On the correlation function of the characteristic polynomials of the hermitian Wigner ensemble

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Abstract

We consider the asymptotics of the correlation functions of the characteristic polynomials of the hermitian Wigner matrices $H_n = n^{-1/2}W_n$. We show that for the correlation function of any even order the asymptotic coincides with this for the GUE up to a factor, depending only on the forth moment of the common probability law $Q$ of entries $\Im W_{jk}$, $\Re W_{jk}$, i.e. that the higher moments of $Q$ do not contribute to the above limit.

1 Introduction

Characteristic polynomials of random matrices have been actively studied in the last years. The interest was initially stimulated by the similarity between the asymptotic behavior of the moments of characteristic polynomials of a random matrix from the Circular Unitary Ensemble and the moments of the Riemann $\zeta$-function along its critical line (see \cite{8}). But with the emerging connections to the quantum chaos, integrable systems, combinatorics, representation theory and others, it has become apparent that the characteristic polynomials of random matrices are also of independent interest. This motivate the asymptotic study of the moments of characteristic polynomials for other random matrix ensembles (see e.g. \cite{10}, \cite{3}).

In this paper we consider the hermitian Wigner Ensembles with symmetric entries distribution, i.e. hermitian $n \times n$ random matrices

$$H_n = n^{-1/2}W_n$$

(1.1)

with independent (modulo symmetry) and identically distributed entries $\Re W_{jk}$ and $\Im W_{jk}$ such that

$$\mathbf{E}\{W_{jk}\} = \mathbf{E}\{(W_{jk})^2\} = 0, \quad \mathbf{E}\{|W_{jk}|^2\} = 1, \quad j, k = 1, \ldots, n, \quad l \in \mathbb{N}. \quad (1.2)$$

Denote by $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ the eigenvalues of random matrix and define their Normalized Counting Measure (NCM) as

$$N_n(\Delta) = \#\{\lambda_j^{(n)} \in \Delta, j = 1, \ldots, n\}/n, \quad N_n(\mathbb{R}) = 1,$$  

(1.3)
where $\Delta$ is an arbitrary interval of the real axis. The global regime of the random matrix theory, centered around the weak convergence of the Normalized Counting Measure of eigenvalues, is well-studied for many ensembles. It is shown that $N_n$ converges weakly to a non-random limiting measure $N$ known as the Integrated Density of States (IDS). The IDS is normalized to unity and is absolutely continuous in many cases

$$N(\mathbb{R}) = 1, \quad N(\Delta) = \int_\Delta \rho(\lambda) d\lambda. \quad (1.4)$$

The non-negative function $\rho$ in (1.4) is called the limiting density of states of the ensemble. In the case of Wigner hermitian ensemble it is well-known (see, e.g.,[11]) that

$$\rho(\lambda) = \rho_{sc}(\lambda) = \frac{1}{\pi} \sqrt{4 - \lambda^2}. \quad (1.5)$$

The mixed moments or the correlation functions of characteristic polynomials are

$$F_{2m}(\Lambda) = \int_{\mathcal{H}_n} \prod_{j=1}^{2m} \det(\lambda_j - H) P_n(dH_n), \quad (1.6)$$

where $\mathcal{H}_n$ is the space of hermitian $n \times n$ matrices,

$$dH_n = \prod_{j=1}^{n} dH_{jj} \prod_{1 \leq j < k \leq n} \Re H_{j,k} \Im H_{j,k} \quad (1.7)$$

is the standard Lebesgues measure on $\mathcal{H}_n$, $P_n(dH_n)$ is probability law of the $n \times n$ random matrix $H_n$, and $\Lambda = \{\lambda_j\}_{j=1}^{2m}$ are real or complex parameters that may depend on $n$.

We are interested in the asymptotic behavior of (1.6) for matrices (1.1) as $n \to \infty$ for

$$\lambda_j = \lambda_0 + \frac{\xi_j}{n\rho_{sc}(\lambda_0)}, \quad j = 1, \ldots, 2m,$$

where $\lambda_0 \in (-2, 2)$, $\rho_{sc}$ is defined in (1.5) and $\hat{\xi} = \{\xi_j\}_{j=1}^{2m}$ are real number varying in a compact set $K \subset \mathbb{R}$.

In the case of hermitian matrix model, i.e. the matrices with $P_n(dH_n) = Z_n^{-1} e^{-n \text{tr} V(H_n)} dH_n,$

where $V$ is a potential function, the asymptotic behavior of (1.6) is known. Using the method of orthogonal polynomials, it was shown (see [13],[2]) that

$$\frac{1}{(n\rho(\lambda_0))^m} F_{2m} \left( \Lambda_0 + \hat{\xi} / (n\rho(\lambda_0)) \right) \quad (1.8)$$

$$= C_n \frac{m n V(\lambda_0) + \alpha_V(\lambda_0) \sum_{j=1}^{2m} \xi_j}{\Delta(\xi_1, \ldots, \xi_m) \Delta(\xi_{m+1}, \ldots, \xi_{2m})} \det \left\{ \frac{\sin(\pi(\xi_i - \xi_{m+j}))}{\pi(\xi_i - \xi_{m+j})} \right\} \left(1 + o(1)\right), \quad n \to \infty,$$

where $\Lambda_0 = (\lambda_0, \ldots, \lambda_0) \in \mathbb{R}^{2m},$

$$\alpha_V(\lambda) = \frac{V''(\lambda)}{2\rho(\lambda)}.$$
\(\rho\) is a density of (1.4), \(\lambda_0\) is such that \(\rho(\lambda_0) > 0\) and \(\Delta(x_1, \ldots, x_m)\) is the Vandermonde determinants of \(x_1, \ldots, x_m\).

Unfortunately, the method of orthogonal polynomials can not be applied to the general case of hermitian Wigner Ensembles. Thus, to find the asymptotic behavior of (1.6) other methods should be used. In [7] Gotze and Kosters use the exponential generating function to study this behavior for the second moment, i.e. for the case \(m = 1\) in (1.6). In this case it was shown for matrices (1.1) that

\[
\frac{1}{n\rho(\lambda_0)}F_2(\lambda_0 + \xi/(n\rho_{sc}(\lambda_0)), \lambda_0 + \xi_2/(n\rho(\lambda_0)))
\]

\[
= 2\pi \exp \left\{n(\lambda_0^2 - 2)/2 + \alpha(\lambda_0)(\xi_1 + \xi_2) + |\kappa_4| \right\} \frac{\sin(\pi(\xi_1 - \xi_2))}{\pi(\xi_1 - \xi_2)}(1 + o(1)),
\]

where

\[
\alpha(\lambda) = \frac{\lambda}{2\rho_{sc}(\lambda)}, \quad \kappa_4 = \mu_4 - 3/4, \quad (1.8)
\]

and \(\mu_4\) is the forth moment of the common probability law \(Q\) of entries \(\Im W_{jk}, \Re W_{jk}\).

In this paper we consider the general case \(m \geq 1\) of (1.6) for the random matrices (1.1). Set

\[
D^{(m)}(\xi) = \frac{1}{n\rho(\lambda_0)}F_2(\lambda_0 + \frac{\xi}{n\rho_{sc}(\lambda_0)}, \lambda_0 + \frac{\xi}{n\rho_{sc}(\lambda_0)})
\]

\[
= 2\pi \exp \left\{\frac{n}{2}(\lambda_0^2 - 2) + 2\alpha(\lambda_0)\xi + \kappa_4 \right\} (1 + o(1)).
\]

The main result of the paper is

**Theorem 1.** Let the entries \(\Im W_{jk}, \Re W_{jk}\) of matrices (1.1) has the symmetric probability distribution with \(4m\) finite moments. Then we have for \(m \geq 1\)

\[
\lim_{n \to \infty} \frac{1}{(n\rho_{sc}(\lambda_0))^{m^2}} \prod_{i=1}^{2m} D^{(m)}(\xi_i)
\]

\[
= \frac{\exp \left\{m(m - 1)\kappa_4(\lambda_0^2 - 2)/2 \right\}}{\pi^{2m(m - 1)\Delta(\xi_1, \ldots, \xi_m)\Delta(\xi_{m+1}, \ldots, \xi_{2m})}} \det \left\{ \frac{\sin(\pi(\xi_i - \xi_{m+j}))}{\pi(\xi_i - \xi_{m+j})} \right\}_{i,j=1}^{m},
\]

where \(F_{2m}\) and \(\rho_{sc}(\lambda)\) are defined in (1.6) and (1.7), \(\Lambda_0 = (\lambda_0, \ldots, \lambda_0) \in \mathbb{R}^{2m}\), \(\lambda_0 \in (-2, 2)\), \(\hat{\xi} = \{\xi_j\}_{j=1}^{2m}\), and \(\alpha(\lambda)\) and \(\kappa_4\) are defined in (1.8).

The theorem shows that the above limit for the mixed moments of characteristic polynomials for random matrices (1.1) coincide with those for the GUE up to a factor, depending only on the forth moment of the common probability law \(Q\) of entries \(\Im W_{jk}, \Re W_{jk}\), i.e. that the higher moments of \(Q\) do not contribute to the above limit. This is a manifestation of universality of the limit, that can be composed with universality of the local bulk regime for Wigner matrices (see [5]).

The paper is organized as follows. In Section 2 we obtain a convenient integral representation for \(F_{2m}\) in the case of symmetric probability distribution of entries with \(4m\) finite moments by using the integration over the Grassmann variables and Harish Chandra/Itzykson-Zuber formula for integrals over the unitary group. In Section 3 we prove Theorem [1] by applying the steepest descent method to the integral representation.

We denote by \(C, C_1, \ldots\) and \(c, c_1, \ldots\) various \(n\)-independent constants below, which can be different in different formulas. Integrals without limits denote the integrals over whole real axis.
2 The integral representation.

In this section we obtain the integral representation for the correlation functions $F_{2m}$ of characteristic polynomials. To this end we use the integration over the Grassmann variables. The integration was introduced by Berezin and widely used in the physics literature (see [1] and [4]). For the reader convenience we give an outline of this technique here.

Let us consider the two sets of formal variables \{ψ\}_j=1^n, \{ψ\}_j=1^n, which satisfy the following anticommutation conditions

$ψ_jψ_k + ψ_kψ_j = ψ_jψ_k + ψ_kψ_j = 0, j, k = 1, ..., n.$

In particular, for $k = j$ we obtain $ψ^2_j = ψ^2_j = 0$.

These two sets of variables \{ψ\}_j=1^n and \{ψ\}_j=1^n generate the Grassmann algebra Λ. Taking into account that $ψ^2_j = 0$, we have that all elements of Λ are polynomials of \{ψ\}_j=1^n and \{ψ\}_j=1^n. We can also define functions of Grassmann variables. Let $χ$ be an element of Λ. For any analytical function $f$ by $f(χ)$ we mean the element of Λ obtained by substituting $χ$ in the Taylor series of $f$ near zero. Since $χ$ is a polynomial of \{ψ\}_j=1^n, \{ψ\}_j=1^n, there exists such $l$ that $χ^l = 0$, and hence the series terminates after a finite number of terms and so $f(χ) ∈ Λ$.

Following Berezin [1], we define the operation of integration with respect to the anti-commuting variables in a formally way:

$$\int dψ_j = \int dψ_j = 0, \quad \int ψ_j dψ_j = \int ϰ_j dψ_j = 1.$$ 

This definition can be extend on the general element of Λ by the linearity. A multiple integral is defined to be repeated integral. The "differentials" $dψ_j$ and $dψ_k$ anticommute with each other and with the variables $ψ_j$ and $ψ_k$.

Therefore, if

$$f(χ_1, ..., χ_m) = a_0 + \sum_{j_1=1}^{m} a_{j_1}χ_{j_1} + \sum_{j_1<j_2} a_{j_1,j_2}χ_{j_1}χ_{j_2} + ... + a_{1,2,...,m}χ_1...χ_m,$$

then

$$\int f(χ_1, ..., χ_m)dχ_m...dχ_1 = a_{1,2,...,m}.$$ 

Let $A$ be an ordinary hermitian matrix. The following Gaussian integral is well-known

$$\int \exp \left\{ -\sum_{j,k=1}^{n} A_{j,k} z_j z_k \right\} \prod_{j=1}^{n} dRe z_j dIm z_j = \frac{1}{\det A}.$$

(2.1)

One of the important formulas of the Grassmann variables theory is an analog of formula (2.1) for Grassmann algebra (see [1]):

$$\int \exp \left\{ \sum_{j,k=1}^{n} A_{j,k} ψ_jψ_k \right\} \prod_{j=1}^{n} dψ_j dψ_j = \det A,$$

(2.2)
where \( \{ \psi_j \}_{j=1}^n \) and \( \{ \bar{\psi}_j \}_{j=1}^n \) are the Grassmann variables. Besides, we have

\[
\int \prod_{p=1}^{q} \bar{\psi}_{l_p} \psi_{s_p} \exp \left\{ \sum_{j,k=1}^{n} A_{j,k} \bar{\psi}_j \psi_k \right\} \prod_{j=1}^{n} d \bar{\psi}_j d \psi_j = \det A_{t_1,..,t_q; s_1,..,s_q}, \tag{2.3}
\]

where \( A_{t_1,..,t_q; s_1,..,s_q} \) is a \((n-q) \times (n-q)\) minor of the matrix \( A \) without rows \( l_1,..,l_q \) and columns \( s_1,..,s_q \).

### 2.1 Asymptotic integral representation for \( F_2 \)

In this subsection we obtain the asymptotic integral representation of \( (1.6) \) for \( m = 1 \). The corresponding asymptotic formula was obtained in \( [7] \) by using the exponential generating function. We give here a detailed proof based on the Grassmann integration to show the basic ingredients of our technique. The technique will be elaborated in the next subsection to obtain the asymptotic integral representation of \( (1.6) \) for \( m > 1 \).

Set

\[
D_2 = \prod_{l=1}^{2} \sqrt{D^{(n)}(\xi_l)}, \tag{2.4}
\]

where \( D^{(n)}(\xi) \) is defined in \( (1.9) \). Note also that

\[
\sqrt{D^{(n)}(\xi)} = e^{\frac{\epsilon^2(\lambda_0)\xi+\kappa_2/2}{2}} \sqrt{\int (t-i\lambda_0/2)^n e^{-\frac{n}{2}((t+i\lambda_0/2)^2) d \max(1+o(1))} \tag{2.5}
\]

as \( n \to \infty \).

Using \( (2.2) \), we obtain from \( (1.6) \)

\[
D_2^{-1} F_2(\Lambda) = D_2^{-1} E \left\{ \int e^{\frac{\sum_{j,k=1}^{n} (\lambda_j-H)_{j,k} \bar{\psi}_j \psi_k}} \prod_{r=1}^{2} \prod_{q=1}^{n} d \bar{\psi}_{qr} d \psi_{qr} \right\}
\]

\[
= D_2^{-1} E \left\{ \int e^{\sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{r=1}^{2} (\bar{\psi}_{rj} \psi_{kl} - \bar{\psi}_{rj} \psi_{kl}) - \sum_{j<k} \sum_{l=1}^{n} \psi_{jl} \bar{\psi}_{kl}} \right\}
\]

where \( \{ \psi_{jl} \}_{j,l=1}^{n} \) are the Grassmann variables (\( n \) variables for each determinant in \( (1.6) \)).

Denote

\[
\chi_{j,k}^+ = \sum_{l=1}^{2} (\bar{\psi}_{jl} \psi_{kl} + \bar{\psi}_{kl} \psi_{jl}), \quad \chi_{j,k}^- = \sum_{l=1}^{2} (\bar{\psi}_{jl} \psi_{kl} - \bar{\psi}_{jl} \psi_{kl}), \quad j \neq k, \tag{2.7}
\]

Using that \( (\chi_{j,k}^\pm) = 0 \) for \( s > 4 \), \( j, k = 1,..,n \) (since \( \bar{\psi}_{js}^2 = \psi_{js}^2 = 0 \) for any \( j = 1,..,n \), \( s = 1,2 \)), we expand the second exponent under the integral in \( (2.6) \) into the series and...
We get then

\[
D_2^{-1} F_2(\Lambda) = D_2^{-1} \int e^{\sum_{p=1}^{n} \psi_p \bar{\psi}_p} \prod_{j<k} \left(1 + \frac{(\chi_{j,k}^+)^2}{4n} + \frac{\mu_4}{4!n^2} (\chi_{j,k}^+)^4 \right) d\psi_q^* d\psi_q, \tag{2.8}
\]

where \(\mu_4\) is 4-th moment of the common probability law \(Q\) of the entries \(3W_{jk}\), \(\Re W_{jk}\) of (1.2). Note that

\[
1 \pm \frac{1}{4n} (\chi_{j,k}^\pm)^2 + \frac{\mu_4}{4!n^2} (\chi_{j,k}^\pm)^4 = \exp \left\{ \pm \frac{1}{4n} (\chi_{j,k}^\pm)^2 + \frac{\kappa_4}{4!n^2} (\chi_{j,k}^\pm)^4 \right\}, \quad j \neq k,
\]

\[
1 + \frac{1}{2n} (\chi_{j,j}^+)^2 = \exp \left\{ \frac{1}{2n} (\chi_{j,j}^+)^2 \right\}, \quad j, k = 1, \ldots, n,
\]

where \(\kappa_4\) is defined in (1.8). Thus, (2.8) yields

\[
D_2^{-1} F_2(\Lambda) = D_2^{-1} \int e^{\sum_{p=1}^{n} \psi_p \bar{\psi}_p} \prod_{j<k} \left(1 + \frac{1}{2n} (\chi_{j,j}^+)^2 \right) d\psi_q^* d\psi_q, \tag{2.9}
\]

where

\[
\sigma_1 = -\frac{1}{2} \sum_{j<k} ((\chi_{j,k}^+)^2 - (\chi_{j,k}^-)^2) - \sum_{j=1}^{n} (\chi_{j,j}^+)^2 \tag{2.10}
\]

\[
\sigma_2 = \frac{1}{4} \sum_{j<k} ((\chi_{j,k}^+)^4 + (\chi_{j,k}^-)^4) = \left( \sum_{j=1}^{n} \psi_j^\dagger \psi_{j1} \psi_{j2} \right)^2.
\]

Now we use the formulas

\[
\sqrt{\frac{\pi}{a}} \exp\{ab^2\} = \int \exp\{-ax^2 - 2abx\} dx, \tag{2.11}
\]

\[
\frac{\pi}{a} \exp\{abc\} = \int \exp\{-a\overline{u}u - ab\overline{u}u - acu\} d\Re u \Im u,
\]

where \(b, c\) are complex numbers or even Grassmann variables (i.e. sums of the products of even number of Grassmann variables), and \(a\) is a positive number. For the case of even Grassmann variables this formulas can be obtained by expanding the exponent into the series and integrating of each term. Therefore, (2.10) – (2.11) imply

\[
\exp \left\{ \frac{1}{2n} \sigma_1 \right\} = \frac{n^2}{2\pi^2} \int_{\mathbb{H}_2} \exp \left\{ -\frac{n}{2} \left( \sum_{q=1}^{2} \tau_q^2 + \sum_{1 \leq a < b \leq 2} \overline{u}_{ab} u_{ab} \right) \right\}
\]

\[
\prod_{j=1}^{n} \exp \left\{ \sum_{p=1}^{2} i\tau_p \overline{\psi}_j \psi_j + \sum_{1 \leq c < d \leq 2} \left( iu_{cd} \overline{\psi}_{jc} \psi_{jd} + i\overline{u}_{cd} \overline{\psi}_{jd} \psi_{jc} \right) \right\} dQ, \tag{2.12}
\]
where

\[ Q = \begin{pmatrix} \tau_1 & u_{12} \\ \bar{u}_{12} & \tau_2 \end{pmatrix} \]

(2.13)

\( \mathcal{H}_2 \) is the space of \( 2 \times 2 \) hermitian matrices and \( dQ \) is given in (1.17) for \( n = 2 \). Write the formula

\[ \exp \left\{ \frac{\kappa_4}{n^2\sigma_2} \right\} = \sqrt{\frac{\kappa_4}{\pi}} \int \exp\{-|\kappa_4|p^2\} \prod_{j=1}^{n} \exp \left\{ \frac{2p\varepsilon(\kappa_4)}{n} \psi_{j1} \psi_{j2} \psi_{j1} \psi_{j2} \right\} dp \]

(2.14)

with

\[ \varepsilon(x) = \begin{cases} x, & x > 0, \\ -ix, & x < 0. \end{cases} \]

(2.15)

Substituting (2.12) – (2.14) in (2.9) and using (2.2) – (2.3) we can integrate in (2.9) over the Grassmann variables. We obtain

\[ D_2^{-1} F_2(\Lambda) = Z_2 \int dQ e^{-\frac{i}{2} \text{tr} Q^2 - |\kappa_4|p^2} \left( \det(Q - i\Lambda) + \frac{2p\varepsilon(\kappa_4)}{n} \right) \]

(2.16)

where \( Q \) is defined in (2.13) and

\[ \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad Z_2 = \frac{(-1)^n n^2 D_2^{-1}}{2\pi^2 \sqrt{\pi |\kappa_4|}}. \]

(2.17)

Recall that we are interested in \( \Lambda = \Lambda_0 + \tilde{\xi}/n\rho_{sc}(\lambda_0) \), where \( \Lambda_0 = \text{diag}\{\lambda_0, \lambda_0\} \) and \( \tilde{\xi} = \text{diag}\{\xi_1, \xi_2\} \). Change variables to \( \tau_j - i\lambda_0/2 - i\xi_j/n\rho_{sc}(\lambda_0) \rightarrow \tau_j, j = 1, 2 \) and note that we can move the integration with respect to \( \tau_j \) from line \( \Im z = \lambda_0/2 + \xi_j/n\rho_{sc}(\lambda_0) \) back to the real axis. Indeed, consider the contour \( C_{jR} \), which is the rectangle with vertices at \((-R, 0), (-R, \lambda_0/2 + \xi_j/n), (R, \lambda_0/2 + \xi_j/n) \) and \((R, 0)\). Since the integrand in (2.16) is analytic in \( \{\tau_j\}_{j=1}^{2} \), the integral with respect to \( \tau_j \) of this function over \( C_{jR} \) is equal to 0. Besides, the integral over the segments of lines \( \Im z = \pm R \) tends to 0 as \( R \rightarrow \infty \), since the integrand in (2.16) is a polynomial of \( \tau_j \) multiplied by \( \exp\{-n\tau_j^2/2\} \). Thus, setting \( R \rightarrow \infty \), we obtain that the integral with respect to \( \tau_j \) over the line \( \Im z = \lambda_0/2 + \xi_j/n \) is equal to the integral over the real axis. Hence, we obtain in new variables

\[ D_2^{-1} F_2(\Lambda) = Z_2 \int dQ e^{-\frac{i}{2} \text{tr} (Q + i\frac{\lambda_0}{2})^2 - \frac{1}{n\rho_{sc}(\lambda_0)} \text{tr}(Q + i\frac{\lambda_0}{2}) \tilde{\xi} - |\kappa_4|p^2 - \frac{1}{2n} \text{tr} \frac{\tilde{\xi}^2}{\rho_{sc}(\lambda_0)^2}} \]

\[ \times \left( \det(Q - i\frac{\lambda_0}{2}) + \frac{2p\varepsilon(\kappa_4)}{n} \right)^n = Z_2 \int dQ \exp \left\{ -\frac{i}{n\rho_{sc}(\lambda_0)} \text{tr}(Q + i\frac{\lambda_0}{2}) \tilde{\xi} \right\} \]

\[ -|\kappa_4|p^2 - \frac{1}{2n} \text{tr} \frac{\tilde{\xi}^2}{\rho_{sc}(\lambda_0)^2} \right\} \mu_n(Q) \left( 1 + \frac{2p\varepsilon(\kappa_4)}{n\det(Q - i\Lambda_0/2)} \right)^n, \]

(2.18)

where \( Q \) is again the hermitian (see (2.13)) and

\[ \mu_n(Q) = \det^n(Q - i\Lambda_0/2)e^{-\frac{i}{2} \text{tr} (Q + i\Lambda_0/2)^2}. \]

(2.19)

Let \( q_1, q_2 \) be the eigenvalues of \( Q \). Set

\[ \tilde{\Omega}_n = \{(Q, p) : a \leq |q_l - i\lambda_0/2| \leq A, l = 1, 2, |p| \leq \log n\}, \]

\[ \Omega_n^{Q} = \{Q \in \mathcal{H}_2 : a \leq |q_l - i\lambda_0/2| \leq A\}. \]

(2.20)
for sufficiently small \( a \) and sufficiently big \( A \) (note that if \( |\lambda_0| \geq \delta \), then \( |q_l - i\lambda_0/2| \geq \delta^2/4 \) and we can omit the first inequality in (2.20)). Note that the integral in (2.18) over the domain \( \max_{l=1,2} |q_l| \geq A \) is \( O(e^{-nA^2/4}) \), \( A \to \infty \) and the integral over the domain \( \min_{l=1,2} |q_l| \leq a \) is \( O(e^{-n\log a^{-1}}) \), \( a \to 0 \). If \( a \leq |q_l - i\lambda_0/2| \leq A \) and \( |p| \geq \log n \), then according to (2.4), (2.5) and (2.13), the corresponding integral is bounded by

\[
Z_2 \int_{\Omega_n^Q} |\mu_n(Q)| dQ \int_{|p| \geq \log n} (1 + C/p/n)^n e^{-|\kappa_4|p^2} dp = O(e^{-C\log^2 n}),
\]

and we can write

\[
D_2^{-1} F_2(\Lambda) = Z_2 \int_{\tilde{\Omega}_n} e^{-i\text{tr}(Q + i\Lambda_0/2) \tilde{\xi}/\rho_{sc}(\lambda_0)} |\kappa_4| p^2 - \frac{1}{2n} \text{tr} \frac{\tilde{\xi}^2}{\rho_{sc}(\lambda_0)^2} + 2p \varepsilon(\kappa_4) \det^{-1}(Q - i\Lambda_0/2)
\times \mu_n(Q) (1 + f_n(\det(Q - i\Lambda_0/2), p)) dp dQ + O(e^{-c\log^2 n}),
\]

where

\[
f_n(\det(Q - i\Lambda_0/2), p) = e^{-2p \varepsilon(\kappa_4) \det^{-1}(Q - i\Lambda_0/2)} \left( 1 + \frac{2p \varepsilon(\kappa_4)}{n \det(Q - i\Lambda_0/2)} \right)^n - 1.
\]

Note that \( f_n \) is an analytic function of \( p \) and entries of \( Q \), and we have on \( \tilde{\Omega}_n \)

\[
|f_n(\det(Q - i\Lambda_0/2), p)| \leq \log^k n/n,
\]

where \( k \) is independent of \( n \). It is easy to check that

\[
I := \int_{|p| \leq \log n} e^{-|\kappa_4|p^2 + 2p \varepsilon(\kappa_4) \det^{-1}(Q - i\Lambda_0/2)} dp = \sqrt{\pi} e^{\kappa_4 \det^{-2}(Q - i\Lambda_0/2) - 1}.
\]

and we obtain that \( |I| > C_3 > 0 \) on \( \tilde{\Omega}_n \) (see (2.20)). Thus, (2.22) yields

\[
D_2^{-1} F_2(\Lambda) = \frac{n^2 D_2^{-1}}{(-1)^n 2\pi^2} \int_{\tilde{\Omega}_n^Q} \mu_n(Q) \exp \left\{ -i\text{tr}(Q + i\Lambda_0/2) \tilde{\xi}/\rho_{sc}(\lambda_0) \right. \\
\left. + \kappa_4 \det^{-2}(Q - i\Lambda_0/2) \right\} \left( 1 + f_n^{(1)}(\det(Q - i\Lambda_0/2)) \right) dQ + O(e^{-c\log^2 n}),
\]

where

\[
f_n^{(1)}(\det(Q - i\Lambda_0/2)) = e^{-\frac{1}{2n} \text{tr} \frac{\tilde{\xi}^2}{\rho_{sc}(\lambda_0)^2}} - 1 \\
+ I^{-1} e^{-\frac{1}{2n} \text{tr} \frac{\tilde{\xi}^2}{\rho_{sc}(\lambda_0)^2}} \int_{|p| \leq \log n} e^{-|\kappa_4|p^2 + 2p \varepsilon(\kappa_4) \det^{-1}(Q - i\Lambda_0/2)} f_n(\det(Q - i\Lambda_0/2), p) dp.
\]

According to (2.24), we get that \( f_n^{(1)}(\det(Q - i\Lambda_0/2)) \) is analytic in elements of \( Q \) on \( \tilde{\Omega}_n^Q \) and

\[
|f_n^{(1)}(\det(Q - i\Lambda_0/2))| \leq \log^k n/n,
\]

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\[
|f_n^{(1)}(\det(Q - i\Lambda_0/2))| \leq \log^k n/n,
\]
where $k$ is independent of $n$.

Let us change variables to $Q = U^*TU$, where $U$ is a unitary matrix and $T = \text{diag}\{t_1, t_2\}$. Then $dQ$ of (1.17) for $n = 2$ transforms to $(t_1 - t_2)^2 dt_1 dt_2 d\mu(U)$, where $\mu(U)$ is the normalized to unity Haar measure on the unitary group $U(2)$ (see e.g. [9], Section 3.3). Hence, since functions $\det(Q - i\Lambda_0/2)$ and $\text{tr} (Q + i\Lambda_0/2)^2$ are unitary invariant, (2.25) implies

\[
D_2^{-1}F_2(\Lambda) = \frac{n^2(-1)^n}{2\pi^2 D_2} \int_{U(2)} d\mu(U) \int_{L_2^A \times L_2^A} dt_1 dt_2 \prod_{l=1}^2 (t_l - i\lambda_0/2)^n
\]

\[
\times e^{-\frac{1}{2} \sum_{s=1}^n (t_s + \frac{i\lambda_0}{2})^2 - \text{tr} U^*(T + \frac{i\lambda_0}{2})U^* \Delta^2(B) f(B) dB}
\]

\[
\times \left(1 + f_n^{(1)}(\det(T - \frac{i\lambda_0}{2}))\right) + O(e^{-c\log^2 n}),
\]

where

\[
L_a^A = \{t \in \mathbb{R} : a \leq |t - i\lambda_0/2| \leq A\}. \tag{2.28}
\]

The integral over the unitary group $U(2)$ can be computed using the well-known Harish Chandra/Itsykson-Zuber formula (see e.g. [9], Appendix 5).

**Proposition 1.** Let $A$ be the normal $n \times n$ matrix with distinct eigenvalues $\{a_i\}_{i=1}^n$ and $B = \text{diag}\{b_1, \ldots, b_n\}$. Then we have

\[
\int_{U(n)} \int \exp\left\{-\frac{1}{2} \text{tr}(A - U^*BU)^2 \right\} \Delta^2(B) f(B) dB dU dB
\]

\[
= \pi^{n/2} \int \exp\left\{-\frac{1}{2} \text{tr}(a_j - b_j)^2 \right\} \frac{\Delta(B)}{\Delta(A)} f(b_1, \ldots, b_n) dB, \tag{2.29}
\]

where $f(B)$ is any symmetric function of $\{b_j\}_{j=1}^n$, $dB = \prod_{j=1}^n db_j$ and $\Delta(A), \Delta(B)$ are Vandermonde determinants for the eigenvalues $\{a_i\}_{i=1}^n$, $\{b_j\}_{j=1}^n$ of $A$ and $B$.

Hence, we obtain finally from (2.27)

\[
D_2^{-1}F_2(\Lambda) = \frac{i\rho_{sc}(\lambda_0)n^2}{2\pi(-1)^n D_2} \int_{L_2^A \times L_2^A} \prod_{l=1}^2 (t_l - i\lambda_0/2)^n e^{-\frac{t_1 + t_2}{2} \sum_{l=1}^2 (t_l + \frac{i\lambda_0}{2})^2 - \sum_{l=1}^2 \frac{i\lambda_2}{2}}
\]

\[
\times e^{\frac{t_1 - t_2}{\xi_1 - \xi_2}} \left(1 + f_n^{(2)}(T)\right) dt_1 dt_2 + O(e^{-c\log^2 n}), \tag{2.30}
\]

where $L_a^A$ is defined in (2.28) and $f_n^{(2)}(T) = f_n^{(1)}(\det(T - i\Lambda_0/2))$ is an analytic function bounded by $\log^k n/n$ if $t_l \in L_a^A, l = 1, 2$.

This asymptotic integral representation is used in the section 3 to prove the theorem for $m = 1$.

### 2.2 Asymptotic integral representation for $F_{2m}$.

Set

\[
D_{2m} = \prod_{l=1}^{2m} \sqrt{D^{(n)}(\xi_l)}, \tag{2.31}
\]
where $D^{(n)}(\xi)$ is defined in (1.9). Using (2.2), we obtain from (1.6) (cf. (2.6))

$$D_{2m}^{-1}F_{2m}(\Lambda) = D_{2m}^{-1}E\left\{ \int e^{\sum_{\ell=1}^{2m} \sum_{j,k=1}^{n} (\lambda_\ell - H)_{j,k} \overline{\psi}_{j\ell} \psi_{k\ell} \prod_{r=1}^{2m} \prod_{q=1}^{n} d\overline{\psi}_{qr} d\psi_{qr} \right\}$$

$$= D_{2m}^{-1}E\left\{ \int e^{\sum_{\ell=1}^{2m} \lambda_\ell \sum_{p,s=1}^{n} \overline{\psi}_{ps} \psi_{ps} - \sum_{j<k} \Re w_{jk} \overline{\psi}_{j\ell} \psi_{k\ell} + \sum_{j=1}^{2m} \overline{\psi}_{j\ell} \psi_{j\ell} } \prod_{r=1}^{2m} \prod_{q=1}^{n} d\overline{\psi}_{qr} d\psi_{qr} \right\},$$

(2.32)

where $\{\psi_{j\ell}\}_{j,\ell=1}^{n,2m}$ are the Grassmann variables ($n$ variables for each determinant). As in (2.7) we denote

$$\chi_{j,k}^+ = \sum_{l=1}^{2m} (\overline{\psi}_{j\ell} \psi_{kl} + \overline{\psi}_{kl} \psi_{j\ell}), \quad \chi_{j,k}^- = \sum_{l=1}^{2m} (\overline{\psi}_{j\ell} \psi_{kl} - \overline{\psi}_{kl} \psi_{j\ell}), \quad j \neq k,$$

$$\chi_{j,j}^+ = \sum_{l=1}^{2m} \overline{\psi}_{j\ell} \psi_{jl}, \quad j, k = 1, \ldots, n.$$  

(2.33)

Using that $(\chi_{j,k}^\pm)^s = 0$ for $s > 4m$, $j, k = 1, \ldots, 2m$ (since $\overline{\psi}_{j\ell} = \psi_{j\ell} = 0$ for any $j = 1, \ldots, n$, $l = 1, \ldots, 2m$), we expand the exponent under the integral in (2.32) into the series and integrate with respect to the measure (1.2). We get then similarly to (2.9)

$$D_{2m}^{-1}F_{2m}(\Lambda) = D_{2m}^{-1} \int e^{\sum_{\ell=1}^{2m} \lambda_\ell \sum_{k=1}^{n} \overline{\psi}_{k\ell} \psi_{k\ell} - \frac{\kappa_2}{2} \sum_{p=2}^{n} \sigma_p \prod_{r=1}^{2m} \prod_{q=1}^{n} d\overline{\psi}_{qr} d\psi_{qr}},$$

(2.34)

where $\kappa_2$ is cumulants of the probability distribution of entries $\Re w_{jk}$, $\Im w_{jk}$ of (1.2), i.e. the coefficients in the expansion

$$l(t) := \log E\{e^{it\Re w_{jk}}\} = \sum_{q=0}^{s} \frac{\kappa_q}{q!} (it)^q + o(t^s), \quad t \to 0.$$  

The function $\sigma_1$ in (2.34) is the same as in (2.10) (but with $\chi_{j,k}^\pm$ of (2.33) and the sums from 1 to $2m$ instead of from 1 to 2),

$$\sigma_2 = \frac{1}{4!} \sum_{j<k} ((\chi_{j,k}^+)^4 + (\chi_{j,k}^-)^4) + \frac{2}{4!} \sum_{j=1}^{n} (\chi_{j,j}^+)^4$$

(2.35)

$$= 2 \sum_{l_1<l_2<s_1<s_2} \sum_{j=1}^{n} \overline{\psi}_{j\ell_1} \overline{\psi}_{j\ell_2} \overline{\psi}_{j\ell_1} \overline{\psi}_{j\ell_2} \cdot \sum_{k=1}^{n} \overline{\psi}_{k\ell_1} \overline{\psi}_{k\ell_2} \psi_{s_1} \overline{\psi}_{k\ell_2}$$

$$+ \frac{1}{4} \sum_{l_1 \neq s_1, l_2 \neq s_2} \sum_{j=1}^{n} \overline{\psi}_{j\ell_1} \overline{\psi}_{j\ell_1} \psi_{j\ell_2} \psi_{j\ell_2} \cdot \sum_{k=1}^{n} \overline{\psi}_{k\ell_1} \psi_{k\ell_1} \overline{\psi}_{k\ell_2} \psi_{k\ell_2}$$

and for $p \geq 3$ we have

$$\sigma_p = \frac{1}{(2p)!} \sum_{j<k} ((\chi_{j,k}^+)^{2p} + (-1)^p (\chi_{j,k}^-)^{2p}) + \frac{2}{(2p)!} \sum_{j<k} (\chi_{j,j}^+)^{2p}$$

$$= \sum_{l_1, \ldots, l_{2p}=1}^{2m} \sum_{s=0}^{[\frac{p}{2}]} \sum_{j=1}^{n} \overline{\psi}_{j\ell_1} \cdots \overline{\psi}_{j\ell_{2p}} \psi_{j\ell_{2p}+2s} \cdots \psi_{j\ell_{2p}+2s+1} \cdots \overline{\psi}_{j\ell_{2p}},$$
where $c_{s,t}^{(p)}$ are $n$-independent positive coefficients and $l = (l_1, ..., l_{2p})$. Using (2.11) we have

$$e^{-2\kappa_4 \sigma_2} = C'_2 \int e^{-|\kappa_4|} \left( \sum_{1 \leq l_2 < s_2} \bar{m}_{l_2 s_2} w_{l_1 l_2 s_1 s_2} + \sum_{1 \neq s_1, l_2 \neq s_2} \bar{m}_{l_2 s_2} m_{l_1 l_2 s_1 s_2} \right) d W d V,$$

where

$$d W = \prod_{l_1 < l_2 < s_1 < s_2} d \Re w_{l_1 l_2 s_1 s_2} d \Im w_{l_1 l_2 s_1 s_2},$$

$$d V = \prod_{l_1 \neq l_2, s_1 \neq s_2} d \Re v_{l_1 l_2 s_1 s_2} d \Im v_{l_1 l_2 s_1 s_2},$$

and

$$C'_2 = \left( \frac{\pi}{2|\kappa_4|} \right)^{(\frac{m}{2})} \left( \frac{\pi}{|\kappa_4|} \right)^{-(2m)^2 (2m-1)^2}.$$

As well, (2.11) yields for $p \geq 3$

$$\exp \left\{ \frac{\kappa_{2p}}{n^p} \sigma_p \right\} = C'_p \int \exp \left\{ -|\kappa_{2p}| \left( \sum_{1 \leq l_2 \leq s_2} \left| \sum_{s=0}^{2m} \bar{m}_{l,s} r_{l,s} \right| \right) \right\}$$

$$\prod_{j=1}^{n} \exp \left\{ \frac{\varepsilon(\kappa_{2p})}{n^p} \sum_{l_1 < l_2 \leq s_2} \sum_{q=0}^{2m} \left| c_{l_1 q}^{(p)} \right|^{1/2} (r_{l,q} \bar{\psi}_{jl_1} \bar{\psi}_{jl_1+2q} \bar{\psi}_{jl_1+2q+1} \bar{\psi}_{jl_2} + \bar{\psi}_{jl_1} \bar{\psi}_{jl_1+2q} \bar{\psi}_{jl_1+2q+1} \bar{\psi}_{jl_2}) \right\} d R,$$

with $l = (l_1, ..., l_{2p})$ and

$$d R = \prod_{l_1, ..., l_2p = 1}^{2m} \prod_{s=0}^{[\frac{n}{2}]} d \Re r_{l,s} d \Im r_{l,s},$$

and

$$C'_p = \left( \frac{\pi}{|\kappa_{2p}|} \right)^{-(\frac{n}{2})^2 (2m)^2}, \quad p \geq 3.$$

Substituting (2.36) - (2.38) and (2.12) with sums from 1 to 2$m$ instead of from 1 to 2 in (2.34) and using (2.2) - (2.3) we can integrate over Grassmann variables in (2.34). We get

$$D_{2m}^{2m} F_2(\Lambda) = Z_m \int_{\mathcal{H}_{2m}} d Q \int d V d R d W e^{-\frac{4\pi Q^2}{\kappa_4}} \nu_\kappa(v, w, r) \Phi^n(iQ + \Lambda, v, w, r),$$

where $\mathcal{H}_{2m}$ is the space of hermitian $2m \times 2m$ matrices,

$$v = \{ v_{a_1 a_2 b_1 b_2} | a_1 \neq b_1, a_2 \neq b_2, a_1, a_2, b_1, b_2 = 1, ..., 2m \},$$

$$w = \{ w_{a_1 a_2 b_1 b_2} | a_1 < a_2 < b_1 < b_2, a_1, a_2, b_1, b_2 = 1, ..., 2m \},$$

$$r_p = \{ r_{l,s} | l_1, ..., l_{2p} = 1, ..., 2m, s = 0, ..., [p/2] \},$$

$$r = (r_3, ..., r_{2m}),$$

and

$$\tilde{\nu}_\kappa(v, w, r) = \exp \left\{ -|\kappa_4| \bar{v} v - 2|\kappa_4| \bar{w} w - \sum_{p=3}^{2m} |\kappa_{2p}| r_p r_p \right\}.$$
$dQ,dV,dR$ and $dW$ are defined in (1.7) for $n=2m$, (2.37) and (2.39), and

$$Q = \begin{pmatrix}
\tau_1 & u_{12} & u_{13} & \ldots & u_{1,2m-1} & u_{1,2m} \\
\tau_1 & u_{12} & u_{23} & \ldots & u_{2,2m-1} & u_{2,2m} \\
\tau_1 & u_{13} & \tau_2 & \ldots & u_{3,2m-1} & u_{3,2m} \\
& \sin\ldots\sin & \sin\ldots\sin & \ldots & \sin & \sin \\
\tau_1 & u_{2m,1} & u_{2m,2} & u_{2m,3} & \ldots & \tau_{2m}
\end{pmatrix} \tag{2.43}$$

is obviously hermitian. We denote also

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2m}\}, \quad Z_m = D_{2m}^{-1} \frac{n^{2m^2}}{2^{2m^2}} \prod_{p=2}^{2m} C_p'. \tag{2.44}$$

According to (2.2) – (2.3) $\Phi(iQ + \Lambda, v, w, r)$ in (2.40) is a polynomial of the entries of $iQ + \Lambda$ and of $\{v_{1,l_2s_1s_2}/n\}, \{w_{1,l_2s_1s_2}/n\}, \{r_{1,..l_2s_1}/n^{p/2}\}$ with $n$-independent coefficients and degree at most $2m$ and

1. the degree of each variable in $\Phi(iQ + \Lambda, v, w, r)$ is at most one;
2. $\Phi(iQ + \Lambda, v, w, r)$ does not contain terms $C(iQ + \Lambda)w_{1,l_2s_1s_2}/n$ or $C(iQ + \Lambda)\overline{w}_{1,l_2s_1s_2}/n$, since the terms $\overline{\psi}_{jl_1}\overline{\psi}_{jl_2}\psi_{s_1}\psi_{s_2}$ or $\overline{\psi}_{jl_1}\overline{\psi}_{jl_2}\psi_{s_1}\psi_{s_2}$ cannot be completed to $\prod_{l=1}^{2m} \overline{\psi}_{jl}\overline{\psi}_{jl}$ only by terms $\overline{\psi}_{jl}\overline{\psi}_{jl}$;
3. $\Phi(iQ + \Lambda, v, w, r)$ can be written as

$$\Phi(iQ + \Lambda, v, w, r) = \det(iQ + \Lambda) - \frac{2\varepsilon(\kappa_4)}{n} \sigma_1' + \tilde{f}_n(iQ + \Lambda, v/n, w/n, r_p/n^{p/2}), \tag{2.45}$$

where $\tilde{f}_n(iQ + \Lambda, v/n, w/n, r_p/n^{p/2})$ contains all terms of $\Phi(iQ + \Lambda, v, w, r)$ which are $O(n^{-3/2})$ as $n \to \infty$ and as $Q, v, w, r$ are fixed, and $\sigma_1'$ contains linear with respect to $v$ terms. In view of (2.3)

$$\sigma_1' = \sum_{l_1\neq s_1, l_2\neq s_2} (v_{1,l_2s_1s_2}q_{1,s_1l_1s_2} + \overline{w}_{1,l_2s_1s_2}q_{1,s_2l_1s_1}), \tag{2.46}$$

where $q_{s,l,p,r}$ is $(2m-2) \times (2m-2)$ minor of the matrix $iQ + \Lambda$ without rows with numbers $s$ and $l$ and columns with numbers $p$ and $r$.

Recall that we are interested in $\Lambda = \Lambda_0 + \tilde{\xi}/n\rho_{sc}(\lambda_0)$, where $\Lambda_0 = \text{diag}\{\lambda_0, \ldots, \lambda_0\}$ and $\tilde{\xi} = \text{diag}\{\xi_1, \ldots, \xi_{2m}\}$. Shift now $\tau_j - i\lambda_0/2 - i\xi_j/n\rho_{sc}(\lambda_0) \to \tau_j, j = 1, \ldots, 2m$. Then similarly to (2.18) we obtain in new variables

$$D_{2m}^{-1} F_{2m}(\Lambda) = Z_m \int dQ \int H_{2m} \tilde{\nu}_n(v, w, r) \Phi^n(iQ + \Lambda_0/2, v, w, r) \tag{2.47}$$

$$\times e^{-\frac{1}{4} \text{tr}(Q + \lambda_0)^2} - i\text{tr}(Q + \lambda_0)\tilde{\xi}/\rho_{sc}(\lambda_0) - \frac{1}{4} \text{tr} \frac{\tilde{\xi}^2}{\rho_{sc}(\lambda_0)} dV dR dW.$$
where \( Q \) is the hermitian matrix of (2.43) and \( dQ, dV, dR \) and \( dW \) are defined in (1.7) for \( n = 2m \), (2.37) and (2.39). The (1) condition of \( \Phi \) yields

\[
|\Phi(iQ + \Lambda_0, v, w, r)| \leq \prod_{q,s} (1 + C |(iQ + \Lambda_0/2)_{qs}|) \prod_{l_1 \neq s_1, l_2 \neq s_2} \left( 1 + C \left| \frac{v_{l_1 l_2 s_1 s_2}}{n} \right| \right)
\times \prod_{a_1 < a_2 < b_1 < b_2} \left( 1 + C \left| \frac{w_{a_1 a_2 b_1 b_2}}{n} \right| \right) \prod_{p=3}^{2m} \prod_{l_1, \ldots, l_{2p}=1}^{2m} \left( 1 + C \left| \frac{r_{l_1 s_1}}{n^{p/2}} \right| \right)
\quad (2.48)
\]

with \( n \)-independent \( C \), and (2.43) yields

\[
|\Phi(iQ + \Lambda_0, v, w, r)| \leq |\det(iQ + \Lambda_0/2)| \prod_{l_1 \neq s_1, l_2 \neq s_2} \left( 1 + C(Q) \left| \frac{v_{l_1 l_2 s_1 s_2}}{n} \right| \right)
\times \prod_{l_1 < l_2 < s_1 < s_2} \left( 1 + C(Q) \left| \frac{w_{l_1 l_2 s_1 s_2}}{n} \right| \right) \prod_{p=3}^{2m} \prod_{l_1, \ldots, l_{2p}=1}^{2m} \left( 1 + C(Q) \left| \frac{r_{l_1 s_1}}{n^{p/2}} \right| \right).
\quad (2.49)
\]

Here \( C(Q) \) is bounded if \( a \leq |q_l - i\lambda_0/2| \leq A, \ l = 1, \ldots, 2m \) and \( \{q_l\}_{l=1}^{2m} \) are the eigenvalues of \( Q \). Note that if \( |\lambda_0| > \delta > 0 \), then \( |q_l - i\lambda_0/2| \geq \delta^2 \) everywhere. Denote

\[
\Omega_n = \{(Q, v, w, r) : a \leq |q_s - i\lambda_0/2| \leq A, \ |v_{l_1 l_2 s_1 s_2}| \leq \log n, \ |w_{l_1 l_2 s_1 s_2}| \leq \log n, \ |r_{l_1 s_1}| \leq \log n \},
\quad (2.50)
\]

\[
\Omega_n^Q = \{Q \in H_{2m} : a \leq |q_s - i\lambda_0/2| \leq A, \ s = 1, \ldots, 2m \}.
\]

According to (2.48) the integral in (2.47) over the domain \( \max_{l=1, \ldots, 2m} |q_l - i\lambda_0/2| \geq A \) is \( O(e^{-n\lambda^2/4}) \), \( A \to \infty \) and the integral over the domain \( \min_{l=1, \ldots, 2m} |q_l - i\lambda_0/2| \leq a \) is \( O(e^{-n\log a^{-1}}) \), \( a \to 0 \). Moreover, the bound (2.49) implies that this integral over the domain, where the absolute value of at least one of \( \{v_{l_1 l_2 s_1 s_2}\}, \{w_{l_1 l_2 s_1 s_2}\} \) or \( \{r_{l_1 s_1}\} \) is greater than \( \log n \) but \( a \leq |q_s - i\lambda_0/2| \leq A, \ s = 1, \ldots, 2m \), can be bounded by \( e^{-c\log^2 n} \) (similarly to (2.21)). Therefore, using (2.50), (2.31), and (2.44) to bound the integral with \( |\mu_n(Q)| \), we can write

\[
D_{2m}^{-1} F_{2m}(A) = Z_m \int_{\Omega_n} \mu_n(Q) e^{-\text{tr} (Q + \Lambda_0/2) - 2 \epsilon Q \text{det}(Q + \Lambda_0/2) \sigma_1' - \frac{1}{2m} \text{tr} \frac{\epsilon^2}{\text{det}(Q + \Lambda_0/2)^2}}
\times \nu_n(v, w, r) (1 + f_n(Q, v, w, r)) dQ dV dR dW + O(e^{-c\log^2 n}),
\quad (2.51)
\]

where \( \mu_n, \sigma_1' \) and \( \nu_n(v, w, r) \) are defined in (2.19), (2.16) and (2.42) respectively, and

\[
f_n(Q, v, w, r) = e^{2\epsilon Q \text{det}^{-1}(Q + \Lambda_0/2) \sigma_1'} \left( 1 + \frac{f_n(iQ + \Lambda_0/2, v, w, r) - 2\epsilon \sigma_1'/n}{\text{det}(iQ + \Lambda_0/2)} \right)^{-1}.
\quad (2.52)
\]

Note that \( f_n \) is an analytic function of the entries of \( Q \), and in view of (2.49) we have on \( \Omega_n \)

\[
|f_n(Q, v, w, r)| \leq n^{-1/2} \log^k n,
\quad (2.53)
\]

where \( k \) is independent of \( n \). It is easy to check that

\[
I := \int_{\Omega_n} \nu_n(Q, v, w, r) dV dR dW
\quad (2.54)
\]

\[
= \prod_{p=2}^{2m} (C_p')^{-1} e^{\epsilon \sigma_4 (iQ + \Lambda_0/2) \text{det}^{-1}(iQ + \Lambda_0/2)} + O(e^{-c\log^2 n}),
\]

where \( C_p' \) are the entries of \( Q \) and \( \epsilon \sigma_4 = \epsilon \sigma_4 Q \text{det}(Q + \Lambda_0/2) \sigma_1' \).
where
\[
\nu_n(Q, v, w, r) = \exp\{-2\varepsilon(\kappa_4)\det^{-1}(iQ + \Lambda_0/2)\sigma_1\} \tilde{\nu}_n(v, w, r),
\]
\[
\sigma(iQ + \Lambda_0/2) = \sum_{l_1 \neq s_1, l_2 \neq s_2} q_{l_1,s_1,l_2,s_2} q_{l_2,s_2,l_1,s_1}
\]
(2.55)

with \( q_{l_1,s_1,l_2,s_2} \) defined in (2.40) (but for the matrix \( iQ + \Lambda_0/2 \) instead of \( iQ + \Lambda_0 \)). Note also that according to the Cauchy-Binet formula (see [6]), we have that \( \sigma(iQ + \Lambda_0/2) \) is the sum \( S_{2m-2}(A) \) of principal minors of order \((2m - 2) \times (2m - 2)\) of the matrix
\[
A = (iQ^* + \Lambda_0/2)(iQ + \Lambda_0/2) = U^*(iT_0 + \Lambda_0/2)^2U,
\]
where \( U \) is a unitary \( 2m \times 2m \) matrix diagonalizing \( Q \) and \( T_0 = \text{diag}\{q_1, \ldots, q_{2m}\} \), i.e. \( Q = U^*T_0U \). Since \( S_{2m-2}(A) \) is a coefficient under \( \lambda^2 \) in the characteristic polynomial \( \det(A - \lambda I) \), \( S_{2m-2}(A) \) is unitary invariant, and thus \( \sigma(iQ + \Lambda_0/2) \) is unitary invariant too. Therefore, we have on \( \Omega_n \) of (2.50)
\[
\left| \sigma(iQ + \frac{\Lambda_0}{2})\det^{-2}(iQ + \frac{\Lambda_0}{2}) \right| = \left| \sum_{1 \leq s < l \leq 2m} \frac{1}{(iq_s + \frac{\Lambda_0}{2})(iq_l + \frac{\Lambda_0}{2})^2} \right| \leq C,
\]
and hence \(|I| > C > 0\). This, (2.51) and (2.54) yield
\[
D_{2m}^{-1}F_{2m}(A) = \frac{n^{2m^2} D_{2m}^{-1}}{2^{m^2} \pi^{2m^2}} \int_{\Omega_n^2} e^{-\text{tr} (Q + i\Lambda_0) \rho_{\mu(\lambda_0)} + \kappa_4 \sigma(iQ + \Lambda_0)/\det(iQ + \Lambda_0/2)^2}
\]
\[
\times \mu_n(Q) \left(1 + f^{(1)}_n(Q)\right) dQ + O(e^{-c\log^2 n}),
\]
(2.56)

where \( \mu_n \) is defined in (2.19) and
\[
f^{(1)}_n(Q) = e^{-\frac{1}{2m} \text{tr} \frac{\rho_{\mu(\lambda_0)}}{\rho_{\mu(\lambda_0)}^2}} - 1
\]
\[
+ I^{-1} e^{-\frac{1}{2m} \text{tr} \frac{\rho_{\mu(\lambda_0)}}{\rho_{\mu(\lambda_0)}^2}} \int_{\Omega_n} \nu_n(Q, v, w, r) f^{(1)}_n(Q, v, w, r) dV dR dW
\]
(2.57)

with \( I \) of (2.54) and \( \nu_n \) of (2.55). According to (2.52) and bound from below of \(|I|\) on \( \Omega_n \), we have
\[
|f^{(1)}_n(Q)| \leq \log^k n / n^{1/2}.
\]
(2.58)

Besides, \( f^{(1)}_n(Q) \) is analytic in elements of \( Q \).

Let us change variables to \( Q = U^*TU \), where \( U \) is a unitary \( 2m \times 2m \) matrix and \( T = \text{diag}\{t_1, \ldots, t_{2m}\} \). The differential \( dQ \) in (2.56) transforms to \( \Delta^2(T) dT d\mu(U) \), where
\[
dT = \prod_{l=1}^{2m} dt_l, \quad \Delta(T) \text{ is a Vandermonde determinant of } \{t_l\}_{l=1}^{2m}, \text{ and } \mu(U) \text{ is the normalized}
\]
to unity Haar measure on the unitary group \( U(2m) \) (see e.g. [9], Section 3.3). Functions \( \det(iQ + \frac{\Lambda_0}{2}), \tr (Q + \frac{\Lambda_0}{2})^2 \) and \( \sigma(iQ + \frac{\Lambda_0}{2}) \) (as we proved before) are unitary invariant. Hence, (2.56) implies
\[
D_{2m}^{-1}F_{2m}(A) = \frac{(-1)^{nm} n^{2m^2}}{2^{m^2} \pi^{2m^2} \mu(U(2m))} \int d\mu(U) \int dT \prod_{l=1}^{2m} (t_l - \frac{i\Lambda_0}{2})^n
\]
\[
\times e^{-\frac{1}{2} \sum_{l=1}^{2m} (t_l + \frac{i\Lambda_0}{2})^2 - \text{tr} U^*(T + \frac{i\Lambda_0}{2}) U \frac{\rho_{\mu(\lambda_0)}}{\rho_{\mu(\lambda_0)}^2} + \kappa_4 \sum_{1 \leq l < s \leq 2m} (t_l - \frac{i\Lambda_0}{2})^{-2} (t_s - \frac{i\Lambda_0}{2})^{-2}}
\]
\[
\times \Delta^2(T) \left(1 + f^{(1)}_n(U^*TU)\right) + O(e^{-c\log^2 n}),
\]
(2.59)
where $L_a^A$ is defined in (2.28). Using Proposition 1 we have

$$D_{2m}^{-1}F_{2m}(\Lambda) = \frac{(-1)^{mn}n^{2m^2}}{D_{2m}^{2m^2n^{2m^2-m}}} \int \frac{(t_l+i\lambda_0/2)^n}{(L_a^2)^m} \left( e^{-\frac{2}{\Delta(T)}} \prod_{l=1}^{2m} \left( t_l - \frac{i\lambda_0}{2} \right)^n \Delta(T) \Delta(-i\xi/\rho_{sc}(\lambda_0)) \right) dt + O(e^{-c\log^2 n}),$$

where

$$\Delta^2(T) \prod_{l=1}^{2m} \left( t_l - \frac{i\lambda_0}{2} \right)^n \left( \pi^m e^{i\sum_{l=1}^{2m} \left( t_l + \lambda_0/2 \right)} \rho_{sc}(\lambda_0) \right) \frac{\Delta(T)\Delta(-i\xi/\rho_{sc}(\lambda_0))}{(1+\Delta(T)\Delta(-i\xi/\rho_{sc}(\lambda_0)))} f^{(2)}_n(T) dT.$$

According to (2.58), we get that

$$|f^{(2)}_n(T)| \leq n^{-1/2} \log^k n, t_l \in L_a^A, l = 1,..,2m.$$ (2.61)

Hence, we obtain finally

$$D_{2m}^{-1}F_{2m}(\Lambda) = \frac{(-1)^{mn}n^{2m^2}}{D_{2m}^{2m^2n^{2m^2-m}}} \int \frac{(t_l+i\lambda_0/2)^n}{(L_a^2)^m} \left( e^{-\frac{2}{\Delta(T)}} \prod_{l=1}^{2m} \left( t_l - \frac{i\lambda_0}{2} \right)^n \Delta(T) \Delta(-i\xi/\rho_{sc}(\lambda_0)) \right) dt + O(e^{-c\log^2 n}),$$

where

$$f^{(2)}_n(T) = \Delta(T)\Delta(-i\xi/\rho_{sc}(\lambda_0)) e^{\sum_{l=1}^{2m} \left( t_l + \lambda_0/2 \right)} f^{(2)}_n(T).$$

$f^{(2)}_n(T)$ is an analytic function bounded by $n^{-1/2} \log^k n$ if $t_l \in L_a^A, l = 1,..,2m$.

### 3 Asymptotic analysis.

In this section we prove Theorem 1 passing to the limit $n \to \infty$ in (2.62) for $\lambda_j = \lambda_0 + \xi_j/n\rho_{sc}(\lambda_0)$, where $\rho_{sc}$ is defined in (1.3), $\lambda_0 \in (-2, 2)$ and $\xi_j \in [-M, M] \subset \mathbb{R}, j = 1,..,2m$.

To this end consider the function

$$V(t, \lambda_0) = \frac{t^2}{2} + \frac{i\lambda_0}{2} t - \log(t - i\lambda_0/2) - \frac{4 - \lambda_0^2}{8}.$$ (3.1)

Then (2.62) yields

$$\frac{D_{2m}^{-1}}{(n\rho_{sc}(\lambda_0))^{m^2}} F_{2m}(\Lambda) = Z_{mn} \int_{(L_a^2)^m} W_n(t_1, \ldots, t_{2m}) dt + O(e^{-c\log^2 n}),$$ (3.2)
where $D_{2m}$ is defined in (2.31).

$$W_n(t_1, \ldots, t_{2m}) = e^{-n \sum_{i=1}^{2m} V(t_i, \lambda_0) - i \sum_{i=1}^{2m} \frac{\sqrt{g}}{\rho_{sc}(\lambda_0)} t_i \Delta(T)} \frac{\Delta(\xi)}{\triangle(T)}$$

$$\times e^{\kappa_4 \sum_{1 \leq l < s \leq 2m} (t_l - \frac{i \lambda_0}{2})^{-2} (t_s - \frac{i \lambda_0}{2})^{-2} \left(1 + f_{n(2)}^2(T)\right)}, \quad (3.3)$$

and

$$Z_{m,n} = \left(-\frac{1}{2}\right)^{m} n^{m} \rho_{sc}(\lambda_0)^{m(m-1)} e^{-m \kappa_4}$$

$$\left(-i\right)^{m(2m-1)} 2^{2m} \pi^{2m^2}. \quad (3.4)$$

Now we need

**Lemma 1.** The function $\Re V(t, \lambda_0)$ for $t \in \mathbb{R}$ has the minimum at the points

$$t = x_\pm := \pm \frac{\sqrt{4 - \lambda_0^2}}{2}. \quad (3.5)$$

Moreover, if $t \not\in U_n(x_\pm) := (x_\pm - n^{-1/2} \log n, x_\pm + n^{-1/2} \log n)$, then we have for sufficiently big $n$

$$\Re V(t, \lambda_0) \geq C \frac{\log^2 n}{n}. \quad (3.6)$$

**Proof.** Note that for $t \in \mathbb{R}$

$$\Re V(t, \lambda_0) = \frac{1}{2} \left(t^2 - (4 - \lambda_0^2)/4 - \log(t^2 + \lambda_0^2/4)\right), \quad (3.7)$$

and thus

$$\frac{d}{dt} \Re V(t, \lambda_0) = t - \frac{t}{t^2 + \lambda_0^2/4}, \quad (3.8)$$

$$\frac{d^2}{dt^2} \Re V(t, \lambda_0) = 1 - \frac{1}{t^2 + \lambda_0^2/4} + \frac{2t^2}{(t^2 + \lambda_0^2/4)^2}. \quad (3.9)$$

Therefore, $t = x_\pm$ of (3.5) are the minimum points of $\Re V(t, \lambda_0)$. Note that

$$V_+ := V(x_+, \lambda_0) = \frac{i \lambda_0 \sqrt{4 - \lambda_0^2}}{4} - i \arcsin(-\lambda_0/2), \quad (3.10)$$

$$V_- := V(x_-, \lambda_0) = -\frac{i \lambda_0 \sqrt{4 - \lambda_0^2}}{4} + i \arcsin(-\lambda_0/2) - i \pi. \quad (3.11)$$

Thus we have

$$\Re V(x_\pm, \lambda_0) = 0.$$

Expanding $\Re V(t, \lambda_0)$ into the Taylor series, we obtain for $t \in U_n(x_\pm)$, using (3.8) - (3.9)

$$\Re V(t, \lambda_0) = \frac{4 - \lambda_0^2}{4} (t - x_\pm)^2 + O(n^{-3/2} \log^3 n), \quad (3.10)$$

where $x_\pm$ is defined in (3.5). Hence, for $t \not\in U_n(x_\pm)$ we get

$$\Re V(t, \lambda_0) \geq \frac{C \log^2 n}{n}, \quad (3.11)$$

which completes the proof of the lemma.
Next note that since $|t_j - i\lambda_0/2| > a$ for $t_j \in L^A\alpha$, $j = 1, \ldots, 2m$, we have
\[
\left| \exp \left\{ \kappa_4 \sum_{1 \leq l < s \leq 2m} (t_t - i\lambda_0/2)^{-2}(t_s - i\lambda_0/2)^{-2} \right\} \right| \leq C. \tag{3.11}\]
This, the inequality $|\Delta(T)/\Delta(\hat{\xi})| \leq C_1$ for $|t_j| \leq A$, $j = 1, \ldots, 2m$ and distinct $\{\xi_j\}_{j=1}^{2m}$, (3.6) and (3.11) yield
\[
\left| \int_{L^A\alpha(U_+ \cup U_-)} \int_{L^A\alpha} \int_{\mathbb{R}} \sum_{n,j} W_n(t_1, \ldots, t_{2m}) dT \right| \leq C_1 n^m e^{-C_2 \log^2 n},
\]
where $L^A\alpha$, $W_n$ and $Z_{m,n}$ are defined in (2.28), (3.3) and (3.4) respectively, and $U_\pm = \{ t \in \mathbb{R} : |t - x_\pm| \leq n^{-1/2} \log n \}$ (3.12)
with $x_\pm$ of (3.5).

Note that we have for $t \in U_\pm$ in view of (3.1) and (3.9) as $n \to \infty$
\[
V(t, \lambda_0) = V_\pm + \left( 1 + \frac{1}{(t_\pm - i\lambda_0/2)^2} \right) \left( \frac{(t - x_\pm)^2}{2} + f_\pm(t - x_\pm) \right), \tag{3.13}
\]
where $f_\pm(t - x_\pm) = O((t - x_\pm)^3)$. Shifting $t_j - x_\pm \to t_j$ for $t_j \in U_\pm$ we obtain using (3.9) that the r.h.s. of (3.2) can be rewritten as
\[
Z_{m,n} \sum_{\alpha} \sum_{(U_\alpha)^{2m}} \prod_{j=1}^{2m} t_j \prod_{j=1}^{2m} e^{-\sum_{j=1}^{2m} \alpha_j t_j^2 - \sum_{j=1}^{2m} \frac{i\xi_j}{n\rho_{sc}(\lambda_0)} - n\alpha_j} \frac{\Delta(t_1 + x_{\alpha_1}, \ldots, t_{2m} + x_{\alpha_{2m}})}{\Delta(\xi_1, \ldots, \xi_{2m})} \tag{3.14}
\]
\[
= \sum_{1 \leq l < s \leq 2m} (t_t + p_{\alpha_l})^{-2}(t_s + p_{\alpha_s})^{-2} \sum_{j=1}^{2m} (nV_j + i\rho_{sc}(\xi_j)\rho_{sc}(\lambda_0)) \frac{2m}{(1 + f_2^{(2)}(T)) + O(e^{-C_2 \log^2 n})},
\]
where sum is over all collection $\alpha = \{\alpha_j\}_{j=1}^{2m}$, $\alpha_j = \pm$, $j = 1, \ldots, 2m$ and
\[
c_\pm = 1 + p_{\pm}^2, \quad p_\pm = x_\pm - i\lambda_0/2, \quad U_n = (-2n^{-1/2} \log n, 2n^{-1/2} \log n). \tag{3.15}
\]

Note that
\[
I := \int_{(U_n)^{2m}} \prod_{j=1}^{2m} e^{-\sum_{j=1}^{2m} \frac{n\alpha_j t_j^2}{2} - \sum_{j=1}^{2m} \frac{i\xi_j}{n\rho_{sc}(\lambda_0)} - n\alpha_j} \Delta(t_1 + x_{\alpha_1}, \ldots, t_{2m} + x_{\alpha_{2m}}) \prod_{j=1}^{2m} d t_j \tag{3.16}
\]
\[
= \det \left\{ \int_{(U_{n,j}^2)^{2m}} \left( t_j + x_{\alpha_j} \right) - \frac{i\xi_j}{n\rho_{sc}(\lambda_0)c_{\alpha_j}} \right\}^{k-1} e^{-\sum_{j=1}^{2m} \frac{n\alpha_j t_j^2}{2} - n\alpha_j (t_j - \frac{i\xi_j}{n\rho_{sc}(\lambda_0)c_{\alpha_j}}) d t_j \right\}^{2m}_{j,k=1},
\]
where
\[
U_{j,n} = \left( -2n^{-1/2} \log n + \frac{i\xi_j}{n\rho_{sc}(\lambda_0)c_{\alpha_j}}, 2n^{-1/2} \log n + \frac{i\xi_j}{n\rho_{sc}(\lambda_0)c_{\alpha_j}} \right).
\]
Since \( f_\pm(t) = O(t^3) \), changing variables to \( \sqrt{n} t_j \to t_j \), expanding \( \exp\{-nf_{\alpha_j}(t_j/\sqrt{n} - i\xi_j/n\rho_{\text{sc}}(\lambda_0)c_{\alpha_j})\} \) in (3.14), and keeping the terms up to the order \( n^{-4m^2} \), we obtain as \( n \to \infty \)

\[
I = \prod_{j=1}^{2m} \sqrt{\frac{2\pi}{nc_{\alpha_j}}} \det \left\{ (x_{\alpha_j} - \frac{i\xi_j}{n\rho_{\text{sc}}(\lambda_0)c_{\alpha_j}})^{k-1} + \frac{1}{n} P_{k,m}^{(\alpha_j)}(\xi_j/n) \right\}^{2m}_{j,k=1} (1 + o(1)), \tag{3.17}
\]

where \( P_{k,m}^{(\alpha_j)} \) and \( P_{k,m}^{(-\alpha_j)} \) are polynomials with \( n- \) and \( j- \) independent (but \( k- \) dependent) coefficients of degree at most \( 4m^2 \). Consider

\[
D(\xi/n, \lambda) = \det \left\{ (x_{\alpha_j} - \frac{i\xi_j}{n\rho_{\text{sc}}(\lambda_0)c_{\alpha_j}})^{k-1} + \lambda P_{k,m}^{(\alpha_j)}(\xi_j/n) \right\}^{2m}_{j,k=1}. \tag{3.18}
\]

Note that \( D(\xi/n, \lambda) \) is a polynomial of \( \{\xi_j/n\rho_{\text{sc}}(\lambda_0)\}_{j=1}^{2m} \), and \( \lambda \). Without loss of generality, let \( \alpha_1 = \ldots = \alpha_s = +, \alpha_{s+1} = \ldots = \alpha_{2m} = - \). Then it is easy to see that if \( \xi_j = \xi_l \) for \( j, l = 1, \ldots, s \) or \( j, l = s+1, \ldots, 2m \), then \( D(\xi/n, \lambda) = 0 \). Thus,

\[
D(\xi/n, \lambda) = \Delta(\xi_1/n, \ldots, \xi_s/n)\Delta(\xi_{s+1}/n, \ldots, \xi_{2m}/n)(C_0 + \lambda F(\xi/n, \lambda)), \tag{3.19}
\]

where \( F(\xi/n, \lambda) \) is a polynomial with bonded coefficients. Substituting \( \lambda = 0 \) in (3.18) and computing the Vandermonde determinant, we obtain

\[
C_0 = \left( \frac{-i}{\rho_{\text{sc}}(\lambda_0)c_+} \right)^{\frac{s(s-1)}{2}} \left( \frac{-i}{\rho_{\text{sc}}(\lambda_0)c_-} \right)^{(2m-s)(2m-s-1)} \left( \frac{2i\xi_j}{n\rho_{\text{sc}}(\lambda_0)c_+} \right)^{s(k-1)} \left( \frac{2i\xi_k}{n\rho_{\text{sc}}(\lambda_0)c_-} \right)^{2m-s-1} \left( x_+ - x_- \right)^{s(2m-s)} (1 + o(1)).
\]

Hence, for \( \alpha_1 = \ldots = \alpha_s = +, \alpha_{s+1} = \ldots = \alpha_{2m} = - \) we get from (3.19) as \( n \to \infty \)

\[
\frac{n^{m^2} I}{\Delta(\xi)} = \frac{2^m\pi^m \rho_{\text{sc}}(\lambda_0)^m (m-1)^{m(m-1)+m-s^2} n^{-m-s(\sqrt{4-\lambda_0^2})^{s(2m-s)}}}{(c_+)^{s^2/2} (c_-)^{(2m-s)/2} \prod_{j=1}^{2m} \prod_{l=s+1}^{s} (\xi_j - \xi_l)} (1 + o(1)). \tag{3.20}
\]

This expression has the order at most \( O(1) \), and for \( s \neq m \) it is of order \( o(1) \). Hence, the terms of (3.14) are always of order \( O(1) \) and the equality holds only if \( m \) of \( \{\alpha_j\}_{j=1}^{2m} \) are pluses, and \( m \) last ones are minuses. Consider one of such terms in (3.14), for example \( \alpha_1 = \ldots = \alpha_m = 1, \alpha_{m+1} = \ldots = \alpha_{2m} = -1 \). Substituting the expressions (3.4), (3.15) and (3.20) with \( s = m \) we can rewrite this term as

\[
\frac{e^{m(m-1)\pi(\lambda_0^2-2)/2\ pi^m(m-1)+m(\xi_{m+1}+\ldots+\xi_{2m}-\xi_1-\ldots-\xi_m)}}{\pi^{2m^2-2m}} = \frac{(2i\pi)^m \prod_{i,j=1}^{m} (\xi_i - \xi_{m+j})}{m!} \tag{3.21}
\]

In view of identity

\[
\frac{\sin(\pi(\xi_j - \xi_{m+k}))}{\pi(\xi_j - \xi_{m+k})} = \frac{\sin(\pi(\xi_j - \xi_{m+k} - \xi_j))}{\pi(\xi_j - \xi_{m+k})} \tag{3.22}
\]

\[
\frac{\sin(\pi(\xi_j - \xi_{m+k}))}{\pi(\xi_j - \xi_{m+k})} \prod_{i=1}^{m} (\xi_i - \xi_{m+j}) = \frac{\sin(\pi(\xi_j - \xi_{m+k} - \xi_j))}{\pi(\xi_j - \xi_{m+k})} \prod_{i=1}^{m} (\xi_i - \xi_{m+j}). \tag{3.23}
\]

In view of identity

\[
\frac{\sin(\pi(\xi_j - \xi_{m+k}))}{\pi(\xi_j - \xi_{m+k})} \prod_{i=1}^{m} (\xi_i - \xi_{m+j}) = \frac{\sin(\pi(\xi_j - \xi_{m+k} - \xi_j))}{\pi(\xi_j - \xi_{m+k})} \prod_{i=1}^{m} (\xi_i - \xi_{m+j}). \tag{3.24}
\]
the determinant in the l.h.s. is the sum of \( \exp\{i\pi \sum_{j=1}^{2m} \alpha_j \xi_j \} \) over the collection \( \{\alpha_j\}_{j=1}^{2m} \), in which \( m \) of elements are pluses, and \( m \) last ones are minuses, with certain coefficient. In view of the identity (see [12], Problem 7.3)

\[
(-1)^{\frac{m(m-1)}{2}} \prod_{k<l}^{m} (a_k - a_l)(b_k - b_l) = \det \left[ \frac{1}{a_k - b_j} \right]_{k,j=1}^{m}.
\]

the coefficient under \( \exp\{i\pi (\xi_{m+1} + .. + \xi_{2m} - \xi_1 - .. - \xi_m)\} \) is equal to

\[
\frac{\det \left\{ \frac{1}{\xi_{m+k} - \xi_j} \right\}_{j,k=1}^{m}}{(2i\pi)^m \Delta(\xi_1,..,\xi_m) \Delta(\xi_{m+1},..,\xi_{2m})} = \frac{(-1)^{\frac{m(m-1)}{2}}}{(-1)^{m^2} (2i\pi)^m \prod_{i,j=1}^{m} (\xi_i - \xi_{m+j})}.
\]

Other coefficients can be computed analogously. Thus, restricting the sum in (3.14) to that over the collection \( \{\alpha_j\}_{j=1}^{2m} \), in which \( m \) of elements are pluses, and \( m \) last ones are minuses, and using (3.21), we obtain Theorem [1] after certain algebra.

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