MAXIMAL DIGRAPHS WITH RESPECT TO PRIMITIVE POSITIVE CONSTRUCTABILITY

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We study the class of all finite directed graphs (digraphs) up to primitive positive constructibility. The resulting order has a unique maximal element, namely the digraph $P_1$ with one vertex and no edges. The digraph $P_1$ has a unique maximal lower bound, namely the digraph $P_2$ with two vertices and one directed edge. Our main result is a complete description of the maximal lower bounds of $P_2$; we call these digraphs submaximal. We show that every digraph that is not equivalent to $P_1$ and $P_2$ is below one of the submaximal digraphs.

1. Introduction

A homomorphism from a directed graph $G$ to a directed graph $H$ is a map from the vertices of $G$ to the vertices of $H$ which maps each edge of $G$ to an edge of $H$. Two directed graphs $G$ and $H$ are called homomorphically equivalent if there is a homomorphism from $G$ to $H$ and from $H$ to $G$. The study of the homomorphism order on the class of all finite directed graphs (or short: digraphs), factored by homomorphic equivalence, has a long history in graph theory. It is known to have a quite complicated structure; we refer to Nešetřil and Tardif [1] and the references therein.

A classical topic in graph homomorphisms is the $H$-coloring problem, which is the computational problem of deciding whether a given finite digraph $G$ maps homomorphically to $H$. The computational complexity of this problem has been classified for finite undirected graphs $H$ by Hell and

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Nešetřil [2] in 1990: they are either in the complexity class L (i.e., they can be decided deterministically with logarithmic work space) or NP-complete. Feder and Vardi [3] proved that every finite-domain CSP is polynomial-time equivalent to an $H$-coloring problem for a finite directed graph $H^1$, and they conjectured that each of these problems are either in P or NP-complete. This dichotomy conjecture was eventually solved in 2017 by Bulatov and, independently, by Zhuk [5,6]. However, other long-standing open problems about the complexity of $H$-coloring for finite digraphs $H$ remain open, for example, the characterisation of when this problem is in L, or in NL [7,8,9].

The border between polynomial-time tractable and NP-complete $H$-colouring problems can be described in terms of primitive positive (pp) constructions, which is a concept that has been introduced by Barto, Opršal, and Pinsker [10] in the setting of general relational structures. The idea is that if $G$ has a pp construction in $H$, then, intuitively, ‘$G$ can be simulated by $H$’, and the $G$-coloring problem reduces (in logarithmic space) to the $H$-coloring problem. In particular, $H$-coloring is NP-hard if $K_3$ has a pp construction in $H$, where $K_3$ is the clique with three vertices, by reduction from the NP-hard three-colorability problem. It follows from the dichotomy theorem of Bulatov and Zhuk that otherwise $H$-coloring is in P. Note that pp constructibility can also be used to study the question of which $H$-coloring problems are in L or in NL. The surprising power of pp constructions is the motivation for studying pp constructions on finite digraphs more systematically.

For digraphs $G$ and $H$ that have at least one edge, the definition of pp constructions takes the following elegant combinatorial form: $G$ has a pp construction in $H$ if there exists a digraph $K$ and $a, b \in V(K)^d$ for some $d \in \mathbb{N}$ such that $G$ is homomorphically equivalent to the digraph with vertices $V(H)^d$ and where $(u, v)$ forms an edge if there is a homomorphism from $K$ to $H$ that maps $a_1, \ldots, a_d, b_1, \ldots, b_d$ to $u_1, \ldots, u_d, v_1, \ldots, v_d$, respectively. We write $H \leq G$ if $G$ has a pp construction in $H$; we deliberately chose the symbol $\leq$ rather than $\geq$; the motivation will become clear in Section 2. It can be shown that $\leq$ is transitive (Corollary 3.10 in [10]) and so it gives rise to a partial order $\mathcal{P}_{\text{Digraphs}}$ on the class of all finite digraphs (where we take the liberty to identify two digraphs $G$ and $H$ if they pp construct each other). Since all finite digraphs have a pp construction in $K_3$ (see, e.g. [11]), it is the smallest element of the poset $\mathcal{P}_{\text{Digraphs}}$. For $n \geq 1$ the directed path of length $n$ is the digraph $P_n := (\mathbb{Z}_n, \{(u, u+1) \mid 0 \leq u < n-1\})$. The poset $\mathcal{P}_{\text{Digraphs}}$ also has a greatest element, namely the digraph $P_1$. The digraph $P_1$ has a unique maximal lower bound, namely the digraph $P_2$, which is, in

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1 This result has been sharpened in [4].
\(\mathcal{P}_{\text{Digraphs}}\), equivalent to \(P_n\) for any \(n \geq 2\); this is not hard to see and will be shown in Section 3.

In this article, we present a complete description of the maximal lower bounds of \(P_2\) in \(\mathcal{P}_{\text{Digraphs}}\); we call these digraphs submaximal. We also prove that every finite digraph which does not pp constructs \(P_2\) is smaller than one of the submaximal digraphs (Theorem 3.5; also see Figure 1). The submaximal digraphs are:

- The directed cycles \(C_p\) for \(p\) prime. (For \(k \in \mathbb{N}^+\), the directed cycle \(C_k\) is defined to be the digraph \((\mathbb{Z}_k, \{(u, u + 1 \mod k) | u \in \mathbb{Z}_k\})\).)
- \(T_3 := (\{0, 1, 2\}, <)\), the transitive tournament with three vertices.

\[\begin{align*}
P_1 & \equiv C_1 \\
P_2 & \equiv P_3 \equiv P_4 \equiv \cdots \\
T_3 & \quad C_2 \quad C_3 \quad C_5 \quad \cdots \\
& \cdots \quad \vdots \quad \vdots \quad \cdots \\
& K_3
\end{align*}\]

**Figure 1.** The pp constructibility poset on finite digraphs

Related work

The pp constructibility poset for smooth digraphs, i.e., digraphs where every vertex has indegree at least one and outdegree at least one (digraphs without sources and sinks), has been described in [11]. The pp constructibility poset on general relational structures over a two-element set has been described in [12].

2. Minor conditions

Primitive positive constructibility has a universal algebraic characterisation; this characterisation plays a role in our proof, so we present it here. If \(H = \)
\((V, E)\) is a digraph, then \(H^k\) denotes the \(k\)-th direct power of \(H\), which is the digraph with vertex set \(V^k\) and edges set
\[
\{((u_1, \ldots, u_k), (v_1, \ldots, v_k)) \mid (u_1, v_1) \in E, \ldots, (u_k, v_k) \in E\}.
\]
A polymorphism of \(H\) is a homomorphism \(f\) from \(H^k\) to \(H\), for some \(k \in \mathbb{N}\), which is called the \textit{arity} of \(f\). We write \(\text{Pol}(H)\) for the set of all polymorphisms of \(H\). This set contains the projections and is closed under composition.\(^2\) An operation \(f\) is called \textit{idempotent} if \(f(x, \ldots, x) = x\) for all \(x \in V\).

A central topic in universal algebra are \textit{minor conditions}. If \(f: V^k \to V\) is an operation and \(\sigma: \{1, \ldots, k\} \to \{1, \ldots, n\}\) is a function, then \(f_\sigma\) denotes the operation
\[
(x_1, \ldots, x_n) \mapsto f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}),
\]
and \(f_\sigma\) is called a \textit{minor} of \(f\). A \textit{minor condition} is a set \(\Sigma\) of expressions of the form \(f_\sigma = g_\tau\) where \(f\) and \(g\) are function symbols (\(f\) and \(g\) might be the same symbol) and \(\sigma: \{1, \ldots, k\} \to \{1, \ldots, n\}\), \(\tau: \{1, \ldots, \ell\} \to \{1, \ldots, n\}\) are functions.

\textbf{Example 2.1.} An operation \(f: V^n \to V\) is called \textit{cyclic} if for all \(x_1, \ldots, x_n \in V\)
\[
f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1).
\]
This condition can be expressed by the minor condition
\[
\Sigma_n := \{f_{\text{id}} = f_\tau\},
\]
where \(\text{id}\) denotes the identity function on \(\{1, 2, \ldots, n\}\) and \(\tau\) denotes the cyclic permutation \((1, 2, \ldots, n)\) on \(\{1, \ldots, n\}\).

If a minor condition \(\Sigma\) contains several expressions, then different expressions in \(\Sigma\) might share the same function symbols.

\textbf{Example 2.2.} An idempotent operation \(f\) is called a \textit{Maltsev operation} if for all \(x, y \in V\)
\[
f(y, y, x) = f(x, x, x) = f(x, y, y).
\]
This condition can be expressed by the minor condition
\[
\Sigma_M := \{f_\sigma = f_\tau, f_\tau = f_\rho\},
\]
where \(\sigma, \tau, \rho: \{1, 2, 3\} \to \{1, 2\}\) are given by \(\sigma(1, 2, 3) = (2, 2, 1)\), \(\tau(1, 2, 3) = (1, 1, 1)\), and \(\rho(1, 2, 3) = (1, 2, 2)\).

\(^2\) Sets of operations with these properties are called \textit{clones} in universal algebra.
A set of operations $F$ satisfies a minor condition $\Sigma$ if the function symbols in $\Sigma$ can be replaced by operations from $F$ so that all the expressions in $\Sigma$ hold; in this case we write $F \models \Sigma$. If $H$ is a digraph, then $\Sigma(H)$ denotes the class of all minor conditions that are satisfied in $\text{Pol}(H)$.

**Theorem 2.3 (Barto, Opršal, and Pinsker [10]).** Let $G$ and $H$ be finite digraphs. Then

$$H \text{ pp constructs } G \quad \text{if and only if} \quad \Sigma(H) \subseteq \Sigma(G).$$

### 3. The pp construction poset

We have already defined pp constructibility for digraphs in the introduction, but present an equivalent description here which is convenient when specifying pp constructions, and which is also closer to the presentation of Barto, Opršal, and Pinsker [10]. A **primitive positive formula** is a formula $\phi(x_1,\ldots,x_k)$ of the form

$$\exists y_1,\ldots,y_n(\psi_1 \land \cdots \land \psi_m),$$

where each of the formulas $\psi_1,\ldots,\psi_m$ is of the form $\bot$ (for false), of the form $z_1 = z_2$, or of the form $E(z_1,z_2)$ where $z_1, z_2$ are variables from $\{x_1,\ldots,x_k,y_1,\ldots,y_n\}$.

**Definition 3.1.** Let $H = (V,E)$ be a digraph. A digraph $G$ with vertex set $V^d$ is called a **pp power of $H$ of dimension $d$** if there exists a primitive positive formula $\phi(x_1,\ldots,x_d,y_1,\ldots,y_d)$ such that the edge set of $G$ equals

$$\{((u_1,\ldots,u_d),(v_1,\ldots,v_d)) \mid \phi(u_1,\ldots,u_d,v_1,\ldots,v_d) \text{ holds in } H\}.$$

It follows from the definitions that $H \leq G$ if and only if $G$ is homomorphically equivalent to a pp power of $H$. We write

- $H \equiv G$ if $H \leq G$ and $G \leq H$;
- $H < G$ if $H \leq G$ and not $G \leq H$.

**Lemma 3.2.** $P_1$ is the greatest element of $\mathcal{P}_{\text{Digraphs}}$. Moreover, $P_1 \equiv C_1$.

**Proof.** Let $G$ be a finite digraph. Consider the pp power of $G$ of dimension one given by the formula $\phi(x,y) := \bot$. The resulting digraph has no edges and is therefore homomorphically equivalent to $P_1$. Now consider the pp power of $G$ of dimension one given by the formula $\phi(x,y) := (x = y)$. The resulting digraph is homomorphically equivalent to the digraph $C_1$ with one vertex and a loop, which implies the statement.  

\[\Box\]
In the proof of the following lemma we need the fundamental concept of cores from the theory of graph homomorphisms (see, e.g., [13]). A digraph $H = (V, E)$ is called a core if every endomorphism of $H$ (i.e., every homomorphism from $H$ to $H$) is an embedding (i.e., an isomorphism between $H$ and an induced subgraph of $H$; for background, see, e.g., [14]). It is easy to see that every finite digraph $H$ is homomorphically equivalent to a core digraph, and that all finite core digraphs $G$ that are homomorphically equivalent to $H$ are isomorphic to each other; we therefore call $G$ the core of $H$. When studying $\mathfrak{P}_{\text{Digraphs}}$ we may therefore restrict our attention to core digraphs; the big advantage of cores is the following useful lemma.

**Lemma 3.3 (follows from Lemma 3.9 in [10]).** Let $H = (V, E)$ be a finite core digraph. Then $H \leq G$ if and only if $G$ is homomorphically equivalent to a pp power of $H$ where the primitive positive formula might additionally contain conjuncts of the form $x = c$ where $x$ is a variable and $c \in V$ is a constant.

**Lemma 3.4.** We have $P_2 < P_1$. Moreover, $P_2$ is the only coatom of $\mathfrak{P}_{\text{Digraphs}}$, i.e., $P_2$ is the unique maximal lower bound of $P_1$ in $\mathfrak{P}_{\text{Digraphs}}$.

**Proof.** We have already seen that $P_2 \leq P_1$. To prove that $P_2 \not\leq P_1$, first observe that $P_1$ has constant polymorphisms, while $P_2$ does not. Let $\Sigma_c := \{f_\rho = f_\sigma\}$ where $f$ is a unary function symbol, $\rho: \{1\} \rightarrow \{1, 2\}$, $1 \mapsto 1$ and $\sigma: \{1\} \rightarrow \{1, 2\}$, $1 \mapsto 2$. Then $\text{Pol}(P_1) \models \Sigma_c$, but $\text{Pol}(P_2) \not\models \Sigma_c$. Then (the easy direction of) Theorem 2.3 implies that $P_1 \leq P_2$ does not hold.

For the second statement, let $G$ be a finite digraph such that $G < P_1$. We have to show that $G \leq P_2$. Without loss of generality we may assume that $G$ is a core. Hence, by Lemma 3.3, we can use constants in pp constructions. Note that $G$ must have at least two different vertices $u$ and $v$. The pp power of $G$ of dimension one given by the formula $\phi(x, y) := (x = u) \land (y = v)$ is a digraph that has exactly one edge, and this edge is not a loop; therefore the graph is homomorphically equivalent to $P_2$.

The following theorem is our main result and will be shown in the remainder of the article; see Figure 1.

**Theorem 3.5.** The submaximal elements of $\mathfrak{P}_{\text{Digraphs}}$ are precisely $T_3$, $C_2$, $C_3$, $C_5$, ... If $G$ is a finite digraph that does not have a pp construction in $P_2$, then $G \leq T_3$ or $G \leq C_p$ for some prime $p$. 
4. Submaximal digraphs and minor conditions

We first discuss which of the minor conditions that we have encountered are satisfied by the polymorphisms of the digraphs that appear in Theorem 3.5. The facts presented in this section are well-known; we present the proof for the convenience of the reader.

Lemma 4.1. Let $p$ and $q$ be primes. Then $\text{Pol}(C_p) \models \Sigma_q$ (introduced in Example 2.1) if and only if $p \neq q$.

**Proof.** If $p \neq q$, then there is an $n \in \mathbb{N}^+$ such that $q \cdot n = 1 \pmod{p}$. The map $(x_1, \ldots, x_q) \mapsto n \cdot (x_1 + \ldots + x_q) \pmod{p}$ is a polymorphism of $C_p$ satisfying $\Sigma_q$.

Now suppose that $p = q$ and assume for contradiction that $f$ is a polymorphism of $C_p$ satisfying $\Sigma_p$. Then $f(0, \ldots, p - 2, p - 1) = a = f(1, \ldots, p - 1, 0)$ and hence $(a, a) \in E$, which is impossible since $C_p$ has a loop only if $p = 1$. ■

Lemma 4.2. $\text{Pol}(C_n) \models \Sigma_M$ for every $n \in \mathbb{N}$.

**Proof.** The ternary operation $(x_1, x_2, x_3) \mapsto x_1 - x_2 + x_3 \pmod{n}$ is a Maltsev polymorphism of $C_n$. ■

Let $H = (V, E)$ be a finite digraph, $u, v \in V$, and $k \in \mathbb{N}$. A directed walk of length $k$ from $u$ to $v$ is a $k$-tuple $(v_0, \ldots, v_{k-1}) \in V^k$ such that $v_0 = u$, $v_{k-1} = v$, and $(v_i, v_{i+1}) \in E$ for all $i \in \{0, \ldots, k-2\}$. The digraph $H$ is called $k$-rectangular if whenever $H$ has directed walks of length $k$ from $a$ to $b$, from $c$ to $b$, and from $c$ to $d$, then also from $a$ to $d$. See Figure 2. A digraph $H$ is called totally rectangular if it is $k$-rectangular for all $k \geq 1$. The following well-known lemma connects total rectangularity with $\Sigma_M$.

Lemma 4.3. A finite digraph $H$ is totally rectangular if and only if it has a Maltsev polymorphism. A finite core digraph $H$ has a Maltsev polymorphism if and only if $\text{Pol}(H) \models \Sigma_M$.

**Proof.** The first part of the statement is Corollary 4.11 in [15]. For the second statement, let $H = (V, E)$ be a core digraph which has a polymorphism $f$ that satisfies $f(x, y, y) = f(x, x, x) = f(y, y, x)$ for all $x, y \in V$; we have to find a polymorphism that is additionally idempotent. Note that the function $x \mapsto f(x, x, x)$ is an endomorphism; since $H$ is a core, the endomorphism is
injective. Since $H$ is finite the endomorphism must in fact be an automorphism, and has an inverse $i$ which is an endomorphism as well. Then the operation $(x_1, x_2, x_3) \mapsto i(f(x_1, x_2, x_3))$ is idempotent and a Maltsev operation.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,1) {$b$};
\node (c) at (1,0) {$c$};
\node (d) at (1,-1) {$d$};
\draw (a) -- (b);
\draw (c) -- (d);
\end{tikzpicture}
\caption{Rectangularity in digraphs}
\end{figure}

\textbf{Lemma 4.4.} $\text{Pol}(T_3) \models \Sigma_n$ for every $n \in \mathbb{N}$, but $\text{Pol}(T_3) \nmid \Sigma_M$.

\textbf{Proof.} The operation $(x_1, \ldots, x_n) \mapsto \max(x_1, \ldots, x_n)$ is a polymorphism of $T_3$ that satisfies $\Sigma_n$. On the other hand, $T_3 = (\{0, 1, 2\}, E)$ is not 1-rectangular, witnessed by $(1, 2), (0, 2), (0, 1) \in E$ but $(1, 1) \notin E$; the second statement therefore follows from Lemma 4.3. \hfill \square

The following theorem states that the digraph $P_2$ is the unique smallest element of $\mathcal{P}_{\text{Digraphs}}$ that satisfies $\Sigma_M$ and $\Sigma_p$ for all $p$ prime.

\textbf{Theorem 4.5.} Let $G$ be a finite digraph that satisfies $\Sigma_M$ and $\Sigma_p$ for all primes $p$. Then $P_2 \leq G$.

In the proof of Theorem 4.5 we make use the following result of Carvalho, Egri, Jackson, and Niven [15], which guides us in our further proof steps.

\textbf{Theorem 4.6 (Lemma 3.10 in [15]).} If $G$ is totally rectangular, then $G$ is homomorphically equivalent to either a directed path or a disjoint union of directed cycles.

Before we come to the proof of Theorem 5.4 we show that $P_2$ can pp construct all other directed paths.

\textbf{Lemma 4.7.} The digraph $P_2$ pp constructs $P_k$ for all $k \in \mathbb{N^+}$.

\textbf{Proof.} Clearly, $P_2 \leq P_1$ and $P_2 \leq P_2$. Let $k \geq 3$ and consider the pp power $G$ of $P_2$ of dimension $k-1$ given by the following formula
Then $G$ contains the following path with $k$ vertices

$$(0, 0, \ldots, 0) \rightarrow (1, 0, \ldots, 0) \rightarrow (1, 1, \ldots, 0) \rightarrow \cdots \rightarrow (1, 1, \ldots, 1),$$

which shows that there exists a homomorphism from $P_k$ to $G$. Note that whenever there is an edge from $u$ to $v$ in $G$, then the tuple $v$ contains exactly one 1 more than the tuple $u$. Therefore, the function $V(G) ightarrow \{0, \ldots, k-1\}$ that maps $v$ to the number of 1’s in $v$ is a homomorphism from $G$ to $P_k$. Hence $P_2 \leq P_k$.

Proof of Theorem 4.5. Let $G$ be a finite digraph satisfying $\Sigma_M$ and $\Sigma_p$ for every prime $p$. By Lemma 4.3 and Theorem 4.6 there are two cases to consider: the first is that $G$ is homomorphically equivalent to $P_k$ for some $k$. Then $P_2 \leq G$ by Lemma 4.7.

The second case is that $G$ is homomorphically equivalent to a disjoint union of directed cycles. Without loss of generality we may assume that $G$ is a disjoint union of directed cycles. Let $(a_0, \ldots, a_{\ell-1})$ be a shortest cycle in $G$. Let $p$ be a prime and $k \in \mathbb{N}^+$ such that $p \cdot k = \ell$, and let $f \in \text{Pol}(G)$ be a function that witnesses that $\text{Pol}(G) \models \Sigma_p$. Then

$$f(a_0, a_k, \ldots, a_{(p-1) \cdot k}) = a = f(a_k, a_{2k}, \ldots, a_0).$$

Note that there are directed walks of length $k$ from $a_{(p-1) \cdot k}$ to $a_0$ and from $a_{i \cdot k}$ to $a_{(i+1) \cdot k}$ for $i \in \{0, \ldots, p-2\}$. Since $f$ is a polymorphism of $G$ there is a directed walk of length $k$ from $a$ to $a$. Thus, $G$ contains a directed cycle whose length divides $k$, which contradicts the assumption that $\ell$ is the length of the shortest directed cycle in $G$. Therefore, $\ell$ has no prime divisors, and $\ell = 1$. So $G$ contains a loop and hence is homomorphically equivalent to $C_1$; it follows that $P_2 \leq G$.

5. Proof of the main result

We use the following general result about when a finite digraph can construct a finite disjoint union of cycles. If $C$ is a finite disjoint union of cycles and $c \in \mathbb{N}$, then $C \vdash c$ denotes the union of cycles which contains for every cycle of length $n$ in $C$ a cycle of length $n / \gcd(n, c)$. If $G$ is any directed graph with vertices $u_1, \ldots, u_n$ and edges $e_1, \ldots, e_m$, then $\Sigma_G$ denotes the minor condition $f_\sigma = f_\tau$, where $\sigma, \tau: \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ are such that if $e_i = (u_p, u_q)$, then $\sigma(i) = p$ and $\tau(i) = q$. Note that whether $\text{Pol}(H)$ satisfies
$\Sigma_G$ does not depend on the choice of the above enumerations of the edges and vertices of $G$. In particular, the condition $\Sigma_{C_k}$ is equivalent to the condition $\Sigma_k$.

**Lemma 5.1 (Lemma 6.8 in [11]).** Let $C$ be a finite disjoint union of cycles and let $G$ be a finite digraph. Then

$$G \leq C \quad \text{iff} \quad \text{Pol}(G) \models \Sigma_{C \prec c} \text{ implies } \text{Pol}(C) \models \Sigma_{C \prec c} \text{ for all } c \in \mathbb{N}^+.$$  

For the special case that $C = C_p$, there are only two conditions of the form $\Sigma_{C \prec c}$, namely $\Sigma_1$, which is trivial and hence satisfied by both $\text{Pol}(G)$ and $\text{Pol}(C)$, and $\Sigma_p$, which is not satisfied by $C_p$. Hence, we obtain the following result.

**Theorem 5.2.** Let $G$ be a finite digraph. If $p$ is a prime number such that $\text{Pol}(G) \not\models \Sigma_p$, then $G \leq C_p$.

We also need a similar result for $\Sigma_M$ instead of $\Sigma_p$.

**Lemma 5.3.** Let $G$ be a finite digraph. If $\text{Pol}(G) \not\models \Sigma_M$, then $G \leq T_3$.

**Proof.** Since $\leq$ is transitive we may assume without loss of generality that $H = (V,E)$ is a core. By Lemma 4.3, $H$ is not totally rectangular. Hence, there are vertices $a,b,c,d \in V$ such that in $G$ there are directed walks of length $k$ from $a$ to $b$, from $c$ to $b$, from $c$ to $d$, and there is no directed walk of length $k$ from $a$ to $d$. Note that by Lemma 3.3 we are allowed to use constants in pp constructions. We write $x \xrightarrow{k} y$ as a shortcut for the primitive positive formula $\exists u_1, \ldots, u_{k-1} (E(x,u_1) \land E(u_1,u_2) \land \cdots \land E(u_{k-1},y))$. Consider the pp power of $G$ of dimension two given by the formula

$$\phi(x_1,x_2,y_1,y_2) := x_1 \xrightarrow{k} y_2 \land (x_2 = d) \land (y_1 = a).$$

Let $H$ be the resulting digraph. Consider the vertices $v_0 = (c,d), \ v_1 = (a,d)$, and $v_2 = (a,b)$ of $H$. Note that the only vertex of $H$ that can have incoming and outgoing edges is $v_1$. Since there is no directed walk of length $k$ from $a$ to $d$ the vertex $v_1$ does not have a loop. Furthermore, $H$ has the edges $(v_0,v_1), (v_1,v_2)$, and $(v_0,v_2)$ (see Figure 3). Hence, $i \mapsto v_i$ is an embedding of $T_3$ into $H$. Let $V_0$ be the set of all vertices in $H$ that have outgoing edges and $V_2$ be the set of all vertices in $H$ that have incoming edges. Let $V_1$ denote the set $(V_0 \cap V_2) \cup (V(H) \setminus (V_0 \cup V_2))$. Note that $V_1$ consists of $v_1$ and
all isolated vertices. Clearly, $V_0 \setminus V_2$, $V_1$, and $V_2 \setminus V_0$ form a partition of $V(H)$ and the map
\[ v \mapsto \begin{cases} 
2 & \text{if } v \in V_2 \setminus V_0 \\
1 & \text{if } v \in V_1 \\
0 & \text{if } v \in V_0 \setminus V_2 
\end{cases} \]
is a homomorphism from $H$ to $T_3$. Hence $G \leq T_3$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (a) at (0,0) {$(a,b)$};
  \node (d) at (0,-2) {$(c,d)$};
  \node (b) at (-1,-1) {$(a,d)$};
  \draw[->] (a) -- (d);
  \draw[->] (a) -- (b);
  \draw[->] (b) -- (d);
\end{tikzpicture}
\caption{The primitive positive construction of $T_3$ in the proof of Lemma 5.3}
\end{figure}

**Proof of Theorem 3.5.** Let $G$ be a digraph such that $P_2 \not\leq G$. Theorem 4.5 implies that either $\Pol(G)$ does not satisfy $\Sigma_M$ or that it does not satisfy $\Sigma_p$ for some prime $p$. In the first case $G \leq T_3$, by Lemma 5.3. In the second case $G \leq C_p$, by Theorem 5.2. Hence, all submaximal elements of $\mathfrak{P}_{\text{Digraphs}}$ are contained in $\{T_3, C_2, C_3, C_5, \ldots\}$. Lemma 4.1, Lemma 4.2, and Lemma 4.4 in combination with Theorem 2.3 imply that these digraphs form an antichain in $\mathfrak{P}_{\text{Digraphs}}$, and hence each of these digraphs is submaximal.

Note that our result implies the following.

**Corollary 5.4.** If a finite digraph $G$ satisfies $\Sigma_M$, $\Sigma_2$, $\Sigma_3$, $\Sigma_5$, $\ldots$, then any minor condition satisfied by $\Pol(P_2)$ is also satisfied by $\Pol(G)$.

The statement of Corollary 5.4 may also be phrased as
\[ \{\Sigma_M, \Sigma_2, \Sigma_3, \Sigma_5, \ldots\} \subseteq \Sigma(G) \quad \Rightarrow \quad \Sigma(P_2) \subseteq \Sigma(G). \]

**Remark 5.5.** We do not know whether Corollary 5.4 holds for arbitrary clones of operations on a finite set, instead of just clones of the form $\Pol(G)$ for a finite digraph $G$. However, the statement is false for clones of operations on an infinite set, as illustrated by the clone of operations on $\mathbb{Q}$ of the form
(x₁, . . . , xₙ) ↦ a₁x₁ + · · · + aₙxₙ for a₁, . . . , aₙ ∈ Q such that a₁ + · · · + aₙ = 1. This clone satisfies Σₙ for every n ∈ N, and contains the function (x₁, x₂, x₃) ↦ x₁ − x₂ + x₃, so it also satisfies Σ₃. However, it is easy to see that it does not contain an operation f that satisfies

\[ f(x, x, y) = f(y, y, x) = f(x, y, y) = f(y, x, x) \]

for all x, y ∈ Q; however, this minor condition is satisfied by Pol(P₂) (for example, by f = max).

Remark 5.6. Many, but not all the statements that we have shown also apply to infinite digraphs. Clearly, P₁ is still the greatest element in the respective poset. In Theorem 2.3, only the forward direction holds if G and H are infinite; however, in this text we only used (e.g., in Lemma 3.4) the forward direction of this theorem. In the proof that P₂ is the unique lower bound of P₁ we used the fact that every finite graph has a core, which is no longer true for infinite digraphs [16].

For the maximal lower bounds of P₂, the situation looks as follows. In the proof that digraphs G such that Pol(G) ̸|= Σ₃ pp construct T₃, we needed to work with an expansion of G by constants; expansions by constants are pp constructible in G if G is countably infinite and an ω-categorical model-complete core; see [10,14]. Every digraph with a Maltsev polymorphism is totally rectangular even if the digraph is infinite. The proof of Theorem 4.6 of Carvalho, Egri, Jackson, and Niven can be generalised to show that every infinite digraph which is totally rectangular is homomorphically equivalent to an infinite disjoint union of cycles or to one of the infinite paths P∞ := (N, {(u, u + 1) | u ∈ N}), P∞ := (N, {(u + 1, u) | u ∈ N}), the disjoint union P∞ + P∞ of P∞ and P∞, and P∞ := (Z, {(u, u + 1) | u ∈ Z}). (All of these graphs have a Maltsev polymorphism.)

An infinite disjoint union of cycles C is not maximal below P₂; to see this, let k be the length of a shortest cycle in C. Observe that the pp power of C of dimension one given by the formula φ(x, y) := x → y ∧ x k → x is homomorphically equivalent to Cₖ. If k = 1, then C is homomorphically equivalent to C₁. If k > 1, then C ≤ Cₖ < P₂. Since finite structures can only pp construct finite structures, we have that P₂ cannot pp construct the core digraphs P∞, P∞, P∞ + P∞, and P∞. Conversely, these graphs can pp construct P₂ with the same formula φ(x₁, x₂, y₁, y₂) := E(y₁, x₁) ∧ E(x₂, y₂) ∧ x₁ = x₂. Clearly, P∞ and P∞ pp construct each other. We do not know whether these graphs are maximal lower bounds of P₂ in the class of all digraphs.
6. Concluding remarks

Primitive positive constructibility orders finite digraphs \( H \) by their ‘strength’ with respect to the \( H \)-coloring problem. Many deep combinatorial statements about graphs and digraphs can be phrased in terms of this order. We showed that at least the top region of the resulting poset can be described completely. A full description of the entire poset \( \mathcal{P}_{\text{Digraphs}} \) would be highly desirable.

We already mentioned that the pp constructibility poset on disjoint unions of cycles has been described in [11]; in particular, it contains no infinite ascending chains and is a lattice. Note that this result combined with the result of the present paper shows that for exploring \( \mathcal{P}_{\text{Digraphs}} \) it remains to explore the interval between \( K_3 \) and \( T_3 \): if a finite digraph \( H \) does not have a Maltsev polymorphism, then we proved that it is below \( T_3 \) (and above \( K_3 \)); otherwise, it is homomorphically equivalent to a directed path or a disjoint union of cycles and hence falls into the region that has already been completely described.

We state three concrete open problems.

1. Is \( \mathcal{P}_{\text{Digraphs}} \) a lattice? (Primitive positive constructibility is known to form a meet semilattice on the class of all finite relational structures factored by pp interconstructibility, but it is not clear to the authors whether the clone product construction for the meet used there can be carried out in the category of digraphs.)

2. Does \( \mathcal{P}_{\text{Digraphs}} \) contain infinite ascending chains? (We have seen an infinite antichain in this article; an infinite descending chain of digraphs with a Maltsev polymorphism can be found in [11] and the existence of infinite descending chains of digraphs without a Maltsev polymorphism follows from results of [17], and also from results in [18].)

3. What are the maximal lower bounds of \( T_3 \) in \( \mathcal{P}_{\text{Digraphs}} \)?

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