INFINITE MASS BOUNDARY CONDITIONS FOR DIRAC OPERATORS

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Abstract. We study a self-adjoint realization of a massless Dirac operator on a bounded connected domain $\Omega \subset \mathbb{R}^2$ which is frequently used to model graphene. In particular, we show that this operator is the limit, as $M \to \infty$, of a Dirac operator defined on the whole plane, with a mass term of size $M$ supported outside $\Omega$.

1. Introduction

Consider a bounded domain $\Omega \subset \mathbb{R}^2$. It is known that a Dirac operator $H$ cannot be self-adjointly realized in $L^2(\Omega, \mathbb{R}^2)$ by imposing Dirichlet boundary conditions. In 1987, Berry and Mondragon initiated the study of self-adjoint realizations of Dirac operators under the condition that the normal projection of the current density vanishes at the boundary $\partial \Omega$ \cite{BerryMondragon}. This condition can be mathematically stated as

\[ n(x) \cdot (\varphi(x), \sigma \varphi(x))_{C^2} = 0, \quad x \in \partial \Omega, \]

where $n \in \mathbb{R}^2$ is the outward normal vector to $\partial \Omega$, $\varphi \in L^2(\Omega, \mathbb{R}^2)$, and $\sigma = (\sigma_1, \sigma_2)$ is a vector formed by the usual Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

Equation (1) gives rise to a whole family of different boundary conditions (see Equation (2) below). In this present work we focus on one of these self-adjoint realizations, denoted by $H_\infty$, which corresponds to the so-called infinite mass boundary conditions. In the physics literature, the operator $H_\infty$ has gained renewed interest due to its application to model quantum dots in graphene \cite{6, 7, 13, 14, 12, 2}.

Let $H_M$ be the Dirac operator defined on $\mathbb{R}^2$ with a mass $M$ on $\mathbb{R}^2 \setminus \Omega$, and 0 inside $\Omega$. In \cite{4} it was shown that certain plane-wave solutions of the eigenvalue equation $H_M \psi = E \psi$, in the limit $M \to \infty$, satisfy the same boundary conditions as the eigenfunctions of $H_\infty$. The main result of this work, Theorem 1, is the convergence, in the sense of spectral projections, of $H_M$ towards $H_\infty$.

1.1. Definitions and main result. Let us introduce some notation used throughout this article. We denote by $\Omega \subset \mathbb{R}^2$ a bounded connected domain with boundary $\partial \Omega \in C^3$ of length $L > 0$. We parametrize $\partial \Omega$ by the curve $\gamma : [0, L] \to \partial \Omega$ in its arc-length, i.e., $|\gamma'(s)| = 1$. For a given self-adjoint operator $H$, we denote by $\sigma(H)$ its spectrum, and by $E_I(H)$ its spectral projection on the set $I \subset \mathbb{R}$. We use the symbols $\langle \cdot , \cdot \rangle$ and $(\cdot , \cdot)$ to denote the scalar products in $L^2$ and $C^2$, respectively.

\[ Key \ terms \ and \ phrases. \ Dirac \ operator, \ Berry \ Mondragon, \ graphene, \ infinite \ mass \ boundary \ conditions. \]
Moreover, we use \( \| \cdot \| \) for the \( L^2 \)-norms in \( \mathbb{R}^2 \), \( \Omega \), and \( \partial \Omega \), respectively. We drop the indication to the domain of integration if it is clear from the context. In particular,

\[
\| \varphi \|_{\partial \Omega}^2 = \int_{\partial \Omega} |\varphi(x)|^2 \, d\omega(x) = \int_0^L |\varphi(\gamma(s))|^2 \, ds.
\]

Let \( T \) be the differential expression associated with the massless Dirac operator, i.e.,

\[
T = \frac{1}{i} \sigma \cdot \nabla = \frac{1}{i} (i\partial_1 \sigma_1 + i\partial_2 \sigma_2) = \frac{1}{i} \left( \begin{array}{cc} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{array} \right).
\]

It is interesting to identify the boundary conditions needed to realize \( T \) as a self-adjoint operator in \( L^2(\Omega, \mathbb{C}^2) \): For \( \varphi \in C^\infty(\overline{\Omega}, \mathbb{C}^2) \), we compute

\[
\langle \varphi, T \varphi \rangle = \langle \varphi, \frac{1}{i} \sigma \cdot \nabla \varphi \rangle = \langle \frac{1}{i} \sigma \cdot \nabla \varphi, \varphi \rangle = -i \int_{\Omega} \nabla \cdot (i\sigma \varphi(x), \varphi(x)) \, dx
\]

\[
= \langle T \varphi, \varphi \rangle - i \int_{\partial \Omega} J_\varphi(x) \cdot n \, d\omega(x),
\]

where, in the last equality, we use Green’s formula. Here, \( J_\varphi(x) := (\varphi(x), \sigma \varphi(x)) \), and \( n \) is the outward normal vector of \( \Omega \). Hence, any self-adjoint realization of \( T \) must satisfy

\[
\int_{\partial \Omega} J_\varphi \cdot n \, d\omega(x) = 0.
\]

Note that the commutator \([T, x_j] = \sigma_j \). Thus, in view of Heisenberg’s evolution equation, we may interpret \( J_\varphi(x) \) as the current density. As noted in \([4]\), it is straightforward to see that \( J_\varphi(x) \) vanishes pointwise if and only if the components of \( \varphi \) satisfy

(2) \quad \varphi_2(\gamma(s)) = iB(s)e^{i\alpha(s)}\varphi_1(\gamma(s)), \quad s \in [0, L),

for some real function \( B \), or when \( \varphi_1 \) equals zero at the boundary. Here \( \alpha(s) \) is the turning angle, i.e., the angle between \( n \) and the \( x_1 \)-axis at the point \( \gamma(s) \in \partial \Omega \).

In this article we focus on the case \( B = 1 \). In order to define the operator, let us first write the corresponding condition (2) in a more compact form that will become useful later on. For \( s \in [0, L) \) define \( a(s) := ie^{i\alpha(s)} \) and consider the matrix

(3) \quad A(s) := \left( \begin{array}{cc} 0 & a(s)^* \\ a(s) & 0 \end{array} \right).

Clearly, \( A(s) \) has eigenvalues 1 and \(-1\). We define the corresponding eigenprojections as

(4) \quad P_{\pm}(s) = (1 \pm A(s))/2.

It is easy to see that condition (2), for \( B = 1 \), is equivalent to \( P_-(s)\varphi(\gamma(s)) = 0 \). Let

\[ D_\infty := \{ \varphi \in H^1(\Omega, \mathbb{C}^2) : P_-(s)\varphi(\gamma(s)) = 0, s \in [0, L) \}. \]

We define the operator

\[
H_\infty : D_\infty \subset L^2(\Omega, \mathbb{C}^2) \to L^2(\Omega, \mathbb{C}^2),
\]

\[
H_\infty \varphi = T \varphi.
\]
It is known that $H_\infty$ is self-adjoint and that its spectrum is purely discrete (see Proposition 1, Remark 2, and Proposition 3 from Section 2 for further details).

In order to state the main result of the work at hand, Theorem 1 below, we introduce the Dirac operator defined on $\mathbb{R}^2$ with a mass term supported outside $\Omega$. For $M > 0$, we define

$$H_M : H^1(\mathbb{R}^2, \mathbb{C}^2) \subset L^2(\mathbb{R}^2, \mathbb{C}^2) \to L^2(\mathbb{R}^2, \mathbb{C}^2),$$

$$H_M \psi = T \psi + \sigma_3 M (1 - 1_\Omega) \psi,$$

where $1_\Omega$ is the characteristic function on $\Omega$ and $\sigma_3 = i \sigma_2 \sigma_1$. It is easy to see that $H_M$ is self-adjoint and has purely discrete spectrum on the interval $(-M, M)$ (see Lemma 4 below).

We are now in position to state the main result of our work.

**Theorem 1** (Convergence of Spectral Projections). Let $\Omega$ be a connected bounded domain with a $C^3$-boundary. Let $\lambda \in \mathbb{R}$ be an eigenvalue of $H_\infty$. Then, for any $0 < \varepsilon < \text{dist}(\lambda, \sigma(H_\infty) \setminus \{\lambda\})$, we have

$$\left\| \tilde{E}_{(\lambda)}(H_\infty) - E_{(\lambda - \varepsilon, \lambda + \varepsilon)}(H_M) \right\| \to 0 \quad \text{as} \quad M \to \infty,$$

where $\tilde{E}_{(\lambda)}(H_\infty) = E_{(\lambda)}(H_\infty) \oplus \{0\}$ with respect to the splitting $\mathcal{H} = L^2(\Omega, \mathbb{C}^2) \oplus L^2(\mathbb{R}^2 \setminus \Omega, \mathbb{C}^2)$. In particular, as $M \to \infty$, the eigenvalues of $H_M$ converge towards the eigenvalues of $H_\infty$ and any eigenvalue of $H_\infty$ is the limit of eigenvalues of $H_M$.

**Remark 1.** (i) The required $C^3$-regularity of the boundary is due to the application of our regularity result Theorem 2 below.

(ii) One can easily see that $H_{-M}$ converges, as $M \to \infty$, to the Dirac operator with the boundary condition (2) with $B = -1$. This can be shown using the antunitary transformation $U = i \sigma_2 \mathcal{C}$. Indeed, $U H_M U^{-1} = H_{-M}$ and if $\varphi \in D_\infty$ then $U \varphi = \tilde{\varphi}$ with $\tilde{\varphi}_2(\gamma(s)) = -i e^{i \alpha(s)} \tilde{\varphi}_1(\gamma(s))$ holds.

Let us briefly describe the strategy of the proof of the main result. We start by observing that both operators $H_\infty$ and $H_M$ have symmetric spectra with respect to zero (see Proposition 2 and Lemma 3). This enables us to study, instead, the spectra of the positive operators $H_\infty^2$ and $H_M^2$ and to apply the minimax principle.

Next, we give a lower bound for the quadratic form $\langle H_M \psi, H_M \psi \rangle$, which allows us to show that a function $\psi \in E_{(-A, A)}(H_M) L^2(\mathbb{R}^2, \mathbb{C}^2)$, for fixed $A > 0$ and $M \to \infty$, should satisfy $\|\psi\|_{L^2(\Omega)} \to 0$ and that $\|P_- \psi\|_{\sigma_1} \to 0$. In other words, in the limit $M \to \infty$, the function $\psi$ is supported inside $\Omega$ and satisfies the infinite mass boundary conditions (see lemmas 1 and 5 and Corollary 1).

The next step goes as follows: Given $A \notin \sigma(H_\infty)$ and $\varphi$ from the range of $E_{(-A, A)}(H_\infty)$, we construct a trial function $\psi_M \in D(H_M)$ with $\psi_M |_{\Omega^c} = \varphi, \psi_M$ exponentially small outside $\Omega$, and having the property $\langle H_M \psi, H_M \psi \rangle < A^2 + \epsilon(M)$ with $\epsilon(M)$ tending to zero as $M \to \infty$ (see Lemma 6). For sufficiently large $M > 1$, this implies that the dimension of the range of $E_{(-A, A)}(H_M)$ is at least as large as that of $E_{(-A, A)}(H_\infty)$. This construction uses some regularity properties of eigenfunction of the operator $H_\infty$ presented in Theorem 2.

To get the converse statement, we construct in Lemma 7 a function $\varphi \in D_\infty$ from a given $\psi \in E_{(-A, A)}(H_M) L^2(\mathbb{R}^2, \mathbb{C}^2)$, with $(H_\infty \psi, H_\infty \psi) < A^2 + \epsilon(M)$, such that $\|\varphi - \psi_M\|_{\Omega^c}$ and $\epsilon(M)$ tend to zero as $M \to \infty$. Finally, Lemma 4 completes the proof of the theorem.
2. Properties of $H_\infty$

We start by stating some general facts on $H_\infty$, namely its self-adjointness and the discreteness of its spectrum.

**Proposition 1.** Let $\Omega$ be a domain with a $C^2$-boundary. Then the operator $H_\infty$ defined above is self-adjoint on $D_\infty$.

**Remark 2.** A similar statement, for a more general class of Dirac operators in domains with $C^\infty$-boundaries, can be found in [13, Lemma 1]. Note however that the most difficult part of the proof, namely to show that the domain of the adjoint operator is contained in $H^1(\Omega, \mathbb{C}^2)$, can be found in [5] and in the references therein.

A more direct proof, which holds for $C^2$-boundaries, is given in [3].

Due to the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ we have that the spectrum of $H_\infty$ is discrete. Moreover, it is straightforward to see that $\sigma(H_\infty)$ is symmetric with respect to zero. Indeed, define $U := \sigma_1C$ where $C$ is the complex conjugation on $L^2(\Omega, \mathbb{C}^2)$. It is clear that $U$ is antiunitary and leaves $D_\infty$ invariant. That the spectrum is symmetric now follows from the relation $UH_\infty\varphi = -H_\infty U\varphi, \varphi \in D_\infty$.

We summarize these observations in the following statement.

**Proposition 2.** The operator $H_\infty$ has purely discrete spectrum and its spectrum is symmetric with respect to zero, that is,

$$ E \in \sigma(H_\infty) \text{ if and only if } -E \in \sigma(H_\infty). $$

The proof of the next result can be found in Appendix A.

**Lemma 1.** For any $\varphi \in D_\infty$ we have

$$ \|H_\infty\varphi\|^2 = \int_\Omega |\nabla \varphi(x)|^2\,dx + \frac{1}{2} \int_0^L \alpha'(s)|\varphi(\gamma(s))|^2\,ds. \quad (6) $$

**2.1. Regularity of eigenfunctions.** For a fix $0 < \delta < 1/\|\alpha'\|_\infty$ we define a neighbourhood $Q_0$ of $\partial\Omega$ as

$$ Q_0 := \{x \in \mathbb{R}^2 : \text{dist}(x, \partial\Omega) < \delta\}. \quad (7) $$

In this set we can use the direction of normal and tangent vectors to $\partial\Omega$ as local system of coordinates $(r, s)$. Indeed, the coordinates map is given by

$$ \kappa : (-\delta, \delta) \times [0, L) \to \mathbb{R}^2 \quad (8) $$

$$ \kappa(r, s) = \gamma(s) + r\mathbf{n}(s). $$

Using that

$$ \mathbf{n}(s) = (\cos \alpha(s), \sin \alpha(s)) \text{ and } \gamma'(s) = (-\sin \alpha(s), \cos \alpha(s)), $$

we readily obtain that

$$ \mathbf{n}'(s) = \alpha'(s)\gamma'(s) $$

and

$$ \partial_r\kappa(r, s) = \mathbf{n}(s) $$

$$ \partial_s\kappa(r, s) = \gamma'(s)(1 + r\alpha'(s)). $$

The Jacobian of the coordinates map is $(1 + \alpha' r)$. Thus $\kappa$ is a $C^1$-diffeomorphism whenever $\delta < 1/\|\alpha'\|_\infty$. Let us now relate derivatives in different coordinates. We have

$$ \partial_r + i\partial_s = \frac{\partial\kappa_1}{\partial r} + i\frac{\partial\kappa_1}{\partial s}\partial_1 + \frac{\partial\kappa_2}{\partial r} + i\frac{\partial\kappa_2}{\partial s}\partial_2 $$

$$ = e^{-i\alpha(s)}(\partial_1 + i\partial_2) + i\alpha'(s)(\cos \alpha(s)\partial_2 - \sin \alpha(s)\partial_1). $$

(9)
Analogously we obtain that
\( \partial_r - i\partial_s = e^{ia(s)}(\partial_1 - i\partial_2) - ir\alpha'(s)(\cos\alpha(s)\partial_2 - \sin\alpha(s)\partial_1). \)

This can be further simplified using the identity \( i(\cos\alpha(s)\partial_2 - \sin\alpha(s)\partial_1) = \frac{1}{2}[e^{-ia(s)}(\partial_1 + i\partial_2) - e^{ia(s)}(\partial_1 - i\partial_2)]. \) We obtain that
\[
\begin{pmatrix}
\partial_r \\
(1 + r\alpha')\partial_s
\end{pmatrix} = \begin{pmatrix}
\cos\alpha & \sin\alpha \\
-\sin\alpha & \cos\alpha
\end{pmatrix} \begin{pmatrix}
\partial_1 \\
\partial_2
\end{pmatrix}.
\]

Our next result is on the regularity of solutions \( \varphi \in D_\infty \) of the following eigenvalue problem
\[
\begin{align*}
H_\infty \varphi &= E\varphi & \text{in } L^2(\Omega, \mathbb{C}^2) \\
P^+\varphi &= 0 & \text{for almost all } s \in [0, L].
\end{align*}
\]

**Theorem 2.** Let \( \Omega \) be a domain with \( C^3 \)-boundary. If \( \varphi \in D_\infty \) is a solution of the eigenvalue problem (12), then \( \varphi \in H^2(\Omega, \mathbb{C}^2) \).

**Proof.** We define the following operator in \( L^2(Q_0, \mathbb{C}^2) \)
\[
(P\varphi)(x) = \frac{1}{2}(1 - A(s))\varphi(\kappa(r, s)), \quad x = \kappa(r, s) \in Q_0 \cap \Omega,
\]
where \( A \) is the matrix function defined in (3). Then, for \( \varphi \) satisfying (12), we have
\[
\begin{align*}
H_\infty \varphi &= E\varphi & \text{in } L^2(\Omega, \mathbb{C}^2) \\
P^+\varphi &= 0 & \text{on } \partial\Omega.
\end{align*}
\]

Since \( -\Delta \varphi = E^2 \varphi \) holds, in a distributional sense, we have using [9, Theorem 8.8] that \( \varphi \) is in \( H^2 \) on the interior of \( \Omega \). For \( x_0 \in \partial\Omega \) we denote by \( B_\rho(x_0) \) the open ball around \( x_0 \) with radius \( \rho > 0 \). Let \( \chi \in C^\infty(\mathbb{R}^2, [0, 1]) \) supported on \( B_{2\rho}(x_0) \) with \( \chi = 0 \) on \( \mathbb{R}^2 \setminus B_{2\rho}(x_0) \) and \( \chi = 1 \) on \( B_{\rho}(x_0) \). We choose \( \rho < \delta/2 \).

Next we show, using the eigenvalue equation (14), that
\[
\begin{align*}
-\Delta(P\chi \varphi) &= f & \text{in } \Omega \\
P\chi \varphi &= 0 & \text{on } \partial\Omega,
\end{align*}
\]
holds for some \( f \in L^2(\Omega, \mathbb{C}^2) \). To this end we note that
\[
H_\infty(P\chi \varphi) = EP\chi \varphi + [H_\infty, P\chi] \varphi
\]
\[
= (EP\chi + \frac{1}{2}[H_\infty, \chi]) \varphi - \frac{1}{2}[H_\infty, A\chi] \varphi,
\]
where \([\cdot, \cdot]\) denotes the commutator. Since \( EP\chi + \frac{1}{2}[H_\infty, \chi] \) is continuously differentiable and \( \varphi \in H^1(\Omega, \mathbb{C}^2) \) we see that the first term above is in \( H^1(\Omega, \mathbb{C}^2) \). A direct computation shows
\[
[H_\infty, A\chi] = \begin{pmatrix}
d^*\beta - \beta^*d \\
0 \\
0 & d\beta^* - \beta d^*
\end{pmatrix},
\]
where \( d := -i(\partial_1 + i\partial_2) \) and \( \beta := a\chi \) (see (3)). We have that
\[
H_\infty[H_\infty, A\chi] \varphi = \{H_\infty, [H_\infty, A\chi]\} \varphi - E[H_\infty, A\chi] \varphi,
\]
where \( \{\cdot, \cdot\} \) denotes the anticommutator. Observe that the second term on the right hand side of (17) is obviously square integrable. Moreover, for the first term we find that
\[
\{H_\infty, [H_\infty, A\chi]\} = \begin{pmatrix}
0 & [-\Delta, \beta] \\
[-\Delta, \beta^*] & 0
\end{pmatrix}.
\]
Since \( \alpha \in C^2 \) we get using (17) that \( H_\infty[H_\infty, A_\chi]|\varphi \in L^2(\Omega, C^2) \). Applying \( H_\infty \) to the l.h.s. and the r.h.s. of (16) yields (15) for
\[
f = H_\infty (EP \chi + \frac{1}{2}[H_\infty, \chi]) \varphi - \frac{1}{2} H_\infty[H_\infty, A_\chi]|\varphi \in L^2(\Omega, C^2).
\]
Equation (15) implies by [9] Theorem 8.12 that \( P_\chi \varphi \in H^2_0(\Omega, C^2) \). As a consequence we get
\[
(19) \quad \chi(\varphi - ie^{i\alpha} \varphi_1) \in H^2_0(\Omega).
\]
In particular, \( \varphi - ie^{i\alpha} \varphi_1 \in H^2(B_\rho(x_0) \cap \Omega) \). Since the boundary can be covered by finitely many balls and interior regularity holds we get, writing \( U := Q_0 \cap \Omega \),
\[
(20) \quad \varphi_2 - ie^{i\alpha} \varphi_1 \in H^2(U).
\]
According to [3] we have that \( E \neq 0 \). Substituting the eigenvalue equation \( \varphi = E^{-1} H \varphi \) in (20) we get
\[
(\partial_1 + i \partial_2) \varphi_2 - ie^{i\alpha}(\partial_1 - i \partial_2) \varphi_2 \in H^2(U).
\]
It follows from this
\[
(21) \quad e^{-i\alpha}(\partial_1 + i \partial_2) \varphi_2 - i(\partial_1 - i \partial_2) \varphi_2 \in H^2(U).
\]
Since \( (\partial_1 - i \partial_2)(\varphi_2 - ie^{i\alpha} \varphi_1) \in H^1(\Omega) \) by (20), we get that
\[
e^{-i\alpha}(\partial_1 + i \partial_2) \varphi_1 + i(\partial_1 - i \partial_2)(\varphi_2 - ie^{i\alpha} \varphi_1)
\]
belongs to \( H^1(\Omega) \). Since \( \alpha \in C^2 \) we find that
\[
e^{-i\alpha}(\partial_1 + i \partial_2) \varphi_1 + e^{i\alpha}(\partial_1 - i \partial_2) \varphi_1 \in H^1(\Omega).
\]
Finally in view of equations (19) and (11) we see that the latter expression equals \( 2 \partial_\gamma \varphi_1 \). This implies that \( \partial^2_\gamma \varphi_1 \) and \( \partial_\gamma \partial_\gamma \varphi_1 \) are square integrable in \( U \). Since \( -\Delta \varphi_1 = E^2 \varphi_1 \) holds we obtain also that \( \partial^2_\gamma \varphi_1 \in L^2(U) \). That \( \varphi_1 \in H^2(\Omega) \) follows from this and interior regularity. The analog statement for \( \varphi_2 \) can be deduced from the latter together with (20). This completes the proof. \( \square \)

3. Properties of \( H_M \)

We start by computing the quadratic energy of \( H_M \).

**Lemma 2.** For and \( \psi \in H^1(\mathbb{R}^2, C^2) \) we have
\[
\|H_M \psi\|^2 = \int_{\mathbb{R}^2} |\nabla \psi(x)|^2 dx + M^2 \int_{\mathbb{R}^2 \setminus \Omega} |\psi(x)|^2 dx
\]
\[
- M \int_0^L [\|P_+ \psi(\gamma(s))\|^2 - \|P_- \psi(\gamma(s))\|^2] ds,
\]
where \( P_\pm \) are the projections defined in (11).

**Proof.** For \( \psi = (\psi_1, \psi_2)^T \in H^1(\mathbb{R}^2, C^2) \), a direct computation shows that
\[
\|H_M \psi\|^2 = \int_{\Omega} |\nabla \psi(x)|^2 dx + M^2 \int_{\mathbb{R}^2 \setminus \Omega} |\psi(x)|^2 dx + 2M \Re \{ \frac{1}{2} \sigma \cdot \nabla \psi, \sigma_3(1 - 1_\Omega) \psi \}.
\]
Applying Green's identity we find that
\[
2M \text{Re}(\frac{1}{\gamma} \mathbf{\sigma} \cdot \nabla \psi, \sigma_3(1 - \mathbb{1}_\Omega)\psi) = iM \sum_{j=1}^2 \int_{\mathbb{R}^2 \setminus \Omega} \partial_j (\sigma_j \psi(x), \sigma_3 \psi(x)) dx
\]
\[
= -iM \int_{\partial \Omega} L \cdot \mathbf{n} d\omega,
\]
where \( L = (L_1, L_2) \) with \( L_j(s) = (\sigma_j \psi(\gamma(s)), \sigma_3 \psi(\gamma(s))) \) and \( \mathbf{n}(s) = (\cos \alpha(s), \sin \alpha(s)) \) is the outward normal vector. Therefore,
\[
\|H_M \psi\|^2 = \int_{\mathbb{R}^2} |\nabla \psi(x)|^2 dx + M^2 \int_{\mathbb{R}^2 \setminus \Omega} |\psi(x)|^2 dx
\]
\[
- 2M \text{Im}\{ \int_0^L \psi_2(\gamma(s)) \psi_1(\gamma(s)) e^{i\alpha(s)} ds \}.
\]
Using (22) we have, for any \( \psi \in H^1(\Omega) \),
\[
\|H_M \psi\|^2 \geq \int_{\Omega} |\nabla \psi(x)|^2 dx + \frac{1}{2} \int_0^L |\psi(\gamma(s))|^2 \alpha'(s) ds
\]
\[
+ M \int_0^L |P_+ \psi(\gamma(s))|^2 ds - \frac{C}{M} \int_0^L |\psi(\gamma(s))|^2 ds.
\]

**Lemma 3.** The operator \( H_M \) has purely discrete spectrum between \((-M, M)\). Moreover, \( E \in \sigma(H_M) \) if and only if \(-E \in \sigma(H_M)\).

**Proof.** The operator \( \tilde{H}_M := \frac{1}{\gamma} \mathbf{\sigma} \cdot \nabla + \sigma_3 M \) has a spectral gap in \((-M, M)\). Since the difference between \( H_M \) and \( \tilde{H}_M \) is relatively compact with respect to \( \tilde{H}_M \) we get that \( \sigma_{\text{ess}}(H_M) = \sigma_{\text{ess}}(\tilde{H}_M) \). That the spectrum of \( H_M \) is symmetric follows from the identity \( UH_M = -H_M U \), where \( U = \sigma_2 C \).

3.1. Energy estimates. In the remainder of this work we frequently use the Trace Theorem stated in the following form. For the proof see e.g. [8, Theorem 5.5.1].

**Proposition 3.** For each \( \varepsilon > 0 \) there is a constant \( C_\varepsilon > 0 \) such that for all \( \varphi \in H^1(\Omega, C^2) \)
\[
\|\varphi\|^2_{L^2(\Omega)} \leq \varepsilon \|\nabla \varphi\|^2_{L^2} + C_\varepsilon \|\varphi\|^2_{H^1}.
\]

**Lemma 4.** There exist constants \( C, M_0 > 0 \) such that for any \( \psi \in H^1(\mathbb{R}^2, C^2) \) and \( M > M_0 \) holds
\[
\|H_M \psi\|^2 \geq \int_{\Omega} |\nabla \psi(x)|^2 dx + \frac{1}{2} \int_0^L |\psi(\gamma(s))|^2 \alpha'(s) ds
\]
\[
+ M \int_0^L |P_+ \psi(\gamma(s))|^2 ds - \frac{C}{M} \int_0^L |\psi(\gamma(s))|^2 ds.
\]

**Proof.** Using (22) we have, for any \( \psi \in H^1(\Omega, C^2) \),
\[
\|H_M \psi\|^2 \geq \int_{\Omega} |\nabla \psi(x)|^2 dx + M \int_0^L |P_+ \psi(\gamma(s))|^2 ds
\]
\[
+ \int_{\mathbb{R}^2 \setminus \Omega} |\nabla \psi(x)|^2 dx + M^2 \int_{\mathbb{R}^2 \setminus \Omega} |\psi(x)|^2 dx
\]
\[
- M \int_0^L |\psi(\gamma(s))|^2 ds.
\]
We further set $\psi$ on $\mathbb{R}^2 \setminus (\Omega \cup Q_0)$. We find by the IMS localization formula

$$
\int_{\mathbb{R}^2 \setminus \Omega} |\nabla \psi(x)|^2 \, dx
\geq
\int_{\mathbb{R}^2 \setminus \Omega} [|\nabla \psi_u(x)|^2 + |\nabla \psi_v(x)|^2 - c^2|\psi_u(x)|^2 - c^2|\psi_v(x)|^2] \, dx,
$$

where $c = \max\{|\nabla u|_\infty, |\nabla v|_\infty\}$. Moreover, for $M > \sqrt{2c}$, we get

$$
(\text{27})
\int_{\mathbb{R}^2 \setminus \Omega} (M^2 - c^2)|\psi_u(x)|^2 \, dx \geq \frac{M^2}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\psi_u(x)|^2 \, dx.
$$

Thus,

$$
(\text{28})
\int_{\mathbb{R}^2 \setminus \Omega} |\nabla \psi(x)|^2 \, dx + M^2 \int_{\mathbb{R}^2 \setminus \Omega} |\psi(x)|^2 \, dx \geq F[\psi_u] + \frac{M^2}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\psi_u(x)|^2 \, dx,
$$

where

$$
F[\psi_u] := \int_{\mathbb{R}^2 \setminus \Omega} |\nabla \psi_u(x)|^2 \, dx + (M^2 - c^2) \int_{\mathbb{R}^2 \setminus \Omega} |\psi_u(x)|^2 \, dx.
$$

Our next goal is to estimate $F[\psi_u]$. Clearly,

$$
F[\psi_u] \geq \int_0^\delta \int_0^L \left[ |\partial_r \psi_u(\mathbf{k}(r, s))|^2 + (M^2 - c^2)|\psi_u(\mathbf{k}(r, s))|^2 \right] \left( 1 + ra'(s) \right) ds \, dr.
$$

In order to estimate the integral in $dr$ above we apply Lemma [11] (from Appendix [B]) with $k = \sqrt{M^2 - c^2}$, $\beta = \alpha'(s)$ and $f = \psi_u(\mathbf{k}(\cdot, s))$. To this end we set

$$
(\text{29})
I(s) := \int_0^\delta |\psi_u(\mathbf{k}(r, s))|^2 \, dr
$$

and define (compare with (70))

$$
(\text{30})
\mathcal{R}(s) := R[\psi_u(\mathbf{k}(\cdot, s))] = \frac{M^2 - c^2}{16} I(s) \mathbb{I}_K(s),
$$

where the set $K \subset \mathbb{R}^2$ is defined as

$$
(\text{31})
K := \text{supp} \left[ \max\{I(s) - 2|\psi(\gamma(s))|/\sqrt{M^2 - c^2}, 0\} \right].
$$

From Lemma [11] we get using $\psi_u(\mathbf{k}(0, s) = \psi(\gamma(s))$ that for $M$ sufficiently large

$$
F[\psi_u] \geq \sqrt{M^2 - c^2}||\psi||^2_{\mathcal{H}^1} + \frac{1}{2} \int_0^L |\psi(\gamma(s))|^2 \alpha'(s) ds + \int_0^L \mathcal{R}(s) ds + \mathcal{O}\left( \frac{1}{M} \right) \|\psi\|^2_{\mathcal{H}^1}
$$

$$
= M ||\psi||^2_{\mathcal{H}^1} + \frac{1}{2} \int_0^L |\psi(\gamma(s))|^2 \alpha'(s) ds + \int_0^L \mathcal{R}(s) ds + \mathcal{O}\left( \frac{1}{M} \right) ||\psi||^2_{\mathcal{H}^1}.
$$
Thus, combining the latter estimate with (26) and (28) yields

\[ \|H_M\psi\|^2 \geq \int_{\Omega} |\nabla \psi(x)|^2 \, dx + \frac{1}{2} \int_0^L |\psi(\gamma(s))|^2 \alpha'(s) \, ds + M\|P_\psi\|^2_{\partial\Omega} \]

(32)

\[ + \frac{M^2}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\psi_N(x)|^2 \, dx + \int_0^L R(s) \, ds + O\left(\frac{1}{M}\right)\|\psi\|^2_{\partial\Omega}. \]

Thus, we obtain (25) dropping the fourth and fifth term on the right hand side of (32).

Note that in the above proof we did not use the full strength of (32). However, we do so in the proof of the next lemma.

**Lemma 5.** For any \( A > 0 \) there are constants \( C, M_0 > 0 \) such that for any \( \psi \in E_{(-A,A)}(H_M)L^2(\mathbb{R}^2, \mathbb{C}^2) \) and any \( M > M_0 \)

\[ \|\psi\|^2_{H^1(\Omega)} \leq C\|\psi\|^2, \]

(33)

\[ \|\psi\|^2_{2\Omega} \leq C\|\psi\|^2, \]

(34)

\[ \|\psi\|^2_{2\Omega \setminus \Omega} \leq \frac{C}{M}\|\psi\|^2. \]

(35)

**Proof.** Note that (34) is a consequence of (33) and (24). According to (32) we have, for sufficiently large \( M > 0 \),

\[ \|H_M\psi\|^2 \geq \int_{\Omega} |\nabla \psi(x)|^2 \, dx - \|\alpha'\|_\infty \|\psi\|^2_{3\Omega} + \frac{M^2}{2}\|\psi\|^2. \]

(36)

Using (24) we get, for some \( c_1 > 0 \),

\[ \|H_M\psi\|^2 \geq \frac{1}{2}\|\nabla \psi\|^2_{\Omega} + \frac{M^2}{2}\|\psi\|^2 - c_1\|\psi\|^2. \]

(37)

Since \( \|H_M\psi\| \leq |A|\|\psi\| \) we obtain (33). Moreover, using again (37) we get for some constant \( c_2 > 0 \)

\[ \|\psi_N\|^2 \leq \frac{c_2}{M^2}\|\psi\|^2. \]

In order to prove (35) it suffices to show that \( \|\psi_N\|^2_{2\Omega \setminus \Omega} \leq C\|\psi\|^2/M. \) First note that since \( \delta < 1/\|\alpha'\|_\infty \) (see (7))

\[ \|\psi_N\|^2_{2\Omega \setminus \Omega} = \int_0^L \int_0^\delta |\psi_N(\kappa(r,s)))|^2(1 + r\alpha'(s)\) \, ds \, dr \leq 2 \int_0^L I(s) \, ds, \]

where \( I(s) \) is defined in (29). Using (24) as above we get from (32) that

\[ |A|^2\|\psi\|^2 \leq \|H_M\psi\|^2 \geq \int_0^L R(s) \, ds - c_1\|\psi\|^2. \]

(40)

This together with (30) and (31) imply that

\[ \int_0^L I(s)\mathbb{1}_K(s) \, ds \leq \frac{16(c_1 + |A|^2)}{M^2 - c^2}\|\psi\|^2. \]

(41)

Using again the definition of \( K \) (31) and (34) we further obtain that

\[ \int_0^L I(s)(1 - \mathbb{1}_K(s)) \, ds \leq \frac{2}{\sqrt{M^2 - c^2}}\int_0^L |\psi(\gamma(s))|^2 \, ds \leq \frac{2C\|\psi\|^2}{\sqrt{M^2 - c^2}}. \]

Thus, combining the latter inequality with (41) and (39) we obtain (35). \( \square \)
Corollary 1. For any $A > 0$ there are constants $C, M_0 > 0$ such that for any $\psi \in E_{(-A, A)}(H_M)L^2(\mathbb{R}^2, \mathbb{C}^2)$ and any $M > M_0$

\begin{align}
(42) \quad &\|\psi\|^2_{H^1(\Omega)} \leq C\|\psi\|^2_{L^2} , \\
(43) \quad &\|H_M\psi\|^2 \geq \int_{\Omega} |\nabla \psi(x)|^2 dx + \int_{\partial \Omega} |\psi(\gamma(s))|^2 \alpha'(s) ds \frac{d}{\phi} - \frac{C}{M} \|\psi\|^2 , \\
(44) \quad &\|P_{-}\psi\|^2_{\partial \Omega} \leq \frac{C}{M} \|\psi\|^2 .
\end{align}

Proof. The estimate (42) follows from (33) and (35). From (34) and (25) we obtain (43). Finally (44) is a consequence of (23), (34), and the fact that $\|H_M\psi\| \leq |A|\|\psi||$.

4. PROOF OF THE MAIN THEOREM

Lemma 6. For $A \notin \sigma(H_\infty)$ assume that

$$\dim\text{Ran}E_{(-A, A)}(H_\infty) = N.$$

Then there is $M_0 > 0$ such that for all $M > M_0$ we find $L_N \equiv L_N(M) \subset H^1(\mathbb{R}^2, \mathbb{C}^2)$ with $\dim L_N = N$ and

$$\|H_M\varphi\|^2 < A^2\|\varphi\|^2 , \quad \varphi \in L_N .$$

I.e., $H_M$ has at least $N$ eigenvalues in $(-A, A)$ for all $M > M_0$.

Proof. Recall the definition of $Q_0$ from (17). Let

$$c_1 := \max\{\lambda \in [0, A) : \lambda \text{ is an eigenvalue of } H_\infty\} .$$

For any $\varphi \in M_N := \text{Ran}E_{(-A, A)}(H_\infty)$ normalized we define

$$\psi_M(x) := \begin{cases} 
\varphi(x), & x \in \Omega \\
\varphi(\gamma(s))e^{-M_r}\zeta(r), & x = \kappa(r, s) \in Q_0 \cap \{x : x \notin \Omega\} , \\
0, & x \notin Q_0 \cup \Omega .
\end{cases}$$

where $\zeta \in C^2(\mathbb{R}, [0, 1])$ with $\zeta(r) = 1$ for $r \in [0, \delta/2]$ and vanishes for $r > \delta$.

We denote by $L_N$ the linear subspace of all such $\psi_M$ with $\varphi \in M_N$. Clearly, $\dim L_N = \dim M_N = N$. Since $\varphi \in H^2(\Omega, \mathbb{C}^2)$, by Theorem 2 we see that $\varphi(\gamma(s)), \varphi(\gamma(s))' \in L^2(\partial \Omega, \mathbb{C}^2)$. In particular, $L_N \subset H^1(\mathbb{R}^2, \mathbb{C}^2)$. Let $(\varphi_j)_{j \in \mathbb{N}}$ be a basis in $M_N$ orthonormal in the $L^2$ sense. Defining

$$\beta_N := \max_{j=1, \ldots, N} \|\varphi_j\|^2_{H^1(\partial \Omega, \mathbb{C}^2)},$$

we get, for any normalized $\varphi \in M_N$,

$$\|\varphi\|^2_{H^1(\partial \Omega, \mathbb{C}^2)} \leq N\beta_N .$$

Let $\psi_M \in L_N$ with $\psi_M 1_{\Omega} = \varphi \in M_N$ be such that $\|\varphi|| = \|\psi_M\|_{\Omega} = 1$. We first show that

$$\|\psi_M\|^2 = 1 + \mathcal{O}(1/M) \quad \text{as} \quad M \to \infty .$$

This follows from the estimate

$$\|\psi_M\|^2_1 \leq 2\|\varphi\|^2_{\partial \Omega} \int_0^\infty e^{-2M_r} dr = \|\varphi\|^2_{\partial \Omega} / M \leq N\beta_N / M ,$$

as $M \to \infty$. \\

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since
\[ \|\psi_M\|^2 = 1 + \int_0^\delta \int_0^L |\varphi(\gamma(s))| e^{-2Mr_\alpha} \zeta^2(r)(1 + \alpha(s)r)dsdr. \]

Next we estimate the \( \|H_M\psi_M\|^2 \). Using \( (22) \) (see also \( (11) \)) we get
\[
\|H_M\psi_M\|^2 = \|\nabla \varphi\|^2_\Omega + M^2 \int_0^\delta \int_0^L |\varphi(\gamma(s))| e^{-2Mr_\alpha} \zeta^2(r)(1 + \alpha(s)r)dsdr
\]
\[
+ \int_0^\delta \int_0^L [\partial_s \varphi(\gamma(s)) e^{-Mr_\alpha} \zeta(r)]^2 (1 + \alpha(s)r)^{-1} ds dr
\]
\[
+ \int_0^\delta \int_0^L [\partial_r \varphi(\gamma(s)) e^{-Mr_\alpha} \zeta(r)]^2 (1 + \alpha(s)r) ds dr - M \|\varphi\|^2_\Omega
\]
\[
=: \|\nabla \varphi\|^2_\Omega + I_1 + I_2 + I_3 - M \|\varphi\|^2_\Omega,
\]
where we used that \( P_+(s) \varphi(\gamma(s)) = \varphi(\gamma(s)) \) for \( s \in [0, L] \). We estimate the terms in the right hand side of the previous equality. For \( I_2 \) we have
\[
I_2 \leq cN \beta_N \int_0^\infty e^{-2Mr_\alpha} dr = O(1/M),
\]
for some positive constant \( c \). Furthermore,
\[
I_3 = M^2 \int_0^\delta \int_0^L |\varphi(\gamma(s))| e^{-Mr_\alpha} \zeta^2(r)(1 + \alpha(s)r)dsdr
\]
\[
- 2M \int_0^\delta \int_0^L |\varphi(\gamma(s))| e^{-Mr_\alpha} \zeta'(r) \zeta(r)(1 + \alpha(s)r)dsdr
\]
\[
+ \int_0^\delta \int_0^L |\varphi(\gamma(s))| e^{-Mr_\alpha} \zeta'(r)^2(1 + \alpha(s)r) ds dr
\]
\[
=: I_{3,1} + I_{3,2} + I_{3,3}.
\]

Using that \( \zeta'(r) = 0 \) for \( r \in [0, \delta/2] \) we get
\[
I_{3,2} \leq 4M \|\varphi\|^2_\Omega \|\zeta'|_\infty e^{-M\delta_\alpha} = O(Me^{-M\delta}).
\]

Similarly, we see that
\[
I_{3,3} = O(e^{-M\delta}).
\]

Noting that \( I_{3,1} = I_1 \) we have altogether
\[
(49) \quad \|H_M\psi_M\|^2 = \|\nabla \varphi\|^2_\Omega + 2I_1 - M \|\varphi\|^2_\Omega + O(1/M).
\]

Finally we estimate \( I_1 \) as follows
\[
2I_1 = 2M^2 \|\varphi\|^2_\Omega \int_0^\delta e^{-2Mr_\alpha} \zeta^2(r) dr
\]
\[
+ 2M^2 \int_0^L |\varphi(\gamma(s))|^2 \alpha'(s) ds \cdot \int_0^\delta e^{-2Mr_\alpha} \zeta^2(r) r dr
\]
\[
\leq M \|\varphi\|^2_\Omega + 2M^2 \int_0^\delta e^{-2Mr_\alpha} \zeta^2(r) r dr \cdot \int_0^L |\varphi(\gamma(s))|^2 \alpha'(s) ds.
\]
In addition we have that
\[\int_0^\delta e^{-2M r^2}r dr = \int_0^{\delta/2} e^{-2M r^2}r dr + \int_{\delta/2}^\delta e^{-2M r^2}r dr\]
\[= \frac{1}{4M^2} \int_0^\infty e^{-u}udu - \frac{1}{4M^2} \int_{\delta/2}^\infty e^{-u}udu + \int_{\delta/2}^\delta e^{-2M r^2}r dr\]
\[= \frac{1}{4M^2} + O(e^{-M\delta/2}).\]
This implies that
\[2I_1 \leq M\|\varphi\|^2_{\partial\Omega} + \frac{1}{2} \int_0^L |\varphi(\gamma(s))|^2 a'(s)ds + O(M^2 e^{-M\delta/2}).\]
Therefore, according to (49), we find
\[\liminf_{j \to \infty} \|H_M \psi_M\|^2 = \|\nabla \varphi\|^2_{\Omega} + \frac{1}{2} \int_0^L |\varphi(\gamma(s))|^2 a'(s)ds + O(1/M) \leq c_1^2 + O(1/M),\]
where in the last inequality we use (39) and (43). This together with (48) implies that
\[\|H_M \psi_M\|^2/\|\psi_M\|^2 \leq c_1^2 + O(1/M), \quad \psi_M \in L_N.\]
Since \(c_1 < A\) we get, by the Spectral Theorem, that \(\dim \text{Ran}(-A,A)(H_M) \geq N.\)

**Lemma 7.** Let \(0 < A \notin \sigma(H_\infty)\) be fixed. Then there is a constant \(M_0 > 0\) such that
\[\dim \text{Ran} E_{(-A,A)}(H_\infty) = \dim \text{Ran} E_{(-A,A)}(H_M),\]
for any \(M > M_0.\)

**Proof.** Let \(N := \dim \text{Ran} E_{(-A,A)}(H_\infty).\) That
\[\dim \text{Ran} E_{(-A,A)}(H_M) \geq N\]
follows from Lemma 6 for large \(M > 0.\) Assume that the reverse inequality does not hold. Due to our assumption there exists a sequence \((M_j)_{j \in \mathbb{N}}\) with \(M_j \to \infty\) such that
\[\dim \text{Ran} E_{(-A,A)}(H_{M_j}) \geq N + 1.\]
Hence we can find a normalized function \(\psi_j \in \text{Ran} E_{(-A,A)}(H_{M_j})\) which is orthogonal to the eigenfunctions of \(H_\infty\) with eigenvalues in \((-A,A)\) (extended by zero in \(\mathbb{R}^2 \setminus \Omega)\). Define \(\varphi_j := \psi_j \chi_\Omega.\) Due to equation (35), (44), and (42) we have that \(\|\varphi_j\|_{H^1(\Omega)}\) is bounded uniformly in \(M_j\) and, moreover, as \(j \to \infty\)
\[\|\varphi_j\|_{\Omega} \to 1 \quad \text{and} \quad \|P_- \varphi_j\|_{\partial\Omega} \to 0.\]
In particular, by the Theorem of Banach-Alaoglu, the sequence \((\varphi_j)_{j \in \mathbb{N}}\) contains a subsequence (also called \((\varphi_{j_k})\)) such that, as \(j \to \infty\),
\[\varphi_{j_k} \rightharpoonup \varphi \quad \text{in} \quad H^1(\Omega, \mathbb{C}^2),\]
with
\[\|\varphi\|_{H^1(\Omega, \mathbb{C}^2)} \leq \liminf_{j \to \infty} \|\varphi_{j_k}\|_{H^1(\Omega, \mathbb{C}^2)}.\]
In addition, using the Theorem of Rellich-Kondrachov, see [1, Theorem 6.2 (4)],
\begin{equation}
\| \varphi_j - \varphi \|_\Omega \to 0, \quad \| \varphi_j - \varphi \|_{\partial \Omega} \to 0, \quad j \to \infty,
\end{equation}
which implies that
\[ \| \varphi \|_\Omega = 1 \quad \| P_- \varphi \|_{\partial \Omega} = 0. \]

Therefore, \( \varphi \in D(H_\infty) \) and satisfies \( \varphi \perp \text{Ran} E_{(-A,A)}(H_\infty) \). Let \( \lambda_n > 0 \) the largest eigenvalue of \( H_\infty \) in \((-A,A)\) and \( \lambda_{n+1} > A \) be the next positive eigenvalue of \( H_\infty \). Define \( \varepsilon = (\lambda_{n+1}^2 - A^2)/2 \). Then, for \( j \) large enough, we have in view of (53) and (56) that
\[ \| H_\infty \varphi \|^2 = \| \varphi \|_{H^1(\Omega, C^2)}^2 - \| \varphi \|_{\partial \Omega}^2 + \frac{1}{2} \int_{\partial \Omega} |\varphi(\gamma(s))|^2 \alpha'(s) ds \
\leq \| \varphi_j \|_{H^1(\Omega, C^2)}^2 - \| \varphi_j \|_{\partial \Omega}^2 + \frac{1}{2} \int_{\partial \Omega} |\psi_j(\gamma(s))|^2 \alpha'(s) ds + \frac{\varepsilon}{2} \\
= \int_{\Omega} |\nabla \psi_j(x)|^2 dx + \frac{1}{2} \int_{\partial \Omega} |\psi_j(\gamma(s))|^2 \alpha'(s) ds + \frac{\varepsilon}{2} \\
\leq \| H_M \psi_j \|^2 + \varepsilon,
\]
where in the last inequality we used (43). The last inequality contradicts the assumption that \( \varphi \perp \text{Ran} E_{(-A,A)}(H_\infty) \) since \( \| H_M \psi_j \|^2 + \varepsilon < \lambda_{n+1}^2 \).

**Corollary 2.** As \( M \to \infty \) the eigenvalues of \( H_M \) converge uniformly on each bounded spectral interval \((A,B)\) against the eigenvalues of \( H_\infty \). More precisely, if \( \lambda_j \) and \( \lambda_{j+1} \) are two subsequent eigenvalues of \( H_\infty \), \( \lambda_{j+1} > \lambda_j \), then for each \( \varepsilon > 0 \) there exists \( M_0 > 0 \) such that for all \( M > M_0 \)
\[ \dim \text{Ran} E_{\lambda_j}(H_\infty) = \dim \text{Ran} E_{(\lambda_j - \varepsilon, \lambda_j + \varepsilon)}(H_M), \]
and
\[ E_{(\lambda_j + \varepsilon, \lambda_{j+1} - \varepsilon)}(H_M) = \emptyset. \]

**Proof.** This follows from Lemma 7 and the symmetry of the spectra of \( H_M \) and \( H_\infty \).

**Lemma 8.** Let \( 0 < A \notin \sigma(H_\infty) \) be fixed. Then,
\[ \| \tilde{E}_{(-A,A)}(H_\infty) - E_{(-A,A)}(H_M) \| \to 0, \quad \text{as} \quad M \to \infty. \]
Here \( \tilde{E}(H_\infty) = E(H_\infty) \oplus \{ 0 \} \) with respect to the splitting \( H = L^2(\Omega, C^2) \oplus L^2(\mathbb{R}^2 \setminus \Omega, C^2) \).

**Proof.** Let \( N := \dim \text{Ran} \tilde{E}_{(-A,A)}(H_\infty) \) and let \( \varphi_j, j = 1, 2, \ldots, N, \) be an orthonormal basis on the range of \( \tilde{E}_{(-A,A)}(H_\infty) \). For each \( \varphi_j \) we define \( \psi_j^M \) according to (46). Due to (48) we have, for \( M > 1 \) large enough, that
\[ \| \varphi_j - \psi_j^M \| = O(M^{-1/2}). \]
In addition,
\[ (\psi_j^M, \psi_k^M) = O(M^{-1}), \quad j \neq k. \]
Let
\[ L^M := \text{span} \{ \psi_1^M, \ldots, \psi_N^M \}. \]
Clearly \(\dim L = N\). Let \(P^M\) be the orthogonal projection onto \(L \subset L^2(\mathbb{R}^2, \mathbb{C}^2)\). Due to (57) holds that
\[
\|E_{(-A,A)}(H_\infty) - P^M\| \to 0, \quad \text{as} \quad M \to \infty.
\]
Next we show that \(\|P^M - E_{(-A,A)}(H_M)\| \to 0\) as \(M \to \infty\). Let \(0 \leq |\lambda_1| < |\lambda_2| < \cdots < |\lambda_J| < \cdots\) be the absolute values of the eigenvalues of \(H_\infty\). (We allow these eigenvalues to be degenerate.) Define \((A_j)_{j \in \mathbb{N}}\) as \(A_j := (|\lambda_j| + |\lambda_{j+1}|)/2\). Let \(\varphi_1, \ldots, \varphi_{p_j}\) be an orthonormal basis of eigenfunctions of \(H_\infty\) on the range of \(E_{(-A_j,A_j)}(H_\infty)\). Using the functions \(\varphi_1, \ldots, \varphi_{p_j}\) we construct \(\psi^M_1, \ldots, \psi^M_{p_j}\) as in (40) and denote by \(P^M_j\) the orthogonal projection onto the span\{\(\psi^M_1, \ldots, \psi^M_{p_j}\)\}.

We now show, using induction, that
\[
\|P^M_j - E_{(-A_j,A_j)}(H_M)\| \to 0 \quad \text{as} \quad M \to \infty,
\]
for each \(j < N\), where \(N > 0\) is some arbitrary fixed number. We set \(I_j := (-A_j, A_j)\)
\[
\mu_j(M) := \min\{|\lambda| : \lambda \in \sigma(H_M) \cap I_j \setminus I_{j-1}\},
\]
with the convention \(I_0 = \emptyset\). Notice that \(\mu_j(M) \to |\lambda_j|\) as \(M \to \infty\). Due to Corollary 2 we may assume that \(M > 0\) is so large that, for all \(j < N\),
\[
\dim \text{Ran} P^M_j = \dim \text{Ran} E_{I_j}(H_M).
\]
Then, according to [10] Theorem 1.6.34, the norm in (61) equals the norms \(\|(1 - E_{I_j}(H_M)) P^M_j\| = \|E_{I_j}(H_M)(1 - P^M_j)\|\).

**Induction start:** We write \(P^\perp := 1 - P\) for an orthogonal projection \(P\). Let \(\psi \in P^\perp L^2(\mathbb{R}^2, \mathbb{C}^2)\). By the spectral theorem we have
\[
\|H_M \psi\|^2 \geq \mu_1^2(M) \|E_{I_1}(H_M) \psi\|^2 + A_1^2 \|E_{I_1}(H_M) \psi\|^2.
\]
According to (52) we have
\[
\|H_M \psi\|^2 \leq \lambda_1^2 \|\psi\|^2 + O(1/M) \|\psi\|^2
\]
\[
= \lambda_1^2 \|E_{I_1}(H_M) \psi\|^2 + \lambda_1^2 \|E_{I_1}(H_M) \psi\|^2 + O(1/M) \|\psi\|^2.
\]
A combination of the above inequality and (52) together with the fact that \(\mu_1(M) \to |\lambda_1|\) as \(M \to \infty\) implies that \(\|E_{I_1}(H_M) P^M_j\| \to 0\) as \(M \to \infty\).

**Induction step:** Assume that the statement holds for the interval \(I_j\). Let \(\epsilon > 0\) and \(\psi \in (1 - P^M_j) P^M_{j+1} L^2(\mathbb{R}^2, \mathbb{C}^2)\). We have
\[
\|H_M \psi\|^2 \geq \|H_M E_{I_{j+1} \setminus I_j}(H_M) \psi\|^2 + A_{j+1}^2 \|E_{I_{j+1} \setminus I_j}(H_M) \psi\|^2.
\]
Using that \(\psi = (1 - P^M_j) \psi\) we find that
\[
\|H_M E_{I_{j+1} \setminus I_j}(H_M) \psi\|^2 \geq \|H_M E_{I_{j+1} \setminus I_j}(H_M) \psi\|^2 - |A_j| \|E_{I_j}(H_M)(1 - P^M_j)\| \|\psi\|^2.
\]
The last term above converges to zero as \(M \to \infty\) due to the induction hypothesis. Therefore, we get for sufficiently large \(M\),
\[
\|H_M \psi\|^2 \geq \mu_{j+1}(M)^2 \|E_{I_{j+1} \setminus I_j}(H_M) \psi\|^2 + A_{j+1}^2 \|E_{I_{j+1} \setminus I_j}(H_M) \psi\|^2 - \epsilon \|\psi\|^2.
\]
Using (52) we obtain
\[
\|H_M \psi\|^2 \leq \lambda_{j+1}^2 \|\psi\|^2 + O(1/M) \|\psi\|^2
\]
\[
\leq \lambda_{j+1}^2 \|E_{I_{j+1} \setminus I_j}(H_M) \psi\|^2 + \lambda_{j+1}^2 \|E_{I_{j+1} \setminus I_j}(H_M) \psi\|^2 + (\epsilon + O(1/M)) \|\psi\|^2.
\]
A combination of this with (54) gives that \( \|E_{l_{j+1}}(H_M)\|_{1}/\|\psi\| \to 0 \), since \( \mu_{j+1}(M) \to |\lambda_{j+1}| \). From this follows that

\[
(65) \quad \|E_{l_{j+1}}(H_M)^{1\over 2}(1-P^M_{j+1})\| \to 0, \quad \text{as} \quad j \to 0.
\]

Finally, the above equation (65), the identity

\[
E_{l_{j+1}}(H_M)^{1\over 2}P^M_{j+1} = E_{l_{j+1}}(H_M)^{1\over 2}E_{l_1}(H_M)^{1\over 2}P^M_{j+1} + E_{l_{j+1}}(H_M)^{1\over 2}(1-P^M_{j+1})P^M_{j+1},
\]

and the induction hypothesis imply the claim. \( \square \)

**Lemma 9.** Let \( \lambda \in \sigma(H_\infty) \). Then, for any \( \varepsilon > 0 \), holds

\[
(66) \quad \|E(\lambda) - E(\lambda-\varepsilon,\lambda+\varepsilon)(H_M)\| \to 0 \quad \text{as} \quad M \to \infty,
\]

where \( E(\lambda) \) is defined as in Lemma 8.

**Proof.** We show the statement by contradiction. Recall that \( \lambda \neq 0 \). If (66) does not hold there exists an eigenfunction \( \phi_M \) of \( H_M \) with eigenvalue \( \lambda_M \) belonging to the range of \( (\lambda-\varepsilon,\lambda+\varepsilon) \) such that \( \|\phi_M - \phi\| \to 0 \), \( M \to \infty \), where \( \phi \) is an eigenfunction of \( H_\infty \) with eigenvalue \( -\lambda \). Using the test function \( \psi_M \) constructed from \( \phi \) as in (46) we have

\[
\langle \phi_M, \varphi \rangle = -\lambda^{-1}\langle \phi_M, H_\infty \varphi \rangle = -\lambda^{-1}\int_\Omega \overline{\phi_M(x)}[H_M\psi_M](x)dx
\]

\[
= -\lambda^{-1}\int_{\mathbb{R}^2\setminus\Omega} \overline{\phi_M(x)}[H_M\psi_M](x)dx + \lambda^{-1}\int_{\mathbb{R}^2\setminus\Omega} \overline{\phi_M(x)}[H_M\psi_M](x)dx
\]

\[
\leq -\lambda_M \lambda^{-1}\langle \phi_M, \psi_M \rangle + |\lambda^{-1}|\|H_M\psi_M\|_\lambda \|\phi_M\|_{\mathbb{R}^2\setminus\Omega}.
\]

Since \( \|H_M\psi_M\| \) is uniformly bounded by (52), \( \|\phi_M\|_{\mathbb{R}^2\setminus\Omega} \) and \( \|\varphi - \psi_M\| \) converge to zero, and \( \lambda_M \to \lambda \) we find a contradiction when taking the limit \( M \to \infty \) of the above inequality. \( \square \)

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**Appendix A. Proof of Lemma 1**

**Lemma 10.** For any \( \varphi \in \mathcal{D}_\infty \) there is a sequence \( (\varphi_n)_{n\in\mathbb{N}} \subset C^1(\overline{\Omega},\mathbb{C}^2) \) with \( \varphi_n \to \varphi \) in the \( H^1 \)-norm, such that \( \varphi_n \) satisfies the boundary conditions, i.e., \( P^-\varphi_n = 0 \) for all \( n \in \mathbb{N} \).

**Proof.** Recall the definition of \( Q_0 \) in (7). Let \( \chi \in C^\infty((0,1]) \) be a smooth characteristic function with \( \chi(r) = 0 \) for \( r \in (-\delta/4,0] \) and \( \chi(r) = 1 \) for \( r \in (-\delta, -\delta/2) \). We define the function \( \tilde{\alpha}: \overline{\Omega} \to \mathbb{R} \)

\[
\tilde{\alpha}(x) = \chi(r)\alpha(s) \quad \text{for} \quad x = \kappa(r,s) \in Q_0 \cap \overline{\Omega},
\]

and being zero otherwise.
For \( \varphi \in D_\infty \) we define
\[
\psi_1 := \varphi_2 - ie^{i\alpha} \varphi_1 \in H^1_0(\Omega), \\
\psi_2 := \varphi_2 + ie^{i\alpha} \varphi_1 \in H^1(\Omega).
\]

There exist sequences \((\psi_1)_n \in C^\infty(\Omega)\) and \((\psi_2)_n \in C^\infty(\overline{\Omega})\) converging to \(\psi_1\) and \(\psi_2\) in the \(H^1\)-norm, respectively. We define further, for \(n \in \mathbb{N}\), the following \(C^1\)-functions
\[
\varphi_{1,n} := \frac{e^{-i\alpha}}{2i}(\psi_{2,n} - \psi_{1,n}), \\
\varphi_{2,n} := \frac{1}{2}(\psi_{1,n} + \psi_{2,n}).
\]

Clearly \(\varphi_{1,n}\) and \(\varphi_{2,n}\) converge to \(\varphi_1\) and \(\varphi_2\) in the \(H^1\)-norm, respectively. Moreover, since \(\psi_{1,n}|_{\partial\Omega} = 0\), one easily verifies that \(\varphi_n := (\varphi_{1,n},\varphi_{2,n})^T\) satisfies the boundary conditions.

**Proof of Lemma 4** We compute, for \(\varphi \in C^1(\Omega,\mathbb{C}^2)\) satisfying the boundary conditions,
\[
\|H_\infty \varphi\|^2 := \int_{\Omega} \left( \frac{1}{i} \sigma \cdot \nabla \varphi(x), \frac{1}{i} \sigma \cdot \nabla \varphi(x) \right) dx = \int_{\Omega} \left( \sigma \cdot \nabla \varphi(x), \sigma \cdot \nabla \varphi(x) \right) dx,
\]
\[
= \sum_{j \neq k} \int_{\Omega} (\sigma_j \partial_j \varphi(x), \sigma_k \partial_k \varphi(x)) dx + \sum_{j=1}^3 \int_{\Omega} (\sigma_j \partial_j \varphi(x), \sigma_j \partial_j \varphi(x)) dx
\]
\[
=: T_1 + T_2.
\]

For the second term above we have \(T_2 = \|\nabla \varphi\|^2_{\Omega}\). Moreover, using that \(\sigma_1 \sigma_2 = -\sigma_2 \sigma_1\) and Green’s identity we obtain
\[
T_1 = \int_{\Omega} (\partial_1 \varphi(x), \sigma_1 \sigma_2 \partial_2 \varphi(x)) dx + \int_{\Omega} (\partial_2 \varphi(x), \sigma_2 \sigma_1 \partial_1 \varphi(x)) dx
\]
\[
= \int_{\Omega} \left[ \partial_1 (\varphi(x), \sigma_1 \sigma_2 \partial_2 \varphi(x)) + \partial_2 (\varphi(x), \sigma_2 \sigma_1 \partial_1 \varphi(x)) \right] dx
\]
\[
= \int_0^L \left[ (\varphi(\gamma(s)), \sigma_1 \sigma_2 \partial_2 \varphi(\gamma(s))) \cos \alpha(s) + (\varphi(\gamma(s)), \sigma_2 \sigma_1 \partial_1 \varphi(\gamma(s))) \sin \alpha(s) \right] ds.
\]

For \(s \in [0,L]\) we set
\[
S_1(s) := (\varphi(\gamma(s)), \sigma_1 \sigma_2 \partial_2 \varphi(\gamma(s))) \cos \alpha(s) + (\varphi(\gamma(s)), \sigma_2 \sigma_1 \partial_1 \varphi(\gamma(s))) \sin \alpha(s).
\]

A simple computation yields (in a slight abuse of notation we write \(\varphi\) for \(\varphi(\gamma(\cdot))\))
\[
S_1 = (\varphi_1 \partial_2 \varphi_1 - i \varphi_2 \partial_2 \varphi_2) \cos \alpha + (-i \varphi_1 \partial_1 \varphi_1 + i \varphi_2 \partial_1 \varphi_2) \sin \alpha
\]
\[
= \frac{1}{2} \left[ -e^{i\alpha} \varphi_1 (\partial_1 - i \partial_2) \varphi_1 + e^{-i\alpha} \varphi_1 (\partial_1 + i \partial_2) \varphi_1 + e^{i\alpha} \varphi_2 (\partial_1 - i \partial_2) \varphi_2 - e^{-i\alpha} \varphi_2 (\partial_1 + i \partial_2) \varphi_2 \right].
\]

Using (9) and (10) we see that at the boundary
\[
\partial_1 \pm i \partial_2 = e^{\pm i\alpha} (\partial_1 \pm i \partial_2).
\]

Therefore,
\[
S_1 = \frac{1}{2} \left[ -\varphi_1 (\partial_1 - i \partial_2) \varphi_1 + \varphi_1 (\partial_1 + i \partial_2) \varphi_1 + \varphi_2 (\partial_1 - i \partial_2) \varphi_2 - \varphi_2 (\partial_1 + i \partial_2) \varphi_2 \right]
\]
\[
=i(\varphi_1 \partial_1 \varphi_1 - \varphi_2 \partial_2 \varphi_2).
\]
Using the boundary conditions we obtain
\[ \overline{\varphi_2} \partial_s \varphi_2 = -ie^{-i\alpha} \overline{\varphi_1} \partial_s (ie^{i\alpha} \varphi_1) = e^{-i\alpha} \overline{\varphi_1} \partial_s (e^{i\alpha} \varphi_1) = \overline{\varphi_1} (i\alpha \varphi_1 + \partial_s \varphi_1). \]
This implies that
\[ S_1(s) = \alpha'(s) |\varphi_1(\gamma(s))|^2, \quad s \in [0, L]. \]
Thus, we obtain that
\[ \|H_\infty \varphi\|^2 = \int_\Omega |\nabla \varphi(x)|^2 dx + \int_0^L \alpha'(s) |\varphi_1(\gamma(s))|^2 ds. \]
From this follows \[ \text{(iii)}, \] since \( \varphi \in \mathcal{D}_\infty \) and hence \( |\varphi_1(\gamma(s))|^2 = |\varphi(\gamma(s))|^2/2 \). Thanks to Lemma \[ \text{(10)} \] the statement remains true for any \( \varphi \in H^1(\Omega, \mathbb{C}^2) \).

**Appendix B. Lower bound for an auxiliary functional**

**Lemma 11.** For \( \delta > 0 \) let \( f : [0, \delta] \to \mathbb{R} \in H^1 \) with \( f(\delta) = 0 \) and \( \beta, k \in \mathbb{R} \) with \( |\beta| < 1 \) and \( \delta|\beta| < 1/4 \). Define
\[ L[f] := \int_0^\delta (f'(t)^2 + k^2 f(t)^2)(1 + \beta t) dt. \]
Then, as \( k \to \infty \), we have
\[ L[f] \geq f(0)^2 [k + \beta/2] + f(0)^2 O(e^{-k\delta}) + R[f], \]
where
\[ R[f] = \left\{ \begin{array}{ll} \frac{\beta^2}{16} \|f\|^2, & \|f\|^2 > \frac{2}{k} f(0)^2 \\ 0, & \|f\|^2 \leq \frac{2}{k} f(0)^2. \end{array} \right. \]

**Proof.** We do the substitution \( y = kt \) and write \( \hat{f}(y) = f(y/k) \) in the integral in \[ \text{(68)} \] to get
\[ L[f] = \tilde{L}[\hat{f}] := k \int_0^{k\delta} (\hat{f}'(y)^2 + \hat{f}(y)^2) dy + \beta \int_0^{k\delta} y(\hat{f}'(y)^2 + \hat{f}^2(y)) dy \]
\[ =: kL_1[\hat{f}] + \beta L_2[\hat{f}] \]
Let \( g_0 \) be the minimizer of \( L_1 \) in \( C^2([0, k\delta]) \) subject to the boundary conditions \( g_0(0) = f(0), g_0(k\delta) = 0 \). Define \( h = \hat{f} - g_0 \) and note that \( h(0) = h(k\delta) = 0 \). Then
\[ \tilde{L}[g_0 + h] \geq \tilde{L}[g_0] + \tilde{L}[h] + 2k \int_0^{k\delta} (g_0(y)h'(y) + g_0(y)h(y)) dy \]
\[ + 2\beta \int_0^{k\delta} (g_0'(y)h'(y) + g_0(y)h(y)) dy. \]
Since \( g_0 \) satisfies the Euler-Lagrange equations integration by parts yields
\[ \int_0^{k\delta} (g_0'(y)h'(y) + g_0(y)h(y)) dy = 0, \]
\[ 2\beta \int_0^{k\delta} (g_0'(y)h'(y) + g_0(y)h(y)) dy = -2\beta \int_0^{k\delta} g_0'(y)h(y) dy. \]
By Schwarz inequality we get
\[ 2\beta \int_0^{k\delta} |g_0(y)||h(y)| dy \leq \frac{2\beta^2 \|g_0\|^2}{k} + \frac{k\|h\|^2}{2}. \]
Therefore,
\begin{equation}
\hat{L}[g_0 + h] \geq \hat{L}[g_0] - \frac{2\beta^2 \|g_0'\|^2}{k} + \hat{L}[h] - \frac{k\|h\|^2}{2}.
\end{equation}
Since $\delta \beta < 1/4$ holds $|\beta L_2[h]| \leq k/4L_1[h]$. Using this together with the fact that $L_1[h] \gg \|h\|^2$ we have that
\[\hat{L}[h] - \frac{k\|h\|^2}{2} \geq \frac{3k}{4}L_1[h] - \frac{k\|h\|^2}{2} \geq \frac{k}{4}\|h\|^2.\]
In addition, we have
\[\hat{L}[g_0] - \frac{2\beta^2 \|g_0'\|^2}{k} \geq (k - \frac{2\beta^2}{k})L_1[g_0] + \beta L_2[g_0].\]
Next we use that $g_0(y) = c_1(k)e^{-y} + c_2(k)e^{y}$, where the constants (which are determined from the boundary conditions $g_0(0) = f(0)$ and $g_0(k\delta) = 0$) are given by
\[c_1(k) = \frac{e^{\delta k}f(0)}{e^{\delta k} - e^{-\delta k}}, \quad c_2(k) = \frac{-e^{-\delta k}f(0)}{e^{\delta k} - e^{-\delta k}}.\]
We note that, as $k \to \infty$,
\begin{equation}
\|g_0\|^2 = 1/2f(0)^2(1 + O(e^{-k\delta})),
\end{equation}
\[L_1[g_0] = c_1^2(k)(1 - e^{-k\delta}) + c_2^2(k)(1 - e^{+k\delta}) = f(0)^2(1 + O(e^{-k\delta})),
\end{equation}
\[L_2[g_0] = \frac{1}{2}c_1^2(k)(1 - e^{-k\delta}) - \frac{1}{2}c_2^2(k)(1 - e^{+k\delta}) = f(0)^2(\frac{1}{2} + O(e^{-k\delta})).\]
Therefore, altogether gives
\begin{equation}
\hat{L}[g_0 + h] \geq \frac{k}{4}\|h\|^2 + (k - \frac{2\beta^2}{k})f(0)^2(1 + O(e^{-k\delta}))
+ \beta f(0)^2(\frac{1}{2} + O(e^{-k\delta}))
= (k + \frac{\beta}{2})f(0)^2 + f(0)^2O(e^{-k\delta}) + \frac{k}{4}\|h\|^2.
\end{equation}
Notice that if $\|\hat{f}\|^2 > 2f(0)^2$ then according to (73) $\|h\|^2 \geq \|\hat{f}\|^2/4 + O(e^{-k\delta}) = k\|\hat{f}\|^2/4 + O(e^{-k\delta})$ which together with (74) implies the statement of the lemma. 

References

[1] R. A. Adams and J. F. Fournier. Sobolev spaces, volume 140 of Pure and Applied Mathematics. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] A. R. Akhmerov and C. W. J. Beenakker. Boundary conditions for Dirac fermions on a terminated honeycomb lattice. Phys. Rev. B, 77(8):085423, 2008.

[3] R. Benguria, S. Fournais, E. Stockmeyer, and H. Van den Bosch. Spectral gaps of Dirac operators with boundary conditions relevant for graphene. Preprint arXiv:1601.06607, 2016.

[4] M. V. Berry and R. J. Mondragon. Neutrino billiards: time-reversal symmetry-breaking without magnetic fields. Proc. Roy. Soc. London Ser. A, 412(1842):53–74, 1987.

[5] B. Booß-Bavnbek, M. Lesch, and C. Zhu. The Calderón projection: new definition and applications. J. Geom. Phys., 59(7):784–826, 2009.

[6] A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, and A.K. Geim. The electronic properties of graphene. Reviews of modern physics, 81:109–162, 2009.

[7] H. De Raedt and M.I. Katsnelson. Electron energy level statistics in graphene quantum dots. JETP letters, 88(9):607–610, 2009.

[8] L.C. Evans. Partial differential equations, volume 19 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, second edition, 2010.
[9] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[10] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[11] G. Marko and T. Milan. Electronic states and optical transitions in a graphene quantum dot in a normal magnetic field. *Serbian Journal of Electrical Engineering*, 8:53–62, 2011.

[12] T. Paananen and R. Egger. Finite-size version of the excitonic instability in graphene quantum dots. *Phys. Rev. B*, 84:155456, 2011.

[13] M. Prokhorova. The Spectral Flow for Dirac Operators on Compact Planar Domains with Local Boundary Conditions. *Comm. Math. Phys.*, 322(2):385–414, 2013.

[14] S. Schnez, K. Ensslin, M. Sigrist, and T. Ihn. Analytic model of the energy spectrum of a graphene quantum dot in a perpendicular magnetic field. *Phys. Rev. B*, 78:195427, 2008.

[15] M. Zarenia, A. Chaves, G.A. Farias, and F.M. Peeters. Energy levels of triangular and hexagonal graphene quantum dots: A comparative study between the tight-binding and dirac equation approach. *Phys. Rev. B*, 84(24):245403, 2011.

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