SCALING OF THE DIFFRACTION MEASURE OF 
k-FREE INTEGERS NEAR THE ORIGIN

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Abstract. Asymptotics are derived for the scaling of the total diffraction intensity for the 
set of $k$-free integers near the origin, which is a measure for the degree of patch fluctuations.

1. Introduction

Given a point set $\Lambda$ in Euclidean $d$-space, an immediate natural question to ask is that of 
the existence of its density, and further, if the density does exist, how it fluctuates locally. 
More generally, one can consider point patches and define means and variances for their 
appearance. When $\Lambda$ is a lattice, there is no fluctuation (in the sense that each patch repeats 
lattice-periodically), while a random set such as the positions of an ideal gas (as modelled by 
a Poisson process) shows a lot of fluctuation. What intermediate regimes exist is a natural 
aspect of spatial order, and such considerations have lately garnered much interest.

More recently, due to general progress in the theory of aperiodic order, see [2] and references 
therein for background, fluctuation or variance considerations have been extended to such 
systems, both in the projection realm and in the context of inflation systems [3, 17, 18]. The 
key to many of these investigations is understanding the scaling behaviour of the diffraction 
measure $\hat{\gamma}$ in the vicinity of the origin (in reciprocal space). In one dimension, the scaling 
behaviour of $Z(x) := \hat{\gamma}((0,x])/\hat{\gamma}(\{0\})$ as $x \to 0^+$ is an important tool for this [16, 3].

While one gets $Z(x) = x$ for the homogeneous Poisson process of unit density in $\mathbb{R}$, power 
laws of the form $Z(x) \sim x^\alpha$ with $\alpha > 1$, as $x \to 0^+$, are typical for aperiodically ordered sets. 
Systems such as the Thue–Morse chain show a decay that is faster than any power [3, 10], 
and a lattice displays a function $Z(x)$ that drops down to 0 for sufficiently small $x > 0$, thus 
signifying a perfect (in fact, periodic) repetition of patches of any size.

Naturally, one can also address such questions for point sets of number-theoretic origin, 
such as the square-free integers on the line and their various generalisations; see [4, 14, 5] and 
references therein for systematic examples. Here, we consider the set $V_k$ of $k$-free integers 
with fixed $k \geq 2$, that is, the elements of $\mathbb{Z}$ that are not divisible by a $k$-th power of any 
(rational) prime number. The sets $V_k$ are examples of weak model sets of maximal density, 
and are thus approachable by the projection method [5, 12]. The diffraction measure of $V_k$ is 
known to be a pure point measure [7], which is explicitly computed in terms of elementary 
number-theoretic functions. So, we can investigate the function $Z_k(x)$ for this family of point 
sets in $\mathbb{Z} \subset \mathbb{R}$. With the knowledge of the diffraction measure, which will be recalled below, 
our task can be termed as follows.
For any integer \( q \neq 0 \), let \( \overline{q} := \ker(q) \) denote the square-free kernel of \( q \), that is, its square-free divisor of largest modulus. For \( q \in \mathbb{N} \), set

\[
(1.1) \quad f_k(q) := \begin{cases} \prod_{p \mid \overline{q}} \frac{1}{p^k - 1}, & \text{if } q \text{ is } (k+1)\text{-free}, \\ 0, & \text{otherwise}. \end{cases}
\]

Clearly, \( f_k(1) = 1 \), and \( f_k(q) = f_k(\overline{q}) \) when \( q \) is \((k+1)\)-free. Also, write

\[
(1.2) \quad \varphi(x, q) := \text{card}\{1 \leq m \leq qx : \gcd(m, q) = 1\}
\]

for the 2-parameter totient function, where both \( x > 0 \) and \( q \in \mathbb{N} \) are assumed. Note that \( \varphi(x, q) = 0 \) whenever \( qx < 1 \). In this article, we are interested in determining the asymptotics, as \( x \to 0^+ \), of

\[
(1.3) \quad Z_k(x) = \sum_{q \in \mathbb{N}^{(k+1)} \atop q \geq 1/x} \varphi(x, q)f_k(q)^2,
\]

where \( \mathbb{N}^{(j)} \) with \( j \geq 2 \) denotes the set of \( j \)-free positive integers; that is, \( \mathbb{N}^{(j)} = V_j \cap \mathbb{N} \).

As our main result, we shall establish the following bounds, where \( \zeta(s) \) denotes Riemann’s zeta function; see [1, 11] for general background.

**Theorem 1.1.** Let \( k \geq 2 \) be a fixed positive integer. As \( x \to 0^+ \), we have

\[
\frac{x^{2-\frac{1}{k}+o(1)}}{(k-1)\zeta(2)}(1 + o(1)) \leq Z_k(x) \leq \frac{x^{2-\frac{1}{k}-o(1)}}{(k-1)\zeta(2)}(1 + o(1)).
\]

**Remark 1.2.** Note that

\[
\frac{1}{(k-1)\zeta(2)} = O\left(\frac{1}{k}\right) \quad \text{as } k \to \infty,
\]

so, as \( k \) increases, the asymptotic behaviour from Theorem 1.1 looks more and more like \( x^2 \), but with a prefactor that tends to 0. This is consistent with the fact that, as \( k \to \infty \), our point set \( V_k \) approaches the lattice \( \mathbb{Z} \), where \( Z(x) = 0 \) for all sufficiently small \( x > 0 \). ♦

The article is organised as follows. In Section 2, we recall the essential results on the diffraction measure of the \( k \)-free integers and then state some results of elementary or asymptotic nature that we require for the proof of Theorem 1.1 in Section 3. Finally, we briefly comment on potential further steps in Section 4.

## 2. Preliminaries and known results

Let us begin by recalling some results from [7] on the diffraction of \( k \)-free integers. It is well known that the set \( V_k \) has natural density \( 1/\zeta(k) \), which can be obtained as a limit along any averaging sequence of growing intervals around an arbitrary, but fixed centre. This is called *tied density* in [7]. If any such averaging sequence is fixed, say \( ([n, n])_{n \in \mathbb{N}} \) for instance, also all patch frequencies exist; in particular, the 2-point correlations within \( V_k \) exist. If
\[ \eta_k(m) \text{ is the frequency of occurrence of two points within } V_k \text{ at distance } m \in \mathbb{Z} \setminus \{0\}, \text{ and } \eta_k(0) = 1/\zeta(k) \text{ is the density of } V_k, \text{ one obtains the autocorrelation measure of } V_k \text{ as} \]

\[ \gamma_k = \sum_{m \in \mathbb{Z}} \eta_k(m) \delta_m, \]

where \( \delta_x \) denotes the normalised Dirac measure at \( x \); compare [2, Ch. 9 and Sec. 10.4].

The autocorrelation measure is positive definite, and hence Fourier transformable as a measure [2, Prop. 8.6]; \( \hat{\gamma}_k \) is known as the diffraction measure of \( V_k \). The main result of [7] on \( \hat{\gamma}_k \) can be summarised as follows.

**Theorem 2.1.** The diffraction measure \( \hat{\gamma}_k \) of \( V_k \) is a pure point measure, supported on

\[ L_k^\oplus := \{ y \in \mathbb{Q} : y = \frac{m}{q} \text{ with } m \in \mathbb{Z}, q \in \mathbb{N}^{(k+1)} \text{ and } \gcd(m, q) = 1 \}, \]

which is a subgroup of \( \mathbb{Q} \). More precisely, the diffraction measure reads

\[ \hat{\gamma}_k = \sum_{z \in L_k^\oplus} I_k(z) \delta_z, \]

where, for any \( z = \frac{m}{q} \in L_k^\oplus \), the intensity is given by

\[ I_k(z) = \left( \frac{f_k(q)}{\zeta(k)} \right)^2 \]

with the function \( f_k(q) \) as defined in Eq. (1.1). \( \square \)

Note that \( 0 = \frac{0}{1} \in L_k^\oplus \), with \( I_k(0) = \zeta(k)^{-2} = \hat{\gamma}_k(\{0\}) \). Clearly, the diffraction measure is reflection symmetric with respect to the origin, which means that the scaling near 0 can be determined from the positive side. Here, for \( x > 0 \), one gets

\[ Z_k(x) := \frac{\hat{\gamma}_k((0, x])}{\hat{\gamma}_k(\{0\})} = \zeta(k)^2 \sum_{z \in L_k^\oplus} I_k(z) = \sum_{q \geq 1/x} \sum_{1 \leq m \leq qx, \gcd(m,q)=1} f_k(q)^2, \]

which leads immediately to Eq. (1.3). Now, according to the definition of \( \varphi(x, q) \), without changing the value of \( Z_k(x) \), we can enlarge the summation set in (1.3) to include all \( q \in \mathbb{N}^{(k+1)} \) subject to the weaker condition \( q^k \geq 1/x \). This is possible since, whenever \( q \in \mathbb{N}^{(k+1)} \), we have both \( q \leq \bar{q}^k \) and \( \varphi(x, q) = 0 \) for all terms with \( q < 1/x \); the latter statement is true because the interval \([1, qx]\) is empty under this condition. Consequently, we have

\[ Z_k(x) = \sum_{q \in \mathbb{N}^{(k+1)}} \varphi(x, q)f_k(q)^2. \]

**Remark 2.2.** The set of \( k \)-free integers defines a topological dynamical system \((X_k, Z)\), where \( X_k \) is the orbit closure of \( V_k \) under the natural, continuous shift action of \( \mathbb{Z} \) in the local topology. The frequency measure \( \nu \) with respect to the chosen averaging sequence is well defined and known as the Mirsky measure. It is ergodic, and \( V_k \) is a generic set for it; see [5] and references therein for details. The measure-theoretic dynamical system \((X_k, Z, \nu)\)
has pure point dynamical spectrum, where the latter (in additive notation) is precisely the
Abelian group $L_k^{\oplus}$ from above. This follows from [7] via the equivalence theorem on pure
dynamical spectra [6]; compare [5]. For the case $k = 2$, it has also been derived by explicit means
in [8]. Alternatively, it systematically follows from Keller’s new approach [12, 13].

To continue, we require the following preliminary results, where, for $n \in \mathbb{N}$, we use $d(n)$ to
denote the number of positive divisors of $n$, $\sigma(n)$ for the sum of those divisors, and $\varphi(n)$ for
Euler’s totient function, that is, $\varphi(n) = \varphi(1, n)$ with the function from (1.2).

**Lemma 2.3.** For any integers $q, k \geq 2$ with $q \in \mathbb{N}^{(k+1)}$, we have

$$f_k(q) = \frac{1}{\varphi(q) \sigma(q^{k-1})} \quad \text{and} \quad \frac{1}{q^k} < f_k(q) < \frac{\zeta(k)}{q^k}.$$  

**Proof.** Under our conditions, the equality $f_k(q) = (\varphi(q) \sigma(q^{k-1}))^{-1}$ follows easily from the
standard definitions of $\varphi$ and $\sigma$. Then, following the argument of [11, Thm. 329], we have

$$\sigma(q^{k-1}) = \prod_{p | q} p^{k-1} = q^{1-1} \prod_{p | q} 1 - p^{-k}.$$  

Since $\varphi(q) = \bar{q} \prod_{p | q} (1 - p^{-1})$ and $\zeta(k) = \prod_{p} (1 - p^{-k})^{-1}$, we clearly get

$$\bar{q}^k f_k(q) = \frac{\bar{q}^k}{\varphi(q) \sigma(q^{k-1})} = \prod_{p | \bar{q}} \frac{1}{1 - p^{-k}} \in (1, \zeta(k)),$$

which is the desired result. \hfill \Box

**Lemma 2.4.** Let $x > 0$ be fixed, $q \in \mathbb{N}^{(k+1)}$, and let $\ell = q / \bar{q}$. Then, one has

$$\varphi(x, q) = \varphi(\ell x, \bar{q}).$$  

Moreover, for square-free $q$ — which means $q = \bar{q} \in \mathbb{N}^{(2)}$ — we have

$$\varphi(x, q) = x \varphi(q) + O(d(q)).$$  

Furthermore, we have the inequality

$$\varphi(x, q) \leq x \varphi(q) + d(q).$$  

**Proof.** The first claim follows from the definition of $\varphi(x, q)$ in (1.2), since $\gcd(m, q) = 1$ if and
only if $\gcd(m, \bar{q}) = 1$, where $q = \ell \bar{q}$ with $\ell | \bar{q}^k$ under our assumption. The second and third
relations follow from [9, Secs. 1 and 2]. \hfill \Box

**Lemma 2.5.** Let $\beta > 0$, and let $k \geq 2$ be an integer. Then, as $x \to 0^+$, one has

$$\sum_{q \geq x^{-\beta}} \frac{|\mu(q)|}{q^k} = \frac{x^{\beta(k-1)}}{(k-1)\zeta(2)} + O(x^{\beta(k-1)}) = \frac{x^{\beta(k-1)}}{(k-1)\zeta(2)} \left(1 + O(x^{\beta/2})\right).$$
Proof. Let $M(t) := \sum_{q \leq t} |\mu(q)|$. By partial summation, we obtain
\[
\sum_{q \geq x^{-\beta}} \frac{|\mu(q)|}{q^k} = k \int_{x^{-\beta}}^{\infty} M(t) \frac{dt}{tk+1} - x^{\beta k} \sum_{q < x^{-\beta}} |\mu(q)|.
\]
Since $M(t) = \frac{t}{\zeta(2)} + O(\sqrt{t})$ as $t \to \infty$, see [11, Thm. 333], we have
\[
\sum_{q \geq x^{-\beta}} \frac{|\mu(q)|}{q^k} = k \int_{x^{-\beta}}^{\infty} \left( \frac{t}{\zeta(2)} + O(\sqrt{t}) \right) \frac{dt}{tk+1} - x^{\beta k} \left( \frac{x^{-\beta}}{\zeta(2)} + O(x^{-\beta/2}) \right)
\]
\[
= \frac{k}{\zeta(2)} \int_{x^{-\beta}}^{\infty} \frac{dt}{tk} - \frac{x^{\beta(k-1)}}{\zeta(2)} + O\left( \int_{x^{-\beta}}^{\infty} \frac{dt}{tk+1} \right) + O(x^{\beta(k-\frac{1}{2})})
\]
\[
= \frac{x^{\beta(k-1)}}{(k-1)\zeta(2)} + O(x^{\beta(k-\frac{1}{2})}).
\]
We now have all prerequisites to turn to our main result.

3. PROOF OF THEOREM 1.1

Let $\alpha \in (0, k]$. Then, by Eq. (2.1) and several applications of Lemmas 2.3 and 2.4, we get
\[
Z_k(x) = \sum_{\substack{q \in \mathbb{N}(k+1) \setminus (1/x) \qquad \sum_{q \in \mathbb{N}(2) \setminus (1/x)}}} \varphi(x,q) f_k(q)^2 = \sum_{q \in \mathbb{N}(2) \setminus (1/x)} f_k(q)^2 \sum_{\ell | q} \varphi(\ell x, q)
\]
\[
\geq \sum_{q^\alpha \geq 1/x} |\mu(q)| f_k(q)^2 \sum_{\ell | q} \varphi(\ell x, q)
\]
\[
= \sum_{q^\alpha \geq 1/x} |\mu(q)| f_k(q)^2 \sum_{\ell | q} (\ell x \varphi(q) + O(d(q)))
\]
\[
= x \sum_{q^\alpha \geq 1/x} |\mu(q)| \varphi(q) \sigma(q^{-k+1}) f_k(q)^2 + O\left( \sum_{q^\alpha \geq 1/x} |\mu(q)| d(q) d(q^{-k+1}) f_k(q)^2 \right)
\]
\[
= x \sum_{q^\alpha \geq 1/x} |\mu(q)| f_k(q) + O\left( \sum_{q^\alpha \geq 1/x} |\mu(q)| d(q) d(q^{-k+1}) f_k(q)^2 \right)
\]
\[
\geq x \sum_{q \geq x^{-1/\alpha}} \frac{|\mu(q)|}{q^k} + O\left( \sum_{q \geq x^{-1/\alpha}} |\mu(q)| d(q) d(q^{-k+1}) q^{2k-\alpha(1)} \right).
\]
As $q$ grows, we have $d(q) d(q^{-k+1}) = q^r(q)$, where $r(q)$ represents a positive function that goes to zero as $q \to \infty$. Since $q \xrightarrow{x \to 0} \infty$, we see that $r(q)$ goes to zero with $x$. Thus, we are left with
\[
Z_k(x) \geq x \sum_{q \geq x^{-1/\alpha}} \frac{|\mu(q)|}{q^k} + O\left( \sum_{q \geq x^{-1/\alpha}} |\mu(q)| q^{2k-\alpha(1)} \right).
\]
Applying Lemma 2.5, we obtain

\[ Z_k(x) \geq \frac{x^{\frac{k-1}{\alpha} + 1}}{(k-1) \zeta(2)} + O\left( x^{\frac{2k-1}{2x} + 1} \right) + O\left( x^{\frac{2k-1-o(1)}{2k}} \right). \]

The first big-O term on the right is an error term for any choice of \( \alpha \). To ensure the second big-O term is a true error term, we must choose \( \alpha \) so that \( k-1 + \alpha < 2k - 1 - o(1) \). Then, as long as \( \alpha < k - o(1) \), the big-O terms in the preceding inequality are both asymptotically smaller than the main term. Thus, let \( \varepsilon > 0 \) be small and set \( \alpha = k(1 - \varepsilon) \). Then,

\[ Z_k(x) \geq \frac{x^{1+(1-\varepsilon)(1)}(k-1)}{(k-1)\zeta(2)} + O\left( x^{(2-\varepsilon)(1)} - o(1) \right). \]

Letting \( \varepsilon \) go to zero at a slightly slower rate than the function represented by \( o(1) \) proves the lower bound of the theorem.

For the upper bound, we may now employ the last assertion of Lemma 2.4, saying that \( \varphi(x, q) \leq x \varphi(q) + d(q) \) for positive square-free integers \( q \). Together with Lemma 2.3, we get

\[
\begin{align*}
Z_k(x) &= x \sum_{q^k \geq 1/x} |\mu(q)| \varphi(q) \sigma(q^{k-1}) f_k(q)^2 + \sum_{q^k \geq 1/x} |\mu(q)| d(q) d(q^{k-1}) f_k(q)^2 \\
&= x \sum_{q^k \geq 1/x} |\mu(q)| f_k(q) + \sum_{q^k \geq 1/x} |\mu(q)| d(q) d(q^{k-1}) f_k(q)^2 \\
&\leq x \frac{\zeta(k)}{x} \sum_{q \geq x^{-1/k}} |\mu(q)| \frac{q^k}{q} + \sum_{q \geq x^{-1/k}} |\mu(q)| q^{2k-o(1)},
\end{align*}
\]

where, as before, we have used that \( d(q) d(q^{k-1}) = q^{o(1)} \), with \( o(1) \) again representing a positive function that goes to zero with \( x \). Applying Lemma 2.5, we have

\[
\begin{align*}
Z_k(x) &\leq \frac{x^{\frac{k-1}{\alpha} + 1} \zeta(k)}{(k-1) \zeta(2)} + O\left( x^{\frac{2k-1}{2x} + 1} \right) + \frac{x^{\frac{2k-1-o(1)}{k}}}{(k-1) \zeta(2)} + O\left( x^{2-1+o(1)} \right) \\
&= \frac{x^{\frac{2k-1-o(1)}{k}}}{(k-1) \zeta(2)} \left( 1 + \zeta(k) x^{o(1)} \right) + O\left( x^{2-\frac{1}{2k}} \right)
\end{align*}
\]

as \( x \to 0^+ \). We can improve on the last step by observing that \( x^{o(1)} \xrightarrow{x \to 0^+} 0 \), which holds because the \( o(1) \) term is asymptotically decreasing slower than \( x \) in the right way (see below for some details on this function). This then gives

\[
Z_k(x) \leq \frac{x^{\frac{2k-1-o(1)}{k}}}{(k-1) \zeta(2)} \left( 1 + o(1) \right) + O\left( x^{2-\frac{1}{2k}} \right),
\]

which completes the proof of the theorem.
4. Further developments

The interested reader will note that the $o(1)$ terms in the exponents can be made explicit by using the fact that $d(q)d(q^{k-1}) = (2k)^{\omega(q)}$ if $q \in \mathbb{N}$ is square-free, where $\omega(q)$ denotes the number of distinct prime factors of $q$, along with the asymptotic

$$\omega(q) = \frac{\log(q)}{\log \log(q)} \left( 1 + O\left( \frac{1}{\log \log(q)} \right) \right) \text{ as } q \to \infty.$$ 

One then applies partial summation to the sums concerning the divisor functions in (3.1) and (3.2). Another option would be to use partial summation, again in (3.1) and (3.2) to the divisor function sums, but using instead the well-known result (suggested by Ramanujan [15] in 1915, and proved by Wilson [19] in 1922) that

$$\sum_{n \leq y} d(n)^k \sim C_k y (\log(y))^{2k-1},$$

for some positive constant $C_k$, where $k$ is fixed and $y \to \infty$. Unfortunately, either option will only allow the replacement of the $o(1)$ with a function resembling $1/\log^A(1/x)$, and as far as we can tell, methods like this will not allow one to remove the $o(1)$ terms completely.

It is immediate to ask if it is even possible to remove the $o(1)$ terms in the exponents of Theorem 1.1. For square-free numbers $q$, set

$$\Delta(x, q) := \varphi(x, q) - x \varphi(q).$$

Then, see [9, p. 133],

$$\Delta(x, q) = -\mu(q) \sum_{d|q} \mu(d) \{dx\},$$

where $\{y\}$ indicates the fractional part of the real number $y$. To address the question above, one is led to getting good estimates for $\sum_{d|q} \mu(d) \{dx\}$ for large square-free $q$, which seems an interesting question in its own right.

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