SEMICLASSICAL ESTIMATES FOR PSEUDODIFFERENTIAL OPERATORS
AND THE MUSKAT PROBLEM IN THE UNSTABLE REGIME

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Abstract. We obtain new semiclassical estimates for pseudodifferential operators with low
regular symbols. Such symbols appear naturally in a Cauchy Problem related to recent weak
solutions to the unstable Muskat problem constructed via convex integration in [CCF16]. In
particular, our new estimates reveal the tight relation between the speed of opening of the
mixing zone and the regularity of the interphase.

1. Introduction and main results

The evolution of a two fluids through a porous media where one of the fluid is above the other
is known as the Muskat problem [Mus37]. The physical derivation builds on the conservation
of mass, the incompressibility of the flow and the Darcy law, which relates the velocity with
the forces, namely the pressure and gravity. Such system is known as the IPM system. Let us
further assume that the fluids have constant densities and equal viscosities, the permeability of
the medium is also constant, and the initial data is given by the graph of a function \( f_0 : \mathbb{R} \to \mathbb{R} \).

That is

\[
\rho|_{t=0} = \rho_1 \chi_{\Omega_M} + \rho_2 (1 - \chi_{\Omega_M}),
\]

where \( \Omega_M \) is the epigraph of \( f_0 \). If we make the ansatz that as time evolve the fluid still consists
of two fluids separated by an smooth interphase \( f(t,x) \), then \( g = \partial^4 f \) has to solve a nonlinear
and non local equation

\[
\partial_t g = p_M(x,D)g + T[f](\partial_x g) + R[f](x,t).
\]

Here,

\[
p_M(x,\xi) = -(\rho_2 - \rho_1) \frac{\xi}{1 + |f'(x)|^2},
\]

and \( p_M(x,D) \) stands for the canonical quantization of \( p_M \) (see (12) below), \( T[f] \) can be thought
as a smooth function, and \( R \) is a lower order remainder.

It turns out that if \( \rho_1 < \rho_2 \) then the system is well posed for sufficiently regular \( f \) (see [CG07,
CGS16, CGSV17, Mat19]). However, if \( \rho_2 > \rho_1 \), the system is ill posed [CCF12, CG07].

In this situation, starting with the pioneering work of Saffman and Taylor [ST58], numerics
and experiments [AT83, MH95] predict a fingering pattern in the evolution and the existence
of an evolving in time mixing zone \( \Omega_{\text{mix}} \), where the fluids mix chaotically and the pointwise
(microscopical) pattern of the two fluids is practically unpredictable. However, as pointed out
by Otto (see among others [Ott97, Ott99, Ott01]) several aspects of the mixing zone and of the
mixing pattern of the fluids can be derived from the relaxation of the system.

Recently, the IPM system and the Muskat problem have been revisited using DeLellis-
Székelyhidi program to apply convex integration in hidrodynamics [CFG11, Szč12, CCF16,
FSz18, CFM19] (see [DSz12] for a review of the method). In particular, the various constructions
of weak solutions to the Muskat problem yield an explicit description of the mixing zone.

In [CCF16] the mixing zone is described as a neighborhood of width \( tc(t,x) \) of a pseudo-
interphase \( f(t,x) \) evolving in time \( t \in [0,T] \). More precisely, the map

\[
\mathbf{x} : \mathbb{R} \times [-1,1] \times [0,T] \to \mathbb{R}^2
\]

\[
(x,\lambda,t) \mapsto (x,f(t,x) + \lambda tc(t,x))
\]
defines the mixing zone $\Omega_{\text{mix}}$ as
\begin{equation}
\Omega_{\text{mix}}(t) = x(\mathbb{R} \times (-1, 1), t), \quad t \in (0, T).
\end{equation}
Moreover, in [CCF16, CFM19] it has been proven that if $f$ and $c$ are suitable coupled through the equation
\begin{equation}
\partial_t f = \mathcal{M}[c, f]f,
\end{equation}
where $\mathcal{M}[c, f]$ is a nonlinear integro-differential operator acting on $f$, then there exist infinitely many weak solutions to the IPM system compatible with such mixing zone (called mixing solutions). For $g = \partial^4 f$ it can be checked that
\begin{equation}
\partial_t g = p_{\text{mix}}(x, D)(g) + T[f] \partial_x g + R[f],
\end{equation}
where, analogously to [2], $T[f]$ is a smooth function and $R$ are lower order terms. The symbol $p_{\text{mix}}$ has a rather cumbersome explicit expression, see [CCF16], but it satisfies that $p_{\text{mix}}(x, \xi) \approx d_t(x, \xi)$, where the symbol $d_t(x, \xi) = t^{-1}d(x, t\xi)$ and
\begin{equation}
d(x, \xi) := \frac{|\xi|}{1 + c(x)|\xi|}.
\end{equation}
Here $c(x)$ is a smooth function which is assumed to not depend on $t$ for simplicity. It turns that $p_{\text{mix}}$ in (6) is slightly better than $p_M$ in [2] and this is the reason why (5) can be solved as is proved in [CCF16].

This paper is focused on the study of the equation
\begin{equation}
(\text{IVP}) \quad \left\{ \begin{array}{l}
\partial_t f(t, x) = d_t(x, D) f(t, x), \\
 f(0, \cdot) = f_0 \in L^2(\mathbb{R}).
\end{array} \right.
\end{equation}
which captures the main difficulties in order to get a new energy estimate for (6) which allows us to show local existence for (5) with an improvement of the regularity with respect to [CCF16].

Notice that if $c$ is identically constant, then (IVP) has a global-in-time solution which can be computed explicitly using the Fourier transform. Indeed,
\begin{equation}
f(t, x) = p(tD)f_0(x) = \int_{\mathbb{R}} \left(1 + c t|\xi|\right)^{-\frac{1}{2}} \hat{f}_0(\xi)e^{2\pi i x \xi}d\xi, \quad t > 0.
\end{equation}

In particular, the following conservation law holds:
\begin{equation}
\frac{d}{dt} \|p^{-1}(tD)f(t)\|^2_{L^2} = 0, \quad \forall t \geq 0,
\end{equation}
and $f(1, x)$ is comparable to the $c^{-1}$ derivative of the initial data. Thus the regularity of the solution seems to decreases as the width of the mixing zone is thinner. In [CCF16] a Gårding-inequality is used to deal with a variable with $c(x)$ and this yields a loss of one derivative with respect to the initial data. Here we frontal attack (IVP) via a suitable new commutator estimates. This new strategy gives us a gain of local regularity.

Notice also that the operator $d_t(x, D)$, written in (IVP) as the canonical quantization of $d_t$, can be also put in the form $d_t(x, D) = t^{-1}d(x, tD)$, where now $d(x, tD)$ denotes the semiclassical quantization of $d$ with semiclassical parameter $t$ (see [13] below). This precision may seem at first view merely cosmetic. However, one of the goals of this work is to clarify that semiclassical calculus, and not only pseudodifferential calculus, is essential to obtain analogous conservation laws to (10) for the solution of (IVP) in the non-constant case.

The study of such evolutions with low regular and variable growth symbols seems to be new in the semiclassical picture and might find applications elsewhere. Interest in low regular symbols appear in other problems in fluid mechanics [La06, Tex07]. In [La06], Lannes studies in a very careful way the action of pseudodifferential operators $a(x, D)$ on Sobolev spaces $H^s$, with symbols $a(x, \xi)$ having limited regularity in the position variable $x$ and in momentum variable $\xi$ near the origin, via the use of paradifferential calculus. In [Tex07], Texier extends some of the techniques of [La06] to deal with semiclassical pseudodifferential operators having only low
regularity in the $x$ variable and being smooth in the $\xi$ variable. Our result is indeed related to these, but we need to use semiclassical calculus with symbols having very low regularity in $x$ and in $\xi$. This, up to our view, entails certain obstructions to extend the techniques of [La06] to the semiclassical framework through the use of paradifferential calculus, and for this reason we only use it tangentially. Alternatively, our approach is strongly concerned with the techniques of [Hw87, CCF16].

We present two semiclassical theorems in the form of conservation laws that predict the $c(x)^{-1}$ loss of regularity with respect to the local regularity of the initial data, in contrast with the loss of one derivative obtained in [CCF16]. To this aim, we define the symbol

\[ p(x, \xi) := (1 + c(x)|\xi|)^{-\frac{1}{c(x)}}. \]

In order to state our results, we need to impose some regularity assumptions on $c(x)$. The precise class of admissible functions $c(x)$ considered in this work is fixed in Definition 4 below. We first state a local-in-time conservation law in terms of the pseudo-inverse of $p(x, tD)$:

**Theorem 1.** Let $c_1, c_2 > 0$ be an admissible pair satisfying [14]. Let $c(x)$ be $(c_1, c_2)$-admissible. Then there exists $T > 0$ such that

\[ \frac{d}{dt} \| p^{-1}(x, tD) f(t) \|_{L^2(\mathbb{R})}^2 \leq C_T \| p^{-1}(x, tD) f(t) \|_{L^2(\mathbb{R})}^2, \quad \forall t \in (0, T], \]

where the constant $C_T$ depends only on $T$ and on $c$. In particular, $\| p^{-1}(x, D) f(t) \|_{L^2(\mathbb{R})}$ remains bounded for $t \in [0, T]$ if $f_0 \in L^2(\mathbb{R})$.

**Remark 1.** Theorem 1 illustrates how the size of $c(x)$ is linked with the regularity of the pseudo-interface $f(x, t)$ . Precisely, as the coefficient $c_1$ is smaller, the regularity of $c(x)$ is required to increase, while the loss of derivatives of $f(t, x)$ with respect to $f(0, x)$ becomes larger. This means that the regularity of $f(t, x)$ in the $x$ variable is related to $c(x)^{-1}$ via the pseudo-inverse $p^{-1}(x, tD)$, and hence it appears as a pseudo-local feature.

Our next result explains the local smoothing properties of $p^{-1}(x, tD)$ around any fixed point $x_0 \in \mathbb{R}$ in terms of local Sobolev regularity. We take $\varepsilon > 0$ small and define

\[ c_+ := \sup_{x \in I_\varepsilon} c(x) + \varepsilon; \quad c_\varepsilon := \inf_{x \in I_\varepsilon} c(x) - \varepsilon, \]

where $I_\varepsilon = (x_0 - \varepsilon, x_0 + \varepsilon)$.

**Theorem 2.** Let $c_1, c_2 > 0$ satisfying [14]. Let $c(x)$ be $(c_1, c_2)$-admissible. Set $m_1 = c_1^{-1}$, $s_- := -1/c_\varepsilon^-$ and $s_+ := -1/c_\varepsilon^+$. Then, for every smooth bump function $\chi_\varepsilon(x)$ supported on $I_\varepsilon$, there exist constants $C_0, C_1, C_2 > 0$ such that

\[ C_1 \| \chi_\varepsilon f \|_{H_{s_-}^m(\mathbb{R})} - tC_0 \| f \|_{H_{s_-}^{m-1}(\mathbb{R})} \leq \| \chi_\varepsilon(x)p^{-1}(x, tD)f \|_{L^2} \leq C_2 \| \chi_\varepsilon f \|_{H_{s_+}^m(\mathbb{R})} + tC_0 \| f \|_{H_{s_-}^{m-1}(\mathbb{R})}, \]

for every $f \in H_{s_-}^{m-1}(\mathbb{R})$ and $0 < t \leq T$.

In principle these theorems combined with the strategy from [CCF16] should yield the corresponding $c(x)^{-1}$ loss of regularity for the more complicated equation [14] as well as the price of reproducing some of the heavy computations of [CCF16] (we remark that this is consistent with the fact that the unstable Muskat problem $c \equiv 0$ is ill posed). A solution to [14] combined with [CCF16, Theorem 4.2] yields the existence of a subsolution and the h-principle [CFM19] yields the corresponding weak solutions which mixed the two fluids in the mixing zone proportionally to the distance to the corresponding fluid. Let us remark that the construction of the mixing zone (and of the corresponding subsolutions and solutions) is highly non unique and in these various problems selecting a one which prevales above the others based on physical principles (diffusion, surface tension, entropy rate maximizing) is perhaps the most challenging problem.
In section 2 we revisit the notation of function spaces, pseudodifferential operators and discuss the admissible opening speeds for the mixing zone. In Section 3 we prove the key commutator estimates, in Section 4 we show how these estimates give information about the smoothing properties of our operators in the Sobolev spaces $H^s$, and in Section 5 we give the proofs of the main theorems.

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2. Admissible symbol classes

2.1. Symbols with limited smoothness. We will consider the following classes of symbols. First we consider symbols having a finite number of derivatives in $L^\infty(\mathbb{R}_x \times \mathbb{R}_\xi)$.

**Definition 1.** Let $m \in \mathbb{R}$ and let $j, k \in \mathbb{N}_0$. A symbol $a(x, \xi)$ belongs to the class $\mathcal{M}^{m}_{j,k}$ if:

- $a \in W^{k,\infty}_{loc}(\mathbb{R}_\xi; W^j,\infty(\mathbb{R}_x))$.
- Moreover,

$$M^{m}_{j,k}(a) := \sup_{\alpha \leq j, \beta \leq k} \sup_{(x,\xi) \in \mathbb{R}^2} (1 + |\xi|)^{|\beta| - m} |\partial_\xi^\alpha \partial_\xi^\beta a(x,\xi)| < \infty.$$ 

If $m = k = 0$, we denote simply $M_j(a) := M^0_{j,0}(a)$.

We will also consider symbols that belong to $H^s(\mathbb{R}_x)$ in the $x$ variable, while in the $\xi$ variable have a finite number of bounded derivatives.

**Definition 2.** Let $m \in \mathbb{R}$, $k \in \mathbb{N}_0$ and $s > 1/2$. A symbol $a(x, \xi)$ belongs to the class $\mathcal{N}^{m}_{s,k}$ if:

- $a \in W^{k,\infty}_{loc}(\mathbb{R}_\xi; H^s(\mathbb{R}_x))$.
- Moreover,

$$N^{m}_{s,k}(a) := \sup_{\beta \leq k} \sup_{\xi \in \mathbb{R}} (1 + |\xi|)^{|\beta| - m} ||\partial_\xi^\beta a(\cdot, \xi)||_{H^s(\mathbb{R})} < \infty.$$ 

If $m = k = 0$, we denote simply $N_s(a) := N^0_{s,0}(a)$.

Given a symbol $a(x, \xi)$, the canonical quantization $a(x, D)$ is defined acting on Schwartz functions by

$$(12) \quad a(x, D)f(x) := \int_\mathbb{R} e^{2\pi i x \xi} a(x, \xi) \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}),$$

where $\hat{f}$ denotes the Fourier transform, with the convention

$$\hat{f}(\xi) = \int_\mathbb{R} e^{-2\pi i x \xi} f(x) dx.$$ 

The symbols under consideration will also depend on time $t \geq 0$, which will play the role of semiclassical parameter. The semiclassical quantization $a(x, tD)$ is defined by

$$(13) \quad a(x, tD)f(x) := \int_\mathbb{R} e^{2\pi i x \xi} a(x, t\xi) \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}).$$

We are interested in the action of $a(x, tD)$ on the Sobolev spaces $H^s(\mathbb{R})$. Since the decay properties of $a$ in the $\xi$ variable scale in terms of the semiclassical parameter $t$, it is useful to include the semiclassical parameter also in the Sobolev spaces $H^s(\mathbb{R})$. To this aim, we recall the following definition of semiclassical Sobolev space (see for instance [Zwo12 Sect. 8.3, eq. (8.3.5)] or [Tex07 Sect. 2.1]):
Definition 3. Let $s \in \mathbb{R}$. We define
\[ H^s_t(\mathbb{R}) := \{ f \in \mathcal{D}'(\mathbb{R}) : (1 + it\xi)^s \hat{f}(\xi) \in L^2(\mathbb{R}) \}, \quad t \in (0, 1]. \]
For $t = 1$, we have that $H^s_t(\mathbb{R}) = H^s(\mathbb{R})$.

Remark 2. Notice that the operator
\[ U_t : H^s \rightarrow H^s \]
\[ u(x) \mapsto U_t u(x) = t^{-1/2} u(tx) \]
is unitary and $U_t^* a(x,tD) U_t = a^*(x,D)$ with $a^*(x,\xi) = a(tx,\xi)$.

From now on, we consider a function $c(x)$ belonging to the following class.

Definition 4. Let $c_1, c_2 > 0$ satisfying that
\[ 0 < c_1 \leq c_2 \leq 2, \quad c_1^{-1} - c_2^{-1} \leq 1. \]
We say that a function $c : \mathbb{R} \rightarrow \mathbb{R}$ is $(c_1, c_2)$-admissible if:
\[ (C1) \quad c_1 \leq \inf_{x \in \mathbb{R}} c(x) \leq \sup_{x \in \mathbb{R}} c(x) \leq c_2. \]
Moreover, $c$ satisfies at least one of the following conditions.
\[ (C2) \quad c \in W^{N,\infty}(\mathbb{R}), \text{ for some } N \in \mathbb{N} \text{ satisfying that } N > \max\{3/2 + c_1^{-1}, 1 + \lfloor c_1^{-1} \rfloor \}. \]
\[ (C2') \quad \text{There exists } c_0 \in [c_1, c_2] \text{ such that } v_1(x) := c(x) - c_0 \text{ and } v_2(x) := c(x)^{-1} - c_0^{-1} \text{ satisfy } \]
\[ v = (v_1, v_2) \in H^{s}(\mathbb{R}) \times H^{s}(\mathbb{R}), \quad s > \max\{3/2 + c_1^{-1}, 1 + \lfloor c_1^{-1} \rfloor \}. \]

Remark 3. In the particular case $(c_1, c_2) = (1, 2)$, for $(C2)$ to hold it is sufficient that $N \geq 3$, while for $(C2')$ it is sufficient that $s > 5/2$.

We define the symbol $d_t(x, \xi) = t^{-1} d(x, t\xi)$, where $d$ is given by \([4]\). With condition $(C2)$, one has $d \in \mathcal{M}^0_{N,1}$. Otherwise, assuming $(C2')$ and following \([La06]\), we can write
\[ d(x, \xi) = \Pi_d(x, \xi) + \Sigma_d(0, \xi) \]
for some $\Pi_d \in \mathcal{N}^0_{s,1}$ and some Fourier multiplier $\Sigma_d(0, \xi) \in \mathcal{M}^0_{\infty,1}$. Precisely,
\[ d(x, \xi) = \frac{\xi}{1 + (c(x) - c_0)\xi + c_0\xi} = \frac{\xi}{1 + v_1(x)\xi + c_0\xi} =: \Sigma_d(v(x), \xi), \]
where $\Sigma_d \in C^\infty(\mathbb{R}^2)$, $W^{1,\infty}(\mathbb{R}^2)$, with $U \subset \mathbb{R}^2$ being a neighborhood of the image of $v$. Moreover, $\Pi_d(x, \xi) := \Sigma_d(v(x), \xi) - \Sigma_d(0, \xi)$ satisfies that $\Pi_d \in \mathcal{N}^0_{s,1}$ due to Moser’s inequality, provided that $\Sigma_d$ is smooth in the image of $v$ (see for instance \([Tay10, Prop. 3.9]\)).

We next consider the symbol
\[ p(x, \xi) := (1 + c(x)|\xi|)^{1/c(x)}. \]
We will denote
\[ m_j := c_j^{-1}, \quad j = 1, 2. \]
Notice that $p \in \mathcal{M}^{m_1}_{\infty,1}$, provided that $(C2)$ holds. Otherwise, in the case of $(C2')$, as we have done for $d$, we can write
\[ p(x, \xi) = (1 + v_1(x)\xi + c_0\xi)^{m_2(x)\xi + c_0} =: \Sigma_p(v(x), \xi), \]
and $\Pi_p(x, \xi) = \Sigma_p(v(x), \xi) - \Sigma_p(0, \xi)$ satisfies that $\Pi_p \in \mathcal{N}^{m_1}_{s,1}$ provided that $\Sigma_p$ is smooth in $U$, while $\Sigma_p(0, \xi) \in \mathcal{M}^{m_1}_{\infty,1}$. Similarly, its inverse $p^{-1} \in \mathcal{M}^{-m_2}_{\infty,1}$ if $(C2)$, (resp. $p_{-1} \in \mathcal{N}^{-m_2}_{s,1}$ and $\Sigma_{p^{-1}}(0, \xi) \in \mathcal{M}^{-m_2}_{\infty,1}$ if $(C2')$).

3. Commutator estimates

In this section we revisit some commutator estimates obtained in \([CCF16]\) and extend them for the proofs of Theorems \([\Pi]\) and \([2]\). The main ideas come from \([Hw87]\).
3.1. Preliminary lemmas.

Lemma 1. [CCF16 Lemma 5.6] Let $f \in L^2(\mathbb{R})$, $\gamma \in H^s(\mathbb{R})$ for every $s \in \mathbb{R}$, and define

\[ \Gamma_{f,\gamma}^\pm(y, \eta) := \int_{\mathbb{R}} e^{\pm 2\pi i y \eta} \gamma(y \mp \xi) f(\xi) d\xi. \]

Then, for every $n \geq 0$,

\[ \| \partial_n^\pm \Gamma_{f,\gamma} \|_{L^2(\mathbb{R}^2)} = \| \gamma(n) \|_{L^2} \| f \|_{L^2}. \]

Moreover, let $p(y, \eta) \in M_{n,0}^0$, then

\[ \| \partial_n^\pm (p \Gamma_{f,\gamma}^\pm) \|_{L^2(\mathbb{R}^2)} \leq C M_n(p) \| \gamma \|_{H^n} \| f \|_{L^2}. \]

Proof. By Plancherel in the $y$-variable, we have

\[ \int_{\mathbb{R}^2} |\partial_n^\pm \Gamma_{f,\gamma}^\pm(y, \eta)|^2 d\eta = \int_{\mathbb{R}^2} |\partial_n^\pm \Gamma_{f,\gamma}^\pm(\cdot, \eta) \gamma(\lambda)|^2 d\lambda d\eta. \]

On the other hand,

\[ \hat{\partial_n^\pm \Gamma_{f,\gamma}^\pm}(\cdot, \eta)(\lambda) = \hat{\gamma}(\lambda) \mathcal{F}_\pm f(\lambda + \eta), \]

where $\mathcal{F}_+$ is the Fourier transform and $\mathcal{F}_-$ its inverse. By Fubini and Plancherel, we conclude \[17\]. To show \[18\], we use the product rule to write

\[ \partial_n^\pm (p(y, \eta) \Gamma_{f,\gamma}^\pm(y, \eta)) = \sum_{j=0}^n \binom{n}{j} \partial_n^j p(y, \eta) \partial_n^{n-j} \Gamma_{f,\gamma}^\pm(y, \eta). \]

Hence \[18\] follows applying \[17\] and

\[ \| \partial_n^\pm (p(y, \eta) \Gamma_{f,\gamma}^\pm(y, \eta)) \|_{L^2(\mathbb{R}^2)} \leq \| \partial_n^\pm p \|_{L^\infty(\mathbb{R}^2)} \| \partial_n^{n-j} \Gamma_{f,\gamma}^\pm \|_{L^2(\mathbb{R}^2)}. \]

\[ \square \]

Lemma 2. [CCF16 Lemma 5.9] Let $Q(x, \xi)$, define

\[ A_Q(x) := \int_{\mathbb{R}} e^{2\pi i x \xi} Q(x, \xi) d\xi. \]

Then

\[ \| A_Q \|_{L^2(\mathbb{R})} \leq C \| (1 - \partial_x) Q \|_{L^2(\mathbb{R}^2)}. \]

Proof. We will prove the Lemma by duality. Let $g \in L^2(\mathbb{R})$, we have

\[ \int_{\mathbb{R}} A_Q(x) g(x) dx = \int_{\mathbb{R}^2} e^{2\pi i x \xi} Q(x, \xi) g(x) d\xi dx \]

\[ = \int_{\mathbb{R}^3} e^{2\pi i x \xi} Q(x, \xi) \hat{g}(\lambda) d\lambda d\xi dx. \]

Integrating by parts in the $x$-variable, we obtain

\[ \int_{\mathbb{R}} A_Q(x) g(x) dx = \int_{\mathbb{R}^2} (1 - \partial_x) Q(x, \xi) \Gamma_{\frac{\partial}{\partial \gamma}}^-(x, \xi) e^{2\pi i x \xiug } d\xi dx, \]

where $\gamma$ is given by \[24\]. Then, applying the Cauchy-Schwartz inequality and Lemma \[11\] we conclude that

\[ \left| \int_{\mathbb{R}} A_Q(x) g(x) dx \right| \leq C \| (1 - \partial_x) Q \|_{L^2(\mathbb{R}^2)} \| g \|_{L^2(\mathbb{R})}. \]

\[ \square \]
3.2. Commutator estimates. Let $p_1$ and $p_2$ be two symbols, we define:

\begin{equation}
\mathcal{C}(p_1, p_2) := p_1(x, D)p_2(x, D) - p_1p_2(x, D).
\end{equation}

**Lemma 3.** \textbf{[CCF16] Thm. 5.2} Let $p_1 \in \mathcal{M}^{0}_{1, 1}$ and $p_2 \in \mathcal{M}^{0}_{2, 1}$. Then

\begin{equation}
\|\mathcal{C}(p_1, p_2)\|_{L^2(\mathbb{R}^2)} \lesssim M_1(\partial_x p_1)M_2(p_2) + M_1(p_1)M_1(\partial_x p_2) + M_1(\partial_x p_1)M_1(\partial_x p_2).
\end{equation}

**Remark 4.** An extended proof of Lemma 3 is given in \textbf{[CCF16]}. We next rewrite the same proof in a more compact form, because some of the ideas will be used later on.

**Proof.** We start by writing the expressions of $p_1(x, D)p_2(x, D)f$ and $p_1p_2(x, D)f$:

\begin{align*}
p_1(x, D)p_2(x, D)f(x) &= \int_{\mathbb{R}^3} e^{2\pi i(x\xi - \xi y + \eta y)}p_1(x, \xi)p_2(y, \eta)\hat{f}(\eta)d\eta dyd\xi,
\end{align*}

while

\begin{align*}
p_1p_2(x, D)f(x) &= \int_{\mathbb{R}^3} e^{2\pi i\eta}p_1(x, \eta)p_2(x, \eta)\hat{f}(\eta)d\eta.
\end{align*}

Therefore, using that

\begin{align*}
p_1(x, \xi)p_2(y, \eta) &= (p_1(x, \xi) - p_1(x, \eta))p_2(y, \eta) + p_1(x, \eta)(p_2(y, \eta) - p_2(x, \eta)) + p_1(x, \eta)p_2(x, \eta),
\end{align*}

and the fact that, in the sense of distributions,

\begin{align*}
\int_{\mathbb{R}^3} e^{2\pi i(x - y)\xi}d\xi = \delta(x - y),
\end{align*}

we obtain

\begin{equation}
\mathcal{C}(p_1, p_2)f(x) = \int_{\mathbb{R}^4} e^{2\pi i(x\xi - \xi y + \eta y)}(p_1(x, \xi) - p_1(x, \eta))p_2(y, \eta)\hat{f}(\eta)d\eta dyd\xi.
\end{equation}

By the Fourier inversion formula,

\begin{equation}
\mathcal{C}(p_1, p_2)f(x) = \int_{\mathbb{R}^4} e^{2\pi i(x\xi - \xi y + \eta y - \eta z)}(p_1(x, \xi) - p_1(x, \eta))p_2(y, \eta)f(z)dzd\eta dyd\xi.
\end{equation}

Using next the identities

\begin{align*}
\frac{1}{1 + 2\pi i(y - z)}(1 + \partial_y)e^{2\pi i(y - z)\eta} &= e^{2\pi i(y - z)\eta},
\end{align*}

\begin{align*}
\frac{1}{1 + 2\pi i(x - y)}(1 + \partial_x)e^{2\pi i(x - y)\xi} &= e^{2\pi i(x - y)\xi},
\end{align*}

\begin{align*}
\frac{1}{1 + 2\pi i(\xi - \eta)}(1 + \partial_\eta)e^{2\pi i(\xi - \eta)\eta} &= e^{2\pi i(\xi - \eta)\eta},
\end{align*}

we integrate by parts in $\eta, \xi$ and $y$ successively to get

\begin{equation}
\mathcal{C}(p_1, p_2)f(x) = \int_{\mathbb{R}^3} e^{2\pi i\sigma(x, \xi, y, \eta)}\gamma(\xi - \eta)D_y(\gamma(x - y)\Gamma^+_{f, \gamma}(y, \eta))D_\eta((p_1(x, \xi) - p_1(x, \eta))p_2(y, \eta))d\eta dyd\xi,
\end{equation}

where $\sigma(x, \xi, y, \eta) = x\xi - \xi y + \eta y$, the differential operators $D_\omega = 1 - \partial_\omega$ act on all the functions on its right (via the product rule), and

\begin{equation}
\gamma(u) = \frac{1}{1 + 2\pi iu}.
\end{equation}

Expanding the derivatives by the product rule, we reach to a sum of terms of the form:

\begin{equation}
T_uf(x) = \int_{\mathbb{R}^3} e^{2\pi i\sigma(x, \xi, y, \eta)}\gamma_1(\xi - \eta)\gamma_2(x - y)a_1(x, \xi, \eta)a_2(y, \eta)\Gamma^+_{f, \gamma_3}(y, \eta)d\eta dyd\xi,
\end{equation}
Finally, using Lemma 1, we conclude that
\[ \| T_1 f \|_{L^2(\mathbb{R})} \leq \| D_x G_1 \|_{L^2(\mathbb{R})}, \]
where
\[ G_1(x, \xi) = \int_{\mathbb{R}^2} e^{2\pi i (\eta - \xi) y} \gamma_1^e(\xi - \eta) \gamma_2^e(x - y) a_1^e(x, \xi, \eta, \eta) \Gamma_1^+ \gamma_3^e(y, \eta, \eta) dy d\eta. \]

By the Cauchy-Schwartz inequality, we get
\[ \| D_x G_1 \|_{L^2(\mathbb{R}^2)} \leq \| \gamma_1^e \|_{L^2} \| e^{2\pi i (\eta - \xi) y} D_x \gamma_2^e(x - y) a_1^e(x, \xi, \eta) a_2^e(y, \eta) \Gamma_1^+ \gamma_3^e(y, \eta, \eta) dy \|^2 d\eta. \]

Expanding the derivatives in \( x \), we obtain
\[ \| D_x G_1 \|_{L^2(\mathbb{R}^2)} \leq \| \gamma_1^e \|_{L^2} \| D_x a_1^e \|_{L^\infty(\mathbb{R}^2)} I_1, \]
where
\[ I_1 = \int_{\mathbb{R}^3} \left| \int e^{2\pi i (\eta - \xi) y} D_x \gamma_2^e(x - y) a_1^e(y, \eta) \Gamma_1^+ \gamma_3^e(y, \eta, \eta) dy \right|^2 d\eta dx d\xi. \]

We next do Plancherel in \( x \), then Fubini to integrate first with respect to \( \xi \) and conclude again with Plancherel with real variable \( y \) and Fourier variable \( \xi \):
\[ I_1 = \int_{\mathbb{R}^3} \left( \int e^{2\pi i (\eta - \lambda - \xi) y} a_2^e(y, \eta) \Gamma_1^+ \gamma_3^e(y, \eta, \eta) dy \right)^2 d\xi d\lambda d\eta \]
\[ = \int_{\mathbb{R}^2} \left( e^{2\pi i (\eta - \lambda - \xi) y} a_2^e(y, \eta) \Gamma_1^+ \gamma_3^e(y, \eta, \eta) \right)^2 dy d\lambda d\eta \]
\[ \leq C \| \gamma_2^e \|^2_{H^1} \int_{\mathbb{R}^2} \left| a_2^e(y, \eta) \Gamma_1^+ \gamma_3^e(y, \eta) \right|^2 dy d\eta. \]

Finally, using Lemma 2, we conclude that
\[ I_1 \leq C \| \gamma_2^e \|^2_{H^1} M_0(a_2^e)^2 \| f \|^2_{L^2}. \]

Notice also that, in some of the terms \( T_i \), it appears
\[ a_1^e(x, \xi, \eta) = p_1(x, \xi) - p_1(x, \eta). \]

In order to estimate this factor in terms of \( \partial_\xi p_1 \), it is necessary to integrate by parts one more time in \( y \), using the identity
\[ \frac{1}{2\pi i (\xi - \eta)} \partial_y e^{2\pi i (\xi - \eta) y} = e^{2\pi i (\xi - \eta) y}, \]

to obtain a new function
\[ a_1^e(x, \xi, \eta) = \frac{p_1(x, \xi) - p_1(x, \eta)}{2\pi i (\xi - \eta)}, \]

which, by the mean value theorem, satisfies
\[ \| D_x a_1^e \|_{L^\infty(\mathbb{R}^3)} \leq M_1(\partial_\xi p_1). \]

Taking into account all of the symbols \( p_1 \) and \( p_2 \) we have performed in each term \( T_i \), we obtain
\[ \| \mathcal{C}(p_1, p_2) f \|_{L^2} \leq \left( M_1(\partial_\xi p_1) M_2(p_2) + M_1(p_1) M_1(\partial_\xi p_2) + M_1(\partial_\xi p_1) M_1(\partial_\xi p_2) \right) \| f \|_{L^2}, \]
as we wanted to prove. \( \square \)
We next explain how Lemma 3 applies to the semiclassical framework. To this aim, let us introduce first the following notation for the semiclassical non-principal part of the composition of two semiclassical operators \( p_1(x, tD) \) and \( p_2(x, tD) \):

\[
(30) \quad \mathcal{E}_t(p_1, p_2) := p_1(x, tD)p_2(x, tD) - p_1p_2(x, tD), \quad t \in (0, 1].
\]

We will also consider a localized version of \( \mathcal{E}_t(p_1, p_2) \) near the diagonal of \( \mathbb{R}_x \times \mathbb{R}_y \). Denoting by

\[
B := \{ \varphi \in C_c^\infty(\mathbb{R}) : 0 \leq \varphi \leq 1 \}
\]

the set of bump functions, and given \( \varphi \in B \), we define:

\[
(32) \quad \mathcal{E}_{t, \varphi}(p_1, p_2)(x) := \int_{\mathbb{R}^3} e^{2\pi i (x \xi - y \eta + y \eta)} \varphi(t(\xi - \eta)) (p_1(x, t \xi) - p_1(x, t \eta)) p_2(y, t \eta) f(y) dy dy d\xi.
\]

Let

\[
(33) \quad I_\varphi := \text{Conv} (\text{supp} \varphi \cup \{0\})
\]

be the closed interval obtained as the convex hull of \( \text{supp} \varphi \cup \{0\} \), and let \( |I_\varphi| \) be the Lebesgue measure of \( I_\varphi \). The following holds:

**Corollary 1.** Let \( m_1, m_2 \in \mathbb{R} \) such that \( m_1 + m_2 - 1 \leq 0 \). Set \( \mu = \max\{m_1, m_2, 0\} \). Let \( p_1 \in \mathcal{M}_{1,1}^{m_1} \) and \( p_2 \in \mathcal{M}_{2,1}^{m_2} \). Then, for every \( \varphi \in B \),

\[
(34) \quad \| \mathcal{E}_{t, \varphi}(p_1, p_2) \|_{L^2(\mathbb{R}^3)} \lesssim t |I_\varphi|^\mu \mathcal{M}(p_1, p_2),
\]

where

\[
\mathcal{M}(p_1, p_2) = M_{1,0,1}^{m_1 - 1}(\partial_\xi p_1)M_{2,0,2}^{m_2 - 1}(p_2) + M_{1,0,1}^{m_1} (p_1)M_{1,0,1}^{m_2 - 1}(\partial_\xi p_2) + M_{1,0,1}^{m_1 - 1}(\partial_\xi p_1)M_{1,0,1}^{m_2 - 1}(\partial_\xi p_2).
\]

**Remark 5.** The implicit constant in (34) depends on the \( L^\infty \)-norm of the first derivative of \( \varphi \), but not on \( p_1 \) nor \( p_2 \).

**Remark 6.** The presence of derivatives in \( \xi \) in all the terms in the right-hand-side of (20) allow us to bring the factor \( t \) in the semiclassical estimate (34).

**Proof.** The proof mimics the one of Lemma 3 but in this case, we replace \( p_1(x, \xi) - p_1(x, \eta) \) in (21) by

\[
(35) \quad a_1(x, \xi, \eta) = \varphi(\xi - \eta) (p_1(x, \xi) - p_1(x, \eta))(1 + 2\pi i \eta)^{m_2},
\]

and \( p_2(y, \eta) \) by \( p_2(y, \eta) = (1 + 2\pi i \eta)^{-m_2} p_2(y, \eta) \). Using next the identities (22), (23) and (24), we integrate by parts in \( \eta, \xi \) and \( y \) successively to get

\[
\mathcal{E}_{t, \varphi}(p_1, p_2)f(x)
\]

\[
= \int_{\mathbb{R}^3} e^{2\pi i \sigma(x, \xi, \eta, y)} \gamma(\xi - \eta) D_y (\gamma(x - y) \Gamma_\gamma^\dagger(\gamma(y, \eta) D_\xi D_\eta (a_1(x, t \xi, t \eta) p_2(y, t \eta)))) dy dy d\xi,
\]

where \( \sigma(x, \xi, \eta, y) = x \xi - \xi \eta + y \eta \). Expanding the derivatives by the product rule, we reach again to a sum of terms of the form (20) after the obvious substitutions. In particular, when no derivatives in \( \xi \) nor \( \eta \) are performed in \( a_1 \), a further use of integration by parts in the \( y \) variable, as we did to obtain (29), allow us to replace \( a_1' = a_1 \) by

\[
a_1'(x, t \xi, t \eta) = \frac{\varphi((t(\xi - \eta)) (p_1(x, t \xi) - p_1(x, t \eta))(1 + 2\pi i t \eta)^{m_2}}{2\pi i (\xi - \eta)}
\]

\[
= t \quad \frac{\varphi((t(\xi - \eta)) (p_1(x, t \xi) - p_1(x, t \eta))(1 + 2\pi i t \eta)^{m_2}}{2\pi i (\xi - \eta)}
\]

We then estimate each of the terms \( J_i \) obtained similarly as we did in the proof of Lemma 3. Here we only remark the main differences and changes required in this case, which appear only when bounding the \( L^\infty \) norms of \( a_1' \) and \( a_2' \). In fact, it is sufficient to indicate how the term \( J_i \) involving \( a_1' \) and \( a_2' \) is managed, since the others can be bounded in a completely analogous way.
We consider the set
\[ \Omega_\varphi := \{ \xi, \eta \in \mathbb{R}^2 : \xi - \eta \in I_\varphi \}, \]
and we use that \( m_1 + m_2 - 1 \leq 0 \) and the mean-value theorem to get
\[
\sup_{(x, \xi, \eta) \in \mathbb{R}^3} |\mathcal{D}_x a_1^x(x, t\xi, t\eta)| = t \cdot \sup_{(x, \xi, \eta) \in \mathbb{R}^3} \frac{\varphi(t(\xi - \eta)) \mathcal{D}_x (p_1(x, t\xi) - p_1(x, t\eta)) (1 + 2\pi i t\eta)^{m_2}}{|t(\xi - \eta)|}
\]
\[
= t \cdot \sup_{(x, \xi, \eta) \in \mathbb{R}^3} \frac{\varphi(\xi - \eta) \mathcal{D}_x (p_1(x, \xi) - p_1(x, \eta)) (1 + 2\pi i \eta)^{m_2}}{|\xi - \eta|}
\]
\[
\leq t \|\varphi\|_{L^\infty} \sup_{x \in \mathbb{R}} |\mathcal{D}_x \partial_\xi p_1(x, \xi)| ||1 + 2\pi i \eta|^{m_2}
\]
\[
\leq t M_{m_1-1}^{m_1-1}(\partial_\xi p_1) \sup_{(\xi, \eta) \in \Omega_\varphi} (1 + |\xi|)^{m_1-1}(1 + |\eta|)^{m_2}
\]
\[
\leq t |I_\varphi| \mu^{m_1-1}(\partial_\xi p_1).
\]
Moreover, for every \( 0 \leq \alpha \leq 2 \),
\[ (37) \quad \sup_{(y, \eta) \in \mathbb{R}^2} |\partial_y^\alpha p_2(y, t\eta)| \leq M_{2,0}^{m_2}(p_2). \]
These and analogous estimates, depending on whether the derivatives \( \partial_\xi \) and \( \partial_\eta \) act on the factors \( \varphi, p_1 \) or \( p_2 \), together with the ones given in the proof of Lemma 4 suffice to bound all the terms \( T_i \).

We next deal with symbols in the classes \( N_{s,1}^0 \). Since we already have \( L^2 \)-decay in the \( x \) variable, we do not need to integrate by parts in the momentum variables \( \xi, \eta \). This simplifies the proof.

**Lemma 4.** Let \( p_1 \in N_{1,1}^0 \) and \( p_2 \in N_{2,0}^0 \). Then
\[ \|\mathcal{E}(p_1, p_2)\|_{L^2} \lesssim N_1(\partial_\xi p_1) N_2(p_2). \]

**Proof.** We now have
\[ \mathcal{E}(p_1, p_2) f(x) = \int_{\mathbb{R}^3} e^{2\pi i \sigma(x, \xi, y, \eta)} \gamma(\xi - \eta)(p_1(x, \xi) - p_1(x, \eta)) \mathcal{D}_y p_2(y, \eta) \hat{f}(\eta) d\eta dy d\xi. \]
We then integrate by parts one more time in \( y \), using the identity
\[ \frac{1}{2\pi i (\xi - \eta)} \partial_y e^{2\pi i (\xi - \eta)y} = e^{2\pi i (\xi - \eta)y}, \]
to obtain
\[ \mathcal{E}(p_1, p_2) f(x) = \int_{\mathbb{R}^3} e^{2\pi i \sigma(x, \xi, y, \eta)} \gamma(\xi - \eta) \left( \frac{p_1(x, \xi) - p_1(x, \eta)}{2\pi i (\xi - \eta)} \right) \partial_y \mathcal{D}_y p_2(y, \eta) \hat{f}(\eta) d\eta dy d\xi. \]
Considering
\[ a_1^x(x, \xi, \eta) := \frac{p_1(x, \xi) - p_1(x, \eta)}{2\pi i (\xi - \eta)} = \frac{1}{2\pi i} \int_0^1 \partial_\xi p_1(x, \eta + s(\xi - \eta)) ds, \]
and using Lemma 3 yields that
\[ \|\mathcal{E}(p_1, p_2) f\|_{L^2} \leq \|\mathcal{D}_x G\|_{L^2_{x} \times \mathbb{R}^2)}, \]
where
\[ G(x, \xi) = \int_{\mathbb{R}^2} e^{2\pi i (-\xi y + \eta y)} \gamma(\xi - \eta) a_1^x(x, \xi, \eta) \partial_y \mathcal{D}_y p_2(y, \eta) \hat{f}(\eta) d\eta dy. \]
The end of the proof follows by similar arguments of those of Lemma 3. By the Cauchy-Schwartz inequality, we have

\[ |D_x G(x, \xi)|^2 \leq \|\|_2 \int_{\mathbb{R}} |D_x a_1(x, \xi, \eta) \int_{\mathbb{R}} e^{2\pi i (\eta - \xi)y} \partial_y D_g p_2(y, \eta) \tilde{f}(\eta) dy| \, d\eta. \]

Moreover, by the Minkowski integral inequality,

\[
\int_{\mathbb{R}} |D_x a_1(x, \xi, \eta)|^2 \, dx \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |D_x \partial_\xi p_1(x, \eta + s(\xi - \eta))|^2 \, dx \right)^{1/2} \, d\xi \leq \sup_{\xi} \int_{\mathbb{R}} |D_x \partial_\xi p_1(x, \xi)|^2 \, dx = N_1(\partial_\xi p_1)^2.
\]

Hence, using this and (38), we obtain

\[
\|D_x G\|_{L^2(\mathbb{R}^2)} \lesssim \|\|_{L^2} N_1(\partial_\xi p_1)^2 \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} e^{2\pi i (\eta - \xi)y} \partial_y D_g p_2(y, \eta) \tilde{f}(\eta) dy \right|^2 \, d\eta.
\]

Finally, using Plancherel in the \( \xi \)-variable, we conclude that

\[
\|D_x G\|_{L^2(\mathbb{R}^2)} \lesssim \|\|_{L^2} N_1(\partial_\xi p_1)^2 \int_{\mathbb{R}^2} |\partial_y D_g p_2(y, \eta) \tilde{f}(\eta)|^2 \, dy \, d\eta 
\lesssim \|\|_{L^2} N_1(\partial_\xi p_1)^2 N_2(p_2)^2 \|f\|_{L^2}^2.
\]

The following corollary is a semiclassical and localized version of Lemma 4.

**Corollary 2.** Let \( p_1 \in N_{1,1}^{m_1} \) and \( p_2 \in N_{2,0}^{m_2} \) with \( m_1 + m_2 - 1 \leq 0 \). Set \( \mu = \max\{m_1, m_2, 0\} \).

Then, for every \( \varphi \in \mathcal{B} \),

\[
\|\mathcal{C}_{\xi, \varphi}(p_1, p_2)\|_{L^2(\mathbb{R}^2)} \lesssim t|I_{\varphi}|^{\mu} N_{1,0}^{m_1-1}(\partial_\xi p_1) N_{2,0}^{m_2}(p_2).
\]

Combining the two previous lemmas, we also have:

**Lemma 5.** Let \( p_1 \in M_{1,1}^{0} \) and \( p_2 \in N_{2,0}^{0} \). Then

\[
\|\mathcal{C}(p_1, p_2)\|_{L^2} \lesssim M_1(\partial_\xi p_1) N_2(p_2).
\]

Similarly, let \( p_1 \in N_{1,1}^{0} \) and \( p_2 \in M_{2,1}^{0} \). Then

\[
\|\mathcal{C}(p_1, p_2)\|_{L^2} \lesssim N_1(\partial_\xi p_1) M_2(p_2) + N_1(p_1) M_1(\partial_\xi p_2).
\]

**Corollary 3.** Let \( p_1 \in M_{1,1}^{m_1} \) and \( p_2 \in N_{2,0}^{m_2} \) with \( m_1 + m_2 - 1 \leq 0 \). Set \( \mu = \max\{m_1, m_2, 0\} \).

Then, for every \( \varphi \in \mathcal{B} \),

\[
\|\mathcal{C}_{\xi, \varphi}(p_1, p_2)\|_{L^2} \lesssim t|I_{\varphi}|^{\mu} M_{1,0}^{m_1-1}(\partial_\xi p_1) N_{2,0}^{m_2}(p_2).
\]

Similarly, let \( p_1 \in N_{1,1}^{m_1} \) and \( p_2 \in M_{2,1}^{m_2} \). Then

\[
\|\mathcal{C}_{\xi, \varphi}(p_1, p_2)\|_{L^2} \lesssim t|I_{\varphi}|^{\mu} (N_{1,0}^{m_1-1}(\partial_\xi p_1) M_{2,0}^{m_2}(p_2) + N_{1,0}^{m_1}(p_1) M_{1,0}^{m_2-1}(\partial_\xi p_2)).
\]

We next improve the previous lemmas when the supports of \( p_1 \) and \( p_2 \) in the \( \xi \) variable are disjoint. To this aim, let us define, for any \( p(x, \xi) \),

\[
supp\xi p := \bigcup_{x \in \mathbb{R}} \text{supp} p(x, \cdot).
\]
Lemma 6. Let \( p_1 \in \mathcal{M}_{1,1}^{m_1} \) and \( p_2 \in \mathcal{M}_{N,1}^{m_2} \) with \( N \geq 2 \). Assume that
\[
d := \text{dist} \left( \text{supp}_x p_1, \text{supp}_x p_2 \right) > 0.
\]

Then
\[
\|\mathcal{E}(p_1, p_2)\|_{L^2(\mathbb{R}^2)} \lesssim d^{-(N-3/2)} \left( M_1(\partial_x p_1)M_N(p_2) + M_1(p_1)M_{N-1}(\partial_x p_2) + M_1(\partial_x p_1)M_{N-1}(\partial_x p_2) \right).
\]

Proof. The proof follows the same lines of the one of Lemma 3. As in that case, we write again the expression
\[
\mathcal{E}(p_1, p_2) f(x) = \int e^{2\pi i (x, \xi, \eta)} \gamma_1(\xi - \eta) \mathcal{D}_y \left( \gamma(x - y) \Gamma^+_{f,\gamma} \right) \mathcal{D}_\eta \left( (p_1(x, \xi) - p_1(x, \eta))p_2(x, \eta) \right) \, d\eta \, dy \, d\xi,
\]
where \( \sigma(x, \xi, \eta) = x \xi - \xi y + \eta \), and the differential operators \( \mathcal{D}_\omega \) act on all the functions on its right. We next do \((N-1)\)-integrations by parts in the \( y \)-variable, using the identity
\[
\frac{1}{2\pi i (\xi - \eta)} \partial_\gamma e^{2\pi i (\xi - \eta) y} = e^{2\pi i (\xi - \eta) y},
\]
to bring a factor \((2\pi i (\xi - \eta))^{-(N-1)}\). Using the definition of \( d > 0 \), we observe that
\[
\left( \frac{1}{2\pi i (\xi - \eta)} \right) \leq \frac{1}{d}
\]
Observe also that in this case, the use of Cauchy-Schwartz as before (27) allows us to obtain
\[
|\mathcal{D}_x G_i(x, \xi)|^2
\]
\[
= \left| \int_{\|\eta - \xi\| \leq d} \int \mathcal{D}_x \gamma \mathcal{D}_\eta \mathcal{D}_y \mathcal{D}_\xi \left( \frac{A_1(x, \xi, \eta)}{2\pi i (\xi - \eta)^{N-2}} A_2(y, \eta) \Gamma^+_{f,\gamma} \right) \, d\eta \, dy \, d\xi \right|^2
\]
\[
\leq \|\gamma\|_{L^2(\mathbb{R}^2)}^2 \sup_{\|\xi - \eta\| \geq d} \left( \frac{\mathcal{D}_x A_1(x, \xi, \eta)}{2\pi i (\xi - \eta)^{N-2}} \right) \lesssim d^{-(N-3/2)} \|\mathcal{D}_x A_1\|_{L^\infty(\mathbb{R}^2)} \lesssim d^{-(N-3/2)} M_1(\partial_x p_1),
\]
and
\[
\|\gamma\|_{L^2(\mathbb{R}^2)} = \left( \int_{\|\eta\| \geq d} \frac{d\eta}{1 + \eta^2} \right) \leq \pi - 2 \arctan(d)^2 = O \left( d^{-\frac{1}{2}} \right), \quad \text{as } d \to \infty.
\]
The rest of the proof mimics the proof of Lemma 3. \( \square \)

From this, we obtain the following corollary which is a semiclassical and localized version of Lemma 6

Corollary 4. Let \( N \geq 2 \). Let \( p_1 \in \mathcal{M}_{1,1}^{m_1} \) and \( p_2 \in \mathcal{M}_{N,1}^{m_2} \) with \( m_1 + m_2 - 1 \leq 0 \). Set \( \mu = \max \{m_1, m_2, 0\} \). Let \( \varphi \in \mathbf{B} \), assume that
\[
d := \text{dist} (0, \text{supp} \varphi) > 0.
\]
Then
\[ \| \mathcal{E}_{t\varphi}(p_1, p_2) \|_{L^2} \lesssim t^{N-1} d^{-\left(N - \frac{3}{2}\right)} |I_\varphi|^p \mathfrak{M}_N(p_1, p_2), \]
where
\[ \mathfrak{M}_N(p_1, p_2) = M_{1,0}^{m_1-1}(\partial_\xi p_1) M_{N,0}^{m_2}(p_2) + M_{1,0}^{m_1}(p_1) M_{N-1,0}^{m_2-1}(\partial_\xi p_2) + M_{1,0}^{m_1-1}(\partial_\xi p_1) M_{N-1,0}^{m_2-1}(\partial_\xi p_2). \]

**Proof.** The proof mimics the one of Lemma 3 but with some changes analogous to those referred in the proof of Corollary 1 with respect to the terms \( J_i \). To highlight the required changes, we consider again \( a_1 \) given by (35) and use identity (28) and integration by parts with respecto the \( y \) variable to replace \( a_1 \) by
\[ a_1^*(x, t\xi, t\eta) = t \frac{\varphi(t(x, t\xi)) \left( p_1(x, t\xi) - p_1(x, t\eta) \right) (1 + 2\pi t\eta)^{m_2}}{2\pi t(\xi - \eta)}. \]

To bound the term \( J_i \) involving this symbol, we observe that
\[
\sup_{(x, \xi, \eta) \in \mathbb{R}^3} \left| \frac{D_x a_1^*(x, t\xi, t\eta)}{(2\pi i(\xi - \eta))^{N/2}} \right| \lesssim t^{N-1} \sup_{(x, \xi, \eta) \in \mathbb{R}^3} \left| \frac{\varphi(t(\xi - \eta)) D_x \left( p_1(x, t\xi) - p_1(x, t\eta) \right) (1 + 2\pi t\eta)^{m_2}}{|t(\xi - \eta)|^{N-1}} \right|
\]
\[
\lesssim t^{N-1} d^{-N-2} \sup_{(x, \xi, \eta) \in \mathbb{R}^3} \left| \frac{\varphi(\xi - \eta) D_x \left( p_1(x, \xi) - p_1(x, \eta) \right) (1 + 2\pi i\eta)^{m_2}}{|\xi - \eta|} \right|
\]
\[
\lesssim t^{N-1} d^{-N-2} \| \varphi \|_{L^\infty} \sup_{x \in \mathbb{R}^3} \sup_{(\xi, \eta) \in \Omega_\varphi} |D_x \partial_\xi p_1(x, \xi)| (1 + 2\pi i\eta)^{m_2}
\]
\[
\lesssim t^{N-1} d^{-N-2} M_{1,0}^{m_1-1}(\partial_\xi p_1) \sup_{(\xi, \eta) \in \Omega_\varphi} (1 + |\xi|)^{m_1-1} (1 + |\eta|)^{m_2}
\]
\[
\lesssim t^{N-1} d^{-N-2} |I_\varphi|^p M_{1,0}^{m_1-1}(\partial_\xi p_1).
\]

Taking into account (37), (41), and the rest of estimates of the proof of Lemma 3, we can manage all the terms \( T_i \), and the result follows. \( \Box \)

The following lemma extends the previous one allowing fractional derivatives of \( p_2 \) in the \( x \) variable.

**Lemma 7.** Let \( p_1 \in \mathcal{N}_{1,0}^s \) and \( p_2 \in \mathcal{N}_{s,0}^0 \) with \( s > 3/2 \). Assume that
\[ d = \text{dist}\left( \text{supp}_x p_1, \supp_x p_2 \right) > 0. \]

Then
\[ \| \mathcal{E}(p_1, p_2) \|_{L^2} \lesssim d^{-(s-3/2)} N_1(\partial_\xi p_1) N_s(p_2). \]

**Proof.** The proof is analogous to the one of Lemma 4. We have
\[
\mathcal{E}(p_1, p_2) f(x) = \int_{\mathbb{R}^3} e^{2\pi i \sigma(x, \xi, y, \eta)} \gamma(\xi - \eta) \left( p_1(x, \xi) - p_1(x, \eta) \right) D_y p_2(y, \eta) \hat{f}(\eta) d\eta dy d\xi.
\]

We then integrate by parts one more time in \( y \), using the identity
\[
\frac{1}{2\pi i(\xi - \eta)} \partial_y e^{2\pi i(\xi - \eta)y} = e^{2\pi i(\xi - \eta)y},
\]
to obtain
\[
\mathcal{E}(p_1, p_2) f(x) = \int_{\mathbb{R}^3} e^{2\pi i \sigma(x, \xi, y, \eta)} \gamma(\xi - \eta) \left( \frac{p_1(x, \xi) - p_1(x, \eta)}{2\pi i(\xi - \eta)} \right) \partial_y D_y p_2(y, \eta) \hat{f}(\eta) d\eta dy d\xi.
\]
Moreover, we also have
\[
\mathcal{E}(p_1, p_2) f(x)
\]
\[
= \int_{\mathbb{R}^3} e^{2\pi i \sigma(x, \xi, y, \eta)} \gamma(\xi - \eta) \left( \frac{p_1(x, \xi) - p_1(x, \eta)}{2\pi i(\xi - \eta)} \right) \partial_y D_y^{s-1} p_2(y, \eta) \hat{f}(\eta) d\eta dy d\xi,
\]
where
\[
\gamma_s(\xi - \eta) := \frac{1}{(1 + 2\pi i(\xi - \eta))^{s-1}}.
\]
Therefore, it is sufficient to use the fact that
\[
\|\gamma_s\|_{L^2(B_d)} = \left(\int_{|\eta| \geq d} \frac{d\eta}{(1 + (2\pi i)^2)^{s-1}}\right)^{\frac{1}{2}} = O\left(d^{-(s-\frac{1}{2})}\right),
\]
and the rest of estimates given in the proof of Lemma \ref{lem:estimates}.

**Corollary 5.** Let \( s > 3/2 \). Let \( p_1 \in \mathcal{M}_{1,1}^{m_1} \) and \( p_2 \in \mathcal{N}_{s,0}^{m_2} \) with \( m_1 + m_2 - 1 \leq 0 \). Set \( \mu = \max\{m_1, m_2, 0\} \). Given \( \varphi \in C_c^\infty \), assume that
\[
d = \text{dist}(0, \text{supp} \varphi) > 0.
\]
Then
\[
\|\mathcal{C}(p_1, p_2)\|_{L^2} \lesssim t^{s-1/2} d^{-(s-3/2)} |I_\varphi|^\mu \mathcal{N}_{1,0}^{m_1-1}(\partial_\xi p_1) \mathcal{N}_{s,0}^{m_2}(p_2).
\]

**Lemma 8.** Let \( p_1 \in \mathcal{M}_{1,1}^{0} \) and \( p_2 \in \mathcal{N}_{s,0}^{0} \) with \( s > 3/2 \). Assume that \( d = \text{dist}(\text{supp}_\xi p_1, \text{supp}_\xi p_2) > 0 \).

Then
\[
\|\mathcal{C}(p_1, p_2)\|_{L^2} \lesssim d^{-(s-3/2)} M_1(\partial_\xi p_1) N_s(p_2).
\]

**Proof.** The proof is similar to the one before, but we need to use integration by parts in the \( \xi \) variable to obtain decayment in the \( x \) variable (since now \( p_1 \) is only bounded in this variable). We have
\[
\mathcal{C}(p_1, p_2)f(x) = \int_{\mathbb{R}^3} e^{2\pi i \sigma(x, \xi, y, \eta)} \gamma(x - y) D_\xi Q_s(x, \xi, \eta) \partial_y D_y^{-1} p_2(y, \eta) \hat{f}(\eta) d\eta dy d\xi,
\]
where
\[
Q_s(x, \xi, \eta) = \gamma_s(\xi - \eta) \left(\frac{p_1(x, \xi) - p_1(x, \eta)}{2\pi i(\xi - \eta)}\right),
\]
which is differentiable in the \( \xi \) variable provided that \( |\xi - \eta| \geq d \). Precisely,
\[
\partial_\xi Q_s(x, \xi, \eta) = \partial_\xi \gamma_s(\xi - \eta) \left(\frac{p_1(x, \xi) - p_1(x, \eta)}{2\pi i(\xi - \eta)}\right) - \gamma_s(\xi - \eta) \left(\frac{p_1(x, \xi) - p_1(x, \eta)}{2\pi i(\xi - \eta)^2}\right) + \gamma_s(\xi - \eta) \frac{\partial_\xi p_1(x, \xi)}{2\pi i(\xi - \eta)}.
\]
Then, using the mean-value theorem, one has
\[
\sup_{|\xi - \eta| \geq d} |\mathcal{D}_x \mathcal{D}_\xi Q_s(x, \xi, \eta)| \lesssim d^{-(s-2)} M_1(\partial_\xi p_1).
\]
As in the previous proofs, we use that \( \|\mathcal{C}(p_1, p_2)f\|_{L^2} \leq \|\mathcal{D}_x G\|_{L^2(\mathbb{R}^2)} \), where
\[
G(x, \xi) = \int_{\mathbb{R}^2} e^{2\pi i \sigma(x, \xi, y, \eta)} \gamma(x - y) D_\xi Q_s(x, \xi, \eta) \partial_y D_y^{-1} p_2(y, \eta) \hat{f}(\eta) d\eta dy.
\]
By the Cauchy-Schwarz inequality, \eqref{eq:Cauchy-Schwarz} and \eqref{eq:mean-value-theorem}, we get
\[
|\mathcal{D}_x G(x, \xi)|^2 \lesssim (d^{-(s-3/2)})^2 M_1(\partial_\xi p_1)^2 \int_{\mathbb{R}^2} \int e^{2\pi i (\eta - \xi) y} D_\xi \gamma(x - y) \partial_y D_y^{-1} p_2(y, \eta) \hat{f}(\eta) d\eta dy \left|\frac{d\eta}{d\xi}\right|^2 d\eta.
\]
By Plancherel and Fubini, as in the end of the proof of Lemma \ref{lem:estimates} we conclude that
\[
\|\mathcal{D}_x G\|_{L^2(\mathbb{R}^2)} \lesssim d^{-(s-3/2)} M_1(\partial_\xi p_1) N_s(p_2) \|f\|_{L^2}.
\]
Corollary 6. Let \( s > 3/2 \). Let \( p_1 \in \mathcal{M}^{m_1}_{J,1} \) and \( p_2 \in \mathcal{N}^{m_2}_{s,0} \) with \( m_1 + m_2 - 1 \leq 0 \). Set \( \mu = \max\{m_1, m_2, 0\} \). Given \( \varphi \in \mathcal{C}^\infty(\mathbb{R}) \), assume that
\[
d = \text{dist}(0, \text{supp} \varphi) > 0.
\]
Then
\[
\|\mathcal{C}_{t, \varphi}(p_1, p_2)\|_{L(L^2)} \lesssim t^{s-1/2} d^{-(s-3/2)} |I_{\varphi}|^\mu M^{-1}_{1,0}(\partial_\xi p_1) N^{m_2}_{s,0}(p_2).
\]

4. Semiclassical estimates on Sobolev spaces

In this Section we establish some semiclassical estimates concerning the action of our operators \( d(x, tD) \), \( p(x, tD) \) and \( p^{-1}(x, tD) \), as well as certain commutators between them, on the Sobolev spaces \( H^s_t(\mathbb{R}) \). Despite we focus on these particular operators, we will only use their properties as operators having symbols in the classes introduced in Section 2, so the techniques below can be used elsewhere.

Some techniques of paradifferential calculus are useful to extend the results of [La06] and [Tex07] concerning commutators require more regularity in \( \xi \) at the origin, in order to get satisfactory semiclassical estimates. Notice that our symbols have only one derivative in the \( \xi \) variable bounded in \( L^\infty \) near the origin. We avoid the use of paradifferential calculus in our commutator estimates by requiring a bit more of regularity in the \( x \) variable (see (C2') in Definition 3).

We first show the following lemma that link the seminorms of semiclassical symbols with those of non-semiclassical ones.

Lemma 9. Let \( m \geq 0 \) and let \( a \in \mathcal{M}^m_{J,k} \). For every \( t \in (0, 1] \), set \( a_t(x, \xi) := a(x, t\xi) \). Then
\[
\sup_{t \in (0, 1]} M^m_{J,k}(a_t) \leq M^m_{J,k}(a).
\]

Analogously, if \( a \in \mathcal{N}^m_{s,k} \), then
\[
\sup_{t \in (0, 1]} N^m_{s,k}(a_t) \leq N^m_{s,k}(a).
\]

Proof. For every \( \alpha \leq j \) and \( \beta \leq k \), one has
\[
|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq M^m_{J,k}(a)(1 + |\xi|)^{m-\beta}.
\]
Then
\[
(1 + |\xi|)^{\beta-\mu} |\partial_\xi^\alpha \partial_x^\beta a_t(x, \xi)| \leq t^\beta (1 + |\xi|)^{\beta-\mu} |\partial_\xi^\alpha \partial_x^\beta a(x, t\xi)|
\]
\[
\leq M^m_{J,k}(a) t^\beta \left( \frac{1 + |\xi|}{1 + t|\xi|} \right)^{\beta-\mu}.
\]
If \( m - \beta \geq 0 \), the latter expression is uniformly bounded by \( M^m_{J,k}(a) \) for all \( t \in (0, 1] \). Otherwise, if \( \beta - m > 0 \), then the function
\[
u(t) = t^\beta \left( \frac{1 + |\xi|}{1 + t|\xi|} \right)^{\beta-\mu}
\]
satisfies that \( u(0) = 0, u(1) = 1 \) and \( u'(t) \geq 0 \) provided that \( m \geq 0 \). This proves the first assertion. The second follows in the same way. \( \square \)

Proposition 1. Assume (C2). Let \( m = \lfloor m_1 \rfloor \). Then the operator \( p(x, tD) : L^2(\mathbb{R}) \rightarrow H^{-m}_t(\mathbb{R}) \) is continuous for every \( t \in (0, 1] \), and
\[
\|p(x, tD) f\|_{H^{-m}_t(\mathbb{R})} \leq M^m_{m+1,1}(p) \|f\|_{L^2(\mathbb{R})}.
\]
(45)
Proof. Let us denote \( p_t(x, \xi) = p(x, t \xi) \). Using integration by parts, for any \( f \in L^2(\mathbb{R}) \), we have:

\[
p(x, tD)f(x) = A_Q(x; t) = \int_\mathbb{R} e^{2\pi i x \xi} Q(x, \xi; t) d\xi,
\]

where

\[
Q(x, \xi; t) = D_x p_t(x, \xi) \Gamma_f^+(x, \xi),
\]

and \( \gamma \) is given by (25). We estimate \( \|A_Q\|_{H^{-m}_t} \) by duality, as in the proof of Lemma 2. Take \( g \in H^{m}_t(\mathbb{R}) \) and write

\[
\int_\mathbb{R} A_Q(x; t) g(x) dx = \int_{\mathbb{R}^2} e^{2\pi i x \xi} Q(x, \xi; t) g(x) d\xi dx
\]

which \( Q^t(x, \xi; t) = Q(x, \xi; t)(1 + 2\pi it \xi)^{-m} \). We next integrate by parts in \( x \) to get

\[
\int_\mathbb{R} A_Q(x; t) g(x) dx = \int_{\mathbb{R}^2} e^{2\pi i x \xi} D_x^{m+1} Q^t(x, \xi; t) \Gamma_g(x, \xi; t) d\xi dx,
\]

where

\[
\Gamma_g(x, \xi; t) = \int_\mathbb{R} (1 + 2\pi it \xi)^m \hat{g}(\lambda) e^{2\pi i \lambda x} (1 + 2\pi i (\lambda + \xi))^{m+1} d\lambda
\]

\[
= \sum_{j=0}^m (-1)^j \binom{m}{j} \int_\mathbb{R} (1 + 2\pi it (\lambda + \xi))^{-j} (1 + 2\pi i (\lambda + \xi))^{m+1} d\lambda.
\]

Using Cauchy-Schwartz inequality and Lemma 1 we obtain that

\[
\left| \int_\mathbb{R} A_Q(x) g(x) dx \right| \lesssim \|D_x^{m+1} Q\|_{L^2(\mathbb{R}^2)} \|g\|_{H^{m}_t(\mathbb{R})}
\]

\[
\lesssim M_{m+1,1}^0(\|f\|_{L^2(\mathbb{R})}) \|g\|_{H^{m}_t(\mathbb{R})},
\]

where \( p_t^\dagger(x, \xi) = p^\dagger(x, t \xi) \) and \( p^\dagger(x, \xi) = p(x, \xi)(1 + 2\pi i \xi)^{-m} \). Finally, to obtain \( \|Q\|_{L^2(\mathbb{R}^2)} \), it is sufficient to use Lemma 3. \( \square \)

**Corollary 7.** Assume (C2') and set \( m = \lfloor m_1 \rfloor \). Then

\[
\|p(x, tD)f\|_{H^{-m}_t(\mathbb{R})} \lesssim \|v\|_{H^s} \|H^s \|_{L^2(\mathbb{R})}
\]

Proof. We write \( p(x, \xi) = \Pi_p(x, \xi) + \Sigma_p(0, \xi) \). Since \( \Sigma_p(0, \xi) \in M^m_{\infty,1} \) is a Fourier multiplier, one has

\[
\|\Sigma_p(0, tD)f\|_{H^{-m}_t(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}
\]

It remains to show that

\[
\|\Pi_p(x, tD)f\|_{H^{-m}_t(\mathbb{R})} \lesssim \|v\|_{H^s} \|H^s \|_{L^2(\mathbb{R})}
\]

To do this, we write

\[
\Pi_p(x, tD)f(x) = A_K(x; t) = \int_\mathbb{R} e^{2\pi i x \xi} K(x, \xi; t) d\xi,
\]

where \( K(x, \xi; t) = \Pi_p(x, t\xi) \hat{f}(\xi) \). We estimate \( \|A_K\|_{H^{-m}_t(\mathbb{R})} \) by duality, exactly as in the proof of Proposition 1 with \( K \) instead of \( Q \). Take \( g \in H^{m}_t(\mathbb{R}) \). Next integrate by parts in \( x \) to get

\[
\int_\mathbb{R} A_K(x; t) g(x) dx = \int_{\mathbb{R}^2} e^{2\pi i x \xi} D_x^{m+1} K^\dagger(x, \xi; t) \Gamma_g(x, \xi; t) d\xi dx.
\]
where $K^\dagger(x, \xi; t) = K(x, \xi; t)(1 + 2\pi it\xi)^{-m}$ and
$$
\Gamma_g(x, \xi; t) = \int_{\mathbb{R}} \frac{(1 + 2\pi it\xi)^m \hat{g}(\lambda)e^{2\pi i\lambda x}}{(1 + 2\pi i(\lambda + \xi))^{m+1}} d\lambda.
$$
$$
= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \int_{\mathbb{R}} \frac{(1 + 2\pi it(\lambda + \xi))^{m-j}(2\pi it\lambda)^j \hat{g}(\lambda)e^{2\pi i\lambda x}}{(1 + 2\pi i(\lambda + \xi))^{m+1}} d\lambda.
$$
Using Cauchy-Schwartz inequality and Lemma 1, we obtain that
$$
\left| \int_{\mathbb{R}} A_K(x)g(x)dx \right| \lesssim \| D_x^{m+1} K^\dagger \|_{L^2(\mathbb{R}^2)} \| g \|_{H^m(\mathbb{R})}
$$
$$
\lesssim N_{m+1}(\Pi_p^\dagger) \| f \|_{L^2(\mathbb{R})} \| g \|_{H^m(\mathbb{R})},
$$
where $\Pi_p^\dagger(x, \xi) = \Pi_p^\dagger(x, t\xi)$ and $\Pi_p(x, \xi) = \Pi_p(x, (1 + 2\pi i\xi)^{-m}$. Finally, making use of Lemma 9 and the fact that $\Pi_p$ is smooth in the image of $v$, hence Moser’s inequality applies, we conclude that
$$
N_{m+1}(\Pi_p^\dagger) \lesssim N_{m+1,0}(\Pi_p) \leq C(\| v \|_{L^\infty})(1 + \| v \|_{H^*H^*}),
$$
provided that $s \geq 2$.

Notice that the proofs of Proposition 1 and Corollary 2 are particularly simple due to the use of the Leibniz rule as after [45]. The use of paradifferential calculus as in [La06] and [Tex07] allows us to improve this result to the case $m_1$ being non-integer.

**Proposition 2.** Assume (C2'). Then
$$
\| p(x, tD)f \|_{H^{m_1}(\mathbb{R})} \lesssim \| v \|_{H^*H^*} \| f \|_{L^2(\mathbb{R})},
$$
provided that $s > m_1$.

**Proof.** We write $p(x, \xi) = \Pi_p(x, \xi) + \Sigma_p(0, \xi)$. Since $\Sigma_p(0, tD)$ is a Fourier multiplier with symbol belonging to $\mathcal{M}_{\infty,1}$, the it satisfies (49) trivially. It is then sufficient to prove the result for $\Pi_p(x, tD)$.

To this aim, set $\sigma(x, \xi) := \Pi_p(x, \xi)$ and consider the decomposition of [La06]:
$$
\sigma = \sigma_I + \sigma_I + \sigma_{II} + \sigma_R.
$$
We will estimate each of these terms separately. First, for the low-frequency term $\sigma_I$ we just observe that $\sigma_I \in \mathcal{N}_{s,1}^m$ for every $m \in \mathbb{R}$. Then we can use [Hw87] Corollary 2.2 together with Lemma 9 and Moser’s inequality to get
$$
\| \sigma_I(x, tD)f \|_{H^{m_1}(\mathbb{R})} \leq \| \sigma_I(x, tD)f \|_{L^2(\mathbb{R})} \lesssim N_1(\sigma_I) \| f \|_{L^2(\mathbb{R})} \lesssim \| v \|_{H^*H^*} \| f \|_{L^2(\mathbb{R})}.
$$
The second term $\sigma_I$ is smooth in both variables, so one can estimate $\| \sigma_I(x, tD) \|_{H^{m_1}}$ by the right hand side of (49) just using classical tools. We refer for instance to [Tex07] Prop. 23.

In order to bound the terms $\sigma_{II}$ and $\sigma_R$, we adapt the proof of [La06] Prop. 25, (ii)] to our context. For the term $\sigma_{II}$, we proceed as follows. Using Remark 2 we have that
$$
\| \sigma_{II}(x, tD)f \|_{H^{-m_1}} = \| \sigma_{II}^\dagger(x, D)U_t^*f \|_{H^{m_1}}.
$$
Moreover, by the second estimate of [La06] Prop. 20], we have
$$
\| \sigma_{II}^\dagger(x, D)U_t^*f \|_{H^{-m_1}} \lesssim N_{s-k,2}^m(\nabla^k_\sigma^t) \| U_t^*f \|_{L^2} = N_{s-k,2}^m(\nabla^k_\sigma^t) \| f \|_{L^2},
$$
for every $k < s$. Using that
$$
N_{s-k,2}^m(\nabla^k_\sigma^t) \leq t^{k-1/2}N_{s-k,2}^m(\sigma),
$$
and Moser’s inequality, we obtain the desired estimate.

\[1\] One can also mimic the proof of Corollary 7 with $m = 0$ instead of $m_1$. 

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Finally, the term $\sigma_R$ can be bounded in a similar way. Using again Remark 2 we have
\[
\|\sigma_R(x,tD)f\|_{H^{-m_1}_t} = \|\sigma_R^*(x,D)U^*_t f\|_{H^{-m_1}_t}.
\]
Moreover, using the first estimate of \cite[Prop. 23]{La06} we get
\[
\|\sigma_R^*(x,D)U^*_t f\|_{H^{-m_1}_t} \lesssim N_{s-k,2}^{m_1}(\nabla_x^s \sigma')\|U^*_t f\|_{L^2} = N_{s-k,2}^{m_1}(\nabla_x^s \sigma')\|f\|_{L^2},
\]
for every $k < s$. By a further use of \cite[(50)]{51} and Moser’s inequality, we conclude.

We next deal with semiclassical commutator estimates.

**Proposition 3.** Assume \((C2)\). Then there exists $C_2 > 0$ such that, for every $t \in (0,1]$:
\[
(51) \quad \|t^{-1} \mathcal{C}_t(p^{-1},d)\|_{\mathcal{L}(H^{-m_1}_t;L^2)} \leq C_2.
\]

**Proof.** Let $f \in H^{-m_1}_t(\mathbb{R})$, define $g \in L^2(\mathbb{R})$ by $\tilde{f}(\xi) = (1 + 2\pi it \xi)^{m_1} \tilde{g}(\xi)$. We have
\[
t^{-1} \mathcal{C}_t(p^{-1},d) f(x) = t^{-1} \mathcal{C}_t(p^{-1},d^t) g(x),
\]
where $d^t(x,\xi) = d(x,\xi)(1 + 2\pi \xi)^{m_1}$. We observe that $d^t \in \mathcal{M}^{m_1}_{N,1}$, while $p^{-1} \in \mathcal{M}^{-m_2}_{N,1}$. Then, to prove \cite[(51)]{51}, it is sufficient to show that
\[
(52) \quad \|t^{-1} \mathcal{C}_t(p^{-1},d^t) g\|_{L^2(\mathbb{R})} \leq C_2\|g\|_{L^2}.
\]
To this aim, we consider a partition of unity as follows: Let $\varphi, \psi \in C^\infty_c(\mathbb{R})$ so that
\[
\text{supp} \varphi \subset \left\{ \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad \text{supp} \psi \subset \{|\xi| < 1\},
\]
and such that, setting
\[
(53) \quad \left\{ \begin{array}{l}
\varphi_{-1}(\xi) := \psi(\xi) \\
\varphi_j(\xi) := \varphi\left(\frac{\xi}{2^j}\right), \quad j \geq 0,
\end{array} \right.
\]
one has: $1 = \sum_{j=-1}^{\infty} \varphi_j(\xi)$. We then write
\[
\mathcal{C}_t(p^{-1},d) = \sum_{j=-1}^{\infty} \mathcal{C}_{t,\varphi_j}(p^{-1},d^t),
\]
where the terms $\mathcal{C}_{t,\varphi_j}(p^{-1},d^t)$ are defined by \cite[(32)]{32}. By Corollary 1, we have
\[
\|\mathcal{C}_{t,\varphi_{-1}}(p^{-1},d^t)\|_{\mathcal{L}(L^2)} \lesssim t \mathfrak{M}(p^{-1},d^t).
\]
For $j \geq 0$, we use Corollary 1 together with condition $N - 3/2 > m_1$, to obtain
\[
\|\mathcal{C}_{t,\varphi_j}(p^{-1},d^t)\|_{\mathcal{L}(L^2)} \lesssim t^{N-1/2-j(N-3/2-m_1)} \mathfrak{M}(p^{-1},d^t).
\]
Summing in $j$, we obtain the claim provided that $N - 3/2 > m_1$.

Proposition 3 allows us to improve Proposition 1 in the following way:

**Corollary 8.** Assume \((C2)\). Then the operator $p(x,tD) : L^2(\mathbb{R}) \rightarrow H^{-m_1}_t(\mathbb{R})$ is continuous for every $t \in (0,1]$, and
\[
(54) \quad \|p(x,tD)f\|_{H^{-m_1}_t(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}.
\]

**Proof.** Denoting $\langle \xi \rangle = 1 + 2\pi t \xi$, and $\langle tD \rangle$ its semiclassical quantization, we have:
\[
\|p(x,tD)f\|_{H^{-m_1}_t} = \|\langle tD \rangle^{-m_1} p(x,tD) f\|_{L^2} \leq \|p(\xi)^{-m_1}(x,tD)f\|_{L^2} + \|\mathcal{C}_t(p,\xi^{-m_1})f\|_{L^2}.
\]
The first term is bounded by Calderón-Vaillancourt Theorem \cite{Hw87} Thm. 2. The second one is also bounded by \cite{Hw87}, after replacing $p^{-1}$ by $p$ and $d^j$ by $(\xi)^{-m_1}$.

\begin{corollary}
Assume (C2'). Then there exists $C_2 > 0$ such that, for every $t \in (0,1]$:
\begin{equation}
\|t^{-1} \mathcal{C}_t(p^{-1}, d)\|_{L(H^{-m_1}; L^2)} \leq C_2.
\end{equation}
\end{corollary}

\begin{proof}
We reason as in the proof of Proposition 3. Let $f \in H^{-m_1}_t(\mathbb{R})$, define $g \in L^2(\mathbb{R})$ by $\hat{f}(\xi) = (1 + 2\pi i t \xi)^m \hat{g}(\xi)$. We have again
\begin{equation}
t^{-1} \mathcal{C}_t(p^{-1}, d) f(x) = t^{-1} \mathcal{C}_t(p^{-1}, d^j) g(x),
\end{equation}
where
\begin{equation}
d^j(x, \xi) = d(x, \xi)(1 + 2\pi i \xi)^{m_1}.
\end{equation}
In this case, we have $\Pi_{d^j} \in \mathcal{N}_{s,1}^{-1}$ and $\Pi_{p^{-1}} \in \mathcal{N}_{s,2}^{-m_2}$. We aim at proving that
\begin{equation}
\|t^{-1} \mathcal{C}_t(p^{-1}, d^j) g\|_{L^2(\mathbb{R})} \leq C_2\|g\|_{L^2}.
\end{equation}
To this aim, we will localize the commutators with the partition of unity \cite{Hw87}, and we will use Corollaries 2, 3, 5 and 6 instead of Corollaries 1 and 4. Let us write
\begin{equation}
d^j(x, \xi) = \Pi_{d^j}(x, \xi) + \Sigma_{d^j}(0, \xi),
\end{equation}
\begin{equation}
p^{-1}(x, \xi) = \Pi_{p^{-1}}(x, \xi) + \Sigma_{p^{-1}}(0, \xi).
\end{equation}
Hence
\begin{equation}
\mathcal{C}_t(p^{-1}, d^j) = \mathcal{C}_t(\Pi_{p^{-1}}, \Pi_{d^j}) + \mathcal{C}_t(\Sigma_{p^{-1}}(0, \cdot), \Pi_{d^j}) + \mathcal{C}_t(\Pi_{p^{-1}}(0, \cdot), \Sigma_{d^j}(0, \cdot)) + \mathcal{C}_t(\Sigma_{p^{-1}}(0, \cdot), \Sigma_{d^j}(0, \cdot))
\end{equation}
\begin{equation}
= A_1 + A_2 + A_3 + A_4.
\end{equation}
First, we have that
\begin{equation}
A_3 = \mathcal{C}_t(\Pi_{p^{-1}}(0, \cdot), \Sigma_{d^j}(0, \cdot)) = 0,
\end{equation}
\begin{equation}
A_4 = \mathcal{C}_t(\Sigma_{p^{-1}}(0, \cdot), \Sigma_{d^j}(0, \cdot)) = 0.
\end{equation}
To estimate $A_1$, instead of Corollaries 1 and 4 we use Corollaries 2 and 4. Notice that we can replace the decayment $2^{-j(N-3/2)}$ by $2^{-j(s-3/2)}$ associated with the distance between 0 and the support of $\varphi_j$. The condition $s - 3/2 > m_1$ suffices then to obtain the claim.

Finally, to deal with $A_2$, we use Corollary 3 (the first statement) instead of Corollary 1 and Corollary 4 instead of Corollary 4. \hfill \square

\begin{proposition}
Assume (C2). Then, for every $0 \leq m \leq [m_1]$
\begin{equation}
\|\mathcal{C}_t(p, p^{-1})\|_{L(H^{-m}; H^{-m}_t)} \leq tC_1,
\end{equation}
for every $t \in (0,1]$.
\end{proposition}

\begin{proof}
By interpolation, we can assume that $m$ is an integer. Let $f \in H^{-m}_t(\mathbb{R})$, we define $g \in L^2(\mathbb{R})$ via $\hat{f}(\xi) = (1 + 2\pi i t \xi)^m \hat{g}(\xi)$. Then
\begin{equation}
\mathcal{C}_t(p, p^{-1}) f = \mathcal{C}_t(p, q) g,
\end{equation}
where $q(x, \xi) = p^{-1}(x, \xi)(1 + 2\pi i \xi)^{m}$ belongs to $\mathcal{M}^{m_2}_{N,1}$. It is then sufficient to show that
\begin{equation}
\|\mathcal{C}_t(p, q) g\|_{H^{-m}_t(\mathbb{R})} \leq tC_1\|g\|_{L^2(\mathbb{R})}.
\end{equation}
Using the partition of unity \cite{Hw87}, we split the sum as
\begin{equation}
\|\mathcal{C}_t(p, q) g\|_{H^{-1}_t(\mathbb{R})} \leq \sum_{j=-1}^{\infty} \|\mathcal{C}_t\varphi_j(p, q) g\|_{H^{-1}_t(\mathbb{R})}.
\end{equation}
We claim that there exist $\alpha > 0$ such that, for every $h \in H^m_t(\mathbb{R})$,
\begin{equation}
\left| \int_{\mathbb{R}} \mathcal{C}_t\varphi_j(p, q) g(x) h(x) dx \right| \lesssim t^{2^{-\alpha j}}\|g\|_{L^2(\mathbb{R})}\|h\|_{H^m_t(\mathbb{R})}.
\end{equation}
To show this, we write
\[
\int_{\mathbb{R}} E_{t,\phi_j}(p, q)g(x)h(x)\,dx = \int_{\mathbb{R}^2} e^{2\pi i x \xi} Q_j^i(x, \xi)h(x)d\xi dx,
\]
where
\[
Q_j^i(x, \xi) = \int_{\mathbb{R}^2} e^{2\pi i (\xi y + \eta \xi)} \phi_j(t(\xi - \eta))(p(x, t\xi) - p(x, t\eta))g(y, t\eta)\hat{g}(\eta)d\eta dy.
\]
Moreover,
\[
\int_{\mathbb{R}^2} e^{2\pi i x \xi} Q_j^i(x, \xi)h(x)d\xi dx = \int_{\mathbb{R}^3} e^{2\pi i x (\xi + \lambda)} \frac{Q_j^i(x, \xi)}{(1 + 2\pi i t\xi)^m}(1 + 2\pi it\xi)^m \hat{h}(\lambda)d\lambda
\]
\[
= \sum_{k=0}^{m} \binom{m}{k} \int_{\mathbb{R}^3} Q_j^i(x, \xi)(1 + 2\pi it(\lambda + \xi))^{m-k}(2\pi it\lambda)^k \hat{h}(\lambda)d\lambda
\]
\[
= \sum_{k=0}^{m} I_k.
\]
Using integration by parts in the \(x\) variable, and the following identity,
\[
\frac{1}{(1 + 2\pi i (\xi + \lambda))^{m+1}}(1 + \partial_x)^{m+1} e^{2\pi i x (\xi + \lambda)} = e^{2\pi i x (\xi + \lambda)},
\]
as in the proof of Proposition 1, we obtain, for \(0 \leq k \leq m\), that
\[
|I_k| \leq \left\| \mathcal{D}_x^{m+1} Q_j^i(x, \xi) \right\|_{L^2(\mathbb{R}^2)} \left\| h \right\|_{H^k(\mathbb{R})}.
\]
Recalling the proofs of Corollary 4 and Proposition 3 the claim is obtained in a similar way. Notice that
\[
t \sup_{(x, \xi, \eta) \in \mathbb{R}^3} \left| \frac{\varphi_1(t(\xi - \eta))}{(1 + 2\pi it\xi)^m} \mathcal{D}_x^{m+1}(p(x, t\xi) - p(x, t\eta)) \right|
\]
\[
= t \sup_{(x, \xi, \eta) \in \mathbb{R}^3} \left| \frac{\varphi_1(\xi - \eta)}{(1 + 2\pi i \xi)^m} \mathcal{D}_x^{m+1}(p(x, \xi) - p(x, \eta)) \right|
\]
\[
\leq t \left\| \varphi_1 \right\|_{L^\infty} \sup_{x \in \mathbb{R}} \sup_{(\xi, \eta) \in \Omega_{\varphi_1}} \left| \mathcal{D}_x^{m+1}\partial_\eta p(x, \eta) \right| (1 + 2\pi i \xi)^m
\]
\[
\leq t M_{m+1, 1, 0}^{m+1}(\partial_\xi p) \sup_{(\xi, \eta) \in \Omega_{\varphi_1}} (1 + |\eta|)^m (1 + |\xi|)^{-m}
\]
\[
\lesssim t.
\]
Similarly, for $j \geq 0$,

\[
    t^{N-1} \cdot \sup_{(x,\xi,\eta) \in \mathbb{R}^3} \left| \varphi_j(t(\xi - \eta)) \frac{D_x^{m+1}(p(x, t\xi) - p(x, t\eta))}{t(2\pi i(\xi - \eta))^{N-1}} \right| 
\]

\[
    \lesssim t^{N-2-j(N-2)} \cdot \sup_{(x,\xi,\eta) \in \mathbb{R}^3} \left| \varphi_j(\xi - \eta) \frac{D_x^{m+1}(p(x, \xi) - p(x, \eta))}{(1 + 2\pi i\xi)^m} \right| 
\]

\[
    \lesssim t^{N-2-j(N-2)} \sup_{x \in \mathbb{R}} \sup_{(\xi,\eta) \in \Omega_{x_j}} \left| \frac{D_x^{m+1} \partial_\eta p(x, \eta)}{(1 + 2\pi i\xi)^m} \right| 
\]

\[
    \lesssim t^{N-2-j(N-2)} M_{m+1,j}(\partial_\xi p) \sup_{(\xi,\eta) \in \Omega_{x_j}} (1 + |\eta|)^m(1 + |\xi|)^{-m} 
\]

\[
    \lesssim t^{N-2-j(N-2)\frac{m-1}{2}}. 
\]

Thus, using (41), that $p \in \mathcal{M}_{N,1}$ and $q \in \mathcal{M}_{N,1}^{m-2}$, condition $N - 3/2 - m_1 > 0$, and the ideas above, is sufficient to finish the proof.

Assembling the previous ideas together with those of the proof of Corollary 3, we get also the following:

**Corollary 10.** Assume (C2'). Then, for every $0 \leq m \leq \lceil m_1 \rceil$,

\[
    (57) \quad \| \mathcal{E}_t(p, p^{-1}) \|_{\mathcal{L}(H^{-m}_t, H^{-m}_t)} \leq tC_1, \quad t \in (0, 1]. 
\]

We will also require the following coercivity property for $p^{-1}(x, tD)$:

**Proposition 5.** Let $\langle tD \rangle$ be the semiclassical quantization of the symbol $\langle \xi \rangle = 1 + 2\pi i\xi$. Then there exists $C > 0$ and $0 < T \leq 1$ such that

\[
    \| f \|_{H^{-m_1}_t(\mathbb{R})} = \| \langle tD \rangle^{-m_1} f \|_{L^2(\mathbb{R})} \leq C \| p^{-1}(x, tD) f \|_{L^2(\mathbb{R})}, \quad f \in \mathcal{D}'(\mathbb{R}),
\]

for every $0 < t < T$.

**Proof.** We formally have

\[
    \langle tD \rangle^{-m_1} f(x) = \langle tD \rangle^{-m_1} (p(x, tD)p^{-1}(x, tD))^{-1} p(x, tD)p^{-1}(x, tD)f(x). 
\]

By Corollary 8 with hypothesis (C2) (resp. Proposition 2 with (C2')), $p(x, tD) : L^2(\mathbb{R}) \rightarrow H^{-m_1}_t(\mathbb{R})$ is continuous, uniformly in $t \in (0, 1)$. Then it is sufficient to show that the operator $p(x, tD)p^{-1}(x, tD)$ is invertible in $H^{-m_1}_t(\mathbb{R})$ with continuous inverse, uniformly in $t \in (0, T]$ for some $T > 0$. To this aim, notice that

\[
    p(x, tD)p^{-1}(x, tD) = I + \mathcal{E}_t(p, p^{-1}). 
\]

Using Proposition 4 together with hypothesis (C2) (resp. Corollary 10 together with (C2'))), we have that

\[
    \| \mathcal{E}_t(p, p^{-1}) \|_{\mathcal{L}(H^{-m_1}_t)} \leq tC_1. 
\]

Then, there exists $T > 0$ sufficiently small such that $\| \mathcal{E}_t(p, p^{-1}) \|_{\mathcal{L}(H^{-m_1}_t)} < 1$ for $t \in (0, T]$. Then, the operator $p(x, tD)p^{-1}(x, tD)$ is invertible and has continuous inverse in $H^{-m_1}_t(\mathbb{R})$, uniformly bounded for $t \in (0, T]$. \(\square\)

5. **Proof of Theorems 1 and 2**

**Proof of Theorem 7.** First observe that

\[
    \frac{d}{dt} p^{-1}(x, t\xi) = -t^{-1} d(x, t\xi)p^{-1}(x, t\xi). 
\]
In view of (IVP) and \( (10) \), we have
\[
\frac{d}{dt} \| p^{-1}(x, tD)f(t) \|_{L^2}^2 = \frac{d}{dt} \langle p^{-1}(x, tD)f(t), p^{-1}(x, tD)f(t) \rangle_{L^2} = 2 \Re \langle t^{-1} \mathcal{C}_t(p^{-1}, d)f(t), p^{-1}(x, tD)f(t) \rangle_{L^2}.
\]

Finally, using Proposition 3 and Lemma 3, we conclude that
\[
\| t^{-1} \mathcal{C}_t(p^{-1}, d)f(t) \|_{L^2(\mathbb{R})} = \| t^{-1} \mathcal{C}_t(p^{-1}, d)(tD)^{m_1}(tD)^{-m_1}f(t) \|_{L^2(\mathbb{R})} \leq \| t^{-1} \mathcal{C}_t(p^{-1}, d) \|_{L_t(H_t^{-m_1}, L^2)} \| (tD)^{m_1} \|_{L_t(\mathbb{R})} \| (tD)^{-m_1}f \|_{L^2(\mathbb{R})} \leq C_T \| p^{-1}(x, tD)f(t) \|_{L^2(\mathbb{R})}.
\]

\[\square\]

Proof of Theorem 2. We first prove the inequality on the left. To this aim, we define a new function \( c_\epsilon : \mathbb{R} \to \mathbb{R} \) satisfying (C2*), such that \( c_\epsilon(x) = c(x) \) in \( I_\epsilon \), and such that \( c_\epsilon^- \leq c_\epsilon(x) \leq c_\epsilon^+ \) for every \( x \in \mathbb{R} \). We then define a new symbol \( p_\epsilon^{-1} \) by
\[
p_\epsilon^{-1}(x, \xi) = (1 + c_\epsilon(x)|\xi|)^{-1/c_\epsilon(x)}.
\]
Notice that \( \chi_\epsilon(x)p_\epsilon^{-1}(x, tD)f(x) = \chi_\epsilon(x)p_\epsilon^{-1}(x, tD)f(x) \)
and
\[\langle tD \rangle^{s-}p_\epsilon(x, tD)f(x) = \chi_\epsilon(x)p_\epsilon^{-1}(x, tD)(\chi_\epsilon f)(x) - \mathcal{C}_t(p_\epsilon^{-1}, \chi_\epsilon)f(x).\]

By Corollary 10 (notice that we can replace \( p^{-1} \) by \( p_\epsilon^{-1} \) and \( d \) by \( \chi_\epsilon \) in the statement),
\[
\| \mathcal{C}_t(p_\epsilon^{-1}, \chi_\epsilon)f \|_{L^2(\mathbb{R})} \leq C_0 t \| f \|_{H_t^{-m_1}(\mathbb{R})}.
\]

On the other hand, notice that
\[
\langle tD \rangle^{s-}p_\epsilon(x, tD) = p_\epsilon(\xi)^{s-}(x, tD) + \mathcal{C}_t(\langle \xi \rangle^{s-}, p_\epsilon),
\]
where \( \langle \xi \rangle = 1 + 2\pi i \xi \). Hence, using that \( \langle \xi \rangle^{s-} \in \mathcal{M}_{1, \infty}^s \), and the commutator estimate 50 with \( p_\epsilon \) replacing \( d^1 \) and \( \langle \xi \rangle^{s-} \) replacing \( p^{-1} \) yields that
\[
\| p_\epsilon(x, tD)f \|_{H_t^{s-}} = \| \langle tD \rangle^{s-}p_\epsilon(x, tD) \|_{L^2} \leq \| p_\epsilon(\xi)^{s-}(x, tD)f \|_{L^2} + \| \mathcal{C}_t(\langle \xi \rangle^{s-}, p_\epsilon) \|_{L^2} \leq \| p_\epsilon(\xi)^{s-}(x, tD)f \|_{L^2} + C t \| f \|_{L^2}.
\]

By the Calderón-Vaillancourt theorems [HW87, Thm. 2] and [HW87] Corol. 2.2, we also obtain that
\[
\| p_\epsilon(\xi)^{s-}(x, tD)f \|_{L^2} \leq C \| f \|_{L^2}, \quad t \in (0, 1).
\]
Moreover, Corollary 10 remains valid for \( p_\epsilon^{-1} \) instead of \( p^{-1} \). By Corollary 10 we also have that
\[
\| \mathcal{C}_t(p_\epsilon, p_\epsilon^{-1}) \|_{L_t(H_t^{-m_1}, H_t^{s-})} \leq t C.
\]

Thus, we can apply Proposition 3 (notice that we can replace \( p^{-1} \) by \( p_\epsilon^{-1} \) and \( H_t^{-m_1} \) by \( H_t^{s-} \)), to obtain, for every \( 0 < t \leq T \),
\[
\| p_\epsilon^{-1}(x, tD)(\chi_\epsilon f) \|_{L^2} \geq C_1 \| \chi_\epsilon f \|_{H_t^{s-}(\mathbb{R})}.
\]
It remains to show the inequality on the right. To this aim, we use again 58 and 59, the facts that
\[
p_\epsilon^{-1}(x, tD)(\chi_\epsilon f)(x) = p_\epsilon^{-1}(x, tD)\langle tD \rangle^{-s+}\langle tD \rangle^{s+}(\chi_\epsilon f)(x),
\]
and that \( p_\epsilon^{-1}(x, tD)\langle tD \rangle^{-s+} \) is the semiclassical quantization of the symbol \( p_\epsilon^{-1}(x, \xi)\langle \xi \rangle^{-s+} \). It is then sufficient to use [HW87, Thm. 2] and [HW87] Corol. 2.2 to show that \( p_\epsilon^{-1}(x, tD)\langle tD \rangle^{-s+} \) is bounded from \( L^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \) uniformly for \( 0 < t \leq T \). This finishes the proof. \[\square\]
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