experiments. We have provided the details in the electronic supplementary material, SI.

(iii) Model

The response of the system for $f > f_c$ and $f < f_c$ is qualitatively different. For $f < f_c$, the coin and the plate behave as a single rigid object, i.e. $x_c(t) = x_p(t)$, which is unlike the situation for $f > f_c$. The description below is limited to the latter case only.
Experiments suggest that the coin never gets stuck to the plate at the smallest time scale over which the motion of the coin and the plate is measured, i.e. 5.5 ms. Hence, to explain the experimental observations, we propose that the coin has access to (i) a ‘high-friction’ state (which in a limiting case becomes the stuck state) and (ii) a ‘low-friction’ state (or a slip state). Without the presence of a ‘high-friction’ state, the coin would never experience the external driving applied to the plate and hence the response of the coin would never reflect the waveform of the drive. However, over a large parameter regime, we find that the amplitude of the coin is smaller than that of the plate. Thus, it is reasonable to assume that the ‘high-friction’ state survives over a time scale that is smaller than the smallest time scale of measurement and that during this time scale, \( \tau_p \), a fraction of the momentum of the plate gets transferred to the coin. Here, it becomes necessary to differentiate between a ‘stuck’ state (observed for \( f < f_c \)) and a ‘high-friction’ state. In contrast to a ‘high-friction’ state, in a ‘stuck’ state, the entire momentum and hence the entire force would be transferred from the plate to the coin. We assume the transitions between the high-friction and low-friction states to occur randomly in time. We thus propose a stochastic variant of the Coulomb friction model for the motion of the coin on the vibrated plate, which captures the essential qualitative features of its dynamics observed in the experiments described above.

To be consistent with the experiment, we choose to describe the motion of the centre of mass of the coin in an inertial frame, i.e. in the laboratory frame. With respect to such a frame, the plate on which the coin is placed is being vibrated sinusoidally in time, so that its velocity at time \( t \) reads \( v_p(t) = v_{p0} \sin(2\pi ft) \). When the coin is in the ‘high-friction’ state, the equation of motion of its centre of mass is the usual Newton’s equation of motion

\[
m\partial_t v = F(t); \quad F(t) \equiv \frac{m_p v_p(t)}{\tau_p},
\]

where \( v \) is the velocity of the coin, while \( F(t) \) is the force experienced by the coin due to transfer of momentum from the plate to the coin. Here, the quantity \( m_p \) is the mass of the plate, and \( \tau_p \) as defined earlier is the time scale over which the ‘high-friction’ state survives and during which the momentum of the vibrating plate is transferred to the coin. On the other hand, when the coin is in the ‘low-friction’ (slip) state, it is detached from the plate and consequently it moves in the presence of an effective damping force that dissipates in time the initial momentum of the coin at the instant of detachment from the plate. Hence, in the ‘low-friction’ state, the equation of motion of the centre of mass is given by \( m\partial_t v = -\gamma(v - v_p) \), where \( \gamma > 0 \) is a phenomenological dissipation constant. The coin toggles randomly in time between the ‘high-friction’ and ‘low-friction’ states. Introducing a random variable \( \xi \) taking on values 1 or 0 corresponding to the ‘high-friction’ and ‘low-friction’ states, respectively, on the basis of the foregoing one may write down the equation of motion of the centre of mass as a stochastic differential equation, a Langevin-like equation, of the form

\[
m\partial_t v = -\gamma(v - v_p) + \xi(t)(F(t) + \gamma(v - v_p)).
\]

Equation (2.2) has to be supplemented by another equation describing the time evolution of the instantaneous \( \xi \), namely, \( \xi(t) \). While derivation of an exact form of the latter would invariably involve a detailed modelling of the dynamics of the contact region via which the friction force is transmitted, we offer here a phenomenological description for the evolution of \( \xi \) in terms of a stochastic Markov process, namely, between times \( t \) and \( t + dt \), and the variable \( \xi(t) \) is updated to read \( \xi(t + dt) = 1 \) with probability \( \beta \, dt \), while \( \xi(t + dt) = 0 \) with the complementary probability \( 1 - \beta \, dt \). Here, \( \beta > 0 \) is a dynamical parameter. It then follows that the random time \( \tau \) between two successive occurrences of the value unity for \( \xi \) is distributed as an exponential: \( p(\tau) = \beta \, e^{-\beta \tau}; \tau \in [0, \infty) \), and that the average \( \tau \) is given by \( \langle \tau \rangle = 1/\beta \). As a function of time, \( \xi(t) \) appears as a set of impulses distributed randomly in time, as shown in figure 1c. For a given drive, the parameter \( \beta \) is a phenomenological parameter. To fit our experimental data, we require \( \gamma \propto 1/(f - f_c) \). This suggests strain-rate-induced weakening of the contact zone, which could possibly arise from the reduction of the contact area with the increasing strain rate [37]. A reduction in \( \gamma \) results in re-scaling of the relaxation time \( \tau_p \sim m/\gamma \propto (f - f_c) \), which thereby influences the momentum transfer process via \( F(t) = m_p v_p/\tau_p \). For the coin to slip, the contact
zone must reach a critical size [38]. Further, given $\tau_p$ describes the dynamics at the contact, it is also likely to influence the time scale associated with the growth of the slip region of the contact zone. We thus assume $\beta^{-1}$ to be a function of $\tau_p$. This forms the rationale for assuming a linear dependence between $1/\beta$ and $(f - f_c)$, which is used to fit the data. The specific forms of $\gamma$ and $\beta$ as functions of $f$ are given in the electronic supplementary material, SI.

It is interesting and pertinent that we draw a parallel between equation (2.2) and the usual form of the Langevin equation that one encounters in describing, say, the paradigmatic Brownian motion. Besides the nature of the stochastic noise, which in the latter is a Gaussian white noise and in equation (2.2) is a dichotomous noise, one very important difference is the following. In the case of Brownian motion, the strength of the noise term is a constant that does not depend on the value of the dynamical variable in question (more precisely, the constant is by virtue of the fluctuation–dissipation theorem related to the equilibrium temperature of the ambient medium). By contrast, in equation (2.2), the strength of the noise term is explicitly dependent on the value of the dynamical variable $v$ being studied. Another point worth mentioning is that the noise in the case of Brownian motion is considered stationary, while in our case, the noise is non-stationary.

Another interesting parallel that one may draw is between the dynamics equation (2.2) and the dynamics of a driven, damped harmonic oscillator evolving in the presence of noise. In such an approach, one may consider the noise to be multiplicative, in the sense that it appears as a coefficient of the damping term in the dynamics, i.e. $m\ddot{v} = -\gamma \dot{v}(v - v_p) + F(t)$. The problem with this approach is that even when the coin is slipping, there will be the unphysical dynamics of momentum transfer between the coin and the plate. By contrast, the model presented here is nevertheless founded on physical reasoning, as presented in the paragraph preceding equation (2.2). Note that in the dynamics described by equation (2.2), the force experienced by the coin is being continually reset in time between the pure sinusoidal drive and the entirely dissipative form as the random variable $\xi$ toggles in time between its two possible values. Equation (2.2) may be considered representative of a class of stochastic dynamical systems in which the force resets stochastically in time between a set of possible values, with the two-state process considered here being the simplest one may conceive.

The dynamics described by equation (2.2) involve two time scales $1/\beta$, setting the average time between two impulses, and $1/f$. We work in the regime in which $1/\beta \ll 1/f$. Consequently, in the long-time limit, we expect the velocity $v$ to vary sinusoidally over time with an occasional jump in values as an effect of toggling acting on a time scale that is faster compared with the sinusoidal variation. As far as the parameter $\gamma$ is concerned, its magnitude would set the cut-off scale of the velocity in the slip state.

Figure 2b–e compares the results obtained from numerical simulation of the dynamics equation (2.2) with the experimental findings. In panel (b), we see that consistent with our expectations mentioned above, the velocity represents an almost sinusoidal variation over time and, correspondingly, the velocity distribution in (c) exhibits a peak at a value around the amplitude value of the sinusoidal variation in (b). The toggling phenomenon manifests itself as deviations from a pure sinusoidal variation in (b) and contributes to a width around the peak in the distribution in (c). It is remarkable that our phenomenological model described by equation (2.2) is able to reproduce in a rather striking manner the experimental features of the coin dynamics in (a) and the velocity distribution in (c). Moreover, the distribution has a tail that is exponential, as seen in panel (c). In simulations, this is a result of dissipation set by $\gamma$ in the dynamics equation (2.2). Moreover, the amplitude and the phase response obtained from the simulations are in good agreement with the experimental data (figure 2d,e). As for the coin-trick problem that was proposed in the introduction, we expect the coin to move with a constant velocity. The magnitude of the velocity would depend on the coefficient of friction and the acceleration of the card being pulled out.

(b) A sphere on an orbital shaker

In the above experiment, the stochasticity in the forcing term arises from the toggling between the ‘high-friction’ and ‘low-friction’ states of the coin. In the second experiment that we describe...
now, we place a stainless steel ball (a sphere of mass $m$, radius $a$ and moment of inertia $I$) on a plate connected to an orbital shaker. The schematic set-up of the experiment is shown in figure 1b.

Here, the system can toggle between the rolling (without slipping) and the sliding states. With an orbital shaker, the plate moves in an orientation-preserving manner such that each point on the plate moves on a circle of the same radius; a given point $Q(x_p, y_p)$ on the plate moves as $x_p = x_o + r \cos(2\pi ft)$, $y_p = y_o + r \sin(2\pi ft)$, where $(x_o, y_o)$ is the centre of the circle about which the chosen point $Q$ moves. Each point on the plate has a centre associated with it. The motion of the sphere comes from the force that is exerted at the point of contact between the sphere and the moving plate. This frictional force affecting the velocity $\hat{v}$ of the centre of mass of the ball as $m\hat{v} = -\mu mg \hat{v}_p$, with $\mu$ the coefficient of friction, $g$ the acceleration due to gravity and $\hat{v}_p$ being the velocity of a point on the plate, exerts a torque $\Gamma$ about an axis parallel to the platform and passing through the centre of mass of the sphere, i.e.

$$I\dot{\omega}_b = -a\hat{k} \times m\hat{v} = a\mu mg(\hat{k} \times \hat{v}_p),$$

where $\omega_b$ is the angular velocity of the sphere. An isolated sphere on the plate performs a swirling motion [6,39,40]. An example of such a motion for $f = 2.1$ Hz in the reference frame of the moving plate is shown in figure 3a. In experiments, the long-time trajectory of both the position and the velocity of the sphere is not deterministic.

(i) Experimental results

Effectively, this is a two-dimensional version of the previous problem, wherein the particle–substrate interaction generates the randomness in the motion. Individually, both the $x$- and the $y$-components of the motion are approximately sinusoidal in nature. The abrupt alteration in the trajectory marked as $P_1$ in figure 4a is an example of such scattering. Occasional collisions with the boundary lead to additional randomness in the trajectories. An instance of this kind of scattering is shown as $P_2$ in figure 4b. These scattering events introduce a random phase difference between the two components, which results in the randomization of the trajectories. The probability distribution of the magnitude of the $x$-component of the velocity, $|v_x|$, of a rolling sphere for a representative value of $f$ is plotted in figure 3c. A similar distribution is seen for $|v_y|$ also. In the tails, the velocity distribution varies as $P(|v|) \sim \exp(-|v|^\alpha)$, with $\alpha \approx 1$. This is very different from the well-known Maxwell distribution, in which case $\alpha = 2$. 

Figure 3. Panel (a) shows the experimental trajectory of a single spherical particle on an orbital shaker driven at frequency $f = 2.1$ Hz, in a reference frame co-moving with the orbital shaker. Panel (b) shows the simulated trajectory based on equation (2.4). The colourbar in (a) and (b) indicates time. Panel (c) shows the probability distribution of the magnitude of the $x$-component $v_x$ of the velocity of the centre of mass of the ball in the co-moving reference frame. The red-filled symbols are from the experiment, while the black solid line is from the simulated trajectory. (Online version in colour.)
Figure 4. (a) Transitions between stuck to moving states and moving to stuck states occur at different frequencies. (b) The trajectory of a single sphere shows the two scattering events $P_1$ (particle-substrate) and $P_2$ (particle-wall). (c) The trajectories $\pi_1$ and $\pi_2$ in the figure show the formation and destabilization of dyad-like structures. In the dyad state, the trajectories are almost parallel to each other. Panels (d) and (e) indicate formation and destabilization of the triad (a three-particle cluster). Panel (f) shows the sequence of formation and dissolution of different particle clusters. The size $n$ of these clusters is marked beside each plot. There are 11 particles on the plate and $f = 1.9$ Hz. (Online version in colour.)

(ii) Model

Similar to the previous experiment, we postulate that the noise in the trajectory arises from occasional slipping of the ball with respect to the plate. These partial slips have a concomitant toggling between the sliding and rolling states of the system, and hence similar to the previous example, the resulting noise can also be considered dichotomous. Thus, the Langevin-like equation of motion of the velocity of the centre of mass of the ball has the form

$$m \ddot{v}_\sigma = -\gamma (v_\sigma - v_{pa}) + \xi(t)(F_\sigma(t) + \gamma (v_\sigma - v_{pa})), \quad (2.4)$$

with $\sigma = \{x, y\}$. Here, $\xi(t) = 0$ and 1 correspond to sliding and rolling states, respectively. It is to be noted that contrary to the coin on the plate problem, here momentum transfer from the plate to the sphere happens in the rolling (‘low-friction’) state. This is due to the fact in a pure rolling motion (without slipping), the contact point of the sphere is temporarily at rest with respect to the plate. On the other hand, in the sliding state, which in this case is a relatively ‘high-friction’ state, the sphere moves in the presence of the damping force.

However, when there is more than one particle in the system, the inter-particle collisions also contribute to the overall noise in the trajectories. In the following section, we consider the case of many spheres on an orbital shaker.

(c) Collection of spheres on an orbital shaker

In the third experiment, we put $N$ number of balls on the same set-up. Care has been taken to ensure that the driven plate is kept horizontal as much as possible, which can be seen from the inset in figure 5b. It shows the averaged intensity of a stack of 600 images taken at an interval of 1 s. The homogeneous distribution of particle density clearly shows that particles are mostly concentrated at the centre of the plate. This also indicates that particle–wall collisions occur rarely, compared with inter-particle collisions (the wall is indicated by the dotted circular red line).
Figure 5. Panels (a), (b) and (c) show the distribution of $|v_j|$ for different numbers of particles $N$ when the orbital shaker is driven at $f = 1.2\, Hz, 1.8\, Hz$ and $2.1\, Hz$, respectively. The solid black lines in (a–c) are obtained by solving the phenomenological model (equation (2.4)) for the respective frequencies. The parameters used for the numerical simulation are provided in the figure. Inset (b) shows the homogeneous distribution of particles concentrated around the centre of the plate away from the wall (red dotted line). The corresponding distribution from simulation of the phenomenological model (equation (2.4)) for $f = 1.2\, Hz$ is shown in panel (d) for different values of $\beta$. (Online version in colour.)

(i) Experimental results

The trajectories shown in figure 4b,c are obtained by averaging a sequence of images, each taken with a time interval of 100 ms. The micrographs in figure 4c,d highlight the influence exerted by one particle on the others. When two spheres collide tangentially, two events can occur: (i) the spheres alter their direction and continue swirling, or (ii) they form a bound dyad-like pair, moving together mostly in a reciprocating manner, rolling back and forth in the direction that is perpendicular to the line joining the centres of the two balls. In the direction that is along the line joining the two centres, the pair exhibit a sliding motion. Figure 4c shows two trajectories $\pi_1$ and $\pi_2$ that correspond to a collision and formation of a transiently stable dyad structure. In its dyad form, the two trajectories move parallel to each other. This structure is short lived and often spontaneously breaks apart. When it does so, the trajectories $\pi_1$ and $\pi_2$ depart from each other. The destabilization of the dyads can be brought about either by a particle–substrate or a particle–particle scattering event. In the case of the trajectory shown in figure 4c, it is the particle–substrate scattering that destabilizes the dyad.

Occasionally, a third ball bumps into the two-particle pair. This can either destabilize the dyad or form a compact three-particle triad. For spheres in contact, rolling in the same direction causes shearing of the contact region [3]. Thus, a triad cannot roll and hence it either becomes static or exhibits a small sliding motion. Figure 4d shows a sequence of micrographs that captures the event corresponding to the formation of the triad. These micrographs are obtained by using an exposure time of 10 ms. This long exposure produces a motion-induced dilation effect of the objects. Thus, moving objects in these micrographs will appear extended and their shape tells us about the nature of motion, e.g. a roller’s motion will appear as a curved line and dyads will appear as parallel straight lines. For static objects, motion-induced elongation is absent and the objects appear as dots. The leftmost frame shows a dyad and a nearby rolling isolated sphere. The other
two particles in the frame are present to provide a reference. After collision, the three spheres form a static triad. The absence of substantial motion for this triad can be inferred from the lack of motion-induced dilation (see the second frame of the micrograph in figure 4d and the first frame in figure 4e). For any higher-order structure involving more than three spheres that can form, the only available mode of movement is sliding. So, the transitions in these structures are limited to stuck to sliding states. For the values of \( f \) reported in this paper, the clusters (dyads, triads and higher-order structures) are only transiently stable. Structures involving more than three particles are destabilized by particle collisions. An instance of this destabilization can be seen in figure 4e.

Figure 4f shows the corresponding sequence of formation and dissolution of particle clusters for a system of 11 particles that is being driven. The system begins with no clustering, i.e. \( n = 1 \), and then quickly shows clustering. We find that dyads \( n = 2 \) are the most frequently formed clusters and that these can be stable for a few seconds. Occasionally, we observe triads \( n = 3 \), but these structures are relatively less stable compared with the dyads.

The very fact that two moving particles can come to a halt after a collision points us to instances where the momentum is not conserved. Moreover, the transitions between the rolling, sliding and stuck states are hysteretic in nature, i.e. the frequency at which a moving particle makes a transition to the stuck state is lower than the corresponding transition from a stuck to a moving state. Figure 4a shows this hysteresis between stuck and rolling states for a single sphere.

With each hysteretic transition, a certain amount of kinetic energy is lost. It would thus be natural to expect higher particle density to result in higher frictional dissipation and hence lower velocities. However, contrary to this, we observe in figure 5a–c the tail of the velocity distribution \( P(|v|) \sim \exp(-|v|^\alpha) \), with \( \alpha \approx 1 \), to increase with the density.

(ii) Model

There is as yet no agreement on the value of \( \alpha \), with different experiments showing different values, e.g. references [41,42] report \( \alpha \approx 3/2 \) as a universal parameter, while other experiments [43–46] find \( \alpha \) to depend on system parameters, where \( \alpha \) could take on values close to 2. Multiple models exist to explain this deviation, e.g. presence of clustering [47], non-uniformity of granular temperature [48,49] and nature of noise injected by the drive [49,50]. In most experiments, the strength of the noise is assumed to be a parameter controlled externally, e.g. energizing a shaker more injects more noise into a granular gas, and it is assumed to be independent of the density of the particles. In the present situation, noise is injected when the particle toggles its state. Thus, with increasing density, the toggling increases in ways that are described in figure 4 and hence the strength of the noise increases.

In our model dynamics equation (2.4), which we now invoke as an effective single-particle dynamics to describe qualitatively the features observed in our experiment, the parameter \( \beta \) parametrizes the frequency of this toggling. The model does well in reproducing the single-particle velocity distribution functions for different values of the drive frequency. The data obtained from the numerical model are shown as black lines in figure 5a–c. For the many-particle case, it is mainly the initial part of the tail that fits well with the data.

(d) The uniqueness of the stationary state

We now provide justification for claiming that the stationary state obtained for the experiments mentioned in this paper is qualitatively different from that observed in other driven systems. There is a large class of driven systems where the energy injection happens at the boundary [42,43,45,46,51–60], e.g. balls on an electromagnetic shaker vibrating vertically or balls in a sinusoidally driven box. The height gained by the ball increases with the increase in frequency of the shaker (provided the peak displacement of the shaker does not change with the frequency). This is qualitatively opposite to what we observe for the coin on the moving plate, where the coin’s displacement reduces with the increasing frequency of the moving plate. It may be noted here that for the coin on the plate, the point of energy injection is not spatially separated from
where the energy is dissipated. It is the point of contact with the plate through which the system gains energy through momentum transfer and it is also the same point through which energy is dissipated via friction. Similarly, the velocity distribution obtained for the spheres on the orbital shaker is also qualitatively different from that obtained in other experiments involving vibrated granular materials. For most low-density systems, the tail of the velocity distribution varies as $P(|v|) \approx \exp(-|v|^\alpha)$. There is a large spread in the value of $\alpha$ reported in the literature, ranging from 3/2 to 2. By contrast, we find that for the present method of driving, the tail of the velocity distribution has an exponential decay, i.e. $\alpha \approx 1$. Moreover, the strength of the tail grows with increasing density.

3. Conclusion

In conclusion, we first summarize the main experimental findings of the paper, then enumerate the salient features and the assumptions inherent in developing the state-dependent driving model of the systems, and comment on the success, and shortcomings of the model.

(a) Experiments

Across all the three experiments we find the following essential features: (i) abrupt alterations in the dynamics are associated with toggling of the frictional state, and (ii) the tail of the velocity distribution has an exponential nature. In the context of many-particle systems, this tail gains weight with increasing density.

(b) Model

We have constructed a one-particle theoretical model that captures the essence of the experimental findings. Within our model, we treat the injection of energy into the system to be dependent on its dynamical state. In writing down the model we have made the following assumptions:

(i) We have treated the frictional coupling parameter $\xi$ as a stochastic variable whose toggling in time is taken to be a Markov process that has two states: on ($\xi = 1$) and off ($\xi = 0$). This Markovian assumption is a simplification of the underlying frictional process that is known to have memory associated with it.

(ii) There is a single time scale in the problem given by the constant parameter $\beta$ whose inverse sets the time scale of toggling between the frictional states.

(iii) In the absence of a microscopic model that establishes the functional dependence between $\gamma$ and $1/\beta$, we have chosen both to linearly vary with $f - f_c$.

In spite of these assumptions, we find that our state-dependent energy injection model does well in capturing the features of the experiments that are enumerated above. However, in terms of exact matching, this model leaves room for further improvement for many-particle systems—see figure 5a–d—by relaxing the assumptions mentioned above, for example.

Systems with multiple components maintain themselves in a stable state by constantly regulating energy flow and dissipation, e.g. homeostasis in a biological setting is maintained by constant regulation of chemical processes, or the governor in a combustion engine uses the inertial forces acting on it to limit the fuel injection. In physical terms, one can think of the system to be consisting of two parts: the body and the environment. The body is the place where the energy is dissipated and to maintain the processes in the body, energy has to flow from the environment to the body. The rate of dissipation is a function of the inward energy flux and by controlling this flux, the body maintains a desired stationary state. Though such self-sustained stationary states are common to biological, chemical and system science settings, there seems to be a gap in realizing this regulation process in a more prosaic physical setting, particularly in situations where the energy injection and dissipation processes are clearly identified. In this paper, we
have shown that for multiple experimental settings that span from a single particle to multiple particles, a state-dependent energy injection process can lead to a non-trivial stationary state in driven frictional systems. This stationary state is maintained by continued toggling between the different frictional states of the system. The energy injected into any of these states is a function of the state itself. We also provide a simple, single-parameter theoretical description of the various experimental realizations. Our work provides a new paradigm for finding routes to achieve non-equilibrium stationary states. It is only natural that future work in this direction would be to understand the conditions under which the stationary state is maintained, and the processes by which a given stationary state becomes unstable and a new stationary state is arrived at. This could possibly open new avenues to understand modes of failure leading to destabilization in complex systems.

4. Method

(a) Experiment 1

A brass coin of diameter 20 mm is placed on a groove on an aluminium plate. The groove has a length of 300 mm and a thickness slightly larger than the diameter of the coin so that the coin is restricted to move in the x-direction only. The plate is kept on top of a shaker executing simple harmonic motion in the x-direction (figure 1a). The amplitude of the vibration ($x_0$) is 5.4 mm. The motion of both the coin and the plate is observed simultaneously at 180 fps from a camera fixed in the laboratory frame for a duration of 120 s. The coefficient of static friction measured between the coin and the plate is $\approx 0.42$. The critical frequency below which the coin is completely stuck to the plate is calculated to be $f_c = 1/2\pi \sqrt{\mu g/x_0} \approx 4.4$ Hz. Image processing is done in Matlab. We have performed the experiments for other pairs of materials, namely an aluminium coin on an aluminium plate, an acrylic coin on an aluminium plate and an acrylic coin on an acrylic plate. The experimental results and the results from numerical simulation for these pairs are provided in electronic supplementary material, SI.

(b) Experiment 2

A single stainless steel ball of diameter 800 µm is confined on a cylindrical container of diameter 150 mm and height 1 mm made of anodized aluminium, with a glass plate used as a lid for the container to enforce a two-dimensional geometry for the problem. The container is placed on a square platform and the entire assembly along with a camera (used for imaging) is made to perform an orientation-preserving horizontal circular motion (orbital motion). The amplitude of the circular motion ($r$) is 12 mm. Images are captured at 110 fps at a resolution of 6.6 MP for a duration of 600 s. Image processing is done in Matlab and ImageJ.

(c) Experiment 3

$N$ number of stainless steel balls are placed on the same set-up as Experiment 2. Particle tracking is performed by using the algorithm mentioned in [61].

Data accessibility. All of the information required to reproduce the paper is contained within the paper.

Authors' contributions. S.D.: conceptualization, data curation, formal analysis, investigation, methodology, validation, visualization, writing—original draft, writing—review and editing; S. Ghosh: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, supervision, validation, visualization, writing—original draft, writing—review and editing; S. Gupta: conceptualization, formal analysis, supervision, writing—original draft, writing—review and editing.

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