NON-FIBERED L-SPACE KNOTS

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Abstract. We construct an infinite family of knots in rational homology spheres with irreducible, non-fibered complements, for which every non-longitudinal filling is an L-space.

1. Introduction

The Heegaard Floer homology of a three-manifold $Y$ is an abelian group $\widehat{HF}(Y)$ satisfying $\text{rk} \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$. When equality is realized in this bound, $Y$ is called an L-space, and any knot in $Y$ admitting a non-trivial L-space surgery is called an L-space knot. A result of Ghiggini [6] and Ni [9] shows that L-space knots in the three-sphere must be fibered. Since manifolds with finite fundamental group provide examples of L-spaces, this result implies that a knot $K$ in $S^3$ admitting a finite filling must be fibered. This observation should be compared with other restrictions related to finite fillings such as the Cyclic Surgery Theorem [5] and its extensions [4].

The restriction to knots in $S^3$ is not necessary: it is shown in [1] that a primitive knot in an irreducible L-space admitting a non-trivial L-space surgery must be fibered. Irreducibility is required, as removing an unknotted torus from an embedded three-ball in any L-space produces a non-fibered manifold with non-trivial L-space fillings. However, in the general setting of knots in rational homology spheres with irreducible complements fibered is not a necessary condition:

**Theorem 1.** There exist infinitely many irreducible, non-fibered knot complements such that all non-longitudinal Dehn fillings are L-spaces. Moreover, these examples arise as knots in manifolds with finite fundamental group.

2. Preliminaries

We begin by fixing terminology. Fibrations will always be locally trivial surface bundles over a circle and we say the total space fibers. To avoid confusion, we will refer to Seifert fibrations as Seifert structures; these are foliations of a manifold by circles. The base orbifold is the leaf space of such a foliation, where the (possibly empty) collection of cone points records the multiplicities of the exceptional fibers in the Seifert structure. A circle bundle is a Seifert structure for which there are no exceptional fibers.

Given a slope $\alpha$ on a manifold $M$ with torus boundary, we use $M(\alpha)$ to denote Dehn filling along $\alpha$. If $\partial M = T_1 \cup T_2$, for tori $T_i$, then we denote $\alpha$-filling on $T_1$ (respectively $T_2$) by $M(\alpha, -)$ (respectively $M(-, \alpha)$). When $M$ admits a Seifert structure, the slope given by a regular fiber in the boundary is called the fiber slope. For background on Seifert structures and Dehn filling we refer the reader to Boyer [2]. A key fact is that Dehn filling a Seifert manifold with torus boundary along any slope $\alpha$ other than the fiber slope results in a Seifert manifold with an additional singular fiber. The multiplicity of this new fiber is $\Delta(\alpha, \phi)$, the distance between the slopes $\alpha$ and $\phi$ [7].

In general, we will consider oriented manifolds $M$ with torus boundary for which $H_1(M; \mathbb{Q}) \cong \mathbb{Q}$. These arise as the complements of knots in rational homology spheres. As such, there is always a

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Ozsváth and Szabó show that manifolds admitting elliptic geometry are L-spaces [10]; the Geometrization Theorem [8] implies that three-manifolds with finite fundamental group admit elliptic geometry.
preferred slope given by the rational longitude, characterized by the property that some number of like-oriented parallel copies bounds a properly embedded surface in $M$. We will refer to this slope as the longitude.

Let $N$ denote the twisted $I$-bundle over the Klein bottle; a Heegaard diagram for this manifold with torus boundary is given in Figure 1. As there is a unique line bundle over the Klein bottle with oriented total space, the manifold $N$ is unique. This manifold can be given two different Seifert structures. The first is by treating $N$ as a circle bundle over the Möbius band. We denote the fiber slope in this Seifert structure by $\phi_0$. This slope coincides with the longitude of $N$, and this circle bundle gives $N$ the structure of an annulus fibration over the circle. The other Seifert structure has base orbifold $\mathbb{R}P^2(2, 2)$ (a disk with two cone points each of order 2); the fiber slope here is denoted $\phi_1$. These conventions are consistent with [3, Section 3]. It can be shown that $\Delta(\phi_0, \phi_1) = 1$ and any filling $N(\alpha)$ for which $\alpha \neq \phi_0, \phi_1$ admits a pair of Seifert structures with base orbifolds $\mathbb{R}P^2(\Delta(\alpha, \phi_0))$ and $S^2(2, 2, \Delta(\alpha, \phi_1))$. We point out that these manifolds always admit elliptic geometry [11].

Since $N$ is homotopy equivalent to a Klein bottle, we have $\pi_1(N) = \langle a, b | a^2 b^2 \rangle$. Note that this presentation may be easily deduced from the Heegaard diagram in Figure 1. The longitude of $N$ is homotopic to the element $ab$ (this element has order two in the abelianization of $\pi_1(N)$).

Now consider a knot $K_0$ in $N$ that is isotopic to $\phi_0$. Define $M$ by removing a neighborhood of $K_0$ from $N$; by construction $M$ inherits a Seifert structure (the base orbifold is a punctured Möbius band). Now $\partial M = T_1 \cup T_2$ where $T_2$ denotes the boundary of a regular neighborhood of $K_0$.

The fundamental group of $M$ is presented by

$$\pi_1(M) = \langle a, b, t | a^2 b^2, [t, ab] \rangle.$$  

To see this, consult Figure 1 and notice that $M$ may be constructed by identifying (disjoint neighborhoods of) each boundary component of the annulus with core $ab$ in $\partial N$. This gives rise to the HNN extension presented above. Notice that $M(-, \mu) \cong N$ for any slope satisfying $\Delta(\mu, \phi_0) = 1$. A preferred choice for $\mu$ is given by a representative of the homotopy class of $t$ in the above presentation. Notice that $\phi_0$, as a regular fiber in $M$, also gives a slope on $T_2$. Using a self-diffeomorphism of $M$ which exchanges $T_1$ and $T_2$, $M(\alpha, -)$ is also homeomorphic to $N$ if $\Delta(\alpha, \phi_0) = 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{heegaard_diagram.png}
\caption{A Heegaard diagram for the twisted $I$-bundle over the Klein bottle $N$. With $a$ and $b$ generating the fundamental group of the genus two handlebody, $N$ is obtained by attaching a handle along a curve in the boundary representing $a^2 b^2$ so that $\phi_0 \simeq ab$ and $\phi_1 \simeq b^2$. An annulus in the boundary with core representing the element $\phi_0 \simeq ab$ may be used to find the fundamental group of $M$, the complement of a regular fiber in the interior of $N$, via HNN extension.}
\end{figure}
Our interest is in the family of manifolds $M(-,\alpha)$ for any slope $\alpha$ with $\Delta(\alpha,\phi_0) > 1$. Notice that each of these manifolds admits a Seifert structure with base orbifold a Möbius band with a single cone point of order $\Delta(\alpha,\phi_0)$. Since $M(\phi_1,\alpha)$ admits a Seifert structure with base orbifold $S^2(2,2,n)$ it follows that $M(-,\alpha)$ is the complement of a knot in an elliptic manifold.

3. The proof of Theorem 1

Lemma 2. Fix a slope $\alpha$ on $T_2$ with $\Delta(\alpha,\phi_0) = p$. Then

$$M(\phi_0,\alpha) = \begin{cases} S^2 \times S^1 \# S^2 \times S^1 & \text{if } p = 0, \\ S^2 \times S^1 \# L(p,q) & \text{if } p > 1, \\ S^2 \times S^1 & \text{if } p = 1. \end{cases}$$

Proof. Since

$$\pi_1(M) \cong \langle a,b,t|a^2b^2,[t,ab]\rangle$$

and $\phi_0 \cong ab$, we have that

$$\pi_1(M(\phi_0,-)) \cong \langle a,b,t|a^2b^2,[t,ab]/\langle\langle ab\rangle\rangle\rangle$$

$$\cong \langle a,b,t|ab\rangle.$$ 

In other words, $\pi_1(M(\phi_0,-)) \cong \mathbb{Z} \ast \mathbb{Z}$. If $\alpha = p\mu + q\phi_0$, then

$$\pi_1(M(\phi_0,\alpha)) \cong \langle a,b,t|ab/\langle\langle t^p(ab)^q\rangle\rangle\rangle$$

$$\cong \mathbb{Z} \ast \mathbb{Z}/p.$$ 

By Whitehead’s proof of Kneser’s conjecture [12], $M(\phi_0,\alpha)$ is a connect-sum of closed manifolds $Y_1$ and $Y_2$ with $\pi_1(Y_1) \cong \mathbb{Z}$ and $\pi_1(Y_2) \cong \mathbb{Z}/p$. Geometrization now establishes the lemma. 

![Figure 2. The branch set for the manifold $M = M(-,-)$ with branch sets for the fillings $N = M(\phi_1,-)$ and $M(\phi_0,-)$. Notice that the latter manifold is reducible, containing an $S^2 \times S^1$ summand.](image)

Remark 3. Alternatively, Lemma 2 follows from considering $M(\phi_0,-)$ as the double branched cover of a tangle as in Figure 2. The unknotted component gives rise to the $S^2 \times S^1$ summand. Dehn filling corresponds to attaching a rational tangle, which (ignoring the unknotted component) produces a two-bridge link and exhibits the lens space connect-summand.

Proposition 4. For any $\alpha$ on $T_2$ with $\Delta(\alpha,\phi_0) \neq 1$, the manifold $M(-,\alpha)$ does not fiber.

Proof. Suppose that $M(-,\alpha)$ fibers. Since $\phi_0$ is the longitude, this is the only filling that extends the fibration on $M(-,\alpha)$, as any other filling of $M(-,\alpha)$ results in a rational homology sphere. By Lemma 2 $M(\phi_0,\alpha) \cong S^2 \times S^1 \# L(p,q)$ for $p = \Delta(\phi_0,\alpha) \geq 2$, and $M(\phi_0,\phi_0) \cong S^2 \times S^1 \# S^2 \times S^1$. Because $M(\phi_0,\alpha)$ is fibered and $\pi_2(M(\phi_0,\alpha)) \neq 0$, the fiber surface $F$ must also have $\pi_2(F) \neq 0$, and hence $F$ must be $S^2$ or $\mathbb{R}P^2$. However, $\pi_1(M(\phi_0,\alpha))$ is not the fundamental group of such a fibration, since it does not admit a surjective homomorphism onto $\mathbb{Z}$ with finite kernel. 

\[\square\]
Proof of Theorem 1. Fix $\alpha$ with $\Delta(\alpha, \phi_0) \geq 2$. As the fiber slope of the Seifert structure on $M(-, \alpha)$ is the longitude, all non-longitudinal fillings will extend the Seifert structure, yielding a base orbifold $\mathbb{R}P^2$ with two cone points. By [3, Proposition 5], such manifolds are always L-spaces. Proposition 4 shows that $M(-, \alpha)$ is not fibered. Furthermore, $M(-, \alpha)$ is irreducible, since the only orientable, reducible Seifert manifolds are $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$ (and in particular, are closed). Finally, $M(-, \alpha)$ is the complement of a knot in an elliptic manifold as observed in Section 2.

Remark 5. Further examples may be constructed in an analogous way by removing a regular fiber from any manifold which has a Seifert structure with base orbifold $\mathbb{R}P^2$ with any positive number of singular fibers. It is also possible to construct examples, in a similar manner, admitting Sol geometry. The main observation is that every Sol rational homology sphere is an L-space [3, Theorem 2]. Since every such L-space arises by identifying two twisted I-bundles along the boundary tori, one may consider the complement of the knot $K_0$ in one of the twisted I-bundles. In this setting, our construction goes through almost verbatim, having noticed that the obvious essential torus must be horizontal to the purported fibration of the exterior of $K_0$.

Question 6. All of our examples have non-hyperbolic exterior. Do there exist examples of hyperbolic, non-fibered knots for which every non-longitudinal surgery is an L-space?

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